Phase structure of the $O(2)$ ghost model with higher-order gradient term

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The phase structure and the infrared behaviour of the Euclidean 3-dimensional $O(2)$ symmetric ghost scalar field model with higher-order derivative term has been investigated in Wegner and Houghton’s renormalization group framework. The symmetric phase in which no ghost condensation occurs and the phase with restored symmetry but with a transient presence of a ghost condensate have been identified. Finiteness of the correlation length at the phase boundary hints to a phase transition of first order. The results are compared with those for the ordinary $O(2)$ symmetric scalar field model.

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I. INTRODUCTION

Scalar fields with negative quadratic gradient term may have relevance in various gravity theories as the dilaton field in Einstein, Einstein-Hilbert and Einstein-Maxwell gravity as well as gravitating phantom scalar fields coupled to gravity and/or the electromagnetic field. The dilaton field, being a one-component real scalar field, occurs by making the conformal degree of freedom of the metric explicit. The usual kinetic term of the dilaton field as well as that of the phantom fields is of ‘wrong’ sign and this implies that those fields are repulsively coupled to gravity. In quantum field theory the phantom fields are the ghost fields.

A great amount of astronomical observational data collected from type Ia supernovae, large scale structures, and cosmic microwave background anisotropy support that our Universe is under accelerated expansion [1–3]. There is observational evidence [4] that at about 70 percent of the contents of our Universe consists of dark energy, some exotic energy with negative pressure that is responsible for the accelerated expansion by its gravitational repulsion. In various cosmological models the dilaton field and the gravitating phantom scalar fields are possible candidates for producing dark energy [5, 6]. Also the cosmological evolution in scalar-tensor gravity may show up phantom epochs that occur as a result of dynamics [7]. Nonstationary models for gravitational collapse of gravitating phantom fields and formation of phantom black holes have been widely investigated recently [8], including the description of the internal structure of phantom black holes [9], their thermodynamical properties [10], as well as the consequences of the no-hair theorem stating that perturbations outside of the black hole are either gravitationally radiated away to infinity or swallowed by the black hole [11].

It is the main goal of the present paper to determine the phase structure and the infrared scaling behaviour of the Euclidean 3-dimensional $\phi^4$ scalar model with internal $O(2)$ symmetry in the particular case when it possesses the kinetic energy operator $\Omega(-\Box) = -Z\Box + Y\Box^2$ with negative wavefunction renormalization constant, $Z < 0$ ($\Box = \partial_\mu\partial_\mu$ denotes the d’Alembert operator).

Scalar fields with the ‘wrong’ sign of the usual kinetic term (like the dilaton field and phantom scalar fields) may condensate when suitable higher-order gradient terms are present in the model [12]. The situation is similar to the case of the spontaneously symmetry broken phase of usual $\phi^4$ theory, when the scalar field acquires a nonvanishing vacuum expectation value and the contribution of the small fluctuations around the new minimum of the energy remains bounded. Gradient terms in the Lagrangean of the type $\phi\Omega(-\Box)\phi$ with $Z < 0$ and $Y > 0$ may provide an inhomogeneous condensate with deeper minimum of energy as that would be for a homogeneous field configuration. Even for such theories in Euclidean space, the stationary-wave modes of the field $\phi$ with momenta $p^2 \leq -Z/Y$ have negative kinetic contribution to the action, and the corresponding spatially inhomogeneous configurations may produce a deeper minimum of the action than that would be produced by any homogeneous field configuration minimizing the potential. For a detailed analysis of the ghost-condensation mechanism see e.g. [13, 14]. This inhomogeneous field configuration can be stable on the quantum level and provide a negative pressure component. Ghost condensation and its relation to cosmological evolution has recently been investigated intensively [15], and kinematically driven acceleration of the Universe has been proposed in various frameworks [16].

Instead of turning to some realistic dilaton or phantom field model of gravity, we shall here study the Euclidean, $O(2)$ symmetric scalar $\phi^4$ model, with modified kinetic energy term as a toy model, similar to that investigated in [13]. The modification $-\Box \rightarrow -\Box + Y\Box^2$ of the kinetic energy operator would introduce unboundedness of the Euclidean action for $Y = 0$, but for $Y > 0$ it still remains bounded from below. Nevertheless the opposite signs of the quadratic and quartic gradient terms may result in occurring ghost condensation. For later convenience let us call scalar fields with $Z = 1$ and $Z = -1$ ordinary and ghost fields, respectively. In this paper we are going to use the Wegner-Houghton (WH) renormalization group (RG) equation [17] and below the singularity scale $k_c$ – if there is any – we deploy the tree-level renormalization
(TLR), which is also called instability induced renormalization [18]. The advantage of the WH RG scheme is the clearcut differentiation of handling the ultraviolet (UV) and the infrared (IR) modes of the field variable. Its main drawback is however that it is restricted to the local potential approximation (LPA), so that the gradient terms do not exhibit any RG flow. Thus in our present approach the dynamics providing the minimum of the Euclidean action is governed by the interplay of the bare gradient terms and the RG flow of the blocked local potential. Functional RG schemes enabling one to go beyond the LPA may reveal more dynamics due to the RG evolution of both the gradient terms and the local potential.

As compared to the analysis in [13], our approach does not include the RG flow of the kinetic piece of the blocked action, but it takes with the flow of the full local potential including the quadratic mass term, the quartic as well as the higher-order polynomial terms. Therefore, it does not provide the possibility to replace the operator $\Omega(-\Box)$ by an effective one in the IR limit. On the other hand, we shall perform the TLR in detail in order to determine the IR scaling laws even in phases exhibiting spinodal instability, when those cannot be obtained from the WH RG equation.

It is well-known that with the usual kinetic term the ordinary $O(N)$ models in Euclidean space with a number of dimensions $d = 3$ exhibit a symmetric phase and a symmetry broken one, the scaling laws at the Gaussian, Wilson-Fisher, and IR fixed points serve as well-sounded test ground for any RG approach [19]. For $N = 1$ the discrete $\phi \rightarrow -\phi$ symmetry, for $N > 1$ the continuous $O(N)$ symmetry is broken spontaneously by the nonvanishing homogeneous vacuum field configuration and there occur $N - 1$ Goldstone bosons. In the symmetric case the RG trajectories can be followed up by means of the numerical solution of the WH RG equation, moving the cutoff scale $k$ from the UV momentum cutoff $\Lambda$ down to the IR scale $k = 0$. The IR limit of the blocked local potential keeps its polynomial form with the minimum at the field configuration $\phi = 0$. In the symmetry broken phase there occurs a singularity of the logarithmic term of the WH RG equation at a finite momentum scale $k_c$ [18]. This happens due to vanishing of the restoring force (the term of the Euclidean action quadratic in the field variable) in the exponent of the integrand of the path integral. The system then starts to develop a spinodal instability. The WH RG equation loses its validity for scales $k < k_c$ and the IR behaviour of the RG trajectories should be determined by the TLR procedure taking explicitly into account the finite amplitude of the inhomogeneous mode that minimizes the blocked action at the moving scale $k$. As a result the ‘Mexican hat’ like local potential becomes convex in the IR limit reproducing the well-known Maxwell cut. Moreover, the approach to the singularity at the scales approaching $k_c$ from above can be revealed as an approach to an IR fixed point of the appropriately rescaled WH RG equations [20, 21].

In our case the modified kinetic energy operator $\Omega(-\Box)$ with $Z = -1$ may make the dynamics more rich. Namely, an inhomogeneous field configuration of finite amplitude may develop at a given scale $k < k_c$ even if no potential were present. The interplay of the gradient and the potential energy terms decides the optimal amplitude $\rho_k$ of the inhomogeneous mode developed at the scale $k$. While for ordinary models with usual kinetic term a nonvanishing amplitude $\rho_k$ of the spinodal instability develops due to the interplay of the positive kinetic term and the negative mass term of the blocked potential, for ghost models with modified kinetic term $\Omega(-\Box)$ with $Z = -1$ the opposite signs of the various gradient terms may also represent for developing a finite amplitude $\rho_k$ of the spinodal instability. TLR enables one to study whether the amplitude $\rho_k$ of the inhomogeneous mode developed by the system does survive the IR limit $k \rightarrow 0$ or not. In our paper a particular emphasis is given to the determination of the IR behaviour of the model by means of the TLR approach. In order to check our numerical procedure for TLR it has also been applied to the symmetry broken phase of Euclidean one-component ordinary scalar field theory and numerical results obtained supporting the theoretical analysis given in [18]. As a further test, application of the numerical TLR procedure to the (ordinary) sine-Gordon model also reproduced the well-known IR behaviour in the molecular phase of the model [22, 23].

It is another peculiarity of the $O(2)$ ghost scalar model with the kinetic operator $\Omega(-\Box)$ that the coefficient $Y$ of the $-\Box^2$ term is of natural dimension mass$^{-2}$. Furthermore the coupling $Y$ is UV irrelevant, i.e. a nonrenormalizable one. Nevertheless, the ghost condensation when it takes place at some scale $k^2 \approx (1/Y)$ plays a definitively decisive role in the IR physics of the model. Therefore the coupling $Y$ may become IR relevant. The rather general way of distinguishing the various phases of the model is opened up by investigating the sensitivity of IR physics to the bare parameters of the model. Such kind of study is possible even if the dimensionful coupling $Y$ is kept constant at its bare value, like in the present work. Then the dimensionless coupling $\tilde{Y} = Yk^2$ scales down during the RG flow and its interplay with the flow of the mass term of the potential severely affects the scale at which the dimensionless inverse propagator may vanish. In this approach no fixed points can be determined. Nevertheless, the global RG flow enables one to identify the phases determining their different IR scaling behaviour and/or sensitivity to the bare parameters of the model. The massive sine-Gordon model was successfully treated in a similar approach in [24].

It should be mentioned that one could have chosen another approach when the the dimensionless coupling $\tilde{Y}$ would have been kept constant at its bare value during the RG flow. This would imply the blow up of the dimensionful coupling $Y = \tilde{Y}k^{-2}$ in the IR limit. Then the zero of the dimensionless inverse propagator might have been occur only due to the flow of the dimension-
less mass parameter of the blocked potential. Such an approach, on the one hand, would have enabled us to determine the fixed points of the model and to draw up the usual type of phase diagrams for various values of $\tilde{Y}$. On the other hand, in that approach one would have loose the possibility to detect the effect of ghost condensation by decreasing the gliding cutoff $k$ downwards. The inverse propagator in momentum representation, $G^{-1}(k^2) = k^2(-1 + \tilde{Y}) + v_1(k)$ (with $v_1(k)$ the mass squared at the scale $k$) shows that the nonvanishing constant value of $\tilde{Y}$ is equivalent with a renormalization $Z = -1$ to $Z = -1 + \tilde{Y}$. Therefore the eventual presence of the ghost condensation would have remained hidden by the RG procedure.

The paper is organized as follows. In Sec. II we test our numerical TLR procedure applying it to the symmetry broken phase of the ordinary 3-dimensional Euclidean one-component scalar field model and to the molecular phase of the 2-dimensional sine-Gordon model. In Sec. III we identify the phases of the $O(2)$ symmetric ghost scalar model, determine the IR behaviour of its various phases, and determine the behaviour of the correlation length at the phase boundary. A comparison is also given to the ordinary counterpart of the model. Finally, the results are summarized in Sec. IV.

II. ONE-COMPONENT SCALAR FIELD MODELS

A. Blocking transformation

In the LPA the blocked action for the one-component scalar field $\phi(x)$ in 3-dimensional Euclidean space has the form

$$ S_k[\phi] = \int d^3x \frac{1}{2} \phi \Omega(-\Box) \phi + \int d^3x U_k(\phi^2), $$

where $k$ is the gliding cutoff, $Z = 1$ or $-1$, $Y \geq 0$, and $U_k(\phi)$ stands for the blocked potential. The blocking transformation corresponding to integrating out the modes of the field in the thin momentum shell $[k - \Delta k, k]$ is given as

$$ e^{-S_{k-\Delta k}[\phi]} = \int \mathcal{D} \phi' e^{-S_k[\phi + \phi']}, $$

where the fields $\phi$ and $\phi'$ contain Fourier modes with momenta $p < k - \Delta k$, and $k - \Delta k < p < k$, respectively. (Throughout this paper we set $\hbar = 1$ for the sake of simplicity.) This blocking transformation consists of integrating out the high-frequency Fourier components $\phi'$ of the field variable. In every infinitesimal step of the blocking given via Eq. (2), the integral is evaluated with the saddle point approximation. Two qualitatively different situations may occur depending on whether the second functional derivative of the blocked action $(i)$ is positive definite or $(ii)$ it starts to develop zero eigenvalues. In case $(i)$ the saddle point is at $\phi' = 0$ and in the limit $\Delta k \to 0$ one arrives to the WH equation

$$ k \partial_k U_k(\Phi) = -\alpha k^3 \ln \left( \Omega(k^2) + \partial^2_k U_k(\Phi) \right) $$

in the LPA, where the field variable $\phi = \Phi = \text{const.}$ is taken to be homogeneous, $\Omega(k^2) = Zk^2 + Yk^4$ and $\alpha = \Omega_3/[2(2\pi)^3] = 1/(4\pi^2)$ with the solid angle $\Omega_3 = 4\pi$.

In the case of unbroken $Z_2$ symmetry the RG trajectories can be followed up by means of Eq. (3) from the UV scale $k = \Lambda$ down to the IR limit $k \to 0$. In the symmetry broken phase at some finite scale $k_c$, the situation $(ii)$ appears. It is signalled by vanishing the argument of the logarithm in the right-hand side of Eq. (3). Now a non-trivial saddle point can be developed by the system, that minimizes the blocked action. Then Eq. (3) loses its validity and one has to turn to the TLR procedure and rewrite the blocking relation (2) into the form

$$ S_{k-\Delta k}[\phi] = \min_{\phi'} S_k[\phi + \phi'], $$

where the loop integral has been neglected completely. Reducing the functional space of the saddle-point configurations $\phi'$ to those of periodic ones of the form

$$ \psi_k(x) = 2\rho \cos(kn_\mu(k)x_\mu + \theta(k)), $$

the action $S_k[\phi + \psi_k]$ becomes a function of the amplitude $\rho$. Here $n_\mu(k)$ and $\theta(k)$ are a spatial unit vector and a phase shift, respectively. Let $\rho_k$ be the value of the amplitude of the saddle-point configuration for which the action $S_k[\phi + \psi_k]$ takes its minimum value. Various saddle points of the system corresponding to various values of $n_\mu(k)$ and $\theta(k)$ are physically inequivalent but are expected to belong to the same minimal value of the blocked action. Inserting ansatz (5) into the tree-level blocking relation (4) one finds

$$ U_{k-\Delta k}(\Phi) = \min_{\rho} \left( \Omega(k^2)\rho^2 + \frac{1}{2} \int_{-1}^{1} du U_k(\Phi + 2\rho \cos(\pi u)) \right). $$

Due to spatial $O(3)$ symmetry the expression in the braces in the right-hand side of Eq. (6) only depends on $\rho$, as expected.

B. Polynomial potential

For the local potential chosen in the Taylor-expanded form

$$ U_k(\Phi) = \sum_{n=0}^{M} \frac{v_n}{n!} \Phi^n = \sum_{n=0}^{M} \frac{g_{2n}}{(2n)!} \Phi^{2n} $$

with $r = \frac{1}{2} \Phi^2$, $g_{2n} = \frac{(2n)!}{n!^2} v_n$ and truncated at $n = M$ the WH RG equation (3) can be rewritten as a set of coupled ordinary, first order differential equations for the
dimensionless running couplings $\tilde{v}_n(k)$. For $M = 2$ those are

$$k\partial_k \tilde{v}_1 = -a\alpha \frac{\tilde{v}_2}{v_1 + Z + Y k^2} - 2\tilde{v}_1,$$

$$k\partial_k \tilde{v}_2 = b\alpha \frac{\tilde{v}_2^2}{(v_1 + Z + Y k^2)^2} - \tilde{v}_2$$

(8)

with $a = 1$ and $b = 3$. The dimensionless couplings $\tilde{v}_1$ and $\tilde{v}_2$ are defined by $v_1 = k^2 \tilde{v}_1$ and $v_2 = k \tilde{v}_2$. Since the phase structure and the scaling laws in the various scaling regimes do not alter qualitatively with increasing truncation $M$, we shall work with $M = 2$ when solving the WH RG equations and choose $\Lambda = 1$ throughout this work.

As discussed above the RG trajectories belonging to the symmetry broken phase can be followed by the WH RG equation down to the scale $k_c$ at which the right-hand side of Eq. (3) becomes singular. The IR scaling regimes do not alter qualitatively with increasing scale $k < k_c$ spinodal instability occurs. For scales $k < k_c$ and background fields $\Phi \in [-\Phi_c, \Phi_c]$ one turns to the tree-level blocking relation (6) and inserting the ansatz (7) into it, one obtains the recursion relation

$$U_{k-k_{c}}(\Phi) = \min_{\{\rho\}} \left( U_k(\Phi) + (Z + Y k^2)k^2 \rho^2 + \sum_{n=1}^{M} \frac{\rho^{2n}}{(n!)^2} \partial_k^{2n} U_k(\Phi) \right)$$

(14)

for the running couplings [18]. For given scale $k$ with given couplings $v_n(k)$ and for given homogeneous field $\Phi \in [-\Phi_c, \Phi_c]$, one determines the value $\rho(\Phi)$ minimizing the right-hand side of Eq. (14). Then one repeats this minimization for various $\Phi$ values and determines the corresponding $U_{k-k_{c}}(\Phi)$ values. Finally these discrete values of $U_{k-k_{c}}(\Phi)$ are fitted by the polynomial (7) in the interval $\Phi \in [-\Phi_c, \Phi_c]$ in order to read off the new $v_n(k - k_{c})$ of the couplings. In such a manner the behaviour of the RG trajectories can be investigated in the deep IR region. This numerical procedure generally converges for sufficiently small values of the ratio $\Delta k/k$. The blocked potential $U_{k-k_{c}}(\Phi)$ outside of the interval $-\Phi_c \leq \Phi \leq \Phi_c$ can be taken identical to $U_{k_{c}}(\Phi)$ with good accuracy, because there it suffers no tree-level renormalization [18]. In Sec. III A we shall argue that the TLR of the ghost scalar field with $O(2)$ symmetry can be reduced to the case of the TLR of the real one-component ghost scalar field when the nontrivial saddle-point configuration is looked for in an appropriately reduced functional space. Numerical study of that case shall be pursued in Sec. III B.

Here we concentrate on the test of our numerical procedure for TLR, applying it to the 3-dimensional Euclidean polynomial model of the ordinary one-component scalar field with $Z_2$ symmetry, considering the case with $Z = 1$, $Y = 0$. Choosing the truncation of the polynomial potential at $M = 10$, we achieved good numerical convergence for $\Delta k/k = 0.001$ and the least square fitting procedure with the number 60 of equidistant grid points in the interval $-\Phi_c(k) \leq \Phi \leq \Phi_c(k)$. It has been checked that the results are stable against increasing the number of grid points. In order to achieve a better least square fit, the TLR procedure has been performed in the wider interval $\Phi \in [-\bar{\Phi}, \Phi]$ with $\bar{\Phi} = \sqrt{-2v_1(k_c)/3v_2(k_c)}$. The latter is a good estimate of $\Phi_c(k)$ for $k \ll k_c$ [18]. It has been established numerically that the blocked potential
The restricted equality $1 + \alpha_1 - \alpha_2 = 0$ should hold implying that $\alpha_1 \approx 0$ with high accuracy. It is known that $\tilde{v}_1(k) \to -1$ and $\tilde{v}_{n>1} \to 0$ in the IR limit $k \to 0$ and that limit corresponds to the RG invariant effective potential $\tilde{U}_{k \to 0}(\Phi) = -\frac{1}{2} \tilde{\Phi}^2$ in the interval $[-\tilde{\Phi}_c, \tilde{\Phi}_c]$ [18]. In our numerical calculations, $\tilde{v}_1(k)$ tends to a constant value close to $-1$, but this value turned out to decrease linearly with decreasing step size $\Delta k/k$. In order to fix the IR limiting value of $\tilde{v}_1$ numerically, we calculated it for five different step sizes, and the extrapolation to $\Delta k/k \approx 0$, shown in Fig. 4, yielded the extrapolated value $\tilde{v}_1^{extr}(0) = -1.005$. We conclude that

our numerical results for the IR behaviour of the ordinary one-component scalar field completely reproduce those reported and argued for in Ref. [18].

C. Tree-level renormalization of the sine-Gordon model

Another verification of our numerical apparatus for TLR has been obtained from its application to the 2
dimensional Euclidean sine-Gordon model given by the classical action

\[ S[\phi] = \int d^2 x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + u_1 \cos(\beta \phi) \right]. \tag{15} \]

The parameter region \( \beta^2 < 8\pi \) belongs to the spontaneously broken phase of the model, while for \( \beta^2 > 8\pi \) we can find the symmetric phase. The results of the TLR of the sine-Gordon model are well known [20, 22, 23] and provide another test of our numerical procedure. The tree-level blocking relation (6) for the ansatz

\[ U_k(\Phi) = \sum_{n=0}^{M} u_n(k) \cos(n\beta \Phi) \tag{16} \]

can be rewritten now in the form of the recursion equation

\[ U_{k-\Delta k}(\Phi) = \min_{\rho} \left[ k^2 \rho^2 + \sum_{n=0}^{M} u_n(k) \cos(n\beta \rho) J_0(2n\beta \rho) \right] \tag{17} \]

(see Ref. [22]). Here \( J_0 \) stands for the Bessel function, and the potential is truncated at the \( M \)-th upper harmonic.

For the numerical calculations we set \( \beta^2 = 4\pi \) and \( M = 10 \). The outcome of our numerical calculations is in complete agreement with the literature. Under the scale \( k_c \), where the spinodal instability occurs, it is known that the amplitude \( \rho(k) \) of the periodic field configurations, which minimizes the action, is given by \( \rho(k) = -\frac{1}{2} \left( (\Phi) - \frac{2}{5} \right) \) [22]. Similarly to the spontaneously broken phase of the one-component scalar field theory with polynomial interaction, the amplitude of the spinodal instability survives the IR limit again. The comparison of our numerical result for \( \rho_k(\Phi) \) to the one obtained in Ref. [22] can be seen in Fig. 5. We also plotted

\[ \Phi \]

in Fig. 6 the magnitude of the first four dimensionless couplings, which in fact, are renormalizable and tend to a constant value in the \( k \to 0 \) limit. This means that the dimensionful effective potential becomes vanishing in accordance with the requirements of convexity and periodicity [22]. Thus we conclude that our numerical procedure applied for the TLR yields results for the IR behaviour of the sine-Gordon model in complete agreement with those obtained in Refs. [20, 22, 23].

III. \( O(2) \) SCALAR MODEL

A. Application of the WH RG approach

Let us turn now to our main goal, the study of the 3-dimensional, Euclidean, \( O(2) \) symmetric model for the ghost scalar field \( \phi \) with polynomial potential, using the LPA ansatz

\[ S_k[\phi] = \frac{1}{2} \int d^3 x \phi^T \Omega(-\Box) \phi + \int d^3 x U_k(\phi^T \phi), \tag{18} \]

for the blocked action, where \( \phi = (\phi_1, \phi_2) \) denotes the two-component real scalar field and \( U_k(\phi^T \phi) \) stands for the blocked potential. For the latter we shall use the ansatz (7) with \( r = \frac{\phi^T \phi}{\mu^2} \). For comparison, we shall also discuss the behaviour of the model for ordinary scalar field with \( Z = 1 \).

In order to determine the phase diagram of the \( O(2) \) symmetric scalar ghost model, we apply the WH RG method again. In Appendix B 1 we derive the WH RG equation

\[ k \partial_k U_k(r) = -\alpha k^2 \left[ \ln[\Omega(k^2)] + U_k'(r) + 2r U_k''(r) \right] + \ln[\Omega(k^2)] + U_k'(r) \tag{19} \]

with \( U_k'(r) = \partial_r U_k(r) \) and \( U_k''(r) = \partial_r^2 U_k(r) \). Eq. (19) for \( Z = 1, Y = 0 \) is just the particular case \( N = 2 \) of the
WH equation for $O(N)$ symmetric $\phi^d$ models,

$$k\partial_k U_k(\Phi) = -\alpha k^3 \left[ \ln[k^2 + \partial_k^2 U_k(\Phi)] + \ln[k^2 + \frac{1}{\Phi} \partial_k U_k(\Phi)]^{N-1} \right],$$

(20)
given in Ref. [18]. To reveal the complete agreement of our result with Eq. (20) for $N = 2$, one has to make the substitution $\Phi = \sqrt{2r}$. It is trivial that the $U(1)$ symmetric ansatz

$$S_k[\phi^*, \phi] = \int d^4x \phi^* \Omega(-\Box) \phi + \int d^4x U_k(\phi^* \phi)$$

(21)
for the blocked action of the one-component complex scalar field $\phi = \frac{1}{2\sqrt{2}}(\phi_1 + i\phi_2)$ is equivalent with the ansatz (18). In Appendix B 2 it is shown that both of the blocked actions given by Eqs. (18) and (21) yield the same WH RG equation for the blocked potential. In order to avoid numerical work with complex numbers, we shall apply numerically the WH RG scheme to the $O(2)$ symmetric case.

The applicability of the WH RG equation may break down at some scale $k_c$ because the argument of the logarithm in the right-hand side of Eq. (19) can eventually reach zero. This occurs, when either of the conditions $s_-(k) = \left[ \Omega(k^2) + U'_k(r) \right] \leq 0$ or $s_+(k) = \left[ \Omega(k^2) + U'_k(r) + 2r U''_k(r) \right] \leq 0$ is fulfilled [18]. This is the case of a spontaneously broken symmetry. These conditions mean, that the loop expansion is inapplicable when $k \leq k_c$. The expression $s_-(k)$ is the inverse propagator of the lightest excitations of the field, the Goldstone-bosons. In the $O(N)$ symmetric model with a homogeneous vacuum field configuration pointing into a given direction of the internal space, there are $N - 1$ transversal excitations or Goldstone-bosons, as it can be seen from the power $N - 1$ of the eigenvalue $s_-(k)$ under the logarithm in the right-hand side of Eq. (20). As mentioned before, in the case of the one-component scalar field, i.e., in the case with $N = 1$, the vanishing of $s_+(k)$ drives the occurrence of spindal instability. For $N \geq 2$ the vanishing of $s_-(k)$ takes over that role. The critical scale $k_c$ is given by $s_-(k_c)|_{\Phi=0} = 0$ implying $Z + Y k^2 + \tilde{v}_1(k_c) = 0$, just like in the case $N = 1$. For local potentials monotonically increasing for asymptotically large values of $|\Phi|$ and for scales $k < k_c$ the interval $0 \leq |\Phi| \leq \Phi_c(k)$ (with $\Phi_c(k) = \sqrt{\bar{K}_c(k)}$) of instability may open up determined via the vanishing of $s_-(k)$ as

$$\Phi_c(k) = \sqrt{\frac{2[Z + Y k^2 + \tilde{v}_1(k)]}{\tilde{v}_2(k)}}.$$  

(22)

Supposing that a nontrivial saddle point $\phi' = \tilde{\psi}_k$ appears in the integrand on the right-hand side of Eq. (B1), the integral can be approximated by the contribution of that saddle point. Thus one finds the relation, the generalization of Eq. (4),

$$S_{k-\Delta k}[\phi] = \min_{[\phi']} S_k[\phi + \phi'] = S_k[\phi + \tilde{\psi}_k],$$

(23)
where $\tilde{\psi}_k(x) \neq 0$ represents the nontrivial saddle-point configuration minimizing the action $S_k[\phi + \phi']$. For practical purpose we restrict ourselves to looking for nontrivial saddle-point configurations in a particular subspace of the configuration space, say to periodic configurations of the type given in Eq. (5).

In the $O(N)$ case there are, however, possibilities to choose the nontrivial saddle-point configuration with various orientations in the internal space. Being restricted to LPA by the WH approach, the background configuration should be chosen homogeneous, $\phi = \Phi = \Phi_k$ pointing to some particular direction given by the unit vector $\xi$ in the internal space. In general, the nontrivial saddle-point configuration might have components parallel and orthogonal to the direction $\xi$. The question arises how these components should be chosen in order to minimize the value of the action. It was argued in Ref. [18] that the TLR of ordinary $O(N)$ models for $N \geq 2$ can be reduced to the TLR of the ordinary one-component scalar model. The argument is based on the positivity of the quadratic gradient term. Without loss of generality, the field configuration $\Phi + \tilde{\psi}_k$ can be rewritten as

$$\Phi_k + \tilde{\psi}_k(x) = \eta_k(x) \bar{R}(x) \xi$$

(24)
in terms of an appropriately chosen amplitude function $\eta_k(x)$ with the $SO(N)$ matrix $\bar{R}(x)$. Now the quadratic gradient term of the action takes the form

$$\frac{1}{2} \int d^4x \left( \partial_\mu |\eta_k(x)\bar{R}(x)\xi\right)^T \left( \partial_\mu |\eta_k(x)\bar{R}(x)\xi\right)$$

(25)
$$= \frac{1}{2} \int d^4x \left[ \partial_\mu \eta_k(x) \right] \left[ \partial_\mu \eta_k(x) \right] + \eta_k^2(x) \delta_{\mu\nu} \left[ \partial_\mu \bar{R}(x)\xi \right] \left[ \partial_\nu \bar{R}(x)\xi \right],$$

where the identities $\left[ \bar{R}(x)\xi \right]^T \bar{R}(x)\xi = 1$ and $\left[ \bar{R}(x)\xi \right]^T \partial_\mu \left[ \bar{R}(x)\xi \right] = 0$ have been used. This means that any inhomogeneity of the vector $\bar{R}(x)\xi$ yields a positive contribution to the action, so that the nontrivial saddle point should be such that $\bar{R}(x)\xi$ were homogeneous. Then the relation (24) implies that both of the vectors $\bar{R}(x)\xi$ and $\tilde{\psi}_k(x)$ should be parallel to the direction $\xi$ of the background field. The periodic ansatz for the nontrivial saddle-point configuration, similar to the one given by Eq. (5), is then

$$\tilde{\psi}_k(x) = e^{2\rho_k \cos(kn_\mu(x)x_\mu + \theta_k)}.$$  

(26)
Inserting it into the tree-level blocking relation (23) one arrives at Eq. (6) that can be recasted in the form of the recursion relation (14). So the TLR procedure of the ordinary $O(N)$ model reduces to the that for the ordinary $O(1)$ model.
For $Z = -1$, i.e., for the $O(N)$ ghost models with negative quadratic gradient term the above given argumentation fails because the terms in Eq. (25) acquire negative signs and no conclusion can be made that $R_{\xi}$ were homogeneous. Here we shall make the ansatz (26) for the nontrivial saddle point again. It might happen however that similar periodic saddle-point configurations with more sophisticated orientation in the internal space could give smaller value of the blocked action. When the ansatz (26) is used, the TLR of the $O(2)$ ghost model reduces to the TLR of the polynomial model of the one-component scalar field, except that the interval of constant background fields in which the spinodal instability occurs is now determined by the critical value $\Phi_c(k)$ given in Eq. (22) instead of Eq. (A3). The tree-level blocking relation (23) results in the recursion equation (14) for the blocked potential, again.

### B. Phase structure and IR scaling laws

#### 1. Identification of the phases

The WH RG flow is mapped numerically for keeping the dimensionful coupling $Y$ at various given constant values. The phases in the parameter plane $(\tilde{v}_1(\Lambda), \tilde{v}_2(\Lambda))$ of the bare dimensionless couplings can be distinguished by considering the global RG flow started from the various points of that plane both for the ghost and ordinary $O(2)$ models. In the parameter space the symmetric phase of the model, called here phase I, corresponds to the points such that the RG trajectories started from them can be followed by the WH RG equation (19) down to the IR scale $k \to 0$, i.e., the inverse propagator $G^{-1}(k) \equiv s_-(k)|_{\Phi=0} = Zk^2 + Yk^4 + v_1(k)$ remains positive along those RG trajectories. As opposed to that, phase II is characterized by RG trajectories along which the inverse propagator either develops a zero at some finite scale $k_c$, $G^{-1}(k_c) = 0$, at which the right-hand side of Eq. (19) becomes singular, or it is negative already at the UV scale, $G^{-1}(\Lambda) < 0$. Therefore the region corresponding to phase II may consists of region IIA with $G^{-1}(k) > 0$ for $\Lambda \geq k > k_c > 0$ and region IIB with $G^{-1}(\Lambda) = \Lambda^2 + Y\Lambda^4 + v_1(\Lambda) < 0$, i.e., $-Z - Y > v_1(\Lambda)$ for $\Lambda = 1$. Restricting ourselves to the bare parameter values satisfying the inequalities $|v_1(\Lambda)| < \Lambda^2 = 1$ and $Y > 0$, region IIB never occurs for the ordinary $O(2)$ model, but it occurs for the ghost model for $v_1(\Lambda) \leq 1 - Y$ when $Y < 2$.

Our first task is to find numerically the regions corresponding to phases I and IIA in the parameter plane. The points of region IIA can be identified by solving the WH RG equation (19) and detecting that the inverse propagator vanishes at a finite scale $k_c$. For local potentials given in (7) Eq. (19) reduces to the coupled set of ordinary first-order differential equations which has the form (8) with $a = 3$ and $b = 5$ for $M = 2$. In order to find region IIA numerically, 1000 random starting points of the RG trajectories have been been generated in the parameter region $(\tilde{v}_1(\Lambda), \tilde{v}_2(\Lambda)) \in [-1, 1] \otimes [0, 10]$. The phase diagrams are shown in Fig. 7 for various values of the higher-derivative coupling $Y$; the empty, dotted, and shadowed regions correspond to regions IIA and IIB of phase II, respectively.

![FIG. 7. Phase diagrams in the plane $(\tilde{v}_1(\Lambda), \tilde{v}_2(\Lambda))$ for various given values $Y$ for the ghost (to the left) and the ordinary (to the right) $O(2)$ models. The empty regions correspond to the symmetric phase I, the dotted and shadowed regions correspond to regions IIA and IIB of phase II, respectively.](image-url)
For the ordinary \(O(2)\) model phase II contains only region IIA. The phase diagrams of the ordinary and ghost models are compared in Fig. 7. The phase boundary \(\tilde{v}_u(Y, \tilde{v}_2(\Lambda))\) depends approximately linearly on \(\tilde{v}_2(\Lambda)\) as \(\tilde{v}_u \approx -c(Y)\tilde{v}_2(\Lambda)\) where \(c(Y)\) is monotonically increasing with increasing value of the higher-derivative coupling \(Y\). Therefore the phase boundary is at \(\tilde{v}_u \approx 0\) for \(\tilde{v}_2(\Lambda) \ll 1\) for all values \(0 \leq Y \leq 2\).

2. IR scaling in phase I

For phase I the IR scaling laws have been determined by solving the WH RG Eq. (19). The IR limits have been compared on RG trajectories started at various given ‘distances’ \(t = \tilde{v}_1(\Lambda) - \tilde{v}_u\) from the phase boundary for \(\tilde{v}_2(\Lambda) = 0.01, 0.1\) and all investigated values of \(Y\). It has been found that the dimensionful couplings \(v_u(k)\) of the local potential tend to constant nonvanishing values in the IR limit \(k \to 0\) for both the ghost and the ordinary \(O(2)\) models. The effective potential in phase I is convex and paraboloid like for both the ghost and the ordinary models and sensitive to the choice of the bare potential. For the ghost model the linear relation

\[
v_1(0) = at + b(Y)
\]

has been established where the slope \(a\) is independent of \(Y\), whereas the mass squared \(b(Y)\) at the phase boundary for \(t \to 0\) monotonically decreases with the increasing higher-derivative coupling \(Y\) (see Fig. 8). The coupling \(v_2(0)\) decreases with decreasing coupling \(Y\) for given \(t\), i.e., bare coupling \(\tilde{v}_1(\Lambda)\) and approaching the phase boundary (i.e., for \(t \to 0\) it tends to zero independently of the value of the coupling \(Y\)). For the ordinary counterpart of the model the effective potential seems to be insensitive to the value of the higher-derivative coupling in the range \(0 \leq Y < 2\), but keeps its sensitivity to the parameters of the bare potential.

3. IR scaling laws in phase II

The RG trajectories belonging to region IIA can be followed by the WH RG equation (19) from the UV scale \(\Lambda\) to the scale \(k_c\) of the singularity and the scaling of the couplings in the deep IR region \(k < k_c\) should be obtained by TLR which has been started from the initial potential obtained at the critical scale \(k_c\) by the solution of Eq. (19). In order to find the RG trajectories belonging to region IIB TLR should be started at the UV scale. In both cases the ansatz (7) with the truncation \(M = 10\) has been used for the potential. In our numerical TLR procedure the scale \(k\) has been decreased from either the critical one \((k_c)\) for region IIA or from the UV scale \(\Lambda\) for region IIB by 3 orders of magnitude with the step size \(\Delta k/k = 0.01\) during the numerical tree-level blocking. Generally \(\approx 500\) iteration steps have been numerically performed at each value of the constant background \(\Phi\) for the minimization of the blocked potential \(U_k(\rho, \Phi)\) with respect to the amplitude \(\rho\) of the spinodal instability. The TLR procedure is quantitatively sensitive to the choice of the interval \(|\Phi| \leq \Phi_c\) in which the minimization of the potential \(U_k(\rho, \Phi)\) with respect to the amplitude \(\rho\) of the spinodal instability and the least square fit of the blocked potential at scale \(k - \Delta k\) are performed. For ‘Mexican hat’ like potential \(U_{k_c}(\Phi)\) for region IIA or \(U_{\Lambda}(\Phi)\) for region IIB the choice \(\Phi \approx 1.5\Phi_m\) has been made where \(\pm \Phi_m\) are the positions of the local minima of the potential with \(\Phi_m = \sqrt{-2v_1(k_c)/v_2(k_c)}\) or \(\Phi_m = \sqrt{-2v_1(\Lambda)/v_2(\Lambda)}\), respectively. For convex potentials with \(v_1(k_c) > 0\) for region IIA or \(v_1(\Lambda) > 0\) for region IIB the choice \(\Phi \gtrsim 30\) has been made. It has been observed numerically that the blocked potential does not acquire tree-level corrections outside of the interval \(|\Phi| \leq \Phi_c\) with \(\Phi_c\) given by Eq. (22), but the choice of the larger interval makes the minimization and fitting numerically stable.

For each given value of the higher-derivative coupling \(Y\) and both values of the bare coupling \(\tilde{v}_2(\Lambda) = 0.01\) and 0.1 we have determined the RG trajectories for 3 to 5 bare values of \(\tilde{v}_1(\Lambda)\) distributed uniformly in the interval \(-1 < \tilde{v}_1(\Lambda) < \tilde{v}_u\). It was found that the couplings of the dimensionful blocked potential tend to constant values in the IR limit. Moreover, it has been observed that for any given value of the higher-derivative coupling \(Y\) the effective potential is universal in the sense that it does not depend on at which point \((v_1(\Lambda), v_2(\Lambda))\) the RG trajectories have been started. Therefore we have determined the mean values \(v_1(0)\) and \(v_2(0)\) of the couplings \(v_1(0)\) and \(v_2(0)\) with their variances via averaging them over all evaluated RG trajectories belonging to a given value of the coupling \(Y\). It turned out that the mean value \(v_1(0)\) decreases strictly monotonically with increasing values of the higher-derivative coupling \(Y\) as shown in Fig. 9. The mean values \(\tilde{v}_2(0)\) take randomly positive and negative small values with variances comparable with their magnitudes when the coupling \(Y\) is altered. Thus we concluded that the quartic coupling of the effective potential

![FIG. 8. The parameters \(v_1(0) = at + b\) (to the left) and \(v_2(0)\) (to the right) vs. the ‘distance’ \(t = \tilde{v}_1(\Lambda) - \tilde{v}_u\) from the phase boundary for \(v_2(\Lambda) = 0.01\). The dots, boxes, and rombs correspond to \(Y = 0.7, 1.0, 1.5\), respectively, the lines are for guiding the eyes. The dependence of the coefficient \(b\) on the higher-derivative coupling \(Y\) is shown in the inset.](image-url)
vanishes, an averaging over all considered values of the coupling $Y$ yields $\langle v_2(0) \rangle = 0.004 \pm 0.01$. Therefore the dimensionful effective potential is an upsided paraboloid with its minimum at $\Phi = 0$ in phase II. Moreover, the nonrenormalizable, UV irrelevant coupling $Y$ turns out to be IR relevant in phase II.

For each RG trajectory we have also evaluated the ratio $r$ characterizing how large part of the sum of the negative terms is cancelled totally or partially by the positive higher derivative term in the inverse propagator $G^{-1}(k_s)$,

$$r = \begin{cases} \frac{Y k_s^4}{v_1(k_s) + 1}, & \text{if } v_1(k_s) \geq 0 \\ 1, & \text{if } v_1(k_s) < 0 \end{cases}$$

where $k_s = k_c$ and $k_s = \Lambda$ for regions IIA and IIB, respectively. In this manner the ratio $r$ characterizes how significant is the role played by the ghost condensation in this cancellation either at scale $k_s$ where TLR should be started. In the cases with $v_1(k_s) < 0$ the ghost condensation is the only mechanism that can be responsible for the above mentioned cancellation. Numerics has shown that the value of the ratio is at $r \approx 1$ in most of the parameter region belonging to phase II, but it generally decreases suddenly when $\tilde{v}_1(\Lambda)$ approaches the phase boundary at $\tilde{v}_a$. The IR couplings in the effective potential seem, however, to be insensitive to the value of $r$, i.e., to the importance of the ghost condensation at the scale $k_s$. As argued for below, rather the role of the ghost condensation during the *global* RG flow during TLR makes its imprint on the value of $v_1(0)$ via its dependence on the coupling $Y$.

**FIG. 9.** The dependence of mean values $\overline{v_1(0)}$ on the higher-derivative coupling $Y$ in phase II. The lines are only to guide the eyes.

The numerical TLR procedure has shown that although the amplitude $\rho_k$ of the spinodal instability suddenly acquires a large value just below either the scale $\Lambda$ for region IIB (cases $Y = 0$ and 0.7 in Fig. 10) or the scale $k_c$ for region IIA (cases $Y = 1.0$ and 1.5 in Fig. 10), but after relatively few ($\sim 30$) blocking steps it is rapidly washed away and does not survive the IR limit. The vanishing of $\rho_k(0)$ is accompanied by the saturation of the value of $v_1(k)$ at its IR limiting value $v_1(0)$. In Fig. 10 the plots belong to RG trajectories characterized by the ratio $r \approx 1$. This indicates that basically the ghost condensation should be responsible for the occurrence of the finite amplitude $\rho_k(\Phi)$ of the spinodal instability when TLR is started. However, as numerics shows, the RG flow of the local potential starts to dominate the IR scaling after a relatively small decrement of the scale $k$. Nevertheless, this would-be condensate seems to left behind its footprint on the curvature of the effective potential through the dependence of the mass parameter $v_1(0)$ on the higher-derivative coupling $Y$. As seen in Fig. 9, the exponential dependence of $v_1(0)$ on $Y$ changes its slope at around $Y \approx 1$. Fig. 10 seems to support the conjecture that the ghost condensation plays the most significant role during the global RG flow when the higher-derivative coupling is at around $Y \approx O(\Lambda^{-2} = 1)$. Namely, Fig. 10 shows that the width of the $k$-interval in which $\rho_k(\Phi = 0)$ is nonvanishing increases for $Y$ increasing from 0 towards 1, but it remains unaltered for $Y \geq 1$. This may be connected with the following circumstances.

**FIG. 10.** The TLR flow of $v_1(k)$ and the corresponding $\rho_k(\Phi = 0)$ for $t = \tilde{v}_2(\Lambda) = 0.1$ and $Y = 0$, 0.7, 1.0, and 1.5 (from the top to the bottom) in phase II of the ghost $O(2)$ model.
The kinetic piece $\Omega(k^2)$ of the inverse propagator is an up-sided parabola with zeros at $k^2 = 0$ and $k^2 = 1/Y$ and the minimum at $k^2 = 1/(2Y)$. If $Y > \Lambda^{-2} = 1$ then the modes which can give negative contributions to the action by ghost condensation represent a small amount of the modes below the UV cutoff $\Lambda = 1$. In the extreme limit $Y \to \infty$ these modes are restricted to an interval of vanishing size at zero UV momentum, and the spinodal instability is governed by the potential. In the other extreme with $Y \ll \Lambda^{-2} = 1$ all modes below the UV cutoff are available for ghost condensation, but for $1/(2Y) \gg 1$ all they may give rather small negative contribution to the action and in the limit $Y \to 0$ this contribution becomes negligible. By this naive argumentation one concludes that the ghost condensation may only play a significant role in forming the IR value $v_1(0)$ for $Y \approx O(\Lambda^{-2}) = 1$.

It has also been established numerically that the range $\Phi_c(k)$ of the homogeneous background field in which spinodal instability occurs opens up gradually when the scale $k$ decreases from $k_s$, its width reaches a maximum and then suddenly decreases to zero at some finite scale $k_0$, where also the amplitude $\rho_2$ vanishes and the couplings $v_1(k)$ and $v_2(k)$ reach their IR values. No more TLR corrections appear below that scale $k_0$. This behaviour is rather different of that shown by the ordinary $O(2)$ symmetric model in its broken symmetric phase.

The ghost condensate occurring at the scale $k_s$ breaks internal $O(2)$ symmetry as well as Euclidean rotational symmetry in the 3-dimensional space and translational symmetry in the $x_1$ direction in the Euclidean space. These symmetries are, however, restored in the IR limit. Thus we have to conclude that even phase II of the ghost model is a symmetric one. The distinction between phases I and II can only be done by considering the global RG flow: the effective potential exhibits no sensitivity to the couplings of the bare potential in phase II, as opposed to phase I, where such a sensitivity is essential.

It has also been checked numerically that for phase II of the ordinary $O(2)$ model with nonvanishing higher-derivative coupling $Y$ the TLR reproduces the Maxwell-cut for the dimensionful effective potential, as expected.

4. Correlation length

Finally, let us determine the behaviour of the correlation length $\xi \sim 1/k_c$ approaching the boundary of phases I and II from the side of phase II for the ghost model. This is only possible for the RG trajectories belonging to region IIA, when the singularity scale $k_s$ can be detected by solving the WH RG equation (19). Fig. 7 makes it plausible that for given $\tilde{v}_2(\Lambda)$ the ‘distance’ $\tilde{v}_u - \tilde{v}_1(\Lambda)$ measures how far an RG trajectory belonging to phase II runs from the boundary of the phases I and II. Therefore, one can identify $\tilde{v}_u - \tilde{v}_1(\Lambda)$ with the reduced temperature $t$ up to a constant factor. In order to determine the dependence of the correlation length $\xi$ on the difference $\tilde{v}_u - \tilde{v}_1(\Lambda)$, we have solved the WH RG equation with various initial conditions $\tilde{v}_2(\Lambda) = 0.01, 0.1$ and $\tilde{v}_1, (\Lambda) = [1 - (i/100)]\tilde{v}_u (i = 1, 2, \ldots)$ for each value of $Y = 1.0, 1.5, 2.0, 4.0, 10.0$. It has been established that the correlation length increases linearly with decreasing reduced temperature,$\quad \xi \sim 1/k_c = \xi_0 - \kappa (\tilde{v}_u - \tilde{v}_1(\Lambda))$, \quad (29)

for any fixed values the coupling $Y$, as shown in Fig. 11. The coefficient $\kappa$ seems to rise nearly linearly with increasing higher-derivative coupling $Y$ (see Fig. 12). Although the correlation length increases approaching the phase boundary from the side of phase II, but it remains finite, while it is infinite in the symmetric phase I. This signals that the phase transition of the ghost $O(2)$ model is of first order, as opposed to the ordinary $O(2)$ model where the correlation length blows up according to the power law $\xi \sim t^{-\nu}$ (see Fig. 13), indicating a continuous phase transition.

IV. SUMMARY

In the present paper the phase structure and the IR behaviour of the $O(2)$ symmetric ghost scalar field model with the kinetic energy operator $\Omega(-\square) = -Z\square + Y\square^2$ with $Z = -1$, $Y > 0$ has been investigated in 3-dimensional Euclidean space in the framework of Wegner and Houghton’s renormalization group (WH RG).
scheme. A particular emphasis has been laid on tree-level renormalization (TLR) in order to obtain the deep IR scaling of the blocked potential. The opposite signs of the wavefunction renormalization $Z$ and the higher-derivative coupling $Y$ enable the system to lower the value of its action for inhomogeneous field configurations as compared to that for homogeneous ones. The corresponding nontrivial saddle-points of the path integral have been looked for in sinusoidal form of wavelength $2\pi/k$ in one spatial direction, where $k$ is the gliding sharp cutoff scale of the WH RG approach. The occurrence of such a periodic field configuration breaks – among others – the internal $O(2)$ symmetry spontaneously. The WH RG approach has been applied by keeping the dimensionful higher-derivative coupling $Y$ constant.

Our numerical TLR procedure has been tested by successfully reproducing well-known results for the symmetry broken phase of ordinary ($Z = +1$, $Y = 0$) one-component real scalar field model in 3-dimensional Euclidean space and for the molecular phase of (ordinary) sine-Gordon model in 2-dimensional Euclidean space.

It has been established that the 3-dimensional $O(2)$ symmetric ghost scalar model has two phases. In phase \( I \), i.e., in the symmetric phase no ghost condensation occurs; the couplings of the dimensionful blocked potential tend to constant values in the IR limit $k \to 0$, their values depend on the couplings of the bare potential and on the higher-derivative coupling $Y$. The dimensionful effective potential for phase \( I \) is of paraboloid-like shape. In phase \( II \) spinodal instability occurs along the RG trajectories at some finite scale $k_\ast$ which was shown to occur basically due to ghost condensation. Numerics has revealed, however, that neither the amplitude of the condensate nor the interval of the homogeneous background field in which it occurs survive the IR limit. This means that the symmetries broken at intermediate scales are restored in the IR limit. Nevertheless, this would-be condensate makes a significant imprint in the effective potential which becomes insensitive to the couplings of the bare potential. The curvature of the effective potential decreases monotonically with increasing higher-derivative coupling $Y$, while the couplings of its higher-order terms vanish, so that the effective potential of phase \( I \) with restored symmetry is of paraboloid shape. The identification of the scale of spinodal instability in phase \( II \) with the reciprocal of the correlation length has shown that approaching the phase boundary the correlation length increases to a finite value, indicating that the ghost model exhibits a phase transition of the first order.

The phase structure obtained for the 3-dimensional ghost $O(2)$ model has been compared to that of the ordinary ($Z = +1$, $Y > 0$) $O(2)$ model. The latter has a symmetric phase and a symmetry broken one. In the latter the effective potential is insensitive to the bare potential and reproduces the Maxwell cut. The correlation length has been found to scale according to a continuous phase transition.

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Appendix A: Tree-level renormalization of Euclidean one-component scalar field theory with polynomial potential

Here we would like to remind the reader how TLR works in one-component scalar field theory with ordinary kinetic term. More detailed discussion can be found in Ref. [18]. For scales $k < k_c$ spinodal instability occurs when the logarithm in the right-hand side of Eq. (3) satisfies the inequality

$$Z + \tilde{v}_1(k_c) + \frac{3}{2} \tilde{v}_2(k_c) \Phi^2 \leq 0.$$  \hfill (A1)

Since the last term in the left-hand side of the inequality (A1) is never negative, the critical scale is given via the equation

$$Z + \tilde{v}_1(k_c) = 0.$$  \hfill (A2)

Moreover, one can estimate the interval $\Phi \in [-\Phi_c(k),\Phi_c(k)]$ in which instability occurs for scales $k < k_c$ from inequality (A1) as

$$\tilde{\Phi}_c(k) = \sqrt{-\frac{2[Z + \tilde{v}_1(k)]}{3\tilde{v}_2(k)}}, \quad \Phi_c(k) = \sqrt{k} \tilde{\Phi}_c(k).$$  \hfill (A3)

For scales $k < k_c$ and background fields $\Phi \in [-\Phi_c(k),\Phi_c(k)]$ one turns to the tree-level blocking relation (6) and inserting the ansatz (7) into it, one obtains the recursion relation

$$U_{k - \Delta k}(\Phi) = \min_{\{\rho\}} \left( U_k(\Phi) + Z k^2 \rho^2 + \sum_{n=1}^M \rho_0^n (n! k)^2 \Phi^n U_k(\Phi) \right),$$  \hfill (A4)

for the blocked potential. For given scale $k$ with given couplings $v_n(k)$ and for given homogeneous field $\Phi_c(\{\rho_k(\Phi)\}$, one determines the value $\rho_k(\Phi)$ minimizing the right-hand side of Eq. (A4). Then one repeats this minimization for various $\Phi$ values and determines the corresponding $U_{k - \Delta k}(\Phi)$ values. Finally these discrete values of $U_{k - \Delta k}(\Phi)$ are fitted by the polynomial (7) in the interval $\Phi \in [-\Phi_c(k),\Phi_c(k)]$ in order to read off the new couplings $v_n(k - \Delta k)$. In such a manner the behaviour of the RG trajectories can be investigated in the deep IR region. This numerical procedure generally converges for sufficiently small values of the ratio $\Delta k/k$. It was shown in Ref. [18] that for $Z = 1$ the amplitude $\rho_k(\Phi)$ of the spinodal instability is a linear function of the homogeneous background $\Phi$, $2 \rho_k(\Phi) = -\Phi + \Phi_c(\Phi)$. Outside of the interval $-\Phi_c(\Phi) \leq \Phi \leq \Phi_c(\Phi)$ the dimensionful blocked potential $U_{k - \Delta k}(\Phi)$ can be taken identical to $U_{k_c}(\Phi)$. In the IR limit $k \to 0$ and in the interval $-\Phi_c(0) \leq \Phi \leq \Phi_c(0)$ the tree-level blocking results in
the downsided parabola $\hat{U}_{k-0}(\Phi) = -\frac{1}{2}\Phi^2$ for the dimensionless blocked potential corresponding to $\hat{v}_1(0) = -1$, $\hat{v}_n(0) = 0$ for $n \geq 2$. Therefore, the dimensionful potential flattens out taking a constant value in the interval $-\Phi_-(k) \leq \Phi \leq \Phi_+(k)$ that represents the so-called Maxwell cut.

### Appendix B: WH RG equations for scalar $\phi^4$ models with $O(2)$ and $U(1)$ symmetry

#### 1. Case of $O(2)$ symmetry

Here we derive the WH RG equation for the real, two-component scalar field theory using the ansatz for the blocked action (18) exhibiting $O(2)$ symmetry. The blocking relation

$$ e^{-S_{k-\Delta k}[\phi]} = \int \mathcal{D}\phi' e^{-S_k[\phi'+\phi]} $$

is the straightforward generalization of the relation (2) for the 2-component scalar field. Since the WH equation is applicable only in the LPA, the lowest order of the gradient expansion, it is sufficient to Taylor-expand the WH equation $S_k[\phi+\phi']$ in the exponent of the integrand around the homogeneous field configuration $\phi(x) = \Phi$.

$$ S_k[\Phi + \phi'] = S_k[\Phi] + \frac{1}{2} \int d^d x \phi^T \left( S_{k-\Delta k}[\Phi] \right) \phi' + O((\phi')^3), $$

where the matrix of the second functional derivative of the blocked action has been introduced as

$$ S_{k-\Delta k}[\Phi] = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \delta(x-y). $$

Abandoning the terms of order $O(\phi'^3)$ and higher, we can perform the Gaussian path integral and reduce Eq. (B1) to the blocking relation for the blocked action

$$ S_{k-\Delta k}[\Phi] = S_k[\Phi] + \frac{\hbar}{2} \text{tr} \ln S_{k-\Delta k}[\Phi]. $$

As it is well-known, in the limit $\Delta k/k \to 0$ the neglected terms of higher order in $\phi'$ give vanishing contributions and one arrives at the exact WH RG equation

$$ \partial_{k} S_k[\Phi] = -\lim_{\Delta k \to 0} \frac{\hbar}{2\Delta k} \text{tr} \ln S_{k\Delta k}[\Phi]. $$

In order to cast Eq. (B5) into a more explicit form, we have to evaluate the trace log in its right-hand side. Fortunately, the matrix $S_{k\Delta k}[\Phi]$ is diagonal in momentum space consisting of $2 \times 2$ block matrices in the internal space. For the purpose of the determination of the elements of those block matrices for given momentum $p$ let us make the LPA ansatz

$$ S[\phi] = \frac{1}{2} \sum_{a=1}^{2} \int \frac{d^d p}{(2\pi)^d} \phi_{a,-}(p^2) \phi_{a,p} + \int d^dx U_{k}(\phi^T \phi) $$

for the blocked action. According to this, the matrix elements are

$$
S_{11} = \Omega(p^2) + U_r^r(r) + \Phi^2_{p} U_{k}^{rr}(r),
S_{22} = \Omega(p^2) + (U_k^r(r) + \Phi^2_{p} U_{k}^{rr}(r)),
S_{12} = \Phi_1 \Phi_2 U_{k}^{rr}(r) = S_{21},
$$

with $U_r^r(r) = \partial_r U_k(r)$ and $U_k^r(r) = \partial^2_r U_k(r)$ and $r = \frac{1}{2} \Phi^T \Phi = \frac{1}{2}(\Phi^2 + \Phi^2_{p})$. The eigenvalues $s_+$ and $s_-$ of the block matrices of $S_{k\Delta k}[\Phi]$ can be determined from the vanishing of the determinant of the corresponding eigenvalue equations $s^2 - s(S_{11} + S_{22}) + S_{11} S_{22} - S_{12}^2 = 0$ and are

$$ s_+(p) = \Omega(p^2) + U_r^r(r) + 2r U_k^r(r),
S_-(p) = \Omega(p^2) + U_k^r(r). $$

The trace log of the matrix $S_{k\Delta k}[\Phi]$ is the sum of the logarithms of the eigenvalues of the matrix. The trace operation in the right-hand side of Eq. (B5) can be carried out by summation over the internal space degrees of freedom and integrating over the modes in the infinitesimally thin momentum shell $|p| \leq |k - \Delta k|$. Thus the WH RG equation

$$ k \partial_k U_k(r) = -\alpha k \delta \ln[s_+(k)s_-(k)] $$

is obtained. From this one obtains the WH RG equation (19).

#### 2. Case of $U(1)$ symmetry

Let us now derive the WH RG equation for the $U(1)$ symmetric model given by the ansatz (21). Splitting the field variable at scale $k$ again into the sum of the contribution $\phi$ of the IR modes with momenta $p$ such that $|p| \leq k - \Delta k$ and that of $\phi'$ of the UV modes with momenta from the infinitesimal momentum shell $k - \Delta k \leq |p| \leq k$, we write the blocking relation as

$$ e^{-S_{k-\Delta k}[\phi'^* \phi]} = \int \mathcal{D}\phi' e^{-S_k[\phi'^* \phi + \phi + \phi']} $$

Let us expand the exponent in the integrand of the path integral around the constant background configuration $\Phi$

$$ S_k[\Phi + \phi' \phi + \phi + \phi'] = S_k[\Phi, \Phi] + \frac{1}{2} \left( \delta(x, x') \cdot S_k[\Phi, \Phi] \cdot \delta(x, x') \right) + O((\phi')^3). $$

(B11)
with the second functional derivative matrix

\[ S^{(2)}_k[\Phi^*, \Phi] = \begin{pmatrix} \delta^2 S & \delta^2 S \\ \delta^2 S & \delta^2 S \end{pmatrix} \]

\[ = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} (2\pi)^d \delta^{(d)}(p - p'), \] (B12)

where

\[ S_{11} = U''_k(r)(\Phi^*^2, \Phi^2), \]
\[ S_{22} = U''_k(r), \]
\[ S_{12} = S_{21} = \Omega(p^2) + rU''_k(r) + U'_k(r) \] (B13)

and \( r = \Phi^*\Phi \), the repeated prime over \( U_k \) denotes repeated derivations with respect to the variable \( r \).

Assuming that the path integral in the right-hand side of Eq. (B10) exhibits the trivial saddle point \( \phi' = \phi'^* = 0 \), the first order term vanishes in the expansion (B11). Neglecting the terms of the orders higher than quadratic and performing the Gaussian path integral, we get from (B10) the equation

\[ S_k - \Delta_k[\Phi^*, \Phi] = S_k[\Phi^*, \Phi] + \frac{1}{2} \text{tr} \ln S^{(2)}_k[\Phi^*, \Phi] \] (B14)

for the blocked action. Here the trace in the right-hand side is taken over momenta from the infinitesimal momentum shell \( k - \Delta k \leq |p| \leq k \) as well as over the internal-space matrix. The former is trivial since \( S^{(2)}_k \) is diagonal in the momentum space, so that we only need the matrix elements \( S_{11}, \text{etc.} \) at momentum \( p = k \). In order to take the trace over the internal space, we perform diagonalization. The corresponding eigenvalues \( s_\pm \) of the matrix \( S_{ij} \) \((i, j = 1, 2)\) are given by the roots of the second order algebraic equation

\[ s^2 - s(S_{11} + S_{22}) + S_{11}S_{22} - S_{12}^2 = 0, \]

\[ s_\pm^2 = \frac{1}{2} \left\{ U''_k(r)(\Phi^*^2 + \Phi^2) \pm \left[ U''_k(\Phi^2 + \Phi'^2)^2 \right] \right. \]
\[ + 4\Omega^2(k^2) + 8\Omega(k^2)(U' + ru'' + 4U'' + 8ru'U'' + 8U'^2 + 8ru''U''') \right\}; \] (B15)

Using the above eigenvalues, making the momentum integral over the infinitesimal momentum shell explicit and inserting the ansatz (21) we can rewrite the limit \( \Delta k \to 0 \) of the blocking relation (B14) as

\[ k\partial_k U_k(r) = -\alpha d \ln(s + s -), \] (B16)

where we find with trivial but somewhat lengthy algebraic manipulations that

\[ s + s - = [\Omega(k^2) + U''_k(r)] \times [\Omega(k^2) + U'_k(r) + 2ru''_k(r)] \] (B17)

Then we recover just the same WH RG equation (19) for the local potential which has been obtained for the \( O(2) \) symmetric model.