$L^p - L^{p'}$ estimates for matrix Schrödinger equations*†

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Abstract
This paper is devoted to the study of dispersive estimates for matrix Schrödinger equations on the half-line with general boundary condition, and on the line. We prove $L^p - L^{p'}$ estimates on the half-line for slowly decaying selfadjoint matrix potentials that satisfy $\int_0^\infty (1 + x)|V(x)| \, dx < \infty$ both in the generic and in the exceptional cases. We obtain our $L^p - L^{p'}$ estimate on the line for a $n \times n$ system, under the condition that $\int_{-\infty}^\infty (1 + |x|)|V(x)| \, dx < \infty$, from the $L^p - L^{p'}$ estimate for a $2n \times 2n$ system on the half-line. With our $L^p - L^{p'}$ estimates we prove Strichartz estimates.

Keywords: matrix Schrödinger equation; general-boundary conditions; dispersive estimates; scattering theory; Jost solutions methods.

1 Introduction.
In this paper we consider the matrix Schrödinger equation on the half-line with general selfadjoint boundary condition

$$\begin{cases}
    i\partial_t u(t,x) = (-\partial_x^2 + V(x)) u(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^+, \\
    u(0,x) = u_0(x), \quad x \in \mathbb{R}^+, \\
    -B^\dagger u(t,0) + A^\dagger (\partial_x u)(t,0) = 0,
\end{cases} \tag{1.1}$$

where $\mathbb{R}^+ := (0, +\infty)$, $u(t,x)$ is a function from $\mathbb{R} \times \mathbb{R}^+$ into $\mathbb{C}^n$, $A, B$ are constant $n \times n$ matrices, the potential $V$ is a $n \times n$ selfadjoint matrix-valued function of $x$, i.e.

$$V(x) = V^\dagger(x), \quad x \in \mathbb{R}^+, \tag{1.2}$$

where the dagger denotes the matrix adjoint. We suppose that $V$ is in the Faddeev class $L^1_1$, i.e. that it is a Lebesgue measurable matrix-valued function and,

$$\int_{\mathbb{R}^+} (1 + x)|V(x)| \, dx < \infty, \tag{1.3}$$

where by $|V|$ we denote the matrix norm of $V$. The more general selfadjoint boundary condition at $x = 0$ can be expressed in several equivalent ways [3], [4], [5], [19], [20], [21], [27], and [28]. See also [39] for further results on general selfadjoint boundary conditions. We find it convenient to state the boundary condition following [3], [4], and [5], as in (1.1) where the matrices $A$ and $B$ satisfy (see Section 2.1),

$$-B^\dagger A + A^\dagger B = 0, \tag{1.4}$$

and

$$A^\dagger A + B^\dagger B > 0. \tag{1.5}$$

We denote by

$$H := H_{A,B,V},$$

the selfadjoint operator in $L^2(\mathbb{R}^+)$ associated to the initial-boundary problem (1.1).

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Currently there is a considerable interest in matrix Schrödinger equations. In part, this is because of the importance of these equations for quantum mechanical scattering of particles with internal structure, and also since a quantum star graph is a particular case of a matrix Schrödinger equation with a diagonal potential matrix. There is an extensive literature in quantum graphs. See, for example, [3]-[10], [13], [27]-[33], and the references therein. The matrix Schrödinger equation with a diagonal potential matrix corresponds to a star graph which describes the behavior of $n$ connected very thin quantum wires that form a star graph, that is, a graph with only one vertex and a finite number of edges of infinite length. The boundary condition in (1.1) restrict the value of the wave function and of its derivative at the vertex. The problem is relevant from the physical point of view. For instance, it appears in the design of elementary gates in quantum computing and nanotubes for microscopic electronic devices, where, for example, strings of atoms may form a star-shaped graph. The consideration of the most general boundary condition at the vertex and not only, for say, Dirichlet boundary condition is also physically motivated: for quantum graphs the relevant boundary conditions are the ones that link the values, at the different edges, of the wave function and of the first derivative. An important example is the Kirchhoff boundary condition. In fact, a quantum graph is an idealization of wires with a small cross section that meet at vertices. It is obtained as the limit when the cross section goes to zero. The boundary conditions on the graph’s vertices depends on how the limit is taken. In principle, all the boundary conditions in (1.1) can appear in this limit procedure. This motivates the study of the more general selfadjoint boundary condition.

The purpose of this paper is to obtain $L^p - L^{p'}$ estimates for the initial boundary problem (1.1).

Notation. We denote by $L^p(U; \mathbb{C}^n)$, for $1 \leq p \leq \infty$, and $U = \mathbb{R}^+$ or $U = \mathbb{R}$, $(\mathbb{R}^+ = (0, \infty))$, the standard Lebesgue spaces of $\mathbb{C}^n$ valued functions, where $\mathbb{C}$ denotes the complex numbers. For an integer $m \geq 0$ and a real $1 \leq p \leq \infty$, $W_{m,p}(U; \mathbb{C}^n)$ is the standard Sobolev space. (See e.g. [1] for the definitions and properties of these spaces.) If there is no place for confusion, we shall omit $\mathbb{C}^n$ or both $U$ and $\mathbb{C}^n$ in writing the above spaces. $W_{m,p}^{(0)}(U)$ is the closure of $C^\infty_0(U)$ in the space $W_{m,p}(U)$. We denote the Fourier transform by,

$$\mathcal{F} f := \int_{\mathbb{R}} e^{ikx} f(x) \, dx,$$

and the inverse Fourier transform by,

$$\mathcal{F}^{-1} f := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} f(k) \, dk.$$

We designate,

$$\mathcal{F} \{L^1(\mathbb{R})\} := \{f \in L^\infty(\mathbb{R}) : f = \mathcal{F}g, \ g \in L^1(\mathbb{R})\}.$$

By $\mathbb{C}^+$ we denote the open upper-half complex plane. For any pair of Banach spaces $X, Y$ we denote by $B(X, Y)$ the Banach space of all bounded operators from $X$ into $Y$. When $X = Y$ we use the notation $B(X)$. For any operator $G$ in a Banach space $X$ we denote by $D[G]$ the domain of $G$. For a bounded below selfadjoint operator, $G$, the quadratic form domain of $G$ is the domain of its associated quadratic form [24]. By $0_m$ and $I_m, m = 1, 2, \ldots$, we designate the $m \times m$ zero and identity matrices, respectively. Finally, we shall denote by $C$ a generic positive constant, which does not has to take the same value when it appears in different places.

1.1 Main results.

In order to present our results, let us first define the function spaces we will work with. We can diagonalize the boundary condition in (1.1) to get $n$ equations $\cos \theta_j \psi_j(x) + \sin \theta_j \psi'_j(x) = 0, 0 < \theta_j \leq \pi, j = 1, 2, \ldots, n$, (see (2.7) below). Let us define the space $\widetilde{W}_j^p(\mathbb{R}^+)$, for $1 \leq p \leq \infty$, which is the Sobolev space $W_{1,p}^{(0)}(\mathbb{R}^+)$, in the case of Dirichlet boundary condition, $\theta_j = \pi$, and $\widetilde{W}_j^p(\mathbb{R}^+) = W_{1,p}(\mathbb{R}^+)$, in the case of Neumann, $\theta_j = \pi/2$, or mixed, $\theta_j \neq \pi, \theta_j \neq \pi/2$, boundary conditions (see (2.9) below for the precise definition). We consider $\widetilde{W}_{1,p}(\mathbb{R}^+) = \bigoplus_{j=1}^n \widetilde{W}_j^p(\mathbb{R}^+)$. Then, the quadratic form domain of the Hamiltonian $H := H_{A,B}$ that corresponds to the general boundary condition in (1.1) is given by $W_{1,2}^{A,B}(\mathbb{R}^+)$ with

$$W_{1/2}^{A,B}(\mathbb{R}^+) = M \widetilde{W}_{1,2}(\mathbb{R}^+) \subset W_{1,2}(\mathbb{R}^+),$$

where $M$ is a unitary matrix (see (2.10) below). Let $P_c$ denote the projector onto the continuous subspace of $H$. We observe that $P_c = I - P_p$, where $P_p$ is the projector onto the subspace of $L^2$ generated by the eigenvectors corresponding to the bound states of $H$. We also note that under our assumptions on $V$, the number of negative bound states of $H$ is finite and that $H$ has no positive or zero bound states. Hence, the subspace generated by the eigenvectors is finite-dimensional. We now present our results.
Theorem 1.1 (The $L^p - L^{p'}$ estimate). Suppose that the potential $V$ satisfies $[L,2]$ and $[L,3]$. Then, for any $p \in [1,2]$ and $p'$ such that $1/p + 1/p' = 1$, the estimates

$$
\|e^{-itH}Pc\|_{B(L^p(R^+), L^{p'}(R^+))} \leq \frac{C}{|t|^{1/p-1/2}},
$$

and

$$
\|e^{-itH}Pc\|_{B(W^{1,p}_x(R^+), W^{1,p'}_x(R^+))} \leq \frac{C}{|t|^{1/p-1/2}},
$$

hold for all $t \in \mathbb{R}\setminus\{0\}$.

Theorem 1.2 (Strichartz estimates). Suppose that the potential $V$ satisfies $[L,2]$ and $[L,3]$. Let $(q,r)$ be an admissible pair, that is, $2/q = 1/2 - 1/r$ and $2 \leq r \leq \infty$. Then, for every $\varphi \in L^2(\mathbb{R}^+)$, the function $t \to e^{-itH}Pc\varphi$ belongs to $L^q(\mathbb{R}^+, L^r(\mathbb{R}^+)) \cap C(\mathbb{R}, L^2(\mathbb{R}^+))$. Moreover, there exists a constant $C > 0$ such that

$$
\|e^{-itH}Pc\varphi\|_{L^q(\mathbb{R}^+, L^r(\mathbb{R}^+))} \leq C \|\varphi\|_{L^2(\mathbb{R}^+)},
$$

for every $\varphi \in L^2(\mathbb{R}^+)$. Moreover, let $I \subset \mathbb{R}$ be an interval. For an admissible pair $(\gamma, \rho)$, let $f \in L^{q'}(I, L^{r'}(\mathbb{R}^+))$, where $1/\gamma + 1/\gamma' = 1$ and $1/\rho + 1/\rho' = 1$. Then, for $t_0 \in I$, the function

$$
t \to \Phi_f(t) = \int_{t_0}^t e^{i(t-s)H}Pc\varphi(s) \, ds, \quad t \in I,
$$

belongs to $L^q(I, L^{r'}(\mathbb{R}^+)) \cap C(I, L^2(\mathbb{R}^+))$ and

$$
\|\Phi_f\|_{L^q(I, L^{r'}(\mathbb{R}^+))} \leq C \|f\|_{L^{q'}(I, L^{r'}(\mathbb{R}^+))}, \quad \text{for every } f \in L^{q'}(I, L^{r'}(\mathbb{R}^+)),
$$

where the constant $C$ is independent of $I$.

1.1.1 The matrix Schrödinger equation on the full-line.

Following [7] we show that a $2n \times 2n$ matrix Schrödinger equation on the half-line is unitarily equivalent to a $n \times n$ matrix Schrödinger equation on the full-line with a point interaction at $x = 0$. We define the unitary operator $U$ from $L^2(\mathbb{R}^+; \mathbb{C}^{2n})$ onto $L^2(\mathbb{R}; \mathbb{C}^n)$ by

$$
\phi(x) = U\psi(x) := \begin{cases} 
\psi_1(x), & x \geq 0, \\
\psi_2(-x), & x < 0,
\end{cases}
$$

for a vector-valued function $\psi = (\psi_1, \psi_2)^T$, ($T$ denotes the matrix transpose) where $\psi_j \in L^2(\mathbb{R}^+; \mathbb{C}^n)$, $j = 1,2$. Let the potential in $[1,1]$ be the diagonal matrix

$$
V(x) := \text{diag}\{V_1(x), V_2(x)\},
$$

where $V_j, j = 1,2$ are selfadjoint $n \times n$ matrix-valued functions that satisfy $V_j \in L^1(\mathbb{R}^+)$. Under $U$ the Hamiltonian $H$ is transformed into the following Hamiltonian in the full-line,

$$
H_\mathbb{R} := UHU^\dagger, \quad D[H_\mathbb{R}] := \{ \phi \in L^2(\mathbb{R}; \mathbb{C}^n) : U\phi \in D[H] \}.
$$

The operator $H_\mathbb{R}$ is a selfadjoint realization in $L^2(\mathbb{R}; \mathbb{C}^n)$ of the formal differential operator $-\partial_x^2 + Q(x)$ where,

$$
Q(x) = \begin{cases} 
V_1(x), & x \geq 0, \\
V_2(-x), & x < 0.
\end{cases}
$$

Further, the quadratic form domain of $H_\mathbb{R}$ is given by $W^{R,A,B}_{1,2}$ where,

$$
W^{R,A,B}_{1,2} := UW^{A,B}_{1,2} \subset W^{1,2}_{1,2}(-\infty, 0) \oplus W^{1,2}_{1,2}(0, \infty).
$$

Let us write the $2n \times 2n$ matrices $A, B$ as follows,

$$
A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
$$

(1.11)
with $A_j, B_j, j = 1, 2$, being $n \times 2n$ matrices. We have that the functions in the domain of $H_\mathbb{R}$ satisfy the following transmission condition at $x = 0$,

$$-B_1^j \phi(0+) - B_2^j \phi(0-) + A_1^j (\partial_x \phi)(0+) - A_2^j (\partial_x \phi)(0-) = 0.$$  \hspace{1cm} (1.12)

Then, $u(t, x)$ is a solution of the problem (1.1) if and only if $v(t, x) := U u(t, x)$ is a solution of the following $n \times n$ system in the full-line,

$$\begin{cases}
i \partial_t v(t, x) = (-\partial_x^2 + Q(x)) v(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \\
v(0, x) = v_0(x), \quad x \in \mathbb{R}, \\
-B_1^j v(t, 0+) - B_2^j v(t, 0-) + A_1^j (\partial_x v)(t, 0+) - A_2^j (\partial_x v)(t, 0-) = 0.
\end{cases}$$  \hspace{1cm} (1.13)

For example, let us take,

$$A = \begin{bmatrix} 0_n & I_n \\ 0_n & I_n \end{bmatrix}, \quad B = \begin{bmatrix} -I_n & \Lambda \\ I_n & 0_n \end{bmatrix},$$

where $\Lambda$ is a self-adjoint $n \times n$ matrix. These matrices satisfy (1.4) (1.5). Moreover, the transmission condition in (1.13) is given by,

$$v(t, 0+) = v(t, 0-) = v(t, 0), \quad (\partial_x v)(t, 0+) - (\partial_x v)(t, 0-) = \Lambda v(t, 0).$$  \hspace{1cm} (1.14)

This transmission condition corresponds to a Dirac delta point interaction at $x = 0$ with coupling matrix $\Lambda$. If $\Lambda = 0$, $v(t, x)$ and $(\partial_x v)(t, x)$ are continuous at $x = 0$ and the transmission condition corresponds to the matrix Schrödinger equation on the full-line without a point interaction at $x = 0$.

Using Theorem 1.2 and the unitary operator $U$, as above, we deduce the following result concerning the Cauchy problem (1.13).

**Corollary 1.3.** (The full-line case) Let $n \in \mathbb{N}$. Suppose that $Q(x), x \in \mathbb{R}$, is a $n \times n$ self-adjoint matrix-valued function such that $Q \in L_1^1(\mathbb{R}; \mathbb{C}^n)$. Then, for any $p \in [1, 2]$ and $p'$ such that $1/p + 1/p' = 1$, the estimates

$$\|e^{-itH_\mathbb{R}} P_{t, \mathbb{R}}\|_{B(L^p(\mathbb{R}; \mathbb{C}^n), L^{p'}(\mathbb{R}; \mathbb{C}^n))} \leq \frac{C}{|t|^{1/p - 1/2}},$$

and

$$\|e^{-itH_\mathbb{R}} P_{t, \mathbb{R}}\|_{B\left(W^{2, A, A, b}_{r, p, r, p'}(\mathbb{R}; \mathbb{C}^n)\right)} \leq \frac{C}{|t|^{1/p - 1/2}},$$

hold for all $t \in \mathbb{R}\setminus\{0\}$, where $P_{t, \mathbb{R}}$ is the projector onto the continuous subspace of $H_\mathbb{R}$. Moreover, let $(q, r)$ be an admissible pair, that is, $2/q = 1/2 - 1/r$ and $2 \leq r \leq \infty$. Then, the conclusions of Theorem 1.2 are true with $\mathbb{R}^+$ replaced by $\mathbb{R}$ and with $H_\mathbb{R}, P_{t, \mathbb{R}}$ instead, respectively, of $H$ and $P_t$.

**Comments on the results and on the literature.**

In the case of star graphs with potential identically zero, and with general boundary conditions, $L^p - L^{p'}$ estimates, and Strichartz estimates were obtained by [23]. Moreover, for a star graph with the Kirchhoff boundary condition and a potential that satisfies $\int_0^\infty dx(1 + x)\gamma|V(x)| < \infty, \gamma > 5/2$, $L^p - L^{p'}$ estimates, and Strichartz estimates were proven in [30]. Note that Theorems 1.1 and 1.2 and Corollary 1.3 hold under the same conditions in the generic and exceptional cases. Recall that we are in the generic case if the Jost matrix is invertible at zero energy and that we are in the exceptional case if the Jost matrix is not invertible at zero energy. In the exceptional case there is a resonance (or half-bound state) with zero energy, and in the generic case there is no resonance at zero energy. In other words, the validity of the dispersive estimates is independent of the existence of a resonance with zero energy.

In order to obtain the $L^p - L^{p'}$ estimates, we follow the approach of [14]. For this purpose, we use the scattering theory for the matrix Schrödinger equation on the half-line developed in [2], [3], [5], [6], [7] and [45]. From the spectral representation for the matrix Schrödinger operator $H$, we get a representation (see (2.39) below) for the continuous part (which corresponds to the scattering process) of the evolution group $e^{-itH}$ in terms of the Jost solutions for the stationary matrix Schrödinger equation. Then, we can estimate the large-time behavior of $e^{-itH} P_t$ by using the low- and high-energy behaviours of the Jost solutions and the scattering matrix. For this purpose, we need to estimate the difference between the scattering matrix and its high-energy limit and to show that the Fourier transform of the difference is integrable on the whole real line. This is Theorem 2.5 below. This result, which is interesting by its own, is crucial for obtaining the $L^p - L^{p'}$ estimates for such general perturbations as $V \in L_1^1(\mathbb{R}^+)$. We prove Theorem 2.5 by adapting the arguments of [2] for the Dirichlet boundary condition, which involve the well-known Wiener theorem, to the case of general self-adjoint boundary condition in (1.11). The key technical tools that allow us to prove that the Fourier transform of the scattering matrix minus its high-energy limit is integrable, under this generality, are the sharp results on the low-energy behavior of the Jost matrix, including a formula for the Jost matrix at zero energy, that where obtained in [3] and the precise estimate of the high-energy behavior of the
scattering matrix of \([3]\). We observe that an alternative method for obtaining the \(L^p - L^{p'}\) estimates is developed in \([42]\). This approach requires a more detailed and subtle study of the low-energy properties of the scattering data. Hence, it needs stronger conditions.

There is a very extensive literature on dispersive estimates. For surveys see \([15]\) and \([40]\). We will only comment on results in one dimension. The \(L^p - L^{p'}\) estimates on the line were first proven in the scalar case by Weder \([42]\) under the condition

\[
\int_{\mathbb{R}} (1 + |x|)^\gamma |V(x)| \, dx < \infty,
\]

with \(\gamma > 3/2\) in the generic case and \(\gamma > 5/2\) in the exceptional case. This was generalized by M. Goldberg and W. Schlag \([17]\) to, respectively \(\gamma = 1\) and \(\gamma = 2\), and by Egorova, Kopylova, Marchenko, and Teschl \([14]\) to \(\gamma = 1\) in the generic and the exceptional cases. D’Ancona and Selberg \([15]\) considered a potential that satisfies \((1.15)\) with \(\gamma = 2\) plus a step potential. Note that Corollary \([1.3]\) with the point interaction at \(x = 0\) is new in the scalar case. We are not aware of any result on \(L^p - L^{p'}\) estimates on the line for matrix Schrödinger equations.

The \(L^p - L^{p'}\) estimates on the half-line, in the scalar case and with Dirichlet boundary condition was proven by Weder \([44]\) under the condition \(\int_0^\infty x |V(x)| \, dx < \infty\) in the generic and the exceptional cases. It was actually in this paper that it was discovered that the \(L^p - L^{p'}\) estimates hold under the same condition in the generic and the exceptional cases. The case of the spherical Schrödinger equation was considered by Holzleitner, Kostenko and Teschl \([22]\) and by Kostenko, Teschl and Toloz\([20]\). The case of the one-dimensional Klein-Gordon equation with a potential was studied by Weder \([43]\), Egorova, Kopylova, Marchenko, and Teschl \([14]\) and by Prill \([37]\). Kopylova and Teschl \([25]\) considered one-dimensional discrete Dirac equations.

The paper is organized as follows. In Section 2 we consider results concerning the scattering theory for the matrix Schrödinger equation on the half-line, which play a crucial role in the proof of our dispersive estimates. In particular, in Subsection 2.1 we construct the self-adjoint extension \(H\) associated to the matrix Schrödinger equation \((1.1)\). In Subsection 2.2 we introduce the relevant solutions for the stationary matrix Schrödinger equation. Using these solutions, in Subsection 2.3 we construct the spectral representations for the operator \(H\) via the generalized Fourier transforms. In Section 2.4 we prove that the Fourier transform of the scattering matrix minus its high-energy limit is integrable on the line. We use the results of Section 2 in Section 3 to prove the \(L^p - L^{p'}\) and Strichartz estimates for the matrix Schrödinger equation.

2 Scattering for Matrix Schrödinger Equations.

2.1 The Schrödinger equation on the half-line.

Let \(n \in \mathbb{N}\). Consider the stationary matrix Schrödinger equation on the half-line

\[
-\psi'' + V(x) \psi = k^2 \psi, \quad x \in \mathbb{R}^+,
\]

where the prime denotes the derivative with respect to the spatial coordinate \(x\), \(k^2\) is the complex-valued spectral parameter, \(V(x)\) satisfies \((1.2)\) and is such that

\[
V \in L^1(\mathbb{R}^+).
\]

The wavefunction \(\psi(k, x)\) appearing may be either a \(n \times n\) matrix-valued function or it may be a column vector with \(n\) components. As mentioned at the beginning of the introduction, the more general self-adjoint boundary condition at \(x = 0\) can be expressed in terms of two constant \(n \times n\) matrices \(A\) and \(B\) as

\[
- B^\dagger \psi(0) + A^\dagger \psi'(0) = 0, \tag{2.3}
\]

where \(A\) and \(B\) satisfy

\[
- B^\dagger A + A^\dagger B = 0, \tag{2.4}
\]

\[
A^\dagger A + B^\dagger B > 0. \tag{2.5}
\]

We observe that \([5]\) provides the explicit steps to go from any pair of matrices \(A\) and \(B\) appearing in the selfadjoint boundary condition \((2.3)-(2.5)\) to a pair \(\tilde{A}\) and \(\tilde{B}\), given by

\[
\tilde{A} = - \text{diag}[\sin \theta_1, ..., \sin \theta_n], \quad \tilde{B} = \text{diag}[\cos \theta_1, ..., \cos \theta_n],
\]

with appropriate real parameters \(\theta_j \in (0, \pi]\), which still satisfy \((2.3)-(2.5)\). For the matrices \(\tilde{A}, \tilde{B}\), the boundary conditions \((2.3)-(2.5)\) are given by

\[
\cos \theta_j \psi_j(0) + \sin \theta_j \psi_j'(0) = 0, \quad j = 1, 2, ..., n, \tag{2.7}
\]
Moreover, by (2.13) there are positive constants $C$.

Similarly, $H$.

We denote by $H$.

The special case $\theta_j = \pi$ corresponds to the Dirichlet boundary condition and the case $\theta_j = \pi/2$ corresponds to the Neumann boundary condition. In general, there are $n_N \leq n$ values with $\theta_j = \pi/2$ and $n_D \leq n$ values with $\theta_j = \pi$, and hence there are $n_M$ remaining values, with $n_M = n - n_N - n_D$ such that those $\theta_j$-values lie in the interval $(0, \pi/2)$ or $(\pi/2, \pi)$, i.e., they correspond to mixed boundary conditions. In fact, it is proven in [5] that for any pair of matrices $(A, B)$ that satisfy (2.3) there is a pair of matrices $(\tilde{A}, \tilde{B})$ as in (2.4), a unitary matrix $M$ and two invertible matrices $T_1, T_2$ such

\[ A = M \hat{A} T_1 M^\dagger T_2, \quad B = M \hat{B} T_1 M^\dagger T_2. \]

We construct a selfadjoint realization of the matrix Schrödinger operator $-\partial_x^2 + V(x)$ by quadratic forms methods. For the following discussion see [7] and [14]. Let $\theta_j$ be given by equations (2.7). For $1 \leq p \leq \infty$, we denote

\[ \bar{W}_j^p := W_{1,p}^{(0)}, \text{ if } \theta_j = \pi, \text{ and } \bar{W}_j^p := W_{1,p}, \text{ if } \theta_j \neq \pi. \]

We put

\[ \bar{W}_1^p := \otimes_{j=1}^n \bar{W}_j^p. \]

We write

\[ \Theta := \text{diag}[\cot \theta_1, \ldots, \cot \theta_n], \]

where $\cot \theta_j = 0$, if $\theta_j = \pi/2$, or $\theta_j = \pi$, and $\cot \theta_j = \cot \theta_j$, if $\theta_j \neq \pi/2, \pi$. Suppose that the potential $V$ satisfies (2.2) and (2.2). The following quadratic form is closed, symmetric and bounded below,

\[ h(\phi, \psi) := (\phi', \psi')_{L^2} - \langle M \Theta M^\dagger \phi(0), \psi(0) \rangle + (V \phi, \psi)_{L^2}, \quad Q(h) := W_{1,2}^{A,B}, \]

where by $Q(h)$ we denote the domain of $h$ and

\[ W_{1,p}^{A,B} := M \bar{W}_{1,p} \subset W_{1,p}. \]

Further, by $\langle \cdot, \cdot \rangle$ we designate the scalar product in $\mathbb{C}^n$. We denote by $H_{A,B,V}$ the selfadjoint bounded below operator associated to $h$ [24]. The operator $H_{A,B,V}$ is the selfadjoint realization of $-\partial_x^2 + V(x)$ with the selfadjoint boundary condition (2.3). When there is no possibility of misunderstanding we will use the notation $H$, i.e., $H \equiv H_{A,B,V}$. It is proven in [7] and [15] that

\[ H_{A,B,V} = MH_{A,B,M^\dagger M} M^\dagger. \]

We denote by $H_{D,N,0}$ the selfadjoint bounded below operator associated to the quadratic form (2.10) with $V \equiv 0$ and the $\theta_j$ corresponding to the $n_M$ mixed boundary conditions replaced by $\theta_j = \pi/2$, i.e. with the mixed boundary conditions replaced by Neumann boundary conditions. Note that the quadratic form domain of $H_{D,N,0}$ is $W_{1,2}^{A,B}$. Take $L > 1$ such that $H + L > I$ and $H_{D,N,0} + L > I$. Hence, since the domains of $\sqrt{H + L}$ and of $\sqrt{H_{D,N,0} + L}$ are equal to $W_{1,2}^{A,B}$ we have that

\[ \left( \sqrt{H + L} \right) \left( \sqrt{H_{D,N,0} + L} \right)^{-1} \in B(L^2), \quad \left( \sqrt{H_{D,N,0} + L} \right) \left( \sqrt{H + L} \right)^{-1} \in B(L^2). \]

Denote by $\mathcal{H}$ the domain of $\sqrt{H + L}$ endowed with the norm,

\[ \| \phi \|_{\mathcal{H}} := \| \sqrt{H + L} \phi \|_{L^2}, \quad \phi \in \mathcal{H}. \]

In other words, $\mathcal{H}$ consists of $W_{1,2}^{A,B}$, but with the norm (2.14). Observe that it follows from (2.10) that

\[ \| \phi \|^2_{\mathcal{H}} = (\phi', \phi')_{L^2} - \langle M \Theta M^\dagger \phi(0), \phi(0) \rangle + (V \phi, \phi)_{L^2} + L (\phi, \phi)_{L^2}, \quad \phi \in \mathcal{H}. \]

Similarly,

\[ \| \sqrt{H_{D,N,0} + L} \phi \|^2_{L^2} = (\phi', \phi')_{L^2} + L (\phi, \phi)_{L^2} = \| \phi \|^2_{W_{1,2}^{A,B}} + (L-1)\| \phi \|^2_{L^2}, \quad \phi \in W_{1,2}^{A,B}. \]

Moreover, by (2.13) there are positive constants $C_1, C_2$ such that

\[ C_1 \| \phi \|^2_{W_{1,2}^{A,B}} \leq \| \phi \|_{\mathcal{H}} \leq C_2 \| \phi \|^2_{W_{1,2}^{A,B}}. \]
2.2 The Jost and scattering matrices.

Below, in Propositions 2.1, 2.2 and 2.3 we state results in special solutions to the matrix Schrödinger equation that we use. The interested reader can consult the monographs, [12, 34, 35], and the references quoted there, for similar results in the scalar case.

We now introduce the special solutions for that play a crucial role in our analysis. By [2], [3], [7] we have that:

**Proposition 2.1.** Suppose that the potential \( V \) satisfies (2.2). For each fixed \( k \in C^+ \setminus \{0\} \) there exists a unique \( n \times n \) matrix-valued Jost solution \( f(k,x) \) to equation (2.1) satisfying the asymptotic condition

\[
f(k,x) = e^{ikx}(I + o(1)), \quad f'(k,x) = e^{ikx}[ikI + o(1)], \quad x \to +\infty.
\]

For each \( k \in C^+ \setminus \{0\} \), the quantity \( f(k,x) \) and its \( x \)-derivative \( f'(k,x) \) are continuous in \( x \in [0, +\infty) \). Moreover, for any fixed \( x \in [0, \infty) \), \( f(k,x) \) and \( f'(k,x) \) are analytic in \( k \in C^+ \) and continuous in \( k \in C^+ \setminus \{0\} \).

Note that [2] proves a result that is slightly different from the one given in Proposition 2.1 because they use Jost solutions analytic in \( C^- \). Furthermore, [2] considers potentials such that \( x^{1+\delta}V(x) \) is integrable for \( \delta \geq 0 \), but they obtain a sharper error bound that depends on \( \delta \). Proposition 2.1 as we state it above, for Jost solutions analytic in \( C^+ \), and for potentials that satisfy (2.2) is given in [3, 7].

Given the boundary matrices \( A \) and \( B \) satisfying (2.3), (2.4), from the Jost solution we construct the Jost matrix \( J(k) \), which is a \( n \times n \) matrix-valued function of \( k \),

\[
J(k) = f(-k^*,0)^\dagger B - f'(-k^*,0)^\dagger A, \quad k \in C^+,
\]

where the asterisk denotes complex conjugation. The following proposition is proven in [2, 3, 7, 20].

**Proposition 2.2.** Suppose that the potential \( V \) satisfies (2.2) and (2.3). Then, the Jost matrix \( J(k) \) is analytic for \( k \in C^+ \), continuous for \( k \in C^+ \setminus \{0\} \) and invertible for \( k \in R \setminus \{0\} \). If furthermore, the potential satisfies (1.3), then the Jost matrix is continuous for \( k \in C^+ \).

Note that [2] proves a result that is slightly different from the one given in Proposition 2.2 because they use Jost solutions analytic in \( C^- \), moreover, they always assume that \( xV(x) \) is integrable. Proposition 2.2 as we state it above, for Jost solutions analytic in \( C^+ \), and for general boundary condition is given in [3, 7], and [20]. In [20] it is always assumed that (1.3) holds.

For \( x \geq 0 \), let \( K(x,y) \) be defined as

\[
K(x,y) = (2\pi)^{-1} \int_{-\infty}^{\infty} [f(k,x) - e^{ikx}I]e^{-iky}dk.
\]

Let us define the functions

\[
\sigma(x) = \int_{0}^{x} |V(y)|dy, \quad \sigma_1(x) = \int_{0}^{x} y|V(y)|dy, \quad x \geq 0.
\]

We observe that for potentials satisfying (1.3), both \( \sigma(0) \) and \( \sigma_1(0) \) are finite, and moreover,

\[
\int_{0}^{\infty} \sigma(x)dx = \sigma_1(0) < \infty.
\]

The following proposition is given in [2].

**Proposition 2.3.** Suppose that the potential \( V \) satisfies (1.3) and (2.3). Then, the matrix \( K(x,y) \) is continuous in \((x,y)\) in the region \( 0 \leq x \leq y \), and is related to the potential via

\[
K(x,x^+) = \frac{1}{2} \int_{x}^{\infty} V(z)dz, \quad x \in [0, +\infty).
\]

The Jost solution \( f(k,x) \) has the representation

\[
f(k,x) = e^{ikx}I + \int_{x}^{\infty} e^{iky}K(x,y)dy.
\]

(2.18)

The matrix \( K(x,y) \) satisfies,

\[
K(x,y) = 0, \quad y < x, y, x \in [0, \infty),
\]

\[
|K(x,y)| \leq \frac{1}{2} e^{\sigma_1(x)}\sigma \left( \frac{x + y}{2} \right), \quad x, y \in [0, \infty),
\]

(2.19)
\[ \partial_x^i \partial_y^j K(x, y) = 0, \quad y < x, x, y \in [0, \infty), \]
\[ |\partial_x^i \partial_y^j K(x, y)| \leq \frac{1}{4} \left| V \left( \frac{x + y}{2} \right) \right| + \frac{1}{2} e^{\sigma(x)} \sigma \left( \frac{x + y}{2} \right) \sigma(x), \quad 0 < x < y, \quad i + j = 1. \quad (2.20) \]

Note that \([2]\) states the result in a slightly different form from the one in Proposition \([2,3]\) because they use Jost solutions analytic in \(C^-\).

We observe that the Jost matrix \(J(k)\) can be expressed in terms of \(K\) as
\[ J(k)^\dagger = B^\dagger + i k A^\dagger + A^\dagger K(0, 0) + \int_0^\infty e^{-ikz} \left( B^\dagger K(0, z) - A^\dagger K_x(0, z) \right) dz, \quad k \in \mathbb{R}. \quad (2.21) \]

From the Jost matrix \(J(k)\) we construct the scattering matrix \(S(k)\), which is an \(n \times n\) matrix-valued function of \(k\) given by
\[ S(k) = -J(-k) J(k)^{-1}, \quad k \in \mathbb{R}. \quad (2.22) \]

In the exceptional case where \(J(0)\) is not invertible the scattering matrix is defined by \((2.22)\) only for \(k \neq 0\). However, it is proven in \([3]\) that for potentials satisfying \((1.2)\) and \((1.3)\) the limit \(S(0) := \lim_{k \to 0} S(k)\) exists in the exceptional case and, moreover, a formula for \(S(0)\) is given. Actually, the low-energy analysis of \([3]\) plays a crucial role in the proof of Theorem \((2.5)\).

Further, it is proven in \([3]\) that the relation
\[ S(-k) = S(k)^\dagger = S(k)^{-1}, \quad k \in \mathbb{R}, \quad (2.23) \]
holds. In particular, the scattering matrix \(S(k)\) is unitary for \(k \in \mathbb{R}\) and
\[ S(-k) = -\left( J(k)^\dagger \right)^{-1} (J(-k))^\dagger. \quad (2.24) \]

In terms of the Jost solution \(f(k, x)\) and the scattering matrix \(S(k)\) we construct the physical solution \([5]\)
\[ \Psi(k, x) = f(-k, x) + f(k, x) S(k), \quad k \in \mathbb{R}. \quad (2.25) \]

For the definition of the physical solution in the scalar case the reader can consult \([12, 33, 15]\). Observe that \([2]\) gives a definition of the physical solution in the case with Dirichlet boundary condition that is different from \((2.25)\). Recall that they use Jost solutions analytic in \(C^-\). Observe that by \((2.18), (2.19)\) and the unitarity of \(S(k)\) we have
\[ |\Psi(k, x)| \leq C. \quad (2.26) \]

The physical solution \(\Psi\) is the basis to construct the generalized Fourier maps for the absolutely continuous subspace of \(H\). We observe that in the case when \(V = 0, f(k, x) = e^{ikx} I\) and then, it follows from \((2.17)\) and \((2.22)\) that
\[ J_0(k) = B - ik A, \quad J_0^{-1}(k) = (B - ik A)^{-1}, \quad (2.27) \]
\[ S_0(k) = -(B + ik A) (B - ik A)^{-1}, \quad (2.28) \]

where the zero index refers to the zero potential. In the diagonal form \(\tilde{A}\) and \(\tilde{B}\) given by \((2.6)\), the Jost and the scattering matrices take the form
\[ \tilde{J}_0(k) = \tilde{B} - ik \tilde{A} = \text{diag} \left[ \cos \theta_1 + ik \sin \theta_1, ..., \cos \theta_{n_M} + ik \sin \theta_{n_M}, -I_{n_D}, i k I_{n_N} \right], \quad (2.29) \]
\[ \tilde{J}_0^{-1}(k) = \text{diag} \left[ \left( \cos \theta_1 + ik \sin \theta_1 \right)^{-1}, ..., \left( \cos \theta_{n_M} + ik \sin \theta_{n_M} \right)^{-1}, -I_{n_D}, (ik)^{-1} I_{n_N} \right], \quad (2.30) \]
\[ \tilde{S}_0(k) = -\tilde{J}_0(-k) \tilde{J}_0(k)^{-1} = \text{diag} \left[ -\cos \theta_1 + ik \sin \theta_1, ..., -\cos \theta_{n_M} + ik \sin \theta_{n_M}, -I_{n_D}, I_{n_N} \right], \quad (2.31) \]

Furthermore, \(J_0(k)^{-1}\) is related to the corresponding \(\tilde{J}_0^{-1}(k)\) by the relation \((5)\)
\[ J_0(k)^{-1} = T_2^{-1} M T_1^{-1} \tilde{J}_0^{-1}(k) M^\dagger, \quad (2.32) \]

where \(M\) is a unitary matrix and \(T_1, T_2\) are invertible. Similarly,
\[ S_0(k) = M \tilde{S}_0(k) M^\dagger. \quad (2.33) \]
2.3 Generalized Fourier transforms.

We now turn to the definition of the generalized Fourier transforms [7] and [15]. Using the physical solution $\Psi$ we define

$$(F^\pm \psi)(k) = \sqrt{\frac{1}{2\pi}} \int_0^\infty (\Psi(\mp k, x))^* \psi(x) \, dx,$$

for $\psi \in L^2 \cap L^1$. For any Borel set $O$ let $E(O)$ be the spectral projector of $H$ for $O$. Then, (7, 15)

$$\|F^\pm \psi\|_{L^2} = \|E(\mathbb{R}^+)\psi\|_{L^2}. \quad (2.34)$$

Thus, $F^\pm$ extend to bounded operators on $L^2$ that we also denote by $F^\pm$.

The following spectral result for $H$ are proven in [7], [15].

**Proposition 2.4.** Suppose that the potential $V$ satisfies (1.2) and (2.2). Then, the Hamiltonian $H$ has no positive bound states, and the negative spectrum of $H$ consists of isolated bound states of multiplicity smaller or equal than $n$, that can accumulate only at zero. Furthermore, $H$ has no singular continuous spectrum and its absolutely continuous spectrum is given by $(0, \infty)$. The generalized Fourier maps $F^\pm$ are partially isometric with initial subspace $\mathcal{H}_{ac}(H)$ and final subspace $L^2$. Moreover, the adjoint operators are given by

$$(F^\pm)^\dagger \varphi(x) = \sqrt{\frac{1}{2\pi}} \int_0^\infty \Psi(\mp k, x) \varphi(k) \, dk,$$

for $\varphi \in L^2 \cap L^1$. Furthermore,

$$F^\pm H (F^\pm)^\dagger = \mathcal{M} \quad (2.35)$$

where $\mathcal{M}$ is the operator of multiplication by $k^2$. If, in addition, $V \in L^1$, there is no bound state at $k = 0$ and the number of bounded states of $H$ is finite.

We observe that in particular (2.34) implies that

$$F^\pm e^{-itH} (F^\pm)^\dagger = e^{-it\mathcal{M}}. \quad (2.36)$$

Note that by (2.34) $(F^\pm)^\dagger F^\pm$ is the orthogonal projector onto $\mathcal{H}_{ac}(H)$. Since the singular continuous spectrum is absent we get

$$(F^\pm)^\dagger F^\pm = P_c, \quad (2.37)$$

with $P_c$ the projector onto the continuous subspace of $H$. Therefore, from (2.36) and (2.37) it follows

$$e^{-itH} P_c = (F^\pm)^\dagger e^{-it\mathcal{M}} F^\pm. \quad (2.38)$$

Equation (2.38) is the starting point for the proof of our main results.

From (2.38) (with the negative sign), for $\psi \in \mathcal{S}$ ($\mathcal{S}$ denoting the Schwartz class) we have

$$e^{-itH} P_c \psi = (2\pi)^{-1} \int_0^\infty \Psi(k, x) e^{-itk^2} \left( \int_0^\infty (\Psi(k, y))^\dagger \psi(y) \, dy \right) \, dk.$$

Using the definition (2.23) of $\Psi$, and as by (2.23) $S(k) S^\dagger(k) = I$, $S^\dagger(-k) = S(k)$, for $k \in \mathbb{R}$, we get

$$e^{-itH} P_c \psi = (2\pi)^{-1} \int_0^\infty T(x,y) \psi(y) \, dy, \quad (2.39)$$

where

$$T(x,y) = \int_{-\infty}^\infty e^{-itk^2} \left( f(-k,x) f(-k,y))^\dagger + f(k,x) S(k) (f(-k,y))^\dagger \right) \, dk. \quad (2.40)$$

2.4 The Fourier transform of $S(k) - S_\infty$

We define below a set $\kappa_j$, $j = 1, ..., l$ of $l$ distinct positive numbers related to the bound-state energies $-\kappa_j^2$, $j = 1, ..., l$, and a set $M_j$, $j = 1, ..., l$ of constant $n \times n$ matrices related to the normalization of matrix-valued bound-state eigenfunctions. These positive numbers and matrices were first introduced by [2] in the case of Dirichlet boundary condition, and later by [20] for general boundary condition (see also [8], [7]). As we mentioned in Proposition 2.2 the Jost matrix $J(k)$ is analytic for $k \in \mathbb{C}^+$, continuous for $k \in \mathbb{C}^+$ and invertible for $k \in \mathbb{R} \setminus \{0\}$. Further, it is proved in [2] and [20] (see also [9], [7]), that...
the determinant $\det[J(k)]$ is nonzero in $\mathbb{C}^+$ except perhaps at a finite number of distinct $k$-values on the positive imaginary axis, that we denote by $\kappa_j, j = 1, ..., l$. In the case that $\det[J(k)]$ has no zeros in $\mathbb{C}^+$ we take $l = 0$. We use $m_j$ to denote the multiplicity of the zero of $\det[J(k)]$ at $k = i\kappa_j, j = 1, ..., l$. Each $m_j$ satisfies $1 \leq m_j \leq n, j = 1, ..., l$. The bound-state energies of the Schrödinger operator $H$ are given by $-\kappa_j^2$, and they have multiplicity $m_j, j = 1, ..., l$. Moreover, we denote by $\text{ker}[J(i\kappa_j)^\dagger]$ the kernel of the $n \times n$ constant matrix $J(i\kappa_j)^\dagger$, and we designate by $P_j$ the orthogonal projection matrix onto $\text{ker}[J(i\kappa_j)^\dagger]$, for $j = 1, ..., l$.

Let us define the constant $n \times n$ matrices $A_j, B_j$, and $M_j$ as follows,

$$A_j := \int_0^\infty dx f(i\kappa_j, x)^\dagger f(i\kappa_j, x), \quad j = 1, \ldots, l,$$

$$B_j := (I - P_j) + P_j A_j P_j, \quad j = 1, \ldots, l,$$

$$M_j := B_j^{-1/2} P_j, \quad j = 1, \ldots, l.$$

The matrices $B_j, j = 1, \ldots, l$ are invertible. The normalized matrix-valued bound-state eigenfunctions are given by,

$$\Psi_j(x) := f(i\kappa_j, x) M_j, \quad j = 1, \ldots, l.$$

Let us denote by $F_s$ the Fourier transform of $(2\pi)^{-1/2}(S(k) - S_\infty)$, that is

$$F_s(y) = \frac{1}{2\pi} \int_{-\infty}^\infty [S(k) - S_\infty] e^{iky} dk, \quad y \in \mathbb{R}.$$ 

where (5),

$$S_\infty := \lim_{|k|\to\infty} S(k) = \lim_{|k|\to\infty} S_0(k) = MZ_0 M^\dagger, \quad (2.42)$$

$Z_0 := \text{diag}[I_{nM}, -I_{nD}, I_{nN}], \text{the numbers} \ n_M, n_D, n_N \text{are defined below (2.21) and} \ M \text{is the unitary} \ M \text{in (2.8). Here we denote by} \ I_m \text{the} \ m \times m \text{identity matrix. We define}$

$$F(y) := F_s(y) + \sum_{j=1}^l M_j^2 e^{-ny}, \quad y \in \mathbb{R}^+.$$ 

In the case when $l = 0$ we take $F = F_s$. It is proved in [2], [6] and [11] that,

$$F \in L^1 (0, \infty) \cap L^\infty (0, \infty). \quad (2.43)$$

Moreover, by [2], [6] and [7], the function $K(x, y)$ satisfies the Marchenko equation

$$K(x, y) + F(x + y) + \int_x^\infty K(x, t) F(t + y) dt = 0, \quad 0 \leq x < y. \quad (2.44)$$

We now prove the following result concerning $F_s$.

**Theorem 2.5.** Suppose that the potential $V$ satisfies (2.22) and (2.23). Then,

$$F_s \in L^1 (\mathbb{R}). \quad (2.45)$$

**Remark 2.6.** We observe that this result is known in the case of the Dirichlet boundary condition (see Theorem 5.6.2 on page 137 of [2]). In what follows, we aim to extend (2.45) to the case of the most general boundary condition (2.24).

In order to prove Theorem 2.5, we prepare some results. We begin by proving the following adaptation of Lemma 5.6.2 on page 132 of [2] to our settings:

**Proposition 2.7.** Suppose that the $n \times n$ matrix $P_0$ satisfies $P_0 J^\dagger(0) = 0$. Then, the matrix $k^{-1} P_0 J(k)^\dagger$ fulfills

$$k^{-1} P_0 J(k)^\dagger = FP_0 G + iP_0 A^\dagger, \quad (2.46)$$

where $G(t) \in L^1 (\mathbb{R})$ and it is equal to zero for $t > 0$. 

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Proof. Integrating the Marchenko equation (2.44) on \((z, \infty)\), with \(z \geq x \geq 0\), we have

\[
\int_{z}^{\infty} K(x, y) \, dy + \int_{z+x}^{\infty} F(y) \, dy + \int_{x}^{\infty} K(x, t) \int_{x}^{\infty} F(y) \, dy \, dt = 0. 
\]  
(2.47)

Evaluating in \(x = 0\) we get

\[
K_1(z) + \int_{z}^{\infty} F(y) \, dy + \int_{0}^{\infty} K(0, t) \left( \int_{x}^{\infty} F(y) \, dy \right) \, dt = 0,
\]  
(2.48)

where we denote

\[
K_1(z) = \int_{z}^{\infty} K(0, y) \, dy.
\]

Moreover, differentiating (2.47) with respect to \(x\) (this is possible due to (2.19), (2.20) and (2.43)) and taking \(x = 0\) we have

\[
K_2(z) - F(z) - K(0, 0) \int_{z}^{\infty} F(y) \, dy + \int_{0}^{\infty} K_x(0, t) \int_{z}^{\infty} F(y) \, dy \, dt = 0,
\]  
(2.49)

with

\[
K_2(z) = \int_{z}^{\infty} K_x(0, y) \, dy.
\]

Note that \(K_1\) and \(K_2\) are well-defined due to (2.19) and (2.21). Observe that \(K'_1(t) = -K(0, t)\) and \(K'_2(t) = -K_x(0, t)\). Then, integrating by parts in the last integral in the left-hand side of (2.48) and (2.49) we get,

\[
K_1(z) + (I + K_1(0)) \int_{z}^{\infty} F(y) \, dy - \int_{0}^{\infty} K(t) \left( F(z + t) + F(0) \right) \, dt = 0,
\]  
(2.50)

\[
K_2(z) - F(z) - (K(0, 0) - K_2(0)) \int_{z}^{\infty} F(y) \, dy - \int_{0}^{\infty} K_2(t) \left( F(z + t) + F(0) \right) \, dt = 0.
\]  
(2.51)

Multiplying from the left (2.50) by \(B^\dagger\) and (2.51) by \(A^\dagger\) and considering the difference between the resulting equations we get

\[
B^\dagger K_1(z) - A^\dagger K_2(z) + (B^\dagger (I + K_1(0)) + A^\dagger (K(0, 0) - K_2(0))) \int_{z}^{\infty} F(y) \, dy = -A^\dagger F(z) + B^\dagger \int_{0}^{\infty} K(t) \left( F(z + t) + F(0) \right) \, dt - A^\dagger \int_{0}^{\infty} K_2(t) \left( F(z + t) + F(0) \right) \, dt.
\]  
(2.52)

From the representation (2.21) for \(J(k)^\dagger\) we see that

\[
J(0)^\dagger = B^\dagger (I + K_1(0)) + A^\dagger (K(0, 0) - K_2(0)).
\]  
(2.53)

By (2.50) we get

\[
F(z) = -K(0, z) - \int_{0}^{\infty} K(0, t) \left( F(z + t) + F(0) \right) \, dt.
\]  
(2.54)

Hence, from (2.52) via (2.53) and (2.54), we deduce

\[
B^\dagger K_1(z) - A^\dagger K_2(z) + J(0)^\dagger \int_{z}^{\infty} F(y) \, dy = A^\dagger K(0, z) + \int_{0}^{\infty} \left( A^\dagger K(0, t) + B^\dagger K_1(t) - A^\dagger K_2(t) \right) \left( F(z + t) + F(0) \right) \, dt.
\]

Letting act \(P_0\) from the left on the last equation and using that by assumption \(P_0 J^\dagger(0) = 0\) we get

\[
\mathcal{K}(z) = \int_{0}^{\infty} \mathcal{K}(t) \left( F(z + t) + F(0) \right) \, dt + P_0 \left( A^\dagger K(0, z) + \int_{0}^{\infty} A^\dagger K(0, t) \left( F(z + t) + F(0) \right) \, dt \right),
\]  
(2.55)

where we denote

\[
\mathcal{K}(z) = P_0 \left( B^\dagger K_1(z) - A^\dagger K_2(z) \right).
\]

From the estimates (2.19), (2.20) for \(K\) it follows that \(K(0, z) \in L^1 (0, \infty) \cap L^\infty (0, \infty)\) and \(K_1, K_2 \in L^\infty (0, \infty)\). In particular,

\[
\mathcal{K} \in L^\infty (0, \infty).
\]  
(2.56)
Let us prove that \( K \in L^1(0, \infty) \). We proceed similarly to the proof of Lemma 3.3.2 on page 72 of [2]. Since \( K(0, z) \in L^1(0, \infty) \cap L^{\infty}(0, \infty) \) and by (2.55) \( F \in L^1(0, \infty) \), we see that the second term in the right-hand side of (2.55) belongs to \( L^1(0, \infty) \cap L^{\infty}(0, \infty) \). Moreover, by the density of the Schwartz class \( S \) in \( L^1(0, \infty) \), we can find \( \hat{F} \in S \) such that

\[
\left\| F - \hat{F} \right\|_{L^1(0,\infty)} < 1.
\]  

(2.57)

Then, we write (2.55) as

\[
\mathcal{K}(z) + \int_0^\infty \mathcal{K}(t) F_1(z + t) \, dt = g(z),
\]

(2.58)

where \( F_1 = \hat{F} - F \) and \( g \in L^1(0, \infty) \cap L^{\infty}(0, \infty) \). Here we used that by (2.50) and \( \hat{F} \in S \),

\[
\left\| \int_0^\infty \mathcal{K}(t) \hat{F}(z + t) \, dt \right\|_{L^1} + \left\| \int_0^\infty \mathcal{K}(t) F(z + t) \, dt \right\|_{L^\infty} \leq C \left\| \mathcal{K} \right\|_{L^\infty} \int_z^\infty (1 + t) \left| \hat{F}(t) \right| \, dt \leq C.
\]

By (2.57) \( \| F_1 \|_{L^1} < 1 \). Then, by the method of successive approximations we see that there is a unique solution \( \mathcal{K}_1 \in L^1 \cap L^{\infty} \) to equation (2.58). Since \( \mathcal{K} \) satisfies (2.55) and (2.58), we prove that \( \mathcal{K} \equiv \mathcal{K}_1 \). Therefore, \( \mathcal{K} \in L^1(0, \infty) \). Let

\[
\hat{\mathcal{K}}(k) = \int_0^\infty e^{-ikz} \mathcal{K}(z) \, dz.
\]

(2.59)

Integrating by parts in the last integral we get

\[
\hat{\mathcal{K}}(k) = \frac{1}{ik} P_0 \left( (B^\dagger K_1(0) - A^\dagger K_2(0)) - \int_0^\infty e^{-ikz} (B^\dagger K(0, z) - A^\dagger K_x(0, z)) \, dz \right).
\]

Then, by using (2.21) and (2.59), since \( P_0 J(0)^\dagger = 0 \) we get

\[
\hat{\mathcal{K}}(k) = \frac{1}{ik} P_0 \left( J(0)^\dagger - J(k)^\dagger \right) + P_0 A^\dagger = -\frac{1}{ik} P_0 J(0)^\dagger + P_0 A^\dagger.
\]

(2.60)

Denoting \( G(t) := \mathcal{K}(-t), t < 0 \) and \( G(t) = 0, t > 0 \), we obtain (2.40) from (2.59) and (2.60). This completes the proof. 

In order to present our next result, we need the sharp small energy behaviour of \( J(k) \), obtained in [3]. We have

\[
J(k) = \mathcal{G} P_2^{-1} \begin{bmatrix} kA_1 + o(k) & kB_1 + o(k) \\ kC_1 + o(k) & D_0 + o(1) \end{bmatrix} P_1 \mathcal{G}^{-1},
\]

where the matrices \( A_1, D_0, G, P_1, P_2 \) are invertible. Let us introduce the notation

\[
\alpha := \mathcal{G} P_2^{-1} \quad \text{and} \quad \beta := P_1 \mathcal{G}^{-1}.
\]

Then, it follows from (2.61) that

\[
J(0) = \alpha \begin{bmatrix} 0 & 0 \\ 0 & D_0 \end{bmatrix} \beta
\]

and

\[
J(0)^\dagger = \beta^\dagger \begin{bmatrix} 0 & 0 \\ 0 & P_0^\dagger \end{bmatrix} \alpha^\dagger.
\]

(2.62)

We let

\[
P_0 = \beta^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} (\beta^\dagger)^{-1}.
\]

(2.63)

Since \( P_0 J(0)^\dagger = 0 \), \( P_0 \) satisfies the assumptions of Proposition 2.4. We observe that \( P_0^\dagger \) is a projection onto the null space of \( J(0) \). Using this operator \( P_0 \) we define

\[
D(k) = \left( I - P_0 + \frac{1}{ik} P_0 \right) J(k)^\dagger.
\]

(2.64)

Let us show that this matrix is non-singular. We prove the following:

**Proposition 2.8.** For all \( k \in \mathbb{R} \) we have

\[
det D(k) \neq 0.
\]

(2.65)
Proof. Since \( P_0^2 = P_0 \), the equation \((I - P_0 + \frac{1}{ik} P_0) \psi = 0\), for \( k \neq 0 \), implies that both \((I - P_0) \psi = 0\) and \( P_0 \psi = 0\) are satisfied. It follows that \( \det (I - P_0 + \frac{1}{ik} P_0) \neq 0\), for \( k \neq 0 \). Moreover, using Proposition 2.10 we have

\[
\det D(k) = \det \left( I - P_0 + \frac{1}{ik} P_0 \right) \det J(k) \neq 0, \tag{2.66}
\]

for \( k \neq 0 \). Thus, we need to consider \( D(0) \). Using (2.61) and (2.63) we see that

\[
\frac{1}{ik} P_0 J(k) \uparrow = \beta \uparrow \left[ i^{-1} A_1^\dagger + o(1) \begin{bmatrix} i^{-1} C_1^\dagger + o(1) \end{bmatrix} \alpha \uparrow.\right.
\]

Then,

\[
\lim_{k \to 0} \frac{1}{ik} P_0 J(k) \uparrow = \beta \uparrow \begin{bmatrix} i^{-1} A_1^\dagger & i^{-1} C_1^\dagger \end{bmatrix} \alpha \uparrow. \tag{2.67}
\]

Moreover, from (2.62) we calculate

\[
\lim_{k \to 0} (I - P_0) J(k) \uparrow = (I - P_0) J(0) \uparrow = \beta \uparrow \begin{bmatrix} 0 & 0 \\ 0 & D_0^\dagger \end{bmatrix} \alpha \uparrow. \tag{2.68}
\]

Therefore, by (2.67), (2.68) we get

\[
D(0) = \lim_{k \to 0} \left( I - P_0 + \frac{1}{ik} P_0 \right) J(k) \uparrow = \beta \uparrow \begin{bmatrix} i^{-1} A_1^\dagger & i^{-1} C_1^\dagger \end{bmatrix} \alpha \uparrow.
\]

Hence,

\[
\det D(0) = \det \left( \alpha \uparrow \right) \det \left( \beta \uparrow \right) \det \left( i^{-1} A_1^\dagger \right) \det D_0^\dagger.
\]

Since \( \alpha, \beta, A_1, D_0 \) are invertible, we show that \( \det D(0) \neq 0 \).

This relation together with (2.61) imply (2.65). \( \square \)

We prepare the following remark. We denote by \( f * g \) the convolution of \( f \) and \( g \),

\[
(f * g)(x) := \int f(x - y) g(y) \, dy.
\]

Remark 2.9. Suppose that \( f, g \in L^1(\mathbb{R}) \). Then, since for \( f, g \in L^1(\mathbb{R}) \), \((\mathcal{F} f)(\mathcal{F} g) = \mathcal{F} (f \ast g) \), and \( f \ast g \in L^1(\mathbb{R}) \), we have that \((\mathcal{F} f)(\mathcal{F} g) \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \). That is to say, \( \mathcal{F} \left( L^1(\mathbb{R}) \right) \) is closed under products.

We also need the Wiener-Lévy theorem (see 6.1.8 on page 262 of [11]):

**Proposition 2.10** (Wiener-Lévy theorem). Suppose that \( f \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \). Let \( F \) be an analytic function on an open set of \( \mathbb{C} \) which contains the range of \( f \). Then \( F \circ f \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \).

In fact, what we actually use is the following corollary of the Wiener-Lévy theorem:

**Corollary 2.11.** Suppose that \( f \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \). Given \( l \in \mathbb{C} \), if \( f(x) \neq l \), for all \( x \in \mathbb{R} \), then \( \frac{f}{f - l} \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \).

**Proof.** In the Wiener-Lévy theorem take \( F(z) = (z - l)^{-1} \). Then, \((f - l)^{-1} \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \) and by Remark 2.9 \( f(f - l)^{-1} \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \). \( \square \)

Finally, we present a local Wiener theorem (see Theorem 229 on page 290 of [16]):

**Proposition 2.12.** Suppose that \( f, g \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \), that \( g(x) \neq 0 \), for all \( x \in \mathbb{R} \), and that \( f(x) = 0 \), for \( |x| > \lambda > 0 \). Then, \( \frac{f}{g} \in \mathcal{F} \left( L^1(\mathbb{R}) \right) \).

We have now all the necessary ingredients to prove (2.45).
Proof of Theorem 2.5. We depart from the definition of the scattering matrix $S(k)$. Let $\chi \in C^\infty(\mathbb{R}), 0 \leq \chi \leq 1$, be such that $\chi(k) = 1$, for $|k| \leq 1$ and $\chi(k) = 0, for |k| \geq 2$. For $a > 0$, we set $\chi_a(k) := \chi(\frac{k}{a}), k \in \mathbb{R}$. We consider $S(-k)$. Using (2.21) we decompose

$$S(-k) - S_\infty = S_1(k) + S_2(k),$$

where

$$S_1(k) := -\chi_a(k) \left( (J(k)^\dagger)^{-1} \chi_{2a}(k) J(-k)^\dagger + S_\infty^\dagger \right),$$

$$S_2(k) := -(1 - \chi_a(k)) \left( (J(k)^\dagger)^{-1} J(-k)^\dagger + S_\infty^\dagger \right).$$

Using (2.64) we write

$$S_1(k) = -\chi_a(k) \left( (D(k)^{-1} (I - P_0 + \frac{1}{ik} P_0) (I - P_0 - \frac{1}{ik} P_0)^{-1} \chi_{2a}(k) D(-k) + S_\infty^\dagger \right)$$

$$= -\chi_a(k) \left( (D(k)^{-1} (I - 2P_0) \chi_{2a}(k) D(-k) + S_\infty^\dagger \right).$$

Observe that

$$\mathcal{F}^{-1} \chi_a, \mathcal{F}^{-1} (k \chi_a(k)) \in L^1 \cap L^2, a > 0.$$ 

Moreover by Remark 2.9 we see that $\mathcal{F} \left( L^1(\mathbb{R}) \right)$ is closed by products. Therefore, it follows from (2.19), (2.20) and (2.21) that

$$\chi_a(k) J(k)^\dagger \in \mathcal{F} \left( L^1(\mathbb{R}) \right), a > 0.$$ 

Hence, from Proposition 2.7 we show that all the elements of the matrix $\chi_{2a}(k) D(-k)$ belong to $\mathcal{F} \left( L^1(\mathbb{R}) \right)$. On the other hand, all the entries of the matrix $\chi_a(k) (D(k))^{-1}$ can be represented as

$$\frac{\chi_a(k) L(k)}{\det D(k)},$$

with $L \in \mathcal{F} \left( L^1(\mathbb{R}) \right)$. Due to the cut-off function $\chi_a$, we express the last relation as

$$\frac{\chi_a(k) L(k)}{\det D(k)} = \frac{\chi_a(k) L(k)}{\chi_{3a}(k) \det D(k) + (1 - \chi_{3a}(k)) G(k)^\dagger},$$

with any $G \in \mathcal{F} \left( L^1(\mathbb{R}) \right), G(k) \neq 0$, for all $k \in \mathbb{R}$ (for example, taking any non-vanishing function from the Schwartz class). By Proposition 2.7 and 2.73, $\chi_{3a}(k) \det D(k) \in \mathcal{F} \left( L^1(\mathbb{R}) \right)$. By 2.65, $\det D(k) \neq 0$, for all $k \in \mathbb{R}$. Then, on the support of $\chi_{3a}$, the function $\chi_{3a}(k) \det D(k)$ has a definite sign. We take $G$ with a definite sign such that $\chi_{3a}(k) \det D(k) + (1 - \chi_{3a}(k)) G(k) \neq 0$, for all $k \in \mathbb{R}$. Then, each element of the matrix $\chi_a(k) (D(k))^{-1}$ is of the form $\chi_a(k) \frac{f}{g}$, with $f, g \in \mathcal{F} \left( L^1(\mathbb{R}) \right)$, such that $g(k) \neq 0$, for all $k \in \mathbb{R}$, and $f(k) = 0$, for all $|k| \geq 2a$. By Proposition 2.12, the entries of $\chi_a(k) S_\infty$ are functions in $\mathcal{F} \left( L^1(\mathbb{R}) \right)$, we conclude that $S_1(k)$ is a matrix which elements can be represented as Fourier transform of functions in $L^1(\mathbb{R})$. Next, we consider $S_2(k)$. We put $a > 2$ in (2.71). Using (2.21) we have

$$J(k)^\dagger = (B^\dagger + i k A^\dagger) \left( I + (B^\dagger + i k A^\dagger)^{-1} (A^\dagger K(0,0) + G_1(k)) \right),$$

where the elements of the matrix $G_1$ belong to $\mathcal{F} \left( L^1(\mathbb{R}) \right)$. By (2.32)

$$(B^\dagger + i k A^\dagger)^{-1} = M \left( J_0(k)^\dagger \right)^{-1} (T_1^\dagger)^{-1} M^\dagger \left( T_2^\dagger \right)^{-1},$$

where $J_0^{-1}(k)$ is given by the diagonal matrix (2.30). Then, as by Proposition 2.2, $J(k)^\dagger$ is invertible for $k \neq 0$, we see that

$$\det \left( I + (B^\dagger + i k A^\dagger)^{-1} (A^\dagger K(0,0) + G_1(k)) \right) \neq 0, k \in \mathbb{R} \backslash \{0\}.$$ 

Therefore, using (2.28) and (2.74) we decompose

$$- (1 - \chi_a(k)) \left( J(k)^\dagger \right)^{-1} J(-k)^\dagger = (1 - \chi_a(k)) \left( I + MJ_\chi(k) \left( T_1^\dagger \right)^{-1} M^\dagger \left( T_2^\dagger \right)^{-1} (A^\dagger K(0,0) + G_1(k)) \right)^{-1} \times S_0(k)^\dagger \left( I + MJ_\chi(-k) \left( T_1^\dagger \right)^{-1} M^\dagger \left( T_2^\dagger \right)^{-1} (A^\dagger K(0,0) + G_1(-k)) \right) ,$$

(2.76)
with $J_\chi (k) = \text{diag} \left[ (\cos \theta_1 - ik \sin \theta_1)^{-1}, \ldots, (\cos \theta_{n_M} - ik \sin \theta_{n_M})^{-1}, -I_{n_D}, -(ik)^{-1} (1 - \chi (k)) I_{n_N} \right]$, 

where we can introduce the functions $1 - \chi (k)$ in the entries of $\left( J_0 (k)^1 \right)^{-1}$ corresponding to the Neumann boundary conditions without modifying the equality thanks to the cut-off function $1 - \chi_a (k)$ (we put $a > 2$). We now observe that 

$$\mathcal{F}^{-1} (\cos \theta_j - ik \sin \theta_j)^{-1} \in L^1 (\mathbb{R}), \ j = 1, \ldots, n_M,$$

and 

$$\mathcal{F}^{-1} \left( \frac{1 - \chi (k)}{k} \right) \in L^1 (\mathbb{R}),$$

where the first relation is due to the Jordan lemma and contour integration and the second one follows by integration by parts. Then, we show that the entries of $MJ_\chi (k) \left( T_1^1 \right)^{-1} M^1 \left( T_2^1 \right)^{-1} G_1 (k)$ belong to $\mathcal{F} \left( L^1 (\mathbb{R}) \right)$. Using (2.8) we calculate 

$$MJ_\chi (k) \left( T_1^1 \right)^{-1} M^1 \left( T_2^1 \right)^{-1} A^1 K (0,0) = MJ_\chi (k) \tilde{A} M^1 K (0,0).$$

Since $J_\chi (k) \tilde{A} = \text{diag} \left[ (\cos \theta_1 - ik \sin \theta_1)^{-1}, \ldots, (\cos \theta_{n_M} - ik \sin \theta_{n_M})^{-1}, 0, -(ik)^{-1} (1 - \chi (k)) I_{n_N} \right]$, 

it also follows from (2.77) and (2.78) that the elements of $MJ_\chi (k) \left( T_1^1 \right)^{-1} M^1 \left( T_2^1 \right)^{-1} A^1 K (0,0)$ belong to $\mathcal{F} \left( L^1 (\mathbb{R}) \right)$. Thus, we can write 

$$I + MJ_\chi (k) \left( T_1^1 \right)^{-1} M^1 \left( T_2^1 \right)^{-1} A^1 K (0,0) + G_1 (k)) = I + G_2 (k),$$

where the elements of $G_2$ are in $\mathcal{F} \left( L^1 (\mathbb{R}) \right)$. Then, from (2.79) we get 

$$- (1 - \chi_a (k)) \left( J (k)^1 \right)^{-1} J (-k)^1 = (1 - \chi_a (k)) \left( I + G_2 (k) \right)^{-1} S_0 (k)^1 \left( I + G_2 (-k) \right).$$

Since $\mathcal{F} \left( L^1 (\mathbb{R}) \right)$ is closed by products, we write 

$$\det (I + G_2 (k)) = 1 + g_2 (k),$$

with $g_2 \in \mathcal{F} \left( L^1 (\mathbb{R}) \right)$. We observe that on the support of $1 - \chi_a (k)$ we can represent 

$$1 + g_2 (k) = 1 + (1 - \chi_a/2 (k)) g_2 (k).$$

By Riemann–Lebesgue lemma $g_2 (k) \to 0$, as $|k| \to \infty$. Then, we can take $a > 2$ sufficiently large in a way that $1 + (1 - \chi_a/2 (k)) g_2 (k) \neq 0$, for all $k \in \mathbb{R}$. By (2.77) $(1 - \chi_a/2) g_2 \in \mathcal{F} \left( L^1 (\mathbb{R}) \right)$. Then, by Corollary (2.11) we show that 

$$\left( 1 + (1 - \chi_a/2 (k)) g_2 (k) \right)^{-1} = 1 + g_3 (k),$$

where $g_3 \in \mathcal{F} \left( L^1 (\mathbb{R}) \right)$. Hence, from (2.80), (2.81) and (2.82) it follows that 

$$\frac{1}{\det (I + G_2 (k))} = 1 + g_3 (k),$$

for all $|k| \geq a$. Using the last expression in (2.77) we get 

$$- (1 - \chi_a (k)) \left( J (k)^1 \right)^{-1} J (-k)^1 = (1 - \chi_a (k)) \left( I + G_3 (k) \right) S_0 (k)^1 \left( I + G_2 (-k) \right),$$

where the elements of $G_3$ are in $\mathcal{F} \left( L^1 (\mathbb{R}) \right)$. From (2.31), (2.38), via (2.77) and (2.78) we show that all the elements of the matrix $(1 - \chi_a (k)) \left( S_0 (k)^1 - S_{\infty}^1 \right)$ belong to $\mathcal{F} \left( L^1 (\mathbb{R}) \right)$. Therefore, from (2.83) we conclude that the entries of $S_2 (k)$ are in $\mathcal{F} \left( L^1 (\mathbb{R}) \right)$. Finally, from (2.69) we obtain (2.45).
3 The $L^p - L^{p'}$ estimate for the matrix Schrödinger equation.

This section is devoted to the proof of the $L^p - L^{p'}$ and Strichartz estimates for the matrix Schrödinger equation. We begin by proving the following $L^1 - L^\infty$ estimate.

**Proposition 3.1.** Suppose that the potential $V$ satisfies \( L_2 \) and \( L_3 \). Then, the estimates

\[
\|e^{-itH}P_c\|_{\mathcal{B}(L^1, L^\infty)} \leq \frac{C}{\sqrt{|t|}},
\]

and

\[
\|e^{-itH}P_c\|_{\mathcal{B}(W_1, W_1)} \leq \frac{C}{\sqrt{|t|}},
\]

are true for all \( t \in \mathbb{R} \setminus \{0\} \).

**Proof.** We depart from the spectral representation (2.39, 2.40) for \( e^{-itH}P_c \). We decompose \( T(x, y) \) as follows

\[
T(x, y) = \sum_{j=0}^{5} T_j(x, y),
\]

with

\[
\begin{align*}
T_0 &:= \int_{-\infty}^{\infty} e^{-itk^2} e^{-ik(x+y)} dk + S_\infty \int_{-\infty}^{\infty} e^{-itk^2} e^{ik(x+y)} dk, \\
T_1 &:= \int_{-\infty}^{\infty} e^{-itk^2} \left( e^{-ikx} d(-k, y)^\dagger + e^{-iky} d(k, x) \right) dk \\
&\quad + \int_{-\infty}^{\infty} e^{-itk^2} \left( e^{-ikx} S_\infty d(k, y)^\dagger + e^{-iky} d(-k, x) S_\infty \right) dk, \\
T_2 &:= \int_{-\infty}^{\infty} e^{-itk^2} \left( d(-k, x) d(-k, y)^\dagger + d(k, x) S_\infty d(-k, y)^\dagger \right) dk, \\
T_3 &:= \int_{-\infty}^{\infty} e^{-itk^2} e^{ik(x+y)} T(k) dk, \\
T_4 &:= \int_{-\infty}^{\infty} e^{-itk^2} \left( e^{ikx} T(k) d(-k, y)^\dagger + e^{iky} d(k, x) T(k) \right) dk, \\
T_5 &:= \int_{-\infty}^{\infty} e^{-itk^2} d(k, x) T(k) d(-k, y)^\dagger dk,
\end{align*}
\]

where we denote,

\[
d(k, x) := f(k, x) - e^{ikx},
\]

and

\[
T(k) := S(k) - S_\infty.
\]

Recall that,

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} e^{-ikz} dk = \frac{e^{iz^2/4t}}{\sqrt{4\pi it}},
\]

with the Fourier transform understood in the sense of distributions. Using (3.6) we have that

\[
T_0 = \sqrt{\frac{\pi}{it}} \left( e^{i(x-y)^2/4t} + e^{i(x+y)^2/4t} S_\infty \right).
\]

We observe that \( T_0 + T_3 \) corresponds to thefree evolution \( V \equiv 0 \). Moreover, if in the diagonal representation (2.19) there are only Dirichlet and Neumann boundary conditions, and \( V \equiv 0 \) it follows from (2.28) that \( T \equiv 0 \) and then, \( T_3 \equiv 0 \) in this case. By (2.19), for fixed \( x \geq 0 \), \( K(x, \cdot) \in L^2(x, \infty) \). Then, using (2.18) we get

\[
d(k, x) = \int_{-\infty}^{\infty} e^{ikz} K(x, z) dz,
\]
(recall that $K(x, z) = 0$, for $z < x$). Hence, by the convolution theorem for the Fourier transform and \((3.8)\) we obtain
\[
\mathcal{T}_1 = \frac{(2\pi)}{4\pi i t} \left( \int_{-\infty}^{\infty} e^{i(x-z)^2/4t} K(y, z) dz + \int_{-\infty}^{\infty} e^{i(y-z)^2/4t} K(x, z) dz \right.
\]
\[
+ \int_{-\infty}^{\infty} e^{i(y+z)^2/4t} K(x, z) S_\infty dz + \int_{-\infty}^{\infty} e^{i(x+z)^2/4t} S_\infty K(y, z) dz \bigg). 
\]

Hence, from \((2.10)\) it follows that
\[
|\mathcal{T}_1| \leq C \frac{1}{\sqrt{|t|}}. 
\]

Moreover, differentiating \((3.8)\) with respect to $x$, noting that $\partial_x e^{i(x\pm z)^2/4t} = \pm \partial_x e^{i(x\pm z)^2/4t}$ and using \((3.13), (3.14))$, we prove that
\[
|\partial_x \mathcal{T}_1| \leq C \frac{1}{\sqrt{|t|}}. 
\]

Next, we consider $\mathcal{T}_2$. By Parseval’s identity and the convolution theorem, via \((3.8)\) we get
\[
\mathcal{T}_2 = \frac{2\pi}{4\pi i t} \left( \int_{-\infty}^{\infty} e^{i(x+z)^2/4t} K(x, -z_1) K(y, z_1 - z_2) dz_2 dz_1 \right)
\]
\[
+ \int_{-\infty}^{\infty} e^{i(x+z)^2/4t} K(x, z_1) S_\infty K(y, z_1 - z_2) dz_2 dz_1 \bigg). 
\]

Then, by \((2.11)\) and \((2.20)\) we get
\[
|\partial_x^j \mathcal{T}_2| \leq C \frac{1}{\sqrt{|t|}}, \quad j = 0, 1. 
\]

Next, we consider $\mathcal{T}_3$. By the convolution theorem we have
\[
\mathcal{T}_3 = \frac{2\pi}{4\pi i t} \int_{-\infty}^{\infty} e^{i(x+y-z)^2/4t} F_s(z) dz,
\]
where $F_s$ is given by \((2.41)\). Further, by \((2.45)\) we prove that
\[
|\mathcal{T}_3| \leq C \frac{1}{\sqrt{|t|}}. 
\]

Denote by $\mathcal{I}_3$ the integral operator from $L^1(\mathbb{R}^+)$ into $L^\infty(\mathbb{R}^+)$ with integral kernel $\mathcal{T}_3(x, y),$
\[
(I_3 \phi)(x) := \int_{0}^{\infty} \mathcal{T}_3(x, y) \phi(y) dy.
\]

By \((3.13)\)
\[
\|I_3 \phi\|_{L^\infty} \leq C \frac{1}{\sqrt{|t|}} \|\phi\|_{L^1}. 
\]

Moreover, deriving the right-hand side of \((3.13)\), using \((3.12)\), noting that $\partial_x e^{i(x+y-z)^2/4t} = \partial_y e^{i(x+y-z)^2/4t}$, integrating by parts in $y$ and using \((2.41)\), by Sobolev embedding theorem we show that
\[
\|\partial_x I_3 \phi\|_{L^\infty} \leq C \frac{1}{\sqrt{|t|}} (|\phi(0)| + \|\phi\|_{L^1}) \leq C \frac{1}{\sqrt{|t|}} \|\phi\|_{W^{1,1}}. 
\]

Further, by \((3.13)\) and \((3.10)\),
\[
\|I_3\|_{B(W^{1,1},W^{1,\infty})} \leq C \frac{1}{\sqrt{|t|}}. 
\]

Next, we turn to $\mathcal{T}_4$. By the convolution theorem
\[
\mathcal{T}_4 = \frac{2\pi}{4\pi i t} \int_{-\infty}^{\infty} e^{i(x-z)^2/4t} \int_{-\infty}^{\infty} F_s(z - z_1) K(y, z_1) dz_1 dz
\]
\[
+ \frac{2\pi}{4\pi i t} \int_{-\infty}^{\infty} e^{i(y-z)^2/4t} \int_{-\infty}^{\infty} K(x, z_1) F_s(z - z_1) dz_1 dz. 
\]
Then, using (2.19) and (2.45) we prove that
\[ |T_4| \leq C_1 \sqrt{|t|}, \] (3.19)
Moreover, differentiating (3.18) with respect to \( x \), noting that the following identities are true \( \partial_x e^{i(x-z)^2/4t} = -\partial_x e^{i(x-z)^2/4t} \) and \( \partial_z F_s(z-z_1) = -\partial_z F_s(z-z_1) \), integrating by parts first in \( z \) and then in \( z_1 \) in order to make the derivative act on \( K \), and using (2.19), (2.20) and (2.45) we show
\[ |\partial_x T_4| \leq C_1 \sqrt{|t|}. \] (3.20)
Finally, we look to \( T_5 \). Again, using Parseval’s identity and the convolution theorem we calculate
\[
T_5 = \frac{(2\pi)^2}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, z_2) \left( \int_{-\infty}^{\infty} F_s(z - z_2 - z_1) K(y, -z_1)^3 dz_1 \right) dz_2 dz.
\]
Then, using (2.19), (2.20) and (2.45) we show that
\[ |\partial_x T_5| \leq C_1 \sqrt{|t|}, j = 0, 1. \] (3.21)
By means of (3.7) and the estimates (3.9, 3.10, 3.11, 3.15, 3.17, 3.19, 3.20, 3.21) we deduce from (2.39, 2.40), and (3.3) that
\[ \| e^{-itH} P_c \psi \|_{L^\infty} \leq \frac{C}{\sqrt{t}} \| \psi \|_{L^1}, \] and
\[ \| e^{-itH} P_c \psi \|_{W^{1,\infty}} \leq \frac{C}{\sqrt{t}} \| \psi \|_{W^{1,1}}, \] for all \( t \neq 0 \).

Let us now prove the \( L^2 - L^2 \) estimate for \( e^{-itH} P_c \).

**Proposition 3.2.** Suppose that the potential \( V \) satisfies (1.2) and (1.3). Then, there is \( C > 0 \), such that
\[ \| e^{-itH} P_c \|_{B(L^2)} \leq 1, \] (3.22)
and
\[ \| e^{-itH} P_c \|_{B(W^{1,\infty})} \leq C, \] (3.23)
holds for all \( t \in \mathbb{R} \).

**Proof.** Estimate (3.22) is consequence of the unitarity of \( e^{-itH} \) in \( L^2 \). Further, by (2.14) and since
\[ \sqrt{H} + L e^{-itH} P_c = e^{-itH} P_c \sqrt{H} + L, \]
we have that,
\[ \| e^{-itH} P_c \|_{B(H^1)} \leq C. \] (3.24)
Then, using (2.11b) we obtain (3.23).

**Proof of Theorem 1.1** The general estimate (1.6) is an interpolation (use the Riesz-Thorin theorem, see [38]) between the estimates (1.11) and (1.17). Interpolating between estimates (1.17) and (3.23), we attain (1.7).

**Proof of Theorem 1.2** The Strichartz estimates are deduced from the \( L^p - L^{p'} \) estimates in Theorem 1.1. See the proof of Theorem 2.3.3 on page 33 of [11] for further details.

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