ABSTRACT

The helicity-effect problem for a single compressible flow is transformed to the pure-background-field-effect one, i.e., to the situation with background uniform rotation and/or magnetic field but without any helicity, reducing to the pure (compressible-version) Taylor-Proudman effect (TPE) and/or its magnetic analogue. A chiral base flow/field (CBF) with two-dimensional horizontal velocity components is used to materialize the screw-and-knot scenario for the ‘tightening-up’ notion, raised from the statistical result of helicity-reducing-turbulence-compressibility effect found earlier for the neutral-gas case. In the CBF, the existence of helicity nontrivially indicates a kind of mean rotation, naturally invoking the compressible version of the TPE, with horizontal-compressibility reduction but without direct constraint on the vertical velocity, which serves as the genuine mechanism, i.e., the underlying element of the statistical effect. The statistical fluctuations in the compressibility reduction effect may be due to that of the variation of the vertical derivative of the vertical velocity. A minimal working model with disorders in the CBFs bridges the single-flow and statistical arguments, completing the story. I further argue a posteriori that recent data, of the superfluid and Bose-Einstein condensate model, also agree the result from my previous a priori analysis. Statistical mechanical analyses, of compressible magnetohydrodynamics (MHD) and extended MHD for the ionized gas flows, show that helicities may reduce compressive and density modes relevant to the compressibility, i.e., tightening up the ionized-gas turbulence, implying the universality of the notion. And, a unified view is offered, with substantial extensions of the CBF and the analogue of the compressible TPE for a strong background magnetic field encapsulating the geometry of the Alfvén theorem for the (magnetised) plasma.

Keywords: helicity, compressibility, Taylor-Proudman effect, magnetosonic noise, aeroacoustics, superfluid/quantum hydrodynamics turbulence, (rotating) Bose-Einstein Condensate, plasma turbulence

1. INTRODUCTION; A PRIMA FACIE EVIDENCE FROM QUANTIZED VORTEXES’ AND BEYOND

In a previous paper (Zhu 2016, hereafter referred to as I), a helicity effect of depressing the compressibility relevant modes was obtained, with the analysis of the polarized absolute equilibrium and dissipation, and used to make the conjecture of turbulence ‘noise’ reduction (in the context of aeroacoustics). Numerical data have convincingly verified such a prediction directly (Yang et al. 2019). Actually, it appears, in a looser sense, that the prediction had already been indirectly verified in the numerical turbulence data of a nonlinear Schrödinger (Gross-Pitaevskii) equation for the low-temperature superfluid or Bose-Einstein condensate by Clark di Leoni, Mininni & Brachet (2016): They show that for some amount of time (before \( t = 4 \) there) both the energy and helicity, by their regularization, are well conserved in the simulation as presented in their figure 6, and when the constraint of the helicity calculated there is eased (\( t > 4 \)), compressive and quantum modes (‘noise’ in the context of quantum acoustics) are greatly released with the reduction of the incompressible mode. Given the a priori arguments of Zhu (2016) for helical barotropic flows and the study by Clark di Leoni, Mininni & Brachet (2016), an a posteriori analysis can be made to explain their observations and to predict that if the helicity (as they defined or some other working analog) may somehow be ‘manipulated’ in a nice way, the quantum ‘noise’ can also be controlled.
The scenario that vortex knots (Moffatt 1969) or (streamline) screws (Betchov 1961) tighten up the field then emerges (and will be materialized below). Helicity can be conserved in the Eulerian sense (Betchov 1961) for a ‘fixed’ frame with appropriate boundary conditions or along with the flow in the Lagrangian sense (Moreau 1961). Knot-theory/topological notion has led to heuristic observation and phenomenological theory for the energy of knots and links (Moffatt 1990) with preceding and paralleling series of systematic results by Freedman and He (c.f., Arnold & Khesin (1998) and references therein), but for incompressible turbulence (Frisch 1995), some very fundamental insights pertain to the persistent thermalization effects (Lee 1952; Kraichnan 1955, 1973; Frisch et al. 1975, 2008, 2012; Zhu, Yang & Zhu 2014) of the Galerkin-truncated systems. While the nature of compressible turbulence is under debate (Éyink & Drivas 2018), a fundamental knowledge, such as turbulence aeroacoustics concerning energy partition (Kraichnan 1955), is wanted; and, the first helical statistical analysis of compressible flows (Zhu 2016) thus should be carried further. We are discussing helicity in turbulence statistical mechanics, thus unifying the ideas of Betchov (1961) and Moffatt (1969) by further developing the analysis of paper I. Synthesizing and developing the works of Lee (1952), Kraichnan (1955), hereafter referred to as K55), Frisch et al. (1975), Zhu, Yang & Zhu (2014), Miloshevich, Lingam & Morrison (2017) and paper I, this work will show a quite universal mechanism of helicities reducing the compressibility of (‘tightening-up’) various mediums. For the focused ionized gases, magnetic b ropes are of significance, but as mentioned in Zhu, Yang & Zhu (2014), in the kinetic view of the gyro-motion frequencies of charged particles, b ropes can actually be imagined to be the vortex ropes representing the ‘microscopic’ motion. Such a unified view may help excluding possible surprise about the universal mechanism from the beginning. The heuristic is of course best depicted with “discrete vortex field” (as termed by Moffatt 1969) which is perfectly represented by the phase defect in quantum fluid, thus we now slightly elaborate the analysis of the latter, before which we should briefly summarize the relevant results of paper I:

We work in a cyclic box of dimension $2\pi$, thus $V = (2\pi)^3$, and apply the Fourier representation any variable $v(\mathbf{r}) \rightarrow \hat{v}(\mathbf{k})$, thus $u(\mathbf{r}) = \sum_k \hat{u}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r})$ with $i^2 = -1$. Helmholtz decomposition of $\hat{u}(\mathbf{k})$ results in transverse/vortical and compressive components respectively perpendicular and parallel to $\mathbf{k}$, and the transverse velocity field is further decomposed into left and right-handed chiral modes (Cambon & Jacquin 1989; Waleffe 1992; Chen, Chen & Éyink 2003; Biferale, Musacchio & Toschi 2013)

$$
\hat{u}(\mathbf{k}) = \hat{u}_+(\mathbf{k})\hat{h}_+(\mathbf{k}) + \hat{u}_-(\mathbf{k})\hat{h}_-(\mathbf{k}) + \hat{u}_i(\mathbf{k})\mathbf{k}/k,
$$

with $i\mathbf{k} \times \hat{h}_s = sk\hat{h}_s$ and $s = \pm$ (denoting opposite — right- v.s. left-handed — screwing directions, or chiralities, around $k$), and that the one-dimensional (1D) kinetic energy spectrum

$$
E(k) = \frac{1}{2} \sum_{|k|=k} (|\hat{u}_+|^2 + |\hat{u}_-|^2 + |\hat{u}_i|^2) =: E_- + E_+ + E_i.
$$

Each of the definitions for the right hand side is self-evident. Working with $p = c^2\rho$ (with the sound speed $c = 1$ hereafter for simplicity) and applying the extra constraint of helicity ($\mathcal{H} = \frac{1}{2} \sum_k k(|\hat{u}_+|^2 - |\hat{u}_-|^2)$), the absolute equilibrium analysis found (Zhu 2016) that more energy (than that without helicity constraint Kraichnan 1955) is normalized (than without helicity constraint Kraichnan 1955) to the statistical average of the vortical modes $\langle |\hat{u}_+|^2 \rangle + \langle |\hat{u}_-|^2 \rangle$. Since the absolute values depend on the choice of physical units and since the overall energy levels are different when comparing different cases, which requires normalization, the more general conclusion is that the fractions of the compressive mode $\langle |\hat{u}_i|^2 \rangle$ and the density mode $\langle |\rho|^2 \rangle$ (or the potential energy of pressure or the work part of the internal energy (Kraichnan 1955)) are reduced, which can persist in the turbulent state and could even be partly strengthened by the polarized damping when the viscous effects are carefully examined (Zhu 2016) with theoretical considerations favoring and disfavoring Kraichnan (1955); in words, the turbulence compressibility is reduced, or the turbulent gas is tightened up. We now are ready to address the discrete vortexes of quantum turbulence and the statistical tightening-up effect of helicity, but readers not interested in this can skip this and go to Sec. 2.1 to check Fig. 1 for the caricature.

Below I will argue that Clark di Leoni, Mininni & Brachet (2016) have ‘indirectly’ confirmed, not only prima facie but probably essentially, the previous statistical prediction (Zhu 2016), for which I need to show the calculation of paper I can be carried over, mutatis mutandis: —

We know that the equation

$$
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla \left( \frac{\rho}{2} \right) - \frac{1}{\rho} \nabla (\rho \nabla^2 \ln \rho^{-1}) =: -\nabla p
$$

(3)
is obtained from the cubic nonlinear Schroedinger (Gross-Pitaevskiiii) equation

\[ i \partial_t \psi + \nabla^2 \psi - |\psi|^2 \psi = 0, \]  

(4)

by the Madelung transform connecting the quantum order field and the fluid field

\[ \psi = \sqrt{\rho} \exp\{i \phi\}, \quad u = \nabla \phi, \]  

(5)

which is singular for \( \rho = 0 \) on some lines in the 3-space. On such vortex lines \( \phi \) is not well-defined by \( \psi \) itself, and curdles the vorticity with quantized circulation. We have written everything in nondimensional form and taken the relevant parameters to be unit(s). See, e.g., Nore, Abid & Brachet (1997) for particularly explicit demonstration of the physical dimensions and various parameters. The second term in the barotropic pressure \( p \) of Eq. (3) is called the quantum pressure.

First, we need to check the Liouville theorem: Due to no explicit presence of \( u \), the new quantum pressure does not break the original Liouville theorem of the phase flow in \( k \)-space.

Then we check that the energy is in the appropriate form that suitable for carrying over the previous analysis: Now, the potential energy (only this ‘work’ part of the internal energy is relevant for such barotropic flows: e.g., Ziman 1953)

\[ w = \int_{\rho_0}^\rho \frac{p - p_0}{\rho^2} d\rho. \]  

(6)

[Alternatively, we can use the thermodynamics variables from the well-known action formulation (e.g., Lund 1991)]. Taking \( \rho = \rho_0 \exp(\zeta) \) and \( \rho_0 = 1 \), with appropriate scaling of the variables for simplicity of algebra, we immediately found, to leading order for ‘weak-excitation’/small \( \zeta \) as K55,

\[ w = \frac{\zeta^2}{2} - \int_0^\zeta \nabla^2 \zeta d\zeta \]  

(7)

Averaging over the volume (\( V \)) and working in a cyclic box of dimension \( 2\pi \), and noting that a constant in the thermodynamic variable is not relevant and indeed is not involved in the absolute equilibrium calculation (as can also be seen in K55), we have the following quadratic expression in both \( r \)- and \( k \)-space [with the Fourier coefficients \( \zeta(k) \)]

\[ W := \frac{1}{V} \int w d^3 r = \frac{1}{2V} \int (\zeta^2 + |\nabla \zeta|^2) d^3 r = \frac{1}{2} \sum_k (1 + k^2) |\zeta(k)|^2. \]  

(8)

As for the helicity, there is no consensus on the \textit{a priori} determination of the helicity corresponding to the classical one (Scheeler et al. 2014; Zucher & Ricca 2015; Clark di Leoni, Mininni & Brachet 2016): Due to the singularity of the vortex lines as the phase defects, each regularization of quantum flows (implicitly) introduces some model of the ‘internal’ structure of the vortex line or phase twist, and has its own significance for definite physical process(es). A \textit{a posteriori} analysis is necessary at this point. And, in a recent study by Clark di Leoni, Mininni & Brachet (2016) it is found that the energy and helicity, as they defined, are both well conserved before some time and the quantum energy is much smaller, and that the helicity starts to ‘decay’ “shortly before the incompressible kinetic energy” does (together with the increase of the compressible modes: Clark di Leoni, Mininni & Brachet 2016, Fig. 6 and relevant discussions there), just the ‘tightening-up’ scenario! It then appears that the helicity they calculated is physically relevant and suitable for our analysis. Note that the necessary information is actually not their precise regularization scheme but the existence of some helicity which presents the relevant physical effect and which can be obtained from the field with appropriate regularization (probably non-unique). Such a helicity of course can be written appropriate as a quadratic form in Fourier space. The quantum vortex lines being discrete and not space filling, thus of zero 3-space volume with no contribution to kinetic energy according to the scaling of density (\( \rho \sim r^2 \)) and velocity (\( v \sim r^{-1} \), thus \( \rho r v^2 \) regular) as the distance to the vortex line \( r \to 0 \) (e.g. Nore, Abid & Brachet 1997), the regularization of velocity does not affect the calculation of the mean kinetic energy as in paper I, so the quadratic form (8) as in K55 assures that the analysis in paper I can be carried over.

Thus, to this point, we see, and will further develop in our analyses for ionized gases, that, with the velocity replaced by that of Clark di Leoni, Mininni & Brachet (2016) or other working regularizations with similar properties, all analyses and results of paper I are basically unchanged (thus not elaborated here) for such a quantum fluid. \( W \) is
2. HELICITIES TIGHTEN UP THE TURBULENT FLOWS OF IONIZED GASES (PLASMAS)

2.1. Demonstration and model of the tightening-up scenario

Before embarking on extending the statistical mechanics calculation of paper I to the ionized gases, it is helpful to materialize the tightening-up mechanism in the screw-and-knot scenario. I choose to start with a chiral base flow (CBF) which is helical and the two velocity (‘horizontal’) components of which is two-dimensional (2D). As note in (Zhu 2019), while the slightly more special but most familiar two-dimensional-three-component (2D3C) flow is the asymptotic state of the Taylor-Proudman effect (TPE) in the fast rotation limit of incompressible fluid, compressible TPE leads to the two-dimensionalisation of the horizontal flow with also additionally horizontal incompressibility. The reason of choosing CBF is also because that the structure is both simple and generic enough to make the tightening-up notion crystal clear.

Notion crystal clear. The reason of choosing CBF is also because that the structure is both simple and generic enough to make the tightening-up mechanism in the screw-and-knot scenario. I choose to start with a chiral base flow (CBF) which is helical and the two velocity (‘horizontal’) components of which is two-dimensional (2D). As note in (Zhu 2019), while the slightly more special but most familiar two-dimensional-three-component (2D3C) flow is the asymptotic state of the Taylor-Proudman effect (TPE) in the fast rotation limit of incompressible fluid, compressible TPE leads to the two-dimensionalisation of the horizontal flow with also additionally horizontal incompressibility. The reason of choosing CBF is also because that the structure is both simple and generic enough to make the tightening-up notion crystal clear.
2.1.1. CBF and TPE of tightening up with intrinsic rotation

As sketched in Fig. 1, for the vorticity field \( \omega = \nabla \times \mathbf{u} \) and the velocity field \( \mathbf{u} = \mathbf{u}_h + \mathbf{u}_v \) our CBF with uniformly \( \partial_x \mathbf{u}_h = 0 \), i.e., \( \partial_x u_x = \partial_y u_y = 0 \) with z-axis chosen to be the vertical one, \( \omega \) can be decomposed into the horizontal component

\[
\omega_h = (\partial_y u_z, -\partial_x u_z, 0) = \nabla \times \mathbf{u}_v \text{ lying in the } x-y \text{ plane (loops with arrows)}
\]

and the vertical component

\[
\omega_v = (0, 0, \partial_x u_y - \partial_y u_x) = \nabla \times \mathbf{u}_h \text{ along the } z \text{ axis (straight lines with arrows)}.
\]

Unlike Moffatt (1969) who further let \( \nabla \cdot \mathbf{u} = 0 \) and \( \partial_z \mathbf{u} = 0 \) but with ‘helicity’ (typical 2D3C helical incompressible flow), here neither \( \nabla \cdot \mathbf{u} \) nor \( \partial_z \mathbf{u} \) is required to vanish. However, in the barotropic case, both \( \omega_v \) and \( \omega_h \) are still ideally frozen-in to the flow (in Helmholtz’ sense Zhu 2018b) with the invariant “average (over the volume \( V \)) helicity density \( h = \omega \cdot u \), or simply the helicity (Betchov 1961; Moreau 1961)

\[
\mathcal{H} := \frac{1}{2V} \int \omega \cdot u d^3r = \frac{1}{V} \int \omega_v \cdot u_v d^3r,
\]

just as in the 2D3C case, described in the discrete case by the mutual linkage between the horizontal and vertical vortex loops (the latter understood to be closed at infinity Zhu 2018b): The second equality holds with no boundary contribution from the integration by parts.

We introduce such a kind of ‘mean rotation’ rate \( \Omega \) along the vertical direction that the helicity

\[
\mathcal{H}' = \frac{1}{2V} \int \nabla \times \mathbf{u}' \cdot \mathbf{u}' d^3r = 0,
\]

for the relative motion with velocity \( \mathbf{u}' = \mathbf{u} - \mathbf{\Omega} \times r = (u_x + y\Omega, u_y - x\Omega, u_z) \) and relative vorticity \( \omega' = \nabla \times \mathbf{u}' = \omega - 2\mathbf{\Omega} \).

After some algebra, we see that

\[
\mathcal{H} = \mathbf{\Omega} \cdot \frac{\int [2u_v + r \times (\nabla \times \mathbf{u}_v)] d^3r}{2V}.
\]

Thus the \( \Omega \)-frame is a ‘zero-frame’ of the helicity for the relative motion. Our strategy is to reduce the helicity effect on \( \mathbf{u} = \mathbf{u}' + \mathbf{\Omega} \times r \) to the ‘relative motion’ with no helicity but only rotation. In the above (and below), we choose such CBF that the \( \mathbf{\Omega} \) is well-defined, with the integration in Eq. (13) non-vanishing, say, and that \( \mathbf{u}' \) is appropriate ('slow') to make the TPE to be discussed below indeed work. Note also that

\[
\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{u}', \text{ and } \partial_z \mathbf{u} = \partial_z \mathbf{u}',
\]

which is precisely what we will exploit.

In the discussions of helicity effect on incompressible isotropic turbulence with mean rotation \( \Omega_0 \) (e.g., Pouquet & Mininni 2010), it is supposed that

\[
\mathcal{H}_0 := \frac{\Omega_0}{2V} \int u d^3r = 0, \text{ because } \frac{\int u d^3r}{2V} = 0, \text{ or very small in practice.}
\]

Thus, it makes sense to talk about the helicity effect, with the anisotropic centrifugal force being balanced by part of the pressure (we will come back to this for our compressible turbulence with mean rotation \( \Omega_0 \) and guiding field \( \mathbf{B}_0 \)). One should not confuse our reasoning and notion with such studies, and that is why we resist put the index ‘0’ in our \( \mathbf{\Omega} \), and similarly later in the corresponding \( \mathbf{B} \). There is a slightly subtle conceptual difference to our consideration:

\[\mathcal{H} \text{ in Eq. (11) is fixed, while } \mathcal{H}' = 0;\]

that is, we transform the effect-of-helicity (with or without \( \Omega_0 \)) problem to the pure-rotation-effect-without-helicity one relevant to the Taylor-Proudman effect (TPE), as we will show below.]

In the CBF, if the pressure term does not contribute (as in the barotropic case), for any material circuit \( c(t) \) with horizontal projection area \( A(t) = \int c(t) xdy - ydx \), we have the invariant Kelvin circulation:

\[
\oint_{c(t)} \mathbf{u} \cdot d\mathbf{r} = \oint_{c(t)} (u'_x - y\Omega)dx + (u'_y + x\Omega)dy + u'_z dz = \oint_{c(t)} \mathbf{u}' \cdot d\mathbf{r} + 2\Omega A.
\]
Figure 1. Chiral base flow. Left panel: the vorticity field; right panel: the velocity field. Nonvanishing helicity implies the zero-frame $\Omega$ for the relative motion. Closed lines are the (projected) horizontal vorticity or velocity streamlines, perpendicular to which are the straight vertical ones. Screws are for the velocity helical streamlines. Arrows point to the forward direction of time. Unlike $\partial_z \omega'_v = 0$, $u'_v$ can depend on $z$, as indicated by the changing thicknesses of the vertical lines for $u'_v$. And, unlike $\partial_z u'_h = 0$, $\omega'_h$ can depend on $z$ thus have richer structures, as indicated by the projection of a trefoil knot. According to TPE, such structures may be stabilized by the Coriolis force and/or the centrifugal force with (fast) rotation. The colors are chosen purely for esthetic preference.

If $\oint_c u' \cdot dr$ varies comparatively slowly, $A$ changes little, which is the geometrical argument to prove the Taylor-Proudman Theorem (TPE) by Taylor (1921) for the two-dimensionalization of incompressible rotating flows:

$$\text{For } \nabla \cdot u = 0 \text{ and } \Omega \to \infty, \frac{dA}{dt} \to 0 \Rightarrow \partial_z u \to 0.$$ (18)

The above geometrical proof of Taylor (1921) looks like a detour than the popular argument in most textbooks, and is either neglected or abandoned, except for the half-way inheriting by Chandrasekhar (1961). Such geometrical content is actually fundamental and unifies the fast rotation treatment of any dimension $d$ which can be larger than 3 (Zhu 2019), and it now turns out to be crucial for the very basic mechanism of our ‘tightening-up’ notion:

We consider compressible flows and take the (projected) circuit to be any of the velocity streamlines screwing around the vertical axis probably closed at infinity or at the periodic boundary with finite circulation (in the stream-screw scenario of Betchov 1961), or, to be any of the horizontal vorticity loops binding the vertical ones (in the vortex-knot scenario of Moffatt 1969), both caricatured clearly in Fig. 1. TPE requires each $A$ invariant, geometrically meaning that the screws and knots tighten the gas. Indeed, the asymptotic TPE for compressible flows would imply also

$$\partial_z u_h \to 0 \text{ and the horizontal incompressibility } \nabla_h \cdot u_h := \partial_x u_x + \partial_y u_y \to 0,$$ (19)

but with no constraint on $\partial_z u_z$, (20)

which led to the anticipation that “horizontal compressibility would be reduced by rotation” (Zhu 2019) for a time-dependent flow. $\partial_z u_z$, if not set to be zero here, is not supposed to alter so much as to fully compensate the divergence loss of the horizontal flow, then the total compressibility is reduced. Note that TPE itself in the compressible and incompressible flows are actually the same, except that the latter adds the extra constraint $\partial_z u_z = 0$, so strictly there is no ‘compressible-version TPE’. However, since the matter appears to be not so familiar to all audiences and even imprecisely stated (e.g., Eq. 2.49, correct for the incompressible case, of ?, is not correct for the compressible case discussed there, probably due to the misrecognition of the left hand side of Eq. 2.42, which is correct, as the Eulerian time derivative, rather than the material derivative), to emphasize the difference we still use the latter terminology for convenience.
We then conclude that $\mathcal{H} \neq 0$ underlies the rotating-gas mechanism, through Eq. (13), responsible for ‘tightening up’ the above CBF. The sign of the $\mathcal{H}$ should not enter the measure of (averaged) compressibility reduction $\mathcal{D}$ (say, computed through the parallel-mode or density spectrum of a turbulence), so presumably $\mathcal{D} = \mathcal{D}(|\mathcal{H}|)$.

Such a one-flow mechanism may be the cornerstone of an earlier statistical prediction (Zhu 2016) that helicity could reduce the compressibility of turbulence, as will be demonstrated with a minimal working model.

2.1.2. A working minimal model

As mentioned, we choose to start with our CBF because the crew-and-knot scenario in the tightening-up notion is most clearly and simply materialized, and consistent with TPE. In any case, TPE is for a single flow with given (large) $\Omega$. In particular, TPE is dynamical and anisotropic, so we need some superposition of the dynamics for turbulence (c.f., the discussion of the relevance with particularly remarks on the time scales of different physical processes Zhu 2019), particularly isotropic turbulence. Due to statistical fluctuations, the density or compressive modes is not supposed to be smaller with higher $|h|$ uniformly in space and time of turbulence, not even for each sample with given $\mathcal{H}$. Hence, we now need a ‘working’ model to unite the statistical-mechanics and prototypical-flow analyses.

A flow as a whole, say, the one in a cyclic box, is nonlinearly and nonlocally interacting, thus, the very tempting idea of summing up (linearly) the decomposed sub-domains’ different local helicities and their respective effects on local compressibility does not work. We then shall treat the field and the helicity (of value $\mathcal{H}$) holistically.

We introduce disorder, with equiprobability, to the vertical direction of the CBF in Fig. 1 to have our isotropic model. Statistical fluctuations in compressibility reduction, can be added to the variation of $\partial_z u_z$ in the CBF, so that the horizontal compressibility is strictly reduced in accordance with TPE, with the $\partial_z u_z$ distribution unaffected by the helicity. Simple average of such an ensemble of our CBFs is made eligible by letting them be (linearly) carried by an independent $\mathbf{V}$, (Lie) transporting the mass and the (screwing) velocity streamlines (Arnold & Khesin 1998):

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{V}), \quad \partial_t \mathbf{u} = (\mathbf{u} \cdot \nabla)\mathbf{V} - (\mathbf{V} \cdot \nabla)\mathbf{u}. \quad (21)$$

to the latter of which $(\nabla \cdot \mathbf{V})\mathbf{u}$ can be added for compressible $\mathbf{V}$ and a linear damping on $\mathbf{u}$ can also be added, resulting in a linear problem allowing superposition of the carried fluids, with, say, a barotropic $p$. Note that a linear transport of the vorticity would not be enough for describing a compressible flow. Now, the barotropic case $p \propto \rho$ makes particular sense, because the flow is then completely defined and precisely superposable, including the pressure. $p \propto \rho^2$ is also interesting, for then the usual pressure term $-(\nabla p)/\rho$ can be included without breaking the linearity of the system. It is possible to let $\mathbf{V}$ be of some ‘generalized flow’ (not necessarily differentiable, and could even be of values with some distribution as in the weak solution theory of partial differential equation) to cover more general cases. For example, in the integral form, letting $\mathbf{V}$ contain some discontinuity, with an integrated difference over it, may be used to eliminate the extra term, if exists, that otherwise would break the linearity in the ‘classical’ (smooth-$\mathbf{V}$) form of Eq. (21). According to the phenomenology of $\mathcal{D} = \mathcal{D}(|\mathcal{H}|)$ for the helicity effect on compressibility reduction, the nonlinear dependence on $\mathcal{H}$ of $\mathcal{D}$ suggests that our ‘minimal’ model should have each constituent CBF bear the same $\mathcal{H}$ (otherwise the ensemble average of $\mathcal{D}$ would need extra treatment.) The model is ‘working’ in the sense that, once the (horizontal) compressibility reduction is quantitatively given by the measure $\mathcal{D}$ for each CBF according to the qualitative TPE, together with the probabilities of the direction and $\partial_z u_z$ variation randomness, the notion of tightening up effect can be systematically demonstrated by linear superpositions of the results from the sample CBFs.

We should remark that such a model should not be understood to be a systematic one for turbulence, but a reasonable toy model to help demonstrating the genuine mechanism, as a completion of the story.

2.2. Turbulent ionized gases

Now we are well prepared to extend the calculation and analysis of paper I for neutral gas flows to (magnetised) plasma flows, before coming back to the CBFs for the final unification.

2.2.1. Compressible MHD

Kraichnan (1955) considered a gas whose pressure is given by $p = c^2 \rho$ (again we can take $c = 1$ with appropriate scaling), a particular barotropy which is widely used in modern simulations (e.g., Cho & Lazarian 2015) and which actually is nonessential for our analysis as shown in the introductory discussions for quantum turbulence. The ideal MHD equation then read with the transformation $\rho = \rho_0 \exp\{\zeta\}$ for the density with a background $\rho_0$ ($= 1$ again)

$$\partial_t \zeta + (\mathbf{u} \cdot \nabla)\zeta + \nabla \cdot \mathbf{u} = 0, \quad (22)$$
\[ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + c^2 \nabla \zeta + \frac{\nabla \times \mathbf{b} \times \mathbf{b}}{\rho_0 \exp\{\zeta\}} = 0, \]  

\[ \partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}). \]

Now the magnetic helicity \( \mathcal{H}_M = \int \mathbf{a} \cdot \mathbf{b} d^3r/(2V) \) for \( \nabla \times \mathbf{a} = \mathbf{b} \) and the cross helicity \( \mathcal{H}_C = \int \mathbf{u} \cdot \mathbf{b} d^3r/(2V) \) (but not the kinetic helicity) are ideal invariants. We refer to the final discussion section the consideration of the background mean guiding field \( \mathbf{B}_0 \) and the solid-body rotation \( \Omega_0 \), which change the ideal conservation laws (see, e.g., recently, Shebalin 2006).

All the idea and method of paper I carry over, mutatis mutandis. For example, since the Lorentz force and the magnetic induction is supposed to exchange the magnetic and kinetic energies but not to violate the conservativeness of the system. The invariant total — kinetic (vortical and compressive) plus potential and plus the magnetic — mean energy per unit volume reads in the “small excitation (Kraichnan 1955)” approximation (with \( c = 1 \) and \( \rho_0 = 1 \))

\[ \mathcal{E} = \frac{1}{2} \sum_k |\tilde{u}_+|^2 + |\tilde{u}_-|^2 + |\tilde{\zeta}|^2 + |\tilde{\eta}_+|^2 + |\tilde{\eta}_-|^2, \]  

(25)

where the helical Fourier representation for the magnetic field is also applied. In the above, the compressional potential energy \( w = \int_1^\rho \frac{\rho - 1}{\rho^2} d\rho \) has been estimated to the second order in the “small excitation” \( \zeta \). The magnetic and cross helicities write:

\[ \mathcal{H}_M = \frac{1}{2} \sum_k (|\tilde{\eta}_+|^2 - |\tilde{\eta}_-|^2)/k, \]

\[ \mathcal{H}_C = \frac{1}{2} \sum_k (\tilde{\eta}_+ \tilde{\zeta}_+ + \tilde{\eta}_- \tilde{\zeta}_- + c.c.) \]

(26)

(27)

To formally carry out the absolute equilibrium analysis (Kraichnan 1955; Zhu 2016) for the Galerkin-truncated system (with only finite modes kept), we have to check two things: (1) the Liouville theorem, i.e., the incompressibility of the phase dynamics in the space spanned by the real and imaginary parts of all the Fourier coefficients, and (2) the (approximate) conservation laws, which should be rugged after the Galerkin truncation. Letting \( \chi_i \), with \( i = 1, ..., N < \infty \), represent the real or imaginary parts of the variables appearing in Eq. (25), we can put down the abstract form for the Galerkin-truncated \( k \)--space version of Eqs. (22, 23, 24):

\[ \partial_t \chi_i + \mathcal{F}(\chi) = 0. \]

(28)

Then it is direct to check that the Liouville theorem \( \sum_i \frac{\partial}{\partial \chi_i} \partial_t \chi_i = 0 \) indeed holds. Actually, comparing to the incomparable MHD (Lee 1952) and the neutral gas (Kraichnan 1955), c.f. also Zhu, Yang & Zhu (2014) and Zhu (2016) for the helical mode representation, the only new thing now is the appearance of the varying density in the Lorentz force which does not add anything to \( \frac{\partial}{\partial \chi_i} \partial_t \chi_i \), neither to the (approximate) energy conservation law in the following sense: Given that \( \mathcal{E} \) in Eq. (25) is approximate invariant for the un-truncated system, then the Galerkin truncation means for all \( i > N \) that \( \chi_i = 0 \) and thus \( \chi_i \partial_t \chi_i = \partial_t \chi_i^2/2 = 0 \), assuring the ruggedness of \( \mathcal{E} \) invariance after truncation. The turbulent system is then supposed to tend to, but of course not to really arrive at, the thermalized state constrained by the ‘constant of motion’ \( \mathcal{C} = \alpha \mathcal{E} + \beta \mathcal{M} \mathcal{H}_M + \beta \mathcal{C} \mathcal{H}_C \), i.e., the absolute equilibrium (usually argued with the assumption of ‘ergodicity’: Lee 1952).

We then calculate the absolute equilibrium spectra density from the canonical distribution \( \sim \exp\{-\mathcal{C}\} \) of the Galerkin truncated system (Kraichnan 1955):

\[ U_{K}^{\pm}(k) := \langle |\tilde{u}_{\pm}|^2 \rangle = \frac{4(\alpha k \pm \beta_M)}{(4\alpha^2 - \beta^2)k \pm 4\alpha \beta_M}, \]

\[ U_{M}^{\pm}(k) := \langle |\tilde{\eta}_{\pm}|^2 \rangle = \frac{4(\alpha k)}{(4\alpha^2 - \beta^2)k \pm 4\alpha \beta_M}, \]

\[ Z(k) := \langle |\tilde{\zeta}|^2 \rangle = \frac{1}{\alpha} = \langle |\tilde{\eta}_| \rangle =: U_{K}^{\pm}(k). \]

(29)

(30)

(31)

Note that \( Q_{K}^{\pm}(k) = \pm \frac{1}{2} U_{M}^{\pm}(k), Q_{K}^{\pm}(k) = \pm k U_{M}^{\pm}(k) \). The chiral parts (29, 30) are the same as in the incompressible case (Zhu, Yang & Zhu 2014), decomposing those of Frisch et al. (1975), and the longitudinal/compressive and density spectra now appear in Eq. (31) for comparative analysis.
When $\beta_C \neq 0$ (thus $\mathcal{H}_C \neq 0$),

$$U_K^+ > U_K^+ = Z = 1/\alpha, \text{ thus } U_K^+ := U_K^+ + U_K^- > U_K^+ + Z =: U^\sim$$

(32)
even for $\beta_M = 0$ with then $U_M^+ = U_K^+$. As said, more general and precise expression is that the fractions of $U_K^+ = 0$ (thus $\mathcal{H}_M \neq 0$),

$$U_M := U_M^+ + U_M^- = \frac{2}{\alpha \pm \beta_M/(\alpha k)} > \frac{2}{\alpha} = U^\sim = U_K^+,$$

(33)
indicating that the energy partition is favoring the magnetic, but not the kinetic vortical mode, especially at small $k$, against all the kinetic and density modes: The dynamo in such compressible case may be easier with more ‘preys’. Comparing the results with vanishing and non-vanishing $\beta_M$, we can infer that the cross-helicity effect should be more pronounced in the case without magnetic helicity, as the larger fraction of the ‘dynamo’ effect of the latter is screened out. When the realistic physical condition enables such nonlinear effects to reveal itself, the appropriate mutual linkage (the degree of which is measured by $\mathcal{H}_C$, Moffatt 1969) of the vorticity and magnetic fluxes and the knottedness/linkage (the degree of which is measured by $\mathcal{H}_M$, Moffatt 1969) of the magnetic flux(es) will tighten up the plasmas.

Note that our results should not be confused with the dynamics of linear waves (Alfvénic, fast and slow: c.f, Appendix A of Cho & Lazarian 2015, good also for general fluid dynamists), but a prediction of reducing turbulent magnetosonic ‘noise’ may be reasonable. Eqs. (29,30) also indicate higher efficacy of the cross-helicity effect at larger $k$. The dissipation of turbulence at very large $k$ however may enter to reduce such nonlinear effect there. Viscous dissipation effect in neutral fluid turbulence have been carefully discussed in paper I. Now, similar discussion can be carried out. However, extra carefulness is necessary for nonunit Prandtl number and won’t be elaborated here. Concerning the energy of the electric field given by the Ohm’s law $E = -\nabla \times b$, it seems that we can not conclude any helicity effect (but see below remarks on two-fluid model).

A direct indication is that for statistical models the (generalized) cross-helicity, particularly set to zero in the EDQNM model by Pouquet et al. (1976) for incompressible MHD, should be included in studying the density fluctuation relevant dynamics. It is also hoped that our results can promote new ideas and methods in observation-based studies (e.g., Bi et al. 2016; Zhang & Brandenburg 2018) to help understand the photospheric plasma. Our result directly connecting the density, velocity and magnetic fields may also be helpful in understanding the solar and interstellar plasma dynamics, where the very definition of the heliopause is under controversy (e.g., Gurnett et al. 2013). Numerical checks of cross-helicity reducing the compressibility would be more subtle than the neutral fluid case. For example, if the kinetic and magnetic energy injections are different, other effects besides that due to the cross-helicity may enter, requiring careful design of the normalization scheme for a ‘fair’ or reasonable comparison.

2.2.2. Compressible XMHD

One may ask whether the more general fluid models of plasma dynamics, such as the extended MHD (XMHD) and the two-fluid model with much more complete (small-scale) plasma physics, present similar mechanisms. The XMHD dynamics have recently been intensively studied. For example, for information closer to our problem, Lingam, Miloshevich & Morrison (2016) have exposed the nice local structure which has been exploited by Besse & Frisch (2017) for test application of their generalized Cauchy invariants theorem; Miloshevich, Lingam & Morrison (2017), for understanding cascade directions, calculated the absolute equilibrium whose finer chiral structure has been exposed in Zhu (2017) to address the solar wind chirality issue. Relevant investigations with similar developments of the two-fluid model can also be found in Zhu, Yang & Zhu (2014) and Zhu (2018a). Now we focus on XMHD with the same assumptions (barotropic relation, weak excitation etc.) for MHD in the above. Barotropic XMHD equations in Alfvénic units, for momentum and induction, besides the continuity (mass conservation) equation (22), read

$$\partial_t \mathbf{u} = -\nabla \left[ \Pi + \frac{u^2}{2} + \frac{(d_e \nabla \times \mathbf{b})^2}{2\rho^2} \right] + \mathbf{u} \times (\nabla \times \mathbf{u}) + \frac{(\nabla \times \mathbf{b}) \times \mathbf{b}}{\rho},$$

(34)
\[\frac{\partial_t \mathbf{b}}{\partial_t} = \nabla \times (\mathbf{u} \times \mathbf{b}) - \frac{d_i}{d_e} \nabla \times \left[ \frac{(\nabla \times \mathbf{b}) \times \mathbf{b}}{\rho} \right] + d_e^{-2} \nabla \times \left[ \frac{(\nabla \times \mathbf{b}) \times (\nabla \times \mathbf{u})}{\rho} \right], \tag{35}\]

with \(\nabla \cdot \mathbf{b} = \nabla \cdot \mathbf{b} = 0\), \(\mathbf{b} = \mathbf{b} + d_e^2 \nabla \times (\nabla \times \mathbf{b})\), and \(\Pi\) the enthalpy as the barotropic pressure (electron plus ion) uniquely determined by the density \(\rho\). Potential vectors \(\nabla \times \mathbf{a} = \mathbf{b}\) and \(\nabla \times \mathbf{a} = \mathbf{b}\) are also introduced. Formally, but with all kinds of subtleties, taking the limit of \(d_i \to 0\), one arrives at the **inertial MHD** relevant for sub-electron scales, say, of the Earth’s magnetosphere and solar wind plasmas; with \(d_e \to 0\), the well-known Hall MHD; and, the ideal single-fluid MHD we have discussed is recovered with both \(d_i\) and \(d_e\) taken to be zero. In some sense, ‘\(X\)’ in XMHD is about the new physical ingredients coming with the ion and electron skin depths, \(d_i\) and \(d_e\). We could also put down the equations in the Fourier space, and the chiral parts was already explicitly written in Zhu (2017). Now, corresponding to the previously discussed MHD the arguments for whom concerning the Liouville theorem and conservation laws also apply here, with correspondingly

\[
\mathcal{E} = \frac{1}{2} \sum_k |\hat{u}_+|^2 + |\hat{u}_-|^2 + |\hat{\zeta}|^2 + \frac{|\hat{b}_+|^2 + |\hat{b}_-|^2}{(1 + d_e^2 k^2)}, \tag{36}\]

\[
\mathcal{H}_M = \frac{1}{2} \sum_k k d_e^2 |\hat{u}_+|^2 - k d_e^2 |\hat{u}_-|^2 + \frac{|\hat{b}_+|^2}{k} - \frac{|\hat{b}_-|^2}{k}, \tag{37}\]

and

\[
\mathcal{H}_C = \frac{1}{2} \sum_k \hat{u}_+ \hat{b}_+ + \hat{u}_- \hat{b}_- + c.c. + k d_i (|\hat{u}_+|^2 - |\hat{u}_-|^2). \tag{38}\]

We are again justified to check the absolute equilibrium with the distribution \(\sim \exp\{-\alpha \mathcal{E} - \beta_M \mathcal{H}_M - \beta_C \mathcal{H}_C\}\), as in MHD; and, the analysis are similar. For example, we can just write down one of the unchanged chiral parts (given in Zhu 2017) spectral density for illustration

\[
U_K^\pm(k) = \frac{1}{\alpha \pm k (\beta_M d_e^2 + \beta_C d_i)} \frac{\beta_C}{\alpha + \frac{\beta_C}{d_i}}, \tag{39}\]

with again Eq. (31) for the compressive and density modes.

Comparing Eqs. (39) and (31), we see, mostly easily by letting \(\beta_M = 0\) but \(\beta_C \neq 0\), or the other way round, to find the increase of \(U_K(k)\), qualitative similar features as in MHD emerge but with quantitative differences coming from the \(d_i\) and \(d_e\) terms: the latter always are accompanied with \(k\) factors, indicating the dependence of scales relative to \(d_i\) and \(d_e\). The detailed roles of XMHD \(\mathcal{H}_M\) and \(\mathcal{H}_C\) are also different to the corresponding ones of MHD. For instance, letting \(\beta_C = 0\) but \(\beta_M \neq 0\), we see that \(U_K(k)\) is enhanced now (in a way actually as the enhancement due to the kinetic helicity), while in MHD only magnetic energy is enhanced. Nevertheless, the overall tightening effect, relatively reducing the compressive and density modes, of helicities are the same. Thus, we make a more general conclusion that, under appropriate conditions to allow such nonlinear effects to reveal themselves, “helicity can tighten up the turbulent flows of the ionized gas (plasma).”

Indeed, as familiar readers would quickly see, without going too much into the details, or do it as an exercise to check that the statement can be further supported by the two-fluid model analysis, with, again, the chiral parts of the absolute equilibrium ensemble presented in Zhu, Yang & Zhu (2014) being unchanged, and the additional compressive and density modes of the two fluids uncorrelated. Note however that the electric field \(\mathbf{E}\) of the chiral parts presented in Zhu, Yang & Zhu (2014) is already uncorrelated to other modes in the canonical ensemble and thus equipartitioned ((|\(E|)^2 = \frac{1}{2})). Then, presumably electric field would also be reduced in turbulence, say, in the situation where the typical dynamical time scale of \(\mathbf{E}\) variation is comparable with others. As said, unlike here for two-fluid model, we however could not claim from the Ohm’s law in MHD whether the electric field energy can be reduced by the helicities or not, neither from the even more complicated generalized Ohm’s law (e.g., Miloshevich, Lingam & Morrison 2017) in XMHD. Such details are more involved ‘plasma physics’ and will not be elaborated here. The word “tighten” here has double meanings: one is that the medium appears ‘hardened’; the other is related to the fact that the topological interpretation of the helicity is related to the knottedness/linkage of the frozen-in (generalized) vorticity rope(s) or the screwing of the streamlines of the (generalized) momentums.

### 2.2.3. Further discussions
A background guiding field $B_0$ removes $\mathcal{H}_M$, but not $\mathcal{H}_C$, from the ideal precise conservation law of the fluctuation fields, as in the incompressible case (Shebalin 2006), thus the implication of reduced compressive and density modes, still applies. The relevance of magnetic helicity in the presence of $B_0$ depends on how approximately $\mathcal{H}_M$ is still conserved or the time scale of the term involving $B_0$ that breaks the $\mathcal{H}_M$ invariance. The ergodicity issue raised by Shebalin (2006) and references therein is also subtle in the sense that statistical analysis of the physical relevance may not necessarily need ergodicity. Nevertheless, with $B_0$ appearing to be typically introduced in many discussions, it deserves to emphasize the cross-helicity effect. It is then reasonable to expect applications in much more complex realistic situations, with all the caveats relevant to the assumptions and approximations, and, other subleties such as the anisotropy issue [e.g., Goldreich & Sridhar (1995) and recently, e.g., Wang, Tu & He (2019)], kept in mind.

Actually, even when a solid-body rotation $\Omega_0$ presents (as considered in Yokoi 1999) with the centrifugal force $\Omega_0 \times r \times \Omega_0$ being non-periodic, but, as for the incompressible case, if part of the barotropic pressure term $\nabla p/\rho = \nabla H$ balances $\Omega_0 \times r \times \Omega_0 = \nabla(|\Omega \times r|^2/2)$, the Fourier analysis of the system may still apply. [Some authors (e.g., Brandenburg et al. 2008) even just simply remove the centrifugal force term for weak compressibility simulations.] That is, our calculation may still be of some value as long as the correct invariance law is used: Shebalin (2006) showed that when $\sigma B_0 = \Omega_0$ with a real number $\sigma \neq 0$, a ‘parallel helicity’ $\mathcal{H}_P = \mathcal{H}_C - \sigma \mathcal{H}_M$ is invariant, which makes Eqs. (29,30) et al., with $\beta_M = -\sigma \beta_C$ and $\beta_C$ replaced by $\beta_P$, still relevant and similar $\mathcal{H}_P$ tightening effect favorable. Though not the focus of this note, it still deserves to remark, as a side note, that now

$$Q_M^\pm (k) = \pm \frac{1}{k} U_M^\pm (k) = \pm \frac{4\alpha}{(4\alpha^2 - \beta_P^2)k} \mp \frac{\sigma \alpha \beta_P}{4\alpha^2 - \beta_P^2}$$  \hspace{2cm} (40)

indicates a possible $\mathcal{H}_P$ dynamo in a way argued by Frisch et al. (1975) and Pouquet et al. (1976) for the magnetic helicity inverse cascade. And, this dynamo will also take energy from the compressive and density or pressure modes to tighten up the flow.

3. UNIFICATION AND CONCLUSION

For magnetised plasmas, it is well-known that the background mean magnetic field have analogous two-dimensionalization effect (Montgomery & Turner 1981; Cho & Lazarian 2015; Zank et al. 2017), like the background rotation in Fig. 1 where the CBF thus can have a counterpart in magnetohydrodynamics (c.f., Fig. 2). And, because the magnetic field $b$, proportional to the gyrofrequency ($\omega \propto b$) of charged particles, can be viewed (Zhu, Yang & ZHU 2014) as a kind of macroscopic vorticity of the microscopic gyromotions (with a flavor of, but much more concret than the notion of ‘molecular vortices’ as discussed by Rankin 1854), with all kinds of helicities, thus the mechanisms, unified for our multidisciplinary universal law of tightening compressible flows.

The above very succinct statement with only very recent references may sound somewhat superficial to some audiences before sufficient ponderation. So, it is important to elaborate and materialize them more definitely, which resembles and extends that in Sec. 2.1, especially Sec. 2.1.1. It would be helpful for readers to get familiar with early works of J. Hartmann, H. Alfvén and B. Lehnert, among others, for basic facts of background magnetic field concerning laminar flow, wave and turbulence, but it appears that some of the discussions of the monograph by Chandrasekhar (1961, Chaps. 36–40) should be enough to grasp the main idea and results.

3.1. Technical details for unification

First, we extend the analogue of TPE for MHD with a strong background uniform magnetic field $B$ to the compressible case. In the incompressible case, Chandrasekhar (1961) now presented in the trivial way as in most text books, without invoking the geometry of the Alfvén theorem as the analogue of the Helmholtz-Kelvin theorems, with the result that $(B \cdot \nabla)u = 0$: for $B$ lying the $z$ axis, $\partial_z u = 0$. Now, for the compressible but the barotropic case, with $a$ (and $a'$) in

$$\text{the Alfvén theorem} \int_{c(t)} \mathbf{a} \cdot d\mathbf{r} = \int_{\mathbf{b} \cdot d\mathbf{s}} \mathbf{b} \cdot d\mathbf{s} = \text{const.} \hspace{2cm} (41)$$

replaced with $u$ (and $u'$) in Eq. (17), and $B$ with $\Omega$ there, the same geometrical argument of Taylor (1921), with $A := (\nabla \times)^{-1}(2B) = (\nabla \times B, 0, 0) + \nabla \varphi$ whose gradient of potential $\varphi$ does not contrite to the loop integral, leads to the same compressible TPE formula as in Eq. (20). Alternatively, in the analysis of Chandrasekhar (1961), an extra term $B(\nabla \cdot u)$ from the compressibility transforms his Eq. (83) to our Eq. (20). Actually, the treatment in terms of differential forms in Zhu (2019), extending TPE to compressible and higher-dimensional cases with the
Figure 2. Chiral base fields. Left panel: — Nonvanishing magnetic helicity implies a kind of mean background field $B$, thus *tightens up* the flow according to the magnetic analogue of the compressible TPE. Closed lines are the (projected) horizontal magnetic or potential streamlines, perpendicular to which are the straight vertical ones, all similar to Fig. 1 (thus most of the caption there carries over, *mutatis mutandis*, and we won’t offer redundant explanations for things that are self-evident, such as the indication of the projections of the trefoil knots.) Right panel: — Nonvanishing cross-helicity implies a kind of aligned mean background field $B$ and rotation $\Omega$, chosen to be aligned here as a special case.

georigic notion, carries over to the magnetic field corresponding to a 2-form $B = dA$ (thus the projected area in Taylor’s argument naturally emerges). The emphasis here however is the materialization of the tightening-up notion with such geometrical ‘screws’ and ‘ropes’ in the TPE mechanism:—

Note that, as in the left panel of Fig. 2 for the chiral base field (‘CBF’ again, without confusion), now for magnetic case with the two-dimensionality of the horizontal potential $\partial_z a_h = 0$, we can make the replacement with the above prescription. We can equally introduce $B$, from nonvanishing magnetic helicity $H_M$, as $\Omega$ in Eq. (13). Thus, the above analogue of compressible TPE indeed geometrically materializes the tightening-up effect of $H_M$ for such a CBF. Similarly, the cross-helicity tightening-up scenario is caricatured in the right panel of Fig. 2 for the case with $\Omega$ and $B$ aligned: $\Omega$ can be accordingly introduced, not necessarily aligned with $B$, to make the relative cross-helicity vanishing, thus pure effect of rotation and guiding fields in the CBF. Alfvén’s theorem again provides the geometrical materialization of the tightening-up effect with the compressible TPE and its analogue.

Just to iterate, we have transformed the helicity-effects (with or without $B_0$ and/or $\Omega_0$: see later discussions) to pure $B$-and/or-$\Omega$-effect one. The generalizations of the above discussions to CBFs of XMHD, its reduction as the Hall MHD, or the more complete two-fluid model are straightforward: It is just that the $u$ and $b$ should then be replaced with the two generalized momentums, whose curls (generalized vorticities) are ideally frozen-in to two flows, the compressibility of the velocities of which are reduced by the respective generalized helicities. The definite physical discussions would be different and interesting, but beyond the scope of this note.

The statistical minimal working model for neutral gas turbulence then can be simply carried over with extension for the ionized case, by adding a term linear in $b$, say, $\nabla \times V \times b$ or simply $V \cdot \nabla b$ (the exact form should not matter for the present purpose), mimicking the Lorentz force to the momentum equation and complementing the system with an extra transport, say, $\partial_t a = b \times B + \nabla \phi$ (with linear addable $\phi$) or $\partial_t b = \nabla \times (b \times V)$ for the magnetic field. Thus, with the (imagined) gyrokinetic view of magnetic field as the vorticity field of the ‘micro-gyrofluid’ velocity, we indeed have unified the universal law in a concrete sense and made the first paragraph of this section more vivid and substantial.

3.2. Conclusion

If our CBF and the minimal working toy model indeed underly the helicity-reducing-compressibility effect (I believe so!), it may appear somewhat ironic that the genuine fundamental mechanism of the ‘global’ tightening-up effect of
Hardening an ionized gas

helicity is hidden right under our noses — in the most familiar rotation coming with the TPE (but the probably not-so-familiar compressible version.) This is so fundamental that it can be extended to the magnetic and cross helicities for compressible MHD and more general plasma fluids with again the compressible version of the TPE and its analogue for the magnetic field, with even a unified view. For the incompressible case, TPE simply leads to anisotropy which, due to the arbitrariness of the direction of the axis of the intrinsic rotation in the isotropic case, is cancelled out in the superposition for turbulence, thus no gross or global effect. Reduction of compressibility however is not cancelled as such, but added up to reveal the global effect. The method of absolute-equilibrium analysis for turbulence appears to encapsulate all such ‘details’.

For the ‘microscopic’ mechanism, knot-theoretic thinking and relevant decomposition and construction techniques (e.g., Kedia et al. 2016; Kleckner, Kauffman & Irvine 2016; Scheeler et al. 2017) may lead to the speculation that different constituents of helicity play different roles in tightening-up the gas, which sounds interesting but so far not clear to be trackable in turbulence: Supposed that such constituents can be categorized in $k$-space, like three categories in the helical representation we have been using, then more statistical information would reveal. Historically, complicatedly knotted/linked field with zero or very small degree of knottedness/linkage (Arnold & Khesin 1998) does not fit perfectly the paradigm of “helicity bounds the energy” (Arnold 1974), and later systematic works (Freedman 1988; Freedman & He 1991a,b), together with the beautiful heuristic observation and the phenomenological theory of the knot energy spectrum (Moffatt 1990), established the notion of “topological obstruction” (Arnold & Khesin 1998). Turbulent gas (ionized, quantum, or not) dynamics appear to be much less tractable, but we are now satisfied, at least for a while, with the truth of the CBF, statistical mechanics inference and minimal working model, which most clearly materializes (with geometry of velocity/potential and vorticity/magnetic streamlines — gaseous screws and ropes, compressible TPE and its analogue) and unifies (with the combined fluid and gyrokinetic point of views) the mechanisms of the universal law of “helicities tightening up the turbulent gases”.

To conclude, screws generally work better than nails, or good knots of ropes better than naive ones for fastening solid matters. Gases appear more timeless to be ‘tightened up’ as such. But, the flow reveals a relevant physical quantity, the helicity, which characterizes the screwing strength of the velocity field and the degree of the knottedness (or linkage) of the vorticity rope(s), and which is conserved by the ideal barotropic flow. The helicity of our chiral base flow (CBF) nontrivially indicates a kind of mean rotation in which there is no helicity, and we adapt Kelvin’s circulation theorem to materialize the above tightening-up notion in the CBF, as the element of ‘helicity-reducing-compressibility’, a mechanism predicted in paper I. A minimal working model adding disorders to the CBFs bridges the single-flow and statistical arguments, completing the story. We have managed to argue that, recent data, of the superfluid and Bose-Einstein condensate model, and the absolute equilibrium analysis, of ionized gases extending previous incompressible calculations, both support the universality of the notion. The magnetic field can be (imagined to be) identified with an adhered string of nuclei with the ‘atmosphere’ of ‘molecular vortices’ and the ‘magnetic pressure’ with the centrifugal force, as the “the constant endeavour of natural philosophers, by conceiving the other phenomena of nature as modifications of motion and force, to reduce the other physical sciences to branches of mechanics” (Rankin 1854), so a unified view, with substantial extensions of CBF and compressible TPE analogue enclosing the geometry of the Alfvén theorem, can indeed be taken. Multidisciplinary implications include but not limit to, in the context of acoustics, helicity offering a handle for controlling the turbulence aeroacoustic, quantum and magnetosonic noises.

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REFERENCES

Arnold, V.I. 1974 The asymptotic Hopf invariant and its applications. Proc. Summer School in Diff. Equations at Dilizhan, 1973; English transl.: 1986 Sel. Math. Sov. 5, 327–345.

Arnold, V.I. & Khesin, B.A. 1998 Topological methods in hydrodynamics. Springer.

Barenghi, C. F., Galantucci, L., Parker, N.G., and Baggaley, A. W. 2018 Classical helicity of superfluid helium. arXiv:1805.09005 [physics.flu-dyn].

Berger, M. A. & Field, G. B. 1984 The topological properties of magnetic helicity. J. Fluid Mech. 147, 133–148.
Besse, N. & Frisch, U. 2017 Geometric formulation of the Cauchy invariants for incompressible Euler flow inflat and curved spaces. J. Fluid Mech. 825, 412–478.

Betchov, R. 1961 Semi-isotropic turbulence and helicoidal flows. Phys. Fluids 4, 925–926.

Bi, Y., Jiang, Y., Yang, J., Hong, J., Li, H., Yang, B., Xu, Z. Observation of a reversal of rotation in a sunspot during a solar flare. Nature Communication 7, 13798.

Biferale, L., Musacchio, S. & Toschi, F. 2013 Inverse energy cascade in three-dimensional isotropic turbulence. J. Fluid Mech. 730, 309–327.

Brandenburg, A., Rädler, K.-H., Rheinhardt, M. & Käpylä, P. J. 2008 Magnetic diffusivity tensor and dynamo effects in rotating and shearing turbulence. Astrophys. J. 676, 740.

Cambon C. & Jacquin, L. 1989 Spectral approach to non-isotropic turbulence subjected to rotation. J. Fluid. Mech. 202, 295–317.

Chandrasekhar S. 1961 Hydrodynamic and Hydromagnetic Stability. Dover, New York.

Chen, Q., Chen, S. & Eyink, G. 2003 The joint cascade of energy and helicity in three-dimensional turbulence. Phys. Fluids 15, 361–374.

Cho, J., & Lazarian, A. 2005 Generation of compressible modes in MHD turbulence. Theoret. Comput. Fluid Dynamics 19, 127–157.

Clark di Leoni, P., Mininni P.D. & Brachet M.-E. 2016, Helicity, topology and Kelvin waves in reconnecting quantum knots. Phys. Rev. A 94, 043605.

Cruz-Pacheco, G., Levermore, C. D. & Luce, B. P. 1995 Melnikov Methods for PDEs: Applications to Perturbed Nonlinear Schrődinger-Equations. CNLS NEWSLETTER LALP-95-012-114.

Dennis, M. R., Hannay, J. H. 2005 Geometry of Călugăreanu’s theorem. Proc. R. Soc. A. 461, 3245–3254.

Eyink, G. L. & Drivas, T. D. 2018 Cascades and Dissipative Anomalies in Compressible Fluid Turbulence. Phys. Rev. X 8, 011022.

Freedman, M. H. 1988 A note on topology and magnetic energy in incompressible perfectly conducting fluids. J. Fluid Mech. 194, 549–551.

Freedman, M.H. and He, Z.-X. 1991a Links of tori and the energy of incompressible flows. Topology 30, 283–287.

Freedman, M.H. and He, Z.-X. 1991b Divergence-free fields: energy and asymptotic crossing number. Annals of Math. 134, 189–229.

Frisch, U. 1995 Turbulence: The Legacy of A. N. Kolmogorov. Cambridge University Press, Cambridge, England.

Frisch, U., Kurien, S., Pandit, R., Pauls, W., Ray, S., Wirth, A. & Zhu, J.-Z. 2008 Hyperviscosity, Galerkin Turbomation and Bottleneck of Turbulence. Phys. Rev. Lett. 101, 114501–114504.

Frisch, U., Pomyalov, A., Procaccia, I., Ray, S. S. 2012 Turbulence in Noninteger Dimensions by Fractal Fourier Decimation. Phys. Rev. Lett. 108, 074501.

Frisch, U., Pouquet, A., Leorat, J. & Mazure, A. 1975 Possibility of an inverse magnetic helicity cascade in magnetohydrodynamic turbulence. J. Fluid Mech. 68, 769–778.

Goldreich P. & Sridhar, S. 1995 Toward a theory of interstellar turbulence. 2: Strong Alfvénic turbulence, Astrophys. J. 438, 763.

Gurnett, D. A., Kurth, W. S., Burlaga, L. F. & Ness, N. F. 2013 In situ observations of interstellar plasma with Voyager 1. Science 341, 1489–1492.

Kauffman, L. H. 1991 Knots and physics. World Scientific Publishing.

Kedia, H., Foster, D., Dennis, M. R. & Irvine, W. T. M. 2016 Weaving Knotted Vector Fields with Tunable Helicity. Phys. Rev. Lett. 117, 274501.

Kleckner, D., Kauffman, L. H. & Irvine, T. M. 2016 How superfluid vortex knots untie. Nature Physics 12, 650–655.

Kraichnan, R. H. 1955 On the statistical mechanics of an adiabatically compressible fluid. J. Acoust. Soc. Am. 27, 438–441.

Kraichnan, R. H. 1973 Helical turbulence and absolute equilibrium. J. Fluid Mech. 59, 745–752.

Kurien, S. 2003 The reflection-antisymmetric counterpart of the Kármán-Chowarth dynamical equation. Physica D 175, 167–176.

Lee, T.-D. 1952 On some statistical properties of hydrodynamic and hydromagnetic fields. Q. Appl. Math. 10, 69–74).

Lingam, M., Miloshevich, G. & Morrison, P. 2016 Concomitant Hamiltonian and topological structures of extended magnetohydrodynamics. Phys. Lett. A 380, 2400–2406.

Lund, F. 1989 Response of a filamentary vortex to sound. Phys. Fluids 1, 1521–1531.

Lund, F. 1991 Defect dynamics for the nonlinear Schrödinger equation derived from a variational principle. Phys. Lett. A 159, 245–251.

Miloshevich, G., Lingam, M., & Morrison, P. 2017 On the structure and statistical theory of turbulence of extended magnetohydrodynamics. New J. Phys. 19, 015007.
Moffatt, H. K. 1969 The degree of knottedness of tangled vortex lines. J. Fluid Mech. 35, 117–129.
Moffatt, K. 1990 The energy spectrum of knots and links. Nature 347, 369–369.
Montgomery, D. & Turner, L. 1981 Anisotropic magnetohydrodynamic turbulence in a strong external magnetic field. Phys. Fluids 24, 825–831.
Moreau, J. J. 1961 Constantes d’un îlot tourbillonnaire en fluide parfait barotrope. (French) C. R. Acad. Sci. Paris. 252, 2810–2812.
Moses, H. E. 1971 Eigenfunctions of the curl operator, rotationally invariant Helmholtz theorem and applications to electromagnetic theory and fluid mechanics. SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. 21, 114–130.
Panagiotou, E. 2015 The linking number in systems with Periodic Boundary Conditions. J. Comp. Phys. 300, 533–573.
Pouquet, A., Frisch, U. & Léorat, J. 1976 Strong MHD helical turbulence and the nonlinear dynamo effect. J. Fluid Mech. 77, 321–354.
Pouquet A & Mininni PD. 2010 The interplay between helicity and rotation turbulence: implications for scaling laws and small-scale dynamics. Phil. Trans. R. Soc. A 368, 1635–1662.
Rankine, William John Macquorn (C.E. F.R.SS. Lond Edinb.) 1854 I. On the mechanical action of heat, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 7, 1–21.
Salman, H. 2017 Helicity conservation and twisted Seifert surfaces for superfluid vortices. Proc. R. Soc. A 473, 20160853.
Scheeler, M.W., Kleckner, D., Proment, D., Lindlmann, G.L., Irvine, W.T.M. 2014 Helicity conservation by flow across scales in reconnecting vortex links and knots. Proc. Natl Acad. Sci. USA 111, 350–355.
Scheeler M. W., van Rees, W. M., Kedia H., Kleckner, D & Irvine W. T. M. 2017 Complete measurement of helicity and its dynamics in vortex tubes. Science 357, 487–491.
Shebalin, J. V. 2006 Ideal homogeneous magnetohydrodynamic turbulence in the presence of rotation and a mean magnetic field. J. Plasma Phys. 72, 507–524.
Taylor, G. I. 1921 Experiments with Rotating Fluids. P. Roc. R. Soc. Lond. A. 100, 114–121.
Waleffe, F. 1992 The nature of triad interactions in homogeneous turbulence. Phys. Fluids A 4, 350–363.
Wang, X., Tu, C. & He, J. 2019 2D Isotropic Feature of Solar Wind Turbulence as Shown by Self-correlation Level Contours at Hour Timescales. Astrophys. J. 871, 93.
Woltjer, L. 1958 On Hydromagnetic Equilibrium. Proc. Nat. A. Soc., 44, 833–841.
Nore, C., Abid, M. & Brachet M. E. 1997 Decaying Kolmogorov turbulence in a model of superflow. Phys. Fluids 9, 2644.
Yang, J., Xu, J.-X., Yang, Y. & Zhu, J.-Z. 2019 Helicity hardens the gas. arXiv:1901.00423 [physics.flu-dyn]
Yokoi, N. 1999 Magnetic-field generation and turbulence suppression due to cross-helicity effects. Phys. Fluids 11, 2307–2316.
Zank, G. P., Adhikari, L., Hunana, P., Shiota, D., Bruno, R., Telloni, D. & Avinash, K. 2017 The Theory of Nearly Incompressible Magnetohydrodynamic Turbulence: Homogeneous Description. J. Phys.: Conf. Ser. 900, 012023
Zhang, H. & Brandenburg, A. 2018 Solar Kinetic Energy and Cross Helicity Spectra. Astrophys. J. Lett. 862, L17
Zhu, J.-Z. 2016 Isotropic polarization of compressible flows. J. Fluid Mech., 787, 440
Zhu, J.-Z. 2017 Chirality, extended magnetohydrodynamics statistics and topological constraints for solar wind turbulence. Mon. Notes Roy. Astronomy Soc. 470, L87–L91
Zhu, J.-Z. 2018a Local invariants in non-ideal flows of neutral fluids and two-fluid plasmas. Phys. Fluids 30, 037104.
Zhu, J.-Z. 2018b Vorticity and helicity decompositions and dynamics with real Schur form of the velocity gradient. Phys. Fluids 30, 031704.
Zhu, J.-Z. 2019 Taylor-Proudman theorems in $E_3$, $E_4$ and $E_5$. arXiv:1905.11783v1 [math.AP].
Zhou, J.-Z., Yang W. & Zhu, G.-Y. 2014 Purely helical absolute equilibria and chirality of (magneto)fluid turbulence. J. Fluid. Mech. 739, 479–501.
Ziman J. M. 1953 Quantum Hydrodynamics and the Theory of Liquid Helium. Proc. Roy. Soc. A 219, 257.
Zuccher, S., Ricca, R.L. 2015 Helicity conservation under quantum reconnection of vortex rings. Phys. Rev. E 92, 061001(R).