On the convergence rate improvement of a splitting method for finding the resolvent of the sum of maximal monotone operators *

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Abstract

This paper provides a new way of developing the splitting method which is used to solve the problem of finding the resolvent of the sum of maximal monotone operators in Hilbert spaces. By employing accelerated techniques developed by Davis and Yin (in Set-Valued Var. Anal. 25(4):829-858, 2017), this paper presents an implementable, strongly convergent splitting method which is designed to solve the problem. In particular, we show that the distance between the sequence of iterates and the solution converges to zero at a rate \(O(1/k)\) to illustrate the efficiency of the proposed method, where \(k\) is the number of iterations. Then, we apply the result to a class of optimization problems.

Keywords: splitting method, maximal monotone, resolvent, fixed point, nonexpansive, Hilbert space

MSC2010: 47H05, 47H09, 47H10, 47J25, 90C25

1 Introduction

The purpose of this paper is to present a new splitting method for solving the following monotone inclusion problem:

\[
\text{find } u \in H \text{ such that } z \in (I + A + B)(u),
\]

\(\text{(1)}\)

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where $A: H \rightharpoonup H$ and $B: H \rightharpoonup H$ are maximal monotone operators on a real Hilbert space $H$, $I$ is the identity mapping on $H$ and $z \in H$ is given. Problem (1) has been widely studied in various fields such as sparse signal recovery and best approximation problems; see [32, 14, 15, 16, 20, 7, 21, 24, 9, 22, 23] and the references therein. If the solution set of problem (1) is nonempty (which is assured if, for example $A + B$ is maximal monotone), the monotonicity of $A + B$ guarantees uniqueness of the solution. We denote by $u = J_{A+B}(z) := ((I+A+B)^{-1}(z))$ the solution of problem (1) and $J_{A+B}$ is called the resolvent of $A + B$ (see, e.g., [31, Subchapter 4.6], [7, Definition 23.1]). Throughout this paper we assume that the resolvents of $A$ and $B$ are easy to compute.

An interesting way of dealing with problem (1) is to transform it into a problem with a particular structure. Aragón Artacho and Campoy have shown that problem (1) can be transformed into a problem of finding a zero of the sum of two maximal strongly monotone operators [3, Proposition 3.2]. They then developed the averaged alternating modified reflections method for solving problem (1) by applying the Douglas-Rachford splitting method [25] to the latter problem [3, Theorem 3.1]. The main advantage of their method is that it generates the sequence of iterates which converges strongly to the solution. However, it seems that the estimate of convergence rate for the method has not been considered and the Douglas-Rachford splitting method can be slow [4, Sections 6 and 7] (see also [18, Subsection 3.4] for related results). Motivated by this fact, new techniques should be developed for analyzing the convergence of iterative methods for solving (1).

The goal of this paper is to propose an implementable, strongly convergent method for solving problem (1) which has global convergence rates. In order to present a method, the structure in the transformed problem of (1) needs to be exploited. This can be done by using ideas from [3,18]. The contributions of this paper can be summarized as follows. Firstly, we investigate the properties of an operator $(A_z)^{(\beta)}: H \rightharpoonup H$ defined by

$$(A_z)^{(\beta)} := 2(1 - \beta)A \left( \frac{1}{\beta} I + z \right) + \frac{1 - \beta}{\beta} I,$$  \hspace{1cm} (2)$$

for some $\beta \in (0, 1)$. The operator $(A_z)^{(\beta)}$ can be viewed as a modification of the inner $z$-perturbation and the $\beta$-strengthening of $A$ introduced in [3, Definitions 2.3 and 3.2]. It can be shown that $(A_z)^{(\beta)}$ is $\frac{1-\beta}{\beta}$-strongly monotone, and $J_{A+B}(z) = \frac{1}{\beta}v + z$ if and only if $0 \in ((A_z)^{(\beta)} + (B_z)^{(\beta)}) (v)$ [3, Propositions 3.1 and 3.2]. Thus, problem (1) can be transformed into the

In [3], the operator $A \left( \frac{1}{\beta} I - z \right) + \frac{(1-\beta)}{\beta}$ was treated as the inner $z$-perturbation and the $\beta$-strengthening $(A_z)^{(\beta)}$ of $A$. We use [2] for notational convenience.
problem of finding a zero of the sum of two maximal strongly monotone operators \((A_z)^{(\beta)}\) and \((B_z)^{(\beta)}\). We consider the relation between the resolvents of \(A\) and \(r(A_z)^{(\beta)}\), where \(r > 0\). In particular, we will show that the resolvent of \(r(A_z)^{(\beta)}\) can be obtained by evaluating the resolvent of \(A\) in the original problem and thus the resolvent of the translated problem can be calculated.

Secondly, we consider an accelerated variant of the three operator splitting method developed in [18, Algorithm 3], which is designed to solve inclusion problems with three maximal monotone operators. Their method is conceptually very simple, but seems to be implementable only for the limited classes of problems where at least one operator is strongly monotone. We present a strongly convergent splitting method which is designed to solve problem \((\Pi)\) by applying the method in [18, Algorithm 3] to the translated problem. By exploiting special properties of the operators \((A_z)^{(\beta)}\) and \((B_z)^{(\beta)}\), the method can be applied without modifying the properties of \(A\) and \(B\) in the original problem. Moreover, the proposed method involves the evaluation of the resolvents \(r_k(A_z)^{(\beta)}\) and \(r_k(B_z)^{(\beta)}\), and contains the parameter \(\{r_k\}\) which has to vary at each step to get better efficiency. This is in contrast with the averaged alternating modified reflections algorithm [3], which uses similar resolvents with a constant parameter, for solving problem \((\Pi)\). It follows from the fact mentioned above that the resolvent \(r_k(A_z)^{(\beta)}\) (resp. \(r_k(B_z)^{(\beta)}\)) can be obtained by evaluating the resolvent of \(A\) (resp. \(B\)). Thus the proposed splitting method can be implemented and may be considered as a modification of the method in [18]. In particular, we can provide a \(O(1/k)\) rate of convergence and a strong convergence result for the sequence of iterates.

Finally, we apply the results to a class of optimization problems. Our theoretical analysis is general and can handle convex minimization problems with three objective functions. Note that two of the objective functions are not necessary differentiable. As important applications we consider the problem of minimizing the sum of a nonsmooth strongly convex function and a nonsmooth weakly convex function under the assumption that the strong convexity constant is larger than the weak convexity constant, and the best approximation problem since these problems possess a special structure. The convergence results based on the Douglas-Rachford splitting method applied to these problems were obtained in [22, Theorems 5.1 and 5.2] and [1, Theorem 4.1], respectively. However, it does not seem obvious how to estimate the distance between the sequences of iterates and the solutions. As a whole, the proposed method can be implemented and may be considered an improved version of the methods given in [22, 1]. Indeed, we can show that the distance between the sequence of iterates and the solution converges to zero at a rate \(O(1/k)\), where \(k\) is the number of iterations.

The rest of this paper is organized as follows. In section 2, we recall some
definitions and known results for further analysis. Then, we investigate some properties of the mapping \( r(A_x)^{(j)} \) in section 3, where \( r > 0 \) and \((A_x)^{(j)}\) is defined by [2]. A new splitting method is presented, the convergence of the method is shown, and the rate of convergence is derived in section 4. Then, concrete examples of (1) are given and we show how the proposed method can be applied in section 5. Finally, we draw some conclusions in section 6.

2 Basic definitions and preliminaries

The following notation will be used in this paper: \( \mathbb{R} \) denotes the set of real numbers; \( \mathbb{N} \) denotes the set of nonnegative integers; \( H \) denotes a real Hilbert space; for any \( x, y \in H \), \( \langle x, y \rangle \) denotes the inner product of \( x \) and \( y \); for any \( z \in H \), \( \|z\| \) denotes the norm of \( z \), i.e., \( \|z\| = \sqrt{\langle z, z \rangle} \); for mappings \( T : H \to H \) and \( U : H \to H \), \( T \circ U \) denotes the composition of \( T \) and \( U \); for any \( C \subset H \) and mapping \( U : C \to C \), \( \text{Fix}(U) \) denotes the fixed point set of \( U \), i.e., \( \text{Fix}(U) = \{ x \in C : U(x) = x \} \); \( d(x, C) = \inf \{ \|x - y\| : y \in C \} \) denotes the distance from any \( x \) to \( C \); \( \text{int}C \) denotes the interior of set \( C \); \( \text{cone}(C) \) denotes the conical hull of \( C \); \( \text{sri}C \) denotes the strong relative interior of \( C \), i.e., \( \text{sri}C := \{ x \in C : \text{cone}(C - x) \text{ is a closed linear subspace of } H \} \); for any set-valued operator \( A : H \rightrightarrows H \), \( \text{dom}(A) \) denotes the domain of \( A \), i.e., \( \text{dom}(A) = \{ x \in H : A(x) \neq \emptyset \} \); \( \text{ran}(A) \) denotes the range of \( A \), i.e., \( \text{ran}(A) = \bigcup \{ A(x) : x \in \text{dom}(A) \} \); \( G(A) \) denotes the graph of \( A \), i.e., \( G(A) = \{ (x, x^*) : x^* \in A(x) \} \); The set of zero points of \( A \) is denoted by \( A^{-1}(0) \) i.e., \( A^{-1}(0) = \{ z \in \text{dom}(A) : 0 \in A(z) \} \).

A mapping \( U : C \to C \) is said to be

(i) \textbf{nonexpansive} if

\[ \|U(x) - U(y)\| \leq \|x - y\| \quad (x, y \in C); \]

(ii) \textbf{firmly nonexpansive} if

\[ \|U(x) - U(y)\|^2 \leq \langle x - y, U(x) - U(y) \rangle \quad (x, y \in C). \]

In particular, \( U \) is firmly nonexpansive if and only if \( 2U - I \) is nonexpansive [7, Proposition 4.2].

A set-valued operator \( A : H \rightrightarrows H \) is said to be

(i) \textbf{monotone} if

\[ \langle x - y, x^* - y^* \rangle \geq 0 \quad ((x, x^*), (y, y^*) \in G(A)); \]
(ii) maximal monotone if $A$ is monotone and $A = B$ whenever $B : H \rightrightarrows H$ is a monotone mapping such that $G(A) \subset G(B)$.

The maximal monotonicity of $A$ implies that $\text{ran}(I + rA) = H$ for all $r > 0$. Then, we can define the resolvent $J_{rA}$ of $rA$ by

$$J_{rA}(x) = \{z \in H : x \in z + rA(z)\} = (I + rA)^{-1}(x)$$

(3)

for all $x \in H$. It is well-known that the resolvent is firmly nonexpansive and hence is Lipschitz continuous (see, e.g., [7, 31]). The following is a useful characterization of zeros of the sum of two maximal monotone operators.

**Proposition 2.1.** [4, Proposition 25.1] Let $A$ and $B$ be monotone operators on $H$, and let $r > 0$. Then

$$(A + B)^{-1}(0) = J_{rA}(\text{Fix}((2J_{rB} - I) \circ (2J_{rA} - I)))$$

Let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous convex function. The domain of function $f$ is $\text{dom} f := \{x \in H : f(x) < \infty\}$. The epigraph of $f$ is the set $\text{epi} f$ defined by $\text{epi} f = \{(x, r) \in H \times \mathbb{R} : f(x) \leq r\}$. $f$ is said to be strongly convex with constant $\beta > 0$ if for any $x, y \in H$ and for any $\lambda \in (0, 1)$, we have

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) - \frac{\beta \lambda(1 - \lambda)}{2} \|x - y\|^2.$$  

$f$ is said to be weakly convex if for some $\omega > 0$, the function $f + \frac{\omega}{2} \| \cdot \|^2$ is convex. The conjugate function of $f$ is the function $f^* : H \to \mathbb{R} \cup \{\infty\}$ defined by $f^*(v) = \sup\{\langle x, v \rangle - f(x) : x \in \text{dom} f\}$ for $v \in H$. The subdifferential of $f$ at $x \in E$ is defined by

$$\partial f(x) = \{x^* \in H : f(y) \geq f(x) + \langle y - x, x^* \rangle \text{ for all } y \in H\}.$$  

We know the subdifferential of a proper, lower semicontinuous and convex function is maximal monotone (see, e.g., [28, 7, Theorem 20.40]). Using the properties of subdifferentials, we can write (3) equivalently as

$$J_{r\partial f}(x) = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2r} \|y - x\|^2 \right\}$$  

(4)

and (4) is known as the proximal mapping of $f$ [7, Proposition 16.34]. In particular, we denote by $\text{prox}_r f(x)$ the proximal mapping of parameter $r$ at $x$ (i.e., $\text{prox}_r f(x) := J_{r\partial f}(x)$).
Let $C \subset H$ be a nonempty set. The indicator function of $C$, $i_C : H \to \mathbb{R} \cup \{\infty\}$, is the function which takes the value 0 on $C$ and $+\infty$ otherwise. The support function $\sigma_C$ is defined by $
abla \sigma_C(x) = \sup_{c \in C} \langle c, x \rangle$ for $x \in H$.

The subdifferential of the indicator function is the normal cone of $C$, that is $N_C(x) = \{u \in H : \langle u, y - x \rangle \leq 0 \ (\forall y \in C)\}$, if $x \in C$ and $N_C(x) = \emptyset$ for $x \notin C$. The proximal mapping is indeed an extension of the metric projection. In fact, let $f(x) = i_C(x)$, it holds

$$J_{r\sigma_C}(x) = J_{\eta_C}(x) = J_{\partial i_C}(x) = P_C(x) \quad \text{for any} \ r > 0,$$

where $P_C : H \to C$ denotes the metric projection (see [7, Example 23.3 and Example 23.4]).

We state the Stolz-Césaro theorem, which will be used.

**Theorem 2.1.** *(Stolz-Cesáro theorem)* Let $\{a_k\}$ and $\{b_k\}$ be two sequences of real numbers. If $b_k$ is positive, strictly increasing and unbounded and the following limit exists:

$$\lim_{k \to \infty} \frac{a_{k+1} - a_k}{b_{k+1} - b_k} = l,$$

then the limit $\lim_{k \to \infty} \frac{a_k}{b_k}$ exists and it is equal to $l$.

### 3 Some properties of $(A_z)^{(\beta)}$

In this section, we investigate the properties of $(A_z)^{(\beta)}$ defined by (2). Let $S$ be the set of solutions of problem (1), i.e., $S = \{u \in H : z \in (I + A + B)(u)\}$. Under $S \neq \emptyset$, the monotonicity of $A + B$ guarantees the uniqueness of the solution of problem (1) and hence $S = \{J_{A+B}(z)\}$.

We introduce some fundamental properties for $(A_z)^{(\beta)}$ defined by (2).

**Proposition 3.1.** [3, Propositions 3.1 and 3.2] Let $A$ and $B$ be operators on $H$ and let $\beta \in (0, 1)$, let $z \in H$, and let $(A_z)^{(\beta)}$ (resp. $(B_z)^{(\beta)}$) be the mapping defined by (2). Then

If $A$ is monotone, then $(A_z)^{(\beta)}$ is $\frac{1-\beta}{\beta}$-strongly monotone;

If $A$ is maximal monotone, then $(A_z)^{(\beta)}$ is maximal monotone;

$J_{A+B}(z) = \frac{1}{\beta}v + z$ if and only if $0 \in ((A_z)^{(\beta)} + (B_z)^{(\beta)})(v)$.

We consider the resolvent of $r(A_z)^{(\beta)}$ with $r > 0$. Our method in the next section need to vary the parameter $r$ at each step. The following result is important to present an implementable method which is designed to find the solution to problem (1).
Proposition 3.2. Let $A$ be a maximal monotone operator on $H$ and let $\beta \in (0, 1)$, let $z \in H$, let $(A_\beta)_{(\beta)}$ be the mapping defined by (2), and let $r > 0$. Then for any $x \in H$, $J_{r(A_\beta)(\beta)}(x) = \beta J_{2r(1-\beta)} \frac{1}{\beta + r(1-\beta)} x + z - \beta z$.

Proof. Let $u := J_{r(A_\beta)(\beta)}(x) = (I + r(A_\beta)(\beta))^{-1}(x)$. This together with the definition of $(A_\beta)_{(\beta)}$ implies that

\[ x \in u + 2r(1-\beta)A \left( \frac{1}{\beta} u + z \right) + \frac{1 - \beta}{\beta} ru \]

\[ = \frac{\beta + r(1-\beta)}{\beta} u + 2r(1-\beta)A \left( \frac{1}{\beta} u + z \right) \]

\[ = (\beta + r(1-\beta)) \left( \frac{1}{\beta} u + z \right) + 2r(1-\beta)A \left( \frac{1}{\beta} u + z \right) - (\beta + r(1-\beta))z. \]

Thus we have

\[ \frac{1}{\beta + r(1-\beta)} x + z \in \frac{1}{\beta} u + z + \frac{2r(1-\beta)}{\beta + r(1-\beta)} A \left( \frac{1}{\beta} u + z \right) \]

\[ = \left( I + \frac{2r(1-\beta)}{\beta + r(1-\beta)} A \right) \left( \frac{1}{\beta} u + z \right), \]

and hence

\[ u = \beta \left( I + \frac{2r(1-\beta)}{\beta + r(1-\beta)} A \right)^{-1} \left( \frac{1}{\beta + r(1-\beta)} x + z \right) - \beta z \]

\[ = \beta J_{2r(1-\beta)} \frac{1}{\beta + r(1-\beta)} x + z - \beta z. \]

Remark 3.1. Aragón Artacho and Campoy [3, Proposition 3.1] showed that the resolvent of $A \left( \frac{1}{\beta} I \right) + \frac{1-\beta}{\beta} I$ is $\beta J_A$. Proposition 3.2 enhances this result.

We next consider the existence of the solution of problem (1). Using Propositions 2.1 and 3.1, we establish a new connection between the existence of fixed points for nonexpansive mappings and the solvability of problem (1).

Theorem 3.1. Let $A$ and $B$ be maximal monotone operators on $H$ and let

\[ T := (2J_{r(B_\beta)(\beta)} - I) \circ (2J_{r(A_\beta)(\beta)} - I), \tag{6} \]

where $J_{r(A_\beta)(\beta)}$ (resp. $J_{r(B_\beta)(\beta)}$) is the resolvent of $r(A_\beta)(\beta)$ (resp. $r(B_\beta)(\beta)$) for some $\beta \in (0, 1)$ and $r > 0$. Then
(i) $\text{Fix}(T) \neq \emptyset$ if and only if $S \neq \emptyset$;

(ii) $S = \beta \left( J_{r(A_z)}(x)(\text{Fix}(T)) - z \right)$.

**Proof.** (1) Let $u \in \text{Fix}(T)$. It follows from Proposition 2.1 that

$\left( (A_z)^{\beta} + (B_z)^{\beta} \right)^{-1}(0) = J_{r(A_z)}(\text{Fix}(T))$.

Let $v := J_{r(A_z)}(u) \in \left( (A_z)^{\beta} + (B_z)^{\beta} \right)^{-1}(0)$. From Proposition 3.1 (3), we have

$J_{A+B}(z) = \frac{1}{\beta} v + z$,

and hence $\{\frac{1}{\beta} v + z\} = S$.

For the converse, let $u \in S$. Then define $v := \beta (J_{A+B}(z) - z)$. It follows from Propositions 2.1 and 3.1 that

$v \in \left( (A_z)^{\beta} + (B_z)^{\beta} \right)^{-1}(0) = J_{r(A_z)}(\text{Fix}(T))$.

Therefore, we conclude that $\text{Fix}(T) \neq \emptyset$.

(2) From the arguments in the proof of (1), the result is obtained. \(\square\)

**Remark 3.2.** Theorem 3.1 provides a new necessary and sufficient condition that guarantees the existence of $J_{A+B}(z)$. The advantage of our results is that the existence of solution of problem (7) can be interpreted as a fixed point problem for nonexpansive mapping $(2J_{r(B_z)} - I) \circ (2J_{r(A_z)} - I)$. Hence, some existing results depending on the nonexpansiveness of a mapping are applicable.

By employing the classical result in [11, Theorem 1] (see also [31, Theorem 3.1.6]), we prove the following result.

**Corollary 3.1.** Let $A$ and $B$ be maximal monotone operators on $H$ and let $T$ be defined by (6). Then the following are equivalent:

(i) There exists $x \in H$ such that $\{T^k(x)\}$ is bounded;

(ii) $S \neq \emptyset$.

**Proof.** Since $T = (2J_{r(B_z)} - I) \circ (2J_{r(A_z)} - I)$ is nonexpansive, by using the result in [11, Theorem 1], $\{T^k(x)\}$ is bounded if and only if $\text{Fix}(T) \neq \emptyset$. It follows from Theorem 3.1 that $\{T^k(x)\}$ is bounded if and only if $S \neq \emptyset$. \(\square\)

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Remark 3.3. The maximal monotonicity of $A + B$ guarantees the existence of the solution of problem (1). Various qualification conditions have been presented in the literature to prove maximality of the sum of two maximal monotone operators (see [27, Theorems 1 and 2], [7, Subchapter 24.1]). For example, if

$$0 \in \text{sri}(\text{dom}A - \text{dom}B)$$

(7) holds, then $A + B$ is maximal monotone. Moreover, (7) holds if one of the following condition holds

1. $\text{dom}B = H$;
2. $\text{dom}A \cap \text{int} \text{dom}B \neq \emptyset$;
3. $0 \in \text{int}(\text{dom}A - \text{dom}B)$;

(see, e.g., [7, Corollary 24.4]). Thus, these conditions guarantee the existence of the solution of (1). However, the solution set $S$ may be empty when (7) does not hold and the difficulty is how to check that such condition holds. Corollary 3.1 shows that $\{T_k(x)\}$ can be used to determine the existence of the solution of problem (1).

4 Convergence analysis

In this section, we will propose a splitting method to solve problem (1). Let $A$ and $B$ be maximal monotone operators on $H$. Assume that $\beta \in (0, 1)$, $z_0 \in H$, $x_0 = \beta J_{\frac{1}{\beta + r_0(1-\beta)}} A \left( \frac{1}{\beta + r_0(1-\beta)} z_0 + z \right) - \beta z_0$, $y_0 = (1/r_0)(z_0 - x_0)$ and let $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ be the sequences generated by

$$
\begin{align*}
  x_k &= \beta J_{\frac{1}{\beta + r_k(1-\beta)}} A \left( \frac{1}{\beta + r_k(1-\beta)} (z_{k-1} + r_{k-1} y_{k-1}) + z \right) - \beta z, \\
  y_k &= \frac{1}{r_k-1}(z_{k-1} + r_{k-1} y_{k-1} - x_k), \\
  z_k &= \beta J_{\frac{1}{\beta + r_k(1-\beta)}} B \left( \frac{1}{\beta + r_k(1-\beta)} (x_k - r_k y_k) + z \right) - \beta z.
\end{align*}
$$

(8)

where $\{r_k\}$ is a sequence of positive real numbers such that

$$r_0 \in (0, 2(1 - \beta)/\beta) \text{ and } r_{k+1} = r_k/\sqrt{1 + 2r_k(1 - \beta)/\beta}.$$  

We can provide convergence results and rates for the sequence $\{x_k\}$ in (8).
4.1 Connections to other existing methods

In this subsection, we present the connections of the proposed iterative method \((8)\) to existing iterative methods.

By Proposition 3.1 \((8)\) can be stated equivalently as \(x_0 = J_{r_0(A_2)^{(β)}}(z_0)\), \(y_0 = (1/r_0)(I - J_{r_0(A_2)^{(β)}})(z_0)\) and

\[
\begin{align*}
x_k &= J_{r_k(A_2)^{(β)}}(z_{k-1} + r_{k-1}y_{k-1}), \\
y_k &= (1/r_{k-1})(z_{k-1} + r_{k-1}y_{k-1} - x_k), \\
z_k &= J_{r_k(B_2)^{(β)}}(x_k - r_ky_k), \quad k = 1, 2, \ldots \tag{10}
\end{align*}
\]

\((10)\) can be considered as an instance of the iterative method for solving the problem of finding a zero of the sum of monotone operators developed by Davis and Yin \([18\text{, Algorithm 3}]\). More precisely, we apply their method to the problem of finding a point \(v \in H\) such that

\[
0 \in ((A_2)^{(β)} + (B_2)^{(β)})(v), \tag{11}
\]

where \((A_2)^{(β)}\) and \((B_2)^{(β)}\) are defined by \((2)\). The main difficulties in implementing \((10)\) lies in the fact that it involves the evaluation of the resolvents \(J_{r_k(A_2)(β)}\) and \(J_{r_k(B_2)(β)}\), and contains the parameter \(\{r_k\}\) which has to be adjusted adaptively at each iteration. Using Proposition 3.1 \((10)\) can be implemented by using the resolvents of \(A\) and \(B\). In particular, we will show that the sequence \(\{(1/β)x_k + z\}\) converges strongly to \(J_{A+B}(z)\), and \(\|((1/β)x_{k+1} + z - J_{A+B}(z))\| = O(1/k)\) holds under condition \((9)\). Thus \((8)\) can considered as the modification of the method in \([18]\).

Next, we consider the connection between \((8)\) and the Douglas-Rachford splitting method \([25]\). The Douglas-Rachford splitting method has the following form:

\[
w_{k+1} = w_k + \lambda_k(J_γB \circ (2J_γA - I)(w_k) - J_γA(w_k)) \tag{12}
\]

where \(w_0 \in H, \gamma \in (0, \infty)\) and \(\{\lambda_k\} \subset [0, 2]\). The iterative scheme \((12)\) can be applied to solve the inclusion \(0 \in (A + B)(u)\). A general discussion on the Douglas-Rachford method can be found in \([7\text{, Subchapter 25.2}]\). In \((10)\), we use a fixed parameter \(r_k := r > 0\). Now we define \(u_{k+1} := z_k + ry_k\). Then we have

\[
u_{k+1} = z_k + ry_k
\]

\[
= J_{r(B_2)^{(β)}}(x_k - ry_k) + z_{k-1} + ry_{k-1} - x_k
\]

\[
= u_k + J_{r(B_2)^{(β)}}(2J_{r(A_2)^{(β)}}(u_k) - u_k) - J_{r(A_2)^{(β)}}(u_k)
\]

\[
= u_k + J_{r(B_2)^{(β)}}(2J_{r(A_2)^{(β)}}(u_k) - I)(u_k) - J_{r(A_2)^{(β)}}(u_k).
\]

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Thus, the sequence \( \{u_k\} \) can be viewed as a special case of (12) for solving (11) when we keep the parameter \( r_k \) fixed.

On the other hand, (12) is closely related to the averaged alternating modified reflections algorithm in [3]. Aragón Artacho and Campoy considered the following iterative scheme:

\[
v_{k+1} = (1 - \lambda_k) v_k + \lambda_k \left( 2J_{\frac{\gamma}{\sqrt{1-\beta}}} B_z^{(\beta)} - I \right) \circ \left( 2J_{\frac{\gamma}{\sqrt{1-\beta}}} A_z^{(\beta)} - I \right) (v_k),
\]

where \( v_0 \in H, \gamma > 0 \) and \( \{\lambda_k\} \subset [0, 1] \) such that \( \sum_{j=0}^{\infty} \lambda_j (1 - \lambda_j) = \infty \). Note that (13) is equivalently written as

\[
v_{k+1} = v_k + 2\lambda_k \left( J_{\frac{\gamma}{\sqrt{1-\beta}}} B_z^{(\beta)} - J_{\frac{\gamma}{\sqrt{1-\beta}}} A_z^{(\beta)} (v_k) - J_{\frac{\gamma}{\sqrt{1-\beta}}} A_z^{(\beta)} (v_k) \right)
\]

(see [7, Proposition 4.21]), and hence the averaged alternating modified reflections algorithm can be viewed as a special case of (12) applied to solve

\[
0 \in \left( \frac{\gamma}{2(1-\beta)} A_z \right)^{(\beta)} + \left( \frac{\gamma}{2(1-\beta)} B_z \right)^{(\beta)}(v).
\]

It is shown in [3, Theorem 3.1] that \( \{J_{\gamma A}(v_k + z)\} \) converges strongly to \( J_{\frac{\gamma}{\sqrt{1-\beta}}}(A+B)(z) \) when \( z \in \text{ran } (I + (\gamma/2(1-\beta))(A+B)) \). Instead of fixing the parameter, the varying sequence of parameters \( \{r_k\} \) is used in our proposed method (8). Thus, (8) is different but closely related to the averaged alternating modified reflections algorithm.

### 4.2 Convergence of (8)

The following theorem concerns the strong convergence and convergence rate of the sequence \( \{(1/\beta)x_{k+1} + z\} \), where \( \{x_k\} \) is generated by (8). We first prove a proposition which plays important roles in the convergence analysis.

**Proposition 4.1.** Let \( A \) and \( B \) be maximal monotone operators such that \( S \neq \emptyset \), and let \( \{x_k\}, \{y_k\} \) and \( \{z_k\} \) be the sequences generated by (8) (or equivalently (10)). Then the following inequality holds:

\[
(1/r_k^2)\|x_{k+1} - v\|^2 + \|y_{k+1} - v_A\|^2 \leq (1/r_k^2)\|x_k - v\|^2 + \|y_k - v_A\|^2,
\]

where \( r > 0, u \in \text{Fix}(T), v = J_{r(A_z^{(\beta)})}(u), v_A = (1/r)(u - v) \) and \( v_B = (1/r)(v - u) \) such that \( v_A \in (A_z^{(\beta)}(v) \) and \( v_B \in (B_z^{(\beta)}(v) \).
Proof. The proof is similar to [18 Proposition 3.1], however, for the convenience of the reader, we sketch it here. From the definition of \( \{ z_k \} \) in (10), we have

\[
x_k - r_k y_k \in z_k + r_k (B \varepsilon)^{(\beta)} (z_k).
\]

Let

\[
v_k := (1/r_k)(x_k - r_k y_k - z_k) \in (B \varepsilon)^{(\beta)} (z_k).
\]

It follows from the definitions of \( \{ v_k \} \) and \( \{ y_k \} \), we have

\[
\begin{align*}
r_k(y_k + v_k) &= z_k + r_k y_k - x_k + x_k - r_k y_k - z_k = x_k - x_k + 1, \tag{16} \\
r_k(y_{k+1} + v_k) &= z_{k+1} + r_k y_k - x_k + x_k - r_k y_k - z_k = x_k - x_k + 1, \tag{17} \\
\end{align*}
\]

and

\[
r_k(y_k + v_k) = r_k y_k + x_k - r_k y_k - z_k = x_k - z_k. \tag{18}
\]

By using (16), (17) and (18), we have

\[
\begin{align*}
2r_k \left( \langle z_k - v, v_k \rangle + \langle x_{k+1} - v, y_{k+1} \rangle \right) &= 2r_k \left( \langle z_k - x_{k+1}, v_k \rangle + \langle x_{k+1} - v, y_{k+1} + v_k \rangle \right) \\
&= 2r_k \left( \langle z_k - x_{k+1}, v_k + y_k \rangle - \langle z_k - x_{k+1}, y_k \rangle \right) + 2 \langle x_{k+1} - v, x_k - x_{k+1} \rangle \\
&= 2 \langle z_k - x_{k+1}, x_k - z_k \rangle + 2 \langle x_{k+1} - v, x_k - x_{k+1} \rangle \\
&\quad + 2r_k \langle z_k - x_{k+1}, v_A - y_k \rangle - 2r_k \langle z_k - x_{k+1}, v_A \rangle \\
&= 2 \langle z_k - x_{k+1}, x_k - z_k \rangle + 2 \langle x_{k+1} - v, x_k - x_{k+1} \rangle \\
&\quad + 2r_k^2 \langle y_{k+1} - y_k, v_A - y_k \rangle - 2r_k \langle z_k - x_{k+1}, v_A \rangle. \tag{19}
\end{align*}
\]

Applying the relation

\[
2 \langle a - b, c - a \rangle = \| b - c \|^2 - \| a - c \|^2 - \| b - a \|^2
\]

to (19), we have

\[
\begin{align*}
2r_k \left( \langle z_k - v, v_k \rangle + \langle x_{k+1} - v, y_{k+1} \rangle \right) &= \| x_{k+1} - x_k \|^2 - \| z_k - x_k \|^2 - \| x_{k+1} - z_k \|^2 \\
&\quad + \| v - x_k \|^2 - \| x_k - u \|^2 - \| v - x_{k+1} \|^2 \\
&\quad + r_k^2 (\| y_k - v_A \|^2 - \| y_k - v_A \|^2 - \| y_k - y_{k+1} \|^2) - 2r_k \langle z_k - x_{k+1}, v_A \rangle \\
&= \| x_k - v \|^2 - \| x_k - u \|^2 - \| z_k - x_k \|^2 \\
&\quad + r_k^2 \| y_k - v_A \|^2 - r_k^2 \| y_{k+1} - v_A \|^2 + 2r_k \langle x_{k+1} - z_k, v_A \rangle. \tag{20}
\end{align*}
\]
On the other hand, by (19) and strong monotonicity of \((A_z)^{(β)}\) and \((B_z)^{(β)}\), we have
\[
2r_k(\langle z_k - v, v_k \rangle + \langle x_{k+1} - v, y_{k+1} \rangle)
\geq 2r_k(\langle z_k - v, v_B \rangle + (1 - β)/β\|z_k - v\|^2
+ \langle x_{k+1} - v, v_A \rangle + (1 - β)/β\|x_{k+1} - v\|^2)
= 2r_k(\langle x_{k+1} - z_k, v_A \rangle + (1 - β)/β\|z_k - v\|^2 + (1 - β)/β\|x_{k+1} - v\|^2).
\tag{21}
\]
By using (20) and (21) we obtain
\[
2r_k(\langle x_{k+1} - z_k, v_A \rangle + (1 - β)/β\|z_k - v\|^2 + (1 - β)/β\|x_{k+1} - v\|^2)
\leq \|x_k - v\|^2 - \|x_{k+1} - v\|^2 - \|z_k - x_k\|^2
+ r_k^2\|y_k - v_A\|^2 - r_k^2\|y_{k+1} - v_A\|^2 + 2r_k\langle x_{k+1} - z_k, v_A \rangle,
\]
and hence
\[
(1 + 2r_k(1 - β)/β)\|x_{k+1} - v\|^2 + r_k^2\|y_{k+1} - v_A\|^2
\leq \|x_k - v\|^2 + r_k^2\|y_k - v_A\|^2.
\]
Multiplying the inequality by \(r_k^2\) and using (19), we get (15). \hfill \Box

We prove the strong convergence of the sequence \(\{x_k\}\) generated by (8).

**Theorem 4.1.** Let \(A\) and \(B\) be maximal monotone operators and let \(\{x_k\}, \{y_k\}\) and \(\{z_k\}\) be the sequences generated by (8). If \(S \neq \emptyset\), then \(\{(1/β)x_k + z\}\) converges strongly to \(J_{A+B}(z)\). In particular, the following holds:
\[
\|(1/β)x_{k+1} + z - J_{A+B}(z)\| = O(1/k).
\tag{22}
\]

**Proof.** We know that \(0 < r_{k+1} < r_k < r_0 < 2(1 - β)/β \ (\forall k \in \mathbb{N})\). It follows that the sequence \(\{r_k\}\) has the limit. Moreover, it follows from (9) that \(\lim_{k \to \infty} r_k = 0\). Hence, we can further get
\[
\lim_{k \to \infty} \frac{r_k}{r_{k+1}} = \lim_{k \to \infty} \sqrt{1 + 2r_k(1 - β)/β} = 1.
\]
This implies that
\[
\frac{(k + 2) - (k + 1)}{r_{k+1} - r_k} = \frac{r_k r_{k+1}}{r_k - r_{k+1}} = \frac{r_k r_{k+1}(r_k + r_{k+1})}{r_k^2 - r_{k+1}^2}
= \frac{r_k r_{k+1}(r_k + r_{k+1})}{2r_k r_{k+1}(1 - β)/β}
= \frac{r_k + r_{k+1}}{2r_{k+1}(1 - β)/β}
= \frac{r_k}{r_{k+1} + 1}/(2(1 - β)/β),
\]

13
and thus
\[ \lim_{k \to \infty} \frac{(k + 2) - (k + 1)}{r_{k+1} - r_k} = \beta \frac{1}{1 - \beta}. \]

So, we can use the Stolz-Cesáro theorem with \( a_k := k + 1 \) and \( b_k := \frac{1}{r_k} \) to conclude that \( \{(a_k - a_{k-1})/(b_k - b_{k-1})\} \) and \( \{a_k/b_k\} \) have the same limit.

On the other hand, let \( r > 0, u \in \text{Fix}(T) \) and \( v := J_{r(A_n)e_0}(u) \). It follows from (15) that the following inequality holds for all \( k \in \mathbb{N} \cup \{0\} \):
\[ (1/r_{k+1}) \|x_{k+1} - v\|^2 + \|y_{k+1} - v_A\|^2 \leq (1/r_k^2) \|x_k - v\|^2 + \|y_k - v_A\|^2. \]

Thus, we have
\[ \|x_{k+1} - v\|^2 \leq r_{k+1}^2((1/r_k^2) \|x_0 - v\|^2 + \|y_0 - v_A\|^2) = O(1/k^2). \]

From Theorem 3.1 (2), we have
\[ \|(1/\beta)x_{k+1} + z - J_{A+B}(z)\| = \|(1/\beta)x_{k+1} + z - ((1/\beta)v + z)\| = O(1/k). \]

The proof is complete. \( \square \)

5 Applications

In this section, we provide some concrete problems that reduce to problem (1). We apply the proposed method (8) to a class of optimization problems consisting of the sum of three functions. Let \( z \in H \) and let \( f, g : H \to (-\infty, \infty] \) be proper, lower semicontinuous and convex functions. We consider the following problem:
\[ \text{minimize } \frac{1}{2} \|x - z\|^2 + f(x) + g(x). \]...

We refer the reader to [16, 17] for more details and applications of problem (23) and its useful variants in image processing. The solution set of problem (23) coincides with the solution set of the monotone inclusion problem
\[ \text{find } u \in H \text{ such that } z \in (I + \partial(f + g))(u). \]

Under the condition that \( \text{dom}f \cap \text{dom}g \neq \emptyset \), the maximal monotonicity of \( \partial(f + g) \) guarantees the existence and uniqueness of the solution of problem (23), denoted by \( \text{prox}_{f+g}(z) \) [31, Theorem 4.6.5], [7, Proposition 16.35]. It is important to point out that it holds \( \text{prox}_{f+g}(z) = J_{\partial f + \partial g}(z) \) when \( J_{\partial f + \partial g}(z) \) exists [6, Remark 3.4].

From the discussion in Sections 3 and 4 we get the following result. The proof is similar to that of Corollary 3.1 and Theorem 3.1 and thus is omitted.
Corollary 5.1. Let \( z \in H \) and let \( f, g : H \to (-\infty, \infty] \) be proper, lower semicontinuous and convex functions with \( \text{dom} f \cap \text{dom} g \neq \emptyset \). Assume that \( \beta \in (0,1) \) and \( \{r_k\} \subset (0,2(1-\beta)/\beta) \) such that (24) holds. Let \( \{x_k\}, \{y_k\} \) and \( \{z_k\} \) be the sequences generated by \( z_0 \in H \), \( x_0 = \beta \text{prox} \frac{\frac{1}{\beta} + r_0(1-\beta)}{z_0 + z} - \beta z_0 \), \( y_0 = (1/r_0)(z_0 - x_0) \) and

\[
\begin{align*}
x_k &= \beta \text{prox} \frac{z_{k-1}(1-\beta)}{\beta + r_k - 1(1-\beta)} \left( \frac{1}{\beta + r_k - 1(1-\beta)} (z_{k-1} + r_{k-1}y_{k-1}) + z \right) - \beta z, \\
y_k &= \frac{1}{r_k - 1}(z_{k-1} + r_{k-1}y_{k-1} - x_k), \\
z_k &= \beta \text{prox} \frac{z_k(1-\beta)}{\beta + r_k(1-\beta)} \left( \frac{1}{\beta + r_k(1-\beta)} (x_k - r_k y_k) + z \right) - \beta z.
\end{align*}
\]

The following assertions hold:

(i) \( J_{\beta f + \beta g}(z) \) exists if and only if there exists \( x \in H \) such that \( \{T^k(x)\} \) is bounded, where \( T := \partial f \) with \( A := \partial f \) and \( B := \partial g \);

(ii) If \( J_{\beta f + \beta g}(z) \) exists, then \( \{(1/\beta)x_k + z\} \) converges strongly to \( \text{prox}_{f+g}(z) \), and the convergence rate estimate \( \| (1/\beta)x_{k+1} + z - \text{prox}_{f+g}(z) \| = O(1/k) \) holds.

Remark 5.1. Burachik and Jeyakumar [12] showed that the subdifferential sum formula \( \partial(f + g)(x) = \partial f(x) + \partial g(x) \) (\( \forall x \in \text{dom} f \cap \text{dom} g \)) holds whenever \( \text{epi} f^* + \text{epi} g^* \) is weakly closed [12, Theorem 3.1]. Furthermore, it was shown that \( 0 \in \text{sri}(\text{dom} f - \text{dom} g) \) implies \( \text{epi} f^* + \text{epi} g^* \) is weakly closed [12, Proposition 3.2]. Note that under the subdifferential sum formula, the assumption of the existence of \( J_{\beta f + \beta g}(z) \) in Corollary 5.1 can be removed.

5.1 Minimizing the sum of a strongly convex function and a weakly convex function

We apply (24) to the minimization of two functions, where one is strongly convex and the other is weakly convex. Consider the following minimization problem:

\[
\text{minimize } \tilde{f}(x) + \tilde{g}(x),
\]

where \( \tilde{f} : H \to (-\infty, \infty] \) is proper lower semicontinuous strongly convex with constant \( \gamma > 0 \), and \( \tilde{g} : H \to (-\infty, \infty] \) is proper lower semicontinuous weakly convex with constant \( \omega > 0 \). (25) contains signal and image processing problems; see, e.g., [26, 8, 9, 22, 23].

The convergence of the Douglas-Rachford splitting method for (25) was established in [22] under the assumption \( \gamma > \omega \). In this case, problem (25) has the unique solution and we denote the unique minimizer by \( x^* \). The
Remark 5.2. In [22], Guo, Han and Yuan showed the following assertions hold:

where \( f \) is a proper lower semicontinuous strongly convex function, \( \tilde{g} := (1/(\gamma - \omega))(\tilde{f} - (\gamma/2)) \cdot \| \cdot \|^2 \) is convex so that the method (24) can be applied to problem (26). By letting \( f := (1/(\gamma - \omega))(\tilde{f} - (\gamma/2)) \cdot \| \cdot \|^2 \) and \( \beta := (1/(\gamma - \omega))(\tilde{g} + (\omega/2)) \cdot \| \cdot \|^2 \) and \( z := 0 \), it holds that \( J_{\partial f + \partial g}(0) = x^* \) when \( J_{\partial f + \partial g}(0) \) exists and hence \( J_{\partial f + \partial g}(0) \) is a solution of (25). Now, we get the following result.

**Corollary 5.2.** Let \( \tilde{f} : H \to (-\infty, \infty) \) be proper lower semicontinuous strongly convex with constant \( \gamma > 0 \), and \( \tilde{g} : H \to \mathbb{R} \cup \{\infty\} \) be proper lower semicontinuous weakly convex with constant \( \omega > 0 \). Assume that \( \gamma > \omega \), \( \beta \in (0,1) \) and \( \{r_k\} \subset (0, 2(1-\beta)/\beta) \) such that (27) holds. Let \( \{x_k\}, \{y_k\} \) and \( \{z_k\} \) be the sequences generated by by \( z_0 \in H \), \( x_0 = \beta \text{prox}_{\frac{r_k(1-\beta)}{\beta+r_k(1-\beta)}} f \left( \frac{1}{\beta+r_k(1-\beta)} x_0 \right) + \beta z_0 \), \( y_0 = (1/r_0)(z_0 - x_0) \) and

\[
\begin{align*}
x_k &= \beta \text{prox}_{\frac{r_k(1-\beta)}{\beta+r_k(1-\beta)}} f \left( \frac{1}{\beta+r_k(1-\beta)} (z_{k-1} + r_{k-1} y_{k-1}) \right), \\
y_k &= (1/r_{k-1})(z_{k-1} + r_{k-1} y_{k-1} - x_k), \\
z_k &= \beta \text{prox}_{\frac{r_k(1-\beta)}{\beta+r_k(1-\beta)}} \left( \frac{1}{\beta+r_k(1-\beta)} (x_k - r_k y_k) \right),
\end{align*}
\]

where \( f := (1/(\gamma - \omega))(\tilde{f} - (\gamma/2)) \cdot \| \cdot \|^2 \) and \( g := (1/(\gamma - \omega))(\tilde{g} + (\omega/2)) \cdot \| \cdot \|^2 \). The following assertions hold:

(i) \( J_{\partial f + \partial g}(0) \) exists if and only if there exists \( x \in H \) such that \( \{T^k(x)\} \) is bounded, where \( T \) is defined by (27) with \( z := 0 \), \( A := \partial f \) and \( B := \partial g \);

(ii) If \( J_{\partial f + \partial g}(0) \) exists, then \( \{(1/\beta)x_k\} \) converges strongly to \( x^* \), and the convergence rate estimate \( \| (1/\beta)x_{k+1} - x^* \| = O(1/k) \) holds, where \( x^* \) is the unique minimizer of (25).

**Remark 5.2.** In [22], Guo, Han and Yuan showed the \( o(1/\sqrt{k}) \) rate of asymptotic regularity for the Douglas-Rachford operator [22 Theorems 5.1 and 5.2]. However, it does not seem obvious how to estimate the distance
between the sequences of iterates and the solutions. Furthermore, under the metric subregularity assumption [19, p. 183] on (25), they established the local linear convergence rate [22, Theorem 6.1]. In Corollary 5.2, we show that \( \| (1/\beta) y_k - x^* \| \) converges to zero at the rate of \( O(1/k) \) without any additional restrictions on \( \tilde{f} \) and \( \tilde{g} \).

### 5.2 Best approximation problems

Let \( C \) and \( D \) be closed convex subsets in \( H \) with nonempty intersection and let \( z \in H \). Problem (23) contains as a special case the best approximation problem:

\[
\text{minimize } \frac{1}{2} \| x - z \|^2 + i_C(x) + i_D(x),
\]

where \( i_C \) and \( i_D \) are the indicator functions of the sets \( C \) and \( D \). It is important to point out that it holds \( P_{C\cap D}(z) = J_{N_C+N_D}(z) \) when \( J_{N_C+N_D}(z) \) exists. (28) contains a wide variety of problems such as covariance design, constrained least-squares matrix and signal recovery problems, and the analytic expressions for the metric projections onto the constraints sets of these problems were developed; see, e.g., [30, 21, 14, 16] and the references therein.

Now let us apply (24) to problem (28). By letting \( f := i_C \) and \( g := i_D \), (24) is reduced to \( z_0 \in H, x_0 = \beta P_C \left( \frac{1}{\beta+r_k(1-\beta)} z_0 + z \right) - \beta z_0 \), \( y_0 = (1/r_0)(z_0 - x_0) \) and

\[
\begin{align*}
  x_k &= \beta P_C \left( \frac{1}{\beta+r_k(1-\beta)} (z_{k-1} + r_k y_{k-1}) + z \right) - \beta z, \\
  y_k &= (1/r_k)(z_{k-1} + r_k y_{k-1} - x_k), \\
  z_k &= \beta P_D \left( \frac{1}{\beta+r_k(1-\beta)} (x_k - r_k y_k) + z \right) - \beta z.
\end{align*}
\]

The evaluation of \( P_{C\cap D}(z) \) is in general difficult, but each step of our method requires only the projections \( P_C \) and \( P_D \) onto the sets \( C \) and \( D \) respectively. We get the following result.

**Corollary 5.3.** Let \( z \in H \) and let \( C \) and \( D \) be closed convex subsets in \( H \) with nonempty intersection. Assume that \( \beta \in (0, 1) \) and \( \{r_k\} \subset (0, 2(1 - \beta)/\beta) \) such that (4) holds. Let \( \{x_k\}, \{y_k\} \) and \( \{z_k\} \) be the sequences generated by (27). The following assertions hold:

(i) \( J_{N_C+N_D}(z) \) exists if and only if there exists \( x \in H \) such that \( \{T^k(x)\} \) is bounded, where \( T \) is defined by (7) with \( A := N_C \) and \( B := N_D \);

(ii) If \( J_{N_C+N_D}(z) \) exists, then \( \{(1/\beta)x_k + z\} \) converges strongly to \( P_{C\cap D}(z) \), and the convergence rate estimate \( \| (1/\beta)x_{k+1} + z - P_{C\cap D}(z) \| = O(1/k) \) holds.
Remark 5.3.

(i) Burachik and Jeyakumar [13] showed that the normal cone intersection formula
\[ N_{C \cap D}(x) = N_C(x) + N_D(x) \quad (\forall x \in C \cap D) \]
holds whenever \( \text{epi} \sigma_C + \text{epi} \sigma_D \) is weakly closed [13, Theorem 3.1]. Furthermore, it was shown that \( 0 \in \text{sri}(C - D) \) implies \( \text{epi} \sigma_C + \text{epi} \sigma_D \) is weakly closed [13, Proposition 3.1]. Aragón Artacho and Campoy [1, Proposition 4.1] showed that the normal cone intersection formula is equivalent to the following condition:
\[ q - P_{C \cap D}(q) \in (N_C + N_D)(P_{C \cap D}(q)) \quad (\forall q \in H). \]

Note that under the normal cone intersection formula, the assumption of the existence of \( J_{N_C + N_D}(z) \) in Corollary 5.3 can be removed.

(ii) The convergence results of the averaged alternating modified reflections method for solving problem (28) were obtained in [1, Theorem 4.1] (see, also [3, Corollary 3.1]). This method is based on the Douglas-Rachford splitting method and generates the sequence of iterates which converges strongly to \( P_{C \cap D}(z) \). However, it seems that the estimate of convergence rate for the method has not been considered. On the other hand, it is shown that the sequence generated by the Douglas-Rachford splitting method converges to the solution to (28) with a linear rate when \( C \) and \( D \) are closed subspaces, and \( C + D \) is closed [4]. However, it is worth mentioning that the Douglas-Rachford splitting method can be slow without such requirements [4, Section 6] (see also [18, Subsection 3.4] for related results). (27) can provide \( O(1/k) \) convergence rate estimate for the distance between the sequence of iterates and \( P_{C \cap D}(z) \).

6 Conclusion

In this paper, we proposed a splitting method for finding the resolvent of the sum of two maximal monotone operators. Our method is based on the accelerated variant of the three operator splitting method developed in [18]. The method was proved to be strongly convergent to the solution and the \( O(1/k) \) convergence rate estimate was also established. Finally, we gave some concrete examples and showed how the method can be applied to such examples.

The behavior of the averaged alternating modified reflections algorithm can be estimated from the computational experience reported for the best approximation problem of two subspaces [1, 2] and the continuous-time optimal control problem [3]. Numerical results show a very good performance.
from the algorithm, compared to the other existing methods. In particular, the numerical results in [2] show that the algorithm exhibits a linear rate of convergence. Hence, it is natural to ask if the linear convergence holds when (29) is applied to two subspaces, or to give a counter example, if it does not. We leave this as one of our future research topics.

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