Level correspondence of $K$-theoretic $I$-function in Grassmannian duality

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Abstract

In this paper, we prove a class of nontrivial q-Pochhammer symbol identities with extra parameters by iterated residue method. Then we use these identities to find relations of the quasi-map $K$-theoretical $I$-functions with level structure between Grassmannian and its dual Grassmannian. Here we find an interval of levels within which two $I$-functions are the same, and on the boundary of that interval, two $I$-functions are intertwining with each other. We call this phenomenon level correspondence in Grassmannian duality.

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1 Introduction

A fundamental relation between 3d supersymmetric gauge theories and quantum K-theory on the moduli space is established by the works of Nekrasov [7] and Nekrasov and Shatashvili [9] [8], amongst many others. For the concrete case of massless theories with a non-trivial UV-IR flow, H. Jockers shows a 3d gauge theory/quantum K-theory correspondence [5], connecting the BPS partition functions of specific N = 2 supersymmetric gauge theories to Givental's permutation equivariant quantum K-theory which is introduced by Givental [4] and Y.P. Lee [6] decades ago.

Recently, Givental shows that q-hypergeometric solutions represent \( K \)-theoretic Gromov-Witten invariants in the toric case [3] and Ruan-Zhang [11] introduce the level structures and there is a serendipitous discovery that some special toric spaces with certain level structures result in Mock theta functions. Nevertheless, beyond the toric case, much less is known.

Grassmannian \( Gr(r,V) \) is isomorphic to dual Grassmannian \( Gr(n-r,V^*) \) geometrically, without miss of understanding, we will use \( Gr(r,n) \) and \( Gr(n-r,n) \) to denote Grassmannian and its dual respectively. However, they encode very different combinatorial data. A long-standing problem is to match their combinatorial data directly. For example, the presentation of quantum \( K \)-theoretic \( I \)-functions depend on their combinatorial data, and it is tough to see why the \( I \)-function of Grassmannian equals the \( I \)-function of dual Grassmannian. In this paper, we give the explicit formula of K-theoretic \( I \)-function of Grassmannian with level structure by using abelian/non-abelian correspondence [14] as follows

\[
I_{T,d}^{Gr(r,n),E_r,l} = \sum_{d_1+d_2+\ldots+d_r=d} Q^d \prod_{i,j=1}^{r} \left( 1 - q^{k_i} L_i L_j^{-1} \right) \prod_{k=1}^{n} \left( 1 - q^{L_i L_j} \right) \prod_{m=1}^{n} \left( 1 - q^{L_i L_j} \right)
\]

and

\[
I_{T,d}^{Gr(n-r,n),E_{n-r},l} = \sum_{d_1+d_2+\ldots+d_{n-r}=d} Q^d \prod_{i,j=1}^{n-r} \left( 1 - q^{k_i} L_i L_j^{-1} \right) \prod_{k=1}^{n} \left( 1 - q^{L_i L_j} \right) \prod_{m=1}^{n} \left( 1 - q^{L_i L_j} \right)
\]

We want to remark here that the isomorphism between Grassmannian and its dual would imply the equivalence of \( J \)-function when level \( l \) is 0. In fact, \( I \)-function is known to be different from \( J \)-function with negative levels.

In this paper, we use Proposition [21] to show the relations of the equivariant \( I \)-function between Grassmannian \( Gr(r,n) \) and that of dual Grassmannian \( Gr(n-r,n) \) with level structures, here we find an interval of levels within which two \( I \)-functions with levels are the same, and on the boundary of that interval, two \( I \)-functions with levels are intertwining with each other. We call this phenomenon level correspondence in Grassmannian duality.

**Theorem 1.1 (Level Correspondence)** For Grassmannian \( Gr(r,n) \) and its dual Grassmannian \( Gr(n-r,n) \) with standard \( T = (\mathbb{C}^*)^n \) torus action, let \( E_r, E_{n-r} \) be the standard representation of \( GL(r,\mathbb{C}) \) and \( GL(n-r,\mathbb{C}) \), respectively. Consider the following equivariant \( I \)-function

\[
I_T^{Gr(r,n),E_r,l} = 1 + \sum_{d=1}^{\infty} I_{T,d}^{Gr(r,n),E_r,l} Q^d
\]

and

\[
I_T^{Gr(n-r,n),E_{n-r},-l} = 1 + \sum_{d=1}^{\infty} I_{T,d}^{Gr(n-r,n),E_{n-r},-l} Q^d
\]
Then we have following relations between $I_{T,d}^{Gr(r,n),E_{r,l}}$ and $I_{T}^{Gr(n-r,n),E_{n-r,-l}}$ in $K^1_{T}(Gr(r,n)) \otimes \mathbb{C}(q) \cong K^1_{T}(Gr(n-r,n)) \otimes \mathbb{C}(q)$:

- For $1 - r \leq l \leq n - r - 1$, we have
  \[ I_{T,d}^{Gr(r,n),E_{r,l}} = I_{T,d}^{Gr(n-r,n),E_{n-r,-l}} \]

- For $l = n - r$, we have
  \[ I_{T,d}^{Gr(r,n),E_{r,l}} = \sum_{s=0}^{d} C_s(n - r, d) I_{T,d-s}^{Gr(n-r,n),E_{n-r,-l}} \]
  where $C_s(k, d)$ is defined as
  \[ C_s(k, d) = \frac{(-1)^k}{(q; q)_s q^{s(d - s + k)} (\bigwedge^{top} S_{n-r})^s} \]
  and $S_{n-r}$ is the tautological bundle of $Gr(n-r,n)$

- For $l = -r$, we have
  \[ I_{T,d}^{Gr(n-r,n),E_{n-r,-l}} = \sum_{s=0}^{d} D_s(r, d) I_{T,d-s}^{Gr(r,n),E_{r,l}} \]
  \[ D_s(r, d) = \frac{(-1)^r}{(q; q)_s q^{s(d - s)} (\bigwedge^{top} S_r)^s} \]
  and $S_r$ is the tautological bundle of $Gr(r,n)$

here we use $q$-Pochhammer symbol notation:

\[
(a; q)_d := \begin{cases} 
(1-a)(1-q^a)\cdots(1-q^{d-a}) & d > 0 \\
1 & d = 0 \\
\frac{1}{(1-q^{-a})\cdots(1-q^{-d})} & d < 0
\end{cases}
\]

A key step in our proof is the following series of non-trivial $q$-Pochhammer symbol identities which are of independent interest.

**Proposition 1.1** Denoted by $[n]$ the set of elements $\{1, \ldots, n\}$, let $\emptyset \neq I \subsetneq [n]$ be a subset of $[n]$, $|I|$ be its cardinality, and denoted by $^c I$ the complementary set of $I$ in $[n]$. For constant positive integers $d$, $n$ and integer $l$ with restriction: $1 - |I| \leq l \leq n - |I| - 1$, let $A_d(\vec{x}, I, l)$ and $B_d(\vec{x}, I, l)$ be two rational functions in $\vec{x}$ and $q$ with an extra data $l$

\[
A_d(\vec{x}, I, l) = \sum_{|d_I| = d} \frac{\left(\prod_{i \in I} x_i^{d_i} q^{d_i(d_i+1)}}{\left|\prod_{i,j \in I} (q^{d_{ij}+1}x_{ij}; q)_{d_j} \prod_{i \in I} (q^{d_{ij}}x_{ij}; q)_{d_i}\right|^l}
\]

\[
B_d(\vec{x}, I, l) = \sum_{|d_I| = d} \frac{\left(\prod_{i \in I} x_i^{-d_i} q^{d_i(d_i+1)}}{\left|\prod_{i,j \in I} (q^{d_{ij}+1}x_{ij}; q)_{d_j} \prod_{i \in I} (q^{d_{ij}}x_{ij}; q)_{d_i}\right|^l}
\]
where \( \vec{d} \) is \(|I|\)-tuple of non negative integers, and \(|\vec{d}| := \sum_{i \in I} d_i\). \( x_i, i = 1, \ldots, n \) are parameters. For convenience, we use the notation \( x_{ij} := x_i/x_j \) and \( d_{ij} := d_i - d_j \). Then we have

\[
A_d(\vec{x}, I, l) = B_d(\vec{x}, I^c, -l)
\]

This paper is arranged as follows. In subsection 2.1, we prove Proposition 2.1 by constructing a rational function (3) and then using iterated residue method which is useful in Nekrosov partition function [2]. In the following subsection 2.2, we provide two explicit examples to explain the proof and also provide a non-trivial identity by using Proposition 2.1. In subsection 2.3, we expand the restriction to the boundary, i.e. \( l = -|I| \) and \( l = n - |I| \). In section 3, we first revisit \( K \)-theoretic quasi-map theory in which we review some basic definitions and theorems, especially, the formula of equivariant \( I \)-function of Grassmannian \( Gr(r, n) \), finally, we apply Proposition 2.1 to obtain the level correspondence of \( I \)-function in Grassmannian duality.

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2 The class of q-Pochhammer symbol identities

2.1 The proof of identities

Now we prove this proposition for one case \( I = \{1, \cdots, r\} \) by constructing the following symmetric complex rational function \( f(w_1, \cdots, w_d) \) with parameters \( q \) and \( x_1, \cdots, x_n \), we made following assumptions of parameters

\[
|q| < 1 \quad x_i x_j^{-1} \neq q^k \quad \forall i \neq j \in [n], \forall k \in \mathbb{Z}
\]

Furthermore there exists some \( \rho > 0 \) such that

\[
\max_{i \in [n]} |x_i| < \rho < \min_{i \in [n]} |q|^{-1} |x_i|
\]

where \([n] := \{1, \cdots, n\}\) and general situations follow from analytic continuation. Let \( f(w_1, \cdots, w_d) \) be as follows

\[
f(w_1, \cdots, w_d) = \frac{1}{(1 - q)^d d!} \prod_{i \neq j} \frac{d}{w_i - w_j} \prod_{i=1}^{d} \prod_{j=1}^{d} (1 - x_j/w_i) \prod_{j=r+1}^{n} (1 - qw_i/x_j)
\]

\[
= g(w_1, \cdots, w_d) \prod_{i=1}^{d} \left( \prod_{u \in U} \frac{w_i - q^{-1} u}{w_i - u} \prod_{i < j} \frac{(w_i - w_j)^2}{(w_i - qw_j)(qw_i - w_j)} \right)
\]
where $U$ is a set of complex numbers all contained in open disk $|w|<\rho$, at the moment $U=\{x_1,\cdots,x_r\}$ and $g$ is a symmetric function of the form
\[
g(\bar{w}) = \frac{1}{(1-q)d!} \prod_{i=1}^{d} w_{j}^{l+r-1} \prod_{j=r+1}^{d} (1-qw_j/x_j)
\]
from condition (2) and the restriction of $l$ we know $g$ is analytical in the polydiscs $\{w_i,\cdots,w_n\} : |w_i|\leq\rho, \forall i \in [n]$ and $g$ can only have possible zeros for some $w_j = 0$.

We consider the following integration
\[
E_d := \int_{C_\rho}\frac{dw_{j_1}}{2\pi\sqrt{-1}} \cdots \int_{C_\rho}\frac{dw_{j_d}}{2\pi\sqrt{-1}} f(w_1,\cdots,w_d)
\]
where $(w_1,\cdots,w_d)$ is any arrangement of $\{1,\cdots,d\}$ and the integration contour $C_\rho$ for each variable $w_i$ is the circle centered at origin with radius $\rho$ and takes counter-clockwise direction. The condition (2) ensures that there isn’t a pole on the integration contour. By Fubini’s theorem, we could permute the order of integration variables and since $f(w_1,\cdots,w_d)$ is a symmetric function, we can change $(w_1,\cdots,w_d)$ to other order, like, $(w_{i_1},\cdots,w_{i_d})$.

Suppose we have the following evaluating sequence for some $S_1 \leq d$ by induction,
\[
\hat{w}_1 = qw_2, \hat{w}_2 = qw_3, \cdots, \hat{w}_{S_1-1} = qw_{S_1}
\]
which are all simple poles inside $|w|<\rho$, then we have
\[
\begin{aligned}
\text{Res}_{\hat{w}_{S_1-1}=qw_{S_1}} \cdots \text{Res}_{\hat{w}_{2}=qw_3,\hat{w}_1=qw_2} f \\
= \prod_{i=S_1+1}^{d} \left( \prod_{u \in U} \frac{w_i-q^{-1}u}{w_i-u} \prod_{i\neq j} \frac{(w_i-w_j)^2}{(w_i-qw_j)(qw_i-w_j)} \right) \\
\cdot w_{S_1}^{S_1-1} \prod_{k=0}^{S_1-1} q^k w_{S_1} - q^{-1}u \cdot \prod_{S_1\leq j} \frac{(w_{S_1}-w_j)(q^{S_1-1} w_{S_1}-w_j)}{(w_{S_1}-qw_j)(q^{S_1-1} w_{S_1}-qw_j)} \\
\cdot \frac{(q-1)^{S_1-1} q^{-(S_1-1)(d-S_1)} g(q^{S_1-1} w_{S_1}, q^{S_1-2} w_{S_1}, \cdots, w_{S_1}, w_{S_1+1}, \cdots, w_d)}{q^{S_1-1}-1}
\end{aligned}
\]
now integrating variable $w_{S_1}$, we pick up residue as $\hat{w}_{S_1} = q^{-k_1} u$ for some $0 \leq k_1 < S_1$ and $u_1 \in U = \{x_1,\cdots,x_r\}$, due to the condition (2), $|\hat{w}_{S_1}|<\rho$ implies that $k_1 = 0$. Evaluating $\hat{w}_{S_1} = u_1$, we get
\[
\begin{aligned}
\text{Res}_{\hat{w}_{S_1}=u_1} \text{Res}_{\hat{w}_{S_1-1}=qw_{S_1}} \cdots \text{Res}_{\hat{w}_2=qw_3,\hat{w}_1=qw_2} f \\
= \prod_{i=S_1+1}^{d} \left( \frac{w_i-q^{S_1-1} u_1}{w_i-q^{S_1} u_1} \prod_{u \in U \setminus \{u_1\}} \frac{w_i-q^{-1}u}{w_i-u} \prod_{i\neq j} \frac{(w_i-w_j)^2}{(w_i-qw_j)(qw_i-w_j)} \right) \\
\cdot u_1^{S_1-1} \prod_{k=0}^{S_1-1} q^k u_1 - q^{-1}u \cdot \frac{q^{S_1-1} u_1-q^{-1}u}{q^k u_1-u} \cdot \frac{(q-1)^{S_1} q^{S_1-1} u_1}{q^{S_1-1}-1} \\
\cdot g(q^{S_1-1} u_1, q^{S_1-2} u_1, \cdots, q u_1, u_1, w_{S_1+1}, \cdots, w_d)
\end{aligned}
\]
(7)
where

\[ \tilde{U} = \mathbb{U} \setminus \{ u_1 \} \cup \{ q^{S_1} u_1 \} \quad (9) \]

all elements of \( \tilde{U} \) are still all in the open disk \(|w| < \rho\), and

\[
\tilde{g}(w_{S_1+1}, \ldots, w_d) = u_1^{S_1} \prod_{k=0}^{S_1-1} \prod_{w \in \mathbb{U} \setminus \{ u_1 \}} \frac{q^k u_1 - q^{-1} u}{q^k u_1 - u} \cdot (q - 1)^{S_1} q^{S_1(S_1-1-d)}
\]

\[
\cdot g(q^{S_1-1} u_1, q^{S_1-2} u_1, \ldots, q u_1, u_1, w_{S_1+1}, \ldots, w_d) \quad (10)
\]

so we just write \( \tilde{f} := \text{Res} \frac{\text{Res}_{w_{S_1-1} =qw_{S_1}} \ldots \text{Res}_{w_2 =qw_3} \text{Res}_{w_1 =qw_2}}{w_{S_1} =qw_{S_1-1}} f \) into the same pattern as in the original form \( (1) \). One could check that setting \( S_1 = 1 \) in equation \( (7) \) is valid.

If one takes the following evaluation sequence of simple poles by induction

\[ \hat{w}_1 = u_1, \hat{w}_2 = q u_1, \ldots, \hat{w}_{S_1-1} = q^{S_1-2} u_1, \hat{w}_{S_1} = q^{S_1-1} u_1 \quad (11) \]

we get

\[
\text{Res}_{w_{S_1} =q^{S_1-1} u_1} \ldots \text{Res}_{w_2 =q u_1} \text{Res}_{w_1 =u_1} f = \prod_{S_1 < i} \left( \frac{w_i - q^{S_1-1} u_1}{w_i - q^{S_1} u_1} \prod_{w \in \mathbb{U} \setminus \{ u_1 \}} \frac{w_i - q^{-1} u}{w_i - u} \prod_{i < j} \frac{(w_i - w_j)^2}{(w_i - q w_j)(q w_i - w_j)} \right)
\]

\[
\cdot u_1^{S_1-1} \prod_{k=0}^{S_1-1} \prod_{w \in \mathbb{U} \setminus \{ u_1 \}} \frac{q^k u_1 - q^{-1} u}{q^k u_1 - u} \cdot (q - 1)^{S_1} q^{S_1(S_1-1-d)} g(u_1, q u_1, \ldots, q^{S_1-1} w_{S_1}, w_{S_1+1}, \ldots, w_d) \quad (12)
\]

which agrees with the equation \( (7) \), since \( g \) is a symmetric function. That is to say, we get the same results from two different evaluation sequences

\[
\text{Res}_{w_{S_1} =q^{S_1-1} u_1} \ldots \text{Res}_{w_2 =q u_1} \text{Res}_{w_1 =u_1} f = \text{Res}_{w_{S_1-1} =qw_{S_1}} \ldots \text{Res}_{w_2 =qw_3} \text{Res}_{w_1 =qw_2} f \quad (13)
\]

As the evaluation process for sequence \( (6) \), we now picking up residues of \( \tilde{f} \) in the following sequence

\[ w_{S_1+1} = qw_{S_1+2} \quad w_{S_1+2} = qw_{S_1+3} \quad \ldots \quad w_{S_1+S_2-1} = qw_{S_1+S_2} \quad (14) \]

Suppose \( \hat{w}_{S_1+S_2} = u_2 \), we have two cases here, i.e. \( u_2 \neq q^{S_1} u_1 \) or \( u_2 = q^{S_1} u_1 \). By a little bit of computation, we obtain

**Case 1:** \( u_2 \neq q^{S_1} u_1 \),

\[
\text{Res}_{w_{S_1+S_2} =u_2} \text{Res}_{w_{S_1+S_2-1} =qw_{S_1+S_2}} \ldots \text{Res}_{w_{S_1+2} =qw_{S_1+3}} \text{Res}_{w_{S_1+1} =qw_{S_1+2}} \text{Res}_{w_{S_1} =u_1} \text{Res}_{w_{S_1-1} =qw_{S_1}} \ldots \text{Res}_{w_2 =qw_3} \text{Res}_{w_1 =qw_2} f
\]

\[ = \text{Res}_{w_{S_1+S_2} =u_1} \text{Res}_{w_{S_1+S_2-1} =qw_{S_1+S_2}} \ldots \text{Res}_{w_{S_1+2} =qw_{S_1+3}} \text{Res}_{w_{S_1+1} =qw_{S_1+2}} \text{Res}_{w_{S_1} =u_2} \text{Res}_{w_{S_1-1} =qw_{S_2}} \ldots \text{Res}_{w_2 =qw_3} \text{Res}_{w_1 =qw_2} \quad (15)
\]
Case 2: \( u_2 = q^{s_1}u_1 \),

\[
\begin{align*}
\text{Res} \frac{\hat{w}_{s_1 + s_2 = q^{s_1}u_1}}{\hat{w}_{s_1 + s_2 - 1 = q^{u_2}w_{s_2}}} \cdots \text{Res} \frac{\hat{w}_{s_1 + 2} = q^{u_2}w_{s_1 + 1}}{\hat{w}_{s_1 - 1 = q^{u_2}w_{s_1}}} \text{Res} \frac{\hat{w}_1 = q^{u_2}w_{1}}{\hat{w}_1 = q^{u_2}} f
\end{align*}
\]

\[
\begin{align*}
\frac{\hat{w}_{s_1 + s_2 = q^{s_1}u_1}}{\hat{w}_{s_1 + s_2 - 1 = q^{u_2}w_{s_2}}} \cdots \text{Res} \frac{\hat{w}_{s_1 + 2} = q^{u_2}w_{s_1 + 1}}{\hat{w}_{s_1 - 2 = q^{u_2}w_{s_1}}} \text{Res} \frac{\hat{w}_1 = q^{u_2}w_{1}}{\hat{w}_1 = q^{u_2}} f
\end{align*}
\]

(16)

To summarize all above, we can repeat using above arguments to integrating all variables for
the integrand of the form as in (1) with one variable less each time.

When there is only one variable left

\[
f(w) = g(w) \prod_{u \in U} \frac{w - q^{-1}u}{w - u}
\]

(17)

we still update the set \( U \) to \( U \setminus \{u\} \cup \{qu\} \) after choosing pole at \( \hat{w} = u \in U \). Using same argument
to get (15) and (16), after picking up poles for all \( w_i, i \in [d] \), the results only depends on the final
set \( U \), and final set \( U \) must be of the form

\[
\{q^{d_1}x_1, \ldots, q^{d_r}x_r\}
\]

(18)

where \( d_1 + \cdots + d_r = d \), which means for each sequence, the final result can be indexed by a \( r \)-tuple partition of \( d \).

Suppose there is a sequence with final set \( \{q^{d_1}x_1, \ldots, q^{d_r}x_r\} \), then we can compute the result
by following sequence

\[
(\hat{w}_1, \ldots, \hat{w}_d) = (x_1, qx_1 \cdots, q^{d_1-1}x_1, x_2 \cdots, x_r, \ldots q x_r, q^{d_r-1}x_r)
\]

(19)

and note that we can actually do permutation on the right side, so for each partition \( |\vec{d}| = d \), we
have \( d! \) possible evaluation sequences.

In all we get following lemma to compute \( E_d \):

Lemma 2.1 We can write \( E \) as

\[
E_d = \sum_{|\vec{d}| = d} d! E_{\vec{d}}
\]

(20)

where

\[
E_{\vec{d}} = \lim_{w_d \to \hat{w}_d} \cdots \lim_{w_1 \to \hat{w}_1} \left( \prod_{i=1}^{n} (w_i - \hat{w}_i) f(\vec{w}) \right)
\]

(21)

here

\[
(\hat{w}_1, \ldots, \hat{w}_d) = (x_1, qx_1, \ldots, q^{d_1-1}x_1, x_2, qx_2, \ldots, q^{d_2-1}x_2, \ldots, x_r, \ldots, q^{d_r-1}x_r)
\]

and the order to take limit is from \( w_1 \) to \( w_d \).
We now evaluate one specific configuration of these simple pole residues for given $\vec{d}$ by changing of variables:

$$u_{n_i}^i = x_i q^{n_i-1} z_{n_i}^i, \quad i = 1, \ldots, r \quad n_i = 1, \ldots, d_i$$

**Notations:** From now on, we would frequently use the following notations

$$x_{ij} := x_i/x_j \quad n_{ij} := n_i - n_j$$

so

$$f(\vec{w}) = \frac{1}{(1-q)^d d!} \prod_{i,n_i} z_{n_i}^i \cdot \prod_{i=1}^r \frac{d_i}{d_i} \prod_{n_i \neq n_j} 1 - q^{n_{ij}} \frac{z_{n_i}^i/z_{n_j}^j}{1 - q^{n_{ij}+1} z_{n_i}^i/z_{n_j}^j}$$

$$\times \prod_{i,j=1|i \neq j}^{r} \prod_{n_i=1}^{d_i} \prod_{n_j=1}^{d_j} \frac{1 - q^{n_{ij}} z_{n_i}^i/z_{n_j}^j x_{ij}}{1 - q^{n_{ij}+1} z_{n_i}^i/z_{n_j}^j x_{ij}}$$

$$\times \prod_{i,j=1|i \neq j}^{r} \prod_{n_i=1}^{d_i} \frac{1}{(1-x_{ij} q^{1-n_i}/z_{n_i}^i)} \cdot \frac{1}{\prod_{i=1}^{r} \prod_{j=r+1}^{n} \prod_{n_i=1}^{d_i} (1-x_{ij} q^{n_i} z_{n_i}^i)}$$

now obtain grid of the simple pole terms and evaluate the function with $z_i = 1$, note that

$$\lim_{z_{d_i}^i \to 1} \cdots \lim_{z_{d_i}^i \to 1} \left( \prod_{n_i=1}^{d_i} z_{n_i}^i - 1 \right) \cdot \frac{1}{(1-(z_d^d)^{-1}) (1-z_{d_i}^i/z_{d_i}^i) \cdots (1-z_{d_i}^i-1/z_{d_i}^i) z_{d_i}^i \cdots z_{d_i}^i} = 1$$

where the order to take limits is from $z_{d_i}^i$ to $z_{d_i}^i$. So this specific configuration of residues is

$$\frac{1}{(1-q)^d d!} \prod_{i=1}^r \left( \prod_{n_i \neq n_j | n_i j \neq -1}^{d_i} \frac{1 - q^{n_{ij}}}{1 - q^{n_{ij}+1}} \prod_{n_i=2}^{d_i} \frac{1 - q^{-1}}{1 - q^{1-n_i}} \right)$$

$$\times \prod_{i,j=1|i \neq j}^{r} \prod_{n_i=1}^{d_i} \prod_{n_j=1}^{d_j} \frac{1 - q^{n_{ij}} x_{ij}}{1 - q^{n_{ij}+1} x_{ij}}$$

$$\times \prod_{i,j=1|i \neq j}^{r} \prod_{n_i=1}^{d_i} \frac{1}{(1-x_{ij} q^{1-n_i})} \cdot \frac{1}{\prod_{i=1}^{r} \prod_{j=r+1}^{n} \prod_{n_i=1}^{d_i} (1-x_{ij} q^{n_i})}$$

and the factor only with $x_{ij}$ with $i = 1, \ldots, r$ and $j = r + 1, \ldots, n$ is

$$A := \frac{1}{\prod_{i=1}^{r} \prod_{j=r+1}^{n} \prod_{n_i=1}^{d_i} (1-x_{ij} q^{n_i})} = \frac{1}{\prod_{i=1}^{r} \prod_{j=r+1}^{n} (q x_{ij}; q) d_i}$$

the factors do not involve with any $x_{ij}$ is

$$B := \frac{1}{(1-q)^d} \cdot \prod_{i=1}^r \left( \prod_{n_i \neq n_j | n_i j \neq -1}^{d_i} \frac{1 - q^{n_{ij}}}{1 - q^{n_{ij}+1}} \prod_{n_i=2}^{d_i} \frac{1 - q^{-1}}{1 - q^{1-n_i}} \right)$$

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and define \( P_d \) as
\[
P_d := \left\{ \prod_{i \neq j | i - j \neq -1}^{d} \frac{1 - q^{i-j}}{1 - q^{i-j+1}} \cdot \prod_{i=2}^{d_i} \frac{1 - q^{-1}}{1 - q^{1-i}} \right\}^{d > 1}
\]
\[
d = 0, 1
\]
by simple induction, it is easy to show that
\[
\frac{P_d}{(1-q)^d} = \frac{1}{(q;q)_d} d \geq 0
\]
and
\[
B = \prod_{i=1}^{r} \frac{P_{d_i}}{(1-q)^{d_i}} = \prod_{i=1}^{r} \frac{1}{(q;q)_{d_i}}
\]
the factors left is
\[
C := \left( \prod_{i,j=1}^{r} \prod_{n_i=1}^{d_i} \prod_{n_j=1}^{d_j} \frac{1 - q^{n_i n_j x_{ij}}}{1 - q^{n_j n_j+1} x_{ij}} \right) \prod_{i,j=1}^{r} \prod_{n_i=1}^{d_i} \prod_{n_j=1}^{d_j} \frac{(x_i q^{n_i-1})^l}{(1 - x_j q^{1-n_j})}
\]
\[
= \prod_{i,j=1}^{r} \prod_{n_j=1}^{d_j} \left( \prod_{n_i=1}^{d_i} \frac{1 - q^{n_j n_j} x_{ij}}{1 - q^{n_i n_i+1} x_{ij}} \right) \cdot \prod_{i=1}^{r} x_i^{l d_i} \cdot \prod_{i=1}^{r} q^{(l d_i - (d_i - 1))}
\]
\[
= \prod_{i=1}^{r} x_i^{l d_i} \cdot \prod_{i,j=1}^{r} \prod_{n_j=1}^{d_j} \prod_{n_i=1}^{d_i} \frac{1}{(q^{d_i+1} - q)_{d_j}}
\]
The above equations prove that the summand in (20) corresponding to given \( \vec{d} \) equals to one summand in \( A_d(\vec{x}, I, l) \), thus, we have
\[
A_d(\vec{x}, I, l) = \sum_{|\vec{d}|=d} d! E_d \equiv E_d
\]
Now calculate the integration with clockwise direction.
\[
E_d := \int_{C_{\rho}'} \frac{dw_i}{2\pi \sqrt{-1}} \cdots \int_{C_{\rho}'} \frac{dw_i}{2\pi \sqrt{-1}} f(w_{i_1}, \ldots, w_{i_d})
\]
The assumption with \( l \) ensures that when doing integration by any order, for each variable \( w \) the residue at infinity is 0, by definition, we can calculate this integration by taking the sum of residues outside the circle \( |w_i| = \rho \).

The iterated residues in this case are similar to the previous counter-clockwise direction. Similar arguments as (2.1) shows
\[
E_d = \sum_{|\vec{d}|=d} d! E_{\vec{d}}
\]

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where

\[ E_d' = \lim_{w_d \to \hat{w}_d} \cdots \lim_{w_1 \to \hat{w}_1} \left( \prod_{i=1}^{n} (w_i - \hat{w}_i) f(\hat{w}) \right) \]

Here

\[ \{\hat{w}_1, \ldots, \hat{w}_d\} = \{x_{r+1}q^{-1}, x_{r+1}q^{-2}, \ldots, x_{r+1}q^{-d_r+1}, \ldots, x_nq^{-1}, x_nq^{-2}, \ldots, x_nq^{-d_n}\} \]

and the order to take limit is from \(w_1\) to \(w_d\).

We now doing the following the change of variables calculate the residues

\[ w_i^j = x_iq^{-n_i}z_i^j \quad i = r + 1, \ldots, n \quad n_i = 1, \ldots, d_i \]

\[ f(\hat{w}) = \frac{1}{(1-q)d!} \prod_{i=1}^{n} \frac{1 - q^{n_i}z_i^i / z_i^j}{1-q^{n_i+1}z_i^i / z_i^j} \]

\[ \times \prod_{i,j=r+1}^{n} \frac{1 - q^{n_{ij}z_i^j / z_{ij}}}{1-q^{n_{ij}+1}z_i^j / z_{ij}} x_{ij} \]

\[ \times \prod_{i,j=r+1}^{n} \frac{1}{\prod_{n=1}^{d_i} (1-x_{ij}q^{-n_i}z_i^j)} \]

Note that

\[ \lim_{z_i^j \to 1} \lim_{z_i^j \to 1} \left( \prod_{n=1}^{d_i} (z_i^j - 1) \cdot \frac{1}{(1-z_i^j) (1-z_i^j/z_i^j) \cdots (1-z_i^j/z_i^j-1) z_i^j \cdots z_i^j} \right) = (-1)^{d_i} \]

where the order to take limits is from \(z_i^j\) to \(z_i^j\). So the residues for one specific configuration of residues of type \(d_i^j\) is

\[ \frac{(-1)^d}{(1-q)d!} \prod_{i=r+1}^{n} \left( \prod_{n=1}^{d_i} \frac{1 - q^{n_i}z_i^j / z_i^j}{1-q^{n_i+1}z_i^j / z_i^j} \cdot \prod_{n=2}^{d_i} \frac{1 - q^{-1}}{1 - q^{1-n_i}} \right) \]

\[ \times \prod_{i,j=r+1}^{n} \frac{1 - q^{n_{ij}z_i^j / z_{ij}}}{1-q^{n_{ij}+1}z_i^j / z_{ij}} x_{ij} \]

\[ \times \prod_{i,j=r+1}^{n} \frac{1}{\prod_{n=1}^{d_i} (1-x_{ij}q^{-n_i})} \]

\[ \times \prod_{i,j=r+1}^{n} \frac{1}{\prod_{n=1}^{d_i} (1-x_{ij}q^{-n_i})} \]

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after almost same computation as for $E_d^\alpha$, we can simplify the above equation to

\[
(-1)^d \prod_{i=r+1}^{n} x_i^{d_i} q^{\frac{d_i(d_i+1)}{2}} \prod_{i=r+1}^{n} x_{ij}^{d_{ij}+1} q^{d_{ij}+1} \prod_{i=r+1}^{n} \prod_{j=i+1}^{n} (q x_{ij}; q)_{d_{ij}}
\]

which proves

\[
E_d' = (-1)^d B(\vec{x}, \vec{I}, -l)
\]

Since the residue at infinity is zero, using Cauchy Residue Theorem $d$ times,

\[
\int_{C_p} \cdots \int_{C_p} f(\vec{w}) \frac{dw_1}{2\pi \sqrt{-1} w_1} \cdots \frac{dw_d}{2\pi \sqrt{-1} w_d} = (-1)^d \int_{C_p} \cdots \int_{C_p} f(\vec{w}) \frac{dw_1}{2\pi \sqrt{-1} w_1} \cdots \frac{dw_d}{2\pi \sqrt{-1} w_d}
\]

then we arrive at (24) (25) and (26) of the following proposition stated in the introduction:

**Proposition 2.1** Denoted by $[n]$ the set of elements \{1, ..., n\}, let $\emptyset \neq I \subseteq [n]$ be a subset of $[n]$, $|I|$ be its cardinality, and denoted by $I^c$ the complementary set of $I$ in $[n]$. Then for constant positive integers $d$, $n$ and integer $l$ with restriction: $1 - |I| \leq l \leq n - |I| - 1$, let $A_d(\vec{x}, I, l)$ and $B_d(\vec{x}, I, l)$ be two rational functions in $\vec{x}$ and $q$ with an extra data $l$.

\[
A_d(\vec{x}, I, l) = \sum_{|d_I| = d} \prod_{i \in I} x_i^{d_i} q^{\frac{d_i(d_i-1)}{2}} \prod_{i \in I} x_{ij}^{d_{ij}+1} q^{d_{ij}+1} \prod_{j \in I^c} (q x_{ij}; q)_{d_{ij}}
\]

\[
B_d(\vec{x}, I, l) = \sum_{|d_I| = d} \prod_{i \in I} x_i^{-d_i} q^{\frac{d_i(d_i+1)}{2}} \prod_{i \in I^c} x_{ij}^{d_{ij}+1} q^{d_{ij}+1} \prod_{j \in I} (q x_{ij}; q)_{d_{ij}}
\]

where $d_I$ is $|I|$-tuple of non negative integers, and $|d_I| := \sum_{i \in I} d_i$. $x_i, i = 1, ..., n$ are parameters. For convenience, we use the notation $x_{ij} := x_i/x_j$ and $d_{ij} := d_i - d_j$. Then we have

\[
A_d(\vec{x}, I, l) = B_d(\vec{x}, I^c, -l)
\]

**2.2 Examples**

In the following two examples, we show how the proof of Proposition 2.1 works.

**Example 2.1 (d=1)** For the case $l=0$, $d=1$, $r=2$, $n=3$. (3) becomes the following simple form

\[
f(w) = \frac{1}{(1-q)(1-x_1/w)(1-x_2/w)(1-qw/x_3)^{-1}}
\]

Consider integration (5), then there are simple poles of type $(0,1)$ and $(0,1)$ in the contour $C_p$:

- **type $(1,0)$**: $w = x_1$
• type (0,1): $w = x_2$

Then the residue for each type comes as follows:

• type (1, 0):

$$E_{(1,0)} = \text{Res}_{\hat{w} = x_1} f = \frac{1}{(1 - q)(1 - x_{21})(1 - qx_{13})}$$

• type (0, 1):

$$E_{(0,1)} = \text{Res}_{\hat{w} = x_2} f = \frac{1}{(1 - q)(1 - x_{12})(1 - qx_{23})}$$

and there is only one simple pole $w = q^{-1}x_3$ in the counter $C'_\rho$, so

• type 1:

$$E'_1 = \text{Res}_{\hat{w} = q^{-1}x_3} f = \frac{-1}{(1 - q)(1 - qx_{13})(1 - qx_{23})}$$

and it is easy to obtain

$$\frac{1}{(1 - q)(1 - x_{21})(1 - qx_{13})} + \frac{1}{(1 - q)(1 - x_{12})(1 - qx_{23})} = \frac{1}{(1 - q)(1 - qx_{13})(1 - qx_{23})}$$

which agrees with (26).

**Example 2.2 (d=2)** For the case $l=0$, $d=2$, $r=2$, $n=3$. (3) becomes the following simple form

$$f(\vec{w}) = \frac{1}{2(1 - q)^2} \prod_{i \neq j} \frac{1 - w_i/w_j}{1 - qw_i/w_j} \prod_{i=1}^2 \frac{1}{\prod_{j=1}^2 (1 - x_j/w_i) \cdot (1 - qw_i/x_3)}$$

Consider integration (5), then there are simple poles of type $(2, 0)$, $(1, 1)$ and $(0, 2)$ in the counter $C'_\rho$:

• type $(2,0)$: $\{w_1, w_2\} = \{x_1, x_1q\}$

• type $(1,1)$: $\{w_1, w_2\} = \{x_1, x_2\}$

• type $(0,2)$: $\{w_1, w_2\} = \{x_2, x_2q\}$

Then the residue for each type comes as follows:

• type $(2,0)$:

$$2!E_{(2,0)} = \text{Res}_{\hat{w}_2 = qx_1 \hat{w}_1 = x_1} f + \text{Res}_{\hat{w}_3 = x_1 \hat{w}_1 = qx_2} f$$

$$= \frac{1}{2(1 - q)^2 (1 + q)(1 - qx_{13})(1 - q^2x_{13})(1 - x_{21})(1 - qx_{21})} + \frac{1}{2(1 - q)^2 (1 + q)(1 - qx_{13})(1 - q^2x_{13})(1 - qx_{21})(1 - x_{21})}$$

$$= \frac{1}{(1 - q)^2 (1 + q)(1 - q^2x_{13})(1 - qx_{13})(1 - qx_{21})(1 - x_{21})}$$
• type \((1, 1)\):

\[
2!E_{(1, 1)} = \left[ \text{Res}_{w_2=x_2w_1=x_1} \text{Res}_{w_2=x_1w_1=x_2} f \right] + \left[ \text{Res}_{w_2=q_2w_1=q_1w_1} \text{Res}_{w_2=q_1w_1=q_2w_1} f \right]
\]

\[
= \frac{1}{2(1-q)^2(1-qx_{12})(1-qx_{21})(1-qx_{13})(1-qx_{23})} + \frac{1}{2(1-q)^2(1-q^{-1}x_{12})(1-q^{-1}x_{21})(1-q^{-1}x_{13})(1-q^{-1}x_{23})}
\]

Then the residue for each type 2 comes as follows:

• type 2: \(\{w_1, w_2\} = \{q^{-1}x_3, q^{-2}x_3\}\)

Consider integration \((23)\), then there are simple poles of type 2 in the counter \(C'_{\rho_i}\):

• type 2:

\[
(-1)^2 2!E_2' = \left[ \text{Res}_{w_2=q^{-2}x_3w_1=q^{-1}x_3} \text{Res}_{w_2=q^{-1}x_3w_1=q^{-2}x_3} f \right] + \left[ \text{Res}_{w_2=q^{-1}x_3w_1=q^{-2}x_3} \text{Res}_{w_2=q^{-2}x_3w_1=q^{-1}x_3} f \right]
\]

\[
= \frac{1}{(1+q)(1-q)^2(1-q^{-2}x_{13})(1-q^{-1}x_{13})(1-q^{-2}x_{23})(1-q^{-1}x_{23})}
\]

by a little bit computation, we have

\[
E_2 = 2!E_{(2, 0)} + 2!E_{(1, 1)} + 2!E_{(0, 2)} = E_2'
\]

**Example 2.3** From Proposition \(2.7\), if we take \(n = 3, l = 0\) and \(I = [2]\), we know that \(A_d(\bar{x}, [2], 0) = B_d(\bar{x}, [3]|\{2\}, 0)\). By the following computation, there is a phenomenon that we could extracting from \(A_d(\bar{x}, [2], 0)\) to get \(B_d(\bar{x}, [3]|\{2\}, 0)\) times another factor, when \(d = 1, 2\), i.e. \(A_d(\bar{x}, [2], 0) = B_d(\bar{x}, [3]|\{2\}, 0) \times G(\bar{x})\), \(d = 1, 2\), thus we can conclude that \(G(\bar{x}) = 1\). Furthermore, this is a general phenomenon for all \(d\), see following Corollary \(2.7\).

By definition \(\bar{x} = \{x_1, x_2, x_3\}\), so

\[
A_d(\bar{x}, [2], 0) = \sum_{d_1+d_2=d} \frac{1}{(q_1; q_1)_d (q_2; q_2)_d (q_3; q_3)_d (q_{d+1}; q_{d+1})_d (q_{d+2}; q_{d+2})_d \cdots (q_{d+2k}; q_{d+2k})_d}
\]

\[
B_d(\bar{x}, [3]|\{2\}, 0) = \frac{1}{(q; q)_d (q_{d+1}; q_{d+1})_d (q_{d+2}; q_{d+2})_d}
\]
For $d = 1$, i.e., $(d_1, d_2) = (1, 0)$ or $(0, 1)$, we have

$$A_1(\vec{x}, [2], 0) = \sum_{d_1 + d_2 = 1} \frac{1}{(q; q)_{d_1} (q; q)_{d_2} (q^{d_12+1}x_{12}; q)_{d_2} (q^{d_22+1}x_{12}; q)_{d_1} (q^{x_{13}}; q)_{d_1} (q^{x_{23}}; q)_{d_2}}$$

$$= \sum_{d_1 + d_2 = 1} \frac{1}{(q; q)_{d_1} (q^{x_{13}}; q)_{d_1} (q^{x_{23}}; q)_{d_2}} \cdot \frac{(q; q)_{d_2} (q^{d_22+1}x_{12}; q)_{d_2} (q^{d_22+1}x_{12}; q)_{d_1} (q^{x_{13}}; q)_{d_1} (q^{x_{23}}; q)_{d_2}}{(q; q)_{d_1} (q^{d_12+1}x_{12}; q)_{d_2} (q^{d_12+1}x_{12}; q)_{d_1} (q^{x_{13}}; q)_{d_1} (q^{x_{23}}; q)_{d_2}}$$

$$= B_1(\vec{x}, [3]\setminus[2], 0) \times \sum_{(d_1, d_2) = (1, 0), (0, 1)} \frac{(q; q)_{d_2} (q^{d_22+1}x_{12}; q)_{d_2} (q^{d_22+1}x_{12}; q)_{d_1} (q^{x_{13}}; q)_{d_1} (q^{x_{23}}; q)_{d_2}}{2 \prod_{j \neq i} (q^{d_j+1}x_{ij}; q)_{d_j}}$$

$$= B_1(\vec{x}, [3]\setminus[2], 0) \times \left(\frac{1 - qx_{23}}{1 - x_{21}} + \frac{1 - qx_{13}}{1 - x_{12}}\right)$$

$$= B_1(\vec{x}, [3]\setminus[2], 0)$$

for $d = 2$, i.e. $(d_1, d_2) = (2, 0), (1, 1)$ or $(0, 2)$, similarly, we have

$$A_2(\vec{x}, [2], 0) = B_2(\vec{x}, [3]\setminus[2], 0) \times \sum_{(d_1, d_2) = (2, 0), (1, 1), (0, 2)} \frac{(q; q)_{d_2} (q^{d_22+1}x_{12}; q)_{d_2} (q^{d_22+1}x_{12}; q)_{d_1} (q^{x_{13}}; q)_{d_1} (q^{x_{23}}; q)_{d_2}}{2 \prod_{j \neq i} (q^{d_j+1}x_{ij}; q)_{d_j}}$$

$$= B_2(\vec{x}, [3]\setminus[2], 0) \times \left(\frac{(1 - q^{x_{13}})(1 - q^2x_{13})}{(1 - qx_{12})(1 - x_{21})} + \frac{(1 + q)(1 - q^2x_{13})(1 - q^2x_{23})}{(1 - qx_{21})(1 - q^2x_{21})} + \frac{(1 - qx_{13})(1 - q^2x_{13})}{(1 - q^2x_{12})(1 - x_{12})}\right)$$

$$= B_2(\vec{x}, [3]\setminus[2], 0)$$

More generally, we have the following Corollary,

**Corollary 2.1**

$$\sum_{d_1 + d_2 = d} \frac{(q; q)_{d}}{(q; q)_{d_1} (q; q)_{d_2} (q^{d_1+1}x_{ij}; q)_{d_1} (q^{d_2+1}x_{ij}; q)_{d_2} (q^{x_{13}}; q)_{d_1} (q^{x_{23}}; q)_{d_2}}{2 \prod_{j \neq i} (q^{d_j+1}x_{ij}; q)_{d_j}} = 1$$

**Proof**

Set $l = 0$, $r = 2$, $n = 3$ in (26), we have

$$A_d(\vec{x}, [2], 0) = \sum_{d_1 + d_2 = d} \prod_{i, j = 1}^{2} \frac{1}{(q^{d_j+1}x_{ij}; q)_{d_j}} \prod_{i = 1}^{2} \frac{1}{(q^{d_j+1}x_{ij}; q)_{d_j}} \frac{1}{(q^{x_{13}}; q)_{d_1}}$$

$$= \sum_{d_1 + d_2 = d} \prod_{i = 1}^{2} \frac{1}{(q^{d_j+1}x_{ij}; q)_{d_j}} \frac{1}{(q^{d_j+1}x_{ij}; q)_{d_j}} \cdot \frac{1}{(q^{x_{13}}; q)_{d_1}}$$

$$= \sum_{d_1 + d_2 = d} \left(\frac{q^{d_1+1}x_{13}; q}_{(q^{d_1+1}x_{13}; q)} (q^{d_2+1}x_{23}; q)_{d_2} \prod_{j \neq i} \frac{1}{(q^{d_j+1}x_{ij}; q)_{d_j}} \right)$$

$$= \sum_{d_1 + d_2 = d} \left(\frac{q^{d_1+1}x_{13}; q}_{(q^{d_1+1}x_{13}; q)} (q^{d_2+1}x_{23}; q)_{d_2} \prod_{j \neq i} \frac{1}{(q^{d_j+1}x_{ij}; q)_{d_j}} \right)$$

$$= \sum_{d_1 + d_2 = d} \left(\frac{q^{d_1+1}x_{13}; q}_{(q^{d_1+1}x_{13}; q)} (q^{d_2+1}x_{23}; q)_{d_2} \prod_{j \neq i} \frac{1}{(q^{d_j+1}x_{ij}; q)_{d_j}} \right)$$

since we know $A_d(\vec{x}, I, 0)$ equals to $B_d(\vec{x}, I^c, 0)$, we get the conclusion. \qed

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2.3 Boundary cases

For \( l = -|I|, l = n - |I| \), (26) no longer holds, since the residue at infinity is nonzero, but we can compare the behavior of some special limit in (26) to obtain following results.

**Corollary 2.2**

- For \( l = n - |I| \), we have

\[
A_d(\vec{x}, I, l) = \sum_{s=0}^{d} C_s(\vec{x}, \vec{I}, d) B_{d-s}(\vec{x}, q, \vec{I}, -l)
\]

where \( C_s(\vec{x}, I, d) \) is defined as

\[
C_s(\vec{x}, I, d) = \frac{(-1)^{|I|+s}}{(q; q)_s q^s(d-s+|I|)}
\]

- For \( l = -|I| \), we have

\[
B_d(\vec{x}, \vec{I}, -l) = \sum_{s=0}^{d} D_s(\vec{x}, I, d) A_{d-s}(\vec{x}, q, I, l)
\]

\[
D_s(\vec{x}, I, d) = \frac{(-1)^{|I|+s}}{(q; q)_s q^s(d-s)}
\]

**Proof**

Consider \([n+1], I \subset [n+1], \{n+1\} \notin I, \ l = n - |I| \) in (26), then we have

\[
\sum_{|d|\leq d} \frac{\left(\prod_{i \in I} x_i^{-d_i} q^{d_i(d_i-1)/2}\right)^{|I|}}{\prod_{j \in I} (q^{d_j+1} x_j; q)_{d_j} \prod_{j \in I} (q x_j; q)_{d_j}}
\]

\[
= \sum_{|d|\leq d} \frac{\left(\prod_{i \in I} x_i^{-d_i} q^{d_i(d_i+1)/2}\right)^{-l}}{\prod_{j \in I} (q^{d_j+1} x_j; q)_{d_j} \prod_{j \in I} (q x_j; q)_{d_j}}
\]

It is easy to see taking \( \lim_{x_{n+1} \to \infty} \) in (31), we obtain

\[
\lim_{x_{n+1} \to \infty} (9) = A_d(\vec{x}, I, l), \ \text{for} \ l = n - |I|
\]

Now let’s take limit \( \lim_{x_{n+1} \to \infty} \) in (32)

\[
\sum_{|d|\leq d} \frac{1}{\prod_{j \in I} (q^{d_j+1} x_j; q)_{d_j} \prod_{j \in I} (q x_j; q)_{d_j}} \cdot \frac{1}{\prod_{j \in [n], I} (q^{d_{n+1}-d_j+1} x_{j,n+1}; q)_{d_j}}
\]

\[
\times \frac{(x_{d_1+1} q^{-d_1(d_1+1)/2})^l}{\prod_{i \in [n], I} (q^{d_i+1} x_i; q)_{d_i}}
\]

\[
\times \prod_{i \in [n], I} \left( \frac{1}{\prod_{j \in [n], I} (q^{d_i-d_j+1} x_{j}; q)_{d_j}} \cdot \left(\frac{x_i^{d_1-1} q^{-d_1(d_1+1)/2}}{\prod_{j \in I} (q x_j; q)_{d_i}}\right)^l \right)
\]
the limits of last two terms in (33) equal 1, and by a little bit computation, we obtain (34) equals
\[-1\]^{d_{n+1}(n-r)} \cdot q^{-\sum_{i \in \{n \mid I\}} d_i} \prod_{i \in \{n \mid I\}} x_i^{d_{n+1}}
then we obtain
\[
\lim_{x_{n+1} \to \infty} \sum_{|d| = d \in I^c} \prod_{|d_i| = d_i \in I} \left( \frac{1}{\prod_{j \in I} (q^{d_i - d_{i+1}} x_{j}; q)_{d_j}} \cdot \frac{x_i^{d_i} q^{-d_i d_{i+1}/2}}{\prod_{j \in I} (q x_{j}; q)_{d_j}} \right) = \sum_{|d| = d \in I^c} \prod_{|d_i| = d_i \in I} \left( \frac{1}{\prod_{j \in I} (q^{d_i - d_{i+1}} x_{j}; q)_{d_j}} \cdot \frac{x_i^{d_i} q^{-d_i d_{i+1}/2}}{\prod_{j \in I} (q x_{j}; q)_{d_j}} \right) \cdot \prod_{i \in \{n \mid I\}} x_i^{-d_{n+1}} \cdot (1 - q^{d_{n+1}(n-|I|)}) \cdot \prod_{i = r+1}^n x_i^{-d_{n+1}} \cdot B_\alpha (\vec{x}, I^c, -l)
\]
we obtain the conclusion.

Similarly, consider \(A_d (x \cup x_{n+1}, \vec{I}, l)\) and \(B_d (x \cup x_{n+1}, \vec{F}, -l)\), for \(\vec{I} = I \cup \{n + 1\}\) and \(l = -|I|\), from (33), we have
\[
\sum_{|d| = d \in \vec{I}} \prod_{|d_i| = d_i \in \vec{I}} \left( \frac{1}{\prod_{j \in \vec{I}} (q^{d_i - d_{i+1}} x_{j}; q)_{d_j}} \cdot \frac{x_i^{d_i} q^{d_i (d_i+1)/2} |I|}{\prod_{j \in \vec{I}} (q x_{j}; q)_{d_j}} \right) \cdot \prod_{i \in \{n \mid I\}} x_i^{-d_{n+1}} \cdot (1 - q^{d_{n+1}(n-|I|)}) \cdot \prod_{i = r+1}^n x_i^{-d_{n+1}} \cdot B_\alpha (\vec{x}, I^c, -l)
\]
It is easy to see that after taking \(\lim_{x_{n+1} \to \infty}\) in (33), we obtain
\[
B_d (\vec{x}, I^c, l), \text{ for } l = -|I|
\]
First, rewrite (38) as follows,
\[
\sum_{|d| = d \in \vec{I}} \prod_{|d_i| = d_i \in \vec{I}} \left( \frac{1}{\prod_{j \in \vec{I}} (q^{d_i - d_{i+1}} x_{j}; q)_{d_j}} \cdot \frac{x_i^{d_i} q^{d_i (d_i+1)/2} |I|}{\prod_{j \in \vec{I}} (q x_{j}; q)_{d_j}} \right) \cdot \prod_{i \in \{n \mid I\}} x_i^{-d_{n+1}} \cdot (1 - q^{d_{n+1}(n-|I|)}) \cdot \prod_{i = r+1}^n x_i^{-d_{n+1}} \cdot B_\alpha (\vec{x}, I^c, -l)
\]
called universal cotangent line bundles. The fiber of $L_p$ over $C$ given by evaluation at the $i$-th marked point. There are line bundles $\pi_Q$ where $\rho$ is a section and $\gamma$ is called the virtual structure sheaf \[6\]. And $g,n,d$ $\text{vir}$ $K$-theoretic $\text{pushforward along the projection}$ $\pi_* : [Q_{g,n}(X,d)/S_n] \to [pt]$
and \( \{ \phi_\alpha \} \) is a basis in \( K^0 (X) \otimes Q \) and \( t_{k, \alpha} \) are formal variables. The last term in (36) is the level \( l \) determinant line bundle over \( Q^{\xi}_{g,n} (X, d) \) defined as

\[
D^{R,l} := (\det R^* \pi_* (\mathcal{P} \times_G R))^{-l}
\]

the bundle \( \mathcal{P} \times_G R \) is the pullback of the vector bundle \( [V \times R/G] \to [V/G] \) along the evaluation map to the quotient stack \([V/G]\).

Similarly, we can define quasimap graph space \( \mathcal{Q}^{\xi}_{0,n} (X, \beta) \) which parametrizes quasimaps with \( \mathcal{T} \)-parametrized component \( \mathbb{P}^1 \), so there is a natural \( \mathbb{C}^* \)-action on quasimap graph space. Denoted by \( F_{0, \beta} \) the special fixed loci in \( (\mathcal{Q}^{\xi}_{0,n} (X, \beta)) \mathbb{C}^* \), and denoted by \( q \) the weight of cotangent bundle at \( 0 := [1, 0] \) of \( \mathbb{P}^1 \). For details, see [1].

**Definition 3.1** [10] The permutation-equivariant \( K \)-theoretic \( J^{R,l, \epsilon} \)-function of \( V//G \) of level \( l \) is defined as

\[
J^{R,l, \epsilon}_{S_{\chi}} (t(q), Q) := \sum_{k \geq 0, \beta \in \text{Eff}(V, G, \beta)} Q^\beta (ev_\epsilon)_* [\text{Res}_{F_{0, \beta}} (\mathcal{Q}^{\xi}_{0,n} (V//G, \beta))_0]^{\text{vir}} \otimes D^{R,l} \otimes \prod_{i=1}^n t(L_i)]_{n}^S n
\]

\[
:= 1 + \frac{t(q)}{1 - q} + \sum_{a, \beta \neq 0} Q^\beta \left( F_{0, \beta}, \mathcal{O}_{F_{0, \beta}}^{\text{vir}} \otimes ev_\epsilon^* (\phi_a) \otimes \left( \frac{\text{tr} \mathcal{C}^* \otimes D^{R,l}}{\lambda_{-1}^{\text{vir}} N_{F_{0, \beta}}} \right) \right) \phi^a
\]

\[
+ \sum_{a, n \geq 1} \sum_{(n, \beta) \neq (1, 0)} Q^\beta \left( \frac{\phi_a}{(1 - q)(1 - qL)}, t(L), \ldots, t(L) \right) \right)_{0,n+1, \beta} w^a
\]

where \( \{ w_a \} \) is a basis of \( K^0 (V//G) \) and \( \{ w^a \} \) is the dual basis with respect to twisted pairing \( ( , )^{R,l} \).

**Definition 3.2** [10] When taking \( \epsilon \) small enough, denoted by \( \epsilon = 0^+ \), we call \( J^{R,l, 0^+} (0) \) the small \( l \)-function of level \( l \), i.e,

\[
I^{R,l} (q; Q) := J^{R,l, 0^+}_{S_{\chi}} (0, Q) = 1 + \sum_{\beta \geq 0} Q^\beta (ev_\epsilon)_* \left( \mathcal{O}_{F_{0, \beta}}^{\text{vir}} \otimes \left( \frac{\text{tr} \mathcal{C}^* \otimes D^{R,l}}{\lambda_{-1}^{\text{vir}} N_{F_{0, \beta}}} \right) \right)
\]

3.2 Level correspondence in Grassmannian duality

Let \( V \) be \( r \times n \) matricies \( M_{r \times n}, G \) be the general linear group \( GL_r \) and let \( \theta \) be the \( \text{det} : GL_r \to \mathbb{C}^* \), then we have

\[
V//\text{det} G = M_{r \times n}//\text{det} G = Gr(r, n)
\]

There is a natural \( T = (\mathbb{C}^*)^n \)-action \( \mathbb{C}^n \) with weights \( \mathbb{C}^n = \Lambda_1 + \cdots + \Lambda_n \), then deducing an action on \( \text{Gr}(r, n) \) by \( T \cdot A = AT, A \in M_{r \times n} \). Using general abelian/non-abelian correspondence in [14] for \( \text{Gr}(r, n) \), we have

\[
I^{\text{Gr}(r, n)}_T = \sum_{d} \sum_{\omega} \left[ \prod_{1 \leq i < j \leq r} \prod_{1 \leq m < d_i - d_j} \left( 1 - L_i L_j^{-1} q^m \right) \prod_{1 \leq i < j \leq r} \left( 1 - L_i^{-1} L_j \right) \prod_{i=1}^r \prod_{k=1}^d \prod_{m=1}^n \frac{1}{(1 - q^k L_i \Lambda_m^{-1})} \right] Q^d
\]

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where \( \vec{d} = \{ d_1 \leq d_2 \leq \cdots \leq d_r \} \) such that \( d_1 = d_2 = \cdots = d_{r_1} < d_{r_1+1} = \cdots = d_{r_1+r_2} < d_{r_1+\cdots+r_h} = \cdots = d_{r_1+\cdots+r_h+r_{h+1}} \), i.e. \( r_1 + \cdots + r_{h+1} = r \). \( \omega \) is the Weyl group acting on \( L_i \) to change the index, \( \{ L_i \}_i=1 \) come from the filtration of tautological bundle \( S_r \) of \( Gr(r,n) \). We could rewrite the equivariant \( I \)-function in the following way

\[
I_T^{Gr(r,n)} = \sum_{d} \sum_{d_1,d_2,\ldots,d_r} \omega \left[ \prod_{i,j=1}^{r} \frac{1}{(1-q^k L_j L_j^{-1})^{i}} \prod_{i=1}^{n} \frac{1}{(1-q^k L_i \Lambda^{-1})} \right] Q^d
\]

(39)

suppose \( \omega \) changes \( i_1 \) to \( i_2 \) and \( j_1 \) to \( j_2 \), then one of the factors changes from

\[
\prod_{k=-\infty}^{d_{i_1}-d_{i_2}} (1-q^k L_{i_1} L_{j_1}^{-1}) \prod_{k=-\infty}^{0} (1-q^k L_{i_2} L_{j_2}^{-1})
\]

(40)

to

\[
\prod_{k=-\infty}^{d_{i_1}-d_{i_2}} (1-q^k L_{i_2} L_{j_2}^{-1}) \prod_{k=-\infty}^{0} (1-q^k L_{i_1} L_{j_1}^{-1})
\]

(41)

since \( \omega \in S_r/S_{r_1} \times \cdots \times S_{r_{h+1}} \), we have \( d_{i_1} \neq d_{i_2}, d_{j_1} \neq d_{j_2} \). In (39) we have an order of partition \( \vec{d} \), from (40) to (41), one could see \( \omega \)-action is just rearrange \( \{ d_i \} \) without changing the form. There is an unique \( \omega \in S_r/S_{r_1} \times \cdots \times S_{r_{h+1}} \) whose inverse \( \omega^{-1} \) arranges \( (d_1, \ldots, d_r) \) in nondecreasing order \( d_1 \leq d_2 \leq \cdots \leq d_r \) and then we have:

\[
I_T^{Gr(r,n)} = \sum_{d} \sum_{d_1+d_2+\cdots+d_r=d} Q^d \prod_{i,j=1}^{r} \frac{1}{(1-q^k L_i L_j^{-1})} \prod_{i=1}^{n} \frac{1}{(1-q^k L_i \Lambda^{-1})}
\]

(42)

note that in [12] where the author claimed a version of mirror theorem with a different \( I \)-function.

If we consider the standard representation of \( GL_r \), denoted by \( E_r \), then the associated bundle \( \mathcal{P} \times G \mathcal{R} \) can be identified with \( \bigoplus_{i=1}^{r} L_i \otimes \mathcal{O}_{\mathcal{P}^1}(-d_i) \)

\[
\mathcal{D}^{E_r,l} = \det^{-l} R^* \pi_* (\bigoplus_{i=1}^{r} L_i \otimes \mathcal{O}_{\mathcal{P}^1}(-d_i))
\]

\[
= \det^{-l} (\bigoplus_{i=1}^{r} [L_i \otimes R^1 \pi_* \mathcal{O}_{\mathcal{P}^1}(-d_i)])^{-1}
\]

\[
= \bigotimes_{i=1}^{r} \left( L_i^{d_i-1} \cdot q^{\frac{d_i(d_i-1)}{2}} \right)^l
\]

Similarly, if we take dual standard representation, denoted by \( E_r^* \), then

\[
\mathcal{D}^{E_r^*,l} = \det^{-l} (\bigoplus_{i=1}^{r} L_i^{-1} \otimes R^0 \pi_* \mathcal{O}_{\mathcal{P}^1}(d_i))
\]

\[
= \bigotimes_{i=1}^{r} \left( L_i^{d_i+1} \cdot q^{\frac{d_i(d_i+1)}{2}} \right)^l
\]

so the equivariant \( I \)-function of \( Gr(r,n) \) with level structure is as follows

\[
I_T^{Gr(r,n), E_r,l} = \sum_{d_1+d_2+\cdots+d_r=d} Q^d \prod_{i,j=1}^{r} \frac{1}{(1-q^k L_i L_j^{-1})} \prod_{i=1}^{n} \frac{1}{(1-q^k L_i \Lambda^{-1})}
\]

(42)
and

\[ I_{T,d}^{Gr(r,n),E_{e_{-i}},l} = \sum_{d_1 + d_2 + \ldots + d_r = d} Q^d \prod_{i,j=1}^{n-r} \prod_{k=-\infty}^{d_i-d_j} (1 - q^k L_i \tilde{L}_j^{-1}) \prod_{i=1}^{r} L_i^d q^{\frac{d(d+1)}{2}} \frac{1}{l} \]

notice that one factor \( R^{-l} \) in (42) and (43) disappears due to the change of pairings in the twisted Lagrange cone (see Convention 4.4 in [10]).

**Remark 3.1** For the dual Grassmannian \( Gr(n-r, n) \), the \((\mathbb{C}^*)^n\)-action on \( \mathbb{C}^n \) is the dual action, so the weights are \( \Lambda_1^{-1} + \ldots + \Lambda_n^{-1} \), the deduced action on \( Gr(n-r, n) \) as follows, \( T \cdot B = BT, B \in M_{n-r \times n} \). So the corresponding equivariant I-function is as follows,

\[ I_{T,d}^{Gr(n-r,n),E_{e_{-i}},l} = \sum_{d_1 + d_2 + \ldots + d_r = d} Q^d \prod_{i,j=1}^{n-r} \prod_{k=-\infty}^{d_i-d_j} (1 - q^k L_i \tilde{L}_j^{-1}) \prod_{i=1}^{r} L_i^d q^{\frac{d(d+1)}{2}} \frac{1}{l} \]

and

\[ I_{T,d}^{Gr(n-r,n),E_{e_{-i}},l} = \sum_{d_1 + d_2 + \ldots + d_r = d} Q^d \prod_{i,j=1}^{n-r} \prod_{k=-\infty}^{d_i-d_j} (1 - q^k L_i \tilde{L}_j^{-1}) \prod_{i=1}^{r} L_i^d q^{\frac{d(d+1)}{2}} \frac{1}{l} \]

where \( \tilde{L}_i \) for \( i = 1, \ldots, n-r \) come from the filtration of tautological bundle \( S_{n-r} \) over \( Gr(n-r, n) \).

Let \( T \) act on Grassmannian \( Gr(r, n) \) as before, then there are \( \binom{n}{r} \) fixed points, i.e. denoted by \( \{e_1, \ldots, e_n\} \), the basis of \( \mathbb{C}^n \), then the subspace \( V \) spanned by \( \{e_{i_1}, \ldots, e_{i_r}\} \) is a \( T \)-fixed point. Let

\[ I_{e} : K_T \left( Gr(r, n)^T \right) \rightarrow K_T(Gr(r, n)) \]

the kernel and cokernel are \( K_T(pt) \)-modules and have some support in the torus \( T \). From a very general localization theorem of Thomason [13], we know

\[ \text{supp } \text{Coker } I_{e} \subset \bigcup_{\mu} \{t^\mu = 1\} \]

where the union over finitely many nontrivial characters \( \mu \). The same is true of ker \( I_{e} \), but since

\[ K_T \left( Gr(r, n)^T \right) = K(Gr(r, n)) \otimes_{\mathbb{Z}} K_T(pt) \]

has no such torsion, this forces ker \( I_{e} = 0 \), so after inverting finitely many coefficients of the form \( t^\mu - 1 \), we obtain an isomorphism, i.e.

\[ K_T^{loc}(Gr(r, n)^T) \cong K_T^{loc}(Gr(r, n)) \]

we denote \( K_T^{loc}(-) \) by

\[ K_T^{loc}(-) = K_T(-) \otimes_{R(T)} R \]

where \( R \cong \mathbb{Q}(t_1, \ldots, t_n) \) and \( \{t_i\} \) are the characters of torus \( T \).
Similarly, $T = (\mathbb{C}^*)^n$-action on $Gr(n - r, n)$ also has $(\binom{n}{n-r})$ isolated fixed points, which is indexed by $(n - r)$-element subsets of $[n]$, so identification of $Gr(r, n)^T$ with $Gr(n - r, n)^T$ gives an $\mathcal{R}$-module isomorphism of $K^\text{loc}_T(Gr(r, n))$ with $K^\text{loc}_T(Gr(n - r, n))$. Indeed, suppose $W$ is a subspace of dimension $r$ in a vector space $V$ of dimension $n$, then we have a natural short exact sequence

$$0 \rightarrow W \rightarrow V \rightarrow V/W \rightarrow 0$$

taking the dual of this short exact sequence yields an inclusion of $(V/W)^*$ in $V^*$ with quotient $W^*$

$$0 \rightarrow (V/W)^* \rightarrow V^* \rightarrow W^* \rightarrow 0$$

so $\psi : W \rightarrow (V/W)^*$ gives a canonical equivariant isomorphism $Gr(r, V) \cong Gr(n - r, V^*)$, where action of $T = (\mathbb{C}^*)^n$ on $V^*$ is induced from action of $T$ on $V$, thus, $\psi$ gives the canonical identification of fixed points

$$\psi : Gr(r, n)^T \rightarrow Gr(n - r, n)^T \quad <e_j>_{j \in I}\mapsto <e^j>_{j \in I'}$$ \hspace{1cm} (44)

where $I$ is a set of $[n]$ with $|I| = r$, and $\{e^i\}_{i=1}^n$ is the dual basis of $\{e_i\}_{i=1}^n$. Now we can state the following Level correspondence in Grassmannian duality

**Theorem 3.1 (Level Correspondence)** For Grassmannian $Gr(r, n)$ and its dual Grassmannian $Gr(n - r, n)$ with standard $T = (\mathbb{C}^*)^n$ tours action, let $E_r, E_{n-r}$ be the standard representation of $GL(r, \mathbb{C})$ and $GL(n - r, \mathbb{C})$, respectively. Consider the following equivariant $I$-function

$$I^{Gr(r, n), E_r, l}_{T, d} = 1 + \sum_{d=1}^{\infty} I^{Gr(r, n), E_r, l}_{T, d} Q^d,$$

$$I^{Gr(n-r, n), E_{n-r}, -l}_{T, d} = 1 + \sum_{d=1}^{\infty} I^{Gr(n-r, n), E_{n-r}, -l}_{T, d} Q^d.$$

Then we have the following relations between $I^{Gr(r, n), E_r, l}_{T, d}$ and $I^{Gr(n-r, n), E_{n-r}, -l}_{T, d}$ in $K^\text{loc}_T(Gr(r, n)) \otimes \mathbb{C}(q)$ (which equals to $K^\text{loc}_T(Gr(n - r, n)) \otimes \mathbb{C}(q)$):

- For $1 - r \leq l \leq n - r - 1$, we have
  $$I^{Gr(r, n), E_r, l}_{T, d} = I^{Gr(n-r, n), E_{n-r}, -l}_{T, d}$$

- For $l = n - r$, we have
  $$I^{Gr(r, n), E_r, l}_{T, d} = \sum_{s=0}^{d} C_s(n - r, d) I^{Gr(n-r, n), E_{n-r}, -l}_{T, d - s}$$

where $C_s(k, d)$ is defined as

$$C_s(k, d) = \frac{(-1)^k q^s (d - s + k) \left( \bigwedge_{n-r}^{\text{top}} S_{n-r} \right)^s}{(q-1)^s q^{s(d-k)}}$$

and $S_{n-r}$ are the tautological bundle of $Gr(n - r, n)$
For $l = -r$, we have
\[
I_{T,d}^{Gr(n-r,n),E_{n-r},-l} = \sum_{s=0}^{d} D_s(r, d) I_{T,d-s}^{Gr(r,n),E_{r,l}}
\]
\[
D_s(r, d) = \frac{(-1)^s}{(q; q)_s q^{s(d-s)} \left( \bigwedge_{\text{top}} S_r \right)^s}
\]
and $S_r$ are the tautological bundle of $Gr(r,n)$.

**Proof** Form the discussion above, we prove the above identity by comparing $i_{dI}^* I_{dI}^{E_{r,l}}$ and $i_{dI}^* I_{dI}^{E_{n-r},-l}$.

Let $I = (j_1, \ldots, j_r)$ be the subset of $[n] = \{1, \ldots, n\}$, with $|I| = r$. Denote $v_1, v_2, \ldots, v_r$ the fiber coordinates in the fiber of $S$ at fixed point $< e_j > j \in I$, with weights $C^n = \Lambda_1 + \cdots + \Lambda_n$ and
\[
(t_1, \ldots, t_n) \cdot (e_{j_1}, \ldots, e_{j_r}; v_1, v_2, \ldots, v_r) = (t_{j_1} e_{j_1}, \ldots, t_{j_r} e_{j_r}; v_1, v_2, \ldots, v_r)
\]
so the weights respect to $S_r$ are $\{\Lambda_{i,j}\}_{i \in I}$ and the weights of $i_{dI}^* S_{n-r}$ are $\{\Lambda_{i,j}^{-1}\}_{i \in I}$. Since the $I$-function is symmetric, we then could take any choice of weights
\[
i_{dI}^* I_{dI}^{Gr(n-r,n),E_{r,l}} = \sum_{|dI| = d, j \in I} \prod_{k=-\infty}^{d_i - d_j} \prod_{k=-\infty}^{d_i - d_j} \prod_{k=-\infty}^{d_i - d_j} \prod_{i \in I} \frac{1}{(q; q)_s q^{s(d-s)} \left( \bigwedge_{\text{top}} S_r \right)^s}
\]
and
\[
i_{dI}^* I_{dI}^{Gr(n-r,n),E_{n-r},-l} = \sum_{|dI| = d, j \in I} \prod_{k=-\infty}^{d_i - d_j} \prod_{k=-\infty}^{d_i - d_j} \prod_{k=-\infty}^{d_i - d_j} \prod_{i \in I} \frac{1}{(q; q)_s q^{s(d-s)} \left( \bigwedge_{\text{top}} S_r \right)^s}
\]
using notation $\Lambda_{i,j} = \Lambda_i \Lambda_j^{-1}$, and the following Lemma 3.1, we obtain
\[
i_{dI}^* I_{dI}^{Gr(n-r,n),E_{r,l}} = \sum_{|dI| = d, j \in I} \prod_{i \in I} \left( \frac{1}{\prod_{j \in I} (q^{d_i - d_j + 1} - \Lambda_{i,j}; q d_i)} \cdot \frac{(\Lambda_{i,j}^{d_j} q^{-d_j(d_j-1)/2})^l}{\prod_{j \in I} (q^{d_j - d_i + 1} - \Lambda_{j,i}; q d_i)} \right)
\] (45)
and
\[
i_{dI}^* I_{dI}^{Gr(n-r,n),E_{n-r},-l} = \sum_{|dI| = d, j \in I} \prod_{i \in I} \left( \frac{1}{\prod_{j \in I} (q^{d_j - d_i + 1} - \Lambda_{i,j}; q d_i)} \cdot \frac{(\Lambda_{i,j}^{d_j} q^{-d_j(d_j+1)/2})^l}{\prod_{j \in I} (q^{d_j - d_i + 1} - \Lambda_{j,i}; q d_i)} \right)
\] (46)
Comparing (45) and (46) with (24) and (25), we obtain the conclusion.

**Lemma 3.1** Let $I$ be the subset of $[n] = \{1, \ldots, n\}$. We have
\[
\prod_{i,j \in I} \left( \frac{1}{\prod_{k=-\infty}^{d_i} (1 - q^k x_{ij})} \frac{1}{\prod_{k=-\infty}^{d_i} (1 - q^k x_{ij})} \frac{1}{\prod_{k=-\infty}^{d_i} (1 - q^k x_{ij})} \right) = \prod_{i,j \in I} \frac{1}{(q^{d_j + 1} x_{ij}; q d_i)}
\]

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Proof It is sufficient to consider one term, if $d_i \geq d_j$, then

\[
LHS = \frac{\prod_{k=1}^{d_i} (1 - q^{k} x_{ij})}{\prod_{k=1}^{d_i} (1 - q^{k} x_{ij})} = \frac{1}{\prod_{k=d_j+1}^{d_i} (1 - q^{k} x_{ij})} = RHS
\]

If $d_i \leq d_j$, then

\[
LHS = \frac{1}{\prod_{k=d_j+1}^{d_i} (1 - q^{k} x_{ij})} \prod_{k=1}^{d_i} (1 - q^{k} x_{ij}) = \frac{1}{\prod_{k=d_j+1}^{d_i} (1 - q^{k} x_{ij})} = RHS
\]

\[\square\]

References

[1] Ionut Ciocan-Fontanine and Bumsig Kim. Wall-crossing in genus zero quasimap theory and mirror maps. arXiv e-prints, page arXiv:1304.7056, April 2013.

[2] Giovanni Felder and Martin Müller-Lennert. Analyticity of nekrasov partition functions. Communications in Mathematical Physics, 364(2):683–718, 2018.

[3] Alexander Givental. Permutation-equivariant quantum K-theory V. Toric q-hypergeometric functions. arXiv e-prints, page arXiv:1509.03903, Sep 2015.

[4] Alexander B Givental. On the wddv-equation in quantum k-theory. arXiv preprint math/0003158, 2000.

[5] Hans Jockers, Peter Mayr, Urmi Ninad, and Alexander Tabler. Wilson loop algebras and quantum k-theory for grassmannians, 2019.

[6] Y-P Lee et al. Quantum k-theory, i: Foundations. Duke Mathematical Journal, 121(3):389–424, 2004.

[7] N. Nekrasov. Four dimensional holomorphic theories. PhD thesis, 06 1996.

[8] N. Nekrasov and S. Shatashvili. Quantization of integrable systems and four dimensional gauge theories. 08 2009.

[9] N. Nekrasov and S. Shatashvili. Supersymmetric vacua and bethe ansatz. Nuclear Physics B - Proceedings Supplements, 192, 01 2009.

[10] Yongbin Ruan and Ming Zhang. The level structure in quantum k-theory and mock theta functions. arXiv preprint arXiv:1804.06552, 2018.

[11] Yongbin Ruan and Ming Zhang. The level structure in quantum K-theory and mock theta functions. arXiv e-prints, page arXiv:1804.06552, April 2018.

[12] Kaisa Taipale. K-theoretic J-functions of type A flag varieties. arXiv e-prints, page arXiv:1110.3117, October 2011.
[13] R. W. Thomason. Une formule de lefschetz en $k$-thorie quivariante algébrique. *Duke Math. J.*, 68(3):447–462, 12 1992.

[14] Yaoxiong Wen. K-Theoretic $I$-function of $V/\mathfrak{g}G$ and Application. *arXiv e-prints*, page arXiv:1906.00775, May 2019.