New Anisotropic Sudden Singularities and Dimensional Reduction

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Abstract

We demonstrate the existence of sudden, finite-time singularities, with constant scale factor, expansion rate and density, in expanding Bianchi type IX universes with free anisotropic pressures. A new type of non-simultaneous anisotropic sudden singularity arises because of the divergences of the pressures, which may be of barrel or pancake type. The effect of one or more directions of expansion hitting a sudden singularity is tantamount to dimensional reductions as the non-singular directions continue expanding and can see the sudden singularity in their past.

1 Introduction

Sudden cosmological singularities, first introduced in ref. [1] and developed systematically in refs. [2, 3, 4, 5, 6, 7, 8], and reviewed in refs. [9], have attracted widespread interest. They appear in a wide range of gravity theories and their solutions. They typically occur when the pressure and scale factor acceleration diverge at a finite time, $t_s$, while the scale factor, density and Hubble expansion rate remain finite. Thus all terms in the Friedman equation, or its equivalent in other theories of gravity, remain finite whilst finite-time singularities occur in the acceleration and conservation equations.

Sudden singularities and their generalised counterparts [3] are weak singularities in the senses of Tipler [10] and Krolak [11] and their conformal diagrams have been constructed in ref. [12]. Geodesics are unscathed by sudden singularities, [13], and the general behaviour of the Einstein and geodesic equations in their neighbourhood was found in refs. [5, 6, 14]. This behaviour appears robust in the presence of quantum particle production [7]. The first examples were existence proofs that required unmotivated pressure-density relations so that the density could remain finite while the pressure diverged. However, more
recently, generalised singularities of this sort have been found by Barrow and Graham [15] and appear in simple isotropic Friedman universes with a scalar field having power-law self-interaction potentials for a scalar field, \( \phi \), of the form:

\[
V(\phi) = V_0 \phi^n, \quad 0 < n < 1.
\]

They always develop a finite-time singularity where the Hubble rate and its first derivative are finite, but its second derivative diverges. For non-integer \( n > 1 \), there is a class of models with even weaker singularities. Infinites first occur at a finite time in the \((k + 2)^{th}\) time-derivative of the Hubble expansion rate, where \( k < n < k + 1 \) and \( k \) is a positive integer [15]. These models inflate but inflation ends in a singular fashion.

In this paper we study a new effect in anisotropic cosmological models experiencing non-simultaneous sudden singularities in all, or some, of their directional scale factors. This can create a form of dimensional reduction in which some directional scale factors experience sudden singularities, while others do not. Those that do not experience the singularities continue expanding as if in a lower-dimensional universe. We use the Bianchi type IX, 'Mixmaster', universe expanding away from the initial strong curvature at \( t = 0 \) to illustrate this point and derive the general forms of the evolution of the three expansion scale factors.

In what follows we set \( c = 1 = 8\pi G \). Planck’s constant does not appear and our study is entirely classical. Quantum features can be studied using our paper ref. [7].

2 The Mixmaster model equations

The spatially homogeneous diagonal Bianchi IX metric is [16],

\[
ds^2 = dt^2 - \gamma_{ab}(t)\epsilon^a_\mu e^b_\nu dx^\mu dx^\nu, \tag{1}
\]

where

\[
\gamma_{ab}(t) = \text{diag}[a^2(t), b^2(t), c^2(t)], \tag{2}
\]

and

\[
\epsilon^a_\mu = \begin{pmatrix}
\cos z & \sin z \sin x & 0 \\
-\sin z & \cos z \sin x & 0 \\
0 & \cos x & 1
\end{pmatrix}. \tag{3}
\]

The general relativistic field equations in vacuum Bianchi type IX with scale factors \( a(t), b(t), c(t) \) and matter with density \( \rho \) and anisotropic pressures \( p_1, p_2 \) and \( p_3 \), [16,17,18], are most simply expressed by introducing the \( \tau \) time defined in terms of the comoving proper time, \( t \), by

\[
d\tau = \frac{dt}{abc}, \tag{4}
\]

and \( ' \) denotes \( d/d\tau \). The field equations are
\[
\frac{1}{2} \left( \frac{a'b'}{ab} + \frac{b'c'}{bc} + \frac{a'c'}{ac} \right) = a^4 + b^4 + c^4 - 2a^2b^2 - 2c^2(b^2 + a^2) + 4a^2b^2c^2 \rho. \tag{6}
\]

We see that when \( a = b = c \) it reduces to

\[
12 \frac{a^2}{a^2} = 4a^6 \rho - 3a^4.
\]

Restoring the cosmic time derivative \( a^2 \dot{a} = da/d\tau \), we have (overdot denotes \( d/dt \)) the standard closed isotropic universe’s Friedman equation after the coordinate transform \( a \rightarrow \frac{4}{a} \):

\[
3 \frac{\dot{a}^2}{a^2} = \frac{\rho}{3} - \frac{1}{a^2}.
\]

When all the quartic terms are dropped in eqs. (5) then they are just like the Bianchi I equations with a curvature term that will dominate the matter at late times so long as \( p_i > -\rho/3 \). We looked at the flat Bianchi I and VII_0 models in ref.([4]).

### 3 The sudden singularity scale factor evolutions

In order to establish the existence of anisotropic sudden singularities in the Mixmaster metric at late time we look for the following forms for the scale factors on approach to a finite-time singularity \( s \rightarrow t_s \) from below. The scale factors, their first time derivatives and the density, \( \rho \), will be assumed to be finite at \( t_s \), but second derivatives of the scale factors, first derivatives of the density, and the principal pressures will be allowed to diverge. Thus, we assume that the asymptotic forms of the scale factors as \( t \rightarrow t_s \) have the form that we know is part of the general solution of the Einstein equations [5, 6]. So, all terms in eq. (6) will be finite and eqs. (5) reduce asymptotically to:

\[
(\ln a^2)'' = -a^2b^2c^2 \rho = -C_a p_a, \tag{7}
\]

et cycl. In this \( t \rightarrow t_s \) limit we also have

\[
a'' \rightarrow \ddot{a} a^2b^2c^2, \tag{8}
\]

et cycl., and so we have the simple system of asymptotic equations:

\[
(\ln a^2) \rightarrow -p_a. \tag{9}
\]

Therefore, explicitly in the limit \( t \rightarrow t_s \) we have,
\[
\left( \begin{array}{c}
\ddot{a} \\
\ddot{b} \\
\ddot{c}
\end{array} \right) \rightarrow -\frac{1}{2}(p_a, p_b, p_c).
\]  
(10)

However, we want to allow the sudden singularity to arise at different times for the motions in the directions of the different scale factors, so we introduce three sudden singularity times, \( t_{sa}, t_{sb} \) and \( t_{sc} > 0 \), thus:

\[
a(t) = \left( \frac{t}{t_{sa}} \right)^{q_a} (a_{sa} - 1) + 1 - \left( 1 - \frac{t}{t_{sa}} \right)^{n_a},
\]
(11)

\[
b(t) = \left( \frac{t}{t_{sb}} \right)^{q_b} (a_{sb} - 1) + 1 - \left( 1 - \frac{t}{t_{sb}} \right)^{n_b},
\]
(12)

\[
c(t) = \left( \frac{t}{t_{sc}} \right)^{q_c} (a_{sc} - 1) + 1 - \left( 1 - \frac{t}{t_{sc}} \right)^{n_c}.
\]
(13)

Here, the constants \( 0 < q_a, q_b, q_c < 1 \) and \( 1 < n_a, n_b, n_c < 2 \). Therefore,

\[
\ddot{a} = \frac{q_a(q_a - 1)}{t_{sa}^2} \left( \frac{t}{t_{sa}} \right)^{q_a-2} - n_a(n_a - 1)(t_{sa} - t)^{n_a-2}
\]
(14)

and the forms for \( \ddot{b}(t) \) and \( \ddot{c}(t) \) are given by in the same form after substitution of the \( q \)'s and \( n \)'s. The values of \( \ddot{a}, \ddot{b}(t), \) and \( \ddot{c}(t) \) can each diverge if the values of \( n_a, n_b, n_c \) are less than two and greater than one, as assumed. This will create complementary divergences in the values of the pressures because of eq. (10). When \( n_a = n_b = n_c \), this is similar to the behaviour in the Friedman model, first described in (3). However, it is possible to create new types of anisotropic sudden singularity in acceleration and associated principal pressure by making \( n_a \neq n_b \neq n_c \), or allow some directions to avoid a sudden singularity while others experience it. The possibilities are:

1. \( 1 < n_a, n_b, n_c < 2 \) : sudden singularities in all directions and their associated principal pressures at different times if \( t_{sa}, t_{sb} \) and \( t_{sc} \) are unequal, simultaneously if they are equal.

2. \( 1 < n_a, n_b < 2 \) and \( n_c > 2 \) and similar for the other two permutations: sudden singularities in two directions and their associated principal pressures (but not in the third).

3. \( 1 < n_a < 2 \) and \( n_b, n_c > 2 \) and similar for the other two permutations: sudden singularity in one direction and its principal pressure (but not in the third).

\(^{1}\)Technically, we can create a slightly more general but rather cumbersome form by including powers of logarithms. Thus, for \( a(t) \), we would have

\[
a(t) = \left( \frac{t}{t_{sa}} \right)^{q_a} (a_{sa} - 1) + 1 - \left( 1 - \frac{t}{t_{sa}} \right)^{n_a} \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} a_{jk}(t_{sa} - t)^j/Q \log^k[t_{sa} - t]
\]

where \( N_j \leq j \) is a positive integer and \( Q \) is a positive rational. For the corresponding expressions for \( b(t) \) and \( c(t) \) we replace \( a_{jk}, N_j, \) and \( Q \) by different independent constants satisfying the same inequalities.
For example, in the second case (2), we have a non-simultaneous sudden singularity in the \( a \) and \( b \) directions, with

\[
\begin{align*}
a(t) &= \left( \frac{t}{t_{sa}} \right)^{qa} (a_{sa} - 1) + 1 \rightarrow a_{sa}, \\
\end{align*}
\]

\[
\begin{align*}
b(t) &= \left( \frac{t}{t_{sa}} \right)^{qb} (a_{sb} - 1) + 1 \rightarrow a_{sb},
\end{align*}
\]

while the expansion parallel to the \( c \) direction continues with \( n_c > 2 \), and there is no singularity in \( c(t) \) as \( t \rightarrow t_{sc} \) and so it continues past the singularities affecting particles moving parallel to the \( a \) and \( b \) directions for \( t > t_{sc} \),

\[
\begin{align*}
c(t) &= \left( \frac{t}{t_{sc}} \right)^{qc} (a_{sc} - 1) + 1 - \left( 1 - \frac{t}{t_{sc}} \right)^{nc} \rightarrow \left( \frac{t}{t_{sc}} \right)^{qc} (a_{sc} - 1) + 1 - \left( 1 - \frac{t}{t_{sc}} \right)^{nc} \\
\end{align*}
\]

The relative values of \( q_c \) and \( n_c \) determine this evolution. Typically we expect \( n_c > q_c \) and so

\[
\begin{align*}
c(t) &\rightarrow \left( \frac{t}{t_{sc}} \right)^{qc} (a_{sc} - 1) + 1 - \left( 1 - \frac{t}{t_{sc}} \right)^{nc} \rightarrow 1 - \left( \frac{t}{t_{sc}} \right)^{nc} (1)_{nc} \end{align*}
\]

For example, for odd \( n_c \), we have for \( t > t_{sc} \),

\[
\begin{align*}
c(t) &\rightarrow \left( \frac{t}{t_{sc}} \right)^{nc} .
\end{align*}
\]

For even \( n_c \), we have for \( t > t_{sc} \), from (17),

\[
\begin{align*}
c(t) &\rightarrow \left( \frac{t}{t_{sc}} \right)^{qc} (a_{sc} - 1) - 1 + \left( \frac{t}{t_{sc}} \right)^{nc} ,
\end{align*}
\]

and the possibility of a switch between the two asymptotic time-dependencies.

4 Dimensional reduction and its physical interpretation

The possibility of sudden singularities occurring anisotropically at different times is a new feature of this phenomenon. There are no strong curvature singularities associated with any of the sudden singularities and we expect geodesics to be unscathed by the experience unless the underlying expansion anisotropy contributed strong tidal forces [13]. The appearance of finite-time singularities for the motion of only some of the expansion scale factors is created by the anisotropic pressures. It means that particles moving in the singular directions will hit the pancake or barrel-like sudden singularity, leaving those moving in
the directions orthogonal to them unscathed. This has an interesting consequence. Suppose that we repeated our calculations for anisotropic cosmological models with many space dimensions, $N$, then we might have $S$ of those dimensions experiencing sudden singularities (not necessarily all at the same time), eventually leaving $N-S$ to continue expanding. In effect, this is a cosmological dimensional reduction process. If $N-S = 3$ then we would be left with a 3-dimensional expanding space. Observers in that space could in principle look back down their past lightcones and see consequences of the sudden singularities in the other non-evolving dimensions. This will have consequences for the constants of Nature. If the true constants are defined in the $N$-dimensional space then in all subspaces of lesser dimension, the apparent constants in their space will be seen to evolve in time on the same timescale that the extra dimensions change on. Observers in an $(N-S)$-dimensional expanding subspace will at first see small variations in quantities like their local fine structure 'constant', or the Newtonian gravitation 'constant', following the overall volume expansion. But when the extra dimensions hit their finite-time singularities there could be dramatic evolution of the local 3-dimensional constants [20, 21, 22, 23]. However, the sudden singularities are characterised by the scale factors tending to constant values at the singularity. Since the evolution of the local 'constants' is determined by inverse powers of the mean scale of the extra dimensions, there will no dramatic evolution of the values of local constants towards zero or infinity [24, 25]. The global structure of singular cosmological models of this type may prove to be interesting and quite different to that accompanying strong curvature singularities. We don’t expect quantum particle production effects, or their classical analogue of bulk viscosity, because the local expansion rates on which these processes depend are assumed constant on approach to the sudden singularity [7]. Higher-order versions of sudden singularities [3, 15] will also be possible in these cosmological models.

5 Conclusions

We have studied the presence of sudden singularities, with finite scale factors, expansion rates and matter densities in the most general closed spatially homogeneous universes of Bianchi type IX. They permit divergences in scale factor accelerations an pressures at finite-time singularities, where no curvature invariants diverge. In the presence of anisotropic pressures we found a new variety of non-simultaneous directional sudden singularity which can occur in all or any of the expanding directions. This allows expansion in some directions to end at a sudden singularity while those in other non-singular directions do not. This creates a new form of dimensional reduction driven by the anisotropic pressures, some of which may diverge at finite time while others remain finite. The expansion continues unaffected in the non-singular directions and the sudden singularities and their consequences could be observed in the past of the non-singular directions.

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References

[1] J.D. Barrow, Galloway G. and Tipler F.J., Mon. Not. R. Astron. Soc. 223, 835 (1986)
[2] J.D. Barrow, Class. Quantum Gravity 21, L79 (2004)
[3] J.D. Barrow, Class. Quantum Gravity 21, 5619 (2004)
[4] J.D. Barrow and C.G. Tsagas, Class. Quantum Gravity 22, 1563 (2005)
[5] J.D. Barrow, S.T. Cotsakis and A.Tsokaros, Class. Quantum Gravity 27, 165017 (2010)
[6] J.D. Barrow, and S.T. Cotsakis, Phys. Rev. D 88, 067301 (2013)
[7] J.D. Barrow, A.B. Batista, J.C. Fabris and S. Houndjo, Phys. Rev. D 78, 123508 (2008)
[8] J.D. Barrow, Classical and Quantum Gravity 37, 065014 (2020)
[9] S. Nojiri and S.D. Odintsov, Phys. Rev. D 78, 046006 (2008)
[10] F.J. Tipler, Phys. Lett. A 64, 8 (1977)
[11] A. Krolak, Class. Quantum Gravity 3, 267 (1986)
[12] M.P. Dąbrowski and K. Marosek, arXiv:1806.00601
[13] L. Fernandez-Jambrina and R. Lazkoz, Phys. Rev. D 70, 121503(R) (2004)
[14] J.D. Barrow and S.Z.W. Lip, Phys. Rev. D 80, 043518 (2009)
[15] J.D. Barrow, and A.A.H. Graham, Phys. Rev. D 91, 083513 (2015)
[16] L. Landau and E.M. Lifshitz, The Classical Theory of Fields, Pergamon Press, Oxford, 4th rev. edn., (1974)
[17] V.A. Belinski and M. Henneaux, The Cosmological Singularity, Cambridge Univ. Press, Cambridge, (2018)
[18] C.W. Misner, Phys. Rev. Lett. 22, 1071 (1969)
[19] A.G. Doroshkevich, V.N. Lukash and I.D. Novikov, Sov. Phys. JETP 37, 739 (1973)
[20] J.D. Barrow, Phys. Rev. D 35, 1805 (1987)
[21] C.H Brans and R.H. Dicke, Phys. Rev. 124, 925 (1961)
[22] W. Marciano, Phys. Rev. Lett. 52, 489 (1984)
[23] J.-P. Uzan, Varying constants, gravitation and cosmology, Living Reviews in Relativity, https://arxiv.org/abs/1009.5514

[24] H.B. Sandvik, J.D. Barrow and J. Magueijo, Phys. Rev. Lett. 88, 031302 (2002)

[25] J.D. Barrow and J. Magueijo, Phys. Rev. D 72, 043521 (2005)