EMBEDDED CURVES AND GROMOV-WITTEN INVARIANTS OF THREE-FOLDS

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Abstract. Associated with a prime homology class \( \beta \in P_2(X, \mathbb{Z}) \) (i.e. \( \beta = p\alpha \) and \( \alpha \in H_2(X, \mathbb{Z}) \) imply \( p = 1 \) or \( p \) is an odd prime) on a symplectic three-manifold with vanishing first Chern class, we count the embedded perturbed pseudo-holomorphic curves in \( X \) of a fixed genus \( g \) to obtain certain integer valued invariants analogous to Gromov-Witten invariants of \( X \).

1. Introduction and main theorems

The aim of the work in this paper, is to construct some integer valued invariants of the symplectic threefolds with vanishing first Chern class, along the lines that Gromov-Witten invariants are defined by Ruan and Tian \([13, 14]\). The motivation for this project is the conjecture of Gopakumar and Vafa, which writes the Gromov-Witten invariants of a Calabi-Yau threefold \( X \) in terms of some (not mathematically defined) integer valued invariants, called the Gopakumar-Vafa invariants.

In their paper \([4]\), Gopakumar and Vafa introduce certain counts of the so-called BPS-states to get integer valued invariants of Calabi-Yau threefolds. These are integers \( n_h(\alpha) \) associated with a genus \( h \) and a homology class \( \alpha \in H_2(X, \mathbb{Z}) \), called Gopakumar-Vafa invariants.

The mathematical definition of these invariants is yet to be understood. There has been an attempt by Hosono, Saito and Takahashi \([5]\) to define these numbers through some intersection cohomology construction. They are not able to show the invariance of these numbers and also it is not clear that they satisfy the Gopakumar-Vafa equation predicted in \([4]\).

The Gopakumar-Vafa equation is a generating function equation, which relates Gopakumar-Vafa invariants with Gromov-Witten invariants. More
precisely, it reads as:

\[
\sum_{g \geq 0} N_g(\beta)q^g \lambda^{2g-2} = \sum_{k>0} \frac{1}{k} \sum_{h \geq 0} n_h(\alpha) (2 \sin(\frac{k\lambda}{2}))^{2h-2} q^{k\alpha}.
\]

One may look at this equation as a definition for Gopakumar-Vafa invariants (see [3]) and then it is still to be shown that these numbers are in fact integers.

One other way to follow the motivation provided by Gopakumar and Vafa, is trying to write Gromov-Witten invariants in terms of some other integer valued invariants of \((X, \omega)\), when \(X\) is a symplectic 3-fold with vanishing first Chern class, or more, a Calabi-Yau 3-fold. For instance, the relation between Gromov-Witten invariants and Donaldson-Thomas invariants suggested in [11] follows these lines.

Fix a genus \(g > 1\). As an attempt toward such a construction, we introduce the invariants

\[
I_g : P_2(X, \mathbb{Z}) \rightarrow \mathbb{Z},
\]

which assign integer numbers to any prime homology class \(\beta \in P_2(X, \mathbb{Z})\). A prime homology class is defined to be any homology classes \(\beta \in H_2(X, \mathbb{Z})\) such that if \(\beta = p\alpha\) with \(1 < p \in \mathbb{Z}\) and \(\alpha \in H_2(X, \mathbb{Z})\), then \(p \neq 2\) is a prime number.

We will embed the coarse universal curve \(\mathcal{C}_g\) of the moduli space \(\mathcal{M}_g\) (of genus \(g\) curves) in a projective space \(\mathbb{P}^N\). Correspondingly we will consider a generic almost complex structure \(J\) on the tangent bundle of the three-fold \(X\) and a generic perturbation term

\[
v \in \Gamma(\mathbb{P}^N \times X, p_1^* \Omega_{\mathbb{P}^N}^{0,1} \otimes J p_2^* T' X).
\]

If the choice of \((J, v)\) is generic enough, there are only finitely many pairs \((f, j\Sigma)\) of a complex structure \(j\Sigma \in \mathcal{M}_g\) on a surface \(\Sigma\) of genus \(g\) and a map \(f : \Sigma \rightarrow X\), such that \(f_*[\Sigma]\) is the fixed prime homology class \(\beta\) and such that

\[
\overline{\partial}_{j\Sigma, J} f = (\pi_{\Sigma} \times f)^* v.
\]

Here \(\pi_{\Sigma} : (\Sigma, j\Sigma) \rightarrow \mathcal{C}_g \subset \mathbb{P}^N\) is the map from \((\Sigma, j\Sigma)\) to the universal curve.
In fact if we denote the space of all possible pairs \((J,v)\), with \(J\) an almost complex structure on the tangent bundle of \(X\), compatible with \(\omega\), and with \(v\) a perturbation term on \(\mathbb{P}^N \times X\) with respect to \(J\), by \(\mathcal{P}\), then we show

**Theorem 1.1.** Given \(\beta \in P_2(X,\mathbb{Z})\) and \(g > 1\), there is a Bair subset \(\mathcal{P}_{\text{reg}} \subset \mathcal{P}\) such that for any \((J,v) \in \mathcal{P}_{\text{reg}}\), the above problem has only finitely many solutions. The linearization \(L\) of the perturbed Cauchy-Riemann equation

\[
\mathcal{L}_{J^\Sigma,J} f = (\pi_\Sigma \times f)^* v
\]

at any solution will have a trivial kernel and a trivial cokernel. Moreover, to any such solution is assigned a sign

\[
\epsilon(f, j_\Sigma) \in \{-1, +1\},
\]

coming from the spectral flow \(SF\), which connects \(L\) to a complex \(\overline{\partial}\) operator.

Using the claim of this theorem, we define

**Definition 1.2.** Suppose that \(\beta \in P_2(X,\mathbb{Z})\) and that \(g > 1\) is a fixed genus. Fix a pair \((J,v) \in \mathcal{P}_{\text{reg}}\) as above and define

\[
\mathcal{I}_g(\beta) = \sum_{(f,j_\Sigma)} \epsilon(f, j_\Sigma),
\]

where \(\epsilon(f, j_\Sigma)\) is understood to be zero if \((f, j_\Sigma)\) is not a solution to the above problem.

Furthermore, we show

**Theorem 1.3.** If \((J_0,v_0)\) and \((J_1,v_1)\) are in \(\mathcal{P}_{\text{reg}}\), then the number \(\mathcal{I}_g(\beta)\) as computed using \((J_0,v_0)\), is the same as the number \(\mathcal{I}_g(\beta)\) as computed using \((J_1,v_1)\). Moreover, this number does not depend on the the special embedding of \(\mathcal{U}_g\) in the specific projective space \(\mathbb{P}^N\), and is independent of the of the choice of the symplectic form \(\omega\) in its isotopy class.

We hope that this construction can be extended to all the homology classes in \(H_2(X,\mathbb{Z})\). Once this is done, the Gromov-Witten invariants may be written down in terms of these counts of embedded curves in the symplectic three-folds with vanishing first Chern class, giving an equation similar to equation (1) above.

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2. A Review of Gromov-Witten Invariants

Suppose that $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$ are homology classes in a symplectic $n$-manifold $X$ and $\alpha \in H_2(X, \mathbb{Z})$ is a class in the second homology of $X$ with $\mathbb{Z}$ coefficients. Let

$$\Phi^X_{g, \alpha}(\alpha_1, \ldots, \alpha_k | \beta_1, \ldots, \beta_l)$$

denote the genus $g$ Gromov-Witten invariant associated with these elements of $H_*(X, \mathbb{Z})$ as defined by Ruan and Tian in [13]. In fact, to define these invariants, consider some fixed pseudo-manifold representatives of the homology classes $\alpha_i$ and $\beta_j$, which we still denote by $\alpha_i$ and $\beta_j$, for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Also fix a complex structure $j$ on a surface $\Sigma$ of genus $g$, together with $k$ marked points $x_1, \ldots, x_k$ on $\Sigma$. Associated with the symplectic structure $\omega$ on $X$ is the space of compatible almost complex structures $J$ on the tangent bundle of $X$, which will be denoted by $J$. With $j$ and $J \in J$ fixed, by a perturbation term we mean an element

$$v \in \Gamma(\Sigma \times X, \pi_1 \Omega^0_{J}(\Sigma) \otimes_{J} \pi_2 T^* X).$$

Ruan and Tian fix $\Sigma, j, x_1, \ldots, x_k$ and choose a generic pair $(J, v)$ of an almost complex structure and a corresponding perturbation term, and count the number of solutions to the following problem:

\begin{equation}
\begin{aligned}
f : \Sigma & \longrightarrow X, \\
f_*[\Sigma] & = \alpha \in H_2(X, \mathbb{Z}), \\
\overline{\partial}_{j, f} f & = df + J \circ df \circ j = (\text{Id} \times f)^* v, \\
f(x_i) & \in \alpha_i, \quad i = 1, \ldots, k,
\end{aligned}
\end{equation}

such that the image $f(\Sigma)$ intersects all of $\beta_j$’s. With an appropriate choice of the sign, this will give an invariant of the isotopy class of $\omega$ and the homology classes $\alpha, \alpha_i, \beta_j$, and the genus $g$, if $X$ is a semi-positive symplectic manifold and

$$\sum_{i=1}^{k} (2n - \deg(\alpha_i)) + \sum_{j=1}^{l} (2n - 2 - \deg(\beta_i)) = 2c_1(X)\alpha + 2n(1 - g).$$

Using these invariants, they define the quantum cup product and show that it is associative (see [13]).

Later, they defined the higher genus Gromov-Witten invariants coupled with gravity. The main idea is to relax the almost complex structure $j$ on the punctured Riemann surface

$$\Sigma - \{x_1, x_2, \ldots, x_k\}$$

in the equation (2) to vary in the moduli space $\mathcal{M}_{g,k}$ of $k$-pointed Riemann surfaces of genus $g$. This way, the classes $\alpha_i$ and $\beta_j$ will be treated in a uniform way, so we may assume that $l = 0$. The result is an invariant
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$$
\Psi_{X}^{\alpha,g,k} : (H_{*}(X,\mathbb{Z}))^{k} \rightarrow \mathbb{Q},
$$
which is zero unless the dimension criterion is satisfied:

$$
\sum_{i=1}^{k} (2n - 2 - \deg(\alpha_i)) = 2c_{1}(X)\alpha + 2(n - 3)(1 - g).
$$

These are rational numbers because of the orbifold structure on the universal curve $\mathcal{C}_{g,k}$ of the moduli space $\mathcal{M}_{g,k}$ of genus $g$ Riemann surfaces with $k$ marked points.

In fact, in the definition of Gromov-Witten invariants coupled with gravity by Ruan and Tian [14], they consider the compactified moduli space of $k$ pointed genus $g$ Riemann surfaces $\overline{\mathcal{M}}_{g,k}$ and the universal curve

$$
\pi_{g,k} : \overline{\mathcal{C}}_{g,k} \rightarrow \overline{\mathcal{M}}_{g,k}
$$
over it. The moduli spaces $\overline{\mathcal{M}}_{g,k}, \overline{\mathcal{C}}_{g,k}$ are both projective varieties sitting in a projective space which we denote by $\mathbb{P}^{N}$. A natural candidate for the perturbation terms associated with an almost complex structure $J$ on $X$ can be an element of

$$
\text{Hom}_{J}^{0,1}(T\mathbb{P}^{N}, TX) = \Gamma(\mathbb{P}^{N} \times X, p_{1}^{*}\Omega_{\mathbb{P}^{N}}^{0,1} \otimes_{J} p_{2}^{*}T'X).
$$

Then, one will consider the maps $f$ from a Riemann surface $(\Sigma, j_{\Sigma}, x_1, \ldots, x_k) \in \overline{\mathcal{M}}_{g,k}$ to $X$ satisfying $f_{*}[\Sigma] = \alpha \in H_{2}(X,\mathbb{Z})$, with certain constrains $f(x_i) \in \alpha_{i}$ as above, which also satisfy

$$
\bar{\partial}_{J,j_{\Sigma}} f = (\pi_{\Sigma} \times f)^{*}v.
$$
Here $\pi_{\Sigma} : (\Sigma, j_{\Sigma}, x_1, \ldots, x_k) \rightarrow \overline{\mathcal{C}}_{g,k}$ is the map to the universal curve.

However, these perturbation terms do not rule out the technical difficulties in the construction of Ruan and Tian [14]. The problem is that if a Riemann surface with marked points $(\Sigma, j_{\Sigma}; x_1, \ldots, x_k)$ admits an automorphism group $\mathfrak{G}$, then the map $\pi_{\Sigma}$ will factor through $\overline{\Sigma^{\mathfrak{G}}}$ and the transversality and compactness arguments fail to work.

One is forced to go to a finite "fine" cover of the moduli space $\overline{\mathcal{C}}_{g,k}$ to achieve the transversality and compactness. Naturally, one has to divide the final count of pseudo-holomorphic curves by the degree of this covering. This will cook up invariants of the symplectic manifold which are rational numbers.

In particular for a Calabi-Yau threefold, the only chance for $\Psi$ to be non-zero is for $k = 0$. Here associated with each $\alpha \in H_{2}(X,\mathbb{Z})$ and each genus $g$, we will get a number $N_{g}(\alpha) = \Psi_{X}^{\alpha,g,0}()$ which is naturally a rational number.
as noted above.

In this paper, we will try to avoid this detour to fine covers of the moduli space \( \mathcal{M}_g \). The special nature of dimension three and the vanishing of the first Chern class \( c_1(X) \) will help us out of some of the technical difficulties, at least for the prime homology classes.

The main idea we want to pursue in this paper comes from the work of Taubes in relating Seiberg-Witten invariants to some Gromov type invariants, which we will discuss now.

Suppose that \( X \) is a symplectic manifold of real dimension 4 with a symplectic form \( \omega \). Taubes has defined in [15], an invariant

\[
(3) \quad \text{Gr} : H_2(X, \mathbb{Z}) \to \mathbb{Z}
\]

which assigns an integer to any homology class in \( H_2(X, \mathbb{Z}) \). The definition of these invariants relies on counting the pseudo-holomorphic submanifolds of \( X \) with respect to a generic almost complex structure \( J \), which is compatible with \( \omega \). The special nature of dimension 4 is used to exclude the convergence of a sequence of pseudo-holomorphic submanifolds to a singular pseudo-holomorphic curve, or a convergence of the curves in a class \( n\beta \) to a curve, multiply covered by this sequence, and representing the class \( \beta \in H_2(X, \mathbb{Z}) \). The special case of the convergence of tori, with topologically trivial normal bundle, and representing a class \( n\beta \), to a torus in class \( \beta \), is more delicate to exclude. As is clear from the above discussion, these invariants are pretty much similar to the Gromov-Witten invariants of Ruan and Tian (the "coupled with gravity" version).

Taubes assigns certain weights to each of these tori, which may be different from \( \pm 1 \). These signs enable him to conclude that a passage from an almost complex structure \( J_0 \), which is generic enough so that there are only finitely many pseudo-holomorphic curves in a class \( \beta \in H_2(X, \mathbb{Z}) \) with respect to \( J_0 \), to another generic almost complex structure \( J_1 \), does not change the total count.

These invariants are shown to be equivalent to the Seiberg-Witten invariants for manifolds with \( b_2^+ > 1 \) in the celebrated papers [16, 17] of Taubes, and the non triviality of them is a result of the non triviality of the Seiberg-Witten invariants.

We will start a similar project in dimension 6 (complex dimension 3) through the next couple of sections.
3. Construction of the invariants

In this section we will introduce an invariant

\[ I_g : P_2(X, \mathbb{Z}) \rightarrow \mathbb{Z} \]

for any genus \( g > 1 \), which assigns an integer to any prime homology class \( \beta \in P_2(X, \mathbb{Z}) \). A prime homology class is defined to be any homology classes \( \beta \in H_2(X, \mathbb{Z}) \) such that if \( \beta = p\alpha \) with \( 1 < p \in \mathbb{Z} \) and \( \alpha \in H_2(X, \mathbb{Z}) \), then \( p \neq 2 \) is a prime number. We hope that this construction can be extended to all the homology classes in \( H_2(X, \mathbb{Z}) \).

Let us begin by embedding the moduli space of genus \( g \) Riemann surfaces with one marked point in some projective space \( \mathbb{P}^N \). More precisely, suppose that \( \mathcal{M}_g \) and \( \overline{\mathcal{M}}_g \) denote the moduli space of genus \( g \) curves and its compactification. The coarse universal curve over \( \mathcal{M}_g \) and \( \overline{\mathcal{M}}_g \) will be denoted by \( \mathcal{C}_g \) and \( \overline{\mathcal{C}}_g \), respectively. It is a well-known fact that these moduli spaces are in fact projective varieties. Thus, we may assume that \( \overline{\mathcal{C}}_g \) is embedded in some projective space \( \mathbb{P}^N \). Fix such an embedding. For a complex curve \( (\Sigma, \gamma_\Sigma) \), note that there is a natural holomorphic map \( \pi_\Sigma \) defined as

\[ \pi_\Sigma = q_\Sigma \circ \tau_\Sigma : (\Sigma, \gamma_\Sigma) \xrightarrow{\tau_\Sigma} \overline{\mathcal{C}}_g \subset \mathbb{P}^N, \]

where \( \tau_\Sigma \) is the map going from \( \Sigma \) to its quotient by its automorphism group and \( q_\Sigma \) is the embedding of the quotient in the universal curve \( \overline{\mathcal{C}}_g \).

If \( (X, \omega) \) is a symplectic threefold with vanishing first Chern class, i.e. \( c_1(X) = 0 \), we may denote the space of almost complex structures, compatible with \( \omega \), by \( \mathcal{J} = \mathcal{J}_\omega \). The elements of \( \mathcal{J}_\omega \) are the homomorphism

\[ J : TX \rightarrow TX \]

with the property that \( J \circ J = -\text{Id} \) and \( \omega(\zeta, J\zeta) > 0 \) for any \( \zeta \in T_xX \).

Given any compatible almost complex structure \( J \in \mathcal{J} \), we may consider the space of associated perturbation terms

\[ \text{Hom}^{0,1}_J(T\mathbb{P}^N, TX) = \Gamma(\mathbb{P}^N \times X, p_1^*\Omega_{\mathbb{P}^N}^{0,1} \otimes_J p_2^*TX), \]

where \( p_1, p_2 \) denote the projections to the first and the second factor, respectively.
For a fixed genus $g > 1$ and a prime homology class $\beta \in P_2(X, \mathbb{Z})$, we are interested in counting the solutions to the following enumerative problem: Find the number of all somewhere injective maps $f$ such that

$$f : (\Sigma, J_\Sigma) \longrightarrow X$$

$$f_*[\Sigma] = \beta \in P_2(X, \mathbb{Z})$$

$$\overline{\partial}_{J_\Sigma, J} f = (\pi_\Sigma \times f)^* v,$$

where $J$ is a fixed generic almost complex structure on $X$, compatible with $\omega$ and $v$ is a fixed generic perturbation term $v \in \text{Hom}^{0,1}_J(T\mathbb{P}^N, TX)$.

Let us denote the space of all possible pairs $(J, v)$ by $\mathcal{P}$. One may think of $\mathcal{P}$ as an infinite dimensional bundle over $J$. The first major theorem of this chapter is:

**Theorem 3.1.** Given $\beta \in P_2(X, \mathbb{Z})$, there is a Bair subset $\mathcal{P}_{\text{reg}} \subset \mathcal{P}$ such that for any $(J, v) \in \mathcal{P}_{\text{reg}}$, the above problem has only finitely many solutions. The linearization $L$ of the perturbed Cauchy-Riemann equation

$$\overline{\partial}_{J_\Sigma, J} f = (\pi_\Sigma \times f)^* v$$

at any solution will have a trivial kernel and a trivial cokernel. Moreover, to any such solution is assigned a sign

$$\epsilon(f, j_\Sigma) \in \{-1, +1\},$$

coming from the spectral flow $SF$, which connects $L$ to a complex $\overline{\partial}$ operator.

Using the claim of this theorem, we define

**Definition 3.2.** Suppose that $\beta \in P_2(X, \mathbb{Z})$ and that $g > 1$ is a fixed genus. Fix a pair $(J, v) \in \mathcal{P}_{\text{reg}}$ as above and define

$$\mathcal{I}_g(X, \beta) = \sum_{(f, j_\Sigma)} \epsilon(f, j_\Sigma),$$

where $\epsilon(f, j_\Sigma)$ is understood to be zero if $(f, j_\Sigma)$ is not a solution to the above problem.

The second major result is the following:

**Theorem 3.3.** If $(J_0, v_0)$ and $(J_1, v_1)$ are in $\mathcal{P}_{\text{reg}}$, then the number $\mathcal{I}_g(X, \beta)$ as computed using $(J_0, v_0)$, is the same as the number $\mathcal{I}_g(X, \beta)$ as computed using $(J_1, v_1)$. Moreover, this number does not depend on the the special embedding of $\overline{\mathcal{T}}_g$ in the specific projective space $\mathbb{P}^N$, and is independent of the choice of $\omega$ in its isotopy class.

Once we can prove the first claim, the other claims are completely standard in the theory of Gromov-Witten invariants. Thus we will not come back to this issue.
4. Transversality

In this section we will formulate the above moduli problem in terms of zeros of a certain section of an infinite dimensional bundle over some Banach space. In fact, the spaces we will be dealing with are not Banach spaces, but one can take Sobolov completions of these spaces and prove regularity results, as is standard in the theory of Gromov-Witten invariants (c.f. [13, 14]). We choose to hide this detour to Banach spaces, in order to simplify the descriptions.

To begin, note that the moduli spaces $\mathcal{C}_g$ and $\mathcal{M}_g$ are stratified according to the automorphism group of the curves $(\Sigma, j_\Sigma) \in \mathcal{M}_g$. For a fixed topological action of a group on a (possibly nodal) Riemann surface $\Sigma$, denoted by $G$, let $\mathcal{M}_g^G$ denote the (open) subvariety of $\mathcal{M}_g$, consisting of the curves $(\Sigma, j_\Sigma)$ with automorphism group $G$, with the topological type of the action being specified by $G$.

Denote the part of the coarse moduli space $\mathcal{C}_g$ that lies over $\mathcal{M}_g^G$ by $\mathcal{C}_g^G$.

Let $\mathcal{X}$ denote the space of maps $f$ from $\Sigma$ to $X$, which are somewhere injective and represent the prime homology class $\beta \in P_2(X, \mathbb{Z})$. Fix the topological action $\mathfrak{G}$ and denote the topological quotient by $C = \Sigma \mathfrak{G}$. Denote by $\mathcal{X}^{\mathfrak{G}}$ the space of maps representing the class $\beta$ which factor through the quotient map $\tau_\Sigma$:

$$f = g \circ \tau_\Sigma : \Sigma \xrightarrow{\tau_\Sigma} C = \Sigma \mathfrak{G} \xrightarrow{\mathfrak{G}} X,$$

with $g$ somewhere injective. In particular, the domain of any map in $\mathcal{X}^{\mathfrak{G}}$ will be contained in the closure of $\mathcal{C}_g^{\mathfrak{G}}$. It is then implied that $\beta = [\mathfrak{G}] \cdot \alpha$ for some class $\alpha \in H_2(X, \mathbb{Z})$. So, either the action is trivial, or the underlying group of the action is $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime number $p \neq 2$. We will keep denoting this action by $\mathfrak{G}$, except if it is necessary to do otherwise, for two reasons. The first one is to distinguish the underlying group, from the topological action, which contains more combinatorial information. The second reason is that, most of the work in this paper does not use the fact that this group is $\mathbb{Z}_p$, and may be used in a more general context.

We make a remark that for the moduli space $\mathcal{X}^{\mathfrak{G}}$, we will also include the maps of the form

$$f = g \circ \tau : (\Sigma, j_\Sigma) \xrightarrow{\tau} \Sigma \mathfrak{G} \xrightarrow{\mathfrak{G}} X$$

where $\mathfrak{G}$ is a quotient of the automorphism group $\mathfrak{H}$ of $(\Sigma, j_\Sigma)$, but the map $g$ can not be factored through any other nontrivial quotient of $\Sigma \mathfrak{G}$. Moreover
the topological action of \( G \) on \((\Sigma, j_\Sigma)\) is assumed implicit in the notation.

Over the appropriate Banach versions of these moduli spaces of maps (consisting of \((k, p)\)-maps) one may construct a Banach bundle as follows: First of all let \( J \) denote the space of almost complex structures on \( X \). Let \( P \) be the bundle over \( J \) defined over a point \( J \in J \) by

\[
P_J = \text{Hom}^{0, 1}_J(T^N P, TX).
\]

Let \( \mathcal{Y}^G \) be the space \( P \times \mathcal{M}_g^G \times \mathcal{X}^G \). For a point

\[
\mu = ((J; v), j_\Sigma, (f : \Sigma \xrightarrow{\tau_\Sigma} \Sigma \xrightarrow{g} X)) \in \mathcal{Y}^G
\]

define the bundle \( E^G \) by setting

\[
[E^G]_\mu := \Gamma_G(\Sigma, \Omega^{0, 1}_{j_\Sigma} \otimes f^*TX),
\]

where \( \Gamma_G \) denotes the \( G \)-invariant sections of the corresponding bundle. In reality, \( E \) should be chosen to be the bundle of sections in an appropriate Sobolov space. More precisely, we should consider \((k, p)\)-sections of these bundles and then prove some regularity results. The details of these arguments will be left to the reader.

The Cauchy-Riemann operator defines a section of this bundle by setting:

\[
\overline{\partial}_G : \mathcal{Y}^G \longrightarrow \mathcal{E}^G,
\]

\[
\overline{\partial}_G(\mu) = \overline{\partial}_j_{j_\Sigma, jf} - (\pi_\Sigma \times f)^*v,
\]

where \( \mu = ((J; v), j_\Sigma, (f : \Sigma \xrightarrow{\tau_\Sigma} \Sigma \xrightarrow{g} X)) \) is the above point of \( \mathcal{Y}^G \). It is easy to use the fact that \( g \) is somewhere injective and that the target bundle is the space of \( G \)-invariant sections to show that the section \( \overline{\partial}_G \) is everywhere transverse to the zero section. The proof of this fact is identical to the transversality arguments of [13, 14] and we choose not to repeat them here.

Denote the intersection of \( \overline{\partial}_G \) with the zero section by \( \mathcal{M}_g^G(X, \beta) \). There is a projection map from \( \mathcal{M}_g^G(X, \beta) \) to the parameter space \( P \) which we will denote by \( \text{proj}_G \). This projection map is a Fredholm operator on the Banach versions of these moduli spaces. To compute the index, note that the kernel and the cokernel of the linearization of \( \text{proj}_G \) are isomorphic to the kernel and the cokernel of the the linearization of \( \overline{\partial}_G \) restricted to a fiber of \( \text{proj}_G \). This index computation is a special case of the index computation in the Appendix. It is implied that the index of \( \text{proj}_G \) is zero. As a result, for a generic choice of the parameters \((J, v) \in P \), \( \text{proj}_G^{-1}\{(J, v)\} \) will be a set of isolated solutions to the Cauchy-Riemann equation ??.
The parameters \((J, v)\) which are generic in the above sense are not yet generic enough to rule out the possibility that a sequence of embedded solutions converges to a multiply covered one. Namely, a sequence in \(\text{proj}^{-1}_G\{ (J, v) \}\) can converge to a curve in \(\text{proj}^{-1}_H\{ (J, v) \}\), if \(G\) is a normal subgroup of \(H\). In particular, although the solutions in each stratum are isolated, there can still exist infinitely many of them. A careful study of the behavior of sequences in \(\text{proj}^{-1}_G\{ (J, v) \}\) is the next step toward the definition of the embedded curve invariants.

5. Compactness

In this section we will fix an almost complex structure \(J\) on \(TX\) and a perturbation term \(v \in \text{Hom}^{0,1}_J(T\mathbb{P}^N, TX)\), without any genericity assumptions. The problem we want to study, is the behavior and the possible limits of a sequence of maps \(f_n : (\Sigma, j_n) \to (X, J)\) which satisfy the equations

\[
\overline{\partial}_{j_n,J} f_n = (\pi_n \times f_n)^* v,
\]

where \(\pi_n : (\Sigma, j_n) \to \overline{\mathcal{C}}_g \subset \mathbb{P}^N\).

As usual we will assume that \((f_n)_* [\Sigma]\) is a fixed homology class \(\beta \in H_2(X, \mathbb{Z})\).

By Gromov compactness theorem, there is a subsequence of \(\{f_n\}\), which we will identify by the sequence itself, which converges to a map \(f = f_\infty : (\Sigma, j) \to X\) satisfying

\[
\overline{\partial}_{j,J} f = (\pi \times f)^* v, \quad \pi : (\Sigma, j) \to \overline{\mathcal{C}}_g \subset \mathbb{P}^N.
\]

Here \(\Sigma\) can lie in the boundary components of the moduli space \(\overline{\mathcal{M}}_g\) consisting of nodal curves.

If we fix the topological type of the nodal curve \(\Sigma = \{\Sigma^1, ..., \Sigma^n\}\), where \(\Sigma^i\) are the irreducible components of \(\Sigma\), and the automorphism group \(\mathcal{G}\) of the domain, then the moduli of maps \(f = f_\infty : (\Sigma, j) \to X\) as above, which solve the perturbed Cauchy-Riemann equation, may be described as the intersection of a certain \(\overline{\partial}\) section, and the zero section of a bundle, similar to the construction of the previous section. The index of the projection map will be \(2 - 2n\). This implies that for a generic choice of \((J, v)\), we can not have a solution with a singular domain. Moreover, if \((J_t, v_t)\) is a generic choice of a path of parameters (with the fixed generic end points) then we may assume that the singular solutions do not exist for any of \((J_t, v_t)\). For a more careful discussion of these boundary components we refer the reader to \[13\, 14\].
The above discussion shows that we only need to worry about the convergence of a sequence \( \{f_n\} \), as above, to a solution \( f = f_\infty : (\Sigma, j) \to X \) with \( j_n \to j \) as \( n \) goes to infinity. If the map \( f = f_\infty : (\Sigma, j) \to X \), is somewhere injective, the usual transversality results will work to find a non trivial element in the kernel of the linearization of the Cauchy-Riemann operator. More precisely, if

\[
\mu_n = ((J; v), j_n, (\Sigma \xrightarrow{f_n} X)) \in \mathcal{M}_g(X, \beta),
\]

where \( \mathcal{M}_g(X, \beta) = \mathcal{M}_g^\Theta(X, \beta) \) for \( \Theta = \{\text{Id}\} \), then \( \mu_n \) will converge to

\[
\mu = ((J; v), j, (\Sigma \xrightarrow{f} X)).
\]

This \( \mu \) will also be in \( \mathcal{M}_g(X, \beta) \) if \( f \) does not factor through a nontrivial quotient map \( \tau : \Sigma \to \mathcal{M}_g(X, \beta) \). This is a contradiction if \((J, v)\) is generic in the sense of previous section, so that the elements of \( \text{proj}_{\text{Id}}^{-1}(J, v) \) are isolated.

If a sequence \( \mu_n = ((J; v), j_n, (\Sigma \xrightarrow{f_n} X)) \) converges to some \( \mu = ((J; v), j, (\Sigma \xrightarrow{f} X)) \) which is not in \( \mathcal{M}_g(X, \beta) \), then \( \mu \) will be in \( \mathcal{M}_g^\Theta(X, \beta) \) for some group \( \Theta \), which is a normal subgroup of the automorphism group of \( (\Sigma, j) \). Thus, \( f \) will be decomposed as

\[
f : \Sigma \xrightarrow{\tau\Sigma} \mathcal{M}_g \xrightarrow{\Theta} X.
\]

After choosing a connection, for large values of \( n \), we may write \((f_n, j_n)\) as the image of some elements

\[
(\zeta_n \oplus \eta_n) \in \Gamma(\Sigma, f^*TX) \oplus H^1(\Sigma, T\Sigma)
\]

under the exponential map

\[
\exp : \Gamma(\Sigma, f^*TX) \oplus H^1(\Sigma, T\Sigma) \to \mathcal{X} \times \mathcal{M}_g.
\]

Note that the linearization of the map

\[
\overline{\mathcal{D}}_{J,v} : \mathcal{M}_g \times \mathcal{X} \to \mathcal{E},
\]

\[
\overline{\mathcal{D}}_{J,v}(j_\Sigma, f) = \overline{\mathcal{D}}_{j_\Sigma, f} - (\pi_\Sigma \times f)^*v
\]

may be composed with the projection of

\[
T_{(\nu,0)}\mathcal{E} = \mathcal{E}_\nu \oplus T_{\nu}(\mathcal{M}_g \times \mathcal{X}), \quad \nu = (j_\Sigma, (f : \Sigma \to X))
\]

to the first component to give a map

\[
d\overline{\mathcal{D}}_{J,v} : T_{\nu}(\mathcal{M}_g \times \mathcal{X}) = \Gamma(\Sigma, f^*TX) \oplus H^1_{j_\Sigma}(\Sigma, T\Sigma)
\]

\[
\to \mathcal{E}_\nu = \Gamma(\Sigma, \Omega^0_{\Sigma} \otimes f^*T'X).
\]
This decomposition of $E_\nu$ can be made since we are looking at a zero of the map $\overline{\partial}_{J,\nu}$.

The map $d\overline{\partial}_{J,\nu}$ is a Fredholm operator. As a result, it has a finite dimensional kernel and a finite dimensional cokernel. In fact, the index of $d\overline{\partial}_{J,\nu}$ is zero, which says that the kernel and the cokernel have the same dimension.

Since the sections $(\zeta_n \oplus \eta_n)$ are in the domain of $d\overline{\partial}_{J,\nu}$, we may look at the projection of these sections to the kernel of $D = d\overline{\partial}_{J,\nu}$. Let us write

$$(\zeta_n \oplus \eta_n) = (\zeta_n^0 \oplus \eta_n^0) + (\zeta_n^1 \oplus \eta_n^1),$$

with $(\zeta_n^0 \oplus \eta_n^0)$ the projection on the kernel of $D$, and $(\zeta_n^1 \oplus \eta_n^1)$ in the image of the adjoint operator $D^*$. The following lemma reproves the non triviality of the kernel of $D$:

**Lemma 5.1.** Suppose that $(f_i, j_i) = \exp(f_{j_i})(\zeta \oplus \eta) = \exp(f_{j_i})(\theta_i)$ for $i = 1, 2$, be two elements of $X \times M_g$ such that $\overline{\partial}_{j_i, J} f_i = (\pi_i \times f_i)^* v$, where $\pi_i : (\Sigma, j_i) \to \mathbb{C}^*_{\Sigma} \subset \mathbb{P}^N$. Furthermore, assume that $(\zeta_1 - \zeta_2) \oplus (\eta_1 - \eta_2)$ is in the image of $D^*$. Then $\theta_1 = \theta_2$ if $\|\zeta_i\|$ and $\|\eta_i\|$ are assumed to be small enough.

**Proof.** Define the function

$$\mathcal{F} : \Gamma(\Sigma, f^*TX) \oplus H^1(\Sigma, T\Sigma) \to \Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes f^*T'X)$$

for $\theta = (\zeta, \eta)$ by the formula

$$\mathcal{F}(\theta) = \Phi_\theta(\overline{\partial}_{j, J}(\exp f(\zeta)) - (\pi_{exp f}(\eta) \times \exp f(\zeta))^* v),$$

where $\Phi_\theta$ is the parallel transport map along the curve

$$\exp(f_{j, j})(t\theta) = (\exp f(t\zeta), \exp_j(t\eta))$$

for $t \in [0, 1]$. A simple computation shows that the differential of $\mathcal{F}$ at zero is $d\mathcal{F}(0) = D$.

Assume that $\theta_1 = Q\gamma + \theta_2 = \hat{\theta} + \theta_2$. Here

$$Q : \text{Im}(D) \to \text{Im}(D^*)$$

is a right inverse for $D$. It is clearly possible to write the difference of $\theta_1, \theta_2$ in this form since $Q$ is surjective.
Since $f_i$ are $(J, v, j_i)$-holomorphic, we know that $F(\theta_i) = 0$, $i = 1, 2$. Suppose that $\|Q\| = c_0$. Then:

\[
\|\hat{\theta}\| = \|Q\gamma\| \\
\leq c_0\|\gamma\| \\
= c_0\|D\hat{\theta}\| \\
(7) \quad = c_0\|dF(0)\hat{\theta}\| \\
= c_0\|F(\theta_1) - F(\theta_2) - dF(0)(\theta_1 - \theta_2)\| \\
\leq c_0\|F(\theta_1) - F(\theta_2) - dF(\theta_2)(\theta_1 - \theta_2)\| + c_0\|dF(\theta_2) - dF(0)\||\hat{\theta}| \\
\leq c_0c_1\|\hat{\theta}\|_{L^\infty}||\hat{\theta}|| + c_0c_2\|\theta_2\| ||\hat{\theta}||.
\]

Here the constants only depend on $(f, j)$ and on $(J, v)$. If $\|\hat{\theta}\|_{L^\infty}$ and $\|\theta_2\|$ are small enough then this implies that $\|\hat{\theta}\|$ is zero, or equivalently, $\theta_1 = \theta_2$. This completes the proof.

As noted before, the first consequence of this lemma is that if we have a sequence of maps

\[\mu_n = ((J; v), j_n, (\Sigma \xrightarrow{J_n} X)) \in \mathcal{M}_g(X, \beta)\]

for some $(J, v)$, converging to a multiply covered one

\[\mu = ((J; v), j, (\Sigma \xrightarrow{\pi} G \xrightarrow{\varnothing} X)) \in \mathcal{M}^\varnothing_g(X, \beta),\]

then we obtain a nonzero element in the kernel of the linearization $d\overline{\partial}_{J,v}$ (note that this is different from the linearization of the equivariant section $\overline{\partial}_{\varnothing}$).

This completes our first step toward understanding the convergence of sequences in $\mathcal{M}_g(X, \beta)$.

6. A STRUCTURE THEOREM FOR PARAMETERS

In this section we will consider some local models that describe the set of acceptable pairs $(J, v)$ of almost complex structures $J$ on $X$ and perturbation terms $v$ associated with $J$. To begin, fix a real rank-6 bundle $E$ on the Riemann surface $\Sigma$ with $c_1(E) = 0$, a topological action of a group $\varnothing$ on $\Sigma$ giving a quotient $C = \Sigma/\varnothing$, and an isomorphism class of an almost complex structure on $E$. The bundle $E \to \Sigma \xrightarrow{\pi} C$ will be assumed to be of the form $\pi^*F$, where $F \to C$ is a bundle over $C$ with trivial first Chern class. Note that the map $\pi : \Sigma \to C$ is obtained as a result of fixing the action of $\varnothing$ on $\Sigma$.

Let $\mathcal{J}^\varnothing_E$ denote the space of all almost complex structures $J : E \to E$, which are invariant under the action of $\varnothing$, or saying in a different way, the
space of almost complex structures on \( F \). Consider the following bundle over \( \mathcal{M}_g^\Theta \times \mathcal{F}^\Theta \), where \( \mathcal{M}_g^\Theta \) is the part of \( \mathcal{M}_g \) consisting of the complex structures on \( \Sigma \) which have \( \mathcal{G} \) as a subgroup of their automorphism group. Let \( \mathcal{G}^\Theta \) be the bundle given by

\[
\mathcal{G}^\Theta_{(j_\Sigma, J)} = \Gamma_{\mathcal{G}}(\Sigma \times E, p_1^* \Omega_{j_\Sigma}^{0,1} \otimes_J p_2^* E)
\]

for any complex structure \( j_\Sigma \) on \( \Sigma \) and any almost complex structure \( J \) on \( F \). Here \( E \) is considered as a bundle over itself. We will identify some "bad locus variety" in the total space \( \mathcal{G}^\Theta \).

A point in \( \mathcal{G}^\Theta \) will be of the form \((j_\Sigma, J; v)\) where \( j_\Sigma \) is a complex structure on \( \Sigma \) with automorphism group \( \mathcal{G} \) such that the map to the quotient \((\Sigma, j_\Sigma) \to \Sigma_{\mathcal{G}}\) is in fact the fixed map \( \pi \), \( J \) is a complex structure on \( F \) (inducing one on \( E = \pi^* F \)) and \( v \) is a \( \mathcal{G} \)-invariant perturbation term as above. Associated with this data we will look at the linearization operator

\[
L = L(j_\Sigma, J; v) : \Gamma(\Sigma, E) \oplus H^1(\Sigma, T\Sigma) \to \Gamma(\Sigma, \Omega_{(\Sigma, j_\Sigma)}^{0,1} \otimes_J E)
\]

defined for a section \( \omega \) of \( T\Sigma \) to be

\[
L(\zeta, \eta)(\omega) = \nabla_\omega \zeta + J \nabla_{j_\Sigma} \omega \zeta + (J \circ d\pi \circ \eta)(\omega) + \frac{1}{2} \left\{ (\nabla_\zeta J)(d\pi \circ j_\Sigma) - (\nabla_{(\zeta \oplus \eta)} v) \right\}.
\]  

(8)

This is a Fredholm operator of index zero. We are interested in excluding the locus of \((j_\Sigma, J; v)\) where the operator has a nontrivial kernel (c.f. computation of the Appendix).

If \( \theta = (\zeta \oplus \eta) \in \text{Ker}(L(j_\Sigma, J; v)) \), and \( g : \Sigma \to \Sigma \) is an element of the automorphism group \( \mathcal{G} = \text{Aut}(\Sigma, j_\Sigma) \), then \( \theta_g = g^* \theta \) is also an element in the kernel of \( L \).

Define \( \mathcal{O}_\theta \) to be the subspace of \( \text{Ker}(L) \) generated by \( \{\theta_g\}_{g \in \mathcal{G}} \), and denote by \( m_\theta \) the ideal of the group ring \( \mathbb{R}^\mathcal{G} \) consisting of all elements

\[
\alpha = \sum_{g \in \mathcal{G}} a_g g^{-1} \in \mathbb{R}^\mathcal{G}, \quad a_g \in \mathbb{R},
\]

such that

\[
\sum_{g \in \mathcal{G}} a_g \theta_g = 0.
\]

It is easy to check that \( m_\theta \) is a left ideal of the group ring \( \mathbb{R}^\mathcal{G} \).

In this part we will have a discussion of the maximal ideals of \( \mathbb{R}^\mathcal{G} \) and a decomposition of \( \text{Ker}(L) \) into the orbits \( \mathcal{O}_\theta \), with associated ideal being maximal, for the case where \( \mathcal{G} \) is \( \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \), and \( p \) is prime.
Suppose that $p \neq 2$ is prime. The group $\mathbb{Z}_p$ may be identified with the multiplicative group of the elements

$$\{1, \lambda, \lambda^2, \ldots, \lambda^{p-1}\}$$

where $\lambda = e^{\frac{2\pi i}{p}}$. Then the formal sums

$$a_k = \sum_{i=0}^{p-1} \text{Re}(\lambda^{ki}).\lambda^i, \quad k = 0, 1, \ldots, \frac{p-1}{2}$$

and

$$b_k = \sum_{i=0}^{p-1} \text{Im}(\lambda^{ki}).\lambda^i, \quad k = 1, 2, \ldots, \frac{p-1}{2}$$

generate $\mathbb{R}_\Theta$ as a vector space over $\mathbb{R}$. Moreover if $I_k = \langle a_k, b_k \rangle_\mathbb{R}$ is the subspace generated by $a_k, b_k$, then in the decomposition

$$\mathbb{R}_\Theta = I_0 \oplus I_1 \oplus \ldots \oplus I_{\frac{p-1}{2}},$$

the space of elements which are zero in the $k$-th component forms an ideal $m^k$ of $\mathbb{R}_\Theta$, which is in fact maximal.

For any section $\theta \in \text{Ker}(L)$, the sections

$$\theta^k = \sum_{i=0}^{p-1} \text{Re}(\lambda^{ki}).\theta^i$$

and

$$\overline{\theta}^k = \sum_{i=0}^{p-1} \text{Im}(\lambda^{ki}).\theta^i$$

have the associated ideal $m^k$.

It is interesting to note that if $\theta^k$ and $\overline{\theta}^k$ are zero for $k = 1, \ldots, \frac{p-1}{2}$, then $m_\theta = m^0$.

Before going further, define a local product on $\text{Ker}(L)$, associated to a point $p \in C$, as follows: Let $\{p_g\}_{g \in \Theta}$ denote the points in the pre-image $\pi^{-1}(p)$ of $p$, such that $h(p_g) = p_{hg}$. Then define

$$\langle \theta, \overline{\theta} \rangle_p = \sum_{g \in \Theta} \langle \theta(p_g), \overline{\theta}(p_g) \rangle,$$

using some metric $\langle \cdot, \cdot \rangle$ on the bundle $E$.

Now start from a section $\theta$ in $\text{Ker}(L)$. By changing $\theta$ with one of $\theta^k$ or $\overline{\theta}^k$ if necessary, we may assume that the associated ideal $m_1 = m_\theta$ is one of $m^i, i = 0, \ldots, \frac{p-1}{2}$. Call this section $\overline{\theta}_1$. Choose a point $p_1$ such that $\overline{\theta}_1$ is not identically zero at $(p_1)_g$'s. Then look at the orthogonal complement of
\( \mathcal{O}_{\mathcal{O}} \) in \( \text{Ker}(L) \), with respect to \( \langle \cdot, \cdot \rangle_{p_1} \). Choose another section \( \bar{\theta}_2 \) with the corresponding ideal \( m_2 \) among \( m^i \)'s in this orthogonal complement, together with a point \( p_2 \) in \( C \), different from \( p_1 \). Furthermore, choose \( p_2 \) such that \( \bar{\theta}_2 \) is nonzero above \( \pi^{-1}(p_2) \). Then look at the intersection of orthogonal complements of \( \mathcal{O}_{\mathcal{O}} \) with respect to \( \langle \cdot, \cdot \rangle_{p_i}, i = 1, 2, \ldots \). This process will give us a decomposition

\[
\text{Ker}(L) = \mathcal{O}_{\mathcal{O}} \oplus \mathcal{O}_{\mathcal{O}} \oplus \ldots \oplus \mathcal{O}_{\mathcal{O}}
\]

\( m_{\bar{\theta}_1} = m_1, m_{\bar{\theta}_2} = m_2, \ldots, m_{\bar{\theta}_\ell} = m_\ell, \)

together with \( \ell \) points \( p_1, p_2, \ldots, p_\ell \). Note that the ideals \( m_i \) appearing in this decomposition (the whole collection) is independent of the way we decompose.

Fix the values \( (j_\Sigma, J; \nu) \) as above such that \( L = L(j_\Sigma, J; \nu) \) has an element in its kernel. Note that there is an action of \( \mathfrak{G} \) on the spaces

\[
\Gamma(\Sigma, E) \oplus H^1(\Sigma, T\Sigma)
\]

and

\[
\Gamma(\Sigma, \Omega^{0,1}_{(\Sigma, j_\Sigma)} \otimes J E).
\]

One may also consider the space of those sections which have the associated ideal equal to a given ideal \( \mathfrak{m} \) (denoted by \( \Gamma_\mathfrak{m}(\bullet) \)). It is easy to see that the operator \( L \) restricts to an operator \( L_\mathfrak{m} \),

\[
L_\mathfrak{m} = L_\mathfrak{m}(j_\Sigma, J; \nu) : \Gamma_\mathfrak{m}(\Sigma, E) \oplus H^1_\mathfrak{m}(\Sigma, T\Sigma) \longrightarrow \Gamma_\mathfrak{m}(\Sigma, \Omega^{0,1}_{(\Sigma, j_\Sigma)} \otimes J E)
\]
on the space of such "\( \mathfrak{m} \)-sections". Similarly the adjoint operator \( L^* \) restricts to give an adjoint operator \( L_\mathfrak{m}^* \) which goes in the other direction.

The index computation of the Appendix shows that for any ideal \( \mathfrak{m} \), the operator \( L_\mathfrak{m} \) is Fredholm of index zero. This fact may be used to show that, parallel to the above process of decomposing \( \text{Ker}(L) \) into orbits \( \mathcal{O}_{\mathcal{O}} \), one may also find elements \( \mathcal{P}_1, \ldots, \mathcal{P}_\ell \) in \( \text{Coker}(L) \) with the property that

\[
\text{Coker}(L) = \mathcal{O}_{\mathcal{P}_1} \oplus \mathcal{O}_{\mathcal{P}_2} \oplus \ldots \oplus \mathcal{O}_{\mathcal{P}_\ell}
\]

\( m_{\mathcal{P}_1} = m_1, m_{\mathcal{P}_2} = m_2, \ldots, m_{\mathcal{P}_\ell} = m_\ell \)

with a similar orthogonality assumption (using the same set of points \( p_1, \ldots, p_\ell \)).

The goal is to show that the space \( \mathcal{D}^{\mathfrak{G}}_{\{m_1, \ldots, m_\ell\}} \) of the tuples \( (j_\Sigma, J; \nu) \) such that the kernel \( \text{Ker}(L(j_\Sigma, J; \nu)) \) has a decomposition as above, is locally a submanifold of \( \mathcal{G}^{\mathfrak{G}} \).
Theorem 6.1. For odd prime numbers \( p \), the space \( D_{\{\text{m}_1, \ldots, \text{m}_\ell\}}^{\text{G}} \) is an analytic submanifold of \( \text{G}^{\text{G}} \), of codimension \( 2\ell - q \), if \( q \) of \( \text{m}_i \)'s are equal to \( \text{m}_0 \).

Proof. We give the proof for the case where \( \ell = 1 \) and \( \text{m}_1 \neq \text{m}_0 \). Then we will make a remark on the other cases. Suppose that \( \text{Ker}(L) = \mathcal{O}_\pi \) is generated by \( \theta_1, \theta_2 \). This implies that \( \text{Coker}(L) = \mathcal{O}_\pi \) is generated by some \( \mu_1, \mu_2 \). We will assume that \( \theta_i \)'s are orthogonal to each other and that the same is true for \( \mu_i \)'s. Furthermore, assume that \( \text{m}_\pi = \text{m}_\pi = \text{m} \).

Suppose that \( (j_{\Sigma}', J'; v') \) is another element in \( D_{\text{m}}^{\text{G}} \) which is very close to \( (j_{\Sigma}, J; v) \). Assume that \( \theta_k' = \theta_k + \theta_k^0 \) are the corresponding sections of \( \Gamma(\Sigma, E) \oplus H^1(\Sigma, T\Sigma) \). We may assume that \( \theta_k^0 \)'s are orthogonal to \( \text{Ker}(L) \), hence to all \( \theta_i \)'s, \( i, k \in \{1, 2\} \). One may write \( j_{\Sigma}' = j_{\Sigma} + \delta + F_1(\delta) \), \( J' = J + Y + F_2(Y) \) and \( v' = v + Z + F_3(\delta, Y, Z) \). Then we should have the extra condition that

\[
\begin{align*}
\delta j_{\Sigma} + j_{\Sigma} \delta &= 0 \\
YJ + JY &= 0 \\
JZ + Zj_{\Sigma} + Y v + v \delta &= 0.
\end{align*}
\]

(11)

The functions \( F_1, F_2, F_3 \) will be analytic functions of their variables. The vanishing of \( L(j_{\Sigma}', J'; v') \) at

\[
\theta_k' = (\zeta_k' = \zeta_k + \zeta_k^0) \oplus (\eta_k' = \eta_k + \eta_k^0)
\]

can be written as

\[
0 = L'(\theta_k')(\omega) = L(\theta_k^0)(\omega) + Y \nabla_{j_{\Sigma} \omega} \zeta_k + Y \circ d\pi \circ \eta_k \\
+ \frac{1}{2} \{(\nabla_{\zeta_k} Y)(d\pi \circ j_{\Sigma}) + (\nabla_{\zeta_k} J)(d\pi \circ \delta) - \nabla \theta_k Z\}
\]

+ terms of higher order, \( k = 1, 2 \),

where \( L' = L(j_{\Sigma}', J'; v') \) and \( L \) is as before.

In particular, we have to solve two equations of the form \( L(\theta_k^0) = \bullet \). This is not something that one can have any hope to do in general, since \( L \) is not surjective. However, let us follow Taubes \([15]\) and denote by \( \Pi \) the projection

\[
\Gamma(\Sigma, \Omega^{0,1}_{(\Sigma, j_{\Sigma})} \otimes J E) \longrightarrow \text{Coker}(L),
\]
and denote the projection over the image of $L$ by $\Pi^c$. Then $\text{Id} = \Pi + \Pi^c$ and we obtain the following two equations:

\[
L(\theta^0_k) + \Pi^c\{Y \nabla_{j_\Sigma} \omega_k \zeta_k + Y \circ d\pi \circ \eta_k
\]
\[
+ \frac{1}{2}[(\nabla_{\zeta_k} Y)(d\pi \circ j_\Sigma) + (\nabla_{\zeta_k} J)(d\pi \circ \delta) - \nabla_{\theta_k} Z]
\]
\[
+ \text{higher order}\} = 0
\]
\[
\Pi\{Y \nabla_{j_\Sigma} \omega_k \zeta_k + Y \circ d\pi \circ \eta_k
\]
\[
+ \frac{1}{2}[(\nabla_{\zeta_k} Y)(d\pi \circ j_\Sigma) + (\nabla_{\zeta_k} J)(d\pi \circ \delta) - \nabla_{\theta_k} Z]
\]
\[
+ \text{higher order}\} = 0
\]

(12)

For given values for $(j_\Sigma', J'; v')$, or rather for given values for $(\delta, Y; Z)$, the first equation may be solved uniquely to give a value for $\theta^0_i$ as an analytic function of $(\delta, Y; Z)$. The second set of equations may then be thought of as an analytic function

\[
F : B(j_\Sigma, J; v) \subset G^\Theta \longrightarrow \text{Hom}(\text{Ker}(L), \text{Coker}(L))
\]

where $B(j_\Sigma, J; v)$ denotes a small neighborhood of $(j_\Sigma, J; v)$ in $G^\Theta$.

The map $F$ will be an analytic function and its derivative at zero will be given explicitly as follows: It will take a tangent vector $(\delta, Y; Z)$ satisfying the equation (11) and will give the matrix with $(i, j)$ entry equal to:

\[
[dF]_{ij} = \int_\Sigma \langle \gamma_i, \mu_j \rangle,
\]

\[
\gamma_i = Y \nabla_{j_\Sigma} \omega_i \zeta_i + Y \circ d\pi \circ \eta_i
\]
\[
+ \frac{1}{2}[(\nabla_{\zeta_i} Y)(d\pi \circ j_\Sigma) + (\nabla_{\zeta_i} J)(d\pi \circ \delta) - \nabla_{\theta_i} Z].
\]

We will choose $\delta$ and $Y$ to be zero and $Z$ will be any arbitrary section satisfying $JZ + Z j_\Sigma = 0$. Clearly, if such tuples give us the required surjectivity for $dF$, then $F^{-1}(0)$ will be a subvariety. More coherently, we may think of $dF(Z) = dF(0, 0; Z)$ as a constant multiple of the pairing

\[
\rho_Z : \text{Ker}(L) \times \text{Coker}(L) \longrightarrow \mathbb{R}
\]

defined by

\[
\rho_Z(\theta, \mu) = \int_\Sigma \langle \nabla \theta Z, \mu \rangle.
\]

It is important to note that $\rho_Z(g^* \theta, g^* \mu) = \rho_Z(\theta, \mu)$ for any element $g$ of $\mathfrak{g}$. This implies that the only independent relations given by $\rho_Z = 0$ are

\[
\rho_Z(\theta_i, \mu_i) = 0, \quad i = 1, 2.
\]
The equation $F = 0$ is implied from $Q(F) = 0$ where $Q$ is the projection on the first row of $\text{Hom}(\text{Ker}(L), \text{Coker}(L))$. The differential of $Q(F)$ may be described on $(0, 0; Z)$ as the map taking $Z$ to

$$(\rho_Z(\theta_1, \mu_1), \rho_Z(\theta_1, \mu_2)).$$

We will show that this map is surjective. As a result, $(Q \circ F)^{-1}(0) = F^{-1}(0)$ is locally an analytic submanifold of codimension 2 near the point $(j_{\Sigma}, J; v)$ of $G^\Theta$. This is equivalent to showing that the map taking $Z$ to $(\rho_Z(\vartheta, \overline{\mu}_g))_{g \in \Theta}$ has rank 2.

Denote $\nabla_Z Z$ by $G(\vartheta)$, where $G$ is a function depending on $Z$. For a generic choice of $p \in C$, let $\{p_g\}_{g \in \Theta}$ be the points in $\tau^{-1}_\Sigma(p)$. Fix an identification of the fibers $F_p \cong E_p$ with $\mathbb{C}^3$ such that $J$ becomes the standard complex structure, and an identification of $T_p \Sigma \cong T_p C$ with $\mathbb{C}$, such that $j_\Sigma$ is identified with $i$. Denote the $i$-th component of the image of $\overline{\pi}(p_g)(1) \in \mathbb{C}^3 \cong \mathbb{R}^6$ by $\overline{\mu}_i(g)$, $g \in \Theta, i = 1, \ldots, 6$.

Look at the values for $Z$ such that they are supported over the point $p$ (i.e. invariantly supported on $p_g$'s). The corresponding section $G(\overline{\vartheta})$ will be supported on $p_g$'s as well. We may assume that

$$\int_{\Sigma} \langle G(\overline{\vartheta}), \overline{\mu}_g \rangle = \sum_{h \in \Theta} f_{\overline{\mu}}(p_h) \overline{\mu}_i(gh),$$

where $f_{\overline{\mu}} : \{p_g\} \to \mathbb{R}$ is a function that depends on $Z$. Our freedom in choosing $Z$ guarantees that any such function may be obtained, except if there is a relation

$$\sum_{g \in \Theta} a_g \vartheta(p_g) = 0$$

for the section $\vartheta$, when we have the similar relation

$$\sum_{g \in \Theta} a_g f_{\vartheta}(p_g) = 0.$$

If there is a relation between $\rho_Z(\overline{\vartheta}, \overline{\mu}_g)$ of the form

$$\sum_{g \in \Theta} a_g \rho_Z(\overline{\vartheta}, \overline{\mu}_g) = 0, \quad \forall Z$$

then in particular we get

$$0 = \sum_{g \in \Theta} \sum_{h \in \Theta} a_g f_{\overline{\mu}}(p_h) \overline{\mu}_i(gh) = \sum_{h \in \Theta} b_h f_{\overline{\mu}}(p_h).$$

Thus, it is implied that $\sum_{h \in \Theta} b_h \vartheta(p_h) = 0.$
Lemma 6.2. If \( p \) (and consequently the collection \( \{ p_i \} \) \( i \in \mathcal{G} \)) come from a generic choice, and \( \sum_{h \in \mathcal{G}} b_h \bar{\theta}(p_h) = 0 \) for \( \beta = \sum_{h \in \mathcal{G}} b_h h^{-1} \) as above, then \( \beta \in \mathfrak{m} \).

**Proof.** (of the lemma). Suppose that \( \beta \) is not in the ideal \( \mathfrak{m} \), which has a rank (as a vector space over \( \mathbb{R} \)) equal to \( p - 2 = |\mathcal{G}| - 2 \). Then there are \((p - 1)\) independent relations between the vectors \( \{ \bar{\theta}(p_h) \} \) \( h \in \mathcal{G} \). This implies that \( \bar{\theta}(p_h) \) are mutually linearly dependent. Without loss of generality we may assume that \( \bar{\theta}(p_h) = a(p_h) \bar{\theta}(p_e) \) where \( e \in \mathcal{G} \) is the identity element.

Note that if the above claim is not true for generic \( p \), it will not be true for any \( p \). As a result, \( a \) defines a function \( a : \Sigma \rightarrow \mathbb{R}, \quad \bar{\theta}_h = a_h \bar{\theta} = h^*(a) \bar{\theta} \).

On the other hand, the above relation implies that

\[
0 = L(\bar{\theta}_h)(\omega) = L(a_h \bar{\theta})(\omega) = a_h L(\bar{\theta}) + (\omega.a_h) \bar{\theta} + [(j_\Sigma \omega).a_h](J \bar{\theta}).
\]

As a result \((\omega.a_h) \bar{\theta} + [(j_\Sigma \omega).a_h](J \bar{\theta}) = 0\) which can not be the case for \( \bar{\theta} \neq 0 \) and real valued function \( a_h \), unless \( a_h \) is constant. If \( a_h \) is constant, the assumptions \( p \neq 2 \) and \( a_h \in \mathbb{R} \) imply that \( a_h = 1 \). Thus \( \bar{\theta} \) is invariant under the action of \( \mathcal{G} \), contradicting our assumption. This completes the proof of the lemma.

We conclude that \( \sum_{h \in \mathcal{G}} b_h h^{-1} \in \mathfrak{m}_{\bar{\theta}} = \mathfrak{m} \). But on the other hand

\[
\sum_{h \in \mathcal{G}} b_h h^{-1} = \left( \sum_{h \in \mathcal{G}} \bar{\mu}(h).h^{-1} \right) \left( \sum_{h \in \mathcal{G}} a_h.h \right)
\]

Note that this is true for all components of \( \bar{\mu} \) corresponding to \( i = 1, \ldots, 6 \). By considering the fact that \( \mathfrak{m}_{\bar{\mu}} = \mathfrak{m} \) as well, the only possible way for

\[
\left( \sum_{h \in \mathcal{G}} \bar{\mu}(h).h^{-1} \right) \left( \sum_{h \in \mathcal{G}} a_h.h \right)
\]

to be in \( \mathfrak{m} \) for a generic choice of \( p \), is when \( \sum_{h \in \mathcal{G}} a_h.h \in \mathfrak{m} \). This proves that the only relations between \( \rho_Z(\bar{\theta}, \bar{\mu}) \)'s are those corresponding to the elements of \( \mathfrak{m} \), and completes the proof of the theorem for the case \( \ell = 1 \).

When \( \ell > 1 \), but the ideals are different from \( \mathfrak{m}^0 \), the proof is just slightly more complicated. In the definition of orthogonality, one should choose the points \( p_i \) (and correspondingly \( (p_i)_g \)'s) to be generic for \( \bar{\theta}_i, \bar{\mu}_i \), in the sense of the above lemma. Then the independence of the equations

\[
\rho_Z(\theta_1^i, \mu_1^i) = 0, \quad \rho_Z(\theta_1^i, \mu_2^i) = 0, \quad i = 1, 2, \ldots, \ell.
\]

follows from the orthogonality, and a discussion similar to the above one.
For the ideal $m^0$ of $R$, the corresponding sections $\theta, \mu$ are the generators of their orbits as a $R$-vector space. Here the claim is that $\rho_Z(\theta, \mu)$ is not identically zero. Similar to the above arguments, if $\rho_Z(\theta, \mu) = 0$ for all $Z$, then at a point $p \in C$, and the corresponding points $\{p_g\}_{g \in \mathfrak{G}}$, we will have

$$\sum_{g \in \mathfrak{G}} \theta(p_g) \cdot \mu(p_g) = 0$$

where $\mu^i$ denotes the $i$-th component of $\mu$. If $\sum_{g \in \mathfrak{G}} \mu^i(p_g) \cdot g^{-1} \not\in m = m^0$, then there are $p$ independent linear relations between $\{\theta(p_g)\}_{g \in \mathfrak{G}}$, which implies that they are all zero. Since this can not be true for generic $p$,

$$\sum_{g \in \mathfrak{G}} \mu^i(p_g) \cdot g^{-1} \in m.$$

This is also a contradiction, since it implies that $\sum_{g \in \mathfrak{G}} \mu^i(p_g) \cdot g^{-1}$ is annihilated by all elements of $R$ (it is already annihilated by elements of $m$, and being in $m$ gives one more relation independent of the previous ones). The passage from one ideal to the $\ell > 1$ case is similar to the above discussion.

\[\square\]

7. Finiteness

The goal of this section is to show that for a generic choice of an almost complex structure $J$ on $X$ and a perturbation term $v \in \text{Hom}_{J}^{0,1}(T\mathbb{P}^N, TX)$, the somewhere injective solutions to the equations

$$f : (\Sigma, j_\Sigma) \longrightarrow X,$$

$$\pi_\Sigma : (\Sigma, j_\Sigma) \rightarrow \mathcal{C}_g \subset \mathbb{P}^N,$$

$$f_*[\Sigma] = \beta \in H_2(X, \mathbb{Z}),$$

$$\overline{\partial}_{j_\Sigma, J} f = (\pi_\Sigma \times f)^* v,$$

are isolated and finite. Our discussion on transversality will assign a sign to each such solution coming from the transverse intersection of the moduli spaces and the spectral flow to a complex $\overline{\partial}$ operator. For a more careful treatment of signs, we refer the reader to [13, 14].

Write the homology class $\beta \in P_2(X, \mathbb{Z})$ as $p\alpha$, where $\alpha$ is a primitive homology class in $H_2(X, \mathbb{Z})$ (i.e. $\alpha$ is not a multiple of some other class). Note that when $\beta$ itself is primitive, then the standard arguments in Gromov-Witten theory rules out the possibility of convergence of a sequence of embedded solutions to any solution with singular domain, and we can not have a multiply covered solution. So the set of embedded solutions for a generic choice of parameters $(J, v)$ consists of isolated points and is compact, thus
finite. The independence of the signed count of these solutions from the parameters is also standard. So we will assume that \( p \neq 2 \) is a prime number.

Suppose that \( (J, v, (f : (\Sigma, j_{\Sigma}) \xrightarrow{\tau_{\Sigma}} C \xrightarrow{h} X)) \) is an element in \( \mathcal{M}_g^\Theta(X, \beta) \). We will get a map from a neighborhood of

\[
(J, v, (f : (\Sigma, j_{\Sigma}) \xrightarrow{\tau_{\Sigma}} C \xrightarrow{h} X))
\]

in \( \mathcal{M}_g^\Theta(X, \beta) \) to the local model \( \mathcal{G}^\Theta \) associated with the bundle \( E = f^*TX \). The map takes this point to \( (j_{\Sigma}, \bar{J}, \bar{v}) \), where \( \bar{J} \) is the induced almost complex structure on \( f^*TX \) and \( \bar{v} \) is the perturbation term induced by \( v \) on a an identification of \( h^*TX \) with a tubular neighborhood of \( h(C) \). Denote this map by

\[
q : B(J, v, (f, j_{\Sigma})) \subset \mathcal{M}_g^\Theta(X, \beta) \longrightarrow B(j_{\Sigma}, \bar{J}, \bar{v}) \subset \mathcal{G}^\Theta.
\]

Here \( \mathcal{G} \) is the automorphism group of \((\Sigma, j_{\Sigma})\). Construct a map \( F \) from \( B(j_{\Sigma}, \bar{J}, \bar{v}) \) to \( \text{Hom}(\ker(L), \text{coker}(L)) \) as in the proof of theorem 6.1, where \( L = L(j_{\Sigma}, \bar{J}, \bar{v}) \). Also construct the projection \( Q \) as in the proof of theorem 6.1. Since in the proof of surjectivity of \( d(j_{\Sigma}, \bar{J}, \bar{v}) \) \((Q \circ F)\), we only used the perturbation of \( \bar{v} \), it can be easily concluded that the composition \( Q \circ F \circ q \) has a surjective derivative at \( (J, v, (f, j_{\Sigma})) \).

This observation implies that we obtain submanifolds

\[
\mathcal{D}^\Theta_{\{m_1, \ldots, m_\ell\}} \subset \mathcal{M}_g^\Theta(X, \beta)
\]

consisting of the points \((J, v, (f, j_{\Sigma}))\) such that the linearization operator

\[
L = L(J, v, (f, j_{\Sigma})) : H^1(j_{\Sigma}(T\Sigma) \oplus \Gamma(\Sigma, f^*TX)
\]

\[
\longrightarrow \Gamma(\Sigma, \Omega_{(\Sigma, j_{\Sigma})}^{0,1} \otimes J f^*TX)
\]

defined by

\[
L(\zeta, \eta)(\omega) = \nabla_\omega \zeta + J\nabla_{j_{\Sigma}\omega} \zeta + (J \circ df \circ \eta)(\omega)
\]

\[
+ \frac{1}{2} \{ (\nabla_\xi J)(df \circ j_{\Sigma}) - (\nabla_{(\zeta \oplus \eta)} \xi) \},
\]

has a kernel \( \ker(L) = O_{_{\mathcal{G}_1}} \oplus \ldots O_{_{\mathcal{G}_k}} \). Here we assume that \( m_{\mathcal{G}_i} = m_i \).

The submanifold \( \mathcal{D}^\Theta_{\{m_1, \ldots, m_\ell\}} \) will have a codimension equal to \( \dim(\ker(L)) \), using theorem 6.1.

After setting up the above notation, we are now ready to prove the following theorem (compare with theorem 3.1), which shows that the claimed counts of the embedded solutions of the perturbed Cauchy-Riemann equation, are in fact meaningful.
Theorem 7.1. For a Bair subset $P_{\text{reg}} \subset P$, the following is true: If $(J; v) \in P_{\text{reg}}$, then the space $M_g(X, \beta; (J; v))$ of the somewhere injective solutions $f : (\Sigma, j_\Sigma) \to X$ to the perturbed Cauchy-Riemann equation

$$\overline{\partial}_{j_\Sigma} f = (\tau_\Sigma \times f)^* v, \quad f_* [\Sigma] = \beta,$$

is finite. Moreover, at any such solution, $L = L(J, v, (f, j_\Sigma))$ has a trivial kernel. Finally, to any such solution $(J, v, (f, j_\Sigma))$ is assigned a sign $\epsilon(f, j_\Sigma) = \epsilon(J, v, (f, j_\Sigma))$ coming from the spectral flow from $L$ to a complex $\overline{\partial}$ operator.

Proof. Consider all of the possible actions of the group $\mathbb{Z}_p = \mathbb{Z}_p \mathbb{Z}_p$ on the surface $\Sigma$. Denote such an action by $\mathcal{G}$ and fix the quotient map $\pi : \Sigma \to \Sigma_G$. There are finitely many such group actions. For any such action $\mathcal{G}$, consider the manifold $M_g^\mathcal{G}(X, \beta)$ and the projection map

$$q_\mathcal{G} : M_g^\mathcal{G}(X, \beta) \to P.$$

It is easy to check that $q_\mathcal{G}$ is in fact a Fredholm operator of index zero.

For any set $\{m_1, ..., m_\ell\}$ of ideals, define $d(\{m_1, ..., m_\ell\})$ to be the dimension of the corresponding kernel. Then if we restrict the map $q_\mathcal{G}$ to the submanifold $\mathcal{D} = \mathcal{D}_{\{m_1, ..., m_\ell\}}$, the index of this restriction will be equal to $-d(\{m_1, ..., m_\ell\})$. The set of regular values for all the projection maps $q_\mathcal{G}$ and $q_\mathcal{G}|_{\mathcal{D}}$, for different choices of the group action $\mathcal{G}$ and the ideals $\{m_1, ..., m_\ell\}$, will still be a Bair subset $P_{\text{reg}} \subset P$ (note that in particular we consider the case where $\mathcal{G}$ is the trivial group). If $(J, v)$ is a regular value of $q_\mathcal{G}|_{\mathcal{D}}$, then $q_\mathcal{G}^{-1}(J, v) \cap \mathcal{D} = \emptyset$, since the index of $q_\mathcal{G}|_{\mathcal{D}}$ is negative.

Suppose that $(J; v) \in P_{\text{reg}}$. Then the points in $M_g(X, \beta; (J; v))$ will be isolated, and at any such point, the kernel of the linearization map is trivial. We will be done if we can show that $M_g(X, \beta; (J; v))$ has only finitely many points, since the signs $\epsilon(J, v, (f, j_\Sigma))$ may be assigned as in [13, 14].

Suppose that this is not the case and $\{f_n : (\Sigma, j_{\Sigma_n}) \to X\}$ is a sequence of somewhere injective solutions to the perturbed Cauchy-Riemann equation above. This sequence will have a convergent subsequence. The limit will be a map

$$f = f_\infty : (\Sigma, j_\Sigma) \to X$$

which solves the same equation. We have already argued that $\Sigma$ can not be singular and we may identify it with $\Sigma$. 
If \( f \) is somewhere injective then \((f, j_\Sigma) \in \mathcal{M}_g(X, \beta; (J; v))\). By lemma \ref{lemma:kernel_nontrivial}, \( L(J, v, (f, j_\Sigma)) \) has to have a nontrivial kernel, which contradicts our assumption on \((J; v)\).

So, \( f \) will factor as
\[
f : \Sigma \xrightarrow{\tau} \Sigma \xrightarrow{\mathcal{G}} \frac{\mathbb{Z}}{p\mathbb{Z}}, X
\]
where \( \mathcal{G} \) denotes a group action on \((\Sigma, j_\Sigma)\). But \( \beta = p\alpha \), and \( \alpha \) is primitive. This implies that the underlying group of \( \mathcal{G} \) is \( \mathbb{Z}_p = \mathbb{Z}_{p^n} \). So \((J, v, (f, j_\Sigma)) \in \mathcal{M}^\mathcal{G}_g(X, \beta)\). Again the convergence of a sequence of solutions to \((f, j_\Sigma)\) implies that the kernel of the linearized operator is nontrivial. Thus
\[
(J, v, (f, j_\Sigma)) \in \mathcal{D}^\mathcal{G}_{\{m_1, \ldots, m_\ell\}}
\]
for some choice of \(\{m_1, \ldots, m_\ell\}\). This is a contradiction, completing the proof of the theorem.

Using the information given by the above theorem, we define:

**Definition 7.2.** For a prime homology class \( \beta \in \mathcal{P}_2(X, \mathbb{Z}) \), choose a point \((J, v) \in \mathcal{P}_{\text{reg}}\). Define
\[
\mathcal{I}_g(\beta) = \sum_{(f, j_\Sigma) \in \mathcal{M}_g(X, \beta; (J; v))} \epsilon(f, j_\Sigma)
\]

8. **Invariance**

This section will be devoted to the proof of the invariance of
\[
\mathcal{I}_g : \mathcal{P}_2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}
\]
from the choice of the regular values \((J, v) \in \mathcal{P}_{\text{reg}}\). We begin this section by setting up a way of thinking of the moduli space of solutions corresponding to paths between two regular values \(\gamma_i = (J_i, v_i) \in \mathcal{P}_{\text{reg}}\) for \(i = 0, 1\). Namely, denote by \(\mathcal{Q}(\gamma_0, \gamma_1)\) the moduli space of the paths
\[
\gamma : [0, 1] \longrightarrow \mathcal{P}, \quad \text{with } \gamma(i) = \gamma_i, \ i = 0, 1.
\]
There is a section
\[
\overline{\gamma} : \mathcal{Z} = [0, 1] \times \mathcal{Q}(\gamma_0, \gamma_1) \times \mathcal{M}_g \times \mathcal{X} \longrightarrow \mathcal{E}
\]
defined by \(\overline{\gamma}(t, \gamma, j_\Sigma, f) = \overline{\gamma}(\gamma(t), j_\Sigma, f)\). Here we will think of \(\mathcal{E}\) as the bundle pulled back to \(\mathcal{Z}\) from \(\mathcal{Y}\), via the map \(\mathcal{Z} \rightarrow \mathcal{Y}\) defined by
\[
(t, \gamma, j_\Sigma, f) \rightarrow (\gamma(t), j_\Sigma, f).
\]
Again, we may also define the equivariant versions of these, giving the maps
\[ \partial_{\Theta} : \mathcal{Z}^g = [0, 1] \times \mathcal{Q}(\gamma_0, \gamma_1) \times \mathcal{M}^g \times \mathcal{N}^g \to \mathcal{E}^g. \]

One may argue, using an argument similar to those in the section on transversality, that the sections \( \partial \) and \( \partial_{\Theta} \) are transverse to the zero section. As a result, we obtain a smooth submanifold of \( \mathcal{Z} \), consisting of the zeros of \( \partial \), which will be denoted by \( \mathcal{N}_g(X, \beta) \). Similarly we may define \( \mathcal{N}_g^g(X, \beta) \) as a submanifold of \( \mathcal{Z}^g \).

Note that theorem 8.1 is a direct corollary of the following theorem:

**Theorem 8.1.** There is a Bair subset

\[ \mathcal{Q}(\gamma_0, \gamma_1)_{\text{reg}} \subset \mathcal{Q}(\gamma_0, \gamma_1), \]

such that for \( \gamma \in \mathcal{Q}(\gamma_0, \gamma_1)_{\text{reg}}, \) the moduli space \( \mathcal{N}_g(X, \beta; \gamma) \) of the somewhere injective solutions associated with the path \( \gamma(t) = (J_t, v_t) \), forms a compact 1-manifold, giving a cobordism between \( \mathcal{M}_g(X, \beta; (J_0, v_0)) \) and \( \mathcal{M}_g(X, \beta; (J_1, v_1)) \). In particular \( \mathcal{I}_g(\beta) \), as computed using \( (J_0, v_0) \) is equal to \( \mathcal{I}_g(\beta) \), as computed using \( (J_1, v_1) \).

**Proof.** There are projection maps from \( \mathcal{N}_g(X, \beta) \) to \( \mathcal{Q}(\gamma_0, \gamma_1) \), and from \( \mathcal{N}_g^g(X, \beta) \) to \( \mathcal{Q}(\gamma_0, \gamma_1) \), which are Fredholm maps of index 1. As a result there is a Bair set of regular values of these projection maps which we will denote by \( \mathcal{Q}(\gamma_0, \gamma_1)_{\text{reg}} \). The pre-image of our submanifolds \( \mathcal{D}_{\Theta}^{m_1, \ldots, m_\ell} \) will be submanifolds of \( \mathcal{N}_g^g(X, \beta) \), and the restriction of the projection map will typically have negative index on these submanifolds. The exception is \( \mathcal{D}_{\Theta}^{m_\ell} \), where the index is zero. We may choose the set of regular values so that they are regular values of these restriction maps as well.

With this choice of \( \mathcal{Q}(\gamma_0, \gamma_1)_{\text{reg}} \), if \( \gamma \in \mathcal{Q}(\gamma_0, \gamma_1)_{\text{reg}} \) then \( \mathcal{N}_g^g(X, \beta; \gamma) \) will be a smooth 1-dimensional manifold and for any \( (t, \gamma, j, f) \in \mathcal{N}_g^g(X, \beta; \gamma) \) the linearization \( L(\gamma(t), (f, j)) \) will have a trivial kernel, except for an isolated set of such points where the kernel is 1-dimensional.

We are interested in a study of a neighborhood of the points \( (t, \gamma, j, f) \) giving \( (J, v; (f, j)) \) (with \( (J, v) = \gamma(t) \)) where the kernel of \( L = L(J, v; (f, j)) \) is one dimensional.

Suppose that this kernel is generated by \( \theta = (\zeta, \eta) \) and that the corresponding cokernel is denoted by \( \mu \). At the time \( t + \epsilon \), a nearby point corresponding to the parameter \( \gamma(t + \epsilon) = (J_\epsilon, v_\epsilon) \) may be described as follows: If \( (Y, Z) \) denotes the derivative \( \gamma'(t) \), then \( J_\epsilon = J + \epsilon Y + G_1(\epsilon), v_\epsilon = v + \epsilon Z + G_2(\epsilon) \). Here \( G_1(\epsilon) \) and \( G_2(\epsilon) \) are in \( o(|\epsilon|) \).

Any solution to the perturbed Cauchy-Riemann equation corresponding to \( (J_\epsilon, v_\epsilon) \) which is close to \( (f, j) \) will be of the form \( (f_\epsilon, j_\epsilon) = \exp(f, j)(\theta) \).
\( \theta \) may be written as \( \theta = s\theta_0 + \theta_0 \), where \( \theta_0 = (\zeta_0, \eta_0) \) is orthogonal to the kernel of \( L \). The equation

\[
\mathcal{D}(J, v, (f, j_\Sigma)) = 0
\]

may be reformulated as

\[
L(s\theta_0 + \theta_0) + \epsilon(Y \circ df \circ j_\Sigma - (\tau_\Sigma \times f)^* Z) + \text{higher order terms} = 0
\]

Again let \( \Pi, \Pi^c \) be the projection over the kernel of \( L^* \) and image of \( L \), respectively. Then we may rewrite the above equation as the following two equations:

\[
\begin{align*}
L(\theta_0) + \Pi^c\{\epsilon(Y \circ df \circ j_\Sigma - (\tau_\Sigma \times f)^* Z) + \text{higher order terms}\} &= 0, \\
\Pi\{\epsilon(Y \circ df \circ j_\Sigma - (\tau_\Sigma \times f)^* Z) + \text{higher order terms}\} &= 0.
\end{align*}
\]

The first equation gives \( \theta_0 \) uniquely as an analytic function of \( s, \epsilon \) with no linear terms in \( s \). As a result, the second equation will be an analytic function of \( s, \epsilon \) which we denote by \( g(s, \epsilon) \). An argument similar to the one used by Taubes in [15] shows that the Taylor expansion of \( g \) starts as

\[
g(s, \epsilon) = r_1 \epsilon + r_2 s^2 + \text{higher order terms}.
\]

Note that \( r_1 \) is in fact computed as

\[
r_1 = \int_{\Sigma} \langle (Y \circ df \circ j_\Sigma - (\tau_\Sigma \times f)^* Z), \mu \rangle
\]

Our assumption on the regularity of \( \gamma \) for all projection maps implies that this pairing is nonzero. An argument similar to that of [15], shows that \( r_2 \) is also proportional to the differential of the map \( Q \circ F \), which is pulled back to the submanifold \( N_g(X, \beta; \gamma) \). The regularity implies again, that \( r_2 \) is also nonzero. This gives a description of the neighborhood of \((t, \gamma, j_\Sigma, f)\) in \( N_g(X, \beta; \gamma) \).

Note that this discussion may be followed even if \((t, \gamma, j_\Sigma, f)\) is a weak limit point of a sequence in \( N_g(X, \beta; \gamma) \) that lies in \( N_g^\mathcal{G}(X, \beta; \gamma) \).

If \((t, \gamma, j_\Sigma, f)\) is in \( N_g(X, \beta; \gamma) \), then the projection

\[
pr : N_g(X, \beta; \gamma) \longrightarrow [0, 1]
\]

will locally be like the map going from \( \{(s, \epsilon) | r_1 \epsilon + r_2 s^2 = 0\} \) to \([0, 1]\) which sends \((s, \epsilon)\) to \( t + \epsilon \). As a result \((t, \gamma, j_\Sigma, f)\) is a critical point of the projection map \( pr \) and nothing interesting happens at this point.

Now suppose that \((t, \gamma, j_\Sigma, f)\) is in \( N_g^\mathcal{G}(X, \beta; \gamma) \), with \( \mathcal{G} = \mathbb{Z}_p \). Since the kernel is nontrivial, \((J, v, (f, j_\Sigma))\) is forced to be in \( \mathcal{D}^\mathcal{G}_m \). Note that the other
submanifolds \( \mathcal{D}_{m_1,\ldots,m_\ell} \) are excluded by the regularity. As a result, the sections \( \theta, \mu \) are \( \mathcal{G} \)-invariant and the whole argument above may be done for the \( \mathcal{G} \) invariant sections, and inside \( \mathcal{N}_g^\mathcal{G}(X, \beta; \gamma) \). The result is that a neighborhood of \( (t, \gamma, j_\Sigma, f) \) in \( \mathcal{N}_g^\mathcal{G}(X, \beta; \gamma) \) is described by a similar function \( g'(s, \epsilon) = 0 \). A version of the uniqueness lemma 5.1 may be used to conclude that this neighborhood of \( (t, \gamma, j_\Sigma, f) \) in \( \mathcal{N}_g^\mathcal{G}(X, \beta; \gamma) \) is in fact identical to the previous neighborhood of it (as a weak limit point) in \( \mathcal{N}_g(X, \beta; \gamma) \). This is a contradiction which proves the theorem.  

\[ \square \]

**Remark 8.2.** When \( p = 2 \), there is a possibility that at a point \( (t, \gamma, j_\Sigma, f) \) of \( \mathcal{N}_2^\mathcal{G}(X, \beta; \gamma) \) where the linearization has a kernel with the corresponding ideal \( m^1 \), a solution in \( \mathcal{N}_g(X, \beta; \gamma) \) is blown up for the times \( t + \epsilon \) with \( \epsilon > 0 \), while there is no such solution for the times \( t - \epsilon \) which is close to \( (t, \gamma, j_\Sigma, f) \). This means that to count the embedded solutions in a class \( \beta = 2\alpha \), we will always get contributions from the curves in the class \( \alpha \). As a result a passage from the arguments of this paper to the case where \( p = 2 \) requires a weighted count of the solutions similar to the counts in [15]. We postpone such wall-crossing formulas to a future paper.

9. **Appendix: A Riemann-Roch Formula**

**Introduction**

Suppose that \( (\Sigma, j = j_\Sigma) \) is a Riemann surface with automorphism group \( \mathcal{G} \). Denote the quotient curve \( \Sigma/\mathcal{G} \) by \( C \). Fix a divisor \( D \) on \( \Sigma \) which is invariant under the action of \( \mathcal{G} \). This means that for any \( \sigma : \Sigma \to \Sigma \) in the automorphism group \( \mathcal{G} \) we have \( \sigma^* (D) = D \).

For any section \( \eta \) of the line bundle \([D]\) over \( \Sigma \), note that the pull back \( \eta_\sigma = \sigma^* \eta \) is also a section of \([D]\). Let \( m_\eta \) be the ideal of the group ring \( \mathbb{R}_\mathcal{G} \) consisting of all elements

\[ \alpha = \sum_{\sigma \in \mathcal{G}} a_\sigma \sigma^{-1}, \quad a_\sigma \in \mathbb{R}, \]

such that

\[ \sum_{\sigma \in \mathcal{G}} a_\sigma \eta_\sigma = 0. \]

It is easy to check that \( m_\eta \) is a left ideal of the group ring \( \mathbb{R}_\mathcal{G} \).

Fix an ideal \( m \) of the group ring \( \mathbb{R}_\mathcal{G} \) and let

\[ H^i_m([D]) := \left\{ \eta \in H^i(\Sigma, [D]) \mid m \subset m_\eta \right\}, \quad i = 0, 1. \]

Define

\[ h^i_m([D]) = \dim_{\mathbb{R}} (H^i_m([D])), \]
and let $\chi_m([D]) = h_m^0([D]) - h_m^1([D])$.

Our goal in this note is to obtain some results on the behavior of $\chi_m([D])$ in terms of the topological properties of $[D]$ and the covering

$$\pi : \Sigma \to C = \frac{\Sigma}{G}.$$  

Let us start with a consideration of the holomorphic tangent bundle $T_\Sigma$ of $\Sigma$. If $B$ denotes the branching divisor of the covering map $\pi : \Sigma \to C$, then we will have

$$T_\Sigma = \pi^* T_C \otimes [-B].$$

This gives the short exact sequence:

$$0 \to T_\Sigma \to \pi^* T_C \to \mathcal{O}_B \to 0.$$

If we consider the sections corresponding to the left ideal $m$ of $R_G$, we will get the following long exact sequence:

$$0 \to H^0_m(T_\Sigma) \to H^0_m(\pi^* T_C) \to H^0_m(\mathcal{O}_B) \to H^1_m(T_\Sigma) \to H^1_m(\pi^* T_C) \to H^1_m(\mathcal{O}_B) = 0.$$  

As a result

$$(16) \quad \chi_m(T_\Sigma) = \chi_m(\pi^* T_C) - h_m^0(\mathcal{O}_B).$$  

In the next step we consider line bundles of the form $\pi^* L$ where $L \to C$ is a line bundle on the quotient curve $C$. Any such line bundle may be written as $L = [E]$ where $E$ is a divisor on $C$ such that its support is disjoint from the branched locus. Note that if $p$ is a point on $C$ which is not in the branched locus of $\pi : \Sigma \to C$ then there is a short exact sequence:

$$0 \to \pi^* L = [\pi^* E] \to [\pi^*(E + p)] \to \mathcal{O}_{\pi^* p} = \bigotimes_{\sigma \in G} \mathcal{O}_{p_{\sigma}} \to 0,$$

where $\{p_{\sigma}\}_{\sigma \in G}$ is the pre-image $\pi^{-1}\{p\}$.

Again taking the long exact sequence corresponding to the ideal $m$ gives

$$(17) \quad \chi_m(\pi^*[E + p]) = \chi_m(\pi^*[E]) + h_m^0([\pi^* p]).$$  

Note that both $R_G$ and $m$ are vector spaces over $\mathbb{R}$. The dimension of the first one is $g = |G|$ and the dimension of the second one, we denote by $r$. It is easy to check that since $p$ is a generic point, $h_m^0([\pi^* p]) = 2(g - r)$. As a result of these two observations

$$(18) \quad \chi_m(\pi^* L) = 2(g - r).\deg(L) + \chi_m(\mathcal{O}_\Sigma).$$
In the last step, we compare the line bundle $\mathcal{O}_\Sigma$ with the canonical bundle $K_\Sigma$. Note that $K_\Sigma = \pi^*K_C + [B]$ where $B$ is the branching divisor introduced earlier. From the short exact sequence

$$0 \longrightarrow \pi^*K_C \longrightarrow K_\Sigma \longrightarrow \mathcal{O}_B \longrightarrow 0,$$

and the consideration of the global sections, we obtain

$$\chi_m(K_\Sigma) = \chi_m(\pi^*K_C) + h^0_m(\mathcal{O}_B)$$
$$= 2(g - r).\deg(K_C) + \chi_m(\mathcal{O}_\Sigma) + h^0_m(\mathcal{O}_B)$$
$$= 2(g - r)(2h - 2) + \chi_m(\mathcal{O}_\Sigma) + h^0_m(\mathcal{O}_B).$$

On the other hand, by Serre duality $\chi_m(\mathcal{O}_\Sigma) + \chi_m(K_\Sigma) = 0$. This implies that

$$-h^0_m(\mathcal{O}_B) = 2(g - r)(2h - 2) + 2\chi_m(\mathcal{O}_\Sigma).$$

Combining with the information on $T_\Sigma$ and the fact that $h^0_m(T_\Sigma) = 0$, this implies that

$$h^1_m(T_\Sigma) = -3\chi_m(\mathcal{O}_\Sigma).$$

### General invariant bundles; The index computation

Now suppose that $E \rightarrow C$ is a bundle over the quotient surface, of rank $n$. The bundle $\pi^*E$ will be invariant under the action of the automorphism group $\mathfrak{G}$ and one may consider the global sections associated with a left ideal $m$ of the group ring $\mathbb{R}_\mathfrak{G}$. Namely:

$$H^i_m(\Sigma, \pi^*E) := \{ \eta \in H^i(\Sigma, \pi^*E) \mid m \subset m_\eta \}.$$  

The Euler characteristic $\chi_m(\pi^*E)$ may be defined similarly.

For any such bundle, we may formally break it down to the line bundles:

$$E = L_1 \oplus L_2 \oplus ... \oplus L_n.$$  

From this presentation

$$\chi_m(\pi^*E) = \sum_{i=1}^n \chi_m(\pi^*L_i)$$
$$= \sum_{i=1}^n [2(g - r).\deg(L_i) + \chi_m(\mathcal{O}_\Sigma)]$$
$$= 2(g - r).c_1(E) + n(\chi_m(\mathcal{O}_\Sigma))$$

In the last part of this note we will consider the index computation associated with the ideal $m$ of the group ring $\mathbb{R}_\mathfrak{G}$.

In our moduli problems, there is a kernel isomorphic to

$$H^1_m(\Sigma, T_\Sigma) \oplus H^0_m(\Sigma, \pi^*(TX|_C)),$$
where our quotient curve \( C \) is embedded in an almost complex symplectic manifold \( (X, \omega, J) \), and \( TX \) is the tangent bundle of \( X \).

The cokernel will be isomorphic to

\[
H^0_m(\Sigma, \pi^*(TX|_C)).
\]

This implies that the index \( I \) of our operator is equal to

\[
I = h^1_m(T\Sigma) + \chi_m(\pi^*(TX|_C)) = 2(c_1(X).[C])(g - r) + (n - 3)\chi_m(\mathcal{O}_\Sigma),
\]

where \([C]\) represents the homology class represented by the curve \( C \) in \( H_2(X, \mathbb{Z}) \) and \( n \) is the complex dimension of \( X \).

In particular if \( c_1(X) = 0 \) and \( n = 3 \) then \( I = 0 \):

**Corollary 9.1.** The index of the linearized operator associated with any left ideal \( m \) of the group ring \( \mathbb{R}_G \) is equal to zero, as far as the target manifold is a symplectic 3-fold with vanishing first Chern class \( c_1(X) = 0 \), the kernel is isomorphic to

\[
H^1_m(\Sigma, T\Sigma) \oplus H^0_m(\Sigma, \pi^*(TX|_C))
\]

and the cokernel is isomorphic to

\[
H^0_m(\Sigma, \pi^*(TX|_C)).
\]

**References**

[1] Aspinwall, P.S., Morrison, D.R. Topological field theory and rational curves, *Comm. in Math. Phys.* 151 (1993) 245-262

[2] Bryan, J., Katz, S., Leung, N.C. Multiple covers and the integrality conjecture in Calabi-Yau threefolds, *J. Algebraic Geom.* 10 (2001) no.3, 549-568

[3] Bryan, J., Pandharipande, R., BPS-states of curves in Calabi-Yau 3-folds, *Geom. Topol.* 5 (2001) 287-318 (electronic)

[4] Gopakumar, R., Vafa, C., M theory and topological strings II, (1998), arxiv:hep-th/9812127

[5] Hosono, S., Saito, M-H., Takahashi, A., Relative Lefschetz action and BPS state counting, *Internat. Math. Res. Notices*, (2001), no.15, 783-816

[6] Ionel, E. N., Parker, T., The Gromov invariants of Ruan-Tian and Taubes, *Math. Res. Lett.* 4 (1997), no. 4, 521-532.

[7] Kontsevich, M., Enumeration of rational curves via torus action, *Prog. Math. 129, 335-368*

[8] Lian, B.H., Liu, K., Yau, S-T, Mirror principle I, *Asian J. Math.* 1(4) (1997) 729-763

[9] Liu, G., Tian, G., On the equivalence of multiplicative structures in Floer homology and quantum cohomology, *Acta Math. Sin. (Engl. Ser.)* 15 (1999), 53-80

[10] Manin, I., Generating functions in algebraic geometry and sums over trees, in *The moduli space of curves*, Dijkgraaf, Faber, van der Geer (editors) Birkhauser (1995) 401-417

[11] Maulik, D., Nekrasov, N., Okounkov, A., Pandharipande, R., Gromov-Witten theory and Donaldson-Thomas theory, *ITEP-TH-61/03, IHES/M/03/67*
[12] McDuff, D., Salamon, D., *Introduction to symplectic topology*, Oxford Sci. Pub., Oxford, 1994

[13] Ruan, Y., Tian, G., A mathematical theory of quantum cohomology, *J. Differential Geom.*, 42 (1995) no.2, 258-367

[14] Ruan, Y., Tian, G., Higher genus symplectic invariants and sigma model coupled with gravity, *Turkish J. Math.*, 20 (1996) no.1, 75-83

[15] Taubes, C.H., Counting pseudo-holomorphic submanifolds in dimension 4, *J. Differential Geom.* 44 (1996) no.4, 818-893

[16] Taubes, C.H., SW ⇒ Gr: From Seiberg-Witten invariants to pseudo-holomorphic curves, *J. American Math. Soc.* 9 (1996) 845-918

[17] Taubes, C.H., Gr ⇒ SW: From pseudo-holomorphic curves to the Seiberg-Witten solutions, *J. Differential Geom.* 51 (1999), no. 2, 203-334.

[18] Voisin, C., A mathematical proof of a formula of Aspinwall and Morrison, *Composito Math.* 104 (1996), no.2, 135-151

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