LOCAL EXACT CONTROLLABILITY TO POSITIVE TRAJECTORY FOR PARABOLIC SYSTEM OF CHEMOTAXIS

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ABSTRACT. In this paper, we study controllability for a parabolic system of chemotaxis. With one control only, the local exact controllability to positive trajectory of the system is obtained by applying Kakutani’s fixed point theorem and the null controllability of associated linearized parabolic system. The positivity of the state is shown to be remained in the state space. The control function is shown to be in $L^\infty(Q)$, which is estimated by using the methods of maximal regularity and $L^p$-$L^q$ estimate for parabolic equations.

1. Introduction and main results. Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with sufficiently smooth boundary $\partial \Omega$. Let $\omega$ be a nonempty open subset of $\Omega$ and let $T > 0$. Throughout the paper, we denote by $Q = \Omega \times (0, T)$, $\Sigma = \partial \Omega \times (0, T)$, $Q_\omega = \omega \times (0, T)$; $C^{2,1}(\overline{Q})$, $W^{s,q}(\Omega)$, $W^{2,1}_q(Q)$, and $C^\alpha(\overline{\Omega})$ $(s, \alpha \geq 0, 1 \leq q \leq \infty)$ the usual Sobolev spaces ([16]). Set $H^m(\Omega) = W^{m,2}(\Omega)$ for $m \in \mathbb{N}$ and denote by $L^p(\Omega)$ and $L^p(Q)$ $(1 \leq p \leq \infty)$ the usual Lebesgue function spaces with the norms $| \cdot |_p$ and $\| \cdot \|_p$, respectively. In addition, let

\[
V^1(Q) = \{ y | y \in L^2(0, T; H^1(\Omega)), \partial_t y \in L^2(0, T; H^1(\Omega)^*) \},
\]

\[
V^2(Q) = \{ y | y \in L^2(0, T; H^2(\Omega)), \partial_t y \in L^2(Q) \},
\]
be equipped with their graph norms, where $H^1(\Omega)^*$ stands for the dual space of $H^1(\Omega)$. The duality between $H^1(\Omega)^*$ and $H^1(\Omega)$ is represented by $\langle \cdot, \cdot \rangle$.

In this paper, we are concerned with the following controlled parabolic system with the state functions $u \equiv u(x, t)$ and $v \equiv v(x, t)$:

$$
\begin{aligned}
\partial_t u &= \nabla \cdot (\nabla u - \chi u \nabla v) + 1_{\omega} f \quad \text{in } Q, \\
\partial_t v &= \Delta v - \gamma v + \delta u \quad \text{in } Q, \\
\partial_\nu u &= 0, \partial_\nu v = 0 \quad \text{on } \Sigma, \\
u(x, 0) &= u_0(x) \quad v(x, 0) = v_0(x) \quad x \in \Omega,
\end{aligned}
$$

(1)

where $\partial_t = \partial/\partial t$, $\partial_\nu = \partial/\partial \nu$ denotes the derivative with respect to the outer normal $\nu$ of $\partial \Omega$, $1_{\omega}$ represents the characteristic function of $\omega$, $f \equiv f(x, t)$ is the control function, $u_0(\cdot)$ and $v_0(\cdot)$ are the initial values, and $\chi, \gamma$, and $\delta$ are given positive constants.

**Definition 1.1.** A pair of functions $(u, v)$ with

$$u \in V^1(Q) \cap L^\infty(Q), \quad v \in V^1(Q) \cap L^\infty(Q)$$

is said to be a weak solution to Equation (1), if for all $\varphi \in L^2(0, T; H^1(\Omega))$, the following identities hold:

$$\int_0^T (\partial_t u, \varphi) dt + \int_Q [(\nabla u - \chi u \nabla v) \cdot \nabla \varphi - 1_{\omega} f \varphi] dx dt = 0,$$

$$\int_Q \varphi \partial_t v dx dt + \int_Q [\nabla v \cdot \nabla \varphi + (\gamma v - \delta u) \varphi] dx dt = 0.$$

We write the free system (1) (i.e. in the absence of $f$) as follows:

$$
\begin{aligned}
\partial_t \Omega &= \nabla \cdot (\nabla \Omega - \chi \Omega \nabla \Omega) \quad \text{in } Q, \\
\partial_t \Omega &= \Delta \Omega - \gamma \Omega + \delta \Omega \quad \text{in } Q, \\
\partial_\nu \Omega &= 0, \partial_\nu \Omega = 0 \quad \text{on } \Sigma, \\
\Omega(x, 0) &= \Omega_0(x) \quad \Omega(x, 0) = \Omega_0(x) \quad x \in \Omega.
\end{aligned}
$$

(2)

The system (2) is a prototype chemotaxis system, called the Keller-Segel model which describes the aggregation process of slime mold resulting from chemotactic attraction. In Equation (2), $\Omega(\cdot, \cdot)$ represents the density of the cellular slime mold and $\Omega(\cdot, \cdot)$ is the density of the chemical substance ([15]). In the last decade, there are a large number of works attributed to the mathematical analysis of the Keller-Segel system. Several topics on the Keller-Segel model for chemotaxis such as the aggregation, the blow-up solutions, and the chemotactic collapse have been addressed and some significant results have been achieved from different discipline perspectives. We refer to [11] and [12] and the references therein for a detailed introduction of mathematical problems on the Keller-Segel model of chemotaxis. Generally speaking, the blow-up of solutions of the Keller-Segel system in finite or infinite time depends strongly on the space dimension. In 1-d case, a finite time blow-up never occurs, and the global solution exists and converges to the stationary solution as times goes to infinity ([20]). However, the blow-up may occur in finite or infinite time in $n$-dimensional case for $n \geq 3$ ([13]). For the 2-d case, several thresholds have been found for the existence of the global solution. When the mass of the initial data is below some threshold value, the solution exists globally and its $L^\infty$-norm is uniformly bounded for all time, while the mass of the initial data is larger than some threshold value, the solution would blow up either in finite or
in infinite time ([4, 8, 23]). For more results on chemotaxis equations, we refer to [19, 26].

Since the densities of the cell and the chemical substance are usually positive, it is reasonable to consider the positivity of the solutions. In fact, it is easy to see that when \((u_0, v_0) \geq 0\), the corresponding solution \((u, v) \geq 0\) holds for Equation (2) ([4, 13]). In addition, the total mass of \(u\) is conserved:

\[ \int_{\Omega} u(x,t) \, dx = \int_{\Omega} u_0(x) \, dx \geq 0, \quad \forall \, t \in [0, T]. \]

The strict positive solution to Equation (2) is claimed by the following Theorem 1.2.

**Theorem 1.2.** Suppose that \(u_0 \in C(\Omega)\) and \(v_0 \in W^{1,q}(\Omega)\) \((q > n)\) satisfy \(u_0 \geq k_0\) and \(v_0 \geq k_0\) for some constant \(k_0 > 0\). Then, there exist \(T = T(u_0, v_0) > 0\) and a positive constant \(\delta_1\) such that Equation (2) admits a unique classical solution \((\pi, \tau) \in (C(\overline{Q}) \cap C^{2,1}(\Omega \times (0, T))) \times C^{2,1}(\overline{Q})\) that satisfies

\[ \pi(x,t) \geq e^{-\delta_1 T} k_0, \quad \tau(x,t) \geq k_0, \quad \forall \, (x,t) \in Q. \]  

On account of the blow-up feature of solutions of the Keller-Segel model, it is significant to consider controllability for system (1).

**Definition 1.3.** Let \((\pi, \tau)\) be a positive trajectory of system (2) corresponding to an initial value \((u_0, v_0)\). We say that the system (1) is **locally exactly controllable** to the trajectory \((u, v)\) at time \(T\), if there is a neighborhood \(O\) of \((u_0, v_0)\) such that for any initial value \((u_0, v_0) \in O\), the corresponding solution \((u, v)\) of Equation (1) driven by some control function \(f\) satisfies

\[ u(x,T) = \pi(x,T), \quad v(x,T) = \tau(x,T), \quad x \in \Omega \text{ a.e.} \]

**Remark 1.** The Theorem 1.2 implies the existence of local solution to Equation (1). We take \(T = T(u_0, v_0) < T_{\max}\) in Theorem 1.2, where \(T_{\max}\) is the maximal time for the existence of solution to Equation (2). When \(t \to T_{\max}\), the solution of (2) may blow up. If system (1) is locally exactly controllable, then we can drive the state of system (1) by some control force to a given trajectory at time \(T\) before \(T_{\max}\) to avoid blow-up.

Controllability for parabolic equations attracts intensive attention in the last few years. We refer [5], [7] and other literature. However, very few results, to the best of our knowledge, are available on the control problems of system (1). In [22], an optimal control problem for the system (1) with the control to be distributed on the second equation of Equation (1) is considered. We believe that the present paper is a first work on controllability for system (1). Compared with the coupled parabolic systems aforementioned, the mathematical difficulty for the control problems of system (1) is brought by the chemotactic term \(-\chi \nabla \cdot (u \nabla v)\). The techniques presented in this paper would be useful for other forms of chemotaxis system such as the parabolic-elliptic chemotaxis system and even for other coupled systems like drift-diffusion equations from the semiconductor device.

The idea of obtaining the controllability of system (1) is somehow classical: We first establish null controllability for the linearized system and then apply a fixed point theorem. Now, we consider null controllability for the linearized system of
system (1), which reads as follows:
\[
\begin{aligned}
\partial_t y &= \Delta y - \nabla \cdot (By) - \nabla \cdot (a \nabla z) + 1_\omega f & \quad & \text{in } Q, \\
\partial_t z &= \Delta z - \gamma z + \delta y & \quad & \text{in } Q, \\
\partial_y y &= 0, \partial_y z = 0 & \quad & \text{on } \Sigma, \\
y(x,0) = y_0(x) \quad z(x,0) = z_0(x) & \quad & x \in \Omega,
\end{aligned}
\]
where \( a(\cdot, \cdot) \in L^\infty(Q), B(\cdot, \cdot) \in L^\infty(Q)^N \) with \( B \cdot \nu = 0 \) on \( \Sigma \), \( f(\cdot, \cdot) \in L^2(Q) \) is the control force, and \( y(\cdot), z(\cdot) \in L^2(\Omega) \) are given initial data. We state our first result on null controllability for linear parabolic system (4).

**Theorem 1.4.** Let \( T > 0 \). For any \( (y_0, z_0) \in L^2(\Omega) \times L^2(\Omega) \), there exists a control \( f \in L^\infty(Q) \) such that the solution \( (y, z) \) of system (4) corresponding to \( f \) satisfies \( (y, z) \in V^1(Q) \times V^1(Q) \) and \( y(x, T) = z(x, T) = 0 \) for almost all \( x \in \Omega \). In addition, the control \( f \) satisfies
\[
\| f \| \leq e^{C \kappa} (|y_0|_2 + |z_0|_2),
\]
where \( C \) is a positive constant depending only on \( \Omega \) and \( \omega \), and
\[
\kappa = (1 + \|a\|_\infty^2 + \|B\|_\infty^2) T + \frac{1}{T} + 1 + \|a\|_\infty + \|B\|_\infty.
\]

To the best of our knowledge, there are two methods to build \( L^\infty \) controls. The paper [24] (see also [6]) proposes a direct way to realize \( L^\infty \) control for a single heat equation by applying the \( L^1 \) observability inequality from an \( L^2 \) observability inequality and the well-posedness of adjoint system. Here we adopt a different way (see, e.g., [2]) that the \( L^\infty \) control is obtained by minimizing a cost function and applying the boot-strap method, which involves \( L^p \)-\( L^q \) estimate and maximal regularity of semigroups in order to improve the estimation by giving an explicit representation of the upper bound of controls with respect to \( T \). This method can also be adopted to build control in space \( W^{1,2}([0,T]; L^2(\Omega)) \) (see, e.g., [25]).

The main result of this paper is the following Theorem 1.5.

**Theorem 1.5.** Let \( p > N + 2 \) and let \((\pi, \tau)\) be a positive trajectory of system (2) corresponding to \((\bar{u}_0, \bar{v}_0)\), which satisfies (3). Then, there exists a positive constant \( c_1 \) independent of \( T \) such that for each \((u_0, v_0)\) that satisfies \( u_0 \geq 0, v_0 \geq 0 \), and
\[
|u_0 - \bar{u}_0|_\infty + |v_0 - \bar{v}_0|_{W^{2(1-\frac{1}{p})^\ast, p}(\Omega)} \leq e^{-c_1(1+T+\frac{1}{T})},
\]
there is a control \( f \in L^\infty(Q) \) such that system (1) admits a unique solution \((u, v)\) satisfying
\[
u(x, t) \geq 0, v(x, t) \geq 0 \text{ for all } (x, t) \in Q \text{ and } u(x, T) = \pi(x, T), v(x, T) = \tau(x, T) \text{ for almost all } x \in \Omega.
\]

For notational simplicity, we use \( C \) or \( C(\Omega, \omega) \) to denote, throughout the paper, a positive constant that is independent of time \( T \) yet depends on \( \Omega \) and \( \omega \) without specification.

We proceed as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the proofs of the main results.
2. Preliminaries. In this section, we collect some results that are needed in next section. These results are particularly useful in establishment of the regularity of linear parabolic systems and the $L^\infty$-estimate of controls.

For $p \in (1, \infty)$, let $A := A_p$ denote the sectorial operator given by

$$A_p u := -\Delta u, \forall u \in D(A_p) := \{ u \in W^{2,p}(\Omega) ; \partial_u|_{\partial\Omega} = 0 \}. \quad (8)$$

Suppose that $\gamma$ is a positive constant.

**Proposition 1.** Let $A$ be given by (8). Then the following assertions hold, where (iv) is a direct consequence of succeeding Equations (11) and (13).

(i). Let $\alpha \geq 0$ and $D((A+\gamma)^\alpha)$ be the function space endowed with the graph norm. Then $D((A+\gamma)^\alpha)$ is a Banach space with the following embedding properties ([9, p.39])

$$D((A+\gamma)^\alpha) \hookrightarrow W^{1,p}(\Omega) \; \text{if} \; \alpha > \frac{1}{2}, \quad (9)$$

$$D((A+\gamma)^\alpha) \hookrightarrow C^0(\overline{\Omega}) \; \text{if} \; 0 \leq \gamma < 2\alpha - \frac{n}{p}. \quad (10)$$

(ii). Let $\{e^{-tA}\}_{t \geq 0}$ and $\{e^{-t(A+\gamma)}\}_{t \geq 0}$ be the analytic $C_0$-semigroups generated by $-A$ and $-(A+\gamma)$ on $L^p(\Omega)$ $(1 < p < \infty)$, respectively. By standard $C_0$-semigroup theory ([21])

$$|e^{-tA}u|_q \leq Cm(t)^{-\frac{n}{p}(\frac{1}{2} - \frac{1}{p})}|u|_p, \forall u \in L^p(\Omega), t > 0, 1 < p \leq q < \infty, \quad (11)$$

where $m(t) = \min\{1,t\}$, and

$$|(A+\gamma)^\alpha e^{-t(A+\gamma)}|_q \leq Ct^{-\frac{n}{p}(\frac{1}{2} - \frac{1}{p})-\alpha}|u|_p, \forall u \in L^p(\Omega), t > 0, 1 < p \leq q < \infty, \quad (12)$$

(iii). Let $\alpha \geq 0$ and $1 < p < \infty$. Then, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending on $\Omega, \varepsilon$, and $p$ such that ([13, Lemma 2.1])

$$|(A+\gamma)^\alpha e^{-tA\nabla} \cdot u|_p \leq C_\varepsilon t^{-\alpha-\frac{1}{2} - \varepsilon}|u|_p, \forall u \in L^p(\Omega), t > 0. \quad (13)$$

(iv). For any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending on $\Omega, \varepsilon$, and $p$ such that

$$|e^{-tA\nabla} \cdot u|_q \leq C_\varepsilon m(t)^{-\frac{1}{2} - \varepsilon-\frac{n}{p}(\frac{1}{2} - \frac{1}{p})}|u|_p \quad (14)$$

for all $u \in L^p(\Omega)$, $t > 0, 1 < p \leq q < \infty$.

(v). (Maximal regularity) Let $1 < p < \infty$. If $F \in L^p(\Omega)$ and $u_0 \in W^{2(1-\frac{1}{2})p}(\Omega)$ with $\partial_v u_0 = 0$ on $\partial\Omega$, then there exists a unique solution to the equation of the following

$$\frac{du}{dt} = (A+\gamma)u + F, \forall t \in (0,T) \; \text{a.e.,} \; u(0) = u_0$$

and the solution satisfies

$$\left\| \frac{du}{dt} \right\|_p^p + \|(A+\gamma)u\|_p^p + \|u\|_p^p \leq C \left( \|F\|_p^p + \|u_0\|_{W^{2(1-\frac{1}{2})p}(\Omega)}^p \right), \quad (15)$$

where $C$ is a positive constant independent of $T$ and $F$.

**Remark 2.** Inequality (15) is first established as Theorem 9.1 of [16] in Chapter IV, but the independency of $C$ with respect to $T$ is given later as Theorem 1.1 of [17].
Now, we consider the well-posedness of the following linear parabolic system which covers system (4) as its special case:

\[
\begin{align*}
\partial_t y &= \Delta y - \nabla \cdot (By) - \nabla \cdot (\alpha \nabla z) + F \quad \text{in } Q, \\
\partial_t z &= \Delta z - \gamma z + \delta y \quad \text{in } Q, \\
\partial_x y &= 0, \partial_x z = 0 \quad \text{on } \Sigma, \\
y(x, 0) = y_0(x) \quad z(x, 0) = z_0(x) \quad x \in \Omega.
\end{align*}
\]

(16)

**Proposition 2.** Let \( a \in L^\infty(Q) \) and \( B \in L^2(Q) \) with \( B \cdot \nu = 0 \) on \( \Sigma \). Then the following assertions hold.

(i). If \( y_0, z_0 \in L^2(\Omega) \) and \( F \in L^2(Q) \), then Equation (16) admits a unique solution \((y, z) \in V^1(\Omega) \times V^1(\Omega)\) satisfying

\[
\|y\|_{V^1(\Omega)}^2 + \|z\|_{V^1(\Omega)}^2 \leq e^{C\kappa} \left( \|y_0\|_2^2 + \|z_0\|_2^2 + \|F\|_2^2 \right). \tag{17}
\]

(ii). Let \( 2 \leq p < \infty \). If \( F \in L^p(\Omega) \), \( y_0 \in L^p(\Omega) \), and \( z_0 \in W^{2(1 - 1/p)}, p(\Omega) \) with \( \partial_z z_0 = 0 \) on \( \partial \Omega \), then Equation (16) admits a unique solution \((y, z) \in L^p(\Omega) \times W^{2, 1} \) satisfying

\[
\|y\|_p^p + \|z\|_{W^{2, 1}(\Omega)}^p \leq e^{C\kappa} \left( \|y_0\|_p^p + \|z_0\|_{W^{1, p}(\Omega)}^p + \|F\|_p^p \right). \tag{18}
\]

(iii). Let \( p > N + 2 \). If \( F \in L^\infty(Q) \), \( y_0 \in L^\infty(\Omega) \), and \( z_0 \in W^{1, p}(\Omega) \) with \( \partial_z z_0 = 0 \) on \( \partial \Omega \), then Equation (16) admits a solution \((y, z) \in L^\infty(\Omega) \times L^\infty(\Omega)\) satisfying

\[
\|y\|_\infty + \|z\|_\infty \leq e^{C\kappa} \left( \|y_0\|_\infty + \|z_0\|_{W^{1, p}(\Omega)} + \|F\|_\infty \right), \tag{19}
\]

where \( \kappa \) is given by Equation (6) and \( C = C(\Omega) \).

**Proof.** The existence of solution with respect to \((y_0, z_0, F)\) in corresponding function spaces can be deduced similarly as [16] for which we omit here. We only show the required estimations with respect to time \( T \). Since the proof for (17) is similar to (18), we need only show (18). Multiply the first equation of Equation (16) by \(|y|^{p-2} y\) and integrate over \( \Omega \) to obtain

\[
\frac{d}{dt} \|y\|_p^p + \int_\Omega |\nabla y|^2 |y|^{p-2} \, dx \leq C \left( 1 + \|a\|_{\infty}^2 + \|B\|_{\infty}^2 \right) |y|_p^p + C \|a\|_{\infty}^2 \|\nabla z\|_p^p + C |F|_p^p. \tag{20}
\]

In the same way, we can obtain, from the second equation of Equation (16), that

\[
\frac{d}{dt} |z|^p + \int_\Omega |\nabla z|^2 |z|^{p-2} \, dx + |z|_p^p \leq C |y|_p^p. \tag{21}
\]

Differentiate \(|\nabla z|_p^p\) with respect to \( t \) and take the second equation of Equation (16) into account again to obtain

\[
\frac{d}{dt} |\nabla z|_p^p + \int_\Omega |\nabla z|^{p-2} |\Delta z|^2 \, dx \leq C |\nabla z|_p^p + C \left( |y|_p^p + |z|_p^p \right). \tag{22}
\]

The inequalities (20)–(22) together with Gronwall’s inequality lead to

\[
|y(\cdot, t)|_p^p + |z(\cdot, t)|_p^p + |\nabla z(\cdot, t)|_p^p \leq e^{C\kappa} \left( |y_0|_p^p + \|z_0\|_{W^{1, p}(\Omega)}^p \right), \quad \forall \, t \in [0, T]. \tag{23}
\]

On the other hand, by the maximal regularity (15) for the second equation of Equation (16), it follows that

\[
\|\partial_t z\|_p^p + \|\Delta z\|_p^p + \|z\|_p^p \leq C \left( \|z_0\|_{W^{2(1 - 1/p)}, p(\Omega)}^p + \|y\|_p^p + \|z\|_p^p \right),
\]
which, together with Equation (23), yields (18).

Now, we turn to the $L^\infty$-estimate (19). We first assume that $y_0 \in C(\bar{\Omega})$ and $F \in C(\bar{Q})$. Let $A$ be defined by (8), and let $\{e^{-tA}\}_{t \geq 0}$ and $\{e^{-t(A+\gamma)}\}_{t \geq 0}$ be the analytic $C_0$-semigroups generated by $-A$ and $-(A+\gamma)$ in $L^p(\Omega)$, $1 < p < \infty$, respectively. Then the solution $(y, z)$ of Equation (16) can be represented as

$$
y(\cdot, t) = e^{-tA}y_0 + \int_0^t e^{-(t-s)A} [-\nabla \cdot (By) - \nabla \cdot (a \nabla z) + F](\cdot, s) ds, \quad (24)$$

$$
z(\cdot, t) = e^{-(A+\gamma)z_0} + \delta \int_0^t e^{-(t-s)(A+\gamma)} y(\cdot, s) ds. \quad (25)$$

First, take $C(\bar{\Omega})$-norm on both sides of (24) to obtain

$$\|y(\cdot, t)\|_{C(\bar{\Omega})} \leq \|y_0\|_{C(\bar{\Omega})}, \quad \|e^{-tA}y_0\|_{C(\bar{\Omega})} + \int_0^t \|e^{-(t-s)A} [-\nabla \cdot (By) - \nabla \cdot (a \nabla z) + F](\cdot, s)\|_{C(\bar{\Omega})} ds$$

$$+ \int_0^t \|e^{-(t-s)A}F(\cdot, s)\|_{C(\bar{\Omega})} ds. \quad (26)$$

To estimate (26), we observe that the operator $-A$ generates a bounded analytic semigroup on $C(\bar{\Omega})$ ([1]). By the maximum principle, for any $0 \leq s < t \leq T$,

$$\|e^{-tA}y_0\|_{C(\bar{\Omega})} \leq \|y_0\|_{C(\bar{\Omega})}, \quad \|e^{-tA}F(\cdot, s)\|_{C(\bar{\Omega})} \leq \|F(\cdot, s)\|_{C(\bar{\Omega})}. \quad (27)$$

Since $p > N + 2$, we can take $\varepsilon$ and $\alpha$ so that

$$0 < \varepsilon < \frac{p - N - 2}{2p} \quad \text{and} \quad \frac{N}{2p} < \alpha < \frac{1}{2} - \frac{1}{p} - \varepsilon.$$

Then, with the help of (10) and (13), and the Hölder inequality, we obtain

$$\int_0^t \|e^{-(t-s)A} \nabla \cdot (By + a \nabla z)(\cdot, s)\|_{C(\bar{\Omega})} ds$$

$$\leq \int_0^t \|(A + \gamma)^\alpha e^{-(t-s)A} (By + a \nabla z)(\cdot, s)\|_p ds$$

$$\leq C \int_0^t (t-s)^{-\alpha - \frac{1}{p} - \varepsilon} \|e^{-(t-s)(A+\gamma)}(\cdot, s)\|_p ds$$

$$\leq C (1 + \|a\|_\infty + \|B\|_\infty) (\|y\|_p + \|\nabla z\|_p) T^{\frac{1}{2} - \alpha - \frac{1}{p}} \varepsilon, \quad \forall t \in [0, T]. \quad (28)$$

By (26)-(28), and (23)

$$\|y\|_\infty \leq e^{CT} \left( \|y_0\|_\infty + \|z_0\|_{W^{1,p}(\Omega)} + \|F\|_\infty \right). \quad (29)$$

Next, take $W^{1,p}(\Omega)$-norm on both sides of (25) to derive that for $0 \leq t \leq T$,

$$\|z(\cdot, t)\|_{W^{1,p}(\Omega)} \leq \|e^{-(t)(A+\gamma)z_0}\|_{W^{1,p}(\Omega)} + \delta \int_0^t \|e^{-(t-s)(A+\gamma)} y(\cdot, s)\|_{W^{1,p}(\Omega)} ds. \quad (30)$$

To estimate (30), we first notice that

$$\|e^{-(t)(A+\gamma)z_0}\|_{W^{1,p}(\Omega)} \leq e^{CT} \|z_0\|_{W^{1,p}(\Omega)}. \quad (31)$$
which can be obtained by the same energy method used in proving (23). Let $1/2 < \alpha < 1 - 1/p$. By (9) and (12) with application of the Hölder inequality, we have
\[
\int_0^t \left\| e^{-(t-s)(A+\gamma)} y(\cdot, s) \right\|_{W^{1,p}(\Omega)} ds \leq C \int_0^t \left| (A+\gamma)^\alpha e^{-(t-s)(A+\gamma)} y(\cdot, s) \right|_p ds \\
\leq C \int_0^t (t-s)^{-\alpha} |y(\cdot, s)| ds \\
\leq C \|y\|_p T^{-\alpha + \frac{1}{2}}, \forall t \in [0, T].
\]
This, together with (23), (30)-(31), and the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow C(\Omega)$ for $p > N$, we obtain
\[
\|z\|_\infty \leq e^{C_\kappa} \left( \|y_0\|_\infty + \|z_0\|_{W^{1,p}(\Omega)} + \|F\|_\infty \right).
\]

Finally, let us consider the general case that $y_0 \in L^\infty(\Omega)$ and $F \in L^\infty(Q)$. Let \( \{y_{n}\}_{n=1}^\infty \subset C(\Omega) \) and \( \{F_n\}_{n=1}^\infty \subset C(Q) \) be such that \( y_{n} \to y_0 \) in \( L^2(\Omega) \), \( F_n \to F \) in \( L^2(Q) \) and \( \|y_{n}\|_\infty \leq \|y_0\|_\infty, \|F_n\|_\infty \leq \|F\|_\infty \). For each $n$, let \( (y_n, z_n) \) be a solution of Equation (16) corresponding to \( (y_{n}, z_{0}, F_{n}) \), which satisfies the inequalities (17) and (19) with \( (y, z) \) replaced by \( (y_n, z_n) \). We can extract a subsequence of \( (y_n, z_n) \) such that it converges to \( (y, z) \) which is a weak solution of (16) corresponding to \( (y_0, z_0, F) \). Moreover, \( (y, z) \) satisfies the inequality (19). This ends the proof of the proposition. 

3. Proof of main results. To prove Theorem 1.2, we need the following Lemma 3.1.

Lemma 3.1. Let $u_0 \in L^2(Q)$ with $u_0 \geq k_0$ for some positive constant $k_0$, $F \in L^2(Q)$ with $F \geq 0$, $B_1 \in L^\infty(Q)^N$, $B_2 \in L^\infty(0, T; W^{1,\infty}(\Omega))^N$ with $B_2 \cdot \nu = 0$ on $\Sigma$, and $a \in L^\infty(Q)$. Then there is a positive constant $\delta_0$ depending on $\|a\|_\infty$ and $\|\nabla \cdot B_2\|_\infty$ such that the weak solution of the following equation
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + B_1 \nabla u + \nabla \cdot (B_2 u) + au &= F \quad \text{in } Q, \\
\frac{\partial u}{\partial t} &= 0 \quad \text{on } \Sigma, \\
u(x, 0) &= u_0(x), \quad x \in \Omega.
\end{aligned}
\]
satisfies $u \in L^2(0, T; H^1(\Omega))$ and $u(x, t) \geq e^{-\delta_0 T} k_0$ for $(x, t) \in Q$. In addition, if $B_2 \equiv 0$, then $u(x, t) \geq k_0$ for $(x, t) \in Q$.

Proof. We split the proof into two steps.

Step 1. Suppose that $a \geq 0$ and $u_0 \geq 0$. We show that $u(x, t) \geq 0$ for $(x, t) \in Q$.

To this end, let $u_\sim = \max\{-u, 0\}$. Multiply both sides of Equation (33) by $-u_\sim$ and perform integration over $\Omega$ to give
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_\sim|^2 dx + \int_\Omega |\nabla u_\sim|^2 dx + \int_\Omega a|u_\sim|^2 dx \leq \int_\Omega (u_\sim)(B_2 - B_1) \nabla u_\sim dx \\
\leq \frac{1}{2} \int_\Omega |\nabla u_\sim|^2 dx + (\|B_1\|_\infty^2 + \|B_2\|_\infty^2) \int_\Omega |u_\sim|^2 dx.
\]

Apply Gronwall’s lemma to above inequality to obtain
\[
\int_\Omega |u_\sim|^2 dx \leq e^{(\|B_1\|_\infty^2 + \|B_2\|_\infty^2) T} \int_\Omega |(u_0)_-|^2 dx.
\]
Since \((u_0)_- = 0\), it follows that \(u_- = 0\) and hence \(u(x,t) \geq 0\) as claimed.

**Step 2.** Suppose that \(a \in L^\infty(Q)\) and \(u_0 \geq k_0\). Set \(u = (v + k_0 e^{-\lambda t}) e^\mu t\), where \(\lambda\) and \(\mu\) are specified later. Then Equation (33) is transformed into

\[
\begin{align*}
\partial_t v - \Delta v + B_1 \nabla v + \nabla \cdot (B_2 v) + (\mu + a)v & = (\lambda - \mu - a - \nabla \cdot B_2)k_0 e^{-\lambda t} + F e^{-\mu t}, \\
\partial_t v & = 0 \text{ on } \Sigma, \\
v(x,0) & = u_0 - k_0 \geq 0 \text{ for } x \in \Omega.
\end{align*}
\]

Choose \(\mu \geq \|a\|_\infty\) and \(\lambda \geq \mu + \|a\|_\infty + \|\nabla \cdot B_2\|_\infty \geq 2 \|a\|_\infty + \|\nabla \cdot B_2\|_\infty\). Then the right-hand side of the first equation of Equation (34) is nonnegative and thus the conditions of Step 1 verify. Therefore, \(v(x,t) \geq 0\) and \(u(x,t) \geq e^{-\delta_0 T} k_0\) by taking \(\delta_0 \geq 2 \|a\|_\infty + \|\nabla \cdot B_2\|_\infty\).

Finally, if \(B_2 = 0\), then taking \(u\) in Step 2 as \(u = (v + k_0) e^\mu t\), we can obtain \(u(x,t) \geq k_0\) for \((x,t) \in Q\) with the same arguments. \(\square\)

**Remark 3.** Although the strict positivity of solution may be obtained by the maximum principle and the Hopf lemma, the diffusion coefficient \(B_2(\cdot, \cdot)\) and the potential \(a(\cdot, \cdot)\) are usually required to satisfy some positivity conditions (see, e.g., \([18, \text{Theorem 6.25, p.128}]\) and \([16, \text{Theorem 7.2, p.188}]\)). Here, we only require \(B_2(\cdot, \cdot)\) and \(a(\cdot, \cdot)\) to be in \(L^\infty\) space.

**Proof of Theorem 1.2.** By Theorem 3.1 of [13], for any given initial data \((\bar{u}_0, \bar{v}_0)\) with \(\bar{u}_0 \in C(\overline{\Omega})\), \(\bar{v}_0 \in W^{1,q}(\Omega)\) \((q > n)\), \(\bar{u}_0 \geq 0\), and \(\bar{v}_0 \geq 0\), there exists \(T = T(\bar{u}_0, \bar{v}_0)\) such that Equation (2) admits a classical solution \((\bar{u}, \bar{v}) \in (C(\overline{\Omega}) \cap C^{2,1}(\overline{\Omega} \times (0,T))) \times C^{2,1}(\overline{\Omega})\) satisfying \(\bar{u} \geq 0\), \(\bar{v} \geq 0\), and

\[
\|\bar{u}\|_{C(\overline{\Omega})} + \|\bar{v}\|_{C^{2,1}(\overline{\Omega})} \leq \varpi_0,
\]

for some positive constant \(\varpi_0\).

Since \(v_0 \geq k_0\), applying Lemma 3.1 to the second equation of Equation (2) which is rewritten here

\[
\begin{align*}
\partial_t \bar{u} - \Delta \bar{u} + \gamma \bar{u} & = \delta \bar{u} \text{ in } Q, \\
\partial_t \bar{v} & = 0 \text{ on } \Sigma, \\
\bar{u}(x,0) & = \bar{v}_0(x) \text{ for } x \in \Omega,
\end{align*}
\]

we obtain

\[
v(x,t) \geq k_0, \quad \forall (x,t) \in Q.
\]

We then formulate the first equation of Equation (2) as

\[
\begin{align*}
\partial_t \bar{u} - \Delta \bar{u} + \nabla \cdot (\bar{u} B_\bar{v}) & = 0 \text{ in } Q, \\
\partial_t \bar{v} & = 0 \text{ on } \Sigma, \\
\bar{u}(x,0) & = \bar{v}_0(x) \text{ for } x \in \Omega,
\end{align*}
\]

where \(B_\bar{v} = \chi \nabla \bar{v}\) and \(\bar{v}\) is the solution of (2). By Equation (35) or standard Schauder estimate for linear parabolic equations, \(\|\nabla \cdot B_\bar{v}\|_{C(\bar{u})} \leq C \varpi_0\), where the constant \(\varpi_0\) is given by (35). Applying Lemma 3.1 once again, we obtain a positive constant \(\delta_1\) depending on \(\varpi_0\) such that \(\bar{u}(x,t) \geq e^{-\delta_1 T}\) for \((x,t) \in Q\). \(\square\)
To study null controllability for system (4), we consider observability for the adjoint system of system (4):

\[
\begin{align*}
-\partial_t \phi &= \Delta \phi + B \nabla \phi + \delta \theta & \text{in } Q, \\
-\partial_t \theta &= \Delta \theta - \gamma \theta - \nabla \cdot (a \nabla \phi) & \text{in } Q, \\
\partial_x \phi &= 0, \partial_x \theta = 0 & \text{on } \Sigma, \\
\phi(x, T) &= \phi^T(x), \theta(x, T) = \theta^T(x) & x \in \Omega,
\end{align*}
\] (37)

where \( \phi^T, \theta^T \in L^2(\Omega) \). In order to obtain null controllability for Equation (4), we need to establish an observability inequality, which can be derived as a consequence of a global Carleman inequality for the adjoint system (37).

Let \( \omega' \subset \omega \), that is, \( \overline{\omega'} \subset \omega \), and \( \omega' \) is a nonempty open subset of \( \omega \). Then, there is a function \( \beta \in C^2(\overline{\Omega}) \) such that \( \beta(x) > 0 \) for all \( x \in \Omega \), and \( \beta|_{\partial \Omega} = 0, |\nabla \beta(x)| > 0 \) for all \( x \in \Omega \setminus \omega' \) (see e.g., [7, Lemma 1.1]). For \( \lambda > 0 \), set

\[
\varphi = e^{\lambda \beta}, \quad \alpha = \frac{e^{\lambda \beta} - e^{2 \lambda \|\beta\|_{C(\overline{\omega})}}}{t(T-t)},
\] (38)

and

\[
\gamma(\lambda) = e^{2 \lambda \|\beta\|_{C(\overline{\omega})}}. \quad (39)
\]

**Lemma 3.2.** Let \( f_i \in L^2(Q), i = 0, 1, \ldots, N \) and let \( d \) be a real number. Then there exists a constant \( \lambda_0 = \lambda_0(\Omega, \omega', d) > 1 \), such that for all \( \lambda \geq \lambda_0 \) and \( s \geq \gamma(\lambda)(T + T^2) \),

\[
\int_{Q} \left[ \lambda^2 (s\varphi)^{1+d} |\nabla z|^2 + \lambda^4 (s\varphi)^{3+d} |z|^2 \right] e^{2s\alpha} \, dx \, dt \leq C \left( \int_{Q} (s\varphi)^{d} e^{2s\alpha} |f_0|^2 \, dx \, dt + \sum_{i=1}^{N} \int_{Q} \lambda^2 (s\varphi)^{2+d} e^{2s\alpha} |f_i|^2 \, dx \, dt \right) + \int_{Q\omega'} \lambda^2 (s\varphi)^{3+d} e^{2s\alpha} |z|^2 \, dx \, dt
\] (40)

where \( z(x, t) \) is the weak solution of the following equation:

\[
\begin{align*}
-\partial_t z - \Delta z &= f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} & \text{in } Q, \\
\partial_x z &= 0 & \text{on } \Sigma, \\
z(x, T) &= z_T(x) & x \in \Omega,
\end{align*}
\] (41)

Here, \( \partial f_i / \partial x_i, i = 1, \ldots, N \) are the weak derivatives of \( f_i \) (\( i = 1, \ldots, N \)), \( C = C(\Omega, \omega', d) \), and \( \gamma(\lambda) \) is given by Equation (39).

Since the essential part of the proof for inequality (40) is very similar to [14], where instead of \( w = e^{s\alpha} u \), we can introduce \( w = (s\varphi)^d e^{s\alpha} u \) and estimate in a similar way to obtain (40), we put the detailed proof in the Appendix.

For notation simplicity in the sequel, we introduce

\[
I_{s, \lambda}^{\phi}(\phi) = \int_{Q} \left[ \lambda^2 (s\varphi)^{3} |\nabla \phi|^2 + \lambda^4 (s\varphi)^{5} |\phi|^2 \right] e^{2s\alpha} \, dx \, dt, \quad (42)
\]

and

\[
I_{s, \lambda}^{\theta}(\theta) = \int_{Q} \left[ \lambda^2 s\varphi |\nabla \theta|^2 + \lambda^4 (s\varphi)^{3} |\theta|^2 \right] e^{2s\alpha} \, dx \, dt. \quad (43)
\]
To establish observability for the adjoint system (37), we need the following type of Carleman estimate (44). The key step to establishing this inequality is to bound the energy of $\theta$ by the partial energy of $\phi$, for which we use the multiplier method and the fact that $\theta$ can be represented by $\phi$. More precisely, $\theta = \delta^{-1}(-\partial_t \phi - \Delta \phi - B \nabla \phi)$. If we substitute $\theta$ in the second equation of (37), then the adjoint system (37) is actually a non-local equation with respect to $\phi$.

**Lemma 3.3.** There exists a positive constant $\lambda_1 = C(\Omega, \omega') (1 + \|a\|_\infty^2 + \|B\|_{\infty}^2)$ satisfying $\gamma(\lambda_1) \geq \lambda_1 > 1$ such that for any $\lambda \geq \lambda_1$, $s \geq \gamma(\lambda)(T + T^2)$ and $\phi^T, \theta^T \in L^2(\Omega)$, the associated solution $(\phi, \theta)$ to Equation (37) satisfies

$$I_{s, \lambda}^1(\phi) + I_{s, \lambda}^2(\theta) \leq C_1 \int \int_{Q_{\omega'}} \lambda^{14} (s \varphi)^9 e^{2s \alpha} |\phi|^2 \, dx \, dt,$$

where $C_1 = C_1(\Omega, \omega', \omega)$.

**Proof.** Applying Lemma 3.2 to the first equation of Equation (37) with $d = 2$ and the second one with $d = 0$, respectively, we obtain that there exist positive constants $c_0(\Omega, \omega')$ and $\lambda_1^0$ satisfying

$$\gamma(\lambda_1^0) \geq \lambda_1^0 = c_0(\Omega, \omega') \left(1 + \|a\|_\infty^2 + \|B\|_{\infty}^2 \right) > 1$$

such that for all $\lambda \geq \lambda_1^0$ and $s \geq \gamma(\lambda)(T + T^2)$,

$$I_{s, \lambda}^1(\phi) + I_{s, \lambda}^2(\theta) \leq c_1 \int \int_{Q_{\omega'}} \lambda^4 \left[ (s \varphi)^5 |\phi|^2 + (s \varphi)^3 |\theta|^2 \right] e^{2s \alpha} \, dx \, dt$$

for any solution $(\phi, \theta)$ of Equation (37) with $\phi^T, \theta^T \in L^2(\Omega)$, and in what follows, the symbols $c_i, i = 1, 2, 3$ denote some positive constants depending on $\Omega, \omega'$ and $\omega$.

Next, let $\xi \in C_0^\infty(\Omega)$ be such that $\xi = 1$ in $\omega'$, $\xi = 0$ in $\Omega \setminus \overline{\omega}$, $0 \leq \xi \leq 1$ in $\omega$, and

$$\Delta \xi \cdot \xi^{-1/2} \in L^\infty(\Omega), \nabla \xi \cdot \xi^{-1/2} \in L^\infty(\Omega)^N.$$  

The existence of such a function $\xi$ is easy to be obtained. Set

$$\eta = \lambda^4 (s \varphi)^3 e^{2s \alpha}.$$

Multiply the first equation of Equation (37) by $\theta \eta \xi$ to obtain

$$\delta \int \int_Q \lambda^4 (s \varphi)^3 e^{2s \alpha} |\theta|^2 \xi \, dx \, dt = \int \int_Q \eta \xi \theta [-\partial_t \phi - \Delta \phi - B \nabla \phi] \, dx \, dt$$

$$= \int \int_Q \{ \eta \xi \phi [-\Delta \theta + \gamma \theta + \nabla \cdot (a \nabla \phi)] + \phi \theta \xi (\partial_\theta \eta) + \eta \xi \theta (-\Delta \phi - B \nabla \phi) \} \, dx \, dt.$$

Performing the integration by parts gives

$$\delta \int \int_Q (s \varphi)^3 e^{2s \alpha} |\theta|^2 \xi \, dx \, dt = J_1 + J_2,$$

where

$$J_1 = \int \int_Q \phi [(\partial_\theta \eta + \gamma \eta) \xi \theta + \nabla (\eta \xi) \cdot \nabla \theta] \, dx \, dt,$$

$$J_2 = \int \int_Q [((\theta - a \phi) \nabla (\eta \xi)) + \eta \xi (2 \nabla \theta - B \theta) \cdot \nabla \phi] \, dx \, dt.$$

To estimate these integrals, we first observe by (38) and (47) that

$$|\partial_\theta \eta| \leq \lambda^4 (s \varphi)^5 e^{2s \alpha}; \quad |\nabla (\eta \xi)| \leq C \lambda^4 (\xi^{1/2} s^3 \varphi^3 + s \lambda s^4 \varphi^4) e^{2s \alpha}.$$
These together with Cauchy’s inequality give the estimations of $J_1$ and $J_2$ as follows:

$$J_1 \leq \varepsilon_1 \left[ I_{s,\lambda}^1 (\phi) + I_{s,\lambda}^2 (\theta) \right] + \frac{C(a \|\epsilon\|_\infty^2 + 1)}{\varepsilon_1} \iint_Q \lambda^8 (s \phi)^7 e^{2s\alpha} |\phi|^2 \xi \, dx \, dt,$$  

$$J_2 \leq \varepsilon_2 I_{s,\lambda}^2 (\theta) + \frac{c_2 (1 + \|a\|_\infty^2 + \|B\|_\infty^2)}{\varepsilon_2} \iint_Q \lambda^6 (s \phi)^5 e^{2s\alpha} |\nabla \phi|^2 \xi \, dx \, dt,$$  

where $\varepsilon_1$ and $\varepsilon_2$ are arbitrary positive constants which are determined later. Now, we estimate the integral on the right-hand side of inequality (50). Let

$$\bar{\eta} = \lambda^6 (s \phi)^5 e^{2s\alpha}.$$

Multiply the first equation of Equation (37) by $\bar{\eta} \xi \phi$ and perform integration by parts over $Q$ to obtain

$$\iint_Q \lambda^6 (s \phi)^5 e^{2s\alpha} |\nabla \phi|^2 \xi \, dx \, dt$$

$$= \iint_Q \left\{ \phi \xi B - \nabla (\bar{\eta} \xi) \cdot \nabla \phi + (\delta \eta \theta - \frac{1}{2} \phi \partial_t \bar{\eta}) \xi \phi \right\} \, dx \, dt,$$

and by Cauchy’s inequality again,

$$\iint_Q \lambda^6 (s \phi)^5 e^{2s\alpha} |\nabla \phi|^2 \xi \, dx \, dt$$

$$\leq \varepsilon_3 I_{s,\lambda}^1 (\phi) + \frac{C(\|B\|_\infty^2 + 1)}{\varepsilon_3} \iint_Q \lambda^{12} (s \phi)^9 e^{2s\alpha} |\phi|^2 \xi \, dx \, dt.$$  

Finally, we take

$$\varepsilon_1 = \varepsilon_2 = \delta (8c_1)^{-1}, \quad \varepsilon_3 = \delta^2 |64c_1^2 c_2 (1 + \|a\|_\infty^2 + \|B\|_\infty^2)|^{-1}$$

to obtain, from Equations (48)-(51), that

$$I_{s,\lambda}^1 (\phi) + I_{s,\lambda}^2 (\theta) \leq c_3 (1 + \|a\|_\infty^2 + \|B\|_\infty^2)^2 \iint_{Q_w} \lambda^{12} (s \phi)^9 e^{2s\alpha} |\phi|^2 \, dx \, dt.$$  

Thus, there is a positive constant

$$\gamma (\lambda_1) \geq \lambda_1 = c_3 \left(1 + \|a\|_\infty^2 + \|B\|_\infty^2\right) \geq \lambda^0_1 > 1$$

such that for any $\lambda \geq \lambda_1$ and $s \geq \gamma (\lambda) (T + T^2)$, the inequality (44) holds, where $\lambda^0_1$ is given by (45).

It is well known that null controllability for system (4) with $L^2(Q)$ control is equivalent to the “observability inequality” for system (37):

$$|\phi(\cdot,0)|_2^2 + |\theta(\cdot,0)|_2^2 \leq C \iint_{Q_w} |\phi|^2 \, dx \, dt$$

for every solution $(\phi, \theta)$ of Equation (37). However, in order to make the control in the space $L^\infty(Q)$, we need to establish instead an “improved observability inequality” as the following Proposition 3.

**Proposition 3.** There exist positive constants $\lambda$ and $s$ such that, for all $T > 0, \phi^T, \theta^T \in L^2(\Omega)$, the solution $(\phi, \theta)$ of Equation (37) satisfies

$$|\phi(\cdot,0)|_2^2 + |\theta(\cdot,0)|_2^2 \leq e^{C_\kappa} \iint_{Q_w} e^{\frac{2s\alpha}{2}} |\phi|^2 \, dx \, dt,$$  

where $\kappa$ is given by Equation (6).
Proof. By performing the integration by parts, we observe from (37) that
\[ |φ(·,0)|_2^2 + |θ(·,0)|_2^2 \leq e^{C[(1+∥a∥_∞^2+∥B∥_∞^2)T]} \left( |φ(·,T)|_2^2 + |θ(·,T)|_2^2 \right), \quad ∀ t \in (0, T). \]
Integrating on both sides of (53) over \([T/4, 3T/4]\) leads to
\[ |φ(·,0)|_2^2 + |θ(·,0)|_2^2 \leq \frac{2}{T} e^{C[(1+∥a∥_∞^2+∥B∥_∞^2)T]} \int_0^{T/4} \int_Ω |φ|^2 + |θ|^2 \, dxdt. \]
Since
\[ (sφ)^{-5} e^{-2sα} \leq e^{\frac{C}{T^2}} \quad \text{in} \quad Ω × \left[ T/4, 3T/4 \right], \]
it follows from (44) that
\[ |φ(·,0)|_2^2 + |θ(·,0)|_2^2 \leq \frac{2C}{T} e^{C[(1+∥a∥_∞^2+∥B∥_∞^2)T]} + \frac{C}{T} \lambda \int_0^{T/4} \int_Ω e^{2sα} |φ|^2 \, dxdt. \]
Setting \( λ = C \left( 1 + ∥a∥_∞^2 + ∥B∥_∞^2 \right), s = C \left( 1 + ∥a∥_∞^2 + ∥B∥_∞^2 \right)(T + T^2) \),
we obtain (52). This completes the proof of the proposition. \( \square \)

Now we are ready to prove Theorem 1.4 by showing that the linear system (4) is null controllability with \( L^∞ \) controls.

Proof of Theorem 1.4. For \( ε > 0 \), let us consider the following optimal control problem:

\[ \text{Minimize} \left\{ \int_{Q_ω}|f|^2 e^{-\frac{2}{ε}sα} \, dxdt + \frac{1}{ε} \left( |y(·,T)|_2^2 + |z(·,T)|_2^2 \right) \right\} \quad (54) \]

subject to all \( f ∈ L^2(Q) \), where \( (y, z) \) is the solution of Equation (4) corresponding to \( f \). The existence of an optimal pair \( (f_ε, y_ε, z_ε) \) to the optimal control problem (54) follows from the standard argument. By the Pontryagin maximum principle (3),

\[ f_ε = 1_ω φ_ε e^{\frac{3}{2}sα}, \quad (55) \]

where \( (φ_ε, θ_ε) \) is the solution of the adjoint system following

\[ \begin{cases} -∂_t φ_ε = Δ φ_ε + B∇ φ_ε + δθ_ε & \text{in} \ Q, \\ -∂_t θ_ε = Δ θ_ε - γθ_ε - ∇ \cdot (a∇ φ_ε) & \text{in} \ Q, \\ ∇ φ_ε = 0, ∇ θ_ε = 0 & \text{on} \ Σ, \\ (φ_ε, θ_ε)(x, T) = -\frac{1}{2}(y_e, z_e)(x, T) & x ∈ Ω. \end{cases} \quad (56) \]

Here, \( (y_e, z_e) \) is the solution of (4) with \( f = f_ε \). By Equations (4), (55), and (56) and Proposition 3, it follows that

\[ \int_{Q_ω}|φ_ε|^2 e^{\frac{3}{2}sα} \, dxdt + \frac{1}{ε} \left( |y(·,T)|_2^2 + |z(·,T)|_2^2 \right) \leq e^{Cκ} \left( |y_0|_2^2 + |z_0|_2^2 \right), \quad (57) \]

where and in the rest of the proof \( C \) denotes different positive constant depending only on \( Ω \) and \( ω \). We can simply get, from (55) and (57), that the control function \( f_ε \) satisfies

\[ ||f_ε||_2 \leq e^{Cκ} \left( |y_0|_2 + |z_0|_2 \right). \]

Next we show that \( f_ε \) can be taken in \( L^∞(Q) \). To this end, let \( τ \) be a sufficiently small positive constant and let \( \{τ_j\}_{j=0}^{M+1} \) be a finite increasing sequence such that
0 < \tau_j < \tau, j = 0, 1, \ldots, M, \tau_{M+1} = \tau. Let \{p_i\}_{i=0}^{M} be another finite increasing sequence such that \(p_0 = 2, p_M > (N + 2)/2\) and,
\[
\left( \frac{N}{2} + 1 \right) \left( \frac{1}{p_i} - \frac{1}{p_{i+1}} \right) < \frac{1}{4}, \quad i = 0, 1, \ldots, M - 1.
\]
Moreover, let \(\alpha_0 = \min_{\Gamma} \alpha^1\) Then
\[
\alpha_0 \leq \alpha \leq \alpha_0 [1 + \gamma(\lambda)^{-1/2}]^{-1} < 0,
\]
where \(\gamma(\lambda)\) is given by (39). We can assume at the first beginning that the parameters \(\lambda\) and \(s\) are large enough so that the “observability inequality” (52) and
\[
\gamma(\lambda) > 4
\]
hold.

For each \(i, i = 0, 1, \ldots, M, M + 1\), define
\[
\zeta_i(x, t) = e^{(s + \tau_i)\alpha_0} \phi_c(x, T - t), \quad \phi_i(x, t) = e^{(s + \tau_i)\alpha_0} \theta_c(x, T - t),
\]
\[
G_i(x, t) = [\partial_t (e^{(s + \tau_i)\alpha_0})] \phi_c(x, T - t), \quad H_i(x, t) = [\partial_t (e^{(s + \tau_i)\alpha_0})] \theta_c(x, T - t),
\]
\[
\tilde{a}(x, t) = a(x, T - t), \quad \tilde{B}(x, t) = B(x, T - t).
\]
Then \((\zeta_i, \phi_i)\) solves the following equation:
\[
\begin{align*}
\partial_t \zeta_i - \Delta \zeta_i &= \tilde{B} \nabla \zeta_i + \delta \phi_i + G_i & \text{in } Q, \\
\partial_t \phi_i - \Delta \phi_i &= -\gamma \phi_i - \nabla \cdot (\tilde{a} \nabla \zeta_i) + H_i & \text{in } Q, \\
\partial_e \zeta_i = 0, \partial_e \phi_i = 0 & \text{on } \Sigma, \\
\zeta_i(x, 0) = 0, \phi_i(x, 0) = 0 & x \in \Omega.
\end{align*}
\]

Now, we apply the \(L^p-L^q\) estimate to Equation (60). By the semigroup theory, the solution \((\zeta_i, \phi_i)\) of Equation (60) can be represented as
\[
\begin{align*}
\zeta_i(\cdot, t) &= \int_0^t e^{-(t-s)A} \left[ \tilde{B} \nabla \zeta_i + \delta \phi_i + G_i \right] (\cdot, s) ds, \\
\phi_i(\cdot, t) &= \int_0^t e^{-(t-s)A} \left[ -\gamma \phi_i - \nabla \cdot (\tilde{a} \nabla \zeta_i) + H_i \right] (\cdot, s) ds.
\end{align*}
\]
Firstly, applying \(L^{p-1}-L^p\) estimate (11) to (61) and (62), we calculate involving Young’s convolution inequality to obtain
\[
\|\zeta_i\|_{p_1} + \|\phi_i\|_{p_2} \leq e^{C\kappa} \left( \|\phi_i\|_{p_{1-1}} + \|\nabla \zeta_i\|_{p_{1-1}} + \|G_i\|_{p_{1-1}} + \|H_i\|_{p_{1-1}} \right).
\]
Secondly, we estimate the energy of solution \((\zeta_i, \phi_i)\) to give the following \(L^{p-1}\) estimate:
\[
\|\zeta_i\|_{p_{1-1}} + \|\phi_i\|_{p_{1-1}} + \|\nabla \zeta_i\|_{p_{1-1}} \leq e^{C\kappa} \left( \|G_i\|_{p_{1-1}} + \|H_i\|_{p_{1-1}} \right).
\]
Thirdly, it is easy to check that
\[
\|G_i\|_{p_{1-1}} \leq C T \|\zeta_{i-1}\|_{p_{1-1}}, \quad \|H_i\|_{p_{1-1}} \leq C T \|\phi_{i-1}\|_{p_{1-1}},
\]
Thus, it follows from (63), (64), and (65) that
\[
\|\zeta_i\|_{p_1} + \|\phi_i\|_{p_2} \leq e^{C\kappa} \left( \|\zeta_{i-1}\|_{p_{1-1}} + \|\phi_{i-1}\|_{p_{1-1}} \right),
\]
Therefore, iterating the inequality (66) from 0 to \(M\) and using (57) we obtain that
\[
\|\zeta_M\|_{p_M} + \|\phi_M\|_{p_M} \leq e^{C\kappa} (\|\zeta_0\|_2 + \|\phi_0\|_2) \leq e^{C\kappa} (\|y_0\|_2 + \|z_0\|_2).
On the other hand, we apply $L^p$-maximal regularity (15) for the first equation of Equation (60) for $\zeta_{M+1}$ to obtain
\[ \|\zeta_{M+1}\|_{W^{2,1}_p(Q)} \leq C \left( \|B\|_\infty \|\nabla \zeta_{M+1}\|_{pM} + \|\vartheta_{M+1}\|_{pM} + \|G_{M+1}\|_{pM} \right). \]
This, by taking (65) and (64) into account leads to
\[ \|\zeta_{M+1}\|_{W^{2,1}_p(Q)} \leq e^{C\kappa} \left( \|\zeta_{M}\|_{pM} + \|\vartheta_{M}\|_{pM} \right). \]  
Hence, by the embedding inequality ([16, Lemma 3.3, Ch.II]), it follows that
\[ \|\zeta_{M+1}\|_{C(Q)} \leq e^{C(1+T+\frac{1}{\kappa})} \|\zeta_{M+1}\|_{W^{2,1}_p(Q)} \]  
and by (67), (68), and (58) we obtain
\[ \|f_\varepsilon\|_\infty \leq \left\| e^{[\frac{s}{2}-(\gamma(\lambda)^{-1/2})+\tau(1+\gamma(\lambda)^{-1/2})]} \|f_\varepsilon\|_\infty \right\| \leq e^{C\kappa} (|y_0|_2 + |z_0|_2). \]  
by choosing $\tau$ small enough so that $-s \left( 2^{-1} - \gamma(\lambda)^{-1/2} \right) + \tau \left( 1 + \gamma(\lambda)^{-1/2} \right) < 0$, because $\gamma(\lambda) > 4$ by (59).

The inequality (69) enables us to extract subsequences of $f_\varepsilon$, still denoted by itself, such that $f_\varepsilon \rightarrow f$ weakly in $L^2(Q)$, weak* in $L^\infty(Q)$ as $\varepsilon \rightarrow 0$, and $f \in L^\infty(Q)$. Let $(y_\varepsilon, z_\varepsilon)$ be the solution of system (1) corresponding to $f_\varepsilon$. Then, by Proposition 2, we see that both $y_\varepsilon$ and $z_\varepsilon$ are bounded in $V^1(Q)$. Thus, there exist subsequences of $y_\varepsilon$ and $z_\varepsilon$, still denoted by themselves, such that $y_\varepsilon \rightarrow y, z_\varepsilon \rightarrow z$ weakly in $V^1(Q)$. Since $(y, z) \in V^1(Q) \cap C([0,T];L^2(\Omega))$ is the weak solution of system (1) corresponding to $f \in L^\infty(Q)$ and $y(x,T) = z(x,T) = 0$ for almost all $x \in \Omega$, we have proved the required result.

\[ \square \]

**Proof of Theorem 1.5.** Let $(\pi, \varpi)$ be a trajectory of system (2) with the initial value $(\pi_0, \varpi_0)$ which satisfies (3). Set $u = \pi + y, v = \varpi + z, y_0 = u_0 - \pi_0, z_0 = v_0 - \varpi_0$. Then, $(y, z)$ solves the following parabolic system
\[
\begin{align*}
\partial_x y &= \Delta y - \chi \nabla \cdot (y \nabla \varpi) - \chi \nabla \cdot ((\pi + y) \nabla z) + 1_\omega f \quad \text{in } Q, \\
\partial_t z &= \Delta z - \gamma z + \delta y \quad \text{in } Q, \\
\partial_x y &= 0, \partial_t z = 0 \quad \text{on } \Sigma, \\
y(x,0) &= y_0(x), z(x,0) = z_0(x) \quad \text{in } \Omega.
\end{align*}
\]
(70)

The local exact controllability of system (1) is equivalent to the local null controllability of system (70).

Let $K = \{ \eta \in L^\infty(Q) \mid \|\eta\|_\infty \leq e^{-\delta_0 T k_0} \}$, where $\delta_0$ and $k_0$ are the positive constants defined in Theorem 1.2. For each $\eta \in K$, we consider the following linearized system:
\[
\begin{align*}
\partial_x y &= \Delta y - \nabla \cdot (By) - \nabla \cdot (a_\eta \nabla z) + 1_\omega f \quad \text{in } Q, \\
\partial_t z &= \Delta z - \gamma z + \delta y \quad \text{in } Q, \\
\partial_x y &= 0, \partial_t z = 0 \quad \text{on } \Sigma, \\
y(x,0) &= y_0(x), z(x,0) = z_0(x) \quad \text{in } \Omega,
\end{align*}
\]
(71)

where $a_\eta = \chi(\pi + \eta)$ and $B = \chi \nabla \varpi$. (3), we see that $a_\eta \in L^\infty(Q), B \in L^\infty(Q)^N$ with $B \cdot \nu = 0$ on $\Sigma$.

Hence system (71) is casted into the exact framework of system (4). By Theorem 1.4, for each $\eta \in K$, there exists a pair $(y, z, f)$ which solves Equation (71) with $y(x,T) = z(x,T) = 0$ for almost all $x \in \Omega$. Here and in what follows, we denote
by \((y, z)\) the solution to Equation (71) corresponding to \(f\) and \(\eta\) if there is no ambiguity. By (5), we see that the control functions are bounded:

\[
\|f\|_\infty \leq e^{C\kappa_0} (|y_0|_2 + |z_0|_2), \quad \kappa_0 = c_0 \left(1 + \frac{1}{T}\right).
\]  
(72)

By (19) and (72), we have the following estimate:

\[
\|y\|_{V^1(Q)} + \|z\|_{V^2(Q)} + \|y\|_\infty + \|z\|_\infty \leq e^{C\kappa_0} \left(|y_0|_\infty + \|z_0\|_{W^{1,\infty}(\Omega)}\right).
\]  
(73)

For \(\eta \in K\), define a multi-valued mapping \(\Lambda : K \to 2^{L^2(Q)}\) by

\[
\Lambda(\eta) = \left\{ y \in L^2(Q) \mid \exists f \text{ satisfying (72) such that } (y, z) \text{ is the solution to Equation (71) corresponding to } \eta \text{ and } f, \right. \\
\left. \text{and } y(x, T) = z(x, T) = 0 \text{ a.e. in } \Omega \right\}.
\]

We apply Kakutani’s fixed-point theorem ([3, p.7]) to the map \(\Lambda\) to prove Theorem 1.5. First, it is clear that \(K\) is a convex subset of \(L^2(Q)\). By the argument aforementioned, we see that \(\Lambda(\eta)\) is nonempty and convex for each \(\eta \in K\). Moreover, by (73), \(\Lambda(\eta)\) is bounded in \(V^1(Q)\) for each \(\eta \in K\) and hence \(\Lambda(\eta)\) is a compact subset of \(L^2(Q)\) by the Aubin-Lions lemma ([3, p.17]).

Next, we show that \(\Lambda\) is upper semi-continuous. To this purpose, let \(\{\eta_n\}_{n=1}^\infty\) be a sequence of functions in \(K\) such that \(\eta_n \to \eta\) strongly in \(L^2(Q)\), and let \(y_n \in \Lambda(\eta_n)\) for each \(n\). Then, by the definition of \(\Lambda(\eta_n)\), there exists \(f_n\) for each \(n\) such that \((y_n, z_n)\) solves the following equation:

\[
\begin{aligned}
\partial_t y_n &= \Delta y_n - \nabla \cdot (B y_n) - \nabla \cdot (a_{\eta_n} \nabla z_n) + 1_\omega f_n \quad \text{in } Q, \\
\partial_t z_n &= \Delta z_n - \gamma z_n + \delta y_n \quad \text{in } Q, \\
\partial_v y_n &= 0, \quad \partial_v z_n = 0 \quad \text{on } \Sigma, \\
y_n(x, 0) &= y_0(x), \quad z_n(x, 0) = z_0(x) \quad \text{for almost all } x \in \Omega,
\end{aligned}
\]  
(74)

and \(y_n(x, T) = z_n(x, T) = 0\) for almost all \(x \in \Omega\). Moreover, the control \(f_n\) satisfies

\[
\|f_n\|_\infty \leq e^{C\kappa_0} \left(|y_0|_2 + |z_0|_2\right).
\]  
(75)

By (75) and Proposition 2, we obtain

\[
\|y_n\|_{V^1(Q)} + \|z_n\|_{V^2(Q)} \leq e^{C\kappa_0} \left(|y_0|_2 + \|z_0\|_{W^{1,2}(\Omega)}\right).
\]  
(76)

By (75) and (76) and applying the Aubin-Lions lemma again, we can obtain that \(f \in L^\infty(Q), \ y \in V^1(Q), \) and \(z \in V^2(Q), \) and the subsequences of \(f_n, y_n, z_n, \) still denoted by themselves, such that

\[
\begin{align*}
\quad f_n &\to f \text{ weak }^* \text{ in } L^\infty(Q) \text{ and weakly in } L^2(Q); \\
y_n &\to y \text{ weakly in } V^1(Q) \text{ and strongly in } L^2(Q); \\
z_n &\to z \text{ weakly in } V^2(Q) \text{ and strongly in } L^2(0, T; H^1(\Omega)).
\end{align*}
\]

Passing to the limit as \(n \to \infty\) in Equation (74), we obtain that \((y, z)\) is a weak solution of Equation (74) corresponding to \(\eta\). We claim that \(y \in \Lambda(\eta)\). Actually, let \(Y_n = y_n - y, \ Z_n = z_n - z, \) and \(F_n = 1_\omega(f_n - f)\). Then \((Y_n, Z_n)\) solves the following equation:

\[
\begin{aligned}
\partial_t Y_n &= \Delta Y_n - \nabla \cdot (B Y_n) - \nabla \cdot [a_{\eta_n} \nabla Z_n + (a_{\eta_n} - a_0) \nabla z] + F_n \quad \text{in } Q, \\
\partial_t Z_n &= \Delta Z_n - \gamma Z_n + \delta Y_n \quad \text{in } Q, \\
\partial_v Y_n &= 0, \quad \partial_v Z_n = 0 \quad \text{on } \Sigma, \\
Y_n(x, 0) &= 0, \quad Z_n(x, 0) = 0 \quad \text{for almost all } x \in \Omega.
\end{aligned}
\]  
(77)
Since \(\|a_n\|_{\infty} \leq C\) for all \(n\), we obtain by energy estimate that
\[
\left| Y_n(\cdot, t) \right|^2 + \left| Z_n(\cdot, t) \right|^2 + \left| \nabla Z_n(\cdot, t) \right|^2 \\
\leq e^{C(1 + \|\theta\|_{L^2}^2)T} \left( \int_{\Omega} \left| \eta_n - \eta \right|^2 |\nabla z|^2 dx + \int_{\Omega} F_n Y_n dx \right).
\]
(78)
On the other hand, since \((y, z)\) solves Equation (71), by (ii) of Proposition 2, we obtain that
\[
\|z\|_{W^{2,1}(Q)} \leq C \left( |y_0|_p + \|z_0\|_{W^{2(1 - \frac{1}{p})},p(\Omega)} + \|1_\omega f\|_p \right).
\]
Since \(1_\omega f\) is bounded as in (72), we obtain by the Sobolev embedding \(W^{2,1}_p(Q) \hookrightarrow C^1(\overline{Q})\) \((p > N + 2)\) that
\[
\|\nabla z\|_{C(\overline{Q})} \leq \|z\|_{W^{2,1}(Q)} \leq C \left( |y_0|_p + \|z_0\|_{W^{2(1 - \frac{1}{p})},p(\Omega)} \right).
\]
(79)
Since \(\eta_n \to \eta\) strongly in \(L^2(Q), Y_n \to 0\) strongly in \(L^2(Q),\) and \(F_n \to 0\) weakly in \(L^2(Q),\) by (79), we see that the right-hand side of (78) tends to 0 as \(n \to \infty.\) Hence, \(|Y_n(\cdot, t)|_2 \to 0, |Z_n(\cdot, t)|_2 \to 0\) for all \(t \in [0, T].\) Since \(y_n(x, T) = z_n(x, T) = 0\) for almost all \(x \in \Omega,\) we have \(y(x, T) = z(x, T) = 0\) for all most all \(x \in \Omega.\) This implies that \(y \in \Lambda(\eta).\) Therefore, \(\Lambda\) is upper semi-continuous.

It remains to show that \(\Lambda(K) \subset K.\) By Proposition 2, for any \(y \in \Lambda(K),\)
\[
\|y\|_{\infty} \leq e^{c_1 K_0} \left( |y_0|_{\infty} + \|z_0\|_{W^{2(1 - \frac{1}{p})},p(\Omega)} \right),
\]
where \(c_1^0\) is a positive constant. Take \(e^{-c_1 K_0} \leq e^{-c_1 K_0 - \delta_0 T k_0}\) so that if \(|y_0|_{\infty} + \|z_0\|_{W^{2(1 - \frac{1}{p})},p(\Omega)} \leq e^{-c_1 K_0}\) which is just (7), then \(|y|_{\infty} \leq e^{-\delta_0 T k_0}\) and hence \(\Lambda(K) \subset K.\) Thus, by the Kakutani’s fixed point theorem, if the initial data \((u_0, v_0)\) satisfies (7), then there exists at least one fixed point \(y,\) which together with \(z,\) consists of the solution of Equation (1) corresponding with some control \(f\) and satisfies \(y(x, T) = z(x, T) = 0\) for almost all \(x \in \Omega.\) Therefore, \((u, v) = (y + \pi, z + \pi)\) is the solution of Equation (1) corresponding to \(f\) with \(u(x, T) = \overline{u}(x, T), v(x, T) = \overline{v}(x, T)\) for \(x \in \Omega.\) Moreover, since \(|y|_{\infty} \leq e^{-\delta_0 T k_0},\) it follows that \(u = y + \pi \geq 0\) in \(Q.\) Applying Lemma 3.1 to the second equation of Equation (1), we obtain that \(v \geq 0\) in \(Q.\) This completes the proof of the theorem.

With the same argument as that in the proof of Theorem 1.5, we obtain null controllability for system (1).

**Corollary 1.** Let \(p > N + 2.\) Then there exists a positive constant \(c_1\) independent of \(T\) such that for each \((u_0, v_0)\) that satisfies
\[
|u_0|_{\infty} + \|v_0\|_{W^{2(1 - \frac{1}{p}),p(\Omega)}} \leq e^{-c_1(1 + T + \frac{k}{\pi})},
\]
(80)
there is a control \(f \in L^\infty(Q)\) such that system (1) admits a solution \((u, v)\) satisfying
\[u \in V^1(\Omega) \cap L^\infty(Q), v \in V^2(\Omega) \cap L^\infty(Q),\]
and \(u(x, T) = v(x, T) = 0\) for almost all \(x \in \Omega.\)

To end this paper, we indicate some interesting problems for the Keller-Segel equations. First, if the control forces are acted on both states through the same portion \(\Gamma\) of the boundary \(\partial \Omega,\) i.e.,
\[
\partial_{\nu} u = 1_{\Gamma} f_1 \quad \text{and} \quad \partial_{\nu} v = 1_{\Gamma} f_2 \quad \text{on} \quad \Sigma,
\]
the controllability for such boundary problem may be solved by a method of inflating boundary, see, for instance, [7]. If the control is acted only on one state through the boundary, i.e.,
\[ \partial_{\nu} u = 1_{\Gamma} f \quad \text{and} \quad \partial_{\nu} v = 0 \quad \text{on} \ \Sigma, \]
the current augment is not applied directly. Such a problem remains open on controllability. Another problem is to locate the control function only on the second equation of (1). Unfortunately, since \( \phi \) can not be represented directly by \( \theta \), our strategy to prove the Carleman inequality (44) may not apply to establish the associated observability estimate for the adjoint equation (37).

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Appendix. In this Appendix, we present a proof for Lemma 3.2.

Lemma A. Let \( f \in L^2(Q) \) and \( d \) be a real number. Then there is a constant \( \lambda_0 = \lambda_0(\Omega, \omega', d) > 1 \) such that for all \( \lambda \geq \lambda_0 \) and \( s \geq \gamma(\lambda)(T + T^2) \),
\[
\int_Q \left[ \lambda^2(s \varphi)^{1+2d} |\nabla z|^2 + \lambda^4(s \varphi)^{3+2d} |z|^2 \right] e^{2s\lambda} dxdt
\]
\[
\leq C(\Omega, \omega', d) \left( \int_Q (s \varphi)^{2d} e^{2s\lambda} |f|^2 dxdt + \int_Q \lambda^4(s \varphi)^{3+2d} e^{2s\lambda} |z|^2 dxdt \right),
\]
where \( z \) is the weak solution of the following equation:
\[
\begin{align*}
-\partial_t z - \Delta z &= f \quad \text{in } Q, \\
\partial_\nu z &= 0 \quad \text{on } \Sigma, \\
z(x, T) &= z_T(x) \in L^2(\Omega) \quad \text{on } \Omega.
\end{align*}
\]

Proof. Let \( z = \varphi^{-d} e^{-s\lambda} w \). Then the equation \(-\partial_t z - \Delta z = f\) can be transformed into
\[
L_1 w + L_2 w = f_\lambda,
\]
where
\[
L_1 w = \Delta w + \lambda^2(s \varphi)^2 |\nabla \beta|^2 w - s\alpha_t w - d\varphi \varphi^{-1} w,
\]
\[
L_2 w = -2\lambda(d + s \varphi) \nabla \beta \cdot \nabla w + w_t,
\]
and
\[
f_\lambda = \left[ (d + s \varphi) \Delta \beta + \lambda^2 s \varphi |\nabla \beta|^2 - \lambda^2(2d s \varphi + d^2) |\nabla \beta|^2 \right] w + e^{s\lambda} \varphi^d.
\]

Compute by performing integration by parts to obtain
\[
(L_1 w, L_2 w)_2 \geq 3 \int_Q \lambda^4(s \varphi)^3 |\nabla \beta|^4 w^2 dxdt - \int_Q \lambda^2 s \varphi |\nabla \beta|^2 |\nabla w|^2 dxdt
\]
\[
- C \int_Q \lambda(d + s \varphi) |\nabla w|^2 dxdt - C \int_Q \lambda^3(s \varphi)^3 w^2 dxdt
\]
\[
+ D_1(s, \lambda, w),
\]
where and in what follows, \( D_i(s, \lambda, w) \), \( i = 1, 2, 3 \), are used to denote the boundary terms and \( C \) is a constant depending only on \( \Omega, \omega' \), and \( d \). Since \( |\nabla w|^2 = \frac{1}{2} \Delta w^2 - w \Delta w \), by (84), we have
\[
\int_Q \lambda^2 s \varphi |\nabla \beta|^2 |\nabla w|^2 dxdt - \int_Q \lambda^4(s \varphi)^3 |\nabla \beta|^4 w^2 dxdt
\]
\[
= \frac{1}{2} \int_Q \lambda^2 s \varphi |\nabla \beta|^2 \Delta w^2 dxdt - \int_Q \lambda^2 s \varphi |\nabla \beta|^2 w(L_1 w) dxdt
\]
\[
- \int_Q \lambda^2 s \varphi |\nabla \beta|^2 (s\alpha_t + d\varphi \varphi^{-1}) w^2 dxdt,
\]
which is estimated by integration by parts and application of Cauchy’s inequality to be

$$\frac{3}{4} \iint_Q \lambda^2 s \varphi |\nabla \beta|^2 |\nabla w|^2 \, dxdt - \iint_Q \lambda^4 (s \varphi)^3 |\nabla \beta|^4 w^2 \, dxdt$$

$$\leq \frac{1}{8} \|L_1 w\|_2^4 + C \iint_Q \lambda^3 (s \varphi)^3 w^2 \, dxdt + D_2(s, \lambda, w). \tag{88}$$

Hence, by (87), (88), and \( \|L_1 w\|^2 + \|L_2 w\|^2 + 2(L_1 w, L_2 w)_2 = \|f_\lambda\|^2 \), we have

$$\iint_Q \lambda^3 (s \varphi)^3 |\nabla \beta|^4 w^2 \, dxdt + \frac{1}{2} \iint_Q \lambda^2 s \varphi |\nabla \beta|^2 |\nabla w|^2 \, dxdt + \frac{1}{4} \|L_1 w\|_2^2$$

$$\leq C_1 \iint_Q \lambda s \varphi |\nabla w|^2 \, dxdt \tag{89}$$

$$+ C \iint_Q [\lambda^3 (s \varphi)^3 + (d + d^2) \lambda^4 (s \varphi)^2] w^2 \, dxdt + CF + X_0, \tag{90}$$

where \( C_1 = C_1(\Omega, \omega', d) \),

$$F = \iint_Q \varphi^{2d} e^{2s \alpha} f^2 \, dxdt,$$

and \( X_0 = -D_1(s, \lambda, w) + 2D_2(s, \lambda, w) \). More precisely,

$$X_0$$

$$= \iint_{\Omega} \lambda (d + s \varphi) \left[ \lambda^2 (s^2 \varphi^2 + 2s \varphi) |\nabla \beta|^2 - (s \alpha + d \varphi \varphi^{-1}) \right] + \frac{1}{2} \varphi \lambda t \frac{\partial \beta}{\partial \nu} w^2 \, d\sigma dt$$

$$+ \iint_{\Omega} \lambda (d + s \varphi) \left[ 2(\nabla w \cdot \nu)(\nabla w \cdot \nabla \beta) - (\nabla \beta \cdot \nu) |\nabla w|^2 \right] \, dxdt.$$

To eliminate the gradient term in (90), we use \( |\nabla w|^2 = \frac{1}{2} \Delta w^2 - w \Delta w \) and (84) again to obtain

$$\iint_Q s \lambda \varphi |\nabla w|^2 \, dxdt \leq \frac{1}{8C_1} \|L_1 w\|_2^4 + C \iint_Q \lambda^3 (s \varphi)^3 w^2 \, dxdt + D_3(s, \lambda, w), \tag{91}$$

and

$$D_3(s, \lambda, w) = 2 \iint_{\Sigma} s^2 \lambda^2 (d + s \varphi) \varphi \frac{\partial \beta}{\partial \nu} w^2 \, d\sigma dt \leq 0,$$

because \( \frac{\partial \beta}{\partial \nu} \leq 0 \) on \( \Sigma \). Thus, it follows from (90) and (91) that

$$\iint_Q \lambda^4 (s \varphi)^3 |\nabla \beta|^4 \, dxdt + \iint_Q \lambda^2 s \varphi |\nabla \beta|^2 |\nabla w|^2 \, dxdt + \frac{1}{8} \|L_1 w\|_2^2$$

$$\leq C \iint_Q [\lambda^3 (s \varphi)^3 + (d + d^2) \lambda^4 (s \varphi)^2] w^2 \, dxdt + CF + 2X_0. \tag{92}$$

To eliminate the boundary term \( X_0 \), we let

$$\bar{\varphi} = e^{-\lambda \beta \frac{t}{T - t}}, \quad \bar{\alpha} = \frac{e^{-\lambda \beta} - e^{2\lambda \beta} \|c(a)\}}{t(T - t)}, \quad \bar{w} = \varphi^{d} e^{s \bar{\alpha} z},$$

and then estimate in the same way as above to obtain that

$$\iint_Q \lambda^4 (s \varphi)^3 |\nabla \beta|^4 \bar{w}^2 \, dxdt + \iint_Q \lambda^2 s \bar{\varphi} |\nabla \beta|^2 |\nabla \bar{w}|^2 \, dxdt + \frac{1}{8} \|L_1 \bar{w}\|_2^2$$

$$\leq C \iint_Q [\lambda^3 (s \bar{\varphi})^3 + (d + d^2) \lambda^4 (s \bar{\varphi})^2] \bar{w}^2 \, dxdt + C \bar{F} + 2X_0. \tag{93}$$
where $\check{X}_0$ satisfies $\check{X}_0 = -X_0$ on $\Sigma$. The addition of the inequalities (92) and (93) therefore leads to

$$
\iint_Q \lambda^4(s\varphi)^3 |\nabla \beta|^4 w^2 dxdt + \|L_1 w\|_2^2 \\
\leq C \iint_Q \left[ 3 \lambda^3(s\varphi)^3 + (d + d^2) \lambda^4(s\varphi)^2 \right] w^2 dxdt + C \iint_Q \varphi^{2d} e^{2s\alpha} f^2 dxdt. \quad (94)
$$

Now, we take $\lambda_0 = \lambda_0(\Omega, \omega', d)$ so that $\lambda_0 \geq \max\{1, d\}$. Then for all $\lambda \geq \lambda_0$ and $s \geq \gamma(\lambda)(T + T^2)$, $s\varphi \geq \max\{\lambda, d, 1\}$ and $C(\lambda^3(s\varphi)^3 + (d + d^2) \lambda^4(s\varphi)^2) \leq \frac{1}{2} \lambda^4(s\varphi)^3$ hold. Hence,

$$
\iint_Q \lambda^4(s\varphi)^3 w^2 dxdt + \|L_1 w\|_2^2 \leq C \iint_Q \lambda^4(s\varphi)^3 w^2 dxdt + C \iint_Q \varphi^{2d} e^{2s\alpha} f^2 dxdt. \quad (95)
$$

Apply (90) and (84) again to obtain

$$
\iint_Q \lambda^2 s\varphi |\nabla w|^2 dxdt \leq \|L_1 w\|_2^2 + C \iint_Q \lambda^4(s\varphi)^3 w^2 dxdt.
$$

This inequality, together with (95), gives (82) by changing back to the original variables.

**Proof of Lemma 3.2.** Set

$$
J(y, u) = \frac{1}{2} \iint_Q (s\varphi)^{-2d} e^{-2s\alpha} y^2 dxdt + \frac{1}{2} \iint_Q \lambda^{-4}(s\varphi)^{-3-2d} e^{-2s\alpha} u^2 dxdt.
$$

We consider the following optimal control problem:

Minimize $J(y, u)$, \hspace{1cm} (96)

subject to $(y, u) \in V^1(Q) \times L^2(Q)$ satisfying

$$
\begin{cases}
\partial_t y - \Delta y = \lambda^4(s\varphi)^3 + 2d e^{2s\alpha} z + 1_w u & \text{in } \Omega \times (0, T), \\
\partial_n y = 0 & \text{on } \partial \Omega \times (0, T), \\
y(x, 0) = y(x, T) = 0 & \text{in } \Omega,
\end{cases}
$$

where $z$ is the weak solution of (41). We apply the Lagrange principle to the problem (96) to obtain the optimal pair $(\hat{y}, \hat{u})$ which satisfies

$$
L\hat{y} = \partial_t \hat{y} - \Delta \hat{y} = \lambda^4(s\varphi)^3 + 2d e^{2s\alpha} z + 1_w \hat{u}, \partial_n \hat{y}|_{\Sigma} = 0, \hat{y}(-, 0) = \hat{y}(\cdot, T) = 0, \quad (98)
$$

$$
L^* p = -\partial_t p - \Delta p = -(s\varphi)^{-2d} e^{-2s\alpha} \hat{y}, \partial_n p|_{\Sigma} = 0, \quad (99)
$$

$$
\hat{u} = \lambda^4(s\varphi)^3 + 2d e^{2s\alpha} p 1_w. \quad (100)
$$

Next, applying (82) in Lemma A to the solution $p$ of (99) gives

$$
\iint_Q \lambda^4(s\varphi)^{3+2d} e^{2s\alpha} p^2 dxdt \leq 2C J(\hat{y}, \hat{u}) = -C \iint_Q \lambda^4(s\varphi)^{3+2d} e^{2s\alpha} p(z) dxdt
$$

$$
\leq C \left( \iint_Q \lambda^4(s\varphi)^{3+2d} e^{2s\alpha} p^2 dxdt \right)^{1/2} \left( \iint_Q \lambda^4(s\varphi)^{3+2d} e^{2s\alpha} z^2 dxdt \right)^{1/2}.
$$

Hence,

$$
\iint_Q \lambda^4(s\varphi)^{3+2d} e^{2s\alpha} p^2 dxdt \leq C \iint_Q \lambda^4(s\varphi)^{3+2d} e^{2s\alpha} z^2 dxdt,
$$
and
\[ \iint_Q (s\varphi)^{-2d} e^{-2\alpha s} \hat{y}^2 \, dx dt + \iint_Q \lambda^{-4} (s\varphi)^{-3-2d} e^{-2\alpha s} \hat{z}^2 \, dx dt \leq C \iint_Q \lambda^4 (s\varphi)^{3+2d} e^{2\alpha s} \hat{z}^2 \, dx dt. \] (101)

Multiplying (98) by \( \lambda^{-2} (s\varphi)^{-2-2d} e^{-2\alpha s} \hat{y} \) and integrating by parts, we obtain
\[ \iint_Q \lambda^{-2} (s\varphi)^{-2-2d} e^{-2\alpha s} |\nabla \hat{y}|^2 \, dx dt \]
\[ = \frac{1}{2} \iint_Q \lambda^{-2} \left( (s\varphi)^{-2-2d} e^{-2\alpha s} \right) \hat{y}^2 \, dx dt - \iint_Q \lambda^{-2} \hat{y} \nabla \hat{y} \cdot \nabla \left( (s\varphi)^{-2-2d} e^{-2\alpha s} \right) \, dx dt \]
\[ + \iint_Q \lambda^2 s\varphi \hat{y} z \, dx dt + \iint_Q \hat{\omega} \lambda^{-2} (s\varphi)^{-2-2d} e^{-2\alpha s} \hat{y} \, dx dt. \]

Then, by (101) and Cauchy’s inequality, it follows that
\[ \iint_Q \lambda^{-2} (s\varphi)^{-2-2d} e^{-2\alpha s} |\nabla \hat{y}|^2 \, dx dt \leq C \iint_Q \lambda^4 (s\varphi)^{3+2d} e^{2\alpha s} \hat{z}^2 \, dx dt. \] (102)

Now, by the weak solution of (41) in the sense of transposition, we see that
\[ \iint_Q z (\partial_t \hat{y} - \Delta \hat{y}) \, dx dt = \iint_Q f_0 \hat{y} \, dx dt - \sum_{i=1}^N \iint_Q \partial_x_i \hat{y} f_i \, dx dt, \]
which implies
\[ \iint_Q \lambda^4 (s\varphi)^{3+2d} e^{2\alpha s} \hat{z}^2 \, dx dt = \iint_Q \hat{\omega} \lambda^{-2} (s\varphi)^{-2-2d} e^{-2\alpha s} \hat{y} \, dx dt + \iint_Q f_0 \hat{y} \, dx dt - \sum_{i=1}^N \iint_Q \partial_x_i \hat{y} f_i \, dx dt. \]

Therefore, by (101), (102), and Hölder’s inequality, we obtain
\[ \iint_Q \lambda^4 (s\varphi)^{3+2d} e^{2\alpha s} \hat{z}^2 \, dx dt \]
\[ \leq C \iint_Q (s\varphi)^{2d} e^{2\alpha s} f_0 \, dx dt + C \sum_{i=1}^N \iint_Q \lambda^2 (s\varphi)^{2+2d} e^{2\alpha s} f_i^2 \, dx dt \]
\[ + C \iint_{Q', \varphi} \lambda^4 (s\varphi)^{3+2d} e^{2\alpha s} \hat{z}^2 \, dx dt. \] (103)

By the weak solution of (100) again, we have
\[ \iint_Q \lambda^2 (s\varphi)^{1+2d} e^{2\alpha s} |\nabla z|^2 \, dx dt \]
\[ = -\frac{1}{2} \iint_Q [\lambda^2 (s\varphi)^{1+2d} e^{2\alpha s}] |z|^2 \, dx dt - \iint_Q z \nabla z \cdot \nabla \left( \lambda^2 (s\varphi)^{1+2d} e^{2\alpha s} \right) \, dx dt \]
\[ + \iint_Q \lambda^2 (s\varphi)^{1+2d} e^{2\alpha s} z f_0 \, dx dt - \sum_{i=1}^N \iint_Q \partial_x_i \left( \lambda^2 (s\varphi)^{1+2d} e^{2\alpha s} z \right) f_i \, dx dt \]
which leads to the following estimate
\[ \iint_Q \lambda^2 (s\varphi)^{1+2d} e^{2\alpha s} |\nabla z|^2 \, dx dt \leq C \iint_Q \lambda^4 (s\varphi)^{3+2d} e^{2\alpha s} \hat{z}^2 \, dx dt \]
\[ + C \int \int_Q (s\varphi)^{2d} e^{2\sigma} f_0^2 dx dt + C \sum_{i=1}^{N} \int \int_Q \lambda^2 (s\varphi)^{2+2d} e^{2\sigma} f_i^2 dx dt. \]

This inequality, together with (103), leads to (40) with 2d being replaced by d. This completes the proof of Lemma 3.2.

\[ \square \]

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