The Optimal Environment Selection Strategy of Risk Model With Perturbed Diffusion

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Abstract. In this paper, the authors consider the optimal investment of the risk model with perturbed diffusion. Insurance companies invest the surplus in risky asset and risk-free asset. This paper discusses the problem of minimizing the ruin probability of insurance company. By solving the corresponding Hamilton-Jacobi-Bellman equations, the optimal investment portfolio and the upper bound of Lundberg of the minimal ruin probability are obtained. Especially, when the claim distribution is exponential distribution, asymptotic optimality and asymptotic uniqueness of strategy A and R are obtained.

1. Introduction
In order to ensure the interests of policyholders, insurance companies should minimize the ruin probability. So, finding the optimal investment is a hot issue in risk theory. Browne[1] first considered this issue, obtained the display solutions of the optimal investment strategy and the minimal ruin probability under the terms of surplus process was perturbed Brownian motion, risky asset was only one kind, claims followed a geometric Brownian motion and no reinsurance. Taksarand Markus-sen [2], LUO [3] reconsidered the model in Browne[1] in the case of making proportional reinsurance for compensation, also obtained clear expressions of optimal strategies and optimal value function. Scheer N, Schmidli H [4] considered a classical risk model with dividend payments and capital injections in the presence of both fixed and proportionals administration costs. Liu X, Chen Z, Ming R [5] considered the dividend problems in the perturbed compound Poisson risk model. Hipp and Plum [6] considered the traditional risk model and supposed surplus process could invest one kind of risky asset that the price followed a geometric Brownian motion, obtained the Hamilton-Jacobi-Bellman equations of minimal ruin probability in the case of none reinsurance, proved the existence of solutions and presented numerical results. LIU and YANG [7] reconsidered the model in Plum[6] in the case of surplus could invest no risky asset. Mataramvura S, Ksendal, Bernt [8] considered the effect of demand on price where the objective of the pricing model is to maximize the expected utility of the insurer's terminal wealth. Mao H, Carson J M, Ostaszewski K M, et al [9] considered the effect of demand on price where the objective of the pricing model is to maximize the expected utility of the insurer's terminal wealth. YANG and ZHANG [10] considered the optimal investment problem of jump-diffusion model, discussed the numerical results of the optimal investment and the minimal ruin probability under the terms of claim settlement followed exponential distribution, and Gamma distribution. Yuen K, Lu Y, Wu R[11] considered the compound Poisson process perturbed by a diffusion with a threshold dividend.
strategy. This paper consider the classical risk model with perturbed diffusion and dividends, research the optimal investment problem, and obtain the optimal investment proportion of minimal ruin probability. When the claim distribution is exponential distribution, asymptotic optimality and asymptotic uniqueness of strategy $A^*$ and $R$ are obtained.

2. Model Introduction
Let $(\Omega, A, P)$ be a complete probability space containing all objects defined at the following. We consider the continuous time risk time:

$$U(t) = u + (c - d)t - \sum_{i=1}^{N(t)} X_i + \delta W(t)$$  \hspace{1cm} (1)$$

where $u$ is the initial reserve of the insurance company, the premiums are received continuously at a constant rate $c$, per unit time; $d$ is dividend payout ratio, $N(t)$ is Poisson process with intensity $\lambda$, $\{X_i\}_{i=1}^{\infty}$ are non-negative, i.i.d., the distribution function of $Y(t) = \sum_{i=1}^{N(t)} X_i$ is $F(x)$, $Y(t)$ and $\{X_i\}_{i=1}^{\infty}$ are i.i.d, $\delta$ is constant, $W(t)$ is a standard Brownian motion, which is independent with $Y(t)$.

We consider the investment of insurance company when surplus is $X(t)$ at the moment of $t$. We assume the risky asset is $(1 - a)X(t)$ $(0 < a < 1)$, rate of return is $r_0$ $(r_0 > 0)$, the percentage of the risk-free asset is $a$, the situation invested in risk project $S(t)$ is

$$dS(t) = \mu S(t)dt + \delta S(t)dB(t), t \geq 0$$  \hspace{1cm} (2)$$

where $\mu$, $\delta > 0$ are fixed constants; $\mu$ is the risk expectation; $\delta$ is volatility of risk; $B(t)$ is a standard Brownian motion, which is independent with $U(t)$.

As a result

$$dX(t) = dU(t) + a\mu X(t)dt + (1 - a)r_0 X(t)dt + a\delta X(t)dB(t)$$  \hspace{1cm} (3)$$

Thus $X_0 = u, u > 0$, $t \geq 0$, the correlation between $B(t)$ and $W(t)$ is

$$dW(t)dB(t) = \rho dt$$  \hspace{1cm} (4)$$

Definition of the ruin probability is

$$\Psi(x) = P(T < \infty \mid X_0 = x)$$  \hspace{1cm} (5)$$

where $T = \inf\{t : X_t < 0\}$ is a stopping time.

There are many researches on model (1), this paper will obtain the HJB equations of ruin probability by dynamic programming method in literature [12], then obtain the optimal risky investment and an upper bound of Lundberg for ruin probability.

3. Intgerodifferential Equation of Ruin Probability
Suppose ruin time of the surplus process that with initial surplus $x$ and investment strategy $A$ is

$$\tau(x, A) = \inf\{t \geq 0 : X(t, A) \leq 0\}$$  \hspace{1cm} (6)$$
ruin probability is

\[ \Psi(x, A) = P(\tau(x, A) < \infty) \]  

(7)

admissible set of research strategy is \( A, A \in \Lambda \), \( A \) is \( \sigma \)-manifold measurable.

We can obtain the HJB equations of ruin probability \( \Psi(x) \) through formula and regularization method:

\[
\inf \left\{ \frac{1}{2} \left[ \delta_1^2 + 2 \rho \delta_1 \delta_2 A + \delta_2^2 A^2 \right] \Psi'(x) + \left[ (\mu - r_0)A + c - d + r_0 \mu \right] \Psi(x) + \lambda E[\Psi(u-x) - \Psi(u)] \right\} = 0
\]

(8)

the boundary conditions are \( \Psi(0) = 1 \), \( \Psi(\infty) = 0 \).

The optimal investment strategy can be got through equation (8):

\[
A^*(u) = \frac{-(\mu - r_0)\Psi'(u) - \rho \delta_1 \delta_2 \Psi'(u)}{\delta_2^2 \Psi'(u)}
\]

(9)

4. Upper Bound of Lundberg

In order to discuss the Lundberg upper bound, we suppose the distribution function of the claim settlement followed is exponential decline, which means moment generating function:

\[
M(r) = E[e^{rY}] = \int_0^\infty e^{ry} dF(y)
\]

(10)

We assume that existing \( r_0 \in (0, \infty) \), then \( r \uparrow r_0 \), \( M(r) \uparrow \infty \). It is easy to know that \( M(0) = 1 \) and \( M \) is a continuously increasing convex function.

Assuming the solution of equation (2) is \( \Psi(x) = e^{-Rx}, x \geq 0 \), and supposing \( R(A) \) is the solution of equation \( \frac{1}{2} \left( \delta_1^2 + 2 \rho \delta_1 \delta_2 A + \delta_2^2 A^2 \right) r^2 - \left[ (\mu - r_0)A + c - d + r_0 \mu \right] r + \lambda[M(r) - 1] = 0 \), then it is the definition of Lundberg index that with investment strategy \( A \) (constant), where Lundberg index is defined as \( R = \sup R(A) \). If strategy \( A \) is optimal, then the ruin probability achieve a minimum under this strategy. For when Lundberg index is larger and initial funding increased, the speed of ruin probability decline becomes faster, so we can obtain the asymptotic optimal investment strategy through the method that find the largest Lundberg index.

Hence \( R \) is the solution of equation below

\[
\inf \left\{ \frac{1}{2} \left[ \delta_1^2 + 2 \rho \delta_1 \delta_2 A + \delta_2^2 A^2 \right] r^2 - \left[ (\mu - r_0)A + c - d + r_0 \mu \right] r + \lambda[M(r) - 1] \right\} = 0
\]

(11)

denotes the left of equal as

\[
f(A, r) = \frac{1}{2} \left( \delta_1^2 + 2 \rho \delta_1 \delta_2 A + \delta_2^2 A^2 \right) r^2 - \left[ (\mu - r_0)A + c - d + r_0 \mu \right] r + \lambda[M(r) - 1]
\]

(12)

Note: For any \( A \geq 0 \), \( f(A, r) \) is a convex function with variable \( R \). Since \( R \geq R(A) \), \( f(A, r) \geq f(A, R(A)) = 0 \). Therefore optimal investment strategy \( A^* \), corresponds to the largest
Lundberg index \( r \) satisfies \( f(A, r) \geq f(A^*(u), R) = 0 \), that means \( R \) is the solution of equation \( \inf\{f(A, r)\} = 0 \).

Now what we need to do is to find \( R \) and \( A^* \). Assuming \( R \) is known, then

\[
f(A, R) = \frac{1}{2}(\delta_1^2 + 2\rho\delta_1\delta_2 A + \delta_2^2 A)R^2 - [(\mu - r_0)A + c - d + r_0\mu]R + \lambda[M(R) - 1] \tag{13}
\]

If and only if \( A^* = \frac{\mu - \rho\delta_1\delta_2 R}{\delta_2^2 R^2} \) that \( f(A, r) \) becomes minimum.

We put \( A^* \) into \( f(A, R) \), it shows that \( R \) is the solution of the equation in the following:

\[
\frac{1}{2}(1 - \rho^2) \delta_1^2 r^2 + \lambda M(r) = \lambda + cr + \frac{\mu^2}{2\delta_2^2} - \frac{\mu \rho \delta_1 r}{\delta_2} \tag{14}
\]

**Lemma 1** Suppose \( x \geq 0, \mu \neq 0, \delta_2 \neq 0 \), then there exists a unique \( r, 0 < r < r_0 \), satisfied equation below:

\[
\frac{1}{2}(1 - \rho^2) \delta_1^2 r^2 + \lambda M(r) = \lambda + cr + \frac{\mu^2}{2\delta_2^2} - \frac{\mu \rho \delta_1 r}{\delta_2} \tag{15}
\]

**Proof:** Denoting \( h_1(r) = \frac{1}{2}(1 - \rho^2) \delta_1^2 r^2 + \lambda M(r), h_2(r) = \lambda + cr + \frac{\mu^2}{2\delta_2^2} - \frac{\mu \rho \delta_1 r}{\delta_2} \), then \( h_1(0) = \lambda < h_2(0) = \lambda + \frac{\mu^2}{2\delta_2^2} \).

Obviously, \( h_1'(r) > 0 \), and \( \lim_{r \to r_0} h_1(r) = \infty \). Therefore there exists a unique \( r \), such that \( h_1(r) = h_2(r) \).

The proof completes.

**Lemma 2** \( R \) and \( A^* \) are defined as above, then \( M_t := \exp\{-RX_{t,A^*}^\tau\} \) is martingale.

**Proof:** Obviously, \( M_t \) adapts to \( F_t \). On the other hand, for any \( 0 < s \leq t < \infty \),

\[
E[M_t|F_s] = E[\exp\{-RX_{t,A^*}^\tau\}|F_s]
= E[\exp\{-R[X_{t}^{x,A^*} - X_{s}^{x,A^*}]\} \exp\{-RX_{s,A^*}^\tau\}|F_s]
= M_s \exp\{f(A^*, R)(t-s)\}
= M_s
\]

The proof completes.

**Theorem 2** for investment strategy \( A^*(u) = \frac{\mu - \rho\delta_1\delta_2 R}{\delta_2^2 R^2} \) (constant), \( \Psi(x) \) has an upper bound.

\( \Psi(x, A^*) \leq e^{-Rt}, x \geq 0 \)

**Proof:** Since \( \{M_t\}_{t \geq 0} \) is Non-negative martingale, stopping time process \( M_{\tau(x,A^*)} = M_{\tau(x,A^*)} \) is martingale. Hence

\[
e^{-RX} = E[M_{\tau(x,A^*)}^\tau] = E[M_{\tau(x,A^*)}^{\tau(x,A^*)}] + E[M_{\tau(x,A^*)}^{\tau(x,A^*)}]_{\{\tau(x,A^*) \leq t\}} + E[M_{\tau(x,A^*)}^{\tau(x,A^*)}]_{\{\tau(x,A^*) > t\}} \tag{17}
\]
Where \( I_C \) is the characteristic function of \( C \). By monotone convergence theorem, we have

\[
e^{-R_t} \geq E[M^{\tau(x,A^*)}] \tau(x,A^*) < \infty] P(\tau(x,A^*) < \infty) \tag{18}
\]

then,

\[
\Psi(x, A^*) = P(\tau(x,A^*) < \infty) \leq \frac{e^{-R_t}}{E[M^{\tau(x,A^*)}] \tau(x,A^*) < \infty]} \tag{19}
\]

by the definition of \( M^{\tau(x,A^*)} \), we can easily obtain the denominator

\[
E[M^{\tau(x,A^*)}] \tau(x,A^*) < \infty] > 1 \tag{20}
\]

The proof completes.

5. The Asymptotic Optimality and Asymptotic Uniqueness of Strategy \( A^* \) And \( R \)

When claim distribution is exponential distribution, the asymptotic optimality and asymptotic uniqueness of strategy \( A^* \) and \( R \) are essential.

Definition 1 Suppose \( 0 < r < r_e \), we say the variable \( Q \) had uniform exponential moment in the tail distribution for \( R \), in the event of \( \sup_{y \geq 0} E[e^{-r(y-q)} | Q > y] < \infty \)

The risk ratio of claim \( Q \):

\[
h(y) = \frac{g_Q(y)}{1-G_Q(y)} > 0 \text{ satisfies } \lim_{y \to \infty} h(y) > R \tag{21}
\]

\( G_Q(y) \) is distribution function and \( g_Q(y) \) is density function of random variable \( Q \).

Lemma 3 Suppose \( Q \) has uniform exponential moment in the tail distribution for \( R \), so for any \( A \in \mathcal{A} \), the process \( \{e^{-(R_t+y)}\}_{t \geq 0} \) is uniformly integrable.

Lemma 4 If variable \( Q \) has uniform exponential moment in the tail distribution for \( R \), so for any \( A \in \mathcal{A} \) and \( x \in (0, \infty) \), when \( t \to \infty \), the process \( Y_t^{x,A} \) with set \( \tau(x,A) = \infty \) almost everywhere converges to \(+\infty\).

Theorem 3 Suppose \( Q \) has uniform exponential moment in the tail distribution for \( R \), so for any admissible strategy \( A \in \mathcal{A} \), ruin probability satisfies \( \Psi(x,K) \geq L e^{-Rt} \)

\[
L = \inf_{y \geq 0} \frac{\int_{y}^{\infty} dG_Q(u)}{\int_{y}^{\infty} e^{-R(y-q)} dG_Q(u)} = \frac{1}{\sup_{y \geq 0} E[e^{-r(y-q)} | Q > y]} > 0 \tag{22}
\]

Lemma 5 In theorem 3, if there is \( \varepsilon > 0 \) and \( x_\varepsilon \geq 0 \), such that,

\[
|A(x) - A^*| \geq \varepsilon \text{ with all } x > x_\varepsilon \text{ are established, then there exist } r_\varepsilon \text{ and } A_\varepsilon \text{ satisfied } \Psi(x,A) \geq A_\varepsilon e^{-r_\varepsilon x}
\]

Theorem 4 In theorem 3, suppose \( A(x) \) is defined function of \( A \) . Then, \( \lim_{x \to \infty} A(x) = A^* \).
**Proof:** Suppose \( \lim_{x \to x_c} A(x) \neq A^+ \) then there exists \( \epsilon, \ x_\epsilon > 0 \), such that \( |A(x) - A^+| \geq \epsilon \) with all \( x > x_\epsilon \). So by lemma 5, we can get \( \Psi(x) \geq A\_z \ e^{\epsilon x + r_\epsilon} \) with one of \( r_\epsilon < R \), then we get the conclusion which contradicts with the optimality of \( A^+ \). \( \lim_{x \to x_c} \frac{\Psi(x)}{e^{-R\epsilon}} = \infty \).

The proof completes.

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