THE SUPERSINGULAR LOCUS IN SIEGEL MODULAR VARIETIES WITH IWAHORI LEVEL STRUCTURE

ULRICH GÖRTZ AND CHIA-FU YU

Abstract. We study moduli spaces of abelian varieties in positive characteristic, more specifically the moduli space of principally polarized abelian varieties on the one hand, and the analogous space with Iwahori type level structure, on the other hand. We investigate the Ekedahl-Oort stratification on the former, the Kottwitz-Rapoport stratification on the latter, and their relationship. In this way, we obtain structural results about the supersingular locus in the case of Iwahori level structure, for instance a formula for its dimension in case \( g \) is even.

1. Introduction

Fix an integer \( g \), a prime number \( p \), and let \( k \) be an algebraic closure of the field \( \mathbb{F}_p \) with \( p \) elements. Denote by \( \mathcal{A}_g \) the moduli space of principally polarized abelian varieties of dimension \( g \) (with a suitable level structure away from \( p \)), and by \( \mathcal{A}_I \) the moduli space of abelian varieties of dimension \( g \) “with Iwahori level structure at \( p \)”, i.e. the space of chains \( A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_g \) of isogenies of order \( p \), satisfying certain further conditions. See Subsection 2.1 for details.

Whereas the space \( \mathcal{A}_g \) is a well-studied object, much less is known about \( \mathcal{A}_I \). For instance, there are a number of structural results about the supersingular locus \( \mathcal{S}_g \) in \( \mathcal{A}_g \), but not even the dimension of the supersingular locus \( \mathcal{S}_I \) inside \( \mathcal{A}_I \) is known in general. We prove (see Corollary 8.10, Propositions 8.11 and 8.12, and [13], Proposition 4.6 and Theorem 6.3):

**Theorem 1.1.**

1. If \( g \) is even, then \( \dim \mathcal{S}_I = \frac{g^2}{2} \).
   If \( g \) is odd, then
   \[
   \frac{g(g-1)}{2} \leq \dim \mathcal{S}_I \leq \frac{(g+1)(g-1)}{2}.
   \]

2. Suppose \( g \) is even. Then every top-dimensional irreducible component of \( \mathcal{S}_I \) is isomorphic to the full flag variety of the group \( \text{Sp}_{2g} \times \text{Sp}_{2g} \) (over \( k \)).

3. Suppose \( g \) is odd. Then \( \mathcal{S}_I \) has irreducible components which are isomorphic to the flag variety of \( \text{SL}_g \) (over \( k \)). (So these components are of dimension \( g(g-1)/2 \).)

However, one must keep in mind that in the Iwahori case, as soon as \( g \geq 2 \), the supersingular locus is not equidimensional.

Let us mention that the supersingular locus is expected to be of considerable interest from the point of view of the Langlands program. In fact, the supersingular locus is the unique closed Newton stratum, and the Newton stratification, which

\begin{footnotesize}

Öörtz was partially supported by a Heisenberg grant and by the SFB/TR 45 “Periods, Moduli Spaces and Arithmetic of Algebraic Varieties” of the DFG (German Research Foundation).

Yu was partially supported by grants NSC 97-2115-M-001-015-MY3 and AS-98-CDA-M01.

\end{footnotesize}
is given by the isogeny type of the underlying $p$-divisible group, provides a natural way to split up the space $A_I$, or more generally the special fiber of a Shimura variety of PEL type. This can be seen, for instance, by looking at Kottwitz’ method of counting the points of the special fiber of a “simple Shimura variety”, and by his conjecture on the cohomology of Rapoport-Zink spaces [30] which should provide a local approach to the local Langlands correspondence. Conjecturally, in the non-supersingular locus, induced representations should be realized, whereas in the supersingular locus the supercuspidal representations are found. (Of course, here one will have to use deeper level structure.) See [2], [6], [16], [18], [31] for details and further references.

The key method to obtain our results is to study the Kottwitz-Rapoport stratification (KR stratification) on $A_I$. Insofar, the current paper is a continuation of [13], and in fact, we prove the two conjectures we made in (the preprint version of) [13]. The first one concerns the dimension of the $p$-rank 0 locus in $A_I$, and we have (Theorem 1.2):

**Theorem 1.2.** The dimension of the $p$-rank 0 locus in $A_I$ is $[g^2/2]$.

An important ingredient of the proof is the result of Ngô and Genestier that the $p$-rank is constant on KR strata (together with a formula for the $p$-rank of a given stratum), but the task to make a list of maximal strata of $p$-rank 0 and to relate the formula of Ngô and Genestier to the dimension of these strata, though purely combinatorial, turns out to be quite involved. The theorem in particular gives us an upper bound on the dimension of $S_I$. Again, note that the $p$-rank 0 locus is usually not equidimensional.

Note that for $g ≥ 3$, the supersingular locus is not a union of KR strata. For instance, the stratum given by $x = s_3 s_1 s_0$ in $A_I$, $g = 3$, intersects the supersingular locus without being contained in it ([36]). We call a KR stratum **supersingular** if it is contained in the supersingular locus.

The second conjecture says that all supersingular KR strata a very specific form, and hence can be described in very concrete terms. Let us recall the following definitions:

**Definition 1.3.**

1. An abelian variety over a field is called **superspecial**, if it is isomorphic to a product of supersingular elliptic curves over some extension field.

2. A KR stratum $A_x$ is called **superspecial**, if there exists $i$ with $0 ≤ i ≤ [g/2]$ such that for all $k$-valued points $(A_i, \lambda_0, \lambda_g)$ in $A_x$, the abelian varieties $A_i$ and $A_{g-i}$ are superspecial, and the isogeny $A_i → A_{g-i}^\vee$ is isomorphic to the Frobenius morphism of $A_i$. (Cf. [13], Section 4.)

In particular, every superspecial abelian variety is supersingular, and every superspecial KR stratum is supersingular. The main result of [13] is an explicit geometric description of these superspecial KR strata in terms of Deligne-Lusztig varieties. Here we prove (Corollary 7.5):

**Theorem 1.4.** All supersingular KR strata are superspecial.

Looking at superspecial strata, we get a lower bound on $\dim S_I$, and in fact a more precise version of Theorem [13] is that the irreducible components referred to in (2) and (3) are closures of components of superspecial KR strata of the maximal
dimension. Along the way, we prove the following results about KR strata, which are of independent interest (see Theorems 5.4 and 7.4).

**Theorem 1.5.**

1. All KR strata are quasi-affine.
2. All KR strata which are not superspecial, are connected.

The key ingredient which we did not use in [13] is a comparison of the KR stratification with the Ekedahl-Oort stratification on $\mathcal{A}_g$. This allows us to use results about the latter: for instance, part (2) of the previous theorem ultimately follows from the fact (proved by Harashita [17]) that all EO strata which are contained in the supersingular locus are disconnected.

The paper at hand (and Ekedahl and van der Geer’s results [5]) clarifies the relationship between the KR stratification and the EO stratification. The precise relationship between our description of supersingular KR strata and Hoeve’s description of EO strata contained in $S_g$ [19] is explained in [12].

We were inspired at several places by Ekedahl and van der Geer [5], and generalize some of their ideas. The relationship between the situations considered here and in [5], respectively, is explained in detail in Section 9; see in particular Proposition 9.6. See also van der Geer [7], where many of the results relevant for us are already present.

Let us mention one point which facilitates the work of Ekedahl and van der Geer in comparison with our situation (and allows them to go further than we can). Namely, their main object of study, the bundle of flags inside the Hodge filtration of the universal abelian scheme over $\mathcal{A}_g$, has a natural compactification (because one can extend this bundle to the Baily-Borel compactification of $\mathcal{A}_g$). No similarly well understood compactification is available for $\mathcal{A}_I$.

Another way to compute the dimension of the supersingular locus (or of an arbitrary Newton stratum) is to use the theory of affine Deligne-Lusztig varieties. (In the non-supersingular case one would also have to study the leaves in the Newton stratum in the sense of Oort.) However, even if one ignores the problem that the results about affine Deligne-Lusztig varieties which are currently available in the Iwahori case relate to the function field case, the combinatorial complexity of the algorithm to compute the dimension of such varieties given in [11] is so high that computing the dimension of $S_I$ is entirely out of sight for $g > 4$.

Let us survey the content of the individual sections. We start, in Section 2, collecting the definitions and all the relevant results we need about the $p$-rank stratification (on either $\mathcal{A}_g$ or $\mathcal{A}_I$), the Ekedahl-Oort stratification (on $\mathcal{A}_g$) and the Kottwitz-Rapoport stratification (on $\mathcal{A}_I$). Section 3 explains that the image in $\mathcal{A}_g$ of each KR stratum is a union of EO strata. This is a simple but extremely important fact. In Section 4, we construct a morphism from $\mathcal{A}_I$ to the bundle of symplectic flags in the first de Rham cohomology of the universal abelian scheme over $\mathcal{A}_g$. This morphism is almost an embedding, but not quite: it is finite, and a universal homeomorphism, but is inseparable. Using this map, and a version of Raynaud’s trick, we prove in Section 5 that all KR strata are quasi-affine. The main result of Section 6 is that connected components of KR strata are never closed in $\mathcal{A}_I$, unless they are 0-dimensional (and hence are components of the minimal KR stratum). This is important for Section 7, where we show that non-superspecial KR strata are connected. As a corollary we obtain (using Harashita’s results about the number of connected components of EO strata in the supersingular
locus) a proof that all supersingular KR strata are superspecial. Section 8 contains the computation of the dimension of the \( p \)-rank 0 locus in \( A_I \) (which gives us an upper bound on the dimension of the supersingular locus in \( A_I \)); this computation is independent of the rest of the paper. At the end of this section, we derive consequences about the supersingular locus in \( A_I \) from our results. Finally, in Section 9 we discuss the relationship to the work of Ekedahl and van der Geer \[5\].

2. Preliminaries about EO and KR stratifications

In this section we set up the notation and recall a number of results on the \( p \)-rank stratification, the Ekedahl-Oort stratification, and the Kottwitz-Rapoport stratification. No claim to originality is made.

2.1. Notation. We fix a prime \( p \) and an integer \( g \geq 1 \). Let \( k \) be an algebraic closure of \( \mathbb{F}_p \). We denote by \( A_g \) the moduli space of principally polarized abelian varieties of dimension \( g \) over \( k \). We can either consider \( A_g \) as an algebraic stack, or instead consider the moduli space of \( g \)-dimensional principally polarized abelian varieties with a symplectic level \( N \)-structure (with respect to some fixed primitive \( N \)-th root of unity), where \( N \geq 3 \) is an integer coprime to \( p \). In this way we obtain a quasi-projective scheme over \( k \) of dimension \( g(g+1)/2 \). As long as it does not matter which point of view we choose (and mostly, it will not), we will not be very precise about this.

Inside \( A_g \) we have the supersingular locus, a closed subset of \( A_g \), which we denote by \( S_g \). All its irreducible components have dimension \( n \geq g/2 \), and it is known to be connected if \( g > 1 \). See the book by Li and Oort \[22\]. Furthermore, we denote by \( A_I \) the moduli space of tuples

\[
(A_0 \to A_1 \to \cdots \to A_g, \lambda_0, \lambda_g),
\]

where all \( A_i \) are abelian varieties of dimension \( g \), the maps \( A_i \to A_{i+1} \) are isogenies of degree \( p \), and \( \lambda_0, \lambda_g \) are principal polarizations on \( A_0 \) and \( A_g \), respectively, such that the pull-back of \( \lambda_g \) to \( A_0 \) is \( p\lambda_0 \). Here \( I \) stands for Iwahori type level structure at \( p \). It can also be seen as the index set \( I = \{0, \ldots, g\} \) indicating that we consider full chains. Instead we could consider partial chains and would obtain more general parahoric level structure at \( p \). As above, whenever appropriate, we could in addition consider a level structure outside \( p \), in order to obtain an honest scheme.

The dimension of \( A_I \) is \( g(g+1)/2 \), as well. Inside it, we have the supersingular locus, i.e., the locus where one or equivalently all abelian varieties in the chain are supersingular. Its dimension is not known in general (but see \[8,10\]).

We have a natural projection map \( \pi: A_I \to A_g \) which maps \((A_*, \lambda_0, \lambda_g)\) to \((A_0, \lambda_0)\). This map is proper and surjective. It is not at all flat, however: The fiber dimension jumps, and the behavior is quite complicated.

The points in \( A_I \) are chains \( A_0 \to \cdots \to A_g \) of isogenies of order \( p \). Given such a chain, we can extend it “by duality” to a chain \( A_0 \to \cdots \to A_{2g} \), i.e., for \( i = g, \ldots, 2g \) we let \( A_{2g-i} = A_i^\vee \) (and use \( \lambda_g \) to identify \( A_g \) with \( A_g^\vee \)). Then \( \lambda_0 \) gives us an identification of \( A_{2g} \) with \( A_0 \). The isogeny \( A_{2g-i} \to A_{2g-i+1} \) is the dual isogeny for \( A_{i+1} \to A_i \).

All of the above is explained in more detail in \[13\], Section 2.
2.2. Group-theoretic notation. I. In this section, we fix the notation related to the (extended affine) Weyl group and the (affine) root system of the group \( G = \text{GSp}_2g \) of symplectic similitudes. For a more comprehensive account, we refer to [13], Subsections 2.1–2.3.

We use the Borel subgroup of upper triangular matrices and the maximal torus \( T \) of diagonal matrices with respect to the embedding \( G \subset \text{GL}_2g \) induced by the alternating form \( \psi \) such that for the standard basis vectors \( e_1, \ldots, e_{2g} \),

\[
\psi(e_i, e_{2g-i+1}) = 1 = -\psi(e_{2g-i+1}, e_i), \quad 1 \leq i \leq g,
\]

and all other pairs of standard basis vectors pair to 0. We denote by \( \widehat{W} \) the extended affine Weyl group for \( G \). We often regard it as a subgroup of the extended affine Weyl group for \( \text{GL}_2g \) which we can identify with the semidirect product \( \mathbb{Z}^{2g} \rtimes S_{2g} \). Inside \( \widehat{W} \), we have the affine Weyl group \( W_a \), an (infinite) Coxeter group generated by the simple affine reflections \( s_0, \ldots, s_g \).

The finite Weyl group \( W \) of \( G \) is the subgroup of \( W_a \) generated by \( s_1, \ldots, s_g \). For a translation element \( \lambda \in X_+(T) \), we denote by \( t^\lambda \) the corresponding element of \( \widehat{W} \).

For a subset \( J \subseteq \{0, \ldots, g\} \) we use the following somewhat unusual notation (as in [13]): \( W_J \) denotes the subgroup generated by the simple reflections \( s_i, \ i \in \{0, \ldots, g\} \setminus J \). The case most relevant for us will be \( J = \{i, g-i\} \) for some \( 0 \leq i \leq g/2 \).

Since \( W_a \) is a Coxeter group, the choice of generators \( s_0, \ldots, s_g \) gives rise to a length function \( \ell \) and to the Bruhat order \( \leq \) on \( W_a \). Both can be extended to \( \widehat{W} \) in a natural way and these extensions will be denoted by the same symbols. The extended affine Weyl group is the semi-direct product \( \widehat{W} = W_a \rtimes \Omega \), where \( \Omega \cong \pi_1(\text{GSp}_{2g}) \cong \mathbb{Z} \) is the subgroup of \( \widehat{W} \) of elements of length 0.

2.3. The \( p \)-rank stratification. Denote by \( A^{(i)}_g \) the locally closed subset where the \( p \)-rank of the underlying abelian variety is \( i \) (i. e. where \( A[p](k) \cong (\mathbb{Z}/p\mathbb{Z})^i \)). Likewise, let \( A^{(i)}_j = \pi^{-1}(A^{(i)}_g)_{\text{red}} \) be the \( p \)-rank \( i \) locus inside \( A_j \). The \( p \)-rank \( g \) locus is the ordinary locus. The closure of \( A^{(i)}_g \) is the union of all \( A^{(j)}_g \), \( j \leq i \), but the closure of a “stratum” \( A^{(i)}_j \) in general is not a union of \( p \)-rank strata, as can be seen already for \( g = 2 \).

**Proposition 2.1.** The \( p \)-rank 0 locus \( A^{(0)}_g \) is complete and equidimensional of dimension \( g(g-1)/2 \). The locus \( A^{(0)}_i \) is proper over \( k \).

**Proof.** See the papers by Koblitz [20], Theorem 7, and Oort [28], Theorem 1.1 (a) for the assertions about \( A^{(0)}_g \). The properness of \( A^{(0)}_i \) follows immediately.

More generally, Koblitz shows that the \( p \)-rank \( i \) locus \( A^{(i)}_g \) inside \( A_g \) is equidimensional of dimension \( g(g-1)/2 + i \). Furthermore, it is known that \( A^{(0)}_g \) has the minimal possible codimension which a complete subvariety of \( A_g \) can have; see [7], Corollary 2.7. We will prove later that \( \dim A^{(0)}_i = [g^2/2] \) (see Theorem 8.8).

2.4. Results about EO strata. The Ekedahl-Oort stratification was defined in [29] (where it is called the canonical stratification). It is the stratification on \( A_g \) given by the isomorphism type of the \( p \)-torsion points \( A[p] \). We denote by ES the
set of strata. For \( w \in ES \), we denote by \( EO_w \) the corresponding stratum. Each \( EO_w \) is locally closed in \( A_g \), and the closure of a stratum is a union of strata.

In the literature, there are three ways to describe the set \( ES \). Let us briefly discuss how they are related. In [29], Oort classifies \( EO \) strata by elementary sequences, i.e. maps \( \varphi : \{0, \ldots, g\} \to \{0, \ldots, g\} \) such that \( \varphi(0) = 0 \) and \( \varphi(i) \leq \varphi(i+1) - \varphi(i) + 1 \). (Ekedahl and) van der Geer ([7], see also [5], 2.2) use the set of minimal length representatives in \( W \) for the cosets \( W/S_g \). This is the description we will use, too, so we make the following definition.

**Definition 2.2.** Let \( S_g \subset W \) denote the subgroup generated by \( s_1, \ldots, s_{g-1} \) (this subgroup is isomorphic to the symmetric group on \( g \) letters, as the notation indicates). An element \( w \in W \) is called final, if it is the unique element of minimal length in the coset \( wS_g \). We denote the set of final elements by \( W_{\text{final}} \).

Explicitly, an element \( w \in W \) is in \( W_{\text{final}} \) if and only if

\[
 w(1) < w(2) < \cdots < w(g).
\]

The bijection \( W_{\text{final}} \cong ES \) is given by

\[
 w \mapsto \nu_w, \quad \nu_w(i) = i - \# \{ a \in \{1, \ldots, g\}; w(a) \leq i \}.
\]

Finally, Moonen and Wedhorn [23], [25], see also [33], 5.2, 6.2, describe the set of \( EO \) strata as the quotient \( S_g \backslash W \), or equivalently as the set of minimal length representatives in \( W \) for the cosets in \( S_g \backslash W \). This description is related to the description by the set \( W_{\text{final}} \) by the map \( w \mapsto w^{-1} \). Correspondingly, using the description of Moonen and Wedhorn, an element \( w \in W \) corresponds to the elementary sequence

\[
 w \mapsto \varphi_w, \quad \varphi_w(i) = \# \{ a \in \{1, \ldots, i\}; w(a) > g \}.
\]

**Proposition 2.3.**

1. There is a natural bijection between the set \( ES \) of \( EO \) strata and the set \( W_{\text{final}} \) of final elements in \( W \).
2. Every \( EO \) stratum is quasi-affine.
3. Let \( w \in W_{\text{final}} \), and let \( \varphi \) be the corresponding elementary sequence. Then \( EO_w \) is equidimensional of dimension

\[
 \dim EO_w = \ell(w) = \sum_{i=1}^{g} \varphi(g).
\]

**Proof.** For (1), see [23], Theorem 4.7. Part (2), as well as the dimension formula in (3), is proved in [29], Theorem 1.2. For the equidimensionality in (3), see [24], Corollary 3.1.6. See also [5].

There is a unique 0-dimensional \( EO \) stratum; it is precisely the locus of superspecial abelian varieties in \( A_g \). There is a unique one-dimensional \( EO \) stratum. See [29], §1. Finally, there is also a unique open (and hence dense) stratum. The open stratum is equal to the ordinary locus \( A_g^{(g)} \).

**Proposition 2.4.**

1. The \( EO \) stratification is a refinement of the \( p \)-rank stratification.
2. For \( w \in W_{\text{final}} \), the \( p \)-rank on \( EO_w \) is

\[
 \# \{ i \in \{1, \ldots, g\}; w(i) = g + i \},
\]
where we consider \( w \) as an element of the symmetric group \( S_{2g} \) by the natural embedding \( W_{\text{final}} \subset W \subset S_{2g} \).

**Proof.** Part (1) is obvious from the definition of the EO stratification. For part (2), see [5], Lemma 4.9 (i).

**Proposition 2.5.** Let \( w \in W_{\text{final}} \).

1. The stratum \( EO_w \) is contained in \( S_g \) if and only if \( w(i) = i \) for \( i = 1, \ldots, g - \lfloor g/2 \rfloor \) (where we consider \( w \) as an element of the symmetric group \( S_{2g} \) as in Proposition 2.4 (2)).
2. If \( EO_w \) is contained in \( S_g \) and \( N \geq 4 \), then \( EO_w \) is not connected.
3. If \( EO_w \) is not contained in \( S_g \), then \( EO_w \) is irreducible.
4. Every EO stratum is smooth. In particular, every connected component of \( EO_w \) is irreducible.
5. Let \( X \subseteq EO_w \) be a connected component. Then the closure of \( X \) meets the superspecial locus of \( A_g \).

**Proof.** Harashita ([17], Proposition 5.2) proves the following result which is due to Oort: The EO stratum associated to an elementary sequence \( \varphi \) is contained in \( S_g \) if and only if \( \varphi(g - \lfloor g/2 \rfloor) = 0 \). For the corresponding element \( w \in W_{\text{final}} \), this means \( \{1, \ldots, g - \lfloor g/2 \rfloor\} \subseteq w(\{1, \ldots, g\}) \), and because \( w(1) < \cdots < w(g) \), this is equivalent to the condition stated in (1).

Part (3) was proved by Ekedahl and van der Geer [5], Theorem 11.5. In fact, in order to align this with the condition given in (1), it is easier to start from their Theorem 11.4 and Lemma 11.3. By (1), if \( EO_w \) is not contained in \( S_g \), then there exists \( i \in \{1, \ldots, g - \lfloor g/2 \rfloor\} \) with \( w(i) \neq i \). Because \( w \) is final, this implies that all simple reflections \( s_{g-\lfloor g/2 \rfloor}, \ldots, s_g \) are less than \( w \) in the Bruhat order (i.e., occur in any reduced expression of \( w \)). The results in [5] then imply that \( EO_w \) is irreducible.

The smoothness claimed in Part (4) can be extracted from Oort’s paper [29]. It was shown in a different way by Wedhorn in [32]; see Corollary 3.5 and Theorem 6.4 (at least if \( p > 2 \)). It also follows from the results in [3] (in particular Corollary 8.4 (iii)). The second statement follows immediately.

To prove (2), by the mass formulas (cf. [13], Proposition 3.9) and Theorem 6.17 in Harashita [17], the number of the irreducible components of \( EO_w \) has the form \( |\text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})|L_w(p) \) for any prime \( p \) and prime-to-\( p \) positive integer \( N \geq 3 \), where \( L_w(X) \) is a polynomial over \( \mathbb{Q} \) which is positive and increasing for positive numbers. As the formula gives a natural number for \( N = 3 \) and \( p = 2 \), it is enough to show that for \( N \geq 4 \) one has \( |\text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})| \geq 2 |\text{Sp}_{2g}(\mathbb{Z}/3\mathbb{Z})| \). The latter is easily verified for \( N = 4 \) and \( N = 5 \) and holds for general \( N \geq 4 \).

Finally, we prove part (5). If \( g = 1 \), then there is nothing to prove. By [29], Theorem 1.3, the closure of \( X \) contains a component of the closure of the unique one-dimensional EO stratum, which is contained in the \( p \)-rank 0 locus as soon as \( g > 1 \). Since every such component is quasi-affine by [29], Theorem 4.1, and of dimension 1, it is not proper and hence cannot be closed in the \( p \)-rank 0 locus. Therefore it contains a superspecial point. Alternatively, (5) also follows from part (3), Proposition 2.3 (2) and the fact that \( A_g^{(0)} \) is proper.

### 2.5. Results about KR strata

On \( A_i \), we have the **Kottwitz-Rapoport stratification** which is given by the relative position of the chains \( H^1_{DR}(A_i) \) and \( \omega(A_i) \),
the latter denoting the Hodge filtration inside $H^1_{DR}(A_i)$. This relative position is an element in $\tilde{W}$; the set of possible relative positions is called the $\mu$-admissible set and denoted by $\text{Adm}(\mu)$. It is a finite set, closed under the Bruhat order, and its maximal elements with respect to the Bruhat order are the translation elements $t^{w(\mu)}$, where $w \in W$ and $\mu = (1^{(g)}, 0^{(g)}) \in \mathbb{Z}^{2g}$. There exists a unique element $\tau \in \tilde{W}$ of length 0 such that $\text{Adm}(\mu) \subset W_\tau$ (and $\tau \in \text{Adm}(\mu)$). We denote the finite part of $\tau$ by $w_\emptyset$. This is at the same time the longest final element. For $x \in \text{Adm}(\mu)$, we denote the finite part of $\tau$ by $\mu^x$. This is the longest final element. For $x \in \text{Adm}(\mu)$, we denote the KR stratum associated with $x$ by $A_x$.

The KR stratification was first considered by Genestier and Ngô, [27]. See also [16] for a detailed exposition. We refer to Section 3 below and to [13], Section 2, for further details.

More or less by definition the KR stratification is smoothly equivalent to the so-called local model $M^{\text{loc}}$ (for $G$ and $\mu$) with its natural stratification given by orbits under the action of the Iwahori group scheme. In [9] it is shown that the special fiber of the local model is isomorphic to the union

$$\bigcup_{x \in \text{Adm}(\mu)} C_x$$

of Schubert varieties in the affine flag variety for $G$. In particular we obtain

**Proposition 2.6.**

1. For $x \in \text{Adm}(\mu)$, the stratum $A_x$ is smooth of dimension $\ell(x)$.
2. The closure relations between KR strata are given by the Bruhat order: We have $A_x \subseteq A_y$ if and only if $x \leq y$.
3. All closures of KR strata are normal and equidimensional.

**Proof.** We have a “local model diagram”

$$A_I \leftarrow \tilde{A}_I \rightarrow M^{\text{loc}},$$

where both morphisms are smooth of the same relative dimension; see [31] or [16]. By definition of the KR stratification on $A_I$, this diagram restricts to a corresponding diagram where on the right hand side we have the corresponding Schubert cell (or its closure, resp.), and on the left hand side, we have the KR stratum (or its closure, resp.), and where again both arrows are smooth. The assertions in (1), (2) and (3) then follow from the same statements for Schubert cells, as soon as we know that all KR strata are non-empty.

By the above, it is enough to show that the stratum $A_\tau$ is non-empty. But it is not hard to write down chains $(A_i)_i$ lying in $A_\tau$ explicitly, starting from a product $A_0$ of supersingular elliptic curves; see Genestier [8], Proposition 1.3.2.

**Example 2.8.** The figure shows the 13 alcoves in the $\mu$-admissible set for $g = 2$. The black dot marks the origin; the gray dots show the translations in the $W$-orbit of $\mu$. The dark gray alcoves are those of $p$-rank 0. Note that the picture is very
similar to the one in [16], Fig. 2. The reduced expressions of the admissible elements coincide, but in our setup, the base alcove $\tau$ lies in the anti-dominant chamber.

3. The image of KR strata in $A_g$

3.1. Let $\pi : A_I \to A_g$ be the natural projection. We will show below that each KR stratum is mapped to a union of EO strata under $\pi$, a fact which will be of great importance in the following sections. We start with a lemma which says in particular that the KR stratum of a point $(A_i)_i \in A_I(k)$, which a priori depends on the chains $(H^1_{DR}(A_i))_i$, $(\omega(A_i))_i$, is in fact already determined by the flags inside $H^1_{DR}(A_0)$ which arise as the images of these chains. Let us make this more precise by introducing some more notation. See also [13], Section 2.

Relative position of chains. Given a point $(A_i)_i \in A_I(k)$, we have the chains $(H^1_{DR}(A_i))_i$ of $2g$-dimensional $k$-vector spaces, and $(\omega(A_i))_i$ of $g$-dimensional subspaces. By using duality, we get chains indexed by $i = 0, \ldots, 2g$. We will express their relative position in terms of a tuple of vectors in $\mathbb{Z}^{2g}$. For $v \in \mathbb{Z}^{2g}$, we denote by $v(1), \ldots, v(2g)$ the entries of $v$. Denote by $\omega_i = ((-1)^{i}, 0^{(2g-i)}) \in \mathbb{Z}^{2g}$. The relative position of the chains $(H^1_{DR}(A_i))_i$, and $(\omega(A_i))_i$ is the tuple $(x_i)_{i=0, \ldots, 2g}$, $x_i \in \mathbb{Z}^{2g}$, such that there exist $k$-bases $e_i^1, \ldots, e_i^{2g}$ of $H^1_{DR}(A_i)$, $i = 0, \ldots, 2g$, with the properties

1. With respect to these bases, the map $H^1_{DR}(A_i) \to H^1_{DR}(A_{i-1})$ is given by the diagonal matrix $\text{diag}(1, \ldots, 1, 0, 1, \ldots, 1)$, the 0 being in the $2g-i+1$-th place.
2. For all $i, j$, we have $(x_i - \omega_i)(j) \in \{0, 1\}$.
3. The vectors $e_j^i$ with $(x_i - \omega_i)(j) = 0$ are a basis of the $g$-dimensional subspace $\omega(A_i) \subset H^1_{DR}(A_i)$.
One checks that the tuple \((x_i)_i\) does not depend on the choice of basis. The strange-looking normalization implies that \((x_i)_i\) is an \textit{extended alcove} in the sense of [24], using the normalization of [13] Section 2. The extended affine Weyl group \(\widetilde{W}\) acts simply transitively on the set of extended alcoves. If we write \(x \in \widetilde{W}\) as \(x = t^\lambda w\), \(w \in W\), then \(x\) acts on an alcove \((y_i)_i\) by \(x.(y_i)_i = (\lambda + w(y_i))_i\). It is clear by construction that all alcoves which arise as relative positions in the above situation are permissible (or equivalently, admissible); see [24] or [13], Definition 2.4.

Relative position of extended flags. Consider a 2\(g\)-dimensional symplectic vector space \((V, \langle \cdot, \cdot \rangle)\) over \(k\) with a complete self-dual flag \(0 = V_0 \subset V_1 \subset \cdots \subset V_{2g-1} \subset V\) and an “extended flag”, by which we mean a system \(0 = F_0 \subset F_1 \subset \cdots \subset F_{2g}\) of \(V\), such that \(F_{2g}\) is a maximal totally isotropic subspace (in particular \(\dim F_{2g} = g\)), \(F_i \subset V_i\), and \(\dim F_i/F_{i-1} \in \{0, 1\}\) for all \(i\). The relative position of such a system is the tuple \((\tilde{x}_i)_{i=0,...,2g}, \tilde{x}_i \in \mathbb{Z}^{2g}\), such that there exists a \(k\)-basis \(e_1, \ldots, e_{2g}\) of \(V\) with the properties

1. For every \(i\), \(V_i\) is spanned by \(e_1, \ldots, e_i\).
2. For all \(i, j\), we have \(\tilde{x}_i(j) \in \{0, -1\}\).
3. For all \(i\), \(F_i\) is the subspace of \(V\) spanned by those \(e_j\) with \(\tilde{x}_i(j) = -1\).

We obtain a system like this in the above setting by defining \(V = H^1_{DR}(A_0)\), \(V_i = \text{im}(H^1_{DR}(A_{2g-i}) \to H^1_{DR}(A_0))\), \(F_i = \text{im}(\omega(A_{2g-i}) \to H^1_{DR}(A_0))\). From the relative position \((x_i)_i\) of chains we obtain the relative position of extended flags by setting \(\tilde{x}_i(j) = \min(x_i(j), 0)\).

Relative position of flags. As before, fix a 2\(g\)-dimensional symplectic vector space \((V, \langle \cdot, \cdot \rangle)\) over \(k\) with a complete self-dual flag \(0 = V_0 \subset V_1 \subset \cdots \subset V_{2g-1} \subset V\). Now fix a second complete self-dual flag, or what amounts to the same, a partial flag \(0 = F'_0 \subset F'_1 \subset \cdots \subset F'_{2g}\) with \(\dim F'_i = i\), \(F'_{2g}\) maximal totally isotropic. The relative position of such a system is the tuple \((x'_i)_{i=0,...,g}, x'_i \in \mathbb{Z}^{2g}\), such that there exists a \(k\)-basis \(e_1, \ldots, e_{2g}\) of \(V\) with the properties

1. For every \(i\), \(V_i\) is spanned by \(e_1, \ldots, e_i\).
2. For all \(i, j\), we have \(x'_i(j) \in \{0, -1\}\).
3. For all \(i\), \(F'_i\) is the subspace of \(V\) spanned by those \(e_j\) with \(x'_i(j) = -1\).

Of course, we can also express the relative position as an element of the Weyl group \(W\): it is the unique element \(v \in W\) such that \(v \omega_i = x'_i\) for \(i = 0, \ldots, g\). Given an extended flag as above, and the corresponding relative position \((\tilde{x}_i)_i\), we obtain a usual flag by forgetting the steps where we have equality, and obtain the relative position of flags by analogously forgetting multiple occurrences of vectors among the \(\tilde{x}_i\)’s.

Lemma 3.1. Let \((x_i)_{i=0,...,2g} \in \text{Adm}(\mu)\) be an admissible extended alcove. Let \(x \in \widetilde{W}\) with \(x(\omega_i) = (x_i)_i\), and write \(x = t^\lambda w = wt^{\rho(x)}\) for \(w \in W\), \(\lambda(x), \rho(x) \in X_*(T)\). Define \((\tilde{x}_i)_{i=0,...,2g}\) by \(\tilde{x}_i(j) = \min(x_i(j), 0)\).

1. The extended alcove \((x_i)_i\) is determined by the datum \((\tilde{x}_i)_{i=0,...,2g}\). We can recover \(w, \lambda, \rho\) as follows: \(\lambda(x) = \tilde{x}_{2g} + (1, \ldots, 1)\), \(\rho(x)\) is given by

\[
\rho(x)(i) = \begin{cases} 
1 & \text{if } \tilde{x}_{i-1} = \tilde{x}_i \\
0 & \text{if } \tilde{x}_{i-1} \neq \tilde{x}_i,
\end{cases}
\quad i = 1, \ldots, 2g.
\]

Finally \(w \in W \subset S_{2g}\) is the unique permutation such that for every \(i\) with \(\tilde{x}_{i-1} \neq \tilde{x}_i\) the difference \(\tilde{x}_i - \tilde{x}_{i-1}\) has its unique \(-1\) in the \(w(i)\)-th place.
(2) Now define \((x'_i)_{i=0,...,g}\) by letting \(x'_i\) be the unique vector among the \(\tilde{x}_s\) with \(\sum_j x'_i(j) = -i\) (this vector may occur several times among the \(\tilde{x}_s\)). Let \(v \in W\) be the unique element such that \(v(\omega_i) = x'_i, \ i = 0, \ldots, g\). Denote by \(w_0 \in W\) the longest element, and let \(w_{\rho(x)}\) be the unique element of minimal length with \(w_{\rho(x)}(\mu) = w_0(\rho(x))\). (Here the minimality is equivalent to requiring that \(w_{\rho(x)}\) be final.) Then \(v = w_{\rho(x)}\).

(3) Continuing with the above notation, if \(\rho(x) = \mu\), then \(x = wt^\mu = \tau\).

Proof. We start with (1). First, \(\lambda(x) = x_0 = x_{2g} + (1, \ldots, 1) = \tilde{x}_{2g} + (1, \ldots, 1)\). Furthermore, for all \(i, x_i - x_{i-1} = (0, \ldots, 0, -1, 0, \ldots, 0)\), the \(-1\) being in the \(w(i)\)-th place. This implies the desired formulas for \(\rho(x)\) and \(w\). The description in (2) is also easy to check. For (3), use that with the above notation, \(\tau = w_{\mu} t^\mu\).

The lemma can also be seen in the spirit of the numerical characterization of KR strata as in [13]. Now recall that \(W_{\text{final}}\) (or equivalently, the set of elementary sequences) parametrizes the set of EO strata on \(A_g\). For \(w \in W_{\text{final}}\), we denote by \(EO_w\) the corresponding EO stratum.

**Proposition 3.2.** If \(\pi(A_{I,x}) \cap EO_w \neq \emptyset\) for some \(x \in \text{Adm}(\mu)\) and \(w \in W_{\text{final}}\), then \(\pi(A_{I,x}) \supset EO_w\).

Proof. An Iwahori level structure on an object \((A, \lambda, \eta)\) in \(A_g\) is the same as a flag \(H_\bullet\) of finite flat subgroup schemes of \(A[p]\) satisfying certain properties and the Kottwitz-Rapport invariant is determined by this structure. In particular if two objects \((A, \lambda, \eta, H_\bullet)\) and \((A', \lambda', \eta', H_\bullet')\) in \(A_I\) have the property that \((A[p], e_\lambda, H_\bullet) \simeq (A'[p], e'_\lambda, H_\bullet')\), where \(e_\lambda\) denotes the pairing on \(A[p] \times A[p]\) defined by \(\lambda\), then they lie in the same KR stratum. This follows from Lemma 3.1 (2).

Let \((A, \lambda, \eta, H_\bullet) \in A_{I,x}\) be a point such that \((A, \lambda, \eta) \in EO_w\). Let \((A', \lambda', \eta')\) be a point in \(EO_w\). One has an isomorphism \(\alpha : (A'[p], e_{\lambda'}) \simeq (A[p], e_{\lambda})\). Set \(H'_\bullet := \alpha^{-1}(H_\bullet)\). We get a point \((A', \lambda', \eta', H'_\bullet)\) in \(A_I\) such that there is an isomorphism \(\alpha : (A'[p], e_{\lambda'}, H'_\bullet) \simeq (A[p], e_{\lambda}, H_\bullet)\).

Therefore, \((A', \lambda', \eta', H'_\bullet)\) lies in \(A_{I,x}\) and hence \(\pi(A_{I,x}) \supset EO_w\).

Note that in [5], Proposition 9.6, the case of KR strata for \(w\tau, w \in W\), is treated. For \(x \in \text{Adm}(\mu)\), put

\[\text{ES}(x) := \{w \in W_{\text{final}} : \pi(A_{I,x}) \cap EO_w \neq \emptyset\}\]

**Corollary 3.3.** For \(x \in \text{Adm}(\mu)\), one has

\[\pi(A_{I,x}) = \coprod_{w \in \text{ES}(x)} EO_w\]

Let \(w \in W_{\text{final}}\). Then \(w\tau\) is an element of the admissible set: We have \(w \leq w_0\), because \(w_0\) is the longest final element, and therefore \(w\tau = w_0\tau = t^\mu\). Ekedahl and van der Geer [5] show that the image of \(A_{w\tau}\) in \(A_g\) is the EO stratum \(EO_w\), in other words \(\text{ES}(w\tau) = \{w\}\). Furthermore, they show that the restriction of \(\pi\) induces a finite étale surjective map \(A_{w\tau} \to EO_w\) (although in [5] this is phrased in a different way; see Section 9).

It seems to be an interesting problem to describe the sets \(\text{ES}(x)\), where \(x \in \text{Adm}(\mu)\). See [5], Section 13, for some results in the case when \(x \in W\tau\).
Example 3.4. In the case \( g = 2 \) (see the example in the previous section), we have 4 EO strata, which can be described as the superspecial locus, the supersingular locus minus the superspecial points, the \( p \)-rank 1 locus and the \( p \)-rank 2 locus, and which correspond to the following final elements

\[
w_{\text{asp}} = \text{id}, \ w_{\text{ssi}} = s_2, \ w_1 = s_1 s_2, \ w_2 = s_2 s_1 s_2.
\]

In this case, it is not hard to write down all the sets \( \text{ES}(x) \). As an abbreviation, we write \( s_{120} := s_1 s_2 s_0 \) etc.

| \( x \) | \( \text{ES}(x) \) | \( p - \text{rank} \) |
|------|----------------|----------|
| \( \tau, s_1 \tau \) | \{w_{\text{asp}}\} | 0 |
| \( s_0 \tau, s_2 \tau \) | \{w_{\text{ssi}}\} | 0 |
| \( s_0 \tau \) | \{w_{\text{asp}}, w_{\text{ssi}}\} | 0 |
| \( s_{10} \tau, s_{21} \tau, s_{10} \tau, s_{12} \tau \) | \{w_1\} | 1 |
| \( s_{120} \tau, s_{101} \tau, s_{212} \tau, s_{121} \tau \) | \{w_2\} | 2 |

4. A flag bundle over \( \mathcal{A}_g \)

4.1. Definition of \( \text{Flag}^+(\mathbb{H}) \). Let \( f: \mathcal{A}^{\text{univ}} \rightarrow \mathcal{A}_g \) denote the universal abelian scheme. We write \( \mathbb{H} := H^1_{\text{DR}}(\mathcal{A}^{\text{univ}}/\mathcal{A}_g) := R^1 f_{\ast} \Omega^1 \). This is a locally free \( \mathcal{O}_{\mathcal{A}_g} \)-module of rank \( 2g \). Since \( \mathcal{A}^{\text{univ}} \) is equipped with a principal polarization, \( \mathbb{H} \) carries a non-degenerate alternating pairing. Furthermore, Frobenius and Verschiebung on \( \mathcal{A}^{\text{univ}} \) induce operators \( F: \mathbb{H}^{(p)} \rightarrow \mathbb{H}, V: \mathbb{H} \rightarrow \mathbb{H}^{(p)} \) (where \( \mathbb{H}^{(p)} \) denotes the pull-back of \( \mathbb{H} \) under Frobenius).

Definition 4.1. Denote by \( p: \text{Flag}^+(\mathbb{H}) \rightarrow \mathcal{A}_g \) the variety of symplectic flags in \( \mathbb{H} \).

4.2. The map \( \mathcal{A}_I \rightarrow \text{Flag}^+(\mathbb{H}) \). The flag bundle \( \text{Flag}^+(\mathbb{H}) \) is closely related to the moduli space \( \mathcal{A}_I \) with Iwahori level structure. There are several ways to establish a relationship (see Section 9 and [5] for a slightly different one). Here we use the following naive approach.

Definition 4.2. Denote by \( \iota \) the morphism \( \mathcal{A}_I \rightarrow \text{Flag}^+(\mathbb{H}) \) defined as follows. For \( (\mathcal{A}_I)^{(S)} \), the image point in \( \text{Flag}^+(\mathbb{H})(S) \) is given by the pair \( (A_0, F_{\ast}) \), where \( F_{\ast} \) is the flag

\[
0 = \alpha(H^1_{\text{DR}}(A_{2g})) \subset \alpha(H^1_{\text{DR}}(A_{2g-1})) \subset \cdots \subset \alpha(H^1_{\text{DR}}(A_1)) \subset \alpha(H^1_{\text{DR}}(A_0)) = \mathbb{H}.
\]

Here for each \( i \), \( \alpha \) denotes the natural map \( H^1_{\text{DR}}(A_i) \rightarrow H^1_{\text{DR}}(A_0) \).

Lemma 4.3. The morphism \( \iota \) is universally injective and finite. In particular, it is a homeomorphism of \( \mathcal{A}_I \) onto a closed subset of \( \text{Flag}^+(\mathbb{H}) \).

Proof. For an algebraically closed field \( K \), the morphism induces an injection on \( K \)-valued points: if \( A_0 \) is given, then we can recover the chain of Dieudonné modules, and hence the chain \( (A_i) \) from the flag \( F_{\ast} \). Since \( \mathcal{A}_I \) is proper over \( \mathcal{A}_g \), it is clear that \( \iota \) is proper; being proper and quasi-finite, it is finite. 

Note however that the induced maps on tangent spaces is not injective. In particular, \( \iota \) is not a closed immersion. It is easy to describe the image of \( \iota \) (in terms of points over an algebraically closed field): We just must ensure that the flag gives rise to a chain of Dieudonné modules, i.e. that all its members are stable.
under Frobenius and Verschiebung. This is not to say, however, that the geometry of this locus would be easy to understand.

**Example 4.4.** Let us discuss the case \( g = 1 \). This is not a typical case because it is too simple in some respects, but it is nevertheless instructive. The Iwahori type moduli space \( \mathcal{A}_I \) can be described as the union of 2 copies of the smooth curve \( \mathcal{A}_g \), glued at the finitely many supersingular points, so that they intersect transversely in these points. More precisely, we have closed subschemes \( \mathcal{A}, \mathcal{A}' \) of \( \mathcal{A}_I \), both isomorphic to \( \mathcal{A}_g \), which we can describe as

\[
\mathcal{A} = \{ (E \xrightarrow{F} E^{(p)}) \}
\]
\[
\mathcal{A}' = \{ (E^{(p)} \xrightarrow{V} E) \}.
\]

The intersection of \( \mathcal{A} \) and \( \mathcal{A}' \) inside \( \mathcal{A}_I \) can be identified with the discrete set of supersingular points of either \( \mathcal{A} \) or \( \mathcal{A}' \).

The projection \( \pi : \mathcal{A}_I \to \mathcal{A}_g \) is the identity map when restricted to \( \mathcal{A} \) (and hence induces isomorphisms on the tangent spaces of non-supersingular points), and is the Frobenius morphism when restricted to \( \mathcal{A}' \) (and hence induces the zero map on the tangent spaces of non-supersingular points). If \( x \in \mathcal{A}_I(k) \) is a supersingular point, we can identify its tangent space with the product

\[
T_x \mathcal{A} \times T_x \mathcal{A}' \to \mathcal{T}_\pi(x) \mathcal{A}_g = T_x \mathcal{A}'.
\]

Now let us investigate the morphism \( \iota \). Let \( x \in \mathcal{A}_I(k) \). We can write \( T_\iota(x) \Flag^+ \mathbb{H} \) as a product of \( T_\pi(x) \mathcal{A}_g \) and the tangent space \( T_f \) of \( \iota(x) \) within the fiber over \( \mathcal{A}_g \). Then the map

\[
T_x \mathcal{A}_I \xrightarrow{d\iota} T_\iota(x) \Flag^+ \mathbb{H} \xrightarrow{proj.} T_f
\]

is the zero map. The analogous statement is true for all \( g \), because for an abelian variety \( A \) over \( k \), all lifts \( \tilde{A} \) of \( A \) to \( k[\varepsilon]/\varepsilon^2 \) give rise to “the same” \( H^1_{DR}(\tilde{A}) \)—the crystal of \( A \) evaluated at \( k[\varepsilon]/\varepsilon^2 \).

In this case \( \iota \) induces a closed immersion \( \mathcal{A} \to \Flag^+ \mathbb{H} \), but it induces the zero map on the tangent spaces of the non-supersingular points in \( \mathcal{A}' \). In particular, even in the case \( g = 1 \), \( \iota \) is not a closed immersion.

5. KR strata are quasi-affine

In this section we prove that all KR strata are quasi-affine. We follow the method used in [5] Lemma 6.2, based on “Raynaud’s trick”.

5.1. Ample line bundles on \( \mathcal{A}_I \). We use the following lemmas to produce an ample sheaf on \( \mathcal{A}_I \).

**Lemma 5.1.** Let \( X \) be a scheme, let \( V \) be a vector bundle on \( X \) of rank \( r \), and let \( \mathcal{F} = \Flag(V) \) be the \( X \)-scheme of full flags in \( V \). Denote by

\[
\mathcal{F}^\univ = (0 = \mathcal{F}_0^\univ \subset \mathcal{F}_1^\univ \subset \cdots \subset \mathcal{F}_r^\univ = V \times_X \mathcal{F})
\]

the universal flag, and for each \( i \), by \( \mathcal{L}_i \) the line bundle \( \mathcal{F}_i^\univ/\mathcal{F}_i^\univ \) on \( \mathcal{F} \). Then the tensor product

\[
\bigotimes_{i=1}^r \mathcal{L}_i \xrightarrow{(1+n)} \bigotimes_{i=1}^r \mathcal{F}_i^\univ / \mathcal{F}_i^\univ
\]

is very ample (for \( \mathcal{F} \to X \)).
Proof. Embed the flag scheme into a product of Grassmannians over \(X\), and then use the Plücker embedding and the Segre embedding to embed it into projective space over \(X\). The pull-back of \(\mathcal{O}(1)\) under this embedding is the line bundle given in the lemma.

The following result was proved by Moret-Bailly; see Theorem 1.1 in the Introduction of \cite{20}.

Lemma 5.2. The line bundle \(\bigwedge^{\text{top}} \omega(\mathcal{A}^{\text{univ}})\) on \(\mathcal{A}_g\) is ample.

5.2. As a special case of Lemma 3.1 (2), we have

\[
\text{Lemma 5.3. Let } x \in \text{Adm}(\mu), \text{ write } x = wt^p(x), \text{ and fix a chain } (A_i)_i \text{ in the KR stratum associated with } x. \text{ We write } \rho(x) = (\rho(x)(1), \ldots, \rho(x)(2g)) \in \mathbb{Z}^{2g}.
\]

Furthermore fix \(i \in \{1, \ldots, 2g\}\).

Then \(\alpha(\omega_i) \subseteq \alpha(\omega_{i-1}) \subset H^1_{DR}(A_0)\) is a strict inclusion if and only if \(\rho(x)(i) = 1\).

Now we can prove the main result of this section:

Theorem 5.4. All KR strata in \(A_i\) are quasi-affine.

Proof. Let \(x \in \text{Adm}(\mu)\). We construct an ample line bundle on \(A_x\) which is a torsion element in the Picard group. This shows that the structure sheaf is ample, which by EGA II Proposition 5.1.2 is equivalent to \(A_x\) being quasi-affine. For \(i \in \{1, \ldots, 2g\}\), let

\[
\mathcal{L}_i := H^1_{DR}(A_{i-1})/\alpha(H^1_{DR}(A_i)),
\]

where \((A_i)\) denotes the universal chain of abelian schemes, and where as usual \(\alpha\) denotes the canonical map. So \(\mathcal{L}_i\) is a line bundle on \(A_x\) (or even on \(A_i\)). Write \(x = wt^p(x), w \in W, \rho(x) \in X_*(T)\).

1. We have

\[
\mathcal{L}_i \cong (\omega(A_{v(i)-1})/\alpha(\omega(A_{v(i)})))^{\varepsilon(i)}
\]

where

\[
v(i) = \begin{cases} w^{-1}(i) & \text{ if } \rho(x)(w^{-1}(i)) = 1 \\ 2g - w^{-1}(i) + 1 & \text{ if } \rho(x)(w^{-1}(i)) = 0 \end{cases}
\]

\[
\varepsilon(i) = \begin{cases} 1 & \text{ if } \rho(x)(w^{-1}(i)) = 1 \\ -1 & \text{ if } \rho(x)(w^{-1}(i)) = 0 \end{cases}
\]

2. If \(\rho(x)(i) = 1\), then

\[
\mathcal{L}_i \cong (\omega(A_{i-1})/\alpha(\omega(A_i)))^{\otimes p}
\]

To prove (1), first note that using duality we may restrict to the case where \(\varepsilon(i) = 1\).

It is easy to check that the analogue of the claimed equality holds at the unique \(T\)-fixed point of the stratum of the local model corresponding to \(x\) (cf. \cite{13}, Subsection 2.6). This implies that it holds pointwise, at every point of \(A_x\). More precisely, we see (from the permissibility condition) that \(i \geq v(i)\), and that the inclusion \(\omega(A_{v(i)-1}) \rightarrow H^1_{DR}(A_{v(i)-1})\) factors through \(\alpha(H^1_{DR}(A_i))\). This is true globally, so that we get a morphism \(\omega(A_{v(i)-1})/\alpha(\omega(A_{v(i)})) \rightarrow \mathcal{L}_i\) globally on \(A_x\). It follows that this is an isomorphism by the pointwise result, and this gives (1).

Furthermore, (2) follows immediately from the fact that the Hodge filtration is the image of the Verschiebung morphism, more precisely:

\[
\omega(A)^{(p)} = \text{im}(V: H^1_{DR}(A) \rightarrow H^1_{DR}(A)^{(p)}).
\]
Putting (1), (2) and Lemma 6.3 together, we obtain first that all the line bundles \( \omega(A_{i-1})/\omega(A_i) \) (restricted to \( A_i \)) are of finite order, and using (1) once more, that all the \( L_i \) are of finite order. By Lemma 6.1 \( a \) suitable tensor product of \( L_i \)'s is very ample for the projection from \( A_x \) to \( A_g \). The pull-back of the line bundle \( \Lambda^{\text{top}} \omega(A_{\text{univ}}) \) on \( A_g \) to \( A_f \) is \( \Lambda^{\text{top}} \omega(A_0) \), which can also be expressed as a tensor product of \( L_i \)'s, because \( \omega(A_0) \) has a filtration whose subquotients are of the form \( L_i \). Using Lemma 6.2 altogether we obtain an ample invertible sheaf of finite order on \( \iota(A_x) \), and also on \( A_x \) since \( \iota \) is finite. \( \blacksquare \)

5.3. Affineness. Compare the results about affineness of KR strata: Using Proposition 9.6 \( (\text{ii}) \) we get from \( [5] \), Proposition 10.5 \( (\text{ii}) \) that for \( p \) large and \( w \in W \) such that \( w \tau \) is admissible and has \( p \)-rank 0, the KR stratum \( A_{w \tau} \) is affine. It would be interesting to generalize this result to all KR strata of \( p \)-rank 0, maybe using the map \( \iota \) defined above together with a Pieri formula for the affine flag variety.

In \( [13] \) we showed that for \( p \geq 2g \), every superspecial KR stratum is affine.

6. The closure of components of KR strata

6.1. Sections. Note that although for any two points in a fixed EO stratum \( EO_w \) the \( p \)-torsion group schemes are isomorphic, one cannot expect to find a flat, surjective cover \( X \to EO_w \) such that the pull-back of the finite flat group scheme \( A_{\text{univ}}[p] \) to \( X \) is constant. However, we at least have the following lemma (whose first part is taken from \([5]\)).

Lemma 6.1. Fix an EO type \( w \in W_{\text{final}} \).

1. There exists a flat surjective map \( X \to EO_w \) such that the data

\[
(H^1_{DR}(A^\text{univ}_{|X}), \omega(A^\text{univ}_{|X}), F, V, \langle \cdot, \cdot \rangle)
\]

is constant.

2. Let \( X \) be as in (1), let \( x \in \text{Adm}(\mu) \), such that \( \pi(A_x) \cap EO_w \neq \emptyset \), and fix \( a \in A_x \) such that \( \pi(a) \in EO_w \). There is a morphism \( f_a : X \to \text{Flag}^+(\mathbb{H}) \) of schemes over \( A_g \), such that \( f_a(X) \subseteq \iota(A_x) \) and \( \iota(a) \in f_a(X) \).

Proof. Part (1) is proved in the proof of Proposition 9.6 \( (\text{ii}) \) in \([5]\). We prove part (2). Let \( H \) be the pull-back of \( H^1_{DR}(A_{x(a)}) \) to \( X \) (where \( A_{x(a)} \) denotes the abelian scheme over \( k \) corresponding to \( \pi(a) \)). Denote the map \( X \to A_g \) by \( q \). We have to define, for every \( S \) and every \( s \in X(S) \), a flag inside \( H^1_{DR}(A_{q(s)}) \). However, by assumption \( H^1_{DR}(A_{q(s)}) \) is trivial, and we can identify it with the pull-back of \( H \) to \( S \). In particular, the flag in \( H \) induced from the point \( a \) gives us a flag in \( H^1_{DR}(A_{q(s)}) \), and this is the one we use to define the morphism \( f_a \). Clearly, \( \iota(a) \in f_a(X) \). Furthermore, the assertion that \( f_a(X) \subseteq \iota(A_x) \), which is meant purely set-theoretically, follows immediately from Lemma 6.1. \( \blacksquare \)

As an application, we have

Lemma 6.2. Let \( x \in \text{Adm}(\mu) \), let \( Z \) be a connected component of \( A_x \). Then the image \( \pi(Z) \) is a union of connected components of EO strata.

Proof. Let \( w \) be an EO type and \( EO_w^c \) be a connected component of the EO stratum \( EO_w \), such that \( \pi(Z) \cap EO_w^c \neq \emptyset \). We must show that \( EO_w^c \subseteq \pi(Z) \). Let \( q : X \to EO_w^c \) be the restriction of a map as in Lemma 6.1 to \( EO_w^c \). Let \( \{X_i\} \)
be the connected components of $X$. We show that whenever $\pi(Z) \cap q(X_i) \neq \emptyset$, one has $q(X_i) \subset \pi(Z)$. Let $a \in Z$ such that $\pi(a) \in q(X_i)$, and consider the map $f_a : X \to \text{Flag}^+(\mathbb{H})$ from Lemma 6.1. Recall that $p$ denotes the projection $\text{Flag}^+(\mathbb{H}) \to A_g$. Since $f_a(X_i) \subset i(Z)$ and $p \circ i = \pi$, we get $q(X_i) = p(f_a(X_i)) \subset p(i(Z)) = \pi(Z)$. Since $g$ is flat and surjective, $\{q(X_i)\}$ is an open covering of $EO^c_w$, and hence $U_1 := \pi(Z) \cap EO^c_w$ is the union of those $q(X_i)$ with $q(X_i) \cap \pi(Z) \neq \emptyset$. The scheme $EO^c_w$ is a disjoint union of $U_1$ and the open subset $U_2$ which is the union of those $q(X_i)$ which are disjoint from $\pi(Z)$. The connectedness of $EO^c_w$ implies that $\pi(Z) \supset EO^c_w$. \(\blacksquare\)

Remark 6.3. As we know, all non-supersingular EO strata are connected (and we will show below that the same is true for all non-supersingular KR strata; see Theorem 7.3). However, in the supersingular case, the image $\pi(Z)$ of a connected component $Z$ of $\mathcal{A}_{w_0, \tau}$ contains one connected component of $EO^c_{w, \text{suppl}}$ and $p^2 + 1$ connected components (which simply are points) of $EO^c_{w, \text{suppl}}$.

6.2.

Theorem 6.4. Let $x \in \text{Adm}(\mu)$, let $Z$ be a connected component of $\mathcal{A}_x$, and let $\overline{Z}$ be its closure in $\mathcal{A}_1$. Then $\overline{Z} \cap \mathcal{A}_x \neq \emptyset$.

This was proved in the special case that $x$ is superspecial ([13], Lemma 6.4). It follows from [4], Theorem 6.1 and the identification of $\mathcal{U}_w$ and $\mathcal{A}_{w, \tau}$ that this is true for elements $w \in W$ such that $w \tau \in \text{Adm}(\mu)$. We start with a simple lemma which will allow us to prove the theorem by induction.

Lemma 6.5. With the notation of the theorem, assume that for some $y \in \text{Adm}(\mu)$, the intersection $\mathcal{A}_y \cap \overline{Z}$ is not empty. Then this intersection is a union of connected components of $\mathcal{A}_y$.

Proof. Assume that $V$ is a connected component of $\mathcal{A}_y$ which meets $\overline{Z}$. We have to show that it is contained in $\overline{Z}$. The assumption implies that the closure $\overline{\mathcal{A}_x}$ contains $\mathcal{A}_y$, so in particular it contains $V$. However, if $Z, Z'$ are different connected components of $\mathcal{A}_x$, then their closures do not intersect, because $\overline{\mathcal{A}_x}$ is normal. \(\blacksquare\)

Because of this lemma, the theorem follows from

Proposition 6.6. Let $x \in \text{Adm}(\mu)$ be an element of length $> 0$, i.e. such that $\dim \mathcal{A}_x > 0$. Let $Z$ be a connected component of $\mathcal{A}_x$. Then $Z$ is not closed in $\mathcal{A}_1$.

Proof. If $\mathcal{A}_x$ is contained in the $p$-rank 0 locus, then $Z$ being closed would imply that $Z$ is proper, since the $p$-rank 0 locus is proper (Proposition 2.1). However, since $Z$ is quasi-affine, it cannot be proper unless it is finite, which contradicts our hypothesis, because $\dim Z = \dim \mathcal{A}_x$.

Now assume that $Z$ is closed in $\mathcal{A}_1$. We will prove that the $p$-rank on $Z$ is necessarily 0 in this case, so that we can conclude by the previous step.

By Lemma 6.2, $\pi(Z)$ is a union of connected components of EO strata. Since $\pi$ is proper and $Z$ is assumed to be closed, $\pi(Z)$ is closed, too. However, whenever $V$ is a connected component of any EO stratum, its closure meets the superspecial.
locus, the unique zero-dimensional EO stratum. This shows that \( \pi(Z) \) meets the superspecial locus, hence the \( p \)-rank on \( Z \) is 0.

7. Connectedness of KR strata

The aim of this section is to prove that all non-superspecial KR strata are connected, or equivalently irreducible. We follow the strategy of proof of [5], Theorem 11.4 (ii).

7.1. Unions of one-dimensional strata. Here, and below, we silently exclude the case \( g = 1 \) which is uninteresting with respect to the results discussed in the sequel, and would often require a separate treatment. We start by considering unions of KR strata of dimension \( \leq 1 \). All of these are superspecial by [13], Proposition 4.4, which we recall here for the convenience of the reader, because it will be used below in several places:

**Proposition 7.1.** The KR stratum associated with \( x \in \text{Adm}(\mu) \) is superspecial if and only if \( w \in \bigcup_i W_{(i,g-i)} \tau \).

In particular, for all \( w \in \bigcup_i W_{(i,g-i)} \), the KR stratum associated with \( w\tau \) is supersingular.

So for all KR strata of dimension \( \leq 1 \) we have the description of superspecial KR strata given in [13] at our disposal. To make use of this description, we use the following lemma (the notation used in its statement is independent from our notation fixed above).

**Lemma 7.2.** Let \( G \) be a connected reductive group over the finite field \( \mathbb{F}_q \). Let \( T \subset B \subset G \) be a maximal torus and a Borel subgroup over \( \mathbb{F}_q \). Denote by \( W \) the absolute Weyl group, and denote by \( \sigma \) the automorphism of \( W \) induced by the Frobenius automorphism \( \sigma \) of \( \mathbb{F}_q/\mathbb{F}_q \) (and \( G(\overline{\mathbb{F}_q}) \) etc.). Let \( S \subset W \) be the set of simple reflections determined by \( B \), and let \( I \subseteq S \) be a subset which is not contained in any proper \( \sigma \)-stable subset of \( S \). Then the union

\[
X(I) := X(\text{id}) \cup \bigcup_{s \in I} X(s)
\]

of Deligne-Lusztig varieties is connected.

**Proof.** See [10]. In the case of a unitary group, which is the case we need below, this result is proved in [5], Lemma 7.6 (ii).

As a side note we remark that the lemma implies Lusztig’s connectedness criterion for Deligne-Lusztig varieties (see e.g. [11]) much in the same way as we derive the connectedness of non-superspecial KR strata below. Namely, we use the facts that Deligne-Lusztig varieties are quasi-affine (this implies that no irreducible component can be closed in \( G/B \)), and they have normal closures. Then proceed as in the proof of Theorem 7.4. See [10] for more details.

**Theorem 7.3.** Let \( x \in \text{Adm}(\mu) \), and assume that \( x \) is not superspecial. Let \( S(x) \subseteq \{0, \ldots, g\} \) be the set of indices \( i \) such that the simple reflection \( s_i \) is less or equal
than $x^{r-1}$ with respect to the Bruhat order (in other words: $s_i$ occurs in any, or equivalently: every, reduced word expression for $x^{r-1}$). Then

$$A_{x,1} := \bigcup_{i \in S(x)} \overline{A}_{s_i}$$

is connected, where $\overline{A}_{s_i} = A_{s_i} \cup A_{s_{r}}$ is the closure of $A_{s_i}$.

**Proof.** Saying that $x$ is not superspecial is equivalent to saying that for every $i = 0, \ldots, [g/2]$, $S(x)$ contains at least one of $s_i$, $s_{g-i}$. In particular $A_{x,1}$ contains $\overline{A}_{s_0}$ or $\overline{A}_{s_g}$. Both these have one-dimensional image in $A_g$, because the image, which is a union of EO strata, strictly contains the 0-dimensional EO stratum (which is the set of superspecial points). This implies that in both cases, the image is the closure of the unique 1-dimensional EO stratum, which is connected (as proved by Oort [29], Proposition 7.3). This implies that for any two superspecial points in $A_g$, there exists a connected component of $\overline{A}_{s_0}$ (and likewise for $s_g$) connecting a point of $A_r$ in the fiber of $\pi: A_I \to A_g$ over the first point with a point of $A_r$ in the fiber over the second point.

Therefore it is enough to show that any two points of $A_r$ lying in the same fiber $\pi^{-1}(A_0)$ can be connected by a sequence of lines in $A_{x,1}$. Here $A_0 \subset A_g(k)$ is some superspecial abelian variety. We obtain a point $(A_0 \to A_g)$ of $A_{\{0,g\}}(k)$ by setting $A_g := A_0^{(p)}$, and taking Frobenius as the isogeny. Denote the projection $A_I \to A_{\{0,g\}}$ by $\pi_{\{0,g\},1}$. We have

$$A_r \cap \pi^{-1}(A_0) = A_r \cap \pi_{\{0,g\},1}^{-1}((A_0 \to A_g)).$$

Using the description of $\pi_{\{0,g\},1}^{-1}((A_0 \to A_g))$ given in [13], Theorem 6.3, we are in the situation of the previous lemma in the special case of the unitary group over $\mathbb{F}_p$, given by the Dynkin diagram of type $A_{g-1}$, on which Frobenius acts by the non-trivial automorphism (if $g > 2$). The theorem follows.

---

### 7.2. Non-superspecial KR strata are connected.

**Theorem 7.4.** Let $x \in \text{Adm}(\mu)$, and assume that $x$ is not superspecial. Then $A_x$ is irreducible.

Recall that, in order to avoid technicalities, we equip $A_g$ and $A_I$ with a full symplectic level $N$-structure with respect to a fixed primitive $N$-th root of unity, so that these spaces are connected.

**Proof.** It is equivalent to show that the closure $\overline{A}_x$ is irreducible, and because this closure is normal, it is even enough to show that it is connected. By theorem 6.4, every connected component meets the minimal KR stratum $A_r$, hence it meets the locus $A_{x,1}$ defined in Theorem 7.3. Since $A_{x,1}$ is connected by the theorem and contained in $\overline{A}_x$, $\overline{A}_x$ itself is connected.

Compare [5], Theorem 11.4. Note that if $w$ has $p$-rank $g$ or $g-1$, this result is known by the work of the second author [31], [32]. On the other hand, for superspecial KR strata, there is a formula for the number of connected components in [13].
7.3. All supersingular KR strata are superspecial. As a corollary, we can prove conjecture 4.5 in [13].

Corollary 7.5. Every KR stratum which is entirely contained in the supersingular locus, is superspecial, i.e. of the form \( A_x \) for \( x \in \bigcup_{i=0}^{[g/2]} W_{(i,g-i)} \).

Proof. We may clearly assume that \( N \) is large (by passing to a suitable étale extension, if necessary), so that we can assume that all EO strata contained in the supersingular locus \( S_g \) are disconnected. Let \( x \in \text{Adm}(\mu) \) such that \( A_x \subseteq S_I \). Its image under the projection \( \pi: A_I \to A_g \) is a union \( \bigcup_{\psi \in \text{ES}(x)} EO_{\psi} \) of EO strata by Proposition 3.2. Clearly, all these EO strata are entirely contained in the supersingular locus of \( A_g \). It follows from Proposition 2.5 that they are not irreducible. Hence the union cannot be irreducible. This means however that \( A_x \) is not irreducible, so by the theorem \( A_x \) is superspecial.

8. The \( p \)-rank 0 locus

8.1. Group-theoretic notation, II. We need some more notation. We embed \( \text{Sp}_{2g} \subset \text{SL}_{2g} \) in the standard way (see Section 2.2). In this setup, the positive roots are

\[
\beta_{ij}^1, \quad \text{where } \beta_{ij}^1(k) = \begin{cases} 1, & k = i \text{ or } k = 2g - j + 1, \\ -1, & k = j \text{ or } k = 2g - i + 1, \\ 0, & \text{otherwise,} \end{cases} \quad 1 \leq i < j \leq g,
\]

\[
\beta_{ij}^2, \quad \text{where } \beta_{ij}^2(k) = \begin{cases} 1, & k = i \text{ or } k = j, \\ -1, & k = 2g - i + 1 \text{ or } k = 2g - j + 1, \\ 0, & \text{otherwise,} \end{cases} \quad 1 \leq i < j \leq g,
\]

\[
\beta_i^3, \quad \text{where } \beta_i^3(k) = \begin{cases} 1, & k = i, \\ -1, & k = 2g - i + 1, \\ 0, & \text{otherwise,} \end{cases} \quad 1 \leq i \leq g.
\]

So there are \( g^2 \) positive roots. The simple roots are \( \beta_{i,i+1}^1, i = 1, \ldots, g - 1 \) and \( \beta_g^3 \).

We also need the Iwahori-Matsumoto formula for the length of an element in \( \tilde{W} \) which expresses the fact that the length is equal to the number of affine root hyperplanes separating the alcove in question from the base alcove. We use the following version:

\[
(8.1) \quad \ell(t^\lambda w) = \sum_{\beta \geq 0 \atop w^{-1}\beta > 0} |\langle \beta, \lambda \rangle| + \sum_{\beta > 0 \atop w^{-1}\beta < 0} |\langle \beta, \lambda \rangle| + 1, \quad \lambda \in X_*(T), w \in W,
\]

which is easily checked to be equivalent to formula (2.1) given in [13].

8.2. Dimension of the \( p \)-rank 0 locus. Let \( W^{(0)} \subset W \) be the subset of elements which have no fixed point, and let \( \text{Adm}(\mu)^{(0)} \) be the set of admissible elements which give rise to a stratum on which the \( p \)-rank is 0. Proposition 2.7 shows that the projection \( \tilde{W} \to W \) induces a map

\[ \text{Adm}(\mu)^{(0)} \to W^{(0)}. \]
Lemma 8.1. The map $\text{Adm}(\mu)(0) \to W(0)$ defined above is a bijection. Its inverse is given by $w \mapsto t^{\lambda(w)}w$ with

$$\lambda(w)(i) = \begin{cases} 0, & w^{-1}(i) > i \\ 1, & w^{-1}(i) < i \end{cases}, \quad i = 1, \ldots, 2g.$$  

Proof. This is easy to check writing $x$ as an extended alcove (see [13], Subsections 2.5, 2.6), and is also contained in [14], Proof of 8.2. Note however that Haines uses a different normalization from ours!  

Now write $W = (\mathbb{Z}/2)^g \rtimes S_g$, where corresponding to the embedding $W \subset S_{2g}$, an element $\sigma \in S_g$ corresponds to the element $w \in W$ with $w(i) = \sigma(i)$, $w(2g - i + 1) = 2g - \sigma(i) + 1$, $i = 1, \ldots, g$, and an element $v \in (\mathbb{Z}/2)^g$ corresponds to the permutation $w$ with $w(i) = i$ if $v(i) = 0$, and $w(i) = 2g - i + 1$, if $v(i) = 1$, $i = 1, \ldots, g$.

Fix $\sigma \in S_g$. We say that a vector $v \in (\mathbb{Z}/2)^g$ is $\sigma$-admissible, if $v \sigma \in W(0)$. Clearly, $v$ is $\sigma$-admissible if and only if $v(i) = 1$ for all fixed points $i$ of $\sigma$, so the number of $\sigma$-admissible vectors is $2^{2g-f}$, where $f$ is the number of fixed points of $\sigma$.

For $v, v' \in (\mathbb{Z}/2)^g$, we write $v' \leq v$ if $v'(i) \leq v(i)$ (where we set $0 < 1$) for all $i$.

Lemma 8.2. Let $\sigma \in S_g$, let $v, v' \in (\mathbb{Z}/2)^g$ be $\sigma$-admissible, and assume that $v' \leq v$. Denote by $\lambda(\sigma)$ the translation element such that $t^\lambda(\sigma)(v \sigma) \in \text{Adm}(\mu)(0)$, and likewise for $v'$. Then we have $t^\lambda(v \sigma)(v \sigma) \leq t^\lambda(v' \sigma)(v' \sigma)$ with respect to the Bruhat order.

Proof. We may assume that $v$ and $v'$ differ at only one place, say $d \in \{1, \ldots, g\}$, where we have $v'(d) = 0$, $v(d) = 1$. In particular, $\sigma(d) \neq d$. We write $\lambda = \lambda(v \sigma)$, $\lambda' = \lambda(v' \sigma)$ and $w = v \sigma$, $w' = v' \sigma \in W$. Our assumption says precisely that $w' = t_dw$, where $t_d$ is the reflection $(0, \ldots, 0, 1, 0, \ldots, 0) \sigma \in (\mathbb{Z}/2)^g \rtimes S_g = W$ associated with $\beta_d^\lambda$ (the 1 is in the $d$-th place).

We have $\lambda(d) = 0$, and $\lambda' = \lambda$ if $\sigma^{-1}(d) > d$ and $\lambda' = t_d \lambda$ if $\sigma^{-1}(d) < d$. So we get

$$t^\lambda w' = \begin{cases} t^\lambda t_d w, & \sigma^{-1}(d) > d \\ t^\lambda t_d w, & \sigma^{-1}(d) < d \end{cases} = \begin{cases} t^\lambda w t_{\sigma^{-1}(d)}, & \sigma^{-1}(d) > d \\ t_d t^\lambda w, & \sigma^{-1}(d) < d \end{cases}.$$  

We see that the two elements $t^\lambda w'$, $t^\lambda w$ are related with respect to the Bruhat order.

First case: $\sigma^{-1}(d) < d$. The two elements differ by the application of $t_d$ on the left, and we need to check which of the elements is on the same side of the wall corresponding to $\beta_d^\lambda$ as the base alcove (this will be the smaller element). Since $\langle \beta_d^\lambda, \lambda \rangle = -1$, we see that the alcove corresponding to $t^\lambda w$ is the smaller of the two (recall our normalization of putting the base alcove in the anti-dominant chamber).

Second case: $\sigma^{-1}(d) > d$. Instead of comparing the two elements directly, we compare their inverses. One easily computes that $(t^\lambda w')^{-1} = (t^\lambda w t_{\sigma^{-1}(d)})^{-1} = t_{\sigma^{-1}(d)} t^{-w^{-1}} t^{-1}$. It suffices to check that $\langle \beta_{\sigma^{-1}(d)}^\lambda, -w^{-1} \lambda \rangle < 0$. We have

$$\langle \beta_{\sigma^{-1}(d)}^\lambda, -w^{-1} \lambda \rangle = -\lambda(w(\sigma^{-1}(d))) + \lambda(2g - w(\sigma^{-1}(d)) + 1)$$  

$$= -\lambda(2g - d + 1) + \lambda(d) = -1 < 0$$  

as desired.  

As a consequence of the two lemmas, we have the following description.

**Proposition 8.3.** Let \( x \in \text{Adm}(\mu)^{(0)} \) be an admissible element of p-rank 0 which is maximal with respect to the Bruhat order within this set. Let \( \sigma \) be the \( S_g \)-component of its image in \( W^{(0)} \). Then \( x = t^{\lambda_\sigma}(v_\sigma \sigma) \) with

\[
v_\sigma(i) = \begin{cases} 
1, & \sigma(i) = i \\
0, & \sigma(i) \neq i
\end{cases}, \quad i = 1, \ldots, g.
\]

and

\[
\lambda_\sigma(i) = \begin{cases} 
0, & \sigma^{-1}(i) \geq i \\
1, & \sigma^{-1}(i) < i
\end{cases}, \quad i = 1, \ldots, 2g.
\]

To express the length of elements of the form \( t^{\lambda_\sigma}(v_\sigma \sigma) \), we make the following definition:

**Definition 8.4.** Let \( \sigma \in S_g \). We define

\[
A_\sigma = \#\{(i, j) \in \{1, \ldots, g\}^2; \ i < j < \sigma(j) < \sigma(i)\}.
\]

We have the following estimate involving the length of \( \sigma \) and the quantities \( A_\sigma, A_{\sigma^{-1}} \). This is an elementary statement about the symmetric group. The proof we give is quite intricate, even if entirely elementary. It would be interesting to formulate it in a way which generalizes to Weyl groups of arbitrary reductive groups; maybe that would also lead to a more conceptual proof.

**Lemma 8.5.** Let \( \sigma \in S_g \). Then

\[
\ell(\sigma) - 2(A_\sigma + A_{\sigma^{-1}}) \geq g - \frac{\#\{i; \ \sigma(i) = i\}}{2}.
\]

**Proof.** It is not hard to derive this statement from the results of Clarke, Stein-grimsson and Zeng [4], in particular the statement \( \text{INV}_{\text{MV}} = \text{INV} \); see [4], Proposition 9. To save the reader the work of tracing through the notation of [4], we will explain how to reduce our claim to the following lemma, which is Lemma 8 in [4]; see also Lemma 3 in Clarke [3].

**Lemma 8.6.** Let \( \sigma \in S_g \) be a permutation. Write \( a_i := \sigma(i) \). Then

\[
\#\{(i, j); \ i \leq j < a_i, a_j > j\} = \#\{(i, j); \ a_i < a_j \leq i, a_j > j\},
\]

\[
\#\{(i, j); \ i \leq j < a_i, a_j \leq j\} = \#\{(i, j); \ a_i < a_j \leq i, a_j \leq j\}.
\]

Using the lemma, let us prove Lemma 8.5. Let \( \sigma \in S_g \), and again write \( a_i = \sigma(i) \) to shorten the notation. Writing \( \ell(\sigma) \) as the number of inversions, i. e. as the
number of pairs \((i, j)\) such that \(i < j, a_i > a_j\), we have

\[
\ell(\sigma) - A_\sigma - A_{\sigma^{-1}} = \#\{(i, j); i < j, a_i > a_j, i \leq a_i, j \geq a_j\}
\]

\[
= \#\{(i, j); i < j, a_i > a_j, i \leq a_i, j \geq a_j, j < a_i\}
\]

\[
= \#\{(i, j); j \geq a_i > a_j, i \leq a_i\}
\]

\[
+ \#\{(i, j); i < j < a_i, j \geq a_j\}
\]

\[
\geq \#\{(i, j); j \geq a_i > a_j, i \leq a_i\}
\]

\[
+ \#\{(i, j); j < a_i\}
\]

\[
= \#\{(i, j); j < a_i\}
\]

So, for every \(\sigma \in S_g\), we get

\[
\ell(\sigma) - 2(A_\sigma + A_{\sigma^{-1}}) \geq \#\{i; i < \sigma(i)\}.
\]

Since the quantity on the left is the same for \(\sigma\) and \(\sigma^{-1}\), we even get that it is greater or equal than

\[
\max(\#\{i; i < \sigma(i)\}, \#\{i; i < \sigma^{-1}(i)\}).
\]

But

\[
\#\{i; i < \sigma(i)\} + \#\{i; i < \sigma^{-1}(i)\} = g - \#\{i; \sigma(i) = i\},
\]

and we finally get the desired inequality.

The next lemma, whose proof is unfortunately quite technical and long, is the heart of the computation of the dimension of \(A_g^{(0)}\).

**Lemma 8.7.** For \(\sigma \in S_g\), let \(v_\sigma\), \(\lambda_\sigma\) be as in the proposition.

1. For all \(\sigma\), we have

\[
\ell(t^{\lambda_\sigma}(v_\sigma)) = g(g + 1)/2 + 2A_\sigma + 2A_{\sigma^{-1}} - \ell(\sigma) - \#\{i; \sigma(i) = i\}.
\]

2. Furthermore, for all \(\sigma\),

\[
\ell(t^{\lambda_\sigma}(v_\sigma)) \leq [g^2/2].
\]

3. Let \(\sigma = (12)(34) \cdots (g-1, g)\) if \(g\) is even, and \(\sigma = (12)(34) \cdots (g-2, g-1)\) if \(g\) is odd. Then

\[
\ell(t^{\lambda_\sigma}(v_\sigma)) = [g^2/2].
\]

**Proof.** We prove part (1) using the Iwahori-Matsumoto formula [8.1]. The proof is not hard, but quite long, so we restrict ourselves to illustrating the method by discussing a few cases. We write \(w = v_\sigma, \lambda = \lambda_\sigma\) and \(x = t^\lambda w\).
Contributions to the sum from roots $\beta_{ij}^1$. First, we need to check when $w^{-1}\beta_{ij}^1$ is positive. We get

$$w^{-1}\beta_{ij}^1 \begin{cases} > 0, & \sigma^{-1}(i) < \sigma^{-1}(j), \\ < 0, & \text{otherwise}. \end{cases}$$

Now, for each of these cases, we compute the contributions to the length of $x$. Assume that $v_\sigma(i) = v_\sigma(j) = 0$. So $w^{-1}\beta_{ij}^1 > 0$ if and only if $\sigma^{-1}(i) < \sigma^{-1}(j)$. We have

$$\langle \beta_{ij}^1, \lambda \rangle = \lambda(i) - \lambda(j) = \begin{cases} 1, & \sigma^{-1}(i) < i, \sigma^{-1}(j) > j, \\ 0, & \sigma^{-1}(i) > i, \sigma^{-1}(j) < j \text{ or } \sigma^{-1}(i) < i, \sigma^{-1}(j) < j, \\ -1, & \sigma^{-1}(i) > i, \sigma^{-1}(j) < j. \end{cases}$$

The overall contribution to the final sum is (we always sum over $i, j \in \{1, \ldots, g\}$)

$$\# \{(i, j): i < j, \sigma^{-1}(i) < i, \sigma^{-1}(j) > j, \sigma^{-1}(i) < \sigma^{-1}(j)\}$$

$$+ \# \{(i, j): i < j, \sigma^{-1}(i) > i, \sigma^{-1}(j) < j, \sigma^{-1}(i) < \sigma^{-1}(j)\}$$

$$+ \# \{(i, j): i < j, \sigma^{-1}(i) > i, \sigma^{-1}(j) < j, \sigma^{-1}(i) > \sigma^{-1}(j)\}$$

$$+ \# \{(i, j): i < j, \sigma^{-1}(i) < i, \sigma^{-1}(j) < j, \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$  

One gets similar (but a little simpler) contributions from the other cases.

Contributions to the sum from roots $\beta_{ij}^2$. This is similar to the case of $\beta_{ij}^1$.

Sum of contributions from roots $\beta_{ij}^1$, $\beta_{ij}^2$. Summing up all the contributions we have so far, one gets

$$\begin{aligned} &\# \{(i, j): \sigma^{-1}(i) < \sigma^{-1}(j)\} \\
&+ 2 \cdot \# \{(i, j): \sigma^{-1}(j) < \sigma^{-1}(i) < i < j\} \\
&+ 2 \cdot \# \{(i, j): i < j < \sigma^{-1}(j) < \sigma^{-1}(i)\}. \end{aligned}$$

Since

$$\ell(\sigma) = \ell(\sigma^{-1}) = \# \{(i, j): i < j, \sigma^{-1}(i) > \sigma^{-1}(j)\},$$

the term in the first row is just $\frac{g(g-1)}{2} - \ell(\sigma)$. In the second row we have $2A_\sigma$, and in the third row $2A_{\sigma^{-1}}$, so summing up we get, as the contribution from roots $\beta_{ij}^1$, $\beta_{ij}^2$:

$$g(g-1) \allowbreak - \ell(\sigma) + 2A_\sigma + 2A_{\sigma^{-1}}.$$  

(8.2)

Contributions to the sum from roots $\beta_{ij}^3$. We have $w^{-1}\beta_{ij}^3 > 0$ if and only if $v_\sigma(i) = 0$, and only this case gives a contribution. Since $\langle \beta_{ij}^3, \lambda \rangle = 1$, independently of $i$, the contribution we get is

$$\# \{i: \sigma(i) \neq i\} = g - \# \{i: \sigma(i) = i\}.$$

(8.3)

Summing up. Summing up the terms in (8.2) and (8.3), we get the desired result.

The estimate in part (2) of the lemma follows from the formula established in (1) and Lemma 8.4.

To prove part (3), we apply the formula in part (1). We have $\ell(\sigma) = \lfloor g/2 \rfloor$, $A_\sigma = A_{\sigma^{-1}} = 0$, and the number of fixed points is 0 if $g$ is even, 1 if $g$ is odd. So

$$\ell(t^{\lambda_\sigma}(v_\sigma\sigma)) = \frac{g(g+1)}{2} - \left\lfloor \frac{g+1}{2} \right\rfloor = \left\lfloor \frac{g^2}{2} \right\rfloor.$$
Altogether we get a formula for the dimension of $A_l^{(0)}$.

**Theorem 8.8.** The dimension of the $p$-rank $0$ locus is

$$\dim A_l^{(0)} = \max_{x \in \text{Adm}(\mu)^{(0)}} \ell(x) = \left\lfloor \frac{g^2}{2} \right\rfloor.$$

8.3. Comparison of superspecial KR locus and $p$-rank $0$ locus. We recall the following result from [13] (Proposition 4.6).

**Proposition 8.9.** The dimension of the union of all superspecial KR strata is $g^2/2$ if $g$ is even, and $g(g-1)/2$ if $g$ is odd. There is a unique superspecial KR stratum of this maximal dimension, namely the one corresponding to $w\tau$, where $w$ is the longest element of $W_{(g/2)}$, if $g$ is even, and $w$ is the longest element of $W_{(0,g)}$ if $g$ is odd.

Combining this with the theorem of the previous section, we have

**Corollary 8.10.**

1. If $g$ is even, the dimension of the supersingular locus inside $A_l$ is

   $$\dim S_l = g^2/2.$$

2. If $g$ is odd, then

   $$(g^2 - g)/2 \leq \dim S_l \leq (g^2 - 1)/2.$$

8.4. Top-dimensional components of $S_l$. Now let us look at the top-dimensional irreducible components of the union of superspecial KR strata, and in particular of the supersingular locus $S_l$.

**Proposition 8.11.** Let $X$ be an irreducible component of the closure of the maximal-dimensional superspecial KR stratum. Then $X$ is an irreducible component of $A_l^{(0)}$, and hence in particular an irreducible component of $S_l$.

**Proof.** Let us first assume that $g$ is odd. The maximal-dimensional superspecial KR stratum is $A_{w\tau}$, where $w$ is the longest element of $W_{(0,g)} = S_g$, so

$$w\tau = t^{(0(g),1(g))}w_0,$$

where $w_0 \in W$ is the longest element of $W$. We must show that $x := w\tau$ is maximal in $\text{Adm}(\mu)^{(0)}$ with respect to the Bruhat order. So assume that $x < x'$ for some $x' \in \bar{W}$, with $\ell(x') = \ell(x) + 1$. By (a suitable) definition of the Bruhat order, this means that $x' = tx$, for some reflection $t \in W_w$. Because $x\tau^{-1}$ is the longest element of $W_{(0,g)}$, $t$ must be conjugate to $s_0$ or to $s_g$. This implies that the finite part $t_{\text{fin}}$ of $t$ acts as some transposition $(i,2g-i+1)$. But then the finite part of $x'$ will have a fixed point, so even if it is admissible, it cannot have $p$-rank $0$.

If $g$ is even, the result follows from a similar consideration, or, even easier, directly from the dimension counts.

In case $g$ is even, we can prove that every top-dimensional irreducible component of $S_l$ is an irreducible component of the closure of the maximal-dimensional superspecial KR stratum.
Proposition 8.12. Let \( g \) be even. Then every top-dimensional irreducible component of the supersingular locus is an irreducible component of the union of superspecial KR strata. More precisely, it is an irreducible component of the closure of the unique maximal-dimensional superspecial KR stratum \( A_{w\tau} \), where \( w \) is the longest element of \( W_{(g/2)} \).

Proof. Let \( Z \) be an irreducible component of the supersingular locus of maximal dimension. By Corollary 8.10 (1), \( Z \) has dimension \( g^2/2 \), and is hence an irreducible component of the \( p \)-rank 0 locus. In particular, it is an irreducible component of the closure of some KR stratum \( A_x \). By Theorem 7.4, if \( A_x \) is not superspecial, then it is irreducible—a contradiction. 

We do not expect the analogous statement to be true for odd \( g \). Evidence from the theory of affine Deligne-Lusztig varieties predicts that it fails even for \( g = 3 \).

9. The relationship to the work of Ekedahl and van der Geer

9.1. As in [5], and as above, let \( \mathbb{H} = H^{DR}_{DB}(A_{univ}/\mathbb{A}_g) \), where \( A_{univ} \to \mathbb{A}_g \) is the universal abelian scheme. Let \( E \subset \mathbb{H} \) be the Hodge filtration; this is locally a direct summand of rank \( g \), which is totally isotropic. We can extend any flag in \( E \) in a unique way to a symplectic flag in \( \mathbb{H} \). In this way, we embed the bundle \( \mathcal{F} := \text{Flag}(E) \) of flags in \( E \) into \( \text{Flag}^+(\mathbb{H}) \). The fibers of \( \text{Flag}(E) \) over \( \mathcal{A}_g \) are flag varieties for \( SL_g \). The space \( \text{Flag}(E) \) is denoted \( \mathcal{F}_g \) in [5].

Let \((E_i)_i\) be a point of \( \mathcal{F} \). Ekedahl and van der Geer ([5], 3.1) define its conjugate flag as the unique point \((D_i)_i \in \text{Flag}^+(\mathbb{H})\) with

\[ D_{g+i} = V^{-1}(E_i^{(p)}) \]

We get a stratification

\[ \mathcal{F} = \coprod_{w \in W} \mathcal{U}_w \]

by locally closed subsets, given by the relative position of the flag to its “conjugate flag” (both considered as points of \( \text{Flag}^+(\mathbb{H}) \)). Denoting a flag and its conjugate flag by \((E_i)_i, (D_i)_i\) as above, we use \((D_i)_i\) as the base point to determine the relative position, i.e., with our usual notation, \( w = \text{inv}((D_i)_i, (E_i)_i) \). Here \( W \), as above, denotes the finite Weyl group of the symplectic group \( \text{Sp}_{2g} \). This stratification is similar to the stratification of the flag variety by Deligne-Lusztig varieties: however here the Frobenius varies along the base, and we also have the shift by \( g \) (i.e., \( D_{g+i} \) is defined in terms of \( E_i \)).

9.2. The map \( \mathcal{A}_{t^\mu} \to \mathcal{F} \). The closure \( \mathcal{A}_{t^\mu} \) of the KR stratum corresponding to \( t^\mu \) is an irreducible component of \( \mathcal{A}_t \) (see [34]; see also Theorem 7.4 above). We construct a closed embedding \( \mathcal{A}_{t^\mu} \to \mathcal{F} \). First note the following lemma which characterizes the KR strata inside \( \mathcal{A}_{t^\mu} \).

Lemma 9.1.

\[ W t^\mu \cap \text{Adm}(\mu) = W \tau \cap \text{Adm}(\mu) = \{ x \in \tilde{W}; \ x \leq t^\mu \} \]

Proof. The decomposition of \( \tau \) according to the decomposition \( \tilde{W} = X_\tau(T) \times W \) is \( \tau = w_\tau t^\mu \), so \( t^\mu = w_\tau \tau \). This proves the first equality. Further, it shows that for \( x \in W \tau \), we have \( x \leq t^\mu \) if and only if \( x\tau^{-1} \leq w_\tau \), by the definition of the Bruhat order on \( \tilde{W} \). It is now clear that the right hand side is contained in the left.
hand side. To see the converse, use that for every \( x \in \text{Adm}(\mu) \), \( x \leq t^{\rho(x)} \), where \( \rho(x) \in X_0(T) \) is such that \( x = wt^{\rho(x)} \), \( w \in W \). This is proved in \cite{15}, Proof of Proposition 4.6 (it amounts to the validity of the statement called Hyp(\(\lambda\)). Note that at this point it is irrelevant that Haines uses a slightly different normalization (he puts the base alcove in the dominant chamber).

From this combinatorial statement, we can derive the following characterization in terms of abelian varieties.

**Proposition 9.2.** Let \( K \) be an algebraically closed field, and let \( A = (A_i) \) be a \( K \)-valued point of \( A_t \). Then the following are equivalent:

(i) We have \( A \in \mathcal{A}_\mu \).

(ii) We have \( \text{im}(H^1_{DR}(A_2) \to H^1_{DR}(A_g)) = \omega_g \), the Hodge filtration inside \( H^1_{DR}(A_g) \).

(iii) There is an isomorphism \( A_g \cong A_0^{(p)} \) which identifies the given isogeny \( A_0 \to A_g \) with the Frobenius morphism \( A_0 \to A_0^{(p)} \).

If these conditions are satisfied, then

(1) The images \( \alpha(\omega_i) \subset H^1_{DR}(A_0) \) of the Hodge filtrations of \( A_i \), \( i = 0, \ldots, g \), form a complete flag inside \( \omega_0 \), i.e. all the inclusions \( \alpha(\omega_{i+1}) \subseteq \alpha(\omega_i) \) are strict.

(2) The finite group scheme \( \text{ker } A_0 \to A_g \) is connected.

**Proof.** To see that conditions (i) and (ii) are equivalent, it is easiest to use the extended alcove notation (see \cite{13}). Say \( A \) lies in the KR stratum \( A_x \). Condition (i) is equivalent to \( x \leq t^{\mu} \). By the previous lemma, this holds if and only if \( x \in W_\tau \), and this is easily seen to be equivalent to \( x_g = (0, \ldots, 0) \) (where \( x = (x_0, \ldots, x_2g) \) as an extended alcove). So in terms of lattice chains, this says that the \( g \)-th element of the chain corresponding to \( A \) is the standard lattice, and this means precisely that condition (ii) is satisfied.

We have a natural identification of \( A_0 \) and \( A_{2g} \), and hence of the Verschiebung morphism (which induces the Verschiebung of \( A_0 \) in cohomology) with a morphism \( A_0^{(p)} \to A_{2g} \). The Hodge filtration is the image of Verschiebung, i.e. \( \omega(\alpha(\omega_0)) = \text{im}(H^1_{DR}(A_2) \to H^1_{DR}(A_0^{(p)})) \). Hence condition (iii) implies condition (ii). Now assume that the latter condition is satisfied. Consider the Dieudonné modules of \( A_g \) and \( A_0^{(p)} \) as submodules of the Dieudonné module \( M(A_0) \) of \( A_0 \). For both of them we know that the image of Verschiebung is equal to \( pM(A_0) \). This implies that they coincide, and therefore condition (iii) holds true.

If these equivalent conditions are satisfied, then (1) follows immediately from \( \ref{3.1} \) (2). Alternatively, one can use the numerical characterization of KR strata in terms of abelian varieties; see \cite{13}, Corollary 2.7. Finally, (2) is an obvious consequence of (iii).

This proposition shows that \( \mathcal{A}_\mu \) is the subscheme denoted by \( S(g, p)^\circ \) in \cite{5}. See also Lemma \( \ref{5.3} \) for a more precise version of (1).

**Definition 9.3.** We define a morphism \( i : \mathcal{A}_\mu \to \mathcal{F} \) as follows. To an \( S \)-valued point \( ((A_i)_i, \lambda_0, \lambda_g) \), we associate the element of \( \mathcal{F}(S) \) given by \( (A_0, \lambda_0) \) and the following flag inside \( \omega_0 := \omega(A_0) \):

\[
0 = \alpha(\omega(A_g)) \subset \alpha(\omega(A_{g-1})) \subset \cdots \subset \alpha(\omega(A_1)) \subset \omega_0.
\]
The proposition shows that this indeed defines a morphism. Note that it is different from the morphism \( \iota \) defined above. The following lemma shows that this is the same map as the map \( \mathcal{S}(g,p)^\circ \to \mathcal{F} \) considered in \([5]\), Section 14.

**Lemma 9.4.** We can also describe the map \( \iota \) as follows: a point \( A \) as above is mapped to the unique symplectic flag extending the inverse image of

\[
0 = \operatorname{Lie} H_0 \subset \operatorname{Lie} H_1 \subset \cdots \subset \operatorname{Lie} H_g = \operatorname{Lie} A_0
\]

in \( H^1_{DR}(A_0) \) under the projection \( H^1_{DR}(A_0) \cong H^1_{DR}(A_0') \to \operatorname{Lie} A_0 \). Here \( H_i \) is defined as the kernel of the isogeny \( A_0 \to A_i \), and we use the given principal polarization of \( A_0 \) to identify \( H^1_{DR}(A_0) \) and \( H^1_{DR}(A_0') \).

**Proof.** This follows from the equalities

\[
\operatorname{Lie} H_i = \ker(\operatorname{Lie} A_0 \to \operatorname{Lie} A_i) = \ker(\operatorname{Lie} H^1_{DR}(A_0') \to \operatorname{Lie} A_i)/\omega(A_0') = \alpha^{-1}(\omega(A_i'))/\omega(A_0') \quad \text{with} \quad \alpha: H^1_{DR}(A_0') \to H^1_{DR}(A_i')
\]

\[
\alpha(\omega(A_i'))/\omega(A_0') \quad \text{with} \quad \alpha: H^1_{DR}(A_i) \to H^1_{DR}(A_0)
\]

Let \( \mathcal{F}' \subset \mathcal{F} \) denote the closed subscheme of \( V \)-stable flags.

**Lemma 9.5.** We have \( \overline{\mathcal{U}_{w_0}} \subseteq (\mathcal{F}')_{\text{red}} \), where \( (\cdot)_{\text{red}} \) denotes the underlying reduced subscheme, and as before \( w_0 \in W \) denotes the finite part of \( \tau \).

**Proof.** This (and even \( \overline{\mathcal{U}_{w_0}} = (\mathcal{F}')_{\text{red}} \)) follows from \([5]\), Proposition 4.3 (iii).

The proof of the following proposition shows in particular that \( \mathcal{F}' \) is reduced, and is hence equal to \( \overline{\mathcal{U}_{w_0}} \).

**Proposition 9.6.** The map \( \iota \) is a closed embedding which identifies \( \overline{\mathcal{A}_\mu} \) with \( \overline{\mathcal{U}_{w_0}} \).

For each \( w \in W \), we have \( w \leq w_0 \) if and only if \( w \tau \in \operatorname{Adm}(\mu) \), and in this case the isomorphism \( \overline{\mathcal{A}_\mu} \cong \overline{\mathcal{U}_{w_0}} \) restricts to an isomorphism

\[
\overline{\mathcal{A}_{w \tau}} \cong \mathcal{U}_w.
\]

**Proof.** We will show that \( \iota \) induces an isomorphism \( \overline{\mathcal{A}_\mu} \cong \mathcal{F}' \). First, it is clear that \( \iota \) factors through \( \mathcal{F}' \).

There is an inverse morphism \( \mathcal{F}' \to \overline{\mathcal{A}_\mu} \): For a connected finite flat group scheme \( G \) of height 1 (i.e. such that the Frobenius morphism is zero), every \( V \)-stable subspace of \( \operatorname{Lie} G \) gives rise to a subgroup scheme, and hence we can use the description of the morphism \( \iota \) given in Lemma 9.4 to obtain the desired inverse. In particular, we see that \( \mathcal{F}' \) is reduced.

To prove the compatibility between the stratifications note that restricting the morphism \( \mathcal{F} \to \operatorname{Flag}^+(\mathbb{H}) \) which maps a flag in \( \mathbb{E} \) to its conjugate flag to \( \overline{\mathcal{A}_\mu} \) gives us precisely the morphism

\[
\overline{\mathcal{A}_\mu} \to \operatorname{Flag}^+(\mathbb{H}), \quad (A_\mu)_\bullet \mapsto (H^1_{DR}(A_{2g-\bullet}))_\bullet.
\]

Now we can use Lemma 9.4 (3).
9.3. Relation between KR stratification and EO stratification. Combining Proposition 9.6 and [5], Corollary 8.4 (iii), we have

**Corollary 9.7.** Let \( w \in W_{\text{final}} \). Then the morphism \( \pi \) restricts to a finite étale surjective morphism \( A_{\tau w} \rightarrow EO_w \).

**Acknowledgments.** We thank Gerard van der Geer for emphasizing that there is a relationship between our previous paper [13] and the article [5] he had written with Ekedahl, and from which we learnt a lot. We also thank Michael Rapoport for his helpful remarks on a preliminary version of this paper. Most of the results were obtained, and much of the paper was written during the stay of the first author at Academia Sinica in March 2008. He would like to thank Academia Sinica for its hospitality, generous support, and for providing an excellent working environment. The stay was also supported by the SFB/TR 45 *Periods, moduli spaces, and arithmetic of algebraic varieties*.

**References**

[1] C. Bonnafé and R. Rouquier, On the irreducibility of Deligne-Lusztig varieties. *C. R. Acad. Sci. Paris Sér. I Math.* 343 (2006), 37–39.

[2] P. Boyer, Monodromie du faisceau pervers des cycles vanescents de quelques varits de Shimura simples, *Invent. Math.* 177 (2009), no. 2, 239–280.

[3] R. Clarke, A short proof of a result of Foata and Zeilberger, *Adv. in Applied Math.* 16 (1995), 129–131.

[4] R. Clarke, E. Steingrímsson and J. Zeng, New Euler-Mahonian Statistics on Permutations and Words, *Adv. in Applied Math.* 18 (1997), 237–270.

[5] T. Ekedahl, G. van der Geer, Cycle classes of the E-O stratification on the moduli of abelian varieties, arXiv:math.AG/0412272v2. To appear in *Arithmetic, Algebra and Geometry–Manin- Festschrift*, Birkhäuser Verlag.

[6] L. Fargues and E. Mantovan, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales, *Astérisque* 291, 2004.

[7] G. van der Geer, Cycles on the moduli space of abelian varieties, in *Moduli of curves and abelian varieties*, Eds: C. Faber, E. Looijenga, Aspects of Math. E33, Vieweg 1999, 65–89.

[8] A. Genestier, Un modèle semi-stable de la variété de Siegel de genre 3 avec structures de niveau de type \( \Gamma_0(p) \), *Compositio Math.* 123 (2000), no. 3, 303–328.

[9] U. Görtz, On the flatness of local models for the symplectic group, *Adv. Math.* 176 (2003), 89–115.

[10] U. Görtz, On the connectedness of Deligne-Lusztig varieties. *Represent. Theory* 13 (2009), 1–7.

[11] U. Görtz, T. Haines, R. Kottwitz and D. Reuman, Affine Deligne-Lusztig varieties in affine flag varieties, arXiv:0805.0045v2.

[12] U. Görtz, M. Hoeve, Ekedahl-Oort strata and Kottwitz-Rapoport strata, arXiv:0808.2537.

[13] U. Görtz, C.-F. Yu, Supersingular Kottwitz-Rapoport strata and Deligne-Lusztig varieties, arXiv:0802.3260v2. To appear in *Journal de l’Institut de Math. de Jussieu*.

[14] T. Haines, The combinatorics of Bernstein functions. *Trans. Amer. Math. Soc.* 353 (2001), no. 3, 1251–1278

[15] T. Haines, Test functions for Shimura varieties: the Drinfeld case. *Duke Math. J.* 106 (2001), no. 1, 19–40.

[16] T. Haines, Introduction to Shimura varieties with bad reduction of parahoric type. *Harmonic analysis, the trace formula, and Shimura varieties*, 583–642, Clay Math. Proc., 4, Amer. Math. Soc., 2005.

[17] S. Harashita, Ekedahl-Oort strata contained in the supersingular locus and Deligne-Lusztig varieties. To appear *J. Algebraic Geom.*

[18] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*. With an appendix by V. G. Berkovich. Annals of Math. Studies, 151. Princeton University Press, 2001.
[19] M. Hoeve, Ekedahl-Oort strata in the supersingular locus, arXiv:0802.4012v1.
[20] N. Koblitz, $p$-adic variant of the zeta function of families of varieties defined over finite fields, Compositio Math. 31 (1975), 119–218.
[21] M. Kottwitz and M. Rapoport, Minuscule alcoves for $GL_n$ and $GSp_{2n}$, Manuscripta Math. 102 (2000), 403–428.
[22] K.-Z. Li and F. Oort, Moduli of Supersingular Abelian Varieties, Lecture Notes in Mathematics 1680, Springer (1998).
[23] B. Moonen, Group schemes with additional structures and Weyl group cosets, in Moduli of abelian varieties (Texel Island, 1999), 255–298, Progr. Math. 195, Birkhäuser 2001.
[24] B. Moonen, A dimension formula for Ekedahl-Oort strata, Ann. Inst. Fourier (Grenoble) 54 (2004), no. 3, 666–698.
[25] B. Moonen, T. Wedhorn, Discrete invariants of varieties in positive characteristic, Intern. Math. Res. Not. 2004, no. 72, 3855–3903.
[26] L. Moret-Bailly, Pinceaux de variétés abéliennes, Astérisque 129 (1985).
[27] B.C. Ngô and A. Genestier, Àlcôves et $p$-rang des variétés abéliennes, Ann. Inst. Fourier (Grenoble) 52 (2002), 1665–1680.
[28] F. Oort, Subvarieties of moduli spaces, Invent. Math. 24 (1974), 95–119.
[29] F. Oort, A stratification of a moduli space of abelian varieties, in: Moduli of abelian varieties (Texel Island, 1999), Progr. Math. 195, 345–416, Birkhäuser 2001.
[30] M. Rapoport, Non-Archimedean period domains, in Proc. Int. Cong. Math., (Zürich, 1994), Birkhäuser, 1995, 423–434.
[31] M. Rapoport, A guide to the reduction modulo $p$ of Shimura varieties, in: Automorphic forms. I. Astérisque 298 (2005), 271–318.
[32] T. Wedhorn, The dimension of Oort strata of Shimura varieties of PEL-type, in: Moduli of abelian varieties (Texel Island, 1999), Progr. Math. 195, 441–471, Birkhäuser 2001.
[33] T. Wedhorn, De Rham cohomology of varieties over fields of positive characteristic, Preprint 2007.

http://www2.math.uni-paderborn.de/de/people/torsten-wedhorn/

[34] C.-F. Yu, Irreducibility of the Siegel moduli spaces with parahoric level structure, Intern. Math. Res. Not. 2004, No. 48, 2593–2597.
[35] C.-F. Yu, Irreducibility and $p$-adic monodromies on the Siegel moduli spaces, Adv. Math. 218 (2008), 1253–1285.
[36] C.-F. Yu, Kottwitz-Rapoport strata of the Siegel moduli spaces. To appear in Taiwanese J. Math.

(Görtz) Universität Duisburg-Essen, Institut für Experimentelle Mathematik, Ellernstr. 29, 45326 Essen, Germany
E-mail address: ulrich.goertz@uni-due.de

(Yu) Institute of Mathematics, Academia Sinica, 128 Academia Rd. Sec. 2, Nankang, Taipei, Taiwan, and NCTS (Taipei Office)
E-mail address: chiafu@math.sinica.edu.tw