Black Hole Decay as Geodesic Motion

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\textbf{Abstract}

We show that a formalism for analyzing the near-horizon conformal symmetry of Schwarzschild black holes using a scalar field probe is capable of describing black hole decay. The equation governing black hole decay can be identified as the geodesic equation in the space of black hole masses. This provides a novel geometric interpretation for the decay of black holes. Moreover, this approach predicts a precise correction term to the usual expression for the decay rate of black holes.

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The discovery of a conformal group structure in the near-horizon region of a black hole has led to interesting developments in quantum gravity [1, 2]. The emergence of such a structure is closely related to the holographic nature of the system [3, 4]. Recently we have shown that by studying the algebraic structure of an operator that governs the near-horizon dynamics of a scalar probe in the background of a massive Schwarzschild black hole, the underlying conformal group structure can be explicitly revealed [5]. Our analysis was based on the requirement of unitarity of the conformal theory and self-adjointness [6] of the associated near-horizon Hamiltonian [7, 8]. This led to the introduction of the self-adjoint extension of the near-horizon Hamiltonian, described by a real parameter $z$. In this approach, the holographic nature of the black hole is thus realized by the self-adjoint extension. It was also shown that for the formalism to be self-consistent, $z$ must be a small positive number. This constraint on $z$ essentially incorporates the correct boundary conditions required for holography.

In a subsequent paper [9] we showed that further confirmation of the near-horizon conformal structure emerged when $z$ was related to the mass $M$ of the Schwarzschild black hole as

$$z = \frac{a}{M^2},$$

where $a$ is a nonzero positive constant. For the Schwarzschild background it was found that $a = \frac{1}{8}$. For the moment we consider a general positive value of $a$. It should be stressed that our analysis is valid only for massive black holes, i.e. for small values of $z$. Once the identification given by Eqn. (1) is made, it was shown in Ref. [9] that our approach naturally leads to the characteristic logarithmic correction to the Bekenstein-Hawking entropy [10, 11, 12, 13]. It is known that such a logarithmic correction term is universally present in any calculation of black hole entropy within a conformal field theory framework [13].

In this Letter we demonstrate that our formalism [5, 9] for analyzing the near-horizon conformal structure is also capable of describing black hole decay. For this purpose, we consider the space of all self-adjoint extensions which we denote by $\mathcal{M}$. As indicated in Eqn. (1), the self-adjoint extension parameter $z$ is related to the mass $M$ of the black hole. $M$ can therefore be taken as a good coordinate to describe $\mathcal{M}$. The central idea here is to study geodesic motion in the space of self-adjoint extensions. We find that within our formalism, there is a natural way to give the space $\mathcal{M}$ a metric. The geodesic motion in $\mathcal{M}$ is then calculated using this metric. We find that the equation for geodesic motion in $\mathcal{M}$ agrees with the equation for black hole decay [14, 15] for a suitable choice of the affine parameter. Once the affine parameter has been so chosen, our approach predicts a precise correction term in the equation for black hole decay arising from the logarithmic correction to the black hole entropy.

We start by recalling some of the basic results obtained in references [5] and [9] which shall be useful for our analysis. The action $S$ for a massless scalar field coupled to a Schwarzschild black hole of mass $M$ in 3+1 dimensions is given by

$$S = -\frac{1}{2} \int \sqrt{|g|} g^{ij} \partial_i \phi \partial_j \phi,$$

where the Schwarzschild metric in the spherical polar coordinates has the form

$$g_{ij} dx^i dx^j = -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2.$$
In the near-horizon limit, the corresponding Klein-Gordon operator for the time-independent, zero frequency and zero angular momentum modes of a massless scalar field is given by [8]

\[ H = -\frac{d^2}{dx^2} - \frac{1}{4x^2}, \]  

(4)

where \( x = r - 2M \) is the near-horizon coordinate. In order to study the quantum properties of \( H \), we consider the equation

\[ H\psi = \mathcal{E}\psi, \quad \psi(0) = 0, \]

(5)

where \( \mathcal{E} \) is the eigenvalue and \( \psi \in L^2[r^+, dr] \) is the associated wavefunction. \( H \) belongs to the class of unbounded linear operators on a Hilbert space [6] and it admits a one-parameter family of self-adjoint extensions labelled by a \( U(1) \) parameter \( e^{iz} \), where \( z \) is real. The normalized bound state eigenfunction and eigenvalue of Eqn. (5) are given by \(^3\) [7, 8]

\[ \psi(x) = \sqrt{2Ex}K_0\left(\sqrt{Ex}\right) \]

(6)

and

\[ \mathcal{E} = -E = -\exp\left[\frac{\pi}{2}\cot\frac{z}{2}\right] \]

(7)

respectively, where \( K_0 \) is the modified Bessel function. In our formalism, this solution is interpreted as bound state excitation of the black hole due to the capture of the scalar field.

As mentioned before, the requirement of near-horizon conformal symmetry places an important constraint on \( z \) [5]. To see this, consider a band-like region \( \Delta = [x_0 - \delta/\sqrt{E}, x_0 + \delta/\sqrt{E}] \), where \( x_0 \sim \frac{1}{\sqrt{E}} \) and \( \delta \sim 0 \) is real and positive. When \( z > 0 \) and satisfies the condition \( z \sim 0 \), we see that \( x_0 \approx 0 \). Under this condition, \( \Delta \) belongs to the near-horizon region of the black hole. At any point \( x \in \Delta \) the leading behaviour of the wavefunction is given by [9]

\[ \psi = A\sqrt{E}x, \]

(8)

where \( A = \sqrt{2(\ln2 - \gamma)} \) and \( \gamma \) is Euler’s constant. Taking a typical value of \( x = x_0 \), we can write the wavefunction as

\[ \psi = AE^{1/4} \approx Ae^{\frac{x}{\sqrt{E}}} \]

(9)

for \( z \sim 0 \). Substituting the value of \( z \) from Eqn. (1) in Eqn. (9), the wavefunction can be written as a function of \( M \) as

\[ \psi(M) = Ae^{cM^2} \equiv Ag(M^2). \]

(10)

The function \( g(M^2) = e^{cM^2} \) captures the \( M \) dependence of the wavefunction and \( c = \frac{\pi}{4a} \) is a positive constant. For the Schwarzschild background we have \( c = 2\pi \).

The wave function of a system is a natural object to examine in order to understand any symmetry that might be present in the system. To this end, consider the set \( G \equiv \{g(M^2)|M^2 \in \mathbb{R}\} \). The elements of \( G \) are the functions \( g \) defined in Eqn. (10) with different values of \( M \) corresponding to different elements of \( G \). For any two elements of \( G \) given by \( g(M_i^2) = e^{cM_i^2} \)

\(^3\) It may be noted that the results of our earlier analysis [9] as well as the present one depends only on the existence of the \( n = 0 \) bound state of ref. [7, 8].
and \( g(M_2^2) = e^{cM_2^2} \), we can define a composition law as \( g(M_1^2) \cdot g(M_2^2) \equiv g(M_1^2 + M_2^2) \in G \). Similarly, for any \( g(M^2) \in G \), we can define the inverse element as \( g^{-1}(M^2) \equiv g(-M^2) \in G \).

With respect to the composition law defined above, the set \( G \) has the structure of a continuous abelian group.

The presence of the continuous abelian group \( G \) allows us to describe the way the black hole mass changes in a geometric fashion. To do this we need to construct a group invariant metric on the space \( \mathcal{M} \). There is a well known procedure for doing this. We begin with the observation that the group \( G \) has a natural action on the space \( \mathcal{M} \). Under the action of \( G \), a point \( M_0 \in \mathcal{M} \) transforms as \( M_0 \rightarrow e^{cM_2^2}M_0 \in \mathcal{M} \). \( G \) therefore acts as a group of transformations on the space \( \mathcal{M} \). On a continuous group \( G \), the group invariant metric can be written as \( ds^2 = \text{Trace} \left( g^{-1}dg \right)^2 \). (11)

In our case, \( G \) is abelian and using Eqns. (10) and (11), we obtain the expression of the metric on \( \mathcal{M} \) as

\[
ds^2 = [d(\log g)]^2 = 4c^2M^2(dM)^2 \equiv h_{MM}(dM)^2, \tag{12}\]

where \( h_{MM} = 4c^2M^2 \).

We now have all the ingredients to calculate the geodesic equation of motion in \( \mathcal{M} \). For this purpose, consider a parametrised curve \( M(\lambda) \in \mathcal{M} \) where \( \lambda \in \mathbb{R} \) is taken as the affine parameter. Using the metric in Eqn. (12), the geodesic equation of motion in \( \mathcal{M} \) can be written as

\[
\frac{d^2 M}{d\lambda^2} + \Gamma_{MM}^M \left( \frac{dM}{d\lambda} \right)^2 = 0, \quad \text{(no sum over } M), \tag{13}\]

where

\[
\Gamma_{MM}^M = \frac{1}{2} h_{MM} \frac{dh_{MM}}{dM} = \frac{1}{2} \frac{d}{dM}(\log h_{MM}) = \frac{1}{M} \tag{14}\]

and \( h_{MM}^{-1} = h_{MM}^{-1} = (4c^2)^{-1}M^{-2} \). In terms of the variable \( v = \frac{dM}{d\lambda} \), Eqn. (13) can be expressed as

\[
\frac{dv}{dM} + \frac{v}{M} = 0. \tag{15}\]

Integrating Eqn. (15) we get,

\[
v = \frac{dM}{d\lambda} = \text{constant } \frac{M}{M}. \tag{16}\]

Eqn. (16) describes how the mass of the black hole changes with respect to the affine parameter \( \lambda \) up to an overall undetermined constant. It is chosen to be given by \( \lambda = M^{-1}t + b \) where \( t \) is time and \( b \) is a constant, Eqn. (16) reduces to

\[
\frac{dM}{dt} = -k \frac{M}{M^2}, \tag{17}\]

where \( k > 0 \) is a constant. The sign of the constant \( k \) has been chosen based on the expectation that the back-reaction effects will cause the mass of the black hole to decrease in order to compensate for this energy loss [14, 15]. With these identifications, Eqn. (17) agrees with the standard result of black hole decay. The assumption of large black hole mass used in our formalism is a feature present in the usual approach to black hole decay as well [14, 15]. It may
be noted that the decay rate obtained above is independent of the constant $\alpha$ which appears in the relation between $z$ and $M$ in Eqn. (1). We thus have a robust description of the decay of black holes which is independent of the microscopic details of our formalism.

The idea outlined above can be applied elsewhere as well. There is, for instance, a well known example in non-relativistic quantum mechanics where the mass $m$ appears as a parameter in the projective representation of the Galilean Group [17]. Interpreting the set of Galilean transformations as an abelian group on the space of masses and following the steps outlined above we get \( \frac{dm}{d\lambda} = \alpha \), where $\alpha$ is a constant. Again, with the choice of $\lambda = rt + s$ where $t$ is time and $r$, $s$ are constants leads to \( \frac{dm}{dt} = \alpha r \). Choosing $\alpha = 0$ we see that this is consistent with conservation of mass.

In ref. [9] we showed that the near-horizon conformal structure leads to a logarithmic correction to the Bekenstein-Hawking entropy. We would now demonstrate that this logarithmic contribution to the entropy generates a corresponding correction term for decay rate of black holes. In our formalism, density of states for the black hole was related to the modulus square of the wavefunction as $\rho \sim |\psi|^2$ [9]. The logarithmic correction to the Bekenstein-Hawking entropy is given by $-\frac{3}{2} \log M^2$. A change in the entropy due to this term would lead to a corresponding change in the density of states. Let $\chi(M)$ denote the effective wavefunction associated with this new density of states. In our formalism, we then have

$$|\chi(M)|^2 \sim e^{2cM^2 - \frac{3}{2} \log M^2},$$

which gives

$$\chi(M) \sim \frac{1}{M^{\frac{3}{2}}} e^{cM^2}.$$ (18)

Next we observe that $\chi(M)$ can be written as

$$\chi(M) = p(M)g(M^2),$$ (20)

where $p(M) = \frac{1}{M^{\frac{3}{2}}}$. The set $P = \{p(M)|M \neq 0\}$ forms an abelian group with respect to the composition law $p(M_1) \cdot p(M_2) = p(M_1M_2)$ with $p^{-1}(M) = p(M^{-1})$. The wavefunction $\chi(M)$ thus belongs to the direct product $P \otimes G$ which is again an abelian group. Following the analysis presented above, we can write the metric on $P \otimes G$ as

$$ds^2 = (gp)^{-1}d(gp),$$ (21)

where $g \in G$ and $p \in P$. In the limit of large black hole mass, this metric has the form

$$ds^2 \approx 4c^2M^2 \left[ 1 - \frac{3}{4c^2M^2} \right].$$ (22)

The corresponding geodesic equation of motion gives

$$\frac{dv}{dM} + v \left[ \frac{1}{M} + \frac{3}{4c^2M^3} \right] = 0$$ (23)

where $v = \frac{dM}{d\lambda}$. Eqn. (23) can be integrated to give

$$vMe^{-\frac{3}{4c^2M^2}} = \text{constant.}$$ (24)
Recall that the affine parameter $\lambda$ has already been identified with $M^{-1}t + b$. With this identification, and for large black hole mass, the decay rate of black hole is obtained as

$$\frac{dM}{dt} = -k \left[ \frac{1}{M^2} + \frac{3}{8cM^4} \right].$$

(25)

Note that using the same logic as discussed before, the constant in Eqn. (24) has been written as $-k$ with $k > 0$. Both terms in the r.h.s. of Eqn. (25) contribute to the decay with the same sign. The first term on the r.h.s. of Eqn. (25) is identical to what was already obtained before through appropriate identification of the affine parameter $\lambda$. The main result of this Letter is that once $\lambda$ is chosen in the fashion indicated here, the formalism naturally leads to a precise correction term for the black hole decay rate, determined solely by the logarithmic correction to the Bekenstein-Hawking entropy. Such a logarithmic term appears universally whenever the Bekenstein-Hawking entropy is calculated within a conformal field theory framework [13]. It is thus expected that the correction obtained for the black hole decay should share this same universal property as well.

To summarize we note that the presence of a conformal structure in the near-horizon region of a black hole is a consequence of the holographic principle. In our formalism, the appropriate condition for realizing holography is encoded in the self-adjoint extension parameter $z$. The parameter $z$, or equivalently the black hole mass $M$, then has the natural interpretation as a moduli. It is known that the geodesic motion in the moduli space of certain physical system provides an appropriate description for the corresponding dynamics [18]. Implementing this idea with an appropriate identification of the affine parameter is consistent with the standard description of black hole decay. Moreover, the approach then predicts a correction term to the black hole decay rate arising from the logarithmic correction to the Bekenstein-Hawking entropy. We are thus led to the surprising conclusion that even the decay of black hole can be given a novel geometric interpretation within the context of the holographic principle.

It has been claimed that the near-horizon conformal symmetry is associated with a large class of black holes in arbitrary dimensions [2, 19]. It seems plausible that probing the near-horizon geometry of these black holes would lead to an operator of the form of $H$, possibly with a different coefficient for the inverse square term [8]. It is thus likely that the analysis presented above for the massive Schwarzschild black hole could be generalized to include other cases as well.
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