Condensate of $\mu$-Bose gas as a model of dark matter

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Abstract

Though very popular, Bose-Einstein condensate models of dark matter have some difficulties. Here we propose the so-called $\mu$-Bose gas model ($\mu$-BGM) as a model of dark matter, able to treat weak points. Within $\mu$-BGM, the $\mu$-dependence of thermodynamics arises through the respective $\mu$-calculus (it generalizes usual differential calculus) and enters the partition function, total number of particles, internal energy, etc. We study thermodynamic geometry of the $\mu$-BGM and find singular behavior of (scalar) curvature, confirming Bose-like condensation. The critical temperature of condensation $T_c(\mu)$ for $\mu \neq 0$ is higher than the boson $T_c$. We find other important virtues of $\mu$-thermodynamics versus usual bosons and conclude: the condensate of $\mu$-Bose gas can serve as (an effective) model of galactic-halos dark matter.

Keywords: boson condensate, deformed Bose gas model, nonstandard statistics, critical temperature, thermodynamic geometry, dark matter halo

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1. Introduction

Among the approaches to model dark matter, those exploiting Bose-Einstein condensate (BEC) are very popular, see e.g. [1, 2, 3, 4, 5, 6] and the review [7]. They share nice features of cold dark matter, show a number of advantages, but also encounter their own difficulties, e.g. the problem of gravitational collapse [8], overestimated dark halo mass [3] etc.

As the true nature of dark matter constituents is unknown, many exotic candidates were considered, e.g. axionic [9] or even stringy ones [10]. In some papers, the authors exploit certain models of nonstandard thermostatistics with aim to describe basic objects of quantum cosmology [11], the physics of dark matter [12, 13] or black holes [14, 15, 16]. It is worth to examine various nontrivial models of nonstandard thermostatistics as possible candidates for modeling, at least effectively, main properties of dark matter in order to choose most adequate one.

In this paper we explore the $\mu$-Bose gas model as a possible model of dark matter, and come to important facts. The $\mu$-Bose gas model was first proposed in [17, 18] where the correlation functions intercepts of 2nd and higher order were derived. The study of $\mu$-Bose gas thermodynamics started in [19] and used the special so-called $\mu$-calculus. That allowed to explore basic quantities, e.g. $\mu$-analogues of elementary and special functions.

There are diverse deformed Bose gas models, see e.g. [17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. As usual, the deformations are based on respective deformed oscillator (DO) models such as $q$-oscillators [30, 31] or the 2-parameter $p, q$-deformed (or Fibonacci) oscillators [32]. A plenty of non-standard one-parameter DOs exist [33], along with polynomially deformed ones [34]. Among the so-called quasi-Fibonacci oscillators [35] we find the $\mu$-oscillator [36]. Note that unlike DOs of polynomial type, the $\mu$-oscillator (and $\mu$-bosons) belong to the less studied class of rational type DOs. The DO models often possess unusual properties, e.g. energy level degeneracies, nontrivial recurrent relations for energy spectra etc. Nontrivial features of DOs motivate their application in diverse fields of quantum physics.

Physical meaning of deformation parameter(s) of deformed model depends on its specific application to physical system. Say, when the model of ideal gas of deformed bosons is applied to compute the intercepts of the momentum correlation functions [17, 25, 26], one effectively takes into account the non-zero proper volume of particles [37], or their internal structure, or compositeness [38, 39]. The experimental data on the two-pion correlation-
function intercepts unravel [40] non-Bose type behavior of pions, and the use of deformed BGM has shown its efficiency [26, 41, 42]. There exists an application of $q$-Bose gas setup to description of the phonon spectrum of $^4$He, and the agreement with experiment is obvious [23].

In the $\mu$-BGM and other deformed analogs of Bose gas model, quantum statistical interaction gets modified [29, 43]. Moreover, deformation can also absorb [28] an interaction present in the initially non-deformed system.

About the plan of our paper. In Sec. 2 we give a setup of the $\mu$-deformed Bose gas model and of $\mu$-calculus. Thermodynamical quantities are considered in Sec. 3: the total number of particles is given explicitly and from it – the partition function (all with explicit $\mu$-dependence). In Sec. 4, within geometric approach to thermodynamics (see e.g. [44, 45, 46, 47]) we confirm the existence of Bose-like condensation in $\mu$-Bose gas. Critical temperature of condensation and its dependence on the deformation parameter $\mu$ are studied. Other aspects or thermodynamical functions, useful for the application, are considered in the next section. Discussion of most important features and virtues of $\mu$-Bose gas enabling its application to model dark matter, and the concluding remarks, are given in the final section of the paper.

2. Deformed analogs of Bose gas model

Like in other works on deformed oscillators, see e.g. [23, 28, 27], we deal in fact with the (system of) deformed bosons. The important virtues of such deformation is its ability to provide effective account of interaction between particles, their non-zero volume, their inner (composite) structure etc.

The $\mu$-deformed BGM associated with $\mu$-oscillator [36] was introduced in [17, 18]. Therein and in this paper the thermal average of the operator $\mathcal{O}$ is determined by the formula

$$\langle \mathcal{O} \rangle = \frac{Tr(\mathcal{O}e^{-\beta H})}{Z},$$

$Z$ being the grand canonical partition function. Its logarithm is

$$\ln Z = -\sum_i \ln(1 - ze^{-\beta \varepsilon_i})$$

with the fugacity $z = e^{\beta \tilde{\mu}}$ ($\tilde{\mu}$ is chemical potential). The familiar formula

$$N = z \frac{d}{dz} \ln Z$$

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for the number of particles will be modified (deformed), see below.

To study the \( \mu \)-BGM as the model for the system of deformed bosons, we use the Hamiltonian with chemical potential \( \tilde{\mu} \)

\[
H = \sum_i (\varepsilon_i - \tilde{\mu}) N_i. \tag{4}
\]

Here \( \varepsilon_i \) is kinetic energy of particle in the state "i" and \( N_i \) the particle number (occupation number) operator corresponding to state "i".

Elements of \( \mu \)-calculus. To develop the \( \mu \)-analog of BGM, we extend (deform) the notion of derivative. So-called \( \mu \)-derivative, introduced in [19], differs from the known Jackson or \( q \)-derivative [48] and its \( p, q \) extension (used in [29]). The easiest way to define the \( \mu \)-extension is to apply the rule

\[
\mathcal{D}_x^{(\mu)} x^n = [n]_\mu x^{n-1}, \quad [n]_\mu \equiv \frac{n}{1 + \mu n} \quad (\mu\text{-bracket}) \tag{5}
\]

so that the \( \mu \)-derivative involves \( \mu \)-bracket from the work on \( \mu \)-oscillator [36]. If \( \mu \to 0 \), then \([n]_\mu \to n\) and the \( \mu \)-extension \( \mathcal{D}_x^{(\mu)} \) reduces to ordinary \( d/dx \).

Formula for \( \mu \)-derivative acting on monomials \( x^m \) is enough for us in this work, while the general rule for \( \mu \)-derivative action on a function \( f(x) \) is

\[
\mathcal{D}_x^{(\mu)} f(x) = \int_0^1 dt f'_x(t^\mu x), \quad f'_x(t^\mu x) = \frac{df(t^\mu x)}{dx}. \tag{6}
\]

Clearly, formula (5) stems from this general definition. Note that the inverse \( (\mathcal{D}_x^{(\mu)})^{-1} \) of the \( \mu \)-derivative \( \mathcal{D}_x^{(\mu)} \) in (5) and (6) is also known (we omit it).

For \( k \)th power of \( \mu \)-derivative acting on \( x^n \) we have

\[
(\mathcal{D}_x^{(\mu)})^k x^n = \frac{[n]_\mu!}{[n-k]_\mu!} x^{n-k}, \quad [n]_\mu! \equiv \frac{n!}{(n; \mu)} \tag{7}
\]

where \((n; \mu) \equiv (1 + \mu)(1 + 2\mu)...(1 + n\mu)\).

There exist certain \( q, \mu \)- or \((p, q; \mu)\)-deformed extensions of \( \mu \)-derivative: instead of \((d/dx)f(t^\mu x)\) in (6) take \( \mathcal{D}_x^{q} f(t^\mu x) \) or \( \mathcal{D}_x^{(p,q)} f(t^\mu x) \). The two extensions correspond to the quasi-Fibonacci \((q; \mu)\)-DO or \((p, q; \mu)\)-DO in [35].

So, to develop the \( \mu \)-Bose gas thermodynamics we replace, where necessary, the usual \( d/dz \) with \( \mu \)-derivative \( \mathcal{D}_x^{(\mu)} \). Due to the \( \mu \)-derivative, basic parameter \( \mu \) enters the treatment: the system gets \( \mu \)-deformed. For \( \mu \ll 1 \), the usual and \( \mu \)-deformed derivatives of a function have similar behavior,
that is easily seen by acting with $\mu$-derivative and usual one on the monomial, logarithmic, exponential function, etc. Such property of $\mu$-derivative can justify its use in developing thermodynamics of $\mu$-Bose gas.

The $\mu$-bracket $[n]_\mu$ and $\mu$-factorial $[n]_\mu!$, see (5), (7), generate $\mu$-deformed analogs [19] of familiar functions: $\mu$-exponential $\exp_\mu(x)$, $\mu$-logarithm $\ln_\mu(x)$ (with $\mu$-numbers $[n]_\mu$ and $\mu$-factorial $[n]_\mu! = [n]_\mu [n-1]_\mu ... [2]_\mu [1]_\mu$). New special functions e.g. $\mu$-polylogarithms do also appear, see [19] and below.

3. Thermodynamics of $\mu$-Bose gas model

Thermodynamics of $\mu$-BGM is based on $\mu$-calculus. We consider the gas of non-relativistic particles for the regimes [49] of both high and low temperatures.

Total number of particles. The usual relation for total number of Bose gas particles is given in (3). For $\mu$-BGM, the total number of particles $N \equiv N^{(\mu)}$ is defined as $N^{(\mu)} = Z D^{(\mu)} \ln Z$ using $\mu$-derivative $D^{(\mu)}$ from (5). Applying it, for $\mu \geq 0$, to the (log of) partition function in (2) we get

$$N^{(\mu)} = \sum_i \sum_n [n]_\mu^n e^{-\beta \varepsilon_i n} z^n. \quad (8)$$

Set $0 \leq |ze^{-\beta \varepsilon_i}| < 1$ in (8). For non-relativistic particles the energy $\varepsilon_i$ is

$$\varepsilon_i = \frac{\vec{p}_i^2}{2m} = \frac{p_i^2}{2m}, \quad (9)$$

involving 3-momentum $\vec{p}_i$ of particle of mass $m$ in the $i$-th state.

At $z \to 1$ the summand in (8) diverges if $p_i = 0, i = 0$. Suppose that the $i = 0$ ground state admits macroscopically large occupation number. For

\footnote{These regimes are respectively given by the inequalities $\frac{v}{\lambda} \gg 1$ and $\frac{v}{\lambda} \ll 1$ which involve the specific volume $v = V/N$ and thermal wavelength $\lambda$, see (11) and below.}

\footnote{Similarly to refs. [27, 29] and others, we initially take the particle kinetic energy as that of non-relativistic free particle. But, the very particles in the model are not the usual bosons, because of the deformation of thermodynamics that uses $\mu$-calculus. As result, all thermodynamical quantities including the mean kinetic energy of particle become dependent on the deformation parameter $\mu$.}
For $z \neq 1$ we as well separate the term with $p_i = 0$ from the remaining sum:

$$N^{(\mu)} = \sum_i \sum_{n=1}^\infty \frac{[n]_{\mu}}{n} (e^{-\beta \xi_i})^n z^n + \sum_{n=1}^\infty \frac{[n]_{\mu}}{n} z^n. \quad (10)$$

The symbol $\sum_i'$ means that the $i = 0$ term is dropped from the sum. For large volume $V$ and large $N$ the spectrum of single-particle states is almost continuous so we replace the sum $\sum_i \rightarrow V(2\pi\hbar)^3 \int d^3 p$ (the ground state, $p_0$, does not contribute to the integral, so it starts from zero).

Further calculation of $N^{(\mu)}$ proceeds (in $d = 3$) similarly to the case of $p, q$-Bose gas [29] (see also ref. [19] for details), and the final result reads:

$$N^{(\mu)} = \frac{V}{\lambda^3 g_{3/2}^{(\mu)}(z)} + g_0^{(\mu)}(z), \quad g_0^{(\mu)}(z) = N_0^{(\mu)}. \quad (11)$$

Here $\lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}}$ is the thermal wavelength; $g_0^{(\mu)}(z)$ and $g_{3/2}^{(\mu)}(z)$ are $\mu$-analogs of polylogarithm $g_l(z) = \sum_{n=1}^\infty z^n/n^l$, or $\mu$-polylogarithms, defined as

$$g_l^{(\mu)}(z) = \sum_{n=1}^\infty \frac{[n]_{\mu}}{n^{l+1}} z^n. \quad (12)$$

For real $\mu > 0$ the convergence properties are not spoiled (like for the usual $g_l(z)$, there should be $|z| < 1$). If $\mu \rightarrow 0$, we recover polylogarithm $g_l(z)$.

For further needs, it is convenient to rewrite the expression (11) in terms of volume per particle $v = V/N^{(\mu)}$.

**Deformed grand partition function.** In $\mu$-BGM, we use the relations between thermodynamic functions similar to those of usual Bose gas thermodynamics, but in our case all the thermodynamic functions like the partition function etc., become $\mu$-dependent.

The deformed partition function $\ln Z^{(\mu)}$ satisfies

$$N^{(\mu)} = z \frac{d}{dz} \ln Z^{(\mu)} \quad \text{or} \quad \ln Z^{(\mu)} = \left( z \frac{d}{dz} \right)^{-1} N^{(\mu)}. \quad (13)$$

To apply $\left( z \frac{d}{dz} \right)^{-1}$, we use the property $f \left( z \frac{d}{dz} \right) z^k = f(k)z^k$ for a function $f(X)$ which admits power series expansion. From (13), (11), (12) we infer

$$\ln Z^{(\mu)} = -\frac{V}{\lambda^3} \sum_{n=1}^{\infty} \frac{[n]_{\mu}}{n^{5/2}} (n)_{-1} z^n + \sum_{n=1}^{\infty} \frac{[n]_{\mu}}{n} (n)_{-1} z^n. \quad (14)$$
In a more compact form,

\[ Z^{(\mu)}(z, T, V) = \exp \left( \frac{V}{\lambda^3} g_{5/2}^{(\mu)}(z) + g_1^{(\mu)}(z) \right). \]  

Formulas (14)-(15) provide the \( \mu \)-partition function and play basic role: from these we can derive other thermodynamical functions and relations.

4. Geometric approach to \( \mu \)-Bose gas model

We study the \( \mu \)-thermodynamics using thermodynamic geometry in the space of two parameters \( \beta, \gamma \), where \( \gamma = -\beta \bar{\mu} \) and \( \bar{\mu} \) is chemical potential. The (scalar) curvature in the thermodynamic parameters space provides the efficient and elegant tools for exploring thermodynamical properties of the system under study \[44, 45, 47]. This geometric construction is formed, see below, by the derivatives of a thermodynamic potential that determines a surface in the space of thermodynamic parameters. The curvature reveals the extremal points of this surface, identified \[46\] with phase transitions: its singularity gives a sufficient condition for the existence of the phase transition point(s). Moreover, the curvature is nothing but a thermodynamic measure of interaction within the system, and its sign indicates attractive or repulsive character of this interaction.

The components of (symmetric) metric in the Fisher-Rao representation are

\[ G_{\beta\beta} = \frac{\partial^2 \ln Z^{(\mu)}}{\partial \beta^2}, \quad G_{\beta\gamma} = \frac{\partial^2 \ln Z^{(\mu)}}{\partial \gamma \partial \beta}, \quad G_{\gamma\gamma} = \frac{\partial^2 \ln Z^{(\mu)}}{\partial \gamma^2}. \]  

Using the thermodynamic relations, we also have

\[ G_{\beta\beta} = -\left( \frac{\partial U}{\partial \beta} \right)_\gamma, \quad G_{\beta\gamma} = -\left( \frac{\partial N}{\partial \beta} \right)_\gamma, \quad G_{\gamma\gamma} = -\left( \frac{\partial N}{\partial \gamma} \right)_\beta. \]

Remembering (15), we arrive at the metric (given by \( \mu \)-polylogarithms):

\[ G_{\beta\beta} = \frac{15}{4} \frac{V}{\lambda^3 \beta^2} g_{5/2}^{(\mu)}(z), \quad G_{\beta\gamma} = \frac{3}{2} \frac{V}{\lambda^3 \beta} g_{3/2}^{(\mu)}(z), \quad G_{\gamma\gamma} = \frac{V}{\lambda^3} g_{1/2}^{(\mu)}(z) + g_{-1/2}^{(\mu)}(z). \]  

The determinant of the metric \( g \equiv \det |G_{ij}| \) results as (let \( \beta \leftrightarrow 1, \gamma \leftrightarrow 2 \))

\[ g = \frac{3V}{4\lambda^3 \beta^2} \left( 5g_{3/2}^{(\mu)}(z)g_{-1/2}^{(\mu)}(z) + \frac{V}{\lambda^3} \left( 5g_{3/2}^{(\mu)}(z)g_{1/2}^{(\mu)}(z) - 3g_{3/2}^{(\mu)}(z)g_{1/2}^{(\mu)}(z) \right) \right) \]
and the inverse metric is $G^{11} = G_{22}/g$, $G^{12} = -G_{12}/g$, $G^{22} = G_{11}/g$.

As the metric components are given by the derivatives of partition function, the Christoffel symbols and the Riemann tensor are found as

$$\Gamma_{\lambda\sigma\nu} = \frac{1}{2} (\ln Z^{(\mu)})_{,\lambda\sigma\nu},$$

(20)

$$R_{\lambda\sigma\nu\rho} \equiv G^{\kappa\tau} (\Gamma_{\kappa\lambda\rho} \Gamma_{\tau\sigma\nu} - \Gamma_{\kappa\lambda\nu} \Gamma_{\tau\sigma\rho}).$$

(21)

Calculation of the Christoffel symbols yields

$$\Gamma_{\beta\beta\beta} = -\frac{105}{16} \frac{V}{\lambda^3 \beta^3} g_{\frac{3}{2}}^{(\mu)}(z),$$

(22)

$$\Gamma_{\beta\gamma\beta} = \Gamma_{\beta\beta\gamma} = \Gamma_{\gamma\beta\beta} = -\frac{15}{8} \frac{V}{\lambda^3 \beta^2} g_{\frac{3}{2}}^{(\mu)}(z),$$

(23)

$$\Gamma_{\beta\gamma\gamma} = \Gamma_{\gamma\beta\gamma} = \Gamma_{\gamma\gamma\beta} = -\frac{3}{4} \frac{V}{\lambda^3 \beta} g_{\frac{3}{2}}^{(\mu)}(z),$$

(24)

$$\Gamma_{\gamma\gamma\gamma} = -\frac{1}{2} \left( \frac{V}{\lambda^3} g_{\frac{3}{2}}^{(\mu)}(z) + g_{-\frac{1}{2}}^{(\mu)}(z) \right).$$

(25)

Since the scalar curvature in 2-dimensional space is determined by one component of Riemann tensor $R = 2R_{\beta\gamma\beta\gamma}/g$, we obtain our main result:

$$R = \frac{5}{2} \left( \frac{g_{\frac{3}{2}}^{(\mu)} g_{\frac{1}{2}}^{(\mu)} g_{-\frac{1}{2}}^{(\mu)} - 7 g_{\frac{3}{2}}^{(\mu)} g_{\frac{1}{2}}^{(\mu)} g_{-\frac{1}{2}}^{(\mu)} + 2 g_{\frac{3}{2}}^{(\mu)} g_{\frac{1}{2}}^{(\mu)} g_{-\frac{1}{2}}^{(\mu)} + \frac{V}{\lambda^3} W \right),$$

(26)

$$W \equiv 2 g_{\frac{3}{2}}^{(\mu)} g_{\frac{1}{2}}^{(\mu)} g_{-\frac{1}{2}}^{(\mu)} - 4 g_{\frac{3}{2}}^{(\mu)} g_{\frac{1}{2}}^{(\mu)} g_{-\frac{1}{2}}^{(\mu)} + 2 g_{\frac{3}{2}}^{(\mu)} g_{\frac{1}{2}}^{(\mu)} g_{-\frac{1}{2}}^{(\mu)}. $$

The $\mu$-polylogarithm $g_l^{(\mu)}(z)$ involved in $R$ is singular at $z \to 1$, for $l \leq 1$ in the case $\mu = 0$ and for $l \leq 0$ when $\mu \neq 0$. Curvature $R(z)$ in isothermal process has characteristic properties as a function of fugacity. In the case $v/\lambda^3 \ll 1$, $R(z)$ has no singularities as seen in Fig. 1 (left panel).

If $v/\lambda^3$ is sufficiently large (see Fig 1, right panel) to neglect the terms which do not contain this factor, the curvature $R(z)$ is singular at $z \to 1$. We conclude that, in the latter situation, the system undergoes phase transition, and hence Bose-like condensation takes place. That is, the $\mu$-Bose gas model satisfies basic necessary property.

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*4Positive sign of the curvature corresponds to attraction among particles, and the magnitude of curvature is growing function of $\mu$. That is natural since the parameter $\mu$ of deformation provides effective account of quantum-statistical inter-particle interactions.*
5. Critical temperature of condensation

For low temperatures and high density we can obtain \[19\], like for \( p, q \)-Bose gas \[29\], the critical temperature \( T_{c}^{(\mu)} \) of condensation in the considered \( \mu \)-BGM. At vanishing \( N_{0}^{(\mu)} \) in (11), the critical temperature \( T_{c}^{(\mu)} \) of \( \mu \)-Bose gas is determined by the equation \( \lambda^{3}/v = g^{(\mu)}(1) \) that gives \[19\]

\[
T_{c}^{(\mu)} = \frac{2\pi \hbar^{2}/mk}{(vg^{(\mu)}(1))^{2/3}} \quad \text{and} \quad \frac{T_{c}^{(\mu)}}{T_{c}} = \left(\frac{2.61}{g^{(\mu)}(1)}\right)^{2/3}, \tag{27}
\]

the latter being the ratio of \( \mu \)-critical \( T_{c}^{(\mu)} \) to the critical \( T_{c} \) of Bose gas \[49\]. As is seen, the ratio \( T_{c}^{(\mu)}/T_{c} \) has an important feature: the stronger is deformation (measured by \( \mu \)) the higher is \( T_{c}^{(\mu)} \). Say, for \( \mu = 0.06 \) we have \( T_{c}^{(\mu)} \approx 1.22 \cdot T_{c} \). If \( \mu \to 0 \) (no-deformation limit), the ratio is \( T_{c}^{(\mu)}/T_{c} = 1 \), i.e. the \( \mu \)-critical temperature tends to usual one, \( T_{c}^{(\mu)} \to T_{c} \) (a kind of consistency). The existence of condensate of \( \mu \)-bosons is the crucial property for the use of \( \mu \)-BGM in the modeling of dark matter.

Let \( T \) be in the interval \( 0 < T < T_{c}^{(0)} \leq T_{c}^{(\mu)} \). In the \( \mu \)-deformed case we have \( \frac{T^{(\mu)}}{T} = \frac{2}{5}c_{v}^{(\mu)} = \frac{3}{5}S^{(\mu)} \) in similarity with pure Bose case, i.e. \( \frac{T}{T} = \frac{2}{5}c_{v} = \frac{3}{5}S \). From these we infer two useful relations: \( \frac{T^{(\mu)}}{c_{v}^{(\mu)}} = \frac{T}{c_{v}} \) and \( \frac{T^{(\mu)}}{S^{(\mu)}} = \frac{T}{S} \).

6. Other issues important for modeling dark matter

Above, exploiting thermodynamic geometry, we verified main property of \( \mu \)-BGM needed for its ability to model dark matter. The appearance of Bose-like condensation as such property, is confirmed.
As mentioned in [3], the dark matter surrounding dwarf galaxies must be "strongly coupled, dilute system of particles". In case of \( \mu \)-Bose gas, we emphasize that the interaction between \( \mu \)-bosons (of quantum-statistical origin) is also attractive, and due to deformation can even be stronger than that for pure bosons. To witness, compare the two 2nd virial coefficients: \( \mu \)-dependent \( V_2^{(\mu)} \) [19], and the standard \( V_2^{(\text{Bose})} = 2^{-5/2} \) (drop the "−" sign):

\[
V_2^{(\mu)} - V_2^{\text{Bose}} = 2^{-7/2} \left( \frac{[2]_{\mu}}{[1]_{\mu}^2} - 2 \right) = 2^{-5/2} \frac{\mu^2}{1 + 2\mu} > 0.
\]

The enhanced attraction (of quantum origin) implies that the \( \mu \)-bosons are "more bosonic" than usual bosons. This property is good for providing strongly coupled system of (quasi)particles. We see that at large \( \mu \) one has \( g_i^{(\mu)}(z) \to \mu^{-1}g_i^{(0)}(z) \equiv \mu^{-1}g_{i+1}(z) \), where \( g_i(z) \) is the polylogarithm. Then, the internal energy per particle would not depend on \( \mu \) while the total one does, due to scale factor.

However, unlimited growth of \( \mu \)-parameter (strength of attraction) could lead to a collapse of the studied quantum system. To prevent that, we can find some bound on the values of \( \mu \), say, requiring to forbid negative pressure. We take (virial expansion of) the equation of state [19] to second order i.e.

\[
\frac{P v}{kT} = 1 - \frac{[2]_{\mu}}{2^{7/2}[1]_{\mu}^2} \frac{\lambda^3}{v}.
\]

Impose \( P = 0 \) or \( 2^{-7/2}[2]_{\mu} \frac{\lambda^3}{v} = 1 \) and find critical strength \( \bar{\mu} \) of deformation:

\[
\bar{\mu} = \kappa - 1 + \sqrt{(\kappa - 1)\kappa}, \quad \kappa \equiv 2^{5/2} \frac{v}{\lambda^3}.
\]

With \( \mu \leq \bar{\mu} \) we avoid the collapse. Obviously, \( \kappa = 1 \) means \( \bar{\mu} = 0 \) and so \( \mu = 0 \), that is pure Bose case. On the other hand, the bound taken as say \( \bar{\mu} = 1 \) with \( 0 < \mu \leq 1 \) corresponds to the value \( \kappa = \frac{4}{3} \).

Equation of state (28) in its application to the dark matter involves the temperature in the range \( 0 < T \leq T_{c(\mu)} \), with non-vanishing pressure \( P \) of \( \mu \)-BEC. At the same time, the dark matter pressure within the ordinary BEC concept is also supposed to be non-zero: \( P = 2\pi a h^2 \rho^2 / m^3 \), even at \( T = 0 \), because of the supposed scattering with length \( a \) of bosons with the mass density \( \rho \). Since the effect of \( \mu \)-deformed statistics and the scattering represent two non-identical processes, the both can be taken together into account within an extended, unified model of dark model in a future study.
There exists some other, "characteristic" value $\mu_0$ of $\mu$ for which the deformed entropy (see eq. (41) and Fig. 6 in [19]) becomes $\frac{S(\mu_0)\lambda^3}{k_B} = 1$ or just $S(\mu_0)\lambda^3 = k_B$. With the $\mu_0$, we bound our interval as $0 < \mu \leq \mu_0$. At this deformation strength $\mu_0 \simeq 1.895$, we obtain the relation

$$g_{3/2}(1) = 3.3535 \ g_{3/2}^{(\mu=\mu_0)}(1)$$

for the polylogarithm and $\mu$-polylogarithm. Then we find that the critical volume-per-particle (divided by cube of thermal wavelength) in the $\mu$-BGM is related with similar quantity of the usual Bose gas by the formula

$$\left(\frac{v^c}{\lambda^3}\right)_{\mu=\mu_0} = 3.3535 \left(\frac{v^c}{\lambda^3}\right)_{\text{Bose}}.$$

Hence, we can estimate the ratio in our model using respective estimates from Bose-condensate case [3].

In Ref. [3], the BEC concept, using the Gross-Pitaevskii equation in the Thomas-Fermi approximation, is applied for finding a space distribution or density profile of the dark matter. It gives the radius and total mass of the dark matter halo:

$$R = \pi \sqrt{\frac{\hbar^2 a}{Gm^3}}; \quad M = \frac{4}{\pi} R^3 \rho^{(c)}; \quad (31)$$

dependent on the (repulsive) s-wave scattering length $a$, the gravitational constant $G$, mass $m$ of constituent particle and the central mass density $\rho^{(c)}$.

With varying $R$ and $\rho^{(c)}$, their model is able to reproduce characteristics of the rotational curves in the outer region of the galaxies and the total mass $M$. However, it neglects a supplementary interaction in the inner region and leads to some overestimation of $M$, compared with the observational data e.g. those discussed in [3].

To resolve this, we attempt to account for additional, as compared to the BEC model, attraction earned due to $\mu$-deformed statistics and influencing the value $\rho^{(c)}$. Omitting here the dynamics aspects and determining $\rho^{(c)}$ by the ratio $m/v$, an effective interaction is included by defining the (critical) volume-per-particle in the case of $\mu$-deformed thermodynamics: $v = \lambda^3/g_{3/2}^{(\mu)}(1)$. It means that $\rho^{(c)}(\mu) = \left(g_{3/2}^{(\mu)}(1)/g_{3/2}^{(0)}(1)\right)\rho^{(c)}$ and therefore

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5 The issues concerning respective modification of the Gross-Pitaevskii equation and its implications are under study, and we hope to report on that in near future.
$M^{(\mu)} = \left(\frac{g_{3/2}^{(\mu)}(1)}{g_{3/2}^{(0)}(1)}\right) M$ will play the role of new corrected characteristics. Since $g_{3/2}^{(\mu)}(1) < g_{3/2}^{(0)}(1)$ at $\mu > 0$, these predictions can give instead of $M^{\text{BEC}} \equiv M^{(0)}$ a better agreement with the observational data.

7. Concluding remarks

Thermodynamics of the $\mu$-Bose gas – total mean number of particles $N^{(\mu)}$, the (log of) $\mu$-deformed partition function etc., involve $\mu$-generalization $g_k^{(\mu)}(z)$ of polylogarithms $g_k(z)$. This fact influences other results in the paper. Metric tensor, Christoffel symbols and hence the Riemann curvature get expressed through $\mu$-polylogarithms. Positive sign of curvature and its divergence as $z \to 1$ witness the attraction between $\mu$-particles and confirm Bose-like condensation. Formula for the $\mu$-critical temperature $T_c^{(\mu)}$ is compared with usual Bose case, with infinite statistics system, and with $T_c^{(p,q)}$ of $p,q$-Bose gas model [29] (therein, $T_c^{(p,q)}$ can be both higher and lower, depending on the values of parameters $p$ and $q$). The ratio $T_c^{(\mu)}/T_c$ as a function of $\mu$-parameter shows: critical temperature $T_c^{(\mu)}$ exceeds critical $T_c$ of usual Bose gas, in contrast with the system of infinite statistics showing lower critical temperature [12] than the usual Bose $T_c$. We consider such property as one of the virtues, from the viewpoint of applying $\mu$-BGM to modeling dark matter: in our case stability of the condensate extends higher in temperature than with usual $T_c$. Other facts important for the modeling and valid for the infinite statistics system [12], e.g. the smallness of particle mass, can be demonstrated for $\mu$-bosons as well.

Further remarkable properties of $\mu$-Bose gas model (say, the falling behavior of entropy-per-volume versus the $\mu$-parameter – that means decreasing chaoticity with growing deformation strength), when used for modeling dark matter, can also lead to interesting implications.

In the context of dark matter, the inner structure of its constituents at a given deformation plays a remarkable role in their response to extrinsic perturbation. The parameter $\mu$ is determined by specific conditions of the dark matter existence in each galaxy or a local region of Universe. Moreover, since deformed critical temperature obeys the condition $T_c^{(\mu)} \geq T_c^{(0)} > T$, the

\footnote{Concerning this property, it would be very interesting to give detailed analysis of diverse species of cold atoms (Rb, K, Ca etc.), for which the existence of BEC is already confirmed. However, this goes beyond the scope of present paper.}
The present model of dark matter may be also valid for interstellar environment where the temperature $T$, concentration of dark matter particles $\rho$ ($T_c \sim \rho^{2/3}$), and parameter $\mu$ can vary.

Thus, the $\mu$-Bose gas model has its own virtues and like infinite statistics system can be used to effectively model basic properties of dark matter. In that domain, the present $\mu$-Bose gas model may turn out to be just as successful as the $(\tilde{\mu}, q)$-deformed analog of Bose gas model has shown itself in the effective description, see Fig. 5 in [42], of the observed (in the STAR/RHIC experiments) non-Bose like behavior of the intercepts of two-pion correlations. Next, like in [42], compositeness of (quasi-)particles which constitute dark matter may be of importance and can be accounted for by extending the $\mu$-BGM using the results of [38, 39]. Clearly, further study is needed to find new arguments in favor of the proposed model of dark matter.

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Appendix A. Infinite Statistics

A gas of particles obeying [50, 51] infinite statistics (with parameter $p$) was proposed in [12] as a model of dark matter. Authors considered thermodynamical geometry of this gas, and showed with their fig. 1 that condensation does occur for such system.\footnote{Note that the result of [12] is presented there as the figure without explicit expressions of calculated metric components, Christoffel symbols and curvature. For the sake of comparison with the results obtained in our $\mu$-Bose gas model, we reproduce here explicitly the necessary geometric quantities, all being expressed in terms of Lerch transcendent [52]}

In [12], the number of particles and internal energy for infinite statistics gas are given in integral form as $N = 4Apz\beta^{-d/2-1}I(pz, d/2 - 1)$ and $U = 4Apz\beta^{-d/2-1}I(pz, d/2)$ respectively, where $I(\xi, \alpha) = \int_0^{\infty} e^{x}x^{\alpha}(e^{2x} - \xi^2)^{-1}dx$.\footnote{Note that the result of [12] is presented there as the figure without explicit expressions of calculated metric components, Christoffel symbols and curvature. For the sake of comparison with the results obtained in our $\mu$-Bose gas model, we reproduce here explicitly the necessary geometric quantities, all being expressed in terms of Lerch transcendent [52].}
We express these functions through the Lerch transcendent \[52\]:

\[ \Phi(z, s, \alpha) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-\alpha t}}{1-ze^{-t}} dt \]  

(A.1)

hereafter \( \Gamma(x) \) is the usual gamma-function.

Let \( \xi \equiv p z, \Phi_k(\xi) \equiv \Phi(\xi^2, d/2 + k, 1/2), \) and \( C_k \equiv 8(2\beta)^{-d/2-k}\Gamma(d/2+k) \).

The expressions for \( N \) and \( U \) are rewritten as

\[ N = \frac{A}{4\beta} C_0 \xi \Phi_0(\xi), \quad U = \frac{A}{2} C_1 \xi \Phi_1(\xi). \]  

(A.2)

We calculate the metric components in the space of two parameters \((\beta, \gamma)\) in the Fisher-Rao representation, using the known identity \[52\] for Lerch transcendent \( \Phi(z, s-1, \alpha) = (\alpha + z\partial_z) \Phi(z, s, \alpha) \). One has

\[ G_{\beta\beta} = AC_1 \xi \Phi_1(\xi), \quad G_{\beta\gamma} = AC_1 \xi \Phi_0(\xi), \quad G_{\gamma\gamma} = AC_0 \xi \Phi_{-1}(\xi). \]  

(A.3)

The Christoffel symbols are found to be

\[ \Gamma_{\beta\beta\beta} = -AC_3 \xi \Phi_1(\xi), \quad \Gamma_{\beta\beta\gamma} = \Gamma_{\gamma\beta\beta} = -AC_2 \xi \Phi_0(\xi), \quad \Gamma_{\beta\gamma\gamma} = \Gamma_{\gamma\beta\gamma} = -AC_1 \xi \Phi_{-1}(\xi), \quad \Gamma_{\gamma\gamma\gamma} = -AC_0 \xi \Phi_{-2}(\xi). \]

The resulting expression for the thermodynamical curvature is given as

\[ R = \frac{4\beta}{A \xi} C_0 C_1 C_2 \frac{-2\Phi_1 \Phi_{-1}^2 + \Phi_0^2 \Phi_{-1} + \Phi_{-2} \Phi_1 \Phi_0}{(C_2 C_0 \Phi_{-1} \Phi_1 - C_1^2 \Phi_0^2)^2}. \]  

(A.4)

As it is seen from Fig. 1 in \[12\] and also follows from (A.4), in three dimensional space \((d = 3)\) sending \( zp \to 1 \) results in \( R \to \infty \) and thus leads to the phase transition (Bose-like condensation). This fact, and some other properties of particles with infinite statistics (their mass, weakness of their interaction etc.) has allowed the authors of \[12\] to conclude in favor of ability of such system as possible model of dark matter.

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