AN EXPRESSION FOR THE HOMFLYPT POLYNOMIAL
AND SOME APPLICATIONS

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ABSTRACT
Associated with each oriented link is the two variable Homflypt polynomial. Formulas
for the coefficient polynomials of the three lowest v-degrees are presented that shows they
are determined by the writhe of any braid diagram for the link, the Conway polynomial
for the link, and the remaining coefficient polynomials. This is used to show the Jones
and Homflypt polynomials distinguish the same three-braid links.

These Homflypt coefficient polynomials in z satisfy a system of linear equations with
coefficients in \( \mathbb{Z}[z] \). The Conway polynomial is essentially the unique Laurent polynomial
that represents such a linear combination and is also a link invariant; any other is merely
the product of the Conway polynomial and an arbitrary second polynomial. Two other
independent functions that represent such a linear combination are determined by the
writhe and are not link invariants.

Keywords: Homflypt polynomial, Conway polynomial, Jones polynomial, skein relation,
brads and braid groups, Markov stabilization

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1. Introduction
The focus of this paper is to develop and display a relationship between the coef-
ficient polynomials of the Homflypt polynomial. The background required for this
paper is a very basic understanding of skein relations, skein polynomials, and braid
groups. Such material may be found in any introductory textbook, as for example
K. Murasugi’s, [11], and may also be found in many excellent surveys, [1].

The attraction in using a braid word as the starting point for naming a link is
that it not only immediately informs the reader of the exact link under discussion,
but the braid word itself may be manipulated using the braid relations for the braid
group. While there are many braid words that describe the same link, there is still
much to be gained from the braid approach, as this paper hopes to demonstrate.

The remainder of this section will introduce the terms and notation used in
the paper. The second section will introduce the primary results, whose proofs will
appear in the third section. Applications (Section 2.1) include the result that the
Jones and Homflypt polynomials distinguish the same three-braid links. The third
section will also introduce the necessary tools and formulas to facilitate the proofs.
1.1. Definitions and Notation

The conventions used in this paper largely follow those in, [10], to which the reader is referred for expanded discussion. A brief review of the standard symbols and terminology used in the paper is given here for reference.

The braid group on \( n \) strands, \( B_n \), has \( n - 1 \) standard generators, \( \sigma_i \).

Definition 1.1. If a braid word, \( \beta \in B_n \), has the expression \( \prod_{k=1}^{m} \sigma_{\epsilon_k}^{\epsilon_k} \), with \( \epsilon_k = \pm 1 \) for each subscript, \( k \),

(i) the braid length is \( m \) and is denoted \( |\beta| \),
(ii) the exponent sum, or writhe, denoted \( w \) or \( w(\beta) \), is \( \sum_{k=1}^{m} \epsilon_k \).
(iii) a syllable of a braid word is a maximal subword of identical subscript \( s \) (compare [12]). A syllable is reduced when all its exponents have the same sign. A braid word is in standard form when all syllables are reduced.

The writhe is actually associated with the braid diagram, rather than the braid word itself, but this imprecision in usage causes no problems. An \( n \)-braid word whose twist exponents are all zero is defined to have zero length and writhe.

Definition 1.2. If a braid word, \( \beta \in B_n \), is in standard form and has the expression \( \prod_{k=1}^{m} \sigma_{\epsilon_k}^{\epsilon_k} \), with \( \epsilon_k = \pm 1 \) for each subscript, \( k \),

(i) any generator that appears in a unique syllable is called a clustered generator,
(ii) any generator that appears exactly once in \( \beta \) is called a trivial generator,
(iii) a set of generators with subscripts in a range, \( (a, b) \), with \( 0 \leq a \leq b \leq n \), yields a new braid word, \( \Psi(a, b, \beta) = \prod_{i \in (a, b)} \sigma_{i-a}^{\epsilon_i} \), that belongs to \( B_{b-a} \). \( \Psi(a, b, \beta) \) is defined as the identity for \( B_1 \) when \( b = a \) or \( b = a + 1 \).

The link associated with the standard closure of a braid, \( \beta \), is denoted \( \hat{\beta} \). The mirror image of a link, \( L \), is denoted \( L^\ast \). The number of link components in \( L \) is denoted \( \mu(L) \). \( O_n \) denotes the trivial link with \( n \) components.

The link diagrams referenced in this paper will typically be associated with a braid word, but the skein relation, (1.1), for the Homflypt polynomial, \( P \), is defined more generally to allow any valid oriented diagram, \( D \), for the link. The diagrams, \( D_+ \), \( D_0 \), and \( D_- \) below refer to the usual diagrams for the link with positive crossing, null (smoothed) crossing, and negative crossing, respectively.

\[
P_{D_+}(v, z) = vzP_{D_0}(v, z) + v^2P_{D_-}(v, z). \tag{1.1}
\]

The importance of the Homflypt polynomial derives from the fact it is the same for all diagrams of a given link, and so is an invariant of the oriented link. The major skein polynomials and their definitions are: the Conway polynomial, \( \nabla_D(z) = P_D(1, z) \), the Jones polynomial, \( V_D(t) = P_D(t, (t - 1)/\sqrt{t}) \), and the Alexander polynomial, \( \Delta_D = P_D(1, (t - 1)/\sqrt{t}) \).

The torus links are much studied due to their uniformity, symmetry, and other properties. The torus links are characterized by the number of full rows of twists, \( p \),
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together with the number of strands, \( q \), denoted \( K(p, q) \). Each row may be realized by a braid word, \( \prod_{i=q-1}^{1} \sigma_i = \sigma_{q-1} \cdots \sigma_1 \), hereafter called \( \alpha_{p,q} \). Hence \( K(p, q) \), may be represented by the braid word, \( \alpha_{p,q}^{q-1} \). The elementary torus links, denoted \( T_p \), are those with two strands. The corresponding Conway polynomial is so central to the results of this paper that the lengthy expression, \( \nabla_{K(p,2)}(z) \), will be shortened to \( C_p(z) \) or even \( C_p \). Among many other interesting properties of the Conway polynomial (Section 3.2), \( C_p(1) \) is the \( p \)-th Fibonacci number.

For integers \( x \) we may write \( \varepsilon_x \) in place of \((-1)^x\). The usual convention that \( \sum_{j=a}^{b} f_j = \binom{b}{a} = 0 \) when \( a > b \) is followed.

2. The Primary Results

Given that the Homflypt polynomial is a function of two variables, one obvious way to organize its expression is as a polynomial in a single variable, with coefficients that are simply polynomials in the second variable. The question as to which variable should be the primary organizing factor may depend on circumstances, but a powerful motivation exists to chose \( v \) from (1.1). This is due to the remarkable results, [2], [7], known as the Morton-Franks-Williams inequality, that give bounds on the possible values for the highest and lowest powers of \( v \).

The first step is to observe that the Homflypt polynomial for a link with a given braid representation (\( \beta \in B_n \)) may be organized in a standard form, (2.1), in which the \( p_j \) are ordinary polynomials with integer coefficients. An equivalent form appears in (2.2), in which the \( h_j \), are Laurent polynomials with integer coefficients, and \( h_j = p_j / z^{n-1} \). When multiple braid words are under discussion, the symbols \( p_j, \beta \), or \( h_j, \beta \), may be used for clarity; note that the argument, \( z \), may be omitted. Both forms highlight the number of strands in the braid, and the writhe, \( w \), of the braid diagram (exponent sum of the braid word). Theorem 2.1 describes the relations among the coefficient polynomials, \( p_j \), defined by (2.1).

\[
P_{\beta}(v, z) = \frac{v^w \sum_{j=0}^{n-1} p_j(z) v^{2j}}{(vz)^{n-1}},
\]

(2.1)

\[
P_{\beta}(v, z) = v^{w-n+1} \sum_{j=0}^{n-1} h_j(z) v^{2j}.
\]

(2.2)

The Homflypt formula for the elementary torus links is much simpler:

\[
P_{K(p,2)}(v, z) = \frac{v^p \{ C_{p+1}(z) - C_{p-1}(z) v^2 \}}{(vz)^{p-1}}.
\]

(2.3)

The Conway polynomial for the elementary torus links has an expression:

\[
C_p(z) = \sum_{j=0}^{\frac{p-1}{2}} \binom{p-1-j}{j} z^{p-2j-1}, \text{ for } p > 0.
\]

(2.4)
When \( p \) is negative, we have \( C_p(z) = (-1)^{p+1} C_{-p}(z) \), and \( C_0(z) = 0 \).

In 1987, V.F.R. Jones published a number of results relating properties of Hecke algebras to knot theory, [5]. This landmark research displayed, among other insights, how the two variable skein polynomial could be calculated from the Burau representation of a braid. Unfortunately for knot theorists, the expression for the two variable skein polynomial is of nontrivial complexity. A further wrinkle in using the methodology is it requires knowledge of the number of link components, or at least whether this number is even or odd. For three-braid links, this paper first established the Homflypt polynomial is dependent only on the writhe and the Alexander polynomial. H. Murakami, [8], showed how to calculate \( P_D(v, z) \) for a diagram with \( n \) Seifert circles based on knowledge of \( P_D(v_j, z) \) at \( n \) independent points \( v_j \).

Theorem 2.1 shows that the lowest \( v \)-degree coefficient polynomials, \( p_0(z) \), \( p_1(z) \), and \( p_2(z) \) are determined by the writhe, Conway polynomial and the \( p_j(z) \) for \( j \geq 3 \).

**Theorem 2.1.** For an arbitrary braid, \( \beta \), of \( n \geq 1 \) strands, the Homflypt polynomial for \( \hat{\beta} \) is given by (2.1) with:

(i) \( p_0 = z^{n-3} \frac{C_w + 4 - n}{z} q_0 \), with \( q_0 = \sum_{j=3}^{n-1} z^{-2}(C_{2j-3} - 1) p_j \),

(ii) \( p_1 = z^{n-1} \frac{C_w}{z} - p_0 - p_2 - \sum_{j=3}^{n-1} p_j \),

(iii) \( p_2 = z^{n-3} \frac{C_w + 2 - n}{z} q_2 \), with \( q_2 = \sum_{j=3}^{n-1} z^{-2}(C_{2j-1} - 1) p_j \).

In the formula for \( q_0 \), we have:

\[
\frac{C_{2j-3} - 1}{z^2} = \sum_{i=0}^{j-3} \binom{2j - 4 - i}{i} z^{2j - 6 - 2i}, \text{ for } j \geq 3.
\]

Similarly, in the formula for \( q_2 \), we have:

\[
\frac{C_{2j-1} - 1}{z^2} = \sum_{i=0}^{j-2} \binom{2j - 2 - i}{i} z^{2j - 4 - 2i}, \text{ for } j \geq 1.
\]

A few comments are in order about this theorem. First, the formula for the trivial knot on one strand depends on a writhe of zero. Second, the formula for two strands is equivalent to (2.3). Third, \( p_1 \) is exactly what it has to be in order for the identity, \( P_{\hat{\beta}}(1, z) = \nabla_{\beta}(z) \), to be satisfied.

A closer inspection of the forms for \( p_j \), \( q_0 \), and \( q_2 \), shows a new set of relations is satisfied. It is expected that the sum of the Laurent coefficient polynomials will be the Conway polynomial, but more is true. The following theorem introduces a system of linear equations, with coefficients in the ring of polynomials over the integers, that is satisfied by the Laurent coefficient polynomials, \( h_j \).

**Theorem 2.2.** The Laurent coefficient polynomials for a link, \( \hat{\beta} \), with \( \beta \in B_n \), satisfy the following relations. These are independent for \( n > 1 \).
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\[ n - 1 \sum_{j=0}^{n-1} C_{2j-3} h_j = C_{4+w-n}, \quad (2.5) \]

\[ n - 1 \sum_{j=0}^{n-1} C_{2j-1} h_j = C_{2+w-n} \cdot (2.6) \]

Any set of \( n \) functions, \( \{f_j\}_{0}^{n-1} \), that satisfy both equations, will also satisfy the following family of equations for each integer, \( \kappa \):

\[ n - 1 \sum_{j=0}^{n-1} C_{2j-1-\kappa} f_j = (-1)^\kappa C_{\kappa+2+w-n}. \quad (2.7) \]

In the prior theorem, independent means that for a fixed choice of \( w \) and \( n \), an arbitrary set of Laurent polynomials with integer coefficients, \( \{f_j\} \), that satisfy one of the equations need not satisfy the other equation. In fact, these equations are independent over the smaller set of ordinary polynomials with integer coefficients.

There are thus three independent equations satisfied by the \( n \) Laurent coefficient polynomials; two from the Theorem 2.2 and the simple equation relating the sum of the coefficient polynomials to the Conway polynomial. It would clearly be desirable to find any further such relations.

Suppose some linear equation exists in the form of Theorem 2.2 for each value of \( n \), with coefficients that are Laurent polynomials over the integers. As each new strand is added, a new coefficient \( A_{n-1} \) arises to match the new Homflypt coefficient polynomial, \( h_{n-1} \). This assumption yields a relation for each \( n \):

\[ n - 1 \sum_{j=0}^{n-1} A_j h_j = \Omega_{n,\beta}, \text{ with } A_j \in \mathbb{Z}[z, z^{-1}]. \quad (2.8) \]

As any two conjugate braid words in \( B_n \) produce the same set of \( h_j \), the value of \( \Omega_{n,\beta} \) must be identical for them. Any pair of braid words that generate the same braided element in \( B_n \) also have this same property, so \( \Omega_{n,\beta} \) is a function on the conjugacy classes of braided elements. In fact, by (2.2), any two braided elements, even in different braid groups, with identical \( v_i - n_i \) and whose closure is the same link, will have \( \Omega_{n_1,\beta_1} = \Omega_{n_2,\beta_2} \). As the first \( n-1 \) coefficient polynomials, \( h_j \), are identical for \( \beta \in B_n \) and \( \beta\sigma_n \in B_{n+1} \), and \( h_{n,\beta\sigma_n} = 0 \), the left side of (2.8) is invariant under positive Markov stabilization, so \( \Omega_{n,\beta} = \Omega_{n+1,\beta\sigma_n} \).

If \( \Omega_{n,\beta} \) is also invariant under negative Markov stabilization, \( \Omega_{n,\beta} \) is a link invariant. In this case, observe that for the trivial knot, \( A_0 = \Omega_{2,\sigma_1} \), and \( A_1 = \Omega_{2,\sigma_1^{-1}} \), so \( A_0 = A_1 \). This quickly leads to the conclusion all \( A_j \) are equal to \( A_0 \), so (2.8) becomes \( A_0 \nabla_{\beta} = \Omega_{n,\beta}, \) i.e. \( \Omega_{n,\beta} \) is a multiple of \( \nabla_{\beta} \).

If \( \Omega_{n,\beta} \) is not invariant under negative Markov stabilization, a new family of relations is created. First observe that by Prop. 3.2, \( \sum_{j=0}^{n-1} A_{j+1} h_j = \Omega_{n+1,\beta\sigma_n^{-1}} \),
for every \( \beta \in B_n \). By repeating the negative Markov stabilization \( \kappa \) times, we have

\[
\sum_{j=0}^{n-1} A_j + \kappa h_j = \Omega_{n+\kappa}^n \prod_{i=1}^{n-1+\kappa} \sigma_i^{-1}.
\]

Also observe that the functions, \( \Omega_{n,\beta} \), obey the same skein relation as the Conway polynomial due to linearity of the relations, and the skein property for \( h_j \).

There are multiple representations of the trivial knot in \( B_n \) with length \( n-1 \) and constant writhe. Critically, all representations with the same writhe have the same set of coefficient polynomials, \( h_j \). Observe that any such choice of braid word, \( \omega_j \), with \( j \) trivial negative generators, and \( n-1-j \) trivial positive generators, satisfies \( A_j = \Omega_{n,\omega_j} \), since \( h_j, \hat{\omega}_j = \delta_{j,k} \).

It is instructive to investigate the behavior of \( \Omega_{n,\beta} \) for the elementary torus links. Eq. 2.3 shows that for \( T_{\kappa+2} \), we have \( h_0 = C_{\kappa+3}/z \) and \( h_1 = -C_{\kappa+1}/z \), and

\[
\frac{A_0 C_{\kappa+3} - A_1 C_{\kappa+1}}{z} = \Omega_{2,\sigma_1^{\kappa+2}}.
\]

(2.9)

Suppose \( \Omega_{n,\beta} \) depends only on \( w-n \), as is true for the braid words of length \( n-1 \) with all trivial generators. Then \( \Omega_{n,\beta} = \Omega_{2,\sigma_1^{\kappa+2}} \), with \( \kappa = w-n \) and \( A_j = \Omega_{n,\omega_j} = \{A_0 C_{-2j+2} - A_1 C_{-2j}\}/z \). Substitution of these values in (2.8) and use of \( \kappa = 1, -1 \) in Eq. 2.7 reveals an identity, i.e. any \( A_0, A_1 \in \mathbb{Z}[z, z^{-1}] \) are valid. In other words, \( \Omega_{n,\beta} \) is merely a linear combination (with weights \( A_0/z, -A_1/z \)) of functions in Thm. 2.2 and the same is true for each \( A_j \).

The prior discussion is summarized in:

**Theorem 2.3.** Any function, \( \Omega_{n,\beta} \), defined by a linear relation in the \( \{h_j\} \) as in (2.8), satisfies one of the following three conditions:

(i) when \( \Omega_{n,\beta} \) is a link invariant, \( \Omega_{n,\beta} \) must be a multiple of the Conway polynomial for \( \hat{\beta} \); in this case, \( \Omega_{n,\beta} = A_0 \nabla_\beta \), and each \( A_j \) equals \( A_0 \in \mathbb{Z}[z, z^{-1}] \).

(ii) when \( \Omega_{n,\beta} \) is not a link invariant, but depends only on \( w-n \), \( \Omega_{n,\beta} \) and \( A_j \) are linear combinations of the Conway polynomials described in Thm. 2.2.

(iii) \( \Omega_{n,\beta} \) is not a link invariant and depends on parameters other than, or in addition to, \( w-n \). Some parameter must not be a link invariant.

### 2.1. Skein polynomials for three-braid links

It is convenient to restate Thm. 2.1 for three-braids. For three-braid knots Eq. 8.4, p. 356, [5], is equivalent to Prop. 2.4, which is valid for all three-braid links.

**Proposition 2.4.** Three-braid links, \( \hat{\beta} \), with \( \beta \in B_3 \), satisfy (2.1) with

\[
\text{(i) } p_0 = C_{w+1} - \nabla_\hat{\beta},
\]

\[
\text{(ii) } p_1 = z^2 \nabla_\hat{\beta} - p_0 - p_2,
\]

\[
\text{(iii) } p_2 = C_{w-1} - \nabla_\hat{\beta}.
\]
An equivalent form is Lemma 2.5, with Props. 2.6 and 2.7 as easy corollaries.

**Lemma 2.5.** The Homflypt polynomial for a link, $\hat{\beta}$, with $\beta \in B_3$ is

$$P_{\hat{\beta}} = P_{T_w P_{O_2}} - \nabla_{\hat{\beta}} v^w (P_{O_3} - 1). \tag{2.10}$$

**Proposition 2.6.** If three-braid words, $\beta$ and $\gamma$, have the same writhe, and $\hat{\beta}$ and $\hat{\gamma}$ have the same Conway, Jones, or Alexander polynomials, we have $P_{\hat{\beta}} = P_{\hat{\gamma}}$.

**Proposition 2.7.** When three-braid words, $\beta$, $\gamma$, satisfy $w(\beta) \geq w(\gamma)$, we have $P_{\hat{\beta}} = P_{\hat{\gamma}}$ exactly when $\nabla_{\hat{\beta}} = \nabla_{\hat{\gamma}}$ and one of the following is true:

(i) $w(\beta) = w(\gamma)$,
(ii) $w(\beta) = w(\gamma) + 2$, and $\nabla_{\hat{\beta}} = C_{w(\beta)-1}$ or
(iii) $w(\beta) = 2$, $w(\gamma) = -2$, and $\nabla_{\hat{\beta}} = 1$.

The Jones polynomial for three-braid links may be derived from Eq. 2.10:

$$V_{\hat{\beta}}(t) = t^{(w-2)/2} \{t^{w+1} + \epsilon_w (1+t+t^2)\} - (1+t+t^2) t^{w-1} \Delta_{\hat{\beta}}(t), \text{ for } \beta \in B_3. \tag{2.11}$$

An interesting pattern in the Homflypt coefficient polynomials is that in plotting the degree of $p_j$ on the vertical axis, and the subscript, $j$, on the horizontal axis ($0 \leq j \leq n-1$), the points outline what looks like the profile of a landscape. In the analogy at hand, $p_j = 0$ corresponds to a point below sea level. The remarkable feature of this landscape is that there are, apparently, no valleys. For the case of three-braids, this is a consequence of Prop. 2.4.

**Proposition 2.8.** For each braid word, $\beta \in B_3$, the subscripts of $p_j$ (zero, one, two) are the union of two disjoint subranges, $L$ and $R$, with the following property:

(i) the degree of $p_j$ is non-decreasing on $L$ and when $L$ is non-empty, $0 \in L$,
(ii) the degree of $p_j$ is non-increasing on $R$ and when $R$ is non-empty, $2 \in R$.

The prior three-braid results may be combined with the extensive analysis by K. Murasugi, [9], to determine all cases when the Homflypt polynomial of a three-braid link matches that of an elementary torus link, or when the $v$-span has a given value. Research in [12] suggests the Jones polynomial may distinguish the same three-braid links as the Homflypt polynomial. This is confirmed by Prop. 2.12.

In [9], K. Murasugi defines a collection of seven disjoint sets, $\Omega_i$, of three-braid words and shows (Prop. 2.1 p. 7) that each three-braid word is conjugate to exactly one element in some $\Omega_i$. As the numbering and definition of the Artin braid generators differs from that in the later text, [10], we present a similar partitioning, $\Omega^*_i$,.
consistent with the latter. The Alexander polynomial for a typical member may be derived by use of Prop. 2.3 and is included below (here \( G = t^2 + t + 1 \)).

**Proposition 2.9.** Each three-braid is conjugate to a unique element in some \( \Omega^*_b \). In the sets below, \( d \) is any integer, while \( e, E, r, e_k, E_k \) range over all positive integers.

For \( \beta \in \Omega^*_b \) with \( j \neq 6 \), we have \( \max \deg \Delta_{\beta} = (w(\beta) - 2)/2 \).

(i) \( \Omega^*_0 = \{ \alpha^{3d}_2 \} \), with \( \Delta_{\alpha^{3d}_2} = t(t^{3d} - 1)^2/(Gt^{3d}) \),

(ii) \( \Omega^*_1 = \{ \alpha^{3d+1}_2 \} \), with \( \Delta_{\alpha^{3d+1}_2} = t(t^{6d+2} + t^{3d+1} + 1)/(Gt^{3d+1}) \),

(iii) \( \Omega^*_2 = \{ \alpha^{3d+2}_2 \} \), with \( \Delta_{\alpha^{3d+2}_2} = t(t^{6d+1} + 3t^{3d+2} + 1)/(Gt^{3d+2}) \),

(iv) \( \Omega^*_3 = \{ \alpha^{3d+1}_2 \sigma_2 \} \), with \( \Delta_{\alpha^{3d+1}_2 \sigma_2} = t(t^{6d+3} - 1)/(Gt^{3d+1}/\sqrt{t}) \),

(v) \( \Omega^*_4 = \{ \alpha^{3d}_2 \sigma_2^e \} \), with \( \Delta_{\alpha^{3d}_2 \sigma_2^e} = t(t^{3d} - 1)/(Gt^{4d+e}/2) \),

(vi) \( \Omega^*_5 = \{ \alpha^{3d}_2 \sigma_2^e F_k \} \), with \( \Delta_{\alpha^{3d}_2 \sigma_2^e F_k} = t(t^{3d} - 1)/(Gt^{4d+e}/2 - \epsilon E)/(Gt^{4d+e}/2) \),

(vii) \( \Omega^*_6 = \{ \alpha^{3d}_2 \eta ; \eta \in B_3, with \eta = \prod_{k=1}^l \sigma_2^{-e_k} \sigma_1^F \} \) \( \Delta_{\alpha^{3d}_2 \eta} = \Delta_{\hat{\eta}} + t(t^{3d} - 1)(t^{3d+k+1} - w(\eta))/(Gt^{4d+k+1}/2) \).

A. Stoimenow asks in Question 4.1, p. 18 [12], whether "any two 3-braid links with the same \( V \) (or \( \Delta \)) have also equal \( P \) ?". Ex. 2.10 shows \( \Delta \) doesn’t have this property. Lemma 2.11 and the Murasugi classification show that the Jones and Homflypt polynomials distinguish the same three-braid links (Prop. 2.12).

**Example 2.10.** Set \( \beta = \alpha^{3x}_2 \sigma_2^{-3x-3y-1} \sigma_1, \gamma = \alpha^{3y}_2 \sigma_2^{-3x-3y-1} \sigma_1 \), with \( x > y \geq 0 \) and \( x \equiv y \mod 2 \). Then \( \nabla_{\beta} = \nabla_{\gamma} \neq 0 \), by Prop. 2.3 but \( P_\beta \neq P_\gamma \) by Prop. 2.7.

**Lemma 2.11.** Assume \( \beta, \gamma \) are three-braids. Set \( a = w(\beta) \) and \( b = w(\gamma) \). Suppose \( a > \max(b, -1) \) and \( V_\beta = V_\gamma \).

We have \( \mu(\hat{\beta}) \neq 3 \) and \( a = b + 2k \). For \( k = 1 \), we have \( \nabla_{\beta} = C_{w(\beta) - 1} \).

When \( \hat{\beta} \) is a knot with \( k > 1 \), we have \( k = 2 \) and \( a \geq 2 \). If \( a = 2 \), we have \( \nabla_{\beta} = 1 = \nabla_{\gamma} \). For \( a > 2 \), we have \( a = 2(4p + 1) \) for some \( p > 0 \) and

\[
\mu(a-2)/2 \Delta_{\beta}(t) = \sum_{j=0}^{2p-1} \epsilon_j t^{kj}(1 - t + t^3) + t^{8p}, \tag{2.12}
\]

\[
\omega(a-2)/2 \Delta_{\gamma}(t) = 1 - t^2 + t^3 \sum_{j=0}^{2p-2} \epsilon_j t^{kj}(1 - t + t^3). \tag{2.13}
\]

When \( \mu(\hat{\beta}) = 2 \) with \( k > 1 \), we have \( k \) odd, \( a = k(4p + 3) \), for \( p \geq 0 \) and

\[
\mu(a-2)/2 \Delta_{\beta}(t) = \sum_{j=0}^{p-1} t^{4kj} \sum_{x=0}^{k-1} t^{3x}(-1 + t) + t^{3k-1}(-1 + t^k) + t^{4kp} \sum_{x=0}^{k-1} t^{3x}(-1 + t), \tag{2.14}
\]
\[ t^{b/2} \Delta_\gamma(t) = -(1 - t^k) + t^{k+1} \sum_{j=0}^{p-1} t^{4kj} \left( t^3 (-1 + t) + t^{3k-1} (-1 + t^k) \right). \] (2.15)

**Proposition 2.12.** If \( \beta, \gamma \in B_3 \) have \( \hat{V}_\beta = \hat{V}_\gamma \), we have \( \hat{P}_\beta = \hat{P}_\gamma \).

3. Proofs, Lemmas, and Tools

Theorems or equations whose proofs were not described above, and that don’t follow by simple induction, are proven in a subsection, ”Proof of . . .”.

3.1. Proof of (2.4) (Conway polynomial formula for 2-braid links)

**Proof.** Use the following result relating to Pascal’s triangle and induction:

\[
\binom{p}{j} = \binom{p-1}{j} + \binom{p-1}{j-1}
\]

3.2. Properties of the Conway polynomial

3.2.1. Properties of the Conway polynomial for n-braid links

The following formula shows how the Conway polynomial can be calculated using braid words of shorter length. For any braid word, \( \beta \in B_n \), and any integer, \( e \):

\[
\nabla_\beta \sigma_i^e = C_e \nabla_\beta \sigma_i^{e+1} + C_{e+1} \nabla_\beta.
\] (3.1)

3.2.2. Properties of the Conway polynomial for two-braid links

The Conway polynomial, \( C_p \), is central to the results of this paper, and its properties are numerous and remarkable. Most of its properties are most easily derived from the recursive relation, \( C_{p+1} = z C_p + C_{p-1} \), rather than by using (2.4).

It is helpful in calculations and proofs to know the following:

\[
z^m C_p = \sum_{j=0}^{m} \binom{m}{j} \epsilon_j C_{m+p-2j}, \text{ for } m > 0.
\]

In particular, when \( p = 1 \), there is an expression for \( z^m \).

The following important identities are critical tools to prove the main results. Assume integers, \( x, y, p, q, \) and \( \kappa \), are chosen with \( x + y = p + q \) in the equations below. Eq. (3.4) is generally applied with \( \kappa = q \). We have:

\[
C_{x+y} = C_x C_{y+1} + C_{x-1} C_y,
\] (3.2)

\[
C_x C_k = \sum_{j=0}^{k-1} \epsilon_j C_{x+k-1-2j}, \text{ for } k > 0,
\] (3.3)

\[
C_x C_y - C_p C_q = \epsilon_\kappa \{ C_{x-\kappa} C_{y-\kappa} - C_{p-\kappa} C_{q-\kappa} \}.
\] (3.4)
Proposition 3.1. There are no common roots over the complex numbers for $C_p$ and $C_{p+1}$, for any integer $p$, hence $\gcd(C_p, C_{p+1}) = 1$.

For any integers $a \neq 0$, and $b$, we have $C_a C_{ab} = C_b$.

Furthermore, when $\gcd(a, b) = g$, we have $\gcd(C_a, C_b) = C_g$.

Proof. The first claim follows from the relation, $C_{p+1} = zC_p + C_{p-1}$. Next, observe that $C_{ab} = C_a C_{ab-a+1} + C_{a-1} C_{ab-a}$ and use induction. For the third claim, there are integers, $\lambda, \mu$ so that $\lambda a + \mu b = g$. Hence $C_g = C_{\lambda a + \mu b} = C_{\lambda a} C_{\mu b} + C_{\lambda b} C_{\mu a}$.

This implies $C_a$ and $C_b$ have no common roots in $\mathbb{C}$ other than the roots of $C_g$. \hfill \Box

3.3. Properties of the Homflypt polynomial for $n$-braid links

The Homflypt polynomial shares many properties of the Conway polynomial. The following formulas show how the Homflypt polynomial can be calculated using braids of shorter length. For any braid word, $\beta \in B_n$, and any integer, $e$:

\[ P_{\beta \sigma_i^e} = v^{1-e} C_e P_{\beta \sigma_i} + v^{e} C_{e-1} P_{\beta}, \quad (3.5) \]

\[ P_{\beta \sigma_i^{e+1}} = v^{1+e} C_e P_{\beta \sigma_i^{e+1}} + v^{e} C_{e+1} P_{\beta}, \quad (3.6) \]

In particular, when $\sigma_i$ is a clustered generator in $\beta$, with exponent, $e$, there is:

\[ P_{\beta} = P_{\Psi(i,n,\beta)} P_{\Psi(0,i,\beta)}; \quad (3.7) \]

The following result is an easy consequence of Equations (3.4) and (3.7).

Proposition 3.2. The coefficient polynomials, $p_j$ and $h_j$ obey the same skein relation and reduction formulas, (3.7), as the Conway polynomial.

The coefficient polynomials of a link and its mirror image are related as follows:

\[ p_j(z) = \epsilon_{n-1} p_{n-1-j}(-z) = \epsilon_{w+n-1} p_{n-1-j} L(z). \]

When $\beta \in B_n$ is extended to $\beta \sigma_n \in B_{n+1}$ we have $h_{n, \beta \sigma_n} = 0 = p_{n, \beta \sigma_n}$. For all other subscripts, we have $h_{j, \beta \sigma_n} = h_{j, \beta} \sigma_n$ and $p_{j, \beta \sigma_n} = z p_{j, \beta}$.

When $\beta \in B_n$ is extended to $\beta \sigma_{n+1} \in B_{n+1}$, we have $h_{0, \beta \sigma_{n+1}} = 0 = p_{0, \beta \sigma_{n+1}}$. For all other subscripts, we have $h_{j, \beta \sigma_{n+1}} = h_{j-1, \beta} \sigma_{n+1}$ and $p_{j, \beta \sigma_{n+1}} = z p_{j-1, \beta}$.

3.4. Proof of Theorem (Expression for Laurent coefficients)

Proof. If we start with (3.1) and choose $z = s - s^{-1}$ and $v = 1, s, s^{-1}$ we obtain:

\[
\begin{pmatrix}
\Delta P(s^2) \\
\epsilon_{w-n+1} \\
1
\end{pmatrix} =
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
sw-n+1 & sw-n+3 & \cdots & sw+n-1 \\
s-w+n-3 & s-w+n-1 & \cdots & s-w+n+1
\end{bmatrix}
\begin{pmatrix}
h_0(s - s^{-1}) \\
\cdots \\
h_{n-1}(s - s^{-1})
\end{pmatrix}.
\]
If we designate by $A$ the $3 \times 3$ matrix consisting of the first three columns, and by $B$ the matrix consisting of the remaining $n - 3$ columns we have
\[
\begin{pmatrix}
\Delta_\beta(s^2) \\
\epsilon_{w-n+1} \\
1
\end{pmatrix} = A \begin{pmatrix}
h_0(s - s^{-1}) \\
h_1(s - s^{-1}) \\
h_2(s - s^{-1})
\end{pmatrix} + B \begin{pmatrix}
h_3(s - s^{-1}) \\
\cdots \\
h_{n-1}(s - s^{-1})
\end{pmatrix}.
\]
We may rewrite this as
\[
\begin{pmatrix}
h_0(s - s^{-1}) \\
h_1(s - s^{-1}) \\
h_2(s - s^{-1})
\end{pmatrix} = A^{-1} \begin{pmatrix}
\Delta_\beta(s^2) \\
\epsilon_{w-n+1} \\
1
\end{pmatrix} - A^{-1}B \begin{pmatrix}
h_3(s - s^{-1}) \\
\cdots \\
h_{n-1}(s - s^{-1})
\end{pmatrix}.
\]

It is readily seen that with $d(s) = (s^4 - 1)(s^2 - 1)$ we have
\[
d(s)A^{-1} = \begin{pmatrix}
-s^2(1 + s^2) & s^{-w+n-1} & s^{w-n+7} \\
(1 + s^2)(1 + s^4) - s^{-w+n-1}(1 + s^2) & -s^{w-n+5}(1 + s^2) \\
-s^2(1 + s^2) & s^{-w+n+1} & s^{w-n+5}
\end{pmatrix}.
\]

This implies
\[
d(s)h_0(s - s^{-1}) = -s^2(1 + s^2)\Delta_\beta(s^2) + s^{-w+n-1}\epsilon_{w-n+1} + s^{w-n+7} - \sum_{j=3}^{n-1}(s^{2k} - s^4 - s^{6-2k})h_j,
\]
\[
d(s)h_1(s - s^{-1}) = (1 + s^2)(1 + s^4)\Delta_\beta(s^2) - s^{-w+n-1}(1 + s^2)\epsilon_{w-n+1} - s^{w-n+5}(1 + s^2) + \sum_{j=3}^{n-1}(1 + s^2)(s^{2k} - s^4 - 1 + s^{4-2k})h_j,
\]
\[
d(s)h_2(s - s^{-1}) = -s^2(1 + s^2)\Delta_\beta(s^2) + s^{-w+n+1}\epsilon_{w-n+1} + s^{w-n+5} - \sum_{j=3}^{n-1}(s^{2+2k} - s^4 - s^{4-2k})h_j.
\]

This is exactly the result implied by Thm. 2.1. As $z = s - s^{-1}$ is invertible for $|s| > 1$ in the complex plane, $P_\beta(v, z)$ has the form claimed. \hfill \Box

3.5. Proof of Theorem 2.2 (Relations for Laurent coefficients)

Proof. The proof of (2.5), (2.6) follows by substitution of the values from Thm. 2.1.

To show (2.5), (2.6) are independent when $n > 1$, it suffices to find two sets of $n$ polynomials each of which satisfies one relation, but not the other.

In case $w \neq n - 1$, Eq. 3.2 implies $h_0 = C_{w+2-n}$, $h_1 = z C_{w+1-n}$, and $h_j = 0$, for $j > 1$ satisfy (2.5), but these don’t satisfy (2.6). When $w = n - 1$, the choice $h_1 = C_3$ and $h_j = 0$, for $j \neq 1$ satisfies (2.5), but does not satisfy (2.6).

In case $w \neq n - 3$, observe that $h_1 = C_{w+4-n}$, and $h_j = 0$, for $j \neq 1$ satisfy (2.6), but they don’t satisfy (2.5). When $w = n - 3$, the choice $h_0 = 1$, and $h_j = 0$, for $j \neq 0$ satisfies (2.6), but does not satisfy (2.5).
The final point is to show whenever a function satisfies (2.5, 2.6), it also satisfies (2.7). First take the difference of (2.5, 2.6), and divide by $z$ to obtain (2.7) with $\kappa = 1$. Now multiply both sides of this expression by $C$, multiply both sides of (2.6) by $-C_{\kappa = 1}$, and add these two products. Apply (3.2) to the right side of the sum and (3.3) to the left side to obtain (2.7).

### 3.6. Properties of the skein polynomials for 3-braid links

**Proposition 3.3.** When $\gamma \in B_3$ and $a > 0$ we have:

\[
\nabla_{\alpha_2^3} = \nabla_{\hat{\gamma}} + C_{w(\gamma)+6j-6} - C_{w(\gamma)+j+1},
\]

\[
\nabla_{\alpha_2^3 \gamma} = \nabla_{\hat{\gamma}} + \sum_{j=1}^{a} C_{w(\gamma)+6j-6} - \sum_{j=1}^{a} C_{w(\gamma)+j+1},
\]

\[
\nabla_{\alpha_2^3 \gamma} = \nabla_{\hat{\gamma}} + \sum_{j=1}^{a} C_{w(\gamma)-6j+6} - \sum_{j=1}^{a} C_{w(\gamma)-j+1}.
\]

If $\gamma = \prod_{k=1}^{r} \sigma_2^{-e_{k,2}} \sigma_1^{e_{k,1}}$, with $r, e_{k,2}, e_{k,1} > 0$, and $1 < E_i = \sum_{k=1}^{r} e_{k,i}$, we have

\[
\nabla_{\hat{\gamma}} = (-1)^{E_2+1} \{ C_{E_2+E_1} - r C_{E_2+E_1-4} + o(C_{E_2+E_1-3}) \},
\]

(3.8)

\[
\Delta_{\hat{\gamma}} = (-1)^{E_2+1} \{ t^{E_2+E_1-2}/2 - (r+1) t^{E_2+E_1-4} + \ldots \},
\]

(3.9)

\[
\nabla_{\hat{\gamma}} = (-1)^{E_2+1} \{ C_{E_2} C_{E_1} - \prod_{k=1}^{2} C_{e_{k,2}} C_{e_{k,1}} \}, \text{ when } r = 2.
\]

Eq. (3.9) is essentially given by Prop. 4.2, p. 13-14 [8].

**Proof.** Induction, plus (3.1) and (3.2) for the first and last equations. Eq. (3.4) is also helpful to prove the order of magnitude result in (3.8).

**Proposition 3.4.** When $\gamma \in B_3$ and $a > 0$ we have:

\[
P_{\alpha_2^3 \gamma} = v^6 P_{\hat{\gamma}} + P_{T_{w(\gamma)+5}} - v^6 P_{T_{w(\gamma)+1}},
\]

\[
P_{\alpha_2^3 \gamma} = v^{6a} P_{\hat{\gamma}} + \sum_{j=1}^{a} v^{6a-6j} P_{T_{w(\gamma)+6j-6}} - \sum_{j=1}^{a} v^{6a-6j} P_{T_{w(\gamma)+6j-5}},
\]

\[
P_{\alpha_2^3 \gamma} = v^{6a} P_{\hat{\gamma}} + \sum_{j=1}^{a} v^{6j-6a} P_{T_{w(\gamma)-6j+6}} - \sum_{j=1}^{a} v^{6j-6a} P_{T_{w(\gamma)-6j+5}}.
\]

**Proof.** Induction, plus Prop. 3.3 and Prop. 2.4 for the first equation.
3.7. Proof of Prop. 2.7 (When do 3 braid links have $P_\beta = P_\gamma$)

**Proof.** Suppose first that $P_\beta = P_\gamma$. Since $w(\beta) = w(\gamma)$ is valid, assume $w(\beta) > w(\gamma)$. Prop. 2.3 shows that $p_{2,\beta} = 0$ and $p_{0,\gamma} = 0$, i.e. $\nabla_\beta = C_{w(\beta)-1}$ and $\nabla_\gamma = C_{w(\gamma)+1}$. Since $P_\beta = P_\gamma$, we have $C_{w(\beta)-1} = C_{w(\gamma)+1}$, i.e. $w(\beta) = w(\gamma) + 2$ or $w(\beta) = -w(\gamma)$. These correspond to the second and third outcomes.

Conversely, when $\nabla_\beta = \nabla_\gamma$, the first condition implies $P_\beta = P_\gamma$ by Prop. 2.6. Prop. 2.4 shows the remaining two cases also imply $P_\beta = P_\gamma$. □

3.8. Proof of Proposition 2.8 (3-Braid Degree Result)

**Proof.** It suffices to prove either $\deg p_0 \leq \deg p_1$, or $\deg p_2 \leq \deg p_1$, and to consider only the case in which $p_0$ and $p_2$ are both nonzero. By Prop. 3.2, the writhe may be assumed to be non-negative. Separate consideration of the cases $\nabla_\beta = 0$, $w = 0$, $w = 1$, $\deg \nabla_\beta > w$, $\deg \nabla_\beta = w$, $\deg \nabla_\beta = w - 2$, and $\deg \nabla_\beta < w - 2$ shows the result is true in all cases. □

3.9. Proof of Lemma 2.11 ($V_\beta = V_\gamma$ properties for 3-braid links)

**Proof.** Observe that $V_\beta = V_\gamma$ implies $\mu(\beta) = \mu(\gamma)$ by (12.1), p. 368 [5], which states that $V_L(1) = (-2)^{a-1}$. Since $w \equiv 1 + \mu \mod 2$ for odd strand number, we have $a = b + 2k$, with $k > 0$.

Comparison of the Jones polynomials, (2.4), for $\beta$ and $\gamma$ yields an expression, (3.10), for $\Delta_\beta(t)$ in terms of $\Delta_\gamma(t)$. Using $\Delta_L(1/t) = (-1)^{\mu(L)-1}\Delta_L(t)$, p. 6 [12], we obtain an expression, (3.11), for $\Delta_\beta(t)$. A more useful form, (3.12), arises by dividing (3.11) by $t^{a/2} - t^{b/2}$ and simplifying.

$$t^b \Delta_\beta(t) = (t^{3b/2} - t^{3a/2}) t/(1 + t + t^2) + \epsilon_a(t^{b/2} - t^{a/2}) + t^a \Delta_\beta(t). \quad (3.10)$$

$$(t^{2a} - t^{2b}) \Delta_\beta(t) = \{t^{5a/2} - t^{(2a+3b)/2} + \epsilon_a(t^{(2a+b)/2} - t^{(4b-a)/2})\} t/(1 + t + t^2)$$

$+ \epsilon_a(t^{3a/2} - t^{(2a+b)/2}) + \epsilon(t^{2a+3b}/2 - t^{(4b+a)/2}). \quad (3.11)$$

$$t^{a/2}(t^k + 1)(t^k + 1) \Delta_\beta(t) = \{(t^{a+k} + \epsilon_a(t^{2k} + t^k + 1)) t/G$$

$$+ (t^a + \epsilon_a t^{3k}), \text{ for } k \neq 0. \quad (3.12)$$

When $\mu(\beta) = 3$ we must have $(t-1)^2|\Delta_\beta$, p. 6 [12], contradicting (3.12). When $k = 1$, substitute in (3.12) to see $\Delta_\beta(t) = \Delta_{L_{-1}}(t)$.

A calculation for $k \geq 2$ using (3.12) reveals that there are no solutions for $a = 0$ or $a = 1$, so $a > 1$. Assume now that $a$ is even, so $\beta$ and $\gamma$ are knots. Eq. 3.12 at $t = e^{\pi i/k}$ and $t = e^{\pi i/2k}$ show that we must have $k = 2$ and $a = 2(4p+1)$, for $p \geq 0$. For $p = 0$ we have $\Delta_\beta(t) = 1$ and for $p \geq 1$ Eqs. 2.12, 2.13 follow. Similarly for odd $a$ we have $a = k(4p+3)$, for $p \geq 0$ and Eqs. 2.14, 2.15.
3.10. Proof of Prop. 2.12 (V is equivalent to P for 3-braid links)

Proof. As in Lemma 2.11 write \( a = w(\beta) \) and \( b = w(\gamma) \) with \( a = b + 2k \). By Prop. 2.6 we may assume \( a > b \). We may also assume \( a \geq 0 \) (indeed \( a > 0 \)), since \( \Delta_T(t) = \Delta_L(1/t) \) and \( V_T(t) = V_L(1/t) \). Lemma 2.11 and Prop. 2.7 show that three-links and link pairs with \( k = 1 \) or \( a = k = 2 \) satisfy the result. It suffices to show that the link pairs described by Eqs. 2.12-2.15 are not three-braid link pairs.

When \( d = 0 \) in \( \Omega^k \), all coefficients of \( \Delta_\eta \) are non-zero and alternate in sign (Prop. 4.2, p. 13 [9]) as is true for \( \Delta_{\alpha_2^{\pm 3}} \sigma_2^{-e_1} \sigma_2^{e_1} \), when \( \min(e_1, E_1) = 1 \). The braids \( \alpha_2^{\pm 3} \sigma_2^{-e_1} \sigma_2^{e_1} \) with \( \min(e_1, E_1) = 1 \) also do not satisfy Eqs. 2.12-2.15.

Eqs. 2.13 and 2.15 show \( \deg b^{b/2} \Delta_{\eta} = b \). Prop. 2.9 thus implies \( \gamma \in \Omega^k \). Write \( \gamma = \alpha_2^{3d_2} \eta_2 \). By 2.11 we have \( \max \deg t^{b/2} \Delta_{\eta_2} = (b + |\eta_2| - 2)/2 \). Write \( F = t^{b/2}((t^{3d_2} - 1)(t^{3d_2} + w(\eta_2) - \epsilon_w(\eta_2)))/(t^{3d_2}t^{|\eta_2|/2}). \) When \( d_2 < 0 \), we find \( \max \deg F \) is \(-1+3d_2+w(\eta_2)\), which exceeds \( b \). This implies \(-1+3d_2+w(\eta_2) = (b+|\eta_2| - 2)/2\), so \( w(\eta_2) = |\eta_2| \). This is impossible for \( \eta_2 \), an alternating braid. Similarly, when \( d_2 > 0 \), we have \( \max \deg F \) is \( \max(b-1, 3d_2 - 1) \). When \( 3d_2 + w(\eta_2) \geq 0 \), the value is \( b-1 \) which forces \( b = (b+|\eta_2| - 2)/2 \). Eqs. 5.9 2.15 2.15 show this is impossible. We now have \( d_2 > 0 \) and \( 3d_2 + w(\eta_2) < 0 \), for which the same techniques show there are no solutions.

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