SETVALED DYNAMICAL SYSTEMS FOR STOCHASTIC EVOLUTION
EQUATIONS DRIVEN BY FRACTIONAL NOISE

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Abstract. We consider Hilbert-valued evolution equations driven by Hölder paths with Hölder index greater than 1/2, which includes the case of fractional noises with Hurst parameters in (1/2,1). The assumptions of the drift term will not be enough to ensure the uniqueness of solutions. Nevertheless, adopting a multivalued setting, we will prove that the set of all solutions corresponding to the same initial condition generates a (multivalued) nonautonomous dynamical system $\Phi$. Finally, to prove that $\Phi$ is measurable (and hence a (multivalued) random dynamical system), we need to construct a new metric dynamical system that models the noise with the property that the set space is separable.

1. Introduction

In this article we aim at investigating the following type of evolution equation
\begin{equation}
    du(t) = (Au(t) + F(u(t)))dt + G(u(t))d\omega(t), \quad u(0) = u_0 \in V,
\end{equation}
in a Hilbert space $V$, where the driving input is given by a Hölder continuous function with Hölder index greater than 1/2, the operator $A$ generates an analytic semigroup, and the nonlinear mappings $F$ and $G$ will be Lipschitz continuous.

The main example will be given by stochastic evolution equations driven by a fractional Brownian motion $B^H$ with Hurst parameter $H \in (1/2,1)$.

One of the main features of this paper is that the scarce regularity of $G$ will not be enough to ensure uniqueness of solutions of (1), hence we shall adopt the setvalued setting. When $\omega$ is a Brownian motion (which corresponds to the case $B^{1/2}$), this kind of problems have been already tackled. However, the corresponding solutions of these problems are defined only almost surely, which contradicts the cocycle property. In other words, the well-known Itô integral produces exceptional sets that depend on the initial condition and it is not known therefore how to define a random dynamical system if more than countably many exceptional sets occur. As a result, it is not known in general if a stochastic evolution equation driven by Brownian motion generates a random dynamical system, even in the univalued case where uniqueness of solutions holds true. This general open problem has been solved in very particular situations where the noise is additive or linear multiplicative. In those cases, the technique consists in transforming the stochastic equation into a random equation where the noise acts just as a parameter, which in turns implies that deterministic tools can be used to deal with the random equation. This method came out in the 90’s of the last century and since then has been exploited in many papers, as for instance, [5], [16], [17], [18] in the case of uniqueness of solutions, and [4], [6] in the multivalued setting.

Recently, many efforts have been done to go beyond the Brownian motion case and to consider other types of noises that exhibit different properties, as is the case of a fractional Brownian motion $B^H$ with Hurst parameter $H \in (0,1)$. When $H = 1/2$ this process reduces to the Brownian motion but, in the rest of cases, $B^H$ is not a semimartingale and it is not a Markov process. It would be impossible to mention here the huge amount of papers that during the last decade came out with different investigations related to equations driven by $B^H$. To name a few, for the case of regular fractional Brownian motions that corresponds to $H > 1/2$ (that will be the considered case in this paper), we refer to [19],

\begin{flushright}
\textit{2000 Mathematics Subject Classification.} Primary: 60H15; Secondary: 26A33, 37H05.
\textit{Key words and phrases.} fractional Brownian motion, fractional derivatives, multivalued random dynamical systems.
\end{flushright}
To our knowledge, only in one of these papers, to be more precise in [24], there is one result corresponding to existence and not uniqueness of solutions, but assuming between other assumptions that the initial condition is more regular (see Section 3 for a deeper discussion).

In spite of the non-uniqueness of solutions, the assumptions on the nonlinear mapping $G$ will make it possible for us to define a pathwise integral against $\omega$, based on a generalization of the Young integral given by the so-called fractional derivatives, which does not produce exceptional sets in contrast to Ito integral. We will take advantage of this property in order to analyze the dynamical system generated by the solutions of the above problem without making a previous transformation of it into a random equation, that is to say, we will always work with (1). In the univalued setting there have been some progresses in this direction, namely, once the existence and uniqueness of solutions have been established, the investigations of the generation of the (univalued) random dynamical system have been carried out, see [7], [12], [13] among others. This property opens the door to investigate the longtime behavior of solutions by studying the random attractor, invariant stable/unstable manifolds, random fixed points, etc., using for that the theory of random dynamical systems, see the monograph [1] for a comprehensive description. In the univalued case the reader could find some papers dealing with these objects, as [11] and [13], where the random attractor have been studied, or [9] and [14] where the exponential stability of the trivial solution have been considered.

No matter the univalued or the multivalued settings, the random dynamical system consists of an ergodic metric dynamical system, that models the noise, and a cocycle mapping given by the solution(s) associated to an initial condition. There have been some papers in which the ergodicity of the metric dynamical system associated with the fractional Brownian motion has been investigated, see for instance [15] and [25]. But as far as the multivalued setting is concerned these previous investigations cannot be applied, since the set $\Omega$ where $\omega$ belongs should be, on the one hand, a space of Hölder continuous functions allowing the construction of the stochastic integral, and, on the other hand, a separable space that makes it possible to prove the measurability of the cocycle. Therefore, this paper contains the construction of a new ergodic metric dynamical system modeling the fractional Brownian motion, valid for any Hurst parameter, with the mentioned separability property.

The paper is structured as follows. In Section 2 we introduce the integral with integrator given by a Hölder function with Hölder index bigger than 1/2, as well as its main properties. Section 3 addresses the existence of solutions to (1) and for that we shall apply Schauder’s fixed point theorem. Further, Section 4 is concerned with a metric dynamical systems that model the fractional Brownian motion. In particular, we will construct an ergodic metric dynamical system with the property that the set space is separable. Finally, Section 5 investigates the generation of a multivalued nonautonomous dynamical system by the set of all solutions to (1). This multivalued mapping will be shown to be measurable, and hence it is a multivalued random dynamical system, for which we need to study its upper semicontinuity with respect to all its variables, a result that relies upon the separability of the metric dynamical system built in Section 4. The paper finally includes an example of a stochastic parabolic partial differential equation for which our theory can be applied.

2. Preliminaries

In this section we introduce the assumptions of the different terms of equation (1) and give the definition and some properties of the integral for Hölder continuous integrators with Hölder exponent bigger than 1/2, for which we borrow the construction carried out recently in [17].

Throughout this paper $(V, \| \cdot \|, (\cdot, \cdot))$ is a separable Hilbert space.

Assume that $-A$ is a strictly positive and symmetric operator with a compact inverse, generating an analytic semigroup $S$ on $V$. In particular the spaces $V_\delta := D((-A)^\delta)$ with norm $\| \cdot \|_{V_\delta}$ for $\delta \geq 0$ are well-defined. Note that $V = V_0$ and $V_{\delta_1}$ is compactly embedded in $V_{\delta_2}$ for $\delta_1 \geq \delta_2 \geq 0$. Denote by
the complete orthonormal base in $V$ generated by the eigenvalues of $-A$ with associated eigenvalues $(\lambda_i)_{i \in \mathbb{N}}$.

Let $L(V, V')$ denote the space of continuous linear operators from $V$ into $V'$. Then there exists a constant $c_S > 0$ such that

\begin{align}
\|S(t)\|_{L(V, V')} &= \|(-A)^{\gamma}S(t)\|_{L(V)} \leq c_S e^{-\lambda \gamma t} \quad \text{for } \gamma > 0,
\end{align}

\begin{align}
\|S(t) - \text{id}\|_{L(V, V')} &\leq c_S e^{t - \theta}, \quad \text{for } \theta \geq 0, \quad \sigma \in [\theta, 1 + \theta].
\end{align}

The constant $c_S$ may depend on $t$ as well as the parameters $\gamma$, $\sigma$ and $\theta$, and can change from line to line. In general, we will only emphasize the dependence on the semigroup. Moreover, in (2) $\lambda$ is a positive constant such that $\lambda \leq \lambda_1$. From these two inequalities it is straightforward to derive that

\begin{align}
\|S(t - r) - S(t - q)\|_{L(V, V')} &\leq c(t - q)^{\alpha}(t - r)^{-\alpha - \gamma + \delta},
\end{align}

\begin{align}
\|S(t - r) - S(s - r) - S(t - q) + S(s - q)\|_{L(V)} &\leq c(t - s)^{\beta}(r - q)^{\gamma}(s - r)^{-\delta + \gamma},
\end{align}

for $0 \leq q < r \leq s \leq t$. As usual, $L(V)$ denotes the space $L(V, V)$.

Let $C^\beta([T_1, T_2], V)$ be the Banach space of H"older continuous functions with exponent $\beta > 0$ having values in $V$. A norm on this space is given by

\begin{align}
\|u\|_{\beta} = \|u\|_{\beta, T_1, T_2} = \|u\|_{\infty, T_1, T_2} + \|u\|_{\beta, T_1, T_2} := \sup_{t \in [T_1, T_2]} \|u(t)\| + \sup_{T_1 \leq s < t \leq T_2} \frac{\|u(t) - u(s)\|}{|t - s|^\beta}
\end{align}

$C([T_1, T_2], V)$ denotes the space of continuous functions on $[T_1, T_2]$ with values in $V$ with finite supremum norm. Since the properties on the semigroup do not ensure Hölder continuity at zero, we will work with a modification of the spaces of Hölder continuous functions, in the sense that the considered norm will be given by

\begin{align}
\|u\|_{\beta, \beta} = \|u\|_{\beta, T_1, T_2} = \|u\|_{\infty, T_1, T_2} + \|u\|_{\beta, T_1, T_2} := \|u\|_{\infty, T_1, T_2} + \sup_{T_1 \leq s < t \leq T_2} (s - T_1)^{\beta} \frac{\|u(t) - u(s)\|}{|t - s|^\beta}
\end{align}

and denote by $C^\beta([T_1, T_2], V)$ the set of functions $u \in C([T_1, T_2], V)$ such that $\|u\|_{\beta, \beta} < \infty$. It is known that $C^\beta([T_1, T_2], V)$ is a Banach space, see \cite{7} and \cite{23}.

For every $\rho > 0$ we can consider the equivalent norm

\begin{align}
\|u\|_{\beta, \beta; \rho} = \|u\|_{\beta, \beta; \rho, T_1, T_2} = \sup_{s \in [T_1, T_2]} e^{-\rho(s - T_1)} \|u(s)\| + \sup_{T_1 \leq s < t \leq T_2} (s - T_1)^{\beta} e^{-\rho(t - T_1)} \frac{\|u(t) - u(s)\|}{|t - s|^\beta}
\end{align}

Consider now the separable Hilbert space $L_2(V)$ of Hilbert-Schmidt operators from $V$ into $V$ with the usual norm $\| \cdot \|_{L_2(V)}$ and inner product $(\cdot, \cdot)_{L_2(V)}$.

In order to define the integral with respect to a Hölder integrator $\omega$, we introduce fractional derivatives of order $0 < \alpha < 1$ of a sufficiently regular function $g$ and the right sided fractional derivative of order $1 - \alpha$ of $\omega_{1-}(\cdot) := \omega(\cdot) - \omega(t)$, given by the expressions

\begin{align}
D^\alpha_{s+} g[r] &= \frac{1}{\Gamma(1 - \alpha)} \left( \frac{g(r)}{(t - s)^\alpha} + \alpha \int_s^r \frac{g(r) - g(q)}{(r - q)^{\alpha + 1}} dq \right),
\end{align}

\begin{align}
D^{1-\alpha}_{t-} \omega_{1-}[r] &= \frac{(-1)^{1 - \alpha}}{\Gamma(\alpha)} \left( \frac{\omega(r) - \omega(t)}{(t - r)^{\alpha}} + (1 - \alpha) \int_r^t \frac{\omega(r) - \omega(q)}{(q - r)^{\alpha + 1}} dq \right),
\end{align}

where $0 \leq s \leq r \leq t$ and $\Gamma(\cdot)$ denotes the Gamma function. We assume that $1 - \beta' < \alpha < \beta$ and let $[s, t] \ni r \mapsto g(r) \in L_2(V)$, $[s, t] \ni r \mapsto \omega(r) \in V$ be measurable functions such that $g \in C^\beta([0, T], L_2(V))$, $\omega \in C^{\beta'}([0, T], V)$ and

\begin{align}
\|D^\alpha_{s+} g[r]\|_{L_2(V)} \|D^{1-\alpha}_{t-} \omega_{1-}[r]\| &\leq \int_s^t \|g(r)\|_{L_2(V)} \|\omega(r) - \omega(t)\|_{L^1(V)} \frac{dr}{(t - s)^{\alpha}}
\end{align}
is Lebesgue integrable. Then for $0 \leq s \leq t \leq T$ we can define
\begin{equation}
\int_s^t g(r) d\omega(r) := (-1)^n \sum_{j \in \mathbb{N}} \left( \sum_{\iota \in \mathbb{N}} \int_s^t D_{s+}^{\iota}(e_j, g(e_\iota)) \nu [r] D_{t-}^{1-\iota}(e_\iota, \omega(\cdot)) \nu [r] dr \right) e_j.
\end{equation}
This integral is well-defined and it is given by a generalization of the pathwise integral introduced by Zähle [32], which was given as an extension of the Young integral (see [31]).

In the following result we collect some interesting properties of the integral with integrator $\omega$. In the sequel, we assume the following constraints for the different parameters:

\[ \frac{1}{2} < \beta < \beta', \quad 1 - \beta' < \alpha < \beta. \]

Further, we will also consider $\beta' < H$, where $H$ denotes the Hurst parameter of a fractional Brownian motion.

**Lemma 1.** Let $T > 0$, $\omega \in C^{\beta'}([0, T], V)$ and $g \in C^\beta_{\beta}([s, t], L_2(V))$, for $0 \leq s \leq t \leq T$. Then (5) is well-defined and satisfies the following properties:

(i) The norm can be estimated as
\[ \left\| \int_s^t g(r) d\omega(r) \right\| \leq \int_s^t \| D_{s+}^\iota g[r] \|_{L_2(V)} \| D_{t-}^{1-\iota} \omega_{t-}[r] \| dr \]

where $c$ is a positive constant depending only on $t$, $\beta$, $\beta'$ (see [7]).

(ii) The integral is additive (see [32] Theorem 2.5):
\[ \int_s^t g(r) d\omega(r) + \int_s^t g(r) d\omega(r) = \int_s^t g(r) d\omega(r) \quad \text{for} \ s < \tau < t. \]

(iii) For any $\tau \in \mathbb{R}$ it yields
\[ \int_s^{t-\tau} g(r) d\omega(r) = \int_{s-\tau}^{t-\tau} g(r + \tau) d\theta_\tau \omega(r), \]

where \( \theta_\tau \omega(\cdot) := \omega(t + \cdot) - \omega(t) \) is known as the Wiener shift.

To end this section we introduce the nonlinear mappings of (1). We assume that $F : V \mapsto V$ is a continuous function with at most linear growth, that is, there exist $c_F$, $L_F > 0$ such that
\[ \| F(u) \| \leq c_F + L_F \| u \|, \quad \text{for} \ u \in V, \]
and $G : V \mapsto L_2(V)$ is Lipschitz continuous with Lipschitz constant $L_G$. Denoting $c_G = \| G(0) \|_{L_2(V)}$ we then have
\[ \| G(u) \|_{L_2(V)} \leq c_G + L_G \| u \|, \quad \text{for} \ u \in V. \]

### 3. Existence of solutions

Our main goal in this section is to study the existence of solutions to (1). First of all, we introduce the exact definition of a solution to that problem.

**Definition 2.** Under the assumptions on $A$, $F$ and $G$ described in Section 2, given $T > 0$ we say that $u$ is a mild pathwise solution to (1) corresponding to the initial condition $u_0 \in V$ if $u \in C^\beta_{\beta}([0, T], V)$ and satisfies for $t \in [0, T]$ the equation
\begin{equation}
\int_s^t S(t - r) F(u(r)) dr + \int_s^t S(t - r) G(u(r)) d\omega.
\end{equation}
The first integral on the right hand side is a standard Lebesgue integral, while the integral with respect to $\omega$ is interpreted in the sense of the previous section.

Existence and uniqueness of solutions for this kind of equations has been investigated in [7] and in [24]. In both references the nonlinear mapping $G$ is assumed to be twice Fréchet differentiable with bounded first and second derivatives, from which the uniqueness of solutions can be derived. However, in this article we get rid of such strong regularity and only assume $G$ to be Lipschitz continuous, which is not sufficient to ensure the uniqueness of solutions.

We would like to mention that Theorem 3.1 in [24] also deals with existence– and not uniqueness– of a problem like the above one. To be more precise, under Lipschitz regularity of $G$, assuming that the initial condition is more regular (belonging to a particular space $V_\kappa$ instead of $V$), the authors prove existence of solutions in $W^{\alpha,\infty}([0,T],V)$, the space of measurable functions $x : [0,T] \to V$ such that

$$
\|x\|_{\alpha,\infty} := \sup_{t \in [0,T]} \left( \|x(t)\| + \int_0^t \frac{\|x(t) - x(s)\|}{(t-s)^{1+\alpha}} ds \right) < \infty.
$$

Recall that $\alpha < \beta$, which in turn implies that $C^\beta_\beta([0,T],V) \subset W^{\alpha,\infty}([0,T],V)$.

In our setting, we shall apply Schauder’s theorem to establish the existence of solutions to (7). To this end, for $\omega \in C^\beta([0,T],V)$ and $u_0 \in V$ consider the operators

$$
\mathcal{T}(\cdot,\omega, u_0) : C^\beta([0,T],V) \to C^\beta([0,T],V),
$$

$$
\mathcal{T}^I(\cdot,\omega) : C^\beta([0,T],V) \to C^\beta([0,T],V)
$$

defined by

$$
\mathcal{T}(u,\omega, u_0)(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r))dr + \int_0^t S(t-r)G(u(r))d\omega,
$$

$$
\mathcal{T}^I(u,\omega)(t) = \int_0^t S(t-r)F(u(r))dr + \int_0^t S(t-r)G(u(r))d\omega.
$$

The application of Schauder’s theorem will be based on suitable estimates given in the following lemmas. In the different proofs, $c$ will denote a generic constant that may differ from line to line. Sometimes we will write $c_T$ or $c_S$ when we want to stress the dependence on $T$ or on the semigroup.

We start by stating the following technical result:

**Lemma 3.** Let $a > -1$, $b > -1$ and $a + b \geq -1$, $d > 0$ and $t \in [0,T]$. If for $\rho > 0$ we define

$$
H(\rho) := \sup_{t \in [0,T]} \int_0^1 e^{-\rho(1-v)} v^a (1-v)^b dv,
$$

then we have that $\lim_{\rho \to \infty} H(\rho) = 0$.

The proof can be found in [7], where it can be checked that $H(\rho)$ is related to the Kummer or hypergeometric function.

For the sake of presentation we denote $\|\omega\|_{\beta',0,T}$ by $\|\omega\|_{\beta'}$ (and $\|\mathcal{T}(u,\omega, u_0)\|_{\beta,\beta',0,T}$ by $\|\mathcal{T}(u,\omega, u_0)\|_{\beta,\beta',\rho}$), when the time interval does not produce any confusion.

**Lemma 4.** For any $T > 0$ there exists a $c_T > 0$ (that also depends on the constants related to $F$, $G$ and the semigroup) such that for $\omega \in C^\beta([0,T],V)$, $u_0 \in V$ and $u \in C^\beta([0,T],V)$

$$(8) \quad \|\mathcal{T}(u,\omega, u_0)\|_{\beta,\beta',\rho} \leq c_S \|u_0\| + c_T \|\omega\|_{\beta'} K(\rho)(1 + \|u\|_{\beta,\beta',\rho})$$

where $K(\rho)$ is such that $\lim_{\rho \to \infty} K(\rho) = 0$ and $c_S \geq 1$ is a constant depending on the semigroup $S$. 
Proof. Despite the fact that a quite similar result was proved in [7] (but in that paper there was not any drift), for the sake of completeness we give the proof here. First of all, according to (4) we have that
\[
\|S(\cdot)u_0\|_{\beta,\lambda,\rho} = \sup_{t \in [0,T]} e^{-\rho t} \|S(t)u_0\| + \sup_{0 \leq s < t \leq T} s^\beta e^{-\rho t} \|S(t)u_0 - S(s)u_0\| \leq c_S \|u_0\| + c_S \sup_{0 \leq s < t \leq T} s^\beta e^{-\rho t} \|s^{\lambda s}(t-s)^\beta\|u_0\| \leq c_S \|u_0\|.
\]
Note that the last constant \(c_S\) above is bigger than one, which follows from the fact that in particular
\[
\|u_0\| = \|S(0)u_0\| \leq \sup_{t \in [0,T]} e^{-\rho t} \|S(t)u_0\| \leq c_S \|u_0\|.
\]
On the other hand,
\[
\left\| \int_0^t S(\cdot - r)F(u(r))dr \right\|_{\beta,\beta,\rho} \leq \sup_{t \in [0,T]} e^{-\rho t} \left\| \int_0^t S(t-r)F(u(r))dr \right\|
\]
\[
+ \sup_{0 \leq s < t \leq T} s^\beta e^{-\rho t} \left\| \int_s^t S(t-r)F(u(r))dr \right\|
\]
\[
+ \sup_{0 \leq s < t \leq T} s^\beta e^{-\rho t} \left\| \int_0^t (S(t) - S(s - r))F(u(r))dr \right\|.
\]
Due to the at most linear growth of \(F\),
\[
\sup_{t \in [0,T]} e^{-\rho t} \left\| \int_0^t S(t-r)F(u(r))dr \right\| \leq c_{S,F} \sup_{t \in [0,T]} e^{-\rho t} \int_0^t (1 + \|u(r)\|)dr
\]
\[
= c_{S,F} \sup_{t \in [0,T]} \left( e^{-\rho t} t + \|u\|_{\beta,\beta,\rho} \frac{1 - e^{-\rho t}}{\rho} \right)
\]
\[
\leq c_{S,F} k_1(\rho) \left( 1 + \|u\|_{\beta,\beta,\rho} \right),
\]
where \(k_1(\rho) = \frac{1}{\rho} \). Above we have used that \(\max_{t \geq 0} e^{-\rho t} t = \frac{e^{-\rho}}{\rho} \leq \frac{1}{\rho} \). Moreover,
\[
\frac{s^\beta e^{-\rho t}}{(t-s)^\beta} \left\| \int_s^t S(t-r)F(u(r))dr \right\| \leq c_{S,F,T} \left( e^{-\rho t} (t-s)^{1-\beta} + \|u\|_{\beta,\beta,\rho} \frac{t^{1-\beta} e^{-\rho (t-r)}dr}{(t-s)^\beta} \right)
\]
\[
\leq c_{S,F,T} \left( e^{-\rho t} (t-s)^{1-\beta} + \|u\|_{\beta,\beta,\rho} \frac{1}{\rho^{1-\beta}} \right) \leq c_{S,F,T} k_2(\rho) \left( 1 + \|u\|_{\beta,\beta,\rho} \right),
\]
where \(k_2(\rho) = \frac{1}{\rho^{1-\beta}} \). Note that now we have used that
\[
\max_{t \geq 0} e^{-\rho t} (t-s)^{1-\beta} \leq \frac{e^{-\rho(1-\beta)}}{\rho^{1-\beta}} \leq \frac{1}{\rho^{1-\beta}}.
\]
For the last term we have
\[
\frac{s^\beta e^{-\rho t}}{(t-s)^\beta} \left\| \int_0^s (S(t) - S(s - r))F(u(r))dr \right\| \leq c_S \frac{s^\beta e^{-\rho t}}{(t-s)^\beta} \int_0^s (t-s)^\beta (c_F + L_F \|u(r)\|)dr
\]
\[
\leq c_{S,F}(1 + \|u\|_{\beta,\beta,\rho}) s^\beta \int_0^s e^{-\rho (t-r)} (s-r)^{-\beta} dr
\]
\[
\leq c_{S,F,T} k_3(\rho) (1 + \|u\|_{\beta,\beta,\rho}),
\]
for all \(0 \leq s < t \leq T\).
with \( k_3(\rho) \) defined as in Lemma \([3]\). For the stochastic integral we have a similar splitting than before, namely

\[
\left\| \int_0^t S(\cdot-r)G(u(r))d\omega \right\|_{\beta,\beta,\rho} \leq \sup_{t \in [0,T]} e^{-\rho t} \left\| \int_0^t S(t-r)G(u(r))d\omega \right\| \\
+ \sup_{0<s<t \leq T} \frac{s^\beta}{(t-s)^\beta} \left\| \int_s^t S(t-r)G(u(r))d\omega \right\| \\
+ \sup_{0<s<t \leq T} \frac{s^\beta}{(t-s)^\beta} \left\| \int_0^s (S(t-r) - S(s-r))G(u(r))d\omega \right\|.
\]

(10)

From the definition of the fractional derivative of order \( 1 - \alpha \) it follows easily that

\[
\| D_{1-\alpha}^{\omega} \| \leq c \| \omega \|_{\beta'} (t-r)^{\alpha + \beta' - 1},
\]

hence, thanks also to \([4]\) we obtain

\[
\left\| \int_s^t S(t-r)G(u(r))d\omega \right\| \\
\leq c_s \frac{s^\beta}{(r-s)^\beta} \left( \int_s^t e^{-\rho r} e^{-\rho (t-r)} (c_G + L_G) \| u(r) \| (r-q)^\beta \left( (r-q)^\alpha + (r-q)^{1+\alpha} - (r-q)^\beta \right) \| \omega \|_{\beta'} (t-r)^{\alpha + \beta' - 1} dr \\
+ \int_s^t e^{-\rho t} e^{-\rho (t-r)} (c_G + L_G) \| u(r) \| (r-q)^\beta \left( (r-q)^{1+\alpha} + (r-q)^{1+\alpha} q^\beta \right) \| \omega \|_{\beta'} (t-r)^{\alpha + \beta' - 1} dr \\
\right)
\]

(11)

Performing a change of variable, it is easy to see that

\[
(t-s)^\beta \int_s^t e^{-\rho (t-r)} (r-s)^{-\alpha} (t-r)^{\alpha - 1} dr \\
=(t-s)^{\beta'} (t-s)^\beta \int_0^1 e^{-\rho (t-s)(1-v)} v^{-\alpha} (1-v)^{\alpha - 1} dv = (t-s)^\beta k_4(\rho)
\]

with \( \lim_{\rho \to \infty} k_4(\rho) = 0 \), taking in Lemma \([3]\) \( a = -\alpha, b = \alpha - 1, d = \beta' - \beta \) and \( t-s \) as the corresponding \( t \) there. The second integral on the right hand side may be rewritten in the same way, since

\[
\int_s^t e^{-\rho (t-r)} (r-s)^{-\alpha} (t-r)^{\alpha + \beta' - 1} dr \leq (t-s)^{\beta'} \int_s^t e^{-\rho (t-r)} (r-s)^{-\alpha} (t-r)^{\alpha - 1} dr.
\]
Therefore, from (11) we obtain

\[ s^\beta e^{-\rho t} \left\| \int_s^t S(t-r)G(u(r))d\omega \right\| \leq c_{S,G,T} \left\| \omega \right\|_{\beta'} (t-s)^\beta k_4(\rho)(1 + \|u\|_{\beta,\beta,\rho}). \]

In a similar manner than before, for the first expression on the right hand side of (10) we obtain

\[ e^{-\rho t} \left\| \int_0^t S(t-r)G(u(r))d\omega \right\| \leq c_{S,G,T} \left\| \omega \right\|_{\beta'} k_5(\rho)(1 + \|u\|_{\beta,\beta,\rho}), \]

with \( \lim_{\rho \to \infty} k_5(\rho) = 0. \) Finally, for the third term on the right hand side of (10) we follow similar steps than above when deriving the inequality (11). Indeed, for \( 0 < \alpha < \alpha' < 1 \) such that \( \alpha' + \beta < \alpha + \beta' \), applying (4)

\[
\begin{aligned}
& s^\beta e^{-\rho t} \left\| \int_0^s (S(t-r) - S(s-r))G(u(r))d\omega \right\| \\
& \leq c(t-s)^\beta \left\| \omega \right\|_{\beta'} T^\beta \left( \int_0^s e^{-\rho(t-r)} \frac{e^{-\rho r}(c_G + L_G\|u(r)\|)}{r^\alpha (s-r)^\beta} (s-r)^{\alpha + \beta' - 1}dr \\
& + \int_0^s \int_0^r e^{-\rho(t-r)} \frac{e^{-\rho r}(c_G + L_G\|u(r)\|)(r-q)^{\alpha'}}{(s-r)^{\alpha' + \beta'}(r-q)^{1+\alpha}} dq(s-r)^{\alpha + \beta' - 1}dr \\
& + \int_0^s \int_0^r e^{-\rho(t-r)} \frac{e^{-\rho r}L_G\|u(r) - u(q)\|q^3(r-q)^{\beta}}{(s-r)^{\beta}(r-q)^{1+\alpha}q^3(r-q)^{\beta}} dq(s-r)^{\alpha + \beta' - 1}dr \\
& \leq c_{S,G,T} (t-s)^\beta \left\| \omega \right\|_{\beta'} (1 + \|u\|_{\beta,\beta,\rho}) \int_0^s e^{-\rho(t-r)} r^{-\alpha}(s-r)^{\alpha + \beta' - 1 - \beta}dr \\
& + c_{S,G,T} (t-s)^\beta \left\| \omega \right\|_{\beta'} (1 + \|u\|_{\beta,\beta,\rho}) \int_0^s e^{-\rho(t-r)} r^{\alpha'}(s-r)^{\alpha + \beta' - 1 - \alpha' - \beta}dr \\
& + c_{S,G,T} (t-s)^\beta \left\| \omega \right\|_{\beta'} \|u\|_{\beta,\beta,\rho} \int_0^s e^{-\rho(t-r)} r^{-\alpha}(s-r)^{\alpha - \beta + \beta' - 1}dr.
\end{aligned}
\]

Collecting all the above estimates the proof is complete. \( \square \)

**Lemma 5.** For \( \omega \in C^\beta([0,T],V) \) and \( u \in C^\beta_t([0,T],V) \) the mapping \( T^t(u,\omega)(t) \in V_\delta \) for every \( t \geq 0 \) and \( \delta \in [0,\beta') \). Moreover, there exists a constant \( c \) depending on \( S, F \) and \( G \) such that

\[ \|T^t(u,\omega)(t)\|_{V_\delta} \leq c(t^{\beta' - \delta} \|\omega\|_{\beta'} + t^{1-\delta})(1 + \|u\|_{\beta,\beta}). \]

**Proof.** For the deterministic integral we directly obtain

\[
\left\| \int_0^t S(t-r)F(u(r))dr \right\|_{V_\delta} \leq c_{S,F} \int_0^t (t-r)^{-\delta}(1 + \|u(r)\|)dr \leq c_{S,F} t^{1-\delta}(1 + \|u\|_{\beta,\beta}).
\]
For the stochastic integral we have
\[
\left\| \int_0^t S(t-r)G(u(r))d\omega \right\|_{\mathcal{V}_3} \\
\leq c \int_0^t \left( \left\| S(t-r) \right\|_{L^2(V_3)} \left\| G(u(r)) \right\|_{L^2(V_3)} \right) \frac{dr}{r^\alpha} \\
+ \int_0^t \left( \left\| S(t-r) - S(t-q) \right\|_{L^2(V_3)} \left\| G(u(r)) \right\|_{L^2(V_3)} \right) \frac{dq}{(r-q)^{1+\alpha}} \\
+ \int_0^t \left( \left\| S(t-q) \right\|_{L^2(V_3)} \left\| G(u(r)) - G(u(q)) \right\|_{L^2(V_3)} \right) \frac{dq}{(r-q)^{1+\alpha}} \\
\leq c_S \left\| \omega \right\|_{\beta'} \left( \int_0^t \left( c_G + L_G \left\| u(r) \right\| \right) \frac{dr}{r^\alpha} \right) \\
+ \int_0^t \left( \frac{c_G + L_G \left\| u(r) \right\|}{(r-t)^{\alpha+\delta}} \right) d\omega(t) \frac{dr}{r^\alpha} \\
+ \int_0^t \left( \frac{L_G \left\| u(r) - u(q) \right\| q^\beta(r-q)\beta}{(r-q)^{1+\alpha} q^\beta(r-q)\beta} \right) d\omega(t) \frac{dr}{r^\alpha} \\
\leq c_{S,G} \left\| \omega \right\|_{\beta'} \left( 1 + \left\| u \right\|_{\beta,\beta} \right) t^\alpha \frac{dr}{r^\alpha},
\]
where \( \alpha' \) has been chosen such that \( 0 < \alpha < \alpha' < 1 \) with \( \alpha' + \delta < \alpha + \beta' \).

\[\square\]

**Corollary 6.** If \( \omega \in C^\beta([0,T],V) \), \( u_0 \in V \) and \( \omega \in C^\beta_\alpha([0,T],V) \) the mapping \( T(u,\omega,u_0)(t) \in V_3 \) for every \( t > 0 \) and \( \delta \in [0,\beta') \). Moreover,
\[
\left\| T(u,\omega,u_0)(t) \right\|_{\mathcal{V}_3} \leq c_S t^{-\delta} \left\| u_0 \right\| + c (t^{\alpha-\delta} \left\| \omega \right\|_{\beta'} + t^{1-\delta}) (1 + \left\| u \right\|_{\beta',\beta}),
\]
where the constant \( c \) depends on the semigroup and also the constants related to \( F \) and \( G \).

**Proof.** For \( t > 0 \) we trivially have \( \left\| S(t)u_0 \right\|_{\mathcal{V}_3} \leq c t^{-\delta} \left\| u_0 \right\| \). To conclude the result it suffices to take into account Lemma 5.

\[\square\]

We have also the following result:

**Theorem 7.** Denote by \( B := \hat{B}_{C^\beta_\alpha}(0,R) \) and \( \hat{B} := \hat{B}_{C^\beta}(0,R) \) the closed balls in \( C^\beta_\alpha([0,T];V) \) and \( C^\beta([0,T],V) \), respectively, with radius \( R \) and center 0. Let \( \mathcal{K} \) be a compact set in \( V \). Then \( T^I(B,\hat{B}) \) and \( T(B,\hat{B},\mathcal{K}) \) are relatively compact in \( C^\beta([0,T],V) \) and \( C^\beta_\alpha([0,T],V) \), respectively.

**Proof.** For \( u \in B \), \( \omega \in \hat{B} \) and \( T \geq t_2 \geq t_1 \geq 0 \) there exists a \( \gamma \in (\beta,\beta') \) such that
\[
\left\| T^I(u,\omega)(t_2) - T^I(u,\omega)(t_1) \right\| \leq c \left\| \omega \right\|_{\beta'} (1 + \left\| u \right\|_{\beta',\beta} (t_2 - t_1)^\gamma),
\]
where \( c \) is a positive constant that depends on the constants related to \( S \), \( F \) and \( G \), and \( T \). The method to obtain the above estimate is similar to the calculations in Lemma 4 setting \( \rho = 0 \), see also Chen et al. 7, hence we omit the proof here.

As a result, the set \( T^I(B,\hat{B}) \) is equicontinuous and bounded in the space \( C^\gamma([0,T],V) \).

On the other hand, in virtue of Lemma 5 we also have that \( T^I(B,\hat{B})(t) \in B_3 \), for all \( t \in [0,T] \), where \( B_3 \) is a bounded set in \( V_3 \) with \( 0 < \delta < \beta' \). As we know that \( V_3 \subset V \) compactly, we obtain that the set \( T^I(B,\hat{B})([0,T]) \) belongs to a compact set of \( V \). By Lemma 4.5 in 24 we have that \( T^I(B,\hat{B}) \) is relatively compact in \( C^\beta([0,T],V) \) if \( \beta \in (\alpha,\gamma) \) and then in \( C^\beta_\alpha([0,T],V) \) as well, since \( C^\beta([0,T],V) \subset C^\beta_\alpha([0,T],V) \) continuously.

Finally, let \( u^n_0 \in \mathcal{K} \). Then up to a subsequence \( u^n_0 \to u_0 \) in \( V \), and therefore
\[
\left\| (S(t) - S(s))(u^n_0 - u_0) \right\| \leq \left\| S(t-s) - Id \right\|_{L(V_3,V)} \left\| S(s) \right\|_{L(V_3,V)} \left\| u^n_0 - u_0 \right\| \\
\leq c_S s^{-\beta} (t-s)^\beta \left\| u^n_0 - u_0 \right\|, \ 0 < s < t \leq T,
\]
implies that
\[ S(\cdot)u_0^n \rightarrow S(\cdot)u_0 \] in \( C^\beta([0,T],V) \).

Hence, \( S(\cdot)K \) is relatively compact in \( C^\beta([0,T],V) \).

Joining the two results we obtain that \( T(B,\bar{B},K) \) is relatively compact in \( C^\beta([0,T],V) \). \( \square \)

In the next result we address the existence of solutions to (7).

**Theorem 8.** Under the above conditions on \( A, F, G \) and \( G \), given \( T > 0 \), for \( \omega \in C^\beta([0,T],V) \) and \( u_0 \in V \) there exists at least one mild pathwise solution \( u \in C^\beta([0,T],V) \) to the equation (7) given by (E).

**Proof.** We choose \( \rho_0 \) large enough such that \( c_T \| \omega \|_\beta \cdot K(\rho_0) < \frac{1}{2} \). Therefore, from Lemma \( \square \) \( T(\cdot, \omega, u_0) \) maps the ball \( B := B(0, R) = \{ u \in C^\beta([0,T],V) : \| u \|_{\beta, \beta, \rho_0} \leq R \} \), with \( R := 1 + 2c_S\| u_0 \| \) into itself, that is, \( T(B, \omega, u_0) \subset B \).

It is clear that \( B \) is convex, bounded and closed and we know by Theorem \( \square \) that the operator \( T(\cdot, \omega, u_0) \) is compact. In order to apply Schauder’s theorem, it only remains to check that \( T(\cdot, \omega, u_0) : B \rightarrow B \) is continuous. Assume that \( (u^n)_{n \in \mathbb{N}} \subset B \) is such that \( u^n(0) = u_0 \) for every \( n \in \mathbb{N} \), and \( u^n \rightarrow u \) in \( C^\beta([0,T],V) \). We shall check that for \( t \in [0,T], \)
\[
\lim_{n \to \infty} T(u^n, \omega, u_0)(t) - T(u, \omega, u_0)(t) = 0.
\]

We only need to consider the integral terms. First of all, due to the continuity of \( F \),
\[
\left\| \int_0^t S(t-r)(F(u^n(r)) - F(u(r)))dr \right\| \leq c_{S,F} \int_0^t \| F(u^n(r)) - F(u(r)) \| dr
\]
and \( f^n(r) := \| F(u^n(r)) - F(u(r)) \| \rightarrow 0 \) when \( n \rightarrow \infty \), for \( r \in [0,t] \). Moreover, it has a trivial integrable majorant, given by \( 2c_F + L_F(\| u^n \|_{\beta, \beta} + \| u \|_{\beta, \beta}) \leq 2c_F + 2L_FR \).

For the stochastic integral
\[
\left\| \int_0^t S(t-r)(G(u^n(r)) - G(u(r)))dr \right\|
\leq c_{\omega} \int_0^t \left( \frac{\| S(t-r) \|_{L(V)}}{r^\alpha} \| G(u^n(r)) - G(u(r)) \|_{L_2(V)} \right)dr
+ \int_0^r \frac{\| S(t-r) - S(t-q) \|_{L(V)}}{(r-q)^{1+\alpha}} \| G(u^n(r)) - G(u(r)) \|_{L_2(V)} dq
+ \int_0^r \frac{\| S(t-q) \|_{L(V)}}{(r-q)^{1+\alpha}} \| G(u^n(r)) - G(u(r)) - (G(u^n(q)) + G(u(q))) \|_{L_2(V)} dq
\]
\[= A_1 + A_2 + A_3.\]

Using the properties of \( S \) and \( G \) we have that the first two terms in the last inequality can be estimated as follows:
\[
A_1 + A_2 \leq c_S \| \omega \|_{\beta', 0, T} \left( \int_0^t \frac{L_G \| u^n(r) - u(r) \|}{r^\alpha} (t-r)^{\alpha + \beta' - 1} dr \right)
+ \int_0^t \int_0^r \frac{L_G \| u^n(r) - u(r) \| (r-q)^\beta}{(t-r)^{2}(r-q)^{1+\alpha}} dq (t-r)^{\alpha + \beta' - 1} dr
\leq c_{S,G} t^{\beta'} \| \omega \|_{\beta', 0, T} \| u^n - u \|_{\beta, \beta}
\]
thus we get that \( A_1 + A_2 \rightarrow 0 \) as \( n \rightarrow \infty \).
For the term $A_3$ we define the functions

$$h^n(r, q) = \frac{\|G(u^n(r)) - G(u(r)) - G(u^n(q)) + G(u(q))\|_{L^2(V)}}{(r - q)^{1+\alpha}}(t - r)^{\alpha + \beta' - 1}.$$ 

The Lipschitz property of $G$ implies that

$$h^n(r, q) \to 0 \text{ for a.a. } (r, q) \in D = \{(r, q) : 0 \leq q \leq r \leq t\}.$$ 

On the other hand, we construct a majorant in the usual way:

$$h^n(r, q) \leq L_G \frac{u^n(r) - u^n(q)}{(r - q)^{1+\alpha}} + \|u(r) - u(q)\|_{C^\beta} q^\beta (t - r)^{\alpha + \beta' - 1}$$

$$\leq L_G \frac{\|u^n\|_{C^\beta} + \|u\|_{C^\beta}}{(r - q)^{1+\alpha}} q^\beta (t - r)^{\alpha + \beta' - 1}$$

$$\leq 2RL_G \frac{(t - r)^{\alpha + \beta' - 1}}{(r - q)^{1+\alpha - \beta} q^\beta} = f(r, q) \in L^1(D).$$

By Lebesgue’s theorem and Fubini’s theorem we have that $A_3 \to 0$.

Finally, applying Schauder’s fixed point theorem, the problem \ref{eq:SPDE} has at least one mild pathwise solution given by \ref{thm:Schauder}.

\begin{remark}
As it can be easily seen in the proof of the Lemma \ref{lem:holder}, the integrals are well-defined just considering the norm of the space $C^\beta([0, T], V)$. As we already pointed out in Section \ref{sec:holder}, the factor $s^\beta$ that appears in the norm of the space $C^\beta([0, T], V)$ is only required due to the fact that the semigroup $S$ is not Hölder continuous at zero. In fact, if the initial condition were in the space $V_\beta$, then we could simply work in the space of Hölder continuous functions with exponent $\beta$, since then

$$\|S(\cdot)u_0\|_{C^\beta} \leq \sup_{0 \leq s < t \leq T} \frac{\|S(s)\|_{C^\beta} \|S(t - s) - 1\|_{L(V_\beta, V)}}{(t - s)^{1+\beta}} \|u_0\|_{V_\beta} \leq c_S \sup_{0 \leq s < t \leq T} \frac{(t - s)^{\beta}}{(t - s)^{1+\beta}} < \infty.$$ 

Next we would like to establish, on the base of a concatenation procedure, that every mild solution can be extended to be a globally defined mild solution, that is, it exists for any $t \geq 0$. Denote by $u_1 \in C^\beta([0, T_1], V)$ the mild solution obtained in Theorem \ref{thm:concat} corresponding to the initial condition $u_0 \in V$ and the driving path $\omega \in C^\beta([0, T_1], V)$. Since by Corollary \ref{cor:holder} we know that $u_1(T_1) \in V_\beta$, then considering as initial condition $u_2(0) := u_1(T_1)$ and taking the new driving path $\theta_T \omega \in C^\beta([0, T_2], V)$, thanks to Theorem \ref{thm:concat} and the above discussion we are able to obtain a mild solution $u_2 \in C^\beta([0, T_2], V)$.

\begin{lemma}
(Concatenation) Let $u_1 \in C^\beta([0, T_1], V)$, $u_2 \in C^\beta([0, T_2], V)$ be mild solutions to \ref{eq:SPDE} with $u_1(0) \in V$, $u_2(0) = u_1(T_1)$, for $\omega \in C^\beta([0, T_1], V)$ and $\theta_T \omega \in C^\beta([0, T_2], V)$, respectively, and let

$$u(t) = \begin{cases} u_1(t) & \text{if } t \in [0, T_1], \\ u_2(t - T_1) & \text{if } t \in [T_1, T_1 + T_2]. \end{cases}$$

Then $u \in C^\beta([0, T_1 + T_2], V)$ is a mild solution to \ref{eq:SPDE} on $[0, T_1 + T_2]$.

\begin{proof}
First, we need to show that $u \in C^\beta([0, T_1 + T_2], V)$. Trivially

$$\|u\|_{C^\beta([0, T_1 + T_2], V)} \leq \|u_1\|_{C^\beta([0, T_1], V)} + \|u_2\|_{C^\beta([0, T_2], V)}.$$

Finally, applying Schauder’s fixed point theorem, the problem \ref{eq:SPDE} has at least one mild pathwise solution given by \ref{thm:Schauder}.
\end{proof}
\end{remark}
On the other hand, since \( u_2(0) = u_1(T_1) \),
\[
\|u\|_{\beta,\beta,0,T_1+T_2} \leq \max \left( \sup_{0<s<t \leq T_1} s^\beta \frac{\|u_1(t) - u_1(s)\|}{(t-s)^\beta}, \sup_{T_1 \leq t \leq T_1+T_2} (t-s)^\beta \frac{\|u_1(t) - u_1(s)\|}{(t-s)^\beta} \right)
\]
\[
\leq \max \left( \sup_{0<s<t \leq T_1} s^\beta \frac{\|u_1(t) - u_1(s)\|}{(t-s)^\beta}, (T_1 + T_2)^\beta \|u_1\|_{\beta,0,T_1+T_2} \right)
\]
\[
\leq \max (\|u_1\|_{\beta,\beta,0,T_1}, (T_1 + T_2)^\beta \|u_1\|_{\beta,0,T_1+T_2}) < \infty.
\]

Second, we will prove that \( u \) satisfies the integral equality (\( I \)). Since this is obvious if \( t \in [0,T_1] \), let \( t \in (T_1,T_1+T_2) \). Thus, in virtue of properties (ii) and (iii) of Lemma 1, \( u(t) = u_2(t-T_1) \) is such that
\[
u(t)=S(t-T_1)u_1(T_1)+\int_0^{t-T_1}S(t-T_1-r)F(u_2(r))dr+\int_0^{T_1}S(t-T_1-r)G(u_2(r))d\theta_{T_1}d\omega.
\]
\[
u(t)=S(t-T_1)(u_1(0)+\int_0^{T_1}S(t-T_1-r)F(u_1(r))dr+\int_0^{T_1}S(t-T_1-r)G(u_1(r))d\omega)
\]
\[
= S(t)u_1(0)+\int_0^{T_1}S(t-r)F(u_1(r))dr+\int_0^{T_1}S(t-r)G(u_1(r))d\omega
\]
\[
+\int_0^{T_1}S(t-r)F(u_2(r-T_1))dr+\int_0^{T_1}S(t-r)G(u_2(r-T_1))d\omega
\]
\[
= S(t)u(0)+\int_0^{T_1}S(t-r)F(u(r))dr+\int_0^{T_1}S(t-r)G(u(r))d\omega.
\]

The above method can be repeated in such a way that the corresponding solutions to (\( I \)) are globally defined.

We also prove that the solutions satisfy the translation property.

**Lemma 11.** (Translation) Let \( u(\cdot) \) be a mild solution to (I) on \([0,T]\) with \( u_0 \in V \) for \( \omega \in C^{\beta'}([0,T], V) \) and let \( 0 < s < T \). Then the function \( v(\cdot) = u(\cdot + s) \) is a mild solution on \([0,T-s]\) with \( v(0) = u(s) \) for the driving path \( \theta_{s,\omega} \).

For the proof of this lemma we can apply the techniques from the last lemma.

4. **Multivalued non-autonomous and random dynamical systems**

We start this section by introducing the general concept of multivalued non-autonomous and random dynamical systems. Later we will apply it to the set of solutions of the problem (I).

Let \( \Omega \) be some set. On \( \Omega \) we define a flow of non-autonomous perturbations \( \theta : \mathbb{R} \times \Omega \to \Omega \) by
\[
\theta_0 \omega = \omega, \quad \theta_t \circ \theta_s = \theta_{t+s}, \quad t, s, \tau \in \mathbb{R},
\]
for \( \omega \in \Omega \).

We now give the definition of a metric dynamical system, that is a general model for a noise. On a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we consider a \( \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F} \)-measurable flow \( \theta \) such that \( \theta_t \mathbb{P} = \mathbb{P} \) for every
t ∈ ℜ. Often it is also assumed that ℙ is ergodic with respect to the flow θ, which means that, in addition to the invariance property of ℙ defined above, given an invariant set A ∈ ℱ (that is, ℙ(A) = A, for all t ∈ ℜ), we have either ℙ(A) = 0 or ℙ(A) = 1. Then the quadruple (Ω, ℱ, ℙ, θ) is called an ergodic metric dynamical system.

**Remark 12.** For the following results we only need a semiflow instead of a flow θ, that is, defined on ℜ⁺. However, for further considerations regarding the existence of random attractors in a forthcoming paper, we will need to deal with a flow as introduced above. For the existence of a flow defined on ℜ we refer to [8] Page 240.

Denote by P_f(V) the set of all non-empty closed subsets of V.

**Definition 13.** Consider a flow of non-autonomous perturbations θ : ℜ × Ω → Ω. A multivalued mapping Φ : ℜ⁺ × Ω × V → P_f(V) is called a multivalued non-autonomous dynamical system (MNDS) if:

i) Φ(0, ω, ·) = id_V,

ii) Φ(t + r, ω, x) ⊂ Φ(t, θ_r, ω, Φ(r, ω, x)) (cocycle property) for all t, r ∈ ℜ⁺, x ∈ V, ω ∈ Ω.

It is called a strict MNDS if Φ(t + r, ω, x) = Φ(t, θ_r, ω, Φ(r, ω, x)) for all t, r ∈ ℜ⁺, x ∈ V, ω ∈ Ω. Assume now that (Ω, ℱ, ℙ, θ) is an (ergodic) metric dynamical system. An MNDS is called a multivalued random dynamical system (MRDS) if the multivalued mapping (t, ω, x) → Φ(t, ω, x) is ℬ(ℜ⁺) ⊗ ℱ ⊗ ℬ(V) measurable, i.e.

\[ \{ (t, ω, x) : Φ(t, ω, x) ∩ O ≠ ∅ \} ∈ ℬ(ℜ⁺) ⊗ ℱ ⊗ ℬ(V) \]

for every open set O of V.

A suitable concept of continuity in the setting of multivalued dynamical systems is the following one.

**Definition 14.** Φ(t, ω, ·) is called upper semicontinuous at x₀ if for every open neighborhood O ⊂ V of the set Φ(t, ω, x₀) there exists δ > 0 such that if ∥x₀ – y∥ < δ then Φ(t, ω, y) ∈ O. Φ(t, ω, ȧ) is called upper semicontinuous if it is upper semicontinuous at every x₀ in V.

This definition can be extended to the one of upper semicontinuity with respect to all variables assuming that Ω is a Polish space. We are now able to formulate a general condition ensuring that an MNDS defines an MRDS. For the proof, see Lemma 2.5 in [3].

**Lemma 15.** Let Ω be a Polish space and let ℱ be the associated Borel σ-algebra. Suppose that (t, ω, x) ↦ Φ(t, ω, x) is upper semicontinuous. Then Φ is measurable in the sense of Definition [3].

As a result, an MNDS Φ that it is upper semicontinuous with respect to its three variables becomes an MRDS, provided that the system of non-autonomous perturbations is a metric dynamical system. But we stress once again that above we have requested the separability of Ω.

In what follows we present two examples of metric dynamical systems describing a Gauß noise given by the fractional Brownian motion with any Hurst parameter H ∈ (0, 1) (fBm to short). Given H ∈ (0, 1), a continuous centered Gaußian process \( β^H(t), t ∈ ℜ \), with the covariance function

\[ \mathbb{E}β^H(t)β^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s ∈ ℜ \]

is called a two-sided one-dimensional fractional Brownian motion, and H is the Hurst parameter. Assume that Q is a bounded and symmetric positive linear operator on V which is of trace class, i.e., for a complete orthonormal basis \( \{ e_i \}_{i ∈ ℤ} \) in V there exists a sequence of nonnegative numbers \( \{ q_i \}_{i ∈ ℤ} \) such that \( trQ := \sum_{i=1}^{∞} q_i < ∞ \). Then a continuous V-valued fractional Brownian motion \( B^H(t) \) with covariance operator Q and Hurst parameter H is defined by

\[ B^H(t) = \sum_{i=1}^{∞} \sqrt{q_i}e_iβ_i^H(t), \quad t ∈ ℜ, \]
where \( \{ \beta^H_i(t) \}_{i \in \mathbb{N}} \) is a sequence of stochastically independent one-dimensional fBm.

In virtue of Kolmogorov’s theorem we know that \( B^H \) has a continuous version, see [2] Theorem 39.3. Hence we can consider the canonical interpretation of an fBm: let \( C_0 := C_0(\mathbb{R}, V) \) be the space of continuous functions on \( \mathbb{R} \) with values in \( V \). Here and below the subindex means that these functions are zero at zero, equipped with the compact open topology. Let \( \mathcal{F} = B(C_0(\mathbb{R}, V)) \) be the associated Borel-\( \sigma \)-algebra, \( \mathbb{P} \) the distribution of the fBm \( B^H \) and \( \{ \theta_t \}_{t \in \mathbb{R}} \) the flow of Wiener shifts given by \( \theta_t \).

In that way, \( (C_0(\mathbb{R}, V), B(C_0(\mathbb{R}, V)), \mathbb{P}, \theta) \) is an ergodic metric dynamical system, see [25] and [15]. The foundation of that property can be found in [2] Theorem 38.6. Furthermore, this (canonical) process has a version \( \omega \in C^\gamma_0 := C^\gamma_0(\mathbb{R}, V) \), that is, \( \omega(0) = 0 \) and it is \( \gamma \)-Hölder continuous on any interval \([-n, n] \) for all \( \gamma < H \), see [2], Theorem 39.4. This regularity of the fractional Brownian motion makes this process to be the main example fitting our abstract setting.

However, the space \( C^\gamma_0 \) is not suitable for our further purposes since it is not separable, and we need a Polish space (see Lemma 15 above). Nevertheless, in order to give a meaning to the stochastic integrals we still need to consider Hölder continuous functions and the continuous dependence of the integral with respect to the integrators in some subspace of Hölder continuous functions. In fact, in what follows we consider a second example consisting of a subspace of Hölder continuous functions that is separable.

For \( n \in \mathbb{N} \) and a general parameter \( \gamma \in (0, 1) \) we define

\[
C^{0,\gamma}([-n, n], V) := C^\infty([-n, n], V) \cap C^\gamma([-n, n], V)
\]

which is a closed linear subspace of \( C^\gamma([-n, n], V) \) and in particular a separable Banach space, see Friz and Victoir [10], Proposition 5.36. Let us denote by \( C^{0,\gamma}_0 := C^{0,\gamma}_0(\mathbb{R}, V) \) the Fréchet space given by the subset of \( C^\gamma_0 \) whose elements restricted to \([-n, n]\) are in \( C^{0,\gamma}([-n, n], V) \), so it is also a separable complete metric space. The generating norms are the \( \gamma \)-Hölder norms of functions on \([-n, n]\).

According to the Wiener’s characterization, see [10], Theorem 5.31, the space \( C^{0,\gamma}([-n, n], V) \) can be also expressed as

\[
C^{0,\gamma}([-n, n], V) = \left\{ x \in C^\gamma([-n, n], V) : \lim_{\delta \to 0} \sup_{\| t-s \| < \delta} \frac{\| x(t) - x(s) \|}{| t-s |^\gamma} = 0 \right\}
\]

(12)

Next we define a new ergodic metric dynamical system modeling the fractional Brownian motion where the set \( \Omega = C^{0,\gamma}_0 \) is separable. Needless to say that this result has its own interest, since to our knowledge the path space of the metric dynamical system modeling the fBm considered in the literature has been given so far either by \( C_0 \) or \( C^\gamma_0 \).

Consider the ergodic metric dynamical system \((C_0, B(C_0), \mathbb{P}, \theta)\) where \( \theta \) is the Wiener shift on \( C_0 \) and \( \mathbb{P} \) is the distribution of the fBm. We would like to prove that defining \( \mathbb{P}'(A) = \mathbb{P}(B) \), \( A = B \cap C^{0,\gamma}_0 \), \( B \in B(C_0) \) and \( \theta' \) being the restriction of \( \theta \) to \( \mathbb{R} \times B(C^{0,\gamma}_0) \), the quadruple \((C^{0,\gamma}_0, B(C^{0,\gamma}_0), \mathbb{P}', \theta')\) is also an ergodic metric dynamical system.

We start establishing the connections between the two Borel-\( \sigma \)-algebras \( B(C^{0,\gamma}_0) \) and \( B(C_0) \).

**Lemma 16.** We have \( B(C^{0,\gamma}_0) = B(C_0) \cap C^{0,\gamma}_0 \).

**Proof.** Since \( C^{0,\gamma}_0 \) is continuously embedded in \( C_0 \), the inclusion \( B(C_0) \cap C^{0,\gamma}_0 \subset B(C^{0,\gamma}_0) \) follows from Vishik and Fursikov [29] Theorem II.2.1, so we need to check that \( B(C^{0,\gamma}_0) \subset B(C_0) \cap C^{0,\gamma}_0 \).

A countable generator of \( B(C^{0,\gamma}_0) \) consists of all open sets of the form

\[
V(\omega_0; n, \varepsilon_1, \varepsilon_2) = \{ \omega \in C^{0,\gamma}_0 : \| \omega - \omega_0 \|_{\infty, -n, n} < \varepsilon_1, \| \omega - \omega_0 \|_{\gamma, -n, n} < \varepsilon_2 \},
\]
Lemma 17. We have that $C_0^{0,\gamma} \in \mathcal{B}(C_0)$.
Proof. First of all, for $a < b$, for $\omega \in C_0$ we define the following mapping:

$$f_{a,b}(\omega) = \begin{cases} \|\omega\|_{\gamma,a,b} & : \text{if } \omega \in C^\gamma([a,b], V), \\ \infty & : \text{if } \omega \in C([a,b], V) \setminus C^\gamma([a,b], V). \end{cases}$$

Then the mapping $f_{a,b} : (C_0, B(C_0)) \to (\mathbb{R}^+, B(\mathbb{R}^+))$ is measurable. In order to prove this statement, we consider

$$f_{a,b,k}(\omega) = \sup_{a \leq s < t < b, \ t - s \geq \frac{1}{k}} \frac{\|\omega(t) - \omega(s)\|}{|t - s|^{\gamma}},$$

which is a continuous mapping on $C_0$ with values in $\mathbb{R}^+$. If we prove that the following property

(14) $$\lim_{k \to \infty} f_{a,b,k}(\omega) = f_{a,b}(\omega).$$

holds true, then the measurability of $f_{a,b}$ follows, since the pointwise limit of measurable functions is also a measurable function.

Note that the sequence $(f_{a,b,k})_{k \in \mathbb{N}}$ is non decreasing, so that there exists its limit in $\mathbb{R}^+$ and this limit is smaller than or equal to $f_{a,b}(\omega)$ for every $\omega \in C_0$. On the other hand, by the definition of supremum, there exists a sequence $(s_n, t_n)_{n \in \mathbb{N}}$ with $a \leq s_n < t_n \leq b$ such that

$$\lim_{n \to \infty} \frac{\|\omega(t_n) - \omega(s_n)\|}{|t_n - s_n|^{\gamma}} = f_{a,b}(\omega).$$

We can select an increasing subsequence $(k_n)$ such that $1/k_n \leq t_n - s_n$ and hence

$$\frac{\|\omega(t_n) - \omega(s_n)\|}{|t_n - s_n|^{\gamma}} \leq f_{a,b,k_n}(\omega) \leq f_{a,b}(\omega)$$

and the left hand side converges to $f_{a,b}(\omega)$, which shows (14).

Now, for $\omega \in C_0$ consider the mapping

$$g_{k,n}(\omega) = \sup_{-n \leq s < t \leq n, \ s, t \in \mathbb{Q}, \ t - s < \frac{1}{k}} \ f_{s,t}(\omega).$$

Then the mapping $g_{k,n}$ is $(C_0, B(C_0)), (\mathbb{R}^+, B(\mathbb{R}^+))$-measurable, which follows since this supremum is taken over countably many measurable elements. In addition, we can prove that

(15) $$g_{k,n}(\omega) = \sup_{-n \leq s < t \leq n, \ t - s < \frac{1}{k}} \ f_{s,t}(\omega).$$

Straightforwardly, the right hand side of (15) is larger than or equal to the left hand side. Conversely, for fixed $n$, let $(s_m^n, t_m^n)$ with $-n \leq s_m^n < t_m^n \leq n$ and $t_m^n - s_m^n < 1/k$ such that

$$\lim_{m \to \infty} f_{s_m^n, t_m^n}(\omega) = \sup_{-n \leq s < t \leq n, \ t - s < \frac{1}{k}} \ f_{s,t}(\omega).$$

Then we find $s_m^n, t_m^n \in \mathbb{Q}$ such that $-n \leq s_m^n \leq s_m^n < t_m^n \leq t_m^n \leq n, \ t_m^n - s_m^n < 1/k$. Hence

$$f_{s_m^n, t_m^n}(\omega) \leq f_{s_m^n, t_m^n}(\omega)$$

which gives the opposite inequality in (15). Note that if $s_m^n = -n$ we can set $s_m^n = -n \in \mathbb{Q}$ and similarly for $t_m^n = n$. We finally define the mapping

$$h_n(\omega) = \lim_{k \to \infty} \sup_{-n \leq s < t \leq n, \ t - s < \frac{1}{k}} \ g_{k,n}(\omega) \in \mathbb{R}^+$$

which is measurable in $B(C_0)$. We stress that if $h_n(\omega) = 0$ then we indeed have

$$0 = \lim_{k \to \infty} g_{k,n}(\omega) \in \mathbb{R}^+.$$

Hence

$$A_n := h_n^{-1}((0)) \in B(C_0).$$
and then, by the Wiener’s characterization and \([15]\), we finally obtain
\[
A_n = \{ \omega \in C_0 : \omega|_{[-n,n]} \in C_0^{\theta}([-n,n], V) \}
\]
that implies
\[
C_0^{\theta} = \bigcap_n A_n \in B(C_0).
\]

Before proving the main result of this section, let us remind here several properties that are necessary.

**Definition 18.** Given a metric dynamical system \((\Omega, F, P, \theta)\), \(A\) is invariant mod \(P\) if \(P(A \Delta \theta_t A) = 0\), for every \(t \in \mathbb{R}\).

The following result can be found in Walters [30], Theorem 1.5:

**Lemma 19.** The metric dynamical system \((\Omega, F, P, \theta)\) is ergodic if and only if every \(\theta\)-invariant set mod \(P\) has measure zero or one.

Now we can prove the main theorem of this section.

**Theorem 20.** The quadruple \((C_0^{\theta}, B(C_0^{\theta}), \mathbb{P}', \theta')\) is an ergodic metric dynamical system where
\[
\mathbb{P}'(A) = \mathbb{P}(B), A = B \cap C_0^{\theta}, B \in B(C_0)
\]
and \(\theta'\) is the restriction of \(\theta\) to \(\mathbb{R} \times B(C_0^{\theta})\).

**Proof.** Notice that \(\mathbb{P}(C_0^{\theta}) = 1\), which follows from the property that the fBm has paths in \(C_0^{\theta}\). In fact, from [2] Theorem 39.4 and [22] Theorem 1.4.1, we know that \(\omega \in C_0^{\theta} \) for any \(\gamma < \gamma' < H\). Hence, for every \(n \in \mathbb{N}\) in particular \(\omega \in C^{\gamma}([-n, n], V)\) and
\[
\lim_{\delta \to 0} \sup_{-n \leq s < t \leq n} \frac{||\omega(t) - \omega(s)||}{|t-s|^\gamma} = \lim_{\delta \to 0} \sup_{-n \leq s < t \leq n} \frac{||\omega(t) - \omega(s)||}{|t-s|^\gamma'} 
\]
\[
\leq \lim_{\delta \to 0} \sup_{-n \leq s < t \leq n} \frac{||\omega(t) - \omega(s)||}{|t-s|^\gamma'} 
\]
\[
\leq ||\omega||_{\gamma',-n,n} \lim_{\delta \to 0} \delta^{-\gamma'} = 0.
\]

Furthermore, by Lemma [16] and Lemma [17] \(\mathbb{P}'\) is defined on \(B(C_0^{\theta})\). To check that \(\mathbb{P}'\) is well-defined, let \(A \in B(C_0^{\theta})\) be such that for \(B_1, B_2 \in B(C_0)\) we have
\[
A = B_i \cap C_0^{\theta}, \quad i = 1, 2.
\]

It is easy to check that
\[
(B_1 \cap C_0^{\theta}) \Delta (B_2 \cap C_0^{\theta}) = C_0^{\theta} \cap (B_1 \Delta B_2),
\]
thus, because the symmetric difference of a set with itself is the empty set and \(\mathbb{P}(C_0^{\theta}) = 1\), we have
\[
0 = \mathbb{P}((B_1 \cap C_0^{\theta}) \Delta (B_2 \cap C_0^{\theta})) = \mathbb{P}(B_1 \Delta B_2),
\]
so that \(\mathbb{P}(B_1) = \mathbb{P}(B_2)\) and therefore \(\mathbb{P}'(A) = \mathbb{P}(B_i)\). This property, together with the \(\sigma\)-additivity and the fact that trivially \(\mathbb{P}'(C_0^{\theta}) = 1\), implies that \(\mathbb{P}'\) is a probability measure.

We now prove that \(\theta'\) has values in \(C_0^{\theta}\) for \(t \in \mathbb{R}\). Suppose that \(\omega \in C_0^{\theta}\). Then for any \(n \in \mathbb{N}\) there exists a sequence \((\omega_m^t)_{m \in \mathbb{N}}\) converging to \(\omega\) in \(C^{\gamma}([-n, n], V)\) where \(\omega_m^t \in C^{\infty}([-n, n], V)\), \(\omega_m^t(0) = 0\). For some \(t \in \mathbb{R}\) we consider the sequence \((\omega_m^{[t]+1+n})\), which converges to \(\omega\) on \(C^{\gamma}([-|t| - 1 - n, |t| + 1 + n], V)\), where for \(t \geq 0\) the value \(|t|\) is the largest integer less or equal than \(t\) and for \(t < 0\) the value \(|t|\) is the smallest integer larger or equal than \(t\). Then \((\theta_t \omega_m^{[t]+1+n}(-n,n))_{m \in \mathbb{N}}\) converges to \(\theta_t \omega\).
in $C^γ([-n, n], V)$, so that $θ_t ω ∈ C_0^0, γ$, and hence $θ'_t C_0^0, γ ⊂ C_0^0, γ$, for $t ∈ R$. Since $θ_t$ is a bijection with inverse $θ_{-t}$, we also obtain that

\begin{equation}
θ_t C_0^0, γ = θ'_t C_0^0, γ = C_0^0, γ \quad \text{for all } t ∈ R.
\end{equation}

The flow property of $θ'$ follows easily form the flow property of $θ$.

We prove now that $θ'$ is $B(\mathbb{R}) ⊗ B(C_0^0, γ)$ measurable. Note that for $A ∈ B(C_0^0, γ)$, as a consequence of (16),

\begin{equation}
(θ')^{-1}(A) = (θ')^{-1}(B ∧ C_0^0, γ) = θ^{-1}(B) ∧ θ^{-1}(C_0^0, γ) ∈ (B(\mathbb{R}) ⊗ B(C_0)) ∩ (R × C_0^0, γ).
\end{equation}

Indeed the second equality follows by

\[
θ^{-1}(B ∩ C_0^0, γ) = \{(t, ω) ∈ R × C_0 : θ_t ω ∈ B ∩ C_0^0, γ\}.
\]

but when $θ_t ω ∈ C_0^0, γ$ so $ω ∈ C_0^0, γ$. Applying Vishik and Fursikov [29] Theorem II.2.1 to (17), we obtain that

\[
(θ')^{-1}(A) ∈ B(\mathbb{R} × C_0^0, γ) = B(\mathbb{R}) ⊗ B(C_0^0, γ).
\]

Note that the last equality above follows by the separability of $\mathbb{R}$ and $C_0^0, γ$, see [2], Chapter 35, Remark 1.

Next we prove the invariance of the measure $Π'$, which is derived from the invariance of the measure $Π$ and the following chain of equalities: for all $A ∈ B(C_0^0, γ)$,

\[
Π'(θ^{-1}_t A) = Π'(θ^{-1}_t (B ∩ C_0^0, γ)) = Π'(θ^{-1}_t B ∩ C_0^0, γ) = Π(θ^{-1}_t B) = Π(B) = Π'(A).
\]

Finally, we prove the ergodicity of $Π'$. Let $A ∈ B(C_0^0, γ)$ be a $θ'$ invariant set. Then for any $B ∈ B(C_0)$ such that $A = B ∩ C_0^0, γ$, since $Π(C_0^0, γ) = 1$,

\[
Π((θ^{-1}_t B) Δ B) = Π'(C_0^0, γ ∩ ((θ^{-1}_t B) Δ B)) = Π'((θ^{-1}_t B ∩ C_0^0, γ)) Δ (B ∩ C_0^0, γ)
\]

\[
= Π'((θ^{-1}_t B ∩ C_0^0, γ)) Δ (B ∩ C_0^0, γ) = Π'((θ^{-1}_t A) Δ A) = Π(0) = 0.
\]

By Definition 18 we conclude that $B$ is $θ$ invariant mod $Π$, hence the ergodicity of $Π$ implies that either $Π(B) = 0$ or 1, see Lemma 19. As a result, taking once more into account that $Π(C_0^0, γ) = 1$ we conclude that either $Π'(A) = 0$ or 1, and therefore $Π'$ is ergodic.

5. MULTIVALUED NON-AUTONOMOUS AND RANDOM DYNAMICAL SYSTEMS FOR \( f \)

In this section we deal with the multivalued dynamical system related to problem (11). First of all, we emphasize that Lemma 11 implies in particular that every mild pathwise solution can be extended to a globally defined one, that is, it exists for any $t ≥ 0$.

Denote by $F(u_0, ω)$ the set of all globally defined mild solutions with initial value $u_0 ∈ V$ for $ω ∈ C_0^{0, γ'}$ and consider the ergodic metric dynamical system $(C_0^{0, γ'}, B(C_0^{0, γ'}), Π')$ introduced in Theorem 20 for $γ = γ'$.

Denoting by $P(V)$ the set of all non-empty subsets of $V$, we define the (possibly) multivalued operator $Φ : R^+ × Ω × V → P(V)$ by

\[
Φ(t, ω, u_0) = \{u(t) : u ∈ F(u_0, ω)\}.
\]

**Lemma 21.** $Φ(t + s, ω, u_0) = Φ(t, θ_s ω, Φ(s, ω, u_0))$, for all $t, s ≥ 0$, $ω ∈ Ω$, $u_0 ∈ V$. That is, $Φ$ is a strict MRDS.

The proof follows easily from Lemmas 10 and 16.

Next we would like to prove that the norm of any solution $u ∈ F(u_0, ω)$ is uniformly bounded in any interval $[0, T]$. 
Lemma 22. Let us consider the balls $B_V(0, R)$ and $B_{C^0_{0, \alpha'}}(0, \hat{R})$. There exists $C(R, \hat{R}, T)$ such that for any $u_0 \in B_V(0, R)$, $\omega \in B_{C^0_{0, \alpha'}}(0, \hat{R})$ and $u \in F(u_0, \omega)$ one has

$$
\|u\|_{\beta, \beta_0, T} \leq C(R, \hat{R}, T).
$$

Proof. For $u \in F(u_0, \omega)$, according to (8) we have that

$$
\|u\|_{\beta, \beta_0} \leq c_S \|u_0\| + c_T \|\omega\|_\beta K(\rho)(1 + \|u\|_{\beta, \beta_0}),
$$

where $\lim_{\rho \to \infty} K(\rho) = 0$ and $c_T$ is a constant that also depends on the constants related to $F$, $G$ and $S$. Therefore, we can choose $\rho$ large enough such that

$$
c_T \|\omega\|_\beta K(\rho) \leq c_T \hat{R}K(\rho) < \frac{1}{2},
$$

and thus, for all $\omega \in B_{C^0_{0, \alpha'}}(0, \hat{R})$, we have

$$
K(\rho) < \frac{1}{2c_T \hat{R}}.
$$

Plugging this information in the estimate (18) we have

$$
\|u\|_{\beta, \beta_0, T} \leq (c_S \|u_0\| + c_T \hat{R}K(\rho)) (1 - c_T \hat{R}K(\rho))
$$

and therefore

$$
\|u\|_{\beta, \beta_0, T} \leq 2c_S R + 1.
$$

Since $\rho = \rho(T, \hat{R})$ and $\|u\|_{\beta, \beta_0, T}$ is equivalent to $\|u\|_{\beta, \beta_0, T}$, we obtain the result. \qed

Theorem 23. Let $u^n_0 \to u_0$ in $V$ and $\omega \in C^0_{0, \alpha'}$ be fixed. Then every sequence $(u^n)_n \in F(u^n_0, \omega)$ possesses a subsequence $u^{nk}$ such that

$$
u^{nk}|_{[0, T]} \to u|_{[0, T]} \quad \text{in} \quad C^0_\beta([0, T], V),
$$

for all $T > 0$, where $u \in F(u_0, \omega)$.

Proof. First we fix $T > 0$. It follows from Lemma 22 that $u^n$ is bounded in $C^0_\beta([0, T], V)$, hence by Theorem 7 the sequence of mappings

$$
[0, T] \ni t \mapsto \int_0^t S(t - r)F(u^n(r))dr + \int_0^t S(t - r)G(u^n(r))d\omega
$$

is relatively compact in $C^0_\beta([0, T], V)$. On the other hand, the inequality

$$
\|(S(t) - S(s))(u^n_0 - u_0)\| \leq c_S s^{-\beta} (t - s)^\beta \|u^n_0 - u_0\|, \quad 0 < s < t \leq T,
$$

implies that

$$
S(\cdot)u^n_0 \to S(\cdot)u_0 \quad \text{in} \quad C^0_\beta([0, T], V),
$$

and therefore we have that $u^n(\cdot)$ is relatively compact in $C^0_\beta([0, T], V)$. We conclude that up to a subsequence

$$
u^n \to u \quad \text{in} \quad C^0_\beta([0, T], V).
$$

By a diagonal argument the result is true for an arbitrary $T > 0$. It remains to check that $u \in F(u_0, \omega)$. In view of

$$
\|S(t)u^n_0 - S(t)u_0\| \to 0,
$$

it suffices to verify that for any $t \in [0, T]$

$$
\int_0^t S(t - r)F(u^n(r))dr + \int_0^t S(t - r)G(u^n(r))d\omega \to \int_0^t S(t - r)F(u(r))dr + \int_0^t S(t - r)G(u(r))d\omega,
$$

which has a similar proof as the continuity of $T$ in Theorem 8. \qed

Corollary 24. The map $\Phi$ has compact values, and thus $\Phi$ is a strict MNDS.
As a consequence of the previous results, we can also establish the following property, which will be crucial when looking at the existence of attractors for (1).

**Corollary 25.** The map \( u_0 \mapsto \Phi(t, \omega, u_0) \) is upper semicontinuous.

**Proof.** If this is not true, for some \( t, \omega, u_0 \) there exists a neighborhood \( U \) of \( \Phi(t, \omega, u_0) \) in \( V \) and sequences \( y^n \in \Phi(t, \omega, u^n_0), u^n_0 \to u_0, \) such that \( y^n \notin U \). But \( y^n = u^n(t) \) with \( u^n \in \mathcal{F}(u^n_0, \omega), \) so by Theorem 7 we have that up to a subsequence \( y_n \to y \in \Phi(t, \omega, u_0), \) which is a contradiction with \( y_n \notin U. \)

We are now ready to prove upper semicontinuity with respect to all variables.

**Theorem 26.** Let \( u^n_0 \to u_0 \) in \( V \) and \( \omega^n \to \omega. \) Then every sequence \( u^n \in \mathcal{F}(u^n_0, \omega^n) \) possesses a subsequence \( u^{n_k} \) such that

\[
u^{n_k} \to u \in \mathcal{F}(u_0, \omega) \text{ in } C^0([0, T], V),\]

where \( T > 0 \) is arbitrary.

**Proof.** In view of Lemma 22 the sequence \( (u^n)_{n \in \mathbb{N}} \) is bounded in \( C^0([0, T], V) \) for any \( T > 0. \) Hence, Theorem 7 implies the existence of \( u \) such that up to a subsequence

\[
u^n \to u \in C^0([0, T], V) \text{ for any } T > 0.\]

It remains to prove that \( u \in \mathcal{F}(u_0, \omega). \) This is equivalent to checking that \( u = \mathcal{T}(u, \omega, u_0), \) which will follow from

\[
\mathcal{T}(u^n, \omega^n, u^n_0)(t) \to \mathcal{T}(u, \omega, u_0)(t) \text{ in } V \text{ for any } t \in [0, T].
\]

As it is clear that \( S(t)u^n_0 \to S(t)u_0 \) in \( V, \) we only need to consider the integral terms. For the deterministic integral we can follow the same steps as in Theorem 8. For the stochastic integral, we split the difference in the following way:

\[
\left\| \int_0^t S(t-r)G(u^n(r))d\omega^n - \int_0^t S(t-r)G(u(r))d\omega \right\|
\leq \left\| \int_0^t S(t-r)(G(u^n(r)) - G(u(r)))d\omega^n \right\|
+ \left\| \int_0^t S(t-r)G(u(r))d(\omega^n - \omega) \right\| =: I_1 + I_2.
\]

Arguing as in the proof of Theorem 8 we obtain that \( I_1 \to 0. \) For \( I_2 \) we deduce that

\[
\left\| \int_0^t S(t-r)G(u(r))d(\omega^n - \omega) \right\|
\leq c \left\| \omega^n - \omega \right\|_{\beta', 0, T}
\int_0^t \left( \frac{\|S(t-r)\|_{L(V)} \|G(u(r))\|_{L^2(V)}}{r^\alpha} \right)
\frac{\|S(t-r) - S(t-q)\|_{L(V)} \|G(u(r))\|_{L^2(V)}}{(r-q)^{1+\alpha}}dq
+ \int_0^t \frac{\|S(t-q)\|_{L(V)} \|G(u(r)) - G(u(q))\|_{L^2(V)}}{(r-q)^{1+\alpha}}dq (t-r)^{\alpha+\beta'-1}dr =: A_1 + A_2 + A_3.
\]

The first term is estimated by

\[
A_1 \leq c_{S,G} \left\| \omega^n - \omega \right\|_{\beta', 0, T}(1 + \|u\|_{\infty, 0, T}) \int_0^t r^{-\alpha}(t-r)^{\beta'+\alpha-1}dr,
\]

so \( A_1 \to 0. \)
We consider $f$ where

$$
L \text{ where}
$$

Now we introduce the diffusion term. In order to do that, let

$$
g \text{ drift}
$$

$F$

$S$

operator

$A$

assume that

Theorem 28.

We finally can establish the main result of this section.

$$
t, \omega, x \in 27 \text{ it follows that the map } (\cdot) \Phi(\cdot, \omega, x) \text{ is a strict MNDS. On the other hand, from Corollary 24 we already know that } \Phi \text{ is a strict MNDS. On the other hand, from Corollary 27 it follows that the map } (t, \omega, x) \rightarrow \Phi(t, \omega, x) \text{ is upper semicontinuous in the multivalued sense, and, since } C^{0,\beta}_0 \text{ is separable, this property implies the measurability of } \Phi, \text{ see Lemma 15.} \text{ In other words, } \Phi \text{ is a strict MRDS.}
$$

Theorem 28. The mapping $(t, \omega, x) \rightarrow \Phi(t, \omega, x)$ is $B(\mathbb{R}^+) \otimes F \otimes B(V)$ measurable. Hence, $\Phi$ is a MRDS, where $F = B(C^{0,\beta}_0)$.

Proof. From Corollary 24, we already know that $\Phi$ is a strict MNDS. On the other hand, from Corollary 27 it follows that the map $(t, \omega, x) \rightarrow \Phi(t, \omega, x)$ is upper semicontinuous in the multivalued sense, and, since $C^{0,\beta}_0$ is separable, this property implies the measurability of $\Phi$, see Lemma 15. In other words, $\Phi$ is a strict MRDS.

Finally, we give an example of a parabolic partial differential equation whose set of solutions generates a multivalued random dynamical system.

Example 29. Let $D \subset \mathbb{R}^d$ be a bounded domain with regular enough boundary. Consider the space $V = L^2(D)$ with usual norm denoted by $\|\cdot\|_V$ and a complete orthonormal base given by $(e_i)_{i \in \mathbb{N}}$. Assume that $A$ is given by the Laplacian on $D$ with homogeneous Dirichlet boundary condition. The operator $-A$ with domain $D(-A) = H^2(D) \cap H^1_0(D)$ is a strictly positive and symmetric operator with a compact inverse, generating an analytic semigroup $S$ in $V$.

We consider $f : \mathbb{R} \rightarrow \mathbb{R}$ to be a continuous mapping with at most linear growth. We define the nonlinear drift $F : V \rightarrow V$ as the corresponding Nemytskii operator given by

$$
F(u)(x) = f(u(x)), \text{ for } u \in V, x \in D.
$$

Now we introduce the diffusion term. In order to do that, let $g : D \times D \times \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz continuous function in the following sense:

$$
|g(x, y, z_1) - g(x, y, z_2)| \leq L(x)|z_1 - z_2|, \quad x, y \in D, \quad z_1, z_2 \in \mathbb{R},
$$

where $L \in V$. Now we define

$$
G(u)(v)[x] = \int_D g(x, y, u(y)v(y)dy, \text{ for } u, v \in V.
$$
We can see that $G$ is well defined as a mapping $G : V \mapsto L_2(V)$, that is, with values in the Hilbert-Schmidt operators from $V$ into $V$. In fact, for $u \in V$,

$$
\|G(u)\|^2_{L_2(V)} = \sum_j \|G(u)(e_j)\|^2_V = \sum_j \int_D |G(u)(e_j)(x)|^2 \, dx = \sum_j \int_D \left( \int_D g(x,y,u(y)) e_j(y) \, dy \right)^2 \, dx
$$

$$
= \int_D \sum_j \left( \int_D g(x,y,u(y)) e_j(y) \, dy \right)^2 \, dx \leq \int_D \|g(x,\cdot,u(\cdot))\|^2_V \, dx < \infty,
$$

where above we have applied Parseval’s inequality. Furthermore, $G$ is Lipschitz continuous, since for $u_1, u_2 \in V$, in a similar way as before we obtain

$$
\|G(u_1) - G(u_2)\|^2_{L_2(V)} = \sum_j \int_D \left( \int_D (g(x,y,u_1(y)) - g(x,y,u_2(y))) e_j(y) \, dy \right)^2 \, dx
$$

$$
\leq \int_D \|g(x,\cdot,u_1(\cdot)) - g(x,\cdot,u_2(\cdot))\|^2_V \, dx
$$

$$
\leq \left( \int_D L^2(x) \, dx \right) \|u_1 - u_2\|^2_V = \|L\|^2_V \|u_1 - u_2\|^2_V.
$$

Acknowledgement

M.J. Garrido-Atienza was partially supported by FEDER and Spanish Ministerio de Economía y Competitividad, project MTM2015-63723-P and by Junta de Andalucía under Proyecto de Excelencia. J. Valero was partially supported by FEDER and Spanish Ministerio de Economía y Competitividad, projects MTM2015-63723-P and MTM2016-74921-P.

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