On the stability of gravity in the presence of a non-minimally coupled scalar field

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ABSTRACT

We show that Einstein’s gravity coupled to a non-minimally coupled scalar field is stable even for high values of the scalar field, when the sign of the Einstein-Hilbert action is reversed. We also discuss inflationary solutions and a possible new mechanism of reheating.

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Introduction

Non-minimal couplings of a scalar field to spacetime curvature appear very frequently in theoretical physics, in applications ranging from alternative theories of gravity [1] to attempts to quantize gravity [2] to, more recently, scalar field models of “dark energy” [3, 4]. The implications of non-minimal coupling were also extensively investigated in connection with inflation [5].

Recently, some of us studied a previously unexplored sector of Einstein’s gravity with a non-minimally coupled (NMC) scalar, and found a class of new inflationary solutions [6]. A distinguishing feature of this sector is that the coupling to curvature becomes so important that it leads to new features such as a graceful dynamics from flat space to inflation and ending in a Friedman-Robertson-Walker spacetime, and “superinflation” (i.e. $dH/dt > 0$, where $H$ is the expansion rate). However, ominously, the model also brings a dynamical reversal of the sign of the gravitational action.

This last feature should alarm those familiar with General Relativity: the “wrong” sign for the Einstein-Hilbert action means that the excitations of the gravitational field have negative-energy modes. Therefore, this “sign-reversal” of the action could be a hint that the theory is unstable in that sector [4], since the positive-energy scalar field would feed the negative-energy gravitons and vice-versa, leading to an explosive process.

Non-minimal coupling then seemed to be in a predicament: on the one hand it is an unavoidable interaction [7, 8] which produces a very attractive cosmology [6]. On the other hand, the theory appears to be unstable just in the sector which contains the interesting new dynamics.

This Letter shows that in fact there is no predicament: Einstein’s theory of gravity with a scalar field is in fact stable on all sectors for $\xi \leq 0$ and $\xi \geq 1/6$. The stability of the theory follows from the fact that, for this range of $\xi$, both the pure gravitational and the pure scalar degrees of freedom of the theory simultaneously reverse the signs of their actions. This opens the way to novel inflationary scenarios, and may even suggest a new mechanism for reheating the Universe after inflation. In a forthcoming paper [9] we show that the inflationary model gives phenomenologically sound predictions about the strength and scale-dependence of the spectrum of anisotropies of the cosmic microwave background radiation.

Stability of Einstein’s theory with a scalar field

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The system is Einstein’s General Relativity with the addition of a scalar field:

\[ L \equiv \sqrt{-g} \left\{ -\frac{1}{2} \left( 1 - \xi \psi^2 \right) R + \frac{1}{2} g^{\mu\nu} \psi_{,\mu} \psi_{,\nu} - V(\psi) \right\}, \quad (1) \]

where in our conventions \(8\pi G = 1\) and the metric has timelike signature \((+, -, -, -)\). \(V(\psi)\) is an arbitrary scalar self-interaction potential. The scalar coupling to curvature \(\xi\) can in principle assume any value: \(\xi = 0\) corresponds to no (“minimal”) coupling to curvature, while \(\xi = 1/6\) is the case of “conformal” coupling.

The “conformal factor” \(F(\psi) \equiv 1 - \xi \psi^2\) in the Lagrangean (1) multiplies the usual Einstein-Hilbert term \(-\sqrt{-g}R/2\). Because the scalar field multiplies the scalar curvature, it is clear that the physical (helicity two) degrees of freedom of the gravitational field are intertwined with the (helicity zero) degrees of freedom of the scalar field.

The diagonalization of the system (1) can be achieved through a conformal transformation of the metric:

\[ \bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \quad (2) \]

This change of variable induces the following transformation in the Ricci scalar (in four dimensions):

\[ \bar{R} \equiv R[\bar{g}] = \Omega^{-2} \left[ R - 6 g^{\mu\nu} D_\mu D_\nu (\log \Omega) - 6 g^{\mu\nu} D_\mu (\log \Omega) D_\nu (\log \Omega) \right], \quad (3) \]

where \(D_\mu[\bar{g}]\) are covariant derivatives. Using this expression in Eq. (1) we obtain, after neglecting total derivatives:

\[ L_> = \sqrt{-\bar{g}} \left\{ -\frac{F(\psi)}{2\Omega^2} \bar{R} + \frac{1}{2} \bar{g}^{\mu\nu} \left( \frac{\psi_{,\mu} \psi_{,\nu}}{\Omega^2} + 6 \frac{\Omega_{,\mu} \Omega_{,\nu}}{\Omega} \right) - \frac{V(\psi)}{\Omega^4} \right\}. \quad (4) \]

We want to rewrite this Lagrangian in such a way that it resembles as much as possible the Einstein-Hilbert action plus a minimally coupled scalar field \(\Box \psi\). Hence, let us take \(\Omega^2 = F(\psi)\) assuming for the moment that \(F(\psi) > 0\), which leads to:

\[ L_> = \sqrt{-\bar{g}} \left\{ -\frac{1}{2} \bar{R} + \frac{1}{2} \bar{g}^{\mu\nu} \psi_{,\mu} \psi_{,\nu} \frac{1 + \xi(6\xi - 1)\psi^2}{F^2(\psi)} - \frac{V(\psi)}{F^2(\psi)} \right\}. \quad (5) \]
If we now introduce the conformally transformed field and effective potential
\[ d\tilde{\psi} \equiv \frac{\sqrt{1 + \xi(6\xi - 1)\psi^2}}{F(\psi)} d\psi, \quad \tilde{V} \equiv \frac{V(\psi)}{F^2(\psi)}, \]
and substitute these expressions into Eq. (5) we obtain:
\[ L > = \sqrt{-\tilde{g}} \left\{ -\frac{1}{2} \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\psi}_{,\mu} \tilde{\psi}_{,\nu} - \tilde{V} \right\} . \]
Because the Lagrangian (7) has the form of the Einstein-Hilbert action plus a scalar term, it is common to call the tilded variables the “Einstein frame”. Conversely, the original action (1) is referred to as the action in the “Jordan frame”. Notice the minus sign in front of the Einstein-Hilbert action both in the Jordan as well as in the Einstein frames: it guarantees, in our conventions, that in the ultraviolet (or geometrical optics) limit gravitons behave as positive-energy free fields [11].

Of course, for the transformation (6) to make sense the terms inside the square root should be positive, that is:
\[ 1 + \xi(6\xi - 1)\psi^2 \geq 0 . \]
For \( F(\psi) \geq 0 \) it is easy to check that this condition is always satisfied.

However, \( F \) can be negative if \( \xi \geq 0 \) and the value of the scalar field is sufficiently high (\( \psi^2 > 1/\xi \)). In this case, we would like to use a conformal transformation of the metric which preserves its original signature, so we take \( \Omega^2 = -F > 0 \). With this choice we have the counterpart to Eq. (8):
\[ \tilde{R} \equiv R[\tilde{g}] = -\frac{1}{F} \left[ R - 6g^{\mu\nu} D_\mu D_\nu (\log \Omega) - 6g^{\mu\nu} D_\mu (\log \Omega) D_\nu (\log \Omega) \right] . \]
Neglecting total derivatives once again, we obtain an expression very similar to Eq. (4):
\[ L < = \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \frac{\psi_{,\mu} \psi_{,\nu}}{F} + \frac{3}{2} \frac{F_\mu F_\nu}{F^2} - \frac{V(\psi)}{F^2} \right\} . \]
Again, the term between square brackets simplifies:
\[ L < = \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\psi}_{,\mu} \tilde{\psi}_{,\nu} \frac{1 + \xi(6\xi - 1)\psi^2}{F^2} - \frac{V(\psi)}{F^2} \right\} . \]
Notice that both the Einstein-Hilbert and the scalar kinetic terms have a switched sign with respect to (8), as opposed to the scalar potential term which does not. We therefore use the field redefinitions:

\[ d\tilde{\psi} \equiv \frac{\sqrt{1 + \xi(6\xi - 1)\psi^2}}{F(\psi)}d\psi, \quad \tilde{V} \equiv -\frac{V(\psi)}{F^2(\psi)}, \tag{12} \]

Substituting these expressions into Eq. (11) we obtain:

\[ L_{<} = -\sqrt{-\tilde{g}} \left\{ -\frac{1}{2}\tilde{R} + \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\psi}_\mu\tilde{\psi}_\nu - \tilde{V} \right\}. \tag{13} \]

The crucial fact to notice now is that the Lagrangian (13) has an overall minus sign that appeared as a consequence of the change of variables with \( F < 0 \). Therefore, the excitations of both the scalar and gravitational fields carry energy with the same sign, even though the sign happens to be negative in this case. But an overall sign in front of the Lagrangian is irrelevant, so the theory is in effect stable for \( F < 0 \).

We should again take care that the field transformation (12) is well-defined, and the condition is still given by inequality (8). For \( \xi \leq 0 \) and \( \xi \geq 1/6 \) that condition is always satisfied if \( F \leq 0 \). However, for \( F < 0 \) and \( 0 < \xi < 1/6 \) the condition is violated when the scalar field lies outside the range \( 1 \leq \xi\psi^2 \leq 1/(1 - 6\xi) \). In that case, the scalar kinetic term in the Lagrangian (11) reverses its sign. What this means is that for \( 0 < \xi < 1/6 \) there is an unstable sector of the theory corresponding to large values of the scalar field\(^1\).

The consequences of a kinetic term which can reverse its sign depending on the sector of the theory are quite severe: this introduces negative-energy states which make the whole theory highly unstable. Even if the value of the field is in the stable sector in a certain region of space initially, the existence of the unstable sector means that the scalar field will tunnel from the stable sector into the unstable sector. It is highly doubtful whether such a theory could exist even for a moment.

This means that the cases of minimal coupling, \( \xi = 0 \), and of conformal coupling, \( \xi = 1/6 \), are threshold systems: for the sake of the global stability\(^2\),

\(^1\)Of course, the sign of the gravity-scalar Lagrangian is only irrelevant if this theory exists by itself: if other matter fields are included, the relative signs of their Lagrangians should be the same as that of the gravity-scalar sector in order to avoid instabilities.

\(^2\)Similar conditions were also reached by [13].
of gravity with a scalar field, either $\xi \leq 0$ or $\xi \geq 1/6$. The case of minimal coupling case is stable simply because the excitations of the gravitational and scalar fields are always positive. The conformal coupling case is stable because the excitations of the gravitational and scalar fields both carry the same sign — positive or negative.

There is also a trivial method by which the “effective” Lagrangian (13) could be derived in the case of negative $F$: suppose that we let the metric change its signature if $F$ changes sign, so that the conformal transformation into the Einstein frame is $\Omega^2 = F$ in any case. This corresponds to a change of the lorentzian signature — from timelike to spacelike or vice-versa, depending on the convention. The Lagrangian would still be expressed exactly as in Eq. (7). However, the fact that the signature of the metric is switched with respect to the original definition means that both $\tilde{R}$ and $\tilde{g}^{\mu\nu}\tilde{\psi}_{,\mu}\tilde{\psi}_{,\nu}$ have in fact switched signs with respect to the original (positive) Lagrangian. If we wanted to restore the usual signature, we would have to let $\tilde{g}_{\mu\nu} \to -\tilde{g}_{\mu\nu}$, and the signature restoration would lead precisely to Eq. (13), with the definitions (12).

There are, however, two problems with this global description of a NMC scalar field plus gravity. First, that it ignores the rest of the world (we will discuss this point at the conclusion.) And second, that the scalar potential remains unchanged while all other terms of the Lagrangian change sign, so its rôle is reversed if the signature changes.

Scalar self-interactions, inflation and superinflation

We henceforth consider only minimal and conformal couplings, since all other cases are qualitatively equivalent to either one of those. The case of minimal coupling is of course trivial: any potential with a lower bound gives a physically sensible (i.e., stable) theory.

However, in the conformally coupled case, if $F < 0$ ($\psi^2 > 6$) then by Eq. (12) the Einstein frame potential becomes $\tilde{V} = -V/F^2$. Hence, a physical (Jordan frame) scalar potential which is bounded from below but unbounded from above for large values of the scalar field could become, in the $F < 0$ sector of the Einstein frame, bounded from above but unbounded from below. We therefore come to the conclusion that in the case of Einstein’s gravity with a conformally coupled scalar field, the large scalar field sector must have a Jordan-frame potential which is bounded from above, and not necessarily
bounded from below!

Take the simplest scalar potential:

\[ V(\psi) = \frac{1}{2} m^2 \psi^2 - \frac{\lambda}{4} \psi^4 , \quad (14) \]

where \( \lambda > 0 \) in the conformally coupled case. The Einstein-frame scalar field is obtained by direct integration of Eq. (6):

\[ \tilde{\psi} = \begin{cases} 
\sqrt{6} \tanh^{-1} \frac{\psi}{\sqrt{6}} , & \psi^2 < 6 , \\
\sqrt{6} \tanh^{-1} \frac{\sqrt{6}}{\psi} , & \psi^2 > 6 .
\end{cases} \quad (15) \]

(Notice that \( \tilde{\psi} \) is not monotonic in \( \psi \).) The potential in the Einstein frame is, by Eqs. (3) and (12):

\[ \tilde{V} = \begin{cases} 
3m^2 \cosh \frac{\tilde{\psi}}{\sqrt{6}} \sinh \frac{\tilde{\psi}}{\sqrt{6}} \left( 1 - \frac{3\lambda}{m^2} \tanh \frac{\tilde{\psi}}{\sqrt{6}} \right) , & \psi^2 < 6 , \\
-3m^2 \cosh \frac{\tilde{\psi}}{\sqrt{6}} \sinh \frac{\tilde{\psi}}{\sqrt{6}} \left( 1 - \frac{3\lambda}{m^2} \coth \frac{\tilde{\psi}}{\sqrt{6}} \right) , & \psi^2 > 6 .
\end{cases} \quad (16) \]

The potentials are plotted in Fig. 1, where the left and right panels correspond respectively to the sectors \( F > 0 (\psi^2 < 6) \) and \( F < 0 (\psi^2 > 6) \). For the range of parameters \( 1/6 < \lambda/m^2 < 1/3 \), the potential has a maximum in the \( F < 0 \) sector (right panel, solid line.) This corresponds to a de Sitter fixed point in the Einstein frame, which is itself the image of a de Sitter fixed point in the Jordan frame in the \( F < 0 \) sector first exhibited in [6]. Notice that the Einstein-frame potential in the \( F < 0 \) sector becomes exponentially negative when \( \tilde{\psi} \to \infty (\psi \to \sqrt{6}) \). This is of course due to the fact that \( F(\psi) \), which appears in the denominator of \( \tilde{V} \), vanishes when \( \psi = \sqrt{6} \).

It is interesting to consider what would happen if the scalar runs to the right of the fixed point at the maximum of the potential in the right panel of Fig. 1. Under the approximation of an exponential potential, the solutions for a homogeneous scalar field in Einstein’s gravity are those of power-law inflation [14], and in this particular case what would happen is that the field \( \tilde{\psi} \to \infty (\text{or}, \psi \to \sqrt{6}) \) in a finite time. But then the sector changes from \( F < 0 \) to \( F > 0 \), and the field finds itself in the potential of the left panel of Fig. 1. The field then runs down to the bottom of the potential (again in a
finite time), and relaxes there leading to a mild Friedman-Robertson-Walker expansion [6].

The problem with this particular scenario is that although the homogeneous solutions in the Jordan frame are perfectly finite and well-behaved as $F \to 0$ [4], inhomogeneous fluctuations diverge in a few cases, as the system approaches that point [15, 9]. The cause of these divergences is that the infinitely negative values of the “effective” potential $\tilde{V}$ are reached over a finite period of time. Although the background quantities remain finite, because the divergence factors out from the diagonal Einstein’s equations, this is not a priori true for the perturbations, which are determined by the full Einstein’s equations (more precisely, it is the anisotropic stress which is causing the divergence.)

Divergences can be avoided by taking a potential $\tilde{V}$ which is well-behaved everywhere — i.e., bounded from below at $|\tilde{\psi}| < \infty$. But then the transition between sectors $F < 0$ and $F > 0$ seem unlikely. In a forthcoming paper we will discuss the problem of the inhomogeneous perturbations, as well as realistic cosmological scenarios with $F < 0$ which are explicitly free of singularities [9].

Conclusion
Einstein’s gravity with a conformally coupled scalar leads to an attractive alternative to usual (minimally coupled) inflationary models. New scenarios with a unique dynamics have been found, which cannot be reduced to minimally coupled scalar-driven inflation. However, the stability of these models has been a matter of some controversy. Since the conformal factor $F(\psi)$ multiplying the gravitational action can assume negative values in these models, it has been suggested that they are unstable and therefore must be ruled out [4].

In this paper we have shown that this is not so: although the gravitational action indeed acquires a negative sign when $F(\psi) < 0$, the same happens to the scalar degree of freedom. As a result, gravitons cannot decay into scalars, or vice versa.

Of course, one should eventually consider some type of matter besides gravity and the NMC scalar field $\psi$. An additional scalar field $\varphi$, for example, would be unstable with respect to gravity and the scalar field $\psi$, since the sign of the kinetic energy term of $\varphi$ is insensitive to the sign of $F(\psi)$. The same seems to hold for all bosonic and fermionic fields (bosons and fermions do not “switch” energy states when the metric changes lorentzian signature.) Therefore, the presence of additional (positive-energy) fields at the time of inflation, when $F(\psi) < 0$, could trigger exactly the types of instabilities that we tried so hard to avoid [3].

Apart from being evidently a potential disaster, this suggests an interesting, although very speculative, mechanism for reheating the Universe after inflation: suppose that initially all fields were in negative energy states. This can be achieved by switching the signs of the actions of all matter fields except gravity and the NMC scalar. As inflation nears its end, $F(\psi) \rightarrow 0$ and the gravitational and scalar sectors switch the signs of their actions due to the effect that we discussed in this paper. But now this means that a huge instability develops at the end of inflation, between gravity and the inflaton on the one hand, and the rest of the matter fields on the other hand. The outcome of such an instability has to be an explosive process which ends in some stable ground state — which is presumably the state in which we live until now.

A concrete example can be worked out if we consider gravity, the NMC field $\psi$.

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3One way to avoid this would be to suppose that only gravity and the NMC scalar field were present at the energy scales corresponding to inflation.
scalar field and fermions. Suppose that initially the system is in the $F < 0$ sector and all fields are in negative energy eigenstates, so that the whole theory is stable. For this to work out, the fermionic Lagrangian would carry a minus sign with respect to the usual one. It can be thought that the fermions initially occupy the Dirac sea of positive energy eigenstates, while the negative eigenstates are essentially empty (in other words, the Dirac seas are reversed with respect to the usual picture.)

As inflation ends and the field passes through the point $\psi^2 = 1/\xi$, the energy eigenstates of the gravitational and scalar degrees of freedom become positive, while the fermionic degrees of freedom are still negative. This instability endures for a very short time, during which the fermionic states jump out of their positive-energy holes and fill in the negative eigenstates of the Dirac sea. While this explosive process happens, much energy can be extracted from the gravitational and scalar degrees of freedom, and thrown into the fermionic degrees of freedom. In this manner, the Universe can be filled with radiation — i.e., reheated.

In a forthcoming paper we will show that inflationary models which pass from the $F < 0$ to the $F > 0$ sectors are acceptable phenomenologically, provided that the transition between these sectors is free of singularities and that a final stable ground state can be reached after the transition. In that paper we will also discuss theories where the whole cosmological evolution, up to and including the present time, takes place in the $F < 0$ sector.

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