DEFINING RELATIONS ASSOCIATED WITH THE PRINCIPAL
\textit{sl}(2)-SUBALGEBRAS OF SIMPLE LIE ALGEBRAS

PAVEL GROZMAN, DIMITRY LEITES

Department of Mathematics, University of Stockholm

Abstract. The notion of defining relations is, clearly, well-defined for any nilpotent Lie algebra. Therefore, a conventional way to present a simple Lie algebra \( g \) is by splitting it into the direct sum of a commutative Cartan subalgebra and two maximal nilpotent subalgebras \( g_\pm \) (positive and negative). Though there are many (about \((3 \cdot \text{rank} \ g)^2\)) relations between the \( 2 \cdot \text{rank} \ g \) generators of \( g_\pm \) (separately), they are neat; they are called Serre relations. The generators of \( g_\pm \) generate \( g \) as well.

It is possible to define the notion of relations for generators of different type. For instance, with the principal embeddings of \( \text{sl}(2) \) into \( g \) one can associate only two elements that generate \( g \); we call them Jacobson's generators. We explicitly describe the associated with the principle embeddings of \( \text{sl}(2) \) presentations of simple Lie algebras, all finite dimensional and certain infinite dimensional ones; namely, of the Lie algebra “of matrices of a complex size” realized as a subalgebra of the Lie algebra of differential operators in 1 indeterminate.

The relations obtained are rather simple, especially for non-exceptional algebras. In contradistinction with the conventional presentation there are just 9 relations between Jacobson’s generators for \( \text{sl}(\lambda) \) series and not many more for other finite dimensional algebras.

Our results might be of interest in applications to integrable systems (like vector-valued Liouville (or Leznov-Saveliev, or 2-dimensional Toda) equations and KdV-type equations) based on the principal subalgebras \( \text{sl}(2) \). They also indicate how to \( q \)-quantize the Lie algebra of matrices of complex size.

Introduction

This is our paper published in: Dobrushin R., Minlos R., Shubin M. and Vershik A. (eds.) Contemporary Mathematical Physics (F.A. Berezin memorial volume), Amer. Math. Soc. Transl. Ser. 2, vol. 175, Amer. Math. Soc., Providence, RI (1996) 57–68. We just wish to make it more accessible.

This paper continues the description of presentations of simple Lie superalgebras. It is the direct continuation of [LSe] and [LP], where the case of the simplest (for computations) base is considered and where non-Serre relations are first described, though in a different setting.

In what follows we describe some “natural” generators and relations for \textit{simple finite dimensional Lie algebras} over \( \mathbb{C} \). The answer is important in questions when it is needed.

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to identify an algebra given its generators and relations. (Examples of such are Eastbrook-Vahlquist prolongations, Drienfield’s quantum algebras, symmetries of differential equations, integrable systems, etc.).

If $g$ is nilpotent, the problem of its presentation has a natural and unambiguous solution: representatives of $g/[g, g]$ are generators of $g$ and $H_2(g)$ describes relations.

On the other hand, if $g$ is simple, then $g = [g, g]$ and there is no “most natural” way to select generators of $g$. The choice of generators is not unique.

Still, among algebras with the property $g = [g, g]$, the simple ones are distinguished by the fact that their structure is very well known. By trial and error people discovered that, for finite dimensional simple Lie algebras, there are certain “first among equal” sets of generators:

1) Chevalley generators corresponding to positive and negative simple roots;
2) a pair of generators that generate any finite dimensional simple Lie algebra associated with the principal $\mathfrak{sl}(2)$-subalgebra (considered below).

The relations associated with Chevalley generators are well-known, see e.g., [OV]. These relations are called Serre relations.

The possibility to generate any simple finite dimensional Lie algebra by two elements was first claimed by N. Jacobson (an exercise in [J]); for the first (as far as we know) proof, see [BO]. We do not know what generators Jacobson had in mind; [BO] take for them linear combinations with generic coefficients of positive and negative root vectors, respectively; nothing like a “natural” choice of what we suggest to refer to as Jacobson’s generators was ever proposed: to generate a simple algebra with only two elements is tempting but nobody did so explicitly, yet. To check whether the relations between these elements are nice-looking without a computer (cf. an implicit description in [F]). We did it with the help of a Mathematica-based package developed by Grozman [GL]. As far as we could test, the relations for any other pair of generators chosen in a way distinct from ours are more complicated.

One of our aims was to decipher [F]. Certain statements from [F] are clarified (also with the help of a computer) in [PH] appeared as [PH1]; we use some of these clarifications in §2.

In what follows we explicitly list the relations between Jacobson’s generators; well, actually, for beautification of relations we introduce a third generator. Throughout the paper $g$ is a simple Lie algebra.

§1. The case of a finite dimensional $g$

1.1. Principal embeddings. There exists only one (up to equivalence) embedding $\rho : \mathfrak{sl}(2) \rightarrow g$ such that $g$, considered as $\mathfrak{sl}(2)$-module, splits into rank $g$ irreducible modules. (The reader may consider this statement as an exercise or consult [D], [LS] or [OV].) This embedding is called principal and, sometimes, minimal because for other embeddings (there are plenty of them) the number of irreducible $\mathfrak{sl}(2)$-modules is $> \text{rank } g$. Example: for $g = \mathfrak{sl}(n)$, $\mathfrak{sp}(2n)$ or $\mathfrak{o}(2n + 1)$, the principal embedding is the one corresponding to the irreducible representation of $\mathfrak{sl}(2)$ of dimension $n$, $2n$, $2n + 1$, respectively.

For completeness, let us recall how the irreducible $\mathfrak{sl}(2)$-modules with highest weight look like. (They are all of the form $M^\mu = L^\mu$, where $\mu \notin \mathbb{N}$, and $L^n$, where $n \in \mathbb{N}$, described
below.) Select the following basis in \(\mathfrak{sl}(2)\):
\[
X^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

The \(\mathfrak{sl}(2)\)-module \(M^n\) is illustrated with a graph whose nodes are eigenvectors \(l_i\) of \(H\) with the weight indicated;
\[
\cdots \leftarrow \mu \circ - \mu + 2 \circ - \cdots - \mu - 2 \circ - \mu \rightarrow \circ
\]
the edges depict the action of \(X^-\) (the action of \(X^+\) is directed to the right, that of \(X^-\) to the left): \(X^- l_{\mu} = l_{\mu - 2}\) and
\[
(1.1) \quad X^+ l_{\mu - 2i} = X^+((X^-)^i l_{\mu}) = i(\mu - i + 1)l_{\mu - 2i + 2}; \quad X^+(l_{\mu}) = 0.
\]

As follows from (1.1), the module \(M^n\) for \(n \in \mathbb{N}\) has an irreducible submodule isomorphic to \(M^{-n-2}\); the quotient, obviously irreducible as follows from the same (1.1), will be denoted by \(L^n\).

As the \(\mathfrak{sl}(2)\)-module corresponding to the principle embedding, a simple finite dimensional Lie algebra \(\mathfrak{g}\) is as follows (cf. [OV], Table 4):

**Table 1.1. \(\mathfrak{g}\) as the \(\mathfrak{sl}(2)\)-module**

| \(\mathfrak{g}\) | the \(\mathfrak{sl}(2)\)-spectrum of \(\mathfrak{g} = L^2 \oplus L^{k_1} \oplus L^{k_2} \ldots\) |
|----------------|------------------------------------------------------------------|
| \(\mathfrak{sl}(n)\) | \(L^2 \oplus L^4 \oplus L^6 \ldots \oplus L^{2n-2}\) |
| \(\mathfrak{o}(2n+1), \mathfrak{sp}(2n)\) | \(L^2 \oplus L^6 \oplus L^{10} \ldots \oplus L^{4n-2}\) |
| \(\mathfrak{o}(2n)\) | \(L^2 \oplus L^6 \oplus L^{10} \ldots \oplus L^{4n-2} \oplus L^{2n-2}\) |
| \(\mathfrak{g}_2\) | \(L^2 \oplus L^{10}\) |
| \(\mathfrak{f}_4\) | \(L^2 \oplus L^{10} \oplus L^{14} \oplus L^{22}\) |
| \(\mathfrak{e}_6\) | \(L^2 \oplus L^8 \oplus L^{10} \oplus L^{14} \oplus L^{16} \oplus L^{22}\) |
| \(\mathfrak{e}_7\) | \(L^2 \oplus L^{10} \oplus L^{14} \oplus L^{18} \oplus L^{22} \oplus L^{26} \oplus L^{34}\) |
| \(\mathfrak{e}_8\) | \(L^2 \oplus L^{14} \oplus L^{22} \oplus L^{26} \oplus L^{34} \oplus L^{38} \oplus L^{46} \oplus L^{58}\) |

One can show that \(\mathfrak{g}\) can be generated by two elements: \(x := \nabla_+ \in L^2 = \mathfrak{sl}(2)\) and a lowest weight vector \(z := l_{-r}\) from an appropriate module \(L^r\) other than \(L^2\) from Table 1.1. For the role of this \(L^r\) we take either \(L^{k_1}\) if \(\mathfrak{g} \neq \mathfrak{o}(2n)\) or the last module \(L^{2n-2}\) in the above table if \(\mathfrak{g} = \mathfrak{o}(2n)\). (Clearly, \(z\) is defined up to proportionality; we will assume that a basis of \(L^r\) is fixed and denote \(z = t \cdot l_{-r}\) for some \(t \in \mathbb{C}\) that can be fixed at will.)

The exceptional choice for \(\mathfrak{o}(2n)\) is occasioned by the fact that by choosing \(z \in L^r\) for \(r \neq 2n - 2\) instead, we generate \(\mathfrak{o}(2n - 1)\).

We call the above \(x\) and \(z\), together with \(u := \nabla_- \in L^2\) for good measure, *Jacobson’s generators*. The presence of \(u\) considerably simplifies the form of the relations, though slightly increases their number. (One might think that taking the symmetric to \(z\) element \(l_r\) will improve the relations even more but in reality just the opposite happens.)
1.2. Relations between Jacobson’s generators. First, observe that if an ideal of a free Lie algebra is homogeneous (with respect to the degrees of the generators of the algebra), then the number and the degrees of the defining relations (i.e., the generators of the ideal) is uniquely defined provided the relations are homogeneous. This is obvious.

A simple Lie algebra \( g \), however, is the quotient of a free Lie algebra \( F \) modulo a nonhomogeneous ideal, \( I \), the ideal without homogeneous generators. Therefore, we can speak about the number and the degrees of relations only conditionally. Our conditions is the possibility for any element \( x \in I \) to be expressible via the generators \( g_1, ... \) of \( I \) by a formula of the form

\[
(*) \quad x = \sum c_i g_i, \quad \text{where } c_i \in F \text{ and } \deg c_i + \deg g_i \leq \deg x \text{ for all } i.
\]

(The degree is calculated with respect to that of the generators of \( F \).)

Under condition \((*)\) the number and the degree of relations is uniquely defined. Now we can explain the necessity for the extra generator \( y \): without it the weight relations would have been of very high degree.

We divide the relations between Jacobson’s generators into the types corresponding the number of occurrence of \( z \) in them:

0. Relations in \( L^2 = \mathfrak{sl}(2) \);
1. Relations coming from \( L^2 \otimes L^{k_1} \);
2. Relations coming from \( L^{k_1} \wedge L^{k_1} \);
\[ \geq 3 \]. Relations coming from \( L^{k_1} \wedge L^{k_1} \wedge L^{k_1} \wedge \ldots \) with \( \geq 3 \) factors; among the latter relations we distinguish one: of type \( \infty \) — the relation bounding the dimension. (For small rank \( g \), the relation of type \( \infty \) can be of the above types.)

Observe that, apart from relations of type \( \infty \), the relations of type \( \geq 3 \) are those of type 3 except for \( \tau_7 \) which satisfies stray relations of type 4 and 5.

The relations of type 0 are the well-known relations in \( \mathfrak{sl}(2) \)

\[
(0) \quad \begin{align*}
0.1. \quad [x, y], x] &= 2x, \\
0.2. \quad [x, y], y] &= -2y.
\end{align*}
\]

The relations of type 1 express that the space \( L^{k_1} \) is the \((k_1 + 1)\)-dimensional \( \mathfrak{sl}(2) \)-module. To simplify notations we denote: \( z_i = (\text{ad } x)^i z \).

\[
(1) \quad 1.1. \quad [y, z] = 0, \quad 1.2. \quad [[x, y], z] = -k_1 z, \quad 1.3. \quad z_{k_1+1} = 0 \quad \text{with } k_1 \text{ from Table 1.1}.
\]

1.3. Theorem. For the simple finite dimensional Lie algebras, all the relations between Jacobson’s generators are the above relations \((0), (1)\) and the relations from Table 3.1.
This observation is used now and again; Feigin applied it in [F]. As deciphered in [PH], Feigin wrote, actually, that setting
\[ X^- = \frac{d}{du}, \quad H = 2u \frac{d}{du} - (\lambda - 1), \quad X^+ = u^2 \frac{d}{du} - (\lambda - 1)u \]
we obtain a realization of \( \mathfrak{sl}(2) \) by differential operators. This realization can be extended to a morphism of associative algebras: \( R : U(\mathfrak{sl}(2)) \rightarrow \mathbb{C}[u, \frac{d}{du}] \). The kernel of \( R \) is the ideal generated by \( \Delta - \lambda^2 + 1 \), where \( \Delta = 2(X^+X^- + X^-X^+) + H^2 \). Observe, that this morphism is not an epimorphism, either. Though not so easy to describe as \( U(\mathfrak{g}) \) in the PBW theorem, the image of this morphism turned out to be very interesting; this is our Lie algebra of matrices of “complex size”.

**Remark.** In their proof of certain statements from [F] that we will recall, [PH] make use of the well-known fact that the Casimir operator \( \Delta \) acts on the irreducible \( \mathfrak{sl}(2) \)-module \( L^\mu \) with highest weight \( \mu \) (i.e., \( H \cdot l_\mu = \mu \cdot l_\mu \) and \( X^+l_\mu = 0 \)) as the scalar operator of multiplication by \( \mu^2 + 2\mu \). The passage from [PH]'s \( \lambda \) to [F]'s \( \mu \) is done with the help of a shift by the weight \( \rho \), the half-sum of positive roots. Since \( \rho \) for \( \mathfrak{sl}(2) \) can be identified with 1, we have \( \lambda - 1 = \mu \), and \( (\lambda - 1)^2 + 2(\lambda - 1) = \lambda^2 - 1 \).

Consider the Lie algebra \( U(\mathfrak{sl}(2))_L \) associated with the associative algebra \( U(\mathfrak{sl}(2)) \). (We denote by the subscript \( _L \) the functor that sends an associative algebra to the Lie algebra with the bracket determined by the commutator.) It is easy to see that, as \( \mathfrak{sl}(2) \)-module,

\[ U(\mathfrak{sl}(2))_L/(\Delta - \lambda^2 + 1) \simeq L^0 \oplus L^2 \oplus L^4 \oplus \cdots \oplus L^{2n} \oplus \cdots \simeq \mathbb{C}[R(\mathfrak{sl}(2))]_L \subset \mathbb{C}[x, \frac{d}{dx}] \]

Observe the crucial difference between the associative algebra \( \mathbb{C}[\mathfrak{sl}(2)] \) generated by \( \mathfrak{sl}(2) \) and the associative algebra \( \mathbb{C}[R(\mathfrak{sl}(2))] \) generated by the image of \( \mathfrak{sl}(2) \) under \( R \). The associated graded algebras are generated by 3 and 2 generators, respectively.

It is not difficult to show (for details, see [PH]) that the Lie algebra
\[ U_n = U(\mathfrak{sl}(2))_L/(\Delta - (n^2 - 1)) \]
contains an ideal \( I_n \) for \( n \in \mathbb{N} \setminus \{0, 1\} \), and the quotient \( U_n = /I_n \) is the conventional \( \mathfrak{gl}(n) \). In [PH] it is proved that, for \( \lambda \neq \mathbb{Z} \), the Lie algebra \( U_\lambda = U(\mathfrak{sl}(2))_L/(\Delta - \lambda^2 + 1) \) has only one ideal — the space of constants. This justifies Feigin’s suggestive notations
\[ \mathfrak{sl}(\lambda) = \mathfrak{gl}(\lambda)/<1>, \text{ where } \mathfrak{gl}(\lambda) = \begin{cases} U_\lambda & \text{for } \lambda \not\in \mathbb{N} \setminus \{0, 1\} \\ U_n/I_n & \text{for } n \in \mathbb{N} \setminus \{0, 1\} \end{cases} \]

The definition directly implies that \( \mathfrak{sl}(-\lambda) \cong \mathfrak{sl}(\lambda) \), so speaking about real values of \( \lambda \) we can confine ourselves to the nonnegative values.

**2.2. There is no analog of Cartan matrix for \( \mathfrak{sl}(\mu) \).** Are there Chevalley generators, i.e., elements \( X^\pm_\lambda \) of degree \( \pm1 \) and \( H_i \) of degree 0 (the degree is the weight with respect to the \( \mathfrak{sl}(2) = L^2 \subset \mathfrak{sl}(\mu) \)) such that
\[ [X^+_i, X^-_j] = \delta_{ij}H_i, \quad [H_i, H_j] = 0? \]
The answer is **NO**: \( \mathfrak{sl}(\mu) \) is too small. We can complete it by considering infinite sums of its elements, but the completion erases the difference between different \( \mu \)’s:
Proposition. The completion of $\mathfrak{sl}(\mu)$ generated by Jacobson’s generators (see Table 3.2) is isomorphic to $\mathfrak{diff}(1)$.

2.3. The invariants of the map

$$X \mapsto -X^T \quad \text{for} \quad X \in \mathfrak{gl}(n).$$

constitute $\mathfrak{o}(n)$ if $n \in 2\mathbb{N} + 1$ or $\mathfrak{sp}(n)$ if $n \in 2\mathbb{N}$. By analogy, Feigin defined $\mathfrak{o}(\lambda)$ and $\mathfrak{sp}(\lambda)$ as subalgebras of $\mathfrak{gl}(\lambda) = \oplus_{k \geq 0} L^2k$ invariant with respect to the involution

$$X \mapsto \begin{cases} -X & \text{if } X \in L^{4k} \\ X & \text{if } X \in L^{4k+2} \end{cases},$$

the analogue of (2.3). Since $\mathfrak{o}(\lambda)$ and $\mathfrak{sp}(\lambda)$ — the subalgebras of $\mathfrak{gl}(\lambda)$ singled out by the involution (2.4) — differ by a shift of the parameter $\lambda$, it is natural to denote them uniformly (but so as not to confuse with the Lie superalgebras of series $\mathfrak{osp}$), namely, by $\mathfrak{o}/\mathfrak{sp}(\lambda)$. For integer values of the parameter it is clear that $\mathfrak{o}/\mathfrak{sp}(\lambda) = \begin{cases} \mathfrak{o}(\lambda) & \text{if } \lambda \in 2\mathbb{N} + 1 \\ \mathfrak{sp}(\lambda) & \text{if } \lambda \in 2\mathbb{N} \end{cases}$.

In the realization of $\mathfrak{sl}(\lambda)$ by differential operators the above involution is the passage to the adjoint operator; hence, $\mathfrak{o}/\mathfrak{sp}(\lambda)$ is a subalgebra of $\mathfrak{sl}(\lambda)$ consisting of self-adjoint operators.

2.4. The Lie algebra $\mathfrak{sl}(\lambda)$ as a subalgebra of $\mathfrak{sl}_+(\infty)$. Recall that $\mathfrak{sl}_+(\infty)$ often denotes the Lie algebra of infinite (in one direction; index + indicates that) matrices with nonzero elements inside a (depending on the matrix) strip along the main diagonal and containing it. The subalgebras $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ of $\mathfrak{sl}(\infty)$ are naturally defined.

The realization 2.1 provides with an embedding $\mathfrak{sl}(\lambda) \subset \mathfrak{sl}_+(\infty) = “\mathfrak{sl}(M^\lambda)”$, so for $\lambda \notin \mathbb{N}$ the Verma module $M^\lambda$ with highest weight $\mu$ is an irreducible $\mathfrak{sl}(\lambda)$-module.

Proposition. The completion of $\mathfrak{sl}(\lambda)$ (generated by the elements of degree $\pm 1$ with respect to $H \in \mathfrak{sl}(\lambda)$) is isomorphic for any non-integer $\lambda$ to $\mathfrak{sl}_+(\infty) = “\mathfrak{sl}(M^\lambda)”$.

2.5. The Lie algebras $\mathfrak{sl}(\ast)$ and $\mathfrak{o}/\mathfrak{sp}(\ast)$, for $\ast \in \mathbb{CP}^1 = \mathbb{C} \cup \{\ast\}$. The “quantization” of the relations for $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$ (see Table 3.2) is performed by passage to the limit as $\lambda \rightarrow \infty$ under the change:

$$t \mapsto \begin{cases} \frac{1}{\lambda} & \text{for } \mathfrak{sl}(\lambda) \\ \frac{1}{t} & \text{for } \mathfrak{o}/\mathfrak{sp}(\lambda). \end{cases}$$

So the parameter $\lambda$ above can actually run over $\mathbb{CP}^1 = \mathbb{C} \cup \{\ast\}$, not just $\mathbb{C}$. In the realization with the help of deformation, cf. 2.7 below, this is obvious. Denote the limit algebras by $\mathfrak{sl}(\ast)$ and $\mathfrak{o}/\mathfrak{sp}(\ast)$ in order to distinguish them from $\mathfrak{sl}(\infty)$ and $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$ from sec. 2.4.

It is clear that $\mathfrak{sl}(\ast)$ and $\mathfrak{o}/\mathfrak{sp}(\ast)$ are subalgebras of the whole “plane” algebras $\mathfrak{sl}(\infty)$ and $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$, it is impossible to embed $\mathfrak{sl}(\ast)$ and $\mathfrak{o}/\mathfrak{sp}(\ast)$ into the “quadrant” algebra $\mathfrak{sl}_+(\infty)$. 
2.6. Main Theorem. For Lie algebras $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$, where $\lambda \in \mathbb{CP}^1$, all the relations between Jacobson’s generators are the relations of types 0, 1 with $k_1$ found from Table 1.1 and the relations from Tables in §3.

2.7. ([PH]). Now, consider another realization of $\mathfrak{sl}(2)$: before the factorization (2.2). This realization was a starting point of [F], so we give it for completeness. Take the Lie algebra $\mathfrak{po}(2)_{ev}$ of even degree polynomials $\mathbb{C}[q,p]_{ev}$ with respect to the Poisson bracket. Set $X^- = \frac{1}{2}q^2$, $X^+ = \frac{1}{2}p^2$, and notice that $\langle q,p \rangle$ is the identity $\mathfrak{sl}(2)$-module. Observe that, as $\mathfrak{sl}(2)$-modules, the Lie algebras $\mathfrak{po}(2)_{ev}$ and its deform — the result of the quantization, a subalgebra of the Lie algebra $\mathfrak{diff}(1)$ of differential operators on the line — also have spectrum (2.1). So it is natural to look at the deforms for various values of the parameter of deformation; this is done in [PH].

Observe that the deforms of $\mathfrak{po}(2)$ are all isomorphic for nonzero values of the parameter of deformation, unlike the deforms of subalgebra $\mathfrak{po}(2)_{ev}$; indeed apart from the isomorphism $\mathfrak{sl}(-\lambda) \cong \mathfrak{sl}(\lambda)$ all the deforms are non-isomorphic.

§3. Tables. Jacobson’s Generators and Relations between them

Table 3.1. Finite dimensional algebras. In what follows $E_{ij}$ are the matrix units; $X_i^\pm$ stand for the conventional Chevalley generators of $\mathfrak{g}$.

$\mathfrak{sl}(n)$ for $n \geq 3$. Generators:

\[
x = \sum_{1 \leq i \leq n-1} i(n - i)E_{i,i+1}, \quad y = \sum_{1 \leq i \leq n-1} E_{i+1,i}, \quad z = t \sum_{1 \leq i \leq n-2} E_{i+2,i}.
\]

Relations:

2.1. $3[z_1, z_2] - 2[z, z_3] = 24t^2(n^2 - 4)y$,
3.1. $[z, [z, z_1]] = 0$,
3.2. $4[z_3, [z, z_1]] - 3[z_2, [z, z_2]] = 576t^2(n^2 - 9)z$.

$\infty = n - 1$. (ad $z_1$)$^{n-2}z = 0$

For $n = 3, 4$ the degree of the last relation is lower than the degree of some other relations, this yields a simplification:

$n = 4$:

2.1. $3[z_1, z_2] - 2[z, z_3] = 288t^2y$,
3.1. $[z, [z, z_1]] = 0$,
3.2. $[z_3, [z, z_1]] = -576t^2z$.

$\infty = 3$. (ad $z_1$)$^2z = 0$.

$n = 3$: $\infty = 2$. [z_1, z] = 0, 2.1. [z_1, z_2] = 24t^2y$.

$\mathfrak{o}(2n + 1)$ for $n \geq 3$. Generators:

\[
x = n(n + 1)(E_{n+1,2n+1} - E_{n,n+1}) + \sum_{1 \leq i \leq n-1} i(2n + 1 - i)(E_{i,i+1} - E_{n+i+1,n+i+1}),
\]
\[
y = (E_{2n+1,n+1} - E_{n+1,n}) + \sum_{1 \leq i \leq n-1} (E_{i+1,i} - E_{n+i+1,n+i+2}),
\]
\[
z = t((E_{2n-1,n+1} - E_{n+1,n-2}) - (E_{2n+1,n-1} - 2E_{2n,n}) + \sum_{1 \leq i \leq n-3} (E_{i+3,i} - E_{n+i+1,n+i+4})).
\]

Relations:

2.1. $2[z_1, z_2] - [z, z_3] = 144t(2n^2 + 2n - 9)z$,
2.2. $9[z_2, z_3] - 5[z_1, z_4] = 432t(2n^2 + 2n - 9)z_2 + 1728t^2(n-1)(n+2)(2n-1)(2n+3)y,$
3.1. $[z, [z, z_1]] = 0,$
3.2. $7[z_3, [z, z_1]] - 6[z_2, [z, z_2]] = 2880t(n-3)(n+4)[z, z_1],$
$\infty = n. (\text{ad } z_1)^{n-1}z = 0.$

$\text{sp}(2n)$ for $n \geq 3.$ Generators:

$$
\begin{align*}
x &= n^2E_{n,2n} + \sum_{1 \leq i \leq n-1} i(2n-i)(E_{i,i+1} - E_{n+i+i+1}), \\
y &= E_{2n,n} + \sum_{1 \leq i \leq n-1} (E_{i+1,i} - E_{n+i,n+i+1}), \\
z &= t \left((E_{2n,n-2} + E_{2n-2,n}) - E_{2n-1,n-1} + \sum_{1 \leq i \leq n-3} (E_{i+3,i} - E_{n+i,n+i+3})\right).
\end{align*}
$$

Relations:

2.1. $2[z_1, z_2] - [z, z_3] = 72t(4n^2 - 19)z,$
2.2. $9[z_2, z_3] - 5[z_1, z_4] = 216t(4n^2 - 19)z_2 + 1728t^2(n-1)(4n^2 - 9)y,$
3.1. $[z, [z, z_1]] = 0,$
3.2. $7[z_3, [z, z_1]] - 6[z_2, [z, z_2]] = 720t(4n^2 - 49)[z, z_1],$
$\infty = n. (\text{ad } z_1)^{n-1}z = 0.$

$\mathfrak{g}_2.$ Generators:

$$
\begin{align*}
x &= 6X_1^+ + 10X_2^+, \quad y = X_1^- + X_2^-, \quad z = \frac{6}{129600}[[X_1^-, X_2^-], [X_1^-, [X_1^-, X_2^-]]].
\end{align*}
$$

Relations:

2.1. $[z, z_1] = 0,$
2.2. $[z_1, z_2] = 0,$
2.3. $[z_2, z_3] = -6tz,$
2.4. $[z_3, z_4] = -8t_2z_2,$
2.5. $[z_4, z_5] = -8tz_4 + 6t^2y.$

$\mathfrak{f}_4.$ Generators:

$$
\begin{align*}
x &= 16X_1^+ + 30X_2^+ + 42X_3^+ + 22X_4^+, \quad y = X_1^- + X_2^- + X_3^- + X_4^-,
\end{align*}
$$

$$
\begin{align*}
z &= \frac{6}{907200}([X_1^-, X_2^-], [X_3^-] - [X_1^-, X_3^-]) + [X_1^-, X_2^-] + [X_3^-] - [X_1^-, X_3^-],
\end{align*}
$$

$$
\begin{align*}
2([X_1^-, X_2^-], [X_4^-], [X_2^-, X_3^-]) - [[X_3^-], [X_4^-], [X_2^-, X_3^-]]).
\end{align*}
$$

Relations:

2.1. $[z, z_1] = 0,$
2.2. $4[z_2, z_3] - 9[z_1, z_4] = 42tz,$
2.3. $5[z_3, z_4] - 6[z_2, z_5] = 28tz_2,$
2.4. $13[z_4, z_5] - 14[z_3, z_6] = 56tz_4 + 306t^2y.$

$\mathfrak{e}_6.$ Generators:

$$
\begin{align*}
x &= 16X_1^+ + 30X_2^+ + 42X_3^+ + 30X_4^+ + 16X_5^+ + 22X_6^+,
\end{align*}
$$

$$
\begin{align*}
y &= X_1^+ + X_2^+ + X_3^+ + X_4^+ + X_5^+ + X_6^-,
\end{align*}
$$

$$
\begin{align*}
z &= \frac{6}{5100}([[X_1^-, X_2^-], [X_3^-], X_4^-] - [[X_1^-, X_2^-], [X_3^-], X_6^-]) +
\end{align*}
$$

$$
\begin{align*}
[[X_2^-, X_3^-], [X_4^-, [X_5^-, [X_6^-]] + [[X_2^-, X_3^-], [X_4^-, [X_5^-, X_6^-]] + [[X_2^-, X_3^-], X_4^-]]).
\end{align*}
$$

Relations:

2.1. $50[z_2, z_3] + 14[z_1, z_5] - 35[z_1, z_4] = 0,$
2.2. $20[z_3, z_4] - 15[z_2, z_5] + 7[z_1, z_6] = 14t^2y.$
3.1. \([z_1, [z, z_1]] = 0,\]
3.2. \([z_2, [z, z_1]] = 0,\]
3.3. \(4[z_3, [z, z_1]] + 7[z_1, [z_1, z_2]] = 0,\]
3.4. \(5[z_3, [z_2, z_1]] + [z_4, [z, z_1]],\]
3.5. \(8[z_4, [z, z_2]] + 5[z_3, [z_1, z_2]] = 0,\]
3.6. \(3[z_4, [z_1, z_2]] + 4[z_4, [z, z_3]] = 0,\]
3.7. \(51[z_5, [z_1, z_2]] + 4[z_5, [z, z_3]] = -384t^2z.\]

\(\varepsilon_5\) Generators:

\[x = 27X_1^* + 52X_2^* + 75X_3^* + 96X_4^* + 66X_5^* + 34X_6^* + 49X_7^*,\]
\[y = X_1^* + X_2^* + X_3^* + X_4^* + X_5^* + X_6^* + X_7^*,\]
\[z = \frac{13}{18}([X_2^*, [X_5^*, X_6^*]], [X_5^*, [X_4^*, X_7^*]] + [X_8^*, [X_7^*, X_6^*]], [X_7^*, [X_8^*, X_5^*]]) + [X_9^*, [X_8^*, X_6^*]] + [X_7^*, [X_9^*, X_8^*]] + [X_8^*, [X_7^*, X_9^*]] + [X_6^*, [X_5^*, X_7^*]] + [X_5^*, [X_6^*, X_7^*]] + [X_4^*, [X_3^*, X_7^*]] + [X_3^*, [X_4^*, X_7^*]] + [X_2^*, [X_1^*, X_7^*]] + [X_1^*, [X_2^*, X_7^*]] + [X_7^*, [X_1^*, X_2^*]] - 3[[X_7^*, [X_5^*, X_6^*]], [X_5^*, [X_7^*, X_6^*]]].\]

Relations:

2.1. \(3[z, z_2] - 9[z_1, z_4] + 14[z_2, z_3] = -2868tz,\)
2.2. \(18[z_1, z_6] - 50[z_2, z_5] + 75[z_3, z_4] = -9560tz_2,\)
2.3. \(14[z_2, z_7] - 35[z_3, z_6] + 50[z_4, z_5] = -4780tz_4 + 49335t^2y;\)
3.1. \([z, [z, z_1]] = 0,\]
3.2. \(9[z_1, [z, z_2]] - 4[z_2, [z, z_1]] = 0,\]
3.3. \(330[z_2, [z, z_2]] - 425[z_3, [z, z_1]] - 1458[z_1, [z_1, z_2]] = 0,\)
3.4. \(665[z_3, [z, z_2]] - 640[z_4, [z, z_1]] - 1134[z_2, [z_1, z_2]] = 0,\)
3.5. \(5485[z_4, [z, z_3]] - 3910[z_4, [z, z_2]] - 3182[z_3, [z_1, z_2]] = 2527815tz, z_1,\)
3.6. \(825[z_4, [z, z_4]] - 508[z_5, [z, z_2]] - 876[z_4, [z_1, z_2]] = 338422z, z_3,\)
3.7. \(1525[z_5, [z, z_3]] - 7524[z_5, [z_1, z_2]] + 2415[z_4, [z_1, z_3]] = 1106875tz, z_3] + 2734746tz, [z_1, z_2,\]
3.8. \(25250[z_6, [z, z_4]] - 94920[z_6, [z, z_1]] + 44252[z_6, [z, z_4]] = -1305480tz, z_6] + 41398712t, z_1, z_4] - 111792500t^2z;\)
4.1. \(12[[z, z_2], [z_1, z_2]] - 5[[z, z_2], [z, z_3]] = 0,\)
\(\infty = 5.\)

\(\varepsilon_5\) Generators:

\[x = 58X_1^* + 114X_2^* + 168X_3^* + 220X_4^* + 270X_5^* + 182X_6^* + 92X_7^* + 136X_8^* - X_1^* + X_2^* + X_3^* + X_4^* + X_5^* + X_6^* + X_7^* + X_8^*,\]
\[y = \frac{13}{18}([X_7^*, [X_5^*, X_6^*]], [X_5^*, [X_4^*, X_7^*]], [X_7^*, [X_8^*, X_5^*]] + [X_8^*, [X_7^*, X_6^*]], [X_7^*, [X_8^*, X_5^*]]) + [X_9^*, [X_8^*, X_6^*]] + [X_7^*, [X_9^*, X_8^*]] + [X_8^*, [X_7^*, X_9^*]] + [X_6^*, [X_5^*, X_7^*]] + [X_5^*, [X_6^*, X_7^*]] + [X_4^*, [X_3^*, X_7^*]] + [X_3^*, [X_4^*, X_7^*]] + [X_2^*, [X_1^*, X_7^*]] + [X_1^*, [X_2^*, X_7^*]] + [X_7^*, [X_1^*, X_2^*]] - 3[[X_7^*, [X_5^*, X_6^*]], [X_5^*, [X_7^*, X_6^*]]], [X_1^*, X_2^*], [X_3^*, X_4^*]].\]

Relations:

2.1. \(91[z, z_5] - 325[z_1, z_4] + 550[z_2, z_3] = 0,\)
2.2. \(13[z_1, z_6] - 45[z_2, z_5] + 75[z_3, z_4] = -268814tz,\)
2.3. \(33[z_2, z_7] - 11[z_3, z_6] + 180[z_4, z_5] = -682374tz_2,\)
2.4. \(11[z_3, z_8] - 35[z_4, z_7] + 56[z_5, z_6] = -186102tz_4,\)
2.5. \(3[z_4, z_9] - 9[z_5, z_8] + 4[z_6, z_7] = -41356tz_6 + 2686866t^2y;\)
3.1. \([z, [z, z_1]] = 0,\)
3.2. 13[z_1, [z, z_2]] - 6[z_2, [z, z_1]] = 0,
3.3. 542[z_2, [z, z_2]] - 639[z_3, [z, z_1]] - 2236[z_1, [z_1, z_2]] = 0,
3.4. 1067[z_3, [z, z_2]] - 950[z_4, [z, z_1]] - 1892[z_2, [z_1, z_2]] = 0,
3.5. 7255[z_3, [z, z_3]] - 4995[z_4, [z, z_2]] - 4527[z_3, [z_1, z_2]] = 0,
3.6. 105460[z_4, [z, z_3]] - 69597[z_5, [z, z_2]] - 119430[z_4, [z_1, z_2]] = 0,
3.7. 844277[z_5, [z, z_3]] + 1556775[z_4, [z_1, z_3]] - 4442058[z_5, [z_1, z_2]] = -17362538193t[z, z_1],
3.8. 334453[z_6, [z, z_4]] + 746586[z_5, [z_1, z_4]] - 1414050[z_6, [z_1, z_3]] = 1120518212t[z, z_3] + 3082429152t[z_1, z_2],
\[\infty = 4\cdot [[[z, z_1], [z, z_2]]] = 0.\]

Table 3.2. \(\mathfrak{sl}(\lambda)\) and \(\mathfrak{o}/\mathfrak{sp}(\lambda)\). \(\mathfrak{sl}(\lambda)\) Generators:
\[x = u^2 \frac{d}{du} - (\lambda - 1)u, \quad y = -\frac{d}{du}, \quad z = t \frac{d^2}{du^2} .\]
Relations:
2.1. \(3[z_1, z_2] - 2[z, z_3] = 24t^2(\lambda^2 - 4)y,\)
3.1. \([z, [z, z_1]] = 0,\)
3.2. \(4[z_3, [z, z_1]] - 3[z_2, [z, z_2]] = 576t^2(\lambda^2 - 9)z.\)

\(\mathfrak{o}/\mathfrak{sp}(\lambda)\) Generators:
\[x = u^2 \frac{d}{du} - (\lambda - 1)u, \quad y = -\frac{d}{du}, \quad z = t \frac{d^3}{du^3}.\]
Relations:
2.1. \(2[z_1, z_2] - [z, z_3] = 72t(\lambda^2 - 19)z,\)
2.2. \(9[z_2, z_3] - 5[z_1, z_4] = 216t(\lambda^2 - 19)z_2 - 432t^2(\lambda^2 - 4)(\lambda^2 - 9)y,\)
3.1. \([z, [z, z_1]] = 0,\)
3.2. \(7[z_3, [z, z_1]] - 6[z_2, [z, z_2]] = 720t(\lambda^2 - 49)[z, z_1].\)

Table 3.3. \(\mathfrak{sl}(\ast)\) and \(\mathfrak{o}/\mathfrak{sp}(\ast)\). \(\mathfrak{sl}(\ast)\)
2.1. \(3[z_1, z_2] - 2[z, z_3] = 24t^2y,\)
3.1. \([z, [z, z_1]] = 0,\)
3.2. \(4[[z, z_1], z_3]] + 3[z_2, [z, z_2]] = -576t^2z.\)

\(\mathfrak{o}/\mathfrak{sp}(\ast)\)
2.1. \(2[z_1, z_2] - [z, z_3] = 72tz,\)
2.2. \(9[z_2, z_3] - 5[z_1, z_4] = 216tz_2 - 432t^2y,\)
3.1. \([z, [z, z_1]] = 0,\)
3.2. \(7[[z, z_1], z_3]] + 6[z_2, [z, z_2]] = -720t[z, z_1].\)

§4. Remarks

4.1. On proof. For the exceptional cases the proof is direct: the quotient of the free Lie algebra generated by \(x, y\) and \(z\) modulo our relations is the needed finite dimensional one. For rank \(g \leq 10\) we similarly computed relations for \(g = \mathfrak{sl}, \mathfrak{o}\) and \(\mathfrak{sp}\); as Post pointed out, together with the result of [PH] on deformation (cf. 2.7) this completes the proof.

Our theorems elucidate Proposition 2 of [F]; we just wrote relations explicitly. Feigin claimed [F] that, for \(\mathfrak{sl}(\lambda)\), the relations of type 3 follow from the decomposition of
\[L^{k_1} \wedge L^{k_2} \subset L^{k_1} \wedge L^{k_1} \wedge L^{k_1}.\]
We verified that this is so not only in Feigin’s case but for all the above-considered algebras except $\mathfrak{e}_6, \mathfrak{e}_7$ and $\mathfrak{e}_8$: for them one should consider the whole $L^{k_1} \wedge L^{k_1} \wedge L^{k_1}$.

4.2. Proposition. (a) For a principal embedding $\mathfrak{sl}(2) \rightarrow \mathfrak{g}$, where $\mathfrak{g} = \mathfrak{o}(2n+1), \mathfrak{sp}(2n)$ or $\mathfrak{o}/\mathfrak{sp}(\lambda)$, where $\lambda \in \mathbb{C}P^1$, there exists an embedding of $\sigma : \mathfrak{g} \rightarrow \mathfrak{sl}(k)$ for an appropriate $k \in \mathbb{C}P^1$ such that the through map is principal.

(b) There is no such $\sigma$ for the exceptional Lie algebras or $\mathfrak{o}(2n)$.

4.3. How to present $\mathfrak{o}(2n)$? Select $z$ as in sec. 1.1. Clearly, the form of $z$ (hence, relations of type 1) and the number of relations of type 3 depend on $n$; this was not the case for the algebras considered above. Besides, the relations are not as neat as for the above algebras. $\mathfrak{o}(2n)$. Generators:

\[
x = \frac{n(n-1)}{2} (E_{n-1,n} - g_{2n,2n-1} + E_{n-2,n-1} - E_{n,2n-1}) + \sum_{1 \leq i < n - 2} (i(2n - 1 - i)(E_{i,i+1} - E_{n+i+1,n+i})),
\]

\[
y = (E_{2n,n-1} - E_{2n-1,n}) + \sum_{1 \leq i < n - 1} (E_{i+1,i} - E_{n+i,n+i+1}),
\]

\[
z = \frac{1}{(2n-2)!} ((E_{n,1} - E_{n+1,2n}) + (E_{n+1,n} - E_{2n,1})).
\]

We can not write the relations in full generality; for small values of $n$ they are:

$n = 4$.

2.1. $3[z, z_5] - 5[z_1, z_4] + 6[z_2, z_3] = \frac{1}{3} y$; 3.5. $[z_3, [z, z_1]] = 0$,

3.1. $[z, [z, z_1]] = 0$; 3.6. $[z_3, [z, z_2]] = 0$,

3.2. $[z_1, [z, z_1]] = 0$; 3.7. $[z_4, [z, z_2]] = z$,

3.3. $[z_2, [z, z_1]] = 0$; 3.8. $[z_4, [z_1, z_2]] = z_1$,

3.4. $[z_1, [z_1, z_2]] = 0$; 3.9. $[z_5, [z_1, z_2]] = z_2$.

$n = 5$. There are 17 relations of type 3; the relation of type 2 is:

2.1. $-4[z, z_7] + 7[z_1, z_6] - 9[z_2, z_5] + 10[z_3, z_4] = \frac{1}{2} y$.

$n = 6$. The relation of type 2 is still more involved and there are 27 relations of type 3.

We should, perhaps, have taken the generators as for $\mathfrak{o}(2n - 1)$ and add a generator from $L^{2n-2}$. We have no guiding idea; to try at random is frustrating.

4.4. How to realize $\mathfrak{sl}(\ast)$ and $\mathfrak{o}/\mathfrak{sp}(\ast)$? We do not know how to answer this question and while this is a research problem we can not express the Jacobson generators in a form as suggestive as for $\lambda \in \mathbb{C}$.

4.5. A relation with integrable differential systems. The Drinfeld–Sokolov’s construction, as well as its generalization to $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$ ([DS], [KM]), hinges on a certain element that can be identified with the image of $X^+ \in \mathfrak{sl}(2)$ under the principal embedding. One might think that only this image is important but the image of the whole $\mathfrak{sl}(2)$ is recovered from the image of $X^+$ whereas to work with $\mathfrak{sl}(2)$ is easier than with a nilpotent element — the image of $X^+$. 
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Roslagsv. 101, Kräftriket hus 6, S-104 05, Stockholm, Sweden; mleitesmath.su.se