Joint universality for Lerch zeta-functions

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Abstract. For $0 < \alpha, \lambda \leq 1$, the Lerch zeta-function is defined by $L(s; \alpha, \lambda) := \sum_{n=0}^{\infty} e^{2\pi i \lambda n} (n + \alpha)^{-s}$, where $\sigma > 1$. In this paper, we prove joint universality for Lerch zeta-functions with distinct $\lambda_1, \ldots, \lambda_m$ and transcendental $\alpha$.

1. Introduction and statement of main result.

For $0 < \alpha, \lambda \leq 1$, we define the Lerch zeta-function by

$$L(s; \alpha, \lambda) := \sum_{n=0}^{\infty} e^{i\lambda n} (n + \alpha)^{-s}, \quad \sigma > 1,$$

where $e(t) = \exp(2\pi it)$. When $\lambda = 1$, the function $L(s; \alpha, \lambda)$ reduces to the Hurwitz zeta-function $\zeta(s, a)$. If $\lambda \neq 1$, the Lerch zeta-function $L(s; \alpha, \lambda)$ is analytically extendable to an entire function. However, the Hurwitz zeta-function $\zeta(s, a)$ is extended to a meromorphic function, which has a simple pole at $s = 1$.

In this paper, we show the following joint universality theorem expected by Mishou [6, Conjecture 1]. In order to state it, put $D := \{ s \in \mathbb{C} : 1/2 < \Re s < 1 \}$ and let $\text{meas}\{A\}$ be the Lebesgue measure on $\mathbb{R}$ of the set $A$.

**Theorem 1.** Suppose that $L(s; \alpha, \lambda_1), \ldots, L(s; \alpha, \lambda_m)$ are Lerch zeta-functions with distinct $\lambda_1, \ldots, \lambda_m$ and transcendental $\alpha$. For $1 \leq j \leq m$, let $K_j \subset D$ be compact sets with connected complements and $f_j(s)$ be continuous function on $K_j$ and analytic in the interior of $K_j$. Then, for every $\epsilon > 0$, we have

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{ \tau \in [0, T] : \max_{1 \leq j \leq m} \max_{s \in K_j} |L(s + i\tau; \alpha, \lambda_j) - f_j(s)| < \epsilon \} > 0.$$

Roughly speaking, this theorem implies that any analytic functions can be simultaneously and uniformly approximated by Lerch zeta-functions with distinct $\lambda_1, \ldots, \lambda_m$. The proof will be written in Sections 2 and 3. We skip the detail of the proofs of results appeared in Section 2 since they do not contain essentially new ideas. In Section 3, we prove the denseness lemma using an orthogonality of Dirichlet coefficients of the zeta-functions. The main idea of our proof was recently observed in [5] by the authors. However, in the present paper we adopt this approach to completely different kind of zeta-functions without Euler product. It proves the conjecture on joint universality for
Lerch zeta-functions put forward by Mishou in [6] and shows that this idea can be applicable to many collections of zeta and L-functions, which independence relies on some orthogonality property of their coefficients.

Now we look back in the history of the joint universality for Lerch zeta-functions. Laurinčikas showed Theorem 1 with orthogonality property of their coefficients. Laurinčikas and Matsumoto proved Theorem 1 with the condition that \( \lambda_j = k_j/l_j \) are distinct rational numbers satisfying \( (k_j,l_j) = 1 \) and \( 0 < k_j \leq l_j \) in [4, Theorem 1] (see also [3, Theorem 6.3.1] or [6, Theorem 2]). In [7, Theorem 17], Nakamura obtained the joint universality of the Lerch zeta-functions with \( \lambda_j = \lambda + k_j/l_j \), where \( 0 < \lambda \leq 1 \) and \( \lambda_j \) are distinct in mod 1. The method in the both papers [4, Theorem 1] and [7, Theorem 17] are based on the observation that

\[
\ell_j^{1/2} e((\lambda_i - \lambda_j)n) = e\left(\frac{k_i\ell_j - k_j\ell_i}{\ell_i\ell_j} n\right)
\]

is a \( (\ell_i\ell_j) \)-th root of unity for each \( i \neq j \) and \( n \in \mathbb{Z} \) so that

\[
|e(\lambda_i n) - e(\lambda_j n)| = |1 - e((\lambda_i - \lambda_j)n)| \geq |1 - e(1/(\ell_i\ell_j))| > 0
\]

or \( e(\lambda_i n) = e(\lambda_j n) \). Recently, Mishou proved in [6, Theorem 4], the joint universality of the Lerch zeta-functions for almost all real numbers \( \lambda_j, 1 \leq j \leq m \) such that \( 1, \lambda_1, \ldots, \lambda_m \) are linearly independent over \( \mathbb{Q} \). His proof is based on some results of discrepancy estimate from uniform distribution theory (see [6, Section 2]). Obviously, Theorem 1 of the present paper is not only an improvement of Mishou’s result [6, Theorem 4] but also the final answer to [6, Conjecture 1].

By using Theorem 1, we get the following corollaries. We omit their proofs since they follow from the standard argument (see for example [3, Section 7.2]).

**Corollary 2.** Let \( \alpha \in (0,1] \) be transcendental and \( \lambda_1, \ldots, \lambda_m \in (0,1] \) be distinct real numbers. For \( N \in \mathbb{N} \) and \( 1/2 < \sigma < 1 \), define the mapping \( h: \mathbb{R} \to \mathbb{C}^{mN} \) by the formula

\[
h(t) := (L(\sigma + it; \alpha, \lambda_1), L'(\sigma + it; \alpha, \lambda_1), \ldots, L^{(N-1)}(\sigma + it; \alpha, \lambda_1),
\]

\[
\ldots, L(\sigma + it; \alpha, \lambda_m), L'(\sigma + it; \alpha, \lambda_m), \ldots, L^{(N-1)}(\sigma + it; \alpha, \lambda_m)).
\]

Then the image of \( \mathbb{R} \) is dense in \( \mathbb{C}^{mN} \).

**Corollary 3.** Let \( \alpha \in (0,1] \) be transcendental and \( \lambda_1, \ldots, \lambda_m \in (0,1] \) be distinct real numbers. Suppose \( N \in \mathbb{N} \) and \( F_l, 0 \leq l \leq k \) are continuous functions on \( \mathbb{C}^{mN} \) and satisfy

\[
\sum_{l=0}^{k} s^l F_l(L(s; \alpha, \lambda_1), L'(s; \alpha, \lambda_1), \ldots, L^{(N-1)}(s; \alpha, \lambda_1),
\]

\[
\ldots, L(s; \alpha, \lambda_m), L'(s; \alpha, \lambda_m), \ldots, L^{(N-1)}(s; \alpha, \lambda_m)) \equiv 0.
\]

Then we have \( F_l \equiv 0 \) for \( 0 \leq l \leq k \).
2. Proof of Theorem 1.

Recall that \( D := \{ s \in \mathbb{C} : 1/2 < \text{Re} s < 1 \} \) and denote by \( H(D) \) the space of analytic function on \( D \) equipped with the topology of uniform convergence on compacta. Let \( \mathfrak{B}(X) \) stand for the class of Borel sets of the space \( X \). Define \( \gamma \) as the unit circle on \( \mathbb{C} \), and let \( \Omega := \prod_{n=0}^{\infty} \gamma_n \), where \( \gamma_n = \gamma \) for all \( n \in \mathbb{N}_0 \). Denoting by \( m_H \) the probability Haar measure on \((\Omega, \mathfrak{B}(\Omega))\), we obtain a probability space \((\Omega, \mathfrak{B}(\Omega), m_H)\). For \( \sigma > 1 \), we define

\[
L(s; \alpha, \lambda; \omega) := \sum_{n=0}^{\infty} \frac{e(\lambda n)\omega(n)}{(n + \alpha)^s}, \quad \omega(n) \in \gamma.
\]

Note that for almost all \( \omega \in \Omega \) the series above converges uniformly on compact subsets of \( D \) (see for instance [3, Lemma 5.2.1]).

Let \( H(D)^m := H(D) \times \cdots \times H(D) \). We define a probability measure \( P_T \) on \((H(D)^m, \mathfrak{B}(H(D)^m))\) by

\[
P_T(A) := \frac{1}{T} \text{meas}\{ \tau \in [0, T] : (L(s + i\tau; \alpha, \lambda_1), \ldots, L(s + i\tau; \alpha, \lambda_m)) \in A \},
\]

where \( A \in \mathfrak{B}(H(D)^m) \). Next define the \( H(D)^m \)-valued random element \( L(s; \omega) \) by

\[
L(s; \omega) := (L(s; \alpha, \lambda_1; \omega), \ldots, L(s; \alpha, \lambda_m; \omega)).
\]

Denote by \( P_L \) the distribution of the random element \( L(s; \omega) \), namely,

\[
P_L(A) := m_H\{ \omega \in \Omega : L(s; \omega) \in A \}, \quad A \in \mathfrak{B}(H(D)^m).
\]

Then we have the following limit theorem proved by Matsumoto and Laurinčikas [4] (see also [3, Theorem 5.3.1] or [6, Section 5]).

**Proposition 4 ([4, Lemma 1]).** Let \( 0 < \alpha < 1 \) be transcendental. Then the probability measure \( P_T \) converges weakly to \( P_L \) as \( T \to \infty \).

The proof of the next lemma shall be written in Section 3 since it contains the most novel part of the present paper.

**Lemma 5.** The set \( \{ L(s; \omega) : \omega \in \Omega \} \) is dense in \( H(D)^m \).

Recall that the minimal closed set \( S_\mathfrak{P} \subset X \) such that \( \mathfrak{P}(S_\mathfrak{P}) = 1 \) is called the support of a probability space \((X, \mathfrak{B}(X), \mathfrak{P})\). The set \( S_\mathfrak{P} \) consists of all \( x \in S \) such that for every neighborhood \( V \) of \( x \) the inequality \( \mathfrak{P}(V) > 0 \) is satisfied. From Lemma 5 and [3, Lemma 6.1.3] or [9, Lemma 12.7], the support of the probability measure \( P_L \) is \( H(D)^m \). First assume that \( h_1(s), \ldots, h_m(s) \in H(D) \) are polynomials. Let \( K_j \) be the same as in Theorem 1 and \( \Phi \) be the set of functions \( \varphi \in H(D)^m \) which satisfy

\[
\max_{1 \leq j \leq m} \max_{s \in K_j} |\varphi_j(s) - h_j(s)| < \varepsilon.
\]
From Proposition 4, the definition of support, Portmanteau theorem (see for instance [9, Theorem 3.1]) and the fact that the support of $P_L$ is $H(D)^m$, we have

$$\liminf_{T \to \infty} P_T(\Phi) \geq P_L(\Phi) > 0.$$ 

Therefore, we obtain

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \max_{1 \leq j \leq m} \max_{s \in K_j} |L(s + i\tau; \alpha, \lambda_j) - h_j(s)| < \varepsilon\} > 0.$$

Hence it suffices to show that polynomials $h_j(s)$ can be replaced by $f_j(s)$ appeared in Theorem 1. It is possible by Mergelyan’s theorem which implies that any function $f(s)$ which is continuous on $K$ and analytic in the interior of $K$, where $K$ is a compact subset with connected complement, is uniformly approximative on $K$ by polynomials. Hence we omit the details since this is easily done by the well-known method (see for example [3, p. 129] or [6, p. 1125]).

3. Proof of Lemma 5.

Let $U$ be a simply connected smooth Jordan domain such that $\overline{U} \subset D$. Let $B^2(U)$ be the Bergman space of all holomorphic square integrable complex functions with respect to the Lebesgue measure on $U$ with the inner product

$$\langle f, g \rangle = \int_U f(s) \overline{g(s)} d\sigma dt, \quad f, g \in H(U).$$

The properties below are well-known (see for instance [8]).

**Lemma 6** ([8, Proposition 7.2.2 and Theorem 7.2.3]). We have the following.

(a) Convergence in $B^2(U)$ implies local uniform convergence on $U$.

(b) $B^2(U)$ is a Hilbert space.

(c) The set of polynomials is dense in $B^2(U)$.

Now let $\mathbb{B}^m := B^2(U) \times \cdots \times B^2(U)$ is the Hilbert space with the inner product given, for $f = (f_1, \ldots, f_m) \in H(U)^m$ and $g = (g_1, \ldots, g_m) \in H(U)^m$ by

$$\langle f, g \rangle = \sum_{j=1}^m \int_U f_j(s) \overline{g_j(s)} d\sigma dt.$$ 

In order to prove Lemma 5, we use (b) of Lemma 6 and the following result appeared, for example, in [9].

**Lemma 7** ([9, Theorem 6.1.16]). Let $H$ be a complex Hilbert space. Assume that a sequence $v_n \in H$, $n \in \mathbb{N}$ satisfies

(i) the series $\sum_n \|v_n\|^2 < \infty$;

(ii) for any element $0 \neq g \in H$ the series $\sum_n |\langle v_n, g \rangle|$ is divergent.

Then the set of convergent series
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\[ \left\{ \sum_n a_n v_n \in H : |a_n| = 1 \right\} \]

is dense in \( H \).

Let \( g = (g_1, \ldots, g_m) \in \mathbb{B}^m \) be a non-zero element and put

\[ v_n(s) := (v_n(s; \alpha, \lambda_1), \ldots, v_n(s; \alpha, \lambda_m)), \quad v_n(s; \alpha, \lambda_j) := \frac{e(\lambda_j n)}{(n + \alpha)^s}. \]

Then for \( \Delta_j(w) := \int_U e^{-sw} g_j(s) d\sigma dt \), one has

\[ \langle v_n(s), g(s) \rangle = \sum_{j=1}^m e(\lambda_j n) \Delta_j(\log(n + \alpha)). \]

We can see that the condition (i) of Lemma 7 is true since \( U \subset D \) and

\[ \langle v_n(s), v_n(s) \rangle = \sum_{j=1}^m \int_U (n + \alpha)^{-s} (n + \alpha)^{-s} d\sigma dt \ll \sup_{s \in U} |(n + \alpha)^{-2s}|. \]

The truth of the condition (ii) in Lemma 7 easily follows from the following crucial lemma.

**Lemma 8.** Assume that \( g(s) = (g_1(s), \ldots, g_m(s)) \in \mathbb{B}^m \) is a non-zero element and for \( j = 1, \ldots, m \), put \( \Delta_j(z) := \int_U e^{-sz} g_j(s) d\sigma dt \). Then the following series

\[ \sum_{n=0}^{\infty} |e(\lambda_1 n) \Delta_1(\log(n + \alpha)) + \cdots + e(\lambda_m n) \Delta_m(\log(n + \alpha))| \]

is divergent.

In order to prove the lemma above, we quote the following.

**Lemma 9 ([5, Corollary 2.7]).** Let \( \|g_j\| \neq 0 \) for \( 1 \leq j \leq m \). Then for every \( A > 0 \) and every \( x > 1 \), there exist sequences \( B_1 > \cdots > B_m > 0 \), \( x_0^{(0)} = x, x_0^{(1)}, \ldots, x_0^{(m)} \) and intervals \( I_j \subset [x, x+1] \) of length \( |I_j| \geq B_j x^{-2j} \) such that \( x_0^{(j)} \in I_j \), \( I_{j+1} \subset I_j \), and for all \( t \in I_j \) we have

\[ \frac{1}{2} |\Delta_j(x_0^{(j-1)})| + O(e^{-Ax}) \leq \frac{1}{2} |\Delta_j(x_0^{(j)})| + O(e^{-Ax}) \leq |\Delta_j(t)| \leq |\Delta_j(x_0^{(j)})| + O(e^{-Ax}). \]

**Proof of Lemma 8.** Without loss of generality, we can assume that \( g_1 \) is a non-zero element since \( \|g\| \neq 0 \) implies that at least one of \( g_j \)'s is a non-zero element.

We shall check the conditions in [1, Lemma 3] for \( \Delta_1(z) \). Obviously, \( \Delta_1(z) \ll e^{C|z|} \) for some positive constant \( C \) depending on \( U \). Let \( \sigma_1 \) and \( \sigma_2 \) be real numbers with
$1/2 < \sigma_1 < \sigma_2 < 1$ such that the vertical strip $\sigma_1 < \text{Re} s < \sigma_2$ contains the simply connected smooth Jordan domain $U$. Then for sufficiently small $\eta = \eta(U) > 0$ and for all complex $z$ with $|\arg(-z)| \leq \eta$, we have $|e^{\sigma_2 z} \Delta_1(z)| \ll 1$. Furthermore, $\Delta_1$ is not identically zero. If it is, we have

$$0 = \Delta_1^{(k)}(0) = \int_U (-s)^k g_1(s) d\sigma dt$$

for any nonnegative integer $k$, which implies that $g_1$ is orthogonal to every polynomial in $B^2(U)$. So $g_1 = 0$ by (c) of Lemma 6, but it contradicts to the assumption $\|g_1\| \neq 0$. Hence, by [1, Lemma 3] we can find a real sequence $x_k$ tending to infinity such that

$$|\Delta_1(x_k)| \gg e^{-\sigma_2 x_k}.$$  

Fix $k$ and put $x = x_k$. Hence, by using Lemma 9, we can see that for every $A > 0$ and $x = x_k$, there exist sequences $B_1 > \cdots > B_m > 0$, $x_0^{(0)} = x, x_1^{(1)}, \ldots, x^{(m)}_0$ and intervals $I_j \subset [x, x+1]$ of length $|I_j| \geq B_j x^{-2j}$ such that $x_0^{(j)} \in I_j$, $I_{j+1} \subset I_j$, and for all $t \in I_j$, the inequalities (1) holds. Now let $I_m := [y, y + B_m y^{-2m}] \subset [x, x+1]$. Since $I_m \subset I_j$ for every $j = 1, 2, \ldots, m$, the inequalities (1) holds also for all $t \in I_m$. In particular, since $x_0^{(0)} = x$, for $t \in I_m$ one has

$$|\Delta_1(t)| \gg |\Delta_1(x_0^{(0)})| \gg e^{-\sigma_2 x}.$$  

Moreover, for every $j = 1, 2, \ldots, m$ we have

$$|\Delta_j(t)| \ll e^{-\sigma_1 x}, \quad t \in [x, x+1].$$

We denote by $\sum^\ast_n$ the sum over integers $n + \alpha \in [e^y, e^y + B_m y^{-2m}]$ in order to obtain $\log(n + \alpha) \in I_m$.

First we consider the following sum

$$S_1(x) := \sum^\ast_n \sum^m_{j=1} |\Delta_j(\log(n + \alpha))|^2.$$ 

Obviously, it holds that

$$e^{y+y^{-2m}} - e^y = e^y(e^{y-2m} - 1) = \frac{e^y}{y^{2m}} \sum_{n=0}^\infty y^{-2mn} \gg \frac{e^y}{y^{2m}}.$$ 

Let $A > 0$ be sufficiently large. Then by using (1), (2), $x \leq y \leq x+1$ and the formula above, we have

$$S_1(x) \gg \sum^\ast_n \sum^m_{j=1} \left(|\Delta_j(x_0^{(j)})|^2 + |\Delta_j(x_0^{(j)})|O(e^{-Ax}) + O(e^{-2Ax})\right)$$

$$\gg \sum^\ast_n \sum^m_{j=1} \left(|\Delta_j(x_0^{(j)})|^2 + O(e^{-Ax})\right) \gg \sum^\ast_n \left(\sum^m_{j=1} |\Delta_j(x_0^{(j)})|^2\right)^2.$$
\[
\sum_{n} e^{-\sigma nx} \sum_{j=1}^{m} \Delta_j(x_0^{(j)}) \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}} \sum_{j=1}^{m} \Delta_j(x_0^{(j)}).
\]

Since the \(\lambda_k\)'s are assumed to be distinct in the interval \((0, 1]\), it is easy to see that for any \(1 \leq k \neq l \leq m\)

\[
\phi_{k,l}(t) := \sum_{n \leq t} e((\lambda_k - \lambda_l)n) \ll \frac{1}{|1 - e(\lambda_k - \lambda_l)|} \ll 1.
\]

Similarly to (3), one can easily get the estimation

\[
\frac{d}{du} \Delta_j(\log u) = \frac{1}{u} \Delta'_j(\log u) \ll u^{-1-\sigma_1}.
\]

From \(\overline{\Delta_j(\log u)} = \langle u^{-s}, g_j(s) \rangle = \langle u^{-\overline{s}}, g_j(s) \rangle\), we obtain

\[
\frac{d}{du} \overline{\Delta_j(\log u)} = \frac{1}{u} \int_{U} -\overline{u^{-\overline{s}}} g_j(s) d\sigma dt = \frac{1}{u} \Delta'_j(\log u) \ll u^{-1-\sigma_1}.
\]

Hence, using partial summation, we have

\[
\sum_{X_1 \leq n \leq X_2} \sum_{1 \leq k \neq l \leq m} e((\lambda_k - \lambda_l)n) \Delta_k(\log(n + \alpha)) \overline{\Delta_l(\log(n + \alpha))}
\]

\[
= \sum_{1 \leq k \neq l \leq m} \int_{X_1}^{X_2} \Delta_k(\log(u + \alpha)) \overline{\Delta_l(\log(u + \alpha))} d\phi_{k,l}(u)
\]

\[
\ll X_1^{-2\sigma_1} + \sum_{1 \leq k \neq l \leq m} \int_{X_1}^{X_2} \left| \Delta_k(\log(u + \alpha)) \overline{\Delta_l(\log(u + \alpha))} \right|' du
\]

\[
\ll X_1^{-2\sigma_1} + \int_{X_1}^{X_2} \frac{du}{u^{1+2\sigma_1}} \ll X_1^{-2\sigma_1}
\]

for sufficiently large \(X_2 > X_1 > 0\). Thus we obtain

\[
S_2(x) := \sum_{1 \leq k \neq l \leq m} \sum_{n} e((\lambda_i - \lambda_k)n) \Delta_k(\log(n + \alpha)) \overline{\Delta_l(\log(n + \alpha))} \ll e^{-2\sigma_1 x}.
\]

We can easily see that

\[
S(x) := \sum_{n} \left| e(\lambda_1 n) \Delta_1(\log(n + \alpha)) + \cdots + e(\lambda_m n) \Delta_m(\log(n + \alpha)) \right|^2
\]

\[
= S_1(x) + S_2(x) \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}} \sum_{j=1}^{m} |\Delta_j(x_0^{(j)})| + O(e^{-2\sigma_1 x})
\]

when \(A\) is sufficiently large. On the other hand, one has

\[
S(x) \ll \sum_{n} \left| \sum_{j=1}^{m} e(\lambda_j n) \Delta_j(\log(n + \alpha)) \right| \sum_{j=1}^{m} |\Delta_j(\log(n + \alpha))|
\]
\[ \ll \sum_n \sum_{j=1}^m e(\lambda_j n) \Delta_j(\log(n + \alpha)) \sum_{j=1}^m |\Delta_j(x_0^{(j)})| + O(e^{-(A+\sigma_1-1)x}). \]

Hence, dividing the last inequalities by \( \sum_{j=1}^m |\Delta_j(x_0^{(j)})| \), we have

\[ \sum_n \sum_{j=1}^m e(\lambda_j n) \Delta_j(\log(n + \alpha)) \gg \frac{e^{x(1-\sigma_2)}}{x^{2m}}, \]

since \( 2\sigma_1 - \sigma_2 > 0 \). Thus, the last inequality implies Lemma 8.

We now prove Lemma 5. Put

\[ v_n(s, \omega(n); \alpha, \lambda_j) := \frac{e(\lambda_j n) \omega(n)}{(n + \alpha)^s}, \quad \omega(n) \in \gamma, \]

\[ v_n(s, \omega(n)) := (v_n(s, \omega(n); \alpha, \lambda_1), \ldots, v_n(s, \omega(n); \alpha, \lambda_m)). \]

Recall \( U \) be a simply connected smooth Jordan domain such that \( \overline{U} \subset D \). Then the set of convergent series

\[ \left\{ \sum_n v_n(s, \omega(n)) : \omega \in \Omega \right\} \]

is dense in the space \( \mathbb{B}^m \) by Lemmas 7 and 8. Thus, for every compact subsets \( K_1, \ldots, K_m \subset U \), we can find \( b(n) \in \gamma \) and \( M \in \mathbb{N} \) satisfying

\[ \max_{1 \leq j \leq m} \max_{s \in K_j} \left| \sum_{n=0}^M v_n(s, b(n); \alpha, \lambda_j) - h_j(s) \right| < \frac{\varepsilon}{2}, \]

\[ \max_{1 \leq j \leq m} \max_{s \in K_j} \left| \sum_{n>M} v_n(s, b(n); \alpha, \lambda_j) \right| < \frac{\varepsilon}{2} \]

from (a) of Lemma 6 and Lemma 8. The inequality above and the assumption \( \overline{U} \subset D \) implies Lemma 5.

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References

[1] J. Kaczorowski and M. Kulas, On the non-trivial zeros off line for L-functions from extended Selberg class, Monatshefte Math., 150 (2007), 217–232.
[2] A. Laurinčikas, The universality of the Lerch zeta function, Lith. Math. Journal, 37 (1997), 275–280.
[3] A. Laurinčikas and R. Garunkštis, The Lerch zeta-function, Kluwer Academic Publishers, Dordrecht, 2002.
[4] A. Laurinčikas and K. Matsumoto, The joint universality and the functional independence for Lerch zeta-functions, Nagoya Math. J., 157 (2000), 211–227.

[5] Y. Lee, T. Nakamura and Ł. Pańkowski, Selberg’s orthonormality conjecture and joint universality of \( L \)-functions, Math. Z. (2016), doi:10.1007/s00209-016-1754-2.

[6] H. Mishou, Functional distribution for a collection of Lerch zeta functions, J. Math. Soc. Japan, 66 (2014), 1105–1126.

[7] T. Nakamura, Applications of inversion formulas to the joint \( t \)-universality of Lerch zeta functions, J. Number Theory., 123 (2007), 1–9.

[8] H. Queffélec and M. Queffélec, Diophantine Approximation and Dirichlet Series, HRI Lecture Notes Series 2, American Mathematical Society, 2013.

[9] J. Steuding, Value-Distribution of \( L \)-functions, Lecture Notes in Math., 1877, Springer, Berlin, 2007.

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