Unifying frequency combs in active and passive cavities:
CW driving of temporal solitons in ring lasers

A. Gatti,1, 2 F. Prati,2 L.A. Lugiato,2 L. Columbo,3 M. Brambilla,4 C. Silvestri,3
M. Gioannini,3 N. Opacač,6 B. Schwarz,5 M. Piccardo,6,7 and F. Capasso7

1Istituto di Fotonica e Nanotecnologie del CNR-IFN, Milano, Italy
2Dipartimento di Scienza e Alta Tecnologia, Università dell’Insubria, Como, Italy
3Dipartimento di Elettronica e Telecomunicazioni, Politecnico di Torino, Torino, Italy
4Dipartimento di Fisica Interateneo and CNR-IFN, Università e Politecnico di Bari, Bari, Italy
5Institute of Solid State Electronics, TU Wien, Vienna, Austria
6Center for Nano Science and Technology, Fondazione Istituto Italiano di Tecnologia, Milano, Italy
7Harvard John A. Paulson School of Engineering and Applied Sciences, Harvard University, Cambridge, MA, USA

(Dated: July 16, 2020)

Frequency combs have become a prominent research area in optics. Of particular interest as integrated comb technology are chip-scale sources, such as semiconductor lasers and microresonators, which consist of resonators embedding a nonlinear medium either with or without population inversion. Such active and passive cavities were so far treated distinctly. Here we propose a formal unification by introducing a general equation that describes both types of cavities. The equation also captures the physics of a hybrid device - a semiconductor ring laser with an external optical drive - in which we show the existence of temporal solitons, previously identified only in microresonators.

Introduction - The discovery of optical frequency combs [1, 2] (OFCs) in high-Q ring microresonators filled with a Kerr medium, such as SiO2, Si3N4 and diamond [3], and driven by an external laser beam activated worldwide attention on the topic of Kerr frequency combs (KFCs), because this avenue offers substantial potential for miniaturization and chip-scale photonic integration [4, 5]. This technology has been applied to numerous areas, including coherent telecommunication, spectroscopy, atomic clocks as well as laser ranging and astrophysical spectrometer calibration [6]. It was recognized later [7, 8] that the physics of KFCs is very accurately described by the Lugiato-Lefever equation (LLE) formulated in 1987 [9]. The spontaneous formation of spatial patterns travelling along the cavity with constant velocity, described in the LLE, is the spatio-temporal equivalent of the frequency combs and governs their features [10]. By varying the frequency detuning θ of the pump laser injecting the microresonator a variety of spatial patterns can form (Fig. 1(a)), such as Turing rolls, breather solitons and stable temporal solitons, each corresponding to a different frequency comb spectrum. A common feature of these spectra is that their envelope is bell-shaped and can be well approximated by a hyperbolic-secant function (sech) [3, 11].

From a mathematical standpoint, the LLE can be defined as a driven, damped and detuned nonlinear Schrödinger equation with periodic boundary condition. A distinctive feature introduced in the LLE is represented by the inhomogeneous driving term, which discloses a richness of phenomenology.

Recently, frequency comb spectra with sech-type envelope were also observed in ring quantum cascade lasers (QCLs) [12, 13]. These are unipolar semiconductor lasers, first realized in 1994 [14], emitting in the mid-infrared and terahertz regions of the electromagnetic spectrum. QCLs have attracted a lot of attention, investigations and applications, especially in mid-infrared spectroscopy and sensing [15, 16], thanks to their tunability via band-gap engineering and unique physical properties [17], such as ultrafast gain dynamics and strong resonant third-order nonlinearity. The study of ring QCLs operating in a unidirectional regime, where only clockwise or counter-clockwise waves exist and spatial hole burning is absent, revealed a number of similarities with KFCs. In addition to the sech-type envelope (Fig. 1(c)), it was also found that the multimode laser instability is produced by the interplay of dispersive and nonlinear effects, as in the modulational instability (MI) of passive microresonators, and the number of localized structures appearing in the spatial patterns varies stochastically with the initial conditions [12] – a phenomenon known as multistability, also occurring in KFCs [3].

This analogy can be formally understood by considering that under conditions of fast material dynamics and of near-threshold operation, the dynamics of the ring QCL is well described by a complex Ginzburg-Landau equation (CGLE) that depends on two coefficients (Fig. 1(c)) essentially determined by two laser parameters, the linewidth enhancement factor (LEF or α-factor) and the group velocity dispersion (GVD) [18]. The CGLE has basically the same form of a cubic equation previously formulated [19] (and analyzed [20]) as an active counterpart of the LLE. Just as in the case of the passive LLE, this cubic equation [19] refers to 2D transverse patterns in the plane orthogonal to the longitudinal direction of light propagation, originated by the interplay of the nonlinearity and radiation diffraction. However, the same equation restricted to 1D also described patterns traveling in the longitudinal direction of the resonator at the speed of light, originated by the interplay of nonlinearity and GVD [21]-[22].

These considerations brought us to formulate a generalized LLE, defined as the simplest equation that includes
FIG. 1. Active and passive frequency comb sources: unification. The generalized Lugliato-Lefever equation captures the physics of three distinct systems in different limits. (a) Schematic of a passive microresonator described by the LLE. The intracavity power depends on the pump laser detuning \( \theta \) resulting in regions with different patterns: Turing rolls (blue), breather solitons (yellow), and stable temporal solitons (red). An example of different intensity distributions along the cavity and corresponding frequency comb spectrum is given in each case. \( L \) denotes the cavity length. Images adapted from [8]. (b) Schematic of an intermediate system between the active and passive case consisting of a ring quantum cascade laser (QCL) injected by an external optical signal. It is introduced in this work and described by a forced complex Ginzburg-Landau equation (CGLE). (c) Schematic of a unidirectional ring QCL described by the CGLE, which depends on two coefficients, related to the linewidth enhancement factor and group velocity dispersion of the laser. The ring laser can undergo a multimode transition in certain regions of the parameter space (red areas). The resulting dynamic behavior can be characterized by phase (PT) or amplitude (AT) turbulence, for which an example of spatial pattern and frequency comb spectrum are shown.

The generalized Lugiato-Lefever equation captures the physics of three distinct systems in different limits. Let us consider the following equation that describes the spatiotemporal evolution of the envelope \( E \) of the electric field in an optical cavity

\[
\tau_p \partial_t E = E_f + (-1 - i\theta_0) E + (d_R + id_I) \partial_z E + \mu (1 - i\Delta) (1 - |E|^2) E,
\]

where \( t \) is the time coordinate, \( \tau_p \) is the cavity photon lifetime and \( z \) is the coordinate along the cavity (longitudinal) axis in a frame moving at the light velocity in the cavity. Electric fields are scaled [23, 24] to present the equation in its simplest form (Supplementary Material).

The meaning of the various terms at the right hand side of Eq. (1) is described in detail. \( E_f \) is the amplitude of a coherent field injected in the cavity, which may or may not be present depending on the system under consideration. In the second term, the -1 accounts for cavity losses and \( \theta_0 \) is a detuning parameter. The meaning of the third term is clear if we consider its effect on a plane wave of the form \( E_k \sim e^{ikz} \): we obtain \(-d_R k^2 E_k - id_I k^2 E_k\). Here the imaginary part is associated with frequency dispersion weighted by the coefficient \( d_I \), while the real part has the form of a diffusion term weighted by the coefficient \( d_R \), which is positive for a laser and can be null in a passive system. The diffusion term is connected to the reaction-diffusion mechanism responsible for pattern formation as described in Turing’s theory of morphogenesis [25], where in our case the reaction is produced by all the other linear and nonlinear terms appearing in Eq. (1). Both \( d_R \) and \( d_I \) may arise from a non-standard adiabatic elimination of the material variables (Supplementary Material) and, in addition, \( d_I \) may contain a contribution due to the GVD of the host medium. The fourth term describes the nonlinear interaction of the electric field with the material and can be obtained by a standard adiabatic elimination of the material variables in a two-level medium under the approximation \(|E|^2 \ll 1\). In this term, \( \mu \) is the unsaturated gain \((\mu > 0)\) or absorption \((\mu < 0)\) parameter, while \( \Delta \) is the atomic detuning parameter but, as we shall see, it can also account for the cubic nonlinearity of the host medium.

Let us consider the passive and the active case in order.

Passive case - In this case we assume that the medium is an absorber \((\mu < 0)\) but absorption is small \((|\mu| \ll 1)\), the atomic detuning is negative and large \((|\Delta| \gg 1)\), and we assume that the resonance curve has a very large bandwidth \((d_I \gg d_R)\). Under those assumptions we can make the approximations \(\mu(1 - i\Delta) \approx -i\mu\Delta\) and \(d_R + d_I \approx d_R\).
\[
\frac{\partial}{\partial z} F = F_I - \left[ 1 + i \left( \theta - |F|^2 \right) \right] F + i \partial^2 \eta F, \tag{2}
\]

with \( \theta = \theta_0 + \mu \Delta \), \( F = \sqrt{\mu \Delta} E \), \( F_I = \frac{E_I}{\sqrt{|r|}} \), \( \tau = |r| t / \tau_p \), and \( \eta = z / \sqrt{d} \), where we assume that the dispersion is anomalous so that \( d_0 > 0 \). Here \( \theta_0 \) must be taken equal to \( \theta_c = (\omega_r - \omega_0) \tau_p \), with \( \omega_r \) being the cold cavity frequency closest to the frequency \( \omega_0 \) of the incident field.

**Active case**—Now we assume that the medium is active and close to the lasing threshold (\( \mu = 1 + r \), \( |r| \ll 1 \)), above threshold for \( r > 0 \) and below for \( r < 0 \) and that \( |E|^2 \) has the same order of magnitude as \( |r| \). We can then make the approximation \( \mu (1 - |E|^2) \approx \mu - |E|^2 \) and obtain an equation which has the form of a forced CGLE

\[
\frac{\partial}{\partial z} F = F_I + \gamma (1 - i \theta) F - (1 - i \Delta) |F|^2 F + (1 + i G) \partial^2 \eta F, \tag{3}
\]

with \( \gamma = r / |r| \), \( \theta = (\theta_0 + \mu \Delta) / r \), \( G = d_I / d_R \), \( F = E_I / \sqrt{|r|} \), \( F_I = E_I / |r|^{3/2} \), \( \tau = |r| t / \tau_p \), and \( \eta = z \sqrt{|r|} / d \).

In the Supplemental Material we show that an equation identical to Eq. (3) can be derived from a full laser model for a QCL with coherent injection in the limit of ultrafast carriers and in proximity to the lasing threshold. In this case we have \( \gamma = 1 \) and

\[
\Delta = \alpha + \beta, \quad G = \alpha + \zeta, \quad \zeta = -(1 + \alpha^2) \frac{\partial^2 \eta}{2 \sqrt{d}} e^\alpha, \tag{4}
\]

where \( \alpha \) is the linewidth enhancement factor (LEF) [26], \( \beta \) is the Kerr coefficient, \( e^\alpha \) is the GVD coefficient, \( e \) and \( \tau_d \) are the speed of light and the polarization dephasing time in the QCL, respectively. Here \( \theta_0 \) must be taken equal to \( \theta_c - \mu \beta \). Above threshold and without an injected field Eq. (3) with \( \theta = 0 \), \( \Delta = c_{\text{NL}} \), and \( G = -c_D \) coincides with the CGLE in [12] [27].

As we have seen, the generalized LLE includes as a special case the LLE for passive Kerr microresonators on the one hand (Fig. 1(a)), and on the other hand the CGLE (as well as the cubic equation of [19]) for the active case of QCLs [12] (Fig. 1(c)). The case of a QCL with injected signal (Fig. 1(b)) that will now be presented constitutes a new configuration - also captured by the generalized LLE - that offers the advantage of introducing two control parameters, the intensity and frequency of the injected field. The dynamics of the electric field in the transverse plane in a similar system, i.e. a class-A laser with injected signal, was studied in [28] using an equation like Eq. (3), with the second order derivative along \( \eta \) replaced by the transverse Laplacian with a purely imaginary coefficient and \( \Delta = 0 \).

An equation identical to Eq. (3) with \( \Delta = G = \alpha \) was introduced in [29] to model the phase solitons observed in a driven interband semiconductor ring laser. Yet, in that case it was shown that such an equation was not sufficient to account for the experimental findings because the slow gain recovery time prevented a full adiabatic elimination of the material variables. This limitation is overcome in the case of a QCL thanks to its fast material dynamics.

Besides the ring geometry considered here, a connection between QCLs and the LLE was also established recently in the case of Fabry-Perot devices [30].

**Temporal cavity solitons without background in a passive microcavity** coupled with an amplifying fiber loop were demonstrated in [31]. This system combines a passive resonator with an external gain medium, thus completing the possible combinations of pump/resonator/gain shown in Fig. 1, but it cannot be described by the generalized LLE because it requires two coupled equations.

**The injected ring QCL** - Let us now turn our attention to the case in which an external coherent field is injected into the QCL (Fig. 1(b)). Just as in the case of passive microresonators, the presence of the injected field allows to have an S-shaped curve of output intensity as a function of input intensity for the uniform stationary solutions - a phenomenon known as optical bistability (see Fig. 2(b), where we denote with SN1 and SN2, respectively, the right and left turning points of the stationary curve, because these points are related to a saddle-node bifurcation in the plane wave limit). In the active case above threshold only a segment of the lower branch of this curve is stable, and this occurs between the injection locking (IL) point, where the lasing frequency is locked to the injected field, and the turning point SN1 (green segment in Fig. 2(b)). At the same time, part of the upper branch of the curve is affected by a MI - a spontaneous symmetry breaking mechanism producing intensity patterns characterized by a high degree of spatial correlation (Fig. 2(d)), or else spatiotemporal turbulence.

As it happens in passive microresonators, a favourable condition for the existence of temporal solitons, referred here also as cavity solitons (CSs) since they are dissipative localized structures formed inside an optical resonator [24], is to have an interval of input intensity where the uniform stationary state in the upper branch is modulationally unstable and coexists with a uniform stable state in the lower branch. In this situation the system might form a localized pattern emerging from the MI on a uniform stable background, and eventually give origin to a CS [24] (see Fig. 2(c.g)). These considerations guided our search for CSs (and the associated OFCs) in QCLs with injected signal and in particular the choice of an experimentally reasonable parameter set.

For the simulations of the injected QCL we consider the following parameters entering Eq. (3): \( \gamma = 1 \), \( \Delta = \alpha = 2 \), \( \theta = 4.7 \), \( G = 3 \). By assuming a transmittance for the injection port of 5%, waveguide power losses of 4 cm\(^{-1} \), \( \tau_p = 50 \) ps and \( \tau_d = 60 \) fs [12], these parameters correspond to a LEF of 2, a frequency detuning between the pump and the reference frequency of approximately 0.2 free spectral ranges (FSRs), and a GVD around \(-300 \) fs\(^2\)mm\(^{-1} \), all realistic values for QCLs [12]. In particular the GVD is known to be controllable by dispersion engineering. We neglect further contributions to \( \Delta \) from the bulk Kerr nonlinearity as this is small in QCLs. The S-shaped curve calculated for the selected
FIG. 2. Spatiotemporal dynamics of the injected ring quantum cascade laser. (a) One-dimensional Turing rolls exhibiting periodic oscillations in the cavity between two values of output intensity corresponding to a pair of blue dots in (b). (b) S-shaped curve of output intensity from the ring quantum cascade laser vs. input pump intensity, \( Y \). Characteristic points for the laser dynamics are marked on the curve: injection locking threshold (IL), saddle nodes (SN\(_1\) and SN\(_2\)) and modulation instability threshold (MI). Different segments of the curve can be stable (green line), unstable (dashed line) or not accessible (grey line). Red and blue dots correspond to Turing patterns and cavity solitons (CSs). (c) A CS with a pedestal and peak intensity that correspond to a point on the stable lower branch of the S-shaped curve and a red dot of (b), respectively. (d)–(g) Spatio-temporal plots and corresponding frequency comb spectra of: (d) Turing rolls; (e) transition from Turing rolls to multiple non-stationary CSs; (f) single CS on an unstable background; (g) single CS on a stable pedestal. The labels \( Y_1 \)–\( Y_4 \) correspond to different pump intensities varied sequentially from \( Y_1 \) to \( Y_4 \) as described in the text and marked in (b). Spectral frequencies are relative to the central mode and normalized to the free spectral range. The dashed line in (g) is a sech\(^2\) fit.

The following scenario emerges from numerical integration [32] of Eq. (3) as we vary the input intensity \( Y = F_I^2 \), that acts as a control parameter in our nonlinear extended system [33]. Starting on the high-intensity spatially-uniform solution, stable to the right of the MI point, and progressively decreasing \( Y \), a global modulated pattern (Fig. 2(a), \( Y_1 = 6.3 \)) bifurcates from MI at \( Y_{\text{MI}} = 7.8 \). The bifurcation is by definition supercritical as the branch that bifurcates remains stable, down to about \( Y = 5.4 \). The modulated pattern corresponds to a one-dimensional Turing roll [25] - a periodic, spatially-extended structure - and its branch is indicated by the blue dots in Fig. 2(b), which mark the maximum and minimum intensity of the spatial modulation. The spatial period of the roll changes with \( Y \), as illustrated by Fig. 2(d). Here the initial condition is the pattern with 18 rolls which is stable at \( Y = 6.4 \), we set \( Y = Y_1 = 6.3 \) and the pattern evolves to a new one with 17 rolls. The optical spectrum of the rolls at \( Y_1 = 6.3 \) is shown in the panel below and exhibits a mode spacing corresponding to a multiple of the cavity FSR (specifically 17th FSR) - a feature also observed in the harmonic frequency combs of free-running Fabry-Perot QCLs [34], attractive for sub-terahertz wireless communication applications [35]. What is particularly appealing in the configuration of the injected ring QCL, is the possibility of tuning the spacing of the harmonic frequency comb simply by changing the intensity of the injected signal, rather than by widely
tuning its frequency as it was done for the control of the harmonic state in Fabry-Perot QCLs [36].

Below $Y = 5.4$ the rolls become unstable and the system undergoes a spontaneous collapse of the roll pattern. It evolves into a number of non-stationary CSs sitting on a turbulent background, which corresponds to the unstable lower branch of the steady-state curve (Fig. 2(e), $Y_2 = 5.3$). A further decrease of $Y$ brings the system in a turbulent regime where any ordered structure disappears. If instead starting with the solution with non-stationary CSs the value of $Y$ is increased in the interval $5.7 \leq Y < 7$ a single CS with turbulent background survives (Fig. 2(f), $Y_3 = 5.8$). The background fluctuations cause a jitter in the soliton shape and intensity maximum. The range of fluctuations of the CS peak are traced by the pairs of red dots in Fig. 2(b). Finally, by following upwards the soliton branch in the interval $7 \leq Y < 7.4$ the pedestal of the single CS becomes stable corresponding to the lower uniform and stable branch of the steady-state curve, as expected, since $Y_{IL} = 6.97$ (Fig. 2(c,g), $Y_4 = 7.4$). Evolution on the scale of $10^8$ roundtrip times (not shown) confirm the persistence of these CS, both with stable and turbulent background, as stable solutions: their spatial shape and correspond-

ing spectra, well approximated by a sech$^2$ envelope (Fig. 2(f,g)), do not change in time. CS are not only predicted by our reduced model but are also observed in our full dynamical model of the QCL consisting in Eqs. (S1)-(S3) of the Supplementary Materials.

CS enjoy most of the common features typical of dissipative localized structures, the most appealing of which, on the application viewpoint, are multistability, independence and plasticity. The CS we report here are not only stable versus considerable fluctuations (e.g. those occurring in the irregular background as shown in Fig. 2(f)) but their intrinsic multistability allows them also to be independently excited by means of short pulses and to co-exist in the cavity field profile. As Fig. 3(a) illustrates, by means of an external addressing channel - here schematized by a straight waveguide evanescently coupled to the ring QCL - a series of subsequent pulses can turn on a CS in different positions of the resonator. We consider a writing pulse width equal to that of the CS, but in principle much wider pulses could be used (Fig. 3(b)), which may be convenient in practical realizations. In Fig. 3(a) the system is initially in the lower branch uniform solution and the first short pulse excites a CS at $z = L/2$. After 700 roundtrips a second identical pulse excites a second CS at $z = 5L/6$ and after another 700 roundtrips the same is repeated to create a third CS at $z = L/6$. The solitons onset and their following evolution prove they are three CSs completely unperturbed one from another. The optical spectra corresponding to the initial and final configurations demonstrates how the addition of CSs allows to manipulate the OFC (Fig. 3(b)). In particular, exciting equally-spaced CSs allows to control the spacing between the OFC lines and realize an externally addressable harmonic frequency comb that could be of interest for future sub-terahertz and pump-probe applications. Besides this, non-equidistant CSs may also be excited, giving a powerful degree of freedom in shaping OFCs. Further simulations confirm that the CSs remain independent within a spatial separation comparable to the period of the global structure bifurcated at the MI point [37], i.e. similar to the pitch of the rolls of Fig. 2(d). Moreover we verified that the CSs can drift unperturbedly along gradients in the input field $F_I$, hence they are spatially reconfigurable even after creation.

Conclusions - The generalized LLE introduced here unifies comb-forming optical systems based on active and passive cavities with cubic nonlinearity, making it possible to connect for the first time from a formal viewpoint Kerr microresonators and QCLs. The system of the injected ring QCL, which we just started exploring, is a direct result of this unification opening a pathway for the realization of new spatiotemporal patterns in QCLs such as Turing rolls and CSs, previously restrained to Kerr combs. The possibility to generate high-contrast ultrashort pulses represented by a CS in a QCL using a CW pump is of particular importance in the framework of the analogy between coherently injected passive microresonators and QCLs, and with respect to the possi-

![FIG. 3. Harmonic spectral shaping by external writing of temporal solitons. (a) Spatio-temporal plot showing the evolution of the intracavity pattern in a ring QCL as multiple cavity solitons (CS) are sequentially excited by the injection of pulses at times $t_1$, $t_2$ and $t_3$. Schematics of the system consisting of a ring cavity coupled to a straight waveguide are also shown for the different injection times. (b) Excitation of a single CS with a wide pulse seed, approximately 20 times wider than the CS. (c) Frequency comb spectra corresponding to the time intervals in (a) in which a single CS and three equidistant CS have been excited.]
bility of exploiting the injected unidirectional ring QCL as a new source of OFCs for a wealth of applications. Although this result comes as a surprise, as it has long been thought that the ultrafast dynamics of QCLs should strongly suppress amplitude modulation [34] in absence of a radiofrequency modulation of the gain [38], here we have shown that an ultrashort pulse regime is possible, thanks to soliton formation triggered by a compensation between dispersion and nonlinear self-phase modulation associated with finite LEF in a resonator with gain driven by an external CW optical signal.

Moreover, this turns out to be in agreement with the experimental and theoretical evidences reported in [29, 39] where a large photon-to-carrier lifetime ratio was shown to represent a necessary condition for the observation of temporal (phase) solitons in a coherently driven unidirectional ring semiconductor laser. While this requirement was fulfilled in [29] using a long external cavity (≈1 m) in standard semiconductor bipolar lasers with nanosecond gain recovery time, in this work we showed how unipolar lasers, due to their ultrafast carrier dynamics, fulfill the same dynamical scale ratio and allow to downscale the device to the millimeter range, with significant impact on chip-scale frequency comb applications.

[1] T. Udem, R. Holzwarth, and T. W. Hnsch, Nature 416, 233 (2002).
[2] D. J. Jones, S. A. Diddams, J. K. Ranka, A. Stentz, R. S. Windeler, J. L. Hall, and S. T. Cundiff, Science 288, 635 (2000).
[3] T. J. Kippenberg, A. L. Gaeta, M. Lipson, and M. L. Gorodetsky, Science 361, eaan8083 (2018).
[4] P. DelHaye, A. Schliesser, O. Arcizet, T. Wilken, R. Holzwarth, and T. J. Kippenberg, Nature 450, 1214 (2007).
[5] B. Shen, L. Chang, J. Liu, H. Wang, Q.-F. Yang, C. Xiang, R. N. Wang, J. He, T. Liu, W. Xie, J. Guo, D. Kinghorn, L. Wu, Q.-X. Ji, T. J. Kippenberg, K. Vahala, and J. E. Bowers, Nature 582, 365 (2020).
[6] T. J. Kippenberg, R. Holzwarth, and S. A. Diddams, Science 332, 555 (2011).
[7] Y. K. Chembo and C. R. Menyuk, Phys. Rev. A 87, 053852 (2013).
[8] T. Herr, V. Brasch, J. D. Jost, C. Y. Wang, N. M. Kondratiev, M. L. Gorodetsky, and T. J. Kippenberg, Nat. Photonics 8, 145 (2014).
[9] L. A. Lugia and R. Lefever, Phys. Rev. Lett. 58, 2209 (1987).
[10] L. A. Lugia, F. Prati, M. L. Gorodetsky, and T. J. Kippenberg, Philos. T. R. Soc. A 376, 20180113 (2018).
[11] S. Coen and M. Erkintalo, Opt. Lett. 38, 1790 (2013).
[12] M. Piccardo, B. Schwarz, D. Kazakov, M. Beiser, N. Opacak, Y. Wang, S. Jia, J. Hillbrand, M. Tamagnone, W. Chen, A. Zhu, L. Columbo, A. Belyanin, and F. Capasso, Nature 582, 360 (2020).
[13] B. Meng, M. Singleton, M. Shahmohammadi, F. Kapsalidis, R. Wang, M. Beck, and J. Faist, Optica 7, 162 (2020).
[14] J. Faist, F. Capasso, D. L. Sivco, C. Sirtori, A. L. Hutchinson, and A. Y. Cho, Science 264, 553 (1994).
[15] A. Hugi, G. Villares, S. Blaser, H. C. Liu, and J. Faist, Nature 492, 229 (2012).
[16] G. Villares, A. Hugi, S. Blaser, and J. Faist, Nat. Commun. 5, 5192 (2014).
[17] H. Choi, L. Diehl, Z.-K. Wu, M. Giovannini, J. Faist, F. Capasso, and T. B. Norris, Phys. Rev. Lett. 100, 167401 (2008).
[18] N. Opacak and B. Schwarz, Phys. Rev. Lett. 123, 243902 (2019).
[19] L. A. Lugia, C. Oldano, and L. M. Narducci, J. Opt. Soc. Am. B 5, 879 (1988).
[20] W. Kaige, N. B. Abraham, and L. A. Lugia, Phys. Rev. A 47, 1263 (1993).
[21] M. Haelterman, S. Trillo, and S. Wabnitz, Opt. Commun. 91, 401 (1992).
[22] F. Castelli, M. Brambilla, A. Gatti, F. Prati, and L. A. Lugia, Eur. Phys. J. D 71, 84 (2017).
[23] L. Columbo, S. Barbieri, C. Sirtori, and M. Brambilla, Opt. Express 26, 2829 (2018).
[24] L. Lugia, F. Prati, and M. Brambilla, Nonlinear Optical Systems (Cambridge University Press, 2015).
[25] A. M. Turing, Philosphical Transactions of the Royal Society of London. Series B, Biological Sciences 237, 37 (1952).
[26] C. Henry, IEEE J. Quantum Electron. 18, 259 (1982).
[27] Note1, In [9] and [19] and in this paper the field envelope $E$ multiplies the factor $e^{-i\omega t}$, whereas in [12] it multiplies the factor $e^{i\omega t}$, so that the $E$ which appears in [12] is the complex conjugate of the $E$ which appears in this paper.
[28] C. Gibson, A. Yao, and G.-L. Oppo, Phys. Rev. Lett. 116, 043903 (2016).
[29] F. Gustave, L. Columbo, G. Tissoni, M. Brambilla, F. Prati, B. Kelleher, B. Tykalewicz, and S. Barland, Phys. Rev. Lett. 115, 043902 (2015).
[30] D. Burghoff, arXiv:2006.12979 (2020).
[31] H. Bao, A. Cooper, M. Rowley, L. Di Lauro, J. S. Totero Gongora, S. T. Chu, B. E. Little, G.-L. Oppo, R. Morandotti, D. J. Moss, B. Wetzel, M. Peccianti, and A. Pasquazi, Nat. Photonics 13, 384 (2019).
[32] Note2, We observe that a stochastic noise term mimicking spontaneous emission in the system was included in all the simulations.
[33] Note3, Taking $F_I$ real merely sets the reference phase for the optical field.
[34] M. Piccardo, P. Chevalier, B. Schwarz, D. Kazakov, Y. Wang, A. Belyanin, and F. Capasso, Phys. Rev. Lett. 122, 253901 (2019).
[35] M. Piccardo, M. Tamagnone, B. Schwarz, P. Chevalier, N. A. Rubin, Y. Wang, C. A. Wang, M. K. Connors, D. McNulty, A. Belyanin, and F. Capasso, PNAS 116, 9181 (2019).
[36] M. Piccardo, P. Chevalier, S. Anand, Y. Wang, D. Kazakov, E. A. Mejia, F. Xie, K. Lascola, A. Belyanin, and F. Capasso, Appl. Phys. Lett. 113, 031104 (2020).
[37] M. Brambilla, L. A. Lugia, and M. Stefani, Europhys. Lett. 34, 109 (1996).
[38] J. Hillbrand, N. Opacak, M. Piccardo, H. Schneider, G. Strasser, F. Capasso, and B. Schwarz, arXiv:2003.04127 (2020).

[39] L. Columbo, I. M. Perrini, T. Maggipinto, and M. Brambilla, New J. Phys. 8, 312 (2006).
I. EFFECTIVE SEMICONDUCTOR MAXWELL-BLOCH EQUATIONS FOR A MULTIMODE QUANTUM CASCADE LASER WITH INJECTED SIGNAL

Recently, a set of Effective Semiconductor Maxwell-Bloch Equations (ESMBEs) originally introduced in [1] was successfully applied [2] to describe the coherent multimode dynamics of a Quantum Cascade Laser (QCL). Contrary to the standard Maxwell-Bloch approach [3], the ESMBEs allow to self-consistently and properly reproduce typical experimental observations of self-generated Optical Frequency Combs (OFCs) in the system together with the alternance among regular and irregular dynamics [4] by sweeping the bias current. The model encompasses a non-linear optical susceptibility that accounts for peculiar features of radiation-matter interaction in semiconductors lasers such as asymmetric gain/dispersion curves and phase-amplitude coupling due to the linewidth enhancement factor (LEF) [5]. Interestingly, in [2] the existence in a unidirectional ring configuration (where spatial hole burning due to carriers grating cannot occur) of a continuous wave instability close to the lasing threshold was demonstrated; an original result in perfect agreement with very recent experimental evidences [6]. In the following we extend the ESMBEs of [2] to include a detuned, coherent driving field; assuming fast medium variables and a laser close to threshold, we derive the Generalized LLE equation (Eq. (1) of the main text in the active case).

The set of ESMBEs in presence of an injected signal $E_I$, in the additional hypothesis of gain peak coincident with the reference frequency (frequency of the injected field) is

$$\frac{\partial E}{\partial t} + \frac{\partial E}{\partial z} = \frac{1}{\tau_p} \left[-(1 + i\theta_c)E + E_I + P + i\beta|E|^2E - i\tilde{c} \tau_p \frac{k''}{2} \frac{\partial^2 E}{\partial t^2}\right] $$

(1)

$$\frac{\partial P}{\partial t} = \frac{1}{\tau_d} (1 - i\alpha) [(1 - i\alpha)ED - P] $$

(2)

$$\frac{\partial D}{\partial t} = \frac{1}{\tau_c} \left[\mu - D - \frac{1}{2} (E^*P + EP^*)\right] $$

(3)

where $\tau_p$, $\tau_d$ and $\tau_c$ are the photon lifetime, the dephasing time and the carriers lifetime respectively, $\mu$ is the pump parameter scaled to its threshold value. The photon lifetime is defined as $\tau_p = (\tilde{c}(\frac{T}{L} + \alpha L))^{-1}$ and accounts for both localized coupling losses, through the (intensity) transmission coefficient $T$, and distributed waveguide (field) absorption losses through the parameter $\alpha L$. The variables $E$, $P$ and $D$ are scaled as in [2] while $E_I$ is introduced as in [7].

In Eq. (1) $\tilde{c} = c/n$ is the phase velocity in the active medium, while $\theta_c = (\omega_c - \omega_0)\tau_p$ where $\omega_0$ is the reference frequency equal to the frequency of the injected field and $\omega_c$ is the closest cavity resonance. The $\alpha$ factor in Eq. (2) is the LEF calculated at the gain peak. Finally, $\beta$ and $k''$ are the Kerr nonlinearity parameter and the second-order dispersion coefficient respectively [8].

Equations (1)-(3) satisfy the periodic boundary condition

$$E(0,t) = E(l,t).$$

(4)
The formal solutions of Eqs. (2) and (3) are

\[ P = (1 - i\alpha) \left(1 + \frac{\tau_d}{1 - i\alpha} \frac{\partial}{\partial t}\right)^{-1} ED, \quad (5) \]

\[ D = \mu - \frac{1}{2} \left(1 + \tau_d \frac{\partial}{\partial t}\right)^{-1} \left( E^* P + EP^* \right). \quad (6) \]

In the realistic hypothesis that the field evolves on the time scale of \( \tau_p \) much bigger than those of the medium \( \tau_d \) and \( \tau_e \) (fast polarization and carrier dynamics), i.e. assuming \( \frac{\partial}{\partial t} \ll \frac{1}{\tau_d} \) and \( \frac{\partial}{\partial t} \ll \frac{1}{\tau_e} \), the time operators can be expanded in power series. For the polarization \( P \) we must include terms up to second order in the expansion in order to keep information about the shape of the susceptibility, and we obtain

\[ P \simeq (1 - i\alpha) E D - \tau_d \frac{\partial (ED)}{\partial t} + \frac{\tau_d^2}{(1 - i\alpha)} \frac{\partial^2 (ED)}{\partial t^2}. \quad (7) \]

It turns out that the temporal operator in Eq. (6) applies to a quantity of order \( |E|^2 \). If the laser is very close to threshold \( |E|^2 \ll 1 \), we can then keep only the zero order term of the operator and we simply get

\[ D \approx \mu (1 - |E|^2) \simeq \mu - |E|^2, \quad (8) \]

where we have taken into account that \( \mu \simeq 1 \) (near threshold operation), and insert it in Eq. (7) to finally obtain

\[ P \simeq (1 - i\alpha) E (\mu - |E|^2) - \tau_d \frac{\partial E}{\partial t} + \frac{\tau_d^2}{(1 - i\alpha)} \frac{\partial^2 E}{\partial t^2}. \quad (9) \]

We remark that in presence of an injected field of amplitude \( E_I \) the approximated expression (8) also requires \( E_I \ll 1 \). In Eq. (9) the time derivative is always multiplied by \( \tau_d \). In order to have the same temporal scaling, when we insert it in Eq. (1) we multiply both sides of that equation by \( \tau_d \) and we obtain

\[ \frac{\partial E}{\partial z} + \tau_d \frac{\partial E}{\partial t} = \frac{\tau_d}{\tau_p} \left[(1 + i\theta_c)E + E_I + \mu(1 - i\alpha)E - (1 - i\alpha)|E|^2 E \right. \]

\[ + \left. \frac{\tau_d^2}{(1 - i\alpha)} \frac{\partial^2 E}{\partial t^2} + i\beta|E|^2 E - i\overline{\epsilon}\tau_p \frac{k''}{2} \frac{\partial^2 E}{\partial z^2}\right], \quad (10) \]

where we have neglected the first order derivative \( \tau_d^2 \frac{\partial^2 E}{\partial t^2} \) with respect to \( \tau_d \frac{\partial E}{\partial t} \) because \( \tau_d \ll \tau_p \). In the same limit we can replace \( \frac{\partial^2 E}{\partial t^2} \) by \( \overline{\epsilon} \frac{\partial^2 E}{\partial t^2} \) and thus we have

\[ \frac{\partial E}{\partial z} + \tau_p \frac{\partial E}{\partial t} = \frac{1}{\tau_p} \left[(1 + i\theta_c)E + E_I + \mu(1 - i\alpha)E - (1 - i\alpha)|E|^2 E \right. \]

\[ + \left. \frac{1}{(1 - i\alpha)} \frac{\partial^2 E}{\partial z^2} + i\beta|E|^2 E - i\overline{\epsilon}\tau_p \frac{k''}{2} \frac{\partial^2 E}{\partial z^2}\right] \quad (11) \]

We now pass to a moving reference frame by setting \( z - \overline{\epsilon}t \rightarrow z \), we define \( \theta_0 = \theta_c - \mu\beta, \Delta = \alpha + \beta, d_R = (\overline{\epsilon}\tau_d)^2/(1 + \alpha^2) \) and \( d_I = d_R(\alpha + \zeta) \) with \( \zeta = -k''\overline{\epsilon}\tau_p (1 + \alpha^2) / (2\tau_d^2) \) and we get

\[ \tau_p \frac{\partial E}{\partial t} = E_I + (1 + i\theta_0)E - (1 - i\Delta)(\mu - |E|^2)E - (d_R + id_I) \frac{\partial^2 E}{\partial z^2}, \quad (12) \]

Finally, introducing the scaling

\[ \tau = rt/\tau_p, \quad \eta = \frac{z}{d_R} \sqrt{\frac{r}{d_R}}, \quad F = \frac{E}{\sqrt{r}}, \quad F_I = \frac{E_I}{r^{3/2}}, \quad (13) \]

and setting

\[ \theta = (\theta_c + \mu\alpha) / r, \quad G = d_I / d_R, \quad (14) \]

Eq. (12) becomes

\[ \frac{\partial F}{\partial \tau} = F_I + (1 - i\theta) F - (1 - i\Delta)|F|^2 F + (1 + iG) \frac{\partial^2 F}{\partial \eta^2} \quad (15) \]
that coincides with Eq. (3) in the main text with $\gamma = 1$.

[1] F. Prati and L. Columbo, Long-wavelength instability in broad-area semiconductor lasers, Phys. Rev. A 75, 053811 (2007).
[2] L. Columbo, S. Barbieri, C. Sirtori, and M. Brambilla, Dynamics of a broad-band quantum cascade laser: from chaos to coherent dynamics and mode-locking, Opt. Express 26, 2829 (2018).
[3] J. Faist, Quantum Cascade Lasers (Oxford University Press, 2013).
[4] H. Li, P. Laffaille, D. Gacemi, M. Apfel, C. Sirtori, J.Leonardon, G. Santarelli, M. Rosch, G. Scalari, M. Beck, J. Faist, W. Hansel, R. Holzwarth, and S. Barbieri, Dynamics of ultra-broadband terahertz quantum cascade lasers for comb operation, Opt. Express 23, 33270 (2015).
[5] W. W. Chow and S. W. Koch, Semiconductor-Laser Fundamentals (Springer, 1999).
[6] M. Piccardo, B. Schwarz, D. Kazakov, M. Beiser, N. Opaak, Y. Wang, S. Jha, J. Hillbrand, M. Tamagnone, W. Chen, A. Zhu, L. Columbo, A. Belyanin, and F. Capasso, Frequency combs induced by phase turbulence, Nature 582, 360 (2020).
[7] L. Lugiato, F. Prati, and M. Brambilla, Nonlinear Optical Systems (Cambridge University Press, 2015).
[8] N. Opaak and B. Schwarz, Theory of Frequency-Modulated Combs in Lasers with Spatial Hole Burning, Dispersion, and Kerr Nonlinearity, Phys. Rev. Lett. 123, 243902 (2019).