QUANTIZATION COEFFICIENTS IN INFINITE SYSTEMS

EUGEN MIHAILESCU AND MRINAL ROYCHOWDHYU

Abstract. We investigate quantization coefficients for probability measures \( \mu \) on limit sets, which are generated by systems \( S \) of infinitely many contractive similarities and by probabilistic vectors. The theory of quantization coefficients for infinite systems has significant differences from the finite case. One of these differences is the lack of finite maximal antichains, and the fact that the set of contraction ratios has zero infimum; another difference resides in the specific geometry of \( S \) and of its non-compact limit set \( J \). We prove that, for each \( r \in (0, \infty) \), there exists a unique positive number \( \kappa_r \), so that the \( \kappa_r \)-dimensional lower quantization coefficient of order \( r \) of \( \mu \) is positive. We also give estimates for the upper quantization coefficients of order \( r \) of \( \mu \). The above results allow then to estimate the asymptotic errors of approximating the measure \( \mu \) in the \( L_r \)-Kantorovich-Wasserstein metric, with discrete measures supported on finitely many points.

1. Introduction and general setting.

The theory of quantization studies the process of approximating probability measures, which are invariant for certain systems, with discrete probabilities having a finite number of points in their support. Of particular interest are the types of behaviors which may be encountered in this quantization process for various measures.

Let us consider in general, a probability measure \( \mu \) on \( \mathbb{R}^d \), a number \( r \in (0, \infty) \) and a natural number \( n \in \mathbb{N} \). Then, the \( n \)-th quantization error of order \( r \) of \( \mu \) is defined by:

\[
V_{n,r}(\mu) := \inf \{ \int d(x, \alpha)^r d\mu(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \},
\]

where \( d(x, \alpha) \) denotes the distance from an arbitrary point \( x \) to the set \( \alpha \) with respect to the Euclidean norm on \( \mathbb{R}^d \) (see [GL1]). If \( \int \|x\|^r d\mu(x) < \infty \), then there exists some set \( \alpha \) for which the infimum is achieved. A set \( \alpha \) for which the infimum is achieved is called an optimal set of \( n \)-means or \( n \)-optimal set of order \( r \), for the probability \( \mu \) and for \( 0 < r < \infty \).

For \( s > 0 \), the \( s \)-dimensional upper, and lower quantization coefficients of order \( r \) for the probability measure \( \mu \), are defined (cf [GL1]) respectively as:

\[
\overline{QC}_{r,s}(\mu) := \limsup_n n^{s/r} V_{n,r}(\mu), \quad \underline{QC}_{r,s}(\mu) := \liminf_n n^{s/r} V_{n,r}(\mu)
\]

We will be interested below in quantization coefficients for self-similar probability measures \( \mu \) for infinite systems of contractive similarities \( S = (S_1, S_2, \ldots) \) and for infinite probability vectors \( p = (p_1, p_2, \ldots) \). In this case, the theory and the techniques of proof from the finite case do not work. In particular, we do not have finite maximal anti-chains, and also the set of the contraction ratios for the maps \( S_i, i \geq 1 \), has zero infimum.

Recall that in the finite case, a finite self-similar system is determined by a set of contractive similarity mappings on \( \mathbb{R}^d \), namely \{\( S_1, S_2, \ldots, S_N \)\} with contraction rates \( s_1, s_2, \ldots, s_N \), for \( N \geq 2 \). By [H] for any probability vector \( (p_1, p_2, \ldots, p_N) \) there exists a unique Borel probability...
measure $\mu$, known as a self-similar measure, and a unique nonempty compact fractal subset $J$ of $\mathbb{R}^d$, which is the support of $\mu$, satisfying the self-similarity conditions:

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} \quad \text{and} \quad J = \bigcup_{j=1}^{N} S_j(J).$$

The finite iterated system $\{S_1, S_2, \ldots, S_N\}$ satisfies the open set condition, if there exists a bounded nonempty open set $U \subset \mathbb{R}^d$ such that $\bigcup_{j=1}^{N} S_j(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for $1 \leq i \neq j \leq N$. The finite iterated system is said to satisfy the strong open set condition if there is an open set $U$ such that $U \cap J \neq \emptyset$, where $J$ is the limit set of the system ($\mathbb{H}$, etc.).

The upper and lower quantization dimensions of order $r$ of $\mu$, are defined respectively as:

$$\overline{D}_r(\mu) := \limsup_{n \to \infty} \frac{r \log n}{-\log V_{n,r}(\mu)}; \quad \underline{D}_r(\mu) := \liminf_{n \to \infty} \frac{r \log n}{-\log V_{n,r}(\mu)}$$

If $\overline{D}_r(\mu)$ and $\underline{D}_r(\mu)$ coincide, we call the common value the quantization dimension of order $r$ of the probability measure $\mu$, and is denoted by $D_r(\mu)$. Quantization processes form a rich and far-reaching mathematical concept, that has been employed also in applications (for eg. GG, GL1, Za).

Under the open set condition, Graf and Luschgy (GL1, GL2) showed that the quantization dimension $D_r(\mu)$ of order $r$ of the probability measure $\mu$ exists, and satisfies the following relation, $\sum_{j=1}^{N} (p_j s_j)^{\overline{D}_r(\mu)} = 1$. In fact they proved more, namely that the quantization dimension $D_r(\mu)$ also satisfies the following growth conditions for the quantization errors (cf. GL3):

$$0 < \liminf_{n} n V_{n,r}^{\underline{D}_r(\mu)}(\mu) \leq \limsup_{n} n V_{n,r}^{\overline{D}_r(\mu)}(\mu) < \infty.$$  

Also under the open set condition, Lindsay and Mauldin (cf. LM) determined the quantization dimension of an $F$-conformal measure $m$ associated with a conformal iterated function system determined by finitely many conformal mappings. They established a relationship between the quantization dimension and the temperature function of the thermodynamic formalism arising in multifractal analysis, and proved that the upper quantization coefficient of $m$ is finite; however, they left it open whether the lower quantization coefficient is positive. Using a class of finite maximal antichains Zhu gave an answer in [Za].

In this paper, we are interested in the different case of infinite systems of similarities $(S_n)_{n \geq 1}$ with similarity ratios $(s_n)_{n \geq 1}$ respectively, satisfying the strong open set condition. This setting presents several challenges different from the finite case. For example in the infinite case, the fractal limit set $J$ of the system is not necessarily compact, by contrast to the finite case. The Hausdorff dimension of the limit set $J$ of an infinite conformal iterated function system is given, in general, only as the infimum of the values which make the pressure negative; there may be no zero of that pressure, unlike in the finite case. Also the boundary at infinity consisting of accumulation points of sequences of type $(S_i(x_i))_i$ with distinct $i$’s, plays a role in the geometric properties of the respective system. There are examples of infinite systems where the lower box dimension $\dim_{\text{B}}(J)$ is strictly larger than $HD(J) = h$; and examples where the $h$-Hausdorff measure $H_h(J)$ is zero, while for others $H_h(J) > 0$. Moreover, pertaining to our problem of quantization processes, we do not have finite maximal anti-chains, and the infimum of the contraction rates is zero, which makes the proofs from the finite case not to work in the infinite situation.

As it turns out, estimating quantization coefficients in the infinite case is also very different from the finite case. By its intrinsic nature, quantization is a procedure of “fitting” a finite set in the non-compact fractal limit set $J$, in such a way that we obtain as much information as possible about the self-similar measure $\mu$ which is supported on $J$. However, when dealing
with an infinite system, usually no finite set $F$ can be placed properly such that every set $S_j(X)$, $j \geq 1$ contains a point from $F$. This makes quantization in this case different than for finite systems with open set condition.

Let then $\mu$ be the self-similar probability generated by the system $(S_n)_{n \geq 1}$ and by the probability vector $(p_n)_{n \geq 1}$ (see [M], etc). The measure $\mu$ satisfies the following recursive formula:

$$\mu = \sum_{j=1}^{\infty} p_j \cdot \mu \circ S_j^{-1}$$

The measure $\mu$ is supported on the compact closure $\overline{J}$ of the associated limit set $J$ (the precise definition will be given below in the General Setting).

We will prove in Theorem 2.1 that, under strong open set condition, for each $r \in (0, \infty)$ there exists a unique $\kappa_r \in (0, \infty)$ so that $\sum_{j=1}^{\infty} (p_j s_j^r)^{\kappa_r/r} = 1$, and the $\kappa_r$-dimensional lower quantization coefficient of order $r$ for the measure $\mu$, satisfies the following asymptotic condition:

$$0 < \liminf_{n \to \infty} n V^{\kappa_r}_n(\mu) \leq \limsup_{n \to \infty} n V^{\kappa_r}_n(\mu)$$

We also show in Theorem 2.1 that for any $\kappa' > \kappa_r$, the $\kappa'$-dimensional upper quantization coefficient of $\mu$ is finite,

$$\limsup_{n \to \infty} n V^{\kappa'}_{n,r}(\mu) < \infty$$

We also provide estimates for the upper quantization coefficient $\overline{Q}C_{r,\kappa'}(\mu)$ in the above setting.

As a consequence of the main results, we will prove in Corollary 2.5 also a result about the asymptotic behaviour in $n$, of the approximations in the $L_r$-Kantorovich-Wasserstein metric of the self-similar probability measure $\mu$, by discrete probability measures $Q$ which are supported on $n$ points.

Towards the end of the paper, we will also give some examples of self-similar measures for infinite systems and of quantization processes for them, in which case we obtain estimates on the quantization coefficients.

**General Setting.**

The $n$-th quantization error for the probability $\mu$ gives, in essence, the minimal average (with respect to $\mu$) distance, from points in the support of $\mu$ to finite sets of cardinality $n$, and is defined (cf. [GL1]) by the formula:

$$V_{n,r}(\mu) := \inf \{ \int d(x, \alpha)^r d\mu(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \},$$

and denote $e_{n,r}(\mu) := V_{n,r}^{1/2}(\mu)$. A set $\alpha \subset \mathbb{R}^d$ with $\text{card}(\alpha) \leq n$ is called an $n$-optimal set of centers for $\mu$ of order $r$ or $V_{n,r}(\mu)$-optimal set whenever we have:

$$V_{n,r}(\mu) = \int d(x, \alpha)^r d\mu(x)$$

Let $X$ be a nonempty compact subset of $\mathbb{R}^d$ with $X = \text{cl}(\text{int} X)$. We call $f : X \to \mathbb{R}^d$ a Lipschitz function if there exists a number $c$ such that $d(f(x), f(y)) \leq c d(x, y)$ for all $x, y \in X$. The infimum of such $c$’s is called the Lipschitz constant of $f$, written as Lip $f$. A Lipschitz function $f : X \to \mathbb{R}^d$ is called a contractive mapping if $0 < \text{Lip} f < 1$.

Let $(S_j)_{j=1}^{\infty}$ be an infinite set of contractive similarity mappings on $X$ whose contraction ratios are respectively $(s_j)_{j=1}^{\infty}$, i.e., $d(S_j(x), S_j(y)) = s_j d(x, y)$ for all $x, y \in X$, $0 < s_j < 1,$
\[ j \geq 1. \text{ We shall assume in the sequel that} \]

\[ s := \sup_{j \geq 1} s_j < 1 \]

A *word* with \( n \) letters in \( \mathbb{N} = \{0, 1, 2, \ldots\} \), \( \omega := \omega_1 \omega_2 \cdots \omega_n \in \mathbb{N}^n \), is said to have *length* \( n \), for \( n \geq 1 \). Define also \( \mathbb{N}^{fin} := \bigcup_{n \geq 1} \mathbb{N}^n \) to be the set of finite words with letters in \( \mathbb{N} \), of any length. For \( \omega = \omega_1 \omega_2 \cdots \omega_n \in \mathbb{N}^n \), define:

\[ S_\omega = S_{\omega_1} \circ S_{\omega_2} \circ \cdots \circ S_{\omega_n} \quad \text{and} \quad s_\omega = s_{\omega_1} s_{\omega_2} \cdots s_{\omega_n} \]

The empty word \( \emptyset \) is the only word of length 0 and \( S_\emptyset = \text{Id}_X \). For \( \omega \in \mathbb{N}^{fin} \cup \mathbb{N}^\infty \) and for a positive integer \( n \) smaller than the length of \( \omega \), we denote by \( \omega|_n \) the word \( \omega_1 \omega_2 \cdots \omega_n \). Notice that given \( \omega \in \mathbb{N}^\infty \), the compact sets \( S_{\omega|_n}(X) \), \( n \geq 1 \), are decreasing and their diameters converge to zero. In fact, we have

\[ \text{diam}(S_{\omega|_n}(X)) = s_{\omega_1} s_{\omega_2} \cdots s_{\omega_n} \text{diam}(X) \leq s^n \text{diam}(X) \]

Hence for an infinite word \( \omega \), the set \( \pi(\omega) := \bigcap_{n=1}^\infty S_{\omega|_n}(X) \) is a singleton, and we can define a map \( \pi : \mathbb{N}^\infty \to X \) which, in view of (2), is continuous. One obtains then the following limit set for the above infinite system of similarities,

\[ J := \pi(\mathbb{N}^\infty) = \bigcup_{\omega \in \mathbb{N}^\infty} \bigcap_{n=1}^\infty S_{\omega|_n}(X) \]

This fractal limit set \( J \) is *not* necessarily compact, by contrast to the finite case. Let \( \sigma : \mathbb{N}^\infty \to \mathbb{N}^\infty \) be the shift map on \( \mathbb{N}^\infty \), i.e \( \sigma(\omega) = \omega_2 \omega_3 \cdots \) where \( \omega = \omega_1 \omega_2 \cdots \). Note that \( \pi \circ \sigma(\omega) = S_{\omega_1}^{-1} \circ \pi(\omega) \), and hence, rewriting \( \pi(\omega) = S_{\omega_1}(\pi(\sigma(\omega))) \), we see that \( J \) satisfies the invariance condition:

\[ J = \bigcup_{i=1}^\infty S_i(J) \]

One says that the above iterated function system satisfies the *open set condition* (OSC) if there exists a bounded nonempty open set \( U \subset X \) (in topology of \( X \)), so that \( S_i(U) \subset U \) for every \( i \in \mathbb{N} \) and \( S_i(U) \cap S_j(U) = \emptyset \) for every pair \( i, j \in \mathbb{N} \) with \( i \neq j \); and the *strong open set condition* (SOSC) if \( U \) can be chosen so that \( U \cap J \neq \emptyset \) (see [H], [M], etc.). We assume that our infinite set of similarities satisfies strong open set condition. In the infinite system case, open set condition and strong open set condition are not equivalent, unlike in the finite case.

Let now \((p_1, p_2, \cdots)\) be an infinite probability vector, with \( p_j > 0 \) for all \( j \geq 1 \). Then there exists a unique Borel probability measure \( \mu \) on \( \mathbb{R}^d \) (see [H], [M], etc.), such that

\[ \mu = \sum_{j=1}^\infty p_j \mu \circ S_j^{-1} \]

This measure \( \mu \) is called the *self-similar measure* induced by the infinite iterated function system of self-similar mappings \((S_j)_{j \geq 1}\) and by the infinite probability vector \((p_1, p_2, \cdots)\), and is obtained as the projection \( \pi_*\nu_{(p_1, p_2, \cdots)} \), where \( \nu_{(p_1, p_2, \cdots)} \) is the product measure on \( \mathbb{N}^\infty \) induced by \((p_1, p_2, \cdots)\). One defines the boundary at infinity \( S(\infty) \) as the set of accumulation points of sequences of type \((S_{i_j}(x_{i_j}))_j\), for distinct integers \( i_j \). In our case the self-similar measure \( \mu \) is supported in the closure \( \overline{J} \) of the limit set \( J \), which is given by

\[ \overline{J} = J \cup \bigcup_{\omega \in \mathbb{N}^{fin}} S_\omega(S(\infty)) \]

For the above fixed probability vector \((p_1, p_2, \cdots)\) and contraction vector \((s_j)_{j \geq 1}\), and for arbitrary \( q, t \in \mathbb{R} \), we define the pressure function:

\[ P(q, t) = \log \sum_{j=1}^\infty p_j^q s_j^t. \]
Assume moreover that for every $q \in [0, 1]$, there exists an $u \in \mathbb{R}$ such that
\begin{equation}
0 \leq P(q, u) < \infty
\end{equation}
In this case, for an arbitrary $q \in \mathbb{R}$, let $\theta(q) = \inf\{t \in \mathbb{R} : \sum_{j=1}^{\infty} p_j^q s_j^t < \infty\}$. Then, for $q \in \mathbb{R}$ and $t \in (\theta(q), \infty)$, we have $P(q, t) < \infty$. This is similar to the condition of finiteness of entropy in the case of endomorphisms of Lebesgue spaces.

A particular case when the pressure is finite, is when the infinite probability vector $(p_1, p_2, \cdots)$ and the contraction ratios $(s_j)_{j \geq 1}$ satisfy the following condition: there exists a constant $a > 0$ such that $\sup_j |\log p_j - a \log s_j| < \infty$. Then there exists a constant $K \geq 1$ such that for $j \geq 1$,
\begin{equation}
K^{-1}s_j^a \leq p_j \leq Ks_j^a
\end{equation}
Condition (4) is then satisfied if we have (5), since we know that (Lemma 1.3).

**Lemma 1.1.** Assuming that condition (4) is satisfied above, it follows that, if $q \in \mathbb{R}$ is fixed, then the function $t \mapsto P(q, t)$ is strictly decreasing, convex and continuous on $(\theta(q), \infty)$.

**Lemma 1.2.** Assume that condition (4) is satisfied. Then for any $q \in [0, 1]$, there exists a unique $t = \beta(q) \in (\theta(q), \infty)$ such that $P(q, \beta(q)) = 0$.

**Proof.** By Lemma 1.1, for a given $q \in [0, 1]$, the function $P(q, t)$ is strictly decreasing and continuous on $(\theta(q), \infty)$. Since $0 < P(q, u) < \infty$ for some $u \in (\theta(q), \infty)$, in order to conclude the proof it therefore suffices to show that $\lim_{t \to \infty} P(q, t) = -\infty$. For $t > u,$
\[
P(q, t) = \log \sum_{j=1}^{\infty} p_j^q s_j^t = \log \sum_{j=1}^{\infty} p_j^q s_j s_j^{t-u} \leq \log \sum_{j=1}^{\infty} p_j^q s_j^{t-u} = P(q, u) + (t-u) \log s.
\]
Since $s < 1$, it follows that $\lim_{t \to \infty} P(q, t) = -\infty$, and thus the lemma is obtained.

**Lemma 1.3.** The function $q \mapsto \beta(q)$ given in Lemma 1.2, is strictly decreasing, convex and continuous on $[0, 1]$.

**Proof.** Let $p = \sup\{p_1, p_2, \cdots\}$. Clearly $p < 1$. For any two points $q, q + \delta \in [0, 1]$, where $\delta > 0$, we have to show that $\beta(q + \delta) < \beta(q)$. If not let $\beta(q + \delta) \geq \beta(q)$. Then
\[
0 = P(q + \delta, \beta(q + \delta)) \leq P(q + \delta, \beta(q)) = \log \sum_{j=1}^{\infty} p_j^{q+\delta} s_j^{\beta(q)} \leq \log \sum_{j=1}^{\infty} p_j^q s_j^{\beta(q)}
\]
hence $0 \leq P(q, \beta(q)) + \delta \log p = \delta \log p < 0$, which is a contradiction; thus $\beta(q + \delta) < \beta(q)$. To show $\beta(q)$ is convex, let $q_1, q_2 \in [0, 1]$ and $a_1, a_2 > 0$ with $a_1 + a_2 = 1$. If $\beta(\cdot)$ is not convex, then there exist $a_1, a_2, q_1, q_2$ such that $\beta(a_1 q_1 + a_2 q_2) > a_1 \beta(q_1) + a_2 \beta(q_2)$. Then using Hölder’s inequality, we have
\[
0 = P(a_1 q_1 + a_2 q_2, \beta(a_1 q_1 + a_2 q_2)) < P(a_1 q_1 + a_2 q_2, a_1 \beta(q_1) + a_2 \beta(q_2))
\]
\[
= \log \sum_{j=1}^{\infty} p_j^{a_1 q_1 + a_2 q_2} s_j^{a_1 \beta(q_1) + a_2 \beta(q_2)} \leq \log \left( \sum_{j=1}^{\infty} p_j^{a_1 q_1} s_j^{\beta(q_1)} \right)^{a_1} \left( \sum_{j=1}^{\infty} p_j^{a_2 q_2} s_j^{\beta(q_2)} \right)^{a_2}
\]
\[
= a_1 P(q_1, \beta(q_1)) + a_2 P(q_2, \beta(q_2)) = 0,
\]
thus contradiction; so $\beta(a_1 q_1 + a_2 q_2) \leq a_1 \beta(q_1) + a_2 \beta(q_2)$ i.e $\beta(q)$ is convex and hence continuous.
The function \( (q, t) \mapsto P(q, t) \) is called the \textit{topological pressure function} corresponding to the given infinite iterated function system. The function \( \beta(q) \), sometimes denoted by \( T(q) \), is called the \textit{temperature function} (as in \[HJKPS\]).

\textbf{Remark 1.4.} If \( q = 0 \) then, from \([3]\) we have \( \sum_{j=1}^{\infty} s_j^{\beta(0)} = 1 \), i.e., \( \beta(0) \) gives the Hausdorff dimension \( \dim_H(J) \) of the infinite self-similar set \( J \) (it was shown in \([M]\) that this is the case). Moreover, \( P(1, 0) = 0 \), which gives \( \beta(1) = 0 \).

\section{The Quantization Coefficients for Self-Similar Measures in the Case of Infinite Systems}

For arbitrary \( r > 0 \), let us define the auxiliary function \( h : (0, 1] \to \mathbb{R} \) by \( h(x) := \frac{\beta(x)}{r_x} \), \( x \in (0,1] \), where \( \beta(\cdot) \) was defined in Section 1, in terms of the pressure function \( P(\cdot) \) of our infinite system. We know that \( \beta(1) = 0 \) and \( \beta(0) = \dim_H(J) \), and so \( h(1) = 0 \) and \( \lim_{x \to 0^+} h(x) = \infty \). Moreover, the function \( h \) is continuous and strictly decreasing on \((0,1]\). Hence there exists a unique \( q_r \in (0,1) \) such that \( h(q_r) = 1 \), i.e., \( \beta(q_r) = r q_r \), hence \( P(q_r, \beta(q_r)) = 0 \).

We assume also condition \([1]\). Then, from the above definitions and lemmas it follows that for every \( r > 0 \) there exists a unique number \( \kappa_r \in (0,\infty) \), \( \kappa_r = \frac{\beta(q_r)}{1-q_r} \), and thus we have the formula

\begin{equation}
P \left( \frac{\kappa_r}{r + \kappa_r}; \frac{r \kappa_r}{r + \kappa_r} \right) = 0
\end{equation}

We now give the main result about quantization coefficients of the self-similar measure \( \mu \), in the infinite system case:

\textbf{Theorem 2.1.} Consider an infinite iterated function system of contractive similarities \( S = \{S_1, S_2, \ldots\} \) which satisfies the strong open set condition, and \( J \) be its possibly non-compact limit set. Consider the infinite vector \((s_1, s_2, \ldots)\) consisting of the contraction rates of \( S \), and also an infinite probability vector \((p_1, p_2, \cdots)\), such that condition \([3]\) above is satisfied. Let us consider \( \mu \) to be the self-similar probability measure associated to \( S \) and to \((p_1, p_2, \cdots)\). Denote by \( P(q, t) \) the corresponding pressure function, and by \( \beta(q) \) the zero of the function \( P(q, \cdot) \), and for \( r > 0 \), let \( \kappa_r = \frac{\beta(q_r)}{1-q_r} \).

Then, for any \( r \in (0, \infty) \) and for any \( \kappa' > \kappa_r \), the following estimates on the lower/upper quantization coefficients of order \( r \) of the self-similar measure \( \mu \) (supported on \( J \)), are true:

\[ 0 < \liminf_{n \to \infty} n \cdot V_{n,r}^{\kappa'/(r)}(\mu), \quad \text{and} \quad \limsup_{n \to \infty} n \cdot V_{n,r}^{\kappa'/(r)}(\mu) = 0 \]

\textbf{Proof.} We first want to show that the \textit{lower quantization coefficient} \( QC_{\kappa',r}(\mu) \) is positive, i.e. that \( \liminf_{n \to \infty} n V_{n,r}^{\kappa'/(r)}(\mu) > 0 \), where \( \mu \) is the self-similar measure associated to \((S_j)_j\) and to the probabilistic vector \((p_j)_{j \geq 1}\); and where \( \kappa_r \) is the unique number satisfying the sum condition:

\[ \sum_{j=1}^{\infty} \left( p_j s_j \right)^{\kappa_r/(r + \kappa_r)} = 1. \]

Let \( \tilde{\nu} \) be the self-similar probability measure corresponding to the infinite system \((S, \gamma)\) where \( S = \{S_1, S_2, \cdots\} \) and \( \gamma = (\gamma_1, \gamma_2, \cdots) \) is the probability vector with \( \gamma_j = (p_j s_j)^{\kappa_r/(r + \kappa_r)} \), \( j \geq 1 \). This measure \( \tilde{\nu} \) can be constructed as the image through the canonical projection \( \pi \), of the product measure \( \nu_{(\gamma_1, \gamma_2, \cdots)} \) on \( \mathbb{N}^\infty \) associated to the probability vector \((\gamma_1, \gamma_2, \cdots)\); so we have \( \tilde{\nu} = \pi_*(\nu_{(\gamma_1, \gamma_2, \cdots)}) \).

Consider now \( U \) to be an open set satisfying the strong open set condition, i.e. \( U \cap J \neq \emptyset \), and \( S_j(U) \subset U \) and \( S_i(U) \cap S_j(U) = \emptyset \) for \( i \neq j \). Then it is easy to show that there exists a
finite sequence of integers $\xi$, such that $J_\xi \subset U$, where we denote by $J_\xi := S_\xi(J)$ for arbitrary finite sequence $\xi$. Let now a finite sequence $\xi$ as above, and define the positive constant $\eta_0 := 1 - \frac{1}{2}^{|\xi|}$. Then, for every nonempty set $V \subset J$ which is open with respect to the induced topology on $J$, it can be proved as in [GL2] that there exists an integer $n \in \mathbb{N}$ and finite sequences $(\sigma(k))_{1 \leq k \leq n}$ in $\mathbb{N}_0^{\text{fin}} \setminus \{\emptyset\}$, such that the sets $J_{\sigma(1)}, \ldots, J_{\sigma(n)}$ are pairwise disjoint in $V$ and satisfy the following condition (saying basically that their union has large $\tilde{\nu}$-measure):

\begin{equation}
\tilde{\nu}(V \setminus \bigcup_{k=1}^{n} J_{\sigma(k)}) \leq \eta_0 \cdot \tilde{\nu}(V).
\end{equation}

Moreover, employing the last inequality, one can then show that there exists a sequence $(\sigma(i))_i$ in $\mathbb{N}_0^{\text{fin}} \setminus \{\emptyset\}$, such that the associated sets $J_{\sigma(i)}$, $i \geq 1$ are pairwise disjoint and satisfy:

\begin{equation}
\sum_{i=1}^{\infty} \tilde{\nu}(J_{\sigma(i)}) = 1.
\end{equation}

We are now ready to prove the lower bound for the quantization coefficients of $\mu$. Let $0 < r < \infty$ be fixed and $\kappa_r$ be as in [8]. Then, we want to show that $\liminf_{n \to \infty} n V_{\kappa_r/r}^{r}(\mu) > 0$.

Let us start with an arbitrary number $s < \kappa_r$. By the formula in [8] and from the mutual disjointness of the sets $J_{\sigma(i)}$, $i \geq 1$, we have:

$$1 = \sum_{i=1}^{\infty} \tilde{\nu}(J_{\sigma(i)}) = \sum_{i=1}^{\infty} \left( p_{\sigma(i)} s_{\sigma(i)}^{r} \right)^{\frac{s}{r+s}}.$$

However $\frac{s}{r+s} < \frac{\kappa_r}{r+s}$, hence there exists a fixed positive integer $m$ independent of $s$, such that

$$\sum_{i=1}^{m} \left( p_{\sigma(i)} s_{\sigma(i)}^{r} \right)^{\frac{s}{r+s}} \geq 1.$$ 

Now notice that for every $n \in \mathbb{N}$, there exists a set $Z_n \subset \mathbb{R}^d$ with $\text{card}(Z_n) \leq n$ and

$$e_{n,r}(\mu) = \int_J d(x, Z_n)^r d\mu(x).$$

Let us define now $\delta_n = \sup_{x \in J} d(x, Z_n)$. Then one has $\lim_{n \to \infty} \delta_n = 0$. But the sets $J_{\sigma(1)}, \ldots, J_{\sigma(m)}$ are compact and pairwise disjoint, therefore we obtain the inequality

$$\delta := \min\{d(J_{\sigma(i)}, J_{\sigma(j)}): 1 \leq i, j \leq m, i \neq j\} > 0,$$

and from the above, $\delta = \delta(m)$ is independent of $s$. Thus, there exists an integer $n_0 \in \mathbb{N}$, which is independent of $s$, such that $\delta_n < \frac{\delta}{2}$ for all $n \geq n_0$.

For $n \geq n_0$ and $i \in \{1, 2, \ldots, m\}$, define the set $Z_{n,i} = \{a \in Z_n : d(a, J_{\sigma(i)}) \leq \delta_n\}$, and denote $k_i(n) = \text{card}(Z_{n,i})$. From definition, $k_i(n) \geq 1$. Since $Z_{n,i}$ ($i = 1, 2, \ldots, m$) are mutually disjoint, we get moreover $\sum_{i=1}^{m} k_i(n) \leq n$, hence $k_i(n) \leq n - 1$ ($i = 1, 2, \ldots, m$). Also we obtain

$$e_{n,r} = e_{n,r}^{s} = \int d(x, Z_{n,i})^{r} d\mu(x) \geq \int_{J_{\sigma(i)}} d(x, Z_{n,i})^{r} d\mu(x) \geq \sum_{i=1}^{m} \int_{J_{\sigma(i)}} d(x, Z_{n,i})^{r} d\mu(x)$$

$$= \sum_{i=1}^{m} p_{\sigma(i)} s_{\sigma(i)}^{r} \int d(x, S_{\sigma(i)}^{-1}(Z_{n,i}))^{r} d\mu(x) \geq \sum_{i=1}^{m} p_{\sigma(i)} s_{\sigma(i)}^{r} e_{k_i(n),r}^{s}.$$ 

Define now $\chi = \min\{ne_{n,r}^{s} : n \leq n_0\}$; then $\chi$ is independent of $s$, and $\chi > 0$. We show by induction that $\chi \leq ne_{n,r}^{s}$ for $n \in \mathbb{N}$. In the induction step, let us assume that $\chi \leq \ell e_{\ell,r}^{s}$ for
\( \ell \leq n - 1 \) and \( n - 1 \geq n_0 \). Then, \( e_{n,r}^r \geq \sum_{i=1}^m p_{\sigma(i)} s_{\sigma(i)}^r \chi^{-\frac{r}{\mu}} k_i(n)^{-\frac{1}{\mu}}. \) Hence by the generalized Hölder’s inequality, we obtain that

\[
\sum_{i=1}^m p_{\sigma(i)} s_{\sigma(i)}^r k_i(n)^{-\frac{1}{\mu}} \geq \left( \sum_{i=1}^m (p_{\sigma(i)} s_{\sigma(i)}^r)^{r+\theta} \right)^{1+\frac{\theta}{r}} \cdot \left( \sum_{i=1}^m k_i(n)^{-\frac{1}{\mu}} \right)^{\frac{\theta}{r}}.
\]

Recall however that \( \sum_{i=1}^m (p_{\sigma(i)} s_{\sigma(i)}^r) \geq 1 \) and \( \sum_{i=1}^m k_i(n) \leq n \), hence \( e_{n,r}^r \geq \chi^{-\frac{r}{\mu}} n^{-\frac{1}{\mu}} \), and \( ne_{n,r}^r \geq \chi \). Then by induction, we obtain \( \inf \{ ne_{n,r}^r : n \in \mathbb{N} \} \geq \chi \). Now since \( s < \kappa_\tau \) was chosen arbitrarily, we deduce that

\[
\lim \inf_{n \to \infty} ne_{n,r}^r \geq \chi > 0,
\]

and thus we obtain the lower estimate for the \( \kappa_\tau \)-lower quantization coefficient of order \( r \) of \( \mu \).

We prove now the upper bound for the upper quantization coefficients \( \mathcal{Q}_r^\tau, \kappa(\mu) \) in the infinite self-similar case, where \( \kappa > \kappa_\tau \) is arbitrary.

Let us first fix a number \( \kappa > \kappa_\tau \), and denote by \( \eta := \frac{\kappa}{r+\kappa} \). Then by the definition of \( \kappa_\tau \), we have \( \sum_{i \geq 1} (p_i s_i^r) \geq \eta = 1. \) So since \( \eta > \frac{\kappa}{r+\kappa} \), there exists some number \( \alpha = \alpha(\eta) \) such that

\[
\sum_{i \geq 1} (p_i s_i^r)^\eta < \alpha < 1
\]

Notice now that since \( J \) is compact, we can find a finite number of contractive similarities \( T_1, \ldots, T_K \) on \( X \) such that \( S_i(X) \subset T_1(X) \cup \ldots \cup T_K(X) \), \( i \geq 1 \). Without loss of generality we cans assume that all sets \( S_j(X) \) are contained in \( T_i(X) \) for all \( j \geq j_0 \), for some large fixed integer \( j_0 \). Since \( \alpha = \alpha(\eta) < 1 \), there exists some integer \( N \geq j_0 \) such that

\[
(\sum_{j>N} p_j)^\eta < \frac{1-\alpha}{2}
\]

As \( \alpha \) depends on \( \eta \), also the above integer \( N = N(\eta) \) depends on \( \eta \). Let us define now the finite system of contractive similarities \( \tilde{S}_i, 1 \leq i \leq N+1 \), where \( \tilde{S}_i = S_i, 1 \leq i \leq N \) and \( \tilde{S}_{N+1} = T_1 \), under the above assumption about \( T_1 \). And define also \( \tilde{p}_i = p_i, 1 \leq i \leq N, \tilde{p}_{N+1} = \sum_{i>N} p_i \). We shall denote by \( \tilde{s}_i \) the contraction ratio of \( \tilde{S}_i \), for \( 1 \leq i \leq N+1 \). Recall that, by our assumption we have \( S_i(X) \subset \tilde{S}_{N+1}(X), \forall i > N \). On the other hand, from the self-similarity condition of the measure \( \mu \), we have the decomposition

\[
\mu = \sum_{i \geq 1} p_i \mu \circ S_i^{-1} = \sum_{i=1}^N p_i \mu \circ S_i^{-1} + \sum_{j>N} p_j \mu \circ S_j^{-1}
\]

For \( \eta \) and \( N \) as above, let us introduce also the following numbers from \( (0,1) \),

\[
\gamma_i := (\tilde{p}_i \tilde{s}_i^r)^\eta, 1 \leq i \leq N+1
\]

Consider now an arbitrary integer \( n \geq 2 \). For a finite set \( \mathcal{F} \) of integers, denote by \( \mathcal{F}^* \) the set of all finite sequences of any length, with elements in \( \mathcal{F} \). For a finite sequence \( \omega = (\omega_1, \ldots, \omega_p) \in \{1, \ldots, N+1\}^* \), \( p \geq 1 \), denote by \( \gamma_\omega := \gamma_1 \cdots \gamma_p \). Also we denote by \( \omega^- = (\omega_1, \ldots, \omega_{p-1}) \) to be the truncation of \( \omega \) obtained by cutting the last element.

We want now to decompose \( \mu \) successively, using \( \Box \) up to certain maximal finite sequences \( \omega \in \{1, \ldots, N+1\}^* \), until we achieve that all the corresponding \( \gamma_\omega \)'s are “almost equal” to \( \frac{1}{n} \).

Let us define then the following set of finite sequences determined by \( N \) and \( n \),

\[
F_n := \{ \omega \in \{1, \ldots, N+1\}^* : \gamma_\omega \leq \frac{1}{n} \cdot \rho(N)^{-1}, \gamma_\omega^- > \frac{1}{n} \rho(N)^{-1} \}
\]
where \( \rho(N) := \inf \{ \gamma_1, \ldots, \gamma_{N+1} \} \). It follows that if \( \omega \in F_n \), then \( \gamma_\omega > \frac{1}{n} \). Also since we assumed that \( \tilde{p}_{N+1}^n < \frac{1}{2} \) and \( \sum_{i=1}^{N} \gamma_i < \alpha \), and recalling the definition of the \( \gamma_i \)'s, we obtain

\[
\sum_{i=1}^{N+1} \gamma_i < 1
\]

Then, recalling that \( \gamma_\omega > \frac{1}{n}, \omega \in F_n \), and since we have \( 1 > \sum_{\omega \in F_n} \gamma_\omega \geq \text{Card}(F_n) \cdot \frac{1}{n} \), we obtain:

\[
\text{Card}(F_n) \leq n
\]

In the identity (11) for \( \mu \), we can then continue decomposing successively until reaching the value \( \frac{1}{n} \) for \( \gamma_\omega \), i.e. we can split \( \mu \) according to all finite sequences \( \omega \in F_n \). In order to see this, let us deduce from (11) the following decomposition:

\[
\mu = \sum_{i=1}^{N} p_i \cdot \left( \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} \right) \circ S_i^{-1} + \sum_{i=1}^{N} p_i \cdot \left( \sum_{j>N} p_j \mu \circ S_j^{-1} \right) \circ S_i^{-1} + \\
+ \sum_{j>N} p_j \cdot \left( \sum_{k=1}^{N} p_k \mu \circ S_k^{-1} \right) \circ S_j^{-1} + \sum_{j>N} p_j \cdot \left( \sum_{k>N} p_k \mu \circ S_k^{-1} \right) \circ S_j^{-1}
\]

Notice that if a set \( B \) has a point in \( S_i \tilde{S}_j(X) \) for some \( i, j \in \{1, \ldots, N\} \), then we have

\[
\int d(x, B) d(\mu \circ S_j^{-1} \circ S_i^{-1}) \leq s_i^r s_j^r C,
\]

for a constant \( C > 0 \). And if \( B \) has a point in \( S_i S_j(X) \) for some \( 1 \leq i \leq N \) and \( j > N \), then

\[
\int d(x, B) d(\mu \circ S_j^{-1} \circ S_i^{-1}) \leq s_i^r \tilde{s}_{N+1} C
\]

since \( S_j(X) \subset \tilde{S}_{N+1}(X) \). If we take a set \( B \) with at least \( (N+1)^2 \) points such that \( B \) has a point in each of the sets \( \tilde{S}_i \tilde{S}_j, i, j \in \{1, \ldots, N+1\} \), then, since \( S_i(X) \subset \tilde{S}_{N+1}(X) \), \( 1 \leq i \leq N \), we obtain the following estimate for the \( n \)-quantization error of order \( r \) of \( \mu \),

\[
V_{n,r}(\mu) \leq C \cdot \left( \sum_{i,j=1}^{N} p_i p_j s_i^r s_j^r + \sum_{i=1}^{N} p_i s_i^r \sum_{j>N} p_j \tilde{s}_N^r + \sum_{j=1}^{N} p_j s_j^r \sum_{i>N} p_i \tilde{s}_N^r + \sum_{j,k>N} p_j p_k \tilde{s}_{N+1}^r \right),
\]

where \( C \) is a positive constant independent of \( N \). Similarly we can do this argument for the set \( F_n \) instead of \( \{1, \ldots, N+1\} \), and we can take a set \( B \) of cardinality \( n \), which has points in each of the sets \( \tilde{S}_\omega(X) \) for \( \omega \in F_n \); this is possible since, as we saw in (13), \( \text{Card}(F_n) \leq n \). It follows then similarly as above that

\[
V_{n,r}(\mu) \leq C \cdot \sum_{\omega \in F_n} \tilde{p}_\omega \tilde{s}_\omega^r = C \cdot \left( \frac{1}{n} \right)^{1-n} \rho(N) \frac{1-n}{n} \cdot \sum_{\omega \in F_n} \gamma_\omega \leq C \cdot \left( \frac{\rho(N)}{n} \right)^{1-n}
\]

Hence recalling that \( N \) depends on \( \eta \) (hence on \( \kappa \)), we obtain the following estimate for the \( \kappa \)-dimensional upper quantization coefficient of order \( r \) of \( \mu \),

\[
\limsup_{n \to \infty} n \cdot V_{n,r}(\mu)^{\kappa/r} \leq C(\kappa) < \infty,
\]

where \( C(\kappa) \) is a positive constant depending on \( \kappa \). In fact if we now take \( \kappa' \) slightly larger than \( \kappa \), and since \( \lim_{n \to \infty} V_{n,r}(\mu) = 0 \), we see that

\[
\limsup_{n \to \infty} n \cdot V_{n,r}(\mu)^{\kappa'/r} = 0
\]
From the above inequalities (9) and (14) we obtain also computable estimates for the lower and the upper quantization coefficients of order $r$ for the probability measure $\mu$. We do not know if the $\kappa_r$-dimensional upper quantization coefficient of $\mu$ of order $r$ is always finite in the case of infinite systems. In particular from the estimates above for lower/upper quantization coefficients of $\mu$, and by taking $\kappa' \to \kappa_r$, we obtain that the quantization dimension of $\mu$ exists and is equal to $\kappa_r$.

**Corollary 2.2.** In the setting of Theorem 2.1, it follows that the quantization dimension $D_r(\mu)$ exists and $D_r(\mu) = \kappa_r$.

**Remark 2.3.** We notice that in order to obtain Corollary 2.2 it is **not possible** to use finite truncations with $M$ elements $\mathcal{S}_M$ of the system and associated self-similar measures $\mu_M$, and then to consider $\log V_{n_k,r}(\mu_M)$ when $n_k \to \infty$, followed by the use of the estimates for the quantization dimension of $\mu_M$ from the finite case. This problem is due to the fact that the speed of convergence in $n_k$, in the quantization dimension of $\mu_M$, actually depends on each $M$ (when $M \to \infty$).

Let us give now some examples of infinite systems, when it is possible to say more about the quantization process.

**Examples:**

Consider a sequence of numbers $(s_i)_{i \geq 1}$ in the interval $(0,1)$, such that $s_i = \gamma^i, i \geq 1$ for some $\gamma \in (0,1/2)$. Let us also take $p_i = s_i^a = \gamma^{ai}, i \geq 1$ and $p = (p_1, \ldots)$; in order to make $p$ a probabilistic vector, we will choose $a = \frac{\log 2}{\log \gamma}$.

We take then the infinite iterated function system $\mathcal{S}$, formed by the sequence of similarities $\mathcal{S} = (S_i)_{i \geq 1}$ of the unit disk $\Delta(0,1)$ having contraction rates $s_i$ respectively and such that the boundary at infinity $\mathcal{S}(\infty)$ is equal to the unit circle $S^1$. Consider also the self-similar probability measure $\mu$, associated to $\mathcal{S}$ and $p$. Then, the self-similar measure $\mu$ is supported on the closure $\overline{\mathcal{J}}$, which in this case is given by:

$$
\overline{\mathcal{J}} = \mathcal{J} \cup \bigcup_{\omega \in \mathbb{N}^{\mathbb{N}}} \mathcal{S}_\omega(\mathcal{S}(\infty)) = \mathcal{J} \cup \bigcup_{\omega \in \mathbb{N}^{\mathbb{N}}} \mathcal{S}_\omega(S^1)
$$

We notice that in this case $HD(J) < 1$, but the lower box dimension of $J$ is larger or equal than 1, since $\dim_B(J) \geq \dim_B(\mathcal{S}(\infty)) = 1$. Now, one wants to estimate the quantization coefficients for the measure $\mu$. According to Theorem 2.1, the quantization dimension of $\mu$ is equal to $\kappa_r$, where $\kappa_r$ satisfies

$$
\sum_{i \geq 1} (p_i s_i^a)^{r s_i^a} = 1
$$

In our case, the above sum is just the sum for a geometric series, hence we obtain with the above expression for $s_i, p_i$ and the above exponent $a$, that

$$
\sum_{i \geq 1} (\gamma^{(a+r)t})^i = 1,
$$

where $t = \frac{\kappa_r}{r+\kappa_r}$. Hence $t = \frac{\log 2}{(a+r)\log \gamma} = \frac{\kappa_r}{r+\kappa_r}$. Therefore, we obtain the quantization dimension

$$
D_r(\mu) = \kappa_r = \frac{r \log 2}{(a+r)|\log \gamma| - \log 2} = \frac{\log 2}{|\log \gamma|}
$$

It is interesting to note that, in this particular case, the quantization dimension $D_r(\mu)$ does not depend on $r$. In general however, if the $p_i$'s are not of the form above, then the quantization
dimension $D_r(\mu)$ should depend on $r$. We have also from Theorem 2.1 that the quantization coefficients $\mathcal{QC}_{r, \kappa_r}(\mu)$ and $\mathcal{QC}_{r, \kappa'}$ of $\mu$ satisfy the following estimates:

$$\liminf_{n} n \cdot V_{n, r}(\mu) \frac{\log 2}{r \log n} > 0 \quad \text{and} \quad \limsup_{n} n \cdot V_{n, r}(\mu) \frac{\kappa'}{r} < \infty, \quad \forall \kappa' > \log 2/|\log \gamma|$$

We notice that this example can be modified so that the images $S_i(\Delta)$ are arranged differently inside $\Delta$, and that the boundary at infinity $\mathcal{S}(\infty)$ is more complicated, for instance we can imagine an example where it is a countable union of concentric circles $C_n, n \geq 1$ centered at 0, with radii $c_n$ going to 0. The corresponding self-similar measure $\mu$ will then be supported on the closure of the limit set $J$, namely on the compact set

$$\mathcal{J} = J \cup \bigcup_{\omega \in \mathbb{N}^{\infty}} S_\omega \left( \bigcup_{n} C_n \cup \{0\} \right)$$

Still, if we keep the same contraction rates $s_i$ and the probability vector $p = (p_1, p_2, \ldots)$ as before, then we will obtain the same quantization dimension $\kappa_r$ and quantization coefficients estimates as above.

We want now to approximate the self-similar measure $\mu$ with discrete measures of finite support. Denote by $\mathcal{M}$ the set of probability measures on the compact set $X \subset \mathbb{R}^d$. Then,

$$d_H(\mu, \nu) := \sup \left\{ \left| \int_X g d\mu - \int_X g d\nu \right| : \text{Lip} \ g \leq 1 \right\}, \quad (\mu, \nu) \in \mathcal{M} \times \mathcal{M},$$

defines a metric on $\mathcal{M}$. Then $(\mathcal{M}, d_H)$ is a compact metric space (cf. [3]). It is known that the $d_H$-topology and the weak topology, coincide on the space of probabilities with compact support ([Ma1]). In our case all measures are compactly supported.

First, since $X$ is compact we have $\int \|x\|^r d\mu(x) < \infty$, for any probability measure $\mu$ on $X$. For $r \in (0, \infty)$ and for two arbitrary probabilities $\mu_1, \mu_2$, the $L_r$-Kantorovich-Wasserstein metric is defined by the following formula (see for eg. [GL1]):

$$\rho_r(\mu_1, \mu_2) = \inf_{\nu} \left( \int \|x - y\|^r d\nu(x, y) \right)^{1/r},$$

where the infimum is taken over all Borel probabilities $\nu$ on $X \times X$ with fixed marginal measures $\mu_1$ and $\mu_2$, i.e such that $(\pi_1)_*(\nu) = \mu_1$ and $(\pi_2)_*(\nu) = \mu_2$ for the canonical projections $\pi_1, \pi_2$ on the first, respectively second coordinates.

Note that the weak topology, the topology induced by $d_H$, and the topology induced by $L_r$-metric $\rho_r$, all coincide on the space $\mathcal{M}$ (see for example [Ru]). Let us notice also that, for $r = 1$, the $\rho_1$ metric is in fact equal to the $d_H$ metric in the compact case, as shown by Kantorovich (see [GL1]).

The next Lemma relates the quantization errors for a probability measure $P$, to the $L_r$-Kantorovich-Wasserstein distances between $P$ and discrete measures:

**Lemma 2.4.** ([GL1] Lemma 3.4) Let $\mathcal{P}_n$ denote the set of all discrete probability measures $Q$ on $X$ with $|\text{supp}(Q)| \leq n$. Then for any probability $P$, we have:

$$V_{n, r}(P) = \inf_{Q \in \mathcal{P}_n} \rho_r^*(P, Q)$$

Now by using Lemma 2.4 and Theorem 2.1 we obtain the following result about the asymptotic behaviour in $n$, of the approximations in $L_r$-metric of $\mu$, with discrete measures supported on $n$ points, when $n$ increases to $\infty$. 
Corollary 2.5. In the setting of Theorem 2.1 let us consider the associated self-similar probability measure $\mu$. Then, it follows that for every $r \in (0, \infty)$, there exists a unique number $\kappa_r \in (0, \infty)$ such that the $L_r$-approximations of $\mu$ with discrete measures on $n$ points, behave asymptotically in the following way:

$$0 < \liminf_{n \to \infty} n^{-\frac{1}{\kappa_r}} \cdot \inf_{Q \in \mathcal{P}_n} \rho_r(\mu, Q), \text{ and } \limsup_{n \to \infty} n^{-\frac{1}{\kappa_r}} \cdot \inf_{Q \in \mathcal{P}_n} \rho_r(\mu, Q) = 0, \quad \forall \kappa' > \kappa_r$$

References

[B] M.F. Barnsley, Fractals Everywhere, Academic Press, Harcourt Brace & Company, 1988.
[F] K. Falconer, The multifractal spectrum of statistically self-similar measures, J. Theoretical Probab., Vol 7. No. 3, 681-701, 1994.
[GG] A. Gersho and R. M. Gray, Vector Quantization and Signal Compression, Kluwer Academic, Boston, 1992.
[GL1] S. Graf and H. Luschgy, Foundations of Quantization for Probability Distributions, Lecture Notes Math. 1730, Springer, Berlin, 2000.
[GL2] S. Graf and H. Luschgy, The quantization dimension of self-similar probabilities, Math. Nachr., 241 (2002), 103-109.
[GL3] S. Graf and H. Luschgy, Asymptotics of the quantization errors for self-similar probabilities, Real Anal. Exchange, Volume 26, Number 2 (2000), 795-810.
[H] J. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J., 30 (1981), 713-747.
[HJKPS] T. Halsey, M. Jensen, L. Kadanoff, I. Procaccia and B. Shraiman, Fractal measures and their singularities: the characterization of strange sets, Phys. Review A, 33 (1986) 1141-1151.
[LM] L.J. Lindsay and R.D. Mauldin, Quantization dimension for conformal iterated function systems, Nonlinearity, 15 (2002), 189-199.
[Mat] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge University Press, 1995.
[M] M. Moran, Hausdorff measure of infinitely generated self-similar sets, Monatsh. Math. 122, 1996, 387-399.
[P] N. Patzschke, Self-conformal multifractal measures, Advances in Applied Mathematics, Vol. 19, 486-513, (1997).
[RM] R.H. Riedi and B.B. Mandelbrot, Multifractal formalism for infinite multinomial measures, Advances in Applied Math., 16, (1995), 132-150.
[Ru] L. Rüschendorf, Wasserstein metric, in Hazewinkel, Michiel, eds. Encyclopedia of Math., Kluwer Acad. Publ. (2001).
[Za] P.L Zador, Asymptotic quantization error of continuous signals and the quantization dimension. IEEE Trans. Inform. Theory 28, (1982), 139-149.
[Z] S. Zhu, The lower quantization coefficient of the F-conformal measure is positive, Nonlinear Analysis, 69 (2008), 448-455.

Eugen Mihailescu,
Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O Box 1-764, RO 014700, Bucharest, Romania.
Email: Eugen.Mihailescu@imar.ro Webpage: www.imar.ro/~mihailes

Mrinal Roychowdury,
Dept Mathematics, Univ Texas-Pan American, West Univ Drive, Edinburg, TX 78539 USA.
Email: roychowdhurymk@utpa.edu