ON THE STRENGTH OF FIRST ORDER PHASE TRANSITIONS

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Electroweak baryogenesis may solve one of the most fundamental questions we can
ask about the universe, that of the origin of matter. It has become clear in the past
few years that it also poses a multi-faceted challenge. In order to compute the tiny
primordial baryonic excess, we probably must invoke physics beyond the standard model
(an exciting prospect for most people), we must push perturbation theory to its “limits”
(or beyond), and we must deal with nonequilibrium aspects of the phase transition. In
this talk, I focus mainly on the latter issue, that of nonequilibrium aspects of first order
transitions. In particular, I discuss the elusive question of “weakness”. What does it
mean to have a weak first order transition, and how can we distinguish between weak
and strong? I argue that weak and strong transitions have very different dynamics; while
strong transitions proceed by the usual bubble nucleation mechanism, weak transitions
are characterized by a mixing of phases as the system reaches the critical temperature
from above. I show that it is possible to clearly distinguish between the two, and discuss
consequences for studies of first order transitions in general.
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I. INTRODUCTION

Since the pioneering work of Becker and Döring on the nucleation of droplets in fluids
[1], the theoretical investigation of nonequilibrium aspects of first order phase transitions
has been of interest to researchers in subjects ranging from meteorology and materials
science to quantum field theory and cosmology. Phenomenological field theories were
developed by Cahn and Hilliard and by Langer [2] in the context of coarse-grained time-
dependent Ginsburg-Landau models, in which an expression for the decay rate of the
metastable stable state was obtained within a steady-state formulation [3]. The study
of metastable decay in zero-temperature quantum field theory was initiated by Voloshin,
Kobzarev and Okun [4] and soon after put onto firmer theoretical ground by Coleman
and Callan [5]. Finite temperature vacuum decay was first analysed by Linde [6] and has received considerable attention in the recent literature [7].

The main motivation for studying metastable decay for quantum fields comes from the possibility that the Universe underwent a series of phase transitions as it expanded from its initial hot, and presumably highly symmetric, state [8]. These cosmological phase transitions have the potential of not only answering many questions left open by the standard big-bang model, but also of serving as windows to high energy physics unaccessible to current (and future) terrestrial accelerators. However, as it has become increasingly clear during the past few years, in order to reliably compute quantities of interest, a deeper understanding of the nonequilibrium aspects of the transitions is needed. And we all know that nonequilibrium statistical mechanics is a hard subject; most of the nice universal properties that appear in equilibrium statistical mechanics are lost, and we are forced to study how nonlinearities will influence the approach to equilibrium of this or that particular system. But things are not all that bad. The very fact that we cannot but embark into the study of nonequilibrium dynamics in the context of finite temperature relativistic quantum field theories is also a blessing; as decades of research on the dynamics of phase transitions in condensed matter systems have shown, the subject is extremely rich, offering a wealth of interesting possibilities. It is tempting to speculate that this will also be the case for cosmological phase transitions and that new, unsuspected phenomena are lurking behind our present level of understanding.

This talk is organized as follows. In the next Section I briefly discuss some of the problems related with the weakness of the electroweak potential, such as its evaluation and possible scenarios for the dynamics of the transition. In Section 3 I present the results of a numerical experiment in which a very clear criterion to distinguish between “weak” and strong first order transitions emerges. In Section 4 I briefly review some of the work on subcritical bubbles emphasizing the qualitative agreement between the subcritical bubbles picture and the results of the numerical experiment of Section 3. In Section 5 I present some concluding remarks.

II. The Issue: The Weakness of the Electroweak Transition

Of the many interesting possibilities raised by primordial phase transitions, the generation of the baryon number of the Universe during the electroweak phase transition has been explored extensively since the seminal work of Kuzmin, Rubakov, and Shaposhnikov [9]. For the purpose of this talk, the important aspect of the electroweak phase transition is that it is, in most scenarios proposed so far (see, e.g. [10] for an alternative approach), a first order phase transition. And, at least within the context of the standard model of particle physics, the transition is very possibly a weak one; the standard computation for nucleation of critical bubbles shows that the thin-wall approximation fails and that the bubbles are rather thick [11]. In fact, this seems to be the case even for extensions of...
the standard model, which according to some authors offer better hope of producing the correct baryon asymmetry. (Contrast, for example, the contributions of Shaposhnikov and Turok in these Proceedings.)

The weakness of the transition poses a tremendous challenge even to the study of equilibrium properties of the system; improving the perturbative evaluation of the effective potential proves to be a very demanding and ungrateful task, as technical difficulties are compounded by the fact that nonperturbative effects must be called for to regulate the perturbative expansion near the symmetric phase [12]. Using alternative methods such as the $\epsilon$-expansion offer an interesting possibility, which, nevertheless relies on the success that these methods have on different systems [13]. Another alternative is to go to the computer and study the equilibrium properties of the standard model on the lattice [14]. Recent results are encouraging inasmuch as they seem to be consistent with perturbative results in the broken phase for fairly small Higgs masses. The transition also seems to be stronger than the perturbative estimates would predict. For larger Higgs masses (for this author, smaller than the 80 GeV claimed in Ref. [14] the interpretation of the results is not very straightforward, as finite-size effects become more important, and distinguishing the two phases by the double-peak structure of the distribution function becomes trickier.

It is clear from the above paragraph that much work must be done before we can claim we understand the equilibrium properties of the electroweak phase transition in the context of the standard model. Given that we do not know the exact shape of the effective potential for realistic Higgs masses, quantities such as the curvature of the potential around the symmetric minimum, the critical temperature of the transition, and maybe even the order of the transition are still unknown. Even if one goes beyond the standard model, as most people prefer, some of these problems persist.

However, one thing seems to be certain; that the transition is weakly first order. What does this mean exactly? “A weak first order phase transition...” In the context of electroweak baryogenesis, the usage of the term weak to characterize the transition is usually identified with the wall thickness of the nucleating bubbles. A weak transition would have typically thick bubbles, in that their radius is not much larger than the wall’s thickness. (Of course, in this case the definition of radius is somewhat blurred.) On the other hand, the transition cannot be too weak or not enough baryon number is generated. This is equated with the discontinuity in the magnitude of the Higgs field at the critical temperature, $\langle \phi(T_c) \rangle / T \lesssim 1$. As I hope will be clear later on, this definition of weak, or not too weak, is not enough. Looking at the effective potential (assuming you know it), identifying a barrier between the symmetric and broken-symmetric phase, and proceeding to use the nucleation rate formula for critical bubbles is not necessarily the right thing to do. The reason for this is simple; the vacuum decay formula is obtained by assuming that there is a nicely behaved, nearly-homogeneous background about which we expand
the partition function in order to obtain the semi-classical approximation. The reader
is referred to Ref. [15] for details. However, for small enough barriers between the two
phases, large amplitude fluctuations will be present, invalidating the assumption of a near-
homogeneous background. Homogeneous nucleation breaks down for these situations.
Two questions immediately come to mind. When does homogeneous nucleation break
down, and what should one do when it does? (Getting drunk on Port is not a very good
long term solution, posing problems immediately after you wake up the next day, if you
wake up the next day.)

With these questions in mind, and motivated by the “weakness” of the electroweak
transition, a few years ago Gleiser and Kolb (GK) proposed a novel mechanism by which
such transitions evolve [16]. Rather than having nucleation of thick bubbles below the
critical temperature of the transition, the transition would be characterized by a substan-
tial phase mixing as the critical temperature is approached from above. GK modelled
the dynamics of this phase mixing by estimating the fraction of the volume occupied
by subcritical (correlation volume) thermal fluctuations of each phase at the critical
temperature. In their initial approach, they neglected the fact that these subcritical fluc-
tuations were unstable and thus were criticized on the grounds that they overestimated
the equilibrium fraction of the volume in the broken-symmetric phase [17]. In their sim-
ples analytical approach the equilibrium fraction of both phases should be an identical 0.5
each when the two phases are degenerate (at \( T_c \)). Their method was recently extended
by the authors of Ref. [18] to include the shrinking of the bubbles. The results of Ref.
[18] indicate that GK are at least qualitatively correct; there will be a regime in which
the transition is weak enough that considerable phase mixing occurs above \( T_c \). (It is, of
course, possible that this interesting regime lies beyond the validity of the perturbative
evaluation of the electroweak effective potential. Presently this does not appear to be so
[18].) I will briefly review the subcritical bubbles method later on. Now I would rather
proceed with the discussion of the strength of the transition.

III. Distinguishing the Weak from the Strong: A Simple Model

In order to sharpen the distinction between weak and strong first order transitions,
I decided to investigate this question within the context of a simple toy model in 2+1
dimensions which could be studied numerically [19]. Due to the complex nonequilibrium
nature of the system, any analytical approach (at least those proposed so far) is bound
to be severely limited. One is justified in regarding these simple models with suspicion.
The need for a numerical investigation of this question is clear. This need is even more
justified by noting that several of the gross features of the electroweak transition may
appear in other unrelated physical systems, such as nematic liquid crystals and certain
magnetic materials below their critical temperature. Moreover, numerical simulations
of first-order transitions in the context of field theories (as opposed to discrete Ising
models [20]) are scarce. Recent work has shown that the effective nucleation barrier is accurately predicted by homogeneous nucleation theory in the context of 2+1-dimensional classical field theory [21]. These results were obtained for strong transitions, in which the nucleation barrier $B$ was large. Nucleation was made possible due to the fairly large temperatures used in the simulations. (Recall that the decay time is proportional to $\exp(B/T)$.)

In order to study how the weakness of the transition will affect its dynamics, the homogeneous part of the free-energy density (the effective potential to some order in perturbation theory) is written as follows

$$U(\phi, T) = \frac{a}{2} (T^2 - T_2^2) \phi^2 - \frac{\alpha}{3} T \phi^3 + \frac{\lambda}{4} \phi^4.$$

(1)

With this parameterization, the free-energy resembles the finite-temperature effective potential used in the description of the electroweak transition, where $\alpha$ is determined by the masses of the gauge bosons and $T_2$, the spinodal instability temperature, is related to the zero-temperature mass of the Higgs boson [9]. Here, we will not be concerned with the limits of validity of this effective potential in the context of electroweak transitions. The goal is to explore the possible dynamics of a transition with free energy given by Eq. 1. This free-energy is also remarkably similar to the de Gennes-Ginzburg-Landau free energy used in the study of the nematic-isotropic transition of liquid crystals [22]. This transition is known to be weakly first-order; there is a discontinuity in the order parameter, even though there is no release of latent heat [23]. In fact, departures from mean field estimates for the correlation length were detected as the degeneracy temperature is approached from above, signalling the presence of “pseudo-universal phenomena”, characterized by long-wavelength fluctuations observed by light-scattering experiments.

In $2+1$ dimensions, it proves convenient to introduce dimensionless variables, $\tilde{x} = x/\sqrt{\alpha T_2}$, $\tilde{t} = t/\sqrt{\alpha T_2}$, $X = \phi/\sqrt{T_2}$, $\theta = T/T_2$, so that we can write the Hamiltonian (free energy) as

$$\frac{H[X]}{\theta} = \frac{1}{\theta} \int d^2 \tilde{x} \left[ \frac{1}{2} | \nabla X |^2 + \frac{1}{2} \left( \theta^2 - 1 \right) X^2 - \frac{\tilde{\alpha}}{3} \theta X^3 + \frac{\tilde{\lambda}}{4} X^4 \right],$$

(2)

where $\tilde{\alpha} = \alpha/(a\sqrt{T_2})$, and $\tilde{\lambda} = \lambda/(a T_2)$. (From now on the tildes will be dropped.) For temperatures above $\theta_1 = (1 - \alpha^2/4\lambda)^{-1/2}$ there is only one minimum at $X = 0$. At $\theta = \theta_1$ an inflection point appears at $X_{\text{inf}} = \alpha \theta_1 / 2 \lambda$. Below $\theta_1$ the inflection point separates into a maximum and a minimum given by $X_{\pm} = \alpha \theta / 2 \lambda \left[ 1 \pm \sqrt{1 - 4 \lambda (1 - 1/\theta^2)} / \alpha^2 \right]$. At the critical temperature $\theta_c = (1 - 2\alpha^2/9\lambda)^{-1/2}$ the two minima, at $X_0 = 0$ and $X_\pm$ are degenerate. Below $\theta_c$ the minimum at $X_+$ becomes the global minimum and the $X_0$-phase becomes metastable. Finally, at $\theta = 1$ the barrier between the two phases disappears. Note that $\alpha^2 < 4 \lambda$ for a solution as described above.
In order to study numerically the approach to equilibrium at a given temperature $\theta$, the coupling of the order parameter $X$ with the thermal bath will be modelled by a Markovian Langevin equation,

$$\frac{\partial^2 X}{\partial t^2} = \nabla^2 X - \tilde{\eta} \frac{\partial X}{\partial t} - \frac{\partial U(X, \theta)}{\partial X} + \tilde{\xi}(x, t) ,$$  

(3)

where $\tilde{\eta} = \eta/\sqrt{a}T_2$ is the dimensionless viscosity coefficient, and $\tilde{\xi} = \xi/aT_2^{5/2}$ is the dimensionless stochastic noise with vanishing mean, related to $\eta$ by the fluctuation-dissipation theorem,

$$\langle \xi(\vec{x}, t)\xi(\vec{x}', t') \rangle = 2\eta T \delta(t - t')\delta^2(\vec{x} - \vec{x}') .$$  

(4)

The viscosity coefficient was set to unity in all simulations. Two important comments are in order. In principle it should be possible to obtain a Langevin-like equation for the slower long-wavelength modes from a microscopic approach by integrating out the short-wavelength modes which have faster relaxation time-scales. This programme is quite complicated in the context of relativistic field theories since dissipation is a two-loop effect and progress has been slow [24]. Recent results indicate that one should expect corrections to the above equation, as noise will in general be colored and multiplicative as opposed to additive as above [25]. It is possible that these corrections will change time-dependent quantities, such as equilibration time-scales and nucleation rates, although they should not affect final equilibrium properties of the system, such as the fractional volume in each phase, or its critical properties, which are most important here. In lack of a better understanding of the microscopic dynamics of such systems, the above equation will be adopted here as a starting point. A second important point is to note that as this equation will be solved on a lattice, the lattice spacing works as an effective hard-momentum cut-off. Therefore, the lattice formulation is an effective coarse-grained formulation of the continuum theory and one should be careful when mapping from the lattice to the continuum theory. If one is to probe physics at short-wavelengths, renormalization counterterms should be included in the lattice formulation so that the results are cut-off independent and a proper continuum limit is obtained. This point was emphasized and renormalization counterterms obtained for a temperature-independent potential in the nucleation study of Alford and Gleiser in Ref. [21]. Here, due to the temperature dependence of the potential, the renormalization prescription of Alford and Gleiser does not work. Instead, the lattice spacing will be fixed at $\ell = 1$. It turns out that in all cases of interest here the mean-field correlation length $\xi^{-2} = U''(X_0, \theta)$ (not to be confused with the random noise) will be sufficiently larger than unity to justify this choice.

The Langevin equation was integrated using the fifth-degree Nordsiek-Geer algorithm which allows for fast integration with high numerical accuracy [26]. The time step used
was $\delta t = 0.2$, and the results discussed here were obtained with a square lattice with $L = 64$. (Comparison with $L = 40$ and $L = 128$ produced negligible differences for our present purposes.) No dependence of the results was found on the lattice length, time-step, random noise generator, and the random noise seed.

The strategy adopted was to study the behavior of the system given by Eq. [2] at the critical temperature when the two minima are degenerate. The reason for this choice of temperature is simple. If at $\theta_c$ most of the system is found in the $X = 0$ phase then as the temperature drops below $\theta_c$ one expects homogeneous nucleation to work; the system is well-localized in its metastable phase. This is what happens when a system is rapidly cooled below its critical temperature (rapid quench) so that it finds itself trapped in the metastable state. The large amplitude fluctuations which will eventually appear are the nucleating bubbles. If at $\theta_c$ one finds a large probability for the system to be in the $X^+\!-$ phase, then considerable phase-mixing is occurring and homogeneous nucleation should not be accurate in describing the transition. Large amplitude fluctuations are present initially in the system, before it is quenched to temperatures below $\theta_c$. For definiteness call the two phases the 0-phase and the $\pm$-phase. The phase distribution of the system can be measured if the idea of fractional area (volume in 3 dimensions) is introduced. As the field evolves according to Eq. [3], one counts how much of the total area of the lattice belongs to the 0-phase with $X \leq X_-$ (i.e. to the left of the maximum), and how much belongs to the $\pm$-phase with $X > X_-$ (i.e. to the right of the maximum). Dividing by the total area one obtains the fractional area in each phase, so that $f_0(t) + f_+(t) = 1$, independently of $L^2$.

The system is prepared initially in the 0-phase, $f_0(0) = 1$ and $f_+(0) = 0$. Thus, the area-averaged value of the order parameter, $\langle X \rangle(t) = A^{-1} \int X dA$ is initially zero. The coupling with the thermal bath will induce fluctuations about $X = 0$. By keeping $\lambda = 0.1$ fixed, the dependence of $f_0(t)$, $f_+(t)$, and $\langle X \rangle(t)$ on the value of $\alpha$ can be measured. Larger values of $\alpha$ imply stronger transitions. This is clear from the expression for $\theta_c$ which approaches unity as $\alpha \rightarrow 0$. That is, for small $\alpha$ the critical temperature approaches the spinodal temperature. (In the electroweak case, the same argument applies, as what is relevant is the ratio $\alpha^2/\lambda$; $\alpha$ is fixed but $\lambda$ increases as the Higgs mass increases.) The results are shown in Figure 1 for several values of $\alpha$ between $\alpha = 0.3$ and $\alpha = 0.4$. Each one of these curves is the result of an ensemble average over 200 runs. The two important features here are the final equilibrium fraction in each phase and the equilibration time-scale.

The approach to equilibrium can be fitted at all times to a slow exponential,

$$f_0(t) = \left(1 - f_0^{EQ}\right) \exp\left[-(t/\tau_{EQ})^\sigma\right] + f_0^{EQ},$$ \hspace{1cm} (5)

where $f_0^{EQ}$ is the final equilibrium fraction and $\tau_{EQ}$ is the equilibration time-scale. In Table 1 the values of $\tau_{EQ}$ and $\sigma$ are listed for different values of $\alpha$. 

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Note that the slot for $\alpha = 0.36$ is empty. For this value of $\alpha$ the approach to equilibrium cannot be fitted at all times to a slow exponential; at large times it must be fitted to a power law,

$$f_0(t) \mid_{\alpha=0.36} \propto t^{-k},$$

where $k$ is the critical exponent controlling the approach to equilibrium. An excellent fit is obtained for $k = 0.25 \pm 0.05$, as shown in Figure 2.

**Figure 1**: The approach to equilibrium for several values of $\alpha$.

**Table 1**: The values of the equilibration time-scales and the exponents for the exponential fit of Eq. 5 for several values of $\alpha$. Also shown are the equilibrium fractions $f_0(\theta_c)$ and $f_+(\theta_c)$. Uncertainties are in the last digit.

| $\alpha$ | $\tau_{EQ}$ | $\sigma$ | $f_0(\theta_c)$ | $f_+(\theta_c)$ |
|----------|------------|--------|-----------------|-----------------|
| 0.30     | 21.0       | 0.80   | 0.505           | 0.495           |
| 0.33     | 40.0       | 0.80   | 0.514           | 0.486           |
| 0.35     | 75.0       | 0.60   | 0.525           | 0.475           |
| 0.36     |            | 0.580  | 0.420           |                 |
| 0.37     | 25.0       | 0.65   | 0.800           | 0.200           |
| 0.38     | 15.0       | 0.80   | 0.870           | 0.130           |
| 0.40     | 5.0        | 1.0    | 0.937           | 0.063           |

The fact that there is a slowing down of the dynamics for $\alpha = 0.36$ is indicative of the presence of a phase transition near $\alpha \simeq 0.36$. This transition reveals itself in a striking way if we define as an order parameter the equilibrium fractional difference $\Delta F_{EQ}$,

$$\Delta F_{EQ} = f^\text{EQ}_0 - f^\text{EQ}_+. \quad (7)$$

**Figure 2**: Fitting $f_0(\theta_c)$ to a power law at large times for $\alpha = 0.36$. 

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In Figure 3 $\Delta F_{\text{EQ}}(\theta_c)$ is plotted as a function of $\alpha$. Clearly, there is a sharp transition in the behavior of the system around $\alpha = \alpha_c \simeq 0.36$. For $\alpha < \alpha_c$ the fractional area occupied by both phases is practically the same at 0.5. There is considerable mixing of the two phases, with the system unable to distinguish between them. One may call this phase the symmetric phase with respect to the order parameter $\Delta F_{\text{EQ}}$. For $\alpha > \alpha_c$ there is a clear distinction between the two phases, with the $+$-phase being sharply suppressed. This may be called the broken-symmetric phase. It is clear from the behavior of $\Delta F_{\text{EQ}}$ with $\alpha$ that this is a second order phase transition. The rounding of the curve about $\alpha_c$ is due to the finite size of the lattice. This curve is to be compared with the behavior of the magnetization as a function of the temperature in Ising models. $\alpha$ plays the rôles of “inverse” temperature; small $\alpha$ means high temperature, when symmetry is restored.

As a consequence of this behavior a very clear distinction between a strong and weak transition is possible. A strong transition has $\alpha > \alpha_c$ so that the system is dominated by the 0-phase at $\theta_c$. For a weak transition neither phase clearly dominates the system, and as argued above, the dynamics should be quite different from the usual nucleation mechanism. A mixing of the two phases occurs as the system approaches the critical temperature from above.

It should be possible to parameterize the behavior of $\Delta F_{\text{EQ}}$ in the neighborhood of $\alpha_c$ with a power law,

$$\Delta F_{\text{EQ}} \propto (\alpha - \alpha_c)^\beta,$$

where $\beta$ is the critical exponent controlling the behavior of $\Delta F_{\text{EQ}}$ near $\alpha_c$. The determination of this critical exponent and of a more precise value of $\alpha_c$ will be left for a future publication as it involves a detailed analysis of finite-size scaling [20].

**Figure 3**: The fractional equilibrium population difference $\Delta F_{\text{EQ}}$ as a function of $\alpha$. 

In order to understand the reason for the sharp change of behavior of the system near $\alpha_c$, in Figure 4 the equilibrium area-averaged order parameter $\langle X \rangle_{\text{EQ}}$ and the inflection point $X_{\text{inf}} = \frac{a_0}{3\lambda} \left[ 1 - \sqrt{1 - 3\lambda(1 - 1/\theta^2)/\alpha^2} \right]$, are shown as a function of $\alpha$. Also shown is the rms amplitude of correlation-size fluctuations $X_{\text{rms}}^2 = \langle X^2 \rangle_T - \langle X \rangle_T^2$, where $\langle \cdots \rangle_T$ is the normalized thermal average with probability distribution $P[X_{\text{sc}}] = \exp[-F[X_{\text{sc}}]/\theta]$. $F[X_{\text{sc}}]$ is the free energy of a gaussian-shape subcritical fluctuation. For details see Ref. [27]. It is clear from this Figure that the transition from weak to strong occurs as $\langle X \rangle_{\text{EQ}}$ drops below $X_{\text{inf}}$. This can be interpreted as an effective Ginzburg criterion for the weak-to-strong transition. It matches quite well the fact that the critical slowing down occurs for $\alpha \simeq 0.36$. This result is in qualitative agreement with the study of Langer et al. contrasting the onset of nucleation vs. spinodal decomposition for binary fluid and solid solutions [28], where it was found that the transition between the two regimes occurs
roughly at the spinodal \((i.e. \text{ at the inflection point})\). Note, however, that here we are dealing with relativistic field theories, while the work of Langer and collaborators had to do with phenomenological models of phase transitions for systems with conserved order parameter. These systems will typically have slower dynamics than the field theories of interest in cosmology. Also, for Langer and his collaborators, as in most applications in condensed matter physics, the dynamics is studied \text{ after} the system is quenched (rapidly cooled) to below its critical temperature. In cosmology, the cooling is provided by the expansion rate of the universe.

Even though \(X_{\text{rms}}\) drops below \(X_{\text{inf}}\) for a smaller value of \(\alpha\), being a much less computer intensive quantity to obtain, it should serve as a rough indicator of the weak-to-strong transition.

Figure 4: Comparison between area-averaged field and location of the inflection point as a function of \(\alpha\). Also shown are the location of the barrier, \(X_{\text{MAX}}\), and the rms fluctuation \(X_{\text{rms}}\).

To summarize the results of this Section, the distinction between a weak and a strong first order transition can be made quantitative by studying the behavior of the effective free energy \text{ at the critical temperature}. A clear change in behavior occurs at a critical value of the parameter used to define the “strength” of the barrier separating the two phases. Here, the cubic coupling was chosen as an example. For values of the parameter larger than the critical value, the system is mostly in the symmetric phase and the transition should be well described by homogeneous nucleation theory. For values of the parameter below its critical value, the system is in a mixed phase and no phase is preferred. (Of course this assumes that the system is being cooled down slowly compared to its typical fluctuation rates. If we rapidly quench the system to below its critical temperature and then keep cooling it further, it will not reach the mixed-phase state. In cosmology this would correspond to a \text{ very early transition}, when the universe’s expansion rate is relatively fast compared to typical fluctuation rates in the system.) In this case, the transition will proceed by a mechanism closer to spinodal decomposition, although more studies are needed to obtain a clear picture of the approach to equilibrium in this case.

The transition between weak and strong is itself a (second order) phase transition, with the equilibrium fractional population difference playing the rôle of the order parameter.

Finally let me stress that these results are not particular to 2+1 dimensions. Indeed, recent work on 3+1 dimensions produced qualitatively identical results. There is a second order transition for a critical value of \(\alpha\) which for a given value of \(\lambda\), will be lower than the 2+1 dimensional value. (Systems fluctuate more in two dimensions.) In order to make (some) contact with the electroweak transition, we should investigate how \(\lambda\) (related to the Higgs mass) varies for fixed \(\alpha\) (related to the mass of the gauge bosons). Even though
we investigated a model with a real scalar field, this should give us an indication of the transition from weak to strong in the electroweak case as well [29].

IV. Subcritical Bubbles: A Simple Approach to Nonequilibrium Dynamics

Now I finally, and briefly, turn to the subject of subcritical bubbles, as a possible method to study the approach to equilibrium in cosmology (and in the laboratory). Consider a system described by an effective (coarse-grained) free-energy density as discussed above, for example. If we prepare the system in the symmetric phase (or, in the example above –with no symmetry– at $\langle \phi \rangle = 0$), at any temperature there will be fluctuations which will probe the other phase. These fluctuations will be suppressed by a Boltzmann factor. The larger the amplitude of the field and the larger the volume of the fluctuation, the larger the suppression. Also, once they appear, they will disappear, unless the system is below the critical temperature and they happen to be larger than the critical fluctuations for nucleation. However, at a given temperature, they will always be there. The whole discussion and results of the previous Section offer convincing evidence that systems fluctuate about their equilibrium values, sometimes quite dramatically so. (The nice thing about numerical –and most– experiments is that they are reproducible. You can, if so disposed, always convince yourself that there is a clear distinction between “weak” and strong first order transitions.) The subcritical bubbles method was proposed in order to obtain a semi-analytic description of these fluctuations so that we can examine their importance. In the light of the previous results, for strong transitions they should be irrelevant, while for weak transitions they are crucial. In passing, I note that subcritical fluctuations of the broken-symmetric phase within the symmetric phase have been observed above the critical temperature in the isotropic-nematic liquid crystal transition [30]. Nematic fluctuations within the isotropic phase were identified and their relaxation time measured, in order to study departures from mean-field theory predictions. As expected, only in the neighborhood of the critical temperature substantial departures from mean-field were observed.

Expanding on the GK approach, Gelmini and Gleiser (GG) obtained a kinetic equation that incorporates both the shrinking of the subcritical bubbles, and their possible “destruction” by thermal noise [18]. If $n(R,t)$ is the number density of subcritical bubbles of radius $R$ of the broken-symmetric phase within the symmetric phase, the rate equation is

$$\frac{\partial n(R,t)}{\partial t} = - \frac{\partial n(R,t)}{\partial R} \left( \frac{dR}{dt} \right) + \left( \frac{V_0}{V} \right) \Gamma_{0\rightarrow+}(R)$$

$$- \left( \frac{V_+}{V} \right) \Gamma_{+\rightarrow0}(R) - \left( \frac{V_+}{V} \right) \Gamma_{TN}(R)$$

Here, $\Gamma_{0\rightarrow+}(R)$ ($\Gamma_{+\rightarrow0}(R)$) is the rate per unit volume for the thermal nucleation of a
bubble of radius $R$ of phase $\phi = \phi_+$ within the phase $\phi = 0$ (phase $\phi = 0$ within the phase $\phi_+$). $\Gamma_{T N}(R) \simeq a T/4 \pi R^3$ is the (somewhat ad hoc) expression used for the thermal destruction rate, with $a$ a constant. Also, $V_+$ must be understood as the volume of the $(\pm)$-phase in bubbles of radius $R$ only, since we are following the evolution of $n(R, t)$.

In order to obtain analytical solutions of this equation, GG solved it only for temperatures just below the temperature when the broken-symmetric minimum appears which ($\theta_1$ in the model above), of course, is above the critical temperature. For this temperature, most of the system will always be in the symmetric phase and we can write $V_0/V \simeq 1$. Another important assumption is that most of the subcritical bubbles will be of correlation volume. This is due to the fact that larger fluctuations will be suppressed, while smaller fluctuations are inconsistent with the coarse-graining procedure. With these assumptions, it is possible to solve the kinetic equation and obtain two crucial quantities; the equilibration time-scale typical for each of the processes that suppress subcritical bubbles (shrinking, thermal nucleation of a subcritical bubble of the symmetric phase inside a region of broken symmetric phase, and thermal destruction), and the equilibrium number density of these bubbles. This way we can distinguish which process is the dominant process for the suppression of subcritical bubbles for different parameters of the free energy density. Applying this formalism to the 1-loop electroweak potential, GG showed that for Higgs masses above 55 GeV or so, considerable phase mixing is occurring even for temperatures above the critical temperature. Thus, the subcritical bubble picture is in excellent qualitative agreement with the numerical results described in the previous Section; for weak enough transitions we should expect substantial departures from the usual vacuum decay mechanism.

V. Concluding Remarks

In this talk I discussed some of the issues related to the dynamics of weak vs. strong first order phase transitions. As I hope was made clear, if indeed the electroweak phase transition is “weak” in the sense defined here, novel aspects of nonequilibrium dynamics will have to be taken into account when dealing with the computation of the net baryon number generated in a given model. Taken at face value, the results here may be bad news for electroweak baryogenesis in the context of the standard model. Even if lattice computations show that for Higgs masses of 80 GeV or so the transition is still strong, the Higgs may weigh much more than that. This being the case, there will always be a regime in which phase mixing will occur and nucleation theory will fail. On the other hand, apart from hand-waiving arguments, it is not clear that the domain coarsening dynamics that will take place in a weak transition will not produce a net baryon number. At present, we simply do not know enough about the nature of the approach to equilibrium to decide on this issue, or set it aside. If baryon number is to be generated in extensions of the standard model, then the results here will provide a useful constraint in the usually
large parameter space of these models. In order to have a strong enough transition, the Ginzburg-like criterion discussed here must be satisfied, and at least one parameter of the model may be eliminated this way. If the transition in these models is weak enough, one should again expect departures from the standard vacuum decay estimates for bubble nucleation rates. This is due to the inescapable conclusion that hot systems fluctuate, and if they can these fluctuations will produce some dramatic effects. Critical opalescence in nematic liquid crystals is but one example of these “pre-transitional” phenomena in nature. It may well be that they will also be of crucial importance in cosmological phase transitions.

The possibility of generating the baryon number of the universe during the electroweak phase transition is a tremendous challenge to present research in the cosmology/high energy physics interface. We most probably must invoke physics beyond the standard model to get enough CP violation, we must tackle hard problems related to infrared divergences in gauge theories, and, last but not least, we must understand the nonequilibrium aspects of phase transitions in the context of field theories in a cosmological background. Judging from what happened during the past 5 years or so, and by the number of people working in this topic, progress will keep coming fast. Maybe in another 5 years, we should all get together again (hopefully in Sintra) to see how far we managed to get.
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