QUANTUM ($\mathfrak{sl}_n, \wedge V_n$) LINK INVARIANT AND MATRIX FACTORIZATION

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Abstract. In this paper, we give a generalization of Khovanov-Rozansky homology. We define a homology associated to the quantum ($\mathfrak{sl}_n, \wedge V_n$) link invariant, where $\wedge V_n$ is the set of fundamental representations of $U_q(\mathfrak{sl}_n)$. In the case of an oriented link diagram composed of $[k, 1]$-crossings, we define a homology and prove that the homology is invariant under Reidemeister II and III moves. In the case of an oriented link diagram composed of general $[i, j]$-crossings, we define a normalized Poincaré polynomial of homology and prove that the normalized Poincaré polynomial is a link invariant.

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§1. Introduction

In this paper, we study a categorification of the quantum ($\mathfrak{sl}_n, \wedge V_n$) link invariant, associated to $U_q(\mathfrak{sl}_n)$ and its fundamental representations $\wedge V_n = \{V_n, \wedge^2 V_n, ..., \wedge^{n-1} V_n\}$, using matrix factorizations. That is, this work is a generalization of a categorification of the ($\mathfrak{sl}_n, V_n$) link invariant via matrix factorizations given by Khovanov and Rozansky [6].

Murakami, Ohtsuki, and Yamada [9] gave the state model of the ($\mathfrak{sl}_n, \wedge V_n$) link invariant using a polynomial invariant of MOY diagrams, which

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Figure 1: MOY diagrams $\Gamma_{L,k}^{[i,j]}$ and $\Gamma_{R,k}^{[i,j]}$

are composed of trivalent planar diagrams colored from the set \{1, 2, \ldots, n\} corresponding to the fundamental representations \{\bigwedge^n V_n, \bigwedge^{n-1} V_n, \ldots\}, under planar isotopy moves and MOY relations. MOY diagrams represent intertwiners between tensor products of some fundamental representations, and MOY relations are equivalent to relations between intertwiners. The state model consists of equations for \([i,j]\)-crossings

\[
\langle i^\downarrow j^\uparrow \rangle_n = \sum_{k=0}^j (-1)^{-j+k} q^{k+i+n-i^2+(i-j)^2+2(i-j)} \langle \Gamma_{L,k}^{[i,j]} \rangle_n,
\]

\[
\langle i^\downarrow j^\uparrow \rangle_n = \sum_{k=0}^i (-1)^{j+k-i} q^{-k-n+j^2-(j-i)^2-2(j-i)} \langle \Gamma_{R,k}^{[i,j]} \rangle_n
\]

and MOY relations, where $\Gamma_{L,k}^{[i,j]}$ and $\Gamma_{R,k}^{[i,j]}$ are MOY diagrams in Figure 1.

Khovanov and Rozansky categorized the \((\mathfrak{sl}_n, V_n)\) link invariant via matrix factorizations. Roughly speaking, they first defined matrix factorizations of $\Gamma_{L,0}^{[1,1]}$ and $\Gamma_{L,1}^{[1,1]}$ satisfying isomorphisms of matrix factorizations corresponding to MOY relations colored by 1 and 2. Then, they categorized (1) and (2) for \([1,1]\)-crossings as a complex of matrix factorizations of $\Gamma_{L,0}^{[1,1]}$ and $\Gamma_{L,1}^{[1,1]}$ and proved that if tangle diagrams are related by a Reidemeister I, II, or III move, then the complexes of matrix factorizations of these diagrams are isomorphic.

Note that there are some similar approaches to a categorification of the \((\mathfrak{sl}_n, V_n)\) link invariant (see [3], [10]). Sussan [12] and Mazorchuk and Stroppel [8] studied the categorification using a Lie theoretic category. Cautis and Kamnitzer [2] and Webster and Williamson [13] studied the categorification using a geometric approach. Mackaay, Stosic, and Vaz [7] studied the categorification using bimodules associated to MOY diagrams.
Our strategy of a categorification of the $(\mathfrak{sl}_n \wedge V_n)$ link invariant is as follows.

(S1) We first define matrix factorizations of $\Gamma_{\text{L},k}^{[i,j]}$ and $\Gamma_{\text{R},k}^{[i,j]}$ satisfying isomorphisms between matrix factorizations corresponding to MOY relations colored from the set $\{1, 2, \ldots, n\}$. (In [14], [16], and [17] Wu and the author independently categorified polynomials associated to MOY diagrams using matrix factorizations. These works are a generalization of [6].)

(S2) We categorify (1) and (2) for $[k, 1]$-crossings, where $k$ is an element of the set $\{1, \ldots, n - 1\}$, as a complex of matrix factorizations of $\Gamma_{\text{L},0}^{[k,1]}$ and $\Gamma_{\text{L},1}^{[k,1]}$ (or $\Gamma_{\text{R},0}^{[k,1]}$ and $\Gamma_{\text{R},1}^{[k,1]}$). Then we show that if colored tangle diagrams composed of $[k, 1]$-crossings are related by a Reidemeister II or III move, then the complexes associated to these diagrams are isomorphic.

(S3) We introduce an approximate $[i,j]$-crossing composed of $[i, 1]$-crossings. Therefore, we can define a complex for the approximate $[i,j]$-crossing by a tensor product of complexes associated to $[i, 1]$-crossings. If colored link diagrams are related by a Reidemeister move, then the complexes of these diagrams are not isomorphic. Thus, we would like to define a complex for an $[i,j]$-crossing by normalizing the complex of the approximate $[i,j]$-crossing and prove that if colored tangle diagrams are related by a Reidemeister I, II, or III move, then the complexes of these diagrams are isomorphic.

Since the structure of morphisms between matrix factorizations of $\Gamma_{\text{L},k}^{[i,j]}$ and $\Gamma_{\text{L},k-1}^{[i,j]}$ ($\Gamma_{\text{R},k}^{[i,j]}$ and $\Gamma_{\text{R},k-1}^{[i,j]}$) is intricate, we have two difficulties. One is to define boundary maps of a complex of the $[i,j]$-crossing explicitly. Another is to show that there exists an isomorphism between complexes for the colored tangle diagrams that are related by a Reidemeister move, after we have defined the complex for the $[i,j]$-crossing explicitly. We consider the above strategy to avoid these difficulties. However, we have not defined a complex for an $[i,j]$-crossing by normalizing the complex of the approximate $[i,j]$-crossing in this paper. We hope to return to this question in a future paper.

Instead of defining the complex for the $[i,j]$-crossing, we consider the following strategy.
(S3') We introduce an approximate \([i, j]-\)crossing composed of \([i, 1]-\)crossings and define a complex for the approximate \([i, j]-\)crossing by a tensor product of complexes associated to \([i, 1]-\)crossings. For a given colored link diagram, we define a normalized Poincaré polynomial of the homology of the approximate link diagram and prove that the polynomial is a link invariant.

Using the above strategy, in Section 4 we define matrix factorizations of MOY diagrams and show isomorphisms between matrix factorizations corresponding to some MOY relations in the homotopy category of matrix factorizations. Note that Sussan [12] and Mazorchuk and Stroppel [8] gave a categorification of MOY diagrams colored from the set \(\{1, \ldots, n\}\) via category \(\cO\). In Section 5 we give a complex of the \([k, 1]-\)crossing in strategy (S2). (We remark that the construction is a generalization of a complex for a \([2, 1]-\)crossing given by Rozansky [11].) Theorem 5.3 is one of the main results.

**Theorem A.** If colored tangle diagrams composed of \([k, 1]-\)crossings are related by a Reidemeister II or III move, then the complexes of matrix factorizations of these diagrams are isomorphic.

A point of this construction is that the boundary map of the complex of a \([k, 1]-\)crossing is described explicitly. Therefore, we can calculate a \((\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})\)-graded homology \(H^{i, j, k}(D)\) for a given colored link diagram \(D\) composed of \([k, 1]-\)crossings.

In Section 6 we introduce the approximate \([i, j]-\)crossing, define a complex of the approximate \([i, j]-\)crossing, and calculate the difference between complexes of colored link diagrams that are related by a Reidemeister I, II, or III move in Theorem 6.5.

The information of Theorem 6.5 is enough to give us a new link invariant for a colored oriented link diagram \(D\). We consider the \((\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})\)-graded homology \(H^{i, j, k}(D)\) through the complex for the approximate diagram of \(D\). Then, we take the Poincaré polynomial of the homology \(H^{i, j, k}(D)\), which we call \(\overline{P}(D)\):

\[
\sum_{i, j, k} t^i q^j s^k \dim_{\mathbb{Q}} H^{i, j, k}(D) \in \mathbb{Q}[t^{\pm 1}, q^{\pm 1}, s]/\langle s^2 - 1 \rangle.
\]

We obtain a link invariant by normalizing the Poincaré polynomial \(\overline{P}(D)\) as follows. Let \(C_r(k)\) \((k = 1, \ldots, n - 1)\) denote the number of \([*, k]-\)crossings
of a colored oriented link diagram $D$. We define a rational function $P(D)$ by

$$P(D) = \prod_{k=1}^{n-1} \frac{1}{([k]_q!)^{C_{rk}(D)}}.$$ 

By Theorem 6.5, we have one of the main results (Theorem 6.7) in Section 6.2.

**Theorem B.** $P(D)$ is an invariant of colored oriented links.

$P(D)$ is the $(\mathfrak{sl}_n, \wedge V_n)$ link invariant if $t$ is specialized to $-1$ and $s$ is specialized to $1$. Therefore, $P(D)$ is a refined link invariant of the $(\mathfrak{sl}_n, \wedge V_n)$ link invariant.

**Remark 1.1.** (1) A power of the parameter $s$ associated to the $\mathbb{Z}_2$-grading is a sum of colorings over each component for a given colored link diagram.

(2) Wu gave a similar result [15, Lemma 13.4]. He defined a morphism from the matrix factorization of $\Gamma_{L,k}$ to the matrix factorization of $\Gamma_{L,k+1}$ ($k = 0, \ldots, j - 1$) and defined the complex of matrix factorization of an $[i,j]$-crossing whose boundaries’ morphisms are associated to these morphisms. The author conjectures that the link invariant $P(D)$, for a colored link diagram $D$, is equal to the Poincaré polynomial of the homology associated to Wu’s complex of matrix factorizations $C^W(D)$. However, he does not have a proof of this claim.

Moreover, we have an interesting question: Is there an isomorphism between Wu’s complex of $[i,j]$-crossing $C^W(Cr_{[i,j]})$ and the complex of the approximate $[i,j]$-crossing $C(Cr_{[i,j]})$ (in Definition 6.4)

$$C(Cr_{[i,j]}) \simeq C^W(Cr_{[i,j]}) \oplus [j]_q!.$$

If such an isomorphism exists, then the above conjecture is obviously true. The complex $C(Cr_{[i,j]})$ has an acyclic direct summand. That is, the complex $C(Cr_{[i,j]})$ is isomorphic to a complex $\overline{M}^\bullet \oplus \overline{A}^\bullet$, where $\overline{M}^\bullet$ is a complex of matrix factorizations that does not have acyclic direct summands and $\overline{A}^\bullet$ is an acyclic direct summand in $C(Cr_{[i,j]})$. It is sufficient to show an isomorphism $\overline{M}^\bullet \simeq C^W(Cr_{[i,j]}) \oplus [j]_q!$. However, it is hard to understand boundary maps in $\overline{M}^\bullet$.

(3) This article is a short version of the author’s Ph.D. thesis [18]. Therefore, the author leaves out detailed calculations of proofs. In the author’s
previous paper [17], propositions in Section 4 are proved in detail. In
the author’s Ph.D. thesis [18], propositions in Sections 5 and 6 are
proved in detail.

§2. $\mathbb{Z}$-graded matrix factorization

In this section, we recall definitions and properties of matrix factorizations
(see [6], [5], [19]).

2.1. $\mathbb{Z}$-graded modules

Let $R = \mathbb{Q}[x_1, \ldots, x_r]$ be a polynomial ring such that the degree $\deg(x_i) \in \mathbb{Z}$
is an even positive integer given for each $i = 1, \ldots, r$. Then, $R$ has a $\mathbb{Z}$-
grading decomposition $\bigoplus i R^i$ such that $R^i R^j \subset R^{i+j}$ and $R^0 = \mathbb{Q}$. We denote
the maximal ideal generated by graded homogeneous polynomials of $R$ by $m$.
We consider a free $R$-module $M$ with a $\mathbb{Z}$-grading decomposition $\bigoplus i M^i$
such that $R^j M^i \subset M^{i+j}$ for any $i \in \mathbb{Z}$.

A $\mathbb{Z}$-grading shift $\{m\} (m \in \mathbb{Z})$ is a functor that shifts the $\mathbb{Z}$-grading by
$m$ on an $R$-module,

$(M\{m\})^i = M^{i-m}$.

For a Laurent polynomial $f(q) = \sum a_i q^i \in \mathbb{N}_{\geq 0}[q, q^{-1}]$, we define $M^{\oplus f(q)}$ by

$$\bigoplus_i (M\{i\})^{\oplus a_i}.$$

2.2. Potential and Jacobian algebra

For a homogeneous $\mathbb{Z}$-graded polynomial $\omega \in R$, we define a quotient
ring $R_\omega$ by $R/I_\omega$, where $I_\omega$ is the ideal generated by partial derivatives
$\partial \omega/\partial x_k$ $(1 \leq k \leq r)$. The quotient ring $R_\omega$ is called the Jacobian algebra of $\omega$.
A homogeneous element $\omega \in m$ is a potential of $R$ if the Jacobian algebra
$R_\omega$ is a finite-dimensional $\mathbb{Q}$-vector space.

2.3. $\mathbb{Z}$-graded matrix factorizations

Assume that the polynomial $\omega$ in $R$ is a potential with of an even homo-
genous $\mathbb{Z}$-grading. The polynomial $\omega$ is allowed to be zero, and $R = \mathbb{Q}$ in
such a case. In this setting, we define a $\mathbb{Z}$-graded matrix factorization with
the potential $\omega$ as follows.

We suppose that a 4-tuple $\mathbb{M} = (M_0, M_1, d_{M_0}, d_{M_1})$ is a two-periodic chain

$$M_0 \xrightarrow{d_{M_0}} M_1 \xrightarrow{d_{M_1}} M_0,$$
where $M_0$ and $M_1$ are $\mathbb{Z}$-graded free $R$-modules (permitted to be infinite rank), and $d_{M_0} : M_0 \to M_1$ and $d_{M_1} : M_1 \to M_0$ are $\mathbb{Z}$-graded homogeneous morphisms (do not assume $\mathbb{Z}$-grade-preserving).

We say that a 4-tuple $\mathcal{M}$ is a $\mathbb{Z}$-graded matrix factorization with a potential $\omega \in m$ (or simply a factorization) if $d_{M_0}$ and $d_{M_1}$ are morphisms with $\mathbb{Z}$-grading $(1/2) \deg \omega$ satisfying $d_{M_1}d_{M_0} = \omega \Id_{M_0}$ and $d_{M_0}d_{M_1} = \omega \Id_{M_1}$.

We define a $\mathbb{Z}$-grading shift $\{ m \}$ ($m \in \mathbb{Z}$) on $\mathcal{M} = (M_0, M_1, d_{M_0}, d_{M_1})$ by

$$\mathcal{M}\{ m \} = (M_0\{ m \}, M_1\{ m \}, d_{M_0}, d_{M_1}).$$

For a Laurent polynomial $f(q) = \sum a_i q^i \in \mathbb{N}_{\geq 0}[q, q^{-1}]$, we define $\mathcal{M}^\oplus f(q)$ by

$$\bigoplus_i (\mathcal{M}\{ i \})^{\oplus a_i}.$$

The translation $\langle 1 \rangle$ changes a factorization $\mathcal{M} = (M_0, M_1, d_{M_0}, d_{M_1})$ into

$$\mathcal{M}\langle 1 \rangle = (M_1, M_0, -d_{M_1}, -d_{M_0}).$$

The translation $\langle 2 \rangle (= \langle 1 \rangle^2)$ is the identity. $\langle 1 \rangle^k$ is denoted by $\langle k \rangle$.

**Definition 2.1.** A matrix factorization $(M_0, M_1, d_0, d_1)$ is finite if $M_0$ and $M_1$ are free $R$-modules of finite rank.

**2.4. The homotopy category of matrix factorizations** $\text{HMF}_{\text{gr}}$

**Definition 2.2.** We define a homotopy category $\text{HMF}_{\text{gr}}^{\text{all}}$ of $\mathbb{Z}$-graded matrix factorizations as follows.

- An object in $\text{HMF}_{\text{gr}}^{\text{all}}$ is a factorization $\mathcal{M} = (M_0, M_1, d_{M_0}, d_{M_1})$ with the potential $\omega$, where $M_0, M_1$ are $R$-modules.

- A morphism in the category $\text{HMF}_{\text{gr}}^{\text{all}}$ from $\mathcal{M} = (M_0, M_1, d_{M_0}, d_{M_1})$ to $\mathcal{N} = (N_0, N_1, d_{N_0}, d_{N_1})$ is a pair $\mathcal{f} = (f_0, f_1)$ of $\mathbb{Z}$-grade-preserving morphisms of $R$-modules satisfying the commutative diagram

$$\begin{array}{ccc}
M_0 & \longrightarrow & M_1 & \longrightarrow & M_0 \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_0 \\
N_0 & \longrightarrow & N_1 & \longrightarrow & N_0
\end{array}$$

up to homotopy.
• The composition \( f \circ g \) of morphisms \( f = (f_0, f_1) \) and \( g = (g_0, g_1) \) is defined by \((f_0 g_0, f_1 g_1)\).

For any matrix factorizations \( \overline{M} \) and \( \overline{N} \), let \( \text{Hom}_{\text{HMF}}(\overline{M}, \overline{N}) \) denote the set of \( \mathbb{Z} \)-grade-preserving morphisms from \( \overline{M} \) to \( \overline{N} \).

**Definition 2.3.** A matrix factorization is contractible if it is isomorphic in \( \text{HMF}^{\text{gr}, \text{all}}_{R, \omega} \) to the zero factorization \((0, 0, 0, 0)\). A factorization is essential if it does not include any contractible factorizations.

**2.5. Cohomology of matrix factorization**

We consider a \( \mathbb{Z} \)-graded polynomial ring \( R = \mathbb{Q}[x_1, \ldots, x_k] \) and its maximal ideal \( m = \langle x_1, \ldots, x_k \rangle \). For a factorization \( \overline{M} = (M_0, M_1, d_0, d_1) \in \text{Ob}(\text{HMF}^{\text{gr}, \text{all}}_{R, \omega}) \), we define a quotient \( \overline{M}/m\overline{M} \) by a two-periodic complex of \( \mathbb{Q} \)-vector spaces

\[
M_0/mM_0 \xrightarrow{d_0} M_1/mM_1 \xrightarrow{d_1} M_0/mM_0.
\]

Let \( H(\overline{M}) = H^0(\overline{M}) \oplus H^1(\overline{M}) \) denote the cohomology of \( \overline{M}/m\overline{M} \), which we call the cohomology of the matrix factorization. We consider a full subcategory whose objects are matrix factorizations with finite-dimensional cohomology of \( \text{HMF}^{\text{gr}, \text{all}}_{R, \omega} \), denoted by \( \text{HMF}^{\text{gr}}_{R, \omega} \). A matrix factorization \( \overline{M} \in \text{Ob}(\text{HMF}^{\text{gr}, \text{all}}_{R, \omega}) \) with finite-dimensional cohomology is a direct sum of an essential finite factorization and a contractible factorization (see [6, Corollary 4]). Therefore, we find that \( \text{HMF}^{\text{gr}}_{R, \omega} \) and the full subcategory of finite matrix factorizations in \( \text{HMF}^{\text{gr}, \text{all}}_{R, \omega} \) are categorically equivalent.

**2.6. Tensor product of matrix factorization**

Let \( X = \{x_1, \ldots, x_r\} \) and \( Y = \{y_1, \ldots, y_s\} \) be two sets of variables. Let \( W = \{w_1, \ldots, w_t\} \) be the set of common variables in \( X \) and \( Y \). We consider \( \mathbb{Z} \)-graded rings generated by \( X = \{x_1, \ldots, x_r\} \), \( Y = \{y_1, \ldots, y_s\} \), and \( W = \{w_1, \ldots, w_t\} \), which we put \( R = \mathbb{Q}[X], R' = \mathbb{Q}[Y], \) and \( S = \mathbb{Q}[W] \). We always take a tensor product of \( R \) and \( R' \) over the ring \( S \) generated by the common variables of \( R \) and \( R' \),

\[
R \otimes S R' = R \otimes_S R'/\{rs \otimes r' - r \otimes sr' \mid r \in R, r' \in R', s \in S\}.
\]

Even if the common variables of \( R \) and \( R' \) are nonempty, we denote a tensor product \( R \otimes S R' \) by \( R \otimes R' \) without notice. For an \( R \)-module \( M \) and an \( R' \)-module \( N \), we also take these tensor products over the ring \( S \),

\[
M \otimes S N = M \otimes_S N/\{ms \otimes n - m \otimes sn \mid m \in M, n \in N, s \in S\}.
\]
For $\overline{M} = (M_0, M_1, d_{M_0}, d_{M_1})$ in $\text{HMF}^{\text{gr}}_{R, \omega}$ and $\overline{N} = (N_0, N_1, d_{N_0}, d_{N_1})$ in $\text{HMF}^{\text{gr}}_{R', \omega'}$, we define a tensor product $\overline{M} \boxtimes \overline{N}$ in $\text{HMF}^{\text{gr}, \text{all}}_{R \otimes R', \omega + \omega'}$ by
\[
\left( \left( \frac{M_0 \otimes N_0}{M_1 \otimes N_1} \right), \left( \frac{M_0 \otimes N_0}{M_1 \otimes N_1} \right), \left( \frac{d_{M_0} - d_{N_1}}{d_{M_1}} \right), \left( \frac{d_{M_1} - d_{N_1}}{d_{M_0}} \right) \right).
\]

We remark that $\overline{M} \boxtimes \overline{N}$ is contractible if either factorization $\overline{M}$ or $\overline{N}$ is contractible. Moreover, the tensor product preserves the condition of finite-dimensional cohomology. Therefore, the tensor product is well defined in $\text{HMF}^{\text{gr}}$. As a bifunctor, the tensor product is viewed
\[
\boxtimes : \text{HMF}^{\text{gr}}_{R, \omega} \times \text{HMF}^{\text{gr}}_{R', \omega'} \longrightarrow \text{HMF}^{\text{gr}}_{R \otimes R', \omega + \omega'}.
\]

**Remark 2.4.** (1) The tensor product $\boxtimes$ is commutative, associative, and compatible with the direct sum. Moreover, there exists a unique unit object for the tensor product (see [20]).

(2) As from here, $\overline{M}_1 \boxtimes \overline{M}_2 \boxtimes \overline{M}_3 \boxtimes \ldots \boxtimes \overline{M}_k$ is defined by $(\ldots((\overline{M}_1 \boxtimes \overline{M}_2) \boxtimes \overline{M}_3) \boxtimes \ldots) \boxtimes \overline{M}_k$:
\[
\overline{M}_1 \boxtimes \overline{M}_2 \boxtimes \overline{M}_3 \boxtimes \ldots \boxtimes \overline{M}_k = (\ldots((\overline{M}_1 \boxtimes \overline{M}_2) \boxtimes \overline{M}_3) \boxtimes \ldots) \boxtimes \overline{M}_k.
\]

We consider a special case of the tensor product of two matrix factorizations. Let $\omega(x_1, \ldots, x_i)$, $\omega'(y_1, \ldots, y_j)$, and $\omega''(z_1, \ldots, z_k)$ be potentials of polynomial rings $R = \mathbb{Q}[x_1, \ldots, x_i]$, $R' = \mathbb{Q}[y_1, \ldots, y_j]$, and $R'' = \mathbb{Q}[z_1, \ldots, z_k]$, respectively. Suppose that we have an object of $\text{HMF}^{\text{gr}, \text{all}}_{R \otimes R', \omega - \omega'}$, denoted by $\overline{M}$, and an object of $\text{HMF}^{\text{gr}, \text{all}}_{R \otimes R''', \omega' - \omega''}$, denoted by $\overline{N}$. The potential of their tensor product $\overline{M} \boxtimes \overline{N}$ is $\omega - \omega''$. However, $\omega - \omega''$ is not a potential of $R \otimes R' \otimes R''$ but is a potential of $R \otimes R''$. Therefore, we consider that the matrix factorization $\overline{M} \boxtimes \overline{N}$ is an object of $\text{HMF}^{\text{gr}, \text{all}}_{R \otimes R''', \omega - \omega''}$. Then, we regard the tensor product as a bifunctor to $\text{HMF}^{\text{gr}, \text{all}}_{R \otimes R'', \omega - \omega''}$ through $\text{HMF}^{\text{gr}, \text{all}}_{R \otimes R' \otimes R'', \omega - \omega''}$:
\[
\boxtimes : \text{HMF}^{\text{gr}, \text{all}}_{R \otimes R', \omega - \omega'} \times \text{HMF}^{\text{gr}, \text{all}}_{R' \otimes R'', \omega' - \omega''} \rightarrow \text{HMF}^{\text{gr}, \text{all}}_{R \otimes R'', \omega - \omega''}.
\]

We have the following proposition.

**Proposition 2.5 ([5, Proposition 13]).** If $\overline{M}$ is a factorization of $\text{HMF}^{\text{gr}}_{R \otimes R', \omega - \omega'}$ and $\overline{N}$ is a factorization of $\text{HMF}^{\text{gr}}_{R' \otimes R'', \omega' - \omega''}$, then the tensor product $\overline{M} \boxtimes \overline{N}$ is also a factorization with finite-dimensional cohomology.
Thus, the tensor product is a bifunctor from $\text{HMF}^{\text{gr}}$ to $\text{HMF}^{\text{gr}}$:

\[
\boxtimes: \text{HMF}^{\text{gr}}_{R \otimes R', \omega - \omega'} \times \text{HMF}^{\text{gr}}_{R' \otimes R'', \omega' - \omega''} \to \text{HMF}^{\text{gr}}_{R \otimes R'', \omega - \omega''}.
\]

2.7. Koszul matrix factorizations

For homogeneous polynomials $a, b$ in a $\mathbb{Z}$-graded polynomial ring $R$ and an $R$-module $M$, we define a matrix factorization $K(a; b)_M$ with the potential $ab$ by

\[
K(a; b)_M := \left( M, M \left\{ \frac{1}{2} (\deg(b) - \deg(a)) \right\}, a, b \right).
\]

For sequences $a = (a_1, a_2, \ldots, a_k)$, $b = (b_1, b_2, \ldots, b_k)$ of homogeneous polynomials in $R$ and an $R$-module $M$, a matrix factorization $K(a; b)_M$ with the potential $\sum_{i=1}^{k} a_ib_i$ is defined by

\[
K(a; b)_M = \bigotimes_{i=1}^{K} K(a_i; b_i)_R \boxtimes (M, 0, 0, 0).
\]

This factorization is called a Koszul matrix factorization (see [6]).

**Theorem 2.6** ([5, Theorem 2.1]). Let $a_i, b_i$, and $b'_i$ ($i = 1, \ldots, m$) be homogeneous polynomials in $R$, and let $M$ be an $R$-module. If $a_1, \ldots, a_m$ form a regular sequence in $R$ and

\[
\sum_{i=1}^{m} a_i b_i = \sum_{i=1}^{m} a_i b'_i,
\]

then there exists an isomorphism

\[
\bigotimes_{j=1}^{m} K(a_j; b_j)_M \simeq \bigotimes_{j=1}^{m} K(a_j; b'_j)_M.
\]

**Corollary 2.7.** We put $R = \mathbb{Q}[x_1, x_2, \ldots, x_k], R_y = R[y]/I$, where $I$ is an ideal generated by a monic polynomial $y_l + \alpha_1 y^{l-1} + \cdots + \alpha_l$ ($\alpha_i \in R$ such that $\deg(\alpha_i) = i \deg(y)$). Let $R$ be the image of $R$ under the obvious inclusion map to $R_y$. Let $a_i$ ($i = 1, \ldots, m$) be a homogeneous polynomial in $R_y$, and let $b_i$ ($i = 2, \ldots, m$) be a homogeneous polynomial in $R$.

(1) Let $b_1, \beta$ be homogeneous polynomials in $R_y$, such that $(y + \beta)b_1 \in \hat{R}$. Assume that these polynomials satisfy the following conditions:

- (i) $(y + \beta)b_1, b_2, \ldots, b_m$ form a regular sequence in $R$,
- (ii) $a_1 b_1(y + \beta) + \sum_{i=2}^{m} a_i b_i(=: \omega)$ is a polynomial in $\hat{R}$.
Then, there exist homogeneous polynomials \( a'_i \in R \) \((i = 1, \ldots, m)\) satisfying the following isomorphism:

\[
K \left( \begin{pmatrix} (y + \beta)a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} ; \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right)_{R_y} \cong K \left( \begin{pmatrix} (y + \beta)a'_1 \\ a'_2 \\ \vdots \\ a'_m \end{pmatrix} ; \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right)_{R_y}.
\]

(2) Let \( b_1 \) and \( \beta \) be homogeneous polynomials in \( R \). Assume that these polynomials satisfy the following conditions:

(i) \( b_1, b_2, \ldots, b_m \) form a regular sequence in \( R \),

(ii) \( a_1b_1(y + \beta) + \sum_{i=2}^m a_ib_i(=: \omega) \) is a polynomial in \( \widehat{R} \).

Then, there exist homogeneous polynomials \( a'_1 \in R_y \) and \( a'_i \in R \) \((i = 2, \ldots, m)\) satisfying \( a'_1(y + \beta) \in \widehat{R} \) and the following isomorphism:

\[
K \left( \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} ; \begin{pmatrix} b_1(y + \beta) \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right)_{R_y} \cong K \left( \begin{pmatrix} a'_1 \\ a'_2 \\ \vdots \\ a'_m \end{pmatrix} ; \begin{pmatrix} b_1(y + \beta) \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right)_{R_y}.
\]

Proof. This corollary is proved by Theorem 2.6 and the equation \( y^l + \alpha_1y^{l-1} + \cdots + \alpha_l = 0 \) in the quotient \( R_y \). \( \square \)

THEOREM 2.8 ([6, Proposition 10], [5, Theorem 2.2]). Let \( R = \mathbb{Q}[x] \), where \( x = (x_1, x_2, \ldots, x_l) \); let \( a_i \) and \( b_i \) \((1 \leq i \leq k)\) be homogeneous polynomials in \( R[y] \), where \( y = (y_1, y_2, \ldots, y_m) \); and let \( M \) be an \( R[y] \)-module (also an \( R \)-module). If \( a = t(a_1, a_2, \ldots, a_k) \) and \( b = t(b_1, b_2, \ldots, b_k) \) satisfy the conditions

(i) \( \sum_{i=1}^k a_i b_i (=: \omega) \in R \);

(ii) there exist homogeneous polynomials \( b_{j_1}(x, y), b_{j_2}(x, y), \ldots, b_{j_r}(x, y) \in R[y] \) such that the sequence \( (b_{j_1}(0, y), b_{j_2}(0, y), \ldots, b_{j_r}(0, y)) \) is regular in \( \mathbb{Q}[y] \),

then there exists the following isomorphism in \( \text{HMF}_{R, \omega}^{\text{gr,all}} \):

\[
K(a; b)_M \cong K_{M/(b_{j_1}, b_{j_2}, \ldots, b_{j_r})M}(a; b). \]
2.8. Complex category over additive category

For an additive category $\mathcal{A}$, let $\text{Kom}^b(\mathcal{A})$ denote the (bounded) complex category over $\mathcal{A}$, let $\mathcal{K}^b(\mathcal{A})$ denote the homotopy category of $\text{Kom}^b(\mathcal{A})$, and let $X^\bullet$ denote a complex in the category

$$
(\cdots \rightarrow d_{cX_i} X_i \rightarrow X_{i-1} \rightarrow \cdots).
$$

A translation functor of a complex category, which we call $[k]$ ($k \in \mathbb{Z}$), changes a complex $X^\bullet$ into $$(X^\bullet[k])^i = X^{i-k}.$$ 

Remark 2.9. This definition of the translation functor is different from the ordinary definition $(X^\bullet[k])^i = X^{i+k}$. This definition matches with the Poincaré polynomial $P(D)$ of $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$-graded homology.

We assume that a category $\mathcal{A}$ has a tensor product structure. For complexes $X^\bullet$ and $Y^\bullet$, we define a tensor product $X^\bullet \otimes Y^\bullet$ as follows:

$$(X^\bullet \otimes Y^\bullet)_k := \bigoplus_{i+j=k} X^i \otimes Y^j,$$

$$d_{c(X^\bullet \otimes Y^\bullet)} = \sum_{i+j=k} (d_{cX_i} \otimes \text{Id}_{Y^j} + (-1)^i \text{Id}_{X^i} \otimes d_{cY^j}).$$

2.9. Complex category of $\mathbb{Z}$-graded matrix factorizations

We consider the complex category $\text{Kom}^b(\text{HMF}^{gr}_{R,\omega})$ and the homotopy category $\mathcal{K}^b(\text{HMF}^{gr}_{R,\omega})$. By Proposition 2.5, we have bifunctors

$$\boxtimes : \text{Kom}^b(\text{HMF}^{gr}_{R \otimes R', \omega - \omega'}) \times \text{Kom}^b(\text{HMF}^{gr}_{R' \otimes R'', \omega' - \omega''}) \rightarrow \text{Kom}^b(\text{HMF}^{gr}_{R \otimes R'', \omega - \omega''}),$$

$$\boxtimes : \mathcal{K}^b(\text{HMF}^{gr}_{R \otimes R', \omega - \omega'}) \times \mathcal{K}^b(\text{HMF}^{gr}_{R' \otimes R'', \omega' - \omega''}) \rightarrow \mathcal{K}^b(\text{HMF}^{gr}_{R \otimes R'', \omega - \omega''}).$$

Finally, we show a proposition for a complex of Koszul factorizations.

Proposition 2.10. Let $a_i, a'_i, b_i$ ($i = 1, \ldots, k$), and $c$ be sequences of homogeneous polynomials in $R$ satisfying that

(C1) $ca_1 b_1 + \sum_{i=2}^k a_i b_i = ca'_1 b_1 + \sum_{i=2}^k a'_i b_i,$

(C2) the sequence $(b_1, \ldots, b_k)$ is regular in $R$. 


Put $S = \bigotimes_{j=2}^{k} K(a_j; b_j)_R$, and put $S' = \bigotimes_{j=2}^{k} K(a'_j; b_j)_R$. We have the following isomorphisms by Corollary 2.7:

$$K(ca_1; b_1)_R \otimes S \xrightarrow{\varphi} K(ca'_1; b_1)_R \otimes S',$$

$$K(a_1; cb)_R \otimes S \xrightarrow{\bar{\psi}} K(a'_1; cb)_R \otimes S'.$$

(i) We have the following commutative diagram of matrix factorizations:

(i) \[\begin{array}{ccc}
K(ca_1; b_1)_R \otimes S & \xrightarrow{(c,1) \otimes \text{Id}_\varphi} & K(a_1; cb)_R \otimes S \{- \deg c\} \\
\downarrow \varphi & & \downarrow \bar{\psi} \\
K(ca'_1; b_1)_R \otimes S' & \xrightarrow{(c,1) \otimes \text{Id}_{\bar{\psi}}'} & K(a'_1; cb)_R \otimes S' \{- \deg c\}. 
\end{array}\]

(ii) We have the following commutative diagram of matrix factorizations:

(ii) \[\begin{array}{ccc}
K(a_1; cb)_R \otimes S & \xrightarrow{(1,c) \otimes \text{Id}_\varphi} & K(ca_1; b_1)_R \otimes S \{- \deg c\} \\
\downarrow \bar{\psi} & & \downarrow \varphi \\
K(a'_1; cb)_R \otimes S' & \xrightarrow{(1,c) \otimes \text{Id}_{\bar{\psi}}'} & K(ca'_1; b_1)_R \otimes S'. 
\end{array}\]

Proof. It suffices to apply the isomorphisms of Theorem 2.6 to the following complex:

$$K\left(\begin{pmatrix} ca_1 \\ a_2 \end{pmatrix}; \begin{pmatrix} cb_1 \\ b_2 \end{pmatrix}\right)_R \xrightarrow{(c,1) \otimes \text{Id}} K\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \begin{pmatrix} cb_1 \\ b_2 \end{pmatrix}\right)_R \{- \deg c\}.$$

By direct calculation of morphism composition, we find that $\varphi \cdot ((c,1) \otimes \text{Id}) \cdot \bar{\psi}^{-1}$ is

$$K\left(\begin{pmatrix} ca'_1 \\ a_2' \end{pmatrix}; \begin{pmatrix} b_1 \\ b_2' \end{pmatrix}\right)_R \xrightarrow{(c,1) \otimes \text{Id}} K\left(\begin{pmatrix} a'_1 \\ a_2' \end{pmatrix}; \begin{pmatrix} cb_1 \\ b_2' \end{pmatrix}\right)_R \{- \deg c\}.$$
§3. Homogeneous polynomial and its generating function

In this section, we give a few special polynomials that are generalized elementary symmetric polynomials and their generating functions. Using these polynomials, we define matrix factorizations for MOY diagrams in Section 4.

3.1. Homogeneous polynomial

We suppose that variables \( t_{1,i}, t_{2,i}, \ldots, t_{m,i} \) where \( i \) is a formal index, have \( \mathbb{Z} \)-grading 2. Let \( T^{(m)}_{(i)} \) be a sequence of \( m \) variables \( t_{l,i} \) \((1 \leq l \leq m)\):

\[
T^{(m)}_{(i)} = (t_{1,i}, t_{2,i}, \ldots, t_{m,i}).
\]

Let \( x_{j,i} = \sum_{1 \leq k_1 < \ldots < k_j \leq m} t_{k_1,i} \cdots t_{k_j,i} \) \((1 \leq j \leq m)\) denote the elementary symmetric polynomials. We find that the \( \mathbb{Z} \)-grading of \( x_{j,i} \) is 2\( j \). In addition, we put \( x_{0,i} = 1 \) for any \( i \). Let \( X^{(m)}_{(i)} \) be a sequence of the elementary symmetric polynomials \( x_{l,i} \) \((1 \leq l \leq m)\):

\[
X^{(m)}_{(i)} = (x_{1,i}, x_{2,i}, \ldots, x_{m,i}).
\]

For a sequence of positive integers \((m_1, m_2, \ldots, m_k)\) and a sequence of indices \((i_1, i_2, \ldots, i_k)\), let \( R^{(m_1,m_2,\ldots,m_k)}_{(i_1,i_2,\ldots,i_k)} \) be a polynomial ring over \( \mathbb{Q} \) generated by symmetric polynomials in sequences \( X^{(m_1)}_{(i_1)}, X^{(m_2)}_{(i_2)}, \ldots, X^{(m_k)}_{(i_k)} \):

\[
R^{(m_1,m_2,\ldots,m_k)}_{(i_1,i_2,\ldots,i_k)} = \mathbb{Q}[x_{1,i_1}, x_{2,i_1}, \ldots, x_{m_1,i_1}, \ldots, x_{1,i_k}, x_{2,i_k}, \ldots, x_{m_k,i_k}].
\]

Let \( s(m) \) be a function that is 1 if \( m \geq 0 \) and \(-1\) if \( m < 0 \). For a sequence of integers \((m_1, m_2, \ldots, m_k)\) and a sequence of indices \((i_1, i_2, \ldots, i_k)\), let \( X^{(m_1,m_2,\ldots,m_k)}_{(i_1,i_2,\ldots,i_k)} \) be a rational function composed of polynomials of \( X^{(m_k)}_{(i_k)} \) \((k = 1, \ldots, l)\),

\[
\prod_{k=1}^{l} (1 + x_{1,i_k} + \cdots + x_{m_k,i_k})^{s(m_k)},
\]

and let \( X^{(m_1,m_2,\ldots,m_l)}_{m,(i_1,i_2,\ldots,i_l)} \) be homogeneous terms with \( \mathbb{Z} \)-grading \( 2m \) of the rational function \( X^{(m_1,m_2,\ldots,m_l)}_{(i_1,i_2,\ldots,i_l)} \). In general, let \( X^{(m_1,m_2,\ldots,m_l)}_{m,(i_1,i_2,\ldots,i_l)} \) denote a sequence of \( X^{(m_1,m_2,\ldots,m_l)}_{m,(i_1,i_2,\ldots,i_l)} \) \((m \in \mathbb{N}_{\geq 1})\):

\[
X^{(m_1,m_2,\ldots,m_l)}_{(i_1,i_2,\ldots,i_l)} = (X^{(m_1,m_2,\ldots,m_l)}_{m,(i_1,i_2,\ldots,i_l)})_{m \in \mathbb{N}_{\geq 1}}.
\]
Remark 3.1. Let $m_j$ $(1 \leq j \leq l)$ be a positive integer.

1. $X^{(m_1,\ldots,m_l)}_{(i_1,\ldots,i_l)}$ is a generating function of elementary symmetric polynomials of variables $T^{(m_1)}_{(i_1)}, \ldots, T^{(m_l)}_{(i_l)}$.

2. $X^{(-m_1,\ldots,-m_l)}_{(i_1,\ldots,i_l)}$ is a generating function of complete symmetric polynomials of variables $T^{(m_1)}_{(i_1)}, \ldots, T^{(m_l)}_{(i_l)}$ up to $\pm 1$.

These polynomials have the following properties.

Proposition 3.2. (1) Let $S_k$ be the symmetric group. For any $\sigma \in S_k$,

$$X^{(m_1,m_2,\ldots,m_k)}_{m,\{i_1,i_2,\ldots,i_k\}} = X^{(m_{\sigma(1)},m_{\sigma(2)},\ldots,m_{\sigma(k)})}_{m,\{i_{\sigma(1)},i_{\sigma(2)},\ldots,i_{\sigma(k)}\}}.$$ 

(2) For any $l \in \{1,2,\ldots,k-1\}$,

$$X^{(m_1,m_2,\ldots,m_k)}_{m,\{i_1,i_2,\ldots,i_k\}} = \sum_{j=0}^{m} X^{(m_1,\ldots,m_l)}_{m-j,\{i_1,\ldots,i_l\}} X^{(m_{l+1},\ldots,m_k)}_{j,\{i_{l+1},\ldots,i_k\}}.$$ 

(3) For any positive integer $m_1$,

$$X^{(-m_1,m_2,\ldots,m_k)}_{m,\{i_1,i_2,\ldots,i_k\}} = X^{(m_2,\ldots,m_k)}_{m,\{i_2,\ldots,i_k\}} - x_{1,i_1} X^{(-m_1,m_2,\ldots,m_k)}_{m-1,\{i_1,i_2,\ldots,i_k\}} - \cdots$$

$$- x_{m_1,i_1} X^{(-m_1,m_2,\ldots,m_k)}_{m-m_1,\{i_1,i_2,\ldots,i_k\}}.$$ 

(4) For any positive integer $m$,

$$\sum_{l=0}^{m} X^{(m_1,\ldots,m_k)}_{m-l,\{i_1,\ldots,i_l\}} X^{(-m_1,\ldots,-m_k)}_{l,\{i_1,\ldots,i_k\}} = 0.$$ 

(5) The ideal in $R^{(m_1,m_2,\ldots,m_k)}_{\{i_1,i_2,\ldots,i_k\}}$ generated by $(X^{(m_1,\ldots,m_k)}_{1,\{i_1,\ldots,i_k\}}, \ldots, X^{(m_1,\ldots,m_k)}_{m,\{i_1,\ldots,i_k\}})$

equals the ideal generated by

$$X^{(m_1,\ldots,m_l)}_{1,\{i_1,\ldots,i_l\}} - X^{(-m_{l+1},\ldots,-m_k)}_{1,\{i_{l+1},\ldots,i_k\}}, \ldots,$$

$$X^{(m_1,\ldots,m_1)}_{m,\{i_1,\ldots,i_l\}} - X^{(-m_{l+1},\ldots,-m_k)}_{m,\{i_{l+1},\ldots,i_k\}}.$$ 

We consider the power sum $t_{1,i}^{n+1} + t_{2,i}^{n+1} + \cdots + t_{m,i}^{n+1}$ in $\mathbb{Q}[t_{1,i},\ldots,t_{m,i}]$. The power sum is represented as a polynomial of the subring $\mathbb{Q}[x_{1,i},\ldots,x_{m,i}]$ generated by the elementary symmetric polynomials, which we call $F_m(x_{1,i}, x_{2,i}, x_{m,i})$ or $F_m(X^{(m)}_{(i)})$ for short:

$$F_m(X^{(m)}_{(i)}) = F_m(x_{1,i}, x_{2,i}, \ldots, x_{m,i}) = t_{1,i}^{n+1} + t_{2,i}^{n+1} + \cdots + t_{m,i}^{n+1}.$$
 Proposition 3.3. (1) We have

\[ F_m(x_1,i, x_2,i, \ldots, x_m,i) = \sum_{k=1}^{m} (-1)^{n+1-k} x_{k,i} X_{n+1-k, (i)}. \]

(2) The sum of \( F_{m_k}(X_{(i_k)}^{(m_k)}) \) \((k = 1, \ldots, j)\) equals \( F_{\sum_{k=1}^{j} m_k} (X_{(i_1,i_2,\ldots,i_j)}^{(m_1,m_2,\ldots,m_j)}) \):

\[ \sum_{k=1}^{j} F_{m_k}(X_{(i_k)}^{(m_k)}) = F_{\sum_{k=1}^{j} m_k} (X_{(i_1,i_2,\ldots,i_j)}^{(m_1,m_2,\ldots,m_j)}). \]

(3) The polynomial \( \sum_{k=1}^{j} F_{m_k}(X_{(i_k)}^{(m_k)}) \) is a potential of \( R_{(i_1,i_2,\ldots,i_j)}^{(m_1,m_2,\ldots,m_j)}. \)

§4. MOY diagrams and matrix factorizations

MOY diagrams represent intertwiners between tensor products of some fundamental representations. The diagrams consist of some elementary planar diagrams colored from the set \( \{1, 2, \ldots, n\} \), which corresponds to the set of the fundamental representations \( \{V_n, \ldots, \wedge^{n-1}V_n\} \) and the trivial representation \( \wedge^n V_n \).

In this section, we give a definition of matrix factorizations of MOY diagrams and show isomorphisms between matrix factorizations corresponding to some MOY relations.

4.1. Potential of MOY diagram

A potential for a MOY diagram is a power sum determined by colorings, orientations of the diagram, and formal indices that we put on ends of the diagram. For a given MOY diagram, we assign a different index \( i \) to each end of the diagram and then assign a power sum to each end as follows. When an edge including an \( i \)-assigned end has a coloring \( m \) and an orientation from inside the diagram to the outside end, we assign

\[ \Lambda^{[m_1,m_2]} \quad V^{[m_1,m_2]} \quad (2 \leq m_3 = m_1 + m_2 \leq n) \]

Figure 2: Elementary MOY diagrams
the polynomial \( +F_m(x_{1,i}, x_{2,i}, \ldots, x_{m,i}) \) to the end, and when an edge has
an opposite orientation from outside to inside, we assign the polynomial
\(-F_m(x_{1,i}, x_{2,i}, \ldots, x_{m,i}). \) The potential of a MOY diagram is defined by the
sum of these assigned polynomials over each end of the diagram.

To each end of the edge with coloring \( m \), we simply assign only a formal
index \( i \) or a sequence of variables \( X^{(m)}(i) \) for convenience (see Figure 3). For
instance, the potential of the diagram in Figure 3 is

\[
F_m(X^{(m)}(1)) + F_m(X^{(m)}(2)) - F_m(X^{(m)}(3)) + F_m(X^{(m)}(4)) + F_m(X^{(m)}(5)).
\]

4.2. Elementary MOY diagrams and matrix factorizations

For elemental pieces of MOY diagrams (see Figure 2), we define matrix
factorizations.

We consider elementary MOY diagram \( L^{[m]}_{(1;2)} \) in Figure 4.

**Definition 4.1.** For the diagram \( L^{[m]}_{(1;2)} \), we define a matrix factorization
\( \overline{L^{[m]}_{(1;2)}} \) by

\[
\bigotimes_{j=1}^{m} K \left( L^{[m]}_{j,(1;2)} : X^{(m)}_{j,(1)} - X^{(m)}_{j,(2)} \right) R^{(m,m)}_{(1,2)},
\]

Figure 4: Elementary MOY diagram \( L^{[m]}_{(1;2)} \)
Figure 5: MOY diagrams $\Lambda^{[m_1,m_2]}_{(3,1,2)}$ and $V^{[m_1,m_2]}_{(1,2,3)}$

where $L^{[m]}_{j,(1;2)}$ is the polynomial

\[
\begin{align*}
&\left(F_{m}(X^{(m)}_{1,(2)}, \ldots, X^{(m)}_{j-1,(2)}, X^{(m)}_{j,(1)}, \ldots, X^{(m)}_{m,(1)}) \\
&\quad - F_{m}(X^{(m)}_{1,(2)}, \ldots, X^{(m)}_{j,(2)}, X^{(m)}_{j+1,(1)}, \ldots, X^{(m)}_{m,(1)})\right) \\
&/(X^{(m)}_{j,(1)} - X^{(m)}_{j,(2)}).
\end{align*}
\]

**Remark 4.2.** For $m \geq n + 1$, we can consider a matrix factorization $L^{[m]}_{j,(1;2)}$ as the above definition. However, we find that such matrix factorizations are contractible.

We consider MOY diagrams $\Lambda^{[m_1,m_2]}_{(3,1,2)}$ and $V^{[m_1,m_2]}_{(1,2,3)}$ in Figure 5.

**Definition 4.3.** For the diagram $\Lambda^{[m_1,m_2]}_{(3,1,2)}$, we define a matrix factorization $\Lambda^{[m_1,m_2]}_{(3,1,2)}$ by

\[
\begin{align*}
&\bigotimes_{j=1}^{m_3} K(\Lambda^{[m_1,m_2]}_{j,(3,1,2)}, X^{(m_3)}_{j,(3)} - X^{(m_1,m_2)}_{j,(1,2)}) R^{(m_1,m_2,m_3)}_{(1,2,3)},
\end{align*}
\]

where $\Lambda^{[m_1,m_2]}_{j,(3,1,2)}$ is the polynomial

\[
\begin{align*}
&\left(F_{m_3}(\ldots, X^{(m_1,m_2)}_{j-1,(1,2)}, X^{(m_3)}_{j,(3)}, X^{(m_3)}_{j+1,(3)}, \ldots) \\
&\quad - F_{m_3}(\ldots, X^{(m_1,m_2)}_{j-1,(1,2)}, X^{(m_1,m_2)}_{j,(1,2)}, X^{(m_3)}_{j,(3)}, \ldots)\right) \\
&/(X^{(m_3)}_{j,(3)} - X^{(m_1,m_2)}_{j,(1,2)}).
\end{align*}
\]

For the diagram $V^{[m_1,m_2]}_{(1,2,3)}$, we define a matrix factorization $V^{[m_1,m_2]}_{(1,2,3)}$ by

\[
\begin{align*}
&\bigotimes_{j=1}^{m_3} K(V^{[m_1,m_2]}_{j,(1,2,3)}, X^{(m_1,m_2)}_{j,(1,2)} - X^{(m_3)}_{j,(3)}) R^{(m_1,m_2,m_3)}_{(1,2,3)} \{-m_1 m_2\},
\end{align*}
\]
where $V_{j,(1,2;3)}^{[m_1,m_2]}$ is the polynomial
\[
(F_{m_3}(\ldots, X_{j-1,(3)}^{(m_3)}, X_j^{(m_1,m_2)} , X_j^{(m_1,m_2)} , \ldots) \\
- F_{m_3}(\ldots, X_{j-1,(3)}^{(m_3)}, X_j^{(m_3)}, X_j^{(m_1,m_2)} , \ldots)) \\
/ (X_{j,(1,2)}^{(m_1,m_2)} - X_{j,(3)}^{(m_3)}).
\]

**Remark 4.4.** For $m_3 \geq n + 1$, we can consider matrix factorizations $L^{[m_1,m_2]}_{(1,2)}$ and $V^{[m_1,m_2]}_{(1,2;3)}$ as the above definition. However, we find that such factorizations are contractible.

### 4.3. Glued MOY diagrams and matrix factorizations

For a MOY diagram $\Gamma$, we define a matrix factorization by gluing matrix factorizations $\Lambda^{[m_1,m_2]}_{(3;1,2)}$ and $V^{[m_1,m_2]}_{(1,2;3)}$. Let $\mathcal{C}(\Gamma)_n$ denote a matrix factorization for $\Gamma$.

**Definition 4.5.** For the diagram $\Gamma$ composed of a disjoint union of two diagrams $\Gamma_1$ and $\Gamma_2$, we define a matrix factorization $\mathcal{C}(\Gamma)$ by the tensor product of $\mathcal{C}(\Gamma_1)_n$ and $\mathcal{C}(\Gamma_2)_n$:

$$\mathcal{C}(\Gamma)_n := \mathcal{C}(\Gamma_1)_n \otimes \mathcal{C}(\Gamma_2)_n.$$

We consider two MOY diagrams that have a consistently oriented common $m$-colored edge (see the left and the middle diagrams in Figure 6). These diagrams $\Gamma_L$ and $\Gamma_R$ are glued at the markings $\{1\}$ and $\{2\}$, and then we obtain the diagram $\Gamma_G$ in Figure 6.

**Definition 4.6.** Let $\omega + F_m(X^{(m)}_{(1)})$ be a potential of $\Gamma_L$, and let $\omega' - F_m(X^{(m)}_{(2)})$ be a potential of $\Gamma_R$. We put $\mathcal{C}(\Gamma_L)_n$ the factorization of $\Gamma_L$ in $\text{Ob}(\text{HMF}^{\text{gr}}_{R^{(m_1,\ldots,m_k,m)}_{(i_1,\ldots,i_k,1)}}, \omega + F_m(X^{(m)}_{(1)})$, and we put $\mathcal{C}(\Gamma_R)_n$ the factorization $\Gamma_R$.
We find that $C(\Gamma_G)_n$ has finite-dimensional cohomology by Proposition 2.5. Therefore, $C(\Gamma_G)_n$ is an object of $\text{HMF}^{\text{gr}}_{R(m_1,\ldots,m_k,m'_1,\ldots,m'_l),(i_1,\ldots,i_k,i'_1,\ldots,i'_l)},\omega+\omega'$. We consider the MOY diagram $\Gamma_T$ and the diagram $\Gamma_C$ obtained by joining ends of edges with the same coloring (see Figure 7).

**Definition 4.7.** In this case let $\omega + F_m(X^{(m)}_{(1)}) - F_m(X^{(m)}_{(2)})$ be a potential of the diagram $\Gamma_T$. For factorization $C(\Gamma_T)_n$ in $\text{Ob}(\text{HMF}^{\text{gr}}_{R(m_1,\ldots,m_k,m'_1,\ldots,m'_l),(i_1,\ldots,i_k,i'_1,\ldots,i'_l)},\omega+\omega')$, a matrix factorization of the diagram $\Gamma_C$ is defined by

$$C(\Gamma_C)_n := C(\Gamma_T)_n|_{X^{(m)}_{(2)}=X^{(m)}_{(1)}}.$$ 

Here, $C(\Gamma_C)_n$ has finite-dimensional cohomology. Therefore, $C(\Gamma_C)_n$ is an object of $\text{HMF}^{\text{gr}}_{R(m_1,\ldots,m_k),(i_1,\ldots,i_k),\omega'}$.

We find that a glued matrix factorization loses potentials at glued ends. Therefore, a potential as a matrix factorization associated to a MOY diagram is compatible with the potential of the diagram.

**Proposition 4.8.** A matrix factorization of a MOY diagram is independent of a decomposition of the diagram in $\text{HMF}^{\text{gr}}$.

### 4.4. MOY relations and isomorphisms between factorizations

We show isomorphisms between factorizations corresponding to MOY relations. For a sequence of integers $(m_1,\ldots,m_k)$, a sequence of indices
(i_1, \ldots, i_k), and \epsilon_j \in \{1, -1\} (j = 1, \ldots, k), let \omega_{(i_1, \ldots, i_k)} denote

\begin{equation}
\sum_{j=1}^{k} \epsilon_j f_{m_j}(X_{(i_j)}^{(m_j)}).
\end{equation}

**Proposition 4.9.** (1) We have the following isomorphisms in the homotopy category \(HMF_{\text{gr}}^{R(1, 2, 3, 4)}(m_1, m_2, m_3, m_4) \rightarrow \omega(1, 2, 3, 4)\):

\[\mathcal{C} \left( \begin{array}{c}
X_{(1)}^{(m_1)} \\
X_{(2)}^{(m_2)} \\
X_{(3)}^{(m_3)} \\
X_{(4)}^{(m_4)}
\end{array} \right) \approx \mathcal{C} \left( \begin{array}{c}
X_{(1)}^{(m_1)} \\
X_{(2)}^{(m_2)} \\
X_{(3)}^{(m_3)} \\
X_{(4)}^{(m_4)}
\end{array} \right) \approx \mathcal{C} \left( \begin{array}{c}
X_{(1)}^{(m_1)} \\
X_{(2)}^{(m_2)} \\
X_{(3)}^{(m_3)} \\
X_{(4)}^{(m_4)}
\end{array} \right) \]

(2) We have the following isomorphisms in \(HMF_{\text{gr}}^{R(1, 2, 3, 4)}(m_1, m_2, m_3, m_4) \rightarrow \omega(1, 2, 3, 4)\):

\[\mathcal{C} \left( \begin{array}{c}
X_{(1)}^{(m_1)} \\
X_{(2)}^{(m_2)} \\
X_{(3)}^{(m_3)} \\
X_{(4)}^{(m_4)}
\end{array} \right) \approx \mathcal{C} \left( \begin{array}{c}
X_{(1)}^{(m_1)} \\
X_{(2)}^{(m_2)} \\
X_{(3)}^{(m_3)} \\
X_{(4)}^{(m_4)}
\end{array} \right) \approx \mathcal{C} \left( \begin{array}{c}
X_{(1)}^{(m_1)} \\
X_{(2)}^{(m_2)} \\
X_{(3)}^{(m_3)} \\
X_{(4)}^{(m_4)}
\end{array} \right) \]

where \(1 \leq m_1, m_2, m_3 \leq n - 2; m_5 = m_1 + m_2 \leq n - 1; m_6 = m_2 + m_3 \leq n - 1; \)

and \(m_4 = m_1 + m_2 + m_3 \leq n\).

**Proposition 4.10.** (1) There exists an isomorphism in \(HMF_{\mathbb{Q}, 0}^{R(1, 2, 3, 4)}(m_1, m_2, m_3, m_4) \rightarrow \omega(1, 2, 3, 4)\)

\[\mathcal{C} \left( \begin{array}{c}
X_{(1)}^{(m_1)} \\
X_{(2)}^{(m_2)} \\
X_{(3)}^{(m_3)} \\
X_{(4)}^{(m_4)}
\end{array} \right) \approx (J_{F_m(X_{(1)}^{(m)})}, 0, 0) \{-mn + m^2\} \langle m \rangle;\]

where \(J_{F_m(X_{(1)}^{(m)})}\) is the Jacobian algebra for the polynomial \(F_m(X_{(1)}^{(m)})\):

\[J_{F_m(X_{(1)}^{(m)})} = R_{(1)}^{(m)} / \left\langle \frac{\partial F_m}{\partial x_{1,1}}, \ldots, \frac{\partial F_m}{\partial x_{m,1}} \right\rangle.\]
(2) There exists an isomorphism in $\text{HMF}^{(m_3,m_3)}_{R_{(1,2)},\omega_{(1,2)}}$ \[ C \left( \begin{array}{c} m_1 \\ m_2 \\ m_3 \\ n \\ 1 \\ 2 \\ m_3 \\ m_2 \end{array} \right) \cong C \left( \begin{array}{c} 1 \\ m_3 \\ n \\ 1 \\ 2 \\ m_3 \\ m_2 \end{array} \right) \oplus \left[ \frac{m_3}{m_1} \right]_q \langle m_2 \rangle, \]

(3) There exists an isomorphism in $\text{HMF}^{(m_3,m_1)}_{R_{(1,2)},\omega_{(1,2)}}$ \[ C \left( \begin{array}{c} m_3 \\ m_2 \\ m_1 \\ n \\ 1 \\ 2 \\ m_3 \\ m_2 \end{array} \right) \cong C \left( \begin{array}{c} 1 \\ m_1 \\ n \\ 1 \\ 2 \\ m_3 \\ m_2 \end{array} \right) \oplus \left[ \frac{n-m_1}{m_2} \right]_q \langle m_2 \rangle, \]

where $1 \leq m_1, m_2 \leq n - 1$, and $m_3 = m_1 + m_2 \leq n$.

**Remark 4.11.** The Jacobian algebra $J_{F_m(X^{(m)}_{(1)})}$ is isomorphic to the cohomology ring of the complex Grassmannian $\text{Gr}(m,n)$ as a graded algebra [4].

The cohomology ring of $\text{Gr}(m,n)$ is isomorphic to $H(\text{Gr}(m,n)) = \mathbb{Q}[e_1,\ldots,e_m]/\langle h_{n+1-m},\ldots,h_n \rangle,$ where $h_i$ is the Jacobi-Trudi determinant

$$\begin{vmatrix} e_1 & e_2 & \cdots \\ 1 & e_1 & \cdots \\ 0 & 1 & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & 1 & e_1 \end{vmatrix}.$$ 

On the other hand, we find that $\frac{\partial F_m(X^{(m)}_{(1)})}{\partial x_{i1}}$ is the $(n + 1 - i)$th complete symmetric function of $T^{(m)}_{(1)}$ up to $(-1)^n$. Since $x_{1,1},\ldots,x_{m,1}$ are the elementary symmetric functions of $T^{(m)}_{(1)}$, the polynomial $\frac{\partial F_m(X^{(m)}_{(1)})}{\partial x_{i1}}$ is represented as the Jacobi-Trudi determinant of $x_{1,1},\ldots,x_{m,1}$. Therefore, the Jacobian algebra $J_{F_m(X^{(m)}_{(1)})}$ is naturally isomorphic to the cohomology ring $H(\text{Gr}(m,n))$. 

Proof of Proposition 4.9. (1) By Theorem 2.8, the left-hand factorization is isomorphic to

\[ \bigotimes_{j=1}^{m_4} K(\Lambda^{[m_5,m_3]}_{j,(4;5,3)}, x_j - X^{(m_5,m_3)}_{j,(5,3)}) Q, \]

where \( Q = R^{(m_1,m_2,m_3,m_4,m_5)}_{(1,2,3,4,5)}/\langle x_1, x_2, \ldots, x_{m_5} \rangle \). In the quotient \( Q \), the polynomial \( \Lambda^{[m_5,m_3]}_{j,(4;5,3)} \) equals

\[
\left( F_{m_4}(\ldots, X^{(m_1,m_2,m_3)}_{j-1,(1,2,3)}, x_j, x_{j+1}, \ldots) - F_{m_4}(\ldots, X^{(m_1,m_2,m_3)}_{j-1,(1,2,3)}, x_j, x_{j+1}, \ldots) \right) /
\left( x_j - X^{(m_1,m_2,m_3)}_{j,(1,2,3)} \right).
\]

We denote this polynomial by \( \Lambda^{[m_1,m_2,m_3]}_{j,(4;1,2,3)} \). Since the quotient \( Q \) is isomorphic to \( R^{(m_1,m_2,m_3,m_4)}_{(1,2,3,4)} \), (7) is isomorphic to the middle factorization of Proposition 4.9(1):

\[ \bigotimes_{j=1}^{m_4} K(\Lambda^{[m_1,m_2,m_3]}_{j,(4;1,2,3)}, x_j - X^{(m_1,m_2,m_3)}_{j,(1,2,3)}) R^{(m_1,m_2,m_3,m_4)}_{(1,2,3,4)}. \]

In a similar way, we find that the right-hand factorization of Proposition 4.9(1) is isomorphic to (8).

We can prove Proposition 4.9(2) in a similar way.

Proof of Proposition 4.10. (1) A matrix factorization of an \( i \)-colored loop is

\[ \bigotimes_{j=1}^{m} \left( R^{(m)}_{(1)} R^{(m)}_{(1)} \{ 2j - 1 - n \}, \frac{\partial F_m(X_{m,1})}{\partial x_{j,1}} \right), 0 \right). \]

By Theorem 2.8, (9) is isomorphic to

\[ (J_{F_m(X_{m,1})}, 0, 0, 0) \{ -mn + m^2 \} \langle m \rangle. \]

(2) By Theorem 2.8, the left-hand factorization is isomorphic to

\[ \bigotimes_{j=1}^{m_3} K(\Lambda^{[m_1,m_2]}_{j,(1;3,4)}, x_{j,1} - x_{j,2}) Q' \{ -m_1 m_2 \}, \]
where \( Q' = R_{(1,2,3,4)}^{m_3,m_3,m_1,m_2} / \langle X_{1,(3,4)}^{(m_1,m_2)} - x_{1,2}, \ldots, X_{m_3,(3,4)}^{(m_1,m_2)} - x_{m_3,2} \rangle \). In the quotient \( Q' \), the polynomial \( \Lambda_{j,(3;2,4)}^{m_1,m_2} \) is equal to \( L_{j,(1;2)}^{m_3} \). We have an isomorphism of \( R_{(1,2)}^{m_3,m_3} \)-modules

\[
Q' \{ -m_1 m_2 \} \simeq (R_{(1,2)}^{m_3,m_3}) \oplus \left[ m_1 \right]_q.
\]

Thus, we obtain the isomorphism of Proposition 4.10(2).

(3) By Theorem 2.8, the left-hand factorization is isomorphic to

\[
\bigotimes_{j=1}^{m_3} K(\Lambda_{j,(3;2,4)}^{m_1,m_2}; X_{j,(1,4)}^{(m_1,m_2)} - X_{j,(2,4)}^{(m_1,m_2)}) R_{(1,2)}^{m_1,m_1,m_2} \{-m_1 m_2\},
\]

where

\[
\Lambda_{j,(3;2,4)}^{m_1,m_2} = \frac{F_{m_3}(\ldots, X_{j-1,(2,4)}^{(m_1,m_2)}, X_{j,(1,4)}^{(m_1,m_2)}, \ldots) - F_{m_3}(\ldots, X_{j,(2,4)}^{(m_1,m_2)}, X_{j+1,(1,4)}^{(m_1,m_2)}, \ldots)}{X_{j,(1,4)}^{(m_1,m_2)} - X_{j,(2,4)}^{(m_1,m_2)}}.
\]

The polynomials \( \langle X_{m_1+1,(1,4)}^{(m_1,m_2)} - X_{m_1+1,(2,4)}^{(m_1,m_2)}, \ldots, X_{m_3,(1,4)}^{(m_1,m_2)} - X_{m_3,(2,4)}^{(m_1,m_2)} \rangle \) are described as a linear sum of the polynomials \( \langle X_{1,(1,4)}^{(m_1,m_2)} - X_{1,(2,4)}^{(m_1,m_2)}, \ldots, X_{1,(1,4)}^{(m_1,m_2)} - X_{1,(2,4)}^{(m_1,m_2)} \rangle \). Then, by Theorems 2.6 and 2.8, (10) is isomorphic to

\[
\bigotimes_{j=1}^{m_3} K(\lambda_{j,(1;2)}^{m_1}; x_{j,1} - x_{j,2}) R_{(1,2)}^{m_1,m_1} \otimes (Q'', 0, 0, 0)^{m_2},
\]

where \( Q'' \) is the \( R_{(1,2)}^{m_1,m_1} \)-module \( R_{(1,2)}^{m_1,m_1,m_2} / \langle \Lambda_{m_1+1,(3;2,4)}^{m_1,m_2}, \ldots, \Lambda_{m_3,(3;2,4)}^{m_1,m_2} \rangle \).

Since \( Q'' \) is isomorphic to \( (R_{(1,2)}^{m_1,m_1}) \oplus \left[ m_1 \right]_q \) as an \( R_{(1,2)}^{m_1,m_1} \)-module, we obtain the isomorphism of Proposition 4.10(3).

**Proposition 4.12.** (1) We have that there exist isomorphisms in \( \text{HMFM}_{(1,2,3,4)}^{\text{GR}} \)

\[
C \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & m & m-1 & m \end{pmatrix}_n \simeq C \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & m \end{pmatrix}_n \oplus C \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & m \end{pmatrix}_n \oplus \left[ m-1 \right]_q
\]
(2) There exist isomorphisms in $\text{HMF}^{gr}_{R(1,2,3,4)}((-1,m,1,-m))$, $\omega(1,2,3,4)$

\[
\mathcal{C} \left( \begin{array}{cccc}
1 & m+1 \\
1 & m \\
m & 1 \\
m+1 & m \\
1 & 4 \\
3 & 4 \\
\end{array} \right) \simeq \mathcal{C} \left( \begin{array}{cccc}
1 & m+1 \\
1 & m \\
m & 1 \\
m+1 & m \\
1 & 4 \\
3 & 4 \\
\end{array} \right) \oplus \mathcal{C} \left( \begin{array}{cccc}
1 & m+1 \\
1 & m \\
m & 1 \\
m+1 & m \\
1 & 4 \\
3 & 4 \\
\end{array} \right) \oplus [n-m-1]_q \langle 1 \rangle.
\]

Proof. We will show the isomorphisms in the proof of Theorem 5.3 (see Remarks 5.5 and 7.1).

§ 5. Complexes of matrix factorizations for $[1,k]$-crossing

In this section, we define complexes of matrix factorizations for a $[1,k]$-crossing and a $[k,1]$-crossing ($k = 1, \ldots, n-1$), and we show that there exist isomorphisms corresponding to Reidemeister II and III moves composed of $[1,k]$-crossings and $[k,1]$-crossings. Note that the definition of complexes of matrix factorizations for a $[1,k]$-crossing and a $[k,1]$-crossing is a generalization of a complex of matrix factorizations for a $[1,2]$-crossing given by Rozansky [11].

In the state model of the $(\mathfrak{sl}_n, \land \mathcal{V}_n)$ link invariant, (see [9]), the $[1,k]$-crossings and $[k,1]$-crossings (see Figure 8) are expanded into a linear sum as follows:

\[
\langle \begin{array}{c}
1 \\
k \\
\end{array} \rangle_n = (-1)^{1-k} q^{k-1} \langle \begin{array}{c}
1 \\
k \\
\end{array} \rangle_n + (-1)^{-k} q^{kn} \langle \begin{array}{c}
k+1 \\
1 \\
\end{array} \rangle_n,
\]

\[
\langle \begin{array}{c}
1 \\
k \\
\end{array} \rangle_n = (-1)^{k-1} q^{-k+1} \langle \begin{array}{c}
1 \\
k \\
\end{array} \rangle_n + (-1)^{-k} q^{-kn} \langle \begin{array}{c}
k+1 \\
1 \\
\end{array} \rangle_n,
\]

\[
\langle \begin{array}{c}
1 \\
k \\
\end{array} \rangle_n = (-1)^{1-k} q^{k-1} \langle \begin{array}{c}
1 \\
k \\
\end{array} \rangle_n + (-1)^{-k} q^{kn} \langle \begin{array}{c}
k+1 \\
1 \\
\end{array} \rangle_n,
\]

\[
\langle \begin{array}{c}
1 \\
k \\
\end{array} \rangle_n = (-1)^{k-1} q^{-k+1} \langle \begin{array}{c}
1 \\
k \\
\end{array} \rangle_n + (-1)^{-k} q^{-kn} \langle \begin{array}{c}
k+1 \\
1 \\
\end{array} \rangle_n.
\]
5.1. Complex for colored tangle diagram with $[1,k]$-crossings

First, we consider diagrams $\Gamma^{[1,k]}_{L,0}$ and $\Gamma^{[1,k]}_{L,1}$ appearing in the state model for $[1,k]$-crossings (see Figure 9).

By Theorem 2.6, the factorization of $\Gamma^{[1,k]}_{L,0}$ $(1 \leq k \leq n - 1)$ is isomorphic to

$$\mathcal{N}^{[1,k]}_{(1,2,3,4)} := S^{[1,k]}_{(1,2,3,4)} \otimes K(u^{[1,k]}_{1,2,3,4})(x_{1,1} - x_{1,4}); X^{(k,-1)}_{k,(2,4)} \circ K(1,2,3,4) \{ -k + 1 \},$$

and the factorization of $\Gamma^{[1,k]}_{L,1}$ $(1 \leq k \leq n - 1)$ is isomorphic to

$$\mathcal{N}^{[1,k]}_{(1,2,3,4)} := S^{[1,k]}_{(1,2,3,4)} \otimes K(u^{[1,k]}_{1,2,3,4})(x_{1,1} - x_{1,4})X^{(k,-1)}_{k,(2,4)} \circ K(1,2,3,4) \{ -k \},$$

where

$$S^{[1,k]}_{(1,2,3,4)} = \bigotimes_{j=1}^{k} K(A^{[1,k]}_{j,(1,2,3,4)}; X^{(1,k)}_{j,(1,2)} - X^{(1,k)}_{j,(3,4)}) \circ K(1,2,3,4) \{ -k \},$$

$$A^{[1,k]}_{j,(1,2,3,4)} = u^{[1,k]}_{j,(1,2,3,4)} - (-x_{1,4})^{k+1-j} u^{[1,k]}_{k+1,1,(1,2,3,4)} \quad (1 \leq j \leq k),$$

$$u^{[1,k]}_{j,(1,2,3,4)}$$

$$= \frac{F_{k+1}(..., X^{(1,k)}_{j-1,(3,4)}, X^{(1,k)}_{j,(1,2)}),..., F_{k+1}(..., X^{(1,k)}_{j,(3,4)}, X^{(1,k)}_{j+1,(1,2)}, ...)}{X^{(1,k)}_{j,(1,2)} - X^{(1,k)}_{j,(3,4)}}.$$

We have two $\mathbb{Z}$-grade-preserving morphisms between these matrix factorizations $\mathcal{M}^{[1,k]}_{(1,2,3,4)}$ and $\mathcal{N}^{[1,k]}_{(1,2,3,4)},$

$$\text{Id}_{\mathcal{N}^{[1,k]}_{(1,2,3,4)}} \otimes (1, x_{1,1} - x_{1,4}) : \mathcal{M}^{[1,k]}_{(1,2,3,4)} \longrightarrow \mathcal{N}^{[1,k]}_{(1,2,3,4)} \{ -1 \},$$

Figure 9: Diagrams $\Gamma^{[1,k]}_{L,0}$ and $\Gamma^{[1,k]}_{L,1}$ assigned indices
\begin{align*}
(12) \quad \text{Id}_{S^{[1,k]}_{(1,2,3,4)}} \boxdot (x_{1,1} - x_{1,4}, 1) : \overline{N}^{[1,k]}_{(1,2,3,4)} & \longrightarrow \overline{M}^{[1,k]}_{(1,2,3,4)} \{ -1 \}.
\end{align*}

**Remark 5.1.** We have
\begin{align*}
\dim \mathbb{Q} \text{Hom}_{\text{HMF}}(\overline{M}^{[1,k]}_{(1,2,3,4)}, \overline{N}^{[1,k]}_{(1,2,3,4)} \{ -1 \}) &= 1, \\
\dim \mathbb{Q} \text{Hom}_{\text{HMF}}(\overline{N}^{[1,k]}_{(1,2,3,4)}, \overline{M}^{[1,k]}_{(1,2,3,4)} \{ -1 \}) &= 1.
\end{align*}

**Definition 5.2.** We define complexes of matrix factorizations for \([k,1]\)-crossings and \([1,k]\)-crossings as follows:

\begin{align*}
C \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} := 0 & \longrightarrow \overline{M}^{[1,k]}_{(1,2,3,4)} \{ kn \} \langle k \rangle \quad \chi^{[1,k]}_{\pm,(1,2,3,4)} \quad \overline{N}^{[1,k]}_{(1,2,3,4)} \{ kn - 1 \} \langle k \rangle \rightarrow 0, \\
& \quad k - 1 \quad k
\end{align*}

\begin{align*}
C \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} := 0 & \longrightarrow \overline{N}^{[1,k]}_{(1,2,3,4)} \{ 1 - kn \} \langle k \rangle \quad \chi^{[1,k]}_{\pm,(1,2,3,4)} \quad \overline{M}^{[1,k]}_{(1,2,3,4)} \{ -kn \} \langle k \rangle \rightarrow 0, \\
& \quad k - 1 \quad k
\end{align*}

\begin{align*}
C \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} := 0 & \longrightarrow \overline{M}^{[1,k]}_{(2,1,4,3)} \{ kn \} \langle k \rangle \quad \chi^{[1,k]}_{\pm,(2,1,4,3)} \quad \overline{N}^{[1,k]}_{(2,1,4,3)} \{ kn - 1 \} \langle k \rangle \rightarrow 0, \\
& \quad k - 1 \quad k
\end{align*}

\begin{align*}
C \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} := 0 & \longrightarrow \overline{N}^{[1,k]}_{(2,1,4,3)} \{ 1 - kn \} \langle k \rangle \quad \chi^{[1,k]}_{\pm,(2,1,4,3)} \quad \overline{M}^{[1,k]}_{(2,1,4,3)} \{ -kn \} \langle k \rangle \rightarrow 0,
\end{align*}

where
\begin{align*}
\chi^{[1,k]}_{\pm,(1,2,3,4)} &:= \text{Id}_{S^{[1,k]}_{(1,2,3,4)}} \boxdot (1, x_{1,1} - x_{1,4}), \\
\chi^{[1,k]}_{\pm,(2,1,4,3)} &:= \text{Id}_{S^{[1,k]}_{(2,1,4,3)}} \boxdot (x_{1,1} - x_{1,4}, 1).
\end{align*}
For a given tangle diagram composed of \([k, 1]\)-crossings and \([1, k]\)-crossings, we decompose the tangle diagram into \([k, 1]\)-crossings, \([1, k]\)-crossings, and colored lines using markings and then assign different indices to the markings and ends of the diagram. Then, we define a complex of matrix factorizations for the tangle diagram by a tensor product of complexes for \([k, 1]\)-crossings, \([1, k]\)-crossings, and colored lines in the decomposition.

5.2. Invariance under Reidemeister moves

In the following section, we show one of the main results, which is a generalization of Khovanov and Rozansky [6, Theorem 2].

**Theorem 5.3.** If tangle diagrams composed of \([k, 1]\)-crossings and \([1, k]\)-crossings are related by a Reidemeister II or III move, then the complexes associated to the diagrams are isomorphic in \(K^b(\text{HMF}^{gr})\). That is, we have the following isomorphisms:

\[
\begin{align*}
(IIa_{1k}) & \quad C \left( \begin{array}{c} 1 \\ k \end{array} \right) \cong C \left( \begin{array}{c} 1 \\ k \end{array} \right) \cong C \left( \begin{array}{c} 1 \\ k \end{array} \right), \\
(IIb_{1k}) & \quad C \left( \begin{array}{c} 1 \\ k \end{array} \right) \cong C \left( \begin{array}{c} 1 \\ k \end{array} \right) \cong C \left( \begin{array}{c} 1 \\ k \end{array} \right), \\
(III_{11k}) & \quad C \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ k \end{array} \right) \cong C \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ k \end{array} \right) \cong C \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ k \end{array} \right).
\end{align*}
\]
5.3. Proof of invariance under Reidemeister IIa and IIb moves

We have

\[
(13)
\]

\[
\begin{pmatrix}
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 6
\end{array}
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
-1 & 0 & 1 \\
\vdots & \vdots & \vdots
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\bar{\phi}_1 \\
\bar{\phi}_2 \\
\bar{\phi}_3 \\
\bar{\phi}_4
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
M_{00}(1) \\
M_{10} \oplus M_{01} \\
M_{11}(-1)
\end{pmatrix},
\]

where

\[
M_{00} = N_{(1,2,5,6)}^{[1,k]} \otimes M_{(6,5,4,3)}^{[1,k]},
\]

\[
M_{01} = N_{(1,2,5,6)}^{[1,k]} \otimes N_{(6,5,4,3)}^{[1,k]},
\]

\[
M_{10} = N_{(1,2,5,6)}^{[1,k]} \otimes M_{(6,5,4,3)}^{[1,k]},
\]

\[
M_{11} = N_{(1,2,5,6)}^{[1,k]} \otimes N_{(6,5,4,3)}^{[1,k]},
\]

\[
\bar{\phi}_1 = (\text{Id}_{N_{(1,2,5,6)}^{[1,k]}} \otimes (x_{1,1} - x_{1,6}, 1)) \otimes \text{Id}_{N_{(6,5,4,3)}^{[1,k]}},
\]

\[
\bar{\phi}_2 = \text{Id}_{N_{(1,2,5,6)}^{[1,k]}} \otimes (\text{Id}_{N_{(6,5,4,3)}^{[1,k]}} \otimes (1, x_{1,6} - x_{1,3})),
\]

\[
\bar{\phi}_3 = \text{Id}_{N_{(1,2,5,6)}^{[1,k]}} \otimes (\text{Id}_{N_{(6,5,4,3)}^{[1,k]}} \otimes (1, x_{1,6} - x_{1,3})),
\]

\[
\bar{\phi}_4 = - (\text{Id}_{N_{(1,2,5,6)}^{[1,k]}} \otimes (x_{1,1} - x_{1,6}, 1)) \otimes \text{Id}_{N_{(6,5,4,3)}^{[1,k]}},
\]

We show that this complex is isomorphic to

\[
\mathcal{L}_{(1,2,4,3)}^{[1,k]} = S_{(1,2,4,3)}^{[1,k]} \otimes K(u_{k+1,(1,2,4,3)}^{[k+1]}; x_{1,1} - x_{1,3}) \otimes B_{(1,2,3,4)}^{(1,k,1,k)}.
\]

Note that the above factorization is isomorphic to the middle factorization of \((H_{a_{1k}}) \mathcal{L}_{(1,3)}^{[1]} \otimes \mathcal{L}_{(2,4)}^{[k]}\) by Theorem 2.6.

To prove the isomorphism, we show the following lemma. If the lemma can be proved, we obtain the isomorphism by a chain homotopy equivalence.

**Lemma 5.4.** We have isomorphisms in \(\text{HMF}^{gr}_{R(1,2,3,4)}(1,k,1,k)\)

\[
(14)\quad M_{00}(1) \simeq (M_{(1,2,4,3)}^{[1,k]})^{[k]} \otimes \{1\},
\]

\[
(15)\quad M_{10} \simeq (M_{(1,2,4,3)}^{[1,k]})^{[k+1]} \otimes \{1\},
\]

\[
(16)\quad M_{01} \simeq (M_{(1,2,4,3)}^{[1,k]})^{[k-1]} \otimes \mathcal{L}_{(1,2,4,3)}^{[1,k]}.
\]
the following matrices $\Phi_i$:

\[
\Phi_1 = \begin{pmatrix}
\Phi_{k-1} & -X^{(k,-1)}_{k,(2,3)} \cdot \Phi_{M^{[1,k]}_{(1,2,4,3)}} \\
E_{k-1} \cdot \Phi_{M^{[1,k]}_{(1,2,4,3)}} & \Phi_{k-1}
\end{pmatrix},
\]

\[
\Phi_2 = \begin{pmatrix}
\Phi_{k-1} & -X^{(k,-1)}_{k,(2,3)} \cdot \Phi_{M^{[1,k]}_{(1,2,4,3)}} \\
E_{k-1} \cdot \Phi_{M^{[1,k]}_{(1,2,4,3)}} & \Phi_{k-1}
\end{pmatrix},
\]

\[
\Phi_3 = \begin{pmatrix}
\Phi_{k-1} & -X^{(k,-1)}_{k,(2,3)} \cdot \Phi_{M^{[1,k]}_{(1,2,4,3)}} \\
E_{k-1} \cdot \Phi_{M^{[1,k]}_{(1,2,4,3)}} & \Phi_{k-1}
\end{pmatrix},
\]

\[
\Phi_4 = \begin{pmatrix}
\Phi_{k-1} & -X^{(k,-1)}_{k,(2,3)} \cdot \Phi_{M^{[1,k]}_{(1,2,4,3)}} \\
E_{k-1} \cdot \Phi_{M^{[1,k]}_{(1,2,4,3)}} & \Phi_{k-1}
\end{pmatrix},
\]

where $E_m(f)$ is the diagonal matrix of polynomial $f$ with order $m$ and $\Phi_m$ is the zero low vector with length $m$.

**Proof of Lemma 5.4.** (I) We show the isomorphism (14).

The matrix factorization $M_{00}\{1\}$ is isomorphic to

\[
S^{[1,k]}_{(1,2,4,3)} \otimes K(u^{[1,k]}_{k+1,(6,5,4,3)}; (x_{1,6} - x_{1,3})X^{(k,-1)}_{k,(5,3)})Q_1\{-2k + 2\},
\]

where

\[
Q_1 = R^{(k,1,k,1,k,1,1)}_{(1,2,3,4,5,6)} / \langle X^{(k,1)}_{1,(1,2)} - X^{(k,1)}_{1,(5,6)} , \ldots , X^{(k,1)}_{1,(1,2)} - X^{(k,1)}_{k,(5,6)} , X^{(k,-1)}_{k,(2,6)} \rangle.
\]

The set

\[
\mathcal{B}_1 = \{1, x_{1,6}, \ldots, x_{k-2}^{(k,-1),2}, X^{(k,-1),2}_{k-1,(2,3,6)} \}
\]

is a basis of $Q_1$ as an $R^{(k,1,k,1,k,1,1)}_{(1,2,3,4,5,6)}$-module. By this basis, the matrix factorization $K(u^{[1,k]}_{k+1,(1,2,4,3)}; (x_{1,1} - x_{1,3})X^{(k,-1)}_{k,(2,3)})Q_1\{-2k + 2\}$ is isomorphic to

\[
(R_1\{-2k + 2\}, R_1\{3 - n\}, E_k(u^{[1,k]}_{k+1,(1,2,4,3)}), E_k((x_{1,1} - x_{1,3})X^{(k,-1)}_{k,(2,3)})),
\]
where $R_1$ is the $R^{1,k,1,k}_{(1,2,3,4)}$-module spanned by $\mathcal{B}_1$. Thus, the matrix factorization (18) is isomorphic to
\[
(\overline{M}^{[1,k]}_{(1,2,4,3)}) \oplus [k]_q \{1\}.
\]

(II) We show the isomorphism (15).

The matrix factorization $\overline{M}_{10}$ is isomorphic to
\[
(20) \quad \overline{S}^{[1,k]}_{(1,2,4,3)} \boxtimes K(u^{[1,k]}_{k+1,(6,5,4,3)}; (x_{1,6} - x_{1,3})X^{(k,-1)}_{k,(5,3)})Q_2\{2k\},
\]
where
\[
Q_2 = R^{1,k,1,k,1,k}_{(1,2,3,4,5,6)} \left/ \langle X^{(k,1)}_{1,(1,2)} - X^{(k,1)}_{1,(5,6),}\ldots, X^{(k,1)}_{k,(1,2)} - X^{(k,1)}_{k,(5,6)}, (x_{1,1} - x_{1,6})X^{(k,-1)}_{k,(2,6)} \rangle \right.,
\]
The set
\[
\mathcal{B}_2 = \{1, x_{1,1} - x_{1,6}, \ldots, x_{1,6}^{-2}(x_{1,1} - x_{1,6}), X^{(k,-1)}_{k,(3,5)}\}
\]
is a basis of $Q_2$ as an $R^{1,k,1,k}_{(1,2,3,4)}$-module. By this basis, the matrix factorization $K(u^{[1,k]}_{k+1,(1,2,4,3)}; (x_{1,1} - x_{1,3})X^{(k,-1)}_{k,(2,3)})Q_2\{2k\}$ is isomorphic to
\[
(21) \quad (R_2\{2k\}, R_2\{1 - n\}, E_{k+1}(u^{[1,k]}_{k+1,(1,2,4,3)}), E_{k+1}((x_{1,1} - x_{1,3})X^{(k,-1)}_{k,(2,3)})),
\]
where $R_2$ is the $R^{1,k,1,k}_{(1,2,3,4)}$-module spanned by $\mathcal{B}_2$. Thus, the matrix factorization (20) is isomorphic to
\[
(\overline{M}^{[1,k]}_{(1,2,4,3)}) \oplus [k+1]_q.
\]

(III) We show the isomorphism (16).

The matrix factorization $\overline{M}_{01}$ is isomorphic to
\[
(22) \quad \overline{S}^{[1,k]}_{(1,2,4,3)} \boxtimes K(u^{[1,k]}_{k+1,(6,5,4,3)}(x_{1,6} - x_{1,3}); X^{(k,-1)}_{k,(5,3)})Q_0\{2k + 2\}.
\]
The set $\mathcal{B}_3 = \{1, x_{1,6} - x_{1,3}, \ldots, x_{1,6}^{-2}(x_{1,6} - x_{1,3})\}$ is a basis of $Q_1$ as an $R^{1,k,1,k}_{(1,2,3,4)}$-module. $K(u^{[1,k]}_{k+1,(1,2,4,3)}(x_{1,6} - x_{1,3}); (x_{1,1} - x_{1,3})X^{(k,-1,1)}_{k-1,(2,3,6)})Q_1 \times$
\{-2k + 2\} is isomorphic to \(\langle R_1 \{-2k + 2\}, R_3 \{1 - n\}, F_1, F'_1 \rangle\), where \(R_3\) is the \(R_{(1,2,3,4)}\)-module spanned by \(\mathfrak{B}_3\) and

\[
F_1 = \begin{pmatrix} \varphi_{k-1} & X_{k,(2,3)}^{k-1} u_{k+1,(1,2,4,3)}^{[1,k]} \\ E_{k-1} u_{k+1,(1,2,4,3)}^{[1,k]} & t' \varphi_{k-1} \end{pmatrix},
\]

\[
F'_1 = \begin{pmatrix} t' \varphi_{k-1} & E_{k-1}((x_{1,1} - x_{1,3}) X_{k,(2,3)}^{(k-1)}) \\ x_{1,1} - x_{1,3} & \varphi_{k-1} \end{pmatrix}.
\]

Thus, the matrix factorization (22) is isomorphic to

\[
\frac{\langle M_{(1,2,4,3)}^{[1,k]} \rangle}{\langle [k-1] \rangle} \oplus \frac{L_{(1,2,4,3)}^{[1,k]}}{\langle \rangle}.
\]

(IV) We show the isomorphism (17).

The matrix factorization \(\overline{M}_{01}\{-1\}\) is isomorphic to

\[
\overline{S}_{(1,2,4,3)}^{[1,k]} \boxtimes K(u_{k+1,(6,5,4,3)}^{[1,k]}(x_{1,6} - x_{1,3}); X_{k,(5,3)}^{(k-1)}) Q_1 \{-2k\}.
\]

The set

\[
\mathfrak{B}_4 = \left\{ 1, x_{1,6} - x_{1,3}, (x_{1,1} - x_{1,6})(x_{1,6} - x_{1,3}), \ldots, \\
x_{1,6}^{k-2}(x_{1,1} - x_{1,6})(x_{1,6} - x_{1,3}) \right\}
\]

is a basis of \(Q_1\) as an \(R_{(1,2,3,4)}^{(1,k,1,k)}\)-module. \(K(u_{k+1,(1,2,4,3)}^{[1,k]}(x_{1,6} - x_{1,3}); X_{k,(5,3)}^{(k-1)}) Q_1 \{-2k\}\) is isomorphic to \(\langle R_2 \{-2k\}, R_4 \{1 - n\}, G_1, G'_1 \rangle\), where \(R_4\) is the \(R_{(1,2,3,4)}^{(1,k,1,k)}\)-module spanned by \(\mathfrak{B}_4\) and

\[
G_1 = \begin{pmatrix} \varphi_{k-1} & X_{k,(2,3)}^{k-1} u_{k+1,(1,2,4,3)}^{[1,k]}(x_{1,1} - x_{1,3}) \\ E_{k-1} u_{k+1,(1,2,4,3)}^{[1,k]} & t' \varphi_{k-1} \end{pmatrix},
\]

\[
G'_1 = \begin{pmatrix} t' \varphi_{k-1} & E_{k-1}((x_{1,1} - x_{1,3}) X_{k,(2,3)}^{(k-1)}) \\ 1 & \varphi_{k-1} \end{pmatrix}.
\]

Thus, the matrix factorization (23) is isomorphic to

\[
\frac{\langle M_{(1,2,4,3)}^{[1,k]} \rangle}{\langle [k] \rangle}.
\]

(V) We show that the morphisms \(\overline{\phi}_i (i = 1, 2, 3, 4)\) induce \(\overline{\Phi}_i\) with respect to the above isomorphisms.
By the isomorphisms of (14) and (15), we find that the morphism \( \overline{\phi}_1 : M_{00} \rightarrow M_{10} \) induces the morphism \( \text{Id}_{S_{(1,2,4,3)}^{([1,k],(1,k,1,k)]}} \otimes (x_{1,1} - x_{1,6}, x_{1,1} - x_{1,6}) \) from the matrix factorization (18)–(20). With respect to the \( R_{(1,2,3,4)}^{(1,k,1,k)} \)-module basis \( B_1 \) of \( Q_1 \), a matrix form of the grade-preserving \( R_{(1,2,3,4)}^{(1,k,1,k)} \)-module morphism \( x_{1,1} - x_{1,6} ; Q_1 \{ 2 \} \rightarrow Q_2 \) is

\[
\begin{pmatrix}
\sigma_{k-1} & -X_{k,(2,3)}^{(k,-1)}

E_{k-1}(1) & t \sigma_{k-1}

\sigma_{k-1} & 1
\end{pmatrix}.
\]

Thus, \( \text{Id}_{S_{(1,2,4,3)}^{([1,k],(1,k,1,k)]}} \otimes (x_{1,1} - x_{1,6}, x_{1,1} - x_{1,6}) \) induces the morphism \( \Phi_1 \) from the factorization (19) to the factorization (21).

In a similar way, we find that \( \overline{\phi}_2 \) induces \( \text{Id}_{S_{(1,2,4,3)}^{([1,k],(1,k,1,k)]}} \otimes (1, x_{1,6} - x_{1,3}) \) from the factorization (18) to the factorization (22), \( \overline{\phi}_3 \) induces \( \text{Id}_{S_{(1,2,4,3)}^{([1,k],(1,k,1,k)]}} \otimes (1, x_{1,6} - x_{1,3}) \) from the factorization (18) to the factorization (22), \( \overline{\phi}_4 \) induces \( \text{Id}_{S_{(1,2,4,3)}^{([1,k],(1,k,1,k)]}} \otimes (x_{1,1} - x_{1,6}, x_{1,1} - x_{1,6}) \) from the factorization (22) to the factorization (23), and then these morphisms are deformed into morphisms \( \Phi_2, \Phi_3, \) and \( \Phi_4 \), respectively.

**Remark 5.5.** We showed the above isomorphism (16). This is the claim of Proposition 4.12(1).

We can prove the other isomorphisms of \( (II_{a_{1k}}) \) and \( (II_{b_{1k}}) \) in a similar way. The isomorphisms of \( (II_{b_{1k}}) \) are shown in Section 7.1.

**5.4. Proof of invariance under Reidemeister III move**

We prepare the following isomorphisms for proof of invariance under Reidemeister III move. Mackaay, Stosic, and Vaz [7, Conjecture 2] conjectured that there exist isomorphisms between complexes of bimodules that are associated to the following diagrams.

**Proposition 5.6.** We have the following isomorphisms in \( K^b(\text{HMF}_{R,\omega}^{gr}) \):

\[
C\left(\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\circ & \circ
\end{array}\right)_{n} \cong C\left(\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\circ & \circ
\end{array}\right)_{n},
\]

\[
C\left(\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\circ & \circ
\end{array}\right)_{n} \cong C\left(\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\circ & \circ
\end{array}\right)_{n}.
\]
Proof. Removing acyclic complex in the left-hand complex, we have the right-hand complex, respectively. We prove this proposition in Section 7.2.

\textbf{Corollary 5.7.} We have the following isomorphisms in $K^b(\text{HMF}_{R,\omega})$:

\[
\begin{align*}
C \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right)_{n} &\simeq C \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right)_{n}, \\
C \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right)_{n} &\simeq C \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right)_{n}, \\
C \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right)_{n} &\simeq C \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right)_{n}.
\end{align*}
\]

Proof of invariance under Reidemeister $III_{11k}$ move (Theorem 5.3($III_{11k}$)).

We have

\[
(24) \quad C \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right) = C^{-k} \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right) \xrightarrow{\chi_{k,1} \otimes \text{Id}} C^{1-k} \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right).
\]

By Proposition 5.6, the above complex is isomorphic in $K^b(\text{HMF}_{R,\omega})$ to

\[
(25) \quad C^{-k} \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right) \xrightarrow{\chi_{k,1} \otimes \text{Id}} C^{1-k} \left( \begin{array}{c}
\begin{array}{c}
\ast \\
\ast \\
\ast
\end{array}
\end{array} \right).
\]
We show that $\chi^{[k,1]}_+ \boxtimes \text{Id} = \chi^{[k,1]}_+ \boxtimes \text{Id}$ up to chain homotopy equivalence. Put

$$C_n \left( \begin{array}{c}
\begin{array}{c}
(1) \\
(2) \\
(3) \\
(4) \\
(9) \\
(7) \\
(8)
\end{array}
\end{array} \right) = \begin{array}{c}
C^{-k-1} \xrightarrow{\Phi} C^{-k} \oplus C^{-k} \\
\begin{array}{c}
\text{Id}_M \boxtimes \chi^{[1,1]}_+(2,3,9,8) \\
\chi^{[1,k]}_+ \boxtimes \text{Id}_N
\end{array}
\end{array}
$$

(26)

where

$$C^{-k-1} = M^{[1,k]}_{(9,1,7,4)} \boxtimes M^{[1,1]}_{(2,3,9,8)} \{(k+1)(n-1)\} \langle k+1 \rangle,$$

$$C^{-k}_1 = M^{[1,k]}_{(9,1,7,4)} \boxtimes N^{[1,1]}_{(2,3,9,8)} \{(k+1)(n-1)\} \langle k+1 \rangle,$$

$$C^{-k}_2 = N^{[1,k]}_{(9,1,7,4)} \boxtimes M^{[1,1]}_{(2,3,9,8)} \{(k+1)(n-1)\} \langle k+1 \rangle,$$

$$C^{-k+1} = N^{[1,k]}_{(9,1,7,4)} \boxtimes N^{[1,1]}_{(2,3,9,8)} \{(k+1)(n-1)\} \langle k+1 \rangle.$$

The morphism $\chi^{[k,1]}_+ \boxtimes \text{Id}$ is composed of a tensor product of a morphism from $C \left( \begin{array}{c}
\begin{array}{c}
(1) \\
(2) \\
(3) \\
(4) \\
(9) \\
(7) \\
(8)
\end{array}
\end{array} \right)_n$ to $C \left( \begin{array}{c}
\begin{array}{c}
(1) \\
(2) \\
(3) \\
(4) \\
(9) \\
(7) \\
(8)
\end{array}
\end{array} \right)_n \{ -1 \}$ and an endomorphism $\Phi$ of the complex (26). The endomorphism $\Phi$ consists of morphisms $\overline{f}$, $\overline{g}$, and $\overline{h}$ in the commutative diagram

$$\text{begin diagram}
\begin{array}{ccccccccc}
0 & \xrightarrow{\Phi} & C^{-k-1} & \xrightarrow{\Phi} & C^{-k} & \oplus & C^{-k} & \xrightarrow{\Phi} & C^{-k+1} & \xrightarrow{\Phi} & 0 \\
\downarrow{\overline{f}} & & \downarrow{\overline{g}} & & \downarrow{\overline{h}} & & \downarrow{\overline{g}} & & \downarrow{\overline{h}} \\
0 & \xrightarrow{\Phi} & C^{-k-1} & \xrightarrow{\Phi} & C^{-k} & \oplus & C^{-k} & \xrightarrow{\Phi} & C^{-k+1} & \xrightarrow{\Phi} & 0.
\end{array}
$$
Since $\Phi$ is derived from $\chi_{(k,1)}$ ⊠ $\text{Id}$, we find that $f \neq 0$, $g_{00} \neq 0$, $g_{11} \neq 0$, $h \neq 0$. Moreover, we have

$$\dim Q \text{Hom}_{\text{HMF}}(C^{-k-1}, C^{-k-1}) = \dim Q \text{Hom}_{\text{HMF}}(C_1^{-k}, C_1^{-k}) = 1,$$

$$\dim Q \text{Hom}_{\text{HMF}}(C_2^{-k}, C_2^{-k}) = \dim Q \text{Hom}_{\text{HMF}}(C^{-k+1}, C^{-k+1}) = 1,$$

$$\dim Q \text{Hom}_{\text{HMF}}(C_1^{-k}, C_2^{-k}) = \dim Q \text{Hom}_{\text{HMF}}(C_2^{-k}, C_1^{-k}) = 0.$$

Therefore, $\Phi$ induces the morphism

$$\left( \ldots, 0, \text{Id}_{C^{-k-1}}, \left( \begin{array}{cc} \text{Id}_{C_1^{-k}} & 0 \\ 0 & \text{Id}_{C_2^{-k}} \end{array} \right), \text{Id}_{C^{-k+1}}, 0, \ldots \right)$$

up to chain homotopy equivalence. Thus, we obtain the first isomorphism of Theorem 5.3(III_{11k}). We can prove the other isomorphisms of Theorem 5.3(III_{11k}) in a similar way.

We prepare isomorphisms in $K^b(\text{HMF}^\text{gr}_{R,\omega})$ for Section 6.

**Proposition 5.8.** We have the following isomorphisms in $K^b(\text{HMF}^\text{gr}_{R,\omega})$:

1. $C \left( \begin{array}{c} \Upsilon_{k+1} \\ 1 \\ k_{k} \end{array} \right) \cong C \left( \begin{array}{c} \Upsilon_{k+1} \\ 1 \\ k_{k} \end{array} \right) \{kn + k\} \langle k \rangle [-k],$

2. $C \left( \begin{array}{c} \Upsilon_{k+1} \\ 1 \\ k_{k} \end{array} \right) \cong C \left( \begin{array}{c} \Upsilon_{k+1} \\ 1 \\ k_{k} \end{array} \right) \{-kn - k\} \langle k \rangle [k],$

3. $C \left( \begin{array}{c} \Upsilon_{k+1} \\ 1 \\ k_{k} \end{array} \right) \cong C \left( \begin{array}{c} \Upsilon_{k+1} \\ 1 \\ k_{k} \end{array} \right) \{kn + k\} \langle k \rangle [-k],$

4. $C \left( \begin{array}{c} \Upsilon_{k+1} \\ 1 \\ k_{k} \end{array} \right) \cong C \left( \begin{array}{c} \Upsilon_{k+1} \\ 1 \\ k_{k} \end{array} \right) \{-kn - k\} \langle k \rangle [k].$

**Proof.** We show the isomorphism (1). The left-hand complex is

$$ \Lambda_{(1;4,5)}^{[k,1]} \boxtimes M_{(5,4,3,2)}^{[1,k]} \{kn\} \langle k \rangle \overset{\text{Id}_{\Lambda_{(1;4,5)}^{[k,1]}} \boxtimes \chi_{(5,4,3,2)}^{[1,k]}}{\longrightarrow} \Lambda_{(1;4,5)}^{[k,1]} \boxtimes M_{(5,4,3,2)}^{[1,k]} \{kn - 1\} \langle k \rangle.$$
We have
\[
\Lambda_{(1;4,5)}^{[k,1]} \boxtimes M_{(5,4,3,2)}^{[1,k]} \cong \Lambda_{(1;2,3)}^{[1,k]} \otimes R_{(1,2,3)}^{(k+1,1,k,1)} (R_{(1,2,3,5)}^{(k+1,1,k,1,3)} / \langle X_{k+1,(2,3,5)}^{(k,1)} \rangle ) \{-k\}
\]
\[
\Lambda_{(1;4,5)}^{[k,1]} \boxtimes \Lambda_{(5,4,3,2)}^{[1,k]} \cong \Lambda_{(1;2,3)}^{[1,k]} \otimes R_{(1,2,3)}^{(k+1,1,k,1)} (R_{(1,2,3,5)}^{(k+1,1,k,1,3)} / \langle X_{k,(3,5)}^{(k,1)} \rangle ) \{-k+1\}.
\]

The boundary map of complex (27) \(\text{Id}_{\Lambda_{(1;4,5)}^{[k,1]} \boxtimes \chi_{+, (5,4,3,2)}} \) induces \(\text{Id}_{\Lambda_{(1;2,3)}^{[1,k]} \otimes 1}\) with respect to the above isomorphisms. By a chain homotopy equivalence, the complex (27) is isomorphic to
\[
\Lambda_{(1;2,3)}^{[1,k]} \{kn + k\} \langle k \rangle [-k].
\]

The other isomorphisms (2), (3), and (4) can be proved in a similar way.

5.5. Example of homology of Hopf link with \([1,k]-coloring\)

We show the Poincaré polynomial of the link homology of a \([1,k]-colored\) Hopf link:

\[
P\left(\begin{array}{c}
1 \\
\end{array}\right) = t^{-2k} s^{k+1} q^{2kn+k} [n]_q [n-k]_q
\]
\[
+ t^{-2k+2} s^{k+1} q^{2kn-n+k-2} [n]_q [k]_q
\]
\[
= t^{-2k} s^{k+1} [n]_q q^{2kn} ([k]_q q^{n+k-2} t^2 + [n-k]_q q^k),
\]
\[
P\left(\begin{array}{c}
1 \\
\end{array}\right) = t^{2k-2} s^{k+1} q^{-2kn+n-k+2} [n]_q [k]_q
\]
\[ + t^{2k}s^{k+1}q^{2kn-k} \begin{bmatrix} n \\ k \end{bmatrix}_q [n - k]_q \]

\[ = t^{2k}s^{k+1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-2kn} ([k]_q q^{n-k+2}t^{-2} + [n - k]_q q^{t-k}). \]

Awata and Kanno [1] calculated a homological Hopf link invariant by refined topological vertex. The evaluation for a \([1, k]\)-colored Hopf link is

\[ \overline{P}_{(k, 1)}(q', t') = q'^{-2n+k^2-k}(-t')^k \begin{bmatrix} n \\ k \end{bmatrix}_{q'} ([k]_{q'} q'^{n+k-2}t'^{-2} + [n - k]_{q'} q'^{2n+k}). \]

Therefore, we find the following relation between these evaluations:

\[ P \begin{pmatrix} \cdot & 1 & k \end{pmatrix} = \overline{P}_{(k, 1)}(q^{-1}, -t)s^{k+1}t^k q^{-2kn+k^2-k}, \]

\[ P \begin{pmatrix} \cdot & 1 & k \end{pmatrix} = \overline{P}_{(k, 1)}(q, -t^{-1})s^{k+1}t^{-k} q^{2kn-k^2+k}. \]

§6. Complexes of matrix factorizations for \([i, j]\)-crossing

6.1. Wide edge and propositions

We introduce a wide edge to define a complex of matrix factorizations for an \([i, j]\)-crossing. The wide edge represents a bunch of 1-colored lines with the same orientation. We represent a \(k\)-colored edge branching into a bunch of \(k\) 1-colored lines as a diagram of a wide edge (see Figure 10).
We naturally consider a crossing of a wide edge and colored edge and a crossing of wide edges. For example,

\[ C \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right) \cong C \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right), \]

\[ C \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right) \cong C \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right). \]

For diagrams with the other crossing, their complexes are isomorphic in \( K^b(\text{HMF}^{\text{gr}}_{R,\omega}) \).

**Proof.** We prove this proposition using Proposition 5.6.
We find the following corollary of this proposition.

**Corollary 6.2.** There exist isomorphisms in $\mathcal{K}^b(\text{HMF}^\text{gr}_{R,\omega})$

\[
\begin{align*}
\mathcal{C} \left( \begin{array}{c}
k \\
\downarrow \\
k+1 \\
\end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c}
k \\
\downarrow \\
k+1 \\
\end{array} \right), \\
\mathcal{C} \left( \begin{array}{c}
k \\
\downarrow k \\
k+1 \\
\end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c}
k \\
\downarrow k \\
k+1 \\
\end{array} \right).
\end{align*}
\]

For diagrams with the other crossing, their complexes are isomorphic in $\mathcal{K}^b(\text{HMF}^\text{gr}_{R,\omega})$.

By Propositions 5.6 and 5.8, we find the following corollary.

**Corollary 6.3.** There exist the following isomorphisms in $\mathcal{K}^b(\text{HMF}^\text{gr}_{R,\omega})$:

\[
\begin{align*}
(1) \quad \mathcal{C} \left( \begin{array}{c}
k \\
\uparrow \\
k+1 \\
\end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c}
k+1 \\
\uparrow \\
k \\
\end{array} \right) \{kn + k\} \langle k \rangle [-k], \\
(2) \quad \mathcal{C} \left( \begin{array}{c}
k \\
\uparrow k \\
k+1 \\
\end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c}
k+1 \\
\uparrow k \\
k \\
\end{array} \right) \{-kn - k\} \langle k \rangle [k], \\
(3) \quad \mathcal{C} \left( \begin{array}{c}
k \\
\uparrow k \\
k+1 \\
\end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c}
k+1 \\
\uparrow k \\
k \\
\end{array} \right) \{kn + k\} \langle k \rangle [-k], \\
(4) \quad \mathcal{C} \left( \begin{array}{c}
k \\
\uparrow k \\
k+1 \\
\end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c}
k+1 \\
\uparrow k \\
k \\
\end{array} \right) \{-kn - k\} \langle k \rangle [k].
\end{align*}
\]

6.2. Approximate complex for $[i, j]$-crossing

We consider an approximate crossing of an $[i, j]$-crossing in Figure 11. This approximate crossing has only $[i, 1]$-crossings. Thus, we define a complex of matrix factorizations for the approximate crossing using the complex of an $[i, 1]$-crossing in Section 5.1.
**Definition 6.4.** We define a complex of matrix factorizations for an 
\([i, j]-\text{crossing}\) as an object of \(K^b(\text{HMF}_{\text{gr}} R, \omega)\):

\[
\begin{align*}
\mathcal{C} \left( \begin{array}{c} i \downarrow j \\ n \end{array} \right) & := \mathcal{C} \left( \begin{array}{c} i \downarrow j \\ n \end{array} \right) \{ -i(i-1)(n+1) \} [i(i-1)] \quad (i \geq j), \\
\mathcal{C} \left( \begin{array}{c} i \downarrow j \\ n \end{array} \right) & := \mathcal{C} \left( \begin{array}{c} i \downarrow j \\ n \end{array} \right) \{ -j(j-1)(n+1) \} [j(j-1)] \quad (i < j), \\
\mathcal{C} \left( \begin{array}{c} i \downarrow j \\ n \end{array} \right) & := \mathcal{C} \left( \begin{array}{c} i \downarrow j \\ n \end{array} \right) \{ j(j-1)(n+1) \} [-j(j-1)] \quad (i \leq j), \\
\mathcal{C} \left( \begin{array}{c} i \downarrow j \\ n \end{array} \right) & := \mathcal{C} \left( \begin{array}{c} i \downarrow j \\ n \end{array} \right) \{ i(i-1)(n+1) \} [-i(i-1)] \quad (i > j).
\end{align*}
\]

**Theorem 6.5.** We have the following isomorphisms in \(K^b(\text{HMF}_{\text{gr}} R, \omega)\):

\[
\begin{align*}
(I) \quad \mathcal{C} \left( \begin{array}{c} \circ \downarrow i \circ \\ n \end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c} i \downarrow \circ \\ n \end{array} \right) \oplus [i]_q! \\
(IIa) \quad \mathcal{C} \left( \begin{array}{c} \circ \downarrow i \circ \\ n \end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c} i \downarrow \circ \\ n \end{array} \right) \oplus [i]_q! [j]_q! \\
(IIb) \quad \mathcal{C} \left( \begin{array}{c} \circ \downarrow j \circ \\ n \end{array} \right) & \simeq \mathcal{C} \left( \begin{array}{c} \circ \downarrow i \circ \\ n \end{array} \right) \oplus [i]_q! [j]_q!.
\end{align*}
\]
\[
\bar{C} \left( \begin{array}{c}
\vdots
\end{array} \right) \simeq \bar{C} \left( \begin{array}{c}
\vdots
\end{array} \right),
\]

\[
\left( \text{III} \right) \quad \bar{C} \left( \begin{array}{c}
\vdots
\end{array} \right) \simeq \bar{C} \left( \begin{array}{c}
\vdots
\end{array} \right).
\]

**Proof.** We show the proof of this theorem in Section 6.4.

For a colored oriented link diagram \( D \), we obtain the homology of \( \bar{C}(D) \). We consider the Poincaré polynomial \( P(D) \) in \( \mathbb{Q}[t^{\pm 1}, q^{\pm 1}, s]/(s^2 - 1) \) of the homology of \( \bar{C}(D) \). We obtain the following corollary of Theorem 6.5.

**Corollary 6.6.** If colored oriented link diagrams are related by a Reidemeister move, we have the following equations of \( P \):

\[
\begin{align*}
\left( \text{I} \right) \quad & P \left( \begin{array}{c}
\vdots
\end{array} \right) = P \left( \begin{array}{c}
\vdots
\end{array} \right), \\
\left( \text{IIa} \right) \quad & P \left( \begin{array}{c}
\vdots
\end{array} \right) = P \left( \begin{array}{c}
\vdots
\end{array} \right), \\
\left( \text{IIb} \right) \quad & P \left( \begin{array}{c}
\vdots
\end{array} \right) = P \left( \begin{array}{c}
\vdots
\end{array} \right), \\
\left( \text{III} \right) \quad & P \left( \begin{array}{c}
\vdots
\end{array} \right) = P \left( \begin{array}{c}
\vdots
\end{array} \right),
\end{align*}
\]

where outsides of colored tangle diagrams in each equation are identical.

**6.3. Polynomial link invariant \( P \)**

We define a new link invariant \( P \) by normalizing the Poincaré polynomial \( P \). For a colored oriented link diagram \( D \), let \( C_{r_k}(D) \) \((k = 1, \ldots, n - 1)\)
denote the number of $[* , k]$-crossings in $D$. We define a polynomial $P(D)$ by
\[
P(D) = \prod_{k=1}^{n-1} \frac{1}{([k]_q!)^{Cr_k(D)}}.
\]
By Corollary 6.6, we have the main theorem of this paper.

**Theorem 6.7.** The polynomial $P$ is an invariant of oriented colored links.

$P(D)$ is the $(\mathfrak{sl}_n , \wedge V_n)$ link invariant if $t$ is specialized to $-1$ and $s$ is specialized to 1. Therefore, the polynomial $P(D)$ is a refined link invariant of the $(\mathfrak{sl}_n , \wedge V_n)$ link invariant.

### 6.4. Proof of Theorem 6.5

**Proof of Theorem 6.5 (I).** By Corollary 6.2, We have
\[
C_{\binom{i}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) = C_{\binom{i}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) \{ -i(i-1)(n+1) \} [i(i-1)]
\]
\[
\simeq C_{\binom{i}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) \{ -i(i-1)(n+1) \} [i(i-1)].
\]
We show the following lemma.

**Lemma 6.8.** We have the following isomorphism in $K^b (HMF^{gr})$:

\[
C_{\binom{i}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) \{ -i(i-1)(n+1) \} [i(i-1)] \simeq C_{\binom{i}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) \oplus [i]_q !.
\]

**Proof.** We prove the lemma by induction on $i$. In the case that $i = 2$, by Theorem 5.3 and Proposition 5.8 we have an isomorphism
\[
C_{\binom{2}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) \{ -2(n+1) \} [2] \simeq C_{\binom{2}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) \{ -2(n+1) \} [2]
\]
\[
\simeq C_{\binom{2}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) = C_{\binom{2}{2}} \left( \begin{array}{c}
\infty
\end{array} \right) \oplus [2]_q .
\]
Also by Theorem 5.3 and Proposition 5.8, we have an isomorphism

\[
\mathcal{C} \left( \begin{array}{c}
\uparrow_k \\
\vdots
\end{array} \right)_{n} \{ -k(k - 1)(n + 1) \} [k(k - 1)]
\]

\[
\simeq \mathcal{C} \left( \begin{array}{c}
\uparrow_k \\
\downarrow_{k-1}
\end{array} \right)_{n} \{ -k(k - 1)(n + 1) \} [k(k - 1)]
\]

\[
\simeq \mathcal{C} \left( \begin{array}{c}
\uparrow_k \\
\downarrow_{k-1}
\end{array} \right)_{n} \{ -(k - 1)(k - 2)(n + 1) \} [(k - 1)(k - 2)].
\]

By induction, the complex (33) is isomorphic to

\[
\mathcal{C} \left( \begin{array}{c}
\uparrow_{k-1} \\
\downarrow_k
\end{array} \right)_{n} \simeq \mathcal{C} \left( \begin{array}{c}
\uparrow_k \\
\downarrow_{k-1}
\end{array} \right)_{n} \oplus \left( k \right)_{q!} n.
\]

By this lemma, we find the first isomorphism of Theorem 6.5(I). We prove the isomorphism for a minus \( i \)-curl in a similar way.

Proof of Theorem 6.5(II).

\[
\mathcal{C} \left( \begin{array}{c}
\uparrow_i \\
\downarrow_{i+1}
\end{array} \right)_{n} = \mathcal{C} \left( \begin{array}{c}
\uparrow_i \\
\downarrow_{i+1}
\end{array} \right)_{n} \simeq \mathcal{C} \left( \begin{array}{c}
\uparrow_i \\
\downarrow_{i+1}
\end{array} \right)_{n} \simeq \mathcal{C} \left( \begin{array}{c}
\uparrow_i \\
\downarrow_{i+1}
\end{array} \right)_{n} \oplus \left( i \right)_{q!} \oplus \left( i \right)_{q!}.
\]

\[
\simeq \mathcal{C} \left( \begin{array}{c}
\uparrow_i \\
\downarrow_{i+1}
\end{array} \right)_{n} \oplus \left( i \right)_{q!} \oplus \left( i \right)_{q!}.
\]

\[
\simeq \mathcal{C} \left( \begin{array}{c}
\uparrow_i \\
\downarrow_{i+1}
\end{array} \right)_{n} \oplus \left( i \right)_{q!} \oplus \left( i \right)_{q!}.
\]

We prove the other isomorphism of (IIa) and isomorphisms of (IIb) in a similar way.
Proof of Theorem 6.5(III). It is sufficient to consider the case $i < j < k$. We similarly prove invariance of the Reidemeister (III) move for the other coloring case.

$$C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} = C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} \{\alpha\}[\beta] \simeq C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} \{\alpha\}[\beta]$$

$$\alpha = (-2k(k-1) - j(j-1))(n+1),$$

$$\beta = 2k(k-1) + j(j-1))$$

$$\simeq C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} \{\alpha\}[\beta] \simeq C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} \{\alpha\}[\beta]$$

On the other side, we have

$$C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} = C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} \{\alpha\}[\beta] \simeq C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} \{\alpha\}[\beta]$$

Thus, we have

$$C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n} \simeq C \left( \begin{array}{c} i \\ j \\ k \end{array} \right) |_{n}$$

§7. Proof of Theorem 5.3(IIb) and Proposition 5.6

7.1. Invariance under Reidemeister IIb move

We show the isomorphism

(33) $$C \left( \begin{array}{c} 1 \\ \bar{5} \\ \bar{6} \\ 3 \\ 4 \end{array} \right) |_{n} \simeq C \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ k \end{array} \right) |_{n}$$
By Definition 5.2, we have

(34)

\[
C \begin{pmatrix}
\begin{array}{ccc}
1 & 2 \\
5 & 6 \\
3 & 4
\end{array}
\end{pmatrix} = \mathcal{N}_{00}\{1\} \xrightarrow{(\mathcal{P}_1, \mathcal{P}_2)} \frac{\mathcal{N}_{01}}{\mathcal{N}_{10}} \xrightarrow{(\mathcal{P}_3, \mathcal{P}_4)} \mathcal{N}_{11}\{-1\},
\]

where

\[
\mathcal{N}_{00} = M_{1,5,2,6,6}^{[1,k]} \boxtimes N_{6,4,5,3}^{[1,k]}, \quad \mathcal{N}_{10} = M_{1,5,2,6,6}^{[1,k]} \boxtimes M_{6,4,5,3}^{[1,k]},
\]

\[
\mathcal{N}_{01} = N_{1,5,2,6,6}^{[1,k]} \boxtimes N_{6,4,5,3}^{[1,k]}, \quad \mathcal{N}_{11} = N_{1,5,2,6,6}^{[1,k]} \boxtimes M_{6,4,5,3}^{[1,k]},
\]

\[
\mathcal{P}_1 = \text{Id}_{M_{1,5,2,6,6}^{[1,k]}} \boxtimes (\text{Id}_{N_{6,4,5,3}^{[1,k]}} \boxtimes (x_{1,6} - x_{1,3}, 1)),
\]

\[
\mathcal{P}_2 = (\text{Id}_{N_{1,5,2,6,6}^{[1,k]}} \boxtimes (1, x_{1,1} - x_{1,6})) \boxtimes \text{Id}_{N_{6,4,5,3}^{[1,k]}},
\]

\[
\mathcal{P}_3 = (\text{Id}_{N_{1,5,2,6,6}^{[1,k]}} \boxtimes (1, x_{1,1} - x_{1,6})) \boxtimes \text{Id}_{M_{6,4,5,3}^{[1,k]}},
\]

\[
\mathcal{P}_4 = -\text{Id}_{N_{1,5,2,6,6}^{[1,k]}} \boxtimes (\text{Id}_{N_{6,4,5,3}^{[1,k]}} \boxtimes (x_{1,6} - x_{1,3}, 1)).
\]

By Theorem 2.8, \( \mathcal{N}_{00}\{1\} \) is isomorphic to

\[
\bigotimes_{i=1}^{k} K(A_{i,(6,4,5,3)}^{[1,k]}, X_{i,(6,4)}^{(1,k)} - X_{i,(1,2,3,6)}^{(-1,k,1,1)})Q_1
\]

\[
\bigotimes_{i=1}^{k} K(u_{k+1,(6,4,5,3)}^{[1,k]}, X_{k,(3,4)}^{(-1,k)}(x_{1,6} - x_{1,3}), X_{k,(3,4)}^{(-1,k)}(x_{1,6} - x_{1,3}), Q_1\{3 - n\}\{1\},
\]

where

\[
Q_1 = R_{(1,2,3,4,5,6)}^{(1,k,1,k,k,1)} / \langle X_{1,(1,5)}^{(1,k)}, X_{1,(2,6)}^{(k,1)}, \ldots, X_{k,(1,5)}^{(1,k)}, X_{k,(2,6)}^{(-1,k)}, u_{k+1,(1,5,2,6)}^{[1,k]} \rangle.
\]

By Corollary 2.7, this matrix factorization is isomorphic to

\[
\mathcal{S}_{(1,4,2,3)}^{[1,k]} \boxtimes K(u_{k+1,(1,4,2,3)}^{[1,k]}, X_{k,(3,4)}^{(-1,k)}(x_{1,1} - x_{1,3}, 1), Q_1\{3 - n\}\{1\},
\]

where \( \mathcal{S}_{(1,4,2,3)}^{[1,k]} \) is a matrix factorization defined in Section 5.1. We also find that \( \mathcal{N}_{10} \) is isomorphic to

\[
\mathcal{S}_{(1,4,2,3)}^{[1,k]} \boxtimes K(u_{k+1,(6,4,5,3)}^{[1,k]} + \alpha(x_{1,6} - x_{1,3})X_{k,(3,4)}^{(-1,k)}Q_1\{1 - n\}\{1\},
\]
where \( \alpha \) is a polynomial with \( \mathbb{Z} \)-grading \( 2n - 2k \) satisfying

\[
(35) \quad (u_{k+1}^{[1,k]} + \alpha)(x_{1,6} - x_{1,3}) = u_{k+1}^{[1,k]}(x_{1,1} - x_{1,3})
\]

in the quotient \( Q_1 \), and that \( \mathbb{N}_{01} \) and \( \mathbb{N}_{11}\{1\} \) are isomorphic to

\[
\mathbb{S}_{(1,4,2,3)}^{[1,k]} \otimes K(u_{k+1}^{[1,k]}(x_{1,1} - x_{1,3}); X_{k,(3,4)}^{(-1,k)}) Q_2 \{1 - n\}\{1\},
\]

\[
\mathbb{S}_{(1,4,2,3)}^{[1,k]} \otimes K(u_{k+1}^{[1,k]} + \alpha; (x_{1,6} - x_{1,3})X_{k,(3,4)}^{(-1,k)}) Q_2 \{-1 - n\}\{1\},
\]

where

\[
Q_2 = R_{(1,2,3,4,5,6)}^{(1,k,1,k,k,1)} \\
/\langle X_{1,(1.5)}^{(1,k)} - X_{1,(2.6)}^{(1,k)}, \ldots, X_{k,(1.5)}^{(1,k)} - X_{k,(2.6)}^{(1,k)}, u_{k+1}^{[1,k]}(x_{1,1} - x_{1,6}) \rangle.
\]

Note that the polynomial \( \alpha \) also satisfies (35) in the quotient \( Q_2 \). We calculate the polynomials \( u_{k+1}^{[1,k]}(1,5,2,6) \) and \( u_{k+1}^{[1,k]}(6,4,5,3) \) to decompose \( \mathbb{N}_{00}, \mathbb{N}_{10}, \mathbb{N}_{01}, \) and \( \mathbb{N}_{11} \) into direct summands of indecomposable matrix factorizations.

Since we have equations in \( Q_1 \) and \( Q_2 \)

\[
X_{i,(5)}^{(k)} = X_{i,(1.2,6)}^{(-1,k,1)} \quad (1 \leq i \leq k),
\]

the polynomial \( u_{k+1}^{[1,k]}(1,5,2,6) \) is equal to

\[
F_{k+1}(X_{1,(2.6)}^{(1,k)}, \ldots, X_{k,(2.6)}^{(1,k)}, X_{k+1,(1.5)}^{(1,k)} - F_{k+1}(X_{1,(2.6)}^{(1,k)}, \ldots, X_{k,(2.6)}^{(1,k)}, X_{k+1,(2.6)}^{(1,k)})
\]

\[
X_{k+1,(1.5)}^{(1,k)} = X_{k+1,(2.6)}^{(1,k)}
\]

\[
= c_1 X_{1,(2.6)}^{(1,k)} n^{-k} + c_2 (X_{1,(2.6)}^{(1,k)}) n^{-k - 2} X_{2,(2.6)}^{(1,k)} + \cdots
\]

\[
= c_1 x_{1,6}^{n-k} + c_3 x_{1,2} x_{1,6}^{n-k-1} + \cdots,
\]

where \( c_1 \) and \( c_2 \) are the coefficients of monomials \( x_{1}^{n-k} \) and \( x_{1}^{n-k} \) in \( F_{k+1}(x_1, x_2, \ldots, x_{k+1}) = c_1 x_{1}^{n-k} x_{k+1} + c_2 x_{1}^{m-k-2} e_{2} x_{k+1} + \cdots \) and \( c_3 = c_1(n - k) + c_2 \). Then, the polynomial \( u_{k+1}^{[1,k]}(6,4,5,3) \) is equal in \( Q_1 \) to

\[
F_{k+1}(\ldots, X_{k+1,(1.4,6)}^{(-1,k,1)}, X_{k+1,(1.4,6)}^{(1,k)}, \ldots) - F_{k+1}(\ldots, X_{k+1,(1.4,6)}^{(-1,k,1)}, X_{k+1,(1.4,6)}^{(-1,k,1)}, \ldots)
\]

\[
X_{k+1,(1.4,6)}^{(1,k)} - X_{k+1,(1.4,6)}^{(-1,k,1)}
\]
\[c_1(X_{1,1}^{(-1,k,1,1)})^n - k + c_2(X_{1,1}^{(-1,k,1,1)})^{n-k-2} X_{2,1}^{(-1,k,1,1)} + \cdots\]
\[= c_1 x_1^{n-k} + c_3 (-x_1 + x_2 + x_3) x_1^{n-k-1} + \cdots\]
\[= -c_3 (x_1 - x_3) x_1^{n-k-1} + \cdots.\]

By the condition (35), we have the equation in \(Q_1\)

\[u_{k+1,1,6,4,5,3} + \alpha = -c_3 (x_1 - x_3) (x_1^{n-k-1} + \beta),\]

where \(\beta\) is a polynomial satisfying \(-(x_1 - x_3)c_3(x_1^{n-k-1} + \beta) = u_{k+1,1,4,2,3}\) in \(Q_1\). Thus, \(\mathcal{N}_{10}\) is isomorphic to

\[\mathcal{S}_{(1,4,2,3)} \otimes K0(-c_3(x_1 - x_3)(x_1^{n-k-1} + \beta); (x_1 - x_3)X_{k,1}^{(-1,k)})Q_1\]
\[\times \{1 - n\} \langle 1 \rangle.\]

The sets

\[\mathcal{B}_1 := \{1, x_1, \ldots, x_1^{n-k}, -c_3(x_1^{n-k-1} + \beta)\},\]
\[\mathcal{B}_1' := \{1, (x_1 - x_3), x_1(x_1 - x_3), \ldots, x_1^{n-k-2}(x_1 - x_3)\}\]

are bases of \(Q_1\) as an \(R_{1,2,3,4}^{(1,k,1,k)}\) module. Let \(R_1\) and \(R_1'\) denote the \(R_{1,2,3,4}^{(1,k,1,k)}\) modules spanned by \(\mathcal{B}_1\) and \(\mathcal{B}_1'\), respectively. The sets

\[\mathcal{B}_2 := \{1, (x_1 - x_1 - x_1), \ldots, x_1^{n-k-2}(x_1 - x_1), (u_{k+1,1,4,2,3} + \alpha)\},\]
\[\mathcal{B}_2' := \{1, (x_1 - x_3), (x_1 - x_1)(x_1 - x_3), \ldots, x_1^{n-k-2}(x_1 - x_3)(x_1 - x_3)\}\]

are bases of \(Q_2\) as an \(R_{1,2,3,4}^{(1,k,1,k)}\) module. Let \(R_2\) and \(R_2'\) denote the \(R_{1,2,3,4}^{(1,k,1,k)}\) modules spanned by \(\mathcal{B}_2\) and \(\mathcal{B}_2'\), respectively.

Using these bases, we find that \(K(u_{k+1,1,4,2,3}(x_1 - x_3); X_{k,1}^{(-1,k)})Q_1 \{3 - n\} \langle 1 \rangle\) is isomorphic to

\[(R_1, R_1 \{2k - n - 1\}, E_{n-k}(u_{k+1,1,4,2,3}(x_1 - x_3)),
E_{n-k}(X_{k,1}^{(-1,k)})) \{3 - n\} \langle 1 \rangle.\]
\( K(-c_3(x_{1,1} - x_{1,3})(x_{1,6}^{n-k-1} + \beta); (x_{1,6} - x_{1,3})X_{k,(3,4)}^{(-1,k)})Q_1 \{ 1 - n \} \{ 1 \} \) is isomorphic to

\[
(R'_1, R_1 \{ 2k - n + 1 \}, g_1, g_2) \{ 1 - n \} \{ 1 \},
\]

\[
g_1 = \begin{pmatrix}
  t \circ_{n-k-2} & E_{n-k-2}(u_{k+1,(1,4,2,3)}^{[1,k]}(x_{1,1} - x_{1,3})) \\
  x_{1,1} - x_{1,3} & \circ_{n-k-2}
\end{pmatrix},
\]

\[
g_2 = \begin{pmatrix}
  \circ_{n-k-2} & X_{k,(3,4)}^{(-1,k)} u_{k+1,(1,4,2,3)}^{[1,k]} \\
  E_{n-k-2}(X_{k,(3,4)}^{(-1,k)}) & t \circ_{n-k-2}
\end{pmatrix}.
\]

\( K(u_{k+1,(1,4,2,3)}^{[1,k]}(x_{1,1} - x_{1,3})X_{k,(3,4)}^{(-1,k)})Q_2 \{ 1 - n \} \{ 1 \} \) is isomorphic to

\[
(R_2, R_2 \{ 2k - n - 1 \}, E_{n-k+1}(u_{k+1,(1,4,2,3)}^{[1,k]}(x_{1,1} - x_{1,3})),
\]

\[
E_{n-k+1}(X_{k,(3,4)}^{(-1,k)}) \} 1 - n \} \{ 1 \}.
\]

\( K(u_{k+1,(6,4,5,3)}^{[1,k]}(x_{1,6} - x_{1,3})X_{k,(3,4)}^{(-1,k)})Q_2 \{ 3 - n \} \{ 1 \} \) is isomorphic to

\[
(R'_2, R_2 \{ 2k - n + 1 \}, g_3, g_4) \{ -1 - n \} \{ 1 \},
\]

\[
g_3 = \begin{pmatrix}
  t \circ_{n-k} & E_{n-k}(u_{k+1,(1,4,2,3)}^{[1,k]}(x_{1,1} - x_{1,3})) \\
  1 & \circ_{n-k}
\end{pmatrix},
\]

\[
g_4 = \begin{pmatrix}
  \circ_{n-k} & X_{k,(3,4)}^{(-1,k)} u_{k+1,(1,4,2,3)}^{[1,k]}(x_{1,1} - x_{1,3}) \\
  E_{n-k}(X_{k,(3,4)}^{(-1,k)}) & t \circ_{n-k}
\end{pmatrix}.
\]

With respect to the above isomorphisms, \( \mathbf{\nu}_1, \mathbf{\nu}_2, \mathbf{\nu}_3, \) and \( \mathbf{\nu}_4 \) induce

\[
\begin{pmatrix}
  E_{n-k}(\text{Id}_{N^{[1,k]}_{(1,4,2,3)}}(1, u_{k+1,(1,4,2,3)}^{[1,k]}) \\
  \circ_{n-k-1} & \circ_{n-k-1}
\end{pmatrix},
\]

\[
\begin{pmatrix}
  \circ_{n-k-1} & E_{n-k}(\text{Id}_{N^{[1,k]}_{(1,4,2,3)}}(-u_{k+1,(1,4,2,3)}^{[1,k]})) \\
  \circ_{n-k-1} & -u_{k+1,(1,4,2,3)}^{[1,k]}
\end{pmatrix},
\]

\[
\begin{pmatrix}
  E_{n-k}(\text{Id}_{N^{[1,k]}_{(1,4,2,3)}}(-u_{k+1,(1,4,2,3)}^{[1,k]})) \\
  \circ_{n-k-1} & \circ_{n-k-1}
\end{pmatrix},
\]

\[
-\begin{pmatrix}
  \circ_{n-k-1} & E_{n-k}(\text{Id}_{N^{[1,k]}_{(1,4,2,3)}}(-u_{k+1,(1,4,2,3)}^{[1,k]})) \\
  \circ_{n-k-1} & -u_{k+1,(1,4,2,3)}^{[1,k]}
\end{pmatrix},
\]
By a chain homotopy equivalence, we obtain the isomorphism (33). We can prove the other isomorphisms of Theorem 5.3(IIb) in a similar way.

**Remark 7.1.** We showed above how to decompose $N_{10}$ into a direct sum of indecomposable matrix factorizations. This corresponds to the MOY relation

$$
\langle k+1 \rangle_n = \langle 1 \rangle_n + [n-k-1]q\langle k-1 \rangle_n.
$$

**7.2. Proof of Proposition 5.6**

We show the isomorphism

$$
C \left( \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 7 & 1 \\
6 & 8 & k & 5 \\
3 & 4 & 4 & 5 \\
\end{array}
\end{array} \right)_{n} \simeq C \left( \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 7 & 1 \\
6 & 8 & k & 5 \\
3 & 4 & 4 & 5 \\
\end{array}
\end{array} \right)_{n}.
$$

The left-hand side is

$$
\mathcal{L}_{00} \{ (k+1)n \} \xrightarrow{\langle \xi_1, \xi_2 \rangle} \mathcal{L}_{10} \{ (k+1)n - 1 \} \oplus \mathcal{L}_{01} \{ (k+1)n - 1 \} \xrightarrow{\langle \xi_3, \xi_4 \rangle} \mathcal{L}_{11} \{ (k+1)n - 2 \},
$$

where

$$
\mathcal{L}_{00} = \mathcal{L}_{(1,6,7)}^{[1,k]} \otimes \mathcal{L}_{(8,6,4,3)}^{[1,1]} \otimes \mathcal{L}_{(2,7,5,8)}^{[1,k]}, \quad \mathcal{L}_{10} = \mathcal{L}_{(1,6,7)}^{[1,k]} \otimes \mathcal{L}_{(8,6,4,3)}^{[1,1]} \otimes \mathcal{L}_{(2,7,5,8)}^{[1,k]}.
$$
By Theorem 2.6, this matrix factorization is isomorphic to

\[ L_{01} = \Lambda^{[1,k]}_{(1,6,7)} \boxtimes M^{[1,1]}_{(8,6,4,3)} \boxtimes \bar{N}^{[1,k]}_{(2,7,5,8)}, \quad L_{11} = \Lambda^{[1,k]}_{(1,6,7)} \boxtimes \bar{N}^{[1,1]}_{(8,6,4,3)} \boxtimes \bar{N}^{[1,k]}_{(2,7,5,8)}, \]

\[ \zeta_1 = \text{Id}_{\Lambda^{[1,k]}_{(1,6,7)}} \boxtimes \left( \text{Id}_{S^{[1,1]}_{(8,6,4,3)}} \boxtimes (1, x_{1,8} - x_{1,3}) \right) \boxtimes \text{Id}_{M^{[1,k]}_{(2,7,5,8)}} , \]

\[ \zeta_2 = \text{Id}_{\Lambda^{[1,k]}_{(1,6,7)}} \boxtimes \text{Id}_{M^{[1,1]}_{(8,6,4,3)}} \boxtimes (1, x_{1,2} - x_{1,8}) , \]

\[ \zeta_3 = \text{Id}_{\Lambda^{[1,k]}_{(1,6,7)}} \boxtimes \text{Id}_{M^{[1,1]}_{(8,6,4,3)}} \boxtimes (1, x_{1,2} - x_{1,8}) , \]

\[ \zeta_4 = - \text{Id}_{\Lambda^{[1,k]}_{(1,6,7)}} \boxtimes \left( \text{Id}_{S^{[1,1]}_{(8,6,4,3)}} \boxtimes (1, x_{1,8} - x_{1,3}) \right) \boxtimes \text{Id}_{M^{[1,k]}_{(2,7,5,8)}} . \]

\[ L_{00} \] is isomorphic to

\[ \boxtimes_{i=1}^{k+1} K (\Lambda^{[1,k]}_{i,(1,6,7)}; x_{i,1} - X^{(1,k)}_{i,(6,7)}) Q_1 , \]

\[ \boxtimes K (u^{[1,k]}_{k+1,(2,7,5,8)}; (x_{1,2} - x_{1,8}) X^{(k,-1)}_{k,(7,8)}) Q_1 \{ -k - 1 \} , \]

where \( Q_1 \) is the \( R^{(k+1,1,1,1,1)}_{(1,2,3,4,5)} \)-module

\[ R^{(k+1,1,1,1,1,1,k,1,1)}_{(1,2,3,4,5,6,7,8)} , \]

\[ / \langle X^{(1,1)}_{1,(8,6)} - X^{(1,1)}_{1,(4,3)} , X^{(-1,1,1)}_{2,(3,6,8)} , X^{(1,k)}_{1,(5,8)} , \ldots , X^{(1,k)}_{k,(2,7)} - X^{(1,k)}_{k,(5,8)} \rangle . \]

By Theorem 2.6, this matrix factorization is isomorphic to

\[ \boxtimes_{i=1}^{k+1} K (\Lambda^{[1,k]}_{i,(1,6,7)}; x_{i,1} - X^{(-1,1,1,1,k)}_{i,(2,3,4,5)}) R^{(k+1,1,1,1,1,k)}_{(1,2,3,4,5)} , \]

\[ \boxtimes K (u^{[1,k]}_{k+1,(2,7,5,8)}; - \Lambda^{[1,k]}_{k+1,(1,6,7)}; (x_{1,2} - x_{1,8}) X^{(-1,k)}_{k,(2,5)}) Q_1 \{ -k - 1 \} . \]

Moreover, by Corollary 2.7, this matrix factorization is isomorphic to

\[ (38) \quad \bar{\mathcal{S}}^{[k+1,1,1,1,1,1]}_{(1,2,3,4,5)} \boxtimes K (B_0; (x_{1,2} - x_{1,8}) X^{(-1,k)}_{k,(2,5)}) Q_1 \{ -k - 1 \} , \]

where \( B_0 \) is a polynomial with degree \( 2n - 2k \) in \( Q_1 \) satisfying \( (x_{1,2} - x_{1,8}) B_0 \), denoted by \( B \), in the image of \( R^{(k+1,1,1,1,1,k)}_{(1,2,3,4,5)} \) under the inclusion map to \( Q_1 \), and \( \bar{\mathcal{S}}^{[k+1,1,1,1,1,1]}_{(1,2,3,4,5)} \) is the matrix factorization

\[ \boxtimes_{i=1}^{k+1} K (B_i; x_{i,1} - X^{(-1,1,1,1,k)}_{i,(2,3,4,5)}) R^{(k+1,1,1,1,1,k)}_{(1,2,3,4,5)} \quad (B_i \in R^{(k+1,1,1,1,1,k)}_{(1,2,3,4,5)}). \]
The sets $\mathcal{B}_1 := \{1, (x_{1,2} - x_{1,8})\}$ and $\mathcal{B}_1' := \{1, (x_{1,8} + x_{1,2} - x_{1,3} - x_{1,4})\}$ are bases of the $R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)}$-module $Q_1$. Let $R_1$ and $R_1'$ denote the $R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)}$-module spanned by $\mathcal{B}_1$ and $\mathcal{B}_1'$, respectively. Using the bases, the matrix factorization $K(B_0; (x_{1,2} - x_{1,8})X_{k,(2,5)}^{(-1,k)})Q_1$ is isomorphic to
\[
\begin{pmatrix}
R_1, R_1' \{2k - n + 1\}, \begin{pmatrix} 0 & B \\ A \end{pmatrix}, \begin{pmatrix} 0 & X_{2,(2,3,4)}^{(-1,1)} X_{k,(2,5)}^{(-1,k)} \\ X_{k,(2,5)}^{(-1,k)} \end{pmatrix}
\end{pmatrix}.
\]

We find that $(x_{1,2} - x_{1,8})X_{k,(2,5)}^{(-1,k)} : R_1' \rightarrow R_1$ is an antidiagonal matrix. Since $(x_{1,2} - x_{1,8})X_{k,(2,5)}^{(-1,k)}B_0 : R_1 \rightarrow R_1$ is a diagonal matrix, $B_0 : R_1 \rightarrow R_1'$ is also an antidiagonal matrix. Therefore, the polynomial $A$ is equal to $B/X_{2,(2,3,4)}^{(-1,1)}$.

Thus, the matrix factorization (38) is isomorphic to
\[
\overline{S}_{(1,2,3,4,5)}^{[k+1,1,1,1,k]} \boxtimes K \left( \begin{pmatrix} B \\ X_{2,(2,3,4)}^{(-1,1)} X_{k,(2,5)}^{(-1,k)} \end{pmatrix} R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)} \{ -k - 1 \} 
+ \overline{S}_{(1,2,3,4,5)}^{[k+1,1,1,1,k]} \boxtimes K(B; X_{k,(2,5)}^{(-1,k)}) R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)} \{ -k + 1 \} \right).
\]

$\overline{L}_{10}$ is isomorphic to
\[
\boxtimes \ K(A_{i,(1,6,7)}^{[1,k]} : x_{i,1} - X_{i,(2,3,4,5)}^{(-1,1,1,k)})Q_2,
\]

$\boxtimes K(u_{k+1,(2,7,5,8)}^{[1,k]} - A_{k+1,(1,6,7)}^{[1,k]} : (x_{1,2} - x_{1,4})X_{k,(2,5)}^{(-1,k)})Q_2 \{ -k \},$

where $Q_2$ is the $R_{(1,2,3,4,5)}^{(k+1,1,1,1,k,1,k,1)}$-module
\[
/\langle X_{1,(8,6)}^{(1,1)} - X_{1,(4,3)}^{(1,1)}, X_{1,(6,3)}^{(1,-1)}, X_{1,(2,7)}^{(1,k)}, X_{1,(5,8)}^{(k,1)}, \ldots, X_{k,(2,7)}^{(1,k)} - X_{k,(5,8)}^{(k,1)} \rangle.
\]

By the isomorphism $Q_2 \simeq R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)}$ and Theorem 2.6, this matrix factorization is isomorphic to
\[
\overline{S}_{(1,2,3,4,5)}^{[k+1,1,1,1,k]} \boxtimes K \left( \frac{B}{(x_{1,2} - x_{1,4})X_{k,(2,5)}^{(-1,k)}} \right) R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)} \{ -k \}.
\]

By Theorem 2.8, we find that $\overline{L}_{01}$ is isomorphic to
\[
(39) \quad \overline{S}_{(1,2,3,4,5)}^{[k+1,1,1,1,k]} \boxtimes K(B; X_{k,(2,5)}^{(-1,k)})Q_1 \{ -k \}.
\]
By a chain homotopy equivalence, the left-hand side of (37) is isomorphic to
\[
\left( R_1, R_1\{2k - n - 1\}, \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} X_{k,(2,5)}^{(-1,k)} & 0 \\ 0 & X_{k,(2,5)}^{(-1,k)} \end{pmatrix} \right),
\]
then the matrix factorization (39) is isomorphic to
\[
\begin{aligned}
&\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)} \boxtimes K(B; X_{k,(2,5)}^{(-1,k)}) R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)} \{-k\} \\
&\oplus \mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)} \boxtimes K(B; X_{k,(2,5)}^{(-1,k)}) R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)} \{-k + 2\}.
\end{aligned}
\]
By Theorem 2.8, \( L_{11} \) is isomorphic to
\[
\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)} \boxtimes K(B; X_{k,(2,5)}^{(-1,k)}) R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)} \{-k + 1\}.
\]
With respect to the above isomorphisms, the morphisms \( \bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3, \) and \( \bar{\zeta}_4 \) induce
\[
\begin{pmatrix}
\text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (1, x_{1,2} - x_{1,3}), & \text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (x_{1,2} - x_{1,4}, 1)
\end{pmatrix},
\]
\[
\begin{pmatrix}
\text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (1, (x_{1,2} - x_{1,3})(x_{1,2} - x_{1,4})), & 0 \\
0 & \text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (1, 1)
\end{pmatrix},
\]
\[
\text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (1, x_{1,2} - x_{1,4}),
\]
\[
- \left( \text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (1, 1), \text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (x_{1,2} - x_{1,4}, x_{1,2} - x_{1,4}) \right).
\]
By a chain homotopy equivalence, the left-hand side of (37) is isomorphic to
\[
-k - 1 \quad \xrightarrow{\text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (1, x_{1,2} - x_{1,3})} \quad -k
\]
\[
\overline{L}_1\{(k + 1)n\}(k + 1) \quad \xrightarrow{\text{Id}_{\mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)}} \boxtimes (1, x_{1,2} - x_{1,3})} \quad \overline{L}_2\{(k + 1)n - 1\}(k + 1),
\]
where
\[
\overline{L}_1 = \mathcal{S}^{[k+1,1,1,1,k]}_{(1,2,3,4,5)} \boxtimes K\left( \frac{B}{X_{2,(2,3,4)}^{(-1,1)}; X_{2,(2,3,4)}^{(-1,1)} X_{k,(2,5)}^{(-1,k)}} R_{(1,2,3,4,5)}^{(k+1,1,1,1,k)} \{-k - 1\},
\right)
\]
\[ \mathcal{T}_2 = \overline{S}_{(1,2,3,4,5)}^{[k+1,1,1,k]} \otimes K\left( \frac{B}{x_{1,2} - x_{1,4}}; (x_{1,2} - x_{1,4})X_{(2,5)}^{(1,k)} \right) R^{(k+1,1,1,1,k)}_{(1,2,3,4,5)} \{ -k \}. \]

The right-hand side of (37) is

\[ -k - 1 \quad \xi \quad -k \]

\[ \overline{M}_{(2,1,6,3)}^{[1,k+1]} \otimes \Lambda_{(6;4,5)}^{[1,k]} \{ (k + 1)n \} \langle k + 1 \rangle \xrightarrow{\xi} \overline{N}_{(2,1,6,3)}^{[1,k+1]} \otimes \Lambda_{(6;4,5)}^{[1,k]} \{ (k + 1)n - 1 \} \langle k + 1 \rangle , \]

where \( \xi = \chi_{+, (2,1,6,3)}^{[1,k+1]} \otimes \text{Id}_{\Lambda_{(6;4,5)}^{[1,k]}} \). By Theorem 2.8, \( \overline{M}_{(2,1,6,3)}^{[1,k+1]} \otimes \Lambda_{(6;4,5)}^{[1,k]} \) is isomorphic to

\[ \bigotimes_{i=1}^{k+1} K(A_{i,(2,1,6,3)}^{[1,k+1]}; X_{i,(1,2)}^{(1,k)} - X_{i,(3,4,5)}^{(1,k)})Q_3 ; \]

\[ \otimes K(u_{k+2,(2,1,6,3)}^{[1,k+1]}; (x_{1,2} - x_{1,3})X_{k+1,(1,3)}^{(k+1,-1)})Q_3 \{ -k - 1 \}, \]

where \( Q_3 \) is the \( R^{(k+1,1,1,1,k+1)}_{(1,2,3,4,5)} \)-module

\[ R^{(k+1,1,1,1,k+1)}_{(1,2,3,4,5,6)} / \langle x_{1,6} - X_{1,(4,5)}^{(1,k)}, \ldots, x_{k+1,6} - X_{k+1,(4,5)}^{(1,k)} \rangle. \]

Moreover, \( Q_3 \cong R^{(k+1,1,1,1,k+1)}_{(1,2,3,4,5)} \). This fact and Theorem 2.6 imply that this matrix factorization is isomorphic to \( \overline{L}_1 \). In a similar way, we find that \( \overline{M}_{(2,1,6,3)}^{[1,k+1]} \otimes \Lambda_{(6;4,5)}^{[1,k]} \) is isomorphic to \( \overline{L}_1 \). With respect to the above isomorphisms, \( \xi \) induces \( \text{Id}_{\overline{L}_1} \otimes (1, x_{1,2} - x_{1,3}) \). Thus, we have the isomorphism (37). In a similar way, we can prove the other isomorphisms of Proposition 5.6.

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