Chebyshev potentials, Fubini–Study metrics, and geometry of the space of Kähler metrics

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Abstract
The Chebyshev potential of a Hermitian metric on an ample line bundle over a projective variety, introduced by Witt Nyström, is a convex function defined on the Okounkov body. It is a generalization of the symplectic potential of a torus-invariant Kähler potential on a toric variety, introduced by Guillemin, that is a convex function on the Delzant polytope. A folklore conjecture asserts that a curve of Chebyshev potentials associated to a subgeodesic in the space of positively curved Hermitian metrics is linear in the time variable if and only if the subgeodesic is a geodesic in the Mabuchi metric. This is classically true in the special toric setting, and in general Witt Nyström established the sufficiency. The main obstacle in the conjecture is that it is difficult to compute Chebyshev potentials, that are currently only known on the Riemann sphere and toric varieties. The goal of this article is to disprove this conjecture. To that end we characterize the geodesics consisting of Fubini–Study metrics for which the conjecture is true on the hyperplane bundle of the projective space. The proof involves explicitly solving the Monge–Ampère equation describing geodesics on the subspace of Fubini–Study metrics and computing their Chebyshev potentials.

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1 | A FOLKLORE CONJECTURE

Consider first the simpler setting of a toric manifold \( X \) equipped with an ample line bundle \( A \to X \) [11, section 3.4]. Being toric means that \( X \) is a projective manifold whose automorphism group contains a complex torus \((\mathbb{C}^\ast)^n\) that acts on \( X \) with a dense open orbit \((\mathbb{C}^\ast)^n \cong X_{\text{open}} \subseteq X\). Such pairs \((X, A)\) are in a one-to-one correspondence with Delzant lattice polytopes (also known as the moment polytopes) \( P \subseteq M_{\mathbb{R}} \cong \mathbb{R}^n \) up to translations, where \( M = \text{Hom}(X_{\text{open}}, \mathbb{C}^\ast) \) is the lattice of characters of \( X_{\text{open}} \cong (\mathbb{C}^\ast)^n \), and \( M_{\mathbb{R}} := M \otimes \mathbb{R} \) [6, Theorem 2.1.2; 9, 11, section 3.4]. Indeed, the primitive inner normal vector \( v_i \) of a facet \( F_i \) of \( P \) in the dual \( N \) of the lattice \( M \), that is, the normal vector pointing inward whose coordinates are coprime integers, is the primitive generator of a 1-dimensional cone of the fan (known as the dual fan of \( P \)) corresponding to \( X \), which then corresponds to an irreducible torus invariant divisor \( D_i \) [6, section 6.4; 11, section 1.5]. Also, a vertex of \( P \) corresponds to an \( n \)-dimensional cone of the fan, which then corresponds to an affine toric chart \( \mathbb{C}^n \) that is used to glue \( X \). Let \( a_i \) denote the inner product of \( v_i \) with a point on \( F_i \). This does not depend on the choice of that point, as \( v_i \) is a normal vector of \( F_i \). Also note that \( a_i \) is an integer, as \( P \) is a lattice polytope, that is, each vertex of \( P \) lies in \( M_{\mathbb{R}} \) [6, section 6.6]. Now, \( A = \mathcal{O}(D) \), where \( D = -\sum a_id_i \). A translation of \( P \) corresponds to a different choice of \( D \), or equivalently, a different local trivialization of \( A \) (see below for more details on what a trivialization means in practice).

By translating \( P \) we may assume one of its vertices, \( u_0 \), is the origin. Let \( e_1, \ldots, e_n \) be the primitive vectors coming out of \( u_0 \) along the \( n \) edges adjacent to \( u_0 \). Recall that \( P \) is a Delzant polytope, that is, \( e_1, \ldots, e_n \) form a \( \mathbb{Z} \)-basis of \( M \). Let \( z_1, \ldots, z_n : X_{\text{open}} \to \mathbb{C}^\ast \) denote the characters of \( X_{\text{open}} \cong (\mathbb{C}^\ast)^n \) represented by \( e_1, \ldots, e_n \). Notice that the vertex \( u_0 \) corresponds to the \( n \)-dimensional cone \( \mathbb{R}^n_{\geq 0} \subseteq N_{\mathbb{R}} \), which then corresponds to an affine toric chart \( X_0 \cong \mathbb{C}^n \) on which \( z_1, \ldots, z_n \) extends to the coordinate functions. Let \( Y \) be the standard coordinate flag \( Y_i = \text{V}(z_1, \ldots, z_i) \), and \( C \) the local trivialization on this chart corresponding to this translation of \( P \).

On the torus \( X_{\text{open}} \subseteq X_0 \) define the map \( \Theta : X_{\text{open}} \to \mathbb{R}^n \),

\[
\Theta : z \mapsto (\log |z_1|, \ldots, \log |z_n|),
\]

whose fibers \( \Theta \) are diffeomorphic to \((S^1)^n\). The restriction to \( X_{\text{open}} \) of any positively curved toric metric \( e^{-\phi} \) on \( A \) (i.e., \( \phi \) is plurisubharmonic (psh) and \((S^1)^n\)-invariant), yields a function \( \phi_\Theta = \phi \circ \Theta^{-1} \) that is well-defined, convex on \( \mathbb{R}^n \), and whose gradient image \( \nabla \phi_\Theta(\mathbb{R}^n) = P \).

The geodesic equation on the space of positively curved Hermitian metrics on \( A \) endowed with the Mabuchi \( L^2 \) metric becomes, as observed by Semmes, and later Donaldson, the homogeneous complex Monge–Ampère equation (3.2). This is an equation for a complex curve \( s + \sqrt{-1}t \mapsto \phi(s) \) with \( \phi \) now psh in all \( n + 1 \) (complex) variables (the psh condition is referred to as \( \phi \) being a subgeodesic, and makes the equation degenerate elliptic). In general, the variable \( t \) will be omitted as none of the objects depend on \( t \). Under the toric symmetry assumption also half of the remaining \( 2n \) real variables can be omitted, and the equation simplifies to the homogeneous real Monge–Ampère equation,

\[
\det \nabla^2_{s,x} \phi_\Theta = 0, \quad \text{on} (0,T) \times \mathbb{R}^n, \quad \nabla \cdot \phi_\Theta(s, \cdot)(\mathbb{R}^n) = P \quad \text{for all} \ s \in [0,T],
\]

for a convex function in all \( n + 1 \) variables. For each fixed time \( s \), the partial Legendre transform \( \mathcal{L} \) maps the convex function \( \phi_\Theta(s, \cdot) \) on \( \mathbb{R}^n \) to a convex function on \( P \). Under this correspondence,
the geodesic equation further reduces to \( \frac{d^2}{ds^2} \mathcal{L}_\Theta(s, \cdot) = 0 \) on \( P \). That is, if \( \phi \) is sufficiently regular as well as convex in all \( n + 1 \) variables, then \( s \mapsto e^{-\phi(s)} \) is a geodesic precisely when \( s \mapsto \mathcal{L}_\Theta(s, \cdot) \) is affine in \( s \). These facts go back to Mabuchi, Semmes, and Donaldson \([10, \text{section 6}; 16, \text{section 3.1}; 23, \text{section 1}] \); see also \([22]\).

In general, let \( X \) be a projective manifold and \( A \to X \) an ample line bundle. Denote by

\[
\mathcal{H}_A
\]

the space of positively curved Hermitian metrics on \( A \). The Chebyshev potential

\[
c_{Y_\cdot, c}[h]
\]

of \( h \in \mathcal{H}_A \) was introduced by Witt Nyström \([25, \text{section 5}] \), as a generalization of the symplectic potential, introduced by Guillemin \([13, \text{section 4}] \), to any projective, but not necessarily toric, manifold. The function \( c_{Y_\cdot, c}[h] \) is defined on the interior of the Okounkov body \( \Delta(X, A, Y_\cdot) \) of the line bundle \( A \), itself a generalization of the Delzant polytope \( P \) (see Definition 2.7). In addition to the hermitian metric \( h \), the Chebyshev potential depends on several pieces of data that are fixed once and for all independently of \( h \). The first is a flag of smooth submanifolds \( Y_\cdot : X = Y_0 \supseteq \cdots \supseteq Y_n = \{p\} \). The second, typically swept under the rug, but crucial for actual computations, is a local trivialization \( C : \pi^{-1}(U) \to U \times \mathbb{C} \) over a chart \( U \hookrightarrow \mathbb{C}^n \) around \( p \) compatible with \( Y_\cdot \) in the sense that

\[
Y_i \cap U = \{ z \in U \mid z_1 = \cdots = z_i = 0 \}, \quad (1.1)
\]

Note that \( C \) encodes the choice of local holomorphic coordinates \( z_1, \ldots, z_n \) on \( U \subset X \) as well as a local nonvanishing holomorphic section \( e \) for \( A \to X \).

In the special case of toric \( (X, A) \) with a toric metric \( h \) given by \( e^{-\phi} \) on \( X_{\text{open}} \), the symplectic and Chebychev potentials of \( h \) are related \([25, \text{section 10.3}] \),

\[
c_{Y_\cdot, c}[h] = 2 \mathcal{L} \left( \frac{1}{2} \phi_\Theta \right), \quad (1.2)
\]

where \( Y_\cdot \) is the standard coordinate flag \( Y_i = V(z_1, \ldots, z_i) \) on the chart \( X_0 \), and \( C \) is the standard trivialization over this chart.

Therefore, it is natural to pose the following conjecture, first explicitly stated by Reboulet \([18, \text{Theorem A}] \), that, by the discussion above, holds in the toric case.

**Conjecture 1.1.** Let \( s \mapsto h(s, \cdot) \in \mathcal{H}_A \) be a subgeodesic, that is, on each chart \( h = e^{-\phi} \) where \( \phi \) is plurisubharmonic in all \( n + 1 \) variables. Then it is a geodesic if and only if \( s \mapsto c_{Y_\cdot, c}[h(s, \cdot)] \) is affine.

In fact, one direction of Conjecture 1.1 is known. The main result of Witt Nyström \([25, \text{Theorem 6.2}] \) shows that the Aubin–Mabuchi energy \( \mathcal{E} : \mathcal{H}_A \times \mathcal{H}_A \to \mathbb{R} \) \([16, (2.3.1)] \),

\[
\mathcal{E}(h_0, h) := \frac{1}{n + 1} \sum_{j=0}^n \int_X (\phi_0 - \phi)(dd^c \phi_0)^j \wedge (dd^c \phi)^{n-j},
\]
where \( h_0 = e^{-\phi_0} \) and \( h = e^{-\phi} \) on each chart (note that this definition is independent of the choice of local trivialization of \( A \)), is already encoded in Chebyshev potentials and the Okounkov body,

\[
\mathcal{E}(h_0, h) = n! \int_{\Delta(X, A, Y)} (c_{Y, C}[h_0] - c_{Y, C}[h]) d\mu,
\]

where \( \mu \) is the standard Lebesgue measure. As first observed by Mabuchi in the smooth setting (and later generalized by Berman–Boucksom \([2, (4.1)]\)), the subgeodesic \( s \mapsto h(s, \cdot) \) is a geodesic if and only if \( s \mapsto \mathcal{E}(h(s, \cdot), h(0, \cdot)) \) is affine \([16, \text{Remark 3.3}]\). This implies that the subgeodesic \( h(s, \cdot) \) is a geodesic if and only if the integral \( \int_{\Delta(X, A, Y)} c_{Y, C}[h(s, \cdot)] \) is affine in \( s \), proving one direction of Conjecture 1.1.

From now on we consider the very special setting of the present paper, the hyperplane line bundle \( \mathcal{O}(1) \) on the complex projective space \( \mathbb{P}^n \), that is, set

\[
X = \mathbb{P}^n, \quad A = \mathcal{O}(1).
\]

Our main result, Theorem 1.2, is an explicit formula for the Chebyshev potential of a Fubini–Study metric on \( \mathbb{P}^n \). To emphasize, these metrics are not necessarily toric, and the computation becomes interesting precisely when they are not (recall (1.2) is only valid when all of \( X, Y, C, \) and \( h \) are toric, and in our case only \( X \) is assumed toric). Despite the admittedly restricted setting of (1.3), this seems to be the first explicit computation of Chebyshev potentials outside of toric metrics and the Riemann sphere \([25, \text{section 10}]\).

**Theorem 1.2.** Let \([Z_0 : \ldots : Z_n]\) be standard homogeneous coordinates on \( \mathbb{P}^n \). Fix the flag \( Y : \mathbb{P}^n = Y_0 \supseteq \cdots \supseteq Y_n = \{p\} \) where

\[
Y_i = V(Z_0, \ldots, Z_{i-1}).
\]

Choose the standard coordinate chart on \( U_n = \{Z_n \neq 0\} \) and the standard local trivialization \( C \) of the hyperplane bundle \( \mathcal{O}(1) \) over \( U_n \). Associate a Fubini–Study metric \( h_H = e^{-\phi_H} \) to any positive definite Hermitian matrix \( H \) of order \( n + 1 \) given by

\[
\phi_H := \log z^T Hz \in \mathcal{H}_{\mathcal{O}(1)},
\]

where \( z = (z_0, \ldots, z_{n-1}, 1) \). Then its Chebyshev potential is given by

\[
c_{Y, C}[h_H](\alpha) = \sum_{i=0}^{n-1} \alpha_i \log \frac{\alpha_i}{\mu_i(H)} + \left(1 - \sum_{i=0}^{n-1} \alpha_i \right) \log \frac{1 - \sum_{i=0}^{n-1} \alpha_i}{\mu_n(H)},
\]

where \( \alpha = (\alpha_0, \ldots, \alpha_{n-1}) \) lies in the interior of the simplex (2.1) \( \Delta(\mathbb{P}^n, \mathcal{O}(1), Y) \subseteq \mathbb{R}^n \), and \( \mu_i \) is defined in (2.4).

Moreover, \( c_{Y, C}[h_{H_1}] = c_{Y, C}[h_{H_2}] \) if and only if there is a lower unitriangular matrix \( L \) such that \( H_1 = L^T H_2 L \). Such \( L \) is unique.

This result is motivated by Conjecture 1.1 but is of independent interest. It has the following consequence.
Corollary 1.3. Assume (1.3). For a geodesic \( t \mapsto h_{H(t)} \in H_{\mathcal{O}(1)} \), \( t \mapsto c_{Y,\mathbb{C}}[h_{H(t)}] \) is affine if and only if there is a lower triangular matrix \( L \in GL(n+1,\mathbb{C}) \) with positive diagonal entries and a real diagonal matrix \( K \) such that

\[
H(t) = L^T e^{tK} L.
\]

Such decomposition is unique.

An interpretation of Corollary 1.3 is discussed in section 5.

Next, we record the following, probably known fact, for which we could not find a reference. It describes geodesics (consisting entirely) of Fubini–Study metrics. Proposition 1.4 can be interpreted as saying that geodesics of Fubini–Study metrics are quite rare in general (indeed, even between Fubini–Study metrics on a general projective variety, the geodesics do not remain in the space of Fubini–Study metrics in general; this is quite specific to projective space for which \( H_{\mathcal{O}(1)} \) is totally geodesic because it coincides with the subspace of Kähler–Einstein metrics [16, Proposition 2.6.1]).

Proposition 1.4. The segment \( s \mapsto h_{H(s)} \in H_{\mathcal{O}(1)} \) is a geodesic if and only if there is an \( A \in GL(n+1,\mathbb{C}) \) and a real diagonal matrix \( D \) such that

\[
H(s) = A^T e^{sD} A,
\]

where \( h_{P(s)} \) is defined in Subsection 2.3.

Combining the previous two results, one obtains explicit counterexamples to Conjecture 1.1 in all dimensions. Perhaps the simplest such is:

Corollary 1.5. Consider the hyperplane bundle \( \mathcal{O}(1) \) over \( \mathbb{P}^1 \). There exists a geodesic \( s \mapsto h(s, \cdot) \in H_{\mathcal{O}(1)} \), such that \( s \mapsto c_{Y,\mathbb{C}}[h(s, \cdot)] \) is not affine.

Proof. Consider

\[
H(s) := \begin{pmatrix}
\cosh s & \sinh s \\
\sinh s & \cosh s
\end{pmatrix} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{pmatrix} \begin{pmatrix}
e^s \\
e^{-s}
\end{pmatrix} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{pmatrix}.
\]

By Proposition 1.4, \( h_{H(s)} \) is a geodesic. However, by (2.4),

\[
\mu_0(H(s)) = \frac{1}{\cosh s}, \quad \mu_1(H(s)) = \cosh s.
\]

Therefore, by Theorem 1.2, \( s \mapsto c_{Y,\mathbb{C}}[h_{H(s)}] \) is not affine.

After completing this work and informing Reboulet of our results, he kindly informed us of an independent and beautiful approach to Conjecture 1.1 using completely different ideas [19] (although without counterexamples to Conjecture 1.1 nor explicit computation of the Chebyshev transform).
Organization

In Subsection 2.1, we recall definitions of the Okounkov body and the Chebyshev potential. As an application, in Subsection 2.3, we compute Chebyshev potentials of Fubini–Study metrics and prove Theorem 1.2. In Section 3, we give the definition of a geodesic. Then we use quantization to find geodesics in the space of Fubini–Study metrics and prove Proposition 1.4. In Section 4, we classify all geodesics whose Chebyshev potentials are affine in $t$, and prove Corollary 1.3. We conclude in Section 5 by discussing possible further directions and an alternative proof of Theorem 1.2 based on reduction to the toric case.

2 CHEBYSHEV POTENTIALS OF FUBINI–STUDY METRICS ON THE HYPERPLANE BUNDLE

2.1 Background on Chebyshev potentials

This section gives some background on Chebyshev potentials.

First recall the definition of the Okounkov body of an ample line bundle with respect to an admissible flag [15, section 1].

**Definition 2.1.** Let $L \to X$ be an ample line bundle, and $Y_\cdot : X = Y_0 \supseteq \cdots \supseteq Y_n = \{p\}$ an admissible flag on $X$, that is, each $Y_i$ is a submanifold of codimension $i$. Choose a local trivialization $C$ on a chart $U$ around $p$ compatible with $Y_\cdot$ (consisting of $z_1, \ldots, z_n$ and $e$, see (1.1)). A section $s \in H^0(X, mL)$ has a Taylor expansion

$$\frac{s}{e} \big|_U = \sum_{\alpha} k_{\alpha} z^\alpha$$
onumber

on $U$. If $s \neq 0$, define

$$\nu(s) := \min \{ \alpha \mid k_{\alpha} \neq 0 \} \in \mathbb{N}^n,$$

with respect to the lexicographic order, that is, $\alpha < \beta$ if $\alpha_i < \beta_i$ for some $i$ and $\alpha_j = \beta_j$ for $j < i$. Let

$$\Delta_m(L) := \{ \nu(s) \mid s \in H^0(X, mL) \setminus \{0\} \} \subseteq \mathbb{N}^n.$$

Define the *Okounkov body* associated to $(X, Y_\cdot, L)$

$$\Delta(X, Y_\cdot, L) = \Delta(L) := \bigcup_m \frac{1}{m} \Delta_m(L) \subseteq \mathbb{R}^n.$$

The following proposition is proved in [15, Lemma 1.4].

**Proposition 2.2.** The set $\Delta_m(L)$ has exactly $\dim H^0(X, mL)$ points.

**Example 2.3.** Consider $X = \mathbb{P}^n$ with the standard admissible flag $\mathbb{P}^n = Y_0 \supseteq \cdots \supseteq Y_n$ where $Y_i = V(Z_0, \ldots, Z_{i-1})$. The local frame around $p = [0 : \cdots : 0 : 1]$ can be chosen to be the standard open set $U_n = \{Z_n \neq 0\}$ with the standard coordinate chart $z_i = \frac{Z_i}{Z_n}$, where $0 \leq i \leq n - 1$. 


Choose the standard local trivialization of the hyperplane bundle $\mathcal{O}(1) \to \mathbb{P}^n$ over $U_n$, that is, a map $C : \pi^{-1}(U_n) \to U_n \times \mathbb{C}$ such that for a section $s = z_i$, $C(s(z_0, \ldots, z_{n-1})) = (z_0, \ldots, z_{n-1}, z_i)$. The sections of the line bundle $\mathcal{O}(m)$ are polynomials in $z_0, \ldots, z_{n-1}$ of degree at most $m$. Hence,

$$\Delta_m(\mathcal{O}(1)) = \left\{ (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}^n \left| \sum_{i=0}^{n-1} \alpha_i \leq m \right. \right\},$$

and the Okounkov body is the simplex

$$\Delta(\mathcal{O}(1)) = \left\{ (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{Q}^n_{\geq 0} \left| \sum_{i=0}^{n-1} \alpha_i \leq 1 \right. \right\} = \left\{ (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{R}^n_{\geq 0} \left| \sum_{i=0}^{n-1} \alpha_i \leq 1 \right. \right\}.$$

The definition of Chebyshev potential involves certain sections called Chebyshev sections [25, sections 5 and 7].

**Definition 2.4.** Notation as in Definition 2.1. The leading term of a section $s = \sum_{\alpha \geq \nu(s)} k_{\alpha} z^\alpha$ is defined to be

$$\ell(s) := k_{\nu(s)} z^{\nu(s)}.$$

For $\alpha \in \frac{1}{m} \Delta_m(L)$, define an affine space

$$A_{m,\alpha} := \left\{ s \in H^0(X, mL) \left| \ell(s) = z^m \alpha \right. \right\}.$$

**Definition 2.5.** Notation as in Definition 2.1. Let $h$ be a Hermitian metric on $L$, and $\mu$ a smooth volume form on $X$. The linear space $H^0(X, mL)$ with Hermitian metric $h^m$ induced from $h$ admits a Hermitian inner product

$$Hilb_m(h, \mu)(s, t) := \int_X h^m(s, t) d\mu.$$

Each affine subspace $A_{m,\alpha}$ has a unique minimizer $Ch_{m,\alpha}$ of the norm function [5, Corollary 5.4]. Such a minimizer is called a Chebyshev section for $(X, Y, L, C, h, \mu, \alpha, m)$.

**Remark 2.6.** The definition of $A_{m,\alpha}$ depends on the choice of the chart and the local trivialization. Indeed, monic sections are not preserved by the scaling of the chart or the local trivialization.

The following definition of Chebyshev potential can be found in [25, Definition 5.5].

**Definition 2.7.** Notation as in Definition 2.1. Fix a smooth volume form $\mu$ and let $Ch_{m,\alpha}$ denote Chebyshev sections as in Definition 2.5. Given a point $\alpha \in \text{int} \Delta(L)$, pick a sequence $\alpha_{m_k} \in \frac{1}{m_k} \Delta_m(L)$ that converges to $\alpha$, where $m_k$ is any strictly increasing sequence of natural
numbers. Define the Chebyshev potential

\[ c_{Y, C}[h](\alpha) := \lim_{k \to \infty} \frac{1}{m_k} \log \left\| C_{h_{m_k, \alpha_{m_k}}} \right\|_{h, \mu}^2. \]

This limit always exists, and is independent of the choice of \( \alpha_{m_k} \) and \( \mu \) [25, Proposition 7.3].

We emphasize that Chebyshev potential does depend on the choice of the flag \( Y_* : X = Y_0 \supseteq \cdots \supseteq Y_n = \{p\} \) and the local trivialization \( C \).

For the proof of Theorem 1.2 it will be necessary to know \( c_{Y, C}[h](\alpha) \) is continuous in \( \alpha \). This follows from [25, Proposition 7.3].

**Proposition 2.8.** Chebyshev potential is convex on \( \text{int} \Delta(X, Y_*, L) \).

### 2.2 A Gram–Schmidt criterion for Chebyshev sections

**Proposition 2.9.** For each \( m \), the collection of all Chebyshev sections \( \{Ch_{m, \alpha}\}_{\alpha \in \frac{1}{m} \Delta_m(L)} \) forms an orthogonal basis for \( H^0(X, mL) \). On the other hand, if \( \{s_\alpha\}_{\alpha \in \frac{1}{m} \Delta_m(L)} \) is an orthogonal basis for \( H^0(X, mL) \) with \( \epsilon'(s_{\alpha}) = z^{m\alpha} \), then \( s_{\alpha} \) is the Chebyshev section in \( A_{m, \alpha} \).

**Proof.** Pick distinct \( \alpha_1, \alpha_2 \in \frac{1}{m} \Delta_m(L) \). One may assume \( \alpha_1 < \alpha_2 \) (recall Definition 2.1), so \( Ch_{m, \alpha_2} \) has higher order than \( \alpha_1 \), and

\[
Ch_{m, \alpha_1} = \frac{Hilb_m(h, \mu) \left( Ch_{m, \alpha_1}, Ch_{m, \alpha_2} \right) Ch_{m, \alpha_2}}{\left\| Ch_{m, \alpha_2} \right\|_{h, \mu}^2} \in A_{m, \alpha_1}.
\]

As \( Ch_{m, \alpha_1} \) is a Chebyshev section,

\[
\left\| Ch_{m, \alpha_1} \right\|_{h, \mu}^2 \leq \left\| Ch_{m, \alpha_1} - \frac{Hilb_m(h, \mu) \left( Ch_{m, \alpha_1}, Ch_{m, \alpha_2} \right) Ch_{m, \alpha_2}}{\left\| Ch_{m, \alpha_2} \right\|_{h, \mu}^2} \right\|_{h, \mu}^2
\]

\[
= \left\| Ch_{m, \alpha_1} \right\|_{h, \mu}^2 - \left\| \frac{Hilb_m(h, \mu) \left( Ch_{m, \alpha_1}, Ch_{m, \alpha_2} \right)}{\left\| Ch_{m, \alpha_2} \right\|_{h, \mu}^2} \right\|_{h, \mu}^2.
\]

This shows that \( Ch_{m, \alpha_1} \) and \( Ch_{m, \alpha_2} \) are orthogonal. By Proposition 2.2, it follows that \( \{Ch_{m, \alpha}\}_{\alpha \in \frac{1}{m} \Delta_m(L)} \) is an orthogonal basis for \( H^0(X, mL) \).

Let \( \{s_\alpha\}_{\alpha \in \frac{1}{m} \Delta_m(L)} \) be an orthogonal basis for \( H^0(X, mL) \), where \( \epsilon'(s_{\alpha}) = z^{m\alpha} \). For any \( \alpha_0 \), one can write

\[
Ch_{m, \alpha_0} = \sum_{\alpha \in \frac{1}{m} \Delta_m(L)} k_{\alpha} s_{\alpha}.
\]
Define
\[ \tilde{\alpha} := \min\{\alpha \mid k_{\alpha} \neq 0\}. \]

Then
\[ z^{m\tilde{\alpha}_0} = \ell \left( Ch_{m,\tilde{\alpha}_0} \right) = \ell \left( \sum_{\alpha \in \frac{1}{m} \Delta_m(L)} k_{\alpha} s_{\alpha} \right) = k_{\tilde{\alpha}} z^{m\tilde{\alpha}}. \]

Therefore, \( \tilde{\alpha} = \alpha_0 \) and \( k_{\alpha_0} = 1 \). That is,
\[ Ch_{m,\alpha_0} = s_{\alpha_0} + \sum_{\alpha > \alpha_0} k_{\alpha} s_{\alpha}. \quad (2.2) \]

As \( Ch_{m,\alpha_0} \) is a Chebyshev section,
\[ \| Ch_{m,\alpha_0} \|_{h,\mu} \leq \| s_{\alpha_0} \|_{h,\mu}. \]

On the other hand, \( \{s_{\alpha}\} \) is an orthogonal basis. Thus,
\[ \| Ch_{m,\alpha_0} \|_{h,\mu}^2 = \| s_{\alpha_0} + \sum_{\alpha > \alpha_0} k_{\alpha} s_{\alpha} \|_{h,\mu}^2 \]
\[ = \| s_{\alpha_0} \|_{h,\mu}^2 + \sum_{\alpha > \alpha_0} |k_{\alpha}|^2 \| s_{\alpha} \|_{h,\mu}^2 \]
\[ \geq \| Ch_{m,\alpha_0} \|_{h,\mu}^2 + \sum_{\alpha > \alpha_0} |k_{\alpha}|^2 \| s_{\alpha} \|_{h,\mu}^2. \]

It follows that \( k_{\alpha} = 0 \) for \( \alpha > \alpha_0 \) and (2.2) becomes \( Ch_{m,\alpha_0} = s_{\alpha_0} \).

\[ \square \]

Remark 2.10. The direction that Chebyshev sections form an orthogonal basis is shown by Witt Nyström [25], between his Definitions 7.1 and 7.2.

2.3  Computing Chebyshev sections and potential

This section is devoted to the computation of Chebyshev potential of Fubini–Study metrics on \( \mathbb{P}^n \).

The following setup from Example 2.3 will be used throughout this section.

Fix a set of homogeneous coordinates
\[ [Z_0 : \ldots : Z_n] \]
on \( \mathbb{P}^n \) and an admissible flag \( Y_\ast : \mathbb{P}^n = Y_0 \supseteq \ldots \supseteq Y_n = \{p\} \), where
\[ Y_i := V(Z_0, \ldots, Z_{i-1}). \]
Choose the standard coordinate chart

\[ z_i := \frac{Z_i}{Z_n} \]

valid on the chart \( U_n = \{ Z_n \neq 0 \} \) around \( p \), and the standard local trivialization \( C \) of the hyperplane bundle \( \mathcal{O}(1) \) over \( U_n \). By Example 2.3, the Okounkov body is the simplex

\[ \Delta(\mathcal{O}(1)) = \left\{ (\alpha_0, \ldots, \alpha_{n-1}) \in [0, +\infty)^n \left| \sum_{i=0}^{n-1} \alpha_i \leq 1 \right. \right\}. \]

Define \( P_n(\mathbb{C}) \) to be the space of positive definite Hermitian matrices of order \( n \). For any \( H \in P_{n+1}(\mathbb{C}) \), define a Fubini–Study metric \( h_H = e^{-\phi_H} \) with

\[ \phi_H := \log z^\top Hz, \]

where \( z = (z_0, \ldots, z_{n-1}, 1)^\top \).

**Definition 2.11.** Let

\[
H = \begin{pmatrix}
    h_{00} & \cdots & h_{0n} \\
    \vdots & \ddots & \vdots \\
    h_{n0} & \cdots & h_{nn}
\end{pmatrix}
\]

be a positive definite Hermitian matrix. For \( 0 \leq i \leq n \) set

\[
\det_i(H) := \det \begin{pmatrix}
    h_{ii} & \cdots & h_{in} \\
    \vdots & \ddots & \vdots \\
    h_{ni} & \cdots & h_{nn}
\end{pmatrix}. \tag{2.3}
\]

For \( i > n \) set \( \det_i(H) := 1 \). As \( H \) is positive definite, \( \det_i(H) > 0 \) for any \( i \geq 0 \) \cite[chapter X, Theorem 3]{12}. Define

\[
\mu_i(H) := \frac{\det_i(H)}{\det_{i+1}(H)}. \tag{2.4}
\]

This is an analogue of the \( i \)th eigenvalue. In fact, if \( H \) is diagonal, then \( \mu_i(H) = h_{ii} \). To compute \( \mu_i(H) \) for general \( H \), Claims 2.14 and 2.15 will be used in Lemma 4.1.

Recall the notion of a unitriangular matrix \cite[p. 30]{21}.

**Definition 2.12.** A matrix is unitriangular if it is triangular and its diagonal entries are 1.

We use the following matrix decomposition \cite[Theorem 4.1.2]{24}.

**Theorem 2.13.** Let \( H \) be positive definite Hermitian. Then there is a unique lower unitriangular matrix \( L \) such that \( H = L^\top DL \), where \( D \) is diagonal.
Claim 2.14. Let $H$ be positive definite Hermitian. For any lower triangular matrix $L$, $\mu_i(L^\top HL) = \mu_i(H) \cdot |l_{ii}|^2$. In particular, if $L$ is unitriangular, then

$$\mu_i(L^\top HL) = \mu_i(H).$$  \hspace{1cm} (2.5)

Moreover, if $\mu_i(H_1) = \mu_i(H_2)$ for any $i$, then there is a unique lower unitriangular matrix $L$ such that $H_1 = L^\top H_2 L$.

Proof. To compute $\mu_i$, subdivide an $(n + 1) \times (n + 1)$ matrix into submatrices of sizes $i \times i$, $i \times (n + 1 - i)$, $(n + 1 - i) \times i$ and $(n + 1 - i) \times (n + 1 - i)$,

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \quad L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}.$$

Note that $\det(H_{22}) = \det_i(H)$, $\det(L_{22}) = \prod_{j=i}^n |l_{jj}|$. By definition,

$$\det_i(L^\top HL) = \det_i \left( \begin{pmatrix} L_{11}^\top & L_{21}^\top \\ 0 & L_{22}^\top \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \right)$$

$$= \det \left( \begin{pmatrix} 0 & L_{22}^\top \\ L_{21}^\top & H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} 0 \\ L_{22} \end{pmatrix} \right)$$

$$= \det(L_{22}^\top H_{22} L_{22})$$

$$= \det_i(H) \cdot \prod_{j=i}^n |l_{jj}|^2,$$

and similarly for $i + 1$. Dividing concludes the proof of (2.5).

Assume $\mu_i(H_1) = \mu_i(H_2)$ for any $i$. By Theorem 2.13, we have the decomposition $H_1 = L_1^\top D L_1$, where $L_1$ is lower unitriangular, and $D = \text{diag}(d_{00}, \ldots, d_{nn})$. Notice that $\mu_i(H_1) = \mu_i(D) = d_{ii}$. As $\mu_i(H_1) = \mu_i(H_2)$, we have the decomposition $H_2 = L_2^\top D L_2$ for some lower unitriangular matrix $L_2$. Now, $L := L_2^{-1} L_1$ is a lower unitriangular matrix, and

$$H_1 = L_1^\top D L_1 = L_1^\top \left( L_2^\top \right)^{-1} H_2 L_2^{-1} L_1 = L^\top H_2 L.$$

For the uniqueness of $L$, assume there is a lower unitriangular matrix $L'$ such that $H_1 = L'^\top H_2 L'$. We have

$$H_1 = L'^\top H_2 L' = L_2^\top L'^\top D L_2 L'.$$

By uniqueness in Theorem 2.13, $L_1 = L_2 L'$. Therefore, $L' = L_2^{-1} L_1 = L$. \hfill \Box

Claim 2.15. Let $H \in \mathcal{P}_n(\mathbb{C})$ and $a > 0$. For $0 \leq i \leq n - 1$,

$$\mu_i \left( \begin{pmatrix} H \\ a \end{pmatrix} \right) = \mu_i(H).$$
Proof. By (2.3), for $0 \leq i \leq n$, one has $\det_i \left( \begin{array}{c} H \\ a \end{array} \right) = a \cdot \det_i(H)$. Using (2.4) we conclude. 

We work on the chart $\{Z_n \neq 0\}$. The space $H^0(\mathbb{P}^n, \mathcal{O}(1))$ has a canonical basis

$$s_0 = z_0, \ldots, s_{n-1} = z_{n-1}, s_n = 1.$$ 

By Theorem 2.13, for $H \in \mathcal{P}_{n+1}(\mathbb{C})$, one can write $H = L^T DL$, where $L$ is lower unitriangular, and $D$ is real diagonal. Indeed, $D = \text{diag}\{\mu_0(H), \ldots, \mu_n(H)\}$ by Claim 2.14.

To simplify the computation, use a new homogeneous coordinate on $\mathbb{P}^n$

$$W := LZ.$$

The standard coordinate chart on $\{W_n \neq 0\}$ is $w_i = \frac{w_i}{W_n}$ with

$$w := (w_0, \ldots, w_{n-1}, 1)^T = \frac{Z_n}{W_n}Lz.$$ (2.6)

Fix a smooth volume form

$$\mu = \left( \frac{-1}{2\pi} \right)^n dw_0 \wedge d\bar{w}_0 \wedge \ldots \wedge dw_{n-1} \wedge d\bar{w}_{n-1} \left( \bar{w}^T D w \right)^{n+1}.$$

For a multi-index $I = (I_0, \ldots, I_n)$ set

$$|I| := I_0 + \cdots + I_n,$$

$$I! := I_0! \cdots I_n!.$$

**Proposition 2.16.** Define the sections of $H^0(X, \mathcal{O}(1))$

$$t_i := s_i + \sum_{j=0}^{i-1} l_{ij}s_j,$$

or in matrix form,

$$t := Ls.$$ (2.7)

Then for any multi-index $I$, the section $t^I \in H^0(X, \mathcal{O}(|I|))$ is the Chebyshev section for $(\mathbb{P}^n, Y., \mathcal{O}(1), C, h_H, \mu, \frac{1}{|I|}(I_0, \ldots, I_{n-1}), |I|)$.

**Proof.** For multi-indices $I$ and $J$ with $|I| = |J| = m$, $t^I$ and $t^J$ are sections of $\mathcal{O}(m)$. Note that by (2.6) and (2.7),

$$e^{-\phi_H} = LZ^T DLz = \left[ \frac{W_n}{Z_n} \right]^2 \overline{w}^T Dw,$$ (2.8)
and
\[ t = Lz = \frac{W_z}{Z_n} w. \quad (2.9) \]

So,
\[
\text{Hilb}_m(h_H, \mu)(t^I, t^J) = \left( \frac{-1}{2\pi} \right)^n \int_{V_n} \frac{w^I \bar{w}^J}{(\overline{w}^T Dw)^m} \cdot \frac{dw_0 \wedge d\bar{w}_0 \wedge \ldots \wedge dw_{n-1} \wedge d\bar{w}_{n-1}}{(\overline{w}^T Dw)^{n+1}} \]

\[ = \left( \frac{-1}{2\pi} \right)^n \int_{V_n} \prod_{j=0}^{n-1} \frac{w_j^I \bar{w}_j^J \cdot dw_0 \wedge d\bar{w}_0 \wedge \ldots \wedge dw_{n-1} \wedge d\bar{w}_{n-1}}{\left( \mu_n(H) + \sum_{j=0}^{n-1} \mu_j(H)|w_j|^2 \right)^{m+n+1}}. \]

Using polar coordinates,
\[
\text{Hilb}_m(h_H, \mu)(t^I, t^J) = \frac{1}{\pi^n} \int_{\mathbb{R}^n_+} \prod_{j=0}^{n-1} \frac{r_j^{I_j+J_j+1} dr_0 \ldots dr_{n-1}}{\left( \mu_n(H) + \sum_{j=0}^{n-1} \mu_j(H)r_j^2 \right)^{m+n+1}} \int_0^{2\pi} e^{i(I_j-J_j)\theta_j} d\theta_j \]

\[ = 2^n \delta_{IJ} \int_{\mathbb{R}^n_+} \prod_{j=0}^{n-1} \frac{r_j^{2I_j+1} dr_0 \ldots dr_{n-1}}{\left( \mu_n(H) + \sum_{j=0}^{n-1} \mu_j(H)r_j^2 \right)^{m+n+1}}. \]

In particular, \( t^I \) and \( t^J \) are orthogonal if \( I \neq J \). Notice that \( t^I \) is monic with \( \varepsilon(t^I) = z_0^I \ldots z_{n-1}^I \). These two facts imply, by Proposition 2.9, that \( t^I \) is Chebyshev section \( CH_{m, \frac{1}{m} \Delta_m(O(1))} \), or equivalently, for \( \alpha \in \frac{1}{m} \Delta_m(\mathcal{O}(1)) \),
\[
CH_{m, \alpha} = t_0^{m\alpha_0} \ldots t_{n-1}^{m\alpha_{n-1}} \cdot t_n^{m(1-\alpha_0-\ldots-\alpha_{n-1})}. \]

For \( x > 0 \), let
\[
\Gamma(x) : = \int_0^{+\infty} e^{-t} t^{x-1} dt. \]

and
\[
B(x_0, \ldots, x_n) : = \frac{\Gamma(x_0) \cdots \Gamma(x_n)}{\Gamma(x_0 + \cdots + x_n)}. \]
The following integral formula for the multivariate beta function is also needed [14, eq. 49.5].

**Lemma 2.17.** Let $k_0, \ldots, k_{n-1} \in \mathbb{R}_{>0}$ and $l > \frac{1}{2} \sum_{i=0}^{n-1} k_i$. Then

$$2^n \int_{\mathbb{R}_+^n} \prod_{j=0}^{n-1} x_j^{k_j-1} d x_0 \cdots d x_{n-1} \left(1 + \sum_{j=0}^{n-1} x_j^2\right)^l = B\left(\frac{k_0}{2}, \ldots, \frac{k_{n-1}}{2}, l - \frac{1}{2} \sum_{i=0}^{n-1} k_i\right).$$

**Proposition 2.18.** The norms of the section $t^l$ is given by

$$\|t^l\|_{h_{H,\mu}}^2 = \frac{l! \left(\prod_{j=0}^{n-1} \mu_j(H) I_{j+1}\right)}{(m+n)! \left(\prod_{j=0}^{n-1} \mu_j(H)^{I_j+1}\right)}.$$

**Proof.** To compute the integral, one can use variables $\rho_j := \sqrt{\frac{\mu_j(H)}{\mu_n(H)}} r_j$ and apply Lemma 2.17,

$$\|t^l\|_{h_{H,\mu}}^2 = 2^n \int_{\mathbb{R}_+^n} \prod_{j=0}^{n-1} \mu_n(H)^{m+n+1}\left(1 + \sum_{j=0}^{n-1} \frac{\mu_j(H)}{\mu_n(H)} r_j^2\right)^{m+n+1} \cdot d r_0 \cdots d r_{n-1} \mu_n(H)^{m+n+1}\left(1 + \sum_{j=0}^{n-1} \rho_j^2\right)^{m+n+1} \cdot d \rho_0 \cdots d \rho_{n-1}$$

$$= \frac{l! \left(\prod_{j=0}^{n-1} \mu_j(H) I_{j+1}\right)}{(m+n)! \left(\prod_{j=0}^{n-1} \mu_j(H)^{I_j+1}\right)}.$$

**Lemma 2.19.** Let $m_0, \ldots, m_n$ be positive integers and $M = m_0 + \cdots + m_n$. Then

$$\lim_{k \to \infty} \frac{1}{k} \log \frac{\prod_{j=0}^{n} (k m_j)!}{(k M)!} = \sum_{j=0}^{n} m_j \log \frac{m_j}{M}.$$
Proof. Recall Stirling’s formula
\[
\log n! = n \log n - n + \log \sqrt{2\pi n} + o(1)
\]
as \(n \to \infty\). For any positive integer \(m\),
\[
\frac{1}{k} \log (km)! = m \log km - m + \frac{1}{k} \log \sqrt{2\pi km} + o\left(\frac{1}{k}\right)
\]
as \(k \to \infty\). Therefore,
\[
\lim_{k \to \infty} \frac{1}{k} \log \left(\prod_{j=0}^{n} (km_j)! \right) = \lim_{k \to \infty} \left( \sum_{j=0}^{n} (m_j \log km_j - m_j) - \sum_{j=0}^{n} m_j \log kM + M \right)
\]
\[
= \sum_{j=0}^{n} m_j \log \frac{m_j}{M}. \quad \square
\]

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. For \(\alpha \in \text{int}(\mathcal{O}(1)) \cap \mathbb{Q}^n\), one can find \(M \in \mathbb{N}\) so that \(M\alpha \in \mathbb{N}^n\). In Definition 2.7, pick \(m_k = M_k\) and \(\alpha_{m_k} = \alpha\). By Propositions 2.16 and 2.18, and Lemma 2.19,
\[
c_{Y_\ast, C}[h_H](\alpha) = \lim_{k \to \infty} \frac{1}{Mk} \log \|C_{\mathcal{H}_{M_k,\alpha}}\|_{h_H, \mu}^2
\]
\[
= \lim_{k \to \infty} \frac{1}{Mk} \log \left( \frac{Mk \left( \prod_{j=0}^{n-1} (1 - \sum_{j=0}^{n-1} \alpha_j) \right)^{n-1}}{(Mk \mu_n(H))^{n-1} \prod_{j=0}^{n-1} \mu_j(H)^{Mk \alpha_j + 1}} \right)
\]
\[
= -\left(1 - \sum_{j=0}^{n-1} \alpha_j \right) \log \mu_n(H) - \sum_{j=0}^{n-1} \alpha_j \log \mu_j(H)
\]
\[
+ \frac{1}{Mk} \lim_{k \to \infty} \frac{1}{k} \log \left( \frac{kM \left( \prod_{j=0}^{n-1} (1 - \sum_{j=0}^{n-1} \alpha_j) \right)^{n-1}}{(kM \mu_n(H))^{n-1} \prod_{j=0}^{n-1} \mu_j(H)^{Mk \alpha_j + 1}} \right)
\]
\[
= -\left(1 - \sum_{j=0}^{n-1} \alpha_j \right) \log \mu_n(H) - \sum_{j=0}^{n-1} \alpha_j \log \mu_j(H)
\]
\[
+ \left(1 - \sum_{j=0}^{n-1} \alpha_j \right) \log \left( \prod_{j=0}^{n-1} (1 - \sum_{j=0}^{n-1} \alpha_j) \right) + \sum_{j=0}^{n-1} \alpha_j \log \alpha_j
\]
\[
= \left(1 - \sum_{j=0}^{n-1} \alpha_j \right) \log \frac{1 - \sum_{j=0}^{n-1} \alpha_j}{\mu_n(H)} + \sum_{j=0}^{n-1} \alpha_j \log \frac{\alpha_j}{\mu_j(H)}. \quad (2.10)
\]
By Proposition 2.8, $c_{Y,c}[h_{H_1}]$ is convex and hence continuous on $\text{int} \Delta(\mathcal{O}(1))$. Therefore (2.10) holds for all $\alpha \in \text{int} \Delta(\mathcal{O}(1))$.

Finally, assume $H_1 = L^T H_2 L$ for some lower unitriangular matrix $L$. By Claim 2.14, $\mu_i(H_1) = \mu_i(H_2)$ for any $i$. Therefore $c_{Y,c}[h_{H_1}] = c_{Y,c}[h_{H_2}]$ by (2.10). On the other hand, by (2.10),

$$c_{Y,c}[h_{H_1}](\alpha) - c_{Y,c}[h_{H_2}](\alpha) = \left(1 - \sum_{i=0}^{n-1} \alpha_i \right) \log \frac{\mu_n(H_2)}{\mu_n(H_1)} + \sum_{i=0}^{n-1} \alpha_i \log \frac{\mu_i(H_2)}{\mu_i(H_1)}$$

$$= \log \frac{\mu_n(H_2)}{\mu_n(H_1)} + \sum_{i=0}^{n-1} \alpha_i \left( \log \frac{\mu_i(H_2)}{\mu_i(H_1)} - \log \frac{\mu_n(H_2)}{\mu_n(H_1)} \right).$$

Assume $c_{Y,c}[h_{H_1}] = c_{Y,c}[h_{H_2}]$. Then both the coefficients and the constant should vanish, implying $\mu_i(H_1) = \mu_i(H_2)$ for any $i$. By Claim 2.14, there is a unique lower unitriangular matrix $L$ such that $H_1 = L^T H_2 L$.  

**Remark 2.20.** The notation $h_H$ depends on the choice of the coordinates $Z$, as is implied by (2.8) and (2.9). Indeed, a section $s$ of $\mathcal{O}(m)$ can be expressed as a polynomial $f(Z)$ of degree $m$. By definition, its norm at a point $p = [Z_0 : \cdots : Z_n]$ is given by

$$\|s(p)\|^2 = \left| f(Z) \right|^2 \left( Z^T H Z \right)^m.$$ 

Note that the right-hand side is well-defined because both the numerator and denominator are homogeneous of degree $2m$. Using a new choice of coordinates $W$ given by $Z = AW$, the same section $s$ is expressed as the polynomial $g(W) := f(AW)$ in $W$, and its norm at the same point $p$ is expressed as

$$\|s(p)\|^2 = \left| f(Z) \right|^2 \left( Z^T H Z \right)^m = \left| g(W) \right|^2 \left( W^T A^T H A W \right)^m.$$ 

Therefore, after switching to the new coordinates, the same metric is $h_{A^T H A}$. 

This will be used in the proof of Proposition 3.9.

### 3 MABUCHI GEODESICS OF FUBINI–STUDY METRICS ON THE HYPERPLANE BUNDLE

This section introduces Bergman geodesics and explains how to use them to solve boundary value problems of the geodesic equation. As an application, all geodesics in the spaces of Fubini–Study metrics on $\mathbb{P}^n$ are classified.

First recall the definition of a geodesic [10, Proposition 3].
**Definition 3.1.** Let \((X^n, \omega)\) be a compact Kähler manifold. Define \(S\) to be the unit strip \(\{\text{Re } s \in (0,1)\}\) and \(\pi_2\) the projection map \(S \times X \to X\). A function \(u : S \times X \to \mathbb{R}\) is called a (Mabuchi) geodesic if \(u(s, \cdot)\) is \(\omega\)-plurisubharmonic, is independent of \(\text{Im } s\), and
\[
\left( \pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} u \right)^{n+1} = 0. \tag{3.1}
\]

**Remark 3.2.** On a chart, we can write \(\omega = \sqrt{-1} \partial \bar{\partial} \phi_0\). This way we can rewrite (3.1) as
\[
\left( \sqrt{-1} \partial \bar{\partial} \phi \right)^{n+1} = 0, \tag{3.2}
\]
where \(\phi = \pi_2^* \phi_0 + u\). Equation (3.2) also defines Mabuchi geodesics in the space \(H_A\), where the metric at time \(s\) is given by \(e^{-\phi(s, \cdot)}\) on this chart. Note that this equation is independent of the choice of the chart.

The boundary value problem of the geodesic equation has [4, Theorem 1.3] a unique [7, Theorem 3, Corollary 2] continuous solution.

**Proposition 3.3.** For smooth boundary conditions \(h_0\) and \(h_1\), there is a unique continuous geodesic \(h\) connecting them, that is, \(h(i, \cdot) = h_i\), \(i = 0, 1\).

To obtain the solution, one approach is to use quantization.

**Definition 3.4.** Let \(X^n\) be a projective Kähler manifold with \(L \to X\) an ample line bundle. Given metrics \(h(0, \cdot)\) and \(h(1, \cdot)\) on \(L\) with positive curvature, one can equip \(H^0(X, mL)\) with Hermitian metrics \(H_{m,0}\) and \(H_{m,1}\) as follows,
\[
H_{m,j}(s, t) = \int_X s \bar{t} e^{-m\phi(j, \cdot)} \left( \sqrt{-1} \partial \bar{\partial} \phi(j, \cdot) \right)^n n!, \tag{3.3}
\]
for \(j = 0, 1\), where \(h(j, \cdot) = e^{-\phi(j, \cdot)}\) on each chart. Using simultaneous diagonalization, one can find orthonormal bases \(\{s_{m,j}\}\) and \(\{e^{ij}s_{m,j}\}\) for \(H_{m,0}\) and \(H_{m,1}\), respectively. Define the Bergman geodesic \(h_m = e^{-\phi_m}\) with
\[
\phi_m(t, \cdot) := \frac{1}{m} \log \sum_j e^{ij} \left| s_{m,j} \right|^2. \tag{3.4}
\]
Note that this definition is independent of the choice of the chart.

**Remark 3.5.** We emphasize that Bergman geodesics are not Mabuchi geodesics in general. Rather they come from a geodesic in the space of Hermitian norms.

The following is proved by Berndtsson [3, Theorem 6.1], improving a result of Phong–Sturm [17, Theorem 1].
Theorem 3.6. Notation as in Definition 3.4. The Mabuchi geodesic $h \in C^0([0, 1], H)$ connecting $h(0, \cdot)$ and $h(1, \cdot)$ is given by

$$h = \lim_{m \to \infty} h_m.$$

Lemma 3.7. Notation as in Theorem 1.2. For a real diagonal matrix $D = \text{diag}\{d_0, \ldots, d_n\}$, $h e^D$ is a Mabuchi geodesic.

Proof. It suffices to compute the geodesic connecting $h(0, \cdot) = e^{-\phi I}$ and $h(1, \cdot) = e^{-\phi eD}$.

First, note that the linear space $H^0(\mathbb{P}^n, O(m))$ is spanned by monomials in $s_0 = z_0, \ldots, s_{n-1} = z_{n-1}, s_n = 1$ of degree $m$. By (3.3), for multi-indices $I$ and $J$ with $|I| = |J| = m$,

$$H_{m,0}(s^I, s^J) = \int_X \frac{z^I \bar{z}^J}{\|z\|^{2m}} \cdot \frac{\sqrt{-1}}{\|z\|^{2m+2}} dz_0 \wedge d\bar{z}_0 \wedge \ldots \wedge dz_{n-1} \wedge d\bar{z}_{n-1}$$

$$= 2^n \int_{\mathbb{R}_+^n} \frac{r^{2n+1}}{\left(1 + \sum_{i=0}^{n-1} r_i^2 \right)^{n+m+1}} dr \int_{[0, 2\pi]^n} e^{i(I-J)\theta} d\theta$$

$$= (2\pi)^n \delta_{IJ} B(I_0 + 1, \ldots, I_n + 1).$$

Similarly, applying a change of variable $W_j = e^{\frac{d_j}{2}} Z_j$ one has

$$H_{m,1} \left( \prod_j \left( e^{\frac{d_j}{2}} s_j \right)^{I_j}, \prod_j \left( e^{\frac{d_j}{2}} s_j \right)^{J_j} \right) = (2\pi)^n \delta_{IJ} B(I_0 + 1, \ldots, I_n + 1).$$

Define

$$s_{m,I} := \frac{z^I}{\sqrt{(2\pi)^n B(I_0 + 1, \ldots, I_n + 1)}}.$$

Then $\{s_{m,I}\}$ and $\{ e^{\frac{1}{2} \sum_{j} d_j} s_{m,J} \}$ are orthonormal bases for $H_{m,0}$ and $H_{m,1}$, respectively. By (3.4), we compute the Bergman geodesic $h_m = e^{-\phi_m}$ where

$$\phi_m(t, z) = \frac{1}{m} \log \sum_{|I| = m} \frac{\sum_{j} I_j d_j}{(2\pi)^n B(I_0 + 1, \ldots, I_n + 1)} |z|^{2I_j}$$

$$= \frac{1}{m} \log \sum_{|I| = m} \prod_j \left( e^{d_j} |z_j|^2 \right)^{I_j} - \frac{n}{m} \log 2\pi$$

$$= \frac{1}{m} \log \sum_{|I| = m} \frac{m!}{I!} \prod_j \left( e^{d_j} |z_j|^2 \right)^{I_j} + \frac{1}{m} \log \frac{(n+m)!}{m!} - \frac{n}{m} \log 2\pi$$
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\[
\frac{1}{m} \log \left( \sum_j e^{d_j t} |z_j|^2 \right)^m + \frac{1}{m} \log \frac{(n+m)!}{m!} - \frac{n}{m} \log 2\pi 
\]

\[
= \phi e^{tD}(z) + \frac{1}{m} \log \frac{(n+m)!}{m!} - \frac{n}{m} \log 2\pi.
\]

Using Theorem 3.6,

\[
h = \lim_{m \to \infty} h_m(t, \cdot) = h e^{tD}
\]
is the desired geodesic. 

Remark 3.8. A curious by-product of the proof of Lemma 3.7 is the explicit formula (3.5) for the Kähler quantization of a Fubini–Study metric on \( \mathbb{C}(1) \). In particular, the quantization of a Fubini–Study metric on \( \mathbb{P}^n \) is the metric itself (and so is the quantization of a Fubini–Study geodesic on the level of metrics; on the level of Hermitian metrics Equation (3.5) gives the exact constant (tending to zero) by which the quantizations differ from the original geodesic).

**Proposition 3.9.** For matrices \( A \in GL(n+1, \mathbb{C}) \) and \( D \) real diagonal, \( H(t) \) is a geodesic, where

\[
H(t) := A^T e^{tD} A.
\]

**Proof.** This follows from Lemma 3.7 and Remark 2.20 by a change of variable \( W = AZ \). 

We are now ready to prove Proposition 1.4.

**Proof of Proposition 1.4.** \( \Rightarrow \): Let \( h_{H(t)} \) be a geodesic. By simultaneously diagonalizing \( H(0) \) and \( H(1) \), one can write \( H(0) = A^T A \) and \( H(1) = A^T e^{D} A \) for some \( A \in GL(n+1, \mathbb{C}) \) and \( D \) real diagonal. Define

\[
\tilde{H}(t) := A^T e^{tD} A.
\]

By Proposition 3.9, \( h_{\tilde{H}(t)} \) is a geodesic connecting \( H(0) \) and \( H(1) \). Hence, by uniqueness \( h_{\tilde{H}(t)} = h_{H(t)} \).

\( \Leftarrow \): This is Proposition 3.9. 

**4 | A COUNTEREXAMPLE**

This section proves Corollary 1.3.

**Lemma 4.1.** For a geodesic \( h_{H(t)} \), the following are equivalent.

1. \( \mu_i(H(t)) = e^{k_i t}, k_i \in \mathbb{R} \).
2. There is a lower unitriangular matrix \( L \) such that \( H(t) = L^T e^{Kt} L \), where \( K = \text{diag}\{k_0, \ldots, k_n\} \), \( k_i \in \mathbb{R} \).

Moreover, the decomposition \( H(t) = L^T e^{Kt} L \) is unique.
Proof. (1)⇒(2): One proceeds by induction on the size of $H$. When $H$ has order 1, the matrix $L$ is 1.

Suppose the statement is true for matrices of order $n$. Let $H(t)$ be of order $n + 1$ with $\mu_i(H(t)) = e^{k_i t}$. By Proposition 1.4, $H(t) = A^T e^{tD} A$, where $D = \text{diag}(d_0, ..., d_n)$. Define

$$I := \{i \mid d_i = k_n\}.$$ 

By Definition 2.11, for any $H \in P_{n+1}(\mathbb{C})$, $\mu_n(H) = h_{nn}$. Thus,

$$\mu_n(H(t)) = \mu_n(A^T e^{tD} A) = \sum_{i=0}^{n} |a_{in}|^2 e^{d_i t}.$$ 

Therefore,

$$\sum_{i=0}^{n} |a_{in}|^2 e^{d_i t} = e^{k_n t}.$$ 

As this is true for all $t$, it follows that $a_{in} = 0$ for $i \not\in I$, and

$$\sum_{i \in I} |a_{in}|^2 = 1.$$ 

In particular, $I \neq \emptyset$. By reordering the rows and columns one may assume $I = \{m, m+1, ..., n\}$. This means the vector $v_n = (a_{mn}, ..., a_{nn})^T \in \mathbb{C}^{n+1-m}$ has unit length, so it can be completed to an orthonormal basis $v_m, ..., v_n$. Define a unitary matrix

$$U := \begin{pmatrix} I_m & 0 \\ V \end{pmatrix},$$

where

$$V := \begin{pmatrix} \overline{v_m}^T \\ \vdots \\ \overline{v_n}^T \end{pmatrix}.$$ 

As $d_m = ... = d_n = k_n$, one has $e^{tD} = \overline{U}^T e^{tD} U$, and

$$H(t) = A^T e^{tD} A = A^T \overline{U}^T e^{tD} U A.$$ 

Write

$$B := UA.$$ 

Recall that $a_{in} = 0$ for $0 \leq i < m$, and the vectors $v_m, ..., v_n \in \mathbb{C}^{n+1-m}$ are orthonormal. Therefore, for $0 \leq i < m$,

$$b_{in} = a_{in} = 0;$$
for \( m \leq i < n \),

\[
b_{in} = \overrightarrow{v_i} \, v_n = 0;
\]

and for \( i < n \),

\[
b_{nn} = \overrightarrow{v_n} \, v_n = 1.
\]

That is,

\[
B = \begin{pmatrix} \overrightarrow{\tilde{B}} & 0 \\ w & 1 \end{pmatrix},
\]

where \( \overrightarrow{\tilde{B}} \in P_n(\mathbb{C}) \) and \( w \in \mathbb{C}^n \). Write

\[
\overrightarrow{D} := \text{diag}\{d_0, \ldots, d_{n-1}\}.
\]

One then has the decomposition

\[
H(t) = \begin{pmatrix} \overrightarrow{B}^T & \overrightarrow{w}^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\overrightarrow{D}t} & e_{k_n t} \\ e_k \overrightarrow{B} & w \end{pmatrix} \begin{pmatrix} \overrightarrow{\tilde{B}} & 0 \\ w & 1 \end{pmatrix}
\]

= \begin{pmatrix} I_n & \overrightarrow{w}^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overrightarrow{\tilde{B}} e^{\overrightarrow{D}t} \overrightarrow{\tilde{B}} & e_{k_n t} \\ e_k \overrightarrow{\tilde{B}} & w \end{pmatrix} \begin{pmatrix} I_n & 0 \\ w & 1 \end{pmatrix}.
\]

By Claims 2.14 and 2.15, for \( 0 \leq i \leq n - 1 \),

\[
\mu_i(H(t)) = \mu_i \left( \begin{pmatrix} \overrightarrow{\tilde{B}} & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} e^{\overrightarrow{D}t} & e_{k_n t} \\ e_k \overrightarrow{\tilde{B}} & w \end{pmatrix} \begin{pmatrix} \overrightarrow{\tilde{B}} & 0 \\ w & 1 \end{pmatrix} \right) = \mu_i \left( \begin{pmatrix} \overrightarrow{\tilde{B}} & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} \overrightarrow{\tilde{D}} & e_{k_n t} \\ e_k \overrightarrow{\tilde{B}} & w \end{pmatrix} \begin{pmatrix} \overrightarrow{\tilde{B}} & 0 \\ w & 1 \end{pmatrix} \right).
\]

By the induction hypothesis, one can write \( \overrightarrow{\tilde{B}} e^{\overrightarrow{D}t} \overrightarrow{\tilde{B}} = \overrightarrow{\tilde{L}} e^{\overrightarrow{K} t} \overrightarrow{\tilde{L}}, \) where \( \overrightarrow{\tilde{L}} \) is lower unitriangular and \( \overrightarrow{\tilde{K}} \) is real diagonal. Then,

\[
H(t) = \begin{pmatrix} I_n & \overrightarrow{w}^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overrightarrow{\tilde{L}} e^{\overrightarrow{K} t} \overrightarrow{\tilde{L}} & e_{k_n t} \\ e_k \overrightarrow{\tilde{L}} & w \end{pmatrix} \begin{pmatrix} I_n & 0 \\ w & 1 \end{pmatrix}
\]

= \begin{pmatrix} \overrightarrow{\tilde{L}} e^{\overrightarrow{K} t} \overrightarrow{\tilde{L}} & e_{k_n t} \\ e_k \overrightarrow{\tilde{L}} & w \end{pmatrix} \begin{pmatrix} I_n & 0 \\ w & 1 \end{pmatrix}
\]

= \begin{pmatrix} \overrightarrow{\tilde{L}}^T & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} e^{\overrightarrow{K} t} & e_{k_n t} \\ e_k \overrightarrow{\tilde{L}} & w \end{pmatrix} \begin{pmatrix} \overrightarrow{\tilde{L}} & 0 \\ w & 1 \end{pmatrix}
\]

= \overrightarrow{\tilde{L}} e^{\overrightarrow{K} t} \overrightarrow{\tilde{L}},
where \[ L = \begin{pmatrix} \tilde{L} & 0 \\ w & 1 \end{pmatrix} \]
is lower unitriangular and \( K = \text{diag}\{\tilde{K}, k_n\} \) is real diagonal.

(2) \( \Rightarrow \) (1): This follows directly from Claim 2.14.

For uniqueness, first notice that \( K \) is determined by \( H(t) \) under the relation \( \mu_i(H(t)) = e^{k_i t} \).

Suppose \( L^T e^{k_i L} = L'^T e^{k_i L'} \) for some lower unitriangular matrices \( L \) and \( L' \). That is, \( L'L^{-1} = e^{-k_i L} e^{k_i L'} \). As \( L' L^{-1} \) is lower unitriangular and \( e^{-k_i L} e^{k_i L'} \) is upper triangular, they are equal only when they both are identity matrices, which means \( L = L' \).

\( \square \)

**Proposition 4.2.** For a geodesic \( h_{H(t)} \), the following are equivalent.

1. \( \mu_i(H(t)) = a_i e^{k_i t} \), where \( a_i > 0 \) and \( k_i \in \mathbb{R} \).
2. There is a lower triangular matrix \( L \) such that \( H(t) = L^T e^{k_i L} \), where \( l_{ii} = \sqrt{a_i} \) and \( K = \text{diag}\{k_0, \ldots, k_n\}, k_i \in \mathbb{R} \).

Moreover, such a decomposition is unique.

**Proof.** Let \( H' = E^T HE \), where \( E = \text{diag}\{ \frac{1}{\sqrt{a_0}}, \ldots, \frac{1}{\sqrt{a_n}} \} \), and apply Lemma 4.1 to \( H' \).

\( \square \)

We are now ready to prove Corollary 1.3.

**Proof of Corollary 1.3.** By Theorem 1.2, the Chebyshev potential \( c_{Y.C}[h_{H(t)}] \) is affine if and only if \( \log \mu_i(H(t)) \) is affine for any \( i \). Using Proposition 4.2, we conclude.

\( \square \)

5 | FURTHER DIRECTIONS AND AN ALTERNATIVE PROOF OF THEOREM 1.2

One may wonder to what extent the results of this article generalize. We discuss two such possible directions and their associated caveats, and then turn to give an alternative proof of Theorem 1.2 based on reduction to the toric case.

First, one might consider arbitrary Mabuchi geodesics instead of Fubini–Study geodesics (by which we mean Mabuchi geodesics consisting entirely of Fubini–Study metrics, see section 3). By the Berndtsson–Phong–Sturm Theorem 3.6, to study the Chebyshev transform of an arbitrary Mabuchi geodesic between two continuous metrics on \( \mathbb{P}^n \), it suffices instead to work on a sequence, indexed by \( m \), of Bergman geodesics connecting two Bergman metrics induced by \( H^0(\mathbb{P}^n, \mathcal{O}(m)) \). Additionally, the Bergman geodesic connecting each such pair has a seemingly simple expression in terms of a matrix exponential \( m \log s^T e^{iD}s \) with \( D \) diagonal and \( s = (s_1, \ldots, s_{N_m}) \) (with \( N_m = \dim H^0(\mathbb{P}^n, \mathcal{O}(m)) = O(m^n) \)) a basis of the space of polynomials of degree at most \( m \) \([3, 17]\). Thus, one could be tempted to believe the Chebyshev transform could be easily computed. However, in general, even after changing coordinates by a transformation in \( PGL(n + 1) \), there is no reason the \( s_i \) should be monomials. If they were, they would coincide with the Chebyshev sections (by Proposition 2.9) with respect to a torus invariant volume form. In general, it seems challenging to find the Chebyshev sections. Even if that were possible, there
is no obvious relation between the orthonormal bases of sections of $H^0(\mathbb{P}^n, O(m))$ for different values of $m$.

In a different direction, one might wonder if our results can be extended to describe the Chebyshev transform along geodesics consisting of Kähler–Einstein metrics on manifolds other than $\mathbb{P}^n$. Indeed, according to Mabuchi [16, Proposition 2.6.1] and Bando–Mabuchi [1, Theorem B] the space of Kähler–Einstein metrics is connected and totally geodesic submanifold of the space of Kähler metrics, and, further, it is isometric to a finite-dimensional symmetric space of the form $\text{Aut}(X)/\text{Iso}(X, \omega)$ where $\omega$ is a Kähler–Einstein form. The geodesics of such symmetric spaces are completely understood. It is tempting to conjecture that some of our results should extend to this more general setting. However, there are several potential pitfalls. First, one would need to extend the Bando–Mabuchi results to the case of Kähler potentials and not just forms (i.e., potentials mod $\mathbb{R}$). This can probably be done following ideas in Darvas–Rubinstein (see, e.g., [8, section 5.2]). Second, and more crucially, one would need to compute Chebyshev sections, even with respect to a conveniently chosen basis of anticanonical sections (assuming, say, that $-K_X$ is very ample).

Finally, we point out that one of key observations in the proof of Theorem 1.2 is that (in our simple setting of $\mathbb{P}^n$ with the standard flag) the Chebyshev sections do not change along the geodesic if and only if the associated geodesic in the symmetric space of positive Hermitian matrices is of the form

$$H(t) = L^T D(t)L,$$

where $L$ is a lower unitriangular matrix independent of $t$, and $D(t)$ is diagonal with positive diagonal entries. This essentially means that while the curve of metrics is not a curve of toric metrics, they are all toric up to a lower unitriangular transformation (by which we mean that after a linear change of coordinates by such a matrix, the metrics become toric in the new coordinates), precisely the transformations that preserve the flag, and hence the Chebyshev sections as well (“unit” in “unitriangular” is needed to preserve the monic leading term, see Remark 2.6). Note that as they are toric metrics in the new coordinates, as mentioned in the introduction, one can also use Legendre transform to compute the Chebyshev potential, by Witt Nyström’s result (1.2). Let us now explain this idea in detail.

In fact, for a metric $h = e^{-\phi}$ with

$$\phi = \log z^T Hz,$$

one can write $H = L^T DL$, where $L$ is lower unitriangular and $D$ is positive diagonal with diagonal element (recall (2.4)) $d_j = \mu_j(H)$. Recall Remark 2.20. Under the new coordinates $w = (w_0, \ldots, w_{n-1}, 1)^T$ given by (2.6), the new expression

$$\phi = \log w^TDw = \log \left( \sum_{j=0}^{n-1} \mu_j(H)|w_j|^2 + \mu_n(H) \right)$$

is torus-invariant with respect to the toric structure corresponding to the (new) $w$ coordinates. After switching to the real coordinates (recall the notation from section 1)

$$u_j = \log |w_j|,$$
we get

$$\phi_\Theta(u) = \log \left( \sum_{j=0}^{n-1} \mu_j(H)e^{2u_j} + \mu_n(H) \right).$$

Combining (1.2) and the definition of the Legendre transform [20, p. 104],

$$c_{Y\cdot C}[h](\alpha) = 2\mathcal{L}\left( \frac{1}{2} \phi_\Theta \right)(\alpha) = \sup_u \left( 2 \sum_{j=0}^{n-1} \alpha_j u_j - \log \left( \sum_{j=0}^{n-1} \mu_j(H)e^{2u_j} + \mu_n(H) \right) \right). \quad (5.1)$$

By Jensen’s inequality applied to the concave function $$f(x) = \log x$$, for any $$x_0, \ldots, x_n > 0$$,

$$\log \left( \sum_{j=0}^{n-1} \alpha_j x_j + \left( 1 - \sum_{j=0}^{n-1} \alpha_j \right) x_n \right) \leq \sum_{j=0}^{n-1} \alpha_j \log x_j + \left( 1 - \sum_{j=0}^{n-1} \alpha_j \right) \log x_n,$$

with equality if and only if $$x_0 = \cdots = x_n$$. In particular, by taking

$$\begin{cases} x_i = \frac{\mu_i(H)}{\alpha_i} e^{2u_i}, & i = 0, \ldots, n-1, \\ x_n = \frac{\mu_n(H)}{1 - \sum_{j=0}^{n-1} \alpha_j}, \end{cases}$$

we get an inequality relating the terms in (5.1),

$$\log \left( \sum_{j=0}^{n-1} \mu_j(H)e^{2u_j} + \mu_n(H) \right) \leq 2 \sum_{j=0}^{n-1} \alpha_j u_j - \left( \sum_{j=0}^{n-1} \alpha_j \log \frac{\alpha_j}{\mu_j(H)} + \left( 1 - \sum_{j=0}^{n-1} \alpha_j \right) \log \frac{1 - \sum_{j=0}^{n-1} \alpha_j}{\mu_n(H)} \right),$$

with equality if and only if $$x_0 = \cdots = x_n$$, that is, if for each $$i = 0, \ldots, n-1$$,

$$u_i = \frac{1}{2} \log \frac{\alpha_i}{\mu_i(H)} \frac{\mu_n(H)}{1 - \sum_{j=0}^{n-1} \alpha_j}.$$

Notice that this inequality gives the supremum in (5.1) as well as when the supremum is achieved. As a result,

$$c_{Y\cdot C}[h](\alpha) = \sum_{j=0}^{n-1} \alpha_j \log \frac{\alpha_j}{\mu_j(H)} + \left( 1 - \sum_{j=0}^{n-1} \alpha_j \right) \log \frac{1 - \sum_{j=0}^{n-1} \alpha_j}{\mu_n(H)},$$

giving an alternative proof of Theorem 1.2.
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