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A Short Introduction to Process Theory

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Preface

Complex computer-based systems have become an essential part of our society. These complex systems are generally composed of a number of components that act concurrently and interact with each other and the environment of the system concerned. The complexity arises to a great extent from the many ways in which the components of the system can interact. Not surprisingly, means for the description and analysis of system behaviour become increasingly important to discover flaws in computer-based systems.

When it comes to description and analysis, there is advantage in treating systems, the components of which they are composed, and the environments with which they interact, on an equal footing. Therefore, we call them all processes. These lecture notes concern process theory, i.e. the theory of process behaviour, but intentionally does not cover the entire field. First of all, we do not consider all possible kinds of processes, but only a kind of frequent occurrence. In particular, we use the term process to mean any system whose behaviour is made up of discrete actions. Each action of a process is either performed synchronously with an action of another process, in which case an interaction takes place between those processes, or it is performed on its own. Moreover, we restrict ourselves to basic concepts for the description of process behaviour.

More concretely, we focus in these lecture notes on the concept of a (labelled) transition system, a concept first introduced in [12]. The reason for this is twofold. Firstly, the concept of a transition system can be considered to be the fundamental concept for the description of process behaviour. Almost all formalisms meant for the description of process behaviour are based on the concept of a transition system (see e.g. [16, 9, 10]). Secondly, although mathematically simple, transition systems can model virtually all relevant properties of processes. The transition system describing the behaviour of a process is generally a suitable basis for checking properties of that process (see e.g. [2]).

Outline of the lecture notes

These lecture notes are organized in six chapters and an appendix in which the desirable background in elementary set theory is shortly reviewed.
Chap. 1, which is an introductory chapter, is primarily meant to acquire a good insight into the concept of a transition system and its relevance to the description of process behaviour. No attention is paid in this chapter to issues material to the application of transition systems for the description of process behaviour. These issues are treated in the subsequent chapters, which all build on Chap. 1.

If we have a process composed of a number of subprocesses that act concurrently and interact with each other, the following important question arises. How do we obtain a transition system describing the behaviour of the whole process from the transition systems describing the behaviours of the subprocesses? Therefore, the issue of concurrency and interaction must be dealt with in the setting of transition systems. This is done in Chap. 2.

Frequently, the behaviour of a process is first described at a high level of abstraction, and then as a process composed of several subprocesses that act concurrently and interact with each other. In order to show that the high-level description is correctly refined by the other one, we have to abstract from the actions added for the interactions between the subprocesses. This issue of abstraction is treated in Chap. 3.

Composing a process of subprocesses that act concurrently and interact with each other is only one way of combining processes. Other ways of combining processes, especially the ones known as sequential composition, alternative composition and iteration, are useful in case of large processes to master their complexity. Chap. 4 deals with the issue of composition in this wider sense.

Transition systems describing the behaviour of real-life processes are generally very large or even infinite. The size can be reduced strongly by using expressions representing the behaviour of processes instead. The operators occurring in such process expressions correspond to ways of combining processes such as the ones treated in Chap. 4. Furthermore, process expressions enable us to define processes by means of recursive specifications. Process expressions and recursive specifications are the subjects of Chap. 5.

There are many interesting topics related to process expressions and recursive specifications which are not treated in Chap. 5. Some selected topics, including structural operational semantics and equational laws for process expressions, are covered in Chap. 6.

All concepts and issues treated in these lecture notes are first introduced by means of simple examples, sometimes not even related to real-life systems, and later on illustrated by more complex examples based on real-life systems. To quicken an intuitive understanding, direct connections with programs and automata are established wherever appropriate. For the interested reader, direct connections with Petri nets are also established. Those connections are relevant because Petri nets are basically generalizations of transition systems that support the direct description of concurrency.
In each chapter, except the last one, it shows that what has been dealt with so far still has certain limitations. Each time, the next chapter is devoted to reducing the limitations concerned. It is worth mentioning that, as a result of this set-up, the notion of a transition system is first defined in Sect. 1.2 and then redefined in Sects. 3.2 and 4.2.

How to use the lecture notes

These lecture notes can be used in courses for undergraduate students in computer science. Some familiarity with set theory is assumed. The desirable background in set theory is shortly reviewed in App. A, which also establishes the terminology and notation concerning sets. Each chapter is a prerequisite for all subsequent chapters. The examples are integrated with the text. They should not be ignored.

History of the lecture notes

In 2002, I was invited to write lecture notes for an introductory course on process theory for first year undergraduate computer science students at Eindhoven University of Technology that could serve as a preparation for an undergraduate course on process algebra based on [3]. This has led to an unpolished version of the current lecture notes. They have been written while consistently applying the following three simple rules: (i) begin with an elementary concept, (ii) introduce additional concepts not until the need for them has been explained clearly, and (iii) stray from the main topic for no other reason than explanation.

The unpolished lecture notes from 2002 have been adapted in 2003 by the lecturer of the course in question to his ideas and preferences without taking the above-mentioned rules fully into account. Those adapted lecture notes and shortened versions thereof are circulated since, mostly under the title “Introduction to Process Theory”.

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1. Transition Systems

The notion of a transition system can be considered to be the fundamental notion for the description of process behaviour. This chapter is meant to acquire a good insight into this notion and its relevance for the description of process behaviour. First of all, we explain informally what transition are systems and give some simple examples of their use in describing process behaviour (Sect. 1.1). After that, we define the notion of a transition system in a mathematically precise way (Sect. 1.2). For a better understanding, we next investigate the connections between the notion of a transition system and the familiar notions of a program (Sect. 1.3) and an automaton (Sect. 1.4). For the interested reader, we also investigate the connections with the notion of a Petri net (Sect. 1.5). Finally, we discuss two equivalences on transition systems, called trace equivalence (Sect. 1.6) and bisimulation equivalence (Sect. 1.6). Those equivalences are useful because they allow us to abstract from details of transition systems that we often want to ignore.

1.1 Informal explanation

Transition systems are often considered to be the same as automata. Both consist of states and labeled transitions between states. The main difference is that automata are primarily regarded as abstract machines to recognize certain languages and transition systems are primarily regarded as a means to describe the behaviour of interacting processes. In the case of transition systems, the intuition is that a transition is a state change caused by performing the action labeling the transition. A transition from a state \( s \) to a state \( s' \) labeled by an action \( a \) is usually written \( s \xrightarrow{a} s' \). This can be read as “the system is capable of changing its state from \( s \) into \( s' \) by performing action \( a \)”. Let us give an example to illustrate that it is quite natural to look at real-life computer-based systems as systems that change their state by performing actions.

Example 1.1.1 (Simple telephone system). We consider a simple telephone system. In this telephone system each telephone is provided with a process, called its basic call process, to establish and maintain connections with other telephones. Actions of this process include receiving an off-hook or on-hook
Transition systems have been devised as a means to describe the behaviour of systems that have only discrete state changes. Despite this underlying purpose of transition systems, they can deal with continuous state changes as well. However, such use of transition systems will not be treated in these lecture notes. Instead, we focus on acquiring a good insight into the basics of transition systems. That is not for pedagogical reasons alone. Systems that have only discrete state changes are still of utmost importance in the practice of developing computer-based systems and will remain so for a long time. Here are a couple of examples of the use of transition systems in describing the behaviour of systems with discrete state changes.

Example 1.1.2 (Bounded counter). We first consider a very simple system, viz. a bounded counter. A bounded counter can perform increments of its value by 1 till a certain value \( k \) is reached and can perform decrements of its value by 1 till the value 0 is reached. As states of a bounded counter, we have the natural numbers 0 to \( k \). State \( i \) is the state in which the value of the counter is \( i \). As actions, we have \( \text{inc} \) (increment) and \( \text{dec} \) (decrement). As transitions of a bounded counter, we have the following:

- for each state \( i \) that is less than \( k \), a transition from state \( i \) to state \( i + 1 \) labeled with the action \( \text{inc} \), written \( i \xrightarrow{\text{inc}} i + 1 \);
- for each state \( i \) that is less than \( k \), a transition from state \( i + 1 \) to state \( i \) labeled with the action \( \text{dec} \), written \( i + 1 \xrightarrow{\text{dec}} i \).

If the number of states and transitions is small, a transition system can easily be represented graphically. The transition system describing the behaviour of the bounded counter is represented graphically in Fig. 1.4 for the case where \( k = 3 \).

Notice that the bounded counter has a finite number of states and a finite number of transitions. Furthermore, the bounded counter will never reach a terminal state, i.e. a state from which no transition is possible. Thus, the.

\[^{1}\] In graphical representations of transition systems, we use circles or ellipses for states and arrows for transitions. We indicate the initial state by an incoming unlabeled arrow.
1.1 Informal explanation

The finiteness of the bounded counter does not keep the counter from making an infinite number of transitions.

The bounded counter can easily be adapted to become a counter modulo \( k \), i.e. a counter whose value becomes 0 by performing an increment by 1 when its value is \( k \) and whose value becomes \( k \) by performing a decrement by 1 when its value is 0. We have the same states and actions as before and we have two additional transitions:

- a transition from state \( k \) to state 0 labeled with the action inc, written \( k \xrightarrow{\text{inc}} 0 \);
- a transition from state 0 to state \( k \) labeled with the action dec, written \( 0 \xrightarrow{\text{dec}} k \).

**Example 1.1.3 (Bounded buffer).** We next consider another simple system, viz. a bounded buffer. A bounded buffer can add new data to the sequence of data that it keeps if the capacity \( l \) of the buffer is not exceeded, i.e. if the length of the sequence of data that it keeps is not greater than \( l \). As long as it keeps data, it can remove the data that it keeps – in the order in which they were added. As states of a bounded buffer, we have the sequences of data of which the length is not greater than \( l \). State \( \sigma \) is the state in which the sequence of data \( \sigma \) is kept in the buffer. As actions, we have add(\( d \)) (add \( d \)) and rem(\( d \)) (remove \( d \)) for each datum \( d \). As transitions of a bounded buffer, we have the following:

- for each datum \( d \) and each state \( \sigma \) that has a length less than \( l \), a transition from state \( \sigma \) to state \( d\sigma \) labeled with the action add(\( d \)), written \( \sigma \xrightarrow{\text{add}(d)} d\sigma \);
- for each datum \( d \) and each state \( \sigma d \), a transition from state \( \sigma d \) to state \( \sigma \) labeled with the action rem(\( d \)), written \( \sigma d \xrightarrow{\text{rem}(d)} \sigma \).

The transition system describing the behaviour of the bounded buffer is represented graphically in Fig. 1.2 for the case where \( l = 2 \) and the only data involved are the natural numbers 0 and 1. Although it has a finite capacity, the bounded buffer will have an infinite number of states and an infinite number of transitions in the case where the number of data involved is infinite.

The bounded buffer can easily be adapted to become unreliable, e.g. to get into an error state by adding a datum when it is full. We have one additional state, say err, no additional actions, and the following additional transitions:
• for each datum \( d \) and each state \( \sigma \) that has a length equal to \( l \), a transition from state \( \sigma \) to state \( \text{err} \) labeled with the action \( \text{add}(d) \), written \( \sigma \xrightarrow{\text{add}(d)} \text{err} \).

Notice that no more transitions are possible when this unreliable bounded buffer has reached the state \( \text{err} \). Thus, the additional feature of this buffer may keep it from making an infinite number of transitions.

It is usual to designate one of the states of a transition system as its initial state. At the start-up of a system, i.e. before it has performed any action, the system is considered to be in its initial state. The expected initial states of the bounded counter from Example 1.1.2 and the bounded buffer from Example 1.1.3 are 0 and \( \epsilon \) (the empty sequence), respectively.

Although bounded counters and buffers arise frequently as basic components in computer-based systems, they are not regarded as typical examples of real-life computer-based systems. In the following example, we consider a simplified version of a small real-life computer-based system, viz. a calculator.

**Example 1.1.4 (Calculator).** We consider a calculator that can perform simple arithmetical operations on integers. It can only perform addition, subtraction, multiplication and division on integers between a certain values, say \(\text{min} \) and \(\text{max} \). As states of the calculator, we have pairs \((i, o)\), where \(\text{min} \leq i \leq \text{max} \) or \(i = \ast \) and \(o \in \{\text{add, sub, mul, div, eq, clr, } \ast\}\). State \((i, o)\) is roughly the state in which the result of the preceding calculations is \(i\) and the operator that must be applied next is \(o\). If \(o = \ast\), the operator that must be applied next is not available; and if in addition \(i = \ast\), the result of the preceding calculations is not available either. As initial state, we have the pair \((\ast, \ast)\). As actions, we have \(\text{rd}(i)\) (read operand \(i\)) and \(\text{wr}(i)\) (write result \(i\)), both for \(\text{min} \leq i \leq \text{max} \), and \(\text{rd}(o)\) (read operator \(o\)), for
\( \in \{ \text{add}, \text{sub}, \text{mul}, \text{div}, \text{eq}, \text{clr} \} \). As transitions of the calculator, we have the following:

- for each \( i \) with \( \min \leq i \leq \max \):
  - a transition \((*, *) \xrightarrow{\text{rd}(i)} (i, *)\),
  - a transition \((i, \text{clr}) \xrightarrow{\text{wr}(0)} (*, *)\),
  - a transition \((i, \text{eq}) \xrightarrow{\text{wr}(i)} (i, *)\);

- for each \( i \) with \( \min \leq i \leq \max \) and \( o \in \{ \text{add}, \text{sub}, \text{mul}, \text{div}, \text{eq}, \text{clr} \} \):
  - a transition \((i, *) \xrightarrow{\text{rd}(o)} (i, *)\);

- for each \( i \) with \( \min \leq i \leq \max \) and \( j \) with \( \min \leq j \leq \max \):
  - a transition \((i, \text{add}) \xrightarrow{\text{rd}(j)} (i + j, *)\) if \( \min \leq i + j \leq \max \),
  - a transition \((i, \text{sub}) \xrightarrow{\text{rd}(j)} (i - j, *)\) if \( \min \leq i - j \leq \max \),
  - a transition \((i, \text{mul}) \xrightarrow{\text{rd}(j)} (i \cdot j, *)\) if \( \min \leq i \cdot j \leq \max \),
  - a transition \((i, \text{div}) \xrightarrow{\text{rd}(j)} (i \div j, *)\) if \( \min \leq i \div j \leq \max \) and \( j \neq 0 \).

The transition system describing the behaviour of the calculator is represented graphically in Fig. 1.3 for the case where \( \min = 0 \) and \( \max = 1 \). Although the extremely small range of integers makes this case actually useless, it turns out to be difficult to represent the transition system graphically. The textual description given above is still intelligible. However, it is questionable whether this would be the case for a more realistic calculator.

Examples like Example 1.1.4 indicate that in the case of real-life systems we probably need a way to describe process behaviour more concisely than by directly giving a transition system. This is one of the issues treated in the remaining chapters of these lecture notes.

### 1.2 Formal definition

With the previous section, we have prepared the way for the formal definition of the notion of a transition system.

**Definition 1.2.1 (Transition system).** A transition system \( T \) is a quadruple \((S, A, \rightarrow, s_0)\) where

- \( S \) is a set of states;
- \( A \) is a set of actions;
- \( \rightarrow \subseteq S \times A \times S \) is a set of transitions;
- \( s_0 \in S \) is the initial state.

If \( S \) and \( A \) are finite, \( T \) is called a finite transition system. We write \( s \xrightarrow{a} s' \) instead of \((s, a, s') \in \rightarrow \). We write \( \text{act}(T) \) for \( A \), i.e. the set of actions of \( T \). The set \( \rightarrow \subseteq S \times A^* \times S \) of generalized transitions of \( T \) is the smallest subset of \( S \times A^* \times S \) satisfying:
Fig. 1.3. Transition system for the calculator

- \( s \xrightarrow{\cdot} s \) for each \( s \in S \);
- if \( s \xrightarrow{a} s' \), then \( s \xrightarrow{aa} s'' \);
- if \( s \xrightarrow{\sigma} s' \) and \( s' \xrightarrow{\sigma'} s'' \), then \( s \xrightarrow{\sigma\sigma'} s'' \).

A state \( s \in S \) is called a reachable state of \( T \) if there is a \( \sigma \in A^* \) such that \( s_0 \xrightarrow{\sigma} s \). A state \( s \in S \) is called a terminal state of \( T \) if there is no \( a \in A \) and \( s' \in S \) such that \( s \xrightarrow{a} s' \).

When a system has reached one of its terminal states, no more transitions are possible. Sometimes, certain terminal states are designated as final states. The convention is to do so if there is a need to make a distinction between terminal states in which the system is considered to terminate successfully and
terminal states in which the system is considered not to terminate successfully. In that case, the final states are the terminal states in which the system is considered to terminate successfully. Final states are also loosely called successfully terminating states. A system that reaches a terminal state different from a final state is said to become inactive. With certain terminal states designated as final states, a transition system is a quintuple \((S, A, \rightarrow, \downarrow, s_0)\), where \(S\), \(A\), \(\rightarrow\), \(\downarrow\), and \(s_0\) are as before and the set \(\downarrow \subseteq S\) of final states or successfully terminating states consists of terminal states only. We will return to such transition systems in Chap. 4.

We will now return to some of the transition systems introduced informally in the previous section.

Example 1.2.1 (Bounded counter). We look again at the bounded counter from Example 1.1.2. Formally, the behaviour of a bounded counter with bound \(k\) is described by the transition system \((S, A, \rightarrow, s_0)\) where

\[
S = \{ i \in \mathbb{N} \mid i \leq k \} ,
\]

\[
A = \{ \text{inc}, \text{dec} \} ,
\]

\[
\rightarrow = \{ (i, \text{inc}, i+1) \mid i \in \mathbb{N}, i < k \} \cup \{ (i+1, \text{dec}, i) \mid i \in \mathbb{N}, i < k \} ,
\]

\[
s_0 = 0 .
\]

All states of this finite transition system are reachable. It does not have terminal states.

Example 1.2.2 (Unreliable bounded buffer). We also look at the unreliable bounded buffer mentioned in Example 1.1.3. Formally, the behaviour of the unreliable bounded buffer with capacity \(l\) is described by the transition system \((S, A, \rightarrow, s_0)\) where

\[
S = \{ \sigma \in D^* \mid |\sigma| \leq l \} \cup \{ \text{err} \} ,
\]

\[
A = \{ \text{add}(d) \mid d \in D \} \cup \{ \text{rem}(d) \mid d \in D \} ,
\]

\[
\rightarrow = \{ (\sigma, \text{add}(d), d\sigma) \mid \sigma \in D^*, |\sigma| < l \}
\]

\[
\cup \{ (\sigma, \text{add}(d), \text{err}) \mid \sigma \in D^*, |\sigma| = l \}
\]

\[
\cup \{ (\sigma d, \text{rem}(d), \sigma) \mid \sigma \in D^*, |\sigma| < l \} ,
\]

\[
s_0 = \epsilon .
\]

All states of this transition system are reachable. It has one terminal state, viz. err.

Henceforth, we will only occasionally introduce transition systems in this formal style.

After the informal explanation and formal definition of the notion of a transition system, we are now in the position to relate it to the notions of a program and an automaton.
1.3 Programs and transition systems

For a better understanding of the notion of a transition system, we now look into its connections with the familiar notion of a program.

The behaviour of a program upon execution can be regarded as a transition system. In doing so, we can abstract from how the actions performed by a program are processed by a machine, and hence from how the values assigned to the program variables are maintained. In that case, we focus on the flow of control. The states of the transition system only serve as the control points of the program and its actions are merely requests to perform actions such as assignments, tests, etc. What we have in view here will be called the behaviour of a program upon abstract execution to distinguish it clearly from the behaviour of a program upon execution on a machine, which applies to the processing by a machine of the actions performed by the program. Here is an example of the use of transition systems in describing the behaviour of programs upon abstract execution.

**Example 1.3.1 (Factorial program).** We consider the following PASCAL [21] program to calculate factorials:

```pascal
PROGRAM factorial(input,output);
VAR i,n,f: 0..maxint;
BEGIN
  read(n);
  i := 0; f := 1;
  WHILE i < n DO
    BEGIN i := i + 1; f := f * i END;
  write(f)
END
```

The behaviour of this program upon abstract execution can be described by a transition system as follows. As states of the factorial program, we have the natural numbers 0 to 7, with 0 as initial state. The states can be viewed as the values of a “program counter”. As actions, we have an action corresponding to each atomic statement of the program as well as each test of the program and its opposite. As transitions, we have the following:

- $0 \xrightarrow{\text{read(n)}} 1$
- $1 \xrightarrow{i:=0} 2$
- $2 \xrightarrow{f:=1} 3$
- $3 \xrightarrow{i<n} 4$
- $4 \xrightarrow{i:=i+1} 5$
- $5 \xrightarrow{f:=f*i} 3$
- $3 \xrightarrow{\text{NOT}i<n} 6$
- $6 \xrightarrow{\text{write(f)}} 7$

The transition system for the factorial program is represented graphically in Fig. [1.4].

Here is another example.

**Example 1.3.2 (Greatest common divisor program).** We consider the following PASCAL program to calculate greatest common divisors:
1.3 Programs and transition systems

Fig. 1.4. Transition system for the factorial program

```
PROGRAM gcd(input,output);
VAR m,n: 0..maxint;
BEGIN
  read(m); read(n);
  REPEAT
    WHILE m > n DO m := m - n;
    WHILE n > m DO n := n - m
    UNTIL m = n;
  write(m)
END
```

The behaviour of this program upon abstract execution can be described by a transition system as follows. As states of the greatest common divisor program, we have the natural numbers 0 to 8, with 0 as initial state. As actions, we have an action corresponding to each atomic statement of the program as well as each test of the program and its opposite. As transitions, we have the following:

- $0 \xrightarrow{\text{read}(m)} 1$, $1 \xrightarrow{\text{read}(n)} 2$
- $2 \xrightarrow{m > n} 3$, $3 \xrightarrow{m := m - n} 2$
- $2 \xrightarrow{\text{NOT} m > n} 4$, $4 \xrightarrow{n > m} 5$, $5 \xrightarrow{n := n - m} 4$, $4 \xrightarrow{\text{NOT} n > m} 6$, $6 \xrightarrow{\text{NOT} m = n} 2$
- $6 \xrightarrow{m = n} 7$, $7 \xrightarrow{\text{write}(f)} 8$

The transition system for the greatest common divisor program is represented graphically in Fig. 1.4.

Notice that the transition systems described in Examples 1.3.1 and 1.3.2 have a single terminal state. In both cases, the program is considered to terminate successfully in its terminal state.

A transition system derived from a program in the way described and illustrated above is reminiscent of a flowchart. However, the underlying idea is that the transition system describes the behaviour of the program upon execution in such a way that it can act concurrently and interact with a machine that processes the actions performed by the program. If it does so, the
combined behaviour can be regarded as the behaviour of the program upon execution on a machine. Interaction between processes is one of the issues treated in the remaining chapters of these lecture notes. We can also directly give a transition system describing the behaviour of the program upon execution on a machine. In that case, we have to take into account that an assignment changes the value of a program variable, the values of the program variables determine whether a test succeeds, etc. This is illustrated in the following couple of examples, which are concerned with the same programs as the previous two examples.

Example 1.3.3 (Factorial program). We consider again the program from Example 1.3.1. The intended behaviour of this program upon execution on a machine can be described by a transition system as follows. As states of the program, we have pairs \((l, s)\), where \(l \in \mathbb{N}\) with \(0 \leq l \leq 7\) and \(s = (i, n, f)\) with \(i, n, f \in \{i \in \mathbb{N} \mid i \leq \text{maxint}\} \cup \{\ast\}\). These states can be viewed as follows: \(l\) is the value of the program counter and \(s = (i, n, f)\) is the storage that keeps the values of the program variables \(i, n,\) and \(f\) in that order. The special value \(\ast\) is used to indicate that a value has not yet been assigned to a program variable. The initial state is \((0, (\ast, \ast, \ast))\). As actions, we have again an action corresponding to each atomic statement of the program as well as each test of the program and its opposite. As transitions, we have the following:

- for each \(n\):
  - a transition \((0, (\ast, \ast, \ast)) \xrightarrow{\text{read}(n)} (1, (\ast, n, \ast))\),

```plaintext
read(m) read(n) m > n
m := m - n
NOT m
n > m
n := n - m
m = n
write(m)

Fig. 1.5. Transition system for the greatest common divisor program
```
1.3 Programs and transition systems

– a transition \((1, (*, n, *)) \xrightarrow{i := 0} (2, (0, n, *))\),
– a transition \((2, (0, n, *)) \xrightarrow{f := 1} (3, (0, n, 1))\);

• for each \(i, n, f\) such that \(i < n\) and \(f = i!\):
  – a transition \((3, (i, n, f)) \xrightarrow{i \leq n} (4, (i, n, f))\),
  – a transition \((4, (i, n, f)) \xrightarrow{i := i + 1} (5, (i + 1, n, f))\),
  – a transition \((5, (i + 1, n, f)) \xrightarrow{f := f \cdot i} (3, (i + 1, n, f \cdot (i + 1)))\);

• for each \(i, n, f\) such that \(i = n\) and \(f = i!\):
  – a transition \((3, (i, n, f)) \xrightarrow{\text{NOT } i < n} (6, (i, n, f))\),
  – a transition \((6, (i, n, f)) \xrightarrow{\text{write}(f)} (7, (i, n, f))\).

There are some noticeable differences between this transition system and the transition system from Example 1.3.1. The two relevant intuitions are as follows. In the same state, reading different numbers does not cause the same state change. In the same state, a test and its opposite do not succeed both.

Not all states are reachable. For example, states \((l, (i, n, f))\) with \(i \neq *\) and \(n \neq *\) for which \(i > n\) holds are not reachable. We did not bother to restrict the transition system to the reachable states: we will see later that the resulting transition system would describe essentially the same behaviour.

The transition system for the factorial program is represented graphically in Fig. 1.6 for the case where \(\text{maxint} = 2\).

Example 1.3.4 (Greatest common divisor program). We also consider again the program from Example 1.3.2. The intended behaviour of this program upon execution on a machine can be described by a transition system as follows. As states of the program, we have pairs \((l, s)\), where \(l \in \mathbb{N}\) with \(0 \leq l \leq 8\) and \(s = (m, n)\) with \(m, n \in \{i \in \mathbb{N} | i \leq \text{maxint}\} \cup \{*\}\). These states are like in Example 1.3.3. The initial state is \((0, (\ast, \ast))\). As actions, we have again an action corresponding to each atomic statement of the program as well as each test of the program and its opposite. As transitions, we have the following:

• for each \(m\):
  – a transition \((0, (\ast, *)) \xrightarrow{\text{read}(m)} (1, (m, *))\);
  – for each \(m, n\):
    – a transition \((1, (m, *)) \xrightarrow{\text{read}(n)} (2, (m, n))\);
  – for each \(m, n\) such that \(m > n\):
    – a transition \((2, (m, n)) \xrightarrow{m \geq n} (3, (m, n))\),
    – a transition \((3, (m, n)) \xrightarrow{m := m - n} (2, (m - n, n))\);

• for each \(m, n\) such that \(m \leq n\):
  – a transition \((2, (m, n)) \xrightarrow{\text{NOT } m \geq n} (4, (m, n))\);

• for each \(m, n\) such that \(m < n\):
  – a transition \((4, (m, n)) \xrightarrow{n \geq m} (5, (m, n))\),
  – a transition \((5, (m, n)) \xrightarrow{n := n - m} (4, (m, n - m))\);

• for each \(m, n\) such that \(m \geq n\):
read(n) \rightarrow (0,(*,*,*)) \rightarrow read(n) \rightarrow (1,(*,0,*))

(1,(*,0,*)) \rightarrow i := 0 \rightarrow (2,0,0,*)

(2,0,0,*) \rightarrow f := 1 \rightarrow (3,0,0,1)

(3,0,0,1) \rightarrow \text{NOT } i < n \rightarrow (6,0,0,1)

(6,0,0,1) \rightarrow \text{write}(f) \rightarrow (7,0,0,1)

(7,0,0,1) \rightarrow (1,(0,0,1)) \rightarrow (1,(0,1,1)) \rightarrow (1,(0,2,1))

Fig. 1.6. Another transition system for the factorial program
For a better understanding of the notion of a transition system, we looked in the previous section into its connections with the familiar notion of a program. For the same reason, we now look into its connections with the familiar notion of an automaton from automata theory (see e.g. [11] for an introduction).

Automata can be regarded as a specialized kind of transition systems. In this section, we restrict ourselves to the kind of automata known as non-deterministic finite accepters. They are illustrative for almost any kind of automata. If no confusion can arise, we will call them simply automata. The difference between automata and transition systems is mainly a matter of intended use. As mentioned in Section 1.1, transition systems are primarily regarded as a means to describe the behaviour of processes and automata are primarily regarded as abstract machines to recognize certain languages. Because of the different intended use, final states are indispensable in the case of automata: reaching a final state means that a complete sentence has been recognized. The final states of automata are usually not required to satisfy the restriction that they are terminal states. This restriction would be harmless in the sense that it would not have any influence on the languages that automata are able to recognize. Automata that satisfy the restriction can be regarded as finite transition systems with designated final states. We do
Fig. 1.7. Another transition system for the greatest common divisor program
not give the standard definition of the notion of an automaton. Our definition underlines the resemblance to transition systems mentioned above.

**Definition 1.4.1 (Automaton).** An *automaton* $M$ is a quintuple $(S, A, \rightarrow, s_0, F)$ where

- $S$ is a finite set of *internal states*;
- $A$ is a finite set of *symbols*, called the *input alphabet*;
- $\rightarrow \subseteq S \times A \times S$ is a set of *transitions*;
- $s_0 \in S$ is the *initial state*;
- $F \subseteq S$ is a set of *final states*.

A state $s \in S$ is called a *terminal* state of $M$ if there is no $a \in A$ and $s' \in S$ such that $s \xrightarrow{a} s'$, just as in the case of transition systems. The set $\rightarrow \subseteq S \times A^* \times S$ of *generalized transitions* of $M$ is also defined exactly as for transition systems. The *language* accepted by $M$, written $L(M)$, is the set $\{\sigma \in A^* | s_0 \xrightarrow{\sigma} s \text{ for some } s \in F\}$.

In the standard definition of the notion of an automaton, we have a *transition function* $\delta: S \times A \rightarrow P(S)$ instead of a set $\rightarrow \subseteq S \times A \times S$ of transitions. If we take $\delta$ such that $s' \in \delta(s, a)$ if and only if $s \xrightarrow{a} s'$, then we get an automaton according to the standard definition.

If we regard symbols as actions of reading the symbols, automata are simply transition systems with designated final states. An automaton can be considered to accept certain sequences of symbols as follows. A transition of an automaton is regarded as a state change caused by reading a symbol. A sequence of symbols $a_1 \ldots a_n$ is accepted if a sequence of consecutive state changes from the initial state to one of the final states can be obtained by reading the symbols $a_1, \ldots, a_n$ in turn. This informal explanation can be made more precise as follows.

Let $M$ be the automaton $(S, A, \rightarrow, s_0, F)$ and let $A'$ be the set of actions $\{\text{read}(a) | a \in A\}$. Suppose that each state in $F$ is a terminal state of $M$. Now consider the transition system $T = (S, A', \rightarrow', s_0)$ where $s_1 \xrightarrow{\text{read}(a)}' s_2$ iff $s_1 \xrightarrow{a} s_2$. The sentences of the language accepted by $M$ are exactly the sequences of symbols that can be consecutively read by $T$ till a terminal state is reached that is contained in $F$.

Let us look at a simple example of the use of automata in recognizing a language.

**Example 1.4.1 (Pidgingol).** We consider a very simple language. A sentence of the language consists of a noun clause followed by a verb followed by a noun clause. A noun clause consists of an article followed by a noun. A noun is either *man* or *machine*. A verb is either *simulates* or *mimics*. An example sentence is *the man mimics a machine*. This language is accepted by the following automaton. As internal states of the automaton, we have pairs $(p, i)$, where $p \in \{\text{left}, \text{right}\}$ and $i \in \mathbb{N}$ with $0 \leq i \leq 2$. The choice of states is not really relevant. We could have taken the natural numbers 0 to 5 equally well,
but the choice made here allows for a short presentation of the automaton. The initial state is (left, 0) and the only final state is (right, 2). The input alphabet consists of a, the, man, machine, simulates and mimics. As transitions, we have the following:

- for $p = \text{left, right}$:
  - a transition $(p, 0) \xrightarrow{a} (p, 1)$,
  - a transition $(p, 0) \xrightarrow{\text{the}} (p, 1)$,
  - a transition $(p, 1) \xrightarrow{\text{man}} (p, 2)$,
  - a transition $(p, 1) \xrightarrow{\text{machine}} (p, 2)$;
- a transition $(\text{left}, 2) \xrightarrow{\text{simulates}} (\text{right}, 0)$;
- a transition $(\text{left}, 2) \xrightarrow{\text{mimics}} (\text{right}, 0)$.

The automaton for our very simple language is represented graphically in Fig. 1.8. It is obvious that this automaton accepts the same sequences of symbols as the finite transition system obtained from this automaton by replacing the symbols a, the, man, machine, simulates and mimics by actions of reading these symbols.

Conversely, we can also view any finite transition system as an automaton by regarding its actions as symbols and its terminal states as final states. This is interesting because the sequences of actions it can consecutively perform are an important aspect of the behaviour of a process. We will get back to that later in Section 1.6. Here is an example that illustrates the potential usefulness of focussing on the sequences of actions that a system can consecutively perform.

**Example 1.4.2 (Unreliable bounded counter).** We consider an unreliable version of the bounded counter with bound $k$ from Example 1.1.2. It gets into an error state by performing an increment by 1 when its bound is reached.
We have one additional state, err, and the additional transition \( k \xrightarrow{inc} err \).

More precisely, the behaviour of the unreliable bounded counter with bound \( k \) is described by the transition system \((S, A, \rightarrow, s_0)\) where

\[
S = \{i \in \mathbb{N} \mid i \leq k\} \cup \{err\},
A = \{inc, dec\},
\rightarrow = \{(i, inc, i + 1) \mid i \in \mathbb{N}, i < k\} \cup \{(k, inc, err)\}
\cup \{(i + 1, dec, i) \mid i \in \mathbb{N}, i < k\},
s_0 = 0
\]

This transition system has only one terminal state, viz. err. The sequences of actions that lead to this state are exactly the sequences \( w \) that satisfy the following conditions:

- \( n_{inc}(w) - n_{dec}(w) = k + 1 \),
- for all proper prefixes \( v \) of \( w \), \( 0 \leq n_{inc}(v) - n_{dec}(v) \leq k \);

where \( n_a(u) \) stands for the number of occurrences of action \( a \) in sequence \( u \).

This description of the sequences of actions that lead to its terminal state may be regarded as the specification of the intended system.

If we designate the terminal state as final state, the transition system can be viewed as an automaton recognizing the language on the alphabet \( \{inc, dec\} \) that consists of the sequences \( w \in \{inc, dec\}^* \) satisfying the conditions just mentioned. When viewing the transition system as an automaton, the point is that inc and dec are considered to be symbols to be read instead of actions to be performed.

The following is known from automata theory. The languages that can be accepted by an automaton as defined here, i.e. a non-deterministic finite accepter, are exactly the regular languages. Intuitively, a regular language has a structure simple enough that a limited memory is sufficient to accept all its sentences. Many actual languages are not regular. Broader language categories include the context-free languages and the context-sensitive languages. They can be accepted by automata of more powerful kinds: non-deterministic pushdown accepters for context-free languages and linear bounded accepters for context-sensitive languages. Those kinds of automata are in turn closely related to restricted kinds of infinite transition systems.

### 1.5 Petri nets and transition systems

For a better understanding of the notion of a transition system, we looked in the previous two sections into its connections with the familiar notions of a program and an automaton. For the interested reader, we now look into its connections with the notion of a Petri net. Sometimes, the notion of a Petri net is considered to be the fundamental notion for the description of process behaviour. We believe that it is too complicated to be acceptable as
a fundamental notion. However, there are many applications of Petri nets in a wide variety of areas. The central developments of more than fifty years of Petri net theory and practice are presented in [19].

The notion of a Petri net is essentially a generalization of the notion of a transition system. In this section, we restrict our attention to the kind of Petri nets known as place/transition nets with arc weight 1. They are illustrative for almost any other kind of Petri nets. If no confusion can arise, we will call them simply nets. The crucial difference between nets and transition systems is the following. In transition systems, choices between behaviours and sequentiality of behaviours are regarded as the basic aspects of process behaviour, whereas in nets, concurrency of behaviours is also regarded as a basic aspect of process behaviour. How concurrency can be dealt with in the setting of transition systems is treated in Chap. 2. Nets support the direct description of concurrency because they can deal with states that are distributed over several places. We do not give the standard definition of the notion of a net. Our definition, which is taken from [17], underlines the similarities between transition systems and nets.

**Definition 1.5.1 (Net).** A net $N$ is a quadruple $(P, A, \rightarrow, m_0)$ where

- $P$ is a set of places;
- $A$ is a set of actions;
- $\rightarrow \subseteq (\mathcal{P}_{\text{fin}}(P) \setminus \emptyset) \times A \times (\mathcal{P}_{\text{fin}}(P) \setminus \emptyset)$ is a set of transitions;
- $m_0 \in \mathcal{P}_{\text{fin}}(P) \setminus \emptyset$ is the initial marking.

Let $t$ be the transition $Q \xrightarrow{a} Q'$. Then the preset of $t$, written $\text{pre}(t)$, is $Q$; the postset of $t$, written $\text{post}(t)$, is $Q'$; and the action of $t$, written $\text{act}(t)$, is $a$.

In the standard definition of the notion of a place/transition net, a net has a set $T$ of transitions which are not necessarily composed of their preset, postset and action. The pre- and postsets of each transition is in the standard definition given by a flow relation $F \subseteq (P \times T) \cup (T \times P)$ and the action of each transition by a labeling function $\ell : T \rightarrow A$. Moreover, there is a arc weight function $W : F \rightarrow \mathbb{N}$ in the standard definition. Because, we restrict ourselves to the case where the arc weight is invariably 1, the arc weight function is superfluous. If we take $T = \rightarrow$, $F$ such that $(p, Q \xrightarrow{a} Q') \in F$ if and only if $p \in Q$ and $(Q \xrightarrow{a} Q', p) \in F$ if and only if $p \in Q'$, and $\ell$ such that $\ell(Q \xrightarrow{a} Q') = a$, then we get a place/transition net according to the standard definition.

If we regard singleton sets of places as states, transition systems are nets where the presets, postsets and initial marking are singleton sets. A net can be considered to distribute the states of a transition system over several places as follows. Each place contains zero, one or more tokens. The numbers of tokens contained in the different places make up the states of a net, also called markings. A transition $t$ is firable in a marking if there is at least one
token in each place from the preset of \( t \). By firing \( t \), one token is removed from each place from the preset of \( t \) and one token is inserted in each place from the postset of \( t \). This informal explanation can be made more precise as follows.

Let \( N \) be the net \( (P,A,\rightarrow,m_0) \). Then a marking of \( N \) is a multiset of places, i.e. a function \( m : P \rightarrow \mathbb{N} \). A transition \( t \) of \( N \) is firable in a marking \( m \) if \( m(p) > 0 \) for all \( p \in \text{pre}(t) \). If transition \( t \) is firable in marking \( m \), the firing of \( t \) in \( m \) produces the unique marking \( m' \) such that for all \( p \in P \):

\[
m'(p) = \begin{cases} 
  m(p) - 1 & \text{if } p \in \text{pre}(t) \text{ and } p \not\in \text{post}(t), \\
  m(p) + 1 & \text{if } p \not\in \text{pre}(t) \text{ and } p \in \text{post}(t), \\
  m(p) & \text{otherwise}.
\end{cases}
\]

The notation \( m \xrightarrow{t} m' \) is used to indicate that firing transition \( t \) in marking \( m \) produces marking \( m' \). A set \( Q \subseteq P \) is identified with the unique marking \( m \) such that \( m(p) = 1 \) if \( p \in Q \) and \( m(p) = 0 \) otherwise.

Let \( N \) be the net \( (P,A,\rightarrow,m_0) \) and \( \sigma \in A^* \). The notation \( m \xrightarrow{\sigma} m' \) is used to indicate that there are markings \( m_1, \ldots, m_{n+1} \) and transitions \( t_1, \ldots, t_n \) such that \( m_1 \xrightarrow{t_1} m_2, \ldots, m_n \xrightarrow{t_n} m_{n+1} \), \( m_1 = m \), \( m_{n+1} = m' \) and \( \sigma = \text{act}(t_1) \ldots \text{act}(t_n) \). A marking \( m \) of \( N \) is called a reachable marking of \( N \) if there is a \( \sigma \in A^* \) such that \( m_0 \xrightarrow{\sigma} m \). Reachable markings make an important link between nets and transition systems.

The transition system describing the behaviour of a net is defined as follows. Let \( N \) be the net \( (P,A,\rightarrow,m_0) \) and \( M \) be the set of reachable markings of \( N \). Then the transition system associated with \( N \) is the transition system \( \mathcal{T}(N) = (M,A,\rightarrow',m_0) \) where \( m_1 \xrightarrow{t} m_2 \) iff there exists a transition \( t \) of \( N \) such that \( m_1 \xrightarrow{t} m_2 \) and \( \text{act}(t) = a \).

Let us look at an example of the use of nets in describing process behaviour.

**Example 1.5.1 (Binary memory cell).** We consider a binary memory cell. A binary memory cell holds at any moment either the value 0 or the value 1. Initially, it holds the value 0. The binary memory cell can store a value and retrieve its value. Its behaviour can be described by a net as follows. As places of the binary memory cell, we have the pairs \( (b, \text{rtr}) \), \( (b, \text{sto}) \) for \( b = 0, 1 \). If its marking includes the place \( (b, \text{rtr}) \), the cell can retrieve the value \( b \). If its marking includes the place \( (b, \text{sto}) \), the cell can store the value \( b \). If its marking includes both \( (b, \text{rtr}) \) and \( (b, \text{sto}) \), the cell can store the value \( 1 - b \). As initial marking, we have \( \{(0, \text{rtr}), (0, \text{sto})\} \). As actions, we have \( \text{sto}(b) \) (store \( b \)) and \( \text{rtr}(b) \) (retrieve \( b \)) for \( b = 0, 1 \). As transitions, we have the following (for \( b = 0, 1 \)):

\[
\begin{align*}
\{(b, \text{rtr})\} & \xrightarrow{\text{rtr}(b)} \{(b, \text{rtr})\}, \\
\{(b, \text{sto})\} & \xrightarrow{\text{sto}(b)} \{(b, \text{sto})\}, \\
\{(b, \text{rtr}), (b, \text{sto})\} & \xrightarrow{\text{sto}(1-b)} \{(1-b, \text{rtr}), (1-b, \text{sto})\}.
\end{align*}
\]
The transition system associated with this net is as follows. As states, we have the markings \{ (b, rtr), (b, sto) \} for b = 0, 1, with \{ (0, rtr), (0, sto) \} as the initial state. As actions, we still have sto(b) and rtr(b) for b = 0, 1. As transitions, we have the following (for b = 0, 1):

\[
\begin{align*}
\{(b, rtr), (b, sto)\} &\xrightarrow{rtr(b)} \{(b, rtr), (b, sto)\}, \\
\{(b, rtr), (b, sto)\} &\xrightarrow{sto(b)} \{(b, rtr), (b, sto)\}, \\
\{(b, rtr), (b, sto)\} &\xrightarrow{sto(1-b)} \{(1-b, rtr), (1-b, sto)\}.
\end{align*}
\]

The transition system for the binary memory cell does not indicate that if both rtr(b) and sto(b) can occur, they can also occur simultaneously. This can be covered as well if we generalize transition systems by taking multisets of actions as labels of transitions. We will not discuss this generalization in these lecture notes.

Let us look at one more example of the use of nets in describing process behaviour.

**Example 1.5.2 (Milner’s scheduling problem).** We consider the system of scheduled processes from Milner’s scheduling problem (see [14]). It consists of processes \( P_1, \ldots, P_n \) \((n > 1)\), each wishing to perform a certain task repeatedly, and a scheduler ensuring that they start their task in cyclic order, beginning with \( P_1 \). The behaviour of this system can be described by a net as follows. As places of the system, we have the pairs \( (i, \text{idle}), (i, \text{busy}), (i, \text{sch}) \) for \( 1 \leq i \leq n \). If its marking includes both \( (i, \text{idle}) \) and \( (i, \text{sch}) \), process \( P_i \) can start performing its task. If its marking includes \( (i, \text{busy}) \), process \( P_i \) can finish performing its task. As initial marking, we have \( \{ (1, \text{idle}), \ldots, (n, \text{idle}), (1, \text{sch}) \} \). As actions, we have start\((i)\) (start task \( i \)) and finish\((i)\) (finish task \( i \)) for \( 1 \leq i \leq n \). As transitions, we have the following (for \( 1 \leq i \leq n \)):

\[
\begin{align*}
\{(i, \text{idle}), (i, \text{sch})\} &\xrightarrow{\text{start}(i)} \{(i, \text{busy}), (nxt(i), \text{sch})\}, \\
\{(i, \text{busy})\} &\xrightarrow{\text{finish}(i)} \{(i, \text{idle})\},
\end{align*}
\]

where \( \text{nxt}(i) = i + 1 \) if \( i < n \) and \( \text{nxt}(n) = 1 \). The behaviour of the system is much easier to grasp from this net than from the transition system associated with the net because the structure of the system is clearly reflected in the net.

The net for the system of scheduled processes is represented graphically in Fig. 1.9 for the case where \( n = 3 \). The places and transitions are represented as follows. Places \( p \) are represented as circles and transitions \( t \) as boxes labeled with \( \text{act}(t) \) and connected via directed arcs to the circles representing the places in \( \text{pre}(t) \) and \( \text{post}(t) \). The initial marking is represented by putting a bullet into the circles representing the places that are in the initial marking.
1.6 Equivalences on transition systems

In this section, we look at a couple of notions that are taken up to abstract from those details of transition systems that are often supposed to be irrelevant.

Usually, transition systems show details that are not considered to be relevant to the behaviour of processes. There are, for example, applications of transition systems where only the sequences of actions that can be performed consecutively starting from the initial state of a transition system, called the traces of the transition system, matter. Here is a simple example of a case where only the traces matter.

**Example 1.6.1 (Bounded counter).** We consider again the bounded counter with bound $k$ from Example 1.1.2. Its traces are exactly the traces $w$ for which the following condition holds: for all prefixes $v$ of $w$, $0 \leq n_{inc}(v) - n_{dec}(v) \leq k$. This description of its traces expresses all we expect from the bounded counter: we regard any transition system that has those traces as a bounded counter. For this reason, only the traces are relevant in this case.

Notice that in all cases where a transition system is used to accept a language, as described in Section 1.4, only the traces are relevant.

In all those cases where only the traces of the transition system matter, it is useful to ignore all other details. This is done by identifying transition systems that have the same set of traces. Such transition systems are called trace equivalent. Here is a precise definition.

**Definition 1.6.1 (Trace).** Let $T = (S, A, \rightarrow, s_0)$ be a transition system. A trace of $T$ is a sequence $\sigma \in A^*$ such that $s_0 \xrightarrow{} \sigma s$ for some $s \in S$. We write traces($T$) for the set of all traces of $T$. Then two transition systems $T$ and $T'$ are trace equivalent, written $T \equiv_{tr} T'$, if traces($T$) = traces($T'$).
1. Transition Systems

We will see below that there are also cases where not only the traces of the transition system matter. In those cases, trace equivalence is obviously not the right equivalence to make use of.

There exist different viewpoints on what should be considered relevant to the behaviour of processes. The equivalence known as bisimulation equivalence is based on the idea that not only the traces of equivalent transition systems should coincide, but also the stages at which the choices of different possibilities occur. Therefore, bisimulation equivalence is said to preserve the branching structure of transition systems. Here is an example of a case where apparently not only the traces matter, but also the stages at which the choices of different possibilities occur.

Example 1.6.2 (Split connection). We consider a split connection between nodes in a network (see e.g. [7, 20]). A split connection has one input port and two output ports. A datum that has been consumed at the input port can be delivered at either of the output ports. That is, the choice of the output ports is resolved after the datum has been consumed. The behaviour of a split connection with input port $k$ and output ports $l$ and $m$ can be described as follows. We assume a set of data $D$. As states of the split connection, we have $\ast$ and the data $d \in D$, with $\ast$ as initial state. As actions, we have $s_i(d)$ (send $d$ at port $i$) and $r_i(d)$ (receive $d$ at port $i$) for $i = k, l, m$ and $d \in D$. As transitions, we have the following:

- for each $d \in D$, a transition $\ast \xrightarrow{r_k(d)} d$;
- for each $i \in \{l, m\}$ and $d \in D$, a transition $d \xrightarrow{s_i(d)} \ast$.

The transition system for the split connection is represented graphically in Fig. 1.10 for the case where $D = \{0, 1\}$. Next we consider a transition system that is trace equivalent to the one just presented. As states, we have the pairs $(i, d)$ for $i = k, l, m$ and $d \in D \cup \{\ast\}$, with $(k, \ast)$ as initial state. As actions, we still have $s_i(d)$ and $r_i(d)$ for $i = k, l, m$ and $d \in D$. As transitions, we have the following:

- for each $i \in \{l, m\}$ and $d \in D$: $(k, \ast) \xrightarrow{r_k(d)} (i, d), (i, d) \xrightarrow{s_i(d)} (k, \ast)$.

This transition system is represented graphically in Fig. 1.11 for the case where $D = \{0, 1\}$. This transition system does not describe the intended
behaviour of the split connection correctly. A datum that has been consumed
cannot be delivered at either of the output ports because the choice of the
output ports is resolved at the instant that the datum is consumed. So, we
do not want to identify this transition system with the previous one. They
are not identified by bisimulation equivalence.

What is exactly meant by “the stages at which the choices of different
possibilities occur” in our intuitive explanation of bisimulation equivalence
becomes clear in the following informal definition. Two transition systems $T$
and $T'$ are bisimulation equivalent if their states can be related such that:

- the initial states are related;
- if states $s_1$ and $s'_1$ are related and in $T$ a transition with label $a$ is possible
  from $s_1$ to some $s_2$, then in $T'$ a transition with label $a$ is possible from $s'_1$
  to some $s'_2$ such that $s_2$ and $s'_2$ are related;
- likewise, with the role of $T$ and $T'$ reversed.

This means that, starting from any pair of related states, $T$ can simulate $T'$
and conversely $T'$ can simulate $T$.

Bisimulation equivalence can also be characterized as follows: it identifies
transition systems if they cannot be distinguished by any conceivable
experiment with an experimenter that is only able to detect which actions
are performed at any stage. The kind of identifications made by bisimulation
equivalence is illustrated with the following example.

**Example 1.6.3 (Merge connection).** We consider a merge connection between
nodes in a network (see e.g. [7, 20]). A merge connection has two input ports
and one output port. Each datum that has been consumes at one of the input
ports is delivered at the output port. The behaviour of a merge connection

![Transition system for the split-like connection](image-url)
with input ports $k$ and $l$ and output port $m$ can be described as follows. We assume a set of data $D$. As states, we have the pairs $(i, d)$ for $i = k, l, m$ and $d \in D \cup \{\ast\}$, with $(m, \ast)$ as initial state. As actions, we have again $s_i(d)$ and $r_i(d)$ for $i = k, l, m$ and $d \in D$. As transitions, we have the following:

- for each $i \in \{k, l\}$ and $d \in D$: $(m, \ast) \xrightarrow{r_i(d)} (i, d), (i, d) \xrightarrow{s_i(d)} (m, \ast)$.

This transition system for the merge connection is represented graphically in Fig. 1.12 for the case where $D = \{0, 1\}$. Next we consider the following transition system. As states, we have $\ast$ and the data $d \in D$, with $\ast$ as initial state. As actions, we still have $s_i(d)$ and $r_i(d)$ for $i = k, l, m$ and $d \in D$. As transitions, we have the following:

- for each $i \in \{k, l\}$ and $d \in D$, a transition $\ast \xrightarrow{r_i(d)} d$;
- for each $d \in D$, a transition $d \xrightarrow{s_m(d)} \ast$.

This transition system is represented graphically in Fig. 1.13 for the case where $D = \{0, 1\}$. This transition system describes the intended behaviour of the merge connection correctly as well. Is this transition system identified
1.6 Equivalences on transition systems

with the previous one by bisimulation equivalence? Yes, it is: relate state \((m, \ast)\) to state \(\ast\) and, for each \(i \in \{k, l\}\) and \(d \in D\), state \((i, d)\) to state \(d\).

Let us now give the formal definition of bisimulation equivalence.

**Definition 1.6.2 (Bisimulation).** Let \(T = (S, A, \rightarrow, s_0)\) and \(T' = (S', A', \rightarrow', s'_0)\) be transition systems such that \(A = A'\). Then a **bisimulation** \(B\) between \(T\) and \(T'\) is a binary relation \(B \subseteq S \times S'\) such that the following conditions hold:

1. \(B(s_0, s'_0)\);
2. whenever \(B(s_1, s'_1)\) and \(s_1 \xrightarrow{\sigma} s_2\), then there is a state \(s'_2\) such that \(s'_1 \xrightarrow{\sigma'} s'_2\) and \(B(s_2, s'_2)\);
3. whenever \(B(s_1, s'_1)\) and \(s'_1 \xrightarrow{\sigma'} s'_2\), then there is a state \(s_2\) such that \(s_1 \xrightarrow{\sigma} s_2\) and \(B(s_2, s'_2)\).

The two transition systems \(T\) and \(T'\) are **bisimulation equivalent**, written \(T \equiv T'\), if there exists a bisimulation \(B\) between \(T\) and \(T'\). A bisimulation between \(T\) and \(T\) is called an **autobisimulation** on \(T\).

Restriction to relations \(B\) between the reachable states of \(T\) and the reachable states of \(T'\) does not change the notion of bisimulation equivalence.

Let us return to the experimenter that is only able to detect which actions are performed at any stage. If performing the same experiment on a system more than once leads to the same outcome for all his (or her) experiments, the system behaves predictably. Such a system is called determinate. This is an important notion in the design of a system. In many cases, we have to arrive at a determinate system from components of which some are not determinate. This is, for example, the case with the simple data communication protocol treated in the next chapter. Here is the precise definition of determinacy.

**Definition 1.6.3 (Determinacy).** Let \(T = (S, A, \rightarrow, s_0)\) be a transition system. Then \(T\) is **determinate** if the following condition holds:

whenever \(s_0 \xrightarrow{\sigma} s\) and \(s_0 \xrightarrow{\sigma'} s'\), then there is an autobisimulation \(B\) on \(T\) such that \(B(s, s')\).

For determinate transition systems trace equivalence and bisimulation equivalence coincide.

**Property 1.6.1 (Determinacy).** Let \(T = (S, A, \rightarrow, s_0)\) and \(T' = (S', A', \rightarrow', s'_0)\) be transition systems such that \(A = A'\). Then the following holds:

if \(T\) and \(T'\) are determinate, then \(T \equiv T'\) if and only if \(T \equiv_{tr} T'\).

The notion of determinism of a transition system is closely related to the notion of determinacy of a transition system.

**Definition 1.6.4 (Determinism).** Let \(T = (S, A, \rightarrow, s_0)\) be a transition system. Then \(T\) is **deterministic** if the following condition holds:
whenever $s_0 \xrightarrow{\sigma} s$ and $s_0 \xrightarrow{\sigma} s'$, then $s = s'$.

It is easy to see that all deterministic transition systems are determinate, but not all determinate transition systems are deterministic. One could say that a determinate transition system is deterministic up to bisimulation.

In this section, we have shortly introduced the use of equivalences for abstraction from details of transition systems that we want to ignore. This plays a prominent part in techniques for the analysis of process behaviour. We will come back to trace and bisimulation equivalence later.
2. Concurrency and Interaction

Complex systems are generally composed of a number of components that act concurrently and interact with each other. This chapter deals with the issue of concurrency and interaction by introducing the notion of parallel composition of transition systems. First of all, we explain informally what parallel composition of transition systems is and give a simple example of its use in describing process behaviour (Sect. 2.1). After that, we define the notion of parallel composition of transition systems in a mathematically precise way (Sect. 2.2). For a better understanding, we next investigate the connections between the notion of parallel composition of transition systems and the more familiar notion of parallel execution of programs (Sect. 2.3). We also describe a typical example of a real-life system composed of components that act concurrently and interact with each other, viz. a simple data communication protocol, using parallel composition of transition systems (Sect. 2.4). For the interested reader, we relate the notion of parallel composition of transition systems with the notion of parallel composition of nets (Sect. 2.5). Finally, we have another look at trace equivalence and bisimulation equivalence (Sect. 2.6).

2.1 Informal explanation

Sending a message to another component and receiving a message from another component are typical examples of the kinds of actions that are performed by a component of a system in order to interact with other components that act concurrently. Synchronous communication of a message between two components is a typical example of an interaction that takes place when a send action of one component and a matching receive action of the other component are performed synchronously. When two actions are performed synchronously, those actions cannot be observed separately. Therefore, the intuition is that only one action is left when two actions are performed synchronously. For instance, when a send action and a matching receive action are performed synchronously, only a communication action can be observed. It does not have to be the case that any two actions can be performed synchronously. Usually, two action can be performed synchronously.
only if they can establish an interaction. That is, for example, not the case for two send actions.

Now consider the use of transition systems in describing the behaviour of systems. In the case where a system is composed of components that act concurrently and interact with each other, we would like to reflect the composition in the description of the behaviour of the system. That is, we would like to use transition systems to describe the behaviour of the components and to be able to describe the behaviour of the whole system by expressing that its transition system is obtained from the transition systems describing the behaviour of the components by applying a certain operation to those transition systems. Parallel composition of transition systems as introduced in this chapter serves this purpose. The intuition is that the parallel composition of two transition systems $T$ and $T'$ can perform at each stage any action that $T$ can perform next, any action that $T'$ can perform next, and any action that results from synchronously performing an action that $T$ can perform next and an action that $T'$ can perform next. Parallel composition does not prevent actions that can be performed synchronously from being performed on their own. In order to prevent certain actions from being performed on their own, we introduce a separate operation on transition systems, called encapsulation. The reason why parallel composition and encapsulation are not combined in a single operation will be explained later at the end of Sect. 2.2. Here is an example of the use of parallel composition and encapsulation in describing the behaviour of systems composed of components that act concurrently and interact with each other.

**Example 2.1.1 (Bounded buffers).** We consider the system composed of two bounded buffers, buffer 1 and buffer 2, where each datum removed from the data kept in buffer 1 is simultaneously added to the data kept in buffer 2. In this way, data from buffer 1 is transferred to buffer 2. We start from the bounded buffers from Example 1.1.3. In the case of buffer 1, we rename the actions $\text{add}(d)$ and $\text{rem}(d)$ into $\text{add}_1(d)$ and $\text{rem}_1(d)$, respectively. In the case of buffer 2, we rename the actions $\text{add}(d)$ and $\text{rem}(d)$ into $\text{add}_2(d)$ and $\text{rem}_2(d)$, respectively. In this way, we can distinguish between the action of adding a datum to the data kept in one buffer and the action of adding the same datum to the data kept in the other buffer, as well as between the action of removing a datum from the data kept in one buffer and the action of removing the same datum from the data kept in the other buffer.

The renamings yield the following. As states of bounded buffer $i$, $i = 1, 2$, with capacity $l_i$, we have the sequences of data of which the length is not greater than $l_i$. As initial state, we have the empty sequence. As actions, we have $\text{add}_i(d)$ and $\text{rem}_i(d)$ for each datum $d$. As transitions of bounded buffer $i$, we have the following:

- for each datum $d$ and each state $\sigma$ that has a length less than $l_i$, a transition $\sigma \xrightarrow{\text{add}_i(d)} d \sigma$.
2.1 Informal explanation

Informal explanation $(\epsilon, \epsilon) (0, \epsilon) (1, \epsilon) (\epsilon, 0) (\epsilon, 1)$ $(1, 0) (1, 1)$

Fig. 2.1. Transition system for parallel composition of bounded buffers

- for each datum $d$ and each state $\sigma_d$, a transition $\sigma_d \xrightarrow{\text{rem}_i(d)} \sigma$.

In the case where, for each datum $d$, the actions $\text{rem}_1(d)$ and $\text{add}_2(d)$ can be performed synchronously, and $\text{trf}(d)$ (transfer $d$) is the action left when these actions are performed synchronously, parallel composition of buffer 1 and buffer 2 results in the following transition system. As states, we have pairs $(\sigma_1, \sigma_2)$ where $\sigma_i (i = 1, 2)$ is a sequence of data of which the length is not greater than $l_i$. State $(\sigma_1, \sigma_2)$ is the state in which the sequence of data $\sigma_i (i = 1, 2)$ is kept in buffer $i$. As initial state, we have $(\epsilon, \epsilon)$. As actions, we have $\text{add}_i(d)$, $\text{rem}_i(d)$ and $\text{trf}(d)$ for each datum $d$ and $i = 1, 2$. As transitions, we have the following:

- for each datum $d$ and each state $(\sigma_1, \sigma_2)$ with the length of $\sigma_1$ less than $l_1$, a transition $(\sigma_1, \sigma_2) \xrightarrow{\text{add}_1(d)} (d \sigma_1, \sigma_2)$;
- for each datum $d$ and each state $(\sigma_1, \sigma_2)$ with the length of $\sigma_2$ less than $l_2$, a transition $(\sigma_1, \sigma_2) \xrightarrow{\text{add}_2(d)} (\sigma_1, d \sigma_2)$;
- for each datum $d$ and each state $(\sigma_1, \sigma_2)$, a transition $(\sigma_1, \sigma_2) \xrightarrow{\text{rem}_1(d)} (\sigma_1, \sigma_2)$;
- for each datum $d$ and each state $(\sigma_1, \sigma_2)$, a transition $(\sigma_1, \sigma_2) \xrightarrow{\text{rem}_2(d)} (\sigma_1, \sigma_2)$;
- for each datum $d$ and each state $(\sigma_1, \sigma_2)$ with the length of $\sigma_2$ less than $l_2$, a transition $(\sigma_1, \sigma_2) \xrightarrow{\text{trf}(d)} (\sigma_1, d \sigma_2)$.

This transition system is represented graphically in Fig. 2.1 for the case where $l_1 = l_2 = 1$ and the only data involved are the natural numbers 0 and 1. For each datum $d$, actions $\text{rem}_1(d)$ and $\text{add}_2(d)$ can still be performed on their
Encapsulation with respect to these actions prevents them from being performed on their own, i.e., it results in the following transition system. We have the same states as before. As actions, we have $\text{add}_1(d)$, $\text{rem}_2(d)$, and $\text{trf}(d)$ for each datum $d$. As transitions, we have the following:

- for each datum $d$ and each state $(\sigma_1, \sigma_2)$ with the length of $\sigma_1$ less than $l_1$, a transition $(\sigma_1, \sigma_2) \xrightarrow{\text{add}_1(d)} (d\sigma_1, \sigma_2)$;

- for each datum $d$ and each state $(\sigma_1, \sigma_2 d)$, a transition $(\sigma_1, \sigma_2 d) \xrightarrow{\text{rem}_2(d)} (\sigma_1, \sigma_2)$;

- for each datum $d$ and each state $(\sigma_1 d, \sigma_2)$ with the length of $\sigma_2$ less than $l_2$, a transition $(\sigma_1 d, \sigma_2) \xrightarrow{\text{trf}(d)} (\sigma_1, d\sigma_2)$.

This transition system is represented graphically in Fig. 2.2 for the case where $l_1 = l_2 = 1$ and the only data involved are the natural numbers 0 and 1. So encapsulation is needed to prevent that the actions $\text{rem}_1(d)$ and $\text{add}_2(d)$ do not lead to transfer of datum $d$ from buffer 1 to buffer 2. The transition system obtained from the two bounded buffers by parallel composition and encapsulation would be bisimulation equivalent (see Sect. 1.6) to a bounded buffer with capacity $l_1 + l_2$ if we could abstract from the internal transfer actions $\text{trf}(d)$. Abstraction from internal actions is one of the issues treated in the remaining chapters of these lecture notes.

Although systems composed of bounded buffers that act concurrently and interact with each other as described above actually arise in computer-based systems, they are not regarded as typical examples of real-life computer-based systems composed of components that act concurrently and interact with each other. Later, in Sect. 2.4 we give a fairly typical example, viz.
2.2 Formal definitions

With the previous section, we have prepared the way for the formal definitions of the notions of parallel composition of transition systems and encapsulation of a transition system.

Whether two actions can be performed synchronously, and if so what action is left when they are performed synchronously, is mathematically represented by a communication function. Here is the definition of a communication function.

**Definition 2.2.1 (Communication function).** Let $A$ be a set of actions. A communication function on $A$ is a partial function $\gamma : A \times A \rightarrow A$ satisfying for $a, b, c \in A$:

- if $\gamma(a, b)$ is defined, then $\gamma(b, a)$ is defined and $\gamma(a, b) = \gamma(b, a)$;
- if $\gamma(a, b)$ and $\gamma(\gamma(a, b), c)$ are defined, then $\gamma(b, c)$ and $\gamma(a, \gamma(b, c))$ are defined and $\gamma(\gamma(a, b), c) = \gamma(a, \gamma(b, c))$.

The reason for the first condition is evident: there should be no difference between performing $a$ and $b$ synchronously and performing $b$ and $a$ synchronously. The reason for the second condition is essentially the same, but for the case where more than two actions can be performed synchronously. Let us give an example to illustrate that it is straightforward to define the communication function needed.

**Example 2.2.1 (Bounded buffers).** We consider again the parallel composition of bounded buffers from Example 2.1.1. In that example, for each datum $d$, the actions $\text{rem}_1(d)$ and $\text{add}_2(d)$ can be performed synchronously, and $\text{trf}(d)$ is the action left when these actions are performed synchronously. This is simply represented by the communication function $\gamma$ defined such that $\gamma(\text{rem}_1(d), \text{add}_2(d)) = \gamma(\text{add}_2(d), \text{rem}_1(d)) = \text{trf}(d)$ for each datum $d$, and it is undefined otherwise.

Let us now look at the formal definitions of parallel composition and encapsulation.

**Definition 2.2.2 (Parallel composition).** Let $T = (S, A, \rightarrow, s_0)$ and $T' = (S', A', \rightarrow', s'_0)$ be transition systems. Let $\gamma$ be a communication function on a set of actions that includes $A \cup A'$. The parallel composition of $T$ and $T'$ under $\gamma$, written $T \parallel \gamma T'$, is the transition system $(S'', A'', \rightarrow'', s''_0)$ where

- $S'' = S \times S'$;
- $A'' = A \cup A' \cup \{\gamma(a, a') | a \in A, a' \in A', \gamma(a, a') \text{ is defined}\}$;
- $\rightarrow''$ is the smallest subset of $S'' \times A'' \times S''$ such that:
- if \( s_1 \xrightarrow{a} s_2 \) and \( s' \in S' \), then \((s_1, s') \xrightarrow{a''} (s_2, s')\);  
- if \( s'_1 \xrightarrow{b'} s'_2 \) and \( s \in S \), then \((s, s'_2) \xrightarrow{\gamma} (s, s'_2)\);  
- if \( s'_1 \xrightarrow{a} s_2 \), \( s'_2 \xrightarrow{b'} s'_3 \) and \( \gamma(a, b) \) is defined, then \((s_1, s'_1) \xrightarrow{\gamma(a, b)''} (s_2, s'_3)\);  

- \( s'_0 = (s_0, s'_0) \).

We use the convention of association to the left for parallel composition to reduce the number of parentheses, e.g. we write \( T_1 || T_2 || T'_3 \) for \( (T_1 || T_2) || T'_3 \).

**Definition 2.2.3 (Encapsulation).** Let \( T = (S, A, \rightarrow, s_0) \) be a transition system. Let \( H \subseteq A \). The *encapsulation* of \( T \) with respect to \( H \), written \( \partial_H(T) \), is the transition system \((S', A', \rightarrow', s_0)\) where:

- \( S' = \{ s \mid \text{for some } \sigma \in (A \setminus H)^*: s_0 \xrightarrow{\sigma} s \} \);
- \( A' = \{ a \in A \setminus H \mid \text{for some } s_1, s_2 \in S': s_1 \xrightarrow{a} s_2 \} \);
- \( \rightarrow' \) is the smallest subset of \( S' \times A' \times S' \) such that:
  - if \( s_1 \xrightarrow{a} s_2 \), \( s_1 \in S' \) and \( a \not\in H \), then \( s_1 \xrightarrow{\sigma a'} s_2 \).

In many applications, \( \gamma(a, b, c) \) is undefined for all \( a, b, c \in A \). That case is called *handshaking communication*. We introduce some standardized terminology and notation for handshaking communication. Transition systems send, receive and communicate data at *ports*. If a port is used for communication between two transition systems, it is called *internal*. Otherwise, it is called *external*. We write:

- \( s_i(d) \) for the action of sending datum \( d \) at port \( i \);
- \( r_i(d) \) for the action of receiving datum \( d \) at port \( i \);
- \( c_i(d) \) for the action of communicating datum \( d \) at port \( i \).

Assuming a set of data \( D \), the communication function is defined such that:

\[
\gamma(s_i(d), r_i(d)) = \gamma(r_i(d), s_i(d)) = c_i(d)
\]

for all \( d \in D \), and it is undefined otherwise.

It is important to remember that handshaking communication is just one kind of communication. It is not required that \( \gamma(a, b, c) \) is undefined for all \( a, b, c \in A \). Here is an example of another kind of communication.

**Example 2.2.2 (Non-handshaking communication).** We consider a kind of communication in which three transition systems participate. A communication of this kind takes place by synchronously performing one send action and two matching receive actions. Using a notation which is reminiscent of the standardized notation for handshaking communication, this ternary kind of communication can be represented by a communication function as follows.

Assuming a set of data \( D \), the communication function is defined such that:

\[
\gamma(r_i(d), r_i(d)) = r_r(d), \quad \gamma(s_i(d), r_i(d)) = \gamma(r_i(d), s_i(d)) = s_r(d), \quad \gamma(s_i(d), r_i(d)) = \gamma(r_i(d), s_i(d)) = c_i(d), \quad \gamma(s_r(d), r_i(d)) = \gamma(r_i(d), s_r(d)) = c_i(d),
\]
for all \( d \in D \), and it is undefined otherwise. The actions \( \text{sr}_i(d) \) and \( \text{rr}_i(d) \) represent the possible partial communications.

An important thing to note about the kind of communication treated in the preceding example is the following. If parallel composition and encapsulation were combined in a single operation that prevents actions that can be performed synchronously from being performed on their own, this kind of communication would be excluded.

2.3 Programs and parallel composition

For about thirty five years, there are programming languages in which it can be expressed that a number of (sequential) subprograms must be executed in parallel. What exactly does that mean? Can it be described in a straightforward way by means of transition systems using parallel composition? It turns out that the answers to these questions do not only depend on whether one abstracts from the processing of actions by a machine, but also on the way in which the programming language used supports interaction between subprograms executed in parallel. Roughly speaking, the basic ways of interaction are:

- by synchronous communication, i.e. communication where the sending subprogram must wait till each receiving subprogram (usually one) is ready to participate in the communication;
- by asynchronous communication, i.e. communication where the sending subprogram does not have to wait till each receiving subprogram (usually one) is ready to participate in the communication;
- via shared variables, i.e. program variables to which more than one subprogram has access.

Some programming languages support a combination of these basic ways. An important thing to note is that, in virtually all programming languages that support synchronous or asynchronous communication, the data communicated may depend on the values of program variables.

In this section, we will look at the questions posed above in more detail. We do so primarily to acquire a better understanding of the notion of parallel composition of transition systems. In line with Sect. 1.3, we like to abstract initially from how the actions performed by subprograms are processed by a machine. That is, we like to focus initially on the flow of control.

Let \( T_{P_1} \) and \( T_{P_2} \) be transition systems describing the behaviour of two subprograms \( P_1 \) and \( P_2 \) upon abstract execution. If the programming language does not support synchronous communication, then the behaviour of \( P_1 \) and \( P_2 \) upon parallel abstract execution can be described by

\[
T_{P_1} \| \gamma \ T_{P_2} ,
\]
where γ is undefined for any two actions.

In order to illustrate by an example how this works, we have to choose a programming language first. Our choice is a simple extension of PASCAL introduced by Ben-Ari back in 1982 (see [5, 6]). The extension concerned simply permits to write statements of the form `COBEGIN P1; ...; Pn COEND`, where P1, ..., Pn are procedures defined in the program, in the program body to express that those procedures must be executed in parallel. Moreover, assignments and tests are indivisible and nothing else is indivisible. That is all. The extension does not support communication in a direct way. Interaction is only possible via shared variables. Let us now turn to the promised example.

**Example 2.3.1 (Peterson's protocol).** We consider a program implementing a simple mutual exclusion protocol. A mutual exclusion protocol concerns the exclusive access by components of a system to a shared resource while using that shared resource. As the saying is, a component is in its critical section while it is using the shared resource. We consider Peterson's protocol for guaranteeing that at most one component of a system is in its critical section (see [18]). The protocol assumes that there are three shared variables c0, c1 and t, with initial value ff, ff and 0, respectively, and that all assignments and tests concerning these variables are indivisible.

The idea behind the protocol is as follows. The components have sequence numbers 0 and 1. The value of t is the sequence number of the component that last started an attempt to enter its critical section. That the value of c0 is ff signifies that component 0 is not in its critical section; and that the value of c1 is ff signifies that component 1 is not in its critical section. If component 0 intends to enter its critical section it must assign the value t to c0 before it checks the value of c1, to prevent situations in which the value of both variables is ff. Analogously for component 1. This may lead to situations in which the value of both c0 and c1 is t. In order to prevent that the system becomes inactive in that case, each component checks whether the other last started an attempt to enter its critical section, and the one of which the check succeeds actually enters its critical section.

In the program that we will give below, we have taken the most simple critical sections for which the mutual exclusion problem is not trivial: a sequence of two indivisible statements. Here is the program.

```plaintext
PROGRAM peterson;
VAR
  c0, c1: boolean;
  t: 0..1;

PROCEDURE p0;
BEGIN
  WHILE true DO
    BEGIN
      c0 := true;
```
2.3 Programs and parallel composition

\[
\begin{align*}
t &:= 0; \\
\text{REPEAT UNTIL } &c_1 = \text{false OR } t = 1; \\
&\text{enter0; \{enter critical section\}} \\
&\text{leave0; \{leave critical section\}} \\
&c_0 := \text{false}; \\
\end{align*}
\]

END

PROCEDURE p1;
BEGIN
\[
\begin{align*}
\text{WHILE true DO} \\
&\text{BEGIN} \\
&c_1 := \text{true}; \\
t &:= 1; \\
&\text{REPEAT UNTIL } c_0 = \text{false OR } t = 0; \\
&\text{enter1; \{enter critical section\}} \\
&\text{leave1; \{leave critical section\}} \\
&c_1 := \text{false}; \\
\end{align*}
\]

END

BEGIN
\[
\begin{align*}
c_0 &:= \text{false}; \\
c_1 &:= \text{false}; \\
t &:= 0; \\
\text{COBEGIN } p_1; p_2 \text{ COEND}
\end{align*}
\]

END

Actually, enter0, leave0, enter1 and leave1 are no real statements. They stand for arbitrary indivisible statements that use the shared resource.

The behaviour of the procedures p0 and p1 upon abstract execution can be described by transition systems in the same way as in Examples 1.3.1 and 1.3.2. As states, we have in either case the natural numbers 0 to 7, with 0 as initial state. As actions, we have in either case an action corresponding to each atomic statement of the procedure as well as each test of the procedure and its opposite. As transitions, we have the following in the case of p0:

\[
\begin{align*}
0 &\rightarrow 1, \quad 1 \\rightarrow 2, \quad 2 \rightarrow 3, \\
3 &\rightarrow 0, \\
4 &\rightarrow 5, \quad 5 \rightarrow 6, \quad 6 \rightarrow 0, \\
0 &\rightarrow 7;
\end{align*}
\]
Fig. 2.3. Transition systems for Peterson’s protocol

and the following in the case of p1:

\[
\begin{align*}
0 & \xrightarrow{\text{true}} 1, \quad 1 \xrightarrow{c1 := \text{true}} 2, \quad 2 \xrightarrow{t := 1} 3, \\
3 & \xrightarrow{\text{NOT}(c0 = \text{false} \text{ OR } t = 0)} 3, \quad 3 \xrightarrow{c0 = \text{false} \text{ OR } t = 0} 4, \\
4 & \xrightarrow{\text{enter1}} 5, \quad 5 \xrightarrow{\text{leave1}} 6, \quad 6 \xrightarrow{c1 := \text{false}} 0, \\
0 & \xrightarrow{\text{NOT true}} 7.
\end{align*}
\]

Here, enter0, leave0, enter1 and leave1 are no real actions. They stand for the actions corresponding to the statements that enter0, leave0, enter1 and leave1 stand for.

The transition systems for the procedures p0 and p1 are represented graphically in Fig. 2.3. We call these transition systems \(T_{p0}\) and \(T_{p1}\), respectively. The behaviour of the procedures p0 and p1 upon parallel abstract execution can be described as follows:

\[T_{p0} \parallel \gamma \parallel T_{p1}\]

where the communication function \(\gamma\) is undefined for any two actions.

Notice that the preceding example is based on the idea that the parallel abstract execution of two subprograms can be reduced to arbitrary interleaving only, i.e. to performing again and again an action that one or the other of the two can perform next. This is obviously problematic in the presence of synchronous communication: simultaneously performing actions is not taken into account. However, if one abstracts from the processing of actions by
2.3 Programs and parallel composition

Let us now, like in Sect. 1.3, take into account how the actions performed by subprograms are processed by a machine and turn to the behaviour of subprograms upon parallel execution on a machine. We can describe the behaviour of machines on which subprograms are executed by transition systems as well. We will give a simple example illustrating this later. Let \( T_{P_1} \) and \( T_{P_2} \) be transition systems describing the behaviour of two subprograms \( P_1 \) and \( P_2 \) upon abstract execution. If we suppose that we also have the transition systems of the appropriate machines available, the behaviour of \( P_1 \) and \( P_2 \) upon parallel execution on a machine can in many cases best be described in one of the following ways, depending on the way in which the programming language used supports interaction between subprograms executed in parallel:

\[
\partial_H((T_{P_1} \parallel \gamma' T_{P_2}) \parallel \gamma T_M)
\]

or

\[
\partial_H(\partial_{H_1}(T_{P_1} \parallel \gamma T_{M_1}) \parallel \gamma \partial_{H_2}(T_{P_2} \parallel \gamma T_{M_2}))
\]

where the communication function \( \gamma' \) is undefined for any two actions, and the communication function \( \gamma \), the sets of actions \( H, H_1 \) and \( H_2 \), and the transition systems \( T_M, T_{M_1} \) and \( T_{M_2} \) all depend on the way in which the programming language used supports interaction between subprograms executed in parallel. The transition systems \( T_M, T_{M_1} \) and \( T_{M_2} \) are supposed to describe the behaviour of appropriate machines.

The first way of description applies if the programming language only supports shared variables as a means to interact. The second way of description applies if the programming language supports synchronous communication or asynchronous communication, but does not support shared variables. Synchronous communication can be fully represented by the communication function \( \gamma \), while asynchronous communication cannot be fully represented by the communication function (as explained below). In the case where only shared variables are supported, \( P_1 \) and \( P_2 \) are executed on the same machine: \( T_M \). In the cases where shared variables are not supported, \( P_1 \) and \( P_2 \) are executed on different machines: \( T_{M_1} \) and \( T_{M_2} \), respectively. The machines process the actions performed by the subprograms. In the case of asynchronous communication, they are also involved in the communication between subprograms. In that case, each machine buffers the data sent to the subprogram that the machine executes till the subprogram consumes the data. Actually, the first way of description can be applied in the case of asynchronous communication as well, but it is rather clumsy.

The second way of description shows that, in the case where no abstraction from the processing of actions by a machine is made, parallel execution of subprograms corresponds directly to (encapsulated) parallel composition if synchronous communication or asynchronous communication is supported.
by the programming language used, and moreover shared variables are not supported. This makes it a compositional way of description, which has advantages in analysis. The compositionality is missing in the first way of description, which applies if only shared variables are supported.

Here is an example that illustrates how the behaviour of machines on which subprograms are executed can be described by transition systems.

**Example 2.3.2 (Peterson’s protocol).** We consider again the program from Example 2.3.1 concerning Peterson’s mutual exclusion protocol. The behaviour of a machine on which the procedures \( p_0 \) and \( p_1 \) can be executed in parallel, after initialization of the program variables \( c_0, c_1, \) and \( t \), is described by a transition system as follows. As states of the machine, we have triples \((c_0, c_1, t)\), where \( c_0, c_1 \in \mathbb{B} \) and \( t \in \{0, 1\} \). These states can be viewed as follows: \((c_0, c_1, t)\) is the storage that keeps the values of the program variables \( c_0, c_1, \) and \( t \) in that order. The initial state is \((\text{ff}, \text{ff}, 0)\). As actions, we have an action corresponding to each atomic statement of the procedures as well as each test of the procedures and its opposite. However, these actions differ from the actions of the transition system describing the behaviour of the procedures upon abstract execution: the former actions are actions of processing the latter actions. The difference is indicated by overlining the former actions. As transitions, we have the following:

- a transition \((c_0, c_1, t)\) if \( c_1 = \text{ff} \) or \( t = 1 \),
- a transition \((c_0, c_1, t)\) if \( c_1 \neq \text{ff} \) and \( t \neq 1 \),
- a transition \((c_0, c_1, t)\) if \( c_0 = \text{ff} \) or \( t = 0 \),
- a transition \((c_0, c_1, t)\) if \( c_0 \neq \text{ff} \) and \( t \neq 0 \).

The transition system for the machine is represented graphically in Fig. 2.4. We call this transition system \( T_M \). The behaviour of the procedures \( p_0 \) and \( p_1 \) upon parallel execution on a machine can now be described as follows:

\[ \partial_H((T_{p_0} \parallel \gamma^* T_{p_1}) \parallel \gamma T_M) \]

where
Fig. 2.4. Transition system for the machine executing Peterson's protocol
the communication function $\gamma'$ is undefined for any two actions, and the communication function $\gamma$ is defined such that

$$\gamma(a, a^n) = a^*$$

for all actions $a \in \text{act}(T_{p0}) \cup \text{act}(T_{p1})$, and it is undefined otherwise.

Notice that most procedures, written in the same programming language as $p0$ and $p1$, cannot be executed on the machine of which the behaviour is described by the transition system $T_M$ presented above. This machine can only deal with actions that can possibly be performed by the procedures $p0$ and $p1$. However, because all actions of the machine are prevented from being performed on their own, $T_M$ can safely be replaced by a transition system for a machine that can also deal with actions that can possibly be performed by other procedures.

2.4 Example: Alternating bit protocol

Here is a fairly typical example of the use of parallel composition and encapsulation in describing the behaviour of systems composed of components that act concurrently and interact with each other. The example concerns the ABP (Alternating Bit Protocol), a data communication protocol first introduced in [4].

The ABP is a simple data communication protocol based on positive and negative acknowledgements. Data are labeled with an alternating bit from $B = \{0, 1\}$. The sender either transmits a new datum or retransmits the most recent datum depending on an acknowledgement represented by a bit. The alternating bit used with the most recent datum is considered to be a positive acknowledgement. The configuration of the ABP is shown in Fig. 2.5. We have a sender process $S$, a receiver process $R$ and two channels $K$ and $L$. The process $S$ waits until a datum $d$ is offered at an external port (port 1). When a datum is offered at this port, $S$ consumes it, packs it with an alternating bit $b$ in a frame $(d, b)$, and then delivers the frame at an internal port used for sending (port 3). Next, $S$ waits until a bit $b'$ is offered at an internal port used for receiving (port 5). When a bit is offered and it is the alternating bit $b$, $S$ goes back to waiting for a datum. When a bit is
offered and it is not the alternating bit \( b \), \( S \) delivers the same frame again and goes back to waiting for a bit. The process \( S \) behaves the same when an error value is offered instead of a bit. The process \( R \) waits until a frame with a datum and an alternating bit \((d, b)\) is offered at an internal port used for receiving (port 4). When a frame is offered at this port, \( R \) consumes it, unpacks it, and then delivers the datum \( d \) at an external port (port 2) if the alternating bit \( b \) is the right one and in any case the alternating bit \( b \) at an internal port for sending (port 6). When instead an error value is offered, \( R \) delivers the wrong bit. After that, \( R \) goes back to waiting for a frame, but the right bit changes if the alternating bit was the right one. The processes \( K \) and \( L \) pass on frames from an internal port of \( S \) to an internal port of \( R \) and bits from an internal port of \( R \) to an internal port of \( S \), respectively. The processes \( K \) and \( L \) may corrupt frames and acknowledgements, respectively. In the case where this happens, \( K \) and \( L \) deliver an error value.

We assume a set of data \( D \). Let \( F = D \times B \) be the set of frames. For \( d \in D \) and \( b \in B \), we write \( d, b \) for the frame \((d, b)\). For \( b \in B \), we write \( \overline{b} \) for the bit \( 1 - b \). We use the standardized notation for handshaking communication introduced in Sect. 2.2.

The behaviour of the sender \( S \) is described by a transition system as follows. As states of the sender, we have triples \((d, b, i)\), where \( d \in D \cup \{\ast\} \), \( b \in B \) and \( i \in \{0, 1, 2\} \), satisfying \( d = \ast \) if and only if \( i = 0 \). State \((d, b, i)\) is roughly a state in which the datum being passed on from the sender to the receiver is \( d \) and the alternating bit is \( b \). If \( d = \ast \), no such datum is available. The initial state is \((\ast, 0, 0)\). As actions, we have \( r_1(d) \) for each \( d \in D \), \( s_3(f) \) for each \( f \in F \), and \( r_5(b) \) for each \( b \in B \). As transitions of the sender, we have the following:

- for each datum \( d \in D \) and bit \( b \in B \):
  - a transition \((\ast, b, 0) \xrightarrow{r_1(d)} (d, b, 1)\),
  - a transition \((d, b, 1) \xrightarrow{s_3(d, b)} (d, b, 2)\),
  - a transition \((d, b, 2) \xrightarrow{r_5(b)} (\ast, \overline{b}, 0)\),
  - a transition \((d, b, 2) \xrightarrow{s_6(\overline{b})} (d, b, 1)\),
  - a transition \((d, b, 2) \xrightarrow{r_5(\ast)} (d, b, 1)\).

The transition system for the sender is represented graphically in Fig. 2.6 for the case where only one datum, say \( d \), is involved.

The behaviour of the receiver \( R \) is described by a transition system as follows. As states of the receiver, we have triples \((d, b, i)\) where \( d \in D \cup \{\ast\} \), \( b \in B \) and \( i \in \{0, 1, 2\} \), satisfying \( d = \ast \) if and only if \( i \neq 1 \). State \((d, b, i)\) is roughly a state in which the datum to be delivered is \( d \) and the right bit is \( b \). If \( d = \ast \), no such datum is available. The initial state is \((\ast, 0, 0)\). As actions, we have \( s_2(d) \) for each \( d \in D \), \( r_4(f) \) for each \( f \in F \), and \( s_6(b) \) for each \( b \in B \). As transitions of the receiver, we have the following:

- for each datum \( d \in D \) and bit \( b \in B \):
The transition system for the receiver is represented graphically in Fig. 2.7 for the case where only one datum, say $d$, is involved.

The behaviour of the data transmission channel $K$ is described by a transition system as follows. As states of the channel, we have pairs $(f, i)$, where $f \in F \cup \{\ast\}$ and $i \in \{0, 1, 2, 3\}$, satisfying $f = \ast$ if and only if $i = 0$. State $(f, i)$ is roughly a state in which the frame to be transmitted is $f$. If $f = \ast$, no such frame is available. The initial state is $(\ast, 0)$. As actions, we have i,
2.4 Example: Alternating bit protocol

Fig. 2.8. Transition system for the data transmission channel

\[ r_3(f) \text{ for each } f \in F, \text{ and } s_4(f) \text{ for each } f \in F \cup \{\ast\}. \] As transitions of the channel, we have the following:

- for each frame \( f \in F \):
  - a transition \((\ast, 0) \xrightarrow{r_3(f)} (f, 1)\),
  - a transition \((f, 2) \xrightarrow{s_4(f)} (\ast, 0)\),
  - a transition \((f, 3) \xrightarrow{s_4(\ast)} (\ast, 0)\);
- for each frame \( f \in F \) and \( i \in \{2, 3\} \):
  - a transition \((f, 1) \xrightarrow{i} (f, i)\).

Note that this transition system is not determinate: for each frame \( f \) we have both \((\ast, 0) \xrightarrow{r_3(f)} (f, 2)\) and \((\ast, 0) \xrightarrow{r_3(f)} (f, 3)\), but the actions that can be performed from \((f, 2)\) and \((f, 3)\) are different. The action \( i \) is an internal action that cannot be performed synchronously with any other action. Thus, the channel cannot be forced to leave all frames uncorrupted. The transition system for channel \( K \) is represented graphically in Fig. 2.8 for the case where only one datum, say \( d \), is involved.

The behaviour of the acknowledgement transmission channel \( L \) is described by a transition system as follows. As states of the channel, we have pairs \((b, i)\), where \( b \in B \cup \{\ast\} \) and \( i \in \{0, 1, 2, 3\} \), satisfying \( b = \ast \) if and only if \( i = 0 \). State \((b, i)\) is roughly a state in which the bit to be transmitted is \( b \). If \( b = \ast \), no such bit is available. The initial state is \((\ast, 0)\). As actions, we have \( s_5(b) \) for each \( b \in B \cup \{\ast\} \), and \( r_6(b) \) for each \( b \in B \). As transitions of the channel, we have the following:

- for each bit \( b \in B \):
  - a transition \((\ast, 0) \xrightarrow{r_6(b)} (b, 1)\),
Fig. 2.9. Transition system for the acknowledgement transmission channel

- a transition \((b, 2) \xrightarrow{s_5(b)} (\ast, 0)\),
- a transition \((b, 3) \xrightarrow{s_6(\ast)} (\ast, 0)\);
- for each bit \(b \in B\) and \(i \in \{2, 3\}\):
  - a transition \((b, 1) \xrightarrow{i} (b, i)\).

Just as the transition system for channel \(K\), the transition system for channel \(L\) is not determinate: for each bit \(b\) we have both \((\ast, 0) \xrightarrow{r_6(b)} (b, 2)\) and \((\ast, 0) \xrightarrow{r_6(\ast)} (b, 3)\), but the actions that can be performed from \((b, 2)\) and \((b, 3)\) are different. Like in the case of channel \(K\), channel \(L\) cannot be forced to leave all acknowledgements uncorrupted. The transition system for channel \(L\) is represented graphically in Fig. 2.9.

The behaviour of the whole system is described as follows:

\[\partial_H(S \parallel \gamma K \parallel \gamma L \parallel \gamma R)\]

where

\[H = \{s_3(f), r_3(f) \mid f \in F\} \cup \{s_4(f), r_4(f) \mid f \in F \cup \{\ast\}\}\]
\[\cup \{s_5(b), r_5(b) \mid b \in B \cup \{\ast\}\} \cup \{s_6(b), r_6(b) \mid b \in B\}\]

and the communication function \(\gamma\) is defined in the standard way for handshaking communication (see Sect. 2.2).

Parallel composition and encapsulation of the transition systems of \(S\), \(K\), \(L\) and \(R\) as described above results in the following transition system. As states, we have quadruples \((s, k, l, r)\), where \(s, k, l\) and \(r\) are states of \(S\), \(K\), \(L\) and \(R\), respectively. As initial state, we have \(((\ast, 0, 0), (\ast, 0, 0), (\ast, 0, 0))\). As actions, we have \(r_1(d)\) and \(s_2(d)\) for each \(d \in D\), \(c_3(f)\) for each \(f \in F\).
\(c_4(f)\) for each \(f \in F \cup \{\ast\}\), \(c_5(b)\) for each \(b \in B \cup \{\ast\}\), \(c_6(b)\) for each \(b \in B\), and \(i\). As transitions, we have the following:

- for each datum \(d \in D\) and bit \(b \in B\):
  - \(((\ast, 0), (\ast, 0), (\ast, 0), (\ast, 0), (\ast, 0), (\ast, 0)) \xrightarrow{r_1(d)} ((d, b, 1), (\ast, 0), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 1), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_2(d,b)} ((d, b, 2), ((d, b, 1), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 2), ((d, b, 1), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_3(d,b)} ((d, b, 2), ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 1), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_4(d,b)} ((d, b, 2), ((d, b, 1), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 2), ((d, b, 2), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_5(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 1), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_6(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 2), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_7(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 1), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_8(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 2), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_9(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 1), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_{10}(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 2), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_{11}(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 1), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_{12}(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)
  - \(((d, b, 2), (\ast, 0), (\ast, 0), (\ast, b, 0)) \xrightarrow{c_{13}(d,b)} ((d, b, 2), (\ast, 0), (\ast, b, 0)),\)

The transition system for the whole protocol is represented graphically in Fig. 2.10 for the case where only one datum, say \(d\), is involved. This transition system does not reflect the configuration of the protocol, but is useful for analysis of the protocol. The transition system for the whole protocol shows, for example, that data are delivered in the order in which they were offered, without any loss, if it is assumed that cycles of communication actions at internal ports and the action \(i\) are eventually left.

### 2.5 Petri nets and parallel composition

For a better understanding of the notion of parallel composition of transition systems, we looked in a previous section into its connections with the familiar notion of parallel execution of programs. Is there a corresponding notion for
Definition 2.5.1 (Parallel composition). Let $N = (P, A, \rightarrow, m_0)$ and $N' = (P', A', \rightarrow', m'_0)$ be nets such that $P \cap P' = \emptyset$. Let $\gamma$ be a communication function on a set of actions that includes $A \cup A'$. The parallel composition of $N$ and $N'$ under $\gamma$, written $N \parallel_N N'$, is the net $(P'', A'', \rightarrow'', m''_0)$ where

- $P'' = P \cup P'$;
- $A'' = A \cup A' \cup \{\gamma(a, a') \mid a \in A, a' \in A', \gamma(a, a') \text{ is defined}\}$;
- $\rightarrow''$ is the smallest subset of $(P_{\text{fin}}(P'') \setminus \emptyset) \times A'' \times (P_{\text{fin}}(P'') \setminus \emptyset)$ such that:
  - if $Q_1 \xrightarrow{a} Q_2$, then $Q_1 \xrightarrow{a''} Q_2$;
  - if $Q'_1 \xrightarrow{b'} Q'_2$, then $Q'_1 \xrightarrow{b''} Q'_2$;
of the scheduler, we have the pairs \( (i, S) \) where \( \leq 1 \) mark, we have \( \gamma \) and the communication function \( \text{nxt} \) where \( \leq n \) for 1 of scheduled processes from Example 1.5.2. It consists of processes \( P \) as follows. As places of \( P \) net \( (P, A), \rightarrow, m_0) \) where
- \( P' = \{ p \in P \mid \text{for some } \sigma \in (A \setminus H)^*, m : P \rightarrow \mathbb{N} : m_0 \xrightarrow{\sigma} m, m(p) \neq 0 \} \);
- \( A' = \{ a \in A \setminus H \mid \text{for some } \sigma \in (A \setminus H)^*, m : P \rightarrow \mathbb{N} : m_0 \xrightarrow{\sigma} m \} \);
- \( \rightarrow' \) is the smallest subset of \( (P' \cap \emptyset) \times A' \times (P' \cap \emptyset) \) such that:
  - if \( Q_1 \xrightarrow{a} Q_2, Q_1 \subseteq P' \) and \( a \notin H \), then \( Q_1 \xrightarrow{a'} Q_2 \).

Let us give a definition of encapsulation on nets as well.

**Definition 2.5.2 (Encapsulation).** Let \( N = (P, A, \rightarrow, m_0) \) be a net. Let \( H \subseteq A \). The *encapsulation* of \( N \) with respect to \( H \), written \( \partial_H(N) \), is the net \( (P', A', \rightarrow', m_0) \) where
- \( P' = \{ p \in P \mid \text{for some } \sigma \in (A \setminus H)^*, m : P \rightarrow \mathbb{N} : m_0 \xrightarrow{\sigma} m, m(p) \neq 0 \} \);
- \( A' = \{ a \in A \setminus H \mid \text{for some } \sigma \in (A \setminus H)^*, m : P \rightarrow \mathbb{N} : m_0 \xrightarrow{\sigma} m \} \);
- \( \rightarrow' \) is the smallest subset of \( (P'_\text{fin}(P')) \times A' \times (P'_\text{fin}(P')) \) such that:
  - if \( Q_1 \xrightarrow{a} Q_2, Q_1 \subseteq P' \) and \( a \notin H \), then \( Q_1 \xrightarrow{a'} Q_2 \).

Here is an example of the use of parallel composition of nets and encapsulation of nets in describing process behaviour.

**Example 2.5.1 (Milner’s scheduling problem).** We consider again the system of scheduled processes from Example 1.5.2. It consists of processes \( P_1, \ldots, P_n \) \( (n > 1) \), each wishing to perform a certain task repeatedly, and a scheduler ensuring that they start their task in cyclic order, beginning with \( P_1 \).

The behaviour of process \( P_i \), for \( 1 \leq i \leq n \), can be described by a net as follows. As places of \( P_i \), we have the pairs \((i, \text{idle})\) and \((i, \text{busy})\). As initial marking, we have \{ \( (i, \text{idle}) \) \}. As actions, we have \text{request}(i) \ (\text{request to start task } i) \) and \text{finish}(i) \ (\text{as transitions, we have the following):}
- \( \{ (i, \text{idle}) \} \xrightarrow{\text{request}(i)} \{ (i, \text{busy}) \} \),
- \( \{ (i, \text{busy}) \} \xrightarrow{\text{finish}(i)} \{ (i, \text{idle}) \} \).

The behaviour of scheduler \( S \) can be described by a net as follows. As places of the scheduler, we have the pairs \((i, \text{sch})\) for \( 1 \leq i \leq n \). As initial marking, we have \{ \( (1, \text{sch}) \) \}. As actions, we have \text{grant}(i) \ (\text{grant to start task } i) \) for \( 1 \leq i \leq n \). As transitions, we have the following (for \( 1 \leq i \leq n \)):

\[
\{ (i, \text{sch}) \} \xrightarrow{\text{grant}(i)} \{ (\text{nxt}(i), \text{sch}) \},
\]

where \( \text{nxt}(i) = i + 1 \) if \( i < n \) and \( \text{nxt}(n) = 1 \). The behaviour of the whole system is described as follows:

\[
\partial_H(P_1 \parallel \gamma \ldots \parallel \gamma P_n \parallel \gamma S)
\]

where

\[
H = \{ \text{request}(i), \text{grant}(i) \mid 1 \leq i \leq n \}.
\]

and the communication function \( \gamma \) is defined such that

\[
\gamma(\text{request}(i), \text{grant}(i)) = \gamma(\text{grant}(i), \text{request}(i)) = \text{start}(i)
\]

for \( 1 \leq i \leq n \), and it is undefined otherwise.

The net obtained from the nets \( P_1, \ldots, P_n \) and \( S \) by parallel composition and encapsulation as described above is the same as the net described in Example 1.5.2.
In Sect. 1.5, we associated a transition system $T(N)$ with each net $N$. It happens that this association is useful in showing the close connection between parallel composition of nets and parallel composition of transition systems, and between encapsulation of nets and encapsulation of transition systems.

Property 2.5.1. Let $N = (P, A, \rightarrow, m_0)$ and $N' = (P', A', \rightarrow', m'_0)$ be nets such that $P \cap P' = \emptyset$, let $\gamma$ be a communication function on a set of actions that includes $A \cup A'$, and let $H \subseteq A$. Then we have that

$$T(N \parallel_{\gamma} N') \equiv T(N) \parallel_{\gamma} T(N')$$

$$T(\partial_H(N)) \equiv \partial_H(T(N))$$

In words, the transition system associated with a parallel composition of nets is up to bisimulation equivalence the same as the parallel composition of the transition systems associated with those nets; and analogously for encapsulation.

Example 2.5.2 (Milner's scheduling problem). We consider once again the system of scheduled processes from Examples 1.5.2 and 2.5.1. Associating a transition system with the net describing the behaviour of process $P_i$ ($1 \leq i \leq n$) is trivial because only singleton sets occur as pre- and postsets of transitions. The resulting transition system, $T(P_i)$, can be described as follows. As states, we have the singleton sets of pairs $\{(i, \text{idle})\}$ and $\{(i, \text{busy})\}$. As initial state, we have $\{(i, \text{idle})\}$. As actions, we have $\text{request}(i)$ and $\text{finish}(i)$. As transitions, we have the following:

$$\{(i, \text{idle})\} \xrightarrow{\text{request}(i)} \{(i, \text{busy})\},$$

$$\{(i, \text{busy})\} \xrightarrow{\text{finish}(i)} \{(i, \text{idle})\}.$$

Associating a transition system with the net describing the behaviour of the scheduler $S$ is equally trivial. The resulting transition system, $T(S)$, can be described as follows. As states, we have the singleton sets of pairs $\{(i, \text{sch})\}$ for $1 \leq i \leq n$. As initial state, we have $\{(1, \text{sch})\}$. As actions, we have $\text{grant}(i)$ for $1 \leq i \leq n$. As transitions, we have the following (for $1 \leq i \leq n$):

$$\{(i, \text{sch})\} \xrightarrow{\text{grant}(i)} \{(\text{nxt}(i), \text{sch})\}.$$

So, the transition systems associated with the nets $P_1, \ldots, P_n$ and $S$ are simply obtained by taking the singleton sets of places as states. In other words, those nets are essentially transition systems. However, their parallel composition as nets yields the net from Example 1.5.2 which is not quite a transition system – because non-singleton sets of places occur as pre- and postsets of transitions. The transition system described by

$$\partial_H(T(P_1) |\gamma \cdots |\gamma T(P_n) |\gamma T(S)),$$

where $H$ and $\gamma$ are as in Example 2.5.1 is bisimulation equivalent to the transition system associated with the net from Example 1.5.2.
2.6 Bisimulation and trace equivalence

An important property of parallel composition and encapsulation of transition systems is that they preserve bisimulation equivalence, by which we mean the following.

Property 2.6.1 (Preservation of bisimulation equivalence). Let $T_1$ and $T_2$ be transition systems with $A$ as set of actions, let $T'_1$ and $T'_2$ be transition systems with $A'$ as set of actions, and let $\gamma$ be a communication function on a set of actions that includes $A \cup A'$. Then the following holds:

- if $T_1 \leftrightarrow T_2$ and $T'_1 \leftrightarrow T'_2$, then $T_1 \parallel_\gamma T'_1 \leftrightarrow T_2 \parallel_\gamma T'_2$;
- if $T_1 \leftrightarrow T_2$, then $\partial_H(T_1) \leftrightarrow \partial_H(T_2)$.

Hence, a parallel composition of transition systems is bisimulation equivalent to a parallel composition of transition systems obtained by replacing the constituent transition systems by ones that are bisimulation equivalent. This property is actually what justifies such replacements. It underlies many techniques for the analysis of process behaviour.

Parallel composition and encapsulation of transition systems also preserve trace equivalence.

Property 2.6.2 (Preservation of trace equivalence). Let $T_1$ and $T_2$ be transition systems with $A$ as set of actions, let $T'_1$ and $T'_2$ be transition systems with $A'$ as set of actions, and let $\gamma$ be a communication function on a set of actions that includes $A \cup A'$. Then the following holds:

- if $T_1 \equiv_{tr} T_2$ and $T'_1 \equiv_{tr} T'_2$, then $T_1 \parallel_\gamma T'_1 \equiv_{tr} T_2 \parallel_\gamma T'_2$;
- if $T_1 \equiv_{tr} T_2$, then $\partial_H(T_1) \equiv_{tr} \partial_H(T_2)$.

If an equivalence is preserved by an operation, the equivalence is called a congruence with respect to the operation. Let us now illustrate how the congruence properties can be used.

Example 2.6.1 (Split and merge connections). We consider again the split and merge connections from Examples 1.6.2 and 1.6.3. Both kinds of connections are used as connections between nodes in networks. Suppose that the behaviour of a particular network is described by

$$\partial_H(T_1 \parallel \gamma \cdots \parallel_\gamma T_k \parallel_\gamma T_{k+1} \parallel_\gamma \cdots \parallel_\gamma T_n),$$

where $T_1, \ldots, T_n$ are transition systems describing the behaviour of the nodes and connections that occur in the network. Suppose further that $T_k$ is the first transition system for a merge connection given in Example 1.6.3 and that $T'_k$ is the second transition system for a merge connection given in Example 1.6.3. Recall that the two transition systems for a merge connection are bisimulation equivalent. Hence, replacement of $T_k$ by $T'_k$ yields a network that is bisimulation equivalent to the original network.
Now, suppose instead that $T_k$ is the transition system for the split connection given in Example 1.6.2 and that $T'_k$ is the transition system for the split-like connection given in Example 1.6.2. Recall that those two transition systems are trace equivalent, but not bisimulation equivalent. Hence, replacement of $T_k$ by $T'_k$ yields a network that is trace equivalent to the original network. However, the networks are not bisimulation equivalent because the replacement causes a premature choice of an output port. Such changes remain unnoticed under trace equivalence, because trace equivalence does not tell us anything about the stages at which the choices of different possibilities occur.

The following properties of parallel composition hold because of the conditions imposed on the communication function (see Def. 2.2.1).

Property 2.6.3 (Commutativity and associativity of parallel composition). Let $T_1$, $T_2$ and $T_3$ be transition systems with $A_1$, $A_2$ and $A_3$, respectively, as set of actions. Let $\gamma$ be a communication function on a set of actions that includes $A_1 \cup A_2 \cup A_3$. Then the following holds:

\[ T_1 \parallel_{\gamma} T_2 \leftrightarrow T_2 \parallel_{\gamma} T_1, \]
\[ (T_1 \parallel_{\gamma} T_2) \parallel_{\gamma} T_3 \leftrightarrow T_1 \parallel_{\gamma} (T_2 \parallel_{\gamma} T_3). \]
3. Abstraction

Preferably, the design of a complex system starts from a description of its behaviour at a high level of abstraction, i.e. a description serving as a specification of the system to be developed, and ends in a description of the behaviour at a low level of abstraction together with a proof that the behaviour described at the start is essentially the same as the behaviour described at the end after abstraction from actions that have been added during the design process. This chapter deals with this issue of abstraction by introducing the notions of abstraction from internal actions and branching bisimulation equivalence. First of all, we explain informally what abstraction from internal actions is and what branching bisimulation equivalence is, and give a simple example of their use in comparing descriptions of process behaviour (Sect. 3.1). After that, we define the notions of abstraction from internal actions and branching bisimulation equivalence in a mathematically precise way (Sect. 3.2). We also use abstraction from internal actions and branching bisimulation equivalence to show that a merge connection with a feedback wire behaves as a sink (Sect. 3.3), and to show that the simple data communication protocol from Sect. 2.4 behaves as a buffer of capacity one (Sect. 3.4). For the interested reader, we define the notions of abstraction from internal actions and branching bisimulation equivalence for nets (Sect. 3.5). Finally, we look at some miscellaneous issues (Sect. 3.6).

3.1 Informal explanation

Abstraction from internal actions is an important notion. Frequently, the behaviour of a system is first described at a high level of abstraction, and then as a system composed of interacting components. It should be shown that the two descriptions are equivalent after abstraction from actions added for the interactions between the components. The need for abstraction from certain actions became already apparent in the preceding chapter, while analyzing systems described using transition systems.

Abstraction from internal actions is a means to express that certain actions must be considered to be unobservable. It turns actions from a certain set into a special action, denoted by $\tau$, which is called the silent step. Un-
Abstraction

like other actions, the act of performing a silent step is considered to be unobservable. Let us give an example of the use of abstraction.

Example 3.1.1 (Bounded buffers). We consider again the system composed of two bounded buffers from Example 2.1.1. In that example, parallel composition and encapsulation of the two buffers, buffer 1 and buffer 2, resulted in the following transition system. As states, we have pairs \((\sigma_1, \sigma_2)\) where \(\sigma_i\) \((i = 1, 2)\) is a sequence of data of which the length is not greater than \(l_i\). As initial state, we have \((\epsilon, \epsilon)\). As actions, we have \(\text{add}_1(d)\), \(\text{rem}_2(d)\) and \(\text{trf}(d)\) for each datum \(d\). As transitions, we have the following:

- for each datum \(d\) and each state \((\sigma_1, \sigma_2)\) with the length of \(\sigma_1\) less than \(l_1\), a transition \((\sigma_1, \sigma_2) \overset{\text{add}_1(d)}{\rightarrow} (d\sigma_1, \sigma_2)\);
- for each datum \(d\) and each state \((\sigma_1, \sigma_2 d)\), a transition \((\sigma_1, \sigma_2 d) \overset{\text{rem}_2(d)}{\rightarrow} (\sigma_1, \sigma_2)\);
- for each datum \(d\) and each state \((\sigma_1 d, \sigma_2)\) with the length of \(\sigma_2\) less than \(l_2\), a transition \((\sigma_1 d, \sigma_2) \overset{\text{trf}(d)}{\rightarrow} (\sigma_1, d\sigma_2)\).

At the end of Example 2.1.1 there was a need to abstract from the internal transfer actions \(\text{trf}(d)\). The following transition system is the result of abstraction from the actions \(\text{trf}(d)\) for \(d \in D\). We have the same states as before. As actions, we have \(\text{add}_1(d)\) and \(\text{rem}_2(d)\) for each datum \(d\). As transitions, we have the following:

- for each datum \(d\) and each state \((\sigma_1, \sigma_2)\) with the length of \(\sigma_1\) less than \(l_1\), a transition \((\sigma_1, \sigma_2) \overset{\text{add}_1(d)}{\rightarrow} (d\sigma_1, \sigma_2)\);
- for each datum \(d\) and each state \((\sigma_1, \sigma_2 d)\), a transition \((\sigma_1, \sigma_2 d) \overset{\text{rem}_2(d)}{\rightarrow} (\sigma_1, \sigma_2)\);
- for each datum \(d\) and each state \((\sigma_1 d, \sigma_2)\) with the length of \(\sigma_2\) less than \(l_2\), a transition \((\sigma_1 d, \sigma_2) \overset{\tau}{\rightarrow} (\sigma_1, d\sigma_2)\).

This transition system is represented graphically in Fig. 3.1 for the case where \(l_1 = l_2 = 1\) and the only data involved are the natural numbers 0 and 1.

As mentioned above, the act of performing a silent step is considered to be unobservable. However, the act of performing a silent step can sometimes be inferred because a process may proceed as a different process after performing a silent step. In other words, the capabilities of a transition system may change by performing a silent step. Let us look at an example of this phenomenon.

Example 3.1.2 (Non-inert silent step). We consider the following two transition systems, of which the second is actually a split connection (see Example 1.6.2). We assume a set of data \(D\). As actions of both transition systems, we have \(r_1(d)\), \(s_2(d)\) and \(s_3(d)\) for each \(d \in D\). As states of the first transition system, we have pairs \((d, i)\), where \(d \in D \cup \{\ast\}\) and \(i \in \{0, 1, 2\}\), with \((\ast, 0)\) as initial state. As transitions of the first transition system, we have the following:
3.1 Informal explanation

Fig. 3.1. Transition system for abstraction of two parallel bounded buffers

- for each \( d \in D \):
  - a transition \( (\ast, 0) \xrightarrow{r_1(d)} (d, 1) \),
  - a transition \( (d, 1) \xrightarrow{s_2(d)} (\ast, 0) \),
  - a transition \( (d, 1) \xrightarrow{\tau} (d, 2) \),
  - a transition \( (d, 2) \xrightarrow{s_3(d)} (\ast, 0) \).

As states of the second transition system, we have pairs \( (d, i) \), where \( d \in D \cup \{\ast\} \) and \( i \in \{0, 1\} \), with \( (\ast, 0) \) as initial state. As transitions of the second transition system, we have the following:

- for each \( d \in D \):
  - a transition \( (\ast, 0) \xrightarrow{r_1(d)} (d, 1) \),
  - a transition \( (d, 1) \xrightarrow{s_2(d)} (\ast, 0) \),
  - a transition \( (d, 1) \xrightarrow{s_3(d)} (\ast, 0) \).

The transition systems given in this example are represented graphically in Fig. 3.2 for the case where \( D = \{0, 1\} \). The first transition system has a state, viz. state \( (d, 2) \), in which it is able to perform action \( s_3(d) \) without being able to perform action \( s_2(d) \) instead; whereas the second transition system does not have such a state. This means the following for the observable behaviour of these transition system. In the case of the first transition system, after \( r_1(d) \) has been performed, two observations are possible. The act of performing \( s_2(d) \) can be observed and, after \( \tau \) has been performed, the act of performing \( s_3(d) \) can be observed. However, before anything has been observed, it may have become impossible to observe the act of performing \( s_2(d) \). In the case of the second transition system, it remains possible to observe the act of
Fig. 3.2. Transition systems of Example 3.1.2

performing $s_2(d)$ so long as nothing has been observed. So the observable behaviour of the two transition systems differ.

The purpose of abstraction from internal actions is to be able to identify transition systems that have the same observable behaviour. The preceding example shows that an equivalence based on the idea to simply leave out all unobservable actions does not work. Still, in many cases, the act of performing a silent step cannot be inferred, because the process concerned proceeds as the same process after performing a silent step. In such cases, we sometimes say that the silent step is inert. Here is an example of an inert silent step.

**Example 3.1.3 (Inert silent step).** We consider the following two transition systems. We assume a set of data $D$. As actions of both transition system, we have $r_1(d)$ and $s_2(d)$ for each $d \in D$. As states of the first transition system, we have pairs $(d,i)$, where $d \in D \cup \{\ast\}$ and $i \in \{0,1,2\}$, with $(\ast,0)$ as initial state. As transitions of the first transition system, we have the following:

- for each $d \in D$:
  - a transition $(\ast,0) \xrightarrow{r_1(d)} (d,1)$,
  - a transition $(d,1) \xrightarrow{\tau} (d,2)$,
  - a transition $(d,2) \xrightarrow{s_2(d)} (\ast,0)$.

As states of the second transition system, we have pairs $(d,i)$, where $d \in D \cup \{\ast\}$ and $i \in \{0,1\}$, with $(\ast,0)$ as initial state. As transitions of the second transition system, we have the following:

- for each $d \in D$:
  - a transition $(\ast,0) \xrightarrow{r_1(d)} (d,1)$,
  - a transition $(d,1) \xrightarrow{s_2(d)} (\ast,0)$. 


The transition systems given in this example are represented graphically in Fig. 3.3 for the case where \( D = \{0, 1\} \). Initially, only the act of performing \( r_1(d) \) can be observed. After this has been observed, only the act of performing \( s_2(d) \) can be observed. There is no way to infer the act of performing the silent step in between. So the observable behavior of these transition systems is the same.

What we understand from the preceding two examples is that a silent step can only be left out if no capabilities get lost by performing it. According to this understanding, we adapt the notion of bisimulation equivalence as follows. Two transition systems \( T \) and \( T' \) are branching bisimulation equivalent if their states can be related such that:

- the initial states are related;
- if states \( s_1 \) and \( s'_1 \) are related and in \( T \) a transition with label \( a \) is possible from \( s_1 \) to some \( s_2 \), then
  - either \( a \) is the silent step and \( s_2 \) and \( s'_1 \) are related,
  - or in \( T' \) a transition with label \( a \) is possible from some \( s''_1 \) to some \( s'_2 \) such that a generalized transition with a sequence of zero or more silent steps as label is possible from \( s'_1 \) to \( s''_1 \), \( s_1 \) and \( s''_1 \) are related, and \( s_2 \) and \( s'_2 \) are related;
- likewise, with the role of \( T \) and \( T' \) reversed.

We could have required \( s_1 \) to be related to all states between \( s'_1 \) and \( s''_1 \) as well, but that turns out to be equivalent. Let us return for a while to the preceding two examples.

**Example 3.1.4 (Non-inert silent step).** We consider again the transition systems of Example 3.1.2. Are those transition systems identified by branching bisimulation equivalence? No, they are not. In order to be able to relate, as
required, the state \((*, 0)\) of the first transition system to the state \((*, 0)\) of the second transition system, the states \((d, 1)\) and \((d, 2)\) \((d \in D)\) of the first transition system have to be related to states of the second transition system as well. However, we cannot relate state \((d, 2)\) because the second transition system has no state from which only a transition with label \(s_3(d)\) is possible.

**Example 3.1.5 (Inert silent step).** We also consider again the transition systems of Example 3.1.3. Are those transition systems identified by branching bisimulation equivalence? Yes, they are: relate state \((*, 0)\) of the first transition system to state \((*, 0)\) of the second transition system, and for each \(d \in D\), relate the states \((d, 1)\) and \((d, 2)\) of the first transition system to state \((d, 1)\) of the second transition system. In this way, the states of the two transition systems are related as required for branching bisimulation equivalence.

### 3.2 Formal definitions

With the previous section, we have prepared the way for the formal definitions of the notions of abstraction from internal actions and branching bisimulation equivalence. However, we have to adapt the definitions of the notions of a transition system, a communication function, parallel composition and encapsulation from Chaps. 1 and 2 to the presence of the silent step first. In the adapted definitions, we write \(A_\tau\) for \(A \cup \{\tau\}\).

**Definition 3.2.1 (Transition system).** A transition system \(T\) is a quadruple \((S, A, \rightarrow, s_0)\) where

- \(S\) is a set of *states*;
- \(A\) is a set of *actions*;
- \(\rightarrow \subseteq S \times A_\tau \times S\) is a set of *transitions*;
- \(s_0 \in S\) is the *initial state*.

The set \(\rightarrow \subseteq S \times A^* \times S\) of *generalized transitions* of \(T\) is the smallest subset of \(S \times A^* \times S\) satisfying:

- \(s \xrightarrow{\cdot} s\) for each \(s \in S\);
- if \(s \xrightarrow{a} s'\), then \(s \xrightarrow{a}\) \(s'\);
- if \(s \xrightarrow{\sigma} s'\) then \(s \xrightarrow{\sigma}\) \(s'\);
- if \(s \xrightarrow{a} s'\) and \(s' \xrightarrow{\sigma} s''\), then \(s \xrightarrow{a\sigma} s''\).

A state \(s \in S\) is called a *reachable* state of \(T\) if there is a \(\sigma \in A^*\) such that \(s_0 \xrightarrow{\sigma} s\). A state \(s \in S\) is called a *terminal* state of \(T\) if there is no \(a \in A\) and \(s' \in S\) such that \(s \xrightarrow{a} s'\).

Notice that transitions labeled with the silent step may be included in the set of transitions of a transition system, although the silent step is never included in the set of actions.
Definition 3.2.2 (Communication function). Let $A$ be a set of actions. A communication function on $A$ is a partial function $\gamma : A \times A \to A$ satisfying for $a, b, c \in A$:

- $\gamma(a, \tau)$ and $\gamma(\tau, a)$ are undefined;
- if $\gamma(a, b)$ is defined, then $\gamma(b, a)$ is defined and $\gamma(a, b) = \gamma(b, a)$;
- if $\gamma(a, b)$ and $\gamma(\gamma(a, b), c)$ are defined, then $\gamma(b, c)$ and $\gamma(a, \gamma(b, c))$ are defined and $\gamma(\gamma(a, b), c) = \gamma(a, \gamma(b, c))$.

Notice that we consider the silent step to be an action that cannot be performed synchronously with other actions. The reason for this is that it would otherwise be observable.

Definition 3.2.3 (Parallel composition). Let $T = (S, A, \rightarrow, s_0)$ and $T' = (S', A', \rightarrow', s'_0)$ be transition systems. Let $\gamma$ be a communication function on a set of actions that includes $A \cup A'$. The parallel composition of $T$ and $T'$ under $\gamma$, written $T \parallel_\gamma T'$, is the transition system $(S'', A'', \rightarrow'', s''_0)$ where

- $S'' = S \times S'$;
- $A'' = A \cup A' \cup \{ \gamma(a, a') \mid a \in A, a' \in A', \gamma(a, a') \text{ is defined} \}$;
- $\rightarrow''$ is the smallest subset of $S'' \times A'' \times S''$ such that:
  - if $s_1 \xrightarrow{a} s_2$ and $s' \in S'$, then $(s_1, s') \xrightarrow{\gamma(a, \tau)} (s_2, s')$;
  - if $s'_1 \xrightarrow{b} s'_2$ and $s \in S$, then $(s, s'_1) \xrightarrow{b, \gamma(a, \tau)} (s, s'_2)$;
  - if $s_1 \xrightarrow{a} s_2$, $s'_1 \xrightarrow{b} s'_2$ and $\gamma(a, b)$ is defined, then $(s_1, s'_1) \xrightarrow{\gamma(a,b)} (s_2, s'_2)$;
- $s''_0 = (s_0, s'_0)$.

Definition 3.2.4 (Encapsulation). Let $T = (S, A, \rightarrow, s_0)$ be a transition system. Let $H \subseteq A$. The encapsulation of $T$ with respect to $H$, written $\partial_H(T)$, is the transition system $(S', A', \rightarrow', s_0)$ where

- $S' = \{ s \mid \text{for some } \sigma \in (A \setminus H)^*, s_0 \xrightarrow{\sigma} s \}$;
- $A' = \{ a \in A \setminus H \mid \text{for some } s_1, s_2 \in S' : s_1 \xrightarrow{a} s_2 \}$;
- $\rightarrow'$ is the smallest subset of $S' \times A' \times S'$ such that:
  - if $s_1 \xrightarrow{a} s_2$, $s_1 \in S'$ and $a \notin H$, then $s_1 \xrightarrow{a'} s_2$.

The definitions of parallel composition and encapsulation are, just like the definition of transition system above, nothing else but simple adjustments of the earlier definitions to cover transitions labeled with the silent step. Here is an example of silent steps in parallel composition and encapsulation.

Example 3.2.1 (Silent steps in parallel composition and encapsulation). We consider the following two transition systems. As actions of the first transition system, we have $s_1(0)$ and $s_2(0)$. As states of the first transition system, we have natural numbers $i \in \{0, 1, 2, 3\}$, with 0 as initial state. As transitions of the first transition system, we have the following:

$$0 \xrightarrow{1}, 0 \xrightarrow{2}, 1 \xrightarrow{s_1(0)} 3, 2 \xrightarrow{s_2(0)} 3.$$
As actions of the second transition system, we have only \( r_1(0) \). As states of the second transition system, we have natural numbers \( i \in \{0, 1\} \), with 0 as initial state. As transitions of the second transition system, we have the following:

\[
0 \xrightarrow{r_1(0)} 1.
\]

The following transition system is the result of the parallel composition of these two transition systems and the subsequent encapsulation with respect to actions \( s_1(0), r_1(0), s_2(0) \) and \( r_2(0) \). As actions, we have only \( c_1(0) \). As states, we have the pairs \((0, 0), (1, 0), (2, 0)\) and \((3, 1)\), with \((0, 0)\) as initial state. As transitions, we have the following:

\[
(0, 0) \xrightarrow{\tau} (1, 0), \quad (0, 0) \xrightarrow{\tau} (2, 0), \quad (1, 0) \xrightarrow{c_1(0)} (3, 1).
\]

This transition system is capable of either first performing a silent step, next performing a communication action, and by doing so getting in a terminal state or first performing a silent step and by doing so getting in a terminal state. In the case where the send actions of the first transition system were not preceded by a silent step, the resulting transition system would only have the first alternative. The parallel composition and the subsequent encapsulation are represented graphically in Fig. 3.4 and 3.5 respectively.

Let us now look at the formal definitions of abstraction from internal actions and branching bisimulation equivalence.
Definition 3.2.5 (Abstraction). Let $T = (S, A, \to, s_0)$ be a transition system. Let $I \subseteq A$. The abstraction of $T$ with respect to $I$, written $\tau_I(T)$, is the transition system $(S, A', \to', s_0)$ where

- $A' = A \setminus I$;
- $\to'$ is the smallest subset of $S \times A'_\tau \times S$ such that:
  - if $s_1 \xrightarrow{a} s_2$ and $a \in I$, then $s_1 \xrightarrow{\tau'} s_2$,
  - if $s_1 \xrightarrow{a} s_2$ and $a \notin I$, then $s_1 \xrightarrow{\tau'} s_2$.

In the definition of branching bisimulation equivalence, we write $s \xrightarrow{I \tau} s'$ for $s \xrightarrow{} s'$. In other words, $s \xrightarrow{I \tau} s'$ indicates that state $s'$ is reachable from state $s$ by performing zero or more silent steps.

Definition 3.2.6 (Branching bisimulation). Let $T = (S, A, \to, s_0)$ and $T' = (S', A', \to', s'_0)$ be transition systems such that $A = A'$. Then a branching bisimulation $B$ between $T$ and $T'$ is a binary relation $B \subseteq S \times S'$ such that the following conditions hold:

1. $B(s_0, s'_0)$;
2. whenever $B(s_1, s'_1)$ and $s_1 \xrightarrow{a} s_2$, then either $a = \tau$ and $B(s_2, s'_1)$ or there are states $s''_1, s''_2$ such that $s'_1 \xrightarrow{I \tau''} s''_1 \xrightarrow{\tau'} s''_2$ and $B(s_1, s''_1)$ and $B(s_2, s''_2)$;
3. whenever $B(s_1, s'_1)$ and $s'_1 \xrightarrow{\tau'} s'_2$, then either $a = \tau$ and $B(s_1, s'_2)$ or there are states $s''_1, s''_2$ such that $s_1 \xrightarrow{\tau} s''_1 \xrightarrow{a} s_2$ and $B(s'_1, s''_1)$ and $B(s_2, s''_2)$.
The two transition systems $T$ and $T'$ are branching bisimulation equivalent, written $T \equiv_b T'$, if there exists a branching bisimulation $B$ between $T$ and $T'$. A branching bisimulation between $T$ and $T$ is called a branching auto-bisimulation on $T$.

Here is an example of transition systems that are branching bisimulation equivalent.

**Example 3.2.2 (Bounded buffers).** We consider again the transition system presented at the end of Example 3.1.1 concerning abstraction of two encapsulated parallel bounded buffers. As states, we have pairs $(\sigma_1, \sigma_2)$ where $\sigma_i (i = 1, 2)$ is a sequence of data of which the length of is not greater than $l_i$.

As actions, we have $\text{add}_1(d)$ and $\text{rem}_2(d)$ for each datum $d$. As transitions, we have the following:

- for each datum $d$ and each state $(\sigma_1, \sigma_2)$ with the length of $\sigma_1$ less than $l_1$, a transition $(\sigma_1, \sigma_2) \xrightarrow{\text{add}_1(d)} (d\sigma_1, \sigma_2)$;
- for each datum $d$ and each state $(\sigma_1, \sigma_2 d)$, a transition $(\sigma_1, \sigma_2 d) \xrightarrow{\text{rem}_2(d)} (\sigma_1, \sigma_2)$;
- for each datum $d$ and each state $(\sigma_1 d, \sigma_2)$ with the length of $\sigma_2$ less than $l_2$, a transition $(\sigma_1 d, \sigma_2) \xrightarrow{\tau} (\sigma_1, d\sigma_2)$.

Next, we consider the following transition system. As states, we have sequences of data of which the length is not greater than $l_1 + l_2$. We have the same actions as before. As transitions, we have the following:

- for each datum $d$ and each state $\sigma$ with the length of $\sigma$ less than $l_1 + l_2$, a transition $\sigma \xrightarrow{\text{add}_1(d)} d\sigma$;
- for each datum $d$ and each state $\sigma d$, a transition $\sigma d \xrightarrow{\text{rem}_2(d)} \sigma$.

These two transition systems are branching bisimulation equivalent. Take the following relation:

$$B = \{((\sigma_1, \sigma_2), \sigma_1 \sigma_2) \mid |\sigma_1| \leq l_1, |\sigma_2| \leq l_2\}.$$ 

It is easy to see that $B$ is a branching bisimulation. The important point here is that, for each transition $(\sigma_1 d, \sigma_2) \xrightarrow{\tau} (\sigma_1, d\sigma_2)$ of the first transition system, the conditions imposed on a branching bisimulation permit that the states $(\sigma_1 d, \sigma_2)$ and $(\sigma_1, d\sigma_2)$ are both related to the state $\sigma_1 d\sigma_2$ of the second transition system.

Just as bisimulation equivalence, branching bisimulation equivalence is preserved by parallel composition and encapsulation. Moreover, it is preserved by abstraction.

**Property 3.2.1 (Preservation of branching bisimulation equivalence).** Let $T_1$ and $T_2$ be transition systems with $A$ as set of actions, let $T'_1$ and $T'_2$ be transition systems with $A'$ as set of actions, and let $\gamma$ be a communication function on a set of actions that includes $A \cup A'$. Then the following holds:
3.3 Example: Merge connection with feedback wire

We consider again the merge connections from Example 1.6.3. Two transition systems describing the behaviour of a merge connection are given in that example. For clearness’ sake, the second one is given here again. We assume a set of data $D$. The behaviour of a merge connection with input ports $k$ and $l$ and output port $m$, $\text{Merge}^{kl,m}$, is described by the following transition system. As states, we have $*$ and the data $d \in D$, with $*$ as initial state. As actions, we have $s_i(d)$ and $r_i(d)$ for $i = k, l, m$ and $d \in D$. As transitions, we have the following:

- for each $d \in D$: $* \xrightarrow{r_k(d)} d$, $* \xrightarrow{r_l(d)} d$, $d \xrightarrow{s_m(d)} *$.

This transition system is represented graphically in Fig. 3.6 for the case where $D = \{0, 1\}$.

Wires, which were not mentioned before, constitute another important kind of connection used between nodes in networks. A wire is reminiscent of a buffer with unbounded capacity. The behaviour of a wire with input port $m$ and output port $l$, $\text{Wire}^{m,l}$, is described by the following transition system. As states, we have all sequences $\sigma \in D^*$, with $\epsilon$ as initial state. As actions, we have $r_m(d)$ and $s_l(d)$ for each $d \in D$. As transitions of a wire, we have the following:

- for each $d \in D$ and $\sigma \in D^*$: $\sigma \xrightarrow{r_m(d)} d \sigma$, $\sigma d \xrightarrow{s_l(d)} \sigma$. 

Fig. 3.6. Transition system for the merge connection

if $T_1 \xrightarrow{\gamma} T_2$ and $T_1' \xrightarrow{\gamma} T_2'$, then $T_1 \parallel \gamma T_1' \parallel \gamma T_2'$.

if $T_1 \xrightarrow{\gamma} T_2$, then $\partial_H(T_1) \xrightarrow{\gamma} \partial_H(T_2)$ and $\tau_I(T_1) \xrightarrow{\gamma} \tau_I(T_2)$.

The definition of the notion of determinacy of a transition system has to be adapted to the presence of the silent step as well.

Definition 3.2.7 (Determinacy). Let $T = (S, A, \rightarrow, s_0)$ be a transition system. Then $T$ is determinate if the following condition holds:

whenever $s_0 \xrightarrow{\sigma} s$ and $s_0 \xrightarrow{\sigma} s'$, then there is a branching autbisimulation $B$ on $T$ such that $B(s, s')$.
This transition system is represented graphically in Fig. \ref{fig:trans-system} for the case where $D = \{0, 1\}$.

Let us look at the following transition system:

$$\tau_I(\partial_H(\text{Merge}^{k,l,m} \parallel \gamma \text{Wire}^{m,l}))$$

where

$$H = \{s_i(d), r_i(d) \mid i \in \{m, l\}, d \in D\},$$

$$I = \{c_i(d) \mid i \in \{m, l\}, d \in D\}$$

and the communication function $\gamma$ is defined in the standard way for handshaking communication (see Sect. \ref{sec:handshaking}), i.e. such that

$$\gamma(s_i(d), r_i(d)) = \gamma(r_i(d), s_i(d)) = c_i(d)$$

for all $d \in D$, and it is undefined otherwise. Thus, the data delivered by the merge connection at port $m$ is feed back to one of its input port, viz. $l$.

\footnote{1 In graphical representations of transition systems, we use grey tones to indicate an infinite progression.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{trans-system.png}
\caption{Transition system for the wire}
\end{figure}
Parallel composition, encapsulation and abstraction of the transition systems $\text{Merge}^{kl,m}$ and $\text{Wire}^{m,l}$ as described above results in the following transition system. As states, we have pairs $(d, \sigma)$ where $d$ and $\sigma$ are states of $\text{Merge}^{kl,m}$ and $\text{Wire}^{m,l}$, respectively. As initial state, we have $(*, \epsilon)$. As actions we have $r_k(d)$ for each $d \in D$. As transitions we have the following:

- for each $d \in D$ and $\sigma \in D^*$: $(*, \sigma) \xrightarrow{r_k(d)} (d, \sigma)$, $(d, \sigma) \xrightarrow{\tau} (*)$, and $(*, \sigma d) \xrightarrow{\tau} (d, \sigma)$.

This transition system is represented graphically in Fig. 3.8 for the case where $D = \{0, 1\}$.

Let us also look at the following transition system. As states, we have only *. Consequently, * is the initial state. As actions we have $r_k(d)$ for each $d \in D$. As transitions we have the following:

- for each $d \in D$, a transition $* \xrightarrow{r_k(d)} *$.

This transition system described the behaviour of a special node in a network, viz. a sink. A sink consumes data, but does not deliver it anywhere.
3. Abstraction

These two transition systems are branching bisimulation equivalent. Take the following relation:

\[ B = \{((d, \sigma), *) \mid d \in D \cup \{\ast\}, \sigma \in D^*\} \]

It is easy to see that \( B \) is a branching bisimulation. The important point here is that, for each transition \((d, \sigma) \xrightarrow{\tau} (*)d\sigma)\) of the first transition system, the conditions imposed on a branching bisimulation permit that the states \((d, \sigma)\) and \((*, d\sigma)\) are both related to the state \(*\) of the second transition system; and for each transition \((*, \sigma d) \xrightarrow{\tau} (d, \sigma)\) of the first transition system, the conditions imposed on a branching bisimulation permit that the states \((*, d\sigma)\) and \((d, \sigma)\) are both related to the state \(*\) of the second transition system.

### 3.4 Example: Alternating bit protocol

We continue with the example of Sect. 2.4 concerning the ABP. At the end of that section, we presented the transition system that was the result of parallel composition and encapsulation of the transition systems of the sender \( S \), the data transmission channel \( K \), the acknowledgement transmission channel \( L \) and the receiver \( R \) as described earlier in that section.

Most transitions of that transition system concern internal actions. The behaviour of the ABP after abstraction from the internal actions is described as follows:

\[ \tau_I(\partial_H(S \parallel_\gamma K \parallel_\gamma L \parallel_\gamma R)) \]

where

\[ I = \{c_3(f) \mid f \in F\} \cup \{c_4(f) \mid f \in F \cup \{\ast\}\} \]
\[ \cup \{c_5(b) \mid b \in B \cup \{\ast\}\} \cup \{c_6(b) \mid b \in B \cup \{\ast\}\} \]

and \( H \) and \( \gamma \) are as in Section 2.4 Parallel composition, encapsulation and abstraction of the transition systems of \( S, K, L \) and \( R \) as described above results in the following transition system. We have the same states as before. As actions, we have \( r_1(d) \) and \( s_2(d) \) for each \( d \in D \). As transitions, we have:

- for each datum \( d \in D \) and bit \( b \in B \):
  - \((*, b, 0), (*, 0), (*, 0), (*, b, 0)) \xrightarrow{r_1(d)} ((d, b, 1), (*, 0), (*, 0), (*, b, 0)),
  - \((d, b, 1), (*, 0), (*, b, 0)) \xrightarrow{s_2(d)} ((d, b, 2), ((d, b, 1), (*, 0), (*, b, 0)),
  - \((d, b, 2), ((d, b, 1), (*, 0), (*, b, 0)) \xrightarrow{r_1(d)} ((d, b, 2), ((d, b, 2), (*, 0), (*, b, 0)),
  - \((d, b, 2), ((d, b, 2), (*, 0), (*, b, 0)) \xrightarrow{s_2(d)} ((d, b, 2), (*, 0), (*, b, 2)),
  - \((d, b, 2), (*, 0), (b, 1), (b, 2)) \xrightarrow{r_1(d)} ((d, b, 2), (*, 0), (b, 1), (b, 0)),
  - \((d, b, 2), (b, 1), (b, 2)) \xrightarrow{s_2(d)} ((d, b, 2), (*, 0), (b, 2), (b, 0)),
  - \((d, b, 2), (b, 1), (b, 2), (b, 0)) \xrightarrow{r_1(d)} ((b, 0), (*, 0), (*, 0), (*, b, 0)),
  - \((d, b, 2), (b, 1), (b, 2), (b, 0)) \xrightarrow{s_2(d)} ((b, 0), (*, 0), (*, 0), (*, b, 0)),
3.4 Example: Alternating bit protocol

The transition system for the whole protocol is represented graphically in Fig. 3.3 for the case where only one datum is involved.

Next, we consider the following transition system. As states, we have $d \in D \cup \{\ast\}$. The initial state is $\ast$. As actions, we have $r_1(d)$ and $s_2(d)$ for each $d \in D$. As transitions, we have the following:

- for each datum $d \in D$:
  - a transition $\ast \xrightarrow{r_1(d)} d$
  - a transition $d \xrightarrow{s_2(d)} \ast$

This transition system describes the behaviour of a bounded buffer with capacity 1, but the actions of a bounded buffer as introduced in Example 1.1.3 have been renamed.

The two transition systems presented above are branching bisimulation equivalent. For each $d \in D$, we define the sets $R(d)$ and $S(d)$ of states of the first transition system to be related to states $d$ and $\ast$, respectively, of the second transition system:

$$R(d) = \{(d, b, 1), (\ast, 0), (\ast, b, 0), (d, b, 2), (d, b, 1), (\ast, 0), (\ast, b, 0),
(d, b, 2), (d, b, 2), (\ast, 0), (\ast, b, 0), (d, b, 2), (\ast, 0), (d, b, 1),
(d, b, 2), (d, b, 1), (\ast, 0), (\ast, b, 0), (d, b, 2), (d, b, 3), (\ast, 0), (\ast, b, 0),
(d, b, 2), (\ast, 0), (\ast, b, 0), (d, b, 2), (d, b, 1), (\ast, 0), (\ast, b, 0),
(d, b, 2), (\ast, 0), (\ast, b, 0), (d, b, 2), (d, b, 3), (\ast, 0), (\ast, b, 0)\};$$
Fig. 3.9. Transition system for the ABP after abstraction from internal actions

\[ S(d) = \{(d, b, 2), (\ast, 0), (\ast, b, 2)\}, ((d, b, 2), (\ast, 0), (b, 1), (\ast, \overline{b}, 0)), ((d, b, 2), (\ast, 0), (\ast, b, 2)), ((d, b, 2), (\ast, 0), (b, 0), (\ast, \overline{b}, 0)), ((d, b, 2), (\ast, 0), (b, 2), (\ast, \overline{b}, 0)), ((d, b, 2), (\ast, 0), (b, 3), (\ast, \overline{b}, 0)), ((d, b, 1), (\ast, 0), (\ast, \overline{b}, 0)), ((d, b, 2), (\ast, 0), (b, 1), (\ast, \overline{b}, 0)), ((d, b, 2), (\ast, 0), (\ast, \overline{b}, 0))\} . \]

In Fig. 3.9, the states from the sets \( R(d) \) and \( S(d) \) are coloured light (yellow) and dark (red), respectively. Next we define the relation \( B \) as follows:

\[ B = \{(s, d) | d \in D, s \in R(d)\} \cup \{(s, \ast) | s \in \bigcup_{d \in D} S(d)\} . \]

It is straightforward to see that the conditions imposed on branching bisimulation equivalence permit that all states in \( R(d) \) are related to state \( d \) and
that all states in \( \bigcup_{d \in D} S(d) \) are related to state \( * \). In other words, the two transition systems presented above are branching bisimulation equivalent. This justifies the claim that, after abstraction from internal actions, the ABP behaves the same as a bounded buffer with capacity 1.

As a corollary, we have that the relation \( B' \) defined by
\[
B' = \bigcup_{d \in D} (R(d) \times R(d)) \cup (\bigcup_{d \in D} S(d) \times \bigcup_{d \in D} S(d))
\]
is a branching autobisimulation on the first transition system presented above. It is easy to show by means of \( B' \) that, although the transition systems for the channels are not determinate, the transition system for the whole protocol is determinate.

3.5 Petri nets and abstraction

For the interested reader, we now show that abstraction and branching bisimulation equivalence can be defined on nets as well.

Like with the definition of encapsulation on nets, the definition of abstraction on nets is similar to the definition of abstraction on transition systems. Here is the definition concerned.

**Definition 3.5.1 (Abstraction).** Let \( N = (P, A, \rightarrow, m_0) \) be a net. Let \( I \subseteq A \). The abstraction of \( N \) with respect to \( I \), written \( \tau_I(N) \), is the net \( (P, A', \rightarrow', m_0) \) where

- \( A' = A \setminus I \);
- \( \rightarrow' \) is the smallest subset of \( (\mathcal{P}_{\text{fin}}(P) \setminus \emptyset) \times A' \times (\mathcal{P}_{\text{fin}}(P) \setminus \emptyset) \) such that:
  - if \( Q_1 \xrightarrow{a} Q_2 \) and \( a \in I \), then \( Q_1 \xrightarrow{\tau'} Q_2 \);
  - if \( Q_1 \xrightarrow{a} Q_2 \) and \( a \notin I \), then \( Q_1 \xrightarrow{\rightarrow'} Q_2 \).

Branching bisimulation equivalence on nets is simply defined as branching bisimulation equivalence on their associated transition systems.

**Definition 3.5.2 (Branching bisimulation equivalence).** Let \( N = (P, A, \rightarrow, m_0) \) and \( N' = (P', A', \rightarrow', m'_0) \) be nets such that \( A = A' \). Then the nets \( N \) and \( N' \) are branching bisimulation equivalent, written \( N \leftrightarrow_b N' \), if \( \mathcal{T}(N) \leftrightarrow_b \mathcal{T}(N') \).

As explained in Sect. 1.5, different from transition systems, nets may indicate that transitions can occur simultaneously. By identifying branching bisimulation equivalent nets, this aspect of process behaviour described by nets is no longer covered. Let us look at an example.

**Example 3.5.1 (Bounded counter).** We consider the bounded counter from Example 1.1.2. In this example, we focus on the behaviour of a bounded counter with bound 2. It can simply be described by the following net. As places of the counter with bound 2, we have the natural numbers 0 to 2. As initial marking, we have \( \{0\} \). As actions, we have inc and dec. As transitions, we have the following:
• for each place $i \in \{0, 1\}$: $\{i\} \xrightarrow{\text{inc}} \{i + 1\}$ and $\{i + 1\} \xrightarrow{\text{dec}} \{i\}$.

It is easy to see that all reachable markings of this net are singleton sets. The marking $\{i\}$ indicates that the value of the counter is $i$. Next, we consider a net that is branching bisimulation equivalent to the one just presented. As places, we have the natural numbers 0 to 3. As initial marking, we have $\{0, 2\}$. As actions, we still have inc and dec. As transitions, we have the following:

• $\{0\} \xrightarrow{\text{inc}} \{1\}$, $\{1, 2\} \xrightarrow{\tau} \{0, 3\}$, $\{3\} \xrightarrow{\text{dec}} \{2\}$.

Unlike the first net, the second net indicates that if both inc and dec can occur, they can occur simultaneously. Nevertheless, these nets are branching bisimulation equivalent.

The association of a transition system $\mathcal{T}(N)$ with each net $N$ is also useful in showing the close connection between abstraction of nets and abstraction of transition systems.

**Property 3.5.1.** Let $N = (P, A, \rightarrow, m_0)$ be a net, and let $I \subseteq A$. Then we have that

$$\mathcal{T}(\tau_I(N)) \equiv \tau_I(\mathcal{T}(N)).$$

Similar connections were already shown for parallel composition and encapsulation in Sect. 2.5.

### 3.6 Miscellaneous

#### Programs and abstraction

In Sects. 1.3 and 2.3, we have seen that the behaviour of programs upon execution can be described in a straightforward way by means of transition systems and parallel composition of transition systems. Is abstraction from internal actions relevant in this area as well? In most programming languages, there are no features related to this kind of abstraction. This is to be expected: programs are primarily intended to be executed, not to be analyzed; whereas transition systems are primarily intended to be analyzed.

#### Trace equivalence

In Sect. 1.6 trace equivalence was defined as follows. Let $T = (S, A, \rightarrow, s_0)$ be a transition system. A trace of $T$ is a sequence $\sigma \in A^*$ such that $s_0 \xrightarrow{\sigma} s$ for some $s \in S$. We write $\text{traces}(T)$ for the set of all traces of $T$. Then two transition systems $T$ and $T'$ are trace equivalent, written $T \equiv_{\text{tr}} T'$, if $\text{traces}(T) = \text{traces}(T')$. With the adapted definition of the generalized transitions of a transition system, this means that in the case of trace equivalence we simply leave out all unobservable actions. Because it does not matter in the case of trace equivalence at which stages choices occur, this is all right.
4. Composition

In Chap. 2 we have seen that, by means of parallel composition, a transition system can be composed of others that act concurrently and interact with each other. This is not the only conceivable way of composition. This chapter treats several basic ways in which transition systems can be composed of others that do not interact with each other. Sequential composition is used to describe that a transition systems is composed of two others that act successively. Alternative composition is used to describe that a transition system is composed of two others that act the one or the other. Iteration is used to describe that a transition system is composed of two others of which the first one acts repeatedly until the second one takes over. Many transition systems can be composed using these three ways of composition. Thus, they support mastering the complexity of large transition systems. First of all, we explain informally what alternative composition, sequential composition and iteration are, and give simple examples of their use in describing process behaviour (Sect. 4.1). After that, we first adapt the definitions of transition system, parallel composition, encapsulation and abstraction from Chap. 3 to the addition of alternative composition, sequential composition and iteration (Sect. 4.2), and then define alternative composition, sequential composition and iteration in a mathematically precise way (Sect. 4.3). We also use these operations to define the components of the simple data communication protocol from Sect. 2.4 (Sect. 4.4). Next, we have another look at bisimulation equivalence and trace equivalence (Sect. 4.5). Finally, we look at some miscellaneous issues (Sect. 4.6).

4.1 Informal explanation

The alternative composition of two transition systems $T$ and $T'$ is a transition system describing that there is a choice between the behaviour described by $T$ and the behaviour described by $T'$. The choice is resolved at the instant that one of them performs its first action. The sequential composition of two transition systems $T$ and $T'$ is a transition system describing that the behaviour described by $T$ and the behaviour described by $T'$ follow each other. The notion of a transition system needs to be adapted in the presence of sequential composition because $T'$ should only take over on successful
termination of $T$. The iteration of transition system $T$ with exit transition system $T'$ is a transition system describing that initially there is a choice between the behaviour described by $T$ and the behaviour described by $T'$, and upon successful termination of $T$ there is this choice again. Often, we need to describe that a transition system simply acts repeatedly for ever. Such a no-exit iteration can be treated as a special case of iteration with exit (see Sect. 4.3). The no-exit iteration of transition system $T$ is a transition system describing that initially there is the behaviour described by $T$, and upon successful termination of $T$ the behaviour is again as initially. Here are a couple of examples.

Example 4.1.1 (Simple telephone system). We consider the simple telephone system from Example 1.1.1. Recall that in this telephone system each telephone is provided with a process, called its basic call process, to establish and maintain connections with other telephones. Actions of this process include receiving an off-hook or on-hook signal from the telephone, receiving a dialed number from the telephone, sending a signal to start or to stop emitting a dial tone, ring tone or ring-back tone to the telephone, and receiving an alert signal from another telephone – indicating an incoming call. Initially, there is a choice between the following two alternatives:

- receiving an off-hook signal from the telephone followed by a process of which the first action is sending a signal to start emitting a dial tone to the telephone;
- receiving an alert signal from another telephone followed by a process of which the first action is sending a signal to start emitting a ring tone to the telephone.

In either case the basic call process goes back to waiting for another off-hook or alert signal after the call is terminated. Therefore, the behaviour of the basic call process of a telephone can be described as the no-exit iteration of a process that is itself the alternative composition of two subprocesses, one reacting to an off-hook signal sent to the basic call process and the other reacting to an alert signal sent to the basic call process. The first one of these subprocesses first goes through a dialling phase and after that through a calling phase. So, the behaviour of this process can itself be described as the sequential composition of a subprocess for the dialling phase and a subprocess for the calling phase. And so forth.

Example 4.1.2 (Telephone answering machine controller). In order to control telephone answering, the control component of an answering machine has to communicate with the recorder component of the answering machine, the telephone network, and the telephone connected with the answering machine. When an incoming call is detected, the answering is not started immediately:

- if the incoming call is broken off or the receiver of the telephone is lifted within a certain period, answering is discontinued;
• otherwise, an off-hook signal is issued to the network when this period has elapsed and after that a pre-recorded message is played.

Upon termination of the message, the recorder is started and a beep signal is issued to the network. The recorder is stopped when:
• either the call is broken off;
• or a certain time period has passed in the case where the call has not been broken off earlier.

Thereafter, an on-hook signal is issued to the network. The behaviour of the control component can be described as the no-exit iteration of a process that is itself the sequential composition of three subprocesses, one checking whether the receiver is not lifted when an incoming call is detected, one controlling the answering with the pre-recorded message, and one controlling the recording of a message from the caller. Each of these subprocesses must respond properly if the call is broken off prematurely. Therefore, the behaviour of each of them can be described as an alternative composition with one of the alternatives reacting to signals indicating that the call is broken off prematurely.

4.2 Adjustment of earlier definitions

In the previous section, we have prepared the way for the formal definitions of the notions of alternative composition of transition systems, sequential composition of transition systems, and iteration of transition systems. Before we give those definitions in the next section, we first adapt the definitions of the notions of a transition system, parallel composition, encapsulation and abstraction from Chap. 3.

We already mentioned that the notion of a transition system needs to be adapted, because sequential composition requires that successful termination is distinguished from becoming inactive.

**Definition 4.2.1 (Transition system).** A transition system $T$ is a quintuple $(S, A, →, ↓, s_0)$ where

• $S$ is a set of states;
• $A$ is a set of actions;
• $→ \subseteq S \times A_r \times S$ is a set of transitions;
• $↓ \subseteq S$, with $s \in ↓$ only if there are no $a \in A_r$ and $s' \in S$ such that $s \xrightarrow{a} s'$, is a set of successfully terminating states;
• $s_0 \in S \setminus ↓$ is the initial state.

We write $s \downarrow$ instead of $s \in ↓$. The set $→ \subseteq S \times A^* \times S$ of generalized transitions of $T$ is the smallest subset of $S \times A^* \times S$ satisfying:

• $s \xrightarrow{a} s$ for each $s \in S$;
• if $s \xrightarrow{\tau} s'$, then $s \xrightarrow{} s'$;
• if $s \xrightarrow{a} s'$, then $s \xrightarrow{a} s'$;
• if $s \xrightarrow{\sigma} s'$ and $s' \xrightarrow{\sigma'} s''$, then $s \xrightarrow{\sigma \sigma'} s''$.

A state $s \in S$ is called a reachable state of $T$ if there is a $\sigma \in A^*$ such that $s_0 \xrightarrow{\sigma} s$. A state $s \in S$ is called a terminal state of $T$ if there is no $a \in A$ and $s' \in S$ such that $s \xrightarrow{a} s'$.

Notice that only terminal states may be successfully terminating states. Moreover, the initial state may not be a successfully terminating state. This excludes transition systems that can terminate successfully without performing any action. Such transition systems are not excluded here because they are problematic in whatever way. However, their inclusion would clutter up the coming definitions.

When looking at those definitions, it is important to take the following into account. A transition system can be made more intelligible by a judicious choice of states. However, the identity of the states of a transition system are never really relevant to the behaviour described by the transition system. For example, transition systems that differ only with respect to the identity of their states are bisimulation equivalent. Hence, we can ignore the identity of the states of a transition system. If transition systems differ only with respect to the identity of their states, they are called isomorphic. Here is the mathematically precise definition.

**Definition 4.2.2 (Isomorphy).** Let $T = (S, A, \rightarrow, \downarrow, s_0)$ and $T' = (S', A', \rightarrow', \downarrow', s'_0)$ be transition systems such that $A = A'$. Then $T$ and $T'$ are isomorphic if there exists a bijective relation $B \subseteq S \times S'$ such that the following conditions hold:

1. $B(s_0, s'_0)$;
2. whenever $B(s_1, s'_1)$ and $B(s_2, s'_2)$, then $s_1 \xrightarrow{a} s_2$ if and only if $s'_1 \xrightarrow{a'} s'_2$;
3. whenever $B(s, s')$, then $s \downarrow$ if and only if $s' \downarrow'$.

We will always consider two transition systems the same if they are isomorphic, and write $T = T'$ if $T$ and $T'$ are isomorphic. Because of this, the disjointness requirement on the sets of states that occurs in the definitions of alternative composition, sequential composition and iteration given below does not cause any loss of generality. Moreover, it does not matter that an arbitrary fresh initial state is chosen in the case of alternative composition and iteration: up to isomorphism the result is independent of the particular choice.

Unreachable states, and transitions between them, are never really relevant to the behaviour described by the transition system. For example, transition systems that differ only with respect to unreachable states are bisimulation equivalent. In fact, we are only interested in connected transition systems.
Definition 4.2.3 (Connected transition system). Let \( T = (S, A, \rightarrow, \downarrow, s_0) \) be a transition system. Then the set of reachable states of \( T \), written \( \text{reach}(T) \), is \( \{ s \in S \mid \text{for some } \sigma \in A^* : s_0 \xrightarrow{\sigma} s \} \); and the set of not immediately reachable states of \( T \), written \( \text{reach}^+(T) \), is \( \{ s' \in S \mid \text{for some } s \in \text{reach}(T) \setminus \{ s_0 \}, \sigma \in A^* : s \xrightarrow{\sigma} s' \} \). The transition system \( T \) is called a connected transition system if \( S = \text{reach}(T) \).

All operations defined in this chapter result in connected transition systems if they are applied to connected transition systems. Notice that either \( \text{reach}(T) = \text{reach}^+(T) \) or \( \text{reach}(T) = \text{reach}^+(T) \cup \{ s_0 \} \), depending on whether \( s_0 \) is reachable from other states than \( s_0 \).

A further restriction to finitely branching or countably branching transition systems and/or to finite or regular transition systems is often made.

Definition 4.2.4 (Classification of transition systems). Let \( T = (S, A, \rightarrow, \downarrow, s_0) \) be a transition system. Then \( T \) is a finitely branching transition system if for all \( s \in \text{reach}(T) \) we have that the set \( \{ (a, s') \in A \times S \mid s \xrightarrow{a} s' \} \) is finite, and \( T \) is a countably branching transition system if for all \( s \in \text{reach}(T) \) we have that the set \( \{ (a, s') \in A \times S \mid s \xrightarrow{a} s' \} \) is countable. Furthermore, \( T \) is a finite transition system if the set \( \{ (\sigma, s) \in A^* \times S \mid s_0 \xrightarrow{\sigma} s \} \) is finite, and \( T \) is a regular transition system if the set \( \{ s \in S \mid \text{for some } \sigma \in A^* : s_0 \xrightarrow{\sigma} s \} \), is finite.

Here is an example showing that it also makes sense to distinguish successfully terminating states in a setting without operations such as sequential composition.

Example 4.2.1 (Factorial and greatest common divisor programs). We consider again the transition systems describing the behaviours of PASCAL programs upon execution from Examples 1.3.1 and 1.3.2. State 7 of the transition system for the factorial program and state 8 of the transition system for the greatest common divisor program are intended to be successfully terminating states. However, this cannot be made explicit with the definition of transition system from Chap. 1. With the definition of transition system given in this chapter, we can designate the above-mentioned states as successfully terminating states of those transition systems. As an aside, we mention that the transition systems from Examples 1.3.1 and 1.3.2 are connected, finitely branching, and regular.

Because successfully terminating states are now distinguished from other terminal states, we have to adapt the definitions of parallel composition, encapsulation and abstraction from Chap. 3 as well. The new definitions are nothing else but simple adjustments of the earlier definitions to cover successfully terminating states.

Definition 4.2.5 (Parallel composition). Let \( T = (S, A, \rightarrow, \downarrow, s_0) \) and \( T' = (S', A', \rightarrow', \downarrow', s'_0) \) be transition systems. Let \( \gamma \) be a communication
function on a set of actions that includes \( A \cup A' \). The \textit{parallel composition} of \( T \) and \( T' \) under \( \gamma \), written \( T \parallel T' \), is the transition system \((S'', A'', \rightarrow'', \downarrow'', s''_0)\) where

- \( S'' = S \times S' \);
- \( A'' = A \cup A' \cup \{ \gamma(a,a') \mid a \in A, a' \in A', \gamma(a,a') \text{ is defined} \} \);
- \( \rightarrow'' \) is the smallest subset of \( S'' \times A'' \times S'' \) such that:
  - if \( s_1 \xrightarrow{a} s_2 \) and \( s' \in S' \), then \((s_1, s') \xrightarrow{a} (s_2, s') \);
  - if \( s'_1 \xrightarrow{b'} s'_2 \) and \( s \in S \), then \((s, s'_1) \xrightarrow{b'} (s, s'_2) \);
  - if \( s_1 \xrightarrow{a} s_2, s'_1 \xrightarrow{b'} s'_2 \) and \( \gamma(a, b) \) is defined, then \((s_1, s'_1) \xrightarrow{\gamma(a,b)} (s_2, s'_2) \);
- \( \downarrow'' \) is the smallest subset of \( S'' \) such that:
  - if \( s \downarrow \) and \( s' \downarrow' \), then \((s, s') \downarrow'' \);
- \( s''_0 = (s_0, s'_0) \).

What is new in this definition of parallel composition, compared with the definition from Chap. 3, concerns successful termination. Notice that the parallel composition of two transition systems \( T \) and \( T' \) can only terminate successfully when both \( T \) and \( T' \) can terminate successfully.

**Definition 4.2.6 (Encapsulation).** Let \( T = (S, A, \rightarrow, \downarrow, s_0) \) be a transition system. Let \( H \subseteq A \). The \textit{encapsulation} of \( T \) with respect to \( H \), written \( \partial_H(T) \), is the transition system \((S', A', \rightarrow', \downarrow', s'_0)\) where

- \( S' = \{ s \mid \text{for some } \sigma \in (A \setminus H)^* : s_0 \xrightarrow{\sigma} s \} \);
- \( A' = \{ a \in A \setminus H \mid \text{for some } s_1, s_2 \in S': s_1 \xrightarrow{a} s_2 \} \);
- \( \rightarrow' \) is the smallest subset of \( S' \times A' \times S' \) such that:
  - if \( s_1 \xrightarrow{a} s_2, s_1 \in S' \) and \( a \not\in H \), then \( s_1 \xrightarrow{a'} s_2 \);
- \( \downarrow' = \downarrow \cap S' \).

Like in the case of parallel composition, what is new in this definition of encapsulation, compared with the definition from Chap. 3, concerns successful termination. Notice that the encapsulation of a transition system \( T \) can only terminate successfully when \( T \) could terminate successfully. Notice further that successfully terminating states of \( T \) may become unreachable by encapsulation. Here is an example of successful termination in parallel composition and encapsulation.

**Example 4.2.2 (Successful termination in parallel composition and encapsulation).** We consider the following two transition systems, which are closely related to the ones of Example 3.2.1. As actions of the first transition system, we have \( s_1(0) \) and \( s_2(0) \). As states of the first transition system, we have natural numbers \( i \in \{0, 1, 2, 3\} \), with 0 as initial state and 3 as only successfully terminating state. As transitions of the first transition system, we have the following:

\[
0 \xrightarrow{\cdot} 1, \quad 0 \xrightarrow{\cdot} 2, \quad 1 \xrightarrow{s_1(0)} 3, \quad 2 \xrightarrow{s_2(0)} 3.
\]

As actions of the second transition system, we have only \( r_1(0) \). As states of the second transition system, we have natural numbers \( i \in \{0, 1\} \), with 0 as
4.2 Adjustment of earlier definitions

4.2.2 Adjustment of earlier definitions

\[ \begin{align*}
0 & \xrightarrow{\tau} 1 \\
1 & \xrightarrow{\tau} 2 \\
2 & \xrightarrow{\tau} 3 \\
3 & \xrightarrow{\tau} 1
\end{align*} \]

\[ \begin{align*}
\partial_H
& \begin{cases}
\gamma = (0,0) \\
\gamma = (1,0) \\
\gamma = (2,0) \\
\gamma = (3,1)
\end{cases}
\end{align*} \]

**Fig. 4.1.** Transition systems of Example 4.2.2

initial state and 1 as only successfully terminating state. As transitions of the second transition system, we have the following:

\[ 0 \xrightarrow{r_1(0)} 1. \]

These two transition systems are represented graphically in Fig. 4.1. Parallel composition of these transitions systems and subsequent encapsulation with respect to actions \( s_1(0), r_1(0), s_2(0) \) and \( r_2(0) \) result in the following transition system. As actions of the resulting transition system, we have only \( c_1(0) \). As states of the resulting transition system, we have the pairs \((0,0), (1,0), (2,0)\) and \((3,1)\), with \((0,0)\) as initial state and \((3,1)\) as only successfully terminating state. As transitions of the resulting transition system, we have the following:

\[ (0,0) \xrightarrow{\tau} (1,0), (0,0) \xrightarrow{\tau} (2,0), (1,0) \xrightarrow{c_1(0)} (3,1). \]

The resulting transition system, which is also represented graphically in Fig. 4.1, is capable of either first performing a silent step, next performing a communication action, and then terminating successfully or first performing a silent step and then becoming inactive. In the case of Example 4.2.1 i.e. in the setting without successful termination, it was not clear from the transition system that states \((3,1)\) and \((2,0)\) are states of a different nature.

**Definition 4.2.7 (Abstraction).** Let \( T = (S, A, \rightarrow, \downarrow, s_0) \) be a transition system. Let \( I \subseteq A \). The **abstraction** of \( T \) with respect to \( I \), written \( \tau_I(T) \), is the transition system \( (S, A', \rightarrow', \downarrow, s_0) \) where

- \( A' = A \setminus I \);
- \( \rightarrow' \) is the smallest subset of \( S \times A' \times S \) such that:
  - if \( s_1 \xrightarrow{a} s_2 \) and \( a \in I \), then \( s_1 \xrightarrow{\tau'} s_2 \),
  - if \( s_1 \xrightarrow{a} s_2 \) and \( a \notin I \), then \( s_1 \xrightarrow{a'} s_2 \).

\(^1\) In graphical representations of transition systems, we indicate the successfully terminating state by an outgoing unlabeled arrow.
Like in the cases of parallel composition and encapsulation, what is new in this definition of abstraction, compared to the definition from Chap. 3, concerns successful termination. Notice that the abstraction of a transition system $T$ can only terminate successfully when $T$ could terminate successfully. Notice further that successfully terminating states of $T$ never become unreachable by abstraction.

### 4.3 New definitions

Let us now look at the formal definitions of alternative composition, sequential composition, and (single-exit) iteration.

**Definition 4.3.1 (Alternative composition).** Let $T = (S, A, \rightarrow, \downarrow, s_0)$ and $T' = (S', A', \rightarrow', \downarrow', s_0')$ be transition systems such that $S \cap S' = \emptyset$. The alternative composition of $T$ and $T'$, written $T + T'$, is the transition system $(S'', A'', \rightarrow'', \downarrow'', s_0'')$ where

- $S'' = \{s_0''\} \cup \text{reach}^+ (T) \cup \text{reach}^+ (T')$;
- $A'' = A \cup A'$;
- $\rightarrow''$ is the smallest subset of $S'' \times A'' \times S''$ such that:
  - if $s_0 \rightarrow a s$, then $s_0'' \rightarrow'' a s$;
  - if $s_0' \rightarrow' a' s'$, then $s_0'' \rightarrow'' a'' s'$;
  - if $s_1 \rightarrow a s_2$ and $s_1 \in S''$, then $s_1 \rightarrow'' a'' s_2$;
  - if $s_1' \rightarrow' a' s_2'$ and $s_1' \in S''$, then $s_1' \rightarrow'' a'' s_2'$;
- $\downarrow''$ is the smallest subset of $S''$ such that:
  - if $s \downarrow$, then $s \downarrow''$;
  - if $s' \downarrow'$, then $s' \downarrow''$;
- $s_0'' \notin S \cup S'$.

The following things should be noted about the definition of alternative composition. The alternative composition of transition systems $T$ and $T'$ has a fresh initial state. This fresh initial state adopts the transitions from the initial state of $T$ and the transitions from the initial state of $T'$. However, the fresh initial state does not replace the initial states of $T$ and $T'$. Thus, transitions to the initial state of $T$ or $T'$ do not lead to transitions to the fresh initial state. The latter transitions would imply that the choice, that should be there only initially, could come back later. Here is an example to illustrate that it is quite natural to look at certain real-life processes as the alternative composition of other processes.

**Example 4.3.1 (Railroad crossing controller).** We consider a simple railroad crossing controller. An approach signal is sent to the controller as soon as a train passes a detector placed backward from the gate. An exit signal is sent to the controller as soon as the train passes another detector placed forward from the gate. The controller is able to receive approach and exit signals from
the train detectors at any time. When the controller receives an approach signal, a lower signal must be sent to the gate. When the controller receives an exit signal, a raise signal must be sent to the gate. Suppose that \( A \) and \( E \) are the transition systems describing the behaviours of the subprocesses dedicated to receiving and handling an approach signal and an exit signal, respectively, in the case where the signal is received at the beginning of a cycle of the controller, i.e. when there is no previous signal being handled. Then the behaviour of one cycle of the controller is described by \( A + E \).

Let us also give an example illustrating the details of alternative composition.

**Example 4.3.2 (Alternative composition).** We assume a set of data \( D \), and two input ports \( k \) and \( l \). For \( d \in D \), let \( R_k(d) \) and \( R_l(d) \) be the transition systems \( (S, A, \rightarrow, \downarrow, s_0) \) and \( (S', A', \rightarrow', \downarrow', s'_0) \) where

\[
S = \{(k, *), (k, d)\}, \quad S' = \{(l, *), (l, d)\}, \quad A = \{r_k(d)\}, \quad A' = \{r_l(d)\},
\]

\[
\rightarrow = \{(k, *) \xrightarrow{r_k(d)} (k, d)\}, \quad \rightarrow' = \{(l, *) \xrightarrow{r_l(d)} (l, d)\},
\]

\[
\downarrow = \{(k, d)\}, \quad \downarrow' = \{(l, d)\}, \quad s_0 = (k, *), \quad s'_0 = (l, *).
\]

The transition system \( R_i(d) \) is capable of receiving \( d \) at port \( i \) and then terminating successfully \((i = k, l)\). The alternative composition \( R_k(d) + R_l(d) \) is the transition system \( (S'', A'', \rightarrow'', \downarrow'', s''_0) \) where

\[
S'' = \{(*, *), (k, d), (l, d)\}, \quad A'' = \{r_k(d), r_l(d)\},
\]

\[
\rightarrow'' = \{(*, *) \xrightarrow{r_k(d)} (k, d), (*, *) \xrightarrow{r_l(d)} (l, d)\},
\]

\[
\downarrow'' = \{(k, d), (l, d)\}, \quad s''_0 = (*, *).
\]

This transition system is capable of receiving datum \( d \) at port \( k \) or \( l \) and then terminating successfully. The alternative composition of \( R_k(d) \) and \( R_l(d) \) is represented graphically in Fig. 4.2.

**Definition 4.3.2 (Sequential composition).** Let \( T = (S, A, \rightarrow, \downarrow, s_0) \) and \( T' = (S', A', \rightarrow', \downarrow', s'_0) \) be transition systems such that \( S \cap S' = \emptyset \). The **sequential composition** of \( T \) and \( T' \), written \( T \cdot T' \), is the transition system \( (S'', A'', \rightarrow'', \downarrow'', s''_0) \) where

- \( S'' = (S \cup S') \setminus \downarrow; \)
- \( A'' = A \cup A'; \)
- \( \rightarrow'' \) is the smallest subset of \( S'' \times A'' \times S'' \) such that:
  - if \( s_1 \xrightarrow{a} s_2 \) and not \( s_2 \xrightarrow{\downarrow} \), then \( s_1 \xrightarrow{a''} s_2; \)
  - if \( s_1 \xrightarrow{a} s_2 \) and \( s_2 \xrightarrow{\downarrow} \), then \( s_1 \xrightarrow{a''} s'_0; \)
  - if \( s'_1 \xrightarrow{a'} s'_2 \), then \( s'_1 \xrightarrow{a''} s'_2. \)
The definition of sequential composition is the first definition of a way in which transition systems can be composed where successfully terminating states are relevant to the transitions of the resulting transition system. Notice that, in the sequential composition of transition systems \( T \) and \( T' \), the initial state of \( T' \) replaces all successfully terminating states of \( T \). However, it does not become a successfully terminating state itself. Here is an example to illustrate that it is quite natural to look at certain real-life processes as the sequential composition of other processes.

**Example 4.3.3 (Railroad crossing controller).** We look again at the railroad crossing controller from Example 4.3.1. Suppose that \( R(\text{appr}) \) and \( R(\text{exit}) \) are the transition systems describing the behaviours of the subprocesses dedicated to receiving an approach signal and an exit signal, respectively. Suppose that \( D \) and \( U \) are the transition systems describing the behaviours of the subprocesses dedicated to handling an approach signal and an exit signal, respectively, that is received at the beginning of a cycle of the controller. Then the behaviour of one cycle of the controller is described by \( (R(\text{appr}) \cdot D) + (R(\text{exit}) \cdot U) \).

Let us also give an example illustrating the details of sequential composition.

**Example 4.3.4 (Sequential composition).** We assume a set of data \( D \) and one output port \( m \). For \( d \in D \), let \( S_m(d) \) be the transition system \( (S', A', \rightarrow', \downarrow', s'_0) \) where

\[
S' = \{(m, d), (m, *)\}, \\
A' = \{s_m(d)\}, \\
\rightarrow' = \{(m, d) \xrightarrow{s_m(d)} (m, *)\}, \\
\downarrow' = \{(m, *)\}, \\
s'_0 = (m, d).
\]
The transition system $S_m(d)$ is capable of sending datum $d$ at port $m$ and then terminating successfully. Let $R_k(d) + R_l(d)$ be as defined in Example 4.3.2. The sequential composition $(R_k(d) + R_l(d)) \cdot S_m(d)$ is the transition system $(S'', A'', \rightarrow'', \downarrow'', s_0'')$ where

$$
S'' = \{(*, *), (m, d), (m, *)\}, \\
A'' = \{r_k(d), r_l(d), s_m(d)\}, \\
\rightarrow'' = \{(*, *), r_k(d) \rightarrow (m, d), (m, d) \rightarrow m(d), (m, d) \rightarrow (m, *)\}, \\
\downarrow'' = \{(m, *)\}, \\
s_0'' = (*, *).
$$

This transition system is capable of receiving datum $d$ at port $k$ or $l$, next sending datum $d$ at port $m$ and then terminating successfully. The sequential composition of $R_k(d) + R_l(d)$ and $S_m(d)$ is represented graphically in Fig. 4.3.

**Definition 4.3.3 (Iteration).** Let $T = (S, A, \rightarrow, \downarrow, s_0)$ and $T' = (S', A', \rightarrow', \downarrow', s_0')$ be transition systems such that $S \cap S' = \emptyset$. The iteration of $T$ with exit $T'$, written $T \uparrow T'$, is the transition system $(S'', A'', \rightarrow'', \downarrow'', s_0'')$ where

- $S'' = \{s_0''\} \cup \text{reach}^+(T) \cup \text{reach}^+(T')$;
- $A'' = A \cup A'$;
- $\rightarrow''$ is the smallest subset of $S'' \times A'' \times S''$ such that:
  - if $s_0 \xrightarrow{a} s_2$ and not $s_2 \downarrow$, then $s_0'' \xrightarrow{a''} s_2''$;
  - if $s_0 \xrightarrow{a} s_2$ and $s_2 \downarrow$, then $s_0'' \xrightarrow{a''} s_0''$;
  - if $s_0' \xrightarrow{a'} s_2'$, then $s_0'' \xrightarrow{a''} s_2''$;
  - if $s_1 \xrightarrow{a} s_2$, $s_1 \in S''$ and not $s_2 \downarrow$, then $s_1 \xrightarrow{a''} s_2'$;
  - if $s_1 \xrightarrow{a} s_2$, $s_1 \in S''$ and $s_2 \downarrow$, then $s_1 \xrightarrow{a''} s_0''$;
  - if $s_1' \xrightarrow{a'} s_2'$ and $s_1' \in S''$, then $s_1' \xrightarrow{a''} s_2''$;

Fig. 4.3. Sequential composition of $R_k(d) + R_l(d)$ and $S_m(d)$
Like in the case of alternative composition, the iteration of transition systems $T$ with exit transition system $T'$ has a fresh initial state that adopts the transitions from the initial state of $T$ and the transitions from the initial state of $T'$. Again, this is needed because otherwise choices could come back unintentionally. Like in the case of sequential composition, successfully terminating states are relevant to the transitions of the resulting transition system. In the case of iteration, the fresh initial state replaces all successfully terminating states of $T$. In this way, the choice, that is there initially, will come back after successful termination of $T$. Here is an example to illustrate that it is quite natural to look at certain real-life processes as the iteration with exit of other processes.

**Example 4.3.5 (Railroad crossing controller).** We look once more at the railroad crossing controller from Examples 4.3.1 and 4.3.3. In this example, we take into account that, because of fault tolerance considerations, approach signals should always cause the gate to go down, and exit signals should be ignored while the gate is going down. Suppose that $S(\text{lower})$ is the transition system describing the behaviour of the subprocess dedicated to sending a lower signal. The behaviour of the subprocess dedicated to handling an approach signal that is received at the beginning of a cycle of the controller is described by $(R(\text{appr}) + R(\text{exit})) \cdot S(\text{lower})$. This is the transition system $D$ referred to in Example 4.3.3.

As mentioned in Sect. 4.1, no-exit iteration can be treated as a special case of iteration with exit. Here follows the precise definition.

**Definition 4.3.4 (No-exit iteration).** The no-exit iteration of $T$, written $T^\omega$, is the transition system $T \ast T'$, where $T'$ is the transition system $((\{s_0\}, \emptyset, \emptyset, \emptyset, s_0))$.

Here is an example to illustrate that it is quite natural to look at certain real-life processes as the no-exit iteration of other processes.

**Example 4.3.6 (Railroad crossing controller).** We look again at the railroad crossing controller from Examples 4.3.1 and 4.3.3. The transition system $(R(\text{appr}) \cdot D) + (R(\text{exit}) \cdot U)$ from Example 4.3.3 describes the behaviour of one cycle of the controller. The behaviour of the controller is described by $((R(\text{appr}) \cdot D) + (R(\text{exit}) \cdot U))^\omega$.

Let us also give an example illustrating the details of (no-exit) iteration.

**Example 4.3.7 (Merge connection).** Let $(R_k(d) \text{ and } R_l(d)) \cdot S_m(d)$ be as defined in Example 4.3.3. The no-exit iteration $((R_k(d) + R_l(d)) \cdot S_m(d))^\omega$ is the transition system $(S', A', \rightarrow', \downarrow', s_0')$ where
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**Definition 4.3.5 (Atomic transition system).** Let \( a \) be an action. The atomic transition system performing \( a \) is the transition system \( (\{s_0, s_1\}, \{a\}, \{s_0 \xrightarrow{a} s_1\}, \{s_1\}, s_0) \) where \( s_0 \) and \( s_1 \) are fresh states. The inactive transition system is the transition system \( (\{s_0\}, \emptyset, \emptyset, \emptyset, s_0) \) where \( s_0 \) is a fresh state. If no confusion can arise, the atomic transition system performing \( a \) is simply denoted by \( a \). The inactive transition system is denoted by \( \delta \).

Bear in mind that it does not matter that arbitrary fresh states are chosen, as up to isomorphism the result is independent of the particular choice. Notice that the inactive transition system \( \delta \) is used in the definition of no-exit iteration: \( T^\omega = T^* \delta \).

This transition system is isomorphic to the second transition system given for a merge connection in Example 1.6.3 in the case where \( D \) is a singleton set. The no-exit iteration of \( (R_k(d) + R_l(d)) \cdot S_m(d) \) is represented graphically in Fig. 4.4.

![Fig. 4.4. No-exit iteration of \((R_k(d) + R_l(d)) \cdot S_m(d)\) ](image)

\[
S' = \{\ast, (m, d)\},
A' = \{r_k(d), r_l(d), s_m(d)\},
\rightarrow' = \{\ast \xrightarrow{r_k(d)} (m, d), \ast \xrightarrow{r_l(d)} (m, d), (m, d) \xrightarrow{s_m(d)} \ast\},
\downarrow' = \emptyset,
s'_0 = \ast.
\]
Like for parallel composition, we use the convention of association to
the left for alternative composition and sequential composition. The need to
use parentheses is further reduced by ranking the precedence of the binary
operations on transition systems. We adhere to the following precedence rules:

- the operation + has lower precedence than all others;
- the operation · has higher precedence than all others;
- all other operations have the same precedence.

For example, we write \( x \cdot z + y \cdot z \) for \((x \cdot z) + (y \cdot z)\).

Here are a couple of examples of the composition of transition systems
starting from atomic transition systems. These examples show a way to
present transition systems that is quite different from the way that we used
before. It looks to be a more convenient way. We will return to this later in
Chapter 5.

**Example 4.3.8 (Bounded buffer).** We consider again the bounded buffer from
Example 1.1.3. We restrict ourselves to the case where its capacity is 1 and it
can only keep bits, i.e. \( D = \{0, 1\} \). Using alternative composition, sequential
composition and iteration, its behaviour can be described as follows:

\[
(\text{add}(0) \cdot \text{rem}(0) + \text{add}(1) \cdot \text{rem}(1))^{\omega}.
\]

**Example 4.3.9 (Split and merge connections).** We consider again the split
connection from Example 1.6.2 and the merge connection from Example 1.6.3.
We restrict ourselves once more to the case where only bits are involved,
i.e. \( D = \{0, 1\} \). Using alternative composition, sequential composition and
iteration, the behaviour of the split connection and the merge connection can
be described as follows:

\[
(r_k(0) \cdot (s_{l}(0) + s_{m}(0)) + r_k(1) \cdot (s_{l}(1) + s_{m}(1)))^{\omega}
\]
and

\[
((r_k(0) + r_l(0)) \cdot s_{m}(0) + (r_k(1) + r_l(1)) \cdot s_{m}(1))^{\omega}.
\]

Here is another example, showing that the behaviour of simple PASCAL
programs upon execution can also be described using alternative composition,
sequential composition and iteration.

**Example 4.3.10 (Factorial program).** We consider again the PASCAL pro-
gram to calculate factorials from Example 1.3.1. Using alternative compo-
sition, sequential composition and iteration, the behaviour of this program
upon execution can be described as follows:

\[
(\text{read}(n)) \cdot (i := 0) \cdot (f := 1) \cdot
(((i < n) \cdot (i := i + 1) \cdot (f := f \ast i)) \ast (\text{NOT } i < n)) \cdot (\text{write}(f)).
\]
For reasons of readability, we have enclosed all atomic transition systems in parentheses. We cannot directly give a transition system describing the behaviour of a program upon execution on a machine by means of atomic transition systems, alternative composition, sequential composition and iteration. Nor we can give a transition system describing the behaviour of the machine on which the program is executed in this way. For the machine, as well as a category of simple programs, it is possible if we use in addition parallel composition, encapsulation and abstraction. However, it requires special tricks. The kind of tricks needed here, will be illustrated later in Example 6.3.1. We will see later in Chapter 5 that we can do better if it is in addition possible to define transition systems recursively.

4.4 Example: Alternating bit protocol

We continue with the example of Sects. 2.4 and 3.4 concerning the ABP. Here, we describe the behaviour of the sender \( S \), the data transmission channel \( K \), the acknowledgement transmission channel \( L \) and the receiver \( R \) using alternative composition, sequential composition and iteration.

We restrict ourselves to the case where the set \( D \) of data is finite. Thus, we will use the following abbreviation. Let \( \mathcal{I} = \{i_1, \ldots, i_n\} \) be an index set and \( T_i \) be a transition system for each \( i \in \mathcal{I} \). Then we write \( \sum_{i \in \mathcal{I}} T_i \) for \( T_{i_1} + \ldots + T_{i_n} \). We further use the convention that \( \sum_{i \in \mathcal{I}} T_i \) stands for \( \delta \) if \( \mathcal{I} = \emptyset \).

The behaviour of the sender \( S \) can be described as follows:

\[
\left( \sum_{d \in D} r_1(d) \cdot s_3(d,0) \cdot \left( (r_5(1) + r_5(\ast)) \cdot s_3(d,0) \ast r_5(0) \right) \right) \ast \left( \sum_{d \in D} r_1(d) \cdot s_3(d,1) \cdot \left( (r_5(0) + r_5(\ast)) \cdot s_3(d,1) \ast r_5(1) \right) \right) \omega
\]

The behaviour of the receiver \( R \) can be described as follows:

\[
\left( \left( \left( \sum_{d \in D} r_4(d,1) + r_4(\ast) \right) \cdot s_6(1) \right) \ast \sum_{d \in D} r_4(d,0) \right) \cdot s_2(d) \cdot s_6(0) \cdot \left( \left( \left( \sum_{d \in D} r_4(d,0) + r_4(\ast) \right) \cdot s_6(0) \right) \ast \sum_{d \in D} r_4(d,1) \right) \cdot s_2(d) \cdot s_6(1) \right) \omega
\]

The behaviour of the data transmission channel \( K \) can be described as follows:

\[
\left( \sum_{f \in F} r_3(f) \cdot (i \cdot s_4(f) + i \cdot s_4(\ast)) \right) \omega
\]

The behaviour of the acknowledgement transmission channel \( L \) can be described as follows:
The transition systems for $S$, $R$, $K$ and $L$ presented above using alternative composition, sequential composition and iteration are bisimulation equivalent to the ones presented in Sect. 2.4. The transition systems for $K$ and $L$ are even isomorphic to the ones presented in Sect. 2.3.

4.5 Bisimulation and trace equivalence

Because successfully terminating states are now distinguished from other terminal states, the definition of branching bisimulation equivalence needs to be adapted as well. The new definition is nothing but a rather simple adjustment of the earlier definition reflecting that successful termination is now a capability that counts as well:

- if states $s$ and $s'$ are related and $s$ is a successfully terminating state in $T$, then there is some successfully terminating state $s''$ in $T'$ such that a generalized transition with a sequence of zero or more silent steps as label is possible from $s'$ to $s''$, and $s$ and $s''$ are related;
- likewise, with the role of $T$ and $T'$ reversed.

Definition 4.5.1 (Branching bisimulation). Let $T = (S, A, \rightarrow, \downarrow, s_0)$ and $T' = (S', A', \rightarrow', \downarrow', s_0')$ be transition systems such that $A = A'$. Then a branching bisimulation $B$ between $T$ and $T'$ is a binary relation $B \subseteq S \times S'$ such that the following conditions hold:

1. $B(s_0, s_0')$;
2. whenever $B(s_1, s'_1)$ and $s_1 \xrightarrow{a} s_2$ then either $a = \tau$ and $B(s_2, s'_1)$ or there are states $s''_1, s'_2$ such that $s''_1 \xrightarrow{\tau} s''_2 \xrightarrow{a'} s'_2$ and $B(s_1, s''_1)$ and $B(s_2, s'_2)$;
3. whenever $B(s_1, s'_1)$ and $s'_1 \xrightarrow{a} s'_2$, then either $a = \tau$ and $B(s_1, s'_2)$ or there are states $s''_1, s_2$ such that $s_1 \xrightarrow{\tau} s''_1 \xrightarrow{a} s_2$ and $B(s_1, s''_1)$ and $B(s_2, s'_2)$;
4. whenever $B(s, s')$ and $s \downarrow$, then there is a state $s''$ such that $s' \xrightarrow{\tau} s'' \xrightarrow{a'} s''$ and $B(s, s''')$;
5. whenever $B(s, s')$ and $s' \downarrow$, then there is a state $s''$ such that $s \xrightarrow{a} s'' \xrightarrow{\tau} s''$ and $B(s', s'')$.

Two transition systems $T$ and $T'$ are branching bisimulation equivalent, written $T \cong_B T'$, if there exists a branching bisimulation $B$ between $T$ and $T'$.

What is new in this definition of branching bisimulation equivalence, compared with the definition from Chap. 3, concerns again successful termination.

However, this generalization introduces an anomaly as we will demonstrate in the following example.
Example 4.5.1 (Non-preservation of branching bisimulation equivalence).
We consider again the transition systems from Example 3.1.2. We restrict ourselves to the case where only bits are involved, i.e. $D = \{0, 1\}$. Using atomic transition systems, alternative composition and sequential composition, they can be presented as follows:

$$r_1(0) \cdot (s_2(0) + \tau \cdot s_3(*)) + r_1(1) \cdot (s_2(1) + \tau \cdot s_3(*))$$

and

$$r_1(0) \cdot (s_2(0) + s_3(*)) + r_1(1) \cdot (s_2(1) + s_3(*)).$$

The second case is the first case with $\tau \cdot s_3(*)$ replaced by $s_3(*)$. The latter two transition systems are branching bisimulation equivalent, but the former two are not as explained in Example 3.1.2. Hence, branching bisimulation equivalence fails to be a congruence with respect to alternative composition.

This anomaly can simply be resolved by requiring that the initial states are related as in the case of standard bisimulation equivalence.

Definition 4.5.2 (Rooted branching bisimulation). Let $T = (S, A, \rightarrow, \downarrow, s_0)$ and $T' = (S', A', \rightarrow', \downarrow', s'_0)$ be transition systems such that $A = A'$. If $B$ is a branching bisimulation between $T$ and $T'$, then we say that a pair $(s_1, s'_1) \in S \times S'$ satisfies the root condition in $B$ if the following conditions hold:

1. whenever $s_1 \xrightarrow{a} s_2$, then there is a state $s'_2$ such that $s'_1 \xrightarrow{a'} s'_2$ and $B(s_2, s'_2)$;
2. whenever $s'_1 \xrightarrow{a'} s'_2$, then there is a state $s_2$ such that $s_1 \xrightarrow{a} s_2$ and $B(s_2, s'_2)$.

The two transition systems $T$ and $T'$ are rooted branching bisimulation equivalent, written $T \xrightarrow{rb} T'$, if there exists a branching bisimulation $B$ between $T$ and $T'$ such that the pair $(s_0, s'_0)$ satisfies the root condition in $B$.

Just as branching bisimulation equivalence, rooted branching bisimulation equivalence is preserved by parallel composition, encapsulation and abstraction. Moreover, it is preserved by alternative composition, sequential composition and iteration.

Property 4.5.1 (Preservation of rooted branching bisimulation equivalence). Let $T_1$ and $T_2$ be transition systems with $A$ as set of actions, let $T'_1$ and $T'_2$ be transition systems with $A'$ as set of actions, and let $\gamma$ be a communication function on a set of actions that includes $A \cup A'$. Then the following holds:

if $T_1 \xrightarrow{rb} T_2$ and $T'_1 \xrightarrow{rb} T'_2$, then $T_1 + T'_1 \xrightarrow{rb} T_2 + T'_2$;

$T_1 \cdot T'_1 \xrightarrow{rb} T_2 \cdot T'_2$, $T_1 \cdot T'_1 \xrightarrow{rb} T_2 \cdot T'_2$ and $T_1 \parallel \gamma T'_1 \xrightarrow{rb} T_2 \parallel \gamma T'_2$;

if $T_1 \xrightarrow{rb} T_2$, then $\partial_H(T_1) \xrightarrow{rb} \partial_H(T_2)$ and $\tau_I(T_1) \xrightarrow{rb} \tau_I(T_2)$. 
If we consider transition systems the same if they are rooted branching bisimulation equivalent, then both parallel composition and alternative composition are commutative and associative, and sequential composition is associative.

**Property 4.5.2 (Commutativity and associativity of binary operations).** Let $T_1$, $T_2$ and $T_3$ be transition systems with $A_1$, $A_2$ and $A_3$, respectively, as set of actions. Let $\gamma$ be a communication function on a set of actions that includes $A_1 \cup A_2 \cup A_3$. Then the following holds:

\[
\begin{align*}
T_1 + T_2 \leftrightarrow_{rb} & T_2 + T_1, \\
(T_1 + T_2) + T_3 \leftrightarrow_{rb} & T_1 + (T_2 + T_3), \\
(T_1 \cdot T_2) \cdot T_3 \leftrightarrow_{rb} & T_1 \cdot (T_2 \cdot T_3), \\
T_1 \parallel_{\gamma} T_2 \leftrightarrow_{rb} & T_2 \parallel_{\gamma} T_1, \\
(T_1 \parallel_{\gamma} T_2) \parallel_{\gamma} T_3 \leftrightarrow_{rb} & T_1 \parallel_{\gamma} (T_2 \parallel_{\gamma} T_3).
\end{align*}
\]

Transition systems can be reduced to connected transition systems as follows.

**Definition 4.5.3 (Reduction).** Let $T = (S, A, \rightarrow, \downarrow, s_0)$ be a transition system. Then the reduction of $T$, written $\text{red}(T)$, is the transition system $(S', A', \rightarrow', \downarrow', s_0)$ where

- $S' = \text{reach}(T)$;
- $A' = \{ a \in A \mid \text{for some } s, s' \in S'; s \xrightarrow{a} s' \}$;
- $\rightarrow' = \rightarrow \cap (S' \times A' \times S')$;
- $\downarrow' = \downarrow \cap S'$.

Any transition system is rooted branching bisimulation equivalent to its reduction, which is a connected transition system.

**Property 4.5.3.** Let $T$ be a transition system. Then the following holds:

\[T \leftrightarrow_{rb} \text{red}(T).\]

The definition of trace equivalence needs to be adapted to the setting with successful termination as well.

**Definition 4.5.4 (Trace equivalence).** Let $T = (S, A, \rightarrow, \downarrow, s_0)$ be a transition system. A terminating trace of $T$ is a sequence $\sigma \in A^*$ such that $s_0 \xrightarrow{\sigma} s$ and $s \downarrow$ for some $s \in S$. We write $\text{lang}(T)$ for the set of all terminating traces of $T$. Then two transition systems $T$ and $T'$ are trace equivalent, written $T \equiv_{tr} T'$, if $\text{traces}(T) = \text{traces}(T')$ and $\text{lang}(T) = \text{lang}(T')$.

In those cases where only the terminating traces of a transition system matter, an equivalence can be used that is even coarser than trace equivalence.

**Definition 4.5.5 (Language equivalence).** Two transition systems $T$ and $T'$ are language equivalent, written $T \equiv_{l} T'$, if $\text{lang}(T) = \text{lang}(T')$.

Obvious the terminology used here is based on viewing a transition system as an automaton by regarding its actions as symbols and its successfully terminating states as final states, cf. Sect. [1](#).
4.6 Miscellaneous

We have seen in Chap. 2 that it is slightly simpler to define parallel composition on nets than it is on transition systems. On the other hand, it is fairly complicated to define alternative composition, sequential composition and iteration on nets. For that reason, we will not show that alternative composition, sequential composition and iteration can be defined on nets as well.
5. Expressions and Recursion

Transition systems describing the behaviour of real-life systems are generally very large or even infinite. They become lightly unintelligible. Succinctness can be gained by using the operations introduced to compose transition systems of others. We have already illustrated this in Chaps. 2 and 4. However, the notation used there was introduced in an ad hoc and informal way. In order to preclude any difference of opinion about the form and intended meaning of the expressions concerned, called process expressions, we give in this chapter a syntax and semantics which describe in a mathematically precise way how to construct process expressions and how to assign meanings to them. In other words, we turn the informal notation used the preceding chapters into a formalized language. The formalization enables us to define transition systems, up to rooted branching bisimulation equivalence, by means of recursive specifications. First of all, we discuss some important issues concerning process expressions and recursive specifications (Sect. 5.1). After that, we first give the syntax of process expressions (Sect. 5.2) and then give the semantics of process expressions (Sect. 5.3). Next, we look at recursive specifications (Sect. 5.4). We also use recursive specification to define the components of the simple data communication protocol from Sect. 2.4 (Sect. 5.5) and to define the components of a workcell in a manufacturing system (Sect. 5.6).

5.1 Introduction

A main purpose of this chapter is to turn the informal notation used in the preceding chapters into a formalized language, and consequently to make the intended meaning of the expressions concerned fully precise. The way in which the notation was introduced does not make it really fully precise. Actually, we have used the same notation in different chapters for slightly different things. In order to make the intended meaning of the expressions concerned fully precise, we have to make the form of the expressions fully precise first. The expressions given in Examples 4.3.8, 4.3.9, 4.3.10 and 4.5.1 from Chap. 4 are all of the form that we will make precise in this chapter.

As already suggested after Example 4.3.10 recursive specifications add to the behaviours that can be defined by means of process expressions. It can
be guaranteed that the solutions of recursive specifications are unique up to
rooted branching bisimulation equivalence by imposing a weak restriction,
known as guardedness, on the shape of recursive specifications. This is one of
the reason why transition systems are no suitable candidates for the meanings
of process expressions. The meaning of each process expression should repre-
sent all transition systems rooted branching bisimulation equivalent to some
transition system. The solution is simply to have sets of transition systems
that are rooted branching bisimulation equivalent to some transition system
as meanings. We will call these meanings processes, hence the name process
expressions. It is easy to lift the operations on transition systems defined in
Chap. 4 to processes because rooted branching bisimulation equivalence is a
congruence with respect to those operations.

Example 5.1.1 (Process expressions). In the informal notation of Chap. 4,
\((r_k(0) + r_l(0)) \cdot s_m(0)\) and \(r_k(0) \cdot s_m(0) + r_l(0) \cdot s_m(0)\) denote different, but
rooted branching bisimulation equivalent, transition systems. As expressions
of the formalized language that is introduced in this chapter, they have the
same meaning. This is shown in detail later in Example 5.3.2.

We do not take all transition systems into consideration. Because un-
reachable states and transitions are not relevant to the behaviour described
by a transition system, we do not consider transition systems that are not
connected. In fact, we consider only finitely branching connected transition
systems. The reason for this is that, with finitely branching connected tran-
sition systems, we still cover all processes that are definable by means of
process expressions or specifiable by means of guarded recursion. An impor-
tant thing to remember here is that the operations on transition systems
defined in Chap. 4 result in connected transition systems if they are applied
to connected transition systems.

It is interesting that there are unguarded recursive specifications of which
all solutions consist of transition systems that are not finitely branching.
However, we do not consider unguarded recursive specifications, because not
all of them have a unique solution.

5.2 Syntax of process expressions

In the previous section, we have prepared the way for the formal definition
of the syntax and semantics of process expressions. We give the definition
of the syntax in this section, and the definition of the semantics in the next
section.

We assume a fixed but arbitrary set \(A\) of actions and a fixed but arbitrary
communication function \(\gamma : A \times A \to A\). The set \(A\) and the function \(\gamma\) can be
regarded as parameters of the language, instantiated for each application of
the language. Moreover, we assume a set \(V\) of process variables.
5.3 Semantics of process expressions

We are now ready to describe in a fully precise way how to construct process expressions. This is done by defining the set of process expressions inductively by formation rules.

**Definition 5.2.1 (Process expression).** The set of process expressions over $A$ is the smallest set $PE(A)$ satisfying:

- $x \in PE(A)$ for each variable $x \in \mathcal{V}$;
- $a \in PE(A)$ for each action $a \in A$;
- $\delta \in PE(A)$;
- if $p \in PE(A)$ and $q \in PE(A)$, then $(p + q) \in PE(A)$, $(p \cdot q) \in PE(A)$, $(p \ast q) \in PE(A)$, and $(p \parallel q) \in PE(A)$;
- if $p \in PE(A)$, then $p^\omega \in PE(A)$, $\partial_H(p) \in PE(A)$ for each $H \subseteq A$, and $\tau_I(p) \in PE(A)$ for each $I \subseteq A$.

A process expression is *closed* if it does not contain variables. We write $CPE(A)$ for the set $\{ p \in PE(A) | p \text{ is closed} \}$.

If the set of actions is clear or irrelevant, we write $CPE$ and $PE$ instead of $CPE(A)$ and $PE(A)$, respectively. Let us give an example of the construction of process expressions.

**Example 5.2.1 (Process expression).** We consider the expression $((r_k(0) + r_l(0)) \cdot s_m(0))$. Let $r_k(0)$, $r_l(0)$, $s_m(0) \in A$. Then $r_k(0)$, $r_l(0) \in PE(A)$. Hence, $(r_k(0) + r_l(0)) \in PE(A)$. Because $s_m(0) \in PE(A)$ as well, $((r_k(0) + r_l(0)) \cdot s_m(0)) \in PE(A)$. In other words, $((r_k(0) + r_l(0)) \cdot s_m(0))$ is a process expression.

All $a \in A$ and $\delta$ are called *constants*, and $+$, $\cdot$, $\ast$, $\parallel$, $^\omega$, $\partial_H$ (for $H \subseteq A$) and $\tau_I$ (for $I \subseteq A$) are called *operators*. Constants and operators are symbols. That is, the nature of constants and operators is purely syntactic.

In order to reduce the need to use parentheses, like in the case of the informal notation used in the preceding chapter, we use the convention of association to the left for the operators $+$, $\cdot$ and $\parallel$, and in addition the following precedence rules:

- the operator $+$ has lower precedence than all others;
- the operator $\cdot$ has higher precedence than all others;
- all other operators have the same precedence.

Moreover, we omit the outermost parentheses. For example, we write $x \cdot y \cdot z + w$ for $(((x \cdot y) \cdot z) + w)$.

**5.3 Semantics of process expressions**

We begin with defining the set $Pr(A)$. This set is the semantic domain for process expressions, which means that the meanings of process expressions are elements of $Pr(A)$. 
Definition 5.3.1 (Process). We consider transition systems as defined in Def. 4.2.1. We write $\mathcal{TS}(A)$ for the set of all finitely branching connected transition systems of which the set of actions is a subset of $A$. A process is an equivalence class of $\mathcal{TS}(A)$ with respect to rooted branching bisimulation. We write $[T]$ for the process $\{T' \in \mathcal{TS}(A) \mid T \xrightarrow{rb} T'\}$. We write $\mathcal{P}(A)$ for $\{[T] \mid T \in \mathcal{TS}(A)\}$, i.e. the set of all processes of which the set of actions is a subset of $A$. If a transition system $T \in \mathcal{TS}(A)$ is a member of a process $P \in \mathcal{P}(A)$, then $T$ is called a representative of $P$. A process $P$ is called a regular process if $P$ has a regular representative.

If the set of actions is clear or irrelevant, we write $\mathcal{TS}$ and $\mathcal{P}$ instead of $\mathcal{TS}(A)$ and $\mathcal{P}(A)$, respectively.

For process expressions that contain variables, the meanings depend on the meanings assigned to the variables. This is done by means of an assignment.

Definition 5.3.2 (Assignment). An assignment is a function $\alpha : \mathcal{V} \rightarrow \mathcal{P}$.

In order to assign meanings to process expressions, we have to give an interpretation to each constant and operator. The interpretation of each constant is a process and the interpretation of each operator is an operation on processes. Those operations on processes correspond to the operations on transition systems defined in Chap. 4. In order to distinguish the operators, the operations on processes corresponding to the operators and the operations on transition systems corresponding to the operations on processes from each other, we will write for each operator, say $o$, $o^P$ for the corresponding operation on processes and $o^{TS}$ for the corresponding operation on transition systems. It is important to remember that the operation denoted in this chapter by $o^{TS}$ was denoted by $o$ in Chap. 4.

Definition 5.3.3 (Interpretation of constants and operators). The interpretations of the constants and operators of $\mathcal{PE}(A)$ are defined as follows:

$$
\begin{align*}
\alpha^{P} & = [a^{TS}] \quad \text{(for each } a \in A) \\
\delta^{P} & = [\delta^{TS}] \\
[T]^{P} + [T']^{P} & = [T +^{TS} T'] \\
[T]^{P} . [T']^{P} & = [T .^{TS} T'] \\
[T]^{P} * [T']^{P} & = [T *^{TS} T'] \\
[T]^{P} || [T']^{P} & = [T ||^{TS} T'] \\
[T]^{P} & = [T^{TS}] \\
\partial_H^{P}([T]) & = [\partial_H^{TS}(T)] \quad \text{(for each } H \subseteq A) \\
\tau_I^{P}([T]) & = [\tau_I^{TS}(T)] \quad \text{(for each } I \subseteq A)
\end{align*}
$$
We say that $p$ assigns meanings to process expressions. This is done by defining evaluation that this is important.

**Definition 5.3.4 (Meaning of process expressions).**

Let $\alpha$ be an assignment. The evaluation function for $\alpha$, $\mathcal{M}_\alpha : \mathcal{P}(A) \to \mathcal{P}(A)$, is recursively defined as follows:

$$
\mathcal{M}_\alpha(x) = \alpha(x)
$$

$$
\mathcal{M}_\alpha(a) = a^\alpha
$$

(for each $a \in A$)

$$
\mathcal{M}_\alpha(\delta) = \delta^\alpha
$$

$$
\mathcal{M}_\alpha(p + q) = \mathcal{M}_\alpha(p) + ^\alpha \mathcal{M}_\alpha(q)
$$

$$
\mathcal{M}_\alpha(p \cdot q) = \mathcal{M}_\alpha(p) \cdot ^\alpha \mathcal{M}_\alpha(q)
$$

$$
\mathcal{M}_\alpha(p \ast q) = \mathcal{M}_\alpha(p) \ast ^\alpha \mathcal{M}_\alpha(q)
$$

$$
\mathcal{M}_\alpha(p \parallel q) = \mathcal{M}_\alpha(p) \parallel ^\alpha \mathcal{M}_\alpha(q)
$$

$$
\mathcal{M}_\alpha(p^\omega) = \mathcal{M}_\alpha(p)^\omega
$$

$$
\mathcal{M}_\alpha(\partial_H(p)) = \partial_H^\alpha(\mathcal{M}_\alpha(p))
$$

(for each $H \subseteq A$)

$$
\mathcal{M}_\alpha(\tau_I(p)) = \tau_I^\alpha(\mathcal{M}_\alpha(p))
$$

(for each $I \subseteq A$)

We say that $p = q$ holds iff $\mathcal{M}_\alpha(p) = \mathcal{M}_\alpha(q)$ for all assignments $\alpha$.

These interpretations of the operators are well-defined because rooted branching bisimulation equivalence is a congruence with respect to the corresponding operations on transition systems, so the choice of a representative from an equivalence class does not matter. Here is an example concerning the application of operations on processes defined above.

**Example 5.3.1 (Application of operations on processes).** We consider the following application of operations on processes defined above: $(r_k(0)^\nu + ^\nu r_l(0)^\nu) \cdot ^\nu s_m(0)^\nu$. It corresponds to one cycle of the merge connection from Example 1.6.3. We calculate the resulting process:

$$(r_k(0)^\nu + ^\nu r_l(0)^\nu) \cdot ^\nu s_m(0)^\nu
= ([r_k(0)^{TS} + ^\nu [r_l(0)^{TS}]) \cdot ^\nu [s_m(0)^{TS}]
= [r_k(0)^{TS} + ^\nu r_l(0)^{TS}] \cdot ^\nu [s_m(0)^{TS}]
= ([r_k(0)^{TS} + ^\nu r_l(0)^{TS}) \cdot ^\nu s_m(0)^{TS}]
= \{(r_k(0)^{TS} \cdot ^\nu r_l(0)^{TS}) \cdot ^\nu s_m(0)^{TS},
\quad r_k(0)^{TS} \cdot ^\nu r_l(0)^{TS} \cdot ^\nu s_m(0)^{TS})\}.
$$

The equivalence class $[[r_k(0)^{TS} + ^\nu r_l(0)^{TS}) \cdot ^\nu s_m(0)^{TS}]$ contains only two transition systems, because we consider transition systems to be the same if they are isomorphic. It is clear that $[r_k(0)^{TS} + ^\nu r_l(0)^{TS}) \cdot ^\nu s_m(0)^{TS}] = [r_k(0)^{TS} \cdot ^\nu r_l(0)^{TS} \cdot ^\nu s_m(0)^{TS}]$. In Example 5.3.2 we will see that this is important.

We are now in the position to describe in a fully precise way how to assign meanings to process expressions. This is done by defining evaluation functions, one for each assignment.
Clearly, the meanings of closed process expressions do not depend on the assignment concerned. Process expressions that contain variables are essential for recursive specification of processes. An important thing to note about process expressions is that all of them denote regular processes. Recursively specified processes need not be regular. We will return to this in Section 5.4.

Here is an example of the evaluation of process expressions.

**Example 5.3.2 (Meaning of process expressions).** We consider the process expression \((r_k(0) + r_l(0)) \cdot s_m(0)\) from Example 5.2.1. We assign a meaning to this process expression as follows:

\[
\begin{align*}
M_\alpha((r_k(0) + r_l(0)) \cdot s_m(0)) &= (M_\alpha(r_k(0)) + P r \ M_\alpha(r_l(0))) \cdot P r M_\alpha(s_m(0)) \\
&= [(r_k(0) + r_l(0)) \cdot s_m(0)]^P r \\
&= \{(r_k(0) + r_l(0)) \cdot s_m(0)\}^P r \\
&= (r_k(0) + r_l(0)) \cdot s_m(0).
\end{align*}
\]

From the third step, we made use of the calculations made in Example 5.3.1. As for any closed process expression, we see that the meaning assigned to \((r_k(0) + r_l(0)) \cdot s_m(0)\) does not depend on the assignment concerned. Similarly, we obtain:

\[
\begin{align*}
M_\alpha(r_k(0) \cdot s_m(0) + r_l(0) \cdot s_m(0)) &= [r_k(0) + r_l(0)] \cdot s_m(0) \\
&= \{(r_k(0) + r_l(0)) \cdot s_m(0)\}^P r \\
&= (r_k(0) + r_l(0)) \cdot s_m(0).
\end{align*}
\]

Thus, as to be expected, the process expressions \((r_k(0) + r_l(0)) \cdot s_m(0)\) and \(r_k(0) \cdot s_m(0) + r_l(0) \cdot s_m(0)\) are assigned the same meaning for all assignments. This means that \((r_k(0) + r_l(0)) \cdot s_m(0) = r_k(0) \cdot s_m(0) + r_l(0) \cdot s_m(0)\) holds.

In the preceding example, the meaning of process expressions is given in terms of atomic transition systems and operations on transition systems. In the following two examples, the meaning of process expressions is given directly in terms of transition systems.

**Example 5.3.3 (Milner’s scheduling problem).** We consider again the system of scheduled processes from Examples 1.5.2, 2.5.1 and 2.5.2. It is easy to see that the process that is the meaning of the process expression \((r_k(0) \cdot f_i(i))\)\(^\omega\) has the transition system for \(P_i\) \((1 \leq i \leq n)\) given in Example 2.5.2 as a representative. It is also easy to see that the process that is the meaning of the process expression \((g_1) \cdot \ldots \cdot g_n\)\(^\omega\) has the transition system for \(S\) given in Example 2.5.2 as a representative.
Example 5.3.4 (Binary memory cell). We consider again the binary memory cell from Example 1.5.1. It is easy to see that the process that is the meaning of the process expression

\[ (((rtr(0) + sto(0)) \cdot sto(1)) \cdot ((rtr(1) + sto(1)) \cdot sto(0)))^\omega \]

has the transition system for the binary memory cell given at the end of Example 1.5.1 as a representative.

5.4 Recursive specification

In this section, we first explain what a recursive specification is and after that we define it in a mathematically precise way.

An equation of the form \( X = t \), where \( X \) is a process variable and \( t \) is a process expression that contains no variables other than \( X \), determines a process, i.e., has a unique solution, if it satisfies a criterion known as guardedness. Roughly speaking, this means that \( X \) is always preceded by an action in \( t \). An equation of the above-mentioned form is called a recursive specification. A recursive specification that satisfies the guardedness criterion is called a guarded recursive specification. A solution for \( X = t \) is a process \( p \) such that \( X = t \) holds if \( X \) stands for \( p \). In case \( X = t \) is a guarded recursive specification, it has a unique solution for \( X \). The capabilities of that solution can be approximated to any finite depth \( n \) by taking \( t \) and replacing \( n \) times all occurrences of \( X \) by \( t \). It is easy to see that in the case where \( X = t \) is not guarded, there are occurrences of \( X \) that will inhibit a definite answer about a part of the capabilities, even to depth 1. Here are a couple of examples about guardedness and uniqueness of solutions.

Example 5.4.1 (Uniqueness of solutions). For any action \( a \in A \), the recursive specifications \( X = X \) and \( Y = a + Y \) are unguarded. Each process is a solution of \( X = X \). Replacing in the right-hand side of this equation the occurrences of \( X \) by the right-hand side, even repeatedly, does not reveal anything about the capabilities of a solution. Each process that has the option to perform action \( a \) and then to terminate successfully is a solution of \( Y = a + Y \). Replacing in the right-hand side of this equation the occurrences of \( Y \) by the right-hand side, even repeatedly, only confirms what was already known, viz. that a solution must be capable of performing action \( a \) and then terminating successfully. Let us now look at the guarded recursive specification \( Z = a \cdot Z \). Its only solution is the process that keeps performing action \( a \) forever. Replacing in the right-hand side of this equation the occurrences of \( Z \) by the right-hand side \( n \) times reveals that a solution must be capable of performing action \( a \) \( n + 1 \) times and then proceeding as \( Z \).

Example 5.4.2 (Existence of solutions). For any action \( a \in A \), the recursive specification \( X = a + X \cdot a \) is unguarded. A solution of this recursive
specification must be capable of either performing action \(a\) once and then terminating successfully, or performing action \(a\) twice and then terminating successfully, or performing action \(a\) three times and then terminating successfully, etc. Hence, a solution cannot be finitely branching. This means that \(X = a + X \cdot a\) has no solution, because only finitely branching transition systems are considered.

All of this extends from one equation to a set of equations where the left-hand sides of the equations are process variables and the right-hand sides of the equations are process expressions that contain only process variables that are among the ones on the left-hand sides of the equations. This allows a number of processes to be defined in terms of each other, which is known as mutual recursion. Let us give an example of mutual recursion.

**Example 5.4.3 (Bounded counter).** We consider once more the bounded counter from Example 1.1.2. We give a recursive specification for the case where the bound is 2:

\[
C_2^0 = \text{inc} \cdot C_2^1, \\
C_2^1 = \text{dec} \cdot C_2^0 + \text{inc} \cdot \text{dec} \cdot C_2^1.
\]

The counter of which the value is 0 (\(C_2^0\)) is defined in terms of the counter of which the value is 1 (\(C_2^1\)); and the counter of which the value is 1 (\(C_2^1\)) is defined in terms of both counters (\(C_2^0\) and \(C_2^1\)).

Let us now turn to the precise definitions of the notions of a recursive specification, a solution of a recursive specification, and guardedness of a recursive specification.

**Definition 5.4.1 (Recursive specification).** A recursive specification is a set of recursive equations \(E = \{X = t_X \mid X \in V\}\) where \(V\) is a set of process variables and each \(t_X\) is a process expression that only contains variables from \(V\). We denote the variables that occur in a recursive specification by \(X, X', Y, Y', \ldots\). Let \(E\) be a recursive specification. Then we write \(\text{var}(E)\) for the set of all variables that occur on the left-hand side of an equation in \(E\). Notice that infinite sets of recursive equations are not excluded.

**Definition 5.4.2 (Solution of recursive specification).** A solution of a recursive specification \(E\) is a set of processes \(\{p_X \in \text{Pr}(A) \mid X \in \text{var}(E)\}\) such that \(M_\alpha(X) = M_\alpha(t_X)\) for all equations \(X = t_X \in E\) if \(\alpha\) is an assignment such that \(\alpha(X) = p_X\) for all \(X \in \text{var}(E)\).

**Definition 5.4.3 (Guarded recursive specification).** Let \(t\) be a process expression containing a variable \(X\). We call an occurrence of \(X\) in \(p\) guarded if \(p\) has a subexpression of the form \(a \cdot q\), where \(a \in A\), with \(q\) a process expression containing this occurrence of \(X\). A recursive specification is called
a guarded recursive specification if all occurrences of variables on the right-hand sides of its equations are guarded or it can be rewritten to such a recursive specification using equations that hold and the equations of the recursive specification.

It is important to remember that guarded recursive specifications have unique solutions. Let us look at an example of guarded recursive specifications of processes.

**Example 5.4.4 (Split and merge connections).** We consider again the split connection from Example 1.6.2 and the merge connection from Example 1.6.3. As in Example 4.3.9, we restrict ourselves to the case where only bits are involved, i.e. \( D = \{0, 1\} \). The split connection and the merge connection can be recursively specified as follows:

\[
\text{Split}^{k,l,m} = (r_k(0) \cdot (s_l(0) + s_m(0))) + r_k(1) \cdot (s_l(1) + s_m(1))) \cdot \text{Split}^{k,l,m}
\]

and

\[
\text{Merge}^{k,l,m} = ((r_k(0) + r_l(0)) \cdot s_m(0) + (r_k(1) + r_l(1)) \cdot s_m(1)) \cdot \text{Merge}^{k,l,m}.
\]

The processes denoted by the process expressions given in Example 4.3.9 are the solutions of these recursive specifications.

Here is another example of guarded recursive specifications of processes.

**Example 5.4.5 (Bounded buffer).** We consider once more the bounded buffer from Example 1.1.3. Like in Example 4.3.8, we restrict ourselves to the case where it can only keep bits, i.e. \( D = \{0, 1\} \). We give guarded recursive specifications for the cases where its capacity is 1 and 2. The buffer with capacity 1 can be recursively specified as follows:

\[
B_1 = (\text{add}(0) \cdot \text{rem}(0) + \text{add}(1) \cdot \text{rem}(1)) \cdot B_1.
\]

The solution of this guarded recursive specifications is the process denoted by the process expression given in Example 4.3.8. The buffer with capacity 2 can be recursively specified as follows:

\[
B_2 = \text{add}(0) \cdot B_2' + \text{add}(1) \cdot B_2' ,
\]

\[
B_2' = \text{rem}(d) \cdot B_2 + \text{add}(0) \cdot \text{rem}(d) \cdot B_2' + \text{add}(1) \cdot \text{rem}(d) \cdot B_2' 
\]

(for every \( d \in \{0, 1\} \)).

The solution of this guarded recursive specification can be denoted by a process expression as well, but it is very clumsy.
5. Expressions and Recursion

It is not the case that the solution of each guarded recursive specification can be denoted by a process expression. In the following couple of examples, we give guarded recursive specifications of which the solution cannot be denoted by process expressions.

Example 5.4.6 (Unbounded counter). We consider an unbounded counter. The difference with a bounded counter is that its value can always be incremented. The unbounded counter can be recursively specified as follows:

\[ C = \text{inc} \cdot C' \cdot C, \]
\[ C' = \text{dec} + \text{inc} \cdot C' \cdot C'. \]

The solution of this guarded recursive specification cannot be denoted by a process expression. A representative of the solution of this guarded recursive specification is represented graphically in Fig. 5.1.

Example 5.4.7 (Unbounded buffer). We consider an unbounded buffer. The difference with a bounded buffer is that new data can always be added to the data that it keeps. Like in Example 5.4.6, we restrict ourselves to the case where it can only keeps bits, i.e. \( D = \{0, 1\} \). The unbounded buffer can be recursively specified as follows:

\[ B = B'_e \]
\[ B'_e = \text{add}(0) \cdot B'_0 + \text{add}(1) \cdot B'_1, \]
\[ B'_e \sigma d = \text{rem}(d) \cdot B'_e + \text{add}(0) \cdot B'_{0 \sigma d} + \text{add}(1) \cdot B'_{1 \sigma d} \]

(for every \( \sigma \in \{0, 1\}^* \) and \( d \in \{0, 1\} \)).

The solution of this guarded recursive specification cannot be denoted by a process expression either. A representative of the solution of this guarded recursive specification is represented graphically in Fig. 5.2.

All process expressions introduced at the beginning of this chapter denote regular processes. On the other hand, the solutions of guarded recursive specifications are not necessarily regular processes. In other words, the processes that are specifiable by means of guarded recursion include processes that are not definable by means of process expressions. For example, we have seen that the unbounded counters and buffers from Examples 5.4.6 and 5.4.7 cannot...
be denoted by process expressions. That is because unbounded counters and buffers are not regular.

In the recursive specifications given in the preceding examples, parallel composition and encapsulation do not appear. However, this is not excluded. Let us give an example.

Example 5.4.8 (Unbounded counter). We consider once more the unbounded counter from Example 5.4.6. It can also be recursively specified as follows:

\[ C = \text{inc} \cdot (\text{dec} \parallel C) . \]

Example 5.4.7 is the first occasion where a guarded recursive specification with a (countably) infinite number of equations is given. It is surprising that, if our language of process expressions is extended with operators for the renaming of actions, it becomes possible to specify the unbounded buffer with two equations. Parallel composition and encapsulation has to be used in that case as well.

A more advanced example of guarded recursive specification of processes is given in the next section.
5.5 Example: Alternating bit protocol

We continue with the example of Sects. 2.4, 3.4 and 4.4 concerning the ABP. Here, we give guarded recursive specifications of the sender $S$, the data transmission channel $K$, the acknowledgement transmission channel $L$ and the receiver $R$.

We restrict ourselves again to the case where the set $D$ of data is finite. Like in Sect. 4.4, we will use the sum notation. Let $\mathcal{I} = \{i_1, \ldots, i_n\}$ be an index set and $p_i$ be a process expression for each $i \in \mathcal{I}$. Then we write $\sum_{i \in \mathcal{I}} t_i$ for $t_{i_1} + \ldots + t_{i_n}$. We also use the convention that $\sum_{i \in \mathcal{I}} t_i$ stands for $\delta$ if $\mathcal{I} = \emptyset$.

As in Sects. 2.4 and 3.4, we write $\overline{b}$ for $1 - b$.

The guarded recursive specification of the sender $S$ consists of the following recursive equations:

$$S = S'_0,$$

$$S'_b = \sum_{d \in D} r_1(d) \cdot s_3(d,b) \cdot S''_{d,b}$$

(for every $b \in B$),

$$S''_{d,b} = (r_5(\overline{b}) + r_5(\ast)) \cdot s_3(d,b) \cdot S''_{d,b} + r_5(b) \cdot S'_b$$

(for every $d \in D$ and $b \in B$).

The guarded recursive specification of the receiver $R$ consists of the following recursive equations:

$$R = R'_0,$$

$$R'_b = (\sum_{d \in D} r_4(d,\overline{b}) + r_4(\ast)) \cdot s_6(\overline{b}) \cdot R'_b$$

$$+ \sum_{d \in D} r_4(d,b) \cdot s_2(d) \cdot s_6(b) \cdot R'_b$$

(for every $b \in B$).

The guarded recursive specification of the data transmission channel $K$ consists of the following recursive equation:

$$K = \sum_{f \in F} r_3(f) \cdot (i \cdot s_4(f) + i \cdot s_4(\ast)) \cdot K.$$

The guarded recursive specification of the acknowledgement transmission channel $L$ consists of the following recursive equation:

$$L = \sum_{b \in B} r_6(b) \cdot (i \cdot s_5(b) + i \cdot s_5(\ast)) \cdot L.$$

The processes denoted by the process expressions given in Sect. 4.4 are the solutions of these guarded recursive specifications.
5.6 Example: Workcell

Here is another example of the use of guarded recursion in describing the behaviour of systems. The example concerns a workcell. CIM (Computer Integrated Manufacturing) systems are usually constructed from several workcells connected to each other via some transport service, and controlled by some supervisor. A workcell is itself constructed from various connected components, including a workcell controller. The workcell described in this section is the same as the workcell with quality check described in [13].

The main purpose of this example, which is to illustrate that it is also possible to describe the behaviour of processes whose actions differ from those commonly found in pure software systems. This is important because many systems are composed of both software and hardware components. In this example, processes are involved that do not only send and receive messages, but also accept and deliver products. Another thing to note about this example is the following. The size and complexity of the system concerned exceed those of systems treated in preceding examples. A corresponding description at the level of transition systems would be fairly unintelligible.

The simple workcell described in this section consists of four components: a workstation, a transport service, a quality checker, and a workcell controller. The workstation accepts products, processes them, and delivers processed products of which the quality is either good or bad. The transport service accepts products at the one end, transports them, and delivers the transported products at the other end. The quality checker determines whether the processed products are good. A good product is passed, while a bad product is removed. When a product is removed, this is signalled to the workcell controller. The workcell controller controls the workcell. It receives instructions to process a certain number of products. When an instruction is received, it directs the workcell to do so. While the processing is going on, the workcell controller counts the number of products removed by the quality checker. When the processing is completed, the workcell controller directs the workcell to process again a number of products to compensate for the removed products. The configuration of the workcell is shown in Fig. 5.3. The four components are connected to each other and the environment by 12 ports. Ports 3 to 8, 10 and 11 are internal ports and ports 1, 2, 9 and 12 are external ports. Ports 1 to 8 are used to communicate data and ports 9 to 12 are used to exchange products.

Along ports 1, 3, 5 and 7 a message \textit{produce}(n) can be sent to indicate to the receiver that the workcell has to produce \textit{n} products. Along ports 2, 4, 6 and 8 a message \textit{ready} can be sent back to indicate that the component has fulfilled its part of the task. Along port 8 a message \textit{reject} can be sent back as well. This message indicates that a product has not been passed to the environment. At port 9 unprocessed products are exchanged. At ports 10, 11 and 12 processed products are exchanged.
We assume a finite set of unprocessed products $P_{in}$. Moreover, we assume that for each $p \in P_{in}$ there are a processed product of good quality denoted by $\text{proc}(p, ok)$ and a processed product of bad quality denoted by $\text{proc}(p, nok)$. We write $P^\text{ok}_{\text{out}}$ for the set $\{\text{proc}(p, ok) \mid p \in P_{in}\}$, $P^\text{nok}_{\text{out}}$ for the set $\{\text{proc}(p, nok) \mid p \in P_{in}\}$, and $P_{\text{out}}$ for the set $P^\text{ok}_{\text{out}} \cup P^\text{nok}_{\text{out}}$. We also assume that there is a bound $N$ on the number of products that the workcell can be requested to produce.

The workstation waits until a message $\text{produce}(n)$ is received from the workcell controller. When such a message is received, it accepts, processes and delivers $n$ products, and then sends the message $\text{ready}$ to the workcell controller. The workstation may deliver products of good quality as well as products of bad quality. After the message $\text{ready}$ has been sent, the workstation goes back to waiting for a message from the workcell controller. The guarded recursive specification of the workstation $W$ consists of the following recursive equations:

$$W = \sum_{n \leq N} r_3(\text{produce}(n)) \cdot W'_n,$$

$$W'_0 = s_1(\text{ready}) \cdot W,$$

$$W'_{n+1} = \sum_{p \in P_{in}} r_9(p) \cdot (i \cdot r_{10}(\text{proc}(p, ok)) + i \cdot r_{10}(\text{proc}(p, nok))) \cdot W'_n,$$

(for every $n < N$).

The action $i$ is again an internal action that cannot be performed synchronously with any other action. Thus, the workstation cannot be forced to produce products of good quality only.

The transport service waits until a message $\text{produce}(n)$ is received from the workcell controller. When such a message is received, it accepts, transports and delivers $n$ products, and then sends the message $\text{ready}$ to the workcell controller. The transport service may have to accept products from the workstation while there are accepted products that it could not deliver.
5.6 Example: Workcell

To the quality checker yet. After the message \textit{ready} has been sent, the transport service goes back to waiting for a message from the workcell controller. The guarded recursive specification of the transport service $T$ consists of the following recursive equations:

\[
T = \sum_{n \leq N} r_5(produce(n)) \cdot T'_{n,e},
\]

\[
T'_{0,e} = s_6(ready) \cdot T,
\]

\[
T'_{n+1,e} = \sum_{q \in P_{out}} r_{10}(q) \cdot T'_{n,q}
\]

(for every $n < N$),

\[
T'_{0,q} = s_{11}(q) \cdot T'_{0,\sigma}
\]

(for every $q \in P_{out}$ and $\sigma \in P_{out^*}$),

\[
T'_{n+1,q} = \sum_{q' \in P_{out}} r_{10}(q') \cdot T'_{n,q'q} + s_{11}(q) \cdot T'_{n+1,\sigma}
\]

(for every $n < N, q \in P_{out}$ and $\sigma \in P_{out^*}$).

The quality checker waits until a message \textit{produce}(n) is received from the workcell controller. When such a message is received, it checks $n$ products and then sends the message \textit{ready} to the workcell controller. After the message \textit{ready} has been sent, the quality checker goes back to waiting for a message from the workcell controller. Checking a product includes accepting the product and delivering the product if its quality is good. Each time that the quality checker encounters a product of which the quality is bad, it sends the message \textit{reject} to the workcell controller. The guarded recursive specification of the quality checker $Q$ consists of the following recursive equations:

\[
Q = \sum_{n \leq N} r_7(produce(n)) \cdot Q'_{n},
\]

\[
Q'_{0} = s_8(ready) \cdot Q,
\]

\[
Q'_{n+1} = \sum_{q \in P_{out}} r_{11}(q) \cdot s_{12}(q) \cdot Q'_{n} + \sum_{q \in P_{out^*}} r_{11}(q) \cdot s_8(reject) \cdot Q'_{n}
\]

(for every $n < N$).

The workcell controller waits until a message \textit{produce}(n) is received from the environment. When such a message is received, it sends the same message to the quality checker, the transport service and workstation, in that order. After that, the workcell controller waits for reception of the message \textit{ready} from the quality checker, the transport service and workstation, again in that order. If it receives instead the message \textit{reject} from the quality checker, it increments a counter of rejections and goes back to waiting for the ready
messages. When the ready messages are received, the workcell controller continues as follows. In the case where there are rejections, say \( n' (0 < n' \leq n) \), it first sends the message \( \text{produce}(n') \) to the quality checker, the transport service and workstation, in that order, and then goes back to waiting for ready messages. In the case where there are no rejections, it sends the message \( \text{ready} \) to the environment. The guarded recursive specification of the workcell controller \( C \) consists of the following recursive equations:

\[
C = \sum_{n \leq N} r_1(\text{produce}(n)) \cdot C'_n, \\
C'_0 = s_2(\text{ready}) \cdot C, \\
C'_{n+1} = s_7(\text{produce}(n+1)) \cdot s_5(\text{produce}(n+1)) \cdot s_3(\text{produce}(n+1)) \cdot C''_0 \\
\text{(for every } n < N), \\
C''_n = r_8(\text{ready}) \cdot r_6(\text{ready}) \cdot r_4(\text{ready}) \cdot C_n + r_8(\text{reject}) \cdot C'_{n+1} \\
\text{(for every } n < N), \\
C''_N = r_8(\text{ready}) \cdot r_6(\text{ready}) \cdot r_4(\text{ready}) \cdot C'_N.
\]

The whole workcell is described by

\[
\tau_1(\partial_H (C \parallel W \parallel T \parallel Q))
\]

where

\[
H = \{s_i(m), r_i(m) \mid i \in \{3, 5, 7\}, m \in \{\text{produce}(n) \mid n \in N\}\} \\
\cup \{s_i(m), r_i(m) \mid i \in \{4, 6, 8\}, m \in \{\text{ready}, \text{reject}\}\} \\
\cup \{s_i(p), r_i(p) \mid i \in \{10, 11\}, p \in P_{\text{out}}\}
\]

and

\[
I = \{c_i(m) \mid i \in \{3, 5, 7\}, m \in \{\text{produce}(n) \mid n \in N\}\} \\
\cup \{c_i(m) \mid i \in \{4, 6, 8\}, m \in \{\text{ready}, \text{reject}\}\} \\
\cup \{c_i(p) \mid i \in \{10, 11\}, p \in P_{\text{out}}\} \cup \{i\}.
\]

The workcell is considered to be correct if it behaves as follows in the case where there is a supplier that delivers an unlimited number of any one unprocessed product. The workcell, together with the supplier, waits until a message \( \text{produce}(n) \) is received from the environment. When such a message is received, it processes and delivers \( n \) products, and then sends the message \( \text{ready} \) to the environment. After the message \( \text{ready} \) has been sent, the workcell goes back to waiting for a message from the environment.

Let \( p_0 \) be a fixed but arbitrary member of \( P_{\text{in}} \). The guarded recursive specification of the supplier \( S \) consists of the following recursive equation:

\[
S = s_9(p_0) \cdot S.
\]
The workcell together with the supplier is described by

\[ \tau_I \cdot (\tau_H (S \parallel \tau_I (\tau_H (C \parallel W \parallel T \parallel Q)))) \]

where

\[ H' = \{ s_9(p), r_9(p) \mid p \in P_{in} \} \]

and

\[ I' = \{ c_0(p) \mid p \in P_{in} \} . \]

We can show that this process is the solution of the following guarded recursive specification:

\[
V = \sum_{n \leq N} r_1(produce(n)) \cdot V'_{n} \cdot V ,
\]

\[
V'_0 = s_2(ready) ,
\]

\[
V'_{n+1} = s_{12}(proc(p_0, ok)) \cdot V'_n
\]

(for every \( n < N \)).

This guarded recursive specification describes exactly what is considered in the preceding paragraph to be the correct behaviour of the workcell.
6. Selected topics

There are many interesting topics related to process expressions and guarded recursive specifications which are not treated in Chap. 5. This chapter treats some selected topics. First of all, we give the semantics of closed process expressions in an alternative way known as structural operational semantics (Sect. 6.1). After that, we give equational laws that hold (Sect. 6.2). We also look briefly at the expressive power of process expressions (Sect. 6.3) and an interesting restricted form of guarded recursive specification (Sect. 6.4).

6.1 Structural operational semantics

We still assume a fixed but arbitrary set $A$ of actions and a fixed but arbitrary communication function $\gamma : A \times A \rightarrow A$.

We associate a transition system with a closed process expression $p$ by taking the closed process expressions as states, with $p$ as initial state, and by defining the transitions using transition rules in the style of Plotkin. The way of giving semantics adopted is called structural operational semantics. The transition rules used to define the transitions have the form

$\phi_1, \ldots, \phi_m \Rightarrow \psi$,

which is to be read as

if $\phi_1$ and $\ldots$ and $\phi_m$, then $\psi$.

As customary, $\phi_1, \ldots, \phi_m$ and $\psi$ are called the premises and the conclusion, respectively. The premises and conclusions of a transition rule are of the form $t \xrightarrow{a} t'$, where $t \in \mathcal{PE}(A)$ and $t' \in \mathcal{PE}(A) \cup \{\sqrt{\cdot}\}$. The transition rules used (see Table 6.1) are actually transition rule schemas: $a$, $b$ and $c$ are placeholders for arbitrary members of $A_\tau$, and $H$ and $I$ are placeholders for arbitrary subsets of $A$. A side-condition is added to some of them to restrict the members of $A_\tau$ for which $a$, $b$ and $c$ are placeholders. In applying the transition rules, the

$^1$ A lot of theory has been developed about structural operational semantics (see e.g. \cite{[15]}).
Table 6.1. Transition rules for process expressions

| Rule | Description |
|------|-------------|
| $a \xrightarrow{\gamma} \checkmark$ | $x \xrightarrow{a} x'$ |
| $x + y \xrightarrow{a} y'$ | $x + y \xrightarrow{a} \checkmark$ |
| $x \xrightarrow{\gamma} \checkmark$ | $x \xrightarrow{\gamma} \checkmark$ |
| $x \xrightarrow{b} y$ | $x \xrightarrow{\gamma} \checkmark$ |
| $x \xrightarrow{a} x'$ | $x \xrightarrow{a} \checkmark$ |
| $x^* y \xrightarrow{a} x^* y'$ | $x^* y \xrightarrow{a} \checkmark$ |
| $x^* y \xrightarrow{a} y'$ | $x^* y \xrightarrow{a} \checkmark$ |
| $x^* y \xrightarrow{a} \checkmark$ | $x^* y \xrightarrow{a} \checkmark$ |
| $x \parallel y \xrightarrow{a} x' \parallel y$ | $x \parallel y \xrightarrow{a} x' \parallel y$ |
| $x \parallel y \xrightarrow{a} \checkmark$ | $x \parallel y \xrightarrow{a} \checkmark$ |
| $x \parallel y \xrightarrow{a} \gamma(a, b) = c$ | $x \parallel y \xrightarrow{a} \gamma(a, b) = c$ |
| $x \parallel y \xrightarrow{a} y'$ | $x \parallel y \xrightarrow{a} \checkmark$ |
| $x \parallel y \xrightarrow{a} \gamma(a, b) = c$ | $x \parallel y \xrightarrow{a} \gamma(a, b) = c$ |
| $x \xrightarrow{a} \checkmark$ | $x \xrightarrow{a} \checkmark$ |
| $\partial_H(x) \xrightarrow{a} \partial_H(x')$ | $\partial_H(x) \xrightarrow{a} \checkmark$ |
| $\tau_I(x) \xrightarrow{a} \tau_I(x')$ | $\tau_I(x) \xrightarrow{a} \checkmark$ |
| $\tau_I(x) \xrightarrow{a} \checkmark$ | $\tau_I(x) \xrightarrow{a} \checkmark$ |

process variables $x, x', y$ and $y'$ may be replaced by any process expression, but not by $\checkmark$.

Let $\rightarrow$ be the smallest subset of $CPE(A) \times \Lambda \times (CPE(A) \cup \{\checkmark\})$ satisfying the transition rules from Table 6.1. We can look at the members of $\rightarrow$ as follows:

- a member of the form $p \xrightarrow{a} p'$, where $p' \neq \checkmark$, indicates that the process denoted by $p$ is capable of first performing action $a$ and then proceeding as the process denoted by $p'$;
- a member of the form $p \xrightarrow{\gamma} \checkmark$ indicates that the process denoted by $p$ is capable of first performing action $a$ and then terminating successfully.

So, $\checkmark$ is introduced to represent successful termination. Notice that $\rightarrow$ has no members of the form $\checkmark \xrightarrow{a} p'$.

The transition rules from Table 6.1 provide an alternative way to assign meanings to process expressions.
Definition 6.1.1 (Meaning induced by the transition rules). Let \( p \in \text{CPE}(A) \). Then the meaning of \( p \) induced by the transition rules from Table 6.1, written \( \mathcal{M}'(p) \), is \( \text{red}(S, A, \rightarrow, \downarrow, s_0) \) where

- \( S = \text{CPE}(A) \cup \{\sqrt{\cdot}\} \);
- \( A = A \);
- \( \rightarrow \) is the smallest subset of \( \text{CPE}(A) \times A \times (\text{CPE}(A) \cup \{\sqrt{\cdot}\}) \) satisfying the transition rules from Table 6.1;
- \( \downarrow = \{\sqrt{\cdot}\} \);
- \( s_0 = p \).

Recall that \( \text{red} \) reduces any transition system to a connected transition system that is rooted branching bisimulation equivalent.

The meaning of a closed process expression induced by the transition rules coincides with its meaning according to Def. 5.3.4.

Property 6.1.1 (Equality of meanings). For every \( p \in \text{CPE}(A) \), we have that \( \mathcal{M}'(p) = \mathcal{M}_\alpha(p) \) for all assignments \( \alpha \).

6.2 Equational laws

In Table 6.2, a number of equations that hold are given. Many equations are actually equation schemas: \( a, b \) and \( c \) are placeholders for arbitrary members of \( A_\tau \cup \{\delta\} \), and \( H \) and \( I \) are placeholders for arbitrary subsets of \( A \). A side-condition is added to some of them to restrict the members of \( A_\tau \cup \{\delta\} \) for which \( a, b \) and \( c \) are placeholders. Notice that, unlike in the transition rules from Table 6.1, \( a, b \) and \( c \) are also placeholders for \( \delta \) in the equations from Table 6.2. Two auxiliary operators appear in Table 6.2: \( \| \) and \( | \). The operator \( \| \) is interpreted as left merge, which is the same as parallel composition except that the left merge of \( p_1 \) and \( p_2 \) starts with performing an action of \( p_1 \). The operator \( | \) is interpreted as communication merge, which is the same as parallel composition except that the communication merge of \( p_1 \) and \( p_2 \) starts with performing an action of \( p_1 \) and an action of \( p_2 \) synchronously. These interpretations are clearly reflected by the additional transition rules for \( \| \) and \( | \) given in Table 6.3. From the equations given in Table 6.2, we can derive many other equations that hold. Actually, we can derive all equations between closed process expressions in which only the operators +, \cdot, \|, \partial_H \) and \( \tau_I \) occur. Let us illustrate by means of a simple example what can be done with equational laws for process expressions.

Example 6.2.1 (Merge connection). We consider once more the merge connection from Example 1.6.3. Let \( T_1 \) and \( T_2 \) be the first and second transition system from Example 1.6.3 for the case where \( D = \{0, 1\} \). According to Definition 5.3.4, we assign to the process expressions

\[
(r_k(0) \cdot s_m(0) + r_l(0) \cdot s_m(0) + r_k(1) \cdot s_m(1) + r_l(1) \cdot s_m(1))^\omega
\]

we
Table 6.2. Equational laws for process expressions

| Equation                  | Law |
|---------------------------|-----|
| \( x + y = y + x \)      | A1  |
| \( (x + y) + z = x + (y + z) \) | A2  |
| \( x + x = x \)          | A3  |
| \( (x + y) \cdot z = x \cdot (z + y) \cdot z \) | A4  |
| \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \) | A5  |
| \( x + \delta = x \)     | A6  |
| \( \delta \cdot x = \delta \) | A7  |
| \( a \mid b = c \) if \( \gamma(a, b) = c \) | CF1 |
| \( a \mid b = \delta \) if \( \gamma(a, b) \) undefined | CF2 |
| \( x \parallel y = (x \parallel y \parallel x) + x \mid y \) | CM1 |
| \( a \parallel x = a \cdot x \) | CM2 |
| \( a \cdot x \parallel y = a \cdot (x \parallel y) \) | CM3 |
| \( (x + y) \parallel z = x \parallel (z + y) \parallel z \) | CM4 |
| \( a \cdot x \parallel b = (a \mid b) \cdot x \) | CM5 |
| \( a \mid b \cdot x = (a \mid b) \cdot x \) | CM6 |
| \( a \cdot x \mid b \cdot y = (a \mid b) \cdot (x \parallel y) \) | CM7 |
| \( (x + y) \mid z = x \mid (z + y) \mid z \) | CM8 |
| \( x \mid (y + z) = x \mid (y + x) \mid z \) | CM9 |

Table 6.3. Additional transition rules for \( \parallel \) and \( \mid \)

| Transition              | Law |
|-------------------------|-----|
| \( x \rightarrow x' \)  | B1  |
| \( x \rightarrow y \)   | B2  |
| \( y \rightarrow y' \)  | BKS1|
| \( x \parallel y \rightarrow x' \parallel y \) | BKS2|
| \( x \mid y \rightarrow x' \mid y' \) | BKS3|
| \( \gamma(a, b) = c \)  | NEI |

\[ ((\tau_h(0) + \tau_l(0)) \cdot s_m(0) + (\tau_h(1) + \tau_l(1)) \cdot s_m(1))^{\omega} \]

the meanings \([T_1]\) and \([T_2]\), respectively. The simplest way to show that \([T_1]\) equals \([T_2]\), is by applying equation A4 from Table 6.2. We do not have to construct a bisimulation, like in Example 1.6.3, to prove this.

The equations given in Table 6.2 constitute the axiom system \( ACP^{\tau^*} \) from [S].
6.3 Expressive power of process expressions

All regular processes can be denoted by process expressions.

Property 6.3.1 (Expressive power). Let \( \alpha \) be an arbitrary assignment. Then, for every \( P \in \mathcal{Pr}(A) \) that is regular, there exists a \( A' \supseteq A \) and a \( p \in \mathcal{CPE}(A') \) such that \( M_\alpha(p) = P \).

Although regular processes can be denoted by process expressions, it may easily become very clumsy. This is nicely illustrated in the following example.

Example 6.3.1 (Bounded counter). We consider once more the bounded counter from Example 1.1.2. In Example 1.1.2, a regular transition system was given for the bounded counter in a direct way. The corresponding recursive specifications was given in Example 5.4.3 for the case where the bound is 2. In this example, we give corresponding process expressions for the cases where the bound is 1 and 2. In the case where the bound is 1, the process expression is:

\[ (\text{inc} \cdot \text{dec})^\omega. \]

However, in the case where the bound is 2, the simplest process expression is:

\[ \tau_i(\partial_{\text{inc}', \text{dec}'})((\text{inc} \cdot \text{dec}')^\omega \parallel (\text{dec} \cdot \text{inc}')^\omega)) \]

where the communication function \( \gamma \) is defined such that \( \gamma(\text{inc}', \text{dec}') = \gamma(\text{dec}', \text{inc}') = i \).

6.4 Linear recursive specifications

Given a finitely branching transition system, we can easily construct a guarded recursive specification that has the process of which that transition system is a representative as its solution. For every reachable state \( s \), we introduce a corresponding process variable \( X_s \). The right-hand side of the recursive equation for \( X_s \) is an alternative composition with an alternative \( a \cdot X_{s'} \) for each transition \( s \xrightarrow{a} s' \) and an alternative \( a \) for each transition \( s \xrightarrow{\cdot} \sqrt{.} \). Here are a couple of examples.

Example 6.4.1 (Binary memory cell). We consider again the binary memory cell from Example 1.5.1. In that example, a transition system was given for the binary memory cell in a direct way. The corresponding recursive specification is as follows:

\[ M = M_0', \]

\[ M'_b = \text{rtr}(b) \cdot M'_b + \text{sto}(b) \cdot M'_b + \text{sto}(1-b) \cdot M'_{1-b} \]

(for every \( b \in \{0, 1\} \)).
Example 6.4.2 ( Calculator). We consider again the simple calculator from Example 1.1.4. In that example, a transition system was given for the calculator in a direct way. The corresponding recursive specification is as follows:

\[ C = C'_{(s,s)}, \]

\[ C'_{(s,s)} = \sum_{i \in \{i\mid \min \leq i \leq \max\}} \text{rd}(i) \cdot C'_{(i,s)}, \]

\[ C'_{(i,s)} = \sum_{o \in \{\text{chr}, \text{eq}, \text{add}, \text{sub}, \text{mul}, \text{div}\}} \text{rd}(o) \cdot C'_{(i,o)}, \]

\[ C'_{(i,\text{clr})} = \text{wr}(0) \cdot C'_{(s,s)}, \]

\[ C'_{(i,\text{eq})} = \text{wr}(i) \cdot C'_{(i,s)}, \]

\[ C'_{(i,\text{add})} = \sum_{j \in \{j\mid \min \leq i+j \leq \max\}} \text{rd}(j) \cdot C'_{(i+j,s)}, \]

\[ C'_{(i,\text{sub})} = \sum_{j \in \{j\mid \min \leq i-j \leq \max\}} \text{rd}(j) \cdot C'_{(i-j,s)}, \]

\[ C'_{(i,\text{mul})} = \sum_{j \in \{j\mid \min \leq i \cdot j \leq \max\}} \text{rd}(j) \cdot C'_{(i \cdot j,s)}, \]

\[ C'_{(i,\text{div})} = \sum_{j \in \{j\mid \min \leq i \div j \leq \max, j \neq 0\}} \text{rd}(j) \cdot C'_{(i \div j,s)}. \]

We refrained from mentioning after each equation schema that there is an instance for every \( i \) such that \( \min \leq i \leq \max \).

Conversely, given a guarded recursive specification consisting of equations whose right-hand sides are alternative compositions of which the alternatives are of the form \( a \) or \( a \cdot X \), we can construct a finitely branching transition system that is a representative of the process that is the solution of that guarded recursive specification. Here is an example.

Example 6.4.3 ( Unbounded buffer). We consider again the unbounded buffer from Example 5.4.7. In that example, a guarded recursive specification was given for an unbounded buffer that can only keep bits, i.e. \( D = \{0, 1\} \). The corresponding transition system is as follows. As states of the unbounded buffer, we have all sequences \( \sigma \in D^* \), with \( \epsilon \) as initial state. There are no successfully terminating states. As actions, we have \( \text{add}(d) \) and \( \text{rem}(d) \) for each \( d \in D \). As transitions of an unbounded buffer, we have the following:

- for each \( d \in D \), a transition \( \epsilon \xrightarrow{\text{add}(d)} d \);
- for each \( d \in D \) and \( \sigma \in D^* \), a transition \( \sigma d \xrightarrow{\text{rem}(d)} \sigma \);
- for each \( d, d' \in D \) and \( \sigma \in D^* \), a transition \( \sigma d \xrightarrow{\text{add}(d')}, d' \sigma d \).

A guarded recursive specification consisting of equations whose right-hand sides are alternative compositions of which the alternatives are of the form \( a \)
or \( a \cdot X \) is called a *linear* recursive specification. The examples given above show the close connection between linear recursive specification and finitely branching transition systems.
A. Set theoretical preliminaries

In this appendix, we give a brief summary of facts from set theory used in these lecture notes. This will at least serve to establish the terminology and notation concerning sets. First of all, we treat elementary sets (Appendix A.1). After that, we look at relations, functions (Appendix A.2) and sequences (Appendix A.3).

A.1 Sets

A set is a collection of things which are said to be the members of the set. A set is completely determined by its members. That is, if two sets $A$ and $A'$ have the same members, then $A = A'$. We write $a \in A$ to indicate that $a$ is a member of the set $A$, and $a \notin A$ to indicate that $a$ is not a member of the set $A$. A set $A$ is a subset of a set $A'$, written $A \subseteq A'$ or $A' \supseteq A$, if for all $x$, $x \in A$ implies $x \in A'$.

If a set has a finite number of members $a_1, \ldots, a_n$, then the set is written as follows:

$$\{a_1, \ldots, a_n\}.$$

Let $P(x)$ be the statement that $x$ has property $P$. Then the set whose members are exactly the things that have property $P$, if such a set exists, is written as follows:

$$\{x \mid P(x)\}.$$

If $A$ is a set and $P(x)$ is the statement that $x$ has property $P$, then there exists a subset of $A$ of which the members are exactly the members of $A$ that have property $P$. This set is denoted by $\{x \in A \mid P(x)\}$:

$$\{x \in A \mid P(x)\} = \{x \mid x \in A \text{ and } P(x)\}.$$

If $A$ is a set, then there exists a set of which the members are exactly the subsets of $A$. This set is called the powerset of $A$ and is denoted by $\mathcal{P}(A)$:

$$\mathcal{P}(A) = \{x \mid x \subseteq A\}.$$
If $A$ is a set of sets, then there exists a set of which the members are exactly the members of the subsets of $A$. This set is called the union of $A$ and is denoted by $\bigcup A$:

$$\bigcup A = \{x \mid \text{for some } A \in A: x \in A\}.$$ 

There exists a set with no members. This set is called the empty set and is denoted by $\emptyset$:

$$\emptyset = \{x \mid x \neq x\}.$$ 

Let $A$ and $A'$ be sets. Then the usual set operations union ($\cup$), intersection ($\cap$) and difference ($\setminus$) are defined as follows:

$$A \cup A' = \{x \mid x \in A \text{ or } x \in A'\},$$

$$A \cap A' = \{x \mid x \in A \text{ and } x \in A'\},$$

$$A \setminus A' = \{x \mid x \in A \text{ and } x \notin A'\}.$$ 

If $A$ and $A'$ are sets, then there exists a set of which the members are exactly $A$ and $A'$. This set is called the unordered pair of $A$ and $A'$ and is denoted by $\{A, A'\}$.

Let $A$ be a set, $a \in A$ and $a' \in A$. Then the ordered pair, or shortly pair, with first element $a$ and second element $a'$, written $(a, a')$, is the set defined as follows:

$$(a, a') = \{(a), \{a, a'\}\}.$$ 

Let $A$ and $A'$ be sets. Then the set operation cartesian product ($\times$) is defined as follows:

$$A \times A' = \{(x, x') \mid x \in A \text{ and } x' \in A'\}.$$ 

This is extended in the obvious way to the cartesian product of more than two sets. An ordered $n$-tuple ($n > 2$), or shortly $n$-tuple, with first element $a_1$, ..., $n$th element $a_n$, written $(a_1, \ldots, a_n)$, is the set defined as follows:

$$(a_1, \ldots, a_n) = (((a_1, \ldots, a_{n-1}), a_n).$$ 

A pair is sometimes also called a 2-tuple. Let $A_1, \ldots, A_n$ be sets. Then the cartesian product of more than two sets is defined as follows:

$$A_1 \times \ldots \times A_n = \{(x_1, \ldots, x_n) \mid x_1 \in A_1, \ldots, x_n \in A_n\}.$$ 

If a set has a finite number of members, the set is said to be finite. We use the following abbreviation. We write $\mathcal{P}_{\text{fin}}(A)$ for $\{x \in \mathcal{P}(A) \mid x \text{ is finite}\}$, the set of all finite subsets of $A$.

As usual, we write $\mathbb{N}$ to denote the set of all natural numbers, and $\mathbb{B}$ to denote the set $\{\text{tt, ff}\}$ of all boolean values.
A.2 Relations and functions

Let $A_1, \ldots, A_n$ be sets. An $n$-ary relation $R$ between $A_1, \ldots, A_n$ is a subset of $A_1 \times \ldots \times A_n$. If $A_1 = \ldots = A_n$, $R$ is called an $n$-ary relation on $A_1$. We often write $R(a_1, \ldots, a_n)$ for $(a_1, \ldots, a_n) \in R$.

Let $A$ be a set and $R$ be a binary relation on $A$. Then we define the following:

- $R$ is reflexive if $R(x, x)$ for all $x \in A$;
- $R$ is symmetric if $R(x, y)$ implies $R(y, x)$;
- $R$ is transitive if $R(x, y)$ and $R(y, z)$ implies $R(x, z)$;
- $R$ is an equivalence relation on $A$ if $R$ is reflexive, symmetric and transitive.

Let $A$ be a set and $R$ be an equivalence relation on $A$. Then, for each $a \in A$, the set $\{ x \in A \mid R(a, x) \}$ is called an equivalence class with respect to $R$. The members of an equivalence class are said to be representatives of the equivalence class.

Let $A$ and $A'$ be sets. Then a function from $A$ to $A'$ is a relation $f$ between $A$ and $A'$ such that for all $x \in A$ there exists a unique $x' \in A'$ with $(x, x') \in f$. This $x'$ is called the value of $f$ at $x$. We write $f : A \rightarrow A'$ to indicate that $f$ is a function from $A$ to $A'$, and we write $f(x)$ for the value of $f$ at $x$.

If $A$, $A'$ and $A''$ are sets, $A \subseteq A'$ and $f : A' \rightarrow A''$, then there exists a set of which the members are exactly the values of $f$ at the members of $A$. This set is denoted by $\{ f(x) \mid x \in A \}$:

$$\{ f(x) \mid x \in A \} = \{ x' \mid \text{for some } x \in A : f(x) = x' \} .$$

Let $\mathcal{I}$ be a set, $A$ be a set of sets. Then a family indexed by $\mathcal{I}$ is a function $A : \mathcal{I} \rightarrow A$. The set $\mathcal{I}$ is called the index set of the family. We write $A_i$ for $A(i)$. If $A$ is a family indexed by $\mathcal{I}$, then we write $\bigcup_{i \in \mathcal{I}} A_i$ for $\bigcup \{ A_i \mid i \in \mathcal{I} \}$.

We also use the following abbreviation. We write $\{ f(x) \mid x \in A, P(x) \}$ for $\{ f(x) \mid x \in \{ x' \in A \mid P(x') \} \}$.

Let $A$ and $A'$ be sets. Then a partial function from $A$ to $A'$ is a relation $f$ between $A$ and $A'$ such that there exist a set $B \subseteq A$ for which $f : B \rightarrow A'$.

For $x \in A$, $f(x)$ is said to be defined if $x$ is a member of the unique set $B \subseteq A$ for which $f : B \rightarrow A'$ and $f(x)$ is said to be undefined otherwise.

A.3 Sequences

Let $A$ be a set and $n \in \mathbb{N}$. Then a (finite) sequence over $A$ of length $n$, is a function $\sigma : \{ i \in \mathbb{N} \mid 1 \leq i \leq n \} \rightarrow A$. If $n > 0$ and $\sigma(1) = a_1, \ldots, \sigma(n) = a_n$, then the sequence is written as follows:

$$a_1 \ldots a_n .$$

The sequence of length 0 is called the empty sequence and is denoted by $\epsilon$. 
Let $A$ be a set. Then the set of all sequences over $A$ is denoted by $A^*$, and the set of all nonempty sequences over $A$ is denoted by $A^+$. For each $\sigma \in A^*$, we write $|\sigma|$ for the length of $\sigma$.

Let $A$ be a set, and $\sigma, \sigma' \in A^*$. Then the sequence operation concatenation ($\cdot$) is defined as follows. $\sigma \cdot \sigma'$ is the unique sequence $\sigma'' \in A^*$ with $|\sigma''| = |\sigma| + |\sigma'|$ such that:

\[
\begin{align*}
\sigma''(i) &= \sigma(i) \quad \text{if } 1 \leq i \leq |\sigma|, \\
\sigma''(i) &= \sigma'(i - |\sigma|) \quad \text{if } |\sigma| + 1 \leq i \leq |\sigma| + |\sigma'|.
\end{align*}
\]

We usually write $\sigma \sigma'$ for $\sigma \cdot \sigma'$.

Let $A$ be a set, and $\sigma, \sigma' \in A^*$. Then $\sigma'$ is a prefix of $\sigma$, written $\sigma' \preceq \sigma$, if there exists a $\sigma'' \in A^*$ such that $\sigma' \sigma'' = \sigma$; and $\sigma'$ is a proper prefix of $\sigma$, written $\sigma' \prec \sigma$, if $\sigma' \preceq \sigma$ and $\sigma' \neq \sigma$. 

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