TEMPERATE DISTRIBUTIONS WITH LOCALLY FINITE SUPPORT AND SPECTRUM ON EUCLIDEAN SPACES

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ABSTRACT

We prove that supports of a wide class of temperate distributions with uniformly discrete support and spectrum on Euclidean spaces are finite unions of translations of full-rank lattices. This result is a generalization of the corresponding theorem for Fourier quasicrystals, and its proof uses the technique of almost periodic distributions.

Introduction

The Fourier quasicrystal may be considered as a mathematical model for atomic arrangements having a discrete diffraction pattern. There are a lot of papers devoted to studying properties of Fourier quasicrystals or, more generally, crystalline measures. For example, one can note collections of papers [2], [26], articles [5]–[9], [11]–[24], and so on.

The following problem is very important in the theory of Fourier quasicrystals [16]:

Problem 1: Let $\mu$ be a measure on $\mathbb{R}^d$ with discrete support $\Lambda$, and its Fourier transform in the sense of distributions $\hat{\mu}$ be a measure with discrete support as well. When is $\Lambda$ contained in a finite union of translates of a full-rank lattice?

The answer to this question differs significantly for $d = 1$ and $d > 1$. In order to present the corresponding results, we need to introduce suitable notations and definitions.
1. Preliminaries

Denote by \( S(\mathbb{R}^d) \) the Schwartz space of test functions \( \varphi \in C^\infty(\mathbb{R}^d) \) with the finite norms

\[
N_n(\varphi) = \sup_{\mathbb{R}^d} \max_{\|k\| \leq n} (1 + |x|)^n |D^k \varphi(x)|, \quad n = 0, 1, 2, \ldots,
\]

where

\[
|x| = (x_1^2 + \cdots + x_d^2)^{1/2}, \quad k = (k_1, \ldots, k_d) \in (\mathbb{N} \cup \{0\})^d,
\]

\[
\|k\| = k_1 + \cdots + k_d, \quad D^k = \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d}.
\]

These norms generate the topology on \( S(\mathbb{R}^d) \). Elements of the space \( S^*(\mathbb{R}^d) \) of continuous linear functionals on \( S(\mathbb{R}^d) \) are called temperate distributions. The Fourier transform of a temperate distribution \( f \) is defined by the equality

\[
\hat{f}(\varphi) = f(\hat{\varphi}) \quad \text{for all} \quad \varphi \in S(\mathbb{R}^d),
\]

where

\[
\hat{\varphi}(y) = \int_{\mathbb{R}^d} \varphi(x) \exp\{-2\pi i \langle x, y \rangle\} dx
\]

is the Fourier transform of the function \( \varphi \). Also,

\[
\tilde{\varphi}(y) = \int_{\mathbb{R}^d} \varphi(x) \exp\{2\pi i \langle x, y \rangle\} dx
\]

means the inverse Fourier transform. Note that the Fourier transform is an isomorphism of \( S(\mathbb{R}^d) \) on \( S(\mathbb{R}^d) \) and, respectively, \( S^*(\mathbb{R}^d) \) on \( S^*(\mathbb{R}^d) \).

We will say that a set \( A \subset \mathbb{R}^d \) is locally finite if the intersection of \( A \) with any ball is finite, \( A \) is relatively dense if there is \( R < \infty \) such that \( A \) intersects with each ball of radius \( R \), and \( A \) is uniformly discrete, if \( A \) is locally finite and has a strictly positive separating constant

\[
\eta(A) := \inf\{|x - x'| : x, x' \in A, x \neq x'\}.
\]

A locally finite set \( A \) is of bounded density if

\[
\sup_{x \in \mathbb{R}^d} \#A \cap B(x, 1) < \infty.
\]

As usual, \( \#E \) means a number of elements of the finite set \( E \), and \( B(x, r) \) means the ball with center in \( x \) and radius \( r \).

An element \( f \in S^*(\mathbb{R}^d) \) is called a crystalline measure if \( f \) and \( \hat{f} \) are complex-valued measures on \( \mathbb{R}^d \) with locally finite supports. The support of \( \hat{f} \) for a distribution \( f \in S^*(\mathbb{R}^d) \) is called the spectrum of \( f \).
Denote by \(|\mu|(A)\) the variation of a complex-valued measure \(\mu\) on \(A\). If both measures \(|\mu|\) and \(|\hat{\mu}|\) have locally finite supports and belong to \(S^*(\mathbb{R}^d)\), we say that \(\mu\) is a **Fourier quasicrystal**. A measure \(\mu = \sum_{\lambda \in \Lambda} a_{\lambda} \delta_{\lambda}\) with \(a_{\lambda} \in \mathbb{C}\) and countable \(\Lambda\) is called **purely point**.

Furthermore, it is well known that every distribution with locally finite support \(\Lambda\) has the form
\[
(2) \quad f = \sum_{\lambda \in \Lambda} \sum_{k} p_k(\lambda) D^k \delta_{\lambda}, \quad k \in (\mathbb{N} \cup \{0\})^d,
\]
where the internal sum is finite for every \(\lambda \in \Lambda\), and \(\delta_{\lambda}\) is the unit mass at the point \(\lambda\).

**Proposition 1:**

(i) If a distribution \(f \in S^*(\mathbb{R}^d)\) has locally finite support, then
\[
(3) \quad f = \sum_{\lambda \in \Lambda} \sum_{\|k\| \leq K} p_k(\lambda) D^k \delta_{\lambda}, \quad k \in (\mathbb{N} \cup \{0\})^d,
\]
where \(K < \infty\) does not depend on \(\lambda\) ([7], Proposition 1),

(ii) if a distribution \(f \in S^*(\mathbb{R}^d)\) has uniformly discrete support, then \(K\) does not depend on \(\lambda\) and for some \(M < \infty\) and all \(k\)
\[
(4) \quad p_k(\lambda) = O(|\lambda|^M) \quad \text{as } \lambda \to \infty
\]
([25], Proposition 1),

(iii) if a distribution \(f \in S^*(\mathbb{R}^d)\) of form (3) with locally finite support \(\Lambda\) and locally finite spectrum \(\Gamma\) has the property
\[
(5) \quad \sum_{|\lambda| \leq r} \sum_{\|k\| = 0} K \left| p_k(\lambda) \right| = O(r^{d+M}) \quad (r \to \infty)
\]
with \(M \geq 0\), then
\[
(6) \quad \hat{f} = \sum_{\gamma \in \Gamma} \sum_{\|m\| \leq M} q_m(\gamma) D^m \delta_{\gamma}, \quad m \in (\mathbb{N} \cup \{0\})^d.
\]
If, in addition, \(M\) is an integer, then for \(|m|=M\)
\[
(7) \quad q_m(\gamma) = O(|\gamma|^K) \quad \text{as } \gamma \to \infty
\]
with the same \(K\) as in (3). If \(M\) is a positive real number, and \(\Gamma\) is uniformly discrete, then (7) is valid for all \(m\) ([7], Proposition 3).
In particular, if $f$ is a measure with uniformly bounded masses, support of bounded density, and locally finite spectrum, then $K = M = 0$ and $\hat{f}$ is a measure with uniformly bounded masses as well.

Remark 1: Analysis of the proof of the Proposition shows that if we replace $O(r^{d+M})$ by $o(r^{d+M})$ in (5), then we may replace the inequality $\|m\| \leq M$ by $\|m\| < M$ in (6).

A lattice is a discrete (locally finite) subgroup of $\mathbb{R}^d$. If $A$ is a lattice or a coset of some lattice in $\mathbb{R}^d$, then its rank $r(A)$ is the dimension of the smallest translated subspace of $\mathbb{R}^d$ that contains $A$. Every lattice $L$ of rank $k$ has the form $T\mathbb{Z}^k$, where $T : \mathbb{Z}^k \to \mathbb{R}^d$ is a linear operator of rank $k$. For $k = d$ we get a full-rank lattice. In this case the lattice

$$L^* = \{ y \in \mathbb{R}^d : <\lambda, y> \in \mathbb{Z} \quad \forall \lambda \in L \}$$

is called the conjugate lattice. It follows from Poisson’s formula

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n), \quad f \in S(\mathbb{R}^d),$$

that for a full-rank lattice $L = T\mathbb{Z}^d$ we have

(8) $$\sum_{\lambda \in L} \delta_{\lambda} = |\det T|^{-1} \sum_{\lambda \in L^*} \delta_{\lambda}.$$ 

2. Introduction, one-dimensional case

We begin with the following result of N. Lev and A. Olevskii [18]:

**Theorem 1**: Let $\mu$ be a crystalline measure on $\mathbb{R}$ with uniformly discrete support and spectrum. Then $\text{supp} \mu$ is a subset of a finite union of translates of a single lattice $L \subset \mathbb{R}$ (i.e., of arithmetic progressions with the same difference). Moreover, $\mu$ has the form

$$\mu = \sum_{j=1}^{N} \sum_{\lambda \in L + \tau_j} P_j(\lambda) \delta_{\lambda}$$

for certain reals $\tau_j$ and trigonometric polynomials $P_j(\lambda)$ ($1 \leq j \leq N$).
Theorem 1 remains valid under a weaker assumption that supp $\mu$ is a relatively dense set of bounded density (not assumed to be uniformly discrete) [20]. However there exist examples of crystalline measures on $\mathbb{R}$ whose supports are not contained in any finite union of translates of a lattice (see, for example, [19], [24], [14]).

In article [21] Theorem 1 was extended to temperate distributions on the line:

**Theorem 2:** Let $f$ be a temperate distribution on $\mathbb{R}$ such that $\Lambda = \text{supp} \, f$ and $\Gamma = \text{supp} \, \hat{f}$ are uniformly discrete sets. Then $f$ can be represented in the form

$$f = \sum_{\tau, \omega, l, p} c(\tau, \omega, l, p) \sum_{\lambda \in L} \lambda^l e^{2\pi i \lambda \omega} \delta_{\lambda+\tau},$$

where $L$ is a lattice on $\mathbb{R}$, $(\tau, \omega, l, p)$ goes through a finite set of quadruples such that $\tau, \omega$ are real numbers, $l, p$ are nonnegative integers, and $c(\tau, \omega, l, p)$ are complex numbers.

The result is still correct if the support $\Lambda$ has bounded density and the coefficients $p_k(\lambda)$ in (2) satisfy (4), while the spectrum $\Gamma$ is uniformly discrete.

### 3. Introduction, multivariate case

Various analogs of Theorem 2 for temperate distributions on $\mathbb{R}^d$ were obtained in [25] under conditions of locally finite set of differences $\Lambda - \Lambda$ and $\Gamma - \Gamma$ and in [7] under conditions of locally finite $\Lambda - \Lambda$, not too fast approaching points from $\Gamma$, and uniformly separated from zero and infinity coefficients $p_k(\lambda)$.

In [18], [20] N. Lev and A. Olevskii proved the following result:

**Theorem 3:** Let $\mu$ be a positive measure on $\mathbb{R}^d$ such that $\Lambda = \text{supp} \, \mu$ and $\Gamma = \text{supp} \, \hat{\mu}$ are uniformly discrete sets. Then $\Lambda$ is contained in a finite union of translates of a lattice of rank $d$. The same is valid if $\Lambda$ is locally finite not assumed to be uniformly discrete.

It was proved in [5] that there is a signed measure on $\mathbb{R}^2$ such that its support and spectrum are both uniformly discrete and simultaneously are unions of two incommensurable full-rank lattices. Therefore neither support nor spectrum can be finite unions of translations of a single lattice.

But there are results of a different type for measures in $\mathbb{R}^d$ with discrete support, in which a finite number of lattices are already involved ([22], [13], [12], [3]). In fact, the following result was proved:
Theorem 4: Let $\mu$ be a measure on $\mathbb{R}^d$ with uniformly discrete support $\Lambda$. If complex masses $\mu(\{\lambda\})$ at points $\lambda \in \Lambda$ take values only from a finite set $F \subset \mathbb{C}$, and the measure $\hat{\mu}$ is purely point and satisfies the condition

$$|\hat{\mu}|(B(0,r)) = O(r^d) \quad (r \to \infty),$$

then $\Lambda$ is a finite union of translations of several, possibly incommensurable, full-rank lattices.

In the paper [21], the following problem was posed:

Problem 2: Let $\mu$ be a measure on $\mathbb{R}^d$ with discrete support $\Lambda$ and its Fourier transform in the sense of distributions $\hat{\mu}$ is a measure with discrete support as well. Does it follow that $\Lambda$ can be covered by a finite union of translates of several lattices?

Note that the periodic structure of a crystalline measure in the multidimensional case follows under extra assumptions on masses ([8], [9]):

Theorem 5: Let $\mu$ be a measure on $\mathbb{R}^d$ with uniformly discrete support $\Lambda$ such that $\inf_{\lambda \in \Lambda} |\mu(\{\lambda\})| > 0$ and the measure $\hat{\mu}$ is purely point and satisfies (9). Then $\Lambda$ is a finite union of translates of several disjoint full-rank lattices.

Moreover, there exist an integer $J$, lattices $L_1, \ldots, L_J$ in $\mathbb{R}^d$ of rank $d$ (some of them may coincide), points $\lambda_1, \ldots, \lambda_J \in \Lambda$, a bounded set

$$\{\alpha_s^j : s \in \mathbb{N}, 1 \leq j \leq J\} \subset \mathbb{R}^d,$$

and functions

$$F_j(y) = \sum_s b_s^j e^{2\pi i \langle y, \alpha_s^j \rangle} \quad \text{with} \quad \sum_s |b_s^j| < \infty, \ j = 1, \ldots, J,$$

such that

$$\mu = \sum_{j=1}^J F_j(\lambda) \sum_{\lambda \in L_j + \lambda_j} \delta_\lambda.$$

The proofs of Theorems 4 and 5 are based on Cohen’s Idempotent Theorem (see, e.g., [28]; the first use of the Idempotent Theorem to study structure of tilings was in [17]):
Theorem 6: Let $G$ be a locally compact Abelian group and $\hat{G}$ its dual group. If $\nu$ is a finite Borel measure on $G$ and is such that its Fourier transform $\hat{\nu}(\lambda)$ takes only values 0 and 1, then the set $\{\lambda : \hat{\nu}(\lambda) = 1\}$ belongs to the coset ring of $\hat{G}$.

Recall that the coset ring of an Abelian topological group $G$ is the smallest collection of subsets of $G$ that is closed under complement, finite unions, and finite intersections, which contains all cosets of all open subgroups of $G$. In particular, if $G = \mathbb{R}^d_{\text{discr}}$, i.e., the space $\mathbb{R}^d$ with respect to the discrete topology, then the coset ring contains all cosets of all subgroups of $G$.

Theorem 7 (M. Kolountzakis [12, Theorem 3]): All elements of the coset ring for $\mathbb{R}^d_{\text{discr}}$, which are discrete in the Euclidean topology of $\mathbb{R}^d$, are finite unions of sets of the type $A \setminus \bigcup_{j=1}^d B_j$, where $A, B_j$ are discrete cosets in the Euclidean topology.

4. Results

Here we get the following extension of Theorems 3 and 5 to temperate distributions:

Theorem 8: Suppose that a temperate distribution $f$ has locally finite support $\Lambda$ of bounded density, a uniformly discrete spectrum $\Gamma$, and all coefficients $p_k(\lambda)$ in (2) are uniformly bounded and non-negative. Then $\Lambda$ is contained in a finite union of translates of several full-rank lattices.

The non-negativity of coefficients $p_k(\lambda)$ can be replaced by the following: for each $k \in (\mathbb{N} \cup \{0\})^d$ there are $\xi_k \in \mathbb{R}$ such that $\text{Arg } p_k(\lambda) = \xi_k$ for all $p_k(\lambda) \neq 0, \lambda \in \Lambda$.

Theorem 9: Suppose that a temperate distribution $f$ of form (2) has both uniformly discrete support $\Lambda$ and spectrum $\Gamma$. If there are constants $c, C$ such that for coefficients in (2) the following inequalities hold,

$$(10) \quad 0 < c \leq \sum_k |p_k(\lambda)| \leq C < \infty, \quad \forall \lambda \in \Lambda,$$

then $\Lambda$ is a finite union of translations of several full-rank lattices. More accurately, there exist integers $J, K$, lattices $L_1, \ldots, L_J$ in $\mathbb{R}^d$ of rank $d$ (some
of them may coincide), points \( \lambda_1, \ldots, \lambda_J \in \Lambda \), and a bounded set of exponents \( \beta_s = \beta_s(j, k) \in \mathbb{R}^d \) such that

\[
f = \sum_{j=1}^J \sum_{\|k\| \leq K} \sum_{\lambda \in L_j + \lambda_j} F_{k,j}(\lambda) D^k \delta_\lambda,
\]

where

\[
F_{k,j}(x) = \sum_s b_s(k, j)e^{2\pi i \langle x, \beta_s \rangle} \quad \text{with} \quad \sum_s |b_s(k, j)| < \infty.
\]

Remark 2: Proposition 1(i) implies that \( f \) has form (3) with \( K < \infty \), therefore we may replace the sum \( \sum_k |p_k(\lambda)| \) by \( \sup_k |p_k(\lambda)| \) in (10). Also, we can replace the right-hand bound in (10) by

\[
p_k(\lambda) = o(|\lambda|) \quad (\lambda \to \infty) \quad \forall k,
\]

because in this case we get again \( p_k(\lambda) = O(1) \) (see Remark 3 in Section 6).

Nevertheless, some upper and lower bounds are required.

Let \( A = \{n \in \mathbb{Z}^2 : n_1 \neq 0\} \). It follows from (1) and (8) that the Fourier transform of the measure with unbounded coefficients

\[
\sum_{n \in A} n_1 \delta_n = \sum_{x \in \mathbb{Z}^2} x_1 \delta_x
\]

is the distribution

\[
\frac{i}{2\pi} \sum_{\lambda \in \mathbb{Z}^2} \frac{\partial}{\partial \lambda_1} \delta_\lambda.
\]

Also, the Fourier transform of the measure whose masses do not separate from zero,

\[
\sum_{n \in A} \sin \left( \frac{2\pi n_1}{\sqrt{5}} \right) \delta_n = \frac{1}{2i} \sum_{x \in \mathbb{Z}^2} (e^{2\pi i x_1/\sqrt{5}} - e^{-2\pi i x_1/\sqrt{5}}) \delta_x,
\]

is equal to

\[
\frac{1}{2i} \sum_{y \in \mathbb{Z}^2} \delta_{y+\alpha} - \frac{1}{2i} \sum_{y \in \mathbb{Z}^2} \delta_{y-\alpha}, \quad \alpha = (1/\sqrt{5}, 0),
\]

hence in both cases the Fourier transforms have uniformly discrete supports. But \( A \) is not a finite union of cosets of several full-rank lattices, because in the opposite case the set \( \mathbb{Z} \setminus \{0\} \), which is the projection of \( A \) on \( \mathbb{R}_{x_1} \), would be a finite union of arithmetical progressions, and that is impossible.

The above theorems are based on the following result for temperate distributions with uniformly discrete spectrum, which is also of independent interest:
Theorem 10: Suppose $f \in S^*({\mathbb{R}}^d)$ of form (2) has locally finite support $\Lambda$ of bounded density and uniformly discrete spectrum $\Gamma$. Set, for $k \in (\mathbb{N} \cup \{0\})^d$,

$$\mu_k = \sum_{\lambda \in \Lambda} p_k(\lambda) \delta_{\lambda}.$$  \hfill (11)

(a) If $p_k(\lambda)$ are uniformly bounded in $k \in (\mathbb{N} \cup \{0\})^d$ and $\lambda \in \Lambda$, then $\hat{\mu}_k \in S^*({\mathbb{R}}^d)$ are measures of the form

$$\hat{\mu}_k = \sum_{\gamma \in \Gamma_k} q_k(\gamma) \delta_{\gamma}$$

with uniformly bounded masses $q_k(\gamma)$ and uniformly discrete supports $\Gamma_k$ such that

$$\eta(\Gamma_k) \geq \eta(\Gamma) \quad \forall k.$$  \hfill (12)

(b) If both $\Lambda$ and $\Gamma$ are uniformly discrete, then $\hat{\mu}_k \in S^*({\mathbb{R}}^d)$ are distributions of the form

$$\hat{\mu}_k = \sum_{\|m\| \leq M} \sum_{\gamma \in \Gamma_k} q_{k,m}(\gamma) D^m \delta_{\gamma}$$

with uniformly bounded coefficients $q_{k,m}(\gamma)$ and uniformly discrete supports $\Gamma_k$, which satisfy (12).

Since $\mu_k$ is the inverse Fourier transform of $\hat{\mu}_k$, we see that $\mu_k \in S^*({\mathbb{R}}^d)$. Also, by Proposition 1(i), $\mu_k \equiv 0$ for $\|k\|$ large enough. Hence,

$$f = \sum_k D^k \mu_k, \quad \hat{f}(y) = \sum_k (2\pi i)^{\|k\|} y^k \hat{\mu}_k,$$  \hfill (14)

where, as usual, $y^k = y_1^{k_1} \cdots y_d^{k_d}$.

5. Almost periodic functions and distributions

We recall here definitions and properties of almost periodic functions and distributions that will be used in what follows. A more complete exposition of these issues is available in [4], [1], [23], [24], [27].

Definition 1: A continuous function $g$ on $\mathbb{R}^d$ is almost periodic if for any $\varepsilon > 0$ the set of its $\varepsilon$-almost periods

$$\{ \tau \in \mathbb{R}^d : \sup_{t \in \mathbb{R}^d} |g(t + \tau) - g(t)| < \varepsilon \}$$

is a relatively dense set in $\mathbb{R}^d$. 
An equivalent definition follows:

**Definition 2:** A continuous function \( g \) on \( \mathbb{R}^d \) is almost periodic if for any sequence \( \{t_n\} \subset \mathbb{R}^d \) there is a subsequence \( \{t'_n\} \) such that the sequence of functions \( g(t + t'_n) \) converges uniformly in \( t \in \mathbb{R}^d \).

Using an appropriate definition, one can prove various properties of almost periodic functions:

- almost periodic functions are bounded and uniformly continuous on \( \mathbb{R}^d \);
- the class of almost periodic functions is closed with respect to taking absolute values and linear combinations of a finite family of functions,
- a limit of a uniformly convergent sequence of almost periodic functions is also almost periodic,
- any finite family of almost periodic functions has a relatively dense set of common \( \varepsilon \)-almost periods,
- for any almost periodic function \( g(x) \) on \( \mathbb{R}^d \) the function \( h(t) = g(tx) \) is almost periodic in \( t \in \mathbb{R} \) for any fixed \( x \in \mathbb{R}^d \); in particular, \( g(t_1, \ldots, t_d) \) is almost periodic in each variable \( t_j \in \mathbb{R} \), \( j = 1, \ldots, d \), if the other variables are held fixed.

Also, we will use the following definition:

**Definition 3:** A distribution \( g \) is **almost periodic** if the function \( (g(y), \varphi(t-y)) \) is almost periodic in \( t \in \mathbb{R}^d \) for each \( C^\infty \)-function \( \varphi \) on \( \mathbb{R}^d \) with compact support. A measure \( \mu \) is **almost periodic** if it is an almost periodic distribution.

Clearly, every linear combination of almost periodic distributions is almost periodic, and each almost periodic distribution has a relatively dense support.

Note that the usual definition of almost periodicity for measures (instead of \( \varphi \in C^\infty \), we consider continuous \( \varphi \) with compact support) differs from the one given above. However, these definitions coincide for nonnegative measures or measures with uniformly discrete support (see [1], [23], [5]).

**Proposition 2:** Let \( \mu = \sum_{\lambda \in \Lambda} p(\lambda) \delta_\lambda, \ p(\lambda) \in \mathbb{C} \), be an almost periodic measure with uniformly discrete support \( \Lambda \subset \mathbb{R}^d \). Then

(i) the masses \( p(\lambda) \) are uniformly bounded,
(ii) the measure \( |\mu| = \sum_{\lambda \in \Lambda} |p(\lambda)| \delta_\lambda \) is almost periodic,
(iii) if \( \inf_{\lambda \in \Lambda} |p(\lambda)| \geq \alpha > 0 \), then the measure \( \delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda \) is almost periodic as well.
(iv) for any $\varepsilon > 0$ and $\lambda \in \mathbb{R}^d$ there is a relatively dense set $T$ such that $|\mu(\lambda') - \mu(\lambda)| < \varepsilon$ for all $\lambda' \in T$,

(v) if $\mu_n, n = 1, \ldots, N$ are almost periodic measures with uniformly discrete supports, then for any vector $\lambda \in \mathbb{R}^d$ and numbers $\varepsilon > 0$, $\rho > 0$ there is a relatively dense set $T$ such that for every $\lambda' \in T$ and some $\lambda^{(n)} \in B(\lambda', \rho)$

\begin{equation}
|\mu_n(\lambda^{(n)}) - \mu_n(\lambda)| < \varepsilon, \quad n = 1, \ldots, N.
\end{equation}

The condition on $\Lambda$ to be uniformly discrete is essential (see [10]).

**Proof of Proposition 2.** (i) Let $\varphi$ be a $C^\infty$ function with support in $B(0, \eta(\Lambda)/2)$ such that $\varphi(0) = 1$. The function $(\mu(y), \varphi(t - y))$ is almost periodic, hence it is uniformly bounded, and the numbers $p(\lambda) = (\mu(y), \varphi(\lambda - y))$ are uniformly bounded as well.

(ii) Every $C^\infty$-function with compact support is a finite linear combination of shifts of nonnegative $C^\infty$-functions with supports in balls of radius $\eta/2$.

Consequently we can check the almost periodicity using only such functions.

Let $\varphi$ be one of them and supp $\varphi \subset B(x, \eta/2)$. For each $t, \tau \in \mathbb{R}^d$ we have

\begin{equation}
|(|\mu|(y), \varphi(t - y)) - (|\mu|(y), \varphi(t + \tau - y))| = \left| \sum_{\lambda \in \Lambda} \varphi(t - \lambda)|p(\lambda)| - \sum_{\lambda' \in \Lambda} \varphi(t + \tau - \lambda')|p(\lambda')| \right|.
\end{equation}

Evidently, here each sum consists of at most one nonzero term. Therefore, for some $\lambda, \lambda' \in \Lambda$ that depends on $t, \tau$ we get

\begin{equation}
|(|\mu|(y), \varphi(t - y)) - (|\mu|(y), \varphi(t + \tau - y))| = |\varphi(t - \lambda)|p(\lambda)| - \varphi(t + \tau - \lambda')|p(\lambda')| \leq |\varphi(t - \lambda)p(\lambda) - \varphi(t + \tau - \lambda')p(\lambda')| = |(\mu(y), \varphi(t - y)) - (\mu(y), \varphi(t + \tau - y))|.
\end{equation}

Hence every $\varepsilon$-period $\tau$ of the function $(\mu(y), \varphi(t - y))$ is an $\varepsilon$-period of the function $(|\mu|(y), \varphi(t - y))$.

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1 Using the Theorem on Partition of Unity, it suffices to prove this for the case of a real-valued $\varphi \in C^\infty$ with support in the ball $B(x, \eta(\Lambda)/4)$. Then we have $\varphi = \psi - (\psi - \varphi)$, where $\psi$ is an arbitrary $C^\infty$ nonnegative function with support in $B(x, \eta(\Lambda)/2)$ such that $\varphi \leq \psi$. 

(iii) Pick $\varepsilon > 0$ and $\beta < \min\{\alpha, \eta(\Lambda)/2\}$ such that $|\varphi(x) - \varphi(x')| < \varepsilon$ for $|x - x'| < \beta$. Let $\psi$ be a $C^\infty$-function such that $\text{supp } \psi(x) \subset B(0, \beta)$ and $\psi(0) = 1$. Let $\tau$ be some $\beta$-almost period of the function $(\mu(y), \psi(t - y))$, and let $t \in B(\lambda, \beta)$ for $\lambda \in \Lambda$. Since $\beta < \eta(\Lambda)/2$, we get for at most one $\lambda' \in \Lambda$ that depends on $t + \tau$

$$|p(\lambda) - p(\lambda')\psi(\lambda + \tau - \lambda')| < \beta.$$ 

The inequality $\beta < \alpha \leq |p(\lambda)|$ implies that

$$\psi(\lambda + \tau - \lambda') \neq 0.$$ 

Hence, $|\lambda + \tau - \lambda'| < \beta$. Therefore, $|\varphi(t - \lambda) - \varphi(t + \tau - \lambda')| < \varepsilon$ and

$$\left| \sum_{\lambda \in \Lambda} \varphi(t - \lambda) - \sum_{\lambda' \in \Lambda} \varphi(t + \tau - \lambda') \right| = |\varphi(t - \lambda) - \varphi(t + \tau - \lambda')| < \varepsilon.$$ 

This inequality is valid for all $\beta$-almost periods $\tau$ from a relatively dense set. Thus the function $(\delta(\lambda, y), \varphi(t - y))$ is almost periodic.

(iv) It is enough to consider the case $\varepsilon \leq |\mu(\lambda)|$. Let $\rho < \eta(\text{supp } \mu)/2$, and let $\varphi$ be a $C^\infty$ function on $\mathbb{R}^d$ such that

$$0 \leq \varphi(t) \leq 1, \quad \varphi(0) = 1, \quad \varphi(t) = \varphi(-t), \quad \text{supp } \varphi \subset B(0, \rho).$$ 

We have $\mu(\lambda) = (\varphi(\lambda - x), \mu(x))$. If $\tau$ is an $\varepsilon/3$-almost period of the almost periodic function $(\varphi(t - x), \mu(x))$, we get

$$|((\varphi(\lambda + \tau - x), \mu(x)) - \mu(\lambda)| < \varepsilon/3.$$ 

Therefore, there is a single point $\lambda' \in B(\lambda + \tau, \rho)$ such that

$$|\mu(\lambda')\varphi(\lambda + \tau - \lambda') - \mu(\lambda)| < \varepsilon/3.$$ 

We get $\lambda' \in \text{supp } \mu$ and $|\mu(\lambda')| > 2\varepsilon/3$. Since $-\tau$ is an $\varepsilon/3$-almost period as well, we see that for some $\lambda'' \in B(\lambda' - \tau, \rho)$

$$|\mu(\lambda'')\varphi(\lambda' - \tau - \lambda'') - \mu(\lambda')| < \varepsilon/3.$$
We get $\lambda'' \in \text{supp} \mu$ and $|\lambda'' - \lambda| < 2\rho$, hence $\lambda'' = \lambda$. Then we have for $t = \lambda + \tau - \lambda'$

$$|\mu(\lambda) - \mu(\lambda')| \leq |\mu(\lambda) - \varphi(t)\mu(\lambda')| + |\mu(\lambda') - \varphi(t)\mu(\lambda')|$$

$$\leq |\mu(\lambda) - \varphi(t)\mu(\lambda')| + (1 - \varphi^2(t))|\mu(\lambda')|$$

$$|\mu(\lambda) - \varphi(t)\mu(\lambda')| + |\mu(\lambda') - \varphi(-t)\mu(\lambda)|$$

$$+ |\varphi(t)\mu(\lambda) - \varphi^2(t)\mu(\lambda')|$$

$$< \varepsilon.$$ 

Recall that almost periods form a relatively dense set.

(v) The proof is very close to the previous one. We take $\rho < \min_n \eta(\mu_n)/2$, the same function $\varphi$, and $\lambda' = \lambda + \tau$, where $\tau$ is a common $\varepsilon/3$-almost period of the functions $(\varphi(\lambda - x), \mu_n(x))$. Then, as $\lambda^{(n)}$ we take points satisfying the conditions

$$|\mu_n(\lambda^{(n)})\varphi(\lambda' - \lambda^{(n)}) - \mu_n(\lambda)| < \varepsilon/3.$$ 

Further, arguing as above, we obtain (15). 

Typical examples of almost periodic functions on $\mathbb{R}^d$ are sums of the form

$$(16) \quad f(t) = \sum_n a_n e^{2\pi i (t, s_n)}, \quad a_n \in \mathbb{C}, \quad s_n \in \mathbb{R}^d, \quad \sum_n |a_n| < \infty.$$ 

It is not hard to prove that

$$\hat{f} = \sum_n a_n \delta_{s_n}.$$ 

Denote by $W$ the class of functions admitting representation (16). It is easy to check that a product of two functions from $W$ belongs to $W$ as well. Also, in Section 7 we will use the following local version of the classical Wiener–Levi Theorem:

**Proposition 3 ([8]):** Let $K \subset \mathbb{C}$ be an arbitrary compact set, $h(z)$ be a holomorphic function on a neighborhood of $K$, and $f \in W$. Then there is a function $g \in W$ such that if $f(x) \in K$ then $h(f(x)) = g(x)$.

Next, the Bohr compactification $\mathcal{R}$ of $\mathbb{R}^d$ is a compact group with the dual $\mathbb{R}^d_{\text{discr}}$, and $\mathbb{R}^d$ is a dense subset of $\mathcal{R}$ with respect to the topology on $\mathcal{R}$. Moreover, restrictions to $\mathbb{R}^d$ of continuous functions on $\mathcal{R}$ are almost periodic functions on $\mathbb{R}^d$ (see, e.g., [28]).
6. Representation of distributions with discrete support and spectrum

First we prove the following lemma:

**Lemma 1:** Let \( g_0(y), g_1(y), \ldots, g_N(y) \) be temperate distributions on \( \mathbb{R}^d \) such that for every \( C^\infty \)-function \( \varphi(y) \) with compact support the functions

\[
(g_n(y), \varphi(t - y)), \quad t = (t_1, t_2, \ldots, t_d), \quad n = 0, \ldots, N,
\]

are almost periodic in \( t_1 \in \mathbb{R} \) for every fixed \( (t_2, \ldots, t_d) \in \mathbb{R}^{d-1} \). If the distribution

\[
F = \sum_{n=0}^{N} y_1^n g_n(y), \quad y = (y_1, \ldots, y_d),
\]

has a uniformly discrete support \( T \), then the set \( \bigcup_n \text{supp } g_n \) is uniformly discrete and \( \eta(\bigcup_n \text{supp } g_n) \geq \eta(T) \).

**Proof.** Note that for every \( C^\infty \)-function \( \varphi(y) \) with compact support we have

\[
(F(y), \varphi(t - y)) = \sum_{n=0}^{N} (g_n(y), y_1^n \varphi(t - y)) = \sum_{m=0}^{N} t_1^m \sum_{n=m}^{N} \Phi_{n,m}(t),
\]

where

\[
\Phi_{n,m}(t) = \binom{n}{m} (g_n(y), (y_1 - t_1)^{n-m} \varphi(t - y)), \quad n \geq m.
\]

We will prove that for every \( n, m \leq N \) and every \( y \in \text{supp } g_n, \tilde{y} \in \text{supp } g_m \) we get \( |y - \tilde{y}| \geq \eta(T) \).

Consider the case \( n = m = N \). Assume the converse. Then there are points \( y', y'' \in \text{supp } g_N \) such that \( 0 < |y' - y''| < \eta(T) \). Pick

\[
\alpha < (1/2) \min\{|y' - y''|, (\eta(T) - |y' - y''|)\}
\]

and \( C^\infty \)-function \( \varphi \) with \( \text{supp } \varphi \subset B(0, \alpha) \) such that

\[
\Phi_{N,N}(y') = (g_N(y), \varphi(y' - y)) \neq 0, \quad \Phi_{N,N}(y'') = (g_N(y), \varphi(y'' - y)) \neq 0.
\]

Take \( \varepsilon > 0 \) such that \( |\Phi_{N,N}(y')| > 2\varepsilon, |\Phi_{N,N}(y'')| > 2\varepsilon \). If \( \tau \) is a common \( \varepsilon \)-almost period of the functions

\[
\Phi_{N,N}(y' + se_1), \quad \Phi_{N,N}(y'' + se_1), \quad \text{where } e_1 = (1, 0, 0 \ldots 0), \quad s \in \mathbb{R},
\]
then \( |\Phi_{N,N}(y' + \tau e_1)| > \varepsilon, |\Phi_{N,N}(y'' + \tau e_1)| > \varepsilon \). We have

\[
(y'_1 + \tau)^{-N} (F(y), \varphi(y' + \tau e_1 - y)) = \Phi_{N,N}(y' + \tau e_1) + \sum_{m=0}^{N-1} (y'_1 + \tau)^{m-N} \sum_{n=m}^{N-1} \Phi_{n,m}(y' + \tau e_1).
\]

Note that all functions \( \Phi_{n,m}(y' + se_1), 0 \leq n, m \leq N \), are almost periodic in \( s \), hence they are uniformly bounded on \( \mathbb{R} \). Therefore if \( \tau \) is large enough then

\[
(y'_1 + \tau)^{-N} (F(y), \varphi(y' + \tau e_1 - y)) \neq 0.
\]

Hence there exists a point \( t' \in B(y' + \tau e_1, \alpha) \cap T \). For the same reason, there exists a point \( t'' \in B(y'' + \tau e_1, \alpha) \cap T \) with the same \( \tau \). We have

\[
(18) \ |t' - t''| \leq |t' - y' - \tau e_1| + |t'' - y'' - \tau e_1| + |y' - y''| < 2\alpha + |y' - y''| < \eta(T).
\]

But this inequality is impossible.

Now suppose that our assumption is proved for all \( n, m \) such that \( n + m > M \) with \( 0 \leq M < 2N \); then we prove it for \( p, q \geq 0 \) such that \( p + q = M \).

Assume the converse. There are points \( y' \in \text{supp}\ g_p, y'' \in \text{supp}\ g_q \) such that \( 0 < |y' - y''| < \eta(T) \). Pick

\[
\alpha < (1/2) \min\{|y' - y''|, (\eta(T) - |y' - y''|)\}
\]

and \( C^\infty\)-function \( \varphi \) with \( \text{supp}\ \varphi \subset B(0, \alpha) \) such that

\[
\Phi_{p,p}(y') = (g_p(y), \varphi(y' - y)) \neq 0, \quad \Phi_{q,q}(y'') = (g_q(y), \varphi(y'' - y)) \neq 0.
\]

Take \( \varepsilon > 0 \) such that \( |\Phi_{p,p}(y')| > 2\varepsilon \) and \( |\Phi_{q,q}(y'')| > 2\varepsilon \). Obviously, for a common \( \varepsilon \)-almost period \( \tau \) of the functions \( \Phi_{p,p}(y' + se_1) \) and \( \Phi_{q,q}(y'' + se_1) \) we get

\[
|\Phi_{p,p}(y' + \tau e_1)| > \varepsilon, \quad |\Phi_{q,q}(y'' + \tau e_1)| > \varepsilon.
\]

If there exists \( n > p \) such that at least one of the numbers \( \Phi_{n,j}(y' + \tau e_1) \), \( j = 0, \ldots, n \) does not vanish, then there is a point \( t' \in B(y' + \tau e_1, \alpha) \cap \text{supp}\ g_n \).

Since \( \Phi_{q,q}(y'' + \tau e_1) \neq 0 \), we see that there is a point \( t'' \in B(y'' + \tau e_1, \alpha) \cap \text{supp}\ g_q \).

By (18), \( |t' - t''| < \eta(T) \), hence this inequality contradicts our assumption \( \eta(\text{supp}\ g_n \cup \text{supp}\ g_q) \geq \eta(T) \) for \( n + q > M \). In the same way, we obtain a contradiction in the case when at least one of the numbers \( \Phi_{n,j}(y'' + \tau e_1) \) for \( n > q, j = 0, \ldots, n \), does not vanish.
Therefore, for $t = y' + \tau e_1$ equality (17) implies
\[
(y'_1 + \tau)^{-p}(F(y), \varphi(y' + \tau e_1 - y)) = \Phi_{p,p}(y' + \tau e_1) + \sum_{m=0}^{p-1} (y'_1 + \tau)^{m-p} \sum_{n=m}^{p-1} \Phi_{n,m}(y' + \tau e_1),
\]
and for $t = y'' + \tau e_1$ implies
\[
(y''_1 + \tau)^{-q}(F(y), \varphi(y'' + \tau e_1 - y)) = \Phi_{q,q}(y'' + \tau e_1) + \sum_{m=0}^{q-1} (y''_1 + \tau)^{m-q} \sum_{n=m}^{q-1} \Phi_{n,m}(y'' + \tau e_1).
\]
If $\tau$ is large enough, then we get
\[
(y'_1 + \tau)^{-p}(F(y), \varphi(y' + \tau e_1 - y)) \neq 0, \quad (y''_1 + \tau)^{-q}(F(y), \varphi(y'' + \tau e_1 - y)) \neq 0,
\]
hence there exists $t' \in B(y' + \tau e_1, \alpha) \cap T$ and $t'' \in B(y'' + \tau e_1, \alpha) \cap T$ with the same $\tau$. By (18), we obtain a contradiction. Thus our assumption is valid for all $n, m \leq N$.

**Proof of Theorem 10.** It follows from Proposition 1(i) that there is $K \in \mathbb{N} \cup \{0\}$ such that $\mu_k \equiv 0$ for $\|k\| > K$.

In the case (a), for any $\varphi \in S(\mathbb{R}^d)$ and $k$ we have
\[
(\hat{\mu}_k(y), \varphi(t - y)) = (\mu_k(x), \hat{\varphi}(x)e^{-2\pi i \langle x, t \rangle}) = \sum_{\lambda \in \Lambda} p_k(\lambda) \hat{\varphi}(\lambda)e^{-2\pi i \langle \lambda, t \rangle}.
\]
Since $\hat{\varphi} \in S(\mathbb{R}^d)$, we get $\hat{\varphi}(x) = o(|x|^{-d-1})$. Hence,
\[
\sum_{\lambda \in \Lambda} |p_k(\lambda)||\hat{\varphi}(\lambda)| \leq C_1 + C_2 \sum_{\lambda \in \Lambda, |\lambda| > 1} |\lambda|^{-d-1} = C_1 + C_2 \int_1^{\infty} s^{-d-1}dn(s),
\]
where
\[
n(s) = \#(\Lambda \cap B(0, s)).
\]
The set $\Lambda$ is of bounded density, therefore, $n(s) = O(s^d)$ and the integral
\[
\int_1^{\infty} s^{-d-1}n(ds) = -n(1) + (d + 1) \int_1^{\infty} \frac{n(s)ds}{s^{d+2}}
\]
is finite. Consequently the sum in (19) is absolutely convergent, and the function $(\hat{\mu}_k(y), \varphi(t - y))$ is almost periodic in $t \in \mathbb{R}^d$. 

In the case (b), by Proposition 1(ii), we get
\begin{equation}
    p_k(\lambda) = O(\vert \lambda \vert^M) \quad \text{as} \quad \lambda \to \infty.
\end{equation}
Since $\hat{\varphi}(x) = o(|x|^{-M-d-1})$ for each $H$, we can repeat the previous arguments and obtain that $\hat{\mu}_k$ are almost periodic distributions.

Further, by (14), we have
\begin{equation}
    \hat{f} = \sum_{k_1=0}^{K} (2\pi i)^{k_1} y_1^{k_1} g_{k_1},
\end{equation}
where
\begin{equation}
    g_{k_1}(y) = \sum_{k_2+\cdots+k_d \leq K-k_1} (2\pi i)^{k_2+k_d} y_2^{k_2} \cdots y_d^{k_d} \hat{\mu}_{k_1,k_2},
\end{equation}
and for any $\varphi \in C^\infty$ with compact support
\begin{equation}
    (g_{k_1}(y), \varphi(t-y)) = \sum_{k_2+\cdots+k_d \leq K-k_1} (2\pi i)^{k_2+k_d} \hat{\mu}_{k_1,k_2} y_2^{k_2} \cdots y_d^{k_d} \varphi(t-y)).
\end{equation}
Note that each term of the last sum can be rewritten as
\begin{equation}
    \sum_{m_2 \leq k_2, \ldots, m_d \leq k_d} c_{m,k} t_2^{k_2-m_2} \cdots t_d^{k_d-m_d} [(\hat{\mu}_k, (t_2 - y_2)^{m_2} \cdots (t_d - y_d)^{m_d} \varphi(t-y))]
\end{equation}
with some constants $c_{m,k}$. Since $\hat{\mu}_k$ are almost periodic distributions, we get that the expressions in square brackets are almost periodic functions in $t \in \mathbb{R}^d$, and hence in $t_1 \in \mathbb{R}$. Therefore the functions $(g_{k_1}(y), \varphi(t-y))$ are almost periodic in $t_1 \in \mathbb{R}$ for any fixed $(t_2, \ldots, t_d) \in \mathbb{R}^{d-1}$. Applying Lemma 1 to distributions $g_{k_1}, k_1 = 0, \ldots, K$, we get that they have uniformly discrete supports and $\eta(\text{supp } g_{k_1}) \geq \eta(\Gamma)$.

For a fixed $k_1$ we have
\begin{equation}
    g_{k_1} = \sum_{k_2=0}^{K-k_1} (2\pi i)^{k_2} y_2^{k_2} g_{k_1,k_2},
\end{equation}
where
\begin{equation}
    g_{k_1,k_2}(y) = \sum_{k_3+\cdots+k_d \leq K-k_1-k_2} (2\pi i)^{k_3+k_d} y_3^{k_3} \cdots y_d^{k_d} \hat{\mu}_{k_1,k_2},
\end{equation}
The functions $(g_{k_1,k_2}(y), \varphi(t-y))$ are almost periodic in $t_2 \in \mathbb{R}$ for any fixed $(t_1, t_3, \ldots, t_d) \in \mathbb{R}^{d-1}$. Applying Lemma 1 to distributions $g_{k_1,k_2}, k_2 = 0, \ldots, N-k_1$ with respect to the variable $y_2$, we get that these distributions have uniformly discrete supports and $\eta(\text{supp } g_{k_1,k_2}) \geq \eta(\Gamma)$. 
After a finite number of steps we get that the support $\Gamma_k$ of the distributions $\hat{\mu}_k$ for all $k \in (\mathbb{N} \cup \{0\})^d$ are uniformly discrete and satisfy (12). If the masses $p_k(\lambda)$ are uniformly bounded, then Proposition 1(iii), yields that $\hat{\mu}_k$ are measures with uniformly bounded masses. If masses $p_k(\lambda)$ satisfy (20), we have
\[
\sum_{|\lambda| \leq r} |p_k(\lambda)| \leq C_1 + C_2 r^M n(r) = O(r^{M+d}) \quad \text{as } r \to \infty.
\]
Applying Proposition 1(iii) with $K = 0$, we obtain that the coefficients $q_{k,j}(\gamma)$ in (13) are uniformly bounded.

Remark 3: Let $\Lambda$ be uniformly discrete. If we replace the condition $p_k(\lambda) = O(1)$ by $p_k(\lambda) = o(|\lambda|)$, then we get for any $\varepsilon > 0$ and appropriate $r(\varepsilon)$
\[
\sum_{|\lambda| \leq r} |p_k(\lambda)| \leq \sum_{|\lambda| \leq r(\varepsilon)} |p_k(\lambda)| + \sum_{r(\varepsilon)<|\lambda| \leq r} |p_k(\lambda)| \leq C(\varepsilon) + \varepsilon r n(r).
\]
Hence, (21) is $o(r^{d+1})$ as $r \to \infty$. Applying Proposition 1(iii) with $K=0$, $M=1$ and taking into account Remark 1, we obtain that $\hat{\mu}_k$ (and, of course, $\hat{\mu}_k$) are measures with uniformly bounded masses. Applying Proposition 1(iii) to the measures $\hat{\mu}_k$, we obtain that the coefficients $p_k(\lambda)$ are uniformly bounded.

7. Proofs of the main theorems

Proof of Theorem 8. By Theorem 10, the measures $\mu_k$ from (11) have uniformly discrete spectra. Applying Theorem 3 to these measures, we get that supp $\mu_k$ is contained in a finite union of translates of a full-rank lattice. To obtain the last assertion of the theorem, we take the measures $e^{-it\xi}\mu_k$ instead of the measures $\mu_k$.

Proof of Theorem 9. Notice that masses of the measures $\mu_k$ from (11) are not assumed to be separated from zero, therefore we cannot make use of the result of Theorem 5.

Let $\varphi(y)$ be a nonnegative $C^\infty$-function with support in the unit ball such that $\int \varphi(t)dt = 1$. Pick $k \in (\mathbb{N} \cup \{0\})^d$, and set $\rho_\varepsilon = \varepsilon^d\varphi(\varepsilon y)\hat{\mu}_k(y)$, where $\hat{\mu}_k$ is the inverse Fourier transform of the measure $\mu_k$. We have
\[
\hat{\rho}_\varepsilon(s) = \int_{\mathbb{R}^d} \hat{\varphi}\left(\frac{s-x}{\varepsilon}\right) \mu_k(dx) = \left(\int_{|s-x|<1} + \int_{|s-x|\geq1}\right) \hat{\varphi}\left(\frac{s-x}{\varepsilon}\right) \mu_k(dx).
\]
Since $\varphi \in S(\mathbb{R}^d)$, we get $\hat{\varphi}(y) = o(|y|^{-d-1})$ as $|y| \to \infty$ and $\hat{\varphi}\left((s-x)/\varepsilon\right) \to 0$ for $s \neq x$ as $\varepsilon \to 0$. Also, $\hat{\varphi}(0) = 1$. Hence, by the Dominated Convergence
Theorem,
\[
\lim_{\varepsilon \to 0} \int_{|s-x|<1} \phi\left(\frac{s-x}{\varepsilon}\right)\mu_k(dx) = \begin{cases} 
\mu_k(\{\lambda\}) = p_k(\lambda), & s = \lambda \in \Lambda, \\
0, & s \notin \Lambda.
\end{cases}
\]

Note that the masses \(\mu_k(\{\lambda\})\) are uniformly bounded. Also,
\[
n_s(t) := \#(\Lambda \cap B(s,t)) = O(t^d) \quad (t \to \infty).
\]

Therefore,
\[
\left| \int_{|s-x|\geq 1} \phi\left(\frac{s-x}{\varepsilon}\right)\mu_k(dx) \right| \leq C \int_1^\infty (t/\varepsilon)^{-d-1} n_s(dt)
\leq \varepsilon^{d+1}(d+1)C \int_1^\infty t^{-d-2} n_s(t)dt,
\]
and the left-hand side tends to zero as \(\varepsilon \to 0\).

Furthermore, by Theorem 10, \(\hat{\mu}_k\) has uniformly bounded masses and support of bounded density. Hence, \(|\hat{\mu}_k|(B(0,r)) = O(r^d)\) as \(r \to \infty\) and the same is valid for the measures \(\hat{\rho}_k\). Therefore,
\[
|\rho_{\varepsilon}|(B(0,r)) \leq \varepsilon^d|\hat{\mu}_k|(B(0,1/\varepsilon)) = O(1) \quad (\varepsilon \to 0).
\]

Since support of the measure \(\rho_{\varepsilon}\) is bounded for each fixed \(\varepsilon\), we see that the measures \(\rho_{\varepsilon}\) form a bounded family of linear functionals on the space of continuous bounded functions on \(\mathbb{R}^d\) and, in particular, on its subspace \(C(\mathcal{R})\). A closed ball in the conjugate space \(C^*(\mathcal{R})\) is a compact set in the weak-star topology. Hence there exists a measure \(\tau_k\) on \(\mathcal{R}\) with a total variation \(\|\tau_k\| < \infty\) such that for every \(f \in C(\mathcal{R})\) there is a subsequence \(\varepsilon' \to 0\), for which we have
\[
\langle \rho_{\varepsilon'}, f \rangle \to \langle \tau_k, f \rangle.
\]

Applying this to every character \(x \in \mathbb{R}^d\) in the place of \(f\), we obtain from (22)
\[
\hat{\tau}_k(x) = \lim_{\varepsilon' \to 0} \hat{\rho}_{\varepsilon'}(x) = \begin{cases} 
p_k(\lambda), & x = \lambda \in \Lambda, \\
0, & x \notin \Lambda.
\end{cases}
\]

Note that \(\hat{\tau}_k(x)\) is a continuous function with respect to the discrete topology on \(\mathbb{R}^d\), and \(|p_k(\lambda)| \leq \|\tau_k\|\) for all \(\lambda \in \mathbb{R}^d\).
Let $\psi$ be a $C^\infty$-function with compact support. Since the inverse Fourier transform of the function $\psi(t - x)$ is $e^{2\pi i (t,y)} \hat{\psi}(y)$, we get

$$(\psi \ast \mu_k)(t) = \int_{\mathbb{R}^d} \psi(t - x) \mu_k(dx) = \int_{\mathbb{R}^d} e^{2\pi i (y,t)} \hat{\psi}(y) \hat{\mu}_k(dy)$$

(23)

$$= \sum_{\gamma \in \text{supp } \hat{\mu}_k} \hat{\mu}_k(\gamma) \hat{\psi}(\gamma) e^{2\pi i (\gamma,t)}.$$

Clearly, $\hat{\psi} \in S(\mathbb{R}^d)$ and $\hat{\psi}(\gamma) = o(|\gamma|^{-d-1})$ as $\gamma \to \infty$. Also, the measure $\hat{\mu}_k$ has uniformly bounded masses and uniformly discrete support, therefore the series in (23) is absolutely converges and $(\psi \ast \mu_k)(t) \in W$. In particular, the measure $\mu_k$ is almost periodic.

Furthermore, the function

$$F(t) = \sum_{\|k\| \leq K} (\psi \ast \mu_k)(t) \overline{(\psi \ast \mu_k)(t)}$$

belongs to $W$ as well. Now suppose that $\psi$ has support in the ball $B(0, \eta(\Lambda)/2)$ and $\psi(0) = 1$. Taking into account that

$$\hat{(\psi \ast \mu_k)}(\lambda) = p_k(\lambda) \quad \text{for } \lambda \in \Lambda,$$

and using (10), we get

$$F(\lambda) = \sum_{\|k\| \leq K} |p_k(\lambda)|^2 \geq \alpha > 0 \quad \forall \lambda \in \Lambda.$$

Using Proposition 3 with $h(z) = 1/z$, we construct the function

$$g(t) = \sum_n c_n e^{2\pi i (t,\tau_n)} \in W$$

with the property $g(\lambda) = 1/F(\lambda)$ for $\lambda \in \Lambda$.

Consider the dual pair $(\mathfrak{R}, \mathbb{R}_{\text{discr}}^d)$. Set

$$a = \sum_{\|k\| \leq K} \tau_k(t) \ast \tau_k(-t), \quad \text{t} \in \mathfrak{R}.$$ 

We get

$$\hat{a}(\lambda) = F(\lambda) \quad \forall \lambda \in \Lambda, \quad \text{and} \quad \hat{a}(x) = 0 \quad \forall x \notin \Lambda.$$

Since $\sum_n |c_n| < \infty$, we see that the series $\sum_n c_n a(y + \tau_n)$ converges with respect to the norm to a measure $\hat{b}(y)$ such that

$$\hat{b}(x) = \sum_n c_n e^{2\pi i \langle x, \tau_n \rangle} \hat{a}(x) = \begin{cases} 1, & x \in \Lambda, \\ 0, & \text{otherwise}. \end{cases}$$
By Theorem 6, \( \Lambda \) belongs to the coset ring of \( \mathbb{R}^d_{\text{discr}} \). By Theorem 7,
\[
(25) \quad \Lambda = \bigcup_{n=1}^{N} \left( A_n \setminus \bigcup_{s=1}^{S_n} B_n^s \right),
\]
where \( A_n, B_n^s \) are cosets of discrete lattices.

Let \( A, A' \) be two arbitrary cosets in \( \mathbb{R}^d \) and \( L, L' \) be the corresponding lattices. Clearly, \( A \cap A' \) is a coset of \( L \cap L' \). If \( r(A \cap A') = d \), then the factor-groups \( L/(L \cap L') \) and \( L'/(L \cap L') \) are finite. Therefore, \( A \setminus A' = A \setminus (A \cap A') \) is a finite union of disjoint cosets of \( L \cap L' \), and the same is valid for \( A \cup A' = (A \setminus (A \cap A')) \cup (A' \setminus (A \cap A')) \cup (A \cap A') \).

We repeat this transformation for each pair of cosets with dimension \( d \) of their intersection. After a finite number of steps, we get a representation similar to (25) such that \( r(B_n^s) < d \) and \( r(A_n \cap A_{n'}) < d \) for \( n \neq n' \).

Put \( \delta_A = \sum_{x \in A} \delta_x \). Using the equalities \( \delta_{A \setminus A'} = \delta_A - \delta_{A \cap A'} \) and \( \delta_{A \cup A'} = \delta_A + \delta_{A'} - \delta_{A \cap A'} \), we get
\[
\delta_{\Lambda} = \sum_{j=1}^{J} \delta_{D_j} + \sum_{j=1}^{J_1} \delta_{E_j} - \sum_{j=1}^{J_2} \delta_{F_j},
\]
where \( D_j, E_j, F_j \) are cosets such that \( r(D_j) = d \) and \( r(E_j) < d, r(F_j) < d \).

Every coset \( D_j \) is a translation of a full-rank lattice, hence \( \delta_{D_j} \) is \( d \)-periodic and \( \sum_{j=1}^{J} \delta_{D_j} \) is almost periodic. Above we showed that all measures \( \mu_k \) are almost periodic. By Proposition 2(ii), the measures \( |\mu_k| \) are almost periodic, hence the measure \( \sum_k |\mu_k| \) with support \( \Lambda \) is almost periodic as well. Using (10) and Proposition 2(iii), we get that the measure \( \delta_{\Lambda} \) is almost periodic. Therefore the left-hand side of the equality
\[
(26) \quad \sum_{j=1}^{J_1} \delta_{E_j} - \sum_{j=1}^{J_2} \delta_{F_j} = \delta_{\Lambda} - \sum_{j=1}^{J} \delta_{D_j}
\]
is an almost periodic measure as well. But its support is contained in a finite union of hyperplanes and isn’t relatively dense. Therefore the measure in the left-hand side of (26) is identically zero, and \( \delta_{\Lambda} = \sum_{j=1}^{J} \delta_{D_j} \). Since the measure \( \delta_{\Lambda} \) has the unit masses at every point of \( \Lambda \), we see that the cosets \( D_j \) are pairwise disjoint and for some full-rank lattices \( L_j \) and \( \lambda_j \in \mathbb{R}^d \)
\[
(27) \quad \Lambda = \bigcup_{j=1}^{J} D_j = \bigcup_{j=1}^{J} (\lambda_j + L_j).
\]
Recall that $\psi \ast \mu_k \in W$. It follows from (24) and (27) that

$$(28) \quad \mu_k = \sum_{j=1}^{J} (\psi \ast \mu_k) \delta_{L_j} + \lambda_j = \sum_{j=1}^{J} \sum_{s} a_{s,k} e^{2\pi i \langle x, \gamma_{s,k} \rangle} \delta_{L_j} + \lambda_j$$

for some $\gamma_{s,k} \in \mathbb{R}^d$ and $a_{s,k} \in \mathbb{C}$ such that $\sum_{s} |a_{s,k}| < \infty$. For every $\gamma \in \mathbb{R}^d$ there is $\beta$ inside the parallelepiped generated by corresponding $L_j^*$ such that $\gamma - \beta \in L_j^*$. Therefore, $e^{2\pi i \langle x, \gamma_{s,k} \rangle} = e^{2\pi i \langle x, \beta_{s,k,j} \rangle}$ for $x \in L_j$, and the set of all points $\beta_{s,k,j}$ is bounded. Therefore (28) can be rewritten as

$$\sum_{j=1}^{J} \sum_{s} \sum_{x \in L_j} b_s(k,j) e^{2\pi i \langle x, \beta_{s,k,j} \rangle} \delta_{x} + \lambda_j \quad \text{with} \quad b_s(k,j) = a_{s,k} e^{2\pi i \langle \lambda_j, \gamma_{s,k} - \beta_{s,k,j} \rangle}.$$ 

By (14), we obtain the statement of Theorem 9.  

Here we actually proved the following statement:

**Proposition 4:** Let $\{\nu_n\}$ be a finite set of measures from $S^*(\mathbb{R}^d)$ such that for each $n$

(a) $\text{supp} \nu_n$ is uniformly discrete,

(b) $\text{supp} \hat{\nu}_n$ is locally finite of bounded density,

(c) $\hat{\nu}_n$ is a measure with uniformly bounded masses.

Set $\mathfrak{L} = \bigcup_{n} \text{supp} \nu_n$. If

$$\inf_{\lambda \in \mathfrak{L}} \max_{n} |\nu_n(\lambda)| > 0,$$

then $\mathfrak{L}$ is a finite union of translates of several full-rank lattices.

Note that the boundedness of masses $\nu_n(\lambda)$ follows from (23) with $\nu_n$ instead of $\mu_k$ and $\psi$ such that $\text{supp} \psi \subset B(0, \eta(\text{supp} \nu_n))$ and $\psi(0) = 1$.

**Remark 4:** It is not hard to check that conditions (b) and (c) can be replaced by

(b') $\hat{\nu}_n$ is a purely point measure,

(c') $|\hat{\nu}_n|(B(0, r)) = O(r^d)$ as $r \to \infty$.

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