Strings, higher curvature corrections, and black holes

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ABSTRACT

We review old and recent results on subleading contributions to black hole entropy in string theory.

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1 Introduction

The explanation of black hole entropy in terms of microscopic states is widely regarded as one of the benchmarks for theories of quantum gravity. The analogy between the laws of black hole mechanics and the laws of thermodynamics, combined with the Hawking effect, suggests to assign to a black hole of area $A$ the ‘macroscopic’ (or ‘thermodynamic’) entropy

$$S_{\text{macro}} = \frac{A}{4}. \quad (1.1)$$

$S_{\text{macro}}$ depends on a small number of parameters which can be measured far away from the black hole and determine its ‘macroscopic’ state: the mass $M$, the angular momentum $J$ and its charges $Q$ with respect to long range gauge forces. A theory of quantum gravity should be able to specify and count the microstates of the black hole which give rise to the same macrostate. If there are $N$ states corresponding to a black hole with parameters $M, J, Q$, then the associated ‘microscopic’ or ‘statistical’ entropy is

$$S_{\text{micro}} = \log N. \quad (1.2)$$

By the analogy to the relation between thermodynamics and statistical mechanics, it is expected that the macroscopic and microscopic entropies agree.\footnote{We work in Planckian units, where $c = \hbar = G_N = 1$. We also set $k_B = 1$.}

The characteristic feature of string theory is the existence of an infinite tower of excitations with ever-increasing mass. Therefore it is natural to take the fundamental strings themselves as candidates for the black hole microstates \cite{1, 2}. In the realm of perturbation theory, which describes strings moving in a flat background space-time at asymptotically small string coupling one has access to the number of states with a given mass. The asymptotic number of states at high mass is given by the famous formula of Hardy-Ramanujan. Taking the open bosonic string for definiteness, the mass formula is $\alpha' M^2 = N - 1$, where $\alpha'$ is the Regge slope (the only independent dimensionful constant of string theory) and $N \in \mathbb{N}$ is the excitation level. For large $N$ the number of states grows like $\exp(\sqrt{N})$, so that the statistical entropy grows like

$$S_{\text{micro}} \approx \sqrt{N}. \quad (1.3)$$

As we will see later, there are examples where they agree in leading order of a semi-classical expansion, but disagree at the subleading level. This is a success rather than a problem because the discrepancies can be explained: the entropies that one compares correspond to different statistical ensembles.
It is clear that when increasing the mass (at finite coupling), or, alternatively, when increasing the coupling while keeping the mass fixed, the backreaction of the string onto its ambient space-time should lead to the formation of a black hole. Roughly, this happens when the Schwarzschild radius $r_S$ of the string equals the string length $l_S = \sqrt{\alpha'}$. Using the relation $l_P = l_S g_S$ between the string length, the Planck length $l_P$ and the dimensionless string coupling $g_S$, together with the fact that a black hole of mass $\alpha' M^2 \approx N$ has Schwarzschild radius $r_S \approx G_N M \approx \sqrt{\alpha'} g_S^2 \sqrt{N}$ and entropy $S_{\text{macro}} \approx \frac{4}{G_N} \approx g_S^2 N$ one finds that \[ r_S \approx l_S \Leftrightarrow g_S^2 \sqrt{N} \approx 1. \tag{1.4} \]

It is precisely in this regime that the entropy of string states $S_{\text{micro}} \approx \sqrt{N}$ equals the entropy $S_{\text{macro}} \approx g_S^2 N$ of a black hole with the same mass, up to factors of order unity. The resulting scenario of a string – black hole correspondence, where strings convert into black holes and vice versa at a threshold in mass/coupling space is quite appealing. In particular, it applies to Schwarzschild-type black holes and makes a proposal for the final state of black hole evaporation, namely the conversion into a highly excited string state of the same mass and entropy. However, this picture is very qualitative and one would like to have examples where one can make a quantitative comparison or even a precision test of the relation between macroscopic and microscopic entropy.

Such examples are available, if one restricts oneself to supersymmetric states, also called BPS states. We will consider four-dimensional black holes, where the setup is as follows: one considers a string compactification which preserves a four-dimensional supersymmetry algebra with central charges $Z$. Then there exist supermultiplets on which part of the superalgebra is realized trivially. These multiplets are smaller than generic massive supermultiplets (hence also called ‘short multiplets’), they saturate a mass bound of the form $M \geq |Z|$, and many of their properties are severely restricted by the supersymmetry algebra. By counting all states of given mass and charges, one obtains the statistical entropy $S_{\text{micro}}$. This can now be compared to the entropy $S_{\text{macro}}$ of a black hole which has the same mass, carries the same charges and is invariant under the same supertransformations. As above, the underlying idea is that by increasing the string coupling we can move from the regime of perturbation theory in flat space to a regime where

\[\text{In this paragraph we have reconstructed the dimensionful quantities } G_N \text{ and } \alpha' \text{ for obvious reasons. All approximate identities given here hold up to multiplicative constants of order unity and up to subleading additive corrections.}\]
the backreaction onto space-time has led to the formation of a black hole. This regime can be analyzed by using the low-energy effective field theory of the massless string modes, which encodes all long range interactions. The corresponding effective action can be constructed using string perturbation theory and is valid for small (but finite) coupling $g_S \leq 1$ and for space-time curvature which is small in units of the string length. One then constructs supersymmetric black hole solutions with the appropriate mass and charge. A black hole solution is called supersymmetric if it has Killing spinors, which are the ‘fermionic analogues’ of Killing vectors. More precisely, if we denote the supersymmetry transformation parameter by $\epsilon(x)$, the fields collectively by $\Phi(x)$, and the particular field configuration under consideration by $\Phi_0(x)$, then $\epsilon(x)$ is a Killing spinor and $\Phi_0(x)$ is a supersymmetric (or BPS) configuration, if the supersymmetry transformation with parameter $\epsilon(x)$ vanishes in the background $\Phi_0(x)$:

$$\left(\delta_\epsilon(x)\Phi\right)|_{\Phi_0(x)} = 0.$$ (1.5)

We consider black holes which are asymptotically flat. Therefore it makes sense to say that a black hole is invariant under ‘the same’ supertransformations as the corresponding string states. In practise, the effective action is only known up to a certain order in $g_S$ and $\alpha'$. Thus we need to require that the string coupling and the curvature at the event horizon are small. This can be achieved by taking the charges and, hence, the mass, to be large.

Having constructed the black hole solution, we can extract the area of the event horizon and the entropy $S_{\text{macro}}$ and then compare to the result of state counting, which yields $S_{\text{micro}}$. Both quantities are measured in different regimes, and therefore it is not clear a priori that the number of states is preserved when interpolating between them. For BPS states, there exist only two mechanism which can eliminate or create them when changing parameters: (i) at lines of marginal stability a BPS multiplet can decay into two or more other BPS multiplets, (ii) BPS multiplets can combine into non-BPS multiplets. It is not yet clear whether these processes really play a role in the context of black hole state counting, but in principle one needs to deal with them. One proposal is that the quantity which should be compared to the entropy is not the state degeneracy itself, but a suitable weighted sum, a supersymmetric index \[3, 4, 5\]. We will ignore these subtleties here and take an ‘experimental’ attitude, by just computing $S_{\text{macro}}$ and $S_{\text{micro}}$ in the appropriate regimes and comparing the results. In fact, we will see

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4This is completely analogous to the concept of a Killing vector $\xi(x)$, which generates an isometry of a given metric $g(x)$, i.e., $(L_{\xi(x)}g)(x) = 0$. 

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that the agreement is spectacular, and extends beyond the leading order.\textsuperscript{5} In particular we will see that higher derivative terms in the effective action become important and that one can discriminate between the naive area law and Wald’s generalized definition of black hole entropy for generally covariant theories of gravity with higher derivative terms. Thus, at least for supersymmetric black holes, one can make precision tests which confirm that the number of microstates agrees with the black hole entropy.

Besides fundamental strings, string theory contains other extended objects, which are also important in accounting for black hole microstates. One particular subclass are the D-branes. In string perturbation theory they appear as submanifolds of space-time, on which open strings can end. In the effective field theory they correspond to black p-brane solutions, which carry a particular kind of charge, called Ramond-Ramond charge, which is not carried by fundamental strings. In string compactifications one can put the spatial directions of p-branes along the compact space and thereby obtain black holes in the lower-dimensional space-time. D-branes gave rise to the first successful quantitative matching between state counting and black hole entropy \[6\]. Here ‘quantitative’ means that the leading contribution to \( S_{\text{micro}} \) is precisely \( \frac{A}{4} \), i.e, the prefactor comes out exactly.

When D-branes and other extended objects enter the game, the state counting becomes more complicated, but the basic ideas remain as explained above. Also note that instead of a flat background space-time one can consider other consistent string backgrounds. In particular one can count the microstates of four-dimensional black holes which arise in string compactifications on tori, orbifolds and Calabi-Yau manifolds.

## 2 The black hole attractor mechanism

In this section we discuss BPS black hole solutions of four-dimensional \( N = 2 \) supergravity. This is the most general setup which allows supersymmetric black hole solutions and arises in various string compactifications, including compactifications of the heterotic string on \( K3 \times T^2 \) and of the type-II superstring on Calabi-Yau threefolds. The \( N = 2 \) supergravity multiplet is a supersymmetric version of Einstein-Maxwell theory: it contains the graviton, a gauge field called the graviphoton, and a doublet of Majorana gravitini. The extreme Reissner-Nordstrom black hole is a solution of this

\textsuperscript{5}Of course, at some level the question whether the black hole entropy corresponds to the true state degeneracy or to an index becomes relevant. However, none of the examples analyzed in \[4,5\] appears to be conclusive.
theory and provides the simplest example of a supersymmetric black hole \cite{7,8}.

In string compactifications the gravity multiplet is always accompanied by matter multiplets. The only type of matter which is relevant for our discussion is the vector multiplet, which contains a gauge field, a doublet of Majorana spinors, and a complex scalar. We will consider an arbitrary number \( n \) of vector multiplets. The resulting Lagrangian is quite complicated, but all the couplings are encoded in a single holomorphic function, called the prepotential \cite{9,10}. In order to understand the structure of the entropy formula for black holes, we need to review some more details.

First, let us note that the fields which are excited in black hole solutions are only the bosonic ones. Besides the metric there are \( n \) scalar fields \( z^A, A = 1, \ldots, n \) and \( n + 1 \) gauge fields \( F^I_{\mu\nu} \). The field equations are invariant under \( Sp(2n+2,\mathbb{R}) \) transformations, which generalize the electric-magnetic duality rotations of the Maxwell theory. These act linearly on the gauge fields and rotate the \( F^I_{\mu\nu} \) among themselves and into their duals. Electric charges \( q_I \) and magnetic charges \( p^I \) are obtained from flux integrals of the dual field strength and of the field strength, respectively. They form a symplectic vector \( (p^I, q_J) \). While the metric is inert, the action on the scalars is more complicated. However, it is possible to find a parametrization of the scalar sector that exhibits a simple and covariant behaviour under symplectic transformations. The scalar part of the Lagrangian is a non-linear sigma-model, and the scalar fields can be viewed as coordinates on a complex \( n \)-dimensional manifold \( M \). The geometry of \( M \) is restricted by supersymmetry, and the resulting geometry is known as ‘special geometry’ \cite{10}. In the context of the superconformal calculus, the coupling of \( n \) vector multiplets to Poincaré supergravity is constructed by starting with \( n + 1 \) superconformal vector multiplets, and imposing suitable gauge conditions which fix the additional symmetries. As already mentioned, the vector multiplet Lagrangian is encoded in a single function, the prepotential, which depends holomorphically on the lowest components of the superconformal vector multiplets. A consistent coupling to supergravity further requires the prepotential to be homogenous of degree 2:

\[
F(\lambda Y^I) = \lambda^2 F(Y^I) .
\]  

(2.1)

Here, the complex fields \( Y^I, I = 0,1,\ldots, n \) are the lowest components of the superconformal vector multiplets.\footnote{See for example \cite{25} for a pedagogical treatment.} The physical scalar fields \( z^A \) are given by

\footnote{In the context of black hole solutions, it is convenient to work with rescaled variables.}
the independent ratios, $z^A = \frac{Y^A}{\nu}$. Geometrically, the $Y^I$ are coordinates of a complex cone $C(M)$ over the scalar manifold $M$. The existence of a prepotential is equivalent to the existence of a holomorphic Lagrangian immersion of $C(M)$ into the complex symplectic vector space $\mathbb{C}^{2n+2}$ [11]. If $(Y^I, F_I)$ are coordinates on $\mathbb{C}^{2n+2}$, then, along the immersed $C(M)$, the second half of the coordinates can be expressed in terms of the first half as $F_I = \frac{\partial F}{\partial Y^I}$, where $F$ is the generating function of the Lagrangian immersion, i.e., the prepotential.\textsuperscript{8} Under symplectic transformations $(Y^I, F_I)$ transforms as a vector. Therefore it is convenient to use it instead of $z^A$ to parametrize the scalar fields.

Let us now turn to static, spherically symmetric, supersymmetric black hole solutions. In $N = 2$ supergravity, which has eight independent real supercharges, one can have 8 or 4 or 0 Killing spinors. Solutions with 8 Killing spinors preserve as many supersymmetries as flat space-time and are regarded as supersymmetric vacua. Besides $\mathbb{R}^4$ the only supersymmetric vacua are $AdS^2 \times S^2$ and planar waves [12].\textsuperscript{9} Supersymmetric black holes are solutions with 4 Killing spinors. Since they preserve half as many supersymmetries as the vacuum, they are called $\frac{1}{2}$ BPS solutions. Since these solutions are asymptotically flat, the number of Killing spinors doubles if one goes to infinity.

One difference between supersymmetric black holes in theories with vector multiplets and the extreme Reissner-Nordstrom black hole of pure $N = 2$ supergravity is that there are several gauge fields, and therefore several species of electric and magnetic charges. The other difference is that we now have scalar fields which can have a non-trivial dependence on the radial coordinate. A black hole solution is parametrized by the magnetic and electric charges $(p^I, q_J)$, which are discrete quantities (by Dirac quantization) and by the asymptotic values of the scalar fields in the asymptotically flat region, $z^A(\infty)$, which can be changed continuously. In particular, the mass of a black hole can be changed continuously by tuning the values of the scalar fields at infinity. The area of the horizon and hence $S_{\text{macro}}$ depends on the charges and on the values of the scalar fields at the horizon. If the latter

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\textsuperscript{8}It is assumed here that the immersion is generic, so that that the $Y^I$ are coordinates on $C(M)$. This can always be arranged by applying a symplectic rotation.

\textsuperscript{9}Presumably this is still true in the presence of neutral matter and including higher curvature corrections. In [19] the most general stationary vacuum solution for this case was shown to be $AdS^2 \times S^2$. 

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\textsuperscript{6}The fields $Y^I$ used in this paper are related to the lowest components $X^I$ of the superconformal vector multiplets by $Y^I = \mathcal{Z} X^I$, where $Z = p^I F_I - q_J X^J$ is the central charge. Using that $F_I$ is homogenous of degree one has $F_I(Y) = \mathcal{Z} F_I(X)$. See [15, 13] for more details.
could be changed continuously, this would be at odds with the intended interpretation in terms of state counting.

What comes to the rescue is the so-called black hole attractor mechanism [14]: if one imposes that the solution is supersymmetric and regular at the horizon, then the values of the scalars at the horizon, and also the metric, are determined in terms of the charges. Thus the scalars flow from arbitrary initial values at infinity to fixed point values at the horizon. The reason behind this behaviour is that if the horizon is to be finite then the number of Killing spinors must double on the horizon. This fixes the geometry of the horizon to be of the form $AdS^2 \times S^2$, with fixed point values for the scalars. In the notation introduced above, the values of the scalar fields can be found from the following black hole attractor equations [14]:

$$
(Y^I - \overline{Y}^I)_{\text{Horizon}} = i p^I,
$$
$$
(F_I - \overline{F}_I)_{\text{Horizon}} = i q_I.
$$

(2.2)

For a generic prepotential $F$ it is not possible to solve this set of equations for the scalar fields in closed form. However, explicit solutions have been obtained for many physically relevant examples, where either the prepotential is sufficiently simple, or for non-generic configurations of the charges (i.e., when switching off some of the charges) [16, 17].

The entropy of the corresponding solution is

$$
S_{\text{macro}} = \frac{A}{4} = \pi |Z|_{\text{Horizon}}^2 = \pi (p^I F_I - q_I Y^I)_{\text{Horizon}}.
$$

(2.3)

Here $Z$ is a particular, symplectically invariant contraction of the fields with the charges, which gives the central charge carried by the solution when evaluated at infinity. At the horizon, this quantity sets the scale of the $AdS^2 \times S^2$ space and therefore gives the area $A$.

Let us take a specific example. We consider the prepotential of a type-II Calabi-Yau compactification at leading order in both the string coupling $g_S$ and the string scale $\alpha'$. If we set half of the charges to zero, $q_A = p^0 = 0$, then the attractor equations can be solved explicitly. To ensure weak coupling and small curvature at the horizon, the non-vanishing charges must satisfy $|q_0| \gg p^A \gg 1.$

The resulting entropy is [16]:

$$
S_{\text{macro}} = 2\pi \sqrt{\frac{1}{6} |q_0| C_{ABC} p^A p^B p^C}.
$$

(2.4)

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10We use the notation of [15], where the scalar fields $X^I$ used in [14] have been rescaled in the way explained above.

11Also note that if one can solve the attractor equations, then one can also find the solution away from the horizon [18].

12In our conventions $q_0 < 0$ under the conditions stated in the text.
Here $C_{ABC}$ are geometrical parameters (triple intersection numbers) which depend on the specific Calabi-Yau threefold used for compactification.

For this example the state counting has been performed using the corresponding brane configuration. The result is \[ \text{[19, 20]}: \]

$$S_{\text{micro}} = 2\pi \sqrt{\frac{1}{6} |q_0| (C_{ABC} p^Ap^B p^C + c_{2A} p^A)},$$

(2.5)

where $c_{2A}$ is another set of geometrical parameters of the underlying Calabi-Yau threefolds (the components of the second Chern class with respect to a homology basis). Since this formula contains a subleading term, which is not covered by the macroscopic entropy \[ \text{[23]}, this raises the question how one can improve the treatment of the black hole solutions. Since we interpret supergravity actions as effective actions coming from string theory, the logical next step is to investigate the effects of higher derivative terms in the effective action, which are induced by quantum and stringy corrections.

### 3 Beyond the area law

There is a particular class of higher derivative terms for which the $N = 2$ supergravity action can be constructed explicitly \[ \text{[24]} \] (see also \[ \text{[22]} \] for a review). These terms are encoded in the so-called Weyl multiplet and can be taken into account by giving the prepotential a dependence on an additional complex variable $\Upsilon$, which is proportional to the lowest component of the Weyl multiplet. The equations of motion relate $\Upsilon$ to the (antiselfdual part of the) graviphoton field strength. The generalized prepotential is holomorphic and homogenous of degree 2:

$$F(\lambda Y^I, \lambda^2 \Upsilon) = \lambda^2 F(Y^I, \Upsilon).$$

(3.1)

Expanding in $\Upsilon$ as

$$F(Y^I, \Upsilon) = \sum_{g=0}^{\infty} F^{(g)}(Y^I) \Upsilon^g,$$

(3.2)

one gets an infinite sequence of coupling functions $F^{(g)}(Y^I)$. While $F^{(0)}(Y^I)$ is the prepotential, the $F^{(g)}(Y^I)$, $g \geq 1$, are coefficients of higher derivative terms. Among these are terms of the form

$$F^{(g)}(Y^I)(C^-_{\mu\nu\rho\sigma})^2 (F^-_{\tau\lambda})^{2g-2} + \text{c.c.},$$

(3.3)

where $C^-_{\mu\nu\rho\sigma}$ and $F^-_{\tau\lambda}$ are the antiselfdual projections of the Weyl tensor and of the graviphoton field strength, respectively. In the context of type-II
Calabi-Yau compactifications the functions $F(y)(y^I)$ can be computed using a topologically twisted version of the theory [23, 24].

Starting from the generalized prepotential one can work out the Lagrangian and construct static, spherically symmetric BPS black hole solutions [13]. It can be shown that the near horizon solution is still determined by the black hole attractor equations, \(^{14}\) which now involve the generalized prepotential [15]:

\[
(Y^I - \tilde{Y}^I)_{\text{horizon}} = i p^I, \\
(F_I(Y, \tilde{Y}) - F_I(Y, Y))_{\text{horizon}} = i q_I .
\] (3.4)

The additional variable $\tilde{Y}$ takes the value $\tilde{Y} = -64$ at the horizon. Since the generalized prepotential enters into the attractor equations, the area of the horizon is modified by the higher derivative terms. Moreover, there is a second modification, which concerns the very definition of the black hole entropy.

The central argument for interpreting the area of the horizon as an entropy comes from the first law of black hole mechanics, which relates the change of the mass of a stationary black hole to changes of the area and of other quantities (angular momentum, charges):

\[
\delta M = \frac{\kappa_S}{8\pi} \delta A + \cdots ,
\] (3.5)

where $\kappa_S$ is the surface gravity of the black hole. \(^{15}\) Comparing to the first law of thermodynamics,

\[
\delta U = T \delta S + \cdots ,
\] (3.6)

and taking into account that the Hawking temperature of a black hole is $T = \frac{\kappa_S}{2\pi}$, one is led to the identification $S_{\text{macro}} = \frac{A}{4}$. This is at least the situation in Einstein gravity. The first law can be generalized to more general gravitational Lagrangians, which contain higher derivative terms, in particular arbitrary powers of the Riemann tensor and of its derivatives [20, 27].

\(^{13}\)In fact, one can construct stationary BPS solutions which generalize the IWP solutions of pure supergravity [43].

\(^{14}\)More precisely, the attractor equations are necessary and sufficient for having a fully supersymmetric solution with 8 Killing spinors at the horizon. The geometry is still $AdS^2 \times S^2$, but with a modified scale.

\(^{15}\)See for example [22] for a review of the relevant properties of black hole horizons.
generally covariant, and that it admits stationary black hole solutions whose horizons are Killing horizons. Then there still is a first law of the form
\[ \delta M = \frac{k_0}{2\pi} \delta S_{\text{macro}} + \cdots , \quad (3.7) \]
but \( S_{\text{macro}} \neq \frac{A}{4} \) in general. Rather, \( S_{\text{macro}} \) is given by the surface charge associated with the horizontal Killing vector field \( \xi \):
\[ S_{\text{macro}} = 2\pi \oint_{\text{horizon}} Q[\xi] , \quad (3.8) \]
which can be expressed in terms of variational derivatives of the Lagrangian with respect to the Riemann tensor [27]:
\[ S_{\text{macro}} = -2\pi \oint_{\text{horizon}} \frac{\delta L}{\delta R_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu\varepsilon_{\rho\sigma}} \sqrt{h} \, d^2 \theta . \quad (3.9) \]
Here \( \varepsilon_{\mu\nu} \) is the normal bivector of the horizon, normalized as \( \varepsilon_{\mu\nu} \varepsilon^{\mu\nu} = -2 \), and \( h \) is the pullback of the metric onto the horizon.\(^ {16} \) From (3.9) it is clear that corrections to the area law will be additive:
\[ S_{\text{macro}} = \frac{A}{4} + \cdots . \quad (3.10) \]
Here the leading term comes from the variation of the Einstein-Hilbert action.

The general formula (3.9) can be evaluated for the special case of \( N = 2 \) supergravity with vector multiplets and higher derivative terms encoded in the generalized prepotential. The result is [15]:
\[ S_{\text{macro}}(q,p) = \pi \left( (p^I F_I - q^I Y^I) + 4 \text{Im}(\Upsilon F_{\Upsilon}) \right)_{\text{horizon}} , \quad (3.11) \]
where \( F_{\Upsilon} = \frac{\partial F}{\partial \Upsilon} \). Thus \( F_{\Upsilon} \), which depends on the higher derivative couplings \( F^{(g)}, g \geq 1 \), encodes the corrections to the area law.

If the prepotential is sufficiently simple, one can find explicit solutions of the attractor equations [15, 35, 47].\(^ {17} \) In particular, we can now compare

\(^ {16} \) In carrying out the variational derivatives one treats the Riemann tensor formally as if it was independent of the metric. At first glance this rule looks ambiguous, because one can perform partial integrations. But the underlying formalism guarantees that the integrated quantity \( S_{\text{macro}} \) is well defined [26, 37] (see also [35] for an alternative proof).

\(^ {17} \) One can also construct the solution away from the horizon, at least iteratively [16, 37].
$S_{\text{macro}}$ to the $S_{\text{micro}}$ computed from state counting (2.5) [15]:

$$
S_{\text{macro}}(q,p) = A^4 + \text{Correction term}
$$

$$
= 2\pi \sqrt{\frac{1}{6} |q_0| (C_{ABC}p^A p^B p^C + c_2 A p^A)} + 2\pi \frac{1}{12} |q_0| c_2 A p^A
$$

$$
= S_{\text{micro}}.
$$

(3.12)

In the second line we can see explicitly how the higher derivative terms modify the area. But, when sticking to the naive area law, one finds that $A^4$ differs from $S_{\text{micro}}$ already in the first subleading term in an expansion in large charges. In contrast, when taking into account the modification of the area law, $S_{\text{macro}}$ and $S_{\text{micro}}$ agree completely. In other words ‘string theory state counting knows about the modification of the area law.’ This provides strong evidence that string theory captures the microscopic degrees of freedom of black holes, at least of supersymmetric ones.

At this point one might wonder about the role of other types of higher derivatives terms. So far, we have only included a very particular class, namely those which can be described using the Weyl multiplet. The full string effective action also contains other higher derivative terms, including terms which are higher powers in the curvature. Naively, one would expect that these also contribute to the black hole entropy. However, as we will see in the next sections, one can obtain an even more impressive agreement between microscopic and macroscopic entropy by just using the terms encoded in the Weyl multiplet. One reason might be the close relationship between the terms described by the Weyl multiplet and the topological string, which we are going to review in the next section. There are two other observation which indicate that the Weyl multiplet encodes all contributions relevant for the entropy.\footnote{For toroidal compactifications of type-II string theory there are no $R^2$-corrections, but the entropy of string states is non-vanishing. This case seems to require the presence of higher derivative terms which are not captured by the Weyl multiplet. See [4] for further discussion.} The first observation is that when one just adds a Gauss-Bonnet term to the Einstein Hilbert action, one obtains the same entropy formula (2.5) as when using the full Weyl multiplet [38, 37]. The second is that (2.5) can also be derived using gravitational anomalies [11, 40].
suggest that the black hole entropy is a robust object, in the sense that it does not seem to depend sensitively on details of the Lagrangian.

One might also wonder, to which extent the matching of microscopic and macroscopic entropy depends on supersymmetry. Here it is encouraging that the derivations of (2.5) in [36, 37] and [40] do not invoke supersymmetry directly. Rather, [36, 37] analyses black holes with near horizon geometry $AdS^2 \times S^2$ in the context of general higher derivative covariant actions, without assuming any other specifics of the interactions. This leads to a formalism based on an entropy function, which is very similar to the one found for supersymmetric black holes some time ago [16], and which we will review in a later section. The work of [40] relates Wald’s entropy formula to the AdS/CFT correspondence.

Finally, it is worth remarking that according to [36, 40] similar results should hold in space-time dimensions other than four. A particularly interesting dimension seems to be five, because there is a very close relationship between four-dimensional supersymmetric black holes and five-dimensional supersymmetric rotating black holes and black rings [42, 38], which holds in the presence of higher curvature terms.

Coming back to (3.12), we remark that it is intriguing that two complicated terms, the area and the correction term, combine into a much simpler expression. This suggests that, although (3.11) is a sum of two terms, it should be possible to express the entropy in terms of one single function. Though it is not quite obvious how to do this, it is in fact true.

4 From black holes to topological strings

The black hole entropy (3.11) can be written as the Legendre transform of another function $F_{BH}$, which is interpreted as the black hole free energy. This is seen as follows [3]. The ‘magnetic’ attractor equation $Y^I = \Upsilon^I = ip^I$ can be ‘solved’ by setting:

$$Y^I = \frac{\phi^I}{2\pi} + \frac{i p^I}{2},$$

(4.1)

where $\phi^I \propto \text{Re}Y^I$ is determined by the remaining ‘electric’ attractor equation. From the gauge field equations of motion in a stationary space-time one sees that $\phi^I$ is proportional to the electrostatic potential (see for example [13]). Now define the free energy

$$F_{BH}(\phi, p) := 4\pi \text{Im} F(Y, \Upsilon)_{\text{horizon}}.$$  

(4.2)

\[19\] We use the notation of [15] which is slightly different from the one of [3].
Observe that the electric attractor equations \( F_I - \mathcal{F}_I = ip^I \) are equivalent to
\[
\frac{\partial \mathcal{F}_{\text{BH}}}{\partial \phi^I} = q_I . \tag{4.3}
\]
Next note that the homogeneity property of the generalized prepotential implies the Euler-type relation
\[
2F = Y^I F_I + 2\Upsilon_{I} . \tag{4.4}
\]
Using this one easily verifies that
\[
\mathcal{F}_{\text{BH}}(\phi, p) - \phi^I \frac{\partial \mathcal{F}_{\text{BH}}}{\partial \phi^I} = S_{\text{macro}}(q, p) . \tag{4.5}
\]
Thus the black hole entropy is obtained from the black hole free energy by a (partial) Legendre transform which replaces the electric charges \( q_I \) by the electrostatic potentials \( \phi^I \).

This observation opens up various routes of investigation. Let us first explore the consequences for the relation between \( S_{\text{macro}} \) and \( S_{\text{micro}} \). The black hole partition function associated with \( \mathcal{F}_{\text{BH}} \) is
\[
Z_{\text{BH}}(\phi, p) = e^{\mathcal{F}_{\text{BH}}(\phi, p)} . \tag{4.6}
\]
Since it depends on \( \phi^I \) rather than on \( q_I \), it is clear that this is not a micro-canonical partition function. Rather it refers to a mixed ensemble, where the magnetic charges have been fixed while the electric charges fluctuate. The electrostatic potential \( \phi^I \) is the corresponding thermodynamic potential. However, the actual state degeneracy \( d(q, p) \) should be computed in the microcanonical ensemble, where both electric and magnetic charges are fixed. Using a standard thermodynamical relation, we see that \( Z_{\text{BH}} \) and \( d(q, p) \) are formally related by a (discrete) Laplace transform:
\[
Z_{\text{BH}}(\phi, p) = \sum_q d(q, p) e^{\phi^I q_I} . \tag{4.7}
\]
We can solve this formally for the state degeneracy by an (inverse discrete) Laplace transform,
\[
d(q, p) = \int d\phi \ e^{\mathcal{F}_{\text{BH}}(\phi, p) - \phi^I q_I} \tag{4.8}
\]
and express the microscopic entropy
\[
S_{\text{micro}}(q, p) = \log d(q, p) \tag{4.9}
\]
in terms of the black hole free energy. Comparing (4.5) to (4.8) it is clear that $S_{\text{macro}}$ and $S_{\text{micro}}$ will not be equal in general. Both can be expressed in terms of the free energy, but one is given through a Laplace transform and the other through a Legendre transform [3]. From statistical mechanics we are used to the fact that quantities might differ when computed using different ensembles, but we expect them to agree in the thermodynamic limit. In our context the thermodynamic limit corresponds to the limit of large charges, in which it makes sense to evaluate the inverse Laplace transform (4.8) in a saddle point approximation:

$$e^{S_{\text{micro}}(q,p)} = \int d\phi e^{F_{\text{BH}}(\phi,p) - \phi^I q_I} \approx e^{F_{\text{BH}}(\phi,p) - \phi^I \frac{\partial F_{\text{BH}}}{\partial \phi^I}} = e^{S_{\text{macro}}(q,p)}. \quad (4.10)$$

Since the saddle point value of the inverse Laplace transform is given by the Legendre transform, we see that both entropies agree in the limit of large charges. Note that already the first subleading correction, which comes from quadratic fluctuations around the saddle point, will in general lead to deviations. We will illustrate the relation between $S_{\text{macro}}$ and $S_{\text{micro}}$ using specific examples later on.

We now turn to another important consequence (4.5). As already mentioned the couplings $F(g)(Y)$ of the effective $N = 2$ supergravity Lagrangian can be computed within the topologically twisted version of type-II string theory with the relevant Calabi-Yau threefold as target space. The effect of the topological twist is roughly to remove all the non-BPS states, thus reducing each charge sector to its ground state. The coupling functions can be encoded in a generating function, called the topological free energy $F_{\text{top}}(Y^I, \Upsilon)$, which equals the generalized prepotential $F(Y^I, \Upsilon)$ of supergravity up to a conventional overall constant. The associated topological partition function

$$Z_{\text{top}} = e^{F_{\text{top}}} \quad (4.11)$$

can be viewed as a partition function for the BPS states of the full string theory. Taking into account the conventional normalization factor between $F_{\text{top}}$ and $F(Y^I, \Upsilon)$ one observes [3]:

$$Z_{\text{BH}} = e^{F_{\text{BH}}} = e^{4\pi \text{Im} F} = e^{F_{\text{top}} + \text{Re} F_{\text{top}}} = e^{F_{\text{top}}} |^{2} = |Z_{\text{top}}|^{2}. \quad (4.12)$$

Thus there is a direct relation between the black hole entropy and the topological partition function, which suggests that the matching between macroscopic and microscopic entropy extends far beyond the leading contributions. Moreover, the relation $Z_{\text{BH}} = |Z_{\text{top}}|^{2}$ suggests to interpret $Z_{\text{top}}$
as a quantum mechanical wave function and $Z_{\text{BH}}$ as the associated probability \[3\]. This can be made precise as follows: $Z_{\text{top}}$ is a function on the vector multiplet scalar manifold, which in type-IIA (type-IIB) Calabi-Yau compactifications coincides with the moduli space of complexified Kähler structures (complex structures). This manifold is in particular symplectic, and can be interpreted as a classical phase space. Applying geometric quantization one sees that $Z_{\text{top}}$ is indeed a wave function on the resulting Hilbert space \[28\]. This reminds one of the minisuperspace approximations used in canonical quantum gravity. In our case the truncation of degrees of freedom is due to the topological twist, which leaves the moduli of the internal manifold as the remaining degrees of freedom. In other words the full string theory is reduced to quantum mechanics on the moduli space. One is not restricted to only discussing black holes in this framework, but, by a change of perspective and some modifications, one can approach the dynamics of flux compactifications and quantum cosmology \[29\].

The link between black holes and flux compactifications is provided by the observation that from the higher-dimensional point of view the near-horizon geometry of a supersymmetric black hole is $AdS^2 \times S^2 \times X^*$, where $X^*$ denotes the Calabi-Yau threefold at the attractor point in moduli space corresponding to the charges ($q^I, p_I$). This can be viewed as a flux compactification to two dimensions. The flux is given by the electric and magnetic fields along $AdS^2 \times S^2$, which are covariantly constant, and compensate for the fact that the geometry is not Ricci-flat. From the two-dimensional perspective the attractor mechanism reflects that the reduction on $S^2$ gives rise to a gauged supergravity theory with a nontrivial scalar potential which fixes the moduli. When taking the spatial direction of $AdS^2$ to be compact, so that space takes the form $S^1 \times S^2 \times X^*$, then vacua with different moduli are separated by barriers of finite energy. As a consequence, the moduli, which otherwise label superselection sectors, can fluctuate. In this context $Z_{\text{top}}$ has been interpreted as a Hartle-Hawking type wave function for flux compactifications \[29\], while \[30\] argued that string compactifications with asymptotically free gauge groups are preferred.

It should be stressed that there are many open questions concerning these proposals, both conceptually and technically. Some of these will be discussed in the next section from the point of view of supergravity and black holes. Nevertheless these ideas are very interesting because they provide a new way to approach the vacuum selection problem of string theory. Moreover there seems to be a lot in common with the canonical approach to quantum gravity and quantum cosmology. This might help to develop new ideas how to overcome the shortcomings of present day string theory concerning time-
dependent backgrounds. By phrasing string theory in the language used in canonical quantum gravity, one would have a better basis for debating the merits of different approaches to quantum gravity.

5 Variational principles for black holes

We will now discuss open problems concerning the formulae (4.7), (4.8) and (4.12) which relate the black hole entropy to the counting of microstates. The following sections are based on [31]. See also [32, 33, 34] for further discussion.

Consider for definiteness (4.8):

\[ d(q, p) = \int d\phi \, e^{\mathcal{F}_{\text{BH}}(\phi, p) - \phi^I q_I}, \quad (5.1) \]

which relates the black hole free energy to the microscopic state degeneracy. This is formally an inverse discrete Laplace transformation, but without specifying the integration contour it is not clear that the integral converges. We will not address this issue here, but treat the integral as a formal expression which we evaluate asymptotically by saddle point methods. The next issue is the precise form of the integrand. As we stressed above various quantities of the effective \( N = 2 \) supergravity, in particular the charges \((p^I, q_J)\), are subject to symplectic rotations. The microscopic state degeneracy \( d(q, p) \) is an observable and therefore should transform covariantly, i.e., it should be a symplectic function. Moreover, string theory has discrete symmetries, in particular T-duality and S-duality, which are realized as specific symplectic transformations in \( N \geq 2 \) compactifications. Since these are symmetries, \( d(q, p) \) should be invariant under them. The transformation properties with respect to symplectic transformations are simple and transparent as long as one works with symplectic vectors, such as \((Y^I, F_J)\) and its real and imaginary parts. By the Legendre transform we now take \( \phi^I \) and \( p^I \) as our independent variables, and these do not form a symplectic vector. Thus manifest symplectic covariance, as it is present in the entropy formula (3.11), has been lost. Moreover, it is clear that if \( d\phi \) is the standard Euclidean measure \( \prod_I d\phi^I \), then the integral cannot be expected to be symplectically invariant. From the point of view of symplectic covariance

20Preliminary results have already been presented at conferences, including the ‘Workshop on gravitational aspects of string theory’ at the Fields Institute (Toronto, May 2005) and the ‘Strings 2005’ (Toronto, July 2005). See

http://online.kitp.ucsb.edu/online/joint98/kaeppeli/ and

http://www.fields.utoronto.ca/audio/05-06/strings/wit/index.html
one should expect that the integration measure is symplectically invariant, while the integrand is a symplectic function. We will now outline a systematic procedure which provides a modified version of (4.8) which has this property.

The starting point is the observation that the entropy of supersymmetric black holes can be obtained from a variational principle \cite{16, 31}. Define the symplectic function

\[ \Sigma(q, p, Y, \bar{Y}) := -K - W + 128iF_Y - 128i\bar{F}_{\bar{Y}}, \quad (5.2) \]

where

\[ K := i(Y^I F_I - \bar{F}_I Y^I) \quad \text{and} \quad W := q_I Y^I - p^I F_I. \quad (5.3) \]

One then finds that the conditions for critical points of \( \Sigma \),

\[ \frac{\partial \Sigma}{\partial Y^I} = 0 = \frac{\partial \Sigma}{\partial Y_I} \quad (5.4) \]

are precisely the attractor equations \cite{34}. Moreover, at the attractor we find that

\[ \pi \Sigma_{\text{attractor}}(q, p) = S_{\text{macro}}(q, p). \quad (5.5) \]

We also note that one can split the extremization procedure consistently into two steps. If one first extremizes \( \Sigma \) with respect to the imaginary part of \( Y^I \), one obtains the magnetic attractor equations. Plugging these back we find

\[ \pi \Sigma(\phi, q, p)_{\text{magnetic attractor}} = F_{\text{BH}}(\phi, p) - \phi^I q_I \quad (5.6) \]

and recover the free energy of \cite{3} at an intermediate level. Subsequent extremization with respect to \( \phi^I \propto \text{Re}Y^I \) gives the electric attractor equations, and by plugging them back we find the entropy. Moreover, while the free energy \( F_{\text{BH}}(\phi, p) \) is related to the black hole entropy \( S_{\text{macro}}(p, q) \) by a partial Legendre transform, the charge-independent part of \( \Sigma \), namely

\[ -K + 128iF_Y - 128i\bar{F}_{\bar{Y}} \]

is its full Legendre transform.

Since \( \pi \Sigma(q, p, Y, \bar{Y}) \) is a symplectic function, which equals \( S_{\text{macro}}(p, q) \) at its critical point, it is natural to take \( \exp(\pi \Sigma) \) to define a modified version of (4.8). This means that we should not only to integrate over \( \phi^I \propto \text{Re}Y^I \), but also over the other scalar fields \( \text{Im}Y^I \). What about the measure? Since it should be symplectically invariant, the natural choice is

\[ d\mu(Y, \bar{Y}) = \prod_{IJ} dY^I d\bar{Y}^J \det(-2i\text{Im}(F_{KL})), \quad (5.7) \]

\[ \text{This follows from inspection of the symplectic transformation rules} \cite{21}. \text{Alternatively, one might note that this measure is proportional to the top exterior power of the natural symplectic form of} \ C(M), \text{the cone over the moduli space.} \]
where $F_{KL}$ denotes the second derivatives of the generalized prepotential with respect to the scalar fields. Putting everything together the proposal of [31] for a modified version of (4.8) is:

$$d(q, p) = \mathcal{N} \int \prod_{IJ} dY^I d\bar{Y}^J \det(-2i \text{Im}(F_{KL})) \exp (\pi \Sigma), \quad (5.8)$$

where $\mathcal{N}$ is a normalization factor. In order to compare to (4.8) it is useful to note that one can perform the saddle point evaluation in two steps. In the first step one takes a saddle point with respect to the imaginary parts of $Y^I$, which imposes the magnetic attractor equations. Performing the saddle point integration one obtains

$$d(q, p) = \mathcal{N}' \int \prod_I d\phi^I \sqrt{\det(-2i(\text{Im}F_{KL}))_{\text{magn. attractor}}} \exp (F_{BH} - \phi^I q_I), \quad (5.9)$$

which is similar to the original (4.8) but contains a non-trivial measure factor stemming from the requirement of symplectic covariance. Subsequent saddle point evaluation with respect to $\phi^I \propto \text{Re} Y^I$ gives

$$d(q, p) = \exp (\pi \Sigma_{\text{attractor}}) = \exp S_{\text{macro}}. \quad (5.10)$$

Let us next comment on another issue concerning (4.8). So far we have been working with a holomorphic prepotential $F(Y^I, Y)$, which upon differentiation yields effective gauge couplings that are holomorphic functions of the moduli. However, it is well known that the physical couplings extracted from string scattering amplitudes are not holomorphic. This can be understood purely in terms of field theory (see for example [45] for a review): if a theory contains massless particles, then the (quantum) effective action (the generating functional of 1PI Greens function) will in general be non-local. In the case of supersymmetric gauge theories, this goes hand in hand with non-holomorphic contributions to gauge couplings, which, in $N = 2$ theories, cannot be expressed in terms of a holomorphic prepotential [46, 48, 47]. Symmetries, such as S- and T-duality provide an efficient way of controlling these non-holomorphic terms. While the holomorphic couplings derived from the holomorphic prepotential are not consistent with S- and T-duality, the additional non-holomorphic contributions transform in such a way that they restore these symmetries. The same remark applies to the black hole entropy, as has been shown for the particular case of supersymmetric black holes in string compactifications with $N = 4$ supersymmetry. There, S-duality is supposed to be an exact symmetry, and therefore physical quantities like gauge couplings, gravitational couplings and the entropy...
must be S-duality invariant. But this is only possible if non-holomorphic contributions are taken into account [49, 47]. In the notation used here this amounts to modifying the black hole free energy by adding a real-valued function $\Omega$, which is homogenous of degree 2 and not harmonic [62, 31]:

$$F_{BH} \rightarrow \hat{F}_{BH} = 4\pi \text{Im}F(Y^I, \Upsilon) + 4\pi \Omega(Y^I, \bar{Y}^I, \Upsilon, \bar{\Upsilon}).$$ \hspace{1cm} (5.11)

Non-holomorphic terms can also be studied in the framework of topological string theory and are then encoded in a holomorphic anomaly equation [23]. The role of non-holomorphic contributions has recently received considerable attention [50, 63, 64, 4, 5] (see also [51]). It appears that these proposals do not fully agree with the one of [31], which was explained above. One way to clarify the role of non-holomorphic corrections is the study of subleading terms in explicit examples, which will be discussed in the next sections.

In the last section we briefly explained how black hole solutions are related to flux compactifications. It is interesting to note that variational principles are another feature that they share. Over the last years it has been realized that the geometries featuring in flux compactifications are calibrated geometries, i.e., one can compute volumes of submanifolds by integrating suitable calibrating forms over them, without knowing the metric explicitly (see for example [51, 43]). Such geometries can be characterized in terms of variational principles, such as Hitchin’s [44]. In physical terms the idea is to write down an abelian gauge theory for higher rank gauge fields (aka differential forms) such that the equations of motion are the equations characterising the geometry. The topological partition function $Z_{\text{top}}$ should then be interpreted as a wave function of the quantized version of this theory. Conversely the variational principle provides the semiclassical approximation of the quantum mechanics on the moduli space.

6 Fundamental strings and ‘small’ black holes

So far our discussion was quite abstract and in parts formal. Therefore we now want to test these ideas in concrete models. As was first realized in [52], the $\frac{1}{2}$ BPS states of the toroidally compactified heterotic string provide an ideal test ground for the idea that there is an exact relation between black hole microstates and the string partition function. Since this compactification has $N = 4$ rather than $N = 2$ supersymmetry, one gets an enhanced control over both $S_{\text{macro}}$ and $S_{\text{micro}}$. For generic moduli, the massless spectrum of the heterotic string compactified on $T^6$ consists of the $N = 4$ supergravity multiplet together with 22 $N = 4$ vector multiplets.
Since the gravity multiplet contains 4 graviphotons, the gauge group is $G = U(1)^2$. There are 28 electric charges $q$ and 28 magnetic charges $p$ which take values in the Narain lattice $\Gamma_{6,22}$. This lattice is even and selfdual with respect to the bilinear form of signature $(6,22)$, and hence it is unique up to isometries. The T-duality group $O(6,22,\mathbb{Z})$ group consists of those isometries which are lattice automorphisms. The S-duality group $SL(2,\mathbb{Z})$ acts as $2 \otimes 1$ on the $(28 + 28)$-component vector $(q,p) \in \Gamma_{6,22} \oplus \Gamma_{6,22} \cong \mathbb{Z}^2 \otimes \mathbb{Z}_{6,22}$, where $2$ denotes the fundamental representation of $SL(2,\mathbb{Z})$ and $1$ the identity map on $\Gamma_{6,22}$.

It turns out that the $N = 2$ formalism described earlier can be used to construct supersymmetric black hole solutions of the $N = 4$ theory [47]. If one uses the up-to-two-derivatives part of the effective Lagrangian, the entropy is given by [53] [47]

$$S_{\text{macro}} = \frac{A}{4} = \pi \sqrt{q^2p^2 - (q \cdot p)^2}. \quad (6.1)$$

Here $a \cdot b$ denotes the scalar product with signature $(6,22)$, so that the above formula is manifestly invariant under T-duality. It can be shown that $(q^2,p^2,q \cdot p)$ transforms in the 3-representation under S-duality, and that the quadratic form $q^2p^2 - (q \cdot p)^2$ is invariant. Therefore $S_{\text{macro}}$ is manifestly S-duality invariant as well. The supersymmetric black hole solutions form two classes, corresponding to the two possible types of BPS multiplets [54, 55] (see also [56] for a review). The $\frac{1}{2}$ BPS solutions with 8 (out of a maximum of 16) Killing spinors are characterized by

$$q^2p^2 - (q \cdot p)^2 = 0 \quad (6.2)$$

and therefore have a degenerate horizon, at least in the lowest order approximation. Particular solutions of (6.2) are $p = 0$ (‘electric black holes’) and $q = 0$ (‘magnetic black holes’).

The $N = 4$ theory also has $\frac{1}{4}$ BPS solutions with only 4 Killing spinors. They satisfy

$$q^2p^2 - (q \cdot p)^2 \neq 0 \quad (6.3)$$

and therefore have a non-vanishing horizon. They are ‘genuinely dyonic’ in the sense that it is not possible to set all electric (or all magnetic) charges to zero by an S-duality transformation. Thus they are referred to as dyonic black holes. We will discuss them in the next section.

\textsuperscript{22}Note that $SL(2,\mathbb{R}) \simeq SO(1,2)$ and that the quadratic form $q^2p^2 - (q \cdot p)^2$ has signature $(1,2)$.
Let us return to the electric black holes. We saw that the horizon area is zero, \( A = 0 \), and so is the black hole entropy \( S_{\text{macro}} = 0 \). Geometrically, the solution has a null singularity, i.e., the curvature singularity coincides with the horizon. One might wonder whether stringy or quantum corrections resolve this singularity. Moreover, a vanishing black hole entropy means that there is only one microstate, and one should check whether this is true.

The candidate microstates for the supersymmetric electric black hole are fundamental strings sitting in \( \frac{1}{2} \) BPS multiplets [57]. These are precisely the states where the left-moving, supersymmetric sector is put into its ground state while exciting oscillations in the right-moving, bosonic sector. Such states take the following form:

\[
\prod_l \alpha^{i_l}_{-m_l} \langle (P_L, P_R) \rangle \otimes |8 \oplus 8\rangle \tag{6.4}
\]

Here the \( \alpha \)'s are creation operators for the right-moving oscillation mode of level \( m_l = 1, 2, \ldots \), which can be aligned along the two transverse space directions, \( i_l = 1, 2 \), along the six directions of the torus, \( i_l = 3, \ldots, 8 \), or along the maximal torus of the rank 16 gauge group of the ten-dimensional theory, \( i_l = 9, \ldots, 24 \). \( P_L \) and \( P_R \) are the left- and right-moving momenta, or, in other words, \( q = (P_L, P_R) \in \Gamma_{6,22} \) are the electric charges. Finally \( |8 \oplus 8\rangle \) is the left-moving ground state, which is a four-dimensional \( N = 4 \) vector supermultiplet with eight bosonic and eight fermionic degrees of freedom. Since the space-time supercharges are constructed out of the left-moving oscillators it is clear that this state transforms in the same way as the left-moving ground state, and therefore is an \( \frac{1}{2} \) BPS state.

Physical states satisfy the mass formula

\[
\alpha' M^2 = N - 1 + \frac{1}{2} P_R^2 + \tilde{N} + \frac{1}{2} P_L^2 ,
\]

where \( N, \tilde{N} \) is the total right- and left-moving excitation level, respectively. Moreover physical states satisfy the level matching condition

\[
N - 1 + \frac{1}{2} P_R^2 = \tilde{N} + \frac{1}{2} P_L^2 .
\]

For \( \frac{1}{2} \) BPS states we have \( \tilde{N} = 0 \) and level matching fixes the level \( N \) and the mass \( M \) in terms of the charges:\footnote{With our conventions \( q^2 \) is negative for physical BPS states with large excitation number \( N \).}

\[
N = \frac{1}{2} (P_L^2 - P_R^2) + 1 = -\frac{1}{2} q^2 + 1 = \frac{1}{2} |q^2| + 1 .
\]
The problem of counting the number of \(\frac{1}{2}\) BPS states amounts to counting partitions of an integer \(N\) (modulo the 24-fold extra degeneracy introduced by the additional labels \(i\)). This is a classical problem which has been studied by Hardy and Ramanujan \[58\]. The number \(d(q)\) of states at of given charge admits the integral representation

\[
d(q) = \int_C d\tau \frac{\exp(i\pi \tau q^2)}{\eta^{24}(\tau)}. \tag{6.8}\]

Here \(\tau\) take values in the upper half plane, \(\eta(\tau)\) is the Dedekind \(\eta\)-function and \(C\) is a suitable integration contour. For large charges \(|q^2| \gg 1\), the asymptotic number of states is governed by the Hardy-Ramanujan formula

\[
d(q) = \exp\left(4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{27}{4} \log |q^2| + \cdots\right). \tag{6.9}\]

The statistical entropy of string states therefore is

\[
S_{\text{micro}}(q) = 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{27}{4} \log |q^2| + \cdots. \tag{6.10}\]

Comparing to the black hole entropy of electric supersymmetric black holes, we realize that there is a discrepancy, because \(S_{\text{macro}}(q) = 0\). One can now reanalyze the black hole solutions and take into account higher derivative terms. As a first step one takes those terms which occur at tree level in the heterotic string theory. These are the same terms that one gets by dimensional reduction of higher derivative terms (\(R^4\)-terms) of the ten-dimensional effective theory. Already this leading order higher derivative term is sufficient to resolve the null singularity and to give the electric black hole a finite horizon with area

\[
\frac{A}{4} = 2\pi \sqrt{\frac{1}{2}|q^2|} = \frac{1}{2}S_{\text{micro}} + \cdots, \tag{6.11}\]

as was shown in \[59\] using the results of \[15, 47\]. The resulting area is large in Planckian but small in string units, reflecting that the resolution is a stringy effect. Hence these black holes are called ‘small black holes’, in contrast to the ‘large’ dyonic black holes, which already have a finite area in the classical approximation.

Again it is crucial to deviate from the area law and to use the generalized definition of black hole entropy \(3.11\), which results in \[59\]:

\[
S_{\text{macro}} = \frac{A}{4} + \text{Correction} = \frac{A}{4} + \frac{A}{4} = \frac{A}{2} = 4\pi \sqrt{\frac{1}{2}|q^2|} = S_{\text{micro}} + \cdots. \tag{6.12}\]
Thus we find that $S_{\text{macro}}$ and $S_{\text{micro}}$ are equal up to subleading contributions in the charges.

We can try to improve on this result by including further subleading contributions to $S_{\text{macro}}$. According to \cite{47, 62} the next relevant term is a non-holomorphic correction to the entropy which gives rise to a term logarithmic in the charges:

$$S_{\text{macro}} = 4\pi \sqrt{\frac{1}{2} q^2} - 6 \log |q^2| .$$

This has the same form as $S_{\text{micro}}$, but the coefficient of the subleading term is different. This is, however, to be expected, if $S_{\text{macro}}$ and $S_{\text{micro}}$ correspond to different ensembles. The actual test consists of the following: take the black hole free energy corresponding to the above $S_{\text{macro}}$, and evaluate the integral \cite[(4.8)]{48}, or any candidate modification thereof like \cite[(5.8)]{58}, in a saddle point approximation and compare the result to $S_{\text{micro}}$. This is, however, not completely straightforward, because the measure in \cite[(5.8)]{58} vanishes identically when neglecting non-holomorphic and non-perturbative corrections. This reflects that the attractor points for electric black holes sit at the boundary of the classical moduli space (the Kähler cone). This boundary disappears in the quantum theory, and non-holomorphic and non-perturbative corrections make the measure finite. But still, the point around which one tries to expand does not correspond to a classical limit. This might explain why there is still disagreement for the term of the form $\log |q^2|$ when taking into account the leading non-holomorphic contribution to the measure \cite[(31)]{31}. Moreover, it appears that \cite{4, 5}, who take a different attitude towards non-holomorphic contributions than \cite{47, 31}, also find a mismatch for the logarithmic term. It is tempting to conclude that the conjecture \cite{47} just does not apply to small black holes. However, it was shown in \cite{4, 5} that there is an infinite series of sub-subleading contributions, involving inverse powers of the charges, which matches perfectly! This suggests that there is a more refined version of the conjecture which applies to small black holes. One possibility is that in order to have a simple relation between the macroscopic and microscopic entropy, the latter has to be defined not with respect to the microcanonical ensemble, but using another ensemble. Concrete proposals are the ‘grand canonical ensemble’ used in \cite{63, 64} and the ‘redefined OSV ensemble’ of \cite{65}.

7 Dyonic strings and ‘large’ black holes

Let us finally briefly discuss $\frac{1}{4}$ BPS black holes. While the leading order black hole entropy is \cite{64}, one can also derive a formula which takes into
account the non-perturbative and non-holomorphic corrections [47, 31]:

\[ S_{\text{macro}}(q, p) = -\pi \left[ \frac{q^2 - i(S - \overline{S}) q \cdot p + |S|^2 p^2}{S + \overline{S}} - 2 \log((S + \overline{S})^6 |\eta(S)|^{24}) \right] \text{horizon (7.1)} \]

Here \( S \) denotes the dilaton and \( \eta(S) \) is the Dedekind \( \eta \)-function. Recalling that the dilaton is related to the string coupling \( g_S \) by \( S = \frac{1}{g_S^2} + i\theta \), and using the expansion of the \( \eta \)-function,

\[ \eta(S) = -\frac{1}{12} \pi S - e^{-2\pi S} + O(e^{-4\pi S}) \text{ (7.2)} \]

we see that (7.1) includes an infinite series of instanton corrections.

In order to show that (7.1) is invariant under T-duality and S-duality, note that \( q^2, p^2, p \cdot q \) and \( S \) are invariant under T-duality. Under S-duality \( (q^2, p^2, p \cdot q) \) transforms in the 3 of \( SL(2, \mathbb{Z}) \), while

\[ S \rightarrow \frac{aS - ib}{icS + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}) . \text{ (7.3)} \]

It is straightforward to see that \((S + \overline{S})^{-1}(1, -i(S - \overline{S}), |S|^2)\) transforms in the 3 of \( SL(2, \mathbb{Z}) \), so that the first term of (7.1) is S-duality invariant. S-duality invariance of the second term follows from the fact that \( \eta^{24}(S) \) is a modular form of degree 12. Observe that the non-holomorphic term \( \sim \log(S + \overline{S}) \) is needed to make the second term of (7.1) S-duality invariant.

The entropy formula (7.1) is not fully explicit, since one cannot solve the attractor equations explicitly for the dilaton as a function of the charges. However, the dilaton attractor equations take the suggestive form

\[ \frac{\partial S_{\text{macro}}(q, p, S, \overline{S})}{\partial S} = 0 . \text{ (7.4)} \]

Now we need to look for candidate microstates [60]. Since these must carry electric and magnetic charge and must sit in \( \frac{1}{4} \) BPS multiplets, they cannot be fundamental string states. However, the underlying ten-dimensional string theory contains besides fundamental strings also solitonic five-branes, which carry magnetic charge. Upon double dimensional reduction to six dimension these can become magnetic strings, which sit in \( \frac{1}{2} \) BPS multiplets. By forming bound states with fundamental strings one can obtain dyonic strings forming \( \frac{1}{4} \) BPS multiplets. Further double dimensional reduction to four dimensions gives dyonic zero-branes, which at finite coupling should correspond to dyonic \( \frac{1}{4} \) BPS black holes.
Based on the conjecture that the world volume theory of the heterotic five-brane is a six-dimensional string theory, one can derive a formula for the degeneracy of dyonic states [60]:

\[ d_{DVV}(q, p) = \oint_{C_3} d\Omega \exp i\pi (Q, \Omega Q) \frac{\Phi_{10}(\Omega)}{C_3}, \] (7.5)

where \( \Omega \) is an element of the rank 2 Siegel upper half space (i.e., a symmetric complex two-by-two matrix with positive definite imaginary part). The vector \( Q = (q, p) \in \Gamma_{6,22} \oplus \Gamma_{6,22} \) combines the electric and magnetic charges, \( \Phi_{10} \) is the degree 2, weight 10 Siegel cusp form, and \( C_3 \) is a three-dimensional integration contour in the Siegel upper half space. This formula is a natural generalization of the degeneracy formula (6.8) for electric 1/2 BPS states.

Recently, an alternative derivation has been given [61], which uses the known microscopic degeneracy of the five-dimensional D5-D1-brane bound state and the relation between five-dimensional and four-dimensional black holes [42].

It has been shown in [60] that the saddle point value of the integral (7.5) gives the leading order black hole entropy [61]. More recently it has been shown in [62] that a saddle point evaluation of (7.5) yields precisely the full macroscopic entropy (7.1). In particular, the conditions for a saddle point of the integrand of (7.5) are precisely the dilaton attractor equations (7.4).

The natural next step is to investigate whether the microscopic state degeneracy (7.5) is consistent with the symplectically covariant version (5.8) of (4.8). This can indeed be shown [31] using the recent result of [66], who have evaluated the mixed partition functions of BPS black holes in \( N = 4 \) and \( N = 8 \) string compactifications. In particular, the non-trivial measure found in [66] can be obtained from (5.8) by taking the limit of large charges [31].

8 Discussion

We have seen that there is an impressive agreement between the counting of supersymmetric string states and the entropy of supersymmetric black holes. In particular, these comparisons are sensitive to the distinction between the naive area law and Wald’s generalized definition. Moreover, there appears to be a direct link between the black hole entropy and the string partition function. This is a big leap forward towards a conceptual understanding of black hole microstates. The stringy approach to black hole entropy also has
its limitations. One is that in order to identify the black hole microstates one has to extrapolate to asymptotically small coupling. In particular, one does not really get a good understanding of the black hole microstates ‘as such,’ but only how they look like in a different regime of the theory. A second, related problem is that one needs supersymmetry to have sufficient control over the extrapolation, the state counting and the construction of the corresponding black hole solutions. Therefore, quantitative agreement has only been established for supersymmetric black holes, which are charged, extremal black holes, and therefore not quite relevant for astrophysics. It should be stressed, however, that the proportionality between microscopic entropy and area can be established by a variety of methods, including the string-black hole correspondence reviewed at the beginning. Moreover, the work of \[36, 37\] and \[40\] suggests that the relation between black hole entropy and string theory states holds without supersymmetry.

The main limitation of string theory concerning quantum gravity in general and black holes in particular is that its core formalism, string perturbation theory around on-shell backgrounds, cannot be used directly to investigate the dynamics of generic, curved, non-stationary space-times. But we have seen that even within these limitations one can obtain remarkable results, which define benchmarks for other candidate theories of quantum gravity. Historically, one important guideline for finding new physical theories has always been that one should be able to reproduce established theories in a certain limit. This is reflected by the prominent role that the renormalization group and effective field theories play in contemporary quantum field theory. Concerning quantum gravity, it appears to be important to keep in touch with classical gravity in terms of a controlled semi-classical limit. Specifically, a theory of quantum gravity should allow one to construct a low energy effective field theory, which contains the Einstein-Hilbert term plus computable higher derivative corrections. Moreover, if microscopic black hole states can be identified and counted, the question as to whether they obey the area law or Wald’s generalized formula should be answered.

The study of black holes has also lead to new ideas which will hopefully improve our understanding of string theory. Notably we have seen that there is a lot in common between supersymmetric black holes and flux compactifications, in particular the role of variational principles. The relation between the black hole partition function and the topological string partition function, and the interpretation of the latter as a wave function shows that there is a kind of minisuperspace approximation, which can be used to investigate the dynamics of flux compactifications and quantum cosmology using a stringy version of the Wheeler de Witt equation. This could
not only improve the conceptual understanding of string theory, but would also increase the overlap between string theory and canonical approaches to quantum gravity.

More work needs to be done in order to further develop the proposal made by [3]. One key question is the relation between the macroscopic and microscopic entropies for small black holes, another one is the role of non-perturbative corrections in general [4, 5, 66, 31]. Future work will have to decide whether the relation discovered by [3] is an exact or only an asymptotic statement. Besides non-perturbative corrections also non-holomorphic corrections are important. We have discussed a concrete proposal for treating non-holomorphic corrections based on [31], but it is not obvious how this relates in detail to the microscopic side, i.e., to the topological string.

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