Interpolation by complete minimal surfaces whose Gauss map misses two points

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Abstract Let $M$ be any open Riemann surface and let $\Lambda \subset M$ be a closed discrete subset. In this paper, we prove the existence of complete conformal minimal immersions $M \to \mathbb{R}^n$, $n \geq 3$, with prescribed values on $\Lambda$ and whose generalized Gauss map $M \to \mathbb{CP}^{n-1}$, $n \geq 3$, avoids $n$ hyperplanes of $\mathbb{CP}^{n-1}$ located in general position. For the case $n = 3$, we obtain complete nonflat conformal minimal immersions whose Gauss map $M \to S^2$ omits two (antipodal) values of the sphere.

This result is deduced as a consequence of an interpolation theorem for conformal minimal immersions $M \to \mathbb{R}^n$ into the Euclidean space $\mathbb{R}^n$, $n \geq 3$, when all the harmonic coordinates but two are prescribed.

Keywords minimal surface, Riemann surface, interpolation theory, harmonic map, Gauss map.

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1. Introduction

Conformal minimal immersions of an open Riemann surface with values in a Euclidean space $\mathbb{R}^n$, $n \geq 3$, are harmonic maps. This fact has greatly influenced the study of minimal surfaces providing powerful techniques coming from complex analysis. One of the most interesting result is Mergelyan theorem [33, 30, 17], which has allowed, among other achievements, the development of a theory of uniform approximation on compact subset [15, 11], tangential approximation on some unbounded subsets [18], and a theory of interpolation on discrete subsets [3, 4, 14] for conformal minimal immersions into the Euclidean spaces.

Let $M$ be any open Riemann surface and let $\partial$ denote the complex linear part of the exterior differential, $d = \partial + \overline{\partial}$ on $M$ where $\overline{\partial}$ denotes the antilinear part. Given a conformal minimal immersion $X : M \to \mathbb{R}^n$, $\partial X$ determines the Kodaira type holomorphic map

$$G_X : M \to \mathbb{CP}^{n-1}, M \ni p \mapsto G_X(p) = [\partial X(p)]$$

which assumes values in the complex hyperquadric

$$Q_{n-2} := \{[z_1, \ldots, z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \ldots + z_n^2 = 0\} \subset \mathbb{CP}^{n-1}$$

and is known as the generalized Gauss map of $X$. Conversely, every holomorphic map $M \to Q_{n-2}$ is the generalized Gauss map of a conformal minimal immersion $M \to \mathbb{R}^n$, see [13]. The study of the (generalized) Gauss map of a conformal minimal immersion is an important topic in the theory of minimal surfaces. Chern and Osserman [19, 20] proved that if $X$ is complete then $X(M)$ is a plane or $G_X(M)$
intersects a dense subset of complex hyperplanes. Moreover, Ru [32] showed that the Gauss map of a complete nonflat conformal minimal immersion omits at most $n(n+1)/2$ hyperplanes in $\mathbb{C}P^{n-1}$ located in general position, generalizing a result of Fujimoto [24]. In addition, this upper bound is sharp for some values of $n \geq 3$, see Fujimoto [23]. Ahlfors [1] proved that if $G: \mathbb{C} \to \mathbb{C}P^{n-1}$ is a holomorphic map avoiding $n+1$ hyperplanes in general position, then $G$ is a degenerate map, that is, its image lies in a hyperplane of $\mathbb{C}P^{n-1}$. Concerning this, Alarcón, Fernández, and López [9] proved that for any open Riemann surface $M$, there exists a complete conformal minimal immersion $X: M \to \mathbb{R}^n$ whose generalized Gauss map $G_X$ is nondegenerate and fails to intersect $n$ hyperplanes in general position; the number $n$ here is the maximum possible by Ahlfors theorem. We show in this paper that one can prescribe the values of such an immersion on a closed discrete subset of $M$.

**Theorem 1.1.** Let $M$ be an open Riemann surface and $\Lambda \subset M$ be a closed discrete subset. Any map $\Lambda \to \mathbb{R}^n$, $n \geq 3$, extends to a complete conformal minimal immersion $X: M \to \mathbb{R}^n$ whose generalized Gauss map $G_X: M \to \mathbb{C}P^{n-1}$ is nondegenerate and fails to intersect $n$ hyperplanes of $\mathbb{C}P^{n-1}$ located in general position.

Note that the assumptions on $\Lambda$ are necessary since, by the Identity Principle, it is not possible to prescribe the values of a conformal minimal immersion $M \to \mathbb{R}^n$ on a subset that has an accumulation point. For the case $n = 3$ we obtain the following.

**Corollary 1.2.** Let $M$ be an open Riemann surface and $\Lambda \subset M$ be a closed discrete subset. Any map $\Lambda \to \mathbb{R}^3$ extends to a complete nonflat conformal minimal immersion $X: M \to \mathbb{R}^3$ whose Gauss map $M \to S^2$ omits two (antipodal) values of the sphere $S^2$.

Theorem 1.1 and Corollary 1.2 are deduced from an extension result for complete minimal surfaces with prescribed coordinates, see Theorem 1.3. Alarcón, Fernández, and López showed in [7, 8, 9] that one may prescribe all but two of the component functions of a complete conformal minimal immersion $M \to \mathbb{R}^n$. On the other hand, Alarcón and Castro-Infantes proved in [3] the existence of a complete conformal minimal immersion that interpolates any given map at a closed discrete subset of $M$. Combining these ideas, we show the following result.

**Theorem 1.3.** Let $M$ be an open Riemann surface and $n \geq 3$ be an integer. Let $\Lambda \subset M$ be a closed discrete subset and let $\bar{F}: M \to \mathbb{R}^{n-2}$ be a nonconstant harmonic map. For any map $F: \Lambda \to \mathbb{R}^2$, there is a complete conformal minimal immersion $X = (X_1, X_2, \ldots, X_n): M \to \mathbb{R}^n$ such that $(X_1, X_2)|_{\Lambda} = F$ and $(X_3, \ldots, X_n) = \bar{F}$.

Here, the assumption on $\Lambda$ is also necessary by the Identity Principle. If one does not care about interpolation, that is, $\Lambda = \emptyset$ in Theorem 1.3 then the conclusion follows from [9, Theorem B].

Theorem 1.3 is deduced from a more general result, see Theorem 5.1 which ensures not only interpolation but also jet-interpolation of any order at each point $p \in \Lambda$, uniform approximation on a Runge subset of $M$, and prescription of the flux map of the examples, see Section 2 for details and definitions. Finally, we obtain Theorem 1.1 as a consequence of Theorem 5.1 (see Theorem 5.2 for a precise statement).
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Our proof relies on the method of period dominating sprays of Weierstrass data which has been recently developed in the theories of approximation and interpolation for minimal surfaces in \( \mathbb{R}^n \); see Section 3 and specially Lemma 3.2. These ideas were first used in the study of minimal surfaces in \( \mathbb{R}^n \) and have allowed to construct not only minimal surfaces in \( \mathbb{R}^n \) but also complete null holomorphic curves in \( \mathbb{C}^n \) with many different global behaviours; see for instance [16, 10, 6, 5, 2] and references therein.

**Organization of the paper.** Section 2 is dedicated to establishing some notation and definitions to the well understanding of the paper. In section 3 we establish some technical result about the existence of sprays of holomorphic functions, see Lemma 3.2; it will be very useful in the proof of the main results. Next, we prove Proposition 4.2 in Section 4 which is crucial to ensure completeness of the examples. Finally, Section 5 contains the proof of Theorem 5.1 and Theorem 5.2 which trivially imply Theorems 1.3 and 1.1.

2. Preliminaries

We denote \( i = \sqrt{-1} \) and \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \). Given an integer \( n \in \mathbb{N} = \{1, 2, 3, \ldots\} \) and \( K \in \{\mathbb{R}, \mathbb{C}\} \), we denote by \( \| \cdot \| \) and by \( \text{length}(\cdot) \) the Euclidean norm and length in \( K^n \), respectively. We use the notation \( | \cdot | \) for the absolute value (or complex modulus) when \( K = \mathbb{R} \) (or \( K = \mathbb{C} \)).

Given a smooth connected surface \( S \) (possibly with nonempty boundary) and a smooth immersion \( X : S \to K^n \), we denote by \( \text{dist}_X : S \times S \to \mathbb{R}_+ \) the Riemannian distance induced on \( S \) by the Euclidean metric of \( K^n \) via \( X \); i.e.,

\[
\text{dist}_X(p, q) := \inf\{\text{length}(X(\gamma)) : \gamma \subset S \text{ arc connecting } p \text{ and } q\}, \quad p, q \in S.
\]

An immersed open surface \( X : S \to \mathbb{R}^n \) \( (n \geq 3) \) is said to be **complete** if the image by \( X \) of any divergent arc on \( S \) has infinite Euclidean length; equivalently, if the Riemannian metric on \( S \) induced by \( \text{dist}_X \) is complete in the classical sense.

2.1. Riemann surfaces and spaces of maps. Throughout the paper every Riemann surface will be considered connected unless the contrary is indicated.

Let \( M \) be an open Riemann surface. Given a subset \( A \subseteq M \) we denote by \( \mathcal{C}(A, K^n) \) the space of continuous functions \( A \to K^n \). We also denote by \( \mathcal{O}(A) \) the space of functions \( A \to \mathbb{C} \) which are holomorphic on an unspecified open neighbourhood of \( A \) in \( M \). We use the notation \( \mathcal{A}(A) \) for the space of holomorphic functions on \( A \) which are continuous up to the boundary, that is, \( \mathcal{A}(A) = \mathcal{O}(A) \cap \mathcal{C}(A, \mathbb{C}) \).

Likewise, by a **conformal minimal immersion** \( A \to \mathbb{R}^n \) we mean a continuous map \( A \to \mathbb{R}^n \) which is a conformal minimal immersion in \( A \).

Throughout the paper, we deal with some special subsets of an open Riemann surface: Runge subsets. A compact subset \( K \) contained in an open Riemann surface \( M \) is said to be **Runge** (also called **holomorphically convex** or \( \mathcal{O}(M) \)-**convex**) if every continuous function \( K \to \mathbb{C} \), holomorphic in the interior \( K \), may be approximated.
uniformly on \( K \) by holomorphic functions \( M \to \mathbb{C} \). By the Runge-Mergelyan theorem \[33, 30, 17\] this is equivalent to that \( M \setminus K \) has no relatively compact connected components in \( M \). The following particular kind of Runge subsets will play a crucial role in our argumentation.

**Definition 2.1.** A nonempty compact subset \( S \) of an open Riemann surface \( M \) is called admissible if it is Runge in \( M \) and of the form \( S = K \cup \Gamma \), where \( K \) is the union of finitely many pairwise disjoint smoothly bounded compact domains in \( M \) and \( \Gamma := S \setminus K \) is a finite union of pairwise disjoint smooth Jordan arcs meeting \( K \) only in their endpoints (if at all) such that their intersections with the boundary \( bK \) of \( K \) are transverse.

Despite most of the upcoming results, in particular Lemma \[3, 2\] may be proved for any admissible subset, we deal in this work with very simple admissible subsets. This kind of subsets are enough for the purpose of the paper and make the proofs more readable. They were first introduced in \[3\].

**Definition 2.2.** Let \( M \) be an open Riemann surface. An admissible subset \( S = K \cup \Gamma \subset M \) (see Definition 2.1) will be said simple if \( K \neq \emptyset \), every component of \( \Gamma \) meets \( K \), \( \Gamma \) does not contain closed Jordan curves, and every closed Jordan curve in \( S \) meets only one component of \( K \). Further, \( S \) will be said very simple if it is simple, \( K \) has at most one non-simply connected component \( K_0 \), which will be called the kernel component of \( K \), and every component of \( \Gamma \) has at least one endpoint in \( K_0 \); in this case we denote by \( S_0 \) the component of \( S \) containing \( K_0 \) and call it the kernel component of \( S \).

Note that a connected admissible subset \( S = K \cup \Gamma \) in an open Riemann surface \( M \) is very simple if and only if \( K \) has \( m \in \mathbb{N} \) components \( K_0, \ldots, K_{m-1} \), where \( K_i \) is simply-connected for every \( i > 0 \), and \( \Gamma = \Gamma' \cup \Gamma'' \cup (\bigcup_{i=1}^{m-1} \gamma_i) \) where

- \( \Gamma' \) consists of components of \( \Gamma \) with both endpoints in \( K_0 \),
- \( \Gamma'' \) consists of components of \( \Gamma \) with an endpoint in \( K_0 \) and the other in \( M \setminus K \), and
- \( \gamma_i \) is a component of \( \Gamma \) connecting \( K_0 \) to \( K_i \) for each \( i = 1, \ldots, m - 1 \).

Observe that, in such a case, \( K_0 \cup \Gamma' \) is a strong deformation retract of \( S \). In general, a very simple admissible subset \( S \subset M \) is of the form \( S = (K \cup \Gamma) \cup K' \) where \( K \cup \Gamma \) is a connected very simple admissible subset and \( K' \subset M \setminus (K \cup \Gamma) \) is a (possibly empty) union of finitely many pairwise disjoint smoothly bounded compact disks.

**2.2. Weierstrass representation formula.** Given an isometric immersion \( X: M \to \mathbb{R}^n \) from a Riemannian surface, \( M \), the metric Laplacian of the immersion satisfies the equation

\[ \Delta X = 2H, \]

where \( H \) is the mean curvature vector field of the immersion. The Laplacian operator on a Riemannian surface depends only on the conformal class of the metric, and so harmonic functions are well defined on a Riemann surface. Thus, if \( M \) is a Riemann surface, then a conformal (angle preserving) immersion \( X = (X_1, \ldots, X_n): M \to \mathbb{R}^n \)
is minimal if and only if $X$ is a harmonic map; in particular, there are no compact minimal surfaces in $\mathbb{R}^n$ with empty boundary.

Therefore, let $M$ be an open Riemann surface and let $X : M \to \mathbb{R}^n$, $n \geq 3$, be a conformal minimal immersion. Denoting by $\partial$ the complex linear part of the exterior differential $d = \partial + \overline{\partial}$ on $M$ (here $\overline{\partial}$ denotes the antilinear part), we have that the 1-form $\partial X = (\partial X_1, \ldots, \partial X_n)$, assuming values in $\mathbb{C}^n$, is holomorphic, has no zeros, and satisfies $\sum_{j=1}^n (\partial X_j)^2 = 0$. Furthermore, the real part $\Re(\partial X)$ of $\partial X$ is an exact 1-form on $M$ and the flux map (or simply, the flux) of $X$ is a group homomorphism denoted by $\operatorname{Flux}_X : H_1(M; \mathbb{Z}) \to \mathbb{R}^n$, of the first homology group of $M$ with integer coefficients. It is defined by

$$\operatorname{Flux}_X(\gamma) = \Im \int_\gamma \partial X = -i \int_\gamma \partial X, \quad \gamma \in H_1(M; \mathbb{Z}),$$

where $\Im$ denotes imaginary part. On the other hand, every holomorphic 1-form $\Phi = (\phi_1, \ldots, \phi_n)$ with values in $\mathbb{C}^n$, vanishing nowhere on $M$, satisfying the nullity condition

$$\sum_{j=1}^n (\phi_j)^2 = 0 \quad \text{everywhere on } M,$$

and whose real part $\Re(\Phi)$ is exact on $M$, determines a conformal minimal immersion $X : M \to \mathbb{R}^n$ with $\partial X = \Phi$ by the classical Enneper-Weierstrass, or simply Weierstrass, representation formula:

$$X(p) = x_0 + \Re \int_{p_0}^p 2 \Phi, \quad p \in M,$$

for any fixed base point $p_0 \in M$ and initial condition $X(p_0) = x_0 \in \mathbb{R}^n$.

If we are given $\theta$ a holomorphic 1-form never vanishing on $M$ (such exists by Oka-Grauert \cite{25,26}, see also Gunning and Narasimhan \cite{27}), then any holomorphic 1-form $\Phi$ satisfying (2.2) may be written as $\Phi = f\theta$ where $f : M \to \mathbb{A}$ is a holomorphic function and

$$\mathbb{A} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + \ldots + z_n^2 = 0\}.$$  

The subset $\mathbb{A}$ is usually called the null quadric and it is an Oka manifold when removing the origin (see \cite{10} Example 4.4). This fact allows us to apply all the theory of maps from Stein manifolds into Oka manifolds to the study of minimal surfaces; recall that an open Riemann surface is an Stein manifold. See \cite{21} for a specialized book on the field and \cite{12} for how this techniques apply to the study of minimal surfaces.

Gunning and Narasimhan proved in \cite{27} that given any open Riemann surface, there exists a never vanishing exact holomorphic 1-form. Kusunoki and Sainouchi generalized this result proving the existence of a holomorphic 1-form with given divisor and periods on any open Riemann surface, see \cite{29}.

For the upcoming reasoning we use an existence result which may be obtained from the result of Kusunoki and Sainouchi by combining the ideas of interpolation introduced in \cite{3} by Alarcón and Castro-Infantes.
Lemma 2.3. Let $M$ be an open Riemann surface and $\Lambda \subset M$ be a closed discrete subset. Let $p : H_1(M; \mathbb{Z}) \to \mathbb{C}^n$ be a group morphism and $F : \Lambda \to \mathbb{C}^n$ be a map. Additionally, for any $p \in \Lambda$, let $\gamma_p \subset M$ be a smooth Jordan curve connecting a fixed point $p_0 \in M \setminus \Lambda$ to $p \in \Lambda$. Then, there exists a never vanishing holomorphic 1-form $\theta$ such that

(i) $\int_{\gamma} \theta = p(\gamma)$ for any closed curve $\gamma \subset M$, and

(ii) $\int_{F(p)} \theta = F(p)$ for any $p \in \Lambda$.

Sketch of the proof of Lemma 2.3. It is well known that $\mathbb{C} \setminus \{0\}$ is an Oka manifold. Therefore, if we are given a holomorphic 1-form $\omega$ never vanishing on $M$, we may apply a standard recursive reasoning to construct a holomorphic function $h : M \to \mathbb{C}_* := \mathbb{C} \setminus \{0\}$ such that $\int_{\gamma} h \omega = p(\gamma)$ for any closed curve $\gamma \subset M$, and $\int_{F(p)} h \omega = F(p)$ for any $p \in \Lambda$.

Fix $p_0 \in M \setminus \Lambda$. We consider an exhaustion of $M$ by compact Runge subsets $\{K_j\}_{j \in \mathbb{N}}$ with $p_0 \in K_1$ and such that for any $j \in \mathbb{N}$ and all $p \in \Lambda \cap K_j$ we have that $\gamma_p \subset K_j$.

We follow the ideas of [29, Theorem 1] to construct a sequence of holomorphic functions $\{h_j : K_j \to \mathbb{C}_*\}_{j \in \mathbb{N}}$ with the following properties:

- $\|h_j(p) - h_{j-1}(p)\| < 1/2^j$ for any $p \in K_{j-1}$.
- $\int_{\gamma} h_j \omega = p(\gamma)$ for any closed curve $\gamma \subset K_j$, and
- $\int_{F(p)} h_j \omega = F(p)$ for any $\gamma_p \subset K_j$.

The limit map $h = \lim_{j \to \infty} h_j : M \to \mathbb{C}_*$ is the desired holomorphic function and $\theta = h \omega$ the desired 1-form. \hfill \square

2.3. Jets of maps. Let $M$ and $\mathcal{N}$ be smooth manifolds without boundary, $x_0 \in M$ be a point, and $f, g : M \to \mathcal{N}$ be smooth maps. $f$ and $g$ have a contact of order $k \in \mathbb{Z}_+$ at point $x_0$ if their Taylor series at this point coincide up to the order $k$. An equivalence class of maps $M \to \mathcal{N}$ which have a contact of order $k$ at point $x_0$ is called a $k$-jet; see e. g. [31, §1] for a basic reference.

In particular, if $\Omega$ is a neighbourhood of a point $p$ in an open Riemann surface $M$ and $f, g : \Omega \to \mathbb{C}^n$ are holomorphic functions, then they have a contact of order $k \in \mathbb{Z}_+$, or the same $k$-jet, at point $p$ if and only if $f - g$ has a zero of multiplicity (at least) $k + 1$ at $p$. If this is the case, for any distance function $d : M \times M \to \mathbb{R}_+$ on $M$ (not necessarily conformal) we have

$$
|f - g|(q) = O(d(q, p)^{k+1}) \quad \text{as } q \to p.
$$

Assume that $f, g : \Omega \to \mathbb{R}^n$ are harmonic maps, as, for instance, conformal minimal immersions. Then we say that they have a contact of order $k \in \mathbb{Z}_+$ (or the same $k$-jet) at point $p \in \Omega$ if $f(p) = g(p)$ and, if $k > 0$, the holomorphic 1-form $\partial(f - g)$
has a zero of multiplicity at least $k$ at $p$. Again, if such a pair of maps $f$ and $g$ have the same $k$-jet at point $p \in \Omega$ then (2.5) formally holds.

Throughout the paper we shall say that a holomorphic function has a zero of multiplicity $k \in \mathbb{N}$ at a point to mean that the function has a zero of multiplicity at least $k$ at the point. We will follow the same pattern when claiming that two functions have the same $k$-jet or a contact of order $k$ at a point.

3. Sprays of holomorphic functions

In this section, we prove an existence result for sprays of holomorphic functions. These techniques are very useful to solve the period problem when constructing minimal surfaces with arbitrary conformal structure. The idea of how to construct the sprays comes by combining the arguments in [8, 9] and [4].

Let $K$ be a Runge subset of an open Riemann surface $M$. Assume that we are given a holomorphic map $(f_1, f_2): K \rightarrow \mathbb{C}^2$ and a holomorphic function $H: M \rightarrow \mathbb{C}$ such that

\[ H = f_1^2 + f_2^2, \]

on $K$. We consider

\[ \eta: K \rightarrow \mathbb{C}, \quad \eta := f_1 - if_2. \]

By (3.1) and (3.2), we have that

\[ f_1 = \frac{1}{2}(\eta + \frac{H}{\eta}) \quad \text{and} \quad f_2 = \frac{i}{2}(\eta - \frac{H}{\eta}). \]

For such a holomorphic function $H: M \rightarrow \mathbb{C}$ verifying (3.1), we consider a map $\Phi$, which depends on a holomorphic function $h \in \mathcal{A}(K)$, and is formally defined as

\[ \Phi(h) := \left( \frac{1}{2}(h + \frac{H}{h}), \frac{i}{2}(h - \frac{H}{h}) \right). \]

Notice that $\Phi(\eta) = (f_1, f_2)$.

The following result is needed.

**Lemma 3.1.** Let $M$ be an open Riemann surface and $\theta$ be a holomorphic 1-form never vanishing on $M$. Let $K \subset M$ be a Runge compact subset, $\Lambda \subset K$ be a finite subset and $p_0 \in K \setminus \Lambda$ be a point. Let $k: \Lambda \rightarrow \mathbb{N}$ be a map and $(f_1, f_2): K \rightarrow \mathbb{C}^2$ be a holomorphic map of class $\mathcal{A}(K)$.

There exists a holomorphic map $(g_1, g_2): K \rightarrow \mathbb{C}^2$ of class $\mathcal{A}(K)$ such that

(i) $g_1$ and $g_2$ are (complex) linear independent.

(ii) $(g_1, g_2)$ approximates $(f_1, f_2)$ uniformly on $K$.

(iii) $g_1^2 + g_2^2 = f_1^2 + f_2^2$.

(iv) $((f_1, f_2) - (g_1, g_2)) \theta$ is an exact 1-form.

(v) $\int_{p_0}^{p} ((f_1, f_2) - (g_1, g_2)) \theta = 0$ for any $p \in \Lambda$.

Note that the integral in (v) does not depends on the chosen curve connecting $p_0$ with $p$ when (iv) holds.

(vi) $(g_1, g_2) - (f_1, f_2)$ has a zero of multiplicity $k(p)$ at any $p \in \Lambda$.  

Proof. If $f_1$ and $f_2$ are complex linear independent we are done. If that is not the case, assume that there exists $z_0 \in \mathbb{C}_*$ such that $f_2 = z_0 f_1$.

Define $H : K \to \mathbb{C}$ as in (3.1) and $\eta : K \to \mathbb{C}$ as in (3.2). Notice that equation (3.3) holds. Set $\Lambda = \{p_1, \ldots, p_m\}$ for $m \in \mathbb{N}$ and take $\gamma_1, \ldots, \gamma_l$ with $l \geq m$ smooth Jordan curves in $K$ such that:

- $\gamma_j$ connects $p_0$ to $p_j$ for every $j = 1, \ldots, m$.
- $\gamma_{m+1}, \ldots, \gamma_l \subset K$ are closed Jordan curves determining a basis of $H_1(K, \mathbb{Z})$.
- $\gamma_i \cap \gamma_j = \{p_0\}$ for every $i \neq j \in \{1, \ldots, l\}$.

We consider the period-interpolation map $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l) : \mathcal{A}(K) \to \mathbb{C}^l$ whose $j$-th coordinate, $\mathcal{P}_j : \mathcal{A}(K) \to \mathbb{C}$, $j = 1, \ldots, l$ is given by

$$\mathcal{P}_j(h) := \int_{\gamma_j} (\Phi(h) - (f_1, f_2)) \theta,$$

where $\Phi : \mathcal{A}(K) \to \mathbb{C}^2$ is the map defined by (3.1). Note that $\mathcal{P}(\eta) = 0$.

Take a holomorphic map $\varphi_0 \in \mathcal{A}(K)$ that has a zero of multiplicity $k(p)$ at any $p \in \Lambda$. Then, for any nonconstant holomorphic function $\varphi \in \mathcal{A}(K)$ contained in a small open neighbourhood of the zero function in $\mathcal{A}(K)$, the map $\exp(\varphi_0 \varphi) \eta$ clearly depends holomorphically on $\varphi$, where here $\exp(\cdot)$ denotes the exponential map. If $\varphi = 0$, then $\mathcal{P}(\exp(\varphi_0 \varphi) \eta) = \mathcal{P}(\eta) = 0$. Hence, since the space $\mathcal{A}(K)$ is of infinite dimension, then there exists a nonconstant function $\varphi \in \mathcal{A}(K)$ arbitrarily close to 0 such that $\mathcal{P}(\exp(\varphi_0 \varphi) \eta) = 0$. For such $\varphi$, the map

$$(g_1, g_2) := \Phi(\exp(\varphi_0 \varphi) \eta)$$

solves the lemma.

Indeed, from (3.6) and (3.1) we have that $g_1$ and $g_2$ are linearly independent if and only if $e^{\varphi}$ and 1 are, take into account that $f_2 = z_0 f_1$. Since $\varphi$ is nonconstant, then $g_1$ and $g_2$ are complex linear independent, and so (i) holds. If $\varphi$ is chosen close to 0 enough, then (ii) holds. A straightforward computation gives (iii) since $H = f_1^2 + f_2^2$. From $\mathcal{P}(\exp(\varphi_0 \varphi) \eta) = 0$ we deduce (iv) and (v), see (3.5). Finally, taking into account (3.6), (3.4), and (3.3) we obtain that $(g_1, g_2) - (f_1, f_2)$ has a zero of multiplicity at least the same that $\varphi_0$ at each $p \in \Lambda$; that is, (vi). \(\square\)

We now prove the main technical result of the paper. It ensures the existence of sprays which are dominating with respect to the curves involved in the period and interpolation problems.

Lemma 3.2. Let $M$ be an open Riemann surface and $\theta$ be a holomorphic 1-form never vanishing on $M$. Let $S = K \cup \Gamma \subset M$ be a very simple admissible subset and $L \subset M$ be a smoothly bounded compact domain such that $S \subset \bar{L}$ and the kernel component $S_0$ of $S$ is a strong deformation retract of $L$ (see Definition 2.2). Let $K_0, \ldots, K_{m'}$, $m' \in \mathbb{Z}_+$ denote the components of $K$ contained in $S_0$, where $K_0$ is the kernel component of $K$. Let $m \in \mathbb{Z}_+$, $m \geq m'$, and let $p_0, \ldots, p_m$ be distinct points in $S$ such that $p_i \in K_i$ for all $i = 0, \ldots, m'$ and $p_i \in K_0$ for all $i = m' + 1, \ldots, m$. Let $k : \{p_1, \ldots, p_m\} \to \mathbb{N}$ be a map. Let $H : L \to \mathbb{C}$ be a nonzero holomorphic map
of class $\mathcal{O}(L)$ and $f = (f_1, f_2): S \rightarrow \mathbb{C}^2$ be a holomorphic map of class $\mathcal{A}(S)$ such that

$$f_1^2 + f_2^2 = H|_S.$$  

(3.7)

There exists a holomorphic map $\tilde{f} = (\tilde{f}_1, \tilde{f}_2): L \rightarrow \mathbb{C}^2$ of class $\mathcal{O}(L)$ such that

(i) $\tilde{f}_1^2 + \tilde{f}_2^2 = H$ on $L$.

(ii) $\tilde{f}$ approximates $f$ uniformly on $S$.

(iii) $(\tilde{f} - f)\theta$ is an exact 1-form on $S$, and hence on $L$. Recall that $S$ is a strong deformation retract of $L$.

(iv) $\int_{p_0}^{p_j} (\tilde{f} - f)\theta = 0$ for any $j = 1, \ldots, m$ (by (iii) this integral is well defined independently of the chosen curve connecting $p_0$ to $p_j$).

(v) $\tilde{f} - f$ has a zero of multiplicity $k(p)$ at any point $p_1, \ldots, p_m$.

Proof. Assume that $S$ is connected. If that is not the case, we just add a curve connecting transversely $K_0$ to each connected component of $S$ different from $S_0$, recall Definition 2.2. Assume also by Lemma 3.1 that $(f_1, f_2)$ are (complex) linear independent.

We define $\eta$ as in (3.2) and by (3.7), equation (3.3) holds, that is,

$$\eta = f_1 - if_2 \quad \text{and} \quad (f_1, f_2) = \left(\frac{1}{2} (\eta + \frac{H}{\eta}), \frac{1}{2} (\eta - \frac{H}{\eta})\right).$$

Take $\gamma_1, \ldots, \gamma_l$ with $l \geq m$ smooth Jordan curves in $S$ such that:

(a) $\eta$ never vanishes at any point of the curves $\gamma_1, \ldots, \gamma_l$.

(b) $\gamma_j$ connects $p_0$ to $p_j$ for every $j = 1, \ldots, m$.

(c) $\gamma_{m+1}, \ldots, \gamma_l \subset S_0$ are closed Jordan curves determining a basis of $\mathcal{H}_1(S_0, \mathbb{Z})$, hence of $\mathcal{H}_1(L, \mathbb{Z})$. Recall that $S_0$ is a strong deformation retract of $L$.

(d) $\gamma_i \cap \gamma_j = \{p_0\}$ for every $i \neq j \in \{1, \ldots, l\}$.

Set $C := \bigcup_{j=1}^{l} \gamma_j$ and notice that $C$ is a Runge subset of $M$ which is a strong deformation retract of $L$.

We consider the period-interpolation map $P = (P_1, \ldots, P_l): \mathcal{A}(S) \rightarrow \mathbb{C}^l$ whose $j$-th coordinate, $j = 1, \ldots, l$ is given by equation (3.5) for the curves $\gamma_1, \ldots, \gamma_l$ recently defined and the functions $f_1$ and $f_2$ in the statement of the Lemma, see (3.3). Notice that for a function $h \in \mathcal{A}(S)$, the value of $P(h) \in \mathbb{C}^l$ only depends on $h|_C$.

Claim 3.3. For each $i = 1, 2$, there exist functions $h_{i,1}, \ldots, h_{i,l} \in \mathcal{O}(L)$ with a zero of order $k(p)$ at each point $p \in \Lambda$ such that the map $\varphi: \mathbb{C}^l \rightarrow \mathcal{O}(S)$ defined by

$$\varphi(\zeta) = \exp \left(\sum_{i=1}^{2} \sum_{j=1}^{l} \zeta_{i,j} h_{i,j}(\cdot)\right) \eta(\cdot),$$

(3.9)

for $\zeta = (\zeta_{i,j}) \in \mathbb{C}^l$, $i = 1, 2$ and $j = 1, \ldots, l$, is a dominating spray with respect to $P$ and has core $\eta$. That is to say, $\varphi(0) = \eta$ and the map $P \circ \varphi: \mathbb{C}^l \rightarrow \mathbb{C}^l$ is a
submersion at $\zeta = 0$. Then, there exists a Euclidean ball $W \subset \mathbb{C}^{2l}$ centred at the origin such that $(P \circ \varphi): W \to (P \circ \varphi)(W)$ is a biholomorphism.

Proof. Fix $j = 1, \ldots, l$ for the upcoming reasoning. Since $f_1$ and $f_2$ are linear independent, we may choose two point, $p_{i,j} \in \gamma_j$, $i = 1, 2$ different from the endpoints, such that the vectors
\[
\left\{ \left( \frac{1}{2}(\eta + \frac{H}{\eta}), \frac{i}{2}(\eta - \frac{H}{\eta}) \right) : (p_{i,j}) \right\}_{i=1,2}
\]
determine a basis of $\mathbb{C}^2$, see (3.3). Consider for each $i = 1, 2$ a continuous function $g_{i,j}: C \to \mathbb{C}$ supported on a small neighbourhood of point $p_{i,j} \in \gamma_j$; the precise value will be specified later. In particular, we have that $g_{i,j}(q) = 0$ for each $q \in C \setminus \gamma_j$.

Given $\zeta = (\zeta_{i,j}) \in \mathbb{C}^{2l}$, we consider a continuous map $\hat{\varphi}(\zeta): C \to \mathbb{C}$, depending holomorphically on $\zeta \in \mathbb{C}^{2l}$, defined by
\[
(3.10) \quad \hat{\varphi}(\zeta) = \exp \left( \sum_{i=1}^{2} \sum_{j=1}^{l} \zeta_{i,j} g_{i,j}(\cdot) \right) \eta(\cdot),
\]
recall that $\exp(\cdot)$ denotes the exponential map. The differential of $P \circ \hat{\varphi}$ with respect to $\zeta_{i,j}$ at $\zeta = 0$ may be expressed for any $i = 1, 2$ as
\[
\frac{\partial P_m \circ \hat{\varphi}}{\partial \zeta_{i,j}} \bigg|_{\zeta=0}(\zeta) = \begin{cases} (0,0), & j \neq m, \\
(0,\int g_{i,j} \left( \frac{1}{2}(\eta - \frac{H}{\eta}), \frac{i}{2}(\eta + \frac{H}{\eta}) \right) \theta, & j = m.
\end{cases}
\]

We affirm that we may choose the values of each function $g_{i,j}$ at the curve $\gamma_j$ in order to the vectors
\[
(3.11) \quad \left\{ \frac{\partial P_{1,m} \circ \hat{\varphi}}{\partial \zeta_{1,j}} \bigg|_{\zeta=0}(\zeta), \frac{\partial P_{l,m} \circ \hat{\varphi}}{\partial \zeta_{2,j}} \bigg|_{\zeta=0}(\zeta) \right\}
\]
determine a basis of $\mathbb{C}^2$ and hence the differential of $P \circ \hat{\varphi}$ at $\zeta = 0$ is surjective. Indeed, let $\gamma_j: [0,1] \to \gamma_j$ be a parametrization of the curve $\gamma_j$, we identify $\gamma_j \equiv \gamma_j([0,1])$. For each $i = 1, 2$ there exists a point $t_{i,j} \in (0,1)$ such that $\gamma_j(t_{i,j}) = p_{i,j}$. Take a positive number $\rho > 0$ small enough such that
\[
[t_{1,j} - \rho, t_{1,j} + \rho] \cap [t_{2,j} - \rho, t_{2,j} + \rho] = \emptyset \quad \text{and} \quad [t_{i,j} - \rho, t_{i,j} + \rho] \subset [0,1], \ i = 1, 2.
\]

We now define each of the function $g_{i,j}$, $i = 1, 2$ such that they are supported on a small neighbourhood of $t_{i,j}$, that is:
\[
(3.12) \quad g_{i,j}(t) = 0 \quad \text{if} \quad t \in [0,1] \setminus [t_{i,j} - \rho, t_{i,j} + \rho]
\]
and also such that
\[
\int_0^1 g_{i,j}(t) \, dt = \int_{t_{i,j} - \rho}^{t_{i,j} + \rho} g_{i,j}(t) \, dt = 1.
\]

We affirm that the functions already defined satisfy the conclusion of the claim except that they are only defined on $C$; we will deal with this later. Indeed, for
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\[ \rho > 0 \text{ sufficiently small we have that} \]

\[ \left. \frac{\partial P_j \circ \tilde{\varphi}}{\partial \zeta_{i,j}} \right|_{\zeta = 0} (\zeta) = \int_{\gamma_j} g_{i,j} \left( \frac{1}{2} (\eta - H \eta), \frac{i}{2} (\eta + H \eta) \right) \theta \]

takes approximately the value

\[ \left( \frac{1}{2} (\eta - H \eta), \frac{i}{2} (\eta + H \eta) \right) (t_{i,j}) \theta (\gamma_j (t_{i,j}), \tilde{\gamma}_j (t_{i,j})) \]

for any \( i = 1, 2 \). Since the 1-form \( \theta \) never vanishes on \( M \) and since

\[ \left( \frac{1}{2} (\eta + H \eta), \frac{i}{2} (\eta - H \eta) \right) = \left( \frac{1}{2} (\eta - H \eta), \frac{i}{2} (\eta + H \eta) \right) \left( 0 \quad i \end{array} \right), \]

we have that the vectors in (3.11) are a basis of \( \mathbb{C}^2 \) provided that \( \rho > 0 \) is small enough.

To finish the claim we have to use Mergelyan Theorem with jet-interpolation (see [21]) to approximate each continuous function \( g_{i,j} : C \rightarrow \mathbb{C} \) by a holomorphic function \( h_{i,j} : L \rightarrow \mathbb{C} \) such that

(e) \( h_{i,j} \) approximates \( g_{i,j} \) uniformly on \( C \).
(f) \( h_{i,j} \) has a zero of order \( k(p) \in \mathbb{N} \) at each point \( p \in \Lambda \).
(g) \( h_{i,j} \) vanishes at the points of \( L \setminus C \) where \( \eta \) vanishes.

Recall that the function \( g_{i,j} \) vanishes at a neighbourhood of any \( p \in \Lambda \). Therefore, if the approximation of (e) is close enough, the map \( \varphi \) defined in (3.9), which is obtaining by replacing in (3.10) each of the function \( g_{i,j} \) by \( h_{i,j} \), is also dominating with respect to \( \mathcal{P} \).

□

Mergelyan Theorem with jet-interpolation applied to \( \eta : S \rightarrow \mathbb{C} \) provides a holomorphic function \( \tilde{\eta} : L \rightarrow \mathbb{C} \) such that

(h) \( \tilde{\eta} \) approximates \( \eta \) uniformly on \( S \).
(i) \( \tilde{\eta} \) has exactly the same zeros of \( \eta \) with the same multiplicity. Hence \( H/\tilde{\eta} \) is holomorphic.

We consider the map \( \tilde{\varphi} : \mathbb{C}^{2l} \rightarrow \mathcal{O}(L) \) given by

\[ \tilde{\varphi}(\zeta) = \exp \left( \sum_{i=1}^{2} \sum_{j=1}^{l} \zeta_{i,j} h_{i,j}(\cdot) \right) \tilde{\eta}(\cdot) \]

and for a close to zero \( \zeta \in \mathbb{C}^{2l} \), we define the map \( \Phi_{\zeta} : L \rightarrow \mathbb{C}^2 \) given by

\[ \Phi_{\zeta} := \Phi(\tilde{\varphi}(\zeta)) = \left( \frac{1}{2} (\tilde{\varphi}(\zeta) + \frac{H}{\tilde{\varphi}(\zeta)}), \frac{i}{2} (\tilde{\varphi}(\zeta) - \frac{H}{\tilde{\varphi}(\zeta)}) \right), \]

recall (3.12). \( \Phi_{\zeta} \) is holomorphic and depends holomorphically on \( \zeta \in \mathbb{C}^{2l} \) by properties (g) and (i). Furthermore, provided that the approximations involved in (e) and (h) are close enough, then the spray \( \tilde{\varphi} \) is dominating with respect to \( \mathcal{P} \). Therefore, there exists a close to zero \( \tilde{\zeta} \in \mathbb{C}^{2l} \) such that

\[ \mathcal{P}(\tilde{\varphi} (\tilde{\zeta})) = \left( \int_{\gamma_j} \Phi_{\zeta} \right)_{j=1,\ldots,l} = 0 \in \mathbb{C}^{2l}. \]
Thus, the map \((\tilde{f}_1, \tilde{f}_2)\): \(L \to \C^2\) defined by
\[
(\tilde{f}_1, \tilde{f}_2) := \Phi \zeta
\]
is holomorphic and verifies conditions (i)–(iv). Indeed, a straightforward computation gives that \(\tilde{f}_1\) and \(\tilde{f}_2\) verify condition (i). Condition (ii) follows from (h) provided that \(\zeta\) is chosen small enough, see (3.13) and (3.14). Equation (3.15) ensures conditions (iii) and (iv). Finally, since each of the function \(h_{i,j}\) has a zero of multiplicity \(k(p)\) at any point \(p \in \Lambda\), see Claim 3.3, then also \(\tilde{f} - f\) has a zero of multiplicity \(k(p)\) at any point \(p \in \Lambda\), that is, condition (v); see (3.8), (3.13), and (3.14). □

4. Completeness

This section is dedicated to prove the technical results needed to ensure completeness of the solutions. We start with the following lemma.

**Lemma 4.1.** Let \(M\) be an open Riemann surface and \(\theta\) be a holomorphic 1-form never vanishing on \(M\). Let \(K \subset L \subset M\) be smoothly bounded connected compact domains such that \(L \setminus K\) is a family of pairwise disjoint compact annuli. Let \(\Lambda \subset K\) be a finite subset, let \(k: \Lambda \to \N\) be a map, and take \(p_0 \in \tilde{K} \setminus \Lambda\). Let also \(H: L \to \C\) be a nonzero holomorphic function and \(f = (f_1, f_2): K \to \C^2\) be a holomorphic map such that \(f_1^2 + f_2^2 = H|_K\).

Given any \(\tau > 0\), there exists a holomorphic map \(\tilde{f} = (\tilde{f}_1, \tilde{f}_2): L \to \C^2\) such that

(i) \(\tilde{f}_1^2 + \tilde{f}_2^2 = H\) on \(L\).
(ii) \(\tilde{f}\) approximates \(f\) uniformly on \(K\).
(iii) \((\tilde{f} - f) \theta\) is an exact 1-form on \(K\), hence on \(L\). Recall that \(L \setminus \tilde{K}\) are annuli.
(iv) \(\int_{p_0}^p (\tilde{f} - f) \theta = 0\) for any \(p \in \Lambda\). The integral is well defined independently of the chosen curve by (iii).
(v) \(\tilde{f} - f\) has a zero of multiplicity \(k(p)\) at any point \(p \in \Lambda\).
(vi) If \(\alpha \subset L\) is a curve connecting \(p_0\) with \(bL\), then
\[
\int_{\alpha} (|\tilde{f}_1|^2 + |\tilde{f}_2|^2 + |H|) \|\theta\| > \tau.
\]

**Proof.** For the clearness of exposition we assume that \(L \setminus \tilde{K}\) is connected and then a single annulus. In the general case, we just apply the upcoming reasoning to each of the annuli.

Recall that \(H: L \to \C\) is a nonzero function and \(\theta\) never vanishes on \(M\). Then, since \(L \setminus K\) is an annulus, there exists a family of pairwise disjoint, smoothly bounded, compact disks \(D_1, \ldots, D_m \subset L \setminus K\) such that if \(\alpha \subset L\) is a curve connecting \(p_0\) with \(bL\) and \(\alpha\) do not cross any of the disk \(D_i, i = 1, \ldots, n\), then
\[
\int_{\alpha} |H| \|\theta\| > \tau.
\]

To construct such disks we use pieces of a labyrinth of Jorge-Xavier type contained in \(L \setminus K\), see [28]. For a detailed description of the disks see [29, Lemma 4.1]. We say
that a curve $\alpha$ crosses the disk $D_i$ if the curve $\alpha \cap D_i$ has length at least the width of the subset $D_i$.

Set $D := \bigcup_{i=1}^n D_i$ and $S := K \cup D$, note that $S$ is a very simple admissible subset and a strong deformation retract of $L$. Pick a map $g = (g_1, g_2) : S \to \mathbb{C}^2$ of class $A(S)$ such that

(A) $g_1^2 + g_2^2 = H$ on $S$.
(B) $(g_1, g_2) = (f_1, f_2)$ on $K$.
(C) If $\alpha \subset L$ is a curve that crosses any of the disks $D_i$, then

\[ \int_{\alpha \cap D_i} (|g_1|^2 + |g_2|^2) \parallel \theta \parallel > \tau. \]

The existence of such a map $(g_1, g_2)$ is clear. Necessarily $(g_1, g_2) = (f_1, f_2)$ on $K$ and take for instance $g_1$ big enough on each $D_i$ in order to property (C) holds and $g_2$ such that property (A) is verified, recall that each $D_i$ is a disk and hence simply connected.

Lemma 3.2 applied to

$M, \theta, S \subset L, \Lambda \subset K \subset S, p_0 \in K, k, H,$ and $g = (g_1, g_2)$ provides a holomorphic map $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) : L \to \mathbb{C}^2$ such that

(D) $\tilde{f}_1^2 + \tilde{f}_2^2 = H$ on $L$.
(E) $\tilde{f}$ approximates $g$ uniformly on $S$.
(F) $(\tilde{f} - g) \theta$ is an exact 1-form on $S$, and hence on $L$. Recall $S$ is a strong deformation retract of $L$.
(G) $\int_{p_0}^p (\tilde{f} - g) \theta = 0$ for any $p \in \Lambda \subset K$.
(H) $\tilde{f} - g$ has a zero of multiplicity $k(p)$ at any point $p \in \Lambda \subset K$.

We claim that $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ solves the lemma. Indeed, (i) equals (D), (B) and (E) imply (ii). Since $K$ is a strong deformation retract of $L$ and by (B) and (F) we have (iii). Furthermore, (iv) and (v) follows from (B), (G), and (H).

Finally, let us check that (vi) holds. Take a Jordan arc $\alpha \subset L$ connecting $p_0$ with $bL$, we distinguish cases depending on $\alpha$ crosses $D$ or not. If $\alpha$ does not cross any subset of $D$, then by (4.1) we have

\[ \int_{\alpha} (|\tilde{f}_1|^2 + |\tilde{f}_2|^2 + |H|) \parallel \theta \parallel \geq \int_{\alpha} |H| \parallel \theta \parallel > \tau, \]

whereas if $\alpha$ crosses any of the subset of $D$, then by (C) and taking into account conditions (E) and (B), we have that

\[ \int_{\alpha} (|\tilde{f}_1|^2 + |\tilde{f}_2|^2 + |H|) \parallel \theta \parallel \geq \int_{\alpha \cap D} (|\tilde{f}_1|^2 + |\tilde{f}_2|^2) \parallel \theta \parallel > \tau. \]

Hence, condition (vi) holds. \qed

To finish the section, let us prove the following result. It is used later to ensure completeness of the solutions.
Proposition 4.2. Let $M$ be an open Riemann surface, $K \subset M$ be a smoothly bounded compact Runge domain, and $\Lambda \subset M$ be a closed and discrete subset. Let $\theta$ be a holomorphic 1-form never vanishing on $M$. Let $\Omega_p \ni p, p \in \Lambda$, be small pairwise disjoint compact neighbourhoods of the points in $\Lambda$. Set $\Omega := \bigcup_{p \in \Lambda} \Omega_p \subset M$. Let $H : M \to \mathbb{C}$ be a nonzero holomorphic function and $F : \Omega \to \mathbb{R}^2$ be a harmonic map. Let also $f = (f_1, f_2) : K \cup \Omega \to \mathbb{C}^2$ be a holomorphic map such that $f_1^2 + f_2^2 = H$ on $K \cup \Omega$ and $f \theta = \partial F$ on $\Omega$. Let also $p_0 \in K \setminus \Omega$ be a point, $k : \Lambda \to \mathbb{N}$ be a map and $p : \mathcal{H}_1(M; \mathbb{Z}) \to \mathbb{R}^2$ be a group morphism such that

$$p(\gamma) = \exists \int_\gamma (f_1, f_2) \theta \text{ for any closed curve } \gamma \subset K. \quad (4.2)$$

Then, there exists a holomorphic map $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) : M \to \mathbb{C}^2$ such that

(i) $\tilde{f}_1^2 + \tilde{f}_2^2 = H$ on $M$.
(ii) $\tilde{f}$ approximates $f$ uniformly on $K$.
(iii) $\Re(\tilde{f} \theta)$ is an exact 1-form and $\exists \int \tilde{f} \theta = p(\gamma)$ for any closed curve $\gamma$ on $M$.
(iv) $\Re \int_{p_0}^p \tilde{f} \theta = F(p)$ for any $p \in \Lambda$.
(v) $\tilde{f} - f$ has a zero of multiplicity $k(p)$ at any point $p \in \Lambda$.
(vi) If $\alpha$ is a divergent arc in $M$, then $\int_{\alpha} (|\tilde{f}_1|^2 + |\tilde{f}_2|^2 + |H|) ||\theta||$ is infinite.

Proof. Up to slightly enlarging $K$ and shrinking the subsets $\Omega_p$ if necessary, we may assume without loss of generality that $bK \cap \Lambda = \emptyset$ and that $\Omega_p \subset K$ or $\Omega_p \cap K = \emptyset$ for any $p \in \Lambda$.

Set $K_0 := K$ and let $\{K_j\}_{j \in \mathbb{N}}$ be an exhaustion of smoothly bounded compact Runge domans on $M$ such that

$$K_0 \Subset K_1 \Subset \cdots \subset \bigcup_{j \in \mathbb{N}} K_j = M \quad \text{and} \quad \chi(K_j \setminus K_{j-1}) \in \{-1, 0\}. \quad \text{Recall that } \chi(A) \text{ denotes the Euler characteristic of } A. \text{ We assume that } bK_j \cap \Lambda = \emptyset \text{ and that for any } p \in \Lambda, \text{ we have that } \Omega_p \subset K_j \text{ or } \Omega_p \cap K_j = \emptyset \text{ for } j \in \mathbb{N}. \text{ The existence of such a sequence is guaranteed by basic topological arguments.}

Set $(f_{0,1}, f_{0,2}) := (f_1, f_2) : K_0 \cup \Omega \to \mathbb{C}^2$. Given a sequence of positive numbers $\{\epsilon_j\}_{j \in \mathbb{N}}$ which will be specified later and the fixed point $p_0 \in K_0 \setminus \Omega$, we recursively construct a sequence of holomorphic maps $\{(f_{j,1}, f_{j,2}) : K_j \cup \Omega \to \mathbb{C}^2\}_{j \in \mathbb{N}}$ with the following properties for any $j \in \mathbb{N}$:

(i) $f_{j,1}^2 + f_{j,2}^2 = H$ on $K_j$.
(ii) $\|f_{j,1,1}f_{j,2,2} - (f_{j-1,1,1}f_{j-1,2,2})(p)\| < \epsilon_j$ for any $p \in K_{j-1}$.
(iii) $\int_\gamma (f_{j,1}, f_{j,2}) \theta = ip(\gamma)$ for any closed curve $\gamma \subset K_j$. 

\[ \int_\gamma (f_{j,1}, f_{j,2}) \theta = ip(\gamma) \]
(iv)} \mathbb{R} \int_{p_0}^p (f_{j,1}, f_{j,2}) \theta = F(p) \text{ for any } p \in \Lambda \cap K_j. \text{ By (iii)} \text{ this is independent of the chosen curve connecting } p_0 \text{ with } p.

(v)} (f_{j,1}, f_{j,2}) - (f_{j-1,1}, f_{j-1,2}) \text{ has a zero of multiplicity } k(p) \text{ at any } p \in \Lambda.

(vi)} \text{ If } \alpha \subset K_j \text{ is a curve connecting } p_0 \text{ with } bK_j \text{ then }

\int_{\alpha} (|f_{j,1}|^2 + |f_{j,2}|^2 + |H|) \|	heta\| > j.

Assume that we have proved the existence of such a sequence. If we choose the positive numbers \{\epsilon_j\}_{j \in \mathbb{N}} decreasing to zero fast enough, then properties (ii)} \text{ for } j \in \mathbb{N} \text{ ensure that there exists a limit holomorphic map }

\tilde{f} : M \to \mathbb{C}^2, \quad \tilde{f} = (\tilde{f}_1, \tilde{f}_2) := \lim_{j \to \infty} (f_{j,1}, f_{j,2})

that approximates } f = (f_1, f_2) \text{ uniformly on } K = K_0, \text{ that is, condition (ii) holds. Clearly, properties (i)} \text{ imply (i)}, \text{ (ii) is a consequence of (iii)} \text{; indeed, the real part of the equation implies that } \Re(\tilde{f}\theta) \text{ is an exact 1-form whereas the imaginary part ensures } \Im \int_{\gamma} \tilde{f}\theta = p(\gamma). \text{ Additionally, (iv) is a consequence of (iv)} \text{, } j \in \mathbb{N}, \text{ whilst (v) follows from (vi)} \text{, } j \in \mathbb{N}. \text{ Finally, properties (vi)} \text{ for any } j \in \mathbb{N} \text{ ensure that any divergent path has infinite length, hence (vi) holds.}

Let us prove the existence of such a sequence to finish the proof. First, the base of the induction is given by \((f_0,1, f_0,2)\). Clearly, (i) holds. Conditions (ii), (vi), and (vi)} \text{ are vacuous. Condition (iii)} \text{ holds since } (4.2) \text{ and (iv)} \text{ follows since } \tilde{f} = \partial F \text{ on } K = K_0.

Assume now that we have already constructed the term \((f_{j-1,1}, f_{j-1,2}) : K_{j-1} \cup \Omega \to \mathbb{C}^2\) and let us construct \((f_{j,1}, f_{j,2}) : K_j \cup \Omega \to \mathbb{C}^2\). Notice that, since } \Lambda \text{ is closed and discrete, then } \Lambda_j := \Lambda \cap K_j \text{ is empty or finite. First of all, it is clear that } \tilde{(f_{j,1}, f_{j,2})|_{\partial \Omega \cup K_j}} := (f_{j-1,1}, f_{j-1,2}) \text{ verifies } (vi) \text{ for any } p \in \Lambda \setminus \Lambda_j. \text{ Hence, it is enough to construct } (f_{j,1}, f_{j,2}) \text{ on } K_j.

Let } S \subset K_j \text{ be a connected very simple admissible Runge subset obtaining by attaching finitely many pairwise disjoint Jordan arc to } K_{j-1} \cup (\Omega \cap K_j) \text{ so that } S \text{ is a strong deformation retract of } K_j \text{ and } \Lambda_j \subset S. \text{ Observe that some of the attached arcs describe the topology of } K_j \setminus K_{j-1}, \text{ if nontrivial, whilst the others connect } K_{j-1} \text{ to any } \Omega_p \text{ for each } p \in \Lambda \cap (K_j \setminus K_{j-1}).

Next, we extend \((f_{j-1,1}, f_{j-1,2})\) to } S \text{ continuously using } (3) \text{ Lemma 3.3] \text{ such that the following properties hold:}

(a) } f^2_{j-1,1} + f^2_{j-1,2} = H \text{ on } S, \text{ recall that } (f^2_{j-1,1} + f^2_{j-1,2})(p) = H \text{ for any } p \in \Omega,

(b) \text{ if } \gamma \subset S \text{ is a closed arc, then }

\int_{\gamma} (f_{j-1,1}, f_{j-1,2}) \theta = i\mathbb{P}(\gamma),

(c) \text{ and if } \gamma_p \subset S \text{ is an arc connecting } p_0 \text{ with } p \in \Lambda_j, \text{ then }

\Re \int_{\gamma_p} (f_{j-1,1}, f_{j-1,2}) \theta = F(p).
Take $L$ a smoothly bounded compact neighbourhood of $S$ which is a strong deformation retract of $K_j$, then Lemma 4.1 provides a holomorphic map $(g_{j,1}, g_{j,2}) : L \to \mathbb{C}^2$ such that

(A1) $g_{j,1}^2 + g_{j,2}^2 = H$ on $M$.
(A2) $\|(g_{j,1}, g_{j,2})(p) - (f_{j-1,1}, f_{j-1,2})(p)\| < \epsilon_j/2$ for any $p \in S$.
(A3) $(g_{j,1}, g_{j,2}) (f_{j-1,1}, f_{j-1,2})$ is an exact 1-form on $S$, hence on $L$, see (b).
(A4) $\int_{p_0}^{p} ((g_{j,1}, g_{j,2}) - (f_{j-1,1}, f_{j-1,2})) \theta = 0$ for any $p \in \Lambda_j$.
(A5) $(g_{j,1}, g_{j,2}) = (f_{j-1,1}, f_{j-1,2})$ has a zero of multiplicity $k(p)$ at any point $p \in \Lambda_j$.

Therefore, Lemma 4.1 applied to $M$, $\theta$, $L \subset K_j$, $\Lambda_j \subset L$, $k$, $H$, $(g_{j,1}, g_{j,2})$, and $j > 0$ provides a holomorphic map $(f_{j,1}, f_{j,2})$ such that

(B1) $f_{j,1}^2 + f_{j,2}^2 = H$ on $K_j$.
(B2) $\|(f_{j,1}, f_{j,2})(p) - (g_{j,1}, g_{j,2})(p)\| < \epsilon_j/2$ for any $p \in L$.
(B3) $(f_{j,1}, f_{j,2}) - (g_{j,1}, g_{j,2})$ is an exact 1-form on $L$, hence on $K_j$.
(B4) $\int_{p_0}^{p} ((f_{j,1}, f_{j,2}) - (g_{j,1}, g_{j,2})) \theta = 0$ for any $p \in \Lambda_j$.
(B5) $(f_{j,1}, f_{j,2}) - (g_{j,1}, g_{j,2})$ has a zero of multiplicity $k(p)$ at any point $p \in \Lambda_j$.
(B6) If $\alpha \subset K_j$ is a curve connecting $p_0$ with $bK_j$, then

$$\int_{\alpha} (|f_{j,1}|^2 + |f_{j,2}|^2 + |H|) \| \theta \| > j.$$  

We claim that the pair $(f_{j,1}, f_{j,2})$ is the desired map. Indeed, it is clear that (i$_j$) equals (B1) and (vi$_j$) equals (B6). Additionally, (A2) and (B2) imply (ii$_j$). (v$_j$) follows from (A5) and (B5). On the other hand, by (iii$_{j-1}$) we have that the real part of $(f_{j-1,1}, f_{j-1,2}) \theta$ is an exact 1-form, hence (A3) and (B3) implies (iii$_j$), recall that $L$ is a strong deformation retract of $K_j$. Finally, (iv$_j$) follows from (c), (A4), and (B4).

**Remark 4.3.** Proposition 4.2 holds simpler if one just ensure interpolation but not jet interpolation. That is, if the map $k(p) = 0$ for any $p \in \Lambda$, then it is not necessary to consider the subsets $\Omega_p$’s defining the jets, and so that $F : \Lambda \to \mathbb{R}^3$ is just a map. In such a case, Proposition 4.2 ensures conditions (i)–(iv) and (vi); whilst (v) does not make sense.

5. Main results and applications

In this section we show how the previous results may be used in order to prove Theorem 1.3. Next, Theorem 1.1 is deduced. We introduce the following notation for clearness of exposition. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-2}$ be the projection into the $n - 2$ last coordinates, that is, if $(x_1, \ldots, x_n) \in \mathbb{R}^n$ then $\pi(x_1, \ldots, x_n) = (x_3, \ldots, x_n) \in \mathbb{R}^{n-2}$.

As stated, Theorem 1.3 is trivially a consequence of the following result.

**Theorem 5.1.** Let $M$ be an open Riemann surface, $K \subset M$ be a smoothly bounded compact Runge domain, and $\Lambda \subset M$ be a closed discrete subset. Let $\Omega_p \ni p$,
p ∈ Λ, be small pairwise disjoint compact neighbourhoods of the points in Λ. Set
Ω = \bigcup_{p \in Λ} Ω_p ⊂ M. Let \( \tilde{\mathfrak{H}} : M \to \mathbb{R}^{n-2} \) be a nonconstant harmonic map and
\( X : K \cup Ω \to \mathbb{R}^n \) be a conformal minimal immersion such that \( π \circ X = \tilde{\mathfrak{H}} \) on \( K \cup Ω \).
Let \( k : Λ \to \mathbb{N} \) be a map and \( p : H_1(M; \mathbb{Z}) \to \mathbb{R}^n \) be a group morphism such that
FluxX = p on K and

\[
3 \int_γ \partial \tilde{\mathfrak{H}} = π \circ p(γ), \text{ for any closed curve } γ \subset M.
\]

There exists a complete conformal minimal immersion \( \tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n) : M \to \mathbb{R}^n \) such that

(a) \( (\tilde{X}_3, \ldots, \tilde{X}_n) = \tilde{\mathfrak{H}} \).
(b) \( \tilde{X} \) approximate \( X \) uniformly on \( K \).
(c) \( \tilde{X} \) and \( X \) have a contact of order \( k(p) \in \mathbb{N} \) at any point \( p \in Λ \).
(d) \( \text{Flux}_{\tilde{X}} = p \) on \( M \).

Proof. Let \( θ \) be a holomorphic 1-form never vanishing on \( M \). Set \( \partial X = f θ \) where
\( f = (f_1, \ldots, f_n) : K \cup Ω \to \mathbb{A}_n = \mathbb{R} \setminus \{0\} \) is a holomorphic map, see \( 2.1 \). Set also
\( \partial \tilde{\mathfrak{H}} = h θ \) where \( h = (h_3, \ldots, h_n) : M \to \mathbb{C}^{n-2} \) is a holomorphic map. Notice that
\( (f_3, \ldots, f_n) = (h_3, \ldots, h_n) \) on \( K \cup Ω \).

Write \( X = (X_1, \ldots, X_n) \) and \( p = (p_1, \ldots, p_n) \). We consider the holomorphic map
\( H : M \to \mathbb{C} \) defined by

\[
H := -(h_3^2 + \cdots + h_n^2) = -(f_3^2 + \cdots + f_n^2),
\]
which is nonzero since \( \tilde{\mathfrak{H}} \) is nonconstant. Fix also a point \( p_0 \in K \setminus Ω \). Proposition
\( 4.2 \) applied to the data
\( K \subset M, Λ \subset Ω \subset M, \theta, p_0 \in K, H, F = (X_1, X_2), f = (f_1, f_2), k, \) and \( (p_1, p_2) \)
provides a holomorphic map \( \tilde{f} = (\tilde{f}_1, \tilde{f}_2) : M \to \mathbb{C}^2 \) with the following properties:

(i) \( \tilde{f}_1^2 + \tilde{f}_2^2 = H \) on \( M \).
(ii) \( (\tilde{f}_1, \tilde{f}_2) \) approximates \( (f_1, f_2) \) uniformly on \( K \).
(iii) \( \Re(\tilde{f} θ) \) is an exact 1-form and \( \Im \int_γ (\tilde{f}_1, \tilde{f}_2) θ = (p_1, p_2)(γ) \) for any closed curve
\( γ \) on \( M \).
(iv) \( \Re \int_{p_0}^p (\tilde{f}_1, \tilde{f}_2) θ = (X_1, X_2)(p) \) for any \( p \in Λ \).
(v) \( (\tilde{f}_1, \tilde{f}_2) - (f_1, f_2) \) has a zero of multiplicity \( k(p) \) at any point \( p \in Λ \).
(vi) If \( \alpha \) is a divergent arc in \( M \), then \( \int_α (|\tilde{f}_1|^2 + |\tilde{f}_2|^2 + |H|)|θ| \) is infinite.

We claim that the conformal minimal immersion \( \tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n) : M \to \mathbb{R}^n \)
defined by

\[
(\tilde{X}_1, \tilde{X}_2) = X(p_0) + \Re \int_{p_0}^p (\tilde{f}_1, \tilde{f}_2) θ, \quad \text{and} \quad (\tilde{X}_3, \ldots, \tilde{X}_n) = \tilde{\mathfrak{H}}
\]
solves the Theorem. Indeed, clearly (a) is satisfied. By first part of property (iii), \( \tilde{X} \)
is a well defined map. By (i) and \( 5.2 \), we have that \( (\tilde{f}_1, \ldots, \tilde{f}_n) \) is a holomorphic
map assuming values into $\mathfrak{a}_+$, hence $\tilde{X}$ is a conformal minimal immersion. (b) follows from (ii). (c) is implied by (iv) and (v). Second part of (iii) implies (d), see \((5.1)\). Finally, if $\alpha$ is a divergent curve in $M$, taking into account (vi) and \((5.2)\) we have that the length of $\alpha$ via $\tilde{X}$ is infinite, and hence $\tilde{X}$ is complete. $\square$

At this point, we deduce Corollary 1.2 from Theorem 1.3 and an existence result for harmonic functions due to Forstnerič \cite{22}. Recall that Theorem 1.3 trivially follows from Theorem 5.1.

Proof of Corollary 1.2. Let $M$ be an open Riemann surface and $\Lambda \subset M$ be a closed discrete subset. Let also $F = (F_1, F_2, F_3) : \Lambda \to \mathbb{R}^3$ be a map. By Forstnerič \cite[Theorem 2.1]{22}, there exists a harmonic function $h : M \to \mathbb{R}$ such that

- $h$ has no critical points, that is, $dh \neq 0$ on $M$, and
- $h|\Lambda = F_3$.

Theorem 1.3 applied to the map $F = (F_1, F_2)$ and $\mathfrak{d} = h$ provides a conformal minimal immersion $X = (X_1, X_2, X_3) : M \to \mathbb{R}^3$ such that $X_3 = h$. Therefore, its Gauss map $M \to S^2$ misses the north and south poles of the sphere. Equivalently, its generalized Gauss map $G_X : M \to \mathbb{CP}^2$, see \((1.1)\), fails to intersect the complex plane $\{(z_1, z_2, z_3) \in \mathbb{CP}^2 : z_3 = 0\}$. $\square$

We conclude the paper proving Theorem 1.1. It is deduced from the following result.

Theorem 5.2. Let $M$ be an open Riemann surface and $\Lambda \subset M$ be a closed discrete subset. Let $Y : \Lambda \to \mathbb{R}^n$, $n \geq 3$, be a map and $p : H_1(M; \mathbb{Z}) \to \mathbb{R}^n$ be a group morphism. Then, there exists a complete conformal minimal immersion $X : M \to \mathbb{R}^n$ such that

- (I) $X$ and $Y$ agrees at the points of $\Lambda$.
- (II) Flux $\mathfrak{X} = p$.
- (III) The generalized Gauss map $G_X : M \to \mathbb{CP}^{n-1}$ of $X$ is nondegenerate and fails to intersect $n$ hyperplanes of $\mathbb{CP}^{n-1}$ in general position.

Proof. We distinguish cases depending on $n \in \mathbb{N}$.

Assume $n \in \mathbb{N}$ is even:

Let $\theta$ be a holomorphic 1-form never vanishing on $M$. Consider $K \subset M \setminus \Lambda$ a small simply connected compact disk. Write $p = ((p_{j,1}, p_{j,2})_{j=1, \ldots, n/2})$ and $Y = ((Y_{j,1}, Y_{j,2})_{j=1, \ldots, n/2})$. Fix $p_0 \in K$ and for any $j = 1, \ldots, n/2$ take a complex number $\zeta_j \in \mathbb{C} \setminus \{0\}$ and a holomorphic map $f_j = (f_{j,1}, f_{j,2}) : K \to \mathbb{C}^2$ such that

\[
\sum_{j=1}^{n/2} \zeta_j^2 = 0.
\]

\[
f_{j,1}^2 + f_{j,2}^2 = \zeta_j^2 \text{ on } K \text{ for any } j = 1, \ldots, n/2.
\]

(a3) The map $((f_j)_{j=1,\ldots,n/2})$ is nondegenerate.
Take into account Lemma 3.1 for (a4).

Next, Proposition 4.2 applied with only interpolation, see Remark 4.3, for any \( j = 1, \ldots, n/2 \), to the data
\[
K \subset M, \quad \Lambda, \quad \theta, \quad H = \zeta_j \quad F = (Y_j, 1, Y_j, 2),
\]
\[
f_j := (f_j, 1, f_j, 2), \quad p_0, \quad p = (p_j, 1, p_j, 2)
\]
provides a holomorphic map \( \tilde{f}_j = (\tilde{f}_j, 1, \tilde{f}_j, 2) : M \to \mathbb{C}^2 \) for any \( j = 1, \ldots, n/2 \) such that, defining \( \tilde{f} := ((\tilde{f}_j, 1, \tilde{f}_j, 2))_{j=1,\ldots,n/2} : M \to \mathbb{C}^n \), we have that
\[
(b1) \quad \tilde{f}_{j,1}^2 + \tilde{f}_{j,2}^2 = \zeta_j^2 \neq 0 \text{ on } M, \quad \text{and so, } \quad \sum_{j=1}^{n/2} (\tilde{f}_{j,1}^2 + \tilde{f}_{j,2}^2) = 0, \text{ see (a1)}.
\]
\( (b2) \quad \tilde{f}_j \text{ approximates } f_j \text{ uniformly on } K. \)
\( (b3) \quad \Re(\tilde{f} \theta) \text{ is an exact 1-form on } M. \)
\( (b4) \quad \int_{\gamma} \tilde{f} \theta = i \mathfrak{p}(\gamma) \) for each closed curve \( \gamma \) in \( M. \)
\( (b5) \quad \Re \int_{p_0}^{p} \tilde{f} \theta = Y(p) \) for each \( p \in \Lambda. \)
\( (b6) \quad \text{If } \alpha \text{ is a divergent arc in } M, \text{ then } \int_{\alpha} \| \tilde{f}(\cdot) \| \| \theta \| \text{ is infinite.} \)

Property (b6) follows easily from (vi) in Proposition 4.2. Note that from (b1) we have that
\[
\| \tilde{f}(p) \| \geq |f_{1,1}(p)|^2 + |f_{1,2}(p)|^2 \geq \frac{|\tilde{f}_{1,1}(p)|^2 + |\tilde{f}_{1,2}(p)|^2 + |\zeta_1|^2}{2}.
\]

Therefore, the conformal minimal immersion \( X : M \to \mathbb{R}^n \) defined by
\[
(5.3) \quad X(p) := \Re \int_{p_0}^{p} \tilde{f} \theta, \quad p \in M
\]
solves the theorem. Indeed, \( X \) is a well defined map by (b3) and is a conformal minimal immersion by (b1). \( X \) verifies (I) by condition (b5) and \( (5.3) \). By property (b6), the length of any divergent curve in \( M \) via \( X \) is infinite, hence \( X \) is complete. Clearly, (II) is implied by (b4), see \( (5.3) \). Additionally, if the approximation of (b2) is close enough and taking into account (a3), then \( G_X \) is nondegenerate. Finally, (b1) implies that the generalized Gauss map \( G_X \) (see (1.1)) fails to intersect the hyperplanes given by
\[
(5.4) \quad \left\{ (z_1, \ldots, z_n) \in \mathbb{C}P^{n-1} : z_{2j-1} + (-1)^{j}\delta z_{2j} = 0 \right\},
\]
for any \( j = 1, \ldots, n/2 \) and each \( \delta = 0, 1 \), which are located in general position.

Assume now that \( n \) is odd.

As in the previous case, take \( K \subset M \setminus \Lambda \) a small simply connected compact subset and fix \( p_0 \in K \). Write also \( p = (p_1, \ldots, p_n) \) and \( Y = (Y_1, \ldots, Y_n) \). By Lemma 2.3 there exists a holomorphic 1-form \( \theta \) never vanishing on \( M \) such that
\[
(5.5) \quad \int_{\gamma} \theta = i \mathfrak{p}_n(\gamma), \quad \text{for any loop } \gamma \subset M,
\]
and also
\[(5.6) \quad \Re \int_{p_0}^p \theta = Y_n(p), \text{ for any point } p \in \Lambda \]

Notice that, since $\theta$ is an exact real 1-form by \((5.5)\), then the values in \((5.6)\) are independent of the election of the curve connecting $p_0$ with $p \in \Lambda$.

Take for any $j = 1, \ldots, (n-1)/2$ a complex number $\zeta_j \in \mathbb{C} \setminus \{0\}$ and a holomorphic map $f_j = (f_{j,1}, f_{j,2}): K \to \mathbb{C}^2$ such that
- $\sum_{j=1}^{n/2} \zeta_j^2 = -1$.
- $f_{j,1}^2 + f_{j,2}^2 = \zeta_j^2$ on $K$ for any $j = 1, \ldots, n/2$.
- The map $((f_j))_{j=1,\ldots,n/2}$ is nondegenerate.

Proposition 4.2 applied to each $f_j := (f_{j,1}, f_{j,2}): K \to \mathbb{C}^2$ provides a holomorphic map $\tilde{f}_j: M \to \mathbb{C}^2$ for any $j = 1, \ldots, (n-1)/2$. Set $\tilde{f} := (\tilde{f}_{1,1}, \tilde{f}_{1,2}, \ldots, \tilde{f}_{n/2,1}, \tilde{f}_{n/2,2}, 1): M \to \mathbb{C}^n$.

Reasoning as in the previous case, the conformal minimal immersion $X: M \to \mathbb{R}^n$ defined by
\[X(p) := \Re \int_{p_0}^p \tilde{f} \theta, \quad p \in M\]
solves the theorem, provided that the approximation involved is close enough. Here the generalized Gauss map $G_X$ of $X$ fails to intersect the hyperplanes given in \((5.4)\) for $j = 1, \ldots, (n-1)/2$ and each $\delta = 0, 1$, and also the hyperplane given by
\[(5.7) \quad \left\{ (z_1, \ldots, z_n) \in \mathbb{C}P^{n-1} : z_n = 0 \right\},
\]
which are located in general position. \hfill \Box

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