Quantum Hall Liquid on a Noncommutative Superplane

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Supersymmetric quantum Hall liquids are constructed on a noncommutative superplane. We explore a supersymmetric formalism of the Landau problem. In the lowest Landau level, there appear spin-less bosonic states and spin-1/2 down fermionic states, which exhibit a super-chiral property. It is shown the Laughlin wavefunction and topological excitations have their superpartners. Similarities between supersymmetric quantum Hall systems and bilayer quantum Hall systems are discussed.

I. INTRODUCTION

Over the past few years, accompanied with the developments of the noncommutative (NC) geometry and string theory, quantum Hall (QH) systems have attracted increasing attentions from particle physicists. (See [1, 2] for instance.) It is well known that the underlying mathematical structure of QH system is NC geometry, and QH systems manifest its exotic properties [3–6]. Based on the second Hopf map, a four dimensional generalization of QH liquid was constructed in Ref. [7]. The system has higher dimensional analogues of the exotic structures of the two dimensional QH system, such as NC geometry, multiply charged excitations, massless edge states, etc. Since then, many efforts are devoted to the understanding of the four dimensional QH liquid [8–16] and the construction of even higher dimensional QH systems [17–21]. The studies of higher dimensional QH systems have brought many fruitful developments in both particle physics and condensed matter physics. Particularly, spherical boundstates of D-branes in string theory were well investigated based on the set-up of the fuzzy spheres in higher dimensional QH systems [22]. Three dimensional reduction of four dimensional QH effect gave a hint to the discovery of the spin-Hall effect [23], which has become one of the most rapidly growing topics in condensed matter physics.

Recently, it was discovered that the non-anticommutative (NAC) field theory is naturally realized on D-branes in Ramond-Ramond field or graviphoton background [24–27]. Also, it has been shown that, in the supermatrix model, fluctuations on a fuzzy supersphere yield supersymmetric NC field theories [27]. Besides, some interesting relations between NAC geometry, Landau problem and QH systems are reported [28–31]. Especially, on a fuzzy supersphere, a supersymmetric extension of QH liquid was explicitly constructed in Ref.[32]. While mathematical properties of NAC theories have been well investigated [33–35], their emergent physical consequences have not been satisfactorily understood yet. The supersymmetric QH system provides a rare “physical” set-up whose underlying mathematics is given by NAC geometry. Since two dimensional and higher dimensional QH systems manifest peculiar properties of NC geometry, it would be reasonable to expect that explorations of supersymmetric QH liquids may reveal yet unknown physical aspects of the NAC geometry. In this paper, by taking a planar limit of the fuzzy supersphere, we construct QH liquids on a NC superplane, and investigate physical properties in a NAC world.

This paper is organized as follows. In Section II, we review a systematic construction of NC superplane from the fuzzy supersphere. It is shown that the NC superplane is realized by introducing the super gauge fields. In Section III, we develop Lagrangian and Hamiltonian formalisms for one-particle system on the NC superplane. The system possesses (complex) $\mathcal{N} = 2$ supersymmetry, one of which is dynamical and the other is non-dynamical. Another approach for one-particle system on a NC plane with supersymmetry is found in Ref.[36, 37], where a higher derivative term is introduced to be invariant under the Galilean boosts transformation. In Section IV, we analyze a supersymmetric Landau problem. In each of the higher Landau levels (LLs), there exists $\mathcal{N} = 2$ supersymmetry, while in the lowest Landau level (LLL), only the $\mathcal{N} = 1$ non-dynamical supersymmetry remains valid. We explicitly construct radially symmetric orbit states, which form a “complete” basis in the LLL. These states are super-holomorphic except for their exponential term, and show a super-chiral property where not only the orbital rotation but also the spin polarization is chiral. In Section V, a Laughlin wavefunction and its superpartner on the NC superplane are derived. In Section VI, we present bosonic and fermionic topological excitations, and investigate their basic properties. In Section VII, we discuss a possible mapping from supersymmetric QH systems to bilayer QH systems. Section VIII is devoted for summary and discussions.

II. NONCOMMUTATIVE SUPERPLANE

Based on Ref.[28], we review an algebra on a NC superplane from the $\mathcal{O}sp(1|2)$ algebra. The $\mathcal{O}sp(1|2)$ algebra consists of five generators $L_a(a=1,2,3)$ and
where we have defined the dimensionless coordinates as $X_i = \frac{\alpha}{\sqrt{\ell^2_B}} X_i$, $\Theta_{\alpha} = \frac{\alpha}{\sqrt{\ell^2_B}} \Theta_{\alpha}$. (More general contractions, including asymmetric scaling, are found in Ref.[28].) The bosonic coordinates and the fermionic coordinates are completely decoupled unlike the fuzzy supersphere case. The algebra (2.5a) is equivalent to that on the NC bosonic plane. The original QH systems on NC bosonic plane have already been well investigated as found in Ref.[38]. In the following, we include the known results on the bosonic NC plane for complete description.

A physical set-up for the NC superplane is realized by introducing super gauge fields. We consider a constant magnetic strength made by a bosonic gauge field and a fermionic gauge field as

\begin{align*}
B &= -i(\sigma_2)_{ij} \partial_j A_j = -\epsilon_{ij} \partial_i A_j, \\
B &= -i(\sigma_3)_{\alpha\beta} \partial_\alpha (C_{\beta\gamma} A_\gamma) = -i(\sigma_1)_{\alpha\beta} \partial_\alpha A_\beta. 
\end{align*}

It is apparent that there exists a $U(1)$ gauge degree of freedom, $A_i \to A_i + \partial_i \xi$ and $A_\alpha \to A_\alpha + \partial_\alpha \xi$. The covariant momenta are given by

\begin{align*}
P_i &= -i(\partial_i + iA_i), \\
P_\alpha &= i(\partial_\alpha + iA_\alpha).
\end{align*}

With these covariant momenta, the center-of-mass coordinates are defined as

\begin{align*}
X_i &= x_i + \ell^2_B (\sigma_2)_{ij} P_j, \\
\Theta_\alpha &= \theta_\alpha - i\ell^2_B (\sigma_1)_{\alpha\beta} P_\beta.
\end{align*}

where $\ell_B \equiv 1/\sqrt{B}$ is the magnetic length.

The center-of-mass coordinates and the covariant momenta are completely decoupled, and satisfy the super Heisenberg-Weyl algebra individually,

\begin{align*}
[P_i, P_j] &= -\frac{1}{\ell^2_B} (\sigma_2)_{ij}, \\
[P_i, P_\alpha] &= 0, \\
\{P_\alpha, P_\beta\} &= \frac{1}{\ell^2_B} (\sigma_1)_{\alpha\beta}.
\end{align*}

and

\begin{align*}
[X_i, X_j] &= \ell^2_B (\sigma_2)_{ij}, \\
[X_i, \Theta_\alpha] &= 0, \\
\{\Theta_\alpha, \Theta_\beta\} &= \ell^2_B (\sigma_1)_{\alpha\beta}.
\end{align*}

The set of algebras (2.10) is consistent with Eq.(2.5). In the LLL limit ($B \to \infty$), it is easily seen from Eq.(2.8) the particle position $(x_i, \theta_\alpha)$ reduces to the center-of-mass coordinate operator $(X_i, \Theta_\alpha)$, and the superplane under the strong super magnetic field is identified with the NC superplane.

The angular momentum (2.4b) can be rewritten in terms of the covariant momenta and the center-of-mass coordinates as

\begin{equation}
L_\perp = \frac{1}{2\ell^2_B} (X_i^2 + \frac{1}{2} C_{\alpha\beta} \Theta_\alpha \Theta_\beta) - \frac{1}{2} \ell^2_B (P_i^2 + \frac{1}{2} C_{\alpha\beta} P_\alpha P_\beta).
\end{equation}

The center-of-mass coordinates $(X_i, \Theta_\alpha)$ and the covariant momenta $(P_i, P_\alpha)$ form a closed algebra with $L_\perp$,
individually,

\[
\begin{align*}
[L_\perp, X_i] &= -(\sigma_2)_{ij} X_j, \quad [L_\perp, \Theta_\alpha] = \frac{1}{\sqrt{2}}(\sigma_3)_{\alpha\beta} \Theta_\beta, \\
[L_\perp, P_i] &= (\sigma_2)_{ij} P_j, \quad [L_\perp, P_\alpha] = -\frac{1}{\sqrt{2}}(\sigma_3)_{\alpha\beta} P_\beta.
\end{align*}
\]

(2.12a)

(2.12b)

Due to the existence of two sets of the super Heisenberg-Weyl algebras, two sets of supersymmetric harmonic oscillators are naturally defined. The bosonic creation and annihilation operators are given by

\[
\begin{align*}
a &\equiv \frac{\ell_B}{\sqrt{2}}(P_x + iP_y), \quad a^\dagger \equiv \frac{\ell_B}{\sqrt{2}}(P_x - iP_y), \\
b &\equiv \frac{1}{\sqrt{2\ell_B}}(X - iY), \quad b^\dagger \equiv \frac{1}{\sqrt{2\ell_B}}(X + iY),
\end{align*}
\]

(2.13a)

(2.13b)

which satisfy \([a, a^\dagger] = [b, b^\dagger] = 1\). Other commutators become zeros. Similarly, the fermionic creation and annihilation operators are given by

\[
\begin{align*}
\alpha &\equiv \ell_B P_\theta, \quad \alpha^\dagger \equiv \ell_B P_{\bar{\theta}}, \\
\beta &\equiv \frac{1}{\ell_B} \Theta_1, \quad \beta^\dagger \equiv \frac{1}{\ell_B} \bar{\Theta}_1,
\end{align*}
\]

(2.14a)

(2.14b)

which satisfy \(\{\alpha, \alpha^\dagger\} = \{\beta, \beta^\dagger\} = 1\). Other anticommutators are zeros. With use of supersymmetric harmonic oscillators, the angular momentum can be written as

\[
L_\perp = (b^\dagger b + \frac{1}{2} \beta^\dagger \beta) - (a^\dagger a + \frac{1}{2} \alpha^\dagger \alpha).
\]

(2.15)

Thus, the \(b\)-quantum acquires the angular momentum by 1, while the \(\beta\)-quantum acquires the angular momentum by 1/2.

It is convenient to fix the gauge freedom as the symmetric gauge,

\[
A_i = i(\sigma_2)_{ij} x_j \frac{B}{2}, \quad A_\alpha = i(\sigma_1)_{\alpha\beta} \bar{\theta}_\beta \frac{B}{2}.
\]

(2.16)

These expressions are obtained by expanding the supermonopole gauge fields [31] around the north pole on the supersphere. The field strengths become

\[
\begin{align*}
F_{ij} &= \partial_i A_j - \partial_j A_i = -iB(\sigma_2)_{ij}, \\
F_{i\alpha} &= \partial_i A_\alpha - \partial_\alpha A_i = 0, \\
F_{\alpha\beta} &= \partial_\alpha A_\beta + \partial_\beta A_\alpha = iB(\sigma_1)_{\alpha\beta}.
\end{align*}
\]

(2.17a)

(2.17b)

(2.17c)

In the symmetric gauge, the creation and annihilation operators (2.13) (2.14) read as

\[
\begin{align*}
a &= -\frac{i}{\sqrt{2}} (z + \partial^*), \quad a^\dagger = \frac{i}{\sqrt{2}} (z^* - \partial), \\
b &= \frac{1}{\sqrt{2}} (z^* + \partial), \quad b^\dagger = \frac{1}{\sqrt{2}} (z - \partial^*),
\end{align*}
\]

(2.18a)

(2.18b)

and

\[
\begin{align*}
\alpha &= -\frac{i}{\sqrt{2}} (\theta - \partial_\theta), \quad \alpha^\dagger = -\frac{i}{\sqrt{2}} (\theta^* - \partial^\theta), \\
\beta &= \frac{1}{\sqrt{2}} (\theta^* + \partial_\theta), \quad \beta^\dagger = \frac{1}{\sqrt{2}} (\theta + \partial^\theta),
\end{align*}
\]

(2.19a)

(2.19b)

where we have used dimensionless complex coordinates and derivatives,

\[
\begin{align*}
z &= \frac{1}{2\ell_B}(x + iy), \quad z^* = \frac{1}{2\ell_B}(x - iy), \\
\partial &= \ell_B(\partial_x - i\partial_y), \quad \partial^* = \ell_B(\partial_x + i\partial_y), \\
\theta &= \frac{1}{2\ell_B}\theta_1, \quad \theta^* = \frac{1}{2\ell_B}\theta_2, \\
\partial_\theta &= \sqrt{2\ell_B}\partial_{\bar{\theta}}_1, \quad \partial^\theta = \sqrt{2\ell_B}\partial_{\bar{\theta}}_2.
\end{align*}
\]

(2.21a)

(2.21b)

III. ONE-PARTICLE HAMILTONIAN AND SUPERSYMMETRY

We develop a Lagrangian formalism for one-particle in the presence of super gauge fields. The Lagrangian may be given by

\[
L = \frac{M}{2}(\dot{x}_i^2 + C_{\alpha\beta}\dot{\theta}_\alpha\dot{\theta}_\beta) - (A_i \dot{x}_i + A_\alpha \dot{\theta}_\alpha).
\]

(3.1)

In the LLL limit, the kinetic term is quenched, and the Lagrangian (3.1) reduces to

\[
L_{\text{eff}} = -A_i \dot{x}_i - A_\alpha \dot{\theta}_\alpha.
\]

(3.2)

The canonical momenta are derived as

\[
\begin{align*}
p_i &= \frac{\partial}{\partial \dot{x}_i} L_{\text{eff}} = -A_i = -i(\sigma_2)_{ij} \dot{x}_j \frac{B}{2}, \\
p_\alpha &= \frac{\partial}{\partial \dot{\theta}_\alpha} L_{\text{eff}} = A_\alpha = i(\sigma_1)_{\alpha\beta} \dot{\theta}_\beta \frac{B}{2},
\end{align*}
\]

(3.3a)

(3.3b)

where the symmetric gauge was used in the last equations. By imposing the commutation relations to canonical variables

\[
\begin{align*}
[x_i, p_j] &= i\delta_{ij}, \\
\{\theta_\alpha, p_\beta\} &= i\delta_{\alpha\beta},
\end{align*}
\]

(3.4a)

(3.4b)

we obtain the NC relations

\[
\begin{align*}
[x_i, x_j] &= \ell_B^2(\sigma_2)_{ij}, \\
\{\theta_\alpha, \theta_\beta\} &= \ell_B^2(\sigma_1)_{\alpha\beta}.
\end{align*}
\]

(3.5a)

(3.5b)

These relations are what we have already obtained in Eq.(2.10). Then, it would be reasonable to adopt Eq.(3.1) as the Lagrangian for the present system.

The equations of motions are derived as

\[
\begin{align*}
M \ddot{x}_i &= \epsilon_{ij} B \ddot{x}_j, \\
M \ddot{\theta}_\alpha &= -i(\sigma_3)_{\alpha\beta} B \ddot{\theta}_\beta,
\end{align*}
\]

(3.6a)

(3.6b)
which represent cyclotron motions for bosonic and fermionic degrees of freedom. As we shall discuss in the next section, the fermionic variables \{\theta_\alpha\} are related to the spin degrees of freedom. With the definition of the spin \(S_a = -i\frac{M}{\hbar}\theta_\alpha(\sigma_aC)_{\alpha\beta}\theta_\beta\), Eq. (3.6b) implies the spin precession motion,

\[
\dot{S}_i = -\epsilon_{ij}S_jB. \tag{3.7}
\]

The Lagrangian (3.1) apparently possesses translational symmetries on both the bosonic plane and the fermionic plane. The Noether charges accompanied by the translational symmetries are obtained as

\[
\mathcal{P}_i = M\dot{x} - B\epsilon_{ij}x_j, \tag{3.8a}
\]

\[
\mathcal{P}_\alpha = MC_{\alpha\beta}\dot{\theta}_\beta + iB(\sigma_1)_{\alpha\beta}\theta_\beta, \tag{3.8b}
\]

which are total momenta. The first terms on the right-hand sides in Eq. (3.8) represent the particle momenta, and the second terms represent the field momenta. The total momenta are related to the center-of-mass coordinates as

\[
\mathcal{P}_i = -B\epsilon_{ij}x_j, \quad \mathcal{P}_\alpha = B(\sigma_1)_{\alpha\beta}\Theta_\beta. \tag{3.9}
\]

Hence, the center-of-mass coordinates are conserved quantities and essentially act as translational generators on the NC superplane.

Next, we develop a Hamiltonian formalism. The canonical momenta are given by

\[
p_i = \frac{\partial}{\partial \dot{x}_i}L = M\dot{x}_i - A_i, \tag{3.10a}
\]

\[
p_\alpha = \frac{\partial}{\partial \dot{\theta}_\alpha}L = MC_{\alpha\beta}\dot{\theta}_\beta + A_\alpha, \tag{3.10b}
\]

and Hamiltonian is constructed as

\[
H = \dot{x}_i p_i + \dot{\theta}_\alpha p_\alpha - L = \frac{1}{2M}(p_i^2 + C_{\alpha\beta}p_\alpha p_\beta), \tag{3.11}
\]

where we have used the covariant momenta (2.7).

With use of creation and annihilation operators, two sets of supercharges are naturally defined as

\[
Q \equiv a^\dagger, \quad Q^\dagger \equiv a^\dagger a, \tag{3.12a}
\]

\[
\tilde{Q} \equiv b^\dagger \beta, \quad \tilde{Q}^\dagger \equiv \beta^\dagger b, \tag{3.12b}
\]

and the Hamiltonian (3.11) is expressed as

\[
H = \omega(a^\dagger a + a^\dagger a) = \omega\{Q, Q^\dagger\}. \tag{3.13}
\]

Thus, the supercharges \((Q, Q^\dagger)\) generate a dynamical supersymmetry. This Hamiltonian commutes with four supercharges, and the system possesses (complex) \(N = 2\) supersymmetry. Some comments are added here. The Hamiltonian (3.13) is identical to the one used in the one-dimensional supersymmetric harmonic oscillator system [39]. However, the one-dimensional harmonic oscillator system possesses \(N = 1\) supersymmetry only, while the present system possesses \(N = 2\) supersymmetry. (See also Sect.VII.) The anticommutator of \((\tilde{Q}, \tilde{Q}^\dagger)\) gives the radius on the NC superplane as

\[
2\ell_B^2\{\tilde{Q}, \tilde{Q}^\dagger\} = 2\ell_B^2(b^\dagger b + \beta^\dagger \beta) = X_i^2 + C_{\alpha\beta}\Theta_\alpha\Theta_\beta \equiv R^2. \tag{3.14}
\]

This expression implies that the eigenvalue of the radius operator \(R^2\) takes a semi-positive value, and the supersymmetry generated by \((Q, Q^\dagger)\) is a non--dynamical one. Since \(R^2\) commutes with the four supercharges, \(N = 2\) supermultiplet has not only an identical energy but also an identical eigenvalue of the radius operator.

The Hamiltonian and the radius operator commute with the angular momentum. Then, the four components of the \(N = 2\) supermultiplet can be taken as simultaneous eigenstates of the angular momentum. The angular momentum and the supercharges satisfy the commutation relations

\[
[L_\perp, Q] = -\frac{i}{2}Q, \quad [L_\perp, Q^\dagger] = \frac{i}{2}Q^\dagger, \tag{3.15a}
\]

\[
[L_\perp, \tilde{Q}] = \frac{i}{2}\tilde{Q}, \quad [L_\perp, \tilde{Q}^\dagger] = -\frac{i}{2}\tilde{Q}^\dagger. \tag{3.15b}
\]

Thus, the supersymmetric transformations change the eigenvalue of the angular momentum by 1/2.

## IV. SUPERSYMMETRIC LANDAU PROBLEM

The energy spectrum of the Hamiltonian (3.13) reads as

\[
E_n = \omega n, \tag{4.1}
\]

where \(n = 0, 1, 2, \cdots\) indicates the LL in the supersymmetric Landau problem. [See Appendix C for detail analysis of the eigenvalue problem of the Hamiltonian (3.13) and the explicit expression for the eigenstates in the symmetric gauge.] The zero-point energy is canceled due to the existence of the supersymmetry. The higher LLs are doubly degenerate compared to the LLL. The eigenvalue of the radius operator (3.14) is given by

\[
R_m = \sqrt{2m\ell_B}, \tag{4.2}
\]

where \(m = 0, 1, 2, \cdots\) indicates the radially symmetric orbits. The four components for the \(N = 2\) supermultiplet with energy (4.1) and radius (4.2) are constructed as

\[
\frac{1}{\sqrt{n!!}}(a^\dagger)^n(b^\dagger)^m|0>, \tag{4.3a}
\]

\[
\frac{1}{\sqrt{n!!(m-1)!}}(a^\dagger)^n\beta^\dagger(b^\dagger)^{m-1}|0>, \tag{4.3b}
\]

\[
\frac{1}{\sqrt{(n-1)!m!}}a^\dagger(a^\dagger)^{n-1}(b^\dagger)^m|0>, \tag{4.3c}
\]

\[
\frac{1}{\sqrt{(n-1)!(m-1)!}}\beta^\dagger(a^\dagger)^{n-1}\beta^\dagger(b^\dagger)^{m-1}|0>. \tag{4.3d}
\]
At the same time, they are eigenstates of the angular momenta $L_\perp$ with different eigenvalues, $l = m - n$, $m - n - \frac{1}{2}$, $m - n + \frac{1}{2}$ and $m - n$, respectively. Here, we give a physical interpretation of these states. Because they have the identical energy and the radius, they may represent four particle states, which are on the same radially symmetric orbit, and rotate around the origin with the same frequency. Hence, they should carry the same orbital angular momentum, while their eigenvalues of the angular momentum $L_\perp$ are different. This discrepancy is solved by noticing that $L_\perp$ represents the total angular momentum, and each of the four particle states carries the intrinsic spin as well as the orbital angular momentum. Namely, the components of the $N = 2$ supermultiplet (4.3) are interpreted as the four particle states which have the identical orbital angular momentum $m - n$, and, simultaneously, have different spins $0$, $-1/2$, $1/2$ and $0$, respectively. Thus, two of them (4.3a),(4.3d) are interpreted as spin-less bosons, and the other two (4.3b),(4.3c) are interpreted as spin-1/2 down and up fermions. As suggested by Eq.(3.15), the $N = 2$ supersymmetry changes their spins by 1/2, and transforms the bosons to the fermions and vice versa [Fig.1]. It is noted that, in general, supersymmetric quantum mechanical models do not deal with a real boson-fermion symmetry [40], while supersymmetric quantum Hall systems deal with a real boson-fermion symmetry.

Each Hilbert space of the higher LL possesses the $N = 2$ supersymmetry, because $n$-th $(n \geq 1)$ LL is spanned by $N = 2$ supermultiplets (4.3) with fixed $n$, while, in the LLL, only the non-dynamical supersymmetry $N = 1$ remains valid, because the LLL is the “vacuum” for the $N = 1$ dynamical supersymmetry. In fact, in the LLL, the Hilbert space is spanned only by the $N = 1$ non-dynamical superpartners

$$|m + 1/2> = \frac{1}{\sqrt{m!}} \beta^l (b^l)^m |0>, \quad \text{(4.4a)}$$

$$|m + 1> = \frac{1}{\sqrt{(m + 1)!}} (b^l)^{m+1} |0>, \quad \text{(4.4b)}$$

(and the vacuum $|0>$). In the symmetric gauge, with expression of the vacuum $\psi_0 = \sqrt{\frac{1}{\pi}} e^{-|z|^2 - \theta \theta^*}$, they are represented as

$$\psi_{m+1/2} = \frac{2^{m+1}}{\pi m!} z^m \theta e^{-|z|^2 - \theta \theta^*}, \quad \text{(4.5a)}$$

$$\psi_{m+1} = \frac{2^{m+1}}{\pi (m+1)!} z^{m+1} e^{-|z|^2 - \theta \theta^*}. \quad \text{(4.5b)}$$

The “complete relation” in the LLL is obtained as

$$\sum_{m \in \mathbb{N}/2} \psi_m(z,z^*,\theta,\theta^*) \psi_m^\dagger(z',z'^*,\theta',\theta'^*) = \frac{1}{\pi} e^{-((|z|^2 + \theta \theta^*) - (|z'|^2 + \theta' \theta'^*) - 2(z^* z + \theta^* \theta)} \quad \text{(4.6)}$$

These states are holomorphic about $z$ and $\theta$, i.e. super-holomorphic except for their exponential term. They have angular momenta $m + 1/2$ and $m + 1$ respectively, and are localized on the same radially symmetric orbit with radius $R_{m+1}$. This reminds the situation where two particles, one of which has spin-0 and the other has spin-1/2 down, rotate on a plane with the same radius [Fig.2]. There appear no spin-1/2 up fermions in the LLL, and the system shows the super-chirality, where not only the orbital rotations but also the spin rotations are chiral. In the higher LLs, there are both spin-1/2 up and down fermions, and the system is non-chiral. (See Fig.1.)

![FIG. 1: The left sector about the vertical dashed axis is a “bosonic sector” for the dynamical supersymmetry, and the right sector is a “fermionic sector”. The curved solid arrows represent the non-dynamical supersymmetry transformation generated by $(\mathcal{Q}, \mathcal{Q}^\dagger)$, while the curved dashed arrows represent the dynamical supersymmetry transformation generated by $(\mathcal{Q}, \mathcal{Q}^\dagger)$. In each of the higher LLs, there are spin-less, spin-1/2 up and spin-1/2 down particles due to the existence of the $N = 2$ supersymmetry, while, in the LLL, the system possesses only $N = 1$ non-dynamical supersymmetry, and there appear only spin-less and spin-1/2 down particles.](image)

![FIG. 2: There are spin-less bosons and spin-1/2 down fermions in the LLL. They are on the radially symmetric orbits, and rotate around the origin with the same frequency.](image)

V. LAUGHLIN WAVEFUNCTION AND ITS SUPERPARTNER

We construct a Laughlin wavefunction in the supersymmetric framework, by demanding following condi-
tions as in the original case [41]. The Laughlin wavefunction (i) is an eigenstate of \( L_\perp \), (ii) possesses the translational symmetries on the superplane up to its exponential factor. We also postulate that the Laughlin wavefunction on the NC superplane is composed of a product of the bosonic part and the fermionic part. It may be natural to use the original Laughlin wavefunction as the bosonic part. With respect to the fermionic part, the Vandermonde determinant vanishes due to the nilpotency of the Grassmann number, \( \prod_{j<i}^N (\theta_i - \theta_j) = 0 \) for \( N \geq 3 \), and \((\theta_i - \theta_j)^m = 0 \) for \( m \geq 2 \). Then, Laughlin wavefunction on the NC superplane is simply given by

\[
\Psi_{\text{Lin}} = \prod_{p<q}^N (z_p - z_q)^m e^{-\sum_a^L (\mathcal{z}_p^a + (\mathbb{g}^a)_{pq})},
\]

where \( N \) denotes the number of particles. Apparently, \( \Psi_{\text{Lin}} \) lives in the LLL, and is an eigenstate of \( L_\perp \) with eigenvalue \( mN(N-1)/2 \). Thus, 1 particles described by \( \Psi_{\text{Lin}} \) are spin-less particles, which rotate on the radially symmetric orbits in order from the origin. Intriguingly, \( \Psi_{\text{Lin}} \) has its superpartner \( \Psi_{s\text{Lin}} \) unlike the Laughlin-Haldane wavefunction on the supersphere [32]. This stems from the decoupling between \( X_i \) and \( \Theta_\alpha \) (2.10b) on the NC superplane. The superpartner \( \Psi_{s\text{Lin}} \) is related to \( \Psi_{\text{Lin}} \) by the non-dynamical supersymmetry, and is explicitly given by

\[
\Psi_{s\text{Lin}} = \sum_{p<q} \left( \frac{\theta_p - \theta_q}{z_p - z_q} \right) \cdot \Psi_{\text{Lin}},
\]

which has the angular momentum \( (mN(N-1)-1)/2 \). It is noted that \( \Psi_{s\text{Lin}} \) is not simply expressed as a product of a bosonic part and a fermionic part. The \( N \)-particle state described by \( \Psi_{s\text{Lin}} \) is a superposition of all possible states where the \( (N-1) \) spin-less particles and one spin-1/2 down particle rotate on the radially symmetric orbits in order from the origin. With the definition of the filling factor \( \nu \equiv \frac{A}{2\pi B\ell_B^2} \) (where \( A \) denotes the area on the superplane), \( \Psi_{\text{Lin}} \) and \( \Psi_{s\text{Lin}} \) may become two degenerate ground states of the supersymmetric QH systems at \( \nu = 1/m \), because they should have an identical energy due to the supersymmetry.

The density of \( \Psi_{\text{Lin}} \) is

\[
\rho_{\Psi_{\text{Lin}}} = e^{-\frac{\pi}{\nu} \Psi_{\text{Lin}}} W,
\]

where \( W \) is interpreted as the supersymmetric extension of the plasma potential,

\[
W = -\frac{m^2}{2} \sum_{p<q} \ln |(x + iy)_p - (x + iy)_q|^2 \\
- \frac{mB}{4} \sum_p |(x + iy)|^2 + 2\theta_1 \theta_2 p.
\]

VI. HALL CURRENTS AND EXCITED STATES

The Hall currents on the superplane are expressed as

\[
I_i = \frac{d}{dt} X_i = -i[X_i, V] = \epsilon_{ij} \ell_B^2 E_j, \quad (6.1a)
\]

\[
I_\alpha = \frac{d}{dt} \Theta_\alpha = -i[\Theta_\alpha, V] = i\ell_B^2 (\sigma_3)_{\alpha\beta} E_\beta, \quad (6.1b)
\]

where \( \{E_i\} \) and \( \{E_\alpha\} \) are bosonic and fermionic electric fields defined by \( E_i = -\partial_i V \) and \( E_\alpha = -C_{\alpha\beta} \partial_\beta V \). The Hall currents are orthogonal to the electric fields individually,

\[
E_i I_i = C_{\alpha\beta} E_\alpha I_\beta = 0. \quad (6.2)
\]

As suggested by the existence of the bosonic and fermionic Hall currents, there are two kinds of quasiholes, one of which is bosonic and the other is fermionic. They are superpartners, and are constructed by operating the creation operators

\[
A_B^i = \prod_p z_p, \quad A_F^i = \prod_p \theta_p, \quad (6.3)
\]

on the Laughlin wavefunction \( \Psi_{\text{Lin}} \). They satisfy the commutation relations with the radius operator as

\[
[R^2, A_B^i] = [R^2, A_F^i] = 2N\ell_B^2. \quad (6.4)
\]

These relations imply that both \( A_B^i \) and \( A_F^i \) push each of the particles on the Laughlin state outwards by \( \delta R = \sqrt{2} \ell_B \), to generate a quasi-hole (or a new magnetic cell of the area \( 2\pi \ell_B^2 \)) at the origin. Hence, the bosonic and the fermionic quasi-holes carry the identical fractional charge \( 1/m \). This may be regarded as a consequence of supersymmetry, because superpartners should have same quantum numbers, such as mass, charge, except for spin. The commutation relations with the angular momentum are different

\[
[L_\perp, A_B^i] = N, \quad [L_\perp, A_F^i] = \frac{N}{2}, \quad (6.5)
\]

which implies that, \( A_B^i \) does not change the spin of each particle, while \( A_F^i \) changes the spin from 0 to \(-1/2 \) [Fig.3].

Similarly, bosonic and fermionic quasi-particle wavefunctions would be constructed by operating the annihilation operators

\[
A_B = \prod_p \frac{\partial}{\partial z_p}, \quad A_F = \prod_p \frac{\partial}{\partial \theta_p}, \quad (6.6)
\]
on the Vandermonde determinant of \( \Psi_{L_{\text{lin}}} \). \( A_B \) and \( A_F \) satisfy the commutation relations with the radius operator

\[
[R^2, A_B] = [R^2, A_F] = -2N l_B^2, \tag{6.7}
\]

and with the angular momentum

\[
[L_\perp, A_B] = -N, \quad [L_\perp, A_F] = -\frac{N}{2}. \tag{6.8}
\]

Thus, \( A_B \) attracts each of the particles on the Laughlin state by \( \delta R = \sqrt{2} l_B \) inwards without changing its spin, and a bosonic quasi-particle with charge \(-1/m\) is generated at the origin. However, the operation of \( A_F \) on the Vandermonde determinant of \( \Psi_{L_{\text{lin}}} \) yields zero, and fermionic quasi-particle excitations do not appear in the LLL. It is because, while \( A_F \) changes the spin of each particle from 0 to +1/2, such spin-1/2 up particles are excluded due to the super-chiral property in the LLL.

\( A_B \) and \( A_B^\dagger \) satisfy the bosonic commutation relations,

\[
[A_B, A_B^\dagger] = 1, \tag{6.9a}
\]

\[
[A_B, A_B] = [A_B^\dagger, A_B^\dagger] = 0, \tag{6.9b}
\]

while \( A_F \) and \( A_F^\dagger \) satisfy “fermionic” commutation relations

\[
A_F A_F^\dagger + (-1)^N A_F^\dagger A_F = 1, \tag{6.10a}
\]

\[
\{A_F, A_F\} = \{A_F^\dagger, A_F^\dagger\} = 0. \tag{6.10b}
\]

### VII. RELATIONS TO BILAYER QH SYSTEMS

It is well known, fermionic harmonic oscillators can be regarded as the spin-1/2 ladder operators in supersymmetric quantum mechanics. In fact, the ladder operators made by Pauli matrices, \( \{\sigma_+, \sigma_-\} = \frac{1}{2}(\sigma_1 + i\sigma_2, \sigma_1 - i\sigma_2) \), satisfy the equations

\[
\{\sigma_+, \sigma_-\} = 1, \quad \sigma_+^2 = \sigma_-^2 = 1, \tag{7.1}
\]

which are equivalent to the properties of the fermionic harmonic oscillators. Due to this identification, it is possible to map a supersymmetric harmonic oscillator system to a spin system. In the supersymmetric QH system, there exist two kinds of fermionic harmonic oscillators, \( (\alpha, \alpha^\dagger) \) and \( (\beta, \beta^\dagger) \). Therefore, in its corresponding spin system, two kinds of “spins” are needed. One possible candidate to meet this requirement is a bilayer QH system, where electrons carry not only their intrinsic spins but also pseudospins which specify double layers. By regarding \( \alpha \)-“spin” as pseudospin and \( \beta \)-“spin” as intrinsic spin, there exists a mapping to bilayer QH systems [Table I]. However, unfortunately, the real boson-fermion symmetry in the supersymmetric QH system act as inter-change of the spin-1/2 up and down fermions.

When, we assign \( \alpha \)-“spin” as

\[
(\alpha, \alpha^\dagger) \leftrightarrow (\tau_+, \tau_-) \equiv \frac{1}{2}(\tau_2 + i\tau_3, \tau_2 - i\tau_3), \tag{7.2}
\]

where \( \{\tau_a\}(a = 1, 2, 3) \) represent Pauli matrices for the pseudospin, the Hamiltonian (3.13) is rewritten as

\[
H = \omega (a^\dagger a + \frac{1}{2}) - \omega \frac{1}{2} \tau_1, \tag{7.3}
\]

which is the non-Coulomb part of the Hamiltonian for bilayer QH systems, with tunneling interaction \( \Delta_{\text{SAS}} = \omega \) and without Zeeman interaction \( \Delta_z = 0 \). The LLL in supersymmetric QH systems can be regarded as the LLL of symmetric layer state in bilayer QH systems. The Hamiltonian (7.3) appears in many different context of supersymmetric quantum mechanical systems, such as Pauli Hamiltonian with gyromagnetic factor 2 [42] and the Jaynes-Cummings model without interaction terms used in quantum optics [43]. However, it must be noted that each of such systems possesses \( \mathcal{N} = 1 \) supersymmetry, while the present QH system has larger \( \mathcal{N} = 2 \).
supersymmetry due to the existence of extra $\mathcal{N}=1$ non-dynamical supersymmetry.

VIII. SUMMARY AND DISCUSSION

Based on the supersymmetric NC algebra, we constructed QH liquids on a NC superplane. The supersymmetric Landau model enjoys (complex) $\mathcal{N}=2$ supersymmetry, one of which is dynamical and the other is non-dynamical. In the LLL, only the $\mathcal{N}=1$ non-dynamical supersymmetry remains valid. Unlike ordinary supersymmetric quantum mechanics, the present supersymmetry represents a real boson-fermion symmetry. The NAC fermionic coordinates are related to spin degrees of freedom, and bring the super-chiral property to the LLL. Since, on the NC superplane, the bosonic and the fermionic center-of-mass coordinates are decoupled, the Laughlin wavefunction and topological excitations have their superpartners unlike the QH liquid on the fuzzy superplane. With use of the identification between the fermionic harmonic operators and the “spin”-1/2 ladder operators, supersymmetric QH systems are mapped to bilayer QH systems. In this mapping, the LLL in supersymmetric QH systems is regarded as the LLL in the symmetric layer state of bilayer QH systems.

While we have clarified bulk properties in the supersymmetric QH liquid, it is also important to study its edge excitations and effective field theory for further understanding of physics of the NAC geometry. We would like to pursue them in a future publication.

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APPENDIX A: MAGNETIC TRANSLATIONS ON THE SUPERPLANE

In this section, we summarize Aharonov-Bohm phase accompanied by magnetic translation on the NC superplane. With use of the center-of-mass coordinates $(X_i, \Theta_a)$, the supersymmetric magnetic translation operator is constructed as

$$\mathcal{T}_K = e^{i(k_1 X_i + \kappa_a \Theta_a)}, \quad (A1)$$

which satisfies

$$\mathcal{T}_K \cdot \mathcal{T}_T = \mathcal{T}_{K + T} e^{-\frac{i}{\hbar^2} \Sigma_{I,J} K_i T_j}, \quad (A2)$$

where $K \equiv (k_1, \kappa_a)$, $T \equiv (t_i, \tau_a)$ and $\Sigma \equiv \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_1 \end{pmatrix}$.

The algebra for the super-magnetic translation is given by

$$[\mathcal{T}_K, \mathcal{T}_T] = -2\mathcal{T}_{K+T} \cdot \sinh\left(\frac{1}{2} \Sigma_{I,J} K_i T_j \right)$$

$$= 2\mathcal{T}_{K+T} \cdot [e^{-\frac{i}{\hbar^2} (\sigma_2)_{I,J} K_i T_j} \sinh\left(\frac{1}{2} \epsilon_B^2 (\sigma_1)_{\alpha\beta} \kappa_\alpha \tau_\beta \right)$$

$$+ i e^{-\frac{i}{\hbar^2} (\sigma_1)_{\alpha\beta} \kappa_\alpha \tau_\beta} \sin\left(\frac{1}{2} \epsilon_B^2 (\sigma_1)_{\alpha\beta} \kappa_\alpha \tau_\beta \right)]. \quad (A3)$$

The round-trip acquires a supersymmetric Aharonov-Bohm phase as

$$\mathcal{T}_- K \mathcal{T}_- \mathcal{T}_K \mathcal{T}_T = e^{-BS}, \quad (A4)$$

where

$$S \equiv \ell_B^2 \Sigma_{I,J} K_i T_j = \ell_B^2 (\sigma_2)_{ij} k_i p_j - (\sigma_1)_{\alpha\beta} \kappa_\alpha \tau_\beta, \quad (A5)$$

which represents the “area” on the superplane.

APPENDIX B: INFINITE SYMMETRIES IN THE LLL

It is well known, in the LLL, infinite conserved charges appear and form the $W_\infty$ algebra [4, 44]. Similarly, a supersymmetric extension of the $W_\infty$ algebra appears in the LLL of the supersymmetric QH systems. It is obvious the following quantities commute with the Hamiltonian (3.13),

$$L_{m,n}^B = (b^\dagger)^{m+1} b^{n+1}, \quad (B1a)$$

$$L_{m,n}^F = (b^\dagger)^{m+1} b^{n+1} \beta, \quad (B1b)$$

$$L_{m,n}^{F\dagger} = (b^\dagger)^{n+1} b^{m+1} \beta^\dagger, \quad (B1c)$$

and $L_{m,n}^{\dagger} = L_{m,n}^B$, where $m, n \geq -1$. In particular, non-dynamical supercharges are identified as $(\bar{Q}, \bar{Q}^\dagger) = (L_{0,-1}^F, L_{0,-1}^{F\dagger})$. $L_{m,n}^B$ and $L_{m,n}^F$ satisfy a supersymmetric extension of the $W_\infty$ algebra as

$$[L_{m,n}^B, L_{k,l}^B] = \sum_{s=0}^{\text{Min}(n,k)} \frac{(n+1)! (k+l)!}{(n-s)! (k-s)! (s+1)!} L_{m+k-s,n+l+s}^B$$

$$- ((m,n) \leftrightarrow (k,l)), \quad (B2a)$$

$$[L_{m,n}^B, L_{k,l}^F] = \sum_{s=0}^{\text{Min}(n,k)} \frac{(n+1)! (k+l)!}{(n-s)! (k-s)! (s+1)!} L_{m+k-s,n+l+s}^F$$

$$- ((m,n) \leftrightarrow (k,l)), \quad (B2b)$$

$$\{L_{m,n}^B, L_{k,l}^F\} = 0. \quad (B2c)$$

$L_{m,n}^B$ and $L_{k,l}^F$ satisfy another supersymmetric $W_\infty$ algebra similar to Eq. (B2). The commutation relations
with the angular momentum and the radius operator are given by

\[ [L_\perp, L^B_{m,n}] = (m-n)L^B_{m,n}, \]  
\[ [L_\perp, L^F_{m,n}] = (m-n-\frac{1}{2})L^F_{m,n}, \]  
\[ [L_\perp, L^{F\dagger}_{m,n}] = (-m+n+\frac{1}{2})L^{F\dagger}_{m,n}, \]

and

\[ [R^2, L^B_{m,n}] = (m-n)L^B_{m,n}, \]  
\[ [R^2, L^F_{m,n}] = (m-n-1)L^F_{m,n}, \]  
\[ [R^2, L^{F\dagger}_{m,n}] = (-m+n+1)L^{F\dagger}_{m,n}. \]

These relations imply that radially symmetric orbits (4.4) are related by \( L^B_{m,n}, L^F_{m,n} \) and \( L^{F\dagger}_{m,n} \) as

\[ |m+1/2> = \frac{1}{\sqrt{m!n!}}L^B_{m-1,n-1}|n+1/2>, \]  
\[ |m+1> = \frac{1}{\sqrt{(m+1)!(n+1)!}}L^B_{m,n}|n+1>, \]

and

\[ |m+1/2> = \frac{1}{\sqrt{m!(n+1)!}}L^{F\dagger}_{n,m-1}|n+1>, \]  
\[ |m+1> = \frac{1}{\sqrt{(m+1)!(n+1)!}}L^F_{m,n-1}|n+1/2>. \]

### APPENDIX C: RADIIALLY SYMMETRIC ORBITS

Since the Hamiltonian for the supersymmetric Landau problem (3.13) is given by a sum of the bosonic oscillators and the fermionic oscillators, the whole supersymmetric Hilbert space is simply constructed by a direct product of bosonic and fermionic Hilbert spaces. In this section, with use of the symmetric gauge, we present explicit forms of the basis in bosonic and fermionic Landau problems.

First, we concisely review the bosonic Landau problem. The Hamiltonian and the angular momentum are given by \( H_B = \omega(a^\dagger a + 1/2) \) and \( L_B = b^\dagger b - a^\dagger a \), respectively. The state in the bosonic Hilbert space with energy \( E_n = \omega(n+1/2) \) and angular momentum \( l = m-n \) is

\[ |n,l> = \sqrt{\frac{1}{n!m!}}(a^\dagger)^m(b^\dagger)^n|0>. \]  

In particular, the Hilbert space in LLL (\( n = 0 \)) is spanned by the basis

\[ |m> = \frac{1}{\sqrt{m!}}(b^\dagger)^m|0>. \]

When we adopt the symmetric gauge, the LLL condition, \( a|\text{LLL}> = 0 \), is denoted as

\[ (z^0 + \partial^0)|\text{LLL}>= 0. \]  

Hence, the wavefunction in LLL is generally expressed as

\[ \phi_{\text{LLL}} = f(z)e^{-|z|^2}, \]  

where \( f(z) \) is an arbitrary holomorphic function and any wavefunction in LLL can be expanded by the radially symmetric orbits,

\[ \phi_m = \sqrt{\frac{2m+1}{\pi m!}}z^m e^{-|z|^2}. \]

They are the position representation of Eq.(C2) and satisfy the orthonormal condition

\[ \int dzdz^* \phi^*_m(z,z^*)\phi_m'(z,z^*) = \delta_{mm'}. \]

The “complete relation” in LLL is calculated as

\[ \sum_{m=0}^\infty \phi_m(z^*, z^0)\phi_m^*(z^0, z^*) = \frac{2}{\pi} e^{-|z|^2-|z^0|^2-2z^0z^*}. \]

The fermionic Landau problem is similarly analyzed. The Hamiltonian and the angular momentum are given by \( H_F = \omega(a^\dagger a + 1/2) \) and \( L_F = 1/2(\beta^\dagger \beta - \alpha^\dagger \alpha) \), respectively. Due to the Pauli exclusion principle, the Hilbert space for fermionic oscillators consists of only four states. There are only two energy levels, LLL and 1-st LL, with energy \( -\omega/2, \omega/2 \), both of which are doubly degenerate. Two states in the LLL with angular momentum 0, 1/2 are given by

\[ |0,0> = |0>, \]  
\[ |0,1/2> = \beta|0>. \]  

where \( |0> \) is defined as \( \alpha|0> = \beta|0> = 0 \). Two states in the 1-st LL with angular momentum \(-1/2, 0 \) are given by

\[ |1,-1/2> = \alpha^\dagger|0>, \]  
\[ |1,0> = \alpha \beta|0> . \]

In the symmetric gauge, the LLL condition, \( \alpha|\text{LLL}> = 0 \), is rewritten as

\[ (\theta - \partial_\theta^*)\varphi_{\text{LLL}} = 0. \]

Hence, the wavefunction in fermionic LLL is generally given by

\[ \varphi_{\text{LLL}} = g(\theta)e^{-\theta \theta^*}, \]  

where \( g(\theta) = g_0 + g_1 \theta \) is an arbitrary holomorphic function. Therefore, any wavefunction in the LLL of the fermionic oscillators can be expanded by the following states

\[ \varphi_{0,0} = \frac{1}{\sqrt{2}} e^{-\theta \theta^*} = \frac{1}{\sqrt{2}}(1 - \theta \theta^*), \]  
\[ \varphi_{0,1/2} = \theta e^{-\theta \theta^*} = \theta. \]
In fact, these are the position representation of $|0,0>$ and $|0,1/2>$. Similarly the position representation of the 1-st LL states, $|1,-1/2>$ and $|1,0>$, are

$$\varphi_{1,-1/2} = \theta^* e^{\theta^*} = \theta^*,$$

$$\varphi_{1,0} = \frac{1}{\sqrt{2}} e^{\theta^*} = \frac{1}{\sqrt{2}}(1 + \theta^*).$$

They satisfy the orthonormal condition

$$\int d\theta d\theta^* (-1)^{(n+1)} \varphi_{n,l}^*(\theta, \theta^*) \varphi_{n',l'}(\theta, \theta^*) = \delta_{n,n'} \delta_{l,l'},$$

where we have included a weight factor and have defined the Grassmann integral as $\int d\theta d\theta^* \equiv \partial_\theta \partial_{\theta^*}$. The complete relation for these states is obtained as

$$\sum_{n,l} (-1)^{n+1} \varphi_{n,l}(\theta, \theta^*) \varphi_{n,l}'(\theta, \theta^*) = \delta(\theta - \theta') \delta(\theta^* - \theta'^*),$$

where we have taken into account the weight factor as in Eq.(C14). The “complete relation” in the fermionic LLL is calculated as

$$\sum_{l=0,1/2} \varphi_{0,l}(\theta, \theta^*) \varphi_{0,l}'(\theta, \theta^*) = \frac{1}{2} e^{-\theta^* - \theta^* - 2\theta^* \delta}.$$

APPENDIX D: VON NEUMANN BASIS ON THE SUPERPLANE

In this section, we briefly discuss von Neumann basis formalism on the superplane. The von Neumann basis is a convenient basis to investigate QH systems, because it is a quantum mechanical analogue of the classical cyclotron orbit and, in a continuum limit, the translational symmetries are expected to be recovered [45].

First, we introduce the supercoherent state as a simultaneous eigenstate of two annihilation operators $\hat{b}$ and $\hat{\beta}$,

$$\hat{b} |b, \beta> = (b |b, \beta>,$$

where $|b, \beta>$ and $|\beta>$ are bosonic and fermionic coherent states given by $|b> = e^{-\frac{1}{2}|b|^2} e^{\hat{b} |0>}$ and $|\beta> = e^{-\frac{1}{2}\beta^* \beta} |0>$. In the symmetric gauge, the supercoherent state is written as

$$\psi_{b,\beta} = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}|b|^2 + \beta^* \beta} e^{\frac{1}{2}|b|^2 + \beta^* \beta} |0> e^{-|\theta|^2 - \theta^*}.$$

We define a super von Neumann basis as a subset of the supercoherent states, whose index takes discrete values

$$b_{mn} = \sqrt{\pi} (m + in),$$

where $m$ and $n$ take integers. It is easily checked that the complete relation for the super von Neumann basis exactly coincides with the “complete relation” in the LLL (4.6). Thus, the super von Neumann basis spans the Hilbert space in the LLL.

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