Phase matching in Spontaneous Parametric Down Conversion

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Abstract
Spontaneous Parametric Down Conversion (SPDC) is one of the most celebrated and widely used processes for the generation of entangled photon pair in the arena of Quantum Entanglement. In this article we study both collinear and non-collinear phase matching of Spontaneous Parametric Down Conversion (SPDC) process by using wave optics and present a theory that incorporate several different aspects of down conversion. Our theory also incorporate several experimentally relevant parameters which are so far not part of the regular phase matching theory. This work has implication for finding the true angular Schmidt spectrum.

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Introduction
The parametric process [1], in the early days before the invention of the laser, was initially developed for low noise amplifier in radio wave and microwave regime [2, 3]. With the advent of time, non-linear optics has been adopted in the optical range [4] of electromagnetic wave. Spontaneous parametric down-conversion (SPDC) [5–8] may be treated as a coherent three-photon process where a crystal which is not centrosymmetric [9] is illuminated by a pump beam. The intensity of the pump beam is such that the non-linearity of the crystal will be significant. The second order interaction results in the annihilation of a pump photon and the creation of two down-converted photons termed the signal and idler. One of the most important characteristics of the down converted photon is that they are entangled in various degrees of freedom [10–14].

The abbreviation SPDC is Spontaneous Parametric Down conversion [15], this process is called ‘spontaneous’ as they are generated by the quantum vacuum fields. There is no signal or idler photon field is used to stimulate the process. It is ‘parametric’ as it depends on the electric fields of input and output not only by intensities which imply the phase relationship between the input and output fields. Conservation of energy enforces that the signal and the idler must have the lower frequency than the pump frequency and the words ‘down conversion’ explain this.

SPDC can be implemented in two ways depending on the polarization state of the signal and idler they are termed as Type-I and Type-II [16]. In this article, we have reviewed the two-photon wave function [17–19] for both type of SPDC, discussed their phase matching conditions, and calculated the coherence width [20] at a particular distance from the crystal front face. The phase matching conditions decides the SPDC will be collinear or non-collinear. Varying the angle between the optic axis of the nonlinear crystal and the propagation of pump beam direction we can tune the phase matching conditions and that can be used for both collinear and non-collinear two-photon source. Experimentally, this has been used for
calculating the angular Schmidt spectrum number [21] for both phase matching conditions for Type-I polarization. In a separate section here we have briefly shown the theoretical calculation of the angular Schmidt spectrum.

1. SPDC two-photon wave function calculation

In this chapter, we will start with the classical description of the non linear optics. Then we will enter into the quantum description of the non-linearity and its interaction in quantised form of Hamiltonian.

1.1 Wave equation:

SPDC is a second-order nonlinear optical effect [22]. From the first principle we can study the behaviour of the light in the non linear crystal. We want to arrive at the wave equation from the Maxwell’s equation [25] with certain assumption that there are no free charges and electric current is present in the medium.

\[
\nabla \cdot \mathbf{D} = 0 \tag{1}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{2}
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}
\]

\[
\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t} \tag{4}
\]

Here \( \mathbf{E} \) is the Electric field, \( \mathbf{B} \) stands for Magnetic flux density, \( \mathbf{D} \) is the electric displacement vector, \( \mu_0 \) is the magnetic permeability. We have considered here that the magnetic field is homogeneous (having the same value at all point) and isotropic (does not depend on the direction of the propagation) in the medium. But the electric field is non linear and anisotropic. The electric charge density \( \rho \) is taken to be 0 as it’s our assumption that there is no free charge in the medium. And also the current density \( \mathbf{J} \) is taken to be zero.

The electric displacement vector \( \mathbf{D} \) can be written as:

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{5}
\]

Here \( \mathbf{P} \) is the polarization vector. For linear crystal, the polarization [22] \( \mathbf{P} = \varepsilon_0 \chi^{(1)} \mathbf{E} \), means polarization depends on the only 1st power(linear) of the electric field. But in non-linear case the form of the polarization [24] is:

\[
\mathbf{P} = \varepsilon_0 \chi^{(1)} \mathbf{E} + \varepsilon_0 \chi^{(2)} \mathbf{E}^2 + \ldots \tag{6}
\]

Here the \( \chi^{(1)} \) and \( \chi^{(2)} \) are the susceptibility tensor of rank 2 and rank 3 respectively. As the SPDC is a second order non linear effect so we should concerned about the term of polarization up to \( \mathbf{E} \mathbf{E} \) and ignore the effect of other higher order electric field. As the value of \( \chi^{(n)} \) (nth order susceptibility) fall down \( 10^n \) times [22] the value of previous order susceptibility \( \chi^{(n-1)} \). Hence we can neglect the higher order electric field term in the equation 6.

From Maxwell’s two equations 3, 4 we get the following differential equation which is called the wave equation in non-linear medium.

\[
\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial^2}{\partial t^2} \left[ \varepsilon_0 (1 + \chi^{(1)}) \mathbf{E} + \mathbf{P}_{NL} \right] \tag{7}
\]

Where \( 1 + \chi^{(1)} = n^2 \), \( n \) is the refractive index of the medium. And the second order non linearity [23] is defined by

\[
\mathbf{P}_{NL} = \chi^{(2)}_{ij} \mathbf{E}_i \mathbf{E}_j
\]

For simplicity we take the scalar field instead of tensor product. Hence we arrive at \( \mathbf{P}_{NL} = \varepsilon_0 \chi^{(2)} \mathbf{E}^2 \).

1.2 Complex Analytic Signal Representation

The mapping of a real random variable to the complex form is called the complex signal analytic representation [27]. This mathematical tool has outstanding advantages as far as the calculations concern because the calculation of the complex variable is easier than the real random variables. And in the classical description the equations are in terms of the physical real quantity but in quantum mechanical treatment every dynamical variable we write for the wavefunction which is necessarily complex. Everything we need the real function for the realisation that is why at the end of the calculation we have to extract the real part of the quantity to get the desired real quantity.

Now the classical monochromatic real electric field can be represented in terms of complex signal analytic representation :

\[
E(t) = \int_{-\infty}^{\infty} \tilde{E}(\omega)e^{-i\omega t} d\omega
\]

\[
= \int_{-\infty}^{0} \tilde{E}(\omega)e^{-i\omega t} d\omega + \int_{0}^{\infty} \tilde{E}(\omega)e^{-i\omega t} d\omega
\]

\[
= E^{(-)} + E^{(+)}
\]

Here \( E^{(+)} \) is the complex signal representation of the real electric field.

1.3 Non linear optics and SPDC

We have the extra term appears in the wave equation No. 7 due to the non-linearity i.e \( \mathbf{P}_{NL} = \varepsilon_0 \chi^{(2)} \mathbf{E}^2 \).

Then consider that classical light of two distinct frequency interacting with the non linear crystal hence the expression for the electric field is:

\[
E = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + cc \tag{9}
\]

Now using this expression for electric field in \( \mathbf{P}_{NL} \), we get:

\[
\mathbf{P}_{NL} = \varepsilon_0 \chi^{(2)} (E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + cc)
\]

\[
\times (E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t} + cc)
\]

\[
= \varepsilon_0 \chi^{(2)} (E_1^2 e^{-2i\omega_1 t} + E_2^2 e^{-2i\omega_2 t} + 2E_1E_2 e^{-i(\omega_1 - \omega_2) t} + 2E_1^*E_2^* e^{-i(\omega_1 + \omega_2) t} + 2E_1E_2^* e^{-i(\omega_1 - \omega_2) t} + |E_1|^2 + |E_2|^2 + cc) \tag{10}
\]
Here in equation No. 10, each component of the $P_{NL}$ refers to the different frequency which is generated due to the second order nonlinearity of the crystal. These components and their acronyms [22] are as follows:

$$P_{NL}(2\omega_1) = \epsilon_0 \chi^{(2)} E_1^2 + cc$$  
(11)

$$P_{NL}(2\omega_2) = \epsilon_0 \chi^{(2)} E_2^2 + cc$$  
(12)

$$P_{NL}(\omega_1 + \omega_2) = 2\epsilon_0 \chi^{(2)} E_1 E_2 + cc$$  
(13)

$$P_{NL}(\omega_1 - \omega_2) = 2\epsilon_0 \chi^{(2)} E_1^* E_2 + cc$$  
(14)

$$P_{NL}(0) = 2\epsilon_0 \chi^{(2)} (|E_1|^2 + |E_2|^2)$$  
(15)

$E^*$ is the complex conjugate of $E$. Equations 11 and 12 implies that it is possible to create an electromagnetic wave with twice the frequency of waves one and two respectively. This process is known as second harmonic generation or SHG. Similarly, Eq.13 says that it is possible to create an electromagnetic wave with a frequency equal to the sum of the input frequencies and Eq.14 means that non-linearity can create an electromagnetic wave with a frequency equal to the difference of the input frequencies. These processes are known as sum frequency generation, SFG, and difference frequency generation, DFG, respectively. Finally, Eq. 15 says that it is possible to create a static polarization in the material with zero frequency, which is known as optical rectification or OR.

For difference frequency generation at $\omega_1 = \omega_2 - \omega_1$ (Fig.1) from the field at frequency $\omega_1$, the field at $\omega_2$ is not necessarily to be present in all case. If the field of $\omega_2$ is present then the difference frequency generation is stimulated by the field at frequency $\omega_2$. The process has a very high efficiency and is called the optical parametric amplification/oscillation. However, even in the absence of $\omega_2$, one can have difference frequency generation as the vacuum mode at frequency $\omega_2$ is always present and this vacuum mode stimulates this process. The efficiency of the process is very low and is called spontaneous parametric down-conversion.

Each photon in this process obeys the conservation of energy and momentum. Hence $\omega_1 = \omega_2 + \omega_3$ and $k_1 = k_2 + k_3$. These constraints make the two-photon entangled in their energy-time [13], position-momentum [12], angular position-orbital angular momentum [14] degree of freedom. This process is very very rare phenomena, around 1 times out of $10^6$.

### 1.4 Hamiltonian for SPDC (Classical)

The classical Hamiltonian for the electromagnetic field [26] is given by:

$$H = \frac{1}{2} \int \left( \mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H} \right) dV$$  
(16)

For the linear medium, the Hamiltonian is well established and its result is known. For our interest in non linear effect, we can take the linear part as unperturbed Hamiltonian ($H_0$). And due to the non-linearity, the additional term can be treated as the perturbing term of the Hamiltonian ($H_1$). So for Spontaneous parametric down conversion the $H_1$ [28] is:

$$H_1 = \frac{\epsilon_0}{2} \int \chi^{(2)}_{ijk}(E_P)(E_S)(E_I) dV$$  
(17)

Using the complex signal analytic representation as in Equation 8 we can write pump field ($E_P$), signal field($E_S$) and the idler($E_I$) in the following form:

$$E_P = E_P^{(+)} + E_P^{(-)}$$

$$E_S = E_S^{(+)} + E_S^{(-)}$$

$$E_I = E_I^{(+)} + E_I^{(-)}$$

So, we get:

$$H_1 = \frac{\epsilon_0}{2} \int \chi^{(2)}_{ijk}(E_P^{(+)} + E_P^{(-)})(E_S^{(+)} + E_S^{(-)})(E_I^{(+)} + E_I^{(-)}) dV$$  
(18)

This equation 18 contains total eight terms but out of only two term $E_P^{(+)} E_S^{(+)} E_I^{(-)}$ and $E_P^{(-)} E_S^{(-)} E_I^{(+)}$ has the contribution to the Hamiltonian for parametric down conversion. as other six terms get averaged out when $H_1$ integrated over time. This is quite analogous to making the rotating-wave approximation as in the case of treating atomic absorption and emission processes (see Ref. [29], Section 2.3). Hence, the perturbed Hamiltonian term for Spontaneous parametric down conversion is:

$$H_1 = \frac{\epsilon_0}{2} \int \chi^{(2)}_{ijk}(E_P^{(+)} E_S^{(-)} E_I^{(-)}) dV + cc$$  
(19)

Where, $E_P^{(-)} E_S^{(+)} E_I^{(+)}$ is the complex conjugate term of $E_P^{(+)} E_S^{(-)} E_I^{(-)}$.

### 1.5 Schrodinger picture

To get the quantum mechanical two-photon Parametric Down Conversion wavefunction from the Perturbing Hamiltonian we have to use the Schrodinger picture of time evolution. That means if we know the wave function initially, then after a certain time $t$ we can obtain the wavefunction by this treatment:

$$\left| \Psi(t) \right\rangle = e^{-i \int_{t_{initial}}^{t} H_1(t') dt'} \left| t_{initial} \right\rangle$$  
(20)
Now we apply the perturbation expansion [30] and the equation 20 looks like:

$$|\Psi(t)\rangle = |\Psi_{\text{initial}}\rangle - \frac{i}{\hbar} \int_{\text{initial}}^{t} H(t') dt' |\Psi_{\text{initial}}\rangle + |\Psi(t)\rangle$$

$$+ \left( \frac{i}{\hbar} \right)^{2} \frac{1}{2!} \int_{t_{\text{initial}}}^{t} \int_{t_{\text{initial}}}^{t} H(t'_{1}) dt'_{1} H(t'_{2}) dt'_{2} |\Psi_{\text{initial}}\rangle + \ldots$$

(21)

In this equation 21 the first term in the right hand side represents the vacuum emission, the second term represents the photon pair emission and the third term represents the four-photon emission term and so on. But we will ignore those higher order term, as well as the four-photon emission term as the higher pair of photon generation, is extremely rare.

For our purpose, we will set the time such that when the pump field just enter the crystal means the initial time of interaction we will take as $-t_{\text{int}}$ and the final time when the photon pair is coming out from the crystal, means just front of the crystal the time is set to $t=0$, now the $|\Psi(0)\rangle$ will be [31, 32]:

$$|\Psi(0)\rangle = | - t_{\text{int}} \rangle - \frac{i}{\hbar} \int_{-t_{\text{int}}}^{0} H(t') dt' | - t_{\text{int}} \rangle$$

(22)

1.6 Quantum scenario of SPDC

In this sub-section, we will discuss the quantum description about the SPDC. First, we write the quantised form of the electric field [33], [34] for getting the expression of interaction Hamiltonian perturbation term.

The quantised electric field can be represented as:

$$E^{+}(r,t) = \frac{i}{\sqrt{V}} \sum_{k,\sigma} e_{k,\sigma} \sqrt{\frac{\hbar \omega}{2 \varepsilon_{0} n^{2}(k, \sigma)}} \hat{a}_{k,\sigma}(\omega) G(\omega) e^{i(k \cdot r - \omega t)}$$

(23)

Where $V$ is the quantisation volume, $e_{k,\sigma}$ represents the polarization vector of light having wave vector $k$ and polarization state $\sigma$, $\varepsilon_{0}$ is the vacuum permittivity, $n$ is the refractive index experienced by the field due to the propagation along $k$ and polarization $\sigma$, $\hat{a}$ is the annihilation operator, $G(\omega)$ is the transmission function of the interference filter. As the probability of generating the SPDC photon is very less (once in a million). So the pump field must be very strong compared to daughter signal and idler photon field. Hence the pump field can be treated as the classical field, which implies that in the expression of electric field instead of the annihilation operator we can replace by just pump field amplitude function $v_{p}$. So, $H_{f}$ is:

$$H_{f} = \frac{1}{V^{3/2}} \sum_{k_{p},\sigma_{p}} \sum_{k_{s},\sigma_{s}} \sum_{k_{i},\sigma_{i}} \chi_{ijk}^{(2)}(e_{k_{p},\sigma_{p}})(e_{k_{s},\sigma_{s}}) (e_{k_{i},\sigma_{i}})$$

$$\times \left[ (g_{k_{p},\sigma_{p}})(g_{k_{s},\sigma_{s}}) (g_{k_{i},\sigma_{i}}) \right] \left[ v_{p} (\omega_{p} k_{p}) \hat{a}_{k_{s},\sigma_{s}}^{\dagger}(\omega_{s}) \hat{a}_{k_{i},\sigma_{i}}(\omega_{i}) \right]$$

$$\times \left[ \int_{V} dV e^{-i(k_{s}+k_{i}-k_{p}) \cdot r} \right] \times e^{i(\omega_{p}+\omega_{s}+\omega_{i})t} + \text{cc}$$

(24)

Here we have distinguished the three field as pump (P), signal (S) and idler(I).

$$g_{k_{j},\sigma_{j}} = \sqrt{\frac{\hbar \omega_{j}}{2 \varepsilon_{0} n^{2}(k_{j}, \sigma_{j})}} G(\omega_{j})$$

(25)

1.7 two-photon wavefunction

Now putting the expression of $H_{f}$ from equation 24, the two-photon wavefunction of the equation No. 22 can be written as:

$$|\psi(0)\rangle = \frac{1}{i \hbar V^{3/2}} \sum_{k_{p},\sigma_{p}} \sum_{k_{s},\sigma_{s}} \sum_{k_{i},\sigma_{i}} \chi_{ijk}^{(2)}(e_{k_{p},\sigma_{p}})(e_{k_{s},\sigma_{s}}) (e_{k_{i},\sigma_{i}})$$

$$\times \left[ (g_{k_{p},\sigma_{p}})(g_{k_{s},\sigma_{s}}) (g_{k_{i},\sigma_{i}}) \right] \left[ v_{p} (\omega_{p} k_{p}) \hat{a}_{k_{s},\sigma_{s}}^{\dagger}(\omega_{s}) \hat{a}_{k_{i},\sigma_{i}}(\omega_{i}) \right]$$

$$\times \left[ \int_{V} dV e^{-i(k_{s}+k_{i}-k_{p}) \cdot r} \right] \times \left[ \int_{-t_{\text{int}}}^{0} e^{i(\omega_{p}+\omega_{s}+\omega_{i})t'} dt' \right] - t_{\text{int}} + \text{cc}$$

(26)

Here the first term $| - t_{\text{int}} \rangle$ is neglected as it is the quantum vacuum state which is by no means simply the empty space. The interaction time $t_{\text{int}}$ is assumed to be much longer than the time scale over which down-conversion takes place and much smaller than the time interval between two consecutive down-conversion events. Then the limit of the integration $dt'$ can be taken as $-\infty$ to $\infty$ [31, 35]. So the time part of the integration gives:

$$\int_{-\infty}^{0} e^{i(\omega_{p}+\omega_{s}+\omega_{i})t'} dt' = \delta(\omega_{s} + \omega_{i} - \omega_{p})$$

(27)

Now we will look for the integration of $dV$. First, we can substitute $\int_{V} dV$ to $\int \prod dx dy dz$ and we will set the co-ordinate for the crystal as such that, mathematically the limit of the integrals will be $(-L_{s}/2 + + L_{s}/2), (-L_{i}/2 to + L_{i}/2)$ and $(-L_{p}/2 to + L_{p}/2)$ and $(-L_{s}/2 to + L_{s}/2)$ and $(-L_{i}/2 to + L_{i}/2)$ and $(-L_{p}/2 to + L_{p}/2)$ see fig.2 We also can write the wave vector, component wise $k_{j} = q_{jx} \hat{x} + q_{jy} \hat{y} + q_{jz} \hat{z}$, where $j$ represents P- pump, S- signal and I- idler.

So, working out on the integration:

$$\int_{V} e^{-i(k_{s}+k_{i}-k_{p}) \cdot r} dV$$

$$= \left[ \int_{-L_{s}}^{L_{s}} e^{-i(q_{sx}+q_{ix}+q_{px})} dx \right] \times \left[ \int_{-L_{i}}^{L_{i}} e^{-i(q_{sx}+q_{ix}+q_{px})} dy \right]$$

$$\times \left[ \int_{-L_{p}}^{L_{p}} e^{-i(q_{sx}+q_{ix}+q_{px})} dz \right]$$

$$= (L_{s}L_{s}) \sin[(q_{sx} + q_{ix} - q_{px}) L_{s}/2] \sin[(k_{sx} + k_{ix} - k_{px}) L_{s}/2]$$

$$\times \sin[(q_{sx} + q_{ix} - q_{px}) L_{s}/2] \sin[(k_{sx} + k_{ix} - k_{px}) L_{s}/2]$$

(28)

With another approximation that the crystal transverse length with respect to the pump field means $L_{x}$ and $L_{y}$ are large.
enough than the pump beam size and this makes two sinc function necessarily into delta function. As we can write \( q_x \hat{x} + q_y \hat{y} = q_z \) and \( L_x L_y L = V_c \), total volume of the crystal, then the previous integration value becomes:

\[
\int_V e^{-i(k_x k_x - k_p)} r \, dV = V_c \text{sinc}[(k_x + k_z - k_p) \frac{L}{2}] [e^{-i(k_x + k_z - k_p) \frac{L}{2}}] \delta(q_x + q_y - q_p)
\]

(29)

And at the time \( -t_{int} \), means at the moment of starting the interaction there is no signal and idler photon. So the two-photon wave function at \( t = -t_{int} \) is nothing but the vacuum state. Mathematically the wavefunction we can write \( | -t_{int} \rangle = |\text{vac}_S \rangle |\text{vac}_I \rangle \).

The operator gives the result:

\[
\hat{a}_{k_S,\sigma_S}(\omega_S)\hat{a}^\dagger_{k_I,\sigma_I}(\omega_I)|\text{vac}_S \rangle |\text{vac}_I \rangle = |k_S,\sigma_S,\omega_S \rangle |k_I,\sigma_I,\omega_I \rangle
\]

(30)

Here the \( |k_S,\sigma_S,\omega_S \rangle \) and \( |k_I,\sigma_I,\omega_I \rangle \) are the single photon fock state in the mode \( (k_S,\sigma_S,\omega_S) \) and \( (k_I,\sigma_I,\omega_I) \).

Now if we think about the complex conjugate part (cc) of \( |\psi(0)\rangle \) in equation No. 26, that will have the operator part \( \hat{a}_{k_S,\sigma_S}(\omega_S)\hat{a}^\dagger_{k_I,\sigma_I}(\omega_I) \).

We know in vacuum state there is no photon so annihilating such state will give nothing.

\[
\hat{a}_{k_S,\sigma_S}(\omega_S)\hat{a}^\dagger_{k_I,\sigma_I}(\omega_I)|\text{vac}_S \rangle |\text{vac}_I \rangle = 0
\]

(31)

With all these result from equation No. 27, 29, 30 and 31 and gathering them in equation No. 26 we obtain:

\[
|\psi(0)\rangle = \frac{V_c}{\hbar V^{1/2}} \sum_{k_p,\sigma_p} \sum_{k_S,\sigma_S} \sum_{k_I,\sigma_I} \chi_{1jk}^{(2)}(k_p,\sigma_p) \langle \hat{e}_{k_p,\sigma_p}(\omega_p) | \hat{e}^\dagger_{k_S,\sigma_S}(\omega_S) | \hat{e}^\dagger_{k_I,\sigma_I}(\omega_I) \
\]

\[
× \left[ \langle k_S + k_I - k_p \rangle \frac{L}{2} \right] \delta(q_S + q_I - q_p) \times [\delta(\omega_S + \omega_I - \omega_p)] |q_S,\sigma_S,\omega_S \rangle |q_I,\sigma_I,\omega_I \rangle
\]

(32)

Where \( |q_j,\sigma_j,\omega_j \rangle \) represents the one photon state in the mode defined by transverse wave vector components \( q_j \), polarization \( \sigma_j \) and frequency \( \omega_j \). Now to get the simplified form of the above equation it is convenient to make some assumption:

- It is considered that \( g_{k_p,\sigma_p} \) and \( \chi_{1jk}^{(2)} \) are slowly varying function of \( k_j \) and \( \omega_j \), so we can take these as constant.

- The quantisation volume [34] is chosen in such a way that we can replace the summation by integration.

- As we are doing the calculation for BBO- negative uniaxial crystal [36], the pump beam is necessarily extraordinary polarized [37].

Hence we can write:

\[
|\psi(0)\rangle = \sum_{\sigma_S,\sigma_I} C_{\sigma_S,\sigma_I} \int G(\omega_S) d\omega_S \int G(\omega_I) d\omega_I \int d\omega \int d\omega_P \int d\omega_{q_I} [\delta(\omega_S + \omega_I - \omega_P)] |q_S,\sigma_S,\omega_S \rangle |q_I,\sigma_I,\omega_I \rangle
\]

(33)

Now integrating to remove the delta function we get:

\[
|\psi(0)\rangle = \sum_{\sigma_S,\sigma_I} C_{\sigma_S,\sigma_I} \int G(\omega_S) d\omega_S \int G(\omega_I) d\omega_I \int d\omega \int d\omega_P \int [\delta(vP(\omega_S,\omega_I))] |q_S,\sigma_S,\omega_S \rangle |q_I,\sigma_I,\omega_I \rangle
\]

(34)

If we make sure us by using the particular frequency filter to get only \( \omega_S \) and \( \omega_I \) then we can make the wavefunction free from integration over \( d\omega_S \) and \( d\omega_I \). Hence our desired two-photon wavefunction for Spontaneous Parametric down conversion is:

\[
|\psi(0)\rangle = \sum_{\sigma_S,\sigma_I} \int d\omega \int d\omega_P [vP(\omega_S + \omega_I, q_S + q_I)] |q_S,\sigma_S,\omega_S \rangle |q_I,\sigma_I,\omega_I \rangle
\]

(35)
where,
\[ \phi_{\sigma_s, \sigma_i} = C_{\sigma_s, \sigma_i} \frac{G(\omega_s)G(\omega_i)G(\omega)}{2\times [e^{-i(k_{Ss}+k_{Iz}+k_p)\frac{L}{2}} \times \text{sinc}[(k_{Ss}+k_{Iz}+k_p)\frac{L}{2}]} \]

Here the \( C_{\sigma_s, \sigma_i} \) is the coupling constant depends on the state of the polarization of the signal and idler both and the non-linear susceptibility term. And the \( G(\omega_s) \) and \( G(\omega_i) \) are the spectral functions defined by the frequency selecting filter placed in front of the detector.

2. Phase matching

Calculation of the three wave down conversion requires the conservation of energy and conservation of momentum, which is termed as the phase matching. As the process is non-resonant, a down converted photon may be emitted over a wide range of wavelengths, so long as the energy and momentum conservation conditions for the pair of photons are met. The individual photons of a pair may also propagate along different directions, this is referred to as non-collinear phase-matching (see fig: 3). Collinear phase-matching, where the incident photon and the output pair of photons propagate in the same direction inside the crystal.

Figure 3. Collinear and non-collinear phase matching diagram.

Spontaneous Parametric down conversion can be achieved in two different varieties depending on the state of polarization of the signal and idler. If both signal and idler are ordinary polarized then it is termed as Type-I [16] SPDC. If the signal and idler photon are orthogonally polarized (signal is extraordinary idler is ordinary or vice versa) then it is called Type-II SPDC [38, 39]. In this chapter, we will find the phase matching condition for both the Type-I and Type-II SPDC. In each case collinearity and non-collinearity appears due to the changes of the parameter of phase matching conditions.

For our calculation pump field is extraordinary polarised. The mathematical expression in equation 36 the sinc function, as well as exponential part, are carrying the effectiveness of phase matching conditions. When they are perfectly matched, which means \( k_{Ss}+k_{Iz} = k_p \), then value of sinc function and the exponential part is maximum equal to 1 and the mismatch of phase \( k_{Ss}+k_{Iz} \neq k_p \) makes less probable to get the SPDC photon pair and hence the intensity is lower.

The direction of the propagation of the pump field is taken as the z direction conventionally. If the anisotropy of the medium is neglected then the longitudinal phase mismatch \( (k_{Ss}+k_{Iz}+k_p) \) can be written as [16]:
\[ \sqrt{|k_{Ss}|^2 - |q_s|^2} + \sqrt{|k_{Iz}|^2 - |q_i|^2} - \sqrt{|k_p|^2 - |q_p|^2} \]

The non-linear crystals used for parametric down-conversion is an anisotropic material, so the each longitudinal component of wave vector \( k_z \) will get modified by the different amount as they will realise different refractive index inside the anisotropic medium. Here we are analysing for only uniaxial anisotropic medium because of: (i) The majority of the work on spatial correlation properties of two-photon states, the photon pairs are generated by spontaneous parametric down-conversion in uniaxial crystals. (ii) The physics of down-conversion in biaxial crystals, though more realistic, but to the best of our knowledge it does not contain any essentially new effect than the uniaxial medium.

Before going to the expressing the \( k_{Iz} \) in terms of the transverse components of the wave vector \( q \) prerequisite are the wave propagation in the anisotropic medium.

2.0.1 Wave propagation in anisotropic medium

We are assuming the BBO crystal is electrically anisotropic and magnetically isotropic medium with the magnetic permeability \( \mu_0 \). Now we can write the electric displacement vector [40] in the following way:
\[
\begin{pmatrix}
D_x \\
D_y \\
D_z
\end{pmatrix} =
\begin{pmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
\]

This \( 3 \times 3 \) matrix is called the dielectric tensor \( \varepsilon \). And \( D = \varepsilon E \). We can choose a coordinate system, such that the dielectric matrix will be diagonal.
\[
\varepsilon =
\begin{pmatrix}
\varepsilon_{xx} & 0 & 0 \\
0 & \varepsilon_{yy} & 0 \\
0 & 0 & \varepsilon_{zz}
\end{pmatrix}
\]

(39)

Let the new coordinates be \( (x', y', z') \). And we define that the pump field is extraordinary polarised and it is propagating parallel to \( z \) co-ordinate (see figure: 4). The optic axis is along this \( z' \) (in new co ordinate system which is connected to our actual frame of reference by rotation of angle \( \theta \) about the \( y \) axis).

For the uniaxial crystal, refractive index experience by ray along parallel to the optic axis is denoted as \( n_e \) and for perpendicular to the optic axis direction, the refractive index is \( n_o \). Here the optic axis is parallel to the \( z' \) direction. Hence we can write \( \varepsilon_{xx} = n_e^2 \) and \( \varepsilon_{yy} = \varepsilon_{zz} = n_o^2 \).

So, the electric displacement vector in the new co-ordinate system is:
\[
\begin{pmatrix}
D_{x'} \\
D_{y'} \\
D_{z'}
\end{pmatrix} = E_0
\begin{pmatrix}
n_e^2 & 0 & 0 \\
0 & n_o^2 & 0 \\
0 & 0 & n_o^2
\end{pmatrix}
\begin{pmatrix}
E_{x'} \\
E_{y'} \\
E_{z'}
\end{pmatrix}
\]

(40)
We now consider that the plane wave is propagating through the anisotropic medium, so the electric field in the original frame can be written as:

\[ E = E_0 e^{i(k \cdot r - \omega t)} \]  

(41)

Here \( k = q_x \hat{x} + q_y \hat{y} + k_z \hat{z} \) and position vector \( r = x \hat{x} + y \hat{y} + z \hat{z} \). But in the new transformed co-ordinate it will be:

\[ E' = E_0' e^{i(k' \cdot r' - \omega t)} \]

(42)

Where the wave vector can be written as \( k' = q_x' \hat{x} + q_y' \hat{y} + k_z' \hat{z} \) and position vector can be represented as \( r' = x' \hat{x} + y' \hat{y} + z' \hat{z} \).

Combining Maxwell equation No. 3 and 4 in the new co-ordinate system we get:

\[ \nabla' (\nabla' \cdot E') - \nabla'^2 E' = -\mu_0 \frac{\partial^2 D'}{\partial t^2} \]  

(43)

Here \( \nabla' = \hat{x}' \frac{\partial}{\partial x'} + \hat{y}' \frac{\partial}{\partial y'} + \hat{z}' \frac{\partial}{\partial z'} \). Also we are taking that \( c^2 = \frac{1}{\mu_0 \varepsilon_0} \).

\[(k' \cdot E')k' - |k'|^2 E' = -\left(\frac{\omega}{c}\right)^2 (\varepsilon_0) E' \]

(44)

From the above vector equation we get the three simultaneous scalar equations:

\[ \left(\frac{n_o \omega}{c}\right)^2 - q_y^2 - k_z^2\] \(E' - q_x q_y E' + q_x k_z E' = 0 \]

(45)

\[ q_x q_y E' + \left(\frac{n_o \omega}{c}\right)^2 - q_y^2 - k_z^2\] \(E' - q_x k_z E' = 0 \]

(46)

\[ q_x k_z E' + q_y k_z E' + \left(\frac{n_o \omega}{c}\right)^2 - q_y^2 - k_z^2\] \(E' = 0 \]

(47)

we can now eliminate the \( E' \hat{x}'E' \hat{y}'E' \hat{z}' \) from the above three equations by using the condition of non trivial solution.

The solution exists if the determinant of their coefficient is zero.

\[ \left| \begin{array}{ccc} \left(\frac{n_o \omega}{c}\right)^2 - q_y^2 - k_z^2 & q_x q_y & q_x k_z \\ q_x q_y & \left(\frac{n_o \omega}{c}\right)^2 - q_y^2 - k_z^2 & q_y k_z \\ q_x k_z & q_y k_z & \left(\frac{n_o \omega}{c}\right)^2 - q_y^2 - k_z^2 \end{array} \right| = 0 \]

(48)

Which gives:

\[ \frac{\omega^2}{c^2} \left[ \left(\frac{\omega}{c} n_o \right)^2 - q_y^2 - k_z^2 \right] \times \left[ \left(\frac{\omega}{c} n_o \right)^2 - (n_o q_x)^2 - (n_o q_y)^2 - (n_o k_z)^2 \right] = 0 \]

(49)

As we know that \( \frac{\omega}{c} \neq 0 \). And from the second factor we can write the following equation. It gives the condition for the ordinary ray:

\[ \frac{q_x^2 + q_y^2 + k_z^2}{n_o^2} = \frac{\omega^2}{c^2} \]

(50)

And the final term gives the extraordinary ray condition equation

\[ (n_o q_x)^2 + (n_o q_y)^2 + (n_o k_z)^2 - \left(\frac{\omega}{c} n_o \right)^2 = 0 \]

(51)

These expressions are all in the new coordinate frame but in general we need them in the actual lab frame of references. For that, the proper transformation relation is necessary to know. We considered that pump field is propagating along \( z \) axis and lets it make angle \( \theta \) with the optic axis of the crystal where the optic axis of the crystal is along the diagonal of the crystal, see figure:4. From the picture we can easily say that we need rotation of angle \( \theta \) along the y axis. So our rotation matrix is:

\[ \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \]

(52)

Now we want to impose one condition that the co-ordinate change will not make the plane wave change any more. To ensure this the equation of constraint is: \( k \cdot r = k' \cdot r' \)

\[ (k_x x + k_y y + k_z z) = k' \left( x \cos \theta + y \sin \theta \right) + k_y y + k_z \left( -x \sin \theta + y \cos \theta \right) \]

(53)

Making the coefficient of \( x, y, z \) equal individually in both sides of the equation we obtain.

\[ k_x = k_x' \cos \theta - k_z' \sin \theta \]

(54)

\[ k_y = k_y' \]

(55)

\[ k_z = k_z' \sin \theta + k_z' \cos \theta \]

(56)
And the two-photon wave function respectively. where θ is the pump propagation angle with optic axis but most importantly the \( n_o \) and \( n_e \) have the value at the pump frequency i.e \( \omega_p \). As they obey the proper dispersion relation. And also for the terms \( k_S \) and \( k_I \) the pump angle is same all time but the \( n_{oS} \) should be equal to \( n_oI \) the refractive index of ordinary ray of signal and idler respectively at their respective frequencies. For SPDC \( n_S = \omega = \frac{\eta}{\theta} \) hence \( n_{oS} = n_{oI} \) according to dispersion relation. And also \( q_{px} = q_{Sx} + q_{Ix} \), \( q_{py} = q_{Sy} + q_{Iy} \), the x,y component of pump beam will equal to the sum of x,y component of signal and idler beam as the SPDC conserves the momentum. Hence we get:

\[
\Delta k = k_S + k_I - k_p = n_oS \omega_S + \omega_p + c \frac{c}{2n_{oI} \omega_I} |q_I|^2
\]

\[
|\psi(0)\rangle = \int dq_q \int dq_p \phi_{q}(q_S, \omega_S) |q_I, \omega_I\rangle
\]

\[
\phi_{qo} = C_{oo} G(\omega_S) G(\omega_I) \times |v_p(\omega_S + \omega_I, q_S + q_I)| \times [e^{-i k \frac{q}{2}}] \times sinc(\frac{\Delta k q}{2})
\]

The \( \Delta k \) is the phase matching term. Various parameter (wavelength, pump angle between pump propagation direction and the optic axis of the crystal, beam waist, thickness of the crystal of this will decide the SPDC will be collinear or non-collinear. In the result section, we show their effectiveness.

### 2.2 Type-II Phase matching

In Type-II system, pump field is extraordinary but the signal and idler are orthogonally polarised. Mathematically, the two-photon wavefunction is the superposition of wavefunction of two conditions (condition 1, condition 2) with equal probabilities. The conditions are: (i) signal is extraordinary the idler is ordinary. (ii) the signal is ordinary and the idler is extraordinary.

#### 2.2.1 Condition 1

In this condition the signal is e-ray and the idler is o-ray. So we can write:

\[
k_{pc} = -\alpha_p q_{px} + \eta_p \omega_p c - c \frac{c}{2\eta_p \omega_p} [\beta_p^2 q_{px}^2 + \gamma_p^2 q_{py}^2]
\]

\[
k_{Sx} = n_{oS} \omega_S c - c \frac{c}{2n_{oI} \omega_I} |q_I|^2
\]

\[
k_{Iz} = n_{oI} \omega_I c - c \frac{c}{2n_{oI} \omega_I} |q_I|^2
\]

Here the constant terms \( \alpha_p, \beta_p, \gamma_p, \eta_p \) are calculated from the equation 65, 66, 67, 68 respectively. where \( \theta \) is the pump propagation angle with optic axis but most importantly the \( n_o \) and \( n_e \) have the value at the pump frequency i.e \( \omega_p \). As they obey the proper dispersion relation. And also for the terms \( k_S \) and \( k_I \) the pump angle is same all time but the \( n_{oS} \) should be equal to \( n_oI \) the refractive index of ordinary ray of signal and idler respectively at their respective frequencies. For SPDC \( n_S = \omega = \frac{\eta}{\theta} \) hence \( n_{oS} = n_{oI} \) according to dispersion relation. And also \( q_{px} = q_{Sx} + q_{Ix} \), \( q_{py} = q_{Sy} + q_{Iy} \), the x,y component of pump beam will equal to the sum of x,y component of signal and idler beam as the SPDC conserves the momentum. Hence we get:

\[
\Delta k = k_S + k_I - k_p = n_oS \omega_S + \omega_p + c \frac{c}{2n_{oI} \omega_I} |q_I|^2
\]

\[
|\psi(0)\rangle = \int dq_q \int dq_p \phi_{q}(q_S, \omega_S) |q_I, \omega_I\rangle
\]

\[
\phi_{qo} = C_{oo} G(\omega_S) G(\omega_I) \times |v_p(\omega_S + \omega_I, q_S + q_I)| \times [e^{-i k \frac{q}{2}}] \times sinc(\frac{\Delta k q}{2})
\]

The \( \Delta k \) is the phase matching term. Various parameter (wavelength, pump angle between pump propagation direction and the optic axis of the crystal, beam waist, thickness of the crystal of this will decide the SPDC will be collinear or non-collinear. In the result section, we show their effectiveness.

### 2.1 Type-I Phase matching

Now we will look at the two-photon wave function for the Type-I phase matching condition and its dependence:

As for Type-I the pump field is extraordinary, and the signal and the idler are both ordinary ray. So,

\[
k_{pc} = -\alpha_p q_{px} + \eta_p \omega_p c - c \frac{c}{2\eta_p \omega_p} [\beta_p^2 q_{px}^2 + \gamma_p^2 q_{py}^2]
\]

\[
k_{Sx} = n_{oS} \omega_S c - c \frac{c}{2n_{oI} \omega_I} |q_I|^2
\]

\[
k_{Iz} = n_{oI} \omega_I c - c \frac{c}{2n_{oI} \omega_I} |q_I|^2
\]

This is the quadratic equation of the \( k_z \) of form \( ak_z^2 + bk_z + c = 0 \). And this leads to the solution [41] of \( k_z \):

\[
k_z = -\alpha q_x + \sqrt{(\eta \omega_n^2 - \beta^2 q_z^2 - \gamma^2 q_y^2)}
\]

\[
\approx -\alpha q_x + \frac{\omega_n}{\eta} - c \frac{c}{2\eta \omega} [\beta^2 q_z^2 + \gamma^2 q_y^2]
\]

Where

\[
\alpha = \frac{(n_e^2 - n_o^2) \sin \theta \cos \theta}{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}
\]

\[
\beta = \frac{n_e^2 \sin \theta \cos \theta}{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}
\]

\[
\gamma = \sqrt{n_e^2 \sin^2 \theta + n_e^2 \cos^2 \theta}
\]

\[
\eta = \frac{n_o \sin \theta}{\sqrt{n_o^2 \sin^2 \theta + n_e^2 \cos^2 \theta}}
\]
Here $\alpha_p, \beta_p, \gamma_p, \eta_p$ are calculated by putting the value of $n_o$ and $n_i$ at the pump frequency. But for calculating the value of $\alpha_S, \beta_S, \gamma_S, \eta_S$ we have to take the value of $n_o$ and $n_i$ at the frequency $\omega_S = \frac{\omega_p}{2}$. But the pump angle $\theta$ will remain the same. And the value of $n_{id}$ in the expression of $k_{Sc}$ is nothing but the ordinary refractive index of idler at the frequency $\omega_I = \frac{\omega_p}{2}$.

So for this condition the phase mismatch term is,

$$\Delta k_{1c} = k_{Sc} + k_{Ic} - k_{pc}$$

$$= -\alpha_S q_{sz} + \eta_S \omega_S - \frac{c}{2\eta_S \omega_S} \left[ \beta_S^2 q_{sz}^2 + \gamma_S^2 q_{sz}^2 \right]$$

$$+ n_{id} \frac{\omega_p}{c} - \frac{c}{2n_{id} \omega_p} |q_i|^2 + \alpha_p (q_{sz} + q_{is}) - \eta_p \omega_p$$

$$\Delta k_{1c} = \frac{c}{2\eta_p \omega_p} \left[ \beta_p^2 (q_{sz} + q_{is})^2 - \gamma_p^2 (q_{sz} + q_{is})^2 \right]$$

(77)

Hence we can write that

$$\phi_{eo} = C_{eo} G(\omega_S) G(\omega_I) \times [v_p(\omega_S + \omega_I, q_S + q_I)]$$

$$\times [e^{-i\Delta k_{1c}}] \times \text{sinc} \left[ \Delta k_{1c} \frac{L}{2} \right]$$

(78)

### 2.2.2 Condition 2

Another condition is: signal is ordinary and idler is extraordinary polarised. So similarly like before we can write :

$$k_{pc} = -\alpha_p q_{pz} + \eta_p \omega_p - \frac{c}{2\eta_p \omega_p} \left[ \beta_p^2 q_{pz}^2 + \gamma_p^2 q_{pz}^2 \right]$$

(79)

$$k_{Sc} = n_{os} \frac{\omega_S}{c} - \frac{c}{2n_{os} \omega_S} |q_s|^2$$

(80)

$$k_{Ic} = -\alpha_I q_{iz} + \eta_I \omega_I - \frac{c}{2\eta_I \omega_I} \left[ \beta_I^2 q_{iz}^2 + \gamma_I^2 q_{iz}^2 \right]$$

(81)

The values of $\alpha_p, \beta_p, \gamma_p, \eta_p$ are calculated with taking the values of $n_o$ and $n_i$ at the frequency of idler $\omega_I = \frac{\omega_p}{2}$. $n_{os}$ is the ordinary ray refractive index at $\omega_I$ frequency.

Now for this the mismatch term is :

$$\Delta k_{2c} = k_{Sc} + k_{Ic} - k_{pc}$$

$$= n_{os} \frac{\omega_S}{c} - \frac{c}{2n_{os} \omega_S} |q_s|^2 - \alpha_I q_{iz} + \eta_I \omega_I$$

$$\Delta k_{2c} = \frac{c}{2\eta_I \omega_I} \left[ \beta_I^2 q_{iz}^2 + \gamma_I^2 q_{iz}^2 \right] + \alpha_p (q_{sz} + q_{is}) - \eta_p \omega_p$$

$$\Delta k_{2c} = \frac{c}{2\eta_I \omega_I} \left[ \beta_I^2 q_{iz}^2 + \gamma_I^2 q_{iz}^2 \right]$$

(82)

Hence we can write that

$$\phi_{oe} = C_{eo} G(\omega_S) G(\omega_I) \times [v_p(\omega_S + \omega_I, q_S + q_I)]$$

$$\times [e^{-i\Delta k_{2c}}] \times \text{sinc} \left[ \Delta k_{2c} \frac{L}{2} \right]$$

(83)

Finally, the two-photon wave function for the Type-II [41] is :

$$|\psi(0)\rangle = \int dq_S \int dq_I [\phi_{eo} + \phi_{oe}] |q_S, \omega_S \rangle |q_I, \omega_I\rangle$$

(84)

### 3. Pump field propagation

We consider a general experimental set up (see fig: 5) and Gaussian pump field where the beam waist is not exactly at the crystal back face. But to calculate the two-photon wavefunction we need to know the pump field amplitude $v_p(\omega_p, q_p)$ in momentum space just before the crystal. Here in this section, we have calculated the $v_p(\omega_p, q_p)$.

From the paraxial wave equation, we have the expression for the electric field of the Gaussian beam [42,43] at the beam waist position:

$$E(x', y'; z') = A e^{-\left(\frac{(x'^2 + y'^2)}{w_0^2}\right)}$$

(85)

Here the $\omega_0$ is the beam waist. And we are choosing our co-ordinate $(x', y', z')$ system such that it has the origin at the beam waist.

The transverse momentum representation of the field is given by :

$$\tilde{E}(q_x, q_y; z') = \int \int E(x', y'; z') e^{i(q_xx' + q_yy')} dx' dy'$$

$$= \int \int A e^{-\left(\frac{(q_x^2 + q_y^2)}{w_0^2}\right)} e^{i(q_xx' + q_yy')} dx' dy'$$

$$= A e^{-\left(\frac{(q_x^2 + q_y^2)}{w_0^2}\right)} \times \text{sinc} \left( \frac{q_xw_0}{\lambda z'} \right) \times \text{sinc} \left( \frac{q_yw_0}{\lambda z'} \right)$$

(86)

The transverse field profile under propagation can be expressed as the Fresnel diffraction integral [44],

$$E(x', y'; z') = \int \frac{e^{ikz'}}{i\lambda z'} \int \int E(x'', y''; z = 0) e^{i\frac{ik}{\lambda z'} (x''^2 + y''^2)} dx'' dy''$$

(87)

This propagation integral is essentially the two dimensional convolution $E(x', y'; z') = E(x', y'; z' = 0) * h(x', y'; z')$, where

$$h(x', y'; z') = \frac{e^{ikz'}}{i\lambda z'} e^{i\frac{k}{\lambda z'} (x'^2 + y'^2)}$$

(88)
Using the convolution theorem for Fourier transforms, we know that
\[ \tilde{E}(q_x, q_y; z') = \mathcal{F}(q_x, q_y; z') = 0 \]

The function \( \tilde{h}(q_x, q_y; z') \) can be evaluated as,
\[
\tilde{h}(q_x, q_y; z') = \int \mathcal{F}(q_x, q_y; z') e^{i(q_x x' + q_y y')} dx' dy' \\
= \frac{e^{ikz'}}{k} \int \exp \left\{ \frac{ik}{2z'} [x'^2 + y'^2] \right\} \exp \left\{ i(q_x x' + q_y y') \right\} dx' dy' \\
= e^{ikz'} \exp \left\{ -i\frac{z'(q_x^2 + q_y^2)}{2k} \right\}
\]

Substituting this in the Equation. 89, we get
\[ \tilde{E}(q_x, q_y; z') = A' e^{ikz'} \exp \left\{ -\left(\frac{q_x^2 + q_y^2}{4} \left( \frac{\omega_0^2}{\omega} + \frac{2i\Delta k}{k} \right) \right) \right\} 
\]

This is the expression for the Gaussian classical pump field at just before the crystal. Now as the distance between the beam waist position and the crystal front face is \( d \) then the desired expression for the Gaussian pump field amplitude in momentum space is:
\[ v_p(\omega_p, q_p) = A' \exp \left\{ -\frac{|q_p|^2}{4} \left( \frac{\omega_0^2}{\omega} + \frac{2i\Delta k}{k} \right) \right\} 
\]

### 4. SPDC by Simulation

We have the expression for the two-photon wavefunction in momentum space at the crystal front face. But the expression is not exactly analytical, it contains two infinite integrals. We computationally have done the integral and got the wavefunction in momentum space. Now to get the intensity pattern of the two-photon field at a large distance (Fraunhofer region) we need to make two dimensional Fourier transform to go to position space. But the field at crystal front face in momentum space, that will be the exactly same at far field distance from the crystal in position space.

\[ I = \langle \Psi(\rho_S, \rho_I) | \Psi(\rho_S, \rho_I) \rangle_{far} = \langle \Psi(q_S, q_I) | \Psi(q_S, q_I) \rangle_{near} \]

For a specific pump frequency \( \omega_p \), the angle of pump field direction with the optic axis of crystal \( \theta \), crystal longitudinal thickness \( L \) the intensity of SPDC photon at far field will be:
\[ I = \sum_{\sigma_S, \sigma_I} \sum_{\sigma_S' \sigma_I'} \int d\mathbf{q}_S \int d\mathbf{q}_I \int d\mathbf{q}_S' \int d\mathbf{q}_I' C_{\sigma_S \sigma_I} C_{\sigma_S' \sigma_I'} C_{\sigma_S' \sigma_I} |G(\omega_p)|^2 |G(\omega_I)|^2 |v_p(\omega_p, q_S + q_I)| |v_p(\omega_p, q_S' + q_I')| e^{i(k_{S} + k_{I} - k_{P}) \frac{L}{2}} e^{i(k_{S} + k_{I} - k_{P}) \frac{L}{2}} \]

\[ \times \text{sinc}[(k_{S} + k_{I} - k_{P}) \frac{L}{2}] \text{sinc}[(k_{S}' + k_{I}' - k_{P'}) \frac{L}{2}] \times \langle q_S', \sigma_S' | \Phi_P | q_S, \sigma_S, \omega_S \rangle \langle q_I', \sigma_I' | \Phi_P | q_I, \sigma_I, \omega_I \rangle \]

\[ \times \langle q_S, \sigma_S | \Phi_P | q_S, \sigma_S, \omega_S \rangle \langle q_I, \sigma_I | \Phi_P | q_I, \sigma_I, \omega_I \rangle \]

Simplifying the expression by using the bra-ket properties we get,
\[ I = \left| \sum_{\sigma_S, \sigma_I} \int d\mathbf{q}_S \int d\mathbf{q}_I G(\omega_p) G(\omega_I) C_{\sigma_S \sigma_I} v_p(\omega_p, q_S + q_I) \right| e^{-i\frac{k_{S} + k_{I} - k_{P}}{2} \frac{L}{2}} \]

\[ \times \text{sinc}[(k_{S} + k_{I} - k_{P}) \frac{L}{2}] \left( \omega_0^2 + \frac{2i\Delta k}{k} \right) \]

\[ \times \text{sinc}[(k_{S} + k_{I} - k_{P}) \frac{L}{2}] \]

Similarly for type-II SPDC, the intensity will be:
\[ I = \left| A' \int d\mathbf{q}_S \int d\mathbf{q}_I \exp \left\{ -\frac{|q_p + q_{p'}|^2}{4} \left( \frac{\omega_0^2}{\omega} + \frac{2i\Delta k}{k} \right) \right\} \right| G(\omega_p) G(\omega_I) \]

\[ \times \text{sinc}[(k_{S} + k_{I} - k_{P}) \frac{L}{2}] \left( \frac{C_{oo} e^{-i\Delta k_{\overrightarrow{z}} \frac{L}{2}} \text{sinc}[(\Delta k_{\overrightarrow{z}} \frac{L}{2})] + C_{oe} e^{-i\Delta k_{\overrightarrow{z}} \frac{L}{2}} \text{sinc}[(\Delta k_{\overrightarrow{z}} \frac{L}{2})]}{2} \right) \]

where \( \Delta k_{\overrightarrow{z}}, \Delta k_{\overrightarrow{z}} \) can be taken from equation 77 and equation 82 respectively.

For the simulation of intensity pattern we have used the dispersion relation for the refractive index of the ordinary polarised light [45] for BBO crystal :
\[ n^2_0(\lambda) = 2.7405 + \frac{0.0184}{\lambda^2} - 0.0155\lambda^2 \]

Also for the refractive index of the extraordinary [45], the dispersion relation is:
\[ n^2_e(\lambda) = 2.3730 + \frac{0.0128}{\lambda^2} - 0.0044\lambda^2 \]

In the both cases the \( \lambda \) is the corresponding wavelength at which the refractive index is required should be in micrometre (\( \mu m \)).

### 5. Results

In this section, we will investigate the intensity profile of the SPDC photon in various phase matching conditions both experimentally and theoretically. To obtain the SPDC, we have
used the BBO; β-barium borate crystal mounting on a goniometer which can be rotated by 0.04 degree in each step to change the external angle between optic axis and the pump field propagation direction. The relation between the external angle and the internal angle ($\theta_p$) of optic axis can be obtained from Snell’s law [34].

![Figure 6](image1.png)

**Figure 6.** (a), (d) Intensity pattern of the type-I SPDC at far field in the transverse plane of the propagation obtained experimentally and theoretically respectively, for Gaussian pump wavelength = 405 nm, $\omega_o=388\mu m$, $d=107.8 cm$, $L=2 mm$, and $\theta_p=28.64^\circ$. (b), (e) The corresponding intensity pattern for $\theta_p=28.74^\circ$. (c), (f) The corresponding intensity for $\theta_p=28.95^\circ$.

In Fig:6(a), (d) the blob represents the collinear phase matching. But as $\theta_p$ is increasing the blob becomes ring and the ring expands, which ensures more non-collinear. The ring has some width, called correlation width has been discussed in the another section. The intensity pattern looks like blob for collinear condition because, the idler photons are at just behind their corresponding signal photons. For non-collinear the points of ring are signal(idler) photons and the corresponding entangled idler(signal) photons are situated just opposite side to it, with respect to the center of the ring. The collinear condition depends on the pump wavelength. Theoretically the $\theta_p$ for 355 nm pump beam is 32.89° where $\omega_o=388\mu m$, $d =107.8 cm$, $L=2 mm$. The figure 7 shows the effect of the thickness of the crystal on phase matching condition.

Type-II SPDC appears at larger angle $\theta_p$ for BBO. We have put the same crystal for type-II which we have used for type-I as well and measured the output intensity (Figure: 8) in EMCCD camera. Here also the blob becomes ring and ring expands as the $\theta_p$ increases. The two ring corresponds to signal and idler with orthogonal polarization.

5.1 polarization Entanglement

In case of the Type-I SPDC both the signal and idler are ordinary polarised. So, Type-I phase matching cannot easily provide us the polarization entangled signal and idler photon pair. Still, it is highly popular for generation of the hyper

![Figure 7](image2.png)

**Figure 7.** Theoretically simulated intensity variation of the type-I down conversion at far field for different thickness(L) of the BBO: β-barium borate crystal;(a)L= 1 mm, (b) L=2 mm, (c)L=3 mm. Where $\lambda_p=405$ nm, $\theta_p=28.74^\circ$, $\omega_o=388\mu m$, and $d=107.8 cm$.

![Figure 8](image3.png)

**Figure 8.** (a), (e) Type-II SPDC intensity distribution measured and theoretical simulation respectively at Fraunhofer region for $\lambda_p=405$ nm, $\omega_o=388\mu m$, $L=2 mm$, $d=107.8 cm$, and $\theta_p=40.48^\circ$. (b), (f) The corresponding intensity profile for $\theta_p=40.99^\circ$. (c), (g) The corresponding intensity for $\theta_p=41.40^\circ$. (d), (h) The corresponding intensity for $\theta_p=41.78^\circ$. 
entangled state in the position-momentum basis and angular displacement -OAM basis.

But people have some technique [45–48] to get polarization entanglement from it as well. If we use two crystal with identical optic axis and place them back to back with the condition that one of the crystal is rotated 90 degree about the axis of the propagation of the beam from other. And the two crystal’s optic axis is chosen as polarization basis then with respect to these bases the polarization of the pump beam is the superposition of the extra ordinary and ordinary polarization. So SPDC may occur if one of the crystal observes the pump beam as extraordinary polarised and down converted it, but the detector cannot fix that which crystal is causing the down converted photon. To ensure that, both crystal is made thin enough so that the ring or blob formed from any of the two crystals are superimposed on each other. That is how the polarization entangled photon generates from Type-I SPDC.

In Type- II SPDC [49] photon can be found in the superposition of two polarization state where the signal and the idler rings are intersecting in the Fig: 8(c). In the intensity pattern, one ring is extraordinary polarised and the other ring is ordinary polarised. At the intersection of the two rings, it can not be determined that photon is representing which ring hence the polarization of the photon is not sure. Which implies that the photon at the intersection points has the superposition state of polarization extra ordinary and ordinary (horizontal or vertical). They are polarization entangled photons. The state of the polarised entangled photon can be written in this form:

$$|\psi\rangle = a|H\rangle_A|V\rangle_B + b|V\rangle_A|H\rangle_B$$  \hspace{1cm} (100)$$

Here $a$, $b$ are the complex number and $|a|^2 + |b|^2 = 1$, the subscript A,B represents the photon A and photon B. This is exactly the EPR-Bohm two-photon entangled polarization state [6].

5.2 Correlation Width

People had studied the spatial correlation properties and its relation with several experimental parameters [50–55] of SPDC two-photon field over the past few year. In the two-photon intensity pattern, the correlation width is defined as the range of $\rho_i$ (or $\rho_s$) for which we get a substantial probability to find signal(or idler). When an idler photon is detected at position $\rho_i$, the corresponding signal photon has an appreciable probability of being detected anywhere inside an area whose center is at $-\rho_i$ and its radius is twice the pump beam radius.

Jha and Boyd has shown in their paper [56] rigorously that with approximating the thickness of the non-linear crystal to be negligible the pump beam waist at a distance D from the crystal the correlation width($\sigma(D)$) will be twice of the pump beam width $\omega(D)$.

Now we will calculate the pump beam width at distance D from the crystal (thickness of crystal is considered negligible). Taking the normal Gaussian pump field whose beam waist at d distance before the crystal and placing a thin lens just before the crystal (see fig:9) the expression of the pump field becomes

$$E(x,y;D) = A' \exp \left\{ -\frac{k_p (x^2 + y^2)}{4D^2} \right\}$$  \hspace{1cm} (101)$$

Here $A'$ is the complex term. $f$ is the focal length of the thin lens, $k_p = \frac{2\pi}{\lambda_p}$ is the wave vector of the pump field. And the $\omega_p$ is the beam waist. Now the intensity of this pump field at $z=D$ is:

$$I = E^*(x,y;D)E(x,y;D) = |A'|^2 \exp \left\{ -\frac{2(x^2 + y^2)}{\omega(D)^2} \right\}$$  \hspace{1cm} (102)$$

Where $\omega(D)$ is the pump beam width at $z=D$. So the expression for the two-photon correlation width at distance D from the non linear crystal is:

$$\sigma(D) = \frac{2}{fk_p\omega_p} \left[ (4d^2 + \frac{\omega_p^2 k_p^2}{2}) (D - f)^2 - 8d f D (D - f) + 4 f^2 D^2 \right]^{\frac{1}{2}}$$  \hspace{1cm} (103)$$

6. Angular Schmidt spectrum of entangled photons

The characterisation and measurement of the angular Schmidt spectrum have been a very challenging problem. Several techniques both theoretically [54, 64, 67] and experimentally [57–63] have been introduced to measure it. Torres et al. [64] came with a theoretical formula for calculating the angular spectrum number for collinear phase matching. Later Zhang and Roux [65] extended it for the non collinear phase matching. Though these methods are measuring the true spectrum
but with some divergence of integral issues. More recently G.Kulkarni et al. [66], have derived the exact formula for calculation of true angular Schmidt spectrum number with no diverging of the integral issues for both collinear as well as non collinear phase matching type-I in the same framework and experimentally calculate the angular spectrum number for them.

The two-photon state \(|\psi_2(t)\rangle\) can be expressed in terms of Laguerre-Gaussian (LG) basis [64], [67–69].

\[ |\psi_2(t)\rangle = \sum l_s \sum l_i \sum p_s \sum p_i C_{l_s p_s}^{l_i p_i} |l_s, p_s\rangle_s |l_i, p_i\rangle_i \]  

(104)

Here the suffixes s, i, and p are stand for signal, Idler, and pump respectively. \(l_s, l_i\) represents the OAM mode index and \(p_s, p_i\) defined by the radial mode of signal, idler respectively. Using momentum basis representations of state \(|l_s, p_s\rangle_s\) [64, 67]: \((LG)^{l_s}_{p_s}(q_s) = |q_s,l_s,p_s\rangle\) and in cylindrical coordinate system \(C_{l_s p_s}^{l_i p_i}\) can be written as:

\[ C_{l_s p_s}^{l_i p_i} = \int_0^{\infty} \int_{-\infty}^{\infty} \Phi(p_s, p_i, \phi) \Phi^*(p_i, p_s, \phi) (LG)^{l_i}_{p_i}(p_i) \times (LG)^{l_s}_{p_s}(p_s) dp_s dp_i d\phi d\phi \]  

(105)

Now the probability of detecting signal and idler photon with OAM modes \(l_s\) and \(l_i\) respectively with all possible radial modes is:

\[ P_l^{l_s} = \sum p_s=0^{\infty} |C_{l_s p_s}^{l_i p_i}|^2 \]  

(106)

The term \((LG)^{l_s}_{p_s}(p_s, \phi_s)\) can be written as \((LG)^{l_i}_{p_i}(p_i) e^{il_i \phi}\).

We have the identity \(\sum_{p_s=0}^{\infty} (LG)^{l_s}_{p_s}(p) (LG)^{l_i}_{p_i}(p) = \frac{1}{\pi} \delta(p^2 - p'^2)\). Also OAM conservation implies \(l_s = -l_i = l\). Hence the angular Schmidt spectrum [66] can be written as:

\[ S_l = P_l^{l_s} = \frac{1}{4 \pi^2} \int_0^{\infty} \int_{0}^{\infty} \Phi(p_s, p_i, \phi, \phi) e^{il\phi} dp_s dp_i d\phi d\phi \]

This is the theoretical exact derivation for calculating true spectrum number. As the angle between the optic axis of the BBO crystal and pump field propagation direction is the tuning factor to get the collinear and non collinear phase matching of entangled photons, this technique has been used to generate experimentally [35] the type-I both collinear and non collinear source for calculating angular Schmidt spectrum number which is characterised by \(K_\alpha = 1/\sum l^2\) and highest reported spectrum number 229 by single shot measurement [60].

7. Conclusion

The usefulness of Spontaneous Parametric Down Conversion (SPDC) as a source of entangled two-photon in the field of quantum information [70], quantum metrology [71,72], quantum imaging [7,73] etc. is historic and self-evident. And for experimentally handle it the phase matching condition has a very crucial contribution. We have tried to make a clear picture of that fact theoretically and experimentally both and realised its collinearity and non-collinearity, their dependencies with various parameters. We got the analytical expression for calculating the correlation width of two-photon at a certain distance from the crystal front face with the thin crystal approximation. Also we have reviewed the theoretical calculations of Angular Schmidt spectrum number here. Using the Parametric down conversion two-photon generating technique and varying the collinearity-non collinearity phase matching conditions by tuning the \(\theta_p\) the angular Schmidt spectrum number for both collinear and non collinear type-I can be measured experimentally.

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