NON-EQUILIBRIUM QCD OF HIGH-ENERGY MULTI-GLUON DYNAMICS

Klaus Geiger
Brookhaven National Laboratory, Upton, N.Y. 11973, U.S.A.

I discuss an approach to derive from first principles, a real-time formalism to study the dynamical interplay of quantum and statistical-kinetic properties of non-equilibrium multi-parton systems produced in high-energy QCD processes. The ultimate goal (from which one is still far away) is to have a practically applicable description of the space-time evolution of a general initial system of gluons and quarks, characterized by some large energy or momentum scale, that expands, diffuses and dissipates according to the self- and mutual-interactions, and eventually converts dynamically into final state hadrons. For example, the evolution of parton showers in the mechanism of parton-hadron conversion in high-energy hadronic collisions, or, the description of formation, evolution and freezeout of a quark-gluon plasma, in ultra-relativistic heavy-ion collisions.

1 Introduction

In general, the study of a high-energy multi-particle system and its quantum dynamics involves three essential aspects: first, the aspect of space-time, geometry and the structure of the vacuum; second, the quantum field aspect of the particle excitations; and third, the statistical aspect of their interactions. These three elements are generally interconnected in a non-trivial way by their overall dynamical dependence. Therefore, in order to formulate a quantum description of the complex non-equilibrium dynamics, one needs to find a quantum-statistical and kinetic formulation of QCD that unifies the three aspects self-consistently. The main tools to achieve this are: the closed-time-path (CTP) formalism (for treating initial value problems of irreversible systems), and (ii) transport theory based on Wigner function techniques (for a kinetic description of inhomogenous non-equilibrium systems).

The common feature of high-energy particle collisions is that they allow a distinction between a short-distance quantum field theoretical scale and a larger distance statistical-kinetic scale, which is essentially an effect of ultra-relativistic kinematics. This advantageous property facilitates the passage from exact QCD field theory of coherent non-abelian gauge fields to an approximate quantum kinetic theory of an ensemble of incoherent gluons. When described in a reference frame, in which the particles move close to the speed of light,
the effects of time dilation and Lorentz contraction separate the intrinsic quantum motion of the individual particles from the statistical correlations among them. On the one hand, the quantum dynamics is determined by the self-interactions of the bare quanta, which dress them up to quasi-particles with a substructure of quantum fluctuations. This requires a fully quantum theoretical analysis including renormalization. On the other hand, the kinetic dynamics can well be described statistical-mechanically by the motion of the quasi-particles that is, by binary interactions between these quasi-particles, and by the possible presence of a coherent mean color field that may be induced by the collective motion of the partons. Such a distinct description of quantum and kinetic dynamics is possible, because the quantum fluctuations are highly concentrated around the light cone, occurring at very short distances, and decouple to very good approximation from the kinetic evolution which is dictated by comparably large space-time scales. As mentioned, the natural two-scale separation is just the consequence of time dilation and Lorentz contraction, and is true for any lightcone dominated process. In fact, at asymptotic energies the quantum fluctuations are exactly localized on the lightcone, and so the decoupling becomes perfect. This observation is the key to formulate a quantum kinetic description in terms of particle phase-space densities, involving a simultaneous specification of momentum space and space-time, because at sufficiently high energy, the momentum scale \( \Delta p \) of the individual particles’ quantum fluctuations and the scale \( \Delta r \) of space-time variations of the system of particles satisfy \( \Delta p \Delta r \gg 1 \), consistent with the uncertainty principle.

In what follows, I am guided by the recent paper and the related literature discussed therein, plus on preliminary results of work in progress. For purpose of lucidity, I will henceforth confine myself to pure Yang-Mills theory, i.e. consider gluons only and ignore the quark degrees of freedom. The latter are straightforward to include.

2 Non-equilibrium techniques for QCD

2.1 Basics of the closed-time-path formalism

As proclaimed, the goal is to describe the time evolution of a non-equilibrium quantum system consisting of an initial ensemble of high-energy gluons at starting time \( t_0 \). In this context, the starting point of non-equilibrium field theory is to write down the CTP *in-in amplitude* \( Z_P \) for the evolution of the initial quantum state \( |\text{in} \rangle \) forward in time into the remote future, in the presence of a medium which described by the density matrix. The amplitude
$Z_P$ is formally given by:

$$Z_P[J, \hat{\rho}] = \langle \text{in} | \text{in} \rangle_J = \text{Tr} \left\{ U^\dagger(t_0, t) U(t, t_0) \hat{\rho}(t_0) \right\}_J,$$

where $J = (J^+, J^-)$ is an external source with components on the + and − time branch. $\hat{\rho}(t_0)$ denotes the initial state density matrix, $U$ is the time evolution operator, and $T (T^\dagger)$ denotes the time (anti-time) ordering. Within the CTP formalism the amplitude $Z_P$ can be evaluated by time integration over the closed-time-path $P$ in the complex $t$-plane. This closed-time path extends from $t = t_0$ to $t = t_\infty$ in the remote future along the positive (+) branch and back to $t = t_0$ along the negative (−) branch. where any point on the + branch is understood at an earlier instant than any point on the − branch. With $Z_P$ defined on this closed-time-path, one may then, as in standard field theory, derive from it the Green functions and their equations of motion. The differences between the CTP and the standard field theory, which are briefly summarized below, arise then solely from the different time contour.

The interpretation of this formal apparatus for the evolution along the closed-time path $P$ is rather simple: If the initial state is the vacuum itself, that is, the absence of a medium generated by other particles, then the density matrix $\hat{\rho}$ is diagonal and in (1), one has $|\text{in}\rangle \rightarrow |0\rangle$. In this case the evolution along the + branch is identical to the anti-time ordered evolution along the − branch (modulo an irrelevant phase), and space-time points on different branches cannot cross-talk. In the presence of a medium however, the density matrix contains off-diagonal elements, and there are statistical correlations between the quantum system and the medium particles (e.g. scatterings) that lead to correlations between space-time points on the + branch with space-time points on the − branch. Hence, when addressing the evolution of a multi-particle system, both the deterministic self-interaction of the quanta, i.e. the time (anti-time) ordered evolution along the + (−) branch, and the statistical mutual interaction with each other, i.e. the cross-talk between + and − branch, must be included in a self-consistent manner. The CTP method achieves this through the time integration along the contour $P$. Although for physical observables the time values are on the + branch, both + and − branches will come into play at intermediate steps in a self-consistent calculation.

The convenient feature of this Green function formalism on the closed-time path is that it is formally completely analogous to standard quantum field theory, except for the fact that the fields have contributions from both time branches, and the path-integral representation of the in-in amplitude (1), contains as usual the classical action $I[A]$ and source terms $J \circ A$, but now for both time branches.
\[ Z_P[\mathcal{J}^+, \mathcal{J}^-, \hat{\rho}] = \int \mathcal{D}A^+ \mathcal{D}A^- \exp \left[ i \left( I[A^+] + \mathcal{J}^+ \cdot A^+ \right) - i \left( I^*[A^-] + \mathcal{J}^- \cdot A^- \right) \right] \mathcal{M}[\hat{\rho}] . \] (2)

From this path-integral representation one obtains the \( n \)-point Green functions \( G^{(n)}(x_1, \ldots, x_n) \), which are now
\[ G^{\alpha_1 \alpha_2 \cdots \alpha_n}(x_1, \ldots, x_n), \quad \alpha_i = \pm, \]
depending on whether the space-time points \( x_i \) lie on the + or - time branch, and it is possible to construct a perturbative expansion of the non-equilibrium Green functions in terms of modified Feynman rules (as compared to standard field theory),

(i) The number of elementary vertices is doubled, because each propagator line of a Feynman diagram can be either of the four components of the Green functions. The interaction vertices in which all the fields are on the + branch are the usual ones, while the vertices in which the fields are on the - branch have the opposite sign. On the other hand, combinatoric factors, rules for loop integrals, etc., remain exactly the same as in usual field theory.

(ii) All local 1-point functions, such as the gauge-field or the color current, are ‘vectors’ with 2 components,
\[ \mathcal{A}(x) = \left( A^+ A^- \right), \quad \mathcal{J}(x) = \left( \mathcal{J}^+ \mathcal{J}^- \right) \] (3)

Similarly, all 2-point functions, as the gluon propagator \( \Delta_{\mu\nu} \) and the polarization tensor \( \Pi_{\mu\nu} \), are \( 2 \times 2 \) matrices with components
\[ \Delta(x_1, x_2) = \begin{pmatrix} \Delta^{++} & \Delta^{+-} \\ \Delta^{-+} & \Delta^{--} \end{pmatrix} \quad \Pi(x_1, x_2) = \begin{pmatrix} \Pi^{++} & \Pi^{+-} \\ \Pi^{-+} & \Pi^{--} \end{pmatrix} . \] (4)

Explicitly, the components of the propagator are
\[ \Delta_{\mu\nu}^{F}(x, y) = \Delta_{\mu\nu}^{++}(x, y) = -i \langle T A^+_\mu(x) A^+_{\nu}(y) \rangle \]
\[ \Delta_{\mu\nu}^{<}(x, y) = \Delta_{\mu\nu}^{+-}(x, y) = -i \langle A^+_\mu(y) A^-_{\nu}(x) \rangle \]
\[ \Delta_{\mu\nu}^{>}(x, y) = \Delta_{\mu\nu}^{-+}(x, y) = -i \langle A^-_{\mu}(x) A^+_{\nu}(y) \rangle \]
\[ \Delta_{\mu\nu}^{<}(x, y) = \Delta_{\mu\nu}^{--}(x, y) = -i \langle T A^-_{\mu}(x) A^-_{\nu}(y) \rangle , \] (5)
where $\Delta^F$ is the usual time-ordered Feynman propagator, $\Delta^{\overline{F}}$ is the corresponding anti-time-ordered propagator, and $\Delta^> (\Delta^<)$ is the unordered correlation function for $x_0 > y_0$ ($x_0 < y_0$). In compact notation,

$$\Delta_{\mu\nu}(x, y) = -i \langle T_P A(x) A(y) \rangle ,$$

where the generalized time-ordering operator $T_P$ is defined as

$$T_P A(x) B(y) := \theta_P(x_0, y_0) A(x) B(y) + \theta_P(y_0, x_0) B(y) A(x) ,$$

with the $\theta_P$-function defined as

$$\theta_P(x_0, y_0) = \begin{cases} 
1 & \text{if } x_0 \text{ succeeds } y_0 \text{ on the contour } P \\
0 & \text{if } x_0 \text{ precedes } y_0 \text{ on the contour } P 
\end{cases} .$$

Higher order products $A(x)B(y)C(z)\ldots$ are ordered analogously. Finally, for later use, let me also introduce the generalized $\delta_P$-function defined on the closed-time path $P$:

$$\delta_P^4(x, y) := \begin{cases} 
+\delta^4(x-y) & \text{if } x_0 \text{ and } y_0 \text{ from positive branch} \\
-\delta^4(x-y) & \text{if } x_0 \text{ and } y_0 \text{ from negative branch} \\
0 & \text{otherwise} 
\end{cases} .$$

Henceforth I will not explicitly label the $+, -$ components, unless it is necessary. Instead a compressed notation is used, in which it is understood that, e.g., 1-point functions such as $A(x)$ or $J(x)$, 2-point functions such as $\Delta_{\mu}(x, y)$ or $\Pi_{\mu\nu}(x, y)$, receive contributions from both $+$ and $-$ time branches.

### 2.2 The generating functional for the non-equilibrium Green functions

The amplitude $Z_P$ introduced in \([\text{III}]\) admits a path-integral representation which gives the generating functional for the CTP Green functions defined on closed-time-path $P$:

$$Z_P[J, \hat{\rho}] = \mathcal{N} \int \mathcal{D}A \det \mathcal{F} \delta(f[A]) \exp \left\{ i \left( I [A, J] \right) \right\} \mathcal{M}(\hat{\rho}) ,$$

where $A = (\mathcal{A}^+, \mathcal{A}^-)$ and $J = (\mathcal{J}^+, \mathcal{J}^-)$ have two components, living on the $+$ and $-$ time branches.

The structure of the functional $Z_P$ in \([\text{IV}]\) is the following:
The functional integral (with normalization $\mathcal{N}$) is over all gauge field configurations with measure $\mathcal{D}A \equiv \prod_{\mu,a} \mathcal{D}A^a_\mu$, subject to the condition of gauge fixing, here for the class of non-covariant gauges defined by

$$f^a[A] := \hat{n} \cdot A^a(x) - B^a(x) \quad \Rightarrow \quad \langle \hat{n}^\mu A^a_\mu(x) \rangle = 0 , \quad (11)$$

where $\hat{n}^\mu \equiv \frac{n^\mu}{\sqrt{|n^2|}}$ and $n^\mu$ is a constant 4-vector, being either space-like ($n^2 < 0$), time-like ($n^2 > 0$), or light-like ($n^2 = 0$). With this choice of gauge class the local gauge constraint on the fields $A^a_\mu(x)$ in the path-integral (10) becomes,

$$\det \mathcal{F} \delta (\hat{n} \cdot A^a - B^a) = \text{const} \times \exp \left\{ -\frac{i}{2\alpha} \int_P d^4x \left[ \hat{n} \cdot A^a(x) \right]^2 \right\} = I_{GF} [\hat{n} \cdot A] , \quad (12)$$

where $\det \mathcal{F}$ is the Fadeev-Popov determinant (which in the case of the non-covariant gauges turns out to be a constant factor), and where $\delta (\hat{n} \cdot A) \equiv \prod_a \delta (\hat{n} \cdot A^a)$. The right side translates this constraint into a the gauge fixing functional $I_{GF}$. The particular choice of the vector $\hat{n}^\mu$ and of the real-valued parameter $\alpha$ is dictated by the physics or computational convenience, and distinguishes further within the class of non-covariant gauges [6,7]:

- homogenous axial gauge : $n^2 < 0 \quad \alpha = 0$
- inhomogenous axial gauge : $n^2 < 0 \quad \alpha = 1$
- temporal axial gauge : $n^2 > 0 \quad \alpha = 0$
- lightcone gauge : $n^2 = 0 \quad \alpha = 0 . \quad (13)$

The exponential $I$ is the effective classical action with respect to both the + and the $-$ time contour, $I[A,J] \equiv I[A^+,J^+] - I^*[A^-,J^-]$, including the usual Yang-Mills action $I_{YM} = \int d^4x L_{YM}$, plus the source $J$ coupled to the gauge field $A$:

$$I[A,J] = -\frac{1}{4} \int_P d^4x \mathcal{F}_{\mu\nu}^a(x) \mathcal{F}^{\mu\nu; a}(x) + \int_P d^4x J_\nu^a(x) A^{\mu; a}(x)$$

$$\equiv I_{YM} [A] + \mathcal{J} \circ A , \quad (14)$$

where $\mathcal{F}_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$.
The form of the initial state at $t = t_0$ as described by the density matrix $\hat{\rho}$ is embodied in the function $\mathcal{M}(\hat{\rho})$ which is the density-matrix element of the gauge fields at initial time $t_0$,

$$\mathcal{M}(\hat{\rho}) = \langle A^+(t_0) | \hat{\rho} | A^-(t_0) \rangle \equiv \exp\left(i \mathcal{K}[A]\right),$$  \hspace{1cm} (15)$$

where $A^\pm$ refers to the + and − time branch at $t_0$, respectively. The functional $\mathcal{K}$ may be expanded in a series of non-local kernels corresponding to multi-point correlations concentrated at $t = t_0$,

$$\mathcal{K}[A] = \mathcal{K}^{(0)} + \int_P d^4x \mathcal{K}^{(1)}_{\mu}(x) A^{\mu,a}(x) + \frac{1}{2} \int_P d^4x d^4y \mathcal{K}^{(2)}_{\mu\nu}(x,y) A^{\mu,a}(x) A^{\nu,b}(y) \ldots \hspace{1cm} (16)$$

Clearly, the sequence of kernels $\mathcal{K}^{(n)}$ contains as much information as the original density matrix. In the special case that $\hat{\rho}$ is diagonal, the kernels $\mathcal{K}^{(n)} = 0$ for all $n$, and the usual ‘vacuum field theory’ is recovered.

The path-integral representation (10) can be rewritten in a form more convenient for the following: First, the gauge-fixing functional $I_{GF}[\hat{n} \cdot A]$ is implemented by using (12). Second, the series representation (16) is inserted into the initial state functional $\mathcal{M}(\hat{\rho})$. Third, $\mathcal{K}^{(0)}$ is absorbed in the overall normalization $\mathcal{N}$ of $Z_P$ (henceforth set to unity), and the external source $\mathcal{J}$ in the 1-point kernel $\mathcal{K}^{(1)}$:

$$\mathcal{K}^{(0)} := i \ln \mathcal{N}, \quad \mathcal{K}^{(1)} := \mathcal{K}^{(1)} + \mathcal{J}. \hspace{1cm} (17)$$

Then (10) becomes,

$$Z_P[\mathcal{J}, \hat{\rho}] \implies Z_P[\mathcal{K}] = \int D[A] \exp \left\{ i \left( I [A, \mathcal{K}] \right) \right\},$$  \hspace{1cm} (18)$$

where, instead of (14),

$$I [A, \mathcal{K}] \equiv I_{YM} [A] + I_{GF} [\hat{n} \cdot A] + \mathcal{K}^{(1)} \circ A + \frac{1}{2} \mathcal{K}^{(2)} \circ (A A) + \frac{1}{6} \mathcal{K}^{(3)} \circ (A A A) + \ldots \hspace{1cm} (19)$$
3 Separating soft and hard dynamics and the equations of motion

The first step in the strategy is a separation of soft and hard physics in the path-integral formalism with Green functions of both the soft and hard quanta in the presence of the soft classical field is induced by and feeding back to the quantum dynamics. The basic idea to split up the gauge field $A_\mu$ appearing in the classical action $I_{YM}[A]$ into a soft (long-range) part $A_\mu$, and a hard (short-range) quantum field $a_\mu$:

\[
A_\mu^a(x) = \int \frac{d^4k}{(2\pi)^4} e^{+ik \cdot x} A_\mu^a(k) \theta(\mu - k^0) + \int \frac{d^4k}{(2\pi)^4} e^{+ik \cdot x} A_\mu^a(k) \theta(k^0 - \mu) \equiv A_\mu^a(x) + a_\mu^a(x).
\]

This is the formal definition of the terms ‘soft’ and ‘hard’. The soft and hard physics are separated by a (at this point arbitrary) space-time scale $\lambda \equiv 1/\mu$, so that one may associate the soft field $A_\mu$ being responsible for long range color collective effects, and the hard field $a_\mu$ embodying the short-range quantum dynamics. Consequently, the field strength tensor receives a soft, a hard part, a mixed contribution,

\[
F_\mu^a(x) \equiv \left( F_\mu^a[A] + f_\mu^a[a] + \phi_\mu^a[A,a] \right)(x).
\]

Now comes physics input. Consider the following physics scenario: The initial state is a (dilute) ensemble of hard gluons of very small spatial extent $\ll \lambda$, corresponding to transverse momenta $k_\perp^2 \gg \mu^2$. By definition of $\lambda$, or $\mu$, the short-range character of these quantum fluctuations implies that the expectation value $\langle a_\mu \rangle$ vanishes at all times. However, the long-range correlations of the eventually populated soft modes with very small momenta $k_\perp^2 \ll \mu^2$ may lead to a collective mean field with non-vanishing $\langle A \rangle$. Accordingly, the following condition on the expectation values of the fields is imposed:

\[
\begin{align*}
\langle A_\mu^a(x) \rangle & \left\{ \begin{array}{ll} = 0 & \text{for } t \leq t_0 \\
\geq 0 & \text{for } t > t_0 \end{array} \right. \\
\langle a_\mu^a(x) \rangle & \equiv 0 \text{ for all } t.
\end{align*}
\]

Furthermore, for simplicity the quantum fluctuations of the soft field are ignored, assuming any multi-point correlations of soft fields to be small,

\[
\langle A_{\mu_1}^{a_1}(x_1) \ldots A_{\mu_n}^{a_n}(x_n) \rangle \ll \langle A_{\mu_1}^{a_1}(x_1) \rangle \ldots \langle A_{\mu_n}^{a_n}(x_n) \rangle,
\]

i.e. take $A_\mu$ as a non-propagating and non-fluctuating, classical field.
When quantizing this decomposed theory by writing down the appropriate in-in-amplitude $Z_P$, one must be consistent with the gauge field decomposition into soft and hard components and with the classical character of the former. $M^{(1)}_\mu = 0$, $M^{(2)}_{\mu\nu} \geq 0$. That is, I restrict in the following to a class of non-equilibrium initial states of Gaussian form (i.e. quadratic in the $a_\mu$ fields) and do not consider possible linear force terms.

Substituting the soft-hard mode decomposition (20) with the condition (22) into (18), the functional integral of the in-in amplitude (18) becomes:

$$Z_P[K] = \int DADa \exp \left\{ i \left( I[A] + I[a] + I[A,a] \right) \right\} ,$$  \hspace{1cm} (24)$$
with a soft, hard, and mixed contribution, respectively.

Introducing the connected generating functional for the connected Green functions, denoted by $G^{(n)}$,

$$W_P[K] = -i \ln Z_P[K] ,$$  \hspace{1cm} (25)$$
from which one obtains the connected Green functions $G^{(n)}$ by functional differentiation, in terms of mixed products of $a_\mu$ and $A_\mu$ fields

$$(-i) G^{(n)}_{\mu_1...\mu_n} (x_1, \ldots, x_n) \equiv \frac{\delta}{i \delta K^{(n)}} W_P[K] \bigg|_{K=0} ,$$  \hspace{1cm} (26)$$
where the superscript $(c)$ indicates the ‘connected parts’. Specifically, one finds

$$G^{(1)}_{\mu} (x) = \langle A_\mu^a (x) \rangle_P^{(c)} \equiv \overline{A}_\mu^a (x)$$

$$G^{(2)}_{\mu\nu} (x, y) = \langle a_\mu^a (x) a_\nu^b (y) \rangle_P^{(c)} \equiv i \hat{\Delta}_{\mu\nu}^{ab} (x, y) .$$  \hspace{1cm} (27)$$
These relations define the soft mean field $\overline{A}$ and the hard propagators $\hat{\Delta}$.

The equations of motions for $\overline{A}$ and $\hat{\Delta}$ follow now as in usual field theory by functional differentiation of the effective action,

$$\Gamma_P \left[ \overline{A}, \hat{\Delta} \right] = W_P[K] - K^{(1)} \circ \overline{A} - \frac{1}{2} K^{(2)} \circ \left( i \hat{\Delta} + \overline{A}^2 \right) .$$  \hspace{1cm} (28)$$
Note that the main approximation at this point is the truncation of the infinite hierarchy of equations for the $n$-point Green functions of the exact theory, to the 1-point function (the soft mean field $\overline{A}(x)$) and the 2-point function (the hard propagator $\hat{\Delta}(x,y)$), with all higher-point functions being combinations of these and connected by the 3-gluon and 4-gluon vertices.
3.1 Yang-Mills equation for the ‘soft’ mean field

The equation of motion for the soft field $A_{\mu}(x)$, is given by $\delta \Gamma_P/\delta A = -\mathcal{K}^{(1)} - \mathcal{K}^{(2)} \circ \overline{A}$, from which one obtains, upon taking into account the initial condition $K^{(1)} = 0$, the Yang-Mills equation for $A$:

$$\left[ D^\lambda, \frac{ab}{\mu\nu} \right] (x) = -\hat{j}_a (x) - \int_P d^4 y \mathcal{K}^{(2)}_{\mu\lambda}(x,y) \overline{A}^{\lambda,b}(y),$$

(29)

where $[D, F] = DF - FD$ with the covariant derivative defined as $D^\lambda \equiv \partial^\lambda - ig^2 \mu$, and $F_{\lambda\mu} \equiv F_{\lambda\mu}[\overline{A}] = [D^\lambda, D^\mu]/(-ig)$. The left hand side of (29) may be written as

$$\left[ D^\lambda, \frac{ab}{\mu\nu} \right] (x) = D^{-1}_{(0) \mu\nu} A^{\lambda,b}(x) + \Xi^a (x),$$

(30)

where, upon taking into account the gauge constraint (11), the $\hat{n}_\lambda \hat{n}_\mu A^\lambda$ does not contribute, because $0 = \langle \hat{n} \cdot A \rangle = \hat{n}^\nu A^\nu$, and where

$$\Xi^a (x) = \Xi^a (1) (x) + \Xi^a (2) (x)$$

(32)

$$\Xi^a_{(1) \mu} (x) = -\frac{g}{2} \int_P \prod_{i=1}^2 d^4 x_i V_{(0) \mu\nu\lambda}(x, x_1, x_2) \overline{A}^{\nu\lambda,c}(x_1) \overline{A}^{\lambda,b}(x_2),$$

(33)

$$\Xi^a_{(2) \mu} (x) = \int_P \prod_{i=1}^3 d^4 x_i W_{(0) \nu\mu\lambda\sigma}(x, x_1, x_2, x_3) \overline{A}^{\nu\lambda,c}(x_1) \overline{A}^{\lambda,b}(x_2) \overline{A}^{\lambda,d}(x_3).$$

(34)

On the right hand side of (29), the current $\hat{j}$ is the induced current due to the ‘hard’ quantum dynamics in the presence of the ‘soft’ field $A$:

$$\hat{j}_a (x) = \hat{j}_a^{(1) \mu}(x) + \hat{j}_a^{(2) \mu}(x) + \hat{j}_a^{(3) \mu}(x)$$

(35)

$$\hat{j}_a^{(1) \mu}(x) = -\frac{i g}{2} \int_P \prod_{i=1}^2 d^4 x_i V_{(0) \mu\nu\lambda}(x, x_1, x_2) \overline{A}^{\nu\lambda,b}(x_1) \overline{A}^{\lambda,c}(x_2)$$

(36)

$$\hat{j}_a^{(2) \mu}(x) = -\frac{g^2}{2} \int_P \prod_{i=1}^3 d^4 x_i W_{(0) \mu\nu\lambda\sigma}(x, x_1, x_2, x_3)$$

(37)
\( A^a_b (x_1) \hat{\Delta}^{\lambda \sigma, cd} (x_2, x_3) \) \hspace{1cm} (37)

\[ \hat{j}^{(3)}_{\mu}(x) = -\frac{ig^3}{6} \int \prod_{i=1}^{3} d^4 x_i d^4 y_i \ W_{(0)}^{abcd} (x, x_1, x_2, x_3) \hat{\Delta}^{\nu' \nu', bb'} (x_1, y_1) \times \hat{\Delta}^{\lambda \lambda', cc'} (x_2, y_2) \hat{\Delta}^{\sigma \sigma', dd'} (x_3, y_3) V_{(0)}^{abcd} (y_1, y_2, y_3). \] \hspace{1cm} (38)

Finally, the second term on the right side of (29) is the initial state contribution to the current, which vanishes for \( t = t_0 > t_0 \).

Notice that the function \( \Xi \) on the left hand side of (29) contains the non-linear self-coupling of the soft field \( A \) alone, whereas the induced current \( \hat{j} \) on the right hand side is determined by the hard propagator \( \hat{\Delta} \), thereby generating the soft field.

3.2 Dyson-Schwinger equation for the ‘hard’ Green function

The equation of motion for the ‘hard’ propagator, \( \hat{\Delta}^{ab}_{\mu \nu} (x, y) \), is \( \delta \Gamma_P / \delta \hat{\Delta} = K^{(2)} / (2i) \), from which one finds after incorporating the initial condition \( K^{(1)} = 0 \), the Dyson-Schwinger equation for \( \hat{\Delta} \):

\[ \left[ \left( \hat{\Delta}^{ab}_{\mu \nu} \right)^{-1} - \left( \Delta^{ab}_{(0) \mu \nu} \right)^{-1} + \Pi^{ab}_{\mu \nu} + \hat{\Pi}^{ab}_{\mu \nu} \right] (x, y) = K^{(2)} ab_{\mu \nu} (x, y), \] \hspace{1cm} (39)

where \( \hat{\Delta} \equiv \hat{\Delta} (A) \) is the fully dressed propagator of the ‘hard’ quantum fluctuations in the presence of the ‘soft’ mean field, whereas \( \Delta^{ab}_{(0)} \) is the free propagator. The polarization tensor \( \Pi \) has been decomposed in two parts, a mean-field part, and a quantum fluctuation part. The mean-field polarization tensor \( \Pi \) incorporates the local interaction between the ‘hard’ quanta and the ‘soft’ mean field,

\[ \Pi^{ab}_{\mu \nu} (x, y) = \Pi^{ab}_{(1) \mu \nu} (x, y) + \Pi^{ab}_{(2) \mu \nu} (x, y) \] \hspace{1cm} (40)

\[ \Pi^{ab}_{(1) \mu \nu} (x, y) = \frac{ig}{2} \delta^{\dagger}_{\mu \nu} (x, y) \int P d^4 z V^{abc}_{(0)} (x, y, z) \overline{A}^{\lambda c} (z) \] \hspace{1cm} (41)

\[ \Pi^{ab}_{(2) \mu \nu} (x, y) = \frac{g^2}{6} \delta^{\dagger}_{\mu \nu} (x, y) \int P d^4 z d^4 w W^{abed}_{(0)} (x, y, z, w) \times \overline{A}^{\lambda c} (z) \overline{A}^{\sigma d} (w). \] \hspace{1cm} (42)

plus terms of order \( g^3 \overline{A}^3 \) which one may safely ignore within the present approximation scheme. The fluctuation polarization tensor \( \hat{\Pi} \) contains the quantum self-interaction among the ‘hard’ quanta in the presence of \( \overline{A} \), and is given
by the variation of 2-loop part $\Gamma_p^{(2)}$, of the effective action, $2i\delta\Gamma_p^{(2)}/\delta \hat{\Delta}$,

$$
\hat{\Pi}^{ab}_{\mu\nu}(x, y) = \left( \hat{\Pi}(1) + \hat{\Pi}(2) + \hat{\Pi}(3) + \hat{\Pi}(4) \right)^{ab}_{\mu\nu}(x, y), \quad (43)
$$

$$
\hat{\Pi}(1)^{ab}_{\mu\nu}(x, y) = -g^2 \int d^4 x d^4 y_1 \ W^{abcd}(x, y_1, y_1) \Delta^{\mu\nu_{\lambda\sigma}}(y_1, x_1) \quad (44)
$$

$$
\hat{\Pi}(2)^{ab}_{\mu\nu}(x, y) = -\frac{i g^2}{2} \int \prod_{i=1}^2 d^4 x_i d^4 y_i \ V^{acd}(x, y_i, x_i) \\ \times \hat{\Delta}^{\mu\nu_{\lambda\sigma}}(x, y_1) \hat{\Delta}^{\mu\nu_{\lambda\sigma}}(x, y_2) \quad (45)
$$

$$
\hat{\Pi}(3)^{ab}_{\mu\nu}(x, y) = -\frac{g^4}{4} \int \prod_{i=1}^3 d^4 x_i d^4 y_i d^4 z_i \ W^{acde}(x, y_i, x_i) \\ \times \hat{\Delta}^{\mu\nu_{\lambda\sigma}}(x, y_i) \hat{\Delta}^{\mu\nu_{\lambda\sigma}}(y_i, y_2) \quad (46)
$$

Note that the usual Dyson-Schwinger equation in vacuum is contained in (39) - (47) as the special case when the mean field vanishes, $\mathcal{A}(x) = 0$, and initial state correlations are absent, $K(x, y) = 0$. In this case, the propagator becomes the usual vacuum propagator, since the mean-field contribution $\hat{\Pi}$ is identically zero, and the quantum part $\hat{\Pi}$ reduces to the vacuum contribution.

4 Transition to quantum kinetics

The equations of motion (29) and (39) are non-linear integro-differential equations and clearly not solvable in all their generality. To make progress, one must be more specific and employ now the details of the proclaimed physics scenario, described above.

4.1 Quantum and kinetic space-time regimes

The key assumption is the separability of hard and soft dynamics in terms of the space-time scale $r(\mu) \propto 1/\mu \approx 1 fm$. This implies that one may characterize the dynamical evolution of the gluon system by a short-range quantum
scale $r_{qua} \ll r(\mu)$, and a comparably long-range kinetic scale $r_{kin} \gtrsim r(\mu)$. Low-momentum collective excitations that may develop at the particular momentum scale $g\mu$ are thus well separated from the typical hard gluon momenta of the order $\mu$, if $g \ll 1$. Therefore, collectivity can arise, because the wavelength of the soft oscillations $\sim 1/g\mu$ is much larger than the typical extension of the hard quantum fluctuations $\sim 1/\mu$. Notice that this separation of scales is not an academic construction, but rather is a general property of quantum field theory. A simple example is a freely propagating electron: In this case, the quantum scale is given its the Compton wavelength $\sim 1/m_e$ in the restframe of the charge, and measures the size of the radiative vacuum polarization cloud around the bare charge. The kinetic scale, on the other hand, is determined by the mean-free-path of the charge, which is infinite in vacuum, and in medium is inversely proportional to the local density times the interaction cross-section, $\sim 1/(n_g \sigma_{int})$. Adopting this notion to the present case of gluon dynamics, let me define $r_{qua}$ and $r_{kin}$ as follows:

**quantum scale $r_{qua}$:** Measures the spatial extension of quantum fluctuations associated with virtual and real radiative emission and re-absorption of a given hard gluon, described by the hard propagator $\hat{\Delta}$. It can thus be interpreted as its Compton wavelength, corresponding to the typical transverse extension of the fluctuations and thus inversely proportional to the average transverse momentum,

$$r_{qua} \equiv \hat{\lambda} \simeq \frac{1}{\langle k_\perp \rangle} \quad \langle k_\perp \rangle \geq \mu , \quad (48)$$

where the second relation is imposed by means of the definition (20) of hard and soft modes. Note that $\hat{\lambda}_C$ is a space-time dependent quantity, because the magnitude of $\langle k_\perp \rangle$ is determined by both the radiative self-interactions of the hard gluons and their interactions with the soft field.

**kinetic scale $r_{kin}$:** Measures the range of the long-wavelength correlations, described by the soft mean-field $\overline{A}$, and may be parametrized in terms of the average momentum of soft modes $\langle q \rangle$, such that

$$r_{kin} \equiv \overline{\lambda} \simeq \frac{1}{\langle q_\perp \rangle} \quad \langle q_\perp \rangle \lesssim g\mu , \quad (49)$$

where $\overline{\lambda}$ may vary from one space-time point to another, because the population of soft modes $\overline{A}(q)$ is determined locally by the hard current $\hat{j}$ with dominant contribution from gluons with transverse momentum $\sim \mu$. 

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The above classification of quantum- (kinetic-) scales specifies in space-time the relevant regime for the hard (soft) dynamics, and the separability of the two scales \( r_{\text{qua}} \) and \( r_{\text{kin}} \) imposes the following condition on the relation between space-time and momentum:

\[
\hat{\lambda} \ll \bar{\lambda}, \quad \text{or} \quad \langle k_\perp \rangle \approx \mu \gg g \mu \approx \langle q_\perp \rangle.
\] (50)

The physical interpretation of (50) is simple: At short distances \( r_{\text{qua}} \ll 1/(g\mu) \) a hard gluon can be considered as an incoherent quantum which emits and partly reabsorbs other hard gluons corresponding to the combination of real bremsstrahlung and virtual radiative fluctuations. Only a hard probe with a short wavelength \( \leq r_{\text{qua}} \) can resolve this quantum dynamics. On the other hand, at larger distances \( r_{\text{kin}} \approx 1/(g\mu) \), a gluon appears as a coherent quasi-particle, that is, as an extended object with a changing transverse size corresponding to the extent of its intrinsic quantum fluctuations. This dynamical substructure is however not resolvable by long-wavelength modes \( \geq r_{\text{kin}} \) of the soft field \( \mathcal{A} \).

Accordingly, one may classify the quantum and kinetic regimes, respectively, by associating with two distinct space-time points \( x^\mu \) and \( y^\mu \) the following characteristic scales:

\[
s^\mu \equiv x^\mu - y^\mu \sim \hat{\lambda} = \frac{1}{g\mu}, \quad \partial_s^\mu = \frac{1}{2} \left( \partial_x^\mu - \partial_y^\mu \right) \sim g\mu
\]

\[
r^\mu \equiv \frac{1}{2} (x^\mu + y^\mu) \sim \bar{\lambda} = \frac{1}{\mu}, \quad \partial_r^\mu = \partial_x^\mu + \partial_y^\mu \sim \mu
\] (51)

The kinetic scale is therefore \( g^2\mu^2 \): The effect of the soft field modes of \( \mathcal{A} \) on the hard quanta involves the coupling \( g\mathcal{A} \) to the hard propagator and is of the order of the soft wavelength \( \bar{\lambda} = 1/(g\mu) \), so that one may characterize the soft field strength by

\[
g\mathcal{A}_\mu(r) \sim g\mu, \quad g\mathcal{F}_{\mu\nu}(r) \sim g^2\mu^2,
\] (52)

plus corrections of order \( g^2\mu^2 \) and \( g^3\mu^3 \), respectively, which are assumed to be small.

The quantum scale on the other hand is \( \mu^2 \), because

\[
\hat{\Delta}_{\mu\nu}^{-1} \sim k_\perp^2 \gtrsim \mu^2 \Rightarrow g^2\mu^2 \sim g\mathcal{F}_{\mu\nu},
\] (53)

and one expects that that the short-distance fluctuations corresponding to emission and reabsorption of gluons with momenta \( k_\perp \geq \mu \), are little affected
by the long-range, soft mean field, because the color force $\sim gF$ acting on a gluon with momentum $k_\perp \sim \mu$ produces only a very small change in its momentum.

Concerning the Yang-Mills equation (29), one finds then immediately from the above scale relations that both the derivative terms $\partial^2 A$ and the self-coupling terms $\Xi$ are of the same order and need to be included consistently in order to preserve the gauge symmetry when performing a perturbative analysis. Of course, if the field is weak, $F_{\mu\nu} \ll g\mu^2$, the nonlinear effects contained in the function $\Xi$ of (29) are subdominant, so that in leading order of $g$, the color fields would then behave like abelian fields.

4.2 The kinetic approximation

The realization of the two space-time scales, short-distance quantum and quasi-classical kinetic, allows to reformulate the quantum field theoretical problem as a relativistic many-body problem within kinetic theory. The key element is to establish the connection between the preceding quantum-theoretical description in terms of Green functions and a probabilistic kinetic description in terms of so-called Wigner functions. Whereas the 2-point functions, such as the propagator or the polarization tensor, depend on two separate space-time points $x$ and $y$, their Wigner transform utilizes a mixed space-time/momentum representation, which is particularly convenient for implementing the assumption of well separated quantum and kinetic scales, i.e., that the long-wavelength field $A$ is slowly varying in space-time on the scale of short-range quantum fluctuations. Moreover, the trace of the Wigner transformed propagator is the quantum analogue of the single particle phase-space distribution of gluons, and therefore provides the basic quantity to make the connection with kinetic theory of multi-particle dynamics.

In terms of the center-of-mass coordinate, $r = \frac{1}{2}(x + y)$, and relative coordinate $s = x - y$, of two space-time points $x$ and $y$, eq. (51), one can express any 2-point function $G(x, y)$, such as $\hat{\Delta}, \Pi$, in terms of these coordinates,

$$G_{ab}^{\mu\nu}(x, y) = G_{ab}^{\mu\nu}(r + \frac{s}{2}, r - \frac{s}{2}) \equiv G_{ab}^{\mu\nu}(r, s) \ , \quad \quad (54)$$

The Wigner transform $\mathcal{G}(r, k)$ is then defined as the Fourier transform with respect to the relative coordinate $s$, being the canonical conjugate to the momentum $k$. In general, the necessary preservation of local gauge symmetry leads to additional constraint, but for the specific choice of gauge (11), the
Wigner transform is simply

\[
G(r, s) = \int \frac{d^4k}{(2\pi)^4} e^{-i \cdot s} G(r, k), \quad G(r, k) = \int d^4s e^{i \cdot s} G(r, s).
\]

(55)

The Wigner representation (55) will facilitate a systematic identification of the dominant contributions of the soft field \(A\) to the hard propagator \(\hat{\Delta}\). First one expands both \(A\) and \(\hat{\Delta}\), then one makes an additional expansion in powers of the soft field and of the induced perturbations \(\Delta^{(0)} \sim g\Delta^{(0)}\). On this basis, one isolates and keep consistently terms up to order \(g^2 \mu^2 \Delta^{(0)}\).

To proceed, recall that the coordinate \(r^\mu\) describes the kinetic space-time dependence \(O(\Delta r_{\text{kin}})\), whereas \(s\) measures the quantum space-time distance \(O(\Delta r_{\text{qua}})\). In translational invariant situations, e.g., in vacuum or thermal equilibrium, \(W(r, s)\) is independent of \(r^\mu\) and sharply peaked about \(s^\mu = 0\). Here the range of the variation is fixed by \(\lambda = 1/\mu\), eq. (48), corresponding to the confinement length \(\approx 0.3 \text{ fm}\) in the case of vacuum, or to the thermal wavelength \(\approx 1/T\) in equilibrium. On the other hand, in the presence of a slowly varying soft field \(A\) with a wavelength \(\lambda = 1/(g\mu)\), eq. (49), the \(s^\mu\) dependence is little affected, while the acquired \(r^\mu\) dependence will have a long-wavelength variation. This suggests therefore to neglect the derivatives of \(G(r, k)\) with respect to \(r^\mu\) of order \(g\mu\), relative to those with respect to \(s^\mu\) of order \(\mu\).

Hence one can perform an expansion of the soft field and the hard propagator and polarization tensor in terms of gradients, and keep only terms up to first order, i.e.,

\[
\begin{align*}
\overline{A}_\mu(x) &= \overline{A}_\mu \left( r + \frac{s}{2} \right) \approx \overline{A}_\mu(r) + \frac{s}{2} \partial_r \overline{A}_\mu(r) \\
\overline{A}_\mu(y) &= \overline{A}_\mu \left( r - \frac{s}{2} \right) \approx \overline{A}_\mu(r) - \frac{s}{2} \partial_r \overline{A}_\mu(r) \\
\Delta^{(0)}_{\mu\nu}(x, y) &= \Delta^{(0)}_{\mu\nu}(0, s) \\
\hat{\Delta}_{\mu\nu}(x, y) &= \hat{\Delta}_{\mu\nu}(r, s) \approx \hat{\Delta}_{\mu\nu}(0, s) + s \cdot \partial_r \hat{\Delta}_{\mu\nu}(r, s) \\
\Pi_{\mu\nu}(x, x) &= \Pi_{\mu\nu}(r) + \frac{s}{2} \cdot \partial_r \Pi_{\mu\nu}(r) \\
\hat{\Pi}_{\mu\nu}(x, y) &= \hat{\Pi}_{\mu\nu}(r, s) \approx \hat{\Pi}_{\mu\nu}(0, s) + s \cdot \partial_r \hat{\Pi}_{\mu\nu}(r, s),
\end{align*}
\]

(56)

and furthermore, in order to isolate the leading effects of the soft mean field \(A\) on the hard quantum propagator \(\hat{\Delta}\), one separates the mean field contribution
from the quantum contribution by writing
\[ \hat{\Delta}(r, k) \equiv \hat{\Delta}_{[\mathbf{r}]}(r, k) = \hat{\Delta}_{[\mathbf{r}]}(k) + \delta\hat{\Delta}_{[\mathbf{r}]}(r, k) \] (57)
with a translation-invariant vacuum quantum contribution and a \(r\)-dependent mean field part, respectively,
\[ \hat{\Delta}_{[0]}^{-1} = \hat{\Delta}_{[0]}^{-1}_{\mathbf{A}} \bigg|_{\mathbf{A}=0} = \Delta_{(0)}^{-1} - \hat{\Pi}_{\mathbf{A}=0} \]
\[ \delta\hat{\Delta}_{[\mathbf{r}]}^{-1} = \Delta_{(0)}^{-1} - \Delta_{(0)}^{-1} = -\hat{\Pi} \] (58)
where \(\Delta_{(0)}\) denotes the free propagator, and the \(\overline{\Delta}\) the mean-field propagator, that is, the free propagator in the presence of the mean field, but in the absence of quantum fluctuations.

Given the ansatz (57), with the feedback of the induced soft field to the hard propagator being contained in \(\delta\hat{\Delta}_{[\mathbf{r}]}\), one can expand the latter in powers of the soft field coupling \(g\), and anticipate that it is at most \(g^2\) times the vacuum piece \(\hat{\Delta}_{[0]}\), that is,
\[ \delta\hat{\Delta}_{[\mathbf{r}]}(r, k) = \sum_{n=1,\infty} \frac{1}{n!} \left(g\mathbf{A}(r) \cdot \partial_k\right)^n \hat{\Delta}_{[0]}(k) \simeq g\mathbf{A}(r) \cdot \partial_k \hat{\Delta}_{[0]}(k) , \] (59)
and, to the same order of approximation,
\[ \partial_{\mu} \delta\hat{\Delta}_{[\mathbf{r}]}^{\mu\nu}(r, k) \simeq g(\partial_{\mu} \mathbf{A}) \partial_{\nu} \hat{\Delta}_{[0]}^{\mu\nu} . \]

Inserting now into eqs. (29) and (39) the decomposition (57) with the approximation (59), and keeping consistently all terms up to order \(g^2\mu^2\hat{\Delta}_{[0]}\), one arrives (after quite some journey) at a set of equations that can be compactly expressed in terms of the kinetic momentum \(K\mu\) rather than the canonical momentum \(k\mu\) (as always in the context of interactions with a gauge field). For the class of gauges gauge (13) amounts to the replacements
\[ k_{\mu} \rightarrow K_{\mu} = k_{\mu} - g\mathbf{A}_{\mu}(r) , \quad \partial_{\mu} \rightarrow D_{\mu} = \partial_{\mu} - g\partial_{\mu}\mathbf{A}_{\nu}(r) \partial_{\nu} . \] (60)
and, within the present approximation scheme, one has \(K^2\hat{\Delta} \gg D_{\mu}^2\hat{\Delta}\). The result of this procedure is:
\[ \left[ D_{\lambda}^{, \mu\nu}, F_{\lambda\mu}^{\lambda} \right](r) = -\hat{j}_{\mu}(r) \]
\[ = -g \int \frac{d^4K}{(2\pi)^2} \text{Tr} \left\{ -K_{\mu} \hat{\Delta}_{[\mathbf{r}]}^{\nu,\mu}(r, K) + \hat{\Delta}_{[\mathbf{r}]}^{\nu,\mu}(r, K) K_{\nu} \right\} \] (61)
\[ \left\{ K^2, \hat{\Delta}_{[0]}^{\mu\nu} \right\}(K) = d^{\mu\nu}(K) + \frac{1}{2} \left\{ \hat{\Pi}^{\mu}_{[0],\sigma}(K), \hat{\Delta}_{[0]}^{\sigma\nu}(K) \right\} \] (62)

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One sees that the original Dyson-Schwinger equation reduces in the kinetic approximation to a coupled set of algebraic equations. Recall that (62) and (63) are still $2 \times 2$ matrix equations mix the four different components of $\hat{\Delta} = (\hat{\Delta}_F, \hat{\Delta}^>, \hat{\Delta}^<, \hat{\Delta}^F)$ and of $\hat{\Pi} = (\hat{\Pi}_F, \hat{\Pi}^>, \hat{\Pi}^<, \hat{\Pi}^F)$. For the following it is more convenient to employ instead an equivalent set of independent functions, namely, the \textit{retarded} and \textit{advanced functions} $\hat{\Delta}^{ret}$, $\hat{\Delta}^{adv}$, plus the \textit{correlation function} $\hat{\Delta}^{cor}$, and analogously $\hat{\Pi}$. This latter set is more directly connected with physical, observable quantities, and is commonly referred to as \textit{physical representation}: 

\[
\hat{\Delta}^{ret} = \hat{\Delta}_F - \hat{\Delta}^> \quad \hat{\Delta}^{adv} = \hat{\Delta}_F - \hat{\Delta}^< \quad \hat{\Delta}^{cor} = \hat{\Delta}^> + \hat{\Delta}^<
\]

Similarly, for the polarization tensor the retarded, advanced and correlation functions are defined as (note the subtle difference to (64)): 

\[
\hat{\Pi}^{ret} = \hat{\Pi}_F + \hat{\Pi}^< \quad \hat{\Pi}^{adv} = \hat{\Pi}_F + \hat{\Pi}^> \quad \hat{\Pi}^{cor} = -\hat{\Pi}^> - \hat{\Pi}^<
\]

Loosely speaking, the retarded and advanced functions characterize the intrinsic quantum nature of a ‘dressed’ gluon, describing its substructural state of emitted and reabsorbed gluons, whereas the correlation function describes the kinetic correlations among different such ‘dressed’ gluons. The great advantage of this physical representation is that in general the dependence on the phase-space occupation of gluon states (the local density) is essentially carried by the correlation functions $\hat{\Delta}^>$, $\hat{\Delta}^<$, whereas the dependence of the retarded and advanced functions, $\hat{\Delta}^{ret}$, $\hat{\Delta}^{adv}$, on the local density is weak. More precisely, the retarded and advanced propagators and the imaginary parts of the self-energies embody the renormalization effects and dissipative quantum dynamics that is associated with short-distance emission and absorption of quantum fluctuations, whereas the correlation function contains both the effect of interactions with the soft mean field and of statistical binary scatterings among the hard gluons. In going over to the physical representation, one arrives at the set of master equations: The Yang-Mills equation (61) reads 

\[
\left[ K \cdot \overline{D}_r, \hat{\Delta}_{\mu} \right] (r, K) = -\frac{i}{2} \left[ \hat{\Pi}_{\sigma} (r, K), \hat{\Delta}_{\mu} \right] (K) - \frac{i}{2} \left[ \hat{\Pi}_{\mu} (K), \hat{\Delta}_{\sigma} \right] (K).
\] (63)
and the renormalization (62) and transport equations (63) become

\[
\begin{aligned}
\left\{ K^2, \Delta_{\text{ret}}^{\text{adv}} \right\}_{\mu\nu} &= -\frac{1}{2} \left\{ M^2, \text{Im} \Delta_{\text{ret}} \right\}_{\mu\nu} - \frac{1}{2} \left\{ \Gamma, \text{Re} \Delta_{\text{ret}} \right\}_{\mu\nu} \\
\left[ K \cdot D r, \Delta_{\text{cor}} \right]_{\mu\nu} &= +\frac{i}{2} \left[ \Pi_{\text{cor}}, \text{Re} \Delta_{\text{ret}} \right]_{\mu\nu} - \frac{1}{4} \left\{ \Pi_{\text{cor}}, \text{Im} \Delta_{\text{ret}} \right\}_{\mu\nu} \\
&\quad + \frac{i}{2} \left[ M^2, \Delta_{\text{cor}} \right]_{\mu\nu} - \frac{1}{4} \left\{ \Gamma, \Delta_{\text{cor}} \right\}_{\mu\nu},
\end{aligned}
\]

where \( \Pi = \Pi + \hat{\Pi} \), and the real and imaginary components of the polarization tensor are denoted by

\[
\begin{aligned}
\Pi^2_{\mu\nu} &\equiv \text{Re} \Pi_{\mu\nu} = \frac{1}{2} \left( \Pi_{\text{ret}} + \Pi_{\text{adv}} \right)_{\mu\nu} \quad \Gamma_{\mu\nu} &\equiv \text{Im} \Pi_{\mu\nu} = i \left( \Pi_{\text{ret}} - \Pi_{\text{adv}} \right)_{\mu\nu}
\end{aligned}
\]

Note also, that are the real and imaginary components of the Hard propagator are given by the sum and difference of the retarded and advanced contributions, respectively,

\[
\begin{aligned}
\text{Re} \Delta_{\mu\nu} &= \frac{1}{2} \left( \Delta_{\text{ret}} + \Delta_{\text{adv}} \right)_{\mu\nu} \quad \text{Im} \Delta = i \left( \Delta_{\text{ret}} - \Delta_{\text{adv}} \right)_{\mu\nu}.
\end{aligned}
\]

The physical significance of the (67) and (68) is the following: Eq. (67) determines the state of a dressed parton with respect to their virtual fluctuations and real emission (absorption) processes, corresponding to the real and imaginary parts of the retarded and advanced self-energies. Eq. (68), on the other hand characterizes the correlations among different dressed parton states, and the self-energies appear here in two distinct ways. The first two terms on the right hand side account for scatterings between quasi-particle states, i.e. dressed partons, whereas the last two terms incorporate the renormalization effects which result from the fact that the dressed partons between collisions do not behave as free particles, but change their dynamical structure due to virtual fluctuations, as well as real emission and absorption of quanta. For this reason \( \Pi^\rightarrow \) is called radiative self-energy, and \( \Pi_{\text{cor}} \) is termed collisional self-energy. It is well known 2 that the imaginary parts of the retarded and advanced Green functions and self-energies are just the spectral density \( \rho = \text{Im} \Delta \), giving the probability for finding an intermediate multi-particle state in the dressed parton, respectively the decay width \( \Gamma \), describing the dissipation of the dressed parton. The formal solution of (67) and (68) for the spectral density \( \rho \) is

\[
\rho(r, k) = \frac{\Gamma}{k^2 - M^2 + (\Gamma/2)^2} = \rho_M^2 + \rho_{\Gamma},
\]

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describing the particle density in terms of the finite width \( \Gamma \) and the dynamical ‘mass term’ \( M^2 \) (which in the ‘free-field’ case are \( \Gamma = M^2 = 0 \), corresponding to an on-shell, classically stable particle). On the right hand side of (71), the second form exhibits the physical meaning more suggestively in terms of the ‘wavefunction’-renormalization \( \rho_{M^2} = \rho_{\Gamma=0} \) due to virtual fluctuations, and the dissipative parts \( \rho_{\Gamma} = \rho_{M^2=0} \) due to real emission (absorption) processes.

5 Outlook

What remains to be done is to solve the set of equations (66)-(68) which is the hardest part. For the case of \( \mathcal{A}_\mu = 0 \), this was discussed in Ref. 4. For the present general case, the coupling between hard gluons (\( \hat{\Delta} \)) and the soft field (\( \mathcal{A} \)), complicates things considerably. A possible iterative scheme of solution, which is currently under study, may be as follows:

a) Specify initial condition in terms of a phase-space density of hard gluons at time \( t = t_0 \). This initial gluon distribution determines \( \hat{\Delta}^0(t = t_0, \vec{r}, k) \).

b) Solve the renomalization equation (62) with \( \mathcal{A}(t_0, \vec{r}) = 0 \), i.e. just as in the case of vacuum 4, except that now \( K = k - g\mathcal{A} \) contains the soft field. Substitute the resulting form of \( \hat{\Delta}^0_{\text{cor}} \) and \( \hat{\Delta}^0_{\text{adv}} \) into the transport equation (63) to get the solution for \( \hat{\Delta}^0_{\text{cor}} \).

c) Insert \( \hat{\Delta}^0_{\text{cor}} \) into the right hand side of the Yang-Mills equation (66), and solve for \( \mathcal{A} \).

d) Repeat from a) but now include the finite contribution from the coupling between \( \hat{\Delta}^0_{\text{adv}} \) and \( \mathcal{A} \).

6 References

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