Quantum spin-Hall edges are envisaged as next-generation transistors, yet they exhibit dissipationless transport only over short distances. Here we show that in a diffusive sample, where charge puddles with odd spin cause back-scattering, a magnetic field drastically increases the mean free path and drives the system into the ballistic regime with a Landauer-Buttiker conductance. A strong non-linear non-reciprocal current emerges in the diffusive regime with opposite signs on each edge, and vanishes in the ballistic limit. We discuss its detection in state-of-the-art experiments.

Quantum spin-Hall insulators are a novel class of materials hosting gapless, topologically protected, counter-propagating edge states. These have opposite spin polarizations and exhibit strong spin-momentum locking due to the dominant role of the spin-orbit interaction [1–9]. Time-reversal symmetry ensures edge locking due to the dominant role of the spin-orbit interaction [1–9]. Materials possessing topological edge states come in Kramers doublets, which cannot be back-scattered by time-reversal invariant perturbations [10–17]. Materials possessing topological edge states include topological insulators such as HgTe and Bi$_2$Se$_3$, Weyl semimetals such as WTe$_2$, and Dirac semimetals such as Na$_3$Bi [3, 18–24].

A ballistic edge has a longitudinal conductance of $e^2/h$ at low temperature, a fact that has led to proposals for using topological edge states as building blocks for next-generation transistors, exploiting electrically tunable topological phase transitions [25]. Nevertheless, following the experimental discovery of topological edge states, it has emerged that puddles with odd numbers of charges, which exist inherently in the host materials due to doping disorder in fabrication, can act as effective magnetic impurities that back-scatter the edge states and significantly reduce their mobility [12, 13, 26–29]. This may explain why ballistic conductance has only been observed over spatial scales of the order of 50 nm [19–21, 23, 26, 30–32]. Whereas initial studies focused on the Kondo effect, the Kondo temperature in current samples is expected to be negligibly small [33–35], while other aspects of transport remain poorly understood [2, 3, 17, 36–42]. The unexpectedly large resistance of topological edge states has emerged as a fundamental question and an obstacle in the development of topological transistors [43, 44].

FIG. 1. Experimental setup in a perpendicular magnetic field $\mathbf{B}$. The current is measured across the sample, while the voltage can be measured at two different terminals along one side. On the opposite side one terminal is grounded. Spin-up electrons are shown in orange, spin-down electrons in green.

Quantum spin-Hall edges have a longitudinal conductance of $e^2/h$ at low temperature, a fact that has led to proposals for using topological edge states as building blocks for next-generation transistors, exploiting electrically tunable topological phase transitions [25]. Nevertheless, following the experimental discovery of topological edge states, it has emerged that puddles with odd numbers of charges, which exist inherently in the host materials due to doping disorder in fabrication, can act as effective magnetic impurities that back-scatter the edge states and significantly reduce their mobility [12, 13, 26–29]. This may explain why ballistic conductance has only been observed over spatial scales of the order of 50 nm [19–21, 23, 26, 30–32]. Whereas initial studies focused on the Kondo effect, the Kondo temperature in current samples is expected to be negligibly small [33–35], while other aspects of transport remain poorly understood [2, 3, 17, 36–42]. The unexpectedly large resistance of topological edge states has emerged as a fundamental question and an obstacle in the development of topological transistors [43, 44].

Bearing in mind the role of magnetic impurities, the first step in overcoming this problem is understanding edge magneto-transport in the presence of puddles. This includes the identification of non-reciprocal currents, since non-linear response probes interactions that are difficult to access in linear response, due to constraints imposed by mirror symmetry and Onsager relations [19, 45–48]. The rich physics underlying non-linear phenomena [22, 49–53] has been manifest in recent discoveries such as Hall effects in time-reversal invariant systems, as well as in unexpected features of topological edges, such as a large uni-directional magneto-resistance at zero magnetic field [19, 20, 22, 48].

In this article we demonstrate that a magnetic field has a drastic effect on both the linear and non-linear response of topological edge states: (i) It enhances the mean free path $l$ by orders of magnitude without opening a gap, eventually driving the system into the ballistic regime; (ii) By breaking mirror symmetry the magnetic field enables a strong unidirectional non-linear electrical response in the diffusive regime. The direction of the current is determined by the magnetic field and the spin quantization axis, and it has a different sign on each edge. Interestingly, the non-reciprocal current vanishes in the ballistic regime. This reflects the fact that, once magnetic impurity scattering is surmounted, the only remaining magnetic interaction is the Zeeman interaction with the out of plane field, which can be gauged away. Whereas a complete description of charge puddles is beyond the scope of this work, modelling the puddles as magnetic impurities is a simple way of capturing the physics that governs their magneto-resistance, which is in excellent agreement with experiment [31].
Referring to the set-up shown in Fig. 1, our main results are summarised in Fig. 2. The current in the channel will be denoted by $I$ and the potentials of the left and right electrodes by $V_L$, $V_R$ respectively. We define the conductance $G$ and the non-linear electrical response function $\chi$ by $I = G(V_L - V_R) + \chi(V_L - V_R)^2$. In Fig. 2-(a) and Fig. 2-(b) we have plotted the conductance $G$ as a function of the applied out-of-plane magnetic field $B$ at small and large values of $B$, where small and large are quantified below. It is seen that $G$ increases with $B$ and eventually reaches the quantized Landauer-Buttiker value of $e^2/h$, indicating that the system reaches the ballistic limit. This opens up the exciting possibility of using a ferromagnet with an out-of-plane magnetization as a practical method to increase the mean free path and to study transport in the ballistic regime. The ferromagnet could couple to the impurities either via a magnetic field or through the exchange interaction. Next, Fig. 2-(c) and Fig. 2-(d) show the non-linear electrical response function $\chi$ at small and large magnetic fields respectively. At small $B$, $\chi$ increases with $B$, but in contrast to the Ohmic term the non-linear signal reaches a maximum beyond which it decreases, tending to zero as the system reaches the ballistic regime. This vanishing response is a characteristic of the Dirac cone, indicating that the non-linear response is a probe of the edge state dispersion, and is a unique experimental signature reflecting chiral conduction in the TI. To generate and detect the second-order response at low-frequency it is sufficient to use an oscillator with angular frequency $\omega$ and read off the signal at $2\omega$.

We focus on Na$_3$Bi as a prototype material, motivated by the observations that ultra-thin films of Na$_3$Bi have a band-gap of $\geq 300$ meV [54], much greater than $k_BT$ at room temperature, are robust to layer-number fluctuations caused by imperfect growth [55], exhibit an electrically driven topological phase transition[56], and show clear evidence of edge transport over millimetre distances, as well as a giant negative magneto-resistance [31]. Our model also applies to topological insulators with inversion symmetry such as Bi$_2$Se$_3$. Materials without inversion symmetry, such as WTe$_2$, exhibit a positive magneto-resistance and a position-dependent spin quantization axis, so they fall outside our scope.

Considering a sample of finite size $d$ a magnetic field $B \parallel \hat{z}$ is applied out of the plane. The full Hamiltonian $H$ can be written as $H = H_0 + H_Z + V + U + U_Z$, where the band Hamiltonian $H_0 = \hbar v_F k_x \sigma_z$ represents the edge state dispersion of Na$_3$Bi; $H_Z = g_0 \mu_B B \sigma_z$ is the Zeeman interaction with the magnetic field. Since in the absence of warping terms the magnetic field does not open a gap in the dispersion the topological character of the states is
Mean free path (nm)

U charge. The random magnetic impurity potential spins have the same mean free path in the system in which localisation is nevertheless not expected meaningfully to 1D topological edge states, a diffusive free path, defined explicitly below, can be applied are either ballistic or localised, the notion of a mean free path. Although conventional 1D systems by the relationship between the magnetic field and the DC limit, where $\omega \tau \ll 1$.

The negative magneto-resistance in Fig. 2 is explained by the relationship between the magnetic field and the mean free path. Although conventional 1D systems are either ballistic or localised, the notion of a mean free path, defined explicitly below, can be applied meaningfully to 1D topological edge states, a diffusive system in which localisation is nevertheless not expected due to topological protection. We note that up and down spins have the same mean free path $l$. Figure 3 gives a diagrammatic example of spin-flip scattering, showing a spin-down electron being flipped to the spin-up channel due to scattering off an impurity. The energy required for this transition is set by the Zeeman splitting of the impurity spin states. As the magnetic field increases the energy cost likewise increases and the transition is suppressed. Figure 4 shows the mean free path $l$ increasing as a function of $B$ until it exceeds the size $d$ of the sample. Based on this we define the diffusive regime as $l \ll d$, and the ballistic regime as $l > d$. We focus on these two limiting cases, in which simplifying approximations can be made. Specifically, in the diffusive regime one may assume a constant electric field across the channel, and the conductance takes the simple general form $G = \frac{e^2}{\hbar l}$, where the entire magnetic field dependence is contained in the mean free path $l(B)$. In the ballistic regime it is straightforward to express the current as a function of the potential difference between the source and drain electrodes, and the potential drop occurs overwhelmingly in the vicinity of the electrodes due to contact resistance, although the exact potential profile is immaterial [58, 59]. The conductance is obtained straightforwardly as $G = \frac{e^2}{h}$. The intermediate region is complicated by potential fluctuations, and is not a focus of current experimental efforts. A full treatment requires accounting for screening thoroughly [60, 61].

A perpendicular magnetic field breaks mirror symmetry and enables a second-order response, which increases as a function of the magnetic field in the diffusive regime due to the reduced efficiency of impurity scattering. Nevertheless, the non-linear response vanishes in the ballistic regime. Once transport becomes ballistic there is no more scattering and the impurities become irrelevant. The effective Hamiltonian becomes simply that of a Dirac cone, $H_0 + H_Z$, whence $B$ can be removed by redefining the origin. The second-order response therefore probes the edge state dispersion: if a non-linear response is detected in the ballistic regime it must come from band structure terms of higher order.

**FIG. 3.** Spin-flip scattering. An electron with spin down is scattered into a spin-up state, which, due to spin-momentum locking, travels in the opposite direction; the energy change is given by the impurity Zeeman splitting $E_{1, z} = g_1 \mu_B B$.

**FIG. 4.** The mean free path as a function of magnetic field. The red dashed lines mark the size of the sample $d = 1000 nm$ compared with the mean free path the system. When the magnetic field is small, the system is diffusive, however a larger magnetic field will enhance the mean free path, driving the system into the ballistic regime, leading to a vanishing non-linear response. In all the plots, we have set the mean free path at zero magnetic field to be 10nm.
in the wave vector, which are challenging to calculate computationally for 1D systems. Although they can be determined by symmetry their magnitude is generally unknown [30] (the details are reserved for a future publication).

The response of the other edge can be found by reflecting the Hamiltonian in the $xz$-plane. The Hamiltonian describing the dispersion for the other edge reads $H_0 = -\hbar v_F k_x \sigma_z$. When the magnetic field is flipped the conductance $G$ does not change, consistent with Onsager symmetry. But the direction of the non-linear response on each edge is set by the spin orientation with respect to the magnetic field and the solution to the second order quantum kinetic equation changes sign for the other edge. Hence $\chi$ changes sign, ensuring time reversal breaking in the non-linear response function [49].

We derive a quantum kinetic equation following the procedure of Refs. 33 and 62, which ensures the Pauli blocking terms are correctly accounted for [62]. The full details are provided in the Supplemental material (SM) [63]. The system is described by the density matrix $\rho$, where $\rho$ satisfies the quantum Liouville equation $\partial \rho / \partial t = -\frac{1}{\hbar} [H, \rho] = 0$. The explicit position dependence must be taken into account due to the finite size of the sample. Following a Wigner transformation [64, 65]

$$\frac{\partial \rho}{\partial t} + \frac{1}{2\hbar} \left\{ \frac{\partial H_0}{\partial k_x}, \frac{\partial \rho}{\partial x} \right\} + J(\rho) = -e \frac{\partial V}{\hbar} \frac{\partial \rho}{\partial x}$$

In Eq. (1), the single particle density matrix $\rho$ takes the form $\text{diag}(f_\uparrow, f_\downarrow)$. We write $f_\uparrow = f_\uparrow^{(0)} + g_\uparrow$ is the non-equilibrium distribution for the spin-up electrons composed of the equilibrium part $f_\uparrow^{(0)}$ and out-of-equilibrium part $g_\uparrow$; similarly, $f_\downarrow = f_\downarrow^{(0)} + g_\downarrow$ is the non-equilibrium distribution for the spin-down electrons, the equilibrium distribution have the form $f^{(0)}(\varepsilon) = [1 + \exp(\beta(\varepsilon - \mu))]^{-1}$ where $\beta = (k_B T)^{-1}$. The last term in Eq. (1) is the scattering term in the Born approximation, which take the form

$$J(g_\uparrow) = \int \left\{ P_{k\uparrow, k'\downarrow}(1 - f_\uparrow) - P_{k\downarrow, k'\uparrow}(1 - f_\downarrow^\prime) \right\} \frac{dk'}{2\pi}$$

$$J(g_\downarrow) = \int \left\{ P_{k\uparrow, k'\downarrow}(1 - f_\downarrow) - P_{k\downarrow, k'\uparrow}(1 - f_\downarrow^\prime) \right\} \frac{dk'}{2\pi}$$

Here $P_i(k' \downarrow \rightarrow k, \uparrow)$ indicates the probability of spin-flip scattering between a spin-up electron at $k$ and an impurity, ending with a spin-down electron at $k'$. Primed quantities indicate the final state following a scattering event. We obtain two coupled Boltzmann equations for the spin-up and spin-down electrons:

$$\frac{\partial g_\uparrow(\varepsilon)}{\partial x} + \Gamma_1(\varepsilon) g_\uparrow(\varepsilon) - \Gamma_2(\varepsilon) g_\downarrow(\varepsilon) = -e \frac{\partial V}{\hbar} \frac{\partial f_\uparrow^{(0)}(\varepsilon)}{\partial x}$$

$$\frac{\partial g_\downarrow(\varepsilon)}{\partial x} - \Gamma_1(\varepsilon) g_\downarrow(\varepsilon) + \Gamma_2(\varepsilon) g_\uparrow(\varepsilon) = -e \frac{\partial V}{\hbar} \frac{\partial f_\downarrow^{(0)}(\varepsilon)}{\partial x}$$

where $\varepsilon = \varepsilon - \varepsilon_Z$ and $\varepsilon_Z$ is the Zeeman energy. The two scattering rates are defined as follows:

$$\Gamma_1(\varepsilon) = \frac{N_i j^2}{\hbar^2 v_F} \left[ \frac{1}{1 + e^{\varepsilon / \alpha}} \left[ 1 - f_\downarrow^{(0)}(\varepsilon) \right] \right] + \frac{1}{1 + e^{\varepsilon / \alpha}} f_\uparrow^{(0)}(\varepsilon)$$

$$\Gamma_2(\varepsilon) = \frac{N_i j^2}{\hbar^2 v_F} \left[ \frac{1}{1 + e^{\varepsilon / \alpha}} f_\uparrow^{(0)}(\varepsilon) \right] + \frac{1}{1 + e^{\varepsilon / \alpha}} \left[ 1 - f_\downarrow^{(0)}(\varepsilon) \right]$$

where $N_i$ is the number of impurities, the dimensionless factor $\alpha = g_1 \mu_B B / (k_B T)^{-1}$, and the change of the Zeeman energy during the spin-flipping interactions $\varepsilon_Z = g_1 \mu_B B$. We solve the coupled Eq. (4) and Eq. (5) by integrating separately over left and right movers, which also ensures the correct solution in the ballistic regime:

$$g_\uparrow^{(1)}(x) = -e \int_0^x \left[ \Gamma_2 / \kappa + \Gamma_1 / \kappa \exp[\kappa(x - x')] \right] \frac{\partial V}{\partial x} f_\uparrow^{(0)}(x') dx' + e \int_x^0 \left[ \Gamma_2 / \kappa \left( \exp[\kappa(x - x')] - 1 \right) \right] \frac{\partial V}{\partial x} f_\downarrow^{(0)}(x') dx'$$

$$g_\downarrow^{(1)}(x) = -e \int_0^x \left[ \Gamma_1 / \kappa + \Gamma_2 / \kappa \exp[\kappa(x - x')] \right] \frac{\partial V}{\partial x} f_\uparrow^{(0)}(x') dx' + e \int_0^x \left[ \Gamma_2 / \kappa \left( \exp[\kappa(x - x')] - 1 \right) \right] \frac{\partial V}{\partial x} f_\downarrow^{(0)}(x') dx'$$

where $\kappa = \Gamma_1 + \Gamma_2$, and the mean free path $l = \kappa^{-1} = (\Gamma_1 + \Gamma_2)^{-1}$. The current density $j = -e \text{Tr}(\nu \rho)$, where $\nu = (1/\hbar)(\partial H_0 / \partial k)$. For spin-up electrons the momentum integration is performed over $k > 0$, and for spin-down electrons over $k < 0$. For the second-order response the sign of the magnetic field in the scattering terms Eq. (6), (7) will change, yet $\Gamma_1$ and $\Gamma_2$ are symmetric in $\alpha$; the sign of the driving term will change similarly to the band dispersion. The mean free path is unchanged and the formal solution to the differential equation is analogous to Eqs. 8-9, with the replacements $\frac{\partial f_{\uparrow}^{(0)}}{\partial x} \rightarrow \frac{\partial g_{\uparrow}^{(0)}}{\partial x}$. These equations cannot be reduced to a simple closed form and are solved iteratively.

In this work we have not discussed dispersions beyond the linear case. In Bi$_{2}$Se$_3$ warping complicates the dispersion, and in the spin eigenstate basis $H_0 = Ak_x \sigma_y + C \sigma_z k_x^3$. A magnetic field $\parallel \vec{y}$ yields a term of the form $\sigma_z B_y$ due to warping, which opens a small gap in the
edge spectrum. However, warping only accounts for up to 10% of the Fermi energy, thus the gap is expected to be small, and will not influence the dynamics in the vicinity of the Fermi energy discussed here. Since the addition of warping complicates significantly the description of the interaction with the impurities, a further derivation is beyond the scope of this paper.

In summary, we have shown that a magnetic field drastically enhances the conductivity of topological edge states and gives rise to an edge-dependent non-linear response which vanishes in the ballistic limit. The magnetic field, as well as proximity to a ferromagnet, can be used to drive the system into the ballistic regime, while the non-linear response probes the edge state dispersion. In the future the transport theory can be extended to the Kondo regime along the lines of 33.

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