Three-BMN correlation functions: integrability vs. string field theory. One-loop mismatch

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ABSTRACT: We compare calculations of the three-point correlation functions of BMN operators at the one-loop (next-to-leading) order in the scalar SU(2) sector from the integrability expression recently suggested by Gromov and Vieira, and from the string field theory expression based on the effective interaction vertex by Dobashi and Yoneya. A disagreement is found between the form-factors of the correlation functions in the one-loop contributions. The order-of-limits problem is suggested as a possible explanation of this discrepancy.

KEYWORDS: Supersymmetric gauge theory, 1/N Expansion, Bethe Ansatz, Integrable Field Theories

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1 Introduction

Search for an exact matching between the perturbation theory calculations of anomalous dimensions in the $\mathcal{N} = 4$ supersymmetric Yang-Mills and string theory has been driving the research in the AdS/CFT correspondence for a long time. It was pointed out by Beisert [1] that the correct comparison of the string and field-theoretical results would take place only when the full non-perturbative expressions are being compared. If one compares any expansions up to a certain degree the comparison may be obstructed by the non-commutativity of the limits. Namely, the string theory naturally admits the thermodynamic limit as its basic assumption and then is decomposed perturbatively in the coupling constant, whereas the field theory intrinsically relies on the coupling constant perturbative decomposition, while the thermodynamic limit is taken afterwards. Thus already in [2] Janik argued that a discrepancy between the string theory and the field theory may be explained in terms of the order-of-limits problem. In the two-point sector however the order-of-limits argument has finally been found redundant, since the originally observed three-loop discrepancy [3] and the breakdown of the BMN scaling at four loops [4] was later cured not by the invocation of the order-of-limits considerations but by the introduction of the correct crossing-symmetric phase factor [5, 6] into the S-matrix.

Thus the order-of-limits argumentation, after having been developed for explaining various discrepancies between the anomalous dimensions on the weak and strong coupling sides, has made place for more physical arguments instead. Now that one is in the possession of the full Bethe Ansatz for any coupling value and any chain length [7], the anomalous
dimension of any operator is effectively known at either weak or strong coupling at arbitrary precision.

The three-point functions in the $\mathcal{N} = 4$ SYM present a new challenge to the AdS/CFT correspondence statement. It has been pointed out by Georgiou [8] that even when an agreement is observed for the structure constants $C_{123}$ at a certain loop order, the agreement should fail at a higher power of the coupling because of the order-of-limits problem. Thus the strong vs weak coupling match or mismatch would be reduced to an issue of a lucky coincidence, and would be devoid of physical meaning. For example, a discrepancy between subleading orders in $\lambda'$ expansion of the weak and strong coupling limits for a heavy-heavy-light three-point correlator of scalars was reported by Bissi, Harmark and Orselli [9]. The natural question was how to interpret this result. On one hand, the argumentation proposed by Harmark, Kristjansson and Orselli [10] claimed that the near-BPS-states in fact must necessarily comply with the string results up to one-loop level. On the other hand, it has been stated in [11] that “from a more modern perspective” the match of the spectra in the Frolov-Tseytlin limit be “a fortunate accident”.

We believe that the issue on whether the matching between the structure coefficients in weakly or strongly coupled sector is an accident still remains a valid question. The whole story of how our knowledge of the spectra (i.e. the anomalous dimensions) of the two-point functions developed is instructive for having eliminated possible formal causes for different discrepancies in exchanging them for a better understanding of the physics behind the integrable chain both at the strong and at the weak coupling limit. In particular, these were the discrepancies between the perturbation theory and semiclassics that drove the discovery of e.g. the dressing phase and the Y-system technique that endow us with the full knowledge of the anomalous dimensions at any coupling and any length. Therefore, we believe that it is of utmost importance to collect the “experimental evidence” for (mis)match of the weak and strong coupling results in the various sectors of the theory even before we can interpret this (mis)match properly, as already implemented for the SO(6) sector in comparing the direct perturbative calculation vs. string field theory [12, 13], and for the SO(6)-extended conjectured version of the Gromov-Vieira formula against the string field theory in the leading order [14].

Here a test is performed at the next-to-leading order level for the correlation functions of three SU(2) BMN operators (with two impurities each) in the weak and strong coupling limits. On the strong coupling side in section 2 our basic approach is the string field theory effective vertex quantum mechanics, using the approach suggested originally in [15–17] and employing the correct prefactor for the effective vertex found in [18]. On the weak coupling side treated in section 3 the basic technique is the integrability-assisted calculation suggested by Gromov and Vieira [19]. What is found as a result of our comparison is the disagreement between the two calculations in the next-to-leading order, that is, in the first order in $\lambda'$. We speculate in the Conclusion on whether the possible physical causes of it should be sought or this mismatch may be considered as a formal artifact related to the order-of-limits problem.
2 String field theory computation

The correlator of the three operators \( \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3) \) is characterized by its structure constant \( C_{123} \) defined as

\[
\langle \mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\mathcal{O}_3(x_3) \rangle = \frac{C_{123}}{|x_1-x_2|^{|\Delta_1+\Delta_2-\Delta_3|}|x_2-x_3|^{|\Delta_3+\Delta_1-\Delta_2|}|x_3-x_1|^{|\Delta_3+\Delta_1-\Delta_2|}},
\]

where \( \Delta_i \) are the dimensions of the operators. In this section the three-point structure constant is calculated from the point of view of string field theory. We follow the recipes of \[20\]. Namely, we start with their expression (2.1)

\[
C_{123} = \frac{1}{\mu(\Delta_1+\Delta_3-\Delta_2)} \left(f \frac{J_1J_3}{J_2}\right)^{-\frac{\Delta_1+\Delta_3-\Delta_2}{2}} \Gamma\left(\frac{\Delta_1+\Delta_3-\Delta_2}{2} + 1\right) \frac{\sqrt{J_1J_2J_3}}{N} \langle 1|\langle 2|\langle 3|H_3\rangle, \tag{2.2}
\]

where

\[
f = \frac{1}{4\pi\mu r(1-r)},
\]

\( J_i \) being the R-charges of the respective chains, \( \Delta_i \) the full dimensions of the corresponding operators, \( \langle 1|\langle 2|\langle 3|H_3\rangle \) the matrix element of the string effective Hamiltonian. The parameter \( \mu \) is related to the Frolov-Tseytlin coupling \( \lambda' \) as

\[
\mu = \frac{1}{\sqrt{\lambda'}}.
\]

The concrete form of the normalization factor on the right hand side of (2.2) was figured out in \[18\] by expanding the result of an integral of three bulk-to-boundary propagators in the strong coupling regime for large \( \Delta_i \) and neglecting all subleading terms. For the holographic string theory dictionary this means that the combinations of \( \Delta_i \)'s in this has to be taken only at the leading order in \( 1/\mu \) and must not be expanded further. The subleading terms in \( 1/\mu \) will come only from the expansion of the matrix element on right and side of (2.2).

The string field theory calculation has the property of yielding always the finite result. Field-theoretically we interpret it as a cancellation of the log divergences of two-particle external-leg normalization with the proper three-particle divergences. Thus the string field theory assumes that our basis is indeed the proper basis of eigenstates in the respective order. It is well known that the extremal correlators require the basis redefinition already in the leading \( \mathcal{O}(1/N) \)-order. The correlator is said to be extremal if for the lengths \( L_1, L_2, L_3 \) of its operators holds

\[
L_i + L_j - L_k = 0. \tag{2.4}
\]

Unlike those, the non-extremal correlators (for which \( L_i + L_j - L_k > 0 \) is always true) feel the basis redefinition only for the subleading corrections. Happily enough, string field theory based on the Dobashi-Yoneya improved vertex knows already about these redefinitions \[20\] and is therefore applicable even to the extremal case. The use of the Dobashi-Yoneya vertex and not of its earlier suggested analogs is justified by the next-leading-order two-point calculation \[21\] that has been proven to be the only vertex to yield the correct two-point subleading correlator.
The string field theory we are interested in is limited to the “tree-level” (leading topology, \( \mathcal{O}(J^2)\)-order) contribution, thus no string diagrams of the type considered in e.g. [22] need to be considered. They certainly do exist, but from [22] it is clear that the loop string field theory effects are \( 1/N \) suppressed. Thus the non-extremal correlator is a very neat object to be analyzed: if the general framework of duality is correct, the tree-level result in SFT is exact to all loops in terms of the gauge theory. The gauge theory result is meant at weak coupling \( g_{YM} \) and small \( \lambda' \), whereas the string theory at \( g_s = g_{YM}^2 \gg 1 \), yet it is also taken at small \( \lambda' \), which allows the comparison to be performed. To obtain the string field theory result at the given loop order one needs to expand expression (2.2) up to the corresponding order in \( 1/\mu \).

The string vertex is organized as

\[
|H\rangle = \sum_{m=0}^{\infty} \sum_{r=1}^{3} \omega_m^{(r)} a_m^{(r)} a_m^{(r)\dagger} |E\rangle,
\]

where the operators \( a_m^{(r)\dagger} \) and \( a_m^{(r)} \) are creation and annihilation operators for the oscillator modes with momentum number \( m \), numeric coefficients\(^1\) are \( a_{(1)} = r, a_{(2)} = -1, a_{(3)} = 1-r \), the frequency is \( \omega_m^{(r)} = \sqrt{m^2 + (\mu a_{(r)})^2} \), and the exponential factor \( |E\rangle \) looks like

\[
|E\rangle = \exp \left[ -\frac{1}{2} \sum_{m,n=-\infty}^{\infty} \sum_{r=1}^{3} a_m^{(r)\dagger} N_{m,n}^{rs} a_n^{(r)} \right] |0\rangle.
\]

Notice the two different bases of creation and annihilation operators used in the same formula, related as

\[
\alpha_n = \frac{a_n - i a_{-n}}{\sqrt{2}}, \\
\alpha_{-n} = \frac{a_n + i a_{-n}}{\sqrt{2}}.
\]

Out of these two bases it is the \( \alpha_n \) oscillators that correspond directly to the impurities in the BMN operators. The \( \tilde{N} \) matrices are taken from the work [17] (all indices \( m, n \) assumed to be positive):

\[
\tilde{N}_{m,n}^{rs} = \tilde{N}_{m,n}^{rs} = \frac{\tilde{N}_{m,n}^{rs} - \tilde{N}_{m,n}^{rs}}{2}, \\
\tilde{N}_{m,n}^{rs} = \tilde{N}_{m,n}^{rs} = \frac{-\tilde{N}_{m,n}^{rs} + \tilde{N}_{m,n}^{rs}}{2},
\]

\(^1\)One should be careful not to confuse the numbers \( a_{(i)} \) with the creation and annihilation operators. We stick to the notation of Dobashi and Yoneya in [20].
where the matrices \( \bar{N}_{m,n} \) are
\[
\bar{N}_{m,n}^{r,s} = \frac{1}{2\pi} \left( \frac{1}{\bar{a}(s)\omega_m^{(r)} + \bar{a}(r)\omega_n^{(s)}} \right) \left[ \frac{a(s)a(r)(\omega_m^{(r)} + \mu a(r)))(\omega_n^{(s)} + \mu a(s))}{\omega_m^{(r)}\omega_n^{(s)}} \right],
\]
and \( \bar{N}_{m,-n}^{r,s} \) are
\[
\bar{N}_{m,-n}^{r,s} = -\frac{1}{2\pi} \left( \frac{1}{\bar{a}(s)\omega_m^{(r)} + \bar{a}(r)\omega_n^{(s)}} \right) \left[ \frac{a(s)a(r)(\omega_m^{(r)} - \mu a(r)))(\omega_n^{(s)} - \mu a(s))}{\omega_m^{(r)}\omega_n^{(s)}} \right].
\]
There are special expressions for the zero mode matrices \( N_{00} \), which we do not mention here since we are not going to encounter zero modes in our calculations. These definitions are \( 1/\mu \) exact up to any perturbative order: only exponentially small corrections \( \sim e^{-\mu} \) could be absent from them.

We will consider a non-extremal 3-BMN case (one operator with four impurities and two with two impurities each):
\[
\langle O_{n_1,-n_1,n_4,-n_4}(x_1) \bar{O}_{n_2,-n_2}(x_2) O_{n_3,-n_3}(x_3) \rangle \tag{2.5}
\]
and an extremal 2-BMN case
\[
\langle O_{n_1,-n_1}(x_1) \bar{O}_{n_2,-n_2}(x_2) O_{(1-r)}(x_3) \rangle \tag{2.6}
\]
where the operators with zero, two and four impurities [23] are given by
\[
O^J(x) = \frac{1}{\sqrt{N^J}} \text{Tr} \, Z^J(x),
\]
\[
O^J_{n,-n}(x) = \frac{1}{\sqrt{N^J+2}} \sum_{p=0}^{J} e^{2\pi i p J} \text{Tr} \, X^p \, Z^{-p}(x),
\]
\[
O^J_{n,-n,m,-m}(x) = \frac{1}{\sqrt{N^J+4}} \sum_{p+r+s+t=J} e^{-2\pi i ((p+r)+(s+t)) J} \text{Tr} \, X^p Z^r X^s Z^t + \text{permutations} \tag{2.7}
\]
\( X \) and \( Z \) are complex scalars in the SU(2) sector of \( N = 4 \) SYM theory with \( X \) being considered as impurity. The subscripts of the operators with two and four impurities give the momenta of the corresponding magnons once the operators are identified with spin-chains.

For the non-extremal 3-BMN case one has
\[
L_1 = Jr + 4, \quad L_2 = J + 2, \quad L_3 = J(1-r) + 2,
\]
and
\[
\Delta_1 = L_1 + \frac{n_1^2 + n_3^2}{\mu^2 r^2}, \quad \Delta_2 = L_2 + \frac{n_2^2}{\mu^2}, \quad \Delta_1 = L_3 + \frac{n_3^2}{\mu^2(1-r)^2}.
\]
For the extremal 2-BMN case (two operators with two impurities each and a vacuum operator) there is a pair of oscillators less, thus
\[
L_1 = Jr + 2, \quad L_2 = J + 2, \quad L_3 = J(1-r),
\]
and
\[ \Delta_1 = L_1 + \frac{n_1^2}{\mu^2}, \quad \Delta_2 = L_2 + \frac{n_2^2}{\mu^2}, \quad \Delta_3 = L_3. \]

To calculate the matrix element \( \langle 1|2\rangle \langle 3|H_3 \rangle \) for the three BMN case all the possible contractions are considered between the four magnons with the momenta \( n_1, -n_1, n_4, -n_4 \) and the other four magnons with momenta \( n_2, -n_2, n_3, -n_3 \). There are 24 such contractions of the type
\[ F_{abcd,a'b'c'd'} \equiv \tilde{N}^{12}_{aa'} \tilde{N}^{13}_{bb'} \tilde{N}^{13}_{cc'} \tilde{N}^{13}_{dd'}, \tag{2.8} \]
where \( a, b, c, d \) take the values of \( n_1, -n_1, n_4, -n_4, a', b', c', d' \) those of \( n_2, -n_2, n_3, -n_3 \) in all possible combinations. It should also be taken into account that the prefactor written in terms of \( \alpha_m, \alpha'_m \) operators looks like
\[ P \equiv \sum_n \sum_{r=1}^3 \frac{\omega_n^{(r)}}{a_n} \left( \alpha_n^{(r)} a_n + \alpha'_{-n} a_{-n} + \alpha'_n a_n + \alpha'_{-n} a_{-n} \right). \tag{2.9} \]

Therefore, while contracting the matrix element
\[ \langle 1|2\rangle \langle 3|H_3 \rangle = \langle \alpha_1^{(1)} \alpha_{-n}^{(1)} \alpha_4^{(1)} \alpha_{-n_4}^{(1)} \alpha_2^{(2)} \alpha_{-n_2}^{(2)} \alpha_3^{(3)} \alpha_{-n_3}^{(3)} |P|E \rangle \]
some of the momenta “change their sign” when they are contracted with \( |E \rangle \) through the prefactor \( P \). Thus denoting auxiliary quantity
\[ F_{1}^{(1)} = F_{n_1,n_2,n_3,n_4} - F_{n_1,n_2,n_3,n_4} + F_{-n_1,-n_2,n_3,n_4} + F_{n_1,-n_2,n_3,n_4} + F_{n_1,n_2,-n_3,n_4} + F_{n_1,n_2,n_3,-n_4} + F_{n_1,-n_2,n_3,-n_4} + F_{n_1,n_2,-n_3,-n_4} + F_{-n_1,-n_2,n_3,-n_4} + F_{-n_1,n_2,-n_3,n_4} + F_{-n_1,n_2,n_3,-n_4} + F_{-n_1,-n_2,-n_3,n_4}, \]
and similarly defining \( F_{2}^{(2)}, F_{3}^{(3)}, F_{4}^{(4)} \) for the permutations of the signs of magnon momenta \( n_2, n_3, n_4 \) respectively one obtains finally the matrix element
\[ \langle 1|2\rangle \langle 3|H_3 \rangle = \sum_{n_1 \in \text{magnons}} \frac{\omega_n^{(r)}}{a_n} F_{1}^{(i)} \]
Combinatorics for the two-BMN case is derived analogously. Taking these matrix elements \( \langle 1|2\rangle \langle 3|H_3 \rangle \) for the two-BMN and three-BMN cases together with the normalization factors of \( (2.2) \), the following results are obtained on the string field theory side.

(A) For the three-BMN case we obtain
\[ C_{SFT,3BMN}|_{n_4 \rightarrow n_1} = \frac{1}{N} \frac{16\sqrt{r}}{\pi^2} \left( \frac{3n_2^2}{\pi r^2} + n_1^2 \right) \frac{\sin^2(\pi n_2 r)}{\pi^2} \times \left[ 1 + \frac{3n_2^2}{\pi r^2} - \frac{2n_3^2}{(r-1)^2} + \frac{12n_2^2n_3^2}{3n_2^2 + n_1^2} \right]. \tag{2.10} \]
The \( n_4 \rightarrow n_1 \) limit should be understood not in the sense of a nonexistent solution of Bethe equations with a double root, but as a purely formal manipulation to simplify the expressions; we could have chosen any other particular condition on \( n_1 \) and \( n_4 \) to increase readability and reduce cluttering of the formulae.
As has been mentioned above, the $1/\mu$-expanded expression for the correlator inherits the $\mu$-dependence from the Neumann matrices and the effective vertex prefactor $P$, yet not from the $\Delta_i$ in the normalization factor

$$\frac{1}{\mu^{(\Delta_1+\Delta_3-\Delta_2)}} \left( f^{J_1J_3}_{J_2} \right)^{-\frac{\Delta_1+\Delta_3-\Delta_2}{2}} \Gamma \left( \frac{\Delta_1+\Delta_3-\Delta_2}{2} + 1 \right)$$

by Dobashi-Yoneya. There has been no weak-coupling calculation for the three-BMN correlator so far, thus it will be compared to the weakly-coupled side after the three-point correlator is computed using integrability in the next section.

(B) For the extremal correlator of two-BMN one-BPS one obtains

$$C_{SF,2BMN} = \frac{2J^{3/2}}{N} \sqrt{r(1-r)n_2^2 \sin^2(\pi n_2 r)} \left[ 1 + \frac{\lambda'}{4} \left( \frac{n_1^2}{r^2} - n_2^2 \right) \right]. \quad (2.11)$$

Comparing this to the result of [24] for the extremal correlator

$$C_{FT,2BMN} = \frac{2J^{3/2}}{N} \sqrt{r(1-r)n_2^2 \sin^2(\pi n_2 r)} \left[ 1 + \frac{\lambda'}{2} \left( \frac{n_1^2}{r^2} - n_2^2 \right) \right], \quad (2.12)$$

one sees that it does not match at next-to-leading order.

For the three-point function to have a proper scaling the operators have to be the eigenstates of the dilatation operator. In general, the eigenstates are mixed states between single and double trace operators. In the case of the extremal correlators the contribution from the double-trace operators could be of the same order in $1/N$ as from the single trace operators. The mixing at $O(\lambda')$-level affects the $O(\lambda')$ and $O(\lambda^0)$-contributions to the three-point function. The one-loop contribution becomes also affected from the mixing at two-loop level.

Having this in mind we should note that the result of [24] does include the mixing only at one-loop level where the string field theory computation should capture the mixing also at two-loop level. This makes the discrepancy at $O(\lambda')$ in (2.11) and (2.12) plausible.

3 Integrability-assisted computation

The direct perturbative calculation of three-point function is straightforward and has been implemented since a long time. An ambitious project to cast the calculation of the three-point functions into the formalism of Bethe Ansatz was proposed in [25] and realized there at the leading order in coupling constant. This “three-point-functions from integrability” framework has been certainly inspired by the success of integrable systems describing the two-point functions (that is, the spectra of anomalous dimensions). In the leading order integrability has provided a combinatorial simplification, which is very important for calculating the correlation functions with more than two excitations per operator. Yet it is the next-to-leading order result of [19] that allows one to fully appreciate the convenience of the Bethe Ansatz calculation compared with the ordinary perturbation theory. Physically, the integrability calculation does not yet provide us with any new information like higher-order Hamiltonians or semiclassical descriptions of highly-excited states. However, a direct perturbative calculation would have required from us an explicit knowledge of the interaction Hamiltonians and the fudge-factors (local wavefunction renormalization factors
as introduced in eq. 41 of [26]), the latter becoming increasingly more complicated with larger numbers of magnons. Surprisingly, integrability becomes a natural language to describe these complicated objects in terms of scalar products of Bethe states; contributions of the fudge-factors and Hamiltonian insertions are shown to be nicely packed into a simple structure of a determinant of a matrix, the size of which is proportional to the number of magnons rather than e.g. to the operator length. This simplifies the problem significantly and de facto proposes a new formalism rather than a rewriting of an old one. Thus, although “integrability calculation” is not independent physically from perturbation theory, it exists at the present level as a very special formalism that can be considered as a separate entry in the register of duality recipes. Therefore the tests done on the weak side of duality are expressly performed as tests of either perturbative theory, or “integrability-assisted” perturbation theory.

As shown by Gromov and Vieira in [19] integrability allows us to build up the expression for the three-point function structure coefficient up to the $O(g^2)$-order out of the following ingredients

$$C_{123} = \text{norms} \times \text{simple} \times \text{involved} + O(g^4).$$

Below we discuss the form and the meaning of each of these ingredients. Roughly speaking they can be understood in the following way: the involved factor contains matrix elements of the operator $\mathcal{O}_3$ between the states 1 and 2. The norms factor precisely corresponds to the norms of Bethe states. The simple factor represents a phase of the wave-function of the third operator that is generated when transforming the rest of the expression into the structure of the $\langle 1|\mathcal{O}_3|2 \rangle$ structure.

To apply this formalism the operators should be the Bethe eigenstates at two-loop level. The operator lengths are denoted as $L_1, L_2, L_3$ with corresponding number of magnons $N_1, N_2, N_3$ and Bethe vectors $u = \{u_i\}, v = \{v_i\}, w = \{w_i\}$. Below we shall use the terms “Bethe vector”, “Bethe state” and “operator” as complete synonyms.

- The first building block, norms, has the form$^2$

$$\text{norms} = \frac{L_1 L_2 L_3}{\sqrt{\langle u|u\rangle \langle v|v\rangle \langle w|w\rangle}}, \quad \text{(3.1)}$$

$^2$In [19] there are two conventions which go by the names of algebraic and coordinate Bethe ansatz normalizations, $\langle u|u\rangle_{\text{co}} = \frac{1}{\mu} \langle u|u\rangle_{\text{al}}$ with

$$\mu = (1 - g^2 \Gamma^2_u) \prod_{j=1}^{N} \left(\frac{x(u_j - i/2)}{x(u_j + i/2)} - 1\right) \prod_{i<j} f(u_i, u_j), \quad f = \left(1 + \frac{i}{u - v}\right) \left(1 + \frac{g^2}{(u^2 + \frac{i}{2})(v^2 + \frac{i}{2})}\right).$$

In this work we use algebraic Bethe ansatz normalization.
where
\[
\langle u|u \rangle = (1 - 2g^2 \Gamma_u - g^2 \Gamma_u^2) \langle \theta; u|\theta; u \rangle, \\
\langle \theta; u|\theta; u \rangle = \prod_{m \neq k} \frac{u_k - u_m + i}{u_k - u_m - i} \partial_{u_j} \left( \frac{L_i}{i} \log \frac{x(u_k + i/2)}{u_k - i/2} + \frac{1}{i} \sum_{m \neq k} \log \frac{u_k - u_m - i}{u_k - u_m + i} \right), \\
\Gamma_u = \sum_{i=1}^{N_1} \frac{1}{u_i^2 + \frac{1}{4}}, \quad x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2g} = \frac{u - g}{u} + \ldots 
\]

\( \cdot \) The expression simple that corresponds to the phases of the third operator’s wavefunction has the form
\[
\text{simple} = (1 - g^2 \Gamma_w) A_{N_3}(p), 
\]
where
\[
A_N(p) = (1 - g^2 \Gamma_w^2) \sum_{\alpha, \bar{\alpha}} (-1)^{|\alpha|} \prod_{k \in \alpha, \bar{k} \in \bar{\alpha}} f(k, \bar{k}) \prod_{\bar{k} \in \bar{\alpha}} e^{iN\bar{k}} 
\]

with \( \alpha, \bar{\alpha} \) all possible partitions of the set of momenta, and \(|\alpha|\) number of the elements in partition \( \alpha \), and
\[
f(k, \bar{k}) = \left( 1 + \frac{i}{w(k) - w(\bar{k})} \right) \left( 1 + \frac{g^2}{(w(k)^2 + \frac{1}{4})(w(\bar{k})^2 + \frac{1}{4})} \right). 
\]

To simplify the building blocks of the final expression of the three-point function we denote the expression for the \( A_N \) without the \( (1 - g^2 \Gamma_w) \) prefactor \( \tilde{A}_N \).

\( \cdot \) The expression involved corresponds to the scalar product of Bethe states. Scalar products of Bethe states lie at heart of the simplification offered by the integrability framework for the three-point correlation functions. The scalar products for Bethe eigenstates of simplest groups (e.g. SU(2)) were built as early as in the eighties (see e.g. [27–30]), later this technique was extended towards more complicated cases (non-compact groups, non-eigenstate vectors in the scalar products); see also references in e.g. [31]. The novelty of [19] was to use the scalar product for calculations of physical quantities, the three-point functions.

The scalar product could be expressed with the help of the so-called theta-morphism, which is a particular linear transformation of a function \( f \) that is related to some

\[ \text{The relation derived in [19] is given in the coordinate Bethe ansatz normalization. Note that the relation between } A_N \text{ in coordinate and algebraic Bethe ansatz normalizations is} \]
\[
A_N^{(alg)} = \mu A_N^{(coord)} 
\]
homogeneous integrable chain of length $L$. Introduce inhomogeneities $\theta_i, i = 1 \ldots L$, one per chain node, into the chain; then the theta-morphism $((f))_\theta$ of the function $f$ is defined as

$$
(f(\theta)) \equiv f \bigg|_{\theta \to 0} + \frac{g^2}{2} \sum_{i=1}^{L} D_i^2 f \bigg|_{\theta \to 0},
$$

(3.6)

where

$$
D_i = \frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_{i+1}}, \quad \text{with} \quad \theta_{L+1} = \theta_1,
$$

see appendix A for more properties. Not going deeply into the physical origin, derivation and the meaning of the theta-morphism itself let us write out the recipe for the “involved” part of the calculation; in combination with the norms of the first two operators it could be written as

$$
\frac{\text{involved}}{\sqrt{\langle u|u\rangle\langle v|v\rangle}} = \frac{\left( \langle \theta^{(1)}; u|\hat{O}|\theta^{(2)}; v \rangle \right)_{\theta^{(1)}}}{\sqrt{\left( \langle \theta^{(1)}; u|\theta^{(1)}; u \rangle \right)_{\theta^{(1)}} \left( \langle \theta^{(2)}; v|\theta^{(2)}; v \rangle \right)_{\theta^{(2)}}}} + \text{pure imaginary term}.
$$

(3.7)

All other building blocks of the final expression of the three-point function are real at $O(g^2)$-order. This allows to absorb the imaginary term into the overall complex phase (the imaginary part being of order $O(g^2)$ could influence the magnitude of the absolute value of the structure coefficient only at the $g^4$-level which we neglect).

Scalar product involving operators 1 and 2 is given by

$$
\langle \theta^{(1)}; u|\hat{O}_3|\theta^{(2)}; v \rangle = \prod_{m<n}^{N_3,N_1} \frac{(u_n - \theta^{(1)}_m + i/2)}{(v_n - \theta^{(1)}_m + i/2)} \prod_{n<m}^{N_2,N_1} \frac{(v_n - \theta^{(1)}_m + i/2)}{(u_n - \theta^{(1)}_m + i/2)} \prod_{n<m}^{N_3,N_1} (\theta^{(1)}_n - \theta^{(1)}_m) \det \left( [G_{nm}] \oplus [F_{nm}] \right).
$$

(3.8)

The parameters $\theta^{(r)}_m$ living on the nodes, where $r = 1, 2, 3, m = 1 \ldots L_r$, are auxiliary quantities necessary to perform the theta-morphism operation. Here where $\theta^{(1)}_m = \hat{\theta}^{(1)}_{L_{1+1}-m}$ and

$$
F_{nm} = \frac{1}{\left( u_n - \hat{\theta}^{(1)}_m \right)^2 + \frac{1}{4}},
$$

$$
G_{nm} = \prod_{a=1}^{L_1} \frac{v_m - \theta^{(1)}_a + i/2}{v_m - \theta^{(1)}_a - i/2} \prod_{k \neq n}^{N_1} \frac{(u_k - v_m + i)}{(u_k - v_m - i)} - \prod_{k \neq n}^{N_1} \frac{(u_k - v_m - i)}{(u_k - v_m + i)}
$$

(3.9)
Combining previously discussed pieces gives at $\mathcal{O}(g^2)$-order up to a complex phase factor
\[
C_{123} = \sqrt{\frac{L_1 L_2 L_3}{16 J^4}} \frac{1 + \frac{g^2}{4} \Gamma_w + \frac{g^2}{2} \Gamma_w^2/2}{\sqrt{\left(\theta; w|\theta; w\right)_{\theta}}} \times \text{simple} \times \frac{\text{involved}}{\sqrt{1/1(2/2)}}
\]
\[
= \frac{\sqrt{\frac{L_1 L_2 L_3}{16 J^4}} \left(\theta(1); w|\theta(1); u\right)_{\theta(1)} \left(1 - \frac{g^2}{4} \Gamma_w/2\right) A_N(w)}{\sqrt{\left(\theta(1); u|\theta(1); u\right)_{\theta(1)} \left(\theta(2); v|\theta(2); v\right)_{\theta(2)} \left(\theta(3); w|\theta(3); w\right)_{\theta(3)}}}.
\] (3.10)

### 3.1 BMN-BMN-BMN correlator

In this section the above formalism is applied to a specific computation of a three point function of three BMN operators. We consider a configuration of three BMN operators with lengths
\[
L_1 = rJ + 4, \quad L_2 = J + 2, \quad L_3 = J(1 - r) + 2,
\] (3.11)
the corresponding numbers of magnons are $N_1 = 4$, $N_2 = 2$, $N_3 = 2$ and the rapidities are
\[
\begin{align*}
\mathcal{O}_1 : & \ u_1, -u_1, u_3, -u_3, \\
\mathcal{O}_2 : & \ v, -v, \\
\mathcal{O}_3 : & \ w, -w.
\end{align*}
\] (3.12)

Inserting all the ingredients into the expression (3.10) (for details see appendix B) one gets
\[
C_{GV,3BMN}\bigg|_{n_4 \rightarrow n_1} = \frac{1}{N \sqrt{J}} \frac{16 \sqrt{J} \sin^2(\pi n_2 r)}{\sqrt{(1 - r)^2}} \frac{n_1^2 + 3n_2^2r^2}{(n_1^2 - n_2^2r^2)^2}
\times \left[1 + \frac{\lambda'}{4} \left(-\frac{2n_3^2}{(1 - r)^2} - \frac{n_2^2}{r^2} - \frac{n_2^2}{r^2} + \frac{8n_3^2n_2^2}{n_1^2 + 3n_2^2r^2}\right)\right],
\] (3.13)
where the limit $n_4 \rightarrow n_1$ has been taken at the very end to keep the expression compact.

We also used $\lambda' = \frac{16\pi^2 r^2}{J^2}$. This matches the SFT result (2.10) at the leading orders, but disagrees with it at the subleading order.

#### 3.1.1 Comparison to the result of the string field theory computation

As demonstrated above there is complete matching of the three-BMN correlators in the leading order. This term with $n_4$-dependency restored has the following form
\[
C_{GV,3BMN}^0 = C_{SFT,3BMN}^0 = \frac{1}{N \sqrt{J}} \frac{8\sqrt{J} \sin^2(\pi n_2 r)}{\sqrt{(1 - r)^2}} \frac{(n_3^2 r^2 (n_1^2 + n_4^2) - 5n_3^2 r^4 (n_1^2 + n_2^2) + n_2^4 (n_2^2 + n_4^2) + 6n_3^2 n_2^4)}{(n_1^2 - n_2^2 r^2)^2 (n_2^2 - n_2^2 r^2)^2}.
\] (3.14)

The mismatch is happening at the $\mathcal{O}(\lambda')$-order. The difference is coming from the terms depending on $n_1$ and $n_4$. To illustrate the difference the $n_4$ is restored in one of these terms, which then takes the form
\[
C_{GV,3BMN}^0 \left(1 + \frac{\lambda'}{4} \left(-\frac{n_1^2}{r^2} + \cdots\right)\right) \longrightarrow C_{GV,3BMN}^0 \left(1 + \frac{\lambda'}{4} \left(-\frac{1}{r^2} n_1^4 + n_4^4 + \cdots\right)\right),
\]
\[
C_{SFT,3BMN}^0 \left(1 + \frac{\lambda'}{4} \left(-\frac{3n_2^2}{r^2} + \cdots\right)\right) \longrightarrow C_{SFT,3BMN}^0 \left(1 + \frac{\lambda'}{4} \left(-\frac{2}{r^2} n_1^4 + n_4^4 + \cdots\right)\right).
\] (3.15)
The different structures between these expressions will not allow to match both expressions by sending \( n_1 \) or \( n_4 \) to zero. However, an interesting feature of this result is the observation that sending both \( n_1 \) and \( n_4 \) at the same time to zero will yield a matching expression on both sides

\[
C_{4BMN} \bigg|_{n_1,n_4 \to 0} = \frac{1}{N\sqrt{J}} \frac{48 \sin^2(\pi n_2 r)}{\sqrt{r^2(1-r)\pi^2}} \frac{1}{n_2^2} \left( 1 - \frac{\lambda'}{2} \left( \frac{n_3^2}{(1-r)^2} + n_2^2 \right) \right).
\] (3.16)

It should be noted here that the operator with \( n_1 \to 0, n_4 \to 0 \) is a descendant of a chiral primary operator. The fact that matching occurs exactly for this physically distinct case is intriguing, yet no explanation thereof has been given so far.

### 3.2 BMN-BMN-BPS correlator

In the formalism by Gromov-Vieira, the case of the extremal three-point correlators, like e.g. BMN-BMN-BPS, could be computed via

\[
C_{123} = \frac{\sqrt{L_1 L_2 L_3} \langle \theta^{(1)}; u|\hat{O}_3|\theta^{(2)}; v \rangle}{\sqrt{\langle \theta^{(1)}; u|\theta^{(1)}; u \rangle \langle \theta^{(2)}; v|\theta^{(2)}; v \rangle}}. 
\] (3.17)

However, there is a subtlety involved concerning the mixing between single and double trace operators. At one loop the operators with well defined scaling dimensions are the mixed states of the single and double trace operators. In the non-extremal case, like e.g. the three point correlation function of three BMN operators, the contribution from the double trace operators at \( O(g^2) \) order is always subleading in \( 1/N \) compared to the one of the single trace operators. In the extremal case, the contribution from the double trace operators is of the same order in \( 1/N \) and becomes relevant already at the \( O(\lambda'^0) \) order.

The formalism of Gromov/Vieira uses mappings between the single trace operators and the Bethe eigenstates which means this formalism cannot give the complete three-point function but only the contribution from the single trace operators. This phenomenon could also be seen explicitly in the case of the BMN-BMN-BPS three point function. The eq. (3.17) is applied to the configuration with the lengths

\[
L_1 = rJ + 2, \quad L_2 = J + 2, \quad L_3 = J(1-r), \nonumber
\]

and the rapidities

\[
\mathcal{O}_1 : u, -u, \nonumber \\
\mathcal{O}_2 : v, -v, \nonumber
\]

and obtain

\[
C_{GV2BMN} = \frac{2 J^{3/2}}{N} \frac{\sqrt{r^2(1-r)n_2^2 \sin^2(\pi n_2 r)}}{\pi^2 (n_1^2 - n_2^2 r^2)^2} \left[ \left( 1 + \frac{n_1^2}{n_2^2 r^2} \right) - \frac{\lambda'}{2} \left( \frac{n_3^2}{(1-r)^2} + n_2^2 \right) \right]. 
\] (3.18)

The mixing with the double trace operators is already relevant at the tree level of the three point function. Note that the extremal three-point function at \( O(\lambda') \)-order might
also get some contributions from the mixing of the operators at two-loop order. As shown in \cite{32}, the inclusion of the double trace operators will change the tree level contribution by a factor

\[ C^{\text{double trace}}_{123} = \left(1 + \frac{n_1^2}{n_2^2 r_2} \right) C^{\text{without double trace}}_{123}. \tag{3.19} \]

With the one-loop mixing contribution taken into account it is clear that the $\mathcal{O}(\lambda'^0)$ of the integrability calculation matches the perturbative computations, see e.g. \cite{24}.

Concerning the $\lambda'$-correction we are aware of two perturbative computations in the above sector. The one by Beisert et al. \cite{24} apparently takes into account the contributions from the mixing of the single trace with double trace operators and that’s why cannot be compared. The authors of \cite{33} compute the $\lambda'$ correction for the tree-point function without taking the mixing into account. Their result is given for $n_1 = 0$ (see their eq.34) which exactly matches the one obtained from the formula (3.17).

4 Conclusion

Let us collect here the results of the calculations:

1. The extremal (two-BMN, one-BPS) correlator from integrability (the Gromov-Vieira formalism) fully coincides at NLO with the purely single-trace part of the perturbative extremal correlator.

2. The extremal correlator from the string field theory with the Dobashi-Yoneya vertex does not match the extremal correlator from the Gromov-Vieira integrability-assisted formalism at NLO.

3. The non-extremal (three-BMN) correlators from the string field theory and from integrability match in the leading order and do not match in the subleading order.

Statement (1) has nothing remarkable in it; it merely says that apparently no obvious mistakes have been done in the calculation and ensures that the one-loop results at weak coupling (the perturbative field theory and the integrability-assisted computation) are equally adequate in describing the weakly-coupled limit for single trace operators.

The formalism of Gromov and Vieira does not capture the contributions from non-single trace operators and thus cannot take the effects related to the operator redefinition into account which become relevant in the case of the extremal correlator. One should note, as had been pointed out already by Beisert et al. \cite{24} that the single-to-double trace mixing matrix might also influence the $\lambda'$-order correlator by the terms of $\lambda'^2$-order in the mixing matrix, therefore the $\lambda'$-terms in the weakly-coupled limit must be considered to be reliable only after the next-next-to-leading order of mixing matrix has been elaborated.

This means that the statement (2) potentially contains an interplay of the effects related to this operator redefinition and to some fundamental mismatch between the weakly and strongly coupled theories. Thus one should rather analyze the mismatch (3) instead to which operator redefinition does not contribute.
What is the possible origin of this mismatch? The first guess would be to invoke the order-of-limits argument that we have discussed in the Introduction. This would be the most natural explanation, yet we have seen in the story with the two-point particle spectra that some discrepancies originally explained via the different order of the limits had eventually found a more physical explanation.

Furthermore, one could try to argue that there is still some non-traced error or typo in the next-leading-order formula by Gromov and Vieira, existence of which is not absolutely excluded, since very few analytic calculations have been implemented using it so far.

However, there could exist in principle a more fundamental mismatch between the strongly and weakly coupled sectors at next-to-leading order. This would certainly be the most intriguing scenario, since it would challenge our current understanding of the AdS/CFT duality for the three-point correlation function sector.

To choose between these logical alternatives, we hope for more tests to be performed in the nearest future, in particular those extending beyond the SU(2) sector, taking into account fermions or considering short operators and going beyond the Frolov-Tseytlin limit.

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A  Theta-morphism

Up to the $O(g^2)$-order the $\theta$-morphism is given by

$$\left(f(\theta)g(\theta)\right)_{\theta} = \left(f(\theta)\right)_{\theta}\left(g(\theta)\right)_{\theta} + g^2 \sum_{i=1}^{L} D_i f D_i g + O(g^4),$$

(A.1)

where

$$D_i = \frac{\partial}{\partial \theta_i} - \frac{\partial}{\partial \theta_{i+1}}, \quad \text{with} \quad \theta_{L+1} = \theta_1.$$

It satisfies

$$\left(f(\theta)g(\theta)\right)_{\theta} = \left(f(\theta)\right)_{\theta}\left(g(\theta)\right)_{\theta} + g^2 \sum_{i=1}^{L} D_i f D_i g.$$

(A.2)
If one of the functions is symmetric
\[ \left( f_{\text{sym}}(\theta)g(\theta) \right)_{\theta} = \left( f_{\text{sym}}(\theta) \right)_{\theta} \left( g(\theta) \right)_{\theta}. \] (A.3)

The property which relates it to the Zhukovsky variable
\[ \left( \sum_{i=1}^{L} \log \frac{u - \theta^i + i/2}{u - \theta^i - i/2} \right)_{\theta} = L \log \frac{x(u + i/2)}{x(u - i/2)} + \mathcal{O}(g^4), \quad x(u) = \frac{u}{g} - \frac{g}{u} + \mathcal{O}(g^3). \] (A.4)

B Details for the integrability-assisted computation of the three BMN correlator

In this appendix we list all the intermediate computational steps necessary for computing the three point function of three BMN operators of lengths
\[ L_1 = rJ + 4, \quad L_2 = J + 2, \quad L_3 = J(1 - r) + 2 \]
and rapidities
\[ \mathcal{O}_1 : u_1, -u_1, u_3, -u_3 \]
\[ \mathcal{O}_2 : v, -v \]
\[ \mathcal{O}_3 : w, -w \] (B.1)

which up to the \( \mathcal{O}(g^2) \)-order are given by
\[ u_1 = \frac{Jr + 3}{2\pi n_1} + g^2 \frac{4\pi n_1}{Jr + 3}, \]
\[ u_3 = \frac{Jr + 3}{2\pi n_4} + g^2 \frac{4\pi n_4}{Jr + 3}, \]
\[ v = \frac{J - 1}{2\pi n_2} + g^2 \frac{4\pi n_2}{J - 1}, \]
\[ w = \frac{J(1 - r) + 1}{2\pi n_3} + g^2 \frac{4\pi n_3}{J(1 - r) + 1}. \] (B.2)

For the computation below we also need to know the momenta of the third operator which is up to the \( \mathcal{O}(g^2) \)-order
\( p^{(3)}_1 = -p^{(3)}_2 = \frac{2\pi n_3}{J(-1 + r)} \left( 1 - \frac{g^2}{J^2} \frac{8\pi n_3^4}{J^2(-1 + r)^4} \right) \) (B.3)

- The norms with the \( \theta \)-morphism are given by
\[ \left( \langle \theta; u | \theta; u \rangle \right)_{\theta} = \frac{(2\pi n_1)^8}{J^4 r^4} \left( 1 + \frac{g^2}{J^2} \frac{8(2\pi n_1)^2}{r^2} \right), \]
\[ \left( \langle \theta; v | \theta; v \rangle \right)_{\theta} = \frac{(2\pi n_2)^4}{J^2} \left( 1 + \frac{g^2}{J^2} 4(2\pi n_2)^2 \right), \]
\[ \left( \langle \theta; w | \theta; w \rangle \right)_{\theta} = \frac{(2\pi n_3)^4}{J^2 (1 - r)^2} \left( 1 + \frac{g^2}{J^2} \frac{4(2\pi n_3)^2}{(1 - r)^2} \right). \] (B.4)
The denominator in the final expression (B.11) is given by

\[
\left(\left(\langle \theta; u|\theta; v \rangle \right)_{\theta} \left(\langle \theta; v|\theta; w \rangle \right)_{\theta} \left(\langle \theta; w|\theta; w \rangle \right)_{\theta}\right)^{-1/2} = (2\pi)^8 \frac{n_1^2 n_2^2 n_3^2 (n_2^2 + n_3^2)}{(Jr)^2 [J(1 - r)] J} \times \\
\times \left(1 + \frac{g^2}{Jr} 8\pi^2 \left(\frac{n_1^2 + n_2^2}{r^2} + n_2^2 + \frac{n_3^2}{(1 - r)^2}\right)\right). \tag{B.5}
\]

- The normalization coefficient $\sqrt{L_1 L_2 L_3}$ in the large $J$-limit is given by

\[
\sqrt{L_1 L_2 L_3} = \sqrt{Jr J (1 - r) J}. \tag{B.6}
\]

- $\tilde{A}$ for the operator with two magnons and rapidities $w, -w$ becomes

\[
\tilde{A}_{N_3} (w) = \left(1 - f(p_1, p_2) e^{2ip_2} - f(p_2, p_1) e^{2ip_1} + e^{2i(p_1 + p_2)}\right),
\]

\[
f(p_i, p_j) = \left(1 + \frac{i}{w(p_i) - w(p_j)}\right) \left(1 + \frac{g^2}{(w(p_i)^2 + \frac{1}{4})(w(p_j)^2 + \frac{1}{4})}\right)
\]

which combined with $(1 - g^2 \Gamma_w)$ gives

\[
\left(1 - g^2 \frac{\Gamma_w}{2}\right) \tilde{A}_{N_3} = \frac{8\pi^2 n_3^2}{J^2 (1 + r)^2} + O(g^4). \tag{B.7}
\]

- Computationally, the most complicated expression is $\langle \langle \theta^1; u|O_3|\theta^{(2)}; v \rangle \rangle$. We use the property of the theta-morphism applied to a product of factors. Then we Taylor expand the determinant expression up to cubic order in $\theta$ and execute the theta-morphism up to $O(g^2)$-order.

\[
\langle \langle \theta^1; u|O_3|\theta^{(2)}; v \rangle \rangle_{\theta^{(1)}} = \left(\text{prefactor } \times \text{det } [G_{nm} \oplus F_{nm}]\right)_{\theta^{(1)}}
\]

\[
= \left(\theta^{(1)}_1 - \theta^{(1)}_2\right) \left(\text{prefactor } \langle \langle \theta^1; u|O_3|\theta^{(2)}; v \rangle \rangle_{\theta^{(1)}}\right)
\]

\[
\times \langle \langle \theta^1; u|O_3|\theta^{(2)}; v \rangle \rangle_{\theta^{(1)}}
\]

\[
+ \text{cross term} \tag{B.8}
\]

with

\[
D^{p,q} = \text{det } \left(\left.G_{nm}\right|_{\theta_{a=0}} \oplus \left(\Phi^p_{n} \Phi^{q+1}_n\right)\right)_{\theta^{(1)}}
\]

\[
= \left(\frac{\pi^3 r^2}{J^3} \frac{n_1 n_2 n_3^2}{(n_1^2 - n_3^2)^2}\right) \left(1 + \frac{g^2 8\pi^2}{J^2 r} \left(n_1^2 + n_2^2 - 3n_3^2 r^2\right)\right).
\tag{B.9}
\]

\[
\left(\langle \theta^{(1)}; u|O_3|\theta^{(2)}; v \rangle \right)_{\theta^{(1)}} \bigg|_{n_4 \to n_1}
\]

\[
= - \frac{32(2\pi)^4 n_1^2 n_2^2 n_3^2}{J^4 r^2} \left(n_1^2 + 3n_2^2 r^2\right) \left(1 + \frac{g^2 (2\pi)^2 3n_1^4 + 17n_2^2 n_3^2 r^2}{n_1^2 + 3n_2^2 r^2}\right).
\tag{B.10}
\]
Combining all the intermediate results together gives

\[
C_{GV,3BMN} = \frac{16\sqrt{\pi}\sin^2(\pi n^2 r)}{\sqrt{J(1-r)^2}} \left( \frac{n_1^2 + 3n_2^2 r^2}{(n_1^2 - 3n_2^2 r^2)^2} \right) \left( 1 + \frac{g^2}{J^2} \frac{4\pi^2}{(1-r)^2} \left( \frac{2n_3^2}{r^2} - \frac{n_1^2}{r^2} - 2n_2^2 + \frac{8n_1^2 n_2^2}{n_1^2 + 3n_2^2 r^2} \right) \right).
\]

(B.11)

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