A sparse version of Reznick’s Positivstellensatz

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Abstract

If \( f \) is a positive definite form, Reznick’s Positivstellensatz [Mathematische Zeitschrift. 220 (1995), pp. 75–97] states that there exists \( k \in \mathbb{N} \) such that \( \|x\|^{2k} \) is a sum of squares of polynomials. Assuming that \( f \) can be written as a sum of forms \( \sum_{l=1}^{p} f_l \), where each \( f_l \) depends on a subset of the initial variables, and assuming that these subsets satisfy the so-called running intersection property, we provide a sparse version of Reznick’s Positivstellensatz. Namely, there exists \( k \in \mathbb{N} \) such that
\[
 f = \sum_{l=1}^{p} \sigma_l / H_l^k,
\]
where \( \sigma_l \) is a sum of squares of polynomials, \( H_l \) is a uniform polynomial denominator, and both polynomials \( \sigma_l, H_l \) involve the same variables as \( f_l \), for each \( l = 1, \ldots, p \). In other words, the sparsity pattern of \( f \) is also reflected in this sparse version of Reznick’s certificate of positivity. We next use this result to also obtain positivity certificates for (i) polynomials nonnegative on the whole space and (ii) polynomials nonnegative on a (possibly non-compact) basic semialgebraic set, assuming that the input data satisfy the running intersection property. Both are sparse versions of a positivity certificate due to Putinar and Vasilescu.

Keywords: Reznick’s Positivstellensatz, sparsity pattern, positive definite forms, running intersection property, sums of squares, Putinar-Vasilescu’s Positivstellensatz, uniform denominators, basic semialgebraic set

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1 Introduction and overview

Before the 1990s, representations of positive polynomials, also known as Positivstellensätze, have been discovered within a purely theoretical branch of real algebraic geometry. More recently, such Positivstellensätze have become a powerful tool in polynomial optimization and control.

Positivstellensätze and polynomial optimization. With $x = (x_1, \ldots, x_n)$, let $\mathbb{R}[x]$ stands for the ring of real polynomials and let $\Sigma[x] \subset \mathbb{R}[x]$ be the subset of sums of squares (SOS) of polynomials. Let us note $\mathbb{R}[x]_d$ and $\Sigma[x]_d$ the respective degree of these two sets to polynomials of degree at most $d$ and $2d$.

SOS decompositions of nonnegative polynomials have a distinguishing feature with important practical implications: Indeed they are tractable and can be determined by solving a semidefinite program. Namely, writing a polynomial $f \in \mathbb{R}[x]_{2d}$ as an SOS boils down to computing the entries of a symmetric (Gram) matrix $G$ with only nonnegative eigenvalues (denoted by “$G \succeq 0$”) such that $f = v_d^T G v_d$, with $v_d$ being the vector of all monomials of degree at most $d$.

Given $f, g_1, \ldots, g_m \in \mathbb{R}[x]$, and the basic semialgebraic set $S(g) := \{x \in \mathbb{R}^n : g_j(x) \geq 0, j = 1, \ldots, m\}$, with $g := \{g_1, \ldots, g_m\}$, polynomial optimization is concerned with computing $f^* := \inf \{f(x) : x \in S(g)\}$. A basic idea is to rather consider $f^* = \sup \{\lambda \in \mathbb{R} : f - \lambda > 0 \text{ on } S(g)\}$ and replace the difficult constraint “$f - \lambda > 0$ on $S(g)$” with a more tractable SOS-based decomposition of $f - \lambda$, thanks to various certificates of positivity on $S(g)$. For instance, if $S(g)$ is compact and satisfies the so-called Archimedean assumption, Putinar’s Positivstellensatz provides the decomposition $f - \lambda = \sigma_0 + \sum_{j=1}^m \sigma_j g_j$, with $\sigma_j \in \Sigma[x]$. Then one obtains the monotone non-decreasing sequence $(\rho_k)_{k \in \mathbb{N}}$ of lower bounds on $f^*$ defined by:

$$
\rho_k := \sup \{\lambda : f - \lambda = \sigma_0 + \sum_{j=1}^m \sigma_j g_j, \sigma_j \in \Sigma[x], \deg(\sigma_j g_j) \leq 2k\}. \quad (1.1)
$$

For each fixed $k$, (1.1) is a semidefinite program and therefore can be solved efficiently. Moreover, by invoking Putinar’s Positivstellensatz, one obtains the convergence $\rho_k \uparrow f^*$ as $k$ increases. In Table 1 are listed several useful Positivstellensätze that guarantee convergence of similar sequences $(\rho_k)_{k \in \mathbb{N}}$ to $f^*$ (where now in 1 one uses the appropriate positivity certificate). However their associated so-called dense hierarchies of linear/SDP programs are only suitable for modest size POPs (e.g., $n \leq 10$ and $\deg(f), \deg(g_j) \leq 10$). Indeed, for instance, even though (1.1) is a semidefinite program, it involves $(\begin{array}{c} n+2m \\end{array})$ variables and semidefinite matrices of size up to $(\begin{array}{c} n+k \end{array})$, a clear limitation for state-of-the-art semidefinite solvers.

Therefore a scientific challenge with important computational implications is to develop alternative positivity certificates that scale well in terms of computational complexity, at least in some identified class of problems.

Fortunately as we next see, we can provide such alternative positivity certificates for the class of problems where some structured sparsity pattern is present in the problem description (as often the case in large-scale problems). Indeed this sparsity pattern can be exploited to yield a positivity certificate in which the sparsity pattern is reflected, thus with potential significant computational savings.

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1 Semidefinite programming (SDP) is an important class of convex conic optimization problems that can be solved efficiently, up to arbitrary precision, fixed in advance; the interested reader is referred to e.g. [5 Chapter 4].

2 There are $\sigma_j \in \Sigma[x]$ such that $S(\sigma_0 + \sum_{j=1}^m \sigma_j g_j)$ is compact.
Table 1: Several Positivstellensätze applicable in practice.

| Author(s)               | Statement                                                                 | Application(s) |
|-------------------------|---------------------------------------------------------------------------|----------------|
| Schmüdgen [20]          | If $f$ is positive on $\mathbb{S}(g)$ and $\mathbb{S}(g)$ is compact, then $F = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha} \prod_{i=1}^{m} y_i^\alpha$ for some $c_{\alpha} \in \mathbb{S}(x)$. | 7              |
| Putinar [21]            | If a polynomial $f$ is positive on $\mathbb{S}(g)$ satisfying Archimedian assumption [2], then $f = \sigma^0 + \sum_{j=1}^{m} \sigma_j g_j$ for some $\sigma_j \in \mathbb{S}(x)$. | 12             |
| Resnick [23]            | If $f$ is a positive definite form, then $\|x\|^2 f \in \mathbb{S}(x)$ for some $k \in \mathbb{N}$. | 11             |
| Polya [20]              | If $f$ is a homogeneous form and $f > 0$ on $\mathbb{R}_+^n \setminus \{0\}$, then $(\sum_{j} x_j)^k f$ has nonnegative coefficients for some $k \in \mathbb{N}$. | 5              |
| Krivine-Stengle [11, 27]| If a polynomial $f$ is positive on $\mathbb{S}(g)$, $\mathbb{S}(g)$ is compact and $g_j \leq 1$ on $\mathbb{S}(g)$, then $f = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha, \beta} \prod_{j=1}^{m} g_j^{\alpha_j}(1 - g_j)^{\beta_j}$ for some $c_{\alpha, \beta} \geq 0$. | 13             |
| Putinar-Vasilescu [22]  | If a polynomial $f$ is nonnegative on $\mathbb{S}(g)$, then for every $c > 0$, there exists $k \in \mathbb{N}$ such that $\theta^k(f + cg^k) = \sigma^0 + \sum_{j} \sigma_j g_j$ for some $\sigma_j \in \mathbb{S}(x)$, where $d := 1 + \lceil \text{deg}(f)/2 \rceil$ and $\theta := \|x\|^2 + 1$. | 14             |

Exploiting sparsity pattern. For $n, m \in \mathbb{N}^n$, let $I := \{1, \ldots, n\}$ and $J := \{1, \ldots, m\}$. For $T \subseteq I$, denote by $\mathbb{R}(x(T))$ (resp. $\mathbb{S}(x(T))$) the ring of polynomials (resp. the subset of SOS polynomials) in the variables $x(T) := \{x_i : i \in T\}$. Also denote by $\mathbb{R}(x(T))_1$ (resp. $\mathbb{S}(x(T))_1$) the restriction of $\mathbb{R}(x(T))$ (resp. $\mathbb{S}(x(T))$) to polynomials of degree at most $t$ (resp. $2t$). For $R \subseteq J$, we note $g_{R} := \{g_j : j \in R\}$.

Designing alternative hierarchies for solving $f^* := \inf \{f(x) : x \in \mathbb{S}(g)\}$, significantly (computationally) cheaper than their dense version [11], while maintaining convergence to the optimal value $f^*$ is a real challenge with important implications.

One first such successful contribution is due to Waki et al. [29] when the input polynomial data $f, g_j$ are sparse, where by sparse we mean the following:

Assumption 1.1. The following conditions hold:

(i) Running intersection property (RIP): $I = \bigcup_{i=1}^{p} I_i$ with $p \in \mathbb{N}^n$, $I_i \neq \emptyset$, $l = 1, \ldots, p$, and for every $l \in \{2, \ldots, p\}$, there exists $n_l \in \{1, \ldots, l - 1\}$, such that $I_l \subseteq I_{n_l}$, where $I_l := I_l \cap \bigcup_{i=1}^{l-1} I_i$. W.l.o.g, set $s_2 := 1$ and $I_1 := \emptyset$. Denote $n_l := |I_l|$ and $n_l := |I_l|$, $l = 1, \ldots, p$.

(ii) Structured sparsity pattern for the objective function [3]: $f = \sum_{i=1}^{p} f_i$ where $f_i \in \mathbb{R}(x(I_i))_{\text{deg}(f)}$, $l = 1, \ldots, p$.

(iii) Structured sparsity pattern for the constraints: $J = \bigcup_{i=1}^{p} J_l$ and for every $j \in J_l$, $g_j \in \mathbb{R}(x(I_l))$, $l = 1, \ldots, p$.

(iv) Additional redundant quadratic constraints: There exists $L > 0$ such that $\|x\|^2 \leq L$ for all $x \in \mathbb{S}(g)$ and $\|x(I_l)|^2 \in g_{I_l}$, $l = 1, \ldots, p$.

With $\tau (\leq n)$ being the maximum number of variables appearing in each index subset $I_l$ of $f, g_j$, i.e., $\tau := \max \{n_l : l = 1, \ldots, p\}$, Table 2 displays the respective computational complexity of the sparse hierarchy of Waki et al. [29] and the dense hierarchy of Lasserre [12] for SDPs with same order $k \in \mathbb{N}$. Obviously the sparse hierarchy provides a potentially high computational saving when compared to the dense one. In addition, convergence of the hierarchy of Waki et al. to the optimal

3If there are $f_I$ in the sum $f$ such that $\text{deg}(f) > \text{deg}(f)$, we can always remove the high degree redundant term in $f_I$ which cancel with each other to make degree of $f_I$ at most $\text{deg}(f)$.
value of the original POP was proved in [13], resulting in the following sparse version of Putinar’s Positivstellensatz:

**Theorem 1.1.** (Lasserre, Waki et al.) Let Assumption 1.1 holds. If a polynomial \( f \) is positive on \( S(g) \), then there exist \( \sigma_{0, i} \in \Sigma[x(I_i)]_{k_i} \), \( \sigma_{j, l} \in \Sigma[x(I_l)]_{k_l - u_j} \) with \( u_j := \lceil \deg(g_j)/2 \rceil \), \( j \in J_l, l = 1, \ldots, p \) such that

\[
f = \sum_{i=1}^{p} \left( \sigma_{0, i} + \sum_{j \in J_i} \sigma_{j, l} g_j \right).
\]

Motivation for sparse representations on non-compact sets. We remark that Theorem 1.1 requires the additional redundant quadratic constraints (Assumption 1.1 (iv)), which is slightly stronger than just assuming the compactness of \( S(g) \). When \( S(g) \) is compact, we can always add these constraints but we need to know the radius \( L > 0 \) of a ball centered at the origin and containing \( S(g) \). In this case, adding such constraints increases the number of positive semidefinite matrices from \( m \) to \( m + p \) in each SDP. In addition, it may be hard to verify compactness of \( S(g) \) and obtain such a radius \( L \).

To the best of our knowledge, in the non-compact case there is still no Positivstellensatz allowing one to build hierarchies for POPs satisfying:

- the RIP and the structured sparsity pattern from Assumption 1.1 (i)-(iii),
- and a guarantee of convergence to the global optimum.

In fact we provide examples 2.2, 2.3, and 2.4 which show that in both unconstrained and constrained cases, there exist sparse nonnegative polynomials which do not have a sparse SOS-based decomposition (1.2) à la Putinar. Such examples have been our motivation to investigate existence of sparse representations in the non-compact case, as well as to construct converging SDP-hierarchies for sparse polynomial optimization in general.

Dense rational SOS representations and non-compact POPs. In his famous and seminal work [8], Hilbert characterized all cases where nonnegative polynomials are SOS of polynomials. In 1927, Artin proved in [2] that every nonnegative polynomial can be decomposed as an SOS of rational functions (or rational SOS for short), thereby solving Hilbert’s 17th problem. Namely, a polynomial \( f \) is nonnegative if and only if there exist \( \sigma_1, \sigma_2 \in \Sigma[x] \) such that \( f = \sigma_1/\sigma_2 \).

Of course one can use Hilbert-Artin’s representation to obtain a hierarchy of lower bounds for unconstrained POPs: \( f^* := \inf_{x \in \mathbb{R}^n} f(x) \), by computing \( \rho_k := \sup \{ \lambda \in \mathbb{R} : \sigma_2(f - \lambda) = \sigma_1, \sigma_j \in \Sigma[x]_k \}, \) for every \( k \in \mathbb{N} \), so that \( \rho_k \leq \rho_{k+1} \leq f^* \) for all \( k \). However for each \( k \) the resulting optimization problem is not an SDP (and not even convex) because of the nonlinear term \( \sigma_2 \lambda \). (Even with an iterative dichotomy procedure on \( \lambda \), one is left with an SDP hierarchy for each fixed \( \lambda \).)
When $f$ is a positive definite form Reznick proposes to select a so-called uniform denominator in the Hilbert-Artin’s representation, namely to replace $\sigma_2$ by some power of $\|x\|^2$ (see Table 1). As a result one obtains a decomposition in SOS of rational functions for any arbitrary small perturbation of a nonnegative polynomial $f$ as follows: For every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\theta^k (f + \varepsilon \theta^d) = \sigma_0$ for some $\sigma_0 \in \Sigma[x]$, with $d := \lceil \deg(f)/2 \rceil$ and $\theta := \|x\|^2 + 1$. For arbitrary $\varepsilon > 0$ fixed, we obtain an SDP-based hierarchy of bounds $\rho_k(\varepsilon) = \sup \{ \lambda \in \mathbb{R} : \theta^k (f - \lambda + \varepsilon \theta^d) = \sigma_0, \sigma_0 \in \Sigma[x,k,d] \}$, for every $k \in \mathbb{N}$. If $f^*$ is attained then the sequence $(\rho_k(\varepsilon))_{k \in \mathbb{N}}$ converges to a value in a neighborhood of $f^*$. A similar idea, now based on Putinar-Vasilescu’s Positivstellensatz [22], can be applied for polynomials nonnegative on non-compact basic semialgebraic sets (see Table 1).

This shows that rational SOS representations with fixed forms for denominators are highly useful and applicable in non-compact POPs.

Contribution. Our contribution is twofold:

- We first provide a rational SOS representation for a positive definite rational form which is a sum of sparse rational functions with uniform denominators, satisfying the structured sparsity pattern and the RIP stated in Assumption 1.1 (i). This representation is provided in Theorem 2.1. As a direct consequence, we obtain a sparse version of Reznick’s Positivstellensatz in Corollary 2.1.

- Then, we provide two positivity certificates for arbitrary small perturbations of – globally nonnegative polynomials in Corollary 2.2 – and polynomials nonnegative on a (possibly non-compact) basic semialgebraic set in Corollary 2.3, when the input data satisfy a similar sparsity pattern. These two certificates are obtained via a sparse version of Putinar-Vasilescu’s Positivstellensatz and do not require the additional constraints from Assumption 1.1 (iv).

Illustrations of such positivity certificates for polynomials nonnegative on non-compact basic semialgebraic sets are provided in Example 2.1, 2.2, 2.3 and 2.4, for which positivity certificates [11] do not exist. The existence of such sparse SOS-representations is proved by combining different tools:

- First, we use an idea similar to that developed in Grimm et al. [6] (in the compact case) to prove that a sparse positive definite form can be decomposed as SOS of sparse positive definite rational forms; as expected the non-compact case is technically more involved. This yields a sparse version of Hilbert-Artin’s representation theorem in the case of positive definite forms.

- Next, we use generalizations of Schm"udgen’s Positivstellensatz presented by Schweighofer [26], Berr-W"ormann [4], Jacobi [9], and Marshall [17, 18], for a finitely generated $\mathbb{R}$-algebra in each term of the sum, to obtain again a sparse version, this time of Reznick’s Positivstellensatz for positive definite forms.

- Finally we combine the homogenization/dehomogenization method that we already used in [16] together with limit tools, to provide the two sparse versions of Putinar-Vasilescu’s Positivstellensatz.

2 Main results

For $(i, j) \in \mathbb{N}^2$, we denote the Kronecker delta function by

$$\delta_{i,j} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$
When Assumption 1.1 holds, define:

\[ \Phi_l := \begin{cases} \|x(I_l)\|_2^{2(1-\delta_l,1)} \prod_{i=1}^p \|x'(I_l)\|_2^{2\delta_i} & \text{if } l = 1, \ldots, p - 1, \\ \|x(I_l)\|_2 & \text{if } l = p. \end{cases} \]

Obviously, one has \( \Phi_l \in \mathbb{R}[x(I)] \), for each \( l = 1, \ldots, p \). Let us state the first main result of this paper which yields a sparse version of Reznick’s Positivstellensatz as a particular case.

**Theorem 2.1.** Let Assumption 1.1 (i) holds. Let \( f \in \mathbb{R}(x) \) be a positive definite rational form of degree \( 2d \) with \( d \in \mathbb{N}^{>0} \) such that

\[ f = \sum_{l=1}^p \frac{p_l}{\|x(I_l)\|_2^{2k_l}}, \]

where \( p_l \in \mathbb{R}[x(I_l)] \) is homogeneous of degree \( 2(d + k_l) \) for some \( k_l \in \mathbb{N} \), \( l = 1, \ldots, p \). Then there exist \( k \in \mathbb{N} \) and \( \sigma_l \in \Sigma[x(I_l)]_{d+k(1+\deg(\Phi_l))/2}, l = 1, \ldots, p \), such that

\[ f = \sum_{l=1}^p \frac{\sigma_l}{H_l}, \]  

(2.3)

The proof of Theorem 2.1 can be found in Section 4.1. As a consequence, we obtain the following sparse version of Reznick’s Positivstellensatz:

**Corollary 2.1.** Let Assumption 1.1 (i) holds. Assume that \( f \) is a positive definite form of degree \( 2d \) with \( d \in \mathbb{N}^{>0} \) and \( f = \sum_{l=1}^p f_l \), where \( f_l \in \mathbb{R}[x(I_l)] \) is homogeneous of degree \( 2d, l = 1, \ldots, p \). Then there exist \( k \in \mathbb{N} \) and \( \sigma_l \in \Sigma[x(I_l)]_{d+k(1+\deg(\Phi_l))/2}, l = 1, \ldots, p \), such that

\[ f = \sum_{l=1}^p \frac{\sigma_l}{H_l}, \]  

(2.4)

where \( H_l := \|x(I_l)\|_2^{2k_l}, l = 1, \ldots, p \).

To prove Corollary 2.1 we apply Theorem 2.1 with \( k_l = 0, l = 1, \ldots, p \). The representation (2.4) can still hold even when \( f \) is not a positive definite form, as illustrated in the following example:

**Example 2.1.** Let \( f = f_1 + f_2 \), where

\[ f_1 := x_2^2(x_1^4x_2^2 + x_2^4x_1^2 + x_1^2x_2^4 - 3x_2^2x_1^2) + x_3^8 \]

is the so-called Delzell’s polynomial and \( f_2 := x_1^3x_2^4x_3^4 \). The polynomial \( f_1 \) is nonnegative, but not SOS as shown in [13, Example 2]. Let \( I_1 := \{1,2,3,4\} \) and \( I_2 := \{1,2,3,5\} \). Then \( f_1 \in \mathbb{R}[x(I_1)] \) and \( f_2 \in \mathbb{R}[x(I_2)] \) are nonnegative and homogeneous of degree 8. Since \( f_1 \) is nonnegative then \( f \) is nonnegative. The following statements hold:

1. \( f \) is a nonnegative form, but is not positive definite;
2. \( f \notin \Sigma[x(I_1)] + \Sigma[x(I_2)] \), but \( f \in \frac{\Sigma[x(I_1)]}{\|x(I_1)\|_2^{\Phi_1}} + \frac{\Sigma[x(I_2)]}{\|x(I_2)\|_2^{\Phi_2}} \).

The first statement follows from the fact that \( f(0,0,0,1,1) = 0 \), ensuring that \( f \) is not a positive definite form.

Proof of the second statement: Assume by contradiction that \( f = \sigma_1 + \sigma_2 \) for some \( \sigma_l \in \Sigma[x(I_l)], l = 1, 2 \). Evaluation at \( x_5 = 0 \) yields \( f_1 = \sigma_1 + \sigma_2(x_1, x_2, x_3, x_4, 0) \), so that \( f_1 \) is an SOS, which is impossible. Thus, \( f \notin \Sigma[x(I_1)] + \Sigma[x(I_2)] \). However, \( (x_1^4 + x_2^4 + x_3^4)f_1 \) is SOS according to [22, Example 4.4], so \( (x_1^4 + x_2^4 + x_3^4)f \in \Sigma[x(I_1)]_{5} + \Sigma[x(I_2)]_{5} \). Note that \( \Phi_1 = \Phi_2 = x_1^4 + x_2^4 + x_3^4 \). Therefore

\[ f \in \frac{\Sigma[x(I_1)]}{\Phi_1} + \frac{\Sigma[x(I_2)]}{\Phi_2} \subset \frac{\Sigma[x(I_1)]}{H_1} + \frac{\Sigma[x(I_2)]}{H_2}. \]
When Assumption 1.1 (i) holds, define the following polynomials, for each \( l = 1, \ldots, p \):
- \( \theta_l := \|x(I_l)\|^2 + 1 \) and \( \bar{\theta}_l := \|x(\bar{I}_l)\|^2 + 1 \);
- \( D_l := \begin{cases} \theta_l^{-1/2} \prod_{j=1}^p \theta_j^{-1/2} & \text{if } l < p, \\ \theta_l^{-1} & \text{if } l = p; \end{cases} \)
- \( \Theta_l := \theta_l D_l \) and \( \omega_l := \deg(\Theta_l)/2 \).

Note that \( \Theta_l \in \mathbb{S}[x(I_l)]_{\omega_l} \), for each \( l = 1, \ldots, p \).

We next state the following sparse version of Putinar-Vasilescu’s Positivstellensatz for polynomials nonnegative on the whole \( \mathbb{R}^n \).

**Corollary 2.2.** Let \( f \) be a nonnegative polynomial such that the conditions (i) and (ii) of Assumption 1.1 hold. Let \( \varepsilon > 0 \) and \( d \geq \deg(f)/2 \). Then there exist \( k \in \mathbb{N} \) and \( \sigma_l \in \mathbb{S}[x(I_l)]_{d+k\omega_l}, \ l = 1, \ldots, p \), such that

\[
f + \varepsilon \sum_{i=1}^p \theta_i^l = \sum_{i=1}^p \frac{\sigma_l}{\Theta_l}.
\]

The proof of Corollary 2.2 is postponed to Section 4.2.

The representation (2.5) can still hold even if \( \varepsilon = 0 \), as illustrated in the following examples:

**Example 2.2.** Let \( f = f_1 + f_2 \), where

\[
f_1 := 8 + \frac{1}{2} x_1^2 x_2^2 + (x_1^2 - 2x_1^2)x_2^2 + (2x_1 + 10x_2^2 + 4x_3^2 + 3x_4^4)x_2^2 + 4(x_1 - 2x_2^2)x_2
\]

is the so-called Leep-Starr’s polynomial and \( f_2 := x_2^2 x_2^2 \). Let \( I_1 := \{1, 2\} \) and \( I_2 := \{1, 3\} \), so that \( f_1 \in \mathbb{R}[x(I_1)] \) and \( f_2 \in \mathbb{R}[x(I_2)] \). As shown in [15, Example 2.2], \( f_1 \) is nonnegative but not an SOS. In addition, \( f \) is nonnegative.

We claim that \( f \notin \mathbb{S}[x(I_1)]_2 + \mathbb{S}[x(I_2)]_2 \). Indeed, assume by contradiction that \( f = \sigma_1 + \sigma_2 \) for some \( \sigma_j \in \mathbb{S}[x(I_j)]_k, k = 1, 2 \). Evaluation at \( x_3 = 0 \), yields \( f_1 = \sigma_1 + \sigma_2(x_2, 0) \), so that \( f_1 \) is an SOS, which is impossible.

However, \((x_1^2 + 1)^2 f_1\) is a sum of squares of polynomials according to [15, Example 2.2], so \((x_1^2 + 1)^2 f \in \mathbb{S}[x(I_1)]_2 + \mathbb{S}[x(I_2)]_2 \). Note that \( D_1 = D_2 = x_1^2 + 1 \). Thus,

\[
f = \sum_{l=1}^p \frac{\mathbb{S}[x(I_l)]_2}{D_l^2} + \sum_{l=1}^p \frac{\mathbb{S}[x(I_l)]_2}{\Theta_l} = \sum_{l=1}^p \frac{\mathbb{S}[x(I_l)]_2}{\Theta_l}.
\]

**Example 2.3.** As shown in [16, Example 5.2], the nonnegative polynomial

\[
f = x_1^2 - 2x_1 x_2 + 3x_2^2 - 2x_1^2 x_2 + 2x_1^2 x_2^2 - 2x_2 x_3 + 6x_3^2 + 18x_2^2 x_3 - 54x_2 x_3^2 + 142x_2 x_3^2
\]

satisfies \( f \in \mathbb{R}[x(I_1)] + \mathbb{R}[x(I_2)] \) and \( f \notin \mathbb{S}[x(I_1)]_2 + \mathbb{S}[x(I_2)]_2 \), with \( I_1 := \{1, 2\} \) and \( I_2 := \{2, 3\} \). However, \( f \notin \mathbb{S}[x(I_1)]_1 + \mathbb{S}[x(I_2)]_1 \), where \( \Theta_1 = (x_2^2 + 1)(x_1^2 + x_2^2 + 1) \) and \( \Theta_2 = (x_2^2 + 1)(x_2^2 + x_3^2 + 1) \). It is due to the fact that \( f = \frac{3x_2}{x_1} + \frac{x_1}{x_2} \), where \( D_1 = D_2 = x_2^2 + 1 \) and \( \sigma_1 \) and \( \sigma_2 \) are SOS polynomials given in Appendix A.

We next state our second main result, namely a sparse version of Putinar-Vasilescu’s Positivstellensatz for polynomials nonnegative on (possibly non-compact) basic semialgebraic sets.

**Corollary 2.3.** Let \( f \in \mathbb{R}[x] \) be nonnegative on \( S(g) \) such that the conditions (i), (ii) and (iii) of Assumption 1.1 hold. Let \( \varepsilon > 0 \) and \( d \geq 1 + \lceil \deg(f)/2 \rceil \). Recall that \( u_j = \lceil \deg(g_j)/2 \rceil \) for all \( j = 1, \ldots, m \). Then there exist \( k \in \mathbb{N} \), \( \sigma_j \in \mathbb{S}[x(I_j)]_{d+k\omega_j} \) and \( \sigma_j \in \mathbb{S}[x(I_j)]_{d+k\omega_j} \), such that

\[
f + \varepsilon \sum_{i=1}^p \theta_i^l = \sum_{i=1}^p \frac{\sigma_j \sigma_j}{\Theta_l}.
\]
The proof of Corollary 2.3 is postponed to Section 4.3.

Example 2.4. Let \( f = f_1 + f_2 \), where \( f_1 = x_1 x_2 \) and \( f_2 = x_2^2 x_3 \). Let \( g = \{ g_1, g_2, g_3 \} \), where \( g_1 = x_1^2 \), \( g_2 = -x_1 \) and \( g_3 = x_3 \). It is not hard to show that \( f = 0 \) on \( S(g) \), so that \( \int g \) depends on \( S(g) \). By noting \( I_1 := \{ 1, 2 \} \) and \( I_2 := \{ 2, 3 \} \), one has \( \{ f_1, g_1, g_2 \} \subset \mathbb{R}[x(I_1)] \) and \( \{ f_2, g_3 \} \subset \mathbb{R}[x(I_2)] \). We claim the following statements:

1. \( f \notin \Sigma[x(I_1)] + g_1 \mathbb{R}[x(I_1)] + \Sigma[x(I_2)] + g_3 \Sigma[x(I_2)] \).
2. for every \( \varepsilon > 0 \),

\[
f + \varepsilon (\theta_1^2 + \theta_2^2) \subset \frac{\Sigma[x(I_1)]_{2k+2} + g_1 \mathbb{R}[x(I_1)]_{4k+1}}{\Theta_1^3} + \frac{\Sigma[x(I_2)]_{2k+2} + g_3 \Sigma[x(I_2)]_{4k+3}}{\Theta_2^3},
\]

for some \( k \in \mathbb{N} \) depending on \( \varepsilon \).

Proof of the first statement: Assume by contradiction that there exist \( \sigma_1 \in \Sigma[x(I_1)] \), \( \psi_1 \in \mathbb{R}[x(I_1)] \) and \( \sigma_2, \sigma_3 \in \Sigma[x(I_2)] \) such that \( f = \sigma_1 + \psi_1 g_1 + \sigma_2 + \sigma_3 g_3 \). Evaluation at \( x_1 = 1 \) and \( x_3 = 0 \) yields

\[
x_2 = \sigma_1(1, x_2) + \psi_1(1, x_2) x_3^2 + \sigma_2(x_2, 0) \in \Sigma[x_2] + x_2^3 \mathbb{R}[x_2],
\]

which is impossible due to [IZ Lemma 3.3 (i)].

Proof of the second statement: With \( \varepsilon > 0 \) fixed,

\[
f_1 + \varepsilon \theta_1^2 = x_1 x_2 + \varepsilon (1 + x_1^2 + x_2^2)^2 = x_1 x_2 + \varepsilon x_1^2 + \sigma_4.
\]

for some \( \sigma_4 \in \Sigma[x(I_1)]_2 \). Let \( k \in \mathbb{N} \geq 2 \) be fixed. Then \( D_k^1 = (1 + x_2^2)^k = 1 + k x_2^2 + \frac{x_2^4}{2} \) for some \( \sigma_5 \in \Sigma[x_2]_{k-2} \), which implies

\[
D_k^1(f_1 + \varepsilon \theta_1^2) = x_1 x_2 + \varepsilon x_1^2 + \varepsilon k x_2^2 + \sigma_6 + \psi_2 x_2^3,
\]

for some \( \sigma_6 \in \Sigma[x(I_1)]_{k+2} \) and \( \psi_2 \in \mathbb{R}[x(I_1)]_{2k+1} \). Assume that \( k \geq 2 \). Then

\[
D_k^1(f_1 + \varepsilon \theta_1^2) = x_1^3 \left( \varepsilon - \frac{1}{4k} \right) + \left( x_2 \varepsilon k + \frac{x_2^3}{4k} \right)^2 + \sigma_6 + \psi_2 x_2^3
\]

\[
\in \Sigma[x(I_1)]_{k+2} + g_1 \mathbb{R}[x(I_1)]_{2k+1},
\]

which implies \( f_1 + \varepsilon \theta_1^2 \in \Sigma[x(I_1)]_{2k+2} + g_1 \Sigma[x(I_1)]_{4k+1} \). We also have

\[
f_2 + \varepsilon \theta_2^2 \in \Sigma[x(I_2)]_{2k+2} + g_3 \Sigma[x(I_2)]_{4k+3}
\]

since \( f_2 \in g_3 \Sigma[x(I_2)]_1 \), proving the second statement.

## 3 Preliminary material

Given \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), we note \( |\alpha| := \alpha_1 + \cdots + \alpha_n \) and \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

Let \( (x^\alpha)_{\alpha \in \mathbb{N}^n} \) be the canonical basis of monomials for \( \mathbb{R}[x] \) (ordered according to the graded lexicographic order) and \( \psi_t(x) \) be the vector of monomials up to degree \( t \), with length \( s(t) = \binom{n+1}{n} \). A polynomial \( h \in \mathbb{R}[x] \) is written as \( h(x) = \sum_{|\alpha| \leq t} h_\alpha x^\alpha \), where \( h_\alpha \) is its vector of coefficients in the canonical basis.

Denote by \( \mathbb{R}^{n-1} := \{ x \in \mathbb{R}^n : \|x\| = 1 \} \) the \( (n - 1) \)-dimensional unit sphere.

A function \( h \) is homogeneous of degree \( t \) if \( h(\lambda x) = \lambda^t h(x) \) for all \( x \in \mathbb{R}^n \) and each \( \lambda \in \mathbb{R} \). Therefore a homogeneous polynomial can be written as \( h = \sum_{|\alpha| = t} h_\alpha x^\alpha \). A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is even if \( f(x) = f(-x) \) for all \( x \). A rational function \( h \) is the ratio of two polynomials and denote by \( \mathbb{R}(x) \) the space of all rational functions. A homogeneous rational function (also called be a rational form, or form in short) can be written as the ratio of two homogeneous polynomials.
The degree-\(d\) homogenization \(\tilde{h}\) of \(h \in \mathbb{R}(x_1, \ldots, x_n)\) is a homogeneous rational function in \(\mathbb{R}(x_1, \ldots, x_{n+1})\) of degree \(d\) defined by \(\tilde{h}(x, x_{n+1}) = x^{d+1}_{n+1} h(x/x_{n+1}).\) A rational positive definite form of degree \(t\) is a homogeneous rational function of degree \(t\) which is positive everywhere except at the origin. Equivalently, a homogeneous rational function \(h\) of degree \(t\) is a rational positive definite form of degree \(t\) if and only if there exists \(\varepsilon > 0\) such that \(h \geq \varepsilon \|x\|_2^t\).

We briefly recall some algebraic tools from generalizations of Schmüdgen’s Positivstellensatz\ [20] which will be used in the sequel. An associative algebra \(A\) is called a finitely generated \(\mathbb{R}\)-algebra if there exists a finite set of elements \(a_1, \ldots, a_n\) of \(A\) such that every element of \(A\) can be expressed as a polynomial in \(a_1, \ldots, a_n\), with coefficients in \(\mathbb{R}\). Let \(A\) be a commutative ring. We denote by \(\Sigma A^2\) the set of all SOS of elements in \(A\). A subset \(T\) of \(A\) is called a preordering if \(T\) contains all squares and is closed under addition and multiplication. The preordering \(T\) generated by elements \(t_1, \ldots, t_m\) (so-called smallest preordering containing \(t_1, \ldots, t_m\)) consists of all elements of the form \(\sum_{\alpha \in \{0,1\}^m} (\sigma_\alpha \prod_{j=1}^m t_j^{\alpha_j})\), with \(\sigma_\alpha \in \Sigma A^2\). The real spectrum of a ring \(A\) with fixed preordering \(T\), denoted by \(\text{Sper}_T \ A\), is defined by

\[
\text{Sper}_T \ A := \{ \varphi \in \text{Hom}(A, \mathbb{R}) : \varphi(T) \subset \mathbb{R}_+ \},
\]

where \(\text{Hom}(A, \mathbb{R})\) is the set of all ring homomorphisms from \(A\) to \(\mathbb{R}\). Let \(A\) be a ring with fixed preordering \(T\). We denote by \(H(A)\) (resp. \(H'(A)\)) the ring of geometrically (resp. arithmetically) bounded elements in \(A\), i.e.,

\[
H(A) := \{ h \in A : \exists K \in \mathbb{N} : K \leq h \geq 0 \text{ on } \text{Sper}_T \ A \}
\]

\[
H'(A) := \{ h \in A : \exists K \in \mathbb{N} : K \leq h \geq 0 \text{ on } T \},
\]

where \(\|h\| \geq 0\) on \(\text{Sper}_T \ A\) means \(\varphi(h) \geq 0\) for all \(\varphi \in \text{Sper}_T \ A\). From \[26\] (1.1),

\[
A = H(A) \Rightarrow A = H'(A) . \tag{3.7}
\]

Let us restate \[26\] Theorem 1.3 as follows:

**Lemma 3.1.** If \(Q \subset A\) and \(A = H'(A)\), then for any \(f \in A\),

\[
f > 0 \text{ on } \text{Sper}_T \ A \Rightarrow f \in T .
\]

Let us note \(\|h\| := \sum_{\alpha} |h_\alpha|\) for a given \(h \in \mathbb{R}[x]\). We start with two preliminary results.

**Lemma 3.2.** For \(k \in \mathbb{N}\) and \(d \in \mathbb{N}^0\), let \(q\) be a form of degree \(2(d+k)\) and \(f = \frac{q}{\|x\|_2^d} \in \mathbb{R}(x)\). Then \(f\) is continuous and homogeneous of degree \(2d\).

**Proof.** The rational function \(f\) is obviously homogeneous of degree \(2d\). To show that \(f\) is continuous, it is sufficient to prove that \(f\) is continuous at zero. Let \(y \in \mathbb{S}^{n-1}\), then one has \(\|y\| \leq 1\), for all \(\alpha\) such that \(|\alpha| = 2(d+k)\). Thus,

\[
|q(y)| = \left| \sum_{\alpha} q_\alpha y^\alpha \right| \leq \sum_{\alpha} |q_\alpha| |y^\alpha| \leq \sum_{\alpha} |q_\alpha| = \|q\|_1 .
\]

From this, one has \(|f(y)| = |q(y)| \leq \|q\|_1\). Let \(x \neq 0\). Since \(f\) is homogeneous of degree \(2d\),

\[
\left| \frac{f(x)}{\|x\|_2^d} \right| = \left| f \left( \frac{x}{\|x\|_2^d} \right) \right| \leq \|q\|_1 .
\]

Hence for all \(x \neq 0\), \(|f(x)| \leq \|q\|_1 \|x\|_2^d\), thus \(\lim_{x \to 0} f(x) = 0\), yielding the conclusion. \(\square\)

**Lemma 3.3.** Let \(h : \mathbb{R}^n \to \mathbb{R}\) be an even function such that \(h\) is continuous on \(\mathbb{S}^{n-1}\). Then there exists a sequence \(\{q_k\}_{k \in \mathbb{N}}\) of homogeneous polynomials, with \(\deg(q_k) = 2k\) for all \(k \in \mathbb{N}\), converging uniformly to \(h\) on \(\mathbb{S}^{n-1}\).
Lemma 3.4. Assume that $I = I_1 \cup I_2$. Let $f \in \mathbb{R}(x)$ be a rational positive definite form of degree $2d$ with $d \in \mathbb{N}^+$ such that $f = f_1 + f_2$ with $f_1 \in \mathbb{R}(x(I_1))$ and $f_2 \in \mathbb{R}(x(I_2))$ being continuous and homogeneous of degree $2d$. Then there exists a continuous rational function $\varphi \in \mathbb{R}(x(I_1 \cap I_2))$ defined by

$$\varphi(y) = \frac{q(y)}{\|y\|^{2d}}, \quad \forall y \in I^{[I_1 \cap I_2]},$$

where $q \in \mathbb{R}[x(I_1 \cap I_2)]$ is a form of degree $2(d + k)$ for some $k \in \mathbb{N}$ (only depending on $d$, $\varepsilon$ and $f_1$) such that

$$f = h_1 + h_2,$$

where $h_1 := f_1 - \varphi \in \mathbb{R}(x(I_1))$ and $h_2 := f_2 + \varphi \in \mathbb{R}(x(I_2))$ are continuous rational positive definite forms of degree $2d$.

Proof. Since $f \in \mathbb{R}(x)$ is a rational positive definite infinite form of degree $2d$, there exists $\varepsilon > 0$ such that

$$f \geq \varepsilon \|x\|^{2d} \quad \text{on} \quad \mathbb{R}^n. \quad (3.8)$$

Let us define the function $h : \mathbb{R}^{[I_1 \cap I_2]} \to \mathbb{R}$ by

$$h(y) := \min\{\psi(\xi, y) : \xi \in I^{[I_1 \cap I_2]}\}, \quad (3.9)$$

where $\psi(\xi, y) := f_1(\xi, y) - \frac{\varepsilon}{2}\|\xi, y\|^{2d}$. To show that $h$ is well-defined, it is sufficient to prove that $\xi \mapsto \psi(\xi, y)$ is coercive on $\mathbb{R}^{[I_1 \cap I_2]}$ with fixed $y \in I^{[I_1 \cap I_2]}$. Indeed, for all $\xi \in I^{[I_1 \cap I_2]}$, by (3.3),

$$\frac{\varepsilon}{2}\|\xi\|^{2d} \leq \frac{\varepsilon}{2}\|\xi, y\|^{2d} \leq f(\xi, y, 0) - \frac{\varepsilon}{2}\|\xi, y\|^{2d} = f_1(\xi, y) - \frac{\varepsilon}{2}\|\xi, y\|^{2d} + f_2(y, 0),$$

so $\psi(\xi, y) \geq \frac{\varepsilon}{2}\|\xi\|^{2d} - f_2(y, 0)$. Moreover, $h$ is homogeneous of degree $2d$. Indeed, for every $t \in \mathbb{R}\setminus\{0\}$, one has

$$h(ty) = \min\{f_1(t, y) - \frac{\varepsilon}{2}\|t, y\|^{2d} : \xi \in I^{[I_1 \cap I_2]}\} = t^{2d}\min\{f_1(\xi/t, y) - \frac{\varepsilon}{2}\|t, y\|^{2d} : \xi \in I^{[I_1 \cap I_2]}\} = t^{2d}h(y).$$

To show that $h$ is continuous, let $y_1, y_2 \in \mathbb{R}^{[I_1 \cap I_2]}$. We choose $\xi_1, \xi_2 \in I^{[I_1 \cap I_2]}$ minimizing $\xi \mapsto \psi(\xi, y_1)$ and $\xi \mapsto \psi(\xi, y_2)$, respectively. Then

$$\psi(\xi_1, y_1) - \psi(\xi_1, y_2) \leq \psi(\xi_1, y_1) - \psi(\xi_2, y_1) \leq \psi(\xi_2, y_1) - \psi(\xi_2, y_2).$$

From this and by (3.9),

$$|h(y_1) - h(y_2)| = |\psi(\xi_1, y_1) - \psi(\xi_2, y_2)| \leq \max\{|\psi(\xi_1, y_1) - \psi(\xi_1, y_2)|, |\psi(\xi_2, y_1) - \psi(\xi_2, y_2)|\}.$$

This shows that $h$ is uniformly continuous on every compact subset of $\mathbb{R}^{[I_1 \cap I_2]}$ because $\psi$ is uniformly continuous on every compact subset of $\mathbb{R}^{[I_1 \cap I_2]}$. Next, we claim that

$$f_1 - h \geq \frac{\varepsilon}{2}\|x(I_1)\|^{2d} \quad \text{on} \quad \mathbb{R}^{[I_1]} \quad \text{and} \quad f_2 + h \geq \frac{\varepsilon}{2}\|x(I_2)\|^{2d} \quad \text{on} \quad \mathbb{R}^{[I_2]}.
\quad (3.10)$$
The first claim is clear by the definition of \( h \). To prove the second one, let \((y,z) \in \mathbb{R}^{|I_2|} = \mathbb{R}^{|I_1 \cap I_2|} \times \mathbb{R}^{|I_2| \setminus I_1|} \), and choose \( \xi \in \mathbb{R}^{|I_1 \cap I_2|} \) such that \( h(y) = f_1(\xi, y) - \frac{\varepsilon}{4} \| (\xi, y) \|_2^2 \). By (3.3), observe that
\[
 f_2(y, z) + h(y) = f_2(y, z) + f_3(\xi, y) - \frac{\varepsilon}{2} \| (\xi, y) \|_2^2 = f(\xi, y, z) - \frac{\varepsilon}{2} \| (\xi, y) \|_2^2 \\
 \geq \varepsilon \| (\xi, y) \|_2^2 - \frac{\varepsilon}{2} \| (\xi, y) \|_2^2 \geq \frac{\varepsilon}{4} \| (y, z) \|_2^2.
\]

Next, we will approximate \( h \) by a form of even degree on \( \mathbb{R}^{|I_1 \cap I_2|} \). Note that \( h \) is continuous and even since \( h \) is homogeneous of even degree. From this and by using Lemma 3.3, there exists \( q \in \mathbb{R}^{[x(I_1 \cap I_2)]} \) homogeneous of degree 2\( K \) for some \( K \geq d \) such that
\[
 |q - h| \leq \frac{\varepsilon}{4} \text{ on } \mathbb{R}^{[x(I_1 \cap I_2)]}.
\]

Since \( 1 = \| x(I_1 \cap I_2) \|_2^2 \) on \( \mathbb{R}^{[x(I_1 \cap I_2)]} \),
\[
 \frac{q}{\| x(I_1 \cap I_2) \|_2^{2(K-d)}} - h \leq \frac{\varepsilon}{4} \text{ on } \mathbb{R}^{[x(I_1 \cap I_2)]}.
\]

From this and since \( h \) is homogeneous of degree 2\( d \), one has
\[
 \frac{q}{\| x(I_1 \cap I_2) \|_2^{2(K-d)}} - h \leq \frac{\varepsilon}{4} \| x(I_1 \cap I_2) \|_2^{2d} \text{ on } \mathbb{R}^{[x(I_1 \cap I_2)] \setminus \{0\}}.
\]

By setting \( \varphi := \frac{q}{\| x(I_1 \cap I_2) \|_2^{2(K-d)}} \) and using Lemma 3.2, \( \varphi \) is continuous on \( \mathbb{R}^{[x(I_1 \cap I_2)]} \).

By setting \( h_1 := f_1 - \varphi \in \mathbb{R}(x(I_1)) \) and \( h_2 := f_2 + \varphi \in \mathbb{R}(x(I_2)) \), one has \( f = h_1 + h_2 \).

Let us prove that \( h_1 \) and \( h_2 \) are both rational positive definite forms of degree 2\( d \). Indeed, by (3.10) and (3.11),
\[
 h_1 = (f_1 - h) + (h - \varphi) \geq \frac{\varepsilon}{4} \| x(I_1) \|_2^{2d} - \frac{\varepsilon}{4} \| x(I_1 \cap I_2) \|_2^{2d} \geq \frac{\varepsilon}{4} \| x(I_1) \|_2^{2d},
\]
\[
 h_2 = (f_2 + h) + (\varphi - h) \geq \frac{\varepsilon}{4} \| x(I_2) \|_2^{2d} - \frac{\varepsilon}{4} \| x(I_1 \cap I_2) \|_2^{2d} \geq \frac{\varepsilon}{4} \| x(I_2) \|_2^{2d}.
\]

Thus, \( h_1 \geq \frac{\varepsilon}{4} \| x(I_1) \|_2^{2d} \) on \( \mathbb{R}^{[x]} \), \( l = 1, 2 \). By setting \( k := K - d \), the conclusion follows.

Building up on Lemma 3.4, the following helpful result provides a non-compact analogue of Grimm et al. [6] and as expected, the non-compact case is much more involved.

**Lemma 3.5.** Let Assumption [I.3] (i) holds. Let \( f \in \mathbb{R}(x) \) be a rational positive definite form of degree 2\( d \) such that \( f = \sum_{i=1}^p f_i \) with \( f_i \in \mathbb{R}(x(I_i)) \) being continuous and homogeneous of degree 2\( d \), \( l = 1, \ldots, p \). Then there exist continuous rational functions \( \varphi_i \in \mathbb{R}(x(I_i)) \), \( l = 2, \ldots, p \), defined by
\[
 \varphi_i(y) = \frac{q_i(y)}{\| y \|_2^{2k_i}}, \forall y \in \mathbb{R}^{k_i}, l = 2, \ldots, p,
\]
where \( q_i \in \mathbb{R}[x(I_i)] \) is homogeneous of degree 2\((d + k_i) \) for some \( k_i \in \mathbb{N}, l = 2, \ldots, p \), such that
\[
 f = \sum_{i=1}^p h_i,
\]
where \( h_i := f_i + \varphi_i - \sum_{j=1}^{i} \delta_{ij} \varphi_j \in \mathbb{R}(x(I_i)) \), with \( \varphi_1 := 0 \), is a continuous rational positive definite forms of degree 2\( d \), for each \( l = 1, \ldots, p \).
Proof. The proof is by induction on $p \in \mathbb{N}^{\geq 2}$. For $p = 2$, the desired result follows from Lemma 3.4. Next, assume that Lemma 3.5 holds for $p = \bar{p} - 1$ and let us prove that it is also true for $p = \bar{p}$. By applying Lemma 3.4 with $I_1 = \bigcup_{j=1}^{\bar{p}-1} I_j$, $I_2 = I_{\bar{p}}$ and $f_1 = \sum_{i=1}^{\bar{p}-1} f_i$, $f_2 = f_{\bar{p}}$, there exists a continuous rational function $\varphi_{\bar{p}} \in \mathbb{R}(x(I_{\bar{p}}))$ defined by

$$\varphi_{\bar{p}}(y) := \frac{q_{\bar{p}}(y)}{\|y\|_{\mathbb{R}^p}^2}, \quad \forall y \in \mathbb{R}^p,$$

where $q_{\bar{p}} \in \mathbb{R}[x(I_{\bar{p}})]$ is homogeneous of degree $2(d + k_{\bar{p}})$ for some $k_{\bar{p}} \in \mathbb{N}$ (only depending on $d$, $\varepsilon$ and $f_1 + \cdots + f_{\bar{p}-1}$) such that

$$f = h_{(1, \ldots, \bar{p}-1)} + h_{\bar{p}},$$

where $h_{(1, \ldots, \bar{p}-1)} := f_1 + \cdots + f_{\bar{p}-1} - \varphi_{\bar{p}} \in \mathbb{R}\{x \left( \bigcup_{j=1}^{\bar{p}-1} I_j \right) \} \text{ and } h_{\bar{p}} := f_{\bar{p}} + \varphi_{\bar{p}} \in \mathbb{R}(x(I_{\bar{p}}))$ are continuous rational positive definite forms of degree $2d$. By the RIP, there exists $s_{\bar{p}} \in \{2, \ldots, \bar{p} - 1\}$ such that $I_{\bar{p}} \subset I_{s_{\bar{p}}}$, so $\varphi_{\bar{p}} \in \mathbb{R}(x(I_{s_{\bar{p}}}))$. Then $h_{(1, \ldots, \bar{p}-1)} = \sum_{j=1}^{s_{\bar{p}}-1} (f_j - \delta_{j,s_{\bar{p}}} \varphi_{\bar{p}})$ satisfies $f_j - \delta_{j,s_{\bar{p}}} \varphi_{\bar{p}} \in \mathbb{R}(x(I_j))$, $j = 1, \ldots, \bar{p} - 1$. From this and by the induction hypothesis, there exist continuous rational functions $\varphi_l \in \mathbb{R}(x(I_l))$, $l = 2, \ldots, \bar{p} - 1$, defined by

$$\varphi_l(y) = \frac{q_l(y)}{\|y\|_{\mathbb{R}^l}^2}, \quad \forall y \in \mathbb{R}^l, \quad l = 2, \ldots, \bar{p} - 1,$$

with $q_l \in \mathbb{R}[x(I_l)]$ being homogeneous of degree $2(d + k_l)$ for some $k_l \in \mathbb{N}$, $l = 2, \ldots, \bar{p} - 1$, such that

$$h_{(1, \ldots, \bar{p}-1)} = \sum_{l=1}^{\bar{p}-1} h_l,$$

where for $l = 1, \ldots, \bar{p} - 1$,

$$h_l := (f_l - \delta_{l,s_{\bar{p}}} \varphi_{\bar{p}}) + \varphi_l - \sum_{j=l+1}^{\bar{p}-1} \delta_{l,s_{\bar{p}}} \varphi_{j} = f_l + \varphi_l - \sum_{j=l+1}^{\bar{p}-1} \delta_{l,s_{\bar{p}}} \varphi_{j} \in \mathbb{R}(x(I_l)),$$

is a continuous rational positive definite form of degree $2d$. Then $f = h_{(1, \ldots, \bar{p}-1)} + h_{\bar{p}} = \sum_{l=1}^{\bar{p}} h_l$, yielding the conclusion. \qed

The following result shows that one may write a sparse rational positive definite form as a rational SOS with uniform denominator.

Lemma 3.6. Let $I = \bigcup_{l=1}^{d} I_l$ and $d \in \mathbb{N}^{\geq 0}$. Let $f \in \mathbb{R}(x)$ be a rational positive definite form of degree $2d$ such that $f = \sum_{l=1}^{d} \left( \frac{q_l}{\|x\|_{\mathbb{R}^l}^2} \right)$, where $q_l \in \mathbb{R}[x(I_l)]$ is homogeneous of degree $2(d + k_l)$ for some $k_l \in \mathbb{N}$, $l = 1, \ldots, p$. Then there exists $\sigma \in \mathbb{R}[x]_{d+k(p+1)}$ for some $k \in \mathbb{N}$ such that

$$f = \frac{\sigma}{\|x\|_{\mathbb{R}^d} \prod_{l=1}^{d} \|x(I_l)\|_{\mathbb{R}^l}^2}.$$  \hfill (3.12)

Proof. Denote by $A$ the $\mathbb{R}$-algebra finitely generated by polynomials $x_j$, $j = 1, \ldots, n$, and rational functions $\frac{x(I_l)}{\|x(I_l)\|_{\mathbb{R}^l}^2}$, $\alpha \in \mathbb{N}^{\alpha}$ such that $|\alpha| = 2(d + k_l)$, $l = 1, \ldots, p$. Let $C(\mathbb{R}^n)$ be the space of all continuous functions on $\mathbb{R}^n$. By Lemma 3.2.1 the function $\frac{x(I_l)}{\|x(I_l)\|_{\mathbb{R}^l}^2}$ is continuous for each $l = 1, \ldots, p$, and $\alpha \in \mathbb{N}^{\alpha}$ with $|\alpha| = 2(d + k_l)$. Then $A$ is a commutative ring and $\mathbb{R}[x] \subset A \subset \mathbb{R}(x) \cap C(\mathbb{R}^n)$.

Denote by $T$ the preordering generated by $\pm (1 - \|x\|_{\mathbb{R}^2}^2)$, i.e., $T$ consists of all elements of the form $\psi \in \Sigma A^2$ and $\psi \in A$. Then $A$ is a preordered ring with fixed preordering $T$.\hfill 12
We first prove that $S^{n-1} = \{ x \in \mathbb{R}^n : h(x) \geq 0, \forall h \in T \}$. Obviously $S^{n-1} \subseteq \{ x \in \mathbb{R}^n : h(x) \geq 0, \forall h \in T \}$. For the other inclusion, assume by contradiction that there exists $a \in \mathbb{R}^n S^{n-1}$ such that $h(a) \geq 0$ for all $h \in T$. Then $1 - \|a\|_2^2 \neq 0$. By selecting $h := -1 \|a\|_2^2 (1 - \|x\|_2^2) \in T$, one obtains the contradiction $0 \leq h(a) = -(1 - \|a\|_2^2)^2 < 0$.

Next, notice that $\text{Sper}_T A$ is a Hausdorff space and contains all mappings $\tilde{a} : A \to \mathbb{R}$, $h \mapsto h(a)$ for $a \in S^{n-1}$ (see [18]). Here $\tilde{a}$ is well-defined by the continuity of each element in $A$. In addition, since $S^{n-1}$ is compact, $(\tilde{a})_{a \in \mathbb{R}^{n-1}}$ is dense in $\text{Sper}_T A$ in the topology induced by the sup-norm, i.e., for each $r > 0$ and for each $\varphi \in \text{Sper}_T A$ there exists $a \in S^{n-1}$ such that $\sup_{h \in A} |\tilde{h}(a) - \varphi(h)| = \sup_{h \in A} |(\tilde{a} - \varphi)(h)| \leq r$ (see [17] Section 2 and [3] Section 2).

Let $H(A)$ (resp. $H'(A)$) be the ring of geometrically (resp. arithmetically) bounded elements in $A$. Since $(\tilde{a})_{a \in \mathbb{R}^{n-1}}$ is dense in $\text{Sper}_T A$,

$$H(A) = \{ h \in A : h \text{ is bounded on } S^{n-1} \} = A.$$  

The latter equality is due to the compactness of $S^{n-1}$ and the inclusion $A \subseteq C(\mathbb{R}^n)$. Combining this together with [3.7], one obtains $A = H'(A)$.

Next we claim that $f > 0$ on $\text{Sper}_T A$. Indeed $f \geq \varepsilon \|x\|_2^{2d}$ on $\mathbb{R}^n$ for some $\varepsilon > 0$, because $f$ is a rational positive definite form of degree $2d$. Therefore $f \geq \varepsilon$ on $S^{n-1}$.

Let $\varphi \in \text{Sper}_T A$ be fixed, arbitrary. By denseness of $S^{n-1}$ in $\text{Sper}_T A$, there exists $a \in S^{n-1}$ such that $|f(a) - \varphi(f)| \leq \varepsilon$. Thus, $\varphi(f) = f(a) - (f(a) - \varphi(f)) \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > 0$, and the result follows.

Next, since $f \in A$ and $f > 0$ on $\text{Sper}_T A$, then by Lemma [3.4] $f \in T$. Therefore $f = \sigma + (1 - \|x\|_2^2)\psi$ for some $\sigma \in \Sigma A^2$ and $\psi \in A$. By replacing $x$ by $\frac{x}{\|x\|_2}$ and noting that $f$ is homogeneous of degree $2d$, $\|x\|_2^{2d} f = \sigma (\frac{x}{\|x\|_2})$. By multiplying both sides with $\|x\|_2^{2(k+\delta)} \prod_{l=1}^p \|x(I_l)\|_2^{2k}$ for some large enough $k$, there exist $r \in \mathbb{N}$ and $h_j, v_j \in \mathbb{R}[x], j = 1, \ldots, r$, such that

\[
\left( \|x\|_2^{2k} \prod_{l=1}^p \|x(I_l)\|_2^{2k} \right) f = \left( \|x\|_2^{2(k+\delta)} \prod_{l=1}^p \|x(I_l)\|_2^{2k} \right) \sigma \left( \frac{x}{\|x\|_2} \right) = \sum_{j=1}^r (h_j + v_j \|x\|_2)^2 = \sum_{j=1}^r (h_j^2 + v_j^2 \|x\|_2^2) + 2 \|x\|_2 \sum_{j=1}^r h_j v_j.
\]

Recall that $f = \sum_{i=1}^m \frac{a_i}{\|x(I_i)\|_2^{2r}}$. Therefore assume that $k$ is large enough to ensure that $\left( \|x\|_2^{2k} \prod_{l=1}^p \|x(I_l)\|_2^{2k} \right) f$ is a polynomial. Then $\sum_{j=1}^r (h_j^2 + v_j^2 \|x\|_2^2) + 2 \|x\|_2 \sum_{j=1}^r h_j v_j$ must be a polynomial. However since $\|x\|_2$ is not a polynomial, then necessarily $\sum_{j=1}^r h_j v_j = 0$. Hence,

\[
\left( \|x\|_2^{2k} \prod_{l=1}^p \|x(I_l)\|_2^{2k} \right) f = \sum_{j=1}^r (h_j^2 + v_j^2 \|x\|_2^2),
\]

which yields [3.12].

**Remark 3.1.** Observe that Reznick’s Positivstellensatz is a particular case of Lemma 3.6 with $p = 1$. Our proof is similar to the one of [24, Theorem 3.7], which addresses the case $p = 1$.

## 4 Proofs

### 4.1 Proof of Theorem 2.1

**Proof.** One has $f = \sum_{l=1}^p f_l$ with $f_l := \frac{a_l}{\|x(I_l)\|_2^{2r}}$, $l = 1, \ldots, p$. By Lemma [2.2], the function $f_l \in \mathbb{R}(x(I_l))$ is continuous and homogeneous of degree $2d$, for each
\( l = 1, \ldots, p \). By applying Lemma 3.5, there exist continuous functions \( \varphi_l \in \mathbb{R}(x(\hat{I})) \), \( l = 2, \ldots, p \), defined by

\[
\varphi_l(y) = \frac{q_l(y)}{\|y\|^2_{\ell_2}}, \quad \forall y \in \mathbb{R}^{n_l}, \ l = 2, \ldots, p,
\]

where \( q_l \in \mathbb{R}[x(\hat{I})] \) is homogeneous of degree \( 2(d+k_l) \) for some \( k_l \in \mathbb{N}, \ l = 2, \ldots, p \), and one has

\[
f = \sum_{l=1}^{p} h_l,
\]

where each \( h_l := f_l + \varphi_l - \sum_{j=l+1}^{p} \delta_{l,s_j} \varphi_j \in \mathbb{R}(x(I_l)), \ l = 1, \ldots, p \) (with \( \varphi_1 := 0 \)) is a continuous rational positive definite form of degree 2d. Then, we apply Lemma 3.6 with the notation \( f \leftarrow h_l, I \leftarrow I_1 \) and \( I_t \leftarrow I_t \cup \{I_j : s_j = l, j = l+1, \ldots, p\} \).

Therefore, there exist \( k_l \in \mathbb{N} \) and \( \psi_l \in \mathbb{R}[x(I_l)]\) such that

\[
h_l = \frac{\psi_l}{\|x(I_l)\|^2_{\ell_2}} \|x(I_l)\|^2_{\ell_2} \prod_{j=l+1}^{p} \|x(I_j)\|^2_{\ell_2}, \ l = 1, \ldots, p.
\]

Let \( k := \max\{k_1, \ldots, k_p\} \) and define for all \( l = 1, \ldots, p \)

\[
\sigma_l := \psi_l ||x(I_l)||^2_{\ell_2} ||x(\hat{I}_l)||^2_{\ell_2} \prod_{j=l+1}^{p} ||x(\hat{I}_j)||^2_{\ell_2}.
\]

Then \( \sigma_l \in \mathbb{R}[x(I_l)]\) and \( 2.8 \) follows, yielding the conclusion. \( \square \)

### 4.2 Proof of Corollary 2.7.2

**Proof.** Let \( \bar{f}(x, x_{n+1}) := x_n^{2d} f(x/x_{n+1}) \) be the degree-2d homogenization of \( f \). Set \( \bar{x} := (x, x_{n+1}), \bar{I} := I \cup \{n+1\}, \bar{I}_l := I_l \cup \{n+1\} \) and \( \bar{I}_t := I_t \cup \{n+1\} \), for all \( l = 1, \ldots, p \). Then \( \bar{f} = \bigcup_{l=1}^{p} \bar{f}_l \) and

\[
\forall l \in \{1, \ldots, p\}, \bar{I}_l = \bar{I}_t \cap \left( \bigcup_{j=1}^{l-1} \bar{I}_j \right) \subset \bar{I}_t \quad \text{and} \quad \bar{f} = \bigcup_{l=1}^{p} \bar{f}_l.
\]

Since \( f \) is nonnegative, \( \bar{f} \) is also nonnegative.

We first prove that \( \bar{f} + \varepsilon \sum_{l=1}^{p} ||\hat{x}(\bar{I}_l)||^2_{\ell_2} \) is a positive definite form. Let \( y \in \mathbb{R}^{n+1} \) such that \( \bar{f}(y) + \varepsilon \sum_{l=1}^{p} ||y(x(I_l))||^2_{\ell_2} = 0 \). By the nonnegativity of \( \bar{f} \) and \( ||x(I_l)||^2_{\ell_2} \),

\[
\bar{f}(y) = ||x(I_l)||^2_{\ell_2} = \cdots = ||x(I_p)||^2_{\ell_2} = 0.
\]

Hence \( y(\hat{I}_l) = 0, \ l = 1, \ldots, p \), and therefore since \( \bar{I} = \bigcup_{l=1}^{p} \bar{I}_l, \ y = 0 \).

By Theorem 2.1, there exist \( l = 1, \ldots, p \), such that

\[
\bar{f}(y) = \sum_{l=1}^{p} \psi_l ||x(I_l)||^2_{\ell_2} \prod_{j=l+1}^{p} ||x(I_j)||^2_{\ell_2}, \quad (4.13)
\]

where \( \hat{\Phi}_l := ||\hat{x}(\bar{I}_l)||^2_{\ell_2} \prod_{j=l+1}^{p} ||\hat{x}(\bar{I}_j)||^2_{\ell_2} \in \mathbb{R}[x(\bar{I}_l)] \) and \( \omega_l := \deg(\hat{\Phi}_l) + 1, \ l = 1, \ldots, p \). Letting \( x_{n+1} := 1 \) in (4.13) yields

\[
f + \varepsilon \sum_{l=1}^{p} \sigma_l = \sum_{l=1}^{p} \hat{\Phi}_l \theta_l D_{\ell_2}^2,
\]

with \( D_l = \hat{\Phi}_l(x, 1) \in \mathbb{R}[x(I_l)] \) and \( \sigma_l := \psi_l(x, 1) \in \mathbb{R}[x(I_l)]\). Hence the conclusion follows since \( \Theta_l = \theta_l D_l, \ l = 1, \ldots, p \). \( \square \)
4.3 Proof of Corollary 2.3

Proof. Recall that \( u_j = \| \text{deg}(g_j)/2 \|, \) for all \( j = 1, \ldots, m. \) Define \( \lambda_j := (\| g_j \|_1 + 1)^{-1}, \) for all \( j = 1, \ldots, m. \) We claim that

\[
|\lambda_j g_j / \theta_j^{p_j} | = \sum_{\alpha \in N^d_{2u_j} : \theta_j^{p_j} a^{x^\alpha}} \leq \sum_{\alpha \in N^d_{2u_j}} |g_j|_1 |x^\alpha| / \theta_j^{p_j} \leq \| g_j \|_1 .
\]

For each \( \alpha \in N^d_{2u_j}, j \in J, l = 1, \ldots, p, \) which implies

\[
|g_j / \theta_j^{p_j}| = \frac{\sum_{\alpha \in N^d_{2u_j} g_j \theta_j^{p_j} a^{x^\alpha}}}{\theta_j^{p_j}} \leq \sum_{\alpha \in N^d_{2u_j}} |g_j|_1 |x^\alpha| / \theta_j^{p_j} \leq \| g_j \|_1 .
\]

Let us show that \( Q_k := f + \sum_{l=1}^p 2 \sum_{j \in J_l} \theta_j^{p_j} - \sum_{l=1}^p \sum_{j \in J_l} (1 - \lambda_j g_j / \theta_j^{p_j})^{2k} (\lambda_j g_j / \theta_j^{p_j})^{2k+1} \)

\[
\geq 1 - \sum_{l=1}^p \sum_{j \in J_l} \left( \left| \theta_j^{p_j} g_j \right| / \| g_j \|_1 \right)^{2k+1} \quad \text{by (4.13)}
\]

\[
\geq 1 - \sum_{l=1}^p \sum_{j \in J_l} \left( \left| \theta_j^{p_j} g_j \right| / \| g_j \|_1 \right)^{2k+1} .
\]

Since \( y \) was arbitrary in \( B(0,M)^c, \)

\[
\inf\{ Q_k(x) : x \in B(0,M)^c \} \geq 1 - \sum_{l=1}^p \sum_{j \in J_l} \left( \left| \theta_j^{p_j} g_j \right| / \| g_j \|_1 \right)^{2k+1} \quad \text{as} \ k \to \infty .
\]

Thus, \( Q_k \) is nonnegative on \( B(0,M)^c \) for some large enough \( k. \)

(II) Then, note that \( \lim_{k \to \infty} (1 - a)2k^2 a^{2k+1} = 0 \) for all \( a \in (0,1) \) and \( \lim_{k \to \infty} (1 + a)^2k^2 a^{2k+1} = \infty \) for all \( a \in (0,1). \) By using (4.14), each term \( (1 - \lambda_j g_j / \theta_j^{p_j})^{2k} (\lambda_j g_j / \theta_j^{p_j})^{2k+1} \)

\[
\text{involved in (4.14)} \text{ can be written either as} \ (1 - a)2k^2 a^{2k+1} \text{ or as} \ (1 + a)^2k^2 a^{2k+1} \text{ when} \ g_j(x) > 0 \text{ or as} \ (1 + a)^2k^2 a^{2k+1} \text{ when} \ g_j(x) < 0. \)

Therefore, \( Q_k \to \infty \) pointwise on \( B(0,M) \cap S(g) \) and \( Q_k \to f + \frac{\sum_{j=1}^p \theta_j^{p_j} g_j}{\| g_j \|_1} \) pointwise on \( B(0,M) \cap S(g). \) By compactness of \( B(0,M) \) and positivity of \( f + \frac{\sum_{j=1}^p \theta_j^{p_j} g_j}{\| g_j \|_1} \) on \( S(g), \) \( Q_k \) is nonnegative on \( B(0,M) \) for large enough \( k. \)

(III) Let \( K \in \mathbb{N} \) be fixed such that \( Q_K \) is nonnegative. Define \( r_j := (2K^2 + 2K + 1)u_j \)

and \( w_{j,l} := (\theta_j^{p_j} - \lambda_j g_j)^{2k^2} (\lambda_j g_j)^{2k+1}, \)

so that \( w_{j,l} \in \mathbb{R}^{|x(l)|/2r_j}, j \in J_l, l = 1, \ldots, p, \)

\[
Q_K = f + \frac{\sum_{l=1}^p \theta_j^{p_j} g_j}{\| g_j \|_1} - \sum_{l=1}^p \sum_{j \in J_l} \frac{\theta_j^{p_j} g_j}{\| g_j \|_1} .
\]
With every $h \in \mathbb{R}(x)$ associate its degree-2d homogenization $\tilde{h}(x, x_{n+1}) := x^{2d}_{n+1} h(x/x_{n+1})$.
Then with same notation $\tilde{x}, \tilde{I}, \tilde{I}_t$ and $\tilde{I}_t$ as in the proof of Corollary 2.22
that
$$
\tilde{Q}_K = \tilde{f} + \frac{\varepsilon}{2} \sum_{l=1}^p \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 = \sum_{l=1}^p \left( \tilde{f}_l + \varepsilon \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 - \sum_{j \in J_l} \frac{x^{2r_{l,j}}_{n+1} w_{l,j}}{\|\tilde{x}(\tilde{I}_l)\|^{2d}_2} \right) = \sum_{l=1}^p F_l,
$$
(4.16)
Equivalently:
$$
\tilde{Q}_K + \frac{\varepsilon}{2} \sum_{l=1}^p \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 = \sum_{l=1}^p \left( \tilde{f}_l + \varepsilon \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 - \sum_{j \in J_l} \frac{x^{2r_{l,j}}_{n+1} w_{l,j}}{\|\tilde{x}(\tilde{I}_l)\|^{2d}_2} \right) = \sum_{l=1}^p F_l,
$$
where $F_l := \tilde{f}_l + \varepsilon \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 - \sum_{j \in J_l} \frac{x^{2r_{l,j}}_{n+1} w_{l,j}}{\|\tilde{x}(\tilde{I}_l)\|^{2d}_2} \in \mathbb{R}(\tilde{x}(\tilde{I}_l))$ is homogeneous of degree 2d, $l = 1, \ldots, p$. In addition, $\tilde{Q}_K$ is nonnegative by nonnegativity of $\tilde{Q}_K$. Then there exists $\varepsilon > 0$ such that $\tilde{Q}_K + \frac{\varepsilon}{2} \sum_{l=1}^p \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 \geq \varepsilon \|\tilde{x}\|^{2d}_2$. Indeed, for $d = 1$, it is trivial. For $d \geq 2$, let $d^*$ be such that $\frac{1}{d} + \frac{1}{d^*} = 1$ and let us use Hölder’s inequality as follows:
$$
\left( \sum_{l=1}^p \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 \right)^{1/d^*} \left( \sum_{l=1}^p 1^{1/d^*} \right) \geq \sum_{l=1}^p \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 \geq \|\tilde{x}\|^{2d}_2,
$$
which implies the desired result for $\varepsilon = \frac{\varepsilon}{2} p^{-d/d^*}$. Therefore $\tilde{Q}_K + \frac{\varepsilon}{2} \sum_{l=1}^p \|\tilde{x}(\tilde{I}_l)\|^{2d}_2$ is a positive definite form of degree 2d. By Theorem 2.11 there exist $k \in \mathbb{N}$ and $\psi_l \in \Sigma(\tilde{x}(\tilde{I}_l))_{d+k\omega_1}$, $l = 1, \ldots, p$ such that
$$
\tilde{Q}_K + \frac{\varepsilon}{2} \sum_{l=1}^p \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 = \sum_{l=1}^p \frac{\psi_l}{\|\tilde{x}(\tilde{I}_l)\|^{2d}_2}.
$$
where $\tilde{D}_l := \|\tilde{x}(\tilde{I}_l)\|^{2(1-d_{l,1})}_2 \prod_{j=2}^p \|\tilde{x}(\tilde{I}_j)\|^{2d_{j-1,j}}_2 \in \mathbb{R}(\tilde{x}(\tilde{I}_j))$, $l = 1, \ldots, p$. From this and (4.10),
$$
\tilde{f} + \varepsilon \sum_{l=1}^p \|\tilde{x}(\tilde{I}_l)\|^{2d}_2 = \sum_{l=1}^p \sum_{j \in J_l} \frac{x^{2r_{l,j}}_{n+1} w_{l,j}}{\|\tilde{x}(\tilde{I}_l)\|^{2d}_2} + \sum_{l=1}^p \frac{\psi_l}{\|\tilde{x}(\tilde{I}_l)\|^{2d}_2} \tilde{D}_l^k.
$$
Letting $x_{n+1} := 1$ yields
$$
f + \varepsilon \sum_{l=1}^p \theta_l^d = \sum_{l=1}^p \sum_{j \in J_l} \frac{\theta_l^d w_{l,j}}{\theta_l^d D_l^k} + \sum_{l=1}^p \frac{\sigma_{0,l}}{\theta_l^d D_l^k},
$$
with $D_l = \tilde{D}_l(x, 1) \in \mathbb{R}(\tilde{x}(\tilde{I}_l))$ and $\sigma_{0,l} := \psi_l(x, 1) \in \Sigma(\tilde{x}(\tilde{I}_l))_{d+k\omega_1}$. For $j \in J_l$, $l = 1, \ldots, p$, by setting $\sigma_j := (\theta_l^d D_l^k - \lambda_j g_j) 2K^2 (\lambda_j g_j)^{2K}$, $\sigma_{j,l} := \Sigma(\tilde{x}(\tilde{I}_l))_{d+k\omega_1}$ and $w_{l,j} = \sigma_j$. By setting $k := \max\{k, r_j : j \in J\}$,
$$
f + \varepsilon \sum_{l=1}^p \theta_l^d = \sum_{l=1}^p \sum_{j \in J_l} \frac{\theta_l^d w_{l,j}}{\theta_l^d D_l^k} + \sum_{l=1}^p \frac{\sigma_{0,l}}{\theta_l^d D_l^k}$$
$$
= \sum_{l=1}^p \sum_{j \in J_l} \frac{\theta_l^d D_l^k - \lambda_j g_j}{\theta_l^d D_l^k} + \sum_{l=1}^p \frac{\sigma_{0,l}}{\theta_l^d D_l^k},
$$
with $\sigma_{0,l} := D_l^{k-h} \theta_l^{d-k} \sigma_{0,l} \in \Sigma(\tilde{x}(\tilde{I}_l))_{d+k\omega_1}$ and
$$
\sigma_{j,l} := \theta_l^{d-r_j} D_l^k \sigma_j \in \Sigma(\tilde{x}(\tilde{I}_l))_{d+k\omega_1}, j \in J_l, l = 1, \ldots, p.
$$
Thus,
$$
f + \varepsilon \sum_{l=1}^p \theta_l^d = \sum_{l=1}^p \frac{\sigma_{0,l} + \sum_{j \in J_l} \theta_l^{d-r_j} D_l^k \sigma_j}{\theta_l^d D_l^k}.
$$
Hence, the conclusion follows since $\Theta_l = \theta_l D_l$, $l = 1, \ldots, p$. \qed
5 Conclusion

In this paper, we have provided:

- a sparse version for both Reznick’s Positivstellensatz (resp. Putinar-Vasilescu’s Positivstellensatz) for positive definite forms (resp. nonnegative polynomials).
- a sparse version of Putinar-Vasilescu’s Positivstellensatz for polynomials that are nonnegative on a possibly non-compact basic semialgebraic set.

All these certificates involve sums of squares of rational functions with uniform denominators and a topic of further research is how to exploit such positivity certificates in polynomial optimization on non-compact basic semialgebraic sets.

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A Appendix

\[ \sigma_1 = 2(x_1^2 + \frac{1}{2}x_1 + \frac{1}{2}) \quad \text{and} \quad \sigma_2 = 142x_2^2 + 6x_2 + 1 + 1314993284459107346080199880406085742469204627868504341150915245917907 \]

and

\[ \sigma_2 = 142x_2^2 + 6x_2 + 1 + 1314993284459107346080199880406085742469204627868504341150915245917907 \]

and

\[ \sigma_2 = 142x_2^2 + 6x_2 + 1 + 1314993284459107346080199880406085742469204627868504341150915245917907 \]

and

\[ \sigma_2 = 142x_2^2 + 6x_2 + 1 + 1314993284459107346080199880406085742469204627868504341150915245917907 \]

and

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and

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and

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and

\[ \sigma_2 = 142x_2^2 + 6x_2 + 1 + 1314993284459107346080199880406085742469204627868504341150915245917907 \]

and

\[ \sigma_2 = 142x_2^2 + 6x_2 + 1 + 1314993284459107346080199880406085742469204627868504341150915245917907 \]

and

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