The adjacency matrix of directed cyclic wheel graph \( \vec{W}_n \)

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Abstract. An adjacency matrix is one of the matrix representations of a directed graph. In this paper, the adjacency matrix of a directed cyclic wheel graph \( \vec{W}_n \) is denoted by \( A(\vec{W}_n) \). From the matrix \( A(\vec{W}_n) \) the general form of the characteristic polynomial and the eigenvalues of a directed cyclic wheel graph \( \vec{W}_n \) can be obtained. The norm of each coefficient of its characteristic polynomial is obtained by calculating the sum of the principal minors of subgraphs of the graph \( \vec{W}_n \). It turns out that each coefficient \( a_i \), \( i = 1, 2, \ldots, (n - 2), n \) equals zero except \( a_{n-1} \) is equal to -1. In addition the matrix \( A(\vec{W}_n) \) has real eigenvalues and also some complex eigenvalues that conjugate each other. The real eigenvalues are obtained by searching the real roots through the characteristic equation, then by factorization we get the polynomial factor that contains the complex roots. The complex eigenvalues of the adjacency matrix of a directed cyclic wheel graph \( \vec{W}_n \) or \( A(\vec{W}_n) \) has a relation with the complex eigenvalues of the antiadjacency matrix of a directed cyclic wheel graph \( \vec{W}_n \) or \( B(\vec{W}_n) \), that is the complex eigenvalues of the matrix \( A(\vec{W}_n) \) are equal to the negative of the complex eigenvalues of the matrix \( B(\vec{W}_n) \).

1. Introduction

Graph theory is a branch of mathematics that develops very rapidly, but there are still many properties of a directed graph that have not been studied further. Those properties can be studied through the matrix representation such as antiadjacency matrix and adjacency matrix of a directed graph. Based on its direction, a directed graph is divided into a directed acyclic graph and a directed cyclic graph. A directed acyclic graph is a graph that does not contain any cycles, while a directed cyclic graph is a graph that contains a cycle. The antiadjacency matrix of directed cyclic wheel graph \( \vec{W}_n \) has been discussed in the previous research by Widiastuti L., Suarsih Utama and Siti Aminah (2018) [1]. To continue the previous research, the topic of this paper is about adjacency matrix of directed cyclic wheel graph \( \vec{W}_n \). The characteristic polynomial and the eigenvalues can be found through the adjacency matrix. Moreover the relation between the complex eigenvalues of antiadjacency matrix and adjacency matrix of directed cyclic wheel graph can be found.

2. Basic Theory

Adjacency matrix of a directed graph \( G \) with \( V(G) = \{v_1, \ldots, v_n\} \) is a matrix \( A = [a_{ij}] \) with size \( n \times n \) where \( a_{ij} = 1 \) if \( v_iv_j \in A(G) \) and \( a_{ij} = 0 \) if \( v_iv_j \notin A(G) \) [2].
A wheel graph $W_n'$ is a graph that contains a cycle $C_{n-1}$ of order $n - 1$, and for which every vertex in the cycle is connected to another graph vertex (which is known as the hub) [3]. Furthermore, a directed cyclic wheel graph $\overrightarrow{W_n'}$ is a wheel graph that has a certain directed path.

Principal submatrix of matrix $A_{n \times n}$ with size $(n-r) \times (n-r)$ is obtained by simultaneously deleting $r$ rows and $r$ columns with the same index of $A_{n \times n}$. Principal minor of matrix $A_{n \times n}$ is the determinant of the principal submatrix of $A_{n \times n}$ [4].

Here are some theorems and lemma from the previous research that used to support research in this paper.

Theorem 2.1 A directed acyclic graph $G$ at least has one vertex with outdegree equals to 0 [5].

Theorem 2.2 A directed acyclic graph $G$ at least has one vertex with indegree equals to 0 [5].

Theorem 2.3 If $A = [a_{ij}]$ is a triangular matrix of size $n \times n$, then $\det(A) = a_{11}a_{22} \cdots a_{nn}$ [6].

Theorem 2.4 If $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n = 0$ is the characteristic equation of $A_{n \times n}$ then $c_i = (-1)^i \sum_{j=1}^{n} A_{i,j}^J, \ i = 1, 2, \ldots, n$, where $[A_{i,j}^J]$ is the principal minor of size $i \times i$ of $A$ and $j = 1, 2, 3, \ldots, w$, and $w$ is the number of the principal minor of size $i \times i$ of $A$ [4].

Theorem 2.5 If $P$ is a polynomial in one variable with real coefficients, and $(a + bi)$ is a root of $P$ with $a$ and $b$ are real numbers, then its complex conjugate $(a - bi)$ is also a root of $P$ [7].

Lemma 2.6 Suppose $G$ is a directed graph and $A(G)$ is the adjacency of $G$. If $G$ is a graf cycle $C_n$, then $\det(A(G)) = -1$ for $n$ even and $\det(A(G)) = 1$ for $n$ odd [8].

3. Main Result

In this section we discuss about the theorems related to the main results. The following is a general form of an adjacency matrix of graph $\overrightarrow{W_n'}$ of size $n \times n$ that denoted by $A(\overrightarrow{W_n'}) = [a_{ij}]$, where $a_{ij} = \begin{cases} 1, & i = 1, \ldots, n - 2, j = i + 1 \text{ or } i = n - 1, j = 1 \text{ or } i = n, j = 1, \ldots, n - 1 \\ 0, & \text{other} \end{cases}$

Theorem 3.1 Let $\overrightarrow{W_n'}$ be a directed cyclic wheel graph and $A(\overrightarrow{W_n'})$ be the adjacency matrix that has characteristic polynomial $P(A(\overrightarrow{W_n'})) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n$. Then $\|a_i\| = 0, i = 1, \ldots, n - 2, n$ and $\|a_{n-1}\| = 1$.

Proof. $\|a_i\|, i = 1, \ldots, n$ can be obtained by expanding the form of $\sum_{j=1}^{w} A_{i,j}^J, i = 1, 2, \ldots, n$ and $j = 1, 2, 3, \ldots, w$ in Theorem 2.4. The form is obtained from the principal submatrices each with size $i \times i$. The principal submatrices represent the subgraphs obtained by deleting any $n - i$ vertices $(i = 1, \ldots, n)$ including the edges that connected to those $n - i$ vertices. The proof is divided into 4 cases:

i. $\|a_1\| = 0$ is obtained by deleting $(n - 1)$ vertices

ii. $\|a_i\| = 0, i = 2, \ldots, n - 2$ is obtained by deleting $(n - i)$ vertices

iii. $\|a_{n-1}\| = 1$ is obtained by deleting $n - (n - 1) = 1$ vertex

iv. $\|a_n\| = 0$ is obtained by deleting $n - n = 0$ vertex

The following is the description of all 4 cases:

i. Because graph $(\overrightarrow{W_n'})$ contains $n$ vertices, after any $n - 1$ vertices are deleted, some acyclic induced subgraphs that contain 1 vertex are obtained, that is as many as $C_1^n = n$ subgraphs. Every
induced subgraph can be represented by principal submatrix with size $1 \times 1$ that has entry equals to 0. Because the principal submatrix is a zero matrix, then the principal minor is equal to 0 and $\sum_j \lambda_j A_i^{(J)} = 0, j = 1, \ldots, n$ with $\|a_i\| = 0$. 

ii. Because graph $(\overline{W}_n)$ contains $n$ vertices, after any $(n - i), i = 2, \ldots, (n - 2)$ vertices are deleted, some acyclic induced subgraphs, each of them contains $i$ vertices, are obtained. There are as many as $c_i^n$ subgraphs. Every induced subgraph can be represented by principal submatrix with size $i \times i$. Because those subgraphs are acyclic, then its principal submatrix has a column and a row that has entry equals to 0, and the determinant of the principal minor is equal to 0 and $\sum_j \lambda_j A_i^{(J)} = 0, j = 2, \ldots, (n - 2), i = 1, \ldots, c_i^n$ with $\|a_i\| = 0, i = 2, \ldots, (n - 2)$. 

iii. Because graph $(\overline{W}_n)$ contains $n$ vertices, after any 1 vertex is deleted, some induced subgraphs that contain $n - 1$ vertices are obtained. There are as many as $c_{n-1}^n = n$ subgraphs.

![Figure 2. a) An induced subgraph that contains $(n - 1)$ vertices after the middle vertex is deleted, and b) Some induced subgraphs that contains $(n - 1)$ vertices after any 1 vertex in its cycle is deleted](image)

Every induced subgraph can be represented by principal submatrix with size $(n - 1) \times (n - 1)$. Figure 2.a represent the induced subgraph that is obtained after the middle vertex is deleted. Those subgraph is a cycle induced subgraph, that is subgraph $c_{n-1}^n$, that can be represented by principal submatrix $A_{n-1}^{(n)}$. According to Lemma 2.6 $\det\left(A(\overline{C}_{n-1})\right) = (-1)^n$, then $|A_{n-1}^{(n)}| = (-1)^n$. In the figure 2.b, after 1 of the vertex in its cycle is deleted, an acyclic induced subgraph is obtained. Because the induced subgraphs is acyclic, then its principal submatrix has a row and a column that has entry equals to 0, so the principal minor is equal to 0, then $\sum_{j=1}^{n-1} A_{n-1}^{(j)} = 0$ and $\sum_{j=1}^{n-1} |A_{n-1}^{(j)}| = \sum_{j=1}^{n-1} |A_{n-1}^{(j)}| + |A_{n-1}^{(n)}| = (-1)^n$ with $\|a_{n-1}\| = 1$.

iv. Because graph $(\overline{W}_n)$ contains $n$ vertices, after 0 vertex is deleted, an induced subgraph that contains $n$ vertices is obtained, that is the graph $(\overline{W}_n)$. The adjacency matrix of graph $(\overline{W}_n)$ has 1 columns that has entry equals to 0 and $\sum_{j=1}^{n} |A_n^{(j)}| = 0, j = 1$ with $\|a_n\| = 0$.

**Corollary 3.2** Let $\overline{W}_n$ be a directed cyclic wheel graph and $A(\overline{W}_n)$ be the adjacency matrix, then the general form of its characteristic polynomial is $P\left(A(\overline{W}_n)\right) = \lambda^n - \lambda$.

**Proof.** Suppose the general form of characteristic polynomial of $A(\overline{W}_n)$ is $P(A(\overline{W}_n)) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n$. 

It will be proved that $a_i = 0, (i = 1, \ldots, n - 2, n)$ whereas $a_{n-1} = -1$. From Theorem 3.1 we get $\sum_{j=1}^{w} A_i^{(j)} = 0, (i = 1, \ldots, n - 2, n)$ and $\sum_{j=1}^{n} |A_{n-1}^{(j)}| = (-1)^n$. By using the equation $a_i = (-1)^i \sum_{j=1}^{w} A_i^{(j)}$ in Theorem 2.4 we obtain, for:

- $i = 1, \ldots, (n - 2), n, a_i = (-1)^i \sum_{j=1}^{w} A_i^{(j)} = (-1)^i 0 = 0$.
- $i = (n - 1), a_{n-1} = (-1)^{n-1} \sum_{j=1}^{w} A_{n-1}^{(j)} = (-1)^{n-1}(-1)^n = (-1)^{2n-1} = -1$. 

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It can be concluded that the general form of the characteristic polynomial of matrix $A(W_n)$ is $P\left(A\left(W_n^r\right)\right) = \lambda^n - \lambda$.

**Theorem 3.3** The real eigenvalues of adjacency matrix of directed cyclic wheel graph $W_n$ are equal to 0 and 1 for $n = 2k$, and they are equal to -1, 0 and 1 for $n = 2k + 1$ where $k \geq 2$.

*Proof.* The proof is divided into 2 cases, that is for $n = 2k$ and $n = 2k + 1$ where $k \geq 2$.

From Corollary 3.3 we get $P\left(A\left(W_{2k}\right)\right) = \lambda^{2k} - \lambda$. By finding the real roots through this equation $\lambda^{2k} - \lambda = 0$, we obtained $\lambda = 0$ and $\lambda^{2k-1} = 1$. Because $(2k - 1)$ is odd, then the real number that satisfies $\lambda^{2k-1} = 1$ is $\lambda = 1$. Finally it can be concluded that the real eigenvalues of matrix $A(W_n)$ are 0 and 1 for $n = 2k, k \geq 2$. From Corollary 3.3 we get $P\left(A\left(W_{2k+1}\right)\right) = \lambda^{2k+1} - \lambda$. By finding the real roots through this equation $\lambda^{2k+1} - \lambda = 0$, we obtained $\lambda = 0$ and $\lambda^{2k} = 1$. Because $(2k)$ is even, then the real number that satisfies $\lambda^{2k} = 1$ are $\lambda = -1$ and $\lambda = 1$. Finally it can be concluded that the real eigenvalues of matrix $A(W_n)$ are -1, 0 and 1 for $n = 2k + 1, k \geq 2$.

**Theorem 3.4** The number of complex eigenvalues of adjacency matrix of graph $W_n$, are always even and conjugate to each other for every $n$.

*Proof.* The proof is divided into 2 cases, that is for $n = 2k$ and $n = 2k + 1, k \geq 2$. The other factors of $P(A(W_{2k}))$ and $P(A(W_{2k+1}))$ will be obtained by using Horner method given in the figure 3.

![Horner method](image)

From those Horner methods we obtained the result as the following.

$P\left(A\left(W_{2k}\right)\right)$ can be factorized as the following: $P\left(A\left(W_{2k}\right)\right) = \lambda (\lambda - 1) (1 + \sum_{i=1}^{2k-2} \lambda^i)$. The complex eigenvalues of $A(W_{2k})$ can be obtained from the equation $1 + \sum_{i=1}^{2k-2} \lambda^i = 0$ which is similar to a finite geometric series form $(a = 1, r = \lambda, n = 2k - 1)$. Then the sum of the series is $\frac{1(\lambda^{2k-1} - 1)}{\lambda - 1} = 0, \lambda \neq 1$.

From the equation (1), it follows that the complex eigenvalues are

$\lambda = e^{i\frac{2\pi m}{2k-1}}; m = 0, ..., 2k - 2$.

Since $\lambda = e^0$ is achieved when $m = 0$ in the equation (2) and since $\lambda \neq 1$ in the equation (1), it follows that there are $(2k - 2)$ complex eigenvalues of $A(W_{2k})$. Hence, the number of complex eigenvalues of $A(W_{2k})$ are even. It can be proved that the eigenvalues $\lambda_{k-j} = e^{i\frac{2k-j}{2k-1}}$ is the conjugate of the eigenvalue $\lambda_{k-j} = e^{i\frac{2k-j}{2k-1}}$.

$P\left(A\left(W_{2k+1}\right)\right)$ can be factorized as the following: $P\left(A\left(W_{2k+1}\right)\right) = (\lambda + 1) \lambda (\lambda - 1) (1 + \sum_{i=1}^{k-1} \lambda^i)$. The complex eigenvalues of $A(W_{2k+1})$ can be obtained from the equation $1 + \sum_{i=1}^{k-1} \lambda^i = 0$ which is similar to a finite geometric series form $(a = 1, r = \lambda^2, n = k)$. Then the sum of the series is...
\[ \frac{1(\lambda^{2k-1})}{\lambda^{2k}-1} = 0, \ \lambda \neq -1 \text{ and } \lambda \neq 1 \]  
(3)

From the equation (3), it follows that the complex eigenvalues are

\[ \lambda = e^{i(\frac{m\pi}{k})}; \ m = 0, ..., 2k - 1 \]  
(4)

Since \( \lambda = e^0 \) & \( \lambda = e^{i\pi} \) are achieved when \( m = 0 \) & \( m = k \) respectively in the equation (4) and since \( \lambda \neq 1 \) & \( \lambda \neq -1 \) in the equation (3), it follows that there are \( 2k - 2 \) complex eigenvalues of \( A(\overline{W_{2k+1}}) \).

Hence, the number of complex eigenvalues of \( A(\overline{W_{2k+1}}) \) are even. It can be proved that the eigenvalues \( \lambda_{k-j} = e^{i\frac{(k-j)\pi}{k}} \) is the conjugate of the eigenvalue \( \lambda_{k+j} = e^{i\frac{(k+j)\pi}{k}}, j = 1, ..., k - 1. \)

The complex eigenvalues of antiadjacency matrix of graph \( \overline{W}_{n} \) that denoted as \( B(\overline{W}_{n}) \) are:

- for \( n = 2k \) (\( k \geq 2 \)), \( \lambda = e^{i\frac{(\frac{\pi}{2} + 2k)n}{2k-1}}; l = 0, ..., k - 2, \) is equal to the negative of \( \lambda \) for 4 cases as the following:
  - i. \( \lambda_{c}(B(\overline{W}_{n})) \) is equal to the negative of \( \lambda \) for 4 cases as the following:
    - ii. \( \lambda_{c}(B(\overline{W}_{n})) = e^{i\frac{\pi}{k}}; l = 1, ..., k - 1, \) is equal to the negative of \( \lambda \) for 4 cases as the following:
  - iii. \( \lambda_{c}(B(\overline{W}_{n})) = e^{i\frac{\pi}{k}}; l = 1, ..., k - 1, \) is equal to the negative of \( \lambda \) for 4 cases as the following:
  - iv. \( \lambda_{c}(B(\overline{W}_{n})) = e^{i\frac{\pi}{k}}; l = 1, ..., k - 1, \) is equal to the negative of \( \lambda \) for 4 cases as the following:

**Theorem 3.5** Let \( \lambda_{c}(B(\overline{W}_{n})) \) and \( \lambda_{c}(A(\overline{W}_{n})) \) be the complex eigenvalues of \( B(\overline{W}_{n}) \) and \( A(\overline{W}_{n}) \) respectively, then \( \lambda_{c}(B(\overline{W}_{n})) = -\lambda_{c}(A(\overline{W}_{n})) \) for 4 cases as the following:

- \( \lambda_{c}(B(\overline{W}_{n})) = -\lambda_{c}(A(\overline{W}_{n})); \) \( m = l + k. \)

**Proof.** If \( \lambda_{c}(B(\overline{W}_{n})) + \lambda_{c}(A(\overline{W}_{n})) = 0, \) then \( \lambda_{c}(B(\overline{W}_{n})) = -\lambda_{c}(A(\overline{W}_{n})). \) It will be shown for every cases that \( \lambda_{c}(B(\overline{W}_{n})) + \lambda_{c}(A(\overline{W}_{n})) = 0. \)

- i. \( \lambda_{c}(B(\overline{W}_{n})) = -\lambda_{c}(A(\overline{W}_{n})); \) \( m = l + k. \)
- ii. \( \lambda_{c}(B(\overline{W}_{n})) = -\lambda_{c}(A(\overline{W}_{n})); \) \( m = l + k. \)
- iii. \( \lambda_{c}(B(\overline{W}_{n})) = -\lambda_{c}(A(\overline{W}_{n})); \) \( m = l + k. \)
- iv. \( \lambda_{c}(B(\overline{W}_{n})) = -\lambda_{c}(A(\overline{W}_{n})); \) \( m = l + k. \)
4. Conclusions

If \( \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n = 0 \) is the characteristic equation of \( A(W_n^*) \), then the value of the coefficients are summarized in table 1 and the eigenvalues of \( A(W_n^*) \) are summarized in table 2.

| The Coefficient | The Value of coefficient |
|-----------------|--------------------------|
| \( a_1 \)      | 0                        |
| \( a_2, \ldots, a_{n-2} \) | 0                        |
| \( a_{n-1} \)  | -1                       |
| \( a_n \)      | 0                        |

**Table 1.** The coefficients of \( P(A(W_n^*)) \)

| The number of \( n \) | The eigenvalues |
|-----------------------|-----------------|
| \( (n = 2k; k \geq 2) \) | \( \lambda = 0 \) and \( \lambda = 1 \) | \( \lambda = e^{i\frac{2m\pi}{2k-1}} \); \( m = 1, \ldots, 2k-2 \) |
| \( (n = 2k + 1; k \geq 2) \) | \( \lambda = -1, \lambda = 0 \) and \( \lambda = 1 \) | \( \lambda = e^{i\frac{m\pi}{k}} \); \( m = 1, \ldots, k-1, k+1, \ldots, 2k-1 \) |

From table 1 we conclude that the norm of every coefficient of \( P(A(W_n^*)) \) does not represent any structure of a directed cyclic wheel graph. Moreover we obtained that the complex eigenvalues of \( B(W_n^*) \) is equal to the negative of the complex eigenvalues of \( A(W_n^*) \).

From the discussion in section 4, it can be concluded about the strength and the weakness of this research. The strength of this research is there is a relation between the complex eigenvalues of \( B(W_n^*) \) and the complex eigenvalues of \( A(W_n^*) \). The weakness of this research is there is not any relation between the coefficient of the characteristic polynomial of the adjacency matrix \( A(W_n^*) \) with the structure of directed cyclic wheel graph \( W_n^* \). For further research, maybe another matrix representation of directed cyclic wheel graph \( W_n^* \) can be used to see whether there is any relation between the coefficients of the characteristic polynomial of this matrix representation with the structure of the graph.

**Acknowledgement**

This research is funded by Hibah PITTA UI [2307/UN2.R3.1/HKP.05.00/2018].

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