The Positive Mass Theorem for Manifolds with Distributional Curvature

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Abstract: We formulate and prove a positive mass theorem for \( n \)-dimensional spin manifolds whose metrics have only the Sobolev regularity \( C^0 \cap W^{1,n}_{\text{loc}} \). At this level of regularity, the curvature of the metric is defined in the distributional sense only, and we propose here a (generalized) notion of ADM mass for such a metric. Our main theorem establishes that if the manifold is asymptotically flat and has non-negative scalar curvature distribution, then its (generalized) ADM mass is well-defined and non-negative, and vanishes only if the manifold is isometric to Euclidian space. Prior applications of Witten’s spinor method by Lee and Parker and by Bartnik required the much stronger regularity \( W^{2,p}_{\text{loc}} \) with \( p > n \). Our proof is a generalization of Witten’s arguments, in which we must treat the Dirac operator and its associated Lichnerowicz-Weitzenböck identity in the distributional sense and cope with certain averages of first-order derivatives of the metric over annuli that approach infinity. Finally, we observe that our arguments are not specific to scalar curvature and also allow us to establish a “universal” positive mass theorem.

1. Introduction

A fundamental problem in Riemannian geometry is to understand generalized notions of curvature restrictions. For example, Toponogov’s theorem motivates a generalized notion of non-negative sectional curvature that makes sense for length spaces, and the theory of these length spaces of non-negative sectional curvature could be thought of as the gold standard for a theory of “singular curvature”.

In this paper we consider the question of whether metrics with non-negative scalar curvature \textit{in the sense of distributions} share any interesting properties with honest-to-goodness \( C^2 \)-regular metrics with non-negative scalar curvature in the classical sense. Our main result (in Theorem 1.1 below) is that the positive mass theorem generalizes to
this setting. One reason to consider weak regularity for the positive mass theorem\(^1\) is for application to stability of the positive mass theorem (cf. [11,12,16] and the references therein).

Recall that the positive mass theorem was established by Schoen and Yau in dimensions \(n\) less than eight [22,23] and Witten for spin manifolds [28], under the assumption that the underlying metric is regular. (See also [18] for some advances in the general case.) Bartnik [2] showed that Witten’s spinor argument works whenever the metric is\(^2\) \(W^{2,p}_{loc}\) with \(p > n\). For the slightly weaker integrability class \(C^0 \cap W^{2,n/2}_{loc}\), see [6]. As far as solely “piecewise regular” metrics are concerned, Miao [20] used a smoothing plus conformal deformation (following Bray [4]) and proved a version of the positive mass theorem for metrics that are singular only along a hypersurface. Similar results were also proved by Shi and Tam [26] (using Witten’s spinor method) and McFeron and Szkeleyhidi [19] (using the Ricci flow). The conformal deformation method was also used by Lee [10] to treat metrics with low-dimensional singular sets.

Our result only assumes that the metric is \(C^0 \cap W^{1,n}_{loc}\) and thereby generalizes all of those previous results in the spin case, as explained in Sect. 5. Our result also fits together with and was motivated by earlier work by LeFloch and collaborators [13–15,17], who defined and investigated the Einstein equations within the broad class of metrics with \(L^\infty \cap W^{1,2}_{loc}\) regularity and established existence results for the Cauchy problem at this level of regularity.

We state here our main result and refer to Sect. 2 below for details.

**Theorem 1.1** (The positive mass theorem for distributional curvature). Let \(M\) be a smooth \(n\)-manifold (\(n \geq 3\)) endowed with a spin structure and a \(C^0 \cap W^{1,n}_{loc}\) regular and asymptotically flat Riemannian metric \(g\) with \(q \geq (n-2)/2\). If the distributional scalar curvature \(R_g\) of \(g\) is non-negative, then its generalized ADM mass, denoted by \(m_{ADM}(M, g)\), is non-negative, that is,

\[
m_{ADM}(M, g) \geq 0,
\]

Moreover, equality occurs only when \((M, g)\) is isometric to Euclidean space.

Note that under the conditions in Theorem 1.1, the mass \(m_{ADM}(M, g)\) exists but could be (positive) infinite; however, we will present a “finiteness” condition at infinity that guarantees that the ADM mass is finite. We also point out that once the appropriate spaces are defined, it follows from the Sobolev embedding theorem that \(W^{1,p}_{-q} \subset C^0 \cap W^{1,n}_{-q}\) for any \(p > n\), so that Theorem 1.1 holds in the class \(W^{1,p}_{-q}\) for any \(p > n\).

Our proof of Theorem 1.1 relies on the following main ideas:

- The first difficulty, dealt with in Sect. 2, is defining the notions required in the statement of the theorem, including the concepts of \(W^{1,n}_{-q}\) asymptotic flatness, distributional curvature, and generalized ADM mass.

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\(^1\) After completion of this work, Cox pointed out to us that Theorem 1.1 implies that if a sequence of smooth complete asymptotically flat metrics of non-negative scalar curvature happens to converge in \(C^1\) and has mass converging to zero, then that limit space must be Euclidean. By applying this argument he can deduce a topological positive mass stability theorem.

\(^2\) We use the standard notation for the Lebesgue spaces \(L^p\) and \(L^p_{loc}\) and the Sobolev spaces \(W^{k,p}\) and \(W^{k,p}_{loc}\).