Kolmogorov complexity of unitary transformations
and quantum operators

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Abstract

We introduce a notion of Kolmogorov complexity of a unit vector in a finite-dimensional Hilbert space, which is based on "qubit complexity" of Berthiaume, W. Van Dam and S. Laplante. Then, the Kolmogorov complexity of quantum density operator is naturally defined as the Kolmogorov complexity of the corresponding purified state.

Inspired by the quantum mechanical relation between the unitary evolution and the Hermitian operator of a quantum system, we introduce a mapping from the unitary transformation to a Hermitian operator first and, subsequently, we map the Hermitian operator to a unit vector in a Hilbert space. Thus, for every unitary transformation, we construct a corresponding unit vector. Then, we define the Kolmogorov complexity of the unitary transformation via the Kolmogorov complexity of the constructed vector and obtain a simple upper bound. Therefore, we establish the duality of unitary transformations and unit vectors and shall measure the descriptional (i.e., Kolmogorov) complexity of a unitary transformation in terms of the unit vector complexity.

Keywords: quantum information theory; information theory; quantum; Kolmogorov complexity; quantum computing; quantum information; unitary; quantum state.

I. INTRODUCTION

Quantum computing has become one of the most fascinating and promising manifestations and applications of quantum mechanics, which is based upon the following main postulates:

Postulate 1 [] Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system’s state space.

Postulate 2 [] The evolution of a closed quantum system is described by a unitary transformation.

Postulate 3 introduces quantum measurements.

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For a comprehensive introduction to quantum computing with linear algebra and quantum mechanics, see, for example, [1]. In quantum computing, quantum states are associated with quantum data, and a unitary evolution of quantum states is associated with a quantum computer program. Thus, the concepts and notions of complexity of a quantum state as well as complexity of the unitary evolution, have become relevant in the quantum computing context.

Kolmogorov complexity [2, 3] of a classical object is defined as the shortest effective binary description of the object. So it is about how much information (binary data) we need to describe, reproduce or recreate the object. The concept of Kolmogorov complexity has been extended from classical objects (strings, files, images, etc.) to quantum domain: the complexity of quantum states [4–7] (i.e. the complexity of quantum data).

In computer science [3] and information theory [8], Kolmogorov complexity of a (classical) string (for example, a binary sequence) or, more generally, of a classical finite object is commonly known as the shortest binary program \( p \), which runs on a Turing machine and computes (in other words, describes) the string or the object. Thus, Kolmogorov complexity is also called descriptive complexity.

There have been several approaches [4–7, 9, 10] to define a complexity of a quantum state. In [4, 5], the concept of Kolmogorov complexity was extended from classical to quantum domain. Such an extension naturally relies upon the concept of a universal quantum Turing machine, which was first proposed by Deutsch [11]. The detailed construction of quantum Turing machines can be found in [4, 12].

In [5], the quantum Kolmogorov complexity, also known as "qubit complexity", is defined as follows: the quantum Kolmogorov complexity of a string of qubits is defined, relative to a universal quantum Turing machine \( M \), as the length of the shortest qubit string which when given as input to \( M \), produces on its output register the qubit string. In this paper, we shall be concerned with the "qubit complexity" type of quantum Kolmogorov complexity.

Let \( \mathcal{H} \) be a 2-dimensional complex vector space (Hilbert space). A qubit is described by a unit vector in \( \mathcal{H} \). Then, for any integer \( n \), the state of \( n \) qubits corresponds to a unit vector in \( n \)-folded space \( \mathcal{H}^\otimes n \). We will use the bra–ket Dirac notation with low case Latin letters to denote vectors in \( \mathcal{H}^\otimes n \), that is \( |x\rangle, |y\rangle \), etc.

Roughly speaking, for any \( |x\rangle \in \mathcal{H}^\otimes n \), the qubit complexity [5] of \( |x\rangle \) is equal to the logarithm in base 2 of the dimension of the smallest Hilbert space (spanned by computational
basis vectors) containing a quantum state $|y\rangle$ that, once fed into a universal quantum Turing machine, makes the universal quantum Turing machine compute the output $|x\rangle$ and halt. Thus, qubit sequence $|y\rangle$ can be seen as a \textit{compressed description} of $|x\rangle$. The upper-bound for the qubit complexity of any $|x\rangle \in \mathcal{H}^\otimes n$ is immediately seen to be $n$.

We note that any unitary transformation $U$ can be implemented\cite{1} as a quantum computation on universal quantum Turing machine (as well as using quantum logic circuits) and, vice versa, any quantum Turing machine computation (without measurements) can be seen as a unitary transformation. Thus, to any unit vector $|x\rangle$ in Hilbert space $\mathcal{H}^\otimes n$, we assign its complexity equal to the quantity $QC(|x\rangle)$, irrespective of the fact whether or not $|x\rangle$ represents a physical quantum system. The complexity of $|x\rangle$ shall be understood as the logarithm of the dimension of the respective Hilbert space.

Our notion of $QC(|x\rangle)$ is "backward-compatible" with the original qubit complexity\cite{5} as follows: If vector $|x\rangle$ represents a quantum state, then the quantity $QC(|x\rangle)$ is also equal to the number of physical qubits as defined in \cite{5} and the quantum state can be fully described/represented by the number $QC(|x\rangle)$ of qubits. \textit{On the other hand, it's important to point out that if vector $|x\rangle$ describes (i.e. maps) a unitary operator, it cannot be described/represented by qubits.}

Our paper is organized as follows: In Section 2, we define the Kolmogorov complexity of a unit vector in a Hilbert space, which is based on "qubit complexity"\cite{5} of Berthiaume, W. Van Dam and S. Laplante. In Section 3, we define the Kolmogorov complexity of a positive operator with trace 1 via the Kolmogorov complexity of the corresponding unit vector (i.e. purified state). In Section 4, we introduce the Kolmogorov complexity of a unitary transformation. In Section 5, we briefly discuss the operator convergence for approximate vector/operator reconstruction. In Section 6, we discuss the optimality of our notion of Kolmogorov complexity in a broad and permissive sense. In the Appendix, we review the properties of the quantum fidelity and its relationship with the trace distance.

\section*{II. KOLMOGOROV COMPLEXITY OF A UNIT VECTOR}

Berthiaume et. al in \cite{5} define "qubit complexity" and use the fidelity measure of how close two quantum states. We use their definition of "qubit complexity" with the following two amendments:
1. We restrict the original definition to pure quantum state states only (which corresponds to unit vectors). We note that, as the quantum Turing machine (QTM) output is a pure state, the input is necessarily pure, too.

2. Instead of fidelity, we use the trace distance as a measure of how close two operators are. As we summarize in the Appendix, in view of relation (26), the trace distance is essentially equivalent to the fidelity measure for quantum states. Moreover, the trace distance is well defined for all operators acting on Hilbert space, including unitary transformations.

The trace distance between operators $\rho$ and $\sigma$ is defined by

$$D(\rho, \sigma) \triangleq \frac{1}{2} \text{tr}|\rho - \sigma|,$$

where $|A| \triangleq \sqrt{A^\dagger A}$. From this definition it follows that the trace distance is a genuine metric on quantum states, with $0 \leq D \leq 1$. Note that in literature, trace distance is sometimes defined as $\text{tr}|\rho - \sigma|$, without factor $\frac{1}{2}$, which directly matches the operator trace norm $\|A\|_1 \equiv \text{tr} (\sqrt{A^\dagger A})$.

**Definition 1** *Kolmogorov complexity with trace distance $\delta$*: For any quantum Turing machine $M$ and pure qubit state $|x\rangle$, the $\delta$-approximation Kolmogorov complexity, denoted $QC^\delta_M(|x\rangle)$, is the length of the shortest qubit string $|y\rangle$ such that, for any $\delta > 0$, we have $D(|x\rangle, M(|y\rangle))) \leq \delta$.

We slightly abuse notation by using vectors as the inputs for the trace distance $D(\cdot, \cdot)$, in which we understand any such vector $|x\rangle$ as operator $|x\rangle\langle x|$. 

**Remark 1** Where the unit vector $|x\rangle$ represents a quantum state of a physical system, the internal workings of the quantum Turing machine become relevant. Overwise, we are only concerned with the input and output of the unitary transformation implemented by QTM. The invariance of Kolmogorov complexity and all the convergence results of remain applicable.

If the trace distance $\delta$ is equal zero, we have the following definition.

**Definition 2** *Kolmogorov complexity with perfect reconstruction*: The perfect reconstruction Kolmogorov complexity is $QC^0_M(|x\rangle)$. 

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Thus, \( QC^1_M(X) \) of [3] is qual to \( QC^0_M(|x\rangle) \) of this paper, where the unit vector \(|x\rangle\) is the state of qubit string \( X \) in [3].

**Remark 2** Thought the paper, we will omit the subscript \( M \) in the notation \( QC^\delta_M(|x\rangle) \). Additionally, we will omit the superscript whenever \( \delta = 0 \), that is for the case of perfect reconstruction.

### III. Kolmogorov Complexity of a Positive Operator with Trace 1

Let \( \rho \) be a positive operator with trace 1 acting on a Hilbert space \( \mathcal{H}^{\otimes n} \). First, we need to "purify"\([1]\) \( \rho \) as follows. We use a common notation \(|i\rangle\) for the eigenvector basis of \( \rho \). Let \( \rho \) have the following orthonormal decomposition:

\[
\rho = \sum_{i=1}^{2^n} p_i |i\rangle \langle i|,
\]

(2)

where \( \sum_{i=1}^{2^n} p_i = 1 \). We note that operator \( \rho \) acts on the Hilbert space \( \mathcal{H}^{\otimes n} \) and we label the underlying system by \( A \). To purify \( \rho \), we introduce another, axillary system, which we label by \( R \) and which has a state space identical with \( A \), with orthonormal basis states \(|i^R\rangle\). We now define an pure state for the combined system with the state space \( \mathcal{H}^{\otimes n} \otimes \mathcal{H}^{\otimes n} \) as follows:

\[
|AR\rangle \triangleq \sum_i \sqrt{p_i} |i^A\rangle |i^R\rangle.
\]

(3)

It’s easy to see that the operator \( \rho \) can be obtained from the pure state \(|AR\rangle\) by tracing out the auxiliary system \( R \) as follows:

\[
\text{tr}_R (|AR\rangle \langle AR|) = \sum_{ij} \sqrt{p_i p_j} |i^A\rangle \langle j^A| \text{tr} (|i^R\rangle \langle j^R|)
\]

\[
\quad = \sum_{ij} \sqrt{p_i p_j} |i^A\rangle \langle j^A| \delta_{ij}
\]

\[
\quad = \sum_i p_i |i^A\rangle \langle i^A|
\]

\[
\quad = \rho.
\]

(4)

To summarize the above construction, for each \( \rho \), there is a corresponding unit vector \(|AR\rangle\):

\[
\rho \xrightarrow{\text{purifying } \rho} |AR\rangle
\]

(5)
Now, we define the Kolmogorov complexity of any positive operator \( \rho \) with trace 1 as the complexity of the corresponding unit vector \( |AR\rangle \):

**Definition 3**

\[
K(\rho) \triangleq QC(|AR\rangle)
\]  

(6)

**Remark 3** The idea of using purification of a mixed quantum state and then defining the complexity of the mixed state via the complexity of the purified state has been previously explored in [7].

**IV. KOLMOGOROV COMPLEXITY OF A UNITARY TRANSFORMATION**

In this section, we provide a new, universal definition of the Kolmogorov complexity of a unitary transformation \( U \), via the Kolmogorov complexity of a (purified state) unit vector.

Let \( U \) be a unitary transformation on \( \mathcal{H}^{\otimes n} \), represented by a \( 2^n \times 2^n \) unitary matrix. It preserves inner product and has a property \( U^\dagger U = I \).

First, we define an operator \( S \):

\[
S \triangleq i \ln U.
\]  

(7)

Since operator \( U \) is unitary, operator \( S \) is necessarily Hermitian. The logarithm of a unitary matrix is not uniquely defined as is the logarithm of a complex number. We can select \( S \) with eigenvalues in the range 0 to \( 2\pi \) as follows. A unitary operator is a diagonalizable operator whose eigenvalues all have unit norm. In the eigenvector basis of \( U \), we have a matrix of the following form:

\[
U = \begin{pmatrix}
    e^{-i\varphi_1} & 0 & \ldots & 0 \\
    0 & e^{-i\varphi_2} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & e^{-i\varphi_{2^n}}
\end{pmatrix},
\]  

(8)

with the eigenvalues of the form \( e^{-i\varphi_i} \). So the eigenvalues of \( S \) will be

\[
\varphi_i = i \ln U_{ii},
\]  

(9)

where \( \varphi_i \in [0, 2\pi] \) and \( i = 1, \ldots, 2^n \).
While $S$ is a non-negative-definite operator with non-negative eigenvalues, the trace $\text{tr}(S)$ is not necessarily equal to 1. Therefore, we aim to modify operator $S$, so that to have trace 1. However, a simple normalization $\frac{S}{\text{tr}(S)}$ will not work because the normalized operator $\frac{S}{\text{tr}(S)}$ would no longer match the unitary transformation. Instead, we will first modify the unitary transformation itself by multiplying $U$ by a global phase factor $e^{i\alpha}$ and select $\alpha$ so that the resulting operator $i \ln (e^{i\alpha}U)$ has trace equal 1. Such a modification of the unitary transformation is appropriate for quantum computations because global phase factors do not change the observed properties of the physical system [1].

So, $\alpha$ must satisfy the equality
\[
1 = \sum_{i=1}^{2^n} (\varphi_i - \alpha) .
\] (10)

Thus, we set $\alpha$ as follows:
\[
\alpha \triangleq \frac{1}{2^n} \left( -1 + \sum_{i=1}^{2^n} \varphi_i \right) .
\] (11)

Now, we define operator $\rho$:
\[
\rho \triangleq i \ln (e^{i\alpha}U)
\] (12)

Thus, in the eigenvector basis of $U$, operator $\rho$ is represented by a matrix of the following form:
\[
\rho \equiv \begin{pmatrix}
\varphi_1 - \frac{\alpha}{2^n} & 0 & \ldots & 0 \\
0 & \varphi_2 - \frac{\alpha}{2^n} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \varphi_{2^n} - \frac{\alpha}{2^n}
\end{pmatrix},
\] (13)

Note that the constructed operator $\rho$ is Hermitian and has trace 1, but some of its eigenvalues may be negative, so $\rho$ may not always represent a physical quantum state.

Now we are going to "purify" $\rho$ as follows. We use a common notation $|i\rangle$ for the eigenvector basis of $U$ and $\rho$, thus, $\rho$ has the following orthonormal decomposition:
\[
\rho = \sum_{i=1}^{2^n} p_i |i\rangle\langle i|,
\] (14)

where $p_i \triangleq \varphi_i - \frac{\alpha}{2^n}$ and $\sum_{i=1}^{2^n} p_i = 1$. We note that operator $\rho$ acts on the Hilbert space $\mathcal{H}^{\otimes n}$ and we label the underlying system by $A$. To purify $\rho$, we introduce another, auxiliary system, which we label by $R$ and which has a state space identical with $A$, with orthonormal
basis states $|i^R\rangle$. We now define an pure state for the combined system with the state space $\mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n$ as follows:

$$|AR\rangle \triangleq \sum_i \sqrt{p_i} |i^A\rangle |i^R\rangle.$$  \hspace{1cm} (15)

Note that some of $\sqrt{p_i}$ can be complex numbers.

It’s easy to see that the operator $\rho$ can be obtained from the pure state $|AR\rangle$ by tracing out the auxiliary system $R$ as follows:

$$\text{tr}_R (|AR\rangle \langle AR|) = \sum_{ij} \sqrt{p_i p_j} |i^A\rangle \langle j^A| \text{tr} (|i^R\rangle \langle j^R|)$$

$$= \sum_{ij} \sqrt{p_i p_j} |i^A\rangle \langle j^A| \delta_{ij}$$

$$= \sum_i p_i |i^A\rangle \langle i^A|$$

$$= \rho.$$ \hspace{1cm} (16)

To summarize the above construction, for each unitary transformation $U$, there is a corresponding unit vector $|AR\rangle$:

$$U \rightarrow \rho \xrightarrow{\text{purifying \rho}} |AR\rangle$$ \hspace{1cm} (17)

Finally, we define the Kolmogorov complexity of a unitary transformation $U$ as the complexity of vector $|AR\rangle$:

**Definition 4**

$$K(U) \triangleq QC (|AR\rangle)$$ \hspace{1cm} (18)

Clearly, the upper bound on $K(U)$ is equal to the logarithm in base 2 of the dimension of the Hilbert space $\mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n$, i.e. $2^n$.

**Remark 4** As vector $|AR\rangle$ describes (i.e. maps) a unitary operator, it cannot be described/represented by qubits. The quantity $QC (|AR\rangle)$ shall be understood as the logarithm of the dimension of the respective Hilbert space as discussed in Chapter 4.

V. OPERATOR CONVERGENCE FOR APPROXIMATE RECONSTRUCTION

So far, we have focused on perfect reconstruction of the vector or operator. In this section, we briefly discuss the convergence for an approximate (imperfect) reconstruction.
For all purifications $\psi$ of $\rho$ and $\varphi$ of $\sigma$, the following inequality holds:

$$D(\rho, \sigma) \leq D(\langle \psi | \psi \rangle, \langle \varphi | \varphi \rangle) \quad (19)$$

The exponential of a normal operator is well defined and continuous operator function. Therefore, the convergence of $|\varphi\rangle$ to $|AR\rangle$ along with inequality (19) implies the convergence of $e^{-i\cdot tr_R(\langle \varphi | \varphi \rangle)}$ to $e^{i\cdot \alpha}U$ with respect to the operator trace norm $\|A\|_1$, where $tr_R(\cdot)$ is the partial trace notation as in (16).

**Remark 5** Strictly speaking, in view of equality (19), the definition (18) of Kolmogorov complexity of a unitary transformation is actually an upper bound on the complexity. However, the bound is tight as discussed in the next section.

**VI. DISCUSSION**

In this section, we shall understand Kolmogorov complexity in the most broad and permissive sense. We emphasise that, despite the presence of word "complexity", the notion of Kolmogorov complexity is not about a computational time or a computational space complexity, but is about the shortest (most compact) possible description or representation of the object or system. In particular, the complexity of one vector or operator shall be equal (up to $O(1)$) to the complexity of another vector or operator if one can be converted to the other using a finite amount of information.

Logarithmic and exponential functions are known to have implementations in Fortran and C. The size of the program text file is 20–100 Kb. The calculation precision can be arbitrary high and depends on hardware, but does not affect the program size. So are the Fortran and C programs for other related calculations. Therefore, the Kolmogorov (descriptive) complexity of mapping

$$U \to \rho \quad (20)$$

is $O(1)$. Similarly, we can argue that the Kolmogorov (descriptive) complexity of mapping

$$\rho \to |AR\rangle \quad (21)$$

is $O(1)$ and such a mapping injects just a constant amount of information into $|AR\rangle$, which does not depend on $n$. 

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Remark 6 To obtain an infinitely high calculation precision we may need an auxiliary Turing machine with an infinite tape. Nevertheless, the program size remains finite and does not depend on $n$.

The $O(1)$ complexity of mapping (20) and the $O(1)$ complexity of mapping (21) imply that, for any consistent Kolmogorov complexity definition, the complexity of unitary transformation $U$ and the complexity of the corresponding vector $|AR\rangle$ are equal up to $O(1)$. Thus, via mapping (17), we have established the duality of the Kolmogorov complexity of unitary transformation and the Kolmogorov complexity of a unit vector.

VII. CONCLUSION

We have defined the Kolmogorov complexity of a quantum density operator and that of a unitary transformation and established its relation with the qubit complexity of a quantum state. Thus, the concept of Kolmogorov complexity has become ever more relevant in quantum computing context. It provides theoretical framework for quantum computing complexity analysis and can be further linked to the complexity of a quantum computer program at high level and low (hardware) level, as well as quantum computer compilers.

VIII. APPENDIX: THE TRACE DISTANCE AND QUANTUM FIDELITY

In quantum computing, most commonly used are the following two measures of how close two quantum states are: quantum fidelity and the trace distance. Below we review their properties, for details, see [1].

The trace distance: The trace distance between quantum states $\rho$ and $\sigma$ is defined by

$$D(\rho, \sigma) \triangleq \frac{1}{2} \text{tr}|\rho - \sigma|,$$

where $|A| \triangleq \sqrt{A^\dagger A}$. From this definition it follows that the trace distance is a genuine metric on quantum states, with $0 \leq D \leq 1$. Trace distance is well defined not only for density operators, but also for all operators acting on Hilbert space, including unitary transformations. Note that in literature, trace distance is sometimes defined as $\text{tr}|\rho - \sigma|$, without factor $\frac{1}{2}$, which directly matches the operator trace norm $\|A\|_1 \equiv \text{tr} \left(\sqrt{A^\dagger A}\right)$. 

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The fidelity between quantum states $\rho$ and $\sigma$ is defined by

$$F(\rho, \sigma) \triangleq \text{tr} \left( \sqrt{\sqrt{\rho} \cdot \sigma \cdot \sqrt{\rho}} \right).$$  \hspace{1cm} (23)$$

If $\rho$ is a pure state $\psi$, and $\sigma$ is a pure state $\varphi$, then the above definition reduces to $|\langle \psi | \varphi \rangle|$. It can be shown that $0 \leq F(\rho, \sigma) \leq 1$. If $F(\rho, \sigma) = 1$, then $\rho = \sigma$, and vice versa.

For all purifications $\psi$ of $\rho$ and $\varphi$ of $\sigma$, the following two inequalities hold:

$$F(\rho, \sigma) \geq F(|\psi \rangle \langle \psi|, |\varphi \rangle \langle \varphi|) \equiv |\langle \psi | \varphi \rangle|$$  \hspace{1cm} (24)

$$D(\rho, \sigma) \leq D(|\psi \rangle \langle \psi|, |\varphi \rangle \langle \varphi|)$$  \hspace{1cm} (25)

Although not a metric, the fidelity upper bounds and lower bounds the trace distance, so whenever the fidelity converges to one, the trace distance converges to zero:

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F^2(\rho, \sigma)}$$  \hspace{1cm} (26)

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