On Binary Cyclic Codes with Five Nonzero Weights

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Abstract

Let $q = 2^n$, $0 \leq k \leq n - 1$, $n / \gcd(n, k)$ be odd and $k \neq n/3, 2n/3$. In this paper the value distribution of following exponential sums

$$\sum_{x \in \mathbb{F}_q} (-1)^{\text{Tr}_q'(\alpha x^{2k+1} + \beta x^{k+1} + \gamma x)} \quad (\alpha, \beta, \gamma \in \mathbb{F}_q)$$

is determined. As an application, the weight distribution of the binary cyclic code $C$, with parity-check polynomial $h_1(x)h_2(x)h_3(x)$ where $h_1(x)$, $h_2(x)$ and $h_3(x)$ are the minimal polynomials of $\pi^{-1}$, $\pi^{-(2^k+1)}$ and $\pi^{-(2^{2k}+1)}$ respectively for a primitive element $\pi$ of $\mathbb{F}_q$, is also determined.

Index terms: Exponential sum, Cyclic code, Moment identity, Weight distribution, Sequence
1 Introduction

Basic knowledge on finite fields could be found in [16]. The following notations are fixed throughout this paper except for specific statements.

- Let \( n \) be a positive integer, \( q = 2^n \), \( \mathbb{F}_q \) be the finite field of order \( q \). Let \( \pi \) be a primitive element of \( \mathbb{F}_q \).

- Let \( \text{Tr}_i^j : \mathbb{F}_{2^i} \to \mathbb{F}_{2^j} \) be the trace mapping, and \( \chi(x) = (-1)^{\text{Tr}_i^1(x)} \) be the canonical additive character on \( \mathbb{F}_q \).

- Let \( k \) be a positive integer, \( 1 \le k \le n-1 \) and \( k \notin \{ \frac{n}{3}, \frac{2n}{3} \} \). Let \( d = \gcd(n, k) \), \( q_0 = 2^d \), and \( s = n/d \). Assume \( s \) is odd.

For cyclic code \( C \) with length \( l \), let \( A_i \) be the number of codewords in \( C \) with Hamming weight \( i \). The weight distribution \( \{ A_0, A_1, \ldots, A_l \} \) is an important research object for both theoretical and application interests in coding theory. Classical coding theory reveals that the weight of each codeword can be expressed by exponential sums so that the weight distribution of \( C \) can be determined if the corresponding exponential sums can be calculated explicitly (see Feng and Luo [8], Kasami [15], Luo and Feng [17], Moisio [21], van der Vlught [27], Wolfmann [28], Zeng, Liu and Hu [31]).

More precise speaking, let \( q = 2^n \), \( C \) be the binary cyclic code with length \( l = q - 1 \) and parity-check polynomial

\[
h(x) = h_1(x) \cdots h_u(x) \quad (u \ge 1)
\]

where \( h_i(x) \) \( (1 \le i \le u) \) are distinct irreducible polynomials in \( \mathbb{F}_2[x] \) with the same degree \( e_i \) \( (1 \le i \le u) \), then \( \dim_{\mathbb{F}_2} C = \sum_{i=1}^u e_i \). Let \( \pi^{-s_i} \) be a zero of \( h_i(x) \), \( 1 \le s_i \le q - 2 \) \( (1 \le i \le u) \). Then the codewords in \( C \) can be expressed by

\[
c(\alpha_1, \ldots, \alpha_u) = (c_0, c_1, \cdots, c_{l-1}) \quad (\alpha_1, \cdots, \alpha_u \in \mathbb{F}_q)
\]

where \( c_i = \sum_{\lambda=1}^u \text{Tr}_i^n(\alpha_\lambda \pi^{i\lambda}) \) \( (0 \le i \le l-1) \). Therefore the Hamming weight of the codeword \( c = c(\alpha_1, \cdots, \alpha_u) \) is

\[
w_H(c) = 2^{n-1} - \frac{1}{2} S(\alpha_1, \cdots, \alpha_u) \quad (1)
\]
where \( f(x) = \alpha_1x^{s_1} + \alpha_2x^{s_2} + \cdots + \alpha_u x^{s_u} \in \mathbb{F}_q[x] \), \( \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\} \), and
\[
S(\alpha_1, \cdots, \alpha_u) = \sum_{x \in \mathbb{F}_q} (-1)^{T_\chi(\alpha_1x^{s_1}+\cdots+\alpha_u x^{s_u})}.
\]

In this way, the weight distribution of cyclic code \( C \) can be derived from the explicit evaluating of the exponential sums \( S(\alpha_1, \cdots, \alpha_u) \) \( (\alpha_1, \cdots, \alpha_u \in \mathbb{F}_q) \).

Let \( h_1(x) \), \( h_2(x) \) and \( h_3(x) \) be the minimal polynomials of \( \pi^{-1}, \pi^{-(2^k+1)} \) and \( \pi^{-(2^k+1)} \) over \( \mathbb{F}_2 \) respectively. Then \( \deg h_i(x) = n \) for \( i = 1, 2, 3 \). Let \( C \) be the binary cyclic codes with length \( l = q - 1 \) and parity-check polynomials \( h_1(x)h_2(x)h_3(x) \). It is a consequence that \( C \) is the dual of the binary cyclic code whose defining zeroes are \( \pi^{2^{2k+1}}, \pi^{2^k+1} \) and \( \pi \). Then the dimensions of \( C \) is \( 3n \) by excluding several special cases: \( k = n/3, 2n/3 \).

For \( \alpha, \beta, \gamma \in \mathbb{F}_q \), define the exponential sum
\[
S(\alpha, \beta, \gamma) = \sum_{x \in \mathbb{F}_q} \chi \left( \alpha x^{2^{2k+1}} + \beta x^{2^k+1} + \gamma x \right).
\]

Then the complete weight distributions of \( C \) can be derived from the explicit evaluation of \( S(\alpha, \beta, \gamma) \).

For \( k = 1 \), \( C^\perp \) or its extended code, is the well-known triple-error correcting BCH code which has been extensively studied. For instance,

1. The weight distribution of \( C \) has been calculated, see MacWilliams and Sloane\(^{[20]} \), pp.669, Kasami\(^{[15]} \) for \( n \) odd and Berlekamp\(^{[2]} \) for \( n \) even.

2. The covering radius of \( C \) is 5, which has been proven in Assmus and Mattson\(^{[1]} \).

3. The coset distribution of \( C^\perp \) has been determined, see Charpin, Helleseth and Zinoviev\(^{[3]} \).

4. The weight of coset leaders to the extended code of \( C^\perp \) has been studied, see Charpin, Helleseth and Zinoviev\(^{[4]}-^{[5]} \) and Charpin, Zinoviev\(^{[6]} \).

Let \( a = (a_\lambda)^{2^n-2}_{\lambda=0} \) and \( b = (b_\lambda)^{2^n-2}_{\lambda=0} \) be two \( m \)-sequences with period \( q - 2 \). The correlation function of \( a \) and \( b \) for a shift \( \tau \) is defined by
\[
M_{a,b}(\tau) = \sum_{\lambda=0}^{2^n-2} (-1)^{a(\lambda)-b(\lambda+\tau)} \quad (0 \leq \tau \leq q - 2).
\]
Binary sequences with low cross correlation and auto-correlation are widely used in Code Division Multiple Access (CDMA) spread spectrum (see Simon, Omura and Scholtz [26]). Pairs of binary \(m\)-sequences with few-valued correlations have been extensively studied for several decades, see Dobbertin, Felke, Helleseth and Rosendahl [7], Helleseth, Kholosha and Ness [10], Helleseth and Kumar [11], Hu, Zeng, Li and Jiang [12], Johansen, Helleseth and Tang [14], Ness and Helleseth [23]–[22], Niho [24], Rosendahl [25], Yu and Gong [29]–[30] and references therein. Recently, the exponential sum \(S(\alpha, \beta, 0)\) with \(n/d\) odd has been studied, in terms of certain combination of exponential sums, see Johansen and Helleseth [13]. In particular, for the case \(k = 1\) and \(n\) odd, the five-valued correlation distribution between two \(m\)-sequences has been determined.

Based on the exponential sum \(S(\alpha, \beta, \gamma)\), we could define a family of \(m\)-sequences \(\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2\) with

\[
\mathcal{F}_1 = \left\{ \left( \text{Tr}^n_1(\alpha \pi^{(2k+1)} + \beta \pi^{(2k+1)} + \pi^\lambda) \right)_{\lambda=0}^{q-2} \big| \alpha, \beta \in \mathbb{F}_q \right\}
\]

and

\[
\mathcal{F}_2 = \left\{ \left( \text{Tr}^n_1(\alpha \pi^{(2k+1)} + \pi^{(2k+1)}) \right)_{\lambda=0}^{q-2} \big| \beta \in \mathbb{F}_q \right\} \cup \left\{ \left( \text{Tr}^n_1(\pi^{(2k+1)}) \right)_{\lambda=0}^{q-2} \right\}.
\]

The sequence family \(\mathcal{F}\) has family size \(2^{2n} + 2^n + 1\).

In this paper, we will focus on the case \(n/d\) is odd and it is presented as follows. In section 2 we introduce some preliminaries. In section 3 we will determine the value distribution of \(S(\alpha, \beta, \gamma)\), and at the same time, the weight distribution of \(\mathcal{C}\). As a corollary, the possible correlation values among the sequences in \(\mathcal{F}\) can be obtained. But unfortunately, we could not determine the correlation distribution. The main techniques we will employ are binary quadratic form theory and the third-power moment identities of \(S(\alpha, \beta, \gamma)\).

## 2 Preliminaries

We follow the notations in section 1. The first technique is quadratic form theory over \(\mathbb{F}_{q_0}\).

Let \(H\) be an \(s \times s\) matrix over \(\mathbb{F}_{q_0}\). For the quadratic form

\[
F : \mathbb{F}_{q_0}^s \to \mathbb{F}_{q_0}, \quad F(x) = XHX^T \quad (X = (x_1, \cdots, x_s) \in \mathbb{F}_{q_0}^s), \quad (3)
\]
define $r_F$ of $F$ to be the rank of the skew-symmetric matrix $H + H^T$. Then $r_F$ is even. We have the following result on the exponential sum of binary quadratic forms (see [19]).

**Lemma 1.** For the quadratic form $F = XHX^T$ defined in (3),

$$\sum_{X \in \mathbb{F}_{q_0}^s} (-1)^{\text{Tr}_d^q(F(X))} = \pm q_0^{s - \frac{r_F}{2}} \text{ or } 0$$

Moreover, if $r_F = s$, then

$$\sum_{X \in \mathbb{F}_{q_0}^s} (-1)^{\text{Tr}_d^q(F(X))} = \pm q_0^s$$

The following result will be used in the study of $S(\alpha, \beta, \gamma)$ (see [27]).

**Lemma 2.** For the fixed quadratic form defined in (3), the value distribution of

$$\sum_{X \in \mathbb{F}_{q_0}^s} (-1)^{\text{Tr}_d^q(F(X) + AX^T)}$$

when $A$ runs through $\mathbb{F}_{q_0}^s$ is shown as following

| value          | multiplicity            |
|----------------|-------------------------|
| 0              | $q_0^s - q_0^{r_F}$     |
| $q_0^{s - \frac{r_F}{2}}$ | $\frac{1}{2}(q_0^{r_F} + q_0^{r_F})$ |
| $-q_0^{s - \frac{r_F}{2}}$ | $\frac{1}{2}(q_0^{r_F} - q_0^{r_F})$ |

Note that the field $\mathbb{F}_q$ is a vector space over $\mathbb{F}_{q_0}$ with dimension $s$. For fixed basis $v_1, \cdots, v_s$ of $\mathbb{F}_q$ over $\mathbb{F}_{q_0}$, each $x \in \mathbb{F}_q$ can be uniquely expressed as

$$x = x_1v_1 + \cdots + x_s v_s \quad (x_i \in \mathbb{F}_{q_0}).$$

Thus we have the following $\mathbb{F}_{q_0}$-linear isomorphism:

$$\mathbb{F}_q \xrightarrow{\sim} \mathbb{F}_{q_0}^s, \quad x = x_1v_1 + \cdots + x_s v_s \mapsto X = (x_1, \cdots, x_s).$$

With this isomorphism, a function $f : \mathbb{F}_q \to \mathbb{F}_{q_0}$ induces a function $F : \mathbb{F}_{q_0}^s \to \mathbb{F}_{q_0}$ where for $X = (x_1, \cdots, x_s) \in \mathbb{F}_{q_0}^s, F(X) = f(x)$ with $x = x_1v_1 + \cdots + x_s v_s$. In this way, function $f(x) = \text{Tr}_d^q(\gamma x)$ for $\gamma \in \mathbb{F}_q$ induces a linear form

$$F(X) = \text{Tr}_d^q(\gamma x) = \sum_{i=1}^{s} \text{Tr}_d^q(\gamma v_i)x_i = A_\gamma X^T$$  \hspace{1cm} (4)
where \( A_\gamma = (\text{Tr}_d^n(\gamma v_1), \cdots, \text{Tr}_d^n(\gamma v_s)) \), and \( f_{\alpha,\beta}(x) = \text{Tr}_d^n(\alpha x^{p^{2k}+1} + \beta x^{p^k+1}) \) for \( \alpha, \beta \in \mathbb{F}_q \) induces a quadratic form \( F_{\alpha,\beta}(X) = XH_{\alpha,\beta}X^T \).

From Lemma 3, we need to determine the rank of \( A \alpha, \beta \) of \( H_{\alpha,\beta} + H_{\alpha,\beta}^T \) over \( \mathbb{F}_{q_0} \).

**Lemma 3.** For \( \alpha, \beta \in \mathbb{F}_q \) and \((\alpha, \beta) \neq (0,0), \) let \( r_{\alpha,\beta} \) be the rank of \( H_{\alpha,\beta} + H_{\alpha,\beta}^T \). Then the possible values of \( r_{\alpha,\beta} \) are \( s-1 \) and \( s-3 \).

**Proof.** For \( Y = (y_1, \cdots, y_s) \in \mathbb{F}_{q_0}^s, \) \( y = y_1 v_1 + \cdots + y_s v_s \in \mathbb{F}_q \), we know that

\[
F_{\alpha,\beta}(X + Y) - F_{\alpha,\beta}(X) - F_{\alpha,\beta}(Y) = 2XH_{\alpha,\beta}Y^T
\]

(5)

is equal to

\[
f_{\alpha,\beta}(x + y) - f_{\alpha,\beta}(x) - f_{\alpha,\beta}(y) = \text{Tr}_d^n(y^{2^{2k}}(\alpha^{2^{2k}} x^{2^{2k}} + \beta^{2^{2k}} x^{2^{2k}} + \beta^{2^{2k}} x^{2^{2k}} + \alpha x))
\]

(6)

Let

\[
\phi_{\alpha,\beta}(x) = \alpha^{2^{2k}} x^{2^{2k}} + \beta^{2^{2k}} x^{2^{2k}} + \beta^{2^{2k}} x^{2^{2k}} + \alpha x.
\]

(7)

Therefore,

\[ r_{\alpha,\beta} = r \Leftrightarrow \text{the number of common solutions of } XH_{\alpha,\beta}Y^T = 0 \text{ for all } Y \in \mathbb{F}_{q_0}^s \text{ is } q_0^{s-r}, \]

\[ \Leftrightarrow \text{the number of common solutions of } \text{Tr}_d^n(y^{2^{2k}} \cdot \phi_{\alpha,\beta}(x)) = 0 \text{ for all } y \in \mathbb{F}_q \text{ is } q_0^{s-r}, \]

\[ \Leftrightarrow \phi_{\alpha,\beta}(x) = 0 \text{ has } q_0^{s-r} \text{ solutions in } \mathbb{F}_q. \]

For a fixed algebraic closure \( \mathbb{F}_{2^\infty} \) of \( \mathbb{F}_2 \), since the degree of \( 2^{2k} \)-linearized polynomial \( \phi_{\alpha,\beta}(x) \) is \( 2^{4k} \) and \( \phi_{\alpha,\beta}(x) = 0 \) has no multiple roots in \( \mathbb{F}_{2^\infty} \) (this fact follows from \( \phi'_{\alpha,\beta}(x) = \alpha \in \mathbb{F}_q^* \)), then the zeroes of \( \phi_{\alpha,\beta}(x) \) in \( \mathbb{F}_{2^\infty} \), say \( V \), form an \( \mathbb{F}_{2^k} \)-vector space of dimension 4. Then \( V \cap \mathbb{F}_{2^n} \) is a vector space on \( \mathbb{F}_{2^{n(d+4)}} \) of dimension less that or equal to 4 since any elements in \( \mathbb{F}_{2^n} \) which are linear independent over \( \mathbb{F}_{2^d} \) are also linear independent over \( \mathbb{F}_{2^k} \) (see [?], Lemma 4). Note that \( s \) is odd and \( r_{\alpha,\beta} \) is always even. Hence the possible values of \( r_{\alpha,\beta} \) are \( s-1 \) and \( s-3 \).

Another technique to determine the value distribution of \( S(\alpha, \beta, \gamma) \) is the third-power moment identity of \( S(\alpha, \beta, \gamma) \).
Lemma 4. For the exponential sum and $S(\alpha, \beta, \gamma)$, we have
\[
\sum_{\alpha, \beta, \gamma \in \mathbb{F}_q} S(\alpha, \beta, \gamma)^3 = (2^{n+d} + 2^n - 2^d) \cdot 2^{3n}.
\]
Proof. We can calculate
\[
\sum_{\alpha, \beta, \gamma \in \mathbb{F}_q} S(\alpha, \beta, \gamma)^3 = \sum_{x,y,z \in \mathbb{F}_q} \sum_{\alpha \in \mathbb{F}_q} \chi(\alpha (x^{22k+1} + y^{22k+1} + z^{22k+1}))) \sum_{\beta \in \mathbb{F}_q} \chi(\beta (x^{2k+1} + y^{2k+1} + z^{2k+1})) \sum_{\gamma \in \mathbb{F}_q} \chi(\gamma (x + y + z))
\]
\[
= M_3 \cdot 2^{3n}
\]
where $M_3$ is the number of solutions to the system of equations
\[
\begin{aligned}
x + y + z &= 0 \\
x^{2k+1} + y^{2k+1} + z^{2k+1} &= 0 \\
x^{22k+1} + y^{22k+1} + z^{22k+1} &= 0
\end{aligned}
\] (8)

- If $xyz = 0$, we may assume $x = 0$ and then $y = z$ which gives $2^n$ solutions. So is $y = 0$ or $z = 0$. Note that $x = y = z = 0$ has been counted 3 times. Hence there are exactly $3 \cdot 2^n - 2$ solutions to (8) satisfying $xyz = 0$.

- If $xyz \neq 0$, then the number of solutions to (8) is equal to $2^n - 1$ multiple of that to the system of equations
\[
x^{2k+1} + y^{2k+1} + 1 = x^{2k+1} + y^{2k+1} + 1 = x + y + 1 = 0
\]
(9)
with $xy \neq 0$. By (8) we have $x^{2k+1} + (x + 1)^{2k+1} + 1 = x^{2k+1} + (x + 1)^{2k+1} + 1 = 0$ which is equivalent to $x^{2k} = x$. Hence $x \in \mathbb{F}_{2^k} \cap \mathbb{F}_{2^n} = \mathbb{F}_{2^d}$ and $y = x + 1$. Since $y \neq 0$, then $x \neq 1$. Therefore (8) has $2^d - 2$ solutions with $xy \neq 0$.

In total, we get $M_3 = 3 \cdot 2^n - 2 + (2^d - 2)(2^n - 1) = 2^{n+d} + 2^n - 2^d$. \qed

3 The Weight Distribution of the Cyclic Code $\mathcal{C}$

In the sequel we will give the the value distribution of $S(\alpha, \beta, \gamma)$ and, at the same time, the weight distribution of binary cyclic code $\mathcal{C}$.
**Theorem 1.** The value distribution of the multi-set \( \{ S(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in \mathbb{F}_q \} \) and the weight distribution of \( C \) are shown as following (Column 1 is the value of \( S(\alpha, \beta, \gamma) \), Column 2 is the weight of \( c(\alpha, \beta, \gamma) \) and Column 3 is the corresponding multiplicity).

| value    | weight    | multiplicity |
|----------|-----------|--------------|
| \( 2(n+d)/2 \) | \( 2^{n-1} - 2(n+d-2)/2 \) | \( \frac{(2^{n-d-1}+2(n-d-2)/2)(2^n-1)(2^n+2d-2n^d+2d)}{2^{2d+1}} \) |
| \( -2(n+d)/2 \) | \( 2^{n-1} + 2(n+d-2)/2 \) | \( \frac{(2^{n-d-1}-2(n-d-2)/2)(2^n-1)(2^n+2d-2n^d+2d)}{2^{2d+1}} \) |
| \( 2(n+3d)/2 \) | \( 2^{n-1} - 2(n+3d-2)/2 \) | \( \frac{(2^{n-3d-1}+2(n-3d-2)/2)(2^n-d-1)(2^n-1)}{2^{2d+1}} \) |
| \( -2(n+3d)/2 \) | \( 2^{n-1} + 2(n+3d-2)/2 \) | \( \frac{(2^{n-3d-1}-2(n-3d-2)/2)(2^n-d-1)(2^n-1)}{2^{2d+1}} \) |
| 0        | \( 2^{n-1} \) | \( (2^n-1)(2^{2n-2n^d-2n^d+2n^d-2n^d}) \) |
| \( 2^n \) | 0         | 1            |

**Proof.** Let \( n_1 \) to be the number of pairs \((\alpha, \beta) \in \mathbb{F}_q \setminus \{(0,0)\} \) such that \( r_{\alpha,\beta} = s - i \). Define

\[
\Xi = \{ (\alpha, \beta, \gamma) \in F_q^3 \mid S(\alpha, \beta, \gamma) = 0 \}
\]

and \( \xi = |\Xi| \).

Since \( n/d \) is odd, from Lemma 2, Lemma 3 and Lemma 4 we have

\[
n_1 + n_3 = 2^{2n} - 1 \tag{10}
\]

\[
n_1 + 2^{2d} \cdot n_3 = 2^{n-d}(2^d + 1)(2^n - 1). \tag{11}
\]

These two equations yield

\[
n_1 = \frac{(2^n-1)(2^{n+2d}-2^n-2n^d+2d^2)}{2^{2d+1}}, \quad n_3 = \frac{(2^{n-d}-1)(2^n-1)}{2^{2d+1}} \tag{12}
\]

Note that \( S(0,0,\gamma)=0 \) unless \( \gamma = 0 \). From Lemma 2 we get that

\[
\xi = 2^n - 1 + (2^n - 2n^d)n_1 + (2^n - 2n^d)n_3
= (2^n - 1)(2^{2n-2n^d} + 2n - 2n^d) + 2^n - 2n^d - 2n^3d + 1 \tag{13}
\]

Then the result follows from (12), (13) by using Lemma 2.
We hereby could give the possible values of correlation function among sequences in $\mathcal{F}$. For example, let $a_{\alpha_1, \beta_1} = \left( \text{Tr}_n^q(\alpha_1 \pi^{\lambda(2k+1)} + \beta_1 \pi^{\lambda(2k+1)} + \pi^\lambda) \right)_{\lambda=0}^{q-2}$ and $a_{\alpha_2, \beta_2} = \left( \text{Tr}_1^n(\alpha_2 \pi^{\lambda(2k+1)} + \beta_2 \pi^{\lambda(2k+1)} + \pi^\lambda) \right)_{\lambda=0}^{q-2}$. Then the correlation function of $a_{\alpha_1, \beta_1}$ and $a_{\alpha_2, \beta_2}$ by a shift $\tau$ ($0 \leq \tau \leq q - 2$) is

$$C_{(\alpha_1, \beta_1), (\alpha_2, \beta_2)}(\tau) = \sum_{\lambda=0}^{q-2} (-1)^{a_{\alpha_1, \beta_1}(\lambda)-a_{\alpha_2, \beta_2}(\lambda+\tau)}$$

$$= \sum_{\lambda=0}^{q-2} (-1)^{\text{Tr}_n^q(\alpha_1 \pi^{\lambda(2k+1)} + \beta_1 \pi^{\lambda(2k+1)} + \pi^\lambda)-\text{Tr}_1^n(\alpha_2 \pi^{\lambda+\tau}(2k+1) + \beta_2 \pi^{\lambda+\tau}(2k+1) + \pi^\lambda+\tau)}$$

$$= S(\alpha', \beta', \gamma') - 1$$

where

$$\alpha' = \alpha_1 - \alpha_2 \pi^{\tau(2k+1)}, \quad \beta' = \beta_1 - \beta_2 \pi^{\tau(2k+1)}, \quad \gamma' = 1 - \pi^\tau. \quad (14)$$

**Remark.**

As a corollary, we have

**Corollary 1.** The non-trivial correlation values of the sequences in $\mathcal{F}$ is $-1$, $\pm 2^{n/2} - 1$ and $\pm 2^{n/2} - 2$.

### 4 Conclusion and Further Study

In this paper we have studied the exponential sums $S(\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \in \mathbb{F}_{2^n}$. After giving the value distribution of $S(\alpha, \beta, \gamma)$, we determine the weight distributions of the cyclic codes $C$.

For the case $n/d$ even, we could get the possible values of $S(\alpha, \beta, 0)$ and $S(\alpha, \beta, \gamma)$. But the first four moment identities $\sum_{\alpha, \beta \in \mathbb{F}_q} S(\alpha, \beta, 0)^i$ and $\sum_{\alpha, \beta \in \mathbb{F}_q} S(\alpha, \beta, \gamma)^i$ for $0 \leq i \leq 3$ is not enough to determine the value distribution of $S(\alpha, \beta, 0)$ and $S(\alpha, \beta, \gamma)$. However, we could get the possible non-trival weights of $C$: $2^{n-1}$, $2^n - 1 \pm \frac{2^n}{2} + d - 1$ and $2^{n-1} \pm \frac{2^n}{2} + 2d - 1$. New machinery and technique should be proposed to attack this problem.

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