MIXED QUIVER ALGEBRAS

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Abstract. In this paper we introduce a new class of $K$-algebras associated with quivers. Given any finite chain $K_r : K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_r$ of fields and a chain $E_r : H_0 \subset H_1 \subset \cdots \subset H_r = E^0$ of hereditary saturated subsets of the set of vertices $E^0$ of a quiver $E$, we build the mixed path algebra $P_{K_r}(E, H_r)$, the mixed Leavitt path algebra $L_{K_r}(E, H_r)$ and the mixed regular path algebra $Q_{K_r}(E, H_r)$ and we show that they share many properties with the unmixed species $P_K(E)$, $L_K(E)$ and $Q_K(E)$.

Introduction

The work in the present paper is instrumental for the constructions developed in [4], where the regular algebra of a finite poset has been introduced in connection with the realization problem for von Neumann regular rings, see also [5], [15] and [3]. The reader is referred to these papers for further information on the realization problem, and to [1], [2], [7], [6] for related work on Leavitt path algebras.

In the following, $K$ will denote a field and $E = (E^0, E^1, r, s)$ a finite quiver (oriented graph) with $E^0 = \{1, \ldots, d\}$. Here $s(e)$ is the source vertex of the arrow $e$, and $r(e)$ is the range vertex of $e$. A path in $E$ is either an ordered sequence of arrows $\alpha = e_1 \cdots e_n$ with $r(e_t) = s(e_{t+1})$ for $1 \leq t < n$, or a path of length 0 corresponding to a vertex $i \in E^0$, which will be denoted by $p_i$. The paths $p_i$ are called trivial paths, and we have $r(p_i) = s(p_i) = i$. A non-trivial path $\alpha = e_1 \cdots e_n$ has length $n$ and we define $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$. We will denote the length of a path $\alpha$ by $|\alpha|$, the set of all paths of length $n$ by $E^n$, for $n > 1$, and the set of all paths by $E^*$.

For $v, w \in E^0$, set $v \geq w$ in case there is a (directed) path from $v$ to $w$. A subset $H$ of $E^0$ is called hereditary if $v \geq w$ and $v \in H$ imply $w \in H$. A set is saturated if every vertex which feeds into $H$ and only into $H$ is again in $H$, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by $H$ (or by $H_E$ when it is necessary to emphasize the dependence on $E$) the set of hereditary saturated subsets of $E^0$.

Date: September 2, 2009.

2000 Mathematics Subject Classification. Primary 16D70; Secondary 06A12, 06F05, 46L80.

Key words and phrases. von Neumann regular ring, path algebra, Leavitt path algebra, universal localization.

Both authors were partially supported by DGI MICIN-FEDER MTM2008-06201-C02-01, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The second author was partially supported by a grant of the Departament de Matemàtiques, Universitat Autònoma de Barcelona.
Let us recall the construction from [5] of the regular algebra \( Q_K(E) \) of a quiver \( E \), although we will follow the presentation in [4] rather than the used in [5]. That is, relations (CK1) and (CK2) below are reversed with respect to their counterparts in [5], so that we are led to work primarily with left modules instead of right modules.

Therefore we recall the basic features of the regular algebra \( Q_K(E) \) in terms of the notation used here. We will only need finite quivers in the present paper, so we restrict attention to them. The algebra \( Q(E) := Q_K(E) \) fits into the following commutative diagram of injective algebra morphisms:

\[
\begin{array}{cccc}
K^d & \longrightarrow & P(E) & \longrightarrow P_{\text{rat}}(E) & \longrightarrow P((E)) \\
\downarrow & & \downarrow \iota_{\Sigma_1} & & \downarrow \iota_{\Sigma_1} \\
\overline{P(E)} & \longrightarrow & L(E) & \longrightarrow Q(E) & \longrightarrow U(E)
\end{array}
\]

Here \( P(E) \) is the path \( K \)-algebra of \( E \), \( \overline{E} \) denotes the inverse quiver of \( E \), that is, the quiver obtained by reversing the orientation of all the arrows in \( E \), \( P((E)) \) is the algebra of formal power series on \( E \), and \( P_{\text{rat}}(E) \) is the algebra of rational series, which is by definition the division closure of \( P(E) \) in \( P((E)) \) (which agrees with the rational closure [5, Observation 1.18]). The maps \( \iota_{\Sigma} \) and \( \iota_{\Sigma_1} \) indicate universal localizations with respect to the sets \( \Sigma \) and \( \Sigma_1 \) respectively. Here \( \Sigma \) is the set of all square matrices over \( P(E) \) that are sent to invertible matrices by the augmentation map \( \epsilon : P(E) \to K^{[E^0]} \). By [5, Theorem 1.20], the algebra \( P_{\text{rat}}(E) \) coincides with the universal localization \( P(E) \Sigma_{-1}^{-1} \). The set \( \Sigma_1 = \{ \mu_v \mid v \in E^0, s^{-1}(v) \neq \emptyset \} \) is the set of morphisms between finitely generated projective left \( P(E) \)-modules defined by

\[
\mu_v : P(E)v \longrightarrow \bigoplus_{i=1}^{n_v} P(E)r(e_i^v)
\]

for any \( v \in E^0 \) such that \( s^{-1}(v) \neq \emptyset \). By a slight abuse of notation, we use also \( \mu_v \) to denote the corresponding maps between finitely generated projective left \( P_{\text{rat}}(E) \)-modules and \( P((E)) \)-modules respectively.

The following relations hold in \( Q(E) \):

\begin{align*}
\text{(V)} & \quad p_v p_{v'} = \delta_{v,v'} p_v \text{ for all } v, v' \in E^0. \\
\text{(E1)} & \quad p_{s(e)} e = e p_{r(e)} = e \text{ for all } e \in E^1. \\
\text{(E2)} & \quad p_{r(e)} \overline{t} = \overline{t} p_{s(e)} = \overline{t} \text{ for all } e \in E^1. \\
\text{(CK1)} & \quad \overline{t} e' = \delta_{e,e'} p_{r(e)} \text{ for all } e, e' \in E^1. \\
\text{(CK2)} & \quad p_v = \sum_{\{e \in E^1 \mid s(e) = v\}} e \overline{t} \text{ for every } v \in E^0 \text{ that emits edges.}
\end{align*}

The Leavitt path algebra \( L(E) = P(E) \Sigma_{-1}^{-1} \) is the algebra generated by \( \{ p_v \mid v \in E^0 \} \cup \{ e, \overline{t} \mid e \in E^1 \} \) subject to the relations (1)–(5) above. By [5, Theorem 4.2], the algebra \( Q(E) \) is a von Neumann regular hereditary ring and \( Q(E) = P(E)(\Sigma \cup \Sigma_1)^{-1} \).
Here the set $\Sigma$ can be clearly replaced with the set of all square matrices of the form $I_n + B$ with $B \in M_n(P(E))$ satisfying $\epsilon(B) = 0$, for all $n \geq 1$.

1. Structure of ideals

The structure of the lattice of ideals of $Q(E)$ can be neatly computed from the graph. Let $H$ be a hereditary saturated subset of $E^0$. Define the graph $E/H$ by $(E/H)^0 = E^0 \setminus H$ and $(E/H)^1 = \{e \in E^1 : r(e) \notin H\}$, with the functions $r$ and $s$ inherited from $E$. We also define $E_H$ as the restriction of the graph $E$ to $H$, that is $(E_H)^0 = H$ and $(E_H)^1 = \{e \in E^1 : s(e) \in H\}$. For $Y \subseteq E^0$ set $p_Y = \sum_{v \in Y} p_v$.

**Proposition 1.1.** (a) The ideals of $Q(E)$ are in one-to-one correspondence with the order-ideals of $M_E$ and consequently with the hereditary and saturated subsets of $E$.

(b) If $H$ is a hereditary saturated subset of $E$, then $Q(E)/I(H) \cong Q(E/H)$, where $I(H)$ is the ideal of $Q(E)$ generated by the idempotents $p_v$ with $v \in H$.

(c) Let $H$ be a hereditary subset of $E^0$. Then the following properties hold:

1. $P(E_H) = p_H P(E) = p_H P(E) p_H$,
2. $P((E_H)) = p_H P((E)) = p_H P((E)) p_H$,
3. $p_{\text{rat}}(E_H) = p_H p_{\text{rat}}(E) = p_H p_{\text{rat}}(E) p_H$,
4. $Q(E_H) \cong p_H Q(E) p_H$.

**Proof.** (a) By [5, Theorem 4.2] we have a monoid isomorphism $\mathcal{V}(Q(E)) \cong M_E$. Since $Q(E)$ is von Neumann regular, we have a lattice isomorphism $L_2(Q(E)) \cong L(M_E)$, where $L_2(Q(E))$ denotes the lattice of two-sided ideals of $Q(E)$ and $L(M_E)$ denotes the lattice of order-ideals of $M_E$, cf. [16, Proposition 7.3]. Now by [8, Proposition 5.2] there is a lattice isomorphism $L(M_E) \cong \mathcal{H}$, where $\mathcal{H}$ is the lattice of hereditary saturated subsets of $E^0$. Given an ideal $I$ of $Q(E)$, the set of vertices $v$ such that $p_v \in I$ is a hereditary saturated subset of $E^0$ which generates $I$ as an ideal.

(b) We shall use some universal properties. Let $H$ be a hereditary saturated subset of $E^0$ and let $I(H)$ be the ideal of $Q(E)$ generated by $H$. By [10, Lemma 2.3], there is a $K$-algebra isomorphism $\varphi: L(E)/J \to L(E/H)$, where $J$ is the ideal of $L(E)$ generated by the idempotents $p_v$ with $v \in H$. The isomorphism $\varphi$ is defined in such a way that it is the identity on $(E/H)^*$ and 0 on $E^* \setminus (E/H)^*$. Write $\Sigma(E)$ (resp. $\Sigma(E/H)$) for the set of matrices of the form $I_n + B$, where $B \in M_n(P(E))$ (resp. $B \in M_n(P(E/H))$) satisfies $\epsilon(B) = 0$. Clearly $\varphi(\Sigma(E)) \subseteq \Sigma(E/H)$ so that the map

$$\tilde{\varphi} = \varphi \circ \pi: L(E) \to L(E)/J \to L(E/H)$$

gives rise to an algebra homomorphism $Q(E) \to Q(E/H)$ which is 0 on $H$, so we get a homomorphism $\rho: Q(E)/I(H) \to Q(E/H)$.

To construct the inverse, consider the map $\psi: L(E/H) \to Q(E)/I(H)$ which is given by the composition of $\varphi^{-1}: L(E/H) \to L(E)/J$ and the natural map $L(E)/J \to Q(E)/I(H)$. Clearly $\psi(\Sigma(E/H))$ is contained in the set of invertible matrices over $Q(E)/I(H)$, because each element in $\Sigma(E/H)$ can be lifted to an element in $\Sigma(E)$. It follows from the universal property of $Q(E/H)$ that there is a unique homomorphism
\( \lambda : Q(E/H) \to Q(E)/I(H) \) extending \( \psi \). Using uniqueness of extensions, it is fairly easy to see that \( \lambda \circ \rho = \text{Id}_{Q(E)/I(H)} \) and \( \rho \circ \lambda = \text{Id}_{Q(E/H)} \).

(c) (1), (2): This is clear from the fact that \( H \) is a hereditary subset of \( E^0 \).

(3) The algebra \( p_H P_{\text{rat}}(E) = p_H P_{\text{rat}}(E)p_H \) is rationally closed in \( p_H P((E))p_H = P((E_H)) \) and contains \( P(E_H) \), so that \( P_{\text{rat}}(E_H) \subseteq p_H P_{\text{rat}}(E) \).

It remains to show that \( p_H P_{\text{rat}}(E) \subseteq P_{\text{rat}}(E_H) \). If \( a \in P_{\text{rat}}(E) \), there exist by [13, Theorem 7.1.2] a row \( \gamma \in P(E) \), a column \( \delta \in P(E)^n \) and a matrix \( B \in M_n(P(E)) \) such that \( \epsilon(B) = 0 \) such that

\[
a = \gamma (I - B)^{-1} \delta.
\]

Note that, since \( H \) is a hereditary subset of \( E^0 \), we have \( p_H \tau = p_H \tau p_H \) for every matrix \( \tau \) over \( P((E)) \). Applying this we get

\[
p_H a = (p_H \gamma p_H)(p_H I_n - (p_H B p_H))^{-1}(p_H \delta p_H),
\]

which shows that \( p_H a = p_H a p_H \in P_{\text{rat}}(E_H) \).

(4) We have a map \( P(E_H) = p_H P(E)p_H \to p_H Q(E)p_H \) which is clearly \( (\Sigma(E_H) \cup \Sigma_1(E_H))^{-1} \)-inverting and thus induces a \( K \)-algebra homomorphism \( Q(E_H) \to p_H Q(E)p_H \). Since this map does not annihilate any basic idempotent \( p_v \), we conclude from (a) that it is injective, so that we can consider \( Q(E_H) \) as a subalgebra of \( p_H Q(E)p_H \).

To show the reverse containment, recall from [5] that an element \( a \in Q(E) \) can be written as a finite sum

\[
a = \sum_{\gamma \in E^*} a_{\gamma} \gamma,
\]

where \( a_{\gamma} \in P_{\text{rat}}(E)p_{s(\gamma)} \). We get

\[
p_H a p_H = \sum_{\gamma \in (E_H)^*} (p_H a_{\gamma} p_H) \gamma
\]

with \( p_H a_{\gamma} p_H = p_H a_{\gamma} \in P_H P_{\text{rat}}(E) = P_{\text{rat}}(E_H) \) by (c). Thus \( p_H a p_H \in Q(E_H) \) as desired.

\[
\]

2. Mixed quiver algebras

Since we will be playing in this section with different fields, it will be convenient that our notation remembers the field we are considering, henceforth we will denote the path \( K \)-algebra by \( P_K(E) \), the regular \( K \)-algebra of the quiver by \( Q_K(E) \), and so on.

Let \( K \subseteq L \) be a field extension and let \( E \) be a finite quiver. There is an obvious \( K \)-algebra homomorphism \( h : Q_K(E) \to Q_L(E) \) which satisfies \( h(p_v) \neq 0 \) for all \( v \in E^0 \). It follows from Proposition [14] that the map \( h \) is injective. Using this map, we will view \( Q_K(E) \) as a \( K \)-subalgebra of \( Q_L(E) \). Let \( H \) be a hereditary saturated subset of \( E^0 \) and consider the idempotent

\[
p_H = \sum_{v \in H} p_v \in Q_K(E) \subseteq Q_L(E).
\]
By Proposition 1.1(c)(4) we have that \( p_H \mathcal{Q}_L(E) p_H \cong \mathcal{Q}_L(E_H) \), where \( E_H \) denotes the restriction of \( E \) to \( H \). The mixed regular path algebra \( \mathcal{Q}_{K \subseteq L}(E, H) \) is defined as the \( K \)-subalgebra of \( \mathcal{Q}_L(E) \) generated by \( \mathcal{Q}_K(E) \) and \( p_H \mathcal{Q}_L(E) p_H \). Observe that

\[
\mathcal{Q}_{K \subseteq L}(E, H) = \mathcal{Q}_K(E) + \mathcal{Q}_K(E)(p_H \mathcal{Q}_L(E) p_H) \mathcal{Q}_K(E)
\]

and that \( I = \mathcal{Q}_K(E)(p_H \mathcal{Q}_L(E) p_H) \mathcal{Q}_K(E) \) is an ideal in \( \mathcal{Q} = \mathcal{Q}_{K \subseteq L}(E, H) \) such that \( \mathcal{Q} / I \cong \mathcal{Q}_K(E / H) \), because \( I \cap \mathcal{Q}_K(E) \) agrees with the ideal \( I_K(H) \) of \( \mathcal{Q}_K(E) \) generated by \( H \).

**Definition 2.1.** Let \( K_0 \subseteq K_1 \subseteq \cdots \subseteq K_r \) be a chain of fields. Let \( E \) be a finite quiver and let \( H_0 \subset H_1 \subset \cdots \subset H_r = E^0 \) be a chain of hereditary saturated subsets of \( E^0 \). We build rings \( R_i, i = 0, 1, \ldots, r \) inductively as follows:

1. \( R_0 = \mathcal{Q}_{K_r}(E_{H_0}) \).
2. \( R_i = \mathcal{Q}_{K_{r-i}}(E_{H_i}) + \mathcal{Q}_{K_{r-i}}(E_{H_i}) p_{H_{i-1}} R_{i-1} p_{H_{i-1}} \mathcal{Q}_{K_{r-i}}(E_{H_i}) \) for \( 1 \leq i \leq r \).

Each \( R_i \) is a unital \( K_{r-i} \)-algebra with unit \( p_{H_i} \) and we have \( \mathcal{Q}_{K_{r-i}}(E_{H_i}) \subseteq R_i \subseteq \mathcal{Q}_{K_r}(E_{H_i}) \).

Before we establish the basic properties of our construction, we simplify notation as follows. A chain of fields of length \( r \) will be denoted:

\[
K_r : K_0 \subseteq K_1 \subseteq \cdots \subseteq K_r.
\]

(Note that the inclusions need not be strict.) Similarly a chain of hereditary saturated subsets of \( E^0 \) of length \( r \) will be denoted:

\[
H_r : H_0 \subset H_1 \subset \cdots \subset H_r = E^0.
\]

(Here we have strict inclusions. The choice of strict/non-strict inclusions is made to gain flexibility in the notation, and in particular with regard to be aligned with the notation used in [4].) Now we denote the \( K_0 \)-algebra \( R_r \) constructed in Definition 2.1 by \( \mathcal{Q}_{K_r}(E; H_r) \). The straightforward proof of the next two results is left to the reader.

**Proposition 2.2.** Let \( K_r, H_r \) and \( \mathcal{Q}_{K_r}(E; H_r) \) be as before. Let \( I_{i-1} \) be the ideal of \( \mathcal{Q}_{K_r}(E; H_r) \) generated by \( p_{H_{i-1}} \). Then

\[
\mathcal{Q}_{K_r}(E; H_r) / I_{i-1} \cong \mathcal{Q}_{K_{r-i}}(E / H_{i-1}; H^{r-i}),
\]

where

\[
K_{r-i} : K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{r-i}
\]

and

\[
H^{r-i} : H_1 \setminus H_{i-1} \subset H_{i+1} \setminus H_{i-1} \subset \cdots \subset H_r \setminus H_{i-1} = (E / H_{i-1})^0.
\]

**Proposition 2.3.** Let \( K_r, H_r \) and \( \mathcal{Q}_{K_r}(E; H_r) \) be as before. Then

\[
p_{H_i} \mathcal{Q}_{K_r}(E; H_r) p_{H_i} \cong \mathcal{Q}_{K_i}(E_{H_i}; H_i),
\]

where

\[
K_i : K_{r-i} \subseteq K_{r-i+1} \subseteq \cdots \subseteq K_r.
\]
and
\[ H_i : H_0 \subset H_1 \subset \cdots \subset H_i. \]

We are going to show that the algebras \( Q_{K_r}(E; H_r) \) above are universal localizations of suitable mixed path algebras. This is analogous to the situation with the usual path algebra of a quiver and its regular algebra [5], and plays an important role in the applications, see [4, Sections 5 and 6].

We retain the above notation. The mixed path algebra \( P_{K_r}(E; H_r) \) is the \( K_0 \)-subalgebra of the usual path \( K_r \)-algebra \( P_{K_r}(E) \) defined inductively as follows. Set \( P_0 := P_{K_r}(E_{H_0}) \), and for \( 1 \leq i \leq r \), put \( P_i := P_{K_{r-i}}(E_{H_i}) + P_{K_{r-i}}(E_{H_i}) p_{H_{i-1}} P_{i-1} \). Then the \( K_0 \)-algebra \( P_{K_r}(E; H_r) \) is by definition the algebra \( P_r \). Observe that this algebra is the usual path algebra whenever all the fields in the chain are equal.

Assume that \( |E^0| = d \). The usual augmentation \( \epsilon : P_{K_r}(E) \to K_r^d \) restricts to a surjective split homomorphism
\[ \epsilon : P_{K_r}(E; H_r) \longrightarrow \prod_{i=0}^r \prod_{v \in H_i \setminus H_{i-1}} K_{r-i} p_v. \]

Similar definitions give the mixed power series algebra over the quiver \( P_{K_r}((E; H_r)) \) and the mixed algebra of rational power series \( P_{K_r}^{rat}(E; H_r) \). For instance, when \( r = 1 \) we have \( P_{K_1}^{rat}(E; H_1) = P_{K_0}^{rat}(E) + P_{K_0}^{rat}(E) p_{H_0} P_{K_1}^{rat}(E_{H_0}) \).

The following generalizes the unmixed case [5, Theorem 1.20].

**Theorem 2.4.** Let \( K_r, H_r \) and \( P_{K_r}(E; H_r) \) be as before. Let \( \Sigma \) denote the set of matrices over \( P_{K_r}(E; H_r) \) that are sent to invertible matrices by \( \epsilon \). Then \( P_{K_r}^{rat}(E; H_r) \) is the rational closure of \( P_{K_r}(E; H_r) \) in \( P_{K_r}((E)) \), and the natural map \( P_{K_r}(E; H_r) \Sigma^{-1} \to P_{K_r}^{rat}(E; H_r) \) is an isomorphism.

**Proof.** We will give the proof in the case \( r = 1 \). An easy induction argument can be used to get the general case.

So assume that we have a field extension \( K \subseteq L \) and a hereditary saturated subset \( H \) of \( E^0 \). We have to show that \( S := P_{K}^{rat}(E) + P_{K}^{rat}(E) p_{H} P_{L}^{rat}(E_H) \) is the rational closure of \( R := P_{K}(E) + P_{K}(E) p_{H} P_{L}(E_H) \) in \( P_{L}((E)) \), the algebra of power series over \( E \) with coefficients in \( L \). Write \( R \) for this rational closure.

We start by showing that \( S \subseteq R \). Since \( P_{K}^{rat}(E) \) is the rational closure of \( P_{K}(E) \) inside \( P_{L}((E)) \), we see that \( P_{K}^{rat}(E) \subseteq R \). Also, note that the algebra \( p_{H} R \subseteq p_{H} R p_{H} \) is inversion closed in \( p_{H} P_{L}(E_H) \) and contains \( p_{H} P_{L}(E_H) \), so it must contain the rational closure of \( p_{H} P_{L}(E_H) \) in \( p_{H} P_{L}(E_H) \) which is precisely \( p_{H} P_{L}^{rat}(E_H) \). It follows that \( P_{K}^{rat}(E) \) and \( p_{H} P_{L}^{rat}(E_H) \) are both contained in \( R \). Since \( R \) is a ring, we get \( S \subseteq R \).

To show the reverse inclusion \( R \subseteq S \), take any element \( a \in R \). There exist a row \( \lambda \in n R \), a column \( \rho \in R^{n} \) and a matrix \( B \in M_{n}(R) \) such that \( \epsilon(B) = 0 \) such that
\[ a = \lambda (I - B)^{-1} \rho. \]
Now the matrix $B$ can be written as $B = B_1 + B_2$, where $B_1 \in P_K(E) \subset R$ and $B_2 \in R$ satisfy that $\epsilon(B_1) = \epsilon(B_2) = 0$, all the entries of $B_1$ are supported on paths ending in $E^0 \setminus H$ and all the entries of $B_2$ are supported on paths ending in $H$. Note that, since $H$ is hereditary, this implies that all the paths in the support of the entries of $B_1$ start in $E^0 \setminus H$ and thus $B_2B_1 = 0$. It follows that

\[(I - B)^{-1} = (I - B_1 - B_2)^{-1} = (I - B_1)^{-1}(I - B_2)^{-1},\]

and therefore $(I - B)^{-1} = (I - B_1)^{-1} + (I - B_1)^{-1}B_2(I - B_2)^{-1} \in M_n(S)$. It follows from (2.1) that $a \in S$, as desired.

Since the set $\Sigma$ is precisely the set of square matrices over $R$ which are invertible over $P_L((E))$, we get from a well-known general result (see for instance Lemma 10.35(3)) that there is a surjective $K$-algebra homomorphism $\phi : R\Sigma^{-1} \to R$. The rest of the proof is devoted to show that $\phi$ is injective. We have a commutative diagram

$$
\begin{array}{cccc}
P_K(E)\Sigma(\epsilon_K)^{-1} & \longrightarrow & R\Sigma^{-1} & \longrightarrow & P_L(E)\Sigma(\epsilon_L)^{-1} \\
\phi_K \downarrow \cong & & \phi \downarrow & & \phi_L \downarrow \cong \\
P_K^{rat}(E) & \longrightarrow & R & \longrightarrow & P_L^{rat}(E)
\end{array}
$$

The map $P_K(E)\Sigma(\epsilon_K)^{-1} \to P_L(E)\Sigma(\epsilon_L)^{-1}$ is injective, so the map $P_K(E)\Sigma(\epsilon_K)^{-1} \to R\Sigma^{-1}$ must also be injective. Hence the $K$-subalgebra of $R\Sigma^{-1}$ generated by $P_K(E)$ and the entries of the inverses of matrices in $\Sigma(\epsilon_K)$ is isomorphic to $P_K^{rat}(E)$. Observe that we can replace $\Sigma$ by the set of matrices of the form $I - B$, where $B$ is a square matrix over $R$ with $\epsilon(B) = 0$. As before we write $B = B_1 + B_2$, where all the entries of $B_1$ end in $E^0 \setminus H$ and all the entries in $B_2$ end in $H$, and thus $B_2B_1 = 0$, so that (2.2) holds in $R\Sigma^{-1}$. An element $x$ in $R\Sigma^{-1}$ is of the form

\[(2.4) \quad x = \lambda(I - B)^{-1}\rho\]

with $\lambda \in nR$ and $\rho \in R^n$, and $\epsilon(B) = 0$.

Claim 1. We have

\[p_H R\Sigma^{-1} = p_H P_L^{rat}(E_H) = p_H P_L(E_H)\Sigma(\epsilon_L^H)^{-1} p_H.\]

Proof of Claim 1. Observe first that we have a natural $L$-algebra homomorphism $P_L(E_H)\Sigma(\epsilon_L^H)^{-1} \to p_H R\Sigma^{-1}$. The composition of this map with the map $R\Sigma^{-1} \to P_L(E_H)\Sigma(\epsilon_L^H)^{-1}$ is injective (since its image is $p_H P_L^{rat}(E_H) \cong P_L^{rat}(E_H) \cong P_L(E_H)\Sigma(\epsilon_L^H)^{-1}$) so the map $P_L(E_H)\Sigma(\epsilon_L^H)^{-1} \to p_H R\Sigma^{-1}$ must be injective. We identify $p_H P_L^{rat}(E_H)$ with its image in $p_H R\Sigma^{-1}$, which is the $L$-subalgebra of $p_H R\Sigma^{-1}$ generated by $p_H P_L(E_H)$ and the entries of the inverses of matrices of the form $p_H I - B$, with $B$ a square matrix over $p_H P_L(E_H)$ with $\epsilon(B) = 0$. For an element $x$ in $R\Sigma^{-1}$, we write it in its canonical form (2.4) and we write $B = B_1 + B_2$ with all the entries in $B_1$ ending in $E^0 \setminus H$ and all the entries of $B_2$ ending in $H$. 


Now multiply (2.4) on the left by $p_H$ and use (2.2) to get
\[
p_H x = p_H \lambda(I - B_1)^{-1} \rho + p_H \lambda(I - B_1)^{-1} B_2(I - B_2)^{-1} \rho
\]
\[
= p_H \lambda p_H(I - B_1)^{-1} \rho + p_H \lambda p_H(I - B_1)^{-1} B_2(I - B_2)^{-1} \rho
\]
\[
= p_H \lambda p_H \rho + p_H \lambda p_H B_2 p_H(I - B_2)^{-1} \rho.
\]

Write $B_2 = B'_2 + B''_2$, where all the entries of $B'_2$ start in $E^0 \setminus H$ and all the entries in $B''_2$ start in $H$ (and so end in $H$ as well). Note that $(I - B'_2)^{-1} = I + B'_2$, because $B'_2 = 0$, so that $p_H(I - B'_2)^{-1} = p_H$. Since $B''_2 B'_2 = 0$ we have $(I - B_2)^{-1} = (I - B'_2)^{-1}(I - B''_2)^{-1}$, and thus
\[
p_H x = p_H \lambda p_H \rho + p_H \lambda p_H B_2 p_H(I - B''_2)^{-1} p_H \rho p_H.
\]
It follows that $p_H x \in p_H P_L^{rat}(E_H)$, as wanted. □

Assume now that $x \in \ker(R \Sigma^{-1} \to \mathcal{R}) = \ker(R \Sigma^{-1} \to P_L(E) \Sigma(\epsilon_L)^{-1})$ and write $x$ as in (2.4), with $B = B_1 + B_2$ as before. Then

(2.5) \[
x = \lambda(I - B_1)^{-1} \rho + \lambda(I - B_1)^{-1} B_2(I - B_2)^{-1} \rho.
\]

Multiplying on the right by $1 - p_H$, we get
\[
x(1 - p_H) = \lambda(I - B_1)^{-1} \rho(1 - p_H) = \lambda(1 - p_H)(I - B_1)^{-1} \rho(1 - p_H) \in P_K(E) \Sigma(\epsilon_K)^{-1}
\]
and $0 = \phi(x(1 - p_H)) = \phi_K(x(1 - p_H))$. Since $\phi_K$ is an isomorphism, we get $x(1 - p_H) = 0$.

Hence we have

(2.6) \[
x = \lambda(I - B_1)^{-1} \rho_2 + \lambda(I - B_1)^{-1} B_2(I - B_2)^{-1} \rho_2,
\]
where $\rho = \rho_1 + \rho_2$ with $\rho_1$ ending in $E^0 \setminus H$ and $\rho_2$ ending in $H$. By Claim 1 we have $p_H x = 0$, because $\phi$ is an isomorphism when restricted to $p_H P_L^{rat}(E_H)$. Now we are going to find a suitable expression for $x = (1 - p_H)x p_H$. Write $\lambda = \lambda_1 + \lambda_2$ with $\lambda_1 = (1 - p_H)\lambda$ and $\lambda_2 = p_H \lambda$. Then

(2.7) \[
(1 - p_H)\lambda(I - B_1)^{-1} \rho_2 = \lambda_1(I - B_1)^{-1} \rho_2.
\]

Similarly $(1 - p_H)\lambda(I - B_1)^{-1} B_2(I - B_2)^{-1} \rho_2 = \lambda_1(I - B_1)^{-1} B_2(I - B_2)^{-1} \rho_2$. Write $B_2 = B'_2 + B''_2$, with $B'_2$ starting in $E^0 \setminus H$ and $B''_2$ starting in $H$. Then $B''_2 B'_2 = 0$ and $(I - B_2)^{-1} = (I - B'_2)^{-1}(I - B''_2)^{-1}$, so that

(2.8) \[
(1 - p_H)\lambda(I - B_1)^{-1} B_2(I - B_2)^{-1} \rho_2 = \lambda_1(I - B_1)^{-1} B_2(I - B_2)^{-1} \rho_2
\]
\[
= \lambda_1(I - B_1)^{-1} B_2(I + B'_2)(I - B''_2)^{-1} \rho_2
\]
\[
= \lambda_1(I - B_1)^{-1} B_2(I - B''_2)^{-1} \rho_2.
\]

Substituting (2.7) and (2.8) in (2.6) we get

(2.9) \[
x = (1 - p_H)x p_H = \lambda_1(I - B_1)^{-1} \rho_2 + \lambda_1(I - B_1)^{-1} B_2(I - B''_2)^{-1} \rho_2.
\]

It follows that $x \in \sum_{i=1}^{k} P_{K}^{rat}(E/H)e_{i}P_{L}^{rat}(E/H)$, where $e_{1}, \ldots, e_{k}$ is the family of crossing edges, that is, the family of edges $e \in E^{1}$ such that $s(e) \in E^{0} \setminus H$ and $r(e) \in H$. Write $x = \sum_{i=1}^{k} \sum_{j=1}^{m_{i}} a_{ij}e_{i}b_{ij}$ for certain $a_{ij} \in P_{K}^{rat}(E/H)$ and $b_{ij} \in P_{L}^{rat}(E/H)$. Then we have

$$0 = \phi(x) = \sum_{i=1}^{k} \sum_{j=1}^{m_{i}} a_{ij}e_{i}b_{ij},$$

this element being now in $P_{L}((E))$. Clearly this implies that $\sum_{j=1}^{m} a_{ij}e_{i}b_{ij} = 0$ in $P_{L}((E))$ for all $i = 1, \ldots, k$. So the result follows from the following claim:

**Claim 2.** Let $e$ be a crossing edge, so that $s(e) \in E^{0} \setminus H$ and $r(e) \in H$. Assume that $b_{1}, \ldots, b_{m} \in P_{r(e)}P_{L}((E/H))$ are $K$-linearly independent elements, and assume that $a_{1}e, \ldots, a_{m}e$ are not all 0, where $a_{1}, \ldots, a_{m} \in P_{K}((E \setminus H))$. Then $\sum_{i=1}^{m} a_{i}e_{i}b_{i} \neq 0$ in $P_{L}((E))$.

**Proof of Claim 2.** By way of contradiction, suppose that $\sum_{i=1}^{m} a_{i}e_{i} = 0$. We may assume that $a_{1}e \neq 0$. Let $\gamma$ be a path in the support of $a_{1}$ such that $r(\gamma) = s(e)$. For every path $\mu$ with $s(\mu) = r(e)$ we have that the coefficient of $\gamma e_{\mu}$ in $a_{1}e_{i}b_{i}$ is $a_{1}(\gamma)b_{i}(\mu)$, so that $\sum_{i=1}^{m} a_{i}(\gamma)b_{i}(\mu) = 0$ for every $\mu$ such that $s(\mu) = r(e)$. Since every path in the support of each $b_{i}$ starts with $r(e)$, we get that

$$\sum_{i=1}^{m} a_{i}(\gamma)b_{i} = 0$$

with $a_{1}(\gamma) \neq 0$, which contradicts the linear independence over $K$ of $b_{1}, \ldots, b_{m}$. \hfill \Box

This concludes the proof of the theorem.

Following [5, Section 2], we define, for $e \in E^{1}$, the right transduction $\tilde{\delta}_{e}: P_{L}((E)) \to P_{L}((E))$ corresponding to $e$ by

$$\tilde{\delta}_{e}(\sum_{\alpha \in E^{*}} \lambda_{\alpha} \alpha) = \sum_{\substack{\alpha \in E^{*} \atop s(\alpha) = r(e)}} \lambda_{\alpha} \alpha.$$ 

Similarly the left transduction corresponding to $e$ is given by

$$\delta_{e}(\sum_{\alpha \in E^{*}} \lambda_{\alpha} \alpha) = \sum_{\substack{\alpha \in E^{*} \atop r(\alpha) = s(e)}} \lambda_{\alpha} \alpha.$$ 

Observe that $\tilde{R} := P_{K}(E; H_{r})$ is closed under all the right transductions, i.e. $\tilde{\delta}_{e}(R) \subseteq R$, but $R$ is not invariant under all the left transductions. Some of the proofs in [5] make use of the fact that the usual path algebra $P_{K}(E)$ is closed under left and right transductions. Fortunately we have been able to overcome the potential problems arising from the failure of invariance of $R$ under left transductions by using alternative arguments.

We are now ready to get a description of the algebra $Q_{K}(E; H_{r})$ as a universal localization of the mixed path algebra $P_{K}(E; H_{r})$. 
Write $R := P_{K_r}(E; H_r)$. For any $v \in E^0$ such that $s^{-1}(v) \neq \emptyset$ we put $s^{-1}(v) = \{e_1^v, \ldots, e_{n_v}^v\}$, and we consider the left $R$-module homomorphism
\[
\mu_v : Rv \longrightarrow \bigoplus_{i=1}^{n_v} R(e_i^v)
\]
\[
r \longmapsto (re_1^v, \ldots, re_{n_v}^v)
\]
Write $\Sigma_1 = \{\mu_v \mid v \in E^0, s^{-1}(v) \neq \emptyset\}$.

**Theorem 2.5.** Let $K_r$ and $H_r$ and $P_{K_r}(E; H_r)$ be as before. Let $\Sigma$ denote the set of matrices over $P_{K_r}(E; H_r)$ that are sent to invertible matrices by $\epsilon$ and let $\Sigma_1$ be the set of matrices defined above. Then we have $Q_{K_r}(E; H_r) = (P_{K_r}(E; H_r))((\Sigma \cup \Sigma_1))^{-1}$. Moreover $Q_{K_r}(E; H_r)$ is a hereditary von Neumann regular ring and all finitely generated projective $Q_{K_r}(E; H_r)$-modules are induced from $P_{K_r}^{\text{rat}}(E; H_r)$.

**Proof.** First observe that the mixed path algebra $P_{K_r}(E; H_r)$ is a hereditary ring and that $\mathcal{V}(P_{K_r}(E; H_r)) = (\mathbb{Z}^+)^d$, where $|E^0| = d$. This follows by successive use of [3, Theorem 5.3].

In order to get that the right transduction $\tilde{\delta}_r : P_{K_r}((E)) \rightarrow P_{K_r}((E))$ corresponding to $e$ is a right $\tau_e$-derivation on $P_{K_r}(E; H_r)$, that is,

\[
\tilde{\delta}_r(rs) = \tilde{\delta}_r(r)s + \tau_e(r)\tilde{\delta}_e(s)
\]
for all $r, s \in P_{K_r}(E; H_r)$, we have to modify slightly the definition of $\tau_e$ given in [5, page 220]. Concretely we define $\tau_e$ as the endomorphism of $P_{K_r}((E))$ given by the composition

\[
P_{K_r}((E)) \rightarrow \prod_{v \in E^0} K_r p_v \rightarrow \prod_{v \in E^0} K_r p_v \rightarrow P_{K_r}((E)),
\]
where the first and third maps are the canonical projection and inclusion respectively, and the middle map is the $K_r$-lineal map given by sending $p_{s(e)}$ to $p_{r(e)}$, and any other idempotent $p_v$ with $v \neq s(e)$ to 0. Observe that this restricts to an endomorphism of $P_{K_r}(E; H_r)$ and that the proof in [5, Lemma 2.4] gives the desired formula (2.10) for $r, s \in P_{K_r}((E))$ and, in particular for $r, s \in P_{K_r}(E; H_r)$. The constructions in [5, Section 2] apply to $R := P_{K_r}^{\text{rat}}(E; H_r)$ (with some minor changes), and we get that

\[
R\Sigma_1^{-1} = R(\overline{E}; \tau, \tilde{\delta})/I,
\]
where $I$ is the ideal of $R(\overline{E}; \tau, \tilde{\delta})$ generated by the idempotents $q_i := p_i - \sum_{e \in s^{-1}(i)} e\overline{e}$ for $i \notin \text{Sink}(E)$. By [5, Remark 2.14], we get that the map

\[
R\Sigma_1^{-1} = R(\overline{E}; \tau, \tilde{\delta})/I \longrightarrow (P_{K_r}^{\text{rat}}(E))(\overline{E}; \tau, \tilde{\delta})/I_2 = Q_{K_r}(E)
\]
is injective, and the image of this map is clearly $Q_{K_r}(E; H_r)$. So we get an isomorphism $R\Sigma_1^{-1} \cong Q_{K_r}(E; H_r)$, which combined with the isomorphism $R \cong P_{K_r}(E; H_r)\Sigma^{-1}$ established in Theorem 2.4 gives $Q_{K_r}(E; H_r) \cong (P_{K_r}(E; H_r))((\Sigma \cup \Sigma_1))^{-1}$. By a result of Bergman and Dicks [12] any universal localization of a hereditary ring is hereditary, thus we get that both $P_{K_r}^{\text{rat}}(E; H_r)$ and $Q_{K_r}(E; H_r)$ are hereditary rings. Since $P_{K_r}^{\text{rat}}(E; H_r)$ is hereditary, closed under inversion in $P_{K_r}((E))$ (by Theorem 2.4), and closed under
all the right transductions $\tilde{\delta}_e$, for $e \in E^1$, the proof of [5, Theorem 2.16] gives that $Q_{K_r}(E; H_r)$ is von Neumann regular and that every finitely generated projective is induced from $P_{K_r}^{rat}(E; H_r)$.

This concludes the proof of the theorem. □

**Remark 2.6.** Theorem 2.16 in [5] is stated for a subalgebra $R$ of $P_K((E))$ which is closed under all left and right transductions (and which is inversion closed in $P_K((E))$). However the invariance under right transductions is only used in the proof of that result to ensure that the ring $R$ is left semihereditary. Since we are using the opposite notation concerning (CK1) and (CK2), the above hypothesis translates in our setting into the condition that $P_{K_r}(E; H_r)$ and $P_{K_r}^{rat}(E; H_r)$ should be invariant under all left transductions, which is not true in general as we observed above. We overcome this problem by the use of the result of Bergman and Dicks ([12]), which guarantees that $P_{K_r}(E; H_r)$ and $P_{K_r}^{rat}(E; H_r)$ are indeed right and left hereditary (see the proof of Theorem 2.5).

Define the mixed Leavitt path algebra $L_{K_r}(E; H_r)$ as the universal localization of $P_{K_r}(E; H_r)$ with respect to the set $\Sigma_1$. Let $M(E)$ be the abelian monoid with generators $E^0$ and relations given by $v = \sum_{e \in s^{-1}(v)} r(e)$, see [8] and [9].

**Theorem 2.7.** With the above notation, we have natural isomorphisms

$$M(E) \cong \mathcal{V}(L_{K_r}(E; H_r)) \cong \mathcal{V}(Q_{K_r}(E; H_r)).$$

**Proof.** The proof that $M(E) \cong \mathcal{V}(L_{K_r}(E; H_r))$ follows as an application of Bergman’s results [11], as in [8] Theorem 3.5.

Note that $R := P_{K_r}^{rat}(E; H_r)$ is semiperfect. Thus we get $\mathcal{V}(R) \cong (\mathbb{Z}^+)^{|E_0|}$ in the natural way, that is the generators of $\mathcal{V}(R)$ correspond to the projective modules $p_v R$ for $v \in E^0$. By Theorem 2.5, we get that the natural map $M(E) \to \mathcal{V}(Q_{K_r}(E; H_r))$ is surjective. To show injectivity observe that we have

$$M(E) \cong \mathcal{V}(Q_{K_0}(E)) \longrightarrow \mathcal{V}(Q_{K_r}(E; H_r)) \longrightarrow \mathcal{V}(Q_{K_r}(E)) \cong M(E),$$

and that the composition of the maps above is the identity. It follows that the map $M(E) \to \mathcal{V}(Q_{K_r}(E; H_r))$ is injective and so it must be a monoid isomorphism. □

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