On a Simultaneous Approach to the Even and Odd Truncated Matricial Hamburger Moment Problems

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Dedicated to Lev Aronovich Sakhnovich on the occasion of his 80th birthday

The main goal of this paper is to achieve a simultaneous treatment of the even and odd truncated matricial Hamburger moment problems in the most general case. In the odd case, these results are completely new for the matrix case, whereas the scalar version was recently treated by V. A. Derkach, S. Hassi and H. S. V. de Snoo [12]. The even case was studied earlier by G.-N. Chen and Y.-J. Hu [9]. Our approach is based on Schur analysis methods. More precisely, we use two interrelated versions of Schur-type algorithms, namely an algebraic one and a function-theoretic one. The algebraic version was worked out in a former paper of the authors. It is an algorithm which is applied to finite or infinite sequences of complex matrices. The construction and investigation of the function-theoretic version of our Schur-type algorithm is a central theme of this paper. This algorithm will be applied to relevant subclasses of holomorphic matrix-valued functions of the Herglotz-Nevanlinna class. Using recent results on the holomorphicity of the Moore-Penrose inverse of matrix-valued Herglotz-Nevanlinna functions, we obtain a complete description of the solution set of the moment problem under consideration in the most general situation.

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1. **Introduction**

The investigation of matrix versions of the Hamburger moment problem was initiated in the second half of the 1940’s by M. G. Krein (see [35, 36]). Mainly, based on new approaches (V. P. Potapov’s method of fundamental matrix inequalities, reproducing kernel Hilbert spaces, Schur analysis) a renaissance of this topic could be observed in the last three decades (see e.g. the papers Kovalishina [34], Katsnelson [32, 33], Dym [17], Bolotnikov [7], Chen/Hu [9] and the monographs Sakhnovich [41] and Bakonyi/Woerdeman [5]).

This paper continues the authors’ investigations on matricial versions of the truncated Hamburger moment problem (see [19, 21, 24]). It was inspired by the recent work by V. A. Derkach, S. Hassi and H. S. V. de Snoo on indefinite versions of truncated moment problems in the class of scalar generalized Nevanlinna functions (see [12]). As a byproduct they obtained a result (see [12, Corollary 5.2]) which seems to be new even for the odd truncated scalar Hamburger moment problem. More precisely, if \( n \) is a non-negative integer and if \( (s_j)_{j=0}^{2n+1} \) is a sequence of real numbers such that the Hankel matrix \( H_n := [s_j+k]^n_{j,k=0} \) is positive Hermitian, then the set of Stieltjes transforms of all solutions of the Hamburger moment problem corresponding to the sequence \( (s_j)_{j=0}^{2n+1} \) could...
be parametrized with the aid of a linear fractional transformation the generating matrix-valued function of which is a $2 \times 2$ matrix polynomial built from the sequence $(s_j)_{j=0}^{2n+1}$. The role of the set of parameter functions was taken by a particular subclass of the class of all functions which are holomorphic in the open upper half plane of the complex plane and have a non-negative imaginary part at each point. The study of such classes of holomorphic functions was initiated by I. S. Kats in [31] and then continued by S. Hassi, H. S. V. de Snoo and A. D. I. Willemsma (see [28, 29]). Using several results originating in the just mentioned papers, V. A. Derkach, S. Hassi and H. S. V. de Snoo [12] were able to derive new Hamburger-Nevanlinna type theorems which played an important role in their approach to the moment problems for generalized Nevanlinna functions. Another cornerstone in the approach of V. A. Derkach, S. Hassi and H. S. V. de Snoo [12] is the application of the Schur-Chebyshev recursion algorithm, studied in the nondegenerate situation by M. Derevyagin [13] (see also [2]). The main goal of this paper is to generalize the assertion of [12, Corollary 5.2] to the matrix case without assuming some conditions of nondegeneracy. Roughly speaking, our strategy to realize this plan is modelled in basic steps along the lines of the approach in [12]. We will use a lot of results from the recent paper [22] on various classes of holomorphic matrix-valued functions, which are generalizations of the scalar theory obtained in [28, 29, 31]. A further basic element in our approach is to use new Hamburger-Nevanlinna type theorems for matrix-valued holomorphic functions which will be derived in Section 6. An other central feature of our approach is the use of Schur analysis methods. We will apply two interrelated versions of matricial Schur type algorithms, namely an algebraic one and a function-theoretic one. The algebraic version was worked out in [24]. It is an algorithm which is applied to a finite or infinite sequences of complex $p \times q$ matrices. An essential feature of this algorithm is that it preserves several properties of block Hankel matrices built from the sequences of complex $q \times q$ matrices under consideration (see Section 7). The construction and investigation of the function-theoretic version of our Schur type algorithm is a central theme of this paper. This algorithm will be applied to relevant classes of holomorphic matrix-valued functions in the open upper half plane. In the scalar case, this algorithm coincides with the classical algorithm which was constructed by R. Nevanlinna [37] in adaptation of the classical Schur algorithm for bounded holomorphic functions, which is due to I. Schur [42]. The idea how to build the algorithm in the matrix case was inspired by some constructions in the paper [9] by Chen and Hu. An essential point of our approach is an intensive use of the interplay between the function-theoretic and algebraic versions of our matricial Schur type algorithms. Both algorithms are formulated in terms of Moore-Penrose inverses of matrices. What concerns the function-theoretic version, it can be said that its effectiveness is mostly caused by recent results from [20, 22] on the holomorphicity of the Moore-Penrose inverse of special classes of holomorphic matrix-valued functions.

In order to describe more concretely the central topics studied in this paper, we introduce some notation. Throughout this paper, let $p$ and $q$ be positive integers. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the set of all positive integers, the set of all non-negative integers, the set of all integers, the set of all real numbers, and the set of all complex numbers, respectively. For every choice of $\rho, \kappa \in \mathbb{R} \cup \{-\infty, +\infty\}$, let $\mathbb{Z}_{\rho, \kappa} := \{k \in \mathbb{Z} | \rho \leq k \leq \kappa\}$. 

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We will write $\mathbb{C}^{p \times q}$, $\mathbb{C}_H^{q \times q}$, $\mathbb{C}_\geq^{q \times q}$, and $\mathbb{C}_\geq^{q \times q}$ for the set of all complex $p \times q$ matrices, the set of all Hermitian complex $q \times q$ matrices, the set of all non-negative Hermitian complex $q \times q$ matrices, and the set of all positive Hermitian complex $q \times q$ matrices, respectively.

We will use $\mathcal{B}_R$ to denote the $\sigma$-algebra of all Borel subsets of $\mathbb{R}$. For all $\Omega \in \mathcal{B}_R \setminus \{\emptyset\}$, let $\mathcal{B}_\Omega := \mathcal{B}_R \cap \Omega$. Furthermore, we will write $\mathcal{M}_\geq^q(\mathbb{R})$ to designate the set of all non-negative Hermitian $q \times q$ matrices, the set of all non-negative Hermitian $q \times q$ measures defined on $\mathcal{B}_R$, i.e., the set of $\sigma$-additive mappings $\mu: \mathcal{B}_R \to \mathbb{C}_\geq^{q \times q}$. We will use the integration theory with respect to non-negative Hermitian $q \times q$ measures which was worked out independently by I. S. Kats [30] and M. Rosenberg [40]. For all $j \in \mathbb{N}_0$, we will use $\mathcal{M}_\geq^{q,j}(\mathbb{R})$ to denote the set of all $\sigma \in \mathcal{M}_\geq^q(\mathbb{R})$ such that the integral

\[
s_j^{[\sigma]} := \int_{\mathbb{R}} t^j \sigma(\dd t)
\]

exists. Furthermore, we set $\mathcal{M}_\geq^{q,\infty}(\mathbb{R}) := \bigcap_{j=0}^{\infty} \mathcal{M}_\geq^{q,j}(\mathbb{R})$.

Remark 1.1. If $k, l \in \mathbb{N}_0$ and $k < l$, then it can be verified, as in the scalar case, that the inclusion $\mathcal{M}_\geq^{q,k}(\mathbb{R}) \subseteq \mathcal{M}_\geq^{q,l}(\mathbb{R})$ holds true.

The central problem in this paper is the truncated version of the following power moment problem of Hamburger type:

**Problem** ($\mathcal{M}[\mathbb{R}; (s_j)^{\kappa=0} = ]$). Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)^{\kappa=0}$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_\geq^{q,\infty}([\mathbb{R}; (s_j)^{\kappa=0} = ]$ of all $\sigma \in \mathcal{M}_\geq^q(\mathbb{R})$ for which $s_j^{[\sigma]} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,\kappa}$.

There is a further matricial version of the truncated Hamburger moment problem:

**Problem** ($\mathcal{M}[\mathbb{R}; (s_j)^{2n=0} \leq ]$). Let $n \in \mathbb{N}_0$ and let $(s_j)^{2n=0}$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_\geq^{q,\infty}([\mathbb{R}; (s_j)^{2n=0} \leq ]$ of all $\sigma \in \mathcal{M}_\geq^q(\mathbb{R})$ for which $s_j^{[\sigma]} = s_j$ is satisfied for each $j \in \mathbb{Z}_{0,2n-1}$ whereas the matrix $s_{2n} - s_{2n}^{[\sigma]}$ is non-negative Hermitian.

The first investigation of Problem $\mathcal{M}[\mathbb{R}; (s_j)^{2n=0} \leq ]$ goes back to I. V. Kovalishina [34] who used V. P. Potapov’s method of fundamental matrix inequalities. In the nondegenerate case, she obtained in [34] Theorem $\mathcal{H}$ a complete description of the Stieltjes transforms of the solution set of Problem $\mathcal{M}[\mathbb{R}; (s_j)^{2n=0} \leq ]$ in terms of a linear fractional transformation. An extension of V. P. Potapov’s method to degenerate situations was worked out by V. K. Dubovoj (see [14],[15]) for the case of the matricial Schur problem. Using a modification of V. K. Dubovoj’s method, Problem $\mathcal{M}[\mathbb{R}; (s_j)^{2n=0} \leq ]$ could be handled by V. A. Bolotnikov (see [7], Theorem 4.6) in the degenerate case. A common method of solving simultaneously the nondegenerate and degenerate versions of Problem $\mathcal{M}[\mathbb{R}; (s_j)^{2n=0} \leq ]$ was presented by Chen and Hu in [9]. Their method is based on the use of a matricial Schur type algorithm involving matrix-valued continued fractions.

What concerns the investigation of interrelations between the two moment problems under consideration, we refer the reader to the papers [9],[19]. A detailed treatment of the history of these two moment problems is contained in the introduction to the paper [19].
In order to state a necessary and sufficient condition for the solvability of each of the above formulated moment problems, we have to recall the notion of two types of sequences of matrices. If \( n \in \mathbb{N}_0 \) and if \((s_j)_{j=0}^{2n}\) is a sequence of complex \( q \times q \) matrices, then \((s_j)_{j=0}^{2n}\) is called *Hankel non-negative definite* if the block Hankel matrix

\[
H_n := [s_{j+k}]_{j,k=0}^n
\]

is non-negative Hermitian. Let \((s_j)_{j=0}^{\infty}\) be a sequence of complex \( q \times q \) matrices. Then \((s_j)_{j=0}^{\infty}\) is called *Hankel non-negative definite* if \((s_j)_{j=0}^{2n}\) is Hankel non-negative definite for all \( n \in \mathbb{N}_0 \). For all \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \), we will write \( \mathcal{H}^{\geq}_{q,2\kappa} \) for the set of all Hankel non-negative definite sequences \((s_j)_{j=0}^{2n}\) of complex \( q \times q \) matrices. Furthermore, for all \( n \in \mathbb{N}_0 \), let \( \mathcal{H}^{\geq}_{q,2n} \) be the set of all sequences \((s_j)_{j=0}^{2n}\) of complex \( q \times q \) matrices for which there exist complex \( q \times q \) matrices \( s_{2n+1} \) and \( s_{2n+2} \) such that \((s_j)_{j=0}^{2(n+1)} \in \mathcal{H}^{\geq}_{q,2(n+1)} \), whereas \( \mathcal{H}^{\geq}_{q,2n+1} \) stands for the set of all sequences \((s_j)_{j=0}^{2n+1}\) of complex \( q \times q \) matrices for which there exist some \( s_{2n+2} \in \mathbb{C}^{q \times q} \) such that \((s_j)_{j=0}^{2(n+1)} \in \mathcal{H}^{\geq}_{q,2(n+1)} \). For each \( m \in \mathbb{N}_0 \), the elements of the set \( \mathcal{H}^{\geq}_{q,m} \) are called *Hankel non-negative definite extendable sequences*. For technical reason, we set \( \mathcal{H}^{\geq}_{q,\infty} := \mathcal{H}^{\geq}_{q,\infty} \). Now we can characterize the situations that the mentioned problems have a solution:

**Theorem 1.2** ([9, Theorem 3.2]). Let \( n \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^{2n}\) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}^{\geq}_{n}(\mathbb{R};(s_j)_{j=0}^{2n},\leq) \neq \emptyset \) if and only if \((s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq}_{q,2n} \).

For an extension of Theorem 1.2 we refer the reader to [19, Theorem 4.16]. This extension says that \( \mathcal{M}^{\geq}_{n}(\mathbb{R};(s_j)_{j=0}^{2n},\leq) \neq \emptyset \) if and only if this set contains a molecular measure (i.e., a measure which is concentrated on a finite set of real numbers). Now we characterize the solvability of Problem \( M[\mathbb{R};(s_j)_{j=0}^{m},\leq] \).

**Theorem 1.3** ([21, Theorem 6.6]). Let \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \) and let \((s_j)_{j=0}^{n}\) be a sequence of complex \( q \times q \) matrices. Then \( \mathcal{M}^{\geq}_{n}(\mathbb{R};(s_j)_{j=0}^{n},\leq) \neq \emptyset \) if and only if \((s_j)_{j=0}^{n} \in \mathcal{H}^{\geq}_{q,\kappa} \).

Note that, in the case of an even integer \( \kappa \), Theorem 1.3 was proved in Chen/Hu [9, Theorem 3.1]. For the case that \( \kappa = 2n \) with some non-negative integer \( n \), Chen and Hu also stated a parametrization of the solution set of Problem \( M[\mathbb{R};(s_j)_{j=0}^{n},=] \) in the language of the Stieltjes transforms (see [9, Theorem 4.1]). The goal of our paper here is to give a parametrization of the solution set of Problem \( M[\mathbb{R};(s_j)_{j=0}^{n},=] \) in the case that \( \kappa \) is an odd integer (see Theorems 12.4 and 12.8). As already mentioned above in the scalar case \( q = 1 \), such a description of the solution set of Problem \( M[\mathbb{R};(s_j)_{j=0}^{m},=] \) with odd integer \( m \) was given by Derkach, Hassi, and de Snoo [12, Corollary 5.2]. During the work at the odd case we observed that a slight modification of our approach also works for the even case. For this reason, have worked out a simultaneous approach to the odd and even versions of the problem. In this way, we will also alternately prove the corresponding results in the case of an arbitrary even integer \( m \) (see Theorems 12.1 and 12.7).
2. The Class \( \mathcal{R}_q(\Pi_+) \)

For all \( A \in \mathbb{C}^{p \times q} \), let \( N(A) \) be the null space of \( A \) and \( R(A) \) be the column space of \( A \). If \( A \in \mathbb{C}^{q \times q} \), then let \( \text{Re} A := \frac{1}{2}(A + A^*) \) and \( \text{Im} A := \frac{1}{2i}(A - A^*) \) be the real part and the imaginary part of \( A \), respectively. Let

\[
\mathcal{R}_{q_+} := \{ A \in \mathbb{C}^{q \times q} \mid \text{Re} A \in \mathbb{C}_{2}^{q \times q} \} \quad \text{and} \quad \mathcal{I}_{q_+} := \{ A \in \mathbb{C}^{q \times q} \mid \text{Im} A \in \mathbb{C}_{2}^{q \times q} \}. \tag{2.1}
\]

If \( A \) and \( B \) are Hermitian complex \( q \times q \) matrices, then we will write \( A \geq B \) or \( B \leq A \) to indicate that the matrix \( A - B \) is non-negative Hermitian, and \( A > B \) means that \( A - B \) is positive Hermitian. As usual, for all \( A \in \mathbb{C}^{p \times q} \), let \( A^\dagger \) be the Moore-Penrose pseudoinverse of \( A \). If \( A \in \mathbb{C}^{p \times q} \), then \( \|A\| \) is the operator norm of \( A \). The notation \( I_q \) (or short \( I \)) stands for the unit matrix which belongs to \( \mathbb{C}^{q \times q} \) and \( 0_{p \times q} \) (or short \( 0 \)) designates the null matrix which belongs to \( \mathbb{C}^{p \times q} \). If \( \nu \) is a non-negative Hermitian \( q \times q \) measure on a measurable space \( (\Omega, \mathfrak{A}, \nu; \mathbb{C}) \), then we will use \( L^1(\Omega, \mathfrak{A}, \nu; \mathbb{C}) \) to denote the space of all \( \mathfrak{A} \)-measurable functions \( f: \Omega \to \mathbb{C} \) for which the integral \( \int_\Omega f \, d\nu \) exists. Let \( \Pi_+ := \{ z \in \mathbb{C} \mid \text{Im} z \in (0, +\infty) \} \) be the open upper half plane of \( \mathbb{C} \). Of central importance to this paper is the class \( \mathcal{R}_q(\Pi_+) \) of all matrix-valued functions \( F: \Pi_+ \to \mathbb{C}^{q \times q} \) which are holomorphic in \( \Pi_+ \) and which satisfy \( F(z) \in \mathcal{I}_{q_+} \) for all \( z \in \Pi_+ \). The elements of \( \mathcal{R}_q(\Pi_+) \) are called \( q \times q \)-matrix-valued Herglotz-Nevanlinna functions. For a comprehensive survey on the class \( \mathcal{R}_q(\Pi_+) \), we refer the reader to [27] and [11, Section 8].

Hermitian-Nevanlinna functions admit a well-known integral representation. To state this, we observe that, for all \( \nu \in \mathcal{M}_1^q(\mathbb{R}) \) and each \( \nu \in \mathcal{C}_0^{q \times q}(\mathbb{R}) \), the function \( h_z: \mathbb{R} \to \mathbb{C} \) defined by

\[
h_z(t) := \frac{1 + tz}{t - z} \quad \text{belongs to} \quad L^1(\mathbb{R}, \mathfrak{B}_\mathbb{R}, \nu; \mathbb{C}).
\]

**Theorem 2.1.** (a) Let \( F \in \mathcal{R}_q(\Pi_+) \). Then there are unique matrices \( \alpha \in \mathbb{C}_H^{q \times q} \) and \( \beta \in \mathbb{C}_G^{q \times q} \) and a unique non-negative Hermitian measure \( \nu \in \mathcal{M}_1^q(\mathbb{R}) \) such that

\[
F(z) = \alpha + z\beta + \int_\mathbb{R} \frac{1 + tz}{t - z} \nu(\mathrm{d}t) \quad \text{for each} \quad z \in \Pi_+. \tag{2.2}
\]

(b) If \( \alpha \in \mathbb{C}_H^{q \times q}, \beta \in \mathbb{C}_G^{q \times q} \) and \( \nu \in \mathcal{M}_1^q(\mathbb{R}) \), then \( F: \Pi_+ \to \mathbb{C}^{q \times q} \) defined by (2.2) belongs to the class \( \mathcal{R}_q(\Pi_+) \).

For all \( F \in \mathcal{R}_q(\Pi_+) \), the unique triple \( (\alpha, \beta, \nu) \in \mathbb{C}_H^{q \times q} \times \mathbb{C}_G^{q \times q} \times \mathcal{M}_1^q(\mathbb{R}) \) for which the representation (2.2) holds true is called the Nevanlinna parametrization of \( F \) and we will also write \( (\alpha_F, \beta_F, \nu_F) \) for \( (\alpha, \beta, \nu) \). In particular, \( \nu_F \) is said to be the Nevanlinna measure of \( F \).

**Example 2.2.** Let \( F: \Pi_+ \to \mathbb{C}^{q \times q} \) be defined by \( F(z) := 0_{q \times q} \). In view of Theorem 2.1, then \( F \in \mathcal{R}_q(\Pi_+) \) and \( (\alpha_F, \beta_F, \nu_F) = (0_{q \times q}, 0_{q \times q}, \nu_0) \), where \( \nu_0 : \mathfrak{B}_\mathbb{R} \to \mathbb{C}^{q \times q} \) is given by \( \nu_0(B) := 0_{q \times q} \).

**Remark 2.3.** Let \( F \in \mathcal{R}_q(\Pi_+) \) with Nevanlinna parametrization \( (\alpha_F, \beta_F, \nu_F) \). If the matrix \( \beta \in \mathbb{C}^{q \times q} \) satisfies \( \beta_F + \beta \in \mathbb{C}_{2}^{q \times q} \), then Theorem 2.1 shows that the function \( G: \Pi_+ \to \mathbb{C}^{q \times q} \) defined by \( G(z) := F(z) + z\beta \) belongs to \( \mathcal{R}_q(\Pi_+) \) and satisfies \( (\alpha_G, \beta_G, \nu_G) = (\alpha_F, \beta_F + \beta, \nu_F) \).
Remark 2.4. Let $n \in \mathbb{N}$ and $(p_k)_{k=1}^n$ be a sequence of positive integers. For each $k \in \mathbb{Z}_{1,n}$ let $F_k \in \mathcal{R}_{p_k}(\Pi_+)$ and $A_k \in \mathbb{C}^{p_k \times q}$. Then it can be easily verified that $F := \sum_{k=1}^n A_k^* F_k A_k$ belongs to $\mathcal{R}_q(\Pi_+)$ and satisfies

$$(\alpha_F, \beta_F, \nu_F) = \left( \sum_{k=1}^n A_k^* \alpha_{F_k} A_k, \sum_{k=1}^n A_k^* \beta_{F_k} A_k, \sum_{k=1}^n A_k^* \nu_{F_k} A_k \right).$$

Proposition 2.5 (see [27, Theorem 5.4 (iv)]). Let $F \in \mathcal{R}_q(\Pi_+)$ with Nevanlinna parametrization $(\alpha_F, \beta_F, \nu_F)$. Then $\alpha_F = \text{Re}[F(i)]$ and $\beta_F = \lim_{y \to +\infty} \frac{1}{y} F(iy)$.

We are particularly interested in the structure of the values of functions belonging to $\mathcal{R}_q(\Pi_+)$.  

Lemma 2.6. Let $F \in \mathcal{R}_q(\Pi_+)$. For each $z \in \Pi_+$, then $R([F(z)]^*) = R(F(z))$, $N([F(z)]^*) = N(F(z))$, and $[F(z)][F(z)]^* = [F(z)]^*[F(z)]$.  

Proof. This follows from Lemma A.9 and Proposition A.8.  

Proposition 2.7 ([22, Proposition 3.7]). Let $F \in \mathcal{R}_q(\Pi_+)$ with Nevanlinna parametrization $(\alpha_F, \beta_F, \nu_F)$. For all $z \in \Pi_+$, then

$$N(F(z)) = N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})), $$

$$R(F(z)) = R(\alpha_F) + R(\beta_F) + R(\nu_F(\mathbb{R})).$$

and

$$N(\text{Im } F(z)) = N(\beta_F) \cap N(\nu_F(\mathbb{R})), \quad R(\text{Im } F(z)) = R(\beta_F) + R(\nu_F(\mathbb{R})).$$

Proposition 2.7 contains essential information on the class $\mathcal{R}_q(\Pi_+)$. It indicates that, for an arbitrary function $F$ belonging to $\mathcal{R}_q(\Pi_+)$, the null space $N(F(z))$ and the column space $R(F(z))$ are independent of the concrete point $z \in \Pi_+$ and, furthermore, in which way these linear subspaces of $\mathbb{C}^q$ are determined by the Nevanlinna parametrization $(\alpha_F, \beta_F, \nu_F)$ of $F$.

In the sequel, we will sometimes meet situations where interrelations of the null space (resp. column space) of a function $F \in \mathcal{R}_q(\Pi_+)$ to the null space (resp. column space) of a given matrix $A \in \mathbb{C}^{p \times q}$ are of interest. More precisely, we will frequently apply the following observation.

Lemma 2.8. Let $A \in \mathbb{C}^{p \times q}$ and let $F \in \mathcal{R}_q(\Pi_+)$ with Nevanlinna parametrization $(\alpha_F, \beta_F, \nu_F)$. Then:

(a) The following statements are equivalent:

(i) $N(A) \subseteq N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R}))$.

(ii) $N(A) \subseteq N(F(z))$ for all $z \in \Pi_+$.

(iii) There exists a number $z \in \Pi_+$ with $N(A) \subseteq N(F(z))$.  

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(iv) \( FA^\dagger A = F \).

(v) \( R(\alpha_F) + R(\beta_F) + R(\nu_F(\mathbb{R})) \subseteq R(A^\ast) \).

(vi) \( R(F(z)) \subseteq R(A^\ast) \) for all \( z \in \Pi_+ \).

(vii) There exists a number \( z \in \Pi_+ \) with \( R(F(z)) \subseteq R(A^\ast) \).

(viii) \( A^\dagger AF = F \).

(b) Let [i] be satisfied. If \( p = q \) and if \( A \in \mathbb{C}^{q \times q}_{\mathcal{EP}} \), then \( AA^\dagger F = F \) and \( R(F(z)) \subseteq R(A) \) for all \( z \in \Pi_+ \).

Proof. (a) \( (\text{ii}) \Rightarrow (\text{iii}) \) and \( (\text{vi}) \Rightarrow (\text{vii}) \): These implications hold true obviously.

\( (\text{i}) \Rightarrow (\text{ii}) \) and \( (\text{iii}) \Rightarrow (\text{i}) \): Use Proposition 2.7.

\( (\text{ii}) \Leftrightarrow (\text{iv}) \): This equivalence follows from part (m) of Remark A.3.

\( (\text{ii}) \Leftrightarrow (\text{vi}) \): Because of Lemma 2.6 we have \( R(F(z)) = [N(F(z))]^\perp \) for all \( z \in \Pi_+ \).

Hence, \( (\text{ii}) \) and \( (\text{vi}) \) are equivalent.

\( (\text{vi}) \Rightarrow (\text{vii}) \) and \( (\text{vii}) \Rightarrow (\text{vi}) \): Use Proposition 2.7.

\( (\text{vi}) \Leftrightarrow (\text{viii}) \): Use Proposition A.2 and part (b) of Remark A.1.

(b) In view of \( A \in \mathbb{C}^{q \times q}_{\mathcal{EP}} \), we have \( R(A^\ast) = R(A) \) and, taking Proposition A.8 into account, furthermore, \( AA^\dagger = A^\dagger A \). Thus, (b) follows from (a).

A generic application of Lemma 2.8 will be concerned with situations where the matrix \( A \) even belongs to \( \mathbb{C}^{\geq q} \).

In our subsequent considerations we will very often use the Moore-Penrose inverse of functions belonging to the class \( \mathcal{R}_q(\Pi_+) \). In this connection, the following result turns out to be of central importance.

**Proposition 2.9** ([22, Proposition 3.8]). Let \( F \in \mathcal{R}_q(\Pi_+) \) with Nevanlinna parametrization \((\alpha_F, \beta_F, \nu_F)\). Then \( -F^\dagger \in \mathcal{R}_q(\Pi_+) \) and \( \alpha_{-F^\dagger} = -[\alpha_F]^\dagger \alpha_F([\alpha_F]^\dagger)^* \).

### 3. On Some Subclasses of \( \mathcal{R}_q(\Pi_+) \)

An essential feature of our subsequent considerations is the use of a whole variety of different subclasses of \( \mathcal{R}_q(\Pi_+) \). In this section, we summarize basic facts about these subclasses under the special orientation of this paper. The first part of these subclasses concerns objects which are already well-studied, whereas the larger remaining part deals with subclasses of \( \mathcal{R}_q(\Pi_+) \), which were introduced and studied very recently by the authors in [22]. The latter collection of subclasses are characterized by growth properties on the positive imaginary axis. It should be mentioned that the scalar versions of the function classes were introduced and studied in the paper [28, 29, 31]. An important subclass of the class \( \mathcal{R}_q(\Pi_+) \) is the set \( \mathcal{R}'_q(\Pi_+) \) of all \( F \in \mathcal{R}_q(\Pi_+) \) for which the function \( g: \mathbb{R} \to \mathbb{R} \) defined by \( g(t) := t^2 + 1 \) belongs to \( \mathcal{L}^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \nu_F; \mathbb{R}) \), where \( \nu_F \) is taken from the Nevanlinna parametrization of \( F \). In view of Remark 1.1, a member \( F \) of the class
\( \mathcal{R}_q(\Pi_+) \) belongs to \( \mathcal{R}_q'(\Pi_+) \) if and only if \( \nu_F \in \mathcal{M}_2^q(\mathbb{R}) \). For all \( F \in \mathcal{R}_q'(\Pi_+) \), then the mapping \( \sigma_F : \mathbb{B}_\mathbb{R} \to \mathbb{C}^{q \times q} \) given by

\[
\sigma_F(B) := \int_B \frac{1}{t^2 + 1} \nu_F(d \, t) \tag{3.1}
\]

is a well-defined non-negative Hermitian measure belonging to \( \mathcal{M}_2^q(\mathbb{R}) \). The measure \( \sigma_F \) is called the spectral measure of \( F \). In this paper, we encounter mostly situations in which, for a given function \( F \in \mathcal{R}_q'(\Pi_+) \), then the spectral measure \( \sigma_F \) plays a more important role than the Nevanlinna measure \( \nu_F \). Observe that the functions of the class \( \mathcal{R}_q'(\Pi_+) \) also admit a special integral representation (see [22, Theorem 4.3]).

Now we will see that each measure \( \sigma \in \mathcal{M}_2^q(\mathbb{R}) \) generates by a special integration procedure, in a natural way, a function belonging to \( \mathcal{R}_q'(\Pi_+) \).

**Proposition 3.1.** Let \( \sigma \in \mathcal{M}_2^q(\mathbb{R}) \).

(a) Let \( \nu_{[\sigma]} : \mathbb{B}_\mathbb{R} \to \mathbb{C}^{q \times q} \) be defined by

\[
B \mapsto \int_B \frac{1}{t^2 + 1} \sigma(d \, t).
\]

Then \( \nu_{[\sigma]} \in \mathcal{M}_2^q(\mathbb{R}) \) and

\[
s_j^{[\nu_{[\sigma]}]}(z) = \int_{\mathbb{R}} \frac{t^j}{t^2 + 1} \sigma(d \, t) \quad \text{for each } j \in \{1, 2\}.
\]

(b) Let \( F_\sigma : \Pi_+ \to \mathbb{C}^{q \times q} \) be defined by

\[
F_\sigma(z) := \int_{\mathbb{R}} \frac{1}{t - z} \sigma(d \, t).
\tag{3.2}
\]

Then \( F_\sigma \) is a matrix-valued function belonging to the class \( \mathcal{R}_q'(\Pi_+) \) with Nevanlinna parametrization \( (\alpha_{F_\sigma}, \beta_{F_\sigma}, \nu_{F_\sigma}) = (s_1^{[\nu_{[\sigma]}]}, 0_{q \times q}, \nu_{[\sigma]}) \) and spectral measure \( \sigma_{F_\sigma} = \sigma \).

**Proof.** This follows immediately from [22, Proposition B.5].

Taking into account that, for each \( z \in \Pi_+ \) and each \( t \in \mathbb{R} \), the identity

\[
\frac{1 + tz}{t - z} = \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) (1 + t^2)
\]

holds true, the assertion of (b) follows by direct computations, using Remark 1.1 and [22, Proposition B.5].

Let \( \sigma \in \mathcal{M}_2^q(\mathbb{R}) \). Then the function \( F_\sigma \) defined in (3.2) is called the Stieltjes transform of \( \sigma \). Part (b) of Proposition 3.1 shows that each \( \sigma \in \mathcal{M}_2^q(\mathbb{R}) \) occurs as spectral measure of an appropriately chosen function belonging to \( \mathcal{R}_q'(\Pi_+) \). Now we want to characterize the set of all Stieltjes transforms of measures belonging to \( \mathcal{M}_2^q(\mathbb{R}) \) by the matricial generalization of a classical result due to R. Nevanlinna [37].
Theorem 3.2. Let
\[ \tilde{R}_{0,q}(\Pi_+):= \left\{ F \in R_q(\Pi_+) \left| \sup_{y \in [1, +\infty)} y \| F(iy) \| < +\infty \right. \right\}. \quad (3.3) \]

Then \( \tilde{R}_{0,q}(\Pi_+) = \{ F_\sigma | \sigma \in M^q_{\geq}(\mathbb{R}) \} \) and the mapping \( \sigma \mapsto F_\sigma \) is a bijective correspondence between \( M^q_{\geq}(\mathbb{R}) \) and \( \tilde{R}_{0,q}(\Pi_+) \).

For each \( F \in \tilde{R}_{0,q}(\Pi_+) \) the unique measure \( \sigma \in M^q_{\geq}(\mathbb{R}) \) satisfying \( F_\sigma = F \) is called the Stieltjes measure of \( F \). Theorem 3.2 indicates that the Stieltjes transform \( F_\sigma \) of a measure \( \sigma \in M^q_{\geq}(\mathbb{R}) \) is characterized by a particular mild growth on the positive imaginary axis.

In view of Theorem 3.2, Problem \( M[R; (s_j)_{j=0}^\kappa, =] \) can be given a first reformulation as an equivalent problem in the class \( \tilde{R}_{0,q}(\Pi_+) \) as follows:

**Problem** \( (R[R; (s_j)_{j=0}^\kappa, =]) \). Let \( \kappa \in \mathbb{N}_0 \cup \{ +\infty \} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( q \times q \) matrices. Describe the set of all matrix-valued functions \( S \in \tilde{R}_{0,q}(\Pi_+) \), the Stieltjes measure of which belongs to \( M^q_{\geq}(\mathbb{R}); (s_j)_{j=0}^\kappa, = \).

In Section 6, we will state a reformulation of the original power moment problem \( M[R; (s_j)_{j=0}^\kappa, =] \) as an equivalent problem of finding a prescribed asymptotic expansion in a sector of the open upper half plane \( \Pi_+ \). Furthermore, we will see that a detailed analysis of the behaviour of the concrete functions of \( F \in R_q(\Pi_+) \) under study on the positive imaginary axis is extremely useful. For this reason, we turn now our attention to some subclasses of \( R_q(\Pi_+) \) which are described in terms of their growth on the positive imaginary axis. First we consider the set
\[ R_q^{[-2]}(\Pi_+):= \left\{ F \in R_q(\Pi_+) \left| \lim_{y \to +\infty} \frac{1}{y} \| F(iy) \| = 0 \right. \right\}. \quad (3.4) \]

In the following, we will use the symbol \( \lambda \) to denote the Lebesgue measure defined on \( \mathbb{B}_R \).

**Example 3.3.** Let \( A \in \mathcal{I}_{q, \geq} \) and let \( F: \Pi_+ \to \mathbb{C}^{q \times q} \) be given by \( F(z) := A \). Then it is immediately checked that \( F \in R_q^{[-2]}(\Pi_+) \). Let \( \mu: \mathbb{B}_R \to [0, +\infty) \) be defined by
\[ \mu(B) := \frac{1}{\pi} \int_B \frac{1}{1+u^2} \lambda(du). \]

Using the residue theorem, it can be verified then by direct computation that
\[ (\alpha_F, \beta_F, \nu_F) = (\mathrm{Re} A, 0_{q \times q}, (\mathrm{Im} A)\mu). \]

**Remark 3.4.** From (3.3) and (3.4) it is obvious that \( \tilde{R}_{0,q}(\Pi_+) \subseteq R_q^{[-2]}(\Pi_+) \).

**Remark 3.5.** In view of (3.4) and Proposition 2.5, we have
\[ R_q^{[-2]}(\Pi_+) = \{ F \in R_q(\Pi_+) | \beta_F = 0_{q \times q} \}. \]
Remark 3.6. Let \( n \in \mathbb{N} \) and let \( (p_k)_{k=1}^n \) be a sequence from \( \mathbb{N} \). For all \( k \in \mathbb{Z}_{\geq n} \), let \( F_k \in \mathcal{R}_q^{-2}(\Pi_+) \) and let \( A_k \in \mathcal{C}^{p_k,q} \). In view of Remark 3.5 and [22, Remark 3.4], then 
\[
\sum_{k=1}^n A_k^* F_k A_k \in \mathcal{R}_q^{-2}(\Pi_+).
\]

Now we state modifications of Proposition 2.7 and Lemma 2.8 for \( \mathcal{R}_q^{-2}(\Pi_+) \).

**Proposition 3.7.** Let \( F \in \mathcal{R}_q^{-2}(\Pi_+) \) and let \((\alpha_F, \beta_F, \nu_F)\) the Nevanlinna parametrization of \( F \). For all \( z \in \Pi_+ \), then
\[
N(F(z)) = N(\alpha_F) \cap N(\nu_F(\mathbb{R})), \quad R(F(z)) = R(\alpha_F) + R(\nu_F(\mathbb{R}))
\]
and
\[
N(\text{Im } F(z)) = N(\nu_F(\mathbb{R})), \quad R(\text{Im } F(z)) = R(\nu_F(\mathbb{R})).
\]

**Proof.** Combine Proposition 2.7 and Remark 3.5.

**Lemma 3.8.** Let \( A \in \mathcal{C}^{p,q} \) and let \( F \in \mathcal{R}_q^{-2}(\Pi_+) \). Then the statements

(i) \( N(A) \subseteq N(\alpha_F) \cap N(\nu_F(\mathbb{R})) \),

and

(x) \( R(\alpha_F) + R(\nu_F(\mathbb{R})) \subseteq R(A^*) \),

are equivalent. Furthermore, (ix) is equivalent to each of the statements (i)–(viii) stated in Lemma 2.8.

**Proof.** Combine Remark 3.5 and Lemma 2.8.

In the following, the notation \( \lambda \) stands for the Lebesgue measure defined on \( \mathcal{B}_{[1,\infty)} \). We now recall some subclasses of \( \mathcal{R}_q(\Pi_+) \), which were introduced and studied in [22]. A further subclass of \( \mathcal{R}_q(\Pi_+) \), which we will need in the following, is the set
\[
\mathcal{R}_q^{-1}(\Pi_+) := \left\{ F \in \mathcal{R}_q(\Pi_+) \left| \int_{[1,\infty)} \frac{1}{y} \| \text{Im } F(iy) \| \lambda(dy) < +\infty \right. \right\}. \tag{3.5}
\]

**Example 3.9.** Let \( A \in \mathcal{I}_{q,\geq} \) and let \( F: \Pi_+ \to \mathbb{C}^{q,q} \) be defined by \( F(z) := A \). Then from Example 3.3 and (3.5) it is obvious that \( F \in \mathcal{R}_q^{-1}(\Pi_+) \) if and only if \( \text{Im } A = 0_{q,q} \).

**Remark 3.10.** From [22, Lemma 5.1] we get more information about the Nevanlinna parametrization of the functions belonging to \( \mathcal{R}_q^{-1}(\Pi_+) \), namely for all \( F \in \mathcal{R}_q^{-1}(\Pi_+) \), we have \( \beta_F = 0_{q,q}, \nu_F \in \mathcal{M}_{2,1}^q(\mathbb{R}) \), and the function \( h: \mathbb{R} \to \mathbb{R} \) defined by \( h(t) := \frac{t^2 + 1}{|t| + 1} \) belongs to \( L^1(\mathbb{R}), \mathcal{B}_\mathbb{R}, \nu_F; \mathbb{R}) \). This implies that, for all \( F \in \mathcal{R}_q^{-1}(\Pi_+) \), the mapping \( \mu_F: \mathcal{B}_\mathbb{R} \to \mathbb{C}^{q,q} \) given by
\[
\mu_F(B) := \int_B \frac{t^2 + 1}{|t| + 1} \nu_F(dt) \tag{3.6}
\]
is a well-defined non-negative Hermitian measure belonging to \( \mathcal{M}_q^\geq(\mathbb{R}) \) and, in view of (1.1) and [22, Remark B.4], that the matrix
\[
\gamma_F := \alpha_F - s_1^{[\nu_F]} \tag{3.7}
\]
satisfies \((\gamma_F)^* = \gamma_F\).

**Remark 3.11.** From Remarks 3.5 and 3.10 we see that \( R_q^{-1}(\Pi_+) \subseteq R_q^{-2}(\Pi_+) \).

The next result indicates that functions which belong to \( R_q^{-1}(\Pi_+) \) admit a particular characterization in terms of a constant Hermitian matrix and an integral representation:

**Theorem 3.12 ([22, Theorem 5.6]).** (a) Each matrix-valued function \( F \) belonging to \( R_q^{-1}(\Pi_+) \) admits, for all \( z \in \Pi_+ \), the representation
\[
F(z) = \gamma_F + \int_{\mathbb{R}} \frac{|t| + 1}{t - z} \mu_F(d t),
\]
where \( \gamma_F \) and \( \mu_F \) are given via (3.7) and (3.6), respectively.

(b) Let \( \gamma \in \mathbb{C}_H^{q \times q} \) and let \( \mu \in \mathcal{M}_q^\geq(\mathbb{R}) \). Then \( F : \Pi_+ \to \mathbb{C}^{q \times q} \) defined by
\[
F(z) := \gamma + \int_{\mathbb{R}} \frac{|t| + 1}{t - z} \mu(d t)
\]
belongs to \( R_q^{-1}(\Pi_+) \) and \((\gamma_F, \mu_F) = (\gamma, \mu)\) holds true.

**Example 3.13.** Let \( F : \Pi_+ \to \mathbb{C}^{q \times q} \) be defined by \( F(z) := 0_{q \times q} \). In view of part (b) of Theorem 3.12, then \( F \in R_q^{-1}(\Pi_+) \) and \((\gamma_F, \mu_F) = (0_{q \times q}, o_q)\), where \( o_q : \mathfrak{B}_\mathbb{R} \to \mathbb{C}^{q \times q} \) is given by \( o_q(B) := 0_{q \times q} \).

The next result describes the asymptotics of the functions belonging to \( R_q^{-1}(\Pi_+) \) on the positive imaginary axis.

**Proposition 3.14 ([22, Proposition 5.8]).** Let \( F \in R_q^{-1}(\Pi_+) \). Then
\[
\lim_{y \to +\infty} \Re F(iy) = \gamma_F, \quad \lim_{y \to +\infty} \Im F(iy) = 0_{q \times q}
\]
and
\[
\lim_{y \to +\infty} F(iy) = \gamma_F. \tag{3.8}
\]

In our scale of subclasses of \( R_q(\Pi_+) \), we next consider the class
\[
R_q^{[0]}(\Pi_+) := \left\{ F \in R_q(\Pi_+) \left| \sup_{y \in [1, +\infty)} y \| F(iy) \| < +\infty \right. \right\}. \tag{3.9}
\]
In view of [22, Proposition 6.4], we have
\[
R_q^{[0]}(\Pi_+) = R_q^{-1}(\Pi_+) \cap R_q'(\Pi_+), \tag{3.10}
\]
where \( R_q^{-1}(\Pi_+) \) is given via (3.5). Now we recall a special characterization for the functions of the class \( R_q^{[0]}(\Pi_+) \).
Theorem 3.15 ([22, Theorem 6.3]). (a) Let \( F \in \mathcal{R}_q^0(\Pi_+) \). Then \( F \) belongs to the class \( \mathcal{R}_q^0(\Pi_+) \) and, if \( \gamma_F \) and \( \sigma_F \) are given via (3.7) and (3.1), respectively, then

\[
F(z) = \gamma_F + \int_{\mathbb{R}} \frac{1}{t-z} \sigma_F(\text{d}t)
\]

for all \( z \in \Pi_+ \).

(b) For all \( \gamma \in \mathbb{C}_{\mathbb{H}}^{q \times q} \) and each \( \sigma \in \mathcal{M}_q^\mathbb{H} \), the function \( F: \Pi_+ \to \mathbb{C}_{\mathbb{H}}^{q \times q} \) given by

\[
F(z) := \gamma + \int_{\mathbb{R}} \frac{1}{t-z} \sigma(\text{d}t)
\]

belongs to \( \mathcal{R}_q^0(\Pi_+) \) and satisfies \( (\gamma_F, \sigma_F) = (\gamma, \sigma) \).

For each \( \kappa \in \{-2, -1, 0\} \), the class \( \mathcal{R}_q^{[\kappa]}(\Pi_+) \) is already defined. In view of (3.10), we have \( \mathcal{R}_q^{[0]}(\Pi_+) \subseteq \mathcal{R}_q'(\Pi_+) \). Thus, the functions \( F \) belonging to \( \mathcal{R}_q^{[0]}(\Pi_+) \) have a well-defined spectral measure \( \sigma_F \), which is given via (3.1). So, for all \( \kappa \in \mathbb{N} \cup \{+\infty\} \), the class

\[
\mathcal{R}_q^{[\kappa]}(\Pi_+) := \left\{ F \in \mathcal{R}_q^0(\Pi_+) \mid \sigma_F \in \mathcal{M}_q^{\mathbb{H}}(\mathbb{R}) \right\} \tag{3.11}
\]

is well-defined. We recall that then the following result holds true:

**Proposition 3.16.** Let \( \kappa \in \mathbb{Z}_{-2, +\infty} \cup \{+\infty\} \). Then

\[
\mathcal{R}_q^{[\kappa]}(\Pi_+) = \left\{ F \in \mathcal{R}_q(\Pi_+) \mid \beta_F = 0_{q \times q} \text{ and } \nu_F \in \mathcal{M}_q^{\mathbb{H}}(\mathbb{R}) \right\},
\]

where \( \beta_F \) and \( \nu_F \) are taken from the Nevanlinna parametrization of \( F \).

**Proof.** Use Remark 3.5 in the case \( \kappa = -2 \) and [22, Proposition 7.3] in the case \( \kappa \geq -1 \).

**Remark 3.17.** Observe that Proposition 3.16 shows that the proper inclusions

\[
\mathcal{R}_q^{[+\infty]}(\Pi_+) \subsetneq \mathcal{R}_q^{[l]}(\Pi_+) \subsetneq \mathcal{R}_q^{[k]}(\Pi_+) \subsetneq \mathcal{R}_q(\Pi_+)
\]

are fulfilled for all \( l \in \mathbb{N}_0 \) and all \( k \in \mathbb{Z}_{-l-1} \).

For all \( \kappa \in \mathbb{Z}_{-2, +\infty} \cup \{+\infty\} \), we now consider the class

\[
\mathcal{R}_{\kappa,q}(\Pi_+) := \left\{ F \in \mathcal{R}_q^{[\kappa]}(\Pi_+) \mid \gamma_F = 0_{q \times q} \right\}. \tag{3.12}
\]

**Remark 3.18.** From (3.12) and Remark 3.17 one sees that

\[
\mathcal{R}_{+\infty,q}(\Pi_+) \subsetneq \mathcal{R}_{l,q}(\Pi_+) \subsetneq \mathcal{R}_{k,q}(\Pi_+) \subsetneq \mathcal{R}_q(\Pi_+)
\]

holds true for all \( l \in \mathbb{N}_0 \) and all \( k \in \mathbb{Z}_{-l-1} \).
Example 3.19. Let $F : \Pi_+ \to \mathbb{C}^{q\times q}$ be defined by $F(z) := 0_{q\times q}$. In view of Example 3.13, then $F \in \mathcal{R}_{-1,q}(\Pi_+)$. The following results complement Proposition 2.7. More precisely, we state now some modifications of Proposition 2.7 for various subclasses of $\mathcal{R}_q(\Pi_+)$. 

**Lemma 3.20.** Let $\kappa \in \mathbb{Z}_{-1, +\infty} \cup \{+\infty\}$ and let $F \in \mathcal{R}_q^{[\kappa]}(\Pi_+)$. For each $z \in \Pi_+$, then

$$ R(F(z)) = R(\gamma F) + R(\mu_F(\mathbb{R})), \quad N(F(z)) = N(\gamma F) \cap N(\mu_F(\mathbb{R})) $$

and

$$ R(\text{Im}[F(z)]) = R(\mu_F(\mathbb{R})), \quad N(\text{Im}[F(z)]) = N(\mu_F(\mathbb{R})). $$

**Proof.** Use [22, Lemma 5.4] and Remark 3.17. □

**Lemma 3.21** ([22, Lemma 8.1]). Let $\kappa \in \mathbb{Z}_{-1, +\infty} \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. For each $z \in \Pi_+$, then

$$ R(F(z)) = R(\mu_F(\mathbb{R})) = R(\text{Im}[F(z)]) $$

and

$$ N(F(z)) = N(\mu_F(\mathbb{R})) = N(\text{Im}[F(z)]). $$

**Lemma 3.22** ([22, Lemma 8.2]). Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. For each $z \in \Pi_+$, then

$$ R(F(z)) = R(\sigma_F(\mathbb{R})) = R(\text{Im}[F(z)]) $$

and

$$ N(F(z)) = N(\sigma_F(\mathbb{R})) = N(\text{Im}[F(z)]). $$

**Remark 3.23.** Let $\kappa \in \mathbb{Z}_{-1, +\infty} \cup \{+\infty\}$ and $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. Then from (3.12) and Proposition 3.14 we see that

$$ \lim_{y \to +\infty} F(iy) = 0_{q\times q}. \quad (3.13) $$

**Remark 3.24.** In view of Proposition 3.16, (3.7) and (3.12), for all $\kappa \in \mathbb{Z}_{-1, +\infty} \cup \{+\infty\}$, we have

$$ \mathcal{R}_{\kappa,q}(\Pi_+) = \left\{ F \in \mathcal{R}_q(\Pi_+) \left| \beta_F = 0_{q\times q} \right. \right\} $$

for all $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and each $m \in \mathbb{Z}_{-1, +\infty}$, the class $\mathcal{R}_{\kappa,q}(\Pi_+)$ is a proper subset of $\mathcal{R}_{m,q}(\Pi_+)$. 

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Lemma 3.25. Let $A \in \mathbb{C}^{p \times q}$, let $\kappa \in \mathbb{Z}_{-1, +\infty} \cup \{+\infty\}$, and let $F \in \mathcal{R}_{\kappa, q}(\Pi_+)$. Then the statements

(i) $N(A) \subseteq N(\mu_F(\mathbb{R}))$.

and

(ii) $R(\mu_F(\mathbb{R})) \subseteq R(A^*)$.

are equivalent. Furthermore, (i) is equivalent to each of the statements (i)–(viii) in Lemma 2.8.

Proof. Combine Remark 3.24 and Lemmas 3.21 and 2.8.

Remark 3.26. From [22, Proposition 6.4] we know that

$\mathcal{R}_{0,q}(\Pi_+) = \mathcal{R}_{-1,q}(\Pi_+) \cap \mathcal{R}'_{q}(\Pi_+)$.

Remark 3.27. From (3.12), (3.5) and Proposition 3.14 we see that

$\mathcal{R}_{-1,q}(\Pi_+) = \{ F \in \mathcal{R}_{q}(\Pi_+) \mid \int_{[1, +\infty)} \frac{1}{y} \| \text{Im} [F(iy)]\| \lambda(dy) < +\infty \text{ and } \lim_{y \to +\infty} \| F(iy) \| = 0 \}$.

Now we get that the classes given in (3.12) and (3.3) coincide.

Proposition 3.28. $\mathcal{R}_{0,q}(\Pi_+) = \tilde{\mathcal{R}}_{0,q}(\Pi_+)$.

Proof. Combine Theorems 3.2 and 3.15.

Corollary 3.29. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$. Then $\mathcal{R}_{\kappa,q}(\Pi_+) \subseteq \tilde{\mathcal{R}}_{0,q}(\Pi_+)$.

Proof. Combine (3.12), Remark 3.17 and Proposition 3.28.

Remark 3.30. Let $n \in \mathbb{N}$, $\kappa \in \mathbb{Z}_{-1, +\infty} \cup \{+\infty\}$, and $(p_k)_{k=1}^n$ be a sequence from $\mathbb{N}$. For $k \in \mathbb{Z}_{1,n}$ let $F_k \in \mathcal{R}_{\kappa, p_k}(\Pi_+)$ and $A_k \in \mathbb{C}^{p_k \times q}$. Then [22, Remark 8.3] shows that

$\sum_{k=1}^n A_k^* F_k A_k \in \mathcal{R}_{\kappa,q}(\Pi_+)$.

4. On the Classes $\mathcal{P}_{q}^{\text{even}}[A]$ and $\mathcal{P}_{q}^{\text{odd}}[A]$

In this section, we consider particular subclasses of the classes $\mathcal{R}_{q}^{[-2]}(\Pi_+)$ and $\mathcal{R}_{-1,q}(\Pi_+)$, which were introduced in (3.4) and (3.12), respectively. We have seen in Proposition 2.7 that, for an arbitrary function $F \in \mathcal{R}_{q}(\Pi_+)$, the null space of $F(z)$ is independent from the concrete choice of $z \in \Pi_+$. For the cases $F \in \mathcal{R}_{q}^{[-2]}(\Pi_+)$ or $F \in \mathcal{R}_{-1,q}(\Pi_+)$, a complete description of this constant null space was given in Proposition 3.7 and Lemma 3.21, respectively. Against to this background, we single out now special subclasses of $\mathcal{R}_{q}^{[-2]}(\Pi_+)$ and $\mathcal{R}_{-1,q}(\Pi_+)$ which are characterized by the interrelation of the
constant null space to the null space of a prescribed matrix \( A \in \mathbb{C}^{p \times q} \). More precisely, for all \( A \in \mathbb{C}^{p \times q} \), let
\[
\mathcal{P}_q^{\text{even}}[A] := \left\{ F \in \mathcal{R}_q^{-2}(\Pi_+) \mid N(A) \subseteq N(\alpha_F) \cap N(\nu_F(\mathbb{R})) \right\}
\tag{4.1}
\]
and let
\[
\mathcal{P}_q^{\text{odd}}[A] := \left\{ F \in \mathcal{R}_{-1,q}(\Pi_+) \mid N(A) \subseteq N(\mu_F(\mathbb{R})) \right\},
\tag{4.2}
\]
where \( \mu_F \) is given via (3.6). The choice of the terminology is caused by the role which the sets introduced in (4.1) and (4.2) will later play in the framework of the even and odd version of the truncated matrixal Hamburger moment problem, respectively. The role of the matrix \( A \) will be taken then by matrices which are generated from the sequence of data matrices of the problem of consideration via a Schur type algorithm.

**Remark 4.1.** If \( A \in \mathbb{C}^{p \times q} \) satisfies \( N(A) = \{0_{q \times 1}\} \), then \( \mathcal{P}_q^{\text{even}}[A] = \mathcal{R}_q^{-2}(\Pi_+) \) and \( \mathcal{P}_q^{\text{odd}}[A] = \mathcal{R}_{-1,q}(\Pi_+) \). In particular, this situation arises in the case that \( p = q \) and \( \det q \neq 0 \) are fulfilled.

**Example 4.2.** Let \( A \in \mathbb{C}^{p \times q} \) and let \( F \colon \Pi_+ \to \mathbb{C}^{q \times p} \) be defined by \( F(z) := 0_{q \times q} \). In view of Examples 3.3 and 2.2, and (4.1), then \( F \in \mathcal{P}_q^{\text{even}}[A] \), and, in view of Examples 3.19 and 3.13 and (4.2), furthermore \( F \in \mathcal{P}_q^{\text{odd}}[A] \).

**Lemma 4.3.** Let \( A \in \mathbb{C}^{p \times q} \) and \( F \colon \Pi_+ \to \mathbb{C}^{q \times p} \). Then:

(a) The following statements are equivalent:

(i) \( F \in \mathcal{P}_q^{\text{even}}[A] \).

(ii) \( F \in \mathcal{R}_q^{-2}(\Pi_+) \) and \( N(A) \subseteq N(F(z)) \) for all \( z \in \Pi_+ \).

(iii) \( F \in \mathcal{R}_q(\Pi_+) \), \( \lim_{y \to +\infty} \frac{1}{y} \|F(iz)\| = 0 \), and \( N(A) \subseteq N(F(z)) \) for all \( z \in \Pi_+ \).

(iv) \( F \in \mathcal{R}_q(\Pi_+) \), \( \beta_F = 0_{q \times q} \), and \( N(A) \subseteq N(\alpha_F) \cap N(\nu_F(\mathbb{R})) \).

(b) The following statements are equivalent:

(v) \( F \in \mathcal{P}_q^{\text{odd}}[A] \).

(vi) \( F \in \mathcal{R}_{-1,q}(\Pi_+) \) and \( N(A) \subseteq N(F(z)) \) for all \( z \in \Pi_+ \).

(vii) \( F \in \mathcal{R}_q(\Pi_+) \), \( \int_{1, +\infty} \frac{1}{y} |\text{Im}[F(iz)]| \lambda(\lambda) \leq +\infty \), \( \lim_{y \to +\infty} \frac{1}{y} \|F(iz)\| = 0 \), and \( N(A) \subseteq N(F(z)) \) for all \( z \in \Pi_+ \).

(viii) \( F \in \mathcal{R}_q(\Pi_+) \), \( \beta_F = 0_{q \times q} \), \( \nu_F \in \mathcal{M}_{\geq 1}(\mathbb{R}) \), \( \alpha_F = s_{1\nu_F}^1 \), and \( N(A) \subseteq N(\nu_F(\mathbb{R})) \).

**Proof.** (a) This follows from (4.1), Lemma 3.8 (3.3), and Remark 3.5. (b) Use (4.2), Lemma 3.25, Remarks 3.27 and 3.24, (3.12), and [22, Remark 5.3].

**Remark 4.4.** Let \( z_0 \in \Pi_+ \). From Lemma 4.3, Proposition 3.7 and Lemma 3.21 one can easily see then that:
Remark 4.5. Let $A \in \mathbb{C}^{p \times q}$ and let $B \in \mathbb{C}^{p \times q}$ with $N(A) \subseteq N(B)$. In view of (4.1) and (4.2), then $P_{q}^{\text{even}}[B] \subseteq P_{q}^{\text{even}}[A]$ and $P_{q}^{\text{odd}}[B] \subseteq P_{q}^{\text{odd}}[A]$.

Proposition 4.6. Let $A \in \mathbb{C}^{p \times q}$.

(a) $P_{q}^{\text{even}}[A] = \{A^{\dagger}AF A^{\dagger}A \in \mathcal{R}_{q}^{[-2]}(\Pi_{+})\}$.

(b) $P_{q}^{\text{odd}}[A] = \{A^{\dagger}AF A^{\dagger}A \in \mathcal{R}_{-1,q}(\Pi_{+})\}$.

Proof. (a) Let $F \in P_{q}^{\text{even}}[A]$. In view of (4.1), we have then

$$F \in \mathcal{R}_{q}^{[-2]}(\Pi_{+}) \quad \text{and} \quad N(A) \subseteq N(\alpha F) \cap N(\nu F(\mathbb{R})). \quad (4.3)$$

Taking (4.3) into account, Lemma 3.8 yields for $z \in \Pi_{+}$ now $F(z)A^{\dagger}A = F(z)$ and $A^{\dagger}AF(z) = F(z)$. Consequently, $A^{\dagger}AF A^{\dagger}A = F$. Combining this with (4.3), we infer

$$P_{q}^{\text{even}}[A] \subseteq \left\{A^{\dagger}AF A^{\dagger}A \mid F \in \mathcal{R}_{q}^{[-2]}(\Pi_{+})\right\}. \quad (4.4)$$

Conversely, let us consider an arbitrary

$$F \in \mathcal{R}_{q}^{[-2]}(\Pi_{+}). \quad (4.5)$$

In view of (4.5) and $(A^{\dagger}A)^{*} = A^{\dagger}A$, from Remark 3.6 we see that $G := A^{\dagger}AF A^{\dagger}A$ fulfills

$$G \in \mathcal{R}_{q}^{[-2]}(\Pi_{+}) \quad (4.6)$$

and $GA^{\dagger} = (A^{\dagger}AF A^{\dagger}A)A^{\dagger}A = G$. Thus, part (a) of Remark 3.3 gives $N(A) \subseteq N(G(z))$ for each $z \in \Pi_{+}$. Combining this with (4.6) and applying part (a) of Lemma 4.3 we conclude $G \in P_{q}^{\text{even}}[A]$. Thus, $P_{q}^{\text{even}}[A] \supseteq \{A^{\dagger}AF A^{\dagger}A \mid F \in \mathcal{R}_{q}^{[-2]}(\Pi_{+})\}$. This inclusion and (4.4) show that part (a) holds.

(b) Let $F \in P_{q}^{\text{odd}}[A]$. In view of (4.2), we have then

$$F \in \mathcal{R}_{-1,q}(\Pi_{+}) \quad (4.7)$$

and $N(A) \subseteq N(\mu F(\mathbb{R}))$. Consequently, from Lemma 3.25 we see that $A^{\dagger}AF A^{\dagger}A = F$ holds true. Thus, (4.7) yields

$$P_{q}^{\text{odd}}[A] \subseteq \left\{A^{\dagger}AF A^{\dagger}A \mid F \in \mathcal{R}_{-1,q}(\Pi_{+})\right\}. \quad (4.8)$$

Conversely, we now consider an arbitrary $F \in \mathcal{R}_{-1,q}(\Pi_{+})$. From Remark 3.30 and $(A^{\dagger}A)^{*} = A^{\dagger}A$ we get then that $G := A^{\dagger}AF A^{\dagger}A$ belongs to $\mathcal{R}_{-1,q}(\Pi_{+})$ and fulfills $N(A) \subseteq N(G(z))$ for each $z \in \Pi_{+}$. Thus, the application of part (b) of Lemma 4.3 yields $G \in P_{q}^{\text{odd}}[A]$. Consequently,

$$P_{q}^{\text{odd}}[A] \supseteq \left\{A^{\dagger}AF A^{\dagger}A \mid F \in \mathcal{R}_{-1,q}(\Pi_{+})\right\}. \quad \square$$

Combining this with (4.8) completes the proof of part (b).
Corollary 4.7. Let $A \in \mathbb{C}^{p \times q}$ and let $F \in \mathcal{P}_q^{even}[A] \cup \mathcal{P}_q^{odd}[A]$. Then $FA^\dagger A = F$ and $A^\dagger AF = F$.

Lemma 4.8. Let $A \in \mathbb{C}^{p \times q}$. Then $\mathcal{P}_q^{odd}[A] \subseteq \mathcal{P}_q^{even}[A]$.

Proof. Let $F \in \mathcal{P}_q^{odd}[A]$. In view of (4.2), we have then $F \in \mathcal{R}_{-1,q}(\Pi_+)$. From Remark 3.24 we get then $P$ and $F$ where the inclusion is due to $R$. Let Lemma 4.8.

Corollary 4.7. Let $A \in \mathbb{C}^{p \times q}$. Then $A^\dagger AF = F$ and $A^\dagger FA = F$.

Proof. Let $F \in \mathcal{P}_q^{odd}[A]$. In view of (4.2), we have then $F \in \mathcal{R}_{-1,q}(\Pi_+)$. From Remark 3.24 we get then $\beta_F = 0_{q \times q}$. Hence, Remark 3.5 yields $F \in \mathcal{R}_q^{[-2]}(\Pi_+)$. Furthermore, we obtain

$$N(A) \subseteq N(\mu_F(\mathbb{R})) = N(F(i)) = N(\alpha_F) \cap N(\nu_F(\mathbb{R})),$$

where the inclusion is due to $F \in \mathcal{P}_q^{odd}[A]$ and (4.2), the 1st equation is due to $F \in \mathcal{R}_{-1,q}(\Pi_+)$ and Lemma 3.21 (resp. (4.1)), and the 2nd equation is due to $F \in \mathcal{R}_q^{[-2]}(\Pi_+)$ and Proposition 3.7. In view of (4.1), we get then $F \in \mathcal{P}_q^{even}[A]$.

The following result contains essential information on the structure of the sets $\mathcal{P}_q^{even}[A]$ and $\mathcal{P}_q^{odd}[A]$, where $A \in \mathbb{C}^{p \times q}$.

Proposition 4.9. Let $A \in \mathbb{C}^{p \times q}$. Then:

(a) If $A = 0_{p \times q}$, then $\mathcal{P}_q^{even}[A] = \{F\}$ and $\mathcal{P}_q^{odd}[A] = \{F\}$, where $F: \Pi_+ \to \mathbb{C}^{q \times q}$ is defined by $F(z) := 0_{q \times q}$.

(b) Suppose that $r := \text{rank } A$ fulfills $r \geq 1$. Let $u_1, u_2, \ldots, u_r$ be an orthonormal basis of $R(A^*)$ and let $U := [u_1, u_2, \ldots, u_r]$. Then

$$\mathcal{P}_q^{even}[A] = \left\{Uf U^* \Big| f \in \mathcal{R}_q^{[-2]}(\Pi_+)\right\}$$

and

$$\mathcal{P}_q^{odd}[A] = \left\{Uf U^* \Big| f \in \mathcal{R}_{-1,r}(\Pi_+)\right\}.$$

(c) If $f, g \in \mathcal{R}_q^{[-2]}(\Pi_+) \cup \mathcal{R}_{-1,r}(\Pi_+)$ are such that $Uf U^* = Ug U^*$, then $f = g$.

Proof. (a) This follows from Proposition 4.6 and Example 4.2.

(b) Let $G \in \mathcal{P}_q^{even}[A]$ (resp. $G \in \mathcal{P}_q^{odd}[A]$). In view of part (a) (resp. part (b)) of Proposition 4.6, there exists an $F \in \mathcal{R}_q^{[-2]}(\Pi_+)$ (resp. $F \in \mathcal{R}_{-1,q}(\Pi_+)$) such that $G = A^\dagger AF A^\dagger$. Let $f := U^* FU$. Because of Remark 3.6 (resp. Remark 3.30), then $f \in \mathcal{R}_q^{[-2]}(\Pi_+)$ (resp. $f \in \mathcal{R}_{-1,r}(\Pi_+)$). In view of Remark A.6 we have $UU^* = A^\dagger A$. Thus, $G = UU^* FUU^* = Uf U^*$. Hence,

$$\mathcal{P}_q^{even}[A] \subseteq \left\{Uf U^* \Big| f \in \mathcal{R}_q^{[-2]}(\Pi_+)\right\}$$

(resp.

$$\mathcal{P}_q^{odd}[A] \subseteq \left\{Uf U^* \Big| f \in \mathcal{R}_{-1,r}(\Pi_+)\right\}.$$
Conversely, let $f \in \mathcal{R}_r^{-2}(\Pi_+)$ (resp. $f \in \mathcal{R}_{-1,r}(\Pi_+)$). In view of Remark 3.6 (resp. Remark 3.30), then

$$ UfU^* \in \mathcal{R}_q^{-2}(\Pi_+) \quad \text{ (resp. } UfU^* \in \mathcal{R}_{-1,q}(\Pi_+)). $$

Now we consider an arbitrary $x \in N(A)$. In view of the construction of $U$ and the relation $[N(A)]^\perp = R(A^*)$, we get $U^*x = 0_{r \times 1}$. Thus, $x \in N(U^*)$. Consequently, for each $z \in \Pi_+$ we get $N(A) \subseteq N((UfU^*)(z))$. The application of part (a) (resp. part (b)) of Lemma 4.3 yields now

$$ \{UfU^* \mid f \in \mathcal{R}_r^{-2}(\Pi_+)\} \subseteq \mathcal{P}^{\text{even}}[A] \quad \text{ (resp. } \{UfU^* \mid f \in \mathcal{R}_{-1,r}(\Pi_+)\} \subseteq \mathcal{P}^{\text{odd}}[A]). $$

This completes the proof of (b).

(c) In view of Remark A.6, we have $U^*U = I_r$. Thus $UfU^* = UgU^*$ implies $f = g$. □

5. The Classes $\mathcal{R}_q^{[\kappa]}(\Pi_+; (s_j)_{j=-1}^{\kappa})$ and $\mathcal{R}_{\kappa,q}(\Pi_+; (s_j)_{j=0}^\kappa)$

In this section, we consider particular subclasses of the class $\mathcal{R}_q^{[\kappa]}(\Pi_+)$, which was introduced in (3.9) for $\kappa = 0$ and in (3.11) for $\kappa \in \mathbb{N} \cup \{+\infty\}$. Because of (3.10) and (3.11), we have the inclusion

$$ \mathcal{R}_q^{[\kappa]}(\Pi_+) \subseteq \mathcal{R}_r^q(\Pi_+). $$

(5.1)

In view of (5.1), for each function $F$ belonging to one of the classes $\mathcal{R}_q^{[\kappa]}(\Pi_+)$ with some $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, the spectral measure $\sigma_F$ given by (3.1) is well-defined. Now taking Remark 1.1 into account, we turn our attention to subclasses of functions $F \in \mathcal{R}_q^{[\kappa]}(\Pi_+)$ with prescribed parameter $\gamma_F$ and prescribed first $\kappa + 1$ power moments of the spectral measure $\sigma_F$.

Taking (3.11) and (3.7) into account, for all $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and each sequence $(s_j)_{j=-1}^{\kappa}$ of complex $q \times q$ matrices, now we consider the class

$$ \mathcal{R}_q^{[\kappa]}(\Pi_+; (s_j)_{j=-1}^{\kappa}) := \left\{ F \in \mathcal{R}_q^{[\kappa]}(\Pi_+) \mid \gamma_F = -s_{-1} \text{ and } \sigma_F \in \mathcal{M}_q^\mathbb{R} \left[ \mathbb{R}; (s_j)_{j=0}^{\kappa}, = \right] \right\} $$

(5.2)

and, for all $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and each sequence $(s_j)_{j=0}^{\kappa}$ from $\mathbb{C}^{q \times q}$, furthermore

$$ \mathcal{R}_{\kappa,q}(\Pi_+; (s_j)_{j=0}^{\kappa}) := \left\{ F \in \mathcal{R}_{\kappa,q}(\Pi_+) \mid \sigma_F \in \mathcal{M}_q^\mathbb{R} \left[ \mathbb{R}; (s_j)_{j=0}^{\kappa}, = \right] \right\}, $$

(5.3)

where $\mathcal{R}_{\kappa,q}(\Pi_+)$ is defined in (3.12).
Remark 5.1. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_j^{\kappa}$ be a sequence from $\mathbb{C}^{q \times q}$. Let $t_{-1} := 0_{q \times q}$ and let $t_j := s_j$ for all $j \in \mathbb{N}_0$. Then taking (5.2), (5.3), and (3.12) into account we see that

$$\mathcal{R}_{\kappa,q} \left[ \Pi_+; (s_j)_{j=0}^{\kappa} \right] = \mathcal{R}_q^{[\kappa]} \left[ \Pi_+; (t_j)_{j=-1}^{\kappa} \right].$$

Remark 5.2. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q \times q}$. If $\iota \in \mathbb{N}_0 \cup \{+\infty\}$ with $\iota \leq \kappa$, then (5.2) and Remark 3.17 show that

$$\mathcal{R}_q^{[\kappa]} \left[ \Pi_+; (s_j)_{j=0}^{\iota} \right] = \mathcal{R}_q^{[\iota]} \left[ \Pi_+; (s_j)_{j=0}^{\iota} \right].$$

In particular,

$$\mathcal{R}_{\kappa,q} \left[ \Pi_+; (s_j)_{j=0}^{\kappa} \right] = \bigcap_{m=0}^{\kappa} \mathcal{R}_{\iota,q} \left[ \Pi_+; (s_j)_{j=0}^{m} \right] = \bigcap_{m=0}^{\kappa} \mathcal{R}_{\iota,q} \left[ \Pi_+; (s_j)_{j=0}^{m} \right].$$

Now we characterize those sequences for which the sets defined in (5.2) and (5.3) are non-empty.

Proposition 5.4. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{q \times q}$. Then:

(a) $\mathcal{R}_q^{[\kappa]} \left[ \Pi_+; (s_j)_{j=0}^{\kappa} \right] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,k}^{\geq,e}$ and $s_{-1} \in \mathbb{C}^{q \times q}$.

(b) $\mathcal{R}_{\kappa,q} \left[ \Pi_+; (s_j)_{j=0}^{\kappa} \right] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,k}^{\geq,e}$.

Proof. (a) Combine (5.2), (3.11), and Theorems 3.15 and 2.1

(b) Combine (5.3), (3.12), and Theorems 3.15 and 2.1

Now we state a useful characterization of the set of functions given in (5.3).

Proposition 5.5. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then

$$\mathcal{R}_{\kappa,q} \left[ \Pi_+; (s_j)_{j=0}^{\kappa} \right] = \left\{ F \in \mathcal{T}_{0,q}(\Pi_+) \left| \sigma_F \in \mathcal{M}_2^{q} \left[ \mathbb{R}; (s_j)_{j=0}^{\kappa}, = \right] \right. \right\}.$$  

Proof. In view of (5.3) and Corollary 3.29 we have

$$\mathcal{R}_{\kappa,q} \left[ \Pi_+; (s_j)_{j=0}^{\kappa} \right] \subseteq \left\{ F \in \mathcal{T}_{0,q}(\Pi_+) \left| \sigma_F \in \mathcal{M}_2^{q} \left[ \mathbb{R}; (s_j)_{j=0}^{\kappa}, = \right] \right. \right\}.$$  

Conversely, now let $F \in \mathcal{T}_{0,q}(\Pi_+)$ be such that

$$\sigma_F \in \mathcal{M}_2^{q} \left[ \mathbb{R}; (s_j)_{j=0}^{\kappa}, = \right].$$  

(5.4)
From (5.4) we get
\[ \sigma_F \in \mathcal{M}_\geq^\| (\mathbb{R}), \] (5.5)
whereas \( F \in \mathcal{R}_{0,q}(\Pi_+) \) and Proposition 3.28 imply \( F \in \mathcal{R}_{0,q}(\Pi_+) \). Hence, (3.12) yields \( F \in \mathcal{R}_{q}[0]_0^{(0)}(\Pi_+) \) and \( \gamma_F = 0_{q \times q} \). From \( F \in \mathcal{R}_{q}[0]^{(0)}(\Pi_+) \), (5.5), and (3.11) we see that \( F \in \mathcal{R}_{q}[0]_0^{(c)}(\Pi_+) \). Combining this with \( \gamma_F = 0_{q \times q} \), we infer from (3.12) that \( F \in \mathcal{R}_{n,q}(\Pi_+) \).

Because of (5.4) and (5.3), then it follows that \( F \) satisfies (5.8), and the corresponding functions \( s^\alpha \) and \( F \) belong to the class \( \mathcal{R}_{q}(\Pi_+) \) and its Nevanlinna parametrization \( (\alpha_F, \beta_F, \nu_F) \) satisfies
\[ N(s_0) = N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})) \] (5.6)
and
\[ R(s_0) = R(\alpha_F) + R(\beta_F) + R(\nu_F(\mathbb{R})). \] (5.7)

Proof. (a) In view of the choice of \( F \), we get from (5.4) that
\[ F \in \mathcal{R}_{n,q}(\Pi_+) \] (5.8)
and \( \sigma_F \in \mathcal{M}_\geq^\| [\mathbb{R}; (s_j)_{j=0}^\infty] \). Thus, we have \( s_{0_{\sigma_F}} = s_0 \). Otherwise, from (1.1) we have \( s_{0_{\sigma_F}} = \sigma_F(\mathbb{R}) \). Hence, \( \sigma_F(\mathbb{R}) = s_0 \). Combining this with (5.8), we obtain from Lemma 3.22 all assertions of (a).

(b) The assertions of (b) follow from (a) by application of Remark A.3.

(c) From (5.4) and Remark 3.24 we get \( F \in \mathcal{R}_{q}(\Pi_+) \). Now the assertions of (c) follow by combination of Proposition 2.7 with (ii).
The next result establishes a connection to the class \( \mathcal{P}_q^{\text{odd}}[s_0] \) introduced in Section 4.

**Lemma 5.8.** Let \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \) and let \((s_j)^{\kappa}_{j=0}\) be a sequence of complex \(q \times q\) matrices. Then \( \mathcal{R}_{\kappa,q}[\Pi_+; (s_j)^{\kappa}_{j=0}] \subseteq \mathcal{P}_q^{\text{odd}}[s_0] \).

**Proof.** Let \( F \in \mathcal{R}_{\kappa,q}[\Pi_+; (s_j)^{\kappa}_{j=0}] \). From (5.3) we get then \( F \in \mathcal{R}_{\kappa,q}(\Pi_+) \). Thus, from Remark 3.17 and (3.12) we infer \( F \in \mathcal{R}_{-1,q}(\Pi_+) \). Let \( z \in \Pi_+ \). In view of \( F \in \mathcal{R}_{-1,q}(\Pi_+) \) Lemma 3.21 yields \( N(F(z)) = N(\mu_F(\mathbb{R})) \), whereas part (c) of Proposition 5.7 gives \( N(F(z)) = N(s_0) \). Hence, \( N(s_0) = N(\mu_F(\mathbb{R})) \). In view of (4.7) and (4.2), this implies \( F \in \mathcal{P}_q^{\text{odd}}[s_0] \). \( \square \)

Now we want to discuss the asymptotic behaviour of functions belonging to \( \mathcal{R}_q(\Pi_+) \). For this reason, we need a particular construction, which will be introduced now.

**Remark 5.9.** Let \( \kappa \in \mathbb{Z}_{-1,+\infty} \cup \{+\infty\} \), let \((s_j)^{\kappa}_{j=-1}\) be a sequence of complex \(p \times q\) matrices, let \( \mathcal{G} \) be a non-empty subset of \( \mathbb{C} \), and let \( F: \mathcal{G} \to \mathbb{C}^{p \times q} \) be a matrix-valued function. For all \( k \in \mathbb{Z}_{-1,\kappa} \), let then \( F_k^{(s)}: \mathcal{G} \to \mathbb{C}^{p \times q} \) be defined by

\[
F_k^{(s)}(z) := z^{k+1} F(z) + \sum_{j=0}^{k+1} z^{k+1-j} s_{j-1}. \tag{5.9}
\]

For every choice of integers \( k \) and \( l \) with \(-1 \leq k < l \leq \kappa \) and each \( z \in \mathcal{G} \), then it is immediately checked that

\[
F_l^{(s)}(z) = z^{-k} F_k^{(s)}(z) + \sum_{j=1}^{l-k} z^{-k-j} s_{k+j} \tag{5.10}
\]

and in the case \( z \neq 0 \) furthermore

\[
F_k^{(s)}(z) = z^{-l} \left[ F_l^{(s)}(z) - \sum_{j=0}^{l-k-1} z^{j} s_{l-j} \right]. \tag{5.11}
\]

In the following, we will often use the construction of Remark 5.9 for the case that \( \mathcal{G} = \Pi_+ \) and that the function \( F \) belongs to particular subclasses of \( \mathcal{R}_q(\Pi_+) \). We start with the case that \( F \) belongs to the class introduced in (5.2) and investigate the associated sequence \((F_k^{(s)})^{\kappa}_{k=-1}\). First we show that these functions admit integral representations with respect to \( \sigma_F \).

**Proposition 5.10.** Let \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \), let \((s_j)^{\kappa}_{j=-1}\) be a sequence of complex \(q \times q\) matrices, and let \( F \in \mathcal{R}_q^{[\kappa]}[\Pi_+; (s_j)^{\kappa}_{j=-1}] \). For each \( k \in \mathbb{Z}_{-1,\kappa} \) and each \( z \in \Pi_+ \), then

\[
F_k^{(s)}(z) = \int_{\mathbb{R}} \frac{t^{k+1}}{t-z} \sigma_F(d\,t), \tag{5.12}
\]

where \( \sigma_F \) is given via (3.1).
Proof. Since $F$ belongs to $\mathcal{R}^{q}_{y}(\Pi_{+};(s_{j})_{j=-1}^{s})$, we have

$$F \in \mathcal{R}^{q}_{y}(\Pi_{+}), \quad \gamma_{F} = -s_{-1} \quad \text{and} \quad \sigma_{F} \in \mathcal{M}_{\geq}^{q} \left[ \mathbb{R}; (s_{j})_{j=-1}^{s} = \right]. \quad (5.13)$$

The last relation in (5.13) implies

$$\int_{\mathbb{R}} t^{l} \sigma_{F}(dt) = s_{j} \quad \text{for each } j \in \mathbb{Z}_{0,k}. \quad (5.14)$$

From the definition (3.11) of the class $\mathcal{R}^{q}_{y}(\Pi_{+})$, we see that $F$ belongs in particular to $\mathcal{R}^{0}_{y}(\Pi_{+})$. Thus, for all $z \in \Pi_{+}$, part (ii) of Theorem 3.15 and the second relation in (5.13) yield

$$F(z) = -s_{-1} + \int_{\mathbb{R}} \frac{1}{t-z} \sigma_{F}(dt) \quad (5.15)$$

and, in view of (5.9) and (5.15), then, in particular, that (5.12) holds true for $k = -1$. Because of (5.9) and (5.15), for all $z \in \Pi_{+}$, we obtain

$$F_{0}^{(s)}(z) = z[F(z) + s_{-1}] + s_{0} = \int_{\mathbb{R}} \frac{z}{t-z} \sigma_{F}(dt) + \int_{\mathbb{R}} 1 \sigma_{F}(dt)$$

$$= \int_{\mathbb{R}} \left( \frac{z}{t-z} + 1 \right) \sigma_{F}(dt) = \int_{\mathbb{R}} \frac{t^{0+1}}{t-z} \sigma_{F}(dt)$$

and, consequently, (5.12) for $k = 0$. It remains to consider the case $k \in \mathbb{Z}_{1,k}$. For all $z \in \Pi_{+}$ and each $t \in \mathbb{R}$, it is readily checked that

$$\frac{z^{j+1}}{t-z} + \sum_{j=1}^{k+1} z^{j+1-j} t^{j-1} = \frac{t^{k+1}}{t-z} \quad (5.16)$$

is true. Using (5.9), (5.15), (5.14), and (5.16), for all $z \in \Pi_{+}$, we get then

$$F_{k}^{(s)}(z) = z^{k+1} \left[ -s_{-1} + \int_{\mathbb{R}} \frac{1}{t-z} \sigma_{F}(dt) \right] + \sum_{j=0}^{k+1} z^{j+1-j} s_{j-1}$$

$$= z^{k+1} \int_{\mathbb{R}} \frac{1}{t-z} \sigma_{F}(dt) + \sum_{j=1}^{k+1} z^{k+1-j} \int_{\mathbb{R}} t^{j-1} \sigma_{F}(dt)$$

$$= \int_{\mathbb{R}} \left( \frac{z^{k+1}}{t-z} + \sum_{j=1}^{k+1} z^{k+1-j} t^{j-1} \right) \sigma_{F}(dt) = \int_{\mathbb{R}} \frac{t^{k+1}}{t-z} \sigma_{F}(dt).$$

Thus, (5.12) is also proved in the case $k \in \mathbb{Z}_{1,k}$.

Remark 5.11. Let $\kappa \in \mathbb{N}_{0} \cup \{ +\infty \}$, let $(s_{j})_{j=-1}^{s}$ be a sequence of complex $q \times q$ matrices, let $k \in \mathbb{Z}_{0,k}$, and let $F \in \mathcal{R}^{q}_{y}(\Pi_{+};(s_{j})_{j=-1}^{s})$. Then (5.10) and Proposition 5.10 show that

$$F_{l}^{(s)}(z) = \sum_{j=0}^{l-k-1} z^{j} s_{l-j} + z^{l-k} \int_{\mathbb{R}} \frac{t^{k+1}}{t-z} \sigma_{F}(dt)$$

holds true for all $l \in \mathbb{Z}_{k+1,k}$ and each $z \in \Pi_{+}$. 

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Now we consider the functions $F_{k}^{(s)}$ occurring in Proposition 5.10 especially for odd numbers $k$.

**Proposition 5.12.** Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_{j})_{j=-1}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Furthermore let $F \in \mathcal{R}_{q}^{(s)}[\Pi_{+}; (s_{j})_{j=-1}^{\kappa}]$ and let $\sigma_{F}$ be given via (5.11). Then:

(a) For all $n \in \mathbb{N}_0$ with $2n - 1 \leq \kappa$, the function $F_{2n-1}^{(s)}$ belongs to $\mathcal{R}_{-1,q}(\Pi_{+})$ and the measures $\nu_{F_{2n-1}^{(s)}}$ and $\mu_{F_{2n-1}^{(s)}}$ admit, for all $B \in \mathcal{B}_{R}$, the representations

\[ \nu_{F_{2n-1}^{(s)}}(B) = \int_{B} \frac{t^{2n}}{t^2 + 1} \sigma_{F}(d\,t) \]  

and

\[ \mu_{F_{2n-1}^{(s)}}(B) = \int_{B} \frac{t^{2n}}{|t| + 1} \sigma_{F}(d\,t). \]  

(b) For all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, it holds $F_{2n-1}^{(s)} \in \mathcal{R}_{\kappa-2n,q}[\Pi_{+}; (s_{2n+j})_{j=0}^{\kappa-2n}]$ and, for all $B \in \mathcal{B}_{R}$, furthermore

\[ \sigma_{F_{2n-1}^{(s)}}(B) = \int_{B} t^{2n} \sigma_{F}(d\,t). \]  

**Proof.** Since $F$ belongs to $\mathcal{R}_{q}^{(s)}[\Pi_{+}; (s_{j})_{j=-1}^{\kappa}]$, we have (5.13). In particular, $\sigma_{F} \in \mathcal{M}_{q}^{(s)}(\mathbb{R})$. Proposition 5.10 shows that, for all $z \in \Pi_{+}$, furthermore

\[ F_{2n-1}^{(s)}(z) = \int_{\mathbb{R}} \frac{t^{2n}}{t - z} \sigma_{F}(d\,t). \]  

**Remark.** Let $n \in \mathbb{N}_0$ with $2n - 1 \leq \kappa$. If $n \geq 1$, then $0 \leq \frac{t^{2n}}{t^2 + 1} \leq t^{2n-2}$ and $|\frac{t^{2n}}{t^2 + 1}| \leq |t^{2n-1}|$ for all $t \in \mathbb{R}$, whereas if $n = 0$, then $0 \leq \frac{t^{2n}}{t^2 + 1} \leq 1$ and $|\frac{t^{2n}}{t^2 + 1}| \leq 1$ for all $t \in \mathbb{R}$. Thus, from $\sigma_{F} \in \mathcal{M}_{q}^{(s)}(\mathbb{R})$ and [22] Proposition B.5 we get that the following statement holds true:

(I) The mapping $\nu: \mathcal{B}_{R} \to \mathbb{C}^{q \times q}$ given by

\[ \nu(B) := \int_{B} \frac{t^{2n}}{t^2 + 1} \sigma_{F}(d\,t) \]  

is a well-defined non-negative Hermitian measure which belongs to $\mathcal{M}_{q}^{(s)}(\mathbb{R})$ and which fulfills

\[ s_{1}^{(s)} = \int_{\mathbb{R}} \frac{t^{2n}}{t^2 + 1} \sigma_{F}(d\,t). \]  

For all $z \in \Pi_{+}$, from (5.20) and (I) we then get

\[ F_{2n-1}^{(s)}(z) = s_{1}^{(s)} + \int_{\mathbb{R}} \frac{t^{2n}}{t - z} \sigma_{F}(d\,t) - \int_{\mathbb{R}} \frac{t^{2n}}{t^2 + 1} \sigma_{F}(d\,t). \]  

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Since [1] and [22, Proposition B.5] provide us
\[
\int_{\mathbb{R}} \frac{t^{2n}}{t-z} \sigma_F(\,dt) - \int_{\mathbb{R}} \frac{t^{2n}}{t^2+1} \sigma_F(\,dt) = \int_{\mathbb{R}} \left( \frac{t^{2n}}{t-z} - \frac{t^{2n}}{t^2+1} \right) \sigma_F(\,dt)
\]
\[
= \int_{\mathbb{R}} \frac{1+tz}{t-z} \frac{t^{2n}}{t^2+1} \sigma_F(\,dt)
\]
\[
= \int_{\mathbb{R}} \frac{1+tz}{t-z} \nu(\,dt)
\]
for all \(z \in \Pi_+\), from (5.22) we conclude that
\[
F^{(s)}_{2n-1}(z) = z_1 + z \cdot 0_{q \times q} + \int_{\mathbb{R}} \frac{1+tz}{t-z} \nu(\,dt)
\]
for all \(z \in \Pi_+\). Since [1,1] and [22, Proposition B.4] show that \((s_1^{[\nu]})^* = s_1^{[\nu]}\), Theorem 2.1 then yields that \(F^{(s)}_{2n-1} \in \mathcal{R}_{q}(\Pi_+)\) with Nevanlinna parametrization
\[
(\alpha^{(s)}_{F_{2n-1}}, \beta^{(s)}_{F_{2n-1}}, \nu^{(s)}_{F_{2n-1}}) = (s_1^{[\nu]}, 0_{q \times q}, \nu).
\] (5.23)

From (5.23) we get
\[
\alpha^{(s)}_{F_{2n-1}} = s_1^{[\nu]}_{F_{2n-1}}
\] (5.24)
and, taking (5.21) into account, also that formula (5.17) is true for all \(B \in \mathfrak{B}_R\) and that
\[
\nu^{(s)}_{F_{2n-1}} \in M^q_{\geq 1}(\mathbb{R}).
\] (5.25)

In view of (5.23), we have \(\beta^{(s)}_{F_{2n-1}} = 0_{q \times q}\). Combining this with (5.21) and (5.25), we see from Remark 3.24 that \(F^{(s)}_{2n-1} \in \mathcal{R}_{-1,q}(\Pi_+)\) and, in view of (3.13), especially \(F^{(s)}_{2n-1} \in \mathcal{R}^{-1}_{q}(\Pi_+)\). Taking (3.6) and (5.17) into account and using [22, Proposition B.5], it follows
\[
\mu^{(s)}_{F_{2n-1}}(B) = \int_B \frac{t^2+1}{|t|+1} \nu^{(s)}_{F_{2n-1}}(\,dt) = \int_B \frac{t^{2n}}{|t|+1} \sigma_F(\,dt)
\]
for all \(B \in \mathfrak{B}_R\). Thus, (5.18) is verified and the proof of part (ii) is complete.

In view of \(F \in \mathcal{R}^q_{q}[\Pi_+; (s_j)_{j=-1}^\kappa]\), we get from (5.2) that
\[
\sigma_F \in M^q_{\geq 2n}(\mathbb{R})
\] (5.26)
and
\[
s_j^{[\sigma_F]} = s_j \quad \text{for each } j \in \mathbb{Z}_{0,n}.
\] (5.27)

Now we assume that \(n \in \mathbb{N}_0\) is such that \(2n \leq \kappa\). Then (5.26) implies \(\sigma_F \in M^q_{\geq 2n}(\mathbb{R})\). Therefore, from [22, Proposition B.5] we see that the mapping \(\sigma : \mathfrak{B}_R \rightarrow \mathbb{C}^{q \times q}\) given by
\[
\sigma(B) := \int_B t^{2n} \sigma_F(\,dt)
\] (5.28)
is a well-defined non-negative Hermitian measure belonging to $\mathcal{M}^q_\geq(\mathbb{R})$ and that
\[
\int_B \frac{1}{t-z} \sigma(d t) = \int_B \frac{t^{2n}}{t-z} \sigma_F(d t)
\]
for all $z \in \Pi_+$. Combining (5.28) and (5.29), we obtain
\[
F_{2n-1}^{(s)}(z) = 0_{q \times q} + \int_{\mathbb{R}} \frac{1}{t-z} \sigma(d t)
\]
for all $z \in \Pi_+$, which, in view of Theorem 5.13, implies
\[
F_{2n-1}^{(s)} \in \mathcal{R}_q^{[0]}(\Pi_+) \quad \text{and} \quad (\gamma_{F_{2n-1}^{(s)}}, \sigma_{F_{2n-1}^{(s)}}) = (0_{q \times q}, \sigma).
\]
Consequently, from (5.28) we get (5.19) for all $B \in \mathcal{B}_\mathbb{R}$. For all $m \in \mathbb{Z}_{0,\kappa-2n}$, we have $m + 2n \leq \kappa$, hence (5.26) and Remark 1.1 imply $\sigma_F \in \mathcal{M}^q_{\kappa,m+2n}(\mathbb{R})$, and, because of (5.28) and [22, Proposition B.5], furthermore, we infer $\sigma \in \mathcal{M}^q_{\kappa,m}(\mathbb{R})$ and
\[
\int_{\mathbb{R}} t^{m+2n} \sigma_F(d t) = \int_{\mathbb{R}} t^m \sigma(d t) = \int_{\mathbb{R}} t^m \sigma(d t).
\]
Thus, $\sigma \in \mathcal{M}^q_{\kappa,2n-2n}(\mathbb{R})$. In view of (5.30), this means $\sigma_{F_{2n-1}^{(s)}} \in \mathcal{M}^q_{\kappa,2n-2n}(\mathbb{R})$. Thus, since (5.30) implies $\gamma_{F_{2n-1}^{(s)}} = 0_{q \times q}$, from (5.12), we get $F_{2n-1}^{(s)} \in \mathcal{R}_{\kappa-2n,q}(\Pi_+)$. For all $m \in \mathbb{Z}_{0,\kappa-2n}$, we have $m + 2n \leq \kappa$, so that (5.30), (5.28), [22, Proposition B.5], (1.1), and (5.27) imply
\[
\int_{\mathbb{R}} t^m \sigma_{F_{2n-1}^{(s)}}(d t) = \int_{\mathbb{R}} t^m \sigma(d t) = \int_{\mathbb{R}} t^m \sigma_{F_{2n-1}^{(s)}}(d t) = \int_{\mathbb{R}} t^m t^{2n} \sigma_F(d t) = s_2^{[\sigma_F]} = s_{2n+m}.
\]
Consequently, $\sigma_{F_{2n-1}^{(s)}} \in \mathcal{M}^q_{\kappa,2n-2n}(\mathbb{R})$. Because of (5.3), then we see that $F_{2n-1}^{(s)}$ belongs to $\mathcal{R}_{\kappa-2n,q}[\Pi_+; (s_{2n}+j)^{\kappa-2n}_{j=0}]$. \[\square\]

It should be mentioned that in the scalar case the membership of $F_{2n-1}^{(s)}$ to $\mathcal{R}_{\kappa-2n,q}(\Pi_+)$, which is contained in part (a) of Proposition 5.12, was already obtained in [29, Theorem 3.2].

The following results complement the theme of Proposition 5.12. They will play an important role in the proof of Theorem 6.6.

**Proposition 5.13.** Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=-1}^{2n+1}$ be a sequence of complex $q \times q$ matrices. Furthermore, let $F \in \mathcal{R}_q^{[2n]}[\Pi_+;(s_j)_{j=-1}^{2n+1}]$ and let $\sigma_F$ be given via (3.11). Then:

(a) For all $z \in \Pi_+$, the matrix-valued function $F_{2n+1}^{(s)}$ can be represented via
\[
F_{2n+1}^{(s)}(z) = s_{2n+1} + z \int_{\mathbb{R}} \frac{t^{2n+1}}{t-z} \sigma_F(d t).
\]
(b) Suppose $s_{2n+1}^* = s_{2n+1}$. Then $F_{2n+1}^{(s)} \in \mathcal{R}_q(\Pi_+)$, and the Nevanlinna parameterization $(\alpha_{F_{2n+1}^{(s)}}, \beta_{F_{2n+1}^{(s)}}, \nu_{F_{2n+1}^{(s)}})$ of $F_{2n+1}^{(s)}$ is given by

$$\alpha_{F_{2n+1}^{(s)}} = s_{2n+1} - \int_{\mathbb{R}} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\,t),$$

$$\beta_{F_{2n+1}^{(s)}} = 0_{q \times q},$$

and

$$\nu_{F_{2n+1}^{(s)}}(B) = \int_B \frac{t^{2n+2}}{t^2 + 1} \sigma_F(d\,t) \quad \text{for each } B \in \mathcal{B}_\mathbb{R}. $$

If $F_{2n+1}^{(s)}$ belongs to $\mathcal{R}_q^{[2n]}(\Pi_+)$, then $F \in \mathcal{R}_q^{[2n+1]}(\Pi_+)$ and

$$s_{2n+1}^{[\sigma_F]} = s_{2n+1} - \gamma_{F_{2n+1}^{(s)}},$$

where $\gamma_{F_{2n+1}^{(s)}}$ is given by (3.7).

**Proof.** [a] Formula (5.31) immediately follows from Remark 5.11.

[b] Since $F$ belongs to $\mathcal{R}_q^{[2n]}(\Pi_+; (s_j)_{j=0}^{2n})$, in view of (5.2), we have

$$F \in \mathcal{R}_q^{[2n]}(\Pi_+), \quad \gamma_F = -s_{-1}, \quad \text{and} \quad \sigma_F \in \mathcal{M}_{\leq}^q [\mathbb{R}; (s_j)_{j=0}^{2n}, =].$$

(5.32)

Because of $0 \leq \frac{t^{2n+2}}{t^2 + 1} \leq t^{2n}$ for all $t \in \mathbb{R}$ and the third relation in (5.32), from [22, Proposition B.5] we then get that $\nu : \mathcal{B}_\mathbb{R} \rightarrow \mathbb{C}^{q \times q}$ given by

$$\nu(B) := \int_B \frac{t^{2n+2}}{t^2 + 1} \sigma_F(d\,t)$$

(5.33)

is a well-defined non-negative Hermitian measure belonging to $\mathcal{M}_{\geq}^q(\mathbb{R})$ for which the identity

$$\int_{\mathbb{R}} \frac{1 + tz}{t - z} \nu(d\,t) = \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{t^{2n+2}}{t^2 + 1} \sigma_F(d\,t), \quad z \in \Pi_+,$$

(5.34)

holds true. If $n \geq 1$, then $|\frac{t^{2n+1}}{t^2 + 1}| \leq |t^{2n-1}|$ is fulfilled for all $t \in \mathbb{R}$. If $n = 0$, then $|\frac{t^{2n+1}}{t^2 + 1}| \leq 1$ for all $t \in \mathbb{R}$. Thus, we see from [22, Lemma B.1] and (5.32) that the integral $\int_{\mathbb{R}} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\,t)$ exists. For every choice of $z$ in $\Pi_+$, from (5.31) and (5.34) we conclude

$$F_{2n+1}^{(s)}(z) - s_{2n+1} = \int_{\mathbb{R}} \frac{1 + tz}{t - z} \nu(d\,t) - \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\,t)$$

$$= \int_{\mathbb{R}} \left( \frac{z^{2n+1}}{t - z} + \frac{t^{2n+1}}{t^2 + 1} \right) \sigma_F(d\,t) - \int_{\mathbb{R}} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\,t)$$

$$= \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{t^{2n+2}}{t^2 + 1} \sigma_F(d\,t) - \int_{\mathbb{R}} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\,t)$$

$$= \int_{\mathbb{R}} \frac{1 + tz}{t - z} \nu(d\,t) - \int_{\mathbb{R}} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\,t)$$

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and, consequently,  
\[ F_{2n+1}(s) = s_{2n+1} - \int_{\mathbb{R}} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\ t) + z \cdot 0_{q \times q} + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \nu(d\ t). \]  
(5.35)

Thanks to \( s_{2n+1}^* = s_{2n+1} \) and [22, Remark B.4], the matrix \( s_{2n+1} - \int_{\mathbb{R}} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\ t) \) is Hermitian. Thus, in view of (5.35), applying Theorem 2.1 yields \( F_{2n+1}^{(s)} \in \mathcal{R}_q(\Pi_+) \) and

\[ (\alpha_{F_{2n+1}^{(s)}}, \beta_{F_{2n+1}^{(s)}}, \nu_{F_{2n+1}^{(s)}}) = \left( s_{2n+1} - \int_{\mathbb{R}} \frac{t^{2n+1}}{t^2 + 1} \sigma_F(d\ t), 0_{q \times q}, \nu \right). \]

Now suppose that \( F_{2n+1}^{(s)} \in \mathcal{R}_q[\Pi_+] \).

In view of (5.32) and the definition (3.11) of the class \( \mathcal{R}_q^{[2n]}(\Pi_+) \), we also have \( F_{2n+1}^{(s)} \in \mathcal{R}_q^{[2n]}(\Pi_+) \). Furthermore, Remark 3.10 yields \( \nu_{F_{2n+1}^{(s)}} \in \mathcal{M}_{q,1}^q(\mathbb{R}) \). Thus, in view of \( \nu_{F_{2n+1}^{(s)}} = \nu \), the integral \( \int_{\mathbb{R}} t \nu(d\ t) \) exists. Because of (5.33) and [22, Proposition B.5], then we see that the integral \( \int_{\mathbb{R}} \frac{t^{2n+3}}{t^2 + 1} \sigma_F(d\ t) \) exists. Hence, since \( |\frac{t^{2n+3}}{t^2 + 1}| \) holds for all \( t \in \mathbb{R} \), from [22, Lemma B.1] we get that \( \sigma_F \in \mathcal{M}_{q,1}^q(\mathbb{R}) \). In view of (3.11) this shows that \( F \) belongs to \( \mathcal{R}_q^{[2n+1]}(\Pi_+) \). Because of (5.36), we infer from Proposition 3.14 that

\[ \lim_{k \to +\infty} \Re \ F_{2n+1}^{(s)}(ik) = \gamma_{F_{2n+1}^{(s)}}. \]  
(5.37)

Using (5.31), \( s_{2n+1}^* = s_{2n+1} \), and [22, Remark B.4], for all \( k \in \mathbb{N} \), we conclude

\[ \Re F_{2n+1}^{(s)}(ik) = s_{2n+1} + \int_{\mathbb{R}} \Re \left( \frac{ikt^{2n+1}}{t - ik} \right) \sigma_F(d\ t) = s_{2n+1} - \int_{\mathbb{R}} \frac{k^2t^{2n+1}}{t^2 + k^2} \sigma_F(d\ t). \]  
(5.38)

Thus, from (5.37) and (5.38) we get

\[ s_{2n+1} - \gamma_{F_{2n+1}^{(s)}} = \lim_{k \to +\infty} \int_{\mathbb{R}} \frac{k^2t^{2n+1}}{t^2 + k^2} \sigma_F(d\ t). \]  
(5.39)

For all \( t \in \mathbb{R} \), we have

\[ \lim_{k \to +\infty} \frac{k^2t^{2n+1}}{t^2 + k^2} = t^{2n+1}. \]  
(5.40)

Since \( |\frac{k^2t^{2n+1}}{t^2 + k^2}| \leq |t^{2n+1}| \) holds for all \( k \in \mathbb{N} \) and all \( t \in \mathbb{R} \), from \( \sigma_F \in \mathcal{M}_{q,1}^q(\mathbb{R}) \), (5.40), [22, Lemma B.1] and Lebesgue's dominated convergence theorem then

\[ \lim_{k \to +\infty} \int_{\mathbb{R}} \frac{k^2t^{2n+1}}{t^2 + k^2} \sigma_F(d\ t) = \int_{\mathbb{R}} t^{2n+1} \sigma_F(d\ t) = s_{2n+1}^{[\sigma_F]} \]  
(5.41)

follows. Combining (5.39) and (5.41) yields \( s_{2n+1}^{[\sigma_F]} = s_{2n+1} - \gamma_{F_{2n+1}^{(s)}}. \)
6. On Hamburger-Nevanlinna Type Results for $\mathcal{R}_q[\kappa][\Pi_+; (s_j)_{j=-1}^\infty]$ 

Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q \times q}$. Then the moment problem $M[\mathbb{R}; (s_j)_{j=0}^{2n}, =]$ can be reformulated as a problem of a prescribed asymptotic expansion for functions in $\tilde{\mathcal{R}}_{0,q}(\Pi_+)$. This is a consequence of a matricial version of a classical result due to Hamburger and Nevanlinna. This matricial version can be found in [34, p. 47] and [9, Lemma 2.1], where it was stated without proof. It can be proved along the lines of the proof of the scalar result which was given in [1, Ch. 3, Sect. 2]. Before formulating the result, we introduce some notation. For all $r \in (0, +\infty)$ and each $\delta \in (0, \pi/2]$, let 

$$
\Sigma_{r,\delta} := \{ z \in \mathbb{C} ||z| \geq r \text{ and } \delta \leq \arg r \leq \pi - \delta \}.
$$

Taking (5.3), Corollary 5.29 and Theorem 5.2 into account, now we can reformulate the matricial version of the Hamburger-Nevanlinna theorem:

**Theorem 6.1.** Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices.

(a) Let $F \in \mathcal{R}_{2n,q}[\Pi_+; (s_j)_{j=0}^{2n}]$. Then

$$
\lim_{r \to +\infty} \sup_{z \in \Sigma_{r,\delta}} \left\| z^{2n+1} \left[ F(z) + \sum_{j=0}^{2n} \frac{1}{z^{j+1}} s_j \right] \right\| = 0.
$$

(b) Let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q \times q}$, and let $F \in \mathcal{R}_q(\Pi_+)$ be such that

$$
\lim_{y \to +\infty} \left\| (iy)^{2n+1} \left[ F(iy) + \sum_{j=0}^{2n} \frac{1}{(iy)^{j+1}} s_j \right] \right\| = 0.
$$

Then $F \in \mathcal{R}_{2n,q}[\Pi_+; (s_j)_{j=0}^{2n}]$.

Part (a) of Theorem 6.1 will be often applied in the following. It contains a sufficient condition which implies that a function $F \in \mathcal{R}_q(\Pi_+)$ is the Stieltjes transform of a solution $\sigma$ of Problem $M[\mathbb{R}; (s_j)_{j=0}^{2n}, =]$. The main goal of this section is to find appropriate generalizations of Theorem 6.1 for the class $\mathcal{R}_q[\kappa][\Pi_+; (s_j)_{j=-1}^\infty]$ with arbitrary $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$. In the scalar case, this theme was treated by [29]. As we will see soon, that similar as in [29], the essential tool in our strategy is the use of the construction introduced in Remark 5.9. In this connection, it should be mentioned that, in view of Remark 5.1, in the case of an affirmative answer to the generalization of part (a) of Theorem 6.1, we would also obtain a sufficient condition for a function $F \in \mathcal{R}_q(\Pi_+)$ to be the Stieltjes transform of a solution $\sigma$ of Problem $M[\mathbb{R}; (s_j)_{j=0}^{2n+1}, =]$. The following result, which in the scalar case goes back to [29, Theorem 3.2], meets our above formulated goal concerning part (a) of Theorem 6.1.
Theorem 6.2. Let \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \), let \((s_j)_{j=-1}^\kappa\) be a sequence of complex \( q \times q \) matrices, and let \( F \in \mathcal{R}^{[\kappa]}_\mathbb{R}[\Pi_+; (s_j)_{j=-1}^\kappa] \). For all \( k \in \mathbb{Z}_{-1,\kappa} \) and each \( \delta \in (0, \frac{\pi}{2}] \), then

\[
\lim_{r \to +\infty} \sup_{z \in \Sigma_{r,\delta}} \| F_k(z) \| = 0 \tag{6.2}
\]

holds true, where \( \Sigma_{r,\delta} \) is given by (6.1).

Proof. The strategy of our proof is inspired by the proof which was given in [1, Ch. 3, Sect. 2] for the scalar case \( q = 1 \) of part (a) of Theorem 6.1. Since the function \( F \) belongs to \( \mathcal{R}^{[\kappa]}_\mathbb{R}[\Pi_+; (s_j)_{j=-1}^\kappa] \), we have \( F \in \mathcal{R}^{[\kappa]}_\mathbb{R}(\Pi_+), \gamma_F = -s-1, \) and \( \sigma_F \in M_{\geq}^q[\mathbb{R}; (s_j)_{j=0}^\kappa] \). From Proposition 5.10 we know that (5.12) holds true for all \( z \in \Pi_+ \). Now we let \( k \in \mathbb{Z}_{-1,\kappa} \), let \( \delta \in (0, \frac{\pi}{2}] \), and let \( \epsilon \in (0, +\infty) \). We consider an arbitrary \( u \in \mathbb{C}^q \). In view of [22, Lemma B.3], then \( \rho_u := u^*\sigma_F u \) is a finite measure on \((\mathbb{R}, \mathcal{B}_\mathbb{R})\), which belongs to \( M_{\geq}^q[\mathbb{R}; (s_j)_{j=0}^\kappa] \). Using additionally (5.12), for all \( z \in \Pi_+ \), we then obtain

\[
|u^*F_k(z)u| = \left| \int_{\mathbb{R}} \frac{t^{k+1}}{t-z} \rho_u(d\,t) \right| \leq \int_{\mathbb{R}} \frac{|t^{k+1}|}{|t-z|} \rho_u(d\,t). \tag{6.4}
\]

First we now consider the case \( k \in \mathbb{Z}_{0,\kappa} \). Then the mapping \( \mu : \mathcal{B}_\mathbb{R} \to \mathbb{C} \) given by

\[
\mu(B) = \int_B |t|^k \rho_u(d\,t) \tag{6.5}
\]

is a well-defined finite measure, i.e., \( \mu \) belongs to \( M_{\geq}^1(\mathbb{R}) \). Obviously,

\[
\lim_{n \to +\infty} \mu(\mathbb{R} \setminus [-n, n]) = 0. \tag{6.6}
\]

Thanks to (6.6), there is an \( N \in \mathbb{N} \) such that

\[
\mu(\mathbb{R} \setminus [-N, N]) < \frac{\epsilon}{2} \sin \delta. \tag{6.7}
\]

Clearly, if we set

\[
R := \frac{2N^{k+1} \rho_u([-N, N])}{\epsilon \sin \delta} + 1,
\]

then \( R \) belongs to \([1, +\infty)\) and we have

\[
\frac{N^{k+1} \rho_u([-N, N])}{R \sin \delta} < \frac{\epsilon}{2}. \tag{6.8}
\]

We consider an arbitrary \( r \in [R, +\infty) \) and an arbitrary \( z \in \Sigma_{r,\delta} \). Then

\[
|z| \in [r, +\infty) \quad \text{and} \quad \arg z \in [\delta, \pi - \delta]. \tag{6.9}
\]

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For all $t \in \mathbb{R}$, we get furthermore

$$|t - z| \geq |\text{Im}(t - z)| = |\text{Im} z| = |z| |\text{sin}(\text{arg } z)| \geq |z| \sin \delta. \quad (6.10)$$

For all $t \in [-N, N]$, we have

$$0 \leq |t^{k+1}| \leq N^{k+1}$$

and, because of (6.10), we conclude

$$|t - z| \geq |z| \sin \delta \geq r \sin \delta \geq R \sin \delta. \quad (6.12)$$

If $t \in \mathbb{R} \setminus [-n, n]$, then

$$|t - z| \geq |e^{-i \text{arg } z}| |t - z| e^{i \text{arg } z} = |te^{-i \text{arg } z} - |z| | \geq |\text{Im} (te^{i \text{arg } z} - |z|)|$$

$$= |t| |\text{sin}(\text{arg } z)| \geq |t| \sin \delta > 0$$

and, consequently,

$$\frac{|t|}{\sin \delta} = \frac{|t^{k+1}|}{|t| \sin \delta} \geq \frac{|t^{k+1}|}{|t - z|}. \quad (6.13)$$

Using (6.4), (6.11), (6.12), (6.13), (6.5), (6.8), and (6.7), we get then

$$\left| u_s F_k^{(s)}(z)u \right| \leq \int_{[-N,N]} \frac{|t^{k+1}|}{|t - z|} \rho_u(dt) + \int_{\mathbb{R}\setminus[-N,N]} \frac{|t^{k+1}|}{|t - z|} \rho_u(dt)$$

$$\leq \int_{[-N,N]} \frac{N^{k+1}}{R \sin \delta} \rho_u(dt) + \int_{\mathbb{R}\setminus[-N,N]} \frac{|t|}{R \sin \delta} \rho_u(dt) \quad (6.14)$$

$$= \frac{N^{k+1}}{R \sin \delta} \rho_u([-N,N]) + \frac{1}{\sin \delta} \mu(\mathbb{R} \setminus [-N,N]) < \epsilon.$$

Now we consider the case $k = -1$. Then

$$R := \frac{\rho_u(\mathbb{R})}{\epsilon \sin \delta} + 1 \quad (6.15)$$

belongs to $[1, +\infty)$. Let $r \in [R, +\infty)$ and $z \in \Sigma_{r, \delta}$. Hence the relations in (6.9) are true and the inequalities in (6.10) follow again. Consequently, (6.12) is fulfilled. Because of (6.4), $k = -1$, (6.12), and (6.15), we get then

$$\left| u_s F_k^{(s)}(z)u \right| \leq \int_{\mathbb{R}} \frac{1}{|t - z|} \rho_u(dt) \leq \int_{\mathbb{R}} \frac{1}{R \sin \delta} \rho_u(dt) = \frac{\rho_u(\mathbb{R})}{R \sin \delta} < \epsilon. \quad (6.16)$$

Thus, in view of (6.14) and (6.16), we proved that, for all $k \in \mathbb{Z}_{-1, \infty}$, for all $\delta \in (0, \frac{\pi}{2}]$, for all $\epsilon \in (0, +\infty)$, and each $u \in \mathbb{C}^q$, there is an $R \in [0, +\infty)$ such that $|u_s F_k^{(s)}(z)u| < \epsilon$ for all $r \in [R, +\infty)$ and each $z \in \Sigma_{r, \delta}$. In other words, for all $k \in \mathbb{Z}_{-1, \infty}$, each $\delta \in (0, \frac{\pi}{2}]$, and each $u \in \mathbb{C}^q$, we have

$$\lim_{r \to +\infty} \sup_{z \in \Sigma_{r, \delta}} \left| u_s F_k^{(s)}(z)u \right| = 0.$$

A standard argument of linear algebra (see, e. g. [16 Remark 1.1.1]) yields then (6.2).
Corollary 6.3. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=-1}^n$ be a sequence of complex $q \times q$ matrices, and let $F \in \mathcal{R}_q[(\Pi_+: (s_j)_{j=-1}^n)]$. For all $k \in \mathbb{Z}_{-1,\kappa}$, then

$$\lim_{y \to +\infty} F_k^{(s)}(iy) = 0_{q \times q} \quad (6.17)$$

and

$$-s_k = \begin{cases} \lim_{y \to +\infty} F(yi) & \text{if } k = -1 \\ \lim_{y \to +\infty} (iy)^{k+1} [F(iy) + \sum_{j=0}^{k} (iy)^{-j}s_{j-1}] & \text{if } k \geq 0 \end{cases} \quad (6.18)$$

Proof. Because of $F \in \mathcal{R}_q[(\Pi_+: (s_j)_{j=-1}^n)]$, we have (6.3). In particular, $F$ belongs to $\mathcal{R}_q[(\Pi_+)]$. Thus, (3.11) gives $F \in \mathcal{R}_q^{(0)}(\Pi_+)$. Taking (3.10) into account, then $F \in \mathcal{R}_q[-1](\Pi_+)$ follows. Consequently, Proposition 5.14 yields (3.8). Let $k \in \mathbb{Z}_{-1,\kappa}$. The limit (6.17) is an immediate consequence of Theorem 6.2. If $k = -1$, then (6.3) and (3.8) imply

$$-s_{-1} = \gamma_F = \lim_{y \to +\infty} F(iy).$$

Thus, (6.17) holds true for $k = -1$. If $k \in \mathbb{Z}_{0,\kappa}$, then (6.17) and (5.9) imply

$$-s_k = \lim_{y \to +\infty} \left[ F_k^{(s)}(iy) - s_k \right] = \lim_{y \to +\infty} (iy)^{k+1} \left[ F(iy) + \sum_{j=0}^{k} (iy)^{-j}s_{j-1} \right]. \quad \square$$

Now we are going to show that part (a) of Theorem 6.1 is an immediate consequence of Theorem 6.2.

Proof of part (a) of Theorem 6.1. Let $t_{-1} := 0_{q \times q}$ and let $t_j := s_j$ for all $j \in \mathbb{Z}_{0,2n}$. Then Remark 5.1 yields $F \in \mathcal{R}_q[(\Pi_+; (t_j)_{j=-1}^{2n})]$. Thus, Theorem 6.2 implies

$$\lim_{r \to +\infty} \sup_{z \in \Sigma_{t_1}} \|F_{2n}^{(t)}(z)\| = 0. \quad (6.19)$$

Because of $t_{-1} = 0_{q \times q}$ we see from (5.9) that

$$F_{2n}^{(t)}(z) = z^{2n+1} \left[ F(z) + \sum_{j=0}^{2n} \frac{1}{z^{j+1}}s_j \right]. \quad (6.20)$$

Now the combination of (6.19) and (6.20) completes the proof of part (a) of Theorem 6.1. \square

Now we state a corresponding generalization of the second part of Theorem 6.1. It should be mentioned that in the scalar case the result goes back to [29, Theorem 3.3].

Theorem 6.4. Let $n \in \mathbb{N}_0$, let $(s_j)_{j=-1}^{2n}$ be a sequence of Hermitian complex $q \times q$ matrices, and let $F \in \mathcal{R}_q(\Pi_+)$ be such that

$$\lim_{y \to +\infty} F_{2n}^{(s)}(iy) = 0_{q \times q} \quad (6.21)$$

Then $F$ belongs to $\mathcal{R}_q[(\Pi_+; (s_j)_{j=-1}^{2n})]$. 

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Proof. Let \( G : \Pi_+ \to \mathbb{C}^{q \times q} \) defined by \( G(z) := s_{-1} \), and let \( S := F + G \). Since \( s^*_{-1} = s_{-1} \) holds, we see from Theorem 5.7 and 22 Remark 3.4 that \( G \) and \( S \) belong to \( R_q(\Pi_+) \). Using (6.21) and (5.9), we obtain then

\[
0_{q \times q} = \lim_{y \to +\infty} (iy)^{2n+1} \left[ F(iy) + \sum_{j=0}^{2n+1} (iy)^{-j} s_{j-1} \right]
\]

Hence, part (b) of Theorem 6.1 shows that \( R \) belongs to \( F \) function.

In particular, from (6.22) we infer then

\[
\text{Using (6.22) we obtain then }
\]

\[
0_{q \times q} = \lim_{y \to +\infty} (iy)^{2n+1} \left[ S(iy) + \sum_{k=0}^{2n} (iy)^{-k-1} s_k \right].
\]

Thus, taking (6.22), (5.11), and (3.12) into account we see that \( S \) belongs to \( R_q^0(\Pi_+) \) and that \( \gamma_S = 0_{q \times q} \) holds. Since Theorem 3.15 shows that \( -G \in R_q^0(\Pi_+) \), that \( \gamma_{-G} = -s_{-1} \), and that \( \sigma_{-G} \) is the zero measure in \( M^q_+(\mathbb{R}) \), we see from (3.10) that \( S \) and \( -G \) both belong to \( R_q^1(\Pi_+) \cap R_q^0(\Pi_+) \). Thus, [22] Remark 4.4 yields \( F \in R_q^1(\Pi_+) \) and \( \sigma_F = \sigma_S \). In particular, from (6.22) we infer then \( \sigma_F \in M^q_+[\mathbb{R}; (s_j)_{j=0}^{2n}] \) and \( F \in R_q^{2n}(\Pi_+) \). Furthermore, [22] Remark 5.7 provides us \( \gamma_F = \gamma_S + \gamma_{-G} = -s_{-1} \). Consequently, \( F \) belongs to \( R_q^{2n}[\Pi_+; (s_j)_{j=-1}^{2n}] \).

Our next aim can be described as follows. Let \( k \in \mathbb{N}_0 \) and let \( (s_j)_{j=-1}^{2n} \) be a sequence of Hermitian complex \( q \times q \) matrices. Then we are looking for appropriate descriptions of the set \( R_q^k[\Pi_+; (s_j)_{j=-1}^{2n}] \). First we consider the case of an even number \( k \).

**Proposition 6.5.** Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=-1}^{2n} \) be a sequence of Hermitian complex \( q \times q \) matrices. Then

\[
R_q^{2n}[\Pi_+; (s_j)_{j=-1}^{2n}] = \left\{ F \in R_q(\Pi_+) \left| \lim_{y \to +\infty} F_{2n}(iy) = 0_{q \times q} \right. \right\}.
\]

Proof. Combine (6.14) and Theorem 0.4.

Now we treat the case of a sequence \( (s_j)_{j=-1}^{2n} \) from \( \mathbb{C}^{q \times q} \) with odd number \( k \).

The following result contains a useful sufficient condition which guarantees that a function \( F \in R_q(\Pi_+) \) belongs to the set \( R_q^{2n+1}[\Pi_+; (s_j)_{j=-1}^{2n+1}] \). In the scalar case, the result goes back to [23] Theorem 3.3.

**Theorem 6.6.** Let \( n \in \mathbb{N}_0 \), let \( (s_j)_{j=-1}^{2n+1} \) be a sequence of Hermitian complex \( q \times q \) matrices, and let \( F \in R_q(\Pi_+) \) be such that

\[
\lim_{y \to +\infty} F_{2n+1}(iy) = 0_{q \times q}.
\]

Then \( F_{2n+1} \) belongs to \( R_q(\Pi_+) \). If \( F_{2n+1} \) even belongs to \( R_q^{[-1]}(\Pi_+) \), then the function \( F \) belongs to \( R_q^{2n+1}[\Pi_+; (s_j)_{j=-1}^{2n+1}] \).

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Proof. For all \( y \in (0, +\infty) \), we get from formula (5.11) in Remark 5.3 that
\[
F_{2n}^\ast(iy) = (iy)^{-1} \left[ F_{2n+1}^\ast(iy) - s_{2n+1} \right].
\]

In view of (6.23), this implies (6.21). Thus, Theorem 6.4 yields \( F \in \mathcal{R}_q^{[2n]}[\Pi_+; (s_j)_{j=1}^{2n}] \).

Part (ii) of Proposition 5.13 shows then that \( F \) belongs to \( \mathcal{R}_q(\Pi_+) \). Now we additionally suppose that \( F_{2n+1}^\ast \in \mathcal{R}_q^{[-1]}(\Pi_+) \). Then part (i) of Proposition 5.13 provides us \( F \in \mathcal{R}_q^{[2n+1]}(\Pi_+) \), in particular \( \sigma_F \in \mathcal{M}_q^{\geq 2n+1}(\mathbb{R}) \), and \( s^\ast_{2n+1} = s_{2n+1} - \gamma_{F_{2n+1}^\ast} \). From \( F \in \mathcal{R}_q^{[2n]}[\Pi_+; (s_j)_{j=1}^{2n}] \) we see \( \gamma_F = -s_{-1} \) and that \( \sigma_F \in \mathcal{M}_q^{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n}] \). Since \( F_{2n+1}^\ast \) belongs to \( \mathcal{R}_q^{[-1]}(\Pi_+) \), Proposition 3.14 and (6.23) yield \( \gamma_{F_{2n+1}^\ast} = \lim_{y \to +\infty} F_{2n+1}^\ast(iy) = 0_{q \times q} \). Thus, we get \( s^\ast_{2n+1} = s_{2n+1} \). Consequently, \( \sigma_F \) belongs to \( \mathcal{M}_q^{\geq}[\mathbb{R}; (s_j)_{j=0}^{2n+1}] \).

Hence, \( F \in \mathcal{R}_q^{[2n+1]}[\Pi_+; (s_j)_{j=1}^{2n+1}] \). \( \square \)

Proposition 6.7. Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=1}^{2n+1} \) be a sequence of Hermitian complex \( q \times q \) matrices. Then
\[
\mathcal{R}_q^{[2n+1]} \left[ \Pi_+; (s_j)_{j=1}^{2n+1} \right] = \left\{ F \in \mathcal{R}_q(\Pi_+) \left| F_{2n+1}^\ast \in \mathcal{R}_q^{[-1]}(\Pi_+) \text{ and } \lim_{y \to +\infty} F_{2n+1}^\ast(iy) = 0_{q \times q} \right. \right\}.
\]

Proof. Combine part (ii) of Proposition 5.12, (6.17), and Theorem 6.6. \( \square \)

7. On a Schur Type Algorithm for Sequences of Complex \( p \times q \) Matrices

In this section, we recall some essential facts on a Schur type algorithm for sequences from \( \mathbb{C}^{p \times q} \), which was introduced and investigated in [24]. The elementary step of this algorithm is based on the use of the construction of the reciprocal sequence of a finite or infinite sequence from \( \mathbb{C}^{p \times q} \). For this reason, we first remember the definition of the reciprocal sequence.

Let \( \kappa \in \mathbb{N}_0 \cup \{ +\infty \} \) and let \( (s_j)_{j=0}^{\kappa} \) be a sequence of complex \( p \times q \) matrices. Then the sequence \( (s_j^\ast)_{j=0}^{\kappa} \) of complex \( q \times p \) matrices, which is given by \( s_0^\ast := s_0^\dagger \) and, for all \( k \in \mathbb{Z}_{1, \kappa} \), recursively by
\[
s_k^\ast := -s_0^\dagger \sum_{j=0}^{k-1} s_{k-j}^\dagger s_j^\ast,
\]
is called the reciprocal sequence corresponding to \( (s_j)_{j=0}^{\kappa} \). For a detailed treatment of the concept of reciprocal sequences, we refer the reader to [23].

Now we explain the elementary step of the Schur type algorithm under consideration. Let \( \kappa \in \mathbb{Z}_{2, +\infty} \cup \{ +\infty \} \) and let \( (s_j)_{j=0}^{\kappa} \) be a sequence from \( \mathbb{C}^{p \times q} \) with reciprocal sequence
$s_j^{(1)} := -s_0 s_j^{\kappa} + s_0$

(7.1)

is said to be the first Schur transform of $(s_j)_{j=0}^{\kappa}$.

**Remark 7.1.** Let $\kappa \in \mathbb{Z}_{2, +\infty} \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Let $(s_j^{(1)})_{j=0}^{\kappa-2}$ be the first Schur transform of $(s_j)_{j=0}^{\kappa}$. Let $m \in \mathbb{Z}_{2, \kappa}$. Then from (7.1) it is obvious that $(s_j^{(1)})_{j=0}^{m-2}$ is the first Schur transform of $(s_j)_{j=0}^{m}$.

The repeated application of the first Schur transform generates in a natural way a corresponding algorithm for (finite or infinite) sequences of complex $p \times q$ matrices:

Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then we call the sequence $(s_j^{(0)})_{j=0}^{\kappa}$, given by $s_j^{(0)} := s_j$ for all $j \in \mathbb{Z}_{0, \kappa}$, the 0-th Schur transform of $(s_j)_{j=0}^{\kappa}$. If $\kappa \geq 2$, then we define recursively, the $k$-th Schur transform: For all $k \in \mathbb{N}$ with $2k \leq \kappa$, the first Schur transform $(s_j^{(k)})_{j=0}^{\kappa-2k}$ of $(s_j^{(k-1)})_{j=0}^{\kappa-2(k-1)}$ is called the $k$-th Schur transform of $(s_j)_{j=0}^{\kappa}$.

One of the central properties of the just introduced Schur type algorithm is that it preserves the Hankel non-negative definite extendability of sequences of matrices. This is the content of the following result, which is proved in [24] Propositions 9.4 and 9.5.

**Proposition 7.2.** Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q, \kappa}^{\geq e}$, and let $k \in \mathbb{N}_0$ with $2k \leq \kappa$. Then the $k$-th Schur transform $(s_j^{(k)})_{j=0}^{\kappa-2k}$ of $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{H}_{q, \kappa-2k}^{\geq e}$.

In our considerations below, the special parametrization of block Hankel matrices introduced in [19, 24] will play an essential role, the so-called canonical Hankel parametrization. For the convenience of the reader, we recall this notion. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence in $\mathbb{C}^{p \times q}$. For every choice of non-negative integers $l$ and $m$ with $l \leq m \leq \kappa$, let

$$y_{l,m} := \begin{bmatrix} s_l \\ s_{l+1} \\ \vdots \\ s_m \end{bmatrix} \quad \text{and} \quad z_{l,m} := [s_l, s_{l+1}, \ldots, s_m].$$

For all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let

$$H_n := [s_{j+k}]_{j,k=0}^{n}$$

and, for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, let

$$K_n := [s_{j+k+1}]_{j,k=0}^{n}.$$

Let

$$M_0 := 0_{p \times q} \quad \text{and} \quad M_n := z_{n,2n-1} H_{n-1}^{\dagger} y_{n+1,2n}.$$
for all \( n \in \mathbb{N} \) with \( 2n \leq \kappa \). Furthermore, let

\[
N_0 := 0_{p \times q} \quad \text{and} \quad N_n := z_{n+1,2n}H_{n-1}^\dagger y_{n,2n-1}
\]

for all \( n \in \mathbb{N} \) with \( 2n \leq \kappa \). Let

\[
\Sigma_0 := 0_{p \times q} \quad \text{and} \quad \Sigma_n := z_{n,2n-1}H_{n-1}^\dagger K_{n-1}H_{n-1}^\dagger y_{n,2n-1}
\]

for all \( n \in \mathbb{N} \) with \( 2n - 1 \leq \kappa \). For all \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), let

\[
\Lambda_n := M_n + N_n - \Sigma_n.
\]

Let

\[
L_0 := s_0 \quad \text{and} \quad L_n := s_{2n} - z_{n,2n-1}H_{n-1}^\dagger y_{n,2n-1} \quad (7.2)
\]

for all \( n \in \mathbb{N} \) with \( 2n \leq \kappa \).

**Definition 7.3.** Let \( \kappa \in \mathbb{N} \cup \{+\infty\} \) and let \( (s_j)_{j=0}^{2\kappa} \) be a sequence in \( \mathbb{C}^{p \times q} \). Then the pair of sequences \( [(C_k)_{k=1}^\kappa, (D_k)_{k=0}^\kappa] \) given by \( C_k := s_{2k-1} - \Lambda_{k-1} \) for all \( k \in \mathbb{Z}_{1,\kappa} \) and by \( D_k := L_k \) for all \( k \in \mathbb{Z}_{0,\kappa} \) is called the canonical Hankel parametrization of \( (s_j)_{j=0}^{2\kappa} \).

**Remark 7.4.** Let \( \kappa \in \mathbb{N} \cup \{+\infty\} \) and let \( (C_k)_{k=1}^\kappa \) and \( (D_k)_{k=0}^\kappa \) be sequences of complex \( p \times q \) matrices. Then one can easily see that there is a unique sequence \( (s_j)_{j=0}^{2\kappa} \) of complex \( p \times q \) matrices such that \( [(C_k)_{k=1}^\kappa, (D_k)_{k=0}^\kappa] \) is the canonical Hankel parametrization of \( (s_j)_{j=0}^{2\kappa} \), namely the sequence given by \( s_0 := D_0 \) and for each \( k \in \mathbb{Z}_{0,\kappa} \), by \( s_{2k-1} := \Lambda_{k-1} + C_k \) and \( s_{2k} := z_{2k,2k-1}H_{2k-1}^\dagger y_{2k,2k-1} + D_k \).

In [19, 24] several important classes of sequences of complex \( p \times q \) matrices were characterized in terms of their canonical Hankel parametrization. From the view of this paper, the class \( H_{q,2\kappa}^{\geq, e} \) of Hankel non-negative definite extendable sequences is of extreme importance (see Theorem 1.3, Proposition 5.4). In the case of a sequence \( (s_j)_{j=0}^{2\kappa} \in H_{q,2\kappa}^{\geq, e} \), the canonical Hankel parametrization can be generated by the above constructed Schur type algorithm. This is the content of the following theorem.

**Theorem 7.5** ([24 Theorem 9.15]). Let \( \kappa \in \mathbb{N} \cup \{+\infty\} \) and let \( (s_j)_{j=0}^{2\kappa} \in H_{q,2\kappa}^{\geq, e} \) with canonical Hankel parametrization \( [(C_k)_{k=1}^\kappa, (D_k)_{k=0}^\kappa] \). Then \( C_k = s_{1}^{(k-1)} \) for all \( k \in \mathbb{Z}_{1,\kappa} \) and \( D_k = s_{0}^{(k)} \) for all \( k \in \mathbb{Z}_{0,\kappa} \).

An essential step in the further considerations of this paper can be described as follows. Let \( \kappa \in \mathbb{Z}_{2,+\infty} \) and let \( (s_j)_{j=0}^{\kappa} \in H_{q,\kappa}^{\geq, e} \). Denote by \( (s_j^{(1)})_{j=0}^{\kappa-2} \) the first Schur transform of \( (s_j)_{j=0}^{\kappa} \). In view of Proposition 5.4 we get then \( (s_j^{(1)})_{j=0}^{\kappa-2} \in H_{q,\kappa-2}^{\geq, e} \). Thus, part 13 of Proposition 5.4 yields that both sets \( \mathcal{R}_{\kappa,q}[\Pi_+;(s_j)_{j=0}^{\kappa}] \) and \( \mathcal{R}_{\kappa-2,q}[\Pi_+;(s_j^{(1)})_{j=0}^{\kappa-2}] \) are non-empty. Then a central aspect of our strategy is based on the construction of a special bijective mapping \( \mathcal{F}_{(+; s_0, s_1)} \) with the property

\[
\mathcal{F}_{(+; s_0, s_1)}(\mathcal{R}_{\kappa,q}[\Pi_+;(s_j)_{j=0}^{\kappa}]) = \mathcal{R}_{\kappa-2,q}[\Pi_+;(s_j^{(1)})_{j=0}^{\kappa-2}].
\]
This mapping $F_{(+;s_0,s_1)}$ will help us to realize the basic step for our construction of a special Schur type algorithm in the class $R_q(\Pi_+)$, which stands in a bijective correspondence to the above described Schur type algorithm for Hankel nonnegative definite extendable sequences. In order to verify several interrelations between the algebraic and function theoretic versions of our Schur type algorithm, we will need various properties of sequences of complex $q \times q$ matrices with prescribed Hankel properties. Now we give a short summary of this material, which is mostly taken from [24].

**Lemma 7.6** ([24, Lemma 3.1]). Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{H}_{q,\kappa}^\geq$. Then:

(a) $s_{2k} \in \mathbb{C}^{q \times q}_{\geq}$ for all $k \in \mathbb{Z}_0$.

(b) $s_l^* = s_l$ for all $l \in \mathbb{Z}_{0,\kappa}$.

(c) $\bigcup_{j=2k}^\kappa R(s_j) \subseteq R(s_{2k})$ and $N(s_{2k}) \subseteq \bigcap_{j=2k}^\kappa N(s_j)$ for all $k \in \mathbb{Z}_0$.

**Lemma 7.7.** Let $\kappa \in \mathbb{N} \cup \{+\infty\}$ and $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^\geq$. Then:

(a) $s_{2k} \in \mathbb{C}^{q \times q}_{\geq}$ for all $k \in \mathbb{Z}_{0,\kappa}$.

(b) $s_l^* = s_l$ for all $l \in \mathbb{Z}_{0,2\kappa}$.

(c) $\bigcup_{j=0}^{2\kappa-1} R(s_j) \subseteq R(s_0)$ and $N(s_0) \subseteq \bigcap_{j=0}^{2\kappa-1} N(s_j)$.

(d) For all $k \in \mathbb{Z}_{0,\kappa}$, the matrix $L_k$ defined in (7.2) is non-negative Hermitian.

**Proof.** The assertions of (a)–(c) follow from [24, Lemma 3.2]. From (7.2) and (a) it follows $L_0 = s_0 \in \mathbb{C}^{q \times q}_{\geq}$. If $\kappa \in \mathbb{N}$, then a standard result on the structure of non-negative Hermitian block matrices (see e.g. [16, Lemma 1.1.9]) implies $L_k \in \mathbb{C}^{q \times q}_{\geq}$ for $k \in \mathbb{Z}_{1,\kappa}$.

Now we recall a class of sequences of complex matrices which, as the consideration in [24] have shown, turned out to be extremely important in the framework of the above introduced Schur algorithm.

**Definition 7.8** ([23, Definition 4.3]). Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. We then say that $(s_j)_{j=0}^\kappa$ is dominated by its first term (or, simply, that it is first term dominant) if

$$N(s_0) \subseteq \bigcap_{j=0}^\kappa N(s_j) \quad \text{and} \quad \bigcup_{j=0}^\kappa R(s_j) \subseteq R(s_0).$$

The set of all first term dominant sequences $(s_j)_{j=0}^\kappa$ of complex $p \times q$ matrices will be denoted by $D_{p \times q,\kappa}$.

For a comprehensive investigation of first term dominant sequences, we refer the reader to the paper [23].

From the view of the Schur type algorithm for sequences of matrices, the following result (see also [24, Proposition 4.24]) proved to be of central importance.
Lemma 7.9. For all \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \), the inclusion \( \mathcal{H}_{q,\kappa}^{>c} \subseteq \mathcal{D}_{q \times q, \kappa} \) holds true.

**Proof.** Apply part (c) of Lemma 7.6. \( \square \)

Now we turn our attention to further important subclasses of the class of all Hankel non-negative definite sequences. Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^{2n} \) be a sequence from \( \mathbb{C}^{q \times q} \). Then \( (s_j)_{j=0}^{2n} \) is called **Hankel positive definite** if the block Hankel matrix \( H_n := [s_{j+k}]_{j,k=0}^{n} \) is positive Hermitian. A sequence \( (s_j)_{j=0}^{\infty} \) from \( \mathbb{C}^{q \times q} \) is called **Hankel positive definite** if for all \( n \in \mathbb{N}_0 \) the sequence \( (s_j)_{j=0}^{2n} \) is Hankel positive definite. For all \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \), we will write \( \mathcal{H}_{q,2\kappa}^> \) for the set of all Hankel positive definite sequences \( (s_j)_{j=0}^{2\kappa} \) from \( \mathbb{C}^{q \times q} \).

**Remark 7.10.** Let \( n \in \mathbb{N}_0 \). Then [19, Remark 2.8] shows that \( \mathcal{H}_{n,2n}^> \subseteq \mathcal{H}_{q,2n}^> \).

**Lemma 7.11.** Let \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \) and let \( (s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^> \). Then:

(a) \( s_{2k} \in \mathbb{C}_q^> \) for all \( k \in \mathbb{Z}_{0,\kappa} \).

(b) \( s_l^* = s_l \) for all \( l \in \mathbb{Z}_{0,2\kappa} \).

(c) For all \( k \in \mathbb{Z}_{0,\kappa} \), the matrix \( L_k \) defined in (7.2) is positive Hermitian.

**Proof.** All assertions are immediate consequences of results on non-negative Hermitian block matrices (see e.g. [16, Lemma 1.1.7]). \( \square \)

Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^{2n} \) be a sequence from \( \mathbb{C}^{q \times q} \). Then \( (s_j)_{j=0}^{2n} \) is called **Hankel positive definite extendable** if there exist matrices \( s_{2n+1} \) and \( s_{2n+2} \) from \( \mathbb{C}^{q \times q} \) such that \( (s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^> \). The symbol \( \mathcal{H}_{q,2n}^{>c} \) stands for the set of all Hankel positive definite extendable sequences \( (s_j)_{j=0}^{2n} \) from \( \mathbb{C}^{q \times q} \).

**Proposition 7.12 ([19, Proposition 2.24]).** Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^{2(n+1)} \) be a sequence from \( \mathbb{C}^{q \times q} \). Then the following statements are equivalent:

(i) \( (s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^> \).

(ii) \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^> \), \( s_{2n+1} \in \mathbb{C}_H^{q \times q} \) and there exists a matrix \( D \in \mathbb{C}_r^{q \times q} \) such that

\[
    s_{2n+2} = z_{n+1,2n+1} H_{j}^{*} y_{n+1,2n+1} + D. \tag{7.3}
\]

**Corollary 7.13.** Let \( n \in \mathbb{N}_0 \). Then \( \mathcal{H}_{q,2n}^{>c} = \mathcal{H}_{q,2n}^> \).

**Proof.** Use Proposition 7.12. \( \square \)

It should be mentioned that Remark 7.10 is also a consequence of Corollary 7.13.

Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^{2n+1} \) be a sequence from \( \mathbb{C}^{q \times q} \). Then \( (s_j)_{j=0}^{2n+1} \) is called **Hankel positive definite extendable** if there exists a matrix \( s_{2n+2} \in \mathbb{C}^{q \times q} \) such that \( (s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^> \). The symbol \( \mathcal{H}_{q,2n+1}^{>c} \) stands for the set of all Hankel positive definite extendable sequences \( (s_j)_{j=0}^{2n+1} \) from \( \mathbb{C}^{q \times q} \).
Proposition 7.14. Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q \times q}$.

(a) The following statements are equivalent:

(i) $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^e$.

(ii) $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e$ and $s_{2n+1} \in \mathbb{C}^{q \times q}$.

(b) Let $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^e$ and $s_{2n+2} \in \mathbb{C}^{q \times q}$. Then the following statements are equivalent:

(iii) $(s_j)_{j=0}^{2(n+1)} \in \mathcal{H}_{q,2(n+1)}^e$.

(iv) There exists a matrix $D \in \mathbb{C}^{q \times q}$ such that (7.3).

Proof. All assertions follow immediately from Proposition 7.12.

8. On a Coupled Pair of Schur Type Transforms

The main goal of this section is to prepare the elementary step of our Schur type algorithm for the class $\mathcal{R}_q(\Pi_+)$. We will be led to a situation which, roughly speaking, looks as follows: Let $A, B \in \mathbb{C}^{p \times q}$, let $G$ be a non-empty subset of $\mathbb{C}^{q \times q}$, and let $F: G \rightarrow \mathbb{C}^{p \times q}$.

Then the matrix-valued functions $F^{(+; A, B)}: G \rightarrow \mathbb{C}^{p \times q}$ and $F^{(-; A, B)}: G \rightarrow \mathbb{C}^{p \times q}$ which are defined by

$$F^{(+; A, B)}(z) := -A \left( zI_q + [F(z)]^\dagger A \right) + B$$

and

$$F^{(-; A, B)}(z) := -A \left( zI_q + A^\dagger [F(z) - B] \right)^\dagger,$$

respectively, will be central objects in our further considerations. The special case $B = 0_{p \times q}$ will occupy a particular role. For this reason, we set

$$F^{(+; A)} := F^{(+; A, 0_{p \times q})}, \quad F^{(-; A)} := F^{(-; A, 0_{p \times q})}.$$  (8.3)

Against to the background of our later considerations, the matrix-valued functions $F^{(+; A, B)}$ and $F^{(-; A, B)}$ are called the $(A, B)$-Schur transform of $F$ and the inverse $(A, B)$-Schur transform of $F$.

The generic case studied here concerns the situation where $p = q$, $A$ and $B$ are matrices from $\mathbb{C}^{q \times q}$ with later specified properties, $G = \Pi_+$, and $F \in \mathcal{R}_q(\Pi_+)$. The use of the transforms introduced in (8.2) and (8.3) was inspired by some considerations in the paper Chen/Hu [9]. In particular, we mention [9 Lemma 2.6 and its proof, formula (2.7)]. Before treating more general aspects we state some relevant concrete examples for the constructions given by formulas (8.1) and (8.2). First we illustrate the transformations given in (8.1) and (8.2) by some examples.

Example 8.1. Let $A, B \in \mathbb{C}^{q \times q}$. In view of (8.1), then:

(a) Let $\alpha \in \mathbb{C}^{q \times q}$ and let $F: \Pi_+ \rightarrow \mathbb{C}^{q \times q}$ be defined by $F(z) := \alpha$. Then $F^{(+; A, B)}(z) = B - A\alpha^\dagger A + z(-A)$ for all $z \in \Pi_+$.  

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(b) Let $\beta \in \mathbb{C}^{q \times q}$ and let $F: \Pi_+ \to \mathbb{C}^{q \times q}$ be defined by $F(z) := z\beta$. Then $F^{(+,A,B)}(z) = B + z(-A) - \frac{1}{2}A\beta A$ for all $z \in \Pi_+$.

(c) Let $M \in \mathbb{C}^{q \times q}$, and let $F: \Pi_+ \to \mathbb{C}^{q \times q}$ be defined by $F(z) := \frac{1+i\pi}{\pi-z}M$. For all $z \in \Pi_+$, then

$$F^{(+,A,B)}(z) = \begin{cases} B + z(AM^\dagger A - A) & \text{if } \tau = 0 \\ B + z(-A) + \frac{1+i(-\frac{1}{2})z}{(-\frac{1}{2})-z} AM^\dagger A & \text{if } \tau \neq 0 \end{cases}$$

For all $\tau \in \mathbb{R}$, let $\delta_\tau$ be the Dirac measure defined on $\mathfrak{B}_\mathbb{R}$ with unit mass at $\tau$. Furthermore, let $\omega_q: \mathfrak{B}_\mathbb{R} \to \mathbb{C}^\mathbb{R}_{\geq q}$ be defined by $\omega_q(B) := 0_{q \times q}$.

**Example 8.2.** Let $A \in \mathbb{C}^{q \times q}$ with $-A \in \mathbb{C}^{q \times q}$ and let $B \in \mathbb{C}^{q \times q}$. In view of Theorem 2.1 and Example 8.1, then:

(a) Let $\alpha \in \mathbb{C}^{q \times q}$ and let $F: \Pi_+ \to \mathbb{C}^{q \times q}$ be defined by $F(z) := \alpha$. Then $F \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_F, \beta_F, \nu_F) = (\alpha, 0_{q \times q}, \omega_q)$. Furthermore, $F^{(+,A,B)} \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_{F^{(+,A,B)}}, \beta_{F^{(+,A,B)}}, \nu_{F^{(+,A,B)}}) = (B - A\alpha^\dagger A, -A, \omega_q)$.

(b) Let $\beta \in \mathbb{C}^{q \times q}$ and let $F: \Pi_+ \to \mathbb{C}^{q \times q}$ be defined by $F(z) := z\beta$. Then $F \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_F, \beta_F, \nu_F) = (0_{q \times q}, \beta, \omega_q)$. Furthermore, $F^{(+,A,B)} \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_{F^{(+,A,B)}}, \beta_{F^{(+,A,B)}}, \nu_{F^{(+,A,B)}}) = (B, -A, \delta_0\beta^\dagger A)$.

(c) Let $M \in \mathbb{C}^{q \times q}$, let $\tau \in \mathbb{R} \setminus \{0\}$ and let $F: \Pi_+ \to \mathbb{C}^{q \times q}$ be defined by $F(z) := \frac{1+i\pi}{\pi-z}M$. Then $F \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_F, \beta_F, \nu_F) = (0_{q \times q}, 0_{q \times q}, \delta_\tau M)$. Furthermore, $F^{(+,A,B)} \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_{F^{(+,A,B)}}, \beta_{F^{(+,A,B)}}, \nu_{F^{(+,A,B)}}) = (B, -A, \delta_{-\frac{1}{2}} A\beta^\dagger A)$.

**Example 8.3.** Let $A \in \mathbb{C}^{q \times q}$, let $B \in \mathbb{C}^{q \times q}$, let $M \in \mathbb{C}^{q \times q}$ with $AM^\dagger A \geq A$ and let $F: \Pi_+ \to \mathbb{C}^{q \times q}$ be defined by $F(z) := -\frac{1}{2}M$. In view of Theorem 2.1 and Example 8.1, then $F \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_F, \beta_F, \nu_F) = (0_{q \times q}, 0_{q \times q}, \delta_0 M)$, and, furthermore, $F^{(+,A,B)} \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_{F^{(+,A,B)}}, \beta_{F^{(+,A,B)}}, \nu_{F^{(+,A,B)}}) = (B, AM^\dagger A - A, \omega_q)$.

**Example 8.4.** Let $A, B, \beta \in \mathbb{C}^{q \times q}$ and let $F: \Pi_+ \to \mathbb{C}^{q \times q}$ be defined by $F(z) := B + z\beta$. In view of (8.2), then $F^{(-,A,B)}(z) = -\frac{1}{2}A(I_q + A^\dagger \beta)^\dagger$ for all $z \in \Pi_+$.

**Example 8.5.** Let $A \in \mathbb{C}^{q \times q}$, let $B \in \mathbb{C}^{q \times q}$, let $\beta \in \mathbb{C}^{q \times q}$ with $A(I_q + A^\dagger \beta)^\dagger \in \mathbb{C}^{q \times q}$ and let $F: \Pi_+ \to \mathbb{C}^{q \times q}$ be defined by $F(z) := B + z\beta$. In view of Theorem 2.1 and Example 8.1, then $F \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_F, \beta_F, \nu_F) = (B, \beta, \omega_q)$, and, furthermore, $F^{(-,A,B)} \in \mathcal{R}_q(\Pi_+)$ and $(\alpha_{F^{(-,A,B)}}, \beta_{F^{(-,A,B)}}, \nu_{F^{(-,A,B)}}) = \left(0_{q \times q}, 0_{q \times q}, \delta_0 A(I_q + A^\dagger \beta)^\dagger \right)$.

A central theme of this paper is to choose, for a given function $F \in \mathcal{R}_q(\Pi_+)$, special matrices $A$ and $B$ from $\mathbb{C}^{q \times q}$ such that the function $F^{(+,A,B)}$ and $F^{(-,A,B)}$, respectively, belongs to $\mathcal{R}_q(\Pi_+)$ or to special subclasses of $\mathcal{R}_q(\Pi_+)$. The following result provides a first contribution to this topic.
Proposition 8.6. Let \( F \in \mathbb{R}^{q \times q}(\Pi_+) \). Further, let \( A \in \mathbb{C}^{q \times q} \) and let \( B \in \mathbb{C}^{p \times q}_H \). Then \( F^{(+;A,B)} \in \mathbb{R}_q(\Pi_+) \).

Proof. Let \( G : \Pi_+ \to \mathbb{C}^{q \times q} \) be defined by

\[
G(z) := B - zA.
\]

(8.4)

In view of \( B \in \mathbb{C}^{p \times q}_H \), \( -A \in \mathbb{C}^{q \times q}_\geq \), and (8.4), we see from part (d) of Theorem 2.1 then

\[
G \in \mathbb{R}^q(\Pi_+).
\]

(8.5)

Taking \( A^* = A \) into account, we infer from Proposition 2.9 and Remark 2.4 then

\[
-AP^\dagger A \in \mathbb{R}^q(\Pi_+).
\]

(8.6)

Because of (8.1) and (8.4), we have

\[
F^{(+;A,B)} = G - AF^\dagger A.
\]

(8.7)

Using (8.5), (8.6), and (8.7), we get \( F^{(+;A,B)} \in \mathbb{R}^q(\Pi_+) \).

Furthermore, we will show that under appropriate conditions the equations

\[
(F^{(+;A,B)})(-;A,B) = F \quad \text{and} \quad (F^{(-;A,B)})(+;A,B) = F
\]

hold true. The formulas in (8.8) show that the functions \( F^{(+;A,B)} \) and \( F^{(-;A,B)} \) form indeed a coupled pair of transformations. Furthermore, it will be clear now our terminologies “\( (A,B) \)-Schur transform” and “inverse \( (A,B) \)-Schur transform”. If all Moore-Penrose inverses in (8.1) and (8.2) would be indeed inverse matrices, then the equations in (8.8) could be confirmed by straightforward direct computations. Unfortunately, this is not the case in more general situations which are of interest for us. So we have to look for a convenient way to prove the equations in (8.8) for situations which will be relevant for us. Now we verify that in important cases the formulas (8.1) and (8.2) can be rewritten as linear fractional transformations with appropriately chosen generating matrix-valued functions. The role of these generating functions will be played by the matrix polynomials \( W_{A,B} \) and \( V_{A,B} \) which are studied in Appendix C.

Lemma 8.7. Let \( A, B \in \mathbb{C}^{p \times q} \) be such that \( N(A) \subseteq N(B) \) and let \( W_{A,B} \) be defined in (C.1). Furthermore, let \( \mathcal{G} \) be a non-empty subset of \( \mathbb{C} \) and let \( F : \mathcal{G} \to \mathbb{C}^{p \times q} \) be a mapping which satisfies for all \( z \in \mathcal{G} \) the conditions \( N(A) \subseteq N(F(z)) \) and \( R(F(z)) \subseteq R(A) \). For all \( z \in \mathcal{G} \), then the relations

\[
F(z) \in \mathbb{Q}_H[-A^\dagger, I_q - A^\dagger A] \quad \text{and} \quad S_{W_{A,B}(z)}^{(p,q)}(F(z)) = F^{(+;A,B)}(z)
\]

hold true, where \( F^{(+;A,B)} \) is defined in (8.1).

Proof. Taking (8.1) and (C.1) into account Lemma C.3 yields all assertions. \( \square \)
Remark 8.8. Let $A \in \mathbb{C}^{p \times q}$ and let $W_A$ be defined by (C.3). Furthermore let $\mathcal{G}$ be a non-empty subset of $\mathbb{C}$ and let $F : \mathcal{G} \rightarrow \mathbb{C}^{p \times q}$ be a mapping which satisfies, for all $z \in \mathcal{G}$, the conditions $N(A) \subseteq N(F(z))$ and $R(F(z)) \subseteq R(A)$. Setting $B = 0_{p \times q}$ in Lemma 8.7, then, for all $z \in \mathcal{G}$, the relations

$$F(z) \in Q_{[-A^t, I_q - A^t A]} \quad \text{and} \quad S_{W_A(z)}^{(p, q)}(F(z)) = F^{(+; A)}(z)$$

hold true, where $F^{(+; A)}$ is defined in (8.3).

The following application of Remark 8.8 and Lemma 8.7 is important for our further considerations.

Lemma 8.9. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^{s_0} \in \mathcal{H}_{q, \kappa}^{\geq, e}$, and let $F \in \mathcal{R}_{\kappa, q}[\Pi^+; (s_j)_{j=0}^{s_0}]$. Further, let $z \in \Pi^+$. Then:

(a) $F(z) \in Q_{[-s_0^t I_q - s_0^t s_0]}$.

(b) $S_{W_0(z)}^{(q, q)}(F(z)) = F^{(+; s_0)}(z)$.

(c) If $\kappa \in \mathbb{N} \cup \{+\infty\}$, then

$$S_{W_0,s_1(z)}^{(q, q)}(F(z)) = F^{(+; s_0, s_1)}(z).$$

Proof. In view of part (a) of Proposition 5.7 we have

$$N(F(z)) = N(s_0) \quad \text{and} \quad R(F(z)) = R(s_0). \quad (8.9)$$

Taking (8.9) into account, we infer from Remark 8.8 the assertions of (a) and (b).

Suppose now that $\kappa \in \mathbb{N} \cup \{+\infty\}$. Then part (c) of Lemma 7.6 gives $N(s_0) \subseteq N(s_1)$. Combining this with (8.9) we see from Lemma 8.7 that (c) holds.

Now we state an important situation where formula (8.2) can be rewritten as linear fractional transformation.

Lemma 8.10. Let $A \in \mathbb{C}^{q \times q}$, $B \in \mathbb{C}^{q \times q}$, and let $V_{A, B}$ be given by (C.2). Let $G \in \mathbb{R}_q(\Pi^+)$ be such that its Nevanlinna parametrization $(\alpha_G, \beta_G, \nu_G)$ satisfies the inclusion $N(A) \subseteq \left[ N(\alpha_G) \cap N(\beta_G) \cap N(\nu_G(\mathbb{R})) \right]$. Let $F := G + B$. For all $z \in \Pi^+$, then

$$F(z) \in Q_{[A^t I_q - A^t A]} \quad \text{and} \quad F^{(-; A, B)}(z) = S_{V_{A, B}(z)}^{(q, q)}(F(z)). \quad (8.10)$$

Proof. Let $z \in \Pi^+$. Since $G \in \mathbb{R}_q(\Pi^+)$ is supposed, we infer from Lemma 7.8 that

$$N(A) \subseteq N(G(z)). \quad (8.11)$$

Since $A \in \mathbb{C}^{q \times q}$ we have $A \in \mathbb{C}^{q \times q}_{\geq}$. Thus, part (b) of Lemma 7.8 yields

$$R(G(z)) \subseteq R(A). \quad (8.12)$$
In view of \( G \in \mathcal{R}_q(\Pi_+) \), we have \( F(z) - B \in \mathcal{I}_{q,\ge} \), whereas the relations (8.11) and (8.12) yield the inclusions \( N(A) \subseteq N(F(z) - B) \) and \( R(F(z) - B) \subseteq R(A) \). Thus, the application of part (d) of Lemma C.5 provides us \( F(z) \in \mathcal{Q}_{[A^1,B^1]} \). Now the application of Remark C.4 yields finally

\[
F(-;A,B)(z) = S^{q(q,q)}_{V,h,B}(z) \left( F(z) \right).
\]

Now we are going to consider the following situation which will turn out to be typical for larger parts of our future considerations. Let \( A \in \mathbb{C}^{q \times q} \) and \( B \in \mathbb{C}^{q \times q} \) be such that \( N(A) \subseteq N(B) \). Further, let \( F \in \mathcal{R}_q(\Pi_+) \) be such that

\[
N(A) \subseteq N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})). \tag{8.13}
\]

Then our aim is to investigate the function \( F(-;A,B) \) given by (8.2). We start with an auxiliary result.

**Lemma 8.11.** Let \( A \in \mathbb{C}^{q \times q} \) and let \( B \in \mathbb{C}^{q \times q} \) be such that \( N(A) \subseteq N(B) \). Further, let \( F \in \mathcal{R}_q(\Pi_+) \) be such that (8.13) holds. Then \( G := F - B \) belongs to \( \mathcal{R}_q(\Pi_+) \) as well and

\[
(\alpha_G, \beta_G, \nu_G) = (\alpha_F - B, \beta_F, \nu_F). \tag{8.14}
\]

Furthermore, for all \( z \in \Pi_+ \), the relations in (8.10) hold true.

**Proof.** Since the matrix \( B \) is Hermitian and since \( F \) belongs to \( \mathcal{R}_q(\Pi_+) \), we see from Theorem 2.1 that \( G := F - B \) belongs to \( \mathcal{R}_q(\Pi_+) \) as well and that (8.11) holds. Because of \( N(A) \subseteq N(B) \), \( N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})) \cap N(B) \)

\[
= N(\alpha_G) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})) = N(\alpha_G) \cap N(\beta_G) \cap N(\nu_G(\mathbb{R})).
\]

Applying Lemma 8.10 completes the proof. \( \square \)

The following two specifications of Lemma 8.11 play an essential role in our subsequent considerations.

**Lemma 8.12.** Let \( (s_j)_{j=0}^1 \in \mathcal{H}_{q,1}^{\ge} \) and let \( F \in \mathcal{P}_q^{\text{odd}}[s_0] \). For each \( z \in \Pi_+ \), then

\[
F(z) \in \mathcal{Q}_{[s_0,s_1]} \quad \text{and} \quad F(-;s_0,s_1)(z) = S^{q(q,q)}_{V,s_0,s_1}(z) \left( F(z) \right).
\]

**Proof.** From Lemma 7.6 we infer

\[
s_0 \in \mathbb{C}^{q \times q}, \quad s_1 \in \mathbb{C}^{q \times q}, \quad \text{and} \quad N(s_0) \subseteq N(s_1). \tag{8.15}
\]

Let \( z \in \Pi_+ \). In view of \( F \in \mathcal{P}_q^{\text{odd}}[s_0] \), part (b) of Lemma 4.3 yields

\[
N(s_0) \subseteq N(F(z)). \tag{8.16}
\]

From (1.2) and Remark 3.24 we see the inclusion \( \mathcal{P}_q^{\text{odd}}[s_0] \subseteq \mathcal{R}_q(\Pi_+) \). Thus, from (8.16) and part (c) of Lemma 2.8 we obtain

\[
N(s_0) \subseteq N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})). \tag{8.17}
\]

In view of (8.15) and (8.17), the application of Lemma 8.11 yields all assertions. \( \square \)
Lemma 8.13. Let \((s_j)_{j=0}^0 \in H_{q,s}^{\geq,0}\) and let \(F \in P_{q}^{\text{even}}[s_0]\). Further, let \(z \in \Pi_+\). Then
\[
F(z) \in Q[|s_0\| z_{\xi}^e] \quad \text{and} \quad F^{(-s_0)}(z) = S_{q,z_0}^{(q,q)}(F(z)).
\]

Proof. From part (iii) of Lemma 7.6 we infer \(s_0 \in C_{q,q}^{\geq,0}\). Clearly \(0_{q,q} \in C_{q,q}^{\geq,0}\) and \(N(s_0) \subseteq N(0_{q,q})\). (8.18)

In view of \(F \in P_{q}^{\text{even}}[s_0]\), we see from (4.1) that \(F \in R_{q}^{[-2]}(\Pi_+)\) and \(N(s_0) \subseteq N(\alpha_F) \cap N(\nu_F)\). (8.19)

In view of \(s_0 \in C_{q,q}^{\geq,0}\), (8.18), (8.19), and (8.3), the application of Lemma 8.11 completes the proof.

Now we formulate the first main result of this section. Assuming the situation of Lemma 8.11, we will obtain useful insights into the structure of the inverse \((A,B)-\text{Schur}\) transform of \(F\).

Proposition 8.14. Let \(A \in C_{q,q}^{\geq,0}\) and \(B \in C_{q,q}^{\geq}\) be such that \(N(A) \subseteq N(B)\). Further, let \(F \in F_{q}(\Pi_+)\) be such that (8.13) holds, and let \(H : \Pi_+ \rightarrow C_{q,q}^{\geq}\) be defined by \(H(z) := -B + zA + F(z)\). Then:

(a) \(H \in R_{q}(\Pi_+)\) and \((\alpha_H, \beta_H, \nu_H) = (\alpha_F - B, \beta_F + A, \nu_F)\).

(b) For each \(z \in \Pi_+\),
\[
N(H(z)) = N(\beta_H) \subseteq N(A) \subseteq N(F(z) - B) \quad \text{(8.20)}
\]
and
\[
\det(zI_q + A^+[F(z) - B]) \neq 0. \quad \text{(8.21)}
\]

(c) \(F^{(-A,B)} = A^+(-H^+)A\).

(d) \(F^{(-A,B)} \in R_{0,q}^{\Pi_+}(t_j)_{j=0}^0\) where \(t_0 := A(A + \beta_F)^+A\). If \(F \in R_{q}^{[-2]}(\Pi_+)\), then \(t_0 = A\).

(e) \(R(F^{(-A,B)}(z)) = R(A)\) and \(N(F^{(-A,B)}(z)) = N(A)\) for all \(z \in \Pi_+\).

(f) \(N(\sigma_F^{(-A,B)}(\mathbb{R})) = N(A)\) and \(R(\sigma_F^{(-A,B)}(\mathbb{R})) = R(A)\).

(g) If \(\det A \neq 0\), then \(\det F^{(-A,B)}(z) \neq 0\) for all \(z \in \Pi_+\).
Proof. Taking \( B \in \mathbb{C}^{n \times q}_H, A \in \mathbb{C}^{n \times q}_\geq, \) and \( F \in \mathcal{R}_q(\Pi_+) \) into account, we see from Theorem 2.1 that \( H \) belongs to \( \mathcal{R}_q(\Pi_+) \) as well and that

\[
(\alpha_H, \beta_H, \nu_H) = (\alpha_F - B, \beta_F + A, \nu_F). \tag{8.22}
\]

From (8.22) we infer

\[
N(\alpha_F) \cap N(B) \subset N(\alpha_F - B) = N(\alpha_H) \tag{8.23}
\]

and

\[
N(\beta_F) \cap N(A) \subset N(\beta_F + A) = N(\beta_H). \tag{8.24}
\]

Consequently, from \( N(A) \subset N(B), \) (8.13), (8.23), and (8.24) then

\[
N(A) = N(A) \cap N(B) \subset N(\alpha_F) \cap N(B) \cap N(\beta_F) \cap N(A) \cap N(\nu_F(\mathbb{R})) \subset N(\alpha_H) \cap N(\beta_H) \cap N(\nu_H(\mathbb{R}))
\]

follows. Let \( G := F - B. \) Thus, Lemma 8.11 shows that \( G \in \mathcal{R}_q(\Pi_+) \) and (8.14) hold true. Proposition 2.7 yields

\[
N(G(z)) = N(\alpha_G) \cap N(\beta_G) \cap N(\nu_G(\mathbb{R})) \tag{8.25}
\]

and

\[
N(H(z)) = N(\alpha_H) \cap N(\beta_H) \cap N(\nu_H(\mathbb{R})) \tag{8.26}
\]

for all \( z \in \Pi_+. \) Since \( A \) and \( \beta_F \) are non-negative Hermitian matrices, we have \( \beta_H = \beta_F + A \geq A \geq 0_{q \times q} \) and, consequently,

\[
N(\beta_H) \subset N(A). \tag{8.27}
\]

The inclusion (8.27) and the assumptions \( N(A) \subset N(B) \) and (8.13) imply together with (8.22) the relations

\[
N(\beta_H) \subset N(A) \subset N(B) \cap N(\alpha_F) \cap N(\nu_F(\mathbb{R})) \subset N(\alpha_F - B) \cap N(\nu_F(\mathbb{R})) = N(\alpha_H) \cap N(\nu_H(\mathbb{R})).
\]

Thus, we see that

\[
N(\alpha_H) \cap N(\beta_H) \cap N(\nu_H(\mathbb{R})) = N(\beta_H) \tag{8.28}
\]

is true. For all \( z \in \Pi_+, \) from (8.26), (8.28), and (8.27) we know that \( N(H(z)) = N(\beta_H) \subset N(A) \) holds for all \( z \in \Pi_+. \) Since \( N(A) \subset N(B) \) is assumed, we get from (8.13), (8.14), and (8.28) that

\[
N(A) \subset N(B) \cap N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})) \subset N(\alpha_F - B) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})) = N(\alpha_G) \cap N(\beta_G) \cap N(\nu_G(\mathbb{R})) = N(G(z)) = N(F(z) - B)
\]

is fulfilled for all \( z \in \Pi_+. \) Thus, (8.20) is true for all \( z \in \Pi_+. \)
Thus, (c) and [22, Remark 3.4] show that $F$ follows. Consequently, (8.20) implies $G$ as well. From Proposition 2.5 we know that matrices $A$ and $v$ we now consider an arbitrary $z \in \Pi_+$. From Lemma 2.8 then

$$AA^\dagger F(z) = F(z)$$

for all $z \in \Pi_+$. Because of $A^* = A^t$, $B^* = B^t$ and $N(A) \subseteq N(B)$ we get from Remark A.3 then

$$AA^\dagger B = B.$$  

Let $z \in \Pi_+$. In order to check that

$$N\left(zI_q + A^\dagger [F(z) - B]\right) = \{0_{q\times 1}\},$$

we now consider an arbitrary $v \in N(zI_q + A^\dagger [F(z) - B])$. From (8.30) and (8.29) then

$$H(z)v = \left[-AA^\dagger B + zA + AA^\dagger F(z)\right]v = A\left(zI_q + A^\dagger [F(z) - B]\right)v = 0_{q\times 1},$$

follows. Consequently, (8.20) implies $|G(z)|v = 0_{q\times 1}$ and, thus,

$$v = \frac{1}{z}(zv + A^\dagger 0_{q\times 1}) = \frac{1}{z}\left[zI_q + A^\dagger G(z)\right]v = \frac{1}{z}\left[zI_q + A^\dagger [F(z) - B]\right]v = 0_{q\times 1}.$$

This shows that (8.31) is true. In other words, (8.21) holds.

(c) Let us now consider again an arbitrary $z \in \Pi_+$. From (8.20) and part (a) of Remark A.3 we conclude $AH^\dagger(z)H(z) = A$. Using this, (8.21), (8.29), (8.30), and the assumption $A^* = A$, we then obtain

$$F^{(-;A,B)}(z) = -A[H(z)]^\dagger H(z)\left(zI_q + A^\dagger [F(z) - B]\right)^{-1}$$

$$= -A[H(z)]^\dagger [zA + F(z) - B]\left(zI_q + A^\dagger [F(z) - B]\right)^{-1}$$

$$= -A[H(z)]^\dagger A\left(zI_q + A^\dagger [F(z) - B]\right)\left(zI_q + A^\dagger [F(z) - B]\right)^{-1}$$

$$= A[-H(z)]^\dagger A = A^t[-H^\dagger(z)]A.$$

(d) Because of $H \in \mathcal{R}_q(\Pi_+)$ and Proposition 2.9, we see that $-H^\dagger$ belongs to $\mathcal{R}_q(\Pi_+)$. Thus, (c) and [22, Remark 3.4] show that $F^{(-;A,B)}$ belongs to $\mathcal{R}_q(\Pi_+)$ as well. Since the matrices $A$ and $\beta_F$ are non-negative Hermitian, the matrix $t_0$ is non-negative Hermitian as well. From Proposition 2.5 we know that

$$\lim_{y \to +\infty} \frac{1}{iy} H(iy) = \beta_H.$$ 

Furthermore, for all $y \in [1, +\infty)$, we get from (8.20) that

$$\text{rank } \left[\frac{1}{iy} H(iy)\right] = q - \dim N(H(iy)) = q - \dim N(\beta_H) = \text{rank } \beta_H.$$ 

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holds. Thus, the application of Lemma A.10 yields
\[ \lim_{y \to +\infty} \left[ \frac{1}{iy} H(iy) \right]^\dagger = \beta_H^\dagger. \] (8.32)

Now, we see from (8.32), parts (ii) and (iii), and Remark 5.9 that
\[ 0_{q \times q} = A^* \left( - \lim_{y \to +\infty} \left[ \frac{1}{iy} H(iy) \right]^\dagger \right) A + A^* \beta_H^\dagger A \]
\[ = \lim_{y \to +\infty} \left[ iy A^* \left( - [H(iy)]^\dagger \right) A + A(A + \beta_F)^\dagger A \right] \]
\[ = \lim_{y \to +\infty} iy \left[ F^{(-;A,B)}(iy) + (iy)^{-1}t_0 \right]. \]

Consequently, in view of \( F^{(-;A,B)} \in \mathcal{R}_q(\Pi_+) \), part (iii) of Theorem 6.1 implies \( F^{(-;A,B)} \in \mathcal{R}_q[t_0]; t_j=0 \). If \( F \in \mathcal{R}_q[-2](\Pi_+) \) we see from Remark 5.9 that \( \beta_F = 0_{q \times q} \). Thus, \( t_0 = AA^\dagger A = A \).

(iii) In view of (8.21), we have
\[ F^{(-;A,B)}(z) = -A \left( zI_q + A^\dagger [F(z) - B] \right)^\dagger = -A \left( zI_q + A^\dagger [F(z) - B] \right)^{-1}. \]

Thus,
\[ R \left( F^{(-;A,B)}(z) \right) = R(A) \quad \text{and} \quad N \left( \left[ F^{(-;A,B)}(z) \right]^* \right) = N(A^*). \]

In view of \( F^{(-;A,B)} \in \mathcal{R}_q(\Pi_+) \), Lemma 2.14 yields
\[ N \left( \left[ F^{(-;A,B)}(z) \right]^* \right) = N \left( F^{(-;A,B)}(z) \right). \]

Taking \( A^* = A \) into account we get
\[ N \left( F^{(-;A,B)}(z) \right) = N(A). \]

(i) In view of (i) and (ii), the application of Lemma 3.22 yields (i).

This follows immediately from (ii). \( \square \)

**Corollary 8.15.** Let \( A \in \mathbb{C}_H^{q \times q} \) and \( B \in \mathbb{C}_H^{q \times q} \) be such that \( N(A) \subseteq N(B) \). Let \( s \in \mathbb{C}_H^{q \times q} \) with \( 0_{q \times q} \leq s \leq A \) and \( \text{rank } s = \text{rank } A \). Further, let \( F: \Pi_+ \to \mathbb{C}_H^{q \times q} \) be defined by \( F(z) := z(A - s)s^\dagger A \). Then \( F \) belongs to \( \mathcal{R}_q(\Pi_+) \) and fulfills \( N(A) \subseteq N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})) \). Moreover, \( F^{(-;A,B)} \in \mathcal{R}_q(\Pi_+) \) and \( s_0[F^{(-;A,B)}] = s \).

**Proof.** Because of \( 0_{q \times q} \leq s \leq A \) and \( \text{rank } s = \text{rank } A \), we have \( N(s) = N(A) \) and \( R(s) = R(A) \), which in view of Lemma A.12 implies
\[ ss^\dagger = AA^\dagger \quad \text{and} \quad s^\dagger s = A^\dagger A. \] (8.33)

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Hence, from (8.33) and Lemma A.7 we obtain
\[(A - s)s^TA = As^TA - ss^TA = A^s s^TA - ss^TA \geq A^s s^TA ss^TA - ss^TA = AA^sAA^sA A^s - AA^sA = 0_{q \times q}.
\]
Thus, Theorem 2.1 yields \(F \in R_q(\Pi_+)\) and \((\alpha_F, \beta_F, \nu_F) = (0_{q \times q}, (A - s)s^TA, o_q)\), where \(o_q: \R \rightarrow \C^{q \times q}\) is given by \(o_q(B) := 0_{q \times q}\). In particular, we have then
\[
N(A) \subseteq N((A - s)s^TA) = N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\R))
\]
and, taking (8.33) into account, furthermore
\[
A + \beta_F = A + (A - s)s^TA = A + As^TA - ss^TA = As^TA.
\]
Thanks to part (1) of Proposition 8.14 we get then \(F^{(-;A,B)} \in R_{o,q}([\Pi_+; (t_j)_{j=0}^0])\), where \(t_0 := A(A + \beta_F)^\dagger A\). In view of (8.33), this implies \(F^{(-;A,B)} \in R_{o,q}(\Pi_+)\) and, because of (8.34) and (8.33), furthermore
\[
s_0^{[F^{(-;A,B)}]} = t_0 = A(A + \beta_F)^\dagger A = A(As^TA)^\dagger A
\]
\[
= AA^sAA^sA(As^TA)^\dagger AA^sAA^sA = ss^TA(As^TA)^\dagger As^ss^sA
\]
\[
= sA^sAs^sA(As^TA)^\dagger As^sA(s^sA)^\dagger = ss^ss^sA
\]
\[
\]
\]

**Corollary 8.16.** Let \(A \in \C_\geq^{q \times q}\) and \(B \in \C_H^{q \times q}\) such that \(N(A) \subseteq N(B)\). Let \(F \in R_q(\Pi_+)\) be such that (8.13) holds. Then \(F^{(-;A,B)} \in R_0,q(\Pi_+)\), and \(0_{q \times q} \leq s_0^{[F^{(-;A,B)}]} \leq A\) and rank \(s_0^{[F^{(-;A,B)}]} = rank A\) hold true.

**Proof.** The application of part (1) of Proposition 8.14 yields \(F^{(-;A,B)} \in R_0,q([\Pi_+; (t_j)_{j=0}^0])\), where \(t_0 := A(A + \beta_F)^\dagger A\). In view of (8.33), this implies \(F^{(-;A,B)} \in R_0,q(\Pi_+)\) and
\[
s_0^{[F^{(-;A,B)}]} = t_0 = A(A + \beta_F)^\dagger A.
\]
Because of \(\beta_F \in \C_\geq^{q \times q}\), we have \(A + \beta_F \geq A \geq 0_{q \times q}\); which in view of Lemma A.7 implies \(A \geq A(A + \beta_F)^\dagger A \geq 0_{q \times q}\) and \(R(A(A + \beta_F)^\dagger A) = R(A)\). Thus, we obtain
\(0_{q \times q} \leq s_0^{[F^{(-;A,B)}]} \leq A\) and rank \(s_0^{[F^{(-;A,B)}]} = rank A\).

Now we indicate some generic situations in which the formulas in (8.8), respectively, are satisfied. We start with formula (8.33).

**Proposition 8.17.** Let \(A, B \in \C^{q \times q}\) and let \(F \in R_q(\Pi_+)\) be such that (8.13) and
\[
R(A) = R(\alpha_F) + R(\beta_F) + R(\nu_F(\R))
\]
are fulfilled. Then \((F^{(+;A,B)})^{(-;A,B)} = F\).

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Proof. Let $z \in \Pi_+$. From (8.36), (8.37), and Proposition 5.7, we know that
\[ N(A) \subseteq N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R})) = N(F(z)) \] (8.36)
and
\[ R(A) = R(\alpha_F) + R(\beta_F) + R(\nu_F(\mathbb{R})) = R(F(z)). \] (8.37)
In view of (8.36) and (8.37), we infer from Lemma A.5 then
\[ [F(z)] [F(z)]^\dagger = AA^\dagger \quad \text{and} \quad [F(z)]^\dagger [F(z)] = A^\dagger A. \] (8.38)
This implies
\[ A^\dagger A [F(z)]^\dagger = [F(z)]^\dagger \] (8.39)
and
\[ [F(z)]^\dagger AA^\dagger [F(z)] = [F(z)]^\dagger [F(z)] = A^\dagger A. \] (8.40)
Using \((I_q - A^\dagger A)A^\dagger = 0_{q \times q}, A(I_q - A^\dagger A) = 0_{q \times q},\) and (8.40), we obtain
\[
\left(I_q - A^\dagger A + \frac{1}{z} [F(z)]^\dagger A \right) \left(I_q - A^\dagger A + zA^\dagger [F(z)] \right) = I_q - A^\dagger A + (I_q - A^\dagger A) \left(zA^\dagger [F(z)] \right) + \frac{1}{z} [F(z)]^\dagger A(I_q - A^\dagger A) + [F(z)]^\dagger AA^\dagger [F(z)] = I_q - A^\dagger A + A^\dagger A = I_q.
\]
Thus,
\[ \det \left(I_q - A^\dagger A + \frac{1}{z} [F(z)]^\dagger A \right) \not= 0 \] (8.41)
and
\[
\left(I_q - A^\dagger A + \frac{1}{z} [F(z)]^\dagger A \right)^{-1} = I_q - A^\dagger A + zA^\dagger [F(z)] .
\]
Using (8.29), (8.31), (8.39), (8.41), and (8.38), we get finally
\[
(F^{(+:A,B)})(^{-:A,B})_j(z) = -A \left(zI_q + A^\dagger \left[F^{(+:A,B)}(z) - B \right] \right)^\dagger = -A \left(zI_q + A^\dagger \left[-A \left(zI_q + [F(z)]^\dagger A \right) \right] \right)^\dagger = -A \left(z(I_q - A^\dagger A) + [F(z)]^\dagger A \right)^\dagger = -\frac{1}{z} A \left(I_q - A^\dagger A + \frac{1}{z} [F(z)]^\dagger A \right)^\dagger = -\frac{1}{z} A \left(I_q - A^\dagger A + zA^\dagger [F(z)] \right) = AA^\dagger [F(z)] = F(z).
\]
\[ \square \]

Corollary 8.18. Let $\kappa \in \mathbb{N} \cup \{+\infty\}, (s_j)_{j=0}^{\infty} \in \mathcal{H}_{q,\kappa}^{\geq 0},$ and $F \in \mathcal{R}_{\kappa,q,[\Pi^+;(s_j)_{j=0}^{\infty}]}$. 
(a) \((F^{(+:0_B)})(^{-:0,B}) = F\) for each $B \in \mathbb{C}^{q \times q}$. 

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(b) If $\kappa \in \mathbb{N} \cup \{+\infty\}$, then \( F^{(+;s_0,s_1)}_{(-;s_0,s_1)} = F \).

(c) \( F^{(+;s_0)}_{(-;s_0)} = F \).

**Proof.** According to part (c) of Proposition 5.7, the function $F$ belongs to $\mathcal{R}_q(\Pi_+)$ and \( (5.3) \) and \( (5.7) \) hold. Thus, the application of Proposition 8.17 yields \( (F^{(+;s_0,B)}_{(-;s_0,B)}) = F \) for each $B \in \mathbb{C}^{q \times q}$. Hence, \( (a) \) is proved. Choosing $B = s_1$ and $B = 0_{q \times q}$ in \( (a) \), we get the assertions of \( (b) \) and \( (c) \), respectively.

Now we turn our attention to formula \( (8.8) \).

**Proposition 8.19.** Let $A \in \mathbb{C}^{q \times q}$ and $B \in \mathbb{C}_H^{q \times q}$ be such that $N(A) \subseteq N(B)$. Further, let $F \in \mathcal{R}_q(\Pi_+)$ be such that \( (8.13) \) holds. Then \( (F^{-;A,B})^{(+;A,B)} = F \).

**Proof.** Let $z \in \Pi_+$. In view of $A \in \mathbb{C}^{q \times q}$ we have $A \in \mathbb{C}_E^{q \times q}$. Thus taking \( (8.13) \) into account, we infer from part (b) of Lemma 2.8 then

\[
AA^\dagger F(z) = F(z). \tag{8.42}
\]

Part (c) of Proposition 8.14 yields $N(F^{-;A,B}(z)) = N(A)$. Thus, from part (c) of Remark A.3 we infer

\[
A \left[ F^{-;A,B}(z) \right]^\dagger F^{-;A,B}(z) = A. \tag{8.43}
\]

Since the matrices $A$ and $B$ are both Hermitian, from $N(A) \subseteq N(B)$ the inclusion $R(B) \subseteq R(A)$ follows. Consequently, part (b) of Remark A.3 yields \( (8.30) \). Part (b) of Proposition 8.14 yields \( (8.21) \). From \( (8.21) \) and \( (8.2) \) we see

\[
F^{-;A,B}(z) = -A \left( zI_q + A^\dagger [F(z) - B] \right)^{-1}. \tag{8.44}
\]

Using \( (8.1) \), \( (8.2) \), \( (8.21) \), \( (8.41) \), \( (8.43) \), \( (8.42) \) and \( (8.30) \) we get

\[
(F^{-;A,B})^{(+;A,B)}(z) = -A \left( zI_q + \left[ F^{-;A,B}(z) \right]^\dagger A \right) + B
\]

\[
= -zA - A \left[ F^{-;A,B}(z) \right]^\dagger A + B
\]

\[
= -zA + A \left[ F^{-;A,B}(z) \right]^\dagger \left[ -A \left( zI_q + A^\dagger [F(z) - B] \right)^{-1} \right]
\]

\[
\times \left( zI_q + A^\dagger [F(z) - B] \right) + B
\]

\[
= -zA + A \left[ F^{-;A,B}(z) \right]^\dagger F^{-;A,B}(z) \left( zI_q + A^\dagger [F(z) - B] \right) + B
\]

\[
= -zA + A \left( zI_q + A^\dagger [F(z) - B] \right) + B = AA^\dagger F(z) - AA^\dagger B + B = F(z). \tag{8.44}
\]

**Corollary 8.20.** Let $A \in \mathbb{C}_E^{q \times q}$ and let $F \in \mathcal{R}_q(\Pi_+)$ be such that \( (8.13) \) holds true. Then \( (F^{-;A})^{(+;A)} = F \).
Proof. Choose $B = 0_{q \times q}$ in Proposition \ref{prop:beta-infty}. \hfill \square

**Corollary 8.21.** Let $m \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^m \in \mathcal{H}_{q,m}^{\geq e}$. Then:

(a) $(F(-s_0,s_1))^{(+;s_0,s_1)} = F$ for each $m \in \mathbb{N} \cup \{+\infty\}$.

(b) $(F(-s_0))^{(+;s_0)} = F$.

Proof. Part (a) of Lemma \ref{lem:beta-infty} yields $s_0 \in \mathbb{C}^{q \times q}_{\geq e}$. In the case $m \in \mathbb{N} \cup \{+\infty\}$ we see from part (b) of Lemma \ref{lem:beta-infty} that $s_1 \in \mathbb{C}^{q \times q}_{H}$ and from part (c) of Lemma \ref{lem:beta-infty} that $N(s_0) \subseteq N(s_1)$. Thus, taking part (a) of Proposition \ref{prop:beta-infty} into account, the application of Proposition \ref{prop:beta-infty} and Corollary \ref{cor:beta-infty} yields (a) and (b), respectively. \hfill \square

**9. On the $(s_0, s_1)$-Schur Transform for the Classes**

\[ \mathcal{R}_{\kappa,q}[\Pi_+; (s_j)_{j=0}^k] \]

The central topic of this section can be described as follows. Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{H}_{q,m}^{\geq e}$. Then part (b) of Proposition \ref{prop:beta-sign} tells us that the class $\mathcal{R}_{m,q}[\Pi_+; (s_j)_{j=0}^m]$ is non-empty. If $F \in \mathcal{R}_{m,q}[\Pi_+; (s_j)_{j=0}^m]$, then our interest is concentrated on the $(s_0, s_1)$-Schur transform $F^{(+;s_0,s_1)}$ of $F$ in the case $m \in \mathbb{N}$ and on the $(s_0, 0_{q \times q})$-Schur transform $F^{(+;s_0)}$ of $F$ in the case $m = 0$. We will obtain a complete description of these objects. In the case $m = 0$, we will show that $F^{(+;s_0)}$ belongs to $\mathcal{P}_q^{\text{even}}[s_0]$ (see Theorem \ref{thm:main}).

In the case $m = 1$, the function $F^{(+;s_0,s_1)}$ belongs to $\mathcal{P}_q^{\text{odd}}[s_0]$ (see Theorem \ref{thm:main}). The proof of the latter result is mainly based on Corollary \ref{cor:even} and Proposition \ref{prop:odd}. Let us now consider the case $m \in \mathbb{Z}_{\geq 0}$. If $(s_j)_{j=0}^{m-2}$ denotes the first Schur transform of $(s_j)_{j=0}^m$, then it will turn out (see Theorem \ref{thm:main}) Corollary \ref{cor:even} and Theorem \ref{thm:main} that $F^{(+;s_0,s_1)}$ belongs to $\mathcal{R}_{m-q}[\Pi_+; (s_j)_{j=0}^{m-2}]$. Our strategy to prove this is based on the application of Hamburger-Nevanlinna type results for the class $\mathcal{R}_q^{-1}[\Pi_+]$, which were developed in Section \ref{sec:Hamburger}. Realizing the proofs, we will observe that there is an essential difference between the case of even and odd numbers $m$. In the even case, we will rely on Theorem \ref{thm:main}. The main tool in the odd case (, which requires much more work,) is Theorem \ref{thm:main}.

Now we start with the detailed treatment of the cases $m = 0$ and $m = 1$.

**Theorem 9.1.** Let $s_0 \in \mathbb{C}^{q \times q}$ and let $F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^0]$. Then $F^{(+;s_0)}$ belongs to $\mathcal{P}_q^{\text{even}}[s_0]$.\hfill \square

Proof. Since $F$ belongs to $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^0]$, we have $F \in \mathcal{R}_{0,q}(\Pi_+)$ and $\sigma_F$ belongs to $\mathcal{M}_0^{\mathbb{R}}[\mathbb{R}; (s_j)_{j=0}^0]$. In particular, $s_0 = s_0^{[\sigma_F]} = \sigma_F(\mathbb{R}) \in \mathbb{C}^{q \times q}$ and Remark \ref{rem:sigma} shows that $F \in \mathcal{R}_q(\Pi_+)$. From Proposition \ref{prop:F-1} we see that $-F^\dagger$ belongs to $\mathcal{R}_q(\Pi_+)$ as well. According to Remark \ref{rem:sigma} then $s_0(-F^\dagger)s_0 \in \mathcal{R}_q(\Pi_+)$. Using Proposition \ref{prop:F-1} we get

\[ \beta_{s_0(-F^\dagger)s_0} = \lim_{y \to +\infty} \left( \frac{1}{iy} s_0 [-F(iy)]^\dagger s_0 \right). \quad (9.1) \]
In view of $F \in \mathcal{R}_{0,q}[\Pi_+;(s_j)_{j=0}^0]$, we conclude from part (ii) of Theorem 6.1 that

$$s_0 = \lim_{y \to +\infty} [-iyF(iy)].$$  \hspace{1cm} (9.2)

From $F \in \mathcal{R}_{0,q}(\Pi_+)$ and Lemma 3.22 we know that $R(F(z)) = R(\sigma_F(\mathbb{R}))$ and, hence $\text{rank } F(z) = \text{rank } \sigma_F(\mathbb{R})$ for all $z \in \Pi_+$. Thus, for all $y \in [1, +\infty)$, we have

$$\text{rank } [-iyF(iy)] = \text{rank } \sigma_F(\mathbb{R}) = \text{rank } s_0.$$  \hspace{1cm} (9.3)

Because of (9.2) and (9.3), we see from Lemma A.10 that

$$\lim_{y \to +\infty} [-iyF(iy)]^\dagger = s_0^\dagger.$$  \hspace{1cm} (9.4)

Combining (9.4) and (9.1), we infer

$$s_0 = s_0 \left( \lim_{y \to +\infty} [-iyF(iy)] \right)^\dagger = \lim_{y \to +\infty} \left( \frac{1}{iy} s_0 [F(iy)]^\dagger s_0 \right) = \beta_{s_0}(-F^\dagger) s_0$$

and, consequently, $\beta_{s_0}(-F^\dagger) s_0 - s_0 = 0_{q \times q} \in \mathbb{C}_{\geq}^{q \times q}$. For all $z \in \Pi_+$, from (8.1) and (8.3) we see that

$$F^{(+;s_0)}(z) = -s_0 \left( z I_q + [F(z)]^\dagger \right)^\dagger \left( -s_0 (-F)^\dagger s_0 \right) (z) + z(-s_0).$$  \hspace{1cm} (9.5)

Since $s_0 (-F)^\dagger s_0$ belongs to $\mathcal{R}_q(\Pi_+)$, we get from (9.5) and Remark 2.3 that then $F^{(+;s_0)}$ belongs to $\mathcal{R}_q(\Pi_+)$ as well and that $\beta_{F^{(+;s_0)}} = 0_{q \times q}$. Thus,

$$F^{(+;s_0)} \in \mathcal{R}_q^{[-2]}(\Pi_+)$$  \hspace{1cm} (9.6)

follows from Remark 3.3. Taking (9.5) into account, we conclude

$$F^{(+;s_0)}(i) = \left[ s_0 (-F)^\dagger s_0 \right](i) - is_0 = \left( -s_0 [F(i)]^\dagger - iI_q \right) s_0$$

and, in particular, $N(s_0) \subseteq N(F^{(+;s_0)}(i))$. Thus from (9.6) and Proposition 3.7 we get $N(s_0) \subseteq N(\alpha_{F^{(+;s_0)}}) \cap N(\nu_{F^{(+;s_0)}}(\mathbb{R}))$. Consequently, $F^{(+;s_0)} \in \mathcal{P}_q^{\text{even}}[s_0]$.

**Corollary 9.2.** Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, $(s_j)_{j=0}^\kappa \in \mathcal{H}_q^{\geq,e}$ and $F \in \mathcal{R}_{\kappa,q}[\Pi_+;(s_j)_{j=0}^\kappa]$. Then $F^{(+;s_0)} \in \mathcal{P}_q^{\text{even}}[s_0]$.

**Proof.** From Remark 5.3 we get $F \in \mathcal{R}_{0,q}[\Pi_+;(s_j)_{j=0}^0]$. Thus, the application of Theorem 9.1 completes the proof.

**Theorem 9.3.** Let $(s_j)_{j=0}^1 \in \mathcal{H}_q^{\geq,e}$ and let $F \in \mathcal{R}_{1,q}[\Pi_+;(s_j)_{j=0}^1]$. Then $F^{(+;s_0,s_1)}$ belongs to $\mathcal{P}_q^{\text{odd}}[s_0]$.  

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Proof. For all \( z \in \Pi_+ \), we see from (8.1) that
\[
F^{(+;s_0,s_1)}(z) = F^{(+;s_0)}(z) + s_1. \tag{9.7}
\]
Because of \( F \in \mathcal{R}_{1,q}[\Pi_+; (s_j)_j^1]=0] \), we have \( F \in \mathcal{R}_{1,q}(\Pi_+) \) and
\[
\sigma_F \in M_{\geq }^q [\mathbb{R}; (s_j)_j^1]=0]. \tag{9.8}
\]
Furthermore, we see from (5.3) that the function \( F \) belongs to \( \mathcal{R}_0[q,\Pi_+; (s_j)_j^0] \) as well. Thus, Theorem 9.1 yields \( F^{(+;s_0)} \in \mathcal{R}_q(\Pi_+) \). Especially, we get then that \( \Phi: [1, +\infty) \to [0, +\infty) \) given by
\[
\Phi(y) := \frac{\| \text{Im} F^{(+;s_0,s_1)}(iy) \|}{y} \tag{9.9}
\]
is continuous and, in particular, Borel measurable. Since \( F \) belongs to \( \mathcal{R}_{1,q}(\Pi_+) \), we have \( F \in \mathcal{R}_q(\Pi_+) \). Consequently, Proposition 2.9 shows that \( -F^\dagger \) belongs to \( \mathcal{R}_q(\Pi_+) \) as well. In particular, \( F^\dagger \) is continuous. Furthermore, from \( F \in \mathcal{R}_q(\Pi_+) \) and (5.9) we see that \( F^{(s)} \) is continuous. Thus, \( \Theta: [1, +\infty) \to [0, +\infty) \) defined by
\[
\Theta(y) := \left\| \text{Re}\left( -\left( iy \right)^{-1}s_0 F^\dagger(iy) \left[ F^{(s)}(iy) - s_1 \right] s_0^\dagger \left[ F^{(s)}(iy) - s_1 \right] \right) \right\| \tag{9.9}
\]
is continuous. Let \( s_{-1} := 0_{q \times q} \). In view of Remark 5.1 we then conclude that \( F \) belongs to the class \( \mathcal{R}_q[\Pi_+; (s_j)_j^1=s_{-1}] \). According to Corollary 6.3 we get
\[
\lim_{y \to +\infty} F^{(s)}(iy) = 0_{q \times q} \tag{9.10}
\]
and
\[
s_0 = \lim_{y \to +\infty} [-iy F(iy)]. \tag{9.11}
\]
Since \( F \) belongs to \( \mathcal{R}_{1,q}(\Pi_+) \) we get from Lemma 8.22 and (9.8) that
\[
N(F(z)) = N(\sigma_F(\mathbb{R})) = N(s_0) \tag{9.12}
\]
and
\[
R(F(z)) = R(\sigma_F(\mathbb{R})) = R(s_0) \tag{9.13}
\]
for all \( z \in \Pi_+ \) and, hence, \( \text{rank} F(iy) = \text{rank} s_0 \) for all \( y \in [1, +\infty) \). Using this fact, we see from (9.11) and Lemma 8.10 that
\[
s_0^\dagger = \left( \lim_{y \to +\infty} [-iy F(iy)] \right)^\dagger = \lim_{y \to +\infty} \left( (-iy)^{-1} \left[ F(iy) \right]^\dagger \right). \tag{9.14}
\]

Because of (9.10) and (9.13), we obtain
\[ s_0s_0^\dagger s_1^0s_1 = s_0s_0^\dagger (0_{q \times q} - s_1)s_0^\dagger (0_{q \times q} - s_1) \]
\[ = \lim_{y \to +\infty} s_0 (-iy)^{-1} [F(iy)]^\dagger [F_1^{(s)}(iy) - s_1] s_0^\dagger [F_1^{(s)}(iy) - s_1] \]  
\[ = \lim_{y \to +\infty} \Theta(y). \]  
and, in view of (9.9), consequently,
\[ \left\| - \text{Re}(s_0s_0^\dagger s_1^0s_1) \right\| = \lim_{y \to +\infty} \Theta(y). \]  
Since \( \Theta \) is continuous, (9.16) implies the existence of a real number \( c \) such that \( \Theta(y) \leq c \) for all \( y \in [1, +\infty) \). According to part (a) of Proposition 5.12, the function \( F_1^{(s)} \) belongs to \( R_{-1,q}(\Pi_+) \), i.e., \( F_1^{(s)} \in R_{q,2}(\Pi_+) \) and \( \gamma_F = 0_{q \times q} \) hold true. In particular,
\[ \int_{[1, +\infty)} \left\| \text{Im} F_1^{(s)}(iy) \right\| \frac{\lambda(dy)}{y} < +\infty, \]
where \( \lambda \) is again the restriction of the Lebesgue measure on \( \mathcal{B}_{[1, +\infty)} \). Thus, we get
\[ \int_{[1, +\infty)} \left\| \text{Im} F_1^{(s)}(iy) \right\| \frac{\lambda(dy)}{y} + \frac{c}{y^2} \lambda(dy) < +\infty. \]  
From (9.12), (9.13), and Remark A.3 we conclude that
\[ s_0 F(z)^\dagger F(z) = s_0 \quad \text{and} \quad s_0s_0^\dagger F(z) = F(z) \]  
hold true for all \( z \in \Pi_+ \). In view of (9.9), we have
\[ F_1^{(s)}(z) = z^2 F(z) + zs_0 + s_1 \]  
for all \( z \in \Pi_+ \). Using (9.19), \( s_{-1} = 0_{q \times q} \), (9.18), and (8.1), we get
\[ F_1^{(s)}(z) + \frac{1}{z} s_0 [-zF(z)]^\dagger \left[ F_1^{(s)}(z) - s_1 \right] s_0^\dagger \left[ F_1^{(s)}(z) - s_1 \right] \]
\[ = F_1^{(s)}(z) - \frac{1}{z^2} s_0 [F(z)]^\dagger \left[ z^2 F(z) + zs_0 \right] s_0^\dagger \left[ z^2 F(z) + zs_0 \right] \]
\[ = F_1^{(s)}(z) - z^2 s_0 [F(z)]^\dagger F(z) s_0^\dagger F(z) - s_0 [F(z)]^\dagger F(z) s_0^\dagger s_0 s_0 \]
\[ - z s_0 [F(z)]^\dagger s_0^\dagger F(z) - s_0 [F(z)]^\dagger s_0^\dagger s_0 s_0 \]  
\[ = F_1^{(s)}(z) - z^2 s_0^\dagger F(z) - z s_0^\dagger s_0 - s_0 [F(z)]^\dagger F(z) - s_0 [F(z)]^\dagger s_0 \]
\[ = z^2 F(z) + zs_0 + s_1 - z^2 F(z) - zs_0 - s_0 [F(z)]^\dagger s_0 \]
\[ = -s_0 \left( zI_q + [F(z)]^\dagger s_0 \right) + s_1 = F^+(s_0, s_1)(z). \]
Because of (9.10), (9.15), and \( \Theta(y) \leq c \) for all \( y \in [1, +\infty) \), we then get

\[
\left\| \text{Im} F^{(+; \sigma_0, s_1)}(iy) \right\|_y \leq \frac{1}{y} \left\| \text{Im} F_1^{(s)}(iy) \right\| \]

\[
+ \frac{1}{y} \left\| \text{Im} \left( \frac{1}{iy} s_0 [-iyF(z)]^\dagger \left[ F_1^{(s)}(iy) - s_1 \right] s_0^\dagger \left[ F_1^{(s)}(iy) - s_1 \right] \right) \right\| \quad (9.21)
\]

\[
= \frac{1}{y} \left\| \text{Im} F_1^{(s)}(iy) \right\| \]

\[
+ \frac{1}{y} \left\| \text{Re} \left( -s_0 [-iyF(z)]^\dagger \left[ F_1^{(s)}(iy) - s_1 \right] s_0^\dagger \left[ F_1^{(s)}(iy) - s_1 \right] \right) \right\| \]

\[
= \frac{1}{y} \left\| \text{Im} F_1^{(s)}(iy) \right\| + \frac{1}{y^2} \Theta(y) \leq \frac{1}{y} \left\| \text{Im} F_1^{(s)}(iy) \right\| + \frac{c}{y^2}
\]

for all \( y \in [1, +\infty) \). Thus, (9.21) and (9.17) imply

\[
\int_{[1, +\infty)} \frac{\left\| \text{Im} F^{(+; \sigma_0, s_1)}(iy) \right\|_y}{y} \lambda(dy) < +\infty. \quad (9.22)
\]

Since \( F^{(+; \sigma_0, s_1)} \) belongs to \( \mathcal{R}_q(\Pi_+) \), inequality (9.22) shows that \( F^{(+; \sigma_0, s_1)} \) belongs to \( \mathcal{R}^{[-1]}_q(\Pi_+) \). From Proposition 3.14 we then know that

\[
\gamma_{F^{(+; \sigma_0, s_1)}} = \lim_{y \to +\infty} F^{(+; \sigma_0, s_1)}(iy). \quad (9.23)
\]

Because of (9.10), (9.15), (9.20), and (9.23), we then have

\[
0_{q \times q} = 0_{q \times q} - 0 \cdot s_0 s_0^\dagger s_1 s_1^\dagger
\]

\[
= \lim_{y \to +\infty} F_1^{(s)}(iy)
\]

\[
+ \left( \lim_{y \to +\infty} \frac{1}{iy} \right) \lim_{y \to +\infty} \left( s_0 [-iyF(iy)]^\dagger \left[ F_1^{(s)}(iy) - s_1 \right] s_0^\dagger \left[ F_1^{(s)}(iy) - s_1 \right] \right) \quad (9.24)
\]

\[
= \lim_{y \to +\infty} F^{(+; \sigma_0, s_1)}(iy) = \gamma_{F^{(+; \sigma_0, s_1)}}.
\]

Since \( F^{(+; \sigma_0, s_1)} \) belongs to \( \mathcal{R}^{[-1]}_q(\Pi_+) \), we see from (9.21) and (3.12) that \( F^{(+; \sigma_0, s_1)} \) belongs to \( \mathcal{R}^{[-1]}_{\mathcal{Q}_0}(\Pi_+) \). In view of (9.3), the function \( E_1: \mathbb{R} \to \mathbb{R} \) defined by \( E_1(t) := t \) belongs to \( \mathcal{L}^1(\mathbb{R}, \mathcal{B}_\mathbb{R}, \sigma_F; \mathbb{R}) \). Taking (22) Lemma B.2 into account, we infer then that

\[
N(\sigma_F(\mathbb{R})) \subseteq N(\int_\mathbb{R} E_1 \ d \sigma_F), \ i.e., \ N(s_0) \subseteq N(s_1^{[\sigma_F]}). \]

Using (9.8) and part (x) of Remark (A.3), we then get \( s_1 s_0^\dagger s_0 = s_1 \). Consequently,

\[
F^{(+; \sigma_0, s_1)}(i) = -s_0 \left( iI_q + [F(i)]^\dagger \right) s_0 + s_1 = -is_0 - s_0 [F(i)]^\dagger s_0 + s_1 s_0^\dagger s_0
\]

\[
= \left( -iI_q - s_0 [F(i)]^\dagger + s_1 s_0^\dagger \right) s_0.
\]
In particular, \( N(s_0) \subseteq N(F^{(+;s_0,s_1)}(i)) \). Since \( F^{(+;s_0,s_1)} \) belongs to \( \mathcal{R}_{-1,q}(\Pi_+) \), from Lemma 3.21 it follows \( N(s_0) \subseteq N(\mu F^{(+;s_0,s_1)}(\mathbb{R})) \). Thus, \( F^{(+;s_0,s_1)} \) belongs to \( \mathcal{P}_{q}^{\text{odd}}[s_0] \).

**Corollary 9.4.** Let \( \kappa \in \mathbb{N} \cup \{ +\infty \} \), let \((s_j)_{j=0}^\kappa \in \mathcal{H}^\ge_{q,\kappa} \) and let \( F \in \mathcal{R}_{\kappa,q}[\Pi_+; (s_j)_{j=0}^\kappa] \). Then \( F^{(+;s_0,s_1)} \in \mathcal{P}_{q}^{\text{odd}}[s_0] \).

**Proof.** From Remark 5.3 we get \( F \in \mathcal{R}_{1,q}[\Pi_+; (s_j)_{j=0}^1] \). Thus, the application of Theorem 9.3 completes the proof. \( \square \)

Now we turn our attention to the case \( m \in \mathbb{Z}_{2,+\infty} \). As already mentioned we will apply several Hamburger-Nevanlinna type results from Section 9. In order to prepare the application of this material, we still need some auxiliary results. First we will compute the functions introduced in Remark 5.9 (in particular, see (5.9)) for the case that the application of this material, we still need some auxiliary results. First we will compute the functions introduced in Remark 5.9 (in particular, see (5.9)) for the case that the function \( F \) is replaced by \( F^{(+;s_0,s_1)} \), whereas the role of the sequence is occupied by the first Schur transform \((s_j^{(1)})_{j=0}^{m-2}\) of the original sequence \((s_j)_{j=0}^m\).

**Lemma 9.5.** Let \( \mathcal{G} \) be a non-empty subset of \( \mathbb{C} \setminus \{ 0 \} \), let \( F: \mathcal{G} \to \mathbb{C}^{p \times q} \), let \( \kappa \in \mathbb{N} \cup \{ +\infty \} \), and let \((s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. In the case \( \kappa \geq 2 \), let \((s_j^{(1)})_{j=0}^{m-2}\) be the first Schur transform of \((s_j)_{j=0}^\kappa\). Let \( s_{-1} := 0_{p \times q} \), let \( s_{-1}^{(1)} := 0_{p \times q} \), let \( m \in \mathbb{Z}_{1,\kappa} \), and let \( \Delta_m: \mathcal{G} \to \mathbb{C}^{p \times q} \) be defined by

\[
\Delta_m(z) := \begin{cases} 
[F_1^{(s)}(z) - s_1]s_0^{\dagger}F_1^{(s)}(z) - s_1 & \text{if } m = 1 \\
[F_1^{(s)}(z) - s_1]s_0^{\dagger}F_m^{(s)}(z) - s_1 & \text{if } m \geq 2 \\
- F_m^{(s)}(z) s_0^{\dagger} (s_1 + \sum_{j=1}^{m-1} z^{-j} s_j^{(1)}) & \text{if } m \geq 2 \\
+ s_m s_0^{\dagger} s_1 + \sum_{k=1}^{m-1} s_{m-k} s_0^{\dagger} s_k^{(1)} & \text{if } m \geq 2 \\
+ \sum_{j=1}^{m-1} (z^{-j} \sum_{k=j}^{m-1} s_{m+j-k} s_0^{\dagger} s_k^{(1)}) & \text{if } m \geq 2 
\end{cases} \tag{9.25}
\]

Suppose that \((s_j)_{j=0}^m\) belongs to \( \mathcal{D}_{p \times q,m} \) given in Definition 7.8 and that \( z \in \mathcal{G} \) is such that \( N(F(z)) = N(s_0) \) and \( R(F(z)) \subseteq R(s_0) \) are fulfilled. Then

\[
(F^{(+;s_0,s_1)})^{(m-1)}_{m-2}(z) = F_m^{(s)}(z) + \frac{1}{z} s_0 [-z F(z)]^{\dagger} \Delta_m(z). \tag{9.26}
\]

**Proof.** Because of \( N(F(z)) = N(s_0) \) and part \( \mathbf{13} \) of Remark \( \mathbf{13} \), we have

\[
s_0^{\dagger} F(z) = s_0 \quad \text{and} \quad F(z)s_0^{\dagger} s_0 = F(z), \tag{9.27}
\]

whereas in view of part \( \mathbf{10} \) of Remark \( \mathbf{10} \) the assumption \( R(F(z)) \subseteq R(s_0) \) yields \( s_0 s_0^{\dagger} F(z) = F(z) \). Since \((s_j_{j=0})^{m}\) belongs to \( \mathcal{D}_{p \times q,m} \) and since \( s_{-1} = 0_{q \times q} \), the equations

\[
s_0^{\dagger} s_0 = s_j \quad \text{and} \quad s_j s_0^{\dagger} s_0 = s_j \] (9.28)
are valid for all \( j \in \mathbb{Z}_{-1,m} \). From [24, Remark 8.5] we know that
\[
s_0 s_0^{(1)} s_j = s_j^{(1)} \tag{9.29}
\]
is true for all \( j \in \mathbb{Z}_{-1,m-2} \). Using (5.9), (9.27), and (9.28), we easily check that
\[
s_0 s_0^{(1)} F_m^{(s)}(z) = F_m^{(s)}(z) \quad \text{and} \quad F_m^{(s)}(z) s_0^\dagger s_0 = F_m^{(s)}(z) \tag{9.30}
\]
are fulfilled. Because of (5.9), (8.1), (9.28), (9.29), and (9.27), we get
\[
(F^{(s)} s_0, s_1)) \cap_{m-2}^{(s(1))} (z) = z^{m-1} \left[ -s_0 \left( z I_q + [F(z)]^\dagger s_0 \right) + s_1 + \sum_{k=0}^{m-1} z^{-k} s_k^{(1)} \right] \\
= z^{m-1} \left( -zs_0 s_0^\dagger s_0 - s_0 [F(z)]^\dagger s_0 + s_0 s_0^\dagger s_1 + \sum_{k=0}^{m-1} z^{-k} s_0 s_0^\dagger s_k^{(1)} \right) \\
= z^{m-1} \left( -zs_0 [F(z)]^\dagger F(z) s_0 s_0 - s_0 [F(z)]^\dagger s_0 + s_0 [F(z)]^\dagger F(z) s_0 s_1 \right. \\
+ \sum_{k=0}^{m-1} z^{-k} s_0 [F(z)]^\dagger F(z) s_0 s_k^{(1)} \right) \\
= -z^{m-1} s_0 [F(z)]^\dagger \left[ z F(z) s_0^\dagger s_0 + s_0 - F(z) s_0^\dagger s_1 - \sum_{k=0}^{m-1} z^{-k} F(z) s_0^\dagger s_k^{(1)} \right] \\
= z^{m} s_0 \left[ -z F(z) \right]^\dagger \left[ s_0 + F(z) s_0^\dagger \left( zs_0 - s_1 - \sum_{k=0}^{m-1} z^{-k} s_k^{(1)} \right) \right].
\]
Taking into account \( s_{-1} = 0_{q \times q} \) and (9.24), we see that
\[
z^{m+1} F(z) = F_m^{(s)}(z) - \sum_{j=1}^{m+1} z^{-j} s_{j-1} \tag{9.32}
\]
holds true. Combining (9.31), (9.32), (9.28), and using (9.29), we conclude

\[
(F^{(+;s_0,s_1)}_{m-2})(z)
= s_0 [-zF(z)]^\dagger \left[ z^m s_0 + F_m^{(s)}(z) s_0^\dagger \left( s_0 - z^{-1} s_1 - \sum_{k=0}^{m-1} z^{-(k+1)} s_k^{(1)} \right) \right]
= s_0 [-zF(z)]^\dagger \left[ F_m^{(s)}(z) s_0^\dagger \left( s_0 - z^{-1} s_1 - \sum_{k=0}^{m-1} z^{-(k+1)} s_k^{(1)} \right) + z^m s_0 
- \sum_{j=1}^{m+1} z^{m+1-j} s_{j-1}s_0^\dagger + \sum_{j=1}^{m+1} z^{m-j} s_{j-1}s_0^\dagger s_1 
+ \sum_{j=1}^{m+1} z^{m-(j+k)} s_{j-1}s_0^\dagger s_k^{(1)} \right]
(9.33)
\]

From (5.9), \(s_{-1} = 0_{q \times q}\), and the first equation in (9.30) we get

\[
\left[ F_1^{(s)}(z) - s_1 \right] s_0^\dagger F_m^{(s)}(z) = [z^2 F(z) + z s_0] s_0^\dagger F_m^{(s)}(z) = z F_m^{(s)}(z) + z^2 F(z) 
= z F_m^{(s)}(z) + z^2 F(z) s_0^\dagger F_m^{(s)}(z).
(9.34)
\]

First we now consider the case \(m = 1\). Thanks to (9.33), \(s_{-1} = 0_{q \times q}\), and \(s_{-1}^{(1)} = 0_{q \times q}\), we then see that

\[
(F^{(+;s_0,s_1)}_{m-2})(z) = s_0 [-zF(z)]^\dagger \left[ F_m^{(s)}(z) s_0^\dagger (s_0 - z^{-1}) + z^{-1} s_1 s_0^\dagger s_1 \right].
(9.35)
\]

Since the first equation in (9.30) and (9.27) yield

\[
F_m^{(s)}(z) = s_0 s_0^\dagger F_m^{(s)}(z) = s_0 [F(z)]^\dagger F(z) s_0^\dagger F_m^{(s)}(z),
(9.36)
\]

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we see from (9.35), the second equation in (9.30), (9.34), and (9.25), and \( m = 1 \) that
\[
\begin{align*}
    z \left[ (F^{(0);s_0,s_1})_{m-2}^{(1)} (z) - F_m^{(s)} (z) \right] \\
    = s_0 \left[ -zF(z) \right]^\dagger \left[ F_m^{(s)} (z) s_0^\dagger (zs_0 - s_1) + s_1 s_0^\dagger s_1 \right] - z s_0 \left[ F(z) \right]^\dagger F(s)^\dagger F_m^{(s)} (z) \\
    = s_0 \left[ -zF(z) \right]^\dagger \left[ z F_m^{(s)} (z) - F_m^{(s)} (zs_0 s_1 + s_1 s_0^\dagger s_1 + z^2 F(z) s_0^\dagger F_m^{(s)} (z) \right] \\
    = s_0 \left[ -zF(z) \right]^\dagger \left[ \left[ F_1^{(s)} (z) - s_1 \right] s_0^\dagger F_m^{(s)} (z) - F_m^{(s)} (z) s_0^\dagger s_1 + s_1 s_0^\dagger s_1 \right] \\
    = s_0 \left[ -zF(z) \right]^\dagger \left[ s_0^\dagger F_1^{(s)} (z) - s_1 \right] s_0^\dagger F_1^{(s)} (z) - s_1 \right] s_0^\dagger \Delta_m (z).
\end{align*}
\]
Thus, (9.26) is proved if \( m = 1 \). Now we consider the case \( m \geq 2 \). By virtue of (9.33) and \( s_{-1} = 0_{q \times q} \), we then conclude that
\[
\begin{align*}
    (F^{(0);s_0,s_1})_{m-2}^{(1)} (z) = s_0 \left[ -zF(z) \right]^\dagger \left[ F_m^{(s)} (z) s_0^\dagger \left( s_0 - z^{-1} s_1 - \sum_{k=1}^{m-1} z^{-(k+1)} s_k^{(1)} \right) \right] \\
    - \sum_{j=3}^{m+1} z^{m+1-j} s_j s_0^\dagger s_1 + \sum_{j=2}^{m+1} z^{m-j} s_0^\dagger s_2 s_1 \\
    + \sum_{k=1}^{m-1} z^{m-j} \left( s_j - s_1 s_0^\dagger s_1 \right) + \sum_{k=1}^{m-1} \sum_{j=2}^{m+1} z^{m-j} s_0^\dagger s_k^{(1)} \\
    = s_0 \left[ -zF(z) \right]^\dagger \left[ F_m^{(s)} (z) s_0^\dagger \left( s_0 - z^{-1} s_1 - \sum_{j=2}^{m} z^{j} s_j^{(1)} \right) \right] + z^{-1} s_m s_0^\dagger s_1 \\
    + \sum_{j=2}^{m} z^{m-j} (s_j - s_1 s_0^\dagger s_1 - s_j) + \sum_{j=2}^{m+1} \sum_{k=1}^{m-1} z^{m-j} s_j s_0^\dagger s_k^{(1)} \right].
\end{align*}
\]
From [24], Proposition 8.23] we know that
\[
\begin{align*}
    s_j^{(1)} &= \begin{cases} 
        s_2 - s_1 s_0^\dagger s_1, & \text{if } j = 0 \\
        s_j + s_1 s_0^\dagger s_1 - \sum_{l=0}^{j-1} s_l s_0^\dagger s_l^{(1)}, & \text{if } j \in \mathbb{Z}_{1,m-2}
    \end{cases}
\end{align*}
\]
holds. If \( m = 2 \), then from (9.37) and (9.38) we can easily check that
\[
\begin{align*}
    (F^{(0);s_0,s_1})_{m-2}^{(1)} (z) = s_0 \left[ -zF(z) \right]^\dagger \left[ F_m^{(s)} (z) s_0^\dagger (s_0 - z^{-1} s_1 - z^{-2} s_0^{(1)}) \right] \\
    + z^{-1} (s_2 s_0^\dagger s_1 + s_1 s_0^\dagger s_1^{(1)} + z^{-1} s_2 s_0^\dagger s_0^{(1)})
\end{align*}
\]
and, consequently, in view of (9.36), the second equation in (9.30), (9.34), and (9.25),
then
\[ z \left[ (F^+(s_0; s_1))^{(s^{(1)})}_{m-2}(z) - F^+_m(z) \right] \]
\[ = s_0 \left[ -zF(z) \right]^\dagger \left[ zF_m(z)s_0^\dagger s_0 - F_m(z)z^\dagger (s_1 + z^{-1}s_0) + s_2s_0^\dagger s_1 \right. \]
\[ + (s_1 + z^{-1}s_2)s_0^\dagger s_0^{(1)} \] 
\[ = s_0 \left[ -zF(z) \right]^\dagger \left[ zF_m(z) - F_m(z)s_0^\dagger (s_1 + z^{-1}s_0) + s_2s_0^\dagger s_1 \right. \]
\[ + (s_1 + z^{-1}s_2)s_0^\dagger s_0^{(1)} + z^2F(z)s_0^\dagger F_m(z) \] 
\[ = s_0 \left[ -zF(z) \right]^\dagger \left( F_1^{(s)}(z) - s_1 \right) s_0^\dagger F_m(z) - F_m(z)s_0^\dagger (s_1 + z^{-1}s_0) \]
\[ + s_2s_0^\dagger s_1 + (s_1 + z^{-1}s_2)s_0^\dagger s_0^{(1)} \] 
\[ = s_0 \left[ -zF(z) \right]^\dagger \Delta_2(z) = s_0 \left[ -zF(z) \right]^\dagger \Delta_m(z) \]
holds as well. Thus, (9.26) is also proved in the case \( m = 2 \). Now let \( m \geq 3 \). Then

\[ \sum_{j=2}^{m+1} \sum_{k=1}^{m-1} z^{m-(j+k)} s_{j-1}^\dagger_{0^r} s_{k-1} = \sum_{l=3}^{m+1} \sum_{r=\max\{m+2,l\}-(m+1)}^{\min\{m-1,l-2\}} z^{m-l} s_{l-r-1}^\dagger_{0^r} s_{r-1} \]
\[ = \sum_{l=3}^{m+1} \sum_{r=1}^{m-1} z^{m-l} s_{l-r-1}^\dagger_{0^r} s_{r-1} + \sum_{r=1}^{m-1} z^{-1} s_{m-r}^\dagger_{0^r} s_{r-1} \]
\[ + \sum_{l=m+2}^{2m} \sum_{r=l-(m+1)}^{2m} z^{m-l} s_{m-r}^\dagger_{0^r} s_{r-1} \]
\[ = \sum_{l=3}^{m+1} \sum_{k=0}^{m-1} z^{m-l} s_{l-k-2}^\dagger_{0^r} s_{k} + \sum_{r=1}^{m-1} z^{-1} s_{m-r}^\dagger_{0^r} s_{r-1} \]
\[ + \sum_{l=m+2}^{2m} \sum_{k=l-m}^{2m} z^{m-l} s_{m-k}^\dagger_{0^r} s_{k-2} \] 
\[ = \sum_{j=3}^{m} z^{m-j} \sum_{k=0}^{(j-2)-1} s_{j-2-k}^\dagger_{0^r} s_{k} + z^{-1} \sum_{r=1}^{m-1} s_{m-1}^\dagger_{0^r} s_{r-1} \]
\[ + \sum_{j=2}^{m} \sum_{k=j}^{m} s_{m-j-k}^\dagger_{0^r} s_{k-2} \]
Consequently, because of (9.37), \( s_{-1} = 0 \times q \), (9.38), and (9.39), we obtain

\[
(F^{(+; s_0, s_1)}_{m - 2}(z))^{(s_{(1)})}_0
\]

\[
= s_0 [-zF(z)] s_0^{\dagger} \left[ F_{m}(s_{0}) z s_0^{\dagger} \left( s_0 - z^{-1} s_1 - \sum_{j=2}^{m} z^{-j} s_{j - 2}^{(1)} \right) \right.
\]

\[
+ \left. z^{m-2} \left( s_{1}^{(1)} + s_1 s_0^{\dagger} s_1 - s_2 \right) + \sum_{j=3}^{m} z^{-j} \left( s_{j - 2}^{(1)} + s_{j-1} s_0^{\dagger} s_1 - s_j \right) \right]
\]

\[
+ z^{-1} s_m s_0^{\dagger} s_1 + \sum_{j=3}^{m} z^{-j} \sum_{k=0}^{(j-2)-1} s_{(j-2)-k} s_0^{\dagger} s_k^{(1)} + z^{-1} \sum_{r=1}^{m-1} s_{m-1} s_0^{\dagger} s_{k - 2}^{(1)} \right] \]

\[
= s_0 [-zF(z)] s_0^{\dagger} \left[ F_{m}(s_{0}) z s_0^{\dagger} \left( s_0 - z^{-1} s_1 - \sum_{j=2}^{m} z^{-j} s_{j - 2}^{(1)} \right) \right.
\]

\[
+ \left. z^{m-2} \left( s_{1}^{(1)} - (s_2 - s_1 s_0^{\dagger} s_1) \right) \right]
\]

\[
+ \sum_{j=3}^{m} z^{-j} \left[ s_{j-2}^{(1)} - \left( s_j - s_{j-1} s_0^{\dagger} s_1 - \sum_{k=0}^{(j-2)-1} s_{(j-2)-k} s_0^{\dagger} s_k^{(1)} \right) \right]
\]

\[
+ z^{-1} \left( s_m s_0^{\dagger} s_1 + \sum_{k=1}^{m-1} s_{m-k} s_0^{\dagger} s_{k-1}^{(1)} \right) + \sum_{j=2}^{m} \sum_{k=j}^{m} z^{-j} s_{m+j-k} s_0^{\dagger} s_{k-2}^{(1)} \right] \]

\[
= s_0 [-zF(z)] s_0^{\dagger} \left[ F_{m}(s_{0}) z s_0^{\dagger} \left( s_0 - z^{-1} s_1 - \sum_{j=2}^{m} z^{-j} s_{j - 2}^{(1)} \right) \right.
\]

\[
+ \left. z^{m-2} \left( s_{1}^{(1)} - (s_2 - s_1 s_0^{\dagger} s_1) \right) \right]
\]

\[
+ \sum_{j=3}^{m} z^{-j} \left[ s_{j-2}^{(1)} - \left( s_j - s_{j-1} s_0^{\dagger} s_1 - \sum_{k=0}^{(j-2)-1} s_{(j-2)-k} s_0^{\dagger} s_k^{(1)} \right) \right]
\]

\[
+ z^{-1} \left( s_m s_0^{\dagger} s_1 + \sum_{k=1}^{m-1} s_{m-k} s_0^{\dagger} s_{k-1}^{(1)} \right) + \sum_{j=2}^{m} \sum_{k=j}^{m} z^{-j} s_{m+j-k} s_0^{\dagger} s_{k-2}^{(1)} \right] .
\]
Using (9.40), the first equation in (9.30), (9.27), and (9.25), we get

\[
\begin{align*}
&z \left[ (F^{(s_0, s_1)})^{(s)}_{m-2}(z) - F^{(s)}_{m}(z) \right] \\
&= s_0 \left[ -z F(z)^{\dagger} \left[ F^{(s)}_{m}(z)s_0^{\dagger} \left( zs_0 - s_1 - \sum_{j=2}^{m} z^{j+1}s^{(1)}_{j-2} \right) + s_ms_0^{\dagger}s_1 \\
&\quad + \sum_{k=1}^{m-1} s_m s_{k}^{(1)} s_{k-1} + \sum_{j=2}^{m} z^{-j} \sum_{k=j}^{m} s_m s_{j-k} s_{k-2} \right] \right] \\
&\quad - zs_0 \left[ F(z)^{\dagger} F(z)s_0^{\dagger} F^{(s)}_{m}(z) \right] \\
&= s_0 \left[ -z F(z)^{\dagger} \left[ F^{(s)}_{m}(z)s_0^{\dagger} s_0 - F^{(s)}_{m}(z)s_0^{\dagger} \left( s_1 + \sum_{j=1}^{m-1} z^{-j}s^{(1)}_{j-1} \right) + s_ms_0^{\dagger}s_1 \\
&\quad + \sum_{k=1}^{m-1} s_m s_{k}^{(1)} s_{k-1} + \sum_{j=1}^{m-1} z^{-j} \sum_{k=j}^{m} s_m s_{j-k} s_{k-2} - z^2 F(z)s_0^{\dagger} F^{(s)}_{m}(z) \right] \right] \\
&= s_0 \left[ -z F(z)^{\dagger} \left[ F^{(s)}_{m}(z) - s_1 s_0^{\dagger} F^{(s)}_{m}(z) - F^{(s)}_{m}(z)s_0^{\dagger} \left( s_1 + \sum_{j=1}^{m-1} z^{-j}s^{(1)}_{j-1} \right) \\
&\quad + s_ms_0^{\dagger}s_1 + \sum_{k=1}^{m-1} s_m s_{k}^{(1)} s_{k-1} + \sum_{j=1}^{m-1} z^{-j} \sum_{k=j}^{m} s_m s_{j-k} s_{k-2} \right] \right] \\
&= s_0 \left[ -z F(z)^{\dagger} \Delta_m(z) \right].
\end{align*}
\]

Thus, (9.26) is also proved in the case \(m \geq 3\). \(\square\)

Now we study the asymptotic behaviour of the function \(\Delta_m\), which was introduced in (9.25).

**Lemma 9.6.** Let \(\theta \in [0, 2\pi)\) and let \(G\) be a subset of \(\mathbb{C} \setminus \{0\}\) with \(\{e^{i\theta}y|y \in [1, +\infty)\} \subseteq G\).

Let \(F: G \to \mathbb{C}^{p \times q}\) be a matrix-valued function, let \(\kappa \in \mathbb{N} \cup \{+\infty\}\), and let \((s_j)_{j = 0}^{\kappa}\) be a sequence of complex \(p \times q\) matrices. In the case \(\kappa \geq 2\) let \((s_j^{(1)})_{j = 0}^{\kappa-2}\) be the first Schur transform of \((s_j)_{j = 0}^{\kappa}\). Let \(s_{-1} := 0_{p \times q}\), let \(m \in \mathbb{Z}_{1, \kappa}\), and let \(\Delta_m: G \to \mathbb{C}^{p \times q}\) be defined by (9.25). Suppose that

\[
\lim_{r \to +\infty} F^{(s)}_{m}(e^{i\theta}r) = 0_{p \times q} \tag{9.41}
\]

holds true. Then

\[
\lim_{r \to +\infty} \Delta_m(e^{i\theta}r) = \begin{cases} 
  s_1 s_0^{\dagger}s_1, & \text{if } m = 1 \\
  s_m s_0^{\dagger}s_1 + \sum_{k=1}^{m-1} s_m s_{k}^{(1)} s_{k-1}, & \text{if } m \geq 2
\end{cases}.
\]

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Proof. In the case $m = 1$, in view of (9.11) and (9.22), we get immediately
\[
s_m s_0 \bar{s}_1 = \left( \lim_{r \to +\infty} F_m^{(s)}(e^{i\theta} r) - s_1 \right) s_0 \left( \lim_{r \to +\infty} F_m^{(s)}(e^{i\theta} r) - s_1 \right)
\]
\[
= \lim_{r \to +\infty} \Delta_m(e^{i\theta} r).
\]

Now assume that $m \geq 2$. For all $r \in [1, +\infty)$, from Remark 5.9 we conclude
\[
F_1^{(s)}(e^{i\theta} r) = (e^{i\theta} r)^{1 - m} \left[ F_m^{(s)}(e^{i\theta} r) - \sum_{j=0}^{m-1} (e^{i\theta} r)^j s_{m-j} \right]
\]
\[
= (e^{i\theta} r)^{-(m-1)} F_m^{(s)}(e^{i\theta} r) - \sum_{j=0}^{m-2} (e^{i\theta} r)^{j-(m-1)} s_{m-j}.
\] (9.42)

In view of $m \geq 2$, the assumption (9.41), and (9.42), we have
\[
0_{p \times q} = \left( \lim_{r \to +\infty} (e^{i\theta} r)^{-(m-1)} \right) \left[ \lim_{r \to +\infty} F_m^{(s)}(e^{i\theta} r) - \sum_{j=0}^{m-2} (e^{i\theta} r)^{j-(m-1)} s_{m-j} \right] s_{m-j}
\]
\[
= \lim_{r \to +\infty} F_1^{(s)}(e^{i\theta} r).
\] (9.43)

Consequently, (9.43), (9.41), and (9.25) yield
\[
s_m s_0 \bar{s}_1 + \sum_{k=1}^{m-1} s_{m-k} s_0 s_{k-1}^{(1)}
\]
\[
= (0_{p \times q} - s_1) s_0 \bar{s}_1 + \sum_{k=1}^{m-1} s_{m-k} s_0 s_{k-1}^{(1)}
\]
\[
= \left( \lim_{r \to +\infty} F_1^{(s)}(e^{i\theta} r) - s_1 \right) s_0 \left( \lim_{r \to +\infty} F_m^{(s)}(e^{i\theta} r) \right) - \sum_{j=1}^{m-1} \left( \lim_{r \to +\infty} (e^{i\theta} r)^{j-1} \right) s_{m-j} - \sum_{j=1}^{m-1} \left( \lim_{r \to +\infty} (e^{i\theta} r)^{j-1} \right) s_{m-j}^{(1)} + s_m s_0 s_1
\]
\[
= \lim_{r \to +\infty} \Delta_m(e^{i\theta} r).
\]
Lemma 9.5. From \((s_j)_{j=0}^{2n} \in H_{q,2n}^{-}\) and let \(F \in R_{2n,q}[\Pi_+;(s_j)_{j=0}^{2n}]. \) Further, let \((s_j^{(1)})_{j=0}^{2n-2}\) be the first Schur transform of \((s_j)_{j=0}^{2n}\). Then \(F^{(s_0,s_1)}\) belongs to 
\(R_{2n-2,q}[\Pi_+;(s_j^{(1)})_{j=0}^{2n-2}].\)

Proof. Since \(F\) belongs to \(R_{2n,q}[\Pi_+;(s_j)_{j=0}^{2n}].\) from (5.3) we see that \(F\) belongs to \(R_{2n,q}(\Pi_+)\) and that
\[
\sigma_F \in M_2^s[R; (s_j)_{j=0}^{2n}, =]
\]
hold true. Furthermore, Remark 5.3 shows that \(F\) belongs to \(R_{1,q}[\Pi_+;(s_j^{1})_{j=0}^{1}].\) Thus, we see from Theorem 9.3 that \(F^{(s_0,s_1)} \in P_{odd}[s_0].\) In view of (3.12), this in particular means that \(F^{(s_0,s_1)}\) belongs to \(R_{-1,q}(\Pi_+).\) Consequently, Proposition 9.13 yields \(F^{(s_0,s_1)} \in R_q(\Pi_+).\) Letting \(s_{-1} := 0_{q \times q},\) from \((s_j)_{j=0}^{2n} \in H_{q,2n}^{-}\) we get in view of part (b) of Lemma 7.6 that \(s_j^\ast = s_j\) for all \(j \in \mathbb{Z}_{-1,2n-1}\) and, because of Proposition 7.2 that the first Schur transform \((s_j^{(1)})_{j=0}^{2n-2}\) of \((s_j)_{j=0}^{2n}\) belongs to \(H_{q,2n-2}^{-}\). In particular, setting \(s_{-1}^{(1)} := 0_{q \times q},\) we see again from part (b) of Lemma 7.6 that
\[
(s_j^{(1)})^\ast = s_j^{(1)}
\]
for all \(j \in \mathbb{Z}_{-1,2n-2}.\) Now we verify that the function \(F\) satisfies the assumptions of Lemma 9.3. From \((s_j)_{j=0}^{2n} \in H_{q,2n}^{-}\) and [23, Lemma 3.1] we know that \((s_j)_{j=0}^{2n} \in D_{q \times q,2n}.\) In view of \(F \in R_{2n,q}[\Pi_+;(s_j)_{j=0}^{2n}].\) we infer from part (a) of Proposition 5.7 that
\[
N(F(z)) = N(s_0) \quad \text{and} \quad R(F(z)) = R(s_0)
\]
for all \(z \in \Pi_.\) Because of \((s_j)_{j=0}^{2n} \in D_{q \times q,2n}\) and (9.13), Lemma 9.3 implies
\[
(F^{(s_0,s_1)})(s_j^{(1)})_{2n-2}(z) = F^{(s_j)}_{2n}(z) + \frac{1}{z} s_0 [-zF(z)]^\dagger \Delta_{2n}(z)
\]
for all \(z \in \Pi_.\) In view of \(s_{-1} = 0_{q \times q},\) the application of Remark 5.1 yields \(F \in R_{q}^{(2n)}[\Pi_+;(s_j)_{j=0}^{2n-1}].\) Corollary 6.3 then yields
\[
\lim_{y \to +\infty} F^{(s_j)}_{1}(iy) = 0_{q \times q}, \quad \lim_{y \to +\infty} F^{(s_j)}_{2n}(iy) = 0_{q \times q}
\]
and
\[
s_0 = \lim_{y \to +\infty} (-iy) [F(iy) + s_{-1}] = \lim_{y \to +\infty} [-iyF(iy)].
\]
Since (9.45) implies \(\text{rank}[-iyF(iy)] = \text{rank} s_0\) for all \(y \in [1, +\infty),\) from (9.48) and Lemma A.10 we get that
\[
\lim_{y \to +\infty} ([-iyF(iy)]^\dagger) = s_0^\dagger.
\]
Because of (9.47), \(\lim_{y \to +\infty} \frac{1}{iy} = 0,\) and (9.25), we have
\[
\lim_{y \to +\infty} \Delta_{2n}(iy) = s_{2n} s_0^\dagger s_1 + \sum_{k=1}^{2n-1} s_{2n-k} s_0^\dagger s_{k-1}^{(1)}.
\]
Consequently, keeping in mind (9.37) again as well as equation (9.49), (9.50), and (9.40), we obtain

\[
q < q \\
= \lim_{y \to +\infty} F_{2n}^{(s)}(iy) + \left( \lim_{y \to +\infty} \frac{1}{iy} \right) s_0 \left[ \lim_{y \to +\infty} \left[ -iyF(iy) \right] \right] \left[ \lim_{y \to +\infty} \Delta_{2n}(iy) \right] \\
= \lim_{y \to +\infty} \left( F_{2n}^{(s)}(iy) + \frac{1}{iy} s_0 \left[ -iyF(iy) \right] \Delta_{2n}(iy) \right) = \lim_{y \to +\infty} (F^{(s_0,s_1)})^{(s_1)}_{2n-2}(iy).
\]

From \( F^{(+;s_0,s_1)} \in \mathcal{R}_q(\Pi_+), \) (9.44), (9.51), and Theorem 9.7 we conclude that \( F^{(+;s_0,s_1)} \) belongs to the class \( \mathcal{R}_{q}^{2n-2}[\Pi_+; \langle s_1^{(1)} \rangle_{j=0}^{2n-2}] \). Because of \( s_1^{(1)} = 0 \), we see from Remark 6.1 that \( F^{(+;s_0,s_1)} \in \mathcal{R}_{2n-2,q}[\Pi_+; \langle s_1^{(1)} \rangle_{j=0}^{2n-2}] \).

**Corollary 9.8.** Let \( (s_j)_{j=0}^{\infty} \in \bar{H}_{q,\infty}^c \) and let \( F \in \mathcal{R}_{\infty,q}[\Pi_+; (s_j)_{j=0}^{\infty}] \). Further, let \( (s_j^{(1)})_{j=0}^{\infty} \) be the first Schur transform of \( (s_j)_{j=0}^{\infty} \). Then \( F^{(+;s_0,s_1)} \) belongs to the class \( \mathcal{R}_{\infty,q}[\Pi_+; \langle s_1^{(1)} \rangle_{j=0}^{\infty}] \).

**Proof.** Combine Remarks 5.3 and 6.1 and Theorem 9.7.

**Theorem 9.9.** Let \( n \in \mathbb{N} \), let \( (s_j)_{j=0}^{2n+1} \in \bar{H}_{q,2n+1}^c \) and let \( F \in \mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] \). Further, let \( (s_j^{(1)})_{j=0}^{2n+1} \) be the first Schur transform of \( (s_j)_{j=0}^{2n+1} \). Then \( F^{(+;s_0,s_1)} \) belongs to \( \mathcal{R}_{2n-1,q}[\Pi_+; (s_j^{(1)})_{j=0}^{2n+1}] \).

**Proof.** Since \( F \) belongs to \( \mathcal{R}_{2n+1,q}[\Pi_+; (s_j^{(1)})_{j=0}^{2n+1}] \), we get \( F \in \mathcal{R}_{2n+1,q}[\Pi_+] \) and

\[
\sigma F \in \mathcal{M}_q^2[\mathbb{R}; (s_j^{(2)})_{j=0}^{2n+1}].
\]

Furthermore, Remark 5.3 shows that \( F \) belongs to \( \mathcal{R}_{1,q}[\Pi_+; (s_j^{(1)})_{j=0}^{1}] \). Thus, we see from Theorem 9.3 that \( F^{(+;s_0,s_1)} \in \mathcal{P}^{\text{odd}}[s_0] \). This in particular means that \( F^{(+;s_0,s_1)} \) belongs to \( \mathcal{R}_{-1,q}(\Pi_+) \). Consequently, \( F^{(+;s_0,s_1)} \in \mathcal{R}_q(\Pi_+) \). Letting \( s_1 := 0 \), \( q_0 \), from \( (s_j)_{j=0}^{2n+1} \in \bar{H}_{q,2n+1}^c \) we get in view of part (b) of Lemma 7.6 that \( s_j^{(1)} = s_j \), holds for all \( j \in \mathbb{Z}_{-1,2n} \), and, because of Proposition 7.2 that the sequence \( (s_j^{(1)})_{j=0}^{2n-1} \) belongs to \( \bar{H}_{q,2n-1}^c \). In particular, setting \( (s_j^{(1)1})_{j=0}^{2n-1} \), we see again from part (c) of Lemma 7.6 that (9.44), i.e., \( (s_j^{(1)})^{(1)} = s_j^{(1)} \), holds for all \( j \in \mathbb{Z}_{-1,2n-1} \). We verify now that the function \( F \) satisfies the assumptions of Lemma 9.5. From \( (s_j)_{j=0}^{2n+1} \in \bar{H}_{q,2n+1}^c \) and Lemma 9.7 we know that

\[
(s_j)_{j=0}^{2n+1} \in \mathcal{D}_{q,2n+1}.
\]

In view of \( F \in \mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] \), we infer from part (c) of Proposition 5.7 that

\[
N(F(z)) = N(s_0) \quad \text{and} \quad R(F(z)) = R(s_0)
\]
for all \( z \in \Pi_+ \). Because of (9.52) and (9.53), Lemma 9.3 implies
\[
(F^{(+;s_0,s_1)})^{(s_1)}_{2n-1}(z) = F^{(s)}_{2n+1}(z) + \frac{1}{z} s_0 \left[ -z F(z) \right] \Delta_{2n+1}(z)
\]  
(9.54)
for all \( z \in \Pi_+ \). In view of (9.56), the limit relation
\[
\lim_{y \to +\infty} F^{(s)}_{2n+1}(iy) = 0_{q \times q}
\]
(9.55)
and
\[
F^{(s)}_1(iy) = \left. \lim_{y \to +\infty} F^{(s)}_{2n+1}(iy) \right|_{y \to +\infty} = 0_{q \times q}
\]
(9.56)
In view of (9.53), Corollary 6.3 then provides us
\[
\lim_{y \to +\infty} F^{(s)}_{2n+1}(iy) = 0_{q \times q}
\]
(9.57)
and
\[
\lim_{y \to +\infty} (-iy) F^{(s)}_1(iy) = -iy F^{(s)}_1(iy)
\]
(9.58)
Since (9.53) implies \( \text{rank}[-iy F^{(s)}_1(iy)] = \text{rank} s_0 \) for all \( y \in [1, +\infty) \), from (9.57) and Lemma A.10, we get that
\[
\lim_{y \to +\infty} (-iy F^{(s)}_1(iy)) = s_0^†.
\]
(9.59)
Because of (9.56), the limit relation \( \lim_{y \to +\infty} \frac{1}{iy} = 0 \) and (9.58), we have
\[
\lim_{y \to +\infty} \Delta_{2n+1}(iy) = s_{2n+1} \xi q \xi s_1 + \sum_{k=1}^{2n} s_{2n+1-k} s_0 s_{k-1}^†.
\]
(9.60)
Consequently, keeping in mind (9.56), the limit relation \( \lim_{y \to +\infty} \frac{1}{iy} = 0 \), equation (9.60), (9.58), and (9.51), we obtain
\[
0_{q \times q}
\]
(9.61)
Thus, Theorem 6.6 provides us \((F^{(+;s_0,s_1)})^{(s_1)}_{2n-1} \in \mathcal{R}_q(\Pi_+)\). In particular, the function \((F^{(+;s_0,s_1)})^{(s_1)}_{2n-1} \) is holomorphic in \( \Pi_+ \). This shows us that \( \Phi: [1, +\infty) \to [0, +\infty) \) defined by
\[
\Phi(y) := \frac{1}{y} \left\| (F^{(+;s_0,s_1)})^{(s_1)}_{2n-1}(iy) \right\|
\]
(9.62)
belongs to \( L^1([1, +\infty), \mathfrak{B}_{[1, +\infty)}, \lambda; \mathbb{R}) \), where \( \lambda \) is again the Lebesgue measure defined on \( \mathfrak{B}_{[1, +\infty)} \). Because of \( F \in \mathcal{R}_{2n+1,q}(\Pi^+; (s_j)_{j=0}^{2n+1}) \), Proposition 5.5 and Theorem 3.2 we have \( F \in \mathcal{R}_q(\Pi^+) \). Thus, we see from (5.9) that the functions \( F_1^{(s)} \) and \( F_2^{(s)}_{2n+1} \) are both holomorphic in \( \Pi^+ \). Therefore, (9.25) shows that \( \Delta_{2n+1} \) is holomorphic in \( \Pi^+ \). Since \( F \) belongs to \( \mathcal{R}_q(\Pi^+) \), we also see from Proposition 2.9 that \( F_1 \) is holomorphic in \( \Pi^+ \). Consequently, \( \Theta: [1, +\infty) \rightarrow \mathbb{R} \) defined by

\[
\Theta(y) := \left\| \text{Re} \left( (iy)^{-1} s_0 [F(iy)]^\dagger \Delta_{2n+1}(iy) \right) \right\|
\]  

(9.63)
is a continuous function. Using (9.58), (9.59), and (9.63), we get that

\[
\| - \text{Re} \left[ s_0 s_0^\dagger \left( s_{2n+1} s_{0}^\dagger + \sum_{k=1}^{2n} s_{2n+1-k} s_{0}^\dagger \right) \right] \| \leq \lim_{y \to +\infty} \left( \| -iy \Delta_{2n+1}(iy) \| \right) = \lim_{y \to +\infty} \Theta(y).
\]

This implies that there exists a non-negative real number \( c \) such that \( |\Theta(y)| \leq c \) for all \( y \in [1, +\infty) \). Since \( \Omega \) belongs to \( L^1([1, +\infty), \mathfrak{B}_{[1, +\infty)}, \lambda; \mathbb{R}) \) the function \( \Psi: [1, +\infty) \rightarrow \mathbb{R} \) given by \( \Psi(y) := \Omega(y) + \frac{c}{y} \) also belongs to the space \( L^1([1, +\infty), \mathfrak{B}_{[1, +\infty)}, \lambda; \mathbb{R}) \). Keeping in mind (9.61), (9.62) and (9.64), we infer

\[
|\Phi(y)| \leq \frac{1}{y} \left\| - \text{Im} \left( F_2^{(s)}(iy) \right) \right\| + \frac{1}{y} \left\| \text{Im} \left( \frac{1}{iy} s_0 [iy F(iy)]^\dagger \Delta_{2n+1}(iy) \right) \right\| = \Omega(y) + \frac{1}{y^2} \left\| \text{Re} \left( s_0 [iy F(iy)]^\dagger \Delta_{2n+1}(iy) \right) \right\| = \Omega(y) + \Theta(y) \leq \Psi(y)
\]

for all \( y \in [1, +\infty) \), and, consequently, \( \Phi \) belongs to \( L^1([1, +\infty), \mathfrak{B}_{[1, +\infty)}, \lambda; \mathbb{R}) \). Since \( (F^{(+;s_0,s_1)})^{(s_1)}_{2n-1} \) belongs to \( \mathcal{R}_q(\Pi^+) \), we thus see from (3.5) and (9.61) that

\[
(F^{(+;s_0,s_1)})^{(s_1)}_{2n-1} \in \mathcal{R}_q^{[-1]}(\Pi^+),
\]

(9.64)

From \( (F^{(+;s_0,s_1)}) \in \mathcal{R}_q(\Pi^+), (9.44), (9.60), (9.63) \) and Theorem 6.6 we conclude that \( F^{(+;s_0,s_1)} \) belongs to the class \( \mathcal{R}_q^{[2n-1]}[\Pi^+; (s_j)_{j=0}^{2n-1}] \). Because of \( s_{-1}^{(1)} = 0_{q \times q} \), then Remark 5.1 shows that the function \( F^{(+;s_0,s_1)} \) belongs to \( \mathcal{R}_{2n-1,q}[\Pi^+; (s_j)_{j=0}^{2n-1}] \). \( \square \)

**Proposition 9.10.** Let \( \kappa \in \mathbb{Z}_{2,+\infty} \cup \{+\infty\} \), let \( (s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^e \), and let \( F \) belong to \( \mathcal{R}_{\kappa,q}[\Pi^+; (s_j)_{j=0}^{\kappa}] \). Then \( F^{(+;s_0,s_1)} \in \mathcal{R}_{\kappa-2,q}[\Pi^+; (s_j)_{j=0}^{\kappa-2}] \), where \( (s_j)_{j=0}^{\kappa-2} \) denotes the first Schur transform of \( (s_j)_{j=0}^{\kappa} \).

**Proof.** Because of \( (s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^e \), we have \( (s_j)_{j=0}^{m} \in \mathcal{H}_{q,m}^e \) for each \( m \in \mathbb{Z}_{0,\kappa} \). Remark 5.3 yields \( F \in \bigcap_{m=0}^{\kappa} \mathcal{R}_{m,q}[\Pi^+; (s_j)_{j=0}^{m}] \). Thus, Theorems 9.7 and 9.9 show then that \( F^{(+;s_0,s_1)} \) belongs to \( \bigcap_{m=0}^{\kappa-2} \mathcal{R}_{m,q}[\Pi^+; (s_j)_{j=0}^{m}] \). Consequently, from Remark 5.3 then \( F^{(+;s_0,s_1)} \in \mathcal{R}_{\kappa-2,q}[\Pi^+; (s_j)_{j=0}^{\kappa-2}] \) follows. \( \square \)
10. On the Inverse \((s_0, s_1)\)-Schur Transform for Special Subclasses of \(\mathcal{R}_q(\Pi_+^+)\)

Against to the background of Propositions 8.17 and 8.19 the considerations of Section 9 lead us to study three inverse problems which will be formulated now. Let \(m \in \mathbb{N}_0\), \((s_j)_{j=0}^m \in \mathcal{H}_{q,m}^e\) and \(G \in \mathcal{R}_{m,q}[\Pi_+;(s_j)_{j=0}^m]\). In the case \(m = 0\), it was shown in Theorem 9.1 that \(G(\pm; s_0)\) belongs to \(\mathcal{P}_q^{\text{even}}[s_0]\). Now we start with a function \(F \in \mathcal{P}_q^{\text{even}}[s_0]\) and will show in Proposition 10.1 that \(F(\pm; s_0)\) belongs to \(\mathcal{R}_{0,q}[\Pi_+;(s_j)_{j=0}^m]\) in the case \(m = 1\), Theorem 9.1 yields that \(G(\pm; s_0, s_1) \in \mathcal{P}_q^{\text{odd}}[s_0]\). For this reason, we will now consider a function \(F \in \mathcal{P}_q^{\text{odd}}[s_0]\) and show in Proposition 10.4 that \(F(\pm; s_0, s_1) \in \mathcal{R}_{1,q}[\Pi_+;(s_j)_{j=0}^m]\). Finally, we investigate the case \(m \in \mathbb{Z}_{2, +\infty}\). Let \((s_j)_{j=0}^m \in \mathcal{H}_{q,m}^e\) be the first Schur transform of \((s_j)_{j=0}^m\), then we know from Theorems 9.7 and 9.9 that \(G(\pm; s_0, s_1) \in \mathcal{R}_{m-2,q}[\Pi_+;(s_j)_{j=0}^m]\). So we are going to verify now that, for a function \(F \in \mathcal{R}_{m-2,q}[\Pi_+;(s_j)_{j=0}^m]\), its inverse \((s_0, s_1)-\text{Schur transform} \ F(\pm; s_0, s_1) \) belongs to \(\mathcal{R}_{m,q}[\Pi_+;(s_j)_{j=0}^m]\) (see Theorems 10.9 and 10.11).

Now we turn our attention to a detailed treatment of the case \(m = 0\). An application of Proposition 8.14 will provide us quickly the desired result.

**Proposition 10.1.** Let \(s_0 \in \mathcal{C}_{\geq 0}^{q \times q}\) and let \(F \in \mathcal{P}_q^{\text{even}}[s_0]\). Then \(F(\pm; s_0)\) belongs to \(\mathcal{R}_{0,q}[\Pi_+;(s_j)_{j=0}^0]\).

**Proof.** Since \(F\) belongs to \(\mathcal{P}_q^{\text{even}}[s_0]\), part (3) of Lemma 4.3 shows that the function \(F\) belongs to \(\mathcal{R}_q(\Pi_+)\) and that the conditions \(\beta_F = 0\) and 

\[
N(s_0) \subseteq N(\alpha_F) \cap N(\nu_F(\mathbb{R})) = N(\alpha_F) \cap N(\beta_F) \cap N(\nu_F(\mathbb{R}))
\]

are fulfilled. Setting \(B := 0_{q \times q}\) and \(t_0 := s_0(s_0 + \beta_F)^{-1} s_0\), we get \(t_0 = s_0\) and part (3) of Proposition 8.14 shows that \(F(\pm; s_0)\) belongs to \(\mathcal{R}_{0,q}[\Pi_+;(s_j)_{j=0}^0]\). \(\square\)

**Corollary 10.2.** Let \((s_j)_{j=0}^0 \in \mathcal{H}_{q,0}^{\geq e}\). For each \(F \in \mathcal{R}_{0,q}[\Pi_+;(s_j)_{j=0}^0]\), let

\[
(F(\pm; s_0))(F) := F(\pm; s_0).
\]

Then \(F(\pm; s_0)\) generates a bijective correspondence between the classes \(\mathcal{R}_{0,q}[\Pi_+;(s_j)_{j=0}^0]\) and \(\mathcal{P}_q^{\text{even}}[s_0]\). The inverse mapping \((F(\pm; s_0))^{-1}\) is given for each \(G \in \mathcal{P}_q^{\text{even}}[s_0]\) by

\[
(F(\pm; s_0))^{-1}(G) := G(\pm; s_0).
\]

**Proof.** Taking part (3) of Corollary 8.18 and part (3) of Corollary 8.21 into account, the combination of Theorem 9.1 and Proposition 10.1 yields all assertions. \(\square\)

Now we consider the case \(m = 1\). First we state a more general result, which is of own interest.
Proposition 10.3. Let $A \in \mathbb{C}^{q\times q}_\geq$ and $B \in \mathbb{C}^{q\times q}_H$ be such that $N(A) \subseteq N(B)$. Let $F \in \mathcal{R}_{q}^{-[-1]}(\Pi_+)$ be such that

$$N(A) \subseteq N(\gamma_F) \cap N(\mu_F(\mathbb{R})).$$

(10.1)

Then $F(-;A,B) \in \mathcal{R}_{1,q}[\Pi_+; (t_j^1)_{j=0}^1]$, where $t_0 := A$ and where $t_1 := B - \gamma_F$.

Proof. Since $F$ belongs to $\mathcal{R}_{q}^{-[-1]}(\Pi_+)$, we get from (3.5) that

$$F \in \mathcal{R}_{q}(\Pi_+).$$

(10.2)

and from (10.1) and Lemma 3.20 that

$$N(A) \subseteq N(F(z)) \quad \text{for each } z \in \Pi_+. \quad (10.3)$$

Using (10.2), (10.3) and Proposition 2.7, we obtain

$$N(A) \subseteq N(\alpha_F) \cap N(\beta_F) \cap N(\mu_F(\mathbb{R})). \quad (10.4)$$

In view of (10.2) and (10.4), we infer from part (d) of Proposition 8.14 then

$$F(-;A,B) \in \mathcal{R}_{q}(\Pi_+)$$

and from (8.21) that

$$\det\left(zI_q + A^\dagger [F(z) - B]\right) \neq 0 \quad \text{for each } z \in \Pi_+. \quad (10.5)$$

Using (10.2), (10.4), and Lemma 2.8 we get

$$A^\dagger AF = F. \quad (10.6)$$

Since $A$ and $B$ are Hermitian matrices with $N(A) \subseteq N(B)$, the application of Remark A.4 yields

$$AA^\dagger B = B. \quad (10.7)$$
Setting \( t_{-1} = 0_{q \times q} \) and using (5.4), (8.2), (10.5), (10.6), and (10.7), we get

\[
\begin{align*}
  z \left( (F^{(-;A,B)})_1^{(t)} (z) - [F(z) - \gamma_F] \right) \\
  = z^3 F^{(-;A,B)}(z) + z^3 t_{-1} + z^2 t_0 + z t_1 - z F(z) + z \gamma_F \\
  = -z^3 A \left( z I_q + A^\dagger [F(z) - B] \right) + z^2 A + z [B - F(z)] \\
  = -z^3 A \left( z I_q + A^\dagger [F(z) - B] \right)^{-1} + z^2 A + z [B - F(z)] \\
  = -z^2 A \left( I_q + \frac{1}{z} A^\dagger [F(z) - B] \right)^{-1} + z^2 A + z [B - F(z)] \\
  = \left[ -z^2 A + (z^2 A + z [B - F(z)]) \left( I_q + \frac{1}{z} A^\dagger [F(z) - B] \right) \right] \\
  \times \left( I_q + \frac{1}{z} A^\dagger [F(z) - B] \right)^{-1} \\
  = \left( z [B - F(z)] + z A A^\dagger [F(z) - B] + [B - F(z)] A^\dagger [F(z) - B] \right) \\
  \times \left( I_q + \frac{1}{z} A^\dagger [F(z) - B] \right)^{-1} \\
  = -[F(z) - B] A^\dagger [F(z) - B] \left( I_q + \frac{1}{z} A^\dagger [F(z) - B] \right)^{-1}.
\end{align*}
\]

This implies

\[
(F^{(-;A,B)})_1^{(t)} (z) = F(z) - \gamma_F - \frac{1}{z} [F(z) - B] A^\dagger [F(z) - B] \left( I_q + \frac{1}{z} A^\dagger [F(z) - B] \right)^{-1}. \tag{10.8}
\]

Since \( F \) belongs to \( \mathcal{R}_q^{[-1]}(\Pi_+) \), from Proposition 3.14 we see that (3.8) is true. From (3.8) we get

\[
I_q = I_q + 0 \cdot A^\dagger (\gamma_F - B) = I_q + \left( \lim_{y \to +\infty} \frac{1}{iy} \right) \left( A^\dagger \left[ \lim_{y \to +\infty} F(iy) \right] - B \right) \\
= \lim_{y \to +\infty} \left( I_q + \frac{1}{iy} A^\dagger [F(iy) - B] \right)
\]

and, consequently,

\[
\lim_{y \to +\infty} \left[ \left( I_q + \frac{1}{iy} A^\dagger [F(iy) - B] \right)^{-1} \right] = I_q \tag{10.9}
\]
Using (3.8), (10.9), and (10.8), it follows

\[
0_{q} = \gamma_{F} - \gamma_{F} - 0 \cdot (\gamma_{F} - B)A(I_{q} - B) \cdot I_{q} = \\
\lim_{y \to +\infty} F(iy) - \gamma_{F} - \left( \lim_{y \to +\infty} \frac{1}{iy} \right) \left( \lim_{y \to +\infty} F(iy) - B \right) A^{\dagger} \left( \lim_{y \to +\infty} \left( I_{q} + \frac{1}{iy} A^{\dagger} [F(iy) - B]^{-1} \right) \right) = \\
\lim_{y \to +\infty} \left( F(iy) - \gamma_{F} - \frac{1}{iy} [F(iy) - B] A^{\dagger} [F(iy) - B] \right) \left( I_{q} + \frac{1}{iy} A^{\dagger} [F(iy) - B]^{-1} \right) = \\
\lim_{y \to +\infty} (F(-; A, B))_{1}^{(t)} (iy).
\]

(10.10)

In view of \( F \in \mathcal{R}_{q}^{-1} (\Pi_{+}) \), we get from Remark 3.10 that

\[
(\gamma_{F})^{*} = \gamma_{F}.
\]

(10.11)

Thus, we see from the definition of the sequence \( (t_{j})_{j=-1}^{1} \) that

\[
t_{j}^{*} = t_{j} \quad \text{for each } j \in \{-1, 0, 1\}.
\]

(10.12)

Hence, taking (10.10) and (10.12) into account, we infer from Theorem 6.6 that

\[
(F(-; A, B))_{1}^{(t)} \in \mathcal{R}_{q} (\Pi_{+}).
\]

(10.13)

Now we are going to show that

\[
(F(-; A, B))_{1}^{(t)} = \mathcal{R}_{q}^{-1} (\Pi_{+}).
\]

Since \( F \) belongs to \( \mathcal{R}_{q} (\Pi_{+}) \), the function \( \Theta : [1, +\infty) \to \mathbb{R} \) defined by

\[
\Theta(y) := \| \text{Re} \left[ [F(iy) - B] A^{\dagger} [F(iy) - B] \left( I_{q} + (iy)^{-1} A^{\dagger} [F(iy) - B]^{-1} \right) \right] \|.
\]

(10.14)

is continuous. From (3.8) and (10.9) we then get

\[
\| \text{Re} \left[ (\gamma_{F} - B)A(I_{q} - B)I_{q} \right] \| = \| \text{Re} \left[ \left( \lim_{y \to +\infty} F(iy) - B \right) A^{\dagger} \left( \lim_{y \to +\infty} F(iy) - B \right) \times \left( \lim_{y \to +\infty} \left( I_{q} + (iy)^{-1} A^{\dagger} [F(iy) - B]^{-1} \right) \right) \right] \| = \lim_{y \to +\infty} \Theta(iy).
\]

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Consequently, there is a non-negative real number \( c \) such that

\[
|\Theta(y)| \leq c \quad \text{for each } y \in [1, +\infty). \tag{10.15}
\]

Since \( F \in \mathcal{R}_q^{-1}(\Pi_+) \) is supposed, the function \( \Psi \colon [1, +\infty) \to \mathbb{R} \) given by

\[
\Psi(y) := \frac{1}{y} \| \text{Im}(iy) \| + \frac{c}{y^2} \tag{10.16}
\]

fulfills

\[
\Psi \in L^1 \left( [1, +\infty), \mathfrak{B}_{[1, +\infty)}, \lambda; \mathbb{R} \right), \tag{10.17}
\]

where \( \lambda \) is the Lebesgue measure defined on \( \mathfrak{B}_{[1, +\infty)} \). For all \( y \in [1, +\infty) \) from (10.13), (10.14), (10.15), and (10.16) we see that \( \Omega \colon [1, +\infty) \to \mathbb{R} \) given by

\[
\Omega(y) := \frac{1}{y} \| \text{Im}(F^{(-;A,B)}_1)(iy) \|
\]

satisfies, for \( y \in [1, +\infty) \), the inequality

\[
|\Omega(y)| = \frac{1}{y} \| \text{Im}(iy) \|
+ \frac{1}{y} \text{Im} \left[ (F(iy) - B) A^\dagger (F(iy) - B) \left( I_q + (iy)^{-1} A^\dagger [F(iy) - B] \right)^{-1} \right] \| 
\leq \frac{1}{y} \| \text{Im}(iy) \| + \Theta(y) \leq \Psi(y). \tag{10.18}
\]

Since (10.13) holds, the function \( \Omega \) is continuous and thus Borel-measurable. Hence, (10.18) and (10.17) yield that \( \Omega \) also belongs to \( L^1([1, +\infty), \mathfrak{B}_{[1, +\infty)}, \lambda; \mathbb{R}) \). Keeping in mind (10.13), this implies \( F^{(-;A,B)}_1 \in \mathcal{R}_q^{-1}(\Pi_+) \). Thus, combining this with (10.12), (10.2), and (10.10), we infer from Theorem 6.6 that \( F^{(-;A,B)} \in \mathcal{R}_q[\Pi_+; (t_j)_j^{-1}] \). In view of \( t_{-1} = 0_{q \times q} \) and Remark 5.1 this implies that \( F^{(-;A,B)} \in \mathcal{R}_q[\Pi_+; (t_j)_{j=0}^{1}] \).\( \square \)

The following result gives a complete answer to the case \( m = 1 \).

**Proposition 10.4.** Let \( (s_j)_{j=0}^{1} \in \mathcal{H}_{q,1}^\geq \) and let \( F \in \mathcal{P}_q^{\text{odd}}[s_0] \). Then \( F^{(-;s_0,s_1)} \) belongs to \( \mathcal{R}_q[\Pi_+; (s_j)_{j=0}^{1}] \).

**Proof.** Because of \( F \in \mathcal{P}_q^{\text{odd}}[s_0] \), we get from (1.12) that \( F \in \mathcal{R}_q^{-1}(\Pi_+) \) and \( N(s_0) \subseteq N(\mu_F(\mathbb{R})) \). Hence, (3.12) yields \( F \in \mathcal{R}_q^{-1}(\Pi_+) \) and \( \gamma_F = 0_{q \times q} \). Thus, we get \( N(s_0) \subseteq N(\gamma_F) \cap N(\mu_F(\mathbb{R})) \). Since \( (s_j)_{j=0}^{1} \) belongs to \( \mathcal{H}_{q,1}^\geq \), we see from Lemma 7.6 that \( s_0 \in \mathbb{C}_{\geq q}^\times \), \( s_1^\dagger = s_1 \), and \( N(s_0) \subseteq N(s_1) \). Applying Proposition 10.3 and using \( s_1 - \gamma_F = s_1 \), we conclude \( F^{(-;s_0,s_1)} \in \mathcal{R}_q[\Pi_+; (s_j)_{j=0}^{1}] \).\( \square \)
Corollary 10.5. Let \((s_j)_{j=0}^\infty \in \mathcal{H}_{q,1}^{\geq 0}\). For \(F \in \mathcal{R}_{1,q}[\Pi_+; (s_j)_{j=0}^\infty]\) let
\[
\mathcal{F}_{(+;s_0,s_1)}(F) := F^{(s_0,s_1)}.
\]
Then \(\mathcal{F}_{(+;s_0,s_1)}\) generates a bijective correspondence between the classes \(\mathcal{R}_{1,q}[\Pi_+; (s_j)_{j=0}^\infty]\) and \(\mathcal{P}_q^{\odd}[s_0]\). The inverse mapping \((\mathcal{F}_{(+;s_0,s_1)})^{-1}\) is given for \(G \in \mathcal{P}_q^{\odd}[s_0]\) by
\[
(\mathcal{F}_{(+;s_0,s_1)})^{-1}(G) := G^{(-;s_0,s_1)}.
\]

Proof. Taking part \((\text{c})\) of Corollary 8.18 and part \((\text{b})\) of Corollary 8.21 into account, the combination of Theorem 9.3 and Proposition 10.4 completes the proof.

Now we turn our attention to the case \(m \in \mathbb{Z}_{2,+\infty}\). Similar as in Section 9, we will use various Hamburger-Nevanlinna type results from Section 6. In order to prepare the application of this material, we still need some auxiliary results, which can be considered as analogues of Lemmas 9.3 and 9.6. First we will compute the functions introduced in Remark 5.9 for the case that the function \(F\) is replaced by \(F^{(-;s_0,s_1)}\).

Lemma 10.6. Let \(\mathcal{G}\) be a non-empty subset of \(\mathbb{C} \setminus \{0\}\), let \(F:\mathcal{G} \to \mathbb{C}^{p \times q}\) be a matrix-valued function, let \(\kappa \in \mathbb{N} \cup \{+\infty\}\), and let \((s_j)_{j=0}^\infty\) be a sequence of complex \(p \times q\) matrices. In the case \(\kappa \geq 2\) let \((s_j^{(1)})_{j=0}^{\kappa-2}\) be the first Schur transform of \((s_j)_{j=0}^\infty\). Further, let \(s_{-1} := 0_{p \times q}\), let \(s_{-1} := 0_{p \times q}\), let \(m \in \mathbb{Z}_{2,+\infty}\), and let \(\nabla_m : \mathcal{G} \to \mathbb{C}^{p \times q}\) be defined by
\[
\nabla_m(z) := \begin{cases} 
[F(z) - s_1]s_0^\dagger [F(z) - s_1] & \text{, if } m = 1 \\
F_m^{(s(1))}(z)s_0^\dagger [F(z) - s_1] - \sum_{j=0}^{m-1} z^{-j}s_{j+1}s_0^\dagger F_{m-2}^{(s(1))}(z) + s_m s_0^\dagger s_1 + \sum_{k=1}^{m-1} s_{m-k}s_0^\dagger s_{k-1}^{(1)} + \sum_{j=1}^{m-1} z^{-j} \sum_{k=j}^{m-1} s_{m+j-k}s_0^\dagger s_{k-1}^{(1)} & \text{, if } m \geq 2
\end{cases}.
\]
Suppose that \((s_j)_{j=0}^m \in \mathcal{D}_{p \times q,m}\) and that \(z \in \mathcal{G}\) is such that \(R(F(z)) \subseteq R(s_0)\) and
\[
\det \left( I_q + z^{-1}s_0^\dagger [F(z) - s_1] \right) \neq 0
\]
are fulfilled. Then
\[
(F^{(-;s_0,s_1)})_m(z) = F_m^{(s(1))}(z) - \frac{1}{z} \nabla_m(z) \left( I_q + z^{-1}s_0^\dagger [F(z) - s_1] \right)^{-1}.
\]

Proof. In view of Remark 5.9 we have
\[
z^{-(m-1)}F_m^{(s(1))}(z) - \sum_{k=0}^{m-1} z^{-k}s_k^{(1)} = F(z).
\]
The assumption $R(F(z)) \subseteq R(s_0)$ and part (1) of Remark \ref{rem:equation} yield $s_0s_0^\dagger F(z) = F(z)$. From \cite[Remark 8.5]{24} we know that \eqref{eq:29} holds true for all $j \in \mathbb{Z}_{-1,m-2}$. Thus, we get
\begin{equation}
  s_0s_0^\dagger F^{(s(1))}_{m-2}(z) = F^{(s(1))}_{m-2}(z).
\end{equation}
Because of $(s_j)_{j=0}^m \in D_{p \times q,m}$, Definition \ref{def:3} and part (3) of Remark \ref{rem:equation} the first equation in \eqref{eq:28} is fulfilled for all $j \in \mathbb{Z}_0,m$. Using $s_{-1} = 0_{p \times q}, s_{-1}^{(1)} = 0_{p \times q}, (5.9), (8.2), (10.20), (10.22)$, and \eqref{eq:23}, we conclude
\begin{align*}
  (F^{(−;s_0,s_1)}^{(s)})_{m}(z) &= z^{m+1} \left[ F^{(−;s_0,s_1)}(z) + \sum_{j=0}^{m+1} z^{-j}s_{j-1} \right] \\
  &= z^{m+1} \left[ -s_0 \left( zI_q + s_0^\dagger [F(z) - s_1] \right)^\dagger 
  + \sum_{j=0}^{m+1} z^{-j}s_{j-1} \left( zI_q + s_0^\dagger [F(z) - s_1] \right) \left( zI_q + s_0^\dagger [F(z) - s_1] \right)^{-1} \right] \\
  &= \left[ -z^{m+1}s_0 + \sum_{j=0}^{m+1} z^{m+1-j}s_{j-1} \left( zI_q + s_0^\dagger [F(z) - s_1] \right) \left( zI_q + s_0^\dagger [F(z) - s_1] \right)^{-1} \right] \times \left( zI_q + s_0^\dagger [F(z) - s_1] \right)^{-1} \\
  &= \left[ -z^m s_0 + \sum_{j=0}^{m+1} z^{m+1-j}s_{j-1} + \sum_{j=0}^{m+1} z^{-j+1}s_{j-1}s_0^\dagger F^{(s(1))}_{m-2}(z) \\
  &\quad - \sum_{j=0}^{m+1} \sum_{k=0}^{m-1} z^{m-(j+k)} s_{j-1}s_0^\dagger s_k^{(1)} - \sum_{j=0}^{m+1} z^{m-j}s_{j-1}s_0^\dagger \right] \times \left( zI_q + z^{-1}s_0^\dagger [F(z) - s_1] \right)^{-1}
\end{align*}
\[
\begin{align*}
&= -z^m s_0 + z^{m+1} s_{-1} + z^m s_0 + \sum_{j=2}^{m+1} z^{m+1-j} s_{j-1} + z^1 s_{-1} s_0 F_{m-2}^{(s^{(1)})} (z) \\
&+ z^0 s_0 s_0 F_{m-2}^{(s^{(1)})} (z) + \sum_{j=2}^{m+1} z^{-j+1} s_{j-1} s_0 F_{m-2}^{(s^{(1)})} (z) - \sum_{k=0}^{m-1} z^{m-k} s_{-1} s_0 s_{k-1} \\
&- \sum_{k=0}^{m-1} z^{m-k} s_0 s_{k-1} s_0 - \sum_{j=2}^{m+1} \sum_{k=0}^{m-1} z^{m-j-k} s_{j-1} s_0 s_{k-1} - z s_1 \\
&- \sum_{j=2}^{m+1} z^{m-j} s_{j-1} s_0 s_1 \left( I_q + z^{-1} s_0 [F(z) - s_1] \right)^{-1} \\
&\quad (10.24)
\end{align*}
\]

In the case \( m = 1 \), from \( s_{-1}^{(1)} = 0_{p \times q} \), and \( F_{m-2}^{(s^{(1)})} (z) = F(z) \), it follows

\[
(F^{(-s_0, s_1)})_{m}^{(s)} = (F^{(-s_0, s_1)})_{1}^{(s)}(z) = F(z) + z^{-1} s_1 s_0 [F(z) - z^{-1} s_1 s_0 s_1] \left( I_q + z^{-1} s_0 [F(z) - s_1] \right)^{-1}
\]

and, consequently,

\[
\begin{align*}
&z \left[ F_{m-2}^{(s^{(1)})} (z) - (F^{(-s_0, s_1)})_{m}^{(s)} (z) \right] \left( I_q + z^{-1} s_0 [F(z) - s_1] \right) \\
&= z \left[ F(z) - \left( F(z) + z^{-1} s_1 s_0 [F(z) - z^{-1} s_1 s_0 s_1] \left( I_q + z^{-1} s_0 [F(z) - s_1] \right)^{-1} \right) \right. \\
&\quad \times \left( I_q + z^{-1} s_0 [F(z) - s_1] \right) \\
&= F(z) [F(z) - s_1] - s_1 s_0 [F(z) - s_1] = \nabla_1 (z) = \nabla_m (z).
\]
Thus, (10.21) is proved in the case $m = 1$. If $m \geq 2$, then (10.24) and $s_{-1}^{(1)} = 0_{p \times q}$ yield
\[
(F^{(-; s_0, s_1)})^{(s)}_m (z) = \sum_{j=3}^{m+1} z^{-m+1-j} s_{j-1} F_{m-2}^{(s^{(1)})} (z) + \sum_{j=0}^{m-1} z^{-j} s_{j-1} s_0^\dagger F_{m-2}^{(s^{(1)})} (z) - z^{-1} s_1^\dagger s_0^\dagger (z) - s_1^\dagger s_0^\dagger (z) (I_q + z^{-1} s_0^\dagger [F(z) - s_1])^{-1}.
\]

Furthermore, if $m \geq 2$, then Proposition 8.23 shows that (9.33) is valid. Thus, in the case $m = 2$, from (10.25) and (9.38) we get
\[
(F^{(-; s_0, s_1)})^{(s)}_2 (z) = (F^{(-; s_0, s_1)})^{(s)}_2 (z) = [s_2 + F_{m-2}^{(s^{(1)})} (z) + \sum_{j=0}^{m-1} z^{-j-1} s_{j+1} s_0^\dagger F_{m-2}^{(s^{(1)})} (z) - s_0^\dagger (z) - z^{-1} s_1^\dagger s_0^\dagger (s_1) - z^{-2} s_2^\dagger s_0^\dagger (s_1)] (I_q + z^{-1} s_0^\dagger [F(z) - s_1])^{-1}
\]

and, consequently,
\[
(F^{(-; s_0, s_1)})^{(s)}_m (z) = [F_{m-2}^{(s^{(1)})} (z) + \sum_{j=0}^{m-1} z^{-j-1} s_{j+1} s_0^\dagger F_{m-2}^{(s^{(1)})} (z) - z^{-1} s_1^\dagger s_0^\dagger (s_1) - z^{-2} s_2^\dagger s_0^\dagger (s_1)] (I_q + z^{-1} s_0^\dagger [F(z) - s_1])^{-1}.
\]

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In case $m \geq 3$, we have

\[
\sum_{j=2}^{m+1} \sum_{k=1}^{m-1} z^{m-(j+k)} s_j^{-1} s_{0k}^{j+1} = \\
\sum_{j=2}^{m-1} \sum_{k=1}^{m-1} z^{m-(j+k)} s_j^{-1} s_{0k}^{j+1} + \sum_{k=1}^{m-1} z^{-k} s_m^{-1} s_{0k}^{0} + \sum_{k=1}^{m-1} z^{-(k-1)} s_m^{-1} s_{0k}^{0} \\
= 2m-2 \min\{m-1, l-2\} \sum_{l=3}^{m-1} \sum_{k=\max\{1, l-(m-1)\}}^{m-1} z^{m-l} s_{l-k}^{-1} s_{0k}^{l} + z^{-1} s_{m-1}^{-1} s_{00}^{0} \\
+ \sum_{k=2}^{m-1} z^{-k} s_m^{-1} s_{0k}^{0} + \sum_{k=1}^{m-1} z^{-(k+1)} s_m^{-1} s_{0k}^{0} + z^{-m} s_m^{-1} s_{0m-2}^{0} \\
= 2m-2 \min\{m+1, l\} - 2 \sum_{l=3}^{m-1} \sum_{k=\max\{m, l\}-(m-1)}^{m-1} z^{m-l} s_{l-k}^{-1} s_{0k}^{l} + z^{-1} s_{m-1}^{-1} s_{00}^{0} \\
+ \sum_{k=2}^{m-1} z^{-k} s_m^{-1} s_{0k}^{0} + \sum_{k=1}^{m-1} z^{-(k+1)} s_m^{-1} s_{0k}^{0} + z^{-m} s_m^{-1} s_{0m-2}^{0} \\
= m \sum_{l=3}^{m-2} \sum_{k=1}^{m-1} z^{m-l} s_{l-k}^{-1} s_{0k}^{l} + \sum_{l=m+1}^{m-1} \sum_{k=l-(m-1)}^{m-1} z^{m-l} s_{l-k}^{-1} s_{0k}^{l} \\
+ \sum_{k=2}^{m-1} \sum_{l=m+1}^{m-1} z^{-k} s_m^{-1} s_{0k}^{0} + \sum_{k=1}^{m-1} \sum_{l=m+1}^{m-1} z^{-(k+1)} s_m^{-1} s_{0k}^{0} + z^{-m} s_m^{-1} s_{0m-2}^{0} \\
= m \sum_{l=3}^{m} \left( z^{(l-2)-1} \sum_{k=0}^{l-1} s_{(l-2)-k}^{-1} s_{0k}^{l} \right) + m-3 \sum_{j=0}^{m-3} \left( z^{j-1} \sum_{k=j+2}^{m-1} s_{j+m-k}^{-1} s_{0k}^{j} \right) \\
+ \sum_{k=2}^{m-1} \sum_{l=m+1}^{m-1} z^{-k} s_m^{-1} s_{0k}^{0} + \sum_{k=1}^{m-1} \sum_{l=m+1}^{m-1} \left( z^{-(k+1)} s_m^{-1} s_{0k}^{0} + z^{-m} s_m^{-1} s_{0m-2}^{0} \right)
\]

(10.27)
and, taking into account (10.25) and (10.27), furthermore,

\[
(F^{(-s_0, s_1)})^{(m)}_{m-2}(z) = \left[ z^{m-2}s_2 + \sum_{j=4}^{m+1} z^{m+1-j}s_{j-1} + F^{(s_1^{(1)})}_{m-2}(z) + \sum_{j=0}^{m-1} z^{-j-1}s_{j+1}s_0^{\dagger} F^{(s_1^{(1)})}_{m-2}(z) \right] - z^{m-2}s_0^{(1)} - \sum_{k=2}^{m-1} z^{m-1-k}s_{k-1}^{(1)} - \sum_{l=3}^{m} z^{m-l} \left( \sum_{k=0}^{(l-2)-1} s_{(l-2)-k}s_0^{\dagger}s_k^{(1)} \right) \\
- \sum_{j=0}^{m-3} \left( z^{-j-1} \sum_{k=j+2}^{m-1} s_{j+m-k}s_0^{\dagger}s_k^{(1)} \right) - z^{-1}s_{m-1}s_0^{\dagger}s_1^{(1)} - \sum_{j=1}^{m-2} z^{-j-1}(s_{m-1}s_0^{\dagger}s_j^{(1)} + s_ms_0^{\dagger}s_{j-1}^{(1)}) - z^{-2}s_{s_m}s_0^{\dagger}s_{m-2} - z^{-1}s_1s_0^{\dagger}s_1 \\
- \sum_{j=3}^{m} z^{m-j}s_{j-1}s_0^{\dagger}s_1 - z^{-1}s_ms_0^{\dagger}s_1 \right] \left( I_q + z^{-1}s_0^{\dagger}[F(z) - s_1] \right)^{-1} \\
= \left[ z^{m-2}s_2 + \sum_{l=3}^{m} z^{m-l}s_{l-1} + F^{(s_1^{(1)})}_{m-2}(z) + \sum_{j=0}^{m-1} z^{-j-1}s_{j+1}s_0^{\dagger} F^{(s_1^{(1)})}_{m-2}(z) \right] - z^{m-2}(s_2 - s_1s_0^{\dagger}s_1) - \sum_{l=3}^{m} z^{m-l}s_{l-1} - \sum_{l=3}^{m} \left( z^{m-l} \sum_{k=0}^{(l-2)-1} s_{(l-2)-k}s_0^{\dagger}s_k^{(1)} \right) \right]
\]

(10.28)
Thus, in the case $m = 3$, from (10.28) we conclude

$$\left( F^{(-s_0,s_1)} \right)^{(s)}_{m-2}(z) = \left( F^{(-s_0,s_1)} \right)^{(s)}_{m-2}(z)$$

$$= \left[ \sum_{j=0}^{m-1} z^{-j} s_j F^{(s)}_{m-2}(z) - z^{-1} \sum_{k=2}^{m-1} s_{m-k} s_{k-1} \right]$$

$$\times \left( I_q + z^{-1} s_0 [F(z) - s_1] \right)^{-1}$$

and, consequently, that (10.26) holds true. If $m \geq 4$, then (10.28) implies

$$\left( F^{(-s_0,s_1)} \right)^{(s)}_{m}(z)$$

$$= \left[ \sum_{j=0}^{m-1} z^{-j} s_j F^{(s)}_{m-2}(z) - z^{-1} \sum_{k=2}^{m-1} s_{m-k} s_{k-1} \right]$$

$$\times \left( I_q + z^{-1} s_0 [F(z) - s_1] \right)^{-1}$$

and, consequently, (10.26). Hence, (10.29) holds true if $m \geq 2$. Thus, in the case $m \geq 2$,
we get from (10.20) and (10.19) that
\[
\begin{align*}
  z \left[ F^{(s)}_{m-2}(z) - (F^{(-s_0,s_1)}_{m})_{m}(z) \right] & \left( I_q + z^{-1}s_0^\dagger [F(z) - s_1] \right) \\
  = z \left[ F^{(s)}_{m-2}(z) - \left( F^{(s)}_{m-2}(z) + \sum_{j=0}^{m-1} z^{-j-1}s_{j+1}s_0^\dagger F^{(s)}_{m-2}(z) \right) \right] \\
  &- z^{-1} \left( s_ms_0^\dagger s_1 + \sum_{k=1}^{m-1} s_{m-k}s_0^\dagger s_{k-1}^{(1)} \right) - \sum_{j=1}^{m-1} \left( z^{-j-1} \sum_{k=j}^{m-1} s_{m+j-k}s_0^\dagger s_{k-1}^{(1)} \right) \\
  \times \left( I_q + z^{-1}s_0^\dagger [F(z) - s_1] \right)^{-1} \left( I_q + z^{-1}s_0^\dagger [F(z) - s_1] \right) \\
  = zF^{(s)}_{m-2}(z) + F^{(s)}_{m-2}(z)s_0^\dagger [F(z) - s_1] - zF^{(s)}_{m-2}(z) \\
  - \sum_{j=0}^{m-1} z^{-j}s_{j+1}s_0^\dagger F^{(s)}_{m-2}(z) + s_ms_0^\dagger s_1 + \sum_{k=1}^{m-1} s_{m-k}s_0^\dagger s_{k-1}^{(1)} \\
  - \sum_{j=1}^{m-1} \left( z^{-j} \sum_{k=j}^{m-1} s_{m+j-k}s_0^\dagger s_{k-1}^{(1)} \right) = \nabla_m(z)
\end{align*}
\]
and, hence, (10.21) are fulfilled. \(\square\)

Now we study the asymptotic behaviour of the function \(\nabla_m\), which was introduced in (10.19).

**Lemma 10.7.** Let \(\theta \in [0, 2\pi)\) and let \(G\) be a of \(\mathbb{C} \setminus \{0\}\) with \(\{e^{i\theta}y|y \in [1, +\infty)\}\) \(\subseteq G\). Let \(F: G \rightarrow \mathbb{C}^{p \times q}\) be a matrix-valued function, let \(\kappa \in \mathbb{N} \cup \{+\infty\}\), let \((s_j)_{j=0}^{\kappa-1}\) be a sequence of complex \(p \times q\) matrices. In the case \(\kappa \geq 2\), let \((s_j^{(1)})_{j=0}^{\kappa-2}\) be the first Schur transform of \((s_j)_{j=0}^{\kappa-1}\). Let \(s_{-1} := 0_{p \times q}\), let \(s_{-1}^{(1)} := 0_{p \times q}\), let \(m \in \mathbb{Z}_{1,\kappa}\) and let \(\nabla_m: G \rightarrow \mathbb{C}^{p \times q}\) be defined by (10.19). Suppose
\[
\lim_{r \rightarrow +\infty} F^{(s_{-1}^{(1)})}_{m-2}(e^{i\theta}r) = 0_{p \times q}.
\] (10.30)

Then
\[
\lim_{r \rightarrow +\infty} \nabla_m(e^{i\theta}r) = \begin{cases} 
  s_1s_0^\dagger s_1, & \text{if } m = 1 \\
  s_ms_0^\dagger s_1 + \sum_{k=1}^{m-1} s_{m-k}s_0^\dagger s_{k-1}^{(1)}, & \text{if } m \geq 2.
\end{cases}
\] (10.31)

**Proof.** Because of \(s_{-1}^{(1)} := 0_{p \times q}\), from (10.9) we have \(F^{(s_{-1}^{(1)})}_{-1}(z) = F(z)\) for all \(z \in G\). Thus, in the case \(m = 1\), from (10.30) and (10.19) we see that
\[
\begin{align*}
  s_1s_0^\dagger s_1 &= \left( \lim_{r \rightarrow +\infty} F^{(s_{-1}^{(1)})}_{m-2}(e^{i\theta}r) - s_1 \right) s_0^\dagger \left( \lim_{r \rightarrow +\infty} F^{(s_{-1}^{(1)})}_{m-2}(e^{i\theta}r) - s_1 \right) \\
  &= \lim_{r \rightarrow +\infty} \left[ F(e^{i\theta}r) - s_1 \right] s_0^\dagger \left[ F(e^{i\theta}r) - s_1 \right] = \lim_{r \rightarrow +\infty} \nabla_m(e^{i\theta}r)
\end{align*}
\]
holds true. Consequently, if \( m = 1 \), then (10.31) is checked. Now assume that \( m \geq 2 \). Then we first observe that Remark 5.9 shows that

\[
F(z) = F_{m-2}^{(s^{(1)})}(z) = z^{1-m} F_{m-2}^{(s^{(1)})}(z) - \sum_{j=0}^{m-2} z^{-j+1} s_{m-2-j}
\]  

(10.32)

for all \( z \in G \). Using (10.30) and (10.32), we conclude

\[
0_p \times q = \left[ \lim_{r \to +\infty} (e^{i \theta} r)^{1-m} \right] \left[ \lim_{r \to +\infty} F_{m-2}^{(s^{(1)})}(e^{i \theta} r) \right] - \sum_{j=0}^{m-2} \left[ \lim_{r \to +\infty} (e^{i \theta} r)^{j-m+1} \right] s_{m-2-j}
\]  

(10.33)

Thus, in view of (10.30), (10.33), and (10.19), we get then

\[
s_m s_0^\dagger s_1 + \sum_{k=1}^{m-1} s_{m-k} s_0^\dagger s_{k-1} = \left[ \lim_{r \to +\infty} F_{m-2}^{(s^{(1)})}(e^{i \theta} r) \right] s_0^\dagger \left[ \lim_{r \to +\infty} F(e^{i \theta} r) - s_1 \right] - s_1 s_0^\dagger \left[ \lim_{r \to +\infty} F_{m-2}^{(s^{(1)})}(e^{i \theta} r) \right]
\]

\[
- \sum_{j=1}^{m-1} \left[ \lim_{r \to +\infty} (e^{i \theta} r)^{j-m+1} \right] s_{j+1} s_0^\dagger \left[ \lim_{r \to +\infty} F_{m-2}^{(s^{(1)})}(e^{i \theta} r) \right] + s_m s_0^\dagger s_1 + \sum_{k=1}^{m-1} s_{m-k} s_0^\dagger s_{k-1} + \sum_{j=1}^{m-1} \left[ \lim_{r \to +\infty} (e^{i \theta} r)^{j-m+1} \right] \sum_{k=j}^{m-1} s_{m+j-k} s_0^\dagger s_{k-1}
\]

\[
= \lim_{r \to +\infty} \left( F_{m-2}^{(s^{(1)})}(e^{i \theta} r) s_0^\dagger \left[ F(e^{i \theta} r) - s_1 \right] - s_1 s_0^\dagger F_{m-2}^{(s^{(1)})}(e^{i \theta} r) \right)
\]

\[
- \sum_{j=1}^{m-1} (e^{i \theta} r)^{j-m+1} s_{j+1} s_0^\dagger F_{m-2}^{(s^{(1)})}(e^{i \theta} r) + s_m s_0^\dagger s_1
\]

\[
+ \sum_{k=1}^{m-1} s_{m-k} s_0^\dagger s_{k-1} + \sum_{j=1}^{m-1} \left[ (e^{i \theta} r)^{j-m+1} \sum_{k=j}^{m-1} s_{m+j-k} s_0^\dagger s_{k-1} \right]
\]

\[
= \lim_{r \to +\infty} \left( F_{m-2}^{(s^{(1)})}(e^{i \theta} r) s_0^\dagger \left[ F(e^{i \theta} r) - s_1 \right] - \sum_{j=1}^{m-1} (e^{i \theta} r)^{j-m+1} s_{j+1} s_0^\dagger F_{m-2}^{(s^{(1)})}(e^{i \theta} r) \right)
\]

\[
+ s_m s_0^\dagger s_1 + \sum_{k=1}^{m-1} s_{m-k} s_0^\dagger s_{k-1} + \sum_{j=1}^{m-1} \left[ (e^{i \theta} r)^{j-m+1} \sum_{k=j}^{m-1} s_{m+j-k} s_0^\dagger s_{k-1} \right]
\]

\[
= \lim_{r \to +\infty} \nabla_m (e^{i \theta} r).
\]

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In preparing the next result, we remember to part (b) of Proposition 5.4.

**Lemma 10.8.** Let \( m \in \mathbb{Z}_{2,+\infty} \), let \((s_j)_{j=0}^m \in \mathcal{H}_{q,m}^e \), let \((s_j^{(1)})_{j=0}^{m-2} \) be the first Schur transform of \((s_j)_{j=0}^m \) and let \( F \in \mathcal{R}_{m-2,q}[\Pi_+; (s_j^{(1)})_{j=0}^{m-2}] \). Then:

(a) \( F \in \mathcal{P}_{q}^{\text{odd}}[s_0] \).

(b) \( F^{(-;s_0,s_1)} \in \mathcal{R}_{1,q}(\Pi_+) \).

(c) For each \( z \in \Pi_+ \) the inclusion \( R(F(z)) \subseteq R(s_0) \) holds.

**Proof.** (a) Lemma 5.8 yields \( F \in \mathcal{P}_{q}^{\text{odd}}[s_0] \).

From (7.1) we get \( N(s_0) \subseteq N(s_0^{(1)}) \). Thus, Remark 4.6 yields \( \mathcal{P}_{q}^{\text{odd}}[s_0] \subseteq \mathcal{P}_{q}^{\text{odd}}[s_0] \).

(b) From \((s_j)_{j=0}^m \in \mathcal{H}_{q,m}^e \) and \( m \geq 2 \) we get \((s_j)_{j=0}^1 \in \mathcal{H}_{q,1}^e \). Hence, Proposition 10.4 implies \( F^{(-;s_0,s_1)} \in \mathcal{R}_{1,q}[\Pi_+; (s_j)_{j=0}^1] \). Thus, from (5.3) and Remark 3.24 we obtain (b).

(c) In view of (a) we infer from part (b) of Lemma 4.3 then \( N(s_0) \subseteq N(F(z)) \), \( z \in \Pi_+ \).

From part (b) of Lemma 7.6 we obtain

\[
s_0^* = s_0.
\]

Because of (6.3) and Remark 6.24 we have \( F \in \mathcal{R}_{q}(\Pi_+) \). Combining this with (10.34) and (10.38) the application of Lemma 2.8 yields (c).

**Theorem 10.9.** Let \( n \in \mathbb{N} \), let \((s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \), let \((s_j^{(1)})_{j=0}^{2n-2} \) be the first Schur transform of \((s_j)_{j=0}^{2n} \) and let \( F \in \mathcal{R}_{2n-2,q}[\Pi_+; (s_j^{(1)})_{j=0}^{2n-2}] \). Then \( F^{(-;s_0,s_1)} \) belongs to \( \mathcal{R}_{2n,q}[\Pi_+; (s_j^{(1)})_{j=0}^{2n}] \).

**Proof.** The strategy of our proof is based on an application of Theorem 6.4. Let \( m := 2n \) and let

\[
s_{-1} := 0_{q \times q}.
\]

From (10.37), \((s_j)_{j=0}^m \in \mathcal{H}_{q,m}^e \) and part (b) of Lemma 7.6 we get

\[
s_j^* = s_j \quad \text{for each } j \in \mathbb{Z}_{-1,m}.
\]

Part (b) of Lemma 10.8 yields

\[
F^{(-;s_0,s_1)} \in \mathcal{R}_{q}(\Pi_+).
\]

Let \( s_{-1}^{(1)} := 0_{q \times q} \). Then, in view of \( F \in \mathcal{R}_{m-2,q}[\Pi_+; (s_j^{(1)})_{j=0}^{m-2}] \), we infer from Remark 5.1 that

\[
F \in \mathcal{R}_{q}[m-2]\left[\Pi_+; (s_j^{(1)})_{j=-1}^{m-2}\right].
\]

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Thus, Corollary [3.3] shows that
\[
\lim_{y \to +\infty} F_{m-2}^{(s(1))}(iy) = 0_{q \times q}. \tag{10.41}
\]

Setting
\[
G := s_m s_0 \hat{s}_1 + \sum_{k=1}^{m-1} s_{m-k} s_0 \hat{s}_k, \tag{10.42}
\]
we get from (10.41) and Lemma [10.7] then
\[
G = \lim_{r \to +\infty} \nabla_m(e^{i \pi r}) = \lim_{y \to +\infty} \nabla_m(iy), \tag{10.43}
\]
where \(\nabla_m\) is defined in (10.19). From the choice of \(F\) and (5.3) we conclude that
\(F \in \mathcal{R}_{m-2,q}(\Pi_+).\) Thus, Remark [3.24] yields (3.13). Consequently,
\[
I_q = I_q + 0 \cdot s_0^\dagger(0_{q \times q} - s_1) = \lim_{y \to +\infty} \left( I_q + (iy)^{-1} s_0^\dagger [F(iy) - s_1] \right), \tag{10.44}
\]
which implies
\[
\lim_{y \to +\infty} \left( I_q + (iy)^{-1} s_0^\dagger [F(iy) - s_1] \right)^{-1} = I_q. \tag{10.45}
\]

Now we are going to apply Lemma [10.6]. In view of \((s_j)_{j=0}^n \in \mathcal{H}_{q,m}^{e,e}\) and Lemma [7.9] we get \((s_j)_{j=0}^n \in \mathcal{D}_{q \times q,m}.\) Part (c) of Lemma [10.8] yields
\[
R(F(z)) \subseteq R(s_0) \quad \text{for each } z \in \Pi_+. \tag{10.46}
\]

In view of \((s_j)_{j=0}^n \in \mathcal{H}_{q,m}^{e,e}\) we infer from parts (a), (b) and (c) of Lemma [7.6] then \(s_0 \in \mathbb{C}^{q \times q}, s_1^\dagger = s_1\) and \(N(s_0) \subseteq N(s_1).\) Thus, part (ii) of Proposition [8.14] yields
\[
\det \left( I_q + z^{-1} s_0^\dagger [F(iy) - s_1] \right) = z^{-q} \det \left( z I_q + s_0^\dagger [F(iy) - s_1] \right) \neq 0 \quad \tag{10.47}
\]
for all \(z \in \Pi_+.\) Since \((s_j)_{j=0}^n\) belongs to \(\mathcal{D}_{q \times q,m},\) from (10.46), (10.47) and Lemma [10.6] we then conclude that
\[
(F_{-s_0,s_1})_{m}^{(s_j)}(z) = F_{m-2}^{(s(1))}(z) - \frac{1}{z} \nabla_m(z) \left( I_q + z^{-1} s_0^\dagger [F(iy) - s_1] \right)^{-1} \tag{10.48}
\]
for all \(z \in \Pi_+.\) By virtue of (10.41), (10.43), and (10.45), from (10.48) we see that
\[
0_{q \times q} = 0_{q \times q} - 0 \cdot G \cdot I_q
\]
holds true. Taking (10.43), (10.49), and \(m = 2n\) into account, Theorem [6.4] then yields that \(F_{-s_0,s_1}^{(s_j)}\) belongs to \(\mathcal{R}_{q}^{2n}[\Pi_+; (s_j)_{j=0}^{2n}].\) In view of (10.37), then Remark [5.1] shows that the function \(F_{-s_0,s_1}^{(s_j)}\) also belongs to \(\mathcal{R}_{2n,q}[\Pi_+; (s_j)_{j=0}^{2n}].\) \(\square\)
Corollary 10.10. Let \((s_j)^\infty_{j=0} \in \mathcal{H}_{\mathbb{Q},\infty}^\geq\), let \((s_j^{(1)})_j^{\infty}_{j=0}\) be the first Schur transform of 
\((s_j)^\infty_{j=0}\) and let \(F \in \mathcal{R}_{\infty,q}[\Pi_+; (s_j^{(1)})_j^{\infty}_{j=0}]\). Then 
\(F^{(-s_0,s_1)}\) belongs to \(\mathcal{R}_{\infty,q}[\Pi_+; (s_j)_j^{\infty}_{j=0}]\).

Proof. Combine Remarks 5.3 and 7.1 and Theorem 10.9.

Theorem 10.11. Let \(n \in \mathbb{N}\), let \((s_j)^{2n+1}_{j=0} \in \mathcal{H}_{\mathbb{Q},2n+1}^\geq\), let \((s_j^{(1)})^{2n-1}_{j=0}\) be the first Schur transform of 
\((s_j)^{2n+1}_{j=0}\) and let \(F \in \mathcal{R}_{2n-1,q}[\Pi_+; (s_j^{(1)})^{2n-1}_{j=0}]\). Then \(F^{(-s_0,s_1)}\) belongs to 
\(\mathcal{R}_{2n-1,q}[\Pi_+; (s_j)^{2n+1}_{j=0}]\).

Proof. The strategy of our proof is based on an application of Theorem 6.6. Let \(m := 2n + 1\) and \(s_{-1} := 0_{q \times q}\). Then, from \((s_j)^m_{j=0} \in \mathcal{H}_{\mathbb{Q},m}^\geq\) and part (i) of Lemma 7.6 we get \((10.38)\). Part (i) of Lemma 10.8 yields \((10.39)\). Let \(s^{(1)}_{-1} := 0_{q \times q}\). In view of \(F \in \mathcal{R}_{m-2,q}[\Pi_+; (s^{(1)})^{m-2}_{j=0}]\), we infer then from Remark 5.3 that \((10.40)\) holds. In view of \((10.40)\), the application of Corollary 6.3 shows that \((10.41)\) is true. Let \(G\) be defined by \((10.42)\). Then we get from \((10.41)\) and Lemma 10.7 then \((10.43)\), where \(\nabla_m\) is defined in \((10.19)\). From the choice of \(F\) and \((5.3)\) we know that \(F \in \mathcal{R}_{m-2,q}[\Pi_+]\). Thus, Remark 5.3 implies that \((5.3)\) holds true. Consequently, \((10.44)\) is true, which implies \((10.45)\). Now we are going to apply Lemma 10.6. In view of \((s_j)^m_{j=0} \in \mathcal{H}_{\mathbb{Q},m}^\geq\) and Lemma 7.2 we get \((s_j)^m_{j=0} \in \mathcal{D}_{q \times q,m}\). Part (ii) of Lemma 10.8 yields \((10.46)\) and \((10.47)\), we then conclude from Lemma 10.6 that

\[
(F^{(-s_0,s_1)}(s))^{2n+1}_{j=0}(z) = F^{(s^{(1)})}_{2n-1}(z) - \frac{1}{z} \nabla_{2n+1}(z) \left(I_q + z^{-1}s_0^1 [F(z) - s_1]\right)^{-1}
\]

(10.50)

for all \(z \in \Pi_+\). By virtue of \((10.41)\), \((10.43)\) and \((10.45)\), we see that \((10.49)\) holds true. Taking \((10.38)\), \((10.39)\), \((10.49)\), and \(m = 2n + 1\) into account, we get from Theorem 6.6 then \((F^{(-s_0,s_1)}(s))^{2n+1}_{j=0} \in \mathcal{R}_q[\Pi_+]\) follows. In particular, \((F^{(-s_0,s_1)}(s))^{2n+1}_{j=0}\) is holomorphic in \(\Pi_+\). This shows us the function \(\Omega: [1, +\infty) \rightarrow \mathbb{R}\) defined by

\[
\Omega(y) := \frac{1}{y} \left\| \text{Im} \left( F^{(-s_0,s_1)}(s) \right)_{2n+1}(iy) \right\|
\]

is continuous. Because of the continuity of the holomorphic function \(F\), \((5.9)\), \((10.19)\), \((10.47)\) and \(m = 2n + 1\), the function \(\Theta: [1, +\infty) \rightarrow \mathbb{R}\) given by

\[
\Theta(y) := \left\| \text{Re} \left[ \nabla_{2n+1}(iy) \left( I_q + (iy)^{-1}s_0^1 [F(iy) - s_1]\right)^{-1} \right] \right\|
\]

(10.51)

is continuous. In view of \((10.43)\), \((10.45)\), \(m = 2n + 1\), and \((10.51)\), we conclude

\[
\|\text{Re} G\| = \|\text{Re}(G \cdot I_q)\|
\]

\[
= \left\| \text{Re} \left[ \lim_{y \rightarrow +\infty} \nabla_{2n+1}(iy) \left( \lim_{y \rightarrow +\infty} \left( I_q + (iy)^{-1}s_0^1 [F(iy) - s_1]\right)^{-1} \right) \right] \right\|
\]

\[
= \lim_{y \rightarrow +\infty} \Theta(y).
\]

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Consequently, there is a real number $c$ such that $|\Theta(y)| \leq c$ for all $y \in [1, +\infty)$. In view of (10.40) and $m = 2n + 1$, we see from part (iii) of Proposition 5.12 that $F_{2n-1}^{(s)}$ belongs to $\mathcal{R}_{-1,q}(\Pi_+)$ and (see 3.12), in particular, to $\mathcal{R}_{q}^{-1}(\Pi_+)$. This means that $F_{2n-1}^{(s)} \in \mathcal{R}_{q}(\Pi_+)$ and that $\Phi: [1, +\infty) \to \mathbb{R}$ defined by $\Phi(y) := \frac{1}{y} \|\text{Im} F_{2n-1}^{(s)}(iy)\|$ belongs to $\mathcal{L}^1([1, +\infty), \mathcal{M}_{[1, +\infty)}, \lambda; \mathbb{R})$, where $\lambda$ is the Lebesgue measure defined on $\mathcal{M}_{[1, +\infty)}$. Hence, $\Psi: [1, +\infty) \to \mathbb{R}$ given by $\Psi(y) := \Phi(y) + \frac{c}{y}$ belongs to $\mathcal{L}^1([1, +\infty), \mathcal{M}_{[1, +\infty)}, \lambda; \mathbb{R})$. Since $(F^{(-;s_0,s_1)})^{(s)}_{2n+1}$ belongs to the class $\mathcal{R}_q(\Pi_+)$, the function $\Lambda: [1, +\infty) \to \mathbb{R}$ given by

$$\Lambda(y) := \frac{1}{y} \|\text{Im} (F^{(-;s_0,s_1)}_{2n+1}(iy))\|$$

is continuous and, because of (10.50), for all $y \in [1, +\infty)$ it satisfies

$$|\Lambda(y)| \leq \frac{1}{y} \left(\|\text{Im} F_{2n-1}^{(s)}(iy)\| + \|\text{Im} \left[\frac{1}{iy} \nabla_{2n+1}(iy) \left(I_q + \frac{1}{iy} s_0 [F(iy) - s_1]^{-1}\right)\right]\right)$$

$$= \frac{1}{y} \|\text{Im} F_{2n-1}^{(s)}(iy)\| + \frac{1}{y^2} \|\text{Re} \left[\nabla_{2n+1}(iy) \left(I_q + \frac{1}{iy} s_0 [F(iy) - s_1]^{-1}\right)\right]\|$$

$$= \Phi(y) + \frac{1}{y^2} \Theta(y) \leq \Psi(y).$$

Thus, the function $\Lambda$ also belongs to $\mathcal{L}^1([1, +\infty), \mathcal{M}_{[1, +\infty)}, \lambda; \mathbb{R})$. Since $(F^{(-;s_0,s_1)})^{(s)}_{2n+1}$ is a member of the class $\mathcal{R}_q(\Pi_+)$, we then see from (10.52) that

$$(F^{(-;s_0,s_1)})^{(s)}_{2n+1} \in \mathcal{R}_q^{-1}(\Pi_+).$$

Taking (10.38), (10.39), (10.49), and $m = 2n + 1$ into account, Theorem 6.3 implies

$$(F^{(-;s_0,s_1)})^{(s)}_{2n+1} \in \mathcal{R}_{q}[2n+1][\Pi_+; (s_j)_{j=-1}^{2n+1}].$$

In view of $s_{-1} := 0_{q \times q}$ and (10.38), Remark 5.1 yields finally

$$(F^{(-;s_0,s_1)})^{(s)}_{2n+1} \in \mathcal{R}_{q}[2n+1][\Pi_+; (s_j)_{j=0}^{2n+1}]. \quad \square$$

**Corollary 10.12.** Let $m \in \mathbb{Z}_{2, +\infty} \cup \{+\infty\}$, let $(s_j)_{j=0}^m \in \mathcal{H}_{q,m}^{r,e}$ and let $(s_j)_{j=0}^{m-2}$ be the first Schur transform of $(s_j)_{j=0}^m$. Denote $F_{(+;s_0,s_1)}(s_j)_{j=0}^m$ the bijective mapping defined in Corollary 10.3. Then $F_{(+;s_0,s_1)}(s_j)_{j=0}^m$ generates a bijective correspondence between the sets $\mathcal{R}_{m,q}[\Pi_+; (s_j)_{j=0}^m]$ and $\mathcal{R}_{m-2,q}[\Pi_+; (s_j)_{j=0}^{m-2}]$. The inverse mapping $(F_{(+;s_0,s_1)})^{-1}$ is given for $G \in \mathcal{R}_{m-2,q}[\Pi_+; (s_j)_{j=0}^{m-2}]$ by $(F_{(+;s_0,s_1)})^{-1}(G) = G^{(-;s_0,s_1)}.$

**Proof.** We will consider the cases of even $m$, odd $m$ and $m = \infty$. In any of these cases, we can apply Corollary 8.13 and part (i) of Corollary 8.21 to verify the shape of the inverse mapping. If $m$ is even, then the assertion follows from Theorems 9.7 and 10.9. If $m$ is odd, then the application of Theorems 9.9 and 10.11 yields the desired result. Finally, if $m = \infty$, then one has to apply Corollaries 9.8 and 10.10. \square
We mention that the investigations in Sections 9 and 10 were influenced to some extent by considerations in Chen/Hu [9]. In particular, [9, Lemma 2.6] played an essential role for the choice of our strategy. This concerns first the development of a Schur type algorithm for sequences of complex matrices and then the construction of an interrelated Schur type algorithm for functions belonging to special subclasses of $\mathcal{R}_q(\Pi_+)$. The contents of [9, Lemma 2.6] are covered by Theorems 9.7 and 10.9. The method of Chen/Hu to prove [9, Lemma 2.6] strongly differs from our approach. It uses results on generalized Bezoutians to Anderson/Jury [3] and Gekhtman/Shmoish [25].

11. A Schur-Nevanlinna Type Algorithm for the Class $\mathcal{R}_{\kappa,q}(\Pi_+)$

The results of Section 9 suggest the construction of a Schur-Nevanlinna type algorithm for the class $\mathcal{R}_{\kappa,q}(\Pi_+)$ with $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$. The main theme of this section is to work out the details of this algorithm. In Section 9 we fixed an $m \in \mathbb{N}_0$ and a sequence $(s_j)_{j=0}^m \in H_{q,m}^\ell$. Proposition 5.4 tells us then that $R_{m,q}[\Pi_+; (s_j)_{j=0}^m] \neq \emptyset$. Let $F \in R_{m,q}[\Pi_+; (s_j)_{j=0}^m]$. Then the central theme of Section 9 was to study the $(s_0, q \times q)$-Schur transform $F(+; s_0)$ of $F$ and furthermore, in case $m \geq 1$, the $(s_0, s_1)$-Schur transform $F(+; s_0, s_1)$ of $F$. The following observation shows that in the case $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ the results of Section 9 can be applied to arbitrary functions belonging to $\mathcal{R}_{\kappa,q}(\Pi_+)$.  

**Lemma 11.1.** Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. Then $\sigma_F \in M_{q, \kappa}(\mathbb{R})$, $(s_j^{[\sigma_F]})_{j=0}^\kappa \in H_{q, \kappa}^{\ell, e}$ and 

$$F \in \mathcal{R}_{\kappa,q}[\Pi_+; (s_j^{[\sigma_F]})_{j=0}^\kappa].$$

**Proof.** From (3.12) and (3.11) we see that $\sigma_F \in M_{q, \kappa}(\mathbb{R})$. Thus, from (5.3) we infer now $F \in \mathcal{R}_{\kappa,q}[\Pi_+; (s_j^{[\sigma_F]})_{j=0}^\kappa]$. Hence, part (b) of Proposition 5.4 yields $(s_j^{[\sigma_F]})_{j=0}^\kappa \in H_{q, \kappa}^{\ell, e}$.

In view of Lemma 11.1 we introduce the following construction of Schur-Nevanlinna transforms (shortly SN-transforms) for functions belonging to the class $\mathcal{R}_{\kappa,q}(\Pi_+)$ with $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$.

**Definition 11.2.** Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. 

(a) The function $F^{(0)} := F$ is called the 0-step SN-transform of $F$.

(b) The function $F^{(1)} := F(+; s_0^{[F]})$ is called the 1-step SN-transform of $F$.

(c) In the case $\kappa \in \mathbb{N} \cup \{+\infty\}$, the function $F^{(2)} := F(+; s_0^{[F]}, s_1^{[F]})$ is called the 2-step SN-transform of $F$.

We relate now the just introduced notions to the inverse Schur transforms studied in Section 8.
and second assertion, respectively.

**Proof.**

**Proposition 11.4.** Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$.

From Lemma 11.1 and parts 11 and 12 of Corollary 8.18 one can see then that $(F^{(1)})^{-1;0,s_1} = F$ and, in the case $\kappa \geq 1$, furthermore

$$(F^{(2)})^{-1;0,[\sigma_{F}],s_1} = F.$$  

**Proposition 11.5.** Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. Then $F^{(1)} \in \mathcal{P}^\text{even}_{\kappa,q}[0,[\sigma_{F}]]$ and, in the case $\kappa \in \mathbb{N} \cup \{+\infty\}$, $F^{(2)} \in \mathcal{P}^\text{odd}_{\kappa,q}[0,[\sigma_{F}]]$.

**Proof.** In view of Lemma 11.1 the application of Corollaries 9.2 and 9.4 yields the first and second assertion, respectively.

**Proposition 11.6.** Let $\kappa \in \mathbb{Z}_{2,+\infty} \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. Let $(s_j^{(1)})_{j=0}^{\kappa-2}$ be the first Schur transform of $(s_j^{[\sigma_{F}]}),_{j=0}^{\kappa}$. Then $F^{(2)} \in \mathcal{R}_{\kappa-2,q}[\Pi_+;(s_j^{(1)})_{j=0}^{\kappa-2}], \sigma_{F^{(2)}} \in \mathcal{M}_{\kappa}^{q}[\{R; (s_j^{(1)})_{j=0}^{\kappa-2}, =\}, \text{and } (s_j^{(1)})_{j=0}^{\kappa-2} = (s_j^{[\sigma_{F^{(2)}}]})_{j=0}^{\kappa-2}].$

**Lemma 11.7.** Let $\kappa \in \mathbb{N} \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. Then $F^{(2)} \in \mathcal{R}_{\kappa-2,q}(\Pi_+)$.  

**Proof.** In the case $\kappa = 1$, the combination of Proposition 11.4 and (4.2) yields

$$F^{(2)} \in \mathcal{P}^\text{odd}_{\kappa,q}[0,[\sigma_{F}]] \subseteq \mathcal{R}_{-1,q}(\Pi_+) = \mathcal{R}_{1-2,q}(\Pi_+).$$

In the case $\kappa \in \mathbb{Z}_{2,+\infty} \cup \{+\infty\}$, the application of Proposition 11.5 and (5.3) brings

$F^{(2)} \in \mathcal{R}_{\kappa-2,q}(\Pi_+).$

In view of Lemma 11.6 we introduce in recursive way the following notions.

**Definition 11.7.** Let $\kappa \in \mathbb{N} \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. For all $k \in \mathbb{N}_0$ with $2k + 1 \leq \kappa$, we will call the 2-step SN-transform $F^{(2k+1)}$ of the $2k$-step SN-transform $F^{(2k)}$ the $(2k+1)$-step SN-transform of $F$.

**Definition 11.8.** Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$. For all $k \in \mathbb{N}_0$ with $2k \leq \kappa$, we will call the 1-step SN-transform $F^{(2k+1)}$ of the $2k$-step SN-transform $F^{(2k)}$ the $(2k+1)$-step SN-transform of $F$.

**Remark 11.9.** Let $\kappa \in \mathbb{Z}_{-1,+\infty} \cup \{+\infty\}$, let $F \in \mathcal{R}_{\kappa,q}(\Pi_+)$, and let $k \in \mathbb{N}_0$ with $2k-1 \leq \kappa$. From Definitions 11.2, 11.7, and 11.8 and Lemma 11.6 we see that:

(a) $F^{(2k)} \in \mathcal{R}_{\kappa-2k,q}(\Pi_+)$. 

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(b) \((F^{(2k)})^{(l)} = F^{(2k+l)}\) for each \(l \in \mathbb{N}_0\) with \(l - 1 \leq \kappa - 2k\).

The content of our next considerations can be described as follows. Let \(\kappa \in \mathbb{Z}_{2,+\infty} \cup \{+\infty\}\) and let \((s_j)_j^\kappa = 0 \in \mathcal{H}_{q,e}^{\geq \kappa}.\) Then we are going to study the above introduced Schur-Nevanlinna type algorithm particularly for functions which belong to the class \(\mathcal{R}_{\kappa,q}[\Pi_+; (s_j)_j^\kappa].\) The next results contain essential information about the algorithm applied to functions belonging to the class \(\mathcal{R}_{\kappa,q}[\Pi_+; (s_j)_j^\kappa]\). For every choice of \(F \in \mathcal{R}_q(\Pi_+)\) and \(s_0, s_1 \in \mathbb{C}^{q \times q},\) we set

\[
\mathcal{F}_{(+;s_0,s_1)}(F) := F^{(+;s_0,s_1)}, \quad \mathcal{F}_{(-;s_0,s_1)}(F) := F^{(-;s_0,s_1)}
\]

and

\[
\mathcal{F}_{(+;s_0)}(F) := F^{(+;s_0)}, \quad \mathcal{F}_{(-;s_0)}(F) := F^{(-;s_0)}.
\]

**Proposition 11.10.** Let \(\kappa \in \mathbb{N} \cup \{+\infty\},\) let \((s_j)_j^\kappa = 0 \in \mathcal{H}_{q,e}^{\geq \kappa},\) and let \(F\) belong to \(\mathcal{R}_{\kappa,q}[\Pi_+; (s_j)_j^\kappa].\) Further, let \(n \in \mathbb{N}_0\) be such that \(2n + 1 \leq \kappa\) and let \(F^{(2(n+1))}\) be the \((2n+1)\)-step SN-transform of \(F.\) For each integer \(m\) with \(0 \leq m \leq \min\{n+1, \frac{\kappa}{2}\},\) let \((s_j)_j^m = 2m\) be the \(m\)-th Schur transform of \((s_j)_j^\kappa = 0.\) Then:

(a) 
\[
F^{(2(n+1))} = [\mathcal{F}_{(+;s_0^{(n)}_1)} \circ \mathcal{F}_{(+;s_0^{(n-1)}_1)} \circ \cdots \circ \mathcal{F}_{(+;s_0^{(0)}_1)}] \circ \mathcal{F}_{(-;s_0^{(0)}_1)}(F)
\]

and

\[
F^{(2(n+1))} \in \begin{cases} \mathcal{R}_{\kappa-2(n+1),q}[\Pi_+; (s_j)_j^{\kappa-(2(n+1))}], & \text{if } 2n + 1 < \kappa, \\ \mathcal{P}_{q_{\text{odd}}[s_j^{(n)}_0]}, & \text{if } 2n + 1 = \kappa. \end{cases}
\]

(b) 
\[
F = [\mathcal{F}_{(-;s_0^{(0)}_1)} \circ \mathcal{F}_{(-;s_0^{(1)}_1)} \circ \cdots \circ \mathcal{F}_{(-;s_0^{(n)}_1)}] \circ \mathcal{F}_{(+;s_0^{(n)}_1)}(F^{(2(n+1)))}.
\]

**Proof.** \([a]\) Formula (11.1) follows immediately from the definition of \(F^{(2(n+1))}\) (see Definition 11.7 and part \((c)\) of Definition 11.2). Using Corollary 9.4, Theorem 9.7, Corollary 9.8 and Theorem 9.9 formula (11.2) follows by induction.

\([b]\) In view of Corollaries 10.5 and 10.12 part \((b)\) is an immediate consequence of \((a).\)

**Proposition 11.11.** Let \(\kappa \in \mathbb{N}_0 \cup \{+\infty\},\) let \((s_j)_j^\kappa = 0 \in \mathcal{H}_{q,e}^{\geq \kappa}\) and let \(F\) belong to \(\mathcal{R}_{\kappa,q}[\Pi_+; (s_j)_j^\kappa].\) Further, let \(n \in \mathbb{N}_0\) be such that \(2n \leq \kappa\) and denote by \(F^{(2n+1)}\) the \((2n+1)\)-step SN-transform of \(F.\) For each \(m \in \mathbb{Z}_{0,n}\) let \((s_j)_j^m = 2m\) be the \(m\)-th Schur transform of \((s_j)_j^\kappa = 0.\) Then the following statements hold true:

...
\( a \)

\[
F^{(2n+1)} = \begin{cases} 
F_{(\pm; s_0)}(n) \circ F_{(\pm; s_0)}(n-1) \circ \cdots \circ F_{(\pm; s_0)}(0)(F) & \text{if } n \in \mathbb{N} \\
F_{(\pm; s_0)}(F) & \text{if } n = 0 
\end{cases}
\] (11.3)

and

\[
F^{(2n+1)} \in P_{\text{even}}[s_0^{(n)}].
\] (11.4)

\( b \)

\[
F = \begin{cases} 
F_{(\pm; s_0)}(2n+1) & \text{if } n = 0 \\
F_{(\pm; s_0)}(n) \circ \cdots \circ F_{(\pm; s_0)}(1 \circ \cdots \circ F_{(\pm; s_0)}(0)(F^{(2n+1)}) & \text{if } n \in \mathbb{N}.
\end{cases}
\]

**Proof.** (a) Formula (11.3) follows immediately from the definition of \( F^{(2n+1)} \) (see Definition 11.8 and parts (b) and (c) of Definition 11.2). Using Corollary 9.2, Theorem 9.7, Corollary 9.8, and Theorem 9.9, formula (11.4) follows by induction.

(b) In view of Corollaries 10.2 and 10.12, part (b) is an immediate consequence of (a).

Our next considerations are aimed to study the inversion of the Schur-Nevanlinna algorithm. The following result can be considered as an inverse statement with respect to Proposition 11.10.

**Proposition 11.12.** Let \( n \in \mathbb{N}_0 \) and \((s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{n} \). For each \( m \in \mathbb{Z}_0, n \), let \((s_j^{(m)})_{j=0}^{2(n-m)+1} \) be the \( m \)-th Schur transform of \((s_j)_{j=0}^{2n+1} \). Further, let \( G \in P_{\text{odd}}[s_0^{(n)}] \) and let

\[
F := [F_{(\pm; s_0^{(0)})} \circ F_{(\pm; s_0^{(1)})} \circ \cdots \circ F_{(\pm; s_0^{(n)})}](G).
\]

Then the following statements hold true:

(a) \( F \in \mathcal{R}_{2n+1,q}[\Pi_+;(s_j)_{j=0}^{2n+1}] \).

(b) The \( 2(n+1) \)-step SN-transform \( F^{(2(n+1))} \) of \( F \) satisfies \( F^{(2(n+1))} = G \).

**Proof.** (a) This follows by combining Proposition 10.4 and Theorem 10.11.

(b) Taking the definition of \( F \) and (a) into account, the application of Corollaries 10.4 and 10.12 yields

\[
G = [F_{(\pm; s_0^{(n)})} \circ F_{(\pm; s_0^{(n-1)})} \circ \cdots \circ F_{(\pm; s_0^{(1)})} \circ F_{(\pm; s_0^{(0)})}](F).
\]

Combining this with (a), we infer from part (a) of Proposition 11.10 that

\[
F^{(2(n+1))} = G.
\]

The combination of Propositions 11.10 and 11.12 gives us now a complete description of the SN algorithm in the class \( \mathcal{R}_{2n+1,q}[\Pi_+;(s_j)_{j=0}^{2n+1}] \).
Theorem 11.13. Let \( n \in \mathbb{N}_0 \) and \( (s_j)_{j=0}^{2n+1} \in \mathcal{H}^{\geq,c}_{q,2n+1} \). For each \( m \in \mathbb{Z}_{0,n} \), let \((s_j)_{j=0}^{2(n-m)+1}\) be the \( m \)-th Schur transform of \( (s_j)_{j=0}^{2n+1} \). Let
\[
F_{(-;s_j)_{j=0}^{2n+1}} := F_{(-;s_0^{(0)},s_1^{(0)})} \circ F_{(-;s_0^{(1)},s_1^{(1)})} \circ \cdots \circ F_{(-;s_0^{(n)},s_1^{(n)})} \tag{11.5}
\]
and let
\[
F_{(+;s_j)_{j=0}^{2n+1}} := F_{(+;s_0^{(0)},s_1^{(0)})} \circ F_{(+;s_0^{(n-1)},s_1^{(n-1)})} \circ \cdots \circ F_{(+;s_0^{(0)},s_1^{(n-1)})} \circ F_{(+;s_0^{(0)},s_1^{(n-1)})} \tag{11.6}
\]
Then the following statements hold true:

(a) The mapping \( F_{(-;s_j)_{j=0}^{2n+1}} \) generates a bijective correspondence between \( \mathcal{P}^{\text{odd}}_q[s_0^{(n)}] \) and \( \mathcal{R}_{2n+1,q}[[\Pi_+;s_j_{j=0}^{2n+1}]] \).

(b) Let \( (F_{(-;s_j)_{j=0}^{2n+1}})^{-1} \) be the inverse mapping of \( F_{(-;s_j)_{j=0}^{2n+1}} \). For each matrix-valued function \( F \in \mathcal{R}_{2n+1,q}[[\Pi_+;s_j_{j=0}^{2n+1}]] \), then
\[
(F_{(-;s_j)_{j=0}^{2n+1}})^{-1}(F) = F_{(+;s_j)_{j=0}^{2n+1}}(F) = F^{(2n+1)}
\]
where \( F^{(2n+1)} \) is the \((2n+1)\)-step SN-transform of \( F \).

Proof. Combine Propositions 11.10 and 11.11 \( \square \)

The following result can be considered as an inverse statement with respect to Proposition 11.11

Proposition 11.14. Let \( n \in \mathbb{N}_0 \) and \( (s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq,c}_{q,2n} \). For each \( m \in \mathbb{Z}_{0,n} \), let \((s_j)_{j=0}^{2(n-m)+1}\) be the \( m \)-th Schur transform of \( (s_j)_{j=0}^{2n} \). Further, let \( G \in \mathcal{P}^{\text{even}}_q[s_0^{(n)}] \) and let
\[
F := \begin{cases} 
F_{(-;s_0^{(n)})}(G) & \text{if } n = 0 \\
[F_{(-;s_0^{(0)},s_1^{(0)})} \circ \cdots \circ F_{(-;s_0^{(n-1)},s_1^{(n-1)})} \circ F_{(-;s_0^{(0)},s_1^{(n-1)})}](G) & \text{if } n \in \mathbb{N}
\end{cases}
\]
Then the following statements hold true:

(a) \( F \in \mathcal{R}_{2n,q}[[\Pi_+;s_j_{j=0}^{2n}]] \).

(b) The \((2n+1)\)-step SN-transform \( F^{(2n+1)} \) of \( F \) satisfies \( F^{(2n+1)} = G \).

Proof. \( \square \) This follows by combining Proposition 10.1 and Theorem 10.9

|a| Taking the definition of \( F \) and \( G \) into account, the application of Corollaries 10.2 and 10.12 yields
\[
G = \begin{cases} 
F_{(+;s_0^{(n)})}(F) & \text{if } n = 0 \\
[F_{(+;s_0^{(0)},s_1^{(0)})} \circ \cdots \circ F_{(+;s_0^{(n-1)},s_1^{(n-1)})}](F) & \text{if } n \in \mathbb{N}
\end{cases}
\]
Combining this with \( G \), we infer from part \( G \) of Proposition 11.11 that \( F^{(2n+1)} = G \). \( \square \)

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The combination of Propositions 11.11 and 11.14 gives us now a complete description of the SN algorithm in the class \( \mathcal{R}_{2n,q}[\Pi_+; (s_j)_{j=0}^{2n}] \).

**Theorem 11.15.** Let \( n \in \mathbb{N}_0 \) and \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{n,\infty} \). For each \( m \in \mathbb{Z}_{0,n} \) let \( (s_j^{(m)})_{j=0}^{2(n-m)} \) be the \( m \)-th Schur transform of \( (s_j)_{j=0}^{2n} \). Let

\[
\mathcal{F}_{(s_j)_{j=0}^{2n}} := \begin{cases} \mathcal{F}_{(s_j^{(0)})} & \text{if } n = 0 \\ \mathcal{F}_{(s_j^{(0)})} \circ \cdots \circ \mathcal{F}_{(s_j^{(n-1)}, s_1^{(n-1)})} \circ \mathcal{F}_{(s_j^{(0)})} & \text{if } n \in \mathbb{N} \end{cases} \quad (11.7)
\]

and

\[
\mathcal{F}_{(+; (s_j)_{j=0}^{2n})} := \begin{cases} \mathcal{F}_{(+; s_0^{(0)})} & \text{if } n = 0 \\ \mathcal{F}_{(+; s_0^{(0)})} \circ \mathcal{F}_{(+; s_1^{(n-1)})} \circ \cdots \circ \mathcal{F}_{(+; s_1^{(0)})} & \text{if } n \in \mathbb{N} \end{cases} . \quad (11.8)
\]

Then the following statements hold true:

(a) The mapping \( \mathcal{F}_{(s_j)_{j=0}^{2n}} \) generates a bijective correspondence between \( \mathcal{P}_q^{\text{even}}[s_0^{(n)}] \) and \( \mathcal{R}_{2n,q}[\Pi_+; (s_j)_{j=0}^{2n}] \).

(b) Let \( (\mathcal{F}_{(s_j)_{j=0}^{2n}})^{-1} \) be the inverse mapping of \( \mathcal{F}_{(s_j)_{j=0}^{2n}} \). For each matrix-valued function \( F \in \mathcal{R}_{2n,q}[\Pi_+; (s_j)_{j=0}^{2n}] \), then

\[
(\mathcal{F}_{(s_j)_{j=0}^{2n}})^{-1}(F) = \mathcal{F}_{(+; (s_j)_{j=0}^{2n})}(F) = F^{(2n+1)}
\]

where \( F^{(2n+1)} \) is the \((2n+1)\)-step SN-transform of \( F \).

**Proof.** Combine Propositions 11.11 and 11.14.

Our next considerations are aimed to rewrite the mappings introduced in Theorems 11.13 and 11.15 as linear fractional transformations of matrices. The essential tool in realizing this goal will be the matrix polynomials introduced in Appendix C. More precisely, we will use finite products of such matrix polynomials.

Let \( \kappa \in \mathbb{N}_0 \cup \{ +\infty \} \) and let \( (s_j)_{j=0}^{2n} \) be a sequence of complex \( p \times q \) matrices. Let \( n \in \mathbb{N}_0 \) be such that \( 2n \leq \kappa \). For all \( m \in \mathbb{Z}_{0,n} \), let \( (s_j^{(m)})_{j=0}^{\kappa-2n} \) be the \( m \)-th Schur transform of \( (s_j)_{j=0}^{2n} \). For all \( n \in \mathbb{N}_0 \) with \( 2n \leq \kappa \), let

\[
\mathcal{Q}((s_j)_{j=0}^{2n}) := \begin{cases} V_0^{(0)} & \text{if } n = 0 \\ V_0^{(0)} s_1^{(0)} V_0^{(1)} s_1^{(1)} \cdots V_0^{(n-1), s_1^{(n-1)}} V_0^{(n)} & \text{if } n \geq 1 \end{cases} \quad (11.9)
\]

and, for all \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), let

\[
\mathcal{Q}((s_j)_{j=0}^{2n+1}) := V_0^{(0)} s_1^{(0)} V_0^{(1)} s_1^{(1)} \cdots V_0^{(n)} s_1^{(n)}. \quad (11.10)
\]
Furthermore, for all \( m \in \mathbb{Z}_{0,\kappa} \), let
\[
\mathfrak{g}^{((s_j)_j=0)^m} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
\]
be the block representation of \( \mathfrak{g}^{((s_j)_j=0)^m} \) with \( p \times p \) block \( v_{11}^{((s_j)_j=0)^m} \).

**Remark 11.16.** Let \( \kappa \in \mathbb{Z}_{2,\infty} \cup \{ +\infty \} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. For all \( n \in \mathbb{N} \) with \( 2n \leq \kappa \) and all \( k \in \mathbb{Z}_{0,n-1} \), one can see then from (11.9), (11.10), and [24] Remark 9.2 that
\[
\mathfrak{g}^{((s_j)_j=0)^{(2(n-k))}} = \mathfrak{g}^{((s_j)_j=0)^{(2(n-k)-1)}} \mathfrak{g}_{s_0}^{(n)},
\]
and
\[
\mathfrak{g}^{((s_j)_j=0)^{(2(n-k)+1)}} = V_{s_0}^{(n)} \mathfrak{g}^{((s_j)_j=0)^{(2(n-k)-1)}}
\]
where \( t_j := s_j^{(k+1)} \) for all \( j \in \mathbb{Z}_{0,2(n-k-1)} \).

**Remark 11.17.** Let \( \kappa \in \mathbb{Z}_{3,\infty} \cup \{ +\infty \} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. Then, for all \( n \in \mathbb{N} \) with \( 2n + 1 \leq \kappa \) and all \( k \in \mathbb{Z}_{0,n-1} \), one can see from (11.10) and [24] Remark 9.2 that
\[
\mathfrak{g}^{((s_j)_j=0)^{(2(n-k))}} = \mathfrak{g}^{((s_j)_j=0)^{(2(n-k)-1)}} \mathfrak{g}_{s_0}^{(n)}
\]
and
\[
\mathfrak{g}^{((s_j)_j=0)^{(2(n-k)+1)}} = V_{s_0}^{(n)} \mathfrak{g}^{((s_j)_j=0)^{(2(n-k)-1)}}
\]
where \( t_j := s_j^{(k+1)} \) for all \( j \in \mathbb{Z}_{0,2(n-k)-1} \).

Let \( \kappa \in \mathbb{N}_0 \cup \{ +\infty \} \) and let \( (s_j)_{j=0}^\kappa \) be a sequence of complex \( p \times q \) matrices. Let \( n \in \mathbb{N}_0 \) be such that \( 2n \leq \kappa \). For all \( m \in \mathbb{Z}_{0,n} \), let \( (s_j)_j=0^m \) be the \( m \)-th Schur transform of \( (s_j)_{j=0}^\kappa \). Let
\[
\mathfrak{M}^{((s_j)_j=0)^{2m}} := \begin{cases} \mathcal{W}_{s_0}^{(0)}, & \text{if } n = 0 \\ \mathcal{W}_{s_0}^{(n)} \mathcal{W}_{s_0}^{(n-1)} \mathcal{W}_{s_0}^{(n-1),s_1^{(n-1)}} \mathcal{W}_{s_0}^{(n-1)} \mathcal{W}_{s_0}^{(n),s_1^{(n)}}, & \text{if } n \geq 1 \end{cases}
\]
For all \( n \in \mathbb{N}_0 \) with \( 2n + 1 \leq \kappa \), let
\[
\mathfrak{M}^{((s_j)_j=0)^{2n+1}} := \mathcal{W}_{s_0}^{(n),s_1^{(n)}} \mathcal{W}_{s_0}^{(n-1),s_1^{(n-1)}} \mathcal{W}_{s_0}^{(n-1),s_1^{(n-1)}} \mathcal{W}_{s_0}^{(n),s_1^{(n)}},
\]
Furthermore, for all \( m \in \mathbb{Z}_{0,\kappa} \), let
\[
\mathfrak{M}^{((s_j)_j=0)^m} = \begin{bmatrix} w_{11}^{((s_j)_j=0)^m} & w_{12}^{((s_j)_j=0)^m} \\ w_{21}^{((s_j)_j=0)^m} & w_{22}^{((s_j)_j=0)^m} \end{bmatrix}
\]
be the block representation of \( \mathfrak{M}^{((s_j)_j=0)^m} \) with \( p \times p \) block \( w_{11}^{((s_j)_j=0)^m} \).
Remark 11.18. Let $\kappa \in \mathbb{Z}_{2, +\infty} \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For all $n \in \mathbb{N}$ with $2n \leq \kappa$ and all $k \in \mathbb{Z}_{0, n-1}$, one can then see from (11.11), (11.12), and [24, Remark 9.2] that
\[
2\mathfrak{M}(s_j)_{j=0}^{2(n-k)} = W_{s_0(s_n)^{(k)}}(s_j)_{j=0}^{2(n-k)}
\]
and
\[
2\mathfrak{M}(s_j)_{j=0}^{2(n-k)+1} = W_{s_0(s_n)^{(k)}}(s_j)_{j=0}^{2(n-k)}
\]
where $t_j := s_j^{(k+1)}$ for all $j \in \mathbb{Z}_{0, 2(n-k)-1}$.

Remark 11.19. Let $\kappa \in \mathbb{Z}_{3, +\infty} \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then, for all $n \in \mathbb{N}$ with $2n + 1 \leq \kappa$ and all $k \in \mathbb{Z}_{0, n-1}$, one can see from (11.12) and [24, Remark 9.2] that
\[
2\mathfrak{M}(s_j)_{j=0}^{2(n-k)+1} = W_{s_0(s_n)^{(k)}}(s_j)_{j=0}^{2(n-k)+1}
\]
and
\[
2\mathfrak{M}(s_j)_{j=0}^{2(n-k)+1} = W_{s_0(s_n)^{(k)}}(s_j)_{j=0}^{2(n-k)+1}
\]
where $t_j := s_j^{(k+1)}$ for all $j \in \mathbb{Z}_{0, 2(n-k)-1}$.

Proposition 11.20. Let $n \in \mathbb{N}_0$, let $(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q, 2n+1}^{\geq e}$ and let $(s_j^{(1)})_{j=0}^1$ be the $n$-th Schur transform of $(s_j)_{j=0}^{2n+1}$. Further, let $G \in \mathcal{P}_q^{\text{odd}}[s_0^{(1)}]$. Then:

(a) For all $z \in \Pi_+$,
\[
\det \left[ (s_j)_{j=0}^{2n+1} \right] G(z) + v_{22}((s_j)_{j=0}^{2n+1}) (z) \neq 0.
\]

(b) Let $\mathcal{F}_{(-:s_j^{(2n+1)})}$ be given via (11.10). Then
\[
\mathcal{F}_{(-:s_j^{(2n+1)})}(G) = S_{q,(s_j^{(2n+1)})}(G).
\]

Proof. For each $m \in \mathbb{Z}_{0, n}$ let $(s_j)_{j=0}^{2(n-m)+1}$ be the $m$-th Schur transform of $(s_j)_{j=0}^{2n+1}$. For each $m \in \mathbb{Z}_{0, n}$, in view of Proposition 7.2, then
\[
(s_j)_{j=0}^{2(n-m)+1} \in \mathcal{H}_{q, 2(n-m)+1}^{\geq e}.
\]
Thus, for each $m \in \mathbb{Z}_{0, n}$, we have
\[
(s_j^{(m)})_{j=0}^1 \in \mathcal{H}_{q, 1}^{\geq e}.
\]
Because of (11.10), we have
\[
\mathcal{F}_{(-:s_j^{(2n+1)})} = \mathcal{F}_{(-:s_j^{(0)})} \circ \mathcal{F}_{(-:s_j^{(1)})} \circ \cdots \circ \mathcal{F}_{(-:s_j^{(n)})}.
\]
For each \( k \in \mathbb{Z}_{0,n} \), we infer from Proposition \( \textbf{10.4} \) and Theorem \( \textbf{10.11} \) inductively
\[
[F_{(s_{0}^{(k)})}(k)_{s_{0}^{(k)}}] \circ F_{(s_{0}^{(k+1)})}(k+1)_{s_{0}^{(k+1)}} \circ \cdots \circ F_{(s_{0}^{(n)})}(n)_{s_{0}^{(n)}}(G) \in \mathcal{R}_{2(n-k)+1,q} \left[ \Pi_{+}^{(k)}(s_{j}^{(k)})_{j=0}^{2(n-k)+1} \right].
\]

For each \( k \in \mathbb{Z}_{0,n} \), then Lemma \( \textbf{5.8} \) shows that
\[
[F_{(s_{0}^{(k)})}(k)_{s_{0}^{(k)}}] \circ F_{(s_{0}^{(k+1)})}(k+1)_{s_{0}^{(k+1)}} \circ \cdots \circ F_{(s_{0}^{(n)})}(n)_{s_{0}^{(n)}}(G) \in \mathcal{P}_{q}^{\text{odd}}[s_{0}^{(k)}].
\]
Taking (11.13), (11.14), and (11.15) into account, we get from Lemma \( \textbf{8.12} \) that
\[
F_{(s_{j})}^{(2n+1)}(G) = (S_{V_{0}^{(0)}},q_{1})_{s_{0}}^{(1)} \circ \cdots \circ S_{V_{0}^{(n)}},s_{0}^{(n)}(G).
\]
In view of (11.16) and (11.10), now Proposition \( \textbf{3.1} \) yields (\( \textbf{m} \)) and (\( \textbf{l} \)).

**Proposition 11.21.** Let \( n \in \mathbb{N}_{0} \) and let \( (s_{j})_{j=0}^{2n+1} \in \mathcal{H}_{q}^{\geq e} \). Further, let \( F \) belong to \( \mathcal{R}_{2n+1,q}[\Pi_{+}^{(s_{j})_{j=0}^{2n+1}}] \). Then the following statements hold true:

(a) For all \( z \in \Pi_{+} \),
\[
\det \left[ w_{21}^{((s_{j})_{j=0}^{2n+1})}(z)F(z) + w_{22}^{((s_{j})_{j=0}^{2n+1})}(z) \right] \neq 0.
\]

(b) Let \( F_{(s_{j})_{j=0}^{2n+1}} \) be given by (11.6). Then
\[
F_{(s_{j})_{j=0}^{2n+1}}(F) = S_{21}^{(s_{j})_{j=0}^{2n+1}}(F).
\]

**Proof.** For each \( m \in \mathbb{Z}_{0,n} \) let \( (s_{j})_{j=0}^{(m)+1} \) be the \( m \)-th Schur transform of \( (s_{j})_{j=0}^{2n+1} \). For each \( m \in \mathbb{Z}_{0,n} \) in view of Proposition \( \textbf{7.2} \) then
\[
(s_{j})_{j=0}^{(m)+1} \in \mathcal{H}_{q,2(n-m)+1}^{\geq e}.
\]
Thus, for every choice of \( m \in \mathbb{Z}_{0,n} \), we have
\[
(s_{j})_{j=0}^{(m)+1} \in \mathcal{H}_{q,1}^{\geq e}.
\]

Because of (11.6), we have
\[
F_{(s_{j})_{j=0}^{2n+1}} = F_{(s_{j})_{j=0}^{(m)+1}} \circ F_{(s_{j})_{j=0}^{(m-1)+1}} \circ \cdots \circ F_{(s_{j})_{j=0}^{0}}.
\]
For \( n = 0 \), the assertions of (\( \textbf{m} \)) and (\( \textbf{l} \)) are an immediate consequence of (11.18), (11.10), and (\( \textbf{c} \)) of Lemma \( \textbf{8.9} \). Now let \( n \in \mathbb{N} \). For each \( k \in \mathbb{Z}_{0,n-1} \), we infer from Theorem \( \textbf{9.9} \) inductively that
\[
[F_{(s_{j})_{j=0}^{(n)+1}}(s_{0}^{(k)})_{s_{0}^{(k)}}] \circ \cdots \circ F_{(s_{j})_{j=0}^{(n)+1}}(s_{0}^{(n)})_{s_{0}^{(n)}}(G) \in \mathcal{R}_{2(n-k)+1,q} \left[ \Pi_{+}^{(s_{j})_{j=0}^{2(n-k)+1}} \right].
\]
Taking (11.17), (11.18), and (11.19) into account, part (\( \textbf{c} \)) of Lemma \( \textbf{8.9} \) yields
\[
F_{(s_{j})_{j=0}^{2n+1}}(F) = (S_{W_{0}^{(0)}},s_{0}^{(1)})_{s_{0}}^{(1)} \circ S_{W_{0}^{(n-1)}},s_{0}^{(n-1)}(G).
\]
In view of (11.20) and (11.12), now Proposition \( \textbf{3.1} \) yields (\( \textbf{m} \)) and (\( \textbf{l} \)).

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Proposition 11.22. Let \( n \in \mathbb{N}_0 \), let \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^e \), and let \( (s_j^{(n)})_{j=0} \) be the \( n \)-th Schur transform of \( (s_j)_{j=0}^{2n} \). Further, let \( G \in \mathcal{T}_{q}^{\text{even}}[s_0^{(n)}] \). Then:

(a) For all \( z \in \Pi_+ \),
\[
\det \left[ v_{21}^{((s_j^{(n)})_{j=0}^{2n})}(z)G(z) + v_{22}^{((s_j^{(n)})_{j=0}^{2n})}(z) \right] \neq 0.
\]

(b) Let \( F_{((s_j^{(n)})_{j=0}^{2n})} \) be given by (11.7). Then
\[
F_{((s_j^{(n)})_{j=0}^{2n})}(G) = S_{q^{((s_j^{(n)})_{j=0}^{2n})}}(G).
\]

Proof. From (11.7) and (11.5) we infer that
\[
F_{((s_j^{(n)})_{j=0}^{2n})} = \begin{cases} 
F_{((s_j^{(n)})_{j=0}^{2n})}, & \text{if } n = 0 \\
F_{((s_j^{(n)})_{j=0}^{2n-1})} \circ F_{((s_j^{(n)})_{j=0}^{2n})}, & \text{if } n \in \mathbb{N}.
\end{cases}
\]

First we consider the case \( n = 0 \). Taking (11.9) and (11.21) into account, Lemma 8.13 yields the assertions of (a) and (b). Now let \( n \in \mathbb{N} \). Then clearly
\[
(s_j^{2n-1})_{j=0} \in \mathcal{H}_{q,2n-1}^e.
\]
Furthermore, Proposition 7.2 implies \( (s_j^{(n)})_{j=0}^{0} \in \mathcal{H}_{q,0}^e \). Then Lemma 8.13 provides us
\[
F_{((s_j^{(n)})_{j=0}^{0})}(G) = S_{V_{n}(0)}(G).
\]
From Corollary 10.2 we see that
\[
F_{((s_j^{(n)})_{j=0}^{0})}(G) \in \mathcal{R}_{0,q} \left[ \Pi_+; (s_j^{(n)})_{j=0}^{0} \right].
\]
Thus, Lemma 5.8 yields
\[
F_{((s_j^{(n)})_{j=0}^{0})}(G) \in \mathcal{T}_{q}^{\text{odd}}[s_0^{(n)}].
\]
Let \( (s_j^{(n-1)})_{j=0}^{2} \) be the \( (n-1) \)-th Schur transform of \( (s_j^{(n)})_{j=0}^{2} \). In view of (11.11), then \( s_0^{(n)} = -s_0^{(n-1)}(s_2^{(n-1)})^t s_0^{(n-1)} \). In particular, \( N(s_0^{(n-1)}) \subseteq N(s_0^{(n)}) \) and Remark 4.5 implies
\[
\mathcal{T}_{q}^{\text{odd}}[s_0^{(n-1)}] \subseteq \mathcal{T}_{q}^{\text{odd}}[s_0^{(n)}].
\]
From (11.24) and (11.25) we get \( F_{((s_j^{(n)})_{j=0}^{0})}(G) \in \mathcal{T}_{q}^{\text{odd}}[s_0^{(n-1)}] \), and taking (11.22) into account, Proposition 11.20 yields
\[
F_{((s_j^{(n)})_{j=0}^{2n-1})} \left( F_{((s_j^{(n)})_{j=0}^{0})}(G) \right) = S_{q^{((s_j^{(n)})_{j=0}^{2n-1})}} \left( F_{((s_j^{(n)})_{j=0}^{0})}(G) \right).
\]
Using (11.21), (11.26), and (11.23), we get
\[
F_{((s_j^{(n)})_{j=0}^{2n})}(G) = (S_{q^{((s_j^{(n)})_{j=0}^{2n-1})}} \circ S_{V_{n}(0)})(G).
\]
In view of Remark 11.16 we have
\[ q_f((s_j)_{j=0}^{2n}) = q_f((s_j)_{j=0}^{2n-1})V_{0}^{(n)}. \] (11.28)

In view of (11.27) and (11.28), Proposition 3.1 yields then (a) and (b).

**Proposition 11.23.** Let \( n \in \mathbb{N}_0 \), let \( (s_j)_{j=0}^{2n} \in \mathcal{H}^{\geq e}_{q,2n} \) and let \( F \in \mathcal{R}_{2n,q}[\Pi_{+}; (s_j)_{j=0}^{2n}] \). Then the following statements hold true:

(a) For all \( z \in \Pi_{+} \),
\[
\det \left[ \begin{smallmatrix} w_{21}((s_j)_{j=0}^{2n})^2(z)F(z) + w_{22}((s_j)_{j=0}^{2n})^2(z) \end{smallmatrix} \right] \neq 0.
\]

(b) Let \( \mathcal{F}_{(+,(s_j)_{j=0}^{2n})} \) be given by (11.8). Then
\[
\mathcal{F}_{(+,(s_j)_{j=0}^{2n})}(F) = S_{q_f((s_j)_{j=0}^{2n})}(F).
\]

**Proof.** Let \( (s_j^{(n)})_{j=0}^{0} \) be the \( n \)-th Schur transform of \( (s_j)_{j=0}^{2n} \). From (11.8) and (11.9) we infer that
\[
\mathcal{F}_{(+,(s_j)_{j=0}^{2n})} = \begin{cases} \mathcal{F}_{(+,(s_j^{(n)})_{j=0}^{0})}, & \text{if } n = 0 \\ \mathcal{F}_{(+,(s_j^{(n)})_{j=0}^{0})} \circ \mathcal{F}_{(+,(s_j_{j=0}^{2n-1})_{j=0}^{1})}, & \text{if } n \in \mathbb{N}. \end{cases}
\] (11.29)

First we consider the case \( n = 0 \). Taking (11.11) and (11.29) into account the application of parts (a) and (b) of Lemma 8.9 yields the assertions of (a) and (b). Now let \( n \in \mathbb{N} \). Then clearly
\[
(s_j)_{j=0}^{2n-1} \in \mathcal{H}^{\geq e}_{q,2n-1}. \] (11.30)

From (5.3) we see that
\[
F \in \mathcal{R}_{2n-1,q}[\Pi_{+}; (s_j)_{j=0}^{2n-1}]. \] (11.31)

In view of (11.30) and (11.31), we conclude from Proposition 11.21 that
\[
\mathcal{F}_{(+,(s_j)_{j=0}^{2n-1})}(F) = S_{q_f((s_j)_{j=0}^{2n-1})}(F) \] (11.32)

and from part (a) of Proposition 11.10 that
\[
\mathcal{F}_{(+,(s_j)_{j=0}^{2n-1})}(F) \in \mathcal{R}_{0,q}[\Pi_{+}; (s_j^{(n)})_{j=0}^{0}]. \] (11.33)

In view of Proposition 7.2 we have \( (s_j^{(n)})_{j=0}^{0} \in \mathcal{H}^{\geq e}_{q,0} \). Then from (11.33) and parts (a) and (b) of Lemma 8.9 we get
\[
\mathcal{F}_{(+,(s_j^{(n)})_{j=0}^{0})}(\mathcal{F}_{(+,(s_j)_{j=0}^{2n-1})}(F)) = S_{W_{s_j}(n)}(\mathcal{F}_{(+,(s_j)_{j=0}^{2n-1})}(F)). \] (11.34)

Using (11.29), (11.34), and (11.32), we obtain
\[
\mathcal{F}_{(+,(s_j)_{j=0}^{2n})}(F) = (S_{W_{s_j}(n)} \circ S_{q_f((s_j)_{j=0}^{2n})})(F). \] (11.35)

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In view of Remark 11.18, we have
\[
\mathfrak{M}^{((s_j)_{j=0}^{2n})}_0 = W^{((s_j)_{j=0}^{2n-1})}_0. 
\] (11.36)

Taking (11.35) and (11.36) into account, the application of Proposition B.1 yields the assertions of (a) and (b).

12. Descriptions of the Sets \( R_{m,q} \)\[\Pi^+ \mid (s_j)_{j=0}^m \]

Let \( m \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^m \in \mathcal{H}_{q,2n}^{>1} \). Then we know from Theorem 1.3 that the solution set \( \mathcal{M}_\mathbb{R}^\mathbb{R} \{ \mathbb{R} \mid (s_j)_{j=0}^m \} \) is non-empty. Furthermore, Corollary 5.6 tells us that \( R_{m,q} \)\[\Pi^+ \mid (s_j)_{j=0}^m \] is exactly the set of Stieltjes transforms of all the measures belonging to \( \mathcal{M}_\mathbb{R}^\mathbb{R} \{ \mathbb{R} \mid (s_j)_{j=0}^m \} \). On the basis of our Schur-Nevanlinna-type algorithm, which was introduced in Section 11, we have obtained important insights into the structure of the set \( R_{m,q} \)\[\Pi^+ \mid (s_j)_{j=0}^m \]. Using Theorem 7.5, we will now rewrite the descriptions of \( R_{m,q} \)\[\Pi^+ \mid (s_j)_{j=0}^m \], which were obtained in Section 11, in a form which is better adapted to the original data sequence \((s_j)_{j=0}^m \). This is our first main theorem.

**Theorem 12.1.** Let \( n \in \mathbb{N}_0 \), let \((s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>1} \), let \( \mathcal{V}^{((s_j)_{j=0}^{2n})} \) be defined by (11.9) and let \( L_n \) be given by (7.2). Then the following statements hold true:

(a) \( R_{2n,q} \)\[\Pi^+ \mid (s_j)_{j=0}^{2n} \] = \( S_{\mathcal{V}^{((s_j)_{j=0}^{2n})}}(\mathcal{P}_q^{even}[L_n]) \).

(b) For each \( F \in R_{2n,q} \)\[\Pi^+ \mid (s_j)_{j=0}^{2n} \], there is a unique \( G \in \mathcal{P}_q^{even}[L_n] \) which satisfies \( S_{\mathcal{V}^{((s_j)_{j=0}^{2n})}}(G) = F \), namely \( G = F^{(2n+1)} \), where \( F^{(2n+1)} \) stands for the \((2n+1)\)-step SN-transform of \( F \).

**Proof.** Let \((s_j)^{(n)}_{j=0} \) be the \( n \)-th Schur transform of \((s_j)_{j=0}^{2n} \). In view of Theorem 7.3 and Definition 7.3, then \( (s_j)^{(n)}_{j=0} = L_n \). Hence, the application of Theorem 11.15 and Proposition 11.22 completes the proof.

It should be mentioned that a result similar to part (a) of Theorem 12.1 is contained in [9, Theorem 4.1].

**Corollary 12.2.** Let \( n \in \mathbb{N}_0 \), let \((s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>1} \) and let \( \mathcal{V}^{((s_j)_{j=0}^{2n})} \) be defined by (11.9). Then
\[
R_{2n,q} \Pi^+ \mid (s_j)_{j=0}^{2n} = S_{\mathcal{V}^{((s_j)_{j=0}^{2n})}} \left( \mathcal{V}^{[-2]}(\Pi^+) \right).
\]

**Proof.** From Remark 7.10 we get
\[
(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>1}.
\] (12.1)
Because of the assumption \((s_j)^{2n}_{j=0} \in \mathcal{H}^{\geq 2n}_{q,2n}\), we infer from part (i) of Lemma [11.11] that the matrix \(L_n\) defined in (7.2) is positive Hermitian. In particular, \(\det L_n \neq 0\). Consequently, Remark [4.1] implies \(\mathcal{P}_{q}^{\text{even}}[L_n] = \mathcal{R}_{q}^{[-2]}(\Pi_+)\). Thus, taking (12.1) into account, the application of part (iii) of Theorem [12.1] completes the proof.

Observe, that under the assumptions of Corollary [12.2] an alternative description of the class \(\mathcal{R}_{2n,q}[\Pi_+; (s_j)^{2n}_{j=0}]\) was obtained in [17] Theorem 8.2 with the aid of the RKHS method.

Let \(n \in \mathbb{N}_0\) and let \((s_j)^{2n}_{j=0} \in \mathcal{H}^{\geq 2n}_{q,2n}\). Then \((s_j)^{2n}_{j=0}\) is called completely degenerate, if \(L_n = 0_{q \times q}\), where \(L_n\) is defined in (7.2). Observe that the set \(\mathcal{H}^{\geq \text{cd}}_{q,2n}\) of all completely degenerate sequences belonging to \(\mathcal{H}^{\geq 2n}_{q,2n}\) is a subclass of \(\mathcal{H}^{\geq \text{e}}_{q,2n}\) (see [19] Corollary 2.14). This class is connected to the case of a unique solution, which was already discussed in [9] Corollary 3.5.

**Theorem 12.3.** Let \(n \in \mathbb{N}_0\) and let \((s_j)^{2n}_{j=0} \in \mathcal{H}^{\geq \text{e}}_{q,2n}\). Then:

(a) The following statements are equivalent:

(i) The set \(\mathcal{R}_{2n,q}[\Pi_+; (s_j)^{2n}_{j=0}]\) consists of exactly one element.

(ii) \((s_j)^{2n}_{j=0} \in \mathcal{H}^{\geq \text{cd}}_{q,2n}\).

(b) If (i) holds true, then \(\det \mathcal{v}_{22}((s_j)^{2n}_{j=0})(z) \neq 0\) for all \(z \in \Pi_+\) and

\[\mathcal{R}_{2n,q}[\Pi_+; (s_j)^{2n}_{j=0}] = \left\{ \mathcal{v}_{12}((s_j)^{2n}_{j=0})(\mathcal{v}_{22}((s_j)^{2n}_{j=0}))^{-1} \right\}.\]

**Proof.**

(i) \(\Rightarrow\) (ii): Use Theorem 3.2 Corollary 3.20, Corollary 5.6 and [19] Theorem 8.7.

(ii) \(\Rightarrow\) (i): Let \(L_n\) be defined in (7.2). Then (ii) means \(L_n = 0_{q \times q}\). Thus, part (ii) of Proposition 4.9 implies that the set \(\mathcal{P}_{q}^{\text{even}}[L_n]\) consists of exactly one element, namely the constant function defined on \(\Pi_+\) with value \(0_{q \times q}\). Using part (iii) of Theorem 12.1 then (i) follows, and we see moreover that (iii) holds true.

Now we formulate our second main theorem.

**Theorem 12.4.** Let \(n \in \mathbb{N}_0\), let \((s_j)^{2n+1}_{j=0} \in \mathcal{H}^{\geq \text{e}}_{q,2n+1}\), let \(\mathcal{Q}((s_j)^{2n+1}_{j=0})\) be defined by (11.10), and let \(L_n\) be given by (7.2). Then the following statements hold:

(a) \(\mathcal{R}_{2n+1,q}[\Pi_+; (s_j)^{2n+1}_{j=0}] = \mathcal{S}_{\mathcal{Q}((s_j)^{2n+1}_{j=0})}((\mathcal{P}_{q}^{\text{odd}}[L_n]))\).

(b) For each \(F \in \mathcal{R}_{2n+1,q}[\Pi_+; (s_j)^{2n+1}_{j=0}]\), there is a unique \(G \in \mathcal{P}_{q}^{\text{odd}}[L_n]\) which satisfies

\[\mathcal{S}_{\mathcal{Q}((s_j)^{2n+1}_{j=0})}(G) = F,\]

namely \(G = F^{(2(n+1))}\), where \(F^{(2(n+1))}\) stands for the 2\((n+1)\)-step SN-transform of \(F\).
Proof. Let \( (s_j^{(n)})_{j=0}^{1} \) be the \( n \)-th Schur transform of \( (s_j)_{j=0}^{2n+1} \). From the choice of \( (s_j)_{j=0}^{2n+1} \) we get \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{e,c} \). Thus, we infer from Theorem \( \ref{thm:main} \) and Definition \( \ref{def:main} \) then \( s_0^{(n)} = L_n \). Hence, the application of Theorem \( \ref{thm:main} \) and Proposition \( \ref{prop:main} \) completes the proof.

In the scalar case \( q = 1 \), the following result goes back to \cite[Corollary 5.2]{12}.

**Corollary 12.5.** Let \( n \in \mathbb{N}_0 \), let \( (s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{e,c} \) and let \( \mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] \) be defined by \cite{11.10}. Then
\[
\mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] = S_{q^{((s_j)_{j=0}^{2n+1})}}(\mathcal{R}_{-1,q}(\Pi_+)).
\]

**Proof.** From \( (s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{e,c} \) we get \( (s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{e,c} \). Because of part (a) of Proposition \( \ref{prop:main} \), we also get \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{e,c} \). Then part (c) of Lemma \( \ref{lem:main} \) yields \( L_n \in \mathbb{C}^{q \times q} \). In particular, \( \det L_n \neq 0 \). Consequently, Remark \( \ref{rem:main} \) implies \( \mathcal{P}^{odd}_q[L_n] = \mathcal{R}_{-1,q}(\Pi_+) \). In view of \( \cite{12.2} \) and part (a) of Theorem \( \ref{thm:main} \) this completes the proof.

Now we derive an analogue of Theorem \( \ref{thm:main} \).

**Theorem 12.6.** Let \( n \in \mathbb{N}_0 \) and let \( (s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{e,c} \). Then:

(a) The following statements are equivalent:

(i) The set \( \mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] \) consists of exactly one element.

(ii) \( (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{e,c} \).

(b) If (i) holds true, then \( \det \mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] \neq 0 \) for all \( z \in \Pi_+ \) and
\[
\mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] = \left\{ \mathcal{V}_{12}^{((s_j)_{j=0}^{2n+1})}((s_j)_{j=0}^{2n+1})^{-1} \right\}.
\]

**Proof.** (i) \( \Rightarrow \) (ii): Use Theorem \( \ref{thm:main} \), Corollary \( \ref{cor:main} \), \( \ref{cor:main} \), Corollary \( \ref{cor:main} \) and \cite[Theorem 8.9]{19}.

(i) \( \Rightarrow \) (i): Let \( L_n \) be defined in \( \cite{17.2} \). Then (ii) means \( L_n = 0_{q \times q} \). Thus, part (a) of Proposition \( \ref{prop:main} \) implies that the set \( \mathcal{P}^{odd}_q[L_n] \) consists of exactly one element, namely the constant function defined on \( \Pi_+ \) with value \( 0_{q \times q} \). Using part (a) of Theorem \( \ref{thm:main} \) then (i) and (ii) follow.

By using Proposition \( \ref{prop:main} \) we are able to derive alternate descriptions of the set \( \mathcal{R}_{m,q}[\Pi_+; (s_j)_{j=0}^{m}] \) if \( m \in \mathbb{N}_0 \) and \( (s_j)_{j=0}^{m} \in \mathcal{H}_{q,m}^{e,c} \) are arbitrarily given. This gives reformulations of our two main results stated in Theorems \( \ref{thm:main} \) and \( \ref{thm:main} \).
Theorem 12.7. Let \( n \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^{2n} \) be defined in (7.2). Suppose that \( r := \text{rank } L_n \) fulfills \( r \geq 1 \). Let \( u_1, u_2, \ldots, u_r \) be an orthonormal basis of \( R(L_n) \) and let \( U := [u_1, u_2, \ldots, u_r] \). Then:

(a) \( \mathcal{R}_{2n,q}[\Pi_+; (s_j)_{j=0}^{2n}] = \mathcal{S}_{q^0(s_j)_{j=0}^{2n}}(\{Uf U^* | f \in \mathcal{R}_r^{-2}(\Pi_+)\}) \).

(b) Let \( F \in \mathcal{R}_{2n,q}[\Pi_+; (s_j)_{j=0}^{2n}] \). Then there is a unique \( f \in \mathcal{R}_r^{-2}(\Pi_+) \) such that

\[ \mathcal{S}_{q^0(s_j)_{j=0}^{2n}}(Uf U^*) = F, \]

namely \( f := U^* F^{(2n+1)U} \), where \( F^{(2n+1)} \) is the \((2n+1)\)-step SN-transform of \( F \).

Proof. Since \((s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq e} \) implies \((s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq e} \), we get from part (a) of Lemma 7.7 that \( L_n \in \mathbb{C}^{q \times q}_r \). In particular, \( L_n^* = L_n \). Thus, part (b) of Proposition 4.9 yields

\[ \mathcal{P}_q^{even}[L_n] = \left\{ Uf U^* \left| f \in \mathcal{R}_r^{-2}(\Pi_+) \right. \right\}, \tag{12.3} \]

whereas Remark A.3 implies \( U^* U = I_r \).

(a) Because of (12.3), we infer (a) from part (a) of Theorem 12.1.

(b) Using (12.3), \( U^* U = I_r \), and (a), we conclude (b) from part (b) of Theorem 12.1.

Theorem 12.8. Let \( n \in \mathbb{N}_0 \) and let \((s_j)_{j=0}^{2n+1} \) be defined in (7.2). Suppose that \( r := \text{rank } L_n \) fulfills \( r \geq 1 \). Let \( u_1, u_2, \ldots, u_r \) be an orthonormal basis of \( [N(s_j)]^{-1} \) and let \( U := [u_1, u_2, \ldots, u_r] \). Then:

(a) \( \mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] = \mathcal{S}_{q^0(s_j)_{j=0}^{2n+1}}(\{Uf U^* | f \in \mathcal{R}_{r-1}(\Pi_+)\}) \).

(b) Let \( F \in \mathcal{R}_{2n+1,q}[\Pi_+; (s_j)_{j=0}^{2n+1}] \). Then there is a unique \( f \in \mathcal{R}_{r-1}(\Pi_+) \) such that

\[ \mathcal{S}_{q^0(s_j)_{j=0}^{2n+1}}(Uf U^*) = F, \]

namely \( f := U^* F^{(2n+1)U} \), where \( F^{(2n+1)} \) is the \((2n+1)\)-step SN-transform of \( F \).

Proof. Since \((s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q+1,2n+1}^{\geq e} \) implies \((s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q+1,2n+1}^{\geq e} \), we get again from part (d) of Lemma 7.7 that \( L_n^* = L_n \). Thus, part (b) of Proposition 4.9 yields

\[ \mathcal{P}_q^{odd}[L_n] = \left\{ Uf U^* \left| f \in \mathcal{R}_{r-1}(\Pi_+) \right. \right\}, \tag{12.4} \]

whereas Remark A.3 implies \( U^* U = I_r \).

(a) In view of (12.4), we infer (a) from part (a) of Theorem 12.4.

(b) Using (12.4), \( U^* U = I_r \), and (a), we get (b) from part (b) of Theorem 12.4.

\[ \square \]
A. Some Results On Moore-Penrose Inverses of Matrices

For the convenience of the reader, we state some well-known and some special results on Moore-Penrose inverses of matrices (see e.g., Rao/Mitra [39] or [16, Section 1]). If $A \in \mathbb{C}^{p \times q}$, then (by definition) the Moore-Penrose inverse $A^\dagger$ of $A$ is the unique matrix $A^\dagger \in \mathbb{C}^{q \times p}$ which satisfies the four equations

$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A.$

Remark A.1. Let $A \in \mathbb{C}^{p \times q}$. Then one can easily check that:

(a) $(A^\dagger)^\dagger = A, \quad (A^\dagger)^* = (A^*)^\dagger$, and $I_p - AA^\dagger \in \mathbb{C}^{q \times q}_{\geq}.$

(b) $R(A^\dagger) = R(A^*),$ rank$(A^\dagger) =$ rank$A,$ and $N(A^\dagger) =$ N$(A^*)$.

Proposition A.2 (see, e.g. [16, Theorem 1.1.1]). If $A \in \mathbb{C}^{p \times q}$ then a matrix $G \in \mathbb{C}^{q \times p}$ is the Moore-Penrose inverse of $A$ if and only if $AG = P_A$ and $GA = P_G,$ where $P_A$ and $P_G$ are respectively, the matrices associated with the orthogonal projection in $\mathbb{C}^p$ onto $R(A)$ and the orthogonal projection in $\mathbb{C}^q$ onto $R(G)$.

Remark A.3. Let $A \in \mathbb{C}^{p \times q}$. Then it is readily checked that:

(a) Let $r \in \mathbb{N}$ and let $B \in \mathbb{C}^{r \times q}$. Then $N(A) \subseteq N(B)$ if and only if $BA^\dagger A = B.$ Furthermore, $N(B) = N(A)$ if and only if $B^\dagger B = A^\dagger A.$

(b) Let $s \in \mathbb{N}$ and let $C \in \mathbb{C}^{p \times s}$. Then $R(C) \subseteq R(A)$ if and only if $AA^\dagger C = C.$ Furthermore, $R(C) = R(A)$ if and only if $CC^\dagger = AA^\dagger.$

Remark A.4. Let $A, B \in \mathbb{C}^{q \times q}_H$. From Remarks A.3 and A.1 one can easily see that the following statements are equivalent:

(i) $N(A) \subseteq N(B)$.

(ii) $BA^\dagger A = B.$

(iii) $AA^\dagger B = B.$

(iv) $R(A) \subseteq R(B).$

Lemma A.5. Let $A, X \in \mathbb{C}^{p \times q}$. Then the following statements are equivalent:

(i) $N(A) \subseteq N(X)$ and $R(A) \subseteq R(X).$

(ii) $N(A) = N(X)$ and $R(A) = R(X).$

(iii) $A^\dagger A = X^\dagger X$ and $AA^\dagger = XX^\dagger.$

(iv) $N(A^\dagger) = N(X^\dagger)$ and $R(A^\dagger) = R(X^\dagger).$
Proof. In view of \( \dim[N(A)] + \dim[R(A)] = q \) and \( \dim[N(X)] + \dim[R(X)] = q \), the implication \((i) \Rightarrow (ii)\) is true. Otherwise, the implication \((ii) \Rightarrow (i)\) is trivial. Part (b) of Remark A.3 yields the equivalence of \((ii)\) and \((iii)\). Taking part (a) of Remark A.1 into account, we see that the equivalence of \((iii)\) and \((iv)\) is an immediate consequence of the already verified equivalence of \((ii)\) and \((iii)\). \( \square \)

Remark A.6. Let \( A \in \mathbb{C}^{p \times q} \setminus \{0_{p \times q}\} \). Let \( r := \text{rank } A \), let \( u_1, u_2, \ldots, u_r \) be an orthonormal basis of \( R(A^*) \), and let \( U := [u_1, u_2, \ldots, u_r] \). Then \( U^*U = I_r \), and, in view of Proposition A.2 and Remark A.1, furthermore \(UU^* = A^t A \).

Lemma A.7. Let \( A, B \in \mathbb{C}_H^{q \times q} \). Then the following statements are equivalent:

(i) \( A \geq B \geq 0_{q \times q} \).

(ii) \( B^\dagger \geq B^1BA^1BB^\dagger \geq 0_{q \times q} \) and \( N(A) \subseteq N(B) \).

(iii) \( B \geq BA^1B \geq 0_{q \times q} \) and \( N(A) \subseteq N(B) \).

(iv) \( \begin{bmatrix} A & B \end{bmatrix} \geq 0_{2q \times 2q} \).

If (i) is fulfilled, then \( N(BA^1B) = N(B) \) and \( R(BA^1B) = R(B) \).

Proof. \((i) \Rightarrow (ii)\): From Remark A.4 and (i) we get \( N(A) \subseteq N(B) \) and

\[
I_q \geq AA^t = \sqrt{A^tA} \sqrt{A^tA} \geq \sqrt{A^tB} \sqrt{A^tB} = (\sqrt{B} \sqrt{A^t})^*(\sqrt{B} \sqrt{A^t}),
\]

which implies \( I_q \geq (\sqrt{B} \sqrt{A^t}) (\sqrt{B} \sqrt{A^t})^* \). Hence,

\[
B^\dagger = \sqrt{B^\dagger} \cdot I_q \cdot \sqrt{B} \geq \sqrt{B^\dagger} (\sqrt{B} \sqrt{A^t}) (\sqrt{B} \sqrt{A^t})^* \sqrt{B} \geq 0_{q \times q}.
\]

In view of \( \sqrt{B} (\sqrt{B} \sqrt{A^t})^* \sqrt{B} = B^\dagger BA^1BB^\dagger \), then (ii) is fulfilled.

\((ii) \Rightarrow (iii)\): From (ii) we get

\[
B = BB^\dagger B \geq BB^\dagger BA^1BB^\dagger B \geq 0_{q \times q}.
\]

In view of \( BB^\dagger BA^1BB^\dagger B = BA^1B \) and (ii) then (iii) is fulfilled.

\((iii) \Rightarrow (iv)\), \((iv) \Rightarrow (i)\): These implications are immediate consequences of a well-known characterization of non-negative Hermitian block matrices (see, e.g., [16, Lemma A.1.9, p. 18]) and the equation \( BB^\dagger B = B \).

Finally, suppose that (i) is fulfilled. Hence, we have \( N(A) \subseteq N(B) \), which, in view of Remark A.4, implies \( AA^1B = B \). Thus, we obtain

\[
N(B) \subseteq N(BA^1B) = N(\sqrt{A^tB} (\sqrt{A^tB})^*) = N(\sqrt{A^tB}) \subseteq N(A \sqrt{A^t \sqrt{A^tB}}) = N(AA^1B) = N(B),
\]

i.e., \( N(BA^1B) = N(B) \). Because of \( A, B \in \mathbb{C}_H^{q \times q} \), this yields \( R(BA^1B) = R(B) \). \( \square \)
Now we state some basic facts on the class
\[ C_{\text{EP}}^{q \times q} := \{ A \in C^{q \times q} | R(A^*) = R(A) \}. \]

**Proposition A.8.** Let \( A \in C^{q \times q} \). Then the following statements are equivalent:

(i) \( A \in C_{\text{EP}}^{q \times q} \).

(ii) \( N(A^*) = N(A) \).

(iii) \( AA^\dagger = A^\dagger A \).

(iv) \( A^\dagger \in C_{\text{EP}}^{q \times q} \).

(v) \( \text{Re}(A^\dagger) = A^\dagger \text{Re}(A)(A^\dagger)^* \) and \( \text{Im}(A^\dagger) = -A^\dagger \text{Im}(A)(A^\dagger)^* \).

(vi) \( \text{Re}(A^\dagger) = (A^\dagger)^* \text{Re}(A)A^\dagger \) and \( \text{Im}(A^\dagger) = -(A^\dagger)^* \text{Im}(A)A^\dagger \).

(vii) \( \text{Re} A = A \text{Re}(A^\dagger)A^* \) and \( \text{Im} A = -A \text{Im}(A^\dagger)A^* \).

(viii) \( \text{Re} A = A^* \text{Re}(A^\dagger)A \) and \( \text{Im} A = -A^* \text{Im}(A^\dagger)A \).

**Proof.** The equivalence of (i), (ii), (iii), and (iv) can be found in Cheng/Tian [10] or Tian/Wang [43]. What concerns the equivalence of (i), (v), (vi), (vii), and (viii) we refer to [20, Proposition A.6]. \( \square \)

**Lemma A.9 ([20, Lemma A.10]).** Let \( A \in I_{\geq q} \) where \( I_{\geq q} \) is given via (2.1). Then \( N(A) \subseteq N(\text{Im} A), R(\text{Im} A) \subseteq R(A), \) and \( A \in C_{\text{EP}}^{q \times q} \).

At the end of this section, we give a slight generalization of a result due to S. L. Campbell and C. D. Meyer Jr. This result can be proved by an obvious modification of the proof given in [8, Theorem 10.4.1].

**Lemma A.10 ([8, Theorem 10.4.1]).** Let \( (A_n)_{n=1}^{\infty} \) be a sequence of complex \( p \times q \) matrices which converges to a complex \( p \times q \) matrix \( A \). Then \( (A_n^\dagger)_{n=1}^{\infty} \) is convergent if and only if there is a positive integer \( m \) such that \( \text{rank} A_n = \text{rank} A \) for each integer \( n \) with \( n \geq m \). In this case, \( (A_n^\dagger)_{n=1}^{\infty} \) converges to \( A^\dagger \).

**B. On Linear Fractional Transformations of Matrices**

In this appendix, we summarize some basic facts on linear fractional transformations of matrices which are needed in the paper. This material is mostly taken from [38] and [16, Section 1.6].

Let \( a \in C^{p \times p}, b \in C^{p \times q}, c \in C^{q \times p}, d \in C^{q \times q}, \) and let
\[ E := \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \]
If the set \( Q_{[c,d]} := \{ x \in \mathbb{C}^{p \times q} | \det(cx + d) \neq 0 \} \) is non-empty then the linear fractional transformation \( S_{E}^{(p,q)}: Q_{[c,d]} \to \mathbb{C}^{p \times q} \) is defined by

\[
S_{E}^{(p,q)}(x) := (ax + b)(cx + d)^{-1}.
\]

The following well-known result shows that the composition of two linear fractional transformations is again a mapping of this type.

**Proposition B.1** (see, e.g. [16, Proposition 1.6.3]). Let \( a_1, a_2 \in \mathbb{C}^{p \times p} \), let \( b_1, b_2 \in \mathbb{C}^{p \times q} \), let \( c_1, c_2 \in \mathbb{C}^{q \times p} \), and let \( d_1, d_2 \in \mathbb{C}^{q \times q} \) be such that \( \text{rank}[c_1, d_1] = \text{rank}[c_2, d_2] = q \). Furthermore, let \( E_1 := [a_1 b_1 \quad c_1 d_1] \), \( E_2 := [a_2 b_2 \quad c_2 d_2] \), \( E := E_2 E_1 \), and \( E = [a \quad b \quad c \quad d] \) be the block representation of \( E \) with \( p \times p \) block \( a \). Then \( Q := \{ x \in Q_{[c_1,d_1]} | S_{E_1}^{(p,q)}(x) \in Q_{[c_2,d_2]} \} \) is a nonempty subset of the set \( Q_{[c,d]} \) and \( S_{E_2}^{(p,q)}(S_{E_1}^{(p,q)}(x)) = S_{E}^{(p,q)}(x) \) holds true for all \( x \in Q \).

We make the following convention: If a non-empty subset \( \mathcal{G} \) of \( \mathbb{C} \) and a matrix-valued function \( V: \mathcal{G} \to \mathbb{C}^{2q \times 2q} \) with \( q \times q \) block partition \( V = [V_{11} \quad V_{12} \mid V_{21} \quad V_{22}] \) and a matrix-valued function \( F: \mathcal{G} \to \mathbb{C}^{q \times q} \) with \( \det[V_{21}(z)F(z) + V_{22}(z)] \neq 0 \) for all \( z \in \mathcal{G} \) are given, then we will use the notation \( S_{V}(F) \) for the function \( S_{V}(F): \mathcal{G} \to \mathbb{C}^{q \times q} \) defined by \( [S_{V}(F)](z) := S_{V(z)}(F(z)) \) for all \( z \in \mathcal{G} \).

### C. The Matrix Polynomials \( V_{A,B} \) and \( W_{A,B} \)

In this appendix, we study special linear \((p+q) \times (p+q)\) matrix polynomials which are intensively used in Section \[8\]. Let \( A, B \in \mathbb{C}^{p \times q} \). Then we define \( W_{A,B}: \mathbb{C} \to \mathbb{C}^{(p+q) \times (p+q)} \) and \( V_{A,B}: \mathbb{C} \to \mathbb{C}^{(p+q) \times (p+q)} \) by

\[
W_{A,B}(z) := \begin{bmatrix} zI_p - BA^\dagger & A \\ -A^\dagger & I_q - A^\dagger A \end{bmatrix}
\]

and

\[
V_{A,B}(z) := \begin{bmatrix} 0_{p \times p} & -A \\ A^\dagger & zI_q - A^\dagger B \end{bmatrix}.
\]

If \( B = 0_{p \times q} \), then we set

\[
W_A := W_{A,0_{p \times q}} \quad \text{and} \quad V_A := V_{A,0_{p \times q}}.
\]

The use of the matrix polynomial \( V_{A,B} \) was inspired by some constructions in the paper [9]. In particular, we mention [9, p. 225, formula (4.12)]. In their constructions Chen and Hu used Drazin inverses instead of Moore-Penrose inverses of matrices. Since both types of generalized inverses coincide for Hermitian matrices (see [21, Proposition A.2]) we can conclude that in generic case the matrix polynomials \( V_{A,B} \) coincide with the objects used in [9].
Remark C.1. Let $A, B \in \mathbb{C}^{p \times q}$ and let $z \in \mathbb{C}$. Then one can easily see that

$$V_{A,B}(z)W_{A,B}(z) = \text{diag} \left[ AA^\dagger, A^\dagger [A - B(I_q - A^\dagger A)] + z(I_q - A^\dagger A) \right]$$

and

$$W_{A,B}(z)V_{A,B}(z) = \begin{bmatrix} AA^\dagger & BA^\dagger A - AA^\dagger B \\ 0_{q \times p} & A^\dagger A + z(I_q - A^\dagger A) \end{bmatrix}.$$ 

Now we are going to study the linear fractional transformation generated by the matrix $W_{A,B}(z)$ for arbitrarily given $z \in \mathbb{C}$.

Lemma C.2. Let $A \in \mathbb{C}^{p \times q}$. Then:

(a) The matrix $-A$ belongs to $\mathcal{Q}_{[-A^\dagger I_q - A^\dagger A]}$. In particular, $\mathcal{Q}_{[-A^\dagger I_q - A^\dagger A]} \neq \emptyset$.

(b) Let $X \in \mathbb{C}^{p \times q}$ be such that $N(A) \subseteq N(X)$ and $R(A) \subseteq R(X)$. Then $X \in \mathcal{Q}_{[-A^\dagger I_q - A^\dagger A]}$ and $(-A^\dagger X + I_q - A^\dagger A)^{-1} = -X^\dagger A + I_q - A^\dagger A$.

Proof. (a) This follows from $-A^\dagger (-A) + I_q - A^\dagger A = I_q$.

(b) In view of Lemma A.5 we have $AA^\dagger = XX^\dagger$ and $A^\dagger A = X^\dagger X$. Hence,

$$(-A^\dagger X + I_q - A^\dagger A)(-X^\dagger A + I_q - A^\dagger A)$$

$$= A^\dagger XX^\dagger A - A^\dagger X + A^\dagger XA^\dagger A + X^\dagger A + I_q - A^\dagger A$$

$$+ A^\dagger AX^\dagger A - A^\dagger A + A^\dagger AA^\dagger A$$

$$= A^\dagger AA^\dagger A - A^\dagger X + A^\dagger XX^\dagger X - X^\dagger A + I_q - A^\dagger A + X^\dagger XX^\dagger A - A^\dagger A + A^\dagger A$$

$$= A^\dagger A - A^\dagger X + A^\dagger X - X^\dagger A + I_q - A^\dagger A + X^\dagger A = I_q.$$

This completes the proof of (b).

Lemma C.3. Let $A, B \in \mathbb{C}^{p \times q}$ and let $W_{A,B}$ be defined via (C.1). Let $X \in \mathbb{C}^{p \times q}$ be such that the inclusions $N(A) \subseteq N(X)$ and $R(A) \subseteq R(X)$ are satisfied. Furthermore, let $z \in \mathbb{C}$. Then:

(a) The matrix $X$ belongs to $\mathcal{Q}_{[-A^\dagger I_q - A^\dagger A]}$. Furthermore,

$$S_{W_{A,B}(z)}^{(p,q)}(X) = -A(zI_q + X^\dagger A) + BA^\dagger A,$$

$$N(A) \subseteq N \left( S_{W_{A,B}(z)}^{(p,q)}(X) \right),$$

and

$$R \left( S_{W_{A,B}(z)}^{(p,q)}(X) - BA^\dagger A \right) \subseteq R(A).$$

(b) If $N(A) \subseteq N(B)$, then

$$S_{W_{A,B}(z)}^{(p,q)}(X) = -A(zI_q + X^\dagger A) + B \quad \text{and} \quad R \left( S_{W_{A,B}(z)}^{(p,q)}(X) - B \right) \subseteq R(A).$$
Proof. In view of part (b) of Lemma C.2 we have $X \in \mathbb{Q}_{[-A^{\dagger}I_q-A^{\dagger}A]}$ and
\[
(-A^{\dagger}X + I_q - A^{\dagger}A)^{-1} = -X^{\dagger}A + I_q - A^{\dagger}A. \tag{C.4}
\]
Because of the choice of $X$, parts (a) and (b) of Remark A.3 yield $X^{\dagger}A = X$ and $XX^{\dagger}A = A$, and, in view of (C.4), then
\[
S_{W_{A,B}(z)}^{(p,q)}(X) = [zI_p - BA^{\dagger}]X + A \left(-A^{\dagger}X + I_q - A^{\dagger}A\right)^{-1}
\]
\[
= \left[zI_p - BA^{\dagger}\right]X + A \left(-X^{\dagger}A + I_q - A^{\dagger}A\right)
\]
\[
= -(zI_p - BA^{\dagger})XX^{\dagger}A + (zI_p - BA^{\dagger})X(X - A^{\dagger}A) - AX^{\dagger}A + A - AA^{\dagger}A \tag{C.5}
\]
\[
= -(zI_p - BA^{\dagger})A + (zI_p - BA^{\dagger})XX^{\dagger}A - AX^{\dagger}A
\]
\[
= -(zI_p - BA^{\dagger})[X(I_q - A^{\dagger}A) - A] - AX^{\dagger}A = -(zI_p - BA^{\dagger})A - AX^{\dagger}A
\]
\[
= -A(zI_q + X^{\dagger}A) + BA^{\dagger}A.
\]
The remaining assertions of (a) are immediate consequences of (C.5). (b) follows from (a) and part (a) of Remark A.3. \qed

Remark C.4. Let $A, B \in \mathbb{C}^{p \times q}$ and let $z \in \mathbb{C}$. Further, let $X \in \mathbb{Q}_{[A^{\dagger}I_q-A^{\dagger}B]}$. Then from (C.2) we see that
\[
S_{V_{A,B}(z)}^{(p,q)}(X) = -A \left[zI_q + A^{\dagger}(X - B)\right]^{\dagger}
\]
and, in view of $\det[zI_q + A^{\dagger}(X - B)] \neq 0$, thus $R(S_{V_{A,B}(z)}^{(p,q)}(X)) = R(A)$.

Lemma C.5. Let $A \in \mathbb{C}^{q \times q}$ and $B \in \mathbb{C}^{p \times q}$. Further, let $X \in \mathbb{C}^{q \times q}$ be such that
\[
X - B \in \mathcal{I}_{q,\geq}, \quad N(A) \subseteq N(X - B), \quad \text{and} \quad R(X - B) \subseteq R(A)
\]
are satisfied. For each $z \in \Pi_+$, then the following statements hold true:
\[(a) \quad \text{Im}(X + zA - B) \supseteq (\text{Im} z)A \in \mathbb{C}^{q \times q}.
\]
\[(b) \quad N(X + zA - B) \subseteq N(A).
\]
\[(c) \quad X + zA - B = A(A^{\dagger}X + zI_q - A^{\dagger}B).
\]
\[(d) \quad \text{The matrix } X \text{ belongs to } \mathbb{Q}_{[A^{\dagger}I_q-A^{\dagger}B]}. \quad \text{Furthermore,}
\]
\[
S_{V_{A,B}(z)}^{(q,q)}(X) = -A(X + zA - B)^{\dagger}A \quad \text{and} \quad N(A) \subseteq N \left(S_{V_{A,B}(z)}^{(q,q)}(X)\right).
\]
Proof. Let \( z \in \mathbb{C} \).

(a) Because of \( A \in \mathbb{C}_{\geq}^{q \times q} \), \( \text{Im}(X - B) \in \mathbb{C}_{\geq}^{q \times q} \), and \( \text{Im} z \in (0, +\infty) \), we have
\[
\text{Im}(X + zA - B) = \text{Im}(X - B) + (\text{Im} z)A \geq (\text{Im} z)A \in \mathbb{C}_{\geq}^{q \times q}.
\]

(b) From (a) and \( \text{Im} z \neq 0 \) we get
\[
N((\text{Im} z)A) = N(A).
\] (C.6)

In view of (a), we have \( X + zA - B \in I_{q, \geq} \). Thus, Lemma A.9 gives
\[
N(X + zA - B) \subseteq N(\text{Im}(X + zA - B)).
\] (C.7)

Now the combination of (C.7) and (C.6) yields (b).

(c) In view of \( R(X - B) \subseteq R(A) \), Remark A.3 gives \( AA^\dagger(X - B) = X - B \). Thus,
\[
X + zA - B = zA + AA^\dagger(X - B) = A(A^\dagger X + zI_q - A^\dagger B).
\]

(d) We first prove that
\[
N(A^\dagger X + zI_q - A^\dagger B) = \{0_{q \times 1}\}.
\] (C.8)

In view of (c), (b) and \( N(A) \subseteq N(X - B) \), we get
\[
N(A^\dagger X + zI_q - A^\dagger B) \subseteq N(X + zA - B) \subseteq N(A) \subseteq N(X - B).
\] (C.9)

Let
\[
v \in N(A^\dagger X + zI_q - A^\dagger B).
\] (C.10)

From (C.10) and (C.9) we get then \( (X - B)v = 0_{q \times 1} \). Thus, taking again (C.10) into account we see
\[
v = \frac{1}{z} \left[ zv + A^\dagger(X - B)v \right] = \frac{1}{z}(A^\dagger X + zI_q - A^\dagger B)v = \frac{1}{z} \cdot 0_{q \times 1} = 0_{q \times 1}.
\]

Thus (C.8) is proved. Hence
\[
X \in Q_{[A, zI_q - A^\dagger B]}.
\] (C.11)

In view of (b) we infer from part (a) of Remark A.3 then
\[
A(X + zA - B)^\dagger(X + zA - B) = A.
\] (C.12)

Using (C.2), (C.11), (C.12) and (c) we get
\[
S_{V, A, B}^{(q, q, q)}(X) = -A(A^\dagger X + zI_q - A^\dagger B)^{-1}
= -A(X + zA - B)^\dagger(X + zA - B)(A^\dagger X + zI_q - A^\dagger B)^{-1}
= -A(X + zA - B)^\dagger A(A^\dagger X + zI_q - A^\dagger B)(A^\dagger X + zI_q - A^\dagger B)^{-1}
= -A(X + zA - B)^\dagger A.
\]

The last equation implies
\[
N(A) \subseteq N\left(S_{V, A, B}^{(q, q)}(X)\right).
\]

The proof is complete. \( \square \)
References

[1] N. I. Akhiezer, The classical moment problem and some related questions in analysis, Translated by N. Kemmer, Hafner Publishing Co., New York, 1965. MR0184042 (32 #1518)
[2] D. Alpay, A. Dijksma, and H. Langer, Factorization of J-unitary matrix polynomials on the line and a Schur algorithm for generalized Nevanlinna functions, Linear Algebra Appl. 387 (2004), 313–342. MR2069282 (2005b:47029)
[3] B. D. O. Anderson and E. I. Jury, Generalized Bezoutian and Sylvester matrices in multivariable linear control, IEEE Trans. Automatic Control AC-21 (1976), no. 4, 551–556. MR0444175 (56 #2533)
[4] Yu. M. Arlinskii, S. V. Belyi, and E. R. Tsekanovskii, Conservative realizations of Herglotz-Nevanlinna functions, Operator Theory: Advances and Applications, vol. 217, Birkhäuser/Springer Basel AG, Basel, 2011. MR2828331
[5] M. Bakonyi and H. J. Woerdeman, Matrix completions, moments, and sums of Hermitian squares, Princeton University Press, Princeton, NJ, 2011. MR2807419 (2012d:47003)
[6] S. V. Belyi and E. R. Tsekanovskii, Realization theorems for operator-valued R-functions, in: New results in operator theory and its applications—The Israel M. Glazman Memorial Volume (I. Gohberg and Yu. I. Lyubich, eds.), Oper. Theory Adv. Appl., vol. 98, Birkhäuser, Basel, 1997, pp. 55–91. MR1478466 (98k:47018)
[7] V. A. Bolotnikov, On degenerate Hamburger moment problem and extensions of nonnegative Hankel block matrices, Integral Equations Operator Theory 25 (1996), no. 3, 253–276, DOI 10.1007/BF01262294. MR1395706 (97k:44018)
[8] S. L. Campbell and C. D. Meyer Jr., Generalized inverses of linear transformations, Dover Publications Inc., New York, 1991. Corrected reprint of the 1979 original. MR1105324 (92a:15003)
[9] G.-n. Chen and Y.-j. Hu, The truncated Hamburger matrix moment problems in the nondegenerate and degenerate cases, and matrix continued fractions, Linear Algebra Appl. 277 (1998), no. 1-3, 199–236. MR1624548 (99j:44015)
[10] S. Cheng and Y. Tian, Two sets of new characterizations for normal and EP matrices, Linear Algebra Appl. 375 (2003), 181–195. MR2013464 (2004m:15006)
[11] A. E. Choque Rivero, Y. M. Dyukarev, B. Fritzsche, and B. Kirstein, A truncated matricial moment problem on a finite interval, in: Interpolation, Schur functions and moment problems (D. Alpay and I. Gohberg, eds.), Oper. Theory Adv. Appl., vol. 165, Birkhäuser, Basel, 2006, pp. 121–173. MR2222521 (2007b:47034)
[12] V. A. Derkach, S. Hassi, and H. de Snoo, Truncated moment problems in the class of generalized Nevanlinna functions, ArXiv e-prints (Dec. 30, 2010). arXiv:1101.0162v1 [math.FA].
[13] M. S. Derevyagin, On the Schur algorithm for indefinite moment problem, Methods Funct. Anal. Topology 9 (2003), no. 2, 133–145. MR1999775 (2004j:30075)
[14] V. K. Dubovoi, Parametrization of a multiple elementary factor of a non full rank, Analysis in Infinite Dimensional Spaces and Operator Theory, 1983, pp. 54–68.
[15] , Indefinite metric in Schur’s interpolation problem for analytic functions. IV, Teor. Funktsii Funktsional. Anal. i Prilozhen. 42 (1984), 46–57 (Russian). MR751390 (86c:47008)
[16] V. K. Dubovoi, B. Fritzsche, and B. Kirstein, Matricial version of the classical Schur problem, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 129, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1992. With German, French and Russian summaries. MR1152328 (93e:47021)
[17] H. Dym, On Hermitian block Hankel matrices, matrix polynomials, the Hamburger moment problem, interpolation and maximum entropy, Integral Equations Operator Theory 12 (1989), no. 6, 757–812. MR1018213 (91c:30005)
18] Y. M. Dyukarev, B. Fritzsche, B. Kirstein, and C. Mädl er, *On truncated matricial Stieltjes type moment problems*, Complex Anal. Oper. Theory 4 (2010), no. 4, 905–951. MR2735313 (2011i:44009)

19] Y. M. Dyukarev, B. Fritzsche, B. Kirstein, and H. C. Thiele, *On distinguished solutions of truncated matricial Hamburger moment problems*, Complex Anal. Oper. Theory 3 (2009), no. 4, 759–834. MR2570113

20] B. Fritzsche, B. Kirstein, A. Lasarow, and A. Rahn, *On Reciprocal Sequences of Matricial Carathéodory Sequences and Associated Matrix Functions*, in: Interpolation, Schur functions and moment problems II (D. Alpay and B. Kirstein, eds.), Oper. Theory Adv. Appl., vol. 226, Birkhäuser/Springer Basel AG, Basel, 2012, to appear.

21] B. Fritzsche, B. Kirstein, and C. Mädl er, *On Hankel nonnegative definite sequences, the canonical Hankel parametrization, and orthogonal matrix polynomials*, Complex Anal. Oper. Theory 5 (2011), no. 2, 447–511. MR2805417

22] *On Matrix-Valued Herglotz-Nevanlinna Functions with an Emphasis on Particular Subclasses*, Math. Nachr., to appear.

23] B. Fritzsche, B. Kirstein, C. Mädl er, and H. C. Thiele, *On distinguished solutions of truncated matricial Hamburger moment problems*, Complex Anal. Oper. Theory 3 (2009), no. 4, 759–834. MR2570113

24] B. Fritzsche, B. Kirstein, and C. Mädl er, *On Hankel nonnegative definite sequences, the canonical Hankel parametrization, and orthogonal matrix polynomials*, Complex Anal. Oper. Theory 5 (2011), no. 2, 447–511. MR2805417

25] M. I. Gekhtman and M. Shmoish, *On invertibility of nonsquare generalized Bezoutians*, Linear Algebra Appl. 223/224 (1995), 205–241. Special issue honoring Miroslav Fiedler and Vlastimil Pták. MR1340693 (96g:15024)

26] F. Gesztesy, N. J. Kalton, K. A. Makarov, and E. R. Tsekanovskii, *Some applications of operator-valued Herglotz functions*, Operator theory, system theory and related topics—The Moshe Livsic Anniversary Volume (Beer-Sheva/Rehovot, 1997) (D. Alpay and V. Vinnikov, eds.), Oper. Theory Adv. Appl., vol. 123, Birkhäuser, Basel, 2001, pp. 271–321. MR1821917 (2002f:47049)

27] F. Gesztesy and E. R. Tsekanovskii, *On matrix-valued Herglotz functions*, Math. Nachr. 218 (2000), 61–138. MR1784638 (2001j:47018)

28] S. Hassi and H. S. V. de Snoo, *On some subclasses of Nevanlinna functions*, Z. Anal. Anwendungen 15 (1996), no. 1, 45–55. MR1376588 (96k:47044)

29] S. Hassi, H. S. V. de Snoo, and A. D. I. Willemsma, *Smooth rank one perturbations of selfadjoint operators*, Proc. Amer. Math. Soc. 126 (1998), no. 9, 2663–2675. MR1451805 (98k:47029)

30] I. S. Kats, *On Hilbert spaces generated by monotone Hermitian matrix-functions*, Har’kov Gos. Univ. Uč. Zap. 34 = Zap. Mat. Otč. Fiz.-Mat. Fak. i Har’kov. Mat. Obšć. (4) 22 (1950), 95–113 (1951) (Russian). MR0080280 (18,222d)

31] I. V. Kovalishina, *Analytic theory of a class of interpolation problems*, Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), no. 3, 455–497 (Russian); English transl., Math. USSR Izvestiya 22 (1984), 419–463. MR703593 (84i:30043)
[35] M. Kreĭn, *Infinite J-matrices and a matrix-moment problem*, Doklady Akad. Nauk SSSR (N.S.) 69 (1949), 125–128 (Russian). MR0034964 (11,670a)

[36] M. G. Kreĭn and M. A. Krasnosel’skiĭ, *Fundamental theorems on the extension of Hermitian operators and certain of their applications to the theory of orthogonal polynomials and the problem of moments*, Uspekhi Matem. Nauk (N. S.) 2 (1947), no. 3(19), 60–106 (Russian). MR0026759 (10,198b)

[37] R. Nevanlinna, *Asymptotische Entwicklung beschränkter Funktionen und das Stieltjessche Momentenproblem*, Ann. Acad. Sci. Fennicae (A) 18 (1922), no. 5, 1–53 (German).

[38] V. P. Potapov, *Linear-fractional transformations of matrices*, Studies in the theory of operators and their applications (Russian), “Naukova Dumka”, Kiev, 1979, pp. 75–97, 177 (Russian). MR566141 (81f:15023)

[39] C. R. Rao and S. K. Mitra, *Generalized inverse of matrices and its applications*, John Wiley & Sons, Inc., New York-London-Sydney, 1971. MR0338013 (49 #2780)

[40] M. Rosenberg, *The square-integrability of matrix-valued functions with respect to a non-negative Hermitian measure*, Duke Math. J. 31 (1964), 291–298. MR0163346 (29 #649)

[41] L. A. Sakhnovich, *Interpolation theory and its applications*, Mathematics and its Applications, vol. 428, Kluwer Academic Publishers, Dordrecht, 1997. MR1631843 (99j:47016)

[42] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind*. I 147 (1917), 205–232; II 148 (1918), 122–145, J. reine u. angew. Math.

[43] Y. Tian and H. Wang, *Characterizations of EP matrices and weighted-EP matrices*, Linear Algebra Appl. 434 (2011), no. 5, 1295–1318, DOI 10.1016/j.laa.2010.11.014. MR2763588 (2011k:15014)