Nodal solutions for Logarithmic weighted $N$-Laplacian problem with exponential nonlinearities

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Abstract. In this article, we study the following problem

$$-\text{div} (\omega(x) |\nabla u|^{N-2} \nabla u) = \lambda f(x,u) \quad \text{in} \quad B, \quad u = 0 \quad \text{on} \quad \partial B,$$

where $B$ is the unit ball of $\mathbb{R}^N$, $N \geq 2$ and $w(x)$ a singular weight of logarithm type. The reaction source $f(x,u)$ is a radial function with respect to $x$ and is subcritical or critical with respect to a maximal growth of exponential type. By using the constrained minimization in Nehari set coupled with the quantitative deformation lemma and degree theory, we prove the existence of nodal solutions.

Keywords: Weighted Sobolev space, $N$-Laplacian operator, Critical exponential growth, Nodal solutions.

2010 Mathematics Subject classification: 35J20, 49J45, 35K57, 35J60.

1 Introduction and Main results

In this paper, we consider the following elliptic problem involving logarithmic weighted $N$-Laplacian:

$$(P_{\lambda}) \quad \begin{cases} -\text{div} (\omega(x) |\nabla u|^{N-2} \nabla u) = \lambda f(x,u) & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B, \end{cases}$$

where $B = B(0,1)$ is the unit open ball in $\mathbb{R}^N$, $N > 2$, $\lambda$ is a positive parameter, the weight function $\omega(x)$ is given by

$$\omega(x) = \left( \frac{1}{|x|} \right)^{\beta(N-1)} \quad \text{or} \quad \omega(x) = \left( \log \frac{e}{|x|} \right)^{\beta(N-1)} \quad \beta \in [0,1). \quad (1.1)$$

We assume that the nonlinearity $f(x,t) : \overline{B} \times \mathbb{R} \to \mathbb{R}$ is a radial in $x$, continuous function and behaves like $\exp\{\alpha t^{(N-1)(1-\beta)}\}$ as $t \to +\infty$, for some $\alpha > 0$ and $\beta \in [0,1)$.

Such an equation may arise in many fields of physics, such as in non-Newtonian fluids, reaction diffusion problem, turbulent flows in porous media and image treatment [1, 3, 18, 20]. Here we just give some references which are close to the problem we consider in this note.
Without the weight $w(x)$, the problem $(P_\lambda)$ has been widely studied by several authors with different nonlinearities. G. M. Figueredo and F. B. M. Nunes in [13], consider the following equation

$$-\text{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = f(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

(1.2)

where $\Omega \subset \mathbb{R}^N$ is bounded, $1 < p < N$, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a superlinear continuous function with exponential subcritical or exponential critical growth and the function $a$ is $C^1$. By using the minimization argument and deformation lemma, the authors proved the existence of a least energy nodal solutions for the equation (1.2) with two nodal domains.

Recently, when $a(|\nabla u|^p) = 1$, X. Sun and Y. Song, see [19], studied the problem (1.2) in an open smooth bounded domain in the Heisenberg group $H^n = \mathbb{C}^n \times \mathbb{R}$. We also mention the work of Y. Zhang, Y Yang and S. Liang, see [24], where they established the existence of changing-sign solutions to the problem (1.2) under logarithmic and exponential nonlinearities.

Weighted $N$-Laplacian elliptic problems of the following type

$$\begin{cases}
-\text{div}(\omega(x)|\nabla u|^{N-2}\nabla u) = f(x, u) \quad \text{in} \quad B \\
u > 0 \quad \text{in} \quad B \\
u = 0 \quad \text{on} \quad \partial B,
\end{cases}$$

where the weight function $\omega(x)$ is given in (1.1) and the nonlinearity $f(x, t) : \overline{B} \times \mathbb{R} \to \mathbb{R}$ is positive, have been investigated in literatures (see [9, 10, 11, 23] and the references therein). We notice that, the influence of weights on limiting inequalities of Trudinger-Moser type has been studied with some detail in [5, 6, 7, 8] and as consequence the weights have an important impact in the Sobolev norm.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $w \in L^1(\Omega)$ be a nonnegative function, the weighted sobolev space is defined as

$$W_0^{1,N}(\Omega, w) = \text{cl}\{u \in C_0^\infty(\Omega) / \int_B |\nabla u|^N \omega(x)dx < \infty\}$$

For different proprieties and embedding results for the weighted Sobolev spaces, we can refer to [16].

Theorem 1.1 [7] Let $\beta \in [0, 1)$ and let $w$ be given by (1.1), then

$$\int_B e^{u\gamma} \, dx < +\infty, \quad \forall \ u \in \mathbf{E}, \quad \text{if and only if} \quad \gamma \leq \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}$$

(1.3)
and

\[
\sup_{u \in E} \int_B e^{\alpha|u|^{N,\beta}} dx < +\infty \iff \alpha \leq \alpha_{N,\beta} = N[\omega_{N-1}^{-1}(1-\beta)]^{\frac{1}{1-\beta}},
\]

(1.4)

where \(\omega_{N-1}\) is the area of the unit sphere \(S^{N-1}\) in \(\mathbb{R}^N\) and \(N'\) is the Hölder conjugate of \(N\).

Let \(\gamma := \gamma_{N,\beta} = \frac{N'}{1-\beta}\), in view of inequalities (1.5) and (1.6), we say that \(f\) has subcritical growth at \(+\infty\) if

\[
\lim_{s \to +\infty} \frac{|f(x,s)|}{e^{\alpha s^{\gamma}}} = 0, \quad \text{for all } \alpha > 0
\]

(1.5)

and \(f\) has critical growth at \(+\infty\) if there exists some \(\alpha_0 > 0\),

\[
\lim_{s \to +\infty} \frac{|f(x,s)|}{e^{\alpha s^{\gamma}}} = 0, \quad \forall \alpha > \alpha_0 \quad \text{and} \quad \lim_{s \to +\infty} \frac{|f(x,s)|}{e^{\alpha s^{\gamma}}} = +\infty, \quad \forall \alpha > \alpha_0.
\]

(1.6)

In this paper, we deal with problem \((P_\lambda)\) under subcritical and critical growth nonlinearities. Furthermore, we suppose that \(f(x,t)\) satisfies the following hypothesis:

\((V_1)\) \(f : B \times \mathbb{R} \to \mathbb{R}\) is \(C^1\) and radial in \(x\).

\((V_2)\) There exist \(\theta > N\) such that we have

\[0 < \theta F(x,t) \leq tf(x,t), \forall (x,t) \in B \times \mathbb{R} \setminus \{0\}\]

where

\[F(x,t) = \int_0^t f(x,s)ds.\]

\((V_3)\) For each \(x \in B\), \(t \mapsto \frac{f(x,t)}{|t|^{N-1}}\) is increasing for \(t \in \mathbb{R} \setminus \{0\}\).

\((V_4)\) \(\lim_{t \to 0} \frac{|f(x,t)|}{|t|^{N-1}} = 0\).

We give an example of such nonlinearity. The nonlinearity \(f(x,t) = |t|^{N-1}t + |t|^N t \exp(\alpha |t|^\gamma)\) satisfies the assumptions \((V_1), (V_2), (V_3)\) and \((V_4)\).

We will consider the following definition of solutions.

**Definition 1.1** We say that a function \(u \in E\) is a weak solution of the problem \((P_\lambda)\) if

\[
\int_B |\nabla u|^{N-2}\nabla u \cdot \nabla \varphi w(x)dx = \lambda \int_B f(x,u)\varphi dx, \quad \forall \varphi \in E.
\]
Let $J_\lambda : E \to \mathbb{R}$ be the functional given by

$$J_\lambda(u) = \frac{1}{N} \int_B |\nabla u|^N w(x) dx - \lambda \int_B F(x,u) dx,$$  \hspace{1cm} (1.7)

where

$$F(x,t) = \int_0^t f(x,s) ds.$$  

The energy functional $J_\lambda$ is well defined and of class $C^1$ since there exist $a, C > 0$ positive constants and there exists $t_1 > 1$ such for that

$$|f(x,t)| \leq Ce^{a|t|}, \quad \forall |t| > t_1,$$

whenever the nonlinearity $f(x,t)$ is critical or subcritical at $+\infty$.

It is quite clear that finding non trivial weak solutions to the problem $(P_\lambda)$ is equivalent to finding non-zero critical points of the functional $J_\lambda$. Moreover, we have

$$\langle J'_\lambda(u), \phi \rangle = J'_\lambda(u) \phi = \int_B \left( \omega(x)|\nabla u|^{N-2}\nabla u \nabla \phi \right) dx - \lambda \int_B f(x,u) \phi dx, \quad \phi \in E.$$

We define the Nehari set as

$$N_\lambda := \{u \in E : \langle J'_\lambda(u), u^+ \rangle = \langle J'_\lambda(u), u^- \rangle = 0, u^+ \neq 0, u^- \neq 0\},$$

where $u^+ = \max\{u(x),0\}, u^- = \min\{u(x),0\}$.

It’s easy to verify the following decomposition

$$J_\lambda(u) = J_\lambda(u^+) + J_\lambda(u^-),$$

and

$$\langle J'_\lambda(u), u^+ \rangle = \langle J'_\lambda(u), u^- \rangle \quad \text{and} \quad \langle J'_\lambda(u), u^- \rangle = \langle J'_\lambda(u), u^- \rangle$$

We also give the following definitions of the so called nodal solutions and least energy sign-changing solution of problem $(P_\lambda)$.

**Definition 1.2**  
- $v \in E$ is called nodal or sign-changing solution of problem $(P_\lambda)$ if $v$ is a solution of problem $(P_\lambda)$ and $v^\pm \neq 0 \text{ a.e in } B$.
- $v \in E$ is called least energy sign-changing solution of problem $(P_\lambda)$ if $v$ is a sign-changing solution of $(P_\lambda)$ and

$$J_\lambda(v) = \inf\{J_\lambda(u) : J'_\lambda(u) = 0, u^\pm \neq 0 \text{ a.e in } B\}$$

Influenced by the works cited above, we try to find a minimize of the energy functional $J_\lambda$ over the following minimization problem,

$$c_\lambda = \inf_{u \in N_\lambda} J_\lambda(u)$$
Our approach is based on the Nehari manifold method, which was introduced in [15] and is by now a well-established and useful tool in finding solutions of problems with a variational structure, see [14]. To our best knowledge, there are few results for the nodal solutions to the N-weighted Laplace equation with critical exponential nonlinearity on the weighted Sobolev space $E$.

Now, we give our main results as follows:

**Theorem 1.2** Let $f(x,t)$ be a function that has a subcritical growth at $+\infty$ $(V_1)$, $(V_2)$, $(V_3)$, and $(V_4)$ are satisfied. For $\lambda > 0$, the problem $(P_\lambda)$ has a least energy nodal (sign-changing) radial solution $\nu \in N_\lambda$.

For a critical growth nonlinearity, the following result holds.

**Theorem 1.3** Assume that $f(x,t)$ has a critical growth at $+\infty$ for some $\alpha_0$ and $(V_1)$, $(V_2)$, $(V_3)$ and $(V_4)$ are satisfied. Then, there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, problem $(P_\lambda)$ has a least energy nodal (sign-changing) radial solution $\nu \in N_\lambda$.

This present work is organized as follows: in section 2, some preliminaries for the compactness analysis are presented. In section 3, we give some technical key lemmas. In section 4 we prove our result in the subcritical case. Section 5 is devoted for the critical case which is more difficult. We use a concentration compactness result of Lions type to prove Theorem 1.3.

Finally, we note that a constant $C$ may change from line to another and sometimes we index the constants in order to show how they change.

## 2 Preliminaries for the compactness analysis

In this section, we will present a number of technical Lemmas for our future use. We begin by the radial Lemma.

**Lemma 2.1** [7]Let $u$ be a radially symmetric function in $C^1_0(B)$. Then, we have

$$|u(x)| \leq \frac{\log(|x|)}{\omega_{N-1}^\frac{1}{(1-\beta)^\frac{1}{N}}} \|u\|,$$

where $\omega_{N-1}$ is the area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$.

It follows that the embedding $E \hookrightarrow L^q(B)$ is continuous for all $q \geq 1$, and that there exists a constant $C > 0$ such that $\|u\|_{N/q} \leq C\|u\|$, for all $u \in E$. Moreover, the embedding $E \hookrightarrow L^q(B)$ is compact for all $q \geq N$.

**Lemma 2.2** Let $(u_k)_k$ be a sequence in $E$. Suppose that, $\|u_k\| = 1$, $u_k \rightharpoonup u$ weakly in $E$, $u_k(x) \to u(x)$ a.e. in $B$, $\nabla u_k(x) \rightharpoonup \nabla u(x)$ a.e. in $B$ and $u \neq 0$. Then

$$\sup_k \int_B e^{p \alpha_{N,\beta}|u_k|^\gamma} \, dx < +\infty,$$

where $\alpha_{N,\beta} = N[\omega_{N-1}^\frac{1}{(1-\beta)}]^{1\gamma}$. 

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for all $1 < p < \mathcal{U}(u)$, where $\mathcal{U}(u)$ is given by:

$$\mathcal{U}(u) := \begin{cases} 
\frac{1}{(1 - \|u\|^N)^\frac{q}{q'}} & \text{if } \|u\| < 1 \\
+\infty & \text{if } \|u\| = 1.
\end{cases}$$

Proof: For $a, b \in \mathbb{R}, q > 1$. If $q'$ its conjugate i.e. $\frac{1}{q} + \frac{1}{q'} = 1$ we have, by young inequality, that

$$e^{a+b} \leq \frac{1}{q} e^{qa} + \frac{1}{q'} e^{q'b}.$$ 

Also, we have

$$(1 + a)^q \leq (1 + \varepsilon) a^q + (1 - \frac{1}{(1 + \varepsilon)^{\frac{1}{q'}}})^{1-q}, \forall a \geq 0, \forall \varepsilon > 0 \text{ and } \forall q > 1.$$ 

So, we get

$$|u_k|^\gamma = |u_k - u + u|^{\gamma} \leq (|u_k - u| + |u|)^{\gamma} \leq (1 + \varepsilon)|u_k - u|^{\gamma} + (1 - \frac{1}{(1 + \varepsilon)^{\frac{1}{q'}}})^{1-\gamma}|u|^{\gamma}$$

which implies that

$$\int_B e^{p'\alpha_N,\beta |u_k|^{\gamma}} dx \leq \frac{1}{q'} \int_B e^{pq'\alpha_N,\beta (1+\varepsilon)|u_k-u|^{\gamma}} dx + \frac{1}{q'} \int_B e^{pq'\alpha_N,\beta (1 - \frac{1}{(1 + \varepsilon)^{\frac{1}{q'}}})^{1-\gamma}|u|^{\gamma}} dx,$$

for any $p > 1$. Since $(1 - \frac{1}{(1 + \varepsilon)^{\frac{1}{q'}}})^{1-\gamma} \leq 1$, then

$$\frac{1}{q'} \int_B e^{pq'\alpha_N,\beta (1 - \frac{1}{(1 + \varepsilon)^{\frac{1}{q'}}})^{1-\gamma}|u|^{\gamma}} dx \leq \frac{1}{q'} \int_B e^{pq'\alpha_N,\beta |u|^{\gamma}} dx = \frac{1}{q'} \int_B e^{(pq'\alpha_N,\beta)^\gamma |u|^{\gamma}} dx.$$ 

From (1.5), the last integral is finite. 
To complete the evidence, we have to prove that for every $p$ such that $1 < p < \mathcal{U}(u)$,

$$\sup_k \int_B e^{pq'\alpha_N,\beta (1+\varepsilon)|u_k-u|^{\gamma}} dx < +\infty,$$ 

for some $\varepsilon > 0$ and $q > 1$. 
In the following, we suppose that $\|u\| < 1$ and in the case of $\|u\| = 1$, the proof is similar. 
When $\|u\| < 1$
and

\[ p < \frac{1}{(1 - \|u\|^{N})^\gamma}, \]

there exists \( \nu > 0 \) such that

\[ p(1 - \|u\|^{N})^\gamma (1 + \nu) < 1. \]

On the other hand, from the Brezis-Lieb’s lemma [4] it holds that

\[ \|u_k - u\|^{N} = \|u_k\|^{N} - \|u\|^{N} + o(1) \quad \text{where } o(1) \to 0 \quad \text{as } k \to +\infty. \]

Then,

\[ \|u_k - u\|^{N} = 1 - \|u\|^{N} + o(1), \]

so,

\[ \lim_{k \to +\infty} \|u_k - u\|^{\gamma} = (1 - \|u\|^{N})^\gamma. \]

Therefore, for every \( \varepsilon > 0 \), there exists \( k_\varepsilon \geq 1 \) such that

\[ \|u_k - u\|^{\gamma} \leq (1 + \varepsilon)(1 - \|u\|^{N})^\gamma, \quad \forall \ k \geq k_\varepsilon. \]

Then, for \( q = 1 + \varepsilon \) with \( \varepsilon = \sqrt{1 + \nu} - 1 \) and for any \( k \geq k_\varepsilon \), we have

\[ pq(1 + \varepsilon)\|u_k - u\|^{\gamma} \leq 1. \]

Consequently,

\[ \int_B e^{pq \alpha_{N,\beta}(1+\varepsilon)\|u_k - u\|^\gamma} \ dx \leq \int_B e^{(1+\varepsilon) pq \alpha_{N,\beta}(\frac{|u_k - u|}{\|u\|})^\gamma\|u_k - u\|^\gamma} \ dx \]

\[ \leq \int_B e^{\alpha_{N,\beta}(\frac{|u_k - u|}{\|u\|})^\gamma} \ dx \]

\[ \leq \sup_{\|u\| \leq 1} \int_B e^{\alpha_{N,\beta}|u|^\gamma} \ dx < +\infty. \]

Now, (2.1) follows from (1.4). This complete the proof. A second important Lemma.

**Lemma 2.3 [14]** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and \( f : \overline{\Omega} \times \mathbb{R} \) be a continuous function. Let \( \{u_n\}_n \) be a sequence in \( L^1(\Omega) \) converging to \( u \) in \( L^1(\Omega) \). Assume that \( f(x, u_n) \) and \( f(x, u) \) are also in \( L^1(\Omega) \). If

\[ \int_{\Omega} |f(x, u_n) u_n| dx \leq C, \]

where \( C \) is a positive constant, then

\[ f(x, u_n) \to f(x, u) \quad \text{in } L^1(\Omega). \]
3 Some technical lemmas

In the following we assume, unless otherwise stated, that the function $f$ satisfies the conditions $(V_1)$ to
$(V_4)$. Let $u \in E$ with $u^+ \neq 0$ a.e. in the ball $B$, and we define the function $\Upsilon_u : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and mapping $L_u : [0, \infty) \times [0, \infty) \to \mathbb{R}^2$ as

$$\Upsilon_u(p, q) = J_\lambda(pu^+ + qu^-),$$

and

$$L_u(p, q) = (\langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle, \langle J'_\lambda(pu^+ + qu^-), qu^- \rangle)$$

Lemma 3.1 (i) For each $u \in E$ with $u^+ \neq 0$ and $u^- \neq 0$, there exists an unique couple $(p_u, q_u) \in (0, \infty) \times (0, \infty)$ such that $p_u u^+ + q_u u^- \in \mathcal{N}_\lambda$. In particular, the set $\mathcal{N}_\lambda$ is nonempty.

(ii) For all $p, q \geq 0$ with $(p, q) \neq (p_u, q_u)$, we have

$$J_\lambda(pu^+ + qu^-) < J_\lambda(p_u u^+ + q_u u^-).$$

Proof. (i)

Since $f$ is subcritical or critical, and From $(V_1)$ and $(V_4)$, for all $\varepsilon > 0$, there exists a positive constant $C_1 = C_1(\varepsilon)$ such that

$$f(x,t) \leq \varepsilon |t|^N + C_1 |t|^s \exp(\alpha |t|^\gamma)$$

for all $\alpha > a_0$, $s > N$. (3.3)

Now, given $u \in E$ fixed with $u^+ \neq 0$ and $u^- \neq 0$. From (3.3), for all $\varepsilon > 0$, we have

$$\langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle = \langle J_\lambda(pu^+), pu^+ \rangle$$

$$= \|pu^+\|^N - \lambda \int_B f(x, pu^+) pu^+ dx$$

$$\geq \|pu^+\|^N - \lambda \int_B |pu^+|^N dx - \lambda C_1 \int_B |pu^+|^s \exp(\alpha |u^+|^\gamma) dx$$

Using the Hölder inequality, with $a, a' > 1$ such that $\frac{1}{a} + \frac{1}{a'} = 1$, and Lemma 2.1, we get

$$\langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle \geq \|pu^+\|^N - \lambda \varepsilon C_2 - \lambda C_1 \left( \int_B |pu^+|^{a'} dx \right)^{\frac{1}{a'}} \left( \int_B \exp(\alpha |u^+|^\gamma) dx \right)^{\frac{1}{a'}}$$

$$\geq (1 - \varepsilon C_2 - \lambda \varepsilon C_1) \|pu^+\|^N - \lambda C_1 \left( \int_B \exp(\alpha |pu^+|\gamma) \left( \frac{|u^+|}{\|u^+\|}\right)^\gamma dx \right)^\frac{1}{a} C_5 \|pu^+\|^s$$

By (1.4), the last integral is finite provided $p > 0$ is chosen small enough such that $\alpha a \|pu^+\| \gamma \leq a_{N, \beta}$.

Then,

$$\langle J'_\lambda(pu^+ + qu^-), pu^+ \rangle \geq (1 - \varepsilon C_2 - \lambda \varepsilon C_1) \|pu^+\|^N - \lambda C_4 \|pu^+\|^s$$

(3.4)
holds. Choosing $\epsilon > 0$ such that $1 - \epsilon C_2 - \lambda \epsilon C_1 > 0$ and for small $p > 0$ and for all $q > 0$ and $s > N$, we get $\langle J'_\lambda (pu^+ + qu^-), pu^+ \rangle > 0$. In the similar way, it can be proved that $\langle J'_\lambda (pu^+ + qu^-), pu^- \rangle > 0$ for $q > 0$ small enough and all $p > 0$. Therefore, it is quite easy to state that there exists $t_1 > 0$ such that

$$\langle J'_\lambda (t_1u^+ + qu^-), t_1u^+ \rangle > 0, \quad \langle J'_\lambda (pu^+ + t_1u^-), t_1u^- \rangle > 0 \text{ for all } p, q > 0. \quad (3.5)$$

From $(V_5)$, we can derive that there exists $C_5, C_6 > 0$ such that

$$F(x, t) \geq C_5 |t|^\theta - C_6. \quad (3.6)$$

Now, choose $p = t_2^* > t_1$ with $t_2^*$ large enough. Then, by using (3.3), (3.6), we get

$$\langle J'_\lambda (t_2^*u^+ + qu^-), t_2^*u^+ \rangle = \langle J'_\lambda (t_2^*u^+), t_2^*u^+ \rangle \leq \|t_2^*u^+\|^N - \lambda \int_B C_5 |t_2^*u^+|^\theta \, dx + \lambda C_6 |B| \leq 0,$$

for $q \in [t_1, t_2^*]$. Also, we can choose $q = t_2^* > t_1$ with $t_2^*$ large enough and then

$$\langle J'_\lambda (t_2^*u^+ + t_2^*u^-), t_2^*u^+ \rangle < 0 \text{ for all } p \in [t_1, t_2^*].$$

Therefore, if $t_2 > t_2^*$ is large enough, then we obtain that

$$J'_\lambda (t_2u^+ + qu^-), t_2u^+ < 0 \quad \text{and} \quad \langle J'_\lambda (pu^+ + t_2u^-), t_2u^- \rangle < 0 \text{ for all } p, q \in [t_1, t_2]. \quad (3.7)$$

Joining (3.5) and (3.7) with Miranda’s Theorem [2], there exists at least a couple of points $(p_0, q_0) \in (0, \infty) \times (0, \infty)$ such that $L_u (p_0, q_0) = (0, 0)$, i.e., $p_0u^+ + q_0u^- \in N_\lambda$.

Now we will show the uniqueness of the couple $(p_0, q_0)$. Indeed, it is sufficient to show that if $u \in N_\lambda$ and $p_0u^+ + q_0u^- \in N_\lambda$ with $p_0 > 0$ and $q_0 > 0$, then $(p_0, q_0) = (1, 1)$. Let us assume that $u \in N_\lambda$ and $p_0u^+ + q_0u^- \in N_\lambda$. We will then get $\langle J'_\lambda (p_0u^+ + q_0u^-), p_0u^+ \rangle = 0$, $\langle J'_\lambda (p_0u^+ + q_0u^-), p_0u^- \rangle = 0$, and $\langle J'_\lambda (u, u^\pm) \rangle = 0$, that is,

$$\|p_0u^+\|^N = \lambda \int_B f(x, p_0u^+)p_0u^+ \, dx \quad (3.8)$$

$$\|b_0u^-\|^N = \lambda \int_B f(x, q_0u^-)q_0u^- \, dx \quad (3.9)$$

$$\|u^+\|^N = \lambda \int_B f(x, u^+)u^+ \, dx \quad (3.10)$$

$$\|u^-\|^N = \lambda \int_B f(x, u^-)u^- \, dx \quad (3.11)$$

Combining (3.8) and (3.10), we deduce that

$$0 = \lambda \int_B \frac{f(x, p_0u^+)}{p_0^N} \, dx - \lambda \int_B f(x, u^+) \, dx.$$
It follows from (V4) that \( t \mapsto \frac{f(x,t)}{t^{N-1}} \) is increasing for \( t > 0 \), which implies that \( p_0 = 1 \). We can also show, using (V4), (3.9) and (3.11), that \( q_0 = 1 \). This completes the proof of (i).

(ii) To prove (ii), it is sufficient to show that \((p_u, q_u)\) is the unique maximum point of \( \Upsilon_u \in [0, \infty) \times [0, \infty) \). From (3.7), (3.8) and \( \theta > N \), we have

\[
\Upsilon_u(p, q) = J_\lambda(pu^+ + qu^-) \\
= \frac{1}{N} \|pu^+ + qu^-\|^N - \lambda \int_B F(x, pu^+ + qu^-) \, dx \\
\leq \frac{p^N}{N} \|u^+\|^N + \frac{q^N}{N} \|u^-\|^N - \lambda C_5 p^\theta \int_B |u^+|^\theta \, dx - \lambda C_5 q^\theta \int_B |u^-|^\theta \, dx + \lambda C_6 |B|
\]

which implies that \( \lim_{\|p,q\| \to \infty} \Upsilon_u(p, q) = -\infty \). Hence, it suffices to see that the maximum point of \( \Upsilon_u \) cannot be realized on the boundary of \([0, \infty) \times [0, \infty) \). We argue by contradiction and assume that \((0, q)\) with \( q \geq 0 \) is a maximum point of \( \Upsilon_u \). Then from (3.5), we have

\[
p \frac{d}{dp} [J_\lambda(pu^+ + qu^-)] = \langle J_\lambda'(pu^+), pu^+ \rangle > 0,
\]

for small \( p > 0 \), which means that \( \Upsilon_u \) is increasing with respect to \( p \) if \( p > 0 \) is small enough. This gives a contradiction. We can similarly deduce that \( \Upsilon_u \) can not realize its global maximum on \((p, 0)\) with \( p \geq 0 \).

**Lemma 3.2** For any \( u \in \mathbf{E} \) with \( u^+ \neq 0 \) and \( u^- \neq 0 \), such that \( \langle J_\lambda'(pu^+, pu^+) \rangle \leq 0 \), the unique maximum point \((p_u, q_u)\) of \( \Upsilon_u \) on \([0, \infty) \times [0, \infty) \) belongs to \((0, 1) \times (0, 1) \).

Proof. Here we will only prove that \( 0 < p_u \leq 1 \). The proof of \( 0 < q_u \leq 1 \) is similar. Since \( pu^+, qu^+ \in N_\lambda \), we have that

\[
\|pu^+\|^N = \lambda \int_B f(x, pu^+)u^+ \, dx
\]

Moreover, by \( \langle J_\lambda'(pu^+, pu^+) \rangle \leq 0 \), we have that

\[
\|u^+\|^N \leq \lambda \int_B f(x, u^+)u^+ \, dx.
\]

Combining (3.12) and (3.13), it follows that

\[
\int_B f(x, u^+)u^+ \, dx \geq \int_B \frac{f(x, pu^+)p_u u^+}{p_u} \, dx.
\]

Now, we suppose, by contradiction, that \( p_u > 1 \). By (V4), \( t \mapsto \frac{f(x,t)}{t^{N-1}} \) is increasing for \( t > 0 \), which contradicts inequality (3.14). Therefore, \( 0 < p_u \leq 1 \).

**Lemma 3.3** For all \( u \in N_\lambda \),

(i) **there exists** \( \kappa > 0 \) **such that**

\[
\|u^+\|, \|u^-\| \geq \kappa;
\]
(ii) \( J_\lambda(u) \geq (\frac{1}{N} - \frac{1}{\theta})\|u\|^N \)

Proof. (i) We argue by contradiction. Suppose that there exists a sequence \( \{u^+_n\} \subset N_\lambda \) such that \( u^+_n \rightarrow 0 \) in \( E \). Since \( \{u_n\} \subset N_\lambda \), then \( \langle J_\lambda'(u_n), u^+_n \rangle = 0 \). Hence, it follows from (3.3), (3.4) and the radial Lemma 2.1 that

\[
\|u^+_n\|^N = \lambda \int_B f(u^+_n)u^+_n \, dx \\
\leq \epsilon \lambda \int_B |u^+_n|^N \, dx + \lambda C_1 \int_B |u^+_n|^s \exp(\alpha |u^+_n|\gamma) \, dx \\
\leq \epsilon \lambda C_6 \|u^+_n\|^N + \lambda C_1 \int_B |u^+_n|^s \exp(\alpha |u^+_n|\gamma) \, dx
\]

Let \( a > 1 \) with \( \frac{1}{a} + \frac{1}{s} = 1 \). Since \( u^+_n \rightarrow 0 \) in \( E \), for \( n \) large enough, we get \( \|u^+_n\| \leq (\frac{\alpha N, \beta}{\alpha a})^s \). From Hölder inequality, (1.4) and again the radial Lemma 2.1, we have

\[
\int_B |u^+_n|^s \exp(\alpha |u^+_n|\gamma) \, dx \leq \left( \int_B |u^+_n|^{sa'} \, dx \right)^{\frac{1}{a'}} \left( \int_B \exp \left( \alpha a \|u^+_n\|^s \frac{|u^+_n|}{\|u^+_n\|^s} \right) \, dx \right)^{\frac{1}{a}} \\
\leq C_7 \left( \int_B |u^+_n|^{sa'} \, dx \right)^{\frac{1}{a'}} \leq C_8 \|u^+_n\|^s
\]

Combining (3.15) with the last inequality, for \( n \) large enough, we obtain

\[
\|u^+_n\|^N \leq \lambda \epsilon C_6 \|u^+_n\|^N + \lambda C_8 \|u^+_n\|^s
\]

Choose suitable \( \epsilon > 0 \) such that \( 1 - \lambda \epsilon C_6 > 0 \). Since \( N < s \), then (3.16) contradicts the fact that \( u^+_n \rightarrow 0 \) in \( E \).

(ii) Given \( u \in N_\lambda \), by the definition of \( N_\lambda \) and (V3) we obtain

\[
J_\lambda(u) = J_\lambda(u) - \frac{1}{\theta} \langle J_\lambda'(u), u \rangle \\
= \frac{1}{N} \|u\|^N + \lambda \left( \int_B \frac{1}{\theta} f(x, u)u - F(x, u) \, dx \right) \\
\geq (\frac{1}{N} - \frac{1}{\theta})\|u\|^N
\]

Lemma 3.3 implies that \( J_\lambda(u) > 0 \) for all \( u \in N_\lambda \). As a consequence, \( J_\lambda \) is bounded by below in \( N_\lambda \), and therefore \( c_\lambda := \inf_{u \in N_\lambda} J_\lambda(u) \) is well-defined.

The following lemma deals with the asymptotic property of \( c_\lambda \).

**Lemma 3.4** Let \( c_\lambda = \inf_{u \in N_\lambda} J_\lambda(u) \), then \( \lim_{\lambda \to \infty} c_\lambda = 0 \)
Proof. Let us fix $u \in E$ with $u^\pm = 0$. Then, by Lemma 3.1, there exists a point pair $(p_\lambda, q_\lambda)$ such that $p_\lambda u^+ + q_\lambda u^- \in \mathcal{N}_\lambda$ for each $\lambda > 0$. Let $\mathcal{T}_u$ be the set defined by

$$\mathcal{T}_u := \{(p_\lambda, q_\lambda) \in [0, \infty) \times [0, \infty) : L_u(p_\lambda, q_\lambda) = (0, 0), \lambda > 0\},$$

where $L_u$ is given by (3.2).

Since $p_\lambda u^+ + q_\lambda u^- \in \mathcal{N}_\lambda$, by assumption $(V_2)$, (3.7) and (3.8), we have

$$p_\lambda^N \|u^+\|^N + q_\lambda^N \|u^-\|^N = \lambda \int_B f(x, p_\lambda u^+ + q_\lambda u^-)(p_\lambda u^+ + q_\lambda u^-)dx$$

$$\geq \lambda \theta C_5 p_\lambda^\theta \int_B |u^+|^\theta dx + \lambda \theta C_5 q_\lambda^\theta \int_B |u^-|^\theta dx - \lambda \theta C_6 |B|.$$ 

Since $\theta > N$, the set $\mathcal{T}_u$ is bounded. Therefore, if $\{\lambda_n\} \subset (0, \infty)$ satisfies $\lambda_n \to \infty$ as $n \to \infty$, then up to subsequence, there exists $\bar{p}, \bar{q} > 0$, such that $p_{\lambda_n} \to \bar{p}$ and $q_{\lambda_n} \to \bar{q}$.

We claim that $\bar{p} = \bar{q} = 0$. We proceed by contradiction and suppose that $\bar{p} > 0$ and $\bar{q} > 0$. For each $n \in \mathbb{N}$, $p_{\lambda_n} u^+ + q_{\lambda_n} u^- \in \mathcal{N}_{\lambda_n}$. So,

$$\|p_{\lambda_n} u^+ + q_{\lambda_n} u^-\|^N = \lambda_n \int_B f(p_{\lambda_n} u^+ + q_{\lambda_n} u^-)(p_{\lambda_n} u^+ + q_{\lambda_n} u^-)dx.$$ 

(3.17)

It should be noted that $p_{\lambda_n} u^+ \to \bar{p} u^+$ and $q_{\lambda_n} u^- \to \bar{q} u^-$ in $E$.

On one hand, $\lambda_n \to 0$ as $n \to \infty$ and $\{p_{\lambda_n} u^+ + q_{\lambda_n} u^-\}$ is bounded in $E$. On the other hand, from (3.17), we have

$$\int_B |\nabla (\bar{p} u^+ + \bar{q} u^-)|^N dx = \left(\lim_{n \to \infty} \lambda_n\right) \lim_{n \to \infty} \int_B f(p_{\lambda_n} u^+ + q_{\lambda_n} u^-)(p_{\lambda_n} u^+ + q_{\lambda_n} u^-)dx$$

which is impossible.

Thus, $\bar{p} = \bar{q} = 0$, so, $p_{\lambda_n} \to 0$ and $q_{\lambda_n} \to 0$ as $n \to \infty$. Finally, by $(V_2)$ and (3.17), we have

$$0 \leq c_{\lambda} = \inf_{\mathcal{N}_\lambda} \mathcal{J}_\lambda(u) \leq \mathcal{J}_\lambda(p_{\lambda_n} u^+ + q_{\lambda_n} u^-) \to 0.$$ 

Consequently, $c_{\lambda} \to 0$ as $\lambda \to \infty$.

**Lemma 3.5** If $u_0 \in \mathcal{N}_\lambda$ satisfies $\mathcal{J}_\lambda(u_0) = c_{\lambda}$, then $\mathcal{J}_\lambda'(u_0) = 0$.

**Proof.** We proceed by contradiction. We assume that $\mathcal{J}_\lambda'(u_0) \neq 0$. By the continuity of $\mathcal{J}_\lambda'$, there exists $\iota, \delta \geq 0$ such that

$$\|\mathcal{J}_\lambda'(v)\|_E \geq \iota \text{ for all } \|v - u_0\| \leq 3\delta.$$ 

(3.18)

Choose $\tau \in (0, \min\{\frac{1}{4}, \frac{\delta}{4\|u_0\|}\})$. Let $D = (1 - \tau, 1 + \tau) \times (1 - \tau, 1 + \tau)$ and define $g : D \to E$, by

$$g(\rho, \vartheta) = \rho u^+ + \vartheta u^- \text{, } (\rho, \vartheta) \in D$$
By virtue of \( u_0 \in \mathcal{N}_\lambda \), \( \mathcal{J}_\lambda(u_0) = c_\lambda \) and Lemma 3.1, it is easy to see that
\[
c_\lambda := \max_{\partial D} \mathcal{J}_\lambda \circ g < c_\lambda. \tag{3.19}
\]

Let \( \epsilon := \min\{\frac{\lambda - \epsilon}{\lambda}, \frac{\epsilon}{\lambda}\} \), \( S_\tau := B(u_0, r), r \geq 0 \) and \( \mathcal{J}_\lambda^{a} := \mathcal{J}_\lambda^{-1}(]-\infty, a]) \). According to the Quantitative Deformation Lemma [[21], Lemma 2.3], there exists a deformation \( \eta \in C([0, 1] \times g(D), E) \) such that:

1. \( \eta(1, v) = v \), if \( v \not\in \mathcal{J}_\lambda^{a}([c_\lambda - 2\epsilon, c_\lambda + 2\epsilon]) \cap S_2\delta \)
2. \( \eta(1, \mathcal{J}_\lambda^{c_\lambda + \epsilon} \cap S_\delta) \subseteq \mathcal{J}_\lambda^{c_\lambda - \epsilon} \)
3. \( \mathcal{J}_\lambda(\eta(1, v)) \leq \mathcal{J}_\lambda(v) \), for all \( v \in E \).

By lemma 3.1 (ii), we have \( \mathcal{J}_\lambda(g(\rho, \theta)) \leq c_\lambda \). In addition, we have,
\[
\|g(s, t) - u_0\| = \|(\rho - 1)u_0^+ + (\vartheta - 1)u_0^-\| \leq |\rho - 1|\|u_0^+\| + |\vartheta - 1|\|u_0^-\| \leq 2\tau\|u_0\|,
\]
then \( g(\rho, \theta) \in S_\delta \) for \( (\rho, \theta) \in D \). Therefore, it follows from (2) that
\[
\max_{(\rho, \theta) \in D} \mathcal{J}_\lambda(\eta(1, g(\rho, \theta))) \leq c_\lambda - \epsilon. \tag{3.20}
\]

In the following, we prove that \( \eta(1, g(D)) \cap \mathcal{N}_\lambda \) is nonempty. And in this case it contradicts (3.20) due to the definition of \( c_\lambda \). To do this, we first define
\[
\bar{g}(\rho, \theta) := \eta(1, g(\rho, \theta)),
\]
\[
\mathcal{Y}_0(\rho, \theta) = ((\mathcal{J}_\lambda^{1}(g(\rho, \theta)), u_0^+), (\mathcal{J}_\lambda^{a}(g(\rho, \theta)), u_0^-))
\]
\[
= ((\mathcal{J}_\lambda^{1}(\rho u_0^+ + \vartheta u_0^-), u_0^+), (\mathcal{J}_\lambda^{a}(\rho u_0^+ + \vartheta u_0^-), u_0^-))
\]
\[
:= (\varphi_0^{1}(\rho, \theta), \varphi_0^{2}(\rho, \theta))
\]
and
\[
\mathcal{Y}_1(\rho, \theta) := (\frac{1}{\rho}(\mathcal{J}_\lambda^{1}(\bar{g}(\rho, \theta))), (\bar{g}(\rho, \theta))^+) + \frac{1}{\vartheta}(\mathcal{J}_\lambda^{a}(\bar{g}(\rho, \theta)), (\bar{g}(\rho, \theta))^-) \).
\]
Moreover, a simple calculation, shows that
\[
\frac{\varphi_0^{1}(\rho, \theta)}{\partial \rho}\bigg|_{(1,1)} = (N - 1)\|u_0^+\|_N - \lambda \int_B f'(x, u_0^+)|u_0^+|^2 dx
\]
\[
= (N - 1)\lambda \int_B f(u_0^+)|u_0^+|^2 dx - \lambda \int_B f'(x, u_0^+) |u_0^+|^2 dx
\]
and
\[
\frac{\varphi_0^{1}(\rho, \theta)}{\partial \vartheta}\bigg|_{(1,1)} = 0.
\]
In the same manner,

\[
\frac{\varphi_2^2(\rho, \vartheta)}{\partial \rho} \bigg|_{(1,1)} = 0
\]

and

\[
\frac{\varphi_2^2(\rho, \vartheta)}{\partial \vartheta} \bigg|_{(1,1)} = (N - 1)\lambda \int_B f(x, u_0^-)u_0^- dx - \lambda \int_B f'(x, u_0^-)|u_0^-|^2 dx
\]

Let

\[
J = \begin{pmatrix}
\frac{\varphi_1^2(\rho, \vartheta)}{\partial \rho} & \frac{\varphi_0^2(\rho, \vartheta)}{\partial \vartheta} \\
\frac{\varphi_1^2(\rho, \vartheta)}{\partial \vartheta} & \frac{\varphi_0^2(\rho, \vartheta)}{\partial \rho}
\end{pmatrix}
\]

Then we have \( \det J \neq 0 \). Therefore, the point \((0, 1)\) is the unique isolated zero of the \( C^1 \) function \( \Upsilon_0 \). By using the Brouwer’s degree in \( \mathbb{R}^2 \), we deduce that \( \deg(\Upsilon_0, D, 0) = 1 \).

Now, it follows from (3.20) and (1) that \( g(\rho, \vartheta) = \overline{g}(\rho, \vartheta) \) on \( \partial D \). For the boundary dependence of Brouwer’s degree (see [12, Theorem 4.5]), there holds \( \deg(\Upsilon_1, D, 0) = \deg(\Upsilon_0, D, 0) = 1 \). Therefore, there exists some \((\overline{\rho}, \overline{\vartheta}) \in D \) such that

\[ \eta(1, g(\overline{\rho}, \overline{\vartheta})) \in N_\lambda. \]

This finish the proof of the Lemma.

**Lemma 3.6** If \( \upsilon \) is a least energy sign-changing solution of problem \( (P_\lambda) \), then \( \upsilon \) has exactly two nodal domains

**Proof.** Assume by contradiction that \( \upsilon = \upsilon_1 + \upsilon_2 + \upsilon_3 \) satisfies

\( \upsilon_1 \neq 0, i = 1, 2, 3, \upsilon_1 \geq 0, \upsilon_2 \leq 0, \) a.e. in \( B \)

\( B_1 \cap B_2 = \emptyset, B_1 := \{x \in B : \upsilon_1(x) > 0\}, B_2 := \{x \in B : \upsilon_2(x) < 0\} \)

\( \upsilon_1 \bigg|_{B \setminus B_1 \cup B_2} = \upsilon_2 \bigg|_{B \setminus B_1 \cup B_2} = \upsilon_3 \bigg|_{B_1 \cup B_2} = 0, \)

and

\( \langle \mathcal{J}_\lambda'(\upsilon), \upsilon_i \rangle = 0 \) for \( i = 1, 2, 3 \), \( (3.21) \)

Let \( \nu = \upsilon_1 + \upsilon_2 \) and it is easy to see that \( \nu^+ = \upsilon_1, \nu^- = \upsilon_2 \) and \( \nu^+ \neq 0 \). From Lemma (3.1), it follows that there exists a unique couple \((p_\nu, q_\nu) \in [0, \infty) \times [0, \infty) \) such that \( p_\nu \upsilon_1 + q_\nu \upsilon_2 \in N_\lambda \). So, \( \mathcal{J}_\lambda(p_\nu \upsilon_1 + q_\nu \upsilon_2) \geq c_\lambda \). Moreover, using (3.21), we obtain that \( \langle \mathcal{J}_\lambda'(\nu), \nu^\pm \rangle = 0 \). Then, by Lemma (3.2), we have \( 0 < p_\nu, q_\nu \leq 1 \).
Now, combining (3.21), (V₃) and (V₄), we have that
\[ 0 = \frac{1}{\theta} \langle J'_\lambda(v), v_3 \rangle = \frac{1}{\theta} \langle J'_\lambda(v_3), v_3 \rangle \]
\[ < J_\lambda(v_3), \]
and
\[ c_\lambda \leq J_\lambda(p_\nu v_1 + q_\nu v_2) = J_\lambda(p_\nu v_1 + q_\nu v_2) - \frac{1}{\theta} \langle J'_\lambda(p_\nu v_1 + q_\nu v_2), p_\nu v_1 + q_\nu v_2 \rangle \]
\[ = (\frac{1}{N} - \frac{1}{\theta}) p_\nu^N \| v_1 \|^N + (\frac{1}{N} - \frac{1}{\theta}) q_\nu^N \| v_2 \|^N \]
\[ + \lambda \int_B \left[ \frac{1}{\theta} f(x, p_\nu v_1) - F(x, p_\nu v_2) \right] dx + \lambda \int_B \left[ \frac{1}{\theta} f(x, q_\nu v_1) - F(x, q_\nu v_2) \right] dx \]
\[ \leq J_\lambda(v_1 + v_2) - \frac{1}{\theta} \langle J'_\lambda(v_1 + v_2), v_1 + v_2 \rangle \]
\[ = J_\lambda(v_1 + v_2) + \frac{1}{\theta} \langle J'_\lambda(v), v_3 \rangle \]
\[ < J_\lambda(v_1 + v_2) + J_\lambda(v_3) = J_\lambda(v) = c_\lambda, \]
which is a contradiction. Therefore, \( v_3 = 0 \) and \( v \) has exactly two nodal domains.

4 The subcritical case

**Lemma 4.1** If \( \{ u_n \} \subset N_{\lambda} \) is a minimizing sequence for \( c_\lambda \), then there exists some \( u \in E \) such that
\[ \int_B f(u_n^\pm) u_n^\pm dx \to \int_B f(u^\pm) u^\pm dx \]
and
\[ \int_B F(u_n^\pm) dx \to \int_B F(u^\pm) dx \]

**Proof.** we will only prove the first result. Since the second limit is a direct consequence of the first one, we omit it here.

Let sequence \( \{ u_n \} \subset N_{\lambda} \) satisfy \( \lim_{n \to \infty} J_\lambda(u_n) = c_\lambda \). It is clearly that \( \{ u_n \} \) is bounded by Lemma (3.3).
Then, up to a subsequence, there exists \( u \in E \) such that
\[
\begin{align*}
  u_n & \to u \text{ in } E, \\
  u_n & \to u \text{ in } L^t(B) \text{ for } t \in [1, \infty), \\
  u_n & \to u \text{ a.e. in } B, \\
  u_n^\pm & \to u^\pm \text{ in } E, \\
  u_n^\pm & \to u^\pm \text{ in } L^t(B) \text{ for } t \in [1, \infty), \\
  u_n^\pm & \to u^\pm \text{ a.e. in } B.
\end{align*}
\] (4.1)

Note that by (3.3), we have
\[
f(x, u_n^\pm(x))u_n^\pm(x) \leq \epsilon |u_n^\pm(x)|^N + C_2|u_n^\pm(x)|^\gamma \exp(\alpha |u_n^\pm(x)|^\gamma) := h(u_n^\pm(x)),
\] (4.3)
for all \( \alpha > \alpha_0 \) and \( q > N \). It is sufficient to prove that sequence \( \{h(u_n^\pm)\} \) is convergent in \( L^1(B) \).

Choosing \( a, a' > 1 \) with \( \frac{1}{a} + \frac{1}{a'} = 1 \), we get that
\[
|u_n^\pm|^s \to |u|^s \text{ in } L^{a'}(B)
\] (4.4)
Moreover, choosing \( \alpha > 0 \) small enough such that \( \alpha a \left( \max_n \|u_n^\pm\|^\gamma \right) \leq \alpha N, \beta \), we conclude from (1.4) that
\[
\int_B \exp \left( \alpha |u_n^\pm(x)|^\gamma \right) \, dx < \infty.
\] (4.5)

Since \( \exp (\alpha |u_n^\pm(x)|^\gamma) \, dx \to \exp (\alpha |u^\pm(x)|^\gamma) \, dx \), a.e. in \( B \). From (4.5) and [[15], Lemma 4.8, chapter 1], we obtain that
\[
\exp (\alpha |u_n^\pm|^\gamma) \, dx \to \exp (\alpha |u|^\gamma) \, dx \text{ in } L^a(B).
\] (4.6)

Hence, by (4.4), (4.6) and [[15], Lemma 4.8, chapter 1] again, we conclude that
\[
\int_B f(u_n^\pm)u_n^\pm \, dx \to \int_B f(u^\pm)u^\pm \, dx.
\]

**Lemma 4.2** There exists some \( v \in \mathcal{N}_\lambda \) such that \( J_\lambda(v) = c_\lambda \).

**Proof.** Let \( \{v_n\} \subset \mathcal{N}_\lambda \) be a sequence such that \( \lim_{n \to \infty} J_\lambda(v_n) = c_\lambda \). It is clearly that \( \{v_n\} \) is bounded by Lemma (3.3). Then, up to a subsequence, there exists \( v \in E \) such that
\[
\begin{align*}
  v_n^\pm & \to v^\pm \text{ in } E, \\
  v_n^\pm & \to v^\pm \text{ in } L^t(B) \text{ for } t \in [1, \infty), \\
  v_n^\pm & \to v^\pm \text{ a.e. in } B.
\end{align*}
\] (4.7)

We claim that \( v^+ \neq 0 \) and \( v^- \neq 0 \). Suppose, by contradiction, \( v^+ = 0 \). From the definition of \( \mathcal{N}_\lambda \), (4.7), (4.3) and Lemma (4.1), we have that \( \lim_{n \to \infty} \|v_n^+\|^N = 0 \), which contradicts Lemma (3.3). Hence, \( v^+ \neq 0 \)
and \( v^- \neq 0 \).

From the lower semi continuity of norm and (4.7), it follows that

\[
\langle J'(\lambda)(v), v^\pm \rangle \leq \lim_{n \to \infty} \langle J'(\lambda)(v_n), v_n^\pm \rangle = 0.
\]  

(4.8)

Then, Lemma (3.2) implies that there exists \((p_v, q_v) \in (0, 1] \times (0, 1]\) such that \( p_v v^+ + q_v v^- \in \mathcal{N}_\lambda \). Thus, by \((V_2)\), \( \lambda \geq 0 \) and Lemma (4.1), we get that

\[
c_\lambda \leq J_\lambda(p_v v^+ + q_v v^-) = J_\lambda(p_v v^+ + q_v v^-) - \frac{1}{\theta} \langle J'_\lambda(p_v v^+ + q_v v^-), p_v v^+ + q_v v^- \rangle \\
\leq J(v) - \frac{1}{\theta} \langle J'_\lambda(v), v \rangle \\
\leq \lim_{n \to \infty} \left[ J_\lambda(v_n) - \frac{1}{\theta} \langle J'_\lambda(v_n), v_n \rangle \right] \\
= \lim_{n \to \infty} J_\lambda(v_n) = c_\lambda.
\]  

(4.9)

Noticing that if \( p_v < 1 \) or \( q_v < 1 \), then the inequality (4.9) is strict. Hence, by bringing together (4.8) and (4.9), we conclude that \( p_v = q_v = 1 \) and \( v \in \mathcal{N}_\lambda \) satisfying \( J(v) = c_\lambda \).

**Proof of Theorem 1.2.** From Lemma 3.5, Lemma 3.6 and Lemma 4.2, we deduce that \( v \) is a least energy sign-changing solution form problem \((P_\lambda)\) with exactly tow nodal domains.

## 5 The critical case

**Lemma 5.1** There exists \( \lambda^* > 0 \) such that if \( \lambda \geq \lambda^* \), and \( \{v_n\} \subset \mathcal{N}_\lambda \) is a minimizing sequence for \( c_\lambda \), then there exists some \( v \in \mathcal{N}_\lambda \) such that \( J_\lambda(v) = c_\lambda \).

**Proof.** Let \( \{v_n\} \subset \mathcal{N}_\lambda \) be a sequence such that \( \lim_{n \to \infty} J_\lambda(v_n) = c_\lambda \). We have

\[
J_\lambda(v_n) \to c_\lambda \text{ and } \langle J'_\lambda(v_n), \varphi \rangle \to 0, \forall \varphi \in \mathcal{E}
\]

that is

\[
J_\lambda(v_n) = \frac{1}{N} \|v_n\|^N - \int_B F(x, v_n)dx \to c_\lambda, \quad n \to +\infty
\]  

(5.1)

and

\[
|\langle J'_\lambda(v_n), \varphi \rangle| = \left| \int_B \omega(x)|\nabla v_n|^{N-2}\nabla v_n \cdot \nabla \varphi dx - \int_B f(x, v_n)\varphi dx \right| \leq \varepsilon_n \|\varphi\|,
\]  

(5.2)

for all \( \varphi \in \mathcal{E} \), where \( \varepsilon_n \to 0 \), as \( n \to +\infty \).

By lemma 3.3, \( v_n \) is bounded in \( \mathcal{E} \). Furthermore, we have from (5.2) and \((V_2)\), that

\[
0 < \int_B f(x, u_n)u_n \leq C
\]  

(5.3)
and

\[ 0 < \int_B F(x, u_n) \leq C. \]

Since by Lemma 3.2, we have

\[ f(x, u_n) \to f(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty, \tag{5.4} \]

then, it follows from \((H_2)\) and the generalized Lebesgue dominated convergence Theorem that

\[ F(x, u_n) \to F(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty. \tag{5.5} \]

Arguing as Lemma 4.2, we have that, up to a subsequence,

\[ v_n \to u \text{ in } \mathbb{E}, \]

\[ v_n \to u \text{ in } L^t(B) \text{ for } t \in [1, \infty), \]

\[ v_n \to u \text{ a.e. in } B, \]

\[ v_n^+ \to u^+ \text{ in } \mathbb{E}, \]

\[ v_n^+ \to u^+ \text{ in } L^t(B) \text{ for } t \in [1, \infty), \]

\[ v_n^\pm \to u^\pm \text{ a.e. in } B. \]

for some \( u \in \mathbb{E}. \)

Noticing that, according to lemma 3.4, there exists \( \lambda^* > 0 \) such that for all \( \lambda > \lambda^* \), we get

\[ c\lambda < \frac{1}{N} \left( \frac{\alpha N, \beta}{\alpha_0} \right)^{\frac{\gamma}{N}}. \]

In the sequel, the results that are valid for \( v_n \) and \( u \), are also valid for \( v_n^\pm \) and \( u^\pm \). Next, we are going to make some Claims.

**Claim 1.** \( \nabla v_n(x) \to \nabla v(x) \text{ a.e. in } B \text{ and } v \text{ is a solution of the problem } (P_\lambda). \)

Indeed, for any \( \xi > 0 \), let \( \mathcal{A}_\eta = \{x \in B, |v_n - v| \geq \xi\}. \) For all \( t \in \mathbb{R} \), for all positive \( c > 0 \), we have

\[ ct \leq e^t + c^2. \]

It follows that for \( t = \alpha N, \beta \left( \frac{|v_n - v|}{\|v_n - v\|} \right)^\gamma, c = \frac{1}{\alpha N, \beta} \|v_n - v\|^\gamma, \) we get

\[ |v_n - v|^\gamma \leq e^{\alpha N, \beta \left( \frac{|v_n - v|}{\|v_n - v\|} \right)} + \frac{1}{\alpha_{N, \beta}} \|v_n - v\|^{2\gamma} \]

\[ \leq e^{\alpha N, \beta \left( \frac{|v_n - v|}{\|v_n - v\|} \right)^\gamma} + C_1(N), \]

where \( C_1(N) \) is a constant depending only on \( N \) and the upper bound of \( \|v_n\| \). So, if we denote by \( \mathcal{L}(A_\xi) \) the Lebesgue measure of the set \( A_\xi \), we obtain

\[ \mathcal{L}(A_\xi) = \int_{A_\xi} e^{\alpha N, \beta \left( \frac{|v_n - v|}{\|v_n - v\|} \right)} dx \leq e^{-\xi^\gamma} \int_{A_\xi} e^{\alpha N, \beta \left( \frac{|v_n - v|}{\|v_n - v\|} \right)} + C_1(N) dx \]

\[ \leq e^{-\xi^\gamma} e^{C_1(N)} \int_B \exp \left( \alpha N, \beta \left( \frac{|v_n - v|}{\|v_n - v\|} \right)^\gamma \right) dx \]

\[ \leq e^{-\xi^\gamma} C_2(N) \to 0 \text{ as } \xi \to +\infty, \]

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where $C_2(N)$ is a positive constant depending only on $N$ and the upper bound of $\|v_n\|$. It follows that
\[
\int_{A_\xi} |\nabla v_n - \nabla v| dx \leq Ce^{-\frac{1}{2}C} \left( \int_B |\nabla v_n - \nabla v|^2 \omega(x) dx \right)^{\frac{1}{2}} \to 0 \text{ as } \xi \to +\infty. \tag{5.8}
\]

We define for $\xi > 0$, the following truncation function
\[
T_\xi(s) := \begin{cases} 
s & \text{if } |s| < \xi \\
\frac{s}{|s|} & \text{if } |s| \geq \xi.
\end{cases}
\]

If we take $\varphi = T_\xi(v_n - v) \in E$, in (5.2) then with $\nabla \varphi = \chi_{A_\xi} (v_n - v)$, we obtain
\[
\left| \int_{B \setminus A_\xi} \omega(x) |\nabla v_n|^{N-2} \nabla v_n \cdot (\nabla v_n - \nabla v) dx \right| \leq \left| \int_{B \setminus A_\xi} \omega(x) |\nabla v|^{N-2} \nabla v \cdot (\nabla v_n - \nabla v) dx \right| + \int_B f(x, v_n) T_\xi(v_n - v) dx + \varepsilon_n |v_n - v| \leq \int_B f(x, v_n) T_\xi(v_n - v) dx + \varepsilon_n |v_n - v|
\]

where $\varepsilon_n \to 0$ as $n \to +\infty$.

Since $v_n \rightharpoonup v$ weakly, then $\int_B \omega(x) |\nabla v|^{N-2} \nabla v \cdot (\nabla v_n - \nabla v) \to 0$. Moreover, by (5.4) and the Lebesgue dominated convergence Theorem, we get
\[
\int_B f(x, v_n) T_\xi(v_n - v) dx \to 0 \text{ as } n \to +\infty.
\]

Using the well known inequality,
\[
\langle |x|^{N-2} x - |y|^{N-2} y, x - y \rangle \geq 2^{2-N} |x - y|^N \forall \ x, y \in \mathbb{R}^N, \ N \geq 2,
\]
\[
\langle \cdot, \cdot \rangle \text{ is the inner product in } \mathbb{R}^N,
\]
one has
\[
\int_{B \setminus A_\xi} \omega(x) |\nabla v_n - \nabla v|^{N} dx \to 0.
\]

Therefore,
\[
\int_{B \setminus A_\xi} |\nabla v_n - \nabla v| dx \leq \left( \int_{B \setminus A_\xi} \omega(x) |\nabla v_n - \nabla v|^N dx \right)^{\frac{1}{N}} \left( \mathcal{L}(B \setminus A_\xi) \right)^{\frac{1}{N}} \to 0 \text{ as } n \to +\infty. \tag{5.9}
\]

From (5.8) and (5.9), we deduce that
\[
\int_B |\nabla v_n - \nabla v| dx \to 0 \text{ as } n \to +\infty.
\]

Therefore, $\nabla v_n(x) \to \nabla u(x)$ a.e. in $B$. 19
On the other hand,
\[
\left| \nabla v_n \right|^{N-2} \nabla v_n \] is bounded in \( (L^{\frac{N}{N-2}}(B, \omega))^N \).

Then, up to subsequence, we can assume that
\[
\nabla v_n \to \nabla \nu weakly in \ (L^{\frac{N}{N-1}}(B, \omega))^N.
\] (5.10)

Therefore, passing to the limit in (5.2) and using (5.4), (5.10), the convergence everywhere of the gradient, we obtain that \( \nu \) is a solution of problem \((P_\lambda)\). Claim 1 is proved.

Claim 2. \( \nu^+ \neq 0 \) and \( \nu^- \neq 0 \). Suppose, by contradiction, \( \nu^+ = 0 \). Therefore, \( \int_B F(x, v_n)dx \to 0 \) and consequently we get
\[
\frac{1}{N} \left\| v_n \right\|_N \to c_\lambda \left( \frac{\alpha_N \beta}{\alpha_0} \right)^{\frac{N}{N\gamma}}.
\] (5.11)

First, we claim that there exists \( q > 1 \) such that
\[
\int_B |f(x, v_n)|^q dx \leq C.
\] (5.12)

By (5.2), we have
\[
\left| \left\| v_n \right\|_N - \int_B f(x, v_n)v_n dx \right| \leq C\varepsilon_n.
\]

So
\[
\left\| v_n \right\|_N \leq C\varepsilon_n + \left( \int_B |f(x, v_n)|^q dx \right)^{\frac{1}{q'}} \left( \int_B |v_n|^{q'} dx \right)^{\frac{1}{q'}},
\]

where \( q' \) is the conjugate of \( q \). Since \((v_n)\) converge to 0 in \( L^q(B) \)
\[
\lim_{n \to +\infty} \left\| v_n \right\|_N = 0.
\]

According to Lemma 3.3, this result cannot occur. Now for the proof of the claim (5.12), since \( f \) has critical growth, for every \( \varepsilon > 0 \) and \( q > 1 \) there exists \( t_\varepsilon > 0 \) and \( C > 0 \) such that for all \( |t| \geq t_\varepsilon \), we have
\[
|f(x, t)|^q \leq Ce^{\alpha_0(\varepsilon + 1)t_\gamma}.
\]

Consequently,
\[
\int_B |f(x, v_n)|^q dx = \int_{\{v_n \leq t_\varepsilon\}} |f(x, v_n)|^q dx + \int_{\{v_n \geq t_\varepsilon\}} |f(x, v_n)|^q dx \leq \omega_{N-1} \max_{B \times [-t_\varepsilon, t_\varepsilon]} |f(x, t)|^q + C \int_B e^{\alpha_0(\varepsilon + 1)|v_n|\gamma} dx.
\]

Since \( Nc_\lambda < \left( \frac{\alpha_N \beta}{\alpha_0} \right)^{\frac{N}{N\gamma}} \), there exists \( \eta \in (0, \frac{1}{N}) \) such that \( Nc_\lambda = (1 - \eta)\left( \frac{\alpha_N \beta}{\alpha_0} \right)^{\frac{N}{N\gamma}} \). On the other hand, \( \| v_n \|_{\gamma} \to (Nc_\lambda)^{\frac{N}{N\gamma}} \), so there exists \( n_\eta > 0 \) such that for all \( n \geq n_\eta \), we get \( \| v_n \|_{\gamma} \leq (1 - \eta)\left( \frac{\alpha_N \beta}{\alpha_0} \right)^{\frac{N}{N\gamma}} \). Therefore,
\[
\alpha_0(1 + \varepsilon) \left( \frac{\| v_n \|_{\gamma} \| v_n \|}{\| v_n \|} \right) \leq (1 + \varepsilon)(1 - \eta)\alpha_N \beta.
\]

\[20\]
We choose $\varepsilon > 0$ small enough to get

$$\alpha_0(1 + \varepsilon\|v_n\|)^{\gamma} \leq \alpha_{N,\beta}.$$  

So, the second integral is uniformly bounded in view of (1.4) and the claim is proved.

Since $(v_n)$ is bounded, up to a subsequence, we can assume that $\|v_n\| \to \rho > 0$. We affirm that $J_\lambda(v) = c_\lambda$. Indeed, by $(V_2)$ and claim 2, we have

$$J_\lambda(v) = \frac{1}{N} \int_B [f(x,v) - NF(x,v)]dx \geq 0. \quad (5.13)$$

Now, using the lower semi continuity of the norm and (5.5), we get,

$$J_\lambda(v) \leq \frac{1}{N} \liminf_{n \to \infty} \|v_n\|^N - \int_B F(x,v)dx = c_\lambda.$$  

Suppose that $J_\lambda(v) < c_\lambda$.

Then

$$\|v\|^N < \rho^N. \quad (5.14)$$

In addition,

$$\frac{1}{N} \lim_{n \to +\infty} \|v_n\|^N = (c_\lambda + \int_B F(x,v)dx), \quad (5.15)$$

which means that

$$\rho^N = N\left(c_\lambda + \int_B F(x,v)dx\right).$$

Set

$$u_n = \frac{v_n}{\|v_n\|} \quad \text{and} \quad u = \frac{v}{\rho}.$$  

We have $\|u_n\| = 1$, $u_n \rightharpoonup u$ in $E$, $u \not\equiv 0$ and $\|u\| < 1$. So, by Lemma 2.2, we get

$$\sup_n \int_B e^{\rho N,\beta |u_n| \gamma} dx < +\infty,$$

provided $1 < p < \left(1 - \|u\|^N\right)^{-\frac{\beta}{\gamma}}$.

By (5.5) and (5.15), we have the following equality

$$Nc_\lambda - N J_\lambda(v) = \rho^N - \|v\|^N.$$  

From (5.13), Lemma 4.1 and the last equality, we obtain

$$\rho^N \leq Nc_\lambda + \|v\|^N < \left(\frac{\alpha_{N,\beta}}{\alpha_0}\right)^{\frac{\beta}{\gamma}} + \|v\|^N. \quad (5.16)$$
Since
\[ \rho^\gamma = \left( \frac{\rho^N - \|u\|^N}{1 - \|u\|^N} \right)^{\frac{1}{N - 1}(1 - \sigma)}, \]
we deduce from (5.16) that
\[ \rho^\gamma < \left( \frac{\alpha_{N,\beta}}{1 - \|u\|^N} \right)^{\frac{N}{N - 1}(1 - \sigma)}. \]  

(5.17)

On one hand, we have this estimate \( \int_B |f(x, v_n)|^q dx < C \). Indeed, since \( f \) has critical growth, for every \( \varepsilon > 0 \) and \( q > 1 \) there exists \( t_\varepsilon > 0 \) and \( C > 0 \) such that for all \( |t| \geq t_\varepsilon \), we have
\[ |f(x, t)|^q \leq Ce^{\alpha_0(\varepsilon + 1)t^\gamma}. \]
So,
\[ \int_B |f(x, v_n)|^q dx = \int_{\{v_n\leq t_\varepsilon\}} |f(x, v_n)|^q dx + \int_{\{v_n > t_\varepsilon\}} |f(x, v_n)|^q dx \]
\[ \leq \omega_{N-1} \max_{B \times [-t_\varepsilon, t_\varepsilon]} |f(x, t)|^q + C \int_B e^{\alpha_0(1+\varepsilon)\|v_n\|^\gamma} dx \]
\[ \leq C\varepsilon + C \int_B e^{\alpha_0(1+\varepsilon)\|v_n\|^\gamma} \frac{1}{\|v_n\|^\gamma} dx \leq C, \]
provided \( \alpha_0(1+\varepsilon)\|v_n\|^\gamma \leq p \alpha_{N,\beta} \) and \( 1 < p < \mathcal{U}(u) = (1 - \|u\|^N)^{-\frac{1}{\gamma}} \).

From (5.17), there exists \( \delta \in (0, \frac{1}{q}) \) such that \( \rho^\gamma = (1 - 2\delta) \left( \frac{\alpha_{N,\beta}}{1 - \|u\|^N} \right)^{\frac{N}{N - 1}(1 - \sigma)} \).

Since \( \lim_{n \to +\infty} \|v_n\|^\gamma = \rho^\gamma \), then, for \( n \) large enough
\[ \alpha_0(1+\varepsilon)\|v_n\|^\gamma \leq (1 + \varepsilon)(1 - \delta) \alpha_{N,\beta} \left( \frac{1}{1 - \|u\|^N} \right)^{\frac{N}{N - 1}(1 - \sigma)}. \]

We choose \( \varepsilon > 0 \) small enough such that \( (1+\varepsilon)(1-\delta) < 1 \) which implies that
\[ \alpha_0(1+\varepsilon)\|v_n\|^\gamma < \alpha_{N,\beta} \left( \frac{1}{1 - \|u\|^N} \right)^{\frac{N}{N - 1}(1 - \sigma)}. \]

Hence, the sequence \( \langle f(x, v_n) \rangle \) is bounded in \( L^q \), \( q > 1 \).

Using the H"older inequality, we deduce that
\[ \int_B f(x, v_n)(v_n - v) dx \leq \left( \int_B |f(x, v_n)|^q dx \right)^{\frac{1}{q}} \left( \int_B |v_n - v|^q' dx \right)^{\frac{1}{q'}} \]
\[ \leq C \left( \int_B |v_n - v|^q' dx \right)^{\frac{1}{q'}} \to 0 \text{ as } n \to +\infty, \]
where \( \frac{1}{q} + \frac{1}{q'} = 1 \).

Since \( \langle J'_\alpha(v_n), (v_n - v) \rangle = o_n(1) \), it follows that
\[ \int_B (\omega(x)|\nabla v_n|^{N-2}\nabla v_n \cdot (\nabla v_n - \nabla v) dx) \to 0. \]
On the other side,
\[ \int_{B} \omega(x) |\nabla v_n|^{N-2} \nabla v_n \cdot (\nabla v_n - \nabla v) \, dx = \|v_n\|^{N} - \int_{B} \omega(x) |\nabla v_n|^{N-2} \nabla v_n \cdot \nabla v \, dx. \]

Passing to the limit in the last equality, we get
\[ \rho^{N} - \|v\|^{N} = 0, \]
therefore \(\|v\|^{N} = \rho\). This is in contradiction with (5.12). Therefore, \(J_{\lambda}(v) = c_{\lambda}\). By Claim 1, \(J'_{\lambda}(v) = 0\) and by Claim 2, \(\nu \neq 0\).

**Proof of Theorem 1.3.** From Lemma 3.6 and Lemma 5.1, we deduce that \(\nu\) is a least energy sign-changing solution for problem \((P_{\lambda})\) with exactly two nodal domains.

**Declaration of competing interest**

the authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Availability of data**

Data openly available in a public repository that issues data sets with DOIs. We also mention that the documentation to support this study are available from Umm Al-Qura University.

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