Combinations without specified separations and restricted-overlap tiling with combs

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Abstract

We consider a type of restricted-overlap tiling of an \(n\)-board (a linear array of \(n\) square cells of unit width) with squares (1 \(\times\) 1 tiles) and combs. A \((w_1, g_1, w_2, g_2, \ldots, g_{r-1}, w_r)\)-comb is composed of \(r\) sub-tiles known as teeth. The \(i\)-th tooth in the comb has width \(w_i\) and is separated from the \((i+1)\)-th tooth by a gap of width \(g_i\). Here we only consider combs with \(w_1, g_i \in \mathbb{Z}^+\). When performing a restricted-overlap tiling of a board with such combs and squares, the leftmost cell of a tile must be placed in an empty cell whereas the remaining cells in the tile are permitted to overlap other non-leftmost filled cells of tiles already on the board. We give a bijection between such tilings of an \((n+q)\)-board and the restricted subsets of \(N_n = \{1, 2, \ldots, n\}\) with \(q \geq 1\) being the largest member of the set \(Q\) of disallowed differences between subset elements. This enables new results on various classes of problem involving combinations lacking specified separations to be obtained in a quick and intuitive manner. In particular, we can obtain recursion relations for the number of all \(i\) \(\geq Q\) elements in the subset and \(F_Q\) of disallowed differences between subset elements. This enables new results on various classes of problem involving combinations lacking specified separations to be obtained in a quick and intuitive manner. In particular, we can obtain recursion relations for the number of \(i\)-subsets for any \(Q\) when \(N_q - Q\) \(\leq 2\). In addition, our tiling bijection facilitates the proof of a bijection between the case \(Q = \{m, 2m, \ldots, jm\}\) with \(j, m \geq 1\) and permutations \(\pi\) of \(N_{n+jm}\) satisfying \(\pi(i) - i \in \{-m, 0, jm\}\) for all \(i \in N_{n+jm}\). We also identify a bijection between another class of strongly restricted permutation and the cases \(Q = \{1, q\}\). Connections between restricted combinations and bit strings and graphs are also noted.

Keywords: combinatorial proof, tiling, combination, directed pseudograph, strongly restricted permutation, well-based sequence

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1. Introduction

The problem of enumerating combinations with disallowed separations is as follows. We wish to find the number \(S_n\) of subsets of \(N_n = \{1, 2, \ldots, n\}\) satisfying the condition \(|x - y| \notin Q\) where \(x, y\) is any pair of elements in the subset and \(Q\) is a given nonempty subset of \(\mathbb{Z}^+\). We also wish to find the number \(S_{n,k}\) of such subsets of \(N_n\) that are of size \(k\). For \(Q = \{1\}\) it is well known that \(S_n = F_{n+2}\) where \(F_j\) is the \(j\)-th Fibonacci number defined by \(F_j = F_{j-1} + F_{j-2}\) with \(F_0 = 0\) and \(F_1 = 1\). The quantity \(\binom{n+1-k}{k}\) is also the number of ways of tiling an \((n+1)\)-board (i.e., an \((n+1)\times1\) board of unit square cells) using unit squares and \(k\) dominoes (2 \(\times\) 1 tiles) [2]. This correspondence can be regarded as a result of the bijection given in [3], generalized in [4], and that we will extend to the most general case here. It also appears to be well known that if \(Q = \{1, \ldots, q\}\) then \(S_n = S_{n-1} + S_{n-q-1} + \delta_{n+q,0}\) with \(S_{n+q,0} = 0\). Expressions for the number of combinations when \(Q = \{2\}\), \(Q = \{q\}\) for \(q \geq 2\), and \(Q = \{m, 2m, \ldots, jm\}\) for \(j, m \geq 1\) are derived in [5], [6, 7, 8], and [9, 4], respectively.

\(S_{n,k}\) also has links with graph theory. A path scheme \(P(n, Q)\) is an undirected graph with vertex set \(V = N_n\) and edge set \(\{(x, y) : |x - y| \in Q\}\) [10]. A subset \(S\) of \(V\) is said to be an independent set (or a stable set) if no two elements of \(S\) are adjacent. The number of independent sets of path scheme \(P(n, Q)\) of
size $k$ is then clearly $S_{n,k}$ (and the total number is $S_n$). The elements $q_i$ for $i = 1, \ldots, |Q|$ of set $Q$ are said to form a well-based sequence if, when ordered so that $q_j > q_i$ for all $j > i$, then $q_1 = 1$ and for all $j > 1$ and $\Delta = 1, \ldots, q_j - 1$, there is some $r_i$ such that $q_j = r_i + \Delta$ [11, 10]. Equivalently, the sequence of elements of $Q$, the largest of which is $q_j$, is well-based if $a = |N_q - Q|$ is zero or if for all $i, j = 1, \ldots, a$ (where $i$ and $j$ can be equal), $p_i + p_j \not\in Q$ where the $p_i$ are the elements of $N_q - Q$. E.g., the only well-based sequences of length 3 are the elements of the sets $\{1, 2, 3\}, \{1, 2, 4\}, \{2, 5\}, \{1, 2, 5\}$. By considering $P(n, Q)$, Kitaev obtained an expression for the generating function of $S_n$ when the elements of $Q$ are a well-based sequence [10]. We will show the result via combinatorial proof and also obtain a recursion relation for $S_{n,k}$.

We define a $(w_1, g_1, w_2, g_2, \ldots, w_{t-1}, g_t)$-comb as a linear array of $t$ sub-tiles (which we refer to as teeth) of dimensions $w_i \times 1$ separated by $t - 1$ gaps of width $g_i$. The length of a comb is $\sum_{i=1}^{t-1}(w_i + g_i) + w_t$. When the $t$ teeth are all of width $w$ and the $t - 1$ gaps are all of width $g$, we refer to the object as a $(w, g; t)$-comb; such combs can be used to give a combinatorial interpretation of products of integer powers of two consecutive generalized Fibonacci numbers [12]. Evidently, a $(w, g; 1)$-comb (or $w$-comb) is just a $w$-omino (and a $(w, 0; n)$-comb is an $n w$-omino). A $(w, g; 2)$-comb is also known as a $(w, g)$-fence. The fence was introduced in [13] to obtain a combinatorial interpretation of the tribonacci numbers as the number of tilings of an $n$-board using just two types of tiles, namely, squares and $(\frac{1}{2}, 1)$-fences. $(\frac{1}{2}, g)$-fences have also been used to obtain results on strongly restricted permutations [14].

In this paper, we start in §2 by giving the bijection between combinations with disallowed separations and the restricted-overlap tilings of boards with squares and combs. Counting these types of tilings requires knowledge of all permissible minimal gapless configurations of square-filled and/or restricted overlapping combs known as metatiles which are introduced in §3. In §4 we derive results from which recursion relations for $S_{n,k}$ (or $S_n$) for various classes of the set of disallowed differences $Q$ can be obtained. Bijections between two classes of strongly restricted permutations and combinations with $Q = \{m, 2m, \ldots, jm\}$ and $Q = \{1, q\}$ are given in §5. Connections between some instances of $Q$ and bit strings are pointed out in §6. The appendix contains a listing of code for efficiently calculating $S_n$ and $S_{n,k}$ for any given $Q$ with largest element, $q \leq 32$.

2. The bijection between combinations with disallowed separations and restricted-overlap tilings with squares and combs

In order to formulate the bijection we first introduce the concept of restricted-overlap tiling of an $n$-board. In this type of tiling, any cell but the leftmost cell of a tile is permitted to overlap any non-leftmost cell of another tile. The $C^2$, $S^2$, and $C^3 S$ metatiles in Fig. 1 are examples where such overlap occurs. It is readily seen that when tiling an $n$-board, only tiles with gaps can overlap in this sense and so restricted-overlap tiling of $n$-boards with just squares and other $w$-ominoes is the same as ordinary tiling.

Let $g$ be the largest element in the set $Q$ of disallowed differences. The comb corresponding to $Q$ is of length $q + 1$, has cells numbered from 0 to $q$, and is constructed as follows. By definition, cells 0 and $q$ are filled (as they are the end teeth or parts of them). For the cells in between, cell $i$ (for $i = 1, \ldots, q - 1$) is filled if and only if $i \in Q$. For example, $Q = \{1, 2, 4\}$ corresponds to a $(3, 1, 1)$-comb. One could also regard it as a $(1, 0, 2, 1, 1)$-comb but for simplicity we insist that all teeth and gaps in a comb corresponding to $Q$ are of positive width. This ensures that there is only one comb that corresponds to a given $Q$.

**Theorem 2.1.** There is a bijection between the $k$-subsets of $N_n$ each pair $x, y$ of elements of which satisfy $|x - y| \not\in Q$ and the restricted overlap-tilings of an $(n + q)$-board using squares and $k$ combs corresponding to $Q$ as described above.

**Proof:** The $(n + q)$-board associated with a subset $S$ satisfying the conditions regarding disallowed differences has cells numbered from 1 to $n + q$. It is obtained by placing a comb at cell $i$ if and only if $i \in S$ and then, after all the comb tiles have been placed, filling any remaining empty cells with squares. $S = \varnothing$ therefore corresponds to a board tiled with squares only. The $n$ singletons correspond to each of the possible places to put a tile of length $q + 1$ on an $(n + q)$-board. If $S$ contains more than one element, then there will be a tiling corresponding to $S$ iff, for any two elements $x < y$ in $S$, the comb representing $x$ (which has its cell
and combs when $0 \leq l$, of another comb in the gap. For instance, in the case of an \((S_1, S_2, S_3)\)-comb, the bijection also applies in the \(Q\) case if we set $q = 0$. The comb corresponding to \(Q\) is then a square (but we still call it a comb to distinguish it from the ordinary squares) so $S_{n,k}$ is the number of tilings of an \(n\)-board using \(k\) square combs (and \(n - k\) ordinary squares) which is $\binom{n}{k}$.

Let $B_n$ be the number of ways to restricted-overlap tile an \(n\)-board with squares and combs corresponding to \(Q\), and let $B_{n,k}$ be the number of such tilings that use \(k\) combs. We choose to make $B_0 = B_{0,0} = 1$.

**Theorem 2.2.** $S_n = B_{n+q}$ and $S_{n,k} = B_{n+q,k}$.

**Proof:** This follows immediately from Theorem 2.1.

**Lemma 2.3.** If the number of tilings of an \(n\)-board with squares and length-\((q + 1)\) combs is given by

$$B_n = \delta_{n,0} + \sum_{m>0} (\alpha_m \delta_{n,m} + \beta_m B_{n-m}), \quad B_{n<0} = 0,$$

then the generating function $G(x)$ for $S_n$ can be written as

$$G(x) = \frac{1 + \sum_{m>0} (\alpha_{m+q} + \sum_{j=1}^q \beta_{m+j}) x^m}{1 - \sum_{m>0} \beta_m x^m}.$$  \hspace{1cm} (1)

**Proof:** The generating function for $B_n$ is $(1 + \sum_{m>0} \alpha_n x^m)/(1 - \sum_{m>0} \beta_m x^m)$. The first $q + 1$ terms of the expansion of this must be $1 + x + \cdots + x^q$ since there is only one way to tile an \(n\)-board with squares and combs when $0 \leq n \leq q$, namely, the all-square tiling. From Theorem 2.2, $S_n = B_{q+n}$, and so

$$G(x) = \frac{1}{x^q} \left( \frac{1 + \sum_{m>0} \alpha_m x^m}{1 - \sum_{m>0} \beta_m x^m} - 1 - x - \cdots - x^{q-1} \right),$$

which simplifies to (1) after first putting the terms inside the parentheses over a common denominator and then in the numerator discarding the $x^r$ terms for $1 \leq r < q$ since they must sum to zero.

It can be seen that restricted-overlap tiling with squares and just one type of \((1, g; t)\)-comb, where $g = 0, 1, 2, \ldots$, will result in no overlap of the combs and so the number of such tilings is the same as for ordinary tiling. For other types of comb, there will be some tilings where overlap occurs. The case $Q = \{m, 2m, \ldots, jm\}$ with $j, m \geq 1$ as studied in [9, 4], which is a generalization of all the other cases for which results for $S_{n,k}$ were obtained previously, corresponds to tiling with squares and \((1, m-1; j+1)\)-combs. Thus any results for $S_{n,k}$ we obtain for nonempty Q not of this form (and so overlap does occur) will be new ones.

3. Metatile

We extend the definition of a metatile given in [13, 14] to the case of restricted-overlap tiling. A **metatile** is a minimal arrangement of tiles with restricted overlap that exactly covers a number of adjacent cells without leaving any gaps. It is minimal in the sense that if one or more tiles are removed from a metatile then the result is no longer a metatile.

When tiling with squares (denoted by $S$) and combs corresponding to $Q$ (denoted by $C$), the two simplest metatiles are the free square (i.e., a square which is not inside a comb), and a comb with all the gaps filled with squares. The symbolic representation of the latter metatile is $CS^g$ where $g = \sum_i g_i$. If a comb has no gaps then it is a \((q + 1)\)-omino and is therefore a metatile by itself.

If the comb contains a gap, we can initiate the creation of at least one more metatile by placing the start of another comb in the gap. For instance, in the case of an \((l, 1, r)\)-comb where $r \geq l$, this is the only other possible metatile and so there are three metatiles in total (Fig. 1a). However, if $r < l$, adding a comb leaves a gap which can be filled either with a square to form a completed metatile ($C^2S$) or with another comb which will leave a further gap, and so on (Fig. 1b). Thus the possible metatiles in this case are $C^mS$ for $m = 0, 1, 2, \ldots$. More generally, we have the following lemma.
Lemma 3.1. Let \( r \) be the length of the final tooth in the comb which has at least one gap. The set of possible metatiles when restricted-overlap tiling a board of arbitrary length with squares and combs is finite if and only if \( 2r \geq q \).

Proof: We can reuse the depiction of tilings in Fig. 1 but with the left tooth of each comb now replaced by arbitrary teeth and gaps (but starting with a tooth). There is no possibility of a ‘chain’ of combs if, when a second comb is placed with the first cell of its first tooth just before the start of the right tooth of the first comb, the start of the right tooth of the second comb is before or aligned with the end of the right tooth of the first comb. This occurs if \( q - r \leq r \). If \( q - r > r \), a third comb can be placed so that its first cell is immediately to the left of the right tooth of the second comb and this can be continued indefinitely.

As with fence tiling [14], we can systematically construct all possible metatiles with the aid of a directed pseudograph (henceforth referred to as a digraph). As before, the 0 node corresponds to the empty board or the completion of the metatile. The other nodes represent the state of the incomplete metatile. The occupancy of a cell in it is represented by a binary digit: 0 for empty, 1 for filled. We label the node by discarding the leading 1s and trailing zeros and so the label always starts with a 0 and ends with a 1. Each arc represents the addition of a tile and any walk beginning and ending at the 0 node without visiting it in between corresponds to a metatile. With our restricted-overlap tiling, all nodes have an out-degree of 2 as a gap may always be filled by a square or the start of a comb. The destination node is obtained by performing a bitwise OR operation on the bits representing the added tile and the label of the current node, and then discarding the leading 1s. Fig. 1 illustrates this for the metatiles involved when tiling with squares and \((l, 1, r)\)-combs.

The most important property of a metatile is its length. This is obtained by summing the contribution to the length associated with each arc in the walk representing the metatile. The contribution to the length associated with an arc is zero if it corresponds to the addition of a square (except in the case of the trivial S metatile) and for a comb arc equals \( q + 1 - d \) where \( d \) is the number of digits in the node label from which the arc emanates except when that node is the 0 node in which case \( d = 0 \). For example, from the digraph in Fig. 1b, for tiling with squares and \((l, 1, r < l)\)-combs we see that the length of a \( C^m S \) metatile where \( m > 0 \) is \( q + 1 + (m - 1)(q - r) = ml + r + 1 \) as in this case \( q = l + r \).

4. Counting restricted-overlap tilings

For brevity, we just give results for \( B_n \) and \( B_{n,k} \) as these are easily converted to recursion relations for \( S_n \) and \( S_{n,k} \) and the generating function for \( S_n \) using Theorem 2.2 and Lemma 2.3. As with ordinary
(non-overlapping) tiling, the following lemma is the basis for obtaining recursion relations for \( B_n \) and \( B_{n,k} \).

**Lemma 4.1.** For all integers \( n \) and \( k \),

\[
B_n = \delta_{n,0} + \sum_{i=1}^{N_m} B_{n-i,i}, \quad (2a)
\]

\[
B_{n,k} = \delta_{n,0}\delta_{k,0} + \sum_{i=1}^{N_m} B_{n-i,k-k_i}, \quad (2b)
\]

where \( N_m \) is the number of metatiles, \( l_i \) is the length of the \( i \)-th metatile and \( k_i \) is the number of combs it contains, and \( B_{n<0} = B_{n,k<0} = B_{n<k,k} = 0 \).

**Proof:** As in [2, 14], we condition on the last metatile on the board. To obtain (2b) we note that if an \( n \)-board tiled with squares and \( k \) combs ends with a metatile of length \( l_i \) that contains \( k_i \) combs then there are \( B_{n-l_i-k_i} \) ways to tile the rest of the board. The \( \delta_{n,0}\delta_{k,0} \) term is from the requirement that \( B_{0,0} = 1 \). This term arises in the sum when a particular metatile containing \( k \) combs completely fills the board; there is only one tiling where this occurs. The derivation of (2a) is analogous but we ignore the number of combs. Alternatively, each term in (2a) is obtained from the corresponding term in (2b) by summing over all \( k \). \( \square \)

**Theorem 4.2.** If \( Q = \{1, \ldots, l-1, q-r+1, \ldots, q-1, q\} \) (or \( Q = \{q-r+1, \ldots, q-1, q\} \) if \( l = 1 \)) where \( l \geq 1, 2r \geq q, \) and \( q + 1 \geq l + r \), then

\[
B_n = \delta_{n,0} + B_{n-1} + B_{n-q-1} + \sum_{j=0}^{q-l-r} f_j^{(l)} B_{n-l-q-1-j},
\]

\[
B_{n,k} = \delta_{n,0}\delta_{k,0} + B_{n-1,k} + B_{n-q-1,k-k_i} + \sum_{j=0}^{q-l-r} \sum_{i=0}^{l/l} \binom{j-(l-i)k}{i} B_{n-l-q-1-j,k-2-i},
\]

where the \((1,l)\)-bonacci number \( f_j^{(l)} = f_{j+1}^{(l)} + f_{j-l}^{(l)} + \delta_{j,0}, f_j^{(l)} < 0 \). The sums are omitted if \( q + 1 = l + r \).

**Proof:** We use Lemma 4.1. If \( q + 1 = l + r \), \( C \) is just a \((q+1)\)-omino and the results follow immediately. Otherwise, \( C \) is an \((l, q+1 - l-r, r)\)-comb. There are two trivial metatiles: \( S \) and \( CS^{q+1-l-r} \) which have lengths of \( 1 \) and \( q+1 \), respectively. For the remaining metatiles, since \( 2r \geq q \), as described in the proof of Lemma 3.1, the final comb in the metatile must start within the gap of the first comb. Number the cells in this gap from \( j = 0 \) to \( q - l - r \). If the left tooth of the final comb in the metatile lies in cell \( j \) of this gap, the length of the metatile is \( l + q + 1 + j \). Cells \( 0 \) to \( j \) of the gap can have either an \( S \) or the left tooth of a \( C \) and we end up with a metatile of this length. The number of ways this can be done is simply the number of ways to tile a \( j \)-board using squares and \( l \)-ominoes which is \( f_j^{(l)} [2] \). Hence there are \( f_j^{(l)} \) metatiles of length \( l + q + 1 + j \) of which \( (j-l)\) have \( 2 + i \) combs. \( \square \)

For instances where the largest element of \( N_q - Q \) (the set of allowed differences less than \( q \)) is \( q-r \) and \( 2r \geq q \) but the other conditions in Theorem 4.2 do not hold, as the number of possible metatiles is finite (by Lemma 3.1), it is straightforward to find the length and number of combs in each of them and then use (2) to obtain recursion relations for \( B_n \) and \( B_{n,k} \).

To enable us to tackle some cases where there are an infinite number of possible metatiles, we begin by reviewing some terminology describing features of the digraphs used to construct metatiles [14]. A cycle is a closed walk in which no node or arc is repeated aside from the starting node. We refer to cycles by the arcs they contain. For example, the digraph in Fig. 1(a) has 3 cycles: \( S_1, C_{[q+1]} S \), and \( C_{[q+1]} C_{[q-r]} \). An inner cycle is a cycle that does not include the 0 node. For example, the digraph in Fig. 1(b) has a single inner cycle, namely, \( C_{[q-r]} \). If a digraph has an inner cycle, there are infinitely many possible metatiles as, once reached, the cycle can be traversed an arbitrary number of times before the walk returns to the 0 node. If all of the inner cycles of a digraph have one node (or more than one node) in common, that node
(or any one of those nodes) is said to be the common node. In the case of a digraph with one inner cycle, any of the nodes of the inner cycle can be chosen as the common node. A common circuit is a simple path from the 0 node to the common node followed by a simple path from the common node back to the 0 node. For example, in the digraph in Fig. 1(b), the common node is 0’ and the common circuit is C_{q+1}[S]. If a digraph has a common node, members of an infinite family of metatiles can be obtained by traversing the first part of the common circuit from the 0 node to the common node and then traversing the inner cycle(s) an arbitrary number of times (and in any order if there are more than one) before returning to the 0 node via the second part of the common circuit. An outer cycle is a cycle that includes the 0 node but does not include the common node. Thus any metatile which is not a member of an infinite family of metatiles has a symbolic representation derived from an outer cycle. E.g., the only outer cycle in the digraph in Fig. 1(b) is S[1].

The following theorem is a restatement of Theorem 5.4 and Identity 5.5 in [14] but with more compact expressions for \( B_n \) and \( B_{n,k} \) and improved proofs. Note that the length of a cycle or circuit is simply the total contributions to the length of the arcs it contains.

**Theorem 4.3.** For a digraph possessing a common node, let \( l_{ci} \) be the length of the \( i \)-th outer cycle (\( i = 1, \ldots, N_o \)) and let \( k_{ci} \) be the number of combs it contains, let \( L_r \) be the length of the \( r \)-th inner cycle (\( r = 1, \ldots, N_c \)) and let \( K_r \) be the number of combs it contains, and let \( l_{ci} \) be the length of the \( i \)-th common circuit (\( i = 1, \ldots, N_c \)) and let \( k_{ci} \) be the number of combs it contains. Then for all integers \( n \) and \( k \),

\[
B_n = \delta_{n,0} + \sum_{r=1}^{N} (B_{n-L_r} - \delta_{n,L_r}) + \sum_{i=1}^{N_o} \left( \sum_{r=1}^{N} B_{n-l_{ci}} - \sum_{i=1}^{N_o} B_{n-l_{ci}} - L_r \right) + \sum_{i=1}^{N_o} B_{n-l_{ci}} - L_r
\]

\[
B_{n,k} = \delta_{n,0} \delta_{k,0} + \sum_{r=1}^{N} (B_{n-L_r, k-K_r} - \delta_{n,L_r, k,K_r}) + \sum_{i=1}^{N_o} \left( \sum_{r=1}^{N} (B_{n-l_{ci}, k-k_{ci}} - \sum_{i=1}^{N_o} B_{n-l_{ci}, k-k_{ci}} - L_r) - K_r \right) + \sum_{i=1}^{N_o} B_{n-l_{ci}, k-k_{ci}} - L_r - K_r
\]

where \( B_{n<0} = B_{n,k<0} = B_{n<k,k} = 0 \).

**Proof:** From Lemma 4.1,

\[
B_n = \delta_{n,0} + \sum_{i=1}^{N_o} B_{n-l_{ci}} + \sum_{i=1}^{N_o} \sum_{j_1, \ldots, j_N \geq 0} \binom{j_1 + \cdots + j_N}{j_1, \ldots, j_N} B_{n-L_i}
\]

\[
B_{n,k} = \delta_{n,0} \delta_{k,0} + \sum_{i=1}^{N_o} B_{n-l_{ci}, k-k_{ci}} + \sum_{i=1}^{N_o} \sum_{j_1, \ldots, j_N \geq 0} \binom{j_1 + \cdots + j_N}{j_1, \ldots, j_N} B_{n-L_i, k-k_i}
\]

with \( B_{n<0} = B_{n<k<0} = B_{n<k,k} = 0 \), where \( \lambda_i = \sum_{i=1}^{N} j_i L_s \) and \( \kappa_i = \sum_{i=1}^{N} j_i K_s \). The multinomial coefficient (which counts the number of arrangements of the inner cycles) results from the fact that changing the order in which the inner cycles are traversed (after the common node is reached via the outgoing path of a common circuit) gives rise to distinct metatiles of the same length. The sum of terms over \( j_1, \ldots, j_N \) in (4a) may be re-expressed as

\[
\sum_{j_1, \ldots, j_N \geq 0} M(\emptyset) B_{n-L_i} = B_{n-l_{ci}} + \sum_{m=0}^{N-1} \sum_{j_1 \in R_m} \sum_{j_1 \geq 0} M(R_m) B_{n-L_i}
\]

where \( M(R) \) denotes the multinomial coefficient \( \binom{j_1 + \cdots + j_N}{j_1, \ldots, j_N} \) with \( j_t \in R = 0 \), and \( R_m \) denotes a set of \( m \) numbers drawn from \( \{1, 2, \ldots, N_c\} \). For example, if \( N > 2 \) an instance of \( R_2 \) is \{1, 2\} in which case \( M(R_2) = \).
\((j_3^{x+\ldots+j_N})\). Replacing \(n\) by \(n - L_r\) in (4a) gives

\[
B_{n-L_r} = \delta_{n,L_r} + \sum_{i=1}^{N_r} B_{n-L_i-L_r} + \sum_{i=1}^{N_r} \sum_{j_1,\ldots,j_N \geq 0} M(\emptyset)B_{n-L_i-L_r}.
\]

After changing \(j_r\) to \(j_r - 1\), the sum of terms over \(j_1,\ldots,j_N\) may be re-expressed as

\[
\sum_{j_r \geq 1}_{j_r \geq 0} M_r(\emptyset)B_{n-L_i} = \sum_{m=0}^{N-1} \sum_{r \in R_m^N, j_r \in R_m^N} \sum_{r \in \bigcup_{s \not= r} R_m^N} M_r(\emptyset)B_{n-L_i},
\]

where \(M_r(\emptyset)\) denotes the multinomial coefficient \(M(\emptyset)\) with \(j_r\) replaced by \(j_r - 1\) (so, for example, \(M_3(\{1,2\}) = (j_3^{x+\ldots+j_N-1})\)) and \(R_m^N\) is a set of \(m\) numbers none of which equal \(r\) drawn from \(N_N\). After subtracting \(\sum_{r=1}^{N} B_{n-L_r}\) from (4a) and using the result for multinomial coefficients that \(M(\emptyset) = \sum_{r \in R} M_r(\emptyset)\), we obtain (3a). Similarly, subtracting \(\sum_{r=1}^{N} B_{n-L_r, k-K}\) from (4b) gives (3b).

The rest of the theorems in this section concern families of \(Q\) whose corresponding digraphs each have a common node. Once the lengths of and the number of combs in each cycle and common circuit have been determined, the recursion relations for \(B_n\) and \(B_{n,k}\) follow immediately from Theorem 4.3.

It can be seen from (3) that the recursion relation for \(B_n\) can be obtained from that for \(B_{n,k}\) by replacing \(B_{n-N,k}\) and \(\delta_{n,k-1}\) by \(B_{n-N,k-1}\) and \(\delta_{n,k}\), respectively, for any \(n_N\) and \(k\). For the remaining theorems we therefore give the recursion relation for \(B_{n,k}\) and only also show the one for \(B_n\) if we use the expression for \(B_n\) elsewhere.

For the more generally applicable theorems that follow, \(B_n\) and \(B_{n,k}\) are given most simply in terms of the elements \(p_i\) of \(N_N - Q\), the set of allowed differences less than \(q\). We order the \(p_i\) so that \(p_i < p_{i+1}\) for all \(i = 1,\ldots,a - 1\), where \(a = |N_N - Q|\), the number of allowed differences less than \(q\). Note that if the comb corresponding to \(Q\) has leftmost and rightmost teeth of widths \(l\) and \(r\), respectively, then \(p_1 = l\) and \(p_a = q - r\). In digraphs, we let \(\sigma_i\) (for \(i = 1,\ldots,a\)) denote the bit string corresponding to filling the first \(i - 1\) empty cells of a comb with squares and discarding the leading 1s. Thus \(\sigma_a\) is always 01\(^r\). It is also easily seen that the comb are leaving the \(\sigma_1\) node is \(C_{[p_1]}\).

**Theorem 4.4.** If the elements of \(Q\) are a well-based sequence then

\[
B_n = \delta_{n,0} + B_{n-1} + B_{n-q-1} + \sum_{i=1}^{a} (B_{n-p_i} - B_{n-p_i-1} - \delta_{n,p_i}),
\]

\[
B_{n,k} = \delta_{n,0} \delta_{k,0} + B_{n-1,k} + B_{n-q-1,k-1} + \sum_{i=1}^{a} (B_{n-p_i,k-1} - B_{n-p_i,k-1} - \delta_{n,p_i} \delta_{k,1}),
\]

If \(Q = N_N\) (and so \(a = 0\)) then the sums over \(i\) are omitted.

**Proof:** If \(Q = N_N\) the comb is a \((q + 1)\)-omino and the results follow immediately. Otherwise, we first need to establish that the comb leaving the \(\sigma_1\) node for any \(i = 1,\ldots,a\) in the digraph (Fig. 2) takes us back to the \(\sigma_1\) node. This is equivalent to there being a gap at position \(p_i + p_j\) (for any \(i, j = 1,\ldots,a\)) of the first comb (or that position being beyond the end of the comb) where \(p_j\) can be viewed as the position of the \(j\)-th empty cell in the comb added at cell \(p_i\) in the first comb. This must be the case by the definition of a well-based sequence; if there were no gap it would mean \(p_i + p_j \in Q\) which is impossible. The digraph has \(a\) inner cycles, namely, \(S^{i-1}C_{[p_i]}\) for \(i = 1,\ldots,a\), which have lengths \(p_i\), respectively. The common node is \(\sigma_1\). There is one common circuit (\(C_{[q+1]}S^q\)), which is of length \(q + 1\), and one outer cycle (\(S^1\)), which is of length 1.

The following corollary was established via graph theory and results concerning bit strings in [10].
Corollary 4.5. The generating function for $S_n$ when the elements $q_i$ of $Q$ form a well-based sequence is given by
\[ G(x) = \frac{c}{(1-x)c-x}, \]
where $c = 1 + \sum_{i=1}^{|Q|} x^n$.

Proof: Applying Lemma 2.3 to (5a) it can be seen that the numerator of (1) reduces to
\[ 1 + \sum_{m=1}^q \left( \sum_{j=1}^{q-m} \left( \sum_{i=1}^a (\delta_{m+j,p_i} - \delta_{m+j,p_i+1}) \right) + 1 \right)x^m, \]
where the +1 inside the brackets results from the fact that $\beta_{q+1}$ appears as a term in the sum over $j$ for every $m$ up to $q$. Note also that we must have $p_i > 1$ and $p_i < q$. When summed over $j$, $\delta_{m+j,p_i} - \delta_{m+j,p_i+1}$ cancels (thus leaving just the +1 multiplying the $x^m$) except if $p_i = m$ in which case $\delta_{m+j,p_i}$ is always zero and the $-\delta_{m+j,p_i+1}$ when $j = 1$ cancels the +1. Hence the numerator simplifies to $c$. The denominator of (1) is, in the present case, $1 - x - x^{q+1} - (1-x)\hat{c}$, where $\hat{c} = \sum_{i=1}^a x^{p_i}$. Using the result that $c + \hat{c} = \sum_{i=0}^q x^i = (1 - x^{q+1})/(1-x)$ it is then easily shown that the denominator can be re-expressed as $(1-x)c - x$. □

The proofs of some of the results that follow (starting with the next lemma which is used in the proof of Theorem 4.7) require combs where the number of gaps depends on the particular instance of $Q$. We therefore extend our notation: an $(l, [g], r)$-comb is a comb of length $l + g + r$ whose left and right teeth have widths of $l$ and $r$, respectively.

Lemma 4.6. When restricted-overlap tiling an $n$-board with $S$ and $C$, where $C$ contains at least one gap and the width of the final tooth is $r$, there is a 1-arc inner cycle $C_{[1-r]s}$ containing the 01$^r$ node iff (a) $q = 2r + 1$ or (b) the final gap is of unit width and the penultimate tooth has a width of at least $q - 2r - 1$.

Proof: If $q < 2r + 1$, by Lemma 3.1, there can be no inner cycles. If $q = 2r + 1$ and so the length of $C$ is $2(r+1)$, we have the situation depicted in Fig. 3(a). If $q > 2r + 1$ the final gap in the comb must be of unit width and the width $w_{t-1}$ of the penultimate tooth cannot be less than $x = q + 1 - 2(r+1)$ (Fig. 3b). □

Theorem 4.7. Let $\theta$ be the bit string representation of $Q$ whereby the $j$-th bit from the right of the $\theta$ is 1 if and only if $j \in Q$. By $[\theta/2^b]$ we mean discarding the rightmost $b$ bits in $\theta$ and shifting the remaining bits to the right $b$ places. Using $\mid$ to denote the bitwise OR operation, if $\theta \mid [\theta/2^{b-1}]$ for each $i = 1, \ldots, a - 1$
is all ones after discarding the leading zeros, \( a \geq 2 \), and \( p_a = q - r \) (which implies that \( r \geq 1 \)), then if (a) \( q = 2r + 1 \) or (b) \( q > 2r + 1 \) and \( 1 \leq p_a - 1 \leq r \), then

\[
B_{n,k} = \delta_{n,0}\delta_{k,0} - \delta_{n,q-1}\delta_{k,1} + B_{n-1,k} + B_{n-2r-1,k-1} - B_{n-2r-1,k-1} + B_{n-q+r-1,k-1} + B_{n-q-1,k-1} + \sum_{i=1}^{a-1}(B_{n-q-1-p_i,k-2} - B_{n-2q+r-1-p_i,k-3}).
\]

(7)

**Proof:** The condition (a) and (b) correspond to those in Lemma 4.6 and thus guarantee a single-comb inner cycle at the 01r node. The condition on \( \theta \) means that placing a comb at an empty cell (other than the final empty cell) will result in all gaps in the combs to the right of this point being filled. On the digraph this means that there is an arc from the \( \sigma_i \) node, where \( i = 1, \ldots, a - 1 \), to the 0 node. Tiling with squares and combs corresponding to \( \mathcal{Q} \) leads to the digraph shown in Fig. 4. There is one inner cycle (C[q-r]) and one common circuit (C[\( q+1 \).Sn]). The outer cycles are \( S[i] \) and \( C[\( q+1 \).Sn] \) for \( i = 1, \ldots, a - 1 \) and their respective lengths are 1 and \( q+1+p_i \).

It is straightforward to verify that the following four classes of \( \mathcal{Q} \) satisfy the conditions for Theorem 4.7 to apply: (i) \( \mathcal{Q} = \{2, \ldots, q-r-1, q-r+1, \ldots, q\} \) where \( r \geq 1 \) and \( q \geq \max(2r+1,4) \) (e.g., for \( q \leq 7 \): \{2,4\}, \{2,3,5\}, \{2,4,5\}, \{2,3,4,6\}, \{2,3,5,6\}, \{2,3,4,5,7\}, \{2,3,4,6,7\}, \{2,3,5,6,7\}); (ii) \( \mathcal{Q} = \{1, \ldots, l-1, l+1, \ldots, q-r-1, q-r+1, \ldots, q\} \) where \( r \geq 1 \) and \( q \geq 2r+1 \) and \( 2l \neq q-r \) (e.g., for \( q \leq 8 \): \{1,3,4,6,7\}, \{1,2,4,6,7,8\}, \{1,3,4,5,7,8\}, \{1,3,4,6,7,8\}); (iii) \( q = 2r+1 \), \( p_1 = l, p_a = r + 1 \), and \( l \leq r \leq 2l+2 \) (e.g., for \( q \leq 9 \): \{1,4,5\}, \{1,2,5,6,7\}, \{1,2,6,7,8,9\}, \{1,2,3,6,7,8,9\}, \{1,2,4,6,7,8,9\}); (iv) \( \mathcal{Q} = \{2,4, \ldots, 2a, 2a+1, \ldots, q\} \) where \( a \geq 3 \) and \( q = 4a - 4, 4a - 3 \) (e.g., for \( q \leq 9 \): \{2,4,6,7,8\}, \{2,4,6,7,8,9\}). These classes cover all cases where the theorem applies for \( q \leq 9 \).

Note that, omitting the sum, Theorem 4.7 holds for the case \( a = 1 \) if \( p_1 = q-r \) and \( q \geq 2r+1 \). It then coincides with Theorem 4.4.

**Theorem 4.8.** Suppose \( p_1 = l \) and \( p_a = 2l \). Then if either (a) \( q = 4l-1 \) or (b) \( p_a-1 \leq q-2l \) where \( q < 4l-1 \), then

\[
B_{n,k} = \delta_{n,0}\delta_{k,0} - \delta_{n,2l}\delta_{k,1} + B_{n-1,k} + B_{n-2l,k-1} - B_{n-2l-1,k-1} + B_{n-4l,k-1} + B_{n-5l,k-2} + B_{n-6l,k-3} - B_{n-7l,k-3} - B_{n-8l,k-4} + \sum_{i=2}^{a-1}(B_{n-4l-p_i,k-2} - B_{n-6l-p_i,k-3}).
\]

(8)

where the sum is omitted if \( a = 2 \).

**Proof:** Tiling with squares and \( (l, [l+1], q-2l) \)-combs leads to the digraph shown in Fig. 5. There is just one inner cycle (C[\( 2l \)]) and one common circuit (C[\( 2l \).Sn]). Their respective lengths are 2l and 4l. The outer cycles are \( S[i], C[\( 2l \)].S[i], \{S, C[\( 2l \)]\} \), and, if \( a \geq 3 \), \( C[\( 2l \)].S[i-1]C[\( 2l \)] \) for \( i = 2, \ldots, a - 1 \). Their respective lengths are 1, 5l, 6l, and 4l + p_i.

The instances of \( \mathcal{Q} \) with \( q \leq 9 \) to which Theorem 4.8 applies are \{3\}, \{1,3,5,6\}, \{1,5,6,7\}, \{1,3,5,6,7\}, and \{1,2,4,5,7,8,9\}. 

Figure 4: Digraph for tiling a board with squares and combs corresponding to \( \mathcal{Q} \) specified in Theorem 4.7.
There are three other ways in which \( p_1 = l, p_a = q - r, l > r \) and (i) \( q = 2l \) or (ii) \( a = 2, q \geq 2l, \) but \( q \neq 2l + r, \) then

\[
B_{n,k} = \delta_{n,0}\delta_{k,0} + B_{n-1,k} + B_{n-2l-1,k-1} + B_{n-3l-1,k-2} + \sum_{i=2}^{n} \left( B_{n-p_i,k-1} - B_{n-p_i-1,k-1} + B_{n-l+p_i,k-2} - B_{n-l+p_i-1,k-2} - \delta_{n,p_i}\delta_{k,1} - \delta_{n,l+p_i}\delta_{k,2} \right)
\]

(9)

**Proof:** Tiling with (i) squares and \((l, [l-r+1], r)\)-combs, where \( l > r, \) or (ii) squares and \((l, 1, m \neq l-1, 1, r)\)-combs, where \( 0 < l - r \leq m + 1 \), leads to the digraph shown in Fig. 6. There are \( 2(a - 1) \) inner cycles: \([S, C_q]S^{l-2}C_{p_i}\) for \( i = 2, \ldots, a \). Their lengths are \( p_i \) and \( l + p_i \). The common node is \( \sigma_1 \) and so the common circuits are \( C_{[2l+1]}[S, C_q]S^{l-1} \) which have lengths of \( 2l + 1 \) and \( 3l + 1 \).

The instances of \( Q \) for which Theorem 4.9 applies when \( q \leq 8 \) are \{1, 4\}, \{1, 2, 6\}, \{1, 2, 4, 6\}, \{1, 2, 5, 6\}, \{1, 3, 4, 6\}, \{1, 2, 4, 6, 7\}, \{1, 3, 4, 5, 7\}, \{1, 2, 3, 8\}, \{1, 2, 3, 5, 8\}, \{1, 2, 3, 6, 8\}, \{1, 2, 3, 5, 6, 8\}, \{1, 2, 3, 5, 7, 8\}, \{1, 2, 4, 5, 6, 8\}, \{1, 3, 4, 5, 6, 8\}.

We end this section by showing that all possible \( Q \) such that \( a \leq 2 \) have been covered by the theorems given here. When \( a = 0 \), the comb \( C \) is a \((q + 1)\)-omino and \( B_{n,k} \) is given by (5b). When \( a = 1 \), \( C \) is an \((l, l, r)\)-comb and the two possible cases are shown in Fig. 1. Then if \( r \geq 1 \), Theorem 4.2 applies. Otherwise, if \( l > r \), Theorem 4.4 applies since \( 2p_1 > q \) (as \( p_1 = l \) and \( q = l + r \)) and hence the elements of \( Q \) are a well-based sequence. When \( a = 2, C \) is either an \((l, 2, r)\)-comb or an \((l, 1, m, 1, r)\)-comb for some \( l, m, r \). In the former case, \( q = l + r + 1 \) and so the condition \( 2r \geq q \) leads to Theorem 4.2 applying when \( l < r \). When \( l = r, C \) is covered by the class (iii) instances of \( Q \) that apply to Theorem 4.7 except when \( l = 1 \) which case Theorem 4.8(a) applies. When \( l = r + 1 \), we have \( q = 2l \) and \( p_2 = l + 1 = 2l - r \) and so Theorem 4.9 applies. The final possibility is if the elements of \( Q \) are a well-based sequence (and the case is then covered by Theorem 4.4) and this occurs if \( 2p_1 > q \) which implies that \( l > r + 1 \). For \((l, 1, m, 1, r)\)-combs, \( q = l + m + r + 1 \) and so the \( 2r \geq q \) case (Theorem 4.2) is when \( l \leq r - m \). Of the \( l \geq m \) cases we first consider those where \( l = m + 1 \). This can arise in two ways. If \( 2p_1 = p_2 \) (which implies \( l = m + 1 \)) and \( p_1 + p_2 > q \) (which implies \( l > r \)) then the elements of \( Q \) are a well-based sequence and Theorem 4.4 applies. When \( l = m + 1 \) and \( l \leq r \) then Theorem 4.8 applies since these conditions can be re-expressed as \( p_2 = 2l \) and \( p_1 \leq r \), respectively. When \( l = 1 \) (and \( \geq r - m \)), the case falls into class (i) to which Theorem 4.7 applies. There are three other ways in which \( l \neq m + 1 \) arises when \( l \geq r - m \). If \( l \leq r \) then we have class (ii) to which Theorem 4.7 applies. If \( r \leq l \leq m + r + 1 \) then Theorem 4.9 applies. Finally, if \( 2p_1 > q \) (which implies \( l > m + r + 1 \)) then the elements of \( Q \) are a well-based sequence and Theorem 4.4 again applies.
5. Bijections between restricted combinations and strongly restricted permutations

A strongly restricted permutation \( \pi \) of the set \( \mathbb{N}_n \) is a permutation for which the number of permissible values of \( \pi(i) - i \) for each \( i \in \mathbb{N}_n \) is less than a finite number independent of \( n \) [15]. Here we give two bijections between types of strongly restricted permutations and restricted combinations. The proof of the first of these requires the following result established in [14].

Lemma 5.1. There is a bijection between the permutations of \( \mathbb{N}_n \) satisfying \( \pi(i) - i \in \mathcal{D} \) for each \( i \in \mathbb{N}_n \) and the tilings of an \( n \)-board using \((\frac{1}{2}, d_j)\)-fences with the left-hand sides aligned with cell boundaries and \( d_j \) equal to the non-negative elements of \( \mathcal{D} \) and \((\frac{1}{2}, -d_k - 1)\)-fences with their gaps aligned with cell boundaries and \( d_k \) equal to the negative elements of \( \mathcal{D} \). If the left-hand side of cell \( i \) of the \( n \)-board is occupied by the left (right) side of a \((\frac{1}{2}, g)\)-fence then \( \pi(i) = i + g \) \( (\pi(i) = i - g - 1) \).

The following bijection is an extension of that noted but not proved in [16]. Recall that an \textit{excedance} of a permutation \( \pi \) is a position \( i \) such that \( \pi(i) > i \).

Theorem 5.2. There is a bijection between restricted combinations of \( \mathbb{N} \) with \( Q = \{m, 2m, \ldots, jm\} \) where \( j, m \geq 1 \) and permutations \( \pi \) of \( \mathbb{N}_{n+jm} \) satisfying \( \pi(i) - i \in \{-m, 0, jm\} \) for all \( i \in \mathbb{N}_{n+jm} \) that results in the number of such \( k \)-subsets being equal to the number of such strongly restricted permutations that contain \( k \) excedances.

Proof: From Lemma 5.1 the number of such permutations is the number of ways to tile an \((n + jm)\)-board using squares aligned with the cell boundaries \((S), (\frac{1}{2}, jm)\)-fences whose left tooth is always placed on the left side of a cell \((F)\), and \((\frac{1}{2}, m - 1)\)-fences whose left tooth must lie on the right side of a cell \((\bar{F})\). Suppose the left tooth of the first \( F \) on the board occupies (the left side) of cell \( i \). The only possible way to fill the right side of cell \( i \) is with the left tooth of an \( \bar{F} \). The right tooth of the \( \bar{F} \) lies on the left side of cell \( i + m \). If \( j = 1 \), the right side of this cell is occupied by the right tooth of the first \( F \). Otherwise it must again be occupied by the left tooth of another \( \bar{F} \) and similarly until the right tooth of the first \( F \) is reached at cell \( i + jm \). Hence this \( F \) has \( j \bar{F} \) placed end-to-end in its interior. Combined, these \( j + 1 \) tiles occupy exactly the same cells as a \((1, m - 1; j + 1)\)-comb. All subsequent \( F \) on the board behave similarly. Note that owing to the fact that any \( \bar{F} \) must have its left tooth on the right of a cell, such tiles can only appear on the board in such a conjunction with \( j - 1 \) other \( \bar{F} \) surrounded by an \( F \). Hence there is a bijection between tilings of an \((n + jm)\)-board using \( k \bar{F} \), \( jk \bar{F} \) and \( n + jm - (j + 1)k \) \( S \) and the tilings of a board of the same length using squares and \( k \) \((1, m - 1; j + 1)\)-combs and \( n + jm - (j + 1)k \) squares. If \( j = 1 \), the right side of this cell is occupied by the right tooth of the first \( F \). Otherwise it must again be occupied by the left tooth of another \( \bar{F} \) and similarly until the right tooth of the first \( F \) is reached at cell \( i + jm \). Hence this \( F \) has \( j \bar{F} \) placed end-to-end in its interior. Combined, these \( j + 1 \) tiles occupy exactly the same cells as a \((1, m - 1; j + 1)\)-comb. All subsequent \( F \) on the board behave similarly. Note that owing to the fact that any \( \bar{F} \) must have its left tooth on the right of a cell, such tiles can only appear on the board in such a conjunction with \( j - 1 \) other \( \bar{F} \) surrounded by an \( F \). Hence there is a bijection between tilings of an \((n + jm)\)-board using \( k \bar{F} \), \( jk \bar{F} \) and \( n + jm - (j + 1)k \) \( S \) and the tilings of a board of the same length using squares and \( k \) \((1, m - 1; j + 1)\)-combs and \( n + jm - (j + 1)k \) squares. If \( j = 1 \), the right side of this cell is occupied by the right tooth of the first \( F \). Otherwise it must again be occupied by the left tooth of another \( \bar{F} \) and similarly until the right tooth of the first \( F \) is reached at cell \( i + jm \). Hence this \( F \) has \( j \bar{F} \) placed end-to-end in its interior. Combined, these \( j + 1 \) tiles occupy exactly the same cells as a \((1, m - 1; j + 1)\)-comb. All subsequent \( F \) on the board behave similarly. Note that owing to the fact that any \( \bar{F} \) must have its left tooth on the right of a cell, such tiles can only appear on the board in such a conjunction with \( j - 1 \) other \( \bar{F} \) surrounded by an \( F \). Hence there is a bijection between tilings of an \((n + jm)\)-board using \( k \bar{F} \), \( jk \bar{F} \) and \( n + jm - (j + 1)k \) \( S \) and the tilings of a board of the same length using squares and \( k \) \((1, m - 1; j + 1)\)-combs and \( n + jm - (j + 1)k \) squares.

Our second bijection concerns a strongly restricted permutation where \( \pi(i) - i \in \{-1, 0, -1\} \) and any \( m \) consecutive \( \pi(i) \) cannot differ from one another by more than \( m \). It was identified thanks to sequence A263710 in the OEIS [17] which gives the case corresponding to restricted combinations with \( Q = \{1, 4\} \). The conjectured recursion relation for the sequence can be proved using Theorem 4.9.

Theorem 5.3. There is a bijection between the \( k \)-subsets of \( \mathbb{N}_n \) such that no two elements of the subset differ by an element of \( Q = \{1, m\} \), where \( m = 2, 3, \ldots \), and the permutations \( \pi \) of \( \mathbb{N}_{n+1} \) that have \( k \) pairs of digits exchanged such that \( (a) \) the pairs do not overlap and \( (b) \) for all \( i = 1, \ldots, n + 2 - m \) the permutation satisfies \( \max_{j=0, \ldots, m-1} \pi(i+j) - \min_{j=0, \ldots, m-1} \pi(i+j) \leq m \).

Proof: The permutation \( \pi \) of \( \mathbb{N}_{n+1} \) corresponding to a subset \( \mathcal{S} \) of \( \mathbb{N}_n \) is formed as follows. If \( i \in \mathcal{S} \) (where \( i = 1, \ldots, n \)) then \( \pi(i) = i + 1 \) and \( \pi(i+1) = i \) (i.e., a pair of exchanged digits) whereas if neither \( i \) nor \( i-1 \) are in \( \mathcal{S} \) (for \( i = 2, \ldots, n + 1 \)) then \( \pi(i) = i \). As \( Q \) contains 1, the pairs of exchanged digits in the permutation cannot overlap. Given that condition \( (a) \) is satisfied, the only way for \( (b) \) to be violated would be if \( \pi(i+1) = i \) and \( \pi(i+m) = i + m + 1 \) but this could only occur if both \( i \) and \( i + m \) were in \( \mathcal{S} \) which is impossible since \( m \in Q \). The process of forming a permutation from a subset is clearly reversible and so the bijection is established.
6. Connections with bit strings

Others have noted that the number of length- \( n \) bit strings (or binary words) having no two 1s that have positions in the string that differ by \( q \) (or, equivalently, are separated from one another by \( q - 1 \) digits) is \( S_n \) when \( Q = \{ q \} \) \cite{17}. This can be generalized by considering the following bit string \( s \) representing subset \( S \): the \( j \)-th bit from the right of \( s \) is 1 if and only if \( j \in S \). (This representation is used in the program in the Appendix.) Then it is clear that \( S_{n,k} \) is the number of length- \( n \) bit strings that contain \( k \) 1s placed in such a way that the difference in positions of any two 1s does not equal an element of \( Q \). We now mention an instance of this that corresponds to A130137 in the OEIS \cite{17}. If one instead places the same restrictions on 0s rather than 1s, for the \( Q = \{ 1 \} \) case, the bit strings (i.e., those containing no 00) are called Fibonacci binary words, so named as the number of such words of length \( n \) is \( F_{n+2} \). A130137 is described as the number of length- \( n \) Fibonacci binary words that do not contain the substring 0110. This is \( S_n \) when \( Q = \{ 1, 3 \} \) since the three other possible disallowed substrings that only prevent the occurrence of two 0s whose positions differ by 3 do not need to be mentioned; they each contain 00 and so cannot be present in Fibonacci binary words.

A search of the OEIS for sequences \((S_n)_{n \geq 0}\) for various \( Q \) also revealed A317669 which coincides with \( S_{n-4} \) for \( Q = \{ 1, 2, 4 \} \) and is described as the number of equivalence classes of binary words of length \( n \) for the subword 10110 where two length- \( n \) binary words are equivalent if the subword occurs in the same position(s) in those words. One can represent the equivalence class by the set of positions, counting the leftmost position as 1. E.g., when \( n = 11 \), words in the class \( \{ 3, 6 \} \) are xx10110110x where each \( x \) can be either 0 or 1. The connection of this sequence to restricted combinations is an instance of the following theorem.

**Theorem 6.1.** For a given length- \( l \) binary subword \( \omega \), \( j \in Q \) if and only if \( \omega \mod 2^{l-j} \neq [\omega/2^j] \) for \( j = 1, \ldots, l-1 \). The number of equivalence classes of binary words of length \( n \) for the subword \( \omega \) is \( S_{n-l+1} \). Furthermore, \( S_{n-l+1,k} \) is the number of such equivalence classes whose representation as a set is of size \( k \).

**Proof:** We show that the \( k \)-subsets of \( \mathbb{N}_{n-l+1} \) satisfying the conditions for disallowed differences specified by \( Q \) are the same as the sets representing the equivalence classes of binary words of length \( n \) in which the subword appears \( k \) times. In general, the possible positions of any subword of length \( l \) in a word of length \( n \) (and therefore the possible elements of the set representing an equivalence class) are \( 1, \ldots, n-l+1 \). From a restricted subset \( S \) of \( \mathbb{N}_{n-l+1} \) we construct an equivalence class of words as follows. Starting with a length- \( n \) word of \( \omega s \) (each \( x \) representing 0 or 1), for each \( s \in S \) we place the start of subword \( \omega \) at position \( s \) in the word. If and only if \( \omega \mod 2^{l-j} = [\omega/2^j] \) (and so \( j \notin Q \)) then, on shifting the digits of the subword by \( j \) places to the right and overlaying them on the original subword, none of the digits of the original subword are changed. This means that we have constructed a valid equivalence class. The reversible nature of the construction establishes the bijection. \( \square \)

7. Discussion

As \( a = |N_{q} - Q| \) increases, so, in general, does the number of inner cycles in the digraph and we find more and more instances (e.g., when \( Q = \{ 1, 5 \} \)) where the digraph has inner cycles but no common node. In the simpler of such cases, it is still possible to derive general recursion relations analogous to (3) \cite{4}. This enables one to find recursion relations for all the \( a = 3 \) cases as we will demonstrate in forthcoming work. For cases where the digraph is more complex, we have not yet managed to formulate a general procedure for obtaining the recursion relations.

Various authors have also considered the number of ways of choosing \( k \) objects from \( n \) arranged in a circle in such a way that no two chosen objects are certain disallowed separations apart \cite{1, 5, 18, 9}. A modified version of our bijection covers such cases if we instead consider restricted-overlap tiling using squares and combs of an \( n \)-bracelet (which is an \( n \)-board with the \( n \)-th cell joined to the first cell). There are, however, subtleties about the rules for overlap which we will address in detail elsewhere.
Appendix: C program for finding $S_n$ and $S_{n,k}$

There are two main features that contribute to the efficiency of the algorithm used in the program listed below. Rather than directly computing whether the difference of each pair $x, y$ of elements of subset $S$ is in $Q$, we perform a bitwise AND operation on bit string representations of $S$ shifted $x$ places and $Q$. If the AND operation gives an answer of 0 then $y - x$ does not equal any element of $Q$ for all $y > x$. We also exploit the result that if $S$ is an allowed subset of $N_n$ then it is an allowed subset of $N_m$ for any $m > n$.

```c
/* rcl.c (restricted combinations on a line) */
/* count subsets of {1,2,...,n} such that no two elements have a difference equal to an element of Q */
/* finds S_n or S_{n,k} for n from 0 to MAXn which is typically 32 */
#include <stdio.h>
#define MAXn 8*sizeof(unsigned int)
int main(int argc,char **argv) {
    // j-th bit from the right in the bit string is 1 iff j is in the set it represents
    unsigned int Q=0,s,ss; // bit strings giving Q, subset, and shifted subset
    char *p; // used for reading in Q (given as a comma-separated list) from command line
    unsigned short int tri, // true if S_{n,k} triangle rather than S_n is wanted
    i,j,k=0,test,lastn=0;
    unsigned int S[MAXn+1][MAXn+2]; // S[n][k] is S_{n,k} if tri true; otherwise S[n][0] is S_n
    if (argc==1) { // rcl called with no arguments
        puts("example usage:	 rcl 1,2,4
		 rcl 1,2,4 t");
        exit(1);
    }
    for (p=argv[1]-1;p;p=strchr(p,',')) Q|=1<<(atoi(++p)-1); // read in Q
    tri=(argc>2 && argv[2][0]=='t'); // true if argument 2 in command line is t
    for (i=0;i<=MAXn;i++) for (j=0;j<=(tri?i+1:1);j++) S[i][j]=j?0:1; // set k=0 totals to 1, zero k>0 totals
    for (s=1;s<~0;s++) { // will test all possible nonempty subsets s
        if (tri) for (k=0,ss=s;ss;ss>>=1) if (ss&1) k++; // if S_{n,k} wanted, get k = #set bits of s
        for (ss=s,i=1;i<=MAXn;i++) { // test subset s
            test=ss&1; // will only test shifted subset against Q if rightmost bit of ss is 1
            if (test && (Q&ss)) break; // disallowed difference found - done with current s
            ss>>=1; // shift shifted subset ss to the right by 1
            if (ss==0) { // s is ok (no disallowed differences found); i is now minimum n for which s is ok
                if (i>lastn) { // total(s) for previous n finished
                    for (j=0;S[lastn][j];j++) printf("%d,",S[lastn][j]); // display S_n or S_{n,k} if nonzero
                    if (tri) puts("\n"); // new line
                    else fflush(stdout); // display immediately
                    lastn=i; // update lastn for next time it is used
                }
                break; // done with current s
            }
            for (j=0;j<MAXn;j++) S[i][j]=S[i][j-1] + (ss&1); // if s is ok for a given n, it is ok for all higher n
            if (test && (Q&ss)) break; // disallowed difference found - done with current s
        }
    }
    for (j=0;S[lastn][j];j++) printf("%d,",S[lastn][j]); // lastn now equals MAXn
    puts("\n");
}
```

References

[1] I. Kaplansky, Solution of the “problème des ménages”, Bull. Am. Math. Soc. 49 (10) (1943) 784–785.
[2] A. T. Benjamin, J. J. Quinn, Proofs That Really Count: The Art of Combinatorial Proof, Mathematical Association of America, Washington, 2003.
[3] M. A. Allen, K. Edwards, On two families of generalizations of Pascal’s triangle, J. Integer Seq. 25 (7) (2022) 22.7.1.
[4] M. A. Allen, On a two-parameter family of generalizations of Pascal’s triangle https://arxiv.org/abs/2209.01377.
[5] J. Konvalina, On the number of combinations without unit separation, J. Combin. Theor. A 31 (2) (1981) 101–107.
[6] H. Prodinger, On the number of combinations without a fixed distance, J. Combin. Theor. A 35 (3) (1983) 362–365. doi:10.1016/0097-3165(83)90019-5.
[7] J. Konvalina, Y.-H. Liu, Subsets without q-separation and binomial products of Fibonacci numbers, J. Combin. Theor. A 57 (2) (1991) 306-310. doi:10.1016/0097-3165(91)90054-K.
[8] J. Konvalina, Y.-H. Liu, Bit strings without q-separation, BIT Numer. Math. 31 (1) (1991) 32–35.
[9] T. Mansour, Y. Sun, On the number of combinations without certain separations, Eur. J. Combinator. 29 (5) (2008) 1200–1206. doi:10.1016/j.ejc.2007.06.024.
[10] S. Kitaev, Independent sets on path-schemes, J. Integer Seq. 9 (2) (2006) 06.2.2.
[11] A. A. Valyuzhenich, Some properties of well-based sequences, J. Appl. Ind. Math. 5 (4) (2011) 612–614.
[12] M. A. Allen, K. Edwards, Connections between two classes of generalized Fibonacci numbers squared and permanents of (0,1) Toeplitz matrices, Lin. Multilin. Algebra https://arxiv.org/abs/2107.02589. doi:10.1080/03081087.2022.2107979.
[13] K. Edwards, A Pascal-like triangle related to the tribonacci numbers, Fibonacci Quart. 46/47 (1) (2008/2009) 18–25.
[14] K. Edwards, M. A. Allen, Strongly restricted permutations and tiling with fences, Discrete Appl. Math. 187 (2015) 82–90. doi:10.1016/j.dam.2015.02.004.
[15] D. H. Lehmer, Permutations with strongly restricted displacements, in: Combinatorial Theory and its Applications II (Proceedings of the Colloquium, Balatonfured, 1969), North-Holland, Amsterdam, 1970, pp. 755–770.
[16] V. Baltic, Applications of the finite state automata for counting restricted permutations and variations, Yugoslav J. Oper. Res. 22 (2) (2012) 183–198.
[17] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences, Published electronically at https://oeis.org, 2010.
[18] W. O. J. Moser, The number of subsets without a fixed circular distance, J. Combin. Theor. A 43 (1) (1986) 130–132.