The flow and heat transfer in a viscous fluid over an unsteady stretching surface

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Abstract

In this paper we have studied the flow and heat transfer in a viscous fluid by a horizontal sheet. The stretching rate and temperature of the sheet vary with time. The governing equations for momentum and thermal energy are reduced to ordinary differential equations by means of similarity transformation. These equations are solved approximately by means of the Optimal Homotopy Asymptotic Method (OHAM) which provides us with a convenient way to control the convergence of approximation solutions and adjust convergence rigorous when necessary. Some examples are given and the results obtained reveal that the proposed method is effective and easy to use.

Keywords: optimal homotopy asymptotic method (OHAM), film flow, heat transfer, unsteady stretching surface.

1. Introduction

The flow and heat transfer in a viscous fluid over a stretching surface is a relevant problem in many industrial and engineering processes. Examples are
manufacture and drawing of plastics and rubber sheets, polymer extrusion, wire drawing, glass-fiber and paper production, crystal growing, continuous casting, and so on. Cooling of stretching surface requires dedicated control of the temperature and consequently knowledge of flow and heat transfer in a viscous fluid. Sakiadis [1], [2], studied the boundary layer flow over a continuous solid surface moving with constant speed. Crane [3] analyzed the stretching problem having in view the fluid flow over a linearly stretching surface. Tsou et al. [4] studied constant surface velocity and temperature. Gupta and Gupta [5] and Maneschy et al. [6] extended the Crane’s work to the stretching problem with a constant surface temperature including suction or blowing and to fluids exhibiting a non-Newtonian behavior, respectively. Grubka and Bobba [7] studied the stretching problem for a surface moving with a linear velocity and with a variable surface temperature. Wang [8] introduced a similarity transformation to reduce time-dependent momentum equation to a third-order nonlinear differential equation. He analyzed the hydrodynamic behavior of a finite fluid body driven by an unsteady stretching surface. The same problem was considered by Usha and Rukamani [9] for the axisymmetric case. Anderson et al. [10] analyzed the accompanying heat transfer in the liquid film driven by unsteady stretching surface. Ali [11] and Magyari et al. [12] considered permeable surfaces and different surface temperature distributions. Vajravelu [13] studied the flow and heat transfer in a viscous fluid over a planar nonlinear stretching sheet. Magyari and Keller [14] applied the Merkin transformation method to the heat transfer problems of steady boundary layer flows induced by stretching surfaces. Elbashbeshy and Bazid [15] studied similarity solution of the laminar boundary layer equations corresponding to an unsteady stretching surface. Dandapat et al. [16] assumed that the stretching surface is stretched impulsively from rest and the effect of inertia of the liquid is considered. The unsteady heat and fluid flow has been investigated by Ali and Magyari [17]. Liu and Anderson [18] explored the thermal characteristics of a viscous film on an unsteady stretching surface. Chen [19] analyzed the problem of MHD mixed convective flow and heat transfer of an electrically conducting, power-low fluid past a stretching surface in the
presence of heat generation/absorption and thermal radiation. Dandapat et al [20] studied a thin viscous liquid film flow over a stretching sheet under different non-linear stretching velocities in presence of uniform transverse magnetic field. Cortell [21] presented momentum and heat transfer for the flow induced in a quiescent fluid by a permeable non-linear stretching sheet with a prescribed power-low temperature distribution.

Analytical solutions to nonlinear differential equations play an important role in the study of flow and heat transfer of different types fluids, but it is difficult to find these solutions in the presence of strong nonlinearity. A few approaches have been proposed to find and develop approximate solutions of nonlinear differential equations. Perturbation methods have been applied to determine approximate solutions to weakly nonlinear problems [22]. But the use of perturbation theory in many problems is invalid for parameters beyond a certain specified range. Other procedures have been proposed such as the Adomian decomposition method [23], some linearization methods [24], [25], various modified Lindstedt-Poincare methods [26], variational iteration method [27], optimal homotopy perturbation method [28], optimal homotopy asymptotic method [29] - [33].

In the present work we propose an accurate approach to nonlinear differential equations of the flow and heat transfer in a viscous fluid, using an analytical technique, namely optimal homotopy asymptotic method. Our procedure, which does not imply the presence of a small or large parameter in the equation or into the boundary/initial conditions, is based on the construction and determination of the linear operators and of the auxiliary functions, combined with a convenient way to optimally control the convergence of the solution. The efficiency of the proposed procedure is proves while an accurate solution is explicitly analytically obtained in an iterative way after only one iteration. The validity of this method is demonstrated by comparing the results obtained with the numerical solution.
2. Equations of motion

Consider an unsteady, two dimensional flow on a continuous stretching surface, with the governing time-dependent equations for the continuity, momentum and thermal energy [8], [10], [15], [17], [18]:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = \frac{\partial^2 u}{\partial y^2} \tag{2}
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial y^2} \tag{3}
\]

where \( u \) and \( v \) are velocity components in the \( x \) and \( y \) directions, respectively, \( T \) is the temperature and \( k \) is the thermal conductivity of the incompressible fluid. The appropriate boundary conditions are:

\[
u = u_0 x l + \gamma t, \quad v = 0, \quad T = T_\infty + \frac{T_0}{(1 + \gamma t)^c} \left(\frac{x}{l}\right)^n \quad \text{at} \; y = 0 \tag{4}\]

\[
u \to 0, \quad T \to T_\infty \quad \text{at} \; y \to \infty \tag{5}\]

where \( u_0, T_0, T_\infty, \gamma \) are positive constants, \( c \) and \( n \) are arbitrary and \( l \) is a reference length.

If \( Re = \frac{u_0}{v} \) and \( Pr = \frac{\nu}{k} \) are the Reynolds number and the Prandl number respectively and if we choose a stream function \( \Psi(x, y) \) such that

\[
u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x} \tag{6}\]

then the equation (1) of continuity is satisfied and the mathematical analysis of the problems (2) and (3) is simplified by introducing the following similarity transformation:

\[
\Psi = \frac{x}{l} \sqrt{Re(1 + \gamma t)^{1/2}} \tag{7}\]
\[ \eta = \sqrt{Re} \frac{y}{l(1 + \gamma t)^{1/2}} \]  

(8)

\[ T = T_\infty + T_0 \left( \frac{x}{l} \right)^n \frac{\theta(\eta)}{(1 + \gamma t)^c} \]  

(9)

\( T_0 \) being a reference temperature. In this way Eqs. (6) can be written in the form:

\[ u(x, y, t) = \frac{u_0}{l} \frac{x}{(1 + \gamma t)} f'(\eta) \]  

(10)

\[ v(x, y, t) = -\frac{u_0}{\sqrt{Re(1 + \gamma t)^{1/2}}} f(\eta) \]  

(11)

where prime denotes differentiation with respect to \( \eta \).

Substituting Eqs. (7), (8), (9), (10) and (11) into Eqs. (2) and (3), we obtain

\[ f''' + ff'' - f'^2 + \Lambda \left( f' + \frac{1}{2} \eta f'' \right) = 0 \]  

(12)

\[ \frac{1}{Pr} \theta'' + f \theta' - n f' \theta + \Lambda \left( c \theta + \frac{1}{2} \eta \theta' \right) = 0. \]  

(13)

Here \( \Lambda = \frac{l \gamma}{u_0} \) is dimensionless measure of the unsteadiness.

The dimensional boundary conditions (4) and (5) become

\[ u = \frac{u_0}{l} \frac{x}{(1 + \gamma t)} f'(0) \quad \text{at} \quad y = 0 \]  

(14)

\[ v = -\frac{u_0}{\sqrt{Re(1 + \gamma t)^{1/2}}} f(0) \quad \text{at} \quad y = 0 \]  

(15)

\[ T = T_\infty + T_0 \left( \frac{x}{l} \right)^n \frac{\theta(0)}{(1 + \gamma t)^c} \quad \text{at} \quad y = 0 \]  

(16)

such that for the dimensionless functions \( f \) and \( \theta \), the boundary/initial conditions become

\[ f(0) = f_w, \quad f'(0) = 1, \quad f'(\infty) = 0 \]  

(17)
\( \theta(0) = 1, \quad \theta(\infty) = 0. \) \hspace{1cm} (18)

In addition to the boundary conditions (17) and (18), the requirements

\[ f'(\eta) \geq 0, \quad \theta(\eta) \geq 0, \quad \forall \eta \geq 0 \] \hspace{1cm} (19)

must also satisfy [17].

3. Basic ideas of optimal homotopy asymptotic method

Eqs. (12) (or (13)) with boundary conditions (17) (or (18)) can be written in a more general form:

\[ N(\Phi(\eta)) = 0 \] \hspace{1cm} (20)

where \( N \) is a given nonlinear differential operator depending on the unknown function \( \Phi(\eta) \), subject to the initial/boundary conditions:

\[ B(\Phi(\eta), \frac{d\Phi(\eta)}{d\eta}) = 0. \] \hspace{1cm} (21)

It is clear that \( \Phi(\eta) = f(\eta) \) or \( \Phi(\eta) = \theta(\eta) \).

Let \( \Phi_0(\eta) \) be an initial approximation of \( \Phi(\eta) \) and \( L \) an arbitrary linear operator such as

\[ L(\Phi_0(\eta)) = 0, \quad B\left(\Phi_0(\eta), \frac{d\Phi_0(\eta)}{d\eta}\right) = 0. \] \hspace{1cm} (22)

We remark that this operator \( L \) is not unique.

If \( p \in [0, 1] \) denotes an embedding parameter and \( F \) is a function, then we propose to construct a homotopy \([29] - [33] \):

\[ \mathcal{H}\left[L\left(F(\eta,p)\right), H(\eta,C_i), N\left(F(\eta,p)\right)\right] \] \hspace{1cm} (23)

with the following two properties:

\[ \mathcal{H}\left[L\left(F(\eta,0)\right), H(\eta,C_i), N\left(F(\eta,0)\right)\right] = L\left(F(\eta,0)\right) = L\left(\Phi_0(\eta)\right) \] \hspace{1cm} (24)
\[ H \left[ L \left( F(\eta,1) \right), \ H(\eta, C_i), \ N \left( F(\eta,1) \right) \right] = H(\eta, C_i)N\left( \Phi(\eta) \right) \] (25)

where \( H(\eta, C_i) \neq 0 \), is an arbitrary auxiliary convergence-control function depending on variable \( \eta \) and on a number of arbitrary parameters \( C_1, C_2, \ldots, C_m \) which ensure the convergence of the approximate solution.

Let us consider the function \( F \) in the form

\[ F(x,p) = \Phi_0(\eta) + p\Phi_1(\eta, C_i) + p^2\Phi_2(\eta, C_i) + \ldots \] (26)

By substituting Eq. (26) into equation obtained by means of the homotopy (23)

\[ H \left[ L \left( F(\eta,p) \right), \ H(\eta, C_i), \ N \left( F(\eta,p) \right) \right] = 0 \] (27)

and equating the coefficients of like powers of \( p \), we obtain the governing equation of \( \Phi_0(x) \) given by Eq. (22) and the governing equation of \( \Phi_1(\eta, C_i) \), \( \Phi_2(\eta, C_i) \) and so on. If the series (26) is convergent at \( p = 1 \), one has:

\[ F(\eta,1) = \Phi_0(\eta) + \Phi_1(\eta, C_i) + \Phi_2(\eta, C_i) + \ldots \] (28)

But in particular we consider only the first-order approximate solution

\[ \Phi(\eta, C_i) = \Phi_0(\eta) + \Phi_1(\eta, C_i), \quad i = 1, 2, \ldots, m \] (29)

and the homotopy (23) in the form

\[ H \left[ L \left( F(\eta,p) \right), \ H(\eta, C_i), \ N \left( F(\eta,p) \right) \right] = L\left( \Phi_0(\eta) \right) + \\
+ p \left[ L\left( \Phi_1(\eta, C_i) \right) - L\left( \Phi_0(\eta) \right) + H(\eta, C_i)N\left( \Phi_0(\eta) \right) \right]. \] (30)

Equating only the coefficients of \( p^0 \) and \( p^1 \) into Eq. (30), we obtain the governing equation of \( \Phi_0(\eta) \) given by Eq. (22) and the governing equation of \( \Phi_1(\eta, C_i) \) i.e.

\[ L\left( \Phi_1(\eta, C_i) \right) = H(\eta, C_i)N\left( \Phi_0(\eta) \right), \]

\[ B\left( \Phi_1(\eta, C_i), \frac{d\Phi_1(\eta, C_i)}{d\eta} \right) = 0, \quad i = 1, 2, \ldots, m. \] (31)
It should be emphasize that $\Phi_0(\eta)$ and $\Phi_1(\eta, C_i)$ are governed by the linear Eqs. (22) and (31), respectively with boundary conditions that come from the original problem, which can be easily solved. The convergence of the approximate solution (29) depends upon the auxiliary convergence-control function $H(\eta, C_i)$. There are many possibilities to choose the function $H(\eta, C_i)$.

Basically, the shape of $H(\eta, C_i)$ must follow the terms appearing in the Eq. (31). Therefore, we try to choose $H(\eta, C_i)$ so that in Eq. (31), the product $H(\eta, C_i)N(\Phi_0(\eta))$ be of the same shape with $N(\Phi_0(\eta))$. Now, substituting Eq. (29) into Eq. (20), it results the following residual

$$R(\eta, C_i) = N(\Phi(\eta, C_i)). \quad (32)$$

At this moment, the first-order approximate solution given by Eq. (29) depends on the parameters $C_1, C_2, ..., C_m$ and these parameters can be optimally identified via various methods, such as the least square method, the Galerkin method, the Kantorowich method, the collocation method or by minimizing the square residual error:

$$J(C_1, C_2, ..., C_m) = \int_a^b R^2(\eta, C_1, C_2, ..., C_m) \, d\eta \quad (33)$$

where $a$ and $b$ are two values depending on the given problem. The unknown parameters $C_1, C_2, ..., C_m$ can be identified from the conditions:

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = ... = \frac{\partial J}{\partial C_m} = 0. \quad (34)$$

With these parameters known (namely convergence-control parameters), the first-order approximate solution (29) is well-determined.

4. Application of OHAM to flow and heat transfer

We use the basic ideas of the OHAM by considering Eq. (12) with the boundary conditions given by Eq. (17). We can choose the linear operator in the form:

$$L_f(\Phi(\eta)) = \Phi'' - K^2\Phi'.$$  \quad (35)
where $K > 0$ is an unknown parameter at this moment.

We mention that the linear operator is not unique. Also, we have freedom to choose:

$$L_f(\Phi(\eta)) = \Phi''' + \frac{3K}{K+1}\Phi''.$$

Eq. (22) becomes

$$\Phi_0''' - K^2\Phi_0' = 0, \quad \Phi_0(0) = f_w, \quad \Phi_0'(0) = 1, \quad \Phi_0'(\infty) = 0.$$

which has the following solution

$$\Phi_0(\eta) = f_w + \frac{1 - e^{-K\eta}}{K}. \quad (36)$$

The nonlinear operator $N_f(\Phi(\eta))$ is obtained from Eq. (12):

$$N_f(\Phi(\eta)) = \Phi'''(\eta) + \Phi(\eta)\Phi''(\eta) - \Phi'(\eta)^2 + \Lambda\left(\Phi'(\eta) + \frac{1}{2}\eta\Phi''(\eta)\right) = 0 \quad (37)$$

such that substituting Eq. (36) into Eq. (37), we obtain

$$N_f(\Phi_0(\eta)) = (\alpha\eta + \beta)e^{-K\eta} \quad (38)$$

where

$$\alpha = \frac{1}{2}KA; \quad \beta = K^2 - 1 - Kf_w - \Lambda. \quad (39)$$

Heaving in view that in Eq. (38) appears an exponential function and that the auxiliary function $H_f(\eta, C_i)$ must follow the terms appearing in Eq. (38), then we can choose the function $H_f(\eta, C_i)$ in the following forms:

$$H_f(\eta, C_i) = C_1 + C_2\eta + (C_3 + C_4\eta)e^{-K\eta} + (C_5 + C_6\eta)e^{-2K\eta} \quad (40)$$

or

$$H_f(\eta, C_i) = C_1 + (C_2 + C_3\eta + C_4\eta^2)e^{-K\eta} \quad (41)$$

or yet

$$H_f(\eta, C_i) = C_1 + C_2\eta + C_3\eta^2 + (C_4 + C_5\eta)e^{-K\eta} + (C_6 + C_7\eta + C_8\eta^2)e^{-2K\eta} \quad (42)$$
and so on, where \( C_1, C_2, \ldots \) are unknown parameters at this moment.

If we choose only the expression (40) for \( H_f(\eta, C_i) \), then by using Eqs. (38), (40) and (31), we can obtain the equation in \( \Phi_1(\eta, C_i) \):

\[
\Phi_1''' - K^2 \Phi_1' = \left[ \beta C_1 + (\alpha C_1 + \beta C_2)\eta + \alpha C_2\eta^2 \right] e^{-K\eta} + \left[ \beta C_3 + (\alpha C_3 + \beta C_4)\eta + \alpha C_4\eta^2 \right] e^{-2K\eta} + \left[ \beta C_5 + (\alpha C_5 + \beta C_6)\eta + \alpha C_6\eta^2 \right] e^{-3K\eta}, \quad \Phi_1(0) = \Phi_1'(0) = \Phi_1'(\infty) = 0. \tag{43}
\]

The solution of Eq. (43) can be found as

\[
\Phi_1(\eta) = M_1 + \left[ N_1 + \left( \frac{7\alpha C_2}{4K^4} + \frac{3\alpha C_1}{4K^2} + \frac{3\beta C_2}{2K^2} \right) \eta + \left( \frac{3\alpha C_2}{4K^4} + \frac{\alpha C_1}{4K^2} + \frac{\beta C_2}{2K^2} \right) \eta^2 + \left( \frac{\alpha C_4}{6K^4} + \frac{\beta C_3}{6K^3} \right) \eta \right] e^{-K\eta} + \left[ -\frac{85\alpha C_4}{108K^5} - \frac{11\alpha C_3}{36K^4} - \frac{11\beta C_4}{36K^3} - \frac{11\alpha C_5}{18K^4} - \frac{13\alpha C_5}{288K^4} - \frac{13\beta C_6}{288K^4} - \frac{\beta C_5}{24K^3} \right] e^{-2K\eta} + \left[ -\frac{11\alpha C_6}{1728K^5} - \frac{13\alpha C_5}{288K^4} - \frac{13\beta C_6}{288K^4} - \frac{\beta C_5}{24K^3} \right] e^{-3K\eta}, \tag{44}
\]

where

\[
M_1 = -\frac{3\alpha + 2K\beta}{4K^4} C_1 - \frac{7\alpha + 3K\beta}{4K^5} C_2 - \frac{5\alpha + 6K\beta}{36K^4} C_3 - \frac{19\alpha + 15K\beta}{108K^5} C_4 - \frac{7\alpha + 12K\beta}{144K^4} C_5 - \frac{37\alpha + 42K\beta}{864K^5} C_6 - \frac{26\alpha + 12K\beta}{27K^5} C_4 + \frac{3\alpha + 4K\beta}{32K^4} C_5 + \frac{7\alpha + 6K\beta}{64K^5} C_6.
\]

The first-order approximate solution (29) for Eqs. (12) and (17) is obtained from Eqs. (39) and (45):

\[
\mathcal{T}(\eta) = \Psi(\eta) = \Phi_0(\eta) + \Phi_1(\eta). \tag{46}
\]

In what follows, we consider Eqs. (13) and (18). In this case, we choose the linear operator in the form

\[
L_\theta(\varphi(\eta)) = \varphi'' + K\varphi' \tag{47}
\]

where the parameter \( K \) is defined in Eq. (35).
Eq. (22) becomes
\[ \varphi_0'' + K \varphi_0' = 0, \quad \varphi_0(0) = 1, \quad \varphi_0(\infty) = 0. \] (48)

Eq. (48) has the solution
\[ \varphi_0(\eta) = e^{-K\eta}. \] (49)

The nonlinear operator \( N_\theta(\varphi(\eta)) \) is obtained from Eq. (12):
\[ N_\theta(\varphi(\eta)) = \frac{1}{Pr} \varphi'' + \Phi \varphi' - n\Phi' \varphi + \Lambda \left[ c \varphi + \frac{1}{2} \eta \varphi' \right]. \] (50)

Substituting Eq. (49) into Eq. (50), we obtain
\[ N_\theta(\varphi_0(\eta)) = (m_1 \eta + m_2) e^{-K\eta} + m_3 e^{-2K\eta} \] (51)

where
\[ m_1 = -\frac{1}{2} KA; \quad m_2 = \frac{K^2}{Pr} - K f_w - 1 + cA; \quad m_3 = 1 - n. \] (52)

The auxiliary function \( H_\theta(\eta, C_i) \) can be choose in the forms:
\[ H_\theta(\eta, C_i) = C_7 + C_8 \eta + (C_9 + C_{10} \eta) e^{-K\eta} + (C_{11} + C_{12} \eta) e^{-2K\eta} \] (53)
or
\[ H_\theta^*(\eta, C_i) = C_7 + C_8 \eta + C_9 \eta^2 + (C_{10} + C_{11} \eta) e^{-K\eta} + C_{13} e^{-2K\eta} \] (54)
or yet
\[ H_\theta^{**}(\eta, C_i) = C_7 + (C_8 + C_9 \eta) e^{-K\eta} + (C_{10} + C_{11} \eta) e^{-2K\eta} \] (55)

and so on, where \( C_7, C_8, \ldots \) are unknown parameters.

If we choose the Eq. (53) for \( H_\theta \), then from Eqs. (51), (53) and (31) we obtain the equation in \( \varphi_1(\eta, C_i) \) as
\[ \varphi_1'' + K \varphi_1' = \left[ m_2 C_7 + (m_1 C_7 + m_2 C_8) \eta + m_1 C_8 \eta^2 \right] e^{-K\eta} + \]
\[ + \left[ m_2 C_9 + m_3 C_7 + (m_1 C_9 + m_2 C_{10} + m_3 C_8) \eta + m_1 C_{10} \eta^2 \right] e^{-2K\eta} + \]
\[ + \left[ m_2 C_9 + m_2 C_{11} + (m_3 C_{10} + m_1 C_{11} + m_2 C_{12}) \eta + m_1 C_{12} \eta^2 \right] e^{-3K\eta} + \]
\[ + (m_3 C_{11} + m_3 C_{12} \eta) e^{-4K\eta}, \quad \phi_1(0) = \phi_1(\infty) = 0. \] (56)
Solving Eq. (56), we obtain

\[ \varphi_1(\eta) = \left[ P_1 - \frac{2m_1C_8}{K^3} + \frac{m_1C_7}{K^2} + \frac{m_2C_8}{K^2}\right] \eta - \left( \frac{m_1C_8}{K^2} + \frac{m_1C_7}{2K}\right) + \frac{m_2C_8}{2K}\eta^2 - \frac{m_1C_8}{3K} \eta^3 \right] e^{-\kappa \eta} + \left[ \frac{7m_1C_{10}}{4K^4} + \frac{m_1C_9}{4K^3} + \frac{3m_2C_{10}}{4K^3} + \frac{3m_3C_8}{4K^3}\right] + \frac{m_2C_9}{2K^2} + \frac{m_3C_7}{2K^2} + \left( \frac{3m_1C_{10}}{2K^3} + \frac{m_1C_9}{2K^2} + \frac{m_2C_{10}}{2K^2} + \frac{m_3C_7}{2K^2}\right) \eta + \frac{m_1C_{10}}{2K^2} \eta^2 e^{-2\kappa \eta} + \left[ \frac{5(m_3C_{10} + m_1C_{11} + m_2C_{12})}{36K^3} + \frac{m_3C_9 + m_2C_{11}}{6K^2}\right] + \frac{m_3C_{10} + m_1C_{11} + m_2C_{12}}{6K^2} + \frac{5m_1C_{12}}{18K^3} \eta + \frac{m_1C_{10}}{6K^2} \eta^2 e^{-3\kappa \eta} + \frac{7m_3C_{11}}{12K^2} + \frac{7m_3C_{12}}{144K^3} + \frac{m_3C_{12}}{12K^2} \eta) e^{-4\kappa \eta}, \right. \\
\left. \varphi_1(0) = \varphi_1(\infty) = 0, \right. \tag{57}

where

\[ P_1 = -\frac{m_3C_7}{2K^2} - \frac{3m_3C_8}{4K^3} - \frac{9m_1 + 6Km_2 + 2Km_3}{12K^3}C_9 - \frac{63m_1 + 27Km_2 + 5Km_3}{36K^4}C_{10} - \frac{5m_1 + K(m_2 + 3m_3)}{36K^3}C_{11} - \frac{20m_2 + 7m_3}{144K^3}C_{12}. \tag{58} \]

In this way, the first-order approximate solution (29) for Eqs. (13) and (18) becomes

\[ \overline{\theta}(\eta) = \overline{\varphi}(\eta) = \varphi_0(\eta) + \varphi_1(\eta,C_i). \tag{59} \]

5. Numerical examples

In order to prove the accuracy of the obtained results, we will determine the convergence-control parameters \( K \) and \( C_i \) which appear in Eqs. (46), (59) by means of the least square method. In this way, the convergence-control parameters are optimally determined and the first-order approximate solutions known for different values of the known parameters \( f_w, \Lambda, Pr, n \) and \( c \). In what follows, we illustrate the accuracy of the OHAM comparing previously obtained approximate solutions with the numerical integration results computed by means of the shooting method combined with fourth-order Runge-Kutta method using Wolfram Mathematica 6.0 software. For some values of the parameters \( f_w, \Lambda, Pr, n \) and \( c \) we will determine the approximate solutions.
Example 5.1.a For the first alternative given in the subsection 4.1, we consider \( f_w = -1, \Lambda = 1, c = \frac{1}{2}, n = 1, Pr = 0.7 \). For Eq. (46), following the procedure described above are obtained the convergence-control parameters:

\[
C_1 = -0.0881661632, \quad C_2 = 0.0159074525, \quad C_3 = 101.5499315816, \\
C_4 = -16.3157319695, \quad C_5 = -99.5951678657, \quad C_6 = -64.5910051875 \\
K = 0.7591636981
\]

and consequently the first-order approximate solution (46) can be written in the form:

\[
\overline{f}(\eta) = 0.4921333156 + (-0.5668554881 + 0.0071536463\eta - 0.0875181038\eta^2 + 0.0017461596\eta^3)e^{-0.7591636981\eta} + (-0.3980837570 - 7.4190413787\eta + 2.3591440003\eta^2)e^{-1.5183273963\eta} + (-0.5271940704 + 6.1764503488\eta + 2.3348551362\eta^2)e^{-2.2774910945\eta}
\]

(60)

Now, for Eq. (59), the convergence-control parameters are:

\[
C_7 = 0.0363993085, \quad C_8 = 0.0363993085, \quad C_9 = -7.1448075448, \\
C_{10} = 4.3237724702, \quad C_{11} = 42.1871800319, \quad C_{12} = 18.0805214975
\]

and therefore the first-order approximate solution (59) becomes:

\[
\overline{\theta}(\eta) = (0.3541683003 + 0.2415876957\eta - 0.0823906873\eta^2 + 0.0060665514\eta^3)e^{-0.7591636981\eta} + (-2.6848440528 + 6.3660521866\eta - 1.4238603876\eta^2)e^{-1.5183273963\eta} + (3.3306757524 - 3.3281240073\eta - 1.9846972772\eta^2)e^{-2.2774910945\eta}
\]

(61)

In Tables 1 and 2 we present a comparison between the first-order approximate solutions given by Eqs. (60) and (61) respectively, with numerical results for some values of variable \( \eta \) and the corresponding relative errors.

Example 5.1.b In this case, we consider \( f_w = -1, \Lambda = 1, c = \frac{1}{2}, n = 1, Pr = 2 \). The solution \( \overline{f}(\eta) \) is given by Eq. (46). The convergence-control
Table 1: Comparison between OHAM results given by Eq. (60) and numerical results for \( f_w = -1, \Lambda = 1 \)

| \( \eta \) | \( f_{\text{numeric}} \) | \( \mathcal{F}_{\text{OHAM, Eq. (60)}} \) | relative error = \( \left| f_{\text{numeric}} - \mathcal{F}_{\text{OHAM}} \right| \) |
|-------|-----------------|-----------------|-----------------|
| 0     | -1              | 0.9999999999    | 1.88 \cdot 10^{-15} |
| 1     | -0.1497942276   | -0.1501421749   | 3.47 \cdot 10^{-4}  |
| 2     | 0.3108643384    | 0.3106655367    | 1.98 \cdot 10^{-4}  |
| 3     | 0.4604991620    | 0.4600705386    | 4.28 \cdot 10^{-4}  |
| 4     | 0.4887865463    | 0.4894195278    | 6.32 \cdot 10^{-4}  |
| 5     | 0.4919308455    | 0.4923942736    | 1.08 \cdot 10^{-4}  |
| 6     | 0.4921389393    | 0.4918417250    | 2.97 \cdot 10^{-4}  |
| 7     | 0.4921471111    | 0.4919782168    | 1.68 \cdot 10^{-4}  |
| 8     | 0.4921472622    | 0.4922151064    | 6.78 \cdot 10^{-5}  |
| 9     | 0.4921472290    | 0.4923440748    | 1.96 \cdot 10^{-4}  |
| 10    | 0.4921472001    | 0.4923640175    | 2.16 \cdot 10^{-4}  |

parameters for Eq. (59) are:

\[
C_7 = 0.3992391297, \quad C_8 = -0.0398233823, \quad C_9 = 13.9195482545, \\
C_{10} = -9.8140323466, \quad C_{11} = -37.1866029355, \quad C_{12} = -48.0634410813
\]

such that the first-order approximate solution (59) becomes:

\[
\mathcal{F}(\eta) = (-2.1207041376 - 0.0561685488\eta + 0.0879369070\eta^2 - 0.0066372303\eta^3)e^{-0.75916369811\eta} + e^{-1.51832739639\eta}(7.9716346529 - 9.1921128520\eta + 3.2318564396\eta^2)e^{-0.75916369811\eta} + (-4.8509305153 + 8.0572360536\eta + 5.2759197606\eta^2)e^{-2.27749109459\eta}
\]  

(62)

In Table 3 we present a comparison between the first-order approximate solutions given by Eq. (62) with numerical results and corresponding relative errors.
Table 2: Comparison between OHAM results given by Eq. (61) and numerical results for \( f_w = -1, \Lambda = 1, c = \frac{1}{2}, n = 1, Pr = 0.7 \)

| \( \eta \) | \( \theta_{\text{numeric}} \) | \( \tilde{\theta}_{\text{OHAM}, \text{Eq. (61)}} \) | relative error = \( \left| \theta_{\text{numeric}} - \tilde{\theta}_{\text{OHAM}} \right| \) |
|---|---|---|---|
| 0 | 1 | 0.9999999999 | 8.88 \times 10^{-16} |
| 1 | 0.5325816311 | 0.5344078544 | 1.82 \times 10^{-3} |
| 2 | 0.2137609331 | 0.2123010871 | 1.45 \times 10^{-3} |
| 3 | 0.0624485224 | 0.0627998626 | 3.51 \times 10^{-4} |
| 4 | 0.0129724736 | 0.0141235139 | 1.15 \times 10^{-3} |
| 5 | 0.0019027817 | 0.0018865990 | 1.61 \times 10^{-5} |
| 6 | 0.0001968219 | -0.0002871987 | 4.84 \times 10^{-4} |
| 7 | 0.0000144329 | -0.0002516924 | 2.28 \times 10^{-4} |
| 8 | 8.27 \times 10^{-7} | 0.0000468974 | 4.60 \times 10^{-5} |
| 9 | 1.08 \times 10^{-7} | 0.0002281868 | 2.80 \times 10^{-4} |
| 10 | 7.47 \times 10^{-8} | 0.0002807824 | 2.80 \times 10^{-4} |

**Example 5.2.a** For \( f_w = 0, \Lambda = 1, c = \frac{1}{2}, n = 1, Pr = 0.7 \), the convergence-control parameters for Eq. (46) are:

\[
C_1 = -1.2640611927, \quad C_2 = 0.1680009020, \quad C_3 = -34.0575215187, \\
C_4 = 30.7898356526, \quad C_5 = 37.1281425060, \quad C_6 = 13.8590545976, \\
K = 1.1203766872
\]

and therefore, the first-order approximate solution (46) can be written in the form:

\[
\bar{f}(\eta) = 0.9662722752 + (1.3563995648 + 0.0351604322\eta - 0.1157605059\eta^2 + 0.0124958644\eta^3)e^{-1.1203766872\eta} + (-2.5490820161 - 1.7111113870\eta - 2.0440805123\eta^2)e^{-2.2407533744\eta} + (0.2264101761 - 0.7552406743\eta - 0.2300192808\eta^2)e^{-3.3611306616\eta}
\]  

(63)
Table 3: Comparison between OHAM results given by Eq. (62) and numerical results for $f_w = -1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$

| $\eta$ | $\theta_{\text{numeric}}$ | $\bar{\theta}_{\text{OHAM}}, \text{Eq. (62)}$ | relative error $= |\theta_{\text{numeric}} - \bar{\theta}_{\text{OHAM}}|$ |
|-------|---------------------------|---------------------------------|----------------------------------|
| 0     | 1                         | 0.99999999999                   | $1.77 \cdot 10^{-15}$           |
| 1     | 0.3341908857              | 0.3295767993                   | $4.61 \cdot 10^{-3}$            |
| 2     | 0.0347540513              | 0.0372485103                   | $2.49 \cdot 10^{-3}$            |
| 3     | 0.0011305745              | -0.0002320988                  | $1.36 \cdot 10^{-3}$            |
| 4     | 0.0000127262              | -0.0002852788                  | $2.98 \cdot 10^{-4}$            |
| 5     | -5.44 $\cdot 10^{-8}$    | 0.0002991504                   | $2.99 \cdot 10^{-4}$            |
| 6     | -9.15 $\cdot 10^{-8}$    | 0.0002884740                   | $2.88 \cdot 10^{-4}$            |
| 7     | -7.96 $\cdot 10^{-8}$    | 0.0001372878                   | $1.37 \cdot 10^{-4}$            |
| 8     | -7.05 $\cdot 10^{-8}$    | -0.0000295212                  | $2.94 \cdot 10^{-5}$            |
| 9     | -6.53 $\cdot 10^{-8}$    | -0.0001505702                  | $1.50 \cdot 10^{-4}$            |
| 10    | -5.99 $\cdot 10^{-8}$    | -0.0002044072                  | $2.04 \cdot 10^{-4}$            |

For Eq. (59), the convergence-control parameters are:

$$C_7 = -1.6947892627, \quad C_8 = 0.2632100295, \quad C_9 = -1.2815579214,$$

$$C_{10} = 2.6268699384, \quad C_{11} = 15.8897673738, \quad C_{12} = 9.5966071873$$

and the first-order approximate solution (59) is:

$$\bar{\theta}(\eta) = (0.2987405005 + 1.1383970916\eta - 0.4581387031\eta^2 + 0.0438683382\eta^3)e^{-1.1203766872\eta} + (-0.1000278783 + 1.4818560845\eta - 0.5861577557\eta^2)e^{-2.2407533744\eta} + (0.8012873778 - 0.5959066165\eta - 0.7137932043\eta^2)e^{-3.3611300616\eta}$$ (64)

In Tables 4 and 5 we present a comparison between the first-order approximate solutions given by Eqs. (63) and (64) respectively, with numerical results and corresponding relative errors.
Table 4: Comparison between OHAM results given by Eq. (63) and numerical results for $f_w = 0, \Lambda = 1$

| $\eta$ | $f_{\text{numeric}}$ | $\tilde{f}_{\text{OHAM}}$, Eq. (63) | relative error = $|f_{\text{numeric}} - \tilde{f}_{\text{OHAM}}|$ |
|--------|-----------------------|--------------------------------------|----------------------------------|
| 0      | -5.50 ·10^{-21}       | 4.44 ·10^{-16}                       | 4.44 ·10^{-16}                   |
| 1      | 0.6894348341          | 0.6894914970                        | 5.66 ·10^{-5}                   |
| 2      | 0.9167696529          | 0.9166682157                        | 1.01 ·10^{-4}                   |
| 3      | 0.9608821303          | 0.9609858144                        | 1.03 ·10^{-4}                   |
| 4      | 0.9659196704          | 0.9659030730                        | 1.65 ·10^{-5}                   |
| 5      | 0.9662619960          | 0.9661631962                        | 9.87 ·10^{-5}                   |
| 6      | 0.9662759513          | 0.9662663279                        | 9.62 ·10^{-6}                   |
| 7      | 0.9662762950          | 0.966395358                         | 6.32 ·10^{-5}                   |
| 8      | 0.9662763018          | 0.966301433                         | 7.38 ·10^{-5}                   |
| 9      | 0.9662763032          | 0.966306688                         | 5.43 ·10^{-5}                   |
| 10     | 0.9662763043          | 0.9663080318                        | 3.17 ·10^{-5}                   |

Example 5.2.b For $f_w = 0, \Lambda = 1, c = \frac{1}{2}, n = 1, Pr = 2$ the first-order approximate solution (46) is given by Eq. (63). The convergence-control parameters for Eq. (59) are determined as:

$$C_7 = 0.4652101281, C_8 = -0.0724620728, C_9 = 14.4924736065,$$

$$C_{10} = -12.1643720147, C_{11} = 0.7277740132, C_{12} = -56.0450834796$$

such that the first-order approximate solution (63) may be written as:

$$\bar{G}(\eta) = (-1.4030799687 + 0.1042610535\eta + 0.0880913412\eta^2 - 0.0120770121\eta^3)e^{-1.1203766872\eta} + (3.1476673319 - 3.8089261057\eta + 2.7143486992\eta^2)e^{-2.2407533744\eta} + (-0.7445873632 + 5.1973912018\eta + 4.1686190694\eta^2)e^{-3.3611300616\eta}$$

(65)

In Table 6 we compare between the first-order approximate solutions given by Eq. (65) with numerical results. The corresponding relative errors are also
Table 5: Comparison between OHAM results given by Eq. (64) and numerical results for $f_w = 0, \Lambda = 1, c = \frac{1}{2}, n = 1, Pr = 0.7$

| $\eta$ | $\theta_{\text{numeric}}$ | $\overline{\theta}_{\text{OHAM, Eq. (64)}}$ | relative error = $|\theta_{\text{numeric}} - \overline{\theta}_{\text{OHAM}}|$ |
|--------|-------------------------|---------------------------------|-------------------------------|
| 0      | 1.00                    | 1.00                            | 2.22 $\cdot 10^{-16}$         |
| 1      | 0.4003127445            | 0.4006174190                    | 3.04 $\cdot 10^{-4}$          |
| 2      | 0.1184290061            | 0.1183366534                    | 9.23 $\cdot 10^{-5}$          |
| 3      | 0.0250358122            | 0.0254649591                    | 4.29 $\cdot 10^{-4}$          |
| 4      | 0.0037386526            | 0.0032572108                    | 4.81 $\cdot 10^{-4}$          |
| 5      | 0.0003935287            | -0.0000242886                   | 4.17 $\cdot 10^{-4}$          |
| 6      | 0.0000291963            | 0.0001165652                    | 8.73 $\cdot 10^{-5}$          |
| 7      | 1.53 $\cdot 10^{-6}$   | 0.0003370019                    | 3.35 $\cdot 10^{-4}$          |
| 8      | 6.39 $\cdot 10^{-8}$   | 0.0003255701                    | 3.25 $\cdot 10^{-4}$          |
| 9      | 8.43 $\cdot 10^{-9}$   | 0.0002261157                    | 2.26 $\cdot 10^{-4}$          |
| 10     | 6.39 $\cdot 10^{-9}$   | 0.0001326393                    | 1.32 $\cdot 10^{-4}$          |

Example 5.3.a We consider $f_w = 1, \Lambda = 1, c = \frac{1}{2}, n = 1, Pr = 0.7$. The convergence-control parameters for Eq. (46) are given by:

\[
C_1 = 0.6287723857, \quad C_2 = -0.1379103919, \quad C_3 = -64.6127509553, \\
C_4 = 52.1862259014, \quad C_5 = 65.7049797786, \quad C_6 = 66.9471031457, \\
K = 1.6976766716.
\]

The first-order approximate solution (46) one can put as:

\[
\overline{f}(\eta) = 1.6119245343 + (-0.1095284867 - 0.0145745495 \eta + 0.0381089526 \eta^2 - 0.0067695650 \eta^3) e^{-1.6976766716 \eta} + (-0.6841717456 + 0.0590171180 \eta - 1.5089146740 \eta^2) e^{-3.3953533432 \eta} + (0.1817756979 - 0.6276022634 \eta - 0.4839278208 \eta^2) e^{-5.0930300148 \eta} \quad (66)
\]
Table 6: Comparison between OHAM results given by Eq. (65) and numerical results for $f_w = 0, A = 1, c = \frac{1}{2}, n = 1, Pr = 2$

| η   | $\theta_{\text{numeric}}$ | $\theta_{\text{OHAM}} \text{ Eq. (65)}$ | relative error = $|\theta_{\text{numeric}} - \theta_{\text{OHAM}}|$ |
|------|---------------------------|---------------------------------|----------------------------------|
| 0    | 1                         | 1                               | 0                                |
| 1    | 0.1197281855              | 0.1187072441                    | $1.02 \cdot 10^{-3}$            |
| 2    | 0.0042249398              | 0.0041010249                    | $1.23 \cdot 10^{-4}$            |
| 3    | $5.09 \cdot 10^{-5}$     | -6.04 $\cdot 10^{-6}$          | $5.69 \cdot 10^{-5}$            |
| 4    | $2.41 \cdot 10^{-7}$     | 0.0001841967                    | $1.83 \cdot 10^{-4}$            |
| 5    | $1.67 \cdot 10^{-8}$     | 0.0000163574                    | $1.63 \cdot 10^{-5}$            |
| 6    | $1.42 \cdot 10^{-8}$     | -0.0001452859                   | $1.45 \cdot 10^{-4}$            |
| 7    | $1.26 \cdot 10^{-8}$     | -0.0001791057                   | $1.79 \cdot 10^{-4}$            |
| 8    | $1.15 \cdot 10^{-8}$     | -0.0001403309                   | $1.40 \cdot 10^{-4}$            |
| 9    | $1.09 \cdot 10^{-8}$     | -0.0000887807                   | $8.87 \cdot 10^{-5}$            |
| 10   | $1.05 \cdot 10^{-8}$     | -0.0000493843                   | $4.93 \cdot 10^{-5}$            |

The convergence-control parameters for Eq. (59), are:

$$C_7 = -2.4317156290, \quad C_8 = 0.4199773974, \quad C_9 = 3.2538989273,$$

$$C_{10} = 2.8819189890, \quad C_{11} = 2.6307320394, \quad C_{12} = 0.0503403055$$

and the first-order approximate solution (59) becomes:

$$\bar{\theta}(\eta) = (0.0191058146 + 1.8994242629\eta - 0.7216780879\eta^2 +$$

$$+0.0699962329\eta^3)e^{-1.6976766716\eta} + (0.9928667494 +$$

$$+0.3712879543\eta - 0.4243916166\eta^2)e^{-3.3953533432\eta} + (-0.0119725641 -$$

$$-0.1258776771\eta - 0.0024710391\eta^2)e^{-5.0930300148\eta}$$

(67)

In Tables 7 and 8 we present a comparison between the first-order approximate solutions given by Eqs. (59) and (67) respectively, with numerical results and corresponding relative errors.
Table 7: Comparison between OHAM results given by Eq. (66) and numerical results for $f_w = 1, \Lambda = 1$

| η  | $f_{\text{numeric}}$ | $\mathcal{F}_{\text{OHAM, Eq. (66)}}$ | relative error = $|f_{\text{numeric}} - \mathcal{F}_{\text{OHAM}}|$ |
|----|---------------------|---------------------------------|----------------------------------|
| 0  | 1.00                | 1.00                            | $2.22 \cdot 10^{-16}$            |
| 1  | 1.5177074192        | 1.5176780223                    | $2.93 \cdot 10^{-5}$             |
| 2  | 1.6030516967        | 1.603050430                     | $1.66 \cdot 10^{-5}$             |
| 3  | 1.6114161917        | 1.6114348208                    | $1.86 \cdot 10^{-5}$             |
| 4  | 1.6119056438        | 1.6119031833                    | $2.46 \cdot 10^{-6}$             |
| 5  | 1.6119228465        | 1.6119073012                    | $1.55 \cdot 10^{-5}$             |
| 6  | 1.6119232066        | 1.6119136285                    | $9.57 \cdot 10^{-6}$             |
| 7  | 1.6119232084        | 1.6119199331                    | $3.27 \cdot 10^{-6}$             |
| 8  | 1.6119232063        | 1.611929505                     | $2.55 \cdot 10^{-7}$             |
| 9  | 1.6119232045        | 1.6119240509                    | $8.46 \cdot 10^{-7}$             |
| 10 | 1.6119232031        | 1.6119243982                    | $1.19 \cdot 10^{-6}$             |

Example 5.3.b For $f_w = 1, \Lambda = 1, c = \frac{1}{2}, n = 1, Pr = 2$ the first-order approximate solution for $\mathcal{F}(\eta)$ is given by Eq. (66).

For Eq. (59) the convergence-control parameters are given by:

$$C_7 = -0.3023817542, C_8 = 0.0519503347, C_9 = -27.1653468182,$$
$$C_{10} = 23.1898068679, C_{11} = -15.3543205956, C_{12} = 31.0226487950.$$  

The first-order approximate solution (59) one retrieves as:

$$\overline{\theta}(\eta) = (0.9205703595 - 0.1921602046\eta - 0.0487183578\eta^2 + +0.0086583891\eta^3)e^{-1.6976766716\eta} + (0.2637846962 + +0.9429053366\eta - 3.4149327807\eta^2)e^{-3.3953533432\eta} + (-0.1843550558 + -2.0986586159\eta - 1.5227992326\eta^2)e^{-5.0930001489\eta}$$  (68)

In Table 9 we present a comparison between the first-order approximate solutions given by Eqs. (68) with numerical results. The corresponding relative
Table 8: Comparison between OHAM results given by Eq. (67) and numerical results for $f_w = 1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 0.7$

| $\eta$ | $\theta_{\text{numeric}}$ | $\theta_{\text{OHAM}}$ Eq. (67) | relative error $= |\theta_{\text{numeric}} - \theta_{\text{OHAM}}|$ |
|--------|--------------------------|-------------------------------|---------------------------------|
| 0      | 1                        | 1                             | 0                               |
| 1      | 0.2625870984             | 0.2626175913                 | 3.04 $\cdot 10^{-5}$           |
| 2      | 0.0500304293             | 0.0500306615                 | 2.32 $\cdot 10^{-7}$           |
| 3      | 0.0067456425             | 0.0067634155                 | 1.77 $\cdot 10^{-5}$           |
| 4      | 0.0006411532             | 0.0006125219                 | 2.86 $\cdot 10^{-5}$           |
| 5      | 0.0000429270             | 0.0000457402                 | 2.81 $\cdot 10^{-6}$           |
| 6      | 2.01 $\cdot 10^{-6}$    | 2.08 $\cdot 10^{-5}$         | 1.88 $\cdot 10^{-5}$           |
| 7      | 5.14 $\cdot 10^{-8}$    | 1.35 $\cdot 10^{-5}$         | 1.34 $\cdot 10^{-5}$           |
| 8      | -1.29 $\cdot 10^{-8}$   | 6.14 $\cdot 10^{-6}$         | 6.16 $\cdot 10^{-6}$           |
| 9      | -1.33 $\cdot 10^{-8}$   | 2.24 $\cdot 10^{-6}$         | 2.25 $\cdot 10^{-6}$           |
| 10     | -1.22 $\cdot 10^{-8}$   | 7.13 $\cdot 10^{-7}$         | 7.25 $\cdot 10^{-7}$           |

 errores are presented.

Fig. 1 Solutions $\overline{f}_{\text{OHAM}}(\eta)$ given by (60), (63) and (66) for different values of $f_w$

—— numerical solution;

...... OHAM solution

Fig. 2 Solutions $\overline{f}_{\text{OHAM}}(\eta)$ obtained from (60), (63) and (66) for different values of $f_w$

—— numerical solution;

...... OHAM solution
Table 9: Comparison between OHAM results given by Eq. (68) and numerical results for $f_w = 1$, $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $Pr = 2$

| $\eta$ | $\theta_{\text{numeric}}$ | $\theta_{\text{OHAM}}, \text{Eq. (68)}$ | relative error = $|\theta_{\text{numeric}} - \theta_{\text{OHAM}}|$ |
|--------|-----------------|------------------|-----------------------|
| 0      | 1               | 1.00             | 2.22 · 10^{-16}       |
| 1      | 0.0288461240    | 0.0286378502     | 2.08 · 10^{-4}        |
| 2      | 0.0002627867    | 0.0004342087     | 1.71 · 10^{-4}        |
| 3      | 1.00 · 10^{-6}  | - 1.91 · 10^{-4} | 1.91 · 10^{-4}        |
| 4      | 1.21 · 10^{-7}  | - 1.46 · 10^{-4} | 1.46 · 10^{-4}        |
| 5      | 1.05 · 10^{-7}  | - 3.96 · 10^{-5} | 3.97 · 10^{-5}        |
| 6      | 9.39 · 10^{-8}  | - 4.54 · 10^{-6} | 4.63 · 10^{-6}        |
| 7      | 8.68 · 10^{-8}  | 1.08 · 10^{-6}   | 9.96 · 10^{-7}        |
| 8      | 7.95 · 10^{-8}  | 8.82 · 10^{-7}   | 8.02 · 10^{-7}        |
| 9      | 7.44 · 10^{-8}  | 3.60 · 10^{-7}   | 2.85 · 10^{-7}        |
| 10     | 6.91 · 10^{-8}  | 1.18 · 10^{-7}   | 4.89 · 10^{-8}        |

In Figs 1 and 2 are plotted the profiles of $\bar{f}(\eta)$ and velocity profile $\bar{f}'(\eta)$
respectively for different values of $f_w$. It is clear that the solution $\bar{f}(\eta)$ increases with an increase of $f_w$ and the velocity decrease with an increase of $f_w$. The condition $\bar{f}(\eta) > 0$ for $\eta > 0$ is satisfied.

In Figs. 3 - 7 are plotted the temperature profiles given for two values of Prandl number $Pr = 0.7$ and $Pr = 2$ respectively and different values of $f_w$. From Figs 3 and 4 it is observe that the temperature $\bar{\theta}(\eta)$ decreases with an increase of the $f_w$ for any values of parameter $Pr$.

From Figs. 5-7 we can conclude that the temperature decrease with of the Prandl number and different values of $f_w$.

![Fig. 5 Plots of $\bar{\theta}_{OHAM}(\eta)$ given by Eqs. (61) and (62) for $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $f_w = -1$ and two values of $Pr$ — numerical solution; ...... OHAM solution](image1)

![Fig. 6 Plots of $\bar{\theta}_{OHAM}(\eta)$ given by Eqs. (64) and (65) for $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $f_w = 0$ and two values of $Pr$ — numerical solution; ...... OHAM solution](image2)
Fig. 7 Plots of $\bar{\theta}_{OHAM}(\eta)$ given by Eqs. (67) and (68) for $\Lambda = 1$, $c = \frac{1}{2}$, $n = 1$, $f_w = 1$ and two values of $Pr$: — numerical solution; OHAM solution.

From Tables 1-9 we can summarize that the results obtained by means of OHAM are very accurate in comparison with the numerical results.

6. Conclusions

In this work, the Optimal Homotopy Asymptotic Method (OHAM) is employed to propose analytical approximate solutions to the flow and heat transfer in a viscous fluid over an unsteady stretching surface. For three values of the suction/injection parameter $f_w$, the problem admits solutions which are compared with numerical solutions computed by means of the shooting method combined with Runge-Kutta method and using Wolfram Mathematica 6.0 software. An analytical expressions for the heat transfer for two values of the Prandl number are obtained. The solution $f(\eta)$ increases with an increase of $f_w$ and velocity decreases with an increase of $f_w$. The temperature $\theta(\eta)$ decreases monotonically with the Prandl number and with the distance $\eta$ from the stretching surface.

Our procedure is valid even if the nonlinear equations of the motion do not contain any small or large parameters. The proposed approach is mainly based on a new construction of the solutions and especially on the involvement of the convergence-control parameters via the auxiliary functions. These parameters lead to an excellent agreement of the solutions with numerical results. This technique is very effective, explicit and accurate for nonlinear approximations rapidly converging to the exact solution after only one iteration. Also, OHAM
provides a simple but rigorous way to control and adjust the convergence of the solution by means of some convergence-control parameters. Our construction of homotopy is different from other approaches especially referring to the linear operator $L$ and to the auxiliary convergent-control function $H_f$ and $H_\theta$ which ensure a fast convergence of the solutions.

It is worth mentioning that the proposed method is straightforward, concise and can be applied to other nonlinear problems.

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