Exact Solution of Dirac Equation for q-Deformed Trigonometric Scarf potential with q-Deformed Trigonometric Tensor Coupling Potential for Spin and Pseudospin Symmetries Using Romanovski Polynomial

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Abstract. The bound state solutions of Dirac equation for q-deformed trigonometric Scarf potential with q-deformed trigonometric cotangent and cosecant tensor coupling potential under spin and pseudospin symmetric limits are investigated using Supersymmetric Quantum Mechanics (SUSY QM) method. The new tensor potentials proposed is inspired by superpotential form in SUSY quantum mechanics. The Dirac equations for Scarf potential coupled by new tensor potential in the pseudospin and spin symmetric cases reduce to Schrodinger type equations for shape invariant potential since the proposed new potentials are equivalent to the superpotential of q-deformed trigonometric Scarf potential. The relativistic wave functions are exactly obtained by using SUSY operator method and the relativistic energy equation are exactly obtained by using SUSY method and the idea of shape invariance in the approximation scheme of centrifugal term. The new tensor potential causes the energy degeneracies is omit both for pseudospin and spin symmetric case.
1. Introduction

The exact solutions of Dirac equations play important roles in relativistic quantum mechanics since they provide all important information of the system under consideration. Dirac equation for central potentials coupled with Yukawa-type and Coulomb-type tensor potentials and its modification have been solved exactly and applied in quantum chemistry and high energy physics. To describe the motion of spin half particles, some authors have applied various solution methods such as Nikiforov-Uvarov (NU) method [1-5], factorization methods and SUSY QM [6], hypergeometric and confluent hypergeometric method [7-8], and asymptotic iteration method [9]. Dirac equation for central potential such as Hulthen potential, Yukawa potential, attractive exponential potential, Poschl-Teller potential, etc. together with new tensor coupling potential have exact solution only for spin orbit quantum number, $\kappa$ is zero, but for non zero $\kappa$ the approximation scheme of centrifugal term has to be taken account to give exact solution [1, 2, 10-15]. The approximation scheme of the centrifugal term was proposed by Greene and Aldrich [16] and this approximation works well for trigonometric, hyperbolic and exponential potentials. From the observation, the expression of the tensor coupling potentials under the approximation scheme for centrifugal term are similar to the expression of the corresponding potential.

The new tensor coupling potentials proposed is inspired by the SUSY algebraic structure for Dirac equation under spin and pseudospin limit for certain potentials such as trigonometric potential, in particular. For these potentials, it is proposed that the tensor coupling potentials are its superpotential. However, these new tensor coupling potentials are so specific therefore can not be widely used as Coulomb-type and Yukawa-type tensor potentials. But it is worthy to be explored to provide new description of the motion of spin-half particles.

The Dirac equations for some potentials have been solved in the cases of spin symmetry and pseudospin symmetry [5, 8, 17-25]. The spin symmetry occurs when the different between repulsive vector potential with the attractive scalar potential is equal to constant, while the pseudospin symmetry arises when the sum of the scalar potential with vector potential is equal to constant. Spin symmetric and pseudospin symmetric concepts have been used to study the aspect of deformed and superdeformation nuclei in nuclear physics. The concept of spin symmetry has been applied to the spectrum of meson and antinucleon [26], and the pseudospin symmetric concept is used to explain the quasi degeneracy of the nucleon doublets [27], exotic nuclei [28], super-deformation in nuclei [29], and to establish an affective nuclear shell-model scheme [30].

In this paper, we propose new tensor coupling potential as a function of trigonometric and hyperbolic terms expressed as

$$U(r) = -a(V_2 \cot \alpha + V_3 \csc \alpha)$$

The negative q-deformed trigonometric cotangent plus cosecant tensor potential is a little more negative than Coulomb-like tensor potential. Screened Coulomb potential was originally used to model strong nuleon-nucleon interactions caused by the exchange in nuclear physics[31-33]. The Dirac equation with this new tensor coupling potential can be solved exactly under approximation scheme of centrifugal term only for q-deformed trigonometric Scarf potential since the combination of tensor potential together with centrifugal term and q-deformed trigonometric Scarf potential resulting shape invariant potential in the Dirac equation. The relativistic energy and wave functions of this new tensor potential together with Scarf potential plus centrifugal term are analyzed using Romanovski polynomials. The trigonometric Scarf potential is potential model used to explain strong and electromagnetic interactions.
Finite Romanovski polynomial is a traditional method which consists of reducing the Schrodinger equation by an appropriate variable substitution to a form of a generalized hypergeometric equation \[34\]. The polynomial was discovered by Sir E. J. Routh [35] and rediscovered 45 years later by V. I. Romanovski[36]. The notion “finite” refers to the observation that, for any given set of parameters (i.e. in any potential) only a finite number of polynomials appear orthogonal \[37-39\]. From the observation only few researchers used Romanovski polynomials to analyze energy spectra and wave functions for certain potentials \[37, 39, 40-42\].

This paper is organized as follows. Basic theory of Dirac equation is presented in section 2, and the Romanovski polynomial method as an analysis method is presented in section 3. Section 4 presents the research results and discussion, and finally brief conclusions and acknowledgement are presented in section 5.

2. Basic Theory

2.1. Basic Equations of Dirac Spinors

The motion of a nucleon with mass \(M\) in a repulsive vector potential \(V_v(r)\) and an attractive scalar potential \(V_s(r)\) and also coupled by a tensor potential \(U(r)\) is described by the Dirac equation given as \[1, 2, 5, 8, 20, 22, 32 \]

\[\{ \alpha \cdot \mathbf{p} + \beta (M + V_s(r)) - i \beta \alpha \cdot \mathbf{r} U(r) \} \psi(r) = \{ E - V_v(r) \} \psi(r) \] \tag{2}

where \(E\) is the relativistic energy and \(\mathbf{p}\) is the three-dimensional momentum operator, \(-i\nabla\).

\[\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \] \tag{3}

and

\[\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \] \tag{4}

with \(\sigma\) is a three-dimensional Pauli matrix, \(I\) is a 2\(\times\)2 identity matrix. Here we consider the matrix potential in equation (1) as spherically symmetric potential, they do not only depend on the radial coordinate \(r = |r|\) and we have taken \(\hbar = 1\), \(c = 1\). The Dirac equation expressed in equation (3) is invariant under spatial inversion, and therefore its eigenstates have definite parity. By writing the spinor as

\[\psi(r) = \begin{pmatrix} \zeta(r) \\ \varphi(r) \end{pmatrix} = \begin{pmatrix} F_{nk}(r) Y_{jm}^l(\theta, \phi) \\ i G_{nk}(r) Y_{jm}^{l*}(\theta, \phi) \end{pmatrix} \] \tag{5}

where \(\zeta(r)\) is Dirac spinor of upper (large) component and \(\varphi(r)\) is Dirac spinor of lower (small) component, \(Y_{jm}^l(\theta, \phi)\) is spin spherical harmonics, \(Y_{jm}^{l*}(\theta, \phi)\) is pseudospin spherical harmonics, \(l\) is orbital quantum number, \(\bar{l}\) pseudo orbital quantum number, and \(m\) is the projection of the angular momentum on the z-axis. The Dirac Hamiltonian in a spherical field commutes with total angular momentum operator \(\hat{J}\) and spin orbit coupling operator \(\hat{K}\) with its spin orbit quantum number \(K = -\beta(\sigma \cdot \hat{L} + 1)\). \(\hat{L}\) is the usual orbital angular momentum. The eigenvalues of the spin orbit coupling operator are \(\kappa = (J + 1/2) > 0\) for unaligned spin \((p_{1/2}, d_{3/2}, ...)\) and...
\( \kappa = -(J + 1/2) < 0 \) for aligned spin \((s_{1/2}, p_{3/2}, \ldots)\). Therefore the conservative quantities are consisting of set of \( H, K, J, J_z \).

By inserting equations (5), and (4), into equation (1) we have

\[
\left\{ \begin{array}{c}
0 \
\sigma
\end{array} \right\} p \left( \begin{array}{c}
\zeta(r) \\
\varphi(r)
\end{array} \right) + \left( \begin{array}{c}
\I \\
\I
\end{array} \right) \left( M + V_S(r) \right) \left( \begin{array}{c}
\zeta(r) \\
\varphi(r)
\end{array} \right) - i \beta \alpha \partial U(r) \left( \begin{array}{c}
\zeta(r) \\
\varphi(r)
\end{array} \right) = \left( E - V_V(r) \right) \left( \begin{array}{c}
\zeta(r) \\
\varphi(r)
\end{array} \right)
\]

From equation (6) we obtain the coupled first order differential equation given as

\[
\left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{nk}(r) = \left( M + E_{nk} - V_V(r) + V_S(r) \right) G_{nk}(r)
\]

and

\[
\left( \frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{nk}(r) = \left( M - E_{nk} + V_V(r) + V_S(r) \right) F_{nk}(r)
\]

where \( F_{nk}(r) \) is the upper component of Dirac spinor and \( G_{nk}(r) \) is the lower component of Dirac spinor. From equations (7) and (8) we get the upper and lower radial part of the Dirac equation,

\[
\left\{ \begin{array}{c}
\frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} + \frac{2\kappa}{r} U(r) - U^2(r) - \frac{dU}{dr} + \frac{d}{dr} \left( \frac{d}{dr} - \frac{\kappa}{r} - U(r) \right) \left( M + E_{nk} - \Delta(r) \right) \right\} F_{nk}(r) + \\
\left( M + E_{nk} - \Delta(r) \right) \left( E_{nk} - M - \Sigma(r) \right) F_{nk}(r) = 0
\]

\[
\left\{ \begin{array}{c}
\frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} + \frac{2\kappa}{r} U(r) - U^2(r) + \frac{dU}{dr} + \frac{d}{dr} \left( \frac{d}{dr} + \frac{\kappa}{r} + U(r) \right) \left( M - E_{nk} + \Sigma(r) \right) \right\} G_{nk}(r)
\]

\[
+ \left( M + E_{nk} - \Delta(r) \right) \left( E_{nk} - M - \Sigma(r) \right) G_{nk}(r) = 0
\]

with \( \kappa(\kappa + 1) = l(l + 1) \) for upper spinor component, \( \kappa(\kappa - 1) = \bar{l}(\bar{l} + 1) \) for lower spinor component, \( l \) is the pseudo orbital quantum number, \( \Sigma(r) = V_V(r) + V_S(r) \) is the sum of scalar and vector potentials, and \( \Delta(r) = V_V(r) - V_S(r) \) is the different between vector potential and scalar potential. There are two special cases for Dirac equation, pseudospin symmetric case and spin symmetric case.

Pseudospin symmetry occurs when \( \Sigma(r) = V_V(r) + V_S(r) = C_{ps} \) with \( C_{ps} \) is constant, therefore \( \frac{d\Sigma}{dr} = 0 \) and the different between vector and scalar potentials \( \Delta(r) \) is set to be equal with the given potential,

\[
\Delta(r) = V_V(r) - V_S(r) = V(r)
\]

In the limit of pseudospin symmetry we have the Dirac equation for lower component of Dirac spinor obtained from equation (8) as

\[
\left\{ \begin{array}{c}
\frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} + \frac{2\kappa}{r} U(r) - U^2(r) + \frac{dU}{dr} \left( M - E_{nk} + C_{ps} \right) \right\} G_{nk}(r) = \left( M + E_{nk} \right) \left( M - E_{nk} + C_{ps} \right) G_{nk}(r)
\]

From \( \kappa(\kappa - 1) = \bar{l}(\bar{l} + 1) \) one gets the values of \( \kappa \) as \( \kappa = \bar{l} = -(j + 1/2) \) for \( \kappa < 0 \) which is associated with aligned spin and \( \kappa = (\bar{l} + 1) = j + 1/2, \kappa > 0 \) for unaligned spin. In general the pseudo orbital quantum number is written as \( \bar{l} = l - \kappa \ell \ell | \kappa | \). These conditions imply that the total angular momentum \( j = \bar{l} \pm \frac{1}{2} \) that causes the state to be degenerated for \( \bar{l} \neq 0 \). The Dirac equation
expressed in equation (12) is Schrodinger-like equation and therefore can be solved exactly only if the effective potential,

\[ V_{ef} = \frac{\kappa(\kappa - 1)}{r^2} - \frac{2\kappa}{r} U(r) + U^2(r) - \frac{dU}{dr} V(r) \left( M - E_{ax} + C_{ps} \right) = \varphi^2(r) - \varphi'(r) - V(r) \left( M - E_{ax} + C_{ps} \right) \]

(13)
is shape invariant, with \( \varphi(r) = \frac{U(r)}{r} \). Equation (13) shows that the effective potential is mixture of two potentials, \( V_L = \varphi^2(r) - \varphi'(r) \), and \( V(r) \left( M - E_{ax} + C_{ps} \right) \) with \( V(r) \) is given potential. Therefore equation (13) will be shape invariant potential if in the approximation scheme of the centrifugal term, \( \varphi(r) \) has the form of the superpotential of the potential system \( V(r) \). So for the tensor potential expressed in equations (1), the suitable given potentials in Dirac equation expressed in equation (12) are the trigonometric.

On the other hand, the spin symmetry occurs when the different between vector and scalar potentials is constant, \( \Delta(r) = V_L(r) - V_S(r) = C_s \), and the sum of vector and scalar potentials is expressed as \( \Sigma(r) = V_L(r) + V_S(r) = V(r) \), therefore in the spin symmetric limit we get the upper component of Dirac spinor obtained from equation (9) given as

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} + \frac{2\kappa}{r} U(r) - U^2(r) - \frac{dU}{dr} V(r) (M + E_{ax} - C_s) \right\} F_{ns}(r) = (M + E_{ax} - C_s)(M - E_{ax}) F_{ns}(r)
\]

(14)

with \( \kappa(\kappa + 1) = l(l + 1) \) that leads to \( \kappa = l = j + \frac{1}{2} \), \( \kappa > 0 \) for unaligned spin and \( \kappa = -(l + 1) = -(j + \frac{1}{2}) \), \( \kappa < 0 \) for aligned spin. The Dirac equation expressed in equation (14) is Schrodinger-like equation and therefore can be solved exactly only if the effective potential, \( V_{ef} \),

\[ V_{ef} = \frac{\kappa(\kappa + 1)}{r^2} - \frac{2\kappa}{r} U(r) + U^2(r) + \frac{dU}{dr} V(r) (E_{ax} + M - C_s) = \varphi^2(r) + \varphi'(r) + V(r) (E_{ax} + M - C_s) \]

(15)
is shape invariant, with \( \varphi(r) = \frac{U(r)}{r} \). As in equation (13), equation (15) is the effective potential of equation (14) which is mixture of two potentials, first part is centrifugal term plus tensor potential, \( V_L = \varphi^2(r) + \varphi'(r) \), and the second term, \( V(r) \), is the sum of the attractive and repulsive potential. In the approximation scheme of the centrifugal term the effective potential \( V_{ef} \) is shape invariance if \( \varphi(r) \) has the form of the superpotential of the potential system \( V(r) \).

The energy spectra and the upper component wavefunction of Dirac spinor for the spin symmetric case is obtainable from the energy spectra and the lower component wave function of Dirac spinor for pseudospin symmetric case [39]. By comparing the Dirac equation for pseudospin symmetry in equation (12) and spin symmetry in equation (14) we obtain mapping parameters produced from energy equations for pseudospin and spin symmetries as follows

\[ G_{ns}(r) \rightarrow F_{ns}(r), \quad \Delta(r) \rightarrow -\Xi(r) \text{ or } V_L \rightarrow -V_0, V_L \rightarrow -V_L, \quad E_{axps} \rightarrow -E_{axs}, \quad C_{ps} \rightarrow C_s \text{ and } \kappa \rightarrow \kappa + 1 \]

(16)

Both Dirac equations for pseudospin symmetry and spin symmetry in equations (12) and (14) are solved using SUSY QM method. In addition, by using equation (16) we can also map the energy equation from pseudospin symmetric case into spin symmetric case.

2.2. Review of q-deformed trigonometric function

For spin and pseudospin symmetry the Dirac equation for q-deformed trigonometric Scarf potential within the q-deformed trigonometric cotangent plus cosecant type tensor reduces to Schrodinger-type equation therefore it can be solved using Romanovski polynomials. Cotangent plus cosecant tensor
potential behaves like Coulomb-like tensor potential therefore it can be used to describe the nucleon-nucleon interaction [31-33]. The q-deformed trigonometric Scarf potential and the q-deformed trigonometric cotangent plus cosecant type tensor given as

\[
V(r) = a^2 \left\{ \frac{V_0}{\sin_q^2 ar} - \frac{V_1 \cos_q ar}{\sin_q^2 ar} \right\}
\]

(17)

\[
U(r) = -a \left( V_2 \cot_q ar + V_3 \csc_q ar \right)
\]

(18)

\(V_0\) and \(V_1\) describe the depth of the trigonometric function well potential and are positives, \(V_1 > V_0\), \(a\) is a positive parameter which to control the width or the range of the potential well, \(q\) is the deformation of the potential, \(q > 0\), \(V_2\) and \(V_3\) are the strength of the nucleon forces, \(a\) is the range of nucleon force, \(M\) is the mass of the particle, and \(0 < r < \infty\).

The q-deformed trigonometric function is formulated in the same way with the formulation of q-deformed hyperbolic function introduced by Arai [44] some years ago, in accordingly the q-deformed trigonometric function is defined as the definition of trigonometric function as follows:

\[
\sin_q ar = \frac{e^{iar} - q e^{-iar}}{2}; \quad \cos_q ar = \frac{e^{iar} + q e^{-iar}}{2}; \quad \sin_q^2 ar + \cos_q^2 ar = q
\]

(19)

\[
\tan_q ar = \frac{\sin_q ar}{\cos_q ar}; \quad \sec_q ar = \frac{1}{\cos_q ar}; \quad 1 + \tan_q^2 ar = q \sec_q^2 ar
\]

(20)

\[
\frac{d \sin_q ar}{dr} = a \cos_q ar; \quad \frac{d \tan_q ar}{dr} = qa \sec_q^2 ar
\]

(21)

By a convenient translation of the spatial variable, one can transform the deformed potentials into the corresponding non-deformed ones or vice-versa. In analogy to the translation of spatial variable for hyperbolic function introduced by Dutra [45] we propose the translation of spatial variable for trigonometric function as follows

\[
r \to r + \ln \frac{\sqrt{q}}{ia}, \text{ and } \quad r \to r - \ln \frac{\sqrt{q}}{ia}
\]

(22)

and then by inserting equation (22) into equations (19) and (20) we have

\[
\sin_q ar \to \sqrt{q} \sin ar; \quad \cos_q ar \to \sqrt{q} \cos ar; \quad \sin ar \to \frac{\sin_q ar}{\sqrt{q}}; \quad \cos ar \to \frac{\cos_q ar}{\sqrt{q}}.
\]

(23)

The translation of spatial variable in equation (22) can be used to map the energy and wave function of non-deformed potential toward deformed potential of Scarf potential [46].

3. Method of Analysis

The method used to solve the Dirac equation in the limit of spin symmetric and pseudospin symmetric cases is the Romanovski polynomials since the Dirac equations for limited condition, when spin and pseudospin symmetry arise, reduce to one dimensional Schrodinger-like equation. The one dimensional second order differential equation satisfied by Romanovski polynomials is developed based on hypergeometric differential equation. One dimensional Schrodinger equation of potential of interest reduces to the differential equation of Romanovski polynomial by appropriate variable and wave function substitutions. The one dimensional Schrodinger equation is given as

\[
- \frac{\hbar^2}{2M} \frac{\partial^2 \Psi(x)}{\partial x^2} + V(x) \Psi(x) = E \Psi(x)
\]

(24)
where \( V(x) \) is an effective potential which is mostly shape invariant potential. By suitable variable substitution \( x = f(s) \) equation (24) changes into generalized hypergeometric type equation expressed as

\[
\frac{\partial^2 \Psi(s)}{\partial s^2} + \bar{\sigma}(s) \frac{\partial \Psi(s)}{\partial s} + \bar{\sigma}(s) \Psi(s) = 0
\]  

(25)

with \( \sigma(s) \) and \( \bar{\sigma}(s) \) are mostly polynomials of order two, \( \bar{\sigma}(s) \) is polynomial of order one, \( \sigma(s) \), \( \bar{\sigma}(s) \), and \( \bar{\sigma}(s) \) can have any real or complex values [47]. Equation (25) is solved by variable separation method. By introducing new wave function in equation (25),

\[
\Psi_n(r) = g_n(s) = (1 + s^2)^{\frac{d}{2}} e^{-\frac{a}{2} s \tan^{-1}s} D_n^{(\beta, \alpha)}(s)
\]  

(26)

We obtain a hypergeometric type differential equation, which can be solved using finite Romanovski polynomials [37, 38, 39, 47] is expressed as

\[
\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0
\]  

(27)

with \( \sigma(s) = as^2 + bs + c; \quad \tau = fs + h \) and \( -\{n(n-1) + 2n(1-p)\} = \lambda = \lambda_n \)

and \( y_n = R_n^{(p,q)}(s) = D_n^{(\beta, \alpha)}(s) \)

(28)

(29)

For Romanovski polynomials, the values of parameters in equation (28) are

\[ a = 1, \quad b = 0, \quad c = 1, \quad f = 2(1-p) \quad \text{and} \quad h = q' \quad \text{with} \quad p > 0 \]

(30)

therefore equation (27) is rewritten as

\[
(1 + s^2)^{\frac{d}{2}} \frac{\partial^2 R_n^{(p,q)}}{\partial s^2} + \{2s(-p+1) + q\} \frac{\partial R_n^{(p,q)}}{\partial s} - \{n(n-1) + 2n(1-p)\} R_n^{(p,q)}(s) = 0
\]

(31)

Equation (31) which is obtained from equation (27) by applying the specific condition for Romanovski polynomials expressed in equation (30) is second order differential equation satisfied by Romanovski polynomials. Equation (27) is described in the textbook by Nikiforov-Uvarov [M] where it is cast into self adjoint form and its weight function, \( w(s) \), satisfies Pearson differential equation

\[
\int_{\infty}^{\infty} w(s) y_n(s) y_m(s) ds = \delta_{nm}
\]

(32)

The weight function, \( w(s) \), is obtained by solving the Pearson differential equation expressed in equation (32) and by applying condition in equations (28) and (30), given as

\[
w^{(p,q)}(s) = (1 + s^2)^{-\frac{d}{2}} e^{\frac{a}{2} s \tan^{-1}s}
\]

(33)

The corresponding polynomials are classified according to the weight function, and are built up from the Rodrigues representation which is presented as

\[
y_n = \frac{B_n}{w(s)} \left( (as^2 + bs + c)^n w(s) \right)
\]

(34)

with \( B_n \) is a normalization constant, and for \( \sigma(s) > 0 \) and \( w(s) > 0 \), \( y_n(s) \)'s are normalized polynomials and are orthogonal with respect to the weight function \( w(s) \) within a given interval \( (s_1, s_2) \), which is expressed as

\[
\int_{s_1}^{s_2} w(s) y_n(s) y_m(s) ds = \delta_{nm}
\]

(35)

This weight function in equation (33) first reported by Routh [35] and then by Romanovski [36]. The polynomial associated with equation (31) are named after Romanovski and will be denoted by \( R_n^{(p,q)}(s) \). Due to the decrease of the weight function by \( s^{-2p} \), integral of the type
\[
\int_{-\infty}^{\infty} w^{(p,q)}(s) R_n^{(p,q)}(s) ds
\]  
will be convergent only if \( n + n < 2p - 1 \)  
This means that only a finite number of Romanovski polynomials are orthogonal, and the orthogonality integral of the polynomial is expressed similar to the equation (35) where \( y_n = R_n^{(p,q)}(s) \).

The Romanovski polynomials obtained from Rodrigues formula expressed in equation (34) with the corresponding weight function in equation (33) is expressed as

\[
R_n^{(p,q)}(s) = D_n^{(\beta,\alpha)}(s) = \frac{1}{(1+s^2)^p} e^{q \tan^{-1}(s)} ds^n \left\{ (1+s^2)^p (1+s^2)^{-p} e^{q \tan^{-1}(s)} \right\}
\]  
If the wave function of the nth level in equation (26) is rewritten as

\[
\Psi_n(r) = \frac{1}{\sqrt{d(s)}} (1+s^2)^{-\frac{p}{2}} e^{\frac{q}{2} \tan^{-1}(s)} R_n^{(p,q)}(s)
\]  
then the orthogonality integral of the wave functions expressed in equation (39) gives rise to orthogonality integral of the finite Romanovski polynomials, that is given as

\[
\int_{0}^{\infty} \Psi_n(r)^* \Psi_n(r) dr = \int_{-\infty}^{\infty} w^{(p,q)} R_n^{(p,q)}(s) R_n^{(p,q)}(s) ds
\]  
In this case the values of \( p \) and \( q' \) are not \( n \)-dependence where \( n \) is the degree of polynomials. However, if either equation (35) or (37) is not fulfilled then the Romanovski polynomials is infinity [37].

4. Results and Discussion

4.1. The solution Dirac equation for q-deformed trigonometric Scarf and trigonometric tensor potential solution for pseudospin symmetric case

By inserting equations (17) and (18) into equation (12) we obtain

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa}{r} a \left( V_2 \cot q a r + V_1 \csc q a r \right) - a^2 \left( V_2 \cot q a r + V_1 \csc q a r \right)^2 + a^2 V_2 q \csc^2 q a r \right\} G_m(r) + a^2 V_2 q \csc^2 q a r G_m(r) + a^2 \left( \frac{V_0}{\sin^2 q a r} - \frac{V_1 \cos q a r}{\sin^2 q a r} \right) (M - E_{ex} + C_m) G_m(r) = \left( M + E_{ex} \right) \left( M - E_{ex} + C_m \right) G_m(r)
\]  
Equation (38) can not be solved exactly except we use an approximation to the \( \frac{1}{r^2} \) term. For small \( a \), \( ra << 1 \) the approximation of \( \frac{1}{r^2} \) is given as [18]

\[
\frac{1}{r^2} \approx \frac{a^2}{(\sin^2 q a r)}
\]
Equation (39) is substituted into equation (38) then we get
\[
\left\{ \frac{d^2}{dr^2} - \frac{a^2 \{ \kappa (\kappa - 1) + 2aV_3 + V_z^2 + V_{r, q}^2 q - V_{\phi, r}^2 \} \sin^2 \varphi}{\sin^2 \varphi - ar} \right\} G_{n_k}(r) = \left\{ (M + E_{n_k}) \left( M - E_{n_k} + C_p \right) - a^2 V_{n_k}^2 \right\} G_{n_k}(r) \tag{43}
\]
By setting
\[
A_{n_k} = \left\{ \kappa (\kappa - 1) + 2aV_3 + V_z^2 + V_{r, q}^2 q - V_{\phi, r}^2 \right\}; \quad \gamma_{n_k} = (M - E_{n_k} + C_p) \tag{44}
\]
\[
B_{n_k} = (V_3 - 2aV_z + V_{r, q} q - V_{\phi, r} q); \quad E'_{n_k} = \left\{ (M + E_{n_k}) \left( M - E_{n_k} + C_p \right) - V_{n_k}^2 \right\} \tag{45}
\]
in equation (40) then equation (40) reduces to one dimensional Schrodinger-type equation
\[
\left\{ \frac{d^2}{dr^2} - \frac{a^2 A_{n_k} \cos \varphi}{\sin^2 \varphi - ar} + \frac{a^2 B_{n_k} \cos \varphi}{\sin^2 \varphi - ar} \right\} G_{n_k}(r) = a^2 E'_{n_k} G_{n_k}(r) \tag{46}
\]
To simplify the solution of equation (46) we introduce new variable, \( \cos \varphi = -i \sqrt{q} \) then equation (46) becomes
\[
\left\{ (1 + x^2) \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} + \frac{A_{n_k}}{q(1 + x^2)} - \frac{B_{n_k} \sqrt{q} x}{q(1 + x^2)} + E_{n_k} \right\} G_{n_k}(x) = 0 \tag{47}
\]
According to equation (26) the solution of equation (47) is set to be
\[
G_{n_k}(x) = g_{n_k}(x) = \left( 1 + x^2 \right)^{-\frac{1}{2}} e^{\frac{-\alpha x}{2} + \frac{\alpha q^2}{2}} D_{n_k}^{(\beta, \alpha)}(x) \tag{48}
\]
so from equations (47) and (48) we get
\[
\left( 1 + x^2 \right) \frac{\partial^2 D_{n_k}^{(\beta, \alpha)}(x)}{\partial x^2} + \left( x(2\beta + 1) - \alpha \right) \frac{\partial D_{n_k}^{(\beta, \alpha)}(x)}{\partial x} + \left\{ \frac{q \beta \alpha - q}{2} \alpha - q \frac{\alpha^2}{4} + q \beta^2 - q \beta - A_{n_k} \right\} D_{n_k}^{(\beta, \alpha)}(x) = 0 \tag{49}
\]
By setting
\[
q \beta \alpha - \frac{q}{2} \alpha + B_{n_k} i \sqrt{q} = 0 \quad \text{and} \quad -\frac{q \alpha^2}{4} + q \beta^2 - q \beta - A_{n_k} = 0 \tag{50}
\]
in equation (49) we obtain differential equation that satisfies Romanovski polynomials given as
\[
\left( 1 + x^2 \right) \frac{\partial^2 D_{n_k}^{(\beta, \alpha)}(x)}{\partial x^2} + \left( x(2\beta + 1) - \alpha \right) \frac{\partial D_{n_k}^{(\beta, \alpha)}(x)}{\partial x} + \left\{ -E_{n_k} - \beta^2 \right\} D_{n_k}^{(\beta, \alpha)}(x) = 0 \tag{51}
\]
By comparing equations (31) and (51) we get
\[
2\beta + 1 = 2(1 - p) \Rightarrow p = -\beta + 1/2; \quad \alpha = -q'; \quad -E_{n_k} - \beta^2 = n(n-1) + 2n(1-p) \tag{52}
\]
By manipulating equations (50) we obtain the values of \( \beta \) and \( \alpha \) that have physical meaning are
\[
\beta = \frac{1}{2} \left( \sqrt{\frac{q}{4} + A_{n_k}} + B_{n_k} \sqrt{q} \right) \quad \text{and} \quad \alpha = \frac{1}{2} \left( \sqrt{\frac{q}{4} + A_{n_k}} - B_{n_k} \sqrt{q} \right) \tag{53}
\]
\[
\alpha = -\frac{B_{\mu i}}{\sqrt{q(\beta - 1/2)}} = -i\omega = -i\left( \sqrt{\frac{q + A_{\mu i}}{q}} + \sqrt{\frac{q + A_{\mu i}}{q}} \right) \]

(54)

By manipulating equations (45), (52), and (53) we obtain the relativistic energy equation as

\[
\left\{ \frac{(M + E_{\text{ke}})(M - C_{\mu i}) + V^2}{a^2} \right\} = \left\{ \sqrt{\frac{q + A_{\mu i}}{4q}} + \sqrt{\frac{q + A_{\mu i}}{4q}} \right\}^2 + n + 1/2 \]

(55)

The relativistic energy \( E_{\text{ke}} \) calculated from equation (55) using MatLab 11 programming is presented in Table 1. It is shown in Table 1 that there is degeneracy energy for \( n, l = l + 1/2, n, l + 2, j = l - 1/2 \) pair, and this degeneracy energy is removed by the presence of the tensor coupling potential. Table 2 shows the relativistic energy as a function of the q-deformed parameter. The system is not deformed for \( q = 1 \) and otherwise the system is underlying deformation.

To determine the wave function of the system, firstly we determine the weight function. By using equations (53) and (54) we obtain

\[
w^{(p, q')}(x) = (1 + x^2)^{\beta - 1/2} e^{i\sqrt{q}(\beta - 1/2)\text{tan}^{-1}(x)} = \left( 1 - \frac{\cos_q ar}{\sqrt{q}} \right)^{\beta - 1/2 - \alpha/2} \left( 1 + \frac{\cos_q ar}{\sqrt{q}} \right)^{\beta - 1/2 + \alpha/2} \]

(56)

\[
R_{\beta}^{(-\beta + 1/2, -\alpha)}(x) = D_{\beta}^{(\beta, \alpha)}(x) = \frac{1}{(1 + x^2)^{\beta - 1/2} e^{\text{arctan}^{-1}(x)}} \frac{d^n}{dx^n} (1 + x^2)^{\beta - 1/2 + n} e^{\text{arctan}^{-1}(x)} \]

(57)

In equation (56) we have applied the trigonometric-hyperbolic relation

\[
\tan^{-1} \left( -\frac{i\cos_q ar}{\sqrt{q}} \right) = -i\tanh^{-1} \left( \frac{\cos_q ar}{\sqrt{q}} \right) = -\frac{i}{2} \ln \left( \frac{1 + (\cos_q ar)\sqrt{q}}{1 - (\cos_q ar)\sqrt{q}} \right) \]

(58)

The wave function of the system obtained from equations (56) and (57) is given as

\[
G_{\text{ke}}(x) = g_{\mu i}(x) = (1 + x^2)^{\beta - 1/2} e^{2\beta\text{ctan}^{-1}x} D_{\beta}^{(\beta, \alpha)}(x) = (1 + x^2)^{\beta - 1/2} e^{2\beta\text{ctan}^{-1}x} R_{\beta}^{(-\beta + 1/2, -\alpha)}(x) \]

(59)

The ground state wave function of lower component of Dirac spinor is obtained from equations (58) and (59), when for \( n = 0 \) the lowest degree of Romanovski polynomials, \( R_{\beta}^{(-\beta + 1/2, -\alpha)}(x) = 1 \), so

\[
G_{0\text{ke}}(x) = G_{0\text{ke}}(r) = C \left( 1 - \frac{\cos_q ar}{\sqrt{q}} \right)^{\beta - \alpha/2} \left( 1 + \frac{\cos_q ar}{\sqrt{q}} \right)^{\beta + \alpha/2} \]

(60)
### Table 1. The relativistic energy $E_{nx}$ with different values of $l$ and $n$ for $V_0=4$ fm$^{-1}$; $V_1=3$ fm$^{-1}$; $a=0.05$ fm$^{-1}$; $M=3$ fm$^{-1}$; $q=1$; and $C_{ps}=-5$ fm$^{-1}$

| $l$ | $n$ | $k$ | $E_{nx}$, $V_2 = 0.6$ | $E_{nx}$, $V_2 = 0.8$ | $E_{nx}$, $V_2 = 0.6$ | $E_{nx}$, $V_2 = 0.8$ |
|-----|-----|-----|---------------------|---------------------|---------------------|---------------------|
| 0   | 0   | -1  | 0.93967362          | -1.99678744         | 0.93967362          | -1.99678744         |
| 0   | 1   | -2  | 0.97729417          | -1.98336363         | 0.97729417          | -1.98336363         |
| 2   | 0   | -3  | 0.961071492         | -1.975177161        | 0.961071492         | -1.975177161        |
| 3   | 0   | -4  | 0.940392072         | -1.957672974        | 0.940392072         | -1.957672974        |
| 0   | 1   | -1  | 0.977559362         | -1.987684954        | 0.977559362         | -1.987684954        |
| 1   | 1   | -2  | 0.96906399          | -1.974854327        | 0.96906399          | -1.974854327        |
| 2   | 1   | -3  | 0.940222876         | -1.9574063          | 0.940222876         | -1.9574063          |
| 3   | 1   | -4  | 0.915947799         | -1.935988592        | 0.915947799         | -1.935988592        |

The upper component of Dirac spinor for pseudospin symmetry obtained using equations (8) and (59) is

$$ F_{nx}(r) = \left( \frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{nx}(r) $$

For exact pseudospin symmetric case when $C_{ps}=0$, the upper spinor exist if $M \neq E_{nx}$ which means that the relativistic energies for pseudospin symmetric case are always negative.

### 4.2. The solution Dirac equation for $q$-deformed trigonometric Scarf and trigonometric tensor potential solution for spin symmetric case

By inserting equations (17) and (18) into equation (12) we obtain

$$ \left\{ \frac{d^2}{dr^2} - \frac{\kappa (\kappa + 1)}{r^2} - \frac{2 \kappa}{r} g^{-1} (V_2 \cot q ar + V_3 \csc q ar) - a^2 (V_2 \cot q ar + V_3 \csc q ar) \right\} F_{nx}(r) $$

$$ - a^2 V_3 \csc q ar \cot q ar F_{nx}(r) - a^2 \left( \frac{V_0}{\sin^2 q ar} - \frac{V_1}{\sin^2 q ar} \right) (M + E_{nx} - C_s) F_{nx}(r) = (M - E_{nx})(M + E_{nx} - C_s) F_{nx}(r) $$

The solution steps of Dirac equation for spin symmetric case is similar with the solution steps for pseudospin symmetric case, therefore by using the approximation scheme expressed in equation (39) into equation (61) we get

$$ \left\{ \frac{d^2}{dr^2} - \frac{a^2 (\kappa (\kappa + 1) + 2 \kappa V_3 + V_3^2 + V_2 q - V_2 q + V_0 \gamma_s)}{\sin^2 q ar} - \frac{a^2 (2V_2 V_3 - V_2 V_2 - V_1 \gamma_s) \cos q ar}{\sin^2 q ar} \right\} F_{nx}(r) $$

$$ = \left\{ (M - E_{nx})(M + E_{nx} - C_s) - a^2 V_2^2 \right\} F_{nx}(r) $$

By setting

$$ A_y = \{ \kappa (\kappa + 1) + 2 \kappa V_3 + V_3^2 + V_2 q - V_2 q + V_0 \gamma_s \}; \quad \gamma_s = (M + E_{nx} - C_s) $$

$$ B_y = (V_3 - 2V_2 V_3 - 2V_2 V_3 + V_1 \gamma_s); \quad E_y = \left\{ \frac{(M - E_{nx})(M + E_{nx} - C_s)}{a^2} - V_2^2 \right\} $$

in equation (63) then equation (63) reduces to one dimensional Schrödinger-type equation.

11
\[
\left\{ \frac{d^2}{dr^2} - \frac{a^2 A_s}{\sin^2 q ar} + \frac{a^2 B_s \cos q ar}{\sin^2 q ar} \right\} F_{ns}(r) = a^2 E F_{ns}(r) \tag{66}
\]

By variable substitution given as \( \cos q ar = i x \sqrt{q} \) in equation (66) we obtain
\[
\left\{ (1+x^2) \frac{d^2}{dx^2} + x \frac{d}{dx} + \frac{A_x}{q(1+x^2)} - \frac{B_i \sqrt{q x}}{q(1+x^2)} + E_x \right\} F_{ns}(x) = 0 \tag{67}
\]

The solution of equation (67) is obtained by setting
\[
F_{ns}(x) = g_n(x) = (1+x^2)^{\frac{\beta}{2} - \frac{a}{2} \tan^{-1} x} D^{(\beta, \alpha)}_n(x) \tag{68}
\]

and equation (67) becomes
\[
(1+x^2) \frac{d^2 D^{(\beta, \alpha)}_n(x)}{dx^2} + \left\{ x(2\beta+1) - \alpha \right\} \frac{d D^{(\beta, \alpha)}_n(x)}{dx} - \left\{ \frac{q \beta \alpha - \frac{q}{2} \alpha - \frac{q \alpha^2}{4} + q \beta^2 - q \beta - A_n}{q(1+x^2)} - E - \beta^2 \right\} D^{(\beta, \alpha)}_n(x) = 0 \tag{69}
\]

By setting
\[
q \beta \alpha - \frac{q}{2} \alpha + B_i \sqrt{q} = 0 \quad \frac{q \alpha^2}{4} + q \beta^2 - q \beta - A_n = 0 \tag{70}
\]

in equation (69) we get
\[
(1+x^2) \frac{d^2 D^{(\beta, \alpha)}_n(x)}{dx^2} + \left\{ x(2\beta+1) - \alpha \right\} \frac{d D^{(\beta, \alpha)}_n(x)}{dx} - \left\{ -E - \beta^2 \right\} D^{(\beta, \alpha)}_n(x) = 0 \tag{71}
\]

By comparing equations (26) and (71) we get
\[
(2\beta+1) = 2(1-p) \rightarrow p = -\beta + 1/2 ; \quad \alpha = -q' \quad \text{and} \quad -E_s - \beta^2 = n(n-1) + 2n(1-p) \tag{72}
\]

The values of \( \beta \) and \( \alpha \) that have physical meaning are
\[
(\beta, -1/2) = \sqrt{\left( \frac{q}{4} + A_s \right) + B_s \sqrt{q}} - \sqrt{\left( \frac{q}{4} + A_s \right) - B_s \sqrt{q}} \tag{73}
\]

and
\[
\alpha_s = -i \omega = -i \sqrt{\left( \frac{q}{4} + A_s \right) + B_s \sqrt{q}} - \sqrt{\left( \frac{q}{4} + A_s \right) - B_s \sqrt{q}} \tag{73}
\]

By using equations (71) and (73) we have
\[
\left\{ (E_{ns} - M) (E_{ns} + M - C_s) \right\} \left\{ V_s^2 \right\} = \left\{ \sqrt{\left( \frac{q}{4} + A_s \right) + B_s \sqrt{q}} - \sqrt{\left( \frac{q}{4} + A_s \right) - B_s \sqrt{q}} + n + 1/2 \right\}^2 \tag{74}
\]

By using equations (33), (58) and (73) we obtain the weight function given as
\[
W^{(p, q)}(x) = (1+x^2)^{-p} e^{q \tan^{-1}(x)} = \left( 1 - \frac{\cos q ar}{\sqrt{q}} \right)^{\beta_s - 1/2 - \alpha_s/2} \left( 1 + \frac{\cos q ar}{\sqrt{q}} \right)^{\beta_s - 1/2 + \alpha_s/2} \tag{75}
\]
and the corresponding Romanovski polynomials is
\[
R_n^{(1/2, -1/2, -\alpha, \beta)}(x) = D_n^{(1/2, \alpha)}(x) = \frac{1}{(1 + x^2)^{1/2}} e^{i\alpha \tan^{-1}(x)} \frac{d^n}{dx^n} \left( (1 + x^2)^{-1/2} e^{i\alpha \tan^{-1}(x)} \right)
\]  
(76)
The wave function of the system obtained from equations (7) and (77) is given as
\[
F_{nx}(x) = g_n(x) = \left(1 + x^2\right)^{-1/2} e^{i\alpha \tan^{-1}(x)} D_n^{(1/2, \alpha)}(x)
\]
(77)
The lower component of Dirac spinor for symmetric case is obtained using equations (7) and (77) as
\[
G_{nx}(r) = \left( \frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{nx}(r)
\]  
(78)
Exact spin symmetry is special case for spin symmetry when \( C_{ps} = 0 \), therefore the lower spinor component exist only if \( M \neq -E_{nx} \) which means that the relativistic energy, \( E_{nx} \), is always positive.

5. Conclusion
The Dirac equation for q-deformed trigonometric Scarf potential with q-deformed trigonometric cotangent and cosecant tensor potential in the scheme of centrifugal term approximation is exactly solved using Romanovski polynomials both for pseudospin and spin symmetric cases. The relativistic energy spectra calculated using Matlab are always positive for spin symmetric case and negative for pseudospin symmetric case. The lower and upper component of Dirac spinors are obtained in the form of Romanovski polynomial. The tensor potential causes the degeneracy energy is removed both for pseudospin and spin symmetries.

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