Some results on higher eigenvalue optimization

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Abstract
In this paper we obtain several results concerning the optimization of higher Steklov eigenvalues both in two and higher dimensional cases. We first show that the normalized (by boundary length) $k$-th Steklov eigenvalue on the disk is not maximized for a smooth metric on the disk for $k \geq 3$. For $k = 1$ the classical result of Weinstock (J Ration Mech Anal 3:745–753, 1954) shows that $\sigma_1$ is maximized by the standard metric on the round disk. For $k = 2$ it was shown by Girouard and Polterovich (Funct Anal Appl 44(2):106–117, 2010) that $\sigma_2$ is not maximized for a smooth metric. We also prove a local rigidity result for the critical catenoid and the critical Möbius band as free boundary minimal surfaces in a ball under $C^2$ deformations. We next show that the first $k$ Steklov eigenvalues are continuous under certain degenerations of Riemannian manifolds in any dimension. Finally we show that for $k \geq 2$ the supremum of the $k$-th Steklov eigenvalue on the annulus over all metrics is strictly larger that that over $S^1$-invariant metrics. We prove this same result for metrics on the Möbius band.

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1 Introduction

In this paper we obtain several results concerning the optimization of higher Steklov eigenvalues both in two and higher dimensional cases. Recall that for a compact Riemannian manifold...
with non-empty boundary we have the Steklov spectrum which consists of the eigenvalues of the Dirichlet to Neumann map. We denote these eigenvalues $\sigma_0 = 0 < \sigma_1 \leq \sigma_2 \ldots$ and they form an infinite discrete sequence tending to infinity. A Steklov eigenfunction $u$ with eigenvalue $\sigma$ is then a non-zero solution of $\Delta u = 0$ in $M$ with $\frac{\partial u}{\partial \nu} = \sigma u$ on $\partial M$ where $\nu$ denotes the outward unit normal to $\partial M$.

A classical result of Hersch et al. [16] from 1975 gives the upper bound $\sigma_k \cdot L(\partial D) \leq 2\pi k$ for all metrics on the disk $D$ and for all $k \geq 1$. In 2010 it was shown by Girouard and Polterovich [14] that this bound is sharp for all $k$ but is not attained by a smooth metric on the disk for $k = 2$. The bound and the result that it is attained by the standard round disk for $k = 1$ is a classical result of Weinstock [29]. In Sect. 2 of this paper we extend the result of [14] to show that the bound is not attained for a smooth metric for all $k \geq 2$. The proof is based on our earlier work [10] on uniqueness of free boundary minimal disks in higher dimensions together with the characterization of maximizing metrics given in [9].

In Sect. 3 we prove a local uniqueness theorem among free boundary minimal surfaces for the critical catenoid in $\mathbb{B}^n$ and for the critical Möbius band in $\mathbb{B}^n$. It is not known whether there are other embedded free boundary minimal annuli besides the critical catenoid in $\mathbb{B}^n$, but we are able to show that there are none which lie in a $C^2$ neighborhood of the critical catenoid except rotates of the critical catenoid. We prove an analogous result for the critical Möbius band. These results are consequences of the work of [11] where it is shown that the critical catenoid is the only free boundary minimal annulus with the coordinate functions being first Steklov eigenfunctions. It is also shown in [11] that the critical Möbius band is the only free boundary minimal Möbius band with coordinate functions being first Steklov eigenfunctions.

In Sect. 4 we consider the question of the degenerations of Riemannian manifolds under which the first $k$ Steklov eigenvalues are continuous. This question is important when one attempts to construct metrics which optimize an eigenvalue. We prove the following result which concerns the case in which a manifold degenerates into a disjoint union of manifolds.

**Theorem 1.1** Let $M_1, \ldots, M_s$ be compact $n$-dimensional Riemannian manifolds with nonempty boundary. Given $\epsilon > 0$, there exists a Riemannian manifold $M_\epsilon$, obtained by appropriately gluing $M_1, \ldots, M_s$ together along their boundaries, such that

$$\lim_{\epsilon \to 0} |\partial M_\epsilon| = |\partial (M_1 \sqcup \cdots \sqcup M_s)| \quad \text{and}$$

$$\lim_{\epsilon \to 0} \sigma_k(M_\epsilon) = \sigma_k(M_1 \sqcup \cdots \sqcup M_s)$$

for $k = 0, 1, 2, \ldots$.

The results of [14] (see also [18]) may be considered as a special case of gluing for surfaces. The issues are slightly different in the cases $n = 2$ and $n \geq 3$. In the case $n = 2$ we use essentially a rectangular neck of approximately equal side and vanishingly small side lengths, while for $n \geq 3$ we use a portion of a catenoidal hypersurface in order to avoid concentration of eigenfunctions on the neck region. We remark that any shape neck for which the Poincaré-type inequalities of Lemma 4.6 can be proven will work for our proof. This theorem may be considered as a Steklov version of certain results of Anné [2] who obtained general gluing results for Laplace eigenvalues of compact manifolds under degenerations. The methods used in [2] differ slightly from those used here though both rely on showing non-concentration of eigenfunctions on and near the singular region. It seems possible that there is a modified version of the methods of [2] for the Steklov case as well; a possibility which was suggested to us by the referee. In the Steklov case the issues of non-concentration both in the interior
and on the boundary are important, so some modifications of [2] are clearly necessary. Since
our approach is direct and self-contained, and we need the result for our applications, we
include our proof.

We also consider the result of interior gluings such as connected sums with small necks.
In this case we prove under quite weak conditions on the neck region the result.

**Theorem 1.2** Let \( M_1, \ldots, M_s \) be compact \( n \)-dimensional Riemannian manifolds with
nonempty boundary. Given \( \epsilon > 0 \) there exists a Riemannian manifold \( M_\epsilon \), obtained by appro-
priately gluing \( M_1, \ldots, M_s \) together along their interiors, such that \( \partial M_\epsilon = \partial (M_1 \sqcup \ldots \sqcup M_s) \)
and

\[
\lim_{\epsilon \to 0} \sigma_k(M_\epsilon) = \sigma_k(M_1 \sqcup \cdots \sqcup M_s)
\]

for \( k = 0, 1, 2, \ldots \).

The fact that the shape of the neck is unimportant in this theorem is consistent with the recent
results of Colbois et al. [4] which show that up to constants the Steklov eigenvalues depend
only on the geometry near the boundary of a manifold.

The combination of these results in the case \( n = 2 \), which is stated in Corollary 4.11,
yields the bounds stated for the supremum of the \( k \)-th Steklov eigenvalue of a surface in the
paper of Petrides [25].

Finally in Sect. 5 of this paper we explore the question of maximizing eigenvalues with
symmetry imposed on the competing metrics versus maximizing over all smooth metrics.
We consider this question in two specific cases of surfaces with \( S^1 \) symmetry group. The
first case is the annulus where one can pose the maximization question over \( S^1 \)-invariant
metrics or over all metrics. In the case of the annulus we showed in our earlier work [11] that
for \( k = 1 \) the global maximizer is \( S^1 \)-invariant, so these maxima are the same. For \( \sigma_k \) with
\( k \geq 2 \) we show that the supremum over all metrics is strictly larger than the supremum over
\( S^1 \)-invariant metrics. It was shown by Fan et al. [6] that for \( S^1 \)-invariant metrics all \( \sigma_k \) for
\( k \neq 2 \) are maximized by a smooth \( S^1 \)-invariant metric. In the case of the annulus we showed in our earlier work [11] that for \( k = 1 \) the maximizer over all metrics exists and is \( S^1 \)-invariant. We show here for \( k \geq 2 \) the supremum
of \( \sigma_k \) over all smooth metrics on the annulus is strictly larger than the supremum over
\( S^1 \)-invariant metrics.

2 Simply connected surfaces

In this section we show that if \( M \) is a simply connected surface with boundary, then for
\( k \geq 2 \), the supremum of the \( k \)-th nonzero normalized Steklov eigenvalue \( \sigma_k(g) L_g(\partial M) \) over
all smooth metrics on \( M \) is not achieved. There are two main ingredients in the proof. The
first is the following characterization of maximizing metrics.
Proposition 2.1 ( [9, Proposition 2.4]) If $M$ is a surface with boundary, and $g_0$ is a metric on $M$ with

$$
\sigma_k(g_0)L_{g_0}(\partial M) = \max_g \sigma_k(g)L_g(\partial M)
$$

where the max is over all smooth metrics on $M$. Then, rescaling the metric such that $\sigma_k(g_0) = 1$, there exist independent $k$-th eigenfunctions $u_1, \ldots, u_n$, for some $n \geq 2$, that give a proper conformal branched immersion $u = (u_1, \ldots, u_n) : M \to \mathbb{B}^n$ that is an isometry on $\partial M$; in particular, $u(M)$ is a free boundary minimal surface.

The second ingredient is the following minimal surface uniqueness theorem.

Theorem 2.2 ( [10, Theorem 2.1]) Let $u : D \to \mathbb{B}^n$ be a proper branched minimal immersion, such that $u(D)$ meets $\partial \mathbb{B}^n$ orthogonally. Then $u(D)$ is an equatorial plane disk.

We now state the theorem:

Theorem 2.3 Let $M$ be a simply connected surface with boundary. For $k \geq 1$, for any smooth metric $g$ on $M$,

$$
\sigma_k(g)L_g(\partial M) \leq 2\pi k.
$$

For $k = 1$, the equality is achieved if and only if $g$ is $\sigma$-homothetic to the Euclidean unit disk. For $k \geq 2$ the inequality is strict, and equality is achieved in the limit by a sequence of metrics degenerating to a union of $k$ touching Euclidean unit disks.

Proof The case $k = 1$ is due to Weinstock [29]. For $k \geq 2$ the upper bound $\sigma_k(g)L_g(\partial M) \leq 2\pi k$ is due to Hersch et al. [16]. Girouard and Polterovich [14] proved that this upper bound is sharp; precisely, they show that the upper bound is achieved in the limit by a sequence of metrics degenerating to a union of $k$ touching Euclidean unit disks. Moreover, for $k = 2$ Girouard and Polterovich [14] proved that the inequality is strict. We now show that the inequality is strict for all $k \geq 2$.

Suppose there exists a smooth metric $g$ such that $\sigma_k(g)L_g(\partial M) = 2\pi k$. Since $\sigma_k L$ is invariant under rescaling of the metric, without loss of generality, assume $\sigma_k(g) = 1$. Then by Proposition 2.1 there exist $k$-th eigenfunctions $u_1, \ldots, u_n$, for some $n \geq 2$, such that

$$
u := (u_1, \ldots, u_n) : M \to \mathbb{B}^n$$

is a proper conformal branched minimal immersion such that $\nu(M)$ meets $\partial \mathbb{B}^n$ orthogonally, and $g$ is the induced metric on $\partial M$. By Theorem 2.2, $\nu(M)$ is an equatorial plane disk. Thus, $g$ is $\sigma$-homothetic (see [11, Definition 2.1]) to the induced metric on the Euclidean unit disk $\mathbb{D}$, and so $\sigma_k(g)L_g(\partial M) = \sigma_k(\mathbb{D})L(\partial \mathbb{D})$. But $\sigma_k(\mathbb{D})L(\partial \mathbb{D}) < 2\pi k$, a contradiction. \hfill $\square$

3 Rigidity of the critical catenoid and Möbius band

The next natural case to consider after the disk is the annulus. In [11] the authors proved that there exists a smooth metric that maximizes the first nonzero normalized Steklov on the annulus. Moreover, the authors proved that any maximizing metric on the annulus is $\sigma$-homothetic (see [11, Definition 2.1]) to the induced metric on the ‘critical catenoid’. The critical catenoid is the unique portion of a suitably scaled catenoid which defines a free boundary surface in $\mathbb{B}^3$. 

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Theorem 3.1 ([11, Theorem 1.3]) For any metric on the annulus $M$ we have

$$\sigma_1 L \leq (\sigma_1 L)_{cc}$$

with equality if and only if $M$ is $\sigma$-homothetic to the critical catenoid.

For higher eigenvalues for the annulus there are upper bounds due to Karpukhin [17] (see also [15]),

$$\sigma_k (g) L_g (\partial M) \leq 2\pi (k + 1),$$

but it is an open question whether these are sharp upper bounds, and whether there exist maximizing metrics for the higher eigenvalues. For the disk, the nonexistence of metrics that maximize higher eigenvalues, Theorem 2.3, uses the minimal surface uniqueness theorem, Theorem 2.2. For the annulus, if there exists a metric that maximizes $\sigma_k L$, then Proposition 2.1 characterizes the maximizing metric as being $\sigma$-homothetic to the induced metric from a free boundary minimal immersion of the annulus into $B^n$ by $k$-th eigenfunctions, for some $n \geq 2$. Although the critical catenoid is the only known free boundary minimal annulus in $B^3$, there are many other known free boundary minimal annuli in $B^4$ [6,7]. The explicit characterization of the metric that maximizes $\sigma_1 L$ in Theorem 3.1 uses the following minimal surface uniqueness theorem that characterizes the critical catenoid as the only free boundary minimal immersion of the annulus into $B^n$ by first eigenfunctions.

Theorem 3.2 ([11, Theorem 1.2]) If $\Sigma$ is a free boundary minimal surface in $B^n$ which is homeomorphic to the annulus and such that the coordinate functions are first Steklov eigenfunctions, then $n = 3$ and $\Sigma$ is congruent to the critical catenoid.

A consequence of Theorem 3.2 is the following local rigidity result for the critical catenoid.

Theorem 3.3 Any free boundary minimal annulus in $B^n$ that is sufficiently $C^2$-close to the critical catenoid is a rotation of the critical catenoid.

Proof Let $\Sigma$ be the critical catenoid, and suppose that $\tilde{\Sigma}$ is a free boundary minimal annulus in $B^n$ that is $C^2$ close to $\Sigma$. We know that $\sigma_0 (\Sigma) = 0$, $\sigma_1 (\Sigma) = \sigma_2 (\Sigma) = \sigma_3 (\Sigma) = 1$, and $\sigma_4 (\Sigma) > 1$. Now $\sigma_0 (\tilde{\Sigma}) = 0$, and since $\tilde{\Sigma}$ is a free boundary minimal surface, the coordinate functions $x^1, \ldots, x^n$ in $B^n$ restricted to $\tilde{\Sigma}$ are Steklov eigenfunctions with eigenvalue 1. Note that the Steklov spectrum varies continuously if we take a $C^2$ perturbation of $\Sigma$, [9, Lemma 2.5]. Therefore, given $\epsilon > 0$, if $\tilde{\Sigma}$ is sufficiently $C^2$-close to $\Sigma$, then $|\sigma_k (\tilde{\Sigma}) - \sigma_k (\Sigma)| < \epsilon$. Choosing $\epsilon$ small, this implies that $n = 3$, and $\sigma_1 (\tilde{\Sigma}) = 1$. Therefore, by Theorem 3.2, $\tilde{\Sigma}$ is congruent to $\Sigma$, and hence is a rotation of $\Sigma$. □

We have a similar local rigidity result for the critical Möbius band. The critical Möbius band is an explicit free boundary minimal embedding of the Möbius band into $B^3$ by first Steklov eigenfunctions (see [11, Section 7]). In [11, Theorem 1.5] the authors proved that the induced metric on the critical Möbius band uniquely (up to $\sigma$-homothety) maximizes the first normalized Steklov eigenvalue among all smooth metrics on the Möbius band. As in the case of the annulus, the characterization of the maximizing metric uses a minimal surface uniqueness theorem, [11, Theorem 7.4], showing that the critical Möbius band is the unique free boundary minimal Möbius band in $B^n$ such that the coordinate functions are first Steklov eigenfunctions. Another consequence of this is the following local uniqueness theorem for the critical Möbius band:

Theorem 3.4 Any free boundary minimal Möbius band in $B^n$ that is sufficiently $C^2$-close to the critical Möbius band is a rotation of the critical Möbius band.

The proof is exactly analogous to the proof of Theorem 3.3.
4 Continuity of Steklov eigenvalues under degenerations

In this section we prove our results showing that the first $k$ Steklov eigenvalues are continuous under certain degenerations. The difficult case is that of degenerations along the boundary.

**Theorem 1.1** Let $M_1, \ldots, M_s$ be compact $n$-dimensional Riemannian manifolds with nonempty boundary. Given $\epsilon > 0$, there exists a Riemannian manifold $M_\epsilon$, obtained by appropriately gluing $M_1, \ldots, M_s$ together along their boundaries, such that

$$\lim_{\epsilon \to 0} |\partial M_\epsilon| = |\partial (M_1 \sqcup \cdots \sqcup M_s)|$$

and

$$\lim_{\epsilon \to 0} \sigma_k(M_\epsilon) = \sigma_k(M_1 \sqcup \cdots \sqcup M_s)$$

for $k = 0, 1, 2, \ldots$.

We also prove an analogous result in the case of interior degenerations in Theorem 1.2.

Our methods involve the addition of a neck with geometry chosen so that it does not contribute spectrally in the limit. Any neck for which the Poincaré-type inequalities of Lemma 4.6 suffices. We remark that there are similar results of this type for closed manifolds ([2], [3, Lemma 3.2]). The methods of [2] give a more precise result where the limiting spectrum consists of that of the component manifolds together with the eigenvalues of the neck with Dirichlet boundary condition. We do not pursue an analogue of this here.

4.1 Preliminaries

Here we collect various useful estimates on domains in a manifold with bounded geometry. This includes some extensions and refinements of results in [27, Section 2.2]. First, it is observed that bounded geometry implies the metric is uniformly equivalent to the Euclidean metric [5] in balls of fixed radius.

In this work we will need slight modifications of the standard Poincaré and Sobolev inequalities for functions in an annulus. We use the notation $B^+_r$ to denote the points of $B_r$ which lie in a half space, say $x_n \geq 0$. We let $A^+ = B^+_{r_0} \setminus B^+_{r_1}$ in $\mathbb{R}^n$. We let $\Gamma_r$ denote the portion of $\partial B^+_r$ on which $x_n = 0$.

We assume that we have an annulus $A = B_{r_0} \setminus B_{r_1}$ in $\mathbb{R}^n$ with a metric $g$ which is uniformly equivalent to the Euclidean metric; specifically for a positive constant $C_1$ and all $a \in \mathbb{R}^n$

$$C_1^{-1} \sum_{i=1}^n a_i^2 \leq \sum_{i,j=1}^n g^{ij} a_i a_j \leq C_1 \sum_{i=1}^n a_i^2.$$

Then the following estimates hold.

**Lemma 4.1** Suppose we have an annulus as above, and assume $n \geq 3$. For any smooth function $f$ on $A$ with $f = 0$ on $\partial B_{r_0}$, there is a constant depending only on $n$ and $C_1$ (independent of $r_0$ and $r_1$) such that,

$$\left( \int_A f^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}} \leq c \int_A |\nabla_g f|^2 \, dv.$$ 

For a proof, see [27, Lemma 2.4]. We also need the following version of a logarithmic cut-off function argument.
Lemma 4.2. Suppose $B_{r_0}$, a ball in $\mathbb{R}^n$, is equipped with a metric equivalent to the Euclidean metric. For any $\epsilon$, there are small $\rho, \rho_1$ with $\rho < \rho_1 \ll r_0$ and a smooth cut-off function $\zeta$, which is 0 for $x$ in $B_{r_0} \setminus B_{\rho_1}$ and 1 for $x$ in $B_{\rho}$, such that the following holds. For any smooth function $u$,

$$
\int_{B_{\rho_1} \setminus B_{\rho}} |\nabla \zeta|^2 u^2 \, dv \leq c \epsilon \left( \int_{B_{r_0} \setminus B_{\rho}} u^2 \, dv + \int_{B_{r_0}} |\nabla u|^2 \, dv \right)
$$

$$
\int_{B_{\rho_1} \setminus B_{\rho}} u^2 \, dv \leq c \epsilon \left( \int_{B_{r_0} \setminus B_{\rho}} u^2 \, dv + \int_{B_{r_0}} |\nabla u|^2 \, dv \right).
$$

Here $c$ is a constant depending on $r_0$ and bounds on the eigenvalues of the metric with respect to the Euclidean metric.

Proof. For $n \geq 3$ we can take $\rho_1 = \sqrt{\rho}$, and let $A = B_{\sqrt{\rho}} \setminus B_{\rho}$ and we set

$$
\zeta(r) = \frac{\log(r/\sqrt{\rho})}{\log(\sqrt{\rho})} \text{ for } \rho \leq r \leq \sqrt{\rho}.
$$

Note that the function $\zeta$ we have chosen is not smooth but only Lipschitz continuous. It is a standard argument to see that such a $\zeta$ can be approximated by smooth functions in the $W^{1,2}$ norm so that we can justify this choice. Also, it suffices to prove the first inequality because, for our choice of $\zeta$, $1 < |\nabla \zeta|$ on $A$.

Since $n \geq 3$ we can use the Hölder inequality to obtain

$$
\int_{A} |\nabla \zeta|^2 u^2 \, dv \leq \left( \int_{A} |\nabla \zeta|^n \, dv \right)^{\frac{2}{n}} \left( \int_{A} u^{2n} \, dv \right)^{\frac{n-2}{n}}.
$$

From the definition of $\zeta$ and the conditions on the metric on the annulus we have

$$
\int_{A} |\nabla \zeta|^n \, dv \leq c |\log(\rho)|^{-n} \int_{\rho}^{\sqrt{\rho}} r^{-1} \, dr \leq c |\log(\rho)|^{-1-\epsilon}.
$$

Thus for any $\epsilon > 0$, when $\rho$ is small enough we have

$$
\int_{B_{\sqrt{\rho}} \setminus B_{\rho}} |\nabla \zeta|^2 u^2 \, dv \leq \epsilon \left( \int_{B_{\sqrt{\rho}} \setminus B_{\rho}} u^{2n} \, dv \right)^{\frac{n-2}{n}}.
$$

Now if $\psi$ is a cut-off function, which is 1 on $B_{\sqrt{\rho}}$ and supported in $B_{r_0}$, then we have

$$
\left( \int_{B_{\sqrt{\rho}} \setminus B_{\rho}} u^{2n} \, dv \right)^{\frac{n-2}{n}} \leq \left( \int_{B_{r_0}} (\psi u)^{2n} \, dv \right)^{\frac{n-2}{n}} \leq c \int_{B_{r_0}} |\nabla (\psi u)|^2 \, dv.
$$

Here we have used the Sobolev inequality, Lemma 4.1, for functions vanishing on the outer boundary of the annulus $B_{r_0} \setminus B_{\rho}$. Since the gradient of $\psi$ is bounded we obtain

$$
\left( \int_{B_{\sqrt{\rho}} \setminus B_{\rho}} u^{2n} \, dv \right)^{\frac{n-2}{n}} \leq c \int_{B_{r_0}} |\nabla (\psi u)|^2 \, dv \leq c \left( \int_{B_{r_0} \setminus B_{\sqrt{\rho}}} u^2 \, dv + \int_{B_{r_0}} |\nabla u|^2 \, dv \right).
$$

Combining with our previous inequality we obtain,

$$
\int_{B_{\sqrt{\rho}} \setminus B_{\rho}} |\nabla \zeta|^2 u^2 \, dv \leq c \epsilon \left( \int_{B_{r_0} \setminus B_{\sqrt{\rho}}} u^2 \, dv + \int_{B_{r_0}} |\nabla u|^2 \, dv \right).
$$
For \( n = 2 \) we can obtain the conclusion in a slightly different way. We let \( t = \log(\log(1/r)) \), with \( t_0 = \log(\log(1/\rho)) \), and choose \( \rho_1 \) such that \( t_0/2 = \log(\log(1/\rho_1)) \).

We now choose \( \zeta \) to be a linear function of \( t \) which is 1 at \( t = t_0 \) and 0 at \( t = t_0/2 \). We then have

\[
\int_{B_{\rho_1} \setminus B_\rho} |\nabla \zeta|^2 u^2 \, dv = c t_0^{-2} \int_{B_{\rho_1} \setminus B_\rho} (r \log(1/r))^{-2} u^2 \, dv.
\]

We observe that since the metric is near Euclidean in an appropriate annulus \( B_{r_0} \setminus B_r \) where \( r_0 \) is a fixed radius, we may do the estimate in the Euclidean metric. In this case, the volume form \((|x| \log(1/|x|))^{-2} dx^1 dx^2 \) is that of the hyperbolic metric on the cylinder \( \mathbb{R} \times S^1 \) given by \( dt^2 + e^{-2t} d\theta^2 \) with coordinates \( t = \log(\log(1/|x|)) \) and the polar coordinate \( \theta \). The annulus now becomes the cylinder \([t_0/2, t_0] \times S^1 \).

Consider the eigenvalue problem with boundary conditions which are Dirichlet at \( t = \log \log(1/r_0) \) and Neumann at \( t = t_0 \). If \( g \) denotes the hyperbolic metric we have \( \Delta_g(t) = -1 \), and so if \( f \) is a function which is zero at \( t = \log \log(1/r_0) \) we have

\[
\int_{B_{r_0} \setminus B_\rho} f^2 \, d\mu_g = - \int_{B_{r_0} \setminus B_\rho} f^2 \Delta_g(t) \, d\mu_g \leq \int_{B_{r_0} \setminus B_\rho} \langle \nabla t, \nabla f \rangle \, d\mu_g,
\]

where we have used the fact that the boundary term on the inner boundary is nonpositive. Using the fact that \(|\nabla t| = 1\) together with the Schwarz inequality we obtain

\[
\int_{B_{r_0} \setminus B_\rho} f^2 \, d\mu_g \leq 2 \int_{B_{r_0} \setminus B_\rho} |f||\nabla f| \, d\mu_g.
\]

Using the Schwarz inequality again and the arithmetic mean - geometric mean inequality we obtain the Poincaré inequality

\[
\int_{B_{r_0} \setminus B_\rho} f^2 \, d\mu_g \leq 4 \int_{B_{r_0} \setminus B_\rho} |\nabla f|^2 \, dv,
\]

where we have used the conformal invariance of the Dirichlet integral.

Choosing \( \psi \) to be a cut-off function of \( t \) which is 0 for \( t \leq \log\log(1/r_0) \) and 1 for \( t \geq \log\log(1/r_0) + 1 \), we may apply the above Poincaré inequality to obtain,

\[
\int_{B_{\rho_1} \setminus B_\rho} (r \log(1/r))^{-2} u^2 \, dv \leq \int_{B_{r_0} \setminus B_\rho} (\psi u)^2 (r \log(1/r))^{-2} \, dv \leq 4 \int_{B_{r_0} \setminus B_\rho} |\nabla (\psi u)|^2 \, dv.
\]

We have chosen \( \psi \) so that it has bounded derivatives, so we obtain,

\[
\int_{B_{\rho_1} \setminus B_\rho} |\nabla \zeta|^2 u^2 \, dv \leq c t_0^{-2} \int_{B_{r_0} \setminus B_\rho} (u^2 + |\nabla u|^2) \, dv.
\]

Since \( t_0 \) is as large as we like when \( \rho \) is chosen small, this completes the proof of the first inequality. The second follows because \(|\nabla \zeta|^2\) is large on \( A \).

\[\square\]

We will also need an analogous result for the case of half balls.

**Lemma 4.3**  Suppose \( B_{r_0} \), a ball in \( \mathbb{R}^n \), is equipped with a metric equivalent to the Euclidean metric. For any \( \epsilon \), there are small \( \rho, \rho_1 \) with \( \rho < \rho_1 \ll r_0 \) and a smooth cut off function \( \zeta \), which is 0 for \( x \) in \( B_{r_0} \setminus B_{\rho} \) and 1 for \( x \) in \( B_{\rho} \), such that the following holds. For any smooth function \( u \) defined on \( B_{r_0}^+ \),

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\[ \int_{B_{r_0}^+} |\nabla^2 u|^2 \, dv \leq c \epsilon \left( \int_{B_{r_0}^+} u^2 \, dv + \int_{B_{r_0}^+} |\nabla u|^2 \, dv \right) \]
\[ \int_{\Gamma_{r_1}} u^2 \, da + \int_{B_{r_1}^+} u^2 \, dv \leq c \epsilon \int_{B_{r_0}^+} (u^2 + |\nabla u|^2) \, dv \]

Here \( c \) is a constant depending on \( r_0 \) and bounds on the eigenvalues of the metric with respect to the Euclidean metric.

**Proof** The first inequality and the following part of the second
\[ \int_{B_{r_0}^+} u^2 \, dv \leq c \epsilon \int_{B_{r_0}^+} (u^2 + |\nabla u|^2) \, dv \]
follow by extending \( u \) to \( B_{r_0} \) by even reflection and applying the previous lemma.

It remains to prove
\[ \int_{\Gamma_{r_1}} u^2 \, da \leq c \epsilon \int_{B_{r_0}^+} (u^2 + |\nabla u|^2) \, dv. \]

We can prove this by using the standard result that on a half ball \( B_{r_0}^+ \) we have the bound for any \( p \) with \( 2 \leq p \leq \frac{2(n-1)}{n-2} \) for \( n \geq 3 \) and \( 2 \leq p < \infty \) for \( n = 2 \)
\[ \left( \int_{\Gamma_{r_0}} u^p \, da \right)^{\frac{2}{p}} \leq c \int_{B_{r_0}^+} (u^2 + |\nabla u|^2) \, dv \]
where \( c \) depends on \( p \) and \( r_0 \). We can then fix \( p > 2 \) depending on \( n \) and use the Hölder inequality to obtain
\[ \int_{\Gamma_{r_1}} u^2 \, da \leq c \rho_1^{(n-1)(p-2)/p} \left( \int_{\Gamma_{r_0}} u^p \, da \right)^{\frac{2}{p}} \leq c \rho_1^{(n-1)(p-2)/p} \int_{B_{r_0}^+} (u^2 + |\nabla u|^2) \, dv. \]

Since \( \rho_1 \) is as small as we wish, this implies the desired bound.

Finally, we will need the following well known result which provides a bound on the \( L^2 \) norm for functions on a manifold with uniform geometry, in terms of the \( L^2 \) norm of the function on the boundary and its energy in the interior.

**Lemma 4.4** For any \( W^{1,2} \) function \( u \) on an \( n \)-dimensional Riemannian manifold \( (M, g) \) with boundary,
\[ \int_M u^2 \, dv \leq C \left( \int_{\partial M} u^2 \, da + \int_M |\nabla u|^2 \, dv \right) \]
where \( C \) is a constant depending on \( M \).

### 4.2 Gluing construction and neck estimate

Let \( (M_1, g_1) \) and \( (M_2, g_2) \) be compact \( n \)-dimensional Riemannian manifolds with nonempty boundary. We glue \( M_1 \) and \( M_2 \) together along their boundaries as follows. Let \( p_1 \in \partial M_1 \) and \( p_2 \in \partial M_2 \). Choose \( r_0 > 0 \) such that the metrics \( g_1, g_2 \) are uniformly equivalent to the Euclidean metric in balls of radius \( r_0 \). To prove the estimates in this section it then suffices
to prove them assuming the metrics are Euclidean on the geodesics balls $B^i_{r_0}(p_i)$ of radius $r_0$ in $M_i$ centered at $p_i$ for $i = 1, 2,$ and that is what we shall do.

For $n \geq 3$, consider a catenoid in $\mathbb{R}^n$; that is, a complete minimal hypersurface of revolution that is not a hyperplane. A catenoid is parametrized by an embedding

$$F : I \times S^{n-2} \to \mathbb{R}^n$$

with $F(t, \omega) = (\phi(t)\omega, t)$, where $\phi : I \to \mathbb{R}$ is a solution, defined on a maximal interval $I = (-a(n), a(n))$, of an ODE corresponding to the minimal surface equation. There exists $l := l(n) < a(n)$, such that the portion of the catenoid corresponding to $-l \leq t \leq l$ is volume minimizing [26, Corollary 3]. Given $\rho > 0$, consider a rescaled portion of the catenoid given by

$$\tilde{F} : [-l, l] \times S^{n-2} \to \mathbb{R}^n$$

with

$$\tilde{F}(t, \omega) = RF(t, \omega)$$

where $R = \rho/\phi(l)$. Then consider the solid catenoidal tube

$$T_\rho := \tilde{F}([-l, l] \times B^{n-1}).$$

Note that the ends of $T_\rho$, corresponding to $t = \pm l$, are Euclidean balls of radius $\rho$, and the catenoid portion $\tilde{F}([-l, l] \times S^{n-2})$ of the boundary of $T_\rho$ is a volume minimizing hypersurface. For $n = 2$, let $T_\rho$ be a Euclidean square of side length $2\rho$. In this case note that the boundary portion of $T_\rho$ consisting of two aligned parallel line segments of length $2\rho$ that are a distance $2\rho$ apart is length minimizing with respect to its boundary points.

We now let $M_\rho$ be the Lipschitz Riemannian manifold obtained by gluing $(M_1, g_1)$ and $(M_2, g_2)$ together along their boundaries using the tube $T_\rho$. Specifically, $M_\rho$ is obtained by identifying one end of $T_\rho$ with $\partial M_1 \cap B^1(p_1)$, and the other end of $T_\rho$ with $\partial M_2 \cap B^2(p_2)$. Given $0 < \rho'' < \rho' < r_0$, let

$$N_{\rho', \rho''} := B^1(p_1) \cup T_{\rho''} \cup B^2(p_2),$$

with the identifications as above. $N_{\rho', \rho''}$ is a Euclidean domain with piecewise smooth boundary.

In the following we fix $\rho$ and $r_0$ with $0 < \rho < \rho_0 < \rho_1 < r_0$, where $\rho_1 := \rho_1(\rho_0)$ is as in the proof of Lemma 4.2, and let $N = N_{\rho_1, \rho}$. An important ingredient in the proof of Theorem 1.1 is that for a sequence of eigenfunctions, the $L^2$ norm on the boundary $\partial M_\rho$ doesn’t concentrate on the boundary of the neck $\partial N \cap \partial M_\rho$ as $\rho \to 0$. In order to prove this, we will need the following two lemmas. The first lemma uses in a key way the geometry of the neck region $N$.

**Lemma 4.5** Let $f : N \to \mathbb{R}$ be a smooth function with $f \geq 0$ on $N$ and $f = 0$ on $\partial B^1_{\rho_1}(p_1) \setminus \partial M_i$ for $i = 1, 2$. Then

$$\text{Vol}(\{x \in \partial N : f(x) > t\}) \leq \text{Vol}(\{x \in N : f(x) = t\}).$$

**Proof** Let $\Omega_t = \{x \in N : f(x) > t\}$ for $t > 0$, and let

$$\Gamma_t = \{x \in \partial N : f(x) > t\}
\Lambda_t = \{x \in N : f(x) = t\} = \partial \Omega_t \setminus \partial N.$$
We first consider the portion of $\Gamma_i$ that lies on $\partial M_i$, $i = 1, 2$. Recall that $B^i_{\rho_1}(p_i)$ is a Euclidean half ball, and observe that the orthogonal projection maps $P_i : B^i_{\rho_1}(p_i) \to B^i_{\rho_1}(p_i) \cap \partial M_i$ for $i = 1, 2$ reduce volumes of hypersurfaces. Also, since $f = 0$ on $\partial B^i_{\rho_1}(p_i) \setminus \partial M_i$, it follows that $\overline{\Omega}_i \cap (\partial B^i_{\rho_1}(p_i) \setminus \partial M_i) = \emptyset$. This, together with the fact that $(B^i_{\rho_1}(p_i) \setminus B^i_{\rho_1}(p_i)) \cap \partial M_i$ lies in a plane, implies that the line orthogonal to $\partial M_i$ through any point $x \in \Gamma_i \cap \partial M_i$ must intersect $\Lambda_i \cap B^i_{\rho_1}(p_i)$. Therefore,

$$\text{Vol}(\Gamma_i \cap \partial M_i) \leq \text{Vol} \left( P_i \left( \Lambda_i \cap (B^i_{\rho_1}(p_i) \setminus C) \right) \right) \leq \text{Vol} \left( (\Lambda_i \cap (B^i_{\rho_1}(p_i) \setminus C) \right) \quad (4.1)$$

where $C$ is the solid cylinder of radius $\rho$ with axis through $p_i$ orthogonal to $\partial M_i$.

We next consider the remaining portion of $\Gamma_i$, which lies on $\partial T_\rho \cap \partial N$. Let $\Omega_i' = \Omega_i \cap T_\rho$. Then clearly, $\text{Vol}(\partial \Omega_i' \setminus \partial N) \leq \text{Vol}((\partial \Omega_i \setminus \partial N) \cap C)$, since $\partial \Omega_i' \setminus \partial N \subset C$ and the portions of $\partial \Omega_i' \setminus \partial N$ that differ from $(\partial \Omega_i \setminus \partial N) \cap C$ consist of subsets of $B^i_{\rho}(p_i) \cap \partial M_i$ that are contained in the orthogonal projection of $(\partial \Omega_i \setminus \partial N) \cap C$ onto the flat ball $B^i_{\rho}(p_i) \cap \partial M_i$. Furthermore, $\partial \Omega_i' \cap (\partial T_\rho \cap \partial N) = \partial \Omega_i \cap (\partial T_\rho \cap \partial N)$. Since $\partial T_\rho \cap \partial N$ is volume minimizing and $\Gamma_i \cap (\partial T_\rho \cap \partial N)$ and a union of connected components of $\partial \Omega_i' \setminus \partial N$ have the same boundary,

$$\text{Vol}(\Gamma_i \cap (\partial T_\rho \cap \partial N)) \leq \text{Vol}(\partial \Omega_i' \setminus \partial N) \leq \text{Vol}((\partial \Omega_i \setminus \partial N) \cap C) = \text{Vol}(\Lambda_i \cap C). \quad (4.2)$$

Combining (4.1) and (4.2), we obtain the desired volume comparison,

$$\text{Vol}([x \in \partial N : f(x) > t]) = \text{Vol}(\Gamma_i) \leq \text{Vol}(\Lambda_i) = \text{Vol}([x \in \partial N : f(x) = t]).$$

□

As a consequence of the previous lemma, we have the following lemma which gives control on Dirichlet and Dirichlet-Steklov eigenvalues of the neck $N = N_{\rho_1, \rho}$. This will imply that functions with bounded energy cannot concentrate on the neck.

**Lemma 4.6** Let $w$ be a smooth function on $N$ with $w = 0$ on $\partial B^i_{\rho_1}(p_i) \setminus \partial M_i$ for $i = 1, 2$. Then

$$\int_N w^2 \, dv \leq C(n) \rho_1^2 \int_N |\nabla w|^2 \, dv$$

$$\int_{\partial N} w^2 \, da \leq C(n) \rho_1 \int_N |\nabla w|^2 \, dv.$$

**Proof** In what follows, $C(n)$ may increase from one inequality to the next, but its dependence will always be only on $n$. Let $f : N \to \mathbb{R}$ be a smooth function with $f \geq 0$ on $N$ and $f = 0$ on $\partial B^i_{\rho_1}(p_i) \setminus \partial M_i$ for $i = 1, 2$. Let $\Omega_i = \{x \in N : f(x) > t\}$, and observe that

$$\partial \Omega_i \subset \{x \in \partial N : f(x) > t\} \cup \{x \in \overline{N} : f(x) = t\}.$$

By the isoperimetric inequality and Lemma 4.5,

$$\text{Vol}(\{x \in N : f(x) > t\}) = \text{Vol}(\partial \Omega_i) \leq C(n) \text{Vol}(\partial \Omega_i)^\frac{n}{n - 1}$$

$$= C(n) 2^\frac{n}{n - 1} \text{Vol}(\{x \in \overline{N} : f(x) = t\})^\frac{n}{n - 1}. $$

This implies the Sobolev inequality (see for example [28, page 90]),

$$\left( \int_N f^n \frac{\pi}{n - 1} \right)^\frac{n - 1}{n} \, dv \leq C(n) \int_N |\nabla f|^2 \, dv.$$
Then using Hölder’s inequality we obtain

\[
\int_N f \, dv \leq C(n) \text{Vol}(N)^{\frac{1}{2}} \int_N |\nabla f| \, dv \leq C(n) \rho_1 \int_N |\nabla f| \, dv.
\]

Applying this to the function \( f = w^2 \), we obtain

\[
\int_N w^2 \, dv \leq C(n) \rho_1 \int_N |\nabla w^2| \, dv = C(n) \rho_1 \int_N 2|w| |\nabla w| \, dv
\]

\[
\leq 2C(n) \rho_1 \left( \int_N w^2 \, dv \right)^{\frac{1}{2}} \left( \int_N |\nabla w|^2 \, dv \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \int_N w^2 \, dv + 2C(n)^2 \rho_1^2 \int_N |\nabla w|^2 \, dv.
\]

Therefore,

\[
\int_N w^2 \, dv \leq C(n) \rho_1^2 \int_N |\nabla w|^2 \, dv.
\] (4.3)

Similarly, by Lemma 4.5 and the co-area formula, we have

\[
\int_{\partial N} f \, da = \int_0^\infty \text{Vol}(\{x \in \partial N : f > t\}) \, dt
\]

\[
\leq \int_0^\infty \text{Vol}(\{x \in N : f = t\}) \, dt
\]

\[
= \int_N |\nabla f| \, dv.
\]

Applying this to the function \( f = w^2 \), we obtain

\[
\int_{\partial N} w^2 \, da \leq \int_N |\nabla w|^2 \, dv \leq \frac{1}{\rho_1} \int_N w^2 \, dv + \rho_1 \int_N |\nabla w|^2 \, dv \leq C(n) \rho_1 \int_N |\nabla w|^2 \, dv.
\]

where we have used (4.3) in the third inequality.

The following estimate on the neck region will be important in the proof of Theorem 1.1.

**Proposition 4.7** For any \( \epsilon > 0 \), there is a small \( \rho_0 \ll r_0 \), such that for any smooth function \( u \) on \( M_\rho \) for \( \rho \leq \rho_0 \),

\[
\int_{\partial N_{\rho_0} \cap \partial M_\rho} u^2 \, dv \leq c \epsilon \left( \int_{N_{\rho_0}\rho} |\nabla u|^2 \, dv + \int_{\bigcup_{i=1}^L B_r(p_i) \setminus B_{\rho_0}(p_i)} u^2 \, dv \right)
\]

\[
\int_{N_{\rho_0}\rho} u^2 \, dv \leq c \epsilon \left( \int_{N_{\rho_0}\rho} |\nabla u|^2 \, dv + \int_{\bigcup_{i=1}^L B_r(p_i) \setminus B_{\rho_0}(p_i)} u^2 \, dv \right).
\]
Proof Let \( w = \zeta u \) where \( \zeta \) is the smooth cut-off function which is 1 on \( N_{\rho_0, \rho} \) and 0 on \( \bigcup_{i=1}^2 M_i \setminus B^i_{\rho_1}(p_i) \), where \( \rho_1 = \rho_1(\rho_0) \), from Lemma 4.3. By Lemma 4.6,

\[
\int_{\partial N_{\rho_1, \rho} \cap \partial M_{\rho}} u^2 \, da \leq \int_{\partial N_{\rho_1, \rho}} (\zeta u)^2 \, da \\
\leq \epsilon \int_{N_{\rho_1, \rho}} |\nabla (\zeta u)|^2 \, dv \\
\leq 2\epsilon \int_{N_{\rho_1, \rho}} |\nabla u|^2 \, dv + 2\epsilon \int_{\bigcup_{i=1}^2 B^i_{\rho_1}(p_i) \setminus B^i_{\rho_0}(p_i)} |\nabla \zeta|^2 u^2 \, dv \\
\leq c\epsilon \left( \int_{N_{\rho_0, \rho}} |\nabla u|^2 \, dv + \int_{\bigcup_{i=1}^2 B^i_{\rho_0}(p_i) \setminus B^i_{\rho_0}(p_i)} u^2 \, dv \right)
\]

where the last inequality follows from Lemma 4.3. The proof of the second inequality is analogous. \( \square \)

4.3 Proof of continuity of Steklov eigenvalues under certain degenerations

In this section we give the proof of Theorem 1.1 and related results. Using Proposition 4.7, which implies that for a sequence of eigenfunctions the \( L^2 \) norm on the boundary of \( M_{\rho} \) doesn’t concentrate on the neck as \( \rho \to 0 \), the proof of the gluing theorem is similar to the proof of Proposition 4.1 of [12].

Proof of Theorem 1.1 We will prove the result for \( s = 2 \), although the same argument works for gluing any number \( s \geq 2 \) of manifolds. Let \( M_{\rho} \) be the Lipschitz Riemannian manifold defined in Sect. 4.2. First we locally smooth the corners of \( M_{\rho} \). Specifically, there exists a bi-Lipschitz map \( F : M_{\rho} \to \hat{M}_{\rho} \), where \( \hat{M}_{\rho} \) is a smooth Riemannian manifold, such that \( F \) and \( F^{-1} \) have bounded Lipschitz constant independent of \( \rho \). Note that the estimates of Lemma 4.6 and Proposition 4.7 carry over to \( \hat{M}_{\rho} \) under the bi-Lipschitz equivalence, since \( F \) and \( F^{-1} \) have bounded Lipschitz constant independent of \( \rho \). For notational simplicity we will write \( M_{\rho} \), instead of \( \hat{M}_{\rho} \), for the smoothed manifold.

Let \( 0 = \sigma_0(M_{\rho}) \leq \sigma_1(M_{\rho}) \leq \sigma_2(M_{\rho}) \leq \cdots \) be the Steklov eigenvalues of \( M_{\rho} \) and let \( u^{(0)}_{\rho} , u^{(1)}_{\rho} , u^{(2)}_{\rho} , \ldots \) be orthonormal eigenfunctions; i.e. \( \| u^{(k)}_{\rho} \|_{L^2(\partial M_{\rho})} = 1 \),

\[
\int_{\partial M_{\rho}} u^{(k)}_{\rho} u^{(l)}_{\rho} \, da = 0 \quad \text{for} \ k \neq l
\]

and

\[
\begin{cases}
\Delta u^{(k)}_{\rho} = 0 & \text{on } M_{\rho} \\
\frac{\partial u^{(k)}_{\rho}}{\partial N_{\rho}} = \sigma_k(M_{\rho}) u^{(k)}_{\rho} & \text{on } \partial M_{\rho}.
\end{cases}
\]

We first show that each \( \sigma_k(M_{\rho}) \) is bounded from above by a constant \( \Lambda_k \) independent of \( \rho \) for \( \rho \) small. To see this we use the variational characterization of \( \sigma_k \)

\[
\sigma_k(M_{\rho}) = \inf_{E} \left\{ \sup_{f \in E, f \neq 0} \frac{\int_{M_{\rho}} |\nabla \tilde{f}|^2}{\int_{\partial M_{\rho}} \tilde{f}^2} : E \text{ of } L^2(\partial M_{\rho}) \right\}
\]

where the infimum is taken over all \( (k + 1) \)-dimensional subspaces \( E \) of \( L^2(\partial M_{\rho}) \), and \( \tilde{f} \) denotes the harmonic extension of \( f \) to \( M_{\rho} \). Thus to get an upper bound we need only exhibit
$k + 1$ linearly independent functions having bounded Rayleigh quotient. We can do this by choosing $k + 1$ fixed such functions which are supported away from the neck region $N_\rho$ and so are valid test functions for any small $\rho$.

Since $u_\rho^{(k)}$ is a Steklov eigenfunction of $M_\rho$ with eigenvalue $\sigma_k(M_\rho)$,

$$
\int_{M_\rho} |\nabla u_\rho^{(k)}|^2 \, dv = \sigma_k(M_\rho) \int_{\partial M_\rho} (u_\rho^{(k)})^2 \, da = \sigma_k(M_\rho) \leq \Lambda_k.
$$

Fix $\rho_0 < r_0$ such that the conclusion of Proposition 4.7 holds for some fixed $\epsilon > 0$. Let $K = (M_1 \setminus B_{\rho_0}^1(p_1)) \cup (M_2 \setminus B_{\rho_0}^2(p_2))$. By Lemma 4.4,

$$
\int_K (u_\rho^{(k)})^2 \, dv \leq C \left( \int_K |\nabla u_\rho^{(k)}|^2 \, dv + \int_{\partial M_\rho \cap K} (u_\rho^{(k)})^2 \, da \right) \leq C \cdot (\Lambda_k + 1) \quad (4.4)
$$

for all $\rho$, where $C = C(K)$. This together with Proposition 4.7 implies that we have a uniform bound (independent of $\rho$) on the $L^2$ norm of $u_\rho^{(k)}$ on $M_\rho$ for all $\rho \leq \rho_0$. Hence, there exists $C > 0$ independent of $\rho$ such that for all sufficiently small $\rho$,

$$
\|u_\rho^{(k)}\|_{W^{1,2}(M_\rho)} \leq C(k, M_1, M_2). \quad (4.5)
$$

Elliptic boundary estimates ([13, Theorem 6.30]) give uniform bounds

$$
\|u_\rho^{(k)}\|_{C^{2,\alpha}(K)} \leq C\|u_\rho^{(k)}\|_{C^0(K)}
$$

for any compact subset $K$ of $(M_1 \setminus \{p_1\}) \cup (M_2 \setminus \{p_2\})$ for all sufficiently small $\rho$, where $C = C(k, \alpha, \Lambda_k, M_1, M_2)$. By Sobolev embedding and interpolation inequalities ([1, Theorem 5.2], [13, (7.10)]),

$$
\|u_\rho^{(k)}\|_{C^0(K)} \leq C \left( \epsilon\|u_\rho^{(k)}\|_{C^2(K)} + \epsilon^{-\mu}\|u_\rho^{(k)}\|_{L^2(K)} \right)
$$

where $\epsilon > 0$ can be taken arbitrarily small, $\mu > 0$ depends on $n$, and $C$ depends on $M_1, M_2$. Hence $\|u_\rho^{(k)}\|_{C^{2,\alpha}(K)} \leq C$ with $C$ independent of $\rho$. By the Arzela-Ascoli theorem and a diagonal argument, there exists a sequence $\rho_i \to 0$ such that for all $k$, $u_\rho^{(k)}$ converges in $C^2(K)$ on compact subsets $K \subset (M_1 \setminus \{p_1\}) \cup (M_2 \setminus \{p_2\})$ to a harmonic function $u^{(k)}$ on $(M_1, g_1) \cup (M_2, g_2)$, satisfying

$$
\frac{\partial u^{(k)}}{\partial v} = \sigma_k u^{(k)} \quad \text{on} \quad (\partial M_1 \setminus \{p_1\}) \cup (\partial M_2 \setminus \{p_2\}),
$$

with $\sigma_k = \lim_{i \to \infty} \sigma_k(M_{\rho_i})$. By standard arguments, $u^{(k)}$ extends to a Steklov eigenfunction on $M_1 \cup M_2$ with eigenvalue $\sigma_k$ (see for example [12, Proof of Proposition 4.1]).

Now observe that $\{u^{(k)}\}_{k=1}^\infty$ are $L^2$-orthonormal on $\partial (M_1 \cup M_2)$. Let $\epsilon > 0$. By (4.5) and Proposition 4.7 there exists $\rho_0 \ll r_0$ such that $\int_{\partial N_{\rho_0}} u_\rho^{(k)} u_\rho^{(l)} \, da < \epsilon/2$ for all $\rho_i < \rho_0$. We may further assume that $\rho_0$ is chosen sufficiently small so that

$$
\left| \int_{\partial M_1 \cup \partial M_2} u_\rho^{(k)} u_\rho^{(l)} \, da - \int_{\bigcup_{j=1}^2 \partial M_j \setminus \partial M_{\rho_0}^j} u_\rho^{(k)} u_\rho^{(l)} \, da \right| < \frac{\epsilon}{4}
$$
Using this, and since \( u_{\rho_i}^{(k)} \) converges to \( u^{(k)} \) in \( C^2 \) on \( (M_1 \setminus B^1_{\rho_0}(p_1)) \cup (M_2 \setminus B^2_{\rho_0}(p_2)) \), if \( i \) is sufficiently large then
\[
\left| \int_{\partial M_1 \cup \partial M_2} u^{(k)} u^{(l)} \, d\alpha - \delta_{kl} \right| \leq \left| \int_{\bigcup_{j=1}^2 \partial M_j \setminus B^j_{\rho_0}(p_j)} u_{\rho_i}^{(k)} u_{\rho_i}^{(l)} \, d\alpha - \delta_{kl} \right| + \frac{\epsilon}{2} = \left| \int_{\partial N_{\rho_0} \cap \partial M_{\rho_i}} u_{\rho_i}^{(k)} u_{\rho_i}^{(l)} \right| + \frac{\epsilon}{2} < \epsilon
\]
where the equality follows since \( \{u_{\rho_i}^{(k)}\}_{k=1}^\infty \) are \( L^2 \)-orthonormal on \( \partial M_{\rho_i} \). Since this holds for any \( \epsilon > 0 \), the result follows. Here the domains are understood to be the corresponding domains under the bi-Lipschitz map \( F \).

Finally, we show that \( u^{(k)} \) is a \( k \)-th eigenfunction of \( M_1 \cup M_2 \); i.e. \( \sigma_k = \sigma_k(M_1 \cup M_2) \). We prove this by induction on \( k \). First, since \( \sigma_0(M_{\rho}) = 0 \), we have that \( \sigma_0 = \lim_{\rho \to 0} \sigma_0(M_{\rho}) = 0 \), and so \( \sigma_0 = \sigma_0(M_1 \cup M_2) \). Now suppose \( \sigma_l = \sigma_l(M_1 \cup M_2) \) for \( l = 1, \ldots, k-1 \), where \( k \geq 1 \). We will show that \( \sigma_k = \sigma_k(M_1 \cup M_2) \). It follows from (4.6) that \( \sigma_k \geq \sigma_k(M_1 \cup M_2) \).

Let \( w \) be a \( k \)-th eigenfunction of \( M_1 \cap M_2 \) with \( \|w\|_{L^2(\partial(M_1 \cup M_2))} = 1 \), and let
\[
w_{\rho} = \varphi_{\rho} w - \sum_{l=1}^{k-1} \left( \int_{\partial M_{\rho}} (\varphi_{\rho} w) u_{\rho}^{(l)} \, d\alpha \right) u_{\rho}^{(l)}
\]
where \( \varphi_{\rho} \) is the logarithmic cut-off function that is equal to 1 on \( M_j \setminus B^j_{\sqrt{\rho}}(p_j) \), equal to zero on \( B^j_{\rho}(p_j) \) and is given by
\[
\varphi_{\rho}(r) = \log r - \log \rho - \log \sqrt{\rho} \quad \text{for} \quad \rho \leq r \leq \sqrt{\rho}
\]
where \( r \) is the radial distance from \( p_j \), for \( j = 1, 2 \). Note that
\[
\int_{M_1 \cup M_2} |\nabla \varphi_{\rho}|^2 \, dv = \frac{C(n)}{(\log \sqrt{\rho})^2} \int_{\rho}^{\sqrt{\rho}} r^{n-3} \, dr = C(n) \epsilon_n(\rho) \to 0 \quad \text{as} \quad \rho \to 0
\]
where \( \epsilon_2(\rho) = 1/(\log(1/\sqrt{\rho})) \) and \( \epsilon_n(\rho) = \rho^{n-2}/((n-2)(\log \sqrt{\rho})^2) \) for \( n \geq 3 \).

We may use \( w_{\delta} \) as a test function in the variational characterization of \( \sigma_k(M_{\rho}) \). First note that
\[
\int_{\partial M_{\rho}} w_{\rho}^2 \, d\alpha = \int_{\partial M_{\rho}} (\varphi_{\rho} w)^2 \, d\alpha - \sum_{l=1}^{k-1} \left( \int_{\partial M_{\rho}} (\varphi_{\rho} w) u_{\rho}^{(l)} \, d\alpha \right)^2.
\]
But
\[
\lim_{i \to \infty} \int_{\partial M_{\rho_i}} (\varphi_{\rho_i} w) u_{\rho_i}^{(l)} \, d\alpha = \int_{\partial(M_1 \cup M_2)} w u^{(l)} \, d\alpha = 0,
\]
using an argument as in (4.6), where the last equality follows since \( w \) is a \( k \)-th eigenfunction of \( M_1 \cup M_2 \). Therefore,
\[
\lim_{i \to \infty} \int_{\partial M_{\rho_i}} w_{\rho_i}^2 \, d\alpha = \lim_{i \to \infty} \int_{\partial M_{\rho_i}} (\varphi_{\rho_i} w)^2 \, d\alpha = \int_{\partial(M_1 \cup M_2)} w^2 \, d\alpha.
\]
On the other hand,

\[ \int_{M_\rho} |\nabla (\varphi \rho w)|^2 \, dv \leq \int_{M_\rho} \varphi^2 |\nabla w|^2 \, dv + C \int_{M_\rho} |\nabla \varphi|^2 \, dv \xrightarrow{\rho \to 0} \int_{M_1 \sqcup M_2} |\nabla w|^2 \, dv \]

using (4.7), where the constant \( C \) depends on a pointwise upper bound on \( w \) and \( |\nabla w| \). Using this together with (4.5) and (4.8) we deduce that

\[
\lim_{\rho \to 0} \int_{M_\rho} \varphi \rho |\nabla \rho w|^2 \, dv \to 0
\]

\[
\int_{M_1 \sqcup M_2} |\nabla w|^2 \, dv.
\]

Combining these estimates, we have

\[
\sigma_k = \lim_{i \to \infty} \sigma_k(M_{\rho_i}) \leq \lim_{i \to \infty} \frac{\int_{M_{\rho_i}} |\nabla w_{\rho_i}|^2 \, dv}{\int_{\partial M_{\rho_i}} w_{\rho_i}^2 \, da} \leq \frac{\int_{M_1 \sqcup M_2} |\nabla w|^2 \, dv}{\int_{\partial(M_1 \sqcup M_2)} w^2 \, da} = \sigma_k(M_1 \sqcup M_2).
\]

Therefore,

\[
\lim_{i \to \infty} \sigma_k(M_{\rho_i}) = \sigma_k(M_1 \sqcup \cdots \sqcup M_2).
\]

Clearly, \( \lim_{\rho \to 0} |\partial M_\rho| = |\partial(M_1 \sqcup M_2)|. \)

We remark that the same argument can be used to glue a single manifold to itself along its boundary.

**Theorem 4.8** Let \( M \) be an \( n \)-dimensional Riemannian manifold with nonempty boundary. Given any \( \epsilon > 0 \) there exists a manifold \( M_\epsilon \) obtained by gluing \( M \) to itself along its boundary, along neighborhoods of distinct boundary points, such that

\[
\lim_{\epsilon \to 0} |\partial M_\epsilon| = |\partial M| \quad \text{and} \quad \lim_{\epsilon \to 0} \sigma_k(M_\epsilon) = \sigma_k(M)
\]

for \( k = 0, 1, 2, \ldots \).

Using similar methods, we obtain an analogous result showing that the first \( k \) Steklov eigenvalues are continuous under certain degenerations along the interior rather than the boundary.

**Theorem 1.2** Let \( M_1, \ldots, M_s \) be compact \( n \)-dimensional Riemannian manifolds with nonempty boundary. Given \( \epsilon > 0 \) there exists a Riemannian manifold \( M_\epsilon \), obtained by appropriately gluing \( M_1, \ldots, M_s \) together along their interiors, such that

\[
\partial M_\epsilon = \partial(M_1 \sqcup \cdots \sqcup M_s)
\]

and

\[
\lim_{\epsilon \to 0} \sigma_k(M_\epsilon) = \sigma_k(M_1 \sqcup \cdots \sqcup M_s)
\]

for \( k = 0, 1, 2, \ldots \).

The proof is analogous to the proof of Theorem 1.1, yet significantly easier, since the delicate neck estimates of Sects. 4.1 and 4.2 are not needed in this case.

**Proof** We will prove the result for \( s = 2 \), although the same argument works for gluing any number \( s \geq 2 \) of manifolds. Let \( (M_1, g_1) \) and \( (M_2, g_2) \) be compact \( n \)-dimensional Riemannian manifolds with nonempty boundary, and let \( p_1 \in \text{Int} M_1 \) and \( p_2 \in \text{Int} M_2 \). Given \( \rho > 0 \) sufficiently small, choose a smooth metric \( g_{i, \rho} \) on \( M_i \) such that \( g_{i, \rho} \) is flat on the geodesics ball \( B^\rho_{i, \rho}(p_i) \) of radius \( \rho \) in \( M_i \) centered at \( p_i \) and equal to \( g_i \) on \( M_i \setminus B^\rho_{2, \rho}(p_i) \).
for \( i = 1, 2 \). Let \( T_\rho = S^{n-1}(\rho) \times [-l, l] \), for any \( 0 < l < \infty \), with the standard product metric, and let \( M_\rho \) be the Lipschitz Riemannian manifold obtained by gluing \( M_1 \setminus B^1_\rho(p_1) \) to \( M_2 \setminus B^2_\rho(p_2) \) using \( T_\rho \), by identifying one end of \( T_\rho \) with \( \partial B^1_\rho(p_1) \) and the other end of \( T_\rho \) with \( \partial B^2_\rho(p_2) \). We may then locally smooth out the corners of \( M_\rho \) to obtain a smooth Riemannian manifold, which we will continue to denote by \( M_\rho \). Note that \( \partial M_\rho = \partial M_1 \sqcup \partial M_2 \).

Let \( 0 = \sigma_0(M_\rho) \leq \sigma_1(M_\rho) \leq \sigma_2(M_\rho) \leq \cdots \) be the Steklov eigenvalues of \( M_\rho \) and let \( u_\rho^{(0)}, u_\rho^{(1)}, u_\rho^{(2)}, \ldots \) be a complete sequence of eigenfunctions that are \( L^2 \)-orthonormal on \( \partial M_\rho \), such that \( u_\rho^{(k)} \) is an eigenfunction of \( \sigma_k(M_\rho) \). As in the proof of Theorem 1.2, \( \int_{M_\rho} |\nabla u_\rho^{(k)}|^2 \, dv = \sigma_k(M_\rho) \leq \Lambda_k \), with \( \Lambda_k \) independent of \( \rho \). Elliptic boundary estimates give uniform bounds

\[
\|u_\rho^{(k)}\|_{C^{2,\alpha}(K)} \leq C\|u_\rho^{(k)}\|_{L^2(K)} \leq C(k, \alpha, \Lambda_k, M_1, M_2)
\]

for any compact subset \( K \) of \( (M_1 \setminus \{p_1\}) \cup (M_2 \setminus \{p_2\}) \). By the Arzela-Ascoli theorem, there exists a sequence \( \rho_i \to 0 \) such that for all \( k \), \( u_\rho^{(k)} \) converges in \( C^2(K) \) on compact subsets \( K \subset (M_1 \setminus \{p_1\}) \cup (M_2 \setminus \{p_2\}) \) to a harmonic function \( u^{(k)} \) on \( (M_1 \setminus \{p_1\}, g_1) \cup (M_2 \setminus \{p_2\}, g_2) \), satisfying

\[
\frac{\partial u^{(k)}}{\partial v} = \sigma_k u^{(k)} \quad \text{on} \quad \partial M_1 \sqcup \partial M_2,
\]

with \( \sigma_k = \lim_{i \to \infty} \sigma_k(M_\rho_i) \). By standard arguments \( u^{(k)} \) extends to a harmonic function on \( M_1 \sqcup M_2 \), and hence to a Steklov eigenfunction with eigenvalue \( \sigma_k \) on \( M_1 \sqcup M_2 \).

We now show that \( u^{(k)} \) is a \( k \)-th eigenfunction of \( M_1 \sqcup M_2 \); i.e. \( \sigma_k = \sigma_k(M_1 \sqcup M_2) \). We prove this by induction on \( k \). First, since \( \sigma_0(M_\rho) = 0 \), we have that \( \sigma_0 = \lim_{\rho \to 0} \sigma_0(M_\rho) = 0 \), and so \( \sigma_0 = \sigma_0(M_1 \sqcup M_2) \). Now suppose \( \sigma_l = \sigma_l(M_1 \sqcup M_2) \) for \( l = 1, \ldots, k-1 \), where \( k \geq 1 \). We will show that \( \sigma_k = \sigma_k(M_1 \sqcup M_2) \). First observe that \( \{u^{(k)}\}_{k=1}^\infty \) are \( L^2 \)-orthonormal on \( \partial(M_1 \sqcup M_2) \), since \( \{u^{(k)}_{\rho_i}\}_{k=1}^\infty \) are \( L^2 \)-orthonormal on \( \partial M_{\rho_i} = \partial(M_1 \sqcup M_2) \). It follows that \( \sigma_k \geq \sigma_k(M_1 \sqcup M_2) \). The proof that \( \sigma_k \leq \sigma_k(M_1 \sqcup M_2) \) follows exactly as in the proof of Theorem 1.1. Therefore, \( \lim_{i \to \infty} \sigma_k(M_\rho_i) = \sigma_k(M_1 \sqcup M_2) \). \( \square \)

**Remark 4.9** The same spectral convergence result holds for more complicated gluing constructions along the interior of manifolds. Specifically, the geometry of the neck region does not affect the spectrum in the limit. All that is needed in the proof of Theorem 1.2 is that as \( \rho \to 0 \), \( M_\rho \setminus \text{Int} T_\rho \) converges to \( (M_1 \setminus S_1, g_1) \sqcup (M_2 \setminus S_2, g_2) \), where \( S_i \subset \text{Int} M_i \) is a set of Hausdorff dimension at most \( n-2 \), for \( i = 1, 2 \). In this case a standard removable singularity argument shows that \( u^{(k)} \) extends from a harmonic function on \( M_i \setminus S_i \) to a smooth harmonic function on \( M_i \), for \( i = 1, 2 \). The rest of the proof carries through unchanged.

The same argument can be used to glue a single manifold to itself at distinct interior points.

**Theorem 4.10** Let \( M \) be an \( n \)-dimensional Riemannian manifold with nonempty boundary. Given any \( \epsilon > 0 \) there exists a manifold \( M_\epsilon \) obtained by appropriately gluing \( M \) to itself near distinct interior points, such that \( \partial M_\epsilon = \partial M \) and

\[
\lim_{\epsilon \to 0} \sigma_k(M_\epsilon) = \sigma_k(M)
\]

for \( k = 0, 1, 2, \ldots \).

We close this section by mentioning an immediate application of the continuity of the first \( k \) Steklov eigenvalues under certain degenerations for surfaces. Given an orientable surface.
\( M \) of genus \( \gamma \) with \( m \) boundary components, let

\[
\sigma_k^*(\gamma, m) = \sup \{ \sigma_k(M, g)L_g(\partial M) : g \text{ a smooth metric on } M \}.
\]

For any surface, there is an upper bound

\[
\sigma_k^*(\gamma, m) \leq 2\pi(\gamma + m + k - 1)
\]

independent of the metric ([17]). However, the exact value of \( \sigma_k^*(\gamma, m) \) is only known in a few cases. As discussed in Sect. 2, \( \sigma_k^*(0, 1) = 2\pi k \) ([14,16,29]), and is achieved by the Euclidean disk for \( k = 1 \), but is not achieved for any \( k \geq 2 \) (Theorem 2.3, and [14] for \( k = 2 \)). The only other sharp upper bounds that are known are for \( k = 1 \) for the annulus and Möbius band. In [11] the authors proved that \( \sigma_1^*(0, 2) = 4\pi/T_{1,0} \) where \( T_{1,0} \approx 1.2 \) is the unique positive solution of \( t = \coth t \), and the supremum is uniquely (up to \( \sigma \)-homothety) achieved by the induced metric on the critical catenoid.

As a consequence of the gluing results of this section, we have the following lower bound for \( \sigma_k^*(\gamma, m) \), as discussed in [25, Equation (0.2)].

**Corollary 4.11**

\[
\sigma_k^*(\gamma, m) \geq \max_{k_1 + \cdots + k_j = k \atop k_j \geq 1 \forall j} \sum_{j=1}^s \sigma_{k_j}^*(\gamma_j, m_j)
\]

where

\[
\gamma_1 + \cdots + \gamma_s - (s-1) \leq m + \gamma
\]

\[
\gamma_1 < \gamma = m_1 + \gamma_1 < m + \gamma
\]

\( s \neq 1 \)

**Proof** Suppose the maximum of the right hand side is achieved for some \( k_1, \ldots, k_s, \gamma_1, \ldots, \gamma_s, \) and \( m_1, \ldots, m_s \). Let \( M_{\gamma_j, m_j} \) be a Riemannian surface of genus \( \gamma_j \) with \( m_j \) boundary components such that \( \overline{\sigma}_{k_j}(M_{\gamma_j, m_j}) \) is arbitrarily close to \( \sigma_{k_j}^*(\gamma_j, m_j) \). By rescaling the metrics on the surfaces we may assume that \( \sigma_{k_j}(M_{\gamma_j, m_j}) = 1 \) for \( j = 1, \ldots, s \). Then \( \sigma_k(M_{\gamma_1, m_1} \sqcup \cdots \sqcup M_{\gamma_s, m_s}) = 1 \) and \( \overline{\sigma}_k(M_{\gamma_1, m_1} \sqcup \cdots \sqcup M_{\gamma_s, m_s}) = \sum_{j=1}^s \overline{\sigma}_{k_j}(M_{\gamma_j, m_j}) \) which is arbitrarily close to \( \sum_{j=1}^s \sigma_{k_j}^*(\gamma_j, m_j) \). Using Theorem 1.1 we glue the surfaces \( M_{\gamma_1, m_1}, \ldots, M_{\gamma_s, m_s} \) together along their boundaries to obtain a connected Riemannian surface \( M \) of genus \( \gamma_1 + \cdots + \gamma_s \) with \( m_1 + \cdots + m_s - (s-1) \) boundary components, and such that \( \overline{\sigma}_k(M) \) is arbitrarily close to \( \overline{\sigma}_k(M_{\gamma_1, m_1} \sqcup \cdots \sqcup M_{\gamma_s, m_s}) \). If \( (m_1 + \cdots + m_s) - (s-1) - m = l > 0 \), then using Theorem 4.8 we glue \( M \) to itself along its boundary, along neighborhoods of two points in distinct boundary components, to reduce the number of boundary components by one and increase the genus by one, while changing the normalized eigenvalues by an arbitrarily small amount. Doing this \( l \) times, we obtain a surface with \( m \) boundary components and genus \( \gamma_1 + \cdots + \gamma_s + l \leq \gamma \). On the other hand, if \( (m_1 + \cdots + m_s) - (s-1) - m = l \leq 0 \), then we remove \( -l \) small disjoint disks from \( M \) to obtain a surface with \( m \) boundary components with genus \( \gamma_1 + \cdots + \gamma_s \leq \gamma \), while changing the normalized eigenvalues by an arbitrarily small amount ([11, Proposition 4.3]). In either case, if the resulting surface has genus less than \( \gamma \), then using Theorem 4.10 we glue the surface to itself between two interior points to increase the genus by one without changing the number of boundary components, while changing the normalized eigenvalues by an arbitrarily small amount. Repeating this as necessary, we obtain a Riemannian surface \( M' \) of genus \( \gamma \) with \( m \) boundary components with \( \overline{\sigma}_k(M') \) arbitrarily close to \( \sum_{j=1}^s \sigma_{k_j}^*(\gamma_j, m_j) \).

\[ \square \]
5 Higher Steklov eigenvalues for the annulus and Möbius band

It is an open question to determine the suprema of the higher Steklov eigenvalues among all smooth metrics on the annulus and Möbius band, and whether the suprema are achieved. For the first nonzero eigenvalue, as discussed in Sect. 3, the authors proved in [11] that there exists a smooth metric on the annulus and on the Möbius band that maximizes the first nonzero normalized Steklov eigenvalue, and explicitly characterized the maximizing metric as the induced metric on the critical catenoid and the critical Möbius band, respectively. The characterization of the maximizing metrics involves a nontrivial argument showing that a metric that maximizes the first nonzero eigenvalue must be \( \sigma \)-homothetic to an \( S^1 \)-invariant metric. The result then follows from an analysis of \( S^1 \)-invariant metrics on the annulus [8, Section 3] and Möbius band [11, Proposition 7.1]. In particular, the supremum of the first nonzero eigenvalue over all metrics is the same as the supremum of the first nonzero eigenvalue among all \( S^1 \)-invariant metrics. One can then ask whether anything like this is true for the higher eigenvalues. [6] and [7] extended the analysis of \( S^1 \)-invariant metrics to higher Steklov eigenvalues, and for each \( k \geq 2 \), determined the supremum of the \( k \)-th nonzero normalized Steklov eigenvalue among all \( S^1 \)-invariant metrics on the annulus and the Möbius band. Moreover, in each case, the supremum is achieved by the induced metric on an explicit free boundary annulus or Möbius band in a Euclidean ball, except for the supremum of second normalized eigenvalue on the annulus, which is not achieved. In summary, in the case of the annulus, Fan-Tam-Yu proved:

**Theorem 5.1** ([6]) Let \( \sigma^S^1_k \) be the supremum of \( k \)-th normalized Steklov eigenvalue among all \( S^1 \)-invariant metrics on the annulus.

(i) \( \sigma^S^1_2 = 4 \pi \). Moreover, \( \tilde{\sigma}_2(g_T) \rightarrow 4 \pi \) as \( T \rightarrow \infty \), where \( g_T = dt^2 + d\theta^2 \) on the cylinder \([0, T] \times S^1\), and the supremum \( 4 \pi \) is not achieved.

(ii) \( \sigma^S^1_{2k-1} = 4k \pi / T_{1,0} \) for all \( k \geq 1 \), where \( T_{1,0} \) is the unique positive solution of \( t = \coth t \), and is achieved by the induced metric on the \( k \)-critical catenoid.

(iii) \( \sigma^S^1_{2k} = 4k \pi \tanh(kT_{k,1}) \) for \( k > 1 \), where \( T_{k,1} \) is the unique positive solution of \( k \tanh(kt) = \coth(t) \), and is achieved by the induced metric from an explicit free boundary minimal immersion of the annulus into \( \mathbb{B}^4 \).

Here we use the notation \( \tilde{\sigma}_k(g) := \sigma_k(g) L_g(\partial M) \) for the \( k \)-th normalized Steklov eigenvalue of a surface \((M, g)\). In the case of the Möbius band, Fraser-Sargent proved:

**Theorem 5.2** ([7]) Let \( \sigma^S^1_k \) be the supremum of the \( k \)-th normalized Steklov eigenvalue among \( S^1 \)-invariant metrics on the Möbius band. For all \( k \geq 1 \),

\[
\sigma^S^1_{2k-1} = \sigma^S^1_{2k} = 4k \pi \tanh(2kT_{2k,1})
\]

and the supremum is achieved by the induced metric from an explicit free boundary minimal embedding of the Möbius band into \( \mathbb{B}^4 \).

It is natural to ask whether the maximizers for the higher eigenvalues among \( S^1 \)-invariant metrics, in Theorem 5.1 on the annulus and Theorem 5.2 on the Möbius band, also maximize among all metrics, as they do for the first eigenvalue when \( k = 1 \). We show that this is not the case for the higher eigenvalues. Specifically, for \( k \geq 2 \), using Theorem 1.1 we construct smooth metrics on the annulus and Möbius band with \( k \)-th eigenvalue strictly bigger than the supremum of the \( k \)-th eigenvalue over \( S^1 \)-invariant metrics.
Theorem 5.3  For $k \geq 2$, the supremum $\sigma^*_k$ of the $k$-th normalized Steklov eigenvalue over all smooth metrics on the annulus (or respectively, M"obius band) is strictly bigger than the supremum $\sigma^*_k$ over $S^1$-invariant metrics on the annulus (or respectively, M"obius band).

Proof  Fix $k \geq 2$. Let $\tilde{M}$ be the disjoint union of the critical catenoid $C$ and $k - 1$ Euclidean unit disks $\mathbb{D}$. Observe that

$$\sigma_0(\tilde{M}) = \sigma_1(\tilde{M}) = \cdots = \sigma_{k-1}(\tilde{M}) = 0, \quad \sigma_k(\tilde{M}) = 1$$

and

$$\tilde{\sigma}_k(\tilde{M}) = L(\partial C) + (k - 1)L(\partial \mathbb{D}) = \frac{4\pi}{T_{1,0}} + 2(k - 1)\pi > \frac{4\pi}{1.2} + 2(k - 1)\pi.$$  

By Theorem 1.1, for any $\epsilon > 0$, there is a smooth metric annulus $M$ obtained by gluing $C$ and $k - 1$ disks $\mathbb{D}$ together, such that $|\tilde{\sigma}_k(\tilde{M}) - \tilde{\sigma}_k(M)| < \epsilon$. We claim that $\tilde{\sigma}_k(M) > \sigma^*_k$, where $\sigma^*_k$ is the supremum of the $k$-th normalized Steklov eigenvalue over all $S^1$-invariant metrics on the annulus. First note that for $k = 2$,

$$\tilde{\sigma}_2(\tilde{M}) > \frac{4\pi}{1.2} + 2\pi > 4\pi = \sigma^*_2.$$  

For $k = 2l - 1$ odd with $l > 1$, we have

$$\tilde{\sigma}_k(\tilde{M}) > \frac{4\pi}{1.2} + 2(2l - 2)\pi = \frac{4\pi l + 0.8\pi(l - 1)}{1.2} > \frac{4\pi l}{1.2} > \frac{4\pi l}{T_{1,0}} = \sigma^*_k.$$  

For $k = 2l$ even with $l > 1$, we have

$$\tilde{\sigma}_k(\tilde{M}) > \frac{4\pi}{1.2} + 2(2l - 1)\pi > 4l\pi > 4l\pi \tanh(lT_{k,1}) = \sigma^*_k.$$  

For each $k \geq 2$, by choosing $\epsilon > 0$ sufficiently small, it follows that $\tilde{\sigma}_k(M) > \sigma^*_k$.

We now consider the case of the M"obius band. In this case, we let $\bar{M}$ be the disjoint union of the critical M"obius band $C$ and $k - 1$ Euclidean disks $\mathbb{D}$. Observe that

$$\sigma_0(\bar{M}) = \sigma_1(\bar{M}) = \cdots = \sigma_{k-1}(\bar{M}) = 0, \quad \sigma_k(\bar{M}) = 1$$

and

$$\bar{\sigma}_k(\bar{M}) = L(\partial C) + (k - 1)L(\partial \mathbb{D}) = 2\pi \sqrt{3} + 2(k - 1)\pi.$$  

By Theorem 1.1, for any $\epsilon > 0$, there is a smooth metric M"obius band $M$ obtained by gluing $C$ and $k - 1$ disks $\mathbb{D}$ together, such that $|\bar{\sigma}_k(\bar{M}) - \bar{\sigma}_k(M)| < \epsilon$. We claim that $\bar{\sigma}_k(M) > \sigma^*_k$, where now $\sigma^*_k$ denotes the supremum of the $k$-th normalized Steklov eigenvalue over all $S^1$-invariant metrics on the M"obius band. For $k = 2l$ even, with $l \geq 1$, this is clear, since

$$\bar{\sigma}_k(\bar{M}) = 2\pi \sqrt{3} + 2(2l - 1)\pi > 4\pi l > 4\pi l \tanh(2lT_{2l,1}) = \sigma^*_k.$$  

For $k = 2l - 1$ odd, with $l > 1$, we need a better approximation of $4\pi l \tanh(2lT_{2l,1})$. First observe that

$$\frac{d}{dt} \coth t = -\frac{1}{\sinh^2 t} > -\frac{1.2}{t^2} = \frac{d}{dt} \left( \frac{1.2}{t} \right)$$

since $\sinh t/t \geq 1$ for all $t$. Also, $T_{2,1} = \ln(2 + \sqrt{3})/2$, and $\coth(T_{2,1}) = \sqrt{3} < 1.2/T_{2,1}$. It follows that $\coth t < 1.2/t$ for all $t < T_{2,1}$. Denote by $t_k$ the unique positive solution of

\[\tag{$\clubsuit$}Springer\]
k \tanh(kt) = 1.2/t. Recall that $T_{k,1}$ is the unique positive solution of $k \tanh(kt) = \coth t$. Since $\coth t < 1.2/t$ for all $t < T_{2,1}$, $T_{k,1} < T_{2,1}$ for $k > 2$ ([6, Lemma 2.3]), and $k \tanh(kt)$ is increasing in $t$, it follows that $T_{k,1} < t_k$. Therefore, if $k > 2$,

$$k \tanh(kT_{k,1}) < k \tanh(kt_k).$$

By definition of $t_k$, $k \tanh(kt_k) = 1.2/t_k$. Therefore $\tanh(kt_k) = 1.2/(kt_k)$ and so $t_1 = kt_k$.

By approximation we have that $t_1 > 1.36$. Finally, for $l > 1$,

$$\sigma_{2l-1}^k = 4\pi l \tanh(2lT_{2l,1}) < 4\pi l \tanh(2lt_{2l}) = 2\pi \frac{1.2}{t_{2l}} = 4\pi l \frac{1.2}{t_1} < 4\pi l \frac{1.2}{1.36} < 2\pi l \cdot (1.77).$$

On the other hand,

$$\tilde{\sigma}_{2l-1}(\tilde{M}) = 2\pi \sqrt{3} + 2(2l - 2)\pi = 2\pi (2l + \sqrt{3} - 2).$$

If $l > 1$, it is straightforward to check that $2l + \sqrt{3} - 2 > 1.77l$, and so $\tilde{\sigma}_{2l-1}(\tilde{M}) > \sigma_{2l-1}^k$.

For each $k \geq 2$, if $\epsilon > 0$ is sufficiently small, then $\tilde{\sigma}_k(M) > \sigma_k^l$. □

Remark 5.4 As in the case of the disk, it might be reasonable to expect that maximizing metrics do not exist for higher eigenvalues on the annulus and M"obius band, and to ask:

(i) Is the supremum of the $k$-th nonzero normalized Steklov eigenvalue among all smooth metrics on the annulus $4\pi/T_{1,0} + 2(k - 1)\pi$, where $T_{1,0} \approx 1.2$ is the unique positive number such that $\coth t = t$?

(ii) Is the supremum of the $k$-th nonzero normalized Steklov eigenvalue among all smooth metrics on the M"obius band $2\pi \sqrt{3} + 2(k - 1)\pi$?

That this might be true is also suggested by results for higher eigenvalues of the Laplacian on the two-sphere and real projective plane [19–24].

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