RATIONAL SINGULARITIES OF COMMUTING VARIETIES OVER SMALL RANK MATRICES

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Abstract. Let $g = \mathfrak{sl}_3$ defined over an algebraically closed field $k$ of characteristic 0, and let $\mathcal{N}$ be the nilpotent cone of $g$. For each $r \geq 1$, define the nilpotent commuting variety $C_r(\mathcal{N})$ to be the subvariety of $\mathcal{N}^r$ consisting of mutually commuting $r$-tuples. We prove in this note that $C_r(\mathcal{N})$ has rational singularities; hence it is Cohen-Macaulay. Our approach is applying the theory of equivariant resolutions of singularities. This strategy also reduces the study of the same property of $C_r(g)$ to that of $C_r(b)$, the commuting variety of lower triangular matrices.

1. Introduction

Given a normal variety (or scheme) $X$, one call it has rational singularities if there is a resolution of singularities of $X$, i.e. a proper birational map $f : Y \to X$ from a smooth variety $Y$, such that $R^if_*\mathcal{O}_Y = 0$ for $i > 0$. Alternatively, a ring $R$ can also be called having rational singularities if $\text{Spec}(R)$ does. Over the last few decades, the study about rings (or varieties) having rational singularities has been central in commutative algebra. Such property of a ring implies Cohen-Macaulayness of the ring, but not as nice as being regular [Ho]. Hence it has applications to many subjects in modern algebra. Well-known examples for such objects include determinantal rings in commutative algebra and the nilpotent cone of a reductive Lie algebra in representation theory of algebraic groups. We study in this note the property of having rational singularities for commuting varieties over 3 by 3 matrices.

In general for any irreducible closed subvariety $V$ of $\mathfrak{gl}_n$, we denote by

$$C_r(V) = \{(x_1, \ldots, x_r) \in V^r \mid [x_i, x_j] = 0\}$$

the commuting variety of $r$-tuples over $V$. In the case when $V = \mathcal{N}$, the nilpotent cone of $\mathfrak{gl}_n$, we then call $C_r(\mathcal{N})$ the nilpotent commuting variety (of $r$-tuples). For an overview of such commuting varieties, readers can refer to the papers [N][GN]. In summary, with respect to the values of $r$ and $n$ both $C_r(\mathfrak{gl}_n)$ and $C_r(\mathcal{N})$ are reducible except for small value of $r$ or $n$ [Si][GN]. Whence they are irreducible, one would like to explore further nice properties of them such as normality, Cohen-Macaulayness, or having rational singularities. Note that $C_2(\mathfrak{gl}_n)$ is conjectured to be Cohen-Macaulay by Artin and Hochster [MS]. Computer calculations only work for the case up to $n = 4$ [Hr]. Recently, Charbonnel has proved a more general result that $C_2(g)$ is normal and Cohen-Macaulay for any reductive Lie algebra $g$ [C]. The analogous problem for commuting varieties of $r$-tuples, on the other hand, remains unknown. Recall that the nilpotent commuting variety $C_r(\mathcal{N})$ for 2 by 2 matrices was verified to have rational singularities by the author in [N]. Indeed, it can be implied from the paper that all commuting varieties of $r$-tuples over 2 by 2 matrices have rational singularities. Furthermore, we proved that the nilpotent commuting variety over rank 2 matrices is Cohen-Macaulay in arbitrary characteristic $p \neq 2$ of $k$. It is pointed out that the obvious defining ideal of $C_r(\mathcal{N})$ in general is not radical. This causes many difficulties in showing Cohen-Macaulayness from methods of commutative algebra. In this paper, to improve our aforementioned results to rank 3 matrices, we develop the theory of equivariant rational singularities in a similar way as in [Br].
Let \( H \) be an algebraic group acting rationally on an affine variety \( X \). Suppose \( f : Y \to X \) is a proper birational map from a smooth variety \( Y \) to \( X \). If in addition \( Y \) is a \( H \)-variety and \( f \) is \( H \)-equivariant (see definitions in Section 2), then it is called a \( H \)-equivariant resolution of singularities of \( X \). The study of such resolutions was stimulated by a result of Sumihiro in the paper [S], where he proved that every normal \( H \)-variety \( X \) has an \( H \)-equivariant completion. Then the theory was developed by works of Abramovich and Wang, Reichstein and Youssin [AW] [RY]. Brion used this idea to investigate singularities of orbit closures in spherical varieties [Br1].

The paper is organized as follows. Section 2 provides definitions and notation of the terminology in the paper and then reviews of some background needed for later arguments. Then we prove the existence of equivariant rational resolutions in Section 3. In particular, we show that if an \( H \)-variety \( X \) has rational singularities, then it has an \( H \)-equivariant rational resolution. Section 3 includes our calculations on the vanishing of \( R^i \text{ind}_B^G(-) \) for certain modules in the case when \( G = \text{SL}_3 \). Our main method in this section is to employ the Koszul resolution as in [So]. Next these vanishing results are applied to prove the normality of the nilpotent commuting variety (Section 5). Here we imitate the strategy of Thomsen in [Th] (see also [KLT]). Finally, the main result on rational singularities of nilpotent commuting varieties is obtained in Section 6. Our proof relies heavily on Grothendieck spectral sequence and calculations from [Br2].

2. Notation

2.1. Algebraic groups and Lie algebras. Let \( k \) be an algebraically closed field of characteristic 0. Let \( G \) be an algebraic group over \( k \). Fix a maximal torus \( T \subset G \), and let \( \Phi \) be the root system of \( T \) in \( G \). Let \( \Phi^+ \) be the corresponding set of positive roots. Let \( B \subseteq G \) be the Borel subgroup of \( G \) containing \( T \) and corresponding to the set of negative roots \( \Phi^- \), and let \( U \subset B \) be the unipotent radical of \( B \). Set \( \mathfrak{g} = \text{Lie}(G) \), the Lie algebra of \( G \), \( \mathfrak{b} = \text{Lie}(B) \), \( \mathfrak{u} = \text{Lie}(U) \).

Given a vector space \( V \), we denote by \( S^n(V) \) and \( \Lambda^n(V) \) the symmetric and exterior space of degree \( n \) over \( V \). Then the direct sums
\[
\bigoplus_{n=0}^{\infty} S^n(V) \quad \text{and} \quad \bigoplus_{n=0}^{\infty} \Lambda^n(V)
\]
are the symmetric algebra and exterior algebra of \( V \). Define \( V^* = \text{hom}_k(V, k) \) the dual space of \( V \).

2.2. Induction functor. Let \( M \) be a \( B \)-module. Then the induced \( G \)-module can be defined as
\[
\text{ind}_B^G M = (k[G] \otimes_k M)^B.
\]
The higher derived functor of \( \text{ind}_B^G(-) \) is denoted by \( R^i \text{ind}_B^G(-) \). We recall two vanishing results of Charbonnel and Zaïer which play important roles in our computation.

Proposition 2.2.1. [CZ] Proposition 2.6 and B.1] For each \( r \geq 1 \) and \( n \geq 0 \), we have
\[
\begin{align*}
\text{(a)} & \quad R^i \text{ind}_B^G S^n(u^*) = 0 \text{ for all } i > 0, \\
\text{(b)} & \quad R^i \text{ind}_B^G S^n(b^*) = 0 \text{ for all } i > 0.
\end{align*}
\]

2.3. Adjoint action. The group \( G \) acts on the Lie algebra \( \mathfrak{g} \) by the adjoint action denoted by “\( \cdot \)”, and called \( G \)-action. Note that \( \mathcal{N} \) is stable under this action and \( \mathfrak{b}, \mathfrak{u} \) are stable under the \( B \)-action, restriction of \( G \)-action on the Borel subgroup \( B \). For every positive integer \( r \), the \( G \)-action on the direct product \( \mathfrak{g}^r \) is defined diagonally, i.e. \( g \cdot (x_1, \ldots, x_r) = (g \cdot x_1, \ldots, g \cdot x_r) \) for all \( g \in G \) and \( x_i \in \mathfrak{g} \). It also restricts to the \( B \)-action on \( \mathfrak{b}^r \) and \( \mathfrak{u}^r \).

Let \( X \) be a closed subvariety of \( \mathfrak{g}^r \) and \( H \) be a subgroup of \( G \) (\( B \) or \( G \)). Then \( X \) is called an \( H \)-variety if it is stable under the \( H \)-action. This action induces that on the ring \( k[X] \) so we call it an \( H \)-algebra. A morphism between two \( H \)-varieties \( f : X \to Y \) is called \( H \)-equivariant if it commutes with the actions on both varieties.
2.4. Basic algebraic geometry conventions. Suppose $X$ is a variety. Then we always write $k[X]$ for the ring of global sections $\mathcal{O}_X(X)$ on $X$. In case $X$ is affine, it coincides with the coordinate ring of $X$.

For each variety (or scheme) $X$, a morphism $\pi : \tilde{X} \to X$ is called a resolution of singularities if the variety $\tilde{X}$ is non-singular and $\pi$ is proper and birational. If in addition $X$ is normal and the higher direct image of $\pi$ vanishes, i.e., $R^i\pi_*\mathcal{O}_{\tilde{X}} = 0$ for all $i > 0$, then we call $\pi$ a rational resolution, or we say that $X$ has rational singularities. Note that this vanishing condition is equivalent to $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for all $i > 0$ provided that $X$ is affine, [Ha Proposition III.8.5]. This notion can also be applied to a commutative ring $R$ if we replace $X$ by $\text{Spec}(R)$. Suppose that $\pi$ is $H$-equivariant, then the resolution above is called $H$-equivariant resolution of singularities (resp. $H$-equivariant rational resolution).

2.5. Invariant theory. Suppose $R$ is a finitely generated algebra with a group $H$ acting rationally on $R$ as an automorphism (e.g. the adjoint action). The notation $R^H$ stands for the invariant algebra of $R$ under this action. The following classical result will be applied later in our paper.

**Proposition 2.5.1.** [BV 7.D(a)] If $R$ is a normal domain, then so is $R^H$.

2.6. Associated bundles. Let $X$ be an $H$-variety. The associated bundle over $G/H \times X$. It is known that the ring of regular functions (global sections) on $G/H \times X$ coincides with the ring of $H$-invariant regular functions on $G \times X$,

$$k[G \times H X] \cong k[G \times X]^H \cong (k[G] \otimes k[X])^H = \text{ind}_B^G k[X].$$

2.7. Commutative algebra. Consider an $m \times n$ matrix

$$M = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

whose entries are independent indeterminates over the field $k$. Let $k[M]$ be the polynomial ring over all the indeterminates of $M$, and let $I_t(M)$ be the ideal in $k[M]$ generated by all $t$ by $t$ minors of $M$. For each $t \geq 1$, the ring

$$R_t(M) = \frac{k[M]}{I_t(M)}$$

is called a determinantal ring. The following is one of the nice properties of determinantal rings.

**Proposition 2.7.1.** [BV] For every $1 \leq t \leq \min(m, n)$, $R_t(M)$ is a reduced, Cohen-Macaulay, normal domain of dimension $(t - 1)(m + n - t + 1)$. Furthermore, it has rational singularities.

We denote by $D_t(M)$ the determinantal variety defined by $I_t(M)$.

3. Equivariant rational resolution

3.1. Existence. We begin with a $H$-variety $X$. The following result is an immediate corollary of the Exe. 7 in [Cut], Section 6.8.

**Proposition 3.1.1.** If $X$ has rational singularities, then there exists a $H$-equivariant rational resolution of $X$.

**Proof.** As $X$ is a $H$-variety, from [Cut] Exe. 7, Section 6.8] there is a $H$-equivariant resolution of singularities $\pi : \tilde{X} \to X$ for $X$. On the other hand, the rational singularities of $X$ and Remark iv (or Lemma 1) in [V] imply that $\pi$ must carry the property of being rational singularities. Hence the result follows. □
Example 3.1.2. Consider $G$ a reductive algebraic group. The first well-known example of a variety having $G$-rational singularities is the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. Of course, this is trivial as the Springer resolution $G \times \mathbf{B} \mathfrak{u} \rightarrow \mathcal{N}$ already shows us everything about $G$-equivariant properties.

A non-trivial example which will be applied in our later context is the commuting variety $C_r(\mathfrak{u})$ with $r \geq 2$. Here we consider $\mathfrak{u}$ as the space of strictly lower triangular $3 \times 3$ matrices in $\mathfrak{sl}_3$. It can be observed that $C_r(\mathfrak{u})$ is a closed subvariety of the affine space $\mathfrak{u}^r$ and that it is stable under the $\mathbf{B}$-action. On the other hand, witting down the commutator equations of $C_r(\mathfrak{u})$, one can see that it can be identified with the determinantal variety of $2 \times 2$ minors over a $2 \times r$ matrix of indeterminates (see [N, Proposition 7.1.1]). So it has rational singularities, see for example [BV, Theorem 11.23]. Then the above proposition confirms the existence of a $\mathbf{B}$-equivariant rational resolution of $C_r(\mathfrak{u})$.

Let assume for the rest of the paper that $G = SL_3$, $\mathfrak{g} = \mathfrak{sl}_3$, and $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$, and that $\mathfrak{b}, \mathfrak{u}$ are Lie algebras of the Borel subgroup of $G$ and its unipotent radical.

4. Vanishing of induced modules

We prove in this section that the nilpotent commuting variety $C_r(\mathcal{N})$ has rational singularities. In fact we will show that the coordinate algebra $k[C_r(\mathcal{N})]$ is isomorphic to $\text{ind}^G_B k[C_r(\mathfrak{u})]$ (see Example 3.1.2(2)). To do this we use methods in [KLT] (see also [Th]) and ingredients in [CZ].

4.1. Vanishing result. Let $\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$ be the set of positive roots of the root system $\Phi$ of $G$. We show in this section a result about the vanishing of certain line bundle cohomology.

Theorem 4.1.1. For each $r \geq 1$, we have

$$R^i \text{ind}^G_B (S^n(\mathfrak{u}^{\star r}) \otimes \lambda) = 0$$

for all $n \geq 0$ and $i > 0$, and $\lambda \in \Phi^+$. 

Proof. We proceed by induction on $n$. Obviously, it is true for $n = 0$. Then suppose that it is true for all $n < m$. Note that the vanishing occurs when $\lambda = 0$ by Proposition 2.2.1. Observe also that if it holds for $\lambda \in \{\alpha, \beta\}$, then so does it for $\lambda = \alpha + \beta$ by exploiting the long exact sequence of the following short exact sequence

$$0 \rightarrow \mathfrak{u}^{\star r} \otimes (\alpha \times \beta) \rightarrow \mathfrak{u}^{\star (r+1)} \otimes (\alpha + \beta) \rightarrow 0.$$ 

So it suffices to prove that $R^i \text{ind}^G_B (S^m(\mathfrak{u}^{\star r}) \otimes \beta) = 0$ for $i > 0$ as the argument for the case when $\lambda = \alpha$ is very similar.

Consider the short exact sequence

$$0 \rightarrow \mathfrak{u}^{\star r} \rightarrow \mathfrak{u}^{\star r} \rightarrow u^{\star r}_\alpha \rightarrow 0.$$ 

The Koszul resolution gives us the long exact sequence

$$0 \rightarrow S^{m-r}(\mathfrak{u}^{\star r}) \otimes \Lambda^r(\alpha^r) \rightarrow S^{m-r+1}(\mathfrak{u}^{\star r}) \otimes \Lambda^{r-1}(\alpha^r) \rightarrow \cdots \rightarrow S^m(\mathfrak{u}^{\star r}) \rightarrow S^m(u^{\star r}_\alpha) \rightarrow 0.$$ 

Tensoring with $\beta$ and splitting into short exact sequences, we have

$$0 \rightarrow S^{m-r}(\mathfrak{u}^{\star r}) \otimes \Lambda^r(\alpha^r) \otimes \beta \rightarrow S^{m-r+1}(\mathfrak{u}^{\star r}) \otimes \Lambda^{r-1}(\alpha^r) \otimes \beta \rightarrow L_1 \rightarrow 0,$$

$$\vdots$$

$$0 \rightarrow L_i \rightarrow S^{m-r+i}(\mathfrak{u}^{\star r}) \otimes \Lambda^{r-i}(\alpha^r) \otimes \beta \rightarrow L_{i+1} \rightarrow 0,$$

$$\vdots$$

$$0 \rightarrow L_r \rightarrow S^m(\mathfrak{u}^{\star r}) \otimes \beta \rightarrow S^m(u^{\star r}_\alpha) \otimes \beta \rightarrow 0.$$ 

\[\text{In order to get } R^i \text{ind}^G_B (S^m(\mathfrak{u}^{\star r}) \otimes (\alpha + \beta)) = 0, \text{ we only need to show that } R^i \text{ind}^G_B (u^{\star r} \otimes (\alpha + \beta)) = 0 \text{ for all } i > 0, \text{ see details in Proposition B.1 in [CZ]}.\]
As $\alpha$ is a one dimensional vector space, we have $\Lambda^j(\alpha^r)$ is a direct sum of $\alpha^{\otimes j}$ so that $\Lambda^j(\alpha^r) \otimes \beta$ is simply a direct sum of copies of $\beta$ or $\alpha + \beta$ (depending upon the parity of $j$). Hence, using the long exact sequence induced from each short exact sequence above for $R^i \text{ind}^G_B(-)$ and combining the inductive hypotheses, we obtain $R^i \text{ind}^G_B(L_r) = 0$ for each $i > 0$. Therefore, $R^i \text{ind}^G_B(S^n(u^r) \otimes \beta) = 0$ by the simple lemma in [KLT, 3.2], and so complete our proof. \hfill $\Box$

4.2. **Analogs for $S^n(b^r)$**. Similar results can be established when replacing $u$ by $b$ as following

**Proposition 4.2.1.** For each $r \geq 1$, we have

$$R^i \text{ind}^G_B(S^n(b^r) \otimes \lambda) = 0$$

for all $n \geq 0$ and $i > 0$, and $\lambda \in \Phi^+$. 

**Proof.** We start our inductive argument by showing $R^i \text{ind}^G_B(b^r \otimes \lambda) = 0$. Indeed, consider

$$0 \rightarrow t^r \otimes \lambda \rightarrow b^r \otimes \lambda \rightarrow u^r \otimes \lambda \rightarrow 0.$$ 

The result follows from the induced long exact sequence by $R^i \text{ind}^G_B(-)$ and the vanishing of $R^i \text{ind}^G_B(u^r \otimes \lambda) = 0$ for $i > 0$. Now suppose $R^i \text{ind}^G_B(S^n(b^r) \otimes \lambda) = 0$ for all $n < m$. Applying Kozul resolution on the above short exact sequence (set $\lambda = 0$), we obtain

$$0 \rightarrow S^{m-t}(b^r) \otimes \Lambda^t(t^r) \rightarrow S^{m-t+1}(b^r) \otimes \Lambda^{t-1}(t^r) \rightarrow \cdots \rightarrow S^n(b^r) \rightarrow S^n(u^r) \rightarrow 0$$

where $t = \min(m, \dim t^r)$. Tensoring with $\lambda$ and splitting into short exact sequences, we have

$$0 \rightarrow S^{m-r}(b^r) \otimes \Lambda^i(t^r) \otimes \lambda \rightarrow S^{m-r+1}(b^r) \otimes \Lambda^{i-1}(t^r) \otimes \lambda \rightarrow L_1 \rightarrow 0,$$

$$\vdots$$

$$0 \rightarrow L_i \rightarrow S^{m-t+i}(b^r) \otimes \Lambda^{t-i}(t^r) \otimes \lambda \rightarrow L_{i+1} \rightarrow 0,$$

$$\vdots$$

$$0 \rightarrow L_t \rightarrow S^n(b^r) \otimes \lambda \rightarrow S^n(u^r) \otimes \lambda \rightarrow 0.$$ 

Note that $R^i \text{ind}^G_B(S^{m-t+i}(b^r) \otimes \Lambda^{t-i}(t^r) \otimes \lambda) = \Lambda^{t-i}(t^r) \otimes R^i \text{ind}^G_B(S^{m-t+i}(b^r) \otimes \lambda) = 0$ from the inductive hypothesis. So the result follows. \hfill $\Box$

5. **Normality of $C_r(\mathcal{N})$**

Recall that the commuting variety (of $r$-tuples) over $X \subset \mathfrak{g}$ is defined by

$$C_r(X) = \{(x_1, \ldots, x_r) \in X^r \mid [x_i, x_j] = 0\}.$$ 

We shall focus in this paper on the commuting varieties with $X = u, b, \mathcal{N}$, and $\mathfrak{g}$. Note that there is a moment map from $G \times^B C_r(u)$ onto $C_r(\mathcal{N})$ which induces an injective $G$-equivariant algebra homomorphism from $k[\mathcal{N}]$ into $k[G \times^B C_r(u)]$. In this specific section, we show that this induced homomorphism is actually onto so that it is an isomorphism. Hence, the normality follows from that of $k[G \times^B C_r(u)]$ (by Proposition 2.5.1).

5.1. We start with a lemma.

**Lemma 5.1.1.** There is a surjective $G$-equivariant homomorphism of rings from $k[G \times^B u^r]$ to $k[G \times^B C_r(u)]$.

**Proof.** First we consider the canonical homomorphism $k[u^r] \rightarrow k[C_r(u)]$ induced by the embedding of varieties $q : C_r(u) \hookrightarrow u^r$. If we identify $k[u^r]$ with the polynomial ring $R = k[x_i, y_i, z_i \mid 1 \leq i \leq r]$, then $k[C_r(u)]$ can be written as the quotient ring $R/I$ where $I$ is an ideal of $R$ generated by the polynomials $x_i z_j - x_j z_i$ for all $1 \leq i \leq j \leq r$, [N] 7.1(6)]. Next, we make another identification by
setting \( R = S^*(u^r) \) and \( I = S^*(u^r) \otimes (\oplus_{1 \leq i, j \leq r} (\alpha + \beta)) \) as each \( x_i z_j - x_j z_i \) is of weight \( \alpha + \beta \). So we have the following short exact sequence of \( B \)-modules
\[
0 \to S^*(u^r) \otimes \left( \bigoplus_{1 \leq i \leq j \leq r} (\alpha + \beta) \right) \to S^*(u^r) \to R/I \to 0.
\]

Apply the induction functor and Theorem 4.1.1, we obtain the short exact sequence of \( G \)-modules
\[
0 \to \text{ind}_B^G S^*(u^r) \otimes \left( \bigoplus_{1 \leq i \leq j \leq r} (\alpha + \beta) \right) \to \text{ind}_B^G S^*(u^r) \to \text{ind}_B^G(R/I) \to 0.
\]

Note that this sequence is also compatible with the ring structure on each term. Note also that \( \text{ind}_B^G S^*(u^r) \) is isomorphic to \( k[G \times B u^r] \) and \( \text{ind}_B^G(R/I) \) is isomorphic to \( k[G \times B C_r(u)] \). We finally have the desired surjective map \( q^* : k[G \times B u^r] \to k[G \times B C_r(u)] \).

The short exact sequence (1) combined with Theorem 4.1.1 and Proposition 2.5.1 immediately give us the following

**Proposition 5.1.2.** For each \( r \geq 1 \), we have \( R^i \text{ind}_B^G k[C_r(u)] = 0 \) for all \( i > 0 \).

5.2. Now we are ready for proving the result of this section.

**Theorem 5.2.1.** For each \( r \geq 1 \), there is a \( G \)-equivariant isomorphism of \( k \)-algebras between \( k[C_r(N)] \) and \( k[G \times B C_r(u)] \).

**Proof.** Consider the commutative diagram of \( G \)-equivariant maps
\[
\begin{array}{ccc}
G \times B C_r(u) & \xrightarrow{m'} & C_r(N) \\
\downarrow{q} & & \downarrow{e} \\
G \times B u^r & \xrightarrow{m} & G \cdot u^r
\end{array}
\]
where \( m \) is the moment map and \( m' \) is its restriction, \( q \) is induced from the embedding \( C_r(u) \hookrightarrow u^r \), and \( e \) is also the embedding from \( G \cdot C_r(u) \) into \( G \cdot u^r \). It follows the commutative diagram
\[
\begin{array}{ccc}
k[G \times B C_r(u)] & \xrightarrow{m^*} & k[C_r(N)] \\
\downarrow{q^*} & & \downarrow{e^*} \\
k[G \times B u^r] & \xrightarrow{m^*} & k[G \cdot u^r].
\end{array}
\]
We have known that \( m^* \) is injective. By the surjectivity of \( q^* \) in Lemma 5.1.1 and the fact that \( m^* \) is an isomorphism \([\text{CZ}, \text{Theorem 4.3(ii)}]\), we obtain the surjectivity of \( m^* \) so that it is an isomorphism.

**Corollary 5.2.2.** The variety \( C_r(N) \) is normal for every \( r \geq 1 \).

**Proof.** Follow immediately from the above theorem and Proposition 2.5.1.

5.3. **Commuting variety** \( C_r(\mathfrak{g}) \). Very similar arguments as in the previous subsection can be applied (hence we omit the proof here) to show the following

**Proposition 5.3.1.** For each \( r \geq 1 \), there is a \( G \)-equivariant isomorphism of \( k \)-algebras between \( k[C_r(\mathfrak{g})] \) and \( k[G \times B C_r(b)] \).

The result suggests that the variety \( C_r(\mathfrak{g}) \) is normal if \( C_r(b) \) is so. However, the normality for \( C_r(b) \) is not known for arbitrary \( r \). It is rather easy in the case when \( r = 2 \) since \( C_2(b) \) is complete intersection. Then computer calculations can establish the normality. For higher \( r \), new techniques are required to tackle this problem.
6. Equivariant rational singularities

6.1. Now we apply methods in [Br1] (see proof of Theorem 5) to prove our main result of the paper.

**Theorem 6.1.1.** For each $r \geq 1$, the variety $C_r(N)$ has $G$-equivariant rational singularities.

**Proof.** For each $r \geq 1$, let $\pi : X \to C_r(u)$ be a $B$-equivariant rational resolution of $X$. We first construct a resolution of singularities for $C_r(N)$ as follows. Consider the composition

$$m' \circ \tilde{\pi} : G \times B X \to G \times B C_r(u) \to C_r(N)$$

where the morphism $\tilde{\pi}$ is induced from $\pi$ and defined by $(g, x) \mapsto (g, \pi(x))$ for all $g \in G$ and $x \in X$. In fact, $\tilde{\pi}$ is a $G$-equivariant resolution of $G \times B C_r(u)$ (from the proof of [Br1] Theorem 5)). Note that the morphism $m'$ is proper birational by Proposition 3.4.3 in [N] and of course, it is $G$-equivariant. Hence the composition map $m' \circ \tilde{\pi}$ is a $G$-equivariant resolution of singularities of $C_r(N)$.

As $C_r(N)$ is normal from the previous section, we only need to verify that

$$R^i(m' \circ \tilde{\pi})_* O_{G \times B X} = 0$$

for all $i > 0$. Indeed, Grothendieck spectral sequence (see [Jan] Proposition I.4.1)) gives us that

$$E_2^{i,j} = R^i m'_* (R^j \tilde{\pi}_* O_{B \times B X}) \Rightarrow R^{i+j} (m' \circ \tilde{\pi})_* O_{G \times B X}.$$

On the other hand, we have

$$E_2^{i,j} = R^i m'_* (G \times B R^j \pi_* O_X).$$

The property of having rational singularities of $C_r(u)$ implies that $R^j \pi_* O_X = 0$ for $n > 0$ and $\pi_* O_X = O_{C_r(u)}$. So the spectral sequence collapses and yields

$$R^i(m' \circ \tilde{\pi})_* O_{G \times B X} \cong R^i m'_* (G \times B O_{C_r(u)}) = R^i m'_* O_{G \times B C_r(u)}.$$

Now applying [Ha, Proposition III.8.5], it is equivalent to showing that

$$H^i(G \times B C_r(u), O_{G \times B C_r(u)}) = 0$$

for all $i > 0$. Now consider

$$H^0(G \times B C_r(u), O_{G \times B C_r(u)}) = k[G \times B C_r(u)] \cong \text{ind}_B^G k[C_r(u)] \cong \text{ind}_B^G H^0(C_r(u), O_{C_r(u)}).$$

So again, using Grothendieck spectral sequence, we have

$$R^i \text{ind}_B^G \circ H^j(C_r(u), O_{C_r(u)}) \Rightarrow H^{i+j}(G \times B C_r(u), O_{G \times B C_r(u)}).$$

As $C_r(u)$ is affine, the higher cohomology on the structure sheaf $O_{C_r(u)}$ vanishes, so that the spectral sequence collapses. Hence applying Proposition 5.1.2 we obtain the vanishing on the right-hand side. This proves our theorem. \qed

As the property of having rational singularities implies Cohen-Macaulayness, this result also gives us the following.

**Corollary 6.1.2.** For each $r \geq 1$, the variety $C_r(N)$ is Cohen-Macaulay.
6.2. Our strategy can be applied for $C_r(b)$ to obtain the following

**Proposition 6.2.1.** If $C_r(b)$ has rational singularities, then so does $C_r(g)$.

**Proof.** The key points for adapting arguments for $C_r(N)$ to $C_r(g)$ are the properness and birationality of the moment map

$$m' : G \times^B C_r(b) \rightarrow C_r(g).$$

In fact, it is true for $g = sl_n$. Let $U$ be the subset of $C_r(g)$ such that the first component consists of regular matrices. Then observe that the set of regular matrices is open in $g$ (see [NS, Proposition 1]) so that $U$ is its preimage under the projection to the first component; hence $U$ is open. Now for each $x \in U$, we have the centralizer $Z_G(x)$ is in $B$. Therefore, it induces an isomorphism on the open sets $m'^{-1}(U)$ and $U$. This proves the birationality. The properness follows from similar argument as in [N, Proposition 3.4.3]. The rest of the proof will follow exactly the same as in the preceding subsection. □

7. Open questions

There are many things to do to complete the study in this paper. We recommend the following.

1. Prove the results for the case of positive characteristics. One should be aware of that the definition of having rational singularities then would be involving much more technical details (which were vanished in characteristic 0).

2. Although the nilpotent commuting variety $C_r(N)$ does not have rational singularities for higher ranks (since it is not irreducible and normal), we could hope the analogous properties hold for its well-known irreducible component $C_r = G \cdot (x_{reg}, z_{reg} \cap N, \ldots, z_{reg} \cap N)$ where $x_{reg}$ is a regular element of $N$ and $z_{reg}$ is the centralizer of $x_{reg}$. In fact, we conjecture that $C_r$ has rational singularities for all $r \geq 1$.

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