Various gauge invariant but non-Yang-Mills dynamical models are discussed: Précis of Chern-Simons theory in $(2+1)$-dimensions and reduction to $(1+1)$-dimensional $B$-$F$ theories; gauge theories for $(1+1)$-dimensional gravity-matter interactions; parity and gauge invariant mass term in $(2+1)$-dimensions.

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I. INTRODUCTION

The successes of present day theories for fundamental processes in Nature offer persuasive evidence that forces between elementary particles obey the principle of local gauge symmetry. Even Einstein’s gravity theory, which thus far has not been incorporated into the quantal formalism that describes all the other fundamental interactions, is seen to enjoy, at least at the classical level, an invariance against local transformations, viz. diffeomorphism invariance.

While the “standard” particle physics model realizes the gauge principle with the Yang-Mills paradigm – the non-Abelian generalization of Maxwell’s electrodynamics – in the last decade we have come to appreciate that there are other forms of dynamics that are gauge invariant, physically relevant, but do not use the Yang-Mills structure. These alternative realizations of gauge invariance play only a small role in elementary particle field theory, but they seem to have many applications to phenomenological descriptions of various collective, emergent phenomena. Also they are mathematically fascinating.

In my lectures here, I shall describe some recent work on non-Yang-Mills gauge theories. Let me begin by setting notation. Vector gauge potentials (connections) and vector gauge fields (curvatures) will be variously presented in component notation \( A^a_\mu, F^{a\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \), or by Lie-algebra valued relations \( A_\mu = A^a_\mu T_a, F_{\mu\nu} = F^{a\mu\nu} T_a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \). The Lie-algebra generators satisfy \([T_a, T_b] = f_{abc} T^c\), thereby defining the structure constants. We shall assume that there exists an invariant, non-singular bi-linear \( \langle T_a, T_b \rangle = \eta_{ab} \), with which group indices can be moved, so that \( f_{abc} \propto f^{abd} \eta_{dc} \) is totally anti-symmetric. Mostly \( \eta_{ab} \) will be the Killing-Cartan metric, but other structures can arise with non-semi-simple groups. Specifically for the \( SU(N) \) fundamental representation, the Pauli/Gell-Mann matrices are used: \( T_a = \lambda_a/2i, \langle T_a, T_b \rangle = trT_aT_b = -\frac{1}{2} \delta_{ab} \). Finally, we shall also use form notation: \( A = A_\mu dx^\mu, F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA + A^2 \), with no explicit indication of the outer product. In Minkowski space-time, our metric tensor is positive in its time-time component.
II. PRÉCIS OF THE CHERN-SIMONS TERM IN THREE DIMENSIONS

The most important and popular non-Yang-Mills gauge structure is the Chern-Simons term. Physicists have been working with this quantity for over a decade, so the subject is well-established, and I need not discuss it here in detail. Nevertheless a few introductory remarks will be made, because transformations of the Chern-Simons term lead to the new models that I wish to describe.

The Chern-Simons density made its first appearance in physics when it was realized that the anomaly in the conservation law for the axial vector current can be written as a divergence,

\[-\frac{1}{64\pi^2} \epsilon^{\mu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta} = \partial_\mu K^\mu\]  

(1a)

\[K^\mu = -\frac{1}{16\pi^2} \epsilon^{\mu\alpha\beta\gamma} \left\{ A^a_\alpha \partial_\beta A^a_\gamma + \frac{1}{3} f_{abc} A^a_\alpha A^b_\beta A^c_\gamma \right\} \]  

(1b)

or in compact notation for 4-forms.

\[\frac{1}{8\pi^2} \langle F, F \rangle = \frac{1}{8\pi^2} d\langle A, dA + \frac{2}{3} A^2 \rangle\]  

(1c)

Because \(K^\mu\) contracts the Levi-Civita \(\epsilon\)-tensor, any one of its component contains no fields in the direction of that component, and when the coordinate of the selected direction is also suppressed, one naturally arrives at a three-dimensional quantity.

\[\Omega(A) = -\frac{1}{16\pi^2} \epsilon^{\alpha\beta\gamma} \left\{ A^a_\alpha \partial_\beta A^a_\gamma + \frac{1}{3} f_{abc} A^a_\alpha A^b_\beta A^c_\gamma \right\} \]  

(2a)

\[= -\frac{1}{32\pi^2} \epsilon^{\alpha\beta\gamma} \left\{ A^a_\alpha F^a_\beta_\gamma - \frac{1}{3} f_{abc} A^a_\alpha A^b_\beta A^c_\gamma \right\} \]

(2b)

\[= -\frac{1}{16\pi^2} \left\{ A^a_\alpha F^{\alpha a} - \frac{1}{6} \epsilon^{\alpha\beta\gamma} f_{abc} A^a_\alpha A^b_\beta A^c_\gamma \right\} \]

(2c)

In (2b) we have introduced the dual field.

\[F^{\alpha a} = \frac{1}{2} \epsilon^{\alpha\beta\gamma} F^a_{\beta\gamma}\]  

(3)
The 3-form $\Omega(A)$ is the Chern-Simons density, while its integral over three-space is the Chern-Simons term $W(A)$.

$$W(A) = \int \Omega(A)$$ (4)

$W(A)$ possess the important property of being invariant against infinitesimal gauge transformation, while changing under finite gauge transformations by the integer winding number $n$ of the group element that effects the transformation.

$$\Omega(A^U) = \Omega(A) + \frac{1}{8\pi^2} d\langle A, dUU^{-1} \rangle - \frac{1}{48\pi^2} \langle dUU^{-1}, dUU^{-1}dUU^{-1} \rangle$$ (5)

$$A^U \equiv U^{-1}AU + U^{-1}dU$$ (6)

$$W(A^U) = W(A) + n$$ (7)

$$n = -\frac{1}{48\pi^2} \int \langle dUU^{-1}, dUU^{-1}dUU^{-1} \rangle$$ (8)

We have assumed that no surface terms contribute: $\int d\langle A, dUU^{-1} \rangle = 0$.

The discontinuous gauge variance (7) of $W(A)$ leads to the first application of the Chern-Simons term in particle physics: the establishment of the QCD vacuum angle. Within a Hamiltonian, Schrödinger representation approach to $(3+1)$-dimensional Yang-Mills quantum theory, states are functionals of $A$, and in this fixed-time formalism the vector potential is defined on three-space. Gauss’ law requires that these functionals be invariant against infinitesimal gauge transformations, while finite gauge transformations, which are symmetries of the theory, must leave states invariant up to a phase. So we immediately see that a physical state can take the form

$$|\Psi\rangle = e^{i\theta W(A)} \Psi(A)$$ (9a)

where $\Psi(A)$ is invariant against all gauge transformations. Consequently, in view of (7) a physical state responds to a gauge transformation $U$ as

$$|\Psi\rangle \xrightarrow{U} e^{i\theta n} |\Psi\rangle$$ (9b)

We see that the existence of the vacuum angle is here established without reference to any instanton/tunnelling approximation.
[A further tantalizing fact is that $e^{\pm 8\pi^2 W(A)}$ solves the Yang-Mills functional Schrödinger equation with zero eigenvalue. Nevertheless this remarkable wave functional cannot represent a physical state, because it grows uncontrollably for large $A$.]

Dynamical utilization of the Chern-Simons term came when it was suggested that $W(A)$ can be a contribution to the action of a three-dimensional field theory. However with non-Abelian gauge groups, the magnitude with which $W(A)$ enters must be quantized, in integer units of $2\pi$, so that the gauge variance of $W(A)$ produces a shift in the action of $2\pi \times$ integer, which would not be seen in the phase exponential of the action. (Sometimes it is claimed that this is also required for Abelian groups, when the base manifold is topologically non-trivial. But this is not true – in the Abelian case, a well-defined quantum theory can be defined for arbitrary coupling strength, regardless of base-space topology.)

When $W(A)$ is added to the usual Yang-Mills term, the excitations of the theory become massive, while retaining gauge invariance, but reflection symmetry is lost. One may couple further matter fields to $A_\mu$ in a gauge invariant fashion. For low-energy phenomenological applications to physical systems confined on the plane, it makes sense to couple nonrelativistic matter. Moreover since the Yang-Mills term contains two derivatives and the Chern-Simons term only one, the latter dominates the former at low energies, and the Yang-Mills kinetic term may be dropped. In this way one is led to the interesting class of non Yang-Mills gauge theoretic models described by

$$I = 2\pi \kappa W(A) + \int \{ i\psi^* D_t \psi - \frac{1}{2m} (D \psi)^* \cdot (D \psi) - V(\rho) \}$$

$$\rho \equiv \psi^* \psi$$

(10)

where $\kappa$ is an integer (in the non-Abelian case), $(D_t, D)$ are (temporal, spatial) gauge covariant derivatives, while $V(\rho)$ describes matter self-interactions. The matter action is Galileo invariant, the Chern-Simons action is topological, i.e. invariant against all coordinate transformations, so $I$ is Galileo invariant.

Models belonging to the general class (10), with both Abelian and non-Abelian gauge groups, have been widely discussed in the past, so I shall not describe that old work here.
Rather I shall consider dimensional reduction of (10) to (1 + 1) dimensions, and thereby expose some new non-Yang Mills, gauge theoretic dynamical systems.

III. DIMENSIONAL REDUCTION OF CHERN-SIMONS THEORY=BF THEORY

One obvious reduction for the dynamics of (10) is to reduce in time. This corresponds to looking for static solutions to the Euler-Lagrange equations arising from (10), and once again this subject is an old one, well reviewed in the existing literature. So I shall not dwell on it, beyond the reminder that with an appropriate choice for \( V \) one can apply the Bogomolny procedure and replace the second-order Euler-Lagrange equations with coupled first order equations, which in turn can be combined into the completely integrable Toda equation (non-Abelian case) or Liouville equation (Abelian case), with well-known soliton solutions.

A. Reduction to non-Linear Schrödinger Equation

Now I shall describe in detail a reduction to one spatial dimension, which results in an interesting reformulation of the non-linear Schrödinger equation. On the plane, with coordinates \((x, y)\), we suppress all \(y\)-dependence and redefine \(A_y\) as \(B\). Then, in the Abelian case, the action (10) becomes

\[
I = \int dt dx \left\{ -\kappa BF + i\psi^*D_t\psi - \frac{1}{2m}|D\psi|^2 - \frac{1}{2m}B^2 \rho - V(\rho) \right\}
\]  

The “kinetic” gauge field term is the so-called “\(B-F\)” expression where \( F = \frac{1}{2} \epsilon^{\mu\nu} F_{\mu\nu} = -\dot{a} - A'_{0} \). [We have re-named \(A_1\) as \(-a\), and dot/dash refer to differentiation with respect to time/space, i.e. \((t/x)\). The covariant derivatives read \(D_t\psi = \dot{\psi} + iA_0\psi, D\psi = \psi' - ia\psi\). Also \(\kappa\) has been rescaled by \(4\pi\); recall that in the Abelian application, the Chern-Simons coefficient is not quantized.] Evidently the 2-dimensional \(B-F\) quantity is a dimensional reduction of the Chern-Simons expression.

Because (11) is first-order in time-derivatives, the action is already in canonical form, and may be analyzed using the symplectic Hamiltonian procedure. We present (11) as
\[ I = \int \left\{ \kappa B \dot{\psi} + i \psi^* \dot{\psi} - A_0 (\kappa B' + \rho) - \frac{1}{2m} |(\partial_x - ia) \psi|^2 - \frac{1}{2m} B^2 \rho - V(\rho) \right\} \]
\[ = \int \left\{ \kappa B \dot{\psi} + i \psi^* \dot{\psi} - A_0 (\kappa B' + \rho) - \frac{1}{2m} |(\partial_x - ia \pm B) \psi|^2 + \frac{1}{2m} B' \rho - V(\rho) \right\} \]

(12)

\[ A_0 \text{ is Lagrange multiplier, enforcing the Gauss law, which in this theory requires} \]
\[ B' = -\frac{1}{\kappa} \rho \]  

(13a)

or equivalently
\[ B(x) = -\frac{1}{2\kappa} \int d\tilde{x} \epsilon(x - \tilde{x}) \rho(\tilde{x}) \]  

(13b)

[The Green’s function, uniquely determined by parity invariance, is the Heaviside \( \pm 1 \) step.]

Thus, after we eliminate \( B \), (12) involves a spatially non-local Lagrangian.

\[ L = -\frac{1}{2} \int dx \int d\tilde{x} \dot{\psi} \epsilon(x - \tilde{x}) \rho(\tilde{x}) + \int dx i \psi^* \dot{\psi} \]
\[ -\frac{1}{2m} \int dx \left| \left( \partial_x - ia \mp \frac{1}{2\kappa} \int d\tilde{x} \epsilon(x - \tilde{x}) \rho(\tilde{x}) \right) \psi(x) \right|^2 \]
\[ + \int dx \left( \pm \frac{1}{2\kappa m} \rho^2 - V(\rho) \right) \]  

(14a)

The \( a \) dependence is removed when \( \psi(x) \) is replaced by \( e^{\frac{i}{2} \int d\tilde{x} \epsilon(x - \tilde{x}) a(\tilde{x})} \psi(x) \), leaving

\[ L = \int dx i \psi^* \dot{\psi} - \frac{1}{2m} \int dx \left| \left( \partial_x \mp \frac{1}{2\kappa} \int d\tilde{x} \epsilon(x - \tilde{x}) \rho(\tilde{x}) \right) \psi(x) \right|^2 \]
\[ + \int dx \left( \pm \frac{1}{2\kappa m} \rho^2 - V(\rho) \right) \]  

(14b)

Finally we choose \( V(\rho) \) to be \( \pm \frac{1}{2\kappa m} \rho^2 \) [this is the same choice that in \( (2 + 1) \)-dimensions leads to static first-order Bogomolny equations] and our reduced Chern-Simons, \( B-F \) theory is governed by the Hamiltonian

\[ H = \frac{1}{2m} \int dx \left( \partial_x \mp \frac{1}{2\kappa} \int d\tilde{x} \epsilon(x - \tilde{x}) \rho(\tilde{x}) \right) \psi(x) \right|^2 \]  

(15)

which implies the first-order, Bogomolny equation

\[ \psi'(x) \mp \frac{1}{2\kappa} \int d\tilde{x} \epsilon(x - \tilde{x}) \rho(\tilde{x}) \psi(x) = 0 \]  

(16)

On the other hand, we can recognize the dynamics described in (15) by expanding the product.
\[ H = \frac{1}{2m} \int dx \left\{ |\psi'|^2 \pm \frac{1}{\kappa} \rho^2 \right\} + \frac{1}{24m\kappa^2} \int dxd\hat{x}d\hat{x}\rho(x)\rho(\hat{x}) \left\{ \epsilon(x - \hat{x})\epsilon(x - \hat{x}) + \epsilon(\hat{x} - x)\epsilon(\hat{x} - x) + \epsilon(\hat{x} - x)\epsilon(\hat{x} - \hat{x}) \right\} \]

(17)

The last term was symmetrized, leading to a sum of step function products, which in fact equals to 1. Consequently the last integral is \( \frac{1}{24m\kappa^2}N^3 \), where \( N = \int dx \rho(x) \), which is conserved in the dynamics implied by (17). Hence this term can be removed by redefining

\[ \psi \rightarrow e^{-i\frac{\lambda^2 t}{8m\kappa^2}} \psi \]

(18)

What is left is recognized as the Hamiltonian for the non-linear Schrödinger equation, with equation of motion

\[ i\dot{\psi} = -\frac{1}{2m} \psi'' - \lambda \rho \psi \]

\[ \lambda \equiv \mp \frac{1}{m\kappa} \]

(19)

The non-linear Schrödinger equation plays a cycle of interrelated roles in mathematical physics. Viewed as a non-linear, partial differential equation for the function \( \psi \), it is completely integrable, possessing a complete spectrum of multi-soliton solutions, the simplest of these being the single soliton at rest. This requires \( \lambda > 0 \), which is always achievable in our reduction by adjustment of \( \kappa \).

\[ \psi_{\text{rest}}^t(t, x) = \pm e^{\frac{i\alpha^2 t}{2m}} \frac{1}{\sqrt{\lambda m \cosh \alpha x}} \]

(20)

(\( \alpha \) is an integration constant.) Because of Galileo invariance, the solution may be boosted with velocity \( v \), yielding

\[ \psi_{\text{moving}}^t(t, x) = \pm e^{imvx} e^{i t \left( \frac{\alpha^2}{2m} - \frac{mv^2}{2} \right)} \frac{1}{\sqrt{\lambda m \cosh \alpha(x - vt)}} \]

(21)

The soliton solutions can be quantized by the well-known methods of soliton quantization. On the other hand, the non-relativistic field theory can be quantized at fixed \( N \), where it describes \( N \) non-relativistic point particles with pair-wise \( \delta \)-function interactions. This quantal problem can also be solved exactly, and the results agree with those of soliton quantization. All these properties are well-known, and will not be reviewed here. [6]
The present development demonstrates that this classical/quantal completely integrable theory possesses a Bogomolny formulation, which is obtained by using two-dimensional $B$-$F$ gauge theory, which in turn descends from three-dimensional Chern-Simons dynamics. Indeed it is clear that (20) also solves the first-order equation (16) – even the phase, which is undetermined by (14), is consistent with (18). \[ \text{[7]} \]

B. Reduction to Modified non-Linear Schrödinger Equation

While the previous development started with $B$-$F$ gauge theory, which descended from a Chern-Simons model, and arrived at an interesting (first-order, Bogomolny) formulation for the familiar non-linear Schrödinger equation, we now further modify the gauge theory and obtain a novel, chiral, non-linear Schrödinger equation.

Let us observe first that the above dynamics is non-trivial solely because we have chosen $V$ to be non-vanishing. Indeed with $V = 0$ in (11), (12) and (14), the same set of steps (removing $B$ and $a$ from the theory) results in a free theory for the $\psi$ field.

To avoid triviality at $V = 0$, we need to make the $B$ field dynamically active by endowing it with a kinetic term. Such a kinetic term could take the Klein-Gordon form; however we prefer a simpler expression that describes a “chiral” Bose field, propagating only in one direction. A Lagrange density for such a field has been known for some time. \[ \text{[8]} \]

\[
\mathcal{L}_{\text{chiral}} = \pm \dot{B}B' + vB'B' \quad (22)
\]

Here $v$ is a velocity, and the consequent equation of motion arising from $\mathcal{L}_{\text{chiral}}$ is solved by $B = B(x \mp vt)$ (with suitable boundary conditions at infinity), describing propagation in one direction, with velocity $\pm v$. Note that $\dot{B}B'$ is not invariant against a Galileo boost, which is a symmetry of $B'B'$ and of (11), (12), (14): performing a Galileo boost on $\dot{B}B'$ with velocity $\tilde{v}$ gives rise to $\tilde{v}B'B'$, effectively boosting the $v$ parameter in $\mathcal{L}_{\text{chiral}}$ by $\tilde{v}$. Consequently, one may drop the $vB'B'$ contribution, thereby selecting to work in a global “rest frame.” Boosting a solution in this rest frame produces a solution to the theory with a $B'B'$ term.

In view of this discussion, we supplement the previous Lagrange density (11), (12), (14) by $\pm \dot{B}B'$, set $V$ to zero, and thereby replace (13) by
\[ \mathcal{L} = -\kappa B \left( a \mp \frac{1}{\kappa} B' \right) + i \psi^* \dot{\psi} - A_0(\kappa B' + \rho) - \frac{1}{2m} |(\partial_x - ia)\psi|^2 - \frac{1}{2m} B^2 \rho \] (23a)

After redefining \( a \) as \( a \pm \frac{1}{\kappa} B' \), this becomes equivalent to

\[ \mathcal{L} = \kappa B \dot{a} + i \psi^* \dot{\psi} - A_0(\kappa B' + \rho) - \frac{1}{2m} |(\partial_x - i a \mp \frac{1}{\kappa} B')\psi|^2 - \frac{1}{2m} B^2 \rho \] (23b)

Now we proceed as before: solve Gauss’ law as in (13), remove \( a \) by a phase-redefinition of \( \psi \), drop the last term in (23b) by a further phase redefinition as in (18). We are then left with

\[ \mathcal{L} = i \psi^* \dot{\psi} - \frac{1}{2m} |(\partial_x \pm i \frac{1}{\kappa^2} \rho)\psi|^2 \] (24)

It has been suggested that this theory may be relevant to modeling quantum Hall edge states. [9]

The Euler-Lagrange equation that follows from (24) reads

\[ i \partial_t \psi = -\frac{1}{2m} \left( \partial_x \pm i \frac{1}{\kappa^2} \rho \right)^2 \psi \pm \frac{1}{\kappa^2} j \psi \] (25)

where the current density \( j \)

\[ j = \frac{1}{m} \text{Im} \psi^* \left( \partial_x \pm i \frac{1}{\kappa^2} \rho \right) \psi \] (26)

is linked to \( \rho \) by the continuity equation.

\[ \partial_t \rho + \partial_x j = 0 \] (27)

Next we redefine the \( \psi \) field by

\[ \psi(t, x) = e^{\pm \frac{i}{\kappa^2} \int^x dy \rho(t, y)} \Psi(t, x) \] (28)

and see that the equations satisfied by \( \Psi \) is

\[ i \dot{\Psi}(t, x) \pm \frac{1}{\kappa^2} \int^x dy \rho(t, y) \Psi(t, x) = -\frac{1}{2m} \Psi''(t, x) \pm \frac{1}{\kappa^2} j(t, x) \Psi(t, x) \] (29a)

But the integral may be evaluated with the help of (27), so finally we are left with [10]

\[ i \dot{\Psi} = -\frac{1}{2m} \Psi'' \pm \frac{2}{\kappa^2} j \Psi \] (29b)
This is a non-linear Schrödinger equation similar to (19) but with the current density 
\[ j = \frac{1}{m} \text{Im} \Psi^* \Psi' \] 
replacing the charge density \( \rho = \Psi^* \Psi \). The equation is not known to be completely integrable but it does possess an interesting soliton solution, which is readily found by setting the \( x \) dependence of the phase of \( \Psi \) to be \( e^{imvx} \). Then \( j = \nu \rho \), and our new equation (29b) becomes the usual non-linear Schrödinger equation (19)

\[ i \dot{\Psi} = -\frac{1}{2m} \Psi'' + \frac{2\nu}{\kappa^2} \rho \Psi \]  
(30)

\( i.e. \) the non-linear coupling strength of (19) is

\[ \lambda = \mp \frac{2\nu}{\kappa^2} \]  
(31)

The (\( \mp \)) sign is inherited from the “chiral” kinetic term, see (22), (23); once a definite choice is made (say +), positive \( \lambda \), which is required for soliton binding, corresponds to definite sign for \( \nu \) (say positive); \( i.e. \) the soliton solving (30) moves in only one direction. Explicitly, with the above choice of signs, the one-soliton solution reads

\[ \Psi_s(t, x) = \pm e^{imvx} e^{it} \left( \frac{x^2}{2m} - \frac{\alpha^2}{2} \right) \frac{\kappa}{\sqrt{2mv \cosh \alpha(x - vt)}} \]  
(32)

We see explicitly that \( \nu \) must be positive; the soliton cannot be brought to rest; Galileo invariance is lost.

The characteristics of the solution are are follows

\[ N_s = \frac{\alpha \kappa^2}{mv} \]  
(33)

The energy is obtained by integrating the Hamiltonian.

\[ E = \int dx \frac{1}{2m} |\Psi'|^2 \]  
(34)

and on the solution (32), takes the value appropriate to a massive, non-relativistic particle.

\[ E_s = \frac{1}{2} M_s \nu^2 \]  
(35)

where

\[ M_s = mN_s \left( 1 + \frac{1}{3\kappa^4 N_s^2} \right) \]  
(36)
The conserved field momentum in this theory reads

\[ P = \int dx (m_j + \frac{1}{\kappa^2} \rho^2) \quad (37) \]

and on the solution \((32)\) its value again corresponds to that of a massive, non-relativistic particle

\[ P_s = M_s v \quad (38) \]

\[ E_s = \frac{P_s^2}{2M_s} \quad (39) \]

As already remarked, the model is not Galileo invariant, but one can verify that it is scale invariant. Indeed one can show that the above kinematical relations are a consequence of scale invariance. \[10\]

The soliton solution \((32)\) can be quantized; also the quantal many body problem, which is implied by \((24)\), can be analyzed. Because the system does not appear integrable, exact results are unavailable, but one verifies that at weak coupling, the two methods of quantization (soliton, many body) produce identical results. \[10,11\]

**IV. MORE B-F THEORIES**

*B-F* theories can of course be extended to non-Abelian groups, with *B* transforming as an adjoint vector, just as *F*, so that their inner product is a group scalar. (More generally, one can even dispense with the group metric, by positing that *B* transforms in the co-adjoint representation.) Two-dimensional Yang-Mills theory is a *B-F* theory: since the Yang-Mills action may be written as \(I_{YM} = \frac{1}{2} \int d^2 x F^a F^a\), where \(F^a = \frac{1}{2} \epsilon^{\mu\nu} F^a_{\mu\nu}\), equivalent dynamics is governed by

\[ I = \int d^2 x (B^a F^a - \frac{1}{2} B^a B^a) \quad (40) \]

Moreover, since the *B-F* contribution is a world scalar – invariant against all coordinate transformations – general coordinate invariance is broken only by the second term, which however is invariant against area-preserving coordinate transformation. Hence the latter are
seen as symmetries of two-dimensional Yang-Mills theory, and this observation aids greatly in unraveling that model on spaces with non-trivial topology. [12] I do not pursue this topic here, but turn to another role for $B$-$F$ theories: gauge theoretic formulations of two-dimensional gravity.

A. Two-Dimensional Gravity

In order to have a gravity theory in two dimensions, where the Einstein tensor vanishes identically and the Einstein-Hilbert action is a surface term, therefore not generating Euler-Lagrange equations of motion, one introduces a further, world scalar variable, these days called the dilaton $\eta$. A class of possible actions is

$$I_{2-d \text{ gravity}} = \int d^2x \sqrt{-g}\left(\eta R - V(\eta)\right)$$

(41)

where different theories are selected by choosing various forms for $V$. Two especially interesting choices are $V(\eta) = \Lambda \eta$ and $V(\eta) = \Lambda$, where $\Lambda$ is a (cosmological) constant. The former is the first-such model that was proposed in 1984 [13]; the second is the popular, string-inspired CGHS model. [14]

It now happens that precisely these two models can be formulated as $B$-$F$ gauge theories, the former based on the $SO(2, 1)$ de Sitter or anti-de Sitter groups, [15] while the latter on the centrally extended Poincaré group $ISO(1, 1)$. [16] These constructions, which I described in my last visit to a Spanish Summer School (Salamanca, 1992), are two-dimensional analogs to the construction of three-dimensional Einstein theory as a Chern-Simons gauge theory – a discovery by the Spanish physicist Ana Achúcarro, among others. [17]

Let me quickly review how this is done. Rather than using metric variables, we introduce the Zweibein $e^A_\mu$ and the spin-connection $\omega^{AB}_\mu = \epsilon^{AB}_\mu \omega_\mu$, which are viewed as independent variables – the relation between them emerges as an equation of motion. [Capital Roman letters refer to the flat two-dimensional tangent space, with metric tensor $\delta_{AB}$, $g_{\mu\nu} = e^A_\mu e^B_\nu \delta_{AB}$.] We view the spin connection as a vector potential associated with the Lorentz rotation generator $J$ on the $(1+1)$-dimensional Minkowski tangent space, while the Zweibeine are vector
potentials associated with translations $P_A$. Next an algebra is postulated: the commutator with $J$ is conventional

$$[P_A, J] = \epsilon_A^B P_B$$  \hspace{1cm} (42)$$

The commutator of the translations takes different forms for $SO(2, 1)$ and for centrally extended $ISO(1, 1)$,

$$[P_A, P_B] = \epsilon_{AB} \Lambda J \hspace{1cm} SO(2, 1) \hspace{1cm} (43)$$

$$[P_A, P_B] = \epsilon_{AB} I \hspace{1cm} ISO(1, 1) \hspace{1cm} (44)$$

Here $I$ is a central element, commuting with $P_A$ and $J$. Consequently in the extended $ISO(1, 1)$ case, the group is enlarged from three parameters to four, since $I$ is taken to be an additional (commuting) generator, and a further vector potential $a_\mu$ is associated with it. (Henceforth we redefine the generators so that $\Lambda$ is scaled to unity.)

The potentials are collected into a group-valued connection

$$A_\mu = A_\mu^a T_a = \epsilon_\mu^A P_A + \omega_\mu J + a_\mu I$$  \hspace{1cm} (45)$$

where the last term is present only in the centrally extended $ISO(1, 1)$ case. The field strength is computed in the usual manner

$$\epsilon_{\mu\nu} F = F_{\mu\nu} = F_\mu^a T_a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$  \hspace{1cm} (46)$$

with the commutator evaluated from (42) - (44). One finds that the component of $F_{\mu\nu}$ along the translation direction is the gravitational torsion, while that along the rotation direction is the gravitational curvature.

The gauge transformation properties are as expected: an infinitesimal gauge parameter $\Theta$ is constructed in the Lie algebra, i.e., with an expansion similar to (43); then $\delta A_\mu = D_\mu \Theta = \partial_\mu \Theta + [A_\mu, \Theta], \delta F = [F, \Theta], \delta F = [F, \Theta]$, and from these one can read off the transformation properties for the component fields.

Finally to form the action, a set of Lagrange multiplier fields is introduced; they transform in the (co-)adjoint representation and allow construction of the scalar $B-F$ quantity ($B$ replaced by $\eta$)
\[ I_{2-d \ gravity} = \int d^2 x \eta_a F^a \]  

(47)

One readily verifies that the equations of motion that follow from (47) coincide with those of (41) in the \( SO(2, 1) \) and extended \( ISO(1, 1) \) cases.

While the geometric formulation (41) is equivalent to the gauge-group formulation (47), the latter is much more readily analyzed, by exploiting gauge invariance. The equations of motion that follow from (47) are

\[ F^a = 0 \]  

(48a)

\[ \left( D_\mu \eta \right)_a = \partial_\mu \eta_a + f^{ab}_c A^c_\mu \eta_c = 0 \]  

(48b)

Classical solution is straightforward. Eq. (48b) requires \( A \) to be a pure gauge, and we may pass to the gauge \( A = 0 \) (assuming that there is no obstruction). Then (48) states that in the chosen gauge \( \eta_a \) is constant. Of course in this gauge the geometry is lost \( \) – vanishing \( A \) means that the \( Zweibeine \) and spin-connection vanish. But we can now return to a non-trivial gauge \( A = U^{-1} dU \), which leads to non-vanishing geometric quantities. On the other hand, an invariant must be constructed solely from \( \eta_a \), since \( F^a \) vanishes. To form \( \eta_a \eta^{ab} \eta_b \) we need a group metric, and with \( SO(2, 1) \) the obvious expression is the indefinite, diagonal Killing-Cartan metric. The Poincaré group, not being semi-simple, does not possess a non-singular invariant metric, but its central extension does: one verifies that \( \eta_a \eta^{ab} \eta_b = \eta_A \delta^{AB} \eta_B - 2 \eta_2 \eta_3 \) is indeed invariant (we label the four indices \( a : 0, 1, 2, 3 \), where \( A : 0, 1 \) refers to the two-dimensional Minkowski tangent space).

The quantum theory is also readily analyzed with the help of the gauge formalism.

Upon writing the action (47) in canonical, first order form

\[ I_{2-d \ gravity} = \int d^2 x \left( \eta_a A^a_1 + A^a_0 (D_1 \eta)_a \right) \]  

(49)

we recognize that \( A^a_1 \) and \( \eta_a \) are the canonically conjugate coordinates and momenta, respectively, and the only dynamical equation is Gauss’ law, which requires physical states to be annihilated by \( (D_1 \eta)_a \). It is convenient to analyze this requirement in the field theoretic, Schrödinger-“momentum” representation, where states are functionals of the canonical momentum, here \( \eta_a, \Phi(\eta) \). Evidently these functionals must satisfy

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\[
\left( \eta_a' + if_{ab} \eta^c \frac{\delta}{\delta \eta_b} \right) \Phi(\eta) = 0 \tag{50}
\]

Solution is immediate. We first observe that \( \Phi \) has support only on constant \( \eta_a \), as is exemplified by contracting (50) with \( \eta^a \), and using the anti-symmetry of \( f_{abc} \) to drop the functional derivative. So we write

\[
\eta = g^{-1} K g
\tag{51}
\]

where \( g \) is a group element and \( K \) is constant and it follows that a solution to (50) is

\[
\Phi(\eta) = e^{iS(\eta)}
\]

\[
S(\eta) = -\int dx \langle K, g' g^{-1} \rangle = -\int dx \langle \eta, g^{-1} g' \rangle
\tag{52}
\]

with \( g \) related to \( \eta \) by (51).

The above structure (52) has another role in mathematical physics, quite distinct from the role in which we encounter it here as the phase of a wave functional. Observe that \( S \) in (52) is given by an integral of the 1-form \( \langle K, dgg^{-1} \rangle \), which one may take as a canonical 1-form for a Lagrangian with dynamical variables \( g \) depending on “time.” It is then further true that the symplectic 2-form, \( d\langle K, dgg^{-1} \rangle = \langle K, dgg^{-1} dgg^{-1} \rangle \) defines Poisson brackets and that the brackets of the quantities \( Q^a = (g^{-1} K g)^a \) reproduce the Lie algebra of the relevant group. This 2-form is associated with the names Kirillov and Kostant. [18] (One recognized here the development that was previously described as occurring in connection with the Chern-Simons term: an expression with interesting gauge transformation properties arises first in physics as the phase of a wave functional; subsequently it acquires its own dynamical role in a lower-dimensional theory.)

**ASIDE ON GAUGE THEORETIC WAVE FUNCTIONALS IN THE MOMENTUM REPRESENTATION**

Let us observe that the wave functional in (52) is not gauge invariant. A gauge transformation on \( \eta \to \eta^U = U^{-1} \eta U \) is effected, according to (51), by \( g \to gU \). Hence we see in (52) that
\begin{align}
\Phi(\eta^U) &= e^{-i\Omega(\eta^U)}\Phi(\eta) \quad (53a) \\
\Omega(\eta, U) &= \int \langle \eta, U'U^{-1} \rangle . \quad (53b)
\end{align}

This feature is a universal property of gauge theoretic wave functionals in the momentum representation – a little-known fact that deserves elaboration.

We consider a typical gauge theory, in any number of dimensions, with a conventional, non-Chern-Simons gauge Gauss law, which ensures that wave functionals in the coordinate representation, \textit{i.e.} depending on $A$, are gauge invariant. (We ignore the vacuum angle that may arise when topologically non-trivial gauge transformations are considered.) Note that the present discussion does \textbf{not} apply in the presence of a Chern-Simons term, because then Gauss’ law becomes unconventional and further effects are present. The momentum representation – we call it the “E” representation, because (for conventional, non-Chern-Simons gauge theories) the $E^i$ field is conjugate to $A_i$ – can be related to the coordinate representation by a (functional) Fourier transform.

\[ \Phi(E) = \int \mathcal{D}A \left( \exp -i \int \langle E^i, A_i \rangle \right) \Psi(A) \quad (54) \]

Now the following sequence of equations holds.

\[ \Phi(E) = \int \mathcal{D}A \left( \exp -i \int \langle E, A \rangle \right) \Psi(U^{-1}AU + U^{-1}\partial U) \]
\[ = \int \mathcal{D}A \left( \exp -i \int \langle E^U, A \rangle \right) \Psi(A + U^{-1}\partial U) \]
\[ = \exp i \int \langle E, \partial U U^{-1} \rangle \int \mathcal{D}A \left( \exp -i \int \langle E^U, A \rangle \right) \Psi(A) \]
\[ = \exp i \Omega(E, U) \Phi(E^U) \quad (55) \]

The first equality is true because $\Psi(A)$ is gauge invariant. In the next equality we have changed integration variables: $A \rightarrow UAU^{-1}$; this has unit Jacobian, and affects the phase by replacing $E$ with its gauge transform $E^U = U^{-1}EU$. In the next step, $A$ is shifted: $A \rightarrow A - U^{-1}\partial U$; this produces the phase $\Omega(E, U)$ seen in the last equality.

\[ \Omega(E, U) = \int d^d r E_a^i (\partial_i U U^{-1})^a \quad (56) \]

Thus from (53), it follows that physical wave functionals in the “E” representation are not gauge invariant. Rather, after a gauge transformation they acquire the phase $\Omega(E, U)$,
\[ \Phi(E^U) = e^{-i\Omega(E,U)}\Phi(E) \]  

(57)

which is recognized to be a 1-cocycle \(i.e.\ \Omega(E,U)\) satisfies

\[ \Omega(E, U_1 U_2) = \Omega(E^{U_1}, U_2) + \Omega(E, U_1) \]  

(58)

as is required by (56) when two gauge transformations are composed.

We conclude therefore that physical functionals in the "\(E\)" representation, which are annihilated by the Gauss law generator \(G_a\), obey (57). In one spatial dimension with \(\eta\) replacing \(E^i\), we regain (53).

**B. Incorporating Matter in Gauge Theories of Gravity**

While the gauge theoretical formulation of gravity in two dimensions (and also in three) proceeds smoothly, \[15\ 17\] incorporating matter poses new problems, because even when the Zweibein/spin connection are employed in the matter Lagrangian, the gauge symmetry is not apparent.

Here I shall discuss the simplest case – a point-particle coupled to gravity; similar ideas apply when matter fields are coupled to gravity.

An action for a point particle of mass \(m\) can be given in first-order form as

\[ I = \int d\tau \left( p_A e^A_{\mu} \dot{x}^\mu + \frac{1}{2} N (p^2 - m^2) \right) \]  

(59)

Upon varying and eliminating \(p_A = -e^A_{\mu} \dot{x}^\mu / N\) and \(N = \sqrt{\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu}/m\) one recognizes the above as the familiar second-order action \(-m \int d\tau \sqrt{\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu}\). Here \(x^\mu(\tau)\) is the particle trajectory, on which the gravitational quantities \(g_{\mu\nu}\) and \(e^A_{\mu}\) depend, while the auxiliary quantities \(p_A\) and \(N\) depend on the parameter \(\tau\), which can be reparametrized at will.

Expression (59) is diffeomorphism invariant. For its response to gauge transformations, we adopt the CGHS model of gravity theory, in the extended \(ISO(1,1)\) gauge group formulation. According to the general discussion given previously, when the infinitesimal gauge function is taken as

\[ \Theta = \theta^A P_A + \alpha J + \beta I \]  

(60)
the components of the gauge theoretic connection transform as

\[
\delta e^A_\mu = -\alpha e^B_\mu e^A_B + \epsilon^A_B \theta^B \omega_\mu + \partial_\mu \theta^A \\
\delta \omega_\mu = \partial_\mu \alpha \\
\delta a_\mu = e^A_\mu e_{AB} \theta^B + \partial_\mu \beta
\]  

(61)

The natural gauge transformation law for the matter variables in (59) is that \( x^\mu \) and \( N \) are scalars, while \( p_A \) responds by

\[
\delta p_A = -\alpha e^B_A p_B
\]  

(62)

But then (59) is not invariant against local translations, generated by \( \theta^A \).

To remedy this situation we proceed as follows. A new variable is introduced \( q^A \), called the Poincaré coordinate, with infinitesimal gauge transformation law

\[
\delta q^A = -\alpha e^B_A q^B - \theta^A
\]  

(63)

i.e. under Lorentz transformations \( q^A \) rotates in the usual way, but is shifted by translations. As a consequence, by a translational gauge transformation, \( q^A \) may be always set to zero.

A gauge invariant action is now constructed by replacing \( e^A_\mu \dot{x}^\mu \) in (59) by \( (D_\tau q)^A \), where

\[
(D_\tau q)^A = \dot{q}^A + e^A_\mu q^B \omega_\mu \dot{x}^\mu + e^A_\mu \dot{x}^\mu
\]  

(64)

\[
I_{\text{invariant}} = \int d\tau \left( p_A (D_\tau q)^A + \frac{1}{2} N (p^2 - m^2) \right)
\]  

(65)

Invariance is established once it is verified from (61), (62) and (64) that

\[
\delta (D_\tau q)^A = -\alpha e^B_A (D_\tau q)^B
\]  

(66)

On the other hand, dynamics has not been changed because the Poincaré coordinate can be set to zero by a gauge transformation and the action (59) is regained from (65).

In a sense this is a Higgs-like mechanism, with \( q^A \) playing the role of a “Goldstone” field, which is needed for a manifestly symmetric formulation. In the “unitary” gauge the Goldstone field is absent, only physical degrees of freedom are visible, but the symmetry is lost. [20]
A manifestly covariant group theoretical formalism is available, and one can also accommodate matter fields by introducing a Poincaré field. Within this formalism, CGHS-matter quantum field theory has been analyzed completely, with interesting results. The most noteworthy of these shows that the constraints, which are present in the theory, acquire quantum mechanical obstructions (anomalies) whose form depends on the ordering prescription used in defining the theory. These anomalies can be removed by further adjustment of the theory, and a “physical” spectrum of states is displayed. [21]

I shall not pursue this topic here further, beyond remarking that once the route to a successful analysis is found within the gauge theoretical formalism, it is then possible to identify analogous paths in the geometric formulation of the theory, as well. [22]

V. PARITY-EVEN MASS FOR THREE-DIMENSIONAL GAUGE FIELDS

I remarked previously that the Chern-Simons expression, when added to the three-dimensional Yang-Mills action, renders the fields massive, while preserving gauge invariance. However, parity symmetry is lost.

A trivial way of maintaining parity with this mass generation is through the doublet mechanism. Consider a pair of identical Yang-Mills actions, each supplemented with their own Chern-Simons term, which enters with opposite signs. The parity transformation is defined to include field exchange accompanying coordinate reflection, and this is a symmetry of the doubled theory.

But is there a way of maintaining reflection symmetry without introducing parity doublets? Here I shall show how this can be done, and I shall use various ideas that I have already discussed. But first, a bit of motivation.

Three dimensional gauge theories possess theoretical/mathematical interest, but they merit study because they describe (1) kinematical processes that are confined to a plane when external structures (magnetic fields, cosmic strings) perpendicular to the plane are present, and (2) static properties of (3 + 1)-dimensional systems in equilibrium with a high-temperature heat bath. An important issue is whether the apparently massless gauge theory
possesses a mass gap. The suggestion that indeed it does gains support from the observation that the gauge coupling constant squared carries dimension of mass, thereby providing a natural mass-scale (as in the two-dimensional Schwinger model). Also without a mass gap, the perturbative expansion is infrared divergent, so if the theory is to have a perturbative definition, infrared divergences must be screened, thereby providing evidence for magnetic screening in the four-dimensional gauge theory at high temperature.

But in spite of the above indications, a compelling theoretical derivation of the desired result is not yet available, even though many approaches have been tried.

Here I shall not address the dynamical question of how such a mass gap can be generated. Rather I present a phenomenological construction: I offer a theory for massive vector fields, which is gauge invariant and parity preserving. [23]

Consider the Lagrange density

$$\mathcal{L} = \text{tr} \left( F^\mu F_\mu + G^\mu G_\mu - 2mF^\mu \Phi_\mu \right)$$  \hspace{1cm} (67)

The first term is the usual Yang-Mills expression, written in terms of the dual field $F^\mu \equiv \frac{1}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta}$; the second describes a charged vector field $\Phi_\mu$ in the same adjoint representation as the gauge potential and interacting with it.

$$G^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta} G_{\alpha\beta}, \quad G_{\alpha\beta} = D_\alpha \Phi_\beta - D_\beta \Phi_\alpha, \quad D_\alpha = \partial_\alpha + [A_\alpha, \ ] \hspace{1cm} (68)$$

Finally the last contribution carries the mass scale $m$ and involves a mixed Chern-Simons-like structure,

$$F^\mu \Phi_\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta} \Phi_\mu$$

Obviously the theory is gauge invariant.

The equations of motion

$$\epsilon_{\mu\alpha\beta} D^\alpha F^\beta - mG_\mu + \epsilon_{\mu\alpha\beta} [G^\alpha, \Phi^\beta] = 0$$  \hspace{1cm} (69a)

$$\epsilon_{\mu\alpha\beta} D^\alpha G^\beta - mF_\mu = 0$$  \hspace{1cm} (69b)

may be combined into a second-order equation
\[ D^2 G_\mu - D^\nu D_\mu G_\nu + m^2 G_\mu - m e_{\mu \alpha \beta} [G^\alpha, \Phi^\beta] = 0 \]  

(70)

The linear part of (70) shows the \( G^\mu \) is massive, and then (69b) shows that \( F^\mu \) is massive as well. By positing that \( \Phi_\mu \) carries odd parity, we ensure that reflection symmetry is maintained.

The theory possesses an interesting symmetry structure. In addition to being invariant against gauge transformations

\[ \delta_1 A_\mu = D_\mu \theta, \quad \delta_1 \Phi_\mu = [\Phi_\mu, \theta] \]  

(71)

there is a another transformation

\[ \delta_2 A_\mu = 0 \quad \delta_2 \Phi_\mu = D_\mu \chi \]  

(72)

which obviously does not affect the Yang-Mills term, and changes the interaction/mixing term by total derivative, because \( F^\mu \) obeys the Bianchi identity \( D_\mu F^\mu = 0 \). However, the kinetic term for the \( \Phi \) fields is not invariant

\[ \delta_2 G^\mu = [F^\mu, \chi] \]  

(73)

To gain further insight, let us record the Lie algebra of the generators for the symmetry group as

\[ [Q_a, Q_b] = f^{ab}_c Q_c \]  

(74)

and recall that the vector potential \( A^a_\mu \) is the connection associated with this group.

What group theoretical role can we assign to \( \Phi^a_\mu \)? Note that this field has as many components as \( A^a_\mu \). Let us therefore postulate an Abelian group with as many parameters as in (74) and generators \( P_a \), satisfying

\[ [P_a, P_b] = 0 \]  

(75)

We shall consider the vector fields \( \Phi^a_\mu \) to be connections in this Abelian group. Moreover, let us further postulate

\[ [Q_a, P_b] = f^{ab}_c P_c \]  

(76)
With these definitions we can unify much of our formalism. Define the connection on both groups as

$$A_{\mu} = A_{\mu}^a Q_a + \Phi_{\mu}^a P_a$$  \hspace{1cm} (77)

The curvature $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}]$ is evaluated from (74)–(77), and nicely decomposes into $F_{\mu\nu}^a$ and $G_{\mu\nu}^a$

$$F_{\mu\nu} = F_{\mu\nu}^a Q_a + G_{\mu\nu}^a P_a$$  \hspace{1cm} (78)

Also the transformations (71), (72) can be collected together by defining the infinitesimal gauge parameter

$$\Theta = \theta^a Q_a + \chi^a P_a$$  \hspace{1cm} (79)

and recognizing that (71), (72) are equivalent to

$$\delta A_{\mu} = D_{\mu} \Theta = \partial_{\mu} \Theta + [A_{\mu}, \Theta]$$  \hspace{1cm} (80)

But in spite of the above unification of formalism, the action associated with (67) remains non-invariant, since $G^2$ is not invariant. [This is because $Q_a Q^a + P_a P^a$ is not an invariant of the algebra (74)–(76).] However, this defect can be fixed, in a manner similar to the construction of a gauge invariant gravity-matter interaction in two dimensions, which I described earlier. To this end, we introduce an additional scalar field multiplet $\rho^a$, which transforms under (71) as an adjoint vector

$$\delta_1 \rho^a = f_{bc}^a \rho_b \theta_c$$  \hspace{1cm} (81)

while the transformation (72) effects a shift

$$\delta_2 \rho^a = -\chi^a$$  \hspace{1cm} (82)

Also, in $\mathcal{L}$ we replace $G^\mu$ by $G^\mu + [F^\mu, \rho]$, which is invariant: $\delta_2 (G^\mu + [F^\mu, \rho]) = [F^\mu, \chi] + [F^\mu, -\chi] = 0$.

Thus we have constructed an invariant, non-Yang-Mills theory governed by the Lagrange density
\[ \mathcal{L}_\rho = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{4} (G^a_{\mu\nu} + f_{abc} F^b_{\mu\nu} \rho^c)(G^{\mu\nu a} + f_{abc} F^{\mu\nu b} \rho^c) + \frac{m}{2} \epsilon^{\alpha\beta\gamma} F_a^{\alpha\beta} \Phi_a^{\gamma} \] (83)

Moreover, the dynamics is the same as that of \( \mathcal{L} \) in (67), because with the gauge transformation (82) one can set \( \rho \) to zero, thereby reducing \( \mathcal{L}_\rho \) to \( \mathcal{L} \), in this “unitary” gauge.

It is worth commenting on the unconventional aspects of this realization for gauge invariant, but non-Yang-Mills dynamics. Usually when considering gauge fields and vector fields that are “charged” with respect to the gauge group, one includes couplings beyond the minimal ones, and embeds everything (gauge fields, vector fields) in a larger non-Abelian gauge group. [For example, electrically charged vector fields and Maxwell gauge fields are endowed with non-minimal interactions and combined into and \( SU(2) \) non-Abelian group, with the “third” direction being electromagnetism and the other “two” referring to the charged degrees of freedom.] On the other hand, we have combined our group degrees of freedom into a semi-direct product structure, (74)-(76), and invariance is achieved without non-minimal coupling, other than introducing the “Goldstone” field \( \rho^a \), which disappears in the “unitary” gauge.

Although formal quantization of the model can be carried out, [23] developing a perturbative calculational method requires further analysis. The point is that the quadratic portion of the kinetic term for the \( \Phi_\mu \) field does not define a propagator, because the derivative operator is transverse and has no inverse. On the other hand, it seems impossible to resolve this problem by a gauge fixing term – as is done for the \( A_\mu \) kinetic term. One possibility is to use a background \( A_\mu \) field to define the \( \Phi_\mu \) propagator. Indeed the existence of the “Goldstone” field, which shifts by a constant under a symmetry transformation, hints at some kind of symmetry breaking.

This model deserves further study, so its properties can be completely understood.
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