Semiclassical Regge trajectories of noncritical string and large-$N$ QCD

Yuri Makeenko*

Institute of Theoretical and Experimental Physics
B. Cheremushkinskaya 25, 117218 Moscow, Russia
E-mail: makeenko@itep.ru

Poul Olesen*

The Niels Bohr International Academy, The Niels Bohr Institute
Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
E-mail: polesen@nbi.dk

ABSTRACT: By properly treating the path integral over the boundary value of the Liouville field (associated with reparametrizations of the boundary contour) in open string theory, we derive consistent off-shell scattering amplitudes in $d = 26$ dimensions. In $d < 26$ we consider a recently proposed boundary ansatz which reproduces a semiclassical correction to the classical string (known as the Lüscher term) and obtain in the semiclassical approximation a linear Regge trajectory with the intercept $(d - 2)/24$. We associate it with the quark-antiquark Regge trajectory in large-$N$ QCD and explain why it dominates over perturbative QCD when $t > -\text{few GeV}^2$.

KEYWORDS: $1/N$ Expansion, Bosonic Strings

Dedicated to the 40th Anniversary of QFT/string correspondence

*Also at the Institute for Advanced Cycling, Blegdamsvej 19, 2100 Copenhagen Ø, Denmark
1. Introduction

It is commonly believed that the Nambu–Goto and Polyakov strings are equivalent. At the classical level this stems from the fact that the intrinsic metric coincides with the induced one. At the quantum level the tree amplitude of scattering of scalar particles with momenta \( \Delta p_m \) reads for the Polyakov formulation of an open bosonic string in the critical dimension \( d = 26 \), where the Liouville field decouples in the bulk:

\[
A ( \{ p_m \} ) = \int D \varphi (s) \int \prod_m ds_m \ e^{\varphi(s_m)/2-\pi\alpha'\Delta p_m^2 G(s_m,s_m)} \prod_{j \neq m} |s_j - s_m|^\alpha'\Delta p_j \cdot \Delta p_m , \quad (1.1)
\]

where \( -\infty < s_m < s_{m+1} < +\infty \) are the Koba–Nielsen variables. The path integration is over the boundary value \( \varphi (s) \) of the Liouville field which remains as a dynamical variable. It is included in invariant vertex operators, which explains its appearance on the right-hand side of Eq. (1.1).

A very subtle matter with Eq. (1.1) is the singularity in the Green function at coinciding arguments, which has to be regularized in an invariant way as [1]

\[
G(s_m,s_m) \to G_\varepsilon (s_m,s_m) = \frac{1}{\pi} \ln \frac{1}{\varepsilon} + \frac{1}{2\pi} \varphi(s_m) , \quad (1.2)
\]
where the appearance of the boundary value \( \varphi(s_m) \) of the Liouville field is required for an invariant regularization. There is now a precise mutual cancellation of \( \varphi(s_m) \) in the exponent on the right-hand side of Eq. (1.1) when \( \alpha' \Delta p_m^2 = 1 \), i.e. for a tachyon. Therefore, the integrand does not depend on \( \varphi(s) \) for the tachyonic scalars or massless vectors, so the path integration over the boundary value \( \varphi(s) \) of the Liouville field decouples and the standard on-shell Koba–Nielsen amplitudes are reproduced \([1, 2]\) for the Polyakov formulation.

In this Paper we shall not impose the on-shell condition and study the amplitude (1.1) by evaluating the path integral over \( \varphi(s) \), utilizing the technique which has been recently developed in Refs. \([3, 4, 5, 6]\). By properly treating the path integral in Eq. (1.1), we remarkably obtain consistent off-shell scattering amplitudes similar to the ones known in the literature as the Lovelace choice \([7, 8]\). In \( d < 26 \) we consider a recently proposed ansatz \([6]\) for the disk amplitude with the Dirichlet boundary conditions, that reproduces a semiclassical correction to the classical string (also known as the Lüscher term). By evaluating the path integral in the semiclassical approximation, we obtain a linear Regge trajectory with the intercept \( (d - 2)/24 \) which remarkably coincides with the well-known result \([3, 10]\) derived from the spectrum of the Nambu–Goto string. We associate it with the quark-antiquark Regge trajectory in large-\( N \) QCD and explain why it dominates over perturbative QCD in the Regge kinematical regime, when \( s \gg -t \) and \( t > -\text{few GeV}^2 \).

### 2. Discretized Green function

In order to calculate the path integral over \( \varphi(s) \) in Eq. (1.1), we approximate it by an \( N \)-dimensional integral, discretizing the real axis by a set of points \( s_i \)'s. Strictly speaking, it is better to discretize an angular variable

\[
\sigma = -2 \arccot s,
\]

(2.1)

which takes the values in the finite interval \([0, 2\pi]\) and whose discretization can be chosen equidistant, but \( \varphi(s) \) vanishes as \( s \to \pm\infty \), so a proper discretization of \( s \) also works.

Having introduced a discretization of \( s \), we simultaneously regularize the Green function \( G(s, s') \). Let \( s \) lies on the \( i \)-th interval \( (s_{i-1} < s \leq s_i) \) and \( s' \geq s \) may then lie either on the \( i \)-th or \( i + 1 \)-th interval \( (s \leq s' < s_{i+1}) \) or outside. The regularized Green function reads

\[
\tilde{G}(s, s') = \begin{cases} 
-\frac{1}{\pi} \ln(s' - s) & \text{for } s' \geq s_{i+1} \\
\frac{1}{\pi} \ln \left( \frac{(s' - s_{i-1})(s_{i+1} - s)}{(s_{i+1} - s_{i-1})} \right) & \text{for } s \leq s' < s_{i+1}
\end{cases}
\]

(2.2)

which possesses the projective co-invariance and the continuity at \( s' = s_{i+1} \). The construction for \( s \geq s' \) is analogous since \( \tilde{G}(s, s') \) is symmetric in \( s \) and \( s' \).

In particular, we have from Eq. (2.2)

\[
\tilde{G}(s, s) = \frac{1}{\pi} \ln \frac{(s_{i+1} - s_{i-1})}{(s - s_{i-1})(s_{i+1} - s)}
\]

(2.3)
which replaces Eq. (1.2).

The appearance of \( s_i \) in the above formulas is in fact related to the discretization of the boundary value of the Liouville field. Comparing Eqs. (1.2) and (2.3), we arrive at

\[
e^{\varphi(s)/2} \propto \frac{(s_{i+1} - s_{i-1})}{(s_{i+1} - s)(s - s_{i-1})}, \quad s_{i-1} < s \leq s_i
\]

which is in the spirit of the formula

\[
e^{\varphi(s)/2} = \sqrt{\dot{x}^2(s)}
\]

for the induced boundary metric.

For \( s = s_i \) we have from Eq. (2.4)

\[
e^{\varphi(s_i)/2} \propto \frac{(s_{i+1} - s_{i-1})}{(s_{i+1} - s_i)(s_i - s_{i-1})}
\]

providing the discretization of \( \varphi(s_i) \) is through Eq. (1.2).

The presence of the ratio on the right-hand side of Eq. (2.4) provides the correct formula

\[
\varphi(s) \to \varphi \left( \frac{as + b}{cs + d} \right) = \varphi(s) + 2 \ln(cs + d)^2
\]

for the projective transformation exactly at finite discretization, which was a guiding principle of Ref. [4] for constructing the discretization.

Equation (2.7) follows from the general law [4] of transformation of the boundary Liouville field \( \varphi(s) \) under the reparametrization \( s \to t(s) \) (\( dt/ds \geq 0 \)):

\[
\varphi(s) \to \varphi(t(s)) = \varphi(s) - 2 \ln \frac{dt(s)}{ds}.
\]

We see from this formula that the path integration over \( \varphi(s) \) can be represented as a path integration over \( t(s) \), i.e. over reparametrizations. The only difference resides in one mode of \( \varphi(s) \), associated with a scale, which can be ignored as is discussed in Sect. 4 below.

3. Projective-invariant scattering amplitude

Equations (2.3), (2.4) fix the additional factor in Eq. (1.1) to be

\[
e^{\varphi(t_m)/2 - \pi\alpha' \Delta \rho_{m}^2 G(t_m,t_m)} = \left[ \frac{(s_{K_m+1} - s_{K_m-1})}{(s_{K_m+1} - t_m)(t_m - s_{K_m-1})} \right]^{-\alpha' \Delta \rho_{m}^2 + 1}
\]

with \( t_m \equiv s_{K_m} \), which has to be integrated over \( s_i \)'s at intermediate points. This can be done using the following discretization of the measure:

\[
D_{\text{diff}} s = \prod_i ds_i \left[ \frac{(s_{i+1} - s_{i-1})}{(s_{i+1} - s_i)(s_i - s_{i-1})} \right],
\]

that supersedes the one

\[
D_{\text{diff}} s = \prod_i ds_i \frac{1}{(s_i - s_{i-1})}
\]
proposed in [4]. How to regularize integrals with the measure (3.2) is discussed in Appendix A. Both measures (3.3) and (3.3) possess the projective invariance at finite discretization. There should be no difference between these two discretizations for smooth trajectories but typical trajectories in the path integral over reparametrizations are discontinuous [3], so this may be not the case. Actually, using the discretization (3.2), we shall obtain a linear Regge trajectory with the unit intercept instead of the one with zero intercept in Ref. [4].

The integration over $s_i$’s at the intermediate points can be repeatedly done using the formula

$$\int_{s_{i1}}^{s_{i2}} \left[ \frac{s_{22}}{s_{32}} \right] \left[ \frac{s_{31}}{s_{43}s_{32}} \right] \left[ \frac{s_{20}}{s_{21}s_{10}} \right] + c \frac{1}{s_{21}} \propto \left[ \frac{s_{41}}{s_{43}s_{31}} \right] \left[ \frac{s_{30}}{s_{31}s_{10}} \right] + (c + 1) \frac{1}{s_{31}}$$

(3.4)

with $s_{ij} \equiv s_i - s_j$, which is valid modulo an (infinite) factor for an analytic regularization of the type of the $\zeta$-function, as is discussed in Appendix A.

Integrating over $s_i$ at all intermediate points except two for each interval: $v_m, u_{m-1} \in (t_{m-1}, t_m)$ with $v_m > u_{m-1}$, we arrive at the amplitude

$$A (\{ \Delta p_m \}) = \prod_m dt_m \mathcal{K}(\{ t_m \}) \prod_{j \neq m} (t_j - t_m)^{\alpha' \Delta p_j} \Delta p_m,$$

(3.5)

where

$$\mathcal{K}(\{ t_m \}) = \prod_m \int_{t_{m-1}}^{t_m} dv_m \int_{t_{m-1}}^{v_m} du_{m-1} \left[ \frac{(u_m - v_m)}{(u_m - t_m)(t_m - v_m)} \right]^{-\alpha' \Delta p^2_m + 1} D(t_m, v_m; u_{m-1}, t_{m-1})$$

(3.6)

with

$$D(t_m, v_m; u_{m-1}, t_{m-1}) = \left[ \frac{(t_m - u_{m-1})}{(t_m - v_m)(v_m - u_{m-1})} \right] \left[ \frac{(v_m - t_{m-1})}{(v_m - u_{m-1})(u_{m-1} - t_{m-1})} \right]$$

$$+ c \frac{1}{(v_m - u_{m-1})^2},$$

(3.7)

which results from the path integral over reparametrizations as is shown in Appendix A. It is obviously projective-invariant and most probably dual.

The simplest case is that of on-shell tachyons with $\alpha' \Delta p^2 = 1$. Integrating over $v_m$’s and $u_{m-1}$’s by the formula

$$\mathcal{K}(\{ t_m \}) = \prod_m \int_{t_{m-1}}^{t_m} dv_m \int_{t_{m-1}}^{v_m} du_{m-1} D(t_m, v_m; u_{m-1}, t_{m-1}) \delta \approx 1,$$

(3.8)

we then get the usual Koba–Nielsen amplitude with $\alpha(0) = 1$.

For the off-shell case with arbitrary $\Delta p^2$, we find explicitly for the 4-point amplitude, fixing projective invariance by setting $t_1 = 0$, $t_2 = x$, $t_3 = 1$, $t_4 = \infty$ in the usual way:

$$A (\{ \Delta p_m \}) = \int_0^1 dx \left[ x^{-\alpha' s} - \alpha' \Delta p^2 (1 - x)^{-\alpha' t - \alpha' \Delta p^2} \right] K(x).$$

(3.9)
The Regge asymptote of the amplitude \((3.9)\) can be analyzed by the standard change of the variable
\[
x = 1 + \frac{y}{\alpha' s}, \quad 0 < y < -\alpha' s.
\]  
(3.10)
The values of \(y\), which are essential in \((3.9)\), are \(\sim 1\) because of the factor
\[
x^{-\alpha' s} = \left(1 + \frac{y}{\alpha' s}\right)^{-\alpha' s} e^{-y},
\]  
(3.11)
so that \((1 - x) \sim 1/\alpha' s\) are essential.

Noting that an additional \(s\)-dependent factor in the amplitude \((3.9)\) comes from the integrals over \(v_3\) and \(u_2\), we have
\[
K(x) \propto \int_x^{v_3} dv_3 \int_x^{u_2} du_2 (1 - v_3)^{\alpha' \Delta p_3^2 - 1} D(1, v_3; u_2, x) (u_2 - x)^{\alpha' \Delta p_3^2 - 1}
\]
\[
\propto (1 - x)^{\alpha' \Delta p_3^2 + \alpha' \Delta p_3^2 - 2}.
\]  
(3.12)
We thus find for the Regge asymptote of \((3.3)\)
\[
A(\{\Delta p_m\}) \propto \int_0^1 dx x^{-\alpha' s} (1 - x)^{-\alpha' t} \Delta p_3^2 - \alpha' \Delta p_3^2 (1 - x)^{\alpha' \Delta p_3^2 + \alpha' \Delta p_3^2 - 2}
\]
\[
= \int_0^1 dx x^{-\alpha' s} (1 - x)^{-\alpha' t - 2},
\]  
(3.13)
resulting in the linear Regge trajectory
\[
\alpha(t) = 1 + \alpha' t.
\]  
(3.14)
It is clear from this derivation that the same result for the Regge asymptote can be obtained directly from the path integral \((1.1)\), employing the discretization \((3.1), (3.2)\), without the integration over the intermediate points.

Let us stress that the amplitude \((3.3)\) is not precisely the Lovelace amplitude \([7, 8]\), but has the same momentum-dependent part. The only difference resides in the measure for the integration over the Koba–Nielsen variables. The amplitude \((3.9)\) is consistent off-shell in \(d = 26\) and can be viewed as a straightforward off-shell extension of the Koba–Nielsen amplitudes, which possesses the projective invariance.

4. Wilson-loop/scattering-amplitude duality

The above scattering amplitude can be inferred from the reparametrization-invariant disk amplitude
\[
W[\{x(t)\}] = \int D_{\text{diff}} t \, e^{-KS[\{x(t)\}]},
\]  
(4.1)
where \(K = 1/2\pi\alpha'\) is the string tension and
\[
S[\{x(t)\}] = \frac{1}{4\pi} \int_{-\infty}^{+\infty} ds_1 \int_{-\infty}^{+\infty} ds_2 \frac{[x(t(s_1)) - x(t(s_2))]^2}{(s_1 - s_2)^2}
\]  
(4.2)
is the Douglas integral \[11\], whose minimum with respect to reparametrizations equals the minimal area. Equation (4.1) was proposed by Polyakov \[12\] (see also Ref. \[13\]) as an ansatz for Wilson loops in large-\(N\) QCD.

The derivation of the scattering amplitudes from the ansatz (4.1) makes use of the equivalence of the path integrals over the boundary value \(\varphi(s)\) of the Liouville field and the reparametrizations \(t(s)\) of the boundary contour. They are the same modulo a scale factor which has to be fixed to provide the correct value \(L\) of the length of the boundary contour. The precise equivalence of the two measures for the path integrations reads

\[
\int \mathcal{D}\varphi(s) \delta \left( \frac{1}{L} \int ds \, e^{\varphi(s)/2} - 1 \right) \cdots = \int \mathcal{D}t \cdots.
\]

(4.3)

We can rephrase this equivalence by saying that the boundary metric \(e^{\varphi(s)/2}\) has to be equal to the induced metric (2.5) modulo reparametrizations.

In our previous works \[3, 4\] it was shown that the functional Fourier transformation of Eq. (4.1)

\[
A[p(\cdot)] = \int \mathcal{D}x \, e^{ip \cdot \dot{x}} W[x(\cdot)],
\]

(4.4)

which defines a momentum-space disk amplitude, looks exactly like the right-hand side of Eq. (4.1) with \(x(t)\) substituted by the “most important” trajectory

\[
x_*(t) = \frac{1}{R} p(t).
\]

(4.5)

For piece-wise constant \(p(t)\) this momentum-space disk amplitude, integrated over reparametrizations \(s(t)\) obeying \(s(t_m) = t_m\), is equal (modulo an infinite factor) to the scattering amplitude (3.5).

The fact that the position-space and momentum-space disk amplitudes coincide, if Eq. (4.5) is satisfied, is in the spirit of the remarkable Wilson-loop/scattering-amplitude duality, recently discovered for the \(\mathcal{N} = 4\) super Yang–Mills \[14, 15\] (for a review see Ref. \[16\]). As is shown in Refs. \[3, 4\], it is applicable to QCD, but only in the Regge kinematical regime of meson scattering for not too large values of \(-t\). Equation (4.5) is the same as Eq. (5) of \[4\], where it was discussed in detail. For piece-wise constant \(p(t)\), \(x_*(t)\) is also a step function which has (if not regularized) discontinuities at \(t(s_m) = t_m\), that have to be smeared by a regularization.

It is evident that a regularization of the Green function \(G\), like the one displayed in Eq. (1.2), can be moved from \(G\) to the momentum loop \(p(t)\). Let us take\(^1\)

\[
\hat{p}(t) = \sum_m \Delta p_m \frac{\varepsilon_m}{\pi \varepsilon_m^2 + (t - t_m)^2},
\]

(4.6)

which reproduces the sum of deltas as \(\varepsilon_m \to 0\). The boundary value of the Liouville field now enters through

\[
\varepsilon_m = \varepsilon e^{-\varphi(t_m)/2}.
\]

(4.7)

\(^1\)This regularization can be viewed as if we slightly move the contour from the real axis into the complex plane \(z = t + i\varepsilon_m\) near the points \(t_m\), where the momentum-space loop \(p(t)\) has discontinuities, by means of a harmonic function.
To obtain the scattering amplitudes, we calculate
\[
\int dt \, d' t' \, \hat{p}(t) \cdot \hat{p}(t') \ln |s(t) - s(t')|\]
in the exponent with thus regularized \(p(t)\):
\[
\int dt \, d' t' \, \hat{p}(t) \cdot \hat{p}(t') \ln |s(t) - s(t')| = \sum_{m,j} \Delta p_m \cdot \Delta p_j \int \frac{\varepsilon_m dt}{\pi [\varepsilon_m^2 + (t - t_m)^2]} \int \frac{\varepsilon_j dt'}{\pi [\varepsilon_j^2 + (t' - t_j)^2]} \ln |s(t) - s(t')|. \tag{4.8}
\]
For \(j \neq m\) and \(|t_m - t_j| \gg \varepsilon_m, \varepsilon_j\) the integral is the same as in the nonregularized case and gives
\[
\sum_{j \neq m} \Delta p_m \cdot \Delta p_j \ln |t_m - t_j|, \tag{4.9}
\]
while for \(j = m\) we have
\[
\int \frac{\varepsilon_m dt}{\pi [\varepsilon_m^2 + (t - t_m)^2]} \int \frac{\varepsilon_m dt'}{\pi [\varepsilon_m^2 + (t' - t_m)^2]} \ln |s(t) - s(t')| = \ln [2 \varepsilon_m \dot{s}(t_m)] + O(\varepsilon_m). \tag{4.10}
\]
To complete the consideration, we note that
\[
\varepsilon_m \dot{s}(t_m) = \varepsilon e^{-\varphi(s_m)/2} \tag{4.11}
\]
in view of Eqs. (2.8), (4.7).

Adding (4.8) and (4.10) and using Eq. (2.6), we reproduce the principal value prescription of Refs. [3, 4] that leads to the above off-shell scattering amplitudes, provided Eq. (2.6) is satisfied, where \(s_{K_m} = t_m\) but \(s_{K_m \pm 1}\) live at the intermediate points as before. Now these \(s_{K_m \pm 1}\)'s are needed only to define the measure for the integration over reparametrizations. We can alternative rewrite it via the Liouville field and then not to introduce \(s_i\)'s at the intermediate points at all. Those are needed, however, to manage the path integral over reparametrizations.

5. The shape of “most important” contours

The “most important” contour (5.3) is given for thus smeared \(p(t)\) by
\[
x_a(t) = \frac{1}{K} p(t) = \frac{1}{K} \sum_m \Delta p_m \left( \frac{1}{2} + \frac{1}{\pi} \arctan \frac{(t - t_m)}{\varepsilon_m} \right) = \frac{1}{\pi K} \sum_m \Delta p_m \arctan \frac{(t - t_m)}{\varepsilon_m}, \tag{5.1}
\]
which is nothing but a regularized Eq. (5) of [4].

For the 4-particle case we express the four-vectors \(\Delta p_m^\mu\) in the center-mass frame through \(E, p = |\vec{p}_m|\) and the scattering angle \(\theta\) by
\[
\Delta p_1 = (E, p, 0, 0)
\]
\[
\Delta p_2 = (E, -p, 0, 0)
\]
\[
\Delta p_3 = (-E, p \cos \theta, p \sin \theta, 0)
\]
\[
\Delta p_4 = (-E, -p \cos \theta, -p \sin \theta, 0). \tag{5.2}
\]
The scattering amplitude involves the integration over the variable

$$x = \frac{(t_2 - t_1)(t_4 - t_3)}{(t_3 - t_1)(t_4 - t_2)}, \quad (5.3)$$

which the integrand does depend on, as well as the integration over $t_1$, $t_3$ and $t_4$, which factorizes. At large Mandelstam’s variables $-s$ and $-t$ ($-t \lesssim -s$) the integral over $x$ is dominated by the saddle point

$$x_s = \frac{s}{s + t}, \quad (5.4)$$

so it is fixed for given $s$ and $t$. On the contrary, the integral over the values of $t_4 - t_1$ and $t_4 - t_3$ gives a factor, which equals the volume of the projective group. We can set, therefore, $t_1$, $t_3$ and $t_4$ equal to some values, keeping in mind that one such contour belongs, in fact, to a whole family of contours, which are equally important.

Choosing momenta in the 4-particle case according to Eq. (5.2) and using the projective invariance to set $t_1 = 0$, $t_3 = 1$, $t_4 = \infty$, that also fixes $t_2 = x_s$ at large $s$ and $t$ according to Eqs. (5.3) and (5.4), we arrive at the “most important” contours of the shape of rectangles, which are depicted for $t/s = -0.4$ and $t/s = -0.04$ in Fig. 1 and Fig. 2, respectively. They degenerate when $t/s \to 0$.

**Figure 1:** Typical loop $C_*$ for $t/s = -0.4$.

**Figure 2:** Typical loop $C_*$ for $t/s = -0.04$. 
The minimal area

\[ KS_{\text{min}}(C) = \alpha' t \log \frac{s}{t}, \]  

(5.5)

spanned by these contours, can be large even for small \( t \). This will be enough to justify the semiclassical approximation below. However, we have to have \( |t| \gtrsim K \) for the transverse size of the contour also to be large, which we be prescribed by a self-consistency of our approach in QCD.

6. A generalization to \( d \) dimensions

We are now in a position to perform the main task of this Paper: to generalize the scattering amplitudes to \( d < 26 \) dimensions.

For \( d \neq 26 \) the Liouville field does not decouple in the bulk, resulting in the additional factor to be inserted in the ansatz (4.1)

\[
W[x(\cdot)] = \int D_{\text{diff}} t \ e^{-KS[x(t)]} \int D\varphi \exp \left[ \frac{(d-26)}{48\pi} \int d^2z \left( \frac{1}{2} \partial_a \varphi \partial_a \varphi + \mu e^\varphi \right) \right].
\]

(6.1)

Here \( z = x + iy \in \text{UHP} \) and \( \varphi \) obeys the boundary condition

\[
\varphi(x = s, y = 0) = \varphi(s) = 2 \ln \frac{dt(s)}{ds}
\]

(6.2)

at the real axis and we ignore the exponential \( e^\varphi \) in the Liouville action in (6.1), setting \( \mu = 0 \). The second equality in Eq. (6.2) is valid again modulo the scale mode \( \varphi = \text{const} \) which does not contribute to the Liouville action in Eq. (6.1) when \( \mu = 0 \).

As was first shown in Refs. [17, 18], the presence of the additional factor in (6.1) is crucial for reproducing the Lüscher term, describing semiclassical fluctuations of a string, for a rectangle when \( d \neq 26 \). An analogous result holds also for an ellipse, the structure of the Lüscher term for which is advocated in Ref. [6].

It is clear that the scattering amplitude which is a counterpart of Eq. (6.1) is given by Eq. (6.1) with the same additional factor inserted in the integrand. The analysis of the path-integral in thus obtained amplitude is beyond the scope of this paper.

We shall use instead a more simply tractable modification of the disk amplitude (4.1), which has been recently proposed by the authors [8]:

\[
W[x(\cdot)] = \int D_{\text{diff}} t \ e^{-KS[x(t)]} \ (\det O)^{-(d-26)/48}
\]

(6.3)

with the Douglas integral \( S[x(t)] \) given by Eq. (4.2) and the determinant of the operator \( O \) given by the Gaussian path integral

\[
(\det O)^{-1/2} = \int D\beta(t) \ e^{-KS_2[\beta]}
\]

(6.4)

\footnote{The curvature term is not present for the upper half-plane (UHP) parametrization.}
with the quadratic action

$$S_2[\beta] = \frac{1}{4\pi} \int ds_1 \int ds_2 \frac{\dot{x}(t(s_1)) \cdot \dot{x}(t(s_2))}{(s_1 - s_2)^2} [\beta(s_1) - \beta(s_2)]^2.$$  \hspace{1cm} (6.5)

While the operator $O$ lives in the boundary, it captures semiclassical transverse fluctuations of the minimal surface reproducing the Lüscher term for a rectangle or an ellipse. In $d = 26$ Eq. (6.3) reduces to the Polyakov ansatz (4.1).

It is important to emphasize that the additional determinant in Eq. (6.3) does not effect the classical limit, which is given by the exponential of the minimal area $S_{\text{min}}$, but contributes to a pre-exponential. The parameter of the semiclassical expansion is thereby $1/KS_{\text{min}}$, so for large loops we can restrict ourselves with a semiclassical approximation.

The functional Fourier transformation (4.4) of (6.3) is again calculable because the path integral over $x$ is Gaussian. In the semiclassical approximation we obtain again

$$A[p(\cdot)] = W[x^*_{\cdot} = p(\cdot)/K],$$  \hspace{1cm} (6.6)

where $W[x(\cdot)]$ is given by Eqs. (6.3), (6.4), (6.5) with $x(t)$ substituted by the “most important” contour (4.1).

### 7. Semiclassical Regge trajectory

Substituting the smeared step function (5.1) into Eq. (6.5) and repeating the above calculation, we get

$$KS_2 = \alpha' \sum_{i,j} \Delta p_i \cdot \Delta p_j \int \frac{dx}{\pi(1 + x^2)} \int \frac{dy}{\pi(1 + y^2)} \left[ \beta(t_i + \varepsilon_i x) - \beta(t_j + \varepsilon_j y) \right]^2$$

$$+ \alpha' \sum_i \frac{\Delta p_i^2}{2} \int \frac{dx}{\pi(1 + x^2)} \int \frac{dy}{\pi(1 + y^2)} \left[ \beta(t_i + \varepsilon_i x) - \beta(t_i + \varepsilon_i y) \right]^2$$

$$\varepsilon \to 0 \alpha' \sum_{m > n} \Delta p_m \cdot \Delta p_n \left[ \beta(t_m) - \beta(t_n) \right]^2 \frac{1}{[s(t_m) - s(t_n)]^2} + \alpha' \sum_m \frac{\Delta p_m^2}{2} \frac{\dot{\beta}^2(t_m)}{s_m^2(t_m)}. \hspace{1cm} (7.1)$$

To treat the path integral over reparametrizations in Eq. (6.4), we replace (7.1) by its discretized version

$$KS_2 = \alpha' \sum_{K_i > K_j} \Delta p_{K_i} \cdot \Delta p_{K_j} \frac{(\beta_{K_i} - \beta_{K_i})^2}{(s_{K_i} - s_{K_j})^2} + \alpha' \sum_{K_i} \frac{\Delta p_{K_i}^2}{2} \frac{(\beta_{K_i+1} - \beta_{K_i-1})^2}{(s_{K_i+1} - s_{K_i-1})^2}, \hspace{1cm} (7.2)$$

where the indices are labeled as before. The first term on the right-hand side of Eq. (7.2) [or Eq. (6.6)] vanishes since $\beta_{K_i} = 0$ because of $s_{\ast}(t_m) = t_m$. We have also verified that the obtained result does not change for a wide class of discretization, in particular, for another discretization of $\dot{\beta}(t_m)$.

The integral over $\beta_i$’s at the intermediate points now decouples from the one over $\beta_{K_i\pm 1}$. This shows the difference between the path integrals over reparametrizations around the minimizing trajectory $s_{\ast}(t)$ for a smooth contour like an ellipse or rectangle, which are
considered in Ref. [3], and the smeared step function [5]. In the former case we have a
Gaussian integral that reproduces the Lüscher term for large $K S_{\text{min}}$, while in the latter
the quadratic action vanishes and we deal with zero modes. The reason is that the only restriction on the minimizing function $s_s(t)$ for a step-wise contour is $s_s(t_m) = t_m$, as
was pointed out in Ref. [4], so $s_s(t)$ is arbitrary for $t_{m-1} < t < t_m$.

If $\Delta p_i^2 = 0$ all $\beta_i$'s can be treated on equal footing as zero modes. In this case the quadratic approximation is not applicable and we have to integrate over reparametrizations with the whole measure $\mathcal{D}_{\text{diff}}$ given by Eq. (3.2). We write, therefore,

$$
\det O^{-1/2} = \prod_{m} \int_{t_{m-1} < t_m} dt_m \frac{(s_{K_m+1} - s_{K_m-1})}{(s_{K_m+1} - t_m)(t_m - s_{K_m-1})} \times ds_{K_m-1} \frac{(t_m - s_{K_m-2})}{(t_m - s_{K_m-1})(s_{K_m-1} - s_{K_m-2})} \times ds_{K_m-2} \frac{(s_{K_m-1} - s_{K_m-3})}{(s_{K_m-1} - s_{K_m-2})(s_{K_m-2} - s_{K_m-3})} \times ds_{K_m-1+2} \frac{(s_{K_m-1+3} - s_{K_m-1+2})}{(s_{K_m-1+3} - s_{K_m-1+1})(s_{K_m-1+2} - s_{K_m-1+1})} \times ds_{K_m-1+1} \frac{(s_{K_m-1+2} - t_{m-1})}{(s_{K_m-1+2} - s_{K_m-1+1})(s_{K_m-1+1} - t_{m-1})}.
$$

The multiple integral in Eq. (7.3) coincides with the one considered in Sect. 3 if we set there $\Delta p_i = 0$. The resulting formula for the integral is of the type of the contribution from the measure to Eq. (3.13) and reads

$$
\det O^{-1/2} \propto (1 - x)^{-1} \sim \alpha' s.
$$

The same result can be obtained also for $\Delta p_i^2 \neq 0$, substituting $s_i = s_{s_1} + \beta_i$ in Eq. (7.4).

The determinant (7.4) results in an additional factor of $s^{(d-26)/24}$ in the amplitude and there are no other such factors because typical values of $(u_m - v_m)$ in Eq. (7.4) are not small. We therefore obtain the following value of the Regge intercept

$$
\alpha(0) = \alpha_0 + \frac{(d - 26)}{24},
$$

where $\alpha_0 = 1$ is the value calculated above in $d = 26$.

Finally we get in $d = 4$

$$
\alpha(0) = \frac{(d - 2)}{24} \approx 0.083,
$$

which is to be compared with the intercept of a quark-antiquark Regge trajectory. The value (7.6) is smaller than the experimental value of $\alpha(0) \approx 0.5$ for the $\rho - A_2 - f$ meson Regge trajectory. A possible explanation of this discrepancy is that the chiral symmetry apparently is not yet spontaneously broken in our consideration as is clear from the fact that the amplitude does not have Adler’s zero (i.e. does not vanish with vanishing $s$ and $t$ in the chiral limit). Then the breaking of the chiral symmetry may shift the intercept up, like this happens [14] in the Lovelace–Shapiro dual models.
The reason why we consider such a quark-antiquark Regge trajectory is because we are dealing with the disk amplitude, which is associated with planar diagrams and correspondingly a quark-antiquark Regge trajectory in large-$N$ QCD, as we shall now discuss.

8. Application to QCD

As is already mentioned, the disk amplitude (6.3) describes the asymptote of Wilson loops in large-$N$ QCD. 

$M$-particle scattering amplitudes in large-$N$ QCD are expressed through the Green functions of $M$ colorless composite quark operators (e.g. $\bar{q}(x_i)q(x_i)$) in terms of the sum over all Wilson loops passing via the points $x_i$ ($i = 1, \ldots, M$), where the operators are inserted. The on-shell $M$-particle scattering amplitudes can be obtained from these Green functions by the standard Lehman–Symanzik–Zimmerman reduction. Representing $M$ momenta of the (all incoming) particles by the differences $\Delta p_i = p_{i-1} - p_i$ and introducing a piecewise constant momentum-space loop

$$p(t) = p_i \quad \text{for } t_i < t < t_{i+1}, \quad (8.1)$$

we obtain

$$A(\Delta p_1, \ldots, \Delta p_M) \propto \int_0^\infty d\mathcal{T} \mathcal{T}^{M-1} e^{-m\mathcal{T}} \int_{-\infty}^{t_{M-1}^{+}} \frac{dt_{M-1}}{1 + t_{M-1}^2} \prod_{i=1}^{M-2} \int_{-\infty}^{t_{i+1}} \frac{dt_i}{1 + t_i^2} \prod_{i=1}^{M-2} \int_{-\infty}^{t_{i+1}} \frac{dt_i}{1 + t_i^2}$$

$$\times \int_{z(-\infty)=z(+\infty)=0} \mathcal{D}z(t) \ e^{i \int dt \mathcal{T} \cdot p(t)} J[z(t)] W[z(t)]. \quad (8.2)$$

Here we do not integrate over $z(-\infty) = z(+\infty)$, which would produce the (infinite) volume factor because of translational invariance.

For spinor quarks and scalar operators the weight for the path integration in Eq. (8.2) is

$$J[z(t)] = \int \mathcal{D}k(t) \ sp \ e^{i \int dt \mathcal{T} \cdot k(t) / (1 + t^2)}$$

where $sp$ and the path-ordering refer to $\gamma$-matrices. In Eq. (8.2) $W(C)$ is the Wilson loop in pure Yang–Mills theory at large $N$ (or quenched), $m$ is the quark mass and $\mathcal{T}$ is the proper time. For finite $N$, correlators of several Wilson loops have to be taken into account.

Substituting the ansatz (4.1) into Eq. (8.2), we can perform the Gaussian integral over $z(t)$ to obtain

$$A(\Delta p_1, \ldots, \Delta p_M) \propto \int_0^\infty d\mathcal{T} \mathcal{T}^{M-1} e^{-m\mathcal{T}} \int_{-\infty}^{t_{M-1}^{+}} \frac{dt_{M-1}}{1 + t_{M-1}^2} \prod_{i=1}^{M-2} \int_{-\infty}^{t_{i+1}} \frac{dt_i}{1 + t_i^2} \prod_{i=1}^{M-2} \int_{-\infty}^{t_{i+1}} \frac{dt_i}{1 + t_i^2}$$

$$\times \int \mathcal{D}k(t) \ sp \ e^{-i \mathcal{T} \cdot \mathcal{K} / (1 + t^2)} W[x_*(t) = \frac{1}{\mathcal{K}} (p(t) + k(t))]. \quad (8.4)$$

As distinct from its stringy counterpart (6.6), the right-hand side of Eq. (8.4) has the additional path integration over $k(t)$, which emerges from Feynman’s disentangling of
the $\gamma$-matrices. But for the case, where $m$ is small and/or $M$ is very large, the integral over $T$ in Eq. (8.2) is dominated by large $T \sim (M - 1)/m$. Noting that typical values of $k \sim 1/T$ are essential in the path integral over $k$ for large $T$, we can disregard $k(t)$ in the argument of $W$ in Eq. (8.4), so the path integral over $k$ factorizes. We finally obtain the product of the momentum-space disk amplitude $A[p(t)]$ times factors which do not depend on $p$. The substitution of the ansatz (6.3) into Eq. (8.2) for $d < 26$ results in a more complicated path integral over $z(t)$ which is, however, Gaussian in the semiclassical approximation, reproducing again Eq. (8.4).

Thus for the ansatz (6.3) $A[p(t)]$ coincides with $W[x_*(t)]$, where $x_*(t)$ is given by Eq. (5.1) and the path integral over reparametrizations in Eq. (6.3) goes over the functions $s(t)$, obeying $s(t_m) = t_m$. Denoting

$$\frac{dt}{ds} = e^{\phi(s)/2}, \quad (8.5)$$

we can finally rewrite Eq. (6.4) in the form displayed in Eq. (1.1). Therefore, Eq. (8.4) exactly reproduces for piecewise constant $p(t)$ the (off-shell) amplitude (3.5) as $m \to 0$ and/or $M \to \infty$! We thus conclude that the quark-antiquark Regge trajectory in large-$N$ QCD is linear in the semiclassical approximation with the intercept given by Eq. (7.6).

9. Separation of pQCD and npQCD

In QCD string is stretched between quarks, when they are separated by large distances. Its emergence is due to nonperturbative effects. Alternatively, perturbative QCD (pQCD) works at small distances, where the reggeization of $\bar{q}q$ is due to double logarithms in quark amplitudes, as was first pointed out by Kirschner and Lipatov [20] and further investigated in Refs. [21, 22]. They found for scattering amplitudes with an exchange of a quark-antiquark pair:

$$\text{pQCD double logs } \propto I_1\left(\omega \ln \frac{|s|}{\mu^2}\right), \quad \omega = \sqrt{\frac{g^2(t)C_F}{2\pi^2}}, \quad (9.1)$$

where $\mu \sim 1$ GeV is an IR cutoff (usually given in pQCD by a transverse mass). In spite of Eq. (9.1) is obtained with the double logarithmic accuracy, the coupling $g^2$ is replaced by $g^2(t)$ because of the anticipated charge renormalization. We then have asymptotically the Regge behavior

$$\text{pQCD double logs } \propto \left(\frac{s}{\mu^2}\right)^{\omega(t)} \quad (9.2)$$

with almost constant $\alpha(t) \approx 0.25 \div 0.5$ for $t \lesssim -1$ GeV$^2$, where pQCD applies.

Experimental data for $\alpha(t)$ extracted for $t < 0$ from inclusive $\pi^0$ production in $\pi^- p$ collisions and from the exclusive process $\pi^- p \to \pi^0 N$ are analyzed, respectively, in Refs. [21, 23]. The data for $t > -2$ GeV$^2$ as well as the spectrum of resonances for $t > 0$ are well described by a linear $\rho$-meson Regge trajectory with the intercept $\alpha(0) \approx 0.5$. Alternatively, the dependence (9.2) for $\alpha(t)$ would be represented by an almost horizontal line for $t \lesssim -1$ GeV$^2$. As is emphasized in Ref. [23], the scattering data do not fit such a behavior of $\alpha(t)$ in pQCD.
In Eq. (8.2), which relates scattering amplitudes and the Wilson loops, pQCD resides in the contribution from small loops, while nonperturbative stringy effects reside in the contribution of large loops, where the ansatz (6.3) is applicable. The total amplitude is obviously a sum of these two contributions, which may be separated by splitting the integral over $\mathcal{T}$ into two regions: $\mathcal{T} < \tau_{\text{max}}$ and $\mathcal{T} > \tau_{\text{max}}$ with $\tau_{\text{max}} \sim K$. We associate them with pQCD and npQCD (nonperturbative QCD), respectively. For the pQCD region $\tau_{\text{max}}$ plays the role of an IR cutoff, as $1/\mu^2$ does in Minkowski space. If nonperturbative effects have not been taken into account, the pQCD diagrams would be cut in infrared at much smaller momenta of the order of the quark mass $m$.

Because the total amplitude is the sum of both pQCD and npQCD contributions which decrease with $t$ as different powers of $s$, the relative coefficient is of most importance at large but finite $s$. The npQCD contribution to Eq. (8.2) has a relative factor of $(\sqrt{K/m})^M$, which is large for small $m$. It is worth analyzing whether or not this would be numerically enough to explain the dominance of the npQCD contribution for $t > -2 \text{ GeV}^2$.

10. Conclusion

We have shown in this Paper how to deal with the path integral over reparametrizations of the boundary of an open string (or, equivalently, over the boundary value of the Liouville field in the Polyakov formulation). This path integration over reparametrizations is crucial to obtain consistent off-shell scattering amplitudes in the critical dimension $d = 26$.

We used the representation of the disk amplitude as the path integral over reparametrizations of the (exponential of the) boundary action, known as the Douglas integral, which is quadratic in the embedding-space coordinates and the functional Fourier transformation can be performed for this reason. Therefore, it was crucial for the success of calculations that all path integrals are Gaussian except for the one over reparametrizations, which can be partially done and the remainder reduces to an integration over the Koba–Nielsen variables.

For $d < 26$ we used a similar representation in the form of a boundary path integral, which is equivalent in the semiclassical approximation to the path integral over the Liouville field in the bulk, that accounts for transverse fluctuations of the minimal surface. We have obtained from it, again in the semiclassical approximation, a Regge-behaved scattering amplitude with a linear Regge trajectory of the intercept $(d - 2)/24$, that remarkably coincides with the one obtained from the spectrum \([9]\) of the long Nambu–Goto string, which is asymptotically consistent \([10]\) even for $d \neq 26$. This apparently illustrates its equivalence to the Polyakov string in the semiclassical approximation.

An application of thus obtained scattering amplitudes of noncritical string theory is the theory of strong interaction of hadrons. In our case a vector excitation is associated with a massive $\rho$-meson rather than a massless gauge field. We have demonstrated how this result applies to the quark-antiquark Regge trajectory in QCD, where the boundary action describes asymptotic behavior of Wilson loops of large size.

\[\text{Sometimes the terms “hard” and “soft” are also used, respectively, for the pQCD and npQCD contributions to Regge trajectories.}\]
A natural question is as to what are the approximations made in the derivation? Remarkable, there are surprisingly few of them. We have to have the limit of large number of colors to justify the quenched approximation. Also small quark mass and/or the large number of external legs of the amplitude essentially simplify the result — otherwise the amplitude involves a rather complicated path integral associated with spin degrees of freedom. The amplitude applies in the Regge kinematical regime of asymptotically large $s$ and fixed $t$, associated with small scattering angle or fixed momentum transfer, where the non-perturbative stringy behavior of the Wilson loop dominate over perturbative QCD. From experiment we expect this should be the case for $t > -2 \text{GeV}^2$, while a future investigation of the relative strength of perturbative and nonperturbative effects is required to justify this domain. For much larger values of $-t \sim s$ there are no longer reasons to expect the nonperturbative stringy effects to dominate over perturbation theory. On the contrary, perturbative QCD should dominate for large $-t$, so the exponential stringy behavior of the differential cross-section is expected to change for a field-theoretical power-like behavior of perturbative QCD at small distances.

In the region, where the area-law behavior of the Wilson loop sets in, we expect large loops to dominate the sum over path on the right-hand side of Eq. (8.2), so that the semiclassical approximation we used to find the Regge intercept is applicable. Therefore, our semiclassical approximation is justified by the Regge kinematics. A phenomenological argument in favor of this picture is that the Regge trajectory is linear in the semiclassical approximation and higher orders would most probably lead to a bending of the Regge trajectory, which is not seen in experiment for the quark-antiquark Regge trajectory.

Our final comment concerns an ambiguity of the scattering amplitudes which is due to different discretizations of the measure in the path integral over reparametrizations. In Ref. [4] we used the discretization (3.3) which resulted in the zero intercept, while the discretization (3.2) of this Paper has resulted in the unit intercept in $d = 26$. We have also verified these two numbers do not change, when next to neighbor points are involved in the discretization of the measure. These are apparently two universality classes. It is worth further investigating of this issue, in particular, of the meaning of Eq. (3.4) from the point of view of an associated stochastic process, like it was done in Ref. [5] for the measure (3.3).

Acknowledgments

We are indebted to Alexander Gorsky, Alexei Kaidalov, Gregory Korchemsky, Andrei Mironov, Niels Obers, and Konstantin Zarembo for encouragement and useful discussions.

A. Integration with alternative measure (3.2)

In this appendix we derive Eq. (3.4) which plays a crucial role in the integration over reparametrizations with the measure (3.2).
Let us represent
\[ \left[ \frac{(s_{i+1} - s_i)}{(s_{i+1} - s_i)(s_i - s_{i-1})} \right] = \left[ \frac{1}{(s_{i+1} - s_i)} + \frac{1}{(s_i - s_{i-1})} \right]. \] (A.1)

Multiplying three brackets, we get on the left-hand side of Eq. (3.4) eight integrals of the type
\[ \int_{s_1}^{s_3} ds_2 s_{32}^{a-1} s_{21}^{b-1} = s_{31}^{a+b-1} B(a, b) \] (A.2)
with
\[ B(a, b) = \int_1^{\infty} dy \frac{(y^{a-1} + y^{b-1})}{(1 + y)^{a+b}}. \] (A.3)

When the integral in Eq. (A.3) is convergent, it is of course just the beta-function, but a regularization is required for \( a, b = 0, -1 \) as in our case. Integrating over \( s_2 \), we obtain
\[ \int_{s_1}^{s_3} ds_2 \left[ \frac{s_{42}}{s_{43}s_{32}} \right] \left\{ \left[ \frac{s_{31}}{s_{32}s_{21}} \right] \left[ \frac{s_{20}}{s_{21}s_{10}} \right] + c \frac{1}{s_{21}^{2}} \right\} \]
\[ = [B(0, 0) + B(1, -1)] s_{43}^{-1} s_{31}^{-1} + [B(0, 1) + B(1, 0)] s_{43}^{-1} s_{10}^{-1} \]
\[ + [B(-1, 0) + B(0, -1)] s_{31}^{-2} + [B(0, 0) + B(-1, 1)] s_{31}^{-1} s_{10}^{-1} \]
\[ + c \left[ B(0, -1) s_{31}^{-2} + B(1, -1) s_{43}^{-1} s_{31}^{-1} \right]. \] (A.4)

The four coefficients read explicitly as
\[ B(0, 0) + B(1, -1) = B(0, 0) + B(-1, 1) = B(-1, 0) = B(0, -1) = \int_{1}^{\infty} dy \left( 1 + \frac{1}{y} \right)^{2} \] (A.5)
and
\[ B(0, 1) + B(1, 0) = 2 \int_{1}^{\infty} \frac{dy}{y}. \] (A.6)

The integral over \( y \) in Eq. (A.6) is logarithmically divergent at large \( y \) and can be regularized, e.g. by
\[ \int_{1}^{\infty} \frac{dy}{y^{1+\delta}} = \frac{1}{\delta}. \] (A.7)

This type of an analytic regularization simultaneously regularizes
\[ \int_{1}^{\infty} dy y^{\delta} = \frac{1}{\delta - 1}. \] (A.8)
in Eq. (A.5) making the difference between the two to be finite. It preserves, therefore, the projective invariance of the integral (A.4) as \( \delta \to 0 \).

We thus find
\[ \int_{s_1}^{s_3} ds_2 \left[ \frac{s_{42}}{s_{43}s_{32}} \right] \left\{ \left[ \frac{s_{31}}{s_{32}s_{21}} \right] \left[ \frac{s_{20}}{s_{21}s_{10}} \right] + c \frac{1}{s_{21}^{2}} \right\} \]
\[ = \frac{1}{\delta} \left\{ \left[ \frac{s_{41}}{s_{43}s_{31}} \right] \left[ \frac{s_{30}}{s_{31}s_{10}} \right] + (c + 1) \frac{1}{s_{31}^{2}} \right\} + O(\delta^0), \] (A.9)
where both terms are manifestly projective-invariant, which proves Eq. (3.4).
We finally note that the formulas (A.7), (A.8) are of the type of the ones for the regularization via the ζ-function
\[
\sum_1^{\infty} \frac{1}{n^{1+\delta}} = \zeta(1 + \delta), \tag{A.10}
\]
since
\[
\zeta(1 + \delta) = \frac{1}{\delta} + O(\delta^0). \tag{A.11}
\]
This is because we can substitute the integration in Eq. (A.7) by a summation, which does not change the divergent part of the integral. Analogously
\[
\zeta(0) = -\frac{1}{2} \tag{A.12}
\]
is the counterpart of Eq. (A.8).

The \(n\)-fold integral over reparametrizations at the intermediate points can be calculated applying Eq. (3.4) \(n\) times to give
\[
\prod_{i=1}^{n} \int_{s_1}^{s_{i+2}} ds_{i+1} \left[ \frac{(s_{n+3} - s_{n+1})}{(s_{n+3} - s_{n+2})(s_{n+2} - s_{n+1})} \right] \prod_{i=1}^{n} \left[ \frac{(s_{i+2} - s_i)}{(s_{i+2} - s_{i+1})(s_{i+1} - s_i)} \right] \times \left[ \frac{(s_2 - s_0)}{(s_2 - s_1)(s_1 - s_0)} \right] \propto \left\{ \left[ \frac{(s_{n+3} - s_1)}{(s_{n+3} - s_{n+2})(s_{n+2} - s_1)} \right] \left[ \frac{(s_{n+2} - s_0)}{(s_{n+2} - s_1)(s_1 - s_0)} \right] + n \frac{1}{(s_{n+2} - s_1)^2} \right\}. \tag{A.13}
\]
We denote the right-hand side of Eq. (A.13) as
\[
D(s_{n+3}, s_{n+2}; s_1, s_0) = \frac{1}{(s_{n+3} - s_{n+2})(s_{n+2} - s_1)} + \frac{1}{(s_{n+3} - s_{n+2})(s_1 - s_0)}
+ \frac{1}{(s_{n+2} - s_1)(s_1 - s_0)} + \left( n + 1 \right) \frac{1}{(s_{n+2} - s_1)^2}. \tag{A.14}
\]
For smooth functions \(s(t)\) only the second term on the right-hand side of Eq. (A.14) would be essential, when \(n\) is large. Alternatively, for discontinuous trajectories of the type discussed in Ref. [5] the fourth term is expected to dominate because of the large factor of \(n\).

With thus defined \(D\) we can write the amplitude in the form (3.5) with \(K(\{t_m\})\) given by Eq. (3.4). In spite of the complicated structure of \(K(\{t_m\})\), the Regge behavior of the amplitude (3.5) can be analyzed and is given by Eq. (3.13) for the general expression (A.14).

References

[1] A. M. Polyakov, *Quantum geometry of bosonic strings*, Phys. Lett. B 103, 207 (1981); *Gauge fields and strings*, (Harwood Acad. Pub., Chur, 1987).

[2] H. Aoyama, A. Dhar and M. A. Namazie, *Covariant amplitudes in Polyakov’s string theory*, Nucl. Phys. B 267, 605 (1986).
[3] Y. Makeenko and P. Olesen, *Implementation of the duality between Wilson loops and scattering amplitudes in QCD*, Phys. Rev. Lett. 102, 071602 (2009) [arXiv:0810.4778 [hep-th]].

[4] Y. Makeenko and P. Olesen, *Wilson loops and QCD/string scattering amplitudes*, Phys. Rev. D 80, 026002 (2009) [arXiv:0903.4114 [hep-th]].

[5] P. Buividovich and Y. Makeenko, *Path integral over reparametrizations: Levy flights versus random walks*, Nucl. Phys. B 834 453, (2010) [arXiv:0911.1083 [hep-th]].

[6] Y. Makeenko and P. Olesen, *Quantum corrections from a path integral over reparametrizations*, Phys. Rev. D 82 v.4, (2010) [arXiv:1002.0055 [hep-th]].

[7] C. Lovelace, *Simple N-Reggeon vertex*, Phys. Lett. B 32, 490 (1970).

[8] P. Di Vecchia, *Multiloop amplitudes in string theories*, in *String Quantum Gravity and Physics at the Planck Energy Scale, Erice 1992*, ed. N. Sanchez, (World Scientific, 1993), p. 16;

L. Cappiello, A. Liccardo, R. Pettorino, F. Pezzella, and R. Marotta, *Prescriptions for off-shell bosonic string amplitudes*, Lect. Notes Phys. 525, 466 (1999) [arXiv:hep-th/9812152];

A. Liccardo, F. Pezzella, and R. Marotta, *Consistent off-shell tree string amplitudes*, Mod. Phys. Lett. A 14, 799 (1999) [arXiv:hep-th/9903027].

[9] J. F. Arvis, *The exact ¯qq potential in Nambu string theory*, Phys. Lett. B 127, 106 (1983).

[10] P. Olesen, *Strings and QCD*, Phys. Lett. B 160, 144 (1985).

[11] J. Douglas, *Solution of the problem of Plateau*, Trans. Am. Math. Soc. 33, 263 (1931).

[12] A. M. Polyakov, Talk at the Workshop “Particles, Fields and Strings”, Vancouver, July 1997, unpublished.

[13] V. S. Rychkov, *Wilson loops, D-branes, and reparametrization path integrals*, JHEP 0212, 068 (2002) [arXiv:hep-th/0204250].

[14] L. F. Alday and J. Maldacena, *Gluon scattering amplitudes at strong coupling*, JHEP 0706, 064 (2007) [arXiv:0705.0303 [hep-th]].

[15] J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, Nucl. Phys. B 795, 385 (2008) [arXiv:0707.0243 [hep-th]]; A. Brandhuber, P. Heslop, and G. Travaglini, *MHV Amplitudes in N=4 super Yang-Mills and Wilson loops*, Nucl. Phys. B 794, 231 (2008) [arXiv:0707.1153 [hep-th]]; J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality*, Nucl. Phys. B 795, 52 (2008) [arXiv:0709.2368 [hep-th]].

[16] L. F. Alday and R. Roiban, *Scattering amplitudes, Wilson loops and the string/gauge theory correspondence*, Phys. Rep. 468, 153 (2008) [arXiv:0807.1889 [hep-th]].

[17] E. S. Fradkin and A. A. Tseytlin, *On quantized string models*, Ann. Phys. 143, 413 (1982).

[18] B. Durhuus, P. Olesen, and J.L. Petersen, *On the static potential in Polyakov’s theory of the quantized string*, Nucl. Phys. B 232, 291 (1984).

[19] C. Lovelace, *A novel application of Regge trajectories*, Phys. Lett. B 28, 264 (1968); J. A. Shapiro, *Narrow-resonance model with Regge behavior for ππ scattering*, Phys. Rev. 179, 1345 (1969).
[20] R. Kirschner and L. N. Lipatov, *Double logarithmic asymptotics of quark scattering amplitudes with flavor exchange*, Phys. Rev. D *26*, 1202 (1982); *Double logarithmic asymptotics and Regge singularities of quark amplitudes with flavor exchange*, Nucl. Phys. B *213*, 122 (1983).

[21] S. J. Brodsky, W. K. Tang, and C. B. Thorn, *The reggeon trajectory in exclusive and inclusive large momentum transfer reactions*, Phys. Lett. B *318*, 203 (1993).

[22] J. Bartels and M. Lublinsky, *Quark antiquark exchange in γ∗γ∗ scattering*, JHEP *0309*, 076 (2003) [arXiv:hep-ph/0308181].

[23] A. B. Kaidalov, *Some problems of diffraction at high energies*, arXiv:hep-th/0612358.