Critical escape velocity for a charged particle around a weakly magnetized naked singularity

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Abstract

We examine the motion of a charged particle in the vicinity of a weakly magnetized naked singularity. The escape velocity and energy of the particle moving around the naked singularity after being kicked by another particle or photon are investigated. Also at the innermost stable circular orbit (ISCO) escape velocity and energy are examined. Effective potential and angular momentum of the particle are also discussed. We discuss the center of mass energy after collision between two particles having same mass and opposite charges moving along the same circular orbit in the opposite direction. It is investigated that under what conditions maximum energy can be produced as a result of collision.

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1 Introduction

It is believed that particles in the surroundings of black holes are significantly influenced by the strong gravitational pull. Although, other forces that are usually supposed to be weaker are also at work near these objects. These include forces applied by the magnetic field and pressure of the infalling hot gases. Recent investigations have brought up the evidence that magnetic forces can be as powerful as gravity near supermassive black holes. But most of the work till now has been investigated for weak magnetic forces around black holes. The phenomenon of jet formation in black holes is widely under observation nowadays. Adequate amount of energy to establish and strengthen jets is yielded by the matter in accretion disk and rotation of the black hole. It is most likely believed that magnetic fields are responsible for the transfer of energy to the jets [1–3].

There are several papers that examine the dynamics of particles moving around weakly magnetized black holes. Motion of a charged particle near weakly magnetized Schwarzschild black hole is analyzed in [4–6]. The chaotic motion of a charged particle around Kerr black hole perturbed by magnetic field is investigated in [7–13]. Circular motion of charged particles around Reissner-Nordström spacetime is discussed in [14]. Chaotic particle motion in the Majumdar-Papapertou metric is discussed in [15,16]. Furthermore, collision between particles and their escape energies after collision around Kiselev black hole [17] and a slowly rotating Kerr black hole [18] have been studied.

There are compelling arguments that black hole candidates in astrophysics could well be naked singularities instead of black holes, and that the distinction between the two possibilities may have observational consequences [19–21]. In this paper we discuss motion of a charged particle around weakly magnetized naked singularity sourced by a massless scalar field. It is an extension of the Schwarzschild geometry when a massless scalar field is added to it which deforms the event horizon into a naked singularity. This solution was first discovered by Fisher [22], and subsequently rediscovered by Janis, Newman and Winicour [23] and others [24–26]. This solution is known in literature as the Fisher solution, or the Janis-Newman-Winicour (JNW) solution. The understanding of the physical properties and other features of this spacetime were expanded by Janis, Robinson and Winicour, [27]; therefore it is also sometimes known as the Fisher-Janis-Robinson-Winicour (FJRW) spacetime. A generalization to higher dimensions was given in [28], and an analysis of its properties in higher dimensions was performed by Abdolrahimi and Shoom [29]. Since this spacetime is referred to under many different names, for the purposes of this paper, we shall follow references related to the context of our paper such as [20, 30] and refer to this solution as the JNW spacetime. The charged generalization of this solution was obtained by Penney [31], and an exact magnetized solution was given by [32], and a further accelerating generalization was given in [33].

In [6] motion of a charged particle is discussed in the vicinity of weakly magnetized
Schwarzschild black hole. The effect of the magnetic field on a charged particle is similar to the effect of the black hole rotation on a neutral particle. Like fast-rotating black holes weakly magnetized black holes behaves as particle accelerators. The case of particle motion around weakly magnetized Kerr black holes was considered recently in [34]. The critical escape energy and velocity of the kicked charged particle with different initial radial velocities have been investigated in [35]. Recently Lim studied the motion of charged particles around an electrified black hole [36].

Geodesics around the JNW spacetime was considered in previously in Refs. [30, 37, 38]. Here, our core interest is to study the motion of a charged particle orbiting in the JNW spacetime after being hit by another particle. The external magnetic fields we are considering are homogenous at spatial infinity, which can be obtained by performing a Harrison-type transformation to the JNW spacetime [32]. After collision the particle may pursue its motion in distinctive trajectories. We examine that, under what circumstances can a particle leave the locality of naked singularity and what causes the particle to fall in the naked singularity.

In Sec. 2, we derive the general equations of motion for a charged particle in a magnetized JNW spacetime. The equations will contain special cases that will be studied in subsequent sections. Particularly, in Sec. 3 we consider the special case of a particle in the absence of a magnetic field. The value of inner most stable circular orbit (ISCO) is investigated. Escape velocity of the particle is determined, also at the ISCO its value is computed. Energy of the particle is brought under analysis and its value at the ISCO is calculated. In Sec. 4, the motion of charged particles in weakly magnetized spacetimes are discussed. To simplify the analysis, dimensionless forms of the equations of motion, effective potential, energy and escape velocity are discussed. Dimensionless angular momentum and magnetic field are also investigated. In Sec. 5, we made an analysis of collision between two particles with same mass and opposite charges revolving around the naked singularity and computed the center-of-mass energy produced as a result of collision.

2 Particles in a magnetized JNW spacetime

2.1 Metric

We will be considering the motion of a charged particle in the magnetized JNW (or FJRW/Fisher) spacetime [32]. This can be obtained by taking the ordinary JNW metric and applying a Harrison-type transformation [32, 39–41] to it. The result is an exact
solution given by the metric

\[ ds^2 = \Lambda^2 \left( -f'' dt^2 + f' dr^2 + r^2 f^{1-\nu} d\theta^2 \right) + \Lambda^{-2} r^2 f^{1-\nu} \sin^2 \theta \, d\phi^2, \]

\[ f = 1 - \frac{r_g}{r}, \quad \Lambda = 1 + \frac{1}{4} B^2 r^2 f^{1-\nu} \sin^2 \theta, \]  

(1)

while the massless scalar field is given by

\[ \varphi = \sqrt{\frac{1 - \nu^2}{2}} \ln f, \]

(2)

and the Maxwell potential is

\[ A = \frac{B r^2 f^{1-\nu} \sin^2 \theta}{2\Lambda} \, d\phi. \]

(3)

The Maxwell tensor accordingly is \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). It can be verified that the metric (1), together with (2) and (3) solves the four-dimensional Einstein-Maxwell-scalar equations.

The parameters \( r_g \) and \( \nu \) are related to the black hole mass \( M \) and scalar charge \( q \) by

\[ M = \frac{1}{2} \nu r_g, \quad q = \frac{1}{2} r_g \sqrt{1 - \nu^2}, \]

(4)

while the parameter \( B \) is related to the strength of the axisymmetric magnetic field. We can see that the case \( \nu = 1 \) (or, equivalently, \( q = 0 \)) reduces to the magnetic Ernst spacetime. On the other hand, for \( B = 0 \) we recover the usual JNW spacetime. Setting both \( B = 0 \) and \( \nu = 1 \) gives the Schwarzschild spacetime. Similar to the unmagnetized case, the metric (1) has a strong curvature singularity at \( r = r_g \). Similar to the Ernst solution, this solution is not asymptotically flat.

### 2.2 Equations of motion

For a test particle of charge per unit mass \( e \), its motion is described by a trajectory \( x^\mu(\tau) \), where \( \tau \) is an appropriate affine parametrization. The motion is determined by the Lagrangian \( L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + e A_\mu \dot{x}^\mu \), where overdots denote derivatives with respect to \( \tau \). The equations of motion satisfied by \( x^\mu(\tau) \) can be derived with the Euler-Lagrange equations \( \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu} \).

In the following, it will be convenient to define

\[ F \equiv f', \quad G \equiv f^{1-\nu}, \]

(5)

such that the Lagrangian describing a charged particle in the magnetized JNW spacetime
is given by
\[ L = \frac{1}{2} \left[ \Lambda^2 \left( -F t^2 + \frac{r^2}{F} + r^2 G \dot{\theta}^2 \right) + \frac{r^2 G \sin^2 \theta}{\Lambda^2} \dot{\phi}^2 \right] + \frac{e B r^2 G \sin^2 \theta}{2\Lambda} \dot{\phi}. \] (6)

Since \( t \) and \( \phi \) are cyclic variables in the Lagrangian, they give rise to constants of motion \( E \) and \( L \), which we interpret as the energy and angular momentum of the particle. The conserved quantities reduce the equations for \( t \) and \( \phi \) into first integrals:
\[ \dot{t} = \frac{\mathcal{E}}{\Lambda^2 F}, \quad \dot{\phi} = \frac{\Lambda^2}{r^2 G \sin^2 \theta} \left( \mathcal{L} - \frac{e B r^2 G \sin^2 \theta}{2\Lambda} \right). \] (7)

Applying the Euler-Lagrange equation to the remaining two coordinates gives
\[ \ddot{r} = \left( \frac{F'}{2F} - \frac{\partial_r \Lambda}{\Lambda} \right) r^2 + FG \left( r + \frac{r \partial_r \Lambda}{2G} + \frac{r^2 G'}{2G} \right) \dot{\theta}^2 - \frac{2\partial_r \Lambda}{\Lambda} \dot{r} \dot{\theta} + \frac{F}{r^3 G \sin^2 \theta} \left( 1 - \frac{r \partial_r \Lambda}{2G} + \frac{r G'}{2G} \right) \left( \mathcal{L} - \frac{e B r^2 G \sin^2 \theta}{2\Lambda} \right)^2 \] 
\[ + \frac{e B F}{r\Lambda} \left( 1 - \frac{r \partial_r \Lambda}{2G} + \frac{r G'}{2G} \right) \left( \mathcal{L} - \frac{e B r^2 G \sin^2 \theta}{2\Lambda} \right) - \left( \frac{\partial_r \Lambda}{\Lambda} + \frac{G'}{2G} \right) \mathcal{E}^2 \Lambda^3, \] (8)

\[ \ddot{\theta} = \frac{\partial_\theta \Lambda}{\Lambda} \left( \frac{\dot{r}^2}{Fr^2 G} - \dot{\theta}^2 \right) - 2 \left( \frac{\partial_r \Lambda}{\Lambda} + \frac{1}{r} + \frac{G'}{2G} \right) \dot{r} \dot{\theta} - \frac{\mathcal{E}^2 \partial_\theta \Lambda}{\Lambda^3 Fr^2 G} \] 
\[ + \frac{1}{r^4 G^2 \sin^3 \theta} \left( \cos \theta - \frac{\sin \theta \partial_\theta \Lambda}{\Lambda} \right) \left( \mathcal{L} - \frac{e B r^2 G \sin^2 \theta}{2\Lambda} \right)^2 \] 
\[ + \frac{e B}{\Lambda r^2 G \sin \theta} \left( \cos \theta - \frac{\sin \theta \partial_\theta \Lambda}{2\Lambda} \right) \left( \mathcal{L} - \frac{e B r^2 G \sin^2 \theta}{2\Lambda} \right). \] (9)

Here, the primes appearing in \( F' \) and \( G' \) denote derivatives with respect to \( r \). The invariance of \( g_{\mu \nu} \cdot \dot{x}^\mu \cdot \dot{x}^\nu \equiv \epsilon \) gives a constraint equation
\[ -\frac{\mathcal{E}^2}{\Lambda^2 F} + \frac{\Lambda^2}{r^2 G \sin^2 \theta} \left( \mathcal{L} - \frac{e B r^2 G \sin^2 \theta}{2\Lambda} \right)^2 + \Lambda^2 \left( \frac{\dot{r}^2}{F} + r^2 G \dot{\theta}^2 \right) = \epsilon. \] (10)

By appropriately rescaling the affine parameter \( \tau \), the magnitude of \( \epsilon \) can be set to unity if it is nonzero. Therefore in the following, we have \( \epsilon = -1 \) for timelike particles and \( \epsilon = 0 \) for photons or null geodesics. The equations of motion (8) and (9), together with the constraint (10) contains various special cases for motion in related spacetimes. For instance, setting \( \nu = 1 \) describes the motion in the magnetic Ernst spacetime studied in Refs. [36, 42–44]. Setting \( B = 0 \) will give the geodesic equations in the JNW spacetime, and finally if \( \nu = 1 \) and \( B = 0 \) we have the well-known equations for Schwarzschild geodesics.
The constraint equation (10) can be also cast in the effective potential formulation

\[ \Lambda^2 \left( r^2 + r^2 F G \dot{\theta}^2 \right) = \mathcal{E}^2 - \mathcal{U}^2, \]  

(11)

where

\[ \mathcal{U}^2 = \frac{\Lambda^4 F}{r^2 G \sin^2 \theta} \left( \mathcal{L} - \frac{e Br^2 G \sin^2 \theta}{2 \Lambda} \right)^2 - \epsilon \Lambda^2 F. \]  

(12)

While in general, the effective potential \( \mathcal{U}^2 \) is a complicated function that depends on parameters \( \mathcal{L}, \nu, B \) and \( r_g \), we can make a few qualitative observations.

Firstly, we note the asymptotic behavior of \( \mathcal{U}^2 \), at large \( r \), is

\[ \mathcal{U}^2 \sim \frac{1}{256} B^6 (r \sin \theta)^6 (\mathcal{L}B - 2e)^2 + \mathcal{O}(r^5). \]  

(13)

Thus, at large distances away from the singularity, the potential increases to the order of \( (r \sin \theta)^6 \). Therefore, for large values of \( r \) where \( r \to \infty \), the potential remains finite if \( \theta \to 0 \) or \( \pi \). Thus a particle with finite energy \( \mathcal{E} \) can escape to infinity if the trajectory takes it along a path close to the polar axis \( \theta = 0 \) or \( \theta = \pi \).

Secondly, at distances close to the \( r = r_g \) singularity, due to the factor \( F/G = f^2 \nu^{-1} \) in front of the parenthesis in (12), we have

\[ \lim_{r \to r_g^+} \mathcal{U}^2 = \begin{cases} 
0, & \text{if } \nu > \frac{1}{2}, \\
+\infty & \text{if } \nu < \frac{1}{2} \text{ and } \left( \mathcal{L} - \frac{e Br^2 G \sin^2 \theta}{2 \Lambda} \right) \neq 0.
\end{cases} \]  

(14)

Therefore for \( \nu < \frac{1}{2} \) and \( \left( \mathcal{L} - \frac{e Br^2 G \sin^2 \theta}{2 \Lambda} \right) \neq 0 \) there exist an infinite potential barrier preventing the particle from reaching the \( r = r_g \) singularity.

### 2.3 Equatorial circular orbits

We can easily check that \( \theta = \pi/2 \) is a trivial solution to Eq. (9), where \( \ddot{\theta} = \dot{\theta} = 0 \). These orbits lie in the equatorial plane. The plots of \( \mathcal{U}^2 \) against \( r \) for various cases are shown in Fig. 1.

For the unmagnetized case \( B = 0 \), the effective potential obtained in [30] is reproduced. In particular, for \( \nu < \frac{1}{2} \) the potential becomes an infinite barrier. Now if the magnetic field is turned on, an additional gravitational attraction is provided by the magnetic field itself, manifesting in a potential well. Thus charged particles in the magnetized JNW spacetime is bound more tightly compared to the unmagnetized case.

For concreteness, we will be mainly be interested in a charged particles which are initially in the innermost stable circular orbit (ISCO), while second particle is coming from infinity. They both collide in the ISCO. We may define circular orbits to be those of
Figure 1: (Color online) Plots of $U^2$ vs $r$ at $\theta = \pi/2$ and $r_g = 2$ for timelike ($\epsilon = -1$) particles of charge $e = 1$, with various values of $B$ and $\mathcal{L}$. Each plot contains curves of increasing values of $\nu$, from bottom to the top, $\nu = 1, 0.9, 0.7, 0.5$ and $0.3$. 

(a) $B = 0, \mathcal{L} = 3.75$. 
(b) $B = 0.03, \mathcal{L} = 3.75$. 
(c) $B = 0.06, \mathcal{L} = 3.75$. 
(d) $B = 0, \mathcal{L} = -3.75$. 
(e) $B = 0.03, \mathcal{L} = -3.75$. 
(f) $B = 0.06, \mathcal{L} = -3.75$.
constant $r$ and $\theta$. We seek such orbits exist in the equatorial plane where $\theta$ is a constant equal to $\pi/2$, thus Eq. (9) is automatically satisfied. By demanding that $r = r_o$ is a constant in (8) and (10), we have,

\[
\mathcal{L}_\pm = \frac{2r_o G \left[ eB^2r_o^4G (6B^2r_oFG + 3B^2r_o^2FG') + F'(4 + B^2r_o^2G) \pm \sqrt{K} \right]}{(4 + B^2r_o^2G) \left[ (3B^2r_o^2G - 4)(2G + r_oG') \right]},
\]

(15)

\[
\mathcal{E}^2_\pm = \frac{F(4 + B^2r_o^2G)^2}{256r_o^2G} \times \left[ (4 + B^2r_o^2G)^2 \mathcal{L}_\pm^2 - 4eBGr_o^2(4 + B^2r_o^2G)\mathcal{L}_\pm + 4Gr_o^2(e^2B^2r_o^2G - 4\epsilon) \right],
\]

(16)

where

\[
K = 2B^2r_o^2G(2G + r_oG')^2 \left( 3r_o^2G\epsilon B^2 - 4\epsilon + 2\epsilon^2 \right) F^2
\]

\[
+ \epsilon \rho_o F' \left( 4 + B^2r_o^2G \right) \left( 5B^2r_o^2G - 4 \right) (2G + r_oG') F + r_o^2G F^2 \epsilon (4 + B^2r_o^2G)^2
\]

(17)

We consider the stability of a circular orbit by perturbing by perturbing about the radius $r_o$ by writing

\[
r(\tau) = r_o + \epsilon r_1(\tau).
\]

(18)

We substitute this into Eq. (8). To first order in $\epsilon$, the equations of motion reduce to

\[
\ddot{r}_1 = -\omega^2 r_1,
\]

(19)

where, in general, the expression for $\omega$ is a complicated function of $\nu$, $B$, $r_g$ and $r_0$ that is too complicated to be shown here, however it can be handled straightforwardly using a standard computer algebra software such as MAPLE.

We can, however, study the qualitative behavior of $\omega^2$ for various spacetime and orbital parameters. Figure 2 shows the values of $\omega^2$ for various $B$ and $\nu$. The circular orbits are stable for $\omega^2 > 0$, and we can see that for the case $B = 0$ and $\nu = 1$ (the solid curve in Fig. 2a) that stable, circular orbits exist for $r_o > 3r_g$. This is the well-known case of circular orbits in the Schwarzschild spacetime.

Turning on the magnetic field, while keeping $\nu = 1$ corresponds to circular orbits in the magnetic Ernst spacetime. We see that the effect of the magnetic field brings the stable range of circular orbits closer to the black hole. (See the dotted, dashed and dash-dotted curves in Fig. 2a and 2d.)

Setting $\nu < 1$, we see that spacetimes with a massless scalar field can support stable circular orbits of even smaller radii, in particular, for $\nu < 1/2$ all circular orbits up to those infinitesimally close to the singularity $r = r_g$ are stable. This is due to the existence of the infinite potential barrier, as shown in (14).
\[ \nu = 1, L = L_+. \]
\[ \nu = 0.7, L = L_+. \]
\[ \nu = 0.4, L = L_+. \]
\[ \nu = 1, L = L_. \]
\[ \nu = 0.7, L = L_. \]
\[ \nu = 0.4, L = L_. \]

Figure 2: (Color online) Plots of \( \omega^2 \) vs \( r_0 \), for \( e = 1 \) and various \( \nu \) in units where \( r_g = 2 \). For each value of \( \nu \), the solid, dotted, dashed and dash-dotted curves respectively correspond to \( B = 1, 0.025, 0.05, \) and \( 0.75 \).

2.4 Escape trajectories

We consider particles initially in an ISCO which subsequently collides with another particle. In general, the collision would alter its energy and angular momentum, as well as its radial and angular velocities. For concreteness and simplicity, we restrict our attention to collisions which only changes the energy of the particle, while the angular momentum remains the same. Furthermore, we assume that the radial velocity remains unchanged, therefore, \( \dot{r} = 0 \) after collision. From (10), this restricts us to the fact that increasing the energy results \( \dot{\theta} \neq 0 \), i.e., the particle gets kicked out from the equatorial plane.

Plotting the curves of \( U^2 = E^2 \) shows the boundary of regions accessible to the particle after collision. To demonstrate a specific example, we consider particles kicked from an ISCO in a spacetime of \( B = 0.005, \nu = 0.9, r_g = 2 \) and \( e = 10 \). Choosing the lower sign in (15), and solving for \( \omega^2 = 0 \) using Eq.(19), we find that the initial ISCO radius is \( r_{\text{(ISCO)}} = 4.9490145 \), and the angular momentum and energy are respectively \( L_{\text{(ISCO)}} = 2.986451488 \) and \( E_{\text{(ISCO)}} = 0.8853807128 \). With these parameters the curves in \( U^2 = E^2 \) are plotted in Cartesian-type coordinates projected on a plane where \( \theta = 0 \), (or, \( y = 0 \)).

The dark regions in Figs.3a–3d indicate regions inaccessible to the particle. In particular, note that Figs.3c and 3d shows that the particle has the ability to escape far from the origin if it travels up or down where \( |z| \gg 1 \). Recalling that, at \( \phi = 0 \), we have \( z = r \sin \theta \),
Figure 3: (Color online) Curves of $U^2 = \mathcal{E}^2$ plotted as the boundary separating the regions accessible (white) and inaccessible (shaded) for a particle kicked from an ISCO into various possible energies. The spacetime parameters are $B = 0.005$, $\nu = 0.9$ and $r_g = 2$, and the charge per unit mass of the particle is $e = 10$. The initial ISCO radius for these spacetime parameters is $r_{(ISCO)} = 4.9490145$, as can be calculated from (19). The corresponding angular momentum is $\mathcal{L}_{-(ISCO)} = 2.986451488$.

hence this is consistent with the observation made in Eq. (14). The actual trajectory of a kicked particle may be obtained by integrating Eqs. (8) and (9) numerically. Continuing the example where $B = 0.005$, $\nu = 0.9$, $r_g = 2$ and $\mathcal{L}_{-(ISCO)}$, we track the motion of the particle after being kicked from the initial orbit at $r = r_{(ISCO)}$ into a new energy $\mathcal{E}$, with the angular momentum remaining the same and $\dot{r} = 0$ just after collision. The initial value of $\dot{\theta}$ just after collision is calculated from (10). With these initial conditions, we plot the trajectories for two possible values of $\mathcal{E} = 1.0$ and $\mathcal{E} = 1.1$, shown in Fig. 4.

As we can see, the particle after the kick may possibly fall into the $r = r_g$ singularity, as shown in Fig. 4a, or escape, shown in Fig. 4b while continuing to execute a cyclotron-like spiral due to the Lorentz interaction with the magnetic field.
Figure 4: (Color online) Trajectory of a particle kicked from an initial ISCO orbit where $B = 0.005$, $\nu = 0.9$, $r_g = 2$, $e = 10$ and $E_{(ISCO)} = 2.986451488$, for post-collision energies of $E = 1.0$ and $E = 1.1$. The sphere indicates the surface of the singularity $r = r_g = 2$, and the blue circles denote the particle’s original ISCO orbit ($r_{ISCO} = 4.9490145$) before the collision.
3 Escape velocity in the unmagnetized JNW spacetime

In this section, we consider the unmagnetized case where $B = 0$. The angular momentum and energy representing circular orbits, Eq. (15) and (16) for $B = 0$ are

$$\mathcal{L}_o = r_o \sqrt{\frac{\nu \left(1 - \frac{r_g}{r_o}\right)^{1-\nu} r_g}{2r_o - (1 + 2\nu)r_g}}, \quad \mathcal{E}_o = \sqrt{\left(1 - \frac{r_g}{r_o}\right)^{\nu} \left(\frac{2r_o - r_g(1 + \nu)}{2r_o - r_g(1 + 2\nu)}\right)}. \quad (20)$$

For $\nu = 1/2$, the orbit exists for $r_o \in \left(\frac{3}{2}r_g, \infty\right)$ and the ISCO is specified by $r_o = \frac{3}{2}r_g$, which corresponds to an inflection point of the effective potential. For the ISCO we have

$$\mathcal{E}_{ISCO} = \frac{3\frac{1}{2}}{\sqrt{2}}, \quad |\mathcal{L}_{zISCO}| = \frac{3\frac{3}{2}}{2\sqrt{2}} r_g. \quad (21)$$

As already described in Sec. 2.4, we consider collisions of particles initially in an ISCO with another particle, where we assume $\dot{r} = 0$ after the collisions and $\mathcal{L}$ remains unaltered, and the motion is mainly determined by the energy $\mathcal{E}$ after the collision. Since this implies $\dot{\theta} \neq 0$ after the collision, the particles obtains a velocity $v_\perp$ in the direction orthogonal to the equatorial plane. The energy then takes the form

$$\mathcal{E} = \sqrt{\mathcal{E}_o^2 + \frac{v_\perp^2}{\left(1 - \frac{r_g}{r_o}\right)^{1-2\nu}}}, \quad (22)$$

If $\mathcal{E} < 1$ particle cannot escape to infinity, it will escape to infinity if $\mathcal{E} \geq 1$ or in other words it will have unbounded motion,

$$v_\perp \geq \sqrt{(1 - \mathcal{E}_o^2) \left(1 - \frac{r_g}{r_o}\right)^{1-2\nu}}. \quad (23)$$

Specifically, for ISCO the escape condition for $\nu = \frac{1}{2}$ is, $|v_\perp| \geq v_{\perp esc} \geq \sqrt{1 - \frac{\sqrt{3}}{2}}$.

4 Particles in a weakly magnetized JNW spacetime

4.1 Equations of motion in the weak magnetization regime

If the magnetic field is sufficiently weak such that they do not influence the particle gravitationally, there is a possibility for a particle to escape after collision. This may occur if $\Lambda \to 1$, or, equivalently, $B \to 0$. Nevertheless, the magnetic field may still influence the particle via the Lorentz force, as the coupling depends on $eB$. Thus, even
for small $B$, the Lorentz force will remain significant for sufficiently large $e$.

Therefore, for timelike particles ($\epsilon = -1$), we shall focus on this weak magnetization regime by defining $B = \beta/e$ and expand the equations of motion (8) and (9) in powers of $1/e$, the result is

\[
\ddot{r} = \frac{\nu r_g}{2r(r - r_g)} \dot{r}^2 + \frac{1}{2} \left(2r - r_g(1 + \nu)\right) \dot{\theta}^2 - \frac{\nu r_g \mathcal{E}^2}{2r(r - r_g)} + \left(\frac{r - r_g}{r}\right)^{2\nu} \frac{[2r - r_g(1 + \nu)] \mathcal{L}^2}{2(r - r_g)^2 r^2 \sin^2 \theta} + \mathcal{O}(1/e^2),
\]

\[
\ddot{\theta} = -\frac{2r - r_g(1 + \nu)}{r(r - r_g)} \dot{r} \dot{\theta} + \left(\frac{r - r_g}{r}\right)^{2\nu} \frac{\mathcal{L}^2 \cos \theta}{r^2(r - r_g)^2 \sin^3 \theta} - \frac{\beta^2}{4} \cos \theta \sin \theta + \mathcal{O}(1/e^2),
\]

and the constraint equation becomes

\[
0 = -\frac{\mathcal{E}^2}{\mathcal{F}} + \frac{\mathcal{L}^2}{r^2 G \sin^2 \theta} - \beta \mathcal{L} + \frac{\dot{\mathcal{F}}}{\mathcal{F}} + r^2 G \dot{\theta}^2 + 1 + \frac{\beta^2}{4} r^2 G \sin^2 \theta + \mathcal{O}(1/e^2).
\]

Substituting Eq. (26) into (24), and expressing the quantities in terms of $f$, we obtain

\[
\ddot{r} = \frac{1}{2} \left(2r - (1 + 2\nu)r_g\right) \left(\dot{\theta}^2 + \frac{\mathcal{L}^2}{r^4 f^2(1-\nu) \sin^2 \theta}\right) + \frac{\nu r_g \mathcal{E}^2}{2r^2 f^{1-\nu} (\beta \mathcal{L} - 1) - \frac{\beta^2}{8} (2r - r_g) \sin^2 \theta},
\]

\[
\ddot{\theta} = -\frac{2}{r} \left(1 + (1 - \nu)r_g/r\right) \dot{r} \dot{\theta} + \frac{\mathcal{L}^2 \cos \theta}{r^4 f^2(1-\nu) \sin^3 \theta} - \frac{\beta^2}{4} \sin \theta \cos \theta.
\]

Accordingly, to first-order in $1/e$, Eq. (7) reduces to

\[
\mathcal{E} = f^\nu \dot{r}, \quad \mathcal{L} = \left(\dot{\phi} + \frac{\beta}{2}\right) r^2 f^{1-\nu} \sin^2 \theta.
\]

Rearranging Eq. (26) we obtain an effective potential equation for the test field case

\[
\mathcal{E}^2 = \dot{r}^2 + r^2 f \dot{\theta}^2 + U_{\text{eff}},
\]

\[
U_{\text{eff}} = f^\nu \left[1 + r^2 f^{(1-\nu)} \sin^2 \theta \left(\frac{\mathcal{L}}{r^2 f^{(1-\nu)} \sin^2 \theta} - \frac{\beta}{2}\right)^2\right].
\]

### 4.2 Dimensionless form of the equations

Following dimensionless quantities are introduced to avoid complications in our analysis regarding motion of the particle after collision,

\[
\sigma = \frac{\tau}{r_g}, \quad \rho = \frac{r}{r_g}, \quad \ell = \frac{L_z}{r_g}, \quad b = \frac{1}{2} \beta r_g, \quad \tau = \frac{t}{r_g}.
\]
The $\rho$ and $\theta$ components of the dynamical equations (27) and (28) are expressed as follows:

\[
\frac{d^2 \rho}{d\sigma^2} = \frac{1}{2}(2\rho - (1 + 2\nu)) \left( \frac{d\theta}{d\sigma} \right)^2 + \frac{\ell^2(2\rho - (1 + 2\nu))}{2\rho^4 \sin^2 \theta \left( 1 - \frac{1}{\rho} \right)^{2(1-\nu)}} + \frac{\nu}{2\rho^2 \left( 1 - \frac{1}{\rho} \right)^{(1-\nu)}} (2\ell b - 1) - \frac{b^2}{2} \sin^2 \theta(2\rho - 1),
\]

\[
\frac{d^2 \theta}{d\sigma^2} = \frac{2}{\rho} \frac{d\theta}{d\sigma} \frac{d\rho}{d\sigma} - \frac{(1 - \nu)}{\rho^2 \left( 1 - \frac{1}{\rho} \right)^{1-\nu}} \frac{d\theta}{d\sigma} \frac{d\rho}{d\sigma} + \frac{\ell^2 \cos \theta}{\rho^4 \sin^3 \theta \left( 1 - \frac{1}{\rho} \right)^{2(1-\nu)}}
\]

\[- b^2 \sin \theta \cos \theta,
\]

where energy $E$ is

\[
E^2 = \left( \frac{d\rho}{d\sigma} \right)^2 + \rho(\rho - 1) \left( \frac{d\theta}{d\sigma} \right)^2 + U_{\text{eff}},
\]

and

\[
U_{\text{eff}} = \left( 1 - \frac{1}{\rho} \right)^{\nu} + \rho^2 \left( 1 - \frac{1}{\rho} \right) \sin^2 \theta \left( \frac{\ell}{\rho^2 \sin^2 \theta \left( 1 - \frac{1}{\rho} \right)^{1-\nu}} - b \right)^2.
\]

When we consider equatorial plane then the particle moving around the naked singularity at radius $\rho_o$ has the following energy,

\[
E^2_o = \left( 1 - \frac{1}{\rho_o} \right)^{\nu} + \rho_o^2 \left( 1 - \frac{1}{\rho_o} \right) \left( \frac{\ell}{\rho_o^2 \left( 1 - \frac{1}{\rho_o} \right)^{1-\nu}} - b \right)^2.
\]

After collision the motion of particle is merely determined by its new energy

\[
E = \sqrt{E^2_o + \frac{(\rho_o - 1)}{\rho_o} \nu^2}.
\]

We want to examine the motion of the particle after collision, initially orbiting in the ISCO, so new dimensionless forms of the angular momentum and the energy represented as $\ell$ and $b$ respectively are given below [4]: $\frac{dU}{d\rho} = U' = 0$ and $\frac{d^2 U}{d\rho^2} = U'' = 0$ are used to
find $\ell$ and $b$,

$$U'_{\text{eff}} = \frac{1}{\rho^2(\rho - 1)^2} \left[ -\ell^2(1 - \frac{1}{\rho})^{2\nu}(2\rho - 1 - 2\nu) - 
2\ell b(\rho - 1)(1 - \frac{1}{\rho})^\nu \rho \nu + \rho(\rho - 1)(b^2 \rho(1 - 3\rho + 2\rho^2) + 
\nu(1 - \frac{1}{\rho})^\nu) \right],$$

(38)

$$U''_{\text{eff}} = \frac{1}{\rho^3(\rho - 1)^3} \left[ 2\ell b(\rho - 1)(1 - \frac{1}{\rho})^{\nu} \rho \nu (2\rho - 1 - \nu) + \rho(\rho - 1)(2b^2 \rho^2(\rho - 1)^2 + (1 - \frac{1}{\rho})^{\nu} \nu (1 - 2\rho + \nu)) + 
2\ell^2(1 - \frac{1}{\rho})^{2\nu}(1 + 3\rho^2 + 3\nu + 2\nu^2 - 3\rho(1 + 2\nu)) \right].$$

(39)

Equating Eq. (38) and Eq. (39) to zero, and solving them simultaneously gives

$$\ell = \pm \sqrt{\frac{-b^2(\rho_o - 1)^2 \rho_o^2(1 + 6\rho_o^2 + \nu - 2\rho_o(3 + \nu))}{(1 - \frac{1}{\rho_o})^{2\nu}(1 + 2\rho_o^2 + 3\nu + 2\nu^2 - 2\rho_o(1 + 3\nu))},}$$

(40)

$$b = \sqrt{\frac{-\left(1 - \frac{1}{\rho_o}\right)^{\nu - 1} \nu (1 + 2\rho_o^2 + 3\nu + 2\nu^2 - 2\rho_o(1 + 3\nu))}{2\rho_o^2(-1 + 8\rho_o^3 - 3\nu - 2\nu^2 - 2\rho_o^2(6 + 7\nu) + 2\rho_o(3 + 7\nu + 2\nu^2) \pm \nu \sqrt{\Sigma}}},$$

(41)

where

$$\Sigma = -[12\rho_o^4 + (1 + \nu)^2(1 + 2\nu) - 8\rho_o^3(3 + 5\nu) + 4\rho_o^2(5 + 15\nu + 6\nu^2) - 4\rho_o(2 + 7\nu + 6\nu^2 + \nu^3)].$$

(42)

For $\nu = \frac{1}{2}$, when $\ell > 0$ we have $\rho_o \in (1, \frac{3}{2}]$, and when $\ell < 0$ $\rho_o \in (1.183, \frac{3}{2}]$. 

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Figure 5: $\ell$ angular momentum is plotted as function of $\rho_o$ where $\nu = 1/2$, $\ell > 0$ and $\rho_o \in (1, 3/2]$.

Figure 6: $\ell$ angular momentum is plotted as a function of $\rho_o$ where $\nu = 1/2$, $\ell < 0$ and $\rho_o \in (1.183, 3/2]$.

Figure 7: $b$ magnetic field as a function of $\rho_o \in (1, 3/2]$ where $\nu = 1/2$. 
In Fig. (5) it is shown that as the particle moves away from the naked singularity the positive angular momentum increases. In Fig. (6) the negative angular momentum increases as a function of radius. In Fig. (7) and Fig. (8) it is evident that magnetic field strength decreases as radius of orbit increases.

Considering the equatorial plane and utilizing the above mentioned dimensionless quantities Eqs. (29) and (30) takes the form,

\[
\left( \frac{d\rho_o}{d\sigma} \right)^2 = \mathcal{E}^2 - U_{\text{eff}}, \quad \frac{dT}{d\sigma} = \frac{\mathcal{E}}{(1 - \frac{1}{\rho_o})^\nu},
\]

\[
\rho_o \frac{d\phi}{d\sigma} = \alpha, \quad \alpha = \frac{\ell}{\rho_o (1 - \frac{1}{\rho_o})^{1 - \nu}} - b\rho_o.
\]

The effective potential given below is same as in Eq. (35) in the equatorial plane but, here we introduce \( \alpha \) which is later used to define the Lorentz gamma factor

\[
U_{\text{eff}} = \left( 1 - \frac{1}{\rho_o} \right)^\nu + \left( 1 - \frac{1}{\rho_o} \right) \alpha^2.
\]

We consider positive charged particle, so that \( b \) is positive as well. Let us suppose that the particle is moving in the circular orbit of radius \( r \). Its momentum is given by

\[
k^\mu = m\gamma (e^\mu_{(t)} + v e^\mu_{(\phi)}),
\]

\[
e^\mu_{(t)} = f^{-\nu/2} S^{\mu}_{(t)},
\]

\[
e^\mu_{(\phi)} = r^{-1} f^{-(1-\nu)/2} S^{\mu}_{(\phi)}.
\]

Velocity of the particle with respect to rest frame is represented by \( v \). Here \( \gamma \) is the Lorentz gamma factor. Using normalization condition \( k^2 = -m^2 \) we obtain \( \gamma = \)
$(1 - v^2)^{-1/2}$. For a positively charged particle and $v > 0$, the Lorentz force is repulsive (i.e., directed outwards the naked singularity), while for $v < 0$, it is attractive.

Using relation $d\phi/d\tau = v\gamma/r$ one gets

$$v\gamma = \alpha. \quad (49)$$

This relation allows one to write

$$\gamma^2 = 1 + \alpha^2, \quad v = \frac{\alpha}{\sqrt{1 + \alpha^2}}. \quad (50)$$

$\ell$ and $b$ are given in Eq. (40) and Eq. (41) respectively, so one obtains value of $\alpha$. Here we discuss for $\nu = 1/2$ only and consider the equatorial plane.

$$\ell = \pm \frac{1}{2} \sqrt{\frac{(\rho_o - 1)\rho_o(\frac{3}{2} - 7\rho_o + 6\rho_o^2)}{(1 - \frac{1}{\rho_o})^{1/2} \left(8\rho_o^3 - 19\rho_o^2 + 14\rho_o - 3 \pm \frac{1}{2}\sqrt{\mathcal{X}}\right)}}, \quad (51)$$

$$b = \frac{1}{2} \sqrt{\frac{(3 - 2\rho_o)(\rho_o - 1)}{(1 - \frac{1}{\rho_o})^{1/2} \rho_o^2(8\rho_o^3 - 19\rho_o^2 + 14\rho_o - 3 \pm \frac{1}{2}\sqrt{\mathcal{X}})}}, \quad (52)$$

where

$$\mathcal{X} = -\frac{9}{2} + \frac{57}{2}\rho_o - 56\rho_o^2 + 44\rho_o^3 - 12\rho_o^4, \quad (53)$$

$$Q = 8\rho_o^3 - 19\rho_o^2 + 14\rho_o - 3 \pm \frac{1}{2}\sqrt{-\frac{9}{2} + \frac{57}{2}\rho_o - 56\rho_o^2 + 44\rho_o^3 - 12\rho_o^4}. \quad (54)$$

For $\nu = \frac{1}{2}$, when $\ell > 0$ $\rho_o \in (1, 3/2]$ and when $\ell < 0$ $\rho_o \in (1.183, 3/2]$.

Figure 9: $Q$ as a function of $\rho_o$. 

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Figure 10: Velocity (with respect to rest frame) of a particle at ISCO as a function of its radius.

Figure 11: For $\ell > 0$ the Lorentz gamma factor $\gamma$ (dotted graph) and $U_{\text{eff}}$ are plotted as function of $\rho_0$. 
Figure 12: For $\ell < 0$ the Lorentz gamma factor $\gamma$ (dotted graph) and $U_{\text{eff}}$ are plotted as function of $\rho_o$.

$Q$ is a function of $\rho_o$ present in the denominator of $\ell$ and $b$. In Fig. (9) $Q$ is plotted as a function of radius. When $\ell > 0$, $Q$ attains zero value at $\rho_o = 1$ and for $\ell < 0$ it acquires zero at $\rho_o = 1.2$. In Fig. (10) velocity of the particle with respect to rest frame is plotted as a function of radius. It decreases as radius of the orbit increases in both cases when angular momentum is positive $\ell > 0$ and when it is negative $\ell < 0$. In Fig. (11) when $\ell > 0$ $U_{\text{eff}}$ monotonically decreases from its value $\sqrt{3}/2$ at $\rho_o = 3/2$ till it reaches $\rho_o = 1$. Lorentz gamma factor $\gamma$ at the ISCO is, $\gamma(3/2) = \sqrt{1 + \frac{\sqrt{3}}{2}}$. Fig. (12) shows that when $\ell < 0$ $U_{\text{eff}}$ and $\gamma$ increases infinitely as $\rho_o$ reaches 1.2.

4.2.1 Trajectories for escape energy

Figure 13: $\mathcal{E}$ energy as a function of $\rho_o \in (1,3/2]$ where $\nu = 1/2$ and $\ell > 0$. 

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Figure 14: $\mathcal{E}$ energy as a function of $\rho_o \in (1.183, 3/2]$ where $\nu = 1/2$ and $\ell < 0$.

Fig. (13) depicts the energy of the particle for $\ell > 0$ required to escape from the neighborhood of the naked singularity. The concrete line represents the minimum amount of energy needed for the particle to leave the vicinity of the naked singularity. The unbound motion is displayed by the filled region while the unfilled region demonstrates the bounded motion of the particle. Fig. (14) also represents energy of the particle for $\ell < 0$. The particle orbiting the naked singularity requires different values of energy to escape as radius of the orbit increases. For positive angular momentum, the particle requires minimum energy to escape as compared to the particle having negative angular momentum. When $\ell > 0$ at $\rho_o = 1.3$ the particle requires energy $\mathcal{E} = 0.824$ and when $\ell < 0$ for the same value of radius it requires energy $\mathcal{E} = 1.281$ to escape.
4.2.2 Trajectories for escape velocity

Figure 15: Escape velocity as a function of $\rho_o \in (1, \frac{3}{2}]$ where $\nu = 1/2$ and $\ell > 0$.

Figure 16: Escape velocity as a function of $\rho_o \in (1.183, \frac{3}{2}]$ where $\nu = 1/2$ and $\ell < 0$.

In Fig. (15) escape velocity for the particle is demonstrated for $\ell > 0$. The concrete line shows the minimum amount of energy required for the particle to escape the vicinity of the naked singularity when $\mathcal{E} = 1$. The filled region shows the escape velocity required for the unbounded motion when $\mathcal{E} > 1$. The unfilled region shows the escape velocity for the bounded motion when $\mathcal{E} < 1$. Fig. (16) also represents the escape velocity for $\ell < 0$. The escape velocity of the particle having positive angular momentum is maximum. When $\ell > 0$ at $\rho_o = 1.48$, $v_{esc} = 0.76$ and when $\ell < 0$ and for the same value of radius $v_{esc} = 0.44$. 
5 Particle collision

Let us consider collision between two particles with same mass $m$ and opposite charges moving along the same circular orbit in opposite directions. Four-momentum of the particles are given as under,

$$k_1^\mu = m\gamma (e^{(t)}_1 + ve^{(\varphi)}_1), \quad k_2^\mu = m\gamma (e^{(t)}_2 - ve^{(\varphi)}_2),$$  \hspace{1cm} (55)$$

$$K^\mu = k_1^\mu + k_2^\mu = 2m\gamma e^{(t)}_1.$$  \hspace{1cm} (56)$$

$K^\mu$ is the four-momentum of the system after collision. Let us denote the center-of-mass energy after collision by $M$,

$$M^2 = -K^\mu K_\mu = -g_{\mu\delta}K^\mu K^\delta = (2m\gamma)^2, \quad M = 2m\gamma.$$  \hspace{1cm} (57)$$

It is obvious that the center-of-mass energy $M$ depends on the Lorentz gamma factor, as the value of $\gamma$ increases large amount of energy is produced as a result of collision. In Fig. (11) when $\ell > 0$, $\gamma$ tends to infinity as radius of the orbit reaches 1 so large amount of $M$ could be achieved. Similarly in Fig. (12) when $\ell < 0$, $\gamma$ approaches to infinity as $\rho_o$ reaches 1.2. In both cases large amount of energy can be obtained in the presence of weak magnetic field. But if we compare energies at the same obit for $\ell > 0$ and $\ell < 0$, it is evident from the figures that more energy can be produced when the particles are having negative angular momentum.

6 Concluding remarks

Below is the summary of the results that are examined in this paper. Results are computed for $\nu = 1/2$.

- Primarily an analysis of the motion of a neutral particle in the JNW spacetime is established in the absence of magnetic field. It is observed, that particle revolving around the naked singularity can escape from the ISCO after collision if it has energy equal to unity, otherwise it observes bounded motion. If energy is greater than unity particle escapes to infinity and observes unbounded motion.

- In the presence of weak magnetic field motion of a charged particle is examined. We obtained dynamical equations that are influenced by the magnetic field. Dimensionless forms of these equations are also obtained to make clear analysis of the particle’s motion.
• Energy of the particle in the presence of weak magnetic field is demonstrated for
the positive angular momentum and negative angular momentum of the particle in
Fig. (13) and Fig. (14) respectively. The upper filled regions shows the unbounded
motion and the unfilled regions shows the motion in the surroundings of the naked
singularity. It is concluded that the particle having positive angular momentum
requires minimum energy to escape from the surroundings of the naked singularity.

• Escape velocity of the particle is demonstrated in the presence of weak magnetic
field for the positive angular momentum and negative angular momentum of the
particle in Fig. (15 and Fig. (16) respectively. It is concluded that the escape
velocity of the particle having positive angular momentum is maximum.

• In the end we investigated the center-of-mass energy produced as a result of collision
of the particles having same mass and opposite charges revolving around the naked
singularity in the opposite directions. An expression for the center-of-mass energy
is computed which clearly shows that it depends on the Lorentz gamma factor. In
Fig. (11) and Fig. (12) the Lorentz gamma factor is shown for the particle having
positive angular momentum and negative angular momentum respectively. It is
concluded that in both cases maximum amount of energy can be achieved. But
more energy can be achieved for the particles having negative angular momentum.

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