The Relativistic Quantum Stationary Hamilton Jacobi Equation for Particle with Spin $1/2$.

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Abstract

For one dimensional motions, we derive from the Dirac Spinors Equation (DSE) the Quantum Stationary Hamilton-Jacobi Equation for particles with spin $1/2$. Then, we give its solution. We demonstrate that the $QSHJES_{1/2}$ have two explicit forms, which represent the two possible projection of the Spin 1/2.

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1- Introduction

During about three years, a new deterministic approach of quantum mechanics was arising from the works of Djama and Bouda [1, 2, 3, 4]. It consist on the introduction of a quantum Lagrangian with which and basing on the Quantum Stationary Hamilton-Jacobi Equation (QSHJE), they derived the Quantum Newton’s Law in one dimension [1]. They also plot the quantum trajectories for different potentials [2]. After, Djama has generalize this formalism to the relativistic cases [3, 4]. First, he took up the relativistic QSHJE (RQSHJE) already introduced by Faraggi and Matone [5].

\[
\frac{1}{2m_0} \left( \frac{\partial S_0}{\partial x} \right)^2 - \frac{\hbar^2}{4m_0} \left[ \frac{3}{2} \left( \frac{\partial S_0}{\partial x} \right)^2 - \left( \frac{\partial^2 S_0}{\partial x^2} \right)^2 - \left( \frac{\partial S_0}{\partial x} \right)^{-1} \left( \frac{\partial^3 S_0}{\partial x^3} \right) \right] + \frac{1}{2m_0 c^2} \left[ m_0^2 c^4 - (E - V)^2 \right] = 0 ,
\]

(1)

and introduced the relativistic quantum Lagrangian written as [5]

\[
L = -m_0 c^2 \sqrt{1 - f(x) \frac{\dot{x}^2}{c^2}} - V(x) .
\]

(2)

From this Lagrangian, and using the least action principle, he deduce the expression of the conjugate momentum in function of the particle’s velocity [5]

\[
\dot{x} \frac{\partial S_0}{\partial x} = E - V(x) - \frac{m_0^2 c^4}{E - V(x)} ,
\]

(3)

from which he derived the Relativistic Quantum Newton’s Law

\[
[(E - V)^2 - m_0^2 c^4]^2 + \frac{\dot{x}^2}{c^2} (E - V)^2 \left[ (E - V)^2 - m_0^2 c^4 \right] + \frac{\hbar^2}{2} \left[ \frac{3}{2} \left( \frac{\dot{x}}{x} \right)^2 - \frac{\dot{x}}{x} \right] .
\]

(4)

In a last paper [6], Djama had drawn the trajectories of a relativistic spinless particles for many potentials, and established the existence of nodes throw which all possible trajectories pass, even the purely relativistic one [6]. It is useful to stress that those results are found in one dimension.

It is clear that the introduction of quantum and relativistic quantum trajectories in the concept of a deterministic approach of quantum and relativistic quantum mechanics is a great step to elaborate a deterministic theory, but it is not sufficient. The generalization to spinning particle case and three dimension problems must be investigated.

The object of this paper is to investigate-in one dimension-the Law of motion of particles with spin 1/2. With this aim, we derive in Sec. 2 the RQSHJE
for Spin $1/2$ (RQSHJES$_{1/2}$) from the Dirac Spinors Equation (DSE) written in one dimension. We demonstrate that the RQSHJES$_{1/2}$ is composed by two components, each one correspond to the projection of the Spin $1/2$ ($m_s = +1/2$ and $m_s = -1/2$). Then, in Sec. 3, we propose a Double solution of the RQSHJES$_{1/2}$ corresponding to $m_s = +1/2$ and $m_s = -1/2$. Finally, in Sec. 4, we discuss our results.

2- Quantum Stationary Hamilton-Jacobi Equation for a particle with Spin $1/2$

In the beginning, the quantum systems are described by the Schrödinger equation

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x) = E.\]  

(5)

But later, after the discovery of the spin by Goodsmith and Pauli, and for studying the spinning particle’s behaviour, Dirac proposed his famous Spinors equation, written with Pauli representation as

\[i \hbar c \bar{\alpha} \cdot \nabla (\Psi) = \left[ E - V(\vec{r}) - \beta m_0 c^2 \right],\]  

(6)

where $\alpha_i, i = (x, y, z)$ and $\beta$ are the Dirac matrices

\[\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\]  

(7)

for which $\sigma_i$ are the Pauli matrices (Eq. (9)), and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

The Dirac equation is written, in one dimension, in the Appendix of Ref. \[10\] as

\[-i \hbar c \sigma_x \frac{d\psi}{dx} = (E - V(x) - \sigma_z m_0 c^2) \psi\]  

(8)

where

\[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\]  

(9)

are the Pauli matrix. $\psi = \begin{pmatrix} \theta \\ \phi \end{pmatrix}$ is a the matrix of the wave functions $\theta$ and $\phi$. Eq. (7) is a matrix equation of two components describing the state of particles of half spin with energy $E$ moving in one dimension under the action of the potential $V(x)$. It contains two scalar equations which can be deduced after decomposing

\[-i \hbar c \frac{d\phi}{dx} = (E - V(x) - m_0 c^2) \theta(x),\]  

(10)

\[-i \hbar c \frac{d\theta}{dx} = (E - V(x) + m_0 c^2) \phi(x).\]  

(11)

Taking the expression of $\theta$ from Eq. (10) and replacing it into Eq. (11), then, taking the expression of $\phi$ from Eq. (11) and replacing it into Eq. (10), we find
\( \hbar^2 c^2 \frac{d^2 \theta}{dx^2} + h^2 c^2 \frac{dV}{dx} \frac{d}{dx} + [(E - V)^2 - m_0^2 c^4] \theta(x) = 0 , \) \( (12) \)

\( \hbar^2 c^2 \frac{d^2 \phi}{dx^2} + h^2 c^2 \frac{dV}{dx} \frac{d}{dx} + [(E - V)^2 - m_0^2 c^4] \phi(x) = 0 . \) \( (13) \)

Now, the next step is to establish the QSHJES. In this order, let us write the two wave functions as

\[ \theta(x) = A(x) \left( \alpha_+ e^{\frac{i}{\hbar} S_0} + \alpha_- e^{-\frac{i}{\hbar} S_0} \right), \]

\[ \phi(x) = B(x) \left( \beta_+ e^{\frac{i}{\hbar} Z_0} + \beta_- e^{-\frac{i}{\hbar} Z_0} \right), \]

where \( A(x), B(x), S_0(x) \) and \( Z_0(x) \) are real functions of \( x \). \( \alpha_+, \alpha_- \) and \( \beta_+, \beta_- \) are real constants.

Replacing Eqs. (14) and (15) into Eqs.(12) and (13) respectively and decomposing it, we find

\[ \frac{\hbar^2 c^2}{A} \frac{d^2 A}{dx^2} - c^2 \left( \frac{dS_0}{dx} \right)^2 + \frac{\hbar^2 c^2}{E - V + m_0 c^2} \frac{dV}{dx} \frac{dA}{dx} + [(E - V)^2 - m_0^2 c^4] = 0 , \]

\( (14) \)

\[ \frac{\hbar^2 c^2}{B} \frac{d^2 B}{dx^2} - c^2 \left( \frac{dZ_0}{dx} \right)^2 + \frac{\hbar^2 c^2}{E - V - m_0 c^2} \frac{dV}{dx} \frac{dB}{dx} + [(E - V)^2 - m_0^2 c^4] = 0 , \]

\( (15) \)

and

\[ A \frac{d^2 S_0}{dx^2} + 2 \frac{dA}{dx} \frac{dS_0}{dx} + \frac{dV}{dx} \frac{dA}{dx} + \frac{1}{E - V + m_0 c^2} \frac{dS_0}{dx} = 0 , \]

\( (16) \)

\[ B \frac{d^2 Z_0}{dx^2} + 2 \frac{dB}{dx} \frac{dZ_0}{dx} + \frac{dV}{dx} \frac{dB}{dx} + \frac{1}{E - V - m_0 c^2} \frac{dZ_0}{dx} = 0 . \]

\( (17) \)

We can show easily that Eqs. (18) and(19) can be integrated to give

\[ A(x) = k_1 (E - V(x) + m_0 c^2)^{\frac{1}{2}} \left( \frac{dS_0}{dx} \right)^{\frac{1}{2}}, \]

\( (20) \)

\[ B(x) = k_2 (E - V(x) - m_0 c^2)^{\frac{1}{2}} \left( \frac{dZ_0}{dx} \right)^{\frac{1}{2}}, \]

\( (21) \)

where \( k_1 \) and \( k_2 \) are two real constants. Replacing Eqs. (20) and (21) into Eqs. (16) and (17), we get

\[ \frac{1}{2m_0} \left( \frac{dS_0}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \{ S_0, x \} + \frac{\hbar^2}{2m_0} (E - V + m_0 c^2)^{\frac{1}{2}}. \]

\( (22) \)

\[ \frac{1}{2m_0} \left( \frac{dZ_0}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \{ Z_0, x \} + \frac{\hbar^2}{2m_0} (E - V - m_0 c^2)^{\frac{1}{2}}. \]

\( (23) \)
\begin{equation}
\{f(x), x\} = \left[ \frac{3}{2} \left( \frac{df}{dx} \right)^{-2} \left( \frac{d^2f}{dx^2} \right)^2 - \left( \frac{df}{dx} \right)^{-1} \left( \frac{d^3f}{dx^3} \right) \right]
\end{equation}

represents the schwarzian derivative of \( f(x) \) with respect to \( x \). Eqs. (22) and (23) represent the two Relativistic Quantum Stationary Hamilton Jacobi Equations for Spinning particle \( (s = \frac{1}{2}) \) (QSHJES\(_{\frac{1}{2}}\)). One of these equations correspond to the projection \( m_s = +\frac{1}{2} \) of the spin, when the other correspond to the projection \( m_s = -\frac{1}{2} \). It follows that the reduced actions \( S_0 \) and \( Z_0 \) correspond to the two projections of the spin.

We can connect the wave functions \( \theta \) and \( \phi \) to the reduced actions as follows

\begin{equation}
\psi = \begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} k_1 (E - V(x) + m_0 c^2)^{\frac{1}{2}} (\frac{dS_0}{dx})^{\frac{1}{2}} (\alpha_x e^{xS_0} + \alpha_x e^{-xS_0}) \\ k_2 (E - V(x) - m_0 c^2)^{\frac{1}{2}} (\frac{dZ_0}{dx})^{\frac{1}{2}} (\beta_x e^{xZ_0} + \beta_x e^{-xZ_0}) \end{pmatrix}
\end{equation}

Now, let us discuss Eqs. (22) and (23). First, remark that, compared to the RQSHJE (Eq. (1)), Eq. (22) and (23) contains additional terms

\begin{equation}
term_1 = \frac{\hbar^2}{2m_0} (E - V + m_0 c^2)^{\frac{1}{2}} \frac{d^2}{dx^2} \left[ (E - V + m_0 c^2)^{-\frac{1}{2}} \right]
\end{equation}

for Eq. (22), and

\begin{equation}
term_1 = \frac{\hbar^2}{2m_0} (E - V - m_0 c^2)^{\frac{1}{2}} \frac{d^2}{dx^2} \left[ (E - V - m_0 c^2)^{-\frac{1}{2}} \right]
\end{equation}

for Eq. (23). Both \( term_1 \) and \( term_2 \) are proportional to \( \hbar^2 \) which means that at the classical limit (\( \hbar \to 0 \)) they vanish and Eqs. (22) and (23) reduce to the Classical Stationary Hamilton-Jacobi Equation (CSHJE). Secondly, these two terms vanish in the case of a constant potential. So, Eqs. (22) and (23) reduce to the QSHJE. More than this, the motion of a spinning particles are not affected by the constant potentials but only by potentials with a not vanishing gradient with respect to \( x \).

By this, we deduce that \( term_1 \) and \( term_2 \) are directly related to the spinning behaviour of quantum particles in one dimension.

**3- The solutions of the RQSHJES\(_{\frac{1}{2}}\)**

Now, let us investigate the form of the reduced actions \( S_0 \) and \( Z_0 \) solutions of the RQSHJES\(_{\frac{1}{2}}\) (Eqs. (22) and (23)). We propose as solutions of Eqs. (22) and (23) the following functions

\begin{equation}
S_0 = \hbar \arctan \left( a \frac{\theta_1(x)}{\theta_2(x)} + b \right),
\end{equation}

\begin{equation}
Z_0 = \hbar \arctan \left( d \frac{\phi_1(x)}{\phi_2(x)} + e \right),
\end{equation}

where \( a, b, d \) and \( e \) are real constants. \( \theta_1 \) and \( \theta_2 \) are two real and independent solutions of Eq. (12). \( \phi_1 \) and \( \phi_2 \) are two real independent solutions of Eq. (13).
Expressions (27) and (28) are analogous to the expression of the reduced action, solution of the RQSHJE and the QSHJE \cite{1,5,7,11}.

Now, we demonstrate our proposition. First, remark that Eqs. (22) and (23) have an analogous form. To write Eq. (23) we can replace $S_0$ by $Z_0$ and $(E - V + m_0c^2)$ by $(E - V - m_0c^2)$. Eqs. (27) and (28) are also analogous. For this reason, in what follows, we exhibit demonstration only for $S_0$ and Eq. (22) (for $Z_0$ and Eq. (23) the demonstration can be done with the same manner). In this order, we introduce the function

$$\theta = a \theta_1 + b \theta_2 . \tag{29}$$

As $\theta_1$ and $\theta_2$ are solutions of Eq. (12), $\theta$ is also solution of it. $S_0$ takes the form

$$S_0 = \hbar \arctan \left( \frac{a \theta_1(x)}{b \theta_2(x)} \right) . \tag{30}$$

The wronskian $W$ of $\theta$ and $\theta_2$ is deduced from Eq. (12)

$$W = \theta \frac{d\theta_2}{dx} - \theta_2 \frac{d\theta}{dx} = \alpha (E - V(x) + m_0c^2) , \tag{31}$$

where $\alpha$ is a real constant. Using this expression of the wronskian and replacing Eq. (30) into Eq. (22), after calculating the schwarzian derivative of $S_0$ with respect to $x$, one find

$$\frac{\hbar^2}{2m_0} \left( \frac{W^2 - \left( \frac{\theta d\theta_2}{dx} - \frac{\theta_2 d\theta}{dx} \right)^2}{(\theta^2 + \theta_2^2)^2} \right) - \frac{\hbar^2}{2m_0} \left( E - V + m_0c^2 \right)^\frac{\ddot{\theta}}{\ddot{\theta}_2} .$$

$$\frac{d^2}{dx^2} \left[ (E - V + m_0c^2)^\frac{\ddot{\theta}}{\ddot{\theta}_2} \right] + \frac{\hbar^2}{2m_0} \left( E - V + m_0c^2 \right)^\frac{\ddot{\theta}}{\ddot{\theta}_2} .$$

$$\frac{d^2}{dx^2} \left[ (E - V + m_0c^2)^{-\frac{\ddot{\theta}}{\ddot{\theta}_2}} \right] - \frac{\hbar^2}{2m_0} \left[ \frac{dV}{dx} \right] \frac{E - V + m_0c^2}{\frac{d\theta}{dx}} .$$

$$\left( \frac{\theta d\theta_2}{dx} - \frac{\theta_2 d\theta}{dx} \right) + \left( \frac{\theta d^2\theta_2}{dx^2} + \frac{\theta_2 d^2\theta}{dx^2} \right) \right)$$

$$\frac{(\theta^2 + \theta_2^2)}{\theta^2 + \theta_2^2} ,$$

which reduces to

$$\theta \left[ \frac{\hbar^2}{2m_0} \frac{d^2\theta}{dx^2} + \frac{\hbar^2}{2m_0} \frac{d\theta}{dx} \frac{dV}{dx} \frac{dx}{dx} + \left( (E - V)^2 - m_0^2c^4 \right) \theta \right]$$

$$+ \theta_2 \left[ \frac{\hbar^2}{2m_0} \frac{d^2\theta_2}{dx^2} + \frac{\hbar^2}{2m_0} \frac{d\theta_2}{dx} \frac{dV}{dx} \frac{dx}{dx} + \left( (E - V)^2 - m_0^2c^4 \right) \theta_2 \right] = 0 . \tag{32}$$

As $\theta$ and $\theta_2$ are two solutions of Eq. (12), Eq. (33) is automatically satisfied. Then, expression given by Eq. (27) is the solution of the RQSHJE given by Eq. (22). We note that, with same manner, we can demonstrate that expression given by Eq. (28) is the solution of the RQSHJE given by Eq. (23).
4- conclusion

In this paper, we have present two new results. The first one is the establishment of the two Relativistic Quantum Stationary Hamilton-Jacobi Equations for a half spinning particle ($s = \frac{1}{2}$) \((\text{QSHJE}_{\frac{1}{2}})\). The second result is the resolution of the RQSHJE_{\frac{1}{2}}.

The RQSHJE_{\frac{1}{2}} is composed of two equations (Eq. (22) and (23)). Each one represent a projection of the spin \((m_s = \frac{1}{2} \text{ and } m_s = -\frac{1}{2})\). both RQSHJE_{\frac{1}{2}} reduce to the classical Hamilton-Jacobi equation when we take the classical limit \((\hbar \to 0)\).

for a free particle the spinning terms (Eqs. (25) and (26)) vanish and both RQSHJE_{\frac{1}{2}} reduce to the RQSHJE. So, for the free particle, we can not distinguish between two projection of spin until it pass throw a non vanishing gradient potential.

Another interesting remark is the non relativistic case. the question is: Can we deduce, from the RQSHJE_{\frac{1}{2}}, the spinning behaviour of a particle for a purely quantum case? The answer is: Yes. Let us review the RQSHJE_{\frac{1}{2}} (Eqs. (22) and (23)). For the non relativistic limit, \(T << m_0c^2\) (where \(T = E - V(x) - m_0c^2\) is the kinetic energy of the particle), Eq. (22) reduces to

\[
\frac{1}{2m_0} \left( \frac{dS_0}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \{S_0, x\} + V(x) - E' = 0
\]  

which is the ordinary QSHJE. \(E'\) is the quantum energy without rest energy \(m_0c^2\).

In the other hand side, Eq. (23) reduces to

\[
\frac{1}{2m_0} \left( \frac{dZ_0}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \{Z_0, x\} + \frac{\hbar^2}{2m_0} (E' - V)^\frac{1}{2} \left. \frac{d^2}{dx^2} \right[ (E' - V)^{-\frac{1}{2}} \right] + V(x) - E' = 0 ,
\]

which is different from the ordinary QSHJE. So, it is clear that Eq. (35) describe one of the two projections of the spin. Thus there is two QSHJE_{\frac{1}{2}} to describe the spinning particle with \(s = \frac{1}{2}\) in the purely quantum cases. This point will be more investigated in a next papers.
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