Regularization and renormalization in effective field theories of the nucleon-nucleon interaction

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Some form of nonperturbative regularization is necessary if effective field theory treatments of the \(NN\) interaction are to yield finite answers. We discuss various regularization schemes used in the literature. Two of these methods involve formally iterating the divergent interaction and then regularizing and renormalizing the resultant amplitude. Either a (sharp or smooth) cutoff can be introduced, or dimensional regularization can be applied. We show that these two methods yield different results after renormalization. Furthermore, if a cutoff is used, the \(NN\) phase shift data cannot be reproduced if the cutoff is taken to infinity. We also argue that the assumptions which allow the use of dimensional regularization in perturbative EFT calculations are violated in this problem. Another possibility is to introduce a regulator into the potential before iteration and then keep the cutoff parameter finite. We argue that this does not lead to a systematically-improvable \(NN\) interaction.

1. EFFECTIVE FIELD THEORIES OF THE NUCLEON-NUCLEON INTERACTION

There exist many nucleon-nucleon potentials which reproduce phase shifts and nuclear properties with remarkable accuracy [1]. Three fundamental features are shared by these potential models: (i) pions dominate at long distances, (ii) there is a source of intermediate-range attraction, and (iii) there is a source of short-distance repulsion. Generally, distinct physical mechanisms in these models account for the same feature of the nuclear force. The agreement with experiment is maintained in spite of this because of the large number of fit parameters. It would be a considerable advance in nuclear physics if a systematic approach to the nucleon-nucleon potential were developed based solely on symmetries and general physical principles.

One reason to hope that this can be achieved in nuclear physics is provided by the pattern of chiral symmetry breaking in QCD. The fact that chiral symmetry is spontaneously broken implies that the pion is light and interacts weakly at low energies. The lightness of the pion guarantees that it should play a fundamental role in nuclear physics. The weakness of pion interactions at low energies allows pion interactions with nucleons to be systematized using chiral perturbation theory. This procedure has proved remarkably successful in describing the interactions of pions with a single nucleon [2].

Systematic approaches to the scattering of strongly interacting particles, such as chiral perturbation theory, are based on the ideas of effective field theory (EFT). Effective field
theory says that for probes of a system at momentum $k \ll M$, details of the dynamics at scale $M$ are unimportant. What is important at low energies is the physics that can be captured in operators of increasing dimensionality which take the form of a power-series in the quantity, $k/M$.

In order to examine the application of EFT to nuclear physics, consider $NN$ scattering in an $S$-wave channel at momentum scales $k \ll m_\pi$. The EFT at these scales involves only nucleons since the pion is heavy relative to the energy scales under consideration and may therefore be “integrated out”. The effective Lagrangian then consists of contact operators of increasing dimensionality constrained by spin and isospin. One might naively expect to be able to calculate the $NN$ scattering amplitude directly from the effective Lagrangian as a power series in $k/m_\pi$:

$$T(k) = C + C_2 k^2 + C_4 k^4 + \ldots,$$

where with the prevailing prejudice of EFT we anticipate that the dimensional coefficients $C_{2n}$ will be “natural”, i.e. of order $m_\pi^{-2-2n}$. We know that in the $^1S_0$ and $^3S_1-^3D_1$ channels there are, respectively, a quasi-bound state and a bound state at low energies. The power series expansion (1) with natural coefficients can only be correct if these bound states are at “natural” energies $k^2 \sim m_\pi^2$. However, in these channels the bound states occur at unnaturally low energies. Therefore, the coefficients in the expansion must be unnatural: they are fixed by the pole positions of the low-lying bound states rather than the scale of the physics that has been integrated out. This limits the usefulness of an expansion in the amplitude to very low energies. On the other hand, if, following Weinberg’s suggestion [4], one makes an EFT expansion of the potential in $S$-wave channels and iterates it via the Lippmann-Schwinger equation:

$$V(p', p) = C + C_2 (p^2 + p'^2) + C_4 (p^4 + p'^4) + C'_4 p^2 p'^2 + \ldots; \quad (2)$$

$$T(p', p; k) = V(p', p) + M \int \frac{d^3 p''}{(2\pi)^3} V(p', p'') \frac{1}{k^2 - p''^2 + i\epsilon} T(p'', p; k), \quad (3)$$

one may hope to generate (quasi-)bound states at the appropriate energies while maintaining natural coefficients in the potential. There will, of course, be unknown coefficients in the potential, but these can be fit to experimental data [5–8] as in ordinary chiral perturbation theory.

At face value this procedure appears promising. For momenta $p, p' \sim k \ll m_\pi$ the potential (2) may be approximated by its first few terms. More generally, beginning from the Lagrangian of chiral perturbation theory, Weinberg has shown how to construct a nucleon-nucleon potential of definite chiral order [4,5]. In this potential pions are explicit degrees of freedom, while all other particles are “integrated out”, i.e. their effects appear in the Lagrangian via a derivative expansion, à la Eq. (2). However, in iterating this potential via Eq. (3) a number of issues arise which are absent in standard EFT treatments.

The main difficulty concerns nonperturbative regularization and renormalization. This is required when iterating the potential (2) to all orders using Eq. (3), since the hard asymptotic behavior inherent to the momentum expansion necessarily introduces divergences. The existence of a procedure to regularize these divergences and renormalize in a sensible way provides a non-trivial condition on the existence of an EFT.
2. THE NN AMPLITUDE AT SECOND ORDER IN A SIMPLE EFFECTIVE FIELD THEORY

Consider an effective field theory for the nucleon-nucleon interaction in which all exchanged particles are integrated out. At second order in this EFT the nucleon-nucleon potential in the $^1S_0$ channel may be written in momentum space as:

$$ V(p', p) = C + C_2(p^2 + p'^2). $$

(4)

Iterating using Eq. (3), we get the following result for the on-shell T-matrix [9]:

$$ \frac{1}{T(k)} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5 + k^2 C_2 (2 - C_2 I_3)} - I_1, $$

(5)

$$ I_5 = -M \int \frac{d^3p''}{(2\pi)^3} p''^2; \quad I_3 = -M \int \frac{d^3p''}{(2\pi)^3}; \quad I_1 = M \int \frac{d^3p''}{(2\pi)^3} \frac{k^2}{k^2 - p''^2 + i\eta}, $$

(6)

where $k$ is the on-shell momentum. The integrals $I_1, I_3, I_5$ all contain power-law divergences. Note that if $C_2 = 0$ this reduces to the expression derived by Weinberg [4] for the case where the potential is just $V(p', p) = C$.

We renormalize the amplitude (5) using the experimental values of the scattering length, $a$, and effective range, $r_e$. In other words, we fix $C$ and $C_2$ by demanding that

$$ \frac{1}{T(k)} = -\frac{M}{4\pi} \left( -\frac{1}{a} + \frac{1}{2} r_e k^2 + O(k^4) - ik \right), $$

(7)

The divergent integrals $I_1, I_3, I_5$ may be regularized in two different ways: either a (sharp or smooth) cutoff can be introduced, or dimensional regularization (DR) can be applied. In DR the power-law divergent pieces of $I_1, I_3$ and $I_5$ all vanish by prescription. The renormalization then yields:

$$ \frac{1}{T_{\text{DR}}(k)} = -\frac{M}{4\pi} \left( -a - \frac{1}{2} a^2 r_e k^2 - ik \right), $$

(8)

for all $a$ and $r_e$, as found by Kaplan et al. [6]. By contrast, if a cutoff is used then, once renormalization is performed and the cutoff is taken to infinity we get [9]

$$ \frac{1}{T_{\text{cutoff}}(k)} = -\frac{M}{4\pi} \left( -\frac{1}{a} + \frac{1}{2} r_e k^2 - ik \right), $$

(9)

but only if $r_e \leq 0$. If $r_e > 0$ the renormalization cannot be performed in the limit that the cutoff is taken to infinity if the coefficients $C$ and $C_2$ are real [9]. This is qualitatively different to the result found using dimensional regularization. Even in the case where $r_e < 0$ the two regularization schemes lead to amplitudes with different dependence on $k$. In this nonperturbative problem the two methods of regularization are inequivalent.

Recall that the equivalence of DR and other forms of regularization has only been proven for perturbative calculations. In such cases all power-law divergences can be absorbed into counterterms, so DR’s neglect of these divergences does not lead to any difference in the renormalized amplitude. However, here the renormalization conditions are highly nonlinear and the two forms of regularization are no longer equivalent.
The result obtained here using cutoff regularization is in accord with that found if the regularization is performed before iteration, as it was in Refs. [5,7,8]. In that case, a corollary [10] of a theorem due to Wigner [11] states that if a potential goes to zero beyond some range \( R \) then
\[
r_e \leq 2 \left( R - \frac{R^2}{a} + \frac{R^3}{3a^2} \right).
\]
In particular, this implies that there is a minimal range for the regulated delta function below which the effective range cannot be reproduced. Furthermore, a minimal range for the regulated contact interactions exists even if pions are explicitly included in the calculation [7]. These results hold whenever the EFT expansion of the potential (2) is truncated at some finite order.

In fact, they merely reflect a very basic fact of nuclear physics: the range of the potential used to model the nuclear force is important. To any finite order in momentum the potential (2) is a zero-range potential in coordinate space. By regularizing this potential we give the interaction some finite range. However, if this range is taken to zero (or, equivalently, the momentum-space cutoff is taken to infinity), Eq. (10) shows that it is impossible to reproduce the observed energy-dependence of the \( S \)-wave \( NN \) phase shifts.

Why then, is the renormalized amplitude obtained using DR not subject to the bound (10)? In fact the power-law divergences which DR discards carry crucial information on the range of the interaction. DR’s removal of these power-law divergences is justified in perturbative EFT calculations, since such calculations are based on the assumption that details of the short-distance physics, such as power-law divergences, are unimportant. However, in this nonperturbative problem physics at short, but finite, distances carries important additional information which affects the renormalized amplitude and so should not be discarded.

3. CUTOFF EFFECTIVE FIELD THEORY FOR THE POTENTIAL

Thus a cutoff cannot be introduced and then taken to infinity if one wishes to reproduce the energy-dependence of the \( NN \) phase shifts, and the assumptions underpinning the use of DR in EFT calculations are not satisfied in this problem.

An alternative is to introduce a regulator into the potential before iteration, and then keep its mass finite. In such an approach renormalization is performed by adjusting the potential to fit the \( NN \) scattering data. This can be done for several different values of the regulator parameter. (For actual examples of this being done in the \( NN \) problem see [5,7,8].) In this section we explain why this approach is not systematic.

Suppose that a new cutoff scale \( \beta < m_\pi \) is introduced. This can be implemented by writing the potential in this cutoff field theory for \( NN \) scattering in the \( ^1S_0 \) channel as
\[
V(p', p) = [C + C_2(p^2 + p'^2) + \ldots] \theta(\beta - p)\theta(\beta - p').
\]
The coefficients \( C, C_2, \) etc. will all be functions of \( \beta \). This effective potential is to be inserted into Eq. (3) and the equation solved for different values of \( \beta \) with the coefficients \( C(\beta), C_2(\beta), \) etc. fit to low-energy scattering data. Of course, as \( \beta \) is varied the coefficients will change significantly.
If power counting for the potential were correct, and the expansion (11) was only used within its regime of validity, then the terms in the potential would get systematically smaller as the “order” is increased. However, in fact this does not happen—at the upper end of the momentum domain all terms contribute equally to the potential. One simple way to see this is by looking at what happens if a different form of regulator is chosen.

Of course, the choice of a theta function to regulate the momentum-space integrals is entirely arbitrary. One could just as well choose a potential of the form

$$V(p', p) = [\tilde{C} + \tilde{C}_2(p^2 + p'^2) + \ldots]g(p^2/\beta^2, p'^2/\beta^2),$$

where $g$ is some suitably well-behaved function. The effective potential should be essentially unaltered by this change in the form of the cutoff. However, this implies that the ratios $\tilde{C}_n/C$ differ from the ratios $C_n/C$ by terms of order $1/\beta^n$. Therefore for a generic cutoff function $g$ the ratio $C_n/C$ must be of order $1/\beta^n$. If this is the case, and the effective potential is to be used in a momentum regime which extends up to $\beta$, then at the upper end of this momentum regime all terms in the expansion are equally important. Therefore one should not truncate Eqs. (11) or (12) at some finite order in $p$ and $p'$. This is not surprising: if a finite momentum-space cutoff is used, the resulting potential contains all orders in momentum, and so is sensitive to physics at arbitrarily short distances.

Therefore, none of the regularization schemes discussed here give a systematic EFT $N\bar{N}$ potential. A paper discussing these issues in more detail is in preparation [12].

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