Progress in automated perturbation theory for heavy quark physics

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We review our perturbative techniques for improved heavy quark actions. A new procedure for computing improvement coefficients is suggested, where the continuum limit of a lattice-regularized theory provides the matching conditions. We also use a gauge-invariant infrared regulator that is well suited to higher-order calculations in lattice gauge theory. We report on preliminary tests of our method, as well as a possible way to reduce systematic errors in these calculations.

1. Introduction

Improved actions are widely used in heavy quark physics. Examples include NRQCD \cite{1} which is very successful in describing $b \bar{b}$ systems, and the Fermilab action \cite{2} which has been applied to both $b$ and $c$ quark systems. With the MILC results for unquenched light quarks, precision determinations of important physical quantities (i.e. $f_B$) are possible with these heavy quark actions \cite{3}.

One of the dominant systematic errors that remains with improved actions is the determination of the improvement coefficients. The hyperfine splittings of the $J/\psi$ system illustrates this clearly. While there have been good results from heavy quark simulations, these hyperfine splittings remain significantly underestimated with tree-level improved actions. This underestimate is seen with both the Fermilab and NRQCD approaches to heavy quark physics. Unquenching \cite{4, 5} does not seem to resolve it (the Fermilab determination is still $\approx 25\%$ low). We expect that this underestimate should be greatly reduced by perturbative matching of the $\Sigma \cdot B$ operator and the use of a highly improved action \cite{6}.

In order to determine this action coefficient ($c_B$) we adopt the simplest matching strategy possible. We will compute the amplitude for quark scattering off of a background chromomagnetic field in lattice perturbation theory, and, to the same order in perturbation theory compute the continuum result, using the same IR regulator. The matching is simply a matter of subtracting one from the other, and tuning $c_B$ to insure that the difference vanishes to whatever order is desired.

In principle, this procedure should work to whatever order in perturbation theory we wish. In this report we will discuss the matching to one-loop order, however for true high precision results the matching should be done to two loops. This is because at typical lattice spacings $\alpha_s (1/a) \approx 0.2$. Our ultimate goal is high precision results, so we want to organize our calculation in a way which makes going to two loop order as easy as possible.

We have discussed our approach to lattice perturbation theory at length in prior LATTICE presentations (\cite{7, 8, 9}). The core of our technique is to generate Feynman rules automatically using the Lüscher - Weisz algorithm \cite{10}. The primary advantage of this method is that we can change the form of the action easily, despite the complexity of the rules.

Many of the individual diagrams we are calculating are infrared divergent, to regulate this we use twisted boundary conditions. This has the effect of turning integrals over the three momentum into restricted sums, as in the following example:

$$\int dk_0 d\vec{k} f(k) \rightarrow \int dk_0 \sum_{\ell} \chi_\ell f(k) \frac{1}{k^2}.$$ 

Here $\chi_\ell$ is a veto function which excludes many lattice momenta, including the IR zero modes.
2. Lattice-to-Lattice Matching

By design, the lattice and continuum theories should be identical in the infrared. The lattice matching computations that have been done to date have typically used a gluon mass to regulate the IR in both the lattice and continuum theories. This method could work for our calculation, however, it may pose significant problems with gauge invariance at higher order.

One solution is to use twisted boundary conditions, which provide a gauge invariant IR regulator. However this regulator is somewhat difficult to combine with dimensional regularization for continuum calculations. This can be avoided by rethinking the continuum UV regulator that we use. We are calculating matching coefficients, so there is no particular reason to use dimensional regularization. In fact, we can use any UV regulator we want.

Following a suggestion of Peter Lepage we have simply used a lattice cutoff to regulate the continuum theory. This allows us to do the “continuum” side of the computation using the same lattice techniques as we have developed for the heavy quark actions. This procedure is straightforward, we can pick a simple lattice theory (Wilson glue + naive quarks) and run the lattice spacing down. As long as the spacing is made small enough, this theory will be very close to the continuum theory. Then we can subtract off our heavy quark results, and get the matching coefficients. This lattice-to-lattice technique should work for matching calculations within QCD itself, such as the perturbative coefficients for operators in the lattice action. This method does not apply for example to matrix elements of operators in the effective Hamiltonian for the weak interactions.

As we discuss below, for one loop matching computations this naive procedure works. To two (and higher) loops the “continuum” and lattice theories will both have to be renormalized to give the same physical values of the various parameters, however this should be a straightforward application of renormalized perturbation theory.
3. Testing

To test this procedure we do the computation of $c_{\Sigma B}$ in the static quark limit. This calculation has previously been performed, using a gluon mass as the IR regulator, and matching to the dimensionally regularized continuum theory, by Flynn and Hill [11]. We compute the amplitude for quark scattering off of a background field for both the heavy quark and “continuum” theories.

In the “continuum” case, we’re using units in which $a = 1$ so we want to run all other scales in the problem down to zero. For matching, we use naive quarks, so the differences from the continuum limit go like $O(m^2)$. Figure 1 shows the results of this lattice to lattice matching for various naive quark masses. The black squares are the results of [11]. Clearly, in the continuum limit, $L \to \infty$ and $m \to 0$, we reproduce the Flynn-Hill result. Figure 1 shows that our matching procedure works, and that the errors will be small enough to reliably extract the dependence on the ultraviolet cutoff.

In addition to this test, we have thoroughly tested our implementation of the Fermilab fermion action. To do this we have duplicated all of the major results in [12] for the one loop rest mass and wavefunction renormalization. Figure 2 shows the (one loop) wavefunction renormalizations of Wilson and clover quarks, at fixed lattice size. The logarithmic dependence on $M_0$ agrees with [12].

4. Periodic Boundary Conditions

We have investigated other means for controlling infrared divergences. One possibility is to integrate the difference between the lattice and the ”continuum” expressions for a matching coefficient. This difference is infrared finite, and hence could be computed without any infrared regulator.

To test this method, we have calculated the difference in fermion wave function renormalizations $Z_{W}^{Wilson} - Z_{SW}^{SW}$ over a range of fermion masses, without any infrared regulator. In the $m_0 \to 0$ limit we obtain 0.228(1) which is in reasonable agreement with 0.023083(7) which is the result presented in [12]. We are currently investigating this technique for determining $c_B$.

5. Conclusions

Doing high precision QCD with improved heavy quark actions requires two loop determinations of the improvement coefficients. In particular the coefficient of the $\sigma \cdot B$ is needed to resolve the underestimate of the hyperfine splitting we have discussed.

To get to two loops it is important to organize the calculation in as efficient a manner as possible. To this end we have introduced a procedure to match two lattice theories onto each other, where the continuum limit of one lattice regularized theory provides the matching criteria for the improvement of the other. Additionally we have used twisted boundary conditions to provide a gauge invariant regulator. As a preliminary test of these techniques we have reproduced an earlier result for the matching of $\sigma \cdot B$ in the lattice HQET theory. We are currently applying these methods to improvement of the Fermilab and NRQCD actions.

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