A mixed singular/switching control problem with gradient constraint and terminal cost for a modulated diffusion∗

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Abstract

In this paper, we study the regularity of the value function associated with a stochastic control problem where two controls act simultaneously on a modulated multidimensional diffusion process. The first is a switching control modeling a random clock. Every time the random clock rings, the generator matrix is replaced by another, resulting in a different dynamic for the finite state Markov chain of the modulated diffusion process. The second is a singular stochastic control that is executed on the process within each regime.

Keywords. Singular/switching stochastic control problem, modulated multidimensional diffusion process, Hamilton-Jacobi-Bellman equations, non-linear partial differential system.

1 Introduction and main results

The goal of this paper is to study the regularity of the value function that is associated with a mixed singular/switching stochastic control problem for a modulated multidimensional diffusion in a bounded domain. Within a regime $\ell \in M := \{1, 2, \ldots, m\}$, a singular stochastic control is executed on a multidimensional diffusion which is modulated by a finite state Markov chain with generator matrix $Q_\ell := (q_\ell(\iota, \kappa))_{\iota, \kappa \in I}$, where $I := \{1, 2, \ldots, n\}$. Here, the criterion is to minimize the expected costs that the singular and switching controls produce every time that they act on the modulated diffusion process, subject to a penalization that is produced at the first moment that the controlled process is outside the bounded set; for more details about it, see the subsection below.

The control problem presented in this work can be applied, for example, in the area of finance if we assume that the cash reserve process of a firm is governed by a modulated one-dimensional process until a ruin time. Considering a fixed family of transition matrices $Q = \{Q_\ell\}_{\ell \in M}$ and an increasing sequence of stopping times $\{\tau_i\}_{i \geq 0}$, and according to the data observed at time $\tau_i$, the firm has the option of changing the transition matrix $Q_{\ell_{i-1}}$ by $Q_{\ell_i}$, with a cost $\vartheta_{\ell_{i-1}, \ell_i}$, in such a way that the Markov chain associated with the modulated

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process can model the times and the states that the reserve process would be well-defined on the interval \([\tau_i, \tau_{i-1})\). Let us define this switching control by \(\varsigma = (\tau_i, \ell_i)_{i \geq 0}\). Within each regime \(\ell_i\), the expenses of the firm, which are paid out from the reserve process, are given by a non-decreasing and right-continuous process \(\varsigma\). Then, under a minimization criterion, the firm wishes to find a strategy \((\varsigma^*, \varsigma^*)\) that reduces the expected costs that the company must assume.

As far as we know, the existing stochastic control literature has not yet considered the problems described above, and they could be a research line of interest for both the stochastic control theory and its applications.

### 1.1 Model formulation

Let \(W = \{W_t : t \geq 0\}\) be a \(k\)-dimensional standard Brownian motion and let \(I^{(l)} = \{I^{(l)}_t : t \geq 0\}\), with \(l \in \mathcal{M}\), be a continuous-time Markov chain with finite state space \(\mathcal{I}\) and generator matrix \(Q_\ell = (q_\ell(t, \kappa))_{\ell, \kappa \in \mathcal{I}}\), i.e.,

\[
\mathbb{P}\{I^{(l)}_{t+\Delta t} = \kappa | I^{(l)}_t = \ell, I^{(l)}_s, s \leq t\} = \begin{cases} q_\ell(t, \kappa) \Delta t + o(\Delta t), & \text{if } \kappa \neq \ell, \\ 1 + q_\ell(t, \kappa) \Delta t + o(\Delta t) & \text{if } \kappa = \ell. \end{cases}
\]

The entries of the generator matrix \(Q_\ell\) satisfy

\[
q_\ell(t, \kappa) \geq 0 \text{ for } \ell, \kappa \in \mathcal{I}, \text{ with } \kappa \neq \ell, \\
q_\ell(t, \omega) = -\sum_{\kappa \in \mathcal{I}} q_\ell(t, \kappa) \text{ for } \omega \in \mathcal{I}. \tag{1.2}
\]

We assume that \(W, I^{(1)}, \ldots, I^{(m)}\) are independent and are defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) be the filtration generated by \(W\) and \(\{I^{(l)}\}_{l \in \mathcal{M}}\).

We consider the triple \((X^{\xi, \varsigma}, J^{\varsigma, l}, I)\) as a stochastic controlled process that evolves as:

\[
X^{\xi, \varsigma}_t = X^{\xi, \varsigma}_{\tau_i} - \int_{\tau_i}^t b(X^{\xi, \varsigma}_s, I^{(l)}_s) ds + \int_{\tau_i}^t \sigma(X^{\xi, \varsigma}_s, I^{(l)}_s) dW_s - \int_{\tau_i}^t \nu_s d\varsigma_s, \\
I_t = I^{(l)}_t \text{ and } J^{\varsigma}_t = \ell_i \text{ for } t \in [\tau_i, \tau_{i+1}) \text{ and } i \geq 0,
\]

where \(X^{\xi, \varsigma}_0 = x_0 \in \mathcal{O} \subset \mathbb{R}^d, J^{(0)}_0 = \ell_0 \in \mathcal{I}, I_0 = I^{(0)}_0 = t_0 = 0, \tau := \{t > 0 : (X^{\xi, \varsigma}_t, I_t) \notin \mathcal{O} \times 0\}\), and \(\tau := \tau_i \land \tau\). The parameters \(b_i := b(\cdot, \ell) : \mathcal{O} \rightarrow \mathbb{R}^d\) and \(\sigma_i := \sigma(\cdot, \ell) : \mathcal{O} \rightarrow \mathbb{R}^d \times \mathbb{R}^k\), with \(\ell \in \mathcal{I}\) fixed, satisfy appropriate conditions to ensure the well-definiteness of the stochastic differential equation (SDE) \((1.3)\); see Assumption \((H4)\).

The control process \((\xi, \varsigma)\) is in \(\mathcal{U} \times \mathcal{S}\) where the singular control \(\xi = (\bar{n}, \zeta)\) belongs to the class \(\mathcal{U}\) of admissible controls that satisfy

\[
\begin{cases}
(\bar{n}, \zeta) \in \mathbb{R}^d \times \mathbb{R}_+, t \geq 0, \text{ such that } X^{\bar{n}, \zeta}_t \in \mathcal{O} \text{ } t \in [0, \tau), \\
(\bar{n}, \zeta) \text{ is adapted to the filtration } \mathcal{F}, \\
\zeta_0 = 0 \text{ and } \zeta \text{ is non-decreasing and is right continuous with left hand limits, } t \geq 0, \text{ and } |\bar{n}_t| = 1 \text{ d}\zeta_t \text{-a.s., } t \geq 0,
\end{cases} \tag{1.4}
\]

and the switching control process \(\varsigma := (\tau_i, \ell_i)_{i \geq 0}\) belongs to the class \(\mathcal{S}\) of switching regime sequences that satisfy

\[
\begin{cases}
\varsigma \text{ is a sequence of } \mathbb{F}\text{-stopping times and regimes in } \mathcal{M}, \text{ i.e.,} \\
\varsigma = (\tau_i, \ell_i)_{i \geq 0} \text{ is such that } 0 = \tau_0 \leq \tau_1 < \tau_2 < \cdots, \tau_i \uparrow \infty \text{ as } i \uparrow \infty \text{ } \mathbb{P}\text{-a.s.,} \\
\text{and for each } i \geq 0, \ell_i \text{ is } \mathcal{F}_{\tau_i}\text{-measurable valued in } \mathcal{M}. \tag{1.5}
\end{cases}
\]
Given the initial state \((x_0, \ell_0, t_0) \in \mathcal{O} \times M \times \mathbb{I}\) and the control \((\xi, \varsigma) \in \mathcal{U} \times \mathcal{S}\), the functional cost of the controlled process \((X^{\xi, \varsigma}, J^{\varsigma}, I)\) is defined by

\[
V_{\xi, \varsigma}(x_0, \ell_0, t_0) := \mathbb{E}_{x_0, \ell_0, t_0} \left[ \int_0^T e^{-r(t)} [h(X^{\xi, \varsigma}_t, I_t) dt + g(X^{\xi, \varsigma}_t, I_t) \circ d\zeta_t] \right] + \sum_{i \geq 0} \mathbb{E}_{x_0, \ell_0, t_0} [e^{-r(\tau_{i+1})} \vartheta_{\ell_i, \ell_{i+1}} \mathbb{1}_{\{\tau_{i+1} < \tau\}}] + \mathbb{E}_{x_0, \ell_0, t_0} [e^{-r(\tau)} f(X^{\xi, \varsigma}_\tau, I_\tau) \mathbb{1}_{\{\tau < \infty\}}],
\]

where \(\mathbb{E}_{x_0, \ell_0, t_0}\) is the expected value associated with \(\mathbb{P}_{x_0, \ell_0, t_0}\), the probability law of \((X^{\xi, \varsigma}, J^{\varsigma}, I)\) when it starts at \((x_0, \ell_0, t_0)\), \(r(t) = \int_0^t c(X^{\xi, \varsigma}_s, I_s) ds\) represents the accumulated interest rate at time \(t\), and

\[
\int_0^t e^{-r(s)} g(X^{\xi, \varsigma}_s, I_s) \circ d\zeta_s := \int_0^t e^{-r(s)} g(X^{\xi, \varsigma}_s, I_s) d\zeta_s
\]

\[
+ \sum_{0 \leq s \leq t} e^{-r(s)} \Delta \zeta_s \int_0^1 g(X^{\xi, \varsigma}_\tau - \lambda \pi, \Delta \zeta_s, I_s) d\lambda,
\]

with \(\zeta^c\) denoting the continuous part of \(\zeta\). We can appreciate in (1.6) that the cost for switching regimes are represented by \(\vartheta_{\ell, \ell'}\), and the terminal cost is given by \(f(X^{\xi, \varsigma}_\tau, I_\tau) \mathbb{1}_{\{\tau < \infty\}}\). Additionally, at time \(t\), we have the costs \(g(X^{\xi, \varsigma}_t, I_t) \circ d\zeta_t\) when the singular control \(\xi\) is executed, and \(h(X^{\xi, \varsigma}_t, I_t)\) if not.

Under the assumption that there is no loop of zero cost (see Eq (1.10)), one of the main goals of this paper is to verify that the value function

\[
V_{\ell_0}(x_0, \ell_0, t_0) := \inf_{\xi, \varsigma} V_{\xi, \varsigma}(x_0, \ell_0, t_0), \quad \text{for} \quad (x_0, \ell_0, t_0) \in \mathcal{O} \times M \times \mathbb{I},
\]

is in \(C^0(\mathcal{O}) \cap W^{2, \infty}_{\text{loc}}(\mathcal{O})\); see Theorem 1.1.

The novelties of this work, in contrast with the existing literature (see, i.e., [12, 13] and references therein), are:

(i) Every time that there is a switching, the generator matrix is replaced by another, resulting in a different dynamic for the finite state Markov chain of the modulated multidimensional diffusion process.

(ii) We add a terminal cost in the value function.

The issues mentioned above are reflected in the corresponding Hamilton-Jacobi-Bellman (HJB) equation with gradient constraint (see (1.12)) of the value function \(V\) in the following way:

(i) The solution of this HJB equation is a matrix function \(u = (u_{\ell, \iota})_{(\ell, \iota) \in M \times 1}\) where the row \(\ell\) represents the matrix transition \(Q_\ell\) with which the states \(\iota\) (the columns) should be interacting with each other. These types of problems can be found in the literature only when the regime set \(M\) is a singleton set, i.e., optimal stochastic control problems with Markov switching; see, i.e., [6, 7, 11].

(ii) The terminal cost is considered in the HJB equation as a boundary condition. The solution \(u\) to the HJB equation is constructed as a limit of a sequence of functions.
\{u^{\varepsilon,\delta}\}_{(\varepsilon,\delta) \in (0,1)^2} when \((\varepsilon, \delta)\) goes to \((0,0)\). The entries of this sequence are classical solutions to a non-linear partial differential system (NPDS), which inherits the same boundary condition (see (1.15)). So, we must first guarantee the existence and uniqueness of \(\{u^{\varepsilon,\delta}\}_{(\varepsilon,\delta) \in (0,1)^2}\), whose entries are in \(C^4(\overline{\mathcal{O}})\), and then verify for each \((\ell, \iota) \in \mathbb{M} \times \mathbb{I}\), that \(\{u^{\varepsilon,\delta}_{\ell,\iota}\}_{(\varepsilon, \delta) \in (0,1)^2}\) is bounded, uniformly in \((\varepsilon, \delta)\), with respect to the norms \(\|\cdot\|_{C^4(\overline{\mathcal{O}})}\), \(\|\cdot\|_{C^1(\overline{\mathcal{O}})}\) and \(\|\cdot\|_{C^2_{\text{loc}}(\mathcal{O})}\), in such a way that \(u\) is well defined. Previous similar studies to ours; see [12, 13] and references therein; have shown that the sequences of functions related to their HJB equations are uniformly bounded with respect to the norms \(\|\cdot\|_{C^4(\overline{\mathcal{O}})}\) and \(\|\cdot\|_{C^2_{\text{loc}}(\mathcal{O})}\) due to their null boundary condition. Existence and uniqueness of the solutions to the HJB equations with gradient constraint and a non-null boundary condition almost everywhere, have been studied by few authors; see, i.e., [10].

### 1.2 Assumptions and main results

In order to see that the value function \(V_\ell(\cdot, \iota)\), defined in (1.8), belongs to \(C^0(\overline{\mathcal{O}}) \cap W^{2,\infty}_{\text{loc}}(\mathcal{O})\) for each \((\ell, \iota) \in \mathbb{M} \times \mathbb{I}\), let us first give necessary conditions to guarantee the existence and uniqueness of the solution \(u_\ell(\cdot, \iota)\) to (1.12) on the same space.

**Assumptions**

\((H1)\) The domain set \(\mathcal{O}\) is an open and bounded set such that its boundary \(\partial \mathcal{O}\) is of class \(C^{4,\alpha}\), with \(\alpha \in (0,1)\) fixed.

\((H2)\) The switching costs sequence \(\{\vartheta_{\ell,\ell'}\}_{\ell,\ell' \in \mathbb{I}}\) is such that \(\vartheta_{\ell,\ell'} \geq 0\) and satisfies

\[
\vartheta_{\ell_1,\ell_2} \leq \vartheta_{\ell_1,\ell_3} + \vartheta_{\ell_2,\ell_3}, \text{ for } \ell_3 \neq \ell_1, \ell_2, \tag{1.9}
\]

which means that it is cheaper to switch directly from regime \(\ell_1\) to \(\ell_3\) than using the intermediate regime \(\ell_2\). Additionally, we assume that there is no loop of zero cost, i.e.,

\[
\text{no family of regimes } \{\ell_0, \ell_1, \ldots, \ell_n, \ell_0\} \text{ such that } \vartheta_{\ell_0,\ell_1} = \vartheta_{\ell_1,\ell_2} = \cdots = \vartheta_{\ell_n,\ell_0} = 0. \tag{1.10}
\]

Let \(\iota\) be in \(\mathbb{I}\). Then:

\((H3)\) The real valued functions \(f_\iota := f(\cdot, \iota)\), \(h_\iota := h(\cdot, \iota)\) and \(g_\iota := g(\cdot, \iota)\) belong to \(C^{2,\alpha}(\overline{\mathcal{O}})\), are non-negative, and \(\|f_\iota\|_{C^{2,\alpha}(\overline{\mathcal{O}})}, \|h_\iota\|_{C^{2,\alpha}(\overline{\mathcal{O}})}, \|g_\iota\|_{C^{2,\alpha}(\overline{\mathcal{O}})}\), are bounded by some finite positive constant \(\Lambda\).

\((H4)\) Let \(\mathcal{S}(d)\) be the set of \(d \times d\) symmetric matrices. The coefficients of the differential part of \(\mathcal{L}_\ell(\cdot)\) (see (1.14)), \(a_{\iota,i} := a(\cdot, \iota) : \overline{\mathcal{O}} \to \mathcal{S}(d), b_{\iota,i} := b(\cdot, \iota) = (b_1(\cdot, \iota), \ldots, b_d(\cdot, \iota)) : \overline{\mathcal{O}} \to \mathbb{R}^d\) and \(c_{\iota,i} := c(\cdot, \iota) : \overline{\mathcal{O}} \to \mathbb{R}\), are such that \(a_{\iota,i,j} : \ell, b_{\iota,i,j} : \ell, c_{\iota,i,j} \in C^{2,\alpha}(\overline{\mathcal{O}})\), \(c_\iota > 0\) on \(\mathcal{O}\) and \(\|a_{\iota,i,j}\|_{C^{2,\alpha}(\overline{\mathcal{O}})}, \|b_{\iota,i,j}\|_{C^{2,\alpha}(\overline{\mathcal{O}})}, \|c_{\iota,i,j}\|_{C^{2,\alpha}(\overline{\mathcal{O}})}\) are bounded by some finite positive constant \(\Lambda\). We assume that there exist a real number \(\theta > 0\) such that

\[
\langle a_{\iota}(x)\zeta, \zeta \rangle \geq \theta|\zeta|^2, \text{ for all } x \in \overline{\mathcal{O}}, \zeta \in \mathbb{R}^d. \tag{1.11}
\]
Taking into account (1.10) and a heuristic derivation from dynamic programming principle, the HJB equation corresponding to the value function $V_{\ell,t} := V(t, \ell)$ is given by

$$\max \left\{ c_i - \mathcal{L}_{\ell,t} u_{\ell,t} - h_i, D^1 u_{\ell,t} - g_i, u_{\ell,t} - \mathcal{M}_{\ell,t} u \right\} = 0, \text{ on } \mathcal{O},$$

$$\text{s.t. } u_{\ell,t} = f_i, \text{ in } \partial \mathcal{O},$$

where for each $(\ell, t) \in \mathbb{M} \times \mathbb{I}$, $u_{\ell,t} = u(t, \ell) : \overline{\mathcal{O}} \rightarrow \mathbb{R}$ and

$$\mathcal{M}_{\ell,t} u(x) := \min_{e \in \mathbb{N} \setminus \{\ell\}} \{ u_{e,t}(x) + \vartheta_{e,t} \},$$

$$\mathcal{L}_{\ell,t} u_{\ell,t}(x) := \text{tr}[a_i(x) D^2 u_{\ell,t}(x)] - \langle b_i(x), D^1 u_{\ell,t}(x) \rangle + \sum_{\kappa \in \mathbb{N} \setminus \{\ell\}} q_i(\ell, \kappa)[u_{\ell,\kappa}(x) - u_{\ell,t}(x)],$$

with $a_i = (a_{i,j})_{d \times d}$ is such that $a_{i,j} := \frac{1}{2}[\sigma_i \sigma_i^T]_{i,j}$. Here $| \cdot |$, $\langle \cdot, \cdot \rangle$ and $\text{tr}[\cdot]$ represent the Euclidean norm, the inner product, and the matrix trace, respectively. The operator $D^k u_{\ell,t}(x)$, with $k \geq 1$ an integer number, represents the $k$-th differential operator of $u_{\ell,t}(x)$ with respect to $x$.

Under assumptions (H1)–(H4), we have the following proposition.

**Proposition 1.1.** The HJB equation (1.12) has a unique non-negative strong solution (in the almost everywhere sense) $u = (u_{\ell,t})_{\mathbb{M} \times \mathbb{I}}$ where $u_{\ell,t} \in C^0(\overline{\mathcal{O}}) \cap W^{2,\infty}_{\text{loc}}(\mathcal{O})$ for each $(\ell, t) \in \mathbb{M} \times \mathbb{I}$.

In addition to the statement in (H1), we need to assume that the domain set is convex, which will permit the verification of the agreement of the value function $V$ and the solution $u$ to (1.12) in $\overline{\mathcal{O}}$.

(H5) The domain set $\mathcal{O}$ is an open, convex and bounded set such that its boundary $\partial \mathcal{O}$ is of class $C^{4,\alpha}$, with $\alpha \in (0, 1)$ fixed.

Under assumptions (H2)–(H5), the main goal obtained in this document is as follows.

**Theorem 1.2.** Let $V$ be the value function given by (1.8). Then $V_{\ell_0}(x_0, \iota_0) = u_{\ell_0}(x_0, \iota_0)$ for $(x_0, \iota_0) \in \overline{\mathcal{O}} \times \mathbb{I}$ and $\ell_0 \in \mathbb{M}$.

In order to verify the results above, first we need to guarantee the existence and uniqueness of the classical solution $u^{\varepsilon,\delta} = (u_{\ell,t}^{\varepsilon,\delta})_{(\ell,t) \in \mathbb{M} \times \mathbb{I}}$ to the following NPDS

$$(1.15)$$

$$|c_i - \mathcal{L}_{\ell,t} u_{\ell,t}^{\varepsilon,\delta} + \psi_\varepsilon(|D^1 u_{\ell,t}^{\varepsilon,\delta}|^2 - q_i^2) + \sum_{e \in \mathbb{N} \setminus \{\ell\}} \psi_\delta(u_{\ell,t}^{\varepsilon,\delta} - u_{e,t}^{\varepsilon,\delta} - \vartheta_{e,t}^\ell) = h_i \text{ on } \mathcal{O},$$

$$\text{s.t. } u_{\ell,t}^{\varepsilon,\delta} = f_i, \text{ in } \partial \mathcal{O},$$

where $\psi_\varepsilon$ is defined by $\psi_\varepsilon(t) = \varphi(t/\varepsilon)$ with $\varepsilon \in (0, 1)$, and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is in $C^\infty(\mathbb{R})$ is such that

$$\begin{align*}
\varphi(t) &= 0, \quad t \leq 0, \quad \varphi(t) > 0, \quad t > 0, \\
\varphi(t) &= t - 1, \quad t \geq 2, \quad \varphi'(t) \geq 0, \quad \varphi''(t) \geq 0.
\end{align*}$$
Then, as an intermediate step, it will be proven that $u^\varepsilon$ defined as limit of $u^{\varepsilon,\delta}$, when $\delta \downarrow 0$, is the unique strong solution to the following HJB equation

$$\max \left\{ \left[ c_i - L_{\ell,i} u^\varepsilon_{\ell,i} + \psi_i (|D^1 u^\varepsilon_{\ell,i}|^2 - g^2) - h_i, u^\varepsilon_{\ell,i} - M_{\ell,i} u^\varepsilon \right] \right\} = 0, \text{ on } \Omega,$$

s.t. $u^\varepsilon_{\ell,i} = f_i$, in $\partial \Omega$. \hfill (1.17)

The reason for doing that is because $u^\varepsilon$ coincides with the value function $V^\varepsilon$, which will be defined later on (see (4.4)), of an $\varepsilon$-penalized absolutely continuous/switching ($\varepsilon$-PACS) control problem; see Section 4. Although the solution $u$ to the HJB equation (1.12) can be constructed directly as a limit of $u^{\varepsilon,\delta}$, when $(\varepsilon, \delta)$ goes to $(0, 0)$, we required first to analyse the properties of the optimal stochastic control associated with the $\varepsilon$-PACS control problem mentioned above, in such a way that we can corroborate the equivalence between $u$ and $V$ in $\bar{\Omega}$.

We would like to mention that the NPDS (1.15), named in the PDE theory as a non-linear elliptic cooperative system, is a problem of interest itself because we can find literature related to this problem only when the regime set $\mathbb{M}$ is a singleton set; see, i.e., [4, 15, 18].

Under assumptions (H1), (H3) and (H4), the following result is obtained.

**Proposition 1.3.** Let $\varepsilon, \delta \in (0, 1)$ be fixed. There exists a unique non-negative solution $u^{\varepsilon,\delta} = (u^{\varepsilon,\delta}_{\ell,i})_{(\ell,i)\in \mathbb{M} \times \mathbb{I}}$ to the NPDS (1.15) where $u^{\varepsilon,\delta}_{\ell,i} \in C^{4,\alpha}(\bar{\Omega})$ for each $(\ell,i) \in \mathbb{M} \times \mathbb{I}$.

Under assumptions (H1)–(H4), the following result is obtained.

**Proposition 1.4.** For each $\varepsilon \in (0, 1)$ fixed, there exists a unique non-negative strong solution $u^\varepsilon = (u^\varepsilon_{\ell,i})_{\ell\times i}$ to the HJB equation (1.17) where $u^\varepsilon_{\ell,i} \in C^0(\bar{\Omega}) \cap W^{2,\infty}_{\text{loc}}(\Omega)$ for each $(\ell,i) \in \mathbb{M} \times \mathbb{I}$.

The rest of this document is organized as follows: in Section 2, using (H1), (H3) and (H4), and by a fixed point argument, the existence and uniqueness of the solution $u^{\varepsilon,\delta}$ to the NPDS (1.15), with $(\varepsilon, \delta) \in (0, 1)^2$ fixed, is proven. Then, in Section 3, some estimations for $u^{\varepsilon,\delta}$ are given. For that aim, we first study the classical solution to a linear elliptic cooperative system; see Lemma 3.2. Afterwards, using Proposition 1.3 and Lemmas 3.3 and 3.4, Arzelà-Ascoli compactness criterion and the reflexivity of $L^p_{\text{loc}}(\Omega)$; see [5, Section C.8, p. 718] and [1, Thm. 2.46, p. 49] respectively, we discuss the existence, regularity and uniqueness of the solutions $u$ and $u^\varepsilon$ to (1.12) and (1.17), respectively. Later, in Section 4, we introduce the $\varepsilon$-PACS control problem and its verification lemma is presented. Afterwards, we give the proof of Theorem 1.1. To finalize this section, let us say that the notations and definitions of the function spaces that are used in this paper are standard and the reader can find them in [1, 3, 5, 8, 9].

## 2 Existence and uniqueness of the solution to the NPDS (1.15)

Let $C^{k}_{m,n}$, $C^{k,\alpha}_{m,n}$ be the sets of $(m \times n)$-matrix functions given by $(C^k(\bar{\Omega}))^{m \times n}$, $(C^{k,\alpha}(\bar{\Omega}))^{m \times n}$, respectively, with $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. Defining $||w||_{C^k_{m,n}} = \max_{(\ell,i)\in \mathbb{M} \times \mathbb{I}} \{||w_{\ell,i}||_{C^k(\bar{\Omega})}\}$ for each $w = (w_{\ell,i})_{(\ell,i)\in \mathbb{M} \times \mathbb{I}} \in C^{k}_{m,n}$, it can be verified that $||\cdot||_{C^k_{m,n}}$, $||\cdot||_{C^{k,\alpha}_{m,n}}$ are norms on $C^{k}_{m,n}$, $C^{k,\alpha}_{m,n}$, respectively, and $(C^{k}_{m,n}, ||\cdot||_{C^k_{m,n}})$, $(C^{k,\alpha}_{m,n}, ||\cdot||_{C^{k,\alpha}_{m,n}})$ are Banach spaces.

Since the arguments to guarantee the existence of the solution to the NPDS (1.15) are based on Schaefer’s fixed point theorem, we will provide the necessary results to obtain the conditions of this theorem (see, i.e., [5, Thm. 4 p. 539]).
Let us define the operators $\widetilde{L}_\ell$, and $\Xi_{\ell,\iota}$ as follows

\begin{equation}
\widetilde{L}_\ell \psi_{\ell,\iota} = \text{tr}[a_i \psi_{\ell,\iota} - (b_i \partial_{\ell,\iota})^T], \quad \Xi_{\ell,\iota} w = \sum_{\iota' \neq \iota} \psi_{\ell,\iota'} (w_{\ell,\iota'} - \partial_{\ell,\iota'} w_{\ell,\iota}) + \sum_{\kappa \neq \iota} q_{\ell,\kappa} [w_{\ell,\kappa} - w_{\ell,\iota}] + \psi_{\ell,\iota} (|D^1 w_{\ell,\iota}|^2 - g^2_{\ell,\iota}).
\end{equation}

We observe then for each $w \in C^{1,\alpha}_{m,n}$ fixed, there exists a unique solution $u = (u_{\ell,\iota})_{(\ell,\iota) \in \mathbb{M} \times \mathbb{I}} \in C^{2,\alpha}_{m,n}$ to the following linear partial differential system (LPDS)

\begin{equation}
[c_i - \widetilde{L}_\ell] u_{\ell,\iota} = h_{\ell,\iota} - \Xi_{\ell,\iota} w_{\ell,\iota}, \quad \text{in } \mathcal{O},
\end{equation}

s.t. $u_{\ell,\iota} = f_{\ell,\iota}$, on $\partial \mathcal{O}$, for $(\ell, \iota) \in \mathbb{M} \times \mathbb{I}$, (2.2)

since (H1)-(H4) hold and $\Xi_{\ell,\iota} w \in C^{0,\alpha}(\overline{\mathcal{O}})$ (see Theorem 6.14 of [9]). Additionally, due to Theorem 1.2.10 of [8], the following inequality can be checked

\begin{equation}
\|u_{\ell,\iota}\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \leq C_2 \left[ 1 + \frac{1}{\delta} + \frac{1}{\varepsilon} + \left[ 1 + \frac{1}{\delta} \right] \|w\|_{C^{0,\alpha}_{m,n}} + \frac{1}{\varepsilon} \|w_{\ell,\iota}\|_{C^{1,\alpha}(\overline{\mathcal{O}})} \right] \quad \text{for } (\ell, \iota) \in \mathbb{M} \times \mathbb{I},
\end{equation}

for some $C_2 = C_2(d, \Lambda, \alpha, \theta)$. Defining the mapping

\begin{equation}
\mathcal{T} : (C^{1,\alpha}_{m,n}, \|\cdot\|_{C^{1,\alpha}_{m,n}}) \longrightarrow (C^{1,\alpha}_{m,n}, \|\cdot\|_{C^{1,\alpha}_{m,n}})
\end{equation}

as $\mathcal{T}[w] = u$ for each $w \in C^{1,\alpha}_{m,n}$, where $u \in C^{2,\alpha}_{m,n} \subset C^{1,\alpha}_{m,n}$ is the unique solution to the LDPS (2.2), we get that, by (2.3) and by Arzelà-Ascoli’s compactness criterion; see [5, Section C.8, p. 718], $\mathcal{T}$ maps bounded sets in $C^{1,\alpha}_{m,n}$ into bounded sets in $C^{2,\alpha}_{m,n}$ which are precompact in $C^{1,\alpha}_{m,n}$. From here and by the uniqueness of the solution to the LPDS (2.2), it can be verified that $\mathcal{T}$ is a continuous and compact mapping from $C^{1,\alpha}_{m,n}$ into itself.

Now, we only need to verify that the set

\begin{equation}
\mathcal{A}_1 := \{ w \in C^{1,\alpha}_{m,n} : w = \varrho \mathcal{T}[w], \text{ for some } \varrho \in [0,1] \}
\end{equation}

is bounded uniformly on the norm $\|\cdot\|_{C^{1,\alpha}_{m,n}}$. Notice that the LPDS associated with $\varrho = 0$ is

\begin{equation}
[c_i - \widetilde{L}_\ell] w_{\ell,\iota} = 0, \quad \text{in } \mathcal{O},
\end{equation}

s.t. $w_{\ell,\iota} = 0$, on $\partial \mathcal{O}$, for $(\ell, \iota) \in \mathbb{M} \times \mathbb{I}$.

which solution is immediately $w \equiv \overline{0} \in C^{1,\alpha}_{m,n}$, with $\overline{0}$ the null matrix function.

**Lemma 2.1.** If $w \in C^{1,\alpha}_{m,n}$ is such that $\mathcal{T}[w] = \frac{1}{\varrho} w = (\frac{1}{\varrho} w_{\ell,\iota})_{(\ell,\iota) \in \mathbb{M} \times \mathbb{I}}$ for some $\varrho \in (0,1)$, then there exists a constant $C_1 > 0$ independent of $\varrho$ and $w$ such that

\begin{equation}
\|w_{\ell,\iota}\|_{C^{1,\alpha}(\overline{\mathcal{O}})} \leq C_1 \left[ 1 + \frac{1}{\delta} + \frac{1}{\varepsilon} [1 + \|w\|_{C^{0,\alpha}_{m,n}}] \right] \quad \text{for } (\ell, \iota) \in \mathbb{M} \times \mathbb{I}.
\end{equation}

**Proof.** Observe that $w \in C^{2,\alpha}_{m,n}$ and

\begin{equation}
[c_i - \widetilde{L}_\ell] w_{\ell,\iota} = \varrho[h_{\ell,\iota} - \Xi_{\ell,\iota} w_{\ell,\iota}], \quad \text{in } \mathcal{O},
\end{equation}

s.t. $w_{\ell,\iota} = \varrho f_{\ell,\iota}$, on $\partial \mathcal{O}$, for $(\ell, \iota) \in \mathbb{M} \times \mathbb{I}$. (2.6)
Defining \( \tilde{w} = (\tilde{w}_{\ell,i})_{(\ell,i) \in \mathbb{M} \times \mathbb{I}} \) as \( \tilde{w}_{\ell,i} = w_{\ell,i} - \rho f_i, \) (2.6) can be rewritten in the following way

\[
\begin{align*}
\Gamma_{i} + 1 \tilde{w}_{\ell,i} = & \mu [1 + |D^1 \tilde{w}_{\ell,i}|^2], \text{ in } \mathcal{O}, \\
s.t. \tilde{w}_{\ell,i} = & 0, \text{ on } \partial \mathcal{O}, \text{ for } (\ell, i) \in \mathbb{M} \times \mathbb{I},
\end{align*}
\]

where \( \Gamma_{i}w_{\ell,i} := -\text{tr}[a_i D^2 w_{\ell,i}] \) and

\[
\mu := \frac{1}{1 + |D^1 \tilde{w}_{\ell,i}|^2} \left[ (1 - c_i) \tilde{w}_{\ell,i} - (b_i, D^1 \tilde{w}_{\ell,i}) + \rho h_i \right. \\
- \left. \frac{1}{\delta} \sum_{\ell' \in \mathbb{M} \setminus \{\ell\}} \psi_{\ell} (\tilde{w}_{\ell,i} - \tilde{w}_{\ell',i} - \theta_{\ell,\ell'}) \right] \\
- \sum_{\kappa \in \mathbb{I} \setminus \{i\}} q_{\ell}(\kappa, \kappa) [\tilde{w}_{\ell,i} + \rho f_i - [\tilde{w}_{\ell,\kappa} + \rho f_{\kappa}]] - \psi_\kappa (|D^1 \tilde{w}_{\ell,i} + \rho f_i|^2 - g_{\kappa}^2). \]

Applying [1, Thm. 4.12, p.85] and [2, Lemma 4] (see also [19, Lemma 2.4]), we get that

\[
\|\tilde{w}_{\ell,i}\|_{C^{1,\alpha}(\bar{\mathcal{O}})} \leq K_{1,1} \|\mu\|_{L^\infty(\mathcal{O})}, \tag{2.7}
\]

for some constant \( K_{1,1} > 0 \) independent of \( \rho \) and \( \tilde{w} \). Meanwhile

\[
\begin{align*}
|\mu| & \leq (1 + c_i) \|\tilde{w}_{\ell,i}\| + |c_i - \tilde{\kappa}f_i| + h_i \\
& + \sum_{\kappa \in \mathbb{I} \setminus \{i\}} q_{\ell}(\kappa, \kappa) [\tilde{w}_{\ell,i} + f_i + \tilde{w}_{\ell,\kappa} + f_{\kappa}] + \frac{1}{\delta} \sum_{\ell' \in \mathbb{M} \setminus \{\ell\}} \left[ |\tilde{w}_{\ell,i}| + |\tilde{w}_{\ell',i}| + \theta_{\ell,\ell'} \right] \\
& + |b_i| \left[ \frac{|D^1 \tilde{w}_{\ell,i}|}{1 + |D^1 \tilde{w}_{\ell,i}|^2} + \frac{1}{\varepsilon} \left[ \frac{2 |D^1 \tilde{w}_{\ell,i}|^2}{1 + |D^1 \tilde{w}_{\ell,i}|^2} + 2 |D^1 f_i|^2 + g_{\kappa}^2 \right] \right] \\
& \leq K_{1,2} \left[ 1 + \frac{1}{\varepsilon} + \frac{1}{\delta} [1 + \|w\|_{C^{1,\alpha}_{x,\kappa}}] \right] \text{ on } \bar{\mathcal{O}},
\end{align*}
\]

for some constant \( K_{1,2} > 0 \) independent of \( \rho \) and \( \tilde{w} \). By (2.7), (A.10) and taking into account that \( \|w_{\ell,i}\|_{C^{1,\alpha}(\bar{\mathcal{O}})} \leq \|w_{\ell,i}\|_{C^{1,\alpha}(\mathcal{O})} + \|f_i\|_{C^{1,\alpha}(\mathcal{O})} \) we see that (2.5) is true.

In view of (2.5), to see \( A_1 \) is bounded uniformly with respect to the norm \( \| \cdot \|_{C^{1,\alpha}_{x,\kappa}} \), it is sufficient to check that \( w \) is uniformly bounded with respect to the norm \( \| \cdot \|_{C^{1,\alpha}_{x,\kappa}} \).

**Lemma 2.2.** If \( w \in C^{1,\alpha}_{x,\kappa} \) is such that \( T[w] = \frac{1}{\rho} w = (\frac{1}{\rho} w_{\ell,i})_{(\ell,i) \in \mathbb{M} \times \mathbb{I}} \) for some \( \rho \in (0,1] \), then

\[
0 \leq w_{\ell,i}(x) \leq \Lambda \max_{(x',\kappa) \in \mathcal{O} \times \mathbb{I}} \left\{ 1, \frac{1}{c_\kappa(x')} \right\} \text{ for } x \in \bar{\mathcal{O}} \text{ and } (\ell, i) \in \mathbb{M} \times \mathbb{I}. \tag{2.9}
\]

**Proof.** Let \((x_o, \ell_o, i_o) \in \bar{\mathcal{O}} \times \mathbb{M} \times \mathbb{I} \) be such that

\[
w_{x_o,i_o}(x_o) = \min_{(x,\kappa) \in \mathcal{O} \times \mathbb{M} \times \mathbb{I}} w_{x,i}(x).
\]

If \( x_o \in \partial \mathcal{O} \), it follows easily that \( w_{\ell,i}(x) \geq w_{x_o,i_o}(x_o) = \rho f_{x_o}(x_o) \geq 0 \) for \( (x, \ell, i) \in \mathcal{O} \times \mathbb{M} \times \mathbb{I} \). Suppose that \( x_o \in \mathcal{O} \). Then,

\[
\begin{align*}
&D^1 w_{x_o,i_o}(x_o) = 0, \ 0 \leq \text{tr}[a_{x_o}(x_o) D^2 w_{x_o,i_o}(x_o)], \\
&w_{x_o,i_o}(x) - w_{x_o,i}(x) \leq 0 \text{ for } \kappa \in \mathbb{I} \setminus \{i_o\}, \\
&w_{x_o,i_o}(x) - w_{x_o,i}(x) \leq 0 \text{ for } \ell' \in \mathbb{M} \setminus \{\ell_o\}.
\end{align*}
\]
From here and using (H3) and (2.6), it gives
\[
0 \leq \text{tr}[a_\varepsilon(x_0)D^2 w_{\varepsilon,\iota}(x_0)]
\]
\[
= c_\varepsilon(x_0)w_{\varepsilon,\iota}(x_0) - \theta h_\varepsilon(x_0) + \psi w_\varepsilon(x_0) - c_\varepsilon(x_0)w_{\varepsilon,\iota}(x_0).
\]
Since \( c_\varepsilon > 0 \) on \( \mathcal{O} \), it follows that \( w_{\varepsilon,\iota}(x_0) \geq 0 \). Therefore \( w_{\varepsilon,\iota}(x) \geq w_{\varepsilon,\iota}(x_0) \geq 0 \), for all \( x \in \mathcal{O} \) and \((\ell,\iota) \in \mathcal{M} \times I\). Supposing now \((x_0,\ell,\iota) \in \partial \mathcal{O} \times \mathcal{M} \times I\) such that
\[
w_{\varepsilon,\iota}(x_0) = \max_{(\ell,\iota) \in \partial \mathcal{O} \times \mathcal{M} \times I} w_{\varepsilon,\iota}(x),
\]
taking into account that
\[
D^1 w_{\varepsilon,\iota}(x_0) = 0,
\]
\[
\text{tr}[a_\varepsilon(x_0)D^2 w_{\varepsilon,\iota}(x)] \leq 0,
\]
\[
w_{\varepsilon,\iota}(x_0) - w_{\varepsilon,\iota}(x_0) \geq 0 \text{ for } \kappa \in I \setminus \{\iota\},
\]
\[
w_{\varepsilon,\iota}(x_0) - w_{\varepsilon,\iota}(x_0) \geq 0 \text{ for } \ell' \in \mathcal{M} \setminus \{\ell^0\}.
\]
and arguing in a similar way as before, the reader can easily see that (2.9) is true. With this remark, we conclude the proof. \( \blacksquare \)

From now on, for simplicity of notation, we shall replace \( u_{\varepsilon,\iota} \) by \( u \) in the proofs of the results that we shall share below.

**Proof of Proposition 1.3. Existence.** By (H1), (H3), (H4), (2.5) and (2.9), it follows that \( A_1 \), is bounded uniformly with respect to the norm \( \| \cdot \|_{C^{1,\alpha}_{m,n}} \). From here and since the mapping \( T \) is continuous and compact, by Schaefer’s fixed point theorem, it yields that there exists a fixed point \( u = (u_{\ell,\iota})(\ell,\iota) \in C^{1,\alpha}_{m,n} \) to the problem \( T[u] = u \). In addition, we have \( u = T[u] \in C^{1,\alpha}_{m,n} \). By Theorem 9.19 of [9], we conclude that \( u \in C^{1,\alpha}_{m,n} \), since (H1), (H3) and (H4) hold and \( \Xi_{\ell,\iota} u \in C^{1,\alpha}(\mathcal{O}) \). Again, repeating the same argument as before, we obtain that \( u \in C^{1,\alpha}_{m,n} \). Finally, The non-negativeness of \( u_{\ell,\iota} \) can be verified using similar arguments seen in the proof of Lemma 2.2. \( \blacksquare \)

**Proof of Proposition 1.3. Uniqueness.** The uniqueness of the solution \( u \) to the NPDS (1.15) is obtained by contradiction. Assume that there are two solutions \( u, v \in C^{1,\alpha}_{m,n} \) to the NPDS (1.15). Let \( \nu = (\nu_{\ell,\iota})(\ell,\iota) \in C^{1,\alpha}_{m,n} \) such that \( \nu_{\ell,\iota} = u_{\ell,\iota} - v_{\ell,\iota} \) for \((\ell,\iota) \in \mathcal{M} \times I\). By (1.15), it implies
\[
[c_{\ell} - L_{\ell,\iota}]\nu_{\ell,\iota} + \psi_\varepsilon[(D^1 u_{\ell,\iota})^2 - g_{\ell}^2] - \psi_\varepsilon[(D^1 v_{\ell,\iota})^2 - g_{\ell}^2] + \sum_{\ell' \in \mathcal{M} \setminus \{\ell\}} [\psi_\varepsilon(u_{\ell,\iota} - u_{\ell',\iota} - \partial_{\ell,\iota}) - \psi_\varepsilon(v_{\ell,\iota} - v_{\ell',\iota} - \partial_{\ell,\iota})] = 0 \text{ on } \mathcal{O},
\]
s.t. \( u_{\ell,\iota} = 0 \), in \( \partial \mathcal{O} \).

Let \((x_0,\ell,\iota) \in \partial \mathcal{O} \times \mathcal{M} \times I\) such that \( \nu_{\varepsilon,\iota}(x_0) = \max_{(x,\ell,\iota) \in \partial \mathcal{O} \times \mathcal{M} \times I} \nu_{\varepsilon,\iota}(x) \). If \( x_0 \in \partial \mathcal{O} \), trivially it yields \( u_{\ell,\iota} - v_{\ell,\iota} \geq 0 \) in \( \partial \mathcal{O} \), for \((\ell,\iota) \in \mathcal{M} \times I\). Suppose that \( x_0 \in \mathcal{O} \). Then,
\[
D^1 \nu_{\varepsilon,\iota}(x_0) = 0, \quad \text{tr}[a_\varepsilon(x_0)D^2 \nu_{\varepsilon,\iota}(x_0)] \leq 0,
\]
\[
[\nu_{\varepsilon,\iota}(x_0) - \nu_{\varepsilon,\iota}(x_0)] \leq 0, \quad \text{for } \kappa \in I \setminus \{\iota\},
\]
\[
u_{\varepsilon,\iota}(x_0) - u_{\varepsilon,\iota}(x_0) \geq \nu_{\varepsilon,\iota}(x_0) - v_{\varepsilon,\iota}(x_0), \quad \text{for } \ell' \in \mathcal{M} \setminus \{\ell^0\}. \quad (2.11)
\]
Then, from (2.10)–(2.11),
\[
0 \geq \text{tr}[a_{\ell,\iota} D^2 \nu_{\ell,\iota}] + \sum_{\kappa \in \mathcal{N}(\ell,\iota)} q_{\ell,\iota}(\nu_{\ell,\iota})[\nu_{\ell,\kappa} - \nu_{\ell,\iota}] \\
= c_{\ell,\iota} \nu_{\ell,\iota} + \sum_{\nu' \in \mathcal{N}(\ell,\iota)} \{ \psi_{\delta}(u_{\ell,\iota} - u_{\ell,\iota} - \vartheta_{\ell,\iota}) - \psi_{\delta}(u_{\ell,\iota} - u_{\ell,\iota} - \vartheta_{\ell,\iota}) \} \\
\geq c_{\ell,\iota} \nu_{\ell,\iota} \quad \text{at } x^0,
\]
(2.12)
because of \(0 \leq \psi_{\delta}(u_{\ell,\iota} - u_{\ell,\iota} - \vartheta_{\ell,\iota}) - \psi_{\delta}(u_{\ell,\iota} - u_{\ell,\iota} - \vartheta_{\ell,\iota}) \) at \(x^0\), for \(\ell' \in \mathcal{M} \setminus \{\ell,\iota\}\). From (2.12) and since \(c_{\ell,\iota} > 0\), we have that \(u_{\ell,\iota}(x) - v_{\ell,\iota}(x) \leq u_{\ell,\iota}(x^0) - v_{\ell,\iota}(x^0) \leq 0\) for \((x,\ell,\iota) \in \overline{\mathcal{O}} \times \mathcal{M} \times \mathcal{I}\). Taking now \(v := v - u\) and proceeding in the same way as before, it follows immediately that \(v_{\ell,\iota} - u_{\ell,\iota} \leq 0\) on \(\overline{\mathcal{O}}\) for \((\ell,\iota) \in \mathcal{M} \times \mathcal{I}\). Therefore \(u = v\) and from here we conclude that the NPDS (1.15) has a unique solution \(u\), whose components belong to \(C^{4,\alpha}(\overline{\mathcal{O}})\). 

\section{Existence and uniqueness of the solutions to the HJB equations (1.12) and (1.17)}

To study the existence and uniqueness of the solutions \(u\) and \(u^\varepsilon\) to the variational inequalities (1.12) and (1.17), respectively, we will proceed in the same way as in [13], i.e., we will first verify that the sequence \(\{u^\varepsilon,\delta\}_{\varepsilon,\delta \in (0,1)^2}\) is bounded, uniformly in \((\varepsilon,\delta)\), with respect to the norms \(\| \cdot \|_{C^0(\overline{\mathcal{O}})}\), \(\| \cdot \|_{C^1_{\text{loc}}(\overline{\mathcal{O}})}\) and \(\| \cdot \|_{C^2_{\text{loc}}(\overline{\mathcal{O}})}\); see Lemmas 3.1–3.4; and then, for each \(\varepsilon \in (0,1)\) fixed, \(u^\varepsilon\) will be taken as limit of \(u^\varepsilon,\delta\) when \(\delta\) goes to zero. Finally, providing that \(\{u^\varepsilon\}_{\varepsilon \in (0,1)}\) is well defined and is bounded uniformly with respect to the norms \(\| \cdot \|_{C^0(\overline{\mathcal{O}})}\), \(\| \cdot \|_{C^1_{\text{loc}}(\overline{\mathcal{O}})}\) and \(\| \cdot \|_{C^2_{\text{loc}}(\overline{\mathcal{O}})}\), we will see \(u\) as a limit of \(u^\varepsilon\), when \(\varepsilon\) goes to zero.

\textbf{Lemma 3.1.} There exists \(v \in C^{4,\alpha}_{m,n}\) independent of \(\varepsilon\) and \(\delta\) such that for each \((\ell,\iota) \in \mathcal{M} \times \mathcal{I}\),
\[
0 \leq u^\varepsilon,\delta_{\ell,\iota} \leq v_{\ell,\iota} \quad \text{on } \overline{\mathcal{O}},
\]
(3.1)

To prove Lemma 3.1, we require first to see the existence and uniqueness of the classical solution \(v = (v_{\ell,\iota})_{(\ell,\iota) \in \mathcal{M} \times \mathcal{I}}\) to the problem
\[
[c_{\ell,\iota} - L_{\ell,\iota}]v_{\ell,\iota} = h_{\ell,\iota}, \quad \text{on } \mathcal{O}, \quad \text{s.t. } v_{\ell,\iota} = f_{\ell,\iota}, \quad \text{in } \partial \mathcal{O}.
\]
(3.2)

\textbf{Lemma 3.2.} If (H1), (H3) and (H4) hold, there exists a unique non-negative solution \(v\) to the Dirichlet problem (3.2) such that \(v_{\ell,\iota} \in C^{4,\alpha}(\overline{\mathcal{O}})\) for each \((\ell,\iota) \in \mathcal{M} \times \mathcal{I}\).

Though due to the results of Sweers [17], under weak assumptions in \(h_{\ell,\iota}\) and the parameters of \(L_{\ell,\iota}\), that guaranteed the existence of the non-negative solution \(v\) to (3.2) in a strong sense, it is possible to verify that, by fixed point arguments, (3.2) has a classical solution under the assumptions imposed above; see (H1), (H3) and (H4). The reader can find the proof of Lemma 3.2 in the appendix; see Subsection A.1, since this is similar to the proof of Proposition 1.4.

\textbf{Proof of Lemma 3.1.} For each \((\ell,\iota) \in \mathcal{M} \times \mathcal{I}\), by Lemma 3.2, let us consider \(v_{\ell,\iota} \in C^{4,\alpha}(\overline{\mathcal{O}})\) as the unique non-negative solution to the Dirichlet problem (3.2). From (1.15), it can be seen
that \([c_t - L_{\ell,\iota}] u_{\ell,\iota} \leq h_\iota\) on \(O\) for each \((\ell, \iota) \in \mathcal{M} \times I\). Then, taking \(\eta_{\ell,\iota} := u_{\ell,\iota} - v_{\ell,\iota}\), we get for each \((\ell, \iota) \in \mathcal{M} \times I\),

\[
[c_t - L_{\ell,\iota}] \eta_{\ell,\iota} \leq 0, \quad \text{in } O, \quad \text{s.t. } \eta_{\ell,\iota} = 0, \quad \text{on } \partial O.
\]

Then, considering \((x^o, \ell^o, \iota^o) \in \overline{O} \times \mathcal{M} \times I\) such that

\[
\eta_{\ell^o,\iota^o}(x^o) = \max_{(x,\ell,\iota) \in \overline{O} \times \mathcal{M} \times I} \eta_{\ell,\iota}(x),
\]

and proceeding in a similar way than in the proof of Proposition \(1.4\) (uniqueness), we get

\[u_{\ell,\iota} \leq v_{\ell,\iota} \] \(\blacksquare\).

**Remark 3.3.** From now on, we consider cut-off functions \(\varpi \in C^\infty_c(O)\) satisfying \(0 \leq \varpi \leq 1\), \(\varpi = 1\) on the open ball \(B_{\beta r} \subset B_{\beta' r} \subset O\) and \(\varpi = 0\) on \(O \setminus B_{\beta' r}\), with \(r > 0\), \(\beta' = \frac{\beta + 1}{2}\) and \(\beta \in (0, 1]\). It is also assumed that \(\|\varpi\|_{C^1(B_{\beta r})} \leq K_2\) where \(K_2 > 0\) is a constant independent of \(\varepsilon\) and \(\delta\).

**Lemma 3.4.** There exist positive constants \(C_2, C_3\) independent of \(\varepsilon, \delta\) such that for each \(x \in \overline{O}\)

\[
\varpi(x)|D^1 u_{\ell,\iota}^{\varepsilon,\delta}(x)| \leq C_2, \quad \text{(3.3)}
\]

\[
\varpi(x)|D^2 u_{\ell,\iota}^{\varepsilon,\delta}(x)| \leq C_3. \quad \text{(3.4)}
\]

The proof is a slight modification of the proof of the similar conclusion in [13], so put it in the appendix; see Subsections A.2 and A.3.

Let \(\varepsilon \in (0, 1)\) and \((\ell, \iota) \in \mathcal{M} \times I\) be fixed. By Lemmas 3.1 and 3.4, using Arzelà-Ascoli compactness criterion (see [5, p. 718]) and that for each \(p \in (1, \infty)\), \((L^p(B_{\beta r}), \|\cdot\|_{L^p(B_{\beta r})})\), with \(B_r \subset O\), is a reflexive space (see [1, Thm. 2.46, p. 49]), we get that there exist a sub-sequence \(\{u_{\ell,\iota}^{\varepsilon,\delta_n}\}_{n \geq 1}\) of \(\{u_{\ell,\iota}^{\varepsilon,\delta}\}_{\delta \in (0, 1)}\) and a function \(w_{\ell,\iota}^\varepsilon\) in \(W^{2,\infty}_{loc}(O)\) such that

\[
u_{\ell,\iota}^{\varepsilon,\delta_n} \to u_{\ell,\iota}^\varepsilon \quad \text{in } C^1_{loc}(O), \quad \text{(3.5)}
\]

\[
\partial_{ij} u_{\ell,\iota}^{\varepsilon,\delta_n} \rightharpoonup \partial_{ij} w_{\ell,\iota}^\varepsilon, \text{ weakly } L^p_{loc}(O), \text{ for each } p \in (1, \infty).
\]

Taking

\[
u_{\ell,\iota}^\varepsilon(x) := \begin{cases} w_{\ell,\iota}^\varepsilon(x) & \text{if } x \in O, \\ f_\iota(x) & \text{if } x \in \partial O, \end{cases} \quad \text{(3.6)}
\]

and considering that \(0 \leq u_{\ell,\iota}^{\varepsilon,\delta_n} \leq u_{\ell,\iota}^\varepsilon\) on \(O\) and \(u_{\ell,\iota}^{\varepsilon,\delta_n} = v_{\ell,\iota} = f_\iota\) in \(\partial O\), it implies that

\[
0 \leq u_{\ell,\iota}^\varepsilon \leq v_{\ell,\iota} \quad \text{on } O \quad \text{and } \quad u_{\ell,\iota}^\varepsilon = f_\iota \quad \text{in } \partial O. \quad \text{(3.7)}
\]

Therefore \(u_{\ell,\iota}^\varepsilon \in C^0(\overline{O}) \cap W^{2,\infty}_{loc}(O)\). Now, using Lemmas 3.1 and 3.4 and by (3.5), the following inequalities hold for each \((\ell, \iota) \in \mathcal{M} \times I\),

\[
\varpi(x)|D^1 u_{\ell,\iota}^\varepsilon(x)| \leq C_4 \quad \text{for } x \in \overline{O},
\]

\[
|| D^2 u_{\ell,\iota}^\varepsilon||_{L^p(B_{\beta r})} \leq C_5 \quad \text{for each } p \in (1, \infty) \quad \text{and } \quad B_{\beta r} \subset O. \quad \text{(3.8)}
\]
for some positive constants $C_4 = C_4(d, \Lambda, \alpha)$ and $C_5 = C_5(d, \Lambda, \alpha)$. Then, from (3.7), (3.8) and by the same criterion in (3.5), we have that there exist a sub-sequence $\{u_{\varepsilon_n}\}_{n \geq 1}$ of $\{u_{\varepsilon_n}\}_{\varepsilon \in (0,1)}$ and a $w_{\varepsilon_n}$ in $C^{0,1}(\bar{\mathcal{O}}) \cap W^{2,\infty}_{\text{loc}}(\mathcal{O})$ such that

$$u_{\varepsilon_n} \rightarrow w_{\varepsilon_n} \text{ in } C^1(\mathcal{O}),$$

$$\partial_{ij} u_{\varepsilon_n} \rightarrow \partial_{ij} w_{\varepsilon_n},$$

weakly $L^p_{\text{loc}}(\mathcal{O})$, for each $p \in (1, \infty)$.

Define

$$u_{\varepsilon,\ell}(x) = \begin{cases} w_{\varepsilon,\ell}(x) & \text{if } x \in \mathcal{O}, \\ f_{\ell}(x) & \text{if } x \in \partial \mathcal{O}. \end{cases}$$

Since (3.7) holds, we get that

$$0 \leq u_{\varepsilon,\ell} \leq v_{\ell,\iota} \text{ on } \mathcal{O} \text{ and } u_{\varepsilon,\ell} = f_\ell \text{ in } \partial \mathcal{O}.$$ 

Therefore $u_{\varepsilon,\ell} \in C^0(\bar{\mathcal{O}}) \cap W^{2,\infty}_{\text{loc}}(\mathcal{O})$.

To finalize this section, let us remark that $u^\varepsilon$ and $u$, taken as in (3.6) and (3.9) are the unique solutions to (1.17) and (1.12), respectively. Since the proofs of the previous asseverations are a slight modification of the proofs of the similar conclusions in [13], so we put them in the appendix; see Subsections A.4 and A.5.

4 $\varepsilon$-PACS control problem and proof of Theorem 1.2

In this section, we shall verify that the value functions $V$ given in (1.8) agrees with the solution $u$ to the HJB equation (1.12) on $\bar{\mathcal{O}}$ under assumptions (H2) and (H3)--(H5). To prove it, firstly, we study an $\varepsilon$-PACS control problem that is closely related to the value function problem seen previously.

The penalized control set $\mathcal{U}^\varepsilon$ is defined by

$$\mathcal{U}^\varepsilon = \{\xi = (\eta, \zeta) \in \mathcal{U} : \zeta_t \text{ is absolutely continuous, } 0 \leq \zeta_t \leq 2C/\varepsilon\}$$

where $\varepsilon \in (0,1)$ is fixed and $C$ is a positive constant independent of $\varepsilon$. Let $(x_0, \ell_0, t_0) \in \bar{\mathcal{O}} \times \mathbb{N} \times \mathbb{I}$ be fixed. The controlled process $(X^{\xi,\ell, I})$ evolves as

$$X_t^{\xi,\ell} = X_{\tau_i}^{\xi,\ell} - \int_{\tau_i}^t [b(X_{s}^{\xi,\ell}, I_s^{(\ell_i)}) + \eta_s \dot{\zeta}] ds + \int_{\tau_i}^t \sigma(X_{s}^{\xi,\ell}, I_s^{(\ell_i)}) dW_s,$$

$$I_t = I_t^{(\ell_i)} \text{ and } J_t^{(\ell)} = \ell_i \text{ for } t \in [\tau_i, \tau_{i+1}) \text{ and } i \geq 0,$$

where $\tau_i = \tau_i \wedge \tau$ and $\tau$ is the first exit time of $X^{\xi,\ell}$ from the set $\mathcal{O}$. Defining the Legendre transform of $H^\ell(\gamma, x) := H^\varepsilon(\gamma, x, \ell) = \psi_{\ell}(|\gamma|^2 - g_\ell^2(x))$, with $(x, \ell) \in \bar{\mathcal{O}} \times I$, as

$$l_{\gamma}(y, x, \ell) := \sup_{\gamma \in \mathbb{R}^d} \{\langle \gamma, y \rangle - H^\ell(\gamma, y)\},$$

the corresponding penalized functional cost for $(\xi, \zeta) \in \mathcal{U}^\varepsilon \times \mathcal{S}$ is defined as

$$V_{\xi,\ell}(x_0, \ell_0, t_0) := E_{x_0, \ell_0, t_0} \left[ \int_{[0,\tau]} e^{-r(t)} [h(X_t^{\xi,\ell}, I_t) + l^\varepsilon(\dot{\zeta}, \eta_t, X_t^{\xi,\ell}, I_t)] dt \right]$$

$$+ \sum_{i \geq 0} E_{x_0, \ell_0, t_0} \left[ e^{-r(\tau_{i+1})} \partial_{t_{\ell},\tau_{i+1}^+} \mathbb{I}_{\{\tau_{i+1}^+ < \tau\}} \right]$$

$$+ E_{x_0, \ell_0, t_0} \left[ e^{-r(\tau)} f(X_t^{\xi,\ell}, I_t) \mathbb{I}_{\{\tau < \infty\}} \right],$$

(4.3)
and the value function is then given by

\[ V^c_{\ell_0}(x_0, \ell_0) = \inf_{\xi} \mathcal{V}_{\xi}(x_0, \ell_0, \ell_0), \]  

where its HJB equation takes the form

\[
\max \left \{ c_i - \mathcal{L}_{\ell,\ell} u^c_{\ell,\ell} + \sup_{y \in \mathbb{R}^d} \{ (D^1 u^c_{\ell,\ell}, y) - l^c_{\ell}(y, \cdot) \} - h_i, u^c_{\ell,\ell} - \mathcal{M}_{\ell,\ell} u^c_{\ell,\ell} \right \} = 0, \text{ on } \mathcal{O},
\]

\[
\text{s.t. } u^c_{\ell,\ell} = f_i, \text{ in } \partial \mathcal{O},
\]

where \( \mathcal{M}_{\ell,\ell} \) and \( \mathcal{L}_{\ell,\ell} \) are as in (1.13). Observe that (4.5) can be rewritten as (1.17) because of \( H^c_\ell(\gamma, x) = \sup_{y \in \mathbb{R}^d} \{ \langle \gamma, y \rangle - l^c_\ell(y, x) \} \).

To facilitate the notation of this section, let us denote \( \mathcal{L}_{\ell,\ell}, \hat{f}_{\ell,\ell}, \mathcal{M}_{\ell,\ell}, \hat{f} \) by \( \mathcal{L}_{\ell,\ell}, \hat{f}_{\ell,\ell}, \mathcal{M}_{\ell,\ell}, \hat{f} \), respectively. Let us start showing a general result which will be helpful for the purposes of the section.

**Lemma 4.1.** Let \((X^{\xi,}, J^c, I)\) evolve as (1.3), with \((\xi, \varsigma) \in \mathcal{U} \times \mathcal{S}\) and initial state \((x_0, \ell_0, \ell_0) \in \mathcal{O} \times \mathbb{M} \times \mathbb{I}\). Let \( \hat{f} = (\hat{f}_1, \ldots, \hat{f}_m) \) be a sequence of real valued function such that \( \hat{f}_\ell(\cdot, \cdot) \in \mathcal{C}^2(\mathcal{O}) \) for \((\ell, \ell) \in \mathbb{M} \times \mathbb{I}\). Take \( \hat{\tau}^q_0 := 0 \) and \( \hat{\tau}^q_1 := \hat{\tau}_\ell \wedge \inf \{ t > \tau_{i-1} : X_t \notin \mathcal{O}_q \} \), with \( \hat{\tau}_i = \tau_i \wedge \tau \), \( i \geq 1 \), \( \mathcal{O}_q := \{ x \in \mathcal{O} \text{ : dist}(x, \partial \mathcal{O}) > 1/q \} \) and \( q \) a positive integer large enough such that \( X_{0-} = x_0 \in \mathcal{O}_q \). Then

\[
\mathbb{E}_{x_0,\ell_0,\ell_0} \left[ e^{-r(\hat{\tau}^q_\ell)} \hat{f}_{\ell_1}(X^{\xi,}_{\hat{\tau}^q_1}, I^q_{\hat{\tau}^q_1}) \mathbb{I}_{\{\hat{\tau}^q_1 < \tau\}} \right] = \mathbb{E}_{x_0,\ell_0,\ell_0} \left[ \left. e^{-r(\hat{\tau}^q_1)} \hat{f}_{\ell_1}(X^{\xi,}_{\hat{\tau}^q_1}, I^q_{\hat{\tau}^q_1}) - \sum_{\hat{\tau}^q_i < \hat{\tau}^q_{i+1}} e^{-r(\hat{\tau}^q_i)} \mathcal{J}[X^{\xi,}_{\hat{\tau}^q_i}, I_i, \hat{f}_{\ell_i}] \right| \right. \\
+ \int_{\hat{\tau}^q_1}^{\hat{\tau}^q_{i+1}} e^{-r(s)} \left[ \left. \left( \xi(X^{\xi,}_s, I_s) \hat{f}_{\ell_i}(X^{\xi,}_s, I_s) \right) \xi_s \right| \right. \\
- \mathcal{L}_{\ell_i, I_i} \hat{f}_{\ell_i}(X^{\xi,}_s, I_s)] ds + \left( D^1 \hat{f}_{\ell_i}(X^{\xi,}_s, I_s), \nu_s \right) d\xi^c_s \mathbb{I}_{\{\hat{\tau}^q_i < \tau\}} \right],
\]

where \( \int_{a+}^b \) defines the integral operator on the interval \([a, b]\), \( \xi^c \) is the continuous part of the process \( \xi \), and

\[
\mathcal{J}[X, I, \hat{f}_{\ell_i}] := \hat{f}_{\ell_i}(X, I_s) - \hat{f}_{\ell_i}(X_{s-}, I_s) = \hat{f}_{\ell_i}(X_{s-} - \nu_s \xi_s, I_s) - \hat{f}_{\ell_i}(X_{s-}, I_s).
\]

From now on, for simplicity of notation, we replace \( X^{\xi,} \) by \( X \) in the proofs of the results.

**Proof.** For each \( i \geq 0 \), we assign \( \rho_{j}^{\ell_i} := \tau^q_{\ell_i} < \rho_1^{\ell_i} < \rho_2^{\ell_i} < \cdots < \rho_{j-1}^{\ell_i} < \tau^q_{\ell_i+1} := \rho_j^{\ell_i} \), for some \( j \geq 0 \), as all the possible random times where the process \( f \) has a jump on the interval time \( [\tau^q_{\ell_i}, \tau^q_{\ell_i+1}] \), i.e. \( \ell_i = \ell^c_{j-1} \) if \( \tau \in [\rho_{j}^{\ell_i}, \rho_{j+1}^{\ell_i}] \), for \( j \in \{0, 1, \ldots, j-1\} \). Using integration by parts and Itô's formula in \( e^{-r(s)} \hat{f}_{\ell_i}(X_t, t^c_j) \) on \( [\rho_{j}^{\ell_i}, \rho_{j+1}^{\ell_i}] \) for \( j \in \{0, 1, \ldots, j-1\} \); see [16, Theorem
Let us consider $\Delta_\nu$, where we have that (4.8)
\[
\begin{aligned}
\int_{\mathbb{R}^+} e^{-r(\tilde{q}_i^{\nu})} \hat{f}_{t_i}(X_{\tilde{q}_i^{\nu}}, I_{\tilde{q}_i^{\nu}}) - e^{-r(\tilde{q}_i^{\nu} + 1)} \hat{f}_{t_i}(X_{\tilde{q}_i^{\nu} + 1}, I_{\tilde{q}_i^{\nu} + 1}) \\
= \int_{\tilde{q}_i^{\nu}}^\rho \int_{\mathbb{R}^+} e^{-r(\nu)} [c(X_s, I_s) \hat{f}_{t_i}(X_s, I_s) - \tilde{L}_{\rho, I \nu} \hat{f}_{t_i}(X_s, I_s)] ds + (D^1 \hat{f}_{t_i}(X_s, I_s), n_s) d\zeta^\nu
\end{aligned}
\]

where $\tilde{L}_{\rho, I \nu}$, with $\rho \neq \nu$, as the consecutive, with respect to the lexicographic ordering on $\mathbb{R}^+ \times \mathbb{R}$, left-closed, right-open intervals of the real line, which have length $q_{\nu} \nu$. Defining $\tilde{h}_{t_i} : \mathbb{R} \to \mathbb{R}$ as
\[
\tilde{h}_{t_i}(t, z) = \sum_{\kappa \in \mathbb{R} \setminus \{t\}} (\kappa - t) \mathbb{1}_{\{z \in \Delta_{t_i}(\kappa)\}},
\]

we have that (1.1) is equivalent to
\[
\begin{aligned}
dI_{t_i}^{(t_i)}(z) = \int_{\mathbb{R}^+} \tilde{h}_{t_i}(I_{t_i}^{(t_i)}, z) N(dt, dz),
\end{aligned}
\]

where $N(dt, dz)$ is a Poisson random measure with intensity $dt \times \nu(dz)$ independent of $W$, and $\nu$ is the Lebesgue measure on $\mathbb{R}$; for more details see, e.g. [20]. From here and recalling that $I$ is governed by $Q_{t_i}$ on $(\tilde{q}_i^{\nu}, \tilde{q}_i^{\nu + 1})$, we have the next equivalent expression for (4.9),
\[
\begin{aligned}
\int_{\mathbb{R}^+} e^{-r(\tilde{q}_i^{\nu})} \hat{f}_{t_i}(X_{\tilde{q}_i^{\nu}}, I_{\tilde{q}_i^{\nu}}) - e^{-r(\tilde{q}_i^{\nu} + 1)} \hat{f}_{t_i}(X_{\tilde{q}_i^{\nu} + 1}, I_{\tilde{q}_i^{\nu} + 1}) \\
= \int_{\tilde{q}_i^{\nu}}^\rho \int_{\mathbb{R}^+} e^{-r(\nu)} [c(X_s, I_s) \hat{f}_{t_i}(X_s, I_s) - \tilde{L}_{\rho, I \nu} \hat{f}_{t_i}(X_s, I_s)] ds + (D^1 \hat{f}_{t_i}(X_s, I_s), n_s) d\zeta^\nu
\end{aligned}
\]
where the process
\[ \mathcal{M}[\bar{\tau}^q_t, t \wedge \bar{\tau}^q_{t+1}; X, I, \hat{f}_t]: = \int_{\bar{\tau}^q_t}^{t \wedge \bar{\tau}^q_{t+1}} e^{-r(s)} \langle D^1 \hat{f}_t(X_s, I_s), \sigma(X_s, I_s) \rangle dW_s \]
\[ + \int_{\bar{\tau}^q_t}^{t \wedge \bar{\tau}^q_{t+1}} \int_{\mathbb{R}} e^{-r(s)} [\hat{f}_t(X_{s-}, I_{s-} + \hat{h}_t(I_{s-}, z)) - \hat{f}_t(X_{s-}, I_{s-})] (N(ds, dz) - ds \times \nu(dz)) \]
is a square-integrable martingale. Therefore, multiplying by \( I_{\{\tau_i < t\}} \) and taking expected value in both sides of (4.10), we get (4.6). 

### 4.1 Verification Lemma for \( \varepsilon \)-PACS control problem

Let \((X^\varepsilon, \mathcal{K}, I)\) evolve as (4.2), with \((\xi, \varsigma) \in \mathcal{U}^\varepsilon \times \mathcal{S}\) and initial state \((x_0, \ell_0, i_0) \in \mathcal{O} \times \mathbb{M} \times \mathcal{I}\). Under assumptions (H2)–(H4), Lemmas 4.2 and 4.7 shall be proven.

**Lemma 4.2** (Verification Lemma for \( \varepsilon \)-PACS control problem. First part). Let \( \varepsilon \in (0, 1) \) be fixed. Then \( u_{\varepsilon}^0(x_0, \ell_0, i_0) \leq V_{\varepsilon}^0(x_0, \ell_0, i_0) \) for each \((x_0, \ell_0, i_0) \in \mathcal{O} \times \mathbb{M} \times \mathcal{I}\).

**Proof.** Take \( u_{\varepsilon, \delta_h} = (u_{\varepsilon, \delta_h}^1, \ldots, u_{\varepsilon, \delta_h}^m) \) satisfying (3.5) which is the unique solutions to the NPDS (1.15), when \( \delta = \delta_h \). By Proposition 1.3, it is known that \( u_{\varepsilon, \delta_h}^i(\cdot, \ell) \in C^{4,\alpha}(\mathcal{O}) \) for \((\ell, \ell) \in \mathbb{M} \times \mathcal{I}\). Then, considering \( \{\bar{\tau}^q_t\}_{t \geq 0} \) as in Lemma 4.1, we get that (4.9) is true when \( \hat{f} = u_{\varepsilon, \delta_h} \). Notice that \( \zeta^c \equiv \zeta^c \) and \( \Delta \zeta = 0 \), due to \( \xi \in \mathcal{U}^\varepsilon \). Then, \( J[X_s, I_s, u_{\varepsilon, \delta_h}^i] = 0 \) for \( s \in (\bar{\tau}^q_t, \bar{\tau}^q_{t+1}] \). On the other hand, by (1.15) and since \( \psi \geq 0 \), it is known that \( c(x, \ell) u_{\varepsilon, \delta_h}^i(x, \ell) - \mathcal{L}_{e, i} u_{\varepsilon, \delta_h}^i(x, \ell) \leq h(x, \ell) - \psi \varepsilon (D^1 u_{\varepsilon, \delta_h}^i(x, \ell))^2 - g(x, \ell)^2 \) for \( x \in \mathcal{O} \) and \((\ell, \ell) \in \mathbb{M} \times \mathcal{I}\), and \( \langle \gamma, y \rangle \leq \psi \varepsilon \langle \gamma \rangle^2 - g(x, \ell)^2 \) for \( x, y \in \mathbb{R}^d \) and \((\ell, \ell) \in \mathbb{M} \times \mathcal{I}\). Then,
\[
\begin{align*}
&c(X_s, I_s) u_{\varepsilon, \delta_h}^i(X_s, I_s) - \mathcal{L}_{e, i} u_{\varepsilon, \delta_h}^i(X_s, I_s) \\
&\leq h(X_s, I_s) - \psi \varepsilon (D^1 u_{\varepsilon, \delta_h}^i(X_s, I_s))^2 - g(X_s, I_s)^2, \\
&\langle D^1 u_{\varepsilon, \delta_h}^i(X_s, I_s), \nabla \zeta^c_s \rangle - \psi \varepsilon (D^1 u_{\varepsilon, \delta_h}^i(X_s, I_s))^2 - g(X_s, I_s)^2 \leq \ell^c \langle \nabla \zeta^c_s, X_s, I_s \rangle,
\end{align*}
\]for \( s \in (\bar{\tau}^q_t, \bar{\tau}^q_{t+1}] \). Hence, it implies that
\[
\mathbb{E}_{x_0, \ell_0, i_0}^{\mathcal{O} \times \mathbb{M} \times \mathcal{I}} [e^{-r(\bar{\tau}^q_t)} u_{\varepsilon, \delta_h}^i(X_{\bar{\tau}^q_t}, I_{\bar{\tau}^q_t}) I_{\{\tau_i < \bar{\tau}_i\}}] \leq \mathbb{E}_{x_0, \ell_0, i_0}^{\mathcal{O} \times \mathbb{M} \times \mathcal{I}} \left\{ e^{-r(\bar{\tau}^q_{t+1})} u_{\varepsilon, \delta_h}^i(X_{\bar{\tau}^q_{t+1}}, I_{\bar{\tau}^q_{t+1}}) \\
+ \int_{\bar{\tau}^q_t}^{\bar{\tau}^q_{t+1}} e^{-r(s)} [h(X_s, I_s) + \ell^c \langle \nabla \zeta^c_s, X_s, I_s \rangle] ds \right\} I_{\{\tau_i < \bar{\tau}_i\}}. \quad (4.11)
\]
Noticing that \( \max_{(x, i) \in \mathcal{O} \times \mathcal{I}} |u_{\varepsilon, \delta_h}^i(x, \ell) - u_{\varepsilon}^i(x, \ell)| \xrightarrow{\delta_h \rightarrow 0} 0 \) for \( \ell \in \mathbb{M} \), \( \bar{\tau}^q_t \uparrow \bar{\tau}_i \) as \( q \rightarrow \infty \), \( \mathbb{P}_{x_0, \ell_0, i_0} \)-a.s., letting \( q \rightarrow \infty \) and \( \delta_h \rightarrow 0 \) in (4.11), and using Dominated Convergence Theorem, it follows that
\[
\mathbb{E}_{x_0, \ell_0, i_0}^{\mathcal{O} \times \mathbb{M} \times \mathcal{I}} [e^{-r(\bar{\tau}_i)} u_{\varepsilon}^i(X_{\bar{\tau}_i}, I_{\bar{\tau}_i}) I_{\{\tau_i < \bar{\tau}_i\}}] \leq \mathbb{E}_{x_0, \ell_0, i_0}^{\mathcal{O} \times \mathbb{M} \times \mathcal{I}} \left\{ e^{-r(\bar{\tau}_{i+1})} u_{\varepsilon}^i(X_{\bar{\tau}_{i+1}}, I_{\bar{\tau}_{i+1}}) \\
+ \int_{\bar{\tau}_i}^{\bar{\tau}_{i+1}} e^{-r(s)} [h(X_s, I_s) + \ell^c \langle \nabla \zeta^c_s, X_s, I_s \rangle] ds \right\} I_{\{\tau_i < \bar{\tau}_i\}}.
\]
The complement $\epsilon$ On the other hand, since the control $\xi$ From here, using (4.12), observe that $u^\varepsilon_\ell(x, \iota) - [u^\varepsilon_\ell(x, \iota) + \vartheta_{\ell, \iota}] \leq u^\varepsilon_\ell(x, \iota) - M_{\ell} u^\varepsilon(x, \iota) \leq 0$ for $x \in \mathcal{O}$ and $(\ell, \iota, \ell) \in \mathbb{M} \times \mathbb{I}$. Then
\[
u^\varepsilon_\ell(X_{\tau_{\ell+1}}, I_{\tau_{\ell+1}}) = f(X_T, I_T)1_{\{\tau_{\ell+1} \leq \tau\}} + [u^\varepsilon_{\ell+1}(X_{\tau_{\ell+1}}, I_{\tau_{\ell+1}}) + \vartheta_{\ell, \ell+1}]1_{\{\tau > \tau_{\ell+1}\}}
+ [u^\varepsilon_{\ell+1}(X_{\tau_{\ell+1}}, I_{\tau_{\ell+1}}) - u^\varepsilon_{\ell+1}(X_{\tau_{\ell+1}}, I_{\tau_{\ell+1}}) + \vartheta_{\ell, \ell+1}]1_{\{\tau > \tau_{\ell+1}\}} \leq f(X_T, I_T)1_{\{\tau_{\ell+1} \leq \tau\}} + [u^\varepsilon_{\ell+1}(X_{\tau_{\ell+1}}, I_{\tau_{\ell+1}}) + \vartheta_{\ell, \ell+1}]1_{\{\tau > \tau_{\ell+1}\}}.
\]
(4.12)

Thus,
\[
E_{x, \ell, t_0, t_0} [e^{-r(\tau)} u^\varepsilon_\ell(X_{\tau}, I_{\tau})] \leq E_{x, \ell, t_0, t_0} \left[ e^{-r(\tau)} f(X_T, I_T)1_{\{\tau_{\ell} < \tau \leq \tau_{\ell+1}\}}
+ e^{-r(\tau_{\ell+1})} [u^\varepsilon_{\ell+1}(X_{\tau_{\ell+1}}, I_{\tau_{\ell+1}}) + \vartheta_{\ell, \ell+1}]1_{\{\tau > \tau_{\ell+1}\}}
\right.
+ 1_{\{\tau_{\ell} < \tau \}} \int_{\tau_{\ell}}^{\tau_{\ell+1}} e^{-r(\iota)} [h(X_{\iota}, I_{\iota}) + l^\varepsilon(n, \zeta_\iota, X_{\iota}, I_{\iota})] d\iota \right].
\]
(4.13)

On the other hand, since the control $\xi$ acts continuously on $X$, we know that $X_0 = X_{t_0} = x_0$. From here, using (4.12) when $i = 0$, and considering recurrently (4.13), we conclude that
\[
u^\varepsilon_0(x_0, t_0) = E_{x, \ell, t_0, t_0} [u^\varepsilon_0(X_{t_0}, I_{t_0})] + E_{x, \ell, t_0, t_0} [u^\varepsilon_0(X_{t_0}, I_{t_0})1_{\{\tau = \tau_{\ell}\}}]
\leq E_{x, \ell, t_0, t_0} \left[ f(x_0, t_0)1_{\{\tau = \tau\}} + e^{-r(\tau)} f(X_T, I_T)1_{\{\tau_{\ell} < \tau \leq \tau_{\ell+1}\}} + e^{-r(\tau)} \vartheta_{\ell, \ell+1}1_{\{\tau > \tau_{\ell+1}\}}
\right.
+ 1_{\{\tau_{\ell} < \tau \}} \int_{0}^{\tau} e^{-r(\iota)} [h(X_{\iota}, I_{\iota}) + l^\varepsilon(n, \zeta_\iota, X_{\iota}, I_{\iota})] d\iota \right]
+ E_{x, \ell, t_0, t_0} [e^{-r(\tau)} u^\varepsilon_\ell(X_{\tau}, I_{\tau})1_{\{\tau > \tau_{\ell+1}\}}]
+ \vdots
\leq E_{x, \ell, t_0, t_0} \left[ e^{-r(\tau)} f(X_T, I_T)1_{\{\tau < \tau\}} + \sum_{i=0}^{\tau} e^{-r(\tau_{i+1})} \vartheta_{\ell, \ell+1}1_{\{\tau_{i+1} < \tau\}}
\right.
+ \int_{0}^{\tau} e^{-r(\iota)} [h(X_{\iota}, I_{\iota}) + l^\varepsilon(n, \zeta_\iota, X_{\iota}, I_{\iota})] d\iota \right] = \nu_{\zeta, \xi}(x_0, \ell, t_0).
\]
(4.14)

Therefore, it yields $u^\varepsilon_\ell(x_0, t_0) \leq \nu_{\zeta, \xi}(x_0, \ell, t_0)$


4.1.1 $\varepsilon$-PACS optimal control problem

Before presenting the second part of the verification lemma, let us first construct the control $(\xi^c, \zeta^c)$ which turns out to be the optimal strategy for the $\varepsilon$-PACS control problem. Let us first introduce the switching regions.

For any $\ell \in \mathbb{M}$, let $S^\varepsilon_\ell$ be the set defined by
\[
S^\varepsilon_\ell = \{(x, \iota) \in \mathcal{O} \times \mathbb{I} : u^\varepsilon_\ell(x, \iota) - M_{\ell} u^\varepsilon(x, \iota) = 0\}.
\]
The complement $C^\varepsilon_\ell$ of $S^\varepsilon_\ell$ in $\mathcal{O} \times \mathbb{I}$, where is optimal to stay in the regime $\ell$, is the so-called continuation region
\[
C^\varepsilon_\ell = \{(x, \iota) \in \mathcal{O} \times \mathbb{I} : u^\varepsilon_\ell(x, \iota) - M_{\ell} u^\varepsilon(x, \iota) < 0\}.
\]
The set $S^\varepsilon_\ell$ satisfies the following property.
Lemma 4.3. Let \( \ell \) be in \( \mathcal{M} \). Then, \( S^e_\ell = \tilde{S}^e_\ell := \bigcup_{\ell' \in \mathcal{M} \setminus \{ \ell \}} S^e_{\ell', \ell} \) where

\[
S^e_{\ell', \ell} := \{ (x, \tau) \in C^e_{\ell'} : u^e_{\ell'}(x, \tau) = u^e_{\ell'}(x, \tau) + \vartheta_{\ell', \ell} \}. 
\]

Proof. We obtain trivially that \( \tilde{S}^e_\ell \subset S^e_\ell \) due to \( u^e_{\ell'}(x, \tau) - u^e_{\ell'}(x, \tau) - \vartheta_{\ell', \ell} \leq u^e_{\ell'}(x, \tau) - M_{\ell'} w^e_{\ell}(x, \tau) \leq 0 \) for \( (x, \tau) \in \mathcal{O} \times \mathbb{I} \) and \( \ell' \in \mathcal{M} \setminus \{ \ell \} \). If \( (x, \tau) \in S^e_{\ell'} \), there is an \( \ell_1 \neq \ell \) where \( u^e_{\ell'}(x, \tau) = u^e_{\ell_1}(x, \tau) + \vartheta_{\ell_1, \ell} \). Notice that \( (x, \tau) \) must belong either \( C^e_{\ell_1} \) or \( S^e_{\ell_1} \). If \( (x, \tau) \in C^e_{\ell_1} \), it yields that \( (x, \tau) \) \( \in S^e_{\ell_1, \ell_1} \subset \tilde{S}^e_\ell \). Otherwise, there is an \( \ell_2 \neq \ell_1 \) such that \( u^e_{\ell_1}(x, \tau) = u^e_{\ell_2}(x, \tau) + \vartheta_{\ell_1, \ell_2} \). It implies \( u^e_{\ell}(x, \tau) = u^e_{\ell_2}(x, \tau) + \vartheta_{\ell_1, \ell_2} \geq u^e_{\ell_2}(x, \tau) + \vartheta_{\ell_1, \ell_2} \), since (1.9) holds. Then, \( u^e_{\ell}(x, \tau) = u^e_{\ell_2}(x, \tau) + \vartheta_{\ell_1, \ell_2} \). Again \((x, \tau) \in S^e_{\ell_2} \) must belong either \( C^e_{\ell_2} \) or \( S^e_{\ell_2} \). If \((x, \tau) \in C^e_{\ell_2} \), it yields that \((x, \tau) \in S^e_{\ell, \ell_2} \subset \tilde{S}^e_\ell \). Otherwise, arguing the same way before and since the number of regimes is finite, it must occur that there is some \( \ell_i \neq \ell \) such that \((x, \tau) \in C^e_{\ell_i} \) and \( u^e_{\ell_i}(x, \tau) = u^e_{\ell_2}(x, \tau) + \vartheta_{\ell_i, \ell_2} \). Therefore \((x, \tau) \in S^e_{\ell, \ell_2} \subset \tilde{S}^e_\ell \). \( \blacksquare \)

Now we construct the optimal control \((\xi^{e,*}, \varsigma^{e,*})\) to the problem (4.4). Let \((x_0, \ell_0, \tau_0) \in \mathcal{O} \times \mathcal{M} \times \mathbb{I} \). The dynamics of the process \((X^{e,*}, I^*) := \{(X^{e,*}_t, I^*_t) : t \geq 0 \} \) and \((\xi^{e,*}, \varsigma^{e,*})\) is given recursively in the following way:

(i) Define \( \tau^*_0 = 0 \) and \( \ell^*_0 = \ell_0 \). If \((x_0, \tau_0) \notin C^e_{\ell_0} \), take \( \tau^*_0 := 0 \) and pass to item (ii) due to Lemma 4.3. Otherwise, the process \((X^{e,*}, I^*) \) evolves as

\[
X^{e,*}_{t \wedge \tau^*_1} = x - \int_0^{t \wedge \tau^*_1} [b(X^{e,*}_s, I^*_s) + v^{e,*}_s \varsigma^{e,*}_s]ds + \int_0^{t \wedge \tau^*_1} \sigma(X^{e,*}_s, I^*_s)dW_s, \quad t > 0, \\
I^{(\tau^*_1)}_{t \wedge \tau^*_1} = I^{(\tau^*_1)}_t, 
\]

with \( X^{e,*}_0 = x_0, I^*_0 = \ell_0, \tau^* := \inf \{ t > 0 : (X^{e,*}_t, I^*_t) \notin \mathcal{O} \}, \)

\( \tau^*_1 := \tau^*_1 \wedge \tau^* \) and \( \tau^*_i := \inf \{ t > 0 : (X^{e,*}_t, I^*_t) \in S^e_{\ell_0} \}, \) (4.16)

The control process \( \xi^{e,*} = (v^{e,*}, \varsigma^{e,*}) \) is defined by

\[
\varsigma^{e,*}_t = \begin{cases} 
D^1 u^{e}_{\ell^*_0}(X^{e,*}_t, I^*_t), & \text{if } |D^1 u^{e}_{\ell^*_0}(X^{e,*}_t, I^*_t)| \neq 0 \text{ and } t \in [0, \tau^*_1], \\
\gamma_0, & \text{if } |D^1 u^{e}_{\ell^*_0}(X^{e,*}_t, I^*_t)| = 0 \text{ and } t \in [0, \tau^*_1].
\end{cases}
\]

(4.17)

where \( \gamma_0 \in \mathbb{R}^d \) is a unit vector fixed, and \( \varsigma^{e,*}_t \equiv \int_0^t \dot{\varsigma}^{e,*}_s ds, \) with \( t \in [0, \tau^*_1] \) and

\[
\dot{\varsigma}^{e,*}_t = 2\psi'_e(|D^1 u^{e}_{\ell^*_0}(X^{e,*}_t, I^*_t)|^2 - g_0(X^{e,*}_t, I^*_t)^2) D^1 u^{e}_{\ell^*_0}(X^{e,*}_t, I^*_t). 
\]

(4.18)

(ii) Recursively, letting \( i \geq 1 \) and defining

\[
\ell^*_i \in \arg \min \{ u^e_{\ell^*_i}(X^{e,*}_t, I^*_t) + \vartheta_{\ell^*_i, \ell^*_i-1}, \}, \\
\tilde{\tau}^{*}_{i+1} = \tau^*_i \wedge \tau^*, \quad \tau^*_i+1 = \inf \{ t > \tau^*_i : (X^{e,*}_t, I^*_t) \in S^e_{\ell^*_i} \},
\]

(4.19)
if $\tau_i^* < \tau^*$, the process $X_{t\wedge \tau_i^*}$ evolves as
\[ X_{t\wedge \tau_i^*} = X_{\tau_i^*}^\varepsilon = \frac{X_{t\wedge \tau_i^*} - \int_{\tau_i^*}^{t\wedge \tau_i^*} \left[ b(X_{s\wedge \tau_i^*}^\varepsilon, I_s^*) + \psi_s(X_{s\wedge \tau_i^*}^\varepsilon, I_s^*) \right] ds + \int_{\tau_i^*}^{t\wedge \tau_i^*} \sigma(X_{s\wedge \tau_i^*}^\varepsilon, I_s^*) dW_s, }{ t \geq \tau_i^*, } \]
\[ I_{t\wedge \tau_i^*} = \int_{t\wedge \tau_i^*}^{\tau_i^*} \] where
\[ \psi_{t\wedge \tau_i^*} = \begin{cases} \frac{D^1 u_{t\wedge \tau_i^*}^\varepsilon(X_{t\wedge \tau_i^*}, I_{t\wedge \tau_i^*})}{|D^1 u_{t\wedge \tau_i^*}^\varepsilon(X_{t\wedge \tau_i^*}, I_{t\wedge \tau_i^*})|^3}, & \text{if } |D^1 u_{t\wedge \tau_i^*}^\varepsilon(X_{t\wedge \tau_i^*}, I_{t\wedge \tau_i^*})| \neq 0 \text{ and } t \in [\tau_i^*, \tau_i^*], \\ \gamma_0, & \text{if } |D^1 u_{t\wedge \tau_i^*}^\varepsilon(X_{t\wedge \tau_i^*}, I_{t\wedge \tau_i^*})| = 0 \text{ and } t \in [\tau_i^*, \tau_i^*], \end{cases} \] (4.21)
with $\gamma_0 \in \mathbb{R}^d$ is a unit vector fixed, and $\zeta_{t\wedge \tau_i^*} = \int_{t\wedge \tau_i^*}^{\tau_i^*} \dot{\zeta}_{s\wedge \tau_i^*} ds$, with $t \in [\tau_i^*, \tau_i^*]$ and
\[ \dot{\zeta}_{t\wedge \tau_i^*} = 2\psi_t^\varepsilon (|D^1 u_{t\wedge \tau_i^*}^\varepsilon(X_{t\wedge \tau_i^*}, I_{t\wedge \tau_i^*})|^3 - g(X_{t\wedge \tau_i^*}^\varepsilon, I_{t\wedge \tau_i^*})^2) |D^1 u_{t\wedge \tau_i^*}^\varepsilon(X_{t\wedge \tau_i^*}, I_{t\wedge \tau_i^*})| \] (4.22)

**Remark 4.4.** Suppose that $\tau_i^* < \tau^*$ for some $i > 0$. We notice that for $t \in [\tau_i^*, \tau_i^*]$, $\psi_{t\wedge \tau_i^*} \zeta_{t\wedge \tau_i^*} = 2\psi_t^\varepsilon (|D^1 u_{t\wedge \tau_i^*}^\varepsilon(X_{t\wedge \tau_i^*}, I_{t\wedge \tau_i^*})|^3 - g(X_{t\wedge \tau_i^*}^\varepsilon, I_{t\wedge \tau_i^*})^2) |D^1 u_{t\wedge \tau_i^*}^\varepsilon(X_{t\wedge \tau_i^*}, I_{t\wedge \tau_i^*})| \Delta \zeta_{t\wedge \tau_i^*} = 0$, and by (1.16) and (3.8), it yields that $\zeta_{t\wedge \tau_i^*} \leq \frac{2e^4}{\varepsilon^4}$. Also we see that $(X_{t\wedge \tau_i^*}, I_t^*) \in C_{t\wedge \tau_i^*}^\varepsilon$ if $t \in [\tau_i^*, \tau_i^*]$ due to Lemma 4.3.

**Remark 4.5.** On the event $\{\tau^* = \infty\}$, $\tau_i^* = \tau_{i^*}$ for $i \geq 0$. From here and by (4.17)–(4.18) and (4.21)–(4.22), it yields that the control process $(\xi_{t\wedge \tau_i^*}, \zeta_{t\wedge \tau_i^*})$ belongs to $\mathcal{U}^\varepsilon \times \mathcal{S}$. On the event $\{\tau^* < \infty\}$, let $i^*$ be defined as $i^* = \max\{i \in \mathbb{N} : t \leq \tau_i^*\}$. Then, taking $\tau_{i^*}^* = \tau^* + i^*$ and $\ell_{i^*} = \ell$, where $\ell \in \mathbb{S}$ is fixed, it follows that $\zeta_{t\wedge \tau_i^*} = (\tau_{i^*}^*, \ell_{i^*})_{i^*} \in \mathcal{S}$. We take $\tau_{i^*}^* = 0$ and $\psi_{t\wedge \tau_i^*} = \gamma_0$, for $t > \tau^*$. In this way, we have that $(\xi_{t\wedge \tau_i^*}, \zeta_{t\wedge \tau_i^*}) \in \mathcal{U}^\varepsilon$.

**Remark 4.6.** Taking $J_{t\wedge \tau_i^*} = \ell_{0} \mathbf{1}_{[0, \tau_i^*]}(t) + \ell_{i^*} \mathbf{1}_{[\tau_{i^*}^*, \tau_{i^*}^*]}(t) + \sum_{i^* + 1}^{\infty} \ell_{i^*} \mathbf{1}_{[\tau_{i^*}^*, \tau_{i^*}^*]}(t)$, we see that it is a càdlàg process.

**Lemma 4.7** (Verification Lemma for $\varepsilon$-PACS control problem. Second part). Let $\varepsilon \in (0, 1)$ be fixed and let $(X_{t\wedge \tau_i^*}, I^*)$ be the process that is governed by (4.15)–(4.22). Then, $u_{t_0}^\varepsilon(x_0, t_0, \omega) = V_{t_0}^\varepsilon(x_0, t_0, \omega) = V_{t_0}^\varepsilon(x_0, t_0, \omega) = V_{t_0}^\varepsilon(x_0, t_0, \omega)$ for each $(x_0, t_0, \omega) \in \Omega \times \mathcal{M} \times \mathbb{S}$.

**Proof.** Taking $\tilde{\tau}_i^*$ in Lemma 4.1, with $\tilde{\tau}_i = \tau_{i^*}$, and considering $u_{t\wedge \tau_i^*}$ which is the unique solution of (1.15) when $\delta = \delta_{\tilde{\tau}_i}$, by Lemma 4.1, we get that
\[ \mathbb{E}_{x_0, t_0, \omega} \left[ e^{-r(\tilde{\tau}_i^*, \omega)} u_{\tilde{\tau}_i^*} (X_{\tilde{\tau}_i^*, \omega}^\varepsilon, I_{\tilde{\tau}_i^*}^\varepsilon) \mathbf{1}_{\{\tilde{\tau}_i^* < \tau^*\}} \right] \]
\[ = \mathbb{E}_{x_0, t_0, \omega} \left[ \left\{ e^{-r(\tilde{\tau}_i^*, \omega)} u_{\tilde{\tau}_i^*} (X_{\tilde{\tau}_i^*, \omega}^\varepsilon, I_{\tilde{\tau}_i^*}^\varepsilon) \right. \right. \]
\[ + \int_{\tilde{\tau}_i^*}^{\tilde{\tau}_i^*} e^{-r(\ell)} \left[ h(X_{s\wedge \tau_i^*}^\varepsilon, I_{s\wedge \tau_i^*}^\varepsilon) - \sum_{t \in \mathbb{N}_{\{\tilde{\tau}_i^*\}}} \psi_{t\wedge \tau_i^*} (u_{t\wedge \tau_i^*}^\varepsilon (X_{t\wedge \tau_i^*}^\varepsilon, I_{t\wedge \tau_i^*}^\varepsilon) - u_{t\wedge \tau_i^*}^\varepsilon (X_{t\wedge \tau_i^*}^\varepsilon, I_{t\wedge \tau_i^*}^\varepsilon) - \psi_{t\wedge \tau_i^*} (|D^1 u_{t\wedge \tau_i^*}^\varepsilon (X_{t\wedge \tau_i^*}^\varepsilon, I_{t\wedge \tau_i^*}^\varepsilon)|^3 - g(X_{t\wedge \tau_i^*}^\varepsilon, I_{t\wedge \tau_i^*}^\varepsilon)^2) |D^1 u_{t\wedge \tau_i^*}^\varepsilon (X_{t\wedge \tau_i^*}^\varepsilon, I_{t\wedge \tau_i^*}^\varepsilon)| \]
Let \( \delta_n \to \infty \), by dominated convergence theorem, we get

\[
\mathbb{E}_{\mathbf{x}_0, \ell_0, t_0}[e^{-r(\tau_+^* - \tau^-)} u_{\ell_0}^\varepsilon(X_{\tau_+^*, \tau^-}, I_{\tau_+^*}^*) \mathbb{1}_{\{\tau_+^* < \tau^*\}}] \\
= \mathbb{E}_{\mathbf{x}_0, \ell_0, t_0} \left[ e^{-r(\tau_+^* + \tau^-)} u_{\ell_0}^\varepsilon(X_{\tau_+^*, \tau^-}, I_{\tau_+^*}^*) \\
+ \int_{\tau_+^*}^{\tau_+^* + \tau^-} e^{-r(s)} \left[ h(X_{s^*, I_s^*}) - \sum_{\ell' \in \mathbb{M} \setminus \{\ell_0\}} \psi_{\ell'} u_{\ell_0}^\varepsilon(X_{s^*, I_s^*}) - u_{\ell'}^\varepsilon(X_{s^*, I_s^*}) - \vartheta_{\ell', \ell_0} \right] \\
- \psi_{\ell_0}^\varepsilon \left( |D^1 u_{\ell_0}^\varepsilon(X_{s^*, I_s^*})|^2 - g(X_{s^*, I_s^*})^2 \right) + \langle D^1 u_{\ell_0}^\varepsilon(X_{s^*, I_s^*}), \nabla_{s^*, \theta} \rangle \right] ds \mathbb{1}_{\{\tau_+^* < \tau^*\}},
\]

because of max_{(x, t) \in \mathbb{C}_x^\varepsilon} \{ |u_{\ell}^\varepsilon - u_{\ell_0}^\varepsilon|(x, t), |D^1(u_{\ell}^\varepsilon - u_{\ell_0}^\varepsilon)|(x, t) \} \to 0 \text{ for } \ell \in \mathbb{M}, \text{ and continuity of } \psi_. \text{ Then, considering that }

\[
\psi_{\ell}^\varepsilon (D^1 u_{\ell_0}^\varepsilon(X_{s^*, I_s^*}), \nabla_{s^*, \theta}) = (\psi_{\ell}^\varepsilon/|\gamma|^2 - g(x, t)^2) |\gamma|^2 - \psi_{\ell_0}^\varepsilon (|\gamma|^2 - g(x, t)^2),
\]

and letting \( q \to 0 \) in (4.23), it can be checked

\[
\mathbb{E}_{\mathbf{x}_0, \ell_0, t_0}[e^{-r(\tau_+^* - \tau^-)} u_{\ell_0}^\varepsilon(X_{\tau_+^*, \tau^-}, I_{\tau_+^*}) \mathbb{1}_{\{\tau_+^* < \tau^*\}}] \\
= \mathbb{E}_{\mathbf{x}_0, \ell_0, t_0} \left[ e^{-r(\tau_+^*)} u_{\ell_0}^\varepsilon(X_{\tau_+^*, \tau^-}, I_{\tau_+^*}) \\
+ \int_{\tau_+^*}^{\tau_+^* + \tau^-} e^{-r(s)} \left[ h(X_{s^*, I_s^*}) + \ell^\varepsilon \left( \nabla_{s^*, \theta} \right) \right] ds \mathbb{1}_{\{\tau_+^* < \tau^*\}} \right].
\]

By (4.24) and noticing that

\[
u_{\ell_0}^\varepsilon(x_0, t_0) = \mathbb{V}_{\varepsilon^*, \varepsilon^*}(x_0, \ell_0, t_0) \geq V_{\ell_0}^\varepsilon(x_0, t_0).
\]

From here and Lemma 4.2, we conclude \( u_{t_0}^\varepsilon(x_0, t_0) = \mathbb{V}_{\varepsilon^*, \varepsilon^*}(x_0, \ell_0, t_0) = V_{\ell_0}^\varepsilon(x_0, t_0) \) for each \((x_0, \ell_0, t_0) \in \mathbb{S} \times \mathbb{M} \times \mathbb{I}\).

\subsection{Proof of Theorem 1.2}

\textbf{Proof.} Let \( \{u_{t_0}^\varepsilon\}_{t_0 \geq 1} \), with \( u_{t_0}^\varepsilon = (u_{t_0}^\varepsilon, \ldots, u_{t_0}^\varepsilon) \), be the sequence of unique strong solutions to the HJB equation (1.17), when \( \varepsilon = \varepsilon_{t_0} \), which satisfy (3.7). From Lemma 4.7, we know that

\[
u_{t_0}^\varepsilon(x_0, t_0) = \mathbb{V}_{\varepsilon^*, \varepsilon^*^*}(x_0, \ell_0, t_0) = V_{t_0}^\varepsilon(x_0, t_0) \quad \text{for } (x_0, \ell_0, t_0) \in \mathbb{S} \times \mathbb{M} \times \mathbb{I},
\]

with \((\xi^\varepsilon_{t_0}, \varepsilon^\varepsilon_{t_0})\) as in (4.16)–(4.19) and (4.21)–(4.22), when \( \ell = \ell_{t_0} \). Notice that \( l^\varepsilon(\beta_\gamma, x, t) \geq \langle g(x, t) \gamma, \beta_\gamma \rangle - \psi_{\varepsilon_{t_0}}(|g(x, t)|^2 - g(x, t)^2) = \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R}^d \) a unit vector.
Then, from here and considering \((X_{\xi_0,\ast}, J^\ast, I^\ast)\) governed by (4.15)-(4.22), it follows that
\[
V_{t_0}(x_0, t_0) \le V_{\xi_0,\ast, \kappa_{t_0}}(x_0, t_0, t_0)
\]
\[
= \mathbb{E}_{x_0, t_0, 0} \left[ \int_0^{\tau^*} e^{-r(s)} \{ h(X_s, I_s^\ast) ds + \zeta_{\xi_0,\ast} g(X_s, I_s^\ast) \} ds 
\right.
\]
\[
+ e^{r(\tau^* - \tau_0)} f(X_{\tau^*}, I_{\tau^*}) \mathbb{1}_{\{\tau^* < \infty\}} + \sum_{i \ge 0} e^{-r(\tau_{i+1}^*)} \partial_{t_i^*}^\ast \mathbb{1}_{\{\tau_{i+1}^* < \tau^*\}} \right]
\]
\[
\le \mathbb{E}_{x_0, t_0, 0} \left[ \int_0^{\tau^*} e^{-r(s)} \{ h(X_s, I_s^\ast) + I_{\xi_0,\ast} \} ds 
\right.
\]
\[
+ e^{r(\tau^* - \tau_0)} f(X_{\tau^*}, I_{\tau^*}) \mathbb{1}_{\{\tau^* < \infty\}} + \sum_{i \ge 0} e^{-r(\tau_{i+1}^*)} \partial_{t_i^*}^\ast \mathbb{1}_{\{\tau_{i+1}^* < \tau^*\}} \right] = u_{t_0}(x_0, t_0).
\]

(4.25)

Letting \(\varepsilon_\ast \to 0\) in (4.25), it yields \(u_{t_0}(x_0, t_0) \ge V_{t_0}(x_0, t_0)\) for each \((x_0, \ell_0, t_0) \in \Omega \times \mathbb{M} \times \mathbb{L}\).

Let us consider \((X, J, I)\) evolving as in (1.3) with initial state \((x_0, \ell_0, t_0) \in \Omega \times \mathbb{M} \times \mathbb{L}\), and the control process \((\xi, \zeta)\) belongs to \(\mathcal{U} \times \mathcal{S}\). Taking \(\hat{f} = u_{\varepsilon_\ast,\delta_\ast}^\ast\), by Lemma 4.1, we get that (4.6) holds. Since \(u_{\varepsilon_\ast,\delta_\ast}^\ast\) is the unique solution to (1.15) when \(\varepsilon = \varepsilon_\ast\) and \(\delta = \delta_\ast\), and \(\psi\) is a positive function, the reader can verify that
\[
\mathbb{E}_{x_0, t_0, 0} \left[ e^{-r(\tilde{q}^\ast)} u_{\tilde{q}^\ast,\delta_\ast}^\ast(X_{\tilde{q}^\ast}, I_{\tilde{q}^\ast}) \mathbb{1}_{\{\tilde{q}^\ast < \tau\}} \right]
\]
\[
\le \mathbb{E}_{x_0, t_0, 0} \left[ \left\{ e^{-r(\tilde{q}^\ast - \tilde{q}^\ast)} u_{\tilde{q}^\ast,\delta_\ast}^\ast(X_{\tilde{q}^\ast}, I_{\tilde{q}^\ast}) - \sum_{\tilde{q}^\ast < s \le \tilde{q}^\ast} e^{-r(s)} \mathcal{J}[X_s^\ast, I_s, u_{\tilde{q}^\ast,\delta_\ast}] \right. \right.
\]
\[
\left. + \int_{\tilde{q}^\ast +}^{\tilde{q}^\ast} e^{-r(s)} \{ h(X_s, I_s) + \langle D_1 u_{\tilde{q}^\ast,\delta_\ast}^\ast(X_s, I_s), \n_\ast \zeta_s^\ast \rangle \} ds \mathbb{1}_{\{\tilde{q}^\ast < \tau\}} \right]. \quad (4.26)
\]

Additionally, considering \(\Delta \zeta_s \neq 0\) and \(X_{\tau_\ast} - \n_\ast \Delta \zeta_s \in \Omega\) for \(s \in (\tilde{q}^\ast, \tilde{q}^\ast +]\), and using mean value theorem, we have that
\[
-\mathcal{J}[X_s, I_s, u_{\tilde{q}^\ast,\delta_\ast}] \le |u_{\tilde{q}^\ast,\delta_\ast}(X_{\tau_\ast} - \n_\ast \Delta \zeta_s, I_s) - u_{\tilde{q}^\ast,\delta_\ast}(X_{\tau_\ast}, I_s)|
\]
\[
\le \Delta \zeta_s \int_0^1 |D_1 u_{\tilde{q}^\ast,\delta_\ast}(X_{\tau_\ast} - \lambda \n_\ast \Delta \zeta_s, I_s)| d\lambda. \quad (4.27)
\]

Recall that \(\max_{(x, \ell) \in \mathcal{O} \times \mathbb{M}} \{|u_{\ell}(x, \ell) - u_{\ell}(x, \ell)|, |D_1 u_{\ell}(x, \ell) - u_{\ell}(x, \ell)|\} \to 0\) for \(\ell \in \mathbb{M}\), due to (3.5) and (3.7). Then, applying (4.27) in (4.26) and letting \(\varepsilon_\ast,\delta_\ast \to 0\), by the dominated convergence theorem, it follows that
\[
\mathbb{E}_{x_0, t_0, 0} \left[ e^{-r(\tilde{q}^\ast)} u_{\tilde{q}^\ast,\delta_\ast}^\ast(X_{\tilde{q}^\ast}, I_{\tilde{q}^\ast}) \mathbb{1}_{\{\tilde{q}^\ast < \tau\}} \right] \le \mathbb{E}_{x_0, t_0, 0} \left[ \left\{ e^{-r(\tilde{q}^\ast + 1)} u_{\tilde{q}^\ast + 1,\delta_\ast}^\ast(X_{\tilde{q}^\ast + 1}, I_{\tilde{q}^\ast + 1}) \right. \right.
\]
\[
\left. + \int_{\tilde{q}^\ast + 1}^{\tilde{q}^\ast + 1} e^{-r(s)} \{ h(X_s, I_s) ds + g(X_{\tau_\ast} - I_s) \mathbb{d} \zeta_s \} \mathbb{1}_{\{\tilde{q}^\ast < \tau\}} \right]. \quad (4.28)
\]
due to \(|D^1 u(\cdot, t)| - g(\cdot, t) \leq 0\) on \(\mathcal{O}\). Letting \(q \to \infty\) in (4.28) and taking into account that (4.12) holds for this case, by the dominated convergence theorem, it implies that

\[
\mathbb{E}_{x_0, \ell_0, t_0}[e^{-r(\tau)} u_{\ell_1}(X_{\tau_1}, I_{\tau_1}) 1_{\{\tau < \tau_1\}}] \leq \mathbb{E}_{x_0, \ell_0, t_0}\left[ e^{-r(\tau)} f(X_\tau, I_\tau) 1_{\{\tau < \tau_1 < \tau_{i+1}\}} \right.
\]
\[
+ e^{-r(\tau_{i+1})} [u_{\ell_{i+1}}(X_{\tau_{i+1}}, I_{\tau_{i+1}}) + \vartheta_{\ell_i, \ell_{i+1}}] 1_{\{\tau > \tau_{i+1}\}}
\]
\[
+ 1_{\{\tau < \tau_1\}} \int_{\tau_1}^{\tau} e^{-r(\tau)} [h(X_s, I_s) + g(X_{s-}, I_s) \circ d\zeta_s] ds \right].
\]

(4.29)

On the other hand, since the control \(\xi\) can act on \(X\) by a jump of \(\zeta\) at time zero, we have that \(X_0 = x_0 - r_0 \Delta \zeta_0\). From here and considering recurrently (4.29), we conclude that

\[
u_{\ell_0}(x_0, t_0) \leq \mathbb{E}_{x_0, \ell_0, t_0}\left[ f(x_0, t_0) 1_{\{\tau_0 = \tau\}} + \Delta \zeta_0 1_{\{\tau_0 < \tau\}} \int_0^1 g(x_0 - \lambda r_0 \Delta \zeta_0, I_0) d\lambda
\]
\[
+ \vartheta_{\ell_0, \ell_1} 1_{\{\tau_0 = \tau_1 < \tau_1\}} + u_{\ell_1}(X_0, I_0) 1_{\{\tau_0 = \tau_1 < \tau\}} + u_{\ell_0}(X_0, I_0) 1_{\{\tau_0 < \tau_1\}} \right]
\]
\[
\leq \mathbb{E}_{x_0, \ell_0, t_0}\left[ f(x_0, t_0) 1_{\{\tau_0 = \tau\}} + e^{-r(\tau)} f(X_\tau, I_\tau) 1_{\{\tau_0 < \tau \leq \tau_1\}} + e^{-r(\tau_1)} \vartheta_{\ell_0, \ell_1} 1_{\{\tau > \tau_1 \geq \tau_0\}}
\]
\[
+ 1_{\{\tau_0 < \tau_1\}} \int_{\tau_1}^{\tau} e^{-r(\tau)} [h(X_s, I_s) + g(X_{s-}, I_s) \circ d\zeta_s] ds + 1_{\{\tau_0 < \tau_1\}} \int_{\tau_0}^{\tau_1} e^{-r(\tau)} [h(X_s, I_s) + g(X_{s-}, I_s) \circ d\zeta_s] ds
\]
\[
+ \mathbb{E}_{x_0, \ell_0, t_0}[e^{-r(\tau_1)} u_{\ell_1}(X_{\tau_1}, I_{\tau_1}) 1_{\{\tau > \tau_1 \geq \tau_0\}}]
\]
\[
\vdots
\]
\[
\leq \mathbb{E}_{x_0, \ell_0, t_0}\left[ e^{-r(\tau)} f(X_\tau, I_\tau) 1_{\{\tau < \infty\}} + \sum_{i \geq 0} e^{-r(\tau_{i+1})} \vartheta_{\ell_i, \ell_{i+1}} 1_{\{\tau_{i+1} < \tau\}}
\]
\[
+ \int_0^\tau e^{-r(\tau)} [h(X_s, I_s) + g(X_{s-}, I_s) \circ d\zeta_s] ds + \mathbb{E}_{x_0, \ell_0, t_0}[\int_{\tau_0}^{\tau_1} e^{-r(\tau)} [h(X_s, I_s) + g(X_{s-}, I_s) \circ d\zeta_s] ds]ight].
\]

(4.29)

Therefore, by the seen before, it is easily to check that \(u_{\ell_0}(x_0, t_0) \leq V_{\ell_0}(x_0, t_0) \leq u_{\ell_0}(x_0, t_0)\) for \((x_0, \ell_0, t_0) \in \mathcal{O} \times \mathcal{M} \times \mathcal{I}\).}

\section{Proofs of some results seen in the article}

\subsection{Proof of Lemma 3.2}

The existence of the solution \(v\) to (3.2) will be argued using Schaefer’s fixed point theorem (see, i.e., [5, Thm. 4 p. 539]). First, by Theorem 6.14 of [9], notice that for each \(w \in C^{0,\alpha}_{m,n}(\mathcal{O})\), there exists a unique \(v_{\ell,i} \in C^{2,\alpha}(\mathcal{O})\), with \((\ell, i) \in \mathcal{M} \times \mathcal{I}\), such that

\[
[c_i - \tilde{c}_i] v_{\ell,i} = \Xi_{\ell,i} w, \quad \text{on } \mathcal{O}, \quad \text{s.t. } v_{\ell,i} = f_i, \quad \text{in } \partial \mathcal{O}, \quad \text{(A.1)}
\]

where

\[
\Xi_{\ell,i} w = h_i - \sum_{\kappa \in \mathcal{I}} q_{\ell,i} [w_{\ell,i} - w_{\ell,\kappa}].
\]
Additionally, using [1, Thm. 4.12, p. 85] and [8, Thm. 1.2.19], we get that
\[ \|v_{t,t}\|_{C^{1,\alpha}(\overline{\mathcal{O}})} \leq C_1 \left[ 1 + \|w\|_{C^{0,\alpha}_{m,n}(\overline{\mathcal{O}})} \right], \quad \text{for } (t,t) \in \mathbb{M} \times \mathbb{I}, \]  
(A.2)
for some \( C_1 = C_1(d, \Lambda, \alpha, \theta) \). Let us define the mapping
\[ T : (C^{0,\alpha}_{m,n}, \| \cdot \|_{C^{0,\alpha}_{m,n}}) \to (C^{0,\alpha}_{m,n}, \| \cdot \|_{C^{0,\alpha}_{m,n}}) \]
as \( T[w] = v \) for each \( w \in C^{0,\alpha}_{m,n} \), where \( v \in C^{2,\alpha}_{m,n} \subset C^{0,\alpha}_{m,n} \) is the unique solution to the Dirichlet problem (A.1). Observe \( T \) maps bounded sets in \( C^{0,\alpha}_{m,n} \) into bounded sets in itself that are precompact in \( C^{0,\alpha}_{m,n} \), since (H1)-(H4) and (A.2) hold. Then, by the uniqueness of the solution to (A.1), it can be checked that \( T \) is a continuous and compact mapping from \( C^{0,\alpha}_{m,n} \) to \( C^{0,\alpha}_{m,n} \).

Therefore, from here, using (A.1), and arguing in a similar way as in the proof of Lemma (2.2), we see that (2.9) is also true for this case.

**Proof of Proposition 1.3. Existence.** By the seen before and arguing in a similar way than in the proof of Proposition 1.3 (existence), it follows immediately that (3.2) has a solution \( v \in C^{4,\alpha}_{m,n} \).

**Proof of Proposition 3.2. Uniqueness.** The proof of uniqueness of the solution \( v \) to (3.2) shall be given by contradiction. Assume that there are two solutions \( \tilde{v}, v \in C^{4,\alpha}_{m,n} \) to (3.2). Let \( \bar{v} = (\bar{v}_{t,t})(t,t) \in \mathbb{M} \times \mathbb{I} \in C^{4,\alpha}_{m,n} \) such that \( \bar{v}_{t,t} := \tilde{v}_{t,t} - v_{t,t} \) for \( (t,t) \in \mathbb{M} \times \mathbb{I} \). Then,
\[ [c_t - L_{t,t}] \bar{v}_{t,t} = 0, \quad \text{on } \mathcal{O}, \quad \text{s.t. } \bar{v}_{t,t} = 0, \quad \text{in } \partial \mathcal{O}. \]  
(A.3)
Let \( (x^0, \ell^0, \iota^0) \) be in \( \mathcal{O} \times \mathbb{M} \times \mathbb{I} \) such that \( \bar{v}_{x,\ell,\iota}(x^0) = \max_{(x,\ell,\iota) \in \mathcal{O} \times \mathbb{M} \times \mathbb{I}} \bar{v}_{t,t}(x) \). If \( x_0 \in \partial \mathcal{O}, \)
\[ \bar{v}_{t,t} - v_{t,t} \leq 0 \text{ in } \mathcal{O} \text{ for } (t,t) \in \mathbb{M} \times \mathbb{I}. \]
If \( x^0 \in \mathcal{O}, \)
\[ D^1 \bar{v}_{x,\ell,\iota}(x^0) = 0, \quad \text{tr}[a_{x,\iota}(x^0)D^2 \bar{v}_{x,\ell,\iota}(x^0)] \leq 0, \]
\[ \bar{v}_{x,\ell,\iota}(x^0) - \bar{v}_{x,\ell,\iota}(x^0) \geq 0 \quad \text{for } \iota \in \mathbb{I} \setminus \{\iota^0\}. \]  
(A.4)
Then, from (A.3) and (A.4),

$$0 \geq \text{tr}[a^o D^2 \bar{v}_{\ell,\iota}^o]$$

$$= c_{\ell,\iota} \bar{v}_{\ell,\iota}^o + \sum_{\kappa \in I \setminus \{\ell\}} q_{\ell,\kappa}[\bar{v}_{\ell,\kappa} - \bar{v}_{\kappa,0}] \geq c_{\ell,\iota} \bar{v}_{\ell,\iota}^o \quad \text{at } x^o. \quad (A.5)$$

From (A.5) and since $c_{\ell,\iota} > 0$, we have that $\bar{v}_{\ell,\iota}(x) - v_{\ell,\iota}(x) \leq \bar{v}_{\ell,\iota}^o(x^o) - v_{\ell,\iota}^o(x^o) \leq 0$ for $x \in \overline{\mathcal{O}}$ and $(\ell, \iota) \in \mathbb{M} \times \mathbb{I}$. Taking now $\bar{v} := v - \hat{v}$ and proceeding the same way than before, it follows immediately that $v_{\ell,\iota} - \hat{v}_{\ell,\iota} \leq 0$ on $\overline{\mathcal{O}}$ for $(\ell, \iota) \in \mathbb{M} \times \mathbb{I}$. Therefore $\hat{v} = v$ and from here we conclude that the system of equation (3.2) has a unique solution $v$, whose components belong to $C^{4,\alpha}(\mathcal{O})$. \hfill \square

### A.2 Proof of Lemma 3.4. Eq. (3.3)

For each $(\ell, \iota) \in \mathbb{M} \times \mathbb{I}$, let us consider the auxiliary function

$$w_{\ell,\iota} := \varpi^2 |D^1 u_{\ell,\iota}^{\varepsilon,\delta}|^2 - \lambda A_{\varepsilon,\delta} u_{\ell,\iota}^{\varepsilon,\delta}, \quad \text{on } \overline{\mathcal{O}}, \quad (A.6)$$

where $\lambda \geq 1$ is a constant that shall be selected later on and

$$A_{\varepsilon,\delta} := \max_{(x,\ell,\iota) \in \overline{\mathcal{O}} \times \mathbb{M} \times \mathbb{I}} \varpi(x) |D^1 u_{\varepsilon,\delta}^{\varepsilon,\delta}(x)|. \quad (A.7)$$

We shall show that $w_{\ell,\iota}$ satisfies (A.8). In particular (A.8) holds when $w_{\ell,\iota}$ is evaluated at its maximum $x_{\lambda} \in \mathcal{O}$, which helps to see that (3.3) is true.

**Lemma A.2.** Let $w_{\ell,\iota}$ be the auxiliary function given by (A.22). Then, there exists a positive constant $C_7 = C_7(\varpi, \Lambda, 1/\theta, K, C_4)$ such that for $(\ell, \iota) \in \mathbb{M} \times \mathbb{I},$

$$-\text{tr}[a_{\varepsilon,\delta} D^2 w_{\ell,\iota}] \leq C_7 |D^1 u_{\ell,\iota}^{\varepsilon,\delta}|^2 + C_7 [1 + \lambda A_{\varepsilon,\delta}] |D^1 u_{\ell,\iota}^{\varepsilon,\delta}| + \lambda A_{\varepsilon,\delta} C_7$$

$$- q_{\ell,\iota}(\cdot)[2 |D^1 u_{\ell,\iota}^{\varepsilon,\delta}| D^1 w_{\ell,\iota}^\iota + \lambda A_{\varepsilon,\delta}] |D^1 u_{\ell,\iota}^{\varepsilon,\delta}|^2 - C_7 |D^1 u_{\ell,\iota}^{\varepsilon,\delta}|^2 A_{\varepsilon,\delta} - C_7 A_{\varepsilon,\delta}]$$

$$- \sum_{\kappa \in \iota \setminus \{\ell\}} q_{\ell,\kappa}[w_{\ell,\kappa} - w_{\kappa,0}] - \sum_{\ell' \in \mathbb{M} \setminus \{\ell\}} \psi_{\delta,\ell',\iota}(\cdot)[w_{\ell',\iota} - w_{\ell',\iota}], \quad (A.8)$$

where $\psi_{\varepsilon,\delta,\ell',\iota}(\cdot)$, $\psi_{\delta,\ell',\iota}(\cdot)$ denote $\psi_{\varepsilon}(|D^1 u_{\ell,\iota}^{\varepsilon,\delta}|^2 - g_{\ell,\iota}^2)$, $\psi_{\delta}(u_{\ell,\iota}^{\varepsilon,\delta} - u_{\ell',\iota}^{\varepsilon,\delta} - \vartheta_{\ell,\iota',\iota})$, respectively.

**Proof of Lemma 3.4. Eq. (3.3).** Without loss of generality, let us assume that $A_{\varepsilon,\delta} > 1$ since if $A_{\varepsilon,\delta} \leq 1$, we obtain a bound for $A_{\varepsilon,\delta}$ that is independent of $\varepsilon, \delta$ and hence, we obtain (3.3). Let $x_{\lambda} \in \overline{\mathcal{O}}$ and $(\ell_{\lambda}, \iota_{\lambda}) \in \mathbb{M} \times \mathbb{I}$ (depending on $\lambda$) be such that

$$w_{\ell_{\lambda},\iota_{\lambda}}(x_{\lambda}) = \max_{(x,\ell,\iota) \in \overline{\mathcal{O}} \times \mathbb{M} \times \mathbb{I}} w_{\ell,\iota}(x).$$

From here, by (3.1) and definition of $w_{\ell,\iota}$; see (A.6), it gives

$$\varpi^2(x)|D^1 u_{\ell,\iota}(x)|^2 \leq \varpi^2(x_{\lambda})|D^1 u_{\ell_{\lambda},\iota_{\lambda}}(x_{\lambda})|^2 + \lambda A_{\varepsilon,\delta} C_1,$$
for $x \in \mathcal{O}$ and $(\ell, \iota) \in \mathbb{M} \times \mathbb{l}$. Then, from here and using (A.7), it yields for each $\varrho > 0$ small enough, there is $x_\varrho \in \mathcal{O}$ such that $[A_{\varepsilon,\delta}-\varrho]^2 \leq \varpi^2(x_\varrho)|D^1 u_{\ell,\iota}(x_\varrho)|^2 \leq \varpi^2(x_\lambda)|D^1 u_{\ell,\iota \lambda}(x_\lambda)|^2 + \lambda A_{\varepsilon,\delta} C_1$. Thus, letting $\varrho \to 0$ and since $A_{\varepsilon,\delta} > 1$,

$$\varpi |D^1 u_{\ell,\iota}| \leq A_{\varepsilon,\delta} \leq \varpi^2(x_\lambda)|D^1 u_{\ell,\iota \lambda}(x_\lambda)|^2 + \lambda C_1 \quad \text{on } \mathcal{O}.$$  

(A.9)

From here, we see that to verify (3.3), it is enough to check that $\varpi(x_\lambda)|D^1 u_{\ell,\iota \lambda}(x_\lambda)|$ is bounded by a non-negative constant which is independent of $\varepsilon$ and $\delta$. If $x_\lambda \in \mathcal{O} \setminus B_{\varpi r}$, $\varpi(x_\lambda)|D^1 u_{\ell,\iota \lambda}(x_\lambda)| = 0$, and therefore $C_5' = \lambda C_1$. Let $x_\lambda$ be in $B_{\varpi r}$. It is well known that at $x_\lambda$,

$$D^1 w_{\ell,\iota \lambda} = 0, \quad \text{tr}[a_{i\ell} D^2 w_{\ell,\iota \lambda}] \leq 0$$

$$[w_{\ell,\iota \lambda} - w_{\ell,\kappa}] \geq 0, \quad \text{for } \kappa \in \mathbb{l} \setminus \{\iota\}$$

$$[u_{\ell,\iota \lambda} - u_{\ell',\iota \lambda}] \geq 0, \quad \text{for } \ell' \in \mathbb{M} \setminus \{\ell\}.$$  

Then, from here and (A.8),

$$0 \leq C_7 |D^1 u_{\ell,\iota \lambda}|^2 + C_7 [1 + \lambda A_{\varepsilon,\delta}] |D^1 u_{\ell,\iota \lambda}| + \lambda C_7$$

$$\quad - \psi'_{\varepsilon,\ell,\iota \lambda}(\cdot)[\lambda A_{\varepsilon,\delta}] |D^1 u_{\ell,\iota \lambda}|^2 - C_7 |D^1 u_{\ell,\iota \lambda}|^2 A_{\varepsilon,\delta} - C_7 A_{\varepsilon,\delta} \quad \text{at } x_\lambda.$$  

(A.10)

On the other hand, notice that either $\psi'_{\varepsilon,\ell,\iota \lambda}(\cdot) < \frac{1}{\varepsilon}$ or $\psi'_{\varepsilon,\ell,\iota \lambda}(\cdot) = \frac{1}{\varepsilon}$ at $x_\lambda$, by definition of $\psi_\varepsilon$, given in (1.16), it follows that $|D^1 u_{\ell,\iota \lambda}|^2 - g^2_{\lambda} \leq 2\varepsilon$ at $x_\lambda$. It implies that $\varpi^2|D^1 u_{\ell,\iota \lambda}|^2 \leq \Lambda^2 + 2$ at $x_\lambda$. Then, from (A.9) and taking $C_5' = 2 + \Lambda^2 + \lambda C_1$, it follows (3.3). Now, assume that $\psi'_{\varepsilon,\ell,\iota \lambda}(\cdot) = \frac{1}{\varepsilon}$ at $x_\lambda$. Then, taking $\lambda > \max\{1,2C_7\}$ fixed, and using (A.10), we get

$$0 \leq [2C_7 - \lambda]|D^1 u_{\ell,\iota \lambda}|^2 + C_7[1 + \lambda]|D^1 u_{\ell,\iota \lambda}| + \lambda C_7 \quad \text{at } x_\lambda.$$  

(A.11)

From here, it yields that $|D^1 u_{\ell,\iota \lambda}(x_\lambda)| < K_3$, for some $K_3 = K_3(d,\Lambda,\alpha)$. Therefore, taking $C_5' = K_3 + \lambda C_1$ and using (A.9), we get (3.3). \hfill \blacksquare

Proof of Lemma A.2. Consider $u_{\ell,\iota}$ as in (A.6) for $(\ell, \iota) \in \mathbb{M} \times \mathbb{l}$. Taking first and second derivatives in $u_{\ell,\iota}$ on $B_{\varpi r}$, it can be checked that

$$\partial_i u_{\ell,\iota} = |D^1 u_{\ell,\iota}|^2 \partial_i \varpi^2 + \varpi^2 \partial_i |D^1 u_{\ell,\iota}|^2 - \lambda A_{\varepsilon,\delta} \partial_i u_{\ell,\iota}$$

$$\quad - \text{tr}[a_{i\ell} D^2 w_{\ell,\iota}] = -|D^1 u_{\ell,\iota}|^2 \text{tr}[a_{i\ell} D^2 \varpi^2] - 2\langle a_{i\ell} D^1 \varpi^2, D^1 |D^1 u_{\ell,\iota}|^2 \rangle$$

$$\quad - \varpi^2 \text{tr}[a_{i\ell} D^2 |D^1 u_{\ell,\iota}|^2] + \lambda A_{\varepsilon,\delta} \text{tr}[a_{i\ell} D^2 u_{\ell,\iota}].$$

From here and noticing that from (1.11),

$$\text{tr}[a_{i\ell} D^2 |D^1 u_{\ell,\iota}|^2] \geq 2\theta |D^2 u_{\ell,\iota}|^2 + 2 \sum_i \partial_i u_{\ell,\iota} \text{tr}[a_{i\ell} D^2 \partial_i u_{\ell,\iota}],$$

it follows that

$$- \text{tr}[a_{i\ell} D^2 w_{\ell,\iota}] \leq -|D^1 u_{\ell,\iota}|^2 \text{tr}[a_{i\ell} D^2 \varpi^2] - 8 \varpi \sum_i \partial_i u_{\ell,\iota} \langle a_{i\ell} D^1 \varpi, D^1 \partial_i u_{\ell,\iota} \rangle$$

$$\quad - \varpi^2 \left[2\theta |D^2 u_{\ell,\iota}|^2 + 2 \sum_i \partial_i u_{\ell,\iota} \text{tr}[a_{i\ell} D^2 \partial_i u_{\ell,\iota}] \right] + \lambda A_{\varepsilon,\delta} \text{tr}[a_{i\ell} D^2 u_{\ell,\iota}].$$  

(A.13)
Meanwhile, from (1.14) and (1.15),

\[
\lambda A_{\varepsilon, \delta} \text{tr}[a_i D^2 u_{\ell,i}] = \lambda A_{\varepsilon, \delta} \left[ \widetilde{D}_1 u_{\ell,i} + \psi_{\ell,\ell,i}(\cdot) + \sum_{\ell' \in \Pi \{i\}} \psi_{\ell,\ell',i}(\cdot) + \sum_{\kappa \in \Pi \{i\}} q_{\ell}(t, \kappa)[u_{\ell,i} - u_{\kappa,i}] \right]
\]

(A.14)

where \( \widetilde{D}_1 u_{\ell,i} := (b_i, D^1 u_{\ell,i}) + c_i u_{\ell,i} - h_i \). Now, differentiating (1.15), multiplying by \( 2\partial_i u_{\ell,i} \) and taking summation over all \( i \)'s, we see that

\[
-2 \sum_i \text{tr}[a_i D^2 \partial_i u_{\ell,i}] \partial_i u_{\ell,i} = \widetilde{D}_2 u_{\ell,i} - 2\psi'_{\ell,\ell,i}(\cdot)\langle D^1 u_{\ell,i}, D^1 [D^1 u_{\ell,i}]^2 - g_i^2 \rangle
- 2 \sum_{\ell' \in \Pi \{i\}} \psi'_{\ell,\ell',i}(\cdot)\langle D^1 u_{\ell,i}, D^1 u_{\ell',i} \rangle - 2 \sum_{\kappa \in \Pi \{i\}} q_{\ell}(t, \kappa)[D^1 u_{\ell,i}]^2 - \langle D^1 u_{\ell,i}, D^1 u_{\kappa,i} \rangle,
\]

(A.15)

where

\[
\widetilde{D}_2 u_{\ell,i} := 2 \sum_i \partial_i u_{\ell,i} \text{tr}[(\partial_a a_i) D^2 u_{\ell,i}] - 2\langle D^1 u_{\ell,i}, D^1 [(b_i, D^1 u_{\ell,i}) + c_i u_{\ell,i} - h_i] \rangle.
\]

(A.16)

Then, from (A.13)–(A.15), it can be shown that

\[
- \text{tr}[a_i D^2 u_{\ell,i}] \leq -2\theta \varpi^2 |D^2 u_{\ell,i}|^2 - |D^1 u_{\ell,i}|^2 \text{tr}[a_i D^2 \varpi^2]
- 8\varpi \sum_i \partial_i u_{\ell,i} \langle a_i D^1 \varpi, D^1 \partial_i u_{\ell,i} \rangle + \varpi^2 \widetilde{D}_2 u_{\ell,i} + \lambda A_{\varepsilon, \delta} \widetilde{D}_1 u_{\ell,i}
- 2\varpi^2 \psi'_{\ell,\ell,i}(\cdot)\langle D^1 u_{\ell,i}, D^1 [D^1 u_{\ell,i}]^2 - g_i^2 \rangle + \lambda A_{\varepsilon, \delta} \psi_{\ell,\ell,i}(\cdot)
- \sum_{\ell' \in \Pi \{i\}} \{2\varpi^2 \psi'_{\ell,\ell',i}(\cdot)\langle D^1 u_{\ell,i}, D^1 u_{\ell',i} \rangle - \langle D^1 u_{\ell,i}, D^1 u_{\kappa,i} \rangle - \lambda A_{\varepsilon, \delta} \psi_{\ell,\ell',\ell,i}(\cdot)\}
- \sum_{\kappa \in \Pi \{i\}} q_{\ell}(t, \kappa)\{2\varpi^2 \langle D^1 u_{\ell,i}, D^1 u_{\kappa,i} \rangle - \langle D^1 u_{\ell,i}, D^1 u_{\kappa,i} \rangle - \lambda A_{\varepsilon, \delta} [u_{\ell,i} - u_{\kappa,i}] \}.
\]

(A.17)

By (H3), (H4) and (3.1), notice that

\[
- 2\theta \varpi^2 |D^2 u_{\ell,i}|^2 - |D^1 u_{\ell,i}|^2 \text{tr}[a_i D^2 \varpi^2]
- 8\varpi \sum_i \partial_i u_{\ell,i} \langle a_i D^1 \varpi, D^1 \partial_i u_{\ell,i} \rangle + \varpi^2 \widetilde{D}_2 u_{\ell,i} + \lambda A_{\varepsilon, \delta} \widetilde{D}_1 u_{\ell,i}
\leq 2 \left[ 2K_2 d^2 + \Lambda d + \frac{1}{4\theta} \Lambda^2 [4K_2 d^2 + 1 + d^2]^2 \right] |D^1 u_{\ell,i}|^2
+ 2\Lambda \left[ 1 + C_1 + \frac{\lambda A_{\varepsilon, \delta}}{2} \right] |D^1 u_{\ell,i}| + \lambda A_{\varepsilon, \delta} \Lambda C_4
\leq K_4 |D^1 u_{\ell,i}|^2 + K_4 \left[ 1 + \lambda A_{\varepsilon, \delta} \right] |D^1 u_{\ell,i}| + \lambda A_{\varepsilon, \delta} K_4.
\]

(A.18)
for some $K_4 = K_4(d, \Lambda, 1/\theta, K_2, C_4)$. On the other hand by (A.12), it can be checked that

\[
-\varpi^2 \langle D^1 u_{\ell, \kappa}, D^1 [D^1 u_{\ell, \kappa}]^2 - g_2^2 \rangle = -\langle D^1 u_{\ell, \kappa}, D^1 w_{\ell, \kappa} \rangle - \lambda A_{\varepsilon, \delta} D^1 u_{\ell, \kappa}^2 - \langle D^1 u_{\ell, \kappa}, D^1 w_{\ell, \kappa} \rangle - \lambda A_{\varepsilon, \delta} D^1 u_{\ell, \kappa}^2 + 2dK_2 D^1 u_{\ell, \kappa}^2 A_{\varepsilon, \delta} + 2d\Lambda A_{\varepsilon, \delta}. \tag{A.19}
\]

Using (A.6) and since $|y_1|^2 - |y_2|^2 = 2||y_1|^2 - \langle y_1, y_2 \rangle| - |y_1 - y_2|^2 \leq 2||y_1|^2 - \langle y_1, y_2 \rangle|$, for $y_1, y_2 \in \mathbb{R}^d$, it yields that

\[
-2\varpi^2 [D^1 u_{\ell, \kappa}]^2 - \langle D^1 u_{\ell, \kappa}, D^1 u_{\ell', \kappa} \rangle \leq -[w_{\ell, \kappa} - w_{\ell', \kappa}] - \lambda A_{\varepsilon, \delta}[u_{\ell, \kappa} - u_{\ell', \kappa}] \quad \text{for } (\ell', \kappa) \in M \times I. \tag{A.20}
\]

Then, from (A.19)–(A.20),

\[
-2\varpi^2 \psi'_{\varepsilon, \ell, \kappa}(\cdot)\langle D^1 u_{\ell, \kappa}, D^1 [D^1 u_{\ell, \kappa}]^2 - g_2^2 \rangle + \lambda A_{\varepsilon, \delta}(\psi'_{\varepsilon, \ell, \kappa}(\cdot)) - \sum_{\ell' \in \mathbb{N} \setminus \{\ell\}} \{2\varpi^2 \psi'_{\varepsilon, \ell', \kappa}(\cdot)[D^1 u_{\ell, \kappa}]^2 - \langle D^1 u_{\ell, \kappa}, D^1 u_{\ell', \kappa} \rangle - \lambda A_{\varepsilon, \delta}(\psi'_{\varepsilon, \ell', \kappa}(\cdot))\}
\]

\[
- \sum_{\kappa \in \mathbb{N} \setminus \{\ell\}} q_\ell(\kappa)\{[D^1 u_{\ell, \kappa}]^2 - \langle D^1 u_{\ell, \kappa}, D^1 u_{\ell', \kappa} \rangle - \lambda A_{\varepsilon, \delta}[u_{\ell, \kappa} - u_{\ell', \kappa}]\}
\]

\[
\leq -2\psi'_{\varepsilon, \ell, \kappa}(\cdot)[\langle D^1 u_{\ell, \kappa}, D^1 w_{\ell, \kappa} \rangle + \lambda A_{\varepsilon, \delta}|D^1 u_{\ell, \kappa}|^2 - 2dK_2 D^1 u_{\ell, \kappa}^2 2d\Lambda^2 A_{\varepsilon, \delta} + \lambda A_{\varepsilon, \delta}(\psi'_{\varepsilon, \ell, \kappa}(\cdot)) - \sum_{\kappa \in \mathbb{N} \setminus \{\ell\}} q_\ell(\kappa)[w_{\ell, \kappa} - w_{\ell', \kappa}]
\]

\[
- \sum_{\ell' \in \mathbb{N} \setminus \{\ell\}} \{[w_{\ell, \kappa} - w_{\ell', \kappa}] + \lambda A_{\varepsilon, \delta}[u_{\ell, \kappa} - u_{\ell', \kappa}]\} - \lambda A_{\varepsilon, \delta}(\psi'_{\varepsilon, \ell', \kappa}(\cdot))\}
\]

\[
\leq -\psi'_{\varepsilon, \ell, \kappa}(\cdot)[2\langle D^1 u_{\ell, \kappa}, D^1 w_{\ell, \kappa} \rangle + \lambda A_{\varepsilon, \delta}|D^1 u_{\ell, \kappa}|^2 - 4dK_2 D^1 u_{\ell, \kappa}^2 2d\Lambda^2 A_{\varepsilon, \delta} - \sum_{\ell' \in \mathbb{N} \setminus \{\ell\}} \psi'_{\varepsilon, \ell', \kappa}(\cdot)[w_{\ell, \kappa} - w_{\ell', \kappa}], \tag{A.21}
\]

due to $\psi(r) \leq \psi'(r)r$, for all $r \in \mathbb{R}$, $g_2^2 \geq 0$ and $\vartheta_{\ell, \ell'} \geq 0$. Therefore, applying (A.18) and (A.21) in (A.17), we get that (A.8) is true for some $C_7 = C_7(d, \Lambda, 1/\theta, K_2, C_4)$. \hfill \blacksquare

### A.3 Proof of Lemma 3.4. Eq. (3.4)

Let us define the auxiliary function $\phi_{\ell, \kappa}$ as

\[
\phi_{\ell, \kappa} := \varpi^2 |D^2 u_{\ell, \kappa}^e|^2 + \lambda A_{\varepsilon, \delta}^1 \varpi \text{tr}[\alpha_{\varepsilon, \delta} D^2 u_{\ell, \kappa}^e] + \mu |D^1 u_{\ell, \kappa}^e|^2 \quad \text{on } O, \tag{A.22}
\]

with $A_{\varepsilon, \delta}^1 := \max_{x, (\ell, \kappa) \in \mathbb{O} \times I} \varpi(x)|D^2 u_{\ell, \kappa}^e(x)|$, $\lambda \geq \max\{1, 2/\theta\}$, $\mu \geq 1$ fixed, and $\alpha_{\varepsilon, \delta} = (\alpha_{ij})_{d \times d}$ be such that $\alpha_{\varepsilon, \delta} = \alpha_{ij}(x_0)$, where $x_0 \in \mathbb{O}$, $(\ell_0, \kappa_0) \in M \times I$ are fixed. We shall show that $\phi_{\ell, \kappa}$ satisfies (A.23). In particular, (A.23) holds when $\phi_{\ell, \kappa}$ is evaluated at its maximum $x_{\mu} \in O$, which helps to see that (3.4) is true.
Lemma A.3. Let $\phi_{\ell, i}$ be the auxiliary function given by (A.22). Then, there exists a positive constant $C_8 = C_8(d, \Lambda, \alpha, K_1)$ such that on $(x, \ell) \in B_{\beta r} \times \mathbb{l}$,
\[
\varpi^2 \text{tr}[a_i D^2 \phi_{\ell, i}] \geq 2\theta [\varpi^4] D^3 u_{\ell, i}^\varepsilon \delta^2 + \mu \varpi^2 [D^2 u_{\ell, i}^\varepsilon \delta^2] - 2\lambda C_8 A_{\varepsilon, \delta}^1 \varpi^2 [D^3 u_{\ell, i}^\varepsilon \delta] \\
- \lambda C_8 [A_{\varepsilon, \delta}^1]^2 - C_8 (\lambda + \mu) A_{\varepsilon, \delta}^1 - C_8 \mu \\
+ \varpi^2 \left\{ \sum_{\ell' \in \ell \setminus \ell} \psi'_{\varepsilon, \ell, \ell'}(\cdot) [\phi_{\ell, i} - \phi_{\ell', i}] + \sum_{k \in \ell \setminus \ell} q_{\ell}(\kappa, \kappa) [\phi_{\ell, i} - \phi_{\ell, k}] \right\} \\
+ \varpi^2 \psi'_{\varepsilon, \ell, i}(\cdot) \left\{ 2\varpi A_{\varepsilon, \delta}^1 [\lambda \theta - 2] [D^2 u_{\ell, i}^\varepsilon \delta^2] - 2\lambda A_{\varepsilon, \delta}^1 C_8 [D^2 u_{\ell, i}^\varepsilon \delta] \\
- (\lambda + \mu) C_8 A_{\varepsilon, \delta}^1 + 2A_{\varepsilon, \delta}^1 (D^1 u_{\ell, i}^\varepsilon \delta, D^1 \phi_{\ell, i}) \right\}. 
\] (A.23)

Proof of Lemma 3.4. Eq. (3.4). Let $\phi_{\ell, i}$ be as in (A.22), where $\lambda \geq \max\{1, 2/\theta\}$ is fixed and $\mu \geq 1$ will be determined later on, and $(x_0, \ell_0, i_0) \in \overline{\varnothing} \times \mathbb{M} \times \mathbb{l}$ satisfies
\[
\varpi(x_0) [D^2 u_{\ell_0, i_0}(x_0)] = A_{\varepsilon, \delta}^1 = \max_{(x, \ell, i) \in \overline{\varnothing} \times \mathbb{M} \times \mathbb{l}} \varpi(x) [D^2 u_{\ell, i}(x)]. 
\] (A.24)

Notice that if $x_0 \in \overline{\varnothing} \setminus B_{\beta r}$, by Remark 3.3 and (A.24), we obtain $\varpi(x) [D^2 u_{\ell, i}(x)] \equiv 0$, for each $(x, \ell, i) \times \overline{\varnothing} \times \mathbb{M} \times \mathbb{l}$. From here, (3.4) is trivially true. So, assume that $x_0$ is in $B_{\beta r}$. Without loss of generality we also assume that $A_{\varepsilon, \delta}^1 > 1$, since if $A_{\varepsilon, \delta}^1 \leq 1$, we get that $\varpi(x) [D^2 u_{\ell, i}(x)] \leq A_{\varepsilon, \delta}^1 \leq 1$ for $(x, \ell, i) \in \overline{\varnothing} \times \mathbb{M} \times \mathbb{l}$. Taking $C_6 = 1$, we obtain the result in (3.4). Let $(x, \ell, i, \mu) \in \overline{\varnothing} \times \mathbb{M} \times \mathbb{l}$ be such that $\phi_{\ell, i, \mu}(x) = \max_{(x, \ell, i) \in \overline{\varnothing} \times \mathbb{M} \times \mathbb{l}} \phi_{\ell, i, \mu}(x)$. If $x_\mu \in \overline{\varnothing} \setminus B_{\beta r}$, from (3.3) and (A.22), it follows that
\[
\varpi^2 [D^2 u_{\ell, i}]^2 \leq -\lambda A_{\varepsilon, \delta}^1 \varpi \text{tr}[a_i D^2 \phi_{\ell, i}] + \mu C_5^2, \quad \text{for} \quad (x, \ell, i) \in \overline{\varnothing} \times \mathbb{M} \times \mathbb{l}. 
\] (A.25)

Evaluating $(x_\ell, l_0, i_0)$ in (A.25) and by (1.14), (H3), (1.16), (1.15) and (3.3), it can be verified that $[A_{\varepsilon, \delta}^1]^2 \leq \lambda \Lambda [1 + C_2] A_{\varepsilon, \delta}^1 + \mu C_2^2$. From here and due to $A_{\varepsilon, \delta}^1 > 1$, we conclude that $\varpi(x) [D^2 u_{\ell, i}(x)] \leq A_{\varepsilon, \delta}^1 \leq \lambda \Lambda [1 + C_2] + \mu C_2^2 =: C_3$, \, (for $(x, \ell, i) \in \overline{\varnothing} \times \mathbb{M} \times \mathbb{l}$). From now, assume that $x_\mu \in \overline{\varnothing} \setminus B_{\beta r}$. Then,
\[
D^1 \phi_{\ell, i, \mu}(x) = 0, \quad \text{tr}[a_i (x) D^2 \phi_{\ell, i, \mu}(x)] \leq 0, \\
\phi_{\ell, i, \mu}(x) - \phi_{\ell, \mu}(x) \geq 0 \quad \text{for} \quad (\ell', \kappa) \in \mathbb{M} \times \mathbb{l}. 
\] (A.26)

Noting that $2\theta \varpi^4 |D^3 u_{\ell, i}|^2 - 2\lambda C_7 A_{\varepsilon, \delta}^1 \varpi^2 |D^3 u_{\ell, i}| \geq -\lambda C_2^2 [A_{\varepsilon, \delta}^1]^2$, with $C_7 > 0$ as in Lemma A.3, and using (A.23) and (A.26), it yields that
\[
0 \geq 2\theta \mu \varpi^2 |D^2 u_{\ell, i, \mu}|^2 - \lambda^2 C_7 \left[ 1 + \frac{C_7}{\theta} \right] [A_{\varepsilon, \delta}^1]^2 - C_7 (\lambda + \mu) A_{\varepsilon, \delta}^1 - C_7 \mu \\
+ A_{\varepsilon, \delta}^1 \varpi^2 \psi'_{\varepsilon, \ell, i}(\cdot) \{ 2\varpi [\lambda \theta - 2] |D^2 u_{\ell, i, \mu}|^2 - 2\lambda C_7 |D^2 u_{\ell, i, \mu}| - (\lambda + \mu) C_7 \}, \quad \text{at} \quad x_\mu. 
\]

From here, we have that at least one of the next two inequalities is true:
\[
2\theta \mu \varpi^2 |D^2 u_{\ell, i, \mu}|^2 - \lambda^2 C_7 \left[ 1 + \frac{C_7}{\theta} \right] [A_{\varepsilon, \delta}^1]^2 - C_7 (\lambda + \mu) A_{\varepsilon, \delta}^1 - C_7 \mu \leq 0, \quad \text{at} \quad x_\mu, \quad (A.27) \\
A_{\varepsilon, \delta}^1 \varpi^2 \psi'_{\varepsilon, \ell, i}(\cdot) \{ 2\varpi [\lambda \theta - 2] |D^2 u_{\ell, i, \mu}|^2 - 2\lambda C_7 |D^2 u_{\ell, i, \mu}| - (\lambda + \mu) C_7 \} \leq 0, \quad \text{at} \quad x_\mu. \quad (A.28)
\]
Suppose that (A.27) holds. Then, evaluating \((x_\mu, \ell_\mu)\) in (A.22), it follows
\[
\phi_{\ell,\mu} \leq \frac{\lambda^2 C_7}{2\theta\mu} \left[ \frac{C_7}{\theta} \right] [A_{\epsilon,\delta}]^2 + \frac{C_7(\lambda + \mu)}{2\theta\mu} A_{\epsilon,\delta}^1 + \frac{C_7}{2\theta} + \mu [C_2]^2
\]
\[
+ \lambda \Lambda A_{\epsilon,\delta}^1 \left\{ \frac{\lambda^2 C_7}{2\theta\mu} \left[ \frac{C_7}{\theta} \right] [A_{\epsilon,\delta}]^2 + \frac{C_7(\lambda + \mu)}{2\theta\mu} A_{\epsilon,\delta}^1 + \frac{C_7}{2\theta} \right\}^{1/2}, \quad \text{at } x_\mu. \tag{A.29}
\]

Meanwhile, evaluating \((x_0, \ell_0, \iota_0)\) in (A.22) and using (1.14) and (1.15), we get
\[
\phi_{\ell_0,\iota_0} \geq [A_{\epsilon,\delta}]^2 - \lambda \Lambda A_{\epsilon,\delta}^1 |C_2 + 1|, \quad \text{at } x_0. \tag{A.30}
\]

Then, taking \(\mu\) large enough such that
\[
\frac{K_7^{(\Lambda)}}{\mu} \leq \left[ \frac{1}{\lambda \Lambda} \left[ 1 - \frac{K_7^{(\Lambda)}}{\mu} \right] \right]^2,
\]
with \(K_7^{(\Lambda)} := \frac{\lambda^2 C_7}{2\theta \mu} \left[ 1 + \frac{C_7}{\theta} \right],\) using (A.29)–(A.30) and since \(\phi_{\ell_0,\iota_0}(x_0) \leq \phi_{\ell,\mu}(x_\mu)\) and \(\lambda, A_{\epsilon,\delta}^1 > 1,\) we have that
\[
\frac{1}{\lambda \Lambda} \left[ 1 - \frac{K_7^{(\Lambda)}}{\mu} \right] A_{\epsilon,\delta}^1 - K_8^{(\mu)} \leq \left\{ \frac{K_7^{(\Lambda)}}{\mu} [A_{\epsilon,\delta}]^2 + \frac{C_7(\lambda + \mu)}{2\theta\mu} A_{\epsilon,\delta}^1 + \frac{C_7}{2\theta} \right\}^{1/2},
\]
with \(K_8^{(\mu)} := \frac{C_7}{2\theta \lambda} \left[ \frac{1}{\mu} + 1 \right] + \frac{C_7}{2\theta} + \mu C_2.\) Then,
\[
\left\{ \frac{1}{\lambda^2 \Lambda^2} \left[ 1 - \frac{K_7^{(\Lambda)}}{\mu} \right]^2 - \frac{K_7^{(\Lambda)}}{\mu} \right\} [A_{\epsilon,\delta}]^2 \leq \left\{ \frac{2K_8^{(\mu)}}{\lambda \Lambda} \left[ 1 - \frac{K_7^{(\Lambda)}}{\mu} \right] + \frac{C_7(\lambda + \mu)}{2\theta\mu} \right\} A_{\epsilon,\delta}^1 + \frac{C_7}{2\theta}.
\]

From here, we conclude there exists a constant \(C_3 = C_3(d, \Lambda, \alpha, K_3)\) such that
\[
\varpi(x) |D^2 u_{\ell,\iota}(x)| \leq A_{\epsilon,\delta}^1 \leq C_3 \quad \text{for } (x, \ell) \in \mathcal{O} \times \mathbb{I}.
\]

Now, assume that (A.28) holds. Then, \(2\varpi[\lambda \theta - 2]|D^2 u_{\ell,\mu}|^2 \leq 2\lambda C_7 \varpi |D^2 u_{\ell,\mu}| + (\lambda + \mu)C_7\) at \(x_\mu\) due to \(\psi'_\varepsilon \geq 0\) and \(\varpi \leq 1.\) From here, we have that \(\varpi |D^2 u_{\ell,\mu}| \leq K_9^{(\Lambda,\mu)}\) at \(x_\mu,\) where \(K_9^{(\Lambda,\mu)}\) is a positive constant independent of \(A_{\epsilon,\delta}^1.\) Therefore, \([A_{\epsilon,\delta}^1]^2 - \lambda \Lambda A_{\epsilon,\delta}^1 [C_2 + 1] \leq \phi_{\ell_0,\iota_0}(x_0) \leq \phi_{\ell,\mu}(x_\mu) \leq [K_9^{(\Lambda,\mu)}]^2 + \lambda \Lambda A_{\epsilon,\delta}^1 K_9^{(\Lambda,\mu)} + \mu [C_2]^2.\) From here, we conclude there exists a constant \(C_3 = C_3(d, \Lambda, \alpha, K_1)\) such that \(\varpi |D^2 u_{\ell,\iota}| \leq A_{\epsilon,\delta}^1 \leq C_3 \) for all \((x, \ell) \in \mathcal{O} \times \mathbb{I}.\) \hfill \(\blacksquare\)

**Proof of Lemma A.3.** Taking first and second derivatives of \(\phi_{\ell,\iota}\) on \(\overline{B}_{3r},\) it can be verified that
\[
\text{tr}[\alpha_{\iota} D^2 \phi_{\ell,\iota}] = |D^2 u_{\ell,\iota}|^2 \text{tr}[a, D^2 \varpi^2] + 2\langle a, D^1 \varpi^2, D^1 |D^2 u_{\ell,\iota}|^2 \rangle + \varpi^2 \text{tr}[a, D^2 |D^2 u_{\ell,\iota}|^2] + \lambda A_{\epsilon,\delta}^1 \text{tr}[\alpha_{\iota} D^2 u_{\ell,\iota}] + 2\lambda \Lambda A_{\epsilon,\delta}^1 \langle a, D^1 \varpi, D^1 |D^2 u_{\ell,\iota}|^2 \rangle
\]
\[
+ \lambda A_{\epsilon,\delta}^1 \varpi \sum_j \alpha_{\iota, j} \text{tr}[a, D^2 \partial_j u_{\ell,\iota}] + \mu \text{tr}[a, D^2 |D^1 u_{\ell,\iota}|^2].
\]
From here and noticing that from (1.11),

\[
\text{tr}[a_i D^2 | D^1 u_{\ell,i}|^2] \geq 2\theta |D^2 u_{\ell,i}|^2 + 2 \sum_i \partial_i u_{\ell,i} \text{tr}[a_i D^2 \partial_i u_{\ell,i}],
\]

\[
\text{tr}[a_i D^2 | D^2 u_{\ell,i}|^2] \geq 2\theta |D^3 u_{\ell,i}|^2 + 2 \sum_{ji} \partial_{ji} u_{\ell,i} \text{tr}[a_i D^2 \partial_{ji} u_{\ell,i}],
\]

it follows that

\[
\text{tr}[a_i D^2 \phi_{\ell,i}] \geq 2\theta [\mathbb{w}^2 |D^3 u_{\ell,i}|^2 + \mu |D^2 u_{\ell,i}|^2] + |D^2 u_{\ell,i}|^2 \text{tr}[a_i D^2 \mathbb{w}^2]
\]

\[
+ 2\{a_i D^1 \mathbb{w}^2 , D^1 |D^2 u_{\ell,i}|^2 \} + \lambda A_{\varepsilon,\delta}^1 \text{tr}[\alpha_{\varepsilon,\delta} D^2 u_{\ell,i}] \text{tr}[a_i D^2 \mathbb{w}]
\]

\[
+ 2\lambda A_{\varepsilon,\delta}^1 \{a_i D^1 \mathbb{w}^2 , D^1 \text{tr}[\alpha_{\varepsilon,\delta} D^2 u_{\ell,i}] \} + 2\mu \sum_i \text{tr}[a_i D^2 \partial_i u_{\ell,i}] \partial_i u_{\ell,i}
\]

\[
\quad + \sum_{ji} [2\mathbb{w}^2 \partial_{ji} u_{\ell,i} + \lambda A_{\varepsilon,\delta}^1 \mathbb{w} \alpha_{\varepsilon,\delta} \partial_{ji} u_{\ell,i}] \text{tr}[a_i D^2 \partial_{ji} u_{\ell,i}] .
\]

(A.31)

Meanwhile, differentiating twice in (1.15), we see that

\[
\text{tr}[a_i D^2 \partial_{ji} u_{\ell,i}]
\]

\[
= \psi_{\varepsilon,\ell,i}''(\cdot) \tilde{\eta}_{\varepsilon,i}^{(j)} + \psi_{\varepsilon,\ell,i}'(\cdot) \partial_{ji} [D^1 u_{\ell,i}]^2 - g_{ij}^2 + \sum_{e^i \in \mathbb{N} \{ \ell \}} \psi_{\varepsilon,\ell,e,i}'(\cdot) \tilde{\eta}_{\varepsilon,e,i}^{(j)}
\]

\[
+ \sum_{e^i \in \mathbb{N} \{ \ell \}} \psi_{\varepsilon,\ell,e,i}'(\cdot) \partial_{ji} [u_{\ell,i} - u_{e,i}] - \sum_{\kappa \in \mathbb{N} \{ \ell \}} q_{\ell}(\iota, \kappa) \partial_{ji} [u_{\ell,i} - u_{\ell,i}] - \text{tr}[[\partial_{ji} a_i] D^2 \partial_i u_{\ell,i}]
\]

\[
- \text{tr}[[\partial_{ji} a_i] D^2 u_{\ell,i}] - \text{tr}[[\partial_i a_i] D^2 \partial_j u_{\ell,i}] - \partial_{ji} [h_\iota - \langle b_i, D^1 u_{\ell,i} \rangle - c_i u_{\ell,i}]
\]

(A.32)

where \( \tilde{\eta}_{\ell,i} = (\tilde{\eta}_{\varepsilon,i}^{(1)}, \ldots, \tilde{\eta}_{\varepsilon,i}^{(d)}) \) and \( \tilde{\eta}_{\ell,e,i} = (\tilde{\eta}_{\varepsilon,e,i}^{(1)}, \ldots, \tilde{\eta}_{\varepsilon,e,i}^{(d)}) \) with \( \tilde{\eta}_{\varepsilon,i}^{(j)} := \partial_i [D^1 u_{\ell,i}]^2 - g_{ij}^2 \) and \( \tilde{\eta}_{\varepsilon,e,i}^{(j)} := \partial_i [u_{\ell,i} - u_{e,i}] \). From (A.15) and (A.31)–(A.32), it follows that

\[
\mathbb{w}^2 \text{tr}[a_i D^2 \phi_{\ell,i}]
\]

\[
\geq 2\theta [\mathbb{w}^4 |D^3 u_{\ell,i}|^2 + \mu \mathbb{w}^2 |D^2 u_{\ell,i}|^2] + \tilde{D}_3 + \tilde{D}_4
\]

\[
+ \mathbb{w}^2 \left\{ \psi_{\varepsilon,\ell,i}''(\cdot) [2\mathbb{w}^2 D^2 u_{\ell,i} + \lambda A_{\varepsilon,\delta}^1 \mathbb{w} \alpha_{\varepsilon,\delta} \tilde{\eta}_{\ell,i}] \right\}
\]

\[
+ \mathbb{w}^2 \psi_{\varepsilon,\ell,i}'(\cdot) \tilde{D}_{\ell,i} + \mathbb{w}^2 \sum_{e^i \in \mathbb{N} \{ \ell \}} \psi_{\varepsilon,\ell,e,i}'(\cdot) \tilde{D}_{\ell,e,i} + \mathbb{w}^2 \sum_{\kappa \in \mathbb{N} \{ \ell \}} q_{\ell}(\iota, \kappa) D_{\ell,\ell}^{\varepsilon,\kappa}
\]

(A.33)

where
\[
\tilde{D}_3 := 2\pi^2\langle a, D^1 \varpi^2, D^1 | D^2 u_{\ell,t}, |^2 \rangle + 2\lambda A_{e,\delta}^1 \varpi^2\langle a, D^1 \varpi, D^1 \text{tr}[a_{\alpha t} D^2 u_{\ell,t}] \rangle \\
- \sum_{ij} [2\varpi^4 \partial_{ij} u_{\ell,t} + \lambda A_{e,\delta}^1 \varpi^3 \alpha_{t ij}] [2 \text{tr}[\partial_j a_{\delta} D^2 \partial_i u_{\ell,t}] - \partial_{ij} \langle h_t, D^1 u_{\ell,t} \rangle],
\]
\[
\tilde{D}_4 := \varpi^2 |D^2 u_{\ell,t}|^2 \text{tr}[a, D^2 \varpi^2] - \mu \varpi^2 \tilde{D}_2 u_{\ell,t} + \lambda A_{e,\delta}^1 \varpi^2 \text{tr}[a_{\alpha t} D^2 u_{\ell,t}] \text{tr}[a, D^2 \varpi] \\
- \sum_{ij} [2\varpi^4 \partial_{ij} u_{\ell,t} + \lambda A_{e,\delta}^1 \varpi^3 \alpha_{t ij}] \{ \text{tr}[\partial_j a_{\delta} D^2 u_{\ell,t}] + \partial_{ij} [h_t - c_i u_{\ell,t}] \},
\]
\[
\tilde{D}_{6,t} := 2\mu \langle D^1 u_{\ell,t}, \tilde{\eta}_{\ell,t} \rangle + \text{tr}[[2\varpi^2 D^2 u_{\ell,t} + \lambda A_{e,\delta}^1 \varpi a_{\alpha t} D^2 [D^1 u_{\ell,t}]^2 - g_t^2]],
\]
\[
\tilde{D}_{6,t} := 2\mu \langle D^1 u_{\ell,t}, D^1 [u_{\ell,t} - u_{\ell,\epsilon}] \rangle + \text{tr}[[2\varpi^2 D^2 u_{\ell,t} + \lambda A_{e,\delta}^1 \varpi a_{\alpha t} D^2 [u_{\ell,t} - u_{\ell,\epsilon}]].
\]

Recall that \(\tilde{D}_2 u_{\ell,t}\) is given in (A.16). To obtain the next inequalities, we shall recurrently use (H3), (H4), Remark 3.3, 3.1, 3.3 and \(\lambda, \mu \geq 1\). Then,

\[
\tilde{D}_3 \geq -2 \{4\lambda K_1 d^4 + \lambda^2 d^4 + d^3 [2 + \Lambda] [d + \Lambda] \lambda A_{e,\delta}^1 \varpi^2 |D^3 u_{\ell,t}| \\
- 2d^2 \lambda A_{e,\delta}^1 [2 + \Lambda] dC_2 \Lambda - 4d^3 \lambda \lambda A_{e,\delta}^1 [2 + \Lambda] \}
\]

and by (1.11),

\[
\tilde{D}_4 \geq -\{2\lambda d^2 K_1 + d^4 \lambda K_2 + d^2 \lambda [2 + \Lambda] [d^2 + 1] \lambda A_{e,\delta}^1 \}^2 \\
- \{2\mu [2C_2 \lambda d^3 + 2d^{1/2} \Lambda C_2] + d^2 \lambda [2 + \Lambda] [2C_2 + C_1] \} A_{e,\delta}^1 \\
- 2\mu [2C_2 \lambda d^2 + 2C_1 C_2 d^{1/2} \Lambda - 2C_2 \Lambda d^{1/2}].
\]

On the other hand, since \(\lambda \geq \frac{2}{d}\) and using (1.11), we have that

\[
\langle [2\varpi^2 D^2 u_{\ell,t} + \lambda A_{e,\delta}^1 \varpi a_{\alpha t}] | \gamma, \gamma \rangle \geq \varpi [\lambda A_{e,\delta}^1 \theta - 2 \varpi |D^2 u_{\ell,t}|] | \gamma|^2 \\
\geq \varpi A_{e,\delta}^1 |\lambda \theta - 2| | \gamma|^2 \geq 0,
\]

for \(\gamma \in \mathbb{R}^d\). From here and since \(\psi''_{\epsilon,\ell} (\cdot) \geq 0\) and \(\psi''_{\delta,\ell} (\cdot) \geq 0\), it follows that

\[
\psi''_{\epsilon,\ell} (\cdot) \langle [2\varpi^2 D^2 u_{\ell,t} + \lambda A_{e,\delta}^1 \varpi a_{\alpha t}] | \tilde{\eta}_{\ell,t}, \tilde{\eta}_{\ell,t} \rangle \\
+ \sum_{\ell' \in \mathbb{N} \setminus \{\ell\}} \psi''_{\delta,\ell,\ell'} (\cdot) \langle [2\varpi^2 D^2 u_{\ell,t} + \lambda A_{e,\delta}^1 \varpi a_{\alpha t}] | \tilde{\eta}_{\ell,t}, \tilde{\eta}_{\ell,t} \rangle \geq 0.
\]

It is easy to verify that

\[
\langle D^1 u_{\ell,t}, D^1 | D^2 u_{\ell,t} |^2 \rangle + \lambda A_{e,\delta}^1 \varpi \langle D^1 u_{\ell,t}, D^1 \text{tr}[a_{\alpha t} D^2 u_{\ell,t}] \rangle + \mu \langle D^1 u_{\ell,t}, D^1 | D^1 u_{\ell,t} |^2 \rangle \\
= \langle D^1 u_{\ell,t}, D^1 \phi_{\ell,t} \rangle - \langle D^1 u_{\ell,t}, D^1 \varpi \rangle | D^2 u_{\ell,t} |^2 - \lambda A_{e,\delta}^1 \text{tr}[a_{\alpha t} D^2 u_{\ell,t}] \langle D^1 u_{\ell,t}, D^1 \varpi \rangle
\]

due to

\[
\partial_t \phi_{\ell,t} = |D^2 u_{\ell,t}|^2 \partial_t \varpi + \varpi |D^2 u_{\ell,t}|^2 \\
+ \lambda A_{e,\delta}^1 \text{tr}[a_{\alpha t} D^2 u_{\ell,t}] \partial_t \varpi + \lambda A_{e,\delta}^1 \varpi \text{tr}[a_{\alpha t} D^2 \partial_t u_{\ell,t}] + \mu \partial_t | D^1 u_{\ell,t} |^2 \text{ on } B_{\eta'}.\]
Applying (A.36) and (A.38),
\[
\tilde{D}_{\ell,t} \geq 2\varpi A_{1,\delta}^1 |\lambda \theta - 2| D^2 u_{\ell,t}|^2 + 2\langle D^1 u_{\ell,t}, D^1 \phi_{\ell,t} \rangle \\
- 4\lambda d^{1/2} C_2 K_1 A_{1,\delta}^1 |D^2 u_{\ell,t}| - 2\lambda A \delta^{5/2} C_2 K_1 A_{1,\delta}^1 |D^2 u_{\ell,t}| \\
- 4\mu A^2 d^{1/2} C_2 - 2\lambda d\Lambda A_{1,\delta}^1 - \Lambda A^2 A_{1,\delta}^1 \\
- 4d^2 \Lambda^2 A_{1,\delta}^1 - 2\Lambda A_{1,\delta}.
\] (A.39)

Using the following properties \(|A|^2 - 2 \text{tr}[AB] + |B|^2 = \sum_{ij} (A_{ij} - B_{ij})^2 \geq 0\) and \(y^2 = \sum_i (y_{1,i} - y_{2,i})^2 \geq 0\) where \(A = (A_{ij})_{d \times d}, B = (B_{ij})_{d \times d}\) and \(y_1 = (y_{1,1}, \ldots, y_{1,d}), y_2 = (y_{2,1}, \ldots, y_{2,d})\) belong \(S(d)\) and \(\mathbb{R}^d\), respectively, and by definition of \(\phi_{\ell,t}\), it is easy to corroborate the following identity
\[
\tilde{D}_{\ell,t} \geq \phi_{\ell,t} - \phi_{\ell',\kappa}, \text{ for } (\ell',\kappa) \in \mathbb{M} \times \mathbb{I}.
\] (A.40)

Applying (A.37)–(A.40) in (A.33) and considering that all constants that appear in those inequalities (i.e. (A.37)–(A.40)) are bounded by an universal constant \(C_8 = C_8(d, \Lambda, \alpha, K_2)\), we obtain the desired result in the lemma above. With this remark, the proof is concluded. \(\blacksquare\)

**A.4 Proof of Proposition 1.4**

*Proof of Proposition 1.4. Existence.* Taking \(\ell' \in \mathbb{M} \setminus \{\ell\}\), and using (1.14), (1.15) and Lemma 3.4, we have that \(\psi_\delta(\xi_{\ell,t} - \xi_{\ell',t} - \vartheta_{\ell,t'})\) is locally bounded, uniformly in \(\delta\). From here and (3.5), it yields that \(u_{\ell,t}^\varepsilon - u_{\ell',t}^\varepsilon - \vartheta_{\ell,t'} \leq 0\) in \(\mathcal{O}\). Then,
\[
u_{\ell,t}^\varepsilon - \mathcal{M}_{\ell,t}u^\varepsilon \leq 0, \quad \text{in } \mathcal{O}. \quad (A.41)
\]

Note that the previous inequality is true on the boundary set \(\partial\mathcal{O}\), since \(u_{\ell,t}^\varepsilon = \xi_{\ell,t} - \vartheta_{\ell,t'} \leq 0\) on \(\partial\mathcal{O}\) and \(\vartheta_{\ell,t'} \geq 0\). Recall that the operator \(\mathcal{M}_{\ell,t}\) is defined in (1.13). On the other hand, since \(u_{\ell,t}^\varepsilon\) is the unique solution to (1.15), when \(\delta = \delta_h\), it follows that
\[
\int_{B_r} \left\{ [c_i - \mathcal{L}_{\ell,t}]u_{\ell,t}^\varepsilon + \psi_\varepsilon(|D^1 u_{\ell,t}^\varepsilon|^2 - g^2) \right\} d\varpi \leq \int_{B_r} h_r d\varpi dx, \quad \text{for } \varpi \in \mathcal{B}(B_r), \quad (A.42)
\]

where
\[
\mathcal{B}(A) := \{ \varpi \in \text{C}^{\infty}_c(A): \varpi \geq 0 \text{ and supp}[\varpi] \subset A \subset \mathcal{O} \}. \quad (A.43)
\]

By (3.5) and letting \(\delta_h \to 0\) in (A.42), we obtain that
\[
[c_i - \mathcal{L}_{\ell,t}]u_{\ell,t}^\varepsilon + \psi_\varepsilon(|D^1 u_{\ell,t}^\varepsilon|^2 - g^2) \leq h_r \quad \text{a.e. in } \mathcal{O}. \quad (A.44)
\]

From (A.41) and (A.44), \(\max \left\{ [c_i - \mathcal{L}_{\ell,t}]u_{\ell,t}^\varepsilon + \psi_\varepsilon(|D^1 u_{\ell,t}^\varepsilon|^2 - g^2) - h_r, u_{\ell,t}^\varepsilon - \mathcal{M}_{\ell,t}u^\varepsilon \right\} \leq 0\) a.e. in \(\mathcal{O}\). We shall prove that if
\[
u_{\ell,t}^\varepsilon(x^*) - \mathcal{M}_{\ell,t}u^\varepsilon(x^*) < 0, \quad \text{for some } x^* \in \mathcal{O}, \quad (A.45)
\]

then, there exists a neighborhood \(\mathcal{N}_{x^*} \subset \mathcal{O}\) of \(x^*\) such that
\[
[c_i - \mathcal{L}_{\ell,t}]u_{\ell,t}^\varepsilon + \psi_\varepsilon(|D^1 u_{\ell,t}^\varepsilon|^2 - g^2) = h_r, \quad \text{a.e. in } \mathcal{N}_{x^*}. \quad (A.46)
\]
Assume (A.45) holds. Then, taking \( \ell' \in \mathcal{M}\setminus\{\ell\} \), we see that \( u_{\ell_0} - u_{\ell_0} - \partial_{\ell_0} \leq u_{\ell_0} - \mathcal{M}_{\ell_0}u^\varepsilon < 0 \) at \( x^* \). Since \( u_{\ell_0} - u_{\ell_0} \) is a continuous function, there exists a ball \( B_{r_{\ell_0}} \subset \mathcal{O} \) such that \( x^* \in B_{r_{\ell_0}} \) and \( u_{\ell_0} - u_{\ell_0} - \partial_{\ell_0} < 0 \) in \( B_{r_{\ell_0}} \). From here and defining \( \mathcal{N}_{x^*} \), \( \mathcal{N}_{x^*} \subset \mathcal{O} \) is a neighborhood of \( x^* \) and

\[
\begin{align*}
  u_{\ell_0} - u_{\ell_0} - \partial_{\ell_0} < 0, \quad \text{in} \mathcal{N}_{x^*}, \quad \text{for} \quad \ell' \in \mathcal{M}\setminus\{\ell\}. \quad \text{(A.47)}
\end{align*}
\]

Meanwhile, observe that

\[
\begin{align*}
  \|[u_{\ell_0} - u_{\ell_0}] - (u_{\ell_0} - u_{\ell_0})\|_{C(\delta)} \rightarrow 0, \quad \text{for} \quad \ell' \in \mathcal{M}\setminus\{\ell\},
\end{align*}
\]

since (3.5) holds. Then, by (A.47)–(A.48), it yields that for each \( \ell' \in \mathcal{M}\setminus\{\ell\} \), there exists a \( \delta'(\ell') \in (0, 1) \) such that if \( \delta_0 \leq \delta'(\ell') \), \( u_{\ell_0} - u_{\ell_0} - \partial_{\ell_0} < 0 \) in \( \mathcal{N}_{x^*} \). Taking \( \delta' = \min_{\ell' \in \mathcal{M}\setminus\{\ell\}} \{\delta'(\ell')\} \), it follows that \( u_{\ell_0} - u_{\ell_0} - \partial_{\ell_0} \) in \( \mathcal{N}_{x^*} \), for all \( \delta_0 \leq \delta' \) and \( \ell' \in \mathcal{M}\setminus\{\ell\} \). From here and since for each \( \delta_0 \leq \delta' \), \( u_{\ell_0} \) is the unique solution to (1.15), when \( \delta = \delta_0 \), it implies that

\[
\begin{align*}
  \int_{\mathcal{N}_{x^*}} \left\{[c - \mathcal{L}_{\ell_0}]u_{\ell_0} + \psi_\varepsilon(1 - u_{\ell_0}^2) + \frac{\gamma_\varepsilon}{\varepsilon} \right\} \omega dx = \int_{\mathcal{N}_{x^*}} h\omega dx, \quad \text{for} \quad \omega \in \mathcal{B}(\mathcal{N}_{x^*}).
\end{align*}
\]

Therefore, (A.46) holds. Hence, we get that for each \( \varepsilon \in (0, 1) \), \( u^\varepsilon = (u_{\ell_0}^\varepsilon)_{(\ell_0) \in \mathcal{M}\times I} \) is a solution to the HJB equation (4.5).

**Proof of Proposition 1.4. Uniqueness.** Let \( \varepsilon \in (0, 1) \) be fixed. Suppose that \( u^\varepsilon = (u_{\ell_0}^\varepsilon)_{(\ell_0) \in \mathcal{M}\times I} \) and \( v^\varepsilon = (v_{\ell_0}^\varepsilon)_{(\ell_0) \in \mathcal{M}\times I} \) are two solutions to the HJB equation (1.17) whose components belong to \( \mathcal{C}^0(\overline{\mathcal{O}}) \cap \mathcal{W}^{2,\infty}_\text{loc}(\mathcal{O}) \). Take \( (x_0, \ell_0, t_0) \in \overline{\mathcal{O}} \times \mathcal{M} \times I \) such that

\[
\begin{align*}
  u_{\ell_0, t_0}(x_0) - v_{\ell_0, t_0}(x_0) = \max_{(x, \ell, t, \kappa) \in \overline{\mathcal{O}} \times \mathcal{M} \times I} \{u_{\ell_0, t}^\varepsilon(x) - v_{\ell_0, t}^\varepsilon(x)\}. \quad \text{(A.49)}
\end{align*}
\]

Notice that by (A.49), we only need to verify that

\[
\begin{align*}
  u_{\ell_0, t_0}(x_0) - v_{\ell_0, t_0}(x_0) \leq 0, \quad \text{(A.50)}
\end{align*}
\]

which is trivially true, if \( x_0 \in \partial \mathcal{O} \), since \( u_{\ell_0, t_0} - v_{\ell_0, t_0} = 0 \) on \( \partial \mathcal{O} \). Let us assume \( x_0 \in \mathcal{O} \). We shall verify (A.50) by contradiction. Suppose that \( u_{\ell_0, t_0} - v_{\ell_0, t_0} > 0 \) at \( x_0 \). Then, by continuity of \( u_{\ell_0, t_0} - v_{\ell_0, t_0} \), there exists a ball \( B_{r_1}(x_0) \subset \mathcal{O} \) such that

\[
\begin{align*}
  c_{\ell_0}[u_{\ell_0, t_0} - v_{\ell_0, t_0}] \geq \min_{x \in B_{r_1}(x_0)} \{c_{\ell_0}(x)[u_{\ell_0, t_0}^\varepsilon(x) - v_{\ell_0, t_0}^\varepsilon(x)]\} > 0, \quad \text{in} \quad B_{r_1}(x_0). \quad \text{(A.51)}
\end{align*}
\]

The last inequality is true because of \( c_{\ell_0} > 0 \) in \( \mathcal{O} \). Additionally, again by (A.49) and by the continuity of \( u_{\ell_0, t_0} - v_{\ell_0, t_0} \) on \( \overline{\mathcal{O}} \), we get that there is a ball \( B_{r_2}(x_0) \subset \mathcal{O} \) such that

\[
\begin{align*}
  \sum_{\kappa \in \mathbb{I} \setminus \{\ell_0\}} q_0(t_0, \kappa) \{u_{\ell_0, t_0}^\varepsilon - v_{\ell_0, t_0}^\varepsilon - [u_{\ell_0, t_0}^\varepsilon - v_{\ell_0, t_0}^\varepsilon]\} \geq 0 \quad \text{in} \quad B_{r_2}(x_0). \quad \text{(A.52)}
\end{align*}
\]

Meanwhile, taking \( \ell_1 \in \mathbb{I} \) such that

\[
\begin{align*}
  \mathcal{M}_{\ell_0, t_0}v^\varepsilon(x_0) = v_{\ell_1, t_0}^\varepsilon(x_0) + \vartheta_{\ell_0, \ell_1}. \quad \text{(A.53)}
\end{align*}
\]
by (1.17) and (A.49), we get that $v^\varepsilon_{t_0,t_0} - (v^\varepsilon_{t_1,t_0} + \vartheta_{t_0,t_1}) = v^\varepsilon_{t_0,t_0} - \mathcal{M}_{t_0,t_0} v^\varepsilon \leq u^\varepsilon_{t_0,t_0} - \mathcal{M}_{t_0,t_0} u^\varepsilon \leq 0$ at $x_0$. If $v^\varepsilon_{t_0,t_0}(x_0) - \mathcal{M}_{t_0,t_0} v^\varepsilon(x_0) < 0$, there exists a ball $B_{r_3}(x_0) \subset \mathcal{O}$ such that $v^\varepsilon_{t_0,t_0} - \mathcal{M}_{t_0,t_0} v^\varepsilon < 0$ in $B_{r_3}(x_0)$. Moreover, from (1.17),

$$
[c_{t_0} - \mathcal{L}_{t_0,t_0}] v^\varepsilon_{t_0,t_0} + \psi_\varepsilon(\|D^1 v^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}) - h_{t_0} = 0, \quad \text{in } B_{r_3}(x_0).
$$

(A.54)

Notice that $\psi_\varepsilon(\|D^1 u^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}) - \psi_\varepsilon(\|D^1 v^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0})$ is a continuous function in $\mathcal{O}$ due to $\partial_t u^\varepsilon_{t_0,t_0}, \partial_t v^\varepsilon_{t_0,t_0} \in C^0(\mathcal{O})$, which satisfies $\psi_\varepsilon(\|D^1 u^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}) - \psi_\varepsilon(\|D^1 v^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}) = 0$ at $x_0$, since $x_0$ is the point where $u^\varepsilon_{t_0,t_0} - v^\varepsilon_{t_0,t_0}$ attains its maximum. Meanwhile, by Bony’s maximum principle (see [14]), it is known that for every $r \leq r_4$, with $r_4 > 0$ small enough,

$$
\text{tr}[a_{t_0} D^2[u^\varepsilon_{t_0,t_0} - v^\varepsilon_{t_0,t_0}]] \leq 0, \quad \text{a.e. in } B_r(x_0).
$$

(A.55)

So, from (A.51), (A.52), (A.54) and (A.55), it yields that for every $r \leq \hat{r} := \min\{r_1, r_2, r_3, r_4\}$,

$$
0 \geq \text{tr}[a_{t_0} D^2[u^\varepsilon_{t_0,t_0} - v^\varepsilon_{t_0,t_0}]]
$$

$$
\geq c_{t_0}[u^\varepsilon_{t_0,t_0} - v^\varepsilon_{t_0,t_0}] + \langle b_{t_0}, D^1[u^\varepsilon_{t_0,t_0} - v^\varepsilon_{t_0,t_0}] \rangle
$$

$$
+ \psi_\varepsilon(\|D^1 u^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}) - \psi_\varepsilon(\|D^1 v^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0})
$$

$$
+ \sum_{\kappa \in \mathfrak{N}(t_0)} q_\kappa(0,\kappa)[u^\varepsilon_{t_0,t_0} - v^\varepsilon_{t_0,t_0}] - [u^\varepsilon_{t_0,\kappa} - v^\varepsilon_{t_0,\kappa}]
$$

$$
\geq \min_{x \in B_{r_3}(x_0)} \{c_{t_0}(x)[u^\varepsilon_{t_0,t_0}(x) - v^\varepsilon_{t_0,t_0}(x)] + \langle b_{t_0}, D^1[u^\varepsilon_{t_0,t_0} - v^\varepsilon_{t_0,t_0}] \rangle
$$

$$
+ \psi_\varepsilon(\|D^1 u^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}) - \psi_\varepsilon(\|D^1 v^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}), \quad \text{a.e. in } B_r(x_0).
$$

Then,

$$
\lim_{r \to 0} \left\{ \inf_{B_r(x_0)} \text{ess}[\psi_\varepsilon(\|D^1 u^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}) - \psi_\varepsilon(\|D^1 v^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0})] \right\}
$$

$$
< - \min_{x \in B_{r_3}(x_0)} \{c_{t_0}(x)[u^\varepsilon_{t_0,t_0}(x) - v^\varepsilon_{t_0,t_0}(x)]\} < 0. \quad \text{(A.56)}
$$

That means $\psi_\varepsilon(\|D^1 u^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0}) - \psi_\varepsilon(\|D^1 v^\varepsilon_{t_0,t_0}\|^2 - g^2_{t_0})$ is not continuous at $x_0$ which is a contradiction. Thus,

$$
0 = v^\varepsilon_{t_0,t_0} - (v^\varepsilon_{t_1,t_0} + \vartheta_{t_0,t_1}) = v^\varepsilon_{t_0,t_0} - \mathcal{M}_{t_0,t_0} v^\varepsilon \leq u^\varepsilon_{t_0,t_0} - \mathcal{M}_{t_0,t_0} u^\varepsilon \leq 0 \quad \text{at } x_0. \quad \text{(A.57)}
$$

It implies that

$$
u^\varepsilon_{t_1,t_0}(x_0) - v^\varepsilon_{t_1,t_0}(x_0) \geq u^\varepsilon_{t_0,t_0}(x_0) - v^\varepsilon_{t_0,t_0}(x_0) > 0, \quad \text{(A.58)}
$$

By (A.49) and (A.58), we have that $u^\varepsilon_{t_1,t_0} - v^\varepsilon_{t_1,t_0}$ attains its maximum at $x_0 \in \mathcal{O}$, whose value agrees with $u^\varepsilon_{t_0,t_0}(x_0) - v^\varepsilon_{t_0,t_0}(x_0)$. Then, replacing $u^\varepsilon_{t_0,t_0} - v^\varepsilon_{t_0,t_0}$ by $u^\varepsilon_{t_1,t_0} - v^\varepsilon_{t_1,t_0}$ above and repeating the same arguments seen in (A.53)–(A.57), we get that there is a regime $\ell_2 \in \mathfrak{L}$ such that

$$
u^\varepsilon_{t_2,t_0}(x_0) = u^\varepsilon_{t_2,t_0}(x_0) - v^\varepsilon_{t_2,t_0}(x_0) = u^\varepsilon_{t_0,t_0}(x_0) - v^\varepsilon_{t_0,t_0}(x_0) > 0,
$$

$$
u^\varepsilon_{t_1,t_0}(x_0) = u^\varepsilon_{t_2,t_0}(x_0) + \vartheta_{t_1,t_2}.
$$
Recursively, we obtain a sequence of regimes \( \{ \ell_i \} \) such that
\[
\begin{align*}
u_{\ell_i,0}^r(x_0) - u_{\ell_i,0}^r(x_0) &= u_{\ell_{i-1},0}^r(x_0) - v_{\ell_{i-1},0}^r(x_0) = \cdots = u_{\ell_0,0}^r(x_0) - v_{\ell_0,0}^r(x_0) > 0, \\
v_{\ell_i,0}^r(x_0) &= v_{\ell_{i+1},0}^r(x_0) + \vartheta_{\ell_i,\ell_{i+1}}.
\end{align*}
\] (A.59)

Since \( \mathbb{M} \) is finite, there is a regime \( \ell' \) that will appear infinitely often in \( \{ \ell_i \} \). Let \( \ell_{\hat{n}} = \ell' \), for some \( \hat{n} > 1 \). After \( \hat{n} \) steps, the regime \( \ell' \) reappears, i.e. \( \ell_{\hat{n} + \hat{n}} = \ell' \). Then, by (A.59), we get
\[
v_{\ell_{\hat{n}},0}^r(x_0) = v_{\ell_{\hat{n}+1},0}^r(x_0) + \vartheta_{\ell_{\hat{n}+1},\ell_{\hat{n}+2}} + \cdots + \vartheta_{\ell_{\hat{n}+\hat{n}+1},\ell'}.
\] (A.60)

Notice that (A.60) contradicts the assumption that there is no loop of zero cost (see Eq. (1.10)). From here we concludes that (A.50) must occur. Taking \( \nu^r - u^r \) and proceeding in the same way as before, it follows that for each \( (\ell, i) \in \mathbb{M} \times \mathbb{I} \), \( u_{\ell,0}^r - u_{\ell,0}^r \leq 0 \) in \( \mathcal{O} \), and hence we conclude that the solution \( u^r \) to the HJB equation (1.17) is unique.

\[ \Box \]

### A.5 Proof of Proposition 1.1

**Proof of Proposition 1.1.** Existence. Now, let \( (\ell, i) \in \mathbb{M} \times \mathbb{I} \) be fixed. Since \( u_{\ell,0}^r \) is the unique strong solution to the HJB equation (1.17) when \( \varepsilon = \varepsilon_n \), which belongs to \( C^0(\overline{\mathcal{O}}) \), it follows that for each \( \ell' \in \mathbb{M} \setminus \{ \ell \} \), \( u_{\ell,0}^r - \left( u_{\ell',0}^r + \vartheta_{\ell',\ell} \right) \leq u_{\ell,0}^r - \mathcal{M}_{\ell,0} u_{\ell,0}^r \leq 0 \) in \( \mathcal{O} \). From here and (3.7), it yields that \( u_{\ell,0} - u_{\ell',0} - \vartheta_{\ell',\ell} \leq 0 \) in \( \mathcal{O} \). Then, \( u_{\ell,0} - \mathcal{M}_{\ell,0} u \leq 0 \), in \( \mathcal{O} \). Also, we know that \( [c_i - \mathcal{L}_{\ell,0}] u_{\ell,0}^r + \psi_{\varepsilon_n}(D^1 u_{\ell,0}^r - g_i^2) \leq h_i \) a.e. in \( \mathcal{O} \). Then,
\[
0 \leq \psi_{\varepsilon_n}(D^1 u_{\ell,0}^r - g_i^2) \leq h_i - [c_i - \mathcal{L}_{\ell,0}] u_{\ell,0}^r, \quad \text{a.e. in } \mathcal{O}.
\] (A.61)

Consequently, by (H3), (3.8) and (A.61), there exists a positive constant \( C_0 = C_0(d, \lambda, \alpha, \varepsilon') \) such that
\[ 0 \leq \int B_r \psi_{\varepsilon_n}(D^1 u_{\ell,0}^r - g_i^2) \varpi dx \leq \int B_r [h_i - [c_i - \mathcal{L}_{\ell,0}] u_{\ell,0}^r] \varpi dx \leq C_0 \] for each \( \varpi \in \mathcal{B}(B_r) \), with \( \mathcal{B}(\cdot) \) as in (A.43). Thus, using definition of \( \psi_{\varepsilon_n} \) (see (1.16)) and since \( D^1 u_{\ell,0}^r - g_i^2 \) is continuous in \( \mathcal{O} \), we have that for each \( B_r \subset \mathcal{O} \), there exists \( \varepsilon' \in (0, 1) \) such that for all \( \varepsilon_n \leq \varepsilon' \), \( |D^1 u_{\ell,0}^r - g_i| \leq 0 \) in \( B_r \). Then, since (3.7) holds, it follows that \( |D^1 u_{\ell,0}^r| \leq g_i \) in \( \mathcal{O} \). From (A.61), we get
\[ \int B_r [c_i - \mathcal{L}_{\ell,0}] u_{\ell,0}^r - h_i \varpi dx \leq 0, \quad \text{for each } \varpi \in \mathcal{B}(B_r). \] From here and (3.7), we obtain that \( [c_i - \mathcal{L}_{\ell,0}] u_{\ell,0} - h_i \leq 0 \) a.e. in \( \mathcal{O} \). Therefore, by the seen previously,
\[ \max \{ [c_i - \mathcal{L}_{\ell,0}] u_{\ell,0} - h_i, D^1 u_{\ell,0} - g_i, u_{\ell,0} - \mathcal{M}_{\ell,0} u \} \leq 0, \quad \text{a.e. in } \mathcal{O}. \] (A.62)

Without loss of generality we assume that \( u_{\ell,0}(x^*) - \mathcal{M}_{\ell,0} u(x^*) < 0 \), for some \( x^* \subset \mathcal{O} \). Otherwise, the equality is satisfied in (A.62). Then, for each \( \ell' \in \mathbb{M} \setminus \{ \ell \} \), \( u_{\ell,0} - (u_{\ell',0} + \vartheta_{\ell',\ell}) \leq u_{\ell,0} - \mathcal{M}_{\ell,0} u < 0 \) at \( x^* \). There exists a ball \( B_{r_1}(x^*) \subset \mathcal{O} \) such that
\[ u_{\ell,0} - (u_{\ell',0} + \vartheta_{\ell',\ell}) \leq u_{\ell,0} - \mathcal{M}_{\ell,0} u < 0, \quad \text{in } B_{r_1}(x^*) \] (A.63)
due to the continuity of \( u_{\ell,0} - u_{\ell',0} \) in \( \overline{\mathcal{O}} \). Now, consider that \( |D^1 u_{\ell,0} - g_i| < 0 \) for some \( x_1^* \subset B_{r_2}(x^*) \). Otherwise, the equality is also satisfied in (A.62). By continuity of \( |D^1 u_{\ell,0} - g_i| \), it yields that for some \( B_{r_2}(x_1^*) \subset \mathcal{O} \), \( |D^1 u_{\ell,0} - g_i| < 0 \) in \( B_{r_2}(x_1^*) \). From here, using (3.7), (A.63) and taking \( \mathcal{N} := B_{r_1}(x^*) \cap B_{r_2}(x_1^*) \), it can be verified that there exists an \( \varepsilon' \in (0, 1) \) small enough, such that for each \( \varepsilon_n \leq \varepsilon' \), \( |D^1 u_{\ell,0}^r - g_i| < 0 \) and \( u_{\ell,0}^r - \mathcal{M}_{\ell,0} u_{\ell,0}^r < 0 \) in \( \mathcal{N} \). Thus, \( [c_i - \mathcal{L}_{\ell,0}] u_{\ell,0}^r = h_i \) a.e. in \( \mathcal{N} \), since \( u_{\ell,0}^r \) is the unique solution to the HJB equation

\[ \Box \]
(1.17), when $\varepsilon = \varepsilon_n$. Then, $\int_N \{ [c_i - L_{\ell,i}] u_{\ell,i}^\varepsilon - h_{\ell,i} \} \varpi \, dx = 0$, for each $\varpi \in B(\mathcal{N})$. Hence, letting $\varepsilon_n \to 0$ and using again (3.7), we get that $u = (u_1, \ldots, u_m)$ is a solution to the HJB equation (1.12).

**Proof of Theorem 1.1. Uniqueness.** Suppose that

$$u = (u_{\ell,i})(\ell,i) \in \mathbb{M} \times \mathbb{I} \quad \text{and} \quad v = (v_{\ell,i})(\ell,i) \in \mathbb{M} \times \mathbb{I}$$

are two solutions to the HJB equation (1.12) whose components belong to $C^{0,1}(\mathcal{O}) \cap W^{2,\infty}_{\text{loc}}(\mathcal{O})$. Take $(x_0, \ell_0) \in \overline{\mathcal{O}} \times \mathbb{I}$ such that

$$u_{\ell_0,i_0}(x_0) - v_{\ell_0,i_0}(x_0) = \sup_{(x,\ell,i) \in \overline{\mathcal{O}} \times \mathbb{I}} \left\{ u_{\ell,i}(x) - v_{\ell,i}(x) \right\}.$$  \hspace{1cm} (A.64)

As before (see Subsection 3), we only need to verify that

$$u_{\ell_0,i_0} - v_{\ell_0,i_0} \leq 0, \quad \text{at} \; x_0 \in \mathcal{O}. \hspace{1cm} (A.65)$$

Assume that $u_{\ell_0,i_0} - v_{\ell_0,i_0} > 0$ at $x_0$. Then, there exists a ball $B_{r_1}(x_0) \subset \mathcal{O}$ such that $c_{i_0} [u_{\ell_0,i_0} - v_{\ell_0,i_0}] \geq \min_{x \in B_{r_1}(x_0)} [c_{i_0} (x) [u_{\ell_0,i_0}(x) - v_{\ell_0,i_0}(x)]] > 0$ in $B_{r_1}(x_0)$ due to the continuity of $u_{\ell_0,i_0} - v_{\ell_0,i_0}$ in $\overline{\mathcal{O}}$ and that $c_{i_0} > 0$ in $\mathcal{O}$. Meanwhile, from (A.64), $v_{\ell_0,i_0} - M_{\ell_0,i_0} v \leq u_{\ell_0,i_0} - M_{\ell_0,i_0} u \leq 0$ at $x_0$. If $v_{\ell_0,i_0} - M_{\ell_0,i_0} v < 0$ at $x_0$, there exists a ball $B_{r_2}(x_0) \subset \mathcal{O}$ such that $v_{\ell_0,i_0} - M_{\ell_0,i_0} v < 0$ in $B_{r_2}(x_0)$. Now, consider the auxiliary function $f_{\varrho} := u_{\ell_0,i_0} - \varrho u_{\ell_0,i_0}$, with $\varrho \in (0, 1)$. Notice that $f_{\varrho} = -\varrho f_{i_0} < 0$ on $\partial \mathcal{O}$, for $\varrho \in (0, 1)$, and

$$f_{\varrho} \uparrow u_{\ell_0,i_0} - v_{\ell_0,i_0} \quad \text{uniformly in} \; \mathcal{O}, \quad \text{when} \; \varrho \downarrow 0. \hspace{1cm} (A.66)$$

Besides, there is a $\varrho' \in (0, 1)$ small enough such that $\sup_{x \in B_{r_2}(x_0)} |f_{\varrho}(x)| > 0$ for all $\varrho \in (0, \varrho')$ because of $u_{\ell_0,i_0} - v_{\ell_0,i_0} > 0$ at $x_0$. By (A.64) and (A.66), there exists $\varrho \in (0, \varrho')$ small enough such that $f_{\varrho}$ has a local maximum at $x_{\varrho} \in B_{r_1}(x_0) \cap B_{r_2}(x_0)$. It follows that $|D^1 v_{\ell_0,i_0}(x_{\varrho})| = [1 - \varrho] |D^1 u_{\ell_0,i_0}(x_{\varrho})| < |D^1 u_{\ell_0,i_0}(x_{\varrho})| \leq g_i(x_{\varrho})$. Thus, there exists a ball $B_{r_2}(x_{\varrho}) \subset B_{r_1}(x_0) \cap B_{r_2}(x_0)$ such that $c_{i_0} [u_{\ell_0,i_0} - L_{\ell_0,i_0}] |u_{\ell_0,i_0} - h_{\ell,0} = 0$ and $c_{i_0} - L_{\ell_0,i_0}] u_{\ell_0,i_0} - h_{\ell_0} \leq 0$ in $B_{r_2}(x_{\varrho})$. Then, by Bony’s maximum principle, we have that $0 \geq \lim_{r \to 0} \inf \{ a_{i_0} D^2 f_{\varrho} \} \geq c_{i_0} f_{\varrho} + \varrho h_{\ell_0}$ at $x_{\varrho}$, which is a contradiction because of $\varrho h_{\ell_0} \geq 0$. Thus, we conclude that $0 = v_{\ell_0,i_0} - M_{\ell_0,i_0} v \leq u_{\ell_0,i_0} - M_{\ell_0,i_0} u \leq 0$ at $x_0$. Using the same arguments seen in the proof of uniqueness of the solution to the HJB equation (1.12) (see Subsection 3), it can be verified that there is a contradiction with the assumption that there is no loop of zero cost (see Eq. (1.10)). From here we conclude that (A.65) must occur. Taking $v = u$ and proceeding in the same way as before, we see $u$ is the unique solution to the HJB equation (1.12).

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