Analytic Bethe ansatz and functional relations related to tensor-like representations of type II Lie superalgebras $B(r|s)$ and $D(r|s)$

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Abstract

An analytic Bethe ansatz is carried out related to tensor-like representations of the type II Lie superalgebras $B(r|s) = \mathfrak{osp}(2r + 1|2s)$ ($r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$) and $D(r|s) = \mathfrak{osp}(2r|2s)$ ($r \in \mathbb{Z}_{\geq 2}$, $s \in \mathbb{Z}_{\geq 1}$). We present eigenvalue formulae of transfer matrices in dressed vacuum forms labeled by Young (super) diagrams. A class of transfer matrix functional relations ($T$-system) is discussed. In particular for $B(0|s) = \mathfrak{osp}(1|2s)$ ($s \in \mathbb{Z}_{\geq 1}$) case, a complete set of functional relations is proposed by using duality among dressed vacuum forms.

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1 Introduction

Solvable lattice models related to Lie superalgebras [1] have been received much attention [2, 3, 4, 5, 6, 7, 8, 9]. To construct eigenvalue formulae of transfer matrices for such models is an important problem in mathematical physics. To achieve this program, Bethe ansatz has been often used. Nowadays, there are many literatures (See for example [3, 10, 11, 12, 13, 14, 15, 16, 17] and references therein.) on Bethe ansatz analysis for solvable lattice models related to Lie superalgebras. However, most of them deal only with models related to simple representations like fundamental ones. Only a few people [13, 15, 17] tried to deal with more complicated models such as fusion models [18] by Bethe ansatz; while there was no systematic study on this subject.

To break through such situations, we have recently executed [19, 20, 21, 22] an analytic Bethe ansatz [23, 24, 25, 26, 27, 28, 29] systematically for the type I Lie superalgebras \( sl(r+1|s+1) \) or \( C(s) \) cases. Namely, we have proposed a set of dressed vacuum forms (DVFs) and a class of functional relations for it. The purpose of this paper is to develop our recent works to type II Lie superalgebras \( B(r|s) = osp(2r+1|2s) \) \((r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 1})\) cases.

We can express \([3, 16]\) the Bethe ansatz equation (BAE) \((3.1)\) by using the representation theoretical data of \( B(r|s) \) \((r \in \mathbb{Z}_{\geq 1}, s \in \mathbb{Z}_{\geq 1})\) or \( D(r|s) \) \((r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1})\), as long as we adopt the distinguished simple root system [1]. On the other hand, \( B(0|s) = osp(1|2s) \) is a peculiar one among the Lie superalgebras. In contrast to other Lie superalgebras, its simple root system is unique. Corresponding to this fact, BAES \((3.2)-(3.5)\) associated with the root system of \( B(0|s) \) will be also unique. Peculiarity of these BAES is that so far a naive description in terms of the simple root system does not exist for the \( s \)-th BAES \((3.4)\) and \((3.5)\), which correspond to the odd root \( \alpha_s \) with \((\alpha_s | \alpha_s) \neq 0 \) (cf. [3, 16]).

We assume, as our starting point, above-mentioned BAES \((3.1)-(3.5)\) for \( B(r|s) \) and \( D(r|s) \), and then carry out an analytic Bethe ansatz systematically to construct a class of DVFs. On constructing DVFs, the pole-freeness under the BAE and the top term hypothesis [26, 27] play important roles.

We introduce skew-Young (super) diagrams \( \lambda \subset \mu \), which are related to tensor-like representations \([1]\) of \( B(r|s) \) or \( D(r|s) \). On these skew-Young (super) diagrams, we define a set of admissible tableaux \( B(\lambda \subset \mu) \) with some semi-standard like conditions. There is a one-to-one correspondence between these conditions for \( B(0|s) \) case and the conditions for \( A_{2s}^{(2)} \) case.

\[ ^1 \text{In this paper, we do not deal with spinorial representations.} \]
In addition, in contrast to $B(r|s)$ case, these conditions for $D(r|s)$ case have non-local nature. Next, we define a function $T_{\lambda \subset \mu}(u)$ of a spectral parameter $u$ as summations over $B(\lambda \subset \mu)$. It will provide us the spectra of a set of transfer matrices for various fusion $B(r|s)$ or $D(r|s)$ vertex models. It contains the top term \cite{26, 27}, which carries the highest weight of the irreducible representation of $B(r|s)$ or $D(r|s)$ labeled by a skew-Young (super) diagram $\lambda \subset \mu$. In particular, the simplest example of $T_{\lambda \subset \mu}(u)$, that is, $T^1(u) = T_1(u) = T_{(1^r)}(u)$ reduces to the eigenvalue formula of the transfer matrix \cite{10} of some vertex model related to the fundamental representation of $B(r|s)$ or $D(r|s)$ after some redefinitions. The BAEs (3.1)-(3.5) are assumed common to all the DVFs for transfer matrices with various fusion types in the auxiliary space as long as they act on a common quantum space. Therefore, we can prove the pole-freeness of $T^a(u) = T_{(1^a)}(u)$ for any $a \in \mathbb{Z}_{\geq 0}$ under the common BAEs (3.1)-(3.5). We further mention a determinant formula, by which $T_{\lambda \subset \mu}(u)$ can be expressed only by the fundamental functions $\{ T^a \}$ and then pole freeness follows immediately. A set of transfer matrix functional relations among DVFs also follows from this formula. It will be a kind of the $T$-system \cite{32} (see also \cite{23, 33, 34, 29, 27, 35, 36, 37, 13, 15, 17, 19, 20, 21, 22, 38}). In particular for $B(0|s)$ case, there is remarkable duality among DVFs (see, Theorem 4.1 and (4.21)). On constructing above-mentioned functional relations, this duality among DVFs plays an important role.

The outline of this paper is as follows. In section 2, we briefly mention the Lie superalgebras $B(r|s)$ and $D(r|s)$. In section 3, we execute an analytic Bethe ansatz based on the BAEs (3.1)-(3.5) associated with distinguished simple root systems. In Section 4, we discuss transfer matrix functional relations. Section 5 is devoted to summary and discussion. In appendix A.1-A.3, we prove the pole-freeness of DVFs. Appendix B provides generating series of $T^a(u)$ and $T_m(u) = T_{(m)}(u)$. In this paper, we adopt similar notation in \cite{26, 27, 19, 20, 21, 22}. Finally we note that we can recover many formulae in \cite{26, 27, 19, 20, 21, 22} for $B_r$ or $D_r$, if we set $s = 0$ and redefine the vacuum parts.

\section{Lie superalgebras}

In this section, we briefly mention the Lie superalgebras $B(r|s)$ and $D(r|s)$ (see for example \cite{11, 39, 40, 41, 42, 43}).

There are several choices of simple root systems and the simplest one is the distinguished simple root system. They read as follows:

$B(r|s)$ $(r, s \in \mathbb{Z}_{\geq 1})$ case (see Figure 1):

$$\alpha_i = \delta_i - \delta_{i+1} \quad i = 1, 2, \ldots, s - 1,$$
Figure 1: Dynkin diagram for the Lie superalgebra $B(r|s) = osp(2r + 1|2s)$ $(r \in \mathbb{Z}_{\geq 1}, s \in \mathbb{Z}_{\geq 1})$ corresponding to the distinguished simple root system: white circles denote even roots; a gray (a cross) circle denotes an odd root $\alpha$ with $(\alpha|\alpha) = 0$.

Figure 2: Dynkin diagram for the Lie superalgebra $B(0|s) = osp(1|2s)$ $(s \in \mathbb{Z}_{\geq 1})$: a black circle denotes an odd root $\alpha$ with $(\alpha|\alpha) \neq 0$.

Figure 3: Dynkin diagram for the Lie superalgebra $D(r|s) = osp(2r|2s)$ $(r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1})$ corresponding to the distinguished simple root system.
\[ \alpha_s = \delta_s - \epsilon_1, \]
\[ \alpha_{s+j} = \epsilon_j - \epsilon_{j+1}, \quad j = 1, 2, \ldots, r - 1, \]
\[ \alpha_{s+r} = \epsilon_r; \quad (2.1) \]

\[ B(0|s) \quad (s \in \mathbb{Z}_{\geq 1}) \text{ case (see Figure 2)}:\]
\[ \alpha_i = \delta_i - \delta_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, s - 1, \]
\[ \alpha_s = \delta_s; \quad (2.2) \]

\[ D(r|s) \quad (r \in \mathbb{Z}_{\geq 2}, \ s \in \mathbb{Z}_{\geq 1}) \text{ case (see Figure 3)}:\]
\[ \alpha_i = \delta_i - \delta_{i+1} \quad i = 1, 2, \ldots, s - 1, \]
\[ \alpha_s = \delta_s - \epsilon_1, \]
\[ \alpha_{s+j} = \epsilon_j - \epsilon_{j+1}, \quad j = 1, 2, \ldots, r - 2, \]
\[ \alpha_{s+r-1} = \epsilon_{r-1} - \epsilon_r, \]
\[ \alpha_{s+r} = \epsilon_{r-1} + \epsilon_r; \quad (2.3) \]

where \( \epsilon_1, \ldots, \epsilon_r; \delta_1, \ldots, \delta_s \) are the bases of the dual space of the Cartan subalgebra with the bilinear form \( (\ | \ ) \) such that \(^2\)

\[ (\epsilon_i|\epsilon_j) = \delta_{ij}, \quad (\epsilon_i|\delta_j) = 0, \quad (\delta_i|\epsilon_j) = 0, \quad (\delta_i|\delta_j) = -\delta_{ij}. \quad (2.4) \]

\{\( \alpha_i \}_{i \neq s} \} \text{ are even roots and } \alpha_s \text{ is an odd root. Note that } (\alpha_s|\alpha_s) = 0 \text{ for } B(r|s) \quad (r, s \in \mathbb{Z}_{\geq 1}) \text{ and } D(r|s) \quad (r \in \mathbb{Z}_{\geq 2}, \ s \in \mathbb{Z}_{\geq 1}) \text{ cases, while } (\alpha_s|\alpha_s) \neq 0 \text{ for } B(0|s) \quad (s \in \mathbb{Z}_{\geq 1}) \text{ case.} \]

Let \( \lambda \subset \mu \) be a skew-Young (super) diagram labeled by the sequences of non-negative integers \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \) such that \( \mu_i \geq \lambda_i : i = 1, 2, \ldots; \lambda_1 \geq \lambda_2 \geq \ldots \geq 0; \mu_1 \geq \mu_2 \geq \ldots \geq 0 \) and \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \) be the conjugate of \( \lambda \). In particular, for \( B(r|s) \quad (r, s \in \mathbb{Z}_{\geq 1}) \), \( \lambda = \phi, \mu_{r+1} \leq s \) case, the Kac-Dynkin label \([b_1, b_2, \ldots, b_{s+r}]\) is related to the Young (super) diagram with shape \( \mu = (\mu_1, \mu_2, \ldots) \) as follows:

\[ b_i = \mu'_i - \mu'_{i+1} \quad \text{for} \quad i \in \{1, 2, \ldots, s - 1\}, \]
\[ b_s = \mu'_s + \eta_1, \quad (2.5) \]
\[ b_{s+j} = \eta_j - \eta_{j+1} \quad \text{for} \quad j \in \{1, 2, \ldots, r - 1\}, \]
\[ b_{s+r} = 2\eta_r, \]

where \( \eta_i = \text{Max}\{\mu_i - s, 0\} \). For \( B(0|s) \quad (s \in \mathbb{Z}_{\geq 1}) \), \( \lambda = \phi \) case, the Kac-Dynkin label \([b_1, b_2, \ldots, b_s]\) is related to the Young (super) diagram with

\(^2\)We normalized the longest simple root as \(|(\alpha|\alpha)| = 2.\)
shape $\mu = (\mu_1, \mu_2, \ldots)$ as follows:

$$
\begin{align*}
  b_i &= \mu'_i - \mu'_{i+1} \quad \text{for} \quad i \in \{1, 2, \ldots, s - 1\}, \\
  b_s &= 2\mu'_s.
\end{align*}
$$

(2.6)

For $D(r|s)$ case, we use only the Young (super) diagram with shape $\mu = (1^a)$ or $\mu = (m^1)$. The Young (super) diagram with shape $\mu = (1^a)$ is related to the Kac-Dynkin label $[b_1, b_2, \ldots, b_{s+r}]$ as follows:

$$
b_j = a\delta_{j1}.
$$

(2.7)

And the Young (super) diagram with shape $\mu = (m^1)$ is related to the Kac-Dynkin label $[b_1, b_2, \ldots, b_{s+r}]$ as follows:

$$
b_j = \begin{cases} 
\delta_{jm} & \text{if } m \in \{1, 2, \ldots, s\}, \\
(m - s + 1)\delta_{js} + (m - s)\delta_{js+1} & \text{if } r \in \mathbb{Z}_{\geq 3}, \; m \in \mathbb{Z}_{\geq s+1}; \\
(m - s + 1)\delta_{js} + (m - s)(\delta_{js+1} + \delta_{js+2}) & \text{if } r = 2, \; m \in \mathbb{Z}_{\geq s+1}.
\end{cases}
$$

(2.8)

An irreducible representation of $B(0|s)$ with the Kac-Dynkin label $[b_1, b_2, \ldots, b_s]$ is finite dimensional \[39\] if and only if

$$
\begin{align*}
  b_j &\in \mathbb{Z}_{\geq 0} \quad \text{for} \quad j \in \{1, 2, \ldots, s - 1\}, \\
  b_s &\in 2\mathbb{Z}_{\geq 0}.
\end{align*}
$$

(2.9)

The dimensionality of the irreducible representation $V[b_1, b_2, \ldots, b_s]$ of $B(0|s)$ with the highest weight labeled by the Kac-Dynkin label $[b_1, b_2, \ldots, b_s]$ is given \[39, 43\] as follows \[3\]

$$
\begin{align*}
  \dim V[b_1, b_2, \ldots, b_s] &= \prod_{1 \leq i < j \leq s} \frac{b_i + b_{i+1} + \cdots + b_{j-1} + j - i}{j - i} \\
  &\times \frac{b_i + b_{i+1} + \cdots + b_{j-1} + 2(b_j + b_{j+1} + \cdots + b_{s-1}) + b_s + 2s - i - j + 1}{2s - i - j + 1} \\
  &\times \prod_{1 \leq k \leq s} \frac{2(b_k + b_{k+1} + \cdots + b_{s-1}) + b_s + 2s - 2k + 1}{2s - 2k + 1}.
\end{align*}
$$

(2.10)

### 3 Analytic Bethe ansatz

We assume, as our starting point, the following type of the Bethe ansatz equations \[30, 31, 3, 14, 27\].

\[3\]We assume that $b_j + b_{j+1} + \cdots + b_{s-1} = 0$ if $j = s$.

\[4\]In this paper, we deal with the case, as an example, that the quantum spaces of the transfer matrices are fundamental representations.
\[ B(r|s) \quad (r, s \in \mathbb{Z}_{\geq 1}) \quad \text{or} \quad D(r|s) \quad (r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1}) \quad \text{case:} \]

\[ - \left\{ \prod_{j=1}^{N} \frac{\Phi(u^{(a)}_k - w_j - 1)}{\Phi(u^{(a)}_k - w_j + 1)} \right\}^{\delta_{a1}} = (-1)^{\deg(\alpha_a)} \prod_{b=1}^{s+r} B^a_b(u^{(a)}_k + (\alpha_a|\alpha_b)) \quad (3.1) \]

\[ B(0|s) \quad (s \in \mathbb{Z}_{\geq 2}) \quad \text{case:} \]

\[ - \prod_{j=1}^{N} \frac{\Phi(u^{(a)}_k - w_j - 1)}{\Phi(u^{(a)}_k - w_j + 1)} = \frac{Q_1(u^{(a)}_k - 2)Q_2(u^{(a)}_k + 1)}{Q_1(u^{(a)}_k + 2)Q_2(u^{(a)}_k - 1)}, \quad (3.2) \]

\[ -1 = \frac{Q_{a-1}(u^{(a)}_k + 1)Q_a(u^{(a)}_k - 2)Q_{a+1}(u^{(a)}_k + 1)}{Q_{a-1}(u^{(a)}_k - 1)Q_a(u^{(a)}_k + 2)Q_{a+1}(u^{(a)}_k - 1)} \quad \text{for} \quad 2 \leq a \leq s - 1, \quad (3.3) \]

\[ 1 = \frac{Q_{s-1}(u^{(s)}_k + 1)Q_s(u^{(s)}_k + 1)Q_{s+1}(u^{(s)}_k - 2)}{Q_{s-1}(u^{(s)}_k - 1)Q_s(u^{(s)}_k - 1)Q_{s+1}(u^{(s)}_k + 2)}, \quad (3.4) \]

\[ B(0|1) \quad \text{case:} \]

\[ - \prod_{j=1}^{N} \frac{\Phi(u^{(1)}_k - w_j - 1)}{\Phi(u^{(1)}_k - w_j + 1)} = \frac{Q_1(u^{(1)}_k + 1)Q_1(u^{(1)}_k - 2)}{Q_1(u^{(1)}_k - 1)Q_1(u^{(1)}_k + 2)}, \quad (3.5) \]

Here \( Q_a(u) = \prod_{j=1}^{N_a} \Phi(u - u^{(a)}_j) \); \( N \in \mathbb{Z}_{\geq 0} \) is the number of the lattice sites; \( N_a \in \mathbb{Z}_{\geq 0}; \) \( u^{(a)}_j, w_j \in \mathbb{C}; \) \( a, k \in \mathbb{Z} \) \( (a \in \{1, 2, \ldots, s + r\} \quad (r = 0 \text{ for } B(0|s) \text{ case}); \ k \in \{1, 2, \ldots, N_a\}); \)

\[ \deg(\alpha_a) = \begin{cases} 0 & \text{for even root} \\ 1 & \text{for odd root} \end{cases} \quad (3.6) \]

\[ \Phi \text{ is a function, which has zero at } u = 0. \text{ For example, } \Phi(u) \text{ has the following form} \]

\[ \Phi(u) = u. \quad (3.7) \]

Remarkable enough, Bethe ansatz equations can be written in terms of root systems of Lie algebras [30] [31] or Lie superalgebras [3] [16]. Martins and Ramos [16] pointed out that \( B(0|s) \) is an exception to this observation (see also [3]). To put it more precisely, an exception lies in the right hand side of (3.4) and (3.5), which correspond to the odd root \( \alpha_s \) with \( (\alpha_s|\alpha_s) \neq 0 \). In fact, one can derive (3.2) and (3.3) from (3.1) and (2.2); while cannot derive (3.4) and (3.5).

Remark: There are compact expressions of BAEs for twisted quantum affine algebras [30]. Moreover the BAEs (3.2)-(3.5) resemble to the BAEs for \( A^{(2)}_{2s} \).
This resemblance will originate from resemblance between $B(0|s)^{(1)}$ and $A_{2s}^{(2)}$. Thus there is a possibility that the BAEs (3.2)-(3.5) are also compactly written in terms of root system of the Lie superalgebra $B(0|s)$ ($s \in \mathbb{Z}_{\geq 1}$). We also point out that the expression (3.1) is not always valid for non-distinguished simple root systems. In fact we have confirmed for several cases that the Bethe ansatz equations corresponding to the odd roots $\alpha$ with $(\alpha|\alpha) \neq 0$ have similar structure to (3.4) or (3.5) by using the correspondence between the particle-hole transformation and the (super) Weyl reflection.

We define the set

$$J = J_+ \cup J_-,$$

where

$$J_- = \{1, 2, \ldots, s, \overline{s}, \ldots, \overline{2}, \overline{1}\}$$

(3.8)

is common for $B(r|s)$ and $D(r|s)$; while $J_+$ is not:

$$J_+ = \{s + 1, s + 2, \ldots, s + r, \overline{s + r}, \ldots, \overline{s + 2}, \overline{s + 1}\} \cup \{0\} \text{ for } B(r|s),$$

$$J_+ = \{s + 1, s + 2, \ldots, s + r, \overline{s + r}, \ldots, \overline{s + 2}, \overline{s + 1}\} \text{ for } D(r|s).$$

On this set $J$, we define the total order

$$1 \prec 2 \prec \cdots \prec s + r \prec 0 \prec \overline{s + r} \prec \cdots \prec \overline{2} \prec \overline{1}$$

for $B(r|s)$ case, and the partial order

$$1 \prec 2 \prec \cdots \prec s + r - 1 \prec \frac{s + r}{s + \overline{r}} \prec \cdots \prec \frac{s + r - 1}{s + \overline{r}} \prec \overline{2} \prec \overline{1}$$

(3.11)

for $D(r|s)$ case. In contrast to $B(r|s)$ case, there is no order between $s + r$ and $\overline{s + r}$ for $D(r|s)$ case. We also define the grading parameter as follows:

$$p(a) = \begin{cases} 0 & \text{for } a \in J_+, \\ 1 & \text{for } a \in J_- \end{cases}$$

(3.12)

For $a \in J$, we define the following functions.

$B(r|s)$ ($r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 1}$) case:

$$\underline{a}_u = \psi_a(u) \frac{Q_{a-1}(u - a - 1)Q_a(u - a + 2)}{Q_{a-1}(u - a + 1)Q_a(u - a)} \text{ for } 1 \leq a \leq s,$$

5 In this paper, we often abbreviate the spectral parameter $u$. 8
Here we assume and (3.14) are given as follows.

\[
\begin{align*}
\tilde{a}_u &= \psi_a(u) \frac{Q_{a-1}(u - 2s + a + 1)Q_a(u - 2s + a - 2)}{Q_{a-1}(u - 2s + a - 1)Q_a(u - 2s + a)} \\
\tilde{0}_u &= \psi_0(u) \frac{Q_{s+r}(u - s + r + 1)Q_{s+r}(u - s + r - 2)}{Q_{s+r}(u - s + r - 1)Q_{s+r}(u - s + r)}, \\
\tilde{a}_u &= \psi\pi(u) \frac{Q_{a-1}(u + 2r - a - 2)Q_a(u + 2r - a - 1)}{Q_{a-1}(u + 2r - a)Q_a(u + 2r - a - 1)} \\
\tilde{a}_u &= \psi\pi(u) \frac{Q_{a-1}(u - 2s + 2r + a - 3)Q_a(u - 2s + 2r + a - 3)}{Q_{a-1}(u - 2s + 2r + a - 2)Q_a(u - 2s + 2r + a - 2)}
\end{align*}
\]

for \(s + 1 \leq a \leq s + r\),

\[
\begin{align*}
\psi_r(s) (r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1}) case:
\end{align*}
\]

\[
\begin{align*}
\tilde{a}_u &= \psi_a(u) \frac{Q_{a-1}(u - a - 1)Q_a(u - a + 2)}{Q_{a-1}(u - a + 1)Q_a(u - a)} \quad \text{for} \quad 1 \leq a \leq s,
\tilde{a}_u &= \psi_a(u) \frac{Q_{a-1}(u - 2s + a + 1)Q_a(u - 2s + a - 2)}{Q_{a-1}(u - 2s + a - 1)Q_a(u - 2s + a)} \\
&\quad \text{for} \quad s + 1 \leq a \leq s + r - 2,
\psi_{r+s-1}(u) &= \frac{Q_{s+r-2}(u - s + r)Q_{s+r-1}(u - s + r - 3)}{Q_{s+r-2}(u - s + r - 2)Q_{s+r-1}(u - s + r - 1)} \\
&\quad \times \frac{Q_{s+r}(u - s + r - 3)}{Q_{s+r}(u - s + r - 1)}.
\psi_{r+s}(u) &= \psi_{r+s}(u) \frac{Q_{s+r-1}(u - a + 1)Q_{s+r-1}(u - a - 3)}{Q_{s+r-1}(u - a + 2)Q_{s+r-1}(u - a - 2)} \\
&\quad \text{for} \quad s + 1 \leq a \leq s + r - 2,
\psi_{r+s-1}(u) &= \psi_{r+s-1}(u) \frac{Q_{s+r-2}(u - s + r - 2)Q_{s+r-1}(u - s + r - 1)}{Q_{s+r-2}(u - s + r)Q_{s+r-1}(u - s + r - 1)} \\
&\quad \times \frac{Q_{s+r}(u - s + r - 1)}{Q_{s+r}(u - s + r - 1)}.
\psi_{r+s}(u) &= \psi_{r+s}(u) \frac{Q_{a-1}(u + 2r - a)Q_a(u + 2r - a)}{Q_{a-1}(u + 2r - a - 1)Q_a(u + 2r - a - 2)} \\
&\quad \text{for} \quad s + 1 \leq a \leq s + r - 2,
\psi_{r+s-1}(u) &= \psi_{r+s-1}(u) \frac{Q_{a-1}(u - 2s + 2r + a - 3)Q_a(u - 2s + 2r + a - 3)}{Q_{a-1}(u - 2s + 2r + a - 2)Q_a(u - 2s + 2r + a - 2)}
&\quad \text{for} \quad 1 \leq a \leq s.
\end{align*}
\]

Here we assume \(Q_0(u) = 1\). The vacuum parts of the functions \(\tilde{a}_u\), (3.13) and (3.14) are given as follows.
For \( B(r|s) \) \( (r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 1}) \) case:
\[
\psi_1(u) = \phi(u-2)\phi(u-2s+2r-1), \\
\psi_2(u) = \phi(u)\phi(u-2s+2r-1) \quad \text{for} \quad 2 \leq a \leq 2, \\
\psi_3(u) = \phi(u)\phi(u-2s+2r+1). \quad (3.15)
\]

\( D(r|s) \) \( (r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1}) \) case:
\[
\psi_1(u) = \phi(u-2)\phi(u-2s+2r-2), \\
\psi_2(u) = \phi(u)\phi(u-2s+2r-2) \quad \text{for} \quad 2 \leq a \leq 2, \\
\psi_3(u) = \phi(u)\phi(u-2s+2r). \quad (3.16)
\]

Here
\[
\phi(u) = \prod_{j=1}^{N} \Phi(u - w_j). \quad (3.17)
\]

Under the BAES \((3.1)-(3.5)\), we have: \(^6\)

For \( B(r|s) \) \( (r, s \in \mathbb{Z}_{\geq 1}) \) case:
\[
\text{Res}_{u=d+s+_{k}^{(s)}}(d, d+1) = 0 \quad \text{for} \quad 1 \leq d \leq s-1, \quad (3.18)
\]
\[
\text{Res}_{u=s+_{k}^{(s)}}(s, s+1) = 0, \quad (3.19)
\]
\[
\text{Res}_{u=2s-d+_{k}^{(s)}}(d, d+1) = 0 \quad \text{for} \quad s+1 \leq d \leq s+r-1, \quad (3.20)
\]
\[
\text{Res}_{u=s-r+_{k}^{(s+r)}}(s+r, 0) = 0, \quad (3.21)
\]
\[
\text{Res}_{u=s-r+1+_{k}^{(s+r)}}(0, s+r) = 0, \quad (3.22)
\]
\[
\text{Res}_{u=d-2r+1+_{k}^{(s)}}(d+1, d) = 0 \quad \text{for} \quad s+1 \leq d \leq s+r-1, \quad (3.23)
\]
\[
\text{Res}_{u=s-2r+1+_{k}^{(s)}}(s+1, s) = 0, \quad (3.24)
\]
\[
\text{Res}_{u=-d+2s-2r+1+_{k}^{(s)}}(d+1, d) = 0 \quad \text{for} \quad 1 \leq d \leq s-1. \quad (3.25)
\]

For \( B(0|s) \) \( (s \in \mathbb{Z}_{\geq 1}) \) case:
\[
\text{Res}_{u=d+_{k}^{(s)}}(d, d+1) = 0 \quad \text{for} \quad 1 \leq d \leq s-1, \quad (3.26)
\]
\[
\text{Res}_{u=s+_{k}^{(s)}}(s, 0) = 0, \quad (3.27)
\]
\[
\text{Res}_{u=s+1+_{k}^{(s)}}(0, s) = 0, \quad (3.28)
\]
\[
\text{Res}_{u=-d+2s+1+_{k}^{(s)}}(d+1, d) = 0 \quad \text{for} \quad 1 \leq d \leq s-1. \quad (3.29)
\]

\(^{6}\) Here \( \text{Res}_{u=a}f(u) \) denotes the residue of a function \( f(u) \) at \( u = a \).
We assign coordinates \((i, j)\) \(\in \mathbb{Z}^2\) on the skew-Young superdiagram \(\lambda \subset \mu\) such that the row index \(i\) increases as we go downwards and the column index \(j\) increases as we go from the left to the right and that \((1, 1)\) is on the top left corner of \(\mu\). We define an admissible tableau \(b\) on the skew-Young superdiagram \(\lambda \subset \mu\) as a set of elements \(b(i, j)\) \(\in J\) labeled by the coordinates \((i, j)\) mentioned above, with the following rule.

The admissible condition for \(B(r|s)\) \(r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1}\):

1. 
   
   \[ b(i, j) \preceq b(i, j + 1), \]

2. 
   
   \[ b(i, j) \preceq b(i + 1, j), \]

3. 
   
   \[ b(i, j) \prec b(i + 1, j) \quad \text{if} \quad b(i, j) \in J_+ \setminus \{0\}, \]

4. 
   
   \[ b(i, j) \prec b(i, j + 1) \quad \text{if} \quad b(i, j) \in J_- \cup \{0\}. \]

The admissible condition for \(D(r|s)\) \(r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1}\); \(\lambda = \phi\); \(\mu = (1^s)\):
1. 

\[ b(k,1) \preceq b(k+1,1) \quad \text{if} \quad b(k+1,1) \in J_- \]

2. 

\[ b(k,1) \prec b(k+1,1) \quad \text{if} \quad b(k+1,1) \in J_+ \]

unless 

\[ (b(k,1), b(k+1,1)) = (s+r, s+r) \quad \text{or} \quad (s+r, s+r). \]

The admissible condition for \( D(r|s) \) \((r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1})\); \( \lambda = \phi; \mu = (m^1) \):

1. 

\[ b(1,k) \preceq b(1,k+1) \quad \text{if} \quad b(1,k+1) \in J_+, \]

2. 

\[ b(1,k) \prec b(1,k+1) \quad \text{if} \quad b(1,k+1) \in J_-, \]

3. \( s + r \) and \( s + r \) do not appear simultaneously.

Let \( B(\lambda \subset \mu) \) be the set of admissible tableaux \(^7\) on \( \lambda \subset \mu \). We shall present a function \( T_{\lambda \subset \mu}(u) \) with a spectral parameter \( u \in \mathbb{C} \) and skew-Young superdiagrams \( \lambda \subset \mu \), which is a candidate of a set of DVFs for various fusion types in the auxiliary spaces \(^8\) of transfer matrices of \( B(r|s) \) or \( D(r|s) \) vertex models. For the skew-Young (super) diagrams \( \lambda \subset \mu \), define \( T_{\lambda \subset \mu}(u) \) as follows

\[
T_{\lambda \subset \mu}(u) = \sum_{b \in B(\lambda \subset \mu)} \prod_{(i,j) \in (\lambda \subset \mu)} (-1)^{p(b(i,j))} b(i,j)_{-\mu_1+\mu'_1-2i+2j}, \tag{3.38}
\]

where the product is taken over the coordinates \((i,j)\) on \( \lambda \subset \mu \).

\(^7\) In contrast to \( B(r|s) \) case, the admissible condition for \( D(r|s) \) case has non-local nature. This property makes it difficult to extend the admissible condition for \( D(r|s) \) to more general skew-Young (super) diagrams.

\(^8\) We assume that they are finite dimensional modules of quantum affine superalgebras (or super Yangians) \(^{15,16}\). Thus \( T_{\lambda \subset \mu}(u) \) is expected to be a kind of a (super) character of such algebras. At present, we can not justify this speculations mathematically in general, since we luck systematic representation theory of such algebras. We hope that mathematically satisfactory account on our formulae appear after the development of representation theory in the future.
We can express $T_{\lambda\mu}(u)$ as determinants over matrices, whose matrix elements are $T^a$ or $T_m^a$ (cf. [27, 25]).

For $B(r|s)$ ($r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 1}$) case, we have

$$T_{\lambda \mu}(u) = \det_{1 \leq i,j \leq m}(T^{\mu_i \lambda_j}) = \det_{1 \leq i,j \leq m}(T^{\mu_i - \lambda_j + i + j - 1}).$$

For $D(r|s)$ ($r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1}$) case, we have

$$T_m(u) = \det_{1 \leq i,j \leq m}(T^{1-i+j}(u - m + i + j - 1)).$$

Note that the function $T^1(u) = T_1(u)$ coincides with the eigenvalue formula of a $B(r|s)$ or $D(r|s)$ vertex model by the algebraic Bethe ansatz [16] after some redefinitions.

We remark that if $\Phi(-u) = \pm \Phi(u)$, is transformed to $\overline{u}$ under the following transformation.

For $B(r|s)$ ($r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\geq 1}$) case:

$$u \rightarrow -(u + 2r - 2s - 1),$$

$$u_j^{(a)} \rightarrow -u_j^{(a)},$$

$$w_j \rightarrow -w_j.$$  \hspace{1cm} (3.42)

For $D(r|s)$ ($r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1}$) case:

$$u \rightarrow -(u + 2r - 2s - 2),$$

$$u_j^{(a)} \rightarrow -u_j^{(a)},$$

$$w_j \rightarrow -w_j.$$  \hspace{1cm} (3.43)

$T_m(u)$ and $T^a(u)$ are invariant under the transformations (3.42) or (3.43).

This invariance may be viewed as a kind of crossing symmetry.

Now we shall present examples of (3.38) for $B(2|1)$, $J_- = \{1, \overline{1}\}$, $J_+ = \{2, 3, 0, \overline{3}, \overline{2}\}$ case:

$$T^1(u) = -[1 + 2] + [3] + [0] + [\overline{3}] + [\overline{2}] - [\overline{1}]$$

$$= -\phi(-2 + u)\phi(1 + u)\frac{Q_1(1 + u)}{Q_1(-1 + u)} + \phi(u)\phi(1 + u)\frac{Q_1(1 + u)Q_2(-2 + u)}{Q_1(-1 + u)Q_2(u)}.$$
\[ T^2(u) = \begin{bmatrix}
1 & 1 & -1 & -1 & 1 & 1 & 2 & 0 & 2 & 2 \\
1 & 2 & 3 & 0 & 1 & 3 & 2 & 0 & 0 & 3 \\
2 & 3 & 0 & 3 & 3 & 2 & 3 & 0 & 0 & 3 \\
3 & 3 & 2 & 3 & 2 & 2 & 2 & 1 & 1 & 1 \\
\end{bmatrix} \\
= \phi(-1 + u)\phi(2 + u)\left(\phi(-3 + u)\phi(u)\frac{Q_1(2 + u)}{Q_1(-2 + u)}\right) \\
+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(-1 + u)Q_1(2 + u)}{Q_1(u)Q_1(1 + u)} \\
+ \phi(1 + u)\phi(4 + u)\frac{Q_1(-1 + u)}{Q_1(3 + u)} \\
- \phi(-1 + u)\phi(u)\frac{Q_1(2 + u)Q_2(-3 + u)}{Q_1(-2 + u)Q_2(-1 + u)} \\
- \phi(1 + u)\phi(2 + u)\frac{Q_1(-1 + u)Q_1(2 + u)Q_2(-1 + u)}{Q_1(u)Q_1(1 + u)Q_2(u)} \\
- \phi(-1 + u)\phi(u)\frac{Q_1(-1 + u)Q_1(2 + u)Q_2(2 + u)}{Q_1(u)Q_1(1 + u)Q_2(u)} \\
+ \phi(u)\phi(1 + u)\frac{Q_1(-1 + u)Q_1(2 + u)Q_2(-1 + u)Q_2(2 + u)}{Q_1(u)Q_1(1 + u)Q_2(u)Q_2(1 + u)} \\
- \phi(1 + u)\phi(2 + u)\frac{Q_1(-1 + u)Q_2(4 + u)}{Q_1(3 + u)Q_2(2 + u)} \\
+ \phi(u)\phi(1 + u)\frac{Q_1(2 + u)Q_3(-2 + u)}{Q_1(u)Q_3(u)} \tag{3.44} \]
\[ T_2(u) = - \frac{\phi(-1 + u)\phi(u)Q_1(2 + u)Q_2(1 + u)Q_3(-2 + u)}{Q_1(u)Q_2(-1 + u)Q_3(u)} - \frac{\phi(-1 + u)\phi(u)Q_1(2 + u)Q_3(-2 + u)Q_3(1 + u)}{Q_1(u)Q_2(1 + u)Q_3(-1 + u)} + \frac{\phi(u)\phi(1 + u)Q_1(2 + u)Q_2(-2 + u)Q_2(-1 + u)Q_3(1 + u)}{Q_1(u)Q_2(u)Q_2(1 + u)Q_3(-1 + u)} - \frac{\phi(-1 + u)\phi(u)Q_1(2 + u)Q_3(-2 + u)Q_3(1 + u)}{Q_1(u)Q_3(-1 + u)Q_3(u)} + \frac{\phi(u)\phi(1 + u)Q_1(2 + u)Q_3(-1 + u)Q_3(-2 + u)Q_3(1 + u)}{Q_1(u)Q_3(1 + u)Q_3(-1 + u)Q_3(u)} - \frac{\phi(1 + u)\phi(2 + u)Q_1(-1 + u)Q_2(3 + u)Q_3(u)}{Q_1(1 + u)Q_2(1 + u)Q_3(2 + u)} + \frac{\phi(u)\phi(1 + u)Q_1(-1 + u)Q_2(2 + u)Q_2(3 + u)Q_3(u)}{Q_1(1 + u)Q_2(1 + u)Q_3(2 + u)} + \frac{\phi(u)\phi(1 + u)Q_2(3 + u)Q_3(-2 + u)Q_3(1 + u)}{Q_2(1 + u)Q_3(-1 + u)Q_3(2 + u)} + \frac{\phi(u)\phi(1 + u)Q_2(-2 + u)Q_2(3 + u)Q_3(u)Q_3(1 + u)}{Q_2(u)Q_2(1 + u)Q_3(-1 + u)Q_3(2 + u)} + \frac{\phi(u)\phi(1 + u)Q_1(-1 + u)Q_3(3 + u)}{Q_1(1 + u)Q_3(1 + u)} - \frac{\phi(1 + u)\phi(2 + u)Q_1(-1 + u)Q_2(u)Q_5(3 + u)}{Q_1(1 + u)Q_2(2 + u)Q_3(1 + u)} + \frac{\phi(u)\phi(1 + u)Q_3(-2 + u)Q_5(3 + u)}{Q_3(-1 + u)Q_5(2 + u)} + \frac{\phi(u)\phi(1 + u)Q_2(-2 + u)Q_5(u)Q_5(3 + u)}{Q_2(u)Q_5(-1 + u)Q_5(2 + u)} - \frac{\phi(1 + u)\phi(2 + u)Q_1(-1 + u)Q_3(u)Q_5(3 + u)}{Q_1(1 + u)Q_3(1 + u)Q_5(2 + u)} + \frac{\phi(u)\phi(1 + u)Q_1(-1 + u)Q_2(2 + u)Q_3(u)Q_5(3 + u)}{Q_1(1 + u)Q_2(2 + u)Q_3(1 + u)Q_3(2 + u)} \right), \quad (3.45) \]
\[
\begin{align*}
&= \phi(u)\phi(1 + u)\left(\phi(-3 + u)\phi(4 + u)\frac{Q_1(u)Q_1(1 + u)}{Q_1(-2 + u)Q_1(3 + u)}\right) \\
&- \phi(-1 + u)\phi(4 + u)\frac{Q_1(u)Q_1(1 + u)Q_2(-3 + u)}{Q_1(-2 + u)Q_1(3 + u)Q_2(-1 + u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(2 + u)Q_2(-3 + u)}{Q_1(-2 + u)Q_2(1 + u)} \\
&- \phi(-3 + u)\phi(2 + u)\frac{Q_1(2 + u)Q_2(-1 + u)}{Q_1(-2 + u)Q_2(1 + u)} \\
&- \phi(-1 + u)\phi(4 + u)\frac{Q_1(-1 + u)Q_2(2 + u)}{Q_1(3 + u)Q_2(u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(-1 + u)Q_2(4 + u)}{Q_1(3 + u)Q_2(u)} \\
&- \phi(-3 + u)\phi(2 + u)\frac{Q_1(u)Q_1(1 + u)Q_2(2 + u)}{Q_1(-2 + u)Q_1(3 + u)Q_2(2 + u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(u)Q_1(1 + u)Q_2(-3 + u)Q_2(4 + u)}{Q_1(-2 + u)Q_1(3 + u)Q_2(-1 + u)Q_2(2 + u)} \\
&- \phi(-1 + u)\phi(4 + u)\frac{Q_1(1 + u)Q_2(1 + u)Q_3(-2 + u)}{Q_1(3 + u)Q_2(-1 + u)Q_2(u)Q_3(u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(1 + u)Q_2(1 + u)Q_2(4 + u)Q_3(-2 + u)}{Q_1(3 + u)Q_2(-1 + u)Q_2(2 + u)Q_3(u)} \\
&- \phi(-1 + u)\phi(4 + u)\frac{Q_1(1 + u)Q_2(-2 + u)Q_3(1 + u)}{Q_1(3 + u)Q_2(u)Q_3(-1 + u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(1 + u)Q_2(-2 + u)Q_2(4 + u)Q_3(1 + u)}{Q_1(3 + u)Q_2(u)Q_3(2 + u)Q_3(-1 + u)} \\
&- \phi(-1 + u)\phi(4 + u)\frac{Q_1(1 + u)Q_3(-2 + u)Q_3(1 + u)}{Q_1(3 + u)Q_3(-1 + u)Q_3(u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(1 + u)Q_2(4 + u)Q_3(-2 + u)Q_3(1 + u)}{Q_1(3 + u)Q_2(u)Q_3(2 + u)Q_3(-1 + u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_2(3 + u)Q_3(-2 + u)}{Q_2(-1 + u)Q_3(2 + u)} \\
&- \phi(-3 + u)\phi(2 + u)\frac{Q_1(u)Q_2(3 + u)Q_3(u)}{Q_1(-2 + u)Q_2(1 + u)Q_3(2 + u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(u)Q_2(-3 + u)Q_2(3 + u)Q_3(u)}{Q_1(-2 + u)Q_2(-1 + u)Q_2(1 + u)Q_3(2 + u)} \\
&+ \phi(-1 + u)\phi(2 + u)\frac{Q_2(-2 + u)Q_3(3 + u)}{Q_2(2 + u)Q_3(-1 + u)}
\end{align*}
\]
Note that DVFs have so called Bethe-strap component (cf. [47]) which include the top term [26, 27] as the examples in [22]). However we have confirmed for several examples the fact that the pseudo-top terms

$$\phi(-1 + u)\phi(2 + u)\frac{Q_2(u)Q_3(-2 + u)Q_3(3 + u)}{Q_2(2 + u)Q_3(-1 + u)Q_3(u)}$$

$$- \phi(-3 + u)\phi(2 + u)\frac{Q_1(u)Q_2(u)Q_3(3 + u)}{Q_1(-2 + u)Q_2(2 + u)Q_3(1 + u)}$$

$$+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(u)Q_2(-3 + u)Q_2(u)Q_3(3 + u)}{Q_1(-2 + u)Q_2(-1 + u)Q_2(2 + u)}$$

$$+ \phi(-1 + u)\phi(2 + u)\frac{Q_2(u)Q_2(1 + u)Q_3(-2 + u)Q_3(3 + u)}{Q_2(-1 + u)Q_3(2 + u)Q_3(1 + u)}$$

$$+ \phi(-1 + u)\phi(2 + u)\frac{Q_1(u)Q_3(u)Q_3(3 + u)}{Q_1(-2 + u)Q_2(-1 + u)Q_3(1 + u)Q_3(2 + u)} \right). \tag{3.46}$$

Thanks to Theorem 3.11 (see later) and the relation (3.39), these DVFs are free of poles under the following BAE:

$$\frac{\phi(u_k^{(1)} - 1)}{\phi(u_k^{(1)} + 1)} = \frac{Q_2(u_k^{(1)} - 1)}{Q_2(u_k^{(1)} + 1)} \quad \text{for} \quad 1 \leq k \leq N_1,$$

$$-1 = \frac{Q_1(u_k^{(2)} - 1)Q_2(u_k^{(2)} + 2)Q_3(u_k^{(2)} - 1)}{Q_1(u_k^{(2)} + 1)Q_2(u_k^{(2)} - 2)Q_3(u_k^{(2)} + 1)} \quad \text{for} \quad 1 \leq k \leq N_2,$$

$$-1 = \frac{Q_2(u_k^{(3)} - 1)Q_3(u_k^{(3)} + 1)}{Q_2(u_k^{(3)} + 1)Q_3(u_k^{(3)} - 1)} \quad \text{for} \quad 1 \leq k \leq N_3. \tag{3.47}$$

Note that DVFs have so called Bethe-strap structures [26, 28], which bear resemblance to weight space diagrams. We have observed for many examples that $T_{\lambda \mu}(u)$ coincides with the Bethe-strap of the minimal connected component (cf. [47]) which include the top term [26, 27] as the examples in Figure 4, Figure 5 and Figure 6. The top term of $T_{\lambda \mu}(u)$ carries a $B(r|s)$ or $D(r|s)$ weight. For example, for $B(r|s)$, $\lambda = \phi$, $\mu_{r+1} \leq s$ case, the term corresponding to the tableau

$$b(i, j) = \left\{ \begin{array}{ll}
j & \text{for} \quad 1 \leq i \leq \mu_j \quad 1 \leq j \leq s \\
i + s & \text{for} \quad 1 \leq i \leq \mu_j \quad s + 1 \leq j \leq \mu_1
\end{array} \right. \tag{3.48}$$

\footnote{Recently we have found curious terms (pseudo-top terms) in many Bethe-straps (cf. [22]). However we have confirmed for several examples the fact that the pseudo-top terms do not influence on connectivity of the Bethe straps (cf. [26, 27, 17]) in the whole.}
carries the $B(r|s)$ weight with the Kac-Dynkin label (2.5) or (2.6). The top term \[^{12}\text{[26]}\] of the DVF (3.38) for $D(r|s)$, $\lambda = \phi, \mu = (1^a)$ will be

\[
(-1)^a \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} a = (-1)^a \frac{Q_1(u + a)}{Q_1(u - a)},
\]

which carries the $D(r|s)$ weight with the Kac-Dynkin label in (2.7). The top term \[^{13}\text{[26]}\] of the DVF (3.38) for $D(r|s)$, $\lambda = \phi, \mu = (m^1)$ will be

\[
(-1)^m \begin{array}{ccccccc} 1 & 2 & \cdots & m \end{array} = \begin{array}{c} (-1)^m \frac{Q_m(u + 1)}{Q_m(u - 1)} \end{array} \text{ if } 1 \leq m \leq s,
\]

\[
(-1)^r \begin{array}{ccccccc} 1 & 2 & \cdots & s & s + 1 & \cdots & s + 1 \end{array} = \begin{array}{c} (-1)^s \frac{Q_s(u + m - s + 1)Q_{s+1}(u - m + s)}{Q_s(u - m + s - 1)Q_{s+1}(u + m - s)} \end{array}
\text{ if } \begin{array}{c} r \geq 3 \text{ and } m \geq s + 1, \\
\end{array}
\]

\[
= \begin{array}{c} (-1)^s \frac{Q_s(u + m - s + 1)Q_{s+1}(u - m + s)Q_{s+2}(u - m + s)}{Q_s(u - m + s - 1)Q_{s+1}(u + m - s)Q_{s+2}(u + m - s)} \end{array}
\text{ if } \begin{array}{c} r = 2 \text{ and } m \geq s + 1, \\
\end{array}
\]

which carries the $D(r|s)$ weight with the Kac-Dynkin label in (2.8).

Remark: There is a supposition (cf [27, 47]) that the auxiliary space of a transfer matrix is a irreducible one as a representation space of the Yangian (or quantum affine algebra) if the Bethe strap of the DVF is connected in the whole. Then a natural question arise: ”Is the Bethe strap of $T_{\lambda \subset \mu}(u)$ always connected in the whole ?” The answer is no. In fact for $D(r|s)$ case, the Bethe strap of $T^a(u)$ is not connected if $0 \leq r - s - 1 \leq a \leq 2(r - s - 1)$. So it is desirable to extract the minimal connected component of the Bethe strap which contains the top term (3.49) from $T^a(u)$. A candidate is as follows:

\[
T^a(u) - h^a(u)T^{-a+2(r-s-1)}(u),
\]

where $h^a(u) = \prod_{j=1}^{r+s-1} \psi_1(u + a - 2j + 1)\psi_1(u - a + 2j - 1)$.

\[^{12}\text{Here we omit the vacuum part.}
^{13}\text{Here we omit the vacuum part.}
^{14}\text{Here the word ”connected” means that any terms in DVF are connected directly (or indirectly) each other by the arrows like graphs in Figures 4.6}
Figure 4: The Bethe-strap structure of $T^1(u)$ (3.44) for $B(2|1) = osp(5|2)$: The pair $(a, b)$ denotes the common pole $u_k^{(a)} + b$ of the pair of the tableaux connected by the arrow. This common pole vanishes under the BAE (3.47). The leftmost tableau corresponds to the ‘highest weight’, which is called the top term. Such a correspondence between certain term in the DVF and a highest weight (to be more precise, a kind of Drinfel’d polynomial (cf [44, 27])) may be called top term hypothesis [26, 27].

For example, for $D(3|1)$ case, $T^2(u)$ consists of 31 terms and they divide into 30 terms whose Bethe strap is connected in the whole and 1 isolated term $h_2(u) = \{\phi(u - 1)\phi(u + 3)\}^2$. Thus Bethe strap of $T^2(u) - h_2(u)$ is connected in the whole. On the other hand for $D(2|2)$ case, $T^2(u)$ does not have such an isolated term and that the Bethe strap is connected in the whole (in this case, $T^2(u)$ has 33 terms). So far this kind of an isolated term is peculiar to $D(r|s)$ case. In fact, we have never yet observed such an isolated term in $B(r|s)$ case. Similarly, the Bethe strap of $T_m(u)$ for $D(r|s)$ seems not to be connected if $0 \leq s - r + 1 \leq m \leq 2(s - r + 1)$. A candidate for the minimal connected component of the Bethe strap which contains the top term (3.50) is

$$T_m(u) - h_m(u)T_{-m+2(s-r+1)}(u),$$

where $h_m(u) = \prod_{j=1}^{m+r-s-1} \psi_j(u - m + 2j - 1)\psi_j(u + m - 2j + 1)$.\[15\]

A remarkable resemblance between Bethe-straips for vector representations and crystal graphs [48, 49] was pointed out in [26]. Whether such resemblance holds true for the Lie superalgebras in general or not will be an interesting question. There is a remarkable coincidence between currents of deformed Virasoro algebra and DVFs [50]. Whether such coincidence holds true for the Lie superalgebras in general or not will be another interesting question.

We can prove (see Appendix A.1-A.3) the following Theorem, which is essential in the analytic Bethe ansatz.

In this case, $-\frac{2}{1}$ is a pseudo-top term (cf [22]).
Figure 5: The Bethe-strap structure of $T^2(u)$ (3.45) for $B(2|1)$: The topmost tableau corresponds to the top term.
Figure 6: The Bethe-strap structure of $T_2(u) \ (3.46)$ for $B(2|1)$: The topmost tableau corresponds to the top term.
Theorem 3.1 For \( a \in \mathbb{Z}_{\geq 0} \), \( T^a(u) \) (3.33) for \( \lambda = \phi, \mu = (1^a) \) is free of poles under the condition that the BAEs (3.1)-(3.5) are valid.\(^{16}\)

In proving Theorem 3.1 we use the following lemmas.

Lemma 3.2 For \( r \in \mathbb{Z}_{\geq 2} \) and \( b \in \{s+1, s+2, \ldots, s+r-1\} \),

\[
\begin{array}{c|c}
  b & v \\
  \hline
  b+1 & v-2 \\
\end{array}
\quad \begin{array}{c|c}
  b+1 & v \\
  \hline
  b & v-2 \\
\end{array} \quad (3.53)
\]

do not contain \( Q_b \).

For \( B(0|s) \) case, we use the following lemma:

Lemma 3.3 For \( b \in \{1, 2, \ldots, s-1\} \),

\[
\begin{array}{c|c}
  b & b+1 \\
  \hline
  u & u+2 \\
\end{array}
\quad \begin{array}{c|c}
  b+1 & b \\
  \hline
  u & u+2 \\
\end{array} \quad (3.54)
\]

do not contain \( Q_b \), and

\[
\begin{array}{c|c|c}
  s & 0 & \Xi \\
  \hline
  u & u+2 & u+4 \\
\end{array} \quad (3.55)
\]

does not contain \( Q_s \).

Then owing to the relation (3.39), \( T_m^a(u) \) for \( B(r|s) \) is also free of poles under the condition that the BAEs (3.1)-(3.5) are valid. Similarly, owing to the relation (3.41), \( T_m(u) \) for \( D(r|s) \) is also free of poles under the condition that the BAE (3.1) is valid.

4 Functional relations among DVFs

Now we introduce the functional relations among DVFs. For \( B(r|s) \) case, the following relation follows from the determinant formulae (3.39) or (3.40).

\[
T_m^a(u-1)T_m^a(u+1) = T_{m-1}^a(u)T_{m+1}^a(u) + T_{m-1}^{a-1}(u)T_{m+1}^{a+1}(u), \quad (4.1)
\]

\(^{16}\) We consider the case that the solutions \( \{u_j^{(a)}\} \) of the BAEs (3.1)-(3.3) have ‘generic’ distribution: We assume that \( u_i^{(a)} - u_j^{(a)} \neq (\alpha_a|\alpha_a) \) for any \( i, j \in \{1, 2, \ldots, N_a\} \) and \( a \in \{1, 2, \ldots, s+r\} \) \((i \neq j)\) in BAEs (3.1)-(3.3). Moreover we assume that the color \( b \) pole (see Appendix A.1-A.3) of \( T^a(u) \) and the color \( c \) pole do not coincide each other if \( b \neq c \). We will need separate consideration for the case where this assumption does not hold. We also note that similar assumption was assumed in [11, 20, 21, 22].
where \( m, a \in \mathbb{Z}_{\geq 1} \); \( T_m^0(u) = T_0^a(u) = 1 \). This functional relation (4.1) is a Hirota bilinear difference equation [51] and can be proved by using the Jacobi identity. There is a constraint to (4.1) follows from the relation (cf. [52, 53] for \( sl(r|s) \) case):

\[ T_{\lambda \subset \mu}(u) = 0 \] if \( \lambda \subset \mu \) contains an \( a \times m \) rectangular subdiagram (\( a: \) the number of row, \( m: \) the number of column) with \( m \in \mathbb{Z}_{\geq 2s+2} \) and \( a \in \mathbb{Z}_{\geq 2r+1} \).

In particular, we have

\[ T_m^a(u) = 0 \quad \text{if} \quad m \in \mathbb{Z}_{\geq 2s+2} \quad \text{and} \quad a \in \mathbb{Z}_{\geq 2r+1}. \tag{4.2} \]

We also note that the determinant formula (3.41) for \( D(r|s) \) reduces to the following functional relation:

\[ T^1(u-1)T^1(u+1) = T_2(u) + T^2(u), \tag{4.3} \]

if we set \( m = 2 \).

In this section, we consider only \( B(0|s) \) (\( s \in \mathbb{Z}_{\geq 1} \)) case from now on. Now we redefine the function \( T_{\lambda \subset \mu}(u) \) as follows:

\[ T_{\lambda \subset \mu}(u) := T_{\lambda \subset \mu}(u)/\left\{ \prod_{j=1}^{\rho'_1} F_{\mu_j-\lambda_j}(u - \mu_1 + \mu'_1 + \mu_j + \lambda_j - 2j + 1) \right\}, \tag{4.4} \]

where

\[ F_m(u) = \prod_{j=1}^{m-1} \phi(u - m + 2j + 1)\phi(u - 2s - m + 2j - 2) \]

\[ \quad \text{for} \quad m \in \mathbb{Z}_{\geq 2}, \tag{4.5} \]

and

\[ F_1(u) = 1, \quad F_0(u) = \{ \phi(u + 1)\phi(u - 2s - 2) \}^{-1}. \tag{4.6} \]

In particular, we have

\[ T_m(u) := T_m(u)/F_m(u), \tag{4.7} \]

\[ T_0^a(u) = \prod_{j=1}^{a} T_0(u + a - 2j + 1) \]

\[ = \prod_{j=1}^{a} \phi(u + a - 2j + 2)\phi(u + a - 2j - 2s - 1). \tag{4.8} \]

There is remarkable duality for \( T_m(u) \).
**Theorem 4.1** For any $m \in \{0, 1, \ldots, 2s + 1\}$, we have

$$T_m(u) = T_{2s-m+1}(u).$$

(4.9)

**Outline of the proof:** At first, we consider the case that the vacuum parts are formally trivial. In proving the relation (4.9), we use the following relations, which can be verified by direct computation.

$$
\begin{array}{cccc}
\bar{a} \\
\hline
u \\
\end{array} \quad \times \quad 
\begin{array}{cccc}
1 \\
u - 2s + 1 \\
2 \\
u - 2s + 3 \\
\cdots \\
u - 2s - 2 \\
a \\
u + 2s - 3 \\
\end{array}
= 
\begin{array}{cccc}
1 \\
u - 2s + 1 \\
2 \\
u - 2s + 3 \\
\cdots \\
u - 2s - 2 \\
a - 1 \\
u + 2s - 3 \\
\end{array},
$$
(4.10)

$$
\begin{array}{cccc}
\alpha \\
\hline
u \\
\end{array} \quad \times \quad 
\begin{array}{cccc}
\bar{a} \\
u - 2a + 2s + 3 \\
\bar{1} \\
u + 2s - 1 \\
\bar{2} \\
u + 2s - 1 \\
\bar{T} \\
u + 2s - 1 \\
\end{array}
= 
\begin{array}{cccc}
\alpha - 1 \\
u - 2a + 2s + 3 \\
\bar{1} \\
u + 2s - 3 \\
\bar{2} \\
u + 2s - 1 \\
\bar{T} \\
u + 2s - 1 \\
\end{array},
$$
(4.11)

and

$$
\begin{array}{cccc}
1 \\
u - 2s \\
2 \\
u - 2s + 2 \\
\cdots \\
u - 2 \\
s \\
u \\
\bar{a} \\
u - 2 \\
\cdots \\
u + 2 \\
\bar{1} \\
u - 2s + 3 \\
\cdots \\
u + 2s - 2 \\
\bar{T} \\
u + 2s - 1 \\
\end{array} = 1,
$$
(4.12)

where $a \in \{1, 2, \ldots, s + 1\}$, the spectral parameter increases (cf. (3.38)) as we go from the left to the right on each tableau. We will show that any term in $T_m(u)$ coincides with a term in $T_{2s-m+1}(u)$. We will consider the signs originated from the grading parameter (3.12) separately. Any term in $T_m(u)$ can be expressed by a tableau $b \in B((m^1))$ such that $b(1, 1) = i_k$ for $1 \leq k \leq \alpha$ ($1 \leq i_1 < \cdots < i_\alpha \leq s$); $b(1, k) = \underline{j_m-k+1}$ for $\alpha + 1 \leq k \leq m$ ($0 \leq \underline{j_m-k+1} < \cdots < \underline{J_1} \leq \underline{T}$); $\alpha \in \mathbb{Z}$. The term corresponding to this tableau is given as follows.

$$
\begin{array}{cccc}
i_1 \\
u - m + 1 \\
\cdots \\
u - m + 2a - 1 \\
i_\alpha \\
u - m + 2a - 1 \\
\underline{j_m-a} \\
u - m + 2a + 1 \\
\cdots \\
u + m + 1 \\
\underline{J_1} \\
\end{array}
$$
(4.13)

\[\text{17} \text{Here we assume } s + 1 = s + 1 = 0.\]

\[\text{18} \text{is 1 if } m = 0.\]
\[
\begin{align*}
\mathbf{1} & = \\
& \begin{bmatrix}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{m} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{J}_{m-\alpha} \\
\mathbf{J}_{m+\alpha} \\
\vdots \\
\mathbf{J}_{1} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{1} \\
\mathbf{2} \\
\vdots \\
\mathbf{s} \\
\end{bmatrix}
\end{align*}
(4.14)
\]

\[
\begin{align*}
\mathbf{1} & = \\
& \begin{bmatrix}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{m} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{J}_{m-\alpha} \\
\mathbf{J}_{m+\alpha} \\
\vdots \\
\mathbf{J}_{1} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{1} \\
\mathbf{2} \\
\vdots \\
\mathbf{s} \\
\end{bmatrix}
\end{align*}
(4.15)
\]

\[
\begin{align*}
\mathbf{1} & = \\
& \begin{bmatrix}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{m} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{J}_{m-\alpha} \\
\mathbf{J}_{m+\alpha} \\
\vdots \\
\mathbf{J}_{1} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{1} \\
\mathbf{2} \\
\vdots \\
\mathbf{s} \\
\end{bmatrix}
\end{align*}
(4.16)
\]

where \( \{ J_{k} \} = \{ 1, 2, \ldots, s, 0 \} \setminus \{ j_{1}, j_{2}, \ldots, j_{m-\alpha} \} \) \((1 \leq J_{1} < \cdots < j_{s+1+\alpha-m} \leq 0)\); \( \{ \mathbf{1}_{k} \} = \{ \mathbf{1}, s-1, \ldots, \mathbf{1} \} \setminus \{ j_{1}, j_{2}, \ldots, j_{1} \} \) \((1 \leq \mathbf{1}_{s-\alpha} < \cdots < \mathbf{1}_{1} \leq \mathbf{1})\).
follows from (4.12); (4.16) follows from (4.10) and (4.11). After repetition of procedures similar to (4.15)-(4.16), we obtain (4.17). Apparently, (4.17) is a term in $T_{2s-m+1}(u)$. Conversely, one can also show that any term in $T_{2s-m+1}(u)$ coincides with a term in $T_m(u)$.

Noting the relation

$$s+1-m+\alpha \sum_{k=1}^{s+1-m+\alpha} p(J_k) + \sum_{k=1}^{s-\alpha} p(I_k) \equiv \sum_{k=1}^{\alpha} p(i_k) \mod 2,$$

(4.18)

we find that the overall sign for (4.13) coincides with that for (4.17).

Finally, we comment on the vacuum parts. From now on, we assume that the vacuum parts are not trivial. Equivalence between the dress parts of $T_m(u)$ and $T_{2s-m+1}(u)$ has already been shown, so we have only to check that the vacuum part of

$$i_1 \ldots i_{a-1} j_{m-\alpha} \ldots j_1 / F_m(u)$$

(4.19)

is equivalent to that of

$$J_1 \ldots J_{s+1-m+\alpha} T_{s-m} \ldots T_1 / F_{2s-m+1}(u).$$

(4.20)

All we have to do is to check this by direct computation for the following four cases: (i) $i_1 = 1$ and $j_1 = \bar{T}$; (ii) $i_1 = 1$ and $j_1 = \bar{T}$; (iii) $1 < i_1$ and $j_1 = \bar{T}$; (iv) $1 < i_1$ and $j_1 = \bar{T}$.\]

Owing to the relation (3.40), we can generalize the relation (4.9) to

$$T_m^a(u) = T_{2s-m+1}^a(u),$$

(4.21)

where $a \in \mathbb{Z}_{>1}$. Taking note on the relations (4.21) and (4.2), we shall rewrite the functional relation (4.1) in a ‘canonical’ form as the original $T$-system.
for the simple Lie algebra $[32]$. Set $T_{m}^{(a)}(u) = T_{a}^{m}(u)$, $T_{2m}^{(s)}(u) = T_{s}^{m}(u)$ and $T_{0}^{(a)}(u) = T_{a}^{0}(u) = T_{0}^{(0)}(u) = 1$ for $a \in \{1, 2, \ldots, s-1\}$ and $m \in \mathbb{Z}_{\geq 1}$, where the subscript $(n, a)$ of $T_{n}^{(a)}(u)$ corresponds to the Kac-Dynkin label $[b_{1}, b_{2}, \ldots, b_{s}]$ for $b_{i} = n \delta_{ia}$ (cf. (2.6)). Then we have

$$T_{m}^{(a)}(u - 1)T_{m}^{(a)}(u + 1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + g_{m}^{(a)}(u)T_{m}^{(a-1)}(u)T_{m}^{(a+1)}(u) \quad \text{for} \quad a \in \{1, 2, \ldots, s-2\}, \quad (4.22)$$

$$T_{m}^{(s-1)}(u - 1)T_{m}^{(s-1)}(u + 1) = T_{m-1}^{(s-1)}(u)T_{m+1}^{(s-1)}(u) + g_{m}^{(s-1)}(u)T_{m}^{(s-2)}(u)T_{2m}^{(s)}(u), \quad (4.23)$$

$$T_{2m}^{(s)}(u - 1)T_{2m}^{(s)}(u + 1) = T_{2m-2}^{(s)}(u)T_{2m+2}^{(s)}(u) + g_{2m}^{(s)}(u)T_{m}^{(s-1)}(u)T_{2m}^{(s)}(u), \quad (4.24)$$

where $g_{m}^{(b)}(u) = \{\prod_{j=1}^{m} T_{0}(u + 2j - m - 1)\} \delta_{s,1}$ if $s \in \mathbb{Z}_{\geq 2}$; $g_{2m}^{(1)}(u) = \{\prod_{j=1}^{m} T_{0}(u + 2j - m - 1)\}$ if $s = 1$. Note that the function $g_{m}^{(b)}(u)$ obey the following relation

$$g_{m}^{(b)}(u + 1)g_{m}^{(b)}(u - 1) = g_{m}^{(b)}(u)g_{m}^{(b-1)}(u) \quad \text{if} \quad s \in \mathbb{Z}_{\geq 2},$$

$$g_{2m}^{(1)}(u + 1)g_{2m}^{(1)}(u - 1) = g_{2m}^{(1)}(u)g_{2m}^{(1-2)}(u) \quad \text{if} \quad s = 1. \quad (4.25)$$

These functional relations (4.22)-(4.24) will be $B(0|s)$ version of the $T$-system. Note that the subscript $n$ of $T_{n}^{(a)}(u)$ can take only even number (cf. (2.9)). By construction, $T_{m}^{(a)}(u)$ can be expressed as a determinant over a matrix whose matrix elements are only the fundamental functions $T_{1}^{(1)}$, $T_{2}^{(1)}$ and $g_{1}^{(1)}$ for $s \in \mathbb{Z}_{\geq 2}$; $T_{2}^{(1)}$ and $g_{2}^{(1)}$ for $s = 1$. This can be summarized as follows:

**Theorem 4.2** For $m \in \mathbb{Z}_{\geq 1}$,

$$T_{m}^{(a)}(u) = \det_{1 \leq i, j \leq m}(T_{a+i-j}(u + m - i - j + 1)) \quad \text{for} \quad a \in \{1, 2, \ldots, s-1\}, \quad (4.26)$$

$$T_{2m}^{(s)}(u) = \det_{1 \leq i, j \leq m}(T_{s+i-j}(u + m - i - j + 1)) \quad (4.27)$$

solves (4.22)-(4.24). Here $T_{a}(u)$ obeys the relation (4.9) and the boundary condition

$$T_{a}(u) = \begin{cases} 0 & \text{if} \quad a < 0, \\ g_{1}^{(1)}(u) & \text{if} \quad a = 0 \quad \text{and} \quad s \in \mathbb{Z}_{\geq 2} \\ T_{1}^{(a)}(u) & \text{if} \quad a \in \{1, 2, \ldots, s-1\} \\ T_{2}^{(s)}(u) & \text{if} \quad a = s, \end{cases} \quad (4.28)$$

where $g_{m}^{(a)}(u) = \{\prod_{j=1}^{m} g_{1}^{(1)}(u + 2j - m - 1)\} \delta_{s,1}$ if $s \in \mathbb{Z}_{\geq 2}$; $g_{2m}^{(1)}(u) = \{\prod_{j=1}^{m} g_{2}^{(1)}(u + 2j - m - 1)\}$ if $s = 1$.  

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Table 1: The number $N_m^{(a)}$ of the terms in $T_m^{(a)}(u)$ for $B(0|2)$.

| $m$ | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|
| $N_m^{(1)}$ | 5 | 15 | 35 | 70 |
| $N_{2m}^{(2)}$ | 10 | 50 | 175 | 490 |

Table 2: The dimensionality of the module $V[b_1, b_2]$ for $B(0|2)$.

| $[b_1, b_2]$ | dim$V[b_1, b_2]$ | $[b_1, b_2]$ | dim$V[b_1, b_2]$ |
|--------------|-----------------|--------------|-----------------|
| 0 0          | 1               | 0 2          | 10              |
| 1 0          | 5               | 0 4          | 35              |
| 2 0          | 14              | 0 6          | 84              |
| 3 0          | 30              | 2 2          | 81              |

Remark: These functional realtions (4.22)-(4.24) resemble to the ones for $A_{2s}^{(2)}$ [29]. This resemblance will originate from resemblance between $B(0|s)^{(1)}$ and $A_{2s}^{(2)}$.

There is a remarkable relation between the number $N_m^{(a)}$ of the terms in $T_m^{(a)}(u)$ and the dimensionality (2.10) of the Lie superalgebra $B(0|s)$. We conjecture that they are related each other as follows:

$$N_m^{(a)} = \sum \dim V[k_1, k_2, \ldots, k_a, 0, \ldots, 0] \quad \text{if} \quad a \in \{1, 2, \ldots, s-1\},$$

$$N_{2m}^{(s)} = \sum \dim V[k_1, k_2, \ldots, k_{s-1}, 2k_s],$$

where the summation is taken over non-negative integers $\{k_j\}$ such that $k_1 + k_2 + \cdots + k_a \leq m$ and $k_j \equiv m\delta_{ja} \mod 2$. For example, for $B(0|2)$ case, we have (cf. Table 1 and Table 2):

$$N_1^{(1)} = \dim V[1, 0],$$
$$N_2^{(1)} = \dim V[2, 0] + \dim V[0, 0],$$
$$N_3^{(1)} = \dim V[3, 0] + \dim V[1, 0],$$
$$N_2^{(2)} = \dim V[0, 2],$$
$$N_4^{(2)} = \dim V[0, 4] + \dim V[2, 0] + \dim V[0, 0],$$
$$N_6^{(2)} = \dim V[0, 6] + \dim V[2, 2] + \dim V[0, 2].$$

These relations seem to suggest a superization of the Kirillov-Reshetikhin formula [54], which gives the multiplicity of occurrence of the irreducible representations of the Lie superalgebra in the Yangian module.
5 Summary and discussion

In this paper, we have carried out an analytic Bethe ansatz based on the Bethe ansatz equations (3.1)-(3.5) with the distinguished simple root systems of the type II Lie superalgebras $B(r|s)$ and $D(r|s)$. We have proposed eigenvalue formulae of transfer matrices in DVFs related to a class of tensor-like representations, and shown their pole-freeness under the BAEs (3.1)-(3.5). The key is the top term hypothesis and the pole-freeness under the BAE. A class of functional relations has been proposed for the DVFs. In particular for $B(0|s)$ case, remarkable duality among DVFs was found. By using this, a complete set of functional relations is written down for the DVFs labeled by rectangular Young (super) diagrams. To the author’s knowledge, this paper is the first trial to construct systematic theory of an analytic Bethe ansatz related to fusion $B(r|s)$ and $D(r|s)$ vertex models.

In the present paper, we have executed an analytic Bethe ansatz only for tensor-like representations. As for spinorial representations, details are under investigation. For example, in relation to the 64 dimensional typical representation of $B(2|1)$, we have confirmed the fact that the Bethe-strap generated by the following top term

$$Q_1(u + \frac{5}{2})Q_3(u - \frac{1}{2})$$

$$Q_1(u - \frac{5}{2})Q_3(u + \frac{1}{2})$$

(5.1)

which carries $B(2|1)$ weight with the Kac-Dynkin label $(\frac{5}{2}, 0, 1)$ consists of 64 terms.

For $D(r|s)$ case, we have proposed DVFs labeled by Young (super) diagrams with one row or one column. It is tempting to extend these DVFs to general Young (super) diagrams. However, this will be a difficult task since we lack of tableaux sum expressions of DVFs labeled by general Young diagrams even for the non-superalgebra $D_r$ case [26]. One way to bypass cumbersome tableaux sum expressions is to construct a complete set of transfer matrix functional relations (a hierarchy of $T$-system). By solving it, we will be able to calculate DVFs.

It is an interesting problem to derive TBA equations from our $T$-system (4.22)-(4.24). This is accomplished by a similar procedure for $sl(r|s)$ case [55] (see also [56]). We are going to report this in the near future.

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\[\text{Here we omit the vacuum part.}\]
Appendix A.1 Outline of the proof of Theorem 3.1: $B(r|s)$ ($r, s \in \mathbb{Z}_{\geq 1}$) case

For simplicity, we assume that the vacuum parts are formally trivial from now on. We prove that $T^a(u)$ is free of color $b$ poles, that is, $\text{Res}_{u=\nu_k^{(s)}} T^a(u) = 0$ for any $b \in \{1, 2, \ldots, s + r\}$ under the condition that the BAE (3.1) is valid. The function (3.13) with $c \in J$ has color $b$ poles only for $c = b, b + 1, b + 1$ or $\overline{b}$ if $b \in \{1, 2, \ldots, s + r - 1\}$; for $c = s + r, 0$ or $\overline{s + r}$ if $b = s + r$, so we shall trace only $b, b + 1, b + 1$ or $\overline{b}$ for $b \in \{1, 2, \ldots, s + r - 1\}$; $s + r, 0$ or $\overline{s + r}$ for $b = s + r$. Let $S_k$ be the partial sum of $T^a(u)$, which contains $k$ boxes among $b, b + 1, b + 1$ or $\overline{b}$ for $b \in \{1, 2, \ldots, s + r - 1\}$; $s + r, 0$ or $\overline{s + r}$ for $b = s + r$. Evidently, $S_0$ does not have color $b$ poles.

Now we examine $S_1$ which is a summation of the tableaux (with sign) of the form

$$
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix}
$$

(A.1.1)

where $\xi$ and $\zeta$ are columns with total length $a - 1$ and they do not involve $Q_b$. $\eta$ is $b, b + 1, b + 1$ or $\overline{b}$ for $b \in \{1, 2, \ldots, s + r - 1\}$; $s + r, 0$ or $\overline{s + r}$ for $b = s + r$. Thanks to the relations (3.18)-(3.25), $S_1$ is free of color $b$ poles under the BAE (3.1). Hereafter we consider $S_k$ for $k \geq 2$.

\* The case $b \in \{1, 2, \ldots, s - 1\}$: $S_k(k \geq 2)$ is a summation of the tableaux (with sign) of the form

$$
\begin{pmatrix}
\xi \\
\eta \\
\zeta
\end{pmatrix}
= \left( \sum_{n_1=0}^{k_1} E_{1n_1} \right) \left( \sum_{n_2=0}^{k_2} E_{2n_2} \right) \times \xi \times \eta \times \zeta
$$

(A.1.2)

where $\xi$, $\eta$ and $\zeta$ are columns with total length $a - k$, which do not

\* This is void for $B(r|1)$ case.
contain \(b, b+1, b+1\) and \(\bar{b}\). \(E_{1n_1}\) is a column of the form:

\[
\begin{array}{c}
\bar{b} \\
\vdots \\
\bar{b} \\
\bar{b} \\
\vdots \\
\bar{b}
\end{array}
\begin{array}{c}
v \\
v-2n_1+2 \\
v-2n_1 \\
v-2k_1+2
\end{array} = \frac{Q_{b-1}(v - b + 1 - 2n_1)Q_b(v - b + 2)}{Q_{b-1}(v - b + 1)Q_b(v - b - 2n_1 + 2)}
\]

(A.1.3)

\[
\times Q_b(v - b - 2k_1)Q_{b+1}(v - b + 1 - 2n_1)Q_b(v - b - 1 - 2k_1)
\]

where \(v = u + h_1\); \(h_1\) is some shift parameter and \(E_{2n_2}\) is a column of the form:

\[
\begin{array}{c}
\bar{b} \\
\vdots \\
\bar{b} \\
\bar{b} \\
\vdots \\
\bar{b}
\end{array}
\begin{array}{c}
w \\
w-2n_2+2 \\
w-2n_2 \\
w-2k_2+2
\end{array} = \frac{Q_{b-1}(w - 2s + 2r + b - 2n_2)}{Q_{b-1}(w - 2s + 2r + b - 2k_2)}
\]

(A.1.4)

\[
\times \frac{Q_b(w - 2s + 2r + b - 2k_2 - 1)}{Q_b(w - 2s + 2r + b - 2n_2 - 1)}
\times \frac{Q_b(w - 2s + 2r + b + 1)Q_{b+1}(w - 2s + 2r + b - 2n_2)}{Q_b(w - 2s + 2r + b - 2n_2 - 1)Q_{b+1}(w - 2s + 2r + b)}
\]

where \(w = u + h_2\); \(h_2\) is some shift parameter; \(k = k_1 + k_2\). For \(b \in \{1, 2, \ldots, s-1\} \), \(E_{1n_1}\) has color \(b\) poles at \(u = -h_1 + b + 2n_1 + u^{(b)}\) and \(u = -h_1 + b + 2n_1 - 2 + u^{(b)}\) for \(1 \leq n_1 \leq k_1 - 1\); at \(u = -h_1 + b + u^{(b)}\) for

21 We assume that \(E_{10} = \frac{b + 1}{v} \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
v \\
v-2k_1+2
\end{array}\) and \(E_{1k_1} = \frac{b}{v} \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
v \\
v-2k_1+2
\end{array}\)

22 We assume that \(E_{20} = \frac{b + 1}{w} \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
w \\
w-2k_2+2
\end{array}\) and \(E_{2k_2} = \frac{b}{w} \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\begin{array}{c}
w \\
w-2k_2+2
\end{array}\)

23 We assume that \(E_{1n_1} = 1\) (resp. \(E_{2n_2} = 1\)) for \(k_1 = 0\) (resp. \(k_2 = 0\)). In this case, \(E_{n_1}\) does not have poles.
Figure 7: Partial Bethe-strap structure of $E_{1n}$ for color $b$ poles ($1 \leq b \leq s-1$): The number $n$ on the arrow denotes the common color $b$ pole $-h_1 + b + n + u_k(b)$ of the pair of the tableaux connected by the arrow. This common pole vanishes under the BAE (3.1).

$E_{10} \leftarrow E_{11} \xrightarrow{2} \ldots \xrightarrow{2n-4} E_{1n-1} \xrightarrow{2n-2} E_{1n} \xleftarrow{2n} E_{1n+1} \xrightarrow{2n+2} \ldots \xrightarrow{2k-2} E_{1k_1}$

$n_1 = 0$; at $u = -h_1 + b + 2k_1 - 2 + u_p(b)$ for $n_1 = k_1$. The color $b$ residues at $u = -h_1 + b + 2n_1 + u_p(b)$ in $E_{1n_1}$ and $E_{1n_1+1}$ cancel each other under the BAE (3.1). Thus, under the BAE (3.1), $\sum_{n_1=0}^{k_1} E_{1n_1}$ is free of color $b$ poles (see Figure 7).

$E_{2n_2}$ has color $b$ poles at $u = -h_2 + 2s - 2r - b + 2n_2 - 1 + u_p(b)$ and $u = -h_2 + 2s - 2r - b + 2n_2 + 1 + u_p(b)$ for $1 \leq n_2 \leq k_2 - 1$; at $u = -h_2 + 2s - 2r - b + 2k_2 - 1 + u_p(b)$ for $n_2 = 0$; at $u = -h_2 + 2s - 2r - b + 2n_2 + 1 + u_p(b)$ in $E_{2n_2}$ and $E_{2,n_2+1}$ cancel each other under the BAE (3.1). Thus, under the BAE (3.1), $\sum_{n_2=0}^{k_2} E_{2n_2}$ is free of color $b$ poles. So is $S_k$.

• The case $b = s$: $S_k(k \geq 2)$ is a summation of the tableaux (with sign) of the form

$$
\begin{array}{ccc}
D_{11} & - & D_{12} \\
\eta & - & \eta \\
D_{21} & + & D_{22}
\end{array}
= (D_{11} - D_{12})(D_{21} - D_{22})\eta
$$

where $\eta$ is a column with length $a - k$, which does not contain $s$ $s + 1$.

24 We assume that these poles at $u = -h_1 + b + 2n_1 + u_i(b)$, and $u = -h_1 + b + 2n_1 - 2 + u_q(b)$ do not coincide each other for any $i, q \in \{1, 2, \ldots, N_b\}$: namely $u_i(b) - u_q(b) \neq 2$.  

32
\(s + 1\) and \([s, D_{11}]\) is a column \(^{25}\) of the form:

\[
\begin{array}{c|c}
  & v \\
  \hline 
  s & Q_{s-1}(v - s - 2k_1 + 3)Q_s(v - s + 2)Q_{s+1}(v - s - 2k_1 + 1) \\
  \vdots & Q_{s-1}(v - s + 1)Q_s(v - s - 2k_1 + 2)Q_{s+1}(v - s - 2k_1 + 3) \\
  s & v - 2k_1 + 4 \\
  s + 1 & v - 2k_1 + 2 
\end{array}
\]

\(D_{12}\) is a column of the form:

\[
\begin{array}{c|c}
  & v \\
  \hline 
  s & Q_{s-1}(v - s - 2k_1 + 1)Q_s(v - s + 2)Q_{s-1}(v - s + 1)Q_s(v - s - 2k_1 + 2) \\
  \vdots & Q_{s-1}(v - s + 1)Q_s(v - s - 2k_1 + 2) \\
  s & v - 2k_1 + 4 \\
  s & v - 2k_1 + 2 
\end{array}
\]

\(D_{21}\) is a column of the form:

\[
\begin{array}{c|c}
  & w \\
  \hline 
  s + 1 & Q_{s-1}(w - s + 2r - 2)Q_{s-1}(w - s + 2r - 2k_2)Q_s(w - s + 2r)Q_{s+1}(w - s + 2r - 1)Q_{s+1}(w - s + 2r - 2) \\
  \vdots & Q_{s-1}(w - s + 2r - 2k_2)Q_s(w - s + 2r - 1)Q_{s+1}(w - s + 2r - 2) \\
  s & w - 2 \\
  s & w - 2r \\
  s & w - 2r - 2 \\
  s & w - 2k_2 + 2 
\end{array}
\]

\(D_{22}\) is a column of the form:

\[
\begin{array}{c|c}
  & w \\
  \hline 
  s & Q_{s-1}(w - s + 2r)Q_s(w - s + 2r - 2k_2 - 1)Q_{s+1}(w - s + 2r)Q_{s+1}(w - s + 2r - 1)Q_{s+1}(w - s + 2r - 2) \\
  \vdots & Q_{s-1}(w - s + 2r - 2k_2)Q_s(w - s + 2r - 1)Q_{s+1}(w - s + 2r - 2) \\
  s & w - 2 \\
  s & w - 2r \\
  s & w - 2r - 2 \\
  s & w - 2k_2 + 2 
\end{array}
\]

where \(v = u + h_1\): \(h_1\) is some shift parameter; \([D_{21}]\) is a column \(^{26}\) of the form:

\[
\begin{array}{c|c}
  & w \\
  \hline 
  s & Q_{s-1}(w - s + 2r - 2)Q_{s-1}(w - s + 2r - 2k_2)Q_s(w - s + 2r)Q_{s+1}(w - s + 2r - 1)Q_{s+1}(w - s + 2r - 2) \\
  \vdots & Q_{s-1}(w - s + 2r - 2k_2)Q_s(w - s + 2r - 1)Q_{s+1}(w - s + 2r - 2) \\
  s & w - 2 \\
  s & w - 2r \\
  s & w - 2r - 2 \\
  s & w - 2k_2 + 2 
\end{array}
\]

where \(w = u + h_2\): \(h_2\) is some shift parameter; \(k = k_1 + k_2\) \(^{27}\). Obviously, the color \(b = s\) residues at \(v = s + 2k_1 - 2 + u_j^{(s)}\) in \((A.1.6)\) and \((A.1.7)\) cancel each other under the BAE \((3.1)\). And the color \(b = s\) residues at \(w = s - 2r + 1 + u_j^{(s)}\) in \((A.1.8)\) and \((A.1.9)\) cancel each other under the BAE

\(^{25}\) We assume that \([D_{11}] = [s, s + 1]\) if \(k_1 = 1\).

\(^{26}\) We assume that \([D_{21}] = [s, s + 1]\) if \(k_2 = 1\).

\(^{27}\) Here we discussed the case for \(k_1 \geq 1\) and \(k_2 \geq 1\); the case for \(k_1 = 0\) or \(k_2 = 0\) can be treated similarly.
Thus $S_k$ does not have color $s$ poles under the BAE (3.1).

- The case $b \in \{s + 1, s + 2, \ldots, s + r - 1\}$: Owing to the admissibility conditions, we have only to consider $S_k$ for $k = 2, 3, 4$.

$S_2$ is a summation of the tableaux (with sign) of the form

$$
\begin{array}{c}
\xi \\
\downarrow \\
\eta \\
\downarrow \\
\zeta
\end{array}
+ 
\begin{array}{c}
\xi' \\
\downarrow \\
\eta' \\
\downarrow \\
\zeta'
\end{array}
$$

(A.1.10)

and

$$
\begin{array}{cccccc}
\xi & \xi & \xi & \xi \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\eta & \eta & \eta & \eta \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\zeta & \zeta & \zeta & \zeta
\end{array}
+ 
\begin{array}{cccccc}
\xi & \xi & \xi & \xi \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\eta & \eta & \eta & \eta \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\zeta & \zeta & \zeta & \zeta
\end{array}
= 
\begin{array}{cccccc}
\xi & \xi & \xi & \xi \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\eta & \eta & \eta & \eta \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\zeta & \zeta & \zeta & \zeta
\end{array}
$$

(A.1.11)

where \{ $\xi$, $\eta$, $\zeta$ \}, \{ $\xi$, $\zeta'$ \} and \{ $\xi'$, $\zeta$ \} are columns with total length $a - 2$, which do not contain \( b \), $b + 1$ and $\overline{b}$. Thus, owing to Lemma 3.2, the relations (3.20) and (3.23), $S_2$ does not have color $b$ poles under the BAE (3.1).

$S_3$ is a summation of the tableaux (with sign) of the form

$$
\begin{array}{c}
\xi \\
\downarrow \\
\eta \\
\downarrow \\
\zeta
\end{array}
+ 
\begin{array}{c}
\xi \\
\downarrow \\
\eta \\
\downarrow \\
\zeta
\end{array}
+ 
\begin{array}{c}
\xi \\
\downarrow \\
\eta' \\
\downarrow \\
\zeta'
\end{array}
+ 
\begin{array}{c}
\xi \\
\downarrow \\
\eta \\
\downarrow \\
\zeta
\end{array}
$$

(A.1.12)

where \{ $\xi$, $\eta$, $\zeta$ \}, \{ $\xi$, $\eta'$, $\zeta'$ \} and \{ $\xi'$, $\zeta$ \} are columns with total length $a - 3$, which do not contain \( b \), $b + 1$, $b + 1$ and $\overline{b}$. Thus, owing to the relations (3.20), (3.23) and Lemma 3.2, $S_3$ does not have color $b$ poles under the BAE (3.1).

\[28\text{This is void for } B(1|s) \text{ case.}\]
$S_4$ is a summation of the tableaux (with sign) of the form

\[
\begin{array}{c}
\xi \\
 b \\
 b+1 \\
 \eta \\
 b+1 \\
 \zeta
\end{array}
\]  

(A.1.13)

where $\xi$, $\eta$ and $\zeta$ are columns with total length $a - 4$, which do not contain $b$, $b + 1$, $b + 1$ and $b$. Thus, owing to the Lemma 3.2, $S_4$ does not have color $b$ poles under the BAE (3.1).

• The case $b = s + r$: $S_k$ ($k \geq 2$) is a summation of the tableaux (with sign)

\[
\begin{array}{c}
\xi \\
 0 \\
 v-2 \\
 0 \\
 \vdots \\
 0 \\
 v-2k+4 \\
 0 \\
 v-2k+2 \\
 \zeta
\end{array}
\quad + \quad
\begin{array}{c}
\xi \\
 s+r \\
 v-2 \\
 0 \\
 \vdots \\
 0 \\
 v-2k+4 \\
 0 \\
 v-2k+2 \\
 \zeta
\end{array}
\quad + \quad
\begin{array}{c}
\xi \\
 0 \\
 v-2 \\
 0 \\
 \vdots \\
 0 \\
 v-2k+4 \\
 0 \\
 v-2k+2 \\
 \zeta
\end{array}
\quad = A(v)B(v) \times \begin{array}{c}
\xi \\
 \eta
\end{array}
\]  

(A.1.14)

where $v = u + h_3$; $h_3$ is some shift parameter; $\xi$ and $\zeta$ are columns with total length $a - k$, which do not contain $s + r$. $\xi$ and $\zeta$ are columns with total length $s + r$.

\[
A(v) = \frac{Q_{s+r}(v-s+r+1)}{Q_{s+r}(v-s+r)} + \frac{Q_{s+r-1}(v-s+r+1)Q_{s+r}(v-s+r-1)}{Q_{s+r-1}(v-s+r-1)Q_{s+r}(v-s+r)},
\]  

(A.1.15)

\[
B(v) = \frac{Q_{s+r}(v-s+r-2k)}{Q_{s+r}(v-s+r-2k+1)} + \frac{Q_{s+r-1}(v-s+r-2k)Q_{s+r}(v-s+r-2k+2)}{Q_{s+r-1}(v-s+r-2k+2)Q_{s+r}(v-s+r-2k+1)}.
\]

One can check $A(v)$ and $B(v)$ are free of color $s + r$ poles under the BAE (3.1). Thus, $S_k$ does not have color $s + r$ poles under the BAE (3.1).
Remark: There is another proof for Theorem 3.1 by the determinant formula (3.40): for \( b \in \{1, 2, \ldots, s - 1\} \), we prove \( T_m(u) \) is free of color \( b \) poles, and then the pole-freeness of \( T^a(u) \) follows from (3.40); while for \( b \in \{s, s + 1, \ldots, s + r\} \), we prove \( T^a(u) \) is free of color \( b \) poles in the same way as the above-mentioned proof. An advantage of this another proof is that we do not encounter an awkward expression like (A.1.2). We note that similar idea is also applicable for \( sl(r|s) \) case [19, 20, 21].

Appendix A.2 Outline of the proof of Theorem 3.1: \( B(0|s) \) (\( s \in \mathbb{Z}_{\geq 1} \) case)

We will show that \( T_m(u) \) is free of color \( b \) poles, namely, 
\[
\text{Res}_{u=b} T_m(u) = 0
\]
for any \( b \in \{1, 2, \ldots, s\} \) under the condition that the BAEs (3.2)-(3.5) is valid. The function \( c \) with \( c \in J \) has color \( b \) poles only for \( c = b, b + 1, b + 1 \) or \( \overline{b} \) if \( b \in \{1, 2, \ldots, s - 1\} \); for \( c = s \) or \( \overline{s} \) if \( b = s \), so we shall trace only \( b, \overline{b}, b + 1, \overline{b + 1} \) or \( \overline{b} \) for \( b \in \{1, 2, \ldots, s - 1\} \); \( \overline{s}, \overline{0} \) or \( \overline{s} \) for \( b = s \). Let \( S_k \) be the partial sum of \( T_m(u) \), which contains \( k \) boxes among \( b, \overline{b}, b + 1, \overline{b + 1} \) or \( \overline{b} \) for \( b \in \{1, 2, \ldots, s - 1\} \); \( \overline{s}, \overline{0} \) or \( \overline{s} \) for \( b = s \). Apparently, \( S_0 \) does not have color \( b \) poles.

Now we examine \( S_1 \), which is a summation of the tableaux (with sign) of the form
\[
\xi \eta \zeta
\]
where \( \xi \) and \( \zeta \) are rows with total length \( m - 1 \) and they do not involve \( Q_b \). \( \eta \) is \( \overline{b}, b + 1, \overline{b + 1} \) or \( \overline{b} \) for \( b \in \{1, 2, \ldots, s - 1\} \); \( \overline{s},\overline{0} \) or \( \overline{s} \) for \( b = s \). Owing to the relations (3.26)-(3.29), \( S_1 \) is free of color \( b \) poles under the BAEs (3.2)-(3.5). From now on, we consider \( S_k \) for \( k \geq 2 \).

• The case \( b \in \{1, 2, \ldots, s - 1\} \): Owing to the admissibility conditions, we have only to consider \( S_2, S_3 \) and \( S_4 \).

\( S_2 \) is a summation of the tableaux (with sign) of the form
\[
\xi b b + 1 \eta + \xi' b + 1 b \zeta
\]
and
\[
\xi b \eta \overline{b} \zeta + \xi b \eta \overline{b + 1} \zeta \\
+ \xi b + 1 \eta \overline{b} \zeta + \xi b + 1 \eta \overline{b + 1} \zeta \\
= \xi b + b + 1 \eta \overline{b} + b + 1 \zeta
\]

This is void for \( B(0|1) \) case.
where \((\xi, \eta'), (\xi', \xi)\) and \((\xi, \eta, \zeta)\) are rows with total length \(m - 2\), which do not contain \(b, b + 1, b + 1\) and \(b\). Thus, owing to Lemma 3.3, the relations (3.26) and (3.29), \(S_2\) does not have color \(b\) poles under the BAE (3.2) and (3.3).

\(S_3\) is a summation of the tableaux (with sign) of the form

\[
\begin{array}{cccccccccccc}
\xi & b & b + 1 & \eta & b & \zeta \\
\end{array}
+ \begin{array}{cccccccccccc}
\xi & b & b + 1 & \eta & b + 1 & \zeta \\
\end{array}
\]

(A.2.4)

and

\[
\begin{array}{cccccccccccc}
\xi & b & \eta' & b + 1 & b & \zeta \\
\end{array}
+ \begin{array}{cccccccccccc}
\xi & b + 1 & \eta' & b + 1 & b & \zeta \\
\end{array}
\]

(A.2.5)

where \((\xi, \eta, \zeta)\) and \((\xi, \eta', \zeta)\) are rows with total length \(m - 3\), which do not contain \(b, b + 1, b + 1\) and \(b\). Thus, owing to Lemma 3.3, the relations (3.26) and (3.29), \(S_3\) does not have color \(b\) poles under the BAE (3.2) and (3.3).

\(S_4\) is a summation of the tableaux (with sign) of the form

\[
\begin{array}{cccccccccccc}
\xi & b & b + 1 & \eta & b + 1 & \bar{b} & \zeta \\
\end{array}
\]

(A.2.6)

where \(\bar{b}\) and \(b\) are rows with total length \(m - 4\), which do not contain \(b, b + 1, b + 1\) and \(\bar{b}\). Thus, owing to Lemma 3.3, \(S_4\) does not have color \(b\) poles.

- The case \(b = s\): Owing to the admissibility conditions, we have only to consider \(S_2\) and \(S_3\).

\(S_2\) is a summation of the tableaux (with sign) of the form

\[
\begin{array}{cccccccccccc}
\xi & s & 0 & \eta \\
\end{array}
= \begin{array}{cccccccccccc}
Q_{s-1}(v - s - 1)Q_s(v - s + 3) \\
Q_{s-1}(v - s + 1)Q_s(v - s + 1)
\end{array}\times \begin{array}{cccccccccccc}
\xi & \eta \\
\end{array}
\]

(A.2.7)

\[
\begin{array}{cccccccccccc}
\xi & s & \bar{s} & \eta \\
\end{array}
= \begin{array}{cccccccccccc}
Q_{s-1}(v - s - 1)Q_s(v - s + 2) \\
Q_{s-1}(v - s + 1)Q_s(v - s)
\end{array}\times \begin{array}{cccccccccccc}
\xi & \eta \\
\end{array}
\]

(A.2.8)

\[
\begin{array}{cccccccccccc}
\xi & 0 & \bar{s} & \eta \\
\end{array}
= \begin{array}{cccccccccccc}
Q_{s-1}(v - s + 2)Q_s(v - s - 2) \\
Q_{s-1}(v - s)Q_s(v - s + 1)
\end{array}\times \begin{array}{cccccccccccc}
\xi & \eta \\
\end{array}
\]

(A.2.9)

where \(\xi\) and \(\eta\) are rows with total length \(m - 2\), which do not contain \(s, 0, \bar{s}\) and \(v = u + h\): \(h\) is a shift parameter. The color \(s\) residues at

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\[ u = u_k^{(s)} + s - 1 - h \] of the functions \( \xi \; s \; 0 \; \eta \) and \( \xi \; s \; \overline{s} \; \eta \) cancel each other under the BAE (3.4) or (3.5). The color \( s \) residues at \( u = u_k^{(s)} + s - h \) of the functions \( \xi \; s \; \overline{s} \; \eta \) and \( \xi \; 0 \; \overline{s} \; \eta \) cancel each other under the BAE (3.4) or (3.5). Thus, \( S_2 \) does not have color \( s \) poles under the BAE (3.4) or (3.5).

\( S_3 \) is a summation of the tableaux (with sign) of the form

\[ \xi \; s \; 0 \; \overline{s} \; \eta \] (A.2.10)

where \( \xi \) and \( \eta \) are rows with total length \( m - 3 \), which do not contain \( \overline{s} \), \( 0 \) and \( \overline{s} \). Thus, owing to Lemma 3.3, \( S_3 \) does not have color \( s \) poles.

Then \( T_m(u) \) is free of poles under the condition that the BAEs (3.2)-(3.5) are valid; owing to the relation (3.40), this also hold true for \( T_{\lambda \subset \mu}(u) \). In particular, the pole-freeness of \( T_a(u) \) follows immediately.

**Appendix A.3 Outline of the proof of Theorem 3.1:** \( D(r \mid s) \) (\( r \in \mathbb{Z}_{\geq 2}, s \in \mathbb{Z}_{\geq 1} \)) case

We prove that \( T^a(u) \) is free of color \( b \) poles, that is, \( \operatorname{Res}_{u=u_{(b)}} \ldots T^a(u) = 0 \) for any \( b \in \{1, 2, \ldots, s+r\} \) under the condition that the BAE (3.1) is valid. The function \( c_u \) (3.14) with \( c \in J \) has color \( b \) poles only for \( c = b, b+1, b+1 \) or \( b \) if \( b \in \{1, 2, \ldots, s+r-1\} \); for \( c = s+r-1, s+r, s+r \) or \( s+r-1 \) if \( b = s+r \), so we shall trace only \( b, b+1, b+1 \) or \( b \) for \( b \in \{1, 2, \ldots, s+r-1\} \); \( s+r-1, s+r, s+r \) or \( s+r-1 \) for \( b = s+r \). Let \( S_k \) be the partial sum of \( T^a(u) \), which contains \( k \) boxes among \( b, b+1, b+1 \) or \( b \) for \( b \in \{1, 2, \ldots, s+r-1\} \); \( s+r-1, s+r, s+r-1 \) or \( s+r-1 \) for \( b = s+r \).

Apparently, \( S_0 \) does not have color \( b \) poles.

Next we consider \( S_1 \), which is a summation of the tableaux (with sign) of the form

\[ \xi \; \eta \; \zeta \] (A.3.1)

where \( \xi \) and \( \zeta \) are columns with total length \( a - 1 \) and they do not contain \( Q_b \). \( \eta \) is \( b, b+1, b+1 \) or \( b \) for \( b \in \{1, 2, \ldots, s+r-1\} \); \( s+r-1, s+r, s+r-1 \) or \( s+r-1 \) for \( b = s+r \). Owing to the relations (3.30)-(3.37), \( S_1 \) is free of color \( b \) poles under the BAE (3.1). From now on we consider \( S_k \) for \( k \geq 2 \).
• The case \( b \in \{1, 2, \ldots, s + r - 2\} \): The proof is similar to \( B(r|s) \ (r \in \mathbb{Z}_{>1}) \) case, so we omit it.
• The case \( b = s + r - 1 \) or \( b = s + r \): \( S_{2n} (k = 2n, n \in \mathbb{Z}_{\geq 2}) \) is a summation of the tableaux (with signs) of the form

\[
\begin{array}{c|c|c|c}
\xi & \xi & \xi & \xi \\
\hline
s + r - 1 & s + r - 1 & s + r & s + r \\
\hline
s + r & s + r & s + r & s + r \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
s + r & s + r & s + r & s + r \\
\hline
s + r - 1 & s + r & s + r & s + r - 1 \\
\hline
\zeta & \zeta & \zeta & \zeta \\
\end{array}
\]

\[= A(v)B(v) \times \xi \times \zeta \tag{A.3.2} \]

and

\[
\begin{array}{c|c|c|c}
\xi & \xi & \xi & \xi \\
\hline
s + r & s + r - 1 & s + r & s + r \\
\hline
s + r & s + r & s + r & s + r \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
s + r & s + r & s + r & s + r \\
\hline
s + r & s + r & s + r & s + r - 1 \\
\hline
\zeta & \zeta & \zeta & \zeta \\
\end{array}
\]

\[= C(v)D(v) \times \xi \times \zeta \tag{A.3.3} \]

where \( v = u + h_1 \): \( h_1 \) is some shift parameter; \( \xi \) and \( \zeta \) are columns with total length \( a - 2n \), which do not contain \( s + r - 1 \) and \( s + r \) and \( s + r - 1 \).

\[
A(v) = \frac{Q_{s+r-1}(v - s + r + 1)}{Q_{s+r-1}(v - s + r - 1)}
\]

\( ^{30} \)The case \( n = 1 \) can be treated similarly.
Apparently, \( A(v) \) and \( B(v) \) (resp. \( C(v) \) and \( D(v) \)) do not contain \( Q_{s+r} \) (resp. \( Q_{s+r-1} \)). One can also check \( A(v) \) and \( B(v) \) (resp. \( C(v) \) and \( D(v) \)) are free of color \( s + r - 1 \) (resp. \( s + r \)) poles under the BAE (3.1).

\( S_{2n+1} \) \((k = 2n + 1, n \in \mathbb{Z}_{\geq 2})\) is a summation of the tableaux (with signs) of the form

\[
\begin{array}{|c|c|c|c|}
\hline
\xi & \xi & \xi & \xi \\
\hline
s + r & s + r & s + r & s + r \\
\hline
s + r & s + r & s + r & s + r \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
s + r & s + r & s + r & s + r \\
\hline
s + r & s + r & s + r & s + r \\
\hline
s + r & s + r & s + r & s + r \\
\hline
\xi & \xi & \xi & \xi \\
\hline
\end{array}
\]

\( = E(v)F(v) \times \xi \times \xi \) (A.3.5)

\[\begin{aligned}
B(v) &= \frac{Q_{s+r-2}(v - s + r)Q_{s+r-1}(v - s + r - 3)}{Q_{s+r-2}(v - s + r - 2)Q_{s+r-1}(v - s + r - 1)} \\
C(v) &= \frac{Q_{s+r}(v - s + r + 1)}{Q_{s+r}(v - s + r - 1)} \\
D(v) &= \frac{Q_{s+r-2}(v - s + r)Q_{s+r}(v - s + r - 3)}{Q_{s+r-2}(v - s + r - 2)Q_{s+r}(v - s + r - 1)},
\end{aligned}\]

\((A.3.4)\)

\[\begin{aligned}
&\frac{Q_{s+r-2}(v - s + r - 4n - 1)}{Q_{s+r-1}(v - s + r - 4n + 1)} \\
&\frac{Q_{s+r-2}(v - s + r - 4n)Q_{s+r}(v - s + r - 4n + 3)}{Q_{s+r-2}(v - s + r - 4n + 2)Q_{s+r}(v - s + r - 4n + 1)}.
\end{aligned}\]

The case \( n = 1 \) can be treated similarly.
and

\[
\begin{array}{c|c|c|c|c}
\xi & \xi & \xi & \xi \\
\hline
s+r & s+1 & s+r & v \\
s+r & s+r & s+r & v-2 \\
s+r & s+r & s+r & v-4 \\
s+r & s+r & s+r & v-6 \\
\vdots & \vdots & \vdots & \vdots \\
s+r & s+r & s+r & v-4n \\
s+r & s+r & s+r-1 & v-4n+2 \\
s+r & s+r & s+r-1 & v-4n+4 \\
\end{array}
\]

\[
= G(v)H(v) \times \begin{bmatrix} \xi \\ \xi \\ \xi \\ \xi \end{bmatrix} \times \begin{bmatrix} \zeta \\ \zeta \\ \zeta \end{bmatrix} \tag{A.3.6}
\]

where \( v = u + h_2 \); \( h_2 \) is some shift parameter; \( \begin{bmatrix} \xi \\ \xi \end{bmatrix} \) and \( \begin{bmatrix} \zeta \\ \zeta \end{bmatrix} \) are columns with total length \( a - 2n - 1 \), which do not contain \( s + r - 1 \), \( s + r \), and \( s + r - 1 \).

\[
E(v) = \frac{Q_{s+r-1}(v-s+r+1)}{Q_{s+r-1}(v-s+r-1)}
\]

\[
F(v) = \frac{Q_{s+r}(v-s+r-4n-3)}{Q_{s+r}(v-s+r-4n-1)}
\]

\[
G(v) = \frac{Q_{s+r}(v-s+r+1)}{Q_{s+r}(v-s+r-1)}
\]

\[
H(v) = \frac{Q_{s+r-1}(v-s+r-4n-3)}{Q_{s+r-1}(v-s+r-4n-1)}
\]

\[
= \frac{Q_{s+r-2}(v-s+r-4n-2)Q_{s+r}(v-s+r-4n+1)}{Q_{s+r-2}(v-s+r-2)Q_{s+r}(v-s+r-4n-1)}.
\] \tag{A.3.7}

Apparently, \( E(v) \) and \( H(v) \) (resp. \( F(v) \) and \( G(v) \)) do not contain \( Q_{s+r} \) (resp. \( Q_{s+r-1} \)). One can also check \( E(v) \) and \( H(v) \) (resp. \( F(v) \) and \( G(v) \)) are free of color \( s + r - 1 \) (resp. \( s + r \)) poles under the BAE (3.1).
Thus, \( S_k \) have neither color \( s + r - 1 \) poles nor color \( s + r \) poles under the BAE (3.1).

Appendix B Generating series for \( T^a(u) \) and \( T_m(u) \)

The functions \( T^a(u) \) and \( T_m(u) \) \((a, m \in \mathbb{Z}; u \in \mathbb{C})\) are determined by the following non-commutative generating series.

**B(\( r|s \)) case:**
\[
(1 + X)^{-1} \cdots (1 + sX)^{-1}(1 + (s + 1)X) \cdots (1 + \sum_{i=1}^{r} X)(1 - 0X)^{-1} \\
\times (1 + \sum_{j=1}^{r} X) \cdots (1 + sX)(1 + \sum_{i=1}^{r} X)^{-1} \cdots (1 + \sum_{j=1}^{r} X)^{-1}
= \sum_{a=-\infty}^{\infty} T^a(u + a - 1)X^a,
\]
(B.1)

\[
(1 - \sum_{s=1}^{r} X) \cdots (1 - sX)(1 - \sum_{s=1}^{r} X)^{-1} \cdots (1 - \sum_{s=1}^{r+1} X)^{-1}(1 - 0X) \\
\times (1 - \sum_{s=1}^{r} X)^{-1} \cdots (1 - sX)(1 - \sum_{s=1}^{r} X)^{-1} \cdots (1 - \sum_{s=1}^{r+1} X)^{-1}
= \sum_{m=-\infty}^{\infty} T_m(u + m - 1)X^m.
\]
(B.2)

**D(\( r|s \)) case:**
\[
(1 + X)^{-1} \cdots (1 + sX)^{-1}(1 + \sum_{i=1}^{r} X) \cdots (1 + \sum_{i=1}^{r} X)
\times (1 - \sum_{s=1}^{r} X) \cdots (1 - sX)(1 - \sum_{s=1}^{r} X)^{-1} \cdots (1 - \sum_{s=1}^{r+1} X)^{-1}
\times (1 + \sum_{s=1}^{r} X) \cdots (1 + sX)(1 + \sum_{i=1}^{r} X)^{-1} \cdots (1 + \sum_{i=1}^{r} X)^{-1}
= \sum_{a=-\infty}^{\infty} T^a(u + a - 1)X^a,
\]
(B.3)

\[
(1 - \sum_{s=1}^{r} X) \cdots (1 - sX)(1 - \sum_{s=1}^{r} X)^{-1} \cdots (1 - \sum_{s=1}^{r+1} X)^{-1}
\times [(1 - \sum_{s=1}^{r} X)^{-1} + (1 - \sum_{s=1}^{r+1} X)^{-1} - 1] \\
\times (1 - \sum_{s=1}^{r} X)^{-1} \cdots (1 - sX)(1 - \sum_{s=1}^{r} X)^{-1} \cdots (1 - \sum_{s=1}^{r+1} X)^{-1}
= \sum_{m=-\infty}^{\infty} T_m(u + m - 1)X^m.
\]
(B.4)

Here \( X \) is a shift operator \( X = e^{2\beta u} \). In particular, we have \( T^0(u) = 1; T_0(u) = 1; T^a(u) = 0 \) for \( a < 0; T_m(u) = 0 \) for \( m < 0 \).
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