Non-autonomous stochastic lattice systems with Markovian switching

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Abstract The aim of this paper is to study the dynamical behavior of non-autonomous stochastic lattice systems with Markovian switching. We first show existence of an evolution system of measures of the stochastic system. We then study the pullback (or forward) asymptotic stability in distribution of the evolution system of measures. We finally prove that any limit point of a tight sequence of an evolution system of measures of the stochastic lattice systems must be an evolution system of measures of the corresponding limiting system as the intensity of noise converges zero. In particular, when the coefficients are periodic with respect to time, we show every limit point of a sequence of periodic measures of the stochastic system must be a periodic measure of the limiting system as the noise intensity goes to zero.

Keywords. Non-autonomous; Markovian switching; Evolution system of measures; Limit measure.

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1 Introduction

Let \( (W_k)_{k \in \mathbb{N}} \) be a sequence of independent standard two-side Wiener processes on a complete filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P) \) satisfying the usual condition and \( r(t), t \in \mathbb{R}, \) be a right continuous Markov chain, independent of the Brownian motion \( (W_k)_{k \in \mathbb{N}}, \) on the probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P) \) taking values in a finite state space \( S = \{1, 2, \ldots, N\} \) with generator \( \Gamma = (r_{ij})_{N \times N} \) given by

\[
P\{r(t + \Delta) = j \mid r(t) = i\} = \begin{cases} r_{ij}\Delta + o(\Delta), & i \neq j; \\ 1 + r_{ij}\Delta + o(\Delta), & i = j, \end{cases}
\]
where $\Delta > 0$ and $\lim_{\Delta \to 0^+} \Delta \to 0, r_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$ and $r_{ii} = -\sum_{i \neq j} r_{ij}$.

It is well known that almost every sample path of $r(t)$ is a right-continuous step function and $r(t)$ is ergodic.

In this paper, we study the limiting behavior of evolution system of measures of the nonautonomous stochastic lattice system with Markovian switching defined on the integer set $\mathbb{Z}$: for $s \in \mathbb{R}$,

$$
du_i(t) - \nu (u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) dt + \lambda(r(t))u_i(t) dt
= (f_i(t, r(t), u_i(t)) + g_i(r(t))) dt
+ \varepsilon \sum_{k=1}^{\infty} (h_{i,k}(r(t)) + \sigma_{i,k}(t, r(t), u_i(t)))dW_k(t), \quad t > s,
$$

with initial data

$$
u u_i(s) = \xi_i \quad \text{and} \quad r(s) = j \in S,
$$

where $u = (u_i)_{i \in \mathbb{Z}}$ is an unknown sequence, $\xi = (\xi_i)_{i \in \mathbb{Z}} \in l^2$ is given, $0 < \varepsilon \leq 1$, $\nu > 0$, for $j \in S$ $\lambda(j) > 0$, $g(j) = (g_i(j))_{i \in \mathbb{Z}}$ and $h(j) = (h_{i,k}(j))_{i \in \mathbb{Z}, k \in \mathbb{N}}$ are given in $l^2$, and $f_i, \sigma_{i,k} : \mathbb{R} \times S \times \mathbb{R} \to \mathbb{R}$ are nonlinear functions for every $i \in \mathbb{Z}$ and $k \in \mathbb{N}$.

We mention that lattice systems have many applications in practice and have been extensively investigated. For stochastic lattice systems without time-dependent forcing, the existence of random attractors was proved in [10, 3, 5, 1, 4]. The existence of random attractors was also obtained in [15, 20, 2] for the systems with time-dependent forcing. The dynamical behavior of invariant measures of stochastic lattice systems was obtained in [6, 19, 17, 16, 12] and the limiting behavior of periodic measures of stochastic lattice systems with periodic forcing term was studied in [11]. The concept of evolution system of measures was introduced by [8]. It is the natural generalization of the notion of an invariant measure to non-autonomous systems. Recently, in [18], Wang et. al. studied the limiting behavior of evolution system of measures of non-autonomous stochastic lattice systems. There is an extensive literature on existence and stability of invariant measure for stochastic differential equations with Markovian switching, see e.g., [21, 22, 9]. However, there is so far no result of evolution system of measures of non-autonomous stochastic lattice systems with Markovian switching.

This paper is concerned with the theory of evolution system of measures of nonhomogeneous Markov processes generated by stochastic differential equations with both non-autonomous deterministic and Markovian switching. We will prove a sufficient condition for existence and pullback (or forward) asymptotic stability in distribution of evolution system of measures for such processes.
We will also show the effect of evolution system of measures for a family of such processes from parameter disturbance. For periodic Markov processes, we prove the evolution system of measures are also periodic. Periodic measures for SPDEs was studied in [7, 13]. As an application of our abstract results, we will investigate the existence, pullback (or forward) asymptotic stability in distribution, and the limiting behavior of evolution system of measures of (1.1)-(1.2) as the noise intensity $\varepsilon \to 0$.

The rest of this paper is organized as follows. Section 2 is devoted to the existence, stability and periodicity of evolution system of measures of time nonhomogeneous Markov processes. In Section 3, we show the limiting behavior of evolution system of measures of time nonhomogeneous Markov processes. Section 4 is devoted to the existence and uniqueness of solutions to the stochastic lattice system (1.1)-(1.2). In Section 5, we derive the uniform estimates of solutions which are needed for proving our main results in later sections. In Section 6, we establish the existence, stability and periodicity of evolution system of measures on $l^2$ for (1.1)-(1.2) and prove the convergence of evolution system of measures of system (1.1)-(1.2) as $\varepsilon \to 0$.

2 Existence and Stability

In what follows, we denote by $X$ a Polish space with a metric $d_X$ and denote by $H$ a separable Banach space with norm $\| \cdot \|_H$, respectively. Define $C_b(X)$ as the space of bounded continuous functions $f : X \to \mathbb{R}$ endowed with the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|,$$

and denote by $L_b(X)$ the space of bounded Lipschitz functions on $X$. That is, of functions $f \in C_b(X)$ for which

$$\text{Lip}(f) := \sup_{x_1, x_2 \in X} \frac{|f(x_1) - f(x_2)|}{\text{dist}_X(x_1, x_2)} < \infty.$$  

The space $L_b(X)$ is endowed with the norm

$$\|f\|_L = \|f\|_\infty + \text{Lip}(f).$$

Let us denote by $P(X)$ the set of probability measures on $(X, \mathcal{B}(X))$. Define a metric on $P(X)$ by

$$d^*_L(\mu_1, \mu_2) = \sup_{\|f\|_L \leq 1} |(f, \mu_1) - (f, \mu_2)|, \quad \mu_1, \mu_2 \in P(X).$$

Given $s, t \in \mathbb{R}$ and $s \leq t$, we let $r_{s,j}(t)$ be the Markov chain starting from state $j \in S$ at $t = s$ and let $u(t, s, \xi, j)$ be a stochastic process with initial conditions $u(s, s, \xi, j) = \xi \in H$ and $r(s) = j$.\]
at initial time \( t = s \). Let \( y(t, s, \xi, j) \) denote the \((H \times S)\)-valued process \((u(t, s, \xi, j), r_{s,j}(t))\) and \( y(t, s, \xi, j) \) be a time nonhomogeneous Markov process. Let \( p(t, s, \xi, j, (dy, k)) \) denote the transition probability of the process \( y(t, s, \xi, j) \). For \( A \subset B \subset S \), let \( P(t, s, \xi, j, A \times B) \) denote the probability of event \( \{y(t, s, \xi, j) \in A \times B\} \) given initial condition \( y(s, s, \xi, j) = (\xi, j) \) at time \( t = s \), i.e.,

\[
P(t, s, \xi, j, A \times B) = \sum_{k \in B} \int_A p(t, s, \xi, j, (dy, k)).
\]

We define the transition evolution operator

\[
P_{s,t} \varphi (\xi, j) = \mathbb{E} [\varphi (u(t, s, \xi, j), r_{s,j}(t))], \quad \varphi \in C_b(H \times S).
\]

Assume that \( P_{s,t} \) is Feller, that is, \( P_{s,t} : C_b(H \times S) \to C_b(H \times S) \), for \( s < t \). Denote by \( P_{s,t}^* : \mathcal{P}(H \times S) \to \mathcal{P}(H \times S) \) the duality operator of \( P_{s,t} \). For \((\xi, j) \in H \times S\), denote by \( \delta_{\xi,j} \) the Dirac measure concentrating on \((\xi, j)\).

In this section, we show existence and stability of an evolution system of measures \((\mu_t)_{t \in \mathbb{R}}\) indexed by \( \mathbb{R} \). An evolution system of measures \((\mu_t)_{t \in \mathbb{R}}\) satisfies each \( \mu_t, t \in \mathbb{R} \), is a probability measure on \( H \times S \) and

\[
\sum_{j \in S} \int_H P_{s,t} \varphi (\xi, j) \mu_s (d\xi, j) = \sum_{j \in S} \int_H \varphi (\xi, j) \mu_t (d\xi, j), \quad \forall \varphi \in C_b(H \times S), \quad s < t.
\]

For \( \varpi > 0 \), the evolution system of measures \( \mu_t, t \in \mathbb{R} \) is \( \varpi \)-periodic, if

\[
\mu_t = \mu_{t+\varpi}, \quad \forall t \in \mathbb{R}.
\]

We now recall the definition of pullback (or forward) asymptotic stability in distribution of the evolution system of measures.

**Definition 2.1.** The evolution system of measures \((\mu_t)_{t \in \mathbb{R}}\) of Markov processes \( y(t, s, \xi, j) \) is said to be pullback asymptotic stability in distribution if for any \( \varphi \in C_b(H \times S) \),

\[
\lim_{s \to -\infty} P_{s,t} \varphi (\xi, j) = \sum_{j \in S} \int_H \varphi (x, j) \mu_t (dx, j), \quad \forall t \in \mathbb{R}, \ (\xi, j) \in H \times S,
\]

and be forward asymptotic stability in distribution if for any \( \varphi \in C_b(H \times S) \),

\[
\lim_{t \to +\infty} \left[ P_{s,t} \varphi (\xi, j) - \sum_{j \in S} \int_H \varphi (x, j) \mu_t (dx, j) \right] = 0, \quad \forall s \in \mathbb{R}, \ (\xi, j) \in H \times S.
\]
Forward asymptotic stability in distribution implies that $P_{s,t} \varphi(x)$ approaches as $t \to +\infty$ a curve, parametrized by $t$, which is independent of $s$ and $(\xi,j)$. This is the natural generalization of the strongly mixing property for an autonomous dissipative system.

In the sequence, we always assume that

(A0) For any $s \in \mathbb{R}$, $T > 0$, bounded set $B \subset H$ and $\eta > 0$, there exists a constant $R = R(\eta,B,T) > 0$, independent of $s$, such that for any $\xi \in B$ and $j \in S$,

$$P\left\{\|u(t,s,\xi,j)\|_H \geq R, t \in [s,s+T]\right\} < \eta.$$  

**Definition 2.2.** The processes $u(t,s,\xi,j)$ are said to have properties:

(A1) If for any $s \in \mathbb{R}$, $\xi \in H$ and $\eta > 0$, there exists a bounded subset $B = (\eta,\xi)$ of $H$, independent of $s$, such that for any $\xi \in H$, $j \in S$ and $t \geq s$,

$$P\{u(t,s,\xi,j) \in B\} > 1 - \eta.$$

(A2) If for any $s \in \mathbb{R}$, $\eta > 0$ and bounded subset $B$ of $H$, there exists a $T = T(\eta,B)$, independent of $s$, such that for $(\xi_1,\xi_2,j) \in B \times B \times S$,

$$P\{\|u(t,s,\xi_1,j) - u(t,s,\xi_2,j)\|_H < \eta\} \geq 1 - \eta, \quad \forall t - s \geq T.$$  

**Definition 2.3.** The processes $u(t,s,\xi,j)$ are said to have properties:

(A1*) If for any $s \in \mathbb{R}$, $\xi \in H$ and $\eta > 0$, there exists a compact $K = (\eta,\xi) \subset H$, independent of $s$, such that for any $\xi \in H$, $j \in S$ and $t \geq s$,

$$P\{u(t,s,\xi,j) \in K\} > 1 - \eta.$$  

(A2*) If for any $s \in \mathbb{R}$, $\eta > 0$ and any compact subset $K$ of $H$, there exists a $T = T(\eta,K)$, independent of $s$, such that for $(\xi_1,\xi_2,j) \in K \times K \times S$,

$$P\{\|u(t,s,\xi_1,j) - u(t,s,\xi_2,j)\|_H < \eta\} \geq 1 - \eta, \quad \forall t - s \geq T.$$  

**Lemma 2.4.** Assume that the processes $u(t,s,\xi,j)$ have property (A2). Then, for any bounded set $B \subset H$,

$$\lim_{s \to -\infty} d^*_L\left(P_{s,t}\delta_{\xi_1,i}, P_{s,t}\delta_{\xi_2,j}\right) = 0$$

uniformly in $\xi_1,\xi_2 \in B$ and $j_1, j_2 \in S$. 

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Proof. Choose any nonotone decreasing sequence \( \{s_n\}_{n=1}^{\infty} \) satisfying \( s_n < t, \forall n \in \mathbb{N} \), and \( s_n \to -\infty, \) as \( n \to \infty \). For any pair of \( j_1, j_2 \in S \) and \( n \in \mathbb{N} \), define the stopping time
\[
\tau_{j_1,j_2}^n = \inf \{ t - s_n | t \geq s_n, r_{s_n,j_1}(t) = r_{s_n,j_2}(t) \}.
\]
Recall that \( r_{s_n,j}(t) \) is the Markov chain starting from state \( j \in S \) at initial time \( t = s_n \) and due to the ergodicity, for any \( n \in \mathbb{N} \), \( \tau_{j_1,j_2}^n = \tau_{j_1,j_2}^n < \infty \) a.s. By homogeneity of the Markov chain, for any \( \eta > 0 \) and \( n \in \mathbb{N} \), there exists a positive number \( T \), independent of \( n \), such that
\[
P\{ \tau_{j_1,j_2} \leq T \} > 1 - \frac{\eta}{8}, \quad \forall j_1, j_2 \in S.
\]
For such \( T \), by \((A_0)\), there is a sufficiently large \( R > 0 \) such that
\[
P(\Omega_{\xi,j}) > 1 - \frac{\eta}{16}, \quad \forall (\xi,j) \in B \times S, \tag{2.1}
\]
where \( \Omega_{\xi,j} = \{ \|u(t,s_n,\xi,j)\|_H \leq R, \forall t \in [s_n,s_n+T] \} \). For any given \( \xi_1, \xi_2 \in B \) and \( j_1, j_2 \in S \), set \( \Omega' = \Omega_{\xi_1,j_1} \cap \Omega_{\xi_2,j_2} \). For any \( \varphi \in L_0(H \times S) \) and \( s_n < t - T \),
\[
\left| \left( \varphi, P^s_{n,t} \delta_{\xi_1,j_1} \right) - \left( \varphi, P^s_{n,t} \delta_{\xi_2,j_2} \right) \right| = \left| \mathbb{E} \varphi (u(t,s_n,\xi_1,j_1), r_{\{s_n,j_1\}}(t)) - \mathbb{E} \varphi (u(t,s_n,\xi_2,j_2), r_{\{s_n,j_2\}}(t)) \right|
\leq 2P \{ \tau_{j_1,j_2} > T \} + \mathbb{E} \left( I_{\{ \tau_{j_1,j_2} \leq T \}} \left| \varphi (u(t,s_n,\xi_1,j_1), r_{\{s_n,j_1\}}(t)) - \varphi (u(t,s_n,\xi_2,j_2), r_{\{s_n,j_2\}}(t)) \right| \right)
\leq \frac{\eta}{4} + \mathbb{E} \left( I_{\{ \tau_{j_1,j_2} \leq T \}} \mathbb{E} \left( \left| \varphi (u(t,s_n,\xi_1,j_1), r_{\{s_n,j_1\}}(t)) - \varphi (u(t,s_n,\xi_2,j_2), r_{\{s_n,j_2\}}(t)) \right| \right) \right)
\leq \frac{\eta}{4} + \mathbb{E} \left( I_{\{ \tau_{j_1,j_2} \leq T \}} \mathbb{E} \left( 2 \mathbb{E} \left( u(t,\tau_{j_1,j_2},\xi_1^*,k) - u(t,\tau_{j_1,j_2},\xi_2^*,k) \right) \right) \right)
\leq \frac{\eta}{4} + 2P \left( \Omega - \Omega' \right) + \mathbb{E} \left( I_{\Omega' \cap \{ \tau_{j_1,j_2} \leq T \}} \mathbb{E} \left( 2 \mathbb{E} \left( u(t,\tau_{j_1,j_2},\xi_1^*,k) - u(t,\tau_{j_1,j_2},\xi_2^*,k) \right) \right) \right), \tag{2.2}
\]
where \( \xi_1^* = u(\tau_{j_1,j_2},s_n,\xi_1,j_1), \xi_2^* = u(\tau_{j_1,j_2},s_n,\xi_2,j_2) \) and \( k = r_{s_n,j_1}(\tau_{j_1,j_2}) = r_{s_n,j}(\tau_{j_1,j_2}) \). Note that given \( \omega \in \Omega' \cap \{ \tau_{j_1,j_2} \leq T \} \), \( \|\xi_1^*\| \geq \|\xi_2^*\| \leq R \). So, by \((A_2)\), there exists a constant \( T_1 > T \) such that
\[
\mathbb{E} \left( 2 \mathbb{E} \left( u(t,\tau_{j_1,j_2},\xi_1^*,k) - u(t,\tau_{j_1,j_2},\xi_2^*,k) \right) \right) < \frac{\eta}{2}, \quad t - s_n > T_1. \tag{2.3}
\]
It therefore follows from \((2.1)-(2.3)\) that
\[
\left| \mathbb{E} \varphi (u(t,s_n,\xi_1,j_1), r_{\{s_n,j_1\}}(t)) - \mathbb{E} \varphi (u(t,s_n,\xi_2,j_2), r_{\{s_n,j_1\}}(t)) \right| \leq \frac{\eta}{4} + \frac{\eta}{2} + \frac{\eta}{4}, \quad t - s_n > T_1.
\]
Since \( \varphi \) is arbitrary, we must have
\[
d^2_L (P^s_{n,t} \delta_{\xi_1,j_1}, P^s_{n,t} \delta_{\xi_2,j_2}) \leq \eta, \quad t - s_n > T_1,
\]
for all $\xi_1, \xi_2 \in B$ and $j_1, j_2 \in S$. By the $\{s_n\}_{n=1}^\infty$ chosen arbitrarily, the proof is completed.

Repeating the scheme used in the proof Lemma 2.4, we get the following result.

**Lemma 2.5.** Assume that the processes $u(t, s, \xi, j)$ have property (A2). Then, for any bounded subset $B$ of $H$, 
\[
\lim_{t \to +\infty} d^*_{L}(P_{s,t}^* \delta_{\xi_1,j_1}, P_{s,t}^* \delta_{\xi_2,j_2}) = 0
\]
uniformly in $\xi_1, \xi_2 \in B$ and $j_1, j_2 \in S$.

**Lemma 2.6.** Assume that the processes $u(t, s, \xi, j)$ have properties (A1) and (A2). Then for any $t \in \mathbb{R}$ and $(\xi, j) \in H \times S$, there exists a $\mu_t \in P(H \times S)$, independent of $(\xi, j)$, such that 
\[
\lim_{s \to -\infty} d^*_{L}(P_{s,t}^* \delta_{\xi,j}, \mu_t) = 0.
\]

**Proof.** Fix any $t \in \mathbb{R}$ and $(\xi, j) \in H \times S$. We first claim that $\{P_{s,t}^* \delta_{\xi,j} : s \leq t\}$ is Cauchy in the space $P(H \times S)$ with metric $d^*_{L}$. To end this, we need to show that for any $\eta$, there is a $T > 0$ such that
\[
d^*_{L}(P_{s-h,t}^* \delta_{\xi,j}, P_{s,t}^* \delta_{\xi,j}) < \eta, \quad \forall s < t - T, \quad h > 0.
\]
This is equivalent to
\[
|\mathbb{E}[\varphi(u(t, s-h, \xi, j), r_{\{s-h,j\}}(t))] - \mathbb{E}[\varphi(u(t, s, \xi, j), r_{\{s,j\}}(t))]| < \eta,
\]
where $\varphi \in L_b(H \times S)$. For any $\varphi \in L_b(H \times S)$ and $h > 0$, compute
\[
\begin{align*}
&|\mathbb{E}[\varphi(u(t, s-h, \xi, j), r_{\{s-h,j\}}(t))] - \mathbb{E}[\varphi(u(t, s, \xi, j), r_{\{s,j\}}(t))]| \\
&= |\mathbb{E}(\mathbb{E}(\varphi(u(t, s-h, \xi, j), r_{\{s-h,j\}}(t)) | F_s)) - \mathbb{E}[\varphi(u(t, s, \xi, j), r_{\{s,j\}}(t))]| \\
&= \left|\sum_{j \in S} \int_H \mathbb{E}[\varphi(u(t, s, z, k), r_{\{s,k\}}(t)) p(s, s-h, \xi, j, (dz, \{k\})) - \mathbb{E}[\varphi(u(t, s, \xi, j), r_{\{s,j\}}(t))]| \\
&\leq \sum_{j \in S} \int_H |\mathbb{E}[\varphi(u(t, s, z, k), r_{\{s,k\}}(t)) - \mathbb{E}[\varphi(u(t, s, \xi, j), r_{\{s,j\}}(t))]| p(s, s-h, \xi, j, (dz, \{k\})) \\
&\leq 2P(s, s-h, \xi, j, B^c_R \times S) \\
&+ \sum_{j \in S} \int_{B^c_R} |\mathbb{E}[\varphi(u(t, s, z, k), r_{\{s,k\}}(t)) - \mathbb{E}[\varphi(u(t, s, \xi, j), r_{\{s,j\}}(t))]| p(s, s-h, \xi, j, (dz, \{k\}))|
\end{align*}
\]

(2.5)
where \( B_R = \{ x \in H \| \| x \|_H \leq R \} \) and \( B_R^C = H - B_R \). By \((A_1)\), there is a positive number \( R \) sufficiently large for
\[
P \left( s, s - h, \xi, j, B_R^C \times S \right) < \frac{\eta}{4}, \quad s < t. \tag{2.6}
\]
On the other hand, by Lemma \[2.4\] there is a \( T > 0 \) such that
\[
\sup_{\varphi \in L_b(H \times S)} \left| E\varphi (u(t, s, z, k), r_{\{s,k\}}(t)) - E\varphi (u(t, s, \xi, j), r_{\{s,j\}}(t)) \right| < \frac{\eta}{2}, \quad s < t - T, \tag{2.7}
\]
whenever \((z, l) \in B_R \times S\). Substituting \(2.6\) and \(2.7\) into \(2.5\) yields
\[
\left| E\varphi (u(t, s - h, \xi, j), r_{\{s-h,j\}}(t)) - E\varphi (u(t, s, \xi, j), r_{\{s,j\}}(t)) \right| < \eta, \quad \forall s < t - T, \quad h > 0.
\]
Since \( \varphi \) is arbitrary, the desired inequality \[2.4\] must hold, i.e., \( \{P_s^* \delta_{\xi,j} : s \leq t\} \) is Cauchy in the space \( \mathcal{P}(H \times S) \) with metric \( d^*_L \). So, for any \( t \in \mathbb{R} \) and \((\xi, j) \in H \times S\) there is a unique \( \mu_t(\xi,j) \in \mathcal{P}(H \times S) \) such that
\[
\lim_{s \to -\infty} d^*_L (P_s^* \delta_{\xi,j}, \mu_t(\xi,j)) = 0.
\]
It remains to show that \( \mu_t(\xi,j) \) is independent of \((\xi, j)\). Now, for any \((\xi, j) \in H \times S\), by Lemma \[2.4\]
\[
\lim_{s \to -\infty} d^*_L (P_s^* \delta_{\xi,j}, \mu_t(0,1)) \leq \lim_{s \to -\infty} d^*_L (P_s^* \delta_{\xi,j}, P_s^* \delta_{0,1}) + \lim_{s \to -\infty} d^*_L (P_s^* \delta_{0,1}, \mu_t(0,1)) = 0,
\]
which implies that \( \mu_t \) is independent of \((\xi, j)\). This completes the proof of the lemma. \[\square\]

**Theorem 2.7.** Assume that the processes \( u(t,s,\xi,j) \) have properties \((A_1)\) and \((A_2)\). Then, the family measures \( \{\mu_t\}_{t \in \mathbb{R}} \) obtained above is an evolution system of measures on \( H \times S \), i.e.,
\[
\sum_{j \in S} \int_H P_{s,t} \varphi (\xi,j) \mu_s (d\xi,j) = \sum_{j \in S} \int_H \varphi (x,j) \mu_t (d\xi,j), \quad \forall \varphi \in C_b(H \times S), \quad s \leq t.
\]

**Proof.** Let \( s < \tau < t \) and \((\xi, j) \in H \times S\). We have from Lemma \[2.6\] for any \( \varphi \in C_b(H \times S) \)
\[
\lim_{s \to -\infty} P_{s,\tau} \varphi (\xi,j) = (\varphi, \mu_\tau). \tag{2.9}
\]
Letting \( s \to -\infty \) in the identity
\[
P_{s,\tau} P_{\tau,t} \varphi (\xi,j) = P_{s,t} \varphi (\xi,j),
\]
recalling Feller property and taking account \[2.9\] yields
\[
(P_{\tau,t} \varphi, \mu_t) = (\varphi, \mu_t).
\]
This completes the proof. \[\square\]
Remark 2.8. Lemma 2.6 and 2.7 means that the evolution system of measures \( \{\mu_t\}_{t \in \mathbb{R}} \) obtained above is pullback asymptotic stability in distribution.

The following result gives information on the asymptotic behaviour of \( P^*_{s,t} \delta_{\xi,j} \) when \( t \to +\infty \).

**Theorem 2.9.** Assume that the processes \( u(t,s,\xi,j) \) have properties \((A_1)\) and \((A_2)\) and the evolution system of measures \( \{\mu_t\}_{t \in \mathbb{R}} \) is obtained above. For any \( s \in \mathbb{R} \) and \((\xi,j) \in H \times S\), we have

\[
\lim_{t \to +\infty} d^*_L \left( P^*_{s,t} \delta_{\xi,j}, \mu_t \right) = 0.
\]

That is, the evolution system of measures \( \{\mu_t\}_{t \in \mathbb{R}} \) is forward asymptotic stability in distribution.

**Proof.** Fix any \( s \in \mathbb{R} \) and \((\xi,j) \in H \times S\). To end the proof, we need to show that for any \( \eta > 0 \), there is a \( T = T(\eta) > 0 \) such that

\[
d^*_L \left( P^*_{s,t} \delta_{\xi,j}, \mu_t \right) < \eta, \quad \forall t > s + T.
\]

Notice that for \( s_1 < s < t \)

\[
d^*_L \left( P^*_{s,t} \delta_{\xi,j}, \mu_t \right) \leq d^*_L \left( P^*_{s,t} \delta_{\xi,j}, P^*_{s_1,t} \delta_{\xi,j} \right) + d^*_L \left( P^*_{s_1,t} \delta_{\xi,j}, \mu_t \right).
\]

It follows from Lemma 2.6 that there is a \( s^* = s^*(\eta) < s \) such that

\[
d^*_L \left( P^*_{s_1,t} \delta_{\xi,j}, \mu_t \right) < \frac{\eta}{2}, \quad \forall s_1 \leq s^*.
\]

It remains to show that there is a \( T = T(\eta) > 0 \) such that

\[
d^*_L \left( P^*_{s,t} \delta_{\xi,j}, P^*_{s^*,t} \delta_{\xi,j} \right) < \frac{\eta}{2}, \quad \forall t > s + T. \tag{2.10}
\]

This is equivalent to

\[
\left| \mathbb{E} \varphi \left( u(t, s^*, \xi, j), r_{\{s^*,j\}}(t) \right) - \mathbb{E} \varphi \left( u(t, s, \xi, j), r_{\{s,j\}}(t) \right) \right| < \frac{\eta}{2},
\]
for any $\varphi \in L_b(\mathcal{H} \times S)$ and $t > s + T$. Compute

$$
\left| \mathbb{E} \varphi (u(t, s^*, \xi, j), r_{\{s^*, j\}}(t)) - \mathbb{E} \varphi (u(t, s, \xi, j), r_{\{s, j\}}(t)) \right|
= \left| \mathbb{E} \left( \mathbb{E} \left( \varphi (u(t, s^*, \xi, j), r_{\{s^*, i\}}(t)) \mid \mathcal{F}_s \right) \right) - \mathbb{E} \varphi (u(t, s, \xi, j), r_{\{s, j\}}(t)) \right|
= \left| \sum_{j \in S} \int_{H} \mathbb{E} \varphi (u(t, s, z, k), r_{\{s, k\}}(t)) p(s, s^*, \xi, j, (dz, k)) - \mathbb{E} \varphi (u(t, s, \xi, j), r_{\{s, j\}}(t)) \right|$$

$$\leq \sum_{j \in S} \int_{H} \left| \mathbb{E} \varphi (u(t, s, z, k), r_{\{s, k\}}(t)) - \mathbb{E} \varphi (u(t, s, \xi, j), r_{\{s, j\}}(t)) \right| p(s, s^*, \xi, j, (dz, k))$$
$$\leq 2P \left( s^*, x, j, \mathcal{B}_R \times S \right)$$
$$\quad + \sum_{j \in S} \int_{\mathcal{B}_R} \left| \mathbb{E} \varphi (u(t, s, z, k), r_{\{s, k\}}(t)) - \mathbb{E} \varphi (u(t, s, \xi, j), r_{\{s, j\}}(t)) \right| p(s, s^*, \xi, j, (dz, k)).$$

By $(A_1)$, there is a positive number $R$ sufficiently large for

$$P \left( s, s^*, \xi, j, \mathcal{B}_R \times S \right) < \frac{\eta}{4}. \quad (2.12)$$

On the other hand, by Lemma 2.5 there is a $T = T(\eta) > 0$ such that

$$\sup_{\varphi \in L_b(\mathcal{H} \times S)} \left| \mathbb{E} \varphi (u(t, s, z, k), r_{\{s, k\}}(t)) - \mathbb{E} \varphi (u(t, s, \xi, j), r_{\{s, j\}}(t)) \right| < \frac{\eta}{4}, \quad t > s + T, \quad (2.13)$$

whenever $(z, l) \in \mathcal{B}_R \times S$. Substituting (2.13) and (2.12) into (2.11) yields

$$\left| \mathbb{E} \varphi (u(t, s^*, \xi, j), r_{\{s^*, h, i\}}(t)) - \mathbb{E} \varphi (u(t, s, \xi, j), r_{\{s, j\}}(t)) \right| < \frac{\eta}{2}, \quad \forall t > s + T.$$

Since $\varphi$ is arbitrary, the desired inequality (2.10) must hold. The proof is completed. \hfill \Box

**Theorem 2.10.** Assume that the processes $u(t, s, \xi, j)$ have properties $(A_1)$ and $(A_2)$ and the $y(t, s, \xi, j)$ are $\varpi$-periodic Markov processes. Then for any $(\xi, j) \in \mathcal{H} \times S$, there exists a unique $\varpi$-periodic evolution system of measures $\{\mu_t\}_{t \in \mathbb{R}}$, independent of $(\xi, j)$, such that

$$\lim_{s \to -\infty} d^s_L \left( P_{s, t}^\varpi \delta_{\xi, j}, \mu_t \right) = 0, \quad \forall t \in \mathbb{R},$$

and

$$\lim_{t \to +\infty} d^s_L \left( P_{s, t}^\varpi \delta_{\xi, j}, \mu_t \right) = 0, \quad \forall s \in \mathbb{R}.$$

**Proof.** It follows from Lemma 2.7, Remark 2.8 and Lemma 2.9 that for any $(\xi, j) \in \mathcal{H} \times S$, there exists an evolution system of measures $\{\mu_t\}_{t \in \mathbb{R}} \subset \mathcal{P}(\mathcal{H} \times S)$, independent of $(\xi, j)$, such that

$$\lim_{s \to -\infty} d^s_L \left( P_{s, t}^\varpi \delta_{\xi, j}, \mu_t \right) = 0$$
and

\[ \lim_{t \to +\infty} d_L^t (P_s \delta_{\xi,j}, \mu_t) = 0. \]

In the following we shall show that \( \mu_t \) is periodic. Take subsequence \( \{P_{s_n,t} \delta_{\xi,j} : s_n = t - n \omega, \ n \in \mathbb{N}\} \) of \( \{P_{s,t} \delta_{\xi,j} : s \leq t\} \) and subsequence \( \{P_{s_n,t+\omega} \delta_{\xi,j} : s_n = t - (n-1) \omega, \ n \in \mathbb{N}\} \) of \( \{P_{s,t+\omega} \delta_{\xi,j} : s \leq t\} \), respectively. Since the processes \( y(t, s, \xi, j) \) are \( \omega \)-periodic, \( P_{s_n,t+\omega} \delta_{\xi,j} = P_{s_n-\omega,t} \delta_{\xi,j} \). This means the sequences \( \{P_{s_n,t} \delta_{\xi,j} : s_n = t - n \omega, \ n \in \mathbb{N}\} \) and \( \{P_{s_n,t+\omega} \delta_{\xi,j} : s_n = t - (n-1) \omega, \ n \in \mathbb{N}\} \) are same. Consequently, for any \( t \in \mathbb{R}, \mu_t = \lim_{s \to -\infty} P_{s,t} \delta_{\xi,j} = \lim_{s \to -\infty} P_{s,t+\omega} \delta_{\xi,j} = \mu_{t+\omega} \). It remains to prove uniqueness. Assume \( \{\mu_t\}_{t \in \mathbb{R}} \) and \( \{\nu_t\}_{t \in \mathbb{R}} \) are the evolution systems of measures of \( y(t, s, \xi, j) \).

By Fubini’s theorem, we have for any \( \varphi \in C_b(H \times S) \) and \( s < t \)

\[
\begin{align*}
|\varphi, \mu_t - (\varphi, \nu_t)| &= |(\varphi, P_{s,t} \mu_s) - (\varphi, P_{s,t} \nu_s)| \quad = |(P_{s,t} \varphi, \mu_s) - (P_{s,t} \varphi, \nu_s)| \\
&= \sum_{j_1 \in S} \int_H E(\varphi(t, s, \xi_1, j_1)) \mu_s(d\xi_{1,j_1}) - \sum_{j_2 \in S} \int_H E(\varphi(t, s, \xi_2, j_2)) \nu_s(d\xi_{2,j_2}) \\
&\leq \sum_{j_1,j_2 \in S} \int_H \int_H E(\varphi(t, s, \xi_1, j_1) - \varphi(t, s, \xi_2, j_2)) \mu_s(d\xi_{1,j_1}) \nu_s(d\xi_{2,j_2}),
\end{align*}
\]

which together with Lemma 2.14 and the Lebesgue dominated convergence theorem implies that

\[
|\varphi, \mu_t - (\varphi, \nu_t)| \leq \lim_{n \to \infty} \sum_{j_1, j_2 \in S} \int_H \int_H E(\varphi(t, -n \omega, \xi_1, j_1) - \varphi(t, -n \omega, \xi_2, j_2)) \mu_{-n \omega}(d\xi_{1,j_1}) \nu_{-n \omega}(d\xi_{2,j_2}) = 0.
\]

Then we have for any \( \varphi \in C_b(H \times S) \)

\[ (\varphi, \mu_t) = (\varphi, \nu_t). \]

The proof is completed. \( \Box \)

**Remark 2.11.** When we replace \((A_1)-(A_2)\) with Condition \((A_1^*)-(A_2^*)\), by minor modifying to the process of proof, the all results above are correct.

### 3  Limits of evolution system of measures

Suppose for every \( \varepsilon \in [0, 1], \ s, t \in \mathbb{R} \) and \( s \leq t, \ u^\varepsilon(t, s, \xi, j) \) be a stochastic process with initial conditions \( u^\varepsilon(s, s, \xi, j) = \xi \in H \) and \( r(s) = j \in S \) at initial time \( t = s \). Let \( y^\varepsilon(t, s, \xi, j) \) denote the \((H \times S)\)-valued process \((u^\varepsilon(t, s, \xi, j), r_s(j))\). \( y^\varepsilon(t, s, \xi, j) \) are time nonhomogeneous Markov process and its probability transition operators are Feller.
We assume that

\((A_3)\) For every compact set \(K \subset H\), \(\varepsilon_0 \in [0,1]\) and \(\eta > 0\),

\[
\lim_{\varepsilon \to \varepsilon_0} \sup_{(x,j) \in K \times S} P \left( \|y^e(t, s, \xi, j) - y^{\varepsilon_0}(t, s, \xi, j)\|_H \geq \eta \right) = 0. \tag{3.1}
\]

**Theorem 3.1.** Assume \((A_3)\) holds and \(\varepsilon_n \to \varepsilon_0 \in [0,1]\). Let \(\{\mu_t\}_{t \in \mathbb{R}}\) be a family of probability measures on \(H \times S\) and let \(\{\mu_t^{\varepsilon_n}\}_{t \in \mathbb{R}}\) be an evolution system of measures of \(y^{\varepsilon_n}(t, s, \xi, j)\). If for any \(t \in \mathbb{R}\) \(\mu_t^{\varepsilon_n} \to \mu_t\) weakly, as \(n \to \infty\), then \(\{\mu_t\}_{t \in \mathbb{R}}\) must be an evolution system of measures of \(y^{\varepsilon_0}(t, s, \xi, j)\).

**Proof.** We only need to verify that for every \(\varphi \in L_b(H \times S)\) and \(s < t\),

\[
\sum_{j \in S} \int_H \mathbb{E}\varphi(u^{\varepsilon_0}(t, s, \xi, j), r_{s,j}(t)) \mu_s(d\xi, j) = \sum_{j \in S} \int_H \varphi(\xi, j) \mu_t(d\xi, j). \tag{3.2}
\]

Since for \(t \in \mathbb{R}\), \(\{\mu_t^{\varepsilon_n}\}\) is tight, we see that for every \(\epsilon > 0\), there exists a compact set \(K = K(\epsilon, t) \subset H\) such that

\[
\mu_t^{\varepsilon_n}(K \times S) \geq 1 - \epsilon \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.3}
\]

By (3.3) we obtain

\[
\left| \sum_{j \in S} \int_H \mathbb{E}\varphi(u^{\varepsilon_0}(t, s, \xi, j), r_{s,j}(t)) \mu_s^{\varepsilon_n}(dx, i) - \sum_{j \in S} \int_H \varphi(\xi, j) \mu_t^{\varepsilon_n}(d\xi, j) \right|
\]

\[
= \left| \sum_{j \in S} \int_H \mathbb{E}\varphi(u^{\varepsilon_0}(t, s, \xi, j), r_{s,j}(t)) \mu_s^{\varepsilon_n}(d\xi, j) - \sum_{j \in S} \int_H \mathbb{E}\varphi(u^{\varepsilon_n}(t, s, \xi, j), r_{s,j}(t)) \mu_s^{\varepsilon_n}(d\xi, j) \right|
\]

\[
\leq \sum_{j \in S} \int_H \mathbb{E}|\varphi(u^{\varepsilon_0}(t, s, \xi, j), r_{s,j}(t)) - \varphi(u^{\varepsilon_n}(t, s, \xi, j), r_{s,j}(t))| \mu_s^{\varepsilon_n}(d\xi, j)
\]

\[
\leq \sum_{j \in S} \int_K \mathbb{E}|\varphi(u^{\varepsilon_0}(t, s, \xi, j), r_{s,j}(t)) - \varphi(u^{\varepsilon_n}(t, s, \xi, j), r_{s,j}(t))| \mu_s^{\varepsilon_n}(d\xi, j)
\]

\[
+ 2\epsilon \sup_{(\xi,j) \in H \times S} |\varphi(\xi, j)|. \tag{3.4}
\]

Since \(\varphi \in L_b(H \times S)\), for every \(\epsilon > 0\), there exists \(\eta > 0\) such that \(|\varphi(x, j) - \varphi(z, j)| < \epsilon\) for all
\( x, z \in H \) with \( \|x - z\|_H < \eta \) and \( j \in S \). Thus we get

\[
\sum_{j \in S} \int_{K} \mathbb{E}[ \varphi (u^{x_0}(t, s, \xi, j), r_{s,j}(t)) - \varphi (u^{x_0}(t, s, \xi, j), r_{s,j}(t)) | \mu^n_s (d\xi, j)] = \sum_{j \in S} \int_{K} \left( \int_Y |\varphi (u^{x_0}(t, s, \xi, j), r_{s,j}(t)) - \varphi (u^{x_0}(t, s, \xi, j), r_{s,j}(t)) | P(d\omega) \right) \mu^n_s (d\xi, j)
\]

\[
+ \sum_{j \in S} \int_{K} \left( \int_{Y^c} |\varphi (u^{x_0}(t, s, \xi, j), r_{s,j}(t)) - \varphi (u^{x_0}(t, s, \xi, j), r_{s,j}(t)) | P(d\omega) \right) \mu^n_s (d\xi, j)
\]

\[
\leq 2 \sup_{(\xi,j) \in H \times S} |\varphi(\xi, j)| \sup_{(\xi,j) \in K \times S} \left( \|u^{x_0}(t, s, \xi, j) - u^{x_0}(t, s, \xi, j)\|_H \geq \eta \right) + \epsilon,
\]

where \( Y = \{ \omega \in \Omega |\|u^{x_0}(t, s, \xi, j) - u^{x_0}(t, s, \xi, j)\|_H \geq \eta \} \).

It follows from \((A_3)\) and \((3.4)-(3.5)\) that

\[
\lim_{n \to \infty} \left| \sum_{j \in S} \int_{H} \mathbb{E}[\varphi (u^{x_0}(t, s, \xi, j), r_{s,j}(t)) | \mu^n_s (d\xi, j)] - \sum_{j \in S} \int_{H} \varphi (\xi, j) \mu^n_s (d\xi, j) \right| \leq \epsilon + 2\epsilon \sup_{(\xi,j) \in H \times S} |\varphi(\xi, j)|.
\]

Since \( \epsilon > 0 \) is arbitrary and \( \mu^n_s \to \mu_t \) weakly, by \((3.6)\) we obtain \((3.2)\) immediately, which shows that \( \{\mu\}_{t \in \mathbb{R}} \) is an evolution system of measures of the process \( y^{x_0}(t, s, \xi, i) \).

If for every \( \varepsilon \in [0, 1] \), \( u^\varepsilon(t, s, \xi, j) \) have property \((A_2)\), we say \( u^\varepsilon(t, s, \xi, j) \) have property \((A_2)\). Moreover, we also assume

\((A_4)\) For any \( \varepsilon \in [0, 1], s \in \mathbb{R}, \xi \in H \) and \( \eta > 0 \), there exists a compact \( K = (\eta, \xi) \subset H \), independent of \( \varepsilon \) and \( s \), such that for any \( j \in S \),

\[
P \{ u^\varepsilon(t, s, \xi, j) \in K, \ t > s \} < 1 - \eta.
\]

**Remark 3.2.** \((A_4)\) is stronger than \((A_1)\).

Given \( \varepsilon \in [0, 1], \) for \( A \subset B(H) \) and \( B \subset S \), let \( P^\varepsilon(t, s, \xi, j, A \times B) \) denote the probability of event \( \{ y^\varepsilon(t, s, \xi, j) \in A \times B \} \) given initial condition \( y^\varepsilon(s, s, \xi, j) = (\xi, j) \) at time \( t = s \). Denote by \( (\mu^\varepsilon_t)_{t \in \mathbb{R}} \) the evolution system of measures of \( y^\varepsilon(t, s, \xi, j) \) obtained in the section above. For each \( \varepsilon \in [0, 1], \) the definitions of operators \( P^\varepsilon_{s,t} \) and \( P^\varepsilon_{s,t}^* \) with respect to \( y^\varepsilon(t, s, \xi, j) \) are the same as that of \( P_{s,t} \) and \( P_{s,t}^* \) in Section 2.

**Theorem 3.3.** Suppose \((A_2)-(A_4)\) hold. Then:
constant \( L \) where \( \| \cdot \| \) Lipschitz in \( s \)

\[
\text{If Markov processes } \{ y_{\varepsilon} \} \in J,
\text{Note that in the present case, by Theorem 2.10, for every }\]

Proof. (i). By \((A_4)\) and the relationship \( P_{\varepsilon}^{\xi,j}(\Gamma \times S) = P^{\varepsilon}(t, s, \xi, j, (\Gamma \times S)) \), for any \( \Gamma \in B(H) \),

it is easy to verify that the set \( \bigcup_{\varepsilon \in [0,1]} \mu_{\varepsilon}^i \) is tight.

(ii). By (i) we know that \( \{ \mu_{\varepsilon}^i \} \), \( t \in R \), is tight, and hence there exists a subsequence \( \varepsilon_{n_k} \) and a probability measure \( \mu_i^\ast \) such that \( \mu_{\varepsilon_{n_k}} \to \mu_i^\ast \) weakly. It follows from Theorem 3.1 and \((A_3)\) that \( \{ \mu_{\varepsilon_{n_k}}^i \}_{i \in R} \) is a evolution system of measures of \( \tilde{y}^{\varepsilon_0}(t, s, \xi, j) \). This completes the proof. \( \square \)

As an immediate consequence of Theorem 3.3 we have the following convergence result.

**Theorem 3.4.** Suppose \((A_2)-(A_4)\) hold. Let \( \varepsilon_n, \varepsilon_0 \in [0,1] \) for all \( n \in N \) such that \( \varepsilon_n \to \varepsilon_0 \). If \( \{ \mu_{\varepsilon_n}^i \}_{i \in R} \) and \( \{ \mu_i^{\varepsilon_0} \}_{i \in R} \) are the unique \( \varpi \)-periodic evolution systems of measures of \( \varpi \)-periodic Markov processes \( y_{\varepsilon_n}(t, s, \xi, j) \) and \( y^{\varepsilon_0}(t, s, \xi, j) \), respectively, then for each \( t \in R, \mu_{\varepsilon_n}^i \to \mu^{\varepsilon_0} \) weakly.

Proof. Note that in the present case, by Theorem 2.10 for every \( \varepsilon \in [0,1] \), \( y^\varepsilon(t, s, \xi, j) \) has a unique \( \varpi \)-periodic evolution system of measures, which along with Theorem 3.3 implies the desired result. \( \square \)

4 Well-Posedness of stochastic lattice systems

In this section, we prove the existence and uniqueness of solutions to system (1.1)-(1.2). We first discuss the assumptions on the nonlinear drift and diffusion terms in (1.1).

Throughout this paper, we assume the sequences \( g(j) = (g_i(j))_{i \in Z} \) and \( h(j) = (h_{i,k}(j))_{i \in Z, k \in N} \), \( j \in S \), belong to \( l^2 \):

\[
\| g(j) \|^2 = \sum_{i \in Z} |g_i(j)|^2 < \infty \quad \text{and} \quad \| h(j) \|^2 = \sum_{i \in Z} \sum_{k \in N} |h_{i,k}(j)|^2 < \infty.
\]  

(4.1)

where \( \| \cdot \| \) is the norm of \( l^2 \). The inner product of \( l^2 \) will be denoted by \( (\cdot, \cdot) \) throughout this paper.

Assume that for \( j \in S \) \( f_i : R \times j \times R \to R \), \( f_i = f_i(\cdot, j, \cdot) \), is continuous in \( R \times R \) and globally Lipschitz in \( s \in R \) uniformly with respect to \( i \in Z, t \in R \) and \( j \in S \); more precisely, there exists a constant \( L_f > 0 \) such that for all \( t, s_1, s_2 \in R \) and \( i \in Z \) and \( j \in S \),

\[
|f_i(t, j, s_1) - f_i(t, j, s_2)| \leq L_f |s_1 - s_2|.
\]

(4.2)
Moreover, $f_i(t, j, s)$ grows linearly in $s \in \mathbb{R}$: for each $i \in \mathbb{Z}$, $t \in \mathbb{R}$ and $j \in S$, there exists $\alpha_i > 0$ such that

$$|f_i(t, j, s)| \leq \alpha_i + \beta_0|s|, \quad \forall \ t, s \in \mathbb{R} \quad \text{and} \quad i \in \mathbb{Z}, \quad (4.3)$$

where $(\alpha_i)_{i \in \mathbb{Z}} \in l^2$ and $\beta_0 : \mathbb{R} \to \mathbb{R}$ is a positive constant.

For the diffusion terms in $(1.1)$, we assume for $j \in S$, $\sigma_{i,k} : \mathbb{R} \times j \times \mathbb{R} \to \mathbb{R}$, $\sigma_{i,k} = \sigma_{i,k}(\cdot, j, \cdot)$, is continuous in $\mathbb{R} \times \mathbb{R}$ and globally Lipschitz in $s \in \mathbb{R}$ uniformly with respect to $i \in \mathbb{Z}$, $t \in \mathbb{R}$ and $j \in S$; more precisely, for every $k \in \mathbb{N}$, there exists a constant $L_k > 0$ such that for all $t, s_1, s_2 \in \mathbb{R}$, $j \in S$ and $i \in \mathbb{Z}$

$$|\sigma_{i,k}(t, j, s_1) - \sigma_{i,k}(t, j, s_2)| \leq L_k|s_1 - s_2|, \quad (4.4)$$

where $(L_k)_{k \in \mathbb{N}} \in l^2$. In addition, we assume $\sigma_{i,k}(t, j, s)$ grows linearly in $s \in \mathbb{R}$; that is, for each $t \in \mathbb{R}$, $j \in S$, $i \in \mathbb{Z}$ and $k \in \mathbb{N}$, there exists $\delta_{i,k} > 0$ and $\beta_k > 0$ such that

$$|\sigma_{i,k}(t, s, s^*)| \leq \delta_{i,k} + \beta_k |s|, \quad \forall s \in \mathbb{R}, \quad (4.5)$$

where $(\delta_{i,k})_{i \in \mathbb{Z}, k \in \mathbb{N}} \in l^2$ and $(\beta_k(\cdot))_{k \in \mathbb{N}} \in l^2$ is a positive continuous function.

When we will investigate the periodic evolution system of measures of system $(1.1)$-$(1.2)$, we assume that

$(P)$ All given time-dependent functions are $\varpi$-periodic in $t \in \mathbb{R}$ for some $\varpi > 0$; that is, for all $t \in \mathbb{R}$, $i \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$f_i(t + \varpi, \cdot, \cdot) = f_i(t, \cdot, \cdot), \quad \sigma_{i,k}(t + \varpi, \cdot, \cdot) = \sigma_{i,k}(t, \cdot, \cdot).$$

The following notation will be used throughout the paper:

$$\alpha = (\alpha_i)_{i \in \mathbb{Z}}, \quad L = (L_k)_{k \in \mathbb{N}}, \quad \beta = (\beta_k)_{k \in \mathbb{N}}, \quad \delta = (\delta_{i,k})_{i \in \mathbb{Z}, k \in \mathbb{N}},$$

$$\|\alpha\|^2 = \sum_{i \in \mathbb{Z}} |\alpha_i|^2, \quad \|L\|^2 = \sum_{k \in \mathbb{N}} |L_k|^2, \quad \|\beta\|^2 = \sum_{k \in \mathbb{N}} |\beta_k|^2, \quad \|\delta\|^2 = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{N}} |\delta_{i,k}|^2.$$ And,

$$\lambda = \min_{j \in S} \lambda(j), \quad \|g\| = \max_{j \in S} \|g(j)\|, \quad \|h\| = \max_{j \in S} \|h(j)\|.$$}

In addition, for $u = (u_i)_{i \in \mathbb{Z}} \in l^2$, we write $f(t, j, u) = (f_i(t, j, u_i))_{i \in \mathbb{Z}}$ and $\sigma_k(t, j, u) = (\sigma_{i,k}(t, j, u_i))_{i \in \mathbb{Z}}$.

It follows from $(4.2)$-$(4.3)$ that for all $t \in \mathbb{R}$, $j \in S$ and $u_1, u_2 \in l^2$,

$$\|f(t, j, u_1) - f(t, j, u_2)\|^2 \leq L_j^2 \|u_1 - u_2\|^2, \quad (4.6)$$
\[ \| f(t, j, u_1) \|_2 \leq 2 \| \alpha \|^2 + 2 \beta_0^2 \| u_1 \|^2. \] (4.7)

Similarly, by (4.4)-(4.5), we have for all \( t \in \mathbb{R}, j \in S \) and \( u_1, u_2 \in l^2 \),
\[ \sum_{k \in \mathbb{N}} \| \sigma_k(t, j, u_1) - \sigma_k(t, j, u_2) \|^2 \leq \| L \|^2 \| u_1 - u_2 \|^2 \] (4.8)
and
\[ \sum_{k \in \mathbb{N}} \| \sigma_k(t, u_1, v_1) \|^2 \leq 2 \| \delta \|^2 + 2 \| \beta \|^2 \| u_1 \|^2. \] (4.9)

For simplicity, define linear operators \( A, B : l^2 \rightarrow l^2 \) by
\[ (Au)_i = -u_{i-1} + 2u_i - u_{i+1}, \quad (Bu)_i = u_{i+1} - u_i, \quad i \in \mathbb{Z}, \quad u = (u_i)_{i \in \mathbb{Z}} \in l^2. \]
Then, system (1.1)-(1.2) can be put into the following form in \( l^2 \) for \( s \in \mathbb{R} \):
\[
\begin{align*}
&du(t) + \nu Au(t)dt + \lambda(r(t))u(t)dt = (f(t, r(t), u(t)) + g(r(t)))dt \\
&+ \varepsilon \sum_{k=1}^{\infty} (h_k(r(t)) + \sigma_k(t, r(t), u(t)))dW_k(t), \quad t > s,
\end{align*}
\] (4.10)
with initial condition
\[ u(s) = \xi. \] (4.11)

Similar to [14] for stochastic equations with Markovian switching in \( \mathbb{R}^n \), under conditions (4.1)-(4.5), we can show that for any \( \xi \in L^2(\Omega, l^2) \), system (4.10)-(4.11) has a unique solution, which is written as \( u(t) \). To highlight the initial values, we let \( r_{s,j}(t) \) be the Markov chain starting from state \( i \in S \) at \( t = s \) and denote by \( u(t, s, \xi, j) \) the solution of Eq. (4.10)-(4.11) with initial conditions \( u(s, s, \xi, j) = \xi \in L^2(\Omega, l^2) \) and \( r(s) = j \). Moreover, for any bounded subset \( B \) of \( l^2 \),
\[ \sup_{(\xi, j) \in B \times S} \mathbb{E} \left[ \sup_{s \leq \tau \leq t} \| u(\tau, s, \xi, j) \|^2 \right] < \infty \quad \forall t \geq s, \]
which together with Chebyshev’s inequality implies that
\[ (A_0) \text{ For any } s \in \mathbb{R}, \ T > 0, \text{ bounded set } B \subset H \text{ and } \eta > 0, \text{ there exists a constant } R = R(\eta, B, T) > 0, \text{ independent of } s, \text{ such that for any } \xi \in B \text{ and } j \in S, \]
\[ P \{ \| u(t, s, \xi, j) \| \geq R, \ t \in [s, s + T] \} < \eta. \]

In the sequence, we will follow the definition in Section 2 and 3. Sometimes, we need to replace \( H \) with \( l^2 \).
5 Uniform estimates

In this section, we derive uniform estimates of the solution of problem (4.10)-(4.11) which are necessary for establishing the existence and stability of evolution system of measures. We assume that

\[ \lambda > 1 + \beta_0^2 + 2 \| \beta \|^2 \] (5.1)

and

\[ \| L \|^2 + 4 \lambda^{-1} L_f^2 < \frac{7}{4} \lambda. \] (5.2)

We first discuss uniform estimates of solutions of problem (4.10)-(4.11).

**Lemma 5.1.** Suppose (4.1)-(4.5) and (5.1) hold. Then for any \( s \in \mathbb{R}, \xi \in l^2 \) and \( \eta > 0 \), there exists a bounded subset \( B = (\eta, \xi) \) of \( l^2 \), independent of \( s \), such that for any \( 0 < \varepsilon \leq 1, \xi \in l^2, j \in S \) and \( t > s \),

\[ P \{ u(t,s,\xi,j) \in B \} > 1 - \eta. \]

**Proof.** By (4.10) and Ito’s formula, we get for \( t > s \)

\[
\begin{align*}
\mathbb{E} \left( \| u(t) \|^2 \right) &+ 2 \nu \int_s^t \mathbb{E}(\| Bu(\tau) \|^2) d\tau + 2 \int_s^t \lambda(r(\tau)) \mathbb{E}(\| u(\tau) \|^2) d\tau \\
&= \mathbb{E} \left( \| u(s) \|^2 \right) + 2 \int_s^t \mathbb{E} (u(\tau), f(\tau, r(\tau), u(\tau))) d\tau \\
&\quad + 2 \int_s^t \mathbb{E} (u(\tau), g(r(\tau))) ds + \varepsilon^2 \sum_{k=1}^{\infty} \int_s^t \mathbb{E} \left( \| h_k(r(\tau)) + \sigma_k(\tau, r(\tau), u(\tau)) \|^2 \right) d\tau.
\end{align*}
\] (5.3)

By (4.7) and (4.9) we get for \( t \geq 0 \)

\[
\begin{align*}
\mathbb{E} \left( \| u(t) \|^2 \right) &\leq \mathbb{E} \left( \| u(s) \|^2 \right) - \varpi_1 \int_s^t E \left( \| u(\tau) \|^2 \right) d\tau + \varpi_2 (t - s),
\end{align*}
\] (5.4)

where

\[ \varpi_1 = 2 \lambda - 2 - 2 \beta_0^2 - 4 \| \beta \|^2 \] (5.5)

and

\[ \varpi_2 = 2 \left( \| \alpha \|^2 + \| g \|^2 + \| h \|^2 + 2 \| \delta \|^2 \right). \]

Then, we get for \( t \geq 0 \),

\[
\begin{align*}
\mathbb{E} \left( \| u(t) \|^2 \right) &\leq \mathbb{E} \left( \| \xi \|^2 \right) e^{-\varpi_1 (t-s)} + \frac{\varpi_2}{\varpi_1},
\end{align*}
\] (5.6)

which together with Chebyshev’s inequality completes the proof. \( \square \)
In the following, we prove that any two solutions of (4.10)-(4.11) converge to each other.

**Lemma 5.2.** Suppose (4.1)-(4.5) and (5.2) hold. Then for any \( s \in \mathbb{R} \), \( \eta > 0 \) and bounded subset \( B \) of \( H \), there exists a \( T = T(\eta, B) \), independent of \( s \), such that for \( (\xi_1, \xi_2, j) \in B \times B \times S \),

\[
P \{ \| u(t, s, \xi_1, j) - u(t, s, \xi_2, j) \| < \epsilon \} \geq 1 - \eta, \quad \forall t - s \geq T.
\]

**Proof.** For simplicity, we set \( u_1(t) = u(t, s, \xi_1, j) \) and \( u_2(t) = u(t, s, \xi_2, j) \) for \( t \geq s \). By (4.10) we get, for \( t \geq s \) and \( \triangle t > 0 \),

\[
E(\| u_1(t + \triangle t) - u_2(t + \triangle t) \|^2) - E(\| u_1(t) - u_2(t) \|^2) \\
\leq -2\lambda E(\int_t^{t+\triangle t} \| u_1(s) - u_2(s) \|^2 \, ds) \\
+ 2E(\int_t^{t+\triangle t} \| u_1(s) - u_2(s) \| \| f(s, r(s), u_1(s)) - f(s, r(s), u_2(s)) \| \, ds) \\
+ \epsilon^2 E\left( \sum_{k=1}^\infty \int_t^{t+\triangle t} \| \sigma_k(s, r(s), u_1(s)) - \sigma_k(s, r(s), u_2(s)) \|^2 \, ds \right). \tag{5.7}
\]

It follows from (4.6) and (4.8) that for all \( t \geq 0 \),

\[
D^+ E(\| u_1(t) - u_2(t) \|^2) \leq -\left( \frac{7}{4}\lambda - \| L \|^2 - \frac{4}{\lambda} L_j^2 \right) E(\| u_1(t) - u_2(t) \|^2).
\]

which implies that

\[
E(\| u(t, s, \xi_1, j) - u(t, s, \xi_2, j) \|^2) \leq E(\| \xi_1 - \xi_2 \|^2) e^{-\gamma(t-s)}, \tag{5.8}
\]

where \( \gamma = \frac{7}{4}\lambda - \| L \|^2 - \frac{4}{\lambda} L_j^2 \). Then the desired inequality follows from (5.8) and Chebyshev’s inequality immediately. \( \square \)

Next, we derive uniform estimates on the tails of the solutions of (4.10)-(4.11) which are crucial for establishing the tightness of the family of probability distributions of the solutions.

**Lemma 5.3.** Suppose (4.1)-(4.5) and (5.1) hold. Then for every \( s \in \mathbb{R} \), \( \xi \in l^2 \) and \( \eta > 0 \), there exists a positive integer \( N = N(\xi, \epsilon) \), independent of \( s \), such that for all \( 0 < \epsilon \leq 1 \), \( n \geq N \) and \( t \geq s \), the solution \( u \) of (4.10)-(4.11) satisfies,

\[
\sum_{|i| \geq n} E(\| u_i(t, s, \xi, j) \|^2) \leq \eta.
\]

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Proof. Let \( \theta : \mathbb{R} \to \mathbb{R} \) be a smooth function such that \( 0 \leq \theta(s) \leq 1 \) for all \( s \in \mathbb{R} \) and

\[
\theta(s) = 0, \quad \text{for } |s| \leq 1, \quad \text{and} \quad \theta(s) = 1, \quad \text{for } |s| \geq 2. \tag{5.9}
\]

Given \( n \in \mathbb{N} \), denote by \( \theta_n = (\theta(\frac{i}{n}))(i \in \mathbb{Z}) \) and \( \theta_n u = (\theta(\frac{i}{n})u_i)(i \in \mathbb{Z}) \) for \( u = (u_i)(i \in \mathbb{Z}) \). By (4.10) we get

\[
d(\theta_n u(t)) = -\theta_n \nu Au(t) dt - \theta_n \lambda(r(t))u(t) dt + \theta_n f(t, u(t)) dt
\]

\[
+ \theta_n g(t) dt + \sum_{k=1}^{\infty} (\theta_n h_k(t) + \theta_n \sigma_k(t, u(t))) dW_k(t), \quad t > s. \tag{5.10}
\]

By (5.10), Ito’s formula and taking the expectation we obtain for all \( t > s \),

\[
\mathbb{E} \left( \| \theta_n u(t) \|^2 \right) = \mathbb{E} \left( \| \theta_n u(s) \|^2 \right) - 2 \int_s^t \mathbb{E} \left( \theta_n Au(\tau), \theta_n u(\tau) \right) d\tau
\]

\[
- 2 \int_s^t \mathbb{E} \left( \theta_n \lambda(r(\tau))u(\tau), \theta_n u(\tau) \right) d\tau
\]

\[
+ 2 \int_s^t \mathbb{E} \left( \theta_n u(\tau), \theta_n f(\tau, r(\tau), u(\tau)) + \theta_n g(\tau(\tau)) \right) d\tau
\]

\[
+ \epsilon^2 \sum_{k=1}^{\infty} \int_s^t \mathbb{E} \left( \| \theta_n h_k(\tau) + \theta_n \sigma_k(\tau, r(\tau), u(\tau)) \|^2 \right) d\tau. \tag{5.11}
\]

By the similar argument of the proof of Lemma 4.2 in [16], we have from (4.7) and (4.9) there exists \( N = N(\xi, \eta) \), independent of \( s \), such that for \( 0 < \epsilon \leq 1, \quad t > s \) and \( n \geq N \),

\[
\mathbb{E} \left( \| \theta_n u(t) \|^2 \right) \leq c_1 \epsilon - \omega_1 \int_s^t \mathbb{E} \left( \| \theta_n u(s) \|^2 \right) ds + c_2 \epsilon, \tag{5.12}
\]

where \( \omega_1 \) is defined as (5.5) and \( c_1, c_2 > 0 \). Then we get for \( t \geq s \) and \( n \geq N \),

\[
\mathbb{E} \left( \| \theta_n u(t) \|^2 \right) \leq c_1 \epsilon e^{-\omega_1(t-s)} + c_2 \frac{\epsilon}{\omega_1} \leq c \epsilon, \tag{5.13}
\]

where \( c \) is independent of \( t, s \) and \( \epsilon \). This completes the proof.

\[\square\]

Lemma 5.4. Suppose (4.11)–(4.15) hold. Then for every compact set \( K \subseteq l^2, \quad t \geq s, \quad \eta > 0 \) and \( \epsilon_1, \epsilon_2 \geq 0 \),

\[
\lim_{\epsilon_2 \to \epsilon_1} \sup_{(t,s,j) \in K \times S} P \left( \| u^{\epsilon_2}(t,s,j) - u^{\epsilon_1}(t,s,j) \| \geq \eta \right) = 0. \tag{5.14}
\]

Proof. The proof is similar as that of Lemma 6.2 in [11], so we omit it here.

\[\square\]

The transition operators \( \{ P_{\tau,t} \}_{s \leq \tau \leq t} \) have the following properties.
Lemma 5.5. Assume that \((4.1) - (4.5)\) hold. Then:

(i) \(\{P_{\tau,t}\}_{s \leq \tau \leq t}\) is Feller;

(ii) Under the additional condition \((P)\), the family \(\{P_{\tau,t}\}_{s \leq \tau \leq t}\) is \(\varpi\)-periodic; that is, for all \(s \leq r \leq t\),

\[
P(r, t, \xi, j, (\cdot, \cdot)) = P(r + \varpi, t + \varpi, \xi, j, (\cdot, \cdot)), \quad \forall (\xi, j) \in l^2 \times S;
\]

(iii) \(y(t, s, \xi, j)\) is a \((l^2 \times S)\)-valued nonhomogeneous Markov process for every \(\xi \in l^2\) and \(j \in S\).

Proof. The properties (i) and (iii) are standard and the proof is omitted. We only prove (ii). By \((4.10)\) we have for \(t \geq \tau \geq s\) and \((\xi, j) \in l^2 \times S\),

\[
u \int_\tau^t Au(s, \tau, \xi, j)ds + \int_\tau^t \lambda(r_{\tau,j}(s))u(s, \tau, \xi, j)ds = \xi + \int_\tau^t (f(s, r_{\tau,j}(s), u(s, \tau, \xi, j)) + g(r_{\tau,j}(s)))ds + \varepsilon \sum_{k=1}^\infty \int_\tau^t (h_k(r_{\tau,j}(s)) + \sigma_k(s, r_{\tau,j}(s), u(s, \tau, \xi, j)))dW_k(s). \tag{5.15}
\]

We also have

\[
u \int_{\tau + \varpi}^{t + \varpi} Au(s, \tau + \varpi, \xi, j)ds + \int_{\tau + \varpi}^{t + \varpi} \lambda(r_{\tau + \varpi,j}(s))u(s, \tau + \varpi, \xi, j)ds = \xi + \int_{\tau + \varpi}^{t + \varpi} (f(s, r_{\tau + \varpi,j}(s), u(s, \tau + \varpi, \xi, j)) + g(r_{\tau + \varpi,j}(s)))ds + \varepsilon \sum_{k=1}^\infty \int_{\tau + \varpi}^{t + \varpi} (h_k(r_{\tau + \varpi,j}(s)) + \sigma_k(s, r_{\tau + \varpi,j}(s), u(s, \tau + \varpi, \xi, j)))dW_k(s),
\]
which implies from \((P)\) that
\[
\begin{align*}
    u(t + \omega, \tau + \omega, \xi, j) + \nu \int^t_r A u(s + \omega, \tau + \omega, \xi, j) & \, ds \\
    \quad = \xi - \int^t_\tau \lambda (r_{\tau+\omega,j}(s + \omega)) u(s + \omega, \tau + \omega, \xi, j) \, ds \\
    & \quad + \int^t_\tau (f(s + \omega, r_{\tau+\omega,j}(s + \omega), u(s + \omega, \tau + \omega, \xi, j)) + g(r_{\tau+\omega,j}(s + \omega))) \, ds \\
    & \quad + \epsilon \sum^{\infty}_{k=1} \int^t_\tau (h_k(r_{\tau+\omega,j}(s + \omega)) + \sigma_k(s + \omega, r_{\tau+\omega,j}(s + \omega), u(s + \omega, \tau + \omega, \xi, j))) \, d\tilde{W}_k(s) \\
    \quad = \xi - \int^t_\tau \lambda (r_{\tau,j}(s)) u(s + \omega, \tau + \omega, \xi, j) \, ds \\
    & \quad + \int^t_\tau (f(s, r_{\tau,j}(s), u(s + \omega, \tau + \omega, \xi, j)) + g(r_{\tau,j}(s))) \, ds \\
    & \quad + \epsilon \sum^{\infty}_{k=1} \int^t_\tau (h_k(r_{\tau,j}(s)) + \sigma_k(s, r_{\tau,j}(s), u(s + \omega, \tau + \omega, \xi, j))) \, d\tilde{W}_k(s),
\end{align*}
\]

(5.16)

where \(\tilde{W}_k(s) = W_k(s + \omega) - W_k(\omega), k \in \mathbb{N}\), are Brownian motions as well. Based on (5.15)-(5.16), one can show that \((u(t + \omega, \tau + \omega, \xi, j), r_{\tau+\omega,j}(t + \omega))\) and \((u(t, \tau, \xi, j), r_{\tau,j}(t))\) have the same distribution law. Consequently, for any \(A \in \mathcal{B}(l^2)\) and \(j^* \in S\)
\[
P \{ y(t + \omega, \tau + \omega, \xi, j) \in (A, j^*) \} = P \{ y(t, \tau, \xi, j) \in (A, j^*) \},
\]
that is
\[
P \{ t + \omega, \tau + \omega, \xi, j, (A, j^*) \} = P \{ t, \tau, \xi, j, (A, j^*) \},
\]
which completes the proof.

6 Main results

We are now in a position to prove the existence and stability of evolution system of measures of (4.10)-(4.11).

**Theorem 6.1.** Suppose (4.1)-(4.5) and (5.1)-(5.2) hold. Then (4.10)-(4.11) has an evolution system of measures \(\{\mu_t\}_{t \in \mathbb{R}}\), which is pullback and forward asymptotic stability in distribution.

**Proof.** The results of Lemma 5.1 and Lemma 5.2 collide with the conditions \((A_1)\) and \((A_2)\) with \(H\) replaced by \(l^2\), respectively. It follows from Lemma 5.5 that \(y^\tau(t, s, \xi, j)\) are Markov processes and
the transition operator of $y^\varepsilon(t,s,\xi,j)$ is Feller. By Theorem 2.7, Remark 2.8 and Theorem 2.9, we complete the proof immediately.

\[ \square \]

**Theorem 6.2.** Suppose $(P)$, (4.1)-(4.5) and (5.1)-(5.2) hold. Then (4.10)-(4.11) has a unique $\varpi$-periodic evolution system of measures $\{\mu_t\}_{t \in \mathbb{R}}$, which is pullback and forward asymptotic stability in distribution.

**Proof.** Under conditions $(P)$ and (4.1)-(4.5), it follows from Lemma 5.5 that $y^\varepsilon(t,s,\xi,j)$ are $\varpi$-periodic Markov processes. By the similar argument as that in Theorem 6.1 and Theorem 2.10, we complete the proof immediately.

\[ \square \]

Next, we discuss the limiting behavior of evolution system of measures of (4.10)-(4.11) as $\varepsilon \to 0$ by applying Theorem 3.3. Note that all results in the previous sections are valid for $\varepsilon = 0$ in which case the proof is actually simpler. For convenience, we now write the solution of (4.10)-(4.11) as $u^\varepsilon(t,s,\xi,j)$ with $\varepsilon \in [0,1]$, $\xi \in l^2$ and $j \in S$ and the evolution system of measures of (4.10)-(4.11) obtained in 6.1 as $\{\mu^\varepsilon_t\}_{t \in \mathbb{R}}$.

**Theorem 6.3.** Suppose (4.1)-(4.5) and (5.1)-(5.2) hold. Then:

(i) For every $t \in \mathbb{R}$, the union $\bigcup_{\varepsilon \in [0,1]} \mu^\varepsilon_t$ is tight.

(ii) If $\varepsilon_n \to \varepsilon_0 \in [0,1]$, then there exists a subsequence $\varepsilon_{n_k}$ and a evolution system of measures $\{\mu^\varepsilon_{n_k}t\}_{t \in \mathbb{R}}$ of $y^{\varepsilon_0}(t,s,\xi,j)$ such that $\mu^\varepsilon_{n_k}t \to \mu^\varepsilon_{t_0}$ weakly.

**Proof.** Since all uniform estimates given in Lemmas 5.1 and 5.3 are uniform with respect to $\varepsilon \in [0,1]$, by the similar proof that of Lemma 4.2 in [11], it is easy to verify that for any $s,t \in \mathbb{R}$, $\xi \in l^2$ and $\eta > 0$, there exists a compact $K = (\eta, \xi) \subset l^2$, independent of $s,t$ and $\varepsilon$, such that for any $j \in S$,

$$P\{u^\varepsilon(t,s,\xi,j) \in K, \ s < t\} > 1 - \eta.$$

This means that the condition $(A_4)$ with $H$ replaced by $l^2$ is satisfied. The results of Lemma 5.2 and Lemma 5.4 coincide with the conditions $(A_2)$ and $(A_3)$ with $H$ replaced by $l^2$, respectively. By Theorem 3.3 we complete the proof.

\[ \square \]

The following result about the limiting behavior of $\varpi$-periodic evolution system of measures of $y^\varepsilon(t,s,\xi,j)$ follows from Theorem 3.4 immediately.
Theorem 6.4. Suppose (P), (4.1)-(4.5) and (5.1)-(5.2) hold. Let $\varepsilon_n, \varepsilon_0 \in [0,1]$ for all $n \in \mathbb{N}$ such that $\varepsilon_n \to \varepsilon_0$. If $\{\mu_t^{\varepsilon_n}\}_{t \in \mathbb{R}}$ and $\{\mu_t^{\varepsilon_0}\}_{t \in \mathbb{R}}$ are the unique $\omega$-periodic evolution systems of measures of $\omega$-periodic of problem (4.10)-(4.11) with $\varepsilon$ replaced by $\varepsilon_n$ and $\varepsilon_0$, respectively, then for each $t \in \mathbb{R}$, $\mu_t^{\varepsilon_n} \to \mu_t^{\varepsilon_0}$ weakly.

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