A NONSEPARABLE INVARIANT EXTENSION OF
LEBESGUE MEASURE - A GENERALIZED AND
ABSTRACT APPROACH

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ABSTRACT: Here using some methods of combinatorial set theory, particularly the ones related to the construction of independent families of sets and some modified version of the notion of small sets originally introduced by Riečan, Riečan and Neubrunn, we give abstract and generalized formulation of a remarkable theorem of Kakutani and Oxtoby relating to nonseparable extension of Lebesgue measure in spaces with transformation groups.

1 INTRODUCTION

Let $(E, \mathcal{S}, \mu)$ be any measure space where $E$ is a nonempty basic set, $\mathcal{S}$ is a $\sigma$-algebra of subsets of $E$ containing all singletons and $\mu$ a nonzero, $\sigma$-finite, diffused (or, continuous) measure on $\mathcal{S}$. The general measure extension problem is concerned about extending $\mu$ to a maximally large class of subsets of $E$. According to famous theorem of Ulam [19], it is consistent with the Axioms of set theory that so long as no cardinal less than card $E$ is an weakly inaccessible cardinal, there can be no extension of $\mu$ which coincides with the power set of $E$. Thus there always exists $X \subseteq E$ such that $X \notin \text{dom}(\mu)$ and $\mu$ can be further extended to $\mu'$ with $X \in \text{dom}(\mu')$. Therefore, under the assumption that there is no large cardinal, we may infer that no maximal extension of $\mu$ is possible.

Investigations into the extension problems in measure theory have useful applications in many other branches of modern mathematics such as axiomatic set theory, general
topology, functional analysis, probability theory, etc and possibly started with the work of Szpiłrajn (Marczewski). In his classical papers [17], [18], he gave several such constructions of invariant extensions and also added to it a list of problems. His method is sufficiently general so as to make it applicable for any complete measure space and is based upon the idea of adding to the $\sigma$-ideal $\mathcal{I}(\mu)$ of $\mu$-null sets, some new sets which are $\mu$-nonmeasurable and having inner $\mu$-measure zero. This yields the $\sigma$-ideal $\mathcal{I}'$ and in turn generates the $\sigma$-algebra $\mathcal{S}'$ from $\mathcal{S}$ and $\mathcal{I}'$. Any arbitrarily chosen element $U$ of $\mathcal{S}'$ admits a representation of the form $U = (X \setminus Y) \cup Z$ where $X \in \mathcal{S}$, and $Y, Z \in \mathcal{I}'$ and on $\mathcal{S}'$ we define a measure $\mu'$ by putting $\mu'(U) = \mu(X)$. It may be observed that this way of extending $\mu$ do not change its metrical structure and the two metric spaces associated with $\mu$ and $\mu'$ have the same topological weights. Therefore, if the original measure $\mu$ is separable (i.e the metric associated with $\mu$ is separable) then so is $\mu'$.

Many years ago, it was asked whether there is some nonseparable invariant extension of Lebesgue measure in $\mathbb{R}$ and researches in this directions were carried out by Kakutani and Oxtoby [10], Kodaira and Kakutani [9] and several others. The extension obtained by Kakutani and Oxtoby has character $2^c$ and that by Kodaira and Kakutani has character $c$ and both are invariant under the groups of all isometries in $\mathbb{R}$. Moreover, the method of Kakutani and Oxtoby can be generalized for Haar measure in infinite compact metrizable groups [2].

Let $(\Omega, \mathcal{L}, \lambda)$ be the Lebesgue measure space where $\Omega = [0, 1)$, $\mathcal{L}$ is the class of Lebesgue measurable sets in $\Omega$, and $\lambda$ the restriction of the usual Lebesgue measure on $\Omega$. We call a measure space $(\Omega, \mathcal{M}, m)$ an extension of $(\Omega, \mathcal{L}, \lambda)$ [10] if $\mathcal{L} \subseteq \mathcal{M}$ and $m(M) = \lambda(M)$ for every $M \in \mathcal{L}$, and, it is a proper extension if $\mathcal{L} \neq \mathcal{M}$. The character [10] of any extension $(\Omega, \mathcal{M}, m)$ is the smallest cardinal for which there exists a subfamily $\mathcal{U}$ of $\mathcal{M}$ of that cardinality and having the property that for every $M \in \mathcal{M}$ and every $\epsilon > 0$, there exists $A \in \mathcal{U}$ such that $m(M \Delta A) < \epsilon$.

The extension is called separable [10] if $\text{card}(\mathcal{U}) \leq \omega_0$ where $\omega_0$ is the ordinal corresponding to the first infinite cardinal. An one-to-one mapping $T : \Omega \to \Omega$ is called $\mathcal{M} - m$ preserving [10] if $M \in \mathcal{M}$ implies that $T(M) \in \mathcal{M}$, $T^{-1}(M) \in \mathcal{M}$ and $m(T(M)) = m(T^{-1}(M)) = m(M)$.

In their remarkable paper [10], Kakutani and Oxtoby proved the following theorem
which serves as a motivation for the present work. Here, we generalize this theorem through some abstract approach, where, instead of measure, more delicate structures in spaces with transformation groups are used.

**THEOREM (KO):** There exists an extension \((\Omega, \mathcal{M}, m)\) of the Lebesgue measure space \((\Omega, \mathcal{L}, \lambda)\) having the following properties:

(i) The character of \((\Omega, \mathcal{M}, m)\) is \(2^c\).

(ii) Every \(\mathcal{L} - \lambda\) preserving transformation is also \(\mathcal{M} - m\) preserving.

2 **PRELIMINARIES AND RESULTS**

For our purpose, we use some methods of combinatorial set theory, particularly the ones that are related to the construction of independent families of sets in infinite base space and also some modified version of the notion of small sets. It is worthwhile to note here that the notion of small sets (or, families) was originally introduced by Riečan, Riečan and Neubrunn [13], [14] (see also [15], [16]) while giving abstract formulations of some well known classical theorems on measure and integration.

By a space \(X\) equipped with a transformation group \(G\) we mean a pair \((X, G)\) where \(X\) is a nonempty set and \(G\) is a subgroup of the symmetric group \(\text{Symm}(X)\) of all bijections from \(X\) onto \(X\) satisfying the following two conditions:

(i) for each \(g \in G\), \(x \rightarrow gx\) is a bijection (or, permutation) of \(X\).

(ii) for all \(x \in X\) and \(g_1, g_2 \in G\), \(g_1(g_2x) = g_1g_2x\).

We say that \(G\) acts freely on \(X\) if \(\{x \in X : gx = x\} = \emptyset\) for all \(g \in G \setminus \{e\}\) where ‘\(e\)’ is the identity element of \(G\) (in fact, \(e : X \rightarrow X\) is the identity transformation on \(X\)) and \(G\) acts transitively on \(X\) if \(Gx = X\) for every \(x \in X\). For any \(g \in G\) and \(E \subseteq X\), we write \(gE\) to denote the set \(\{gx : x \in E\}\) and call a nonempty family (or, class) \(A\) of sets as \(G\)-invariant [3] if \(gE \in A\) for every \(g \in G\) and \(E \in A\). If \(A\) is a \(\sigma\)-algebra, then a measure \(\mu\) on \(A\) is called \(G\)-invariant [3] if \(A\) is a \(G\)-invariant class and \(\mu(gE) = \mu(E)\) for every \(g \in G\) and \(E \in A\). It is called \(G\)-quasiinvariant [3] if \(A\) and the \(\sigma\)-ideal generated by \(\mu\)-null sets are both \(G\)-invariant classes. Obviously, any \(G\)-invariant measure is also \(G\)-quasiinvariant but not conversely. Any set of the form \(Gx = \{gx : g \in G\}\) for some \(x \in X\) is called a \(G\)-orbit of \(x\). The collection of all such \(G\)-orbits give rise to a partition of \(X\) into mutually disjoint sets. A subset \(E\) of \(X\) is called a complete \(G\)-selector (or, simply, a \(G\)-selector) in
$X$ if $E \cap Gx$ consists of exactly one point for each $x \in X$.

Throughout this paper, we identify every infinite cardinal with the least ordinal representing it, and, every ordinal (in the sense introduced by Von Neumann) with the set of all ordinals preceeding it. We write card $A$ and card $\mathcal{A}$ to denote the cardinals of any set $A$ or any class $\mathcal{A}$ of sets and as is usually done elsewhere, express the first infinite and the first uncountable cardinals by the notations $\omega_0$ and $\omega_1$ respectively. For any infinite cardinal, we use symbols like $\xi, \eta, \xi, k$ etc and write $k^+$ for the successor of $k$. Throughout the entire discourse, we stipulate to work within the framework of ZFC.

Let $k$ be any infinite cardinal and suppose card $X \geq k$, we define

**DEFINITION 2.1[1]**: A pair $(\mathcal{S}, \mathcal{I})$ as a $k$-additive measurable structure on $(X, G)$ if

(i) $\mathcal{S}$ is an algebra and $\mathcal{I} (\subseteq \mathcal{S})$ a proper ideal on $X$.

(ii) Both $\mathcal{S}$ and $\mathcal{I} (\subseteq \mathcal{S})$ are $k$-additive in the sense that they are closed with respect to the union of at most $k$ number of sets, and

(iii) $\mathcal{S}$ and $\mathcal{I}$ are $G$-invariant classes.

Henceforth, by a $k$-additive algebra (resp. ideal) on $(X, G)$ we will mean that it is a $k$-additive algebra (resp. ideal) on $X$ and also $G$-invariant. In particular, if $G$ consists only of the identity transformation on $X$, then $(\mathcal{S}, \mathcal{I})$ is simply called a $k$-additive measurable structure on $X$.

**DEFINITION 2.2[1]**: A $k$-additive measurable structure $(\mathcal{S}, \mathcal{I})$ on $(X, G)$ is called $k^+$-saturated if the cardinality of any arbitrary collection of mutually disjoint sets from $\mathcal{S} \setminus \mathcal{I}$ is atmost $k$.

The notion of $\omega_0$-additive measurable structure on a nonempty basic set $E$ was defined by Kharazishvili. In [4], this was referred to as a measurable structure consisting of a pair $(\mathcal{S}, \mathcal{I})$ where $\mathcal{S}$ is a $\sigma$-algebra and $\mathcal{I} (\subseteq \mathcal{S})$ a proper $\sigma$-ideal of sets in $E$. If $E$ is a group and $\mathcal{S}, \mathcal{I}$ are $G$-invariant classes, then $(\mathcal{S}, \mathcal{I})$ according to Kharazishvili is a $G$-invariant measurable structure on $E$. Using the notion of a measurable structure as introduced by him, Kharazishvili proved several interesting results in commutative (or, more generally in solvable) groups [4]. In [5], he used similar type of structures to generalize two classical results of Sierspinski. It may be noted that the notion of a $k$-additive, $k^+$-saturated measurable structure on $(X, G)$ lies in between a $k$-additive measurable structure on $(X, G)$ satisfying countable chain condition (or, Suslin condition) and a $\omega_0$-additive measurable
structure which is $k^+$-saturated.

**DEFINITION 2.3[1]:** In a space $(X, G)$ with a transformation group $G$, a set $E \subseteq X$ is called almost $G$-invariant with respect to some ideal $\mathcal{I}$ if $gE \Delta E \in \mathcal{I}$ for every $g \in G$.

If the ideal $\mathcal{I}$ is $k$-additive, then it can be easily checked that the class of all sets which are almost $G$-invariant with respect to $\mathcal{I}$ forms a $k$-additive algebra on $X$. Further,

**DEFINITION 2.4[1]:** A set $E \subseteq X$ is called $(\mathcal{S}, \mathcal{I})$-thick if $B \subseteq X \setminus E$ and $B \in \mathcal{S}$ implies that $B \in \mathcal{I}$.

**PROPOSITION 2.5:** Assume that the pair $(\mathcal{S}, \mathcal{I})$ is a $k$-additive measurable structure on $(X, G)$ which is also $k^+$-saturated and let $E \subseteq X$ be a set which is almost $G$-invariant with respect to $\mathcal{I}$. Then $E \in \mathcal{S}$ implies either $E \in \mathcal{I}$ or $X \setminus E \in \mathcal{I}$. If $E \notin \mathcal{I}$, then $E$ is $(\mathcal{S}, \mathcal{I})$-thick in $X$.

**Proof:** Let $E \in \mathcal{S}$. If $E \notin \mathcal{I}$, then there is nothing to prove. Suppose $E \notin \mathcal{I}$. Then $X \setminus E \in \mathcal{I}$, for otherwise, it is possible to generate by transfinite recursion a set $\{g_\alpha : \alpha < k\}$ of points in $G$ such that $X \setminus \bigcup_{0 \leq \alpha < k} g_\alpha E \in \mathcal{I}$. But this contradicts the hypothesis.

Now let $E \notin \mathcal{S}$. Then obviously $E \notin \mathcal{I}$ and also $X \setminus E \notin \mathcal{I}$. If $E$ is not $(\mathcal{S}, \mathcal{I})$-thick, then there should exist $B \in \mathcal{S} \setminus \mathcal{I}$ such that $B \subseteq X \setminus E$. By a similar reasoning as given above, there exists a transfinite $k$-sequence $\{h_\alpha : \alpha < k\}$ in $G$ such that $X \setminus \bigcup_{0 \leq \alpha < k} h_\alpha B \in \mathcal{I}$. But then by virtue of $k$-additivity of $\mathcal{I}$, there exists some $\alpha_0 < k$ such that $E \cap h_{\alpha_0} B \notin \mathcal{I}$ which again contradicts the hypothesis.

The notion of a small system was introduced by Riečan [13], Riečan and Neubrunn [15] (see also [14],[16]). As an initial step towards giving an abstract generalization the above theorem(Theorem(KO))of Kakutani and Oxtoby, we build up a $k$-additive measurable structure on $(X, G)$ from a $k$-additive algebra $\mathcal{S}$ on $(X, G)$ and a transfinite $k$-sequence which in the present situation is a modified and generalized version of systems of small sets or small systems originally introduced by Riečan and Neubrunn.

We call it a $k$-small system on $(X, G)$ and define it as
DEFINITION 2.6: A transfinite $k$-sequence $\{N_\alpha\}_{0 \leq \alpha < k}$ where each $N_\alpha$ is a class of sets in $X$ satisfying the following set of conditions:

(i) $\emptyset \in N_\alpha$ for $\alpha < k$.
(ii) Each $N_\alpha$ is a $G$-invariant class.
(iii) If $E \in N_\alpha$ and $F \subseteq E$, then $F \in N_\alpha$.
(iv) $E \in N_\alpha$ and $F \in \bigcap_{0 \leq \alpha < k} N_\alpha$ implies $E \cup F \in N_\alpha$.
(v) For any $\alpha < k$, there exists $\alpha^* > \alpha$ such that for any one-to-one correspondence $\beta \rightarrow N_\beta$ with $\beta > \alpha^*$, $\cup E_\beta \in N_\alpha$ whenever $E_\beta \in N_\beta$.
(vi) For any $\alpha, \beta < k$, there exists $\gamma > \alpha, \beta$ such that $N_\gamma \subseteq N_\alpha$ and $N_\gamma \subseteq N_\beta$.

We further say that

DEFINITION 2.7: A $k$-additive algebra $S$ on $(X, G)$ is admissible with respect to the $k$-small system $\{N_\alpha\}_{0 \leq \alpha < k}$ if

(i) $S \setminus N_\alpha \neq \emptyset$ for some $\alpha < k$ and $S \cap N_\alpha \neq \emptyset$ for every $\alpha < k$.
(ii) $N_\alpha$ has a $S$-base i.e every $E \in N_\alpha$ is contained in some $F \in N_\alpha \cap S$.
(iii) $S \setminus N_\alpha$ satisfies $k$-chain condition, i.e., the cardinality of any arbitrary collection of mutually disjoint sets from $S \setminus N_\alpha$ is at most $k$.

By (i) in the above definition, we mean that $S$ is compatible with $\{N_\alpha\}_{0 \leq \alpha < k}$; by (ii) we mean that $S$ constitutes a base for $\{N_\alpha\}_{0 \leq \alpha < k}$ and by (iii) we express that $S$ satisfies the $k$-chain condition with respect to $\{N_\alpha\}_{0 \leq \alpha < k}$.

We set $N_\infty = \bigcap_{0 \leq \alpha < k} N_\alpha$. By virtue of conditions (ii), (iii) and (v) of Definition 2.6, it is easy to check that $N_\infty$ is a $k$-additive ideal on $(X, G)$. Let $\tilde{S}$ be the $k$-additive algebra on $(X, G)$ generated by $S$ and $N_\infty$, where each element of $\tilde{S}$ admits a representation of the form $(X \setminus Y) \cup Z$ where $X \in S$ and $Y, Z \in N_\infty$. Thus $(\tilde{S}, N_\infty)$ is a $k$-additive measurable structure on $(X, G)$. Moreover,

THEOREM 2.8: If $S$ is admissible with respect to $\{N_\alpha\}_{0 \leq \alpha < k}$, then $(\tilde{S}, N_\infty)$ is $k^+$-saturated.

The existence of an independent family of subsets of an infinite set, with maximal cardinality was solved by Tarski [11]. He showed that such a family exists and has cardinality
The result has many interesting applications. One such is its use in proving that the cardinality of all ultrafilters defined on an arbitrary infinite set $E$ is $2^{\text{card}(E)}$. However, if the cardinality of the basic set is that of the continuum $c$, then the existence of a strictly independent family of subsets of $E$ having cardinality $2^c$ can be proved and this result has an application in the construction of a nonseparable invariant extension of the Lebesgue measure space [2]. Below, we introduce a more general definition for a $k$-independent (resp. strictly $k$-independent) family where $k$ is an arbitrary infinite cardinal.

Definition 2.9 : A family $\{A_i : i \in I\}$ of subsets of $X$ is called $k$-independent (resp. strictly $k$-independent) if for each set $J \subseteq I$ having card $J < k$ (resp. card $J \leq k$) and every function $f : J \to \{0,1\}$, we have $\cap\{A_{f(j)}^j : j \in J\} \neq \emptyset$ where $A_{f(j)}^j = A_j$ if $f(j) = 0$ and $A_{f(j)}^j = X \setminus A_j$ if $f(j) = 1$.

In particular, the definition of an independent or $\omega_1$-independent (in the set theoretic sense) family is already given in [6]. The above general definition is framed in this pattern. For another introduction to $k$-independent (resp. strictly $k$-independent) family, see [12]. However, Definition 2.9 can be further generalized using Definition 2.1.

Definition 2.10 : A family $\{A_i : i \in I\}$ of subsets of $X$ is $k$-independent (resp. strictly $k$-independent) with respect to any $k$-additive measurable structure $(\mathcal{S}, \mathcal{I})$ on $(X, G)$ if for each set $J \subseteq I$ having card $J < k$ (resp. card $J \leq k$) and every function $f : J \to \{0,1\}$, $B \subseteq X \setminus \cap\{A_{f(j)}^j : j \in J\}$ and $B \in \mathcal{S}$ implies that $B \in \mathcal{I}$, where $A_{f(j)}^j = A_f$ if $j \in J$. Note that in the above Definition, condition $(iii)$ of Definition 2.1 plays no role. So we may think of it as a $k$-independent (resp. strictly $k$-independent) family with respect to some $k$-additive measurable structure $(\mathcal{S}, \mathcal{I})$ on $X$. The notion of an independent (resp. strictly independent) family with respect to a measure was earlier given in [7]. So the above Definition is just an extension of this concept given in terms of some $k$-additive measurable structure and viewed further in the context of $(\mathcal{S}, \mathcal{I})$-thick sets (Definition 2.4), we may call a family $\{A_i : i \in I\}$ $k$-independent (resp. strictly $k$-independent) with respect to $(\mathcal{S}, \mathcal{I})$ on $X$ if for each $J \subseteq I$ having card $J < k$ (resp. card $J \leq k$) and each function $f : J \to \{0,1\}$, the set $\cap\{A_{f(j)}^j : j \in J\}$ is $(\mathcal{S}, \mathcal{I})$-thick in $X$. 


We now establish, under the assumption of generalized continuum hypothesis, the existence of a family of sets in $X$ which is strictly $k$-independent with respect to the $k$-additive measurable structure $(\tilde{S}, \mathcal{N}_\infty)$ on $(X, G)$.

**PROPOSITION 2.11[12]** : Assume that the generalized continuum hypothesis holds. Then for any two infinite cardinals $\lambda$, $k$ where $\lambda < k$, we have $k^\lambda = k$ provided $\lambda$ is not cofinal with $k$.

**PROPOSITION 2.12[12]** : Let $E$ be an infinite set satisfying the condition $\text{card}(E)^k = \text{card}(E)$, where $k$ is an infinite cardinal. Then there exists a maximal strictly $k$-independent family $\{A_i : i \in I\}$ of subsets of $E$ such that $\text{card}(I) = 2^{\text{card}(E)}$.

**THEOREM 2.13** : Let $(X, G)$ be a space with transformation group $G$ where $\text{card}(G) = k^+$, where $k$ is an infinite cardinal. Suppose $G$ acts freely on $X$ and there exists a $G$-selector $L \in \mathcal{S}$. Further assume that there exist a $k$-additive algebra $S$ and a $k$-small system $\{N_\alpha\}_{\alpha < k}$ on $(X, G)$ such that $S$ is admissible with respect to $\{N_\alpha\}_{\alpha < k}$. Then under the assumption of generalized continuum hypothesis, there exists a family $\{A_i : i \in I\}$ of sets in $X$ with $\text{card}(I) = 2^{k^+}$ which is strictly $k$-independent with respect to $(\tilde{S}, \mathcal{N}_\infty)$.

**Proof** : We write $G$ in the form $G = \bigcup_{\xi < k^+} G_\xi$, where $\{G_\xi : \xi < k^+\}$ is an increasing family of subgroups of $G$ satisfying $G_\xi \neq \bigcup_{\eta < \xi} G_\eta$ and $\text{card}(G_\xi) \leq k$ for every $\xi < k^+$ (for the above representation, see [6], Exercise 19, Ch 3).

Since $G$ acts freely on $X$, the above increasing family yields a disjoint covering $\{\Omega_\gamma : \gamma < k^+\}$ of $X$ where $\Omega_\gamma = (G_\gamma \setminus \bigcup_{\eta < \gamma} G_\eta) L$. Moreover, as $L \in \mathcal{S}$, $G$ acts freely on $X$ and $\mathcal{S}$ is admissible with respect to $\{N_\alpha\}_{\alpha < k}$, so $(\tilde{S}, \mathcal{N}_\infty)$ is $k^+$-saturated (Theorem 2.8). Therefore $gL \in \mathcal{N}_\infty$ for every $g \in G$.

Now consider the $Ulam(k, k^+)$-matrix [1] $(\Pi_{\xi, \rho})_{\xi < k, \rho < k^+}$ over $k^+$ and set $E_{\xi, \rho} = \bigcup_{\gamma \in \Pi_{\xi, \rho}} \Omega_\gamma$. Then there exists $\xi_0$ and a subset $\Xi$ of $k^+$ having $\text{card}(\Xi) = k^+$ such that $E_{\xi_0, \rho} \notin \mathcal{N}_\infty$ for $\rho \in \Xi$ and are mutually disjoint. This is so because $\mathcal{N}_\infty$ is $k$-additive and $X \notin \mathcal{N}_\infty$. Consequently, there exists $\tilde{\Xi} \subseteq \Xi$ with $\text{card}(\tilde{\Xi}) = k^+$ and $\alpha_0 < k$ such that $E_{\xi_0, \rho} \notin \mathcal{N}_{\alpha_0}$ for every $\rho \in \tilde{\Xi}$. Moreover, each $E_{\xi_0, \rho}$ for $\rho \in \Xi$ and more generally any union of such sets is almost $G$-invariant with respect to $\mathcal{N}_\infty$ which follows from the constructions of the sets $\Omega_\gamma$.  

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Now note that $k$ is not cofinal with $k^+$. This is so because $k$ is not cofinal with $2^k$ and $2^k = k^+$ under the assumption of generalized continuum hypothesis. Hence according to Proposition 2.11 and Proposition 2.12, it follows that there exists a strictly $k$-independent family $\{\tilde{\Xi}_i : i \in I\}$ of subsets of $\tilde{\Xi}$ such that $\text{card}(I) = 2^{k^+}$. This means that for every set $J \subseteq I$ having $\text{card}(J) \leq k$ and every function $f : J \to \{0, 1\}$, $\cap_{j \in J} \tilde{\Xi}_j \neq \emptyset$. Consequently, $\cap_{j \in J, \tilde{\Xi}_j} A_j \neq \emptyset$ where $A_i = \cup_{\tilde{\Xi}_i, \rho} E_{\xi, \rho}$ for $i \in I$ making $\{A_i : i \in I\}$ a strictly $k$-independent family of sets in $X$. Moreover, this family is also strictly $k$-independent and hence $k$-independent with respect to the $k$-additive measurable structure $(\tilde{\mathcal{S}}, \mathcal{N}_\infty)$ on $(X, G)$ which follows from Proposition 2.5 according to which each $E_{\xi, \rho}$ ($\rho \in \Xi$) is $(\tilde{\mathcal{S}}, \mathcal{N}_\infty)$-thick in $X$.

This proves the theorem.

Again from Proposition 2.5 it follows that none of the sets $A_i$ ($i \in I$) are $(\tilde{\mathcal{S}}, \mathcal{N}_\infty)$-measurable; in other words, $A_i \notin \tilde{\mathcal{S}}$ for any $i \in I$. Let us consider the algebra generated by $\tilde{\mathcal{S}}$ and $\{A_i : i \in I\}$. Its individual members are sets of the form $\bigcup_{f \in \{0, 1\}^p} (E_f \cap (\cap_{j=1}^p A_j^{(f)})$, where $E_f \in \tilde{\mathcal{S}}, p$ is any natural number, $\{0, 1\}^p$ is the set of all functions $f : \{1, 2, \ldots, p\} \to \{0, 1\}$ and $A_j^{(f)} = A_j$ if $f(j) = 0$ and $A_j^{(f)} = X \setminus A_j$ if $f(j) = 1$. It is not hard to verify that the above algebra and $G$-invariant. Let $\tilde{\mathcal{S}}$ be the $k$-additive algebra generated by $\tilde{\mathcal{S}}$ and $\{A_i : i \in I\}$. We will prove next that the extension $(\tilde{\mathcal{S}}, \mathcal{N}_\infty)$ of $(\tilde{\mathcal{S}}, \mathcal{N}_\infty)$ serves to yield the desired generalization of Theorem(KO).

We already note from the introduction that a nonseparable extension is that the cardinality of whose character is greater than $\omega_0$. In any measure space $(X, \mathcal{S}, \mu)$, there is an alternative way to formulate this phenomenon. It is done by using the metrical structure of $\mu$. It is an well known fact [8] that the following two assertions are equivalent:

(a) $\mu$ is nonseparable
(b) there exists an $\epsilon > 0$ and a family $\mathcal{H}(\epsilon) = \{Y : Y \in \mathcal{S}\}$ having $\text{card} \mathcal{H}(\epsilon) > \omega_0$ such that for the corresponding metric $d$, $d(Y, Z) \geq \epsilon$ for $Y, Z \in \mathcal{H}(\epsilon)$ and $Y \neq Z$. where $d(Y, Z) = \mu(Y \Delta Z)$.

Since there is no measure space structure available to us, we circumvent this problem by developing an uniform structure on $\tilde{\mathcal{S}}$. This structure is a more modified form of an uniform structure already used in [14].
Let $V = \{U_\alpha\}_{\alpha<k}$ be a $k$-sequence defined by setting $U_\alpha = \{(E, F) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} : E \Delta F \in N_\alpha\}$. Since $\emptyset \in N_\alpha$ for $\alpha < k$, so the diagonal is contained in every member of $V$. Also, it is evident from the definition of $V$ that $U_\alpha^{-1} \in V$ whenever $U_\alpha \in V$.

From conditions (v) and (vi) of Definition 2.6 for any $\beta, \gamma > \alpha^*$, $N_\beta \cup N_\gamma \subseteq N_\alpha$ and there exists $\delta$ such that $N_\delta \subseteq N_\beta, N_\gamma$. Hence, $U_\delta \ast U_\delta = \{(E, F) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} : \text{there exists } G \in \tilde{\mathcal{S}} \text{ such that } (E, G) \in U_\delta, (G, F) \in U_\delta\}$

$= \{(E, F) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} : \text{there exists } G \in \tilde{\mathcal{S}} \text{ such that } E \Delta G \in N_\delta, G \Delta F \in N_\delta\}$

$\subseteq \{(E, F) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} : \text{there exists } G \in \tilde{\mathcal{S}} \text{ such that } E \Delta F \subseteq (E \Delta G) \Delta (G \Delta F) \subseteq (E \Delta G) \cup (G \Delta F) \in N_\delta \cup N_\gamma \subseteq N_\alpha\} \subseteq U_\alpha$. Also, from condition (vi) of Definition 2.6 it follows that for every $U_\alpha, U_\beta$ there exists $U_\gamma$ such that $U_\gamma \subseteq U_\alpha \cap U_\beta$.

Hence $V = \{U_\alpha\}_{\alpha<k}$ forms the base of some uniformity on $\tilde{\mathcal{S}}$.

Thus we draw the conclusion

**THEOREM 2.14**: Let $(X, G)$ be a space with transformation group where $\text{card}(X) = k^+$ and $G$ acts freely and transitively on $X$. Let $\mathcal{S}$ be a diffused, $k$-additive algebra on $(X, G)$ which is admissible with respect to a $k$-small system $\{N_\alpha\}_{\alpha<k}$. Then under the assumption of generalised continuum hypothesis, we can extend the $k$-additive measurable structure $(\tilde{\mathcal{S}}, N_\alpha)$ to some $k$-additive measurable structure $(\tilde{\mathcal{S}}, N_\infty)$ on $(X, G)$ having the following property:

there exists $\alpha_0 < k$, a base $V = \{U_\alpha\}_{\alpha<\alpha_0}$ of some uniformity on $\tilde{\mathcal{S}}$ and a strictly $k$-independent family $\{A_i : i \in I\}$ of sets from $\tilde{\mathcal{S}}$ with $\text{card}(I) = 2^{k^+}$ such that $(A_i, A_j) \notin U_{\alpha_0}$ for $i \neq j$.

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