NON-DELAY LIMIT IN THE ENERGY SPACE FROM THE NONLINEAR DAMPED WAVE EQUATION TO THE NONLINEAR HEAT EQUATION

TAKAHISA INUI AND SHUJI MACHIHARA

Abstract. We consider a singular limit problem from the damped wave equation with a power type nonlinearity to the corresponding heat equation. We call our singular limit problem non-delay limit. Our proofs are based on the argument for non-relativistic limit from the nonlinear Klein-Gordon equation to the nonlinear Schrödinger equation by the second author, Nakanishi, and Ozawa [2]. Nakanishi [14], and Masmoudi and Nakanishi [9]. We can obtain better results for the non-delay limit problem than that for the non-relativistic limit problem due to the dissipation property. More precisely, we get the better convergence rate of the $L^2$-norm and we also obtain the global-in-time uniform convergence of the non-delay limit in the $L^2$-supercritical case.

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1. Introduction

1.1. Background. We consider the following nonlinear damped wave equation.

\[
\begin{aligned}
\tau \partial_t^2 u - \Delta u + \partial_t u + \mu |u|^{p-1}u = 0, \\
(t, x) \in (0, T) \times \mathbb{R}^d, \\
\end{aligned}
\]

where \(\tau > 0, T > 0, d \in \mathbb{N}, 1 < p < 1 + 4/(d-2)\) if \(d \geq 3\) and \(1 < p < \infty\) if \(d = 1, 2\), \(\mu = \pm 1\), and \((f_\tau, g_\tau) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\) are given initial data depending on \(\tau\). The nonlinearity is called focusing when \(\mu = 1\) and defocusing when \(\mu = -1\).

The power of the nonlinearity is called energy subcritical. It is well known that the energy \(E_\tau\) decays, where it is defined by

\[
E_\tau(u_\tau(t)) := \frac{1}{2} \|\nabla u_\tau(t)\|_{L^2} + \frac{\tau}{2} \|\partial_t u_\tau(t)\|_{L^2} - \frac{\mu}{p+1} \|u_\tau(t)\|_{L^{p+1}}.
\]

Indeed, we have

\[
\frac{d}{dt} E_\tau(u_\tau(t)) = -\|u_\tau(t)\|_{L^2} \leq 0,
\]

where \(u_\tau\) is the solution of (NLDW\(_\tau\)). Thus, the global solution of (NLDW\(_\tau\)) decays.

The parameter \(\tau\) denotes time delay. To explain this, we recall the derivation in the current literature of the linear damped wave equation

\[
\tau \partial_t^2 \phi - \Delta \phi + \partial_t \phi = 0.
\]

by the Cattaneo law (see e.g. [15, (1.9)–(1.11)]). It is well known that the heat equation \(\partial_t \phi - \Delta \phi = 0\) is derived from the Fourier law. More precisely, letting \(\phi\) denote temperature and \(q\) denote heat flux, then we have

\[
\partial_t \phi = -\text{div} q.
\]

(1.1)

The Fourier law implies that the flux depends linearly on the derivative of temperature \(\phi\), i.e.

\[
q = -\nabla \phi,
\]

where we take thermal conductivity as 1. Combining (1.1) and (1.2), we obtain the heat equation. On the other hand, the Cattaneo law implies the damped wave equation. The Cattaneo law states that the flux \(q\) does not depend linearly on \(\nabla \phi(t)\) at the same time, but it depends linearly on \(\nabla \phi(t-\tau)\) with a slight time lag \(\tau\). We replace (1.2) by

\[
q(t, x) = -\nabla \phi(t-\tau, x),
\]

where \(\tau > 0\). From this, we have

\[
q(t+\tau, x) = -\nabla \phi(t, x).
\]

By the Taylor expansion of the left hand side at \(\tau = 0\), we obtain

\[
q(t+\tau, x) = \sum_{n=0}^{\infty} \frac{\partial_t^n q(t, x)}{n!} \tau^n.
\]

Since \(\tau\) is small, we ignore the higher terms \((n \geq 2)\) and thus we obtain

\[
q(t, x) + \tau \partial_t q(t, x) = -\nabla \phi(t, x).
\]
Combining this with (1.1), we obtain the damped wave equation
\[ \tau \partial_t^2 \phi + \partial_t \phi = -\tau \text{div} \partial_t q - \text{div} q = \text{div}(\nabla \phi) = \Delta \phi \]
and \( \tau \) denotes the time delay effect, which is called a relaxation time and given by the inverse square of speed of the second sound. If \( \tau \) goes to 0, then the damped wave equation formally goes to the heat equation. In the present paper, the singular limit \( \tau \rightarrow 0 \) is called non-delay limit. The non-delay limit problem for the linear equations is mathematically investigated well (see e.g. [2, 3]). In the present paper, we consider the non-delay limit problem for the nonlinear equation (NLDW). As \( \tau \rightarrow 0 \), we formally obtain the nonlinear heat equation
\[ \partial_t v - \Delta v = \mu |v|^{p-1} v. \]
We will show mathematically that the solution of (NLDW) converges to the solution of the nonlinear heat equation. This problem was partially studied in [12, 13]. On the other hand, singular limit problems for nonlinear relativistic equation to nonlinear dispersive equation are investigated. The second author, Nakanishi, and Ozawa [7] shows that the solution of the energy subcritical nonlinear Klein–Gordon equation goes to that of the energy subcritical nonlinear Schrödinger equation if the speed of the light goes to infinity. More precisely, they consider the nonlinear Klein–Gordon equation
\[ \tau \partial_t^2 w_{\tau} - \Delta w_{\tau} + \tau^{-1} w_{\tau} = \mu |w_{\tau}|^{p-1} w_{\tau} \]
where \( \tau^{-1/2} = c \) denotes the speed of the light. By modulated function \( \psi_{\tau} = e^{-i \tau^{-1} t} w_{\tau} \), they transform the nonlinear Klein–Gordon equation into
\[ \tau \partial_t^2 \psi_{\tau} - \Delta \psi_{\tau} + i \partial_t \psi_{\tau} = \mu |\psi_{\tau}|^{p-1} \psi_{\tau}. \]
They showed that the solution \( \psi_{\tau} \) goes to the solution \( \psi_0 \) of the following nonlinear Schrödinger equation
\[ (\text{NLS}) \quad i \partial_t \psi_0 - \Delta \psi_0 = \lambda |\psi_0|^{p-1} \psi_0 \]
as \( \tau \) goes to 0, where this means the speed of the light \( c \) goes to infinity and it is called non-relativistic limit. More concretely, they showed local-in-time uniform convergence, i.e.
\[ \| \psi_{\tau} - \psi_0 \|_{L^\infty(0,T;H^1)} \rightarrow 0 \quad \text{as} \quad \tau \rightarrow 0 \]
for finite fixed \( T \) less than the maximal existence time of the solution to (NLS). See [8, 10, 1] for other equations.
We will give such a statement for the non-delay limit from (NLDW) to the nonlinear heat equation. We apply the arguments in [7] and the sequel works by Nakanishi [14] and Masmoudi and Nakanishi [9] to our non-delay limit problem.
However, it is not mere application and we obtain better results than them due to the dissipation. More precisely, we obtain the following difference between our equations and their equations. The almost optimal convergence rate of the \( L^2 \)-norm is \( \tau^{1/4} \) in the non-relativistic limit problem. However, we will find that the rate is \( \tau^{1/2} \) for non-delay limit. Moreover, we can obtain global-in-time uniform convergence for the non-delay limit in the \( L^2 \)-supercritical case, i.e. \( p > 1 + 4/d \), though this does not hold for the non-relativistic limit as pointed out in [14].
1.2. Main result. It is well known that there exist $T_\tau = T(\tau, \|f_\tau\|_{H^1}, \tau^{1/2}\|g_\tau\|_{L^2}) > 0$ and the solution $u_\tau \in C([0, T_\tau) : H^1(\mathbb{R}^d))$ to (NLD) for fixed $\tau$.

First, we have uniform boundedness of the solution $u_\tau$.

**Theorem 1.1 (Uniform bound).** Let $u_\tau \in C([0, T_\tau) : H^1(\mathbb{R}^d))$ be a solution to (NLD) and $T^*_\tau$ be the maximal existence time. If the initial data $(f_\tau, g_\tau) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ satisfies

$$\limsup_{\tau \to 0} (\|f_\tau\|_{H^1} + \tau^{1/2}\|g_\tau\|_{L^2}) < \infty,$$

then we have

$$T^* := \liminf_{\tau \to 0} T^*_\tau > 0$$

and $u_\tau$ satisfies that for any $T < T^*$ there exist a constant $C_T > 0$ and $\tau_T > 0$ such that

$$\|u_\tau\|_{L^\infty(0, T; H^1)} + \tau^{1/2}\|\partial_t u_\tau\|_{L^\infty(0, T; L^2)} \leq C_T$$

for any $\tau \in (0, \tau_T)$.

Let $v$ be a solution of

$$\begin{cases}
\partial_t v - \Delta v = \mu|v|^{p-1}v, & (t, x) \in (0, T) \times \mathbb{R}^d, \\
v(0) = f, & x \in \mathbb{R}^d
\end{cases}$$

(NLH)

$(\text{NLH})$ is the maximal existence time. If the initial data $f \in H^1(\mathbb{R}^d)$, $v$ be a solution to (NLH), and $T_{\text{max}}(v)$ is the maximal existence time of the solution $v$ to (NLH). Then, for any $T < \min\{ T^*, T_{\text{max}}(v) \}$, we have

$$\|u_\tau - v\|_{L^\infty(0, T; L^2(\mathbb{R}^d))} \lesssim \|f_\tau - f\|_{L^2} + \tau\|g_\tau\|_{L^2} + \tau^{1/2}.$$ 

for any $\tau \in (0, \tau_T)$, where the implicit constant is independent of $\tau$.

**Remark 1.1.** For the non-relativistic problem, the optimal rate of $L^2$-convergence was reported in \cite{7}. That is $\tau^{1/4}$. On the other hand, we obtain $\tau^{1/2}$ in the non-delay limit problem. This is a main difference between dispersive equations and dissipative equation. To explain roughly this difference, we consider the main parts of the convergence. For the non-relativistic problem, we have

$$\|e^{it\frac{1-\sqrt{1+4\mu\xi^2}}{2\tau}} - e^{it|\xi|^2}f\|_{L^2(|\xi| \leq \tau^{-1/4})} \lesssim \int_0^\tau \|t|\xi|^4 e^{it\frac{1-\sqrt{1+4\mu\xi^2}}{2\tau}} f\|_{L^2(|\xi| \leq \tau^{-1/4})} d\tilde{\tau} \lesssim \tau^{1/2} \|f\|_{H^1}$$

and we have

$$\|e^{it\frac{1-\sqrt{1+4\mu\xi^2}}{2\tau}} - e^{it|\xi|^2}f\|_{L^2(|\xi| > \tau^{-1/4})} \lesssim \tau^{1/2} \|f\|_{H^1},$$

since $|\xi|^{-1} \leq \tau^{1/4}$ in the high frequency region. This estimate implies that the optimal convergence rate is $\tau^{1/4}$. On the other hand, for our problem, we have

$$\|e^{it\frac{1-\sqrt{1+4\mu\xi^2}}{2\tau}} - e^{-it|\xi|^2}f\|_{L^2(|\xi| \leq \tau^{-1/2})} \lesssim \int_0^\tau \|t|\xi|^4 e^{it\frac{1-\sqrt{1+4\mu\xi^2}}{2\tau}} f\|_{L^2(|\xi| \leq \tau^{-1/2})} d\tilde{\tau} \lesssim \tau^{1/2} \|f\|_{H^1}.$$
since \( t|\xi|^2e^{t\sqrt{1+4\xi^2/\tau^2}} \lesssim t|\xi|^2e^{-t|\xi|^2} \lesssim 1 \) and we also have
\[
\|e^{t\sqrt{1+4\xi^2/\tau^2}} - e^{-t|\xi|^2}\|_{L^2(\{|\xi|>|\tau|^{-1/2}\})} \lesssim \tau^{1/2}\|f\|_{H^1}.
\]
This shows that our rate is \( \tau^{1/2} \).

We have the following locally uniform \( H^1 \)-convergence of the non-delay limit.

**Theorem 1.3** (\( H^1 \)-convergence). Let \( f \in H^1(\mathbb{R}^d) \), \( v \) be a solution to \( \text{(NLH)} \), and \( T_{\text{max}}(v) \) is the maximal existence time of the solution \( v \) to \( \text{(NLH)} \). If the initial data \( (f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) satisfies
\[
(f, \tau^{-2}g) \to (f, 0) \text{ in } H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \text{ as } \tau \to 0,
\]
then we have \( T^* \geq T_{\text{max}}(v) \) and we have
\[
\|u_\tau - v\|_{L^\infty(0,T;H^1(\mathbb{R}^d))} + \tau^2\|\partial_t u_\tau\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \to 0
\]
as \( \tau \to 0 \) for any \( T < T_{\text{max}}(v) \).

We remark that the assumptions in Theorem 1.3 are satisfied under the assumptions of Theorem 1.4.

Due to the dissipation, we obtain the following global convergence result in the \( L^2 \)-supercritical case.

**Theorem 1.4** (Global \( H^1 \)-convergence). Let \( 1 + 4/d < p < 1 + 4/(d - 2) \). Assume that the solution \( v \) to \( \text{(NLH)} \) is global and \( \|v(t)\|_{H^1} \) decays to 0 as \( t \to \infty \). If the initial data \( (f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) satisfies
\[
(f, \tau^{-2}g) \to (f, 0) \text{ in } H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \text{ as } \tau \to 0.
\]
Then we have
\[
\lim_{\tau \to 0} \left( \|u_\tau - v\|_{L^\infty(0,\infty;H^1(\mathbb{R}^d))} + \tau^2\|\partial_t u_\tau\|_{L^\infty(0,\infty;L^2(\mathbb{R}^d))} \right) = 0.
\]

**Remark 1.2.** The non-relativistic problem from NLKG to NLS, the global convergence does not hold (see [14]). Theorem 1.4 depends essentially on the dissipation property.

The following theorem relies on the fact that the spatial derivative implies the additional time decay \( t^{-1/2} \) for the damped wave equation and the heat equation.

**Theorem 1.5** (\( \dot{H}^1 \)-decay order). If the assumption in Theorem 1.4 is satisfied, then we have
\[
t^{1/2}\|u_\tau(t)\|_{\dot{H}^1} \to 0 \text{ as } t \to \infty
\]
uniformly in \( \tau \). Especially, we obtain
\[
\lim_{\tau \to 0} \left( t^{1/2}(u_\tau - v) \right)_{L^\infty(0,\infty;\dot{H}^1)} = 0.
\]
1.3. Plan of proofs. We apply the argument in [7]. To show Theorem 1.3, we prepare the uniform boundedness in \( \tau \), Theorem 1.1. The uniform boundedness in the case of \( d = 1, 2 \) can be shown easily. Indeed, it is enough to use the space \( L^\infty_t H^1_x \) in order to obtain the closed estimate. For the high dimensional case \( d \geq 3 \), we need the Strichartz norms and to divide them into the low and high frequency parts. The low frequency of the solution of the damped wave equation behaves like that of the heat equation and the high frequency part of that behaves like that of the wave equation. However, it is not trivial that the low (resp. high) frequency part of the nonlinearity is the nonlinearity of the low (resp. high) frequency part. To overcome this difficulty, we use argument for the non-relativistic limit. The second author, Nakanishi, and Ozawa [7] shows the nonlinear estimate of the frequency decoupling (see [7, Lemma 3.4]). By applying this lemma, we obtain the closed estimate and thus the uniform boundedness. Next, we will show the decoupling (see [7, Lemma 3.4]). By applying this lemma, we obtain the closed estimate by a direct calculation. By using the dissipation, we obtain the better rate of the \( L^2 \)-convergence rate. At last, we will show the \( H^1 \)-convergence by combining \( L^2 \)-convergence with a compactness argument. In this argument, the energy and charge conservation laws were used in [7]. However, the energy decays for the damped wave equation. Thus, we could not apply their argument. We will apply the compactness argument by Masmoudi and Nakanishi [9]. They brushed Lemma 3.4 in [7] up and prove the \( H^1 \)-convergence by a compactness method without the energy conservation laws.

By using the argument of Nakanishi [14], we obtain Theorem 1.3. He showed the convergence of the wave, inverse wave, and scattering operators in the non-relativistic problem. By applying his argument and the dissipation, we obtain the global convergence unlike the non-relativistic problem. Theorem 1.3 relies on the heat-like property that the spatial first-order derivative implies the additional time decay \( t^{-1/2} \).

2. Preliminaries

2.1. Notation. For an exponent \( p \in [1, \infty] \), the Hölder conjugate of \( p \) is denoted by \( p' \).

Let \( \chi_{\leq 1} \in C_0^\infty(\mathbb{R}^d) \) be a radially symmetric cut-off function such that \( \chi_{\leq 1}(\xi) = 1 \) if \( |\xi| \leq 1 \) and \( \chi_{\leq 1}(\xi) = 0 \) if \( |\xi| \geq 2 \). We set \( \chi_{ \leq a}(\xi) := \chi_{\leq 1}(\xi/a) \) for a positive number \( a \). We set \( \chi_{ > a} := 1 - \chi_{ \leq a} \).

We denote the Fourier transform and its inverse by \( \mathcal{F} \) and \( \mathcal{F}^{-1} \), respectively. We also denote the Fourier transform of a function \( f \) by \( \hat{f} \). For a measurable function \( m \), we define the Fourier multiplier \( m(\nabla) \) by \( m(\nabla) := \mathcal{F}^{-1}m(\xi)\mathcal{F} \). Thus, \( \chi_{ \leq \tau^{-1/2}}(\nabla) = \mathcal{F}\chi_{ \leq \tau^{-1/2}}(\xi)\mathcal{F} \). We sometimes omit \( \nabla \), that is, we use \( \chi_{ \leq \tau^{-1/2}} \) instead of \( \chi_{ \leq \tau^{-1/2}}(\nabla) \). We denote \( P_j := \mathcal{F}(\chi_{ \leq 2^j}(\xi) - \chi_{ \leq 2^{j-1}}(\xi))\mathcal{F} \) for \( j \in \mathbb{Z} \). We also write \( P_j \) to denote the symbol of \( P_j \). We set \( \langle a \rangle := (1 + a^2)^{1/2} \).

We use \( A \lesssim B \) to denote the estimate \( A \leq CB \) with some constant \( C > 0 \). The notation \( A \approx B \) stands for \( A \lesssim B \) and \( B \lesssim A \).

For a time interval \( I \), we set the space-time function space \( L_t^q L_x^r(I) \) by \( L_t^q(I; L_x^r) \) whose norm is

\[
\|f\|_{L_t^q L_x^r(I)} := \left\{ \int_I \left( \int_{\mathbb{R}^d} |f(t, x)|^r \, dx \right)^{\frac{q}{r}} \, dt \right\}^{\frac{1}{q}}.
\]
We also may omit the time interval $I$. We define the Sobolev space by
\[ W^{\sigma,p}(\mathbb{R}^d) := \{ f \in \mathcal{S}'(\mathbb{R}^d) : \| f \|_{W^{\sigma,p}} = \| (\nabla)\sigma f \|_{L^p} < \infty \} \]
for $\sigma \geq 0$ and $1 \leq p \leq \infty$. We set $H^\sigma := W^{\sigma,2}$. For $1 \leq p, q \leq \infty$ and $\sigma \in \mathbb{R}$, we define inhomogeneous Besov norm by
\[ \| f \|_{B^\sigma_{p,q}} := \| \chi_{\leq 1} f \|_{L^p} + \| \{ 2^j \sigma \| P_j f \|_{L^p} \}_{j=1}^\infty \|_{l^q} \]
and inhomogeneous Besov space by
\[ B^\sigma_{p,q}(\mathbb{R}^d) := \{ f \in \mathcal{S}' : \| f \|_{B^\sigma_{p,q}} < \infty \}. \]
For a function space $X$, we denote its homogeneous space by $\dot{X}$. For example, $X = L^q_B$ if $X = L^q_B$.

For Banach spaces $X, Y$, we set
\[ \| f \|_{\dot{X}} := \inf \{ \| f \|_X \mid \| f \|_X \leq \| f \|_Y \} \]
and $[X, Y]_\theta$ is an complex interpolation space between $X$ and $Y$ of order $\theta \in (0, 1)$.

We set $\| f \|_{\dot{a}X} := a^{-1} \| f \|_X$.

We define
\[ \mathcal{L} := \{ \alpha L^q(I; L^r(\mathbb{R}^d)) : \alpha > 0, 1 \leq q \leq \infty, 1 < r \leq \infty \} \]
and
\[ \mathcal{B} := \{ \alpha L^q(I; \dot{B}^s_{r,p}(\mathbb{R}^d)) : \alpha, s > 0, 1 \leq p, q \leq \infty, 1 < r \leq \infty \} \]
for an interval $I$. For $Z = \alpha L^q(I; \dot{B}^s_{r,p}(\mathbb{R}^d)) \in \mathcal{B}$, we define
\[ \sigma(Z) := s \]
For $X \in \mathcal{L}$ and $Z \in \mathcal{B}$, we set
\[ X^{p-1}Z := \alpha L^q(I; \dot{B}^s_{r,q}(\mathbb{R}^d)) \]
where $X = \alpha_0 L^q(I; L^r(\mathbb{R}^d))$, $Z = \alpha_1 L^q(I; \dot{B}^{s_1}_{r_1,p_1}(\mathbb{R}^d))$, and
\[ \frac{1}{q} = \frac{p-1}{q_0} + \frac{1}{q_1}, \quad \frac{1}{r} = \frac{p-1}{r_0} + \frac{1}{r_1}, \quad p = p_1, \quad s = s_1, \quad \text{and} \quad \alpha = \alpha_0^{-1} \alpha_1. \]
This roughly means that $\| u \|^{p-1}_X \leq \| u \|^{p-1}_Z$ holds by the fractional Leibniz rule.

For $d \geq 3$, we use $\theta \in (0, 1)$ which is the internal division ratio of $p$ between 1 and $p_1$, i.e.
\[ p = 1 - \theta + p_1 \theta, \]
where $p_1 := 1 + 4/(d - 2)$.

For exponents $(q, r)$ and $(\tilde{q}, \tilde{r})$ satisfying $q, r, \tilde{q}, \tilde{r} \in [2, \infty]$, we define exponents $\alpha$ and $\gamma$ by
\[ \alpha(q, r) := -\frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) + \frac{1}{q}, \]
\[ \gamma(q, r) := \max \left\{ d \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q} \left( \frac{1}{2} - \frac{1}{r} \right), \frac{d+1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \right\} \]
and $\delta((q, r), (\tilde{q}, \tilde{r}))$ in Table 1 below.
2.2. Symbols. The solution propagator of (DW) is given by
\[
\begin{pmatrix}
\phi_\tau(t) \\
\partial_t \phi_\tau(t)
\end{pmatrix}
= \mathcal{A}_\tau(t)
\begin{pmatrix}
f_\tau \\
g_\tau
\end{pmatrix}
\]
where
\[
\mathcal{A}_\tau(t) := \begin{pmatrix}
\frac{1}{\tau}D_\tau(t) + \partial_t D_\tau(t) \\
\frac{1}{\tau^2}\partial_\tau D_\tau(t) + \partial_\tau^2 D_\tau(t)
\end{pmatrix}
\frac{D_\tau(t)}{\partial_\tau D_\tau(t)},
\]
\[
\mathcal{D}_\tau(t) := \frac{1}{\lambda^- - \lambda^+}(-e^{t\lambda^+} + e^{t\lambda^-})F,
\]
and
\[
\lambda^\pm = \lambda^\pm(\xi) := -\frac{1 \pm \sqrt{1 - 4\tau|\xi|^2}}{2\tau}.
\]
See e.g. [11]. Therefore, by the Duhamel formula, the solution of the nonlinear equation (NLDW) is given by
\[
\begin{pmatrix}
u_\tau(t) \\
\partial_t \nu_\tau(t)
\end{pmatrix}
= \mathcal{A}_\tau(t)
\begin{pmatrix}
f_\tau \\
g_\tau
\end{pmatrix}
+ \int_0^t \frac{1}{\tau} \mathcal{A}_\tau(t-s)
\begin{pmatrix}
0 \\
\mathcal{N}(\nu_\tau(s))
\end{pmatrix}
ds,
\]
where we set \(\mathcal{N}(\nu_\tau(s)) = \mu[\nu_\tau(s)]^{p-1}\nu_\tau(s)\).

For fixed \(\xi \in \mathbb{R}^d\), we have, as \(\tau \to 0\),
\[
\lambda^+ = \frac{-1 + \sqrt{1 - 4\tau|\xi|^2}}{2\tau} \to \frac{2|\xi|^2}{-1 - \sqrt{1 - 4\tau|\xi|^2}} \to -|\xi|^2,
\]
\[
\lambda^- = \frac{-1 - \sqrt{1 - 4\tau|\xi|^2}}{2\tau} \to -\infty,
\]
and
\[
\frac{1}{\lambda^- - \lambda^+} = -\frac{\tau}{\sqrt{1 - 4\tau|\xi|^2}} \to 0
\]
\[
\frac{\lambda^+}{\lambda^- - \lambda^+} = -\frac{1 \pm \sqrt{1 - 4\tau|\xi|^2}}{2\sqrt{1 - 4\tau|\xi|^2}} \to \begin{cases} 0 & \text{if } \lambda^+ \\ 1 & \text{if } \lambda^-
\end{cases}
\]
\[
\frac{\lambda^- - \lambda^+}{\lambda^- - \lambda^+} = -\frac{|\xi|^2}{\sqrt{1 - 4\tau|\xi|^2}} \to -|\xi|^2.
\]
Thus, roughly, we have
\[
\frac{1}{\tau}D_\tau(t) + \partial_t D_\tau(t) = \frac{1}{\lambda^- - \lambda^+} \left(\lambda^+ e^{t\lambda^+} - \lambda^+ e^{t\lambda^-}\right) \to e^{t\Delta},
\]
\[
D_\tau(t) = \frac{1}{\lambda^- - \lambda^+}(-e^{t\lambda^+} + e^{t\lambda^-}) \to 0,
\]
and
\[
\frac{1}{\tau}D_\tau(t) = \frac{1}{\tau(\lambda^- - \lambda^+)} (-e^{t\lambda^+} + e^{t\lambda^-}) \to e^{t\Delta}.
\]

From the observation of the symbols, we find that the low frequency part of the propagator is like the heat propagator and the high part is like the wave propagator with exponential decay. Indeed, we have
\[
D_\tau(t) = \tau e^{-\frac{1}{\tau}F^{-1}L_\tau(t, \xi)F},
\]
and

\[ L_r(t, \xi) = \begin{cases} 
\frac{2}{\sqrt{1 - 4\tau^2|\xi|^2}} \sinh \left( t \frac{\sqrt{1 - 4\tau^2|\xi|^2}}{2\tau} \right) & \text{if } |\xi| \leq \frac{1}{2\sqrt{\tau}}, \\
\frac{2}{\sqrt{4\tau^2|\xi|^2 - 1}} \sin \left( t \frac{\sqrt{4\tau^2|\xi|^2 - 1}}{2\tau} \right) & \text{if } |\xi| > \frac{1}{2\sqrt{\tau}}.
\end{cases} \]

Therefore, we need to calculate low and high parts respectively.

2.3. Lemmas. It is easily seen that \( \phi_r(t, x) = \phi_1(\tau^{-1}t, \tau^{-1/2}x) \), where \( \phi_1 \) is a solution of \( \partial_t^2 \phi_1 + \partial_x \phi_1 - \Delta \phi_1 = 0 \). Therefore, by this scaling and the previous result [5 6], we have the following Strichartz estimates:

Lemma 2.1 (Homogeneous estimate for low frequency). Let \( s \in \mathbb{R}, q \in [2, \infty], r \in [2, \infty] \). Assume that \((q, r)\) satisfies

\[ \frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \geq \frac{1}{q}. \]

Then, we have the following.

\[ \left\| \frac{1}{\tau} \partial_t \mathcal{D}_r(t) \chi_{\tau^{-1/2}f} \right\|_{L^q_t \dot{B}^r_{2, 2}(t)} \lesssim \tau^{\alpha(q, r)} \left\| \chi_{\tau^{-1/2}f} \right\|_{\dot{B}^r_{2, 2}}, \]

and

\[ \left\| \partial_t \mathcal{D}_r(t) \chi_{\tau^{-1/2}f} \right\|_{L^q_t \dot{B}^r_{2, 2}(t)} \lesssim \tau^{\alpha(q, r)} \left\| \chi_{\tau^{-1/2}f} \right\|_{\dot{B}^r_{2, 2}}, \]

where the implicit constants are independent of \( \tau \). Moreover, we also have

\[ \left\| \left( \frac{1}{\tau} \partial_t \mathcal{D}_r(t) + \partial_t^2 \mathcal{D}_r(t) \right) \chi_{\tau^{-1/2}f} \right\|_{L^q_t \dot{B}^r_{2, 2}(t)} \lesssim \left\| \nabla^2 \chi_{\tau^{-1/2}f} \right\|_{L^2} \lesssim \tau^{-\frac{1}{2}} \| f \|_{H^1}. \]

Lemma 2.2 (Homogeneous estimate for high frequency). Let \( s \in \mathbb{R}, q \in [2, \infty], r \in [2, \infty] \). Then, we have the following.

\[ \left\| \frac{1}{\tau} \partial_t \mathcal{D}_r(t) \chi_{\tau^{1/2}f} \right\|_{L^q_t \dot{B}^r_{2, 2}(t)} \lesssim \tau^{\alpha(q, r) + \frac{2(q - 1)}{q}} \left\| \nabla \chi_{\tau^{1/2}f} \right\|_{\dot{B}^r_{2, 2}}, \]

and

\[ \left\| \partial_t \mathcal{D}_r(t) \chi_{\tau^{1/2}f} \right\|_{L^q_t \dot{B}^r_{2, 2}(t)} \lesssim \tau^{\alpha(q, r) + \frac{2(q - 1)}{q}} \left\| \nabla \chi_{\tau^{1/2}f} \right\|_{\dot{B}^r_{2, 2}}, \]

where the implicit constants are independent of \( \tau \). Moreover, we also have

\[ \left\| \left( \frac{1}{\tau} \partial_t \mathcal{D}_r(t) + \partial_t^2 \mathcal{D}_r(t) \right) \chi_{\tau^{1/2}f} \right\|_{L^q_t \dot{B}^r_{2, 2}(t)} \lesssim \tau^{-\frac{1}{2}} \| f \|_{H^1}. \]

Moreover, we also have the inhomogeneous Strichartz estimate.

Lemma 2.3 (Inhomogeneous estimate for low frequency). Let \( s \in \mathbb{R}, q \in [2, \infty], r \in [2, \infty] \). Assume that \((q, r)\) and \((\tilde{q}, \tilde{r})\) satisfy

\[ (2.1) \quad \frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \geq \frac{1}{q} \quad \text{and} \quad \frac{d}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) \geq \frac{1}{q}, \]

and additionally assume \( \tilde{q} < q \) if the both equalities hold. Then we have

\[ \left\| \int_0^t \frac{1}{\tau} \mathcal{D}_r(t - s) \chi_{\tau^{-1/2}F(s)} ds \right\|_{L^q_t \dot{B}^r_{2, 2}(t)} \lesssim \tau^{\alpha(q, r) + \alpha(\tilde{q}, \tilde{r})} \left\| \chi_{\tau^{-1/2}F} \right\|_{L^{q'}_t \dot{B}^r_{2, 2}}. \]
where the implicit constant is independent of \( \tau \). Moreover, we also have
\[
\left\| \int_0^t \frac{1}{\tau} \partial_t D_\tau(t - s) \chi_{\tau \leq s} F(s) ds \right\|_{L_t^\infty L_x^2} \lesssim \tau^{-1 + \alpha(q, \tilde{q}) + \frac{1}{q}} \left\| \chi_{\tau \leq 1/2} F \right\|_{L_t^q \dot{W}^s_{r, \ell}}
\]
for \( s \in [0, 1] \).

**Remark 2.1.** In [5][8], we assume that \( 1 < q < q' < \infty \) if the both equalities (2.1) hold. In the above lemma, we may take \( q = \infty \) or \( q' = \infty \). See Appendix A for the proof.

**Lemma 2.4** (Inhomogeneous estimate for high frequency). Let \( s \in \mathbb{R}, q, \tilde{q} \in [2, \infty] \), and \( r, \tilde{r} \in [2, \infty] \). Then, we have the following.
\[
\left\| \int_0^t \frac{1}{\tau} \partial_t D_\tau(t - s) \chi_{\tau \leq s} F(s) ds \right\|_{L_t^q \dot{B}^s_{r, 2}(I)} \lesssim \tau^{\alpha(q, \tilde{q}) + \frac{\gamma(q, \tilde{q}) + \frac{1}{q} - 1 + \gamma(q, \tilde{q}) + \frac{1}{q} - 1}{2}} \left\| \nabla^{\gamma(q, \tilde{q})} \chi_{\tau \leq 1/2} F \right\|_{L_t^q \dot{B}^s_{r, 2}(I)}
\]
where the implicit constant is independent of \( \tau \) and we set \( \delta := \delta((q, r), (\tilde{q}, \tilde{r})) \), which is defined in Table 1 below. Moreover, we also have
\[
\left\| \int_0^t \frac{1}{\tau} \partial_t D_\tau(t - s) \chi_{\tau \leq s} F(s) ds \right\|_{L_t^q \dot{W}^s_{r, \ell}} \lesssim \tau^{-1 + \alpha(q, \tilde{q}) + \frac{\gamma(q, \tilde{q})}{q}} \left\| \nabla^{\gamma(q, \tilde{q})} \chi_{\tau \leq 1/2} F \right\|_{L_t^q \dot{W}^s_{r, \ell}}.
\]

| \( \delta((q, r), (\tilde{q}, \tilde{r})) \) | \( \frac{1}{q} \left( \frac{1}{2} - \frac{1}{r} \right) < \frac{1}{q} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) \) | \( \frac{1}{q} \left( \frac{1}{2} - \frac{1}{r} \right) > \frac{1}{q} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) \) |
|---|---|---|
| \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \geq \frac{1}{q} \) | 0 | 0 |
| \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) > \frac{1}{q} \) | \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) \) | \( \frac{1}{q} \left( \frac{1}{2} - \frac{1}{r} \right) \) |
| \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) < \frac{1}{q} \) | \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) \) | \( \frac{1}{q} \left( \frac{1}{2} - \frac{1}{r} \right) \) |
| \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) \) | \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \) | \( \frac{1}{q} \left( \frac{1}{2} - \frac{1}{r} \right) > \frac{1}{q} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) \) |

Table 1. The value of \( \delta \). (\( \times \) means that the case does not occur.)

**Remark 2.2.** In [5], the first author assumed that \( d \geq 2 \) in the Strichartz estimates for the high frequency part. However, they holds in the one dimensional case. See Appendix A.3

We use the following lemma without notice.
Lemma 2.5. For \( \sigma, a \geq 0 \) and \( 1 < p < \infty \), we have
\[
\| |\nabla|^{\sigma} \chi_{\leq t^{-1/2}} f \|_{L^p} \leq t^{-\sigma/2} \| |\nabla|^{\sigma-a} \chi_{\leq t^{-1/2}} f \|_{L^p}
\]
\[
\| |\nabla|^{\sigma} \chi_{> t^{-1/2}} f \|_{L^p} \leq t^{a/2} \| |\nabla|^{\sigma+a} \chi_{> t^{-1/2}} f \|_{L^p}
\]

Proof. These inequalities follow immediately from \( \chi_{\leq t^{-1/2}} \) and \( \chi_{> t^{-1/2}} \), respectively. \( \square \)

We use the following lemmas by [7].

Lemma 2.6 ([7, Lemma 3.3]). Let \( I \) be a time interval. Take \( X_i \in \mathcal{L} \) for \( i \in \{0, 1, 2, \ldots, N\} \) and assume that \( |u(t, x)| \lesssim \sum_{i=1}^{N} |v_i(t, x)| \) and \( v_i \in X_i \). Then, we have
\[
\|u\|_{\sum_{i=0}^{N} X_i} \sim \sum_{i=0}^{N} \|v_i\|_{X_i}.
\]

This lemma and the Hölder inequality imply the difference estimate of the nonlinear term.

Lemma 2.7 ([9, Lemma 3.4]). Let \( I \) be a time interval and \( N(u) = \lambda |u|^{p-1} u \) with \( p > 1 \). Take \( X_i \in \mathcal{L} \) and \( Z_i \in \mathcal{B} \) for \( i = 0, 1, 2, 3 \). Assume that \( \sigma(Z_i) < \min \{2, p\} \) and \( X_i^{p-1} Z_i \in \mathcal{B} \) for \( i = 0, \ldots, 3 \). Then we have
\[
\|N(u)\|_{\sum_{i=0}^{3} X_i^{p-1} Z_i} \lesssim \inf_{a+b} (\|a\|_{X_0 \cap X_1} + \|b\|_{X_2 \cap X_3})^{p-1} (\|a\|_{Z_0 \cap Z_2} + \|b\|_{Z_1 \cap Z_3}).
\]

Combining Lemmas 2.6 and 2.7 we also have the similar estimate to (2.2) for inhomogeneous Besov spaces.

We also use the following refined version of Lemma 2.7 by Masmoudi and Nakahashi [9].

Lemma 2.8 ([9, Lemma 3.2]). Let \( I \) be a time interval and \( N(u) = \lambda |u|^{p-1} u \) with \( p > 1 \). Take \( X_i \in \mathcal{L} \) and \( Z_i \in \mathcal{B} \) for \( i = 0, 1, 2, 3 \). Assume that \( \sigma(Z_i) < \min \{2, p\} \) and \( X_i^{p-1} Z_i \in \mathcal{B} \) for \( i = 0, \ldots, 3 \). Then we have
\[
\|P_j N(u)\|_{\sum_{i=0}^{3} X_i^{p-1} Z_i} \lesssim \inf_{a+b} (\|a\|_{X_0 \cap X_1} + \|b\|_{X_2 \cap X_3})^{p-1} \times \kappa_j \|P_j a\|_{Z_0 \cap Z_2} + \|P_j b\|_{Z_1 \cap Z_3},
\]
where \( *_j \) denotes the convolution related to \( j \) and
\[
\kappa_j := \max_{i=0, 1, 2, 3} \min \left\{ 2^{\frac{\sigma(Z_i)}{2}}, \frac{2^{\sigma(Z_i)-2}}{2} \right\}.
\]

2.4. Local existence and Uniformly bound. In what follows, we only treat the case of \( d \geq 3 \). The cases \( d = 1, 2 \) are easier (see Section 2.7).

2.4.1. Function spaces. We define the function spaces as follows.
\[
\mathcal{E} := L_t^\infty H^1_x,
\]
\[
\mathcal{H}_1 := \frac{2(2d+3)}{d-2} L_t^{2(d+2)}(\mathbb{R}^d), \quad \mathcal{H}_2 := \frac{2(2d+2)}{d} B_{2(d+2)}^{\frac{d}{2}}(\mathbb{R}^d),
\]
\[
\mathcal{H}_1 := \frac{2(d+2)}{d} L_t^{2(d+1)}(\mathbb{R}^d), \quad \mathcal{H}_2 := \frac{2(d+2)}{d} B_{2(d+2)}^{\frac{d}{2}}(\mathbb{R}^d),
\]
\[
\mathcal{D}_1 := \frac{2(d+2)}{d} B_{2(d+2)}^{\frac{d}{2}}(\mathbb{R}^d).
\]
We also define the function spaces \( R_0, \ldots, R_3 \) by
\[
R_0 := L_t^{\frac{2(d+2)}{d+2}} W_x^{\frac{1}{d+2}} = H^{\frac{2}{d+2}}_1, \\
R_1 := \tau \frac{1}{2} L_t^{\frac{2(d+2)}{d+4}} W_x^{\frac{1}{d+4}} = H^{\frac{2}{d+4}}_1, \\
R_2 := \tau \frac{d+3}{4} L_t^{\frac{2(d+1)}{d+4}} W_x^{\frac{1}{d+4}} = W^{\frac{1}{2}}_1, \\
R_3 := \tau \frac{2(d+1)(d+2)}{2d^2 + 5d + 8} W_x^{\frac{1}{2d^2 + 5d + 8}} = H^{\frac{2}{d+2}}_1.
\]

We will use the following function spaces to construct a contraction mapping.
\[
R_0 := [E, H_0]_0 \cap [E, H_2]_0 \cap [E, D_1]_0 \\
\mathcal{X} := R_1 | W_1.
\]

For the reader’s convenience, we give an explanation of the exponents of the function spaces. We set the exponents \((q, r)\) by
\[
e := (\infty, 2), \\
h_1 := \left( \frac{2(d+2)}{d-2}, \frac{2(d+2)}{d-2} \right), \quad h_2 := \left( \frac{2(d+2)}{d}, \frac{2(d+2)}{d} \right), \\
w_1 := \left( \frac{2(d+1)}{d-2}, \frac{2(d+1)}{d-2} \right), \quad w_2 := \left( \frac{2(d+1)}{d-1}, \frac{2(d+1)}{d-1} \right), \\
w_3 := \left( \frac{(d+1)(d+2)}{d^2 + d - 4}, \frac{(d+1)(d+2)}{d^2 + d - 4} \right).
\]

The exponent \( e \) is related to the energy space \( E \), \( h_1 \) is \( H_1 \), \( h_2 \) is \( H_2 \) and \( D_1 \), \( w_1 \) is \( W_1 \), \( w_2 \) is \( W_2 \). The exponents of \( R_0 \) and \( R_1 \) is \( h_1' \), \( R_2 \) is related to \( w_2' \), \( R_3 \) is \( w_3' \), where \( p' = (p', p') \).

\( h_2 \) satisfies \( \frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) = \frac{1}{q} \), \( w_2 \) satisfies \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) = \frac{1}{q} \), \( w_1 \) and \( w_3 \) satisfies \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) > \frac{1}{q} \). Namely, \( h_2 \) is related to the heat admissible pair and \( w_1 \) and \( w_2 \) are related to the wave admissible pair. \( h_1 \) satisfies \( \frac{d-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) > \frac{1}{q} \). However, since the Sobolev embedding \( \| f \|_{L_t^{2(d+2)} W_x^{2(d+2)}} \lesssim \| f \|_{W_t^{2d+8} W_x^{2d+8}} \) holds, its new exponent \( \tilde{h}_1 := \left( \frac{d-2}{d+2}, \frac{2d+4}{2d+2} \right) \) lies on the heat line \( \frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) = \frac{1}{q} \). Thus, \( h_1 \) is related to the heat admissible pair.

We collect the value of \( \alpha, \gamma, \) and \( \delta \) for these exponents in the following tables.

| \( e \) | \( h_1 \) | \( h_2 \) | \( w_1 \) | \( w_2 \) | \( w_3 \) |
|---|---|---|---|---|---|
| \( \alpha \) | 0 | 0 | -\( \frac{d+4}{3(d+1)} \) | -\( \frac{1}{2(d+1)} \) | -\( \frac{1}{d+1} \) |
| \( \gamma \) | 0 | \( \frac{d+1}{2(d+2)} \) | 1 | \( \frac{1}{2} \) | \( \frac{d+1}{2(d+2)} \) |

Table 2. The value of \( \alpha \) for the exponents.

Table 3. The value of \( \gamma \) for the exponents.
\[ \therefore, \text{we obtain such that} \quad T \leq \frac{d-1}{2d(d+1)} \quad \text{for any} \quad (2.5) \]

By (2.3) and (2.6), since \( \limsup_{T \rightarrow 0} \| u_r \|_{H^1} + \frac{\tau}{\tau^2} \| g_r \|_{L^2} + T^{1-\theta} \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]} \leq 0 \), we have the uniform boundedness. Since we have
\[
\| u_r \|_{E_{\varepsilon, \mathcal{X}}} \leq \| f_r \|_{H^1} + \frac{\tau}{\tau^2} \| g_r \|_{L^2} + T^{1-\theta} \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]} + \tau \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]}.
\]

Therefore, we obtain
\[
\| u_r \|_{E_{\varepsilon, \mathcal{X}}} \lesssim \| u \|_{X_{\varepsilon, \mathcal{X}}}.
\]

Then, by Lemmas 2.6 and 2.7, we have
\[
\| u \|_{P_{\varepsilon, \mathcal{X}}[0,T]} \leq \| u \|_{X_{\varepsilon, \mathcal{X}}}.
\]

Therefore, we obtain
\[
(2.3) \quad \| u_r \|_{E_{\varepsilon, \mathcal{X}}[0,T]} \leq \| f_r \|_{H^1} + \frac{\tau}{\tau^2} \| g_r \|_{L^2} + T^{1-\theta} \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]} + \tau \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]}
\]

By the Strichartz estimate and Lemmas 2.6 and 2.7, we also have
\[
(2.4) \quad \| u_r \|_{E_{\varepsilon, \mathcal{X}}[0,T]} \leq \| f_r \|_{H^1} + \tau \| g_r \|_{L^2} + T^{1-\theta} \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]} + \tau \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]}
\]

By (2.3) and (2.6), since \( \limsup_{T \rightarrow 0} \| f_r \|_{H^1} + \tau \| g_r \|_{L^2} + T^{1-\theta} \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]} \leq 0 \), there exist \( C_0 > 0 \) and \( \tau_0 > 0 \) such that
\[
(2.5) \quad \| u_r \|_{E_{\varepsilon, \mathcal{X}}[0,T]} + \tau \| g_r \|_{L^2} + T^{1-\theta} \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]} \leq C_0 + T^{1-\theta} \| u_r \|_{P_{\varepsilon, \mathcal{X}}[0,T]}
\]

for any \( T < T^{*}_r \) and \( 0 < \tau < \tau_0 \).

Suppose that \( \liminf_{T \rightarrow 0} T^{*}_r = 0 \). Then, there exists a sequence \( \{ \tau_n \} \subset (0, \tau_0) \) such that \( T^{*}_{\tau_n} \rightarrow 0 \). For sufficiently large \( n \), it holds from (2.3) that
\[
(2.6) \quad \| u_{\tau_n} \|_{E_{\varepsilon, \mathcal{X}}[0,T]} + \tau_n \| g_{\tau_n} \|_{L^2} + T^{1-\theta} \| u_{\tau_n} \|_{P_{\varepsilon, \mathcal{X}}[0,T]} \leq C_0
\]

for any \( T < T^{*}_{\tau_n} \). This estimate and the blow-up alternative imply a contradiction. Therefore, we have \( T^* = \liminf_{n \rightarrow 0} T^{*}_{\tau_n} > 0 \) and thus, for any \( T < T^* \), there exist \( C_T > 0 \) and \( \tau_T \) such that \( \| u_{\tau} \|_{E_{\varepsilon, \mathcal{X}}[0,T]} + \tau^{1/2} \| g_{\tau} \|_{E_{\varepsilon, \mathcal{X}}[0,T]} \leq C_T \) for \( \tau < \tau_T \). Consequently, we obtain Theorem 1.1.
2.5. **Proof of $L^2$-convergence.** In this section, we show that the $L^2$-convergence by a direct calculation.

We set the function spaces without derivative by

\[
\mathcal{E}^0 := L^\infty_t L^2_y,
\]

\[
\mathcal{H}^2_0 := L^2_t L^2_x
\]

and

\[
\mathcal{B}^0_0 := L^2_t L^2_x = \mathcal{H}^2_0
\]

\[
\mathcal{B}^0_3 := \tau \frac{1}{L^2_t} L^2_x = \mathcal{H}^2_0
\]

Moreover, we set

\[
\mathcal{Z}_0 := [\mathcal{E}^0, \mathcal{H}^2_0], [\mathcal{E}^0, \mathcal{H}^2_0] \cap [\mathcal{E}^0, \mathcal{H}^2_0]
\]

\[
\mathcal{Y} := [\mathcal{E}^0, \mathcal{H}^2_0]
\]

We decompose $u_\tau - v$ as follows.

\[
[u_\tau(t) - v(t)] = \left( \frac{1}{\tau} D_\tau(t) + \partial_t D_\tau(t) \right) (f_\tau - f)
\]

\[
+ \left( \frac{1}{\tau} D_\tau(t) + \partial_t D_\tau(t) - e^{t\Delta} \right) f
\]

\[
+ D_\tau(t) g_\tau
\]

\[
+ \int_0^t \frac{1}{\tau} D_\tau(t - s) (N(u) - N(v)) ds
\]

\[
+ \int_0^t \left( \frac{1}{\tau} D_\tau(t - s) - e^{(t-s)\Delta} \right) N(v) ds
\]

\[
=: L_1 + L_2 + L_3 + N_1 + N_2
\]

We estimate the $\mathcal{Y}$-norm of these terms. First, we discuss the $\mathcal{Y}$-estimate of the homogeneous parts $L_1, L_2, \text{ and } L_3$. By the embedding $\mathcal{Y} \supset \mathcal{E}^0 \cap \mathcal{H}^2_0$, it is enough to estimate $\mathcal{E}^0 \cap \mathcal{H}^2_0$-norms.

**The $\mathcal{Y}$-estimate of $L_3$:**

By the Strichartz estimate, we obtain

\[
\|D_\tau(t) g_\tau\|_{\mathcal{E}^0 \cap \mathcal{H}^2_0} \lesssim \|D_\tau(t) \chi_{\leq \tau^{-1/2}} g_\tau\|_{\mathcal{E}^0 \cap \mathcal{H}^2_0} + \|D_\tau(t) \chi_{\tau^{-1/2}} g_\tau\|_{\mathcal{E}^0 \cap \mathcal{H}^2_0}
\]

\[
\lesssim \tau \|g_\tau\|_{L^2} + \tau \frac{3(b_2)}{2} \||\nabla|^{(b_2) - 1} \chi_{\tau^{-1/2}} g_\tau\|_{L^2}
\]

\[
\lesssim \tau \|g_\tau\|_{L^2}
\]

**The $\mathcal{Y}$-estimate of $L_1$:**

In the same way as above, by dividing into the low and high frequency parts and applying the Strichartz estimates, we have

\[
\left\| \left( \frac{1}{\tau} D_\tau(t) + \partial_t D_\tau(t) \right) (f_\tau - f) \right\|_{\mathcal{E}^0 \cap \mathcal{H}^2_0} \lesssim \|f_\tau - f\|_{L^2}.
\]

**The $\mathcal{Y}$-estimate of $L_2$:**
First, we consider the high frequency part $|\xi| > (8\tau)^{-1/2}$. The constant is not essential but technical. By the triangle inequality, we have
\[
\left\| \left( \frac{1}{\tau} D_\tau(t) + \partial_t D_\tau(t) - e^{t\Delta} \right) \chi_{>(8\tau)^{-1/2}} f \right\|_{L^2}\leq \left\| \left( \frac{1}{\tau} D_\tau(t) + \partial_t D_\tau(t) \right) \chi_{>(8\tau)^{-1/2}} f \right\|_{L^2} + \left\| e^{t\Delta} \chi_{>(8\tau)^{-1/2}} f \right\|_{L^2}
\]
By the Strichartz estimate, we obtain
\[
\left\| \left( \frac{1}{\tau} D_\tau(t) + \partial_t D_\tau(t) \right) \chi_{>(8\tau)^{-1/2}} f \right\|_{L^2} \leq \|\chi_{>(8\tau)^{-1/2}} f\|_{L^2} \leq \tau^{1/4} \|\chi_{>(8\tau)^{-1/2}} \nabla f\|_{L^2}
\]
which follows from the argument by e.g. \cite{4} and \cite{5}, and thus we omit the detail. By the Strichartz estimate for the heat propagator we have
\[
\left\| e^{t\Delta} \chi_{>(8\tau)^{-1/2}} f \right\|_{L^2} \leq \|\chi_{>(8\tau)^{-1/2}} \nabla f\|_{L^2}
\]
and so the right hand side is $o(\tau^{1/2})$.

Next, we consider the low frequency part $|\xi| \leq (8\tau)^{-1/2}$ . We divide the operators of $L_2$ into three parts as follows.
\[
\frac{1}{\tau} D_\tau(t) + \partial_t D_\tau(t) - e^{t\Delta} = \left\{ \frac{1}{\lambda_\tau - \lambda_\tau^+} \left( \lambda_\tau^- e^{t\lambda_\tau^+} - \lambda_\tau^+ e^{t\lambda_\tau^-} \right) - e^{-i|\xi|^2} \right\} \mathcal{F} = \left( \frac{\lambda_\tau^-}{\lambda_\tau - \lambda_\tau^+} - 1 \right) e^{t\lambda_\tau^+} \mathcal{F} + (e^{t\lambda_\tau^+} - e^{-i|\xi|^2}) \mathcal{F} - \frac{\lambda_\tau^+}{\lambda_\tau - \lambda_\tau^+} e^{t\lambda_\tau^-} \mathcal{F} =: I + J + K
\]
We set $[\xi]_\tau := \sqrt{1 - 4\tau|\xi|^2}$. Since $|\xi| \leq (8\tau)^{-1/2}$, we have $[\xi]_\tau \geq 1/\sqrt{\tau}$. Then, it holds that
\[
(2.7) \quad \frac{\lambda_\tau^-}{\lambda_\tau - \lambda_\tau^+} - 1 = \frac{\lambda_\tau^+}{\lambda_\tau^- - \lambda_\tau^+} = \frac{2\tau|\xi|^2}{[\xi]_\tau (1 + [\xi]_\tau)} \lesssim \tau^{1/2} |\xi|.
\]
Now, we have the following Strichartz type estimate.
\[
\|e^{t\lambda_\tau^\pm(\nabla)} \chi_{\leq 1} f\|_{L^2} \lesssim \|f\|_{L^2}
\]
This follows from the argument by e.g. \cite{11} and \cite{12}, and thus we omit the detail. By the scaling, this estimate implies
\[
\|e^{t\lambda_\tau^\pm(\nabla)} \chi_{\leq \tau^{-1/2}} f\|_{L^2} \lesssim \|f\|_{L^2},
\]
where we note that the scaling order $\alpha(q, r)$ of $\tau$ disappears since $\alpha(\eta) = \alpha(c) = 0$. Thus, we obtain
\[
\|\chi_{>(8\tau)^{-1/2}} f\|_{L^2} \lesssim \tau^{1/2} \|f\|_{H^1}.
\]
By (2.7), we also have
\[ \| \chi_{(\tau)} f \|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} \lesssim \tau^{\frac{1}{2}} \| f \|_{H^1}. \]
To estimate \( J \), we define \( h(\tau) := e^{i\lambda \tau} \). Then, \( J = h(\tau) - h(0) \). Therefore, by the fundamental theorem of calculus, we have
\[ \| (h(\tau) - h(0)) \chi_{(\tau)} f \|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} \]
\[ \leq \int_0^\tau \left\| F^{-1} \left\{ h'(\tau) \chi_{(\tau)} f \right\} \right\|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} d\tau \]
\[ \leq \int_0^\tau \left\| F^{-1} \left\{ h'(\tau) \chi_{(\tau)} f \right\} \right\|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} d\tau \]
where we used \( \dot{\tau} \leq \tau \) in the last inequality. By a simple calculation, we have
\[ h'(\tau) = \frac{-4|\xi|^4}{(\xi)_{\tau}^2(\xi)_{\hat{\tau}}^2} e^{i\lambda \tau} \lesssim \tau^{-\frac{1}{2}} |\xi|^2 e^{i\lambda \tau} \lesssim \tau^{-\frac{1}{2}} |\xi| \]
when \( |\xi| \leq (8\tau)^{-1/2} \). Thus, by the Strichartz estimates, we obtain
\[ \int_0^\tau \left\| F^{-1} \left\{ h'(\tau) \chi_{(\tau)} f \right\} \right\|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} d\tau \lesssim \int_0^\tau \dot{\tau}^{-\frac{1}{2}} d\tau \lesssim \tau^{\frac{1}{2}} \| f \|_{H^1}. \]
As a conclusion, we get
\[ \left\| \left( \frac{1}{\tau} D_\tau (t) + \partial_t D_\tau (t) - e^{i\lambda} \right) \chi_{(\tau)} f \right\|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} \lesssim \| \chi_{(\tau)} f \|_{L^2} \]
\[ \lesssim \tau^{\frac{1}{2}} \| f \|_{H^1}. \]
Next, we consider the estimate of the nonlinear terms \( N_1 \) and \( N_2 \).

**The \( N \)-estimate of \( N_1 \):**
It holds from the Strichartz estimates that
\[ \| N_1 \|_{\mathcal{N}} \lesssim T^{1-\theta} \| N(u_\tau) - N(v) \|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} + \| e^{i\lambda} \|_{\mathcal{E}^0 \cap \mathcal{H}_2^0}. \]
Since we have
\[ |N(u_\tau) - N(v)| \lesssim (|u_\tau|^p + |v|^p) |u_\tau - v|, \]
by dividing \( u_\tau, v \) into low frequency part \( u_{\tau, l}, v_l \) and high frequency part \( u_{\tau, h}, v_h \), we have
\[ |N(u_\tau) - N(v)| \lesssim (|u_{\tau, l}|^p + |u_{\tau, h}|^p + |v_l|^p + |v_h|^p) |u_\tau - v| \]
\[ \lesssim (|u_{\tau, l}|^p + |v_l|^p) |u_\tau - v| + (|u_{\tau, h}|^p + |v_h|^p) |u_\tau - v|. \]
By Lemma 2.7 and the Hölder inequality, we obtain
\[ \| N(u_\tau) - N(v) \|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} \]
\[ \lesssim \left( (|u_{\tau, l}|^p + |v_l|^p) \right) \| u_\tau - v \|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} + \left( (|u_{\tau, h}|^p + |v_h|^p) \right) \| u_\tau - v \|_{\mathcal{E}^0 \cap \mathcal{H}_2^0} \]
\[ + \left( (|u_{\tau, h}|^p + |v_h|^p) \right) \| u_\tau - v \|_{\mathcal{E}^0 \cap \mathcal{H}_2^0}. \]
By the uniform boundedness, we find that \( \|u_{\tau,t}\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \) and \( \|u_{\tau,h}\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \) are bounded uniformly in \( \tau \). Moreover, \( \|v_{l}\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \) and \( \|u_{h}\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \) are also bounded. Thus, we have
\[
\|N(u_{\tau}) - N(v)\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \lesssim \|u_{\tau} - v\|_\mathcal{Y}.
\]
This means \( \|N_1\|_\mathcal{Y} \lesssim T^{1-\theta}\|u_{\tau} - v\|_\mathcal{Y} \).

**The \( \mathcal{Y} \)-estimate of \( N_2 \):** In the same way as the estimate of \( L_2 \), we divide \( N_1 \) into the high, middle, and low frequency parts. First, we treat the high frequency part.
\[
\left\| \int_0^t \left( \frac{1}{\tau} D_{\tau}(t-s) - e^{(t-s)\Delta} \right) \chi_{>(8\tau)^{-1/2}} N(v) ds \right\|_\mathcal{Y} \lesssim \left\| \int_0^t \frac{1}{\tau} D_{\tau}(t-s) \chi_{>(8\tau)^{-1/2}} N(v) ds \right\|_\mathcal{Y} + \left\| \int_0^t e^{(t-s)\Delta} \chi_{>(8\tau)^{-1/2}} N(v) ds \right\|_\mathcal{Y}
\]
By the Strichartz estimate, we have
\[
\left\| \int_0^t \frac{1}{\tau} D_{\tau}(t-s) \chi_{>(8\tau)^{-1/2}} N(v) ds \right\|_\mathcal{Y} \lesssim \tau^{\frac{1}{2}} \|\nabla N(v)\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \lesssim \tau^{\frac{1}{2}} \|v\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \|v\|_{[\mathcal{E},\mathcal{M}]_\theta}.
\]
Moreover, we have
\[
\left\| \int_0^t e^{(t-s)\Delta} \chi_{>(8\tau)^{-1/2}} N(v) ds \right\|_\mathcal{Y} \lesssim \left\| \chi_{1/\sqrt{\tau}} N(v) \right\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \lesssim \tau^{\frac{1}{2}} \|\nabla N(v)\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \lesssim \tau^{\frac{1}{2}} \|v\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \|v\|_{[\mathcal{E},\mathcal{M}]_\theta}.
\]

Next, we consider the low frequency part. The symbol of \( \frac{1}{\tau} D_{\tau}(t) - e^{t\Delta} \) is calculated by
\[
\frac{-1}{\tau(\lambda_{\tau} - \lambda_{\tau}^+)}(e^{t\lambda_{\tau}^{+}} - e^{t\lambda_{\tau}^{-}}) - e^{-t|\xi|^2} = \left( \frac{-1}{\tau(\lambda_{\tau} - \lambda_{\tau}^+) - 1} \right) e^{t\lambda_{\tau}^{+}} + e^{t\lambda_{\tau}^{+}} - e^{-t|\xi|^2}
\]
\[
+ \frac{1}{\tau(\lambda_{\tau} - \lambda_{\tau}^+)} e^{t\lambda_{\tau}^{-}} =: H + J + M
\]
Since we have
\[
\frac{-1}{\tau(\lambda_{\tau} - \lambda_{\tau}^+)} - 1 = \frac{4\tau|\xi|^2}{|\xi|^2(1 + |\xi|^2)} \lesssim \tau^{\frac{1}{2}} |\xi|
\]
when \( |\xi| \leq (8\tau)^{-1/2} \), we can estimate \( H \) by
\[
\left\| \int_0^t H \chi_{<(8\tau)^{-1/2}} N(v) ds \right\|_\mathcal{Y} \lesssim \tau^{\frac{1}{2}} \|v\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \|v\|_{[\mathcal{E},\mathcal{M}]_\theta}
\]
where we used the inhomogeneous Strichartz estimate for \( e^{t\lambda_{\tau}^{\pm}} \). In the similar way to the estimate of \( L_2 \), we obtain
\[
\left\| \int_0^t J \chi_{<(8\tau)^{-1/2}} N(v) ds \right\|_\mathcal{Y} \lesssim \tau^{\frac{1}{2}} T \|v\|_{[\mathcal{E}^0,\mathcal{M}]_\theta} \|v\|_{[\mathcal{E},\mathcal{M}]_\theta}.
\]
We estimate $M$. Since it holds
\[ \left| \frac{1}{\tau (\lambda_\tau^2 - \lambda_r^2)} \right| \lesssim 1, \]
it is enough to estimate
\[ \left\| \int_0^t e^{t \lambda_\tau^2} \chi_{(\tau \leq (\tau \leq 1/2)} N(v) ds \right\|_{\mathcal{Y}}. \]
We denote $(q, r) = (\infty, 2)$ or $(q, r) = (2/(d + 2), 2/(d + 2)/d)$. When $1 < p \leq d/(d - 2)$, we have
\[ \left\| \int_0^t e^{t \lambda_\tau^2} \chi_{(\tau \leq (\tau \leq 1/2)} N(v) ds \right\|_{L^q_r L^r_r} \lesssim \left\| \int_0^t e^{-\frac{t}{\tau}} \left\| \chi_{(\tau \leq (\tau \leq 1/2)} N(v) \right\|_{L^r_r} ds \right\|_{L^q_r}. \]
Therefore, combining these estimates, we have
\[ \tau^{1 - \frac{2}{p} \left( \frac{1}{r} - \frac{1}{\tau} \right)} \lesssim \tau^{1/2}. \]
When $d/(d - 2) < p < (d + 2)/(d - 2)$, we set $1/r_0 = 1/r - 1/d + (d - 2)(p - 1)/(2d)$. Then, we have $r > r_0 > 1$ and, by the Sobolev inequality, we have \( \|N(v)\|_{L^q_r} \lesssim \|v\|_{H^{1/r}} \|v\|_{W^{1, r}} \).
\[ \left\| \int_0^t e^{t \lambda_\tau^2} \chi_{(\tau \leq (\tau \leq 1/2)} N(v) ds \right\|_{L^q_r L^r_r} \lesssim \left\| \int_0^t e^{-\frac{t}{\tau}} \left\| \chi_{(\tau \leq (\tau \leq 1/2)} N(v) \right\|_{L^r_r} ds \right\|_{L^q_r}, \]
and, by the Sobolev inequality and the Bernstein inequality, we have \( \|v\|_{L^q_r} \lesssim \|v\|_{H^{1/r}} \|v\|_{W^{1, r}} \).
\[ \tau^{1 - \frac{2}{p} \left( \frac{1}{r_0} - \frac{1}{\tau} \right)} \lesssim \tau^{1/2}. \]
where we used the Young inequality in the last. Since $r > r_0 > 1$ and, by the Sobolev inequality, we have \( \|N(v)\|_{L^q_r} \lesssim \|v\|_{H^{1/r}} \|v\|_{W^{1, r}} \).
\[ \left\| \int_0^t M \chi_{(\tau \leq (\tau \leq 1/2)} N(v) ds \right\|_{\mathcal{Y}} \lesssim \tau^{1/2} T^{\kappa_0} \|v\|_{H^{1/r} L^r_r} \|v\|_{[\mathcal{E}, \mathcal{H}]_d}. \]
for a positive constant $\kappa_0$. Since $\|v\|_{H^{1/r} L^r_r}$ and $\|v\|_{[\mathcal{E}, \mathcal{H}]_d}$ are bounded on $[0, T]$, we obtain $O(\tau^{1/2})$. 

Combining these estimates, we obtain
\[ \| N_2 \| \gtrsim \tau^{\frac{1}{2}} \]

**Conclusion of the \( \mathcal{Y} \)-estimate:**

Thus, we have
\[ \| u_\tau - v \|_{\mathcal{Y}[0,T]} \lesssim \| f_\tau - f \|_{L^2(\mathbb{R}^d)} + \tau \| g_\tau \|_{L^2(\mathbb{R}^d)} + T^{1-\theta} \| u_\tau - v \|_{\mathcal{Y}[0,T]} + \tau^{\frac{1}{2}} \]

Taking small \( T \), we obtain
\[ \| u_\tau - v \|_{\mathcal{Y}[0,T]} \lesssim \| f_\tau - f \|_{L^2(\mathbb{R}^d)} + \tau \| g_\tau \|_{L^2(\mathbb{R}^d)} + \tau^{\frac{1}{2}} \]

Repeating this, we obtain the estimate for any \( T < T^* \). Therefore, we complete the proof of Theorem 2.2.

2.6. **Compactness method for \( H^1 \)-convergence.** We prove \( H^1 \)-convergence by a compactness method.

We show the following statement.

**Proposition 2.9.** Let \( f \in H^1(\mathbb{R}^d) \), \( v \) be a solution to \( (\text{NLH}) \), and \( T_{\text{max}} \) is the maximal existence time of the solution \( v \) to \( (\text{NLH}) \). If the initial data \( (f_\tau, g_\tau) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) satisfies
\[ (f_\tau, \tau g_\tau) \to (f, 0) \quad \text{in} \quad H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \quad \text{as} \quad \tau \to 0, \]
then we have
\[ \| u_\tau - v \|_{L^\infty(0,T;H^1(\mathbb{R}^d))} + \tau^{\frac{1}{2}} \| \partial_t u_\tau \|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \to 0 \]
as \( \tau \to 0 \) for any \( T < \min\{T^*, T_{\text{max}}\} \).

Once we obtain the above proposition, we can show \( T^* \geq T_{\text{max}} \) as follows. Suppose that \( T^* < T_{\text{max}} \). Then by the proposition, it holds for arbitrary sufficiently small \( \tau \) that
\[ \| u_\tau - v \|_{L^\infty(0,T;H^1(\mathbb{R}^d))} + \tau^{\frac{1}{2}} \| \partial_t u_\tau \|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \lesssim 1 \]
for any \( T < T^* \). Therefore, we have
\[ \| u_\tau \|_{L^\infty(0,T;H^1(\mathbb{R}^d))} + \tau^{\frac{1}{2}} \| \partial_t u_\tau \|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \lesssim \| v \|_{L^\infty(0,T;H^1(\mathbb{R}^d))} + 1 \]
Since \( T^* < T_{\text{max}} \), we have \( \| v \|_{L^\infty(0,T;H^1(\mathbb{R}^d))} + \tau^{1/2} \| \partial_t u_\tau \|_{L^\infty(0,T;L^2(\mathbb{R}^d))} < C \). Thus, \( \| u_\tau \|_{L^\infty(0,T;H^1(\mathbb{R}^d))} < C \) for any \( T < T^* \). This and the blow-up alternative implies that we obtain the solution on \( [0,T^* + \delta) \) for some \( \delta > 0 \). Since the existence time depends only on the norm, we find that \( \delta \) is independent of \( \tau \). We reach contradiction.

**Proof of Proposition 2.9.** To show this, we use a compactness argument. Let \( \varepsilon > 0 \) be fixed arbitrarily. If there exists \( R_\varepsilon > 0 \) independent of \( \tau \) such that \( \| \chi_{> R_\varepsilon} u_\tau \|_{H^1(\mathbb{R}^d)} \lesssim \varepsilon + \| f_\tau - f \|_{H^1(\mathbb{R}^d)} + \tau^{-1/2} \| g_\tau \|_{L^2(\mathbb{R}^d)} \), then by taking \( R > R_\varepsilon \) such that \( \| \chi_{> R} v \|_{H^1(\mathbb{R}^d)} \lesssim \varepsilon \) we have
\[ \| u_\tau - v \|_{H^1(\mathbb{R}^d)} \lesssim \| u_\tau - v \|_{L^2(\mathbb{R}^d)} + \| u_\tau - v \|_{L^2(\mathbb{R}^d)} \lesssim (1 + R) \| u_\tau - v \|_{L^2(\mathbb{R}^d)} + \| f_\tau - f \|_{H^1(\mathbb{R}^d)} + \tau^{-1/2} \| g_\tau \|_{H^1(\mathbb{R}^d)} \leq \| u_\tau - v \|_{L^2(\mathbb{R}^d)} + \| f_\tau - f \|_{H^1(\mathbb{R}^d)} + \tau^{-1/2} \| g_\tau \|_{H^1(\mathbb{R}^d)} \]
By the $L^2$-convergence, we have $\|u_\tau - v\|_{\mathcal{E}_0} \to 0$ as $\tau \to 0$. Thus, by the assumption on the initial data, we obtain

$$\lim_{\tau \to 0} \|u_\tau - v\|_{\mathcal{E}} = 0.$$  

By the above argument, it is enough to show that there exists $R_\varepsilon > 0$ independent of $\tau$ such that $\|\chi_{\tau > R_\varepsilon} u_\tau\|_{\mathcal{E}} \lesssim \varepsilon + \|f_\tau - f\|_{H^1} + \tau^{1/2}\|g_\tau\|_{L^2}$.

Let $R > 0$ and $J \in \mathbb{N}$ satisfy $R > 2^J$. By the Littlewood–Paley decomposition, we have

$$\|\chi_{\tau > R} u_\tau\|_{\mathcal{E}} \lesssim \|P_J \chi_{\tau > R} u_\tau\|_{\mathcal{E}} \lesssim \|P_J u_\tau\|_{\mathcal{E}} \|v_{(\tau > J)}\|_{\mathcal{E}}.$$  

Therefore, it is enough to show that for any $\varepsilon > 0$ there exists $J \in \mathbb{N}$ independent of $\tau$ such that $\|P_J u_\tau\|_{\mathcal{E}} \|v_{(\tau > J)}\|_{\mathcal{E}} < \varepsilon + \|f_\tau - f\|_{H^1} + \tau^{1/2}\|g_\tau\|_{L^2}$. By the Strichartz estimate, we obtain

$$\|P_J u_\tau\|_{\mathcal{E}} \lesssim \|P_J f_\tau\|_{H^1} + \tau^{1/2}\|P_J g_\tau\|_{L^2} + \left\| \int_0^1 \frac{1}{\tau} D_\tau(t-s) P_J N(u_\tau(s)) ds \right\|_{\mathcal{E}} \lesssim \|P_J f_\tau\|_{H^1} + \tau^{1/2}\|P_J g_\tau\|_{L^2} + T^{1-\theta}\|P_J N(u_\tau)\|_{\sum_{i=1}^3 [\mathcal{E}_{\mathcal{E}_0}^{\mathcal{E}_0}]}.$$  

By Lemma, we have the following estimate.

$$\|P_J N(u_\tau)\|_{\sum_{i=0}^3 [\mathcal{E}_{\mathcal{E}_0}^{\mathcal{E}_0}]} \lesssim 2^{-\frac{|\mathcal{E}_E|}{4}} \|u_\tau\|_{\mathcal{E}}^{p-1}\|P_J u_\tau\|_{\mathcal{E}}.$$  

where $*_j$ denotes the convolution over $Z$. Thus, we have

$$\|P_J u_\tau\|_{\mathcal{E}} \lesssim \|P_J f_\tau\|_{H^1} + \tau^{1/2}\|P_J g_\tau\|_{L^2} + T^{1-\theta}\|P_J N(u_\tau)\|_{\sum_{i=1}^3 [\mathcal{E}_{\mathcal{E}_0}^{\mathcal{E}_0}]}.$$  

since $\|u_\tau\|_{\mathcal{E}}^{p-1}$ is uniformly bounded. By convoluting with $2^{-\frac{|\mathcal{E}_E|}{4}}$, we have

$$2^{-\frac{|\mathcal{E}_E|}{4}} \|P_J u_\tau\|_{\mathcal{E}} \lesssim \|P_J f_\tau\|_{H^1} + \tau^{1/2}\|P_J g_\tau\|_{L^2} + T^{1-\theta}\|P_J N(u_\tau)\|_{\sum_{i=1}^3 [\mathcal{E}_{\mathcal{E}_0}^{\mathcal{E}_0}]}$$  

since $2^{-\frac{|\mathcal{E}_E|}{4}} \|P_J u_\tau\|_{\mathcal{E}} \lesssim \|P_J f_\tau\|_{H^1} + \tau^{1/2}\|P_J g_\tau\|_{L^2}$. If $T$ is sufficiently small, we obtain

$$2^{-\frac{|\mathcal{E}_E|}{4}} \|P_J u_\tau\|_{\mathcal{E}} \lesssim \|P_J f_\tau\|_{H^1} + \tau^{1/2}\|P_J g_\tau\|_{L^2}.$$  

It follows from $2^{-\frac{|\mathcal{E}_E|}{4}} < 2^{-\frac{|\mathcal{E}_E|}{4}}$ and substituting the l.h.s. into the r.h.s. that

$$\|P_J u_\tau\|_{\mathcal{E}} \lesssim \|P_J f_\tau\|_{H^1} + \tau^{1/2}\|P_J g_\tau\|_{L^2} + T^{1-\theta}\|P_J N(u_\tau)\|_{\sum_{i=1}^3 [\mathcal{E}_{\mathcal{E}_0}^{\mathcal{E}_0}]}.$$

Taking $l^2$-norm for $j > J$, by $2^{-\frac{|\mathcal{E}_E|}{4}} \in l^1$ and the Young inequality, we have

$$\|P_J u_\tau\|_{\mathcal{E}} \lesssim \|P_J f_\tau\|_{H^1} + \tau^{1/2}\|P_J g_\tau\|_{L^2} + T^{1-\theta}\|P_J N(u_\tau)\|_{\sum_{i=1}^3 [\mathcal{E}_{\mathcal{E}_0}^{\mathcal{E}_0}]}.$$  

By the Littlewood–Paley decomposition, we have

$$\|P_J f_\tau\|_{H^1} \lesssim \|P_J (f_\tau - f)\|_{H^1} \|v_{(\tau > J)}\|_{\mathcal{E}} + \|P_J f\|_{H^1} \|v_{(\tau > J)}\|_{\mathcal{E}} \lesssim \|f_\tau - f\|_{H^1} + \|\chi_{\tau > 2^J} f\|_{H^1}$$  

and

$$\tau^{1/2}\|P_J g_\tau\|_{L^2} \lesssim \tau^{1/2}\|g_\tau\|_{L^2}.$$
Thus, for any $\varepsilon > 0$ there exists $J \in \mathbb{N}$ independent of $\tau$ such that
\[
\| P_J f_\tau \|_{H^1} \lesssim \varepsilon + \| f_\tau - f \|_{H^1} + \tau^{\frac{1}{2}} \| g_\tau \|_{L^2}
\]
since $\| \chi_{>2^J} f \|_{H^1} < \varepsilon$ for large $J$. We obtain the statement for small $T$. Repeating this argument, we obtain the $H^1$-convergence of the solution. At last, we show $\tau^{1/2} \| \partial_t u_\tau \|_{\ell^0[0,T]} \to 0$ as $\tau \to 0$. First we show that
\[
\tau^{\frac{1}{2}} \| \chi_{\leq R} \partial_t u_\tau \|_{\ell^0[0,T]} \to 0
\]
as $\tau \to 0$ for arbitrary fixed $R > 0$. By the Strichartz estimates, we have
\[
\tau^{\frac{1}{2}} \left\| \left( \frac{1}{\tau} \partial_t D_\tau + \partial^2_\tau D_\tau \right) \chi_{\leq R} f_\tau \right\|_{\ell^0[0,T]} \lesssim \tau^{\frac{1}{2}} R \| f_\tau - f \|_{H^1} + \tau^{\frac{1}{2}} \left\| \left( \frac{1}{\tau} \partial_t D_\tau + \partial^2_\tau D_\tau \right) \chi_{\leq R} f \right\|_{\ell^0[0,T]} \lesssim \tau^{\frac{1}{2}} R \| f_\tau - f \|_{H^1} + \tau^{\frac{1}{2}} R \| f \|_{H^1}
\]
and
\[
\| \partial_t D_\tau \chi_{\leq R} g_\tau \|_{\ell^0[0,T]} \lesssim \tau^{\frac{1}{2}} \| g_\tau \|_{L^2}.
\]
We also have the following estimate for the inhomogeneous term.
\[
\left\| \int_0^t \frac{1}{\tau} \partial_t D_\tau (t-s) \chi_{\leq R} \mathcal{N}(u_\tau(s)) ds \right\|_{L^1_T L^2_x} \lesssim \int_0^t \| \xi^2 e^{(t-s)\lambda_\tau^2} \chi_{\leq R} \mathcal{N}(u_\tau(s)) \|_{L^2} ds + \int_0^t \| \tau^{-1} e^{(t-s)\lambda_\tau^2} \chi_{\leq R} \mathcal{N}(u_\tau(s)) \|_{L^2} ds
\]
The second term of the right hand side is calculated as follows.
\[
\int_0^t \| \tau^{-1} e^{(t-s)\lambda_\tau^2} \chi_{\leq R} \mathcal{N}(u_\tau(s)) \|_{L^2} ds \lesssim \tau^{-1} \int_0^t e^{-\frac{t-s}{\tau}} ds \| \chi_{\leq R} \mathcal{N}(u_\tau) \|_{L^p_T L^2_x} \lesssim \| \chi_{\leq R} \mathcal{N}(u_\tau) \|_{L^p_T L^2_x}
\]
By the Bernstein inequality and the Sobolev inequality, we have
\[
\| \chi_{\leq R} \mathcal{N}(u_\tau(s)) \|_{L^2} \lesssim R^d \left( \frac{1}{\tau_0} - \frac{1}{2} \right) \| \chi_{\leq R} \mathcal{N}(u_\tau(s)) \|_{L^p_0} \lesssim R^d \left( \frac{1}{\tau_0} - \frac{1}{2} \right) \| u_\tau \|_{L^{\infty}_0} \lesssim R^d \left( \frac{1}{\tau_0} - \frac{1}{2} \right) \| u_\tau \|_{H^1}^p
\]
where we set
\[
\frac{1}{\tau_0} = \begin{cases} \frac{1}{2} & \text{if } p \leq \frac{d}{\sigma - 2}, \\ \frac{(d-2)p}{2d} & \text{if } p > \frac{d}{\sigma - 2}. \end{cases}
\]
Since the first term can be estimated by $|\xi|^2 \leq R^2$ and the similar argument, we obtain
\[
\tau^{\frac{1}{2}} \left\| \int_0^t \frac{1}{\tau} \partial_t D_\tau (t-s) \chi_{\leq R} \mathcal{N}(u_\tau(s)) ds \right\|_{L^1_T L^2_x} \lesssim \tau^{\frac{1}{2}} \left( R^2 T + R^d \left( \frac{1}{\tau_0} - \frac{1}{2} \right) \right) \| u_\tau \|_{H^1}^p.
\]
Combining the above estimates, we obtain $\tau^{1/2} \| \chi_{\leq R} \partial_t u_\tau \|_{\ell^0[0,T]} \to 0$ as $\tau \to 0$. 
Next, in order to find $R_x > 0$ independent of $\tau$ such that $\tau^{1/2}\|\chi_{R_x} \partial_t u_\tau\eps|_{t=0} < \varepsilon + \|f_\tau - f\|_H^1 + \tau^{1/2}\|g_\tau\|$, it is enough to do the similar argument as above for $\|\chi_{R_x} u_\tau\|_\delta$. We finish the proof. \hfill $\square$

2.7. The case of $d = 1, 2$. We give the proofs in the cases of $d = 1, 2$.

2.7.1. The case of $d = 1$. By the Sobolev embedding $L^\infty \supset H^1$, we have

$$
\|u_\tau\|_{L^\infty H^1} \lesssim \|f_\tau\|_{H^1} + \tau^{1/2}\|g_\tau\|_{L^2} + T\|N(u_\tau)\|_{L^\infty H^1},
$$

where $1/\tilde{q} = 1/2 - 1/\tilde{\gamma}$, $2 < \tilde{\gamma} < 2/(2 - p)$ if $1 < p < 2$, and $2 < \tilde{\gamma} < \infty$ if $p \geq 2$.

Then, by the Hölder inequality, we have

$$
\|N(u_\tau)\|_{L^p B^1_{\tilde{\gamma}, 2}} \lesssim \|u_\tau\|_{L^p L^{p/(p-1)} H^{(p-1)/2}}^{p-1}\|u_\tau\|_{L^\infty H^1},
$$

where $1/r = 1/2 - 1/\tilde{\gamma}$. Since $2 < \tilde{\gamma} < 2/(2 - p)$ if $1 < p < 2$, and $2 < \tilde{\gamma} < \infty$ if $p \geq 2$, we have $r \in (2, \infty)$. Thus, by the Sobolev embedding $L^r \supset H^1$, we obtain

$$
\|N(u_\tau)\|_{L^p B^1_{\tilde{\gamma}, 2}} \lesssim \|u_\tau\|_{L^p L^{p/(p-1)} H^{(p-1)/2}}^{p-1}\|u_\tau\|_{L^\infty H^1} \lesssim T^{1-n\tilde{\gamma}}\|u_\tau\|_{L^\infty H^1},
$$

where $1 - 1/\tilde{q} > 0$. Therefore, we obtain the uniform boundedness. If $T$ is sufficiently small, we obtain the uniform boundedness. For the estimate of the difference, we use $L^2$-norm. The $L^2$-convergence and $H^1$-convergence can be shown in the same method as in the case of $d = 1$.

3. Global $\dot{H}^1$-convergence

In this section, we prove Theorems 1.4 and 1.5.

In the case of $d \geq 3$, we set

$$
\mathcal{V}_\eta := [\mathcal{H}^0_2, \mathcal{H}^1_\eta]\mathcal{H}^0_2, \mathcal{H}^1_1
$$

$$
\mathcal{Z} := \mathcal{Z}_1 = \mathcal{H}^2_2 \cap \mathcal{D}_1
$$

where $p = \eta p_1 + (1 - \eta)p_0$, $p_0 := 1 + 4/d$, and $p_1 := 1 + 4/(d - 2)$.

In the case of $d = 1, 2$, we use

$$
\mathcal{V} := L^4_{t,x} \frac{(d+2)(p-1)}{2}, \quad \mathcal{W} := \mathcal{H}^2_2 \mathcal{D}_1
$$

instead of $\mathcal{V}_\eta$ and $\mathcal{Z}$. 


Proof of Theorem 1.4. First, we consider the case of $d \geq 3$. Assume that $v$ is global and decay to 0, that is, for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that
\[ \|v(t)\|_{H^1} \leq \varepsilon \]
for any $t \geq T_\varepsilon$.

We set
\[ A^1_\varepsilon(t)(f_\tau, g_\tau) := \left( \frac{1}{\tau} D_\tau(t) + \partial_t D_\tau(t) \right) f_\tau + D_\tau(t) g_\tau \]
By the Duhamel formula and Lemma 2.7, we have
\[ \|u_\tau(t) - A^1_\varepsilon(t - T_\varepsilon)(u_\tau(T_\varepsilon), \partial_t u_\tau(T_\varepsilon))\|_{L^2(\mathcal{Y}(T_\varepsilon, T))} \lesssim \int_{T_\varepsilon}^{t} \frac{1}{\tau} |D_\tau(t - s) N(u_\tau(s))| ds \lesssim \|u_\tau\|_{L^2(\mathcal{Y}(T_\varepsilon, T))} \lesssim \|u_\tau\|_{L^2(\mathcal{Y}(T_\varepsilon, T))}, \]
for $T > T_\varepsilon$. Now, we also have
\[ \|A^1_\varepsilon(t - T_\varepsilon)(u_\tau(T_\varepsilon), \partial_t u_\tau(T_\varepsilon))\|_{L^2(\mathcal{Y}(T_\varepsilon, T))} \lesssim \|u_\tau(T_\varepsilon) - v(T_\varepsilon)\|_{H^1} + \|v(T_\varepsilon)\|_{H^1} + \tau^{\frac{1}{2}} \|\partial_t u_\tau(T_\varepsilon)\|_{L^2}. \]
Therefore, if $\tau$ is sufficiently small, we obtain
\[ (3.1) \quad \|u_\tau\|_{L^2(\mathcal{Y}(T_\varepsilon, T))} \lesssim \varepsilon + \|u_\tau\|_{L^2(\mathcal{Y}(T_\varepsilon, T))} \lesssim \varepsilon. \]
By (3.1), (3.2), and the bootstrap argument, we have
\[ (3.3) \quad \|u_\tau\|_{L^2(\mathcal{Y}(T_\varepsilon, \infty))} \lesssim \varepsilon \]
for any small $\tau$. By the Strichartz estimate, we obtain
\[ \|u_\tau\|_{L^\infty(T_\varepsilon, \infty; H^1)} \lesssim \|u_\tau(T_\varepsilon) - v(T_\varepsilon)\|_{H^1} + \|v(T_\varepsilon)\|_{H^1} + \tau^{\frac{1}{2}} \|\partial_t u_\tau(T_\varepsilon)\|_{L^2} + \|u_\tau\|_{L^2(\mathcal{Y}(T_\varepsilon, \infty))} \lesssim \varepsilon. \]
This shows that
\[ \|u_\tau - v\|_{L^\infty(0, \infty; H^1)} \leq \|u_\tau - v\|_{L^\infty(0, T_\varepsilon; H^1)} + \|u_\tau - v\|_{L^\infty(T_\varepsilon, \infty; H^1)} \lesssim \|u_\tau - v\|_{L^\infty(0, T_\varepsilon; H^1)} + \varepsilon \]
for small $\tau$. Thus, we have
\[ \lim_{\tau \to 0} \|u_\tau - v\|_{L^\infty(0, \infty; H^1)} = 0 \]
by Theorem 1.3.

In the case of $d = 1, 2$, by the Strichartz estimates and the nonlinear estimate in Lemma 2.7, we have
\[ \|A^1_\varepsilon(t)(f, g)\|_{L^2(\mathcal{Y}(T_\varepsilon, T))} \lesssim \|f\|_{H^1} + \tau^{\frac{1}{2}} \|g\|_{L^2}. \]
and
\[
\left\| \int_T^t \frac{1}{\tau} D_\tau(t-s)N(u_\tau(s))ds \right\| \lesssim \|N(u_\tau)\|_{\mathcal{H}_0 + \mathcal{H}_1} \lesssim \|u_\tau\|_{\mathcal{H}(T,\infty)}^{p-1} \|u_\tau\|_{\mathcal{H}(T,\infty)}
\]
since we have
\[
\mathcal{H}_0 = \mathcal{V}^{p-1} \mathcal{H}_2, \quad \mathcal{H}_1 = \mathcal{V}^{p-1} \mathcal{P}_1.
\]
Therefore, we obtain the desired statement in the same way as in the higher dimensional case.

We also have the statement for \(\partial t u_\tau\). Indeed, it holds from the Strichartz estimates that
\[
\tau^\frac{1}{2} \|\partial_t u_\tau\|_{L^p_t L^2_x(T_0,\infty)} \lesssim \|u_\tau(T_0) - v(T_0)\|_{H^1} + \|v(T_0)\| + \tau^\frac{1}{2} \|\partial_t u_\tau(T_0)\|_{L^2} + \|u_\tau\|_{p \mathcal{V}(T_0,\infty)}^p \quad \text{if } d \geq 3.
\]
When \(d = 1, 2\), we have the similar estimate. Thus, we obtain the global convergence for \(\partial_t u_\tau\).

**Proof of Theorem 1.6** We show that \(\lim_{t \to -\infty} (t^\frac{1}{2} \|u_\tau(t)\|_{H^1}) = 0\) uniformly in \(\tau\).

We consider the case of \(d \geq 3\). Let \(T > 0\). Since \(t^\frac{1}{2} \lesssim (t-s)^\frac{1}{2} + (s-T)^\frac{1}{2}\) for \(t \geq s \geq T\), we have
\[
\left\| (t-T)^\frac{1}{2} u_\tau \right\|_{\mathcal{V}(T,\infty)} \lesssim \|u_\tau(T)\|_{L^2} + \tau^\frac{1}{2} \|\partial_t u_\tau(T)\|_{L^2} + \left\| \int_T^t \frac{1}{\tau} (t-s)^\frac{1}{2} D_\tau(t-s)N(u_\tau(s))ds \right\|_{\mathcal{V}} + \left\| \int_T^t \frac{1}{\tau} D_\tau(t-s)(s-T)^\frac{1}{2} \mathcal{N}(u_\tau(s))ds \right\|_{\mathcal{V}} \lesssim \varepsilon + \|N(u_\tau)\|_{\mathcal{V}_0 \mathcal{V}(T,\infty)} + \|u_\tau\|_{\mathcal{V}(T,\infty)}\]
\[
\lesssim \varepsilon + \|u_\tau\|_{p \mathcal{V}(T,\infty)} \|u_\tau\|_{\mathcal{V}(T,\infty)} + \|u_\tau\|_{p \mathcal{V}(T,\infty)} \left\| (t-T)^\frac{1}{2} u_\tau \right\|_{\mathcal{V}(T,\infty)}.
\]
Note that we use the fact that the spatial derivative implies the time decay \(t^{-1/2}\) in the linear part of the first and second inequalities (see [5] for example). Combining (3.3) with the above inequality, we obtain
\[
\left\| (t-T)^\frac{1}{2} u_\tau \right\|_{\mathcal{V}(T,\infty)} \lesssim \varepsilon
\]
for large \(T\) independent of \(\tau\). By the Strichartz estimate, we have
\[
(t-T)^\frac{1}{2} \|u_\tau(t)\|_{\dot{H}^1} \lesssim \|u_\tau(T)\|_{L^2} + \tau^\frac{1}{2} \|\partial_t u_\tau(T)\|_{L^2} + \|u_\tau\|_{p \mathcal{V}(T,\infty)} \|u_\tau\|_{\dot{H}^1} \lesssim \varepsilon.
\]
Thus, it holds that
\[
t^\frac{1}{2} \|u_\tau(t)\|_{\dot{H}^1} \lesssim T^\frac{1}{2} \|u_\tau(t)\|_{\dot{H}^1} + (t-T)^\frac{1}{2} \|u_\tau(t)\|_{\dot{H}^1} \lesssim T^\frac{1}{2} \|u_\tau(t)\|_{\dot{H}^1} + \varepsilon
\]
for \( t > T \). Since \( \lim_{t \to \infty} \| u_\tau(t) \|_{H^1} = 0 \) uniformly in \( \tau \), we obtain the desired decay. The convergence for \( \tau \) immediately follows from the same decay estimate of the solution \( v \) to (NLH).

In the case of \( d = 1, 2 \), by using the function spaces \( \mathcal{V} \) and \( \mathcal{V}_\eta \) instead of \( \mathcal{Y}_\eta \) and \( \mathcal{Z}_\eta \), respectively, we obtain the desired statement in the same way as above. The proof is completed. \( \square \)

**Appendix A. Some Lemmas**

A.1. \( L^\infty L^2 - L^q L^r \) estimate and \( L^q L^r - L^1 L^2 \) estimate.

**Lemma A.1** (\( L^\infty L^2 - L^q L^r \) estimate). Let \( \sigma \geq 0 \), \( 2 \leq \tilde{r} < \infty \), and \( 1 \leq \tilde{q} \leq \infty \). Assume that they satisfy

\[
\frac{d}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) = \frac{1}{\tilde{q}},
\]

Then it holds that

\[
\begin{align*}
\left\| (\nabla)_{d}^\sigma \int_0^t D_1(t-s)\chi_{\leq 1} F(s)ds \right\|_{L^\infty(I: L^2(\mathbb{R}^d))} &\lesssim \| F \|_{L^{\tilde{q}'}(I: L^{\tilde{r}'}(\mathbb{R}^d))}, \\
\left\| \int_0^t \partial_t D_1(t-s)\chi_{\leq 1} F(s)ds \right\|_{L^\infty(I: L^2(\mathbb{R}^d))} &\lesssim \| F \|_{L^{\tilde{q}'}(I: L^{\tilde{r}'}(\mathbb{R}^d))},
\end{align*}
\]

where \( I = [0, T) \) and the implicit constant is independent of \( T \).

**Proof.** For simplicity, we set

\[
I(F) = I(F, \sigma) := (\nabla)^\sigma \int_0^t D(t-s)\chi_{\leq 1} F(s)ds.
\]

We have

\[
\| P_j I(F) \|_{L^2(\mathbb{R}^d)} = \left\| \int_0^t P_j (\nabla)^\sigma D(t-s)\chi_{\leq 1} F(s)ds \right\|_{L^2(\mathbb{R}^d)}
\]

\[
= \left\| \int_0^t P_j (\xi)^\sigma e^{-\frac{t-s}{2}} L(t-s, \xi)\chi_{\leq 1} \tilde{F}(s)ds \right\|_{L^2(\mathbb{R}^d)}.
\]

If \( j \geq 2 \), then \( P_j \chi_{\leq 1} = 0 \). The cases \( j = -1, 0, 1 \) are treated later. When \( j \leq -2 \), we have

\[
P_j L(t-s, \xi) = P_j \frac{\sinh((t-s)\sqrt{1/4-\|\xi\|^2})}{\sqrt{1/4-\|\xi\|^2}}.
\]

Therefore, we have

\[
| P_j (\xi)^\sigma e^{-\frac{t-s}{2}} L(t-s, \xi)\chi_{\leq 1} | \lesssim P_j e^{-\frac{t-s}{2}} |\sinh((t-s)\sqrt{1/4-\|\xi\|^2})|\chi_{\leq 1}
\]

\[
\lesssim P_j e^{-\frac{t-s}{2}} e^{(t-s)\sqrt{1/4-\|\xi\|^2}} \chi_{\leq 1}
\]

\[
\lesssim P_j e^{-2(t-s)\|\xi\|^2} \chi_{\leq 1}
\]

\[
\lesssim P_j e^{-2^{-1}(t-s)2^{2j}} \chi_{\leq 1}.
\]
By the Young inequality, we obtain
\[
\left\| \int_0^t P_j (\xi) \sigma e^{-\frac{t-s}{2}} L(t-s, \xi) \chi_{\leq 1} \tilde{F}(s) ds \right\|_{L^q_{t}(\mathbb{R}^d)} \\
\lesssim \left\| \int_0^t P_j e^{-2^{-1} (t-s) 2^j} \chi_{\leq 1} \tilde{F}(s) ds \right\|_{L^q_{t}(\mathbb{R}^d)} \\
\lesssim \|e^{-2^{-1} 2^j} \|_{L^q_{t}} \| P_j \chi_{\leq 1} \tilde{F} \|_{L^q_{t} L^2_{\tilde{q}}(\mathbb{R}^d)} \\
\lesssim 2^{-2j/\tilde{q}} \| P_j \chi_{\leq 1} F \|_{L^q_{t} L^2_{\tilde{q}}(\mathbb{R}^d)},
\]
where \(1 \leq \tilde{q} \lesssim \infty\).

In the case of \(j = 0, 1\), we have
\[
\left| e^{-\frac{t-s}{2}} P_j L(t-s, \xi) \right| = \left| e^{-\frac{t-s}{2}} P_j \frac{\sin((t-s)\sqrt{\|\xi\|^2 - 1/4})}{\sqrt{\|\xi\|^2 - 1/4}} \right| \lesssim e^{-\frac{t-s}{2}} (t-s) \in L^q_{\tilde{q}}(0, t)
\]
and thus we get
\[
\|P_j I(F)\|_{L^q_{t}(\mathbb{R}^d)} \lesssim \|P_j \chi_{\leq 1} F\|_{L^q_{\tilde{q}} L^2_{\tilde{q}}(\mathbb{R}^d)}
\]
for \(j = 0, 1\).

In the case of \(j = -1\), we have
\[
|P_{-1} e^{-\frac{t-s}{2}} L(t-s, \xi) \chi_{\leq 1}| \\
= \begin{cases} 
|P_{-1} e^{-\frac{t-s}{2}} \frac{\sinh((t-s)\sqrt{1/4 - |\xi|^2})}{\sqrt{1/4 - |\xi|^2}} \chi_{\leq 1}| & \lesssim e^{-\frac{t-s}{2}} \in L^q_{\tilde{q}}(0, t) \quad \text{if } 1/4 < |\xi| < 1/2, \\
|P_{-1} e^{-\frac{t-s}{2}} \frac{\sin((t-s)\sqrt{|\xi|^2 - 1/4})}{\sqrt{|\xi|^2 - 1/4}} \chi_{\leq 1}| & \lesssim e^{-\frac{t-s}{2}} \in L^q_{\tilde{q}}(0, t) \quad \text{if } |\xi| > 1/2.
\end{cases}
\]
Thus, we get
\[
\|P_{-1} I(F)\|_{L^q_{t}(\mathbb{R}^d)} \lesssim \|P_{-1} \chi_{\leq 1} F\|_{L^q_{\tilde{q}} L^2_{\tilde{q}}(\mathbb{R}^d)}
\]

We combine the above estimates. Since \(L^2 \approx \tilde{B}^0_{2,2}\), it holds that
\[
\|I(F)\|_{L^2(\mathbb{R}^d)} \approx \left( \sum_{j \in \mathbb{Z}} \|P_j I(F)\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} \\
\lesssim \left( \sum_{j \in \mathbb{Z}} (2^{-2j/\tilde{q}} \|P_j \chi_{\leq 1} F\|_{L^q_{t} L^2_{\tilde{q}}(\mathbb{R}^d)})^2 \right)^{1/2} \\
\approx \left\| \|2^{-2j/\tilde{q}} \|P_j \chi_{\leq 1} F\|_{L^2(\mathbb{R}^d)}\|_{L^q_{t}} \right\|_{L^2_{\tilde{q}}},
\]
Since \(\tilde{q}' \leq 2\), by the Minkowskii integral inequality, we get
\[
\left\| \|2^{-2j/\tilde{q}} \|P_j \chi_{\leq 1} F\|_{L^2(\mathbb{R}^d)}\|_{L^q_{t}} \right\|_{L^2_{\tilde{q}}} \lesssim \left\| \|2^{-2j/\tilde{q}} \|P_j \chi_{\leq 1} F\|_{L^2(\mathbb{R}^d)}\|_{L^2_{\tilde{q}}} \right\|_{L^q_{\tilde{q}}} \\
\approx \|\chi_{\leq 1} F\|_{\tilde{B}_{2,2}^{-2j/\tilde{q}}},
\]
The Sobolev inequality $\dot{B}^{-2/\tilde{q}}_{r,2} \supset \dot{B}^0_{r,2}$, where $\frac{2}{r} \left( \frac{1}{\tilde{q}} - \frac{1}{r} \right) = \frac{1}{q}$, and $\dot{B}^0_{r',2} \supset L^{r'}$ (since $\tilde{r}' \leq 2$) imply that

$$\left\| \chi_{\leq 1} F \right\|_{\dot{B}^{-2/\tilde{q}}_{r,2}} \lesssim \left\| \chi_{\leq 1} F \right\|_{L^{\tilde{q}}_t B^0_{r',2}}.$$

\[ \square \]

**Lemma A.2** ($L^q L^{r'} - L^1 L^2$ estimate). Let $2 \leq r < \infty$ and $1 \leq q \leq \infty$. Assume that they satisfy

$$\frac{d}{2} \left( \frac{1}{2} - \frac{1}{r} \right) = \frac{1}{q},$$

Then it holds that

$$\left\| \int_0^t D_1(t-s) \chi_{\leq 1} F(s) ds \right\|_{L^q(I; L^{r'}(\mathbb{R}^d))} \lesssim \left\| F \right\|_{L^1(I; L^2(\mathbb{R}^d))},$$

and

$$\left\| \int_0^t \partial_t D_1(t-s) \chi_{\leq 1} F(s) ds \right\|_{L^q(I; L^{r'}(\mathbb{R}^d))} \lesssim \left\| F \right\|_{L^1(I; L^2(\mathbb{R}^d))},$$

where $I = [0, T)$ and the implicit constant is independent of $T$.

**Proof.** This follows from $L^\infty L^2$-$L^q L^{r'}$ estimate and the duality argument. \[ \square \]

A.2. The Strichartz estimates for the high frequency part in the 1-d.

We have the following Strichartz estimates for the high frequency part in the one dimensional case.

**Lemma A.3** (Homogeneous Strichartz estimates for the high frequency part in 1-d). Let $d = 1$. Let $s \in \mathbb{R}$, $q \in [2, \infty]$, and $r \in [2, \infty]$. Then, we have the following.

$$\left\| D_1(t) \chi_{> 1} f \right\|_{L^q_t B^s_{r,2}(I)} \lesssim \left\| \nabla |^{\gamma(q,r) - 1} \chi_{> 1} f \right\|_{B^s_{r,2}},$$

and

$$\left\| \partial_t D_1(t) \chi_{> 1} f \right\|_{L^q_t B^s_{r,2}(I)} \lesssim \left\| \nabla |^{\gamma(q,r)} \chi_{> 1} f \right\|_{B^s_{r,2}},$$

where $\gamma(q,r) = 1/2 - 1/r$.

**Proof.** It is enough to consider

$$e^{-t/2} e^{\pm it \sqrt{-\Delta - 1/4}} \chi_{> 1}.$$

As in [6], we have

$$\left\| e^{-t/2} e^{it \sqrt{-\Delta - 1/4}} \chi_{> 1} P_j f \right\|_{L^q(I; L^{r'}(\mathbb{R}^d))} = \left\| e^{-t/2} \left\| e^{it \sqrt{-\Delta - 1/4}} \chi_{> 1} P_j f \right\|_{L^{r'}(\mathbb{R})} \right\|_{L^q(I)} \lesssim \left\| e^{-t/4} \left\| e^{it |\nabla|} \chi_{> 1} P_j f \right\|_{L^{r'}(\mathbb{R})} \right\|_{L^q(I)} \lesssim \left\| e^{it |\nabla|} \chi_{> 1} P_j f \right\|_{L^\infty(I; L^{r'}(\mathbb{R}^d))}.$$
by the Mihlin–Hörmander multiplier theorem and the Hölder inequality. By the Bernstein inequality and the unitarity of the wave propagator, we obtain
\[ \|e^{it\Delta}P_j f\|_{L^\infty(I:L^r(\mathbb{R}^d))} \lesssim 2^{j(\frac{d}{r}-\frac{1}{2})}\|e^{it\chi_{>j}P_j f}\|_{L^\infty(I:L^2(\mathbb{R}))}, \]
Taking the $l^2$-norm for $j$, it holds that
\[ \|\sum_{j} e^{-it/2}e^{it\sqrt{-\Delta^{1/4}}}\chi_{>j} f\|_{L^{n}(I:B_{r,2}^0(\mathbb{R}))} \lesssim \|\nabla|\gamma|f\|_{L^2}. \]
This estimate implies the estimate for $D_1$ and $\partial_1 D_1$. □

**Lemma A.4** (Inhomogeneous estimate for the high frequency in 1-d). Let $d = 1$. Let $s \in \mathbb{R}$, $q, \tilde{q} \in [2, \infty]$, and $r, \tilde{r} \in [2, \infty]$. Then, we have the following.
\[ \left\| \int_0^t D_1(t-s)\chi_{>1}F(s)ds \right\|_{L^q_tB^s_{r,2}(\mathbb{R})} \lesssim \left\| \nabla|\gamma(q,r)+\gamma(\tilde{q},\tilde{r})+\delta-1|\chi_{>1}F \right\|_{L^q_tB^s_{r,2}(\mathbb{R})}, \]
where $\gamma(q,r) = 1/2 - 1/r$.

**Proof.** This follows from the argument in [5] as $d = 1$. □

By scaling these estimate, we obtain Lemmas [2.2] and [2.4] in the case of $d = 1$.

### A.3. Completeness of $X|Y$.
We revisit the completeness of $X|Y$ for the reader's convenience. Let
\[ X|Y := \{ u : \|\chi_{\leq 1}(\nabla)u\|_X + \|\chi_{>1}(\nabla)u\|_Y < \infty \} \]
\[ \|u\|_{X|Y} := \|\chi_{\leq 1}(\nabla)u\|_X + \|\chi_{>1}(\nabla)u\|_Y \]
where $X$ and $Y$ are Banach spaces satisfying $\|\chi_{\leq 1}(\nabla)u\|_Z \leq \|u\|_Z$ and $\|\chi_{>1}(\nabla)u\|_Z \leq \|u\|_Z$ for $Z = X, Y$.

Take a Cauchy sequence $\{u_n\}$ in $X|Y$.

As $m, n \to \infty$, we get
\[ \|\chi_{\leq 1}(\nabla)^2u_n - \chi_{\leq 1}(\nabla)^2u_m\|_X \leq \|\chi_{\leq 1}(\nabla)u_n - \chi_{\leq 1}(\nabla)u_m\|_X \to 0 \]
\[ \|\chi_{\leq 1}(\nabla)\chi_{>1}(\nabla)u_n - \chi_{\leq 1}(\nabla)\chi_{>1}(\nabla)u_m\|_X \leq \|\chi_{\leq 1}(\nabla)u_n - \chi_{\leq 1}(\nabla)u_m\|_X \to 0 \]

By the completeness of $X$, there exist $U^1$ and $U^2$ such that
\[ \chi_{\leq 1}(\nabla)^2u_n \to U^1 \text{ in } X \quad (A.1) \]
\[ \chi_{\leq 1}(\nabla)\chi_{>1}(\nabla)u_n \to U^2 \text{ in } X \quad (A.2) \]
We also have
\[ \|\chi_{\leq 1}(\nabla)u_n - (U^1 + U^2)\|_X = \|\chi_{\leq 1}(\nabla)(\chi_{\leq 1}(\nabla)u_n + \chi_{>1}(\nabla)u_n) - (U^1 + U^2)\|_X \]
\[ \leq \|\chi_{\leq 1}(\nabla)^2u_n - U^1\|_X + \|\chi_{\leq 1}(\nabla)\chi_{>1}(\nabla)u_n - U^2\|_X \to 0 \]

Therefore, we have
\[ \chi_{\leq 1}(\nabla)u_n \to U^1 + U^2 \text{ in } X \quad (A.3) \]
Now, by (A.1)–(A.3), we obtain

\[
\begin{align*}
\chi_{\leq 1}(\nabla)(U^1 + U^2) &= U^1 \\
\chi_{> 1}(\nabla)(U^1 + U^2) &= U^2
\end{align*}
\]

The similar argument works in \( Y \) for the high frequency part. There exist \( V^1, V^2 \) such that

(A.4) \( \chi_{> 1}(\nabla)^2 u_n \to V^1 \) in \( Y \)

(A.5) \( \chi_{\leq 1}(\nabla)\chi_{> 1}(\nabla)u_n \to V^2 \) in \( Y \)

and

(A.6) \( \chi_{> 1}(\nabla)u_n \to V^1 + V^2 \) in \( Y \)

and thus

\[
\begin{align*}
\chi_{> 1}(\nabla)(V^1 + V^2) &= V^1 \\
\chi_{\leq 1}(\nabla)(V^1 + V^2) &= V^2
\end{align*}
\]

Noting (A.2) and (A.5), \( \chi_{\leq 1}(\nabla)\chi_{> 1}(\nabla)u_n \) converges in both \( X \) and \( Y \), and its limits in \( X \) and \( Y \) are \( U^2 \) and \( V^2 \), respectively.

We get

(3) \( U^2 = V^2 \).

**Proof of (3).** In general, let \( f_n \to F \) in \( X \) and \( f_n \to G \) in \( Y \). Then

\[
\|F - G\|_{X+Y} = \|F - f_n + f_n - G\|_{X+Y} \\
\leq \|F - f_n\|_{X+Y} + \|f_n - G\|_{X+Y} \\
\leq \|F - f_n\|_X + \|f_n - G\|_Y \\
\to 0
\]

by the assumption. Therefore, \( F = G \). \( \Box \)

Set \( u = U^1 + U^2 + V^1 + V^2 \). Then, we will show

\[\|u_n - u\|_{X|Y} \to 0.\]

Now, it follows from (1), (2), and (3) that

\[
\|\chi_{\leq 1}(\nabla)u_n - \chi_{\leq 1}(\nabla)u\|_X = \|\chi_{\leq 1}(\nabla)u_n - \chi_{\leq 1}(\nabla)(U^1 + U^2 + V^1 + V^2)\|_X \\
= \|\chi_{\leq 1}(\nabla)u_n - (U^1 + V^2)\|_X \\
= \|\chi_{\leq 1}(\nabla)u_n - (U^1 + U^2)\|_X
\]

By (A.3), the last term goes to 0 as \( n \to \infty \). The same argument works in \( Y \) for the high frequency term. Thus, we get \( \|u_n - u\|_{X|Y} \to 0 \).

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

Email address: inui@math.sci.osaka-u.ac.jp

Department of Mathematics, Faculty of Science, Saitama University, 255 Shimo-Okubo, Sakura-ku, Saitama City 338-8570, Japan

Email address: machihara@rimath.saitama-u.ac.jp