Straggler Mitigation through Unequal Error Protection for Distributed Matrix Multiplication

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Abstract—Large-scale machine learning and data mining methods routinely distribute computations across multiple agents to parallelize processing. The time required for computation at the agents is affected by the availability of local resources: this gives rise to the “straggler problem” in which the computation results are held back by unresponsive agents. For this problem, linear coding of the matrix sub-blocks can be used to introduce resilience toward straggling. The Parameter Server (PS) codes the matrix products and distributes the matrix to the workers to perform multiplication. At a given deadline, it then produces an approximation the desired matrix multiplication using the results of the computation received so far. In this paper, we propose a novel coding strategy for the straggler problem which relies on Unequal Error Protection (UEP) codes. The resiliency level of each sub-block is chosen according to its norm, since blocks with larger norms affect more the approximation of the matrix multiplication. We validate the effectiveness of our scheme both theoretically, as well as through numerical evaluations. We derive a theoretical characterization of the performance of UEP using random linear codes and compare it the case of equal error protection. We also apply the proposed coding strategy to the computation of the back-propagation step in the training a Deep Neural Network (DNN). In this scenario, we investigate the fundamental trade-off between precision of the updates versus time required for their computation.

Index Terms—Distributed computation; Approximate matrix multiplication; Straggling servers; Unequal error protection.

I. INTRODUCTION

Distributed computing clusters are fundamental in many domains, such as machine learning, data-mining, high-precision numerical simulations as they allow parallelization of the computational tasks [1]. The widespread reliance on distributed computation clusters presents several opportunities over traditional computing paradigms, but also offer a new set of challenges. Among the most well-recognized issues is that of the stochasticity in the time required for computation. This gives rise to the phenomena of “stragglers”, that is agents with large response times which delay computation. In this scenario, coding can be applied to reduce the delays in computation resulting from stragglers [2]. In this paper, we propose a novel scheme for distributed computation with straggling which makes use of the variations in the magnitude of the matrix entries which naturally occurs in many applications, such as back-propagation in DNN training. We first identify the matrix sub-products which have the largest norm and use UEP to provide a level of resiliency to straggling which is proportional to this norm. This offers an improved resilience to straggling when considering the approximate reconstruction of a matrix product by a given computation deadline.

A. Literature Review

As matrix multiplication is a fundamental algebraic operation, distributed approximate matrix multiplication has been investigated in many contexts. In the big-data paradigm, computation and storage are distributed, so that computer processing architectures can be devised for efficiently performing this operation [3], [4]. In a cloud-computing setting, distributed matrix computation is investigated in [5], [6]. DNN training through back-propagation involves large matrix multiplication: distributed matrix computation for this scenario is studied in [7], [8]. More recently, the problem of “stragglers” has emerged as a promising research direction. In many distributed computation platforms such as Amazon Web Services Lambda and Google Cloud Functions distributed computation can be held back by a set of workers which take much longer than the median job time [9]. Such random delays decrease the overall computational efficiency of the system. To mitigate the effect of straggling, coding for matrix multiplication can be applied [10]. Since its inception in [11], this research received much attention in the literature. In [12], the authors use the theory of extreme order statistics to analyze how task replication reduces latency. In [13], the authors introduce redundant computations in a coding theory inspired fashion, for computing linear transforms of long vectors. Product codes for distributed matrix multiplication are studied in [14]. A new class of codes, polynomial codes, is proposed in [15], and their optimality is argued for the straggling problem. The above literature focuses on minimizing the time for completing a computation tasks: one can also consider the issue of approximate computation. In [5], the authors consider the approximate matrix computation.
scenario and propose OverSketch, an algorithm that uses matrix sketching to approximate matrix multiplication.

B. Contribution

In our paper, we investigate the trade-off between accuracy and delay in distributed approximate matrix multiplication with stragglers. Multiplication of large sparse matrices is an important problem in implementing machine learning algorithms. For typical machine learning problems, only approximate matrix multiplication results are sufficient, hence we consider a distributed matrix multiplication scheme in which the sub-blocks of the matrices being multiplied are coded using UEP codes and the distributed across different workers. The workers respond with the results of the products (of coded sub-products and the distributed across different workers. The main goal is to produce an approximation of product of the two matrices as quickly as possible, with a more and more accurate approximation with more and more workers responding, i.e., a progressively improving matrix approximation in time, exploiting the code constraints.

Our main contribution is the introduction of UEP codes to improve the quality of the approximation by exploiting the variation in the matrix entries’ magnitude. By carefully matching the matrix sub-products norm with the level of unequal error protection, significant improvements can be attained over the case of equal error protection or un-coded computation. More specifically:

- **Sec. II – System Model**: We formulate the distributed approximate matrix multiplication problem by considering the case in which the multiplication is broken down in a set of row-times-product block products which is distributed to workers with a random computation time.
- **Sec. III – Approximate Matrix Multiplication with UEP Codes**: We present our proposed scheme in which UEP codes are used to code the row and column terms of the block matrix multiplication. In particular, we leverage the construction of in [16] to offer more protection to the sub-products with larger norm and reduce the effect of the randomness in the service time.
- **Sec. IV – Theoretical Analysis**: We use the results in [16] to provide a theoretical evaluation of the expected error in the matrix approximation as a function of the service time distribution.
- **Sec. V – Numerical Evaluations**: We present the results of a DNN training when the back-propagation step is distributed among workers as in the proposed scheme. This example is particularly relevant since the matrices of the different DNN layers have a different level of sparsity and thus result in matrix with very varied matrices to be multiplied. Furthermore, we illustrate the loss performance of the proposed strategy using simple matrix models, and compare it with the Maximum Distance Separable (MDS) codes.

The paper is concluded in Section VI.

**Notation**: Matrices are indicated with bold capital roman letters, i.e. A, column vectors with bold lower-case roman letters, i.e. v. The Frobenius norm of the matrix A is indicated as \(\|A\|_F\).

The PS is to produce an approximate expression for the matrix multiplication \(C = AB\) as \(\hat{C}\) by the loss\(^1\)

\[
L(C, \hat{C}) = \|C - \hat{C}\|_F^2.
\]

To accomplish this, the PS divides the matrix products into sub-products and distributes them across a set of workers. Specifically, as in [17], we partition A/B into row/column sub-blocks \(A_n/B_p\) as

\[
A = [A_1; \cdots; A_n; \cdots; A_N]
\]

\[
B = [B_1, \cdots; B_p; \cdots; B_P],
\]

\(^1\)In the following, we only consider the case of a Frobenius norm; the case of a more general loss is not discussed here for brevity.
where $\mathbf{A}_n \in \mathbb{F}^{U \times M}$ and $\mathbf{B}_p \in \mathbb{F}^{M \times Q}$ for $n \in [N]$, $p \in [P]$, $\mathbf{C}_{np} \in \mathbb{F}^{U \times Q}$ and $NP$ such matrix multiplications are needed to produce $\mathbf{C}$. However, in general, not all the sub-blocks have the same impact on the final product, as some sub-blocks may have larger Frobenius norm. This motivates the use of UEP codes to better protect the more impactful sub-products when distributing the computation to the workers. Accordingly, we consider that the PS sends to each worker $w$ the matrices $\mathbf{W}_w^A$ and $\mathbf{W}_w^B$ obtained as

$$
\mathbf{W}_w^A = f_{\text{enc-}A}(\mathbf{A}_1, \cdots, \mathbf{A}_N) \quad (3)
$$

$$
\mathbf{W}_w^B = f_{\text{enc-}B}(\mathbf{B}_1, \cdots, \mathbf{B}_P),
$$

and sets a time dead-line $T_{\text{max}}$ by which it expects the matrix product $\mathbf{W}_w^A \mathbf{W}_w^B$ to be returned. At time $T_{\text{max}}$, the PS produces the matrix approximation

$$
\mathbf{\hat{C}} = f_{\text{dec-}C}(\mathbf{W}(T_{\text{max}})), \quad (4)
$$

where $\mathbf{W}(T_{\text{max}})$ is the set of matrix products received up to time $T_{\text{max}}$. Using $\mathbf{\hat{C}}$, the loss in (1) can be evaluated. Let us denote this loss as $\mathcal{L}(T_{\text{max}})$. The random set $\mathbf{W}(T_{\text{max}})$ and is obtained as follows. We assume that the computation time of the worker $w$ is equal to the random variable $T_w$ which is identical and independently distributed (i.i.d.) with a cumulative distribution function (CDF) $F(W_t)$. 2 Accordingly, the set $\mathbf{W}(T_{\text{max}})$ contains the products $\mathbf{W}_w^A \mathbf{W}_w^B$ for which $T_w < T_{\text{max}}$.

The problem we consider next is to optimally design the functions $f_{\text{enc-}A}$, $f_{\text{enc-}B}$ and $f_{\text{dec-}C}$ such that the loss in (1) is minimized over some dataset of matrix multiplications $\mathcal{D}(\mathbf{A}, \mathbf{B})$.

III. APPROXIMATE MATRIX MULTIPLICATION WITH UEP CODES

In this section, we describe a distributed and coded approximate matrix multiplication scheme which aims to provide better protection for the matrix sub-products $\mathbf{A}_n, \mathbf{B}_p$ with larger norm, and thus produce a better matrix approximation to within the prescribed time. The coding scheme is parameterized in such a way as to match the distribution of the matrices in the data-set.

A. Importance Level of a Sub-block

Let us begin by classifying the matrix sub-block in (2) according to their norm. For instance, we may select three different levels for each sub-block, i.e., high, medium, and low to classify the norm of $\mathbf{A}_n$ and $\mathbf{B}_p$. Let us refer to these levels as importance level and let there be $S$ such levels. Further, given a matrix $\mathbf{A}/\mathbf{B}$, we have $n_s^A/n_s^B$ for $s \in [S]$ as the number of blocks with level of importance (let the importance be decreasing in $s$). Clearly, $N = \sum_{s \in [S]} n_s^A$, and $P = \sum_{s \in [S]} n_s^B$.

By construction, any $\mathbf{C}_{np}$ is obtained as the multiplication of sub-block in two classes: accordingly $\mathbf{C}_{np}$ has $L$ possible importance levels with $L = S(S+1)/2$. For instance, in the examples of three importance levels for $\mathbf{A}_n$ and $\mathbf{B}_p$, $\mathbf{C}_{np}$ can have importance high × high, high × medium, high × low, medium × medium, etc.

From a high level-perspective, one would want the PS to be able to more quickly recover those products corresponding to the importance level high × high, while the importance level low × low is not particularly urgent. We can obtain this desired behavior by employing UEP codes.

B. UEP Coded Matrix Multiplication

Let us consider the case in which the encoding functions in (3) as the UEP codes described in [16]. Specifically, we consider the use of two different UEP schemes called Non-overlapping Windows (NOW) UEP and Expanding Window (EW) UEP strategies based on Random Linear Codes (RLC) from [16]. Let us briefly introduce these codes next.

- The NOW-UEP coding strategy simply divides the packets into “windows” based on their importance, and applies RLC code for each type independently and separately. The encoding is performed firstly by selecting a window using a window selection polynomial function $\Gamma(\xi) = \sum_{i \in [I]} \Gamma_i \xi^i$, where $I$ is the number of block types, and $\Gamma_i$ is the window selection probability for the $i$-th type. Then, the encoded matrices are generated only from the matrices of the selected type. The PS selects importance levels for both $\mathbf{A}$ and $\mathbf{B}$ independently using predetermined window selection distributions, and encodes the corresponding rows and columns of $\mathbf{A}$ and $\mathbf{B}$ as

$$
\mathbf{W}_w^A = \sum_i \alpha_i^w \mathbf{A}_{\pi^A(i)}
$$

$$
\mathbf{W}_w^B = \sum_j \beta_j^w \mathbf{B}_{\pi^B(j)},
$$

for the $w$-th worker, where $\pi^A(i)/\pi^B(j)$ is the row/column indices of $\mathbf{A}/\mathbf{B}$ at the corresponding levels, respectively.

- The EW-UEP coding strategy also uses a probabilistic window selection polynomial $\Gamma(\xi)$ for row and column class selection of $\mathbf{A}$ and $\mathbf{B}$, respectively; but the window definition is different than the NOW-UEP strategy. The EW-UEP constructs the $i$-th window by including all packets whose importance level is $i$ or higher than $i$. For instance, let us assume that the third importance level is selected according to the window selection distribution. This strategy includes all the source messages from the first, second, and third importance levels. Thus, the EW strategy includes the most important matrices to the encoding process regardless of the importance level of the selected window to provide better protection than others.

In both cases, the decoding function simply places $\mathbf{\hat{C}}_{np} = \mathbf{C}_{np}$ when it is possible to obtain from $\mathbf{W}(T_{\text{max}})$ and $\mathbf{C}_{np}$ equal to the all-zero matrix otherwise.

IV. THEORETICAL ANALYSIS

In [16], the authors give the corresponding decoding probabilities of NOW and EW-UEP strategies for each importance levels as a function of the number of received packets in each
class. In this section, we assume for simplicity that the entries of the matrix are zero mean and with variance \( \sigma^2_{A_k} \) and \( \sigma^2_{B_k} \) in each sub-block, so that \( \mathbb{E}[\|C_{np}\|^2] = MUPQ\sigma^2_{A_k}\sigma^2_{B_k} \). Let us denote the probability of receiving \( w \) packets from \( W \) workers at time \( t \) by \( P_{N(t)}(w) \) which is simply calculated by

\[
P_{N(t)}(w) = \left( \frac{W}{w} \right) (1 - F(t))^{W-w} (F(t))^w.
\]

From [16, eq. 5], we obtain a bound on the decoding probabilities of NOW-UEP strategy for each importance level as a function of received matrices as

\[
P_{d,l}(N) \leq \sum_{(n_1,n_2,...,n_L), n_1 + \cdots + n_L = N} \left( \frac{n!}{n_1!n_2!\cdots n_L!} \Gamma^{n_1}_{1}\Gamma^{n_2}_{2}\cdots\Gamma^{n_L}_{L} \right).
\]

With three classes, \( W = 40 \) workers, and window selection probabilities \((0.35, 0.35, 0.3)\), the decoding probabilities of each class are given in Fig. 1 which clearly illustrates that the most important class is protected better.

Let us denote the number of encoded matrix products received at time \( t \) by \( N(t) \), then we can bound the performance of the coding system in Sec. III as follows.

**Theorem 1. NOW-UEP loss:** Consider the loss minimization problem in Sec. II for the case in which the set of matrix product \( D(\{A, B\}) \) is the set of matrices with i.i.d. entries with variance \( \sigma^2_{A_k} \) and \( \sigma^2_{B_k} \), respectively. The NOW-UEP strategy described in Sec. III is

\[
\mathbb{E}[L(T_{max})] = \sum_{w \in [W]} P_{N(T_{max})}(w)\mathbb{E}[\|C - \hat{C}\|^2_F | w],
\]

where

\[
\mathbb{E}[\|C - \hat{C}\|^2_F | N(t)] = \sum_{i \in [L]} n_i (1 - P_{d,l}(N(t))) MUPQ\sigma^2_{A_i}\sigma^2_{B_i},
\]

and where the expectation in (9) is on the random entries of \( A, B \).

Note that (9) is a bound since (7) is an upper bound, but it is tight as the field size tends to infinity, i.e., the upper bound is asymptotically achievable. The analog of Th. 1 for the EW-UEP is obtained from the results in [16, eq. 4-9] and is not presented here for brevity. Note that, in Th. 1, there exists a “matching” between the probabilistic structure of the matrices to be multiplied and their row/column block partitioning. In actuality, one would not observe such a neat organization of the matrix values. Instead one would have to fit the row/column weight distribution in the data to design the UEP code resulting in the minimal loss.

**V. NUMERICAL EVALUATIONS**

In this section, we apply approximate matrix multiplication scheme using UEP codes proposed in Sec. III. For service time of workers, we use the exponential latency model, i.e., the completion time of each worker for a given task is exponentially distributed with parameter \( \lambda \).

**A. Matrix Approximation Using UEP**

In this section, we present performance assessment examples of distributed approximate matrix multiplication with UEP codes.

We consider the multiplication of \( A \in \mathbb{R}^{NU \times M} \) and \( B \in \mathbb{R}^{M \times PQ} \) with the help of \( W = 40 \) workers whose task completion times are modeled by exponential latency model with parameter \( \lambda = 0.25 \). At time \( t \), the approximation is performed by only using the worker responses up until time \( t \), and the rest are ignored. We select \( U = Q = 5 \), and \( M = 100 \).

Firstly, as discussed earlier, we classify each row and column blocks of \( A \) and \( B \) with importance level high, medium,
and low. The element of each block is i.i.d. and distributed with $\mathcal{N}(0, 10), \mathcal{N}(0, 1),$ and $\mathcal{N}(0, 0.1)$, for high, medium, and low levels, respectively. We assume that both A and B have only one instance of row and column from each level, i.e., $N = 3, P = 3$. We take the multiplication of high and high blocks as class one, high and medium blocks as class two, and the remaining as class three. With this new class definition, we have $(n_1, n_2, n_3) = (1, 2, 6)$ sub-blocks in each class. We select the window selection probabilities for NOW and EW-UEP strategies as $\Gamma_1 = 0.35, \Gamma_2 = 0.35, \Gamma_3 = 0.3$.

The decoding probability for each class are obtained through the formulation given in [16], which is also illustrated for NOW-UEP strategy in Fig. 2. In Fig. 3, these decoding probabilities are used to obtain the normalized loss values as a function of time $t$ along with the performance of MDS codes. Until time $t = 1.7$, the UEP protection with both NOW and EW performs better than that of MDS, since they enable early recovery of important classes with a small number of received packets. For instance, at time $t = 1$ the MDS gives 0.46 normalized loss which is extremely high while the EW-UEP strategy provides loss of 0.088 which is significantly lower than MDS and close to perfect recovery. Hence, if we are interested in the earlier recovery of certain important parts, using UEP codes for matrix approximation is advantageous. After time $t = 1.7$, MDS code starts to perform better than the others since it can fully recover C after receiving nine packets. However, if we wait long enough, the UEP strategy will also fully recover the desired matrix.

For further interpretation, we give the normalized loss values of matrix multiplication with MDS and approximate matrix multiplication using NOW and EW-UEP coding in Fig. 4 as a function of number of received packets. The matrix multiplication with MDS needs to receive $\sum_{l \in L} n_l$ packets to fully recover the multiplication, where $n_l$ is the number of packets in the $l$-th level. Receiving less than $\sum_{l \in L} n_l$ will not provide any partial information, and results in no recovery, hence the normalized loss for MDS is one until it receives nine packets (the minimum required for recovery). However, matrix approximation with NOW and EW-UEP coding strategies start to recover more important classes just after receiving very few packets and continue to provide additional partial information after each received block.

It is also worth noting that we choose window selection distributions for the UEP codes arbitrarily. As a further improvement, the window selection distribution can be optimized to minimize the loss in the matrix approximation.

### B. ML performance with UEP Coded Matrix Multiplication

In this section, we present the accuracy of the DNN models trained using NOW-UEP, EW-UEP, and Block Repetition for the coding and calculations of the gradient matrix. More specifically, we present the classification results for the MNIST dataset, containing 60,000 training samples with the DNN model defined in Table II. The model is trained using Stochastic Gradient Descent (SGD) optimizer, with a learning rate of 0.01, and cross-entropy as loss. The images are passed through the model in mini batches of 64 at a time over two or three epochs, depending on the learning speed.

We assume that the computation arrival distribution is described by the CDF $F(t)$ when using a single worker, furthermore, when the PS distributes the work to $W$ workers, the completion of the tasks is assumed to be a scaled distribution of a single worker, i.e., $F(Wt)$ is the new CDF. The completion of the task by the workers is assumed to be independent of each other. For these simulations, we used $\lambda = 0.5$ with exponential latency model and $T_{\text{max}} = 0.25, 0.5, 1, 2$.

We used the encoding parameters defined in Table III for encoding the gradients in each dense layer shown in Table II. As for ensuring the higher weight values are biased towards the same location of the matrix, the indexes are permuted in descending order of the column/row weights before using the UEP coding. This permutation idea is similar to the one used in [18] which proposes a fast matrix multiplication algorithm. The $W$ value for the uncoded case is used as a reference for scaling the number of workers of other encodings.

As the results for Fig. 6, UEP shows its advantage in not requiring all the workers to complete their tasks and being able to achieve closer approximations in a shorter time span. UEP can be very close to the performance of the baseline strategy where a single worker is used with full precision. Block repetitions, although useful for increasing the probability of receiving a block back, sacrifices the number of workers required and does not ensure a shorter time for block retrieval than the uncoded at any point.
the sub-operations with larger norms more). We validate the effectiveness of the proposed approach with analytical assessments based on simplified models for sparse matrices, and we compare our results with MDS codes via simulations. Furthermore, the proposed coded approximation strategy is applied to the backpropagation steps of a DNN where different matrices are multiplied. Our results clearly show that we can have a performance close to the centralized training by balancing the trade-off between the precision of the updates versus the computation time.

VI. CONCLUSION

In this paper, we study distributed approximate matrix multiplication using UEP codes which aims to mitigate stragglers and speed up the large-scale operations which are common in machine learning and data mining methods. We use UEP codes to provide better protection for the sub-operations which have a higher effect on the resulting matrix (e.g., by protecting

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