GENERALIZED $\gamma$-GENERATING MATRICES AND NEHARI-TAKAGI PROBLEM

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To the Memory of Leiba Rodman

Abstract. Under certain mild assumption, we establish a one-to-one correspondence between solutions of the Nehari-Takagi problem and solutions of some Takagi-Sarason interpolation problem. The resolvent matrix of the Nehari-Takagi problem is shown to belong to the class of so-called generalized $\gamma$-generating matrices, which is introduced and studied in the paper.

1. Introduction

For a bounded function $f$ defined on $T = \{z : |z| = 1\}$ let us set

$$\gamma_k(f) = \frac{1}{2\pi} \int_T e^{ik\theta} f(e^{i\theta}) d\theta \quad (k = 1, 2, \ldots).$$

The Nehari problem consists of the following: given a sequence of complex numbers $\gamma_k (k \in \mathbb{N})$ find a function $f \in L_\infty(T)$ such that $\|f\| \leq 1$ and

$$\gamma_k(f) = \gamma_k, \quad (k = 1, 2, \ldots).$$

By Nehari theorem [22] this problem is solvable if and only if the Hankel matrix $\Gamma = (\gamma_{i+j-1})_{i,j=1}^\infty$ determines a bounded operator in $l_2(\mathbb{N})$ with $\|\Gamma\| \leq 1$. The problem (1.2) is called indeterminate if it has infinitely many solutions. A criterion for the Nehari problem to be indeterminate and a full description of the set of its solutions was given in [1], [2].

In [2] Adamyan, Arov and Krein considered the following indefinite version of the Nehari problem, so called Nehari-Takagi problem $\text{NTP}_\kappa(\Gamma)$: Given $\kappa \in \mathbb{N}$ and a sequence $\{\gamma_k\}_{k=1}^\infty$ of complex numbers, find a function $f \in L_\infty(T)$, such that $\|f\|_\infty \leq 1$ and

$$\text{rank } (\Gamma(f) - \Gamma) \leq \kappa.$$

Here $\Gamma(f)$ is the Hankel matrix $\Gamma(f) := (\gamma_{i+j-1}(f))_{i,j=1}^\infty$. As was shown in [2], the problem $\text{NTP}_\kappa(\Gamma)$ is solvable if and only if the total multiplicity $\nu_-(I - \Gamma^*\Gamma)$ of the negative spectrum of the operator $I - \Gamma^*\Gamma$ does not exceed $\kappa$. In the case when the operator $I - \Gamma^*\Gamma$ is invertible and $\nu_-(I - \Gamma^*\Gamma) = \kappa$, the set of solutions of this problem was parameterized by the formula

$$f(\mu) = (a_{11}(\mu)\varepsilon(\mu) + a_{12}(\mu))(a_{21}(\mu)\varepsilon(\mu) + a_{22}(\mu))^{-1},$$

where $\mathcal{A}(\mu) = (a_{ij}(\mu))_{i,j=1}^2$ is the so-called $\gamma$-generating matrix and the parameter $\varepsilon$ ranges over the Schur class of functions bounded and holomorphic on $\mathbb{D} = \{z : |z| < 1\}$.

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The following basic classes of mvf’s will be used in this paper: $H^p_{2} \times q$ (resp., $H^p_{\infty} \times q$) is the class of $p \times q$ mvf’s with entries in the Hardy space $H_2$ (resp., $H_\infty$); $H^p_{2} := H^p_{2} \times 1$, and $(H^p_{2})^+ = L^p_{\infty} \otimes H^p_{2}$, $S^{p \times q}$ is the Schur class of $p \times q$ mvf’s holomorphic and contractive on $\Omega_+$, $S_{in}^{p \times q}$ (resp., $S_{out}^{p \times q}$) is the class of inner (resp., outer) mvf’s in $S^{p \times q}$:

$$S_{in}^{p \times q} = \{ s \in S^{p \times q} : s(\mu)s(\mu)^* = I_p \text{ a.e. on } \Omega_0 \};$$

$$S_{out}^{p \times q} = \{ s \in S^{p \times q} : sH^p_{2} = H^p_{2} \}.$$

The Nevanlinna class $N^{p \times q}$ and the Smirnov class $N_{+}^{p \times q}$ are defined by

$$N^{p \times q} = \{ f = h^{-1}g: g \in H^{p \times q}_{\infty}, h \in S := S^{1 \times 1} \},$$

$$N_{+}^{p \times q} = \{ f = h^{-1}g: g \in H^{p \times q}_{\infty}, h \in S_{out} := S^{1 \times 1}_{out} \}.$$
For a mvf \( f(\lambda) \) let us set \( f^\#(\lambda) = f(\lambda^*)^* \). Denote by \( \mathcal{H}_f \) the domain of holomorphy of the mvf \( f \) and let \( \mathcal{H}_f^+ = \mathcal{H}_f \cap \Omega_\pm \).

A \( p \times q \) mvf \( f_- \) in \( \Omega_- \) is said to be a pseudocontinuation of a mvf \( f \in \mathcal{N}^{p \times q} \), if

1. \( f^\# \in \mathcal{N}^{p \times q} \),
2. \( \lim_{\nu \downarrow 0} f_-(\mu - i\nu) = \lim_{\nu \downarrow 0} f_+(\mu + i\nu) = f(\mu) \) a.e. on \( \Omega_0 \).

The subclass of all mvf’s \( f \in \mathcal{N}^{p \times q} \) that admit pseudocontinuations \( f_- \) into \( \Omega_- \) will be denoted \( \Pi^{p \times q} \).

Let \( \varphi(\lambda) \) be a \( p \times q \) mvf that is meromorphic on \( \Omega_+ \) with a Laurent expansion

\[
\varphi(\lambda) = (\lambda - \lambda_0)^{-k} \varphi_{-k} + \cdots + (\lambda - \lambda_0)^{-1} \varphi_{-1} + \varphi_0 + \cdots
\]

in a neighborhood of its pole \( \lambda_0 \in \Omega_+ \). The pole multiplicity \( M_\pi(\varphi, \lambda_0) \) is defined by (see [20])

\[
M_\pi(\varphi, \lambda_0) = \text{rank} \, L(\varphi, \lambda_0), \quad T(\varphi, \lambda_0) = \begin{bmatrix} \varphi_{-k} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \varphi_{-1} & \ldots & \varphi_{-k} \end{bmatrix}.
\]

The pole multiplicity of \( \varphi \) over \( \Omega_+ \) is given by

\[
M_\pi(\varphi, \Omega_+) = \sum_{\lambda \in \Omega_+} M_\pi(\varphi, \lambda).
\]

This definition of pole multiplicity coincides with that based on the Smith-McMillan representation of \( \varphi \) (see [10]).

Let \( b_\omega(\lambda) \), be a Blaschke factor \( b_\omega(\lambda) = \frac{\lambda - \omega}{1 - \overline{\omega} \lambda} \) in the case \( \Omega_+ = \mathbb{D} \), \( b_\omega(\lambda) = \frac{\lambda - \omega}{\lambda - \overline{\omega}} \) in the case \( \Omega_+ = \mathbb{C}_+ \), and let \( P \) be an orthogonal projection in \( \mathbb{C}^p \). Then the mvf

\[
B_\alpha(\lambda) = I_p - P + b_\omega(\lambda)P, \quad \omega \in \Omega_+,
\]

belongs to the Schur class \( S^{p \times p} \) and is called the elementary Blaschke–Potapov (BP) factor and \( B(\lambda) \) is called primary if \( \text{rank} \, P = 1 \). The product

\[
B(\lambda) = \prod_{j=1}^\kappa B_{\alpha_j}(\lambda),
\]

where \( B_{\alpha_j}(\lambda) \) are primary Blaschke–Potapov factors is called a Blaschke–Potapov product of degree \( \kappa \).

Remark 2.1. For a Blaschke-Potapov product \( b \) the following statements are equivalent:

1. the degree of \( b \) is equal \( \kappa \);
2. \( M_\pi(b^{-1}, \Omega_+) = \kappa \);

2.2. The generalized Schur class. Let \( \kappa \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \). Recall, that a Hermitian kernel \( K_\omega(\lambda) : \Omega \times \Omega \to \mathbb{C}^{m \times m} \) is said to have \( \kappa \) negative squares, if for every positive integer \( n \) and every choice of \( \omega_j \in \Omega \) and \( u_j \in \mathbb{C}^m \) \( (j = 1, \ldots, n) \) the matrix

\[
((K_{\omega_j}(\omega_k)u_j, u_k))_{j,k=1}^{\kappa}
\]

has at most \( \kappa \) negative eigenvalues, and for some choice of \( \omega_1, \ldots, \omega_\kappa \in \Omega \) and \( u_1, \ldots, u_\kappa \in \mathbb{C}^m \) exactly \( \kappa \) negative eigenvalues (see [20]).
Let $S_{k}^{q \times p}$ denote the generalized Schur class of $q \times p$ mvf’s $s$ that are meromorphic in $\Omega_{+}$ and for which the kernel

$$
\Lambda^{s}_{\kappa}(\lambda) = \frac{I_{p} - s(\lambda)s(\omega)\ast}{\rho_{\omega}(\lambda)}
$$

has $\kappa$ negative squares on $h_{s}^{+} \times h_{s}^{+}$. In the case where $\kappa = 0$, the class $S_{0}^{q \times p}$ coincides with the Schur class $S^{q \times p}$ of contractive mvf’s holomorphic in $\Omega_{+}$. As was shown in [20] every mvf $s \in S_{k}^{q \times p}$ admits factorizations of the form

$$
s(\lambda) = b_{c}(\lambda)^{-1}s_{c}(\lambda) = s_{r}(\lambda)b_{r}(\lambda)^{-1}, \quad \lambda \in h_{s}^{+},
$$

where $b_{c} \in S_{q \times q}$, $b_{r} \in S_{p \times p}$ are Blaschke–Potapov products of degree $\kappa$, $s_{c}, s_{r} \in S^{q \times p}$ and the factorizations (2.3) are left coprime and right coprime, respectively, i.e.

$$
\text{rank } \begin{bmatrix} b_{c}(\lambda) & s_{c}(\lambda) \end{bmatrix} = q \quad (\lambda \in \Omega_{+})
$$

and

$$
\text{rank } \begin{bmatrix} b_{r}(\lambda)^{\ast} & s_{r}(\lambda)^{\ast} \end{bmatrix} = p \quad (\lambda \in \Omega_{+}).
$$

The following matrix identity was established in the rational case in [16], in general case see [13].

**Theorem 2.2.** Let $s \in S_{k}^{q \times p}$ have Kreïn-Langer factorizations

$$
s = b_{c}^{-1}s_{c} = s_{r}b_{r}^{-1}.
$$

Then there exists a set of mvf’s $c_{c} \in H^{q \times q}_{\infty}, d_{c} \in H^{p \times q}_{\infty}, c_{r} \in H^{p \times p}_{\infty}$ and $d_{r} \in H^{q \times q}_{\infty}$, such that

$$
\begin{bmatrix} c_{r} & d_{c} & b_{r} & -d_{e} \\ -s_{e} & b_{c} & s_{r} & c_{c} \end{bmatrix} = \begin{bmatrix} I_{p} & 0 \\ 0 & I_{q} \end{bmatrix}.
$$

2.3. The generalized Smirnov class. Let $R^{p \times q}$ denote the class of rational $p \times q$ mvf’s and let $\kappa \in \mathbb{N}$. A $p \times q$ mvf $\varphi(z)$ is said to belong to the generalized Smirnov class $N_{p \times q}^{\kappa}$, if it admits the representation

$$
\varphi(z) = \varphi_{0}(z) + r(z), \quad \text{where } \varphi_{0} \in N_{p \times q}^{\kappa}, r \in R^{p \times q} \text{ and } M_{\kappa}(r, \Omega_{+}) \leq \kappa.
$$

If $\kappa = 0$, then the class $N_{p \times q}^{0,0}$ coincides with the Smirnov class $N_{p \times q}^{0}$, defined in (2.1). The generalized Smirnov class $N_{p \times q}^{\kappa}$ was introduced in [23] In [13], mvf’s $\varphi$ from $N_{p \times q}^{\kappa}$ were characterized by the following left coprime factorization

$$
\varphi(\lambda) = b_{c}(\lambda)^{-1}\varphi_{c}(\lambda),
$$

where $b_{c} \in S_{c}^{q \times p}$ is a Blaschke–Potapov product of degree $\kappa$, $\varphi_{c} \in N_{p \times q}^{\kappa}$ and

$$
\text{rank } \begin{bmatrix} b_{c}(\lambda) & \varphi_{c}(\lambda) \end{bmatrix} = p \quad \text{for } \lambda \in \Omega_{+}.
$$

Clearly, for $\varphi \in N_{p \times q}^{\kappa}$ there exists a right coprime factorization with similar properties. This implies, in particular, that the class $S_{k}^{q \times p}$ is contained in $N_{p \times q}^{\kappa}$. 


2.4. Generalized $j_{pq}$-inner mvf’s. Let $j_{pq}$ be an $m \times m$ signature matrix

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad \text{where } p + q = m,$$

Definition 2.3. [4] An $m \times m$ mvf $W(\lambda) = |w_{ij}(\lambda)|_{i,j=1}^2$ that is meromorphic in $\Omega_+$ is said to belong to the class $\mathcal{U}_\kappa(j_{pq})$ of generalized $j_{pq}$-inner mvf’s, if:

(i) The kernel

$$K^W_\omega(\lambda) = \frac{j_{pq} - W(\lambda)j_{pq}W(\omega)^*}{\rho_\omega(\lambda)}$$

has $\kappa$ negative squares in $\hbb_W^+ \times \hbb_W^+$;

(ii) $j_{pq} - W(\mu)j_{pq}W(\mu)^* = 0$ a.e. on $\Omega_0$.

As is known [4] Th.6.8., for every $W \in \mathcal{U}_\kappa(j_{pq})$ the block $w_{22}(\lambda)$ is invertible for all $\lambda \in \hbb_W^+$ except for at most $\kappa$ points in $\Omega_+$. Thus the Potapov-Ginzburg transform of $W$

$$(2.8) \quad S(\lambda) = PG(W) := \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}^{-1}$$

is well defined for those $\lambda \in \hbb_W^+$, for which $w_{22}(\lambda)$ is invertible. It is well known that $S(\lambda)$ belongs to the class $S_{\kappa \times m}$, and $S(\mu)$ is unitary for a.e. $\mu \in \Omega_0$ (see [4, 13]).

Definition 2.4. [13] An $m \times m$ mvf $W \in \mathcal{U}_\kappa(j_{pq})$ is said to be in the class $\mathcal{U}_\kappa(j_{pq})$, if

$$(2.9) \quad s_{21} := -w_{22}^{-1}w_{21} \in S_{\kappa}^{q \times p}.$$ 

Let $W \in \mathcal{U}_\kappa(j_{pq})$ and let the Krein-Langer factorization of $s_{21}$ be written as

$$s_{21}(\lambda) = b_\ell(\lambda)^{-1}s_\ell(\lambda) = s_r(\lambda)b_r(\lambda)^{-1} \quad (\lambda \in \hbb_{s_{21}}^+),$$

where $b_\ell \in S_{in}^{q \times q}$, $b_r \in S_{in}^{p \times p}$, $s_\ell, s_r \in S_{out}^{q \times p}$. Then, as was shown in [13], the mvf’s $b_\ell s_{22}$ and $s_{11}b_r$ are holomorphic in $\Omega_+$, and

$$b_\ell s_{22} \in S_{in}^{q \times q} \quad \text{and} \quad s_{11}b_r \in S_{out}^{p \times p}.$$ 

Definition 2.5. [13] Consider inner-outer factorization of $s_{11}b_r$, and outer-inner factorization of $b_\ell s_{22}$

$$(2.10) \quad s_{11}b_r = b_1a_1, \quad b_\ell s_{22} = a_2b_2,$$

where $b_1 \in S_{in}^{p \times p}$, $b_2 \in S_{in}^{q \times q}$, $a_1 \in S_{out}^{p \times q}$, $a_2 \in S_{out}^{q \times q}$. The pair $\{b_1, b_2\}$ of inner factors in the factorizations (2.10) is called the associated pair of the mvf $W \in \mathcal{U}_\kappa(j_{pq})$.

From now onwards this pair $\{b_1, b_2\}$ will be called also a right associated pair since it is related to the right linear fractional transformation

$$(2.11) \quad T_W[\varepsilon] := (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1},$$

see [5, 8, 7]. Such transformations play important role in description of solutions of different interpolation problems, see [2, 5, 10, 9, 12, 14]. In the case $\kappa = 0$ the definition of the associated pair was given in [5].

For every $W \in \mathcal{U}_\kappa(j_{pq})$ and $\varepsilon \in S_{in}^{q \times q}$ the mvf $T_W[\varepsilon]$ admits the dual representation

$$T_W[\varepsilon] = (\varepsilon w_{11}^\# + \varepsilon w_{12}^\#)^{-1}(w_{21}^\# + \varepsilon w_{22}^\#).$$
As was shown in [13], for $W \in \mathcal{U}_c(j_{pq})$ and $c_\ell$, $d_\ell$, $e_\ell$ and $d_\ell$ as in [27], the mvf 
\begin{equation}
\tag{2.12}
K^\circ := (-w_{11}d_\ell + w_{12}e_\ell)(-w_{21}d_\ell + w_{22}c_\ell)^{-1},
\end{equation}

belongs to $H_{\infty}^{p \times q}$. It is clear that $(K^\circ)^\# \in H_{\infty}^{q \times p}(\Omega_-)$.

In the future we will need the following factorization of the mvf $W \in \mathcal{U}_c(j_{pq})$, which was obtained in [13, Theorem 4.12]:
\begin{equation}
\tag{2.13}
W = \Theta^\circ \Phi^\circ \text{ in } \Omega_+,
\end{equation}

where
\[
\Theta^\circ = \begin{bmatrix} b_1 & K^\circ b_2^{-1} \\
0 & b_2^{-1} \end{bmatrix}, \quad \Phi^\circ, (\Phi^\circ)^{-1} \in \mathbb{N}_+.
\]

3. The Takagi-Sarason interpolation problem

Problem TSP$_\kappa(b_1, b_2, K)$ Let $b_1 \in S_{in}^{p \times p}$, $b_2 \in S_{in}^{q \times q}$ be inner mvf’s, let $K \in H_{\infty}^{p \times q}$ and let $\kappa \in \mathbb{Z}$. A $p \times q$ mvf $s$ is called a solution of the Takagi-Sarason

problem TSP$_\kappa(b_1, b_2, K)$, if $s$ belongs to $S_{\kappa}^{p \times q}$ for some $\kappa' \leq \kappa$ and satisfies
\begin{equation}
\tag{3.1}
b_1^{-1}(s - K)b_2^{-1} \in N_{\kappa, \kappa}'^{p \times q}.
\end{equation}

The set of solutions of the Takagi-Sarason problem will be denoted by
\[
\mathcal{T}\mathcal{S}_\kappa(b_1, b_2, K) = \bigcup_{\kappa' \leq \kappa} \{s \in S_{\kappa'}^{p \times q} : b_1^{-1}(s - K)b_2^{-1} \in N_{\kappa, \kappa}'^{p \times q}\}.
\]

The problem TSP$_\kappa(b_1, b_2, K)$ has been studied in [11], in the rational case ($K \in \mathbb{R}^{p \times q}$) the set $\mathcal{T}\mathcal{S}_\kappa(b_1, b_2, K) \cap \mathbb{R}^{p \times q}$ was described in [10]. In the completely indeterminate case explicit formulas for the resolvent matrix can be found in [14], [15]. In the general positive semidefinite case, the problem was solved in [17], [18].

We now recall the construction of the resolvent matrix from [15]. Let
\[
\mathcal{H}(b_1) = H_2^p \oplus b_1 H_2^p, \quad \mathcal{H}_*(b_2) := (H_2^q)^\perp \oplus b_2^*(H_2^q)^\perp
\]

let
\[
\mathcal{H}(b_1, b_2) := \mathcal{H}(b_1) \oplus \mathcal{H}_*(b_2).
\]

and let the operators $K_{11} : H_2^p \to \mathcal{H}(b_1)$, $K_{12} : \mathcal{H}_*(b_2) \to \mathcal{H}(b_1)$, $K_{22} : \mathcal{H}_*(b_2) \to (H_2^q)^\perp$ and $P : \mathcal{H}(b_1, b_2) \to \mathcal{H}(b_1, b_2)$ be defined by the formulas
\begin{align}
K_{11}h_+ &= \Pi_{\mathcal{H}(b_1)} K h_+, \quad h_+ \in H_2^p,
\tag{3.2}
K_{12}h_2 &= \Pi_{\mathcal{H}(b_1)} K h_2, \quad h_2 \in \mathcal{H}_*(b_2),
K_{22}h_2 &= \Pi_{\mathcal{H}_*(b_2)} K h_2, \quad h_2 \in \mathcal{H}_*(b_2),
\end{align}

\begin{align}
P = \begin{bmatrix} I - K_{11} K_{11}^* & -K_{12} \\
-K_{12}^* & I - K_{22}^* K_{22} \end{bmatrix}.
\tag{3.3}
\end{align}

The data set $b_1, b_2, K$ considered in [15] is subject to the following constraints:

(H1) $b_1 \in S_{in}^{p \times p}$, $b_2 \in S_{in}^{q \times q}$, $K \in H_{\infty}^{p \times q}$.

(H2) $\kappa_1 = \nu_-(P) < \infty$.

(H3) $0 \in \rho(P)$.

(H4) $h_{b_1} \cap h_{b_2} \cap \Omega_0 \neq \emptyset$. 


Define also the operator

\[
F = \begin{bmatrix} I & K_{22} \\ K_{11}^* & I \end{bmatrix} : H(b_1) \oplus H_*(b_2) \rightarrow \frac{b_1(H_2^p)^\perp}{\oplus} \oplus \frac{b_2(H_2^q)^\perp}{\oplus} = K.
\]

As was shown in [15] for every \( h_1 \in H(b_1) \) and \( h_2 \in H_*(b_2) \) the vvf’s \((K_{11}^*,h_1)\) and \((K_{22},b_2)\) admit pseudocontinuations of bounded type which are holomorphic on \( h_{b_1} \) and \( h_{b_2}^\perp \), respectively. This allows to define an \( m \times m \) mvf \( \lambda \rightarrow F(\lambda) \) by

\[
F(\lambda) = E(\lambda)F \quad \text{for} \quad \lambda \in h_{b_1} \cap h_{b_2}^\perp
\]

where \( E(\lambda) \) is the evaluation operator \( E(\lambda) : f \in K \rightarrow f(\lambda) \in \mathbb{C}^m \).

Let \( \mu \in h_{b_1} \cap h_{b_2}^\perp \cap \Omega_0 \). Then the mvf \( W(\lambda) \) defined by

\[
W(\lambda) = I - \rho_\mu(\lambda)F(\lambda)P^{-1}F(\mu)^*j_{pq} \quad \text{for} \quad \lambda \in h_{b_1} \cap h_{b_2}^\perp
\]

belongs to the class \( \mathcal{U}_{<\kappa}(j_{pq}) \) of generalized \( j_{pq} \)-inner mvf’s and takes values in \( L_2^{m \times m} \). The following theorem presents a description of the set \( \mathcal{T}_S(\lambda) \). Let \( (H1) \rightarrow (H4) \) be in force and let \( W(\lambda) \) be the mvf, defined by (3.6). Then \( W \in \mathcal{U}_{\kappa}(j_{pq}) \cap L_2^{m \times m} \) and

(1) \( \mathcal{T}_S(\lambda) \neq 0 \Leftrightarrow \nu_-(P) \leq \kappa \).

(2) If \( \kappa_1 = \nu_-(P) \leq \kappa \), then

\[
\mathcal{T}_S(\lambda) = Tw[S_{\kappa_1,\kappa_2}^{p \times q}] := \{ Tw[\varepsilon] : \varepsilon \in S_{\kappa_1,\kappa_2}^{p \times q} \},
\]

where \( Tw[\varepsilon] \) is the linear fractional transformation given by (2.11).

**Proof.** The proof of this statement can be derived from the proof of Theorem 5.7 in [15]. However, we would like to present here a shorter proof based on the description of the set \( \mathcal{T}_S(\lambda) \), given in [14] Theorem 5.17.

As was shown in [15], see Theorem 4.2 and Corollary 4.4, the mvf \( W(z) \) belongs to the class \( \mathcal{U}_{\kappa}(j_{pq}) \) of generalized \( j_{pq} \)-inner mvf’s with the property (2.9) and \( \{b_1, b_2\} \) is the associated pair of \( W \). Moreover, by construction \( W(z) \) takes values in \( L_2^{m \times m} \). Let \( c_\ell \) and \( d_\ell \) be mvf’s defined in Theorem 2.2 and let \( K^\circ \) be given by (2.12). Then \( W \) admits the factorization (2.13) (see [13] Theorem 4.12). This proves that all the assumptions of Theorem 5.17 from [14] with \( K \) replaced by \( K^\circ \) are satisfied and by that theorem

\[
\mathcal{T}_S(\lambda) = Tw[S_{\kappa_1,\kappa_2}^{p \times q}].
\]

On the other hand it follows from [15] Theorem 4.2 that the mvf \( W \) admits the factorization

\[
W = \Theta \Phi = \begin{bmatrix} b_1 & Kb_2^{-1} \\ 0 & b_2^{-1} \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}
\]

with \( \Phi, \Phi^{-1} \in \mathcal{N}^{m \times m} \). Comparing (3.9) with (2.13) one obtains

\[
\begin{bmatrix} I & b_1^{-1}(K - K^\circ)b_2^{-1} \\ 0 & I \end{bmatrix} = \Phi^\ast \Phi^{-1} \in \mathcal{N}^{m \times m}
\]

and hence

\[
b_1^{-1}(K - K^\circ)b_2^{-1} \in \mathcal{N}_{+}^{p \times q}.
\]
This implies the equality $T\mathcal{S}_\kappa(b_1, b_2; K) = T\mathcal{S}_\kappa(b_1, b_2; K^\circ)$, that in combination with (3.8) completes the proof. □

**Remark 3.2.** Alongside with the set $T\mathcal{S}_\kappa(b_1, b_2, K)$ consider also its subset

$$\mathcal{S}_\kappa(b_1, b_2, K) = \{ s \in \mathcal{S}_\kappa^{p \times q} : b_1^{-1}(s - K)b_2^{-1} \in \mathcal{N}_+^{p \times q} \}. $$

A $p \times q$ mvf $s$ is called a solution of the generalized Schur-Takagi problem $\text{GSTP}_\kappa(b_1, b_2, K)$ if $s$ belongs to the set $\mathcal{S}_\kappa(b_1, b_2, K)$. A description of the set $\mathcal{S}_\kappa(b_1, b_2, K)$ was obtained in [13, Theorem 1.2] in the form:

$$\mathcal{S}_\kappa(b_1, b_2, K) = T_W[\mathcal{S}_\kappa^{p \times q}] \cap \mathcal{S}_\kappa^{p \times q}. $$

Notice, that the reasonings of Theorem 3.1 allows to give a shorter proof of the formula (3.12). Indeed, by [14, Theorem 5.17]

$$\mathcal{S}_\kappa(b_1, b_2; K^\circ) = T_W[\mathcal{S}_\kappa^{p \times q}] \cap \mathcal{S}_\kappa^{p \times q}. $$

Next, it follows from (3.10) that

$$\mathcal{S}_\kappa(b_1, b_2; K) = \mathcal{S}_\kappa(b_1, b_2; K^\circ). $$

Now (3.12) is implied by (3.14) and (3.13).

### 4. Generalized $\gamma$-generating mvf’s

**Definition 4.1.** Let $\mathfrak{M}_\kappa(j_{pq})$ denote the class of $m \times m$ mvf’s $\mathfrak{A}(\mu)$ on $\Omega_0$ of the form

$$\mathfrak{A}(\mu) = \begin{bmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) \end{bmatrix},$$

with blocks $a_{11}$ and $a_{22}$ of size $p \times p$ and $q \times q$, respectively, such that:

1. $\mathfrak{A}(\mu)$ is a measurable mvf on $\Omega_0$ and $j_{pq}$-unitary a.e. on $\Omega_0$;
2. the mvf’s $a_{22}(\mu)$ and $a_{11}(\mu)^*$ are invertible for a.e. $\mu \in \Omega_0$ and the mvf

$$s_{21}(\mu) = -a_{22}(\mu)^{-1}a_{21}(\mu) = -a_{12}(\mu)^*(a_{11}(\mu)^*)^{-1}$$

is the boundary value of a mvf $s_{21}(\lambda)$ that belongs to the class $\mathcal{S}^{p \times q}_\kappa$;
3. $a_{11}(\mu)^*$ and $a_{22}(\mu)$, are the boundary values of mvf’s $a_{11}^\sharp(\lambda)$ and $a_{22}(\lambda)$ that are meromorphic in $\mathbb{C}_+$ and, in addition,

$$a_1 := (a_{11}^\sharp)^{-1}b_{\tau} \in \mathcal{S}_{out}^{p \times p}, \quad a_2 := b_{\tau}a_{22}^{-1} \in \mathcal{S}_{out}^{q \times q},$$

where $b_{\tau}$, $b_\tau$ are Blaschke-Potapov products of degree $\kappa$, determined by Krein-Langer factorizations of $s_{21}$.

Mvf’s in the class $\mathfrak{M}_\kappa(j_{pq})$ are called generalized right $\gamma$-generating mvf’s. The class $\mathfrak{M}_\kappa(j_{pq}) := \mathfrak{M}_\kappa^\circ(j_{pq})$ was introduced in [6], in this case conditions (2) and (3) in Definition 4.1 are simplified to:

(2') $s_{21} \in \mathcal{S}^{p \times p};$

(3') $a_1 := (a_{11}^\sharp)^{-1} \in \mathcal{S}_{out}^{p \times p}, \quad a_2 := a_{22}^{-1} \in \mathcal{S}_{out}^{q \times q}.$

Mvf’s from the class $\mathfrak{M}_\kappa(j_{pq})$ play an important role in the description of solutions of the Nehari problem and are called right $\gamma$-generating mvf’s, [6, 7]. Mvf’s in the class $\mathfrak{M}_\kappa(j_{pq})$ will be called generalized right $\gamma$-generating mvf’s.

**Definition 4.2.** [7] An ordered pair $\{b_1, b_2\}$ of inner mvf’s $b_1 \in \mathcal{S}^{p \times p}$, $b_2 \in \mathcal{S}^{q \times q}$ is called a denominator of the mvf $f \in \mathcal{N}^{p \times q}$, if

$$b_1fb_2 \in \mathcal{N}^{p \times q}.$$
Theorem 4.3. Let $\mathfrak{A} \in \Pi_{m \times m} \cap \mathfrak{M}_{r}(j_{pq})$, and let $c_\ell$, $d_\ell$, $c_r$ and $d_r$ be as in Theorem 2.2,

(4.3) \hspace{1cm} f_0 = (-a_{11}d_\ell + a_{12}c_\ell)a_2.

Then the mef $f_0$ admits the dual representation

(4.4) \hspace{1cm} f_0 = a_1(c_\ell a^\#_{21} - d_\ell a^\#_{22}).

If, in addition, $\{b_1, b_2\} \in \text{den}(f_0)$ and

(4.5) \hspace{1cm} W(z) = \begin{bmatrix} b_1 & 0 \\ b_2 & b_2^{-1} \end{bmatrix} \mathfrak{A}(z),

then $W \in \mathcal{U}^c_{r}(j_{pq})$ and $\{b_1, b_2\}$ is the associated pair of $W$.

Conversely, if $W \in \mathcal{U}^c_{r}(j_{pq})$ and $\{b_1, b_2\}$ is the associated pair of $W$, then

$$\mathfrak{A}(z) = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix} W(z) \in \Pi_{m \times m} \cap \mathfrak{M}_{r}(j_{pq}) \quad \text{and} \quad \{b_1, b_2\} \in \text{den}(f_0).$$

**Proof.** Let $\mathfrak{A} \in \Pi_{m \times m} \cap \mathfrak{M}_{r}(j_{pq})$. It follows from (4.1), (4.2) and (2.3) that

$$-a_{21}d_\ell + a_{22}c_\ell = \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = \begin{bmatrix} -a_{22}s_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix}$$

$$= a_{22}b_\ell^{-1} \begin{bmatrix} -s_\ell & b_\ell \end{bmatrix} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix}$$

$$= a_2^{-1}(s_\ell d_\ell + b_\ell c_\ell) = a_2^{-1}.$$

Let $f_0$ be defined by the equation (4.3). Then

$$f_0 = (-a_{11}d_\ell + a_{12}c_\ell)(-a_{21}d_\ell + a_{22}c_\ell)^{-1}.$$

The identity

$$\begin{bmatrix} c_r & -d_r \end{bmatrix} \mathfrak{A}^\#_{r,jpq} \mathfrak{A} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = \begin{bmatrix} c_r & -d_r \end{bmatrix} j_{pq} \begin{bmatrix} -d_\ell \\ c_\ell \end{bmatrix} = 0$$

implies that

$$(c_r a^\#_{11} - d_r a^\#_{12})(-a_{11}d_\ell + a_{12}c_\ell) = (c_r a^\#_{21} - d_r a^\#_{22})(-a_{21}d_\ell + a_{22}c_\ell),$$

and hence that $f_0$ admits the dual representation

$$f_0 = (c_r a^\#_{11} - d_r a^\#_{12})^{-1}(c_r a^\#_{21} - d_r a^\#_{22}).$$

Using the identity

$$\begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} a^\#_{11} \\ a^\#_{12} \end{bmatrix} = \begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} a^\#_{11} \\ -s_{21}a^\#_{11} \end{bmatrix}$$

$$= \begin{bmatrix} c_r & -d_r \end{bmatrix} \begin{bmatrix} I_p \\ -s_{r}b^{-1}_r \end{bmatrix} b_r a^{-1} = a_1^{-1}$$

one obtains the equality

$$f_0 = a_1(c_r a^\#_{21} - d_r a^\#_{22})$$

which coincides with (4.4).

Let $\{b_1, b_2\} \in \text{den}(f_0)$, i.e. $b_1f_0b_2 \in \mathcal{N}^{p \times q}_r$. Since $b_1f_0b_2 \in L^p_{\infty}$ then by Smirnov theorem

$$b_1f_0b_2 \in H^p_{\infty}.$$
Let us find the Potapov-Ginzburg transform $S = PG(W)$ of $W$, see \[28\]. The formula (4.5) implies that
\[
\begin{align*}
(4.6) \quad s_{21} &= -w_{22}^{-1}w_{21} = -a_{22}^{-1}a_{21} = -b_{\ell}^{-1}s_{\ell}, \\
(4.7) \quad s_{22} &= w_{22}^{-1} = a_{22}^{-1}b_{2} = b_{\ell}^{-1}a_{2}b_{2}, \\
(4.8) \quad s_{11} &= w_{11}^* = b_{1}a_{1}^{-1}b_{1}^{-1}w_{11}^* \\
&= b_{1}a_{1}(c_{r}a_{11} - d_{r}a_{12})b_{1}^{-1}w_{11}^* \\
&= b_{1}a_{1}(c_{r}w_{11}^* - d_{r}w_{12}^*)w_{11}^* \\
&= b_{1}a_{1}(c_{r} + d_{r}s_{21}), \\
(4.9) \quad s_{12} &= -w_{11}^{-1}s_{21}^* = b_{1}a_{1}(c_{r}w_{11}^* - d_{r}w_{12}^*)w_{21}^* \\
&= b_{1}a_{1}(c_{r}w_{11}^* - d_{r}w_{22}^* + d_{r}s_{22}) \\
&= b_{1}f_{0}b_{2} + b_{1}a_{1}d_{r}s_{22}.
\end{align*}
\]
The equalities (4.6)-(4.9) lead easily to the formula
\[
S(z) = \begin{bmatrix}
  b_{1}a_{1}c_{r} + b_{1}a_{1}d_{r}s_{21} & b_{1}f_{0}b_{2} + b_{1}a_{1}d_{r}s_{22} \\
  s_{21} & s_{22}
\end{bmatrix}
\]
\[(4.10) = \begin{bmatrix}
b_{1}a_{1}c_{r} & b_{1}f_{0}b_{2} \\
0 & 0
\end{bmatrix} + \begin{bmatrix}b_{1}a_{1}d_{r} \\ I\end{bmatrix} \begin{bmatrix}s_{21} & s_{22}\end{bmatrix}
= T(z) + \begin{bmatrix}b_{1}a_{1}d_{r} \\ I\end{bmatrix} b_{\ell}^{-1} [-s_{\ell} & a_{2}b_{2}],
\]
where $T(z) \in H_{m \times m}^{\infty}$. It follows from (4.10) that
\[
M_{\pi}(S, \Omega_{+}) \leq \kappa.
\]
On the other hand
\[
M_{\pi}(s_{21}, \Omega_{+}) = M_{\pi}(-b_{\ell}^{-1}s_{\ell}, \Omega_{+}) = \kappa,
\]
and, consequently,
\[
M_{\pi}(S, \Omega_{+}) = \kappa.
\]
Thus, $S \in S_{m \times m}^{\kappa}$ and, hence, $W \in \mathcal{U}_{\kappa}(j_{pq})$. \(\square\)

5. A NEHARI-TAKAGI PROBLEM

Let $f \in L_{\infty}^{p \times q}$ and let $\Gamma(f)$ be the Hankel operator associated with $f_{0}$:
\[
(5.1) \quad \Gamma(f) := \Pi_{-}M_{f}|_{H_{2}^{q}}.
\]
where $M_{f}$ denotes the operator of multiplication by $f$, acting from $L_{2}^{q}$ into $L_{2}^{p}$ and let $\Pi_{-}$ denote the orthogonal projection of $L_{2}^{q}$ onto $(H_{2}^{p})^\perp$. The operator $\Gamma(f)$ is bounded as an operator from $H_{2}^{q}$ to $(H_{2}^{p})^\perp$, moreover,
\[
\|\Gamma(f)\| \leq \| f \|_{L_{\infty}^{p \times q}}.
\]
Consider the following Nehari-Takagi problem

**Problem NTP**$_{\kappa}(f_{0})$: Given a mvf $f_{0} \in L_{\infty}^{p \times q}$. Find $f \in L_{\infty}^{p \times q}$, such that
\[
(5.2) \quad \text{rank}(\Gamma(f) - \Gamma(f_{0})) \leq \kappa \quad \text{and} \quad \| f \|_{\infty} \leq 1.
\]
In the scalar case, the problem NTP$_{\kappa}(f_{0})$ has been solved by V.M. Adamyan, D.Z. Arov and M.G. Krein in \[1\] for the case $\kappa = 0$ and in \[2\] for arbitrary $\kappa \in \mathbb{N}$. In the matrix case a description of solutions of the problem NTP$_{0}(f_{0})$ was obtained
Let’s taking into account that condition (5.4) is equivalent to the condition (3.1), i.e.

\[ K \]

Since \( f \in N \) and \( \kappa \in \mathbb{N} \), the condition (5.4) holds and \( \| f \| \leq 1 \).

Moreover, if \( H \) be the operator in \( \mathcal{N}_T \), then \( \kappa \) in the completely indeterminate case by V.M. Adamyan, \( \gamma \), and in the general positive-semidefinite case by A. Kheifets, \( \delta \). The indefinite case (\( \kappa \in \mathbb{N} \)) was treated in \( \theta \) (see also \( \tau \), where an explicit formula for the resolvent matrix was obtained in the rational case).

In what follows we confine ourselves to the case when \( \text{den}(f_0) \neq \emptyset \) and give a description of all solutions of the problem \( \mathbf{NTP}_\kappa(f_0) \). Let us set for arbitrary \( f_0 \in L_\infty^{\infty,q} \)

\[ \mathcal{N}_\kappa(f_0) = \{ f \in L_\infty^{\infty,q} : f - f_0 \in \mathcal{N}_T, \| f \| \leq 1 \} \]

and let us denote the set of solutions of the problem \( \mathbf{NTP}_\kappa(f_0) \) by

\[ \mathcal{N}_T(f_0) = \{ f \in L_\infty^{\infty,q} : \text{rank}(\Gamma(f) - \Gamma(f_0)) \leq \kappa \text{ and } \| f \| \leq 1 \}. \]

By Kronecker Theorem (\( \nu \)), the condition \( f - f_0 \in \mathcal{N}_T \) is equivalent to

\[ \text{rank}(\Gamma(f) - \Gamma(f_0)) = \kappa, \]

Therefore, the set \( \mathcal{N}_T(f_0) \) is represented as

\[ \mathcal{N}_T(f_0) = \bigcup_{\kappa' \leq \kappa} \mathcal{N}_{\kappa'}(f_0). \]

In the following theorem relations between the set of solutions of the Nehari-Takagi problem and the set of solutions of a Takagi-Sarason problem is established in the case when \( \text{den}(f_0) \neq \emptyset \).

**Theorem 5.1.** Let \( f_0 \in L_\infty^{\infty,q}, \Gamma = \Gamma(f_0), \kappa \in \mathbb{N}, \{ b_1, b_2 \} \in \text{den}(f_0) \) and \( K = b_1f_0b_2 \). Then

\[ f \in \mathcal{N}_\kappa(f_0) \iff s = b_1fb_2 \in \mathcal{T}_\kappa(b_1, b_2, K_0). \]

**Proof.** Let \( f \in \mathcal{N}_\kappa(f_0) \). Then the mvfs \( \varphi(\mu) := f(\mu) - f_0(\mu), f(\mu) \) and \( f(\mu) \) admit meromorphic continuations \( \varphi(z), f_0(z) \) and \( f(z) \) on \( \Omega_+ \), such that

\[ M_\pi(f - f_0, \Omega_+) = \kappa. \]

Let \( s = b_1fb_2 \) and \( K = b_1f_0b_2 \). Then the equality (5.4) yields

\[ M_\pi(s - K, \Omega_+) \leq \kappa. \]

Since \( K \in H_\infty^{\infty,q} \), then

\[ \kappa' := M_\pi(s, \Omega_+) = M_\pi(s - K, \Omega_+) \leq \kappa. \]

Taking into account that \( \| s \|_\infty = \| f \|_\infty \leq 1 \), one obtains \( s \in \mathcal{T}_{\kappa'}. \) Moreover, the condition (5.4) is equivalent to the condition (3.1), i.e. \( s \in \mathcal{T}_{\kappa}(b_1, b_2, K_0) \).

Conversely, if \( s \in \mathcal{S}_{\kappa'}^{\infty,q} \) with \( \kappa' \leq \kappa \) and the condition (3.1) is in force, then for \( f = b_1^{-1}s^{-1}b_2^{-1}, f_0 = b_1^{-1}Kb_2^{-1} \) one obtains that (5.4) holds and \( \| f \|_\infty \leq 1 \). Therefore, \( f \in \mathcal{N}_\kappa(f_0) \). \( \Box \)

**Lemma 5.2.** Let \( f_0 \in L_\infty^{\infty,q}, \Gamma = \Gamma(f_0), \{ b_1, b_2 \} \in \text{den}(f_0), K = b_1f_0b_2 \) and let \( P \) be the operator in \( \mathcal{H}(b_1) \oplus \mathcal{H}_\kappa(b_2) \), defined by formulas (3.2) and (3.3). Then

\[ \nu_-(P) = \nu_-(I - \Gamma^*\Gamma). \]

Moreover, if \( \nu_-(I - \Gamma^*\Gamma) < \infty \), then

\[ 0 \in \rho(P) \iff 0 \in \rho(I - \Gamma^*\Gamma). \]
Proof. Let us decompose the spaces $H_2^q$ and $(H_2^p)^\perp$: 

$$H_2^q = b_2(H_2^q) \oplus \mathcal{H}(b_2), \quad (H_2^p)^\perp = \mathcal{H}_+(b_1) \oplus b_1(H_2^p)^\perp$$

and let us decompose the operator $\Gamma : H_2^q \to (H_2^p)^\perp$, accordingly

$$\Gamma \overset{\text{def}}{=} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{pmatrix} : \begin{array}{c} b_2(H_2^q) \\ \mathcal{H}(b_2) \end{array} \to \begin{array}{c} \mathcal{H}_+(b_1) \\ b_1'(H_2^p)^\perp \end{array},$$

where the operators

$$\Gamma_{11} : b_2(H_2^q) \to \mathcal{H}_+(b_1), \quad \Gamma_{12} : \mathcal{H}(b_2) \to \mathcal{H}_+(b_1), \quad \Gamma_{22} : \mathcal{H}(b_2) \to b_1'(H_2^p)^\perp$$

are defined by the formulas

$$\Gamma_{11}h_+ = \Pi_{\mathcal{H}_+(b_1)}Kh_+, \quad h_+ \in b_2(H_2^q),$$

$$\Gamma_{12}h_2 = \Pi_{\mathcal{H}_+(b_1)}Kh_2, \quad h_2 \in \mathcal{H}(b_2),$$

$$\Gamma_{22}h_2 = (b_1^\dagger \Pi_{-b_1})Kh_2, \quad h_2 \in \mathcal{H}(b_2).$$

It follows from (5.5), (5.6) and (3.3) that the operator $\Gamma : H_2^q \to (H_2^p)^\perp$ and the operator

$$K \overset{\text{def}}{=} \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix} : \begin{array}{c} H_2^q \\ \mathcal{H}(b_2) \end{array} \to \begin{array}{c} \mathcal{H}(b_1) \\ (H_2^p)^\perp \end{array}$$

are connected by

$$\Gamma = \mathcal{M}_{b_2^\dagger}\mathcal{M}_{b_1}\mathcal{M}_{b_2}(H_2^q).$$

and, hence, the operators $\Gamma$ and $K$ are unitary equivalent. Now the statements are implied by [13, Lemma 5.10]. \qed

Theorem 5.3. Let $f_0 \in L^p_{\infty,q}$, $\Gamma = \Gamma(f_0)$, $\kappa \in \mathbb{Z}_+$, $\{b_1, b_2\} \in \text{den}(f_0)$, $K = b_1f_0b_2$, let $\mathbf{P}$ be defined by formulas (4.3), let (H1)–(H4) be in force, let the mvf $\mathbf{W}(z)$ be defined by (4.3) and let

$$\mathfrak{A}(\mu) = \begin{bmatrix} b_1(\mu)^{-1} & 0 \\ 0 & b_2(\mu) \end{bmatrix} \mathbf{W}(\mu), \quad \mu \in b_0 \cap b_{2 q} \cap \Omega_0.$$ 

Then:

1. $\mathfrak{A} \in \mathfrak{M}_{k}(\mu_{pq})$;
2. $\mathcal{N}_\kappa(f_0) \neq \emptyset$ if and only if $\kappa \geq \kappa_1 := \nu_-(I - \Gamma^*\Gamma)$;
3. $\mathcal{N}_\kappa(f_0) = T_{\mathfrak{A}}[S_{k-x,q}]$;
4. $\mathcal{N}_\kappa(f_0) = \bigcup_{k=\kappa_1}^{\kappa} T_{\mathfrak{A}}[S_{k-x,q}].$

Proof. (1) By [13, Theorem 4.2] the rows of $\mathbf{W}(z)$ admit factorizations

$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = b_1 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} \in (H_2^{p \times q})^\perp$, $a_{12} \in (H_2^{p \times q})^\perp$, $a_{21} \in H_2^{q \times p}$, $a_{22} \in H_2^{q \times q}$ and

$$s_{21} = -w_{22}^{-1}w_{21} = -a_{22}^{-1}a_{21} \in S_{k-x,q}.$$  

If the mvf’s $b_\ell^{-1}$, $s_\ell$, $b_r$, $s_r$ are determined by Krein-Langer factorizations of $s_{21}$

$$s_{21} = b_\ell^{-1}s_\ell = s_rb_r^{-1},$$

then in accordance with [13, Theorem 4.3] (see (4.26), (4.27))

$$a_2 := b_\ell a_{22}^{-1} \in S_{out}^{q \times q}, \quad a_1 := (a_{11}^{\#})^{-1}b_r \in S_{out}^{p \times p}.$$
Thus
\[ \mathfrak{A}(z) = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \]

belongs to the class \( \mathcal{M}_r(j\pi) \).

(2) By Theorem 5.1, \( \mathcal{N}_r(f_0) \) is nonempty if and only if \( \mathcal{T}\mathcal{S}_r(b_1, b_2, K) \) is nonempty. Therefore (2) is implied by Theorem 5.1 and Lemma 5.2.

(3) The statement (3) follows from the formula (3.7) proved in Theorem 3.1 and by the statement (3).

6. Resolvent matrix in the case of a rational mvf \( f_0 \)

Assume now that \( \Omega_+ = \mathbb{D} \) and \( f_0 \) is a rational mvf with a minimal realization
\[ f_0(z) = C(zI_n - A)^{-1}B, \]
where \( n \in \mathbb{N}, A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times q}, C \in \mathbb{C}^{p \times n}, \)

(6.2)
\[ \sigma(A) \subset \mathbb{D}. \]

Then the corresponding Hankel operator \( \Gamma = \Gamma(f_0) : H_2^1 \to (H_2^p) \) has the following matrix representation \( (\gamma_{j+k-1})_{j,k=1}^{\infty} \) in the standard bases \( \{e^{ijt}\}_{j=0}^{\infty} \) and \( \{e^{-ikt}\}_{k=1}^{\infty} \):
\[ (\gamma_{j+k-1})_{j,k=1}^{\infty} = (CA^{j+k-2}B)_{j,k=1}^{\infty} = \Omega \Xi, \]
where \( \gamma_j \) are given by (1.1).

(6.3)
\[ \Xi = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} CA^0 \\ \vdots \\ CA^{n-1} \end{bmatrix}. \]

Representation (6.1) is called minimal, if the dimension of the matrix \( A \) in (6.1) is minimal. As is known see [10 Thm 4.14] the representation (6.1) is minimal if and only if the pair \( (A, B) \) is controllable and the pair \( (C, A) \) is observable, i.e.

(6.4)
\[ \text{ran} \Xi = \mathbb{C}^n \quad \text{and} \quad \text{ker} \Omega = \{0\}, \]

The controllability gramian \( P \) and the observability gramian \( Q \), defined by
\[ P = \sum_{k=0}^{\infty} A^k BB^* (A^*)^k = \Xi \Xi^*, \quad Q = \sum_{k=0}^{\infty} (A^*)^k CC^* (A)^k = \Omega^* \Omega, \]
are solutions to the following Lyapunov-Stein equations
\[ P - APA^* = BB^*, \quad Q - A^* QA = C^* C. \]
As was shown in [14, Remark 4.2] a denominator of the mvf $f_0(z)$ may be selected as $(I_p, b_2)$, where
\begin{equation}
(6.6) \quad b_2(z) = I_q - (1 - z)B^*(I_n - zA^*)^{-1}P^{-1}(I_n - A)^{-1}B
\end{equation}
Straightforward calculations show that
\begin{equation}
(6.7) \quad (zI_n - A)^{-1}Bb_2(z) = P(I_n - A^*)(I_n - zA^*)^{-1}P^{-1}(I_n - A)^{-1}B.
\end{equation}
Since the mvf $b_2(z)$ is inner, then $b_2(z)^{-1} = b_2(\frac{1}{z})^*$, and hence
\begin{equation}
(6.8) \quad b_2(z)^{-1} = I_q + (1 - z)B^*(I_n - A^*)^{-1}P^{-1}(I_n - A)^{-1}B.
\end{equation}

**Proposition 6.1.** Let $f_0(z)$ be a mvf of the form (6.1), where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times q}$, $C \in \mathbb{C}^{p \times n}$ satisfy (6.2) and (6.4) and let
\begin{align}
(6.9) & \quad M = \begin{bmatrix} -A & 0 \\ 0 & I_n \end{bmatrix}, \quad N = \begin{bmatrix} -I_n & 0 \\ 0 & A^* \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -Q & I_n \\ I_n & -P \end{bmatrix}, \\
(6.10) & \quad G(z) = \begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix}(M - zN)^{-1}.
\end{align}
Assume that $1 \notin \sigma(PQ)$. Then:
1. $N_\kappa(f_0) \neq \emptyset$ if and only if $\kappa_1 := \nu_-(I - PQ) \leq \kappa$;
2. If (1) holds then the matrix $\Lambda$ is invertible and $N_\kappa(f_0) = T[\kappa_s - \kappa_1], \quad \forall \kappa_1 \in N_\kappa(f_0)$,
\begin{equation}
(6.11) \quad \mathfrak{A}(\mu) = I_n - (1 - \mu)G(\mu)\Lambda^{-1}G(1)^*j_{pq};
\end{equation}
3. The mvf $\mathfrak{A}(\mu)$ is a generalized right $\gamma$-generating mvf of the class $\mathcal{M}_\kappa(f_0)$.

By the cancellation lemma from [14, Theorem 5.1.7] these problems have the same resolvent matrix $\mathfrak{A}(\mu)$ are well known from [10, Theorem 20.5.1]. We will deduce now the formula (6.10) from the general formula (6.6) for the resolvent matrix of the problem $\text{TSP}_\kappa(I_p, b_2, K)$ with
\begin{equation}
(6.12) \quad K(z) = f_0(z)b_2(z) = C(zI_n - A)^{-1}Bb_2(z).
\end{equation}

**Proof.** (1) By Theorem 5.1 $f \in N_\kappa(f_0)$ if and only if $s = fb_2 \in T\mathcal{S}_\kappa(I_p, b_2, K)$.

Alongside with the problem $\text{TSP}_\kappa(I_p, b_2, K)$ consider the problem $\text{GSTP}_\kappa(I_p, b_2, K)$ (see Remark 5.2). As is known [14, Theorem 5.17] these problems have the same resolvent matrix. Assume that $s \in \mathcal{S}_\kappa(I_p, b_2, K)$, see (6.11). The conditions $s \in \mathcal{S}_\kappa(f_0)$ and $(s - K)b_2 \in N_\kappa$ are equivalent to the equalities
\begin{equation}
M_\kappa(s, \Omega_+) = M_\kappa((s - K)b_2^{-1}, \Omega_+) = \kappa.
\end{equation}
By the cancellation lemma from [14, Lemma 5.5]
\begin{equation}
(6.13) \quad M_\kappa((s - b_2K)b_2^{-1}, \Omega_+) = M_\kappa(s - b_2K) = 0.
\end{equation}
By (6.10) and (6.12) the expression $(s - b_2K)b_2^{-1}$ takes the form
\begin{equation}
(6.14) \quad s = (I_q + (1 - z)B^*(I_n - A^*)^{-1}P^{-1}(I_n - A)^{-1}B - b_2C(zI_n - A)^{-1}B.
\end{equation}
Since $(1 - z)(zI_n - A)^{-1} = -I_n + (I_n - A)(zI_n - A)^{-1}$, the condition (6.13) can be rewritten as
\begin{equation}
(6.15) \quad \{sB^*(I_n - A^*)^{-1}P^{-1}(I_n - A) - \delta_C\}zI_n - A)^{-1}B \in \mathcal{N}_+.
\end{equation}
Since the pair $(A, B)$ is controllable, then (6.14) is equivalent to
\begin{equation}
(6.15) \quad [b_2 - s] \left[ B^*(I_n - A^*)^{-1}P^{-1}(I_n - A) \right] (zI_n - A)^{-1} \in \mathcal{N}_+.
\end{equation}
Thus, the condition (6.15) can be rewritten as
\begin{equation}
(6.16) \quad [b_t - s_t] F \in \mathcal{N}_+,
\end{equation}
where
\begin{equation}
(6.17) \quad F(z) = \bar{C}(A - zI_n)^{-1}, \quad \bar{C} = \begin{bmatrix} C \\
B^*(I_n - A^*)^{-1}P^{-1}(I_n - A) \end{bmatrix}.
\end{equation}
Thus, the problem \textsf{GSTP} \(_{\kappa}(I_p, b_2, K)\) is equivalent to the one-sided interpolation problem (6.16) considered in [14]. As was shown in [14, (1.14)] the Pick matrix \(\bar{P}\), corresponding to the problem (6.16), is the unique solution of the Lyapunov-Stein equation
\begin{equation}
\begin{aligned}
A^*\bar{P}A - \bar{P} = \bar{C}^*j_{pq}\bar{C}
\end{aligned}
\end{equation}
and the problem (6.16) is solvable if and only if
\begin{equation}
\kappa_1 := \nu_-(\bar{P}) \leq \kappa.
\end{equation}
Since by (6.16)
\begin{equation}
\begin{aligned}
\bar{C}^*j_{pq}\bar{C} &= C^*C - (I_n - A^*)P^{-1}(I_n - A)^{-1}BB^*(I_n - A^*)^{-1}P^{-1}(I_n - A) \\
&= C^*C - (I_n - A^*)P^{-1} - A^*P^{-1}(I_n - A) \\
&= (Q - P^{-1}) - A^*(Q - P^{-1})A,
\end{aligned}
\end{equation}
then
\begin{equation}
\begin{aligned}
\bar{P} &= P^{-1} - Q = P^{-1/2}(I - P^{1/2}QP^{1/2})P^{-1/2}. \\
\end{aligned}
\end{equation}
Notice, that in (6.20) we use the equality
\begin{equation}
-(I_n - A)^{-1}BB^*(I_n - A^*)^{-1} = -(I_n - A)^{-1}P - PA^*(I_n - A^*)^{-1},
\end{equation}
It follows from (6.20) and Theorem 5.1 that \(TS_{\kappa}(I_p, b_2, K) \neq \emptyset\) if and only if
\begin{equation}
\kappa_1 := \nu_-(I - P^{1/2}QP^{1/2}) \leq \kappa.
\end{equation}
Now it remains to note that \(\sigma(I - P^{1/2}QP^{1/2}) = \sigma(I - PQ)\). In view of Theorem 5.1 this proves (1).

(2) By [14] Theorem 3.1 and Theorem 5.17 the resolvent matrix \(\tilde{W}(z)\), which describes the set \(TS_{\kappa}(I_p, b_2, K)\) via the formula (5.8) takes the form
\begin{equation}
\tilde{W}(z) = I_n - (1 - z)F(z)\bar{P}^{-1}F(1)^*j_{pq},
\end{equation}
where \(\bar{P}\) is given by (6.18). As was shown in [14] Lemma 4.8 \(\tilde{W} \in \mathcal{U}_{\kappa}(j_{pq})\). Let us set
\begin{equation}
(6.21) \quad \tilde{\mathfrak{A}}(\mu) := \begin{bmatrix} I_p & 0 \\
0 & b_2(\mu) \end{bmatrix} W(\mu)
\end{equation}
and show that the mvf
\begin{equation}
(6.22) \quad \tilde{\mathfrak{A}}(\mu) := \begin{bmatrix} I_p & 0 \\
0 & b_2(\mu) \end{bmatrix} + (\mu - 1) \begin{bmatrix} I_p & 0 \\
0 & b_2(\mu) \end{bmatrix} F(\mu)\bar{P}^{-1}F(1)^*j_{pq}
\end{equation}
coincides with the mvf \(\mathfrak{A}\) from (6.11). It follows from (6.7) that
\begin{equation}
b_2(\mu)^{-1}B^*(I_n - \mu A^*)^{-1} = B^*(I_n - A^*)^{-1}P^{-1}(\mu I_n - A)^{-1}(I_n - A)^{-1}P.
\end{equation}
and hence
\begin{equation}
b_2(\mu)B^*(I_n - A^*)^{-1}P^{-1}(\mu I_n - A)^{-1}(I_n - A) = B^*(I_n - \mu A^*)^{-1}P^{-1}.
\end{equation}
In view of (6.17), (6.19) and (6.10) this implies

\[ I_p 0 0 b_2(\mu) \]

\[ F(\mu) = \begin{bmatrix} C(\mu I_n - A)^{-1} B^*(I_n - \mu A^*)^{-1} P^{-1} \\ P^{-1} \end{bmatrix} = G(\mu) \begin{bmatrix} I_p \\ P^{-1} \end{bmatrix}. \]

Next, in view of (6.17) and (6.6)

\[ F(1)^* = [(I_n - A)^{-1} C^* P^{-1}(I_n - A)^{-1} B] = [I_n P^{-1}] G(1)^*, \]

\[ \begin{bmatrix} I_p 0 0 b_2(\mu) \end{bmatrix} = I_m - (1 - \mu) \begin{bmatrix} 0 0 \end{bmatrix} B^*(I_n - z A^*)^{-1} P^{-1}(I_n - A)^{-1} B \]

\[ = I_m - (1 - \mu) G(\mu) \begin{bmatrix} 0 0 \\ 0 - P^{-1} \end{bmatrix} G(1)^* j_{pq}. \]

Substituting (6.23), (6.24) into (6.22) one obtains

\[ \tilde{A}(\mu) = I_m - (1 - \mu) G(\mu) \begin{bmatrix} \tilde{P}^{-1} & \tilde{P}^{-1} P^{-1} & \tilde{P}^{-1} P^{-1} \\ P^{-1} \tilde{P}^{-1} & -P^{-1} + P^{-1} \tilde{P}^{-1} P^{-1} \end{bmatrix} G(1)^* j_{pq}. \]

In view of the equality

\[ \begin{bmatrix} \tilde{P}^{-1} & \tilde{P}^{-1} & \tilde{P}^{-1} P^{-1} \\ P^{-1} \tilde{P}^{-1} & -P^{-1} + P^{-1} \tilde{P}^{-1} P^{-1} \end{bmatrix} = \Lambda^{-1} \]

this proves the formula (6.11).

By [14] Theorem 3.1 and Theorem 5.17 and Theorem 3.1 the set \( T S_\kappa(I_p, b_2, K_0) \) is described by the formula

\[ T S_\kappa(b_1, b_2; K) = T W [S_{\kappa-\kappa_1}^{p \times q}] = \{ T W [\epsilon] : \epsilon \in S_{\kappa-\kappa_1}^{p \times q} \}. \]

Therefore, the statement (3) is implied by Theorem 5.3 (3).

(3) By Theorem 5.3 the mvf \( \tilde{A}(\mu) \) defined by (6.21) belongs to the class \( M_{\kappa_1}(j_{pq}). \)

\[ \square \]

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