Bayesian Estimation for the Coefficients of Variation of Birnbaum–Saunders Distributions

Wisunee Puggard, Sa-Aat Niwitpong and Suparat Niwitpong

Department of Applied Statistics, Faculty of Applied Science, King Mongkut’s University of Technology North Bangkok, Bangkok 10800, Thailand; s62040519110056@email.kmutnb.ac.th (W.P.); sa-aat.n@sci.kmutnb.ac.th (S.-A.N.)

Abstract: The Birnbaum–Saunders (BS) distribution, which is asymmetric with non-negative support, can be transformed to a normal distribution, which is symmetric. Therefore, the BS distribution is useful for describing data comprising values greater than zero. The coefficient of variation (CV), which is an important descriptive statistic for explaining variation within a dataset, has not previously been used for statistical inference on a BS distribution. The aim of this study is to present four methods for constructing confidence intervals for the CV, and the difference between the CVs of BS distributions. The proposed methods are based on the generalized confidence interval (GCI), a bootstrapped confidence interval (BCI), a Bayesian credible interval (BayCI), and the highest posterior density (HPD) interval. A Monte Carlo simulation study was conducted to evaluate their performances in terms of coverage probability and average length. The results indicate that the HPD interval was the best-performing method overall. PM 2.5 concentration data for Chiang Mai, Thailand, collected in March and April 2019, were used to illustrate the efficacies of the proposed methods, the results of which were in good agreement with the simulation study findings.

Keywords: confidence interval; Birnbaum–Saunders distribution; coefficient of variation; bootstrap; generalized confidence interval; Bayesian

1. Introduction

When several random variables comprise non-negative values, their dissemination will fit neither a normal nor other symmetrical distributions, and so asymmetrical distributions must be considered instead. Of these, the Birnbaum–Saunders (BS) distribution is receiving considerable attention because of its useful properties and close relationship to the normal distribution. The BS distribution was originally developed to model failure due to the development and growth of cracks in a material subjected to repeated stress cycles [1]. It has two positive parameters: $\alpha$, the shape parameter, and $\beta$, which represents both the scale parameter and the median of the BS distribution. Suppose that random variable $X$ follows a BS distribution with parameters $\alpha$ and $\beta$ (denoted as $X \sim BS(\alpha, \beta)$), then its cumulative distribution function (cdf) is given by

$$F(x) = \Phi\left[\frac{1}{\alpha} \left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}\right)\right], \quad x > 0, \alpha > 0, \beta > 0,$$

where $\Phi(\cdot)$ is the standard normal cdf. Subsequently, the probability density function (pdf) of the BS distribution can be written as

$$f(x, \alpha, \beta) = \frac{1}{2\alpha\beta\sqrt{2\pi}} \left(\frac{\beta}{x}\right)^{\frac{1}{2}} \left(\frac{\beta}{\sqrt{x}}\right)^{\frac{3}{2}} \exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta} + \frac{\beta}{x} - 2\right)\right].$$

The BS distribution has several interesting properties. For example, for $X \sim BS(\alpha, \beta)$, (i) $\alpha^{-1}(\sqrt{x}/\beta - \sqrt{\beta}/x) \sim N(0, 1)$, where $N(0, 1)$ denotes a standard normal distribution with
parameters 0 and 1; (ii) $kX \sim BS(a, k\beta)$; and (iii) $X^{-1} \sim BS(a, \beta^{-1})$. From (i), the expected value and variance of $X$ are given by $E(X) = \beta(1 + \frac{1}{2}a^2)$ and $Var(X) = (a\beta)^2(1 + \frac{5}{4}a^2)$, respectively. The BS distribution has been applied in many areas, such as reliability testing, environmental studies, engineering, finance, and medical sciences. For example, Birnbaum and Saunders [2] fitted it to datasets of the fatigue life of 6061-T6 aluminum coupons. Since then, the BS distribution has been used to model the active repair time of an airborne communication transceiver [3], wind speed [4], water quality [5], and the hardness of a commercially available polymeric bone cement for evaluating the effect of using nanoparticles at different loading levels [6]. Durham and Padgett [7] applied the BS distribution to the carbon fiber fatigue data subjected to increasing stress. Other applications of the BS distribution can be found in [8–12].

The coefficient of variation (CV) is the standard deviation divided by the mean; it is a unit-free measure of the dispersion of data with a high CV value, indicating a high level of dispersion around the mean. Since it is free from units of measurement, it has often been used instead of the standard deviation to compare the variability within and between populations. The CV has been used in many fields (e.g., economics, medicine, biology, engineering, among others). In addition, the difference between the CVs of two independent populations has been extended to compare the dispersion between them. In statistical inference, the confidence interval has been widely used to assess the population CV and the differences between the CVs of two independent populations, since it provides more information about the unknown population parameter of interest than a point estimator. The confidence interval at the nominal level of $100(1 - \gamma)\%$ is defined as the estimated range of values that is likely to include the unknown population parameter of interest $100(1 - \gamma)\%$ of the time. Moreover, it can be used to perform hypothesis testing. Several researchers have focused on constructing confidence intervals for the CV and the difference between the CVs of populations with various distributions. For instance, Tian [13] applied the generalized confidence interval (GCI) approach to construct a confidence interval for the common CV of several normal distributions. Mahmoudvand and Hassani [14] used an approximately unbiased estimator of the population CV and variance to construct confidence intervals for the CV of a normal distribution. Banik and Kibria [15] compared several methods for constructing confidence intervals for the population CV of symmetric and positively skewed distributions. Sangnawakij and Niwitpong [16] constructed confidence intervals for the single CV and the difference between the CVs of two-parameter exponential distributions by using the method of variance estimates recovery (MOVER), GCI, and the asymptotic confidence interval. Thangjai and Niwitpong [17] proposed confidence intervals for the common CV of several normal populations by using the adjusted GCI and computational approaches. Yosboonruang et al. [18] proposed Bayesian credible intervals for the difference between the CVs of delta-lognormal distributions. Recently, La-ongkaew et al. [19] constructed confidence intervals for the difference between the CVs of Weibull distribution by using GCI, Bayesian methods, MOVER based on Hendricks and Robey’s confidence interval, a bootstrap method with standard errors, and a percentile bootstrap method.

Over the years, many other researchers have developed methods to construct confidence intervals for the BS distribution parameters, such as the mean, $\beta$, $a$, quantiles, and reliability. For example, Engelhardt et al. [20] presented hypothesis testing and confidence intervals for the parameters of BS distributions based on the maximum likelihood estimate. Wu and Wong [21] improved interval estimation for a BS distribution by applying a high-order likelihood asymptotic procedure. Ng et al. [22] proposed point and interval estimations for the parameters of BS distribution based on type-II censored samples. Xu and Tang [23] used reference prior functions of the unknown parameters of a BS distribution by using Bayesian inference to obtain Bayesian estimators from the idea of Lindley and Gibbs’ sampling procedure. Subsequently, Wang [24] examined confidence intervals for the $a$, mean, quantiles, and reliability function of BS distributions based on the concept of GCI. Meanwhile, Niu et al. [25] proposed two test statistics, which are the
exact generalized p-value approach and the delta method, to compare the mean, quantile, and reliability functions of several BS distributions. Li and Xu [26] considered two fiducial methods to estimate the parameters of BS distributions, based on the inverse of the structural equation and Hannig’s method. Recently, Guo et al. [27] presented confidence intervals and hypothesis testing for the common mean of several BS distributions. Despite the diverse theoretical and methodological developments for constructing confidence intervals from the functions of parameters of BS distributions, there have not yet been any studies on the single CV and the difference between the CVs of BS distributions. To fill this gap, we propose confidence intervals for these two scenarios by applying the concepts of GCI, the bootstrap confidence interval (BCI), the Bayesian credible interval (BayCI), and the highest posterior density (HPD) interval. Moreover, we applied the proposed methods to datasets of PM2.5 (particulate matter $\leq 2.5 \mu m$) concentration in Chiang Mai, Thailand, collected in March and April 2019, to illustrate their efficacies.

The rest of this paper is organized as follows. The methodology for the construction of confidence intervals for the CV and the difference between the CVs of BS distributions are described in Sections 2 and 3, respectively. A simulation study and results are presented in Section 4. In Section 5, real datasets are used to illustrate the efficacies of the proposed confidence intervals. Finally, conclusions are provided in Section 6.

2. The Confidence Interval for the CV of a BS Distribution

Suppose $X = (X_1, X_2, ..., X_n)$ is a vector of random samples from a BS distribution with parameters $\alpha$ and $\beta$ (denoted as $X \sim BS(\alpha, \beta)$). The cdf of the BS distribution, as given in Equation (1), provides

$$Z_i = \frac{1}{2} \left( \frac{X_i}{\beta} - \sqrt{\frac{\beta}{X_i}} \right) \sim N(0, \frac{\alpha^2}{4}),$$

where $N(\mu, \sigma^2)$ refers to a normal distribution with mean $\mu$ and variance $\sigma^2$. Hence,

$$X_i = \beta \left( 1 + 2Z_i^2 + 2Z_i \sqrt{1 + Z_i^2} \right) \sim BS(\alpha, \beta).$$

Therefore, the BS random variable $X$ can be generated from the random normal variable based on this relationship. The expected value and variance of $X$ are given by $E(X) = \beta(1 + \frac{1}{2}\alpha^2)$ and $Var(X) = (\alpha\beta)^2(1 + \frac{1}{4}\alpha^2)$, respectively. Hence, the CV denoted by $\tau$ can be defined as

$$\tau = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\alpha \sqrt{1 + \frac{3}{4}\alpha^2}}{1 + \frac{1}{2}\alpha^2}. \tag{5}$$

2.1. Generalized Confidence Interval

Weerahandi [28] proposed GCI based on the concept of the generalized pivotal quantity (GPQ). This is a generalization of the usual pivot quantity but different in that GPQ is a function of the nuisance parameter, whereas the usual pivotal quantity can only be a function of the sample and the parameter of interest. GPQ has an unknown parameter that is distribution-free and an observed value that does not depend on the nuisance parameter.

Suppose $X = (X_1, X_2, ..., X_n)$ is a random sample from the BS distribution in Equation (1) with sample size $n$, then

$$Z_i = \left( \frac{X_i}{\beta} - \sqrt{\frac{\beta}{X_i}} \right) \sim N(0, \alpha^2),$$

After denoting $Z = n^{-1} \sum_{i=1}^n Z_i$ and $S^2 = (n - 1)^{-1} \sum_{i=1}^n (Z_i - Z)^2$, it follows that $Z$ and $S^2$ are respectively independently distributed as $Z \sim N(0, \alpha^2/n)$ and $(n - 1)S^2/\alpha^2 \sim \chi^2(n - 1)$, where $\chi^2(n - 1)$ denotes a Chi-squared distribution with $n - 1$ degrees of freedom. Hence,
\[ R(X; \beta) = \sqrt{n} \times \frac{\bar{X}}{\sigma} \] follows a t-distribution with \( n - 1 \) degrees of freedom that is free from unknown parameter \( \beta \). Moreover, \( R(X; \beta) \) is only a function of \( \hat{\beta} \) and is not related to \( \alpha \). Sun [29] proved that \( R(X; \beta) \) is a strictly decreasing function of \( \beta \), thereby leading to a unique solution of the equation \( R(X; \beta) = T \) as follows:

\[
R_\beta := R_\beta(x; T) = \begin{cases} 
\max(\beta_1, \beta_2), & \text{if } T \leq 0; \\
\min(\beta_1, \beta_2), & \text{if } T > 0,
\end{cases}
\tag{7}
\]

where \( T \sim t(n - 1) \) refers to a t-distribution with \( n - 1 \) degrees of freedom and \( \beta_1 \) and \( \beta_2 \) are the two solutions of quadratic equation

\[
\left[ (n - 1)I^2 - \frac{1}{\nu} LT^2 \right] \beta^2 - 2 \left[ (n - 1)IJ - (1 - I)J^2 \right] \beta + (n - 1)I^2 - \frac{1}{\nu} KT^2 = 0,
\tag{8}
\]

where \( I = \frac{1}{n - 1} \sum_{i=1}^{n} X_i, J = \frac{1}{n - 1} \sum_{i=1}^{n} \frac{1}{\sqrt{X_i}}, K = \sum_{i=1}^{n} (\sqrt{X_i} - 1)^2, \) and \( L = \sum_{i=1}^{n} (1/\sqrt{X_i} - J)^2 \). Therefore, \( R_\beta \) is a GPQ for \( \beta \). Subsequently, the GPQ of \( \alpha \) [24] is obtained as

\[
R_\alpha := R_\alpha(x; v, T) = \left[ \frac{S_2 R_\beta(x; T)^2 - 2nR_\beta(x; T) + S_1}{R_\beta(x; T)v} \right],
\tag{9}
\]

where \( S_1 = \sum_{i=1}^{n} X_i, S_2 = \sum_{i=1}^{n} 1/X_i, v \sim \chi^2(n) \) and \( R_\beta(x; T) \) is defined in Equation (7). By applying Equation (5), the pivotal quantity for \( \tau \) can be expressed by

\[
R_\tau = \frac{R_\alpha(x; v, T)}{1 + \frac{1}{2} R_\alpha^2(x; v, T)}.
\tag{10}
\]

Therefore, the \( 100(1 - \gamma)\% \) confidence interval for \( \tau \) based on GCI is

\[
CI_{GCI} = [L_\tau, U_\tau] = [R_\tau(\gamma/2), R_\tau(1 - \gamma/2)],
\tag{11}
\]

where \( R_\tau(v) \) denotes the \( 100v\% \) percentile of \( R_\tau \).

2.2. Bootstrap Confidence Interval

The bootstrap method was originally introduced by Efron [30] as a re-sampling technique based on the random selection of new samples from the original sample. It uses information from the sample to infer certain characteristics from the distribution of a particular statistical estimator. Since \( \tau \) is a function of \( \alpha \), an estimator of \( \alpha \) can be considered. Ng et al. [31] showed that the maximum likelihood estimator (MLE) of \( \alpha \) is biased. The constant-bias-correcting (CBC) parametric bootstrap recommended by Lemonte et al. [32] can be applied to reduce the bias of \( \alpha \), and so we used it to construct the BCI for \( \tau \). Let \( x = (x_1, x_2, ..., x_n) \) be an original random sample of size \( n \) drawn from \( BS(\alpha, \beta) \), which has distribution function \( F = F_{\alpha,\beta}(x) \). \( \hat{\alpha} \) and \( \hat{\beta} \), the MLEs of \( \alpha \) and \( \beta \), respectively, can be obtained by maximizing the log-likelihood function by applying the Broyden–Fletcher–Goldfarb–Shanno (BFGS) quasi-Newton nonlinear optimization algorithm. Thus, the log-likelihood function of a BS distribution without the additive constant becomes

\[
l(\alpha, \beta) = -n \log(\alpha \beta) + \sum_{i=1}^{n} \ln \left[ \left( \frac{\beta}{x} \right)^{\frac{1}{\beta}} + \left( \frac{\beta}{x} \right)^{\frac{1}{\beta}} \right] - \frac{1}{2\alpha^2} \sum_{i=1}^{n} \left( \frac{x_i - \beta}{x} - 2 \right).
\tag{12}
\]

Assume that \( x^* = (x_1^*, x_2^*, ..., x_n^*) \) is a sequence of bootstrap samples of size \( n \) obtained from \( BS(\hat{\alpha}, \hat{\beta}) \). Thus, the bias of estimator \( \hat{\alpha} \) is given by

\[
B(\hat{\alpha}, \alpha) = E(F[\hat{\alpha} - \alpha]) = E(\hat{\alpha}) - \alpha.
\tag{13}
\]
Suppose that $B$ bootstrap samples are available, then their $\hat{\alpha}$ series (denoted by $\hat{\alpha}_1^*, \hat{\alpha}_2^*, ..., \hat{\alpha}_B^*$) can be calculated, where $\hat{\alpha}_j^*$ is a sequence of the bootstrap MLEs of $\alpha$, for $j = 1, 2, ..., B$. The BFGS quasi-Newton nonlinear optimization algorithm is also used to compute the bootstrap MLEs of $\alpha$. Following this, the bootstrap expectation $E(\hat{\alpha})$ can be approximated by using the mean $\bar{\alpha}^*(\cdot) = 1/B \sum_{j=1}^B \hat{\alpha}_j^*$. Therefore, the bootstrap bias estimate based on $B$ replications of $\hat{\alpha}$ is calculated as

$$B(\hat{\alpha}, \alpha) = \bar{\alpha}^*(\cdot) - \hat{\alpha}. \quad (14)$$

By using the bootstrap bias estimate, the correct estimate for $\alpha^*$ (denoted as $\tilde{\alpha}$) is given by

$$\tilde{\alpha} = \hat{\alpha}^* - 2B(\hat{\alpha}, \alpha). \quad (15)$$

Note that we subtract $B(\hat{\alpha}, \alpha)$ twice because the bootstrap samples are generated by using the bias estimate $(\hat{\alpha}, \hat{\beta})$ and the $\alpha^*$’s are biased estimates of $\alpha$ [33]. By applying Equation (5), the bootstrap estimator of $\tau$ can be expressed as

$$\hat{\tau} = \hat{\alpha} \sqrt{\frac{1 + \frac{5}{4} \hat{\alpha}^2}{1 + \frac{1}{2} \hat{\alpha}^2}}. \quad (16)$$

Therefore, the 100$(1 - \gamma)\%$ confidence interval for $\tau$ based on BCI becomes

$$CI_{BCI} = [L_n, U_n] = [\hat{\tau}(\gamma/2), \hat{\tau}(1 - \gamma/2)], \quad (17)$$

where $\hat{\tau}(\nu)$ denotes the 100$\nu\%$ percentile of $\hat{\tau}$.

### 2.3. Bayesian Credible Interval

To guarantee proper posterior distributions, inverse gamma (IG) priors with known hyperparameters denoted as $IG(\beta|a_1, b_1)$ and $IG(\alpha^2|a_2, b_2)$ are utilized as priors for $\beta$ and $\alpha^2$, respectively [34]. Since showing that $\beta$ and $\alpha^2$ are independent of each other is straightforward, then their respective priors can be written as

$$\pi(\beta|a_1, b_1) \propto \frac{b_1^{a_1}}{\Gamma(a_1)} \beta^{-a_1 - 1} \exp\left(-\frac{b_1}{\beta}\right), \quad a_1, b_1 > 0 \quad (18)$$

and

$$\pi(\alpha^2|a_2, b_2) \propto \frac{b_2^{a_2}}{\Gamma(a_2)} (\alpha^2)^{-a_2 - 1} \exp\left(-\frac{b_2}{\alpha^2}\right), \quad a_2, b_2 > 0. \quad (19)$$

Let $x = (x_1, x_2, ..., x_n)$ be a sample from $BS(\alpha, \beta)$, then the likelihood function is given by

$$L(x|\alpha, \beta) \propto \frac{1}{\alpha^{2n} \beta^n} \prod_{i=1}^n \left[ \left( \frac{\beta}{x_i} \right)^2 + \left( \frac{\beta}{x_i} \right)^2 \exp\left[-\sum_{i=1}^n \frac{1}{2\alpha^2} \left( \frac{x_i}{\beta} + \frac{\beta}{x_i} - 2 \right) \right] \right]. \quad (20)$$

From Equations (18) to (20), the joint posterior density of $\beta$ and $\alpha^2$ can be obtained as

$$f(\alpha^2, \beta|x) \propto L(x|\alpha, \beta) \pi(\beta|a_1, b_1) \pi(\alpha^2|a_2, b_2)$$

$$\propto \frac{1}{\alpha^{2n} \beta^n} \prod_{i=1}^n \left[ \left( \frac{\beta}{x_i} \right)^2 + \left( \frac{\beta}{x_i} \right)^2 \exp\left[-\sum_{i=1}^n \frac{1}{2\alpha^2} \left( \frac{x_i}{\beta} + \frac{\beta}{x_i} - 2 \right) \right] \right]$$

$$\times \beta^{-a_1 - 1} \exp\left(-\frac{b_1}{\beta}\right) (\alpha^2)^{-a_2 - 1} \exp\left(-\frac{b_2}{\alpha^2}\right). \quad (21)$$
Hence, it is clear that
\[
\pi(\beta|x) \propto \beta^{-(n+a_1+1)} \exp\left(-\frac{b_1}{\beta}\right) \prod_{i=1}^{n} \left[\left(\frac{\beta}{x_i}\right)^{\frac{n}{2}} + \left(\frac{\beta}{x_i}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \\
\times \left[\sum_{i=1}^{n} \frac{1}{2} \left(\frac{x_i}{\beta} + \frac{\beta}{x_i} - 2\right) + b_2\right]^{-(n+1)/2-a_2}
\] (22)
and
\[
\pi(a^2|x, \beta) \propto IG\left(\frac{n}{2} + a_2, \frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i}{\beta} + \frac{\beta}{x_i} - 2\right) + b_2\right).
\] (23)

The availability of the marginal posterior distribution of \( \beta \) in Equation (22) and the conditional posterior distribution of \( a^2 \) given \( \beta \) in Equation (23) allow the application of a Markov chain-Monte Carlo method to generate the posterior samples. Therefore, the R-package of LearnBayes was applied to obtain the samples by applying Equation (23) while the generalized ratio-of-uniforms method of Wakefield et al. [35] was used to generate the samples by applying Equation (22) [34]. The concept of the generalized ratio-of-uniforms method is covered here.

Let \( C(r) = \left\{ (u, v) : 0 < u \leq \left[\frac{\pi(\beta|x)}{\pi(x)}\right]^{1/(r+1)} \right\} \), where \( r \geq 0 \) is a constant and \( \pi(\cdot|x) \) is given in Equation (22). If \((u, v)\) are random variables uniformly distributed in \( C(r) \), then the ratio \( \tilde{\beta} = v/u^r \) has density \( \tau(\beta|x) / \int \pi(\beta|x) d\beta \). The accept-reject method from the minimal bounding rectangle \([0, a(r)] \times [b^-(r), b^+(r)]\) is used to generate uniform samples inside region \( C(r) \) where
\[
a(r) = \sup_{\beta > 0} \{ \pi(\beta|x) \}^{1/(r+1)}, \quad (24)
\]
\[
b^-(r) = \inf_{\beta > 0} \{ \beta \pi(\beta|x) \}^{r/(r+1)},\quad (25)
\]
and
\[
b^+(r) = \sup_{\beta > 0} \{ \beta \pi(\beta|x) \}^{r/(r+1)}. \quad (26)
\]

Note that \( a(r) \) and \( b^+(r) \) are finite, while \( b^-(r) = 0 \) [34]. Hence, the procedure for using the confidence intervals constructed with BayCI for the CV is as follows. First, set the values for \( a_1, b_1, a_2, b_2 \) and \( r \), then compute \( a(r) \) and \( b^+(r) \) by using Equations (24) and (26), respectively. At the \( i \)th step, generate the values for \( u \) and \( v \) from \( U(0, a(r)) \) and \( U(0, b^+(r)) \), respectively, where \( U(g, h) \) is a uniform distribution with parameters \( g \) and \( h \). Subsequently, compute \( \rho = v/u^r \). If \( u \leq \left[\pi(\rho|x)\right]^{1/(r+1)} \), then set \( \hat{\beta}_{(i)} = \rho \); otherwise, regenerate \( u \sim U(0, a(r)) \) and \( v \sim U(0, b^+(r)) \). Next, generate a new value for \( a^2_{(i)} \) from the IG distribution given in Equation (23) depending on \( \beta_{(i)} \). Hence, the value of \( a_{(i)} \) can be obtained after a simple algebraic transformation. Following this, the Bayesian estimator of \( \tau \) becomes
\[
\hat{\tau}^*_i = \frac{a_{(i)} \sqrt{1 + \frac{2}{3} a^2_{(i)}}}{1 + \frac{2}{3} a^2_{(i)}}, \quad (27)
\]
The preceding process is repeated \( N \) times. Therefore, the 100(1 - \( \gamma \))% confidence interval for \( \tau \) based on BayCI becomes
\[
CI_{BayCI} = [L_\tau, U_\tau] = [\hat{\tau}^*(\gamma/2), \hat{\tau}^*(1 - \gamma/2)], \quad (28)
\]
where \( \hat{\tau}^*(\nu) \) denotes the 100\( \nu \)% percentile of \( \hat{\tau}^* \).

The HPD interval was also considered for constructing the confidence interval for \( \tau \). The HPD interval, explained by Box and Tiao [36], is the shortest interval containing
(1 − γ)100% of the posterior probability. Each point inside the interval has a higher probability density than any point outside of it. Therefore, the package `HDInterval` version 0.2.2 from R version 3.5.1 was utilized to compute the HPD interval for τ in the simulations and computations.

3. Confidence Intervals for the Difference between the CVs of BS Distributions

In this section, we present the confidence interval for the difference between the CVs of BS distributions based on the concepts of GCI, BCI, BayCI, and the HPD interval.

Suppose that \( X = (X_1, X_2, ..., X_n) \) and \( Y = (Y_1, Y_2, ..., Y_m) \) are two random samples of size \( n \) and \( m \) from a BS distribution (denoted as \( BS(\alpha, \beta) \) and \( BS(\alpha_y, \beta_y) \), respectively). The CV of \( X \) can be calculated by applying Equation (3). By using random variable \( Y \), the expected value and variance are given by \( E(Y) = \beta_y(1 + \frac{1}{2}a_y^2) \) and \( Var(Y) = (\alpha_y \beta_y)^2(1 + \frac{5}{2}a_y^2) \), respectively. Thus, the CV of \( Y \) (denoted as \( \tau_y \)) becomes

\[
\tau_y = \frac{\sqrt{Var(Y)}}{E(Y)} = \frac{\alpha_y \sqrt{1 + \frac{5}{2}a_y^2}}{1 + \frac{1}{2}a_y^2},
\]

Since \( X \) and \( Y \) are independent, the difference between the CVs (denoted as \( \eta \)) can be written as

\[
\eta = \tau - \tau_y = \frac{\alpha \sqrt{1 + \frac{5}{2}a^2}}{1 + \frac{1}{2}a^2} - \frac{\alpha_y \sqrt{1 + \frac{5}{2}a_y^2}}{1 + \frac{1}{2}a_y^2}.
\]

3.1. Generalized Confidence Interval

The GPQ of \( \eta \) is considered for constructing the confidence interval based on GCI. Let \( R_\alpha \) and \( R_{\alpha_y} \) be the GPQs of \( \alpha \) and \( \alpha_y \), respectively. Following this, \( R_\alpha \) is calculated by applying Equation (9) and \( R_{\alpha_y} \), can be obtained from

\[
R_{\alpha_y}(y; v, T_y) = \frac{S_{y,2} R_{\beta_y}(y; T_y)^2 - 2m R_{\beta_y}(y; T_y) + S_{y,1}}{R_{\beta_y}(y; T_y)v_y},
\]

where \( S_{y,1} = \sum_{i=1}^{m} Y_i, S_{y,2} = \sum_{i=1}^{m} 1/Y_i, v_y \sim \chi^2(m), T_y \sim t(m - 1) \) and \( R_{\beta_y}(y; T_y) \) is the GPQ of \( \beta_y \), which can subsequently be defined as

\[
R_{\beta_y}(y; T_y) = \left\{ \begin{array}{ll}
\max(\beta_{y,1}, \beta_{y,2}), & \text{if} \quad T_y \leq 0; \\
\min(\beta_{y,1}, \beta_{y,2}), & \text{if} \quad T_y > 0.
\end{array} \right.
\]

Meanwhile, \( \beta_{y,1} \) and \( \beta_{y,2} \) are the two solutions for

\[
[(m - 1)I_y^2 - \frac{1}{m} L_y T_y^2] \beta_y^2 - 2[(m - 1)I_y J_y - (1 - I_y J_y)T_y^2] \beta_y + (m - 1)I_y^2 - \frac{1}{m} K_y T_y^2 = 0,
\]

where \( I_y = m^{-1} \sum_{i=1}^{m} \sqrt{Y_i}, J_y = m^{-1} \sum_{i=1}^{m} 1/\sqrt{Y_i}, K_y = \sum_{i=1}^{m} (\sqrt{Y_i} - I_y)^2, \) and \( L_y = \sum_{i=1}^{m} (1/\sqrt{Y_i} - J_y)^2 \). By applying Equation (30), the GPQ for \( \eta \) can be written as

\[
R_\eta = \frac{R_\alpha(x; v, T) \sqrt{1 + \frac{5}{2}R_{\alpha_y}^2(x; v, T)}}{1 + \frac{1}{2}R_{\alpha_y}^2(x; v, T)} - \frac{R_{\alpha_y}(y; v, T) \sqrt{1 + \frac{5}{2}R_{\alpha_y}^2(y; v, T)}}{1 + \frac{1}{2}R_{\alpha_y}^2(y; v, T)}.
\]

Therefore, the 100(1 − γ)% confidence interval for \( \eta \) based on GCI is

\[
CI_{GCI}^\eta = [L_\eta, U_\eta] = [R_\eta(\gamma/2), R_\eta(1 - \gamma/2)],
\]

where \( R_\eta(v) \) denotes the 100v% percentile of \( R_\eta \).
3.2. Bootstrap Confidence Interval

Suppose that $B$ bootstrap samples are available; then, the bootstrap estimator of $\eta$ (denoted as $\hat{\eta}$) can be defined as

$$\hat{\eta} = \frac{\tilde{\alpha}}{1 + \frac{1}{2} \tilde{\alpha}^2} - \frac{\tilde{\beta}}{1 + \frac{1}{2} \tilde{\beta}^2},$$  
(36)

where $\tilde{\alpha}$ is calculated by applying Equation (15) and $\tilde{\beta}$ can be obtained by executing the following steps. Let $y = (y_1, y_2, ..., y_m)$ be an original random sample of size $m$ drawn from $BS(\alpha_y, \beta_y)$. The MLEs of $\alpha_y$ and $\beta_y$ (denoted as $\hat{\alpha}_y$ and $\hat{\beta}_y$, respectively) are also obtained by maximizing the log-likelihood function by using the BFGS quasi-Newton nonlinear optimization algorithm. Suppose that $\tilde{y}^* = (\tilde{y}_1^*, \tilde{y}_2^*, ..., \tilde{y}_m^*)$ are bootstrap samples of size $m$ drawn from $BS(\hat{\alpha}_y, \hat{\beta}_y)$, then the bootstrap MLEs of $\alpha_y$ given by $\hat{\alpha}_y, 1, \hat{\alpha}_y, 2, ..., \hat{\alpha}_y, B$ can be computed. Next, the bootstrap bias estimate based on $B$ replications of $\hat{\alpha}_y$ is obtained as

$$\hat{B}(\hat{\alpha}_y, \alpha_y) = \hat{\alpha}_y^{(1)} - \hat{\alpha}_y,$$
(37)

where $\hat{\alpha}_y^{(1)} = 1/B \sum_{j=1}^{B} \hat{\alpha}_y^{*}$. Therefore, the corrected estimate for $\hat{\alpha}_y$ can be written as

$$\tilde{\alpha}_y = \hat{\alpha}_y - 2\hat{B}(\hat{\alpha}_y, \alpha_y).$$
(38)

By substituting $\tilde{\alpha}_y$ in Equation (36), the 100$(1 - \gamma)$% confidence interval for $\eta$ based on BCI becomes

$$CI_{BCI}^\eta = [L_\eta, U_\eta] = [\hat{\eta}(\gamma / 2), \hat{\eta}(1 - \gamma / 2)],$$
(39)

where $\hat{\eta}(v)$ denotes the 100$v$% percentile of $\hat{\eta}$.

3.3. Bayesian Credible Interval

Based on the Bayesian method, the IG priors with known hyperparameters denoted as $IG(\beta_y | c_1, d_1)$ and $IG(\alpha^2_y | c_2, d_2)$ are considered as the priors for $\beta_y$ and $\alpha^2_y$, respectively. Consequently, the marginal posterior distribution of $\beta_y$ becomes

$$\pi(\beta_y | y) \propto \beta_y^{-(m+c_1+1)} exp \left( - \frac{d_1}{\beta_y} \right) \prod_{i=1}^{m} \left( \frac{\beta_y}{y_i} \right)^{\frac{1}{2}} + \left( \frac{\beta_y}{y_i} \right)^{\frac{1}{3}}$$
$$\times \left[ \sum_{i=1}^{m} \left( \frac{y_i}{\beta_y} + \frac{\beta_y}{y_i} - 2 \right) - d_2 \right]^{-(m+1/2-c_2)}$$  
(40)

and the posterior conditional distribution of $\alpha^2_y$ given $\beta_y$ becomes

$$\pi(\alpha^2_y | y, \beta_y) \propto IG \left( \frac{n}{2} + c_2, \frac{1}{2} \sum_{i=1}^{m} \left( \frac{y_i}{\beta_y} + \frac{\beta_y}{y_i} - 2 \right) + d_2 \right).$$
(41)

The process for applying BayCI for the difference between the CVs can be summarized in the following steps.

1. Set the values for $a_1, b_1, a_2, b_2, c_1, d_1, c_2, d_2, r_y$, and $r_y$, where $r_y > 0$ is a constant.
2. Compute $a(r_y), b^+(r_y), a(r_y)$ and $b^+(r_y)$, where $a(r_y)$ and $b^+(r_y)$ are given by Equation (24) and (26), respectively, while $a(r_y)$ and $b^+(r_y)$ are respectively given by

$$a(r_y) = \sup_{\beta_y > 0} \{ |\pi(\beta_y | y)|^{1/(r_y + 1)} \}$$
(42)

and

$$b^+(r_y) = \sup_{\beta_y > 0} \{ \beta_y |\pi(\beta_y | y)|^{r_y/(r_y + 1)} \}.$$  
(43)
3. At the $i$th step:
   (a) i. Generate $u \sim U(0, a(r))$ and $v \sim U(0, b^+(r))$, and then compute 
   $\rho = v/u'$. If $u \leq [\pi(\rho|x)]^{1/(r+1)}$, set $\beta_{(i)} = \rho$; otherwise, repeat 
   the process.
   ii. Generate $a^2_{I(i)} \sim IG \left( \frac{a}{2} + a_2, \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\alpha_j}{\beta_{(j)}} + \frac{\beta_{(j)}}{\gamma_j} - 2 \right) + b_2 \right)$ and then 
   $a_{(i)} = \sqrt{a^2_{I(i)}}$.
   (b) i. Generate $u_y \sim U(0, a(r_y))$ and $v_y \sim U(0, b^+(r_y))$, and then compute 
   $\rho_y = v_y/u_y'$. If $u_y \leq [\pi(\rho_y|y)]^{1/(r_y+1)}$, set $\beta_{y,(i)} = \rho_y$; otherwise, repeat 
   the process.
   ii. Generate $a^2_{y,(i)} \sim IG \left( \frac{a}{2} + c_2, \frac{1}{2} \sum_{j=1}^{m} \left( \frac{y_j}{\beta_{y}} + \frac{\beta_{y}}{\gamma_j} - 2 \right) + d_2 \right)$ and then 
   $a_{y,(i)} = \sqrt{a^2_{y,(i)}}$.
   (c) Calculate the Bayesian estimator of $\eta$ by using 
   $$\hat{\eta}_{(i)}^* = \frac{a_{y,(i)} \sqrt{1 + \frac{3}{4}a^2_{y,i}} - a_{(i)} \sqrt{1 + \frac{3}{4}a^2_{I,i}}}{1 + \frac{3}{4}a^2_{y,i}}.$$  (44)

4. Repeat Step 3 $N$ times.
5. Calculate the 100$(1 - \gamma)$% confidence interval for $\eta$ by applying 
   $$CI_{BayCI}^d = [L_{\eta}, U_{\eta}] = [\hat{\eta}^*(\gamma/2), \hat{\eta}^*(1 - \gamma/2)],$$  (45)
   where $\hat{\eta}^*(v)$ denotes the 100v% percentile of $\hat{\eta}^*$. For the HPD interval, the package 
   HDInterval from R was applied to compute the confidence interval for $\eta$.

### 4. Simulation Studies

A Monte Carlo simulation study was conducted by using R statistical software to 
evaluate the performances of the four methods used for constructing confidence intervals 
for the CV and the difference between the CVs of BS distributions under various 
combinations of parameters. We evaluated the performances of GCI, BCI, BayCI, and 
the HPD interval by measuring their coverage probabilities and average lengths based on 
5000 independently generated replications, with 5000 pivotal quantities for GCI, $B = 500$ 
for BCI, and $N = 5000$ for BayCI and the HPD interval. Note that we set hyperparameters 
$a_1 = b_1 = a_2 = b_2 = c_1 = d_1 = c_2 = d_2 = 10^{-4}$ and $r = r_y = 2$ for BayCI and the HPD 
interval [34]. For the nominal coverage level of 0.95, the best-performing method has a 
coverage probability close to or greater than 0.95, and the shortest average length. Since $\beta$ 
is the scale parameter and the median of the BS distribution, $\beta = \beta_y = 1$ without loss of 
generality was applied in this simulation study.

For the CV of a BS distribution, we used sample sizes $n = 10, 20, 30, 35, 50, or 100$ and 
$a = 0.1, 0.25, 0.5, 0.75, 1, or 2$. The simulation results reported in Table 1 show that the 
coverage probabilities of GCI, BayCI, and the HPD interval were greater than or close to 
0.95, even for a small sample size and/or a high value of $a$. Conversely, although BCI had 
the shortest average lengths, its coverage probabilities were the lowest and under 0.95 but 
improved when $n$ was increased. When considering the average lengths of the other three 
methods, the HPD interval outperformed the others in all cases. In addition, the average 
lengths of the four methods tended to decrease and were similar when the sample size 
was increased.
Table 1. The coverage probabilities and average lengths of the 95% confidence intervals for the CV of a BS distribution.

| $\alpha$ | $n$ | Coverage Probability | Average Length | | | | |
|---------|-----|---------------------|----------------|---------|--------|--------|
|         |     | GCI                | BCI            | BayCI   | HPD    | GCI    | BCI    | BayCI  | HPD     |
| 10      | 0.10| 0.952              | 0.881          | 0.950   | 0.945  | 0.1112 | 0.0787 | 0.1089 | 0.1002  |
|         | 0.25| 0.951              | 0.886          | 0.949   | 0.948  | 0.2855 | 0.2010 | 0.2738 | 0.2563  |
|         | 0.50| 0.949              | 0.883          | 0.948   | 0.940  | 0.5574 | 0.3983 | 0.5433 | 0.5060  |
|         | 0.75| 0.950              | 0.878          | 0.948   | 0.941  | 0.7480 | 0.5540 | 0.7278 | 0.6910  |
|         | 1.00| 0.950              | 0.876          | 0.947   | 0.933  | 0.8461 | 0.6497 | 0.8206 | 0.7929  |
|         | 2.00| 0.951              | 0.875          | 0.948   | 0.925  | 0.7678 | 0.6507 | 0.7341 | 0.7258  |
| 20      | 0.10| 0.949              | 0.917          | 0.947   | 0.946  | 0.0693 | 0.0587 | 0.0688 | 0.0658  |
|         | 0.25| 0.948              | 0.912          | 0.945   | 0.942  | 0.1766 | 0.1491 | 0.1752 | 0.1676  |
|         | 0.50| 0.953              | 0.911          | 0.950   | 0.946  | 0.3533 | 0.2994 | 0.3499 | 0.3360  |
|         | 0.75| 0.951              | 0.913          | 0.950   | 0.943  | 0.4939 | 0.4240 | 0.4883 | 0.4731  |
|         | 1.00| 0.953              | 0.915          | 0.953   | 0.942  | 0.5734 | 0.5004 | 0.5654 | 0.5529  |
|         | 2.00| 0.948              | 0.910          | 0.947   | 0.931  | 0.5284 | 0.4858 | 0.5121 | 0.5076  |
| 30      | 0.10| 0.946              | 0.917          | 0.944   | 0.943  | 0.0544 | 0.0487 | 0.0541 | 0.0525  |
|         | 0.25| 0.948              | 0.927          | 0.946   | 0.944  | 0.1386 | 0.1239 | 0.1378 | 0.1334  |
|         | 0.50| 0.951              | 0.923          | 0.951   | 0.945  | 0.2792 | 0.2496 | 0.2775 | 0.2693  |
|         | 0.75| 0.952              | 0.922          | 0.950   | 0.942  | 0.3937 | 0.3546 | 0.3909 | 0.3816  |
|         | 1.00| 0.947              | 0.920          | 0.944   | 0.939  | 0.4621 | 0.4213 | 0.4576 | 0.4495  |
|         | 2.00| 0.946              | 0.919          | 0.942   | 0.933  | 0.4251 | 0.4023 | 0.4160 | 0.4124  |
| 50      | 0.10| 0.954              | 0.940          | 0.952   | 0.951  | 0.0410 | 0.0383 | 0.0408 | 0.0399  |
|         | 0.25| 0.945              | 0.939          | 0.945   | 0.946  | 0.1045 | 0.0974 | 0.1040 | 0.1017  |
|         | 0.50| 0.945              | 0.931          | 0.944   | 0.941  | 0.2104 | 0.1960 | 0.2093 | 0.2048  |
|         | 0.75| 0.954              | 0.935          | 0.950   | 0.947  | 0.2992 | 0.2798 | 0.2974 | 0.2921  |
|         | 1.00| 0.950              | 0.932          | 0.948   | 0.940  | 0.3540 | 0.3339 | 0.3517 | 0.3468  |
|         | 2.00| 0.946              | 0.927          | 0.943   | 0.935  | 0.3256 | 0.3140 | 0.3210 | 0.3184  |
| 100     | 0.10| 0.948              | 0.941          | 0.948   | 0.945  | 0.0284 | 0.0274 | 0.0283 | 0.0279  |
|         | 0.25| 0.950              | 0.942          | 0.949   | 0.948  | 0.0722 | 0.0693 | 0.0718 | 0.0707  |
|         | 0.50| 0.944              | 0.935          | 0.941   | 0.939  | 0.1460 | 0.1404 | 0.1454 | 0.1432  |
|         | 0.75| 0.951              | 0.938          | 0.950   | 0.945  | 0.2088 | 0.2013 | 0.2079 | 0.2052  |
|         | 1.00| 0.951              | 0.938          | 0.947   | 0.944  | 0.2481 | 0.2398 | 0.2468 | 0.2441  |
|         | 2.00| 0.951              | 0.940          | 0.951   | 0.945  | 0.2283 | 0.2233 | 0.2262 | 0.2244  |

For the difference between the CVs of BS distributions, we used sample sizes $(n, m) = (10, 10), (20, 20), (30, 30), (50, 50), (100, 100), (10, 20), (30, 20), (30, 50),$ or $(100, 50)$ and $(\alpha_x, \alpha_y) = (0.25, 0.25), (0.25, 0.50), (0.25, 1.00), (0.25, 2.00), (0.50, 0.50), (0.50, 1.00), (0.50, 2.00), (1.00, 1.00),$ or $(2.00, 2.00)$. The simulation results for equal and unequal sample sizes are summarized in Tables 2 and 3, respectively. The coverage probabilities of GCI, BayCI, and the HPD interval were greater than or close to 0.95 for all cases irrespective of whether $n = m$ (Table 2) or $n \neq m$ (Table 3). For both equal and unequal sample sizes, the coverage probabilities for BCI were the lowest, whereas its average lengths were similar to the others or the shortest in all cases. Meanwhile, the average lengths of the HPD interval were shorter than GCI and BayCI under all circumstances. Moreover, the performances of the four methods in terms of the average length improved and were close to each other when sample sizes $(n, m)$ were increased.
Table 2. The coverage probabilities and average lengths of the 95% confidence intervals for the difference between the CVs of BS distributions with equal sample sizes. ($n = m$).

| (n,m)       | (α, α_y) | Coverage Probability | Average Length |
|-------------|----------|----------------------|----------------|
|             |          | GCI      | BCI     | BayCI  | HPD    | GCI      | BCI     | BayCI  | HPD    |
| (10,10)     | (0.25,0.25) | 0.952  | 0.901  | 0.945  | 0.964  | 0.4265  | 0.2873  | 0.4156  | 0.4094  |
| (20,20)     | (0.25,0.25) | 0.948  | 0.924  | 0.944  | 0.954  | 0.2576  | 0.2126  | 0.2533  | 0.2525  |
| (30,30)     | (0.25,0.25) | 0.950  | 0.933  | 0.947  | 0.952  | 0.2003  | 0.1760  | 0.1989  | 0.1970  |
| (50,50)     | (0.25,0.25) | 0.949  | 0.936  | 0.948  | 0.951  | 0.1494  | 0.1378  | 0.1486  | 0.1473  |
| (100,100)   | (0.25,0.25) | 0.953  | 0.944  | 0.951  | 0.950  | 0.1025  | 0.0981  | 0.1020  | 0.1012  |
Table 3. The coverage probabilities and average lengths of the 95% confidence intervals for the difference between the CVs of BS distributions with unequal sample sizes \((n \neq m)\).

| \((n,m)\) | \((\alpha, \alpha_y)\) | Coverage Probability | Average Length |
| --- | --- | --- | --- |
| | GCI | BCI | BayCI | HPD | GCI | BCI | BayCI | HPD |
| (10,20) | (0.25,0.25) | 0.953 | 0.906 | 0.951 | 0.961 | 0.3479 | 0.2522 | 0.3409 | 0.3323 |
| (25,10) | 0.954 | 0.917 | 0.951 | 0.952 | 0.6537 | 0.5435 | 0.6432 | 0.6361 |
| (25,20) | 0.952 | 0.908 | 0.944 | 0.940 | 0.6075 | 0.5289 | 0.5895 | 0.5830 |
| (50,50) | 0.952 | 0.935 | 0.950 | 0.953 | 0.4027 | 0.3546 | 0.4001 | 0.3964 |
| (100,100) | 0.951 | 0.904 | 0.947 | 0.950 | 1.0429 | 0.8264 | 1.0173 | 1.0049 |
| (200,200) | 0.948 | 0.901 | 0.944 | 0.938 | 1.0105 | 0.8192 | 0.9797 | 0.9633 |

| (30,20) | (0.25,0.25) | 0.948 | 0.925 | 0.944 | 0.950 | 0.2296 | 0.1944 | 0.2278 | 0.2248 |
| (25,10) | 0.956 | 0.925 | 0.953 | 0.955 | 0.3854 | 0.3261 | 0.3820 | 0.3725 |
| (25,20) | 0.954 | 0.921 | 0.952 | 0.945 | 0.5910 | 0.5157 | 0.5831 | 0.5721 |
| (50,50) | 0.953 | 0.930 | 0.951 | 0.953 | 0.6441 | 0.5614 | 0.6362 | 0.6282 |
| (100,100) | 0.951 | 0.917 | 0.949 | 0.942 | 0.8265 | 0.6459 | 0.8108 | 0.8008 |
| (200,200) | 0.956 | 0.928 | 0.952 | 0.945 | 1.0429 | 0.8264 | 1.0173 | 1.0049 |

| (30,50) | (0.25,0.25) | 0.949 | 0.933 | 0.947 | 0.951 | 0.1762 | 0.1580 | 0.1753 | 0.1732 |
| (25,10) | 0.954 | 0.938 | 0.953 | 0.955 | 0.2561 | 0.2335 | 0.2548 | 0.2522 |
| (25,20) | 0.960 | 0.940 | 0.957 | 0.952 | 0.3824 | 0.3569 | 0.3801 | 0.3759 |
| (50,50) | 0.949 | 0.934 | 0.946 | 0.943 | 0.3548 | 0.3379 | 0.3499 | 0.3469 |
| (100,100) | 0.955 | 0.917 | 0.949 | 0.942 | 0.5607 | 0.5348 | 0.5852 | 0.5799 |
| (200,200) | 0.956 | 0.928 | 0.952 | 0.945 | 0.7066 | 0.6484 | 0.6915 | 0.6852 |

| (100,50) | (0.25,0.25) | 0.953 | 0.942 | 0.952 | 0.953 | 0.1279 | 0.1197 | 0.1273 | 0.1259 |
| (25,10) | 0.953 | 0.939 | 0.949 | 0.946 | 0.2236 | 0.2085 | 0.2224 | 0.2187 |
| (25,20) | 0.960 | 0.940 | 0.957 | 0.952 | 0.3611 | 0.3408 | 0.3588 | 0.3540 |
| (50,50) | 0.954 | 0.936 | 0.948 | 0.949 | 0.3339 | 0.3221 | 0.3292 | 0.3265 |
| (100,100) | 0.951 | 0.927 | 0.946 | 0.949 | 0.3581 | 0.3356 | 0.3531 | 0.3492 |
| (200,200) | 0.948 | 0.931 | 0.945 | 0.938 | 0.5760 | 0.5272 | 0.5612 | 0.5550 |
| (200,200) | 0.952 | 0.934 | 0.945 | 0.945 | 0.5369 | 0.5150 | 0.5269 | 0.5226 |

5. An Empirical Application

The BS distribution has been successfully used to analyze air pollution concentration data, as its properties are similar to those of the lognormal distribution. For example, Leiva et al. [37] applied the BS distribution to examine sulfur dioxide concentration data and, later on, PM10 (particulate matter \(\leq 10 \mu m\)) concentration data in Santiago, Chile [38]. In the second study, the authors proposed a criterion based on a control chart attribute for assessing the environmental risk.

In Chiang Mai, Thailand, air pollution from agricultural burning and forest fires from February to May has become a serious problem. We used datasets of PM 2.5 concentration data from Chiang Mai collected in March and April 2019 to illustrate the efficacies of the confidence intervals for the CV and the difference between the CVs of BS distributions derived using GCI, BCI, BayCI, and the HPD interval. The average daily PM 2.5 concentrations were measured at 9:00 AM in the area of Chang Phueak, Chiang Mai, Thailand. The datasets were obtained from the Pollution Control Department [39]. Since the data comprise positive values, they can be fitted to lognormal, exponential, gamma, Weibull, or BS distributions. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) were applied to check the fitting of the data to these distributions. The
results in Tables 4 and 5 show that the AIC and BIC of the BS distribution were the smallest, thereby ensuring its suitability for application to these datasets. To construct the BayCI and the HPD interval for the CV and difference between CVs of BS distributions using real data, we applied $a_1 = b_1 = a_2 = b_2 = 10^{-4}$ and $r = 2$ for both sampling areas. The sample mean, sample variance, and CV of the data are 96.0645, 2896.9290, and 0.5603 for the March dataset and 78.3000, 887.6655, and 0.3805 for the April dataset, respectively. Subsequently, the difference between the CVs was 0.1798. The 95% confidence interval based on GCI, BCI, BayCI, and the HPD interval for the CV and the difference between the CVs for the BS distributions are reported in Table 6.

Table 4. AIC values for the fitting of five asymmetric distributions.

| Distributions | BS   | Lognormal | Exponential | Gamma | Weibull |
|---------------|------|-----------|-------------|-------|---------|
| March         | 328.6784 | 329.4578 | 347.0312 | 330.2267 | 331.8988 |
| April         | 285.6899 | 285.9449 | 323.6329 | 287.0071 | 290.2045 |

Table 5. BIC values for the fitting of five asymmetric distributions.

| Distributions | BS   | Lognormal | Exponential | Gamma | Weibull |
|---------------|------|-----------|-------------|-------|---------|
| March         | 331.5463 | 332.3257 | 348.4652 | 333.0947 | 334.7668 |
| April         | 288.4923 | 288.7473 | 325.0341 | 289.8095 | 293.0068 |

From the results, it can be seen that BCI had the shortest average length, while that of the HPD interval was shorter than GCI and BayCI, which is in agreement with the results from the simulation study. Once again, BCI attained the lowest coverage probability, so it is not recommended for constructing confidence intervals for the CV and the difference between the CVs of the BS distributions of the two real datasets. Meanwhile, the coverage probabilities of the HPD interval for the CV and the difference between the CVs of the BS distributions were greater than or close to 0.95. Thus, under these circumstances, the HPD interval provided the best-performing confidence intervals for the CV and the difference between the CVs of the BS distributions of these two datasets.

Table 6. The 95% confidence intervals for the CV and the difference between the CVs of PM2.5 concentration datasets from Chiang Mai, Thailand.

| Methods   | March Interval | March Length | April Interval | April Length | The Difference of CVs Interval | The Difference of CVs Length |
|-----------|----------------|--------------|----------------|--------------|--------------------------------|-----------------------------|
| GCI       | 0.4723-0.8009  | 0.3287       | 0.2998-0.5157  | 0.2160       | 0.0311–0.4311                  | 0.4001                      |
| BCI       | 0.4538-0.7380  | 0.2842       | 0.2995-0.4784  | 0.1889       | 0.0454–0.3925                  | 0.3471                      |
| BayCI     | 0.4743-0.7707  | 0.2964       | 0.3029-0.5071  | 0.2042       | 0.0275–0.4299                  | 0.4024                      |
| HPD       | 0.4694-0.7587  | 0.2892       | 0.2859-0.4819  | 0.1960       | 0.0403–0.4394                  | 0.3991                      |

6. Conclusions

We proposed confidence intervals for the CV and the difference between the CVs of BS distributions using the concepts of GCI, BCI, BayCI, and the HPD interval. The coverage probabilities and average lengths of the four methods were evaluated through Monte Carlo simulations. The results show that, for all of the scenarios tested, the HPD interval had a reasonable coverage probability, while its average lengths were shorter than those of GCI and BayCI, and so it can be recommended for constructing the confidence intervals for the CV and the difference between the CVs of BS distributions. Although BCI had the shortest average length in all cases, its coverage probabilities were the lowest and under 0.95. Therefore, it cannot be recommended due to this shortcoming. Moreover, the average lengths of GCI, BayCI, and the HPD interval were similar when the sample sizes increased. Hence, GCI and BayCI can be considered as alternative methods to construct the confidence intervals for these two scenarios. In future research, we will investigate the
confidence interval for the mean and the CV, as well as the difference between the means and the CVs of BS distributions with censored data.

**Author Contributions:** Conceptualization, S.N.; Data curation, W.P.; Formal analysis, W.P. and S.N.; Funding acquisition, S.N.; Investigation, S.-A.N. and S.N.; Methodology, W.P., S.-A.N. and S.N.; Project administration, S.-A.N.; Resources, S.-A.N.; Software, W.P.; Supervision, S.-A.N. and S.N.; Writing original draft, W.P.; Writing—review & editing, S.-A.N. and S.N. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received financial support from the National Science, Research, and Innovation Fund (NSRF), and King Mongkut’s University of Technology North Bangkok (Grant No. KMUTNB–FF–65–23).

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Data may be made available by contacting the corresponding author.

**Conflicts of Interest:** The first author acknowledges the generous financial support from the Science Achievement Scholarship of Thailand (SAST).

**References**

1. Birnbaum, Z.W.; Saunders S.C. A new family of life distributions. *J. Appl. Probab.* **1969**, *6*, 319–327. [CrossRef]
2. Birnbaum, Z.W.; Saunders S.C. Estimation for a family of life distributions with applications to fatigue. *J. Appl. Probab.* **1969**, *6*, 326–347. [CrossRef]
3. Chang, D.S.; Tang, L.C. Reliability bounds and critical time for the Birnbaum-Saunders distribution. *IEEE Trans. Reliab.* **1993**, *42*, 464–469. [CrossRef]
4. Leiva, V.; Athayde, M.E.; Azevedo, C.; Marchant, C. Modeling wind energy flux by a Birnbaum-Saunders distribution with an unknown shift parameter. *J. Appl. Stat.* **2011**, *38*, 2819–2838. [CrossRef]
5. Leiva, V.; Sanhueza, A.; Angulo, J.M. A length-biased version of the Birnbaum-Saunders distribution with application in water quality. *Stoch. Environ. Res. Risk Assess.* **2009**, *23*, 299–307. [CrossRef]
6. Leiva, V.; Ruggeri, F.; Saulo, H.; Vivanco, J.F. A methodology based on the Birnbaum–Saunders distribution for reliability analysis applied to nano-materials. *Reliab. Eng. Syst. Saf.* **2017**, *157*, 192–201. [CrossRef]
7. Durham, S.D.; Padgett, W.J. A cumulative damage model for system failure with application to carbon fibers and composites. *Technometrics* **1997**, *39*, 34–44. [CrossRef]
8. Desmond, A.F. Stochastic models of failure in random environments. *Can. J. Stat.* **1985**, *13*, 171–183. [CrossRef]
9. Guiraud, P.; Leiva, V.; Fierro, R. A non central version of the Birnbaum–Saunders distribution for reliability analysis. *IEEE Trans. Reliab.* **2009**, *58*, 152–160. [CrossRef]
10. Leiva, V.; Santos-Neto, M.; Cysneiros, F.J.A.; Barros, M. Birnbaum–Saunders statistical modelling: A new approach. *Stat. Model.* **2014**, *14*, 21–48. [CrossRef]
11. Lio, Y.L.; Tsai, T.R.; Wu, S.J. Acceptance sampling plans from truncated life tests based on the Birnbaum–Saunders distribution for percentiles. *Commun. Stat.-Simul. Comput.* **2010**, *39*, 119–136. [CrossRef]
12. Marchant, C.; Leiva, V.; Cysneiros, F.J.A.; Vivanco, J.F. Diagnostics in multivariate generalized Birnbaum–Saunders regression models. *J. Appl. Stat.* **2016**, *43*, 2829–2849. [CrossRef]
13. Tian, L. Inferences on the common coefficient of variation. *Stat. Med.* **2005**, *24*, 2213–2220. [CrossRef]
14. Mahmoudvand, R.; Hassani, H. Two new confidence intervals for the coefficient of variation in a normal distribution. *J. Appl. Stat.* **2009**, *36*, 429–442. [CrossRef]
15. Banik, S.; Kibria B.M.G. Estimating the Population Coefficient of Variation by Confidence Intervals. *Commun. Stat.-Simul. Comput.* **2011**, *40*, 1236–1261. [CrossRef]
16. Sangnawakij, P.; Niwitpong, S-A. Confidence intervals for coefficients of variation in two-parameter exponential distributions. *Commun. Stat.-Simul. aComput.* **2017**, *46*, 6618–6630. [CrossRef]
17. Thongjitmongkol, W.; Niwitpong, S-A.; Niwitpong, S. Adjusted generalized confidence intervals for the common coefficient of variation of several normal populations. *Commun. Stat.-Simul. Comput.* **2020**, *49*, 194–206. [CrossRef]
18. Yossoothong, N.; Niwitpong, S-A.; Niwitpong, S. Measuring the dispersion of rainfall using Bayesian confidence intervals for coefficient of variation of delta-lognormal distribution: A study from Thailand. *PeerJ* **2019**, *7*, e7344. [CrossRef]
19. La-Ongkaew, M.; Niwitpong, S-A.; Niwitpong, S. Confidence intervals for the mean and the CV, as well as the difference between the means and the CVs of BS distributions with censored data.
21. Wu, J.; Wong A.C.M. Improved interval estimation for the two-parameter Birnbaum-Saunders distribution. *Comput. Stat. Data Anal.* 2004, 47, 809–821. [CrossRef]

22. Ng, H.K.T.; Kundu, D.; Balakrishnan, N. Point and interval estimation for the two-parameter Birnbaum–Saunders distribution based on type-II censored samples. *Comput. Stat. Data Anal.* 2006, 50, 3222–3242. [CrossRef]

23. Xu, A.; Tang, Y. Reference analysis for Birnbaum–Saunders distribution. *Comput. Stat. Data Anal.* 2010, 54, 185–192. [CrossRef]

24. Wang, B.X. Generalized interval estimation for the Birnbaum–Saunders distribution. *Comput. Stat. Data Anal.* 2012, 56, 4320–4326. [CrossRef]

25. Niu, C.; Guo, X.; Zhu, L. Comparison of several Birnbaum–Saunders distributions. *J. Stat. Comput. Simul.* 2014, 84, 2721–2733. [CrossRef]

26. Li, Y.; Xu, A. Fiducial inference for Birnbaum–Saunders distribution. *J. Stat. Comput. Simul.* 2016, 86, 1673–1685. [CrossRef]

27. Guo, X.; Wu, H.; Li, G.; Li, Q. Inference for the common mean of several Birnbaum–Saunders populations. *J. Appl. Stat.* 2017, 44, 941–954. [CrossRef]

28. Weerahandi, S. Generalized confidence intervals. *J. Am. Stat. Assoc.* 1993, 88, 899–905. [CrossRef]

29. Sun, Z.L. The confidence intervals for the scale parameter of the Birnbaum–Saunders fatigue life distribution. *Acta Armamentarii* 2009, 30, 1558–1561. (In Chinese)

30. Efron, B. Bootstrap methods: Another look at the jackknife. *Ann. Stat.* 1979, 7, 1–26. [CrossRef]

31. Ng, H.K.T.; Kundu, D.; Balakrishnan, N. Modified moment estimation for the two-parameter Birnbaum–Saunders distribution. *Comput. Stat. Data Anal.* 2003, 43, 283–298. [CrossRef]

32. Lemonte, A.J.; Simas, A.B.; Cribari-Neto, F. Bootstrap-based improved estimators for the two-parameter Birnbaum–Saunders distribution. *J. Stat. Comput. Simul.* 2008, 78, 37–49. [CrossRef]

33. MacKinnon, J.G.; Smith, J.A.A. Approximate bias correction in econometrics. *J. Econom.* 1998, 85, 205–230. [CrossRef]

34. Wang, M.; Sun, X.; Park, C. Bayesian analysis of Birnbaum-Saunders distribution via the generalized ratio-of-uniforms method. *Comput. Stat.* 2016, 31, 207–225. [CrossRef]

35. Wakefield, J.C.; Gelfand, A.E.; Smith, A.F.M. Efficient generation of random variates via the ratio-of-uniforms method. *Stat. Comput.* 1991, 1, 129–133. [CrossRef]

36. Box, G.E.P.; Tiao, G.C. *Bayesian Inference in Statistical Analysis*; Wiley: New York, NY, USA, 1992.

37. Leiva, V.; Barros, M.; Paula, G.A.; Sanhueza, A. Generalized Birnbaum–Saunders distributions applied to air pollutant concentration. *Environmetrics* 2008, 19, 235–249. [CrossRef]

38. Leiva, V.; Marchant, C.; Ruggeri, F.; Saulo, H. A criterion for environmental assessment using Birnbaum–Saunders attribute control charts. *Environmetrics* 2015, 26, 463–476. [CrossRef]

39. Pollution Control Department Thailand. Available online: http://www.pcd.go.th/ (accessed on 9 January 2021).