Forward-backward and CP-violating asymmetries in rare $B_{d,s} \to (V,\gamma) \ell^+\ell^-$ decays

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We study the forward-backward and the CP-violating asymmetries (both time-independent and time-dependent) in rare semileptonic $B_d \to \rho^0\ell^+\ell^-$, $B_s \to \phi\ell^+\ell^-$ and radiative leptonic $B_{d,s} \to \gamma\ell^+\ell^-$ decays and investigate the sensitivity of these asymmetries to the extensions of the Standard model.

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I. INTRODUCTION

Rare $B$-decays induced by flavor-changing neutral currents provide a valuable possibility of an indirect search of physics beyond the Standard Model (SM): such decays are forbidden at the tree level in the SM and occur through loops which contain contributions of potential new particles. These contributions modify the effective Hamiltonian describing rare $B$-decays and influence, e.g., the differential distributions and CP-violating effects.

CP-violation in beauty sector has been measured for the first time at $B$-factories BaBar and Belle in two-hadron $B$-meson decays [1]. Other interesting reactions, where CP-violating effects may be studied, are rare semileptonic and radiative leptonic decays induced by $b \to (d,s) (\gamma,\ell^+\ell^-)$ transitions (see, e.g., 2 and refs therein).

In the SM one expects the branching ratios at the level $10^{-6} \div 10^{-7}$ for rare semileptonic $B$-decays [3] and $10^{-8} \div 10^{-9}$ for radiative leptonic $B$-decays [3, 5]. Presently, the largest available samples of $B$ mesons are accumulated by $B$-factories BaBar and Belle: in the recent publications these collaborations have announced 384 and 657 million $B\bar{B}$-pairs, respectively [6]. Samples of $BB$-mesons of this size allow one not only to measure integrated partial rates of rare semileptonic and (potentially) radiative leptonic decays, but also to study differential distributions in $B \to (K^*,K)\mu^+\mu^-$ with several hundred events at hand. The shape of one of these differential distributions—the lepton forward-backward asymmetry $A_{FB}(s)$ ($\sqrt{s}$ being the dilepton invariant mass)—is known to be sensitive to the extensions of the SM, in particular to the sign of the Wilson coefficient $C_{7\gamma}$ in the effective Hamiltonian for $b \to (d,s) (\gamma,\ell^+\ell^-)$ transitions. Recent results for $A_{FB}$ from Belle seem to indicate the sign of $C_{7\gamma}$ opposite to the SM prediction; the BaBar results based on lower statistics do not contradict to the results from Belle [6]. Unfortunately, the decisive conclusions cannot be made because of the limited statistics: the “disagreement” between the measurement and the SM-prediction for the asymmetry is at the level of two standard deviations.

Interesting information on the extentsions of the SM may be obtained also from other reactions. In this paper we focus on the CP-violating effects in rare semileptonic and radiative leptonic $B$-decays.

Obviously, an experimental study of CP-violating observables requires greater samples of beauty hadrons than those provided by the B-factories: since CP-violation effects in beauty sector are of order $10^{-3}$, one needs $B$-meson data samples exceeding those from B-factories by at least two orders of magnitude. One expects such data samples of beauty particles at the LHC; e.g., the detector LHCb is expected to register about $10^{12}$ $b\bar{b}$-pairs per year, approximately by four orders of magnitude more than a yearly yield of $b\bar{b}$-pairs at the $B$-factories [4]. A widely discussed Super-$B$ factory project, having a lower yield of $b\bar{b}$-pairs compared to the LHC, will provide much better signal-to-background ratio [5]. Thus, the new accelerators open exciting possibilities of an experimental study of CP-violating effects in rare $B$-decays with two leptons in the final state.

The issue of CP-violation in rare $b$-decays has been already discussed in the literature, but mainly in connection with rare radiative decays: the structure of the helicity amplitudes — the main theoretical tool for the analysis of CP-violating effects — is very simple in this case. Therefore various extentsions of the SM may be analysed. A high sensitivity of the time-dependent CP-asymmetry $A_{CP}(\tau)$ in $B \to (K^*,\rho)\gamma$ to left-right symmetric models has been reported [7].

The time-independent CP-asymmetries in rare semileptonic decays were considered first for inclusive $B$-decays, i.e., for the process $b \to d\ell^+\ell^-$. An asymmetry of the order of a few percent in $b \to d\ell^+\ell^-$ has been predicted; for the $b \to s\ell^+\ell^-$-transitions the asymmetry is expected to be much smaller. Following [10], the time-independent CP-asymmetries in exclusive $B$-decays for several extentsions of the SM have been analyzed [11, 12].

The time-dependent CP-asymmetries [12] were studied for the case of rare semileptonic $B_s \to \phi\ell^+\ell^-$-decays in Ref. [12].
The present work provides the analysis of the asymmetries (forward-backward, time-independent and time-dependent CP-violating asymmetries) in rare semileptonic and radiative leptonic $B_{s,d} \rightarrow (V,\gamma)\ell^+\ell^-$-decays. We make use of the technique of helicity amplitudes developed in Appendix B. Flavour oscillations of the decaying neutral $B$-mesons are taken into account; this effect is shown to influence considerably the resulting CP-violating asymmetries.

The paper is organized as follows: In Section II we recall the structure of the effective Hamiltonian and the relevant amplitudes describing rare $B$-decays. Sections III and IV provide the definitions of the asymmetries and their expressions in terms of the helicity amplitudes. Section V contains the results of the numerical analysis. Conclusion summarizes our findings. Appendices contain all necessary technical details related to the helicity amplitudes for rare semileptonic (Appendix B) and rare radiative leptonic (Appendix C) decays.

II. EFFECTIVE HAMILTONIAN AND FORM FACTORS FOR RARE SEMILEPTONIC AND RADIATIVE LEPTONIC $B$-DECAYS

Let us briefly recall the basic definitions necessary for the calculation of the amplitudes of $B$-decays of interest.

A. The effective Hamiltonian

The effective Hamiltonian describing the $b \rightarrow s\ell^+\ell^-$ transition in the SM has the form \cite{13}

$$H^\text{SM}_{\text{eff}}(b \rightarrow s\ell^+\ell^-) = \frac{G_F}{\sqrt{2}} \frac{\alpha_{\text{em}}}{2\pi} V_{tb} V_{ts}^{*} \left[ -2 \frac{C_{\gamma}(\mu)}{q^2} \right. \cdot \bar{s} i \sigma_{\alpha\beta} q^\beta \left. \{ m_b (1 + \gamma^5) + m_s (1 - \gamma^5) \} \bar{b} \cdot \bar{\ell} \gamma^\alpha \ell \right. \nonumber + C_{\text{eff}}^{(s)}(\mu, q^2) \cdot \bar{s} \gamma^\alpha (1 - \gamma^5) \bar{b} \cdot \bar{\ell} \gamma^\alpha \ell + C_{10}^{(A)}(\mu) \cdot \bar{s} \gamma_5 (1 - \gamma^5) b \cdot \bar{\ell} \gamma^5 \ell \], \quad (2.1)$$

where $m_b$ ($m_s$) is the $b$ ($s$)-quark mass, $V_{ij}$ are the elements of the CKM matrix, $\mu$ is the renormalization scale, and $q$ is the momentum vector of the $\ell^+\ell^-$ pair. The corresponding expression for the case of the $b \rightarrow d$ transition is self-evident.

The Wilson coefficient $C^\text{eff}_{\gamma V}(\mu, q^2)$ contains the contributions of the virtual $u\bar{s}$ and $c\bar{s}$ pairs, which involve the integration over short and long distances. The long-distance effects are described by the neutral vector-meson resonances $\rho, \omega$, and $J/\psi, \psi'$. We make use of the parameterization of $C^\text{eff}_{\gamma V}(\mu, q^2)$ from \cite{10} where the resonance contributions are modeled in a gauge-invariant way \cite{10}.

The effective Hamiltonian for the $b \rightarrow \bar{s}\ell^+\ell^-$ transition is obtained from (2.1) by interchanging $b$ and $s$ fields, i.e., by replacing $V_{qs} V_{qs}^{*} \rightarrow V_{qs} V_{qs}^{*}, \bar{s} \rightarrow \bar{b}$, $b \rightarrow s$, $m_b \leftrightarrow m_s$, $q \rightarrow -q^2$.

B. Form factors for weak $B \rightarrow V$ decays

The form factors for rare semileptonic transitions $\bar{B}(p_1, M_1) \rightarrow \bar{V}(p_2, M_2, \varepsilon)$ are defined in the standard way (see e.g. \cite{17}):

$$\langle \bar{V}(p_2, M_2, \varepsilon) | q \gamma_\mu | \bar{B}(p_1, M_1) \rangle = \frac{2 V(q^2)}{M_1 + M_2} \varepsilon^\mu \epsilon^{\nu \rho \sigma} \epsilon^*_\rho p^\sigma p^\nu_2; \nonumber$$

$$\langle \bar{V}(p_2, M_2, \varepsilon) | q \sigma_{\mu\nu} q^' | \bar{B}(p_1, M_1) \rangle = i \epsilon^*_\rho p^\nu_1 (M_1 + M_2) A_1(q^2) - \left. - i (\varepsilon^* p_1) (p_1 + p_2)_\mu \frac{A_2(q^2)}{M_1 + M_2} \right. \nonumber$$

$$- i \epsilon^*_\rho p^\nu_1 \frac{2 M_2}{q^2} \left. (A_3(q^2) - A_0(q^2)) \right. \right.$$  

$$\langle \bar{V}(p_2, M_2, \varepsilon) | q \gamma_\mu \gamma_5 q' | \bar{B}(p_1, M_1) \rangle = 2 i T_1(q^2) \varepsilon^\mu \epsilon^{\nu \rho \sigma} \epsilon^*_\rho p^\nu_2; \nonumber$$

$$\langle \bar{V}(p_2, M_2, \varepsilon) | q \sigma_{\mu\nu} q' \gamma_5 | \bar{B}(p_1, M_1) \rangle = T_2(q^2) \left. (\epsilon^*_\rho P q - (\varepsilon^* q) P_\mu) \right. \nonumber$$

$$+ T_3(q^2) \left. (\varepsilon^* q) \right. \left( q_\mu - \frac{q^2}{P} P_\mu \right); \quad (2.2)$$

\footnote{We use the following conventions: $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]$, $\varepsilon^{0123} = -1$, $\epsilon = \sqrt{4\pi\alpha_{\text{em}}} > 0$.}

\footnote{Let us notice that the effective Hamiltonian for the $b \rightarrow \bar{s}$ transition is not related to that for the $b \rightarrow s$ transition by the hermitian conjugate; namely, $H_{\text{eff}} (b \rightarrow s) \equiv V_{qs}^{*} V_{qs} T \{ j_\mu (q \rightarrow s) W^\mu \bar{q} O q j^\mu (q \rightarrow b) W^{\mu \nu} \} \equiv H_{\text{eff}} (b \rightarrow \bar{s}) \equiv V_{qs}^{*} V_{qs} T \{ j_\mu (q \rightarrow s) W^\mu \bar{q} O q j^\mu (q \rightarrow b) W^{\mu \nu} \} + \text{hermitian conjugate},$ where $(\bar{q} O q)^\dagger = \bar{q} O q$. The hermitian conjugate of $H_{\text{eff}} (b \rightarrow s)$ does not give $H_{\text{eff}} (b \rightarrow \bar{s})$, since the hermitian conjugate of the $T$-product gives the anti-$T$-product.}
where \( q = p_1 - p_2, \) \( P = p_1 + p_2, \) \( p_1^2 = M_1^2, \) \( p_2^2 = M_2^2, \) \( \varepsilon^* \) is the polarization vector of the final vector meson, \( \varepsilon^* p_2 = 0. \) The form factors satisfy the following conditions

\[
A_3(q^2) = \frac{M_1 + M_2}{2M_2} A_1(q^2) - \frac{M_1 - M_2}{2M_2} A_2(q^2), \quad A_0(0) = A_3(0), \quad T_1(0) = T_2(0).
\]

(2.3)

C. Form factors for weak \( B \to \gamma \) decays

The \( B \to \gamma \) amplitudes are parametrized as follows [4]:

\[
\langle \gamma(k, \epsilon) | \bar{q} q \gamma_5 b | B^0(q, p, M_1) \rangle = i \epsilon_\alpha^* (g_{\mu\alpha}(pk) - p_\alpha k_\mu) \frac{F_A(q^2)}{M_1},
\]

\[
\langle \gamma(k, \epsilon) | \bar{q} q b | B^0(q, p, M_1) \rangle = i \epsilon_\alpha^* \epsilon_{\mu\alpha\xi\eta} p_\xi k_\eta \frac{F_V(q^2)}{M_1},
\]

(2.4)

\[
\langle \gamma(k, \epsilon) | \bar{q} q \gamma_{\mu\nu} \gamma_5 b | B^0(q, p, M_1) \rangle (p - k)_\nu = i \epsilon_\alpha^* [g_{\mu\nu}(pk) - p_\mu k_\nu] F_T(q^2),
\]

\[
\langle \gamma(k, \epsilon) | \bar{q} q b | B^0(q, p, M_1) \rangle (p - k)_\nu = i \epsilon_\alpha^* \epsilon_{\mu\alpha\xi\eta} p_\xi k_\eta F_{TV}(q^2).
\]

Here \( q = p - k, k^2 = 0, \) \( p^2 = M_1^2. \)

III. FORWARD-BACKWARD ASYMMETRY

The lepton forward-backward asymmetry \( A_{FB}(\hat{s}) \) is one of the differential distributions relatively stable with respect to QCD uncertainties and sensitive to the new physics effects [18]. Therefore it has been extensively studied both theoretically and experimentally [15]. We make use of the following definition of \( A_{FB}(\hat{s}) \) for \( B \to f \) decays

\[
A_{FB}(\hat{s}) = \frac{\int_0^1 d\cos \theta \frac{d^2\Gamma(B \to f)}{ds d\cos \theta} - \int_{-1}^0 d\cos \theta \frac{d^2\Gamma(B \to f)}{ds d\cos \theta}}{\frac{d\Gamma(B \to f)}{ds}},
\]

(3.1)

where \( f = V l^+ l^- \) for rare semileptonic decay and \( f = \gamma l^+ l^- \) for rare radiative decay; \( \hat{s} \equiv s/M_B^2, \) \( \sqrt{s} \) being the dilepton invariant mass. The asymmetry is calculated in the rest frame of the lepton pair, and the angle \( \theta \) is defined as the angle between the \( B \)-meson 3-momentum and the 3-momentum of the outgoing negative-charged lepton, \( l^- \). Equivalently, for \( B \to f \) one defines

\[
A_{FB}(\hat{s}) = \frac{\int_0^1 d\cos \theta_+ \frac{d^2\Gamma(B \to f)}{ds d\cos \theta_+} - \int_{-1}^0 d\cos \theta_+ \frac{d^2\Gamma(B \to f)}{ds d\cos \theta_+}}{\frac{d\Gamma(B \to f)}{ds}},
\]

(3.2)

where \( \theta_+ \) is the angle between the \( B \)-meson 3-momentum and the 3-momentum of the outgoing positive-charged lepton, \( l^+ \). If CP-violating effects are neglected, both asymmetries (3.1) and (3.2) are equal to each other.

The decay rate may be expressed in a simple way via the helicity amplitudes describing the \( B \to (V, \gamma) \ell^+ \ell^- \) and \( B \to (\bar{V}, \gamma) \ell^+ \ell^- \) decay:

\[
\Gamma = \frac{1}{2} \int \frac{d\Phi_3}{2M_B} \sum_{\lambda_1, \lambda_2} \left( |\tilde{A}_{\lambda_1, \lambda_2}(\hat{s}, \cos \theta)|^2 + |\tilde{A}_{\lambda_1, \lambda_2}(\hat{s}, \cos \theta)|^2 \right).
\]

(3.3)

Explicit expressions for the helicity amplitudes and the phase-space volume are given in Appendix B for \( B(B) \to \bar{V}(V) \ell^+ \ell^- \) decay and in Appendix C for \( B(B) \to \gamma \ell^+ \ell^- \) decay. Summation in Eq. (3.3) runs over all possible values of the helicity indices (i.e. \( \lambda_{1,2} = L, R \) for fermions, \( \lambda_{1,2} = \pm 1 \) for vector mesons, and \( \lambda_{1,2} = \pm 1 \) for the photon). Making use of these expressions it is straightforward to calculate, e.g., \( d^2\Gamma/d\hat{s} d\cos \theta \).

Our present calculation of the forward-backward asymmetry in the \( B \to \gamma \ell^+ \ell^- \) decays takes into account Bremsstrahlung effects (the expression for the corresponding amplitude can be found in [6]). We checked that the previous results for \( A_{FB} \) from [4,15], where Bremsstrahlung was not considered, is reproduced if the Bremsstrahlung contribution in our formulas is omitted.
IV. CP-VIOLATING ASYMMETRIES

Time-dependent CP-violating asymmetry is defined in the $B$-meson rest frame as follows \[20\]

\[
A_{\text{CP}}^{B_q \rightarrow f}(\tau) = \frac{d \Gamma(B_q^0 \rightarrow f)}{d \tau} - \frac{d \Gamma(B_{q}^0 \rightarrow f)}{d \tau},
\]

(4.1)

where $f$ is the common final state for $B^0$ and $\bar{B}^0$ decays. In this case a pronounced CP violation is expected in interference between the oscillation and decay amplitudes \[21\]. For instance, for leptonic radiative decays $f = \gamma^+ \gamma^-$. Recall that a generic three-particle final state has no definite CP-parity. Nevertheless, the asymmetry (4.1) which involves the integration over the three-particle phase space characterizes the CP violation as it vanishes in a theory with a conserved CP.

Taking into account meson oscillations, one obtains for time-dependent decay rates

\[
\frac{d \Gamma(B_q^0 \rightarrow f)}{d \tau} = \frac{e^{-\Gamma_q \tau}}{2} \left[ A \text{ch} (y \Gamma_q) + B \cos (x \Gamma_q) - C \text{sh} (y \Gamma_q) - 2 D \sin (x \Gamma_q) \right],
\]

(4.2)

\[
\frac{d \Gamma(\bar{B}_q^0 \rightarrow f)}{d \tau} = \frac{e^{-\Gamma_q \tau}}{2} \left[ A \text{ch} (y \Gamma_q) - B \cos (x \Gamma_q) - C \text{sh} (y \Gamma_q) + 2 D \sin (x \Gamma_q) \right],
\]

where the coefficients $A, B, C,$ and $D$ may be expressed via the helicity amplitudes as follows:

\[
A = \int d\hat{s} \hat{A}(\hat{s}) = \int \frac{d\Phi_3}{2 M_1} \sum_{\lambda_1, \lambda_2} \left( |A_{\lambda_1, \lambda_2}^{(q)}(\hat{s}, \cos \theta)|^2 + |\bar{A}_{\lambda_1, \lambda_2}^{(q)}(\hat{s}, \cos \theta)|^2 \right),
\]

(4.3)

\[
B = \int d\hat{s} \bar{B}(\hat{s}) = \int \frac{d\Phi_3}{2 M_1} \sum_{\lambda_1, \lambda_2} \left( |A_{\lambda_1, \lambda_2}^{(q)}(\hat{s}, \cos \theta)|^2 - |\bar{A}_{\lambda_1, \lambda_2}^{(q)}(\hat{s}, \cos \theta)|^2 \right),
\]

\[
C = \int d\hat{s} \bar{C}(\hat{s}) = \int \frac{d\Phi_3}{2 M_1} \sum_{\lambda_1, \lambda_2} \text{Re} \left( e^{-2i\phi_{\text{ckm}}} A_{\lambda_1, \lambda_2}^{(q)*}(\hat{s}, \cos \theta) \bar{A}_{\lambda_1, \lambda_2}^{(q)}(\hat{s}, \cos \theta) \right),
\]

\[
D = \int d\hat{s} \bar{D}(\hat{s}) = \int \frac{d\Phi_3}{2 M_1} \sum_{\lambda_1, \lambda_2} \text{Im} \left( e^{-2i\phi_{\text{ckm}}} A_{\lambda_1, \lambda_2}^{(q)*}(\hat{s}, \cos \theta) \bar{A}_{\lambda_1, \lambda_2}^{(q)}(\hat{s}, \cos \theta) \right).
\]

In the expressions above one should use the appropriate helicity amplitudes and phase-space factors: for the case of semileptonic decays they are given in Appendix B; for radiative leptonic decays – in Appendix C. As already noticed above the helicity amplitudes of Appendix C contain the contributions of the Bremsstrahlung diagrams.

The weak phase $\phi_{\text{ckm}}$ is related to the CP-violating phase of the CKM matrix: $V_{tb}V_{tq} = |V_{tb}V_{tq}| e^{-i\phi_{\text{ckm}}}$. For instance, for the case $q = d, \phi_{\text{ckm}} = \beta = \phi_1$ of Belle \[22\].

We now take into account the existence of two mass-eigenstates of neutral $B$-mesons. These states have the masses $M_\ell$ and $M_h$, and the full widths $\Gamma_\ell$ and $\Gamma_h$, respectively. We use the definitions

\[
M = (M_\ell + M_h)/2, \quad \Delta m = M_h - M_\ell, \quad \Gamma = (\Gamma_\ell + \Gamma_h)/2, \quad \Delta \Gamma = \Gamma_\ell - \Gamma_h, \quad x = \Delta m/\Gamma, \quad y = \Delta \Gamma/\Gamma.
\]

Then, we obtain the following expression for time-dependent asymmetry:

\[
A_{\text{CP}}^{B_q \rightarrow f}(\tau) = \frac{2 D \sin (x \Gamma_q) - B \cos (x \Gamma_q)}{A \text{ch} (y \Gamma_q) - 2 C \text{sh} (y \Gamma_q)} = \frac{S_f \sin (\Delta m \tau) - C_f \cos (\Delta m \tau)}{\text{ch} (\Delta \Gamma \tau) - H_f \text{sh} (\Delta \Gamma \tau)}.
\]

(4.4)

The coefficients $S_f, C_f,$ and $H_f$, which are used for the parameterization of the data, may be expressed via $A, B, C,$ and $D$.

Let us make the following remarks:

(i) For $B_d$-mesons $\Delta \Gamma \ll \Gamma$ and the denominator \[4.3\] is almost equal to one. For $B_s$-mesons the contribution of $H_f$ is essential \[22\].

(ii) A nonzero asymmetry arises from $\bar{B}_q^0 B_q^0$-oscillations, from the contributions of $u\bar{u}$-, $c\bar{c}$-pairs, $\rho, \omega, \phi,$ and $c\bar{c}$ vector resonances ($J/\psi, \psi', \text{etc}$), and the weak annihilation. Since the oscillation frequency for $B_s$-mesons is much larger than that for $B_d$-mesons, and the weak-annihilation in $B_s$-decays is negligible, one finds $A_{\text{CP}}^{B_s}(\tau) \ll A_{\text{CP}}^{B_d}(\tau)$.
Time-independent CP-asymmetry may be represented via \( \tilde{A}(\hat{s}) \), ..., \( \tilde{D}(\hat{s}) \) from Eq. (4.3) as follows:

\[
A^{B_s \to f}_{C_{B}}(\hat{s}) = \frac{d\Gamma(\bar{B}_q \to f)}{d\hat{s}} - \frac{d\Gamma(B_q \to f)}{d\hat{s}} = -\left(1 - \frac{y^2}{1 + x^2}\right) \tilde{A}(\hat{s}) - 2x\tilde{D}(\hat{s})/A(\hat{s}) - 2yC(\hat{s}),
\]

(4.5)

V. NUMERICAL RESULTS

We are going to apply now the formulas derived above and to provide numerical results for the asymmetries. We use the following numerical parameters:

1. Table 1 summarizes the parameters of the \( B^0_{d,s} \)-oscillations which we use for our numerical estimates.

| \( B \)-meson parameters | \( B^0_d \) | \( B^0_s \) |
|--------------------------|-------------|-------------|
| \( B \)-meson mass \( M_1 \) (GeV) | 5.28 | 5.37 |
| Width \( \Gamma \) (ps\(^{-1}\)) | 0.65 | 0.67 |
| Mass difference \( \Delta m \) (ps\(^{-1}\)) | 0.507 | 17.77 |
| Width difference \( \Delta\Gamma \) (ps\(^{-1}\)) | 0.005 | 0.1 |

(ii) The Wilson coefficients for the SM are evaluated at \( \mu = 5 \text{ GeV} \) [15] for \( C_2(M_W) = -1: C_1(\mu) = 0.241, C_2(\mu) = -1.1, a_1(\mu) = -0.126, C_{7\gamma}(\mu) = 0.312, C_{9V}(\mu) = -4.21 \) and \( C_{10A}(\mu) = 4.64 \). Respectively, we use the running quark masses in the MS scheme at the same scale: \( m_b = 4.2 \text{ GeV}, m_s = 60 - 80 \text{ MeV} \), and the \( d \)-quark mass is neglected. For the coefficient \( C_\text{eff}^{\mu\gamma} \) we use the model proposed in [10] which takes into account resonances in a gauge-invariant way.

For the CKM matrix elements we use the values reported in the 2008 edition of PDG [22]: \( A = 0.814, \lambda = 0.226, \tilde{\rho} = 0.135, \tilde{\eta} = 0.35 \).

(iii) We make use of the form factor parameterizations for rare semileptonic decays from [24] and for rare radiative decays from [4, 5]. The accuracy of these predictions for the form factors is expected to be at the level of 10-15%, which influences strongly the predictions for the decay rates. However, the form factor uncertainties cancel to a large extent in the asymmetries which therefore can be predicted with a few percent accuracy [2, 18]. For the decay constants of the \( B \)-mesons we use the values \( f_B = 220 \pm 20 \text{ MeV} \) and \( f_{B_s} = 240 \pm 20 \text{ MeV} \).

(iv) A cut on the Bremsstrahlung photon spectrum at 20 MeV in the \( B \)-meson rest frame is applied. This corresponds to the expected level of the photon energy resolution of the LHCb detector.

A. Forward-backward asymmetry

The calculated forward-backward asymmetries are presented in Figs. 1, 3. The decay \( B_s \to \phi\mu^+\mu^- \) (Fig. 1) is of special interest: the detector LHCb may accumulate sufficient data sample for this decay already after the first few months of operation. Qualitatively, the asymmetry has the same structure as in the \( B \to K^*\mu^+\mu^- \) decays: its behaviour at small \( \hat{s} \) is sensitive to the inverion of the signs of \( C_{7\gamma} \) and \( C_{10A} \) compared to the SM. Fig. 1 shows the influence of the sign inversion in one of the Wilson coefficients compared to the SM.

Figs. 2 and 3 present \( A_{FB} \) for the radiative decays \( \bar{B}_s \to \gamma\mu^+\mu^- \) and \( \bar{B}_d \to \gamma\mu^+\mu^- \), respectively. Qualitatively, the asymmetry behaves at large and intermediate \( \hat{s} \) similarly to the \( B_s \to \phi\mu^+\mu^- \) decay, although has a larger magnitude. At small \( \hat{s} \), however, the asymmetry is influenced by the neutral light vector resonances \( \phi, \omega, \) and \( \rho^0 \): these resonances cause a strong distortion of the full asymmetry compared to a nonresonance asymmetry. In particular, they leads to a visible shift of the “zero-point” compared to its location in the non-resonant asymmetry, which may be reliably calculated in the SM [25].
Fig. 1: $A_{FB}$ for rare semileptonic $\bar{B}_s \to \phi \mu^+ \mu^-$ decays: (a) in the SM; (b) For $C_{7\gamma} = -C_{7\gamma}^{SM}$, (c) For $C_{9V} = -C_{9V}^{SM}$, (d) For $C_{10A} = -C_{10A}^{SM}$. Solid line (black): the full asymmetry which takes into account the $J/\psi$, $\psi'$, etc contributions. Dashed line (red): the non-resonant asymmetry.

Fig. 2: $A_{FB}(\bar{s})$ for $\bar{B}_s \to \gamma \mu^+ \mu^-$ decays: (a) In the SM. (b) For $C_{7\gamma} = -C_{7\gamma}^{SM}$. (c) For $C_{9V} = -C_{9V}^{SM}$. (d) For $C_{10A} = -C_{10A}^{SM}$. Solid line (black): the asymmetry calculated for the full amplitude of Ref. [5]. Dashed line (red): the asymmetry calculated for the amplitude without the contributions of light neutral vector mesons $\phi$, the $c\bar{c}$ resonances ($J/\psi$, $\psi'$, . . .), Bremsstrahlung, and the weak annihilation.
Fig. 3: $A_{FB}(\hat{s})$ for $\bar{B}_d \to \gamma \mu^+ \mu^-$ decays: (a) In the SM. (b) For $C_{5\gamma} = -C_{9\gamma}^{SM}$. (c) For $C_{9V} = -C_{9V}^{SM}$. (d) For $C_{10A} = -C_{10A}^{SM}$. Solid line (black): the asymmetry calculated for the full amplitude of Ref. [5]. Dashed line (red): the asymmetry calculated for the amplitude without the contributions of light neutral vector mesons $\omega, \rho^0$, the $c\bar{c}$ resonances ($J/\psi$, $\psi'$, $\ldots$) Bremsstrahlung, and the weak annihilation.

B. CP-violating asymmetries

We present now the time-independent and the time-dependent CP-asymmetries in $B_{d,s} \to (\rho, \gamma) \mu^+ \mu^-$. Concerning the $B_s \to (\phi, \gamma) \mu^+ \mu^-$ decays we would like to mention the following: we have calculated these asymmetries and found that $A_{CP}(\hat{s})$, mainly due to flavor oscillations of the $B_s$ mesons, is extremely small (smaller than 0.1%) and therefore cannot be studied experimentally; $A_{CP}(\tau)$ is not small but measuring this asymmetry would require time resolution much smaller than the $B_s$ lifetime.

1. Time-independent asymmetry

First, we would like to demonstrate the impact of flavor oscillations of the initial mesons on the resulting CP-violating asymmetries. Fig. 4 shows $A_{CP}(\hat{s})$ for $B_{d,s} \to \gamma \mu^+ \mu^-$ decays. Obviously, flavour oscillations lead to a strong suppression of the the resulting CP-violating asymmetries in $B_d$ decays and to a complete vanishing of $A_{CP}$ in $B_s$ decays.

Figs. 5 and 6 display $A_{CP}(s)$ for $B_d \to \rho \mu^+ \mu^-$ and $B_d \to \gamma \mu^+ \mu^-$ decays, respectively.

For $B_d \to \rho \mu^+ \mu^-$ decays, the asymmetry reaches a 30-40% level in the region of light vector resonances, and a level of 10% between the light and $c\bar{c}$ resonances. Notice that flavor oscillations enhance the asymmetry by a factor 2.

For $B_d \to \gamma \mu^+ \mu^-$ decays the asymmetry is smaller and may be measured only in the region of light vector resonances.

Both for $B_d \to \rho \mu^+ \mu^-$ and $B_d \to \gamma \mu^+ \mu^-$ decays the asymmetry is sensitive to the signs of the Wilson coefficients $C_7$ and $C_9$. The asymmetry is however not sensitive to the inversion of the sign of $C_{10}$.
Fig. 4: The influence of $B$-meson flavor oscillations upon CP-violating asymmetries: (a) $B_d \rightarrow \gamma \mu^+ \mu^-$, (b) $B_s \rightarrow \gamma \mu^+ \mu^-$. Dashed line (red): $A_{CP}$ without resonances and without flavor oscillations; Dotted line (blue): $A_{CP}$ with resonances but without flavor oscillations; Solid line (black): $A_{CP}$ after flavor oscillations have been taken into account.

Fig. 5: Time-independent CP-asymmetry $A_{CP}(\hat{s})$ in $B_d \rightarrow \rho \mu^+ \mu^-$ decays. (a) SM (b) $C_{7\gamma} = -C_{7\gamma}^{SM}$ (c) $C_{9V} = -C_{9V}^{SM}$. Flavor oscillations have been taken into account. Solid line (black) line: full asymmetry. Dashed (red) line: nonresonant asymmetry. Dotted (blue) line shows the asymmetry if flavor oscillations are not taken into account.

Fig. 6: Time-independent CP-asymmetry $A_{CP}(\hat{s})$ in $B_d \rightarrow \gamma \mu^+ \mu^-$ decays. (a) SM (b) $C_{7\gamma} = -C_{7\gamma}^{SM}$ (c) $C_{9V} = -C_{9V}^{SM}$. Solid (black) line: full asymmetry. Dashed (red) line: nonresonant asymmetry. Flavor oscillations have been taken into account.
Fig. 7: Time-dependent asymmetry $A_{CP}(\tau)$: (a) $B_d \rightarrow \rho \mu^+ \mu^-$; (b) $B_d \rightarrow \gamma \mu^+ \mu^-$; (c) $B_s \rightarrow \gamma \mu^+ \mu^-$. Solid line (black): SM. Dashed line (red): $C_7^{\gamma} = -C_7^{SM\gamma}$; Dotted line (blue): $C_9^V = -C_9^{SMV}$.}

2. Time-dependent asymmetry

Fig. 7 plots the time-dependent asymmetry $A_{CP}(\tau)$ for $B_d \rightarrow \rho \mu^+ \mu^-$ (a) $B_d \rightarrow \gamma \mu^+ \mu^-$ (b) and $B_s \rightarrow \gamma \mu^+ \mu^-$ (c) decays. The region around the $J/\psi$ and $\psi'$ resonances $0.33 \leq \hat{s} \leq 0.55$ was excluded from the integration while calculating the time-dependent asymmetries. This procedure corresponds to the analysis of the experimental data. The asymmetry in $B_d$ decays reaches a level of 10% at the time-scale of a few $B$-meson lifetimes ($\tau_{B_d}=1.53$ ps) and may be studied experimentally. It also exhibits a sensitivity to the extentions of the SM.

VI. CONCLUSIONS

We presented the analysis of the forward-backward and the CP-violating asymmetries in rare semileptonic and radiative leptonic $B$-decays. Our results may be summarized as follows:

1. We obtained the analytic results for the time-dependent and time-independent $CP$-asymmetries in rare radiative leptonic $B$-decays $B_{d,s} \rightarrow \gamma \ell^+ \ell^-$. 
2. We presented numerical results for the forward-backward asymmetry in $B_s \rightarrow \phi \mu^+ \mu^-$ decays which may be measured in the near future at the LHCb. This asymmetry, as could be expected, has a very similar shape to the asymmetry in $B_d \rightarrow K^* \mu^+ \mu^-$ decays and thus may be used for “measuring” the signs of the Wilson coefficients $C_{7\gamma}$, $C_{9V}$, and $C_{10A}$. 
3. We studied the forward-backward asymmetry in $B_{d,s} \rightarrow \gamma \ell^+ \ell^-$ decays taking into account the vector resonance contributions, the Bremsstrahlung, and the weak annihilation effects. We noticed that the light neutral vector resonances strongly distort the shape of the asymmetry at small values of the dilepton invariant mass. In particular, in the SM these resonances lead to a sizeable shift of the zero point of the full asymmetry compared to the zero-point of the non-resonant asymmetry. The $A_{FB}$ in this reaction reaches 60% and thus may be studied experimentally at the LHC and the future Super-B factory.

4. We analysed the CP-violating asymmetries (both time-dependent and time-independent) in $B_d \rightarrow \rho \mu^+ \mu^-$, $B_s \rightarrow \phi \mu^+ \mu^-$, and $B_{s,d} \rightarrow \gamma \mu^+ \mu^-$ decays.

The asymmetries in $B_s$ decays are found to be very small and therefore to be of no practical interest. The asymmetries in $B_d$ decays reach measurable values and thus might provide additional tests of the SM and its extentions. These potentially interesting cases are: (i) The time-independent CP-violating asymmetry $A_{CP}(\hat{s})$ in $B_d \rightarrow \rho \mu^+ \mu^-$ decays in the region below $c \bar{c}$ resonances (10-30 % level) and $A_{CP}(\hat{s})$ in $B_d \rightarrow \gamma \mu^+ \mu^-$ in the region of light neutral vector resonances (5-10 % level). (ii) The time-dependent CP-violating asymmetry $A_{CP}(\tau)$ in $B_d \rightarrow (\rho, \gamma) \mu^+ \mu^-$ decays (10% level).

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Appendix A: Helicity components of lepton currents

Helicity components of the vector current \( j^\mu (\ell^-_{\lambda_2}, \ell^+_{\lambda_1}) \equiv \bar{\ell}(\vec{k}_2, \lambda_2) \gamma^\mu \ell(\vec{k}_1, \lambda_1) \) take the form:

\[
\begin{align*}
    j^\mu (\ell^-_R, \ell^+_L) &= 2i \varepsilon (0, \cos \theta \cos \varphi, + i \sin \varphi, \sin \varphi \cos \theta - i \cos \varphi, - \sin \theta); \\
    j^\mu (\ell^-_L, \ell^+_R) &= 2i \varepsilon (0, - \cos \theta \cos \varphi + i \sin \varphi, - \cos \theta \sin \varphi - i \cos \varphi, \sin \theta); \\
    j^\mu (\ell^-_R, \ell^+_L) &= -2i m (0, \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta); \\
    j^\mu (\ell^-_L, \ell^+_R) &= -2i m (0, \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta). 
\end{align*}
\]

For the axial current \( a^\mu (\ell^-_{\lambda_2}, \ell^+_{\lambda_1}) \equiv \bar{\ell}(\vec{k}_2, \lambda_2) \gamma^\mu \gamma^5 \ell(\vec{k}_1, \lambda_1) \) we find

\[
\begin{align*}
    a^\mu (\ell^-_R, \ell^+_L) &= 2i D \varepsilon (0, \cos \theta \cos \varphi + i \sin \varphi, \cos \theta \sin \varphi - i \cos \varphi, - \sin \theta); \\
    a^\mu (\ell^-_L, \ell^+_R) &= 2i D \varepsilon (0, \cos \theta \cos \varphi - i \sin \varphi, \cos \theta \sin \varphi + i \cos \varphi, - \sin \theta); \\
    a^\mu (\ell^-_R, \ell^+_L) &= -2i m (1, 0, 0, 0); \\
    a^\mu (\ell^-_L, \ell^+_R) &= 2i m (1, 0, 0, 0). 
\end{align*}
\]

For the tensor current \( T^{\mu\nu} (\ell^-_{\lambda_2}, \ell^+_{\lambda_1}) = \bar{\ell}(\vec{k}_2, \lambda_2) \sigma^{\mu\nu} \ell(\vec{k}_1, \lambda_1) \) we have:

\[
\begin{align*}
    T^{\mu\nu} (\ell^-_R, \ell^+_L) &= \frac{2m}{2}\left( \begin{array}{cccc}
        0 & -\cos \theta \cos \varphi - i \sin \varphi & -\cos \theta \sin \varphi + i \cos \varphi & \sin \theta \\
        \cos \theta \cos \varphi + i \sin \varphi & 0 & 0 & 0 \\
        \cos \theta \sin \varphi - i \cos \varphi & 0 & 0 & 0 \\
        -\sin \theta & 0 & 0 & 0 
    \end{array} \right); \\
    T^{\mu\nu} (\ell^-_L, \ell^+_R) &= \frac{2m}{2}\left( \begin{array}{cccc}
        0 & -\cos \theta \cos \varphi - i \sin \varphi & \cos \theta \sin \varphi + i \cos \varphi & -\sin \theta \\
        -\cos \theta \cos \varphi + i \sin \varphi & 0 & 0 & 0 \\
        -\cos \theta \sin \varphi - i \cos \varphi & 0 & 0 & 0 \\
        \sin \theta & 0 & 0 & 0 
    \end{array} \right); \\
    T^{\mu\nu} (\ell^-_R, \ell^+_R) &= \frac{2m}{2}\left( \begin{array}{cccc}
        0 & -\sin \theta \cos \varphi & -\sin \theta \sin \varphi & -\cos \theta \\
        \sin \theta \cos \varphi & 0 & iD \cos \theta & -iD \sin \theta \sin \varphi \\
        \sin \theta \sin \varphi & -iD \cos \theta & 0 & iD \sin \theta \cos \varphi \\
        \cos \theta & iD \sin \theta \sin \varphi & -iD \sin \theta \cos \varphi & 0 
    \end{array} \right); \\
    T^{\mu\nu} (\ell^-_L, \ell^+_L) &= \frac{2m}{2}\left( \begin{array}{cccc}
        0 & \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\
        -\sin \theta \cos \varphi & 0 & iD \cos \theta & -iD \sin \theta \sin \varphi \\
        -\sin \theta \sin \varphi & -iD \cos \theta & 0 & iD \sin \theta \cos \varphi \\
        -\cos \theta & -iD \sin \theta \sin \varphi & -iD \sin \theta \cos \varphi & 0 
    \end{array} \right). 
\end{align*}
\]

Appendix B: Helicity amplitudes for rare semileptonic \( \bar{B}(B) \rightarrow \bar{V}(V)\ell^+\ell^- \)-decays

1. Kinematics

We work in the rest frame of the \( \ell^+\ell^- \)-pair and choose z-axis in this reference frame along the 3-momentum of the \( B \)-meson; In this reference frame we have the following expressions for the 4-momenta of the initial meson \( (p_1) \), the final meson \( (p_2) \), the negative-charged lepton \( (k_1) \), the positive-charged lepton \( (k_2) \), and the polarization vector of the final vector meson \( \vec{V} \):

\[
\begin{align*}
    k^\mu_1 &= \frac{M_1 \sqrt{2}}{2} (1, -D \bar{n}), &
    k^\mu_2 &= \frac{M_1 \sqrt{2}}{2} (1, D \bar{n}), \\
    p^\mu_1 &= (E_1, \vec{p}_1) = \frac{M_1 \sqrt{2}}{2\sqrt{8}} \left( 1 - M_2^2 + \delta, 0, 0, \lambda^{1/2} \right), &
    p^\mu_2 &= (E_2, \vec{p}_2) = \frac{M_1 \sqrt{2}}{2\sqrt{8}} \left( 1 - M_2^2 - \delta, 0, 0, \lambda^{1/2} \right), \\
    \epsilon^{\mu}(\vec{p}_2, \lambda^V = \pm 1) &= \frac{1}{\sqrt{2}} (0, \mp 1, i, 0), &
    \epsilon^{\mu}(\vec{p}_2, \lambda^V = 0) &= \frac{1}{2 \sqrt{2} \sqrt{8}} \left( \lambda^{1/2}, 0, 0, 1 - M_2^2 - \delta \right). 
\end{align*}
\]
where $\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, $s = (k_1 + k_2)^2$, $\hat{s} = s/M_1^2$, $\hat{M}_2 = M_2/M_1$, $D = \sqrt{1 - 4\hat{m}/\hat{s}}$, $\hat{\lambda} = \lambda \left(\hat{s}, 1, \hat{M}_2^2\right)$ with $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac$.

 Similarly, the helicity amplitudes for the $B_{q}$-decay ($q = s, d$) 

$$A_{\lambda_V, \lambda_1, \lambda_2}^{(q)} = \langle \bar{V}_V(p_2, M_2, \ell), \ell_{\lambda_1}^+(m, k_1), \ell_{\lambda_2}^-(m, k_2) \mid H_{\text{eff}}^{SM}(b \rightarrow q\ell^+\ell^-) \mid B_{q}^{0}(M_1, p_1) \rangle \quad \text{(B2)}$$

read ($R_{q \bar{q}} = \frac{G_F}{\sqrt{2}} \frac{\alpha_s}{2\pi} V_{iq}^* V_{kb}$) :

$$A_{\pm 1, L, R}^{(q)} = R_{q \bar{q}} M_1^2 \left(1 \mp \cos \theta\right) \sqrt{\frac{s}{2}} \left[ \pm \frac{\hat{\lambda}^{1/2}}{2} (a + d D) - (b + f D) \right].$$

$$A_{\pm 1, R, L}^{(q)} = R_{q \bar{q}} M_1^2 \left(1 \pm \cos \theta\right) \sqrt{\frac{s}{2}} \left[ \pm \frac{\hat{\lambda}^{1/2}}{2} (a - d D) - (b - f D) \right].$$

$$A_{\pm 1, L, R}^{(q)} = R_{q \bar{q}} M_1^2 \sin \theta \frac{\hat{m}}{\sqrt{2}} \left[ \hat{\lambda}^{1/2} a \mp 2b \right].$$

$$A_{0, L, R}^{(q)} = -R_{q \bar{q}} M_1^2 \sin \theta \frac{1}{2 M_2} \left[ \hat{\lambda} (c + g D) - \left(1 - \hat{M}_2^2 - \hat{s}\right) (b + f D) \right].$$

$$A_{0, R, L}^{(q)} = R_{q \bar{q}} M_1^2 \sin \theta \frac{1}{2 M_2} \left[ \hat{\lambda} (c - g D) - \left(1 - \hat{M}_2^2 - \hat{s}\right) (b - f D) \right].$$

$$A_{0, L, R}^{(q)} = -R_{q \bar{q}} M_1^2 \frac{\hat{m}}{M_2 \sqrt{s}} \hat{\lambda}^{1/2} \left[ f - g(1 - \hat{M}_2^2) - h \hat{s} + \cos \theta \left( \hat{\lambda}^{1/2} c - \frac{1 - \hat{M}_2^2 - \hat{s}}{\hat{\lambda}^{1/2}} b \right) \right].$$

$$A_{0, R, L}^{(q)} = R_{q \bar{q}} M_1^2 \frac{\hat{m}}{M_2 \sqrt{s}} \hat{\lambda}^{1/2} \left[ f - g(1 - \hat{M}_2^2) - h \hat{s} - \cos \theta \left( \hat{\lambda}^{1/2} c - \frac{1 - \hat{M}_2^2 - \hat{s}}{\hat{\lambda}^{1/2}} b \right) \right].$$

Here $\lambda_V, \lambda_1 (\lambda_2)$ are the helicities of the vector meson and positive (negative) charged lepton, respectively.

Similarly, the helicity amplitudes for the $B_{q}$-decay ($q = s, d$) 

$$A_{\lambda_V, \lambda_1, \lambda_2}^{(q)} = \langle \bar{V}_V(p_2, M_2, \ell), \ell_{\lambda_1}^+(m, k_1), \ell_{\lambda_2}^-(m, k_2) \mid H_{\text{eff}}^{SM}(b \rightarrow q\ell^+\ell^-) \mid B_{q}(M_1, p_1) \rangle \quad \text{(B3)}$$
have the form \( R_{qb}^* = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi} V_{iq} V_{ib} \)

\[
A^{(q)}_{\pm 1, L, R} = R_{qb}^* M_1^2 (1 \mp \cos \theta) \sqrt{\frac{s}{2}} \left[ \mp \frac{\lambda^{1/2}}{2} (\tilde{a} + d D) - \left( \tilde{b} + f D \right) \right].
\]

\[
A^{(q)}_{\pm 1, R, L} = R_{qb}^* M_1^2 (1 \pm \cos \theta) \sqrt{\frac{s}{2}} \left[ \mp \frac{\lambda^{1/2}}{2} (\tilde{a} - d D) - \left( \tilde{b} - f D \right) \right].
\]

\[
A^{(q)}_{\pm 1, R, R} = A^{(q)}_{\pm 1, L, L} = R_{qb}^* M_1^2 \sin \theta \frac{\tilde{m}}{\sqrt{2}} \left[ -\lambda^{1/2} \tilde{a} \mp 2\tilde{b} \right].
\]

\[
A^{(q)}_{0, R, L} = -R_{qb}^* M_1^2 \sin \theta \frac{1}{2M_2} \left[ \tilde{\lambda} (\tilde{c} + g D) - \left( 1 - \tilde{M}_2^2 - \tilde{s} \right) \left( \tilde{b} + f D \right) \right].
\]

\[
A^{(q)}_{0, R, R} = -R_{qb}^* M_1^2 \sin \theta \frac{1}{2M_2} \left[ \tilde{\lambda} (\tilde{c} - g D) - \left( 1 - \tilde{M}_2^2 - \tilde{s} \right) \left( \tilde{b} - f D \right) \right].
\]

\[
A^{(q)}_{0, L, L} = R_{qb}^* M_1^2 \frac{\tilde{m}}{M_2 \sqrt{s}} \tilde{\lambda}^{1/2}
\times \left[ f - g(1 - \tilde{M}_2^2) - h \tilde{s} + \cos \theta \left( \tilde{\lambda}^{1/2} \tilde{c} - \frac{1 - \tilde{M}_2^2 - \tilde{s}}{\tilde{\lambda}^{1/2}} \tilde{b} \right) \right].
\]

In these formulas

\[
a(\mu, s) = 4 C_{7\gamma}(\mu) \frac{\tilde{m}_b + \tilde{m}_q}{s} T_1(s) + 2 C_{9V}^{\text{eff}}(\mu, s) \frac{V(s)}{1 + \tilde{M}_2},
\]

\[
\tilde{a}(\mu, s) = 4 C_{7\gamma}(\mu) \frac{\tilde{m}_b + \tilde{m}_q}{s} T_1(s) + 2 C_{9V}^{\text{eff}}(\mu, s) \frac{V(s)}{1 + \tilde{M}_2},
\]

\[
b(\mu, s) = \left( 1 + \tilde{M}_2 \right) \left( 2 C_{7\gamma}(\mu) \frac{\tilde{m}_b - \tilde{m}_q}{s} \left( 1 - \tilde{M}_2 \right) T_1(s) + C_{9V}^{\text{eff}}(\mu, s) A_1(s) \right),
\]

\[
\tilde{b}(\mu, s) = \left( 1 + \tilde{M}_2 \right) \left( 2 C_{7\gamma}(\mu) \frac{\tilde{m}_b - \tilde{m}_q}{s} \left( 1 - \tilde{M}_2 \right) T_2(s) + C_{9V}^{\text{eff}}(\mu, s) A_1(s) \right),
\]

\[
c(\mu, s) = \frac{1}{1 - \tilde{M}_2^2} \left( 2 C_{7\gamma}(\mu) \frac{\tilde{m}_b - \tilde{m}_q}{s} \left( 1 - \tilde{M}_2^2 \right) T_2(s) +
\]

\[
+ 2 C_{7\gamma}(\mu) \left( \tilde{m}_b - \tilde{m}_q \right) T_3(s) + C_{9V}^{\text{eff}}(\mu, s) \left( 1 - \tilde{M}_2 \right) A_2(s) \right),
\]

\[
\tilde{c}(\mu, s) = \frac{1}{1 - \tilde{M}_2^2} \left( 2 C_{7\gamma}(\mu) \frac{\tilde{m}_b - \tilde{m}_q}{s} \left( 1 - \tilde{M}_2^2 \right) T_2(s) +
\]

\[
+ 2 C_{7\gamma}(\mu) \left( \tilde{m}_b - \tilde{m}_q \right) T_3(s) + C_{9V}^{\text{eff}}(\mu, s) \left( 1 - \tilde{M}_2 \right) A_2(s) \right),
\]

\[
d(\mu, s) = 2 C_{10A}(\mu) \frac{V(s)}{1 + \tilde{M}_2},
\]

\[
f(\mu, s) = C_{10A}(\mu) \left( 1 + \tilde{M}_2 \right) A_1(s), \quad g(\mu, s) = C_{10A}(\mu) \frac{A_2(s)}{1 + \tilde{M}_2},
\]

\[
h(\mu, s) = \frac{C_{10A}(\mu)}{\tilde{s}} \left( \left( 1 + \tilde{M}_2 \right) A_1(s) - \left( 1 - \tilde{M}_2 \right) A_2(s) - 2 \tilde{M}_2 A_0(s) \right),
\]
Appendix C: Helicity amplitudes for rare radiative leptonic $\bar{B}(B) \to \gamma \ell^+ \ell^-$-decays

1. Kinematics

We work in the rest frame of the $\ell^+\ell^-$-pair and choose $z$-axis in this reference frame along the 3-momentum of the $\bar{B}$-meson. In this reference frame we have the following expressions for the 4-momenta of the initial meson ($p$), the final photon ($k$), the negative-charged lepton ($k_1$), the positive-charged lepton ($k_2$), and the photon polarization vector $\epsilon$:

$$k^\mu = \frac{M_1 \sqrt{s}}{2} (1, -D \bar{n}), \quad k_1^\mu = \frac{M_1 \sqrt{s}}{2} (1, D \bar{n}), \quad p^\mu = (E, 0, 0, \omega) = \frac{M_1 \sqrt{s}}{2} (1 + \hat{s}, 0, 0, 1 - \hat{s}),$$

$$k_2^\mu = (\omega, 0, 0, \omega), \quad \epsilon^\alpha \left( \hat{k}, \lambda_\gamma = \pm 1 \right) = \frac{1}{\sqrt{2}} (0, \mp 1, i, 0).$$

(C1)

where $\bar{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, $s = (k_1 + k_2)^2$, $D = \sqrt{1 - 4m^2/s}$, $\omega = M_1(1 - \hat{s})/2\sqrt{s}$, and now $\lambda^{1/2}/2 = 1 - \hat{s}$.

Three-particle phase space reads

$$d\Phi_3 = \frac{(2\pi)^4 \delta^4(p - k - k_1 - k_2)}{2\varepsilon_1 2\varepsilon_2 2\omega} \frac{dk_1}{(2\pi)^3} \frac{dk_2}{(2\pi)^3} \frac{dk}{(2\pi)^3} = \frac{M_1^2 (1 - \hat{s})}{2\hat{s} \pi^3} Dd\hat{s}d\cos \theta.$$  

(C2)

2. Helicity amplitudes

The helicity amplitudes for the $\bar{B}_q$-decay ($q = s, d$)

$$\bar{A}^{(q)}_{\lambda_\gamma, \lambda_1, \lambda_2} = \left\langle \gamma(k, \lambda_\gamma), \ell^+_1(m, k_1), \ell^-_2(m, k_2) \left| H_{\text{SM}}^\gamma(b \to q\ell^+\ell^-) \right| \bar{B}_q(M_1, p) \right\rangle$$

read ($R_{q\bar{q}} \equiv \frac{G_F}{\sqrt{2}} \alpha V_{iq} V_{ib}$)

$$\bar{A}^{(q)}_{\pm 1, L, R} = \pm |e| R_{q\bar{q}} M^2 \left( 1 \pm \cos \theta \right) \frac{1 - \hat{s}}{2\sqrt{2}}$$

$$\times \left[ \frac{2 \hat{m}_b}{\sqrt{s}} C_{\gamma 7} \left( F_{TV}^{b \to q} \mp F_{TA}^{b \to q} \right) \mp \sqrt{s} \left( C_{\gamma 9V}^{\text{eff}(q)} + D_{C10A} \right) (F_V \mp F_A) \pm \frac{4\hat{m}_b^2}{\sqrt{s}} \left( 1 - \hat{s} \right) \frac{1 - \hat{s}}{\hat{m}_b^2} \frac{1 - \hat{s}}{\hat{m}_b^2} C_{10A} \frac{f_{B_q}}{M_1} \right].$$

(C3)

Similarly, the helicity amplitudes for the $B_q$-decay

$$A^{(q)}_{\lambda_\gamma, \lambda_1, \lambda_2} = \left\langle \gamma(k, \lambda_\gamma), \ell^+_1(m, k_1), \ell^-_2(m, k_2) \left| H_{\text{SM}}^\gamma(b \to q\ell^+\ell^-) \right| B_q(M_1, p) \right\rangle$$

(C4)
have the form \( R_{qb} = \frac{C_{9V} \alpha_{CT}^{-1}}{2\pi} V_{iq} V_{tb} \)

\[
A_{\pm,1,R,L}^{(q)} = \pm |e| R_{qb} M_T^2 \left( 1 \mp \cos \theta \right) \frac{1 - \hat{s}}{2\sqrt{2}}
\]

\[
x \left[ \frac{\hat{m}_b}{\sqrt{s}} C_\gamma \left( F_{Tq}^{\tilde{b} \rightarrow q} \pm F_{TA}^{\tilde{b} \rightarrow q} \right) + \sqrt{\hat{s}} \left( C_{9V}^{\text{eff}(q)} + D C_{10A} \right) (F_V \pm F_A) + \frac{4\hat{m}_q^2}{\sqrt{s}} \frac{1 - \hat{s}}{(t - \hat{m}_b^2)(\bar{u} - \hat{m}_b^2)} C_{10A} \frac{f_{B_q}}{M_1} \right],
\]

\[
A_{\pm,1,R,L}^{(q)} = \pm |e| R_{qb} M_T^2 \left( 1 \pm \cos \theta \right) \frac{1 - \hat{s}}{2\sqrt{2}}
\]

\[
x \left[ \frac{\hat{m}_b}{\sqrt{s}} C_\gamma \left( F_{Tq}^{\tilde{b} \rightarrow q} \pm F_{TA}^{\tilde{b} \rightarrow q} \right) + \sqrt{\hat{s}} \left( C_{9V}^{\text{eff}(q)} - D C_{10A} \right) (F_V \pm F_A) + \frac{4\hat{m}_q^2}{\sqrt{s}} \frac{1 - \hat{s}}{(t - \hat{m}_b^2)(\bar{u} - \hat{m}_b^2)} C_{10A} \frac{f_{B_q}}{M_1} \right].
\]

\[
A_{\pm,1,R,R}^{(q)} = \pm |e| R_{qb} M_T^2 \sin \theta \frac{1 - \hat{s}}{\sqrt{2}} \frac{\hat{m}_q}{\sqrt{s}}
\]

\[
x \left[ \frac{\hat{m}_b}{\sqrt{s}} C_\gamma \left( F_{Tq}^{\tilde{b} \rightarrow q} \pm F_{TA}^{\tilde{b} \rightarrow q} \right) + \sqrt{\hat{s}} C_{9V}^{\text{eff}(q)} (F_V \pm F_A) \pm \frac{\sqrt{\hat{s}}}{(t - \hat{m}_b^2)(\bar{u} - \hat{m}_b^2)} \left( D (1 + \hat{s}) \mp (1 - \hat{s}) \right) C_{10A} \frac{f_{B_q}}{M_1} \right].
\]

\[
A_{\pm,1,R,R}^{(q)} = \pm |e| R_{qb} M_T^2 \sin \theta \frac{1 - \hat{s}}{\sqrt{2}} \frac{\hat{m}_q}{\sqrt{s}}
\]

\[
x \left[ \frac{\hat{m}_b}{\sqrt{s}} C_\gamma \left( F_{Tq}^{\tilde{b} \rightarrow q} \pm F_{TA}^{\tilde{b} \rightarrow q} \right) + \sqrt{\hat{s}} C_{9V}^{\text{eff}(q)} (F_V \pm F_A) \pm \frac{\sqrt{\hat{s}}}{(t - \hat{m}_b^2)(\bar{u} - \hat{m}_b^2)} \left( D (1 + \hat{s}) \mp (1 - \hat{s}) \right) C_{10A} \frac{f_{B_q}}{M_1} \right].
\]

In these formulas

\[
F_{Tq}^{\tilde{b} \rightarrow q}(q^2) = \left( 1 + \frac{m_q}{m_b} \right) (F_{Tq}(q^2,0) + F_{Tq}(0,q^2)) - \frac{16}{3} \frac{V_{ub} V_{uq}^*}{V_{tb} V_{tq}} \frac{V_{cb} V_{cq}^*}{V_{tb} V_{tq}} \frac{a_1}{C_\gamma} \frac{f_{B_q}}{m_b},
\]

\[
F_{TA}^{\tilde{b} \rightarrow q}(q^2) = \left( 1 - \frac{m_q}{m_b} \right) (F_{TA}(q^2,0) + F_{TA}(0,q^2)).
\]

and

\[
F_{Tq}^{\tilde{b} \rightarrow q}(q^2) = \left( 1 + \frac{m_q}{m_b} \right) (F_{Tq}(q^2,0) + F_{Tq}(0,q^2)) + \frac{16}{3} \frac{V_{ub} V_{uq}^*}{V_{tb} V_{tq}} \frac{V_{cb} V_{cq}^*}{V_{tb} V_{tq}} \frac{a_1}{C_\gamma} \frac{f_{B_q}}{m_b},
\]

\[
F_{TA}^{\tilde{b} \rightarrow q}(q^2) = \left( 1 - \frac{m_q}{m_b} \right) (F_{TA}(q^2,0) + F_{TA}(0,q^2)).
\]

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