Research Article

Observer-Based \( H^\infty \) Fuzzy Synchronization and Output Tracking Control of Time-Varying Delayed Chaotic Systems

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1.Introduction

Chaotic behavior exists in many systems, in some cases being problematic where control is needed. In other cases, chaotic behavior is to be induced, whereas synchronization is the goal. The results [1, 2] represent the synchronization as a T-S (Takagi–Sugeno) fuzzy representation. The result [3] achieves robust state estimation influenced by time delay. The result [4] represents that synchronization is achieved by solving LMIs (linear matrix inequalities). However, from a practical point of view, a unified approach to synchronization and control of both discrete-time and continuous chaotic systems considering immeasurable state, time-varying delay, different antecedent variables, and disturbance is needed.

The T-S fuzzy representation of dynamic models [5, 6] renders nonlinear systems as a parametric combination of several linear subsystems. Linear system control theory can then be applied to each subsystem, where once combined provides a total control solution to the nonlinear system. Finally, control criteria are represented as linear matrix inequality problems (LMIPs) [7–11]. For states immeasurable, results [12, 13] use observers to estimate states. For tracking, the result [14] uses a robust approach where the command signal is induced into the closed-loop system as disturbance with tracking error residual attenuated. For regulation, the results [15, 16] use feed-forward compensator from linear control theory. The results [17] use linear regulation theory for T-S represents linear...
subsystems. The results [18–20] have goals of static or varying output. Note that [21] mentions the challenge of coupled plant and controller rules that ultimately leads to complex controller design. The aforementioned problem has been overcome by fuzzy gain scheduling [22] and integral control schemes [23] for regulating output of DC-DC converters as an example. While the result [23] achieves a satisfactory performance, the gain design relies on the pole assignment technique.

Time delay exists in many practical systems leading to an unstable and suboptimal performance posing a complex control problem. Previous results in control can be categorized as feedback with delay [24–28] or without delay [29, 30]. Note that feedback without delay methods do not need any information of the delay where approaches are more suitable for practical applications. Both stability criteria are then represented into linear matrix inequality problems (LMIPs) [7]. However, for time-delay systems applications, most literature deals with stability to an equilibrium.

Here, we introduce a robust observer-based synchronization and control for chaotic systems with time-varying delay and disturbances in both continuous-time and discrete-time domain. First, the chaotic system with time-varying delay is represented as a T–S fuzzy model. Next, a set of reference variables is formulated based on generalized kinematic constraints. Third, combining the synthesis of the reference variable controller and strategy of the observer-based fuzzy control, the overall output tracking controller is developed.

Using Lyapunov’s method, stability conditions are derived using Lyapunov’s stability analysis. The proposed controller is finally validated by satisfactory numerical results.

The remainder of this paper is organized as follows. In the problem statement, we formulate the overall control problem. In the master-slave synchronization, we design the observer for master-slave synchronization. In the output tracking control, we design the output tracking controller. In the guaranteed $H_{\infty}$ performance, we discuss $H_{\infty}$ performance of the controller. In the tracking controller realization, we realize the tracking controller based on generalized kinematic constraints. The simulation results, we carry out numerical simulation to verify the validity of the proposed scheme. Finally, some concluding remarks are made in the conclusions.

2. Problem Formulation

Based on previous modeling methods [5, 31], chaotic systems can be exactly represented by Takagi–Sugeno (T–S) IF-THEN fuzzy rules. Consider a class of chaotic systems with time-varying delay and disturbance which are described by the following T–S fuzzy rules:

Plant Rule $i$:

IF $z_i(t)$ is $F_{ji}$ and $\cdots$ and $z_j(t)$ is $F_{ji}$, THEN

$$sx(t) = A_{ix}(t)z_i(t) + A_{dx}(t - \tau(t)) + Bu(t) + \Gamma + w(t),$$
$$y(t) = Cx(t) + v(t),$$
$$y_c(t) = Dx(t),$$
$$x(t) = \phi(t), \quad t \in [-\tau_0, 0],$$

where $sx(t)$ denotes $x(t + 1)$ for discrete-time system (DTS) or $\dot{x}(t)$ for continuous-time system (CTS); $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and $y_c(t) \in \mathbb{R}$ are accordingly the state, control input, measured output, and controlled output; $A_i$, $A_{di}$, $B$, $C$, and $D$ are matrices with proper dimensions; $z_i(t)$ and $z_j(t)$ are the antecedent variables as a combination of states; $F_{ji}$ ($j = 1, 2, \ldots, f$) are fuzzy sets; $i = 1, 2, \ldots, r$, with $r$ as the number of fuzzy rules; $\phi(t)$ is the initial condition; $\tau(t)$ is the time-varying delay; $w(t)$ and $v(t)$ are accordingly the modeling error (or external disturbance) and measured disturbance; and $\Gamma$ is a constant term.

The inferred output (using the singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier) of the fuzzy system is

$$sx(t) = \sum_{i=1}^{r} \mu_i(z(t))A_{ix}(t)z_i(t) + A_{dx}(t - \tau(t)) + Bu(t) + \Gamma + w(t),$$
$$y(t) = Cx(t) + v(t),$$

where $z(t) = [z_1(t) z_2(t) \cdots z_n(t)]^T$ and $\mu_i(z(t)) = w_i(z(t))/\sum_{i=1}^{r} w_i(z(t))$ with $w_i(z(t)) = \prod_{j=1}^{f} F_{ij}(z(t))$. Here $\sum_{i=1}^{r} \mu_i(z(t)) = 1$ for all $t$, where $\mu_i(z(t)) > 0$, for $i = 1, 2, \ldots, r$, are the weights normalized. For system (21), we straightforwardly assume $B = [b_0 0 \cdots 0]^T$ with scalar $b_0 \neq 0$ and $x \in \Omega$ with a region of interest $\Omega$.

To consider immeasurable states for output tracking control, define $\tilde{x}(t)$ as state estimation from the measured output. To transform the output tracking problem to a stabilization one, we introduce a set of target variables $x_d(t)$ governed by generalized kinematic constraints. We will discuss the analysis and synthesis of the observer and target variables in Sections 4 and 6, respectively. The following assumptions are held for the remainder of the paper.

Assumption 1. Time-varying delay $\tau(t)$ is unknown but bounded by $\tau_0$, i.e., $0 < \tau(t) \leq \tau_0$, with average constant time delay $\tau$ with $0 < \tau \leq \tau_0$.

Assumption 2. There exists a known $\bar{q} > 1$, such that $\|x(t - \tau(t))\| \leq \bar{q}\|x(t)\|$ for $t \in (0, \tau_0]$.

The output synchronization achieves an $H_{\infty}$ performance in accordance to the error $e(t) = x(t) - \tilde{x}(t)$ as follows:
Lemma 1. For any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ and matrix $\mathbf{D}$ with appropriate dimension, the relationship
\[
2\mathbf{x}^T \mathbf{D} \mathbf{y} \leq \mathbf{x}^T \mathbf{x} + \mathbf{y}^T \mathbf{D}^T \mathbf{D} \mathbf{y},
\]
is satisfied.

3. Master-Slave Synchronization

In this section, we consider master-slave synchronization with immeasurable partial states and antecedent variables. Here, the chaotic system (12) is the master system. We then establish a fuzzy synchronizer to reconstruct the states and antecedent variables. The IF-THEN rules of the fuzzy synchronizer can be written as

\begin{align*}
\text{Fuzzy Synchronizer Rule } i: \\
\text{IF } \bar{z}^T (t) = F_{i1} \text{ and } \cdots \text{ and } \bar{z}^T (t) = F_{fi} \text{ THEN } \\
\bar{s} \tilde{x} (t) = A_i \tilde{x} (t) + A_{id} \tilde{x} (t - T) + L_i (y (t) - \bar{y} (t)) + \Gamma, \\
\bar{y} (t) = C_i \tilde{x} (t),
\end{align*}

where $\bar{y} (t)$ estimates $y (t)$; $L_i$ are the gains of the observer; and $\bar{z}^T (t) = \tilde{z}^T (t)$ are the antecedent variables dependent on $\tilde{x} (t)$. In the fuzzy synchronizer design, we require the average time delay $T$ in Assumption 1. The inferred output
\[
\begin{align*}
\bar{s} \tilde{x} (t) &= \sum_{i=1}^{f} \mu_i (\bar{z} (t)) [A_i \tilde{x} (t) + A_{id} \tilde{x} (t - T) + L_i (y (t) - \bar{y} (t))] + \Gamma, \\
\bar{y} (t) &= \sum_{i=1}^{f} \mu_i (\bar{z} (t)) C_i \tilde{x} (t).
\end{align*}
\]

Therefore, the estimation error
\[
\sum_{i=1}^{f} \mu_i (\bar{z} (t)) [A_i \tilde{x} (t) + A_{id} \tilde{x} (t - T) + L_i (y (t) - \bar{y} (t))] + \Gamma.
\]
where $\overline{w}(t) = \omega(t) + h_0(t)$ and unknown time delay lead to

$$h_0(t) = \sum_{i=1}^r \mu_i(\bar{z}(t))[A_i-L_i \xi(t)]A_d[x(t-\tau(t)) - x(t-\tau)];$$

where immeasurable antecedent variables lead to

$$h_1(t) = \sum_{i=1}^r (\mu_i(z(t)) - \mu_i(\bar{z}(t)))[A_i x(t) + A_d x(t-\tau(t))];$$

and $\overline{e}_i(t) = \left[v_i(t) \overline{w}(t)\right]^T$.

Note that Assumption 2 is satisfied when the state $x(t)$ is bounded in the interest region. Although mismatched antecedent variables lead to the residue terms $h_1(t)$, the membership function satisfies Lipschitz-like conditions [31], where

$$h_1^T(t)h_1(t) \leq \overline{e}^T(t)U_1^T U_1 \overline{e}(t),$$

with a known matrix function $U_1$ which is $x(t)$ dependent and $x(t-\tau(t))$ (note that $x_d(t)$ and $x_d(t-\tau(t))$ are bounded in the region of interest).

Properly chosen observer gains $L_i$ can attenuate the undesired terms $h_1(t)$ affecting performance. Note that in practical implementation, the time delay $\tau$ may be hard to determine. We may therefore use a reasonable $U_1$ and combine resulting errors to $\omega_1(t)$.

Consider a Lyapunov–Krasovskii function candidate as follows:

(i) DTS:

$$V(t) = \overline{e}^T(t) Pe(t) + \sum_{q=1}^r \overline{e}^T(t-q) \overline{S} \overline{e}^T(t-q).$$

(ii) CTS:

$$V(t) = \overline{e}^T(t) \overline{P} e(t) + \int_{t-\tau}^t \overline{e}^T(\nu) \overline{S} \overline{e}^T(\nu) d\nu.$$

**Theorem 1.** For DTS (10), if there exist symmetric and positive definite matrices $\overline{P}$, and $\overline{S}$, solved from matrix inequalities in (14) for all $i$, system (10) is therefore stable with guaranteed $H_{\infty}$ synchronization performance in (3) for a given attenuation level $\rho^2$.

$$\begin{bmatrix}
G_i^T \overline{P} G_i + \overline{S} - \overline{P} & (\ast) & (\ast) \\
\overline{M}_i^T \overline{P} G_i & \overline{M}_i^T \overline{P} M_i - \overline{S} & (\ast) \\
E_i^T \overline{P} G_i & E_i^T \overline{P} M_i & E_i^T \overline{P} E_i - \rho^2 I
\end{bmatrix} < 0,$$  \hspace{1cm} (14)

where $(\ast)$ denotes the transposed elements in the symmetric position and $\overline{G}_i = A_i - L_i C_i$, $\overline{M}_i = A_d$, $\overline{H}_{1i} = 2G_i^T \overline{P}G_i + \overline{S} - \overline{P} - \overline{Q} + 4U_1^T \overline{P}U_1$, $\overline{H}_{12} = 2\overline{M}_i^T \overline{P}M_i - \overline{S}$, and $\overline{H}_{13} = 2E_i^T \overline{P}E_i - \rho^2 I$.

**Proof.** The proof is given in Appendix A. \hfill $\Box$

**Theorem 2.** For the CTS (10), if there exist a symmetric and positive definite matrix $\overline{P}$, and $\overline{S}$, solved from matrix inequalities in (15) for all $i$, system (10) is therefore stable with guaranteed $H_{\infty}$ synchronization performance in (4) for a given attenuation level $\rho^2$.

$$\begin{bmatrix}
G_i^T \overline{P} G_i + \overline{Q} + \overline{S} + \overline{P} \overline{P} + 4U_1^T \overline{P}U_1 & (\ast) & (\ast) \\
\overline{M}_i^T \overline{P} & -\overline{S} & (\ast) \\
E_i^T \overline{P} & 0 & -\rho^2 I
\end{bmatrix} < 0.$$

(15)

**Proof.** The proof is given in Appendix B.

In light of the aforementioned analysis, the important task of the synchronization problem is to find the common $\overline{P}$ and $\overline{S}$ from Theorems 1 and 2. However, analytically determining $\overline{P}$ and $\overline{S}$ is nontrivial, where the details of formulating the LMIs will be given in the upcoming sections. \hfill $\Box$

### 4. Output Tracking Control

Here, we consider immeasurable states and antecedent variables. Along with time-varying delay, estimated error dynamics are then coupled with tracking error dynamics. First, we formulate an observer to estimate the states and antecedent variables with IF-THEN rules:

Observer Rule $i$:

IF $\xi(t)$ is $F_{i1}$ and $\cdots$ and $\xi_j(t)$ is $F_{i j}$ THEN

$s(x(t)) = A_i \hat{x}(t) + A_d \hat{x}(t-\tau) + Bu(t) + L_i(y(t) - \hat{y}(t)) + \Gamma,$

$\hat{y}(t) = C_i \hat{x}(t).$  \hspace{1cm} (16)

In the observer design, we require the average time delay $\overline{\tau}$ in Assumption 1. The inferred output
\[
s\ddot{x}(t) = \sum_{i=1}^{r} \mu_i(\ddot{z}(t))\text{sum}[A_i\ddot{x}(t) + A_{d_i}\ddot{x}(t - \tau) + Bu(t) + L_i(y(t) - \ddot{y}(t))] + \Gamma
\]

\[
\ddot{y}(t) = \sum_{i=1}^{r} \mu_i(\ddot{z}(t))C_i\ddot{x}(t).
\]

The IF-THEN rules of the target variables can be written as

Desired variables Rule i:

\[
sx_d(t) = A_i x_d(t) + A_{d_i} x_d(t - \tau) + Bu_r(t) + \Gamma
\]

\[
y_d(t) = C_i x_d(t),
\]

where \(z_{d1}(t) \sim z_{d_f}(t)\) are the antecedent variables dependent on \(x_d(t)\) and \(u_r(t)\) is the desired control force. The inferred output

\[
sx_d(t) = \sum_{i=1}^{r} \mu_i(z_d(t))[A_i x_d(t) + A_{d_i} x_d(t - \tau) + Bu_r(t)] + \Gamma
\]

\[
y_d(t) = \sum_{i=1}^{r} \mu_i(z_d(t))C_i x_d(t),
\]

where \(z_d(t) = [z_{d1}(t) z_{d2}(t) \cdots z_{df}(t)]^T\). The overall controller

\[
u(t) = u_r(t) + u_k(t),
\]

where \(u_k(t)\) is the combined controller for the linear sub-systems. The controller rule follows:

Controller Rule i:

\[
u_x(t) = -K_i(\ddot{x}(t) - x_d(t)),
\]

where \(K_i\) are the gains of control. The controller inferred output

\[
u_k(t) = -\sum_{i=1}^{r} \mu_i(\ddot{z}(t))K_i(\ddot{x}(t) - x_d(t)).
\]

The above is the controller analysis whereas the controller realization will be discussed in Section 6.

The tracking error dynamics:

\[
sx(x) = \sum_{i=1}^{r} \mu_i(z(t))[(A_i - BK_i)x_k(t) + A_{d_i}x_k(t - \tau) + L_i Ce(t) + L_i v(t) + h_2(t)],
\]

where the mismatched antecedent variables between observer and controller are

\[
h_2(t) = \sum_{i=1}^{r} (\mu_i(\ddot{z}(t)) - \mu_i(z_d(t)))[A_i x_d(t) + A_{d_i} x_d(t - \tau)].
\]

We can now express the system

\[
sx_e(t) = \sum_{i=1}^{r} \mu_i(z(t))[G_i x_e(t) + M_i x_e(t - \tau) + E_i w_e(t) + h(t)],
\]

where

\[
x_e(t) = \begin{bmatrix} e(t) \\ x_e(t) \end{bmatrix}, \quad G_i = \begin{bmatrix} A_i - L_i C & 0 \\ L_i C & A_i - B K_i \end{bmatrix},
\]

\[
M_i = \begin{bmatrix} A_{d_i} & 0 \\ 0 & A_{d_i} \end{bmatrix}, \quad E_i = \begin{bmatrix} -L_i & I \\ L_i & 0 \end{bmatrix},
\]

\[
w_e(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}, \quad \bar{h}(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}.
\]

Remark 1. If all antecedent variables are measurable, i.e., \(\mu_i(\ddot{z}(t)) = \mu_i(z(t))\), the target variable

\[
sx_d(t) = \sum_{i=1}^{r} \mu_i(z(t))[A_i x_d(t) + A_{d_i} x_d(t - \tau) + B(u(t) - u_k(t))] + \Gamma,
\]

and controller

\[
u_k(t) = -\sum_{i=1}^{r} \mu_i(z(t))K_i(\ddot{x}(t) - x_d(t)),
\]

and system (25) therefore becomes

\[
sx_e(t) = \sum_{i=1}^{r} \mu_i(z(t))[G_i x_e(t) + M_i x_e(t - \tau) + E_i w_e(t)],
\]

with \(\bar{h}(t) = 0\).

Remark 2. If all antecedent variables are measurable and disturbance free, i.e., \(\mu_i(\ddot{z}(t)) = \mu_i(z(t)), w(t) = 0\) and \(v(t) = 0\), the target variables are (27) and (28) with system
\[ s x_c(t) = \sum_{i=1}^{r} h_i(z(t))[G_i x_c(t) + M_i x_c(t - \tau)] + \tau \tau E_c(t), \]

where
\[ E_c = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \]
\[ w_c(t) = \begin{bmatrix} 0 \\ \overline{w}(t) \end{bmatrix}, \]

with \( \overline{w}(t) = h_0(t) \).

**Remark 3.** Consider \( \tau(t) = \tau \) for error system (30); we conclude that the biased term \( \overline{w}(t) \) will vanish at \( \tau(t) = \tau \) reducing the system to
\[ s x_c(t) = \sum_{i=1}^{r} h_i(z(t))[G_i x_c(t) + M_i x_c(t - \tau)]. \]

This means that known time delay leads to error (32).

From the upcoming controller design, Assumption 2 is satisfied when the state \( x(t) \) is bounded in the interest region. Although mismatched antecedent variables lead to the residue terms \( h_1(t) \) and \( h_2(t) \), the membership functions assume Lipschitz-like conditions [31], where
\[ h_1^T(t) h_1(t) \leq e^T(t) U_1^T U_1 e(t), \]

with known matrix function \( U_1 \) related to \( x(t) \) and \( x(t - \tau(t)) \); and
\[ h_2^T(t) h_2(t) \leq x_h^T(t) U_2^T U_2 x_h(t), \]

with existing matrix function \( U_2 \) (note that \( x_d(t) \) and \( x_d(t - \tau(t)) \) are bounded in the interest region).

Summarizing the above, the inequality
\[ \overline{h}^T(t) \overline{h}(t) \leq x_c^T(t) (t) \overline{U}^T \overline{U} x_c(t), \]

where \( \overline{h}(t) = [h_1^T(t) h_2^T(t)]^T \) and \( \overline{U} = \text{block-diag} \ (U_1, U_2) \).

Properly chosen controller gains \( K_i \) and observer gains \( L_i \) can attenuate the undesired terms \( h_1(t) \) and \( h_2(t) \) affecting control performance. Note that in practical implementation, the time delay \( \tau \) may be hard to determine. We may therefore choose a reasonable large \( \overline{U} \) and merge resulting errors to the term \( E_c w_c(t) \).

### 5. Guaranteed \( H_\infty \) Performance

For the error system (25), mismatched antecedent variables lead to the residues of \( h_1(t) \) and \( h_2(t) \). Assuming that the membership functions satisfy a Lipschitz-like condition, we carry out further stability analysis on these bias as follows.

Consider a Lyapunov–Krasovskii function candidate as follows:

\[ V(t) = x_c^T(t) P x_c(t) + \int_{t-\tau}^{t} x_c^T(v) S x_c(v) dv. \]

**Theorem 3.** For DTS (25), if there exist symmetric and positive definite matrices \( P, S \), and gain matrices \( K_i \) \( (i = 1, 2, \ldots, r) \) solved from matrix inequalities in (38) for all \( i \), the system (25) is therefore stable with guaranteed \( H_\infty \) tracking control performance in (5) for a given attenuation level \( \rho^2 \).

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**Theorem 4.** For the CTS system (29), if there exist a symmetric and positive definite matrix \( P, S \), and gain matrices \( K_i \) \( (i = 1, 2, \ldots, r) \) solved from matrix inequalities in (39) for all \( i \), the system (25) is therefore stable with guaranteed \( H_\infty \) tracking control performance in (6) for a given attenuation level \( \rho^2 \).

**Corollary 1.** Considering all antecedent variables are measurable, Remark 1 leads to \( \mu_i(\overline{z}(t)) = \mu_i(z(t)) \).

For DTS (29), if matrix inequalities in (40) for all \( i \)
\[ [G_i^T P G_i + S - P - Q (\ast) (\ast) \]
\[ M_i^T P G_i \]
\[ E_i^T P G_i \]

have common solutions \( P > 0 \) and \( S > 0 \), system (29) is therefore stable with guaranteed \( H_\infty \) tracking control performance (5) for a given attenuation level \( \rho^2 \).

For the CTS system (29), if matrix inequalities (41) for all \( i \)
have common solutions $P > 0$ and $S > 0$, system (29) is therefore stable with guaranteed $H_{\infty}$ tracking control performance (6) for a given attenuation level $\rho^2$.

**Corollary 2.** Considering all antecedent variables are measurable without disturbance, according to Remark 2, the terms $\mu_i(\tilde{z}(t)) = \mu_i(z(t))$, $w(t) = 0$, and $v(t) = 0$. For DTS (30), if matrix inequalities in (42) for all $i$

$$
\begin{bmatrix}
G_i^T P + PG_i + Q + S & (\ast) & (\ast) \\
M_i^T P & -S & (\ast) \\
E_i^T P & 0 & -\rho^2 I \\
\end{bmatrix} > 0,
$$

have common solutions $P > 0$ and $S > 0$, system (29) achieves guaranteed $H_{\infty}$ tracking control performance (5) for a given attenuation level $\rho^2$.

In system (30) for CTS, the matrix inequalities in (43) for all $i$

$$
\begin{bmatrix}
G_i^T P + PG_i + Q + S & (\ast) & (\ast) \\
M_i^T P & -S & (\ast) \\
E_i^T P & 0 & -\rho^2 I \\
\end{bmatrix} < 0,
$$

have common solutions $P = P^T > 0$ and $S = S^T > 0$, then guaranteed $H_{\infty}$ tracking control performance (6) is achieved for a given attenuation level $\rho^2$.

**Corollary 3.** Considering all antecedent variables are measurable, without disturbance and exactly known time delay, then according to Remark 3, the terms $\mu_i(\tilde{z}(t)) = \mu_i(z(t))$, $w(t) = 0$, $v(t) = 0$, and $\tau(t) = \overline{\tau}$.

In system (32) for DTS, if $P$ and $S > 0$ are the common solution of the matrix inequalities in (44) for all $i$

$$
\begin{bmatrix}
G_i^T P + PG_i + S - P - Q & (\ast) \\
M_i^T P & -S \\
E_i^T P & 0 \\
\end{bmatrix} > 0,
$$

then system (32) is asymptotically stable. In system (32) for DTS, if $P = P^T > 0$ and $S = S^T > 0$ are the common solution of the matrix inequalities in (44) for all $i$

$$
\begin{bmatrix}
G_i^T P + PG_i + Q + S & (\ast) \\
M_i^T P & -S \\
\end{bmatrix} < 0,
$$

then system (32) is asymptotically stable.

Improved tracking performance can be achieved by formulating the following minimization problem:

(i) DTS:

$$
\min_{\rho^2, P, S} \frac{\rho^2}{P, S},
$$

subject to $P > 0, S > 0$ and (38).

(ii) CTS:

$$
\min_{\rho^2, P, S} \frac{\rho^2}{P, S},
$$

subject to $P > 0, S > 0$ and (39).

In light of the above, we must solve the common $P$ and $S$ solution from the minimization problem (46) or (47). However, to analytically determine that $P$ and $S$ are nontrivial, we therefore discuss the procedure in detail.

First, we consider DTS. From Schur complement, the inequality

$$
\begin{bmatrix}
P - S - 4\overline{U}^T P\overline{U} & (\ast) & (\ast) & (\ast) \\
0 & S & (\ast) & (\ast) \\
0 & 0 & \rho^2 I & (\ast) \\
PG_i & PM_i & PE_i & \frac{1}{2} P \\
\end{bmatrix} > 0.
$$

Given $\rho > 0$, $Q = Q^T > 0$, and $\overline{U}$, we must find the observer gain $L_i$, controller gain, $K_i$ for common $P > 0$, and $S > 0$ satisfying (48). In summary, a three-step procedure is given to solve (48). First, we assume $P$ and $\Lambda$ are in diagonal block form; i.e., $P =$block-diag $(P_1, P_2)$ and $S =$block-diag $(S_1, S_2)$. We expand the matrix inequalities (48) to arrive with the inequalities

$$
\begin{bmatrix}
P_1 - Q_1 - S_1 - 4U_1^T P_1 U_1 & (\ast) & (\ast) \\
0 & S_1 & (\ast) & (\ast) \\
0 & 0 & \rho^2 I & (\ast) \\
0 & 0 & 0 & \frac{1}{2} P_1 \\
\end{bmatrix} > 0,
$$

$$
\begin{bmatrix}
P_2 - Q_2 - S_2 - 4U_2^T P_2 U_2 & (\ast) & (\ast) \\
0 & S_2 & (\ast) & (\ast) \\
0 & 0 & \rho^2 I & (\ast) \\
0 & 0 & 0 & \frac{1}{2} P_2 \\
\end{bmatrix} > 0,
$$

where $N_i = P_i L_i$. We summarize the three-step procedure for DTS as follows:

Step D1: given $\rho, Q_1$ and $U_1$, solve (49) to obtain $P_1, S_1$, and $L_i = P_i^{-1} N_i$. 

...
Step D2: given \( \rho, Q_2 \) and \( U_2 \), inequalities (50) are equivalent to
\[
\begin{bmatrix}
X_2 - \Lambda_2 & (\ast) & (\ast) & (\ast) & (\ast)
\end{bmatrix} > 0.
\]
Step D3: solve (48) with matrices satisfying \( \rho^2 I \) to obtain \( \rho \) and \( \Lambda_2 \).

Step C3: once the gains \( \Lambda_i \) are available from Step C1 and Step C2, then according to (48) there exist matrices \( P \) and \( \Lambda \) satisfying the following LMIs:
\[
\begin{bmatrix}
\Delta_i & (\ast) & (\ast) & (\ast) & (\ast) & (\ast)
\end{bmatrix} < 0,
\]
where \( \Delta_i = (A_i - L_i C)^T P_1 + (A_i - L_i C)^T Q_1 + \Lambda_i + P_1 U_i + U_1^T P_1 + (A_i - B_i K_i)^T P_2 + (A_i - B_i K_i)^T Q_2 + \Lambda_i + U_2^T U_2 \).

In an analogous manner, considering CTS, we assume \( P, S, \) and \( Q \) are diagonal block form. According to (48), the inequality
\[
\begin{bmatrix}
\Lambda_i & (\ast) & (\ast) & (\ast) & (\ast) & (\ast)
\end{bmatrix} < 0,
\]
where
\[
\begin{bmatrix}
\Delta_i & (\ast) & (\ast) & (\ast) & (\ast) & (\ast)
\end{bmatrix} < 0,
\]
where \( \Delta_i = (A_i - L_i C)^T P_1 + (A_i - L_i C)^T Q_1 + \Lambda_i + P_1 U_i + U_1^T P_1 + (A_i - B_i K_i)^T P_2 + (A_i - B_i K_i)^T Q_2 + \Lambda_i + U_2^T U_2 \).

We can arrive at the analogous results for error systems stated in Remarks 1–3 and omitted here for brevity.

Remark 4. According to the above LMI formulation procedure, we derive from (14) for DTS and (15) for CTS accordingly the LMIs:
where \( N_i = P_i L_i. \)

### 6. Tracking Controller Realization

We now address in detail the design of target variables \( x_d(t). \) According to the above discussion, we consider the worst case mismatched antecedent variables with time-varying delay. The target variable design for systems stated in Remarks 1–3 can be obtained in an analogous manner.

We determine the target variables from decomposing variables (19). We then arrive at

\[
\begin{bmatrix}
\bar{P} - \bar{Q} - \bar{S} - 4U_1^T P U_1 \\
0 & \bar{S} \\
0 & 0 & \rho^2 I
\end{bmatrix}
\begin{bmatrix}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{bmatrix}
> 0,
\]

\[
\begin{bmatrix}
P A_i \bar{N}_i C & \bar{P} A_i - \bar{N}_i \frac{1}{2} \bar{P}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_i^T P + \bar{P} A_i + \bar{Q} - N_i C - C^T N_i^T + \bar{S} + U_1^T U_1 \\
0 & \rho^2 I
\end{bmatrix}
\begin{bmatrix}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{bmatrix}
< 0,
\]

\[
\begin{bmatrix}
A_{di}^T P & -\Lambda_2 \\
-\rho^2 I & 0
\end{bmatrix}
\begin{bmatrix}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{bmatrix}
< 0,
\]

\[
\begin{bmatrix}
\bar{P}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & -I
\end{bmatrix}
\]

where \( x_d(t) \) is the \( \ell \) th (for \( \ell = 2, \ldots, n \)) element of \( x_d(t) \) and \( A_{i,\ell} \) and \( A_{di,\ell} \) are accordingly the \( \ell \) th rows of \( A_i \) and \( A_{di} \) matrices. We can therefore synthesize the control

\[
x_d(t) = \sum_{i=1}^{r} \mu_i(z_d(t)) \left[ A_i x_d(t) + A_{di} x_d(t - \tau) \right], \quad \ell = 2, \ldots, n,
\]

where \( x_d(t) \) should be properly chosen such that \( x_d(t) \) is constant.

In the following, we discuss two cases of the observer-based output tracking: (i) output regulation and (ii) time-varying output tracking (where \( y_d(t) = x_{d1}(t) \) and \( y_d(t) \neq x_{d1}(t) \)). Note that the desired state \( x_d(t) \) should be appropriately chosen such that \( x_d(t) \) is constant.

#### 6.1. Output Regulation

For the ideal condition where the desired state \( x_d \) is constant vector and \( x(t) = x(t) \) such that \( x_d(t - \tau) = x_d \) and \( x(t - \tau(t)) = x_{d1} \), we assign the desired output as \( y_d(t) = x_{d1} \) (\( x_{d1} \) is constant). This leads to \( x_{d1}(t + 1) = x_{d1} \) or \( x_{d1}(t) = 0 \), where we solve the constant desired state from equations.
(i) DTS:
\[
\bar{x}_d(t) = \sum_{i=1}^{r} \left( \mu_i(z_d) \right) \left[ A_{i} x_d(t) + A_{di} x_d(t - \tau) \right], \quad \ell = 2, \ldots, n.
\]

(ii) CTS:
\[
\sum_{i=1}^{r} \left( \mu_i(z_d) \right) \left[ A_{i} x_d(t) + A_{di} x_d(t - \tau) \right] = 0, \quad \ell = 2, \ldots, n,
\]
where \( x_d(t) = \bar{x}_d \).

6.2. Time-Varying Output Tracking. Consider two cases:

(i) \( y_d(t) = x_{d1}(t) \). Consider a smooth desired output \( y_d(t) = x_{d1}(t) \); we can then solve the target variables from (57). Once \( x_{d1}(t + 1) \) or \( \dot{x}_{d1}(t) \) is available, we can implement the controller as (58).

(ii) \( y_d(t) = x_{d2}(t), i \neq 1 \). Since the desired output \( y_d(t) \) is not \( x_{d1}(t) \), we may derive \( x_{d1}(t + 1) \) or \( \dot{x}_{d1}(t) \) from (57). Note that direct application of signal \( x_{d1}(t + 1) \) and \( \dot{x}_{d1}(t) \) should be avoided, since the control force \( u(t) \) appear in \( x_{d2}(t + 1) \) and \( \dot{x}_{d1}(t) \). To cope with this problem, we can use an approximate signal for CTS \( \dot{x}_{d1}(t) = (x_{d1}(t) - x_{d1}(t - \Delta T))/\Delta T \), where \( \Delta T \) is the sampling period and for DTS \( x_{d1}(t + 1) = (x_{d1}(t) - \tau x_{d1}(t - 1)) \). Note that we assume the desired output is slow time varying when applying the approximation.

We summarize the overall design procedure for output tracking control as follows:

- Step G1: construct T-S fuzzy model for the nonlinear time-delay system as (2)
- Step G2: given an attenuation level \( \rho \) and matrix \( Q \) and \( U \), then follow the three-step procedure to obtain controller and observer gains
- Step G3: synthesize the target variables from (57) and (59) based on a specified form of desired output
- Step G4: implement the observer (17) and controller (58)

7. Simulation Results

To verify the theoretical derivations, we consider both discrete and continuous chaotic systems as examples. We consider here only the most practical case where immeasurable states are considered with mismatched antecedent variables.

Example 1. Consider a discrete-time 3-dimensional chaotic system as follows:
\[
\begin{align*}
\dot{x}_1(t + 1) & = a x_1^2(t) + x_2(t) + c + e x_1(t - \tau(t)) + u(t) + \omega_1(t), \\
\dot{x}_2(t + 1) & = b x_1(t) + x_3(t) + \omega_2(t), \\
\dot{x}_3(t + 1) & = -b x_1(t) + \omega_3(t), \\
y(t) & = x_1(t) + v(t),
\end{align*}
\]
where \( x_1(t) \sim x_3(t) \) and \( u(t) \) are accordingly the states and control input; \( \tau(t) \) is time-varying delay with an upper bound \( \tau_{\text{max}} = 15 \); \( \omega_1(t), \omega_2(t), \omega_3(t), v(t) \) are uniformly distributed external disturbances, all with the amplitude of 0.01; and parameters \( a = -1, b = 0.33, c = 1, e = 0.05 \).

Choose state \( x_1(t) \) as the antecedent variable. According to the fuzzy modeling method [31], the membership functions \( F_1 = 0.5(1 + (x_1(t)/d)) \), \( F_2 = 1 - F_1 \) with \( |x_1(t)| \leq d \). System (62) can then be exactly represented by the T-S fuzzy model with system matrices
\[
A_1 = \begin{bmatrix} a & d & 1 \\ b & 0 & 1 \end{bmatrix}, \\
A_2 = \begin{bmatrix} -a & d & 1 \\ b & 0 & 1 \end{bmatrix}, \\
A_{d1} = A_{d2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\Gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

First, for \( u(t) = 0 \), we show the \( x_1(t) - x_2(t) \) phase portrait, \( x_2(t) - x_3(t) \) phase portrait, and time-varying delay \( \tau(t) \) and \( x_1(t) \) of the chaotic system (62) in Figures 1(a)–1(d).

7.1. Synchronization. The synchronization parameters \( \rho = 0.9, \bar{Q} = 0.11, \) and \( U_1 = 0.21 \). We obtain the observer gains from solving LMI (14) in Theorem 1. Let initial conditions \( x(0) = [1 \ 0 \ 0]^{T} \) and \( \dot{x}(0) = [0 \ 0 \ 0]^{T} \) for \( t_0 \in [-\tau_0, 0] \). The transmitter sent an output \( y(t) \) to the receiver; then the receiver estimated all the states. The results are shown in Figures 2(a)–2(d) \( (x_1(t), e_1(t), e_2(t), e_3(t)) \).
Figure 1: (a) $x_1(t) - x_2(t)$ phase portrait, (b) $x_2(t) - x_3(t)$ phase portrait, (c) time-varying delay $r(t)$, and (d) $x_1(t)$ of the discrete-time chaotic system example.

Figure 2: Continued.
Consider a 3-dimensional continuous chaotic system described as follows:

\[
\dot{x}_1(t) = a[x_2(t) - f_1(x_1(t))] + c + x_1(t - \tau(t)) + u(t) + \omega(t),
\]
\[
\dot{x}_2(t) = x_1(t) - x_3(t) + x_3(t) + \omega_2(t),
\]
\[
\dot{x}_3(t) = -bx_2(t) + \omega_3(t),
\]
\[
y(t) = x_1(t) + v(t),
\]

where \(x_1(t) \sim x_3(t)\) and \(u(t)\) are accordingly the states and control; the nonlinear function \(f_1(x_1(t)) = (2/7)x_1(t) - (3/14)|x_1(t) + 1| - |x_1(t) - 1|\); \(\tau(t)\) is time-varying delay with an upper bound \(\tau_u = 3.54\); \(\omega_1(t), \omega_2(t), \omega_3(t), v(t)\) are uniformly distributed external disturbances all with the amplitude of 0.01; and parameters \(a = 9, b = 14.286, c = 0.1\).

Choose state \(x_1(t)\) as the antecedent variable. According to the fuzzy modeling method [31], the following membership functions \(F_1 = 0.5(1 + (\phi(t)/d)), F_2 = 1 - F_1\) with \(\phi(t) = f_1(x_1(t))/x_1(t)\) where \(|\phi(t)| \leq d, d = 0.26\). System
Figure 4: Tracking control of discrete-time chaotic system with trajectories (a) $x_{d2}(t)$, (b) $e_2(t)$, and (c) $x_{h2}(t)$.

Figure 5: Tracking control of discrete-time chaotic system with trajectories (a) $x_{d3}(t)$, (b) $e_3(t)$, and (c) $x_{h3}(t)$. 
Figure 6: (a) $x_1(t) - x_2(t)$ phase portrait, (b) $x_3(t) - x_3(t)$ phase portrait, and (c) time-varying delay $\tau(t)$ and (d) $x_1(t)$ of system (64) of the continuous-time chaotic system example.

Figure 7: Synchronization of continuous-time chaotic system with (a) $x_1(t)$, (b) $e_1(t)$, (c) $e_2(t)$, and (d) $e_3(t)$. 
Figure 8: Tracking control of continuous-time chaotic system with trajectories (a) $x_{d1}(t)$, (b) $e_1(t)$, and (c) $x_{h1}(t)$.

Figure 9: Tracking control of continuous-time chaotic system with trajectories (a) $x_{d2}(t)$, (b) $e_2(t)$, and (c) $x_{h2}(t)$. 
can be exactly represented by the T-S fuzzy model with system matrices

\[
A_1 = \begin{bmatrix} -a & d & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \\
A_2 = \begin{bmatrix} a & d & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \\
A_{d1} = A_{d2} = \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

Let initial conditions \( x(0) = [1 \ 0 \ 0]^T \) and \( \tilde{x}(0) = [0 \ 0 \ 0]^T \) for \( t_0 \in [-r_0, 0] \). Synchronization and output tracking are discussed. Figures 6(a)–6(d) show the phase portrait, \( x_2(t) - x_1(t) \) phase portrait, and time-varying delay \( \tau(t) \) and \( x_1(t) \) of system (64) with \( u(t) = 0 \).

7.3. Synchronization. The transmitter sent an output \( y(t) \) to receiver; then the receiver estimated the all states. Choose parameters \( \rho = 0.9, Q = 0.1I, U_1 = 0.3I \). From Theorem 2 solving LMI (15), we obtain the observer gains \( L_i \).

7.4. Time-Varying Output Tracking. Consider controlled output is \( y_c(t) = x_1(t) \) and assume \( x_{d1}(t) = \sin(2t) \). The control parameters \( \rho = 0.9, Q = 0.1I, U_1 = 0.3I, U_2 = 0.2I \).

From solving the LMIs (53)–(55) in Theorem 4, we obtain the control gains \( K_i \), observer gains \( L_i, P, \) and \( S \). In this case, \( x_{d2}(t) \) and \( x_{d3}(t) \) may solve the equation \( \dot{x}_{d2}(t) = x_{d1}(t) - x_{d2}(t) + x_{d3}(t) \) and \( \dot{x}_3(t) = -bx_3(t) \). The results are shown in Figures 8–10. The trajectories of \( x_{d1}(t), e_1(t), \) and \( x_{h1}(t) \) are shown in Figures 8(a)–8(c). Figures 9(a)–9(c) show the trajectories of \( x_{d2}(t), e_2(t), \) and \( x_{h2}(t) \). Figures 10(a)–10(c) show the trajectories of \( x_{d3}(t), e_3(t), \) and \( x_{h3}(t) \).

It can be seen that the proposed control method can achieve \( H_{\infty} \) synchronization and output tracking control performance for chaotic time-delay systems.

8. Conclusions and Future Work

Chaotic systems with time-varying delays pose a practical yet challenging control problem. In this paper, we have proposed a unified T-S fuzzy representation of the chaotic system control and synchronization problem with time-varying delays for both discrete and continuous systems. An observer-based output tracking controller was designed considering different antecedent variables between the controller and observer. This is an important consideration.
for complex system due to uncertainties from estimation and controller mismatches. Numerical simulations for all cases of continuous, discrete with delays disturbances have been carried out. Robustness is shown to achieve H-infinity performance. It is worthwhile to note that asymptotic and exponential tracking can also be achieved for more ideal conditions. The method can also be applied to nonlinear system control with limited restrictions. Observer and controller gains are found numerically through LMI toolboxes which further extends the possibility of actual implementations. For future work, error criteria can further evaluate and quantify the performance of the proposed method and provide reference for implementation.

\[ \Delta V(t) = e^T(t + 1)\overline{Pe}(t + 1) - e^T(t)\overline{Pe}(t) + e^T(t)\overline{Se}(t) - e^T(t - \tau)\overline{Se}(t - \tau) \]

\[ = \left[ \sum_{i=1}^{r} \mu_i(\bar{z}(t))\left[ \overline{Ge}(t) + \overline{M}_i e(t - \tau) + \overline{E}_i w_i(t) \right] + h_i(t) \right]^T P \]

\[ \times \left[ \sum_{i=1}^{r} \mu_i(\bar{z}(t))\left[ \overline{Ge}(t) + \overline{M}_i e(t - \tau) + \overline{E}_i w_i(t) \right] + h_i(t) \right] \]

\[ - e^T(t)\overline{Pe}(t) + e^T(t)\overline{Se}(t) - e^T(t - \tau)\overline{Se}(t - \tau) \leq \sum_{i=1}^{r} \mu_i(\bar{z}(t))\overline{x}_i^T(t) \]

\[ \left[ \overline{G}_i^T \overline{P} G_i + \overline{S} - \overline{P}(\ast)(\ast)\overline{M}_i^T \overline{P} \overline{M}_i \overline{P} - S(\ast)\overline{E}_i^T \overline{P} \overline{E}_i \overline{P} \overline{M}_i \overline{P} \overline{E}_i - \rho^2 \overline{I} \right] \overline{x}_i(t) \]

\[ + \sum_{i=1}^{r} \mu_i(\bar{z}(t))2h_i(t)\overline{P}[\overline{G}_i e(t) + \overline{M}_i e(t - \tau) + \overline{E}_i w_i(t)] \]

\[ + h_i^T(t)\overline{P} h_i(t) + \rho^2\overline{w}_i^T(t)w_i(t), \]

where \( \overline{x}_i(t) = [e^T(t) e^T(t - \tau) \overline{w}_i^T(t)]^T \). According to Lemma 1 and the fact \( e^T(t)\overline{U}_1^T \overline{U}_1 e(t) \), we have

\[ \Delta V(t) \leq \sum_{i=1}^{r} \mu_i(\bar{z}(t))\overline{x}_i^T(t) \left[ \begin{array}{ccc} \overline{H}_{11} & * & * \\ \overline{M}_i^T \overline{P} \overline{G}_i & \overline{H}_{12} & * \\ \overline{E}_i^T \overline{P} \overline{G}_i & \overline{E}_i^T \overline{P} \overline{M}_i & \overline{H}_{13} \end{array} \right] \overline{x}_i(t) + \rho^2\overline{w}_i^T(t)w_i(t). \]

From this, we obtain

**Appendix**

**A. Proof of Theorem 1**

Consider a Lyapunov–Krasovskii function candidate as

\[ V(t) = e^T(t)\overline{Pe}(t) + \sum_{q=1}^{r} e^T(t - q)\overline{Se}(t - q). \]  

(A.1)

From (10), we have

\[ \Delta V(t) \leq \sum_{i=1}^{r} \mu_i(\bar{z}(t))\overline{x}_i^T(t) \left[ \begin{array}{ccc} \overline{H}_{11} & * & * \\ \overline{M}_i^T \overline{P} \overline{G}_i & \overline{H}_{12} & * \\ \overline{E}_i^T \overline{P} \overline{G}_i & \overline{E}_i^T \overline{P} \overline{M}_i & \overline{H}_{13} \end{array} \right] \overline{x}_i(t) + \rho^2\overline{w}_i^T(t)w_i(t). \]

(A.3)
\[
\sum_{t=0}^{tf} e^T(t) Q e(t)
\]
\[
= \sum_{t=0}^{tf} \left\{ e^T(t) Q e(t) + e^T(t+1) P e(t+1) - e^T(t) P e(t) + e^T(t) S e(t) \right\}
\]
\[
- e^T(t-\tau) S e(t-\tau) \right\} - \chi(t) P x_c(t) + e^T(t) P x_c(0)
\]
\[
- \sum_{\tau=1}^{\tau} e^T(t+1-q) S e(t+1-q) + \sum_{\tau=1}^{\tau} e^T(-q) S e(-q) + e^T(t) P e(t)
\]
\[
\leq e^T(0) P e(0) + \sum_{\tau=1}^{\tau} e^T(-q) S e(-q) + \sum_{\tau=1}^{\tau} \left[ e^T(t) Q e(t) + \Delta V(t) \right]
\]
\[
\leq e^T(0) P e(0) + \sum_{\tau=1}^{\tau} e^T(-q) S e(-q) + \rho^2 \sum_{\tau=1}^{\tau} w^T(t) w(t).
\]

According to (38), we have (14). Therefore, the $H_{\infty}$ tracking control performance in (3) is achieved with a given $\rho^2$.

**B. Proof of Theorem 2**

Consider a Lyapunov–Krasovskii function candidate as

\[
V = e^T(t) P e(t) + \int_{t-\tau}^{t} e^T(\lambda) S e(\lambda) \, d\lambda. \tag{B.1}
\]

The control objective is required to satisfy

\[
\int_{0}^{t_f} e^T(t) R e(t) \, dt \leq \rho^2 \int_{0}^{t_f} \| w(t) \|^2 \, dt,
\]

where $t_f$ is terminal time of control, $\rho$ is a given value that denotes the effect of $w(t)$ on $e(t)$, and $R$ is a positive definite matrix. Taking the derivative of Lyapunov function (B.1) and applying (B.2) along with (10), it is concluded that

\[
J = e^T(t) R e(t) - \rho^2 w^T(t) w(t) + \dot{V}
\]
\[
= e^T(t) R e(t) - \rho^2 w^T(t) w(t) + 2 \dot{e}^T(t) P h(t)
\]
\[
+ \sum_{i=1}^{r} \mu_i(\tilde{e}(t)) e^T(t) \left[ G_i^T P + P G_i \right] e^T(t) + 2 e^T(t) P M e^T(t-\tau) + 2 e^T(t) P E w(t)
\]
\[
\leq \sum_{i=1}^{r} \mu_i(\tilde{e}(t)) e^T(t) \left[ G_i^T + P M_i S^{-1} M_i P + \rho^2 P E_i E_i^T P \right] e(t),
\]

where $G_i = G_i^T P + P G_i + R + S + P P + U_1^T U_1$ since $e^T(t) P M_i e(t-\tau) \leq e^T(t) P M_i S M_i^T P e(t) + e^T(t-\tau) P e(t-\tau)$ and $2 e^T(t) P E w(t) \leq e^T(t) P P e(t) + e^T(t) U_1^T U_1 e(t)$.

**C. Proof of Theorem 3**

**Proof.** Consider a Lyapunov–Krasovskii function candidate as

\[
V = e^T(t) P e(t) + \int_{t-\tau}^{t} e^T(\lambda) S e(\lambda) \, d\lambda.
\]
\[ V(t) = x_c^T(t)Px_c(t) + \sum_{q=1}^{\tau} x_c^T(t - q)Sx_c(t - q). \]  

(C.1)

\[
\begin{align*}
\Delta V(t) &= x_c^T(t + 1)Px_c(t + 1) - x_c^T(t)Px_c(t) + x_c^T(t)Sx_c(t) - x_c^T(t - \tau)Sx_c(t - \tau) \\
&= \left[ \sum_{i=1}^{r} \mu_i(\overline{z}(t)) [G_i x_c(t) + M_i x_c(t - \tau) + E_i w_c(t)] + \overline{h}(t) \right]^T P \\
&\quad \times \left[ \sum_{i=1}^{r} \mu_i(\overline{z}(t)) G_j x_c(t) + M_j x_c(t - \tau) + E_j w_c(t) + \overline{h}(t) \right] \\
&- x_c^T(t)Px_c(t) + x_c^T(t)Sx_c(t) - x_c^T(t - \tau)Sx_c(t - \tau) \\
&\leq \sum_{i=1}^{r} \mu_i(\overline{z}(t)) \overline{x}_c^T(t) \begin{bmatrix} H_i & * & * \\ M_i^T P G_i & H_{i2} & * \\ E_i^T P G_i & E_i^T P M_i & H_{i3} \end{bmatrix} \overline{x}_c(t) \\
&\quad + \sum_{i=1}^{r} \mu_i(\overline{z}(t)) 2\overline{h}(t)^T(t)P [G_i x_c(t) + M_i x_c(t - \tau) + E_i w_c(t)] \\
&\quad + \overline{h}(t)^T(t)P\overline{n} + \rho^2 w_c^T(t)w_c(t),
\end{align*}
\]

(C.2)

where \( \overline{x}_c(t) = [x_c^T(t) x_c^T(t - \tau) w_c^T(t)]^T \). According to Lemma 1 and the fact \( \overline{h}(t)P\overline{n}(t) \leq x_c^T(t)U^TPUx_c(t) \), we have

\[
\Delta V(t) \leq \sum_{i=1}^{r} \mu_i(\overline{z}(t)) \overline{x}_c^T(t) \begin{bmatrix} H_i & * & * \\ M_i^T P G_i & H_{i2} & * \\ E_i^T P G_i & E_i^T P M_i & H_{i3} \end{bmatrix} \overline{x}_c(t) \\
+ \rho^2 w_c^T(t)w_c(t).
\]

(C.3)

\[
\sum_{t=0}^{tf} x_c^T(t)Qx_c(t) \\
= \sum_{t=0}^{tf} \left[ x_c^T(t)Qx_c(t) + x_c^T(t + 1)Px_c(t + 1) - x_c^T(t)Px_c(t) + x_c^T(t)Sx_c(t) \\
- x_c^T(t - \tau)Sx_c(t - \tau) \right] \\
- \sum_{q=1}^{\tau} x_c^T(t_f + 1 - q)Sx_c(t_f + 1 - q) + \sum_{q=1}^{\tau} x_c^T(-q)Sx_c(-q) + x_c^T(0)Px_c(0) \\
- x_c^T(t_f + 1)Px_c(t_f + 1) - \sum_{q=1}^{\tau} x_c^T(t_f + 1 - q)Sx_c(t_f + 1 - q) + \sum_{q=1}^{\tau} x_c^T(-q)Sx_c(-q) \\
\leq x_c^T(0)Px_c(0) + \sum_{q=1}^{\tau} x_c^T(-q)Sx_c(-q) + \sum_{t=0}^{tf} \left[ x_c^T(t)Qx_c(t) + \Delta V(t) \right] \\
\leq x_c^T(0)Px_c(0) + \sum_{q=1}^{\tau} x_c^T(-q)Sx_c(-q) + \rho^2 \sum_{t=0}^{tf} w_c^T(t)w_c(t).
\]

(C.4)
According to (38), we have (5). Therefore, the $H_{\infty}$ tracking control performance in (5) is achieved with a given $\rho^2$. \hfill \Box

**D. Proof of Theorem 4**

**Proof.** Consider a Lyapunov–Krasovskii function candidate as

$$V = x_e^T(t)Px_e(t) + \int_{t-\tau}^{t} x_e^T(\lambda)Sx_e(\lambda)d\lambda. \quad (D.1)$$

$$J = x_e^T(t)R_s x_e(t) - \rho^2 w_e^T(t)w_e(t) + \dot{V}$$

$$= x_e^T(t)R_s x_e(t) - \rho^2 w_e^T(t)w_e(t) + 2x_e^T(t)P\bar{h}(t)$$

$$+ \sum_{i=1}^{r} \mu_i(\hat{z}(t))x_e^T(t)\left[\overline{G_i}^T P + PG_i\right] x_e^T(t) + 2x_e^T(t)PM_i x_e(t - \tau) + 2x_e^T(t)PE_i w_i(t) \right]$$

$$\leq \sum_{i=1}^{r} \mu_i(\hat{z}(t))x_e^T(t)\left[\overline{G_i}^T P + PG_i + R + S + PP + \overline{U}^T \overline{U}\right] x_e(t), \quad (D.3)$$

where $\overline{G_i} = G_i^T P + PG_i + R + S + PP + \overline{U}^T \overline{U}$ since $x_e^T(t)PM_i x_e(t - \tau) \leq x_e^T(t)PM_i x_e(t) + x_e^T(t - \tau) Sx_e(t - \tau)$ and $2x_e^T(t)PE_i w_i(t) \leq x_e^T(t)PP x_e(t) + x_e^T(t)\overline{U}^T \overline{U} x_e(t)$.

The control objective is required to satisfy

$$\int_{0}^{t_f} x_e^T(t)R_s x_e(t)dt \leq \rho^2 \int_{0}^{t_f} \|w_e(t)\|^2 dt, \quad (D.2)$$

where $t_f$ is terminal time of control, $\rho$ is a given value that denotes the effect of $w_e(t)$ on $x_e(t)$, and $R$ is a positive definite matrix. Taking the derivative of Lyapunov function (D.1) and applying (D.2) along with (25), it is concluded that

$$\int_{0}^{t_f} x_e^T(t)R_s x_e(t)dt \leq x_e^T(0)Px_e(0) + \int_{0}^{t_f} x_e^T(v)Sx_e(v)dv + \rho^2 \int_{0}^{t_f} \|w_e(t)\|^2 dt. \quad (D.4)$$

This means that the overall system has $H_{\infty}$ performance. \hfill \Box

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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