Discrete Crum’s Theorems and Integrable Lattice Equations

Cheng Zhang\textsuperscript{1*}, Linyu Peng\textsuperscript{2,3}, Da-jun Zhang\textsuperscript{1}

ch.zhang.maths@gmail.com \quad l.peng@aoni.waseda.jp \quad djzhang@staff.shu.edu.cn

\textsuperscript{1}\textsc{Department of Mathematics, Shanghai University, Shanghai, 200444, China}

\textsuperscript{2}\textsc{Waseda Institute for Advanced Study, Waseda University, Tokyo 169-8050, Japan}

\textsuperscript{3}\textsc{Department of Applied Mechanics and Aerospace Engineering, Waseda University, Tokyo 169-8555, Japan}

Abstract

In this paper, we develop discrete versions of Darboux transformations and Crum’s theorems for two second order difference equations. The difference equations are discretised versions (using Darboux transformations) of the spectral problems of the KdV equation, and of the modified KdV equation or sine-Gordon equation. Considering the discrete dynamics created by Darboux transformations for the difference equations, one obtains the lattice potential KdV equation, the lattice potential modified KdV equation and the lattice Schwarzian KdV equation, that are prototypes of integrable lattice equations. It turns out that, along the discretisation processes using Darboux transformations, two families of integrable systems (the KdV family, and the modified KdV or sine-Gordon family), including their continuous, semi-discrete and lattice versions, are explicitly constructed. As direct applications of the discrete Crum’s theorems, multi-soliton solutions of the lattice equations are obtained.

\textit{Key words: discrete Crum’s theorem, Darboux transformation, integrable discretisation, discrete Schrödinger equation, integrable lattice equations, multi-dimensional consistency, soliton solutions}

\*Corresponding author, email: ch.zhang.maths@gmail.com
1 Introduction

The original form of Darboux transformation appeared in Darboux’s classic work [1, 2], when he studied the following eigenvalue problem (known as the Sturm-Liouville equation, or the one-dimensional stationary Schrödinger equation),

\[ \mathcal{L} \phi = \left( -\partial_x^2 + u \right) \phi = \lambda \phi. \quad (1.1) \]

The second order differential operator \( \mathcal{L} \) is called Schrödinger operator, and \( \lambda \) acts as a spectral parameter. Darboux stated that, under certain transformation of the functions \( \phi \) and \( u \), the form of the equation was preserved. This characterises Darboux transformation for the Schrödinger equation, and this operation can be repeated an arbitrary number of times. A remarkable feature of Darboux’s results is that the action of an \( N \)-step Darboux transformation, \( i.e. \) applying Darboux transformations to the Schrödinger equation \( N \) times, can be encoded into compact expressions. This is known as the Crum’s theorem [3], and has direct applications in quantum mechanics.

Darboux transformation and the associated Crum’s theorem have played a pivot role in the development of integrable systems. First, they are powerful tools for constructing multi-soliton solutions of a wide range of integrable partial differential equations (see \( e.g. \) the monograph [4]). For instance, in the Lax formulation of the Korteweg-de Vries (KdV) equation [5], in which the Schrödinger equation is the spectral problem, an \( N \)-step Darboux transformation gives rise to an \( N \)-soliton solution of the KdV equation. Repeated Darboux transformations add discrete dynamics to differential systems. As illustrated in the above example, an effect of an \( N \)-step Darboux transformation to the Schrödinger equation is to introduce a discrete variable \( n, n = 1, 2, \ldots, N \), to the functions so that they become dependent of \( n \). Having the eigenfunction \( \phi \) depending on \( x \) and \( n \), the compatibility allows us to obtain the dressing chain equation [6, 7], which is an integrable differential-difference (semi-discrete) equation. This idea of using Darboux transformations to derive integrable semi-discrete equations has been known in the literature [8, 9]. The discrete dynamics of the eigenfunction \( \phi \) created by repeated Darboux transformations can be casted into certain difference equations (by eliminating the continuous variable) which is the discrete analogue of its continuous counterpart. Thus, Darboux transformation is known as an effective method to discretise the original differential system. This process was described by Shabat as “exact” or “integrable discretisation” [10].

Darboux transformation has also found many applications in various aspects in mathematics: it has a nice geometric interpretation originated from the classical theory of surfaces [2] (see also [11, 12]): it initiated researches in bispectral problems of differential equations [13]: it can be used to generate other types of explicit solutions of integrable equations such as rational solutions [4] and finite-gap-type solutions [7] (see also the monograph [14]), etc.

In this paper, we develop Darboux transformations for certain second order difference equations following the factorisation method initially proposed by Darboux [1, 2] (other applications can be found in [15, 16]). The first equation we consider is the discrete Schrödinger equation, obtained using exact discretisation of Eq (1.1). In particular, we are able to express the effect of \( N \)-step Darboux transformations in compact forms. This result is reminiscent of the Crum’s theorem for the Schrödinger equation, and will be referred to as the discrete Crum’s theorem. The second difference equation is obtained using exact discretisation of another second order differential equation [11, 12], which is the spectral problem of the modified KdV equation and of the sine-Gordon equation. Darboux
transformation for this difference equation and the associated discrete Crum’s theorem are also obtained. Moreover, in viewing Darboux transformation as new discrete dynamics to the original difference system, we can obtain some difference-difference systems using the compatibility of the two discrete variables.

The results we are presenting here have a very natural connection to integrable discrete (lattice) equations. Research in integrable discrete equations has enjoyed rapid developments over the past twenty years (see e.g. [17, 18]). Important achievements, such as the introduction of three-dimensional consistency property [19, 20], the classification of integrable lattice equations [21, 22] (known as the Adler-Bobenko-Suris classification), the derivation of multi-soliton solutions [23, 24], etc., have been established.

The difference-difference systems derived using Darboux transformations are closely connected to certain integrable lattice equations. Precisely, the first difference-difference system derived from the discrete Schrödinger equation is connected to the lattice potential KdV (lpKdV) equation; the second system is connected to both the lattice potential modified KdV (lpmKdV) equation and the lattice Schwarzian KdV (lSKdV) equation (also called the cross-ratio equation). These equations are prototypes of lattice integrable equations of the KdV type (see the review paper [23] for their origins and properties), and appeared in the Adler-Bobenko-Suris classification (named H1, H3δ=0 and Q1δ=0 respectively). It turns out, somehow unsurprisingly, that these families of integrable equations including their continuous, semi-discrete and lattice versions are explicitly constructed by the exact discretisation processes using Darboux transformations. Note that examples of using Darboux transformations to derive integrable lattice equations already exist (see e.g. [26]). As interesting by-products, the three-dimensional consistency properties of these lattice equations are well accommodated into the Darboux scheme, and their soliton solutions can be obtained thanks to the discrete Crum’s theorems.

It is worth noting that Darboux transformations and the associated Crum’s theorems for difference spectral problems have been investigated since the early development of integrable systems: the generic case was constructed in the pioneering paper [27] (although explicit formulae of the potential functions were not shown\footnote{On page 219, the author wrote “We do not present here the similar but more complicated formulae for all ϕ[\(m\),k](n,t)”} ), and special reductions, mainly in the context of solving semi-discrete integrable equations, can be found for instance in the monograph [4] and references therein. However, our approach is along the exact discretisation processes, and the so-constructed lattice equations coincide with three-dimensionally consistent equations. On the one hand, this reveals the inherent connections among continuous, semi-discrete and lattice equations from the same integrable family; on the other hand, soliton solutions of these integrable lattice equations can be directly obtained as applications of the Crum’s theorems.

The paper is organised as follows: in Section 2, we derive Darboux transformation and discrete Crum’s theorem for the discrete Schrödinger equation; study of the lpKdV equation including its derivations and soliton solutions is presented in Section 3; in Section 4, Darboux transformation and discrete Crum’s theorem for another second order difference equation is constructed; these results lay the basis for studies of the lpmKdV and lSKdV equations, which are presented in Section 5.
2 The discrete Schrödinger equation and discrete Crum’s theorem

In this section, we consider the following second order difference equation

\[ L\phi = (-T^2 - hT + a)\phi = \lambda\phi. \]  

(2.1)

This equation is the discrete Schrödinger equation \[10\] (\(L\) is also called Shabat operator) for it is obtained through exact discretisation of its continuous counterpart \[1.1\]. Here, \(T\) is the shift operator in \(n\), \(n \in \mathbb{Z}\), \(\phi\) and \(h\) are functions of \(n\), and \(a\) is a parameter also depending on \(n\). For simplicity, we use the \(\tilde{\cdot}\) notation to denote shifts in \(n\). Note that, a discrete inverse scattering transform for this equation was developed \[28, 29\].

2.1 The discrete Schrödinger equation

We briefly recall the derivation of the discrete Schrödinger equation \[2.1\]. Details can be found in \[10\]. Darboux’s derivation of the Darboux transformation lies on the following decomposition of the differential operator \(L\) given by \[1.1\]

\[ L = -\partial_x^2 + u = -(\partial_x + v)(\partial_x - v) + a, \]

(2.2)

where \(v = (\log \psi)_x = \psi_x \psi^{-1}\) with \(\psi\) being a fixed solution of \[1.1\] at \(\lambda = a\). This decomposition is subject to

\[ v_x + v^2 + a = u, \]

(2.3)

which is a Riccati equation for \(v\). By interchanging the two factors in \[2.2\], one can define a new Schrödinger operator \(\tilde{L}\) in the form

\[ \tilde{L} = -\partial_x^2 + \tilde{u} = -(\partial_x - v)(\partial_x + v) + a, \]

(2.4)

which is subject to another constraint

\[ -v_x + v^2 + a = \tilde{u}. \]

(2.5)

Here \(\tilde{u}\) is the newly transformed function in \(\tilde{L}\). Letting

\[ \tilde{\phi} = (\partial_x - v)\phi, \]

(2.6)

one can immediately show \(\tilde{L}\tilde{\phi} = \lambda\tilde{\phi}\). Now Eqs \[2.3\] and \[2.5\] result in \(\tilde{u} = u - 2v_x\), and together with the expression of \(\tilde{\phi}\) in \[2.6\], one obtains the map \((\phi, u) \mapsto (\tilde{\phi}, \tilde{u})\) characterising the Darboux transformation for the Schrödinger equation. It is clear that this map is a direct consequence of the decompositions \[2.2\] and \[2.4\].

A two-step Darboux transformation is to repeat the above construction with respect to the operator \(\tilde{L}\). The function \(\phi\) is mapped to

\[ \tilde{\tilde{\phi}} = (\partial_x - \tilde{v})\tilde{\phi} = (\partial_x - \tilde{v})(\partial_x - v)\phi. \]

(2.7)

Eliminating \(\phi_x\) and \(\phi_{xx}\) in the above expression using the above formulae yields

\[ -\tilde{\tilde{\phi}} - (\tilde{v} + v)\tilde{\phi} + a\phi = \lambda\phi, \]

(2.8)

which is Eq \[2.1\] with \(h = \tilde{v} + v\). An \(N\)-step Darboux transformation creates \(N\) shifts for \(\phi\) and \(u\) in the \(\tilde{\cdot}\) direction. Now \(u\) and \(v\) are functions of variables \(x\) and \(n\). Eliminating \(u\) in \[2.3\] and \[2.7\] leads to the dressing chain equation \[7\]

\[ (\tilde{v} + v)_x = v^2 - \tilde{v}^2 + a - \tilde{a}, \]

(2.9)

which is also the auto-Bäcklund transformation of the modified KdV equation \[6\].
2.2 Darboux transformation for the discrete Schrödinger equation

The construction of Darboux transformation for the discrete Schrödinger equation is also based on certain decompositions of the discrete Schrödinger operator \( L \). Assume that \( L \) can be decomposed into the following form

\[
L = -(T + f)(T - g) + b. \tag{2.10}
\]

Here, \( f \) and \( g \) are functions of \( n \), and \( b \) is a parameter independent of \( n \). For this decomposition to hold, one needs

\[
-f + \tilde{g} + h = 0, \quad fg = a - b. \tag{2.11}
\]

Eliminating \( f \) yields

\[
(\tilde{g} + h)g + b - a = 0, \tag{2.12}
\]

which is a discrete Riccati equation for \( g \). Posing \( g = \tilde{\psi}\psi^{-1} \), with \( \psi \) be a fixed solution of Eq (2.1) at \( \lambda = b \) solves Eq (2.12). The one-step Darboux transformation for (2.1) can be constructed by interchanging the two factors in the decomposition (2.10).

**Theorem 2.1** Consider the discrete Schrödinger equation (2.1). Under the following map

\[
\begin{align*}
\phi & \mapsto \hat{\phi} = \tilde{\phi} - g\phi, \tag{2.13} \\
h & \mapsto \hat{h} = \tilde{h} + \tilde{g} - g, \tag{2.14}
\end{align*}
\]

where \( g = \tilde{\psi}\psi^{-1} \), with \( \psi \) being a fixed solution of Eq (2.1) at \( \lambda = b \), the functions \( \hat{\phi} \) and \( \hat{h} \) satisfy

\[
\hat{L}\hat{\phi} = (-T^2 - \hat{h}T + a)\hat{\phi} = \lambda\hat{\phi}, \tag{2.15}
\]

which is also a discrete Schrödinger equation.

**Proof:** Let \( \hat{L} \) be in the form

\[
\hat{L} = -(T - g)(T + f) + b. \tag{2.16}
\]

Taking account of the decomposition (2.10), direct calculation shows \( \hat{L}\hat{\phi} = \lambda\hat{\phi} \), with \( \hat{\phi} \) given in (2.13). Equating the two expressions of \( \hat{L} \), namely Eqs (2.15) and (2.16), one gets

\[
-\tilde{f} + \tilde{g} + \hat{h} = 0, \quad fg = a - b. \tag{2.17}
\]

Combining these two equations with (2.11) yields the map \( h \mapsto \hat{h} \) given in (2.14).

Again the decompositions (2.10) and (2.16) are essential to characterise Darboux transformation for the discrete Schrödinger equation (2.1). We will use the \( \hat{\cdot} \) notation to denote the shifts created by the repeated Darboux transformations, and a discrete variable \( m \), \( m \in \mathbb{Z} \), is assigned to this direction. Clearly, starting from (2.1), its Darboux transformation adds another discrete variable to the system, and the functions \( h \) and \( \phi \) now depend on both \( n \) and \( m \) (and also on \( x \) as they descend from a differential system).
2.3 N-step Darboux transformation and discrete Crum’s theorem

As an illustration, we first show an explicit construction of a two-step Darboux transformation. Let \( \psi_1 \) and \( \psi_2 \) be two linearly independent solutions of \( (2.1) \). One has

\[
\hat{\psi}_2 = \tilde{\psi}_2 - g_1 \psi_2 ,
\]

as a solution of Eq \((2.15)\) with \( g_1 = \tilde{\psi}_1 \psi_1^{-1} \). Applying again Theorem 2.1 to \( \hat{L} \) yields

\[
\hat{\phi} = \tilde{\phi} - g_2 \hat{\phi} = (T - g_2) (T - g_1) \phi ,
\]

\[
\hat{h} = \tilde{h} + g_2 - g_2 = \tilde{h} + \tilde{g}_2 - \tilde{g}_1 + \tilde{g}_2 - g_2 ,
\]

where \( g_2 = \tilde{\psi}_2 \tilde{\psi}_2^{-1} \). In particular, it is easy to show \( \hat{\psi}_j = 0, \quad j = 1, 2 \). Recursively, given \( N \) linearly independent solutions \( \psi_j \) of \((2.1)\), \( j = 1, 2, \ldots, N \), one can construct the \( N \)-step map \((\phi, h) \mapsto (\phi[N], h[N])\). The next theorem states that \( \phi[N] \) and \( h[N] \) admit compact expressions, similar to the Crum’s theorem.

**Theorem 2.2 (Discrete Crum’s Theorem)** Assuming there are \( N \) fixed solutions \( \psi_j \) of the discrete Schrödinger equation \((2.1)\) associated with \( N \) distinct parameters \( \lambda = b_j, \quad j = 1, 2, \ldots, N \). Then the \( N \)-step Darboux transformation amounts to the following map

\[
\phi \mapsto \phi[N] = \frac{C(\psi_1, \psi_2, \ldots, \psi_N, \phi)}{C(\psi_1, \psi_2, \ldots, \psi_N)},
\]

\[
h \mapsto h[N] = h^{(N)} - s^{(2)} + s_1.
\]

Here the superscript \((N)\) denotes \( N \) shifts in the \( \sim \) direction, \( C(\varphi_1, \varphi_2, \ldots, \varphi_l) \) is the Casorati determinant for functions \( \varphi_1, \varphi_2, \ldots, \varphi_l \), namely

\[
C(\varphi_1, \varphi_2, \ldots, \varphi_l) = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \cdots & \varphi_1^{(l-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_l & \varphi_l^{(1)} & \cdots & \varphi_l^{(l-1)} \end{vmatrix},
\]

and \( s_1 \) is in the form

\[
s_1 = -\frac{\begin{vmatrix} \psi_1 & \psi_1^{(1)} & \cdots & \psi_1^{(N-2)} & \psi_1^{(N)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_N & \psi_N^{(1)} & \cdots & \psi_N^{(N-2)} & \psi_N^{(N)} \end{vmatrix}}{C(\psi_1, \psi_2, \ldots, \psi_N)}.
\]

**Proof:** Setting \( D[N] = (T - g_N)(T - g_{N-1}) \cdots (T - g_1) \), then \( \phi[N] = D[N] \phi \). One can expand \( \phi[N] \) in a general form

\[
\phi[N] = \phi^{(N)} + s_1 \phi^{(N-1)} + s_2 \phi^{(N-2)} + \cdots + s_N \phi,
\]

with functions \( s_1, s_2, \ldots, s_N \) to be determined. Insert this expression to

\[
L[N] \phi[N] = (-T^2 - h[N] T + a) \phi[N] = \lambda \phi[N].
\]
By equating the coefficient of $\phi^{(N+1)}$, one obtains the formula of $h[N]$ in (2.22). It remains to find the expressions of the coefficient functions $s_j$, $j = 1, 2, \ldots, N$. This can be done using the following set of equations

$$D[N] \phi \bigg|_{\phi = \psi_j} = 0, \quad j = 1, 2, \ldots, N, \quad (2.27)$$

or in matrix form

$$
\begin{pmatrix}
\psi_1 & \psi_1^{(1)} & \ldots & \psi_1^{(N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_N & \psi_N^{(1)} & \ldots & \psi_N^{(N-1)}
\end{pmatrix}
\begin{pmatrix}
s_N \\
s_1
\end{pmatrix}
= -
\begin{pmatrix}
\psi_1^{(N)} \\
\psi_N^{(N)}
\end{pmatrix}.
$$

(2.28)

Solving this system of equations using the Cramer’s rule leads to (2.21) and (2.24).

**Remark 2.1** One of the most important properties of an $N$-step Darboux transformation is that the final result of the map $(\phi, h) \mapsto (\phi[N], h[N])$ is independent of the order in which the solutions $\psi_j$, $j = 1, 2, \ldots, N$, are added. This property is known as the Bianchi permutability property (see Fig. 1). This can be easily realised by looking at the determinantal forms of $\phi[N]$ and $h[N]$. Namely, changing the positions of any two elements in the expressions (2.21) and (2.24), e.g. $\psi_i$ and $\psi_j$, exchanges the order of adding $\psi_i$ and $\psi_j$ in the process of the $N$-step Darboux transformation, but the final results remain unchanged.

\[\begin{array}{c}
\phi, h \\
\psi_1 \\
\psi_2 \\
\psi_i \\
\widehat{\psi}_1 \\
\widehat{\psi}_2 \\
\widehat{\phi}, \widehat{h}
\end{array}\]

Figure 1: Bianchi permutability property: given $\psi_1$ and $\psi_2$, there are two ways to make a two-step Darboux transformation from $(\phi, h)$. The final results $(\widehat{\phi}, \widehat{h})$ are the same.

### 3 Connection to the lpKdV equation

In Section 2, we showed that the second order difference operator $L$ is obtained through exact discretisation of the second order differential operator $L$. By developing Darboux transformation for the difference equation (2.1), one adds another discrete dynamics to the system, which is assigned with the discrete variable $m$. This process for the function $\phi$ is depicted as follows

$$
\phi(x; \lambda) \xrightarrow{\text{Darboux transf. for } L \phi = \lambda \phi} \phi(x, n; \lambda) \xrightarrow{\text{Darboux transf. for } L \phi = \lambda \phi} \phi(x, n, m; \lambda).
$$

This section aims to explore the difference-difference systems of variables $n$ and $m$. 

7
3.1 Lax pair

It is natural to consider the compatibility condition for $\phi$ in the difference systems
\begin{align*}
-\tilde{\phi} - h \tilde{\phi} + a \phi &= \lambda \phi, \quad (3.1) \\
\hat{\phi} &= \tilde{\phi} - g \phi. \quad (3.2)
\end{align*}

This amounts to the following system of difference equations involving $g$ and $h$
\begin{align*}
\hat{h} - \tilde{h} &= \tilde{g} - g, \quad (3.3) \\
(h + \tilde{g}) h + \tilde{a} &= (h + \tilde{g}) \hat{h} + a. \quad (3.4)
\end{align*}

Now, assume $g$ and $h$ are expressions of certain function $w$ (also depending on $n$ and $m$) in the forms
\begin{equation*}
h = \tilde{w} - w, \quad g = \tilde{w} - \hat{w}. \quad (3.5)
\end{equation*}

This assumption is perfectly legitimate, thanks to the specific form of Eq (3.3). One can thus reduce the system (3.4) into a lattice equation involving only $w$. Recall Eq (2.11) which also involves $g$ and $h$. The forms of $g$ and $h$ (3.5) impose $f = \tilde{w} - w$, and the relation $fg = a - b$ then reads
\begin{equation*}
(w - \tilde{w})(\tilde{w} - \hat{w}) = a - b, \quad (3.6)
\end{equation*}

which is the lpKdV equation. Here the parameters $a$ and $b$ play the role of lattice parameters, that depend $a$ priori on the $\tilde{\;}$ and $\hat{\;}$ directions respectively, thus the lpKdV equation we derived here is non-autonomous. The system of linear difference equations (3.1) and (3.2) are the Lax pair of the lpKdV equation.

One can easily formulate the Lax pair in matrix forms. Set a vector function $\Phi^T = (\phi, \tilde{\phi})$, then Eqs (3.1) and (3.2) lead to
\begin{equation*}
\tilde{\Phi} = N \Phi = \begin{pmatrix} 0 & 1 \\ a - \lambda & -h \end{pmatrix} \Phi, \quad \hat{\Phi} = M \Phi = \begin{pmatrix} -g & 1 \\ a - \lambda & -\tilde{g} - h \end{pmatrix} \Phi. \quad (3.7)
\end{equation*}

Their compatibility is equivalent to the discrete zero-curvature condition
\begin{equation*}
\tilde{M} N = \hat{N} M. \quad (3.8)
\end{equation*}

Remark 3.1 In the language of Bäcklund transformations, similar derivation of the lpKdV equation exists [6], known as the nonlinear superposition formula.

Remark 3.2 The expressions $f = \tilde{w} - w$ and $g = \tilde{w} - \hat{w}$ yield the identity
\begin{equation*}
-\tilde{f} + f - \tilde{g} + g = 0. \quad (3.9)
\end{equation*}

This equation, together with $fg = a - b$, is the non-autonomous discrete KdV equation.

The lpKdV equation is a quadrilateral equation possessing the three-dimensional consistency property. This means the equation can be consistently embedded onto the six faces of a cube (see Fig. 2). Here one can easily fit the three-dimensional consistency into
the Darboux derivation of the lpKdV equation. Perform another Darboux transformation for (3.1), and assign this step to a third direction, the \( \bar{\imath} \) direction. Thus, one has

\[
\overline{\phi} = \tilde{\phi} - \mathcal{G} \phi, \quad -\mathcal{F} + \mathcal{G} + h = 0, \quad \mathcal{F} \mathcal{G} = a - c, \tag{3.10}
\]

with

\[
\mathcal{F} = \overline{\imath} - w, \quad \mathcal{G} = \overline{\imath} - \overline{\imath}. \tag{3.11}
\]

The parameter \( c \) is associated with the \( \bar{\imath} \) direction. Direct computations show that the compatibility of the two one-step Darboux transformations in the \( \bar{\imath} \) and \( \bar{\omega} \) directions leads to the lpKdV equation

\[
(w - \overline{\imath})(\bar{\imath} - \overline{\imath}) = b - c. \tag{3.12}
\]

It is clear that applying successive Darboux transformations along three directions of the cube, the consistency is guaranteed by the Bianchi permutability property.

\[ \text{Figure 2: Three-dimensional consistency: all six faces of the cube admit the same form of lattice equation.} \]

### 3.2 Soliton solutions

Thanks to the three-dimensional consistency of the lpKdV equation, it is straightforward to apply the discrete Crum’s theorem to compute its soliton solution.

**Proposition 3.1** Assuming there are \( N \) fixed solutions \( \psi_j \) associated with \( N \) distinct parameters \( \lambda = \lambda_j, j = 1, 2, \ldots, N \), of the following system of difference equations

\[
\begin{align*}
-\tilde{\phi} - (\tilde{w} - w)\tilde{\phi} + a \phi &= \lambda \phi, \tag{3.13} \\
\hat{\phi} &= \tilde{\phi} - (\hat{\imath} - \overline{\imath}) \phi. \tag{3.14}
\end{align*}
\]

Their compatibility is the lpKdV equation (3.6), and its \( N \)-soliton solution is in the form

\[
w^{Nss} = w^{(N)} - s_1, \tag{3.15}
\]

where \( s_1 \) is given by Eq (2.24).
Proof: The $N$-step Darboux transformation is applied to both Eqs (3.13) and (3.14) along a third direction, the $\bar{w}$ direction. Using $\tilde{w} = \tilde{w} - w$ and Theorem 2.2, one has

$$\tilde{w}^N_{s_{s}} - w^N_{s_{s}} = w^{(N+2)} - w^{(N)} - s_1^{(2)} + s_1. \quad (3.16)$$

Solving this equation leads to the expression (3.15).

For simplicity, we consider the autonomous case, i.e. $a$ and $b$ are constants. A simple seed solution of the $lpKdV$ equation is chosen as

$$w = -(\alpha n + \beta m + \xi), \quad (3.17)$$

where $\alpha^2 = -a$, $\beta^2 = -b$ and $\xi$ is a constant. Then, the fixed solution $\psi_j$ is in the form

$$\psi_j = \rho_j^+(\alpha + \kappa_j)n(\beta + \kappa_j)^n + \rho_j^-(\alpha - \kappa_j)n(\beta - \kappa_j)^n, \quad (3.18)$$

where $\kappa_j^2 = -\lambda_j$ and $\rho_j^\pm$ are constants. The spectral parameter $\lambda_j$ acts as the lattice parameter in the $\bar{w}$ direction. Therefore, we recover the results obtained in [23, 24].

4 Discrete Crum’s theorem: case II

In this section, we are constructing Darboux transformation for another second order difference equation

$$L \phi = (h T^2 + (1 + \bar{a} h) T + a) \phi = \lambda \phi. \quad (4.1)$$

Here $h$ is a function of $n$, and the parameter $a$ depends on $n$. This equation is obtained through exact discretisation from another second order differential equation (4.2). We develop one-step and $N$-step Darboux transformations for the difference equation, and the results will amount to another example of the discrete Crum’s theorem.

4.1 Derivation of Eq (4.1)

Consider the following Schrödinger-type second order differential equation

$$\mathcal{L} \phi = (\partial_x^2 - 2u_x \partial_x) \phi = \lambda \phi. \quad (4.2)$$

This equation is the spectral problem of the potential modified KdV equation and the sine-Gordon equation in their Lax formulations. Decompose $\mathcal{L}$ into the form

$$\mathcal{L} = (F \partial_x + b) \left( \frac{1}{F} \partial_x - a \right) + \eta. \quad (4.3)$$

Here, the function $F$ and parameters $a$, $b$, $\eta$ are introduced to characterise the Darboux transformation. Define a new operator $\tilde{\mathcal{L}}$ (interchanging the two factors in $\mathcal{L}$),

$$\tilde{\mathcal{L}} = \left( \frac{1}{F} \partial_x - a \right) (F \partial_x + b) + \eta, \quad (4.4)$$

then clearly $\tilde{\mathcal{L}} \tilde{\phi} = \lambda \tilde{\phi}$, with

$$\tilde{\phi} = \left( \frac{1}{F} \partial_x - a \right) \phi. \quad (4.5)$$
Equating the two expressions of $\mathcal{L}$ and of $\tilde{\mathcal{L}}$ (applying a shift to $u_x$) yields
\[- \partial_x (\log F) - aF + \frac{b}{F} = -2u_x, \quad \partial_x (\log F) - a\tilde{F} + \frac{b}{\tilde{F}} = -2\tilde{u}_x, \quad ab = \eta. \quad (4.6)\]

For simplicity, let $b = 1$, thus $\eta = a$. Arranging the above equations, one gets
\[\tilde{u}_x - u_x = -\partial_x (\log F), \quad \tilde{u}_x + u_x = a\tilde{F} - \frac{1}{\tilde{F}}. \quad (4.7)\]

The first expression together with Eq (4.5) define the map $(\phi, u_x) \mapsto (\tilde{\phi}, \tilde{u}_x)$, i.e. Darboux transformation for Eq (4.2). Solving Eq (4.5) and choosing $F$ of the particular form
\[F = e^{(u-\tilde{u})/\sqrt{a}}, \quad (4.8)\]
the second expression of (4.7) becomes
\[\left(\frac{\tilde{u} + u}{2}\right)_x = \sqrt{a} \sinh(u - \tilde{u}), \quad (4.9)\]
which is the semi-discrete version of the sinh-Gordon equation\(^2\)\(^{[2]}\).

Now having obtained the one-step Darboux transformation for Eq (4.2), one can construct the two-step Darboux transformation. A difference system involving $\phi$, $\tilde{\phi}$ and $\hat{\phi}$ can be easily derived using the above formulae
\[(F\tilde{F})\hat{\phi} + \left(1 + \tilde{a}F\tilde{F}\right)\tilde{\phi} + a\phi = \lambda\phi. \quad (4.10)\]
Setting $h = F\tilde{F}$, one recovers Eq (4.1).

### 4.2 One-step Darboux transformation

To construct Darboux transformation for Eq (4.1), one decomposes the operator $L$ into
\[L = (AT + f + c)(BT - g - b) + \sigma, \quad (4.11)\]
where $A, B, f$ and $g$ are functions of both the variables $n$ and $m$, corresponding to the $\sim$ and $\hat{}$ directions respectively, and $b, c$ and $\sigma$ are parameters independent of the $\sim$ direction but depend \textit{a priori} on the $\hat{}$ direction. Equating the two expressions of $L$ leads to
\[A\hat{B} = h, \quad 1 + \tilde{a}h = (f + c)B - (\tilde{g} + b)A, \quad \sigma - a = (f + c)(g + b). \quad (4.12)\]

This allows us to define a Darboux transformation
\[L \mapsto \hat{L} = (BT - g - b)(AT + f + c) + \sigma, \quad (4.13)\]
\[\phi \mapsto \hat{\phi} = (BT - g - b)\phi, \quad (4.14)\]
\(^2\)Note that choosing $F = e^{(u-\tilde{u})/\sqrt{a}}$, one obtains the semi-discrete sine-Gordon equation
\[\left(\frac{\tilde{u} + u}{2}\right)_x = \sqrt{-a} \sin(u - \tilde{u}).\]
obviously subject to \( \hat{L} \hat{\phi} = \lambda \hat{\phi} \), if \( \hat{h} \) satisfies the following constraints

\[
\hat{A} B = \hat{h}, \quad 1 + \tilde{a} \hat{h} = \left( \hat{f} + c \right) B - (g + b) A.
\] (4.15)

At first glance, Eqs (4.12) and (4.15) define a complicated system of equations involving \( A, B, f, g \) and \( h \). However, this system can be significantly simplified.

First, eliminating \( f + c \), one gets

\[
\left( 1 + b \tilde{A} \right) + \tilde{A} \left( \tilde{g} + \tilde{a} \tilde{B} \right) = \frac{(1 + b A) + g A + \tilde{a} \tilde{A} B}{B}.
\] (4.16)

Since we are content here to find a possible solution of the system, we can choose \( g = -a B \), which leaves Eq (4.10) in the form

\[
(T - 1) \left( \frac{1}{B} + \frac{b A}{B} - a A \right) = 0.
\] (4.17)

This implies \( \frac{1}{B} + \frac{b A}{B} - a A = \gamma \), with \( \gamma \) being a constant of summation, which may depend on the \( \hat{\;} \) direction. Substituting this expression back to Eqs (4.12) and (4.15), a last equation involving the parameters is obtained,

\[
\sigma - \gamma b = 0.
\] (4.18)

Without loss of generality, one absorbs \( \gamma \) into \( b \) by letting \( \gamma = 1 \), so that \( \sigma = b \). Thus, we have obtained

\[
g = -a B, \quad A = \frac{B - 1}{b - a B}, \quad f + c = \frac{b - a}{b - a B},
\] (4.19)

and

\[
\hat{h} \tilde{B} = \tilde{h} B,
\] (4.20)

where \( B \) satisfies certain discrete Riccati equation

\[
-(b - a B) \hat{h} + (B - 1) \tilde{B} = 0.
\] (4.21)

Letting

\[
B = \frac{b \psi}{\psi + a \psi},
\] (4.22)

with \( \psi \) being a fixed solution of the eigenvalue problem (4.1) at \( \lambda = b \), one solves the Riccati equation (4.21). We summarise the above analysis into the following theorem, which characterises the one-step Darboux transformation for Eq (4.1).

**Theorem 4.1** Consider the second order difference equation (4.1). Under the following map

\[
\phi \mapsto \hat{\phi} = B \tilde{\phi} + (a B - b) \phi,
\] (4.23)

\[
h \mapsto \hat{h} = \frac{B \tilde{h}}{\tilde{B}},
\] (4.24)

where \( B \) is defined in (4.22) with \( \psi \) being a fixed solution of (4.1) at \( \lambda = b \), the functions \( \hat{\phi} \) and \( \hat{h} \) satisfy

\[
\hat{L} \hat{\phi} = \left( \tilde{h} T^2 + (1 + \tilde{a} \hat{h}) T + a \right) \hat{\phi} = \lambda \hat{\phi},
\] (4.25)

which shares the same form as Eq (4.1).
4.3 N-step Darboux transformation and discrete Crum’s theorem

We can repeat the above Darboux transformation an arbitrary number of times. Interestingly, we are able to express the N-step mapped functions $\phi[N]$ and $h[N]$ in compact forms. These results are referred to as the discrete Crum’s theorem for Eq (4.1).

As an illustration, let us first compute $\phi[2]$ and $h[2]$ explicitly. Assume that $\psi_1$ and $\psi_2$ are two fixed solutions of Eq (4.1) associated with $\lambda = b_1$ and $b_2$ respectively, with $b_1 \neq b_2$. Following Theorem 4.1, a one-step Darboux transformation using $\psi_1$ leads to explicit expressions

$$\phi[1] = b_1 \frac{\psi_1 \tilde{\phi} - \tilde{\psi}_1 \phi}{\psi_1 + a\psi_1}, \quad h[1] = \tilde{h} \frac{\psi_1 \tilde{\psi}_1 + a \tilde{\psi}_1}{\tilde{\psi}_1},$$

(4.26)

and $\phi[1]$ satisfies

$$h[1] \tilde{\phi}[1] + (1 + \tilde{a} h[1]) \tilde{\phi}[1] + a \phi[1] = \lambda \phi[1].$$

(4.27)

Recall that $T$ denotes the shift operator along the \tilde{\phi} direction, and $C(\varphi_1, \varphi_2, \ldots, \varphi_N)$ is the Casorati determinant, we can rewrite (4.26) as

$$\phi[1] = b_1 \frac{C(\psi_1, \phi)}{\det \mathcal{M}_1}, \quad h[1] = \tilde{h} \frac{C(\psi_1)}{\det \mathcal{M}_1} T^2 \left( \frac{\det \mathcal{M}_1}{C(\psi_1)} \right),$$

(4.28)

where the matrix $\mathcal{M}_1$ is in the form

$$\mathcal{M}_1 = \begin{pmatrix} 1 & -a \\ \psi_1 & -\tilde{\psi}_1 \end{pmatrix}.$$

(4.29)

A two-step Darboux transformation is constructed using both $\psi_1$ and $\psi_2$. Precisely, one has

$$\phi[2] = B_2 \tilde{\phi}[1] + (a B_2 - b_2) \phi[1],$$

(4.30)

where

$$B_2 = \frac{b_2 \psi_2[1]}{\psi_2[1] + a \psi_2[1]}, \quad \psi_2[1] = b_1 \frac{C(\psi_1, \psi_2)}{\det \mathcal{M}_1}.$$

(4.31)

Combining these expressions together, we obtain

$$\phi[2] = b_1 b_2 \frac{C(\psi_1, \psi_2) C(\tilde{\psi}_1, \tilde{\phi}) - C(\tilde{\psi}_1, \tilde{\psi}_2) C(\psi_1, \phi)}{C(\psi_1, \psi_2) \det \mathcal{M}_1 + a C(\psi_1, \psi_2) T \det \mathcal{M}_1} = b_1 b_2 \frac{C(\psi_1, \psi_2, \phi)}{\det \mathcal{M}_2},$$

(4.32)

where

$$\mathcal{M}_2 = \begin{pmatrix} 1 & -a & (-a)(-\tilde{a}) \\ \psi_1 & \tilde{\psi}_1 & \tilde{\phi} \\ \tilde{\psi}_1 & \tilde{\psi}_2 & \tilde{\psi}_2 \end{pmatrix}.$$

(4.33)

Similarly, the expression of $h[2]$ is in the form

$$h[2] = C(\psi_1, \psi_2) T^2 \frac{\det \mathcal{M}_2}{C(\psi_1, \psi_2)}.$$

(4.34)

These give us suggestive forms of $\phi[N]$ and $h[N]$, which are stated in the following discrete Crum’s theorem.
Theorem 4.2 Assuming that there are \( N \) fixed solutions \( \psi_j \) associated with \( N \) distinct parameters \( \lambda = b_j, \ j = 1, 2, \ldots, N \), of the second order difference equation \( \text{(4.1)} \). Then the \( N \)-step Darboux transformation amounts to the following map

\[
\phi \mapsto \phi[N] = \left( \prod_{j=1}^{N} b_j \right) \frac{C(\psi_1, \psi_2, \ldots, \psi_N, \phi)}{\det \mathcal{M}_N}, \tag{4.35}
\]

\[
h \mapsto h[N] = h^{(N)}(\frac{\det \mathcal{M}_N}{C(\psi_1, \psi_2, \ldots, \psi_N)}) \tag{4.36}
\]

where the matrix \( \mathcal{M}_N \) is in the form

\[
\mathcal{M}_N = \begin{pmatrix}
1 & -a & \cdots & \prod_{j=1}^{N} (-a^{(j-1)}) \\
\psi_1 & \psi_1^{(1)} & \cdots & \psi_1^{(N)} \\
\psi_2 & \psi_2^{(1)} & \cdots & \psi_2^{(N)} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_N & \psi_N^{(1)} & \cdots & \psi_N^{(N)}
\end{pmatrix} \tag{4.37}
\]

Proof. Step 1. It is obvious that \( \phi[N] \) is linear with respect to \( \phi \) and its shifts, then an \( N \)-step Darboux transformation gives

\[
\phi[N] = (B_N T + a B_N - b_N) \phi[N - 1] = (B_N T + a B_N - b_N) \cdots (B_1 T + a B_1 - b_1) \phi = S_{N,0} \phi^{(N)} + S_{N,1} \phi^{(N-1)} + \cdots + S_{N,N} \phi, \tag{4.38}
\]

with the coefficients \( S_{N,0}, S_{N,1}, \ldots, S_{N,N} \) to be determined. It is easy to show

\[
S_{N,0} = B_N B_{N-1}^{(1)} \cdots B_{1}^{(N-1)}. \tag{4.39}
\]

Since \( \phi[N]|_{\phi=\psi_j} = 0, \ j = 1, 2, \ldots, N, \) we obtain the following system involving all \( \psi_j \)

\[
\begin{pmatrix}
\psi_1 & \psi_1^{(1)} & \cdots & \psi_1^{(N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_N & \psi_N^{(1)} & \cdots & \psi_N^{(N-1)}
\end{pmatrix}
\begin{pmatrix}
S_{N,N} \\
S_{N,1}
\end{pmatrix}
= -S_{N,0}
\begin{pmatrix}
\psi_1^{(N)} \\
\vdots \\
\psi_N^{(N)}
\end{pmatrix}. \tag{4.40}
\]

Using the Cramer’s rule, one obtains

\[
S_{N,N+1-j} = -S_{N,0} \frac{C_j}{C(\psi_1, \psi_2, \ldots, \psi_N)}, \quad j = 1, 2, \ldots, N, \tag{4.41}
\]

where the function \( C_j \) denotes the Casorati determinant \( C(\psi_1, \psi_2, \ldots, \psi_N) \) by replacing its \( j \)-th column by the vector \( (\psi_1^{(N)} \ \psi_2^{(N)} \ \cdots \ \psi_N^{(N)})^T \). Thus, one has

\[
\phi[N] = S_{N,0} \frac{C(\psi_1, \psi_2, \ldots, \psi_N, \phi)}{C(\psi_1, \psi_2, \ldots, \psi_N)}, \tag{4.42}
\]

On the other hand, inserting the expression \( \text{(4.38)} \) into \( L[N] \phi[N] = \lambda \phi[N] \) gives \( h[N] \) in the form

\[
h[N] = h^{(N)} \frac{S_{N,0}}{C_j}, \tag{4.43}
\]
Step 2. It remains to determine $S_{N,0}$ to get explicit expressions of $\phi[N]$ and $h[N]$. It is clear that the $N$-step Darboux transformation yields

$$B_N = \frac{b_N \psi_N[N - 1]}{\psi_N[N - 1] + a \psi_N[N - 1]}.$$  \hspace{1cm} (4.44)

From Eq (4.42), we have

$$\psi_N[N - 1] = S_{N-1,0} \frac{C(\psi_1, \psi_2, \ldots, \psi_N)}{C(\psi_1, \psi_2, \ldots, \psi_{N-1})}.$$  \hspace{1cm} (4.45)

Using $B_N = \frac{S_{N,0}}{S_{N-1,0}}$ (coming from (4.39)), one gets the following identity

$$S_{N,0} \sim R_{N-1} + b_N S_{N-1,0} \sim h_{N-1} = 0,$$  \hspace{1cm} (4.46)

where

$$R_{N-1} = \frac{C(\psi_1, \psi_2, \ldots, \psi_N)}{C(\psi_1, \psi_2, \ldots, \psi_{N-1})}.$$  \hspace{1cm} (4.47)

One can prove by induction that $S_{N,0}$ defined in the form

$$S_{N,0} = \left( \prod_{j=1}^{N} b_j \right) \frac{C(\psi_1, \psi_2, \ldots, \psi_N)}{\det M_N}$$  \hspace{1cm} (4.48)

satisfies (4.46). Substituting $S_{N,0}$ back to Eqs (4.42) and (4.43) completes the proof. \footnote{15}

5 Connections to the lpmKdV and lSKdV equations

In this section, we explore the difference-difference systems derived in the previous section. Thanks to some appropriate substitutions of the functions $h$, $A$ and $B$, we obtain two integrable lattice equations: the lpmKdV equation and the lSKdV equation. Their $N$-soliton solutions will be derived as consequences of Theorem 4.2.

5.1 The lpmKdV equation

Recall the derivation of the semi-discrete sinh-Gordon equation (see Section 4.1). Letting $w = e^u$ and $\sqrt{a} = \alpha$, from the expression Eq (4.8), one has

$$F = \frac{1}{\alpha} \frac{w}{\bar{w}}.$$  \hspace{1cm} (5.1)

Also knowing that $h = F \bar{F}$, it is straightforward to express the functions $A$ and $B$ (defined in Section 4.2) in the forms

$$A = \frac{1}{\alpha \beta} \frac{w}{\bar{w}}, \quad B = \frac{\beta \bar{w}}{\alpha},$$  \hspace{1cm} (5.2)

where $\beta$ is a parameter depending on the $\hat{w}$ direction only. Now consider the relation between $A$ and $B$ defined in Eq (4.19). Substituting (5.1) and (5.2) inside, and setting $a = \alpha^2$ and $b = \beta^2$, one gets

$$\alpha \left( w \bar{w} - \bar{w} \bar{w} \right) - \beta \left( w \bar{w} - \bar{w} \bar{w} \right) = 0,$$  \hspace{1cm} (5.3)
which is the non-autonomous lpmKdV equation.

The three-dimensional consistency of this equation is naturally encoded into the Bianchi permutability property of the Darboux transformation. Namely, perform a Darboux transformation for Eq (4.1) in a third direction, the \( \bar{\hat{\gamma}} \) direction. Assign \( \gamma \) as the lattice parameter in this direction. By computing the compatibility of the two one-step Darboux transformations in the \( \hat{\gamma} \) and \( \bar{\hat{\gamma}} \) directions, one obtains another lpmKdV equation

\[
\alpha \left( w \frac{\partial}{\partial w} - \tilde{w} \frac{\partial}{\partial \tilde{w}} \right) - \gamma \left( w \frac{\partial}{\partial w} - \tilde{w} \frac{\partial}{\partial \tilde{w}} \right) = 0.
\]

(5.4)

Then, the three-dimensional consistency of the lpmKdV equation is a mere consequence of successive Darboux transformations in three directions of a cube. The \( N \)-soliton solution can be easily obtained using Theorem 4.2.

**Proposition 5.1** Assuming there are \( N \) fixed solutions \( \psi_j \) associated with \( N \) distinct parameters \( \lambda = \lambda_j, j = 1, 2, \ldots, N \), of the following system of difference equations

\[
\begin{align*}
\left( \frac{1}{\alpha} - \frac{\partial}{\partial \alpha} \right) \frac{\partial}{\partial \phi} + \left( 1 + \frac{\partial}{\partial \tilde{\alpha}} \right) \frac{\partial}{\partial \tilde{\alpha}} + \alpha^2 \phi &= \lambda \phi, \\
\tilde{\phi} &= \beta \frac{\partial}{\partial \tilde{\phi}} + \left( \alpha \beta \frac{\partial}{\partial \tilde{\phi}} - \beta^2 \right) \phi.
\end{align*}
\]

(5.5)

(5.6)

Their compatibility gives rise to the lpmKdV equation (5.3), and its \( N \)-soliton solution is in the form

\[
w^{Nss} = \left( \prod_{j=1}^{N} \alpha^{(j-1)} \right) w^{(N)} C(\psi_1, \psi_2, \ldots, \psi_N) \det M_N, \quad (5.7)
\]

where the matrix \( M_N \) is defined in Eq (4.37).

**Proof:** The \( N \)-step Darboux transformation is applied to both Eqs (5.5) and (5.6) along a third direction, the \( \bar{\hat{\gamma}} \) direction. From the above derivation of the lpmKdV equation, it is clear that \( h = w/\left( \alpha \tilde{\alpha} \tilde{\phi} \right) \). Then it follows from Theorem 4.2 that

\[
\frac{1}{\alpha \tilde{\alpha}} \frac{\partial}{\partial \alpha \tilde{\alpha}} \frac{\partial}{\partial w^{Nss}} = \frac{1}{\alpha \tilde{\alpha} \tilde{\phi}} \frac{w^{(N)} \alpha^{(N+1)} \alpha}{w^{(N+2)} \alpha \tilde{\alpha} \tilde{\phi}} C(\psi_1, \psi_2, \ldots, \psi_N) \det M_N T^2 \left( \frac{\det M_N}{C(\psi_1, \psi_2, \ldots, \psi_N)} \right). \quad (5.8)
\]

Solving this equation completes the proof. Note that without loss of generality the constant of summation is absorbed into the seed solution \( w \).

For simplicity, consider only the autonomous case, i.e. the parameters \( \alpha \) and \( \beta \) are both constants. A simple seed solution is

\[
w = \rho s^n t^m, \quad (5.9)
\]

where \( s \) and \( t \) are related to the lattice parameters \( \alpha \) and \( \beta \) (see (5.10)), and \( \rho \) is a constant. The spectral parameter \( \lambda_j \) is related to the lattice parameter \( \gamma_j \) via \( \lambda_j = \gamma_j^2 \). The seed solution imposes

\[
\alpha = \frac{2 \theta s}{s^2 - 1}, \quad \beta = \frac{2 \theta t}{t^2 - 1}, \quad \gamma_j = \frac{2 \theta r_j}{r_j^2 - 1}, \quad (5.10)
\]

16
where $\theta$ is a constant\footnote{Here we assume $s, t$ and $\theta$ are real (rather than complex). More details can be found in [24].}. The last term is introduced to express $\gamma_j$ in terms of another constant $r_j$, similarly to the expressions of $\alpha$ and $\beta$. Solving the Lax pair, one gets the fixed solution $\psi_j$ in the form

$$
\psi_j = \rho_j^+ E(n, m; \kappa_j) + \rho_j^- E(n, m; -\kappa_j), \quad (5.11)
$$

where $\rho_j^\pm$ are constants, and the parameter $\kappa_j$ is defined in terms of of $r_j$ as

$$
\kappa_j = \theta \frac{1 + r_j^2}{1 - r_j^2}. \quad (5.12)
$$

The function $E$ is a discrete exponential-type function

$$
E(n, m; \pm \kappa_j) = \left( -\frac{\alpha^2}{2} (1 + s^2) \pm 2\alpha s \kappa_j \right)^n \left( -\frac{\beta^2}{2} (1 + t^2) \pm 2\beta t \kappa_j \right)^m. \quad (5.13)
$$

Without loss of generality, one can reparametrise the constants $\rho_j^\pm$ as

$$
\rho_j^\pm \mapsto \rho_j^\pm \left( \frac{\alpha^2}{2} (1 - s^2) \pm 2\alpha s \kappa_j \right)^l, \quad (5.14)
$$

where the parameter $l$ is understood as a discrete “variable” in the $\bar{\ }$ direction. Now the function $\psi_j$ depends on $n, m$ and $l$ (also on the parameter $\kappa_j$). The advantage of the above parametrisation of $\psi_j$ is that one can link $\bar{\psi}_j$ and $\psi_j$ via

$$
\bar{\psi}_j = -\alpha^2 \psi_j + \psi_j, \quad (5.15)
$$

where $\bar{\psi}_j$ add a shift in the $\bar{\ }$ direction by mapping the variable $l$ to $l + 1$. Knowing that $\alpha^2 = a$, then the expression of $\det \mathcal{M}_N$ can be drastically simplified to

$$
\det \mathcal{M}_N = D(\psi_1, \psi_2, \ldots, \psi_N). \quad (5.16)
$$

The notation $D(\varphi_1, \varphi_2, \ldots, \varphi_N)$ denotes the Casorati determinant defined in the $\bar{\ }$ direction, i.e. $D(\varphi_1, \varphi_2, \ldots, \varphi_N)$ is in the same form as $C(\varphi_1, \varphi_2, \ldots, \varphi_N)$ (see Eq (2.23)) but with $\sim$ shifts replaced by $\bar{\ }$ shifts. Using $\Theta_N$ to denote $D(\psi_1, \psi_2, \ldots, \psi_N)$, one has $\det \mathcal{M}_N = \Theta_N$. Moreover, one can easily obtain

$$
\Theta_N \equiv D(\psi_1, \psi_2, \ldots, \psi_N) = C(\psi_1, \psi_2, \ldots, \psi_N). \quad (5.17)
$$

Therefore, the $N$-soliton solution of the lpmKdV equation is in the from (with the constant factor being absorbed into the seed solution $w$)

$$
w^{Nss} = w^{(N)} \Theta_N \Theta_N^{-1}. \quad (5.18)
$$

We recover the results obtained in [23, 24].
5.2 The lSKdV equation

Interestingly, the lSKdV equation can also be obtained from the Darboux transformation for Eq \((4.1)\). Consider another set of substitutions of the functions \(h, A\) and \(B\):

\[
h = \frac{\tilde{P}}{a \tilde{P}}, \quad B = \frac{P}{\tilde{Q}}, \quad A = \frac{\tilde{Q}}{a \tilde{P}},
\]

(5.19)

where \(P\) and \(Q\) are functions of \(n\) and \(m\). Substituting these back to Eqs (4.12) and (4.15), one gets two difference constraints

\[
\hat{a} \tilde{P} \tilde{P} Q = a \hat{P} P \tilde{Q},
\]

(5.20)

and

\[
Q - \tilde{Q} = P - \hat{P}.
\]

(5.21)

The first difference equation can be solved using

\[
(T - 1)(T + 1) \log Q = (T - 1) \log \left( a \hat{P} P \right),
\]

(5.22)

and one gets

\[
a \hat{P} P = b \tilde{Q} Q,
\]

(5.23)

with \(b\) being a constant of summation independent of the \(\tilde{}\) direction. With the aid of a new function \(z\), the second difference equation (5.21) can be solved using

\[
P = \tilde{z} - z, \quad Q = \hat{z} - z.
\]

(5.24)

Then, it turns out that Eq (5.23) becomes

\[
a \left( \tilde{z} - \hat{z} \right) \left( \tilde{z} - z \right) = b \left( \tilde{z} - \hat{z} \right) \left( \hat{z} - z \right),
\]

(5.25)

which is the non-autonomous lSKdV equation. From the above derivations, it is clear that the lSKdV equation has a Lax pair in the form

\[
\left( \tilde{z} - \hat{z} \right) \tilde{\phi} + \left( a \tilde{z} - \hat{a} \hat{z} + a \hat{z} - a z \right) \hat{\phi} + a \left( a - \lambda \right) \left( \tilde{z} - z \right) \phi = 0,
\]

(5.26)

\[
\left( \hat{z} - z \right) \hat{\phi} + \left( a \tilde{z} - a \hat{z} - b \left( \tilde{z} - z \right) \right) \phi = 0.
\]

(5.27)

The three-dimensional consistency of this equation can be understood in the same way as showed in the cases of the lpKdV equation and the lpmKdV equation. First perform another Darboux transformation in a third direction, the \(\bar{\tilde{\to}}\) direction assigned with a lattice parameter \(c\). Computing the compatibility of the two one-step Darboux transformations in the \(\tilde{\to}\) and \(\bar{\tilde{\to}}\) directions, one obtains another lSKdV equation

\[
a \left( \tilde{z} - \bar{\tilde{z}} \right) \left( \tilde{z} - z \right) = c \left( \tilde{z} - \bar{\tilde{z}} \right) \left( \bar{z} - z \right).
\]

(5.28)

Then, one can conclude that the lSKdV equation is three-dimensionally consistent.

We have just shown that both the lpmKdV and lSKdV equations can be derived from the Darboux transformation for Eq (4.1). From the substitutions (5.1), (5.2) and (5.19), it
follows that the lpmKdV equation and the lSKdV equation are connected using a Miura-type (Bäcklund) transformation [23]

\[ \tilde{z} - z = \frac{1}{\alpha \tilde{w}}, \quad \hat{z} - z = \frac{1}{\beta \hat{w}}. \] (5.29)

Now we can derive \( N \)-soliton solution of the lSKdV equation using that of the lpmKdV equation. Assume that \( \tilde{z}\text{Nss} \) is an \( N \)-soliton solution of the lSKdV equation. In addition, we assume that \( z\text{Nss} \) is a fixed-point solution of \( z\text{Nss} \) [24]. Based on the three-dimensional consistency of both equations as well as the consistency between them (see Fig. 3), one can link \( z\text{Nss} \) and \( z\text{Nss} \) to the \( N \)-soliton solution \( \tilde{w}\text{Nss} \) of the lpmKdV equation via similar Miura transformation

\[ \frac{z\text{Nss} - \tilde{z}\text{Nss}}{\gamma w\text{Nss} \tilde{w}\text{Nss}} = \frac{1}{\gamma w\text{Nss} \tilde{w}\text{Nss}}. \] (5.30)

Here \( \gamma \) is a lattice parameter in the \( \tilde{z} \) direction that eventually depends on \( a \) and \( b \) in order that both the sets \((z\text{Nss}, \tilde{z}\text{Nss}, \hat{z}\text{Nss}, \tilde{w}\text{Nss})\) and \((z\text{Nss}, \tilde{z}\text{Nss}, \hat{z}\text{Nss}, \tilde{w}\text{Nss})\) satisfy the lSKdV equation. This possibly yields another relation between \( z\text{Nss} \) and \( \tilde{z}\text{Nss} \), so that one can obtain an explicit expression of \( z\text{Nss} \) from that of \( w\text{Nss} \). Since the lSKdV equation is fractional-linearly invariant, \( z\text{Nss} \) and \( \tilde{z}\text{Nss} \) are simply connected by a Möbius transformation.

For simplicity, we consider the non-autonomous case. Choosing \( \tilde{z}\text{Nss} = -z\text{Nss} \), and taking account of the expressions (5.18) and (5.30), one gets

\[ z\text{Nss} = \rho s^{2n - 2m} \Theta_N \Theta_N^{-1}. \] (5.31)

Here \( \rho \) is a constant (absorbing other constant factors), and \( s \) and \( t \) are the parameters appearing in the seed solution for \( w\text{Nss} \). Note that this corresponds to the \( N \)-soliton solution of the lSKdV equation with power background obtained in [23, 24].

6 Concluding remarks

By using and developing Darboux transformations for certain differential and difference equations, we manage to derive two families of integrable equations including their continuous, semi-discrete and lattice versions. These are the KdV family, including the KdV

\[^4 \text{These substitutions can be obtained through some dimensional analysis.}\]
equation, the dressing chain equation and the lattice potential KdV equation; and the modified KdV or sine-Gordon family, including the modified KdV and sine-Gordon equation, the semi-discrete sinh-Gordon and sine-Gordon equation, and the lattice potential modified KdV equation and the lattice Schwarzian KdV (cross-ratio) equation. Darboux transformations represent in all cases an integrable discretisation process. The three-dimensional consistency properties of the lattice equations are well fitted into the Darboux scheme. The associated discrete Crum’s theorems allow us to construct explicit solutions of those equations—results that are consistent with the Cauchy matrix approach [23] and the bilinearisation approach [24].

Natural continuations of the present work are to extend our results to 1) other multi-dimensionally consistent equations from the Adler-Bobenko-Suris classification; 2) higher order differential and difference equations. Note that the decomposition (or factorisation) methods used to derive Darboux transformations for second difference equations can also be found in general situations [31]. It would be of interest to examine whether discrete Crum’s theorems exist for those cases.

In the context of one-dimensional quantum mechanics, discrete Crum’s theorem for the tridiagonal discrete Schrödinger equation (obtained using direct discretisation of the Schrödinger equation (1.1)) is known [32] (similar results can be found in [33] with an integrable approach). In [32], the authors questioned whether their version of discrete Crum’s theorem have applications in discrete integrable systems. The second order difference equations (2.1 and 4.1) we consider in this paper are also of Schrödinger-type. Reversely, one natural question would be whether the discrete Crum’s theorems we obtained in the context of discrete integrable systems can be connected to “discrete” quantum mechanics. Note that we are also able to construct various potentials for the discrete Schrödinger equation (2.1) that have continuous (quantum) analogue.5

After uploading this work, we received a comment from Prof. Liu, who, with his collaborator, have already had a different version of Crum’s theorem for the discrete Schrödinger equation (2.1) [34] (the paper is in Chinese). This suggests that (non-trivial) Darboux transformations and the associated Crum’s theorem are actually not unique. It would be interesting to understand the connection of their results to integrable discrete systems, and also look for other possible Darboux transformations for the systems we consider in this paper.

Lastly, it is interesting to observe that no new difference equations can be obtained using Darboux transformations for the difference equations we considered in this paper (Eqs. (2.1) and (4.1)). This is because the difference systems created by Darboux transformations remain the same as the original difference equations—a fact that is believed to be related to the discrete integrability and the multi-dimensional consistency property of the models.

Acknowledgements

C. Zhang is supported by NSFC (No.11601312) and Shanghai Young Eastern Scholar program (2016-2019). L. Peng is partially supported by JSPS Grant-in-Aid for Scientific Research (16KT0024) and Waseda University Special Research Project (2017K-170). D.J. Zhang is supported by NSFC (No.11371241, 11631007). The authors would like to thank Prof. F.W. Nijhoff for private communications.

5Some results will be reported soon.
References

[1] G. Darboux “Sur une Proposition Relative aux équations Linéaires”, *Comptes Rendus de l’Académie des Sciences*, vol. 94, (1882): pp. 1456-1459.

[2] G. Darboux, “Leçons sur la théorie générale des surfaces”, *Gauthier Villars Et Fils*, vol. 2, (1866): pp. 210.

[3] M.M. Crum, “Associated Sturm-Liouville Systems”, *The Quarterly Journal of Mathematics*, 6(1), (1955): pp. 121–127.

[4] V.B. Matveev, M.A. Salle, “Darboux Transformations and Solitons”, *Springer-Verlag, Berlin*, 1991.

[5] P.D. Lax, “Integrals of nonlinear equations of evolution and solitary waves”, *Communications on Pure and Applied Mathematics, 21*(5), (1968): pp. 467–490.

[6] H.D. Wahlquist, F.B. Estabrook, “Bäcklund transformation for solutions of the Korteweg-de Vries equation”, *Physical review letters*, 31(23), (1973): pp. 1386.

[7] A.P. Veselov, A.B. Shabat, “Dressing chains and the spectral theory of the Schrödinger operator”, *Functional Analysis and Its Applications*, 27(2), (1993): pp. 81–96.

[8] D. Levi, R. Benguria, “Bäcklund transformations and nonlinear differential difference equations”, *Proceedings of the National Academy of Sciences*, 77(9), (1980): pp. 5025–5027.

[9] D. Levi, “Nonlinear differential difference equations as Bäcklund transformations”, *Journal of Physics A: Mathematical and General*, 14(5), (1981): pp. 1083.

[10] A. Shabat, “Dressing chains and lattices”, *Proceedings of the Workshop on Nonlinearity, Integrability and All That: Twenty Years after NEEDS*, vol. 79, pp. 331–342, 2000.

[11] C. Rogers, W.K. Schief, “Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory”, *Cambridge University Press, Cambridge*, 2002.

[12] C.H. Gu, H.S. Hu, Z.X. Zhou, “Darboux Transformation in Soliton Theory and its Geometric Applications”, *Springer Science & Business Media, vol. 26*, 2006.

[13] J.J. Duistermaat, F.A. Grünbaum, “Differential equations in the spectral parameter”, *Communications in Mathematical Physics*, 103(2), (1986): pp. 177–240.

[14] E.V. Doktorov, B.L. Sergey, “A Dressing Method in Mathematical Physics”, *Springer Science & Business Media*, 2007.

[15] A.P. Fordy, J. Gibbons. “Factorization of operators I. Miura transformations”, *Journal of Mathematical Physics*, 21(10), (1980): pp. 2508-2510.

[16] A.P. Fordy, J. Gibbons. “Factorization of operators II”, *Journal of Mathematical Physics*, 22(6), (1981): pp. 1170-1175.
[17] A.I. Bobenko, Y.B. Suris, “Discrete Differential Geometry: Integrable Structure”, *Graduate Studies in Mathematics*, vol. 98, AMS, 2008.

[18] J. Hietarinta, N. Joshi, F.W. Nijhoff, “Discrete Systems and Integrability”, *Cambridge University Press, Cambridge*, 2016.

[19] F.W. Nijhoff, “Lax pair for the Adler (lattice Krichever-Novikov) system”, *Physics Letters A*, 297(1), (2002): pp. 49–58.

[20] A.I. Bobenko, Y.B. Suris, “Integrable systems on quad-graphs”, *International Mathematics Research Notices*, 2002(11), (2002): pp. 573–611.

[21] V.E. Adler, A.I. Bobenko, Y.B. Suris, “Classification of integrable equations on quad-graphs. The consistency approach”, *Communications in Mathematical Physics*, 233(3), (2003): pp. 513–543.

[22] V.E. Adler, A.I. Bobenko, Y.B. Suris, “Discrete nonlinear hyperbolic equations. Classification of integrable cases”, *Functional Analysis and Its Applications*, 43(1), (2009): pp. 3–17.

[23] F.W. Nijhoff, J. Atkinson, J. Hietarinta, “Soliton solutions for ABS lattice equations: I. Cauchy matrix approach”, *Journal of Physics A: Mathematical and Theoretical*, 42(40), (2009): pp. 404005.

[24] J. Hietarinta, D.J. Zhang, “Soliton solutions for ABS lattice equations: II. Casoratians and bilinearization”, *Journal of Physics A: Mathematical and Theoretical*, 42(40), (2009): pp. 404006.

[25] F. Nijhoff, H. Capel, “The discrete Korteweg-de Vries equation”, *Acta Applicandae Mathematica*, 39(1-3), (1995): pp. 133-158.

[26] C.W. Cao, G.Y. Zhang, “Lax Pairs for Discrete Integrable Equations via Darboux Transformations”, *Chinese Physics Letters*, 29(5), (2012): pp. 050202.

[27] V.B. Matveev, “Darboux transformation and the explicit solutions of differential-difference and difference-difference evolution equations I”, *Letters in Mathematical Physics*, 3(3), (1979): pp. 217–222.

[28] M. Boiti, F. Pempinelli, B. Prinari, A. Spire, “An Integrable Discretization of KdV at Large Times”, *Inverse Problems*, 17(3), (2001): pp. 515–526.

[29] S. Butler, N. Joshi, “An inverse scattering transform for the lattice potential KdV equation”, *Inverse Problems*, 26(11), (2010): pp. 115012.

[30] R. Hirota, “Nonlinear partial difference equations III; Discrete sine-Gordon equation”, *Journal of the Physical Society of Japan*, 43(6), (1977): pp. 2079-2086

[31] A. Dobrogowska, G. Jakimowicz, “Factorization method applied to the second order difference equations”, *Applied Mathematics Letters*, 74, (2017): pp. 161–166.

[32] S. Odake, R. Sasaki, “Crum’s theorem for discrete quantum mechanics”, *Progress of Theoretical Physics*, 122(5), (2009): pp. 1067–1079.
[33] P. Gaillard, V.B. Matveev, “Wronskian and Casorati determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions”, *Journal of Physics A: Mathematical and Theoretical*, 42(40), (2009): pp. 404009.

[34] Q. Liu, Y. Wang, “A Note on a Discrete Schrödinger Spectral Problem and Associated Evolution Equations”, *Acta Mathematica Scientia*, 26(5), (2006): pp. 773–777.