Torsional oscillations and magnetic cycles in dynamo models with fluctuations

V.V. Pipin\textsuperscript{1}, A.G. Kosovichev\textsuperscript{2,3}

\textsuperscript{1}Institute of Solar-Terrestrial Physics, Russian Academy of Sciences, Irkutsk, 664033, Russia
\textsuperscript{2}Department of Physics, New Jersey Institute of Technology, Newark, NJ 07102, USA
\textsuperscript{3}Center for Computational Heliophysics, New Jersey Institute of Technology, Newark, NJ 07102, USA

ABSTRACT

Using a nonlinear mean-field dynamo model, we study relationships between amplitude of the “extended” mode of migrating zonal flows (“torsional oscillations”) and magnetic cycles, and investigate whether the torsional oscillations can be used for solar cycle prediction. We consider two types of dynamo models: models with regular variations of the alpha-effect, and models with stochastic variations. The regular dynamo models show the two different relationships for the growing and decaying stages of the magnetic cycle variations on centennial time scales. For the dynamo models with fluctuations these relations are merged. We find that for the both types of models the amplitude of solar cycles correlates well with the integral amplitude of the zonal harmonic $\ell = 9$ of the torsional acceleration waves of the preceding cycles in subsurface layers and with the $\ell = 3$ harmonic at the bottom of the convection zone. While these relationships are weaker than the previously known relationship of the cycle magnitude with the poloidal field in the preceding minima, the prediction horizon of the torsional oscillations is greater and may reach the full 11-year activity cycle. In addition, we find that the amplitude of the asymmetric about the equator components of the torsional oscillations can be used as a precursor in forecasts of the hemispheric asymmetry of magnetic activity.

1. INTRODUCTION

According to current knowledge, global hydromagnetic dynamo acting inside the Sun determines the nature of the solar magnetic activity. Parker (1955) showed that the dynamo action involves cyclic transformation of poloidal and toroidal components of the global magnetic field of the Sun. This scenario suggests that magnetic field of bipolar active regions is formed from large-scale toroidal magnetic field that is generated from the axisymmetric poloidal magnetic field by differential rotation deep in the convection zone. Parker (1955) and Krause & Rädler (1980) suggested a mechanism of generation of the large-scale poloidal magnetic field from the toroidal field by means of a turbulent electromotive force excited by cyclonic convection. It is the so-called the ”alpha - effect”. The mechanism of generation of the large-scale poloidal magnetic field is not yet fully established. Several alternative mechanisms of the poloidal field generation can be found in the literature (Babcock 1961; Choudhuri & Dikpati 1999; Charbonneau 2011; Cameron & Schüssler 2017). In general, the dynamo action provides mutual cyclic amplification of the poloidal and toroidal components of the large-scale magnetic field (LSMF). In a stationary regime the dynamo generation saturates due to nonlinear effects, e.g., because of magnetic helicity conservation (Kleedorin & Rogachevskii 1999; Kleedorin et al. 2000), magnetic buoyancy (Parker 1984), and magnetic feedback on the angular momentum and heat transport in the solar convection zone (see reviews of Brandenburg & Subramanian 2005; Brandenburg 2018).

A well-known method of solar-cycle forecast employs an empirical relationship between the amplitude of the generated toroidal field and the strength of the poloidal magnetic field during the preceding solar minima (Schatten et al. 1978). This relationship makes it possible to predict the sunspot maxima from the amplitude of the polar field observed during the preceding solar activity minima. The forecast horizon of this method is approximately half the 11-year solar cycle. This relationship is employed in the flux-transport and Babcock-Leighton types of dynamo models (Choudhuri et al. 2007). One possibility to improve the forecast is to take into account nonlinear relationship between global flows and magnetic fields of the Sun. From analysis of helioseismic measurements of zonal flows migrating during the solar cycle in the convection zone (so-called “torsional oscillations”) Kosovichev & Pipin (2019) argued that the flow amplitude at the base of the convection zone during the solar maxima correspond to the strength of the following sunspot maxima. This relationship can give the solar activity forecast a full 11-year cycle ahead.
In this work, we study theoretical relationships between variations of the torsional oscillations and the amplitude of solar cycles. We exploit parameters of the “extended” mode of torsional oscillations. The extended wave of zonal variations of rotation propagates from high latitudes to the equator during the 22-year “extended” solar cycle (Altrock 1997; Ulrich & Boyden 2005). Recently, Kosovichev & Pipin (2019) and Pipin & Kosovichev (2019) presented observational and theoretical evidences in favor of the global nature of this wave in the solar dynamo process. These results indicated a possibility for using characteristics of the extended mode of the torsional oscillations for solar-cycle forecasting. In this study, we employ a non-linear dynamo model that couples the magnetic field evolution with global dynamic and thermodynamic variations in the convection zone, and provides a realistic description of the torsional oscillations and their extended mode (Pipin & Kosovichev 2019). Variations of the magnetic activity cycles are modeled using fluctuations of the alpha effect. In Section 2 of we describe the model. In Section 3 we present results for a series dynamo models, derive new forecast indices and compare them with the forecast based on correlation of the polar field strength and sunspot cycle amplitude. The paper concludes with a discussion of the main results.

2. BASIC EQUATIONS

2.1. Dynamo model

A detailed description of the dynamo model can be found in our previous paper (Pipin & Kosovichev 2019, hereafter PK19). The model describes the dynamo generation of large-scale magnetic fields (LSMF) in the bulk of the solar convective zone (CZ). The model is based on the mean-field induction equation (Krause & Rädler 1980),

\[ \partial_t \mathbf{B} = \nabla \times (\mathbf{\varepsilon} + \mathbf{U} \times \mathbf{B}), \]

(1)

where the induction vector of the LSMF, \( \mathbf{B} \), is represented as the sum of the toroidal and poloidal components:

\[ \mathbf{B} = \hat{\phi} \mathbf{B} + \nabla \times \frac{A \hat{\phi}}{r \sin \theta}, \]

where \( r \) is the radial distance, \( \theta \) is the polar angle, \( \hat{\phi} \) is the unit vector in the azimuthal direction. The mean electromotive force \( \mathbf{\varepsilon} \) describes the turbulent generation effects, pumping and diffusion:

\[ \mathbf{\varepsilon}_i = (\alpha_{ij} + \gamma_{ij}) \mathbf{B}_j - \eta_{ijk} \nabla_k \mathbf{B}_j, \]

(2)

where the symmetric tensor \( \alpha_{ij} \) stands for the turbulent generation of the LSMF by kinetic and magnetic helicities; the antisymmetric tensor \( \gamma_{ij} \) describes the turbulent pumping effect; the anisotropic (in general case) tensor \( \eta_{ijk} \) is the eddy diffusivity of the LSMF Pipin (2018). The large-scale (LS) flow field, \( \mathbf{U} = \mathbf{U}^m + r \sin \theta \Omega(r, \theta) \hat{\phi} \) produces the LS toroidal magnetic field from the LS poloidal field by means of the differential rotation, \( \Omega(r, \theta) \). The meridional circulation, \( \mathbf{U}^m \), advects the LSMF in the convection zone. The angular momentum conservation and the equation for the azimuthal component of large-scale vorticity, \( \omega = \left( \nabla \times \mathbf{U}^m \right)_\phi \), determine distributions of the differential rotation and meridional circulation in the model:

\[ \frac{\partial}{\partial t} \rho r^2 \sin^2 \theta \Omega = - \nabla \cdot \left( r \sin \theta \left( \frac{\mathbf{\hat{\phi}} \times \nabla \cdot p \mathbf{T}}{r \rho \sin \theta} - \frac{\mathbf{U}^m \omega}{r \sin \theta} \right) + \nabla \cdot \left( r \sin \theta \frac{\mathbf{B} \mathbf{\hat{B}}_\phi}{4 \pi} \right) \right) \]

(3)

\[ \frac{\partial \omega}{\partial t} = r \sin \theta \nabla \cdot \left( \frac{\hat{\phi} \times \nabla \cdot \mathbf{p} \mathbf{T}}{r \rho \sin \theta} - \frac{\mathbf{U}^m \omega}{r \sin \theta} \right) + r \sin \theta \frac{\partial \Omega^2}{\partial \varphi} - \frac{g}{c_{\rho r} \partial \theta} \]

\[ + \frac{1}{4 \pi p} (\mathbf{B} \cdot \nabla) \left( \nabla \times \mathbf{B} \right)_\phi - \frac{1}{4 \pi p} \left( \left( \nabla \times \mathbf{B} \right) \cdot \mathbf{\hat{B}}_\phi \right), \]

where \( \mathbf{T} \) is the turbulent stress tensor:

\[ \hat{T}_{ij} = \left( \langle u_i u_j \rangle - \frac{1}{4 \pi p} \left( \langle b_i b_j \rangle - \frac{1}{2} \delta_{ij} \langle b^2 \rangle \right) \right). \]

(5)
(see detailed description in PK19). Also, $\rho$ is the mean density, $s$ is the mean entropy; $\partial / \partial z = \cos \theta \partial / \partial r - \sin \theta / r \cdot \partial / \partial \theta$ is the gradient along the axis of rotation. The mean heat transport equation determines the mean entropy variations from the reference state due to generation and dissipation of LSMF and large-scale flows (Pipin & Kitchatinov 2000):

$$\rho T \left( \frac{\partial s}{\partial t} + (\mathbf{U} \cdot \nabla) s \right) = -\nabla \cdot (F^c + F^\alpha) - T_i \frac{\partial U_i}{\partial r} - \mathbf{E} \cdot (\nabla \times \mathbf{B}),$$  \hspace{1cm} (6)

where $T$ is the mean temperature, $F^c$ is the radiative heat flux, $F^\alpha$ is the anisotropic convective flux. An analytical mean-field expression of $F^c$ takes into account effect of the Coriolis force, and influence of the LSMF on the turbulent convection (see, PK19). The last two terms in Eq (6) take into account the convective energy gain and sink caused by the generation and dissipation of LSMF and large-scale flows. The reference profiles of mean thermodynamic parameters, such as entropy, density and temperature are determined from the stellar interior model MESA (Paxton et al. 2011, 2013). The radial profile of the typical convective turnover time, $\tau_c$, is determined from the MESA code, as well. We assume that $\tau_c$ does not depend on magnetic field and global flows. The convective RMS velocity is determined from the mixing-length approximation,

$$u_c = \frac{\ell_c}{2} \sqrt{\frac{g}{2\rho_g \partial r}},$$  \hspace{1cm} (7)

where $\ell_c = \alpha_{MLT}H_p$ is the mixing length, $\alpha_{MLT} = 1.9$ is the mixing length parameter, and $H_p$ is the pressure height scale. Eq. (7) determines the reference profiles for the eddy heat conductivity, $\chi_T$, eddy viscosity, $\nu_T$, and eddy diffusivity, $\eta_T$, as follows,

$$\chi_T = \frac{\ell^2}{6} \sqrt{\frac{g}{2\rho_g \partial r}},$$  \hspace{1cm} (8)

$$\nu_T = \nu_{RT} \chi_T,$$  \hspace{1cm} (9)

$$\eta_T = \nu_{RT} \nu_T.$$  \hspace{1cm} (10)

The model gives the best agreement of the angular velocity profile with helioseismology results for $\nu_{RT} = 3/4$ (PK19). Also, the dynamo model reproduces the solar cycle period, $\sim 22$ years, if $\nu_{RT} = 10$.

Figure 1 shows the angular velocity profile, streamlines of the meridional circulation, the radial profiles of the $\alpha$-effect and the eddy diffusivity in the model. The magnitude of the meridional flow on the surface is about 14 m/s. The angular velocity profile agrees well with helioseismology data. A detailed theoretical discussion of the mechanisms generating the differential rotation and meridional circulation in our model can be found in Pipin & Kosovichev (2018) (also see, Kitchatinov & Rüdiger 1999, 2005). The model includes a phenomenological description of the tachocline where the differential rotation is transformed into a solid body rotation. We assume that intensity of turbulent mixing in the tachocline drops exponentially with distance from the bottom of the convection zone, and that the $\alpha$-effect vanishes in the tachocline.

2.2. Magnetic helicity and $\alpha$-effect fluctuations

Similarly to our previous papers we employ the $\alpha$-effect tensor, which represents a combination of kinetic and magnetic helicities, in the following form:

$$\alpha_{ij} = C_\alpha \left( 1 + \xi^{(\alpha)}(t) \right) \psi_{\alpha}(\beta) \alpha_{ij}^{(H)} + \alpha_{ij}^{(M)} \psi_{\alpha}(\beta) \frac{\nabla \tau_c}{4\pi \rho \ell^2},$$  \hspace{1cm} (11)

where $\xi^{(\alpha)}(t)$ is the fluctuating part of kinetic helicity tensor $\alpha_{ij}^{(H)}$ (Fig. 1b shows the radial profile of the tensor components); $X = \mathbf{a} \cdot \mathbf{B}$ is the magnetic helicity density ($\mathbf{a}$ and $\mathbf{b}$ are the turbulent parts of the magnetic vector potential and magnetic field vector), and tensor $\alpha_{ij}^{(M)}$ takes into account the effect of the Coriolis force. Function $\psi_{\alpha}(\beta)$ stands for the “algebraic” saturation of the $\alpha$-effect caused by the small-scale Lorentz force which opposes convective motions across the field lines of the LSMF, where, $\beta = |\mathbf{B}| / \sqrt{4\pi \rho \ell^2}$. For strong LSMF, when $\beta \gg 1$, $\psi_{\alpha}(\beta) \sim \beta^{-3}$. A detailed description of $\alpha_{ij}^{(H)}$, $\alpha_{ij}^{(M)}$ and $\psi_{\alpha}(\beta)$ is given by Pipin (2018). The magnetic helicity evolution follows the conservation law:

$$\frac{\partial X^{(\text{tot})}}{\partial t} = - \frac{\nabla \cdot \mathbf{B}}{R_m \tau_c} - 2\eta \mathbf{J} - \nabla \cdot \mathbf{F} - (\mathbf{U} \cdot \nabla)X^{(\text{tot})} \hspace{1cm} (12)$$
\[ \chi^{(\text{tot})} = \hat{\mathbf{A}} \cdot \mathbf{B} = \hat{\mathbf{A}} \cdot \mathbf{B} + \mathbf{a} \cdot \mathbf{B}, \]  

where \( \mathbf{B} = \nabla \times \mathbf{A} \), \( \hat{\mathbf{A}} \) is the LSMF vector potential; \( R_m \) is the magnetic Reynolds number, (we put \( R_m = 10^6 \)). We assume that eddy diffusivity of the magnetic helicity is isotropic, and that the diffusive helicity flux \( \mathbf{F} = -\eta_\chi \nabla \chi \), where \( \eta_\chi = 0.1 \eta_T \) (Mitra et al. 2010).

In this study we performed several runs of the dynamo model different parameters of the kinetic helicity. In the first three runs, models C1, C2 and C3, we vary the dimensionless \( \alpha \)-effect parameter, \( C_\alpha = \{0.04, 0.05, 0.06\} \). The value \( C_\alpha = 0.04 \) is slightly above the dynamo instability threshold. In model C4 we simulate long-term magnetic activity variations by increasing \( C_\alpha \) from 0.04 to 0.08 after each cycle and then decreasing it back in the same way. Finally, model C5 employs the random variations of the \( \alpha \)-effect in time.

Similarly to Rempel (2005) and Kitchatinov et al. (2018), we model the random parameter \( \xi^{(\alpha)} \) via Ornstein–Uhlenbeck process, i.e., the evolution of the \( \xi^{(\alpha)} \) is governed by the systems of the differential stochastic equations,

\[ \dot{\xi}^{(\alpha)} = -\frac{2}{\tau_\xi} \left( \xi^{(\alpha)} - \xi_1 \right), \]
\[ \dot{\xi}_1 = -\frac{2}{\tau_\xi} \left( \xi_1 - \xi_2 \right), \]
\[ \dot{\xi}_2 = \frac{2}{\tau_\xi} \left( \xi_2 - g \sqrt{\frac{2 \tau_\xi}{\tau_\Theta}} \Theta \right), \]

where \( g \) is a Gaussian random number which is renewed at every time step, \( \tau_\Theta \) is the time step of the numerical simulations, \( \tau_\xi \) is the relaxation time of \( \xi^{(\alpha)} \) and \( \xi_{1,2,3} \) are auxiliary parameters that are introduced to smooth variations of \( \xi^{(\alpha)} \) with its first- and second-order derivatives. To generate the \( \alpha \)-effect randomness in colatitude we introduce the random function \( \Theta \), renewed at each time step as well. It is defined as follows. We generate spatially random Gaussian sequences, \( \Theta (\theta_j) \), where \( \theta_j \) are the collocation points of the Legendre polynomials, and \( \langle \Theta (\theta_j) \rangle = 0, \sigma (\Theta) = 1 \). Then, the sequence \( \Theta (\theta_j) \) is decomposed into the Legendre polynomials. Finally, we filter out all the Legendre modes higher than \( \ell = 5 \) and normalize \( \Theta (\theta) \) to unity. The resulted latitudinal fluctuations of the \( \alpha \)-effect are described by the smooth functions. We use a moderate level of fluctuations with \( \sigma (g) = 0.3 \) and \( \tau_\xi = 1 \text{ yr} \). With this relaxation time the model gets a rather ‘long memory’ in fluctuations of the \( \alpha \)-effect, which maybe unrealistic. Note that the auxiliary parameters \( \xi_{1,2,3} \) enlarge the memory as well. On other hand, the given model become suitable to study the statistical relations between torsional oscillations and the magnetic cycle parameters. Parameters of the models are given in the Table 1.

### 3. RESULTS AND DISCUSSION

Figure 2 shows the time-latitude diagrams evolution of the LSMF and the zonal acceleration at the surface and the bottom of the convection zone for model C2. The time-latitude diagrams are similar to those published in PK19. The wave-like migration of the toroidal LSMF has two branches: polar and equatorial. The equatorial branch near the
Model parameters:

| Model | $C_\alpha$ | $P$ [YR] | $D$ [G] | $B_\phi$ [G] | $\partial_t U_\phi$ [$10^{-8}$ m s$^{-2}$] |
|-------|-----------|---------|--------|-------------|---------------------------------|
| C1    | 0.04      | 12      | 3.2    | 800/2700    | 2.4                             |
| C2    | 0.05      | 10.2    | 4.3    | 990/3100    | 3.4                             |
| C3    | 0.065     | 8.3     | 5.0    | 1350/3500   | 5.8                             |
| C4    | 0.02 - 0.08 | 7.5/14 | 3.2/5.1 | 1500/3600   | 7.0                             |
| C5    | 0.05 × $(1 + \xi^{(\alpha)})$ | 7.5/13 | 3.2/4.9 | 1500/3600  | 10.0                           |

$\sigma (\xi) = 0.3$, $\tau_\xi = 2$ yr

Table 1. Model parameters: $C_\alpha$ is the dimensionless parameter of the $\alpha$-effect; $P$ is dynamo period; $D$ is the maximum strength of the dipole component of the LSMF; $B_\phi$ is the maximum strength of the toroidal LSMF in the subsurface layer $r = 0.9 R$, and near the bottom of the convection zone ($r = 0.74 R$); $\partial_t U_\phi$ is the amplitude of solar-cycle variations of the zonal acceleration.

Figure 2. Model C2, a) variations of the radial magnetic field at the surface (color image) and the strength of the toroidal magnetic field at $r = 0.95 R$ (contour lines cover the interval ±1 kG); b) the zonal acceleration at the surface; c) variations of low order harmonics $U^{(\ell)}_{1,t}$ ($\ell = 1 - 21$) of the zonal acceleration at the surface; d) contours show the toroidal LSMF at $r = 0.95 R$, and the background color image shows the toroidal LSMF evolution near the bottom of the convection zone, $r = 0.73 R$; e) and f) the same as in panels b) and c) for $r = 0.73 R$.

At the surface, the extended wave of torsional oscillations starts propagating from high latitudes to the equator. At the same time, a new wave of the toroidal LSMF starts near the bottom of the convective zone.

For our analysis we consider parameters of the spectrum of zonal acceleration in the form:

$$\partial_t U_\phi = \sum U^{(\ell)}_{1,t} (t) P^1_{\ell} (\cos \theta),$$

(14)

where $P^1_{\ell}$ is a set of normalized associated Legendre polynomials. Figures 2c and 2f show variations of the $U^{(\ell)}_{1,t}$ for the top and the bottom of the dynamo domain. In both cases the spectral harmonics vary with a period equal to half the period of the dynamo cycle ($\simeq 11$ years). Moreover, the harmonic with $\ell = 9$ shows the largest variations among the others. We see that the development of the cycle is accompanied by a phase shift progressing from high to low-order harmonics. The maximum of the $\ell = 9$ harmonic corresponds to initiation of the extended mode of the solar torsional oscillations at high latitudes. Near the bottom of the convection zone the harmonic of $\ell = 3$ shows the largest variations. At the bottom of the convection zone, the torsional oscillation propagates to the equator during about 12 years (Fig. 2e).
In model C4, the $\alpha$-effect coefficient, $C_\alpha$, increases by a constant value after each half-cycle and then decreases below the critical threshold. Figure 3 shows the results. The increase of the LSMF strength results in an increase of the torsional oscillations magnitude. In the strong cycles, the polar branch of the torsional oscillations disappears.

Also, during the long-term increase of the magnetic cycle amplitude the torsional oscillations harmonic $\ell = 9$ shows excess of the positive variations over the negative ones. The opposite effect happens during the long-term decay of the magnetic activity. Note, that at the bottom of the convection zone the $\ell = 3$ harmonic shows the opposite behavior.

Figure 4 shows that at the surface the torsional oscillations harmonic of $\ell = 9$ remains dominant during the periods of high and low activity. At the bottom of the convection zone the $\ell = 3$ harmonic dominates during the long-term maxima, and the $\ell = 5$ harmonic becomes stronger than the $\ell = 3$ harmonic during the activity minima.

For the integral parameters of the magnetic cycle we introduce the total unsigned magnetic flux $F_T$ of the toroidal LSMF in the subsurface layer:

$$F_T = \int_{-1}^{1} \int_{0.89R}^{0.99R} |\mathbf{B}_\phi| rdrd\mu,$$

(15)
where $\mu = \cos \theta$. Similarly, we define the total magnetic flux, $F_B$, of the toroidal magnetic field near the bottom of the convection zone within $r = 0.73 - 0.75R$. Also, we define the strength of the dipole component of the radial magnetic field, $D$. We define the parameters characterizing the energy of the symmetric and antisymmetric parts of the radial magnetic field at the surface:

$$
E^S_B = \frac{1}{4} \int_{-1}^{1} \left| B_r (\mu, t) + B_r (-\mu, t) \right|^2 d\mu,
$$

$$
E^N_B = \frac{1}{4} \int_{-1}^{1} \left| B_r (\mu, t) - B_r (-\mu, t) \right|^2 d\mu.
$$

Then, the parity index, or the reflection symmetry index, for this component of the magnetic activity is:

$$
P_R = \frac{E^S_B - E^N_B}{E^S_B + E^N_B}.
$$

Similarly, we define the parity index, $P_T$, for the subsurface toroidal magnetic field at $r = 0.9R$.

Figures 5 and 6 show variations of magnetic field $B_r$, torsional acceleration $\partial_t U_\phi$, spectral coefficients $U^{(\ell)}_t$, as well parameters $F_T$, $D$, $\xi^{(a)}$ and the parity indexes for model C5. The time interval includes 6 full dynamo cycles, and covers a period of about 150 years. The strength of toroidal LSMF in subsurface layers changes from about 0.5kG during the centennial minima to 1.5 kG during the maxima. Simultaneously the magnitude of the total flux of the toroidal field in upper part of the convection zone, $F_T$ changes in the range of $(0.5 - 1.3) \times 10^{24}$, and the strength of the radial dipole LSMF covers the range of 2-6 G. The cycle duration varies from about 8-9 years for the high amplitude cycles to 12-13 for the weak cycles. The weakest cycle has duration of 16 years. The hemispheric asymmetry of magnetic activity in the model is not strong. The model keeps the antisymmetric about equator distributions of the LSMF. This results in dominance of the odd harmonics in the $U^{(\ell)}_t$ spectrum. Yet, we see sporadic excitation of weak even harmonics, e.g., the harmonic of order $\ell = 8$ at around $t = 360$ years in Fig. 6. This reflects the deviation of magnetic parity from the pure antisymmetric relative to the equator (Fig 6c). The parity indexes are close to zero during the epochs of centennial minima (Sokoloff & Nesme-Ribes 1994). The phase of $P_T$ is shifted relative to $P_R$ similarly to the shift in variations of $F_T$ and $D$. The spectrum of the torsional oscillations at the bottom of the convection zone in this model is qualitatively similar to model C4.
Following the results presented in Figures 3 and 4 we choose the torsional oscillations harmonic of order $\ell = 9$ at the surface and harmonic of order $\ell = 3$ at the bottom of the convection zone as primary integral parameters of the torsional oscillations. The strength of the torsional oscillations is characterized by the integrals of $U_t^{(9)}$ and $U_t^{(3)}$ calculated over the full dynamo cycle.

Following the procedure of Pipin & Kosovichev (2011), we calculate $F_T$ and $F_B$ separately for each cycle from their minima to maxima. Note that the cycles of $F_B$ are shifted ahead of $F_T$. Also, we calculate integrals of $U_t^{(3)}$ for the bottom of the convection zone, and $U_t^{(9)}$ for the top of the domain. The integrals are calculated for the time intervals corresponding to the cycles of $F_T$. For each minimum of the $F_T$ cycles we calculate the absolute magnitude of the radial dipole magnetic field, which is quantified by parameter $D$. Results for models C4 and C5 are shown in Figure 7. The strong magnetic cycles tend to show an asymmetry of the growth and decay phases (Pipin & Kosovichev 2011; Willamo et al. 2020). The radial dipole LSMF changes in anti-phase with $F_T$. The increase of $U_t^{(9)}$ is about two years ahead of $F_T$. This agrees with the observational fact that the torsional oscillations ahead of the sunspot activity by about two years (Howe et al. 2011).

Figure 8 shows correlations between the magnetic cycle parameters. In agreement with expectations of Schatten et al. (1978), we see that the magnitude of $F_T$ correlates with the radial dipole parameter, $D$, of the preceding minimum of magnetic activity. Similar results were previously found in kinematic dynamo models (Choudhuri et al. 2007; Pipin et al. 2012; Iijima et al. 2017). Figure 8b shows the relationship between $U_t^{(9)}$ integrated over the cycle and the toroidal flux, $F_T$, of the subsequent cycle. The results of model C4 with regular variations of the alpha-effect indicate two branches of this relation. During the long-term growth of the magnetic activity we find that for stronger $U_t^{(9)}$ $F_T$ is higher. This relationship disappears during the periods of low magnetic activity.

In model C5 we have no clear regular long-term variations of $F_T$. In this models weak $F_T$ cycles follow strong cycles. This results in merging of the two branches. We find the same behavior for parameter $U_t^{(3)}$, which characterizes the
Figure 7. Left column shows cyclic variations of parameters $F_T$, $D$ and $U_t^{(9)}$ for model C4, right column shows the same for model C5.

torsional oscillations at the bottom of the convection zone. Despite the relative fluctuations of the $\alpha$ effect are uniform along the radius, model C5 shows no clear relation between the the toroidal magnetic field near the bottom of the convection zone and near the top. We find that the branch of the growing cycles of model C4 is located at the left edge of the model C5 cycle parameters. Also, as in the case of parameters $U_t^{(3)}$ and $U_t^{(9)}$, the dispersion along the $F_B$ in model C5 goes in direction of the second branch of model C4. The parameters of the correlations are listed in Table 2.

A similar analysis is done for the parity indexes of the toroidal and radial LSMF, $P_T$ and $P_R$. We find that position of the $P_T$ maximum is not fixed and it can vary between the subsequent minima of the $F_T$ cycle. Similarly, the position of the $P_R$ maxima varies between the subsequent maxima of $F_T$. Therefore, we calculate the maxima of $P_T$ and $P_R$ for the analysis. Also, we calculate the maxima of the first even mode $U_t^{(2)}$ for each $F_T$ cycle. The amplitude of $U_t^{(2)}$ is used to quantify the hemispheric asymmetry of the torsional oscillations. The results shown in Figure 9 reveal correlations between $P_T$ and $P_R$, as well as between $P_T$ and $U_t^{(2)}$. The result for the $P_T$-P$_R$ correlation qualitatively agrees with the flux-transport model simulations of Nepomnyashchikh et al. (2019).

4. CONCLUSIONS

In the study, using the solar-type non-linear dynamo models, we explored relationships between the torsional oscillations and magnetic activity cycles, and investigated potential of these relationships for magnetic cycle forecasting. The idea is not new, for example, Yoshimura & Kamby (1993) found evidence that variations in the total angular momentum on the surface were ahead of the centennial variations in magnetic activity of the Sun. A similar effect was found in models of Knobloch et al. (1998) and Pipin (1999).
Figure 8. a) The magnitude of the radial dipole LSMF, $D$, in the activity minima versus the total flux of subsurface toroidal LSMF, $F_T$, integrated over the subsequent cycles (black squares show model C4 and white - model C5); b) the integral over cycle of magnitude of the surface torsional oscillations parameter, $U_{t}^{(9)}$, versus $F_T$ in the subsequent cycles; c) the total over cycle flux of the toroidal LSMF, $F_B$, near the bottom of the convection zone versus the parameter $F_T$ (note, the cycles of $F_B$ are shifted ahead of $F_T$); d) $U_{t}^{(3)}$, near the bottom of the convection zone versus the parameter $F_T$.

Figure 9. a) The parity of the radial magnetic field during the minima versus the parity of the toroidal magnetic field in the magnetic cycle; b) the maximum magnitude of the first even mode $U_{t}^{(2)}$, of the torsional oscillations in the preceding cycle versus the parity of the toroidal magnetic field in the subsequent magnetic cycle.

Here, we try to exploit properties of the extended mode of the torsional oscillations. The torsional oscillations in our model are caused by influence of the dynamo generated large-scale magnetic field on the heat transport and turbulent angular momentum fluxes inside the convection zone (see, Pipin & Kosovichev 2019). The extended mode of the torsional oscillations covers a period of the full 22-year magnetic cycle. It was found that at the surface the maximum of the spectrum of the torsional variations, represented by the azimuthal velocity acceleration, $\partial_t U_\phi$, corresponds to the relatively high order harmonic of $\ell = 9$. Consequently, the spectrum of $\partial_t \Omega = \partial_t U_\phi / r \sin \theta$ has the maximum for $\ell = 8$. Our models do not show the extended wave of torsional oscillations at the bottom of the convection zone where the spectral maximum is at $\ell = 3$. We considered the surface harmonic $U_{t}^{(9)}$ and the bottom harmonic $U_{t}^{(3)}$ as precursors of the magnetic activity cycles.
The models showed that an increase of the magnetic cycle amplitude follows an increase of the net acceleration of $U_{t}^{(9)}$ during the preceding cycle. However, this relationship does not work in the case of prolonged decreases of the cycle magnitude. In the model with fluctuating $\alpha$ - effect the long-term increase and decrease of the magnetic activity are not monotonic. This results in statistical correlations between the total surface flux of the toroidal magnetic field and parameters of the torsional oscillations at the bottom of the convection zone: $F_{T}^{(n+1)} \propto -0.4U_{t}^{(2)(n)}$, and near the surface: $F_{T}^{(n+1)} \propto 0.5U_{t}^{(9)(n)}$. These relationships are weaker than the relationship between the radial dipole magnetic field at the cycle minimum and the magnitude of the subsequent sunspot cycle. We find that relations of the $F_{T}^{(n+1)}$ with the integral over the cycle parameters, $U_{t}^{(9)(n)}$ and $U_{t}^{(3)(n)}$ have the highest probabilities among similar relations where these parameters are taken for the particular phases of the magnetic cycle. Therefore, we see that these parameters of torsional oscillations have a similar forecast horizon as the predictions made with help of the dipole components of the poloidal magnetic field of the Sun. The similar conclusion can be drawn for the magnetic parity parameters. We find that the amplitude of the asymmetric about the equator components of the torsional oscillations can be used as a precursor in forecasts of the hemispheric asymmetry of magnetic activity. We checked these relationships for other dynamo models without the extended mode, such as model M7 from PK19, in which influence of magnetic field on the heat transport was neglected. We did not find the same precursors of the torsional oscillations in that model. We conclude that the extended mode of the torsional oscillations is crucial for the cycle prediction. This conclusion may be biased by the types of dynamo models employed in the paper.

In summary, it is found that parameters of the extended 22-yr mode of the torsional oscillations during a 11-yr solar cycle can be used for a statistical forecast of the subsequent 11-yr cycle. The reliability of the forecast can be increased using the parameters of the torsional oscillations at the top and the bottom of the convection zone. This theoretical study should be extended using the available observations.

**Acknowledgements** VP thanks support of RFBR under grant 19-02-53045, the project II.16.3 of ISTP SB RAS; AK thanks support of NASA grants: NNX14AB70G and 80NSSC20K0602.

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In Table 2, the parameters of the Spearman rank correlations, $r_S$ in model C5: $p_S$ is the null hypothesis probability, $F_{T}^{(n)}$ is the total over cycle n flux of the subsurface toroidal LSMF, $D^{(n)}$ is the magnitude of the radial dipole LSMF in the minimum of the cycle n, the $U_{t}^{(9)(n)}$ is the total over cycle n magnitude of the surface torsional oscillation parameter, $U_{t}^{(3)(n)}$ is the same for the torsional oscillation parameter near the bottom of the convection zone; $F_{B}^{(n)}$ is the total over cycle flux of toroidal LSMF near the bottom of the convection zone; note, that the cycles of $F_{B}$ are shifted ahead of $F_{T}$; $P_{R}^{(n)}$ is magnitude of the parity of the radial magnetic field during the minima (between subsequent maxims of $F_{T}$), $F_{T}^{(n)}$ is magnitude of the parity of the subsurface toroidal magnetic field in the magnetic cycle; $U_{t}^{(2)(n)}$ the maximum magnitude of the first even mode of the torsional oscillations.

| $F_{T}^{(n+1)} \propto D^{(n)}$ | $r_S$ | $p_S$ |
|--------------------------|-------|-------|
| $F_{T}^{(n+1)} \propto U_{t}^{(9)(n)}$ | 0.74 | 0 |
| $F_{T}^{(n+1)} \propto U_{t}^{(3)(n)}$ | -0.38 | 0.02 |
| $F_{T}^{(n+1)} \propto F_{B}$ | 0.3 | 0.16 |
| $F_{T}^{(n+1)} \propto P_{R}$ | 0.75 | 0 |
| $F_{T}^{(n+1)} \propto U_{t}^{(2)(n)}$ | 0.42 | 0.01 |
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