Perfect quantum state transfers on the Johnson scheme

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Abstract

For any graph $X$ with the adjacency matrix $A$, the transition matrix of the quantum walk at time $t$ is given by the matrix-valued function $H_X(t) = e^{itA}$. We say that there is a perfect state transfer in $X$ from the vertex $u$ to the vertex $v$ at time $\tau$ if $|H_X(\tau)_{u,v}| = 1$. It is an important problem to determine whether perfect state transfers can happen on a given family of graphs. In this paper we characterize all the graphs in the Johnson scheme which have this property. Indeed, we show that the Kneser graph $K(2k, k)$ is the only class in the scheme which admits perfect state transfers.

Keywords: perfect state transfer, association scheme, Johnson scheme

1. Introduction

Let $X$ be a simple graph. The transition matrix of the quantum walk at time $t$ on the graph $X$ is given by the matrix-valued function $H_X(t) = e^{itA}$, where $A = A(X)$ is the adjacency matrix of $X$. We say that there is a perfect state transfer (or a PST) in $X$ from the vertex $u$ to the vertex $v$ at time $\tau$ if $|H_X(\tau)_{u,v}| = 1$ or, equivalently, if $H_X(\tau)e_u = \lambda e_v$, for some $\lambda \in \mathbb{C}$, where $e_u$ is the characteristic vector of the vertex $u$ in $V(X)$. (Note that since $H_X(\tau)$ is unitary, we must have $|\lambda| = 1$.) If the graph $X$ is clear from the context, we may drop the subscript $X$. We, further, say that $X$ is periodic at a vertex $u$ if there is a time $\tau$ such that $|H(\tau)_{u,u}| = 1$, and we say $X$ itself is periodic if there is a time $\tau$ such that $|H(\tau)_{u,v}| = 1$ for all vertices $u$. Note that if there is a PST from $u$ to $v$, then there is also a PST from $v$ to $u$ (so we may just say there is a PST “between” $u$ and $v$), and that $X$ will be periodic at $u$ and at $v$ (see [3]).

It is an important question to ask on which graphs a PST can happen. As an easy example we can show that a PST occurs between the vertices of the path $P_2$ (see Corollary 2.4). We can, also, show directly (see [7]) that a
PST happens between the end-points of the path $P_3$, as well; but this is not the case for any other path. In other words, it has been shown in \cite{2} that if $n > 3$, then there is no PST on the path $P_n$. On the other hand, in the works such as \cite{1,2,6,9}, the existence of a PST in different graphs has been shown. See \cite{7} for a survey on the subject.

In \cite{3} necessary and sufficient conditions have been proposed for graphs in association schemes to have PST and using this result the authors have examined some of the graphs in some association schemes. Since one of the most important association schemes is the Johnson scheme $J(n,k)$, it is quite natural to ask whether any of the classes of this scheme admits a PST.

In this paper, we prove that except for the Kneser graph $K(2k,k)$, which is one of the classes of the Johnson scheme for the case $n = 2k$, no other class has the capability to have a PST.

In the next section, we will provide some brief background on the topic. Then we will prove the main result in Section 3 and, finally, conclude the paper with some further discussions in Section 4.

2. Background

In this section, we provide the reader with some necessary tools and facts in the topic, which will be used in Section 3. For the notation and basic facts of algebraic graph theory and association schemes the reader may refer to \cite{8} and \cite{4}, respectively. All the association schemes are assumed to be symmetric. Let $\mathcal{A} = \{I = A_0, A_1, \ldots, A_d\}$ (where $A_i$ are $v \times v$ matrices) be an association scheme with $d$ classes whose minimal idempotents (i.e. projections to the eigenspaces) are $E = \{E_0, E_1, \ldots, E_d\}$. Since $E$ is a basis for the Bose-Mesner algebra of $\mathcal{A}$, there are coefficients $p_i(j)$ such that

$$A_i = \sum_{j=0}^{d} p_i(j)E_j, \quad \text{for } i = 0, \ldots, d.$$ 

For any $i = 0, \ldots, d$, the (real) numbers $p_i(j)$, $j = 0, \ldots, d$, are the eigenvalues of $A_i$ and the columns of $E_j$ are eigenvectors of $A_i$. We can think of $A_i$ as the adjacency matrix of a regular graph $X_i$ on $v$ vertices of valency $p_i(0)$. We will denote the valency of $X_i$ by $v_i$. On the other hand, the idempotents $E_i$ can also be written as a linear combination of the $A_j$’s; that is, there are coefficients $q_i(j)$ such that

$$E_i = \frac{1}{d} \sum_{j=0}^{d} q_i(j)A_j, \quad \text{for } i = 0, \ldots, d.$$ 

(1)

The numbers $q_i(j)$ are called the dual eigenvalues of the scheme. We note that $q_i(0) = \text{tr}(E_j)$ which is equal to the dimension of the $j$-th eigenspace of the matrices of the scheme which will be denoted by $m_j$. The following fundamental result will be used in Section 3

2
Proposition 2.1. Using the notation above, for any \( i, j = 0, \ldots, d \), we have \( q_j(i)/m_j = p_i(j)/v_i \).

Given an association scheme, the following result gives necessary conditions for a graph belonging to the scheme to have a PST.

Theorem 2.2. Let \( X \) be a graph that belongs to an association scheme with \( d \) classes and with adjacency matrix \( A = A(X) \). If \( X \) admits perfect state transfer at time \( \tau \), then there is a permutation matrix \( T \) with no fixed points and of order two such that \( \mathcal{H}(\tau) = \lambda T \) for some \( \lambda \in \mathbb{C} \). Moreover, \( T \) is a class of the scheme.

Suppose \( E_0, \ldots, E_d \) are the idempotents of the scheme in a class of which a PST occurs. By Theorem 2.2, \( T \) must be one of the classes of the scheme and since the eigenvalues of \( T \) are \( \{-1, +1\} \), for any \( j = 0, \ldots, d \), we have \( T E_j = \pm E_j \). As in [3], we define a partition \((\mathcal{I}^+, \mathcal{I}^-)\) of \( \{0, \ldots, d\} \) as \( j \in \mathcal{I}^+ \) if \( T E_j = E_j \) and \( j \in \mathcal{I}^- \) otherwise. We might drop the subscript \( T \) if \( T \) is clear from the context. Furthermore, for any \( x \in \mathbb{Z} \) we denote by \( \text{ord}_2(x) \) the exponent of 2 in the factorization of \( x \) and by convention we define \( \text{ord}_2(0) = +\infty \).

Theorem 2.3. With the notation above, a PST occurs on \( X \) at time \( t \) only if \( \text{ord}_2(\lambda_0 - \lambda_j) > \text{ord}_2(\alpha) \) for \( j \in \mathcal{I}^+ \) and \( \text{ord}_2(\lambda_0 - \lambda_\ell) = \text{ord}_2(\alpha) \) for \( \ell \in \mathcal{I}^- \).

We conclude this section with a brief introduction of the Johnson scheme. Throughout the paper we will assume that \( n \geq 2k \). For any \( i = 0, 1, \ldots, k \),
we define the graph $J(n, k, i)$ to be the graph whose vertex set is the set of all $k$-subsets of $\{1, \ldots, n\}$ and in which two vertices $A$ and $B$ are adjacent if $|A \cap B| = i$ (in some texts the adjacency in $J(n, k, i)$ is defined as when $|A \cap B| = k - i$, but this makes no difference in our discussion). It can be shown that the set $\mathcal{A} = \{A_0, A_1, \ldots, A_k\}$, where $A_i$ is $A(J(n, k, k - i))$, the adjacency matrix of the graph $J(n, k, k - i)$, is an association scheme (see for example [4]). This scheme is called the Johnson scheme, denoted by $J(n, k)$, and the classes $J(n, k, i)$ are called the generalized Johnson graphs. The special cases of $J(n, k, 0)$ and $J(n, k, k - 1)$ are called the Kneser graph (often denoted by $K(n, k)$), and the Johnson graph, respectively. Note that the adjacency matrix of $J(n, k, k)$ is the identity matrix; that is $A_0 = I$.

It is clear that the generalized Johnson graph $J(n, k, i)$ is a regular graph with $v = \binom{n}{k}$ vertices and valency $v_i = \binom{k}{i}\binom{n-k}{k-i}$. Furthermore, it has been shown (see [4], p. 220) that the eigenvalues of the graphs $J(n, k, i)$ are given by

$$\lambda_j^i = \sum_{\ell=0}^{k-i} (-1)^\ell \binom{j}{\ell} \binom{k-j}{k-i-\ell} \binom{n-k-j}{k-i-\ell}, \quad \text{for } j = 0, \ldots, k. \tag{3}$$

Note that the eigenvalues of the scheme $\mathcal{A}$ are, then, given by $p_i(j) = \lambda_j^{k-i}$. Assuming that $i$ is clear from the context, we will just write $\lambda_j^i$ instead of $\lambda_j^{k-i}$. An immediate consequence of (3) is that all the eigenvalues of the graph $J(n, k, i)$ are integers. Hence, according to Theorem 2.5, we observe the following.

**Corollary 2.6.** All the graphs in the Johnson scheme are periodic. \qed

3. **PST on the Johnson Scheme**

In this section we will prove the main result of the paper; i.e. we address the question of whether perfect state transfers can occur in any class of the Johnson scheme. First, we provide a necessary condition for the existence of a PST in the Johnson scheme.

**Proposition 3.1.** If a graph in the Johnson scheme $J(n, k)$ admits a PST then we must have $n = 2k$.

**Proof.** If there is a PST in $X$, according to Theorem 2.2, there is a graph $J(n, k, i)$ in the scheme such that $A(J(n, k, i)) = T$ is a permutation matrix of order two; hence $v = \binom{n}{k}$ must be even and one of the graphs of the scheme must be the union of $\frac{1}{2}\binom{n}{k}$ copies of $K_2$ edges, that is, $J(n, k, i) \cong \frac{1}{2}\binom{n}{k} K_2$. This implies that the valency $\binom{k}{i} \binom{n-k}{k-i}$ equals 1. Hence, $\binom{k}{i} = 1 = \binom{n-k}{k-i}$. The first equality implies $i = k$ or $i = 0$, but if the former happens, the graph $J(n, k, i)$ will be the empty graph which implies that $T$ is the identity matrix, a contradiction. Thus we can only have $i = 0$. Now the second equality implies $n = 2k$. \qed
Corollary 3.2. The Kneser graph $K(n, k)$ admits a PST if and only if $n = 2k$.

Proof. If $K(n, k) = J(n, k, 0)$ has a PST, then according to Proposition 3.1, $n = 2k$. On the other hand, since $K(2k, k) \cong \frac{1}{2} \binom{2k}{k} K_2$ and there is a PST on $K_2$ (Corollary 2.4), the result follows.

In the rest of this section we will show that the Kneser graph $J(2k, k, 0)$ is the only graph in the Johnson scheme $J(n, k)$ which has PST. In other words, we prove the following theorem, which is the main result of the paper.

Theorem 3.3. There is a PST in the class $J(n, k, i)$ of the Johnson scheme $J(n, k)$ if and only if $n = 2k$ and $i = 0$.

In order to prove Theorem 3.3 we need several preliminary lemmas. First note that, using (3), we can list all the eigenvalues of $J(2k, k, i)$.

Lemma 3.4. For any $i = 0, \ldots, k$, the eigenvalues of $J(2k, k, i)$ are given by

$$
\lambda_j = \sum_{\ell=0}^{k-i} (-1)^\ell \binom{j}{\ell} \binom{k-j}{i-j+\ell}^2, \quad j = 0, \ldots, k.
$$

In some special cases, we can simplify the formula of the eigenvalues as follows.

$$
\begin{align*}
\lambda_0 &= \binom{k}{i}^2, \\
\lambda_1 &= \binom{k}{i}^2 - 2\binom{k}{i} \binom{k-1}{i}, \\
\lambda_k &= (-1)^{k-i} \binom{k}{i}, \\
\lambda_{k-1} &= (-1)^{k-i} \left[ \binom{k}{i} - 2\binom{k-1}{i} \right], \\
\lambda_i &= \sum_{\ell=0}^{k-i} (-1)^\ell \binom{i}{\ell} \binom{k-i}{\ell}^2, \\
\lambda_{i+1} &= \sum_{\ell=0}^{k-i} (-1)^\ell \binom{i+1}{\ell} \binom{k-i-1}{\ell-1}^2.
\end{align*}
$$

In particular, in the scheme $\mathcal{A}$, we have $v_i = \binom{i}{k}^2$. Next, we identify the partition $(\mathcal{I}^-, \mathcal{I}^+)$ for the generalized Johnson graph $J(2k, k, i)$.

Proposition 3.5. For any $i = 1, \ldots, k-1$ and any eigenvalue $\lambda_j$ of $J(2k, k, i)$, we have $j \in \mathcal{I}^+$ if $j$ is even and $j \in \mathcal{I}^-$ if $j$ is odd.
Proof. Let \( \{E_0, \ldots, E_k\} \) be the idempotents of the scheme \( \mathcal{A} \) and \( T = A_k = A(J(2k, k, 0)) \) be the adjacency matrix of the Kneser graph \( J(2k, k, 0) \). Let \( i = 1, \ldots, k-1 \) be arbitrary. Recall that \( \lambda_j = p_{k-i}(j) \). We sort the vertices of the scheme in the lexicographic order of the \( k \)-subsets of \( \{1, \ldots, 2k\} \), so that the first vertex is \( U = \{1, \ldots, k\} \) and the last (i.e. the \( v \)-th) vertex is \( \overline{U} \) so \( T \) is anti-diagonal. As mentioned before Theorem 2.3, for any \( j = 0, \ldots, k \), we have \( TE_j = \pm E_j \). This means that for any \( j \), \( (TE_j)_{1,1} = \pm (E_j)_{1,1} \). On the other hand, we have \( (TE_j)_{1,1} = (E_j)_{1,v} \). Thus \( j \in \mathcal{I}^+ \) if \( (E_j)_{1,v} = (E_j)_{1,1} \) and \( j \in \mathcal{I}^- \) otherwise. Without loss of generality, we may assume that \( (E_j)_{1,1} \) is positive. Now, using (1) and Proposition 2.1, we have

\[
E_j = \frac{1}{v} \sum_{\ell=0}^{k} q_j(\ell) A_\ell = \frac{m_j}{v} \sum_{\ell=0}^{k} \frac{p_\ell(j)}{v_\ell} A_\ell.
\]

Since \( (A_k)_{1,v} = 1 \), we deduce that

\[
(E_j)_{1,v} = \frac{m_j}{v} \frac{p_k(j)}{v_k} (A_k)_{1,v} = \frac{m_j}{v} p_k(j).
\]

But \( p_k(j) \) are the eigenvalues of the Kneser graph and according to Lemma 3.4

\[
p_k(j) = \lambda_j = (-1)^j.
\]

Therefore \( (E_j)_{1,v} \) is positive if and only if \( j \) is even and the result follows. \( \square \)

We first consider the case where \( \binom{k}{i} \) is even.

**Lemma 3.6.** For any \( i = 1, \ldots, k-1 \), if \( \binom{k}{i} \) is even, then there is no PST on \( J(2k, k, i) \).

**Proof.** Using (5), we have \( \lambda_1 = \binom{k}{i}^2 - 2\binom{k}{i}(\binom{k}{i-1}) \) and according to Proposition 3.5 \( 1 \in \mathcal{I}^- \) and

\[
\lambda_0 - \lambda_1 = \binom{k}{i}^2 - \binom{k}{i}^2 + 2\binom{k}{i}(\binom{k}{i-1}) = 2\binom{k}{i}(\binom{k}{i-1}).
\]

(10)

If \( \text{ord}_2(\alpha) = t \), then for any \( j = 1, \ldots, k \), we must have \( 2^t|\lambda_0 - \lambda_j \). In particular, \( 2^t|\lambda_0 - \lambda_k \) and \( 2^t|\lambda_0 - \lambda_{k-1} \). Hence, \( 2^t|\lambda_k - \lambda_{k-1} \). That is, \( t \leq \text{ord}_2(\lambda_k - \lambda_{k-1}) \). On the other hand, using (6) and (7) we have

\[
\lambda_k - \lambda_{k-1} = 2(-1)^{k-i}(\binom{k}{i-1});
\]

and since \( \binom{k}{i} \) is even, we have \( \text{ord}_2(\lambda_k - \lambda_{k-1}) < \text{ord}_2(\lambda_0 - \lambda_1) \). This means that \( \text{ord}_2(\alpha) < \text{ord}_2(\lambda_0 - \lambda_1) \). According to Theorem 2.3, this shows that \( J(2k, k, i) \) cannot have a PST. \( \square \)

To complete the proof, therefore, we should consider the case where \( \binom{k}{i} \) is odd. To this goal, we will need to make use of Lucas’s famous Theorem (see, for example [5]).

6
Theorem 3.7. Suppose $a$ and $b$ are non-negative integers and $p$ is a prime. Also let

$$a = a_r p^r + a_{r-1} p^{r-1} + \cdots + a_1 p + a_0$$

and

$$b = b_r p^r + b_{r-1} p^{r-1} + \cdots + b_1 p + b_0$$

be the base-$p$ representation of $a$ and $b$, respectively. Then

$$\binom{a}{b} \equiv \prod_{\ell=0}^{r} \binom{a_\ell}{b_\ell} \pmod{p}.$$  

For the case $p = 2$, we get the following consequence of Theorem 3.7. We will write $b \preceq a$ if every digit of binary (i.e. base-2) representation of $b$ is less than or equal to the corresponding digit of $a$.

Theorem 3.8. For any non-negative $a$ and $b$, \(\binom{a}{b}\) is odd if and only if $b \preceq a$.  

The following is an easy consequence of Theorem 3.8.

Corollary 3.9. If $a$ is even and $b$ is odd, then \(\binom{a}{b}\) is even.

In what follows, we assume, always, that $i \in \{1, 2, \ldots, k-1\}$.

Lemma 3.10. If \(\binom{k}{i}\) is odd, then, for any $\ell = 1, \ldots, k-i$, either \(\binom{i}{\ell}\) or \(\binom{k-i}{\ell}\) is even.

Proof. Suppose \(\binom{i}{\ell}\) and \(\binom{k-i}{\ell}\) are both odd. Then, according to Theorem 3.8, we have $\ell \preceq i$ and $\ell \preceq k - i$. Since $\ell \neq 0$, there is a digit $\ell_j$ of the binary representation of $\ell$ such that $\ell_j = 1$. Thus we must have $i_j = 1$ and $(k-i)_j = 1$. On the other hand, since \(\binom{k}{i}\) is odd, we have $i \preceq k$, which implies that $i_j \leq k_j$, thus $k_j = 1$. But then we have $(k-i)_j = k_j - i_j = 1 - 1 = 0$, which is a contradiction. Thus the result follows.

Lemma 3.11. If \(\binom{k}{i}\) is odd, then $\text{ord}_2(\alpha) \leq 1$.

Proof. Using (9) and Pascal’s rule, we have

$$\lambda_{i+1} = \sum_{\ell=0}^{k-i} (-1)^\ell \binom{i}{\ell} \binom{k-i-1}{\ell-1}^2 + \sum_{\ell=0}^{k-i} (-1)^\ell \binom{i}{\ell-1} \binom{k-i-1}{\ell-1}^2.$$ 

Since \(\binom{k-i}{\ell}^2 = \binom{k-i-1}{\ell}^2 + \binom{k-i-1}{\ell-1}^2 + 2\binom{k-i-1}{\ell} \binom{k-i-1}{\ell-1}\), we deduce that
\[ \lambda_{i+1} = \sum_{\ell=0}^{k-i} (-1)^\ell \binom{i}{\ell} \binom{k-i}{\ell}^2 - \sum_{\ell=0}^{k-i} (-1)^\ell \binom{i}{\ell} \binom{k-i-1}{\ell} \]
\[ -2 \sum_{\ell=0}^{k-i} (-1)^\ell \binom{i}{\ell} \binom{k-i-1}{\ell} \binom{k-i-1}{\ell-1} \]
\[ + \sum_{\ell=0}^{k-i} (-1)^\ell \binom{i}{\ell-1} \binom{k-i-1}{\ell-1} \]
\[ = \lambda_i - \sum_{\ell=0}^{k-i-1} (-1)^\ell \binom{i}{\ell} \binom{k-i-1}{\ell}^2 + \sum_{\ell=1}^{k-i} (-1)^\ell \binom{i}{\ell-1} \binom{k-i-1}{\ell-1} \]
\[ - 2 \sum_{\ell=0}^{k-i-1} (-1)^\ell \binom{i}{\ell} \binom{k-i-1}{\ell} \binom{k-i-1}{\ell-1} \]
\[ = \lambda_i - \sum_{\ell=0}^{k-i-1} (-1)^\ell \left[ \binom{i}{\ell} \binom{k-i-1}{\ell}^2 + \binom{i}{\ell} \binom{k-i-1}{\ell} \binom{k-i-1}{\ell-1} \right] \]
\[ = \lambda_i - 2 \sum_{\ell=0}^{k-i-1} (-1)^\ell \binom{i}{\ell} \left[ \binom{k-i-1}{\ell} + \binom{k-i-1}{\ell-1} \right] \]
\[ = \lambda_i - 2 \sum_{\ell=0}^{k-i-1} (-1)^\ell \binom{i}{\ell} \binom{k-i}{\ell} \binom{k-i-1}{\ell} \]
\[ = \lambda_i - 2 - 2 \sum_{\ell=1}^{k-i-1} (-1)^\ell \binom{i}{\ell} \binom{k-i}{\ell} \binom{k-i-1}{\ell} \equiv \lambda_i - 2 \pmod{4}. \]

The last line follows from the fact that according to Lemma 3.10, all the summands in the last summation are even. This shows that \( \lambda_i - \lambda_{i+1} \equiv -2 \equiv 2 \pmod{4} \). Now if \( \text{ord}_2(\alpha) > 1 \), then \( 4\lambda_0 - \lambda_i \) and \( 4\lambda_0 - \lambda_{i+1} \) and hence \( 4\lambda_i - \lambda_{i+1} \), which is a contradiction. Therefore \( \text{ord}_2(\alpha) \leq 1 \).

**Lemma 3.12.** If \( \binom{k}{i} \) is odd and \( \binom{k-1}{i} \) is even, then \( J(2k, k, i) \) cannot have a PST.

**Proof.** According to (10), we have \( \text{ord}_2(\lambda_0 - \lambda_1) \geq 2 \); hence using Lemma 3.11, Theorem 2.3 and the fact that \( 1 \in \mathcal{I}^- \), the result follows. \( \square \)
It remains to consider the case where \( \binom{k}{i} \) and \( \binom{k-1}{i} \) are both odd. According to Corollary 3.9, these imply that \( i \) is even. To show the result in this case, we will make use of the following lemmas.

**Lemma 3.13.** If \( \binom{k}{i} \) is odd, then for any \( \ell = 0, 1, \ldots, k - i \), the following product is even:
\[
\binom{k-i}{\ell} \binom{i}{k-i-\ell} \binom{i-2}{k-i-\ell-2}.
\]

**Proof.** Let \( s \) be the smallest index for which \( i_s \), the \( s \)-th digit of \( i \) in the binary representation, is equal to 1. As \( i \lesssim k \), we have \( k_s = 1 \); thus \( \binom{k-i}{s} = 0 \).

Suppose \( \binom{k-i}{\ell} \) and \( \binom{i}{k-i-\ell} \) are both odd. Since \( \ell \lesssim k-i \), it follows that \( \ell_s = 0 \) and the binary representation of \( k-i-\ell \) will be
\[
k-i-\ell = \cdots (k_{s+1} - i_{s+1} - \ell_{s+1})0(k_{s-1} - \ell_{s-1})\cdots (k_0 - \ell_0).
\]

Now \( k-i-\ell \lesssim i \) implies, first, that
\[
k-i-\ell = \cdots (k_{s+1} - i_{s+1} - \ell_{s+1})000.
\]

It, further, implies that for every \( t \geq s+1 \), if \( i_t = 0 \), then \( k_t - i_t - \ell_t = 0 \) and if \( i_t = 1 \), then \( k_t = 1 \) from which we deduce that \( k_t - i_t = \ell_t = 0 \) which, again, implies that \( k_t - i_t - \ell_t = 0 \). Thus \( k-i-\ell = 0 \) and \( \binom{i-2}{k-i-\ell-2} = 0 \); therefore the result follows. \( \square \)

To prove the next result, we will make use of the well-known Vandermonde Convolution identity: let \( a, b \) and \( c \) be non-negative integers. Then
\[
\sum_{\ell=0}^{a} \binom{a}{\ell} \binom{b}{c-\ell} = \binom{a+b}{c}.
\]

(11)

**Lemma 3.14.** If \( \binom{k}{i} \) and \( \binom{k-1}{i} \) are odd and \( k \) is even, then the summation
\[
S = \sum_{\ell=0}^{k-i-2} \binom{k-i}{\ell} \binom{i-2}{k-i-\ell-2},
\]
where \( \ell \) ranges over even numbers, is odd.

**Proof.** Since \( k-i \) is even, according to Corollary 3.9, we have \( S \equiv S' \pmod{2} \), where
\[
S' = \sum_{\ell=0}^{k-i} \binom{k-i}{\ell} \binom{i-2}{k-i-\ell-2},
\]
where \( \ell \) ranges over all the numbers (even and odd). On the other hand, since for any integer \( x \), we have \( x \equiv x^2 \pmod{2} \), we can write
\[
S' \equiv \sum_{\ell=0}^{k-i} \binom{k-i}{\ell} \binom{i-2}{k-i-\ell-2} \pmod{2}.
\]
Then using (11) we deduce that

\[ S' \equiv \binom{k-2}{i} \pmod{2}. \]

On the other hand, by the assumption that \( \binom{k-1}{i} \) is odd and using Corollary 3.9 we have

\[ \binom{k-2}{i} = \binom{k-1}{i} - \binom{k-2}{i-1} \equiv 1 \pmod{2}. \]

Therefore the result follows.

Lemma 3.15. If \( \binom{k}{i} \) and \( \binom{k-1}{i} \) are both odd and \( k \) is even, then

\[ \lambda_{k-i+2} - \lambda_{k-i} \equiv 2 \pmod{4}. \]

Proof. Recall that the assumptions imply that \( i \) is even. We start by writing

\[ \lambda_{k-i+2} - \lambda_{k-i} = \sum_{\ell=0}^{k-i} (-1)^\ell \left[ \binom{k-i+2}{\ell} \binom{i-2}{k-i-\ell}^2 - \binom{k-i}{\ell} \binom{i}{k-i-\ell}^2 \right]. \]

It is obvious (noting Theorem 3.8) that the summands of this summation, for any odd \( \ell \), are divisible by 4; thus we may write

\[ \lambda_{k-i+2} - \lambda_{k-i} \equiv \sum_{\ell=0}^{k-i} \binom{k-i+2}{\ell} \binom{i-2}{k-i-\ell}^2 - \binom{k-i}{\ell} \binom{i}{k-i-\ell}^2 \pmod{4}, \]

where \( \ell \) ranges over even numbers. Using Pascal’s rule and Corollary 3.9 for any such \( \ell \), we can write

\[
\begin{align*}
\binom{k-i+2}{\ell} \binom{i-2}{k-i-\ell}^2 &\equiv \binom{k-i}{\ell} \binom{i-2}{k-i-\ell}^2 + \binom{k-i}{\ell} \binom{i-2}{\ell-2} \binom{k-i-\ell}{2} \\
&\equiv \binom{k-i}{\ell} \binom{i}{k-i-\ell}^2 + \binom{k-i}{\ell} \binom{i-2}{k-i-\ell-2}^2 \\
&- 2 \binom{k-i}{\ell} \binom{i-2}{k-i-\ell} \binom{i-2}{k-i-\ell-2} + \binom{k-i}{\ell-2} \binom{i-2}{k-i-\ell}^2 \pmod{4}.
\end{align*}
\]
Therefore, using Lemma 3.13, we can write

\[ \lambda_{k-i+2} - \lambda_{k-i} \equiv \sum_{\ell=0}^{k-i-2} \binom{k-i}{\ell} \left( \binom{i-2}{k-i-\ell-2} - \binom{i-2}{k-i-\ell} \right) \]

\[ \equiv \sum_{\ell=0}^{k-i-2} \binom{k-i}{\ell} \left( \binom{i-2}{k-i-\ell-2} - \binom{i-2}{k-i-\ell} \right) + \sum_{\ell=2}^{k-i-2} \binom{k-i}{\ell} \left( \binom{i-2}{k-i-\ell-2} - \binom{i-2}{k-i-\ell} \right) \]

\[ \equiv 2 \sum_{\ell=0}^{k-i-2} \binom{k-i}{\ell} \left( \binom{i-2}{k-i-\ell-2} - \binom{i-2}{k-i-\ell} \right)^2 \pmod{4}, \]

where in all the summations, \( \ell \) ranges over even numbers. This, according to Lemma 3.13, completes the proof. \( \square \)

Using a similar approach, one can prove the following result.

**Lemma 3.16.** If \( \binom{k}{i}, \binom{k-1}{i} \) and \( k \) are odd, then

\[ \lambda_{k-i+1} - \lambda_{k-i-1} \equiv 2 \pmod{4}. \] \( \square \)

Now we are ready to prove the result in the remaining case.

**Lemma 3.17.** If \( \binom{k}{i} \) and \( \binom{k-1}{i} \) are both odd, then \( J(2k, k, i) \) cannot have a PST.

**Proof.** Suppose there is a PST on \( J(2k, k, i) \). Then, according to Theorem 2.3, we must have \( \text{ord}_2(\alpha) = \text{ord}_2(\lambda_0 - \lambda_1) \), which implies, using (10), that \( \text{ord}_2(\alpha) = 1 \). If \( k \) is even, then according to Proposition 3.5, we have \( k-i+2, k-i \in \mathbb{I}^+ \) and, according to Theorem 2.3, it must be the case that \( \text{ord}_2(\lambda_0 - \lambda_{k-i+2}) > 1 \) and \( \text{ord}_2(\lambda_0 - \lambda_{k-i}) > 1 \). But then we must have \( 4|\lambda_{k-i+2} - \lambda_{k-i} \) which contradicts Lemma 3.15. Also, if \( k \) is odd, then \( k-i+1, k-i-1 \in \mathbb{I}^+ \) and we get a similar contradiction with Lemma 3.16. This completes the proof. \( \square \)

Now we can summarize the proof of the main result as follows.

**Proof of Theorem 3.3.** The proof follows from Proposition 3.1, Corollary 3.2 and Lemmas 3.6 and 3.12, 3.17, 4.1. \( \square \)

**4. Conclusion**

We considered the problem of existence of a PST on the classes of the Johnson scheme \( J(n, k) \). It was not hard to see that \( n = 2k \) is a necessary condition and also to see that the Kneser graph \( J(2k, k, 0) \), trivially admits PSTs. Although the other classes \( J(2k, k, i) \) \( (1 \leq i \leq k-1) \), display important evidences
of capability of having a PST, using some eigenvalue analyses, we showed that none of these classes can admit a PST.

It seems an interesting problem to consider other schemes as well. For example, given a permutation group $G \leq \text{Sym}(n)$, which classes of the conjugacy class scheme of $G$ can possibly have a PST? It is not hard to see that two major necessary conditions for a $G$ to have a PST on any of the graphs in its conjugacy class scheme are as follows: (a) any conjugacy class of $G$ must be closed under inversion, and (b) $G$ must have a singleton conjugacy class other than the identity class. As an example of such a group, we can name the dihedral group $D_{2n} \leq \text{Sym}(n)$, when $n$ is even. Indeed we can show that there, actually, are PSTs in some of the classes of the conjugacy class scheme of $D_{2n}$ (where it occurs on $K_2$’s) and the examples show that there are groups other than the dihedral groups which have PST. Therefore it, as well, sounds an interesting problem to classify the permutation group, acting on $\{1, \ldots, n\}$, which have the (necessary) conditions (a) and (b).

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