Partial cohomology of groups

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Abstract

We develop a cohomology theory of groups based on partial actions and explore its relation with the partial Schur multiplier as well as with cohomology of inverse semigroups.

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Introduction

In [12] R. Exel initiated a new and successful method to study $C^*$-algebras by showing that a separable $C^*$-algebra possessing a semi-saturated stable action of the circle group is a crossed product by a partial action of the infinite cyclic group. This permitted him to investigate the internal structure, representations and K-theory of such algebras. Exel’s approach is based on new concepts, namely, those of a partial action, the corresponding crossed product and a partial representation, as well as on the interaction between them. Relevant classes of $C^*$-algebras, endowed with the structure of non-trivial crossed product by a partial action, include the Bunce-Deddens and the Bunce-Deddens-Toeplitz algebras [13], the approximately finite dimensional algebras [14], the Toeplitz algebras of quasi-ordered groups, the Cuntz-Krieger algebras [18], [27] and, more recently, the $C^*$-algebras $O_{m,n}$ related to dynamical systems in which $n$ copies of a topological space is homeomorphic to $m$ copies of the same space [1].

The first algebraic results on the above mentioned notions were established in [16], [5], [29], [22] and [4], which together with the development of the partial Galois theory in [8] stimulated an intensive algebraic activity on partial actions, corresponding crossed products and partial representations (see the surveys [3] and [20]). In particular, applications were obtained to graded algebras in [4] and [6], to Hecke algebras in [17] and to Leavitt path algebras in [21].

The general notion of a (continuous) twisted partial action of a locally compact group on a $C^*$-algebra (a twisted partial $C^*$-dynamical system) was given in [15]. The new construction permitted one to show that any second countable $C^*$-algebraic bundle, which satisfies a certain regularity condition (automatically verified if the unit fiber algebra is stable), is a $C^*$-crossed product of the unit fiber algebra by a continuous partial action of the base group. A purely ring theoretic treatment of twisted partial group actions, including an analogue of the above mentioned fact, was given in [3]. It involves a general twisting which satisfies the 2-cocycle identity in some restricted sense, and one naturally wonders what kind of cohomology theory would fit this. It is complicated to give such a theory which would embrace the full generality of [3], since the twisting takes its values in multiplier algebras of products of some ideals indexed by group elements. Nevertheless, imposing a reasonable restriction on the partial action, namely,
that it is unital, makes it possible to define partial cohomology. Actually, this restriction is assumed in almost all algebraic papers around partial actions, because it provides an appropriate technical framework.

One feels more confident in developing such a cohomology theory if one shows that it matches some other concepts in a way how the usual group cohomology does. In particular, the Schur multiplier of a group \( G \) over the complex numbers \( \mathbb{C} \) is isomorphic to \( H^2(G, \mathbb{C}^*) \) with trivial action of \( G \) on \( \mathbb{C}^* \). Thus a theory of partialprojective group representations could be a testing field for our cohomology. Such a theory was developed in [9], [10] and [11], in particular the structure of the partial Schur multiplier was explored. Further results on the latter topic were obtained in [25] and [26].

A twisted partial action of a group \( G \) on a commutative ring \( A \) falls into two parts: a partial action \( \theta \) of \( G \) on \( A \) and its twisting. With an additional assumption that \( \theta \) is unital (which means that the domains involved in \( \theta \) are generated by central idempotents), one can derive the concept of a partial 2-cocycle (the twisting) whose values belong to groups of invertible elements of appropriate ideals of \( A \). The concept of a partial 2-coboundary then follows from that of an equivalence of twisted partial actions introduced in [7]. Replacing \( A \) by a commutative multiplicative monoid, one comes to the definition of the second cohomology group \( H^2(G, A) \).

Thus instead of a usual \( G \)-module we deal with a partial \( G \)-module, which is a commutative monoid \( A \) with a unital partial action \( \theta \) of \( G \) on \( A \). The groups \( H^n(G, A) \) with arbitrary \( n \) are defined in a similar way (see Section [1]). Replacing \( A \) by an appropriate submonoid one may actually assume that \( A \) is inverse (see Remark 3.13). Next one asks how to obtain these groups using, say, projective resolutions. It turns out that the category of partial \( G \)-modules is not abelian since some sets of morphisms may be empty. Fortunately, our cohomology can be related to cohomology of inverse semigroups, as developed by H. Lausch in [23], via the inverse monoid \( S(G) \) introduced by R. Exel in [16] to serve partial actions and partial representations of \( G \). From a partial action of \( G \) one comes to an action of \( S(G) \) and then to an “almost” Lausch’s \( S(G) \)-module. The latter can be seen as a module in the sense of H. Lausch over an epimorphic image of \( S(G) \). Thus our category is made up of abelian “components” which are categories of Lausch’s modules over epimorphic images of \( S(G) \) (see Remark 3.39). This way we are able to define free objects and free resolutions which lead to \( H^n(G, A) \).

The article is organized as follows. After establishing the basic concept in Section 1 we show in Section 2 that the partial Schur multiplier \( pM(G) \) is a union of 2-cohomology groups with values in some, in general non-trivial, partial \( G \)-modules (Theorem 2.13). The passage from partial \( G \)-modules to \( S(G) \)-modules via unital actions of \( S(G) \) is done in Section 3 (Theorem 3.7). Our notion of a module \( A \) over an inverse semigroup \( S \) is more general than that introduced in [23] (Definition 3.3), and the relation with modules in the sense of [23] (strict \( S \)-modules) is given in Proposition 3.30. Here a crucial role is played by the semi-direct product \( A \rtimes S \) (Definition 3.22), which is the semigroup analogue of the crossed product introduced in [19]. If \( S = S(G) \) then \( A \rtimes S \) is isomorphic to the crossed product \( A \ast \theta G \) (Remark 3.32) defined in [9]. In the same section we also define free \( S \)-modules and show their existence in Proposition 3.47 using Lausch’s construction of (strict) free \( S \)-modules. Finally, in Section 4 we give a free resolution in order to obtain the cohomology groups \( H^n(G, A) \) (Theorem 4.11).

There is a very natural point to be clarified, namely, the relation of low dimensional partial group cohomology to extensions. The constraints of the present article do not allow us to treat it here, and it is being considered in a separate paper under elaboration.

1 The notions

Let \( G \) be a group and \( A \) a semigroup. Recall from [9] that a partial action \( \theta \) of \( G \) on \( A \) is a collection of semigroup isomorphisms \( \theta_x : A_{x^{-1}} \to A_x \), where \( A_x \) is an ideal of \( A \), \( x \in G \), such that
(i) \( A_1 = A \) and \( \theta_1 = \text{id}_A \);
(ii) \( \theta_x(A_{x^{-1}} \cap A_y) = A_x \cap A_{xy} \);
(iii) \( \theta_x \circ \theta_y = \theta_{xy} \) on \( A_{y^{-1}} \cap A_{y^{-1}x^{-1}} \).

We consider the case when \( A \) is a commutative monoid and each ideal \( A_x \) is unital, i.e. \( A_x \) is generated by an idempotent \( 1_x = 1_x^A \), which is central in \( A \). In this case we shall say that \( \theta \) is a unital partial action. Then \( A_x \cap A_y = A_x A_y \), so the properties (ii) and (iii) from the above definition can be replaced by

(ii') \( \theta_x(A_{x^{-1}A_y}) = A_x A_{xy} \);
(iii') \( \theta_x \circ \theta_y = \theta_{xy} \) on \( A_{y^{-1}A_{y^{-1}x^{-1}}} \).

Note also that (ii') implies a more general equality

\[
\theta_x(A_{x^{-1}A_{y_1} \cdots A_{y_n}}) = A_x A_{xy_1} \cdots A_{xy_n},
\]

which easily follows by observing that \( A_{x^{-1}A_{y_1} \cdots A_{y_n}} = A_{x^{-1}A_{y_1} \cdots A_{x^{-1}A_{y_n}}} \).

**Definition 1.1.** A commutative monoid \( A \) with a unital partial action \( \theta \) of \( G \) on \( A \) will be called a (unital) partial \( G \)-module.

Recall from [10] that a morphism of partial actions \((A, \theta) \to (A', \theta')\) of \( G \) is a homomorphism of semigroups \( \varphi : A \to A' \) such that \( \varphi(A_x) \subseteq A'_x \) and \( \varphi \circ \theta_x = \theta'_x \circ \varphi \) on \( A_{x^{-1}} \).

**Definition 1.2.** A morphism of (unital) partial \( G \)-modules \( \varphi : (A, \theta) \to (A', \theta') \) is a morphism of partial actions such that its restriction on each \( A_x \) is a homomorphism of monoids \( A_x \to A'_x \).

The category of (unital) partial \( G \)-modules and their morphisms will be denoted by \( \text{pMod}(G) \). Sometimes \((A, \theta)\) will be denoted simply by \( A \).

**Definition 1.3.** Let \( A \in \text{pMod}(G) \) and \( n \) be a positive integer. An \( n \)-cochain of \( G \) with values in \( A \) is a function \( f : G^n \to A \), such that \( f(x_1, \ldots, x_n) \) is an invertible element of the ideal \( A_{(x_1, \ldots, x_n)} = A_{x_1} A_{x_2} \cdots A_{x_1 \cdots x_n} \). By a 0-cochain we shall mean an invertible element of \( A \).

Denote the set of \( n \)-cochains by \( C^n(G, A) \). It is an abelian group under the pointwise multiplication. Indeed, its identity is

\[
e_n(x_1, \ldots, x_n) = 1_{x_11_{x_1}2 \cdots 1_{x_1 \cdots x_n}}
\]

and the inverse of \( f \in C^n(G, A) \) is \( f^{-1}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)^{-1} \), where \( f(x_1, \ldots, x_n)^{-1} \) means the inverse of \( f(x_1, \ldots, x_n) \) in \( A_{(x_1, \ldots, x_n)} \).

**Definition 1.4.** Let \((A, \theta) \in \text{pMod}(G) \) and \( n \) be a positive integer. For any \( f \in C^n(G, A) \) and \( x_1, \ldots, x_{n+1} \in G \) define

\[
(\delta^n f)(x_1, \ldots, x_{n+1}) = \theta_{x_1}(1_{x_1^{-1}f(x_2, \ldots, x_{n+1}))}
\]

\[
\prod_{i=1}^n f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i}
\]

\[
f(x_1, \ldots, x_n)^{(-1)^n+1}.
\]

Here the inverse elements are taken in the corresponding ideals. If \( n = 0 \) and \( a \) is an invertible element of \( A \), we set \((\delta^0 a)(x) = \theta_x(1_{x^{-1}}a)a^{-1} \).
Proposition 1.5. $\delta^n$ is a homomorphism $C^n(G,A) \to C^{n+1}(G,A)$, such that

$$\delta^{n+1}\delta^n f = e_{n+2}$$

(3)

for any $f \in C^n(G,A)$.

Proof. Let $f \in C^n(G,A)$. We check first that $\delta^n f \in C^{n+1}(G,A)$. Indeed, for $x_1,\ldots,x_{n+1} \in G$ the element $f(x_2,\ldots,x_{n+1})$ is invertible in $A_{(x_2,\ldots,x_{n+1})}$. Then being multiplied by $1_{x_1^{-1}}$ it becomes an invertible element of $A_{x_1^{-1}}A_{(x_2,\ldots,x_{n+1})}$. Therefore, $\theta_{x_1}(1_{x_1^{-1}}f(x_2,\ldots,x_{n+1}))$ is invertible in $A_{(x_1,\ldots,x_{n+1})}$, because $\theta_{x_1}$ maps isomorphically $A_{x_1^{-1}}A_{(x_2,\ldots,x_{n+1})}$ onto $A_{(x_1,\ldots,x_{n+1})}$ by (1). Since the product of invertible elements of some ideals is invertible in the product of these ideals, then by (2) the image $(\delta^n f)(x_1,\ldots,x_{n+1})$ is invertible in

$$A_{(x_1,\ldots,x_{n+1})}\left(\prod_{i=1}^{n} A_{(x_1,x_{i+1},\ldots,x_{n+1})}\right) A_{(x_1,\ldots,x_n)} = A_{(x_1,\ldots,x_n)}.$$

As $A$ is commutative, to see that $\delta^n$ is a homomorphism it suffices to note that

$$\theta_{x_1}(1_{x_1^{-1}}f(g(x_2,\ldots,x_{n+1})) = \theta_{x_1}(1_{x_1^{-1}}f(x_2,\ldots,x_{n+1})1_{x_1^{-1}}g(x_2,\ldots,x_{n+1}))$$

$$= \theta_{x_1}(1_{x_1^{-1}}f(x_2,\ldots,x_{n+1}))\theta_{x_1}(1_{x_1^{-1}}g(x_2,\ldots,x_{n+1})).$$

It remains to prove (3). Take arbitrary $x_1,\ldots,x_{n+2} \in G$. The factors in the product $(\delta^{n+1}\delta^n f)(x_1,\ldots,x_{n+2})$ to which the partial action is applied are as follows:

$$\theta_{x_1}(1_{x_1^{-1}}\theta_{x_2}(1_{x_2^{-1}}f(x_3,\ldots,x_{n+2}))),$$

$$\theta_{x_1x_2}(1_{x_2^{-1}}x_1^{-1}f(x_3,\ldots,x_{n+2})^{-1}),$$

$$\theta_{x_1}(1_{x_1^{-1}}f(x_2,\ldots,x_{n+1})^{(-1)^{i+1}}),$$

$$\theta_{x_1}(1_{x_1^{-1}}f(x_2,\ldots,x_{n+1})^{(-1)^{i+2}}),$$

$$\theta_{x_1}(1_{x_1^{-1}}f(x_2,\ldots,x_{i+1},\ldots,x_{n+2})^{(-1)^{i-1}}),$$

$$\theta_{x_1}(1_{x_1^{-1}}f(x_2,\ldots,x_{i+1},\ldots,x_{n+2})^{(-1)^{i}}),$$

$$2 \leq i \leq n + 1.$$

The product of all factors, except the first two, is $e_{n+2}(x_1,\ldots,x_{n+2})$ for $n \geq 1$. For $n = 0$ the product is $e_1(x_1)$. Furthermore,

$$\theta_{x_1}(1_{x_1^{-1}}\theta_{x_2}(1_{x_2^{-1}}f(x_3,\ldots,x_{n+2})))$$

$$= \theta_{x_1}(\theta_{x_2}(1_{x_2^{-1}}x_2^{-1}f(x_3,\ldots,x_{n+2})))$$

$$= \theta_{x_1x_2}(1_{x_2^{-1}}x_1^{-1}f(x_3,\ldots,x_{n+2}))$$

by the property (iii’) from the definition of a partial action. After multiplying this by the second factor we shall obtain

$$\theta_{x_1x_2}(1_{x_2^{-1}}x_1^{-1}e_n(x_3,\ldots,x_{n+2}))$$

$$= 1_{x_1x_2}e_n(x_1x_2x_3, x_4, \ldots, x_{n+2})$$

$$= e_{n+2}(x_1,\ldots,x_{n+2}).$$

Any other factor in $(\delta^{n+1}\delta^n f)(x_1,\ldots,x_{n+2})$ appears together with its inverse, as in the classical case, and multiplying such a pair we obtain a product of some of the idempotents $1_{x_1}, 1_{x_1x_2}, \ldots$. Thus, $(\delta^{n+1}\delta^n f)(x_1,\ldots,x_{n+2}) = e_{n+2}(x_1,\ldots,x_{n+2})$ as desired. \qed
Definition 1.6. The map $\delta^n$ is called a \textit{coboundary homomorphism}. As in the classical case we define the abelian groups $Z^n(G, A) = \ker \delta^n$, $B^n(G, A) = \text{im} \delta^{n-1}$ and $H^n(G, A) = \ker \delta^n / \text{im} \delta^{n-1}$ of partial $n$-cocycles, $n$-coboundaries and $n$-cohomologies of $G$ with values in $A$, $n \geq 1$ ($H^0(G, A) = Z^0(G, A) = \ker \delta^0$).

For example,

\[
H^0(G, A) = Z^0(G, A) = \{a \in \mathcal{U}(A) \mid \theta_x(a1_{x^{-1}}) = a1_x, \forall x \in G\},
\]

\[
B^1(G, A) = \{f \in C^1(G, A) \mid f(x) = \theta_x(a1_{x^{-1}})a^{-1}, \text{ for some } a \in \mathcal{U}(A)\}
\]

(here and below $\mathcal{U}(A)$ denotes the group of invertible elements of $A$). Notice that $H^0(G, A)$ is the subgroup of $\theta$-invariants of $\mathcal{U}(A)$ (see [8, p. 79]). Furthermore,

\[
(\delta^1 f)(x, y) = \theta_x(f(y)1_{x^{-1}})f(xy)^{-1}f(x)
\]

for $f \in C^1(G, A)$, so that

\[
Z^1(G, A) = \{f \in C^1(G, A) \mid f(xy)1_x = f(x)\theta_x(f(y)1_{x^{-1}}), \forall x, y \in G\},
\]

\[
B^2(G, A) = \{g \in C^2(G, A) \mid g(x, y) = \theta_x(f(y)1_{x^{-1}})f(xy)^{-1}f(x)
\]

for some $f \in C^1(G, A)$).

For $n = 2$ we have

\[
(\delta^2 f)(x, y, z) = \theta_x(f(y, z)1_{x^{-1}})f(xy, z)^{-1}f(x, yz)f(x, y)^{-1},
\]

with $f \in C^2(G, A)$, and

\[
Z^2(G, A) = \{f \in C^2(G, A) \mid \theta_x(f(y, z)1_{x^{-1}})f(x, yz) = f(xy, z)f(x, y),
\]

\[
\forall x, y, z \in G\}.
\]

Observe that if one takes a unital twisted partial action (see [6, Def. 2.1]) of $G$ on a commutative ring $A$ then it is readily seen that the twisting is a 2-cocycle with values in the partial $G$-module $A$, and the concept of equivalent unital twisted partial actions from [7, Def. 6.1] is exactly the notion of cohomologous 2-cocycles from Definition 1.6.

Proposition 1.7. The map which sends a partial $G$-module $A$ to the sequence

\[
C^0(G, A) \xrightarrow{\delta^0} C^1(G, A) \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{n-1}} C^n(G, A) \xrightarrow{\delta^n} \ldots
\]

is a functor from $\text{pMod}(G)$ to the category of complexes of abelian groups.

Proof. Let $\varphi : (A, \theta) \to (A', \theta')$ be a morphism of partial $G$-modules and $f \in C^n(G, A)$. Define $\tilde{\varphi}_n f = \varphi \circ f : G^n \to A'$. We need to show that $\tilde{\varphi}_n f \in C^n(G, A')$. Indeed, for any $x_1, \ldots, x_n$ the element $f(x_1, \ldots, x_n)$ is invertible in $A_{(x_1, \ldots, x_n)}$. Since

\[
\varphi(1^A_{x_1}1^A_{x_2} \cdots 1^A_{x_1, \ldots, x_n}) = 1^{A'}_{x_1}1^{A'}_{x_2} \cdots 1^{A'}_{x_1, \ldots, x_n},
\]

then the restriction of $\varphi$ to $A_{(x_1, \ldots, x_n)}$ is a homomorphism of monoids $A_{(x_1, \ldots, x_n)} \to A'_{(x_1, \ldots, x_n)}$. Therefore, $\varphi(f(x_1, \ldots, x_n))$ is invertible in $A'_{(x_1, \ldots, x_n)}$ and its inverse is $\varphi(f(x_1, \ldots, x_n)^{-1})$. Consequently, in order to prove that $\{\tilde{\varphi}_n\}_{n \geq 0}$ is a morphism of complexes, i.e. to verify that $\tilde{\varphi}_{n+1} \circ \delta^n = \delta^n \circ \tilde{\varphi}_n$, we only need to show that

\[
\varphi \circ \theta_{x_1}(1^A_{x_1}, f(x_2, \ldots, x_{n+1})) = \theta'_{x_1}(1^{A'}_{x_1}, \varphi \circ f(x_2, \ldots, x_{n+1})).
\]
By the definition of a morphism of partial $G$-modules,

$$\varphi \circ \theta_{x_1}(1^{A}_{x_1} f(x_2, \ldots, x_{n+1})) = \theta'_{x_1} \circ \varphi(1^{A}_{x_1} f(x_2, \ldots, x_{n+1}))$$

$$= \theta'_{x_1}(\varphi(1^{A}_{x_1}) \varphi(f(x_2, \ldots, x_{n+1})))$$

$$= \theta'_{x_1}(1^{A}_{x_1} \varphi(f(x_2, \ldots, x_{n+1})))$$

as desired.

The remaining properties of a morphism of partial $G$-modules are straightforward.

**Remark 2.1.**

If $\Gamma : \mathcal{K} \rightarrow \mathcal{H}$ induces $(\Gamma^\sigma, \theta^\sigma) \in \text{pMod}(G)$, where $\mathcal{K}$ and $\mathcal{H}$ are $G$-modules, and $\sigma : G \times G \rightarrow \mathcal{K}^*$ is a $G$-valued twisting related to $\mathcal{K}$, then $(\Gamma^\sigma, \theta^\sigma)$ is isomorphic to the trivial $G$-module $\mathcal{K}$ (where $\mathcal{K}$ is considered as a multiplicative semigroup). Moreover, $H^n(G, \mathcal{K}) \cong H^n(G, \mathcal{K}^*)$, where $H^n(G, \mathcal{K}^*)$ is the classical $n$-th cohomology group of $G$ with values in the trivial $G$-module $\mathcal{K}^*$.

**Corollary 1.8.** For any $n \geq 0$ the map $A \mapsto H^n(G, A)$ determines a functor from $\text{pMod}(G)$ to the category $\text{Ab}$ of abelian groups.

## 2 $H^2(G, A)$ and the partial Schur multiplier

In this section we assume that $M$ is a $K$-cancellative monoid $\mathcal{M}$, where $K$ is a field. Recall from [9] Theorem 6] that each partial projective representation $\Gamma : G \rightarrow M$, $\Gamma(1) = 1$, with (partial) factor set $\sigma : G \times G \rightarrow \mathcal{K}^*$ induces $(\Gamma^\sigma, \theta^\sigma) \in \text{pMod}(G)$. Here $\Gamma^\sigma$ is the commutative sub-monoid of $\mathcal{M}$ generated by $\alpha e_x$, where $\alpha \in \mathcal{K}$, and $e_x$ is the idempotent $\Gamma(x) \Gamma(x^{-1}) \sigma(x^{-1}, x)^{-1}$ if $\Gamma(x) \neq 0$ and zero otherwise (we know that $\Gamma(x) \neq 0 \iff \Gamma(x^{-1}) \neq 0 \iff \sigma(x^{-1}, x)$ is defined). The ideal $A_{x_1}^\Gamma$ is generated by $e_x$ and $\theta^\sigma_x(a) = \Gamma(x)a \Gamma(x^{-1})\sigma(x^{-1}, x)^{-1}$ for $a \in A_{x_1}^\Gamma \neq 0$, and if $A_{x_1}^\Gamma = 0$ then $A_{x_1}^\Gamma = 0$, so $\theta^\sigma_x$ is trivial. One extends $\sigma$ to a fully defined map $\sigma : G \times G \rightarrow K$ by setting $\sigma(x, y) = 0$ for each $(x, y) \notin \text{dom } \sigma$.

**Remark 2.1.** If $\Gamma : G \rightarrow M$ is a (global) projective representation, then $(\Gamma^\sigma, \theta^\sigma)$ is isomorphic to the trivial $G$-module $\mathcal{K}$ (where $\mathcal{K}$ is considered as a multiplicative semigroup). Moreover, $H^n(G, \mathcal{K}) \cong H^n(G, \mathcal{K}^*)$, where $H^n(G, \mathcal{K}^*)$ is the classical $n$-th cohomology group of $G$ with values in the trivial $G$-module $\mathcal{K}^*$.

Indeed, by (21) from [9] $\sigma(x, x^{-1}) = \sigma(x^{-1}, x)$, and in the global case $e_x = 1_M$, so $A_{x_1}^\Gamma = A^\Gamma = \mathcal{K} \cdot 1_M$ and $\theta^\sigma_x = \text{id}$ for all $x \in G$. Any $f \in C^n(G, A^\Gamma)$ is identified with a map $G^n \rightarrow \mathcal{K}^*$, i.e. with a classical $n$-cochain, and (2) becomes the classical coboundary homomorphism under the trivial $G$-action on $\mathcal{K}^*$.

**Definition 2.2.** By a $K$-linear partial $G$-module we shall mean $(A, \theta) \in \text{pMod}(G)$ such that $A$ is $K$-cancellative and each $\theta_x : A_{x_1} \rightarrow A_x$ is a $K$-map.

Evidently, for any partial projective representation $\Gamma$ of $G$ in a $K$-cancellative monoid $\mathcal{M}$, $\theta^\Gamma$ gives a structure of a $K$-linear partial $G$-module on $A^\Gamma$.

**Definition 2.3.** Following [10] a K-linear partial $G$-module $A$ will be called adjusted if $A$ is generated by $\alpha 1_x$ ($\alpha \in K$, $x \in G$).

Clearly, for any partial projective representation $\Gamma : G \rightarrow M$ the $K$-linear partial $G$-module $(A^\Gamma, \theta^\Gamma)$ is adjusted. We recall from [9] the next.

**Definition 2.4.** Let $A$ be a $K$-linear partial $G$-module. A $K$-valued twisting related to $A$ is a function $\sigma : G \times G \rightarrow K$ satisfying the following conditions:

(i) $\sigma(x, y) = 0 \iff A_x A_{xy} = 0$ ($x, y \in G$);

(ii) $\sigma(x, 1_G) = \sigma(1_G, x) = 1_K$ for all $x \in G$ such that $A_x \neq 0$;

(iii) $A_x A_{xy} A_{yz} \neq 0 \Rightarrow \sigma(x, y) \sigma(xy, z) = \sigma(y, z) \sigma(x, y) z$ with $x, y, z \in G$. 
By (iv) of Theorem 3 from [10] for any adjusted $A \in \text{pMod}(G)$ and any twisting $\sigma$ related to it there is a partial projective representation $\Gamma : G \rightarrow M$ with factor set $\sigma$ such that $A$ is isomorphic to $A^\Gamma$. Thus, among the partial $G$-modules, the adjusted ones are precisely those, which come from partial projective representations of $G$.

**Lemma 2.5.** Let $A$ be an adjusted ($K$-linear) partial $G$-module and $A_{x_1} \ldots A_{x_n} \neq 0$. Then for any invertible element $a$ of $A_{x_1} \ldots A_{x_n}$ there exists a unique $\alpha \in K^*$ such that $a = \alpha 1_{x_1} \ldots 1_{x_n}$.

**Proof.** The uniqueness of $\alpha$ obviously follows from the $K$-cancellative property of $A$. Suppose that $a = \alpha 1_{x_1} \ldots 1_{x_n} 1_{y_1} \ldots 1_{y_m}$ is invertible in $A_{x_1} \ldots A_{x_n} \neq 0$. Then

$$\alpha 1_{x_1} \ldots 1_{x_n} 1_{y_1} \ldots 1_{y_m} a^{-1} = 1_{x_1} \ldots 1_{x_n}$$

for some $a^{-1} \in A_{x_1} \ldots A_{x_n}$. Since the left-hand side of this equality is stable under the multiplication by $1_{y_1} \ldots 1_{y_m}$, we conclude that

$$1_{x_1} \ldots 1_{x_n} 1_{y_1} \ldots 1_{y_m} = 1_{x_1} \ldots 1_{x_n}$$

and hence $a = \alpha 1_{x_1} \ldots 1_{x_n}$. \hfill $\square$

Note that in fact $a^{-1} = \alpha^{-1} 1_{x_1} \ldots 1_{x_n}$, so the group of invertible elements of $A_{x_1} \ldots A_{x_n} \neq 0$ is isomorphic to $K^*$. Furthermore, if $A_{x_1} \ldots A_{x_n} = 0$ then $A_{x_1} \ldots A_{x_n}$ can be considered as the identity monoid, so it has the unique invertible element $1_{x_1} \ldots 1_{x_n} = 0$.

Given $A \in \text{pMod}(G)$, a partial 2-cocycle $f \in Z^2(G,A)$ will be called normalized if $f(1,1) = 1_A$. Then using $(\delta^2 f)(x,1,1) = (\delta^2 f)(1,1,x) = 1_x$ we readily see that

$$f(1,x) = f(x,1) = 1_x \quad \forall x \in G. \quad (4)$$

The subgroup of $Z^2(G,A)$ formed by the normalized partial 2-cocycles will be denoted by $NZ^2(G,A)$.

**Remark 2.6.** For each $f \in Z^2(G,A)$ there is $\tilde{f} \in NZ^2(G,A)$ which is cohomologous to $f$, i.e. $f = \tilde{f} \cdot \delta^1 g$ for some $g \in C^1(G,A)$.

For one can take $\tilde{f}(x,y) = f(x,y)f(1,1)^{-1}$ and $g(x) = f(1,1)1_x \in \mathcal{U}(A_x)$ for all $x, y \in G$.

Recall from [9] that the factor sets of all partial projective representations of $G$ form a commutative inverse monoid $pm(G)$ under pointwise multiplication. The quotient semigroup $pM(G) = pm(G)/\sim$, where

$$\sigma \sim \tau \Leftrightarrow \sigma(x,y) = \eta(x)\eta(xy)^{-1}\eta(y)\tau(x,y)$$

for some function $\eta : G \rightarrow K^*$, is called the partial Schur multiplier of $G$.

**Proposition 2.7.** Let $(A,\theta)$ be an adjusted $K$-linear partial $G$-module. Then $NZ^2(G,A)$ is isomorphic to the subgroup of $pm(G)$ consisting of the factor sets of all partial projective representations $\Delta : G \rightarrow M$, such that $(A^\Delta,\theta^\Delta)$ is isomorphic to $(A,\theta)$.

**Proof.** Let $f \in NZ^2(G,A)$. Define $\sigma_f : G \times G \rightarrow K$ by

$$\sigma_f(x,y) = \begin{cases} \alpha_{xy}, & \text{if } A_x A_{xy} \neq 0 \text{ and } f(x,y) = \alpha_{xy}1_x 1_{xy}, \\ 0, & \text{if } A_x A_{xy} = 0. \end{cases} \quad (5)$$

In particular, $f(x,y) = \sigma_f(x,y)1_x 1_{xy}$. By Lemma 2.5 our definition of $\sigma_f$ is correct. Note that $\sigma_f$ is a $K$-valued twisting related to $A$. Indeed, (i) of Definition 2.4 immediately follows from (4), whereas (ii) is a consequence of (4). Next, suppose that $A_x A_{xy} A_{xyz} \neq 0$, i.e. $1_x 1_{xy} 1_{xyz} \neq 0$. Next, suppose that $A_x A_{xy} A_{xyz} \neq 0$, i.e. $1_x 1_{xy} 1_{xyz} \neq 0$.\hfill
Using the 2-cocycle equality \((\delta^2 f)(x, y, z) = 1_x1_{xy}1_{xyz}\), the fact that \(\theta_x : A_{x^{-1}} \to A_x\) is a morphism of \(K\)-monoids and the \(K\)-cancellative property, we obtain

\[\sigma_f(y, z)\sigma_f(xy, z)^{-1}\sigma_f(x, yz)\sigma_f(x, y)^{-1} = 1,\]

so (iii) of Definition 2.4 is also true.

Having the adjusted \(K\)-linear partial \(G\)-module \(A\) and the twisting \(\sigma_f\) related to it, by Theorem 8 from [9], one can construct a partial projective representation \(\Delta : G \to M\) whose factor set is \(\sigma_f\). Using \(\Delta\) we obtain the \(K\)-linear partial \(G\)-module \(A\). Since \(A\) is generated by \(a\), as a semigroup \((a \in K, x \in G)\), Corollary 11 from [9] (or (iv) of Theorem 3 from [10]) implies that \(A\) is isomorphic to \(A\).

Conversely, let \(\Delta : G \to M\) be a partial projective representation with factor set \(\tau\), such that \(A\) is isomorphic to \(\Delta\). Define \(g_r : G \times G \to A\) by

\[g_r(x, y) = \begin{cases} 
\tau(x, y)1_x1_{xy}, & \text{if } A_xA_{xy} \neq 0, \\
1_x1_{xy}, & \text{if } A_xA_{xy} = 0. 
\end{cases}\]  

(6)

Note that by Theorem 6 from [9] \(\tau\) is a twisting related to \(A\), so \(\tau(x, y) = 0 \iff A_x^2A_{xy}^2 = 0\) which is equivalent to \(A_xA_{xy} = 0\), because \(A\) is isomorphic to \(A\). Therefore, \(g_r(x, y)\) is an invertible element of \(A_xA_{xy}\) and thus \(g_r \in C^2(G, A)\). We show next that

\[\delta^2 g_r(x, y, z) = 1_x1_{xy}1_{xyz}.\]  

(7)

If \(A_xA_{xy}A_{xyz} \neq 0\), then \(A_x^2A_{xy}^2A_{xyz}^2 \neq 0\), and (7) follows from the 2-cocycle identity for \(\tau\) (see (iii) of Definition 2.4). If \(A_xA_{xy}A_{xyz} = 0\), then \(1_x1_{xy}1_{xyz} = 0\), so both sides of (7) are zero. Thus, \(g_r \in Z^2(G, A)\). Moreover, \(g_r \in NZ^2(G, A)\) because \(\tau(1, 1) = 1\) by (ii) of Definition 2.4.

In order to check that the map \(NZ^2(G, A) \ni f \mapsto \sigma_f \in pm(G)\) is injective, it is enough to notice that by (5) and (6) one has that \(g_{\sigma_f} = f\).

Clearly \(\sigma_{fg} = \sigma_f\sigma_g\), so \(f \mapsto \sigma_f\) is a semigroup monomorphism of \(NZ^2(G, A)\) into \(pm(G)\), by means of which \(NZ^2(G, A)\) can be identified with a subgroup of \(pm(G)\), as desired.

**Remark 2.8.** Given two normalized 2-cocycles \(f, g\) it is readily seen that \(f\) is cohomologous to \(g\) if and only if \(\sigma_f \sim \sigma_g\).

**Corollary 2.9.** The embedding \(f \mapsto \sigma_f\) of \(NZ^2(G, A)\) into \(pm(G)\) induces the embedding of the group \(H^2(G, A)\) into \(pM(G)\).

This follows from Remarks 2.6 and 2.8.

Given an adjusted \(K\)-linear partial \(G\)-module \(A\), denote by \(R(G, A)\) the subgroup of \(pM(G)\) formed by the equivalence classes of \(K\)-valued twistings related to \(A\).

**Remark 2.10.** The image of \(H^2(G, A)\) in \(pM(G)\) coincides with \(R(G, A)\).

Indeed, this immediately follows from the fact that if \(\tau\) is a twisting related to \(\theta\) and \(\tau'\) is a factor set equivalent to \(\tau\), then \(\tau'\) is also a twisting related to \(\theta\). The latter is a direct consequence of Definition 2.4.

**Remark 2.11.** If an element of \(R(G, A)\) belongs to some component \(pM_D(G)\) of \(pM(G)\), then \(R(G, A) \subseteq pM_D(G)\).

For by (i) of Definition 2.4 all twistings related to \(A\) have the same domain \((x, y) \in G^2 \mid A_xA_{xy} \neq 0\).

**Definition 2.12.** Let \(A\) and \(A'\) be \(K\)-linear partial \(G\)-modules. We shall say that \(A\) and \(A'\) are twisting-equivalent if the sets of \(K\)-valued twistings related to \(A\) and \(A'\) coincide.
The relation between the partial Schur multiplier and partial cohomology is summarized in the next.

Theorem 2.13. Each component $pM_D(G)$ of $pM(G)$ is the union of the subgroups $R(G, A) \cong H^2(G, A)$, where $A$ runs over the representatives of twisting-equivalence classes of adjusted $K$-linear partial $G$-modules such that

$$\{(x, y) \in G^2 \mid A_x A_{xy} \neq 0\} = D.$$ 

Proof. Let $\sigma$ be a factor set of a partial projective representation $\Gamma : G \to M$, whose equivalence class belongs to $pM_D(G)$. Then $\sigma$ is a $K$-valued twisting related to $A^\sigma$ whose equivalence class lies in $R(G, A^\sigma)$. Hence $R(G, A^\sigma) \subseteq pM_D(G)$ by Remark 2.11. The partial $G$-module $A^\sigma$ is adjusted by its definition and $\{(x, y) \in G^2 \mid A^\sigma_x A^\sigma_{xy} \neq 0\} = \text{dom } \sigma = D$. Finally, $R(G, A) \cong H^2(G, A)$, thanks to Corollary 2.9 and Remark 2.10. □

3 Partial modules and modules over Exel’s monoid

Recall that a semigroup $S$ is called inverse if for any $s \in S$ there is a unique $s^{-1} \in S$ (called the inverse of $s$) such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. A natural example of an inverse semigroup is the monoid $I(X)$ of all partial bijections (i.e. bijections between subsets) of $X$. The product $\varphi \psi$ in $I(X)$ is the composition $\varphi \circ \psi$ defined for all $x \in X$, such that $\varphi(\psi(x))$ makes sense. More precisely, $\varphi \psi$ is the bijection

$$\varphi \circ \psi : \psi^{-1}(\text{ran } \psi \cap \text{dom } \varphi) \to \varphi(\text{ran } \psi \cap \text{dom } \varphi).$$

Note that the inverse of $\varphi$ is simply $\varphi^{-1} : \text{ran } \varphi \to \text{dom } \varphi$.

Following Exel [16] by an action of an inverse semigroup $S$ on $X$ we mean a homomorphism $S \to I(X)$ (in semigroup theory such homomorphisms are called representations of $S$, see [2, 7.3]). If $S$ is a monoid, then the term “action” will always mean “unital action”, i.e. a homomorphism of monoids $S \to I(X)$. It was proved in [16] that there is a one-to-one correspondence between the partial actions of a group $G$ on a set $X$ and the (unital) actions of $S(G)$ on $X$, where $S(G)$ is the semigroup generated by $[x] \ (x \in G)$ subject to the following relations:

(i) $[x^{-1}][x][y] = [x^{-1}][xy]$;

(ii) $[x][y][y^{-1}] = [xy][y^{-1}]$;

(iii) $[x][1_G] = [x]$.

(it follows that $S(G)$ is a monoid with identity $[1_G]$). It is shown in [16] that $S(G)$ is an inverse semigroup (namely, $[x]^{-1} = [x^{-1}]$ in $S(G)$) and that any element of $S(G)$ can be expressed in the form $\varepsilon_{x_1} \ldots \varepsilon_{x_n} [y]$, where $\varepsilon_{x_i}$ denotes the idempotent $[x_i][x_i^{-1}]$, all $x_i$ are different, non-identity and not equal to $y$. This expression is unique up to a permutation of the idempotents $\varepsilon_{x_i}$ (note that all $\varepsilon_{x}$ commute). It follows that each idempotent of $S(G)$ has the form $\varepsilon_{x_1} \ldots \varepsilon_{x_n}$ for some (uniquely defined) set $x_1, \ldots, x_n \in G$. It is also known that $S(G)$ is isomorphic to the Birget-Rhodes expansion of $G$ [22].

Notice that given a semigroup $A$, the composition of two partial isomorphisms of $A$ (i.e. isomorphisms between arbitrary ideals of $A$) is not a partial isomorphism in general, since the domain of the composition is not necessarily an ideal. However, the set $I_{un}(A)$ of all isomorphisms between unital ideals of $A$ (i.e. ideals generated by central idempotents of $A$) forms an inverse monoid.

Definition 3.1. Let $S$ be an inverse semigroup (monoid). An action of $S$ on a semigroup $A$ is a homomorphism of semigroups (monoids) $S \to I_{un}(A)$. 

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By [16, Proposition 4.1] a partial action of $G$ on a set $X$ can be seen as a partial representation in $I(X)$. This immediately implies that unital partial actions of $G$ on a semigroup $A$ can be identified with the partial representations of $G$ in $I_{ul}(A)$. Then [16, Proposition 2.2] yields the next.

**Proposition 3.2.** There is a one-to-one correspondence between the partial $G$-modules and the (unital) actions of $S(G)$ on commutative monoids.

More precisely, let $(A,\theta) \in p\text{Mod}(G)$. Then the corresponding action $\tau$ of $S(G)$ on $A$ is given as follows. For arbitrary $s = \varepsilon_{x_1} \ldots \varepsilon_{x_n} [y] \in S(G)$, $\tau_s$ is the bijection

$$1_{y^{-1}x_1} \ldots 1_{y^{-1}x_n} 1_{y^{-1}} A \ni a \mapsto \theta_y(a) \in 1_{x_1} \ldots 1_{x_n} 1_y A.$$ 

This correspondence is in fact an isomorphism of categories as explained below.

**Remark 3.3.** Let $A, A' \in p\text{Mod}(G)$, $\tau$ and $\tau'$ be the corresponding actions of $S(G)$ on the semigroups $A$ and $A'$. Denote by $1_s$ and $1'_s$ the identities of $\text{ran} \tau_s$ and $\text{ran} \tau'_s$, respectively ($s \in S(G)$). Then a homomorphism of semigroups $\varphi : A \to A'$ is a morphism of partial $G$-modules if and only if for any $s \in S(G)$ we have

(i) $\varphi(1_s) = 1'_s$ (and so $\varphi(\text{dom} \tau_s) \subseteq \text{dom} \tau'_s$);

(ii) $\varphi \circ \tau_s = \tau'_s \circ \varphi$ on $\text{dom} \tau_s$.

It is reasonable to give the next.

**Definition 3.4.** Let $\tau$ and $\tau'$ be actions of an inverse semigroup $S$ on semigroups $A$ and $A'$. A morphism $\tau \to \tau'$ is a homomorphism $\varphi : A \to A'$, such that for all $s \in S$ the conditions (i) and (ii) from Remark 3.3 are true (as above $\text{ran} \tau_s = 1_s A$ and $\text{ran} \tau'_s = 1'_s A'$).

Thus, actions of an inverse semigroup $S$ on commutative monoids form a category which will be denoted by $A(S)$.

**Proposition 3.5.** The categories $p\text{Mod}(G)$ and $A(S(G))$ are isomorphic.

Indeed, Proposition 3.2 and Remark 3.3 give a functor $p\text{Mod}(G) \to A(S(G))$ which is bijective on objects and identity on morphisms.

We are going to give a characterization of actions of an inverse semigroup $S$ on a semigroup $A$ in terms of endomorphisms of $A$. For any semigroup $A$ denote by $E(A)$ the set of idempotents of $A$. The center of $A$ will be denoted by $C(A)$. Note that $C(A)$ is a subsemigroup of $A$, while $E(A)$ might not be a subsemigroup. But it is necessarily a subsemigroup when the idempotents commute. In particular, $E(A)$ is a subsemigroup of an inverse $A$, and $E(C(A))$ is a subsemigroup of $A$ for arbitrary semigroup $A$.

**Lemma 3.6.** Let $\tau : S \to I_{ul}(A)$, $s \mapsto \tau_s$, be an action of an inverse semigroup $S$ on a semigroup $A$ and $1_s$ denote the identity of $\text{ran} \tau_s$. Then

(i) $\tau_e = \text{id}1_{A}$ for all $e \in E(S)$;

(ii) $\tau_s(1_s^{-1}1_t) = 1_{st}$ for all $s, t \in S$.

**Proof.** Equality (i) follows by observing that $\tau_e$ is a bijection of $\text{dom} \tau_e = \text{ran} \tau_e$, which coincides with its square. For (ii) note that $1_s^{-1}1_t$ is the identity of

$$1_s^{-1}1_tA = 1_s^{-1}A1_tA = 1_s^{-1}A \cap 1_tA = \text{ran} \tau_{s^{-1}} \cap \text{dom} \tau_{t^{-1}} = \tau_{s^{-1}}(\text{dom} \tau_{t^{-1}s^{-1}}).$$

Therefore, $1_s^{-1}1_t = \tau_{s^{-1}}(1_{st})$, i.e. $\tau_s(1_s^{-1}1_t) = 1_{st}$. 

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Theorem 3.7. Let $S$ be an inverse semigroup and $A$ a semigroup. There is a one-to-one correspondence between the actions of $S$ on $A$ and the pairs $(\lambda, \alpha)$, where $\lambda$ is a homomorphism $S \rightarrow \text{End} A$, $s \mapsto \lambda_s$, and $\alpha$ is a homomorphism $E(S) \rightarrow E(C(A))$ such that

(i) $\lambda_e(a) = \alpha(e)a$ for all $e \in E(S)$ and $a \in A$;
(ii) $\lambda_s(\alpha(e)) = \alpha(se^{-1})$ for all $s \in S$ and $e \in E(S)$.

Proof. Let $\tau : S \rightarrow \mathcal{U}_u(A)$, $s \mapsto \tau_s$, be an action of $S$ on $A$. Obviously, the identity $1_s$ of $\text{ran} \tau_s$ belongs to $E(C(A))$, so we may define $\alpha^\tau : E(S) \rightarrow E(C(A))$ by

$$\alpha^\tau(e) = 1_e.$$  

According to (i) and (ii) of Lemma 3.6, we have $1_{ef} = \tau_e(1_e1_f) = 1_e1_f$, hence $\alpha^\tau$ is a homomorphism. Furthermore, for any $s \in S$ and $a \in A$ set

$$\lambda_s^\tau(a) = \tau_s(1_{s^{-1}}a).$$

The right-hand side of (9) makes sense, because $1_{s^{-1}}$ is the identity of $\text{ran} \tau_{s^{-1}} = \text{dom} \tau_s$ and $\text{dom} \tau_s$ is an ideal. Since $1_{s^{-1}}$ is a central idempotent and $\tau_s$ is a homomorphism, we see that

$$\lambda_s^\tau(ab) = \tau_s(1_{s^{-1}}ab) = \tau_s(1_{s^{-1}}a1_{s^{-1}}b) = \tau_s(1_{s^{-1}}a)\tau_s(1_{s^{-1}}b) = \lambda_s^\tau(a)\lambda_s^\tau(b),$$

so $\lambda_s^\tau \in \text{End} A$. To check whether $\lambda^\tau : S \rightarrow \text{End} A$, $s \mapsto \lambda_s^\tau$, is a homomorphism, use (ii) of Lemma 3.6

$$\lambda_s^\tau(\lambda_t^\tau(a)) = \tau_s(1_{s^{-1}}\tau_t(1_{t^{-1}}a)) = \tau_s(\tau_t(1_{t^{-1}}1_{t^{-1}s^{-1}})\tau_t(1_{t^{-1}}a)) = \tau_s(\tau_t(1_{t^{-1}}1_{t^{-1}s^{-1}}a)),$$

which is $\tau_{st}(1_{t^{-1}}1_{t^{-1}s^{-1}}a)$, because $1_{t^{-1}}1_{t^{-1}s^{-1}}a \in \text{dom} \tau_{st}$. It remains to note that

$$\tau_{st}(1_{t^{-1}}1_{t^{-1}s^{-1}}a) = \tau_{st}(1_{t^{-1}}1_{t^{-1}s^{-1}})\tau_{st}(1_{t^{-1}s^{-1}}a) = 1_s\tau_{st}(1_{t^{-1}s^{-1}}a) = \tau_{st}(1_{t^{-1}s^{-1}}a) = \lambda_{st}^\tau(a).$$

Here we again applied (ii) of Lemma 3.6 and the fact that $\text{ran} \tau_{st} \subseteq \text{ran} \tau_s$.

Now we verify that the pair $(\lambda^\tau, \alpha^\tau)$ satisfies (i) and (ii). The property (i) obviously follows from (i) of Lemma 3.6. For (ii) observe that

$$\lambda_s^\tau(1_e) = \tau_s(1_{s^{-1}}1_e) = \tau_s(1_{s^{-1}}\tau_e(1_{s^{-1}}1_e)) = \tau_s(1_{s^{-1}}1_{es^{-1}}) = 1_{ses^{-1}},$$

using (i) and (ii) of Lemma 3.6.

Conversely, given a pair $(\lambda, \alpha)$ with (i) and (ii), define $1_s \in E(C(A))$ by

$$1_s = \alpha(ss^{-1}).$$

Note that in view of (ii)

$$\lambda_s(1_{s^{-1}}) = \lambda_s(\alpha(s^{-1}s)) = \alpha(ss^{-1}ss^{-1}) = \alpha(ss^{-1}) = 1_s$$

and hence $\lambda_s(1_{s^{-1}}A) \subseteq 1_sA$. Moreover, $\lambda_s|_{1_{s^{-1}}A}$ is a bijection of $1_{s^{-1}}A$ onto $1_sA$ with inverse $\lambda_{s^{-1}}|_{1_sA}$, because

$$(\lambda_s \circ \lambda_{s^{-1}})|_{1_sA} = \lambda_{ss^{-1}}|_{1_sA} = \text{id}_{1_sA},$$

by (i) and similarly $(\lambda_{s^{-1}} \circ \lambda_s)|_{1_{s^{-1}}A} = \text{id}_{1_{s^{-1}}A}$. Hence, we may define $\tau^{(\lambda, \alpha)} : S \rightarrow \mathcal{U}_u(A)$ by

$$\tau_s^{(\lambda, \alpha)} = \lambda_s|_{1_{s^{-1}}A}.$$ 

To prove that $\tau^{(\lambda, \alpha)}$ is a homomorphism first observe that

$$\text{dom} (\tau_s^{(\lambda, \alpha)} \circ \tau_t^{(\lambda, \alpha)}) = \lambda_{t^{-1}}(1_{s^{-1}}A \cap 1_tA) = \lambda_{t^{-1}}(1_{s^{-1}}1_tA)$$

$$= \lambda_{t^{-1}}(\alpha(s^{-1}s))1_{t^{-1}}A = \alpha(t^{-1}s^{-1}st)t(-1)A,$$
which is $\alpha(t^{-1}s^{-1}st)A = 1_{t^{-1}s^{-1}}A = \text{dom } \tau_{st}^{(\lambda,\alpha)}$, because $\alpha$ is a homomorphism. Furthermore

$$\lambda_t(1_{t^{-1}s^{-1}}) = \lambda_t(\alpha(t^{-1}s^{-1}st)) = \alpha(t^{-1}s^{-1}st)t^{-1} = \alpha(s^{-1}stt^{-1}) = 1_{s^{-1}t}.$$

Then since $1_{t^{-1}s^{-1}} = 1_{t^{-1}s^{-1}}1_{t^{-1}}$, for any $a \in \text{dom } \tau_{st}^{(\lambda,\alpha)}$ we have

$$(\tau_s^{(\lambda,\alpha)} \circ \tau_t^{(\lambda,\alpha)})(a) = \tau_s^{(\lambda,\alpha)}(\tau_t^{(\lambda,\alpha)}(1_{t^{-1}s^{-1}}a)) = \tau_s^{(\lambda,\alpha)}(\tau_t^{(\lambda,\alpha)}(1_{t^{-1}s^{-1}}1_{t^{-1}}a))$$

$$= \tau_s^{(\lambda,\alpha)}(\alpha(1_{t^{-1}s^{-1}}1_{t^{-1}}a)) = \tau_s^{(\lambda,\alpha)}(1_{s^{-1}}1_{t^{-1}}a))$$

$$= \lambda_s(1_{s^{-1}}1_{t^{-1}}a) = 1_{s^{-1}}\lambda_s(1_{t^{-1}}a)$$

$$= 1_{s^{-1}}\lambda_t(a) = 1_{s^{-1}}\tau_{st}^{(\lambda,\alpha)}(a) = \tau_{st}^{(\lambda,\alpha)}(a).$$

It remains to show that $\tau \mapsto (\lambda^*, \alpha^*)$ and $(\lambda, \alpha) \mapsto \tau^{(\lambda, \alpha)}$ are mutually inverse. Consider $\tau^{(\lambda^*, \alpha^*)}$. For any $s \in S$ the identity of $\text{dom } \tau_s^{(\lambda^*, \alpha^*)}$ is $\alpha^*(s^{-1}s) = 1_{s^{-1}s}$, which is the identity of $\text{dom } \tau_{s^{-1}}$. But using (ii) of Lemma 3.6 we see that

$$1_{s^{-1}} = \tau_{s^{-1}}(1_{s^{-1}}) = \tau_{s^{-1}}(1_s \cdot 1_s) = 1_{s^{-1}s},$$

so $\text{dom } \tau_s^{(\lambda^*, \alpha^*)} = \text{dom } \tau_s$. By our construction

$$\tau_s^{(\lambda^*, \alpha^*)} = \lambda_s^*|_{1_{s^{-1}s}}A = \lambda_s^*|_{1_{s^{-1}1}}A.$$ But $\lambda_s^*(a) = \tau_s(1_{s^{-1}}a)$ and $a = 1_{s^{-1}}a$ for $a \in 1_{s^{-1}}A$, hence $\lambda_s^* = \tau_s$ on $1_{s^{-1}}A$, and consequently, $\tau^{(\lambda^*, \alpha^*)} = \tau$.

Now given $(\lambda, \alpha)$, we show that $\alpha^{(\lambda, \alpha)} = \alpha$ and $\lambda^{(\lambda, \alpha)} = \lambda$. For any $e \in E(S)$ the image $\alpha^{(\lambda, \alpha)}(e)$ is the identity of $\text{ran } \tau_e^{(\lambda, \alpha)}$, which is $\alpha(\alpha(e^{-1}) = \alpha(e))$. Furthermore, using (i) and (ii) we get

$$\lambda_s^{(\lambda, \alpha)}(a) = \tau_s^{(\lambda, \alpha)}(\alpha(s^{-1}s)a) = \lambda_s|_{\alpha(s^{-1}s)}A(\alpha(s^{-1}s)a) = \lambda_s(\alpha(s^{-1}s)a) = \lambda_s(\alpha(s^{-1}s))\lambda_s(a)$$

$$= \alpha(s^{-1}s)\lambda_s(a) = \alpha(s^{-1}s)\lambda_s(a) = \lambda_{s^{-1}}(\lambda_s(a)) = \lambda_{s^{-1}}(a) = \lambda_s(a).$$

\[\square\]

**Definition 3.8.** Given an inverse semigroup $S$, an $S$-module is a triple $(A, \lambda, \alpha)$ (often written shortly as $A$), where $A$ is a commutative semigroup, $\lambda : S \rightarrow \text{End } A$ and $\alpha : E(S) \rightarrow E(A)$ are homomorphisms satisfying (i) and (ii) of Theorem 3.7. If $\alpha$ is an isomorphism, then we say that $A$ is **strict**.

Notice that in [24] the term $S$-module means our strict inverse $S$-module.

**Proposition 3.9.** Let $\tau$ and $\tau'$ be actions of an inverse semigroup $S$ on semigroups $A$ and $A'$. Set $(\lambda, \alpha) = (\lambda^*, \alpha^*)$ and $(\lambda', \alpha') = (\lambda^{*'}, \alpha^{*'})$. Then a homomorphism $\varphi : A \rightarrow A'$ is a morphism $\tau \rightarrow \tau'$ if and only if it satisfies

(i) $\varphi \circ \alpha = \alpha'$ on $E(S)$;

(ii) $\varphi \circ \lambda_s = \lambda'_s \circ \varphi$ for all $s \in S$.

**Proof.** By [10] condition (i) is equivalent to $\varphi(1_s) = 1'_s$, while (ii) is the same as $\varphi(\tau_s(1_{s^{-1}}a)) = \tau'_s(1'_{s^{-1}}\varphi(a))$ for all $a \in A$ by [20]. In view of (i) the latter can be replaced by $\varphi(\tau_s(1_{s^{-1}}a)) = \tau'_s(\varphi(1_{s^{-1}}a))$ for all $a \in A$, i.e. $\varphi \circ \tau_s = \tau'_s \circ \varphi$ on $1_{s^{-1}}A = \text{dom } \tau_s$. \[\square\]

**Definition 3.10.** Let $(A, \lambda, \alpha)$ and $(A', \lambda', \alpha')$ be modules over an inverse semigroup $S$. By a morphism $(A, \lambda, \alpha) \rightarrow (A', \lambda', \alpha')$ we mean a homomorphism $\varphi : A \rightarrow A'$ satisfying (i) and (ii) of Proposition 3.9.
Given an inverse semigroup $S$, denote by $\text{Mod}(S)$ the category of $S$-modules and their morphisms. Then Theorem [3.7] and Proposition [3.9] yield the next.

**Proposition 3.11.** For any inverse semigroup $S$ the categories $A(S)$ and $\text{Mod}(S)$ are isomorphic.

**Corollary 3.12.** The category $\text{pMod}(G)$ is isomorphic to $\text{Mod}(S(G))$.

This can be specified as follows. Given $(A, \theta) \in \text{pMod}(G)$, the corresponding $\lambda^\theta$ is defined on a generator $[x] \in S(G)$ by $\lambda^\theta_1(a) = \theta_x(1_{x^{-1}}a)$ and $\alpha^\theta(\varepsilon_{x_1} \ldots \varepsilon_{x_n}) = 1_{x_1} \ldots 1_{x_n}$. Then Theorem 3.7 and Proposition 3.9 yield the next.

**Remark 3.13.** Let $(A, \theta) \in \text{pMod}(G)$. Define $\tilde{A}$ to be the (inverse) subsemigroup of $A$ formed by the invertible elements of all ideals $1_{x_1} \ldots 1_{x_n} A$, where $x_1, \ldots, x_n \in G$, $n \in \mathbb{N}$. Then $\theta_x(1_{x^{-1}} \tilde{A}) = 1_{x \tilde{A}}$, so $\theta$ restricted to $\tilde{A}$ defines a partial action $\tilde{\theta}$ of $G$ on $\tilde{A}$. Moreover, $(\tilde{A}, \tilde{\theta}) \in \text{Mod}(G)$ and $H^n(G, A) = H^n(G, \tilde{A})$.

For an $A$-module $M$, if $a \in A$ and $b \in B$, then $ab$ is invertible in the product $1_{x_1} \ldots 1_{x_n} A$. The inverse of $a$ in $\tilde{A}$ is its inverse in the corresponding ideal. Furthermore, if $a \in A$, then $\theta_x(1_{x^{-1}}a) \in A$ and since similarly $\theta_x^{-1}(1_{x^{-1}}a)$, we have the equality $\theta_x(1_{x^{-1}}a) = 1_{x \tilde{A}}$. This implies that $
abla \theta_x(1_{x^{-1}}1_{y} \tilde{A}) = \theta_x(1_{x^{-1}}1_{y} \tilde{A}) = 1_{x1_{y} \tilde{A}}$, so the restrictions $\tilde{\theta}_x$ of $\theta_x$ to $1_{x^{-1}} \tilde{A} \subseteq 1_{x^{-1}} A$ are in fact a partial action of $G$ on $\tilde{A}$. By construction $(\tilde{A}, \tilde{\theta}) \in \text{Mod}(G)$. Note that $U(1_{x_1} \ldots 1_{x_n} A) = U(1_{x_1} \ldots 1_{x_n} \tilde{A})$ and hence $C^n(G, A) = C^n(G, \tilde{A})$. Since $\tilde{\theta}$ is the restriction of $\theta$, the coboundary homomorphisms of these two cochain complexes also coincide and thus, their cohomology groups are equal.

Notice that

$$\tilde{A} = \bigcup U(1_{x_1} \ldots 1_{x_n} A),$$

where $x_1, x_2, \ldots, x_n$ run over $G$. Observe that if $e, f \in A$ are idempotents with $e \neq f$ then $U(eA) \cap U(fA) = \emptyset$. Hence the above union can be made disjoint by removing repetitions, and then it will give a representation of $\tilde{A}$ as a semilattice of abelian groups. In particular, any idempotent of $\tilde{A}$, being an element of some group component, is identity of this component, i.e. has a form $1_{x_1} \ldots 1_{x_n}$ for $x_1, \ldots, x_n \in G$.

**Definition 3.14.** A partial $G$-module $(A, \theta)$ is called **inverse** if $A$ is inverse and $E(A)$ is generated by the idempotents $1_x (x \in G)$.

**Remark 3.15.** If $(A, \theta) \in \text{pMod}(G)$, then $(\tilde{A}, \tilde{\theta})$ is inverse.
Another immediate observation is that \(E(A) = \langle 1_x : x \in G \rangle\) if and only if \(\alpha^0 : E(S(G)) \to E(A)\) is an epimorphism.

**Definition 3.16.** An \(S\)-module \((A, \lambda, \alpha)\) is called inverse if \(A\) is inverse and \(\alpha\) is an epimorphism.

**Remark 3.17.** A partial \(G\)-module \((A, \theta)\) is inverse if and only if the corresponding \(S(G)\)-module \((A, \lambda^0, \alpha^0)\) is inverse.

Thus, passing from an inverse partial \(G\)-module to an \(S(G)\)-module, we see that the only reason why we might not obtain an \(S(G)\)-module in the sense of Lausch is that the map \(\alpha^0 : E(S(G)) \to E(A)\) is not injective in general. For example, considering a classical (i.e. usual) \(G\)-module \(A\) (which is readily seen to be a strict inverse \(G\)-module) we have \(|E(A)| = 1\), while \(|E(S(G))| = 2^{|G|-1}\). Nevertheless, an inverse partial \(G\)-module \(A\) may be seen as a strict inverse module over some epimorphic image of \(S(G)\). We shall see that this epimorphic image can be chosen to be \(E(A) \ast_\theta G\) as explained in what follows (see Corollary 3.33). In particular, in the above example this construction leads to the initial strict inverse \(G\)-module \(A\).

Given an inverse semigroup \(S\) and an \(S\)-module \((A, \lambda, \alpha)\) denote by \(L(A, S)\) the set of \(a\delta_s\), where \(s \in S\), \(a \in \alpha(ss^{-1})A\) and \(\delta_s\) is a symbol.

**Proposition 3.18.** The set \(L(A, S)\) is a semigroup under the multiplication \(a\delta_s \cdot b\delta_t = a\lambda_s(b)\delta_{st}\).

**Proof.** It is directly verified that \(L(A, S)\) can be seen as the \(\lambda\)-semidirect product defined in [24, p. 148], so that the above multiplication is well-defined and associative. \(\square\)

**Remark 3.19.** Notice that \(\lambda_s(a)\delta_s\) and \(\lambda_s(a)\delta_{ss^{-1}}\) are elements in \(L(A, S)\) for all \(a \in A\), as \(\lambda_s(a) = \lambda_{ss^{-1}}(a) = \lambda_{ss^{-1}}(\lambda_s(a)) = \alpha(ss^{-1})\lambda_s(a)\).

**Remark 3.20.** \(E(L(A, S)) = \{\alpha(e)\delta_s : e \in E(S)\}\).

Inspired by [19], instead of \(L(A, S)\) we shall deal with a quotient of \(L(A, S)\) which takes into account the natural partial order on \(S\). This idea has important advantages, one of them being the fact that in the \(C^*\)-algebraic context the analogous construction fits the definition of the crossed product by an inverse semigroup given by N. Sieben in [28] (see [19, Theorem 5.6]).

It is well-known that any inverse semigroup \(S\) is partially ordered in a following natural way: \(s \leq t\) if and only if there is \(e \in E(S)\) such that \(s = et\) (and in this case one can take \(e = ss^{-1}\)). This is the same as to say that \(s = tf\) for some \(f \in E(S)\). Notice that for \(e, f \in E(S)\) we have \(e \leq f \iff ef = e\). Denote by \(\sigma\) the congruence generated by \(\leq\). Since \(\leq\) is compatible with multiplication in \(S\) ([2, Lemma 7.2]), \(\sigma\) is in fact the minimal equivalence relation containing the partial order \(\leq\).

**Lemma 3.21.** For any inverse semigroup \(S\) and for all \(s, t \in S\) we have \((s, t) \in \sigma\) if and only if there is \(u \in S\) such that \(u \leq s, t\).

**Proof.** We shall prove the “only if” part (the “if” part is obvious). As it was mentioned before the lemma, \(\sigma\) is the symmetric transitive closure of \(\leq\). Thus, \((s, t) \in \sigma\) means that there are \(s = s_1, \ldots, s_n = t \in S\) such that for all \(i = 1, \ldots, n-1\) we have \(s_i \leq s_{i+1}\) or \(s_{i+1} \leq s_i\). Therefore, \(s_i = e_is_{i+1}\) or \(s_{i+1} = e_is_i\) for some \(e_i \in E(S) (i = 1, \ldots, n-1)\). In any case \(e_is_i = e_is_{i+1}\) for all \(i = 1, \ldots, n-1\). Set \(e = e_1 \cdots e_{n-1}\). Then

\[
es = es_1 = (e_2 \cdots e_{n-1})(e_1s_1) = (e_2 \cdots e_{n-1})(e_1s_2) = (e_1e_3 \cdots e_{n-1})(e_2s_2) = \cdots = (e_1 \cdots e_{n-2})(e_{n-1}s_{n-1}) = (e_1 \cdots e_{n-2})(e_{n-1}s_n) = es_n = et.
\]

So, we can take \(u = es = et \leq s, t\). \(\square\)

This shows that \(\sigma\) is the minimum group congruence on \(S\) (see [24, Sec. 2.4, Theorem 1]). For \(S = \mathcal{S}(G)\) we have \((e_{z_1} \cdots e_{z_n}[y], e_{z_1} \cdots e_{z_m}[w]) \in \sigma\) if and only if \(y = w\) (see [19, Example 1.5]), so the group \(\mathcal{S}(G)/\sigma\) is isomorphic to \(G\).
The "if" part is easy: if aδs, aδt ∈ L(A,S) such that s ≤ t, then we see that

\{(aδ_s, aδ_t) ∈ L(A,S) × L(A,S) \mid s ≤ t\}.

(12)

The quotient semigroup L(A,S)/ρ will be denoted by A × S and called the semidirect product of A and S.

Notice that s ≤ t implies ss⁻¹ ≤ tt⁻¹ and hence α(ss⁻¹)A = α(tt⁻¹)A ⊆ α(uu⁻¹)A.

Remark 3.23. Under the conditions of Definition 3.22, A × S is A ×_ρ(λ,α) S in the sense of Definition 3.5 from [19] modified for actions of inverse semigroups on semigroups.

Remark 3.24. Under the conditions of Definition 3.22 the relation (12) is compatible with multiplication in L(A,S), so the congruence ρ is the equivalence relation generated by (12).

Indeed, given aδ_s, aδ_t ∈ L(A,S), such that s ≤ t, and an arbitrary bδ_u ∈ L(A,S), we see that

(bδ_u · aδ_s, bδ_u · aδ_t) = (hλ_u(a)δ_u.aλ_u(a)δ_{ut}) ∈ ρ,

because us ≤ ut. Similarly,

(aδ_s · bδ_u, aδ_t · bδ_u) = (aλ_u(b)δ_{su}, aλ_u(b)δ_{tu}) ∈ ρ,

if we show that aλ_u(b) = aλ_t(b). Since s = ss⁻¹t, then aλ_u(b) = aλ_{ss⁻¹t}(b) = α(ss⁻¹)λ_t(b) = aλ_t(b), because a = α(ss⁻¹)a.

As an immediate consequence, we notice that (aδ_s, bδ_t) ∈ ρ implies a = b and (s,t) ∈ σ. The converse, however, is not true in general.

Lemma 3.25. For any (A,λ,α) ∈ Mod(S) and for arbitrary aδ_s, bδ_t ∈ L(A,S) we have (aδ_s, bδ_t) ∈ ρ if and only if there is u ∈ S such that u ≤ s, t and a = b ∈ α(uu⁻¹)A.

Proof. The "if" part is easy: if u ≤ s, t and a = b ∈ α(uu⁻¹)A, then (aδ_u, aδ_s) and (aδ_u, aδ_t) belong to the relation defined by (12), so by symmetry and transitivity of ρ we have (aδ_s, aδ_t) ∈ ρ.

Now suppose (aδ_s, aδ_t) ∈ ρ, then as it was mentioned above a = b. Let aδ_s = aδ_{s_1},...,aδ_{s_n} = aδ_t be a sequence of elements of L(A,S) such that s_i = e_i.s_{i+1} or s_i.e_i = s_{i+1} for all i = 1, ..., n−1 (e_i ∈ E(S)). Since a ∈ α(s_i{s_i}⁻¹)A for all i = 1, ..., n, then a ∈ α(e_i)A, i = 1, ..., n−1. Therefore, a ∈ α(e)A, where e = e_1 ... e_{n−1}. Now setting u = es as in the proof of Lemma 3.24, we see that u ≤ s, t, and, moreover, a ∈ α(e)α(ss⁻¹)A = α(uu⁻¹)A.

Example 3.26. Take the cyclic group Z_2 = < a \mid a^2 = 1 > and set S = Z_2 ∪ {0}. Then S is a commutative inverse monoid, whose group components are Z_2 and {0}. We can define a structure of a (strict) S-module on S as follows: λ_s(t) = ts⁻¹t and α(e) = e for all s, t ∈ S and e ∈ E(S). Then (a,1) ∈ σ, but (aδ_u, aδ_1) ∉ ρ.

Clearly, 0 ≤ a, 1, but a and 1 are incomparable, so the only u ∈ S with u ≤ a, 1 is 0. Although a ∈ α(aa⁻¹)S = α(1) · S = S, we cannot find u ≤ a, 1 such that a ∈ α(uu⁻¹)S, since otherwise a = 0. By Lemma 3.24 (aδ_u, aδ_1) ∉ ρ.

Recall that an inverse semigroup S is called E-unitary if es ∈ E(S) implies s ∈ E(S) for all s ∈ S and e ∈ E(S). Equivalently, σ coincides with the compatibility relation ~ on S (see [24] pp. 24, 66). For instance, S(G) is E-unitary for any group G, while the semigroup S from Example 3.26 is not E-unitary, because 0a = 0 ∈ E(S) and a ∉ E(S).

Remark 3.27. If under the conditions of Lemma 3.24 the semigroup S is E-unitary then (aδ_s, bδ_t) ∈ ρ exactly when (s,t) ∈ σ and a = b.
Proposition 3.30. Let \( (A,\lambda,\alpha) \in \text{Mod}(S) \) and \( a\delta_e, b\delta_f \in L(A,S) \) with \( e,f \in E(S) \). Then \( (a\delta_e, b\delta_f) \in \rho \) exactly when \( a = b \).

For \( ef \leq e, f \), and if \( a = b \) then \( a = b \in \alpha(e)\alpha(f)A = \alpha(e)f \).

Taking into account Remark 3.29, we see that \( \rho \) is idempotent-separating, which means that each \( \rho \)-class contains at most one idempotent of \( L(A,S) \). Equivalently, \( \rho^\circ \) restricted to \( E(L(A,S)) \) is injective, where \( \rho^\circ \) denotes the natural epimorphism \( L(A,S) \to L(A,S)/\rho = A \rtimes S \).

For reader’s convenience we also notice the following straightforward fact: if \( \varphi : S_1 \to S_2 \) is an epimorphism of inverse semigroups, then \( E(S_2) = \varphi(E(S_1)) \). Indeed, taking \( e \in E(S_2) \) and \( s \in S_1 \) with \( \varphi(s) = e \) we have \( \varphi(ss^{-1}) = e, ss^{-1} \in E(S_1) \).

Corollary 3.28. If \( (a\delta_s, b\delta_t) \in \rho \), then for any \( c \in A \) we have \( (ca\delta_s, cb\delta_t) \in \rho \).

Indeed, if \( a = b \in \alpha(uu^{-1})A \), then \( ca = cb \in \alpha(uu^{-1})A \).

Remark 3.29. Let \( (A,\lambda,\alpha) \in \text{Mod}(S) \) and \( a\delta_e, b\delta_f \in L(A,S) \) with \( e,f \in E(S) \). Then \( (a\delta_e, b\delta_f) \in \rho \) exactly when \( a = b \).

We need only explain the “if” part. Suppose \( (s,t) \in \sigma \) and \( a = b \in \alpha(ss^{-1}tt^{-1})A \). Since \( S \) is \( E \)-unitary, \( s \sim t \) and hence \( t^{-1}s \in E(S) \). Set \( v = (tt^{-1})s = t(t^{-1}s) \leq s,t \). Note that \( vv^{-1} = ss^{-1}tt^{-1} \) and thus \( a = b \in \alpha(vv^{-1})A \).

Proposition 3.30. Let \( (A,\lambda,\alpha) \) be an \( S \)-module, such that \( \alpha \) is an epimorphism. Then there exist an inverse semigroup \( S' \), an epimorphism \( \pi : S \to S' \) and an \( S' \)-module \( (A,\bar{\lambda},\bar{\alpha}) \), such that

(i) \( (A,\bar{\lambda},\bar{\alpha}) \) is strict;

(ii) \( \bar{\lambda} \circ \pi = \lambda \) on \( S \) and \( \bar{\alpha} \circ \pi = \alpha \) on \( E(S) \).

Conversely, given an epimorphism \( \pi : S \to S' \) and a strict \( S' \)-module \( (A,\bar{\lambda},\bar{\alpha}) \), the maps \( \lambda \) and \( \alpha \), determined by (ii), endow \( A \) with an \( S \)-module structure, and moreover \( \alpha \) is surjective.

Proof. Note that the map \( \varphi : S \to L(A,S) \) defined by

\[ \varphi(s) = \alpha(ss^{-1})\delta_s \]

is a homomorphism, because

\[ \varphi(s)\varphi(t) = \alpha(ss^{-1})\lambda_s(\alpha(tt^{-1}))\delta_st = \alpha(stt^{-1}st^{-1})\delta_st = \varphi(st). \]

Hence \( \text{im} \varphi \) is an inverse subsemigroup of \( L(A,S) \) with \( (\alpha(ss^{-1})\delta_s)^{-1} = \alpha(s^{-1}s)\delta_{s^{-1}} \).

Set \( \pi = \rho^\circ \varphi : S \to A \rtimes S \). It is an epimorphism onto \( S' = \text{im} \pi \). Clearly, \( S' \) is also inverse and

\[ \pi(s)^{-1} = \pi(s^{-1}) = \rho^\circ(\alpha(s^{-1}s)\delta_{s^{-1}}). \quad (13) \]

Note that for any \( a \in A \) and \( s \in S \) we have

\[ \varphi(s) \cdot aa(s^{-1}s)\delta_{s^{-1}} = \alpha(ss^{-1})\delta_s \cdot aa(s^{-1}s)\delta_{s^{-1}} = \alpha(ss^{-1})\lambda_s(aa(s^{-1}s))\delta_{ss^{-1}} = \lambda_s(a)\delta_{ss^{-1}}, \]

using Remark 3.19. Therefore, applying \( \rho^\circ \) we get

\[ \pi(s)\rho^\circ(aa(s^{-1}s)\delta_{s^{-1}}) = \rho^\circ(\lambda_s(a)\delta_{ss^{-1}}) \quad (14) \]

for arbitrary \( s \in S \) and \( a \in A \). If \( \pi(s) = \pi(t) \), then by (13) and Corollary 3.28 we have \( \rho^\circ(aa(s^{-1}s)\delta_{s^{-1}}) = \rho^\circ(aa(tt^{-1})t)\delta_{t^{-1}} \) for arbitrary \( a \in A \). So, the left-hand sides of (14) corresponding to \( s \) and \( t \) are equal, hence the right-hand sides are also equal. Then by the “only if” part of Lemma 3.25, \( \lambda_s = \lambda_t \). Thus, there exists a map \( \bar{\lambda} : S' \to \text{End} A \) with \( \bar{\lambda} \circ \pi = \lambda \).

It is readily seen that \( \bar{\lambda} \) is a homomorphism. In view of the epimorphism \( \pi : S \to S' \) the idempotents of \( S' \) are precisely the classes \( \pi(e) = \rho^\circ(\alpha(e)\delta_e) \), where \( e \in E(S) \), and \( \pi(e) = \pi(f) \) if and only if \( \alpha(e) = \alpha(f) \), because \( \rho \) is idempotent-separating. This defines a monomorphism \( \bar{\alpha} : E(S') \to E(A) \) satisfying \( \bar{\alpha} \circ \pi = \alpha \) on \( E(S) \). It is in fact an isomorphism due to surjectivity of \( \alpha \). We only need to check the properties (i) and (ii) of Theorem 3.7.

In view of the surjectivity of \( \pi \), it suffices to prove...
(i') \( \lambda_\pi(e)(a) = \tilde{\alpha}(\pi(e))a \) for all \( e \in E(S) \) and \( a \in A \),

(ii') \( \lambda_\pi(s)(\tilde{\alpha}(\pi(e))) = \tilde{\alpha}(\pi(s)\pi(e)\pi(s^{-1})) \) for all \( s \in S \) and \( e \in E(S) \).

But these are precisely the properties (i) and (ii) of Theorem 3.7 for \((\lambda, \alpha)\) by the definition of \((\tilde{\lambda}, \tilde{\alpha})\).

The converse is straightforward: if \( \pi : S \to S' \) is an epimorphism, \( \tilde{\lambda} : S' \to \text{End} A \) is a homomorphism and \( \tilde{\alpha} : E(S') \to E(A) \) is an isomorphism such that \((A, \lambda, \tilde{\alpha})\) is a strict \( S' \)-module, then \( \lambda = \tilde{\lambda} \circ \pi : S \to \text{End} A \) and \( \alpha = \tilde{\alpha} \circ \pi : E(S) \to E(A) \) are homomorphisms. Moreover, \( \alpha \) is an surjective, because \( \pi(E(S)) = E(S') \). As shown above, (i) and (ii) of Theorem 3.7 for \((\lambda, \alpha)\) are equivalent to the same properties for \((\tilde{\lambda}, \tilde{\alpha})\). So, \((A, \lambda, \alpha)\) is an \( S \)-module with surjective \( \alpha \).

\[ \square \]

**Definition 3.31.** Given \((A, \theta) \in \text{pMod}(G)\), define the crossed product of \( A \) and \( G \) to be the set \( A \ast_\theta G \) of \( a\delta \), where \( x \in G \), \( a \in 1_x A \) and \( \delta \) is a symbol. It is a semigroup under multiplication \( a\delta \cdot b\delta = a\delta_{x-b} \delta_{xy} \).

The associativity of multiplication follows from the fact that all domains of \( \theta \) are unital ideals of \( A \) (see [4] Corollary 3.2]).

**Remark 3.32.** Let \((A, \theta) \in \text{pMod}(G)\) and \((A, \lambda, \alpha) \in \text{Mod}(S(G))\) the corresponding \( S(G) \)-module. Then \( A \rtimes S(G) \) is isomorphic to \( A \ast_\theta G \).

On one hand, this follows from Remark 3.23 and [19] Theorem 3.7 and, on the other hand, taking into account Remark 3.27 an easy direct verification shows that the map given by \( A \rtimes S(G) \ni \rho^x(a\delta_y \varepsilon_{x_n \vert y}) \leftrightarrow a\delta_y \in A \ast_\theta G \) is an isomorphism.

**Corollary 3.33.** Let \((A, \theta) \) be an inverse partial \( G \)-module, \((A, \lambda^\theta, \alpha^\theta) \) the corresponding \( (S(G)) \)-module. Then there are a strict \( S \)-module \((A, \lambda, \alpha)\), where \( S = E(A) \ast_\theta G \), and the epimorphism \( \pi : S(G) \to S \) defined by \( \pi(\varepsilon_{x_1} \ldots \varepsilon_{x_n \vert y}) = 1_{x_1} \ldots 1_{x_n} 1_y \delta_y \) such that \((\lambda, \alpha)\) satisfies (ii) of Proposition 3.30 for \( S(G) \).

Indeed, there is \( \pi : S(G) \to A \rtimes S(G) \) as in Proposition 3.30 namely

\[ \pi(\varepsilon_{x_1} \ldots \varepsilon_{x_n \vert y}) = \rho^x(\varepsilon_{x_1} \ldots \varepsilon_{x_n \vert y}) \delta_{x_1 \ldots x_n \vert y} = \rho^x(1_{x_1} \ldots 1_{x_n} 1_y \delta_{x_1 \ldots x_n \vert y}) \],

which can be identified with \( 1_{x_1} \ldots 1_{x_n} 1_y \delta_y \in E(A) \ast_\theta G \) by Remark 3.32. Obviously, each element of \( E(A) \ast_\theta G \) has this form.

**Remark 3.34.** Let \( \varphi : A' \to A'' \) be a morphism of \( S \)-modules. Then there is a (unique) homomorphism of semigroups \( \varphi : A' \rtimes S \to A'' \rtimes S \) such that \( \tilde{\varphi}(\rho^\lambda(a\delta_s)) = \rho^\lambda(\varphi(a)\delta_s) \) for any \( a\delta_s \in L(A', S) \).

In order to see this, notice that by (i) and (ii) of Proposition 3.9 the map \( \varphi : L(A', S) \ni a\delta_s \mapsto \varphi(a)\delta_s \subset L(A'', S) \) is a well-defined homomorphism of semigroups, and in view of Lemma 3.25 \( \varphi(\ker \rho^\lambda) \subset \ker \rho^\lambda(\varphi(a)\delta_s) \), so that one can define a homomorphism \( \tilde{\varphi} : A' \rtimes S \to A'' \rtimes S \) by sending \( \rho^\lambda(a\delta_s) \) to \( \rho^\lambda(\varphi(a)\delta_s) \) (recall that for a homomorphism of semigroups \( \varphi : S_1 \to S_2 \), its kernel is defined to be the congruence ker \( \varphi = \{(a, b) \in S_1 \mid \varphi(a) = \varphi(b)\} \).

**Remark 3.35.** Let \((A', \lambda', \alpha')\) and \((A'', \lambda'', \alpha'')\) be \( S \)-modules, \((A', \tilde{\lambda}', \tilde{\alpha}')\) and \((A'', \tilde{\lambda}'', \tilde{\alpha}'')\) the corresponding strict modules over the epimorphic images \( S' \) and \( S'' \) of \( S \) under \( \pi' : S \to S' \) and \( \pi'' : S \to S'' \) as given in Proposition 3.30. Then a homomorphism \( \varphi : A' \to A'' \) is a morphism of \( S \)-modules if and only if

(i) \( \varphi \circ \tilde{\alpha}' \circ \pi' = \tilde{\alpha}'' \circ \pi'' \) on \( E(S) \);

(ii) \( \varphi \circ \tilde{\lambda}'_{\pi'(s)}(a) = \tilde{\lambda}''_{\pi''(s)} \circ \varphi \) for all \( s \in S \).
Moreover, there is a (unique) homomorphism \( \psi : S' \to S'' \) satisfying \( \psi \circ \pi' = \pi'' \), so that (i) and (ii) can be replaced by

(i') \( \varphi \circ \tilde{\alpha}' = \tilde{\alpha}'' \circ \psi \) on \( E(S') \);
(ii') \( \varphi \circ \tilde{\lambda}'_s = \tilde{\lambda}''_{\psi(s)} \circ \varphi \) for all \( s \in S' \).

Indeed, by (ii) of Proposition 3.30 the equalities (i) and (ii) are precisely (i) and (ii) from Proposition 3.39 written for \( \varphi : A' \to A'' \). Furthermore, thanks to the fact that \( \varphi \circ \alpha' = \alpha'' \), the homomorphism \( \tilde{\varphi} : A' \times S \to A'' \times S \) from Remark 3.34 restricts to a homomorphism \( \psi : S' \to S'' \) with \( \psi \circ \pi' = \pi'' \). Finally, replacing \( \pi'' \) with \( \psi \circ \pi' \) in (i) and (ii) by surjectivity of \( \pi' \) we get (i') and (ii')

**Definition 3.36.** Given an inverse semigroup \( S \) and an epimorphism \( \pi : S \to S' \) we define the concept of a \( \pi \)-strict \( S \)-module as a pair \((A, \pi)\) where \( A = (A, \lambda, \alpha) \) is a strict module over \( S' = \pi(S) \).

Note that id-strict \( S \)-modules are precisely strict \( S \)-modules in the sense of Definition 3.38. Sometimes we shall omit \( \pi \) in \( (A, \pi) \) and call \( A \) a \( \pi \)-strict \( S \)-module. Moreover, if \( \pi \) is not specified, we say that \( A \) is an \( \pi \)-strict \( S \)-module.

**Definition 3.37.** Let \((A', \lambda', \alpha')\) and \((A'', \lambda'', \alpha'')\) be \( \pi \)-strict \( S \)-modules under some epimorphisms \( \pi' : S \to S' \) and \( \pi'' : S \to S'' \), respectively. By a morphism of \( \pi \)-strict \( S \)-modules \((A', \pi') \to (A'', \pi'')\) we mean a pair \((\varphi, \psi)\), where \( \varphi : A' \to A'' \) and \( \psi : S' \to S'' \) are homomorphisms of semigroups such that

(i) \( \psi \circ \pi' = \pi'' \);
(ii) \( \varphi \circ \alpha' = \alpha'' \circ \psi \) on \( E(S') \);
(iii) \( \varphi \circ \lambda'_s = \lambda''_{\psi(s)} \circ \varphi \) for all \( s \in S' \).

It is immediately seen that in the case \( S' = S'' \), \( \pi' = \pi'' \) we have \( \psi = \text{id} \) and the equalities (ii)–(iii) give the definition of a morphism of strict \( S' \)-modules.

\( \pi \)-strict \( S \)-modules and their morphisms form a category (under the coordinatewise composition of morphisms) which will be denoted by \( \text{ESMod}(S) \).

**Definition 3.38.** An \( \pi \)-strict \( S \)-module isomorphic to \((A, \bar{\lambda}, \bar{\alpha})\) given in Proposition 3.30 will be called standard.

We are going to describe the standard \( \pi \)-strict \( S \)-modules \((A, \pi)\) in terms of \( \pi \). Note that for \((A, \pi)\) from Proposition 3.30 we have \( \ker \pi \subseteq \sigma \). It turns out that this condition characterizes \((A, \pi)\) as a standard \( \pi \)-strict module, if \( S \) is \( E \)-unitary.

**Proposition 3.39.** Let \( S \) be an \( E \)-unitary inverse semigroup. An \( \pi \)-strict \( S \)-module \((A, \lambda', \alpha')\), \( \pi' : S \to S' \), is standard if and only if \( \ker \pi' \subseteq \sigma \).

Proof. The “only if” part is obvious: if \((\varphi, \psi) : (A', \pi') \to (A'', \pi'')\) is an isomorphism between some \( \pi \)-strict \( S \)-modules, then \( \psi \circ \pi' = \pi'' \) implies \( \ker \pi' = \ker \pi'' \).

For the “if” part consider the \( S \)-module \((A, \lambda, \alpha)\), where \( \lambda = \lambda' \circ \pi' \) and \( \alpha = \alpha' \circ \pi' \). By Proposition 3.30 it can be viewed as a \( \pi'' \)-strict module \((A, \lambda'', \alpha'')\) for the corresponding \( \pi'' : S \to S'' \subseteq A \times S \) (in particular \( \lambda = \lambda'' \circ \pi'' \) and \( \alpha = \alpha'' \circ \pi'' \)). We will show that \( \ker \pi'' = \ker \pi'' \).

For the inclusion \( \ker \pi'' \subseteq \ker \pi' \) we do not need \( S \) to be \( E \)-unitary. Indeed, \( \pi''(s) = \pi''(t) \) means \((\alpha(s^{-1})\delta_s, \alpha(tt^{-1})\delta_t) \in \rho \) and hence by Lemma 3.29 there is \( u \leq s, t \) such that \( \alpha(ss^{-1}) = \alpha(uu^{-1}) = \alpha(ss^{-1}) = \alpha(tt^{-1}) \). Since \( u \leq s, t \) implies \( uu^{-1} \leq ss^{-1}, tt^{-1} \), then the
above equalities can be reduced to $\alpha uu^{-1} = \alpha ss^{-1} = \alpha tt^{-1}$. These are the same as $\pi' uu^{-1} = \pi' ss^{-1} = \pi' tt^{-1}$, because $\alpha = \alpha' \circ \pi'$ on $E(S)$ and $\alpha'$ is a bijection. But then $\pi'(s) = \pi'(ss^{-1})\pi'(s) = \pi'(uu^{-1})\pi'(s) = \pi'(uu^{-1}s) = \pi'(u)$, as $u = uu^{-1}s$. Similarly $\pi'(t) = \pi'(u)$. Thus, $\pi'(s) = \pi'(u)$.

For the converse inclusion suppose $\pi'(s) = \pi'(t)$. Then $(s, t) \in \sigma$, because ker $\pi' \subseteq \sigma$. Moreover, $\pi'(ss^{-1}) = \pi'(s)\pi'(s^{-1}) = \pi'(t)\pi'(t)^{-1} = \pi'(tt^{-1})$ and hence $\alpha(ss^{-1}) = \alpha(tt^{-1})$. Thus, by Remark 3.27 we conclude that $\alpha(s\delta)\alpha(t\delta) \in R$, i.e. $\pi'(s) = \pi'(t)$.

Since ker $\pi' = \ker \pi''$, then there exists a unique isomorphism $\psi : S' \rightarrow S''$ satisfying $\psi \circ \pi' = \pi''$. The equalities $\alpha = \alpha' \circ \pi' = \alpha' \circ \pi''$ and $\lambda = \lambda' \circ \pi' = \lambda'' \circ \pi''$ imply $\alpha' = \alpha'' \circ \psi$ and $\lambda' = \lambda'' \circ \psi$. This means that $(id, \psi) : (A, \pi') \rightarrow (A, \pi'')$ is an isomorphism in ESMod($S$).

**Corollary 3.40.** Under the conditions of Proposition 3.38 if $(A, \pi')$ is standard, then there is an epimorphism of semigroups $\eta : S' \rightarrow S''$ such that $\eta \circ \pi'$ is the natural map $\sigma^2 : S \rightarrow S'$. In particular, if $A$ is the standard $S(G)$-module from Corollary 3.33 then $\eta$ can be viewed as an epimorphism $S' \rightarrow G$ by means of $\eta(1_{x_1} \ldots 1_{x_n} 1_{y_\rho}) = y$.

From now on we shall work only with inverse modules.

**Definition 3.41.** An epi-strict $S$-module $(A, \pi)$ will be called inverse, if $A$ is an inverse semigroup. The subcategory of inverse epi-strict $S$-modules is denoted by InvESMod($S$).

Following Lausch [23] we are going to define a concept of a free object in InvESMod($S$). Let us first agree on what will be a base of a free inverse epi-strict $S$-module.

In the classical case [23] a base of a free $S$-module is a so-called $E(S)$-set. It is a disjoint union of sets indexed by $E(S)$. A morphism of $E(S)$-sets is a map which agrees with the partitions of these sets (i.e. it sends the $e$-component of one set to the subset of the $e$-component of another set for all $e \in E(S)$). Such sets appear in our situation and they will be called strict $E$-sets. However, we need a more general concept of an $E(S)$-set.

**Definition 3.42.** Let $S$ and $S'$ be inverse semigroups and $\pi : S \rightarrow S'$ an epimorphism. A \textit{$\pi$-strict $E(S)$-set} is a pair $(T, \pi)$ where $T$ is a strict $E(S')$-set.

In particular, id-strict $E(S)$-sets can be identified with the “ordinary” (strict) $E(S)$-sets. As above, $\pi$ will be often omitted and a $\pi$-strict $E(S)$-set, whose $\pi$ is not specified, will be called an epi-strict $E(S)$-set. We shall also use the standard notation $T_e$ for the $e$-component of $T$ ($e \in E(S')$).

**Definition 3.43.** Given two epi-strict $E(S)$-sets $(T', \pi') : S \rightarrow S'$ and $(T'', \pi'') : S \rightarrow S''$, by a morphism of epi-strict $E(S)$-sets $(T', \pi') \rightarrow (T'', \pi'')$ we mean a pair $(\varphi, \psi)$, where $\psi : S' \rightarrow S''$ is a homomorphism of semigroups, satisfying $\psi \circ \pi' = \pi''$, and $\varphi : T' \rightarrow T''$ is a map of sets such that $\varphi(T_e) \subseteq T''_{\varphi(e)}$ for each $e \in E(S')$.

If $S' = S''$ and $\pi' = \pi''$, then $(\varphi, \psi)$ is a morphism of $\pi$-strict $E(S)$-sets if and only if $\psi = id$ and $\varphi$ is a classical [23] morphism of (strict) $E(S')$-sets. The category of epi-strict $E(S)$-sets and their morphisms will be denoted by ESSet($S$).

**Remark 3.44.** Each inverse $\pi$-strict $S$-module is a $\pi$-strict $E(S)$-set and each morphism of inverse epi-strict $S$-modules is a morphism of epi-strict $E(S)$-sets.

The first assertion immediately follows from the fact that an inverse $\pi'$-strict $S$-module $(A', \lambda', \alpha')$, $\pi' : S \rightarrow S'$, being an inverse strict $S'$-module, is a strict $E(S')$-set (see [23]). Its $e$-component is $A'_e = \{a \in A' \mid aa^{-1} = \alpha'(e)\} \subseteq E(S')$. Now if $(\varphi, \psi) : (A', \pi') \rightarrow (A'', \pi'')$ is a morphism of inverse epi-strict $S$-modules and $a \in A'_e$ for some $e \in E(S')$, then (ii) of Definition 3.37 implies $\varphi(a) \varphi(a)^{-1} = \varphi(aa^{-1}) = \varphi(\alpha'(e)) = \alpha''(\psi(e))$, so $\varphi(a) \in A''_{\varphi(e)}$. 

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Definition 3.45. A module $F \in \text{InvESMod}(S)$ is free over a set $T \in \text{ESSet}(S)$ if there is a morphism $\langle t, \kappa \rangle : T \to F$ in $\text{ESSet}(S)$ such that for any $A \in \text{InvESMod}(S)$ and any morphism $(\varphi, \psi) : T \to A$ in $\text{ESSet}(S)$ there exists a unique morphism $(\bar{\varphi}, \bar{\psi}) : F \to A$ in $\text{InvESMod}(S)$ such that $(\varphi, \psi) = (\bar{\varphi}, \bar{\psi}) \circ (t, \kappa)$.

We recall Lausch’s construction of a free (strict) $S$-module. Given an $E(S)$-set $T$, define $F = F(T)$ to be the disjoint union of components $F_e$, where each $F_e$ is the free abelian group (written additively) over

$$\{(s, t) \in S \times T \mid ss^{-1} = e, \ t \in T_f \text{ for some } f \geq s^{-1}s\}.$$  

The sum of elements $(s, t) \in F_e$ and $(s', t') \in F_{e'}$ of different components is the formal sum $(e's, t) + (es', t')$ in $F_{ee'}$. The structure of an $S$-module is defined by $\lambda^F_e(s, t) = (s't, s)$ and $\alpha^F_e = 0_e$, where $0_e$ is the zero of $F_e$. The embedding $\iota : T \to F$ is given by

$$T_e \ni t \mapsto (e, t) \in F_e.$$  

Remark 3.46. Notice that each $(s, t)$, where $t \in T_f$, can be written as $\lambda^F_e(f, t)$, because $s^{-1}s \leq f$, so $sf = (ss^{-1})f = s(s^{-1}s)f = s(s^{-1})s = s$ in this case.

Proposition 3.47. For any epi-strict $E(S)$-set $(T, \pi : S \to S')$ there is a free inverse epi-strict $S$-module $F(T)$ over $T$.

Proof. Since $(T, \pi : S \to S')$ is an $E(S')$-set in the sense of Lausch, then by [23, Proposition 3.1] there is the free Lausch's $S'$-module $(F(T), \lambda^F(T), \alpha^F(T))$ which therefore can be considered as a $\pi$-strict $S$-module. We shall show that $F(T)$ is free over $T$ in $\text{InvESMod}(S)$.

Let $\iota : T \to F(T)$ be the embedding (15). As it was noticed after Definition 3.37 the pair $(\iota, \text{id})$ is a morphism in $\text{ESSet}(S)$. Take any module $(A, \lambda, \alpha) \in \text{InvESMod}(S)$ and $(\varphi, \psi) : T \to A$ in $\text{ESSet}(S)$. It will be convenient to use the additive notation for $A$. We need to find $(\bar{\varphi}, \bar{\psi})$ in $\text{InvESMod}(S)$, such that $\bar{\varphi} \circ \iota = \varphi$ and $\bar{\psi} \circ \iota \circ \text{id} = \psi$. The second condition immediately gives $\bar{\psi} = \psi$. The first one uniquely defines $\bar{\varphi}$ as a homomorphism satisfying (iii) of Definition 3.37. Indeed, take $(s, t) \in F(T)_e$ with $t \in T_f$ and note by Remark 3.46 that $\bar{\varphi}(s, t) = \varphi_e \circ \lambda^F_e(f, t)$, which should be equal to $\lambda_{\psi(s)} \circ \bar{\varphi}(f, t) = \psi(s) \circ \bar{\varphi}(t) = \lambda_{\psi(s)}(\varphi(t))$. Therefore

$$\bar{\varphi} \left( \sum a_{s,t}(s, t) \right) = \sum a_{s,t} \lambda_{\psi(s)}(\varphi(t))$$

and (iii) of Definition 3.37 is straightforward. Moreover, since $\varphi(T_e) \subseteq A_{\psi(e)}$ for all $e \in E(S')$, we observe that

$$\lambda_{\psi(s)}(\varphi(t)) \in \lambda_{\psi(s)}(A_{\psi(f)}) \subseteq A_{\psi(s)} \psi(f) \psi(s^{-1}) = A_{\psi(f)} = A_{\psi(s^{-1})} = A_{\psi(e)},$$

so $\varphi$ is automatically a homomorphism from $F(T)_e$ to $A_{\psi(e)}$ for all $e \in E(S')$. Hence, it maps the zero $\alpha^F(T)(e)$ of $F_e$ to the zero $\alpha(\psi(e))$ of $A_{\psi(e)}$ and thus (ii) of Definition 3.37 is also true. It only remains to make sure that $\bar{\varphi}$ is a homomorphism of semigroups (written additively). For $(s, t) \in F(T)_e$ and $(s', t') \in F(T)_{e'}$ we have

$$\bar{\varphi}((s, t) + (s', t')) = \varphi((e's, t) + (es', t')) = \lambda_{\psi(e')}(\varphi(t)) + \lambda_{\psi(es')}(\varphi(t'))$$

$$= \lambda_{\psi(e')}(\lambda_{\psi(s)}(\varphi(t))) + \lambda_{\psi(es')}(\varphi(t')))$$

$$= \alpha(\psi(e')) + \lambda_{\psi(s)}(\varphi(t)) + \lambda_{\psi(es')}(\varphi(t'))$$

$$= \alpha(\psi(e)) + \lambda_{\psi(s)}(\varphi(t)) + \alpha(\psi(es')) + \lambda_{\psi(es')}(\varphi(t'))$$

$$= \lambda_{\psi(s)}(\varphi(t)) + \lambda_{\psi(es')}(\varphi(t')) = \bar{\varphi}(s, t) + \bar{\varphi}(s', t').$$
Moreover, since we have an epimorphism $S$ of all idempotents of $\Gamma(G)$, where $\pi : S \rightarrow S'$, has form $(\varphi, \text{id}_{S'})$, so this component is isomorphic to the category of strict inverse $S'$-modules. In particular, it is an abelian category (see [24]).

Remark 3.50. The free inverse epi-strict module $F(T)$ over $\Gamma(T, \pi : S \rightarrow S')$ belongs to the $\pi$-component of $\text{InvESMod}(S)$. Moreover, it is projective in this component.

Indeed, by Corollary 3.2 from [23] $F(T)$ is projective as a strict inverse $S'$-module.

Corollary 3.51. There are non-standard epi-strict modules.

For an example consider an inverse semigroup $S$ with $\pi \neq S^2$ and the epimorphism $\pi : S \rightarrow 0$ to the zero semigroup. Clearly $S^2 = \ker \pi \subseteq \sigma$. Then for any $\pi$-strict set $T$ the free inverse $\pi$-strict $S$-module $F(T)$ is not standard.

According to Propositions 3.2, 3.30, Theorem 3.7 and Remark 3.17 each inverse partial $G$-module can be seen as an inverse epi-strict $S(G)$-module.

Definition 3.52. Let $A$ be an inverse partial $G$-module and $T$ an epi-strict $E(S(G))$-set. The module $A$ is said to be free over $T$ if the corresponding inverse epi-strict $S$-module is free over $T$ in $\text{InvESMod}(S(G))$.

We shall now examine the construction of a free inverse partial $G$-module in more detail. Let $\Gamma : G \rightarrow M$ be a partial representation of $G$ in a monoid $M$. Denote by $\Gamma(G)$ the submonoid of $M$ generated by all $\Gamma(x), x \in G$. Then $\Gamma(G)$ is an epimorphic image of $S(G)$ and every epimorphic image $\pi(S(G))$ of $S(G)$ can be obtained this way by setting $\Gamma = \pi \circ [\ ]$, where $[\ ] : G \rightarrow S(G)$ is the canonical partial homomorphism $x \mapsto [x]$. Notice that $\Gamma(G)$ is an inverse monoid with $\Gamma(x)^{-1} = \Gamma(x^{-1})$.

Write $e_x = \Gamma(x)\Gamma(x^{-1}) = \pi(e_x)$. Then the elements $e_x$ are pairwise commuting idempotents and one has that

$$\Gamma(x)e_y = e_\gamma \Gamma(x)$$

for all $x, y \in G$ (see [9, Section 7]). Let $E$ be the submonoid of $\Gamma(G)$ generated by the idempotents $e_x$. We know from [9, Theorem 6] that $\Gamma(G)$ is an epimorphic image of $E \ast_\theta G$ where

$$\theta = \theta^\Gamma = \{ \theta_x : E_{x^{-1}} \rightarrow E_x \}$$

is the partial action of $G$ on $E$ which corresponds to $\Gamma$, with $E_x = e_x E$ and $\theta_x(e) = \Gamma(x)e\Gamma(x^{-1}), e \in E_{x^{-1}}$. It follows that $\Gamma(G)$ is a (not necessarily disjoint) union

$$\Gamma(G) = \bigcup_{x \in G} E_x \Gamma(x).$$

Moreover, since we have an epimorphism $S(G) \rightarrow \Gamma(G)$, we obtain that $E = E(\Gamma(G))$, the set of all idempotents of $\Gamma(G)$. Given an arbitrary element $s = e\Gamma(x) (e \in E_x)$ of $\Gamma(G)$, write $e = e_x e_{y_1} \ldots e_{y_k} (y_1, \ldots, y_k \in G)$. Then $ss^{-1} = e$ and

$$s^{-1}s = e_{x^{-1}1} e_{x^{-1}y_1} \ldots e_{x^{-1}y_k} = \Gamma(x^{-1})e\Gamma(x) = \theta_{x^{-1}}(e).$$

The cited results in [9] are stated for a partial projective representation $\Gamma$ of $G$ in a $K$-cancellative monoid $M$. Notice that any monoid $M$ can be transformed into a $K$-cancellative monoid over $K = GF(2)$ by adding a zero $0_M$ to $M$ (if necessary) and setting $0_M a = 0_M, 1_M a = a, a \in M$. Then any partial homomorphism $\Gamma : G \rightarrow M$ becomes a partial projective representation over $K$ with trivial factor set.
Given a strict $E$-set $T = \bigsqcup_{e \in E} T_e$, the free inverse partial $G$-module $F(T)$ over $T$ can be specified as follows. Taking a fixed $e \in E$ set

$$\Gamma_e = \{ e\Gamma(x) \mid x \in G, e \in E_x \}. $$

The condition $e \in E_x$ means that $e$ can be written as a product $e = e_x e_{y_1} \cdots e_{y_k}$ for some $y_1, \ldots, y_k \in G$. Observe that $\Gamma_e$ consists of all $s \in \Gamma(G)$ with $ss^{-1} = e$. Next for an arbitrary $s = e\Gamma(x) \in \Gamma_e$ let $E(s)$ be the set of all $f \in E$ such that $f \geq s^{-1}s$. In particular, $E(s)$ contains all $f \in E$ which can be written in the form $f = (e_x^{-1})^k \prod_{j \in J} e_{y_j}$, where $\nu \in \{0, 1\}$ and $J$ is a (possibly empty) subset of $\{1, \ldots, k\}$ (if $\nu = 0$ and $J = \emptyset$ we assume $f = 1_M$).

Write $T(s) = \bigsqcup_{f \in E(s)} T_f$. Then the $e$-component $F(T)_e$ of $F(T)$ is the free abelian group with free basis $B_e = \{(s, t) \mid s \in \Gamma_e, t \in T(s)\}$. Denote by $1_{(e)}$ the identity element of $F(T)_e$ and write $1_x = 1_{(e)}$, $x \in G$. The elements from $\bigsqcup_{e \in E} B_e$ will be called the canonical generators of $F(T)$. The product of canonical generators $(s, t) \in F(T)_e$ and $(s', t') \in F(T)_{e'}$ from different components is $(e's, t)(e's', t')$ as an element of $F(T)_{e'e'}$.

The corresponding partial action $\theta^F(T)$ of $G$ on $F(T)$ consists of the isomorphisms $\theta^F_x(T) : D_{x^{-1}} \to D_x$, where

$$D_x = 1_x F(T) = \bigsqcup_{e \in E_x} F(T)_e,$$

$x \in G$ and $\theta^F_x(T)(s, t) = (\Gamma(x)s, t)$ for any canonical generator $(s, t)$ which is contained in $D_{x^{-1}}$. If $(s, t) \in D_{x^{-1}}$, then $s = e_x^{-1} e\Gamma(y)$ for some $e \in E$ and $y \in G$ such that $e_x^{-1} ey = e_x^{-1} e$. Then

$$\theta^F_x(T)(e_x^{-1} e\Gamma(y), t) = (\theta^F_x(T)(e_x^{-1} e)\Gamma(xy), t).$$

Writing $e = e_y e_{z_1} \cdots e_{z_k}$, where $z_1, \ldots, z_k \in G$, we have $\theta^F_x(T)(e_x^{-1} e) = e_x e_{xy} e_{z_1} \cdots e_{z_k}$. An arbitrary element $a \in D_{x^{-1}}$ belongs to some component $F(T)_e$ and can be written as a combination of canonical generators of $F(T)_e$:

$$a = (s_1, t_1)^{n_1} \cdots (s_r, t_r)^{n_r},$$

$n_i \in \mathbb{Z}$, and one has

$$\theta^F_x(T)(a) = (\theta^F_x(T)(s_1, t_1))^{n_1} \cdots (\theta^F_x(T)(s_r, t_r))^{n_r}. $$

### 4 Partial cohomology of groups and cohomology of inverse semigroups

**Definition 4.1.** The cohomology groups of an inverse semigroup $S$ with values in an inverse epistict $S'$-module $(A, \pi : S \to S')$ can be defined in the following way: $H^n(S, A) = H^n_{S'}(A)$, where on the right-hand side $A$ is meant to be a strict $S'$-module (see [23]).

Under the conditions of the above definition denote by $\text{Hom}_x(-, A)$ the restriction of $\text{Hom}(-, A)$ to the $\pi$-component of $\text{InvESMod}(S)$. Then it is an additive functor from an abelian category to $\text{Ab}$, and according to [23] the above cohomology is its derived functor, applied to the inverse $\pi$-strict $S$-module $\mathbb{Z}^G$ with $(\mathbb{Z}^G)_x = \{ n_e \mid n \in \mathbb{Z} \}$, $n_e + m_f = (n + m)e_f$, $\alpha^{\mathbb{Z}^G}(n_e) = \pi(n_e)$, $\alpha^{\mathbb{Z}^G}(e) = 0_e$. Hence, $H^n(S, A)$ can be calculated by taking an appropriate projective resolution of $\mathbb{Z}^G$ in the $\pi$-component of $\text{InvESMod}(S)$. It turns out that for $S = \mathcal{S}(G)$ and $(A, \pi : S \to S')$, which comes from an inverse $(A, \theta) \in \text{pMod}(G)$, we can construct a free resolution of $\mathbb{Z}^G$, such that the corresponding cohomology groups with values in $(A, \pi)$ are precisely $H^n(G, A)$. 

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Definition 4.2. Let $\Gamma : G \to S$ be a partial homomorphism of a group $G$ in an inverse monoid $S$. For any positive integer $n$ denote by $V_n$ the strict $E(S)$-set, whose $e$-component is the (possibly empty) set of ordered $n$-tuples $(x_1, \ldots, x_n) \in G^n$, such that $e(x_1,\ldots,x_n) = e$, where $e(x_1,\ldots,x_n)$ is the idempotent
\[ e x_1 e x_2 \ldots e x_n = \Gamma(x_1) \ldots \Gamma(x_n) \Gamma(x_n)^{-1} \ldots \Gamma(x_1)^{-1}. \]

For $n = 0$ define $(V_n)_e$ to be the one-element set \{(\ )\} if $e = 1_S$, and $\emptyset$ otherwise. The free $S$-module over $V_n$ will be denoted by $R_n$.

Lemma 4.3. Let $(A, \theta)$ be an inverse partial $G$-module and $(A, \lambda, \alpha)$ the corresponding (standard) inverse $\pi$-strict $S(G)$-module, $\pi : S(G) \to S$. Set $\Gamma = \pi \circ [ ] : G \to S$ and consider $R_n = R_n(\Gamma)$ as an element of the $\pi$-component of InvESMod$(S(G))$. Then the abelian group $\text{Hom}_\pi(R_n, A)$ is isomorphic to $C^n(G, A)$.

Proof. Since $R_n$ is the free $S$-module over $V_n = V_n(\Gamma)$, each morphism from $R_n$ to $A$ is fully determined by its values on $V_n$. So, $\text{Hom}_\pi(R_n, A)$ can be identified with the set of morphisms of $\pi$-strict $E(S(G))$-sets from $V_n$ to $A$. A pair $(\varphi, \psi)$ is such a morphism if and only if $\psi = \text{id}_S$ and $\varphi(v) \in A_e$ for all $v \in (V_n)_e$. If $n = 0$, then $\varphi$ is identified with the image of ( ) $\in (V_n)_1$, which should belong to $A_{1S}$, i.e. be invertible with respect to
\[ \alpha(1_S) = \alpha \circ \pi(1_S(\Gamma)) = \alpha^\theta(1_S(\Gamma)) = 1_A. \]

Thus, $\text{Hom}_\pi(R_0, A) \cong U(A) = C^0(G, A)$. For $n > 0$ the function $\varphi$ can be viewed as a map $G^n \to A$, such that $\varphi(x_1, \ldots, x_n) \in A_e$, where $e = e(x_1,\ldots,x_n)$. By definition $A_e$ consists of $a \in A$, which are invertible with respect to
\[ \alpha(e) = \alpha^\theta(\varepsilon x_1 \varepsilon x_2 \ldots \varepsilon x_n) = 1_{x_1} 1_{x_2} \ldots 1_{x_n}. \]

So, $A_e = U(A(x_1,\ldots,x_n))$ and hence $\text{Hom}_\pi(R_n, A) \cong C^n(G, A)$. \hfill \Box

Definition 4.4. Under the conditions of Definition 4.2 for all $n \geq 1$ define the $S$-module morphisms $\partial_n : R_n \to R_{n-1}$ by
\[
\partial_n (x_1, \ldots, x_n) = \left( \Gamma(x_1) e(x_2,\ldots,x_n), (x_2, \ldots, x_n) \right) \\
+ \sum_{i=1}^{n-1} (-1)^i (e(x_1,\ldots,x_n), (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots,x_n)) \\
+ (-1)^n (e(x_1,\ldots,x_n), (x_1, \ldots, x_{n-1})), n > 1,
\]

and $\epsilon : R_0 \to \mathbb{Z}_S$ by $\epsilon( ) = 1_{1S}$.

Lemma 4.5. Under the conditions of Lemma 4.3 for any $n \geq 0$ and for arbitrary $f \in C^n(G, A)$ we have $\delta^nf = f \circ \delta_{n+1}$, where $f$ and $\delta^n f$ are considered as homomorphisms of $\pi$-strict $S(G)$-modules $R_n \to A$ and $R_{n+1} \to A$ respectively.

Proof. Denote by $\iota_n$ the natural embedding $V_n \to R_n$ given in (13). If $n = 0$, then $f \in C^0(G, A)$ considered as $f : R_0 \to A$ is identified with $f(\iota_0( )) = a \in U(A)$, where $\iota_0( ) = (1_S, ( ))$. Therefore, for any $x \in G$:
\[
f(\partial_1(x)) = f(\Gamma(x), ( )) f(\epsilon_x, ( ))^{-1} = \lambda_{\Gamma(x)}(a) \lambda_{\epsilon_x}(a)^{-1} = \lambda^\theta_x(a)^{-1} \lambda^\theta_{\epsilon_x}(a)^{-1} = \theta_x(a x_{n-1})^{-1} \theta_x(a x_{n-1})^{-1} 1_x = \theta_x(a x_{n-1}) a^{-1} 1_x = \theta_x(a x_{n-1}) a^{-1} = (\delta^0 a)(x).
\]
Now let \( n > 0 \) and \( f \in C^n(G, A) \). Considering \( f \) as the morphism \( R_n \to A \), we notice that
\[
f(x_1, \ldots, x_n) = f(t_n(x_1, \ldots, x_n)) = f(e(x_1, \ldots, x_n), (x_1, \ldots, x_n)).
\]
Therefore,
\[
f(\partial_{n+1}(x_1, \ldots, x_{n+1})) = \lambda_{\Gamma(x_1)}(f(x_2, \ldots, x_{n+1}))
\]
\[
\prod_{i=1}^{n} \lambda_{\varepsilon(x_1, \ldots, x_{n+1})}(f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}))^{(-1)^i}
\]
\[
\lambda_{\varepsilon(x_1, \ldots, x_{n+1})}(f(x_1, \ldots, x_n))^{(-1)^n+1}.
\]
As we have seen above, \( \lambda_{\Gamma(x)}(a) = \theta_x(a 1_{x-1}) \) and \( \lambda_{\varepsilon}(a) = a 1_x \). Thus,
\[
f(\partial_{n+1}(x_1, \ldots, x_{n+1})) = (\delta^n f)(x_1, \ldots, x_{n+1})1_{x_1}1_{x_2} \ldots 1_{x_1} \ldots x_{n+1} = (\delta^n f)(x_1, \ldots, x_{n+1}),
\]
because \( (\delta^n f)(x_1, \ldots, x_{n+1}) \in A_{(x_1, \ldots, x_{n+1})} \).

**Corollary 4.6.** Under the conditions of Lemma 4.3 for any \( n \geq 2 \) the composition \( \partial_{n-1} \circ \partial_n \) is the zero morphism from \( R_n \) to \( R_{n-2} \) in the \( \pi \)-component of \( \text{InvESMod}(S(G)) \).

Indeed, by construction the module \((A, \pi)\) is standard. Since \( S(G) \) is \( E \)-unitary, by Proposition 3.39 any module from the \( \pi \)-component of the category \( \text{InvESMod}(S(G)) \) is standard, in particular all \( R_n \) have this property. Thus without loss of generality we may assume that \( (R_n, \pi) \) is obtained from some inverse partial \( G \)-module \((R_n, \theta_n)\) for all \( n > 0 \). Now taking \( A' = R_{n-2} \) and \( f = \text{id}_{R_{n-2}} \in \text{Hom}_\pi(R_{n-2}, A') \) by Lemma 4.3 and Proposition 1.3 we see that
\[
\partial_{n-1} \circ \partial_n = f \circ \partial_{n-1} \circ \partial_n = \delta^{n-1}\delta^{n-2}f = e_n.
\]

In view of Lemma 4.3 the cochain \( e_n \) can be seen as the morphism from \( R_n \) to \( R_{n-2} \) which maps each element of \((V_n)_e\) to the zero of \((R_{n-2})_e\), \( e \in E(S) \). So, \( e_n = 0 \in \text{Hom}_\pi(R_n, R_{n-2}) \).

**Definition 4.7.** Under the conditions of Lemma 4.3 let \( \eta : S \to G \) be as in Corollary 3.40 and for all \( n \geq 0 \) define the morphism \( \sigma_n : R_n \to R_{n+1} \) of \( \pi \)-strict \( E(S(G)) \)-sets as follows. On each \( e \)-component of \( R_n \) it is the homomorphism of abelian groups \((R_n)_e \to (R_{n+1})_e\) given by
\[
\sigma_n(s, (x_1, \ldots, x_n)) = (ss^{-1}, (\eta(s), x_1, \ldots, x_n)).
\]
We shall also define \( \tau : \mathbb{Z}_S \to R_0 \) to be the homomorphism \((\mathbb{Z}_S)_e \to (R_0)_e\), such that \( \tau(1_e) = (e, ( \_\_ )) \), \( e \in E(S) \).

The correctness of the definition of \( \sigma_n \) will follow from the next lemma.

**Lemma 4.8.** Under the conditions of Definition 4.7 for all \( x \in G \) and \( s \in S \)
\begin{enumerate}
  \item \( \eta(\Gamma(x)) = x; \)
  \item \( ss^{-1}\Gamma(\eta(s)) = s; \)
  \item \( s\Gamma(\eta(s)^{-1}) = ss^{-1}. \)
\end{enumerate}

**Proof.** Notice that \( \eta \circ \pi \) is the natural epimorphism \( S(G) \to G \), so that \( \eta(\Gamma(x)) = \eta(\pi([x])) = x \), which is (i). If \( s = \pi(\varepsilon_{x_1} \ldots \varepsilon_{x_n}[y]) \), then \( \eta(s) = y \) and \( \Gamma(\eta(s)) = \Gamma(y) = \pi([y]) \), so (ii) and (iii) follow from equalities in \( S(G) \), which can be easily verified. \( \square \)
Now we show that \((ss^{-1}, (\eta(s), x_1, \ldots, x_n))\) in Definition 4.7 is an element of \(R_{n+1}\), i.e. \((ss^{-1})^{-1}(ss^{-1}) = ss^{-1} \leq e(\eta(s), x_1, \ldots, x_n)\). Using (ii) and (iii) of Lemma 4.8 and the fact that \(se(x_1, \ldots, x_n) = s\) (this follows from \(s^{-1}s \leq e(x_1, \ldots, x_n)\)), we see that
\[
ss^{-1}e(\eta(s), x_1, \ldots, x_n) = ss^{-1}\Gamma(\eta(s))e(x_1, \ldots, x_n)\Gamma(\eta(s)^{-1}) = se(x_1, \ldots, x_n)\Gamma(\eta(s)^{-1}) = s\Gamma(\eta(s)^{-1}) = ss^{-1}.
\]

**Lemma 4.9.** Under the conditions of Lemma 4.3

(i) \(\partial_1 \circ \sigma_0 + \tau \circ \epsilon = \text{id}_{R_0}\);

(ii) \(\partial_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial_n = \text{id}_{R_n}\) for all \(n \geq 1\).

**Proof.** It is sufficient to verify (i) on an arbitrary generator \((s, (\ ))\) of \((R_0)_{ss^{-1}}\). Using (ii) and (iii) of Lemma 4.8 we have:
\[
(\partial_1 \circ \sigma_0 + \tau \circ \epsilon)(s, (\ )) = \partial_1(ss^{-1}, (\eta(s))) + \tau(\lambda_{ss^{-1}}(1_{1s}))
\]
\[
= (ss^{-1}\Gamma(\eta(s)), (\ )) - (ss^{-1}e(\eta(s)), (\ )) + \tau(1_{ss^{-1}})
\]
\[
= (s, (\ )) - (ss^{-1}\Gamma(\eta(s))\Gamma(\eta(s)^{-1}), (\ )) + (ss^{-1}, (\ ))
\]
\[
= (s, (\ )) - (s\Gamma(\eta(s)^{-1}), (\ )) + (ss^{-1}, (\ ))
\]
\[
= (s, (\ )).
\]

For (ii) take a generator \((s, (x_1, \ldots, x_n))\) of \((R_n)_{ss^{-1}}\). Taking into account the facts that \(se(x_1, \ldots, x_n) = s, ss^{-1} \leq e(\eta(s), x_1, \ldots, x_n)\) explained above and (ii) of Lemma 4.8 we calculate \(\partial_{n+1} \circ \sigma_n(s, (x_1, \ldots, x_n))\):
\[
\partial_{n+1} \circ \sigma_n(s, (x_1, \ldots, x_n)) = \partial_{n+1}(ss^{-1}, (\eta(s), x_1, \ldots, x_n))
\]
\[
= (ss^{-1}\Gamma(\eta(s))e(x_1, \ldots, x_n), (x_1, \ldots, x_n))
\]
\[
- (ss^{-1}e(\eta(s), x_1, \ldots, x_n), (\eta(s)x_1, x_2, \ldots, x_n))
\]
\[
+ \sum_{i=1}^{n-1} (-1)^{i+1}(ss^{-1}e(\eta(s), x_1, \ldots, x_n), (\eta(s), x_1, \ldots, x_ix_{i+1}, \ldots, x_n))
\]
\[
+ (-1)^{n+1}(ss^{-1}e(\eta(s), x_1, \ldots, x_n), (\eta(s), x_1, \ldots, x_n-1))
\]
\[
= (s, (x_1, \ldots, x_n)) - (ss^{-1}, (\eta(s)x_1, x_2, \ldots, x_n))
\]
\[
+ \sum_{i=1}^{n-1} (-1)^{i+1}(ss^{-1}, (\eta(s), x_1, \ldots, x_ix_{i+1}, \ldots, x_n))
\]
\[
+ (-1)^{n+1}(ss^{-1}, (\eta(s), x_1, \ldots, x_n-1)). \quad (16)
\]

Furthermore, note that \(se = s\) implies \(sf = s\) for all \(f \geq e\), because \(sf = sef = se = s\). Hence,
\[
s\Gamma(x_1)e(x_2, \ldots, x_n) = se(x_1x_2, \ldots, x_n)\Gamma(x_1) = s\Gamma(x_1),
\]
as \(e(x_1x_2, \ldots, x_n) \geq e(x_1, \ldots, x_n)\). Using this observation and (i) of Lemma 4.8 we compute
\[
\sigma_{n-1} \circ \partial_n(s, (x_1, \ldots, x_n)) = \sigma_{n-1}(s\Gamma(x_1)e(x_2, \ldots, x_n), (x_2, \ldots, x_n))
\]
\[
+ \sum_{i=1}^{n-1} (-1)^i\sigma_{n-1}(se(x_1, \ldots, x_n), (x_1, \ldots, x_ix_{i+1}, \ldots, x_n))
\]
\[
+ (-1)^n\sigma_{n-1}(se(x_1, \ldots, x_n), (x_1, \ldots, x_n-1))
\]
\[
= \sigma_{n-1}(s\Gamma(x_1), (x_2, \ldots, x_n))
\]
\[
= \sigma_{n-1}(x_1, \ldots, x_n)\Gamma(x_1).
\]
Under the conditions of Lemma 4.3 we have Corollary 4.12.\[\text{(16)}\]
A whose cohomology groups with values in \(x\) can be similarly obtained from (ii) of Lemma 4.9.\[\text{(17)}\]

Indeed, if \(\epsilon(r) = 0\), for some \(r \in (R_0)_e\), \(e \in E(S)\), then by (i) of Lemma 4.9 and the fact that \(\tau|_{(Z_S)_e}\) is a homomorphism of abelian groups \((Z_S)_e \rightarrow (R_0)_e\), we see that \(r = \delta_1(\sigma_0(r)) + \alpha_{R_0}(e)\). Since \(\delta_1 \circ \sigma_0\) is a morphism of \(\pi\)-strict \(E(S(G))\)-sets (see Remark 3.44 and Definition 4.7), \(\delta_1(\sigma_0(r)) \in (R_0)_e\), so \(\delta_1(\sigma_0(r)) + \alpha_{R_0}(e) = \delta_1(\sigma_0(r))\), whence \(r = \delta_1(\sigma_0(r))\). The second inclusion can be similarly obtained from (ii) of Lemma 4.9.

Theorem 4.11. Under the conditions of Lemma 4.3 the sequence

\[ \ldots \to \delta_n + 1 \to R_n \delta \to \ldots \to \delta_1 \to R_1 \delta \to R_0 \epsilon \to Z_S \to 0, \]

where 0 is the zero of the \(\pi\)-strict component of \(\text{InvESMod}(S(G))\), is a free resolution of \(Z_S\) whose cohomology groups with values in \(A\) are isomorphic to \(H^n(G, A)\).

Proof. In view of Lemmas 4.3, 4.5 and Corollaries 4.10 it only remains to prove the inclusion \(\text{im} \delta_1 \subseteq \ker \epsilon\) (the exactness in \(Z_S\) is obvious, because \(n_e = \epsilon(n_e, (1))\)) for all \(n \in \mathbb{Z}\) and \(e \in E(S)\). This can be done by direct computation of \(\epsilon \circ \delta_1\). For any \(x \in G\) the image \(\epsilon(\delta_1(x))\) is

\[ \epsilon(\Gamma(x), (1)) - \epsilon(e_x, (1)) = x_{\Gamma(x)}(1) \epsilon_S(1) - x_{e_x}(1) = 1_{\Gamma(x)\Gamma(x)} - 1_{e_x} = 0_{e_x}, \]

as \(\Gamma(x)\Gamma(x)^{-1} = e_x\).\[\Box\]

Corollary 4.12. Under the conditions of Lemma 4.3 we have \(H^n(G, A) \cong H^n(S(G), A)\) for all \(n \geq 0\).

References

[1] P. Ara, R. Exel, T. Katsura, Dynamical systems of type \((m, n)\) and their \(C^*\)-algebras, \(Ergodic Theory Dynam. Systems\) (to appear).

[2] A. Clifford, G. Preston, The algebraic theory of semigroups, vol. II, Math. Surveys of the American Math. Soc. 7, Providence R. I., 1967.
[3] M. Dokuchaev, Partial actions: a survey, *Contemp. Math.*, **537** (2011), 173–184.

[4] M. Dokuchaev, R. Exel, Associativity of crossed products by partial actions, enveloping actions and partial representations, *Trans. Amer. Math. Soc.*, **357** (2005), (5), 1931–1952.

[5] M. Dokuchaev, R. Exel, P. Piccione, Partial representations and partial group algebras, *J. Algebra*, **226** (2000), (1), 251–268.

[6] M. Dokuchaev, R. Exel, J. J. Simón, Crossed products by twisted partial actions and graded algebras, *J. Algebra*, **320** (2008), (8), 3278–3310.

[7] M. Dokuchaev, R. Exel, J. J. Simón, Globalization of twisted partial actions, *Trans. Am. Math. Soc.*, **362** (2010), (8), 4137–4160.

[8] M. Dokuchaev, M. Ferrero, A. Paques, Partial Actions and Galois Theory, *J. Pure Appl. Algebra*, **208** (2007), (1), 77–87.

[9] M. Dokuchaev, B. Novikov, Partial projective representations and partial actions, *J. Pure Appl. Algebra*, **214** (2010), 251–268.

[10] M. Dokuchaev, B. Novikov, Partial projective representations and partial actions II, *J. Pure Appl. Algebra*, **216** (2012), 438–455.

[11] M. Dokuchaev, B. Novikov, H. Pinedo, The partial Schur Multiplier of a group, *J. Algebra*, **392** (2013), 199–225.

[12] R. Exel, Circle actions on *C*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences, *J. Funct. Anal.*, **122** (1994), (3), 361–401.

[13] R. Exel, The Bunce-Deddens algebras as crossed products by partial automorphisms, *Bol. Soc. Brasil. Mat. (N.S.)*, **25** (1994), 173–179.

[14] R. Exel, Approximately finite *C*-algebras and partial automorphisms, *Math. Scand.*, **77** (1995), 281–288.

[15] R. Exel, Twisted partial actions: a classification of regular *C*-algebraic bundles, *Proc. London Math. Soc.*, **74** (1997), (3), 417–443.

[16] R. Exel, Partial actions of groups and actions of inverse semigroups, *Proc. Amer. Math. Soc.*, **126** (1998), (12), 3481–3494.

[17] R. Exel, Hecke algebras for protonormal subgroups, *J. Algebra*, **320** (2008), 1771–1813.

[18] R. Exel, M. Laca, J. Quigg, Partial dynamical systems and *C*-algebras generated by partial isometries, *J. Operator Theory*, **47** (2002), (1), 169–186.

[19] R. Exel and F. Vieira, Actions of inverse semigroups arising from partial actions of groups, *J. Math. Anal. Appl.*, **363** (2010), (1), 86–96.

[20] M. Ferrero, Partial actions of groups on algebras, a survey, *São Paulo J. Math. Sci.*, **3** (2009), (1), 95–107.

[21] D. Gonçalves, D. Royer, Leavitt path algebras as partial skew group rings, *Preprint. http://arxiv.org/abs/1202.2704*

[22] J. Kellendonk, M. V. Lawson, Partial actions of groups, *Internat. J. Algebra Comput.*, **14** (2004), (1), 87–114.

[23] H. Lausch, Cohomology of inverse semigroups, *J. Algebra*, **35** (1975), 273–303.
[24] M. V. Lawson, Inverse semigroups. The theory of partial symmetries, World Scientific, 1998.

[25] B. Novikov, H. Pinedo, On the partial Schur multiplier, Comm. Algebra (to appear).

[26] H. Pinedo, On elementary domains of partial projective representations of groups, Algebra Discrete Math., 15 (2013), (1), 63–82.

[27] J. C. Quigg, I. Raeburn, Characterizations of crossed Products by Partial Actions, J. Operator Theory, 37 (1997), 311–340.

[28] N. Sieben, $C^*$-crossed products by partial actions and actions of inverse semigroups, J. Aust. Math. Soc., Ser. A, 63 (1997), (1), 32–46.

[29] B. Steinberg, Partial actions of groups on cell complexes, Monatsh. Math., 138 (2003), (2), 159–170.