IDENTICAL PARTICLES AND PERMUTATION GROUP

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Abstract. Second quantization is revisited and creation and annihilation operators are shown to be related, on the same footing both to the algebra $h(1)$, and to the superalgebra $osp(1|2)$ that are shown to be both compatible with Bose and Fermi statistics. The two algebras are completely equivalent in the one-mode sector but, because of grading of $osp(1|2)$, differ in the many-particle case. The possibility of a unorthodox quantum field theory is suggested.

PACS 11.10.Cd, 03.70.+k, 02.20.Sv

Claiming that a permutation of two particles has been performed requires distinguishability of the particles themselves. An idealized operational procedure to this effect would be for example the following: one first attaches a label to each particle (i.e. a quantum number identifying its state) in order to distinguish it from any other, then one interchanges the particles and, finally, one looks once more at the labels, to make sure that the exchange has been properly performed. However, one of the fundamental hypotheses of quantum field theory is exactly that particles should be treated as identical and indistinguishable. For this reason, the permutation group is not truly related ab initio to second quantization but, as well known, is introduced in the theory only at a second stage, when the $n$-particle states are described in terms of first quantization observables.

This has deep consequences to the effect that the usual connection between algebraic properties of second quantization operators and statistics of the particles turns out to bear some arbitrariness. In order to prove this statement, we shall build explicitly, in terms of anticommuting creation and annihilation operators, a new scheme...
where, by imposing the symmetry or antisymmetry of the particle states, both bosons
and fermions can be simultaneously constructed. As briefly discussed at the end of
the paper, the construction presented should be considered as an example of a much
more general and far-reaching feature: since there is no necessary connection be-
tween the observables over the Fock space and the particle statistics, we are allowed
not only to relate both fermions and bosons to the Weyl-Heisenberg algebra \(h(1)\),
but the scheme is extendable also to more complex relations among observables (e.g.
quanta lgebras and/or exotic statistics (e.g. anyons). All these structures are, in-
deed, compatible. The one exception is the standard structure for fermions (provided
by the superalgebra \(h(1|1)\)) which is consistent with fermions only. It should be men-
tioned that the approach presented in this letter was inspired by the property that
the algebraic structures relevant to second quantization physics are hopf algebras\(^1\);
it is just the coproduct (i.e. the multi-mode description), trivial for Lie algebras and
brought to attention by studies of quantum algebras, which stands at the basis of
our construction and dramatically discriminates the different descriptions.

More formally, let us begin by showing how creation and annihila-
tion operators can be related, on the same footing, both to the algebra \(h(1)\) and to the
superalgebra \(osp(1|2)\). \(h(1)\) is customarily defined\(^2\) to be generated by the four
operators \((a, a^\dagger, 1, N)\), with commutation relations

\[
[a, a^\dagger] = 1 \quad , \quad [N, a] = -a \quad , \quad [N, a^\dagger] = a^\dagger \quad , \quad [1, \bullet] = 0 .
\]

Upon characterizing the unitary representations (i.e. those for which \(N^\dagger = N\),
\((a^\dagger)^\dagger = a\)) with spectrum of \(N\) bounded below by their lowest eigenvalue \(n_0\), one can write

\[
a^\dagger |k + n_0 > = \sqrt{k + 1} |k + n_0 + 1 > \quad , \quad a |k + n_0 > = \sqrt{k} |k + n_0 - 1 > \quad ,
N |k + n_0 > = (k + n_0) |k + n_0 > \quad , \quad k \in \mathbb{N} .
\]

The usual Fock space \(\mathcal{F}\) is obtained for \(n_0 = 0\), usually adopting the relation \(N \equiv a^\dagger a\)
(which is just one of the solutions of the equations \([N, a] = -a\), \([N, a^\dagger] = a^\dagger\)):

\[
a^\dagger |n > = \sqrt{n + 1} |n + 1 > \quad , \quad a |n > = \sqrt{n} |n - 1 > \quad ,
N |n > = n |n > \quad , \quad n \in \mathbb{N} .
\]

A related \(\mathbb{Z}_2\)-graded structure will be considered here, starting from the set of three
operators \(S \equiv (a, a^\dagger, H)\) with \(H\) even and \(a\) and \(a^\dagger\) odd. \(S\) is characterized uniquely
by the relations

\[
\{ a, a^\dagger \} = 2H , \quad [H, a] = -a , \quad [H, a^\dagger] = a^\dagger ,
\]

\[2\]
\(i.e.,\) as \(N\) in (1), \(H\) is assumed not to be a function of \(a\) and \(a^\dagger\) and it is a **subset**, not a **sub-algebra**, of the \(\mathbb{Z}_2\)-graded algebra \(osp(1|2)\)\(^3\). Completion of \(S\) to the whole \(osp(1|2)\) in fact requires the introduction of the additional set \(S' \equiv (J^-, J^+)\) in the even sector, such that

\[
\{a^\dagger, a^\dagger\} = 2J^+ , \quad \{a, a\} = 2J^- .
\]

Eqs. (3) and (4) imply the algebra closure:

\[
[J^+, a] = -2a^\dagger , \quad [J^-, a^\dagger] = 2a , \quad [J^+, a^\dagger] = 0 = [J^-, a] , \quad [J^+, J^-] = -4H , \quad [H, J^\pm] = \pm 2J^\pm .
\]

The bosonic sector \(B \equiv (J^-, J^+, \frac{1}{2}H)\) is, as well known, isomorphic to \(su(1,1)\), in the direct sum of the representations \(\kappa = \frac{1}{4}, \frac{3}{4}\)\(^4\).

An explicit analysis shows that the set \(S\) with relations (3) is sufficient to give rise to unitary representations of \(osp(1|2)\) that have the spectrum of \(H\) bounded below, and can be characterized by the lowest non negative eigenvalue of \(H\), say \(h_0\). Explicitly,

\[
a^\dagger |h_0 + 2k + 1 > = \sqrt{2(k+1)} |h_0 + 2k + 2 > , \\
 a^\dagger |h_0 + 2k > = \sqrt{2(k+h_0)} |h_0 + 2k + 1 > , \\
 a |h_0 + 2k + 1 > = \sqrt{2(k + h_0)} |h_0 + 2k > , \\
 a |h_0 + 2k > = \sqrt{2k} |h_0 + 2k - 1 > , \\
 H |h_0 + k > = (h_0 + k) |h_0 + k > , \quad k \in \mathbb{N} ,
\]

where the partition of states in two classes exhibits the existence of supersymmetric doublets.

The main point of our derivation is the fact that eqs.(6), with \(h_0 = \frac{1}{2}\), read

\[
a^\dagger |h > = \sqrt{h + \frac{1}{2}} |h + 1 > , \\
a |h > = \sqrt{h - \frac{1}{2}} |h - 1 > , \\
 H |h > = h |h > , \quad h \in \mathbb{N} + \frac{1}{2} ,
\]

namely they coincide with eqs.(2), provided the identification \(h = n + \frac{1}{2}\) is implemented. This means that the closed subset \(S\) of \(osp(1|2)\) defined by (3) and, by induction, the whole \(osp(1|2)\) shares the representation (2) with the Weyl-Heisenberg algebra \(h(1)\). In other words, the Fock space \(F\) provides a faithful representation for both \(h(1)\) (for \(n_0 = 0\)) and \(osp(1|2)\) (for \(h_0 = \frac{1}{2}\)).
Second quantization is based, essentially, on the relations (2). We suggest that creation and annihilation operators may therefore be, with equal rights, be interpreted as belonging either to $osp(1|2)$ or to $h(1)$. The key point in our argument is the following: as far as one considers the algebra as generated by the defining commutation relations only, any physical interpretation is contained in eqs. (2), and it turns out to be essentially irrelevant whether one selects $osp(1|2)$ or $h(1)$. However, when one deals with many-particle states, the two schemes lead to self-consistent yet mutually unequivalent descriptions. The reason why this may happen is that $h(1)$ and $osp(1|2)$ are Hopf algebras (more precisely, $osp(1|2)$ is a super Hopf algebra). Any Hopf algebra, say $A$, has, among its defining operations, the coproduct $\Delta : A \rightarrow A \otimes A$ (in fact, in the representation considered here, this is necessary, in that it implies that the action of the algebra is well defined on $F \otimes F$ and, by induction, on $F \otimes^n$). For both $h(1)$ and $osp(1|2)$ $\Delta$ is, of course, primitive.

In $h(1)$ one has:

$$\Delta(a) = \frac{1}{\sqrt{2}} (a \otimes 1 + 1 \otimes a) \equiv \frac{1}{\sqrt{2}} (a_1 + a_2)$$
$$\Delta(a^\dagger) = \frac{1}{\sqrt{2}} (a^\dagger \otimes 1 + 1 \otimes a^\dagger) \equiv \frac{1}{\sqrt{2}} (a_1^\dagger + a_2^\dagger)$$
$$\Delta(N) = N \otimes 1 + 1 \otimes N \equiv N_1 + N_2$$
$$\Delta(1) = 1 \otimes 1 \quad . \quad (8)$$

The coalgebra for the superalgebra $osp(1|2)$ looks quite similar:

$$\Delta(a) = a \otimes 1 + 1 \otimes a \equiv a_1 + a_2$$
$$\Delta(a^\dagger) = a^\dagger \otimes 1 + 1 \otimes a^\dagger \equiv a_1^\dagger + a_2^\dagger$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H \equiv H_1 + H_2$$
$$\Delta(1) = 1 \otimes 1 \quad . \quad (9)$$

However, since $a$ and $a^\dagger$ are odd, whereas $H$ is even, we have for $c, d, e, f \in osp(1|2)$, the multiplication law on $F \otimes F$

$$(c \otimes d)(e \otimes f) = (-)^{p(d)p(e)} ce \otimes df \quad , \quad (10)$$

where $p(d)$ and $p(e) \in \mathbb{Z}_2$ are the degrees (i.e. parities) of $d$ and $e$ respectively. On $F \otimes^n$ composition rules are therefore quite different. Let us denote

$$a_j \equiv 1 \otimes 1 \otimes \ldots \otimes 1 \otimes a \otimes 1 \otimes \ldots \otimes 1 \quad ,$$
$$a_j^\dagger \equiv 1 \otimes 1 \otimes \ldots \otimes 1 \otimes a^\dagger \otimes 1 \otimes \ldots \otimes 1 \quad ,$$
$$H_j \equiv 1 \otimes 1 \otimes \ldots \otimes 1 \otimes H \otimes 1 \otimes \ldots \otimes 1 \quad ,$$
$$N_j \equiv 1 \otimes 1 \otimes \ldots \otimes 1 \otimes N \otimes 1 \otimes \ldots \otimes 1 \quad ,$$

where the multiple $\otimes$-products have $n$ factors, in each of which the only element different from the identity 1 is in the $j-th$ position. One has, for $(a, a^\dagger, N, 1)$ in
the customary relations \([a_i, a_j] = 0, [a_i, a_j^\dagger] = \delta_{ij} 1, [N_i, a_j] = -a_i \delta_{ij}\) (plus their hermitian conjugates), while for \((a, a^\dagger, H)\) in \(osp(1|2)\), the (graded) commutation relations are

\[
\{a_i, a_j^\dagger\} = 2H_i \delta_{ij}, \quad [H_i, a_j] = -a_i \delta_{ij}, \quad [H_i, a_j^\dagger] = a_i^\dagger \delta_{ij},
\]

and, of course, \([a_j, a_j] = 0\).

On the Fock basis of \(F^{\otimes n}\) adoption of \(osp(1|2)\) leads to

\[
\begin{align*}
\langle n_1, \ldots, n_{j-1}, n_j, \ldots, n_n | & = (-1)^{s_j} \sqrt{n_j + 1} | n_1, \ldots, n_{j-1}, n_j + 1, \ldots, n_n >, \\
\langle a_j | n_1, \ldots, n_{j-1}, n_j, \ldots, n_n | & = (-1)^{s_j} \sqrt{n_j} | n_1, \ldots, n_{j-1}, n_j - 1, \ldots, n_n >, \\
\langle N_j | n_1, \ldots, n_{j-1}, n_j, \ldots, n_n | & = n_j | n_1, \ldots, n_{j-1}, n_j, \ldots, n_n >,
\end{align*}
\]

where the phases turn out to be exactly those customarily used in textbooks for fermions \(^5\) \((s_j \equiv \sum_{k=1}^{j-1} n_k)\).

It is worth pointing out that eqs. (11) differ from the usual (bosonic) ones only in the choice of phases and, on \(F^{\otimes n}\), imply \([a_i, a_i^\dagger] = 1\), consistently with the standard formulation, which in turn gives \(\{a_i, a_i^\dagger\} = 2H_i (\equiv 2N_i + 1)\). Nevertheless the subtle and important implication here is that, in order to determine the phases of the basis vectors, an order must be imposed \textit{a priori} in the set of indices \(j\)'s, such that

\[
| n_1, \ldots, n_{j-1}, n_j, \ldots, n_n > \equiv \frac{1}{(a_1^\dagger)^{n_1} \cdots (a_j^\dagger)^{n_j} \cdots (a_n^\dagger)^{n_n}} |0 >,
\]

contrary to the standard bosonic theory, where the creation operators commute and can be applied in any order. Of course, the usual properties of the Fock space, such as completeness

\[
\sum_{\{n\}} | n_1, n_2, \ldots > < n_1, n_2, \ldots | = 1,
\]

and the projection operators on the one- and two-particles states

\[
P_1 \equiv \sum_i | 1_i < 1_i |, \quad P_2 \equiv \sum_{i<j} | 1_i, 1_j < 1_i, 1_j | + \sum_i | 2_i < 2_i |,
\]

where

\[
\begin{align*}
| 1_i > & \equiv | n_1 = 0, n_2 = 0, \ldots, n_{i-1} = 0, n_i = 1, n_{i+1} = 0, \ldots, n_n = 0 >, \\
| 2_i > & \equiv | n_1 = 0, n_2 = 0, \ldots, n_{i-1} = 0, n_i = 2, n_{i+1} = 0, \ldots, n_n = 0 >, \\
| 1_i, 1_j > & \equiv | n_1 = 0, n_2 = 0, \ldots, n_i = 1, \ldots, n_j = 1, \ldots n_n = 0 >,
\end{align*}
\]
do not depend on such phases, and persist. $osp(1|2)$ can, in such a way, be utilized to construct $n$-particle states, leading to a scheme non-equivalent to that derivable from $h(1)$, because of the grading of odd operators.

This possibility of using anticommuting operators, without restriction on the occupation numbers $n_j$’s, also casts a new light on the question of how one should introduce statistics into play.

As stressed in the introduction, second quantization is essentially unrelated with statistics\,[6], which is required as a necessary set of rules to represent isomorphically $n$-particle states in the state space given by the $n$-fold tensorization of the single-particle Hilbert space, proper of first quantization. The relevant point in the analysis of this problem performed by Pauli in [6] is that the symmetry with respect to permutation of two particles does not depend on the prescription adopted to build $\mathcal{F}$ from the vacuum (i.e. on the commutation or anticommutation relations of the $a_i$’s and $a_j^\dagger$’s), but it must rather be imposed as an external constraint aiming to guarantee a correct implementation of the above isomorphism.

In such a perspective, let us consider what happens with two bosons. Independently on whether the algebra is graded or not, one has to consider for such a system a symmetric Hilbert space, namely a generic state vector must be symmetric with respect to the exchange of the two particles (this being the feature which qualifies them as boson)

$$|x_1, x_2 >_B \equiv \frac{1}{\sqrt{2}} (|x_1 > |x_2 > + |x_2 > |x_1 >) ,$$

and, by (12),

$$|x_1, x_2 >_B = \sum_{i<j} |1_i, 1_j><1_i, 1_j|x_1, x_2>_B + \sum_i |2_i><2_i|x_1, x_2>_B .$$

Independently on how the states $|1_i, 1_j>$ and $|2_i>$ are constructed from the vacuum, the symmetry is here automatically implemented, in that

$$<1_i, 1_j|x_1, x_2>_B = \frac{1}{\sqrt{2}} (<1_i, 1_j|(x_1, x_2 > + |x_2, x_1 >)$$

$$= \frac{1}{\sqrt{2}}(<1_i|x_1><1_j|x_2> + <1_i|x_2><1_j|x_1>) ,$$

$$<2_i|x_1, x_2>_B = \frac{1}{\sqrt{2}}(<2_i|(x_1, x_2 > + |x_2, x_1 >) = <1_i|x_1><1_i|x_2> ,$$

manifestly have the invariance with respect to interchange of the two particles.

The feature that statistics has no connection with the algebra is further proved by the fact that $osp(1|2)$, as well as $h(1)$, work equally well with fermions. For two fermions we must consider an antisymmetric Hilbert state-space

$$|x_1, x_2 >_F \equiv \frac{1}{\sqrt{2}} (|x_1 > |x_2 > - |x_2 > |x_1 >) ,$$
and, once more independently on the algebra considered, the antisymmetry in the exchange of the two fermions is guaranteed, as well as the Pauli exclusion principle:

\[
\begin{align*}
<1_i, 1_j | x_1, x_2>^F &= \frac{1}{\sqrt{2}} <1_i, 1_j | (|x_1, x_2> - |x_2, x_1>)
\end{align*}
\]

\[
= \frac{1}{\sqrt{2}} ( <1_i|x_1> <1_j|x_2> - <1_i|x_2> <1_j|x_1> ),
\]

\[
<2_i|x_1, x_2>^F = \frac{1}{\sqrt{2}} (|x_1, x_2> - |x_2, x_1>)
\]

\[
= \frac{1}{\sqrt{2}} ( <1_i|x_1> <1_i|x_2> - <1_i|x_2> <1_i|x_1> ) = 0.
\]

(13) and (14) clearly demonstrate the possibility – besides the customary scheme \[5\] – of constructing bosons with graded operators or fermions with even operators.

It should be stressed that our arguments in this paper are quite different from other procedures whereby "ad hoc" constraints are introduced on the variables in order to generate the statistics. An instance of such different approaches are non-linear transformations, along the lines proposed by Gutzwiller’s projection operator method\[7\]. In the fermionic case, a suggestive example is provided by the new creation and annihilation operators defined by

\[
c_j = a_j Q_j, \quad Q_j \equiv \frac{1}{\sqrt{N_j + \frac{1}{2}}} \left(1 - e^{i\pi N_j}\right) = Q_j^\dagger, \quad c_j^\dagger = Q_j a_j^\dagger. \quad (15)
\]

It is straightforward to check that – because of (2) – (15) leads to both the Pauli exclusion principle, \(c_j^2 = 0 = (c_j^\dagger)^2\), and the customary fermionic anticommutation relations \(\{c_i, c_j^\dagger\} = \delta_{ij} 1\), on the subsector of \(\mathcal{F}^\otimes n\) consisting of paired superdoublets \(|\ldots, 2n_j, \ldots>, |\ldots, 2n_j+1, \ldots>\). The usual fermions can therefore be recovered by restriction to \(n_j = 0\). Of course, in the above procedure no reference or use has been done of grading.

Adoption of a graded algebra has also deep bearings on the structures one can induce in the universal enveloping algebra (\(UEA\)). For example, it is usually assumed that the algebra \(su(1, 1)\) can be constructed in the \(UEA\) of \(h(1)\). This is not true: as stressed before \(su(1, 1)\) can be easily obtained from eqs.(3) as the bosonic sector of the superalgebra \(osp(1|2)\), while one has to use \(n_0 = 0\) (i.e. one needs to impose indirectly the \(osp(1|2)\) properties also) to obtain the same result from eqs. (1). Moreover, the grading property (10) plays an essential rôle in obtaining the coalgebra (of course primitive) of \(su(1, 1)\) from the one of \(S\), while it is impossible to obtain the same result from the coalgebra of \(h(1)\).

Indeed, from (5), (8) and (9), we have, for instance, in \(h(1)\): \(\Delta(J^-) = a^2 \otimes 1 + 1 \otimes a^2 + 2a \otimes a\) whereas in \(osp(1|2)\), because of (11), \(\Delta(J^-) = a^2 \otimes 1 + 1 \otimes a^2 \equiv J^- \otimes 1 + 1 \otimes J^-\), as it should, because \(J^-\) is primitive.
This shows that $su(1,1)$ is contained as a full Hopf algebra in the universal envelope of $S$, while only in the common representation (2), the $su(1,1)$ algebra can be considered as realized in the $UEA$ of $h(1)$ (of course, all these can be extended to $su(2)$ by analytical continuation from $su(1,1)$).

We have now to recall that both $h(1)^{[8]}$ and $osp(1|2)^{[9]}$ have quantum deformations and the whole discussion could be easily extended to them. Actually, it should be kept in mind that, in the scheme proposed, no relation links the algebraic features of the creation and annihilation operators to the symmetry of the states. It is, indeed, possible to study systems of particles, both fermions or bosons, by means of either $h_q(1)$ or $osp_q(1|2)$. The Fock space remains always the same, while differences appear in the relations of composed observables with the single-particle ones.

We finally conjecture that in the present approach, there is room for considering objects with more complex symmetry such as anyons.

Acknowledgements
The authors gratefully acknowledge fruitful discussions with F. Iachello.

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