Black hole perturbation in the most general scalar-tensor theory with second-order field equations II: the even-parity sector

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We perform a fully relativistic analysis of even-parity linear perturbations around a static and spherically symmetric solution in the most general scalar-tensor theory with second-order field equations. This paper is a sequel to Kobayashi et al. [Phys. Rev. D 85, 084025 (2012)], in which the linear perturbation analysis for the odd-parity modes is presented. Expanding the Horndeski action to second order in perturbations and eliminating auxiliary variables, we derive the quadratic action for even-parity perturbations written solely in terms of two dynamical variables. The two perturbations can be interpreted as the gravitational and scalar waves. Correspondingly, we obtain two conditions to evade ghosts and two conditions for the absence of gradient instabilities. Only one in each pair of conditions yields a new stability criterion, as the conditions derived from the stability of the gravitational-wave degree of freedom coincide with those in the odd-parity sector. Similarly, the propagation speed of one of the two modes is the same as that for the odd-parity mode, while the other differs in general from them. Our result is applicable to all the theories of gravitation with an extra single scalar degree of freedom such as the Brans-Dicke theory, $f(R)$ models, and Galileon gravity.

I. INTRODUCTION

Operation of the second-generation gravitational-wave detectors such as Advanced LIGO [1], Advanced Virgo [2], and KAGRA [3] in the coming decade will bring detection of gravitational waves (GWs) and open a new era of gravitational wave astronomy. Since typical GWs come from strong gravity regions such as the vicinity of a black hole, the detection of GWs enables us to test general relativity (GR) and alternative theories of gravity in the strong fields. In order to distinguish GR from other theories of gravity, theoretical understanding of fundamental properties of modified gravity is crucial.

It is interesting to consider, among many directions of extending GR, a class of scalar-tensor theories for which field equations of the scalar field and the metric are at most of second order. This class is known to be described by the Horndeski theory in a generic way, in terms of four arbitrary functions of the scalar field $\phi$ and its kinetic term $(\partial \phi)^2$ [4]. The Horndeski theory includes well studied models such as the Brans-Dicke theory, $f(R)$ theories, Galileon models as specific cases. In Ref. [5], fully relativistic study of linear perturbations around static and spherically symmetric spacetime has been performed, restricting the analysis to the odd-parity perturbations, in the Horndeski theory [1]. Since no particular theory is assumed in its formulation, the perturbation equation and the stability conditions derived in [5] have versatile and wide applicability. It can also help us understand what kinds of differences are expected in the behavior of perturbations in general within the Horndeski theory. In the odd-parity sector, the scalar field is not perturbed and only the metric perturbations enter the game. As in the case of GR, it has been found that there is only one dynamical variable corresponding to the GW degree of freedom. From the second-order Lagrangian for that variable, we have obtained the conditions for the absence of ghost and gradient instabilities, which put constraints on the form of the four functions involved in the Horndeski action. Variation of the second-order Lagrangian yields a wave equation, giving the generalization of the Regge-Wheeler equation.

The aim of this paper is to extend our previous study [5] to the even-parity perturbations. Since the scalar field perturbation participates in the even-parity sector in addition to the metric perturbations, it is expected that linearized field equations reduce to coupled differential equations for two dynamical variables corresponding to the GW and the scalar-field degrees of freedom. We will explicitly demonstrate that this is indeed the case, starting from the second-order Lagrangian and eliminating the auxiliary fields by using the constraint equations. One can see that the resultant wave equations are of second order both in time and radial coordinates, which is reasonable given the fact that the original field equations are of second order in the Horndeski theory. Our second-order Lagrangian provides two conditions for the absence of ghosts, one for the GWs and the other for the scalar wave. The condition

1 See Refs. [4-8] for black hole perturbations in other classes of modified gravity theories.
for the former mode is the same as that for the odd-parity mode, while stability of the latter imposes a completely new condition and thus places an additional restriction to the Horndeski theory. We also find that the propagation speed of the GWs coincides with that in the odd-parity sector, giving a consistency relation that holds in any second-order scalar-tensor theories. However, the propagation speed of the scalar wave is generically different from that of the GWs. This yields yet another condition to avoid gradient instabilities, restricting further the form of the Horndeski action. Combining the results presented in this paper with the ones for the odd-parity perturbations, we can test the viability of a given modified theory of gravity with a single scalar degree of freedom.

The paper is organized as follows. In the next section, we introduce the Lagrangian of the theories in the Horndeski class. In Sec. II we compute the second-order Lagrangian of the even-parity perturbations and derive the stability conditions as well as the propagation speeds. In Sec. III we apply our results to some specific models. Section IV is devoted to the conclusion. Appendices A and B provide the explicit form of the background equations and coefficients for the second-order Lagrangian, respectively.

II. HORNDESKI THEORY AND SPHERICALLY SYMMETRIC BACKGROUND

We start with defining the theory and the background metric we use. The Horndeski theory [4] is the most general scalar-tensor theory with second-order field equations. This theory was rederived recently [9] in the course of generalizing the Galileons [10], and the modern form given in [9] was shown to be equivalent to the original Horndeski theory [11]. The Horndeski theory is described by the following four Lagrangians:

\[ L_2 = K(\phi, X), \]
\[ L_3 = -G_3(\phi, X)\Box \phi, \]
\[ L_4 = G_4(\phi, X)R + G_{4X} \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right], \]
\[ L_5 = G_5(\phi, X)G_{\mu \nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} G_{5X} \left[ (\Box \phi)^3 - 3 \Box \phi (\nabla_\mu \nabla_\nu \phi)^2 + 2 (\nabla_\mu \nabla_\nu \phi)^3 \right], \]

where \( K \) and \( G_i \) are arbitrary functions of \( \phi \) and \( X := - (\partial \phi)^2 / 2 \). Here we used the notation \( G_{iX} \) for \( \partial G_i / \partial X \). Our action is thus

\[ S = \sum_{i=2}^{5} \int d^4x \sqrt{-g} L_i. \]

By choosing the functions \( K \) and \( G_i \) appropriately, one can express any second-order scalar-tensor theory in terms of the Horndeski theory. Some examples are presented in Ref. [11].

We consider a static and spherically symmetric background solution of the Horndeski theory. The background metric can be written as

\[ ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + C(r)r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right). \]

The scalar field is also dependent only on the radial coordinate, \( \phi = \phi(r) \), and hence \( X = -B(\dot{\phi})^2 / 2 \), where the prime denotes differentiation with respect to \( r \). Without loss of generality, we can set \( C(r) = 1 \). Nevertheless, we introduced \( C(r) \) because it is convenient to retain \( C(r) \) when deriving the field equations from the variational principle. We will therefore impose this condition after taking variations to get the background field equations. The background equations are summarized in Appendix A.

In the analysis of linear perturbations around the above background, we will not specify a concrete form of the solution \( A(r), B(r) \), and \( \phi(r) \), as it is not necessary for deriving generic stability conditions against perturbations.

III. FORMULATION OF THE EVEN-PARITY PERTURBATIONS

A. Decomposition of the even-parity perturbations

Given the background equations of motion, we can now derive the quadratic action for perturbations. In the Regge-Wheeler formalism [12], the metric perturbations are decomposed into odd- and even-type perturbations according to their transformation properties under the two-dimensional rotation. Furthermore, each perturbation can be decomposed into the sum of spherical harmonics. Then, at linear order in perturbation equations, or equivalently at
second order in the action for the perturbations, on the static and spherically symmetric background, the perturbation variables having different \( \ell, m \), and parity do not mix each other. This fact drastically simplifies our perturbation analysis. The odd-parity perturbations in the Horndeski theory have been investigated in Ref. [5]. This paper is the sequel of Ref. [2], and we will now concentrate on the even-parity perturbations.

The even-parity metric perturbations can be written as [12]

\[
\begin{align*}
 h_{tt} &= A(r) \sum_{\ell,m} H_{0, \ell m}(t, r) Y_{\ell m}(\theta, \varphi), \\
 h_{tr} &= \sum_{\ell,m} H_{1, \ell m}(t, r) Y_{\ell m}(\theta, \varphi), \\
 h_{rr} &= \frac{1}{B(r)} \sum_{\ell,m} H_{2, \ell m}(t, r) Y_{\ell m}(\theta, \varphi), \\
 h_{ta} &= \sum_{\ell,m} \beta_{\ell m}(t, r) \partial_a Y_{\ell m}(\theta, \varphi), \\
 h_{ra} &= \sum_{\ell,m} \alpha_{\ell m}(t, r) \partial_a Y_{\ell m}(\theta, \varphi), \\
 h_{ab} &= \sum_{\ell,m} K_{\ell m}(t, r) g_{ab} Y_{\ell m}(\theta, \varphi) + \sum_{\ell,m} G_{\ell m}(t, r) \nabla_a \nabla_b Y_{\ell m}(\theta, \varphi). 
\end{align*}
\]

The scalar field \( \phi \) also has an even-parity perturbation,

\[
\phi(t, r, \theta, \varphi) = \phi(r) + \sum_{\ell,m} \delta \phi_{\ell m}(t, r) Y_{\ell m}(\theta, \varphi).
\]

Because of the general covariance, we have gauge degrees of freedom, enabling us to set some of the perturbation variables to zero by performing an infinitesimal coordinate transformation \( x^\mu \rightarrow x^\mu + \xi^\mu \). Out of totally four degrees of freedom of the gauge transformation, three belong to the even-parity sector. The three gauge functions can be written as [12]

\[
\begin{align*}
 \xi_t &= \sum_{\ell,m} T_{\ell m}(t, r) Y_{\ell m}(\theta, \varphi), \\
 \xi_r &= \sum_{\ell,m} R_{\ell m}(t, r) Y_{\ell m}(\theta, \varphi), \\
 \xi_\alpha &= \sum_{\ell,m} \Theta_{\ell m}(t, r) \partial_a Y_{\ell m}(\theta, \varphi),
\end{align*}
\]

where \( T_{\ell m}(t, r), R_{\ell m}(t, r), \) and \( \Theta_{\ell m}(t, r) \) are arbitrary functions of \( t \) and \( r \). The transformation rule for each metric component under these gauge transformations is given by [12]

\[
\begin{align*}
 H_{0, \ell m}(t, r) &\rightarrow H_{0, \ell m}(t, r) + \frac{2}{A} \dot{T}_{\ell m}(t, r) - \frac{A'}{A} R_{\ell m}(t, r), \\
 H_{1, \ell m}(t, r) &\rightarrow H_{1, \ell m}(t, r) + \dot{R}_{\ell m}(t, r) + T_{\ell m}(t, r) - \frac{A'}{A} T_{\ell m}(t, r), \\
 H_{2, \ell m}(t, r) &\rightarrow H_{2, \ell m}(t, r) + 2B \dot{R}_{\ell m}(t, r) + B' R_{\ell m}(t, r), \\
 \beta_{\ell m}(t, r) &\rightarrow \beta_{\ell m}(t, r) + T_{\ell m}(t, r) + \dot{\Theta}_{\ell m}(t, r), \\
 \alpha_{\ell m}(t, r) &\rightarrow \alpha_{\ell m}(t, r) + R_{\ell m}(t, r) + \Theta_{\ell m}(t, r) - \frac{2}{r} \Theta_{\ell m}(t, r), \\
 K_{\ell m}(t, r) &\rightarrow K_{\ell m}(t, r) + \frac{2B}{r} R_{\ell m}(t, r), \\
 G_{\ell m}(t, r) &\rightarrow G_{\ell m}(t, r) + 2 \Theta_{\ell m}(t, r).
\end{align*}
\]

From these transformation rules we see that \( G_{\ell m}, K_{\ell m}, \) and \( \beta_{\ell m} \) can be set to zero by solving the coupled algebraic equations for \( \Theta_{\ell m}, R_{\ell m}, \) and \( T_{\ell m} \). The solution is unique, and hence the condition \( G_{\ell m} = K_{\ell m} = \beta_{\ell m} = 0 \) completely fixes the gauge. In the following, we will use this gauge condition for the calculation of the second-order action.

Note that the above argument about the gauge fixing does not apply to the monopole (\( \ell = 0 \)) and dipole (\( \ell = 1 \)) perturbations. For the monopole perturbations, we have \( \alpha = \beta = G = 0 \) identically and the gauge transformation given by \( \Theta \) is irrelevant. For the dipole perturbations, \( K \) and \( G \) appear in \( h_{ab} \) only through the combination \( K - G \), and hence the decomposition of \( h_{ab} \) into the two components is in fact redundant. The analysis for these two cases will be presented separately after studying higher multipoles with \( \ell \geq 2 \).
B. Brief review of the odd-parity perturbations

Before going into the detailed investigation of the even-parity perturbations, let us take a quick look at the main result of the analysis for the odd-parity perturbations obtained in [2] in order to facilitate the comparison between the odd-parity and even-parity results. The scalar field $\phi$ does not acquire odd-parity perturbations and only the metric is perturbed. After some manipulations of the second-order Lagrangian, it is confirmed that there is only one dynamical variable left in the final Lagrangian. Varying this final Lagrangian, the master equation for the odd-parity perturbations can be derived. All the other perturbation variables are determined once the solution of the master equation is obtained. It is easy to check that the resultant master equation reduces to the Regge-Wheeler equation in the GR limit. To evade ghost and gradient instabilities it is required that all the following conditions must be satisfied simultaneously:

$$F := 2 \left(G_4 + \frac{1}{2}B\phi'X'G_{5X} - XG_{5\phi}\right) > 0,$$

$$G := 2 \left[G_4 - 2XG_{4X} + X \left(\frac{A'}{2A}B\phi'G_{5X} + G_{5\phi}\right)\right] > 0,$$

$$H := 2 \left[G_4 - 2XG_{4X} + X \left(\frac{B\phi'}{r}G_{5X} + G_{5\phi}\right)\right] = -\frac{2A}{B} \frac{\partial E_C}{\partial A^0} > 0.$$

Here, $E_C$ is the quantity introduced as the “left-hand side” of the background equation and is defined in Appendix A. The propagation speed along the radial direction is given by

$$c_{\text{odd}} = \frac{G}{F}.\tag{25}$$

Since $\phi$ is not perturbed in the odd-parity sector, $c_{\text{odd}}$ can be interpreted as the propagation speed of the GWs. This fact can also be understood by noting that $c_{\text{odd}}$ depends only on $G_4$ and $G_5$, the functions coupled to the curvature in the Horndeski action.

C. Even-parity perturbations with $\ell \geq 2$

Substituting into the action [5] both the metric and the scalar-field perturbations with the gauge choice $G_{\ell m} = K_{\ell m} = \beta_{\ell m} = 0$, we find that the second-order Lagrangian is given by

$$\frac{2\ell + 1}{2\pi} \mathcal{L} = H_0 \left[ a_1 \delta \phi'' + a_2 \delta \phi' + a_3 H_2' + j^2 a_4 \alpha' + (a_5 + j^2 a_6) \delta \phi + (a_7 + j^2 a_8) H_2 + j^2 a_9 \alpha \right]
+ j^2 b_1 H_2^2 + H_1 \left( b_2 \delta \phi' + b_3 \delta \phi + b_4 H_2 + j^2 b_5 \alpha \right)
+ c_1 H_2 \delta \phi + H_2 \left[ c_2 \delta \phi'' + (c_3 + j^2 c_4) \delta \phi + j^2 c_5 \alpha \right] + c_6 H_2^2 + j^2 d_1 \alpha^2 + j^2 \alpha (d_2 \delta \phi' + d_3 \delta \phi) + j^2 d_4 \alpha^2
+ e_1 \delta \phi^2 + e_2 \delta \phi' + (e_3 + j^2 e_4) \delta \phi^2,\tag{26}$$

where $j^2 := \ell(\ell + 1)$. Since only the perturbations in one multipole $\ell$, $m$ are considered at one time, the suffixes $\ell$, $m$ of the perturbation variables are omitted without confusion. The explicit expression for the background dependent expansion coefficients $a_1$, $a_2$, $\cdots$ are presented in Appendix B.

The Lagrangian [26] shows that both $H_0$ and $H_1$ are auxiliary fields. In particular, no quadratic term in $H_0$ is present, and hence $H_0$ is a Lagrange multiplier, giving rise to a constraint among the other three variables, $H_2$, $\alpha$, and $\delta \phi$:

$$a_1 \delta \phi'' + a_2 \delta \phi' + a_3 H_2' + j^2 a_4 \alpha' + (a_5 + j^2 a_6) \delta \phi + (a_7 + j^2 a_8) H_2 + j^2 a_9 \alpha = 0.\tag{27}$$

Since $r$ derivatives of all the three variables appear in the above constraint, this equation in its original form cannot be solved for any one of $H_2$, $\alpha$, and $\delta \phi$. In order to resolve this issue, we need to perform a field redefinition and use a new variable $\psi$ defined by

$$H_2 = \frac{1}{a_3} \left( \psi - a_1 \delta \phi' - j^2 a_4 \alpha \right),\tag{28}$$
instead of \( H_2 \). In terms of \( \psi \), the first derivative of \( \alpha \) as well as the second derivative of \( \delta \phi \) can be removed simultaneously from Eq. (27). Thus, the constraint (27) becomes an algebraic equation for \( \alpha \), which can be solved to give

\[
\alpha = \frac{1}{j^2 a_1 [j^2 a_8 + \left( \frac{A}{2A} - \frac{1}{2} \right) a_3]} \left[ a_3 \psi' + j^2 a_8 \psi + \{a_3(a_2 - a_1') - j^2 a_1 a_8\} \delta \phi' + a_3(a_5 + j^2 a_6) \delta \phi \right].
\]

(29)

Because of the existence of the quadratic term in \( H_1 \), variation with respect to \( H_1 \) gives an equation that can be solved for \( H_1 \), yielding

\[
H_1 = -\frac{1}{2j^2 b_1} (b_2 \delta \phi' + b_3 \delta \phi + b_4 H_2 + j^2 b_5 \alpha'),
\]

(30)

where it should be understood that \( H_2 \) and \( \alpha \) appearing in the above equation are replaced by \( \psi \) and \( \delta \phi \) using Eqs. (28) and (29). Putting Eqs. (28), (29), and (30) back into Eq. (26) gives the reduced Lagrangian that depends only on \( \psi \) and \( \delta \phi \). At this stage, the resultant Lagrangian contains several higher derivative terms such as \( \delta \phi \psi' \). However, as it should be from the second-order nature of the Horndeski theory, all such unwanted terms can be removed by performing some integration by parts. As a result, we end up with the following Lagrangian containing at most first derivatives and no mixing terms between \( t \) and \( r \) derivatives:

\[
\frac{2\ell + 1}{2\pi} \mathcal{L} = \frac{1}{2} K_{ij} \dot{\psi}^i \dot{\psi}^j - \frac{1}{2} G_{ij} \psi^i \psi^j - Q_{ij} \dot{\psi}^i \psi^j - \frac{1}{2} M_{i j} \psi^i \psi^j,
\]

(31)

where \( i \) and \( j \) run from 1 to 2 and \( v^1 := \psi, v^2 := \delta \phi \).

Let us first explore the conditions for the absence of ghost instabilities. Since we have the two dynamical variables, the stability conditions we are looking for are derived from

\[
K_{11} > 0, \quad \text{det}(K) > 0.
\]

(32)

Explicitly, the first condition reads

\[
K_{11} = \frac{8\sqrt{AB}(2rH + \Xi \phi')^2}{\ell(\ell + 1)A^2 H^2} \frac{\ell(\ell + 1)P_1 - \mathcal{F}}{(2rH\ell + 1 + \mathcal{P}_2)} > 0,
\]

(33)

where \( P_1 \) and \( P_2 \) are defined respectively as

\[
P_1 := \frac{B(2rH + \Xi \phi')}{2A^2 H^2} \left[ \frac{A\mathcal{R}^4 H^4}{(2rH + \Xi \phi')^2 B} \right]',
\]

(34)

\[
P_2 := -B \left( 2 - \frac{rA'}{A} \right) (2rH + \Xi \phi'),
\]

(35)

and \( \Xi \) as

\[
\Xi := -\frac{r^2}{B} \frac{\partial E_A}{\partial \phi'} = -\frac{2r^2 A}{B} \frac{\partial E_{\phi}}{\partial A'}
\]

\[
= 2r \left[ -XG_{4X} + \frac{2B\phi'}{r} \{G_{4XY} - (XG_{5\phi})_X \} + G_{4\phi Y} - \frac{1}{r^2} XG_{5X} + \frac{B}{r^2} (XG_{5X})_Y \right].
\]

(36)

In the definition of \( \Xi \) we used again the “left-hand side” of the background equations introduced in Appendix A. We also introduced a notation \( f_Y := -2\sqrt{-X} \partial(\sqrt{-X} f)/\partial X \). For instance, \( f_Y := f + 2Xf_X \) and \( (X f)_Y = X (3f + 2X f_X) \). Notice that the \( Y \)-derivative does not commute with the \( X \)-derivative, i.e., \( G_{4XY} \neq G_{4YX} \). In deriving Eq. (36) and other quantities which we shall define later, one can use the following useful relations:

\[
\frac{\partial [(X f)']}{\partial \phi'} = -B\phi' (X f)_X, \quad \frac{\partial [(\phi' X f)']}{\partial \phi''} = (X f)_Y.
\]

(37)

The second stability condition, \( \text{det}(K) > 0 \), can be written explicitly as

\[
\text{det}(K) = \frac{16(\ell - 1)(\ell + 2)(2rH + \Xi \phi')^2 \mathcal{F}(2P_1 - \mathcal{F})}{\ell(\ell + 1)A^2 H^2 \phi'^2 (2rH\ell + 1 + \mathcal{P}_2)} > 0.
\]

(38)
With the first one of the stability conditions for the odd-parity perturbations [24], \( \mathcal{P} > 0 \), it is found that \( \text{det}(\mathcal{K}) \) is positive if and only if
\[
2\mathcal{P}_1 - \mathcal{F} > 0. \tag{39}
\]
It is interesting to note that if Eq. (39) is satisfied then Eq. (33) is satisfied automatically, given that \( \ell \geq 2 \) and \( \mathcal{F} > 0 \). Consequently, only Eq. (39) gives rise to a new independent condition for the absence of ghosts.

The squared propagation speeds of two modes along the radial direction, \( c_{s1}^2 \) and \( c_{s2}^2 \), are derived from the eigenvalues of the matrix \( (AB)^{-1}\mathcal{K}^{-1}\mathcal{G} \) and are given by
\[
c_{s1}^2 = \frac{G}{\mathcal{F}}, \tag{40}
\]
\[
c_{s2}^2 = \frac{2r^2\Gamma\Xi\phi'^2 - G\Xi^2\phi'^2 - 4r^4\Sigma\mathcal{H}^2/B}{(2r\Gamma + \Xi\phi')^2(2\mathcal{P}_1 - \mathcal{F})}, \tag{41}
\]
where \( \Gamma \) and \( \Sigma \) are defined as
\[
\Gamma := \Gamma_1 + \frac{A'}{A} \Gamma_2 = -\frac{2}{B} \frac{\partial \mathcal{E}_C}{\partial \phi'^2}, \tag{42}
\]
\[
\Gamma_1 := 4 \left[ -XG_{5x} + G_{4\phi y} + \frac{B\phi'}{r} \{G_{4xy} - (XG_{5\phi})x\} \right], \tag{43}
\]
\[
\Gamma_2 := 2B\phi' \left[ G_{4xy} - (XG_{5\phi})x - \frac{B\phi'}{2rX}(XG_{5\phi})y \right] = \frac{\partial \mathcal{H}'}{\partial \phi'^2}, \tag{44}
\]
\[
\Sigma := \frac{X}{B} \frac{\partial \mathcal{E}_\phi}{\partial \phi'^2} = \frac{X}{B} \left[ K_{xy} - B\phi' \left( \frac{4}{r} + \frac{A'}{A} \right) (XG_{3x})x - 2G_{3\phi y} + 2 \left( \frac{1 - B}{r^2} - \frac{B A'}{r A} \right) G_{4xy} - \frac{4B}{r} \left( \frac{1}{r} + \frac{A'}{A} \right) (XG_{4xx})y \right.
\]
\[
+2B\phi' \left( \frac{4}{r} + \frac{A'}{A} \right) (XG_{4x})x - \frac{B\phi'(1 - 3B)}{r^4} \frac{A'}{A} (XG_{5x})x + \frac{2B^2\phi' A'}{r^2} \frac{A'}{A} (XG_{5xx})x \]
\[
-2 \left( \frac{1 - B}{r^2} - \frac{B A'}{r A} \right) G_{5\phi y} + \frac{2B}{r} \left( \frac{1}{r} + \frac{A'}{A} \right) (XG_{5\phi x})y
\]
\[
+2XG_{3\phi x} + B\phi' \left( \frac{4}{r} + \frac{A'}{A} \right) G_{4\phi x} + \frac{2}{r^2} (XG_{5\phi x}) \right]. \tag{45}
\]
The stability conditions for the odd-parity modes, Eqs. (22) and (23), ensure that \( c_{s1}^2 > 0 \). Using the no-ghost condition, Eq. (39), we see that \( c_{s2}^2 \) is positive if and only if the following condition is satisfied:
\[
2r^2\Gamma\Xi\phi'^2 - G\Xi^2\phi'^2 - 4r^4\Sigma\mathcal{H}^2/B > 0. \tag{46}
\]
Since \( c_{s1} \) depends only on the two of the Horndeski functions, \( G_4 \) and \( G_5 \), and it is those two functions that are coupled to the curvature in the action, \( c_{s1} \) can be interpreted as the propagation speed of GWs. On the other hand, \( c_{s2} \) involves both \( K \) and \( G_5 \) as well, and hence it is reasonable to interpret \( c_{s2} \) as the propagation speed of a scalar wave. These interpretations are also supported by the propagation speeds of the monopole and dipole perturbations which will be computed shortly: monopole and dipole modes arise entirely due to the scalar degree of freedom, and it will turn out that the two modes indeed propagate at \( c_{s2} \).

Interestingly, \( c_{s1} \) exactly coincides with \( c_{\text{odd}} \), namely, odd-type and even-type GWs propagate at the same speed (though it is not necessarily equal to the speed of light). If future experiments would reveal that the consistency relation, \( c_{\text{odd}} = c_{s1} \), is violated, all the modified gravity theories in the Horndeski class as well as GR could be excluded.

The mass matrix \( M_{ij} \) and the antisymmetric matrix \( Q_{ij} \) provide further conditions for the stability of static and spherically symmetric solutions. However, explicit expressions for \( M_{ij} \) and \( Q_{ij} \) are found to be too complicated to be illuminating, and we have not been able to give sufficiently concise stability conditions from those matrix elements.

**D. Monopole perturbation: \( \ell = 0 \)**

For the monopole perturbations, \( \alpha, \beta, \) and \( G \) identically vanish in Eqs. (10), (11), and (12). The gauge functions that are still meaningful are \( \xi_1 \) and \( \xi_2 \). As is the case for higher multipoles with \( \ell \geq 2 \), \( \xi_2 \) is fixed completely by
setting $K$ to zero. As for $\xi$, it can in principle be used to eliminate either $H_0$ or $H_1$. However, since that is not a complete gauge fixing, we defer it until we derive the perturbation equations from the second-order Lagrangian. Keeping this in mind, the second-order Lagrangian for the monopole perturbations can be obtained by setting $j^2 = 0$ in the Lagrangian (26) as
\[
\frac{2\ell + 1}{2\pi} \mathcal{L} = H_0 \left( a_1\delta\phi' - \frac{A}{2} b_3\delta\phi - \frac{A}{2} b_4 H_2 \right)' + \frac{b_2}{a_1} H_1 \left( a_1\delta\phi' - \frac{A}{2} b_3\delta\phi - \frac{A}{2} b_4 H_2 \right)',
\]
\[+ c_1 H_2 \delta\phi + H_2 (c_2 \delta\phi' + c_3 \delta\phi) + c_2^2 + c_2 \delta\phi'' + c_3 \delta\phi',
\]0
where we have used the background equations to rewrite the first term. We see from Eq. (47) that there are no terms quadratic in $H_0$ and $H_1$ for $\ell = 0$, and hence those two variables are Lagrange multipliers in this case. As a result, we obtain two constraint equations. However, as is clear from Eq. (47), the two constraints are not independent but merge into the following single constraint in the end:
\[
H_2 = -\frac{b_3}{b_4} \delta\phi + \frac{2}{Ab_4} (a_1\delta\phi' + C_0),
\]0
where the integration constant $C_0$ amounts to the shift of one of the integration constants in the background solution. Since we are interested in the perturbations that do not correspond to a mere change of the background solution, we set $C_0 = 0$. Substituting Eq. (48) back into Eq. (47) and performing integration by parts, we arrive at
\[
\frac{2\ell + 1}{2\pi} \mathcal{L} = \frac{1}{2} K_0 \delta\phi^2 - \frac{1}{2} G_0 \delta\phi'^2 - \frac{1}{2} M_0 \delta\phi'^2,
\]
where $K_0$, $c_s^2 := (AB)^{-1} K_0^{-1} G_0$, and $M_0$ are defined as
\[
K_0 = \frac{4}{\sqrt{AB\phi'^2}} (2P_1 - F),
\]
\[c_s^2 = \frac{2r^2 \Gamma H \Xi \phi^2 - G \Xi^2 \phi'^2 - 4r^4 \Sigma H'^2 / B}{(2rH + \Xi\phi'^2)(2P_1 - F)},
\]
\[M_0 = -\frac{1}{a_3^2} \{ 2a_2 a_3 c_6 - a_2 (-4a_1 a_3 c_6 + a_3^2 (2c_3 - c_4^2) + a_3 (a_4 c_3 + 6c_6 a_1 + 2a_1 c_0) )
\]
\[+ a_3 (2a_2 c_3 + c_1 (4a_1 c_2 + 4c_6 a_1') + a_3 (3c_3 a_1' - a_1' c_2 + c_2 (a_2' - a_1' )))
\]
\[+ a_1 (-4a_1 c_6 a_0 + a_2^2 c_3 + a_3 (-a_1 - 2c_6 a_0' + 2a_1' c_0') ) \}.
\]

The no-ghost condition is given by $2P_1 - F > 0$, which is exactly the same as the one for higher multipoles with $\ell \geq 2$. The propagation speed also coincides with $c_{s2}$ given in Eq. (47). Since only the scalar wave is excited as a monopole perturbation, this result allows us to interpret $c_{s1}$ and $c_{s2}$ as the propagation speeds of gravitational and scalar waves, respectively.

## E. Dipole perturbation: $\ell = 1$

In the $\ell = 1$ case, it can be checked that the metric perturbations $h_{ab}$ depend on $K$ and $G$ only through the combination $K - G$. Therefore, we may set $K = 0$ from the outset. By using the gauge transformation $\Theta$, we can set $G = 0$. We can also set $\beta = 0$ by invoking $T$. We still have a freedom to choose $R$, which can be used to set $\delta\phi = 0$, and finally we are left with the four variables, $H_0$, $H_1$, $H_2$, and $\alpha$. Thus, the second-order Lagrangian for $\ell = 1$ is
\[
\frac{2\ell + 1}{2\pi} \mathcal{L} = H_0 [a_3 H_2^2 + 2a_3 \alpha' + (a_7 + a_8) H_2 + 2a_9 \alpha] + 2b_1 H_1^2 + H_1 (b_4 H_2 + 2b_5 \alpha)
\]
\[+ 2c_5 H_2 \alpha + c_6 H_2^2 + 2d_1 \alpha^2 + 2d_3 \alpha^2.
\]
As in the case of $\ell \geq 2$, $H_0$ is a Lagrange multiplier whose variation yields a constraint between $H_2$ and $\alpha$. Since $r$ derivatives of both $H_2$ and $\alpha$ appear in the constraint equation, we perform a field redefinition
\[
H_2 = \frac{1}{a_3} (\psi - 2a_4 \alpha),
\]0
and use the new variable $\psi$ to remove the $r$ derivative terms. Note that Eq. (54) is deduced from Eq. (28) by setting $\delta \phi = 0$. Variation with respect to $H_1$ gives

$$H_1 = -\frac{1}{4b_1}(b_4 \dot{H}_2 + 2b_5 \dot{\alpha}).$$

(55)

Substituting Eqs. (54) and (55) into the Lagrangian (53), we obtain the final result written solely in terms of $\psi$:

$$\frac{2\ell + 1}{2\pi} \mathcal{L} = \frac{1}{2} \mathcal{K}_1 \dot{\psi}^2 - \frac{1}{2} \mathcal{G}_1 \psi'^2 - \frac{1}{2} M_1 \psi^2,$$

(56)

where $\mathcal{K}_1$ and $\mathcal{G}_1$ are given by

$$\mathcal{K}_1 = 4\sqrt{AB}(2rH + 3\phi')^2 \frac{2P_1 - \mathcal{F}}{A^2\mathcal{H}^2 (4rH + P_2)^2},$$

(57)

$$\mathcal{G}_1 = \frac{AB^{3/2}}{\sqrt{A\mathcal{H}^2}} \frac{(2r^2\Gamma H\Xi\phi'^2 - G\Xi^2\phi'^2 - 4r^4\Sigma\mathcal{H}^2 / B)}{(4rH + P_2)^2}.$$

(58)

The no-ghost condition is given by $2P_1 - \mathcal{F} > 0$, which again coincides with the one for the other multipoles. The propagation speed along the radial direction, $c_{s2} := (AB)^{-1}K_1^{-1}\mathcal{G}_1$, is also the same as $c_{s2}$. This is consistent with the interpretation that $c_{s2}$ corresponds to the propagation speed of a scalar wave.

IV. APPLICATION TO SPECIFIC MODELS

A. General relativity

As a first example, let us consider the simplest case, i.e., GR without a scalar field $\phi$. This case amounts to setting $G_4 = M_1^2/2$ and $K = G_3 = G_5 = 0$, leading to $\mathcal{F} = \mathcal{G} = \mathcal{H} = 2P_1 = M_1^2 > 0$. Note that $c_{s2}^{\text{odd}} = 1$, which means that the odd mode propagates the speed of light, and $2P_1 - \mathcal{F} = 0$, which means that one of the even modes does not propagate, as expected.

The background metric is given by the Schwarzschild metric, $A(r) = B(r) = 1 - 2M/r$. Since we do not have the scalar degree of freedom in the case of GR, the number of degrees of freedom is reduced by one from the general case. Nonetheless, all the procedures to arrive at Eq. (31) are well defined in the GR limit. Therefore, Eq. (31) is still valid in GR and reduces to

$$\frac{2\ell + 1}{2\pi} \mathcal{L} = \frac{1}{M_1^2} \left[ \frac{1}{A} \dot{\psi}^2 - A\psi'^2 - \frac{3 + 3j^2 - j^4 + j^6 - 3(1 + j^2)^2A + 9(1 + j^2)A^2 - 9A^3}{r^2(j^2 + 1 - 3A)^2} \psi^2 \right],$$

(59)

where we used a new field $\Psi$ defined by $\psi = j(1 + j^2 - 3A)\Psi$. Introducing the tortoise coordinate, $r_* := \int dr / A(r)$, we find that the equation of motion for $\Psi$ is given by

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial r_*^2} = \frac{A}{r^2(j^2 + 1 - 3A)^2} \left( 3 + j^2 - j^4 + j^6 - 3(1 + j^2)^2A + 9(1 + j^2)A^2 - 9A^3 \right) \Psi = 0.$$

(60)

The Zerilli equation [13] is thus reproduced.

B. Nonminimally coupled scalar field

Let us next consider models in which the scalar field is nonminimally coupled to the Ricci scalar, corresponding to the following choice of the Horndeski functions:

$$K = X, \quad G_3 = 0, \quad G_4 = f(\phi), \quad G_5 = 0.$$

(61)

In this case, we have

$$\mathcal{F} = \mathcal{G} = \mathcal{H} = 2f, \quad 2P_1 - \mathcal{F} = \frac{2r^2 f(3f'_\phi)^2 \phi'^2}{(2f + rf'_\phi \phi')^2}.$$

(62)
Thus, it is sufficient to impose \( f > 0 \) in order to avoid ghost instabilities in the odd-parity and even-parity sectors. This is consistent with the naive expectation that the kinetic term for the graviton have the wrong sign and hence will be plagued by ghosts for \( f < 0 \). Note that the condition \( f > 0 \) depends on the profile of \( \phi \) but not on the concrete form of the metric. As for the propagation speeds, we find

\[
c_{s1}^2 = c_{s2}^2 = 1.
\]  

Thus, perturbations propagate at the speed of light.

### C. Bocharova-Bronnikov-Melnikov-Bekenstein solution

To give an explicit example of a black hole solution with scalar hair, we consider the theory with a conformally coupled scalar field:

\[
S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} R - \int d^4x \sqrt{-g} \left( \frac{1}{2} \partial^{\mu} \phi \partial_{\mu} \phi + \frac{R}{12} \phi^2 \right).
\]  

This is the special case of the previous example:

\[
K = X, \quad G_3 = 0, \quad G_4 = f(\phi) = \frac{M_{\text{Pl}}^2}{2} \frac{\phi^2}{12}, \quad G_5 = 0.
\]  

An exact black hole solution with a nontrivial scalar-field configuration has been found by Bocharova, Bronnikov, and Melnikov [14] and independently by Bekenstein [15]. The solution is given by

\[
ds^2 = - \left( 1 - \frac{M}{r} \right)^2 dt^2 + \frac{dr^2}{\left( 1 - \frac{M}{r} \right)^2} + r^2 d\Omega^2,
\]

\[
\phi = \pm \sqrt{6M_{\text{Pl}}M} \frac{r}{r - M},
\]

where \( M \) is a constant. The metric is exactly the same as the extremal Reissner-Nordström metric and the horizon is located at \( r = M \). Classical stability of the Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) solution against the monopole perturbation has been addressed in [16–18]. For \( \phi^2 > 6M_{\text{Pl}}^2 \) we have \( f < 0 \), giving rise to ghosts. In terms of \( r \), ghosts appear for \( r < 2M \). Thus, the BBMB solution is quantum mechanically unstable for \( r < 2M \). Note that this unstable region is outside the horizon, as the horizon is at \( r = M < 2M \). Explicitly, we have

\[
F = G = H = \frac{M_{\text{Pl}}^2 r(r - 2M)}{(r - M)^2}, \quad 2P_1 - F = \frac{3M_{\text{Pl}}^2 M^2 r(r - 2M)}{(r^2 - 3Mr + 3M^2)^2}.
\]  

### D. Models with a trivial scalar field configuration

In some classes of scalar-tensor theories, no-hair theorems for black holes have been established under certain assumptions [19–22]. In light of this, let us consider models in which the trivial solution with \( \phi = \phi_0 = \text{const} \) exists. Assuming that \( K, K_{\phi}, \cdots \) are not singular at \( \phi' = 0 \), it is verified that \( K = K_{\phi} = 0 \) must be satisfied at \( \phi = \phi_0 \) in order for the asymptotically flat background solution with \( \phi = \phi_0 \) everywhere to exist. In such theories, the background equations of motion [21] shows that \( A \) and \( B \) are uniquely determined as

\[
A(r) = B(r) = 1 - \frac{\mu}{r},
\]  

(69)
where $\mu$ is an integration constant. Thus, the metric takes the form of Schwarzschild and $r = \mu$ is the horizon location. In the present case, $\mathcal{F} = G = \mathcal{H} = 2G_4$ and we obtain the coefficients of the final second-order Lagrangian \cite{footnote1} as:

$$K_{11} = \frac{2(j^2 - 2)r^3}{G_4 j^2(r - \mu)[3\mu + (j^2 - 2)r]^2},$$  
$$\det(K) = \frac{4(j^2 - 2)r^6}{G_4^2 j^2(r - \mu)^2 [3\mu + (j^2 - 2)r]^2}. \tag{70}$$

We find that if $G_4 > 0$, $\mathcal{F}$, $G$ and $\mathcal{H}$ are positive and $K_{11}$ is also positive outside the horizon, $r > \mu$. The only nontrivial no-ghost condition is obtained from $\det(K) > 0$:

$$K_X G_4 - 2G_{3\phi} G_4 + 3G_{4\phi}^2 > 0. \tag{71}$$

The propagation speeds are given by

$$c_{s1}^2 = c_{s2}^2 = 1, \tag{73}$$
i.e., the two modes propagate at the speed of light.

Note that $G_5$ does not appear at all in the second-order Lagrangian in this case. If the background solution has no scalar hair, i.e., $\phi' = 0$, $G_5$ is irrelevant to the background configuration and still lurks at the level of linear perturbations.

E. Derivative coupling to the Einstein tensor

Finally, we provide an example of the scalar-tensor theory with a derivative coupling to the Einstein tensor:

$$S = \int d^4x \sqrt{-g} \left[ \zeta R - \eta \partial^\mu \phi \partial_\mu \phi + \beta G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2\Lambda \right]. \tag{74}$$

This action corresponds to the choice

$$K = 2(\eta X - \Lambda), \quad G_3 = 0, \quad G_4 = \zeta, \quad G_5 = -\beta \phi. \tag{75}$$

A static solution in this theory is given by \cite{23,24,26}

$$A = 1 - \frac{\mu}{r} + \frac{\eta}{3\beta} \frac{2\zeta \eta - \lambda}{2\eta + \lambda} r^2 + \frac{\lambda^2}{4\zeta^2 \eta^2 - \lambda^2} \frac{\arctan(r \sqrt{\eta/\beta})}{r \sqrt{\eta/\beta}}, \tag{76}$$

$$B = \frac{(\beta + \eta r^2)A}{\beta(rA)^3}, \tag{77}$$

$$\phi'^2 = -\frac{r\lambda(r^2A^2)'}{2(\beta + \eta r^2)^2A^2}, \tag{78}$$

where $\lambda = \zeta \eta + \beta \Lambda$ and $\mu$ is an integration constant. For this solution, we find

$$\mathcal{F} = \frac{2\beta \zeta + (3\zeta \eta + \beta \Lambda)r^2}{\beta + \eta r^2}, \tag{79}$$

$$\mathcal{G} = \mathcal{H} = \frac{2\beta \zeta + (\zeta \eta - \beta \Lambda)r^2}{\beta + \eta r^2}, \tag{80}$$

and therefore,

$$c_{s1}^2 = \frac{2\beta \zeta + (\zeta \eta - \beta \Lambda)r^2}{2\beta \zeta + (3\zeta \eta + \beta \Lambda)r^2}. \tag{81}$$

---

\footnote{Although Eq. \cite{footnote1} contains $\phi'$ in the denominator, this potentially dangerous term is canceled out by $2\mathcal{P}_1 - \mathcal{F}$ in the numerator and the final result remains finite. Since the mathematical procedure to derive Eq. \cite{footnote1} involves division by $\phi'$, we need to repeat the derivation of Eq. \cite{footnote1} starting from Eq. \cite{footnote1} by setting $\phi' = 0$ from the outset. All the results in this subsection are derived by such a manipulation.}
Further conditions for stability can be obtained from $2\mathcal{P}_1 - \mathcal{F} > 0$ and $c_{s2}^2 > 0$, but the full expressions are involved so that we only give their large $r$ behavior:

$$2\mathcal{P}_1 - \mathcal{F} \xrightarrow{r \to \infty} -\frac{4(\zeta \eta + \beta \Lambda)^2}{\eta(\zeta \eta + 3\beta \Lambda)},$$

$$c_{s2}^2 \xrightarrow{r \to \infty} \frac{3(-\zeta \eta + \beta \Lambda)}{\zeta \eta + 3\beta \Lambda}.$$

(82) (83)

Let us impose that $t$ is a time-like coordinate at large $r$, which puts the following condition: $A(r) > 0$ at large $r$, namely:

$$\frac{\eta \zeta \eta - \beta \Lambda}{\beta 3\zeta \eta + \beta \Lambda} > 0.$$  

(84)

Now, the stability conditions both for the odd-parity and even-parity perturbations are summarized as

$$\frac{3\zeta \eta + \beta \Lambda}{\eta} > 0, \quad \frac{\zeta \eta - \beta \Lambda}{\eta} > 0, \quad \eta(\zeta \eta + 3\beta \Lambda) < 0.$$  

(85)

The stable parameter region is shown in Fig. 1.

![FIG. 1: The stable region is colored red. $\zeta > 0$ is assumed and $\Lambda < 0$ must be satisfied.](image)

**TABLE I: No-ghost conditions and propagation speeds for some specific models are summarized.**

| Model                  | No-ghost conditions | Propagation speeds | Remarks                          |
|------------------------|---------------------|--------------------|----------------------------------|
| General Relativity     | No ghost            | Speed of light     | Reduces to the Zerilli equation  |
| Nonminimal coupling    | $f > 0$             | Speed of light     |                                  |
| BBMB solution          | Ghost appears for $r < 2M$ | Speed of light   | Horizon at $r = M$               |
| Models with no scalar hair | $K X G_4 - 2G_{\phi\phi} G_4 + 3G_{\phi\phi}^2 > 0$ | Speed of light | $G_5$ is irrelevant |
| $G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ coupling | $\text{[85]}$ | $\text{[81]}$ and $\text{[80]}$ | Allowed region is shown in Fig. 1 |
V. CONCLUSION

We have formulated the linear perturbation theory around static and spherically symmetric spacetime within the framework of the Horndeski theory, i.e., the most general scalar-tensor theory having second-order field equations both for the metric and the scalar field. Following the previous work [2] in which the analysis of the odd-parity perturbations is presented, we have focused in this paper on the even-parity perturbations. Expanding the Horndeski Lagrangian to second order in perturbations and eliminating the auxiliary variables by use of the constraint equations, we have derived the reduced Lagrangian that contains only dynamical variables. The resultant Lagrangian shows that there are two dynamical variables in the even-parity sector, with at most first $t$ and $r$ derivatives acting on them, ensuring that the perturbation equations derived from the Lagrangian are of second order, as it should be due to the second-order nature of the Horndeski theory. We have obtained two conditions for the absence of ghosts: one coincides with the stability condition for the odd-parity perturbations and the other provides a new criterion. The propagation speeds have also been derived. One of them can be interpreted as the propagation speed of gravitational waves and is exactly the same as that of the odd-type perturbation. The other one, the propagation speed of the scalar wave, is in general different from that of gravitational waves. As for the monopole and dipole perturbations, which is absent in general relativity, there is only one dynamical degree of freedom. The no-ghost conditions and the propagation speeds of the monopole and dipole modes are the same as those of the scalar wave with higher multipoles $\ell \geq 2$.

Our formulation can be applied to any theories belonging to the Horndeski class. As a demonstration, we have considered several concrete models including GR, a nonminimally coupled scalar field, a black hole without scalar hair, the BBMB solution, and the derivative coupling to the Einstein tensor. The main results are summarized in Table I.

Acknowledgments

This work is supported in part by JSPS Grant-in-Aid for Young Scientists (B) No. 24740161 (TK), Grant-in-Aid for Scientific Research on Innovative Areas No. 25103505 (TS) from The Ministry of Education, Culture, Sports, Science and Technology (MEXT), and JSPS Postdoctoral Fellowships for Research Abroad (HM).

Appendix A: Background equations

In this Appendix, we summarize the field equations for a static and spherically symmetric background. The following field equations were first derived in Ref. [2].

Taking the gauge $C(r) = 1$, the functions $A(r)$ and $B(r)$ in the background metric can be determined by solving the field equations supplemented with appropriate boundary conditions. Substituting the metric to the action and varying it with respect to $A$, $B$, $C$ and $\phi$, we obtain the background field equations as

\[
\mathcal{E}_A = 0, \quad \mathcal{E}_B = 0, \quad \mathcal{E}_C = 0, \quad \mathcal{E}_\phi = \frac{1}{r^2} \sqrt{\frac{B}{A}} \frac{d}{dr} \left( r^2 \sqrt{AB} J \right) - S = 0,
\]  
(A1)
\[ E_A := K + B\phi' X' G_{3X} - 2X G_{3\phi} + \frac{2}{r} \left( \frac{1 - B}{r} - B' \right) G_4 + \frac{4B}{r} \left( \frac{1}{r} + \frac{X'}{X} + \frac{B'}{B} \right) X G_{4X} + \frac{8B}{r} X X' G_{4XX} \]

\[ - B\phi' \left( \frac{4}{r} + \frac{X'}{X} \right) G_{4\phi} + 4X G_{4\phi\phi} + 2B\phi' \left( \frac{4}{r} - \frac{X'}{X} \right) X G_{4\phi X} \]

\[ + \frac{B\phi'}{r^2} \left[ \left( 1 - 3B \right) \frac{X'}{X} - 2B' \right] X G_{5X} - \frac{2}{r^2} B^2 \phi' X X' G_{5XX} \]

\[ - \frac{2}{r} \left[ \frac{1 + B}{r} + 2B' \frac{X'}{X} + B' \right] X G_{5\phi} - \frac{4}{r} B\phi' X G_{5\phi\phi} + \frac{4B}{r} \left( \frac{1}{r} - \frac{X'}{X} \right) X^2 G_{5\phi X}, \quad (A2) \]

\[ E_B := K - 2X K_X + \left( \frac{4}{r} + \frac{A'}{A} \right) B\phi' X G_{4X} + 2X G_{3\phi} \]

\[ + \frac{2}{r} \left( \frac{1 - B}{r} - B \frac{A'}{A} \right) G_4 - \frac{4}{r} \left( \frac{1 - 2B}{r} - 2B \frac{A'}{A} \right) X G_{4X} + \frac{8B}{r} \left( \frac{1}{r} + \frac{A'}{A} \right) X^2 G_{4XX} \]

\[ - \left( \frac{4}{r} + \frac{A'}{A} \right) B\phi' G_{4\phi} - 2 \left( \frac{4}{r} + \frac{A'}{A} \right) B\phi' X G_{4\phi X} + \frac{B\phi'}{r^2} \left( 1 - 5B \right) \frac{A'}{A} X G_{5X} \]

\[ - \frac{2B^2 \phi' A'}{r^2} X^2 G_{5XX} + \frac{2}{r} \left( \frac{1 - 3B}{r} - 3B \frac{A'}{A} \right) X G_{5\phi} - \frac{4B}{r} \left( \frac{1}{r} + \frac{A'}{A} \right) X^2 G_{5\phi X}, \quad (A3) \]

\[ E_C := K + 2X G_{3\phi} - X \left( B' \phi' + 2B\phi'' \right) G_{3X} \]

\[ - \left[ \frac{1}{r} \sqrt{\frac{B}{A}} \left( \frac{1}{r} \sqrt{\frac{B}{A}} A' \right)' + \frac{B'}{r} \right] G_4 - B\phi' \left( \frac{2}{r} + \frac{A'}{A} + \frac{B'}{B} + 2\frac{\phi''}{\phi'} \right) G_{4\phi} \]

\[ + BX \left( \frac{A'^2}{A^2} + \frac{2B'}{r B} + \frac{A' B'}{A B} + \frac{2(A' + r A'')}{r A} \right) \left( G_{4X} - \frac{1}{2} G_{5\phi} \right) + BX' \left( \frac{2}{r} + \frac{A'}{A} \right) \left( G_{4X} - G_{5\phi} \right) + 4X G_{4\phi\phi} \]

\[ + 2B\phi' \left( \frac{2}{r} + \frac{A'}{A} \right) X G_{4\phi X} + 2r \left( \frac{2}{r} + \frac{A'}{A} \right) X X' G_{4XX} \]

\[ - \frac{B^2 \phi' A'}{2r} \left[ \left( \frac{2}{r} \right) A' \frac{A'^2}{A^2} + \frac{A'}{A} \left( \frac{2B'}{B} + 3 \frac{X'}{X} \right) \right] X G_{5X} - B\phi' X \left( \frac{2}{r} + \frac{A'}{A} \right) G_{5\phi} \]

\[ - \frac{B}{r} \left[ \frac{2X'}{X} - r A' \left( \frac{2}{r} - \frac{X'}{X} \right) \right] X^2 G_{5\phi X} - \frac{B^2 \phi' A'}{r A} X X' G_{5XX}, \quad (A4) \]

\[ J := \phi' K_X + \left( \frac{4}{r} + \frac{A'}{A} \right) X G_{4X} - 2\phi' G_{3\phi} + 2\phi' \left( \frac{1 - B}{r^2} - \frac{B A'}{r A} \right) G_{4X} - \frac{4B}{r} \left( \frac{1}{r} + \frac{A'}{A} \right) X G_{4XX} \]

\[ - \frac{2}{r} \left( \frac{4}{r} + \frac{A'}{A} \right) X G_{4\phi X} + \frac{1 - 3B A'}{r^2} X G_{5X} - \frac{2B A'}{r^2} X^2 G_{5XX} \]

\[ - \frac{2\phi'}{r^2} \left( \frac{1 - B}{r^2} - \frac{B A'}{r A} \right) G_{5\phi} + \frac{2B}{r^2} \left( \frac{1}{r} + \frac{A'}{A} \right) X G_{5\phi X}, \quad (A5) \]

\[ S := -K_\phi + 2X G_{\phi\phi} - B\phi' X' G_{3\phi X} - \left[ \frac{B}{2} \left( \frac{A'}{A} \right)' \right]^2 + \frac{2}{r} \left( \frac{1 - B}{r} - B' \right) - \frac{B}{2} \left( \frac{2A''}{A} + \frac{A'}{A} \left( \frac{4}{r} + \frac{B'}{B} \right) \right) \right] G_{4\phi} \]

\[ + B \left[ \frac{X'}{X} \left( \frac{4}{r} + \frac{A'}{A} \right) + \frac{4}{r} \left( \frac{1}{r} + \frac{A'}{A} \right) \right] X G_{4\phi X} + 2 \left( \frac{1 - B}{r^2} - \frac{B A'}{r A} \right) X G_{5\phi\phi} \]

\[ - \frac{B\phi'}{r^2} \left( \frac{X'}{X} + B \frac{A'}{A} \right) X G_{5\phi X}, \quad (A6) \]

and we have set \( C(r) = 1 \) after varying the action.
Appendix B: Expressions of $a_1, a_2, \ldots$.

The coefficients in the second-order Lagrangian \[26\] are given explicitly by

\begin{align*}
a_1 &= \sqrt{AB} \Xi, 
\quad \text{(B1)} \\
a_2 &= \frac{\sqrt{AB}}{2\phi'} \left[ 2\phi' \Xi' - \left( 2\phi'' - \frac{A'}{A} \phi' \right) \Xi + 2r \left( \frac{A'}{A} - \frac{B'}{B} \right) \mathcal{H} + \frac{2r^2}{B} (\mathcal{E}_B - \mathcal{E}_A) \right], 
\quad \text{(B2)} \\
a_3 &= -\frac{\sqrt{AB}}{2} (\phi' \Xi + 2r \mathcal{H}), 
\quad \text{(B3)} \\
a_4 &= \sqrt{AB} \mathcal{H}, 
\quad \text{(B4)} \\
a_5 &= -\sqrt{\frac{A}{B}} r^2 \frac{\partial \mathcal{E}_A}{\partial \phi} = a'_2 - a''_1, 
\quad \text{(B5)} \\
a_6 &= -\sqrt{\frac{A}{B}} r^2 \frac{1}{\phi'} \left( r \mathcal{H}' + \mathcal{H} - \mathcal{F} \right), 
\quad \text{(B6)} \\
a_7 &= a'_3 + \frac{r^2}{2} \sqrt{\frac{A}{B}} \mathcal{E}_B, 
\quad \text{(B7)} \\
a_8 &= -\frac{a'_4}{2B}, 
\quad \text{(B8)} \\
a_9 &= \sqrt{\frac{A}{B}} \frac{d}{dr} \left( r \sqrt{B} \mathcal{H} \right) = a'_4 + \left( \frac{1}{r} - \frac{A'}{2A} \right) a_4, 
\quad \text{(B9)} \\
b_1 &= \frac{1}{2} \sqrt{\frac{B}{A}} \mathcal{H}, 
\quad \text{(B10)} \\
b_2 &= -2 \sqrt{\frac{B}{A}} \Xi, 
\quad \text{(B11)} \\
b_3 &= \sqrt{\frac{B}{A}} \frac{1}{\phi'} \left[ \left( 2\phi'' + \frac{B'}{B} \phi' \right) \Xi - 2r \left( \frac{A'}{A} - \frac{B'}{B} \right) \mathcal{H} + \frac{2r^2}{B} \mathcal{E}_A \right] = \frac{2}{A} (a'_1 - a_2) + \frac{2r^2}{\sqrt{AB} \phi'} \mathcal{E}_B, 
\quad \text{(B12)} \\
b_4 &= \sqrt{\frac{B}{A}} (\phi' \Xi + 2r \mathcal{H}), 
\quad \text{(B13)} \\
b_5 &= -2b_1, 
\quad \text{(B14)} \\
c_1 &= -\frac{1}{\sqrt{AB}} \Xi, 
\quad \text{(B15)} \\
c_2 &= -\sqrt{AB} \left( \frac{A'}{2A} \Xi + r \Gamma - \frac{r^2 \phi'}{X} \Sigma \right), 
\quad \text{(B16)} \\
c_3 &= \frac{r^2}{2} \sqrt{\frac{A}{B}} \frac{\partial \mathcal{E}_B}{\partial \phi}, 
\quad \text{(B17)} \\
c_4 &= \frac{1}{2} \sqrt{\frac{A}{B}} \Gamma, 
\quad \text{(B18)} \\
c_5 &= -\frac{1}{2} \sqrt{AB} \left( \phi' \Gamma + \frac{A'}{A} \mathcal{H} + \frac{2}{r} \mathcal{G} \right), 
\quad \text{(B19)} \\
c_6 &= \frac{r^2}{2} \sqrt{\frac{A}{B}} \left( \Sigma + \frac{A'B \phi'}{2r^2 A} \Xi + \frac{B \phi'}{r} \Gamma - \frac{1}{2} \mathcal{E}_B + \frac{B}{r^2} \mathcal{G} + \frac{A'B}{rA} \mathcal{H} \right), 
\quad \text{(B20)}
\end{align*}
\[ d_1 = b_1, \]

\[ d_2 = \sqrt{AB} \Gamma, \]

\[ d_3 = \frac{\sqrt{AB}}{r^2} \left[ \frac{2r}{\phi'} \left( \frac{A'}{A} - \frac{B'}{B} \right) \mathcal{H} - r^2 \left( \frac{2}{r} - \frac{A'}{A} \right) \frac{\partial \mathcal{H}}{\partial \phi} + \frac{2}{B} \phi' (F - G) \right] \]

\[ - \frac{r^2}{2} \left( 2 \phi'' + \frac{B'}{B} \phi' \right) \left( \Gamma_1 + \frac{2}{r} \Gamma_2 \right) - \frac{2r^2}{B} \phi' (E_A - E_B) \],

\[ d_4 = \frac{\sqrt{AB}}{r^2} (G - r^2 \mathcal{E}_B), \]

\[ e_1 = \frac{1}{2\sqrt{AB}} \left[ \frac{r^2}{X} (E_A - E_B) - \frac{2}{\phi'} \Xi' + \left( \frac{A'}{A} - \frac{X'}{X} \right) \Xi \phi' + \frac{2B}{X} F - \frac{2rB}{X} \mathcal{H}' - \mathcal{H} - \frac{B^2}{rX} \Xi \right] \left( \frac{r^2A}{B} \right), \]

\[ e_2 = -\sqrt{AB} \Xi, \]

\[ e_3 = r^2 \sqrt{\frac{A}{B}} \frac{\partial \phi}{\partial \phi}, \]

\[ e_4 = -\frac{1}{4} \sqrt{\frac{A}{B}} \frac{\partial \phi}{\partial \phi} \]

where \( \dot{e}_4 \) is defined by

\[ \dot{e}_4 = \frac{2}{X} (E_A - E_B) - \frac{2}{\phi'} \Gamma' - \frac{2}{r^2X} \left( 1 - rB \frac{A'}{A} \right) F + \frac{2B}{r^2X} G + \frac{2}{r^2X} \left( -2rB \frac{A'}{A} + 1 - B + rB' \right) \mathcal{H} \]

\[ - \frac{2B}{rX} \phi' - \frac{B A'}{X} \mathcal{H}' - \frac{2}{r \phi'} \left( 2 - \frac{A'}{A} \right) \frac{\partial \mathcal{H}}{\partial \phi} + \frac{1}{r \phi'} \left( 2 - \frac{A'}{A} \right) \left[ -2r(1 - B) + rB \frac{A'}{A} \right] \Xi \]

\[ + \left[ -\frac{2}{A} A' \phi' + \frac{B}{2} \left( \frac{A'}{A} \right)^2 \phi' - \frac{B'}{B} \phi' + \frac{2}{r} (1 - B) \phi' - 2 \phi'' \right] \Gamma_1 \frac{\phi}{\phi'^2} \]

\[ + \left[ -\frac{2}{A} A' \phi' - r(1 - B) \left( \frac{A'}{A} \right)^2 \phi' - \frac{B'}{B} \phi' + \frac{4}{r} (1 - B) \phi' - 4 \phi'' \right] \frac{\Gamma_2}{r \phi'^2}. \]

We have the following simple relation among \( \Xi, \Gamma_1, \) and \( \Gamma_2: \)

\[ \Xi = \frac{r^2}{2} \Gamma_1 + r \Gamma_2 - 2XG_{5X}. \]
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