The Classical and Quantum Mechanics of Lazy Baker Maps

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Abstract

We introduce and study the classical and quantum mechanics of certain non hyperbolic maps on the unit square. These maps are modifications of the usual baker’s map and their behaviour ranges from chaotic motion on the whole measure to chaos on a set of measure zero. Thus we have called these maps “lazy baker maps.” The aim of introducing these maps is to provide the simplest models of systems with a mixed phase space, in which there are both regular and chaotic motions. We find that despite the obviously contrived nature of these maps they provide a good model for the study of the quantum mechanics of such systems. We notice the effect of a classically chaotic fractal set of measure zero on the corresponding quantum maps, which leads to a transition in the spectral statistics. Some periodic orbits belonging to this fractal set are seen to scar several eigenfunctions.
1 Introduction

We introduce a class of maps of the unit square that can be given the topology of a torus. These maps are discontinuous, non-hyperbolic, area preserving, piecewise linear and include the geometric actions of stretching, folding and rotation. The disparate dynamics due to these actions result in a wide range of dynamic behaviour, ranging from full fledged chaos to exactly periodic motion on the whole measure. The purpose of introducing these maps is that they can be easily quantized, and thus provide some of the simplest quantum models of systems that are not completely hyperbolic. The classical maps in themselves have also proved to be very interesting, even though, as yet, they have no immediate physical applicability.

Previously several maps of the torus onto itself have been studied with a view to adopt them to quantum mechanics. The class of maps called cat maps [1] have proven to have a non-generic quantum mechanics, while the other much studied model of the standard map [2] has the drawback that information on periodic orbits and the other classical features are hard to come by. The classical baker’s map which has also been quantized is completely hyperbolic [7]. The maps studied here are the simplest manifestations of chaos and order existing in the same system. This class of problems has been recognised as important and describe many physical systems where not all KAM torii have been destroyed.

The quantization of the completely hyperbolic baker’s map has proven to be a very useful model of quantum chaology, and the following maps should be considered as modifications of the baker’s map. They range from the action of bakers who thoroughly mix up the dough (phase fluid) to completely lazy bakers who do not mix the dough at all.

2 SRS Maps

We denote by SRS the class of maps defined geometrically by dividing the fundamental unit square into 3 vertical rectangles and stretching the first and third and rotating the second about the center of the square into horizontal rectangles as shown in fig.(1). We denote by $S_1(a)$ the symmetric maps where the first and third vertical rectangles are of equal measure ($a = 1 - b$). Here $a$ and $b$ are the locations of the first and second cut on the horizontal axis;
hence the area of the first and third vertical rectangles are \( a \) \((0 < a < 1/2)\). \( S_1(a) \) are chaotic only on a set of measure zero, and are critical maps in a sense we discuss below. The breaking of symmetry seems to produce ergodic behaviour on a set of finite measure. In this note we do not study this facet of the problem, except through the numerical results of the corresponding quantum models.

2.1 Classical Mechanics of \( SRS \) Maps

The \( S_1(a) \) map is not even ergodic, let alone mixing. We employ simple symbol sequences to extract the dynamics. The map is given by

\[
\begin{align*}
q' &= \frac{q}{a} \\
p' &= \frac{pa}{a} \\
q' &= \frac{1}{q} \\
p' &= \frac{1 - p}{q} \\
q' &= \frac{(q - 1 + a)/a}{pa + 1 - a}
\end{align*}
\]

if \( 0 \leq q \leq a \)

if \( a < q < 1 - a \)

if \( 1 - a \leq q \leq 1 \)

We now analyse the map \( S_1(1/3) \). Later we show that all the symmetric maps \( S_1(a) \) are topologically conjugate to each other. This means that their classical dynamics are essentially identical due to the existence of a one to one transformation between the two maps which commutes with time [5]. We use the ternary representation for \( q \) and \( p \); thus we represent a state in the square by a bi-infinite sequence with the left side specifying the position and the right side specifying the momentum, both being in their ternary basis. The symbols 0, 1 and 2 now represent the first second and third equal vertical or horizontal partition of the unit square. The map’s dynamics translates into the following rules on the bi-infinite sequence. If the most significant bit in the position is 0 or 2 the dynamics is a left shift, but if the most significant bit is 1, position and momentum get interchanged and the new position bits ( or old momentum bits) have their 0 and 2 exchanged (two’s complement ). For example

\[
\cdots 12210.02101022 \cdots \rightarrow \cdots 122100.2101022 \cdots
\]

\[
\rightarrow \cdots 1221002.101022 \cdots \rightarrow \cdots 220101.0221001.
\]
The dynamics is one of pure left shift only for those momenta and positions between 0 and 1 that have only 0 and 2 in their ternary basis representation, this, of course, being the standard middle third Cantor set \((C_{1/3})\) [3]. Therefore the union of \(C_{1/3} \times [0, 1]\) and \([0, 1] \times C_{1/3}\), is chaotic. We can easily show that the complement of the union of these two sets consists entirely of periodic orbits, and since the Lebesgue measure of the middle third Cantor set is zero this map and its relatives are extremely regular. The chaotic set in \(S_1(1/3)\) is a fractal, whose Hausdorff dimension is \(1 + \log 2/\log 3\) [3]. We refer to this set below as the “fractal set”, the “hyperbolic set” or the “fractal hyperbolic set” of the map.

Any state with at least one 1 bit in both position and momentum must traverse the rotating third of phase space. This allows us to follow the fate of states like

\[
\text{anything } 1a_m a_{m-1} \cdots a_1.1 \text{ anything}
\]

where the anythings represent any infinite string comprising of 0, 1, and 2. Let

\[
\gamma_m \equiv a_1 a_2 \cdots a_m, \quad a_i \in \{0, 1\}, m = 0, 1, 2, \ldots
\]

It is easy to see that the above state represents a rectangular island (the \(\gamma\) island) of area \(3^{-m-2}\) and that this island as a whole is periodic(!) with period \(4(m + 1)\). (See Appendix A). There are isolated points within these islands which have period \((m + 1)\) (if \(m\) is even) or \(2(m + 1)\). They represent elliptic periodic orbits with rational rotation numbers and inverse parabolic orbits respectively; actually they are just either rotation by \(\pi/2\) or \(\pi\). The rest of the periodic orbits of the \(\gamma\) island are direct parabolic. Thus periodic orbits occur in continuous families reminiscent of integrable systems. These dynamical systems are non-generic and have not been previously discussed.

The main feature of the transformation (say \(T\)) is that for every integer \(n\) there exists regions of finite measure \(\sigma\) such that \(T^{4n}(\sigma) = 1\), and \(n\) is the least such integer.

To counter the claim that this trivialises the system we must note that the map has periodic orbits of all periods and a chaotic set of measure zero. This chaotic set effectively becomes measure zero only at infinite times in the sense that the fractal set develops with increasing time. We know from previous studies [7,8,9] that finite time orbits, especially the short periodic orbits, seem to strongly influence the spectra and eigenfunctions for a finite, albeit small Planck’s constant. Which means that in quantum mechanics we
may expect this measure zero set to play a role. That this is in fact the case is borne out by the later discussion of the quantum SRS map (Sec.3).

The \( \gamma \) island is part of an island chain amongst which the dynamics is closed and periodic. Given \( m \) and hence the period, there are

\[
\begin{align*}
2^{m-1} & \quad m \text{ odd} \\
2^{m/2-1}(1 + 2^{m/2}) & \quad m \text{ even}
\end{align*}
\]  

(3)
such distinct island chains. (See Appendix A). Further each island chain consists of \( 2(m + 1) \) \((m \text{ odd or } m \text{ even and } \gamma \neq \gamma^\dagger)\) or \( (m + 1) \) \((m \text{ even and } \gamma = \gamma^\dagger)\) number of islands. We use an overbar to denote the complement operation and the dagger to denote “conjugation” or reflection, which we define below. Here \( \gamma^\dagger \) is the 2’s complement of the string \( \gamma \) compounded with a reflection. If the string length is \( m \) then a reflection is defined as \( a_l \rightarrow a_{m-l+1} \). Therefore the total number of period \( 4(m + 1) \) islands is \( 2^m(m + 1) \) irrespectively of, \( m \) being even or odd.

Since each island is of area \( 3^{-m-2} \) and

\[
\sum_{m=0}^{\infty} 2^m(m + 1)3^{-m-2} = 1
\]

the Lebesgue measure of direct parabolic periodic orbits is unity and the periods are all multiples of 4.

### 2.2 Reducing the area of rotation

We now want to study the effect of reducing the area of rotation while preserving the \( R \)-symmetry of the map. \( R \) symmetry is defined as the invariance of the equations of motion under the following transformation which is a reflection about the center of the square:

\[
\begin{align*}
q & \rightarrow 1 - q \\
p & \rightarrow 1 - p.
\end{align*}
\]  

(4)

We can do this by simply increasing \( a \), or by introducing more vertical partitions. Increasing \( a \) gives us a set of maps that are topologically conjugate to each other. This implies that they share identical orbit structures, such as periodic orbits and chaotic sets, even though the measures of these sets might be different. The required homeomorphism ( a strictly continuous one to one
function) \( f \) should commute with the dynamics of \( S_1(a) \) and \( S_1(1/3) \). Thus if \( T \) denotes the transformation \( S_1(1/3) \) and \( T' \) the transformation \( S_1(a) \) then the following must be true:

\[
 f \circ T = T' \circ f. \tag{5}
\]

Such a function is one that maps the middle third Cantor set into the middle \( (1-2a) \)-th Cantor set (defined similar to the middle third set) with the intermediate points being interpolated by straight lines joining the immediate neighbouring points belonging to the Cantor sets. This homeomorphism applied to the position and momentum makes the dynamics of \( S_1(a) \) commute with that of \( S_1(1/3) \), and hence these maps are topologically identical. \( f \) is a Cantor function and is a dynamically generated homeomorphism of the kind previously discussed in ref.[4]. Such a function is one to one and monotonously increasing with a countably infinite number of points where the derivative is not defined. The scaling properties that makes \( f \) the required homeomorphism are

\[
 f(q/a) = \begin{cases} 
 3f(q) & 0 \leq q \leq a \\
 3f(q) - 2 & 1 - a \leq q \leq 1.
\end{cases} \tag{6}
\]

The periodic orbit structure of \( S_1(a) \) is therefore identical to that of \( S_1(1/3) \), and the chaotic set is of measure zero. The quantum spectra, however, are different, and this is due to the changing stability of short periodic orbits belonging to the hyperbolic fractal set that for some time “pretends” to be in a sea of chaos.

Increasing the number of partitions while keeping them all equal produces classically simple dynamical systems. So we consider \( S_k(1/l), k = 1, 2, 3, \ldots \) \( (2k+1=l \text{ say}) \). We divide the fundamental square into \( l \) vertical rectangles and stretch and fold all rectangles except the central one which we choose to rotate. The extensions from \( S_1(a) \) are immediate. The chaotic set in the map is

\[
 C_{1/l} \times [0, 1] \bigcup [0, 1] \times C_{1/l} \tag{7}
\]

where \( C_{1/l} \) is the standard one \( l \) th Cantor set. We illustrate it’s construction with \( l = 5 \) as follows. Take the unit interval and divide it into 5 equal pieces and remove the central piece. The next step is to divide each of the remaining 4 pieces into 5 more equal pieces and removing the central piece.
from each. This process repeated infinite number of times leaves behind a set of Lebesgue measure zero, the set $C_{1/5}$. The dynamics on the complement of

$$C_{1/5} \times [0, 1] \cup [0, 1] \times C_{1/5}$$

again splits into periodic islands of period $4(m + 1)$. There are a total of $(2k)^m(m + 1)$ islands of area $(2k + 1)^{-m-2}$. The area of period $4(m + 1)$ islands therefore being $(2k)^m(m + 1)(2k + 1)^{-m-2}$ and we see that shorter periodic orbits have relatively lower measures in the higher $k$ maps.

### 2.3 Breaking of $R$-Symmetry

The peculiar regularity of the maps $S_1(a)$ with a thin set of chaotic orbits is supported by the $R$-symmetry of these models. The breaking of this symmetry (when $a \neq 1 - b$) leads to the elliptic orbits and inverse parabolic orbits changing into reflecting hyperbolic ones; it also seems to produce ergodic behaviour on a set of non-zero measure. One can establish the ranges of the parameters $a$ and $b$ such that this change would happen at least to the large islands. These were originally direct parabolic of period 8, with an inverse parabolic period 4 orbit at the center. These maps are themselves interesting and are most probably not structurally stable, but their further discussion here is not warranted. It can be readily seen that toroidal boundary conditions are compatible with the breaking of this symmetry.

We show some results of computer iterations in fig.(2) where the transformation of the parabolic island into a hyperbolic region is evident. The extreme case in which the measure of the first partition is one half the phase space and the second partition is a rotation about the centre and the third partition has measure zero ($a = 1/2, b = 1$) is completely chaotic (We call this map $SR$). Such a map however cannot be given the topology of a torus due to the disparate time behaviour of the boundaries; in considering the quantum mechanics of such maps we will therefore allow a thin third vertical partition. We now give some details on the map $SR$ that is chaotic on the whole measure.

The map $SR$ is given by

$$\begin{align*}
q' &= q/2 \\
p' &= 2p
\end{align*}$$

if $0 \leq q \leq 1/2$  \hspace{1cm} (8)
\[
\begin{align*}
q' &= 1 - p \\
p' &= q \\
\end{align*}
\] if \(1/2 < q \leq 1\). \hspace{1cm} (9)

It is ergodic, has a dense set of periodic orbits, and is sensitively dependent on
the initial conditions. It therefore satisfies the requirements in the definition
of chaos [5]. The positive Lyapunov exponent is \(\log 2/2\) almost everywhere,
and its topological entropy is 1.46. The number of periodic orbits of period
\(T\) is

\[
p(T) = c_1a_1^T + c_2a_2^T + c_3a_3^T, \hspace{1cm} (10)
\]

where \(a_1, a_2, a_3\) are roots of the polynomial equation

\[
a^3 - a^2 - 1 = 0 \hspace{1cm} (11)
\]

and the constants \(c_1, c_2, c_3\) are determined from \(p(3) = 4, p(4) = 5, p(5) = 6\). This counts the fixed point at \((0,0)\) but not the period 2 orbit at \((1,1/2)\).

All the periodic orbits are hyperbolic and are either reflecting or ordinary.
We present some proofs in the Appendix B.

It is debatable if any self respecting baker, however lazy, will adopt any
of the SRS maps discussed above. Our analysis of the \(S_1(a)\) map strongly
advises the baker against this map as mixing only a measure zero subset of
the dough can hardly be expected to produce good pastries. However the
intention of introducing these maps is an attempt to quantize the simplest
possible mixed systems. The semi-classics of these maps can then provide
a testing ground for more generic systems. The chaotic set of measure zero
in \(S_1(a)\) maps may prevent us from proceeding to the strict semi-classical
domain, but may provide us a simple mixed system at finite, albeit small
values of the Planck’s constant.

3  Quantum SRS Maps

We therefore turn to the quantum mechanics of these measure preserv-
ing maps, which are unitary operators on a finite dimensional vector space,
having all the classical symmetries and possessing the correct classical limit.

3.1  The Propagators

The quantum mechanical one step propagator for \(S_1(a)\) in the position rep-
resentation is given by:
\begin{equation}
B_1 = G_N^{-1} \begin{pmatrix}
G_N & 0 & 0 \\
0 & I_N(1-2a) & 0 \\
0 & 0 & G_N \\
\end{pmatrix}
\end{equation}

where

\begin{equation}
(G_N)_{mn} = \langle m | n \rangle = \frac{1}{\sqrt{N}} \exp[-2\pi i (m + 1/2)(n + 1/2)] \quad m, n = 0, \ldots, N - 1
\end{equation}

is the discrete Fourier transform matrix on \( N \) sites and \( I_N(1-2a) \) is the \( N(1-2a) \times N(1-2a) \) identity matrix. \( N a \) is required to be an integer and so the classical cut is allowed only at rational points. The notation of inner product indicates that it is the transformation matrix between discrete position states \( |n\rangle \) and discrete momentum states \( |m\rangle \). These eigenstates are repeated anti-periodically, that is

\begin{equation}
|n\rangle = -|n + N\rangle.
\end{equation}

This results in the discrete Fourier matrix on \( N \) sites, the entries being shifted by 1/2 in the exponent factors to allow the classical \( R \)-symmetry to be preserved on quantization [8].

For details on such a quantization see the original quantization of the baker’s map in [7] and in [8]. Here we simply note that a crucial element of maps quantized in this manner is that during one time step vertical rectangles go into horizontal ones. Quantum mechanically this corresponds to the partitioning of the state space into disjoint sets in either the position or in the momentum basis. In reduced units (when the position and momentum can range from 0 to 1), Planck’s constant \( 2 \pi \hbar \) is \( 1/N \). The classical limit is reached as \( N \rightarrow \infty \). We observe that in this classical limit, contrary to the usual one, the phase space remains finite. The operator \( B_1 \) has the classical map’s reflection symmetry about the center of the square, the \( R \)-symmetry. The operator \( B_1 \) commutes with \( R_N \), where \( R_N \) is defined as

\begin{equation}
\langle n | R_N | n' \rangle = \delta(n + n' + 1)\text{mod}(N).
\end{equation}

Here \( |n\rangle \) and \( |n'\rangle \) are position eigenvectors and \( n, n' = 0, 1, \ldots N - 1 \). This results in the eigenfunctions having an odd or even parity. In accordance with the rule of performing statistical analysis on only one symmetry class, we will
separate out the eigenvalues corresponding to each symmetry class, using the method suggested by Saraceno in [8]. The $T$ symmetry of the classical map is an anticanonical time reversal symmetry resulting from an interchange in position and momentum with a step backward in time ($p \leftrightarrow q, t \rightarrow -t$). The quantum map inherits this symmetry as an antiunitary symmetry

$$G_N B_1 G_N^\dagger = (B_1^{-1})^*.$$  

All this is in complete parallel to the usual baker’s map quantization. In fact all the symmetric maps we consider are such that their quantum propagators commute with $R$-symmetry, and all the maps we consider (including the unsymmetric ones) have the antiunitary $T$-symmetry.

The quantization of the other maps is also straightforward. For example $S_2(1/5)$ has a quantum mechanical propagator given by

$$B_2 = G_N^{-1} \begin{pmatrix} G_{N/5} & 0 & 0 & 0 & 0 \\ 0 & G_{N/5} & 0 & 0 & 0 \\ 0 & 0 & I_{N/5} & 0 & 0 \\ 0 & 0 & 0 & G_{N/5} & 0 \\ 0 & 0 & 0 & 0 & G_{N/5} \end{pmatrix}.$$  

The above propagator has the correct classical symmetries and classical limit.

### 3.2 Eigenangle distributions

Since the propagators associated with maps are unitary operators the eigenvalues are specified by eigenangles. The eigenangles of $S_1(1/3)$ are mostly irrational multiples of $2\pi$. The rational multiples correspond to eigenangles of $0, \pi/2, \pi$ and $3\pi/2$ and these states are degenerate. We are interested in the sequence of maps $S_1(a)$ with increasing $a$, when such degeneracies tend to vanish, and most of the states are of a mixed nature.

We recall that the spectral properties of quantum dynamical systems have centered around the nearest neighbour spacing distribution which is usually Wigner-like if the corresponding classical system is completely chaotic and Poisson-like if it is integrable [9,16]. The generic case of mixed dynamics with a measure of chaos and regularity coexisting has not received such a clear answer. While we do not expect $S_k(1/l)$ or $S_1(a)$ to provide us with
such a case, as their dynamics is not an *interlacing* of elliptic and hyperbolic periodic orbits, it does have a set of measure zero chaotic orbits whose effect at a small but finite Planck’s constant is to mimic a transition from the Poisson like distribution to an intermediate one as $k$ or $a$ is increased.

In fig.3(a)(b)(c)(d) we show the nearest neighbour spacing (nns) distribution for $S_k(1/2k + 1), k = 1, 2, 3, 4$ respectively. The slightly different values of Planck's constant arises since $N/2k + 1$ must be an integer. The transition from a Poisson-like (level attraction) distribution when $k = 1$ to an intermediate statistics (level repulsion) when $k = 4$ is apparent. This suggests that the underlying hyperbolic measure zero set of chaotic orbits is contributing to the quantal spectrum. The density of states can be obtained as the time Fourier transform of the trace of the powers of the propagator [9]. The autocorrelation function therefore must show the Cantor set becoming more prominent as the statistics tends to the Wigner-like one. Below we present some results on this.

The fractal set of chaotic orbits affects the quantal spectrum, reflecting the fact that the behaviour of orbits of classical short periodic orbits strongly influences properties of quantum mechanical stationary wavefunctions. The chaotic set of measure zero in the classical model takes an infinite time to develop. The small but finite value of the Planck’s constant makes the relevant time scales finite. The nns for the $S_1(a)$ map is shown in fig(4a), for $N = 300, a = 149/300$. The reflection symmetry of the map permits the statistical analysis of only 150 of them. The level repulsion in the spectrum is apparent.

We may estimate relevant time scales by noting that there exist regions of the phase space that are thin strips along the position or momentum, of width $3^{-n}$ such that till $n$ time steps they undergo only expansion and contraction but no rotation. Thus we may estimate that at least until $n = \log_3(N)$ a “fattened” set that contains the fractal set must function as a hyperbolic set for the purposes of quantal calculations (a “pseudo-hyperbolic region”). By the same arguments the parabolic periodic islands of period $4(m + 1)$ are not all resolved in quantum mechanics. Recall that the $\gamma_m$ island has an area of $3^{-m-2}$. Periodic islands of period $4(M + 1)$, where

$$M > \log_3(N) - 2$$

(18)
cannot therefore be expected to function as such in the quantum mechanics. They cannot for instance be expected to appear in autocorrelation functions.
for high times and cannot probably support quantum eigenstates. In our
discussion of the eigenfunctions we will note that this naive argument gives
us only a rough estimate at best.

We noted in the section on classical mechanics that the breaking of the \( R \)
symmetry produces in most cases a seeming transition to ergodicity and chaos
on finite measures. The map \( SR \) in particular was completely chaotic. We
show the nns of the corresponding quantum models in fig.4(b)(c)(d). Fig(4b)
is the nns for the map \( SRS \) with partitions at \((0.3,0.6), \) with \( N = 300; \)
fig(4c) is for partitions at \((.3, .5) \) and \( N = 300. \) Fig(4d) is the nns for \( SR \)
for \( N=300. \) In this last map however we allowed a thin third strip of width
1/300, making it strictly speaking a \( SRS \) map. This was done to preserve
the toroidal boundary conditions. The propagator chosen was thus

\[
B_{SR} = G_N^{-1} \begin{pmatrix}
G_{N/2} & 0 & 0 \\
0 & I_{N/2-1} & 0 \\
0 & 0 & i
\end{pmatrix},
\]

where \( i \) is the square root of \(-1.\)

The autocorrelations not shown here indicate the “closeness” of these
maps, which may find a more quantitative meaning in the concept of structural
stability. By closeness we imply visual closeness of autocorrelation
functions. It is not clear how topological conjugacy carries over into quantum
maps. Note that we have already quantized a set of maps topologically con-
jugate to each other, and found different statistics for the eigenangles. What
is needed is a sensible specification of distance in the space of quantum maps.

### 3.3 Autocorrelation functions

We use the coherent state representation for finite dimensional vector
spaces, as developed by Saraceno in [8]. The displayed function is a contour
plot of \(|\langle pq|B^n|pq\rangle|, \) with the \( B \) being the appropriate quantum propagator.
Usually the square of this function is plotted as it is the probability that a
wavepacket initially localised at \((q,p), \) in the sense of a minimum uncertainty
packet, returns to its launching point; The square being the quantal auto-
correlation function at the point \((q,p) \) at time \( n. \) We have however plotted
just the absolute value to emphasize fine structures. Fig.(5a) shows the
autocorrelation for \( n = 2 \) and \( N = 48 \) and for the operator \( B_1 \) with \( a = 1/3, \)
i.e the quantum \( S_1(1/3) \). The maxima should occur at the classical periodic
points of period 2. The center of the square is a elliptic fixed point and is so highlighted also as a period two orbit. The corners of the square $(0,0)$ and $(1,1)$ are also fixed points strictly belonging to the fractal hyperbolic set. The other two maxima are due to the period two orbit $(1/4, 3/4), (3/4, 1/4)$ corresponding to the bi-infinite sequence $...020202.020202...$ (or $02.02$ if we denote with an underline the infinite repetitions of a string), and it's orbit, which is simply a pure left shift.

*This particular periodic orbit belongs to the fractal set we have described above, which is of measure zero.* The associated periodic points act as if they were usual hyperbolic periodic points with eigenvalues $(1/3, 3)$, which implies a quantum contribution *similar* to that of an usual hyperbolic periodic orbit. The infinite returns of this orbit that would affect the quantum correlations would however be *different* from the usual case. Considering that these are usually decreasing contributions, it is possible that *these Cantor set of points can affect the quantal spectrum; and in fact they do, as we see from the eigenangle distributions.*

The fig.(5b) shows the autocorrelation for the operator $B_1$, again with $a = 1/3$ and $N = 48$ for $n = 4$. As we expect we see maxima along classical periodic orbits. The whole central region, the square $1/3 < q < 2/3, 1/3 < p < 2/3$ is now filled with direct parabolic orbits and contributes the most to the autocorrelation. The 12 primitive period 4 points belonging to the hyperbolic set are much smaller peaks, but they are still highlighted in the figure. They correspond to the orbits of the sequences $0220.0220, 0002.0002$ and $2220.2220$. For example the peak at $(14.4, 14.4)$ corresponds to the periodic point $0220.0220$ whose numerical value in base 10 is $(3/10,3/10)$ which after scaling (multiplying by 48, the value of $N$) gives us $(14.4,14.4)$. The primitive orbit of period 2 noted above is also visible in this figure, with of course a much reduced recurrence. The peaks around the fixed points are also seen. The period 4 point $(1/2,1/6)$ along with it’s orbit contributes significantly to the autocorrelation and is seen after scaling at the point $(24,8)$. These maxima around the central square are due to inverse parabolic orbits that sit at the center of the period 8 periodic islands. Thus we see that much of the structure of the figure can be explained in terms of the classical periodic orbits.

We notice, however, the peaks at the center of the four edges that have no periodic orbit support. They correspond on the classical torus to the two points $(0,1/2)$ and $(1/2,0)$, which belong to the hyperbolic fractal set.
These points are homoclinic to the fixed point at the origin and are not near any period 4 orbit. For visualising the possibility of such a recurrence it is helpful to think of the fate of a localised distribution on the classical map centered at say \((0,1/2)\). Under the action of the map the distribution tends to elongate along the \(q\) axis and contract along the \(p\) axis, and as a whole move closer to the \(q\) axis and away from the \(p\). At these time scales the fixed point at the origin acts as an ordinary hyperbolic point. After 3 time steps the distribution is sufficiently spread to have a large enough density at the bottom of the middle third region which after one more time step gets rotated to the region \((1,1/2)\); this on the torus being the same as the point \((0,1/2)\).

Such recurrences are known in general dynamical systems and have been called “homoclinic recurrences” [10]. Their contributions to the correlations are to be added to the periodic orbit recurrences and presumably a complete semi-classical study of these maps would do so. This would result in modifications to the periodic orbit sums for the powers of the propagator, leading possibly to “homoclinic sums”. The semi-classics of the usual baker’s map has proven to conform to the periodic orbit sum theory [9] without any homoclinic recurrences [10], [11]. The value of the \(SRS\) maps in regard to these comments depends upon whether a nontrivial semi-classics exists. Our results here indicate at least the presence of an interesting finite Planck’s constant domain and the discussions here suggest that this may be possible for maps with a smaller rotating region, or with more partitions. The map \(SR\) is completely chaotic and it remains to be seen if the semi-classics of this map will also look like a periodic orbit sum. The organization of the dynamics by the periodic points of the usual baker’s map may not be generic.

The classical topological conjugacy of the \(S_1(a)\) maps has the effect that the quantum autocorrelation functions have similar structures, as is apparent on comparing figs.(5a) and (5c). The maxima of the autocorrelations about the hyperbolic Cantor set of fixed points is however increasing because the classical instability of these orbits is decreasing. The higher hyperbolic contribution coupled with a much lower contribution from the dominant parabolic and elliptic regions contributes to the spectral transition. Fig.(5d) shows the autocorrelation \(|\langle pq|B_2^2|pq\rangle|\), for \(N = 70\). The orbits highlighted as maxima are

1. the five fixed points \((0,0, 1,1, 2,2, 3,3, 4,4)\) and
2. the 12 period two orbits composed of the 6 possible combinations of 
the symbols 0, 1, 3, 4,

all of which belong to the chaotic fractal set, except the center. The number 
of such contributions increase with the number of partitions. This generates 
increasing hyperbolic contributions to the autocorrelations, inducing in turn 
the spectral change shown in fig.(3). A semi-classical theory of such maps 
should thereby shed light on the so called mixed systems where chaos and 
order coexist.

3.4 Eigenfunctions

Eigenfunctions of quantum systems with mixed or chaotic classical limits 
are still in the process of being understood. The phenomenon of periodic orbit 
scarring [12] has been observed in several quantum eigenfunctions when the 
corresponding classical system is chaotic. A semiclassical theory towards the 
understanding of this phenomenon has been developed in [13] and [14]. Once 
again our maps provide a complete collection of eigenfunctions for a mixed 
ystem that is completely understood. Fig.(6) are the intensities of some 
eigenfunctions of $B_1$, which is the quantum $S_1(1/3)$ map, for $N = 48$. The 
plot is a contour of the function

$$|\langle p q | \psi \rangle|^2$$

(20)

where $\langle p q | \psi \rangle$ is an eigenfunction of $B_1$ in the coherent state representation. 
Fig(7) shows some classical periodic points of this map belonging to the 
chaotic measure zero set that scar some of the eigenfunctions shown in fig(6). 
The ternary code of these orbits have also been indicated.

Of the 48 eigenfunctions 9 representatives have been selected to show 
classical structures. Of the remaining, many more are similar and many 
do not apparently possess any prominent classical structures. A majority 
of the eigenfunctions are concentrated around either the regular parabolic 
and elliptic periodic orbits or are scarred by periodic orbits belonging to the 
hyperbolic fractal set. We use scarring in the conventional way to indicate 
the presence of classically “improbable orbits”. Figs.6(a)(b)(c)(d) are some 
eigenfunctions scarred by periodic orbits of the hyperbolic fractal set. Fig.(6a) 
shows an eigenfunction scarred by the period 2 orbit corresponding to the
sequence \(02.02\). Fig. (6b) shows the scarring from two period 3 orbits represented by the sequences 202.202, and 020.020. Fig. (6c) shows high intensities at the period 4 orbit 2002.2002. Fig. (6d) shows an eigenstate scarred by an orbit of as high a period as 6, corresponding to the sequence 220002.220002.

As is expected, a large number of the eigenfunctions are dominated by the parabolic islands and elliptic orbits. The trivial central region of the classical phase space forms the support of four eigenstates and fig. (6e) shows one such state. Fig. (6f) shows a state dominated by the period 8 parabolic island at \((1/3 < q < 2/3, 1/9 < p < 2/9)\) and its 3 partners. Fig. (6g) shows two of the three island chains of period 12. They correspond to the sequences A120.1B and A102.1B and their island partners. Here A and B are arbitrary strings, indicating that the sequence represents an area rather than a single point. Each island chain of this type has three islands and are all highlighted here. The missing period 12 island chain from this eigenstate is the one consisting of 6 islands with one of the representatives being A100.1B.

In fig. (6h) we show an eigenstate which has prominent structures apparently not related to any classical periodic orbit. In fact such peaks along the edges of the square were observed by Saraceno [8] for the quantum baker map. The prominence of hyperbolic trajectories surrounding a hyperbolic fixed point has been observed in the eigenfunctions of other systems, like the standard map. The presence of such scars in the eigenstates is once more an indication that for the purposes of quantum mechanics the classical map seems to possess a “pseudo” or effective hyperbolic region of non-zero measure that we have discussed above. The prominent peaks at the centers of the edges are probably due to the presence of a homoclinic recurrence that was noted in the autocorrelattions. Fig. (6i) shows an eigenstate that seems to have no apparent classical support, although it may well correspond to an interference of several longer periodic orbits. This state represents a mixed state, while the majority of the eigenstates are scarred either by the hyperbolic set or dominated by regular structures, but not both. This shows that in such a system the seperation of states into regular and chaotic subsets may be possible.

Thus the simple map \(S_1(1/3)\) has a rich eigenstructure, with a mixture of states dominated by the hyperbolic fractal set and by states dominated by regular parabolic islands, and elliptic orbits. As we decrease the area of rotation, either with the \(S_1(a), a > 1/3\) maps or the \(S_k(1/2k + 1)\) maps we would expect the proportion of eigenstates with support on the hyper-
bolic set to increase. These features may enable us to study the changes in the eigenstructures of such systems tending towards global chaos. We finally note that the periodic orbits of the fractal set that have scarred the eigenfunctions have periods as high as 6. At this stage in the development of the fractal set the hyperbolic strips are of width $3^{-6}$ or $1/729$ which is very small compared with the value of Planck’s constant used, namely $1/48$. This shows that the quantum map is able to resolve structures that are classically fine. The exact nature of the “smoothing” of the phase space due to the finite value of Planck’s contant is not fully understood. Recent work by P.W. O’Connor et. al [10] indicates that in the quantum baker’s map accuracy of the semiclassical traces was observed to be good much beyond the Log time [15], implying that the quantum dynamics explored delicate structures which appeared, however, not in the classical phase-space, but in a mock phase-space used to display objects in the coherent state representation. Similar mechanisms may be causing here the resolution of such delicate structures.

4 Conclusions

In this note we have introduced and studied some maps of the square onto itself. The purpose has been to quantize the simplest possible maps in which regular and irregular orbits coexist. We have found that even a measure zero set of chaotic classical orbits influences the quantal spectra and the eigenfunctions, due to the finiteness of Planck’s constant. Thus we observe a spectral transition as when a system’s phase space acquires a measurably increasing chaotic component. In some of the maps considered here, the class called $S_1(a)$, such a transition occurs even though the underlying chaotic fractal set is of Lebesgue measure zero for all $a$. This interesting behaviour arises because we are looking at the stationary states at a given Planck’s constant.

The formulae for intermediate statistics as developed by Berry and Robnik [17] depend on the ratio of measures of the chaotic and regular regions of the phase space. This work was for the eigenvalues of a continous time Hamiltonian system. For the class of maps $S_1(a)$ that exhibits a spectral transition such a ratio is of course zero. However the arguments developed above suggests the existence of a “pseudo-hyperbolic” region of positive mea-
sure that depends on the Planck's constant. Part of an ongoing work is to verify whether such an area which is culled from partly classical and partly quantum conditions will produce statistics that agrees with the Berry-Robnik formulae. We have observed the scars of periodic orbits belonging to this fractal set on the eigenfunctions which we have shown in a coherent state representation.

There are several other maps that form the general family we have called the $SRS$, and their classical behaviour has proven to be interesting. It remains to be seen if such a transformation which introduces a discontinuous rotation component to the well known baker’s map has any significance in the context of general dynamical systems. The extreme case we have called $SR$ in which only one half of the phase space is stretched, while the other is rotated has been shown to be chaotic. It’s quantum features follow those of completely chaotic systems such as those of the usual baker’s map.

Acknowledgments

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Appendix A  

Here we prove some statements of the text. In this section of the appendix arbitrary infinite strings of 0,1 and 2 are denoted by $A$ and $B$. Finite strings of length $m$ comprising of 0 and 2 we denote by $\gamma^m$. The operations of 2's complement and “conjugation” were defined in sec. 2.1. They are denoted by an overbar and dagger respectively. An underline denotes infinite repetitions of that string. All points in the square are assumed to be in base 3. First we consider the map $S_1(1/3)$.

Let $T$ be the unit square and the set $S$ be

$$T \setminus (C_{1/3} \times [0,1] \cup [0,1] \times C_{1/3}).$$

we first prove that all points in $S$ are periodic. If $x_0 \in S$ then its orbit must at some time $n$ be given by

$$x_n = A_1.a_1a_2 \ldots a_m1B \equiv A_1.\gamma^m1B; \ a_i \in \{0,2\}, \ i = 1, \ldots, m.$$  

(22)

Here $m = 0, 1, 2 \ldots$. Then future orbit of $x_n$ is as follows:

$$x_n = A_1.\gamma^m1B \longrightarrow A_1\gamma^m1B \longrightarrow B1\tilde{\gamma}^m1\bar{A} \longrightarrow B1\tilde{\gamma}^m1\bar{A} \longrightarrow \bar{B}1\bar{\gamma}^m1\bar{A}$$

(23)

$$\longrightarrow \bar{B}1\bar{\gamma}^m1\bar{A} \longrightarrow \bar{B}1\bar{\gamma}^m1\bar{A} \longrightarrow vA1.\gamma^m1B = x_n$$

We have denoted by short arrows a single time step representing a rotation, while a long arrow has $m-1$ intermediate states all undergoing a simple left shift because the strings $\gamma^m$ and $\tilde{\gamma}^m$ have only 0 and 2 in them. This completes the proof, as well as, it tells us that the period of $x_n$ is $4(m + 1)$, though there are isolated points of smaller primitive periods as we will see below. Primitive periodic orbits of period $4(m + 1)$ are evidently continuous and are infinite in number. They are of the direct parabolic kind. We can infer several other features from the orbit of $x_n$, written above. In particular, the point $x_n$ will be of period $(m+1)$ if $\bar{A} = B = A$ and $\gamma^m = \tilde{\gamma}^m$, where equality of two strings means equality of each corresponding bits (We call
such strings $R$-symmetric strings). This implies that $A = B = 1$ and that $m$ must be even. These periodic orbits are rotated once and stretched $m$ times during a cycle, and hence are of the elliptic kind with the following stability matrix:

$$
\begin{pmatrix}
0 & -3^m \\
3^{-m} & 0
\end{pmatrix}.
$$

(24)

The strangeness of the SRS map is in this feature: stretching along the $q$ direction is followed by a rotation which effectively shifts the stretching to the $p$ direction which is then contracted! We note that the usual Baker compounded by a rotation of the whole square by 90 deg is not at all chaotic, it being period 4 as a whole! Of course, such a process is aided by the symmetry of these maps, the breaking of which seems to produce a whole class of mixed maps.

The elliptic orbits are therefore of only odd periods and if the period is $(m + 1)$ the number of such orbits is simply $2^{m/2}$ as one half of the $\gamma$ string determines the other. Also from (23) we see that if $\bar{A} = A, \bar{B} = B$ then the orbit is of period $2(m + 1)$ and this is a primitive orbit if $\gamma_m \neq \bar{\gamma}_m$. It follows that $A$ and $B$ are again simply strings of 1. These orbits have repeated eigenvalues of $-1$ and are therefore inverse parabolic; given $m$ there are $2^m$ such orbits, including the repetitions of period $(m + 1)$ orbits.

We can also see the island structures from (23). Clearly the point $A_1.\gamma_m 1B$ with no restrictions on $A$ and $B$ represent a rectangular region defined by

$$3^{-m-1} < q < 2.3^{-m-1}, \quad 1/3 < p < 2/3,$$

having an area $3^{-m-2}$. The images of this rectangle will retain the area and form a closed chain of period $4(m + 1)$. The number of such distinct island chains and the number of islands in each may also be easily inferred. If $m$ is odd, none of the finite string are $R$-symmetric. Since both the string and it’s $R$-symmetric partner appear in our specification of an island, there are $2^{m-1}$ distinct island chains. If $m$ is odd, there are two possibilities: a string may or may not be $R$-symmetric. For a fixed $m$ there are $2^{m/2}$ strings of the former kind and therefore $(2^m - 2^{m/2})$ of the latter kind. We therefore get the total number of distinct island chains to be

$$2^{m/2} + (2^m - 2^{m/2})/2 = 2^{m-1}(1 + 2^{m/2-1}).$$

(25)
The number of islands in each island is a simple consequence of the above discussion.
Appendix B  

SR Map.

We now prove that the map SR is chaotic on the whole measure. To get a complete picture of the position of the periodic orbits and so forth we use the binary representation even though the dynamics is a subshift of finite type on three symbols. In this part of the Appendix A and B denote arbitrary strings of 0 and 1. Any state in the square is represented as a bi-infinite string of 0 and 1. Let the most significant bit of the position be 0, then the dynamics is one of left shift. If the most significant bit is 1, then it is an interchange of position and momentum combined with the operation of exchanging 0 and 1 (1’s complement) in the new position (or old momentum). We note that, unlike the baker’s map, the doubly infinite sequence without the dot does not describe the trajectory in the sense that it does not also serve as an itinerary. For convenience we divide the square into the 4 equal open subsquares. We denote by $R_1$ the left bottom square, by $R_2$ the left top square, by $R_3$ the right top square and by $R_4$ the right bottom square. Any state in $R_4$ must traverse $R_3$ and $R_2$ in the following two time steps uniquely, these being merely rotations. This evident observation is an useful device when cast in the following form:

$$A0.1B \xrightarrow{2} \overline{A10.\bar{B}}. \tag{26}$$

The map’s dynamics is one of “eventual” left shift with or without a reflection about the center of the square. To see this we give several rules that may be easily verified. Represent an arbitrary initial state as

$$A.\gamma_mB; \ m = 1, 2, \ldots \tag{27}$$

The $\gamma_m$ string is the first $m$ bits of the position, where $m$ is arbitrary. Denote the number of the 0-1, 1-0 neighbours in the string $\gamma_m$ by $n(\gamma_m)$. Let the most significant bits of position and momentum be $a_1$ and $b_1$. The rules are the following.

$$A.\gamma_mB \xrightarrow{K} \overline{A\gamma_m^\dagger}.\bar{B}, \tag{28}$$

if any one of the following conditions is true:
1. $a_1 = 0$ and $n(\gamma_m)$ is odd. Then $K = 2n(\gamma_m) + m - 1$.

2. $a_1 = 1$, $n(\gamma_m)$ is even and $b_1 = 0$. Then $K = 2n(\gamma_m) + m + 1$.

If any one of the following conditions is true.

1. $a_1 = 0$ and $n(\gamma_m)$ is even. Then $K = 2n(\gamma_m) + m - 1$.

2. $a_1 = 1$, $n(\gamma_m)$ is odd and $b_1 = 0$. Then $K = 2n(\gamma_m) + m + 1$.

Here $K$ is the number of intermediate steps. We do not need to specify the rules for the case when the initial point is at $R_3$ as it’s forward iterate falls in $R_2$ and it’s backward iterate in $R_4$, both of which are covered above. Thus we see from the rules that $SR$ leads to a destruction of information with time, one of the key ingredients of deterministic chaos. The time needed to shift $m$ bits in the usual baker map is $m$; here we see this time increasing depending upon the number of 1-0, 0-1 neighbours that have been shifted. This is a measure of the map’s “laziness”.

One may now prove the existence of a dense orbit. The construction is essentially identical to that of the usual baker’s map. Consider the point $x_0$ in $R_4$ such that its position bits are strings of all possible permutations of 0 and 1 of all possible lengths. This is usually constructed by first placing the permutations of a string of length 1, then 2 and so on. Given any point $y$ in $R_1$ or $R_4$ one may approximate it to any arbitrary degree by finite length strings. Say $y = \alpha.\beta$ is the arbitrary point and its approximation is $\alpha_L.\beta_L$ consisting of the first $L$ significant bits. Then the point $x_0$ has somewhere in its position string the substring $\alpha_L.\beta_L$. Since the last bit in the string $\alpha_L$ of length $L$ (i.e. the most significant bit in the momentum of $y$) is 0 the number of 1-0, 0-1 neighbours is necessarily odd. Therefore, from the above rules the orbit of $x_0$ must at a certain time be at $\ldots \alpha_L.\beta_L \ldots$. Thus the orbit of $x_0$ visits any small neighborhood of any point. This orbit is thus dense in $R_1$ and $R_4$. Therefore it is dense on the whole square since the regions $R_2$ and $R_3$ are accessed from $R_4$ by a mere rotation. The existence of a dense orbit proves the ergodicity of the map $SR$.

We also briefly discuss periodic orbits of this map. Evidently there are no periodic orbits lying wholly in the region $R_1$, thus it is sufficient to count
the periodic orbits in $R^4$. There is a period 2 orbit, comprising of the points $(1,1/2)$ and $(1/2,1)$. Apart from these there are no periodic points on the boundary of the square. Since the dynamics is one of eventual left shift, it must be rational.

In $R^4$ the orbit can be written as either $\gamma_m \cdot \gamma_m$ or $\gamma_m \cdot \bar{\gamma}_m$. In both cases $\gamma_m$ alone determines the period. From the rules given above we find the time period to be $T = 2(n(\gamma_m) + 1) + m$. We allow $n(\gamma_m) = 0, 1, 2, \ldots, m - 1$. If $n(\gamma_m)$ is even, the orbit is of the second kind; if it is odd, it is of the first kind.

After some elementary combinatorics we get the number of periodic orbits of period $T$, excluding the period 2 orbit on the boundary, but including the fixed point, as

$$p(T) = 3\left(\sum_j \left(\frac{T - 2j - 3}{j}\right)\right) + \left(\sum_j \left(\frac{T - 2j - 1}{j}\right)\right). \quad (29)$$

for $T > 2, \quad j = 0, 1, 2, \ldots$

The summations end when the combinatorial symbols become meaningless. The above series for $p(T)$ satisfies the recursion relation

$$p(T) = p(T - 1) + p(T - 3), \quad (30)$$

so that $p(T)$ follows a Fibonacci like sequence

$$4, 5, 6, 10, 15, 21, 31, 46, \ldots$$

The first entry being the number of period three states and so on. The stability matrix of a period $T = 2n(\gamma_m) + m + 2$ trajectory is

$$\begin{pmatrix}
2^{m+1}(-1)^n & 0 \\
0 & 2^{-m-1}(-1)^n
\end{pmatrix}. \quad (31)$$

Hence all the periodic orbits are hyperbolic; they are ordinary if $n \equiv n(\gamma_m)$ is even and they are reflecting if $n$ is odd.

We may note that (in keeping with the rather special features of the above maps) the map $SR$ also shows some non-generic features although it is chaotic on the whole measure. The stable and unstable manifolds of a periodic point are known to foliate the phase space and to usually intersect
transversally. The stable manifolds of the fixed point at the origin are the vertical lines in the regions $R_1, R_2$ and $R_4$ with the position being of the form $k/2^l$, for any two integers $k$ and $l$. In the region $R_3$ they are horizontal lines along similar points. This is understandable as the whole region $R_4$ is rotated into $R_3$, but it leads to non transverse intersections of the stable and unstable manifolds and two whole intervals of orbits homoclinic to the fixed point, the intervals defined by the boundary between $R_3$ and $R_4$ and the boundary between $R_3$ and $R_2$. 
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Figure Captions

Figure (1) Definition of the map SRS. Fig.(1a) and fig.(1b) are pictures of the unit square before and after the transformation.

Figure (2) Fig.(2a) shows 20,000 iterations of the point (0.2,0.2), for cuts given by $a = .3$, $b = .5$. Fig(2b) shows 20,000 iterations of the point $(0.16,.5)$ for $a = .33, b = .66$, when the $R$-symmetry has been just broken. This indicates the structural instability of $S_1(1/3)$. Note the change of the periodic point at $(1/2,1/6)$ and its orbit into hyperbolic regions.

Figure (3) Nearest neighbour spacing distribution for the quantum maps of $S_1(1/3), S_2(1/5), S_3(1/7)$, and $S_4(1/9)$ are shown in figs.(3a), (3b), (3c) and (3d) respectively; the corresponding values of Planck's constant, $(1/N)$ are $1/288, 1/290, 1/294, 1/288$. The reason for choosing slightly different values of the Planck's constant is explained in sec.3.2.

Figure (4) Fig.(4a) shows the nearest neighbour spacing distribution for the symmetric map $S_1(a)$ with the rotating region being very small; $a = 149/300$, and $N = 300$. Figs.(4b) and (4c) are the nns distributions for the unsymmetric SRS for $a = .3, b = .6, N = 300$, and for $a = .3, b = .5, N = 300$ respectively. Fig(4d) shows the nns for the map $SR$, for $N = 300$.

Figure (5) Autocorrelation functions for the following quantum maps for different times and $N$ values. $S_1(1/3)$ with $n = 2, N = 48$ in fig.(5a). $S_1(1/3)$ with $n = 4, N = 48$ in fig.(5b). $S_1(11/24)$ with $n = 2, N = 48$ in fig.(5c). Note that the similarity between this and fig.(5a) is due to the classical topological conjugacy discussed in the text. $S_2(1/5)$ with $n = 2, N = 70$, in fig.(5d).

Figure (6) Eigenfunctions of the quantum $S_1(1/3)$ map, for $N=48$. Nine of the 48 are shown.

Figure (7) Some periodic orbits of $S_1(1/3)$ which belong to the chaotic fractal set and which scar some of the eigenfunctions shown in fig.(6).