ON THE EXISTENCE OF THE UNIVERSAL CLASSES
FOR ALGEBRAIC GROUPS

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Abstract. In this note, we apply the ideas developed by M. Chalupnik in [3] to the framework of strict polynomial bifunctors. This allows us to get a new proof of the existence of the ‘universal classes’ originally constructed in [T1].

1. Introduction

Let \( k \) be a field of prime characteristic \( p \), and let \( GL_n, k \) denote the general linear group scheme over \( k \). In [T1], we exhibited a set of ‘universal classes’ \( c[i] \) \( i \in \mathbb{N} \) living in the cohomology of \( GL_n, k \). These classes’ existence was anticipated by van der Kallen [VdK], and they are one of the key ingredients to prove van der Kallen’s conjecture, which is now a theorem:

**Theorem** ([TVdK]). Let \( G \) be a reductive algebraic group scheme over a field \( k \), and let \( A \) be an finitely generated \( k \)-algebra acted on by \( G \) via algebra automorphisms. Then the cohomology \( H^*(G, A) \) is finitely generated as a graded \( k \)-algebra.

The purpose of this note is to give a new proof of the existence of the universal classes \( c[i] \). To be more specific, if \( V \) is a finite dimensional vector space and \( d \geq 1 \), the vector space \( V^\otimes d \) is acted on by the symmetric group \( S_d \). We denote by \( \Gamma^d(V) \) the ‘\( d \)-th divided power of \( V \)’, that is the subspace of invariants \( (V^\otimes d)^{S_d} \). We also denote by \( \mathfrak{gl}_n \) the adjoint representation of \( GL_n, k \) and by \( \mathfrak{gl}_n^{(1)} \) the representation obtained by base change along the Frobenius twist. We give in section 4 a new proof of the following theorem, originally established in [T1] Thm 0.1.

**Theorem 1.1.** Let \( k \) be a field of positive characteristic and let \( n \geq p \) be an integer. There are cohomology classes \( c[d] \in H^{2d}(GL_n, k, \Gamma^d(\mathfrak{gl}_n^{(1)})) \) such that:

1. \( c[1] \in H^2(GL_n, k, \mathfrak{gl}_n^{(1)}) \) is non zero.
2. If \( d \geq 1 \) and \( \Delta_{(1, \ldots, 1)} : \Gamma^d(\mathfrak{gl}_n^{(1)}) \to (\mathfrak{gl}_n^{(1)})^\otimes d \) is the inclusion, then \( \Delta_{(1, \ldots, 1)} \ast c[d] = c[1]^\otimes d \).

In the ‘old proof’ we built the universal classes by computing explicit cycles, using explicit coresolutions of the representation \( \Gamma^d(\mathfrak{gl}_n^{(1)}) \). To achieve this construction, we used two main ingredients: the twist compatible category, constructed in [T1] Section 3, and a result on the combinatorics of tensor products of \( p \)-complexes [T1 Prop 2.4].

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The new proof uses the ideas of the article [C] and involves rather different ingredients. It relies heavily on derived categories, a formality phenomenon discovered in [T3] and the adjunction argument used by M. Chalupnik to prove the collapsing conjecture [T3] Conjecture 8.1 (suitably adapted to the world of strict polynomial bifunctors).

As a common point, the two proofs rely on the very fundamental complexes constructed by Troesch in [Tr] (see also [T3] Section 9) for a slightly different presentation of these complexes).

2. Functors and bifunctors

Our proof of theorem [11] uses the category of strict polynomial bifunctors introduced in [FF]. In this section, we recall the main facts that we will need about this category. As a reference about functors, bifunctors and other generalizations, we refer the reader to [FS, Section 2], [SFB, Section 3], [FF, Section 1], [P, Section 4], [F, Section 3], [T2, Section 2], or [T4, Section 2].

2.1. Functors. Let us first begin with brief recollections of the simpler category of strict polynomial functors introduced by Friedlander and Suslin in [FS]. If \( k \) is a field of prime characteristic \( p \), we denote by \( P^d \) the abelian category of homogeneous strict polynomial functors of degree \( d \) over \( k \). The objects of \( P^d \) are nice endofunctors of the category \( \mathcal{V}_k \) of finite dimensional \( k \)-vector spaces, which naturally arise in representation theory of algebraic groups, and the morphisms of \( P^d \) are some natural transformations between these functors.

Examples of objects of \( P^d \) include the \( d \)-th tensor power \( \otimes^d : V \mapsto V \otimes^d \), the \( d \)-th symmetric power \( S^d : V \mapsto S^d(V) \), and the \( d \)-th divided power \( V \mapsto \Gamma^d(V) = (V \otimes^d)^{\mathbb{Z}_d} \). The \( r \)-th Frobenius twist \( I^{(r)} \in P^p \) is the subfunctor of \( S^p \) such that \( I^{(r)}(V) \) is generated by the elements of the form \( v^p \in S^p(V) \). As usual, if \( F \in P^d \), we denote by \( F^{\sharp} \) the dual of a functor \( F \), that is \( F^{\sharp}(V) := F(V^\vee)^{\vee} \), where the symbol ‘\( \vee \)’ stands for \( k \)-linear duality. We have

\[
\text{Hom}_{P^d}(F, G) = \text{Hom}_{P^d}(G^\sharp, F^\sharp).
\]

If \( F \in P^d \), and \( X \in \mathcal{V}_k \) we let \( F^X \) and \( F_X \) be the functors with parameter \( X \):

\[
F^X : V \mapsto F(\text{Hom}_k(X, V)) , \quad F^X : V \mapsto F(X \otimes V).
\]

The notation \( F^X \) reminds that \( F^X(V) \) is contravariant with respect to \( X \) (compare the usual notation for functional spaces), while \( F_X(V) \) is covariant with respect to \( X \). The \( S^d_X \), \( X \in \mathcal{V}_k \) form an injective cogenerator of \( P^d \) while the \( \Gamma^d_X := (\Gamma^d)^X = (S^d_X)^\mathbb{Z}_d \) form a projective generator of \( P^d \). We recall the isomorphisms, natural in \( F, X \):

\[
\text{Hom}_{P^d}(\Gamma^d_X, F) \simeq F(X) , \quad \text{Hom}_{P^d}(F, S^d_X) \simeq F^d(X).
\]

These isomorphisms are nothing but a disguised form of the Yoneda lemma (cf. [P, Section 4], or [T4, Section 2]) so we simply call them ‘the Yoneda isomorphisms’.
2.2. **Bifunctors.** If $F \in \mathcal{P}_d$, then the vector space $F(k^n)$ is canonically endowed with an action of the group scheme $GL_{n,k}$, and Friedlander and Suslin proved [FS] Cor 3.13 that the evaluation map

$$\text{Ext}^*_d(F, G) \to \text{Ext}^*_{GL_{n,k}}(F(k^n), G(k^n)) = H^*(GL_{n,k}, \text{Hom}_k(F(k^n), G(k^n)))$$

is an isomorphism if $n \geq d$. (This allows to perform Ext-computations in $\mathcal{P}_d$, where computations are surprisingly easier). Strict polynomial bifunctors were used in [FF] to generalize this formula to more general $GL_{n,k}$-representations than the ones of the somewhat restrictive form $\text{Hom}_k(F(k^n), G(k^n))$. If $d, e \geq 0$, we denote by $\mathcal{P}^d_e$ the abelian category of strict polynomial bifunctors which are homogeneous of bidegree $(d, e)$, contravariant with respect to their first variable and covariant with respect to the second variable.

Thus, objects of $\mathcal{P}^d_e$ are nice bifunctors $B : (V, W) \mapsto B(V, W)$, with $V, W \in \mathcal{V}_k$, contravariant in $V$ and covariant in $W$ and taking values in $\mathcal{V}_k$. The vector spaces $B(k^n, k^m)$ are canonically endowed with a left action of $GL_{m,k}$ and a right action of $GL_{n,k}$ which commute. Using the inverse in $GL_{n,k}$, and taking $n = m$ and the diagonal action, we get an action of $GL_{n,k}$ on $B(k^n, k^n)$. For example, let $gl \in \mathcal{P}^1_1$ denote the bifunctor $(V, W) \mapsto \text{Hom}_k(V, W)$. Then $gl(k^n, k^n)$ is nothing but the adjoint representation $gl_n$ of $GL_{n,k}$. There are several ways to construct bifunctors from functors. If $F \in \mathcal{P}_d$ and $G \in \mathcal{P}_e$, we denote by $\text{Hom}(F, G) \in \mathcal{P}^d_e$ the bifunctor:

$$\text{Hom}(F, G) : (V, W) \mapsto \text{Hom}_k(F(V), G(W)) \, .$$

We also denote by $Fgl \in \mathcal{P}^d_e$ the bifunctor:

$$Fgl : (V, W) \mapsto F(gl(V, W)) \, .$$

Franjou and Friedlander proved [FF] Thm 1.5] the following generalization of Friedlander and Suslin’s isomorphism. For all $n \geq 1$ there is a map

$$\text{Ext}^*_d(\Gamma^d_{gl}, B) \to H^*(GL_{n,k}, B(k^n, k^n)) \, ,$$

and this map is an isomorphism if $n \geq d$. For this reason, the extensions on the left hand side are called the ‘bifunctor cohomology of $B$’ and written under the more suggestive notation $H^*_B(B)$.

Let us recall some basic facts about the structure of $\mathcal{P}^d_e$. We define the dual $B^d$ of a bifunctor by letting $B^d(V, W) := B(V^\vee, W^\vee)^\vee$, so that we have

$$\text{Hom}_{\mathcal{P}^d_e}(B, C) = \text{Hom}_{\mathcal{P}^d_e}(C^d, B^d) \, .$$

The functors $\text{Hom}(\Gamma^d_X, S^e_Y)$, $X, Y \in \mathcal{V}_k$ form an injective cogenerator of $\mathcal{P}^d_e$ and their duals $\text{Hom}(S^d_X, \Gamma^e_Y)$, $X, Y \in \mathcal{V}_k$ form a projective generator. There are isomorphisms, natural in $B, X, Y$:

$$\text{Hom}_{\mathcal{P}^d_e}(\text{Hom}(S^d_X, \Gamma^e_Y), B) \simeq B(X, Y) \, ,$$

$$\text{Hom}_{\mathcal{P}^d_e}(B, \text{Hom}(\Gamma^d_X, S^e_Y)) \simeq B^d(X, Y) \, .$$

We shall call these isomorphisms ‘the Yoneda isomorphisms’, as in the case of ordinary functors. Finally, we recall two basic formulas relating morphisms
in \( \mathcal{P}_e^d \) to morphisms in \( \mathcal{P}_d \) and \( \mathcal{P}_e \), namely:

\[
\text{Hom}_{\mathcal{P}_d e}(\text{Hom}(E, F), \text{Hom}(G, H)) \simeq \text{Hom}_{\mathcal{P}_d}(G, E) \otimes \text{Hom}_{\mathcal{P}_e}(F, H),
\]

\[
\text{Hom}_{\mathcal{P}_e e}(\Gamma^dg, \text{Hom}(F, G)) \simeq \text{Hom}_{\mathcal{P}_d}(F, G).
\]

3. The cohomology of twisted bifunctors

In this section, we study the cohomology of twisted bifunctors. Our approach follows the ideas of [C].

3.1. The adjunction argument. In this paragraph, we adapt the adjunction argument of [C, Section 2] to the category of bifunctors. If \( B \in \mathcal{P}_d^d \), we denote by \( B^{(r)} \) its precomposition by the \( r \)-th Frobenius twist on both variables:

\[
B^{(r)} : (V, W) \mapsto B(V^{(r)}, W^{(r)}).
\]

So precomposition by the Frobenius twist induce an exact functor:

\[
\text{Tw}_r : \mathcal{P}_d^d \to \mathcal{P}_e^d.
\]

The following formula gives an explicit expression for its left adjoint \( \ell_r \). We define \( \ell_r(B) \) to be the dual of the bifunctor:

\[
(V, W) \mapsto \text{Hom}_{\mathcal{P}_e^d}(B, \text{Hom}(\Gamma^dg_{V}, S^e_{W})^{(r)}).
\]

Proposition 3.1. The functors \((\ell_r, \text{Tw}_r)\) form an adjoint pair.

Proof. We have to prove an isomorphism, natural in \( B, B' \):

\[
\text{Hom}_{\mathcal{P}_d d}(\ell_r(B), B') \simeq \text{Hom}_{\mathcal{P}_e^d}(B, (B')^{(r)}).
\]

Since the bifunctors \( \text{Hom}_{\mathcal{P}_d d}(-, -) \) and \( \text{Hom}_{\mathcal{P}_e^d}(-, -) \) are left exact with respect to both variables, it suffices to build isomorphism (\( * \)) when \( B \) is a projective generator and \( B' \) is an injective cogenerator (the general result follows by taking resolutions).

But if \( B = \text{Hom}(S^{dp_{V}}, \Gamma_{p^r_{V}, W}) \) and \( B' = \text{Hom}(\Gamma_{X}^{d_{Y}}, S_{Y}^{e}) \), we can identify the right hand side of formula (\( * \)) via the Yoneda isomorphism:

\[
\text{Hom}_{\mathcal{P}_e^d}(B, (B')^{(r)}) \simeq S^{d(r)}((X \otimes V)^{\vee}) \otimes S^{e(r)}(Y \otimes W).
\]

We can also identify the left hand side of the formula via the Yoneda isomorphism:

\[
\text{Hom}_{\mathcal{P}_d d}(\ell_r(B), B') \simeq \ell_r(B)^{d}(X, Y).
\]

Finally, we can compute \( \ell_r(B)^{d}(X, Y) = \text{Hom}_{\mathcal{P}_e^d}(B, \text{Hom}(\Gamma_{X}^{d_{Y}}, S_{Y}^{e})^{(r)}) \), once again thanks to a Yoneda isomorphism:

\[
\ell_r(B)^{d}(X, Y) \simeq S^{d(r)}((X \otimes V)^{\vee}) \otimes S^{e(r)}(Y \otimes W).
\]

Putting the isomorphisms (1), (2) and (3) together, we construct the isomorphism (\( * \)), natural in \( B, B' \) when \( B \) is a projective generator and \( B' \) is an injective cogenerator. This concludes the proof. \( \square \)
ON THE EXISTENCE OF THE UNIVERSAL CLASSES FOR ALGEBRAIC GROUPS 5

Now we work in the bounded above derived category $\mathbf{D}^{-}\mathcal{P}_{e}^{d}$ (see e.g. [W] Chapter 10 or [K]). Since $Tw_{r}$ is exact, its right derived functor:

$$\mathbf{R}Tw_{r}: \mathbf{D}^{-}\mathcal{P}_{e}^{d} \to \mathbf{D}^{-}\mathcal{P}_{e}^{d}_{ep}$$

is simply defined by sending an object $C \in \mathbf{D}^{+}\mathcal{P}_{e}^{d}$ (that is, a bounded above complex $C$ of bifunctors) to the complex $C^{(r)} = Tw_{r}(C)$. Since $\mathcal{P}_{e}^{d}$ has enough projectives, the left derived functor of $\ell_{r}$ is defined on $\mathbf{D}^{-}\mathcal{P}_{e}^{d}_{ep}$

$$\mathbf{L}\ell_{r}: \mathbf{D}^{-}\mathcal{P}_{e}^{d}_{ep} \to \mathbf{D}^{-}\mathcal{P}_{e}^{d}.$$ 

The following lemma is an easy check from the definition of total derived functors.

**Lemma 3.2.** For all $C \in \mathbf{D}^{-}\mathcal{P}_{e}^{d}$ we have a natural isomorphism:

$$\mathbf{L}\ell_{r}(C)(V, W) \simeq \left(\mathbf{R}\mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{P}_{e}^{d}_{ep}}(C, \mathbf{H}\mathbf{o}\mathbf{m}(\Gamma_{d}^{d}(V), S_{W}^{r}(V)))\right)^{r}_{\mathbb{N}}$$

The following statement is a formal consequence of Proposition 3.1 (adapt the proof of [W] Thm 10.7.6 or see [K] Section 13)

**Proposition 3.3.** The functors $(\mathbf{L}\ell_{r}, \mathbf{R}Tw_{r})$ form an adjoint pair.

3.2. **The formality argument.** Now we use the formality phenomenon discovered in [T3, Section 4] to get an explicit computation (in the derived category) of the complex $L_{\ell_{r}}(\Gamma^{d}_{e}(gl))$.

We first need a few notations. If $B \in \mathcal{P}_{e}^{d}$ and $Z \in \mathcal{V}_{k}$, we denote by $B_{Z} \in \mathcal{P}_{e}^{d}$ the bifunctor:

$$B_{Z}: (V, W) \mapsto B(V, Z \otimes W).$$

If $Z$ is a finite dimensional graded vector space, then the functor $B_{Z}$ inherits a grading, defined similarly as in [T3, Section 2.5]. To be more specific, let the multiplicative group $\mathbb{G}_{m}$ act on each $Z^{i}$ with weight $i$, and trivially (i.e. with weight zero) on $V$ and $W$. Then $B(V, Z \otimes W)$ inherits an action of $\mathbb{G}_{m}$. By definition, the elements of $B(V, Z \otimes W)$ of degree $j$ are the elements of weight $j$ under this action of $\mathbb{G}_{m}$.

**Proposition 3.4.** Let $E_{r}$ denote the graded vector space with $(E_{r})_{2i} = \mathbb{R}$ if $0 \leq i < p^{r}$ and $(E_{r})_{j} = 0$ otherwise. Consider the graded functor $\Gamma^{d}_{e}(gl)_{E_{r}}$ as a complex with trivial differential. There is an isomorphism in $\mathbf{D}^{-}\mathcal{P}_{e}^{d}$:

$$\mathbf{L}\ell_{r}(\Gamma^{d}_{e}(gl)) \simeq \Gamma^{d}_{e}(gl)_{E_{r}}.$$

**Proof.** It suffices to show the isomorphism $\mathbf{D}^{-}\mathcal{P}_{e}^{d}$:

$$\mathbf{R}\mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{P}_{e}^{d}_{dp}}(\Gamma^{d}_{e}(gl), \mathbf{H}\mathbf{o}\mathbf{m}(\Gamma_{V}^{d}, S_{W}^{d}(r))) \simeq (\Gamma^{d}_{e}(gl)_{E_{r}})_{\mathbb{N}}(V, W) = S^{d}gl_{E_{r}}(V, W).$$

Let $P$ denote a projective resolution of $\Gamma^{d}_{V}(r)$ and $J$ denote an injective resolution of $S_{W}^{d}(r)$. Then the left hand side is isomorphic to the complex $\mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{P}_{e}^{d}_{dp}}(\Gamma^{d}_{e}(gl), \mathbf{H}\mathbf{o}\mathbf{m}(P, J))$. The latter is isomorphic to the complex $\mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{P}_{e}^{d}_{dp}}(P, J)$, hence to the complex $\mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{P}_{e}^{d}_{dp}}((\Gamma_{V}^{d})^{(r)}, J)$.

Now if we choose for $J$ a direct sum of Troesch complexes, we know from [T3] Lemmas 4.1 and 4.4] an isomorphism in $\mathbf{D}^{-}\mathcal{P}_{e}^{d}$:

$$\mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{P}_{e}^{d}_{dp}}((\Gamma_{V}^{d})^{(r)}, J) \simeq \mathbf{H}\mathbf{o}\mathbf{m}_{\mathcal{P}_{e}^{d}}((\Gamma_{V}^{d})^{(r)}, (S_{W}^{d})_{E_{r}}).$$
To conclude the proof, we use the Yoneda isomorphism
\[ \text{Hom}_{\mathcal{P}_d}(\Gamma_{r}^{d}, (S_{E_r})_{E_r}) \simeq S^{d}(V \otimes W \otimes E_r) \simeq S^{d}\text{gl}_{E_r}(V, W). \]

\[ \square \]

3.3. **Cohomology of twisted bifunctors.** Now we turn to the study of the cohomology of twisted bifunctors, that is, the study of extensions of the form:
\[ H^{*}_{\mathcal{P}}(B^{(r)}) := \text{Ext}^{*}_{P_{dp}}(\Gamma^{dp}, \text{gl}, B^{(r)}). \]
We first need a technical lemma on graded bifunctors.

**Lemma 3.5.** Let \( B, B' \in \mathcal{P}_{r}^{d} \), and let \( Z \) denote a finite dimensional graded vector space, and let \( Z^\vee \) denote its dual, graded so that \( Z^\vee \simeq Z \). There is an isomorphism of graded vector spaces, natural in \( B, B' \):
\[ \text{Hom}_{\mathcal{P}_{d}}(B, B') \simeq \text{Hom}_{\mathcal{P}_{d}}(B, B'_{Z^\vee}). \]

Since the functor \( B \mapsto B_{Z} \) is exact, this isomorphism induces an isomorphism on the level of the derived category:
\[ \text{Hom}_{\mathcal{D}_{-\mathcal{P}_{d}}}(B_{Z}, B'_{Z}) \simeq \text{Hom}_{\mathcal{D}_{-\mathcal{P}_{d}}}(B, B'_{Z^\vee}). \]

**Proof.** By left exactness of the bifunctor \( \text{Hom}_{\mathcal{P}_{d}}(-, -) \) with respect to both variables, it suffices to build isomorphism \((*)\) when \( B \) is a projective generator and \( B' \) is an injective cogenerator (the general result follows by taking resolutions).

But if \( B = \text{Hom}(S^{d,V}_{E}, \Gamma^{r,W}) \) and \( B' = \text{Hom}(\Gamma_{X}^{d}, S_{Y}^{e}) \) we may compose the two Yoneda isomorphisms:
\[ \text{Hom}_{\mathcal{P}_{d}}(B_{Z}, B'_{Z}) \simeq S^{d}(X \otimes V^\vee) \otimes S^{e}(Y \otimes Z^\vee \otimes W), \]
\[ \text{Hom}_{\mathcal{P}_{d}}(B, B'_{Z^\vee}) \simeq S^{d}(X \otimes V^\vee) \otimes S^{e}(Y \otimes Z^\vee \otimes W) \]
to obtain the result. \[ \square \]

We are now ready to prove the generalization of the collapsing result \[C\] Cor 3.3] to the framework of bifunctors.

**Theorem 3.6.** Let \( r \) be a positive integer and let \( E_r \) denote the graded vector space with \( (E_r)_{2i} = k \) if \( 0 \leq i < p^r \) and \( (E_r)_{2i} = 0 \) otherwise. There is an isomorphism of graded vector spaces, natural in \( B \in \mathcal{P}_{d} \) (take the total grading on the right hand side):
\[ H^{*}_{\mathcal{P}}(B^{(r)}) \simeq H^{*}_{\mathcal{P}}(B_{E_r}). \]

**Proof.** Let \( C \) be an object of \( \mathcal{D}_{-\mathcal{P}_{d}} \). We have isomorphisms:
\[ \text{Hom}_{\mathcal{D}_{-\mathcal{P}_{d}}}(\Gamma^{dp}_{E_r}, C^{(r)}) \simeq \text{Hom}_{\mathcal{D}_{-\mathcal{P}_{d}}}(\Gamma^{dp}_{E_r}, C) \]
\[ \simeq \text{Hom}_{\mathcal{D}_{-\mathcal{P}_{d}}}(\Gamma^{dp}_{E_r}, C) \]
\[ \simeq \text{Hom}_{\mathcal{D}_{-\mathcal{P}_{d}}}(\Gamma^{dp}_{E_r}, C_{E_r}). \]
The first isomorphism follows by adjunction (proposition \[3.1\]), the second isomorphism by formality (proposition \[3.4\]) and the last one from lemma \[3.5\] with the isomorphism \( E_{r}^{d} \simeq E_{r} \). If \( B \in \mathcal{P}_{d} \), we apply this isomorphism to \( C = B[i] \) and take homology to get the result. \[ \square \]
4. Proof of theorem 1.1

In this section, we prove theorem 1.1. We first transpose the problem in the framework of strict polynomial functors as in [T1, Section 1.2]. Using the map

$$H^*_{\mathcal{P}}(B) \to H^*(GL_{n,k}, B(k^n, k^n)),$$

together with the fact that for $n \geq p$, it induces an isomorphism

$$H^2_{\mathcal{P}}(gl^{(1)}) \cong H^2(GL_{n,k}, gl^{(1)}_n),$$

we easily see that theorem 1.1 is implied by the following theorem.

**Theorem 4.1.** Let $k$ be a field of positive characteristic $p$. There are cohomology classes $c[d] \in H^2_{\mathcal{P}}(\otimes^d(1)gl)$ satisfying the following conditions.

1. $c[1]$ is non zero.
2. If $d \geq 1$ and $\Delta_{(1,...,1)} : \Gamma^d(1)gl \to \otimes^d(1)gl$ is the inclusion, then

$$\Delta_{(1,...,1)} c[d] = c[1] \cup d.$$

So we are left with the problem of finding classes $c[d] \in H^2_{\mathcal{P}}(\otimes^d(1)gl)$.

Finding $c[1]$ is not a problem. Indeed, it is well-known that $H^2_{\mathcal{P}}(I^{(1)gl}) \cong k \neq 0$ (this results for example from theorem 3.6 but one can find much more elementary proofs of this computation). So we can choose for $c[1]$ a non zero cohomology class in $H^2_{\mathcal{P}}(I^{(1)gl})$.

Now we want to find the classes $c[d]$ for $d \geq 2$. The action of the symmetric group $\mathfrak{S}_d$ on $\otimes^d$ (by permuting the factors of the tensor product) induce an action on the graded vector space $H^*_{\mathcal{P}}(\otimes^d(1)gl)$.

**Lemma 4.2.** For all $d \geq 2$ the cup product $c[1] \cup d \in H^2_{\mathcal{P}}(\otimes^d(1)gl)$ is invariant under the action of the symmetric group.

**Proof.** Recall that the cup product:

$$H^*_{\mathcal{P}}(B) \otimes H^*_{\mathcal{P}}(B') \xrightarrow{\cup} H^{*+j}(B \otimes B')$$

is defined as the composite of the external product of extensions

$$\text{Ext}^i_{\mathcal{P}}(\Gamma^d gl, B) \otimes \text{Ext}^j_{\mathcal{P}}(\Gamma^d gl, B) \to \text{Ext}^{i+j}_{\mathcal{P}}(\Gamma^d gl \otimes \Gamma^d gl, B \otimes B')$$

and the map induced by the comultiplication $\Gamma^{d+e} gl \rightarrow \Gamma^d gl \otimes \Gamma^e gl$. Since $c[1]$ is in even degree and $\Gamma^* gl$ is a cocommutative coalgebra, one easily gets the result from the definition of the action of $\mathfrak{S}_d$ and the definition of the cup product. □

In view of lemma 4.2, it suffices to prove that all the classes of $H^2_{\mathcal{P}}(\otimes^d(1)gl)_{\mathfrak{S}_d}$ are obtained from classes of $H^2_{\mathcal{P}}(I^{d(1)gl})$ through the map

$$\Delta_{(1,...,1)} : H^2_{\mathcal{P}}(I^{d(1)gl}) \to H^2_{\mathcal{P}}(\otimes^d(1)gl).$$

Actually, we can get a slightly more general statement from theorem 3.6.

**Proposition 4.3.** Let $d \geq 2$. Then the map $\Delta_{(1,...,1)}$ induces a surjection

$$H^*_{\mathcal{P}}(I^{d(1)gl}) \rightarrow H^*_{\mathcal{P}}(\otimes^d(1)gl)_{\mathfrak{S}_d}.$$
Proof. Let us first remark that \( I^{(1)} \mathfrak{gl} \simeq \mathfrak{gl}^{(1)} \), so theorem 3.6 yields a commutative diagram:

\[
\begin{array}{ccc}
H^*_P(\Gamma^d(1) \mathfrak{gl}) & \xrightarrow{\Delta_{(1,\ldots,1)}^*} & H^*_P(\otimes^d(1) \mathfrak{gl}) \\
\cong & & \cong \\
H^*_P(\Gamma^d \mathfrak{gl}_{E_1}) & \xrightarrow{\Delta_{(1,\ldots,1)}^*} & H^*_P(\otimes^d \mathfrak{gl}_{E_1})
\end{array}
\]

The image of the top horizontal arrow lives inside \( H^*_P(\otimes^d(1) \mathfrak{gl}) \mathbb{S}_d \), the image of the bottom horizontal arrow lives inside \( H^*_P(\otimes^d \mathfrak{gl}_{E_1}) \mathbb{S}_d \) and by naturality of the vertical arrow on the right, we have an isomorphism \( H^*_P(\otimes^d(1) \mathfrak{gl}) \mathbb{S}_d \simeq H^*_P(\otimes^d \mathfrak{gl}_{E_1}) \mathbb{S}_d \). Thus, to prove proposition 4.3, it suffices to prove that all the elements of \( H^*_P(\otimes^d \mathfrak{gl}_{E_1}) \mathbb{S}_d \) are hit by the bottom horizontal arrow.

But \( \otimes^d \mathfrak{gl} \) (hence the summands of \( \otimes^d \mathfrak{gl}_{E_1} \)) is injective in \( P^d \), whence an equality

\( H^*_P(\otimes^d \mathfrak{gl}_{E_1}) = H^0_P(\otimes^d \mathfrak{gl}_{E_1}) \).

Now the left exactness of the functor \( B \mapsto H^0_P(B_{E_1}) \) implies that the map

\( \Delta_{(1,\ldots,1)}^* : H^0_P(\Gamma^d \mathfrak{gl}_{E_1}) \to H^0_P(\otimes^d \mathfrak{gl}_{E_1}) \mathbb{S}_d = H^*_P(\otimes^d \mathfrak{gl}_{E_1}) \mathbb{S}_d \)

is surjective. This concludes the proof. \( \square \)

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