Yes-go cross-couplings in collections of tensor fields
with mixed symmetries of the type (3, 1) and (2, 2)

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Abstract

Under the hypotheses of analyticity, locality, Lorentz covariance, and Poincaré invariance of the deformations, combined with the requirement that the interaction vertices contain at most two space-time derivatives of the fields, we investigate the consistent cross-couplings between two collections of tensor fields with the mixed symmetries of the type (3, 1) and (2, 2). The computations are done with the help of the deformation theory based on a cohomological approach in the context of the antifield-BRST formalism. Our results can be synthesized in: 1. there appear consistent cross-couplings between the two types of

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field collections at order one and two in the coupling constant such that some of the gauge generators and of the reducibility functions are deformed, and 2. the existence or not of cross-couplings among different fields with the mixed symmetry of the Riemann tensor depends on the indefinite or respectively positive-definite behaviour of the quadratic form defined by the kinetic terms from the free Lagrangian.

Keywords: BRST symmetry; BRST cohomology; mixed symmetry tensor fields.

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1 Introduction

Tensor fields characterized by a mixed Young symmetry type (neither completely antisymmetric nor fully symmetric) attracted the attention lately on some important issues, like the dual formulation of field theories of spin two or higher, a Lagrangian first-order approach to some classes of massless mixed symmetry-type tensor gauge fields, suggestively resembling to the tetrad formalism of General Relativity, or the derivation of some exotic gravitational interactions.

There exist in fact three different dual formulations of linearized gravity in $D$ dimensions: the Pauli–Fierz description, the version based on a massless tensor field with the mixed symmetry $(D - 3, 1)$, and the formulation in terms of a massless tensor field with the mixed symmetry $(D - 3, D - 3)$ [20, 21]. The last two versions are obtained by dualizing on one and respectively on both indices the Pauli–Fierz field [7]. These dual formulations in terms of mixed symmetry tensor gauge fields have been systematically investigated from the perspective of $M$-theory [22, 23, 24].

An important matter related to the dual formulations of linearized gravity is the study of their consistent interactions, among themselves as well as with other gauge theories. The most efficient approach to this problem is the cohomological one, based on the deformation of the solution to the master equation [25]. Since the mixed symmetry tensor fields involved in dual formulations of linearized gravity allow no self-interactions, it was believed that they are also rigid under the introduction of couplings to other gauge theories. Nevertheless, recent results prove the contrary. For instance, it was shown that some theories with massless tensor fields exhibiting the mixed symmetry $(k, 1)$ can be consistently coupled to a vector field $(k = 3)$ [26], to
an arbitrary $p$-form ($k = 3$) \cite{27}, to a topological BF model ($k = 2$) \cite{28}, and to a massless tensor field with the mixed symmetry of the Riemann tensor ($k = 3$) \cite{29}. There is a revived interest in the construction of dual gravity theories, which led to several new results, viz. a dual formulation of linearized gravity in first order tetrad formalism in arbitrary dimensions within the path integral framework \cite{30} or a reformulation of non-linear Einstein gravity in terms of the dual graviton together with the ordinary metric and a shift gauge field \cite{31}.

A major result concerning spin-two fields within the standard formulation of Einstein–Hilbert gravity is the impossibility of cross-couplings in multi-graviton theories, either direct \cite{32} or intermediated by a scalar field \cite{32}, a Dirac spinor \cite{33}, a massive Rarita–Schwinger field \cite{34}, or a massless $p$-form \cite{35}. The same no-go outcome has occurred at the level of multi-Weyl graviton theories \cite{36, 37} and also in relation with dual formulations of linearized gravity \cite{38, 27}. These no-go results on multi-graviton theories are important since they provide new arguments for ruling out $N > 8$ extended supergravity theories, as they would involve more than one graviton.

The aim of this paper is to combine the study of consistent interactions between two different dual formulations of linearized gravity with the analysis of cross-couplings in collections of such dual multi-graviton theories. More precisely, we generate all consistent interactions in a collection of massless tensor fields with the mixed symmetry $(3, 1)$, $\{|t_{\lambda\mu|\kappa}\}_A^{A=1,N}$, and a collection of massless tensor fields with the mixed symmetry of the Riemann tensor, $\{|r^a_{\mu|\kappa\beta}\}_a^{a=1,n}$. Special attention will be paid to the existence of cross-couplings among different spin-two fields (with the mixed symmetry of the Riemann tensor) intermediated by the presence of tensor fields with the mixed symmetry $(3, 1)$. Our analysis relies on the deformation of the solution to the master equation by means of cohomological techniques with the help of the local BRST cohomology, whose component in a single $(3, 1)$ sector has been reported in detail in \cite{39} and in a single $(2, 2)$ sector has been investigated in \cite{40, 41}. The self-interactions in a collection of tensor fields with the mixed symmetry $(3, 1)$ and respectively $(2, 2)$ has been approached in \cite{42}. We work in the standard hypotheses on the deformations: analyticity in the coupling constant, locality, Lorentz covariance, Poincaré invariance, and preservation of the number of derivatives on each field (derivative order assumption). The derivative order assumption is translated here into
the requirement that the interaction vertices contain at most two space-time derivatives acting on the fields at all orders in the coupling constant.

We show that there exists a case where the deformed solution to the master equation outputs non-trivial cross-couplings. It stops at order two in the coupling constant and is defined on a space-time of dimension $D = 6$, i.e. precisely the dimension where the free fields with the mixed symmetry $(3, 1)$ become dual to the linearized limit of Hilbert–Einstein gravity. The interacting Lagrangian action contains only mixing-component terms of order one and two in the coupling constant. Both the gauge transformations and first-order reducibility functions of the tensor fields $(3, 1)$ are modified at order one in the coupling constant with terms characteristic to the $(2, 2)$ sector. On the contrary, the tensor fields with the mixed symmetry $(2, 2)$ remain rigid at the level of both gauge transformations and reducibility functions. The gauge algebra and the reducibility structure of order two are not modified during the deformation procedure, being the same like in the case of the starting free action. The most important result is that the existence of cross-couplings among different fields with the mixed symmetry of the Riemann tensor is essentially dictated by the behaviour of the metric tensor in the inner space of collection indices $a = 1, ..., n$, $\hat{k} = (k_{ab})$ (the quadratic form defined by the kinetic terms from the free Lagrangian density for the fields $\{r^{a}_{\mu\nu|\kappa\beta}\}_{a=1}^{n}$).

Thus, if $\hat{k}$ is positive-definite, then there appear no cross-couplings among different fields from the collection $\{r^{a}_{\mu\nu|\kappa\beta}\}_{a=1}^{n}$. On the contrary, if $\hat{k}$ is indefinite, then there are allowed cross-couplings among different fields from this collection.

2 Brief review of the deformation procedure

There are three main types of consistent interactions that can be added to a given gauge theory: (i) the first type deforms only the Lagrangian action, but not its gauge transformations, (ii) the second kind modifies both the action and its transformations, but not the gauge algebra, and (iii) the third, and certainly most interesting category, changes everything, namely, the action, its gauge symmetries, and the accompanying algebra.

The reformulation of the problem of consistent deformations of a given action and of its gauge symmetries in the antifield-BRST setting is based on the observation that if a deformation of the classical theory can be consis-
tently constructed, then the solution $S$ to the master equation for the initial theory can be deformed into the solution $\bar{S}$ of the master equation for the interacting theory

$$S \rightarrow \bar{S} = S + gS_1 + g^2S_2 + g^3S_3 + g^4S_4 + \cdots,$$

(1)

$$(S, S) = 0 \rightarrow (\bar{S}, \bar{S}) = 0.$$ 

(2)

The projection of (2) for $\bar{S}$ on the various powers of the coupling constant induces the following tower of equations:

$$g^0 : (S, S) = 0,$$

(3)

$$g^1 : (S_1, S) = 0,$$

(4)

$$g^2 : (S_2, S) + \frac{1}{2}(S_1, S_1) = 0,$$

(5)

$$g^3 : (S_3, S) + (S_1, S_2) = 0,$$

(6)

$$g^4 : (S_4, S) + (S_1, S_3) + \frac{1}{2}(S_2, S_2) = 0,$$

(7)

$$\vdots$$

The first equation is satisfied by hypothesis. The second one governs the first-order deformation of the solution to the master equation, $S_1$, and it expresses the fact that $S_1$ is a BRST co-cycle, $sS_1 = 0$, and hence it exists and is local. The remaining equations are responsible for the higher-order deformations of the solution to the master equation. No obstructions arise in finding solutions to them as long as no further restrictions, such as space-time locality, are imposed. Obviously, only non-trivial first-order deformations should be considered, since trivial ones ($S_1 = sB$) lead to trivial deformations of the initial theory, and can be eliminated by convenient redefinitions of the fields. Ignoring the trivial deformations, it follows that $S_1$ is a non-trivial BRST-observable, $S_1 \in H^0(s)$ (where $H^0(s)$ denotes the cohomology space of the BRST differential in ghost number zero). Once the deformation equations (4)–(7), etc.) have been solved by means of specific cohomological techniques, from the consistent, non-trivial deformed solution to the master equation one can extract all the information on the gauge structure of the resulting interacting theory.
3 Free model: Lagrangian formulation and BRST symmetry

We start from a free theory in $D \geq 5$ that describes two finite collections of massless tensor fields with the mixed symmetries $(3,1)$ and respectively $(2,2)$

$$S_0 \left[ t^A_{\lambda \mu \nu | \kappa}, r^a_{\mu \nu | \kappa \beta} \right] = S_0^t \left[ t^A_{\lambda \mu \nu | \kappa} \right] + S_0^r \left[ r^a_{\mu \nu | \kappa \beta} \right],$$

where

$$S^t_0 \left[ t^A_{\lambda \mu \nu | \kappa} \right] = \int \left\{ \frac{1}{2} \left[ (\partial_{\rho} t^A_{\lambda \mu \nu | \kappa}) (\partial_{\rho} t^A_{\lambda \mu \nu | \kappa}) - (\partial_{\kappa} t^A_{\lambda \mu \nu | \kappa}) (\partial_{\lambda} t^A_{\mu \nu | \kappa}) \right] 
- \frac{3}{2} \left[ (\partial_{\lambda} t^A_{\mu \nu | \kappa}) (\partial_{\mu} t^A_{\rho \nu | \kappa}) + (\partial_{\rho} t^A_{\lambda \mu | \kappa}) (\partial_{\lambda} t^A_{\mu \nu | \kappa}) \right] 
+ 3 \left[ (\partial_{\kappa} t^A_{\lambda \mu \nu | \kappa}) (\partial_{\lambda} t^A_{\mu \nu | \kappa}) + (\partial_{\lambda} t^A_{\mu \nu | \kappa}) (\partial_{\lambda} t^A_{\nu | \kappa}) \right] \right\} d^D x,$$

$$S^r_0 \left[ r^a_{\mu \nu | \kappa \beta} \right] = \int \left\{ -\frac{1}{2} \left[ (\partial_{\rho} r^a_{\mu | \kappa \beta}) (\partial_{\rho} r^a_{\mu | \kappa \beta}) + (\partial_{\lambda} r^a_{\nu | \kappa \beta}) (\partial_{\lambda} r^a_{\nu | \kappa \beta}) 
+ (\partial_{\beta} r^a_{\mu \nu | \kappa}) (\partial_{\beta} r^a_{\mu \nu | \kappa}) + \frac{1}{8} \left[ (\partial_{\lambda} r^a_{\nu | \kappa \beta}) (\partial_{\nu} r^a_{\mu | \kappa \beta}) + (\partial_{\nu} r^a_{\lambda | \kappa \beta}) (\partial_{\lambda} r^a_{\mu | \kappa \beta}) \right] 
- (\partial_{\lambda} r^a_{\mu \nu | \kappa}) (\partial_{\beta} r^a_{\nu | \kappa}) + (\partial_{\beta} r^a_{\mu \nu | \kappa}) (\partial_{\beta} r^a_{\mu \nu | \kappa}) \right\} d^D x.$$

Everywhere in this paper we employ the flat Minkowski metric of ‘mostly plus’ signature $\sigma_{\mu \nu} = \sigma_{\mu \nu} = (- + + + + + \ldots)$. The uppercase indices $A, B, \ldots$ stand for the collection indices of the fields with the mixed symmetry $(3,1)$ and are assumed to take discrete values: $1, 2, \ldots, N$. They are lowered with a symmetric, constant, and invertible matrix, of elements $k_{AB}$, and are raised with the help of the elements $k_{AB}$ of its inverse. This means that $t^A_{\lambda \mu \nu | \kappa} = k_{AB} t^B_{\lambda \mu \nu | \kappa}$ and $t^A_{\lambda \mu \nu | \kappa} = k_{AB} t^B_{\lambda \mu \nu | \kappa}$. Each field $t^A_{\lambda \mu \nu | \kappa}$ is completely antisymmetric in its first three (Lorentz) indices and satisfies the identity $t^A_{\lambda \mu \nu | \kappa} \equiv 0$. Here and in the sequel the notation $[\lambda \ldots \kappa]$ signifies complete antisymmetry with respect to the (Lorentz) indices between brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. The notation $t^A_{ \lambda \mu}$ signifies the trace of $t^A_{\lambda \mu \nu | \kappa}$, defined by $t^A_{\lambda \mu} = \sigma^{\nu \kappa} t^A_{\lambda \mu | \kappa}$. The trace components define an antisymmetric tensor, $t^A_{\lambda \mu} = -t^A_{ \mu \lambda}$. The lowercase indices $a, b, \ldots$ stand for the collection indices of the fields with the mixed symmetry $(2,2)$ and
are assumed to take the discrete values 1, 2, \ldots, n. They are lowered with a symmetric, constant, and invertible matrix, of elements \( k_{ab} \), and are raised with the help of the elements \( k^{ab} \) of its inverse, such that \( r_{\mu\nu|\kappa\beta}^a = k_{ab} r_{\mu\nu|\kappa\beta} \) and \( r_{\lambda|\kappa\beta}^a = k^{ab} r_{\lambda|\kappa\beta} \). Each tensor field \( r_{\mu\nu|\kappa\beta}^a \) is separately antisymmetric in the pairs \( \{ \mu, \nu \} \) and \( \{ \kappa, \beta \} \), is symmetric under their permutation \( \{ \mu, \nu \} \leftrightarrow \{ \kappa, \beta \} \), and satisfies the identity \( r_{\mu\nu|\kappa\beta}^a = \sigma^{\mu\kappa} r_{\mu\nu|\kappa\beta}^a \), which are symmetric, \( r_{\mu\nu|\kappa\beta}^a = r_{\nu\mu|\kappa\beta}^a \), while \( r_{\mu\nu|\kappa\beta}^a \) represent their double traces, \( r_{\mu\nu|\kappa\beta}^a = \sigma^{\mu\beta} r_{\mu\nu|\kappa\beta}^a \), which are scalars.

A generating set of gauge transformations of action (8) can be taken as

\[
\delta_{\epsilon, \chi} t^A_{\lambda|\mu\nu\kappa} = 3 \epsilon^A_{\lambda|\mu\nu\kappa} + \partial_\lambda \epsilon^A_{\mu\nu|\kappa} + \partial_\kappa \epsilon^A_{\mu\nu|\kappa}, \\
\delta_{\xi} r_{\mu\nu|\kappa\beta}^a = \xi_a^{\mu|\kappa\beta} + \xi_a^{\nu|\beta\kappa},
\]

where we used the standard notation \( f_\mu = \partial f / \partial x^\mu \). All the gauge parameters are bosonic, with \( \epsilon^A_{\lambda|\mu\nu} \) completely antisymmetric and \( \chi^A_{\mu\nu|\kappa} \) together with \( \xi_{\mu|\kappa} \) defining two collections of tensor fields with the mixed symmetry \((2, 1)\). The former gauge transformations, (11), are off-shell, second-order reducible in the space of all field histories, the associated gauge algebra being Abelian (see [39, 42]), while the gauge symmetries (12) are off-shell, first-order reducible, the corresponding algebra being also Abelian (see [40, 42]). It follows that the free theory (8) is a linear gauge theory with the Cauchy order equal to four. The simplest gauge invariant quantities are precisely the curvature tensors

\[
K^A_{\lambda|\mu\nu\xi|\kappa\beta} = t^A_{\lambda|\mu\nu\xi|\kappa\beta}, \quad F^A_{\mu\nu|\kappa\beta\gamma} = r_{\mu\nu|\kappa\beta\gamma}^a,
\]

and their space-time derivatives. It is easy to check that they display the mixed symmetry \((4, 2)\) and \((3, 3)\) respectively.

The construction of the BRST symmetry for the free model under study debuts with the identification of the algebra on which the BRST differential acts. The ghost spectrum comprises the fermionic ghosts \( \{ \eta^A_{\lambda|\mu\nu|\kappa}, G^A_{\mu\nu|\kappa}, C^a_{\mu\nu|\kappa} \} \) respectively associated with the gauge parameters \( \{ \epsilon^A_{\lambda|\mu\nu}, \chi^A_{\mu\nu|\kappa}, \xi^a_{\mu|\kappa} \} \) from (11) and (12), the bosonic ghosts for ghosts \( \{ C^A_{\mu\nu}, C^A_{\nu\kappa}, C^a_{\mu\nu} \} \) due to the first-order reducibility, and the fermionic ghosts for ghosts for ghosts \( \{ C^A_\nu \} \) corresponding to the maximum reducibility order (two). We ask that \( \eta^A_{\lambda|\mu\nu}, C^A_{\mu\nu}, \) and \( C^a_{\mu\nu} \) are completely antisymmetric, \( G^A_{\mu\nu|\kappa} \) and \( C^a_{\mu\nu|\kappa} \) exhibit the mixed symmetry \((2, 1)\), and \( G^A_\nu \) are symmetric. The antifield spectrum comprises the
antifields $\{t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa\beta}\}$ associated with the original fields and those corresponding to the ghosts, $\{\eta^A_{\lambda\mu\nu|\kappa}, G^A_{\mu\nu|\kappa}, C^a_{\mu\nu|\kappa}\}$, $\{C^{s\mu\nu}, G^{s\mu\nu}_{\nu\kappa}, C^A_a\}$, and $\{C^{s\mu\nu}_A\}$. The antifields are required to satisfy the same symmetry, antisymmetry, or mixed symmetry properties like the corresponding fields/ghosts. Related to the traces of the antifields, we will use the notations $t^A_{\lambda\mu\nu|\kappa} = \sigma_{\nu\kappa} t^A_{\lambda\mu\nu|\kappa}$, $r^a_{\mu\nu|\kappa\beta} = \sigma_{\mu\kappa} r^a_{\mu\nu|\kappa\beta}$, and $r^*_a = \sigma_{\nu\beta} r^*_a$. Since both the gauge generators and reducibility functions for this model are field-independent, it follows that the BRST differential $s$ simply reduces to

$$s = \delta + \gamma,$$

where $\delta$ represents the Koszul–Tate differential, graded by the antighost number $\text{agh}(\delta) = -1$, and $\gamma$ stands for the exterior longitudinal differential, whose degree is named pure ghost number $\text{pgh}(\gamma) = 1$. These two degrees do not interfere ($\text{agh}(\gamma) = 0$, $\text{pgh}(\delta) = 0$). The overall degree that grades the BRST complex is known as the ghost number $\text{gh}$ and is defined like the difference between the pure ghost number and the antighost number, such that $\text{gh}(s) = \text{gh}(\delta) = \text{gh}(\gamma) = 1$. According to the standard rules of the BRST method, the corresponding degrees of the generators from the BRST complex are valued like $\text{pgh}(t^A_{\lambda\mu\nu|\kappa}) = 0 = \text{pgh}(r^a_{\mu\nu|\kappa\beta})$, $\text{pgh}(G^A_{\mu\nu|\kappa}) = \text{pgh}(C^a_{\mu\nu|\kappa}) = 1$, $\text{pgh}(C^{s\mu\nu}) = \text{pgh}(G^{s\mu\nu}_{\nu\kappa}) = \text{pgh}(C^A_a) = 2$, $\text{pgh}(C^{s\mu\nu}_A) = 3$, $\text{pgh}(\Phi^A) = 0 = \text{agh}(\Phi^A)$,

$$\text{agh}(t^A_{\lambda\mu\nu|\kappa}) = 1 = \text{agh}(r^a_{\mu\nu|\kappa\beta}),$$

$$\text{agh}(\eta^A_{\lambda\mu\nu|\kappa}) = \text{agh}(G^{s\mu\nu}_{\nu\kappa}) = \text{agh}(C^A_a) = 2,$$

$$\text{agh}(C^{s\mu\nu}) = \text{agh}(G^{s\mu\nu}_{\nu\kappa}) = \text{agh}(C^A_a) = 3, \quad \text{agh}(C^{s\mu\nu}_A) = 4,$$

where we made the notations

$$\Phi^A = \{t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa\beta}, \eta^A_{\lambda\mu\nu|\kappa}, G^A_{\mu\nu|\kappa}, C^a_{\mu\nu|\kappa}, C^{s\mu\nu}_{\nu\kappa}, C^A_a, C^{s\mu\nu}_A\},$$

$$\Phi^*_A = \{t^*_A{\lambda\mu\nu|\kappa}, r^*_a{\mu\nu|\kappa\beta}, \eta^*_A{\lambda\mu\nu|\kappa}, G^{s\mu\nu}_{\nu\kappa}, C^{s\mu\nu}_{\nu\kappa}, C^{s\mu\nu}_A, G^{s\mu\nu}_A, C^{s\mu\nu}_A, C^{s\mu\nu}_A\}.$$

The Koszul–Tate differential is imposed to realize a homological resolution of the algebra of smooth functions defined on the stationary surface of field...
equations, while the exterior longitudinal differential is related to the gauge symmetries (see relations (11) and (12)) of action (8) through its cohomology at pure ghost number zero computed in the cohomology of $\delta$, which is required to be the algebra of physical observables for the free model under consideration. The actions of $\delta$ and $\gamma$ on the generators from the BRST complex, which enforce all the above mentioned properties, are given by

$$\gamma t^A_{\lambda \mu \nu | \kappa} = -3\partial_\lambda \eta^{\mu \nu | \kappa} + 4\partial (\lambda \eta^{\mu \nu | \kappa} + \partial_\lambda G^A_{\mu \nu | \kappa}; \quad (17)$$

$$\gamma r^a_{\mu \nu | \kappa \beta} = \partial_\mu C^{a}_{\kappa \beta | \nu} - \partial_\nu C^{a}_{\kappa \mu | \beta} + \partial_\beta C^{a}_{\mu \nu | \kappa}; \quad (18)$$

$$\gamma G^A_{\mu \nu | \kappa} = 2\partial_\mu C^A_{\nu | \kappa} - 3\partial_\mu G^A_{\nu | \kappa}, \quad (19)$$

$$\gamma C^A_{\mu \nu | \kappa} = 2\partial_\kappa C^A_{\mu \nu | \kappa} - \partial_\mu G^A_{\nu | \kappa}, \quad \gamma C^A_{\mu \nu} = 0, \quad (20)$$

$$\gamma C^A_{\mu \nu | \kappa} = \partial_\kappa C^A_{\mu \nu}, \quad \gamma C^A_{\mu \nu | \kappa} = -3\partial_\mu C^A_{\nu | \kappa}, \quad \gamma C^A_{\mu \nu} = 0, \quad (21)$$

$$\gamma A_{\mu \nu | \kappa} = -\frac{1}{2} \partial_\lambda C^A_{\mu \nu | \kappa}, \quad \gamma A_{\mu \nu} = \frac{1}{2} \partial_\lambda C^A_{\mu \nu}. \quad \gamma A_{\mu \nu | \kappa} = \partial_\lambda C^A_{\mu \nu | \kappa}, \quad (22)$$

$$\gamma \Phi_\Delta = 0 = \delta \Phi_\Delta, \quad (23)$$

$$\delta t^A_{\lambda \mu \nu | \kappa} = T^A_{\lambda \mu \nu | \kappa}, \quad \delta \eta^A_{\lambda \mu \nu} = -4\partial_\kappa t^A_{\lambda \mu \nu | \kappa}, \quad (24)$$

$$\delta G^A_{\mu \nu | \kappa} = -\partial_\lambda \left(3t^A_{\lambda \mu \nu | \kappa} - t^A_{\kappa \mu \nu | \lambda}\right), \quad \delta G^A_{\mu \nu} = \partial_\lambda G^A_{\mu \nu | \kappa}, \quad (25)$$

$$\delta C^A_{\mu \nu | \kappa} = \partial_\lambda \left(G^A_{\mu \nu | \lambda} - \frac{1}{2} \eta^A_{\mu \nu | \lambda}\right), \quad \delta G^A_{\mu \nu} = \partial_\lambda G^A_{\mu \nu | \kappa}, \quad (26)$$

$$\delta C^A_{\mu \nu | \kappa} = 3\partial_\lambda \left(G^A_{\mu \nu | \lambda} - \frac{1}{2} \eta^A_{\mu \nu | \lambda}\right), \quad \delta G^A_{\mu \nu} = \partial_\lambda G^A_{\mu \nu | \kappa}, \quad (27)$$

$$\delta r^a_{\mu \nu | \kappa \beta} = \frac{1}{4} R^a_{\mu \nu | \kappa \beta}, \quad \delta C^a_{\mu \nu | \kappa \beta} = -4\partial_\kappa r^a_{\mu \nu | \kappa \beta}, \quad \delta C^a_{\mu \nu} = 3\partial_\kappa C^a_{\mu \nu | \kappa}, \quad (28)$$

where $T^A_{\lambda \mu \nu | \kappa} = -\delta S^A_0/\delta t^A_{\lambda \mu \nu | \kappa}$ and $\delta S^A_0/\delta t^A_{\lambda \mu \nu | \kappa} \equiv -(1/4) R^a_{\mu \nu | \kappa \beta}$. By convention, we take $\delta$ and $\gamma$ to act like right derivations.

We note that the action of the Koszul–Tate differential on the antifields with the antighost number equal to two and respectively three from the $(3, 1)$ sector gains a simpler expression if we perform the changes of variables

$$G^A_{\mu \nu | \kappa} = G^A_{\mu \nu | \kappa} + \frac{1}{4} \eta^A_{\mu \nu \kappa}, \quad G^A_{\mu \nu} = G^A_{\mu \nu} - \frac{1}{3} C^A_{\mu \nu \kappa}. \quad (29)$$

The antifields $G^A_{\mu \nu | \kappa}$ are still antisymmetric in their first two indices, but do not fulfill the identity $G^A_{\mu \nu | \kappa} \equiv 0$, and $G^A_{\mu \nu}$ have no definite symmetry or
antisymmetry properties. With the help of relations (24)–(27), we find that \( \delta \) acts on the transformed antifields through the relations

\[
\delta G_A^{*\mu\nu|\kappa} = -3 \partial_\lambda t_A^{*\lambda\mu\nu|\kappa}, \quad \delta G_A^{*\kappa} = 2 \partial_\mu G_A^{*\mu\nu|\kappa}, \quad \delta C_A^{*\nu\kappa} = 6 \partial_\rho G_A^{*\rho\mu\nu}. \tag{30}
\]

The same observation is valid with respect to \( \gamma \) if we make the changes

\[
G_A^{*\mu\nu|\kappa} = G_A^{\mu\nu|\kappa} + 4 \eta_A^{\mu\nu\kappa}, \quad G_A^{*\nu\kappa} = G_A^{\nu\kappa} - 3 C_A^{\nu\kappa}; \tag{31}
\]

in terms of which we can write

\[
\gamma t_A^{*\lambda\mu\nu|\kappa} = -\frac{1}{4} \partial_\lambda G_A^{*\mu\nu|\kappa} + \partial_\lambda G_A^{*\mu\nu|\kappa}, \quad \gamma G_A^{*\kappa} = \partial_\mu G_A^{*\mu\nu|\kappa}, \quad \gamma G_A^{*\nu\kappa} = -6 \partial_\rho C_A^{\nu\kappa}. \tag{32}
\]

Again, \( G_A^{*\mu\nu|\kappa} \) are antisymmetric in their first two indices, but do not satisfy the identity \( G_A^{*\mu\nu|\kappa} \equiv 0 \), while \( G_A^{*\nu\kappa} \) have no definite symmetry or antisymmetry. We have deliberately chosen the same notations for the transformed variables (29) and (31) since they actually form pairs that are conjugated in the antibracket

\[
\left( G_A^{*\mu\nu|\kappa}, G_B^{*\mu_1\nu_1|\kappa_1} \right) = \frac{1}{2} \delta_A^{A\mu\nu_1\nu_1} \delta_B^{\mu_1\nu_1\kappa_1} \delta_A^{\nu_1\mu\kappa_1}, \quad \left( G_A^{*\nu\kappa}, G_B^{*\mu_1\nu_1|\kappa_1} \right) = \delta_A^{A\mu\nu_1\nu_1} \delta_B^{\nu_1\mu\kappa_1}. \tag{33}
\]

The Lagrangian BRST differential admits a canonical action in a structure named antibracket and defined by decreeing the fields/ghosts conjugated with the corresponding antifields, \( s \cdot = (\cdot, S) \), where \( (\cdot, S) \) signifies the antibracket and \( S \) denotes the canonical generator of the BRST symmetry. It is a bosonic functional of ghost number zero, involving both field/ghost and antifield spectra, that obeys the master equation \( (S, S) = 0 \). The master equation is equivalent with the second-order nilpotency of \( s \), where its solution \( S \) encodes the entire gauge structure of the associated theory. Taking into account formulas (17)–(28) as well as the standard actions of \( \delta \) and \( \gamma \) in canonical form, we find that the complete solution to the master equation for the free model under study is given by

\[
S = S^t + S^r, \tag{33}
\]

where

\[
S^t = S^t_0 + \int \left[ t_A^{*\lambda\mu\nu|\kappa} \left( 3 \partial_\kappa \eta_A^{\lambda\mu\nu} + \partial_\nu \eta_A^{\mu\nu|\kappa} + \partial_\lambda G_A^{\mu\nu|\kappa} \right) \right].
\]

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\[-\frac{1}{2} \eta_A^{\lambda \mu \nu} \partial_\lambda C^A_{\mu \nu} + G_A^{\mu \nu | \kappa} (2 \partial_\kappa C^A_{\mu \nu} - \partial_{[\mu} C^A_{\nu] | \kappa} + \partial_{| \mu} G^A_{\nu | \kappa})
\]
\[+ C_A^{\mu \nu} \partial_{[\mu} C^A_{\nu]} - 3 G_A^{\mu \nu | \kappa} \partial_{(\nu} C^A_{\kappa)} \]  

\[d^D x, \quad (34)\]

\[
S^r = S^r_0 \left[ t^a_{\mu \nu | \kappa \beta} \right] + \int \left[ r^a_{\mu \nu | \kappa \beta} \left( \partial_\mu C^a_{\kappa \beta | \nu} - \partial_\nu C^a_{\kappa \beta | \mu} + \partial_\kappa C^a_{\mu \nu | \beta} - \partial_\beta C^a_{\mu \nu | \kappa} \right) \right]
\]
\[+ C^a_{\mu \nu | \kappa} \left( 2 \partial_\kappa C^a_{\mu \nu} - \partial_{| \mu} C^a_{\nu | \kappa} \right) d^D x. \quad (35)\]

4 Computation of basic cohomologies

In the sequel we investigate the consistent couplings that can be added to the free theory (8) without modifying either the field spectrum or the number of independent gauge invariances. In view of this we apply the deformation procedure based on local BRST cohomology exposed in section 2 and solve equations (4)–(7), etc. The space-time locality of the deformations is ensured by working in the algebra of local differential forms with coefficients that are polynomial functions in the fields, ghosts, antifields, and their space-time derivatives (algebra of local forms). In other words, the non-integrated density of the first-order deformation, \(a\), is assumed to be a polynomial function in all these variables (algebra of local functions). The derivative order assumption restricts the interaction Lagrangian to contain only interaction vertices with maximum two space-time derivatives.

It is natural to decompose \(a\) as a sum of three components

\[a = a^t + a^r + a^{\text{int}}, \quad (36)\]

where \(a^t\) denotes the part responsible for the self-interactions of the fields \(t^A_{\lambda \mu \nu | \kappa}\); \(a^r\) is related to the self-interactions of the fields \(r^a_{\mu \nu | \kappa \beta}\); and \(a^{\text{int}}\) signifies the component that describes only the cross-couplings between \(t^A_{\lambda \mu \nu | \kappa}\) and \(r^a_{\mu \nu | \kappa \beta}\), so each term must mix the BRST generators from the two sectors. According to decomposition (36), equation \(sa = \partial_\mu m^\mu\) becomes equivalent with three equations

\[sa^t = \partial_\mu m^\mu_t, \quad sa^r = \partial_\mu m^\mu_r, \quad sa^{\text{int}} = \partial_\mu m^\mu_{\text{int}}. \quad (37)\]

The most general solutions to the first two equations from (37) were approached in [42], where it was shown that

\[a^t = 0, \quad a^r = c_a r^a, \quad (38)\]
with $c_a$ some arbitrary, real constants and $r^a$ the contractions of order two of the fields $r^a_{\mu\nu|\kappa\beta}$. In the sequel we approach the last equation from (37).

Developing $a^{\text{int}}$ according to the antighost number and assuming that this expansion stops at a maximum, finite value $I$ of this degree, we find that the equation $s a^{\text{int}} = \partial_{\mu} m^{\mu}_{\text{int}}$ becomes equivalent to the chain

$$\gamma a^I_{1} = \partial_{\mu} m^{\mu}_{\text{int}},$$  

$$\delta a^I_{1} + \gamma a^I_{-1} = \partial_{\mu} m^{(I-1)}_{\text{int}},$$  

$$\delta a^I_{k} + \gamma a^I_{k-1} = \partial_{\mu} m^{(k-1)}_{\text{int}}, \quad I - 1 \geq k \geq 1.$$  

Equation (39) can be replaced in strictly positive values of the antighost number with

$$\gamma a^I_{1} = 0, \quad \text{agh} (a^I_{1}) = I > 0.$$  

At this stage we notice that equation $s a^{\text{int}} = \partial_{\mu} m^{\mu}_{\text{int}}$ means that $a^{\text{int}} d^D x \in H^{0,D} (s|d)$, while equation (42) shows that for $I > 0$ $a^I_{1} \in H^{*} (\gamma)$ (cohomology algebra of the exterior longitudinal differential $\gamma$ computed in the algebra of local functions mentioned in the above). Consequently, we need to compute $H^{*} (\gamma)$. Combining the results inferred in (42) on the cohomology algebra of the exterior longitudinal differential in each sector, we obtain that the cohomology algebra $H^{*} (\gamma)$ computed in the algebra of local functions is generated on one hand by the antifields (16), the curvature tensors (13), and their space-time derivative and, on the other hand, by the ghosts or ghost combinations $F_{A\lambda\mu\nu\kappa}$, $C_{a\nu}$, $C_{\mu\nu}$, and $\partial_{[\mu} C_{\nu\kappa]}$, where

$$F_{A\lambda\mu\nu\kappa} = \partial_{[\lambda} \eta^{A}_{\mu\nu\kappa]}.$$  

Therefore, the general, local solution to equation (42) is expressed (up to trivial, $\gamma$-exact contributions) by

$$a^I_{1} = \alpha_{I} \left( [K^{A}], [F^{a}], [\Phi^{*}_{\Delta}] \right) \omega^{I} \left( F_{A\lambda\mu\nu\kappa}^{A}, C_{a\nu}^{a}, \partial_{[\mu} C_{\nu\kappa]}^{a}, C_{\nu}^{A} \right).$$  

The notation $f([q])$ means that $f$ depends on $q$ and its derivatives up to a finite order. In the above $\Phi^{*}_{\Delta}$ denote all the antifields (see formula (16)) and $\omega^{I}$ represent the elements of pure ghost number $I$ (and antighost number zero) of a basis in the space of polynomials in $F_{A\lambda\mu\nu\kappa}^{A}, C_{a\nu}^{a}, \partial_{[\mu} C_{\nu\kappa]}^{a}$, and $C_{\nu}^{A}$. The objects $\alpha_{I}$ are non-trivial elements of the space $H^{0} (\gamma)$ and by hypothesis are polynomials in all the quantities on which they depend, so they are
nothing but the invariant polynomials of the free theory \([8]\) in form degree equal to zero.

Replacing solution (44) into equation (40), we get that a necessary condition for the existence of non-trivial solutions \(a^{\text{int}}_{I-1}\) for \(I > 0\) is that the invariant polynomials \(\alpha_I\) appearing in (44) generate non-trivial elements from the characteristic cohomology \(H^D_I(\delta|d)\) in antighost number \(I > 0\), maximum form degree, and pure ghost number equal to zero computed in the algebra of local forms, \(\alpha_I d^D x \in H^D_I(\delta|d)\). As the free model under study is a linear gauge theory of Cauchy order equal to four, the general results from [43] ensure that

\[
H^D_j(\delta|d) = 0, \quad j > 4.
\]  

(45)

Meanwhile, it is possible to prove (see, for instance, Appendix B, Theorem 3, from [39]) that if \(\alpha_J d^D x\) is a trivial element of \(H^D_j(\delta|d)\) for \(j > 4\), then it can be chosen to be trivial also in the local cohomology of the Koszul–Tate differential computed in the space of invariant polynomials in antighost number \(j\) and maximum form degree (invariant characteristic cohomology), \(H^\text{inv}_j(\delta|d)\). This is important since together with (45) ensures that the entire invariant characteristic cohomology is trivial in antighost numbers strictly greater than four

\[
H^\text{inv}_j(\delta|d) = 0, \quad j > 4.
\]  

(46)

With the help of the general results from [42] on the characteristic cohomology in the \((3,1)\) and respectively \((2,2)\) sector, we identify the non-trivial and Poincaré-invariant representatives of the spaces \(H^D_j(\delta|d)\) and \(H^\text{inv}_j(\delta|d)\).

All the coefficients from Table I denoted by \(f\) or \(g\) define some constant, non-derivative tensors. We remark that there is no non-trivial element in \(H^D_j(\delta|d)\) or \(H^\text{inv}_j(\delta|d)\) that effectively involves the curvature tensors and/or their derivatives, and the same stands for the quantities that are more than linear in the antifields and/or depend on their derivatives. In principle, one can construct from the above elements in Table I other non-trivial invariant polynomials from \(H^D_j(\delta|d)\) or \(H^\text{inv}_j(\delta|d)\), which depend on the space-time co-ordinates. For instance, it can be checked by direct computation that \(G^{\mu\nu|\kappa}_{\Lambda} f^{A}_{\mu\nu\kappa, x^D x}\), with \(f^{A}_{\mu\nu\kappa,}\) some completely antisymmetric and constant tensors, generate non-trivial representatives from

\[\text{We recall that the local cohomology } H^D_j(\delta|d) \text{ is completely trivial at both strictly positive antighost and pure ghost numbers (for instance, see [43], Theorem 5.4 and [44]).}\]
Table 1: Non-trivial representatives spanning $H^D_j(\delta|d)$ and $H^{invD}_j(\delta|d)$

| $j$ | $H^D_j(\delta|d)$, $H^{invD}_j(\delta|d)$ |
|-----|---------------------------------------|
| $j > 4$ | none |
| $j = 4$ | $f^A_j C^a_{\delta|d} d^D x$ |
| $j = 3$ | $(f^A_{\nu\kappa} G^a_{\mu\nu\kappa} + g^a_{\mu\nu} C^a_{\nu\mu}) d^D x$ |
| $j = 2$ | $(f^A_{\mu\nu\kappa} G^a_{\mu\nu\kappa} + g^a_{\mu\nu\kappa} C^a_{\mu\nu\kappa}) d^D x$ |

both $H^D_2(\delta|d)$ and $H^{invD}_2(\delta|d)$. However, we will discard such candidates as they would break the Poincaré invariance of the deformations. In contrast to the groups $(H^P_j(\delta|d))_{j \geq 2}$ and $(H^{invP}_j(\delta|d))_{j \geq 2}$, which are finite-dimensional, the cohomology $H^D_1(\delta|d)$ at pure ghost number zero, that is related to global symmetries and ordinary conservation laws, is infinite-dimensional since the theory is free.

5 First-order deformation

The previous results on $H^D_j(\delta|d)$ and $H^{invD}_j(\delta|d)$ are important because they control the obstructions to removing the antifields from the first-order deformation. Indeed, due to (46), it follows that we can successively eliminate all the pieces with the antighost number $j > 4$ from the non-integrated density of the first-order deformation by adding only trivial terms, so we can take, without loss of non-trivial objects, the condition $I \leq 4$ in the first-order deformation. The last representative, $a^{int}_I$, is of the form (44), where the invariant polynomials necessarily generate non-trivial objects from $H^{invP}_I(\delta|d)$ if $I = 2, 3, 4$ and respectively from $H^P_I(\delta|d)$ if $I = 1$. The cases $I = 4$ and $I = 3$ lead to purely trivial solutions and will be analyzed in Appendix A.

Next, we approach the case $I = 2$

$$a^{int} = a^{int}_0 + a^{int}_1 + a^{int}_2,$$ (47)

where $a^{int}_2$ is the general solution to the homogeneous equation $\gamma a^{int}_2 = 0$, and thus of the type (44) for $I = 2$, with $\alpha_2$ an invariant polynomial from $H^{invD}_2(\delta|d)$. With the help of Table 1 for $j = 2$, we obtain that the general
solution fulfilling all the working hypotheses takes the form

\[ a_2^{\text{int}} = G_A^{[*\mu\nu]\beta} \left( P_{a\mu\nu\beta} C_{\lambda\rho}^a + Q_{a\mu\nu\beta}^A \partial_\lambda C_{\rho\sigma}^a \right), \]  

(48)

where \( P_{a\mu\nu\beta}^{\lambda\rho\sigma} \) and \( Q_{a\mu\nu\beta}^{\lambda\rho\sigma} \) are some non-derivative, real constants, with the properties \( P_{a\mu\nu\beta}^{\lambda\rho\sigma} = -P_{a\rho\lambda\beta}^{\mu\nu\sigma} \) and \( Q_{a\mu\nu\beta}^{\lambda\rho\sigma} = Q_{a\mu\nu\beta}^{A[\lambda\rho\sigma]} \). Acting with \( \delta \) on (48), we infer

\[ \delta a_2^{\text{int}} = \gamma \lambda_1 + \partial^\mu k_\mu + t_A^{s*\mu\nu|\beta} P_{a\mu\nu\beta}^{\lambda\rho\sigma} \partial_\tau C_{\lambda\rho}^a, \]  

(49)

where

\[ \lambda_1 = t_A^{s*\mu\nu|\beta} P_{a\mu\nu\beta}^{\lambda\rho\sigma} C_{\lambda\rho}^a + \frac{3}{2} t_A^{s*\mu\nu|\beta} Q_{a\mu\nu\beta}^{\lambda\rho\sigma} \partial_\lambda C_{\rho\sigma}^a |_\tau. \]  

(50)

From (49) we find that \( a_1^{\text{int}} \) as solution to equation (40) for \( I = 2 \) exists if and only if the last term in the right-hand side of (49) is \( \gamma \)-exact modulo \( d \)

\[ t_A^{s*\mu\nu|\beta} P_{a\mu\nu\beta}^{\lambda\rho\sigma} \partial_\tau C_{\lambda\rho}^a = \gamma u_1 + \partial^\mu q_\mu. \]  

(51)

Taking the (left) Euler–Lagrange derivative of the above equation with respect to \( t_A^{s*\mu\nu|\beta} \) and recalling the anticommutativity of this operation with \( \gamma \), we deduce

\[ P_{a\mu\nu\beta}^{\lambda\rho\sigma} \partial_\tau C_{\lambda\rho}^a = \gamma \left( -\frac{\delta L u_1}{\delta t_A^{s*\mu\nu|\beta}} \right), \]  

(52)

The previous equation reduces to the requirement that the object

\[ P_{a\mu\nu\beta}^{\lambda\rho\sigma} \partial_\tau C_{\lambda\rho}^a, \]  

(53)

which is a non-trivial element of \( H^2(\gamma) \) (see relation (44)), must be \( \gamma \)-exact. This holds if and only if \( P_{a\mu\nu\beta}^{\lambda\rho\sigma} = 0 \). The last result replaced in formulas (48)–(50) yields

\[ a_2^{\text{int}} = G_A^{[*\mu\nu]\beta} Q_{a\mu\nu\beta}^{A\lambda\rho\sigma} \partial_\lambda C_{\rho\sigma}^a, \]  

(49)

\[ \delta a_2^{\text{int}} = \gamma \left( \frac{3}{2} t_A^{s*\mu\nu|\beta} Q_{a\mu\nu\beta}^{\lambda\rho\sigma} \partial_\lambda C_{\rho\sigma}^a |_\tau \right) + \partial^\mu k_\mu. \]  

(55)

Equation (55) produces in a simple manner the solution \( a_1^{\text{int}} \) to equation (40) for \( I = 2 \) as

\[ a_1^{\text{int}} = -\frac{3}{2} t_A^{s*\mu\nu|\beta} Q_{a\mu\nu\beta}^{\lambda\rho\sigma} \partial_\lambda C_{\rho\sigma}^a |_\tau + \bar{a}_1^{\text{int}}, \]  

(56)
where \( \tilde{a}_1^{\text{int}} \) means the general solution to the homogeneous equation \( \gamma \tilde{a}_1^{\text{int}} = 0 \). Recalling once more all the working hypotheses, we conclude that

\[
\tilde{a}_1^{\text{int}} = t^{*_a \mu \nu | \kappa \beta} \mathcal{Z}_{A \mu \kappa \beta}^{a \sigma \tau \gamma \delta} \mathcal{F}_A, 
\]

(57)

where \( \mathcal{Z}_{A \mu \kappa \beta}^{a \sigma \tau \gamma \delta} \) denote some real, non-derivative constants, which are completely antisymmetric with respect to the indices \( \{ \sigma, \tau, \gamma, \delta \} \). Due to the mixed symmetry properties of the antifields \( t^{*_a \mu \nu | \kappa \beta} \) and \( r^{*_a \mu \nu | \kappa \beta} \), the only covariant choice of the tensors \( Q_{a \mu \nu \beta}^{A \lambda \rho \sigma} \) and \( Z_{A \mu \kappa \beta}^{a \sigma \tau \gamma \delta} \) in \( D \geq 5 \) that does not end up with trivial solutions reads as

\[
Q_{a \mu \nu \beta}^{A \lambda \rho \sigma} = \frac{4}{3} f_a^{\lambda \rho \sigma} \nabla_{A \mu \nu \beta} \partial_\kappa C^a_{\beta \gamma},
\]

(58)

with \( \varepsilon_{\mu \nu \beta \lambda \rho \sigma} \) the six-dimensional Levi–Civita symbol and \( f_a^{\lambda \rho \sigma} \) some real constants. Inserting (58) in formulas (54) and (56)–(57) and recalling transformations (29), we finally obtain

\[
a_2^{\text{int}} = f_a^{A \lambda \mu \nu \beta \gamma} R^{A \lambda \mu \nu \beta \gamma}_{A \mu \nu \beta} \partial_\kappa C^a_{\beta \gamma},
\]

(59)

\[
a_1^{\text{int}} = -2 f_a^{A \lambda \mu \nu \beta \gamma} t^{*_A \mu \nu | \kappa} \left( \partial_\rho C^a_{\beta \gamma | \kappa} - \frac{1}{4} \delta^{\gamma \rho} \partial_\tau C^a_{\beta \tau | \tau} \right), \quad \tilde{a}_1^{\text{int}} = 0.
\]

(60)

The last term from the right-hand side of \( a_1^{\text{int}} \) is vanishing due to the identity \( t^{*_A \mu \nu | \kappa} \equiv 0 \), but it has been introduced in order to restore the mixed symmetry (3, 1) of the Euler–Lagrange derivatives \( \frac{\delta L}{\delta t^{*_A \mu \nu | \kappa}} \). By means of (60) we infer

\[
\delta a_1^{\text{int}} = \gamma \left[ 2 f_a^{A \lambda \mu \nu \beta \gamma} t^{*_A \mu \nu | \rho} \left( \partial_\sigma \partial_\kappa r_{\beta \gamma | \rho}^{a} - \frac{1}{2} \delta^{\gamma \rho} \partial_\tau \partial_\kappa r_{\beta \tau | \tau}^{a} \right) \right] + \partial_\mu p_\mu.
\]

(61)

The last relation generates the interacting Lagrangian at order one in the coupling constant as the solution \( a_0^{\text{int}} \) of equation (11) for \( k = 1 \)

\[
a_0^{\text{int}} = -2 f_a^{A \lambda \mu \nu \beta \gamma} t^{*_A \mu \nu | \rho} \left( \partial_\sigma \partial_\kappa r_{\beta \gamma | \rho}^{a} - \frac{1}{2} \delta^{\gamma \rho} \partial_\tau \partial_\kappa r_{\beta \tau | \tau}^{a} \right) + \tilde{a}_0^{\text{int}}.
\]

(62)

Here, \( \tilde{a}_0^{\text{int}} \) is the general solution to the ‘homogeneous’ equation

\[
\gamma \tilde{a}_0^{\text{int}} = \partial_\mu \tilde{m}_\mu^{\text{int}},
\]

(63)
which cannot be replaced any longer with the homogeneous one since the antighost number is vanishing, \( I = 0 \). Without entering technical details, we mention that the solution to equation (63) that fulfills all the working hypotheses is also trivial
\[
\bar{a}^{\text{int}}_0 = 0.
\] (64)
The proof of this result is done in Appendix B.

Putting together the results expressed by formulas (59)–(60), (62), and (64), we can state that the most general form of the first-order deformation associated with the free theory (8) reads
\[
S_1 = \int \left[ c_a r^a + f_a^A \varepsilon_{\mu \nu \lambda \beta \gamma} t_A^{\mu \nu \lambda \beta \gamma \kappa} \partial^\lambda C^{\alpha \beta \gamma} - 2f_a^A \varepsilon_{\lambda \mu \rho \beta \gamma} t_A^{\lambda \mu \rho \beta \gamma \kappa} \left( \partial^\rho C^{\alpha \beta \gamma} - \frac{1}{4} \delta^\rho_{\kappa} \partial^\alpha C^{\sigma \beta \gamma} \right) - 2f_a^A \varepsilon_{\lambda \mu \kappa \beta \gamma} t_A^{\lambda \mu \kappa \beta \gamma \rho} \left( \partial_\sigma \partial_\kappa r^a_{\sigma \beta \gamma} - \frac{1}{2} \delta^\sigma_{\rho} \partial^\tau \partial_\kappa r^a_{\tau \beta \gamma} \right) \right] d^6x \] (65)
and is defined on a space-time of dimension \( D = 6 \).

6 Higher-order deformations

In the sequel we approach the higher-order deformation equations. The second-order deformation is controlled by equation (5). After some computations, with the help of relation (65) we arrive at
\[
(S_1, S_1) = s \left[ f_a^A f_b^A \int \left( 10r_a^{\lambda \rho [\kappa \beta, \gamma] \rho b} r_{\lambda \rho [\kappa \beta, \gamma]} b - 12r_a^{[\kappa \beta, \rho] \rho b \lambda \sigma} r_{[\kappa \beta, \sigma]} b \right) d^6x \right],
\] (66)
such that the second-order deformation of the solution to the master equation reduces to
\[
S_2 = f_a^A f_b^A \int \left( -5r_a^{\lambda \rho [\kappa \beta, \gamma] \rho b} r_{\lambda \rho [\kappa \beta, \gamma]} b + 6r_a^{[\kappa \beta, \rho] \rho b \lambda \sigma} r_{[\kappa \beta, \sigma]} b \right) d^6x, \] (67)
where
\[
r_a^{\lambda \rho [\kappa \beta, \gamma]} = \partial^\gamma r_a^{\lambda \rho [\kappa \beta]} + \partial^\beta r_a^{\lambda \rho [\gamma \kappa]} + \partial^\kappa r_a^{\lambda \rho [\beta \gamma]}.
\] (68)
Introducing relations (65) and (67) into the equation corresponding to the third-order deformation, \( (S_1, S_2) = 0 \), it follows that
we can take

\[ S_3 = 0. \]  

(69)

Under these conditions, it is easy to see that all the remaining higher-order deformation equations are fulfilled with the choice

\[ S_k = 0, \quad k > 3. \]  

(70)

In conclusion, the complete deformed solution to the master equation for the model under study, which is consistent to all orders in the coupling constant, reduces to

\[ \bar{S} = S + gS_1 + g^2S_2, \]  

(71)

where \( S \) is the solution to the classical master equation for the free model in \( D = 6 \), (33), and \( S_{1,2} \) are expressed by (65) and respectively (67).

7 Identification of the coupled model

From relations (71), (33), (65), and (67) we deduce the concrete form of the deformed solution to the master equation

\[
\bar{S} = S + g \int \left[ c_a r^a + f_a^A \varepsilon_{\mu\nu\lambda\beta\gamma} r_{A}^{\mu\nu\lambda\beta\gamma} \partial^\lambda C^{a\beta\gamma} \right. \\
- 2 f_a^A \varepsilon_{\lambda\mu\rho\beta\gamma} t_{A}^{\lambda\mu\rho\beta\gamma} \left( \partial^\rho C^{a\beta\gamma} \right) - \frac{1}{4} \delta^\rho_{\rho} \partial^\rho C^{a\beta\gamma} \right] \, d^6 x \\
- 2 f_a^A \varepsilon_{\lambda\mu\rho\beta\gamma} t_{A}^{\lambda\mu\rho\beta\gamma} \left( \partial_\lambda r_{\beta\gamma}^a - \frac{1}{2} \delta^\rho_{\beta} \partial^\rho \partial_\lambda r_{\beta\gamma}^a \right) \right) \, d^6 x \\
- g^2 \int f_a^A f_b^A \left( 5 r_{a}^{\lambda\rho|\kappa\beta,\gamma} \right) \left[ t_{A}^{\beta\gamma|\lambda\rho} - 6 r_{a}^{\beta\gamma|\lambda\rho} t_{A}^{\beta\gamma|\lambda\rho} \right] \, d^6 x. 
\]  

(72)

The last formula enables us to identify the entire information on the gauge structure of the interacting theory. In view of this, we employ the fact that the piece of antighost number zero from \( \bar{S} \) is nothing but the Lagrangian action of the coupled model, the terms of antighost number one furnish the deformed gauge symmetries, and the components of antighost number greater or equal to two offer us information on the associated gauge algebra and the reducibility structure of the generating set of deformed gauge transformations. As a consequence, we deduce the coupled Lagrangian action

\[
\bar{S}_0 \left[ t_{A}^{\lambda\mu\nu|\kappa\beta}, r_{\mu\nu|\kappa\beta}^a \right] = S_0 \left[ t_{A}^{\lambda\mu\nu|\kappa\beta}, r_{\mu\nu|\kappa\beta}^a \right]
\]  

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cross-couplings among different fields from the collection convenient to work with the symmetric matrix \( \hat{\mathbf{M}} \) of the Riemann tensor non-trivial couplings between different tensor fields with the mixed symmetry one and two in the coupling constant. Apparently, it seems that (73) contains

\[
\text{We observe that action (73) contains only mixing-component terms of order}
\]

\[
\text{M matrix}
\]

\[
\text{The appearance of these cross-couplings is dictated by the properties of the}
\]

\[
S_0\left[t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa}\right] \text{ is the free action (8) in } D = 6 \text{ space-time dimensions.}
\]

We observe that action (73) contains only mixing-component terms of order one and two in the coupling constant. Apparently, it seems that (73) contains non-trivial couplings between different tensor fields with the mixed symmetry of the Riemann tensor

\[
-g f^A_A f^B_B \left( 5 r^a_{\lambda\rho|\kappa|\beta} r_{a\rho|\kappa|\beta} - 6 r^a_{\alpha\lambda\rho} r^{b\lambda\sigma} \right) dt^6 \text{x},
\]

(73)

\[
The appearance of these cross-couplings is dictated by the properties of the matrix \( M \) of elements \( M^b_a = f^A_A f^B_B \).
\]

Let us analyze the properties of the quadratic matrix \( M \). It is more convenient to work with the symmetric matrix \( \hat{M} = (M_{ab}) \), of elements \( M_{ab} = f^A_A f^B_B k_{AB} \). From (10) and (73) we observe that there appear effective cross-couplings among different fields from the collection \( \{ r^a_{\mu\nu|\kappa} \}_{a=1}^n \) if and only if the symmetric matrices \( \hat{M} = (M_{ab}) \) and \( \hat{k} = (k_{ab}) \) are simultaneously diagonalizable. We recall \( \hat{k} \) is the quadratic form defined by the kinetic terms of action (10), or, in other words, the metric tensor in the inner space of collection indices \( a = 1, n \). This means that there exists an orthogonal matrix \( \hat{O} = (O^a_b) \) that diagonalizes simultaneously \( \hat{M} \) and \( \hat{k} \), i.e.

\[
O^c_a O^d_b k_{cd} = k_a \delta_{ab}, \quad O^c_a O^d_b M_{cd} = m_a \delta_{ab},
\]

(75)

where \( k_a \) represent the eigenvalues of the matrix \( \hat{k} \) and \( m_a \) those of \( \hat{M} \). Indeed, if there exists a matrix \( \hat{O} \) that satisfies the conditions (75), then action (73) can be brought to the form

\[
\tilde{S}_0 \left[t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa}\right] = \tilde{S}_0^r \left[t^A_{\lambda\mu\nu|\kappa}, r^a_{\mu\nu|\kappa}\right] = \tilde{S}_0^k \left[t^A_{\lambda\mu\nu|\kappa}\right]
\]

\[
+ \frac{1}{2} \sum_{a=1}^n k_a \left\{ \frac{1}{2} \left[ \left( \partial_{\mu} r^{a\mu\nu|\kappa\beta} \right) \left( \partial_{\lambda} r^a_{\mu\nu|\kappa\beta} \right) + \left( \partial_{\lambda} r^{a\mu\nu|\kappa\beta} \right) \left( \partial_{\lambda} r^a_{\mu\nu|\kappa\beta} \right) \right] + \left( \partial_{\mu} r^{a\mu\nu|\kappa\beta} \right) \left( \partial_{\beta} r^a_{\mu\nu|\kappa\beta} \right) \right\} dt^6 \text{x}
\]

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\[ +g \int \left[ c^a r^a_{\mu
u} - 2 f^A_{\alpha} \varepsilon^{\lambda\mu\nu\beta\gamma} t_{\lambda\mu\nu|\rho} \left( \partial_\sigma \partial_\kappa r^a_{\beta\gamma} \right) \sigma^\rho - \frac{1}{2} \delta^\rho_\gamma \partial^\tau \partial_\kappa r^a_{\beta\tau} \right] \]
\[ - g \sum_{a=1}^{n} m_a \left( 5 r^a_{\mu\rho|\kappa|\beta|\gamma} r^a_{\lambda\rho|\kappa|\beta|\gamma} - 6 r^a_{\lambda|\mu\rho|\kappa|\beta|\gamma} \right) \right] d^6x, \quad (76) \]

where we made the transformations
\[ r^a_{\mu\nu|\kappa|\beta} \rightarrow r^a_{\mu\nu|\kappa|\beta} = \bar{O}^a_{\mu\nu|\kappa|\beta}; \quad (77) \]

and used the notations
\[ c'_a = c_\beta O^b a, \quad f'_a = f^A_{\alpha} O^b a. \quad (78) \]

The quantities \( \bar{O}^a_{\mu\nu|\kappa|\beta} \) from (77) denote the elements of the inverse of \( \hat{O} \). These considerations allow us to conclude that:

1. If the matrix \( \hat{k} \) is positive-definite, then the symmetric matrices \( \hat{M} = (M_{ab}) \) and \( \hat{k} = (k_{ab}) \) are simultaneously diagonalizable and hence there appear no cross-couplings among different fields from the collection \( \{ r^a_{\mu\nu|\kappa|\beta}\}_{a=1}^{n} \). Taking \( \hat{k} \) to be positive-definite might be essential for the physical consistency of the theory (absence of negative-energy excitations or stability of the Minkowski vacuum);

2. If the matrix \( \hat{k} \) is indefinite, then the matrices \( \hat{M} \) and \( \hat{k} \) cannot be diagonalized simultaneously (because then the matrix \( \hat{C} = \hat{k}^{-1} \hat{M} \) is not normal \( [45] \)) and therefore there appear cross-couplings among different fields from the collection \( \{ r^a_{\mu\nu|\kappa|\beta}\}_{a=1}^{n} \).

The terms from (72) that are linear in the antifields of the original fields give the gauge transformations of the deformed Lagrangian action, (73), by replacing the ghosts with the corresponding gauge parameters
\[ \bar{\delta}_t \xi^A_{\lambda\mu\nu|\kappa} = 3 \partial_\kappa \xi^A_{\lambda\mu\nu} + \partial_\lambda \xi^A_{\mu\nu|\kappa} + \partial_\mu \xi^A_{\nu|\lambda\kappa} \]
\[ -2 g f^A_{\alpha} \varepsilon^{\lambda\mu\nu|\rho|\beta\gamma} \left( \partial^\rho \xi^A_{\alpha\beta\gamma} |_{\kappa} - \frac{1}{4} \delta^\rho_\gamma \partial^\tau \partial_\kappa \xi^A_{\alpha\beta\tau} |_{\tau} \right), \quad (79) \]
\[ \bar{\delta}_\xi r^a_{\mu\nu|\kappa|\beta} = \partial_\lambda \xi^a_{\mu\kappa|\lambda\beta} |_{\mu} - \partial_\nu \xi^a_{\mu\kappa|\gamma\beta} |_{\mu} + \partial_\kappa \xi^a_{\nu|\mu\beta|\kappa} - \partial_\beta \xi^a_{\mu|\nu\kappa|\beta} = \delta_\xi r^a_{\mu\nu|\kappa|\beta}. \quad (80) \]
It is interesting to note that only the gauge transformations of the tensor fields $(3, 1)$ are modified during the deformation process. This is enforced at order one in the coupling constant by terms linear in the first-order derivatives of the gauge parameters from the $(2, 2)$ sector. From the terms of antighost number equal to two present in (72) we learn that only the first-order reducibility functions are modified at order one in the coupling constant, the others coinciding with the original ones. Consequently, the first-order reducibility relations corresponding to the fields $t^A_{\lambda\mu\nu|\kappa}$ take place off-shell, like the free ones, while the first-order reducibility relations associated with the fields $r^a_{\mu\nu|\kappa\beta}$ remain the original ones. Since there are no other terms of antighost number two in (72), it follows that the gauge algebra of the coupled model is unchanged by the deformation procedure, being the same Abelian one like for the starting free theory. The structure of pieces with the antighost number equal to three from (72) implies that the second-order reducibility functions remain the same, and hence the second-order reducibility relations are exactly the initial ones. It is easy to see from (73)–(80) that if we impose the PT-invariance at the level of the coupled model, then we obtain no interactions at all.

It is important to stress that the problem of obtaining consistent interactions strongly depends on the space-time dimension. For instance, if one starts with action (8) in $D > 6$, then one inexorably gets $\bar{S} = S + g \int c_a r^a d^Dx$, so no cross-interaction term can be added to either the original Lagrangian or its gauge transformations.

8 Conclusions

Results (72)–(80) lead to the following main result of our work: under the hypotheses of analyticity of deformations in the coupling constant, space-time locality, Lorentz covariance, and Poincaré invariance, combined with the requirement that the interaction vertices contain at most two space-time derivatives of the fields, there appear consistent cross-couplings in $D = 6$ between a collection of massless tensor fields with the mixed symmetry $(3, 1)$ and a collection of massless tensor fields with the mixed symmetry of the Riemann tensor, with the property that they modify the free action and its gauge symmetries. The existence of cross-couplings among different fields with the mixed symmetry of the Riemann tensor is essentially dictated by the behaviour of the metric tensor in the inner space of collection indices $a = 1, n,$
\( \hat{k} = (k_{ab}) \). Thus, if \( \hat{k} \) is positive-definite, then there appear no cross-couplings among different fields with the mixed symmetry of the Riemann tensor. On the contrary, if \( \hat{k} \) is indefinite, then there are allowed cross-couplings among different fields from this collection.

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**A Proof of the triviality of the first-order deformation for \( I = 4 \) and \( I = 3 \)**

In order to solve the third equation from (37), we decompose \( a^\text{int} \) along the antighost number and stop at \( I = 4 \)

\[
a^\text{int} = a_0^\text{int} + a_1^\text{int} + a_2^\text{int} + a_3^\text{int} + a_4^\text{int},
\]

where \( a_4^\text{int} \) can be taken as solution to the equation \( \gamma a_4^\text{int} = 0 \), and therefore it is of the form (44) for \( I = 4 \), with \( t_{\lambda\mu\nu\kappa}^A \) and \( r_{\lambda\mu\nu\kappa\beta}^a \) with maximum two space-time derivatives, it follows that the eligible basis elements at pure ghost number equal to four remain

\[
\omega^4 : (C^a_{\kappa\beta}C^b_{\lambda\rho}, C^a_{\kappa\beta}\partial_\lambda C^b_{\rho\sigma}).
\]

So, up to trivial, \( \gamma \)-exact contributions, we have that

\[
a_4^\text{int} = C^a_A \left( M_{A_{ab}}^{\alpha\kappa\beta\lambda\rho} C^a_{\kappa\beta}C^b_{\lambda\rho} + N_{A_{ab}}^{\alpha\kappa\beta\lambda\rho\sigma} C^a_{\kappa\beta}\partial_\lambda C^b_{\rho\sigma} \right),
\]

where \( M_{A_{ab}}^{\alpha\kappa\beta\lambda\rho} = -M_{A_{ab}}^{\beta\kappa\alpha\lambda\rho} = -M_{A_{ab}}^{\kappa\beta\alpha\lambda\rho} = M_{A_{ab}}^{\lambda\rho\kappa\beta} \) and \( N_{A_{ab}}^{\alpha\kappa\beta\lambda\rho\sigma} = N_{A_{ab}}^{\alpha\kappa\beta\lambda\rho\sigma} \) are some non-derivative, real constants. Replacing \( a_4^\text{int} \) into an equation similar to (40) for \( I = 4 \) and computing \( \delta a_4^\text{int} \), it follows that

\[
\delta a_4^\text{int} = \gamma \lambda_3 + \partial_\mu \tau_\mu - 2C_A^{\alpha\kappa\beta\mu} \partial_\nu C^a_{\kappa\beta} \left( 2M_{A_{ab}}^{\alpha\kappa\beta\lambda\rho} C^a_{\lambda\rho} + N_{A_{ab}}^{\alpha\kappa\beta\lambda\rho\sigma} \partial_\lambda C^b_{\rho\sigma} \right),
\]
where
\[ \lambda_3 = -G^A_{a'b'} \left[ 2C^a_{\kappa |b}a \left( 2M_{ab\mu}^{b'\kappa\beta\lambda\rho}C^{b}_{\lambda|\rho} + N_{ab\mu}^{\prime A\kappa\beta\lambda\rho\sigma} \partial[\lambda C_{\rho\sigma|}] \right) \\
+3C^a_{\kappa |b}N_{ab\mu}^{\prime A\kappa\beta\lambda\rho\sigma} \partial[\lambda C_{\rho\sigma|}]a \right]. \] (85)

Thus, \( a_3^{\text{int}} \) exists if and only if the third term in the right-hand side of (84) can be written in a \( \gamma \)-exact modulo \( d \) form
\[ G^A_{a'b'} \partial[\nu C^a_{\kappa |b}] \left( 2M_{ab\mu}^{b'\kappa\beta\lambda\rho}C^{b}_{\lambda|\rho} + N_{ab\mu}^{\prime A\kappa\beta\lambda\rho\sigma} \partial[\lambda C_{\rho\sigma|}] \right) = \gamma u_3 + \partial^\mu \pi_\mu. \] (86)

Taking the (left) Euler–Lagrange derivative of the above equation with respect to \( G^A_{a'|b} \) and recalling the anticommutativity of this operation with \( \gamma \), we obtain
\[ \partial[\nu C^a_{\kappa |b}] \left( 2M_{ab\mu}^{b'\kappa\beta\lambda\rho}C^{b}_{\lambda|\rho} + N_{ab\mu}^{\prime A\kappa\beta\lambda\rho\sigma} \partial[\lambda C_{\rho\sigma|}] \right) = \gamma \left( -\frac{\delta L u_3}{\delta G^A_{a'b'}} \right). \] (87)

The last relation shows that the object
\[ \partial[\nu C^a_{\kappa |b}] \left( 2M_{ab\mu}^{b'\kappa\beta\lambda\rho}C^{b}_{\lambda|\rho} + N_{ab\mu}^{\prime A\kappa\beta\lambda\rho\sigma} \partial[\lambda C_{\rho\sigma|}] \right), \] (88)
which is a non-trivial element of \( H^4 (\gamma) \) (see formula (44)), must be \( \gamma \)-exact. This takes place if and only if \( M_{ab\mu}^{\prime A\kappa\beta\lambda\rho} = 0 = N_{ab\mu}^{\prime A\kappa\beta\lambda\rho\sigma} \), which further implies
\[ a_3^{\text{int}} = 0, \] (89)
and hence the first-order deformation in the cross-coupling sector cannot end non-trivially at antighost number \( I = 4 \).

The case \( I = 3 \) is solved in a similar manner and leads to the result \( a_3^{\text{int}} = 0 \).

**B  Proof of the result (64)**

Next, we investigate the solutions to (63). There are two main types of solutions to this equation. The first type, to be denoted by \( \tilde{a}_0^{\text{int}} \), corresponds to \( \tilde{a}_0^{\mu} = 0 \) and is given by gauge-invariant, non-integrated densities constructed out of the original fields and their space-time derivatives, which,
according to (44), are of the form
\[
\tilde{a}_0^{\text{int}} = \tilde{a}_0^{\text{int}} \left( \left[ K_{\lambda\mu\nu|\kappa\beta}^\Lambda, [F_{\mu\nu|\kappa\beta}]^a \right] \right),
\]
up to the condition that they effectively describe cross-couplings between the two types of fields and cannot be written in a divergence-like form. Such a solution implies at least four derivatives of the fields and consequently must be forbidden by setting \( \tilde{a}_0^{\text{int}} = 0 \).

The second kind of solutions is associated with \( \tilde{m}_\mu^{\text{int}} \neq 0 \) in (63), being understood that we discard the divergence-like quantities and maintain the condition on the maximum derivative order of the interacting Lagrangian being equal to two. In order to solve this equation we start from the requirement that \( \tilde{a}_0^{\text{int}} \) may contain at most two derivatives, so it can be decomposed like
\[
\tilde{a}_0^{\text{int}} = \omega_0 + \omega_1 + \omega_2,
\]
where \( (\omega_i)_{i=0,1,2} \) contains \( i \) derivatives. Due to the different number of derivatives in the components \( \omega_0, \omega_1, \text{ and } \omega_2 \), equation (63) is equivalent to three independent equations
\[
\gamma \omega_k = \partial_\mu j_k^\mu, \quad k = 0, 1, 2.
\]

Equation (91) for \( k = 0 \) implies the (necessary) conditions
\[
\partial_\lambda \left( \frac{\partial \omega_0}{\partial t^{A}_{\lambda\mu\nu|\kappa}} \right) = 0, \quad \partial_\kappa \left( \frac{\partial \omega_0}{\partial t^{A}_{\lambda\mu\nu|\kappa}} \right) = 0, \quad \partial_\mu \left( \frac{\partial \omega_0}{\partial r^a_{\mu\nu|\kappa\beta}} \right) = 0.
\]

The last equation from (92) possesses only the constant solution
\[
\frac{\partial \omega_0}{\partial r^a_{\mu\nu|\kappa\beta}} = k_a \left( \sigma^{\mu\kappa} \sigma^{\nu\beta} - \sigma^{\mu\beta} \sigma^{\nu\kappa} \right),
\]
where \( k_a \) are some real constants, so we find that
\[
\omega_0 = 2k_a r^a + B \left( t^A_{\lambda\mu\nu|\kappa} \right).
\]

Since \( \omega_0 \) provides no cross-couplings between \( t^A_{\lambda\mu\nu|\kappa} \) and \( r^a_{\mu\nu|\kappa\beta} \), we can take
\[
\omega_0 = 0
\]
in (90).

As a digression, we note that the general solution to the equations
\[
\partial_\lambda T^\lambda_{A\mu\nu|\kappa} = 0, \quad \partial_\kappa T^\lambda_{A\mu\nu|\kappa} = 0
\]

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(with $\bar{T}_A^{\lambda\mu\nu|k}$ some covariant tensor fields with the mixed symmetry $(3, 1)$) reads as \[39\]
\[
\bar{T}_A^{\lambda\mu\nu|k} = \partial_\xi \partial_\beta \bar{\Phi}_A^{\lambda\mu\nu|\kappa\beta},
\] (97)
where $\bar{\Phi}_A^{\lambda\mu\nu|\kappa\beta}$ are some tensors with the mixed symmetry $(4, 2)$. A constant solution $C_A^{\lambda\mu\nu|k}$ is excluded from covariance arguments due to the mixed symmetry $(3, 1)$. Along the same line, the general solution to the equations
\[
\partial_\mu \bar{R}^{\mu\nu|\kappa\beta} = 0
\] (98)
(with $\bar{R}^{\mu\nu|\kappa\beta}$ some covariant tensor fields with the mixed symmetry $(2, 2)$) is represented by \[40\]
\[
\bar{R}^{\mu\nu|\kappa\beta} = \partial_\rho \partial_\gamma \bar{\Omega}^{\mu\nu\rho|\kappa\beta\gamma} + k_a (\sigma^{\mu\kappa} \sigma^{\nu\beta} - \sigma^{\mu\beta} \sigma^{\nu\kappa}),
\] (99)
where $\bar{\Omega}^{\mu\nu|\kappa\beta\gamma}$ are some tensors with the mixed symmetry $(3, 3)$ and $k_a$ some arbitrary, real constants. Now, it is clear why the solution to the last equation from (92) reduces to (93): $\partial_\mu \omega_0 / \partial_\mu \bar{r}^{\mu\nu|\kappa\beta}$ display the mixed symmetry $(2, 2)$, but are derivative-free by assumption, so some terms similar to the former ones from the right-hand side of (99) are forbidden.

Equation (91) for $k = 1$ leads to the requirements
\[
\partial_\lambda \left( \frac{\delta \omega_1}{\delta t^{A}_{\lambda\mu\nu|\kappa}} \right) = 0, \quad \partial_\kappa \left( \frac{\delta \omega_1}{\delta t^{A}_{\lambda\mu\nu|\kappa}} \right) = 0, \quad \partial_\mu \left( \frac{\delta \omega_1}{\delta t^{A}_{\mu\nu|\kappa}} \right) = 0,
\] (100)
where $\delta \omega_1 / \delta t^{A}_{\lambda\mu\nu|\kappa}$ and $\delta \omega_1 / \delta t^{A}_{\mu\nu|\kappa}$ denote the Euler–Lagrange derivatives of $\omega_1$ with respect to the corresponding fields. Looking at (97) and (99) and recalling that $\omega_1$ is by hypothesis of order one in the space-time derivatives of the fields, the only solution to equations (100) reduces to
\[
\frac{\delta \omega_1}{\delta t^{A}_{\mu\nu|\kappa}} = \frac{\delta \omega_1}{\delta t^{A}_{\lambda\mu\nu|\kappa}}.
\] (101)
This solution forbids the cross-couplings between the two types of fields, so we can safely take
\[
\omega_1 = 0.
\] (102)

Finally, we pass to equation (91) for $k = 2$, which produces the restrictions
\[
\partial_\lambda \left( \frac{\delta \omega_2}{\delta t^{A}_{\lambda\mu\nu|\kappa}} \right) = 0, \quad \partial_\kappa \left( \frac{\delta \omega_2}{\delta t^{A}_{\lambda\mu\nu|\kappa}} \right) = 0, \quad \partial_\mu \left( \frac{\delta \omega_2}{\delta t^{A}_{\mu\nu|\kappa}} \right) = 0,
\] (103)
with the solutions (see formulas (97) and (99))

\[
\frac{\delta \omega_2}{\delta t_{\lambda \mu \nu |\kappa}} = \partial_{\gamma} \partial_{\sigma} W_{A}^{\lambda \mu \nu |\kappa \sigma}, \quad \frac{\delta \omega_2}{\delta r_{\mu \nu |\kappa \beta}} = \partial_{\gamma} \partial_{\sigma} U_{a}^{\mu \nu |\kappa \beta \sigma}.
\]  

(104)

The tensors \( W_{A}^{\lambda \mu \nu |\kappa \sigma} \) have the mixed symmetry of the curvature tensors \( K_{A}^{\lambda \mu \nu |\kappa \sigma} \) and the tensors \( U_{a}^{\mu \nu |\kappa \beta \sigma} \) exhibit the mixed symmetry of the curvature tensors \( F_{a}^{\mu \nu |\kappa \beta \sigma} \). Both types of tensors are derivative-free since \( \omega_2 \) contains precisely two derivatives of the fields. At this stage it is useful to introduce a derivation in the algebra of the fields and of their derivatives that counts the powers of the fields and of their derivatives

\[
N = \sum_{k \geq 0} \left[ \left( \partial_{\mu_1 \ldots \mu_k} t_{\lambda \mu \nu |\kappa}^{A} \right) \frac{\partial}{\partial \left( \partial_{\mu_1 \ldots \mu_k} t_{\lambda \mu \nu |\kappa}^{A} \right)} \right] + \left( \partial_{\mu_1 \ldots \mu_k} r_{\mu \nu |\kappa \beta}^{a} \right) \frac{\partial}{\partial \left( \partial_{\mu_1 \ldots \mu_k} r_{\mu \nu |\kappa \beta}^{a} \right)},
\]

(105)

so for every non-integrated density \( \rho \) we have that

\[
N \rho = t_{\lambda \mu \nu |\kappa}^{A} \frac{\delta \rho}{\delta t_{\lambda \mu \nu |\kappa}^{A}} + r_{\mu \nu |\kappa \beta}^{a} \frac{\delta \rho}{\delta r_{\mu \nu |\kappa \beta}^{a}} + \partial_{\mu} s^{\mu},
\]

(106)

where \( \delta \rho/\delta t_{\mu \nu |\kappa \beta}^{A} \) and \( \delta \rho/\delta r_{\mu \nu |\kappa \beta}^{a} \) denote the variational derivatives of \( \rho \) with respect to the fields. If \( \rho^{(l)} \) is a homogeneous polynomial of order \( l > 0 \) in the fields \( \left\{ t_{\lambda \mu \nu |\kappa}^{A}, r_{\mu \nu |\kappa \beta}^{a} \right\} \) and their derivatives, then \( N \rho^{(l)} = l \rho^{(l)} \). Using (104) and (106), we find that

\[
N \omega_2 = \frac{1}{8} K_{A}^{\lambda \mu \nu |\kappa \sigma} W_{A}^{\lambda \mu \nu |\kappa \sigma} + \frac{1}{9} F_{a}^{\mu \nu |\kappa \beta \sigma} U_{a}^{\mu \nu |\kappa \beta \sigma} + \partial_{\mu} v^{\mu}.
\]

(107)

We expand \( \omega_2 \) according to the various eigenvalues of \( N \) like

\[
\omega_2 = \sum_{l>0} \omega_2^{(l)},
\]

(108)

where \( N \omega_2^{(l)} = l \omega_2^{(l)} \), such that

\[
N \omega_2 = \sum_{l>0} l \omega_2^{(l)}.
\]

(109)
Comparing (107) with (109), we reach the conclusion that the decomposition (108) induces a similar decomposition with respect to $W^\lambda_{\mu\nu\gamma|\kappa\sigma}$ and $U^\mu_{\nu|\kappa\beta\sigma}$

\[
W^\lambda_{\mu\nu\gamma|\kappa\sigma} = \sum_{l>0} W^\lambda_{\mu\nu\gamma|\kappa\sigma}(l-1), \quad U^\mu_{\nu|\kappa\beta\sigma} = \sum_{l>0} U^\mu_{\nu|\kappa\beta\sigma}(l-1).
\]  

(110)

Substituting (110) into (107) and comparing the resulting expression with (109), we obtain that

\[
\omega^{(l)}_2 = \frac{1}{8l} K^A_{\lambda\mu\nu\gamma|\kappa\sigma} W^\lambda_{\mu\nu\gamma|\kappa\sigma}(l-1) + \frac{1}{9l} F^a_{\mu\nu\gamma|\kappa\beta\sigma} U^\mu_{\nu|\kappa\beta\sigma}(l-1) + \partial_{\mu} \bar{v}^{(l)}_\mu.
\]

(111)

Introducing (111) in (108), we arrive at

\[
\omega^{(l)}_2 = K^A_{\lambda\mu\nu\gamma|\kappa\sigma} \bar{W}^\lambda_{\mu\nu\gamma|\kappa\sigma} + F^a_{\mu\nu\gamma|\kappa\beta\sigma} \bar{U}^\mu_{\nu|\kappa\beta\sigma} + \partial_{\mu} \bar{v}^{(l)}_\mu,
\]

(112)

where

\[
\bar{W}^\lambda_{\mu\nu\gamma|\kappa\sigma} = \sum_{l>0} \frac{1}{8l} W^\lambda_{\mu\nu\gamma|\kappa\sigma}(l-1), \quad \bar{U}^\mu_{\nu|\kappa\beta\sigma} = \sum_{l>0} \frac{1}{9l} U^\mu_{\nu|\kappa\beta\sigma}(l-1).
\]

(113)

Applying $\gamma$ on (112), we infer that a necessary condition for the existence of solutions to the equation $\gamma \omega^{(l)}_2 = \partial_{\mu} j^{(l)}_\mu$ is that the functions $\bar{W}^\lambda_{\mu\nu\gamma|\kappa\sigma}$ and $\bar{U}^\mu_{\nu|\kappa\beta\sigma}$ entering (112) must satisfy the equations

\[
\partial_{\rho} \left( F^b_{\mu\nu\gamma|\kappa\beta\sigma} \frac{\partial \bar{U}^\mu_{\nu|\kappa\beta\sigma}}{\partial t^{\rho|\xi}} + K^B_{\lambda\mu\nu\gamma|\kappa\sigma} \frac{\partial \bar{W}^\lambda_{\mu\nu\gamma|\kappa\sigma}}{\partial t^{\rho|\xi}} \right) = 0,
\]

(114)

\[
\partial_{\chi} \left( F^b_{\mu\nu\gamma|\kappa\beta\sigma} \frac{\partial \bar{U}^\mu_{\nu|\kappa\beta\sigma}}{\partial A^{\rho|\xi}} + K^B_{\lambda\mu\nu\gamma|\kappa\sigma} \frac{\partial \bar{W}^\lambda_{\mu\nu\gamma|\kappa\sigma}}{\partial A^{\rho|\xi}} \right) = 0,
\]

(115)

\[
\partial_{\rho} \left( F^b_{\mu\nu\gamma|\kappa\beta\sigma} \frac{\partial \bar{U}^\mu_{\nu|\kappa\beta\sigma}}{\partial t^{\rho|\xi}} + K^B_{\lambda\mu\nu\gamma|\kappa\sigma} \frac{\partial \bar{W}^\lambda_{\mu\nu\gamma|\kappa\sigma}}{\partial t^{\rho|\xi}} \right) = 0.
\]

(116)

The general solution to equations (114)–(116) reads as

\[
F^b_{\mu\nu\gamma|\kappa\beta\sigma} \frac{\partial \bar{U}^\mu_{\nu|\kappa\beta\sigma}}{\partial t^{\rho|\xi}} + K^B_{\lambda\mu\nu\gamma|\kappa\sigma} \frac{\partial \bar{W}^\lambda_{\mu\nu\gamma|\kappa\sigma}}{\partial t^{\rho|\xi}} = \partial_{\tau} \partial_{\theta} E^{\rho\delta|\xi\theta},
\]

(117)

\[
F^b_{\mu\nu\gamma|\kappa\beta\sigma} \frac{\partial \bar{U}^\mu_{\nu|\kappa\beta\sigma}}{\partial t^{\rho|\xi}} + K^B_{\lambda\mu\nu\gamma|\kappa\sigma} \frac{\partial \bar{W}^\lambda_{\mu\nu\gamma|\kappa\sigma}}{\partial t^{\rho|\xi}} = \partial_{\tau} \partial_{\theta} F^{\rho|\xi\theta},
\]

(118)
where the functions $E^b_{\alpha} \xi^\theta$ and $H^b_{\alpha} \xi^\theta$ are derivative-free and exhibit the mixed symmetries (3, 3) and (4, 2) respectively. By direct computations we deduce

\[
\partial_\tau \partial_\theta E^b_{\alpha} \xi^\theta = \frac{\partial^2 E^b_{\alpha} \xi^\theta}{\partial r^{\rho'} \xi^\theta \partial E^c_{\alpha} \xi^\theta} \left( \partial_\theta t^B_{\rho' \xi^\theta} \right) \left( \partial_\tau t^C_{\rho' \xi^\theta} \right) \left( \partial_\tau E^b_{\alpha} \xi^\theta \right) + \frac{\partial^2 E^b_{\alpha} \xi^\theta}{\partial t^{\rho' \xi^\theta}} \left( \partial_\theta t^B_{\rho' \xi^\theta} \right) \left( \partial_\tau t^C_{\rho' \xi^\theta} \right) \left( \partial_\tau E^b_{\alpha} \xi^\theta \right) \left( \partial_\tau t^B_{\rho' \xi^\theta} \right)
\]

(119)

\[
\partial_\tau \partial_\theta H^b_{\alpha} \xi^\theta = \frac{\partial^2 H^b_{\alpha} \xi^\theta}{\partial r^{\rho'} \xi^\theta \partial E^c_{\alpha} \xi^\theta} \left( \partial_\theta t^B_{\rho' \xi^\theta} \right) \left( \partial_\tau t^C_{\rho' \xi^\theta} \right) \left( \partial_\tau H^b_{\alpha} \xi^\theta \right) + \frac{\partial^2 H^b_{\alpha} \xi^\theta}{\partial t^{\rho' \xi^\theta}} \left( \partial_\theta t^B_{\rho' \xi^\theta} \right) \left( \partial_\tau t^C_{\rho' \xi^\theta} \right) \left( \partial_\tau H^b_{\alpha} \xi^\theta \right) \left( \partial_\tau t^B_{\rho' \xi^\theta} \right)
\]

(120)

Substituting (119)–(120) in (117)–(118) and comparing the left-hand sides with the corresponding right-hand sides of the resulting relations, we find the necessary equations

\[
\frac{\partial^2 E^b_{\alpha} \xi^\theta}{\partial r^{\rho'} \xi^\theta \partial E^c_{\alpha} \xi^\theta} = 0, \quad \frac{\partial^2 E^b_{\alpha} \xi^\theta}{\partial t^{\rho' \xi^\theta}} = 0
\]

(121)

\[
\frac{\partial^2 H^b_{\alpha} \xi^\theta}{\partial r^{\rho'} \xi^\theta \partial E^c_{\alpha} \xi^\theta} = 0, \quad \frac{\partial^2 H^b_{\alpha} \xi^\theta}{\partial t^{\rho' \xi^\theta}} = 0
\]

(122)

\[
\frac{\partial^2 \left( E^b_{\alpha} \xi^\theta + E^b_{\alpha} \xi^\theta \right)}{\partial r^{\rho'} \xi^\theta \partial t^{\rho' \xi^\theta}} = 0, \quad \frac{\partial^2 \left( H^b_{\alpha} \xi^\theta + H^b_{\alpha} \xi^\theta \right)}{\partial r^{\rho'} \xi^\theta \partial t^{\rho' \xi^\theta}} = 0
\]

(123)

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The above relations allow us to write
\begin{equation}
\frac{1}{2} \left( E^\rho_{ab} \xi^\chi^\theta + E^\rho_{ab} \xi^\chi^\tau \right) = C^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau + C^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau,
\end{equation}
where the quantities denoted by \( C \) or \( \tilde{C} \) are some non-derivative, real tensors, with the expressions
\begin{align*}
C^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau &= \tilde{C}^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau, \\
C^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau &= \tilde{C}^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau,
\end{align*}
where two sets of indices are connected by a semicolon, it is understood
that the corresponding tensor possesses independently the mixed symmetries
with respect to the former and respectively the latter set. On the other hand,
it is obvious that
\begin{align*}
\partial_\tau \partial_\theta E^\rho_{ab} \xi^\chi^\theta &= \frac{1}{2} \partial_\tau \partial_\theta \left( E^\rho_{ab} \xi^\chi^\theta + E^\rho_{ab} \xi^\chi^\theta \right), \\
\partial_\tau \partial_\theta H^\rho_{ab} \xi^\chi^\tau &= \frac{1}{2} \partial_\tau \partial_\theta \left( H^\rho_{ab} \xi^\chi^\tau + H^\rho_{ab} \xi^\chi^\tau \right),
\end{align*}
so equations (127)–(128) become
\begin{align*}
F^b_{\mu \nu \gamma |\kappa \beta \sigma} \frac{\partial U^\mu_{\nu \gamma |\kappa \beta \sigma}}{\partial r^a_{\rho \delta \xi^\chi}} + K^{B}_{\lambda \mu \nu \gamma |\kappa \sigma} \frac{\partial W^\lambda_{\mu \nu \gamma |\kappa \sigma}}{\partial r^a_{\rho \delta \xi^\chi}} &= \\
= C^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau \partial_\theta \partial_\theta r^b_{\rho \delta \xi^\chi} + C^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau \partial_\theta r^b_{\rho \delta \xi^\chi}, \\
F^b_{\mu \nu \gamma |\kappa \beta \sigma} \frac{\partial U^\mu_{\nu \gamma |\kappa \beta \sigma}}{\partial H^A_{\rho \delta \xi^\chi}} + K^{B}_{\lambda \mu \nu \gamma |\kappa \sigma} \frac{\partial W^\lambda_{\mu \nu \gamma |\kappa \sigma}}{\partial H^A_{\rho \delta \xi^\chi}} &= \\
= C^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau \partial_\theta r^b_{\rho \delta \xi^\chi} + C^\rho_{ab} \xi^\chi^\theta \rho^\delta \xi^\chi^\tau \partial_\theta r^b_{\rho \delta \xi^\chi},
\end{align*}
Taking the partial derivatives of equations (132) and (133) with respect to
\( \partial_\tau \partial_\theta r^b_{\rho \delta \xi^\chi} \) and \( \partial_\tau \partial_\theta H^A_{\rho \delta \xi^\chi} \), we infer the relations
\begin{align*}
\frac{\partial U^\mu_{\nu \gamma |\kappa \beta \sigma}}{\partial r^a_{\rho \delta \xi^\chi}} &= k^\mu_{\nu \gamma |\kappa \beta \sigma} r^a_{\rho \delta \xi^\chi}, \\
\frac{\partial W^\lambda_{\mu \nu \gamma |\kappa \sigma}}{\partial r^a_{\rho \delta \xi^\chi}} &= \bar{k}^\lambda_{\mu \nu \gamma |\kappa \sigma} r^a_{\rho \delta \xi^\chi},
\end{align*}
\[
\frac{\partial U^\mu\nu|\kappa\beta\sigma}{\partial t^\rho\delta|\chi} = \hat{k}^\mu_{\lambda A} \nu|\kappa\beta\sigma;\rho\delta\xi|\chi, \quad \frac{\partial W^\lambda\mu\nu|\kappa\sigma}{\partial t^\beta\rho\delta|\chi} = \tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi, \quad (135)
\]

where \(k^\mu_{\lambda A} \nu|\kappa\beta\sigma;\rho\delta\xi|\chi\), \(\tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi\), and \(\hat{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi\) denote some non-derivative, constant tensors. By means of relations (134) and (135) we obtain (up to some irrelevant constants)

\[
\bar{U}^\mu\nu|\kappa\beta\sigma;\rho\delta\xi|\chi = k^\mu_{\lambda A} \nu|\kappa\beta\sigma;\rho\delta\xi|\chi \bar{a} + \hat{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \bar{a}, \quad (136)
\]

\[
\bar{W}^\lambda\mu\nu|\kappa\sigma = \tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \bar{a} + \tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \bar{a}, \quad (137)
\]

From the expression of \(\omega_2\) given by (112) we notice that the terms \(k^\mu_{\lambda A} \nu|\kappa\beta\sigma;\rho\delta\xi|\chi \bar{a} \bar{r} \bar{a} \rho\delta\xi|\chi\) and \(\tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \bar{a} \bar{r} \bar{a} \rho\delta\xi|\chi\) appearing in (136) and (137) bring no contributions to cross-interactions. For this reason, we take

\[
k^\mu_{\lambda A} \nu|\kappa\beta\sigma;\rho\delta\xi|\chi = 0, \quad \tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi = 0, \quad (138)
\]

such that (up to a total, irrelevant divergence) \(\omega_2\) takes the form

\[
\omega_2 = \tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \bar{a} \bar{r} \bar{a} \rho\delta\xi|\chi = \tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \bar{a} \bar{r} \bar{a} \rho\delta\xi|\chi, \quad (139)
\]

The most general expression of \(\tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi\) is represented by

\[
\tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi = c^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \left[ 1 - \frac{1}{4} \varepsilon^\lambda\mu\nu\rho\delta (\sigma^{\xi\kappa}\sigma^{\chi\sigma} - \sigma^{\xi\sigma}\sigma^{\chi\kappa}) \right. \\
+ \frac{1}{4} \varepsilon^\lambda\mu\nu\rho\delta (\sigma^{\mu\kappa}\sigma^{\delta\sigma} - \sigma^{\rho\sigma}\sigma^{\delta\kappa}) \\
- \frac{1}{24} \varepsilon^\lambda\mu\nu\rho\delta (\sigma^{\xi\kappa}\sigma^{\chi\sigma} - \sigma^{\xi\sigma}\sigma^{\chi\kappa}) \right], \quad (140)
\]

which then yields

\[
\tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \bar{a} \bar{r} \bar{a} \rho\delta\xi|\chi = c^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi \tilde{k}^\lambda_{\beta B} \mu\nu|\kappa\sigma;\rho\delta\xi|\chi, \quad (141)
\]

with \(c^\lambda_{\beta B}\) some real constants. On the other hand, there exist non-trivial constant tensors of the type \(k^\mu_{\lambda A} \nu|\kappa\beta\sigma;\rho\delta\xi|\chi\), but they all lead in the end to

\[
k^\mu_{\lambda A} \nu|\kappa\beta\sigma;\rho\delta\xi|\chi \bar{a} \bar{r} \bar{a} \rho\delta\xi|\chi = 0 \quad (142)
\]
due to the algebraic Bianchi I identities \( F^{a}_{[\mu \nu \rho | \kappa] \beta \sigma} \equiv 0 \). Such constants have an intricate and non-illuminating form, and therefore we will skip them. Inserting (141) and (142) in (139), we deduce

\[
\omega_2 = c_{Aa} \varepsilon^{\lambda \mu \nu \gamma \rho \delta} \frac{a}{[\rho \delta] | \xi} K^A_{\lambda \mu \nu \gamma} \xi. \tag{143}
\]

Acting with \( \gamma \) on (143), it is easy to see that

\[
\gamma \omega_2 = -2c_{Aa} \varepsilon^{\lambda \mu \nu \gamma \rho \delta} \left( \partial_{[\lambda} K^A_{\mu \nu \gamma] | \xi} \right) C^a_{\rho \delta | \xi} \neq \partial_{\mu} j_2^\mu, \tag{144}
\]

where \( K^A_{\mu \nu \gamma | \tau} \) is the trace of the curvature tensor \( K^A_{\mu \nu \gamma | \kappa \beta,} K^A_{\mu \nu | \kappa | \tau} = \sigma^{\kappa \beta} K^A_{\mu \nu \gamma | \tau \beta,} \). It is worthy to notice that \( \gamma \omega_2 \neq \partial_{\mu} j_2^\mu \) follows from the differential Bianchi II identity \( \partial_\beta K^A_{\lambda \mu \nu \gamma |} \beta \xi = \partial_\lambda K^A_{\mu \nu \gamma |} \xi \). Due to (144), we must take

\[
c_{Aa} = 0, \tag{145}
\]

and hence

\[
\omega_2 = 0. \tag{146}
\]

Replacing (95), (102), and (146) in (90), we finally find (64).

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