GLOBAL $L^2$-BOUNDEDNESS THEOREMS FOR A CLASS OF FOURIER INTEGRAL OPERATORS

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Abstract. The local $L^2$-mapping property of Fourier integral operators has been established in Hörmander [14] and in Eskin [12]. In this paper, we treat the global $L^2$-boundedness for a class of operators that appears naturally in many problems. As a consequence, we will improve known global results for several classes of pseudo-differential and Fourier integral operators, as well as extend previous results of Asada and Fujiwara [1] or Kumano-go [17]. As an application, we show a global smoothing estimate to generalized Schrödinger equations which extends the results of Ben-Artzi and Devinatz [2], Walther [27], and [28].

1. Introduction

We consider (Fourier integral) operators, which can be globally written in the form

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) dy d\xi \quad (x \in \mathbb{R}^n),$$

where $a(x,y,\xi)$ is an amplitude function and $\phi(x,y,\xi)$ is a real phase function of the form

$$\phi(x,y,\xi) = x \cdot \xi + \varphi(y,\xi).$$

Note that, by the equivalence of phase function theorem, Fourier integral operators with the local graph condition can always be written in this form locally. Although, due to the nontriviality of the Maslov cohomology class, globally defined Fourier integral operators cannot be written in this form with a globally defined phase $\phi$, it is nevertheless convenient to still call them Fourier integral operators. It will be clear below how such operators naturally arise in global smoothing problems if we use an adaptation of the Egorov theorem.

Local $L^2$ mapping property of the operator (1.1) has been established in Hörmander [14] and in Eskin [12]. One of the aims of this paper is to establish the global $L^2$-boundedness properties of operators (1.1). Analogous properties can be then easily obtained for adjoint operators as well.

We will try to make as few assumptions as possible in the spirit of global $L^2$-estimates for pseudo-differential operators (see Calderón–Vaillancourt [5], Childs [7], Coifman-Meyer [8], Cordes [9]). In fact, our Corollary 2.4 will not only extend these $L^2$-boundedness results to more general operators (1.1), but will also reduce the number of assumptions on the amplitude in the case of pseudo-differential operators, compared to the above mentioned papers (see also Sugimoto [25]). Global $L^2$-boundedness of operators (1.1) has been previously studied by Asada-Fujiwara [1], Kumano-Go [17]. However, there one had to make a quite restrictive and not
always natural assumption on the boundedness of $\partial_\xi \partial_\xi \phi$, which fails in many important cases. In Coriasco [11] and Boggiato-Buzano-Rodino [4] such results are applied to obtain global estimates of solutions to some classes of hyperbolic equations. Here again one requires quite strong decay properties of derivatives of both phase and amplitude. We will remove all these assumptions and will give general $L^2$-estimates.

In fact, in global estimates of Section 2 we will actually impose only a finite number of conditions on the phase and the amplitude, compared to infinitely many in the above mentioned papers.

As a consequence of our $L^2$-estimates, we can treat canonical transforms. Operators that appear there are of the form (1.1) with phase function

$$\phi(x, y, \xi) = x \cdot \xi - y \cdot p(\xi) \frac{\nabla p(\xi)}{|\nabla p(\xi)|},$$

where $p(\xi)$ is a positively homogeneous function of degree 1. If we take $p(\xi) = |\xi|$, then we have $\phi(x, y, \xi) = x \cdot \xi - y \cdot \xi$, and the operator $T$ defined by (1.1) is a pseudo-differential operator. Furthermore, the operator $T$ with general (1.2) is used to transform the Fourier multiplier

$$L_p = p(D_x)^2 = F^{-1}_{\xi} p(\xi)^2 F_x$$

to the Laplacian $-\Delta$, where $F_x$ ($F^{-1}_{\xi}$ resp.) denotes the (inverse resp.) Fourier transform. In fact, we have a relation

$$T \cdot (-\Delta) \cdot T^{-1} = L_p$$

under a certain condition on $p(\xi)$ if we take 1 as the amplitude function $a(x, y, \xi)$ (see Section 4). The $L^2$-property of the Laplacian is well known in various situations. Our objective is to know the $L^2$-property of the operator $T$, so that we can extract the $L^2$-property of the operator $L_p$ from that of the Laplacian. This approach allows to give a general treatment of several smoothing problems, including those treated by e.g. Ben-Artzi and Klainerman [3], Simon [22], Kato and Yajima [15], or Walther [27].

We should mention here that the global $L^2$-boundedness with example (1.2) is not covered by previous results, for example, Asada and Fujiwara [1], and Kumano-go [17]. The result of [1] is motivated by the construction of fundamental solution of Schrödinger equation in the way of Feynman’s path integral, and it requires the boundedness of all the derivatives of entries of the matrix

$$\left( \frac{\partial_x \partial_y \phi}{\partial_\xi \partial_y \phi} \quad \frac{\partial_x \partial_\xi \phi}{\partial_\xi \partial_\xi \phi} \right).$$

For the details, see [1] and references cited there. With our example (1.2), the boundedness of the entries of $\partial_\xi \partial_\xi \phi$ fails. On the other hand, the result of [17] is used to construct the fundamental solution of hyperbolic equations, and it requires that $J(y, \xi) = \phi(x, y, \xi) - (x - y) \cdot \xi$ satisfies

$$|\partial_\gamma \partial_\delta J(y, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{1-|\beta|}$$

for all $\alpha$ and $\beta$. Our example (1.2) does not satisfy these estimates with $\alpha = 0$. 

2
In this paper, we develop a new \( L^2 \)-theory which does not require these decay assumptions. In particular, it includes the case of example (1.2). For \( m \in \mathbb{R} \), let \( L_m^2(\mathbb{R}^n) \) be the set of functions \( f \) such that the norm
\[
\|f\|_{L_m^2(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\langle x \rangle^m f(x)|^2 \, dx \right)^{1/2}
\]
is finite. The following is a simplified version of our main result (Theorem 3.1) which is expected to have many applications:

**Theorem 1.1.** Let the operator \( T \) be defined by (1.1), where \( \varphi(y, \xi) \in C^\infty(\mathbb{R}_y^n \times \mathbb{R}_\xi^n) \) is a real-valued function, and \( a(x, y, \xi) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\xi^n) \). Assume that
\[
|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0,
\]
and all the derivatives of entries of \( \partial_y \partial_\xi \varphi \) are bounded. Also assume that
\[
|\partial_\xi^\alpha \varphi(y, \xi)| \leq C_\alpha \langle y \rangle \quad \text{for all } |\alpha| \geq 1,
\]
\[
|\partial_x^\beta \partial_y^\delta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha \beta \gamma} \langle x \rangle^{-|\alpha|} \quad \text{for all } \alpha, \beta, \text{ and } \gamma.
\]
Then \( T \) is bounded on \( L_m^2(\mathbb{R}^n) \) for any \( m \in \mathbb{R} \).

This theorem says that, if amplitude functions \( a(x, y, \xi) \) have some decaying properties with respect to \( x \), we do not need the boundedness of \( \partial_\xi \partial_y \varphi \) for the \( L^2 \)-boundedness, as required in [1], and we can have weighted estimates, as well. (The same is true when both phase and amplitude functions have some decaying properties with respect to \( y \). See Theorem 3.1.)

We explain the plan of this paper. In Section 2, we show the global \( L^2 \)-boundedness of a class of oscillatory integral operators, which generalizes a standard local result explained in Stein [23]. By using it, we prove various type of the \( L^2 \)-boundedness of Fourier integral operators. Some of them are extension of previous results on the \( L^2 \)-boundedness of pseudo-differential operators with non-regular symbols. It is worth mentioning that, in general, we do not necessarily need the standard homogeneity assumption for the phase function in the frequency variable. In addition, we impose the boundedness condition on only a finite number of the derivatives of phase functions, while infinitely many in [1] and [17].

In Section 3, we state and prove our main result Theorem 3.1. We remark that it (together with Theorem 2.3) substantially weaken the assumptions for the \( L^2 \)-boundedness of SG pseudo-differential (as in Cordes [10]) and SG Fourier integral operators (as in Coriasco [11]). These operators are used to handle the SG hyperbolic partial differential equations (roughly speaking, certain equations with coefficients of polynomial growth). The class of symbols \( SG^{m_1, m_2} \) is defined as a space of smooth functions \( a = a(y, \xi) \in C^\infty(\mathbb{R}_y^n \times \mathbb{R}_\xi^n) \) satisfying the estimate
\[
|\partial_\xi^\beta \partial_y^\delta a(y, \xi)| \leq C_{\beta \gamma} \langle y \rangle^{m_1 - |\beta|} \langle \xi \rangle^{m_2 - |\gamma|} \quad \text{for all } \beta \text{ and } \gamma.
\]
SG Fourier integral operators are operators of the form
\[
Tu(x) = \int_{\mathbb{R}^n} e^{i\langle x, \xi + \varphi(y, \xi) \rangle} a(y, \xi) u(y) \, d\xi \, dy
\]
(or its adjoint), where \( a \in SG^{m_1,m_2} \) and \( \varphi \in SG^{1,1} \), which also satisfies
\[
C_1 \langle y \rangle \leq \langle \partial_\xi \varphi \rangle \leq C_2 \langle y \rangle, \quad C_1 \langle \xi \rangle \leq \langle \partial_y \varphi \rangle \leq C_2 \langle \xi \rangle,
\]
for some \( C_1, C_2 > 0 \). A result in [10] for SG pseudo-differential and its extension in [11] for SG Fourier integral operators states that under these assumptions on the phase \( \phi \), and for \( a \in SG^{0,0} \), the corresponding operator \( T \) is bounded on \( L^2(\mathbb{R}^n) \).

Without going much into detail, let us mention here that statements of our results replace the strong decay assumptions \( \phi \in SG^{1,1} \), \( a \in SG^{0,0} \), by (a finite number of) boundedness conditions, for \( T \) to be still bounded in \( L^2(\mathbb{R}^n) \).

In Section 4, we exhibit an example of how to use our main result. We mainly focus on the problem of global smoothing property of generalized Schrödinger equations
\[
(1.3) \begin{cases}
(i \partial_t + Q(D)) u(t, x) = 0, \\
u(0, x) = f(x).
\end{cases}
\]

Ben-Artzi and Devinatz [2] showed a global smoothing estimate to equation (1.3), where the symbol \( Q(\xi) \) of \( Q(D) \) is a real polynomial of principal type. Walther [28] consider the case of radially symmetric \( Q(\xi) \). By using our result Theorem 3.1, we can treat more general case (see Theorem 4.2). More refined applications to this subject will be shown in our forthcoming paper [21]. In subsequent work [20], we will establish properties of operators \( I_\varphi \) in weighted Sobolev spaces, which will have several further applications of these results to hyperbolic equations as well as global canonical transforms.

2. Global \( L^2 \)-estimates

First of all, we confirm a basic result on the \( L^2 \)-boundedness of a class of oscillatory integral operators, based on the argument of Fujiwara [13], which is a global version of a proposition in Stein [23, p.377]. Here and hereafter, the capital \( C \) (sometimes with some suffices) always denotes a positive constant which may differ on each occasion.

**Theorem 2.1.** Let the operator \( I_\varphi \) be defined by
\[
(2.1) \quad I_\varphi u(x) = \int_{\mathbb{R}^n} e^{i \varphi(x,y)} a(x,y) u(y) \, dy,
\]
where \( a(x,y) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n) \), and \( \varphi(x,y) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n) \) is a real-valued function. Assume that
\[
|\partial_\xi^\alpha \partial_\zeta^\beta a(x,y)| \leq C_{\alpha\beta},
\]
for \( |\alpha|, |\beta| \leq 2n + 1 \). Also assume that, on \( \text{supp} a(x,y) \),
\[
|\det \partial_\xi \partial_\zeta \varphi(x,y)| \geq C > 0
\]
and each entry \( h(x,y) \) of the matrix \( \partial_\xi \partial_\zeta \varphi(x,y) \) satisfies
\[
|\partial_\xi^\alpha h(x,y)| \leq C_\alpha, \quad |\partial_\zeta^\beta h(x,y)| \leq C_\beta
\]
for \( |\alpha|, |\beta| \leq 2n + 1 \). Then the operator \( I_\varphi \) is \( L^2(\mathbb{R}^n) \)-bounded, and satisfies
\[
\| I_\varphi \|_{L^2 \to L^2} \leq C \sup_{|\alpha|, |\beta| \leq 2n+1} \| \partial_\xi^\alpha \partial_\zeta^\beta a(x,y) \|_{L^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n)}.
\]
Proof. Let \( g \in C_0^\infty(\mathbb{R}^n) \) be a real-valued positive function such that \( \{g_k(x)\}_{k \in \mathbb{Z}^n} \), where \( g_k(x) = g(x - k) \), forms a partition of unity. We decompose the operator \( I_\varphi \) as

\[
I_\varphi = \sum_{(j,k) \in \mathbb{Z}^n \times \mathbb{Z}^n} I_{(j,k)},
\]

where \( I_{(j,k)} = g_j I_\varphi g_k \), that is,

\[
I_{(j,k)} u(x) = g_j(x) \int e^{i\varphi(x,z)} a(x, z) g_k(z) u(z) \, dz.
\]

We denote the adjoint of \( I_{(j,k)} \) by \( I_{(j,k)}^* \), that is,

\[
I_{(j,k)}^* u(z) = g_k(z) \int e^{-i\varphi(y,z)} a(y, z) g_j(y) u(y) \, dy.
\]

Then we have

\[
I_{(j,k)} I_{(l,m)}^* u(x) = \int K_{(j,k),(l,m)}(x, y) u(y) \, dy,
\]

where

\[
K_{(j,k),(l,m)}(x, y) = g_j(x) g_l(y) \int e^{i(\varphi(x,z) - \varphi(y,z))} a(x, z) a(y, z) g_k(z) g_m(z) \, dz.
\]

By integration by parts, we have

\[
\int e^{i(\varphi(x,z) - \varphi(y,z))} a(x, z) a(y, z) g_k(z) g_m(z) \, dz
\]

\[
= \int e^{i(\varphi(x,z) - \varphi(y,z))} L^{2n+1} \left( a(x, z) a(y, z) g_k(z) g_m(z) \right) \, dz,
\]

where \( L \) is the transpose of the operator

\[
t_L = \frac{1}{i} \frac{\partial_z \varphi(x, z) - \partial_z \varphi(y, z)}{|\partial_z \varphi(x, z) - \partial_z \varphi(y, z)|^2} \cdot \partial_z.
\]

From the assumptions, and using that

\[
\partial_z \varphi(x, z) - \partial_z \varphi(y, z) = \partial_x \partial_z \varphi(w, z)(x - y)
\]

for some \( w \), we obtain

\[
|\partial_z \varphi(x, z) - \partial_z \varphi(y, z)| \geq C|x - y|
\]

and

\[
|\partial^\beta_z \varphi(x, z) - \partial^\beta_z \varphi(y, z)| \leq C_\beta |x - y|
\]

for \( 1 \leq |\beta| \leq 2n + 2 \). Hence, we have

\[
|K_{(j,k),(l,m)}(x, y)| \leq CA^2 \frac{g_j(x) g_l(y)}{1 + |x - y|^{2n+1}} h(k - m),
\]

where \( h \in C_0^\infty(\mathbb{R}^n) \) is a positive function \( h(x) = \int g(z - x) g(z) \, dz \), and

\[
A = \sup_{|\alpha|, |\beta| \leq 2n+1} ||\partial_x^\alpha \partial_y^\beta a||_{L^\infty(\mathbb{R}^n_+ \times \mathbb{R}^n_+)}.
\]
Then we have
\[
\sup_x \left| \int K_{(j,k),(l,m)}(x,y) \, dy \right| \leq CA^2 \frac{h(k-m)}{1 + |j-l|^{2n+1}},
\]
\[
\sup_y \left| \int K_{(j,k),(l,m)}(x,y) \, dx \right| \leq CA^2 \frac{h(k-m)}{1 + |j-l|^{2n+1}},
\]
which implies
\[
\| I_{(j,k)} I^*_l(l,m) \|_{L^2 \to L^2} \leq CA^2 \frac{h(k-m)}{1 + |j-l|^{2n+1}}.
\]
Here we have used the following lemma (see Stein [23, p.284]):

**Lemma 2.1.** Suppose \( S \) is given by
\[
(Sf)(x) = \int s(x,y) f(y) \, dy,
\]
where the kernel \( s(x,y) \) satisfies
\[
\sup_x \int |s(x,y)| \, dy \leq 1, \quad \sup_y \int |s(x,y)| \, dx \leq 1.
\]
Then \( \| S \|_{L^2 \to L^2} \leq 1. \)

By the same discussion, we have
\[
\| I_{(j,k)} I^*_l(l,m) \|_{L^2 \to L^2} \leq CA^2 \frac{h(j-l)}{1 + |k-m|^{2n+1}}.
\]
Then we have
\[
\| I_{(j,k)} I^*_l(l,m) \|_{L^2 \to L^2}, \| I^*_l(l,m) I_{(j,k)} \|_{L^2 \to L^2} \leq CA^2 \{ \gamma(j-l,k-m) \}^2,
\]
where
\[
\gamma(j_1,j_2) = \sqrt{\left\{ \frac{h(j_2)}{1 + |j_1|^{2n+1}} + \frac{h(j_1)}{1 + |j_2|^{2n+1}} \right\}}
\]
and it satisfies the estimate
\[
\sum_{(j_1,j_2) \in \mathbb{Z}^n \times \mathbb{Z}^n} \gamma(j_1,j_2) < \infty.
\]
We have the desired result, by the following Cotlar’s lemma (see Calderón and Vaillancourt [5], Stein [23, Chapter VII, Section 2]):

**Lemma 2.2.** Assume a family of \( L^2 \)-bounded operators \( \{ T_j \}_{j \in \mathbb{Z}^r} \) and positive constants \( \{ \gamma(j) \}_{j \in \mathbb{Z}^r} \) satisfy
\[
\| T_i^* T_j \|_{L^2 \to L^2} \leq \{ \gamma(i-j) \}^2, \quad \| T_i T_j^* \|_{L^2 \to L^2} \leq \{ \gamma(i-j) \}^2,
\]
and
\[
M = \sum_{j \in \mathbb{Z}^r} \gamma(j) < \infty.
\]
Then the operator
\[
T = \sum_{j \in \mathbb{Z}^r} T_j
\]
satisfies $\|T\|_{L^2 \to L^2} \leq M$. \hfill \Box

By using Theorem 2.1 on oscillatory integral operators (2.1), we can easily show the $L^2$-boundedness of Fourier integral operators of special forms. Let us begin with the case when the amplitude $a(x, y, \xi)$ is independent of the variable $y$.

**Theorem 2.2.** Let the operator $T$ be defined by

$$
(2.2) \quad Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y, \xi))} a(x, \xi) u(y) dy d\xi,
$$

where $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n_\xi)$ and $\varphi(y, \xi) \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)$. Assume that the pseudo-differential operators $a(X, D)$ defined by

$$
(2.3) \quad a(X, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot y - \xi \cdot y)} a(x, \xi) u(y) dy d\xi
$$

and the oscillatory integral operator $I_\varphi$ defined by

$$
I_\varphi u(\xi) = \int_{\mathbb{R}^n} e^{i\varphi(x, \xi)} \chi_E(\xi) u(y) dy
$$

are both $L^2(\mathbb{R}^n)$-bounded, where $\chi_E$ is the characteristic function of the set

$$
(2.3) \quad E = \bigcup_{x \in \mathbb{R}^n} E_x; \quad E_x = \text{supp} \ a(x, \cdot) \subset \mathbb{R}^n.
$$

Then $T$ is $L^2(\mathbb{R}^n)$-bounded, and satisfies

$$
\|T\|_{L^2 \to L^2} \leq (2\pi)^n \|a(X, D)\|_{L^2 \to L^2} \cdot \|I_\varphi\|_{L^2 \to L^2}.
$$

**Proof.** We remark that $T = (2\pi)^n a(X, D) F^{-1} I_\varphi$, where $F^{-1}$ is the inverse Fourier transform. The $L^2(\mathbb{R}^n)$-boundedness of $T$ is obtained from the assumptions and Plancherel’s theorem. \hfill \Box

As a corollary, we have the result announced in Ruzhansky and Sugimoto [19].

Now we recall the definition of the Besov space $B^{(s, s')}_{p, q}$ for $0 < p, q \leq \infty$ and multi-indices $(s, s')$, where $s = (s_1, \ldots, s_N)$ and $s' = (s'_1, \ldots, s'_{N'})$. Let $n = (n_1, \ldots, n_N)$, $n' = (n'_1, \ldots, n'_N)$ be splitting of $\mathbb{R}^n_x$ and $\mathbb{R}^n_{\xi}$, respectively:

$$
n = n_1 + \ldots + n_N = n'_1 + \ldots + n'_{N'}.
$$

Then $f \in B^{(s, s')}_{p, q} = B^{(s, s')}_{p, q}(\mathbb{R}^{(n, n')})$ if $f = f(x, \xi) \in \mathcal{S}'(\mathbb{R}^{2n})$ and

$$
\|f\|_{B^{(s, s')}_{p, q}} = \left\{ \sum_{j, k \geq 0} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |2^{j+ks'} \mathcal{F}^{-1} \Phi_{j, k} \mathcal{F} f(x, \xi)|^p dx d\xi \right)^{q/p} \right\}^{1/q} < \infty.
$$

Here $j = (j_1, \ldots, j_N)$, $k = (k_1, \ldots, k_{N'})$, $\mathcal{F}$ is the Fourier transform with respect to $(x, \xi)$, $\mathcal{F}^{-1}$ is the inverse Fourier transform with respect to the dual variable $(y, \eta)$, and $\Phi_{j, k} = \Phi_{j, k}(y, \eta) = \Theta_{j_1}(y_1) \cdots \Theta_{j_N}(y_N) \Theta_{k_1}(\eta_1) \cdots \Theta_{k_N}(\eta_{N'})$. Here we split variables...
y, η ∈ \mathbb{R}^n following the splitting n, n'. Functions Θ_i(z) ∈ S form the dyadic system of the corresponding dimension: \text{supp} Θ_0 ⊂ \{z; |z| ≤ 2\}, \text{supp} Θ_i ⊂ \{z; 2^{i-1} ≤ |z| ≤ 2^{i+1}\} for i ∈ \mathbb{N}, \sum_{i=0}^{∞} Θ_i(z) = 1, and 2^{|\alpha|} |\partial^\alpha \Theta_i(z)| ≤ C_α for all i ≥ 0 and all z. A natural modification is needed for p, q = \infty, see [26].

\textbf{Corollary 2.3.} Let 2 ≤ p ≤ \infty. Let the operator T be defined by (2.2), where \varphi(y, ξ) ∈ C^∞(\mathbb{R}_y^n × \mathbb{R}_ξ^n) is a real-valued function, and a(x, ξ) ∈ B^{(1/2−1/p)(n,n')}_{p,1}. Assume that, on \mathbb{R}_y^n × E, \[|\det \partial_y \partial_ξ \varphi(y, ξ)| ≥ C > 0\]
and each entry h(y, ξ) of the matrix \partial_y \partial_ξ \varphi(y, ξ) satisfies \[|\partial_y^\alpha h(y, ξ)| ≤ C_α, \quad |\partial_ξ^\beta h(y, ξ)| ≤ C_β\]
for |\alpha|, |\beta| ≤ 2n + 1, where E is the set defined by (2.3). Then T is \text{L}^2(\mathbb{R}^n)-bounded, and satisfies \[\|Tu\|_{L^2(\mathbb{R}^n)} ≤ C\|a(x, ξ)\|_{B^{(1/2−1/p)(n,n')}} \|u\|_{L^2(\mathbb{R}^n)}.\]

\textit{Proof.} The \text{L}^2\text{-boundedness of } T \text{ follows from Theorems 2.1, 2.2, and the fact that pseudo-differential operators } a(X, D) \text{ with } a(x, ξ) ∈ B^{(1/2−1/p)(n,n')} \text{ are } \text{L}^2\text{-bounded. See Sugimoto [25].} \qed

Corollary 2.3 is rather general but its conditions may be hard to check. On the other hand, conditions of the corollary below can be checked in various situations.

\textbf{Corollary 2.4.} Let the operator T be defined by (2.2), where \varphi(y, ξ) ∈ C^∞(\mathbb{R}_y^n × \mathbb{R}_ξ^n) is a real-valued function. Assume that, on \mathbb{R}_y^n × E, \[|\det \partial_y \partial_ξ \varphi(y, ξ)| ≥ C > 0\]
and each entry h(y, ξ) of the matrix \partial_y \partial_ξ \varphi(y, ξ) satisfies \[|\partial_y^\alpha h(y, ξ)| ≤ C_α, \quad |\partial_ξ^\beta h(y, ξ)| ≤ C_β\]
for |\alpha|, |\beta| ≤ 2n + 1, where E is the set defined by (2.3). Also assume that a(x, ξ) belongs to the symbol class \mathcal{S}^{0,0}_{l_0} (that is, \partial_x^\alpha \partial_ξ^\beta a(x, ξ) ∈ \text{L}^∞(\mathbb{R}_x^n × \mathbb{R}_ξ^n) for all \alpha and \beta). Otherwise assume one of the following conditions:

1. \[\partial_x^\alpha \partial_ξ^\beta a(x, ξ) ∈ \text{L}^∞(\mathbb{R}_x^n × \mathbb{R}_ξ^n) \text{ for } \alpha, \beta ∈ \{0, 1\}^n.\]
2. \[\partial_x^\alpha \partial_ξ^\beta a(x, ξ) ∈ \text{L}^∞(\mathbb{R}_x^n × \mathbb{R}_ξ^n) \text{ for } |\alpha|, |\beta| ≤ [n/2] + 1.\]
3. \[\partial_x^\alpha \partial_ξ^\beta a(x, ξ) ∈ \text{L}^∞(\mathbb{R}_x^n × \mathbb{R}_ξ^n) \text{ for } |\alpha| ≤ [n/2] + 1, \beta ∈ \{0, 1\}^n.\]
4. \[\partial_x^\alpha \partial_ξ^\beta a(x, ξ) ∈ \text{L}^∞(\mathbb{R}_x^n × \mathbb{R}_ξ^n) \text{ for } \alpha ∈ \{0, 1\}^n, |\beta| ≤ [n/2] + 1.\]
5. \[\text{There exist real numbers } \lambda, \lambda' > n/2 \text{ such that } (1 − \Delta_x)^{\lambda/2}(1 − \Delta_ξ)^{\lambda'/2}a(x, ξ) ∈ \text{L}^∞(\mathbb{R}_x^n × \mathbb{R}_ξ^n).\]
6. \[\text{There exist a real number } \lambda > 1/2 \text{ and a constant } C \text{ such that } \]
\[\|\partial_x^\alpha (h) \partial_ξ^\beta (h')a(x, ξ)\|_{\text{L}^∞(\mathbb{R}_x^n × \mathbb{R}_ξ^n)} ≤ C \prod_{i,j=1}^n |h_i|^{|\alpha|} |h_j'|^{|\beta|}.\]
Theorem 2.5. Let the operator $T$ be defined by

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y, \xi))} a(y, \xi)u(y)dyd\xi,$$

where $a(y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $\varphi(y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a real-valued function. Assume that

$$|\partial_y^\alpha \partial_\xi^\beta a(y, \xi)| \leq C_{\alpha \beta},$$

for $|\alpha|, |\beta| \leq 2n + 1$. Also assume that, on $\text{supp} a(y, \xi)$,

$$|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0$$

and each entry $h(y, \xi)$ of the matrix $\partial_y \partial_\xi \varphi(y, \xi)$ satisfies

$$|\partial_y^\alpha h(y, \xi)| \leq C_{\alpha}, \quad |\partial_\xi^\beta h(y, \xi)| \leq C_{\beta}$$

for $|\alpha|, |\beta| \leq 2n + 1$. Then the operator $T$ is $L^2(\mathbb{R}^n)$-bounded, and satisfies

$$\|T\|_{L^2 \to L^2} \leq C \sup_{|\alpha|, |\beta| \leq 2n+1} \left\|\partial_y^\alpha \partial_\xi^\beta a(y, \xi)\right\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}.$$
Proof. We remark that \( T = (2\pi)^n F^{-1} I_\varphi \), where \( F^{-1} \) is the inverse Fourier transform and \( I_\varphi \) is the oscillatory integral operator defined by

\[
I_\varphi u(\xi) = \int e^{i\varphi(x,\xi)} a(y,\xi) u(y) \, dy.
\]

The result is obtained from Theorem 2.1 and Plancherel’s theorem. \( \square \)

As a corollary of Theorems 2.2 and 2.5, we have a result for amplitudes which are of the product type.

**Corollary 2.6.** Let the operator \( T \) be defined by

\[
Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y,\xi))} a(x, y, \xi) u(y) \, dy \, d\xi,
\]

where \( a_1 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \), \( a_2 \in L^\infty(\mathbb{R}^n) \), and \( \varphi(y, \xi) \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi) \) is a real-valued function. Assume that

\[
|\partial_\alpha^a \partial_\beta^\beta a_1(x, \xi)| \leq C_{\alpha\beta}
\]

for \( |\alpha|, |\beta| \leq 2n + 1 \). Also assume that, on \( \mathbb{R}^n \times \tilde{E} \),

\[
|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0,
\]

and each entry \( h(y, \xi) \) of the matrix \( \partial_y \partial_\xi \varphi(y, \xi) \) satisfies

\[
|\partial_\alpha^a h(y, \xi)| \leq C_\alpha, \quad |\partial_\beta^\beta h(y, \xi)| \leq C_\beta
\]

for \( |\alpha|, |\beta| \leq 2n + 1 \), where

\[
\tilde{E} = \bigcup_{x, y \in \mathbb{R}^n} E_{x,y}; \quad E_{x,y} = \text{supp } a(x, y, \cdot).
\]

Then \( T \) is \( L^2(\mathbb{R}^n) \)-bounded, and satisfies

\[
\|T\|_{L^2 \to L^2} \leq C \sup_{|\alpha|, |\beta| \leq 2n + 1} \left\| \partial_\alpha^a \partial_\beta^\beta a(y, \xi) \right\|_{L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)}.
\]

Proof. Note that \( T \) is a product of the multiplication of the function \( a_2 \) and the operator defined by (2.2) or (2.4), which are all \( L^2 \)-bounded by the assumption. \( \square \)
3. Weighted $L^2$-estimates

Asada and Fujiwara [1] proved Corollary 2.6 without the product type assumption for $a(x, y, \xi)$, but assumed the boundedness of all the derivatives of $a(x, y, \xi)$ and that of each entry of the matrix $\partial_\xi \partial_\xi \varphi$. The following theorem, which is a generalized version of Theorem 1.1, says that we do not need the boundedness assumption for $\partial_\xi \partial_\xi \varphi$ if $a(x, y, \xi)$ has a decaying property. In this case, we have weighted estimates as follows. For $m \in \mathbb{R}$, we use the notation

$$\langle x \rangle^m = (1 + |x|^2)^{m/2},$$

and let $L^2_m(\mathbb{R}^n)$ be the set of functions $f$ such that the norm

$$\|f\|_{L^2_m(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\langle x \rangle^m f(x)|^2 \, dx \right)^{1/2}$$

is finite.

**Theorem 3.1.** Suppose $m_1, m_2 \in \mathbb{R}$. Let the operator $T$ be defined by

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \varphi(y, \xi))} a(x, y, \xi) u(y) \, dy \, d\xi,$$

where $a(x, y, \xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_\xi)$, and $\varphi(y, \xi) \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)$ is a real-valued function. Assume that

$$|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0 \quad \text{on } \mathbb{R}^n \times \tilde{E},$$

where

$$\tilde{E} = \bigcup_{x, y \in \mathbb{R}^n} E_{x, y}; \quad E_{x, y} = \text{supp} \, a(x, y, \cdot).$$

Also assume one of the followings:

1. For all $\alpha, \beta$, and $\gamma$,

$$|\partial_\xi^\alpha \partial_\xi^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{m_1 - |\alpha|} \langle y \rangle^{m_2},$$

and for all $|\alpha| \geq 1$ and $|\beta| \geq 1$,

$$|\partial_\varphi^\beta \varphi(y, \xi)| \leq C_{\beta} \langle y \rangle, \quad |\partial_\xi^\alpha \partial_\varphi^\beta \varphi(y, \xi)| \leq C_{\alpha\beta} \quad \text{on } \mathbb{R}^n \times \tilde{E}.$$

2. For all $\alpha, \beta$, and $\gamma$,

$$|\partial_\xi^\alpha \partial_\xi^\beta \partial_\xi^\gamma a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle y \rangle^{m_1} \langle \xi \rangle^{m_2 - |eta|},$$

and for all $\alpha$ and $|\beta| \geq 1$,

$$|\partial_\xi^\alpha \partial_\varphi^\beta \varphi(y, \xi)| \leq C_{\alpha\beta} \langle y \rangle^{1 - |\alpha|} \quad \text{on } \mathbb{R}^n \times \tilde{E}.$$

Then $T$ is bounded from $L^2_{m + m_1 + m_2}(\mathbb{R}^n)$ to $L^2_m(\mathbb{R}^n)$ for any $m \in \mathbb{R}$. 
Remark 3.1. From the assumptions for phase functions $\varphi$ in Theorem 3.1 we obtain the estimate

\begin{equation}
(3.2) \quad C_1\langle y \rangle \leq \langle \partial_\xi \varphi(y, \xi) \rangle \leq C_2\langle y \rangle \quad \text{on} \quad \mathbb{R}^n \times \tilde{E}
\end{equation}

for some $C_1, C_2 > 0$. In fact, the estimate $\langle \partial_\xi \varphi(y, \xi) \rangle \leq C_2\langle y \rangle$ is obtained from any assumptions (1) or (2). Especially, we have $\langle \partial_\xi \varphi(0, \xi) \rangle \leq C$. From the expression

$$\partial_\xi \varphi(y, \xi) - \partial_\xi \varphi(0, \xi) = \partial_y \partial_\xi \varphi(z, \xi) y$$

with some $z \in \mathbb{R}^n$, we obtain

$$|y| \leq C |\partial_\xi \varphi(y, \xi) - \partial_\xi \varphi(0, \xi)|$$

$$\leq C |\partial_\xi \varphi(y, \xi)| + C |\partial_\xi \varphi(0, \xi)|$$

by the assumptions for $\varphi$. Hence we have the estimate $C_1\langle y \rangle \leq \langle \partial_\xi \varphi(y, \xi) \rangle$, as well.

Proof. We show the $L^2$-boundedness of the operator $T_b$ defined by

$$T_b u(x) = \int \int e^{i(x-x+\varphi(y, \xi))} b(x, y, \xi) u(y) dy d\xi,$$

where

$$b(x, y, \xi) = \langle x \rangle^m a(x, y, x) \langle y \rangle^{-(m_1+m_2)}.$$ 

By using the cut-off function $\chi(x) \in C_0^\infty(|x| \leq 1/2)$ which is equal to one near the origin, we decompose $b$ into two parts:

$$b^I(x, y, \xi) = b(x, y, \xi) \chi((x + \partial_\xi \varphi(x, y, \xi))/\langle \partial_\xi \varphi(x, y, \xi) \rangle),$$

$$b^{II}(x, y, \xi) = b(x, y, \xi) (1 - \chi)((x + \partial_\xi \varphi(x, y, \xi))/\langle \partial_\xi \varphi(x, y, \xi) \rangle).$$

The corresponding decomposition of the operator $T_b$ is denoted by $T^I$ and $T^{II}$ respectively.

On the support of $b^I(x, y, \xi)$, we have $|x + \partial_\xi \varphi(x, y, \xi)| \leq (1/2)\langle \partial_\xi \varphi(x, y, \xi) \rangle$, hence we have the estimates

$$|x| \leq \langle \partial_\xi \varphi(x, y, \xi) \rangle + \frac{1}{2} \langle \partial_\xi \varphi(x, y, \xi) \rangle,$$

$$|\partial_\xi \varphi(x, y, \xi)| \leq |x| + \frac{1}{2} \langle \partial_\xi \varphi(x, y, \xi) \rangle.$$

From the first estimate and estimate (3.2), we obtain $\langle x \rangle \leq C \langle y \rangle$. From the second estimate, we obtain $\langle \partial_\xi \varphi(y, \xi) \rangle \leq 2\langle x \rangle + (1/2)\langle \partial_\xi \varphi(y, \xi) \rangle$, hence $\langle \partial_\xi \varphi(y, \xi) \rangle \leq 4\langle x \rangle$, which implies $\langle y \rangle \leq C \langle x \rangle$ by (3.2) again. Thus we have the equivalence of $\langle y \rangle$ and $\langle x \rangle$, and obtain

\begin{equation}
(3.3) \quad |\partial_\xi \partial_y^\alpha \partial_\xi^\gamma b^I(x, y, \xi)| \leq C_{\alpha \gamma} \langle x \rangle^{-|\alpha|}
\end{equation}

or

\begin{equation}
(3.4) \quad |\partial_\xi \partial_y^\alpha \partial_\xi^\gamma b^I(x, y, \xi)| \leq C_{\alpha \gamma} \langle y \rangle^{-|\beta|}
\end{equation}

from the assumptions (1) and (2) respectively.

We assume estimate (3.3). Otherwise, assume (3.4) and just change the role of $x$ and $y$ below. Let real-valued positive functions $\Phi_0(x)$, $\Phi_k(x) = \Phi(x/2^k)$ $(k \in \mathbb{N})$ form a partition of unity which satisfy $\text{supp} \Phi_0 \subset \{x; |x| < 2\}$, $\text{supp} \Phi \subset \{x; 1/2 < |x| < 2\}$.
We remark that by Corollary 2.6. Hence we have
\[ b_k^l(x, y, \xi) = \hat{\Phi}_k(x)b^l(x, y, \xi) \hat{\Psi}_k(y) \]
with functions \( \hat{\Psi}_k \in \mathcal{C}_0^\infty(\mathbb{R}^n) \) which are of the from \( \hat{\Psi}_k(y) = \hat{\Psi}(y/2^k), \hat{\Psi} \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus 0) \)
with large \( k \). Furthermore, we have
\[ b_k^l(2^k x, y, \xi) = \Psi_k(2^k x) \sum_{l \in \mathbb{Z}^n} e^{it \cdot x} b_{kl}(y, \xi) \hat{\Psi}_k(y), \]
where \( \Psi_k \) is the characteristic function of the support of \( \Phi_k \), and
\[ b_{kl}(y, \xi) = \int e^{-it \cdot x} b_k^l(2^k x, y, \xi) \, dx \]
does not depend of \( l \). Furthermore, we have
\[ \left| \frac{\partial^\alpha_x \partial^\beta_\xi b_{kl}(y, \xi)}{\partial^\alpha_x \partial^\beta_\xi} \right| \leq C_{\alpha\beta}(1 + |l|^2)^{-n}, \]
where \( C_{\alpha\beta} \) is independent of \( k, l \in \mathbb{Z}^n \). Thus we have the decomposition
\[ T^l = \sum_{l \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} e^{it \cdot x/2^k} \Psi_k T_{kl} \hat{\Psi}_k, \]
where
\[ T_{kl}v(x) = \int \int e^{i(x + (k - l))} b_{kl}(y, \xi) v(y) \, dy \, d\xi. \]
We remark that
\[ \left\| \sum_{k \in \mathbb{Z}^n} e^{it \cdot x/2^k} \Psi_k T_{kl} \hat{\Psi}_k u \right\|_{L^2}^2 \leq C \sum_{k \in \mathbb{Z}^n} \left\| \Psi_k T_{kl} \hat{\Psi}_k u \right\|_{L^2}^2 \]
\[ \leq C \sup_{k \in \mathbb{Z}^n} \left\| T_{kl} \right\|_{L^2 \to L^2}^2 \sum_{k \in \mathbb{Z}^n} \left\| \Psi_k u \right\|_{L^2}^2 \]
\[ \leq C \left( 1 + |l|^2 \right)^{-2n} \left\| u \right\|_{L^2}^2 \]
by Corollary 2.6. Hence we have
\[ \left\| T^l \right\|_{L^2 \to L^2} \leq C \sum_{l \in \mathbb{Z}^n} (1 + |l|^2)^{-n} \]
\[ \leq C, \]
that is, the \( L^2 \)-boundedness of \( T^l \).
Next, we show the boundedness of \( T^{\xi} \). Let \( \rho \in \mathcal{C}_0^\infty \) be a real-valued function which satisfies
\[ \sum_{k \in \mathbb{Z}^n} \rho(\xi - k) = 1. \]
We decompose \( b^{\xi}(x, y, \xi) \) into the sum of
\[ b_k^{\xi}(x, y, \xi) = b^{\xi}(x, y, \xi) \rho(\xi - k) \]
We claim, we may replace $b_k^I(x, y, \xi)$ by the symbol (denoted by $b_k^H(x, y, \xi)$) again which has the same (or smaller) support and satisfies the estimate

$$\quad |\partial_\xi^\alpha \partial_y^\beta \partial_\xi^\gamma b_k^I(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle x \rangle^{-(n+1)} \langle y \rangle^{-(n+1)},$$

where $C_{\alpha\beta\gamma}$ is independent of $k \in \mathbb{Z}^n$. Indeed, by integration by parts we have

$$T_ku(x) = \int \int e^{i(x\xi + \varphi(y, \xi))} b_k^H(x, y, \xi) u(y) dyd\xi,$$

where $L$ is the transpose of the operator

$$^tL = \frac{x + \partial_\xi \varphi}{i|x + \partial_\xi \varphi|^2} \cdot \partial_{\xi}$$

and $N$ is a positive integer. We have $\langle \partial_\xi \varphi(y, \xi) \rangle \leq C|x + \partial_\xi \varphi(y, \xi)|$ on the support of $b_k^H(x, y, \xi)$, hence we have

$$\langle x \rangle \leq |x + \partial_\xi \varphi(y, \xi)| + 2\langle \partial_\xi \varphi(y, \xi) \rangle \leq C|x + \partial_\xi \varphi(y, \xi)|,$$

$$\langle y \rangle \leq C\langle \partial_\xi \varphi(y, \xi) \rangle \leq C|x + \partial_\xi \varphi(y, \xi)|$$

by estimate (3.2). Thus $|x + \partial_\xi \varphi|^{-1}$ is dominated by $\langle x \rangle^{-1}$ and $\langle y \rangle^{-1}$, and we can justify our claim by taking large $N$.

Let $T_k^*$ be the adjoint of $T_k$, and we have

$$T_k^*T_k u(x) = \int K_{kl}(x, y) u(y) dy, \quad T_k^*T_k u(x) = \int \tilde{K}_{kl}(x, y) u(y) dy,$$

where

$$K_{kl}(x, y) = \int \int e^{i(x\xi - y\eta + \varphi(z, \xi) - \varphi(z, \eta))} b_k^H(x, z, \xi) b_l^H(y, z, \eta) dzd\xi d\eta,$$

$$\tilde{K}_{kl}(x, y) = \int \int e^{i(\varphi(y, \xi) - \varphi(x, \xi)) + (\varphi(z, \eta) - \varphi(z, \xi))} b_k^H(z, y, \xi) b_l^H(z, x, \eta) dzd\xi d\eta.$$
for all $\beta$. From this argument and (3.5), we obtain

$$|K_{kl}(x, y)| \leq C(x)^{-n(n+1)}(y)^{-n(n+1)}(1 + |k - l|^{2n+1})^{-1},$$

where $C$ is independent of $k, l \in \mathbb{Z}^n$. Then we have

$$\sup_x \int |K_{kl}(x, y)| dy \leq C(1 + |k - l|^{2n+1})^{-1},$$

$$\sup_y \int |K_{kl}(x, y)| dx \leq C(1 + |k - l|^{2n+1})^{-1}$$

which implies, by Lemma 2.1

$$\|T_k T_l^*\|_{L^2 \to L^2} \leq C(1 + |k - l|^{2n+1})^{-1}.$$

Similarly, we have

$$\|T_k^* T_l\|_{L^2 \to L^2} \leq C(1 + |k - l|^{2n+1})^{-1}$$

if we take

$$\iota L = \frac{1}{i} \frac{\xi - \eta}{|\xi - \eta|^2} : \partial_z.$$

Then we have

$$\|T_k T_l^*\|_{L^2 \to L^2}, \|T_k^* T_l\|_{L^2 \to L^2} \leq C\{\gamma(k - l)\}^2,$$

where

$$\gamma(j) = (1 + |j|^{2n+1})^{-1/2}$$

and it satisfies the estimate

$$\sum_{j \in \mathbb{Z}^n} \gamma(j) < \infty.$$

By Lemma 2.2, we have the $L^2$-boundedness of $T^H$. \[\square\]

### 4. Applications

In this section, we explain how to use Theorem 3.1 to show the smoothing effect of generalized Schrödinger equations. The main tool is a class of Fourier integral operators of the form

$$T_\psi u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - y \cdot \psi(\xi))} u(y) dy \, d\xi,$$

(4.1)

$$T_\psi^{-1} u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - y \cdot \psi^{-1}(\xi))} u(y) dy \, d\xi,$$

where $\psi, \psi^{-1} : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ are $C^\infty$-maps satisfying $\psi \circ \psi^{-1}(\xi) = \psi^{-1} \circ \psi(\xi) = \xi$, $\psi(\lambda \xi) = \lambda \psi(\xi)$, and $\psi^{-1}(\lambda \xi) = \lambda \psi^{-1}(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbb{R}^n \setminus 0$. We remark that we have

$$T_\psi u(x) = F_\xi^{-1}[(F_x u)(\psi(\xi))](x), \quad T_\psi^{-1} u(x) = F_\xi^{-1}[(F_x u)(\psi^{-1}(\xi))](x),$$

(4.2)

where $F_x$ ($F_\xi^{-1}$ resp.) denotes the (inverse resp.) Fourier transform. Hence, we have $T_\psi^{-1} \cdot T_\psi = T_\psi \cdot T_\psi^{-1} = id$, and the formula

$$T_\psi \cdot a(D) \cdot T_\psi^{-1} = (a \circ \psi)(D),$$

(4.3)
where \( a(D) = F^{-1}_x a(\xi) F_x \). By (4.2) and Plancherel’s theorem, the operators \( T_\psi \) and \( T_{\psi}^{-1} \) are \( L^2 \)-bounded. Furthermore, as a corollary of Theorem 3.1, we have the following:

**Corollary 4.1.** Suppose \( m \in \mathbb{Z} \) and \( |m| < n/2 \). Assume that \( |\det \partial \psi(\xi)| \geq C > 0 \). Then the operators \( T_\psi \) and \( T_{\psi}^{-1} \) defined by (4.1) are \( L^2_m(\mathbb{R}^n) \)-bounded.

**Remark 4.1.** By Corollary 4.1 and the interpolation, we have the \( L^2_m(\mathbb{R}^n) \)-boundedness of \( T_\psi \) and \( T_{\psi}^{-1} \) with \( m \in \mathbb{R} \) such that \( |m| \leq [n/2] \) ([\( k \)] denotes the greatest integer less than \( k \)).

**Proof.** We prove the boundedness of \( T_\psi \), from which the boundedness of \( T_{\psi}^{-1} \) follows. Let \( \chi(\xi) \in C^\infty_0 \) be a cut off function of the origin. By (4.2), we have

\[
(1 - \chi(D)) T_\psi u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - y \cdot \psi(\xi))}(1 - \chi(\xi))u(y)dyd\xi.
\]

Since \( \psi(\xi) \) is smooth away from the origin, \( (1 - \chi(D))T_\psi \) is \( L^2_m \)-bounded by Theorem 3.1. On the other hand, if we note

\[
e^{ix \cdot \xi} = \frac{1 - ix \cdot \partial_x}{\langle x \rangle^2} e^{ix \cdot \xi}, \quad e^{-iy \cdot \xi} = \frac{1 + iy \cdot \partial_y}{\langle y \rangle^2} e^{-iy \cdot \xi},
\]

we have, by change of variables and integration by parts,

\[
\chi(D) T_\psi u(x) = (2\pi)^{-n} \int e^{i(x \cdot \xi - y \cdot \psi(\xi))} \chi(\xi)u(y)dyd\xi
\]

\[
= (2\pi)^{-n} \int e^{i(x \cdot \xi - y \cdot \psi(\xi))} \left( \frac{1 + ix \cdot \partial_x}{\langle x \rangle^2} \chi(\xi) + x \chi(\xi) \partial_x \psi(\xi) \right)u(y)dyd\xi
\]

\[
= \frac{1}{\langle x \rangle^2} T_\psi u + \frac{x}{\langle x \rangle^2} \cdot \partial_x \chi(D) T_\psi u + \frac{x}{\langle x \rangle^2} \chi(D)^{t} \partial_y \psi(D) T_\psi (\xi x u)
\]

and

\[
\chi(D) T_\psi u(x) = (2\pi)^{-n} \int e^{i(x \cdot \psi^{-1}(\xi) - y \cdot \xi)} \chi(\psi^{-1}(\xi)) |\det \partial \psi^{-1}(\xi)| u(y)dyd\xi
\]

\[
= (2\pi)^{-n} \int e^{i(x \cdot \psi^{-1}(\xi) - y \cdot \xi)} \left( \frac{1 + a(\xi) \cdot y + x A(\xi)^{t} y}{\langle y \rangle^2} \right) u(y)dyd\xi
\]

\[
= d(D) T_\psi \left( \frac{x u}{\langle x \rangle^2} \right) + |D|^{-1} d(D) A(\psi(D)) d(D) \cdot T_\psi \left( \frac{x}{\langle x \rangle^2} u \right)
\]

\[
+ x A(\psi(D)) d(D) T_\psi \left( \frac{t x}{\langle x \rangle^2} u \right)
\]

where

\[
A(\xi) = \chi(\psi^{-1}(\xi)) |\det \partial \psi^{-1}(\xi)| \partial \psi^{-1}(\xi), \quad a(\xi) = -i \partial \left\{ \chi(\psi^{-1}(\xi)) |\det \partial \psi^{-1}(\xi)| \right\}, \quad d(\xi) = |\det \partial \psi(\xi)|.
\]
We remember here that $T_\psi$ is $L^2$-bounded. Assume that $T_\psi$ is $L^2_{\pm(k-1)}$-bounded with some $k < n/2$, $k \in \mathbb{N}$. We remark that $\chi(D)$, $d(D)$ and all entries of $\partial \chi(D)$, $\partial \psi(D)$, $A(\psi(D))$, $|D|a(\psi(D))$ are $L^2_{\pm(k-1)}$-bounded, and $|D|^{-1}$ is bounded from $L^2_{-1}$ to $L^2_{-k}$. To justify these boundedness, use the results of Kurtz and Wheeden [18], Stein and Weiss [24]. Using them, we obtain the $L^2_k$-boundedness of $\chi(D)T_\psi$ from (4.4), and $L^2_{-1}$-boundedness from (4.5). Then, by induction, we have the desired result.

Now, let $p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ be a positive function which satisfies $p(\lambda \xi) = \lambda p(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbb{R}^n \setminus 0$, and let

$$L_p = p(D_x)^2 = F^{-1}_p p(\xi)^2 F_x$$

be the corresponding Fourier multiplier. Assume that $\Sigma = \{ \xi; p(\xi) = 1 \}$ has non-vanishing Gaussian curvature. We consider a generalized Schrödinger equation

$$\begin{cases}
(i\partial_t + L_p)u(t, x) = 0, \\
u(0, x) = f(x).
\end{cases}$$

If we take

$$\psi(\xi) = p(\xi) \frac{\nabla p(\xi)}{|\nabla p(\xi)|},$$

we have the relation

$$T_\psi \cdot (-\Delta_x) \cdot T_\psi^{-1} = L_p$$

by (4.8), and the $L^2_{-1}$-boundedness of the operators $T_\psi$ and $T_\psi^{-1}$ by Corollary 4.1. In fact, the curvature condition on $\Sigma$ means that the Gauss map

$$\frac{\nabla p}{|\nabla p|} : \Sigma \to S^{n-1}$$

is a global diffeomorphism and its Jacobian never vanishes (see Kobayashi and Nomizu [16]). Hence, we can construct the inverse $C^\infty$-map $\psi^{-1}(\xi)$ of $\psi(\xi)$, and can justify the assumption of Corollary 4.1. Applying $T_\psi^{-1}$ defined by (4.1) with (4.7) to equation (4.6), and introducing $v = T_\psi^{-1}u$ and $g = T_\psi^{-1}f$, (4.6) can be transformed to the equation

$$\begin{cases}
(i\partial_t - \Delta_x)v(t, x) = 0, \\
v(0, x) = g(x),
\end{cases}$$

by (4.8). It has been already known that classical Schrödinger equation (4.9) has the global smoothing estimate

$$\|\sigma(X, D)v\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C\|g\|_{L^2(\mathbb{R}_x^2)},$$

where $n \geq 3$ and

$$\sigma(X, D) = \langle x \rangle^{-1}\langle D \rangle^{1/2}.$$  

See Ben-Artzi and Klainerman [3], Simon [22], Kato and Yajima [15], or Walther [27]. From this fact, we can extract a similar estimate for generalized Schrödinger equation (4.9). In fact, we have

$$\langle D \rangle^{1/2}u = M(1 + p(D)^2)^{1/4}T_\phi v = MT_\phi \langle D \rangle^{1/2}v.$$
where

\[ M = \langle D \rangle^{1/2} \left( 1 + p(D)^2 \right)^{-1/4}. \]

Here we have used the formula (4.3) with \( a(\xi) = (1 + |\xi|^2)^{1/4} \). Hence we have

\[ \sigma(X, D)u = \langle x \rangle^{-1} MT_\psi \langle x \rangle \sigma(X, D)v. \]

Since \( M \) is \( L^2 \)-bounded by Theorem 1.1, and \( T_\psi \) by Corollary 4.1, we obtain

\[ \| \sigma(X, D)u \|_{L^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)} \]

from estimates (4.10) and

\[ \| g \|_{L^2(\mathbb{R}^n)} = \| T_\psi^{-1} f \|_{L^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}. \]

Thus, we have obtained the following result which was partially proved for a type of polynomial \( p(\xi)^2 \) by Ben-Artzi and Devinatz [2], and fully for radially symmetric \( p(\xi)^2 \) by Walther [28].

**Theorem 4.2.** Suppose \( n \geq 3 \). Assume that \( \Sigma = \{ \xi; p(\xi) = 1 \} \) has non-vanishing Gaussian curvature. Then the solution \( u(t, x) \) to equation (4.6) has the estimate

\[ \| \langle x \rangle^{-1} \langle D \rangle^{1/2} u \|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}. \]

In Theorem 4.2, the order “– 1” for the weight is the best possible one because of the estimate for the low frequency part (Walther [27], [28]). But, if we replace \( \langle D \rangle^{1/2} \) by \( |D|^{1/2} \), we have another type of estimate

\[ \| \langle x \rangle^{-\delta} |D|^{1/2} u \|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)} \]

for \( \delta > 1/2 \). Chihara [6] obtained this type of estimates for rather general \( p(\xi)^2 \). In our forthcoming paper [21], we use our main result Theorem 3.1 to obtain a refinement of this estimate.

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