Pairs of square-free values of the type $n^2 + 1, n^2 + 2$

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Abstract

In the present paper we show that there exist infinitely many consecutive square-free numbers of the form $n^2 + 1, n^2 + 2$. We also establish an asymptotic formula for the number of such square-free pairs when $n$ does not exceed given sufficiently large positive number.

Keywords: Square-free numbers, Asymptotic formula, Kloosterman sum.

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1 Notations

Let $X$ be a sufficiently large positive number. By $\varepsilon$ we denote an arbitrary small positive number, not necessarily the same in different occurrences. As usual $\mu(n)$ is Möbius’ function and $\tau(n)$ denotes the number of positive divisors of $n$. Further $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of $t$. We shall use the convention that a congruence, $m \equiv n \pmod{d}$ will be written as $m \equiv n(d)$. As usual $(m,n)$ is the greatest common divisor of $m$ and $n$. The letter $p$ will always denote prime number. We put

$$\psi(t) = \{t\} - 1/2. \tag{1}$$

Moreover $e(t) = \exp(2\pi it)$. For $x, y \in \mathbb{R}$ we write $x \equiv y \pmod{1}$ when $x - y \in \mathbb{Z}$. For any $n$ and $q$ such that $(n, q) = 1$ we denote by $\pi_q$ the inverse of $n$ modulo $q$. The number of distinct prime factors of a natural number $n$ we denote by $\omega(n)$. For any odd prime number $p$ we denote by $\left(\frac{\cdot}{p}\right)$ the Legendre symbol. By $K(r,h)$ we shall denote the incomplete Kloosterman sum

$$K(r,h) = \sum_{\alpha \leq x < \beta \atop (x,r) = 1} e\left(\frac{h\pi|\alpha|}{r}\right), \tag{2}$$

where

$$h, r \in \mathbb{Z}, \quad hr \neq 0, \quad 0 < \beta - \alpha \leq 2|r|.$$
2 Introduction and statement of the result

In 1931 Estermann \[6\] proved that there exist infinitely many square-free numbers of the form \( n^2 + 1 \). More precisely he proved that for \( X \geq 2 \) the asymptotic formula

\[
\sum_{n \leq X} \mu^2(n^2 + 1) = c_0 X + \mathcal{O}\left( X^{\frac{2}{3}}\log X \right)
\]

holds. Here

\[
c_0 = \prod_{p \equiv 1 (4)} \left( 1 - \frac{2}{p^2} \right).
\]

Afterwards Heath-Brown \[8\] used a variant of the determinant method and improved the remainder term in the formula of Estermann with \( \mathcal{O}\left( X^{7/12+\varepsilon} \right) \).

On the other hand 1932 Carlitz \[1\] showed that there exist infinitely many pairs of consecutive square-free numbers. More precisely he proved the asymptotic formula

\[
\sum_{n \leq X} \mu^2(n)\mu^2(n+1) = \prod_p \left( 1 - \frac{2}{p^2} \right) X + \mathcal{O}\left( X^{\theta+\varepsilon} \right), \tag{3}
\]

where \( \theta = 2/3 \). Formula (3) was sharpened by Heath-Brown \[7\] to \( \theta = 7/11 \) and by Reuss \[10\] to \( \theta = (26 + \sqrt{433})/81 \).

The existence of infinitely many consecutive square-free numbers of a special form was demonstrated by the author in \[2, 3, 4, 5\]. In particular in \[5\] he proved that there exist infinitely many consecutive square-free numbers of the form \( x^2 + y^2 + 1, x^2 + y^2 + 2 \). While in \[5\] the main role was played by the properties of Gauss sums, in this paper we use a bijective correspondence between the number of representations of numbers by binary quadratic form and the the incongruent solutions of quadratic congruence.

Define

\[
\Gamma(X) = \sum_{1 \leq n \leq X} \mu^2(n^2 + 1) \mu^2(n^2 + 2), \tag{4}
\]

\[
S(q_1, q_2) = \{ n \in \mathbb{N} : 1 \leq n \leq q_1 q_2, \ n^2 + 1 \equiv 0 (q_1), \ n^2 + 2 \equiv 0 (q_2) \} \tag{5}
\]

and

\[
\lambda(q_1, q_2) = \sum_{n \in S(q_1, q_2)} 1. \tag{6}
\]

We establish our result by combining the tasks of Estermann and Carlitz. Thus we prove the following theorem.
**Theorem 1.** For the sum $\Gamma(X)$ defined by (4) the asymptotic formula

$$\Gamma(X) = \sigma X + O\left(X^{\frac{5}{2}+\varepsilon}\right)$$

(7)

holds. Here

$$\sigma = \prod_{p>2} \left(1 - \frac{\left(-\frac{1}{p}\right)+\left(-\frac{2}{p}\right)+2}{p^2}\right).$$

(8)

From Theorem 1 it follows that there exist infinitely many consecutive square-free numbers of the form $n^2 + 1$, $n^2 + 2$, where $n$ runs over naturals.

### 3 Lemmas

The first lemma we need gives us important expansions.

**Lemma 1.** For any $M \geq 2$, we have

$$\psi(t) = -\sum_{1 \leq |m| \leq M} \frac{e(mt)}{2\pi im} + O\left(f_M(t)\right),$$

where $f_M(t)$ is a positive function of $t$ which is infinitely many times differentiable and periodic with period 1. It can be expanded into the Fourier series

$$f_M(t) = \sum_{m=-\infty}^{+\infty} b_M(m)e(mt),$$

with coefficients $b_M(m)$ such that

$$b_M(m) \ll \frac{\log M}{M} \quad \text{for all } m$$

and

$$\sum_{|m|>M^{1+\varepsilon}} |b_M(m)| \ll M^{-A}.$$  

Here $A > 0$ is arbitrarily large and the constant in the $\ll$ - symbol depends on $A$ and $\varepsilon$.

**Proof.** See ([11], Theorem 1).

The next lemma we need is well-known.

**Lemma 2.** Let $A, B \in \mathbb{Z} \setminus \{0\}$ and $(A, B) = 1$. Then

$$\frac{\overline{A}}{|B|} + \frac{\overline{B}|A|}{A} \equiv \frac{1}{AB} \quad (1).$$
Lemma 3. For the sum denoted by (2) the estimate
\[ K(r, h) \ll |r|^\frac{1}{2} + \epsilon (r, h)^{\frac{1}{2}} \]
holds.

Proof. Follows easily from A. Weil’s estimate for the Kloosterman sum. See ([9], Ch. 11, Corollary 11.12).

Lemma 4. Let \( n \geq 5 \). There exists a bijective function from the solution set of the equation
\[ x^2 + 2y^2 = n, \quad (x, y) = 1, \quad x \in \mathbb{N}, \quad y \in \mathbb{Z} \setminus \{0\} \tag{9} \]
to the incongruent solutions modulo \( n \) of the congruence
\[ z^2 + 2 \equiv 0 \pmod{n}. \tag{10} \]

Proof. Let \( F \) denote the set of ordered pairs \((x, y)\) satisfying (9) and \( E \) denote the set of solutions of the congruence (10). We consider each residue class modulo \( n \) with representatives satisfying (10) as one solution of (10). Let \((x, y) \in F\). From (9) it follows that \((n, y) = 1\). Therefore there exists a unique residue class \( z \) modulo \( n \) such that
\[ zy \equiv x(n). \tag{11} \]

For this class we have
\[ (z^2 + 2)y^2 \equiv (zy)^2 + 2y^2 \equiv x^2 + 2y^2 \equiv 0(n). \]

From the last congruence and \((n, y) = 1\) we deduce \( z^2 + 2 \equiv 0(n) \) which means that \( z \in E \). We define the map
\[ \beta : F \to E \tag{12} \]
that associates to each pair \((x, y) \in F\) the residue class \( z = xy_n \) satisfying (11).

We will first prove that the map (12) is a injection. Let \((x, y), (x', y') \in F\) that is
\[ \begin{array}{l}
  x^2 + 2y^2 = n \\
  x'^2 + 2y'^2 = n
\end{array}, \quad \begin{array}{l}
  (x, y) = (x', y') = 1
\end{array} \tag{13} \quad \begin{array}{l}
  (x, y) \neq (x', y'). \tag{15}
\end{array} \]
Assume that
\[
\beta(x, y) = \beta(x', y').
\] (16)

Hence there exists \( z \in E \) such that
\[
\begin{align*}
zy & \equiv x(n) \\
zy' & \equiv x'(n)
\end{align*}
\] (17)

The system (17) implies
\[
xy' - x'y \equiv 0(n).
\] (18)

On the other hand (13) and \( n \geq 5 \) yield
\[
\begin{align*}
0 & < x, x' < \sqrt{n} \\
0 & < |y|, |y'| < \sqrt{\frac{n}{2}}.
\end{align*}
\] (19)

We first consider the case \( yy' > 0 \). By (19) we derive
\[
\begin{align*}
0 & < xy' < \frac{n}{\sqrt{2}} \\
0 & < x'y < \frac{n}{\sqrt{2}}
\end{align*}
\] or
\[
\begin{align*}
-\frac{n}{\sqrt{2}} & < xy' < 0 \\
-\frac{n}{\sqrt{2}} & < x'y < 0
\end{align*}
\]

and consequently
\[
-\frac{n}{\sqrt{2}} < xy' - x'y < \frac{n}{\sqrt{2}}.
\] (20)

Now (18) and (20) lead to
\[
xy' - x'y = 0
\]
which together with (14) gives us
\[
x = x', \quad y = y'.
\] (21)

From (15) and (21) we get a contradiction.

Next we consider the case \( yy' < 0 \). By (19) we deduce
\[
\begin{align*}
0 & < xy' < \frac{n}{\sqrt{2}} \\
-\frac{n}{\sqrt{2}} & < x'y < 0
\end{align*}
\] or
\[
\begin{align*}
-\frac{n}{\sqrt{2}} & < xy' < 0 \\
0 & < x'y < \frac{n}{\sqrt{2}}
\end{align*}
\]

and therefore
\[
- n\sqrt{2} < xy' - x'y < n\sqrt{2}.
\] (22)

Now (18) and (22) lead to
\[
xy' - x'y = n \quad \text{or} \quad xy' - x'y = -n.
\] (23)
Raising to the second power one of the equations (23) we deduce

\[ x^2 y'^2 - 2xx' yy' + x'^2 y'^2 = n^2 \]

which is equivalent to

\[ (x^2 + 2y^2)y'^2 + (x'^2 + 2y'^2) - 4y^2 y'^2 - 2xx' yy' = n^2. \]

The last equation, (13) and (14) assure us that

\[ 2(xx' + 2yy') \equiv 0 \pmod{n}. \] (24)

From (19) we get

\[ 0 < xx' < n \]
\[ -n < 2y'y < 0 \]

and therefore

\[ -n < xx' + 2yy' < n. \] (25)

If \( n \) is odd then (24) implies

\[ xx' + 2yy' \equiv 0 \pmod{n}. \] (26)

which together with (25) yield

\[ xx' + 2yy' = 0. \] (27)

On the other hand when \( n \) is odd by (13) it follows that \( x \) and \( x' \) are odd which contradicts (27). Consequently \( n \) cannot be odd.

Let \( n \) be even. Now (13) and (14) give us that \( x, x' \) are even, \( y, y' \) are odd and \( 4 \nmid n \). These considerations and (24) lead to (26) which together with (25) imply (27). But equation (27) for even \( x \) and \( x' \) means that \( yy' \) is even which contradicts (14). Consequently \( n \) cannot be even. The resulting contradictions show that the assumption (16) is not true. This proves the injectivity of \( \beta \).

It remains to show that the map (12) is a surjection. Let \( z \in E \). From Dirichlet’s approximation theorem it follows that there exist integers \( a \) and \( q \) such that

\[ \left| \frac{z}{n} - \frac{a}{q} \right| < \frac{1}{q\sqrt{n}}, \quad 1 \leq q \leq \sqrt{n}, \quad (a, q) = 1. \] (28)

Replace

\[ r = zq - an. \] (29)

Hence

\[ r^2 + 2q^2 = z^2 q^2 - 2zqan + a^2 n^2 + 2q^2 \equiv (z^2 + 2)q^2 \pmod{n}. \] (30)
From (10) and (30) it follows
\[ r^2 + 2q^2 \equiv 0 \pmod{n}. \tag{31} \]

By (28) and (29) we deduce
\[ |r| < \sqrt{n}. \tag{32} \]

Using (28) and (32) we obtain
\[ 0 < r^2 + 2q^2 < 3n. \tag{33} \]

Bearing in mind (31) and (33) we conclude that \( r^2 + 2q^2 = n \) or \( r^2 + 2q^2 = 2n \).

Consider two cases.

Case 1
\[ r^2 + 2q^2 = n. \tag{34} \]

From (29) and (34) we get
\[ n = (zq - an)^2 + 2q^2 = (zq - an)zq - (zq - an)an + 2q^2 = (zq - an)zq - ran + 2q^2 \]
and therefore
\[ ra + 1 = kq, \tag{35} \]
where
\[ k = \frac{z^2 + 2nq}{q - az}. \tag{36} \]

By (10) and (36) it follows that \( k \in \mathbb{Z} \) and taking into account (35) we deduce
\[ (r, q) = 1. \tag{37} \]

Using (34), (37) and \( n \geq 5 \) we establish that \( r \neq 0 \).

Consider first \( r > 0 \). Replace
\[ x = r, \quad y = q. \tag{38} \]

From (34), (37) and (38) it follows that \( (x, y) \in F \). Also (29) and (38) give us (11).

Consequently \( \beta(x, y) = z \).

Next we consider \( r < 0 \). Put
\[ x = -r, \quad y = -q. \tag{39} \]

Again (34), (37) and (39) lead to \( (x, y) \in F \). As well from (29) and (39) follows (11).

Therefore \( \beta(x, y) = z \).

Case 2
\[ r^2 + 2q^2 = 2n. \tag{40} \]
From (29) and (40) we find
\[ 2n = (zq - an)^2 + 2q^2 = (zq - an)zq - (zq - an)an + 2q^2 = (zq - an)zq - ran + 2q^2 \]
and thus
\[ ra + 2 = kq, \quad (41) \]
where \( k \) is denoted by (36). From (41) we conclude
\[ (r, q) \leq 2. \quad (42) \]
By (40), (42) and \( n \geq 5 \) we deduce that \( r \neq 0 \). On the other hand from (40) it follows that \( r \) is even. We replace \( r = 2r_0 \) in (40) and obtain
\[ q^2 + 2r_0^2 = n. \quad (43) \]
We shall verify that
\[ (r_0, q) = 1. \quad (44) \]
If we assume that \( (r_0, q) > 1 \) then (42) gives us
\[ (r_0, q) = 2. \quad (45) \]
From (43) and (45) it follows
\[ n \equiv 0 (4). \quad (46) \]
Finally (10) and (46) imply
\[ z^2 + 2 \equiv 0 (4) \]
which is impossible. This proves (44). No matter whether \( r \) is positive or negative we replace
\[ x = q, \quad y = -r_0. \quad (47) \]
Using (43), (44) and (47) we deduce that \( (x, y) \in F \). By (29) and (47) we get
\[ 2(zy - x) = -2(zr_0 + q) = -zr - 2q = -(z^2 + 2)q + zan \quad (48) \]
From (10) and (48) we conclude
\[ 2(zy - x) \equiv 0 (n). \quad (49) \]
If \( n \) is odd then (49) gives us (11). Consequently \( \beta(x, y) = z \).
Let \( n \) be even. Since (46) is impossible then
\[ n = 2n_0, \quad n_0 \text{ is odd}. \quad (50) \]
By (43) and (50) it follows
\[ q \equiv 0 \pmod{2}, \tag{51} \]
i.e. \( q \) is even. On the other hand (10) and (50) imply that
\[ z \equiv 0 \pmod{2}, \tag{52} \]
i.e. \( z \) is even. Now (47), (51) and (52) give us
\[ zy - x \equiv 0 \pmod{2}, \tag{53} \]
i.e. \( zy - x \) is even. Finally from (49), (50) and (53) we obtain (11). Therefore
\[ \beta(x, y) = z. \]
The lemma is proved.

4 Proof of the theorem

Using (4) and the well-known identity
\[ \mu^2(n) = \sum_{d \mid n} \mu(d) \]
we get
\[ \Gamma(X) = \sum_{d_1, d_2 \leq z} \mu(d_1) \mu(d_2) \sum_{1 \leq n \leq X \atop \substack{n^2 + i \equiv 0 (d_2^2) \\ n^2 + 2 \equiv 0 (d_2^2)}} 1 = \Gamma_1(X) + \Gamma_2(X), \tag{54} \]
where
\[ \Gamma_1(X) = \sum_{d_1, d_2 \leq z} \mu(d_1) \mu(d_2) \Sigma(X, d_1^2, d_2^2), \tag{55} \]
\[ \Gamma_2(X) = \sum_{d_1, d_2 > z} \mu(d_1) \mu(d_2) \Sigma(X, d_1^2, d_2^2), \tag{56} \]
\[ \Sigma(X, d_1^2, d_2^2) = \sum_{1 \leq n \leq X \atop \substack{n^2 + 1 \equiv 0 (d_2^2) \\ n^2 + 2 \equiv 0 (d_2^2)}} 1, \tag{57} \]
\[ \sqrt{X} \leq z < X, \tag{58} \]
where \( z \) is to be chosen later.

4.1 Estimation of \( \Gamma_1(X) \)

Suppose that \( q_1 = d_1^2, q_2 = d_2^2 \), where \( d_1 \) and \( d_2 \) are square-free, \( (q_1, q_2) = 1 \) and \( d_1 d_2 \leq z \). Denote
\[ \Omega(X, q_1, q_2, n) = \sum_{n \leq X \atop n \equiv n (q_1 q_2)} 1. \tag{59} \]
Using (5), (57) and (59) we obtain upon partitioning the sum (57) into residue classes modulo $q_1q_2$

$$\Sigma(X, q_1, q_2) = \sum_{n \in S(q_1, q_2)} \Omega(X, q_1, q_2, n).$$  \hfill (60)

It is easy to see that

$$\Omega(X, q_1, q_2, n) = \frac{X}{q_1q_2} + O(1).$$  \hfill (61)

From (6), (60) and (61) we find

$$\Sigma(X, q_1, q_2) = X\frac{\lambda(q_1, q_2)}{q_1q_2} + O\left(\lambda(q_1, q_2)\right).$$  \hfill (62)

Taking into account (5), (6), Chinese remainder theorem and that the number of solutions of the congruence $n^2 \equiv a (q_1q_2)$ is less than or equal to $\tau(q_1q_2)$ we get

$$\lambda(q_1, q_2) \ll \tau(q_1q_2).$$  \hfill (63)

From (62), (63) and the inequalities

$$\tau(q_1q_2) \ll (q_1q_2)^\varepsilon \ll X^\varepsilon$$

it follows

$$\Sigma(X, q_1, q_2) = X\frac{\lambda(q_1, q_2)}{q_1q_2} + O\left(X^\varepsilon\right).$$  \hfill (64)

Bearing in mind (55), (58) and (64) we obtain

$$\Gamma_1(X) = X \sum_{d_1, d_2 \leq z \atop (d_1, d_2) = 1} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2, d_2^2)}{d_1^2d_2^2} + O\left(zX^\varepsilon\right)$$

$$= \sigma X - X \sum_{d_1, d_2 > z \atop (d_1, d_2) = 1} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2, d_2^2)}{d_1^2d_2^2} + O\left(zX^\varepsilon\right),$$  \hfill (65)

where

$$\sigma = \sum_{d_1, d_2 = 1 \atop (d_1, d_2) = 1} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2, d_2^2)}{d_1^2d_2^2}.\tag{66}$$

Using (63) we find

$$\sum_{d_1, d_2 > z \atop (d_1, d_2) = 1} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2, d_2^2)}{d_1^2d_2^2} \ll \sum_{d_1, d_2 > z \atop (d_1, d_2) = 1} \frac{(d_1d_2)^\varepsilon}{(d_1d_2)^2} \ll \sum_{n > z} \frac{\tau(n)}{n^{2-\varepsilon}} \ll z^{\varepsilon-1}.\tag{67}$$
It remains to see that the product (8) and the sum (66) coincide. From the definition (6) it follows that the function \( \lambda(q_1, q_2) \) is multiplicative, i.e if

\[
(q_1q_2, q_3q_4) = (q_1, q_2) = (q_3, q_4) = 1
\]

then

\[
\lambda(q_1q_2, q_3q_4) = \lambda(q_1, q_3)\lambda(q_2, q_4). \tag{68}
\]

The proof is elementary and we leave it to the reader.

From the property (68) and \( (d_1, d_2) = 1 \) it follows

\[
\lambda(d_1^2, d_2^2) = \lambda(d_1^2, 1)\lambda(1, d_2^2). \tag{69}
\]

Bearing in mind (66) and (69) we get

\[
\sigma = \sum_{d_1=1}^{\infty} \mu(d_1)\lambda(d_1^2, 1) \sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2), \tag{70}
\]

where

\[
f_{d_1}(d_2) = \begin{cases} 1 & \text{if } (d_1, d_2) = 1, \\
0 & \text{if } (d_1, d_2) > 1. \end{cases}
\]

Clearly the function

\[
\frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2)
\]

is multiplicative with respect to \( d_2 \) and the series

\[
\sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2)
\]

is absolutely convergent.

Applying the Euler product we obtain

\[
\sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^2} f_{d_1}(d_2) = \prod_{p|d_1} \left( 1 - \frac{\lambda(1, p^2)}{p^2} \right) \prod_{p|d_1} \left( 1 - \frac{\lambda(1, p^2)}{p^2} \right)^{-1}. \tag{71}
\]

From (70) and (71) it follows

\[
\sigma = \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p} \left( 1 - \frac{\lambda(1, p^2)}{p^2} \right) \prod_{p|d_1} \left( 1 - \frac{\lambda(1, p^2)}{p^2} \right)^{-1}
\]

\[
= \prod_{p} \left( 1 - \frac{\lambda(1, p^2)}{p^2} \right) \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p|d_1} \left( 1 - \frac{\lambda(1, p^2)}{p^2} \right)^{-1}. \tag{72}
\]
Obviously the function
\[
\frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}
\]
is multiplicative with respect to \(d_1\) and the series
\[
\sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^2} \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}
\]
is absolutely convergent.

Applying again the Euler product from (6) and (72) we find
\[
\sigma = \prod_p \left(1 - \frac{\lambda(1, p^2)}{p^2}\right) \prod_p \left(1 - \frac{\lambda(p^2, 1)}{p^2}\right) \left(1 - \frac{\lambda(1, p^2)}{p^2}\right)^{-1}
\]
\[
= \prod_p \left(1 - \frac{\lambda(p^2, 1)}{p^2} + \lambda(1, p^2)\right) = \prod_{p > 2} \left(1 - \frac{-1}{p} + \frac{-2}{p^2} + 2\right). \tag{73}
\]

Bearing in mind (58), (55), (67) and (73) we get
\[
\Gamma_1(X) = \sigma X + \mathcal{O}(z X^\varepsilon), \tag{74}
\]
where \(\sigma\) is given by the product (8).

4.2 Estimation of \(\Gamma_2(X)\)

Using (56), (57) and splitting the range of \(d_1\) and \(d_2\) into dyadic subintervals of the form \(D_1 \leq d_1 < 2D_1\), \(D_2 \leq d_2 < 2D_2\) we write
\[
\Gamma_2(X) \ll (\log X)^2 \sum_{n \leq X} \sum_{D_1 \leq d_1 < 2D_1} \sum_{D_2 \leq d_2 < 2D_2} \sum_{n^2 + 1 \equiv 0 (d_1^2)} \sum_{n^2 + 2 \equiv 0 (d_2^2)} 1, \tag{75}
\]
where
\[
\frac{1}{2} \leq D_1, D_2 \leq \sqrt{X^2 + 2}, \quad D_1 D_2 > \frac{z}{4}. \tag{76}
\]

On the one hand (73) gives us
\[
\Gamma_2(X) \ll X^\varepsilon \Sigma_1, \tag{77}
\]
where
\[
\Sigma_1 = \sum_{n \leq X} \sum_{D_1 \leq d_1 < 2D_1} \sum_{n^2 + 1 \equiv 0 (d_1^2)} 1. \tag{78}
\]
On the other hand (75) implies
\[
\Gamma_2(X) \ll X^\epsilon \Sigma_2, 
\]
where
\[
\Sigma_2 = \sum_{n \leq X} \sum_{D_2 \leq d_2 < 2D_2} \sum_{n^2 + 2a_0(d_2^2)} 1.
\]

**Estimation of \( \Sigma_1 \)**

Define
\[
\mathcal{N}_1(d) = \{ n \in \mathbb{N} : 1 \leq n \leq d, \ n^2 + 1 \equiv 0 (d) \},
\]
\[
\mathcal{N}_1'(d) = \{ n \in \mathbb{N} : 1 \leq n \leq d^2, \ n^2 + 1 \equiv 0 (d^2) \}.
\]

By (78) and (82) we obtain
\[
\Sigma_1 = \sum_{D_1 \leq d_1 < 2D_1} \sum_{n \in \mathcal{N}_1'(d_1)} \sum_{n^2 + 1 = \psi(n)} \left( \left[ \frac{X - n}{d_1^2} \right] - \left[ \frac{-n}{d_1^2} \right] \right)
\]
\[
= \sum_{D_1 \leq d_1 < 2D_1} \sum_{n \in \mathcal{N}_1'(d_1)} \left( \frac{X}{d_1^2} + \psi \left( \frac{-n}{d_1^2} \right) - \psi \left( \frac{X - n}{d_1^2} \right) \right)
\]
\[
\ll X^{1+\epsilon} D_1^{-1} + |\Sigma_1'| + |\Sigma_1''|,
\]
where
\[
\Sigma_1' = \sum_{D_1 \leq d_1 < 2D_1} \sum_{n \in \mathcal{N}_1'(d_1)} \psi \left( \frac{-n}{d_1^2} \right),
\]
\[
\Sigma_1'' = \sum_{D_1 \leq d_1 < 2D_1} \sum_{n \in \mathcal{N}_1'(d_1)} \psi \left( \frac{X - n}{d_1^2} \right)
\]
and \( \psi(t) \) is defined by (11).

Firstly we consider the sum \( \Sigma_1' \). We note that the sum over \( n \) in (84) does not contain terms with \( n = \frac{d_1^2}{2} \) and \( n = d_1^2 \). Moreover for any \( n \) satisfying the congruences \( n^2 + 1 \equiv 0 (d_1^2) \) and such that \( 1 \leq n < \frac{d_1^2}{2} \) the number \( d_1^2 - n \) satisfies the same congruence and we have \( \psi \left( \frac{-n}{d_1^2} \right) + \psi \left( \frac{-d_1^2 - n}{d_1^2} \right) = 0 \). Bearing in mind these arguments for the sum \( \Sigma_1' \) denoted by (84) we have that
\[
\Sigma_1' = 0.
\]

Next we consider the sum \( \Sigma_1'' \) denoted by (85). Let \( D_1 \leq X^{\frac{3}{4}} \). The trivial estimation gives us
\[
\Sigma_1'' \ll \sum_{D_1 \leq d_1 < 2D_1} d_1 \ll X^{\frac{3}{4} + \epsilon}.
\]
Let
\[ D_1 > X^{\frac{3}{4}}. \] 
(88)

From the theory of the quadratic congruences we know that when \( \#N'_1(d) \neq 0 \) then \( d \) is odd and
\[ \#N_1(d) = \#N'_1(d) = 2^\omega(d). \] 
(89)

Denote
\[ k = 2^\omega(d), \] 
(90)
\[ n_1, \ldots, n_k \in N_1(d_1), \; n'_1, \ldots, n'_k \in N'_1(d_1). \] 
(91)

From (81), (82), (88) – (91) and \( d \geq D_1 > X^{\frac{3}{4}} \) it follows
\[
\sum_{n \in N'_1(d_1)} \psi \left( \frac{X - n}{d_1^2} \right) = \sum_{n \in N'_1(d_1)} \left( \frac{X - n}{d_1^2} - \frac{1}{2} \right) \\
= \sum_{n \in N'_1(d_1)} \left( \frac{X}{d_1^2} - \frac{1}{2} \right) - \frac{n'_1 + \cdots + n'_{k/2} + (d_1^2 - n'_1) + \cdots + (d_1^2 - n'_{k/2})}{d_1^2} \\
= \sum_{n \in N_1(d_1)} \left( \frac{X}{d_1^2} - \frac{1}{2} \right) - n_1 + \cdots + n_{k/2} + (d_1 - n_1) + \cdots + (d_1 - n_{k/2}) \\
= \sum_{n \in N_1(d_1)} \left( \frac{X}{d_1^2} - \frac{\sqrt{X}}{d_1} \right) + \sum_{n \in N_1(d_1)} \left( \frac{\sqrt{X} - n}{d_1} - \frac{1}{2} \right) \\
= \sum_{n \in N_1(d_1)} \left( \frac{X}{d_1^2} - \frac{\sqrt{X}}{d_1} \right) + \sum_{n \in N_1(d_1)} \psi \left( \frac{\sqrt{X} - n}{d_1} \right). \] 
(92)

By (85), (88) and (92) we obtain
\[ \Sigma''_1 \ll X^{\frac{3}{4} + \varepsilon} + |\Sigma_3|, \] 
(93)

where
\[ \Sigma_3 = \sum_{D_1 \leq d_1 < 2D_1} \sum_{n \in N_1(d_1)} \psi \left( \frac{\sqrt{X} - n}{d_1} \right). \] 
(94)

Using (94) and Lemma 1 with
\[ M_1 = X^{\frac{1}{4}} \] 
(95)
we find
\[
\Sigma_3 = \sum_{D_1 \leq d_1 < 2D_1} \sum_{n \in \mathcal{N}_1(d_1)} \left( - \sum_{1 \leq |m| \leq M_1} e\left(\frac{m\sqrt{X-n}}{d_1}\right) \right) + O\left(f_{M_1}\left(\frac{\sqrt{X-n}}{d_1}\right)\right).
\]

Arguing as in (Tolev [12], Theorem 17.1.1) we deduce
\[
\Sigma_3 \ll X^{1+\varepsilon}\left(D_1^{-1} + D_1^2 + X^{\frac{1}{2}}D_1^{-\frac{1}{4}}\right).
\]  

(96)

Bearing in mind (83), (86), (87), (93), (95) and (96) we get
\[
\Sigma_1 \ll X^{1+\varepsilon}D_1^{-\frac{1}{4}}.
\]  

(97)

**Estimation of \( \Sigma_2 \)**

Our argument is a modification of (Tolev [12], Theorem 17.1.1) argument.

Define
\[
\mathcal{N}_2(d) = \{n \in \mathbb{N} : 1 \leq n \leq d, \ n^2 + 2 \equiv 0 (d)\}.
\]  

(98)

Working as in \( \Sigma_1 \) from (80) and (98) we find
\[
\Sigma_2 \ll X^{1+\varepsilon}D_1^{-1}
\]  

(99)

for \( D_2 \leq X^{\frac{1}{2}} \) and
\[
\Sigma_2 \ll X^{\frac{1}{2}+\varepsilon} + |\Sigma_4|
\]  

(100)

for
\[
D_2 > X^{\frac{1}{2}},
\]  

(101)

where
\[
\Sigma_4 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}_2(d_2)} \psi\left(\frac{\sqrt{X-n}}{d_2}\right).
\]  

(102)

From (102) and Lemma 1 with
\[
M_2 = X^{\frac{1}{2}}
\]  

(103)

we obtain
\[
\Sigma_4 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathcal{N}_2(d_2)} \left( - \sum_{1 \leq |m| \leq M_2} e\left(\frac{m\sqrt{X-n}}{d_2}\right) \right) + O\left(f_{M_2}\left(\frac{\sqrt{X-n}}{d_2}\right)\right)
\]  

\[
= \Sigma_5 + \Sigma_6,
\]  

(104)
where

\[ \Sigma_5 = \sum_{1 \leq |m| \leq M_2} \frac{\Theta_m}{2\pi im}, \]  

(105)

\[ \Theta_m = \sum_{D_2 \leq d_2 < 2D_2} e \left( \frac{\sqrt{X} m}{d_2} \right) \sum_{n \in \mathbb{N}_2(d_2)} e \left( -\frac{nm}{d_2} \right), \]  

(106)

\[ \Sigma_6 = \sum_{D_2 \leq d_2 < 2D_2} \sum_{n \in \mathbb{N}_2(d_2)} f_{M_2} \left( \frac{\sqrt{X} - n}{d_2} \right). \]  

(107)

By (106), (107) and Lemma 1 it follows

\[ \Sigma_6 \ll \log M_2 \sum_{1 \leq |m| \leq M_1^{1+\varepsilon}} |\Theta_m| + \sum_{|m| > M_2^{1+\varepsilon}} |b_{M_2}(m)| |\Theta_m| \]

\[ \ll \log M_2 \frac{D_2^{1+\varepsilon}}{M_2} + \log M_2 \sum_{1 \leq |m| \leq M_2^{1+\varepsilon}} |\Theta_m| + D_2^{1+\varepsilon} \sum_{|m| > M_2^{1+\varepsilon}} |b_{M_2}(m)| \]

\[ \ll \log M_2 \frac{D_2^{1+\varepsilon}}{M_2} + \log M_2 \sum_{1 \leq |m| \leq M_2^{1+\varepsilon}} |\Theta_m|. \]  

(108)

Using (104), (105) and (108) we get

\[ \Sigma_4 \ll X^\varepsilon \left( \frac{D_2}{M_2} + \sum_{1 \leq |m| \leq M_2^{1+\varepsilon}} \frac{|\Theta_m|}{m} \right). \]  

(109)

Define

\[ \mathcal{F}(d) = \{(u, v) : u^2 + 2v^2 = d, \ (u, v) = 1, \ u \in \mathbb{N}, \ v \in \mathbb{Z} \setminus \{0\} \}. \]  

(110)

According to Lemma 4 there exists a bijection

\[ \beta : \mathcal{F}(d) \rightarrow \mathcal{N}_2(d) \]

from \( \mathcal{F}(d) \) to \( \mathcal{N}_2(d) \) defined by (98) that associates to each couple \( (u, v) \in \mathcal{F}(d) \) the element \( n \in \mathcal{N}_2(d) \) satisfying

\[ nv \equiv u (d). \]  

(111)

Now (111) gives us

\[ n_{u,v} \equiv u\pi_d(d) \]
and therefore
\[
\frac{n_{u,v}}{d} \equiv u \frac{\overline{u}^2 + 2v^2}{u^2 + 2v^2} \quad (1).
\]

Bearing in mind (112) and Lemma 2 we deduce
\[
\begin{align*}
\frac{n_{u,v}}{d} & \equiv \frac{u}{v(u^2 + 2v^2)} - \frac{\overline{u}|v|}{v} \quad (1), \\
\frac{n_{u,v}}{d} & \equiv -\frac{2v}{u(u^2 + 2v^2)} + \frac{\overline{u}u}{u} \quad (1). 
\end{align*}
\]

From (106), (110), (113) and (114) we find
\[
\Theta_m = \sum_{D_2 \leq d_2 < 2D_2} e \left( \frac{m \sqrt{X}}{d_2} \right) \sum_{(u,v) \in F(d_2)} e \left( -\frac{n_{u,v}}{d_2} m \right) \\
= \sum_{D_2 \leq d_2 < 2D_2} e \left( \frac{m \sqrt{X}}{d_2} \right) \sum_{(u,v) \in F(d_2)} e \left( -\frac{mu}{v(u^2 + 2v^2)} + \frac{m|v|}{v} \right) \\
+ \sum_{D_2 \leq d_2 < 2D_2} e \left( \frac{m \sqrt{X}}{d_2} \right) \sum_{0 < |v| < u} e \left( \frac{2mv}{u(u^2 + 2v^2)} - \frac{m|v|}{u} \right) \\
= \sum_{D_2 \leq u^2 + 2v^2 < 2D_2} e \left( \frac{m \sqrt{X}}{u^2 + 2v^2} - \frac{mu}{v(u^2 + 2v^2)} + \frac{m|v|}{v} \right) \\
+ \sum_{D_2 \leq u^2 + 2v^2 < 2D_2} e \left( \frac{m \sqrt{X}}{u^2 + 2v^2} + \frac{2mv}{u(u^2 + 2v^2)} - \frac{m|v|}{u} \right) \\
= \Theta'_m + \Theta''_m, \quad (115)
\]

say. Let us consider \( \Theta'_m \). Denote
\[
f(u) = e \left( \frac{m \sqrt{X}}{u^2 + 2v^2} - \frac{mu}{v(u^2 + 2v^2)} \right), \quad (116)
\]
\[
\eta_1(v) = \sqrt{\max(0, D_2 - 2v^2)}, \quad \eta_2(v) = \sqrt{\min(v^2, 2D_2 - 2v^2)}, \quad (117)
\]
\[
K_{v,m}(t) = \sum_{\eta_1(v) \leq u < t} e \left( \frac{m|v|}{v} \right). \quad (118)
\]
Using (115) – (118) and Abel’s summation formula we obtain

\[
\Theta'_m = \sum_{\sqrt{\frac{D}{3}} \leq |v| < \sqrt{D}} \left( \sum_{\eta_1(v) \leq u < \eta_2(v)} f(u) e \left( \frac{m\theta|v|}{v} \right) \right)
\]

\[
= \sum_{\sqrt{\frac{D}{3}} \leq |v| < \sqrt{D}} \left( f(\eta_2(v)) K_{v,m}(\eta_2(v)) - \int_{\eta_1(v)}^{\eta_2(v)} K_{v,m}(t) \left( \frac{d}{dt} f(t) \right) dt \right)
\]

\[
\ll \sum_{\sqrt{\frac{D}{3}} \leq |v| < \sqrt{D}} \left( 1 + \frac{m\sqrt{X}}{v^2} \right) \max_{\eta_1(v) \leq t \leq \eta_2(v)} |K_{v,m}(t)|.
\]  

(119)

We are now in a good position to apply Lemma 3 because the sum defined by (118) is incomplete Kloosterman sum. Thus

\[
K_{v,m}(t) \ll |v|^{\frac{1}{2} + \epsilon} (v, m)^{\frac{1}{2}}.
\]  

(120)

By (119) and (120) we get

\[
\Theta'_m \ll \sum_{\sqrt{\frac{D}{3}} \leq |v| < \sqrt{D}} \left( 1 + \frac{m\sqrt{X}}{v^2} \right) |v|^{\frac{1}{2} + \epsilon} (v, m)^{\frac{1}{2}}
\]

\[
\ll X^{\epsilon} \left( D_{\frac{1}{2}}^{\frac{1}{2}} + mX^{\frac{1}{2}}D_{\frac{3}{2}}^{\frac{1}{4}} \right) \sum_{0 < v < \sqrt{D}} (v, m)^{\frac{1}{2}}.
\]  

(121)

On the other hand

\[
\sum_{0 < v < \sqrt{D}} (v, m)^{\frac{1}{2}} \leq \sum_{l|m} \frac{l^{\frac{3}{2}}}{\sqrt{l}} \sum_{v \equiv 0 (l)} 1 \ll D_{\frac{3}{2}}^{\frac{1}{2}} \sum_{l|m} l^{-\frac{1}{2}} \ll D_{\frac{3}{2}}^{\frac{1}{2}} \tau(m) \ll X^{\epsilon} D_{\frac{3}{2}}^{\frac{3}{2}}.
\]  

(122)

The estimations (121) and (122) imply

\[
\Theta'_m \ll X^{\epsilon} \left( D_{\frac{1}{2}}^{\frac{1}{2}} + mX^{\frac{1}{2}}D_{\frac{3}{2}}^{\frac{1}{4}} \right).
\]  

(123)

Proceeding in a similar way for \(\Theta''_m\) from (115) we deduce

\[
\Theta''_m \ll X^{\epsilon} \left( D_{\frac{3}{2}}^{\frac{3}{2}} + mX^{\frac{1}{2}}D_{\frac{3}{2}}^{\frac{1}{4}} \right).
\]  

(124)

Now (115), (123) and (124) give us

\[
\Theta_m \ll X^{\epsilon} \left( D_{\frac{1}{2}}^{\frac{1}{2}} + mX^{\frac{1}{2}}D_{\frac{3}{2}}^{\frac{1}{4}} \right).
\]  

(125)
From (109) and (125) it follows
\[ \Sigma_4 \ll X^\varepsilon \left( D_2 M_2^{-1} + D_2^\frac{3}{2} + X^\frac{1}{2} M_2 D_2^{-\frac{3}{4}} \right). \] (126)

Taking into account (103) and (126) we find
\[ \Sigma_4 \ll X^{1+\varepsilon} D_2^{-\frac{1}{4}}. \] (127)

Using (99), (100) and (127) we obtain
\[ \Sigma_2 \ll X^{1+\varepsilon} D_2^{-\frac{1}{4}}. \] (128)

**Estimation of \( \Gamma_2(X) \)**

Summarizing (76), (77), (79), (97) and (128) we get
\[ \Gamma_2(X) \ll X^{1+\varepsilon} z^{-\frac{1}{4}}. \] (129)

### 4.3 The end of the proof

Bearing in mind (54), (74), (129) and choosing \( z = X^\frac{5}{8} \) we establish the asymptotic formula (7).

The theorem is proved.

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