Irreversibility and small-scale generation in 3D turbulent flows

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In three-dimensional turbulent flows energy is supplied at large scales and cascades down to the smallest scales where viscosity dominates. The flux of energy through scales implies the generation of small scales from larger ones, which is the fundamental reason for the irreversibility of the dynamics of turbulent flows. As we showed recently, this irreversibility manifests itself by an asymmetry of the probability distribution of the instantaneous power p of the forces acting on fluid elements. In particular, the third moment of p was found to be negative. Yet, a physical connection between the irreversibility manifested in the distribution of p and the energy flux or small-scale generation in turbulence has not been established. Here, with analytical calculations and support from numerical simulations of fully developed turbulence, we connect the asymmetry in the power distribution, i.e., the negative value of p3, to the generation of small scales, or more precisely, to the amplification (stretching) of vorticity in turbulent flows. Our result is the first step towards a quantitative understanding of the origin of the irreversibility observed at the level of individual Lagrangian trajectories in turbulent flows.

I. INTRODUCTION

The generation of small scales, or large velocity gradients, is one of the most striking physical phenomenon of 3-dimensional (3D) turbulent fluid flows, and is responsible for a flux of energy ε from large to small scales. Remarkably, in the limit of very small viscosity or very large Reynolds number, the third moment of the longitudinal velocity difference between two points separated by a distance x, ⟨Δu(x)3⟩, is related to the energy flux by the relation ⟨Δu(x)3⟩ = −εxx, which is one of the very few exact results in turbulence theory [1]. In elementary terms, two points are more likely to be pushed closer together (repelled) when their relative energy is large (small) [2]. This fundamental asymmetry persists all the way down to very small distances so the third moment of the velocity derivative ∂xu is negative: ⟨(∂xu)3⟩ ≤ 0. In fact, available data from experiments using hot-wire anemometry [3, 4] and from direct numerical simulations (DNS) have led to the conclusion that the normalized third moment of ∂xu, i.e., the skewness, S ∂xu ≡ ⟨(∂xu)3⟩/(⟨(∂xu)2⟩)3/2, is negative and approximately −0.5, with at most a weak dependence on the Reynolds number [3, 4]. In homogeneous isotropic flows, the seminal work of Betchkov [7] shows that the third moment of ∂xu is related to the generation of small scales in turbulence, through amplification of vorticity by vortex stretching.

Because of the existence of an energy flux from large to small scales, turbulence is a non-equilibrium phenomenon, thus intrinsically irreversible. The possibility to probe turbulence by following the motion of individual particles in both numerical and laboratory high-Reynolds-number flows [8–11], leads to new insights on irreversibility and offers new opportunities for quantitatively understanding turbulence [11–13].

Recently, we observed that the energy differences along particle trajectories present an intriguing asymmetry: kinetic energy grows more slowly than it drops along a trajectory [14]. The consequence of this asymmetry is that the third moment of the power p = a·u is negative, where u and a are the velocity and acceleration of the fluid (see [15] for a related discussion). As a possible explanation, one may expect the pressure gradient, which dominates the fluctuations of the power, to provide an explanation for the negative sign of the third moment of p [16–18]. Unexpectedly, however, in 3D, the contribution of the pressure gradient to the third moment of power is very small [16].

Here, we provide a physical relation between the negative third moment of p and the generation of small scales by turbulence, i.e., vortex stretching. In the following, it is convenient to decompose the power as

\[ p = p_L + p_C \]

where \( p_L = u \cdot a_L = u \cdot \partial_t u \) and \( p_C = u \cdot a_C = u \cdot (u \cdot \nabla)u \) are the local and convective parts, respectively. We find that the magnitude of \( p = p_L + p_C \) is much smaller than the magnitudes of its components \( p_L \) and \( p_C \), which implies significant cancellation between \( p_C \) and \( p_L \). On average, the magnitude of \( p_C \) is larger than that of \( p_L \). Note however that the cancellation between \( p_L \) and \( p_C \) does not automatically follow from the well-known cancellation between \( a_L \) and \( a_C \) [19, 21] since \( p_L, p_C \) involve only one component of \( a_L, a_C \). We demonstrate that

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the moments of \( p \), up to the third order, are dominated by the moments of \( p_C \). In particular, the third moment \( \langle p^3 \rangle \) has the same sign as \( \langle p^3_C \rangle \). We show analytically that \( \langle p^3_C \rangle \) is a surrogate for vorticity amplification. This, together with the observation that \( \langle p^3_C \rangle \) determines the sign of \( \langle p^3 \rangle \), leads us to the conclusion that the origin of the negative sign of the third moment \( \langle p^3 \rangle \) comes in fact from small scale generation, thus clearly establishing a relation between the generation of small scales and the observed irreversibility in the flow.

II. NUMERICAL METHODS

A. Direct Numerical Simulation of Navier-Stokes Turbulence

We investigated numerically turbulent flows, obtained by solving directly the Navier–Stokes equations:

\[
\partial_t \mathbf{u}(x,t) + (\mathbf{u}(x,t) \cdot \nabla) \mathbf{u}(x,t) = -\nabla P(x,t) + \nu \nabla^2 \mathbf{u}(x,t) + \mathbf{f}(x,t) \\
\nabla \cdot \mathbf{u}(x,t) = 0
\]

where \( \mathbf{u}(x,t) \) denotes the Eulerian velocity field, \( P \) is the pressure, \( \nu \) is the viscosity, and \( \mathbf{f}(x,t) \) is a forcing term: the mass density is arbitrarily set to unity. Solving \( \mathbf{u}(x,t) \) by solving directly the Navier–Stokes equations:

\[
\text{such a way that the injection rate of energy, } \varepsilon_i, \text{ remains constant:}
\]

\[
f_k = \varepsilon_i \frac{u_k}{\sum_{|k| \leq K_f} |u_k|^2} \text{ if } |k| \leq K_f ,
\]

with \( K_f = 1.5 \). In the code units, the energy injection rate \( \varepsilon_i \) has been set to \( \varepsilon_i = 10^{-3} \). Note that in stationary turbulent flows, the energy injection rate equals the energy dissipation rate, \( \varepsilon_i = \varepsilon \).

The code is fully dealiased, using the 2/3-rule method \[23\]. We have chosen two different resolutions, corresponding to the highest resolved wavenumber of \( k_{\text{max}} = 256 \) and \( k_{\text{max}} = 384 \) (effectively equivalent to 768 and 1152 grid points in each spatial direction), with the corresponding values of the viscosity \( \nu = 1.6 \times 10^{-4} \) and \( 9.0 \times 10^{-5} \), respectively. With these values, the Kolmogorov scale \( \eta = (\nu^3/\varepsilon)^{1/4} \) is such that the product \( k_{\text{max}} \times \eta \) is very close to 2 in both cases, ensuring adequate spatial resolution. The corresponding Reynolds numbers are \( R_\lambda = 193, 275, \) and 275, respectively.

Once expressed in terms of spatial modes, Eq. (2) reduces to a large set of ordinary differential equations, which were integrated using the second-order Adams-Bashforth scheme. The time step \( \delta t \) has been chosen so that the Courant number \( Co = u_{\text{rms}} k_{\text{max}} \delta t \leq 0.1 \), where \( u_{\text{rms}} \) is the root mean square value of one component of velocity.

We also used additional numerical simulation data at \( R_\lambda = 430 \) from the Turbulence Database of the Johns Hopkins University. The flow is documented in \[24\]. We computed the statistics presented here with at the minimum 2 \times 10^5 points.

| \( R_\lambda \) | 193 | 275 | 430 |
|----------------|-----|-----|-----|
| \( \langle p^2 \rangle/\varepsilon^2 \) | \( 3.83 \times 10^2 \) | \( 7.36 \times 10^2 \) | \( 1.32 \times 10^3 \) |
| \( \langle p^2_C \rangle/\varepsilon^2 \) | \( 2.15 \times 10^3 \) | \( 5.00 \times 10^3 \) | \( 1.20 \times 10^4 \) |
| \( \langle p^2_L \rangle/\varepsilon^2 \) | \( 1.78 \times 10^4 \) | \( 4.25 \times 10^4 \) | \( 1.07 \times 10^5 \) |
| \( -\langle p p_C \rangle/\varepsilon^2 \) | \( 1.77 \times 10^4 \) | \( 4.26 \times 10^4 \) | \( 1.07 \times 10^5 \) |
| \( 15 \langle p^2_L \rangle/(\varepsilon^2 R_\lambda) \) | 0.87 | 0.99 | 0.96 |
| \( \beta \) | 0.83 | 0.86 | 0.90 |

| \( R_\lambda \) | 193 | 275 | 430 |
|----------------|-----|-----|-----|
| \( \langle p^3 \rangle/\varepsilon^3 \) | \( 3.87 \times 10^2 \) | \( 1.23 \times 10^2 \) | \( 3.21 \times 10^2 \) |
| \( -\langle p^3_C \rangle/\varepsilon^3 \) | \( 5.39 \times 10^3 \) | \( 2.40 \times 10^4 \) | \( 1.00 \times 10^5 \) |
| \( \langle p^3_L \rangle/\varepsilon^3 \) | \( 4.54 \times 10^4 \) | \( 2.05 \times 10^5 \) | \( 8.99 \times 10^5 \) |
| \( \langle p p^2_C \rangle/\varepsilon^3 \) | \( 4.02 \times 10^4 \) | \( 1.84 \times 10^5 \) | \( 8.29 \times 10^5 \) |
| \( \langle p^2 L \rangle/\langle p^2 \rangle \) | \( 3.44 \times 10^4 \) | \( 1.63 \times 10^5 \) | \( 7.63 \times 10^5 \) |
| \( \zeta = (\langle p^2_L \rangle/\langle p^2 \rangle) \) | 0.108 | 0.088 | 0.066 |
| \( 1 - \beta \) | 0.17 | 0.14 | 0.11 |
| \( \langle p^3 \rangle/(p^2) \) | 0.072 | 0.051 | 0.032 |
| \( 1 - \beta - \zeta \) | 0.061 | 0.052 | 0.037 |
| \( -\langle p^3_L \rangle/\langle p^3 \rangle \) | 0.64 | 0.68 | 0.76 |
| \( \beta - 2\zeta \) | 0.62 | 0.69 | 0.77 |
| \( \langle p p^2_C \rangle/\langle p^2 \rangle \) | 0.75 | 0.77 | 0.83 |
| \( \beta - \zeta \) | 0.72 | 0.77 | 0.83 |
| \( -\langle p^3 L \rangle/\langle p^3 \rangle \) | 0.84 | 0.85 | 0.90 |
| \( \beta \) | 0.83 | 0.86 | 0.90 |

TABLE I. Second moments of the distributions of \( p/\varepsilon \), \( p_C/\varepsilon \) and \( p_L/\varepsilon \) at the three Reynolds numbers studied in this article. The correlation coefficient between \( p_C \) and \( p_L \) is approaching \( -1 \) as Reynolds number increases. The values of \( \beta \) are measured from fitting the conditional averages \( \langle p_L | p_C \rangle = -\beta p_C \).

TABLE II. Third moments of the distributions of \( p/\varepsilon \), \( p_C/\varepsilon \) and \( p_L/\varepsilon \) at the three Reynolds numbers studied in this article.
III. THEORETICAL BACKGROUND

A. Elementary relations

To investigate the moments of \( p, p_C \) and \( p_L \), we first note that \( p_C \) reduces to a simple form that is particularly useful, namely:

\[
p_C = u \cdot (u \cdot \nabla)u = u \cdot S \cdot u ,
\]

where the rate of strain tensor \( S \) is the symmetric part of the velocity gradient tensor \( \nabla u \): \( S = (\nabla u + (\nabla u)^T)/2 \) or \( S_{ij} = (\partial_i u_j + \partial_j u_i)/2 \). Geometrically, the straining motion decomposes into a superposition of compression or stretching along three orthogonal directions, denoted by \( e_i \), with three straining rates, \( \lambda_i \). The vectors \( e_i \) and the straining rates \( \lambda_i \) are the eigenvectors and eigenvalues of \( S \). A positive (respectively negative) value of \( \lambda_i \) corresponds to stretching (respectively compression) in the direction \( e_i \). Volume conservation (incompressibility) imposes that \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \), i.e., the amount of stretching and compression along the three directions \( e_i \) sums up to 0.

Equation (5) shows that in a steady (frozen) flow, the kinetic energy of a fluid element changes only through the action of the rate of strain. The antisymmetric part of the velocity gradient tensor expresses the local rotation \( \omega \) in the flow, and is characterized by the vorticity \( \omega \). The expression for the amplification (stretching) of vorticity in the flow is given by \( \langle \omega \cdot S \cdot \omega \rangle \). In a statistically homogeneous flow, the following identity holds: \( \langle \omega \cdot S \cdot \omega \rangle = -\frac{4}{3} \langle \text{tr}(S^3) \rangle \). Last, we note that in a homogeneous isotropic flow, the second and third moments of \( \partial_x u_x \) can be simply expressed in terms of the moments of \( \text{tr}(S^2) \) and \( \text{tr}(S^3) \): \( \langle (\partial_x u_x)^2 \rangle = \frac{2}{3} \langle \text{tr}(S^2) \rangle \), and \( \langle (\partial_x u_x)^3 \rangle = \frac{4}{15} \langle \text{tr}(S^3) \rangle \).

B. Decomposition of power \( p_C \): order of magnitudes

The magnitudes of the fluctuations of the convective and local components of \( p \) may be estimated from simple dimensional arguments: \( |p_C| \sim |p_L| \sim U^2/\tau_K \), where \( U \) is the typical size of the velocity fluctuations, and \( \tau_K \) is the fastest time scales of the turbulent eddies. Using the known relation \( \tau_K \sim (U^2/\varepsilon)/R \), one finds \( |p_C| \sim |p_L| \sim \varepsilon R \), where \( R \) is the Reynolds number based on the Taylor microscale, and characterizes the intensity of turbulence. The growth of the variances of \( p_C/\varepsilon \) and \( p_L/\varepsilon \) as \( R^3 \), predicted by this simple dimensional argument, is found to be consistent with our DNS results, see Table I. This result sharply contrasts with the fact that the variance of \( p \) is known to grow more slowly with the Reynolds number, as \( R^{4/3} \). This difference in the observed scalings as a function of the Reynolds number is due to a very strong cancellation between \( p_L \) and \( p_C \), see Table I. We observe that the magnitudes of the third moments \( \langle p_C^3 \rangle \), with \( m+n = 3 \), are found to increase with \( m \), see Table III, signaling that the contribution of \( p_C \) to the third moments is more significant than that of \( p_L \). In fact, we will show, the sign of \( \langle p^3 \rangle \) is dominated by \( \langle p_C^3 \rangle \).

Although the cancellation between \( p_C \) and \( p_L \) is reminiscent of the well-documented cancellation between \( a_C \) and \( a_L \), we stress that it cannot be deduced from the results of [20, 21]. In fact, \( p_C \) and \( p_L \) involve the projections along the direction of the velocity \( u \), of \( a_C \) and \( a_L \), respectively. Our results therefore show that the cancellation between \( a_C \) and \( a_L \) affects their components along the velocity direction, which does not result automatically from [20, 21].

In the following subsections, we begin by expressing \( \langle p_C^3 \rangle \) in terms of vortex stretching, before establishing the prevalence of \( p_C \) on \( \langle p^3 \rangle \).

IV. RESULTS

A. Vortex stretching and moments of \( p_C \)

It is convenient to express \( p_C \) (Eq. 5) by projecting the velocity \( u \) and the rate of strain \( S \) in the basis of the three perpendicular unit vectors \( e_i \) characterizing the straining motion. In this basis, the velocity \( u \) is decomposed as: \( u = \sum_{i=1}^3 u_i e_i \), where \( u_i = u \cdot e_i \) is the coordinate of the velocity \( u \) along the direction \( e_i \); and the rate of strain tensor is expressed as \( S = \sum_{i=1}^3 \lambda_i e_i e_i \). Denoting \( \hat{x}_i \) the cosines of the angles between the velocity \( u \) and the unit vectors \( e_i \): \( \hat{x}_i \equiv u \cdot e_i/|u| = u_i/|u| \), the expression of \( p_C \) reduces to:

\[
p_C = \sum_{i=1}^3 \lambda_i u_i^2 = u^2 \sum_{i=1}^3 \lambda_i \hat{x}_i^2 . \tag{6}
\]

In a turbulent velocity field, small wave numbers (or large scales) provide the main contribution to the velocity field, \( u \), whereas the rate of strain \( S \) is determined by the large wave numbers (or small scales). The two fields \( u \) and \( S \) are therefore expected to be only weakly correlated. Let us now assume that \( S \) and \( u \) are uncorrelated. This approximation implies that the three cosines, \( \hat{x}_i \), are uniformly distributed between \(-1 \) and \( 1 \). Geometrically, the three cosines are the coordinates of a point that is uniformly distributed on the unit sphere in 3D. This assumption allows us to compute the averages necessary to evaluate explicitly the third moment of \( p_C \).

Namely, Eq. (6) leads to:

\[
\langle p_C^3 \rangle = \langle \left( \sum_{i=1}^3 \lambda_i \hat{x}_i^2 \right)^3 \rangle = \langle u^6 \rangle \left( \sum_{i=1}^3 \lambda_i \hat{x}_i^2 \right)^3 . \tag{7}
\]

The assumption that \( u \) and \( S \) are uncorrelated also implies that all the cosines \( \hat{x}_i \) are independent of the eigenvalues of \( S \). As a consequence,
\[ \langle x_i^m \hat{x}_i^n \rangle = \langle \lambda_i^m \rangle \langle x_i^n \rangle \] for any \( m \) and \( n \). Using the observation that \( (\hat{x}_1, \hat{x}_2, \hat{x}_3) \) represents the coordinates of a point that is uniformly distributed on the unit sphere, which gives the symmetry relations such as \( \langle \hat{x}_1^6 \rangle = \langle \hat{x}_2^6 \rangle = \langle \hat{x}_3^6 \rangle \), \( \langle \hat{x}_1^2 \hat{x}_2 \rangle = \langle \hat{x}_3 \hat{x}_1^2 \rangle \), etc, we therefore obtain

\[
\left\langle \left( \sum_{i=1}^{3} \lambda_i \hat{x}_i^2 \right)^3 \right\rangle = \left\langle \hat{x}_1^6 \right\rangle \left\langle \sum_{i=1}^{3} \lambda_i^3 \right\rangle \\
+ 3 \left\langle \hat{x}_1^4 \hat{x}_2^2 \right\rangle \left\langle \sum_{i,j=1, i \neq j}^{3} \lambda_i^2 \lambda_j \right\rangle \\
+ 6 \left\langle \hat{x}_1^2 \hat{x}_2^4 \right\rangle \langle \lambda_1 \lambda_2 \lambda_3 \rangle. \tag{8}
\]

The averages of the products of \( \hat{x}_i \) in Eq. (8) can be calculated by using elementary geometrical considerations \[7\] and the results are:

\[
\langle \hat{x}_1^6 \rangle = \frac{1}{7}, \quad \langle \hat{x}_1^4 \hat{x}_2^2 \rangle = \frac{1}{35}, \quad \langle \hat{x}_1^2 \hat{x}_2^4 \rangle = \frac{1}{105} \tag{9}
\]

Substituting Eqs. (9) and (8) into Eq. (7), and using the incompressibility of the flow, \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \), leads to the following expression for the third moment of \( p_C \):

\[
\langle p_C^3 \rangle = \frac{8}{35} \langle |u|^6 \rangle \langle \lambda_1 \lambda_2 \lambda_3 \rangle. \tag{10}
\]

Using the relation \( \langle \lambda_1 \lambda_2 \lambda_3 \rangle = (1/3) (\text{tr}(S^3)) = -(1/4) \langle \omega \cdot S \cdot \omega \rangle \) \[10\] \[11\], one finally obtains:

\[
\langle p_C^3 \rangle = \frac{8}{35} \langle |u|^6 \rangle \langle \text{tr}(S^3) \rangle = -\frac{2}{35} \langle |u|^6 \rangle \langle \omega \cdot S \cdot \omega \rangle. \tag{11}
\]

Thus, Equation (11) relates the third moment of \( p_C \) to vortex stretching in such a way that positive vortex stretching \( \langle \omega \cdot S \cdot \omega \rangle > 0 \) gives rise to a negative value of \( \langle p_C^3 \rangle \).

Furthermore, many experimental and numerical studies show that the probability distributions of individual components of velocity \( u \) are close to Gaussian, with small deviations that can be quantitatively explained (see e.g., \[23\] \[24\]). Assuming a Gaussian distribution of \( u \) allows us to express the 6th moment of velocity in Eq. (11) in terms of the velocity variance \( \langle u^2 \rangle \). Using other known identities, in particular concerning the relation between \( \langle \text{tr}(S^3) \rangle \) and the skewness of the velocity derivative \( S_{u_{x_{u_y}}} \), as explained in Appendix A, Eq. (11) can be written as:

\[
\langle p_C^3 \rangle = \frac{7}{225} S_{u_{x_{u_y}}} R_\lambda^3 \varepsilon^3. \tag{12}
\]

The weak dependence of the velocity derivative skewness on the Reynolds number, \( S_{u_{x_{u_y}}} \propto R_\lambda^4 \) \[11\] \[12\] suggests a small correction to the simple order of magnitude analysis for the third moment: \( \langle p_C^3 \rangle \propto R_\lambda^{3+\delta} \) with \( \delta \approx 0.1 \).

The assumptions of lack of correlation between \( u \) and \( S \), and of a Gaussian distribution of the velocity \( u \), also lead to an exact determination of the variance of \( p_C \): \( \langle p_C^2 \rangle = \frac{1}{15} R_\lambda^4 \varepsilon^2 \), see Appendix A. This expression for the second moments of \( p_C \) provides further justification for the dimensional estimate of the variance of \( p_C \), and is found to be in very good agreement with our DNS results (see Table II).

Having established the relation between the third moment \( \langle p_C^3 \rangle \) and vortex stretching, we now establish that the third moment \( \langle p^3 \rangle \) is dominated by \( \langle p_C^3 \rangle \). To this end, we first consider the cancellation between \( p_C \) and \( p_L \).

![FIG. 1. The joint probability density function (PDF) between \( p_C/\varepsilon \) (horizontal) and \( p_L/\varepsilon \) (vertical) at \( R_\lambda = 275 \), color-coded in a logarithmic scale (see color-bar). Equal-probability contours, separated by factors of 10, are shown. The PDF is concentrated close to the \( p_C + p_L = 0 \) line, indicating that the two quantities \( p_C \) and \( p_L \) are nearly anti-correlated with each other. The black dashed line shows \( (p_C/p_L)/\varepsilon \), which is approximately \( -0.86 \times p_C/\varepsilon \). The white dashed line shows \( (p_C/p_L)/\varepsilon \), which is approximately \( -p_L/\varepsilon \).]

**B. Cancellation between \( p_L \) and \( p_C \)**

We note that in homogeneous and stationary flows, the first moments of \( p \), \( p_C \) and \( p_L \) are all exactly 0. Table I shows that the correlation coefficient between \( p_L \) and \( p_C \): \( \langle p_C p_L \rangle / (\langle p_C^2 \rangle \langle p_L^2 \rangle)^{1/2} \), is approximately \( -0.9 \) and seems to approach \(-1\) as the Reynolds number increases. This strong anti-correlation results in significant cancellation between \( p_C \) and \( p_L \), so the variance of \( p \) is much smaller than those of \( p_C \) and \( p_L \).

Although the range of values of \( R_\lambda \) covered by the present study is not sufficient to reach unambiguous conclusions, our results are generally consistent with the expected scalings: \( \langle p_L^2 \rangle \sim \langle p_C^2 \rangle \sim R_\lambda^2 \), and \( \langle p^2 \rangle \sim R_\lambda^{4/3} \) \[14\].

Further insight into the strong cancellation between \( p_C \) and \( p_L \) can be gained by studying the joint probability density function (PDF) of \( p_C \) and \( p_L \), shown in Fig. I for our flow at \( R_\lambda = 275 \). Fig. I clearly indicates that with a
high probability, the values of $p_C$ and $p_L$ are concentrated close to the line $p_C + p_L = 0$, thus implying a significant cancellation between the two quantities. The observed tendency of the correlation coefficient between $p_C$ and $p_L$ to approach $-1$ as the Reynolds number increases implies that the joint PDF of $p_C$ and $p_L$ becomes increasingly concentrated around the line $p_L + p_C = 0$ at higher Reynolds numbers. In all our numerical simulations, we find an approximately linear relation between the conditional average $\langle p_L | p_C \rangle$ and $p_C$ (shown as the black dashed line in Fig. 1): $\langle p_L | p_C \rangle \approx -\beta(R_\lambda)p_C$, where the dimensionless coefficient $\beta(R_\lambda)$ depends weakly on $R_\lambda$. In agreement with the observed tendency of $p_L$ and $p_C$ to become increasingly anti-correlated as $R_\lambda$ increases, we find that $\beta(R_\lambda)$ slightly increases with the Reynolds number, see Table II. This implies that $\langle p | p_C \rangle \approx (1-\beta)p_C$, where the coefficient $1-\beta$ decreases as $R_\lambda$ increases, from $\approx 0.17$ at $R_\lambda = 193$ to $\approx 0.10$ at $R_\lambda = 430$. We also observe that the average of $p_C$ conditioned on $p_L$, shown as the white dashed line in Fig. 1, is almost exactly equal to $-p_L$, which implies that $\langle p | p_L \rangle \approx 0$.

Fig. 2 shows the joint PDFs of $p_C$ and $p$ (a) and of $p_L$ and $p$ (b). The conditional averages $\langle p | p_C \rangle$ and $\langle p_L | p \rangle$ are shown as black dashed lines, whereas the conditional averages $\langle p_C | p \rangle$ and $\langle p_L | p \rangle$ are shown as white dashed lines. The conditional averages of $p_C$ and $p_L$ on $p$ have the particularly simple forms: $\langle p_C | p \rangle \approx p$ and $\langle p_L | p \rangle \approx 0$. In addition, the joint PDF of $p$ and $p_L$ is almost symmetrical to both $p = 0$ and $p_L = 0$. The power $p$ is therefore well correlated with $p_C$, but not with $p_L$.

C. Prevalence of $p_C$ on the moments of $p$

1. General assumptions

The prevalence of $p_C$ on the statistical properties of $p$ shown by our numerical results leads to the conclusion that the second and third moment of $p$ are expressible in terms of the corresponding moments of $p_C$. To justify this claim, we use the two following results.

A The numerical results shown in Fig. 1 demonstrate that at a fixed value of $p_C$, $p_L | p_C$ fluctuates around the mean value $\langle p_L | p_C \rangle \approx -\beta p_C$. This immediately implies the following relations:

$$\langle p_L p_C \rangle = -\beta \langle p_C^2 \rangle$$

and

$$\langle p_L p_C^2 \rangle = -\beta \langle p_C^3 \rangle,$$

which can be easily justified by writing $p_L$ conditioned on a value of $p_C$ as:

$$p_L | p_C = -\beta p_C + \xi | p_C,$$

where $\xi | p_C$ is a random variable with zero mean and its distribution depends on $p_C$. Eq. 13 is found to be numerically extremely well satisfied, see Table III, as a direct consequence of the quality of the linear dependence between $\langle p_L | p_C \rangle$ and $p_C$.

B The lack of correlation between $p$ and $p_L$, demonstrated in Fig. 2 and manifested by the two relations $\langle p_L | p \rangle \approx 0$ and $\langle p | p_L \rangle \approx 0$, implies that:

$$\langle p p_L \rangle \approx \langle p^2 p_L \rangle \approx \langle pp_L^2 \rangle \approx 0.$$ 

The equalities shown in Eq. 15 are only approximate. In the following, we explore the consequences of the independence between $p$ and $p_L$ by assuming for now that...
these equalities are exactly satisfied, leaving for later a discussion of the errors made.

As we show below the approximations \( A \) and \( B \) above lead to a very accurate prediction of all the second moments of \( pC, pL \) and \( p \), in terms of \( \langle p^2 \rangle \) and \( \beta \). The predictions concerning the third moments, however, are not as accurate as those concerning the second moments, as a consequence of quantitative deviations from the symmetry assumption \( B \).

2. Second moments

Using Eq. (13), \( \langle pLpc \rangle = -\beta \langle p^2 \rangle \), and Eq. (15), \( \langle ppL \rangle = 0 \), we determine the second moment of \( pL \) as a function of \( \langle p^2 \rangle \): 

\[
\langle p^2 \rangle = \frac{1}{\beta} \langle p^2 \rangle.
\]

(16)

We find that the condition of independence \( \langle ppL \rangle = 0 \) is very well satisfied, which implies that Eq. (15) is numerically very accurately satisfied (see Appendix B and Table IV).

3. Third moments

The results from Eq. (12), showing that \( \langle p^3 \rangle \approx -\varepsilon^3 R^3 \), together with the observation that \( \langle p^3 \rangle \approx -\varepsilon^2 R^2 \) [14], also point to a strong cancellation between \( pC \) and \( pL \) in the third moment \( \langle p^3 \rangle \). To relate the properties of the third moments of \( p \) to \( \langle p^3 \rangle \), we begin by noting that Eq. (15) leads to the following expressions for the third order moments: 

\( \langle p^3 \rangle = -\beta \langle p^3 \rangle \) and \( \langle pLpC \rangle = \beta \langle p^3 \rangle \),

and hence to the expression \( \langle p^3 \rangle = (1 - \beta) \langle p^3 \rangle \). These expressions predict simple relations between the various moments \( \langle p^3 \rangle \) with \( m + n = 3 \) and \( \langle p^2 \rangle \), and lead to the correct sign of \( \langle p^3 \rangle \), thus justifying our claim that the assumption of independence between \( p \) and \( pL \) imposes that the sign of \( \langle p^3 \rangle \) is given by \( \langle p^3 \rangle \).

The expressions obtained above, however, are quantitatively not accurate. The reason is that while \( \langle pLp^2 \rangle \) is found to be very small (of the order of 1% of \( \langle p^3 \rangle \)), the numerical values of \( \langle p^2 \rangle \) are found to be much larger, of the order of 10% of \( \langle p^3 \rangle \). The small, but significant error in \( \langle pLp^2 \rangle = 0 \) therefore leads to a significant reduction of the numerical value of \( \langle p^3 \rangle \), consistent with the numerical values shown in Table I. To take the effect of non-zero \( \langle p^2 \rangle \) into account, we denote \( \zeta = \langle p^2 \rangle \rangle \langle p^3 \rangle \), where \( \zeta \) is a positive number of order \( \sim 0.1 \) (see Table I), and decreases when \( R \) increases. This then leads to 

\( \langle p^3 \rangle = (1 - \beta - \zeta) \langle p^3 \rangle \) and \( \langle pL \rangle = -(1 - 2\zeta) \langle p^3 \rangle \),

and consequently:

\[
\langle p^3 \rangle = (1 - \beta - \zeta) \langle p^3 \rangle.
\]

(17)

Using Eq. (17) and the relation between \( \langle p^2 \rangle \) and vortex stretching, Eq. (11), we obtain:

\[
\langle p^3 \rangle = -\frac{2}{35}(1 - \beta - \zeta) \langle |u|^3 \rangle \langle \omega \cdot S \cdot \omega \rangle,
\]

(18)

which establishes a quantitative relation between the time irreversibility, as measured by \( \langle p^3 \rangle \), and vortex stretching, a small-scale generation mechanism in 3D turbulence.

We note that Eq. (10), together with the observed scaling \( \langle p^2 \rangle \propto R^{1/3} \) and \( \langle p^3 \rangle \propto R^3 \), suggests that \( (1 - \beta) \propto R^{-2/3} \). Similarly, the dependence \( \langle p^3 \rangle \propto R^3 \), together with the observation of [14] that \( \langle p^3 \rangle \approx -\varepsilon^3 R^3 \), imply, using Eq. (17), that \( (1 - \beta - \zeta) \propto R^{-1} \) or \( 1 - \zeta/(1 - \beta) \propto R^{-1/3} \). These are consistent with the values obtained numerically. As shown in Table II the value of \( 1 - \zeta/(1 - \beta) \) decreases slightly, from 0.36 to 0.34, when the Reynolds number increases from \( R = 193 \) to 430.

The results presented here thus show that, while the statement of independence of \( p \) and \( pL \) is merely an enticing approximation, taking quantitatively into account the deviations from Eq. (15) does not affect our main conclusion: the third moment of \( p \) is controlled by the third moment of \( pC \).

D. Lack of correlation between \( u \) and \( S \)

We return here briefly to discuss the essential assumption that \( u \) and \( S \) are uncorrelated. Specifically, we examine in this subsection the correlation between the angles of \( u \) and the eigenvectors \( e_i \) and between the magnitude of \( u \) and the eigenvalues \( \lambda_i \). In the following, the values \( \lambda_i \) are sorted in decreasing order: \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \).
shows the PDFs of \( \lambda_i | u^2 | \), made dimensionless by \( \tau = 1/(2\langle \text{S}^2 \rangle)^{1/2} = 1/(2 \sum \lambda_i^2) \). The thin lines with the same color are the corresponding unconditional averages \( \langle \lambda_i \rangle \), all at \( R_\lambda = 275 \). For small values of \( u^2 \), the conditional averages are nearly the same as the corresponding unconditional averages, consistent with the assumption that \( u \) and \( S \) are uncorrelated. For large \( u^2 \), the magnitudes of the conditional averages increase with \( u^2 \), indicating a deviation from the assumption. Note that because the probability of having large values of \( u^2 \) decreases very rapidly with \( u^2 \), the increases of \( \langle \lambda_i | u^2 | \rangle \) at large \( u^2 \) have only very small effects on the unconditional averages \( \lambda_i \).

Fig. 4 shows the PDFs of \( |\hat{x}_i| = |e_i \cdot e_u| \), the absolute value of the cosine of the angle between the eigenvector \( e_u \) and the unit vector in the direction of the velocity \( e_u = u/|u| \) (the sign of this cosine is immaterial) at \( R_\lambda = 275 \). A complete lack of correlation between \( u \) and \( S \) implies that the PDFs of \( |e_i \cdot e_u| \) are constant and equal to 1. Fig. 4 shows that this is close to be true. Namely, the probability of alignment between \( e_i \) and \( u \), i.e., of \( |e_u \cdot e_i| \) being close to 1, is slightly reduced. On the contrary, the probability of alignment between \( e_u \) and \( e_3 \) is slightly increased. The deviations observed numerically are weak, less than \( \sim 10 \% \), compared to the uniform distribution. The cosine between \( e_u \) and \( e_3 \), is very close to being uniformly distributed. The nearly uniform PDF of \( |e_i \cdot e_u| \) indicate that the assumption that \( S \) and \( u \) are uncorrelated, explicitly used in the determination of \( \langle \lambda^2_3 \rangle \), provides a very good first-order approximation.

The assumption that \( u \) and \( S \) are uncorrelated also implies that the conditional averages of the properties of \( S \) should be independent of the magnitude of \( u \).

Fig. 4 shows that the dependence of the conditional average of the eigenvalues of \( S \) on \( u^2 \), \( \langle \lambda_i | u^2 | \rangle \), is weak. Systematic deviations are visible at large values of \( u^2 \), where the magnitudes of the averaged conditional eigenvalues are larger. The probability of large values of \( u^2 \), however, drops very rapidly when \( u^2 \) increases \( \sim 25 \% \), so the effect of this weak dependence of \( \lambda_3 \) on \( u^2 \) has only a small effect on the low-order moments of \( p_{C} \) studied here.

In summary, the results presented here and in the Appendix C show that the assumption of a lack of correlation between \( u \) and \( S \) provides a very good first-order approximation to describe the third moment of \( p_{C} \).

V. DISCUSSION AND CONCLUSION

Our work, aimed at understanding the third moment of the power \( p \) acting on fluid particles \( \langle p^3 \rangle \approx -C^3 R^2 \), and its implication for the physics of turbulent flows \( \langle \omega \cdot S \rangle \), rests on decomposing \( p \) into two parts: a local part, \( p_L = u \cdot \partial_t u \), induced by the change of the kinetic energy at a fixed spatial point, and a convective part, \( p_C = u \cdot \nabla (u^2/2) \), due to the change in kinetic energy along particle trajectories, assuming the velocity field is frozen. We observe that the two terms \( p_C \) and \( p_L \) cancel each other to a large extent, resulting in a much smaller variance of \( p \) compared to those of either \( p_C \) or \( p_L \). This cancellation may be qualitatively explained by invoking a fast sweeping of the small scales of the flow by the large scales \( R \). In physical words, kinetic energy along particle trajectories, is mostly carried (swept) by the flow, and changes far less than it would change by keeping the flow fixed, or by varying the flow with the same position in time. This fact has been documented in a slightly different context \( 31 \). Our results provide a quantitative characterization of how much sweeping reduce the individual contributions of \( p_L \) and \( p_C \).

One of the two main results of our work is that the third moment of \( p_C \), expressed in terms of the rate of strain, \( S \), and the velocity, \( u \), as: \( p_C = u \cdot S \cdot u \), can be exactly determined, by using the physically justified approximation that \( u \) and \( S \) are uncorrelated. Remarkably, we find that \( \langle p^3_C \rangle \) is directly related to vortex stretching, \( \langle \omega \cdot S \cdot \omega \rangle \). In particular, the negative sign of \( \langle p^3_C \rangle \) originates from the positive sign of the vortex stretching, due to small-scale generation by turbulence. This observation provides the first basis for our claim that the third moment of \( p \) is related to the generation of small scales in 3D turbulent flows.

The other main observation of our work is that, despite the strong cancellation between \( p_L \) and \( p_C \), the power \( p \) correlates with \( p_C \), but not with \( p_L \), as revealed by the nearly vanishing conditional averages \( \langle p | p_L \rangle \approx 0 \) and \( \langle p_L | p \rangle \approx 0 \). Assuming these conditional averages are exactly zero leads to a simple relation between \( \langle p^3 \rangle \) and \( \langle p^2_C \rangle \). The (weak) corrections to this simple assumption modify only quantitatively the results.

Taken together, these two observations, namely that \( \langle p^3 \rangle \) is controlled by \( \langle p^2_C \rangle \) and that \( \langle p^2_C \rangle \) is directly linked to vortex stretching, \( \langle \omega \cdot S \cdot \omega \rangle > 0 \), allow us to establish a relation between the third moment of power, \( \langle p^3 \rangle \), and vortex stretching. Thus, the recently observed manifestation of irreversibility in studying the statistics of individual Lagrangian trajectories can be understood as resulting from small-scale generation in 3D turbulent flow.
For lack of essential information concerning the quantities investigated here, our work rests on several assumptions supported by numerical observations. The well-known fact that velocity, $\mathbf{u}$, and the rate of strain, $S$, are dominated by large- and small-scales, respectively, makes it plausible that these two quantities are mostly uncorrelated. Our numerical results confirm this expectation. Although small, and of little relevance for the low-order moments studied here, the deviations observed suggest an interesting structure, which would be worth elucidating. The observation that $p$ and $p_L$ are not correlated, in the sense that the conditional averages $\langle p\rangle_{pL}$ and $\langle pL | p \rangle$ are both very close to zero, rests only on numerical observations, and requires a proper explanation. Understanding and quantifying the weakness of the correlation between $p$ and $p_L$ in 3D turbulent flows may provide important hints not only on higher moments of $p$, but more importantly, on the structure of the flow itself.

We note that studying the cancellation between $p_C$ and $p_L$ by directly focusing on the effect of the pressure gradient, $-\mathbf{u} \cdot \nabla P$ is likely to lead to satisfactory results when studying the second moments of $p$, as the pressure term has been documented to provide the largest contribution to the variance $\langle p^2 \rangle$ [16]. In 3D, however, the third moment $\langle (\mathbf{u} \cdot \nabla P)^3 \rangle$ has been shown to contribute negligibly to $\langle p^3 \rangle$, whose understanding would require an investigation of other correlations [16].

Finally, the arguments provided here to explain the negative third moments of power fluctuations of particles in 3D turbulent flows should not be applied to 2D turbulence, in which the third moment $\langle p \rangle_{pL}$ is also negative, and grows with a similar power of the Reynolds number [11], but the amplification of large velocity gradients is due to entirely different physical processes [32]. Still, one may expect that the manifestations of irreversibility, in 2D turbulence as well as in a broad class of non-equilibrium systems, to be fundamentally related to a flux in the system.

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**APPENDIX A: ALTERNATIVE EXPRESSIONS OF THE THIRD MOMENT OF $p_C$**

In many experiments and numerical simulations, the probability distributions of individual components of velocity $\mathbf{u}$ are close to Gaussian [29]. That observation allows us to estimate explicitly the $6^{th}$ moment of $|\mathbf{u}|$ in terms of the second moment:

$$\langle |\mathbf{u}|^4 \rangle = \frac{5}{3} \langle |\mathbf{u}|^2 \rangle^2, \quad \langle |\mathbf{u}|^6 \rangle = \frac{35}{9} \langle |\mathbf{u}|^2 \rangle^3,$$

(19)

which, when substituted into Eq. (11), gives an expression for $\langle p_C^3 \rangle$ as:

$$\langle p_C^3 \rangle = \frac{8}{27} \langle |\mathbf{u}|^2 \rangle^3 \langle \text{tr}(S^3) \rangle.$$

(20)

The same assumptions and similar elementary algebraic manipulations as those used to establish Eq. (20), lead to an exact expression for the second moment of $p_C$:

$$\langle p_C^2 \rangle = \frac{2}{15} \langle |\mathbf{u}|^4 \rangle \langle \text{tr}(S^2) \rangle = \frac{1}{15} R_L^2 \varepsilon^2,$$

(21)

This relation is found to be in very good agreement with our numerical results, see Table IV.

The known relation between $\text{tr}(S^3)$ and the experimentally accessible moments of $\partial_x u_x$ [7]:

$$\langle \text{tr}(S^3) \rangle = \frac{105}{8} \langle (\partial_x u_x)^3 \rangle,$$

(22)

together with the expression for the second moment of $\partial_x u_x$: $\langle (\partial_x u_x)^2 \rangle = \varepsilon / (15 \nu)$ [25, 26], leads to the following expression for the third moment of $p_C$:

$$\langle p_C^3 \rangle = \frac{7}{225} S_{\partial_x u_x} R_L^3 \varepsilon^3,$$

(23)

where $S_{\partial_x u_x} \equiv \langle (\partial_x u_x)^3 \rangle / \langle (\partial_x u_x)^2 \rangle^{3/2}$ is the skewness of the velocity derivative.

Combining Eq. (21) and (23) gives the following relation between the skewness of $\partial_x u_x$ and the skewness of $p_C$:

$$S_{\partial_x u_x} = \frac{\sqrt{15}}{7} S_{p_C}.$$

(24)

The values of $\frac{\sqrt{15}}{7} S_{p_C}$, as well as the ratio $225 \langle p_C^3 \rangle / (7 \varepsilon^3 R_L^3)$, determined from our numerical simulations, are shown in Table III. The corresponding value of the skewness of $\partial_x u_x$, $S_{\partial_x u_x}$ is found to be approximately $-0.4$, which is well within the range of values of $S_{\partial_x u_x}$ reported from experiments and simulations at comparable Reynolds numbers [3, 9].

| $R_L$ | 193 | 275 | 430 |
|-------|-----|-----|-----|
| $225 \langle p_C^3 \rangle / (7 \varepsilon^3 R_L^3)$ | -0.24 | -0.37 | -0.40 |
| $\frac{\sqrt{15}}{7} S_{p_C}$ | -0.30 | -0.38 | -0.42 |
| $S_{\partial_x u_x} \equiv \langle (\partial_x u_x)^3 \rangle / \langle (\partial_x u_x)^2 \rangle^{3/2}$ | -0.52 | -0.62 | -0.67 |

**TABLE III.** Third moments of the distributions of $p_C / \varepsilon$ and $p / \varepsilon$ at the three Reynolds numbers studied in this article.
Table IV. Parametrization of the second moments of $p_C/ε$, $p_L/ε$ and $p/ε$, compared to the expression in terms of $β$ given by Eqs. (25), (26) and (27), at the three Reynolds numbers studied in this article.

| $R_λ$ | 193  | 275  | 430  |
|-------|------|------|------|
| $⟨p_Lpc⟩/(p_C^2)^{1/2}/(p_L^2)^{1/2}$ | $−0.907$ | $−0.923$ | $−0.944$ |
| $−β^{1/2}$ | $−0.912$ | $−0.928$ | $−0.947$ |
| $⟨p^2⟩/(p_C^2)$ | 0.178 | 0.147 | 0.110 |
| $1−β$ | 0.17 | 0.14 | 0.10 |
| $⟨p_L^2⟩/(p_C^2)$ | 0.823 | 0.851 | 0.892 |
| $β$ | 0.83 | 0.86 | 0.90 |

Appendix B: Prevalence of $p_C$ on the moments of $p$

A. Second moments of $p$

The decomposition of the distribution of $p_L$ conditioned on $p_C$, see Eq. (14), involving a random variable $ξ|p_C$ with a zero mean, together with the assumption of independence between $p$ and $p_L$, $⟨pp_L⟩=0$ (see Eq. (15)), leads to a full description of the second moments of $p_C$, $p_L$ in terms of $β$ only:

$$⟨p_L^2⟩ = β⟨p_C^2⟩$$ (25)

$$⟨p_Lpc⟩ = −β⟨p_C^2⟩ = −β^{1/2}⟨p_C^2⟩^{1/2}$$ (26)

$$⟨p^2⟩ = (1−β)⟨p_C^2⟩$$ (27)

The values of $β$ are measured from the conditional average $⟨p_Lpc⟩ = −βp_C$, Eq. (14). This allows us to check how accurately are Eqs. (25) to (27) satisfied. The numerical results are shown in Table IV. The relations given by Eqs. (25) to (27) are found to be very well satisfied, with very small errors, thus demonstrating that the proposed parametrization in terms of $⟨p_C^2⟩$ and $β$ provides a very good description of the second moments of $p$.

B. Conditional averages $⟨p|p_C⟩$ and $⟨p_C|p⟩$

Crucial to the argument relating the third moment $⟨p^3⟩$ to the third moment of $p_C$ is the observation that the conditional averages $⟨p|p_L⟩$ and $⟨p_L|p⟩$ are close to 0. Figure 2b of the main text shows these averages $(p_L, ⟨p|p_L⟩)$ and $(⟨p_L|p⟩, p)$, which appear as horizontal and vertical straight lines respectively on the scale of the figure. Our argument is then based on identities such as:

$$⟨p_L^2p⟩ = \int_{−∞}^{∞} P(p_L)⟨p|p_L⟩p_L^2dp_L,$$ (28)

where $P(p_L)$ is the PDF of $p_L$. Equation (28) shows that if $⟨p_L|p⟩ = ⟨p|p_L⟩ = 0$, then, $⟨p^2p_L⟩ = ⟨p_L^2p⟩ = 0$.

Possible deviations from zero of the moments $⟨p_L^2p⟩$ and $⟨p^2p_L⟩$ therefore indicate that the conditional averages $⟨p|p_L⟩$ and $⟨p_L|p⟩$ are not exactly zero. These moments can be readily estimated from the various third moments $(p_L^2p^2)$ with $m + n = 3$ shown in Table 2 of the main text:

$$⟨p^2p_L⟩ = ⟨p_C^2p_L⟩ + 2(pc_p^2p_L) + ⟨p_L^3⟩$$ (29)

and

$$⟨p_L^2p⟩ = ⟨p_L^2p⟩ + ⟨p_L^2p⟩.$$ (30)

When compared to $⟨p_L^2⟩$, the value of $⟨p^2p_L⟩$ is approximately zero ( $|⟨p^2p_L⟩/⟨p_L^2⟩| ≤ 2%$), but $⟨p_L^2p⟩/⟨p_L^2⟩ ≤ 10\%$. This points to a departure of $⟨p|p_L⟩$ from being 0, which we explore here.

Figure 4 shows the conditional averages of $⟨p|p_L⟩$ for the three direct numerical simulation (DNS) runs discussed in this article, and also the integrand in Eq. (28). For all three cases, the curves differ weakly but consistently from 0. While for $p_L < 0$, the values of $⟨p|p_L⟩$ are very small, they there noticebly from 0 on the positive $p_L$ side, with an approximately linear dependence on $p_L$. In order to examine the effect of this deviation of $⟨p|p_L⟩$ on $ζ = (p_L^2)/⟨p_C^2⟩$, we non-dimensionalize the variables $p_L$ and $p$ in Fig. 4 by $⟨p_C^2⟩^{1/2}$. In particular, in Fig. 4b), we plot $P(p_L)p_L^2/(p_C^2)^{3/2}$. In this way, the areas under the curves in Fig. 4b) give $⟨p_L^2p⟩/(p_C^2)^{3/2} = ζS_{pc}$, where $S_{pc}$ is the skewness of $p_C$, which depends weakly on the Reynolds number as shown in Table 2 of the main text. Fig. 5b) shows that $⟨p_L^2p⟩/(p_C^2)^{3/2}$ decreases when the Reynolds number increases. This is consistent with the observed decrease of the value of $ζ$ with the Reynolds number.

As shown in Table 2 of the main text, $ζ = 0.11, 0.088$ and 0.066 at $R_λ = 193, 275$, and 430, respectively. In fact, over the limited range of Reynolds number that we studied here, we observed that $ζ/(1−β)$ remains well below unity: $ζ/(1−β) ≈ 0.64$, which ensures that the third moment of $p$, as given by Eq. (12) in the main text, is determined by $⟨p_C^3⟩$:

$$⟨p^3⟩ = (1−β−ζ)⟨p_C^3⟩ = (1−β) \left(1−\frac{ζ}{1−β}\right)⟨p_C^3⟩.$$ (31)

We note that the scaling of $⟨p^3⟩ \sim R_λ^3$ reported before (14), together with the scalings $⟨p_C^3⟩ \sim R_λ^3$ and $(1−β) \sim R_λ^{-2/3}$ obtained in this work, implies that $1−[ζ/(1−β)] \sim R_λ^{−1/3}$. These predictions can only be checked by using DNS at much higher Reynolds numbers, and with adequate statistical resolution.

For comparison, Fig. 6 shows the conditional average of $p_L$ on $p$: $⟨p_L|p⟩$ and the integrand of $⟨p^2p_L⟩$. As it was the case in Fig. 4, the two quantities $p$ and $p_L$ are normalized by $⟨p_C^2⟩^{1/2}$ in the way such that the areas under the curves in Fig. 6b) represent the normalized moment $⟨p^2p_L⟩/⟨p_C^2⟩^{3/2} = (⟨p^2p_L⟩/⟨p_C^3⟩)S_{pc}$. A systematic deviation of $⟨p_L|p⟩$ from being 0 is visible in Fig. 6a). On the other hand, the integrand $P(p)p^2⟨p_L|p⟩$ is noticeably non-zero only in a small range of $p$, which results in a much smaller value of $⟨p^2p_L⟩$ compared to $⟨p_L^2⟩$. 

APPENDIX C: LACK OF CORRELATION BETWEEN $u$ AND $S$

The results presented in the main text give a good indication that assuming $u$ and $S$ are uncorrelated provides an appropriate first-order approximation. This is in fact corroborated by the quantitative agreement between the numerical results and the predictions. Here we present further information concerning the correlations between $u$ and $S$.

The first hint of a correlation between the velocity and strain was provided by Fig. 3 of the main text, which showed weak, but noticeable deviations from a uniform distribution for PDFs of the cosines of the angles between the direction of velocity, $e_u$, and the eigenvectors of strain, especially between $e_u$ and $e_1$ and between $e_u$ and $e_2$.

It may be expected that the alignment between $u$ and the eigenvectors of $S$ depends on the magnitude of $u$. Figure 7 shows the joint probability distribution function between $u^2$ and $|e_u \cdot e_i|$ for $i = 1$ (a), $i = 2$ (b) and...
FIG. 7. Dependence of the alignment between $\mathbf{e}_i$ and $\mathbf{e}_u$ on $u^2$ at $R_\lambda = 275$. Panels (a) (respectively (b) and (c)) show $P(u^2, |\mathbf{e}_u \cdot \mathbf{e}_i|)$, the joint PDF of $u^2$ and $|\mathbf{e}_u \cdot \mathbf{e}_i|$ for $i = 1$ (respectively $i = 2$ and $i = 3$). The indicated colour coding refers to the decimal logarithm of the PDF, i.e., $\log_{10} P(u^2, |\mathbf{e}_u \cdot \mathbf{e}_i|)$. The equal-probability contours, shown as dashed lines, are close to, but deviate from being vertical, which indicates a systematic dependence on $u^2$, especially for $i = 1$ and $i = 2$. Panel (d) shows the conditional average $\langle |\mathbf{e}_i \cdot \mathbf{e}_u| u^2 \rangle$ vs. $u^2$ (also shown as dash-dotted lines in Panels a-c), which weakly deviates from being constant as implied by the assumption of lack of correlation between $\mathbf{S}$ and $\mathbf{u}$.

The results presented in this section thus demonstrate the expectation based on the lack of correlation between $\mathbf{u}$ and $\mathbf{S}$.

The results shown by Fig. 7 thus show small, but systematic deviations of the statistical quantities relevant to $p_C$ from the expected dependence on $u^2$. In comparison, the dependence on the magnitude of $\mathbf{S}$ seems to be much weaker. Figure 8 shows that the value of the average of the cosines of the angles, $|\mathbf{e}_u \cdot \mathbf{e}_i|$, conditioned on $|\mathbf{S}|$, depends significantly less on $|\mathbf{S}|$: the variations shown in Fig. 8 are of the order of $\sim 5\%$, whereas the ones shown in Fig. 7 are of the order of $\sim 20\%$. This difference points to a stronger dependence of the alignment properties on the large scale features of the flow, than on the small scales.

The results presented in this section thus demonstrate
that, while the results obtained in this work by assuming that $\mathbf{u}$ and $\mathbf{S}$ are uncorrelated do provide a good approximation to the third moment of $p_c$, small, but systematic deviations from this assumption are visible. Judging from the present results, the dependence on $|\mathbf{u}|$ seems to be generally more important than the dependence on $|\mathbf{S}|$.  

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{alignment}
\caption{Dependence of the alignment between $\mathbf{e}_i$ and $\mathbf{e}_n$ on $|\mathbf{S}|$ for $i = 1$ (full line), $i = 2$ (dashed line) and $i = 3$ (dash-dotted line) at $R_\lambda = 275$. The variation of the conditional average of $|\mathbf{e}_i \cdot \mathbf{e}_n|$ on $|\mathbf{S}|$ is much weaker than the dependence on $\mathbf{u}^3$ shown in Fig. 7.}
\end{figure}

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