Reliability based design of shaft for gearbox

N H Loc1,2*,
1Faculty of Mechanical Engineering, Ho Chi Minh city University of Technology (HCMUT), Ho Chi Minh City, Vietnam
2VietNam National University of Ho Chi Minh City (VNUHCM), Ho Chi Minh City, Vietnam

*nhloc@hcmut.edu.vn

Abstract. In this article, the author presents research on the reliability-based design methods of mechanical systems, includes 3 stages: the first, the fault tree analysis (FTA) method is utilized for reliability analysis of single and double reducer into the logic diagram; the second that apply the probabilistic design methods, including moment matching method (MMM), first-order reliability method (FORM), and Monte Carlo simulation (MCs) for reliability-based design and to analyze the components of the mechanical transmission systems, and the last sensitivity analysis is performed in the post-design stage after a design solution is identified. The study also aims to set up RADME (Reliability-based analysis and design of the mechanical systems) computer programs applying the aforementioned methods. The reliability design problem was surveyed as reliability-based design optimization with one design variable.

Keywords: Probabilistic design, Reliability-based design, Mechanical system, Shaft, Monte Carlo simulation, Sensitivity analysis, Gear reducer

1. Introduction

In deterministic design, the variables are considered deterministic and calculated values are selected as the worse – case values or nominal values; however, by calculation, they are divided by safety factor. It is evident that the use of elements of safety may be either risky (under-designed) or conservative (over-designed). Overall, uncertainty exists in any engineering system. When it comes to designing complicated systems, a small variation of system input may cause a significant quality loss, and ignorance of such uncertainty may lead to catastrophic failure events. The probabilistic design includes design and analysis of mechanical transmission systems, reliability-based design optimization of machine structure, tolerance analysis, kinematics, and dynamics of the machine. This method ensures given reliability, safety, quality, and profitability of the product.

Three categories of methods exist for the task of uncertainty analysis. The first category is the sampling-based approach such as MCS [10-11]. The second category comprises approximation methods like first order (FORM) and second-order (SORM) reliability methods [1,2,7] The third
category involves the response surface – using surrogate models to replace the performance function by Design of experiments[3,8].

2. Fundamentals of reliability-based design and analysis of shafts

Machine elements for speed reducer include gear drive, shaft, bearing, and housing. In the papers by [6,12] the details of reliability-based design and an analysis of gear drive were presented. The book by Nguyen (2016, 2018), the researchers have focused on the shaft, gear drive and bearing design in accordance with reliability.

2.1. Reliability-based design and analysis of shaft

The shafts must satisfy the condition of strength, stiffness, wear resistance, and vibration stability. Under strength conditions, for fast-rotating shafts, the fatigue limit is decisive. Calculating shaft reliability is essential because the load and fatigue limits have significant dispersion. In fact, fatigue damage accounts for 40 ÷ 50% of shaft failures. When abruptly overloaded, the shafts made of normalized, tempered steel can be impaired by large plastic deformation while those made of brittle and less malleable materials can be damaged by brittle destruction.

The characteristic of calculating fatigue strength is that the bending stress and torsional stress change according to different cycles. Specifically, bending stress alters in symmetrical alternating cycle while torsional stress changes according to the pulsating cycle. The load acting on the shaft is the force that exerts influence on the gear, belt, sprocket, coupling and we determine the reaction forces at the bearings. The force point depends on the bearing structure.

![Figure 1. The probability density of the amplitudes of stress $\sigma_a$ and the fatigue limits $\sigma_r$.](image)

The curves of probability density of the amplitudes of stress $\sigma_a$ and the fatigue limits $\sigma_r$ are illustrated on the graph in Figure 1. Machine elements and mechanical structures are considered safe and reliable when limited value exceeds the calculating value. As can be seen, these curves intersect, which means that failure can occur.

The function $g (\sigma_a, \sigma_r) = \sigma_r - \sigma_a$ is called performance or limit – state function. When $g (\sigma_a, \sigma_r) > 0$, the shaft is considered safe, but when $g (\sigma_a, \sigma_r) \leq 0$, the shaft is considered as unsafe.

$$m_f = m_{\sigma_a} + m_{\sigma_r}; \quad S_f = \sqrt{S_{\sigma_a}^2 + S_{\sigma_r}^2}$$

Equivalent bending stress of shaft with diameter $d$ at gear position is determined by the formula:
\[ \sigma = \frac{32M_{eq}}{\pi d^3} \]  

(2)

For the drive shaft, there are eight cases of forces acting in a gear and shaft as shown in Figure 2. When calculating, the researchers proceed at the dangerous cross section of the shaft according to the maximum equivalent moment \( M_{eq} \):

\[ M_{eq} = \sqrt{M_X^2 + M_Y^2 + 0.75T^2} \]  

(3)

Mean and standard deviation of the equivalent moment are calculated as follows:

\[ \bar{M}_{eq} = \sqrt{\bar{M}_X^2 + \bar{M}_Y^2 + 0.75\bar{T}^2} \]  

(4)

\[ S_{M_{eq}} = \left( \frac{M_X^2 S_{M_X}^2 + M_Y^2 S_{M_Y}^2 + (0.75T)^2 S_{T}^2}{M_X^2 + M_Y^2 + 0.75T^2} \right)^{1/2} \]  

(5)

a) \( F_{r1} \) and \( F_{r2} \) in the opposite directions

b) \( F_{r1} \) and \( F_{r2} \) in the opposite directions

c) \( F_{a1} \) and \( F_{a2} \) in the same direction

d) \( F_{a1} \) and \( F_{a2} \) in the same direction

e) \( F_{r1} \) and \( F_{r2} \) in the opposite directions

f) \( F_{r1} \) and \( F_{r2} \) in the opposite directions
\[ F_{a1} \text{ and } F_{a2} \text{ in the same direction} \]

\[ F_{a1} \text{ and } F_{a2} \text{ in the opposite directions} \]

\[ g) \ F_{r1} \text{ and } F_{r2} \text{ in the same direction} \]

\[ h) \ F_{r1} \text{ and } F_{r2} \text{ in the same direction} \]

\[ F_{a1} \text{ and } F_{a2} \text{ in the opposite directions} \]

**Figure 2.** Forces acting in a gear and shaft for all cases.

Mean of the bending stress is determined by:

\[ \bar{\sigma} = \frac{32\bar{M}_{eq}}{\pi d^3} \]  

(6)

Standard deviation of the bending stress is calculated by the formula:

\[ S_\sigma = \left( \frac{\partial \sigma}{\partial M_{eq}} \right)^2 S_{M_{eq}}^2 + \left( \frac{\partial \sigma}{\partial d} \right)^2 S_d^2 \right)^{1/2} = \left( \frac{32}{\pi d^2} \right)^2 S_{M_{eq}}^2 + \left( \frac{96M_{eq}}{\pi d^2} \right)^2 S_d^2 \]  

(7)

The formula for moments, equivalent moments and standard deviations are listed in the Table 1.

The fatigue limit \( \sigma_{lim} \) of shaft material can be calculated as:

\[ \sigma_{lim} = \frac{\sigma_r \varepsilon \beta}{K_L} \]  

(8)

with \( \sigma_r \) - endurance limit for bending; \( \varepsilon \) - scale factor taking into account the effect of the dimensions of the shaft; \( \beta \) - hardening factor; \( K_L \) - life coefficient; \( K_\sigma \) - effective stress concentration factors for bending.

Mean of the fatigue limit is:

\[ \bar{\sigma}_{lim} = \frac{\bar{\sigma_r} \bar{\varepsilon} \bar{\beta}}{\bar{K_L}} \]  

(9)

The coefficient of variation of the fatigue limit is determined as:

\[ v_{\sigma_{lim}} = \sqrt{\frac{\sigma_r^2 + \varepsilon^2 + \beta^2 + \sigma_{lim}^2}{\sigma_{lim}}} \]  

(10)

Standard deviation is reckoned by

\[ S_{\sigma_{lim}} = v_{\sigma_{lim}} \sigma_{lim} \]  

(11)

Reliability of shaft is determined by the quantile \( z_1 \):

\[ z_1 = -\frac{\bar{\sigma}_{lim} - \bar{\sigma}}{\sqrt{S_{\sigma_{lim}}^2 + S_{\sigma}^2}} = -\frac{\bar{\sigma} - \bar{\sigma}_{lim}}{\sqrt{\bar{\sigma}_{lim}^2 + \sigma^2}} \]  

(12)
with $\bar{n}$ – mean factor of safety which is determined by mean stress and fatigue limit $\bar{n} = \frac{\bar{\sigma}_{\text{lim}}}{\bar{\sigma}}$.

To solve a design problem, the mean of the shaft diameter $\bar{d}$ is found, then $\bar{n}$ and $\nu^2$ are the functions of $\bar{d}$ . Now the job is to replace the expression $\bar{n}$ and $\nu^2$ into formula (12) and solving the equation, the researchers find $\bar{n}$, then $\bar{\sigma} = \frac{\bar{\sigma}_{\text{lim}}}{\bar{n}}$, from equation (6) $\bar{d}$ is found by the formula:

$$\bar{d} = 10 \left( \frac{32M_{\sigma\bar{n}}}{\pi\bar{\sigma}_{\text{lim}}} \right)^{1/3} \quad (13)$$

The fatigue limits of machine elements according to the bending and torsion stresses are interrelated and the common safety factor of $\bar{n}_\sigma$ is much smaller than that of $\bar{n}_\tau$ . Therefore, the coefficient variation of fatigue limit can be taken from the coefficient variation of bending stress.

For stepped shaft, there is stress concentration distribution along the axial length. Usually a reliability assessment is conducted in a dangerous cross section. The reliability probability of the shaft is equal to the reliability probability of the all dangerous cross sections:

$$R_{\text{shaft}} = \prod_{i=1}^{n} R_i \quad (14)$$

### 2.2. Sensitivity analysis

Traditionally, sensitivity analysis is performed in the post-design stage after a design solution is identified. There is also great necessity for designers to conduct sensitivity analysis in the pre-design stage to gain valuable information about the model and its probabilistic behavior (Nam et al., 2006). This is especially important for models with high dimensions as well as for those with high nonlinearity, such as the invisible relationships between inputs and outputs.

The goal of pre-design sensitivity analysis is to reduce the size of the probabilistic design problem by eliminating insignificant random variables. Based on the ranking of all variables, unimportant design variables and noise parameters could be treated as deterministic variables and fixed as constants. In this case, at the initial design stage, a first-order response surface is constructed using all random variables and can be expressed as ($x_i$ is coded design variable at the scale (-1, +1)):

$$y = b_0 + \sum_{i=1}^{k} b_i x_i \quad (15)$$

The global sensitivity index can be calculated as:

$$s_i = \frac{b_i}{\frac{1}{k} \sum_{j=1}^{k} |b_j|} \quad (16)$$
The focus of the sensitivity analysis in the post–design stage is to answer the question of which random uncertainties should be further controlled to gain the largest improvement on the probabilistic performance of the response. The post–design sensitivity analysis is applied to prioritize available resources for variance reduction.

After the sensitivity analysis, with the reduced number of random variables, a high order response surface is constructed. Reliability based design and analysis problems are formulated in this high order response surface.

3. Results of reliability-based design and analysis of reducer shafts

Applying the aforesaid methods, the researchers could establish a RADME (Reliability-based analysis and design of the mechanical system) computer program which includes the three following parts: reliability-based design and analysis of the mechanical structure, transmission system and other.

In this section, RADME is employed to solve the reliability-based design and analysis of machine elements for speed reducer.

![Figure 3](image)

**Figure 3.** Spur double (a) and bevel-spur reducer (b): 1- Input shaft with rotating part; 2- Output shaft with rotating part; 3- Gear pair.

A popular speed reducer problem represents the design of the single reducer (Figure 3a) which is frequently used in many transmission systems such as light airplanes between the engine and propeller to allow each to rotate at its most efficient speed. As transmission power of reducer equals 7.5 kW, speed ratio \( u = 3.15 \) (Table 1), for double reducer \( u_{12} = 3.15, u_{34} = 2.5 \). For more speed ratio, double reducer is utilized (Figure 3b).

| Shaft | I | II | III |
|-------|---|----|-----|
| Torsion moment, Nm | 73.993 | 233.077 | 582.692 |
| Speed, rpm | 968 | 307.3 | 122.9 |

3.1. Fault tree analysis

Fault tree analysis (FTA) method is used for reliability analysis of single and double reducer into a logic diagrams in Figure 4.

Reliability \( R_1, R_2 \) of single and double reducer can be calculated as:

\[
R_1 = R_{gf}^2 R_{gear}^6 R_{bear}^4 = R_{gf}^2 R_{con}^2 R_{bend} R_{bear}^4
\]  
\( (17) \)

\[
R_2 = R_{gf}^3 R_{gear}^6 R_{bear}^6 = R_{gf}^3 R_{con}^2 R_{bend}^2 R_{bear}^6
\]  
\( (18) \)
where $R_{sf}$, $R_{cont}$, $R_{bend}$ and $R_{bear}$ are respectively shaft, gear and bearing reliability with reliability of gear pair $R_{gear} = R_{cont}R_{bend}$ (contact and bending stress).

With the system reliability of 0.95, using fault-tree analysis and the distribution method of system reliability in mechanical design, the researchers have the reliability of each element in Table 2.

**Table 2. Reliability of machine elements.**

| Elements  | Gear | Shaft I | Shaft II | Bearing |
|-----------|------|---------|----------|---------|
| Reliability $R_1$ | 0.999 | 0.999 | 0.999 | 0.988 |

![Fault–tree analysis of system: single reducer (a); spur double, bevel-spur reducer (b)](image_url)

**Figure 4.** Fault–tree analysis of system: single reducer (a); spur double, bevel-spur reducer (b)

### 3.2. Design and analysis of shaft

Radial load in the end of input shaft is $F_{r2} = 800$ N in the opposite direction of load $F_{r1}$. Firstly the researchers design and analyse the shaft I. Input data are determined in Figure 5, reliability of shaft $R = 0.999$.

Using the RADME program, the researchers choose button *Position C* in *Case 1* (loads $F_{r1}$ and $F_{r2}$ contrariwise). Diameter of shaft is found at $d = 26.744$ mm with 10 random variables (Figure 5). The reliability based design optimization problem with one design variable is now solved using the proposed adaptive reduction of random variables and MCS $d = 26.695$ mm. The researchers choose $d = 28$ mm as standard diameter for shaft.

![Dialog box for shaft design with 10 random variables](image_url)

**Figure 5.** Dialog box for shaft design with 10 random variables
Following that, the reliability-based analysis of shaft I is performed with standard diameter $d = 28\text{mm}$. The calculation results from different methods are shown in Table 3.

### Table 3. Reliability of shaft from different methods for single reducer.

| Shaft | Diameter $d$, mm | Matching moment | FORM with MPP | MCS Method |
|-------|------------------|-----------------|---------------|------------|
| Shaft I | 28 | 0.9998527 | 0.999913 | 0.99990274 |

For bevel-spur reducer with transmission power of reducer equals 5.7 kW, torque moment in shaft I $90.1\text{Nm}$, in shaft II $171.2\text{Nm}$. Calculating, designing and analyzing the reliability of 3 shafts of a 2-speed bevel-spur reducer with the given data, we obtained the following results (Table 4).

### Table 4. Reliability of 3 shaft from different methods for bevel-spur reducer.

| Shaft | Diameter $d$, mm | FORM with MPP | MCS Method |
|-------|------------------|---------------|------------|
| Shaft I | 31.5 | 0.9991257 | 0.99946 |
| Shaft II | 35 | 0.9990795 | 0.99997 |
| Shaft III | 42.5 | 0.9993105 | 0.99994 |

At the last design stage, we proceed to analyze the effect of random parameters on the shaft diameter $d$: a first order response surface is constructed, using all random variables. The first order response surface with 13 random variables can be expressed as (for shaft I):

$$y = 46.414795 - 1.902249x_1 - 1.900806x_2 - 7.100178x_3 - 4.956725x_4 - 0.365519x_5 - 0.097684x_6 - 0.097828x_7 + 35.651551x_8 + 10.101273x_9 + 6.060764x_{10} + 9.091146x_{11} - 1.818229x_{12} + 0.491370x_{13}$$

Sensitivity index: $s[1] = -0.02237170$; $s[2] = -0.02386888$; $s[3] = -0.08915865$; $s[4] = -0.06224280$; $s[5] = -0.00458991$; $s[6] = -0.00122664$; $s[7] = -0.00122845$; $s[8] = 0.44768517$; $s[9] = 0.12684413$; $s[10] = 0.07610648$; $s[11] = 0.11415972$; $s[12] = -0.02283194$; $s[13] = 0.00617025$.

If sensitivity index of specific variable is less than 1%, the variable is considered as deterministic and is fixed at its mean value.

Using Minitab software, we have graphs of the dependence of shaft diameters on the change of random parameters. The regression equation is (Coefficients of regression equations $b_0$, $b_1$, $b_{11}$ in Table 5):

$$d = b_0 + b_1S_{cr} + b_{11}S_{cr}^2$$

Thereby we see that the standard deviation of the fatigue limit $S_{cr}$ has the greatest influence. For example when the values of the fatigue limit standard deviation $S_{cr}$ are from 2.55 to 51 MPa (corresponding to the coefficient of variation of fatigue limit $v_{cr}$ is from 0.01 to 0.2), the size of diameter $d$ increases significantly from 29.6276 to 38.0279 mm (Figure 6a). Dimensional accuracy has little effect on reliability. Hence in some cases this influence can be ignored.
Table 5. Coefficients of regression equations.

| Diameter of shaft, mm | b₀   | b₁   | b₁₁  | R-Sq. % |
|-----------------------|------|------|------|---------|
| Shaft I  d₁           | 29.69| -0.01517| 0.003483 | 99.9 |
| Shaft II d₂           | 32.89| -0.00925| 0.003869 | 100.0 |
| Shaft III d₃          | 40.07| -0.02131| 0.004697 | 99.9 |

Similarly for shaft II and shaft III, the change of fatigue limit \( \sigma_r \) also has the biggest effect. With the above data, the shaft diameter I changes in turn according to the values from 32.8429 to 42.5373 MPa and the values for the diameter of the shaft III are from 39.9802 to 51.27 mm (Figure 6b).

When analyzing the reliability of shaft with two methods: FORM and MCS, are effective tools to give accurate results. However, if compared to the matching moment method, these two methods are much more complicated, large number of calculations, require loops and help with a computer, so when calculating the design of the shaft diameter, people often use the moment matching method.

For the FORM, convergence reliability index after 4 loops, results appear less than 1 second, while with the MCS method, the number of trials must be up to 100 000 and it takes 175 seconds for accurate results. However, we find that the MCS method is more accurate than the FORM, especially when testing with extremely large numbers. Today, with the help of computers and software, the computation is no longer too difficult, the MCS method will be an effective tool to help designers get the exact calculation parameters.

Figure 6. The dependence of shaft diameter I according to the change of standard deviation of the fatigue limit \( S_{\sigma_r} \).

Thanks to the values obtained from the reliability-based design and analysis of machine elements, the 3D models for single and double reducers can be built. The 3D modeling is illustrated in Figure 7.

Figure 7. 3D modeling of single and double reducer.
4. Conclusions

In this paper, the researchers have presented the theoretical basis for design calculation and selection of the transmission shafts and gears with 3 stages. At the same time, the RADME program can be employed for reliability-based design and analysis of machine elements and structure:

- The probabilistic design method gives us more precise results, ensuring product quality, high reliability, and safety, depending on the uncertainty of random variables, and can apply RADME computation software in practical industrial design.
- The fault tree analysis (FTA) method is utilized for reliability analysis of single and double reducer into the logic diagrams. From the given reliability of a mechanical system, the researchers can distribute reasonable reliability for each machine element.
- The mechanical properties of a material (fatigue limit) have the highest effect on the reliability and design dimensions of the mechanical system with sensitivity index \( s[8]= 0.44768517 \). The lengths of shaft \( a, b, l \), and diameter \( d \) have the smallest effect on the reliability. Therefore, with regard to reliability-based design and analysis of shaft, these dimensions \( a, b, d, l \) are unique (deterministic). The distribution of external forces dramatically impacts the reliability and design dimensions.
- The FORM, convergence reliability index after 4 loops, results appear less than 1 second and MCS - 175 seconds for accurate results. However, we find that the MCS method is more accurate than the FORM, especially when testing with extremely large numbers. With the help of RADME software, the computation is no longer too difficult, the MCS method will be an effective tool to help designers get the reliability testing of design parameters.
- In the future, we will complete the RADME software to automatically design all the mechanical drive system elements according to the reliability and apply the second-order (SORM) reliability methods.

Acknowledgement

We acknowledge the support of time and facilities from Ho Chi Minh City University of Technology (HCMUT), VNU-HCM for this study.

REFERENCES

[1] Choi, K. K. and Youn, B.D., Selecting Probabilistic Approaches for Reliability-Based Design Optimization (RBDO), AIAA Journal, 42, No. 1 (2004), pp 124–131.
[2] Du, X. and Chen W. A most probable point-based method for efficient uncertainty analysis. Design Manufacturing, No. 4 (2001), pp 47-66.
[3] Harish, A, Reliability-Based Optimization: Formulations and Methodologies, Doctoral thesis, University of Notre Dam, (2004).
[4] Huibin, L., Wei, Ch. and Agus Sudijanto, Probabilistic sensitivity analysis methods for design under uncertainty. 10th AIAA/ISSMO, (2004).
[5] Nam, H. K. and Haoyu, W and Nestor, V. Q., Adaptive reduction of design variables using global sensitivity in reliability – based optimization, International Journal of Reliability and Safety, No. 1(1-2) (2006), pp.102-119.
[6] Nguyen, H. L. Reliability based design and analysis of gear drive, Journal of Science and technology technical universities (in Vietnamese). No. 05 (2006).
[7] Nguyen, H. L., Tran, V. T. and Pham, Q. T., Reliability–based analysis of machine structure using second-order reliability method. Journal of Advanced Mechanical Design, Systems, and Manufacturing, 13, No.3 (2019).
[8] Nguyen, H. L. Reliability based design and analysis of mechanical systems. Viet Nam National University of Hochiminh City Publishing house. 2016 (in Vietnamese)

[9] Nguyen, H. L.. Textbook Fundamentals of machine design. Viet Nam National University of Hochiminh City Publishing house. 2018 (In Vietnamese).

[10] Padmanabhan, D., Agarwal, H., Renaud, J. E., Batill, S. M, Monte Carlo Simulation in Reliability-Based Optimization Using Approximation Concepts, Fourth International Symposium on Uncertainty Modeling and Analysis, (2003).

[11] Qu, X. Reliability – based structural optimization using response surface approximations and probabilistic sufficiency factor. Doctoral Thesis, University of Florida (2004).

[12] Zhang, Y.M., Qiaoling, L., Bangchun, W., Practical reliability-based design of gear pairs, Mechanism and Machine Theory. Volume 38, Issue 12 (2003), pp. 1363-1370.