PHASE TRANSITIONS ON C*-ALGEBRAS ARISING FROM NUMBER FIELDS AND THE GENERALIZED FURSTENBERG CONJECTURE

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ABSTRACT. In recent work, Cuntz, Deninger and Laca have studied the Toeplitz type C*-algebra associated to the affine monoid of algebraic integers in a number field, under a time evolution determined by the absolute norm. The KMS equilibrium states of their system are parametrized by traces on the C*-algebras of the semidirect products \( J_\gamma \times \mathbb{O}_K^\times \) resulting from the multiplicative action of the units \( \mathbb{O}_K^\times \) on integral ideals \( J_\gamma \) representing each ideal class \( \gamma \in \mathcal{O}_K \). At each fixed inverse temperature \( \beta > 2 \), the extremal equilibrium states correspond to extremal traces of \( C^*(J_\gamma \times \mathbb{O}_K^\times) \). Here we undertake the study of these traces using the transposed action of \( \mathbb{O}_K^\times \) on the duals \( J_\gamma^\vee \) of the ideals and the recent characterization of traces on transformation group C*-algebras due to Neshveyev. We show that the extremal traces of \( C^*(J_\gamma \times \mathbb{O}_K^\times) \) are parametrized by pairs consisting of an ergodic invariant measure for the action of \( \mathbb{O}_K^\times \) on \( J_\gamma^\vee \) together with a character of the isotropy subgroup associated to the support of this measure. For every class \( \gamma \), the dual group \( J_\gamma \) is a d-torus on which \( \mathbb{O}_K^\times \) acts by linear toral automorphisms. Hence, the problem of classifying all extremal traces is a generalized version of Furstenberg’s celebrated \( \times_2 \times_3 \) conjecture. We classify the results for various number fields in terms of ideal class group, degree, and unit rank, and we point along the way the trivial, the intractable, and the conjecturally classifiable cases. At the topological level, it is possible to characterize the number fields for which infinite \( \mathbb{O}_K^\times \)-invariant sets are dense in \( J_\gamma^\vee \), thanks to a theorem of Berend; as an application we give a description of the primitive ideal space of \( C^*(J_\gamma \times \mathbb{O}_K^\times) \) for those number fields.

1. INTRODUCTION

Let \( K \) be an algebraic number field and let \( \mathbb{O}_K \) denote its ring of integers. The associated multiplicative monoid \( \mathbb{O}_K^\times := \mathbb{O}_K \setminus \{0\} \) of nonzero integers acts by injective endomorphisms on the additive group of \( \mathbb{O}_K \) and gives rise to the semi-direct product \( \mathbb{O}_K \rtimes \mathbb{O}_K^\times \), the affine monoid (or ‘\( b + ax \) monoid’) of algebraic integers in \( K \).

Let \( (\xi_{(x,w)} : (x,w) \in \mathbb{O}_K \rtimes \mathbb{O}_K^\times) \) be the standard orthonormal basis of the Hilbert space \( \ell^2(\mathbb{O}_K \rtimes \mathbb{O}_K^\times) \). The left regular representation \( L \) of \( \mathbb{O}_K \rtimes \mathbb{O}_K^\times \) by isometries on \( \ell^2(\mathbb{O}_K \rtimes \mathbb{O}_K^\times) \) is determined by \( L_{(b,a)} \xi_{(x,w)} = \xi_{(b+ax,aw)} \). In [2], Cuntz, Deninger and Laca studied the Toeplitz-like C*-algebra \( \mathcal{T}[\mathbb{O}_K] := C^*(L_{(b,a)} : (b,a) \in \mathbb{O}_K \rtimes \mathbb{O}_K^\times) \) generated by this representation and analyzed the equilibrium states of the natural time evolution \( \sigma \) on \( \mathcal{T}[\mathbb{O}_K] \) determined by the absolute norm \( N_a := |\mathbb{O}_K^\times / \langle a \rangle| \) via

\[
\sigma_t(L_{(b,a)}) = N_a^{it} L_{(b,a)}, \quad a \in \mathbb{O}_K^\times, \quad t \in \mathbb{R}.
\]

One of the main results of [2] is a characterization of the simplex of KMS equilibrium states of this dynamical system at each inverse temperature \( \beta \in [0, \infty) \). Here we will be interested in the low-temperature range of that classification. To describe the result briefly, let \( \mathbb{O}_K^\times \) be the group of units, that is, the elements of \( \mathbb{O}_K^\times \) whose inverses are also integers, and recall that by a celebrated theorem of Dirichlet, \( \mathbb{O}_K^\times \cong \mathbb{Z}^{r+s-1} \), where \( \mathbb{W}_K \) (the
group of roots of unity in $\mathcal{O}_K^*$) is finite, $r$ is the number of real embeddings of $K$, and $s$ is equal to half the number of complex embeddings of $K$. Let $\mathcal{O}_K$ be the ideal class group of $K$, which, by definition, is the quotient of the group of all fractional ideals in $K$ modulo the principal ones, and is a finite abelian group. For each ideal class $\gamma \in \mathcal{O}_K$, let $J_\gamma$ be an integral ideal representing $\gamma$. By [2, Theorem 7.3], for each $\beta > 2$ the KMS$_\beta$ states of $C^*(\mathcal{O}_K \rtimes \mathcal{O}_K^*)$ are parametrized by the tracial states of the direct sum of group C*-algebras $\bigoplus_{\gamma \in \mathcal{O}_K} C^*(J_\gamma \rtimes \mathcal{O}_K^*)$, where the units act by multiplication on each ideal viewed as an additive group. It is intriguing that exactly the same direct sum of group C*-algebras also plays a role in the computation of the K-groups of the semigroup C*-algebras of algebraic integers in the work of Cuntz, Echterhoff and Li, see e.g. [3, Theorem 8.2.1]. Considering as well that the group of units and the ideals representing different ideal classes are a measure of the failure of unique factorization into primes in $\mathcal{O}_K$, we feel it is of interest to investigate the tracial states of the C*-algebras $C^*(J_\gamma \rtimes \mathcal{O}_K^*)$ that arise as a natural parametrization of KMS equilibrium states of $C^*(\mathcal{O}_K \rtimes \mathcal{O}_K^*)$.

This work is organized as follows. In Section 2 we review the phase transition from [2] and apply a theorem of Neshveyev’s to show in Theorem 2.2 that the extremal KMS states arise from ergodic invariant probability measures and characters of their isotropy subgroups for the actions $\mathcal{O}_K^* \subset \hat{J}_\gamma$ of units on the duals of integral ideals.

We begin Section 3 by showing that for imaginary quadratic fields, the orbit space of the action of units is a compact Hausdorff space that parametrizes the ergodic invariant probability measures. All other number fields have infinite groups of units leading to ‘bad quotients’ for which noncommutative geometry provides convenient tools of analysis. Units act by toral automorphisms and so the classification of equilibrium states is intrinsically related to the higher-dimensional, higher-rank version of the question, first asked by H. Furstenberg, of whether Lebesgue measure is the only nonatomic ergodic invariant measure for the pair of transformations $\times 2$ and $\times 3$ on $\mathbb{R}/\mathbb{Z}$. Once in this framework, it is evident from work of Sigmund [22] and of Marcus [15] on partially hyperbolic toral automorphisms and from the properties of the Poulsen simplex [13], that for fields whose unit rank is 1, which include real quadratic fields, there is an abundance of ergodic measures, Proposition 3.5, and hence of extremal equilibrium states, see also [11]. We also show in this section that there is solidarity among integral ideals with respect to the ergodicity properties of the actions of units, Proposition 3.6.

In Section 4, we look at the topological version of the problem and we identify the number fields for which [1, Theorem 2.1] can be used to give a complete description of the invariant closed sets. In Theorem 4.11 we summarize the consequences, for extremal equilibrium at low temperature, of the current knowledge on the generalized Furstenberg conjecture. For fields of unit rank at least 2 that are not complex multiplication fields, i.e. that have no proper subfields of the same unit rank, we show that if there is an extremal KMS state that does not arise from a finite orbit or from Lebesgue measure, then it must arise from a zero-entropy, nonatomic ergodic invariant measure; it is not known whether such a measure exists. For complex multiplication fields of unit rank at least 2, on the other hand, it is known that there are other measures, arising from invariant subtori. As a byproduct, we also provide in Proposition 4.9 a proof of an interesting fact stated in [25], namely the units acting on algebraic integers are generic among toral automorphism groups that have Berend’s ID property.

We conclude our analysis in Section 5 by computing the topology of the quasi-orbit space of the action $\mathcal{O}_K^* \subset \hat{\mathcal{O}}_K$ for number fields satisfying Berend’s conditions. As an
application we also obtain an explicit description of the primitive ideal space of the C*-algebra $C(\mathcal{O}_K \times C_k^*)$, Theorem 5.2. For the most part, sections 3 and 4 do not depend on operator algebra considerations other than for the motivation and the application, which are discussed in sections 2 and 5.

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2. From KMS states to invariant measures and isotropy

Our approach to describing the tracial states of the C*-algebras $\bigoplus_{\gamma \in C_k} C^*(\mathcal{J}_\gamma \times C_k^*)$ is shaped by the following three observations. First, the tracial states of a group C*-algebra form a Choquet simplex [23], so it suffices to focus our attention on the extremal traces. Second, there is a canonical isomorphism $C^*(\mathcal{J}) \cong C^*(\mathcal{J} \times C_k^*)$, which we may combine with the Gelfand transform for $C^*(\mathcal{J})$, thus obtaining an isomorphism of $C^*(\mathcal{J} \times C_k^*)$ to the transformation group C*-algebra $C(\hat{\mathcal{J}}) \times C_k^*$, associated to the transposed action of $\mathcal{O}_k^*$ on the continuous complex-valued functions on the compact dual group $\hat{\mathcal{J}}$. Specifically, the action of $\mathcal{O}_k^*$ on $\hat{\mathcal{J}}$ is determined by

\begin{equation}
(u \cdot \chi)(j) := \chi(\hat{u}j), \quad u \in \mathcal{O}_k^*, \quad \chi \in \hat{\mathcal{J}}, \quad j \in \mathcal{J},
\end{equation}

or by $\langle j, u \cdot \chi \rangle = \langle u_j, \chi \rangle$, if we use $\langle \cdot, \cdot \rangle$ to denote the duality pairing of $J$ and $\hat{J}$. This, third, puts the problem of describing the tracial states squarely in the context of Neshveyev's characterization of traces on crossed products, so our task is to identify and describe the relevant ingredients of this characterization. In brief terms, when [16, Corollary 2.4] is interpreted in the present situation, it says that that for each integral ideal $J$, the extremal traces on $C(\hat{\mathcal{J}}) \times C_k^*$ are parametrized by triples $(H, \chi, \mu)$ in which $H$ is a subgroup of $C_k^*$, $\chi$ is a character of $H$, and $\mu$ is an ergodic $C_k^*$-invariant measure on $\hat{J}$ such that the set of points in $\hat{J}$ whose isotropy subgroups for the action of $\mathcal{O}_k^*$ are equal to $H$ has full $\mu$ measure.

Recall that, by definition, an $\mathcal{O}_k^*$-invariant probability measure $\mu$ on $\hat{J}$ is ergodic invariant for the action of $\mathcal{O}_k^*$ if $\mu(A) \in \{0,1\}$ for every $\mathcal{O}_k^*$-invariant Borel set $A \subset \hat{J}$. Our first simplification is that the action of $\mathcal{O}_k^*$ on $\hat{J}$ automatically has $\mu$-almost everywhere constant isotropy with respect to each ergodic invariant probability measure $\mu$.

**Lemma 2.1.** Let $K$ be an algebraic number field with ring of integers $\mathcal{O}_K$ and group of units $\mathcal{O}_K^*$ and let $J$ be a nonzero ideal in $\mathcal{O}_K$. Suppose $\mu$ is an ergodic $\mathcal{O}_K^*$-invariant probability measure on $\hat{J}$. Then there exists a unique subgroup $H_\mu$ of $\mathcal{O}_K^*$ such that the isotropy group $\{\mathcal{O}_K^*\}_\chi := \{u \in \mathcal{O}_K^* : u \cdot \chi = \chi\}$ is equal to $H_\mu$ for $\mu$-a.a. characters $\chi \in \hat{J}$.

**Proof.** For each subgroup $H \leq \mathcal{O}_K^*$, let $M_H := \{\chi \in \hat{J} \mid \{\mathcal{O}_K^*\}_\chi = H\}$ be the set of characters of $\mathcal{J}$ with isotropy equal to $H$. Since the isotropy is constant on orbits, each $M_H$ is $\mathcal{O}_K^*$-invariant, and clearly the $M_H$ are mutually disjoint. By Dirichlet's unit theorem $\mathcal{O}_K^* \cong W_K \times \mathbb{Z}^{r+s-1}$ with $W_K$ finite, and $r$ and $2s$ the number of real and complex embeddings of $K$, respectively. Thus every subgroup of $\mathcal{O}_K^*$ is generated by at most $|W_K| + (r + s - 1)$ generators, and hence $\mathcal{O}_K^*$ has only countably many subgroups. Thus $\{M_H : H \leq \mathcal{O}_K^*\}$ is a countable partition of $\hat{J}$ into subsets of constant isotropy.

We claim that each $M_H$ is a Borel measurable set in $\hat{J}$. To see this, observe:
because $u \cdot \chi = \chi$ iff $\chi^{-1}(u \cdot \chi) = 1$. Since the map $\chi \mapsto \chi^{-1}(u \cdot \chi)$ is continuous on $\hat{J}$, the sets in the first intersection are closed and those in the second one are open. By above, the intersection is countable, so $M_{H\mu}$ is Borel-measurable, as desired.

For every Borel measure $\mu$ on $\hat{J}$, we have

$$\sum_{H \leq O_k^*} \mu(M_{H\mu}) = \mu\left( \bigcup_{H \leq O_k^*} M_{H\mu} \right) = 1,$$

so at least one $M_{H\mu}$ has positive measure. If $\mu$ is ergodic $O_k^*$-invariant, then there exists a unique $H_{\mu} \leq O_k^*$ such that $\mu(M_{H_{\mu}\mu}) = 1$ and thus $H_{\mu}$ is the (constant) isotropy group of $\mu$-a.a points $\chi \in \hat{J}$.

Since each ergodic invariant measure determines an isotropy subgroup, the characterization of extremal traces from [16, Corollary 2.4] simplifies as follows.

**Theorem 2.2.** Let $K$ be an algebraic number field with ring of integers $O_K$ and group of units $O_K^*$ and let $J$ be a nonzero ideal in $O_K$. Denote the standard generating unitaries of $C^*(J \rtimes O_K^*)$ by $\delta_j$ for $j \in J$ and $\nu_u$ for $u \in O_K^*$. Then for each extremal trace $\tau$ on $C^*(J \rtimes O_K^*)$ there exists a unique probability measure $\mu_{\tau}$ on $\hat{J}$ such that

$$\int_{\hat{J}} \langle j, \chi \rangle d\mu_{\tau}(\chi) = \tau(\delta_j) \quad \text{for } j \in J. \quad (2.2)$$

The probability measure $\mu_{\tau}$ is ergodic $O_K^*$-invariant, and if we denote by $H_{\mu_{\tau}}$ its associated isotropy subgroup from Lemma 2.1 then the function $\chi_{\tau}$ defined by $\chi_{\tau}(h) := \tau(\nu_h)$ for $h \in H_{\mu_{\tau}}$ is a character on $H_{\mu_{\tau}}$.

Furthermore, the map $\tau \mapsto (\mu_{\tau}, \chi_{\tau})$ is a bijection of the set of extremal traces of $C^*(J \rtimes O_K^*)$ onto the set of pairs $(\mu, \chi)$ consisting of an ergodic $O_K^*$-invariant probability measure $\mu$ on $\hat{J}$ and a character $\chi \in \hat{H}_{\mu}$. The inverse map $(\mu, \chi) \mapsto \tau_{(\mu, \chi)}$ is determined by

$$\tau_{(\mu, \chi)}(\delta_j\nu_u) = \begin{cases} 
\chi(u) \int_{\hat{J}} \langle j, \chi \rangle d\mu(\chi) & \text{if } u \in H_{\mu} \\
0 & \text{otherwise},
\end{cases} \quad (2.3)$$

for $j \in J$ and $u \in O_K^*$.

**Proof.** Recall that equation (2.1) gives the continuous action of $O_K^*$ by automorphisms of the compact abelian group $\hat{J}$ obtained on transposing the multiplicative action of $O_K^*$ on $J$. There is a corresponding action $\alpha$ of $O_K^*$ by automorphisms of the $C^*$-algebra $C(\hat{J})$ of continuous functions on $\hat{J}$; it is given by $\alpha_u(f)(\chi) = f(u^{-1} \cdot \chi)$.

The characterization of traces [16, Corollary 2.4] then applies to the crossed product $C(\hat{J}) \rtimes_\alpha O_K^*$ as follows. For a given extremal tracial state $\tau$ of $C^*(J \rtimes O_K^*)$ there is a probability measure $\mu_{\tau}$ on $\hat{J}$ that arises, via the Riesz representation theorem, from the restriction of $\tau$ to $C^*(J) \cong C(\hat{J})$ and is characterized by its Fourier coefficients in equation (2.2). By Lemma 2.1 there is a subset of $\hat{J}$ of full $\mu_{\tau}$ measure on which the isotropy subgroup is
automatically constant, and is denoted by $H_{\mu_r}$. The unitary elements $v_u$ generate a copy of $C^*(O_k^e)$ inside $C(\hat{\mathbb{J}}) \rtimes_o O_k^e$ and the restriction of $\tau$ to these generators determines a character $\chi_\tau$ of $H_{\mu_r}$ given by $\chi_\tau(u) := \tau(v_u)$. See the proof of [16, Corollary 2.4] for more details. By Lemma 2.1 the condition of almost constant isotropy is automatically satisfied for every ergodic invariant measure on $\hat{J}$, hence every ergodic invariant measure arises as $\mu_\tau$ for some extremal trace $\tau$. The parameter space for extremal tracial states is thus the set of all pairs $(\mu, \chi)$ consisting of an ergodic $O_k^e$-invariant probability measure $\mu$ on $\hat{J}$ and a character $\chi$ of the isotropy subgroup $H_\mu$ of $\mu$. Formula (2.3) is a particular case of the formula in [16, Corollary 2.4] with $f$ equal to the character function $f(\cdot) = \langle j, \cdot \rangle$ on $\hat{J}$ associated to $j \in J$. Since for a fixed $u \in O_k^e$ the right hand side of (2.3) is a continuous linear functional of the integrand and the character functions span a dense subalgebra, this particular case is enough to imply

$$
\tau_{(\mu, \chi)}(f v_u) = \begin{cases} 
\chi(u) \int f(x) d\mu(x) & \text{if } u \in H_{\mu} \\
0 & \text{otherwise,}
\end{cases}
$$

for every $f \in C(\hat{J})$.

3. The action of units on integral ideals

Combining [2, Theorem 7.3] with Theorem 2.2 above, we see that for $\beta > 2$, the extremal KMS$_\beta$ equilibrium states of the system $(\Sigma[O_k], \sigma)$ are indexed by pairs $(\mu, \kappa)$ consisting of an ergodic invariant probability measure $\mu$ and a character $\kappa$ of its isotropy subgroup relative to the action of the unit group $O_k^e$ on a representative of each ideal class.

If the field $K$ is imaginary quadratic, that is, if $r = 0$ and $s = 1$, then the group of units is finite, consisting exclusively of roots of unity. In this case, things are easy enough to describe because the space of $O_k^e$-orbits in $\hat{J}$ is a compact Hausdorff topological space.

**Proposition 3.1.** Suppose $K$ is an imaginary quadratic number field, let $J \subset O_k$ be an integral ideal and write $W_K$ for the group of units. Then the orbit space $W_K/\hat{J}$ is a compact Hausdorff space and the closed invariant sets in $\hat{J}$ are indexed by the closed sets in $W_K/\hat{J}$. Moreover, the ergodic invariant probability measures on $\hat{J}$ are the equiprobability measures on the orbits and correspond to unit point masses on $W_K/\hat{J}$.

**Proof.** Since $W_K$ is finite, distinct orbits are separated by disjoint invariant open sets, so the quotient space $W_K/\hat{J}$ is a compact Hausdorff space. Since $\hat{J}$ is compact, the quotient map $q : \hat{J} \to W_K/\hat{J}$ given by $q(\chi) := W_K \cdot \chi$ is a closed map by the closed map lemma, and so invariant closed sets in $\hat{J}$ correspond to closed sets in the quotient.

For each probability measure $\mu$ on $\hat{J}$, there is a probability measure $\tilde{\mu}$ on $W_K/\hat{J}$ defined by

$$
\tilde{\mu}(E) := \mu(q^{-1}(E)) \quad \text{for each measurable } E \subseteq W_K/\hat{J}.
$$

This maps the set of $W_K$-invariant probability measures on $\hat{J}$ onto the set of all probability measures on $W_K/\hat{J}$. Ergodic invariant measures correspond to unit point masses on $W_K/\hat{J}$, and their $W_K$-invariant lifts are equiprobability measures on single orbits in $\hat{J}$.

As a result we obtain the following characterization of extremal KMS equilibrium states.
Corollary 3.2. Suppose $K$ is an imaginary quadratic algebraic number field and let $J_{\gamma}$ be an integral ideal representing the ideal class $\gamma \in \mathcal{O}_K$. For $\beta > 2$, the extremal KMS$_\beta$ states of the system $(\mathbb{T} \mathcal{O}_K, \sigma^N)$ are parametrized by the triples $(\gamma, W \cdot \chi, \kappa)$, where $\gamma \in \mathcal{O}_K$, $\chi$ is a point in $\hat{J}_\gamma$, with orbit $W \cdot \chi$ and $\kappa$ is a character of the isotropy subgroup of $\chi$.

Before we discuss invariant measures and isotropy for fields with infinite group of units, we need to revisit a few general facts about the multiplicative action of units on the algebraic integers and, more generally, on the integral ideals. The concise discussion in [25] is particularly convenient for our purposes. As is customary, we let $\mathcal{O}_R$ be the ring of integers $\mathcal{O}_K$ of $K$ over $\mathcal{O}_R$. The number $r$ of real embeddings and the number $2s$ of complex embeddings satisfy $r + 2s = d$. We also let $n = r + s - 1$ be the unit rank of $K$, namely, the free abelian rank of $\mathcal{O}_K^*$ according to Dirichlet's unit theorem. We shall denote the real embeddings of $K$ by $\sigma_j : K \to \mathbb{R}$ for $j = 1, 2, \ldots r$ and the conjugate pairs of complex embeddings of $K$ by $\sigma_{r+j}, \sigma_{r+s+j} : K \to \mathbb{C}$ for $j = 1, \ldots, s$. Thus, there is an isomorphism

$$\sigma : K \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}^r \times \mathbb{C}^s$$

such that

$$\sigma(k \otimes x) = (\sigma_1(k)x, \sigma_2(k)x, \ldots, \sigma_r(k)x, \sigma_{r+1}(k)x, \ldots, \sigma_{r+s}(k)x).$$

The ring of integers $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank $d$, and thus $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d \cong \mathbb{R}^r \oplus \mathbb{C}^s$. We temporarily fix an integral basis for $\mathcal{O}_K$, which fixes an isomorphism $\theta : \mathcal{O}_K \to \mathbb{Z}^d$. Then, at the level of $\mathbb{Z}^d$, the action of each $u \in \mathcal{O}_K^*$ is implemented as left multiplication by a matrix $A_u \in \text{GL}_d(\mathbb{Z})$. Moreover, once this basis has been fixed, the usual duality pairing $\langle \mathbb{Z}^d, \mathbb{R}^d/\mathbb{Z}^d \rangle$ given by $\langle n, t \rangle = \exp 2\pi i (n \cdot t)$, with $n \in \mathbb{Z}^d$, $t \in \mathbb{R}^d$ and $n \cdot t = \sum_{i=1}^d n_i t_i$, gives an isomorphism of $\mathbb{R}^d/\mathbb{Z}^d$ to $\hat{\mathcal{O}}_K$, in which the character $\chi_t \in \hat{\mathcal{O}}_K$ corresponding to $t \in \mathbb{R}^d/\mathbb{Z}^d$ is given by $\chi_t(x) = \exp 2\pi i (\theta(x) \cdot t)$ for $x \in \mathcal{O}_K$. Thus, the action of a unit $u \in \mathcal{O}_K^*$ is

$$(u \cdot \chi_t)(x) = \chi_t(u \cdot x) = \exp 2\pi i (A_u \theta(x) \cdot t) = \exp 2\pi i (\theta(x) \cdot A_u^T t).$$

This implies that the action $\mathcal{O}_K^* \circlearrowleft \hat{\mathcal{O}}_K$ is implemented, at the level of $\mathbb{R}^d/\mathbb{Z}^d$, by the representation $\rho : \mathcal{O}_K^* \to \text{GL}_d(\mathbb{Z})$ defined by $\rho(u) = A_u^T$, cf. [24] Theorem 0.15).

Similar considerations apply to the action of $\mathcal{O}_K^*$ on $\hat{J}$ for each integral ideal $J \subset \mathcal{O}_K$, giving a representation $\rho_J : \mathcal{O}_K^* \to \text{GL}_d(\mathbb{Z})$. For ease of reference we state the following fact about this matrix realization $\rho_J$ of the action of $\mathcal{O}_K^*$ on $\hat{J}$.

Proposition 3.3. The collection of matrices $\{\rho(u) : u \in \mathcal{O}_K^*\}$ is simultaneously diagonalizable (over $\mathbb{C}$) and for each $u \in \mathcal{O}_K^*$ the eigenvalue list of $\rho(u)$ is the list of its archimedean embeddings $\sigma_K(u) : k = 1, 2, \ldots r + 2s$.

See e.g. the discussion in [14] Section 2.1, and [25] Section 2.1 for the details. Multiplication of complex numbers in each complex embedding is regarded as the action of 2x2 matrices on $\mathbb{R} + i\mathbb{R} \cong \mathbb{R}^2$, and the 2x2 blocks corresponding to complex roots simultaneously diagonalize over $\mathbb{C}^d$. The self duality of $\mathbb{R}^r \oplus \mathbb{C}^s$ can be chosen to be compatible with the isomorphism mentioned right after (2.1) in [14] and with multiplication by units. See also [21] Ch7.

When the number field $K$ is not imaginary quadratic, then $\mathcal{O}_K^*$ is infinite and so the analysis of orbits and invariant measures is much more subtle; for instance, most orbits are infinite, some are dense, and the orbit space does not have a Hausdorff topology. We summarize for convenience of reference the known basic general properties in the next proposition.
**Proposition 3.4.** Let \( K \) be a number field with rank(\( \mathcal{O}_K^* \)) \( \geq 1 \), and let \( I \) be an ideal in \( \mathcal{O}_K^* \). Then normalized Haar measure on \( \hat{I} \) is ergodic \( \mathcal{O}_K^* \)-invariant, and for each \( \chi \in \hat{I} \),

1. the orbit \( \mathcal{O}_K^* \cdot \chi \) is finite if and only if \( \chi \) corresponds to a point with rational coordinates in the identification \( \hat{I} \cong \mathbb{R}^d/\mathbb{Z}^d \); in this case the corresponding isotropy subgroup is a full-rank subgroup of \( \mathcal{O}_K^* \);

2. the orbit \( \mathcal{O}_K^* \cdot \chi \) is infinite if and only if \( \chi \) corresponds to a point with at least one irrational coordinate in \( \mathbb{R}^d/\mathbb{Z}^d \);

3. the characters \( \chi \) corresponding to points \((w_1, w_2, \ldots, w_d) \in \mathbb{R}^d \) such that the numbers \( 1, w_1, w_2, \ldots, w_d \) are rationally independent have trivial isotropy.

**Proof.** By Proposition 3.3, for each \( u \in \mathcal{O}_K^* \), the eigenvalues of the matrix \( \rho(u) \) encoding the action of \( u \) at the level of \( \mathbb{R}^d/\mathbb{Z}^d \) are precisely the various embeddings of \( u \) in the archimedean completions of \( K \). Since rank(\( \mathcal{O}_K^* \)) \( \geq 1 \), there exists a non-torsion element \( u \in \mathcal{O}_K^* \), whose eigenvalues are not roots of unity. Hence normalized Haar measure is ergodic for the action of \( \{ \rho(u) : u \in \mathcal{O}_K^* \} \) by [24, Corollary 1.10.1] and the first assertion now follows from [24, Theorem 5.11]. The isotropy is a full rank subgroup of \( \mathcal{O}_K^* \) because \( |\mathcal{O}_K^*/(\mathcal{O}_K^*)_I| = |\mathcal{O}_K^* \cdot x| < \infty \).

Let \( w = (w_1, w_2, \ldots, w_d) \) be a point in \( \mathbb{R}^d/\mathbb{Z}^d \) such that \( 1, w_1, \ldots, w_d \) are rationally independent. Suppose \( w \) is a fixed point for the matrix \( \rho(u) \in \text{GL}_d(\mathbb{Z}) \) acting on \( \mathbb{R}^d/\mathbb{Z}^d \). Then \( \rho(u)w = w \) (mod \( \mathbb{Z}^d \)) and hence \( \rho(u) - I \) is a fixed point for the action of \( \rho(u) \) on \( \mathbb{R}^d \).

\[
[(\rho(u) - I)w]_i = \sum_{j=1}^{d} (\rho(u) - I)_{ij}w_j \in \mathbb{Z}
\]

for all \( 1 \leq i \leq d \). Since \( (\rho(u) - I)_{ij} \in \mathbb{Z} \) for all \( i, j \), the rational independence of \( 1, w_1, \ldots, w_d \) implies that \( \rho(u) = I \), so \( u = 1 \), as desired. \( \square \)

We see next that for the number fields with unit rank 1 there are many more ergodic invariant probability measures on \( \hat{O}_K \) than just Haar measure and measures supported on finite orbits. In fact, a smooth parametrization of these measures and of the corresponding KMS equilibrium states of \( (\mathcal{S}[\hat{O}_K], \sigma) \) seems unattainable.

**Proposition 3.5.** Suppose the number field \( K \) has unit-rank equal to 1, namely, \( K \) is real quadratic, mixed cubic, or complex quartic. Then the simplex of ergodic invariant probability measures on \( \hat{O}_K \) is isomorphic to the Poulsen simplex [13].

**Proof.** The fundamental unit gives a partially hyperbolic toral automorphism of \( \hat{O}_K \), for which Haar measure is ergodic invariant. By [15, 22], the invariant probability measures of such an automorphism that are supported on finite orbits are dense in the space of all invariant probability measures. This remains true when we include the torsion elements of \( \mathcal{O}_K^* \). Since these equiprobabilities supported on finite orbits are obviously ergodic invariant and hence extremal among invariant measures, it follows from [13, Theorem 2.3] that the simplex of invariant probability measures on \( \hat{O}_K \) is isomorphic to the Poulsen simplex. \( \square \)

For fields with unit rank at least 2, whether normalized Haar measure and equiprobabilities supported on finite orbits are the only ergodic \( \mathcal{O}_K^* \)-invariant probability measures is a higher-dimensional version of the celebrated Furstenberg conjecture, according to which Lebesgue measure is the only non-atomic probability measure on \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) that is jointly
ergodic invariant for the transformations $\times 2$ and $\times 3$ on $\mathbb{R}$ modulo $\mathbb{Z}$. As stated, this remains open, however, Rudolph and Johnson have proved that if $p$ and $q$ are multiplicatively independent positive integers, then the only probability measure on $\mathbb{R}/\mathbb{Z}$ that is ergodic invariant for $\times p$ and $\times q$ and has non-zero entropy is indeed Lebesgue measure [18, 7]. Number fields always give rise to automorphisms of tori of dimension at least 2, so, strictly speaking the problem in which we are interested does not contain Furstenberg's original formulation as a particular case. Nevertheless, the higher-dimensional problem is also interesting and open as stated in general, and there is significant recent activity on it and on closely related problems [9, 8, 10]. In particular, see [4] for a summary of the history and also a positive entropy result for higher dimensional tori along the lines of the Rudolph-Johnson theorem. We show next that the toral automorphism groups arising from different integral ideals have a solidarity property with respect to the generalized Furstenberg conjecture.

**Proposition 3.6.** If for some integral ideal $I$ in $\mathcal{O}_k$ the only ergodic $\mathcal{O}_k^I$-invariant probability measure on $\hat{I}$ having infinite support is normalized Haar measure, then the same is true for every integral ideal in $\mathcal{O}_k$.

The proof depends on the following lemmas.

**Lemma 3.7.** Let $J \subseteq I$ be two integral ideals in $\mathcal{O}_k$ and let $r: \hat{I} \rightarrow \hat{J}$ be the restriction map. Denote by $\lambda_\gamma$ normalized Haar measure on $\hat{I}$. For each $\gamma \in \hat{J}$, there exists a neighborhood $N$ of $\gamma$ in $\hat{J}$ and homeomorphisms $h_j$ of $N$ into $\hat{I}$ for $j = 1, 2, \ldots, |I/J|$, with mutually disjoint images and such that

1. $\lambda_\gamma(h_j(E)) = \lambda_\gamma(h_k(E))$ for every measurable $E \subseteq N$ and $1 \leq j, k \leq |I/J|$;
2. $r \circ h_j = \text{id}_N$;
3. $r^{-1}(E) = \bigsqcup J h_j(E)$ for all $E \subseteq N$, that is, the $h_j$'s form a complete system of local inverses of $r$ on $N$.

**Proof.** Let $J^\perp := \{\kappa \in \hat{I}: \kappa(j) = 1, \forall j \in J\}$ be the kernel of the restriction map $r: \hat{I} \rightarrow \hat{J}$. Since $J^\perp$ is a subgroup of order $|I/J| < \infty$, and since $\hat{I}$ is Hausdorff, we may choose a collection $\{A_\kappa : \kappa \in J^\perp\}$ of mutually disjoint open subsets of $\hat{I}$ such that $\kappa \in A_\kappa$ for each $\kappa \in J^\perp$. Define $B_1 := \bigcap_{\kappa \in J^\perp} \kappa^{-1}A_\kappa$ and for each $\kappa \in J^\perp$ let $B_\kappa := \kappa B_1$. Then $\{B_\kappa : \kappa \in J^\perp\}$ is a collection of mutually disjoint open sets such that $\kappa \in B_\kappa$ and $r(B_\kappa) = r(B_1)$ for every $\kappa \in J^\perp$. We claim that the restrictions $r : B_\kappa \rightarrow \hat{J}$ are homeomorphisms onto their image. Since the $B_\kappa$ are translates of $B_1$ and since $r$ is continuous and open, it suffices to verify that $r$ is injective on $B_1$. This is easy to see because if $r(\xi_1) = r(\xi_2)$ for two distinct elements $\xi_1, \xi_2$ of $B_1$, then $\xi_2 = \kappa \xi_1$ for some $\rho \in J^\perp \setminus \{1\}$, and this would contradict $B_1 \cap \kappa B_1 = \emptyset$. This proves the claim. We may then take $N := \gamma r(B_1)$ and define $h_\rho := (r|_{B_\rho})^{-1}$, for which properties (1)-(3) are now easily verified. \hfill $\square$

**Lemma 3.8.** Let $X$ be a measurable space and let $T : X \rightarrow X$ be measurable. Suppose that $\lambda$ is an ergodic $T$-invariant probability measure on $X$. If $\mu$ is a $T$-invariant probability measure on $X$ such that $\mu \ll \lambda$, then $\mu = \lambda$.

**Proof.** Fix $f \in L^\infty(\lambda)$ and define $(A_n f)(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$. Let $S = \{x \in X : (A_n f)(x) \rightarrow \int_X f d\lambda\}$. By the Birkhoff ergodic theorem, we have that $\lambda(S^c) = 0$, and so $\mu(S^c) = 0$ as well, that is, $(A_n f)(x) \rightarrow \int_X f d\lambda$ $\mu$-a.e.. Since $f \in L^\infty(\lambda)$ and $\mu \ll \lambda$, we have that $f \in L^\infty(\mu)$ as well, with $\|f\|_\infty \leq \|f\|_\lambda^{\lambda}$. Observe that $\|A_n f\|_\mu \leq \|f\|_\infty^{\lambda}$ for $\mu$-a.e. $x$, and so by
the dominated convergence theorem, \( \int_X A_n f d\mu \to \int_X (\int_X f d\lambda) d\mu = \int_X f d\lambda \), with the last equality because \( \mu(X) = 1 \).

Because \( \mu \) is \( T \)-invariant, we have that \( \int_X A_n f d\mu = \int_X f d\mu \) for all \( n \). Combining this with the above implies that \( \int_X f d\lambda = \int_X f d\mu \) for all \( f \in L^\infty(\lambda) \). In particular, this holds for the indicator function of each measurable set, and so \( \mu = \lambda \).

**Lemma 3.9.** Let \( J \subseteq I \) be two integral ideals in \( \hat{O}_K \) and let \( r : \hat{\mathfrak{r}} \to \hat{\mathfrak{r}} \) be the restriction map. If \( \mu \) is an ergodic \( O_K^{\mathfrak{r}} \)-invariant probability measure on \( \hat{\mathfrak{r}} \), then \( \hat{\mu} := \mu \circ r^{-1} \) is an ergodic invariant probability measure on \( \hat{\mathfrak{r}} \). Moreover, the support of \( \hat{\mu} \) is finite if and only if the support of \( \mu \) is finite.

**Proof.** Assume \( \mu \) is ergodic invariant on \( \hat{\mathfrak{r}} \) and let \( E \subseteq \hat{\mathfrak{r}} \) be an \( O_K^{\mathfrak{r}} \)-invariant measurable set. Since \( r \) is \( O_K^{\mathfrak{r}} \)-equivariant, \( r^{-1}(E) \) is also \( O_K^{\mathfrak{r}} \)-invariant so \( \hat{\mu}(E) := \mu(r^{-1}(E)) \in \{0,1\} \) because \( \mu \) is ergodic invariant. Thus, \( \hat{\mu} \) is also ergodic invariant. The statement about the support follows immediately because \( r \) has finite fibers.

**Lemma 3.10.** Suppose \( J \subseteq I \) are integral ideals in \( O_K \), and let \( \lambda_I, \lambda_{\hat{\mathfrak{r}}} \) be normalized Haar measures on \( \hat{\mathfrak{r}}, \hat{I} \), respectively. If the only ergodic \( O_K^{\mathfrak{r}} \)-invariant probability measure on \( \hat{\mathfrak{r}} \) with infinite support is \( \lambda_I \), then the only ergodic \( O_K^{\mathfrak{r}} \)-invariant probability measure on \( \hat{\mathfrak{r}} \) with infinite support is \( \lambda_{\hat{\mathfrak{r}}} \).

**Proof.** Let \( \mu \) be an ergodic \( O_K^{\mathfrak{r}} \)-invariant probability measure on \( \hat{\mathfrak{r}} \) with infinite support. By Lemma 3.9 \( \mu \circ r^{-1} \) is an ergodic \( O_K^{\mathfrak{r}} \)-invariant probability measure on \( \hat{\mathfrak{r}} \) with infinite support, and so by assumption must equal \( \lambda_{\hat{\mathfrak{r}}} \). In particular, \( \mu \circ r^{-1} = \lambda_{\hat{\mathfrak{r}}} \).

Since \( \hat{\mathfrak{r}} \) is compact, the open cover \( \{N_\gamma : \gamma \in \hat{\mathfrak{r}}\} \) given by the sets constructed in Lemma 3.7 has a finite subcover, that is, there exist \( \gamma_1, \ldots, \gamma_n \in \hat{\mathfrak{r}} \) so that \( \hat{\mathfrak{r}} = \bigcup_{k=1}^n N_{\gamma_k} \), where \( N_{\gamma_k} \) is a neighborhood of \( \gamma_k \in \hat{\mathfrak{r}} \) satisfying the conditions stated in Lemma 3.7 with corresponding maps \( h_{\gamma_k}^{(j)} \) for \( 1 \leq j \leq |I|/|J| \) and \( 1 \leq k \leq n \).

We will first show that if \( B \subseteq \hat{\mathfrak{r}} \) is such that \( r|_B \) is a homeomorphism with \( r(B) \subseteq N_{\gamma_1} \) for some \( k \), and if \( \lambda|_B(B) = 0 \), then \( \mu(B) = 0 \). Suppose \( B \) is such a set and \( \lambda(B) = 0 \). By part (3) of Lemma 3.7 \( r^{-1}(B) = \bigcup_{j=1}^{|I|/|J|} h_{\gamma_k}^{(j)}(r(B)) \), so \( r^{-1}(B) \) is a disjoint union of \( |I|/|J| \) sets, all having the same measure under \( \lambda \). Moreover, there exists some \( 1 \leq j \leq |I|/|J| \) such that \( h_{\gamma_k}^{(j)}(r(B)) = B \), because the \( h_{\gamma_k}^{(j)} \)'s form a complete set of local inverses for \( r \), and \( r \) is injective on \( B \). Putting these together yields

\[
\lambda_B(r^{-1}(B)) = |I|/|J| \lambda_I(h_{\gamma_k}^{(j)}(r(B))) = |I|/|J| \lambda_I(B) = 0.
\]

Since \( \mu \circ r^{-1} = \lambda_I = \lambda_{\hat{\mathfrak{r}}} \circ r^{-1} \), this implies that \( \mu(r^{-1}(B)) = 0 \) as well, and since \( B \subseteq r^{-1}(r(B)) \), we have that \( \mu(B) = 0 \).

Now, since \( r : \hat{\mathfrak{r}} \to \hat{\mathfrak{r}} \) is a covering map, for each \( \chi \in \hat{\mathfrak{r}} \), there exists an open neighborhood \( U_\chi \) of \( \chi \) such that \( r|_{U_\chi} \) is a homeomorphism. Let \( 1 \leq k \leq n \) be such that \( r(\chi) \in N_{\gamma_k} \), and let \( W_\chi := U_\chi \cap r^{-1}(N_{\gamma_k}) \). This forms another open cover of \( \hat{\mathfrak{r}} \), and so by compactness of \( \hat{\mathfrak{r}} \), there exists a finite subcover \( W_1, \ldots, W_m \).

Finally, let \( A \subseteq \hat{\mathfrak{r}} \) be such that \( \lambda_I(A) = 0 \). Then \( A \cap W_i \) is a set on which \( r \) acts as a homeomorphism, and there exists \( 1 \leq k \leq n \) such that \( r(A \cap W_i) \subseteq N_{\gamma_k} \). Thus, by the above, we conclude \( \mu(A \cap W_i) = \mu(A) = 0 \) for all \( 1 \leq i \leq m \). Since these sets cover \( A \), we have that \( \mu(A) = 0 \), and hence \( \mu \ll \lambda_I \), as desired. By Lemma 3.8 it follows that \( \mu = \lambda_I \).
Proof of Proposition 3.6. Suppose $J$ is an integral ideal such that the only $\mathcal{O}_K^*$-invariant probability measure on $\hat{J}$ having an infinite orbit is normalized Haar measure. By Lemma 3.10 applied to the inclusion $J \subseteq \mathcal{O}_K$, the only ergodic $\mathcal{O}_K^*$-invariant probability measure on $\hat{\mathcal{O}_K}$ with infinite support is normalized Haar measure.

Suppose now $I \subseteq \mathcal{O}_K$ is an arbitrary integral ideal. Since the ideal class group is finite, a power of $I$ is principal and thus we may choose $q \in \mathcal{O}_K^*$ such that $q\mathcal{O}_K \subseteq I$. The action of $\mathcal{O}_K^*$ on $\hat{\mathcal{O}_K}$ is conjugate to the action of $\mathcal{O}_K^*$ on $\hat{q\mathcal{O}_K}$, and so the only ergodic $\mathcal{O}_K^*$-invariant probability measure on $\hat{q\mathcal{O}_K}$ with infinite support is normalized Haar measure. Thus, by Lemma 3.10 again with $q\mathcal{O}_K \subseteq \hat{I}$, we conclude that the only ergodic $\mathcal{O}_K^*$-invariant probability measure on $\hat{I}$ is $\lambda_I$.

In order to understand the situation for number fields with unit rank higher than 1, we review in the next section the topological version of the problem of ergodic invariant measures, namely, the classification of closed invariant sets.

4. Berend’s Theorem and Number Fields

An elegant generalization to higher-dimensional tori of Furstenberg’s characterization [5, Theorem IV.1] of closed invariant sets for semigroups of transformations of the circle was obtained by Berend [1, Theorem 2.1]. The fundamental question investigated by Berend is whether an infinite invariant set is necessarily dense, and his original formula-

Definition 4.1. (cf. [1, Definition 2.1].) Let $G$ be a group acting on a compact space $X$ by homeomorphisms. We say that the action $G \curvearrowright X$ satisfies the ID property, or that it has the infinite invariant dense property, if the only closed infinite $G$-invariant subset of $X$ is $X$ itself.

The first observation is a topological version of the measure-theoretic solidarity proved in Proposition 3.6, namely, if $K$ is a given number field, then the action $\mathcal{O}_K^* \curvearrowright \hat{J}$ has the ID property either for all integral ideals $J$, or for none.

Proposition 4.2. Suppose $K$ is an algebraic number field, and let $J$ be an ideal in $\mathcal{O}_K$. Then the action of $\mathcal{O}_K^*$ on $\hat{J}$ is ID if and only if the action of $\mathcal{O}_K^*$ on $\hat{\mathcal{O}_K}$ is ID.

Proof. Suppose first that $J_1 \subseteq J_2$ are ideals in $\mathcal{O}_K$ and assume that the action of $\mathcal{O}_K^*$ on $\hat{J}_2$ is ID. The restriction map $\tau : \hat{J}_2 \to \hat{J}_1$ is $\mathcal{O}_K^*$-equivariant, continuous, surjective, and has finite fibers. Thus, if $E$ were a closed, proper, infinite $\mathcal{O}_K^*$-invariant subset of $\hat{J}_1$, then $\tau^{-1}(E)$ would be a closed, proper, infinite $\mathcal{O}_K^*$-invariant subset of $\hat{J}_2$, contradicting the assumption that the action of $\mathcal{O}_K^*$ on $\hat{J}_2$ is ID. So no such set $E$ exists, proving that the action of $\mathcal{O}_K^*$ on $\hat{J}_1$ is also ID.

In particular, if the action of $\mathcal{O}_K^*$ on $\hat{\mathcal{O}_K}$ is ID, then the action on $\hat{J}$ is also ID for every integral ideal $J \subseteq \mathcal{O}_K$. For the converse, recall that, as in the proof of Proposition 3.6, there exists an integer $q \in \mathcal{O}_K^*$ such that $q\mathcal{O}_K \subseteq J$, so we may apply the preceding paragraph to this inclusion. Since the action of $\mathcal{O}_K^*$ on $\hat{q\mathcal{O}_K}$ is conjugate to that on $\hat{\mathcal{O}_K}$, this completes the proof. □
In order to decide for which number fields the action of units on the integral ideals is ID, we need to recast Berend’s necessary and sufficient conditions in terms of properties of the number field. Recall that, by definition, a number field is called a complex multiplication (or CM) field if it is a totally imaginary quadratic extension of a totally real subfield. These fields were studied by Remak \[17\], who observed that they are exactly the fields that have a unit defect, in the sense that they contain a proper subfield \(L\) with the same unit rank.

**Theorem 4.3.** Let \(K\) be an algebraic number field and let \(J\) be an ideal in \(\mathcal{O}_K\). The action of \(\mathcal{O}_K^*\) on \(\mathcal{O}_J\) is ID if and only if \(K\) is not a CM field and \(\text{rank } \mathcal{O}_K^* \geq 2\).

For the proof we shall need a few number theoretic facts. We believe these are known but we include the relatively straightforward proofs below for the convenience of the reader.

**Lemma 4.4.** Suppose \(\mathcal{F}\) is a finite family of subgroups of \(\mathbb{Z}^d\) such that \(\text{rank}(\mathcal{F}) < d\) for every \(\mathcal{F} \in \mathcal{F}\). Then there exists \(m \in \mathbb{Z}^d\) such that \(m + \mathcal{F}\) is nontorsion in \(\mathbb{Z}^d/\mathcal{F}\) for every \(\mathcal{F} \in \mathcal{F}\).

**Proof.** Recall that for each subgroup \(\mathcal{F}\) there exists a basis \(\{n_i^\mathcal{F}\}_{i=1,2,\ldots,d}\) of \(\mathbb{Z}^d\) and integers \(a_1, a_2, \ldots, a_{\text{rank}(\mathcal{F})}\) such that
\[
\mathcal{F} = \left\{ \sum_{i=1}^{\text{rank}(\mathcal{F})} k_i n_i^\mathcal{F} : k_i \in \mathbb{Z}, 1 \leq i \leq \text{rank}(\mathcal{F}) \right\}.
\]

The associated vector subspaces \(S_{\mathbb{R}} := \text{span}_{\mathbb{R}}(n_1^\mathcal{F}, \ldots, n_{\text{rank}(\mathcal{F})}^\mathcal{F})\) of \(\mathbb{R}^d\) are proper and closed so \(\mathbb{R}^d \setminus \cup_{\mathcal{F}} S_{\mathcal{F}}\) is a nonempty open set, see e.g. \[19\] Theorem 1.2. Let \(r\) be a point in \(\mathbb{R}^d \setminus \cup_{\mathcal{F}} S_{\mathcal{F}}\) with rational coordinates. If \(k\) denotes the l.c.m. of all the denominators of the coordinates of \(r\), then \(m := kr \in \mathbb{Z}^d\) and its image \(m + \mathcal{F} \in \mathbb{Z}^d/\mathcal{F}\) is of infinite order for every \(\mathcal{F}\) because \(m \notin S_{\mathcal{F}}\). \(\square\)

**Proposition 4.5.** Let \(K\) be an algebraic number field. Then there exists a unit \(u \in \mathcal{O}_K^*\) such that \(K = \mathbb{Q}(u^k)\) for every \(k \in \mathbb{N}^\times\) if and only if \(K\) is not a CM field.

**Proof.** Assume first \(K\) is not a CM field. Then \(\text{rank } \mathcal{O}_K^* < \text{rank } \mathcal{O}_F^*\) for every proper subfield \(F\) of \(K\). Since there are only finitely many proper subfields \(F\) of \(K\), Lemma 4.4 gives a unit \(u \in \mathcal{O}_K^*\) with nontorsion image in \(\mathcal{O}_K^*/\mathcal{O}_F^*\) for every \(F\). Thus \(u^k \notin F\) for every proper subfield \(F\) of \(K\) and every \(k \in \mathbb{N}\).

Assume now \(K\) is a CM field, and let \(F\) be a totally real subfield with the same unit rank as \(K\) \([17]\). Then the quotient \(\mathcal{O}_K^*/\mathcal{O}_F^*\) is finite and there exists a fixed integer \(m\) such that \(u^m \in F\) for every \(u \in \mathcal{O}_K^*\). \(\square\)

**Lemma 4.6.** Let \(k\) be an algebraic number field with rank \(\mathcal{O}_k^* \geq 1\). Then for every embedding \(\sigma : k \to \mathbb{C}\), there exists \(u \in \mathcal{O}_k^*\) such that \(|\sigma(u)| > 1\).

**Proof.** Assume for contradiction that \(\sigma\) is an embedding of \(k\) in \(\mathbb{C}\) such that \(\sigma(\mathcal{O}_k^*) \subseteq \{z \in \mathbb{C} : |z| = 1\}\). Let \(\mathcal{K} = \mathcal{O}_K^*\) and let \(U_k = \sigma(\mathcal{O}_k^*)\). Then \(K \cap \mathbb{R}\) is a real subfield of \(K\) with \(U_{K \cap \mathbb{R}} = \{\pm 1\}\), so \(K \cap \mathbb{R} = \mathbb{Q}\). Also \(K \cap \mathbb{R}\) is the maximal real subfield of \(K\), and since we are assuming \(\text{rank } \mathcal{O}_k^* \geq 1\), \(K\) cannot be a CM field. To see this, suppose that \(K\) were CM. Let \(\ell \subseteq k\) be a totally real subfield such that \(|\ell : \mathbb{Q}| = 2\). Since \(\ell\) is totally real, \(\sigma(\ell) \subseteq \mathbb{R}\), and since \(K \cap \mathbb{R} = \mathbb{Q}\), it must be that \(\sigma(\ell) = \mathbb{Q}\). Then \(\ell = \mathbb{Q}\), so \(k\) is quadratic imaginary, contradicting \(\text{rank } \mathcal{O}_k^* \geq 1\).

By Proposition 4.5, there exists \(u \in U_k\) such that \(K = \mathbb{Q}(u)\). Since \(|u| = 1\), we have that \(\mathbb{K} = \mathbb{Q}(\overline{u}) = \mathbb{Q}(u^{-1}) = \mathbb{Q}(u) = K\), so \(K\) is closed under complex conjugation. Write
Let $u = a + ib$. Then $u + \pi = 2a \in \mathcal{O} \cap \mathbb{R} = \mathbb{Q}$, so $a \in \mathbb{Q}$. Thus, $\mathcal{O} = \mathbb{Q}(u) = \mathbb{Q}(ib)$. Since $|u| = 1$, $a^2 + b^2 = 1$, and so we have that

$$
\mathbb{Q}(ib) \cong \mathbb{Q}(\sqrt{-b^2}) \cong \mathbb{Q}(\sqrt{a^2 - 1}) \cong \mathbb{Q}\left(\sqrt{\frac{m^2 - n^2}{n^2}}\right) \cong \mathbb{Q}(\sqrt{m^2 - n^2}),
$$

where $a = m/n \in \mathbb{Q}$.

Thus, $\mathcal{O}$ is a quadratic field. But it cannot be quadratic imaginary because rank $\mathcal{O}_K \geq 1$, and it cannot be quadratic real because all the units lie on the unit circle. This proves there can be no such embedding. \qed

**Proof of Theorem 4.3.** By Proposition 4.2, it suffices to prove the case $\mathcal{O} = \mathcal{O}_K$. Let $d = [K : \mathbb{Q}]$ and recall that $\mathcal{O}_K \cong \mathbb{T}^d$. All we need to do is verify that Berend’s necessary and sufficient conditions for $\text{ID}$ [1, Theorem 2.1] when interpreted for the automorphic action of $\mathcal{O}_K^\ast$ on $\mathcal{O}_K$, characterize non-CM fields of unit rank 2 or higher. Since the action of $\mathcal{O}_K^\ast$ by linear toral automorphisms $\rho(u)$ with $u \in \mathcal{O}_K^\ast$ is faithful by [10, p. 729], Berend’s conditions are:

1. (totally irreducible) there exists a unit $u$ such that the characteristic polynomial of $\rho(u^n)$ is irreducible for all $n \in \mathbb{N}$;
2. (quasi-hyperbolic) for every common eigenvector of $\{\rho(u) : u \in \mathcal{O}_K^\ast\}$, there is a unit $u \in \mathcal{O}_K^\ast$ such that the corresponding eigenvalue of $\rho(u)$ is outside the unit disc; and
3. (not virtually cyclic) there exist units $u, v \in \mathcal{O}_K^\ast$ such that if $m, n \in \mathbb{N}$ satisfy $\rho(u^m) = \rho(v^n)$, then $m = n = 0$.

Suppose first that the action of $\mathcal{O}_K^\ast$ on $\mathcal{O}_K$ is ID. By [1, Theorem 2.1] conditions (1) and (3) above hold, i.e., the action of $\mathcal{O}_K^\ast$ on $\mathcal{O}_K$ is totally irreducible and not virtually cyclic. By Proposition 4.5, $K$ is not a CM field and since $\rho : \mathcal{O}_K^\ast \to \text{GL}_d(\mathbb{Z})$ is faithful, (3) is a restatement of rank $\mathcal{O}_K^\ast \geq 2$.

Suppose now that $K$ is not CM and has unit-rank at least 2. By Proposition 4.5, there exists $u \in \mathcal{O}_K^\ast$ such that $\mathbb{Q}(u^n) = K$ for every $n \in \mathbb{N}$. Hence the minimal polynomial of $\rho(u^n)$ has degree $d$, and so it coincides with the characteristic polynomial. This proves that condition (1) holds, i.e., the action of $\rho(u)$ is totally irreducible. We have already observed that condition (3) holds iif the unit rank of $K$ is at least 2, so it remains to see that the hyperbolicity condition (2) holds too. In the simultaneous diagonalization of the matrix group $\rho(\mathcal{O}_K^\ast)$, the diagonal entries of $\rho(u)$ are the embeddings of $u$ into $\mathbb{R}$ or $\mathbb{C}$, see e.g. [10, p. 729]. Then condition (2) follows from Lemma 4.6. \qed

**Remark 4.7.** Notice that for units acting on algebraic integers, Berend’s hyperbolicity condition (2) is automatically implied by the rank condition (3).

**Remark 4.8.** Since the matrices representing the actions of $\mathcal{O}_K^\ast$ on $\mathcal{O}_K$ are conjugate over $\mathbb{Q}$, Proposition 4.2 can be derived from the implication (1) \implies (3) in [10, Proposition 2.1]. We may also see that the matrices implementing the action on $\mathcal{O}_K$ have the same sets of characteristic polynomials, so the questions of expansive eigenvalues (condition (2)) and of total irreducibility are equivalent for the two actions. The third condition is independent of whether we look at $\mathcal{O}_K$ or $\mathcal{O}_K^\ast$, so this yields yet another proof of Proposition 4.2.

By Theorem 4.3, for each non-CM algebraic number field $K$ with unit rank at least 2, the action $\mathcal{O}_K^\ast$ on $\mathcal{O}_K$, transposed as $\{\rho(u) : u \in \mathcal{O}_K^\ast\}$ acting on $\mathbb{R}^d/\mathbb{Z}^d$, is an example of an abelian
toral automorphism group for which one may hope to prove that normalized Haar measure is the only ergodic invariant probability measure with infinite support. So it is natural to ask which groups of toral automorphisms arise this way. A striking observation of Z. Wang [25, Theorem 2.12], see also [14, Proposition 2.2], states that every finitely generated abelian group of automorphisms of $\mathbb{T}^d$ that contains a totally irreducible element and whose rank is maximal and greater than or equal to 2 arises, up to conjugacy, from a finite index subgroup of units acting on the integers of a non-CM field of degree $d$ and unit rank at least 2, cf. [25, Condition 1.5]. We wish next to give a proof of the converse, which was also stated in [25].

**Proposition 4.9.** Suppose $G$ is an abelian subgroup of $\text{SL}_d(\mathbb{Z})$ satisfying [25] Condition 2.8]. Specifically, suppose there exist

- a non-CM number field $K$ of degree $d$ and unit rank at least 2;
- an embedding $\phi : G \to \mathcal{O}_K^*$ of $G$ into a finite index subgroup of $\mathcal{O}_K^*$;
- a co-compact lattice $\Gamma$ in $K \otimes \mathbb{Q} \cong \mathbb{R}^d$ invariant under multiplication by $\phi(G)$; and
- a linear isomorphism $\psi : \mathbb{R}^d \to K \otimes \mathbb{Q} \cong \mathbb{R}^d$ mapping $\mathbb{Z}^d$ onto $\Gamma$ that intertwines the actions $G \subset \mathbb{R}^d$ and $\phi(G) \subset (K \otimes \mathbb{Q} \mathbb{R})/\Gamma$.

Then $G$ satisfies [25] Condition 1.5], namely

1. $\text{rank}(G) \geq 2$;
2. the action $g \subset \mathbb{R}^d/\mathbb{Z}^d \cong \mathbb{T}^d$ is totally irreducible for some $g \in G$;
3. $\text{rank} G_1 = \text{rank} G$ for each abelian subgroup $G_1 \subset \text{SL}_d(\mathbb{Z})$ containing $G$.

**Proof.** Suppose $K$ is a non-CM algebraic number field of degree $d$ with unit rank at least 2, and assume $G$ is a subgroup of $\text{SL}_d(\mathbb{Z})$ that satisfies the assumptions with respect to $K$. Part (1) of [25, Condition 1.5] is immediate, because $\phi(G)$ is of full rank in $\mathcal{O}_K^*$.

By Proposition 4.5, there exists a unit $u \in \mathcal{O}_K^*$ such that the characteristic polynomial of $\rho(u^m)$ is irreducible over $\mathbb{Q}$ for all $m \in \mathbb{N}$. This is equivalent to the action of $u^m$ on $\mathcal{O}_K$ being irreducible for all $m \in \mathbb{N}$, see, e.g. [10, Proposition 3.1]. Since $\phi(G)$ is of finite index in $\mathcal{O}_K^*$, there exists $N \in \mathbb{N}$ such that $u^N \in \phi(G)$. We claim that $g := \phi^{-1}(u^N)$ is a totally irreducible element in $G \subset \mathbb{R}^d/\mathbb{Z}^d$. To see this, it suffices to show that the characteristic polynomial of $g^k$ is irreducible over $\mathbb{Q}$ for every positive integer $k$. Since the linear isomorphism $\psi$ intertwines the actions $g^k \subset \mathbb{T}^d$ and $\rho(\phi(g))^k \subset (K \otimes \mathbb{Q} \mathbb{R})/\Gamma$, the characteristic polynomial of $g^k$ equals the characteristic polynomial of $\rho(\phi(g))^k = \rho(u^{kN})$, which is irreducible because it coincides with the characteristic polynomial of $u^{kN}$ as an element of the ring $\mathcal{O}_K$. This proves part (2) of Condition 1.5.

Suppose now that $G_1$ is an abelian subgroup of $\text{SL}_d(\mathbb{Z})$ containing $G$ and apply the construction from [25, Proposition 2.13] (see also [21, 4]) to the irreducible element $g \in G \subset G_1 \subset \mathbb{T}^d$. Up to an automorphism, the resulting number field arising from this construction is $K = \mathbb{Q}(u^N)$, and the embedding $\phi_1 : G_1 \to \mathcal{O}_K^*$ is an extension of $\phi : G \to \mathcal{O}_K^*$. Since $\phi(G) \subset \phi_1(G_1) \subset \mathcal{O}_K^*$ and $\phi(G)$ is of finite index in $\mathcal{O}_K^*$, $\text{rank}(G_1) = \text{rank} \phi_1(G_1) = \text{rank} \mathcal{O}_K^* = \text{rank} \phi(G) = \text{rank} G$, and this proves proves part (3) of Condition 1.5.

As a consequence, we see that the action of units on the algebraic integers of number fields are generic for group actions with Berend’s ID property in the following sense, cf. [25, 14].
Corollary 4.10. If $G$ is a finitely generated abelian subgroup of $\text{SL}_d(\mathbb{Z})$ of torsion-free rank at least 2 that contains a totally irreducible element and is maximal among abelian subgroups of $\text{SL}_d(\mathbb{Z})$ containing $G$, then $G$ is conjugate to a finite-index toral automorphism subgroup of the action of $\mathcal{O}_k^s \subset \hat{\mathcal{O}_k}$ for a non-CM algebraic number field $K$ of degree $d$ and unit rank at least 2.

Finally, we summarize what we can say at this point for equilibrium states of $\text{C}^*$-algebras associated to number fields with unit rank strictly higher than one. If the generalized Furstenberg conjecture is verified, the following result would complete the classification started in Proposition 3.1 and Proposition 3.5.

Let $K$ be a number field and for each $\gamma \in \mathcal{O}_k$ define $F_\gamma$ to be the set of all pairs $(\mu, \chi)$ with $\mu$ an equiprobability measure on a finite orbit of the action of $\mathcal{O}_k^s$ in $\hat{\mathcal{F}}_\gamma$, and $\chi \in \hat{H}_\mu$, where the $\mu$-a.e. isotropy group $H_\mu$ is a finite index subgroup of $\mathcal{O}_k^s$. Also let $(\lambda_1, 1)$ denote the pair consisting of normalized Haar measure on $\hat{\gamma}$ and the trivial character of its trivial a.e. isotropy group. Then the map $(\mu, \chi) \mapsto \tau_{\mu, \chi}$ from Theorem 2.2 gives an extremal tracial state of $\mathcal{C}^*(\mathcal{O}_k \rtimes \mathcal{O}_k^s)$ for each pair $(\mu, \chi) \in F_\gamma \cup \{ (\lambda_1, 1) \}$.

Recall that the map $\tau \mapsto \varphi_\tau$ from [2, Theorem 7.3] is an affine bijection of all tracial states of $\bigoplus_{\gamma \in \mathcal{O}_k} \mathcal{C}^*(\mathcal{O}_k \rtimes \mathcal{O}_k^s)$ onto $\mathcal{K}_\beta$, the simplex of KMS$_\beta$ equilibrium states of the system $(\mathcal{E}(\mathcal{O}_k), \sigma)$ studied in [2].

Theorem 4.11. Suppose $K$ is an algebraic number field with unit rank at least 2 and define $\Phi : (\mu, \chi) \mapsto \varphi_{\tau_{\mu, \chi}}$ to be the composition of the maps from Theorem 2.2 and from [2, Theorem 7.3], assigning a state $\varphi_{\tau_{\mu, \chi}} \in \text{Extr}(\mathcal{K}_\beta)$ to each pair $(\mu, \chi)$ consisting of an ergodic invariant probability measure $\mu$ in one of the $\hat{\mathcal{F}}_\gamma$ and an associated character of the $\mu$-almost constant isotropy $H_\mu$. Let

$$F_K := \bigcup_{\gamma \in \mathcal{O}_k} (F_\gamma \cup \{ (\lambda_1, 1) \})$$

be the set of pairs whose measure $\mu$ has finite support or is Haar measure. Then

1. if $K$ is a CM field, then the inclusion $\Phi(F_K) \subset \text{Extr}(\mathcal{K}_\beta)$ is proper; and
2. if $K$ is not a CM field, and if there exists $\phi \in \text{Extr}(\mathcal{K}_\beta) \setminus \Phi(F_K)$ then the measure $\mu$ on $\hat{\mathcal{F}}_\gamma$ arising from $\phi$ has zero-entropy and infinite support.

Proof. To prove assertion (1), recall that when $K$ is a CM field Berend’s theorem implies that there are invariant subtori, which have ergodic invariant probability measures on the fibers, cf. [8, 9]. These measures give rise to tracial states and to KMS states not accounted for in $\Phi(F_K)$. Assertion (2) follows from [4, Theorem 1.1].

5. Primitive ideal space

The computation of the primitive ideal spaces of the $\text{C}^*$-algebras $\mathcal{C}^*(\mathcal{J} \rtimes \mathcal{O}_k^s)$ associated to the action of units on integral ideals lies within the scope of Williams’ characterization in [27]. We briefly review the general setting next. Let $G$ be a countable, discrete, abelian group acting continuously on a second countable compact Hausdorff space $X$. We define an equivalence relation on $X$ by saying that $x$ and $y$ are equivalent if $x$ and $y$ have the same orbit closure, i.e. if $\overline{G \cdot x} = \overline{G \cdot y}$. The equivalence class of $x$, denoted by $[x]$, is called the quasi-orbit of $x$, and the quotient space, which in general is not Hausdorff, is denoted by $\mathcal{Q}(G \subset X)$ and is called the quasi-orbit space. It is important to distinguish the quasi-orbit of a point from the closure of its orbit, as the latter may contain other points with strictly smaller orbit closure.
Let $\varepsilon_\chi$ denote evaluation at $x \in X$, viewed as a one-dimensional representation of $C(X)$. For each character $\kappa \in \hat{\mathbb{G}}$, the pair $(\varepsilon_\chi, \kappa)$ is clearly covariant for the transformation group $(C(X), \mathbb{G})$, and the corresponding representation $\varepsilon_\chi \times \kappa$ of $C(X) \rtimes \mathbb{G}$ gives rise to an induced representation $\text{Ind}^{\mathbb{G}}_{\chi}(\varepsilon_\chi \times \kappa)$ of $C(X) \rtimes \mathbb{G}$, which is irreducible because $\varepsilon_\chi \times \kappa$ is. Since $\mathbb{G}$ is abelian and the action is continuous, whenever $x$ and $y$ are in the same quasi-orbit, $[x] = [y]$, the corresponding isotropy subgroups coincide: $\mathbb{G}_x = \mathbb{G}_y$. Thus, we may consider an equivalence relation on the product $\mathcal{Q}(\mathbb{G} \curvearrowright X) \times \hat{\mathbb{G}}$ defined by

$$([x], \kappa) \sim ([y], \lambda) \iff [x] = [y] \text{ and } \kappa|_{\mathbb{G}_x} = \lambda|_{\mathbb{G}_x}.$$ 

By [27, Theorem 5.3], the map $(x, \kappa) \mapsto \ker \text{Ind}^{\mathbb{G}}_{\chi}(\varepsilon_\chi \times \kappa)$ induces a homeomorphism of $(\mathcal{Q}(\mathbb{G} \curvearrowright X) \times \hat{\mathbb{G}})/\sim$ onto the primitive ideal space of the crossed product $C(X) \rtimes \mathbb{G}$, see e.g. [12, Theorem 1.1] for more details on this approach.

We wish to apply the above result to actions $\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}}$ for integral ideals $J$ of non-CM number fields with unit rank at least 2, as in Theorem 4.3 Remark that by Proposition 3.4 if the orbit $\mathcal{O}_k^* \cdot \chi$ is finite, then it is equal to the quasi-orbit $[\chi]$. The first step is to describe the quasi-orbit space for the action of units. We focus on the case $J = \mathcal{O}_k^*$; ideals representing nontrivial classes behave similarly because of the solidarity established in Proposition 4.2.

**Proposition 5.1.** Suppose $K$ is a non-CM algebraic number field with unit rank at least 2. Then the quasi-orbit space of the action $\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}}$ is

$$\mathcal{Q}(\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}}) = \{[x] : |\mathcal{O}_k^* \cdot x| < \infty\} \cup \{\omega_\infty\}.$$ 

The point $\omega_\infty$ is the unique infinite quasi-orbit $[\chi]$ of any $\alpha \in \hat{\mathbb{G}} \cong \mathbb{R}^d/\mathbb{Z}^d$ having at least one irrational coordinate. The closed proper subsets are the finite subsets all of whose points are finite (quasi-)orbits. Infinite subsets and subsets that contain the infinite quasi-orbit $\omega_\infty$ are dense in the whole space.

**Proof.** By Theorem 4.3, the closure of each infinite orbit is the whole space. Thus, the points with infinite orbits collapse into a single quasi-orbit

$$\omega_\infty := \{x \in \hat{\mathbb{G}} : |\mathcal{O}_k^* \cdot x| = \infty\} = \{x \in \hat{\mathbb{G}} : \sqrt[k]{\mathcal{O}_k^* \cdot x} = \hat{\mathbb{G}}\}.$$ 

That this is the set of points with at least one irrational coordinate is immediate from [24, Theorem 5.1]. When the orbit of $x$ is finite, it is itself a quasi-orbit, which we view as a point in $\mathcal{Q}(\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}})$. In this case $x \in \hat{\mathbb{G}}$ has all rational coordinates.

To describe the topology, recall that the quotient map $q : \hat{\mathbb{G}} \to \mathcal{Q}(\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}})$ is surjective, continuous and open by the Lemma in page 221 of [6], see also the proof of Proposition 2.4 in [12].

Any two different finite quasi-orbits $[x]$ and $[y]$ are finite, mutually disjoint subsets of $\hat{\mathbb{G}}$ and as such can be separated by disjoint open sets $V$ and $W$, so that $[x] \subset V$ and $[y] \subset W$. Passing to the quotient space, we have $[x] \notin q(V)$ and $[y] \notin q(W)$, so $[x]$ and $[y]$ are T$_1$-separated, which implies that finite sets of finite quasi-orbits are closed in $\mathcal{Q}(\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}})$.

The singleton $\{\omega_\infty\}$ is dense in $\mathcal{Q}(\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}})$ because every infinite orbit in $\hat{\mathbb{G}}$ is dense by Theorem 4.3. If $A$ is an infinite subset of $\mathcal{Q}(\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}})$ consisting of finite quasi-orbits, then $\bigcup_{[x] \in A} [x]$ is an infinite invariant set in $\hat{\mathbb{G}}$, hence is dense by Theorem 4.3. This implies that $\omega_\infty$ is in the closure of $A$, and hence $A$ is dense in $\mathcal{Q}(\mathcal{O}_k^* \curvearrowright \hat{\mathbb{G}})$. 

$\square$
Let $K$ be a non-CM algebraic number field with unit rank at least 2, and let $G = \mathcal{O}_K^\times$.

The primitive ideal space of $C(\hat{\mathcal{O}_K}) \times G$ is homeomorphic to the space
\[
\bigsqcup_{[x]} ([x] \times \hat{G}_x)
\]
in which a net $([x_i], \gamma_i)$ converges to $([x], \gamma)$ iff $[x_i] \to [x]$ in $Q(G \acts \hat{\mathcal{O}_K})$ and $\gamma_i|_{\mathcal{O}_K} \to \gamma|_{\mathcal{O}_K}$ in $\hat{G}_x$.

Notice that if $[x]$ is a finite quasi-orbit, then the net $([x_i])$ is eventually constant equal to $[x]$, and if $[x] = \omega_{\infty}$, then the condition $\gamma_i|_{\mathcal{O}_{\infty}} \to \gamma|_{\mathcal{O}_{\infty}}$ is trivially true because $G_{\infty} = \{1\}$.

**Proof.** Consider the diagram below, where $f$ is the quotient map and the vertical map $g$ is defined by $g([([x], \gamma)]) = ([x], \gamma|_{\mathcal{O}_K})$, where $([x], \gamma)$ denotes the equivalence class of $([x], \gamma)$ with respect to $\sim$.

\[
\begin{array}{ccc}
Q(G \acts \hat{\mathcal{O}_K}) \times \hat{G} & \xrightarrow{f} & Q(G \acts \hat{\mathcal{O}_K}) \times \hat{G} / \sim \\
g \circ f & \downarrow & \\
\bigsqcup_{[x]} ([x] \times \hat{G}_x)
\end{array}
\]

By the fundamental property of the quotient map, see e.g. [26 Theorem 9.4], $g \circ f$ is continuous if and only if $g$ is continuous.

It is clear that $g$ is a bijection. We show next that $g \circ f$ is continuous. Suppose that $([x_i], \gamma_i)$ is a net in $Q(G \acts \hat{\mathcal{O}_K}) \times \hat{G}$ converging to $([x], \gamma)$. Then $[x_i] \to [x]$ in $Q(G \acts \hat{\mathcal{O}_K})$, and $\gamma_i \to \gamma$ in $\hat{G}$. Then clearly also $\gamma_i|_{\mathcal{O}_K} \to \gamma|_{\mathcal{O}_K}$ in $\hat{G}_x$ as well. Hence the net $g \circ f([x_i], \gamma_i)$ converges to $g \circ f([x], \gamma) = ([x], \gamma|_{\mathcal{O}_K})$, as desired.

It remains to show that $g^{-1}$ is continuous, or equivalently, that $g$ is a closed map. Suppose that $W \subseteq Q(G \acts \hat{\mathcal{O}_K}) \times \hat{G} / \sim$ is closed, and suppose that $([x_i], \gamma_i)$ is a net in $g(W)$ converging to $([x], \gamma)$.

Consider any net $(\gamma_i) \in \hat{G}$ such that $\gamma_i|_{\mathcal{O}_K} = \gamma_i$. By the compactness of $\hat{G}$, there exists a convergent subnet $\gamma_{i_n}$ with limit $\tilde{\gamma}$. Then $\gamma_{i_n}|_{\mathcal{O}_K} \to \tilde{\gamma}|_{\mathcal{O}_K}$ as well, so $\gamma_{i_n} \to \tilde{\gamma}|_{\mathcal{O}_K}$. Since $\hat{G}_x$ is Hausdorff, limits are unique, and hence $\gamma|_{\mathcal{O}_K} = \tilde{\gamma}$.

The net $([x_{i_n}], \gamma_{i_n})$ converges to $(x, \gamma)$ in $Q(G \acts \hat{\mathcal{O}_K}) \times \hat{G}$, and since $f$ is continuous, $f([x_{i_n}], \gamma_{i_n}) = f([x], \gamma)$. Moreover, $f([x_{i_n}], \gamma_{i_n}) \in W$ because $g$ is injective and $g([([x_{i_n}], \gamma_{i_n})]) = ([x_{i_n}], \gamma_{i_n}) \in g(W)$ by assumption. Since $W$ is closed, $([x], \gamma) \in W$, and so its image $(x, \gamma) \in g(W)$, as desired. \hfill $\square$

**Remark 5.3.** Recall that $G \equiv W \times \mathbb{Z}^d$ with $W$ the roots of unity in $G$, and that the isotropy subgroup $G_x$ is constant on the quasi-orbit $[x]$ of $x$. If $[x]$ is finite, then $G_x$ is of full rank in $G$, and thus $G_x \cong V_x \times \mathbb{Z}^d$, with $V_x \subseteq W$ the torsion part of $G_x$. Hence, for every finite quasi-orbit $[x]$, we have $\hat{G}_x \cong \hat{V}_{[x]} \times \mathbb{T}^d$. Notice that $\hat{V}_{[x]} \cong V_{[x]}$ (noncanonically) because $V_x$ is finite.

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