Branes in the OSP(1|2) WZNW model

Thomas Creutzig\textsuperscript{a} and Yasuaki Hikida\textsuperscript{b}

\textsuperscript{a}Department of Physics and Astronomy, University of North Carolina, Phillips Hall, CB 3255, Chapel Hill, NC 27599-3255, USA

\textsuperscript{b}Department of Physics, and Research and Education Center for Natural Sciences, Keio University, Hiyoshi, Yokohama 223-8521, Japan

Abstract

The boundary OSP(1|2) WZNW model possesses two types of branes, which are localized on supersymmetric Euclidean AdS\textsubscript{2} and on two-dimensional superspheres. We compute the coupling of closed strings to these branes with two different methods. The first one uses factorization constraints and the other one a correspondence to boundary $\mathcal{N} = 1$ super-Liouville field theory, which we proof with path integral techniques. We check that the results obey the Cardy condition and reproduce the semi-classical computations. For the check we also compute the spectral density of open strings that are attached to the non-compact branes.

\textsuperscript{*}E-mail: creutzig@physics.unc.edu

\textsuperscript{†}E-mail: hikida@phys-h.keio.ac.jp
Contents

1 Introduction 2

2 The semi-classical limit of branes 4
  2.1 Geometry of the branes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
  2.2 The semiclassical boundary states . . . . . . . . . . . . . . . . . . . . . . . 7
    2.2.1 Super AdS$_2$ branes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
    2.2.2 Fuzzy supersphere branes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11

3 OSP(1|2) WZNW model 12
  3.1 OSP(1|2) current algebra . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13
  3.2 Correlation functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
  3.3 Two, Three and four point functions . . . . . . . . . . . . . . . . . . . . . . . 15
  3.4 Degenerate representation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17

4 Boundary OSP(1|2) model 18
  4.1 Super AdS$_2$ branes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
  4.2 Fuzzy supersphere branes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 23

5 Relation to boundary super-Liouville theory 24
  5.1 Boundary super-Liouville theory . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
  5.2 Boundary OSP(1|2) model . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 25
  5.3 Relation between the two theories . . . . . . . . . . . . . . . . . . . . . . . . . . . . 27
  5.4 One point function of bulk operators . . . . . . . . . . . . . . . . . . . . . . . . . . 29

6 Annulus amplitude and Cardy condition 30
  6.1 Boundary states for super AdS brane in the RR-sector . . . . . . . . . . . . . 31
  6.2 Annulus amplitude from open strings . . . . . . . . . . . . . . . . . . . . . . . . 33
  6.3 Boundary states for super AdS branes in the NSNS-sector . . . . . . . . . . . 35
  6.4 Open-closed duality . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 37
  6.5 Fuzzy supersphere brane . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 37

7 Conclusion and discussions 39

A Some super analysis 40

B Four point function with a degenerate operator 41
  B.1 Null equation from the degenerate operator . . . . . . . . . . . . . . . . . . . . . . 41
  B.2 Solutions to the KZ equation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 42
  B.3 Basic three point functions of bulk operator . . . . . . . . . . . . . . . . . . . . . . 46
Conformal field theories (CFTs) with target-space supersymmetry play an important role in areas as superstring theory and condensed matter physics. In particular two-dimensional models are building blocks of superstring theories with AdS-background, which are important for the AdS/CFT correspondence \[1\]. Moreover, the computation of spectral densities and transport properties in systems with random disorder involve theories with internal supersymmetry \[2\]. In both cases orthosymplectic supergroup symmetries appear frequently. Especially the coset model $\text{OSP}(1|2)/\text{U}(1)$ with target-space and world-sheet supersymmetries has been recently proposed in \[3\] to describe the two-dimensional superstring, which is holographically dual to a Hermitian matrix model \[4, 5\].

Two-dimensional conformal field theories with target-space supersymmetry are non-unitary and non-rational. A class of such models are Wess-Zumino-Novikov-Witten (WZNW) models with Lie supergroup target. Recently, a variety of methods were developed to treat these models. The idea is to reduce the model to a simpler problem that is already understood. Most importantly the computation of correlation functions in WZNW models of unitary (type I) supergroups can be reduced to computations on the WZNW model of its bosonic subgroup plus some free fermions \[6\]. Unfortunately these free-field methods do not carry over to the family of orthosymplectic (type II) supergroups. In this case an idea is to generalize the correspondence between Euclidean AdS and bosonic Liouville theory \[7, 8, 9, 10\]. For this the path integral derivation of the correspondence \[10\] is very suitable. Indeed it is shown that the correlation functions of $\text{OSP}(N|2)$ WZNW models can be computed in terms of $\mathcal{N} = (N,N)$ super-Liouville field theory \[11\].

It is a natural task to extend these works to world-sheets with boundary. We will recall some of their features. So far, the boundary $\text{GL}(1|1)$ WZNW model is well understood, i.e. boundary states and correlation functions are computed \[15, 16, 17, 18\]. The computation of correlation functions can be performed in a similar free fermion realization as in the bulk case of type I supergroup. The difficulty in establishing the formalism is to find the

\[1\] Structure constants for $N = 1$ case have been obtained explicitly in \[11\] with the knowledge of those of super-Liouville field theory \[12, 13\], see also \[14\].
appropriate boundary term of the action and it involves additional fermionic boundary
degrees of freedom. This situation is known from world-sheet supersymmetric models as
super-Liouville field theory \cite{14, 19, 20} and in general from matrix factorization \cite{21}. It
may indicate that the OSP(N|2) to $\mathcal{N} = (N,N)$ super-Liouville correspondence also holds
for the boundary models. Beyond the GL(1|1) WZNW models, some boundary spectra
of PSL(2|2) sigma models have been studied \cite{22}. Furthermore, it is known that branes
on super groups are described by twisted super conjugacy classes \cite{23}.

In this work we combine our experience from bulk OSP(N|2) WZNW models and
boundary supergroup models to understand the boundary OSP(1|2) WZNW model. We
find that there are two classes of branes in the OSP(1|2) WZNW model. We call the
first one (super) AdS$_2$ branes, because the geometry of their bosonic subspace is AdS$_2$.
These branes have super-dimension 2|2. The other class are spherical ones, also of super-
dimension 2|2. When the sphere degenerates to a point, there is as well a point-like brane
as a two-dimensional one in the fermionic directions.

Branes are characterized by their coupling to closed strings. These couplings are
expressed in boundary conformal field theory by bulk one-point functions on a disk or
equivalently by the boundary state. One the other hand, with the boundary state we
can compute the partition function of open strings ending on the branes via world-sheet
duality \cite{24}. In general its computation is more involved than in rational boundary CFTs,
since the boundary state involves a non-trivial spectral density. This happens when the
open string spectrum is continuous, which is usually the case in non-compact and non-

In this article, we compute these quantities in two different ways. The first one is to
proceed in analogy with its bosonic cousin, the $H^+_3$ model, or Euclidean AdS$_3$ model. In
order to compute bulk one-point functions on a disk, we utilize two-point functions with
a degenerate operator. They can be factorized into two ways, and constraints can be
obtained by comparing the two expressions. The one-point functions are then obtained
by solving these constraints. We test these one-point functions by careful analysis of their
classical limits. For the super AdS$_2$ branes we determine the spectral density of open
strings between them and check that it is consistent with the one-point functions via
world-sheet duality. The second approach is less direct. We extend the correspondence
between $\mathcal{N} = 1$ super-Liouville field theory and OSP(1|2) model to the boundary case.
We can then use the results of boundary $\mathcal{N} = 1$ super-Liouville field theory in, e.g., \cite{14}.
One advantage to the previous approach is that we can easily obtain one-point functions
of bulk operators in the NSNS-sector as well.\footnote{In this paper we call the NS-sector as the one with the anti-periodic boundary conditions for worldsheet fermions. If we would like to treat fermions as the Grassmann odd coordinates of target-space, then they should satisfy the periodic boundary condition, in other words, they are in the R-sector.}
The article is organized as follows. We start in section 2 with some geometric considerations. In particular, we find a semi-classical expression for the boundary states. This allows us to read off the semi-classical limit of closed string couplings to the branes. In section 3 we review the basic properties of bulk OSP(1|2) WZNW model. In particular, we show how OSP(1|2) symmetry restricts the form of correlation functions generically. Previous results of two and three point functions in [11] are also given. In section 4, they are used to obtain bulk one-point functions in the boundary theory. Then in section 5 we derive a correspondence between the boundary OSP(1|2) theory describing super AdS\textsubscript{2} branes and \(\mathcal{N} = 1\) super-Liouville theory with boundary. In section 6 we verify that our results agree with world-sheet duality. Namely, the bulk one-point functions define the boundary states, whose modular S-transformation leads to the open string partition functions. Section 7 concludes with a summary of results and a list of interesting open problems. The appendices contain detailed computations of correlation functions and a derivation of the action of the OSP(1|2) boundary theory describing super AdS\textsubscript{2} branes.

2 The semi-classical limit of branes

The analysis of the branes’ geometry already provides useful information of the closed string. The boundary states contain the information of the closed string couplings to the brane. In the semi-classical limit this boundary state becomes a delta-distribution localized on the brane. The aim of this section is to rewrite these semi-classical boundary states in such a way that we can identify the semi-classical limit of the closed strings couplings to the brane.

The results are as follows. We will find two types of branes, whose bosonic subspaces are AdS\textsubscript{2} and spherical ones. The geometry of the spherical ones are superconjugacy classes of OSP(1|2), while the AdS\textsubscript{2} branes are twisted superconjugacy classes. In both cases, we can characterize the branes by the position \(a\) of the (twisted) super conjugacy class. The delta-distribution then has the following form

\[
\delta(g - a) = \int dj \int d^2u d^2\lambda \ (\Phi_h(u, \lambda|g))^* \langle \Phi_h(u, \lambda|z) \rangle_a ,
\]

where \(\Phi_h(u, \lambda|g)\) are eigenfunctions of the Laplacian\(^3\) and \(\langle \Phi_h(u, \lambda|z) \rangle_a\) are the corresponding closed string couplings in the semiclassical limit. For the AdS\textsubscript{2} branes they are precisely

\[
\langle \Phi_h(x, \lambda|z) \rangle_{r,+}^{\text{AdS}} \underset{k \to \infty}{\sim} |x + \bar{x} + \lambda \bar{\lambda}|^{-2h} e^{sgn(x+\bar{x})r(-2h+1/2)}
\]

and

\[
\langle \Phi_h(x, \lambda|z) \rangle_{r,-}^{\text{AdS}} \underset{k \to \infty}{\sim} sgn(x + \bar{x})|x + \bar{x} - \lambda \bar{\lambda}|^{-2h} e^{sgn(x+\bar{x})r(-2h+1/2)} ,
\]

\(^3\)The dual function is \(\Phi_h(u, \lambda|g))^* = \Phi_{-h+1/2}(u, \lambda|g)\).
and for the spherical ones

\[
\langle \Phi_h(u, \lambda) \rangle_{\Lambda_0^-}^{\text{sphere}} \xrightarrow{k \to \infty} |1 + u \bar{u} + \lambda \lambda^2|^{-2h} \sinh(\Lambda_0(2h - 1/2)) \quad \text{and}
\]

\[
\langle \Phi_h(u, \lambda) \rangle_{\Lambda_0^+}^{\text{sphere}} \xrightarrow{k \to \infty} |1 + u \bar{u} - \lambda \lambda^2|^{-2h} \cosh(\Lambda_0(2h - 1/2)).
\]

The remainder of this section is the explanation of (2.2) and (2.3). The analysis is similar to the one of its bosonic cousin, the \(H^+_3\) model [25].

2.1 Geometry of the branes

In this subsection we describe the geometry of the branes in OSP(1|2) following the general analysis of branes on supergroups [23]. We want to describe branes that are maximally symmetry preserving, i.e. they preserve conformal symmetry as well as the Lie super algebra current symmetry. Such branes are described by automorphisms \(\omega\) of the Lie superalgebra which respect its invariant metric. The left and right moving currents are glued together along the boundary of the worldsheet with such an automorphism ensuring current and Virasoro symmetry on the boundary. We want to describe branes that are maximally symmetry preserving, i.e. they preserve conformal symmetry as well as the Lie super algebra current symmetry. Such branes are described by automorphisms \(\omega\) of the Lie superalgebra which respect its invariant metric. The left and right moving currents are glued together along the boundary of the world-sheet with such an automorphism ensuring current and Virasoro symmetry on the boundary. Similar to Lie group WZNW models [26] these gluing conditions describe branes localized at twisted super conjugacy classes

\[
C_a^\omega = \{ \omega(b)ab^{-1} \mid b \in G \}.
\]

We turn to OSP(1|2). The Lie superalgebra \(osp(m|2n)\) is most conveniently expressed in a matrix representation as

\[
osp(m|2n) = \{ X \in gl(m|2n) \mid X^{st}B_{m,n} + B_{m,n}X = 0 \},
\]

where the supertranspose is given in (A.4) and

\[
B_{m,n} = \begin{pmatrix} 1_m & 0 \\ 0 & J_n \end{pmatrix}, \quad \text{where } J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.
\]

Thus \(osp(1|2)\) is represented by matrices as

\[
A = \begin{pmatrix} 0 & \theta_- & \theta_+ \\ \theta_+ & a & b \\ -\theta_- & c & -a \end{pmatrix}.
\]
We obtain elements of OSP(1|2) by exponentiation. Moreover, we require that our OSP(1|2) valued fields are super-hermitean, i.e. they satisfy $g = g^\dagger$. The super-hermitean conjugate $\dagger$ is complex conjugation concatenated with supertransposition. Here complex conjugation for Grassmann numbers is the super star operation $\#$, see appendix A. Strictly speaking this means that our model takes values in the coset of super-hermitean matrices. This situation is analogous to Euclidean AdS$_3$ or $H^+_3$, which is the coset of SL(2) consisting of hermitean matrices, and it means that the bosonic subspace of our model is Euclidean AdS$_3$.

Let us turn to gluing automorphisms. All automorphisms of osp(1|2) are inner. Nonetheless, the geometry of branes corresponding to different gluing maps can differ. The reason is that the automorphisms are not related as automorphisms of the coset of super-hermitean elements. The same situation appears in the $H^+_3$ model, and it is explained in [25]. In that case there are essentially two different types of branes, and their geometry is AdS$_2$ and $S^2$. The branes we are going to study are their analogs in OSP(1|2).

The gluing map that we want to consider acts in our matrix representation by conjugation with the matrix

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (2.8)$$

Note that $X$ acts exactly as $(-1) \circ (st)$. This automorphism has order four and its fix points form a one-dimensional space. This implies that all branes have co-dimension one. Moreover the restriction of the twisted super conjugacy class

$$C^X_a = \{ g = XbX^{-1}ab^{-1} \mid b \in G \} \quad (2.9)$$

to its bosonic subspace is Euclidean AdS$_2$. Hence we describe supersymmetric AdS branes of super-dimension 2|2. The geometry of branes corresponding to the gluing automorphism that is conjugation by the inverse of $X$ is also supersymmetric AdS. The open string ends on the brane, which means that the Lie supergroup valued field $g$ describing the string is restricted to its branes world-volume, the twisted super conjugacy class. This allows us to read off the Dirichlet boundary conditions

$$\text{str}(X^{-1}g) = \text{str}(bX^{-1}ab^{-1}) = \text{str}(X^{-1}a) = \text{constant}. \quad (2.10)$$

The second class of branes is described by (untwisted) super conjugacy classes

$$C_a = \{ bab^{-1} \mid b \in G \}. \quad (2.11)$$

In this case three different types of geometries arise. The restriction of these super conjugacy classes to its bosonic subspace are two-spheres, which would degenerate to points at the identity. In this case also the super conjugacy class becomes zero-dimensional in
the fermionic directions. If we instead choose \( X^2 = (-1)^F \) as gluing condition, then the twisted super conjugacy classes are also superspheres of dimension 2[2], but at the identity they degenerate only to a point in the bosonic direction while still extending in the fermionic ones. Note that these non-generic branes resemble those of GL(1|1) \([15]\). In this spherical case the Dirichlet condition is

\[
\text{str}(g) = \text{str}(bab^{-1}) = \text{str}(a) = \text{constant}.
\]

The Dirichlet conditions can be stated more explicitly. We parameterize an OSP(1|2) group element by a real number \( \phi \), a complex number \( (\gamma, \bar{\gamma}) \), and a complex Grassmann number \( (\theta, \theta^\#) \). It then reads

\[
g = \begin{pmatrix}
1 + \theta^2 e^{-\phi} & \theta e^{-\phi} & \theta^2 + \theta \bar{\gamma} e^{-\phi} \\
\theta^2 e^{-\phi} & e^{-\phi} & e^{-\phi} \bar{\gamma} \\
-\theta + \gamma \theta^2 e^{-\phi} & \gamma e^{-\phi} & -\theta^2 + \gamma \bar{\gamma} e^{-\phi} + e^\phi
\end{pmatrix}.
\]

In this parameterizations super AdS branes are described by the following Dirichlet conditions

\[
\pm \theta \theta^\# + \gamma - \bar{\gamma} = i e^\phi.
\]

Here \( \pm \) corresponds to conjugation by \( X^{\pm 1} \). The super spherical ones satisfy

\[
e^{-\phi}(\pm \theta \theta^\# + \gamma \bar{\gamma} + 1) - \theta \theta^\# + e^\phi = c
\]

for some real constants \( c \). Plus corresponds to conjugation with \((-1)^F\) and minus to the trivial gluing automorphism.

### 2.2 The semiclassical boundary states

In this section we rewrite the delta-distributions corresponding to the branes in terms of eigenfunctions of the Laplacian, which allows us to identify the semiclassical limit of bulk one-point functions. We proceed as \([25]\). First, we need a suitable parametrization of eigenfunctions of the Laplacian. The Laplacian is obtained as the Casimir for the invariant vector fields. As such it is clear that the supergroup element \( g \) (2.13) is an eigenfunction. The Laplacian reads explicitly

\[
\Delta = \frac{1}{4} \partial^2_{\phi} - \frac{1}{4} \partial_{\phi} + e^{2\phi} \partial_\gamma \partial_{\bar{\gamma}} - \frac{1}{2} e^{\phi} (\partial_{\theta} + \theta \partial_{\gamma})(\partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{\gamma}}).
\]

Let \( v = (\lambda, u, 1) \) a complex vector, \( \lambda \) Grassmann odd, and \( v^\dagger = (-\lambda^\dagger, \bar{u}, 1)^\dagger \) its super-adjoint. Then we define the field

\[
\Phi_h(u, \lambda | g) = (vgv^\dagger)^{-2h} = (-|\lambda - \theta|^2 + e^{-\phi}|u + \gamma + \lambda \theta|^2 + e^\phi)^{-2h}
\]
and compute that it is indeed an eigenfunction

$$\Delta \Phi_h(u, \lambda | g) = h(h - 1/2) \Phi_h(u, \lambda | g). \quad (2.18)$$

The representation label takes values $h = 1/4 - iP/2$ with $P$ in $\mathbb{R}$ and non-negative.

Let $k$ be an element of the complexification of $\text{OSP}(1|2)$, then for

$$k = \begin{pmatrix} A & \theta^1 & \theta^2 \\ \theta^{1\sharp} & \alpha & \beta \\ \theta^{2\sharp} & \gamma & \delta \end{pmatrix} \quad (2.19)$$

$\Phi_h$ transforms as follows

$$\Phi_h(u, \lambda | kgk^\dagger) = |\lambda \theta^2 + u\beta + \delta|^{-4h} \Phi_h(k \cdot u, k \cdot \lambda | g), \quad (2.20)$$

where

$$k \cdot u = \frac{\lambda \theta^1 + u\alpha + \gamma}{\lambda \theta^2 + u\beta + \delta} \quad \text{and} \quad k \cdot \lambda = \frac{\lambda A + u\theta^{1\sharp} + \theta^{2\sharp}}{\lambda \theta^2 + u\beta + \delta}. \quad (2.21)$$

It is convenient to summarize $k \cdot u$ and $k \cdot \lambda$ in a vector $v' = (k \cdot \lambda, k \cdot u, 1)$. In terms of $v$ this vector reads

$$v' = \frac{vk}{\lambda \theta^2 + u\beta + \delta}. \quad (2.22)$$

Now suppose that $k$ satisfies

$$k^\dagger = Xk^{-1}X^{-1} \quad (2.23)$$

for some $X$ in $\text{OSP}(1|2)$, then we get

$$v'X^{-1}(v')^\dagger = \frac{vkX^{-1}k^\dagger v^\dagger}{|\lambda \theta^2 + u\beta + \delta|^2} = \frac{vX^{-1}v^\dagger}{|\lambda \theta^2 + u\beta + \delta|^2}. \quad (2.24)$$

As a result, we get that the distribution representation

$$D^h[f] = \int d^2ud^2\lambda \ |vX^{-1}v^\dagger|^{2h-1} \text{sgn}^r(ivX^{-1}v^\dagger) \int dk \Phi_h(u, \lambda | k) f(k) \quad (2.25)$$

is invariant under conjugation, that is

$$D^h[f] = D^h[T_g f] \quad (2.26)$$

where $T_gf(k) = f(g^{-1}k(g^{-1})^\dagger)$ for all $g$ satisfying (2.23). This means that we found a distribution which does not depend on the position in the orbit of the twisted super conjugacy class $C^X_a$. Especially we can view the distribution as a distribution in one variable labeling the position of the twisted super conjugacy class, i.e.

$$D^h[f(k)] = D^h[f(a)]. \quad (2.27)$$

Now $f = f(a)$ is a function on the position of the twisted super conjugacy class. Since $\Phi_h$ is an eigenfunction of the Laplacian, the distribution $D^h$ must be as well. But this
only depends on the position of the twisted super conjugacy class \( a = \exp(\psi t) \) (here \( t \) is fixed under conjugation by \( X \)), thus the eigenfunctions \( E_h(\psi) \) satisfy the second order differential equation

\[
\Delta E_h(\psi) = h(h - 1/2)E_h(\psi). \tag{2.28}
\]

Hence, we get

\[
\int d^2u d^2\lambda \ |vX^{-1}v^\dagger|^{2h-1} \text{sgn}^t(ivX^{-1}v^\dagger) \Phi_h(u, \lambda | k) = K_{h, +}^+ E_h^+(\psi) + K_{h, -}^- E_h^-(\psi), \tag{2.29}
\]

where \( E_h^\pm(\psi) \) are two independent solutions to \(2.28\).

Our goal is to rewrite the delta-distribution \( \delta(\psi - r) \) in terms of wave-functions. We have reduced this problem to finding the eigenfunctions \( E_h^\pm(\psi) \), the coefficients \( K_{h, \epsilon}^\pm \) and functions \( f^\epsilon(r) \), such that

\[
\delta(\psi - r) = \int dh \ (K_{h, +}^+ E_h^+(\psi) + K_{h, -}^- E_h^-(\psi)) f^\epsilon(r). \tag{2.30}
\]

This will be done case by case.

### 2.2.1 Super AdS\(_2\) branes

We start with the AdS-type branes with gluing automorphism conjugation by \( X \) \( \tag{2.8} \). We parameterize a group element in twisted super conjugacy class form, i.e. \( g = XbX^{-1}ab^{-1} \) with

\[
a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \psi & i \sinh \psi \\ 0 & -i \sinh \psi & \cosh \psi \end{pmatrix}. \tag{2.31}
\]

In these coordinates the Dirichlet boundary condition \( \tag{2.14} \) reads

\[
\text{str}(X^{-1}g) = 1 - 2i \sinh \psi = 1 + e^{-\phi}(\theta\theta^\dagger + \gamma - \bar{\gamma}) = \text{constant}. \tag{2.32}
\]

The Laplacian modulo \( b \) is

\[
\Delta \psi = \frac{1}{4} \partial_{\psi}^2 + \frac{1}{4} \frac{\sinh \psi}{\cosh \psi} \partial_{\psi} + \frac{i}{4} \frac{1}{\cosh \psi} \partial_{\psi}
\]

\[
= \frac{1}{4} \partial_{\psi}^2 + \frac{1}{4} \frac{\sinh(\psi/2 + i\pi/4)}{\cosh(\psi/2 + i\pi/4)} \partial_{\psi}
\]

\[
= \frac{1}{4} \frac{1}{\cosh(\psi/2 + i\pi/4)} \left( \partial_{\psi}^2 - \frac{1}{4} \right) \cosh(\psi/2 + i\pi/4), \tag{2.33}
\]

and its eigenfunctions with eigenvalue \( h(h - 1/2) \) are

\[
E_h^\pm(\psi) = \frac{e^{\pm(2h-1/2)\psi}}{\cosh(\psi/2 + i\pi/4)}. \tag{2.34}
\]
The coefficients \( K_{h,\epsilon}^\pm \) are fixed by the asymptotic behavior of the functions \( \Phi_h \). We use the following result of \([25]\)

\[
\frac{2h-1}{\pi} (e^\phi + e^{-\phi} |u - \gamma|^2)^{-2h} \phi \rightarrow -\infty \quad \frac{2h-1}{\pi} |\gamma - u|^{-4h} e^{2h\phi} + 2\delta^2(\gamma - u) e^{(2-2h)\phi}. \tag{2.35}
\]

This implies the asymptotic behavior

\[
\Phi_h(u, \lambda |g) \phi \rightarrow -\infty \quad |\gamma - u - \lambda \theta|^{-4h} e^{2h\phi} + \frac{\pi}{2h-1} \delta^2(\gamma - u - \lambda \theta) e^{(2-2h)\phi} + \]

\[
+ |\lambda - \theta|^2 |\gamma - u|^{-4h-2} e^{(2h+1)\phi} + \pi |\lambda - \theta|^2 \delta^2(\gamma - u) e^{(1-2h)\phi}. \tag{2.36}
\]

Another result of \([25]\) is the integral

\[
\int_C \frac{d^2u}{|u + \bar{u}|^a} \sgn^f(u + \bar{u}) |u - \gamma|^{-2a-4} = \frac{\pi}{a+1} (-1)^r |\gamma + \bar{\gamma}|^{-a-2} \sgn^f(\gamma + \bar{\gamma}). \tag{2.37}
\]

This integral together with \( 2 \sinh \psi \phi \rightarrow -\infty e^{i|\psi|} \) as well as \([2.32]\) and \([2.36]\) implies

\[
\frac{1}{1+2h} \int d^2u d^2\lambda |vX^{-1}v^\dagger|^{2h-1} \sgn^f(i vX^{-1}v^\dagger) \phi \rightarrow -\infty
\]

\[
\rightarrow -\infty \quad \sgn^f(i(\gamma - \bar{\gamma})) \left((i(\gamma - \gamma - \lambda \theta^2))^{2h-1} + i(i(\gamma - \gamma - \bar{\lambda} \theta^2))^{-2h}\right)
\]

\[
\phi \rightarrow -\infty \quad e^{-\frac{1}{2} |\psi|^2 + \frac{i\pi}{4}} \left(e^{-|\psi=(-2h+\frac{1}{2})-\frac{i\pi}{4}} + (-1)^r e^{i|\psi=-2h+\frac{1}{2}}+\frac{i\pi}{4}}\right). \tag{2.38}
\]

This allows us to rewrite

\[
e^{(\psi-r)(-2h+1/2)} + e^{-(\psi-r)(-2h+1/2)} = \tag{2.39}
\]

\[
= \kappa_h \int d^2u d^2\lambda |vX^{-1}v^\dagger|^{2h-1} \Phi_h(u, \lambda |g)(d_0(r) - \sgn(i vX^{-1}v^\dagger)d_1(r))
\]

with

\[
d_0(r) = \sgn(r) \cosh(-r(-2h + 1/2) - i\pi/4),
\]

\[
d_1(r) = \sgn(r) \sinh(-r(-2h + 1/2) - i\pi/4) \text{ and} \tag{2.40}
\]

\[
\kappa_h = \frac{\cosh(\psi/2 + i\pi/4)}{(h+1/2)}.
\]

Notice that \( h = 1/4 - iP/2 \) for non-negative real number \( P \). Hence we arrive at the following expression for the delta-distribution of the AdS-like brane

\[
\delta(\psi - r) = \int dP \kappa_h \int d^2u d^2\lambda |vX^{-1}v^\dagger|^{2h-1} \Phi_h(u, \lambda |g)(d_0(r) - \sgn(i vX^{-1}v^\dagger)d_1(r)). \tag{2.41}
\]
Recall that the dual wave function is \((\Phi_h(u, \lambda|g))^* = \Phi_{-h+1/2}(u, \lambda|g)\).

We can finally conclude that the bulk one-point function behaves as follows in the semiclassical limit:

\[
\langle \Phi_h(u, \lambda|z) \rangle_{\text{AdS}}^{r_{+}} \quad k \to \infty \quad |u - \bar{u} + \lambda \lambda^\#|^{-2h} e^{-\text{sgn}(i(u - \bar{u}))r(-2h+1/2)}. \tag{2.42}
\]

There are two choices of lifting complex conjugation to Grassmann numbers, the super star \(\#\) and the bar operation. In the full field theory it is more convenient for us to work with the bar. The two operations are related as in (A.3). With \(u = ix\) the semiclassical limit is

\[
\langle \Phi_h(x, \lambda|z) \rangle_{\text{AdS}}^{r_{+}} \quad k \to \infty \quad |x + \bar{x} + \lambda \lambda^\#|^{-2h} e^{\text{sgn}(x + \bar{x})r(-2h+1/2)}. \tag{2.43}
\]

We can repeat the analysis for conjugation by \(X^{-1}\) as gluing automorphism instead of \(X\). In that case the semiclassical limit behaves as

\[
\langle \Phi_h(x, \lambda|z) \rangle_{\text{AdS}}^{r_{-}} \quad k \to \infty \quad \text{sgn}(x + \bar{x})|x + \bar{x} - \lambda \lambda^\#|^{-2h} e^{\text{sgn}(x + \bar{x})r(-2h+1/2)}. \tag{2.44}
\]

Eventually, as in (4.23) and (4.24), we will see that this is indeed the behavior of the bulk one-point functions in the semiclassical limit.

### 2.2.2 Fuzzy supersphere branes

The fuzzy-sphere like branes are described by the identity gluing automorphism. We parameterize an group element as before \((bab^{-1})\) with

\[
a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^\Lambda & 0 \\ 0 & 0 & e^{-\Lambda} \end{pmatrix}. \tag{2.45}
\]

The branes are characterized by the equation

\[
\text{str}(g) = e^{-\phi}(\theta \bar{\theta} - \gamma \bar{\gamma} - 1) + 1 + \theta \theta^2 - e^\phi = 1 - 2 \cosh(\Lambda) = \text{constant}, \tag{2.46}
\]

where \(\Lambda\) is the radius of the spherical branes and hence we restrict it to be non-negative.

The Laplacian in these coordinates modulo \(b\) is

\[
\Delta_\Lambda = \frac{1}{4} \partial^2_\Lambda + \frac{1}{4} \cosh \Lambda \partial_\Lambda - \frac{1}{4} \frac{1}{\sinh \Lambda} \partial_\Lambda \\
= \frac{1}{4} \partial^2_\Lambda + \frac{1}{4} \frac{\sinh(\Lambda/2)}{\cosh(\Lambda/2)} \partial_\Lambda \\
= \frac{1}{4} \frac{1}{\cosh(\Lambda/2)} (\partial^2_\Lambda - \frac{1}{4}) \cosh(\Lambda/2). \tag{2.47}
\]

\(^4\)Here \(k\) means the level of \(\text{OSP}(1|2)\) current algebra.
Its eigenfunctions with eigenvalue $h(h - 1/2)$ are

$$E^\pm_h(\Lambda) = \frac{e^{\pm(-2h+1/2)\Lambda}}{\cosh(\Lambda/2)}.$$  \hfill (2.48)

In this case a careful analysis of the asymptotics gives

$$\cosh(\Lambda(2h - 1/2)) = \kappa_h \int d^2 u d^2 \lambda \ |v \lambda^\dagger|^{2h-1} \Phi_h(u, \lambda|g)$$  \hfill (2.49)

with

$$\kappa_h = \frac{\cosh(\Lambda/2)}{(h + 1/2)}.$$ \hfill (2.50)

Since $\Lambda$ is non-negative, the delta-distribution for spherically branes is

$$\delta(\Lambda - \Lambda_0) = \int dP \kappa_h \int d^2 u d^2 \lambda \ |v v^\dagger|^{2h-1} \Phi_h(u, \lambda|g) \cosh(\Lambda_0(2h - 1/2)),$$ \hfill (2.51)

We conclude that the bulk one-point function behaves as follows in the semiclassical limit

$$\langle \Phi_h(u, \lambda|z) \rangle_{\Lambda_0,+}^{\text{sphere}} \underset{k \to \infty}{\sim} \ |1 + u \bar{u} - \lambda \lambda^\dagger|^{-2h} \cosh(\Lambda_0(2h - 1/2)).$$ \hfill (2.52)

The analogous result for spherical branes with gluing automorphism conjugation by $X^2$ instead of the identity map is

$$\langle \Phi_h(u, \lambda|z) \rangle_{\Lambda_0,-}^{\text{sphere}} \underset{k \to \infty}{\sim} \ |1 + u \bar{u} + \lambda \lambda^\dagger|^{-2h} \sinh(\Lambda_0(2h - 1/2)).$$ \hfill (2.53)

In this case, the result in the full quantum theory will have a similar form but the parameter $\Lambda_0$ will be imaginary. This is analogous to the spherical branes in $H_3^+$ \cite{25}.

### 3 OSP(1|2) WZNW model

Closed string sector of OSP(1|2) WZNW model has been studied in \cite{11}, and in this section we review the results in a different form with manifest OSP(1|2) symmetry. First we introduce the OSP(1|2) current algebra, then we give some general arguments concerning correlation functions. After this we restrict ourselves to concrete examples, such as two, three and four point functions. In particular, we give explicit forms of two and three-point functions. Finally, we discuss several useful facts on degenerate representations since a degenerate operator will be utilized to compute a specific four point function.

In the SL(2) WZNW model, there are several bases for vertex operators, such as $m$-basis, $x$-basis and $\mu$-basis, which are related by Fourier transforms. Among them, $x$-basis is useful to keep track of the SL(2) symmetry, and it might be interpreted as a coordinate of boundary theory as in, e.g., \cite{27, 28}. In our case we introduce one fermionic parameter $\xi$ along with $x$, which may be interpreted as a super coordinate of boundary theory. The super coordinate $(x, \xi)$ has been used to describe $\mathcal{N} = (1, 1)$ superconformal field theories in two dimension, so we can utilize the methods developed for them. As a good review see for example \cite{29}.
3.1 OSP(1|2) current algebra

Let us start from superalgebra OSP(1|2). The generators of bosonic subalgebra SL(2) are denoted by $E^\pm, H$ and for the other fermionic generators we use $F^\pm$. The relations between these generators are

\[ [H, E^\pm] = \pm E^\pm, \quad [H, F^\pm] = \pm \frac{1}{2} F^\pm, \quad [E^+, E^-] = -2H, \quad (3.1) \]
\[ [E^\pm, F^\mp] = \pm \frac{1}{2} F^\pm, \quad \{F^+, F^-\} = -\frac{1}{2} H, \quad \{F^\pm, F^\mp\} = \frac{1}{2} E^\pm. \]

As mentioned above, here we use the representation labeled by a commuting parameter $x$ and an anti-commuting parameter $\xi$. With these parameters the generators can be written as

\[ D^{E^+} = \partial_x, \quad D^H = -x\partial_x - \frac{1}{2} \xi \partial_\xi - h, \quad D^{E^-} = x^2 \partial_x + x \xi \partial_\xi + 2xh, \]
\[ D^{F^+} = \frac{1}{2} (\partial_\xi + \xi \partial_x), \quad D^{F^-} = \frac{1}{2} x (\partial_\xi + \xi \partial_x) + \xi h. \quad (3.2) \]

The second Casimir

\[ \Delta = HH - \frac{1}{2} (E^+ E^- + E^- E^+) + (F^+ F^- - F^- F^+) \quad (3.3) \]

is $h(h - 1/2)$ in the above representation.\[\square\]

An affine extension of the above superalgebra may be expressed by currents. The bosonic subalgebra is generated by $J^3(z)$ and $J^\pm(0)$, whose OPEs are

\[ J^+(z)J^-(0) \sim \frac{k}{2} \frac{J^3(0)}{z}, \quad J^3(z)J^\pm(0) \sim \pm \frac{J^\pm(0)}{z}, \quad J^3(z)J^3(0) \sim -\frac{k}{2z^2} \quad (3.4) \]

In addition to these bosonic generators, there are fermionic ones with

\[ J^3(z)j^\pm(0) \sim \pm \frac{j^\pm(0)}{2z}, \quad J^\pm(z)j^\mp(0) \sim \pm \frac{j^\mp(0)}{z}, \quad (3.5) \]
\[ j^+(z)j^-(0) \sim \frac{k}{2z^2} - \frac{J^3(0)}{2z}, \quad j^\pm(z)j^\mp(0) \sim \frac{J^\pm(0)}{2z}. \]

Here $k$ represents the level of current algebra. The energy momentum tensor can be given by the Sugawara construction as

\[ T(z) = \frac{1}{k - 3/2} \left[ -J^3(z)J^3(z) + \frac{1}{2} (J^+(z)J^-(z) + J^-(z)J^+(z)) \right] \]
\[ + j^+(z)j^-(z) - j^-(z)j^+(z) \quad (3.6) \]

\[ \text{Here } h \text{ is related to } j + 1/2 = -h \text{ with respect to } j \text{ in } [11]. \]
where normal ordering is assumed for the operators at the same position. The central charge is $c = 2k/(2k - 3)$.

With these currents primary fields can be defined as

$$J^A(z)\Phi_h(x, \xi|w) \sim -\frac{\mathcal{D}^A}{z-w}\Phi_h(x, \xi|w),$$

where $J^H = J^3$, $J^{H\pm} = J^\pm$, and $J^{F\pm} = j^\pm$. The eigenvalue of the energy momentum tensor is given by $\Delta_h = -2b^2h(h - 1/2)$ with $b^2 = 1/(2k - 3)$. The anti-holomorphic part can be defined similarly. It is related to complex conjugation as

$$(J^3(z))^* = -\bar{J}^3(\bar{z}), \quad (J^\pm(z))^* = \bar{J}^\mp(\bar{z}), \quad (j^\pm(z))^* = \pm\bar{j}^\mp(\bar{z}),$$

thus we may define, for instance

$$\bar{\mathcal{D}}_E^- = \partial_{\bar{z}}, \quad \bar{\mathcal{D}}_H = \bar{x}\partial_{\bar{z}} + \frac{1}{2}\bar{\xi}\partial_{\bar{\xi}} + h, \quad \bar{\mathcal{D}}_E^+ = \bar{x}\partial_{\bar{z}} + \bar{\xi}\partial_{\bar{\xi}} + 2\bar{h},$$

$$\bar{\mathcal{D}}_F^- = \frac{1}{2}(\partial_{\bar{\xi}} + \bar{\xi}\partial_{\bar{z}}), \quad \bar{\mathcal{D}}_F^+ = -\frac{1}{2}\bar{x}(\partial_{\bar{\xi}} + \bar{\xi}\partial_{\bar{z}}) - \bar{\xi}h.$$  

### 3.2 Correlation functions

Recall that the Lie superalgebra OSP(1|2) appears as a subalgebra of super Virasoro algebra. If $x$ is treated as a holomorphic coordinate of some (space-time) conformal field theory, then $\xi$ can be thought of as a super-coordinate in a superfield formalism as mentioned above. Just like the form of correlation functions in $\mathcal{N} = (1,1)$ superconformal field theories is restricted due to the superconformal symmetry, the form of correlation functions of OSP(1|2) WZNW model is also fixed to some extent by the OSP(1|2) global symmetry.

Consider a $N$-point function

$$Z_N = \left\langle \prod_{i=1}^{N} \Phi_h(x_i, \xi_i|z_i) \right\rangle,$$

where $z_i$ denote the positions of vertex operators inserted on the worldsheet. The dependence on $z_i$ is fixed in part by SL(2) subalgebra of Virasoro algebra as usual. A novel point is the dependence on $x_i$ and $\xi_i$. Since the superalgebra OSP(1|2) has super-dimension $3|2$, we can fix three of $x_i$ and two of $\xi_i$. In other words, the $N$-point function can be given by a function of $N - 3$ bosonic cross ratios and $N - 2$ fermionic cross ratios. Bosonic ones are given by

$$X_a = \frac{X_{12}X_{3a}}{X_{13}X_{2a}}, \quad X_{ij} = x_i - x_j + \xi_i\xi_j,$$

with $a = 4, 5, \cdots, N$. On the other hand the fermionic ones are

$$\eta_a = (X_{12}X_{1a}X_{2a})^{-1/2}(X_{2a}\xi_1 + X_{a1}\xi_2 + X_{12}\xi_a - \xi_1\xi_2\xi_a)$$
with \( \alpha = 3, 4, \cdots N \). Utilizing the symmetry we can restrict the \( N \)-point function as

\[
\left\langle \prod_{i=1}^{N} \Phi_{h_{1}}(x_{i}, \xi_{i}|z_{i}) \right\rangle = \prod_{i<j} |z_{ij}|^{-2\Delta_{ij}} |X_{ij}|^{-2\gamma_{ij}} F(X_{a}, \bar{X}_{a}, \eta_{a}, \bar{\eta}_{a}; w_{a}, \bar{w}_{a}) ,
\]

where

\[
\sum_{j \neq i} \Delta_{ji} = 2\Delta_{i} , \quad \sum_{j \neq i} h_{ji} = 2h_{i} , \quad w_{a} = \frac{z_{12}z_{3a}}{z_{13}z_{2a}} .
\]

Here \( \Delta_{i} \) represents the conformal weight of \( \Phi_{h_{i}} \).

In WZNW models we can construct the energy momentum tensor in terms of currents by Sugawara construction as in (3.6). This leads to Knizhnik-Zamolodchikov (KZ) equation which the correlation functions satisfy. In our case they are written as

\[
[\kappa \partial_{z_{i}} - \sum_{j \neq i} \frac{Q_{ij}}{z_{i} - z_{j}}] Z_{N} = 0
\]

(3.15)

with \( \kappa = -1/4b^{2} \) and

\[
Q_{ij} = D_{j}^{H}D_{j}^{H} - \frac{1}{2}(D_{i}^{E+}D_{j}^{E-} + D_{i}^{E-}D_{j}^{E+}) + (D_{i}^{F+}D_{j}^{F-} - D_{i}^{F-}D_{j}^{F+}) .
\]

(3.16)

With the explicit form of (3.12) we find

\[
Q_{12} = -\left( \frac{b}{2}x_{12}^{2} + \frac{1}{4}x_{12}\xi_{1}\xi_{2}\right) \partial_{x_{1}}\partial_{x_{2}} + (x_{12}h_{2} + \frac{1}{2}h_{2}\xi_{1}\xi_{2})\partial_{x_{1}} + (x_{21}h_{1} + \frac{1}{2}h_{1}\xi_{2}\xi_{1})\partial_{x_{2}}
\]

\[
+ \frac{h_{1}^2}{4}(\xi_{1} - \xi_{2})\partial_{\xi_{1}} + \frac{h_{2}^2}{4}(\xi_{2} - \xi_{1})\partial_{\xi_{2}} + (\frac{1}{8}x_{12}\xi_{2} - \frac{1}{8}x_{12}\xi_{1})\partial_{x_{1}}\partial_{x_{2}}
\]

\[
+ (\frac{1}{4}x_{21}\xi_{1} - \frac{1}{4}x_{21}\xi_{2})\partial_{\xi_{1}}\partial_{x_{2}} - \frac{1}{4}(x_{12} + \xi_{1}\xi_{2})\partial_{\xi_{1}}\partial_{\xi_{2}} + h_{1}h_{2} ,
\]

(3.17)

and similarly for other \( Q_{ij} \).

### 3.3 Two, Three and four point functions

Two and three point functions of OSP(1|2) model have been obtained explicitly in (1) by utilizing the relation to \( \mathcal{N} = (1, 1) \) super-Liouville field theory. Let us start from the two point function. It can be fixed by symmetry and is given by

\[
\left\langle \Phi_{h}(x, \xi|z)\Phi_{h'}(x', \xi'|z') \right\rangle
\]

(3.18)

\[
= \frac{1}{|z - z'|^{4\Delta_{h}}} \left[ \pi \delta(h + h' - \frac{1}{2})\delta^{(2)}(x - x')|\xi - \xi'|^{2} + \frac{\delta(h - h')D(h)}{|x - x' + \xi \xi'|^{4h}} \right]
\]

with some function \( D(h) \). The function was obtained in (1) as

\[
D(h) = \nu^{-4h+1}\gamma(b^{2}(k - 2h - 1))
\]

(3.19)
with $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. Here $\nu$ is some parameter.

For the three point function, the form is restricted as

$$Z_3 = \left\langle \prod_{i=1}^{3} \Phi_{h_i}(x_i, \xi_i|z_i) \right\rangle = \prod_{i<j} |z_{ij}|^{-2\Delta_{ij}} |x_{ij}|^{-2\gamma_{ij}} (C(h_1, h_2, h_3) + \tilde{C}(h_1, h_2, h_3)\bar{\eta}\bar{\eta})$$ \hspace{1cm} (3.20)

with

$$\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3 , \quad \gamma_{12} = h_1 + h_2 - h_3$$ \hspace{1cm} (3.21)

and so on. Notice that the three point function depends on one fermionic cross ratio

$$\eta = (X_{12}X_{23}X_{13})^{-1/2}(X_{23}\xi_1 + X_{31}\xi_2 + X_{12}\xi_3 - \xi_1\xi_2\xi_3)$$ \hspace{1cm} (3.22)

while the form of three point functions in a bosonic theory can be fixed uniquely by symmetry up to an overall coefficient. The information of three point function is therefore encoded in the two functions $C(h_1, h_2, h_3)$ and $\tilde{C}(h_1, h_2, h_3)$.

The explicit expressions are \[11\]

$$C(h_1, h_2, h_3) = (\nu b^b)^{-2h_2} \frac{\Upsilon_{NS}(0)\Upsilon_R(4bh_1 - b)\Upsilon_R(4bh_2 - b)\Upsilon_R(4bh_3 - b)}{\Upsilon_R(2bh - b)\Upsilon_{NS}(2bh_{12})\Upsilon_{NS}(2bh_{23})},$$ \hspace{1cm} (3.23)

$$\tilde{C}(h_1, h_2, h_3) = b^{-1}(\nu b^b)^{-2h_2} \frac{\Upsilon_{NS}(0)\Upsilon_R(4bh_1 - b)\Upsilon_R(4bh_2 - b)\Upsilon_R(4bh_3 - b)}{\Upsilon_{NS}(2bh - b)\Upsilon_R(2bh_{12})\Upsilon_R(2bh_{23})\Upsilon_R(2bh_{31})}$$

with $h = h_1 + h_2 + h_3$, $h_{12} = h_1 + h_2 - h_3$ and so on. Here we have used the notation in \[13\]:

$$\Upsilon_{NS}(x) = \Upsilon(x)\Upsilon(x + i\Omega), \quad \Upsilon_R(x) = \Upsilon(x)\Upsilon(x + b^{-1})$$ \hspace{1cm} (3.24)

where the $\Upsilon$ function is introduced in \[30, 31\] as follows

$$\ln \Upsilon(x) = \int_0^\infty \frac{dt}{t} e^{-2u} \left[ \left( \frac{Q}{2} - x \right)^2 - \frac{\sinh^2 \left( \frac{Q}{2} - x \right) t}{\sinh bt \sinh \frac{Q}{b}} \right].$$ \hspace{1cm} (3.25)

Under the shifts of their argument, they behave as

$$\Upsilon_{NS}(x + b) = b^{-b\sigma} \Upsilon(x) , \quad \Upsilon_R(x + b) = b^{1-b\sigma} \Upsilon(x)$$ \hspace{1cm} (3.26)

$$\Upsilon_{NS}(x + \frac{1}{b}) = b^{\frac{1}{b}} \Upsilon(x) , \quad \Upsilon_R(x + \frac{1}{b}) = b^{-1+b\sigma} \Upsilon(x)$$ \hspace{1cm} (3.27)

From the above argument we can see that the operator product expansion (OPE) is of the form

$$\Phi_{h_1}(x_1, \xi_1|z_1)\Phi_{h_2}(x_2, \xi_2|z_2) = \int [dh] |z_{12}|^{2\Delta_1-2\Delta_2 |x_{12}|^{2h_1-2h_2} \times$$ \hspace{1cm} (3.28)

$$C(h_1, h_2)\Phi_{h_1}(x_2, \xi_2|z_2) + |X_{12}|^{-1}\tilde{C}(h_1, h_2)\Phi_{h_2}(x_2, \xi_2|z_2)$$
where $[\ast]_{ec}$ represents the terms with even number of Grassmann odd variables and starts from $\Phi_h$. One the other hand $[\ast]_{oo}$ represents the terms with odd number of Grassmann odd variables and starts from $|\xi_1 - \xi_2|^2 \Phi_h$. The rest is fixed by the symmetry. The explicit forms of the first few terms may be found in, e.g., [29].

In a similar way, the four point function can be written in terms of one bosonic cross ratio and two fermionic cross ratios. The four point function takes the form of

$$Z_4 = \left\langle \prod_{i=1}^4 \Phi_h(x_i, \xi_i | z_i) \right\rangle = \prod_{i<j} |z_{ij}|^{-2\Delta_{ij}} |X_{ij}|^{-2\gamma_{ij}} F(X, \bar{X}, \chi_2, \bar{\chi}_2, \chi_3, \bar{\chi}_3; z, \bar{z}) ,$$

and we set

$$\Delta_{24} = -\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 , \quad \Delta_{13} = 2\Delta_3 ,$$

$$\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3 - \Delta_4 , \quad \Delta_{14} = \Delta_1 + \Delta_4 - \Delta_2 - \Delta_3 ,$$

$$(3.30)$$

$$\gamma_{24} = -h_1 + h_2 + h_3 + h_4 , \quad \gamma_{13} = 2h_3 ,$$

$$\gamma_{12} = h_1 + h_2 - h_3 - h_4 , \quad \gamma_{14} = h_1 + h_4 - h_2 - h_3 .$$

$$(3.31)$$

Addition to the worldsheet cross ratio $z = w_4$, the bosonic cross ratio is $X = X_4$ in (3.11) as

$$X = \frac{X_{12}X_{34}}{X_{13}X_{24}} = \frac{(x_1 - x_2 + \xi_1 \xi_2)(x_3 - x_4 + \xi_3 \xi_4)}{(x_1 - x_3 + \xi_1 \xi_3)(x_2 - x_4 + \xi_2 \xi_4)} ,$$

$$(3.32)$$

and the fermionic ones used here are

$$\chi_3 = (X_{12}X_{24}X_{14})^{-1/2}(X_{24}\xi_1 + X_{41}\xi_2 + X_{12}\xi_4 - \xi_1 \xi_2 \xi_4) ,$$

$$\chi_2 = -\sqrt{X}(X_{13}X_{14}X_{34})^{-1/2}(X_{34}\xi_1 + X_{41}\xi_3 + X_{13}\xi_4 - \xi_1 \xi_3 \xi_4) .$$

$$(3.33)$$

We may fix the parameters as $x_1 \to \infty, x_2 \to 1, x_3 \to x, x_4 \to 0$ by setting $\xi_1 \to x_1 \eta, \xi_2 \to 0, \xi_3 \to \xi, \xi_4 \to 0$. Then we can express the four point function with the three parameters $(x, \eta, \xi)$ in a quite simple way. Notice that $(X, \chi_3, \chi_2)$ map to $(x, \eta, \xi - \eta \xi)$. In general it is quite difficult to find out the explicit expression of the four point function. However, if one of the operators is in a degenerate representation, then we may be able to have the explicit form by solving the Knizhnik-Zamolodchikov equation (3.15). In fact we will utilize four point function with a degenerate operator $\Phi_{k/2}$ to obtain one point functions of bulk operator on a disk. The expression of the four point function is given in appendix [B].

### 3.4 Degenerate representation

A state belongs to a degenerate representation, when a descendant constructed by the action of currents becomes a null state. If we require the null state to vanish, then we have
some equations which correlation functions have to satisfy. Along with the KZ equation we may be able to obtain explicit expressions for correlation functions. According to \[32\] (see also \[33\] for instance), there are degenerate representations with

\[-4h_{r,s} + 1 = r + \frac{s}{b^2} .\]  

(3.34)

The integers \(r, s\) satisfy \(r + s = 2\mathbb{Z} + 1\), and \(r > 0, s \geq 0\) or \(r < 0, s < 0\).

We can easily understand the case of \(s = 0\). In this case the representation of zero modes with \(r > 0\) is \(r\)-dimensional, and in particular

\[j_0^+ (J_0^+)^{(r-1)/2} |h_{r,0}\rangle = 0 ,\]  

(3.35)

which yields the constraint

\[(\partial_\xi + \xi \partial_x) \partial^{(r-1)/2} x \Phi_{h}(x, \xi) = 0 .\]  

(3.36)

It looks useful to utilize the simplest one, \(\Phi_{-1/2}\) with \((r, s) = (3, 0)\). However the OPE is of the form as \(\Phi_{h}\Phi_{-1/2} \sim C[\Phi_{h-1/2}]_{ee} + C[\Phi_{h}]_{oo} + C[\Phi_{h+1/2}]_{ee}\), and related to this fact we have to solve the third order differential equations. Similar computations have been done for \(\mathcal{N} = (1, 1)\) super-Liouville field theory in \[34, 35\].

In the analysis of SL(2) WZNW model, degenerate operators with \(r > 0, s \geq 0\) are used to compute four point functions, see e.g. \[7\]. However, there are another type of degenerate operators with \(r < 0, s < 0\), and it might make the analysis simpler to utilize them. One line of degenerate operators is with \((r, s) = (-2n, -1)\), which implies \(h_{-2n,-1} = (k - 1 + n)/2\). This type of operators have been discussed in \[36\] for the SU(2) WZNW model (see also \[37\] for applications to the SL(2) WZNW model). Denoting the corresponding state as \(|(k - 1 + n)/2\rangle\), we can find a null state as

\[|\theta\rangle = (J_{-1}^-)^n |\frac{k-1+n}{2}\rangle .\]  

(3.37)

Later we will use the case with \(n = 1\) since it is the simplest one. In the bosonic case, the OPE is simply given by \(\Phi_{h}\Phi_{k/2} \sim C[\Phi_{k/2-h}]\), however in the super case the corresponding OPE is complicated enough like \(\Phi_{k}\Phi_{k/2} \sim C[\Phi_{k/2-h}]_{ee} + C[\Phi_{k/2-h-1/2}]_{oo}\). More detailed analysis can be found in appendix [13].

4 Boundary OSP(1|2) model

In this section we study boundary OSP(1|2) WZNW model focusing on bulk one point functions on a disk. In the presence of boundary, we have to assign boundary conditions in a proper way as discussed in section 2. In terms of currents the gluing conditions at the boundary \(z = \bar{z}\) are

\[J^\pm = \bar{J}^\mp = (J^\pm)^* , \quad j^3 = -\bar{j}^3 = (J^3)^* , \quad j^\pm = \pm \epsilon \bar{j}^\mp = \epsilon (j^\pm)^* \]  

(4.1)
with $\epsilon = \pm$. These boundary conditions correspond to branes similar to the AdS$_2$-branes in the SL(2) WZNW model. We can assign another type of boundary conditions as

$$J^\pm = \bar{J}^\pm = (J^\mp)^*, \quad J^3 = \bar{J}^3 = -(J^3)^*, \quad j^\pm = \epsilon j^\mp = \pm \epsilon (j^\mp)^*.$$  \hfill (4.2)

This case corresponds to branes similar to the fuzzy sphere branes in the SL(2) WZNW model. Things are quite analogous to the SL(2) WZNW model with boundary, which are analyzed, for example, in \cite{25, 38}.

Overlaps of closed strings to these branes can be read off from one point functions of bulk operators on a disk with the above boundary conditions. For the super AdS brane, one point function is restricted to have the form locally

$$\langle \Phi_h(x, \xi | z) \rangle = \frac{U^\sigma(h; \epsilon)}{|z - \bar{z}|^{2\Delta_h} |x + \bar{x} + \epsilon \xi \bar{\xi}|^{2h}} \hfill (4.3)$$

from the boundary condition. The above form has a singularity at $x + \bar{x} = 0$, thus the coefficient $U^\sigma(h)$ could depend on the sign $\sigma = \text{sgn}(x + \bar{x})$. We remark here that the same singularity appears in the SL(2) WZNW model, see \cite{25, 38}. This function will be obtained below by solving constraint equations coming from the crossing symmetry. For the fuzzy supersphere brane, one point function is restricted as

$$\langle \Phi_h(x, \xi | z) \rangle = \frac{U(h; \epsilon)}{|z - \bar{z}|^{2\Delta_h} |1 + x\bar{x} + \epsilon \xi \bar{\xi}|^{2h}} \hfill (4.4)$$

In this case we do not have any singularity on the complex $x$-plane. The coefficient $U(h; \epsilon)$ will be again obtained by solving constraints from the crossing symmetry.

### 4.1 Super AdS$_2$ branes

We would like to compute bulk one point function on a disk with boundary condition corresponding to super AdS brane. Since the boundary condition (4.1) restricts the form of one point function as (4.3), we just need to know the function $U^\sigma(h; \epsilon)$. In order to compute the function, we obtain two kinds of constraints which the one point function has to satisfy. One comes from the reflection relation which connects $\Phi_h$ to $\Phi_{1/2-h}$. The other originates from the crossing symmetry of four point function. Solutions to these constraints are found and consistency checks are performed in other sections.

First let us study the reflection relation. The information of reflection relation can be read off from the coefficient $D(h)$ (3.19) in the two point function (3.18) as in the cases of Liouville field theory and SL(2) WZNW model. It might be convenient to define

$$V_h(x|z) = \Phi_h(x, 0| z), \quad W_h(x|z) = \partial_\xi \partial_{\bar{\xi}} \Phi_h(x, \xi | z),$$  \hfill (4.5)
then the two point functions are

\[
\langle V_h(x|z)V_{h'}(x'|z') \rangle = \frac{\delta(h - h')D(h)}{|z - z'|^{4\Delta_h} |x - x'|^{4h}},
\]

\[
\langle W_h(x|z)W_{h'}(x'|z') \rangle = \frac{(2h)^2 \delta(h - h')D(h)}{|z - z'|^{4\Delta_h} |x - x'|^{4h+2}},
\]

and moreover

\[
\langle W_{\frac{1}{2}-k}(x|z)V_{h'}(x'|z) \rangle = \frac{\pi \delta(h - h')\delta^{(2)}(x - x')}{|z - z'|^{4\Delta_h}},
\]

\[
\langle V_{\frac{1}{2}-k}(x|z)W_{h'}(x'|z') \rangle = \frac{\pi \delta(h - h')\delta^{(2)}(x - x')}{|z - z'|^{4\Delta_h}}.
\]

From the above forms we can see that there should be a relation between \(V_h\) and \(W_{\frac{1}{2}-k}\) or \(W_h\) and \(V_{\frac{1}{2}-h}\). Explicitly, the reflection relations are given by

\[
V_h(x|z) = \frac{D(h)}{\pi} \int d^2x'|x - x'|^{-4h}W_{\frac{1}{2}-h}(x'|z),
\]

\[
W_h(x|z) = -(2h)^2 \frac{D(h)}{\pi} \int d^2x'|x - x'|^{-4h-2}V_{\frac{1}{2}-h}(x'|z).
\]

Since the vertex operators satisfy the above reflection relations, the one point function \((4.13)\) should respect these relations. This yields a constraint on the function \(U^{\sigma}(h; \epsilon)\)

\[
U^{-\sigma}(h; \epsilon) = \epsilon D(h)U^{\sigma}(\frac{1}{2} - h; \epsilon),
\]

which can be obtained as in \([25, 38]\) with the help of \((2.37)\). Notice that the condition \((4.10)\) relates the function \(U^{\sigma}(h; \epsilon)\) to the function with the opposite sign \(U^{-\sigma}(\frac{1}{2} - h; \epsilon)\).

Next we move to the crossing symmetry. We study the following two point function on a disk as

\[
Z_2^\sigma = \left\langle \Phi_h(x_1, \xi_1|z_1)\Phi_h(x_2, \xi_2|z_2) \right\rangle
\]

with the operator \(\Phi_{k/2}\) belonging to a degenerate representation. In the limit of \(z_1 \to z_2\), we can use the operator product expansion of \(\Phi_{k/2}\) and \(\Phi_h\), thus we obtain a sum of one point functions. On the other hand, in the limit of \(\text{Im} z_2 \to 0\), the two point function can be expanded in terms of boundary operators, and it reduces to a one point function if we pick up the contribution with identity boundary operator. Comparing these two expressions, we obtain a constraint equation for the coefficient \(U^{\sigma}(h; \epsilon)\).

In the limit of \(z_1 \to z_2\), the bulk operators can be expanded as

\[
\Phi_{\frac{1}{2}}(x_1, \xi_1|z_1)\Phi_h(x_2, \xi_2|z_2) = |z_1|^{2\Delta + -2\Delta -2\Delta_h} |X_{12}|^{-4h} C(h)[\Phi_{\frac{1}{2}-h}]_{ee} \]

\[
+ |z_1|^{2\Delta + -\frac{1}{2} -2\Delta -2\Delta_h} |X_{12}|^{-4h-2} \tilde{C}(h)[\Phi_{\frac{1}{2}-h,-\frac{1}{2}]}_{oo},
\]

\[
\right] \right\rangle.
\]
as mentioned before. The coefficients are computed in appendix B.3 as
\[
C(h) = D(h), \quad \tilde{C}(h) = \frac{D(h)}{b^2 \nu \gamma (b^2 (k - 2h - 1)) \gamma (2b^2 h)}.
\] (4.13)

Here \(D(h)\) was given in (3.19). With the help of this operator product expansion, the two point function can be written as
\[
Z_2^\alpha = \frac{|z_1 - \bar{z}_2|^{-4\Delta_h}}{|z_1 - \bar{z}_1|^{2\Delta_h - 2\Delta_{\bar{z}}}} |x_1 + \bar{x}_2 + \epsilon_1 \tilde{z}_2|^{-4h} |x_1 + \bar{x}_1 + \epsilon_1 \tilde{z}_1|^{2h - k} \times \left( C(h) U^{-\sigma} (\frac{h}{2} - h; \epsilon) F_1^S (X, \chi_2, \chi_3; z) + \epsilon \tilde{C}(h) U^{-\sigma} (\frac{h}{2} - h - \frac{1}{2}; \epsilon) F_2^S (X, \chi_2, \chi_3; z) \right).
\] (4.14)

The functions \(F_1^S\) and \(F_2^S\) are defined in (B.16)\footnote{We have two choices of sign in the coefficient \(U^{\pm \sigma}\), and here we adopt the minus sign. Notice that one reflection relation should be applied as in (4.10) when we compute the OPE in (4.12). Similar computation can be done for the SL(2) WZNW model, and the same sign should be used to reproduce the known result.} and behave as in (B.17) and (B.18), respectively. Two point function of bulk operators on a disk can be mapped to four point function of chiral operators on a plane by the standard mirror trick. Because of the boundary conditions (B.11) the parameters are \((h_3, x_3, \xi_3, z_3) = (h, -\bar{x}_2, \epsilon_2, \tilde{z}_2), (h_4, x_4, \xi_4, z_4) = (k/2, -\bar{x}_1, \epsilon_1, \tilde{z}_1)\) along with \(h_1 = k/2\) and \(h_2 = h\). In this set up, the cross ratios are, for example,
\[
z = \frac{|z_1 - \bar{z}_2|^2}{|z_1 - \bar{z}_1|^2}, \quad X = \frac{|x_1 - x_2 + \xi_1 \tilde{z}_2|^2}{|x_1 + \bar{x}_2 + \epsilon_1 \tilde{z}_2|^2}.
\] (4.15)

The fermionic cross ratios \(\chi_2, \chi_3\) can be written in a similar way.

On the other hand, when we take the limit of \(\text{Im} z_2 \to 0\), the two point function can be expanded in terms of boundary operators as intermediate states. The degenerate operator \(\Phi_{k/2}\) can be expanded by boundary operators near \(\text{Im} z_1 \sim 0\), and among them there is the identity operator. Therefore, if we pick up the contribution with the identity operator as an intermediate state, then we have
\[
Z_2^\alpha \sim |z_1 - \bar{z}_1|^{-2\Delta_{\bar{z}}} |x_1 + \bar{x}_1 + \epsilon_1 \tilde{z}_1|^{-k} A^\sigma (\frac{k}{2}; \epsilon) \langle \Phi_h (x_2, \xi_2; z_2) \rangle.
\] (4.16)

For rational conformal field theories, factors like \(A^\sigma (k/2; \epsilon)\) are proportional to one point functions as \(\langle \Phi_{k/2} \rangle\). However, for non-rational conformal field theories, such as, OSP(1|2) model, this is not always true. Therefore, here we treat \(A^\sigma (k/2; \epsilon)\) as parameters. See\footnote{We should replace \(x (1 - \eta \xi)\) and \(\eta \xi\) by the cross ratios \(X\) and \(\chi_3 \chi_2\), respectively.} for more detail.

We can relate these two expressions by utilizing channel duality from \(s\)-channel to \(t\)-channel. With the help of channel duality formula in (B.20), we can rewrite the expression (4.14) in a suitable way to expand around \(\text{Im} z_2 \to 0\). Then, compared with the expression...
We already see that the semiclassical limit (4.10) and (4.17) can be written as

\[ C(h)U^{-\sigma}(\frac{k}{2} - h; \epsilon)F_{11}^{ST} + \epsilon \tilde{C}(h)U^{-\sigma}(\frac{k}{2} - h - \frac{1}{2}; \epsilon)F_{21}^{ST} = U^{\sigma}(h; \epsilon)A^{\sigma}(\frac{k}{2}; \epsilon) \]  \tag{4.17}

In order to solve this equation, it is useful to set

\[ U^{\sigma}(h; \epsilon) = \nu^{-2h + \frac{1}{2}} \Gamma(\frac{1}{2} - b^2) \nu^{2h - \frac{1}{2}} \]  \tag{4.18}

then the constraint coming from the reflection relation (4.10) can be written as

\[ E^{-\sigma}(h; \epsilon) = \epsilon E^{\sigma}(\frac{1}{4} - h; \epsilon) \]  \tag{4.19}

Moreover, the other constraint equation (4.17) reduces to

\[ E^{-\sigma}(\frac{k}{2} - h; \epsilon) - \epsilon E^{-\sigma}(\frac{k}{2} - h - \frac{1}{2}; \epsilon) = \nu^{k-1} \frac{\Gamma(1 + b^2(1 - k))}{\Gamma(-b^2)} A^{\sigma}(\frac{k}{2}; \epsilon) E^{\sigma}(h; \epsilon) \]  \tag{4.20}

A solution satisfying the both constraints (4.10) and (4.17) is given by

\[ E^{\sigma}(h; +1) = A_b e^{-\sigma(2h - \frac{1}{2}) r}, \quad E^{\sigma}(h; -1) = \sigma A_b e^{-\sigma(2h - \frac{1}{2}) r} \]  \tag{4.21}

with

\[ A^{\sigma}(\frac{k}{2}; \epsilon) = \nu^{1-k}(\epsilon e^{\sigma kr} - e^{\sigma(k-1)r}) \frac{\Gamma(-b^2)}{\Gamma(1 + b^2(1 - k))} \]  \tag{4.22}

Here \( A_b \) is a constant.

In this way, we obtain the one point functions of bulk operator on a disk as

\[ \langle \Phi_h(x, \xi | z) \rangle_{r,+1} = A_b \nu^{-2h + \frac{1}{2}} \frac{\Gamma(\frac{1}{2} - b^2(2h - \frac{1}{2})) e^{-\sigma(2h - \frac{1}{2}) r}}{|z - \bar{z}|^{2\Delta_b} |x + \bar{x} + \xi \bar{\xi}|^{2h}} \]  \tag{4.23}

\[ \langle \Phi_h(x, \xi | z) \rangle_{r,-1} = \sigma A_b \nu^{2h + \frac{1}{2}} \frac{\Gamma(\frac{1}{2} - b^2(2h - \frac{1}{2})) e^{-\sigma(2h - \frac{1}{2}) r}}{|z - \bar{z}|^{2\Delta_b} |x + \bar{x} - \xi \bar{\xi}|^{2h}} \]  \tag{4.24}

for \( \epsilon = +1 \) and \( \epsilon = -1 \), respectively. The parameter \( r \) is related to the boundary condition and it represents the position of brane as seen before. Solutions to the constraints are not unique, but we can see the above choice is the proper one by checking it in several ways. We already see that the semiclassical limit \( (k \to \infty \text{ and thus } b \to 0) \) of these one-point functions agrees with our semi-classical analysis (2.43) and (2.44).
4.2 Fuzzy supersphere branes

As mentioned above, we can take another type of boundary condition as in (4.2), and in this case the one point function is restricted to have the form of (4.4) due to the symmetry. In order to obtain the coefficient $U(h; \epsilon)$, we again utilize the constraint, particularly from the crossing symmetry. For this purpose we examine the two point function (4.11) now with different boundary condition (4.2). The two point function of bulk operators can be mapped to four point function of chiral operator as before, but the parameters are $(h_3, x_3, \xi_3, \bar{z}_3) = (h, -1/\bar{x}_2, \epsilon\xi_2/\bar{x}_2, \bar{z}_2)$, $(h_4, x_4, \xi_4, \bar{z}_4) = (k/2, -/\bar{x}_1, \epsilon\xi_1/\bar{x}_1, \bar{z}_1)$ along with $h_1 = k/2$ and $h_2 = h$. In particular, the bosonic cross ratio is given by

$$X = -\frac{|x_1 - x_2 + \xi_1\xi_2|^2}{|1 + x_1\bar{x}_2 + \epsilon\xi_1\xi_2|^2}.$$  \hspace{1cm} (4.25)

As before, we first express the two point function in the limit of $z_1 \to z_2$ as in (4.14). Then, we study the limit of $\text{Im } z_2 \to 0$ as in (4.16). However, here we assume that the parameter $A(k/2; \epsilon)$ is the same as the one point function $A(k/2; \epsilon) = U(k/2; \epsilon)$. With the help of channel duality formula (B.20), we can rewrite the expression for $z_1 \to z_2$ in terms of those for $\text{Im } z_2 \to 0$. Comparing to the other expression we can obtain the constraint equation as for super $\text{AdS}_2$ brane.

The constraint equation corresponding to (4.17) can be found as

$$C(h)U(h; \epsilon)F_{11}^{ST} - e\tilde{C}(h)U(h; \epsilon)F_{21}^{ST} = U(h; \epsilon)U(k/2; \epsilon).$$ \hspace{1cm} (4.26)

As before it is convenient to set

$$U(h; \epsilon) = \nu^{-2h+1}\Gamma(\frac{k}{2} - b^2(2h - \frac{1}{2}))E(h; \epsilon),$$ \hspace{1cm} (4.27)

then the above constraint equation reduces to

$$E(k/2; \epsilon) + \epsilon E(k/2 - h - \frac{1}{2}; \epsilon) = \Gamma(b^2(k - 2))E(k/2; \epsilon)E(h; \epsilon).$$ \hspace{1cm} (4.28)

For $\epsilon = +1$, we can find a solution

$$E(h) = \frac{\sin(s(2h - 1/2))}{\sin(-s/2)\Gamma(b^2(k - 2))},$$ \hspace{1cm} (4.29)

where $s = 2\pi b^2 n$ with a positive integer $n$. On the other hand we have for $\epsilon = -1$

$$E(h) = \frac{\cos(s(2h - 1/2))}{\cos(-s/2)\Gamma(b^2(k - 2))},$$ \hspace{1cm} (4.30)

where $s = 2\pi b^2 (n + 1/2)$ with a non-negative integer $n$. Thus the one point functions are summarized as

$$\langle \Phi_h(x, \xi|z) \rangle_{s;+1} = \frac{\nu^{-2h+1}\Gamma(\frac{k}{2} - b^2(2h - \frac{1}{2}))}{|z - \bar{z}|^{2\Delta_h}|x + \bar{x} + \xi|^{|2h|}} \frac{\sin(s(2h - 1/2))}{\sin(-s/2)\Gamma(b^2(k - 2))},$$ \hspace{1cm} (4.31)

$$\langle \Phi_h(x, \xi|z) \rangle_{s;-1} = \frac{\nu^{-2h+1}\Gamma(\frac{k}{2} - b^2(2h - \frac{1}{2}))}{|z - \bar{z}|^{2\Delta_h}|x + \bar{x} - \xi|^{|2h|}} \frac{\cos(s(2h - 1/2))}{\cos(-s/2)\Gamma(b^2(k - 2))}.$$ \hspace{1cm} (4.32)
for $\epsilon = +1$ and $\epsilon = -1$, respectively. These solutions are not unique either, and we check them by examining the Cardy condition in section 6. In this case the semiclassical analysis with (2.52) and (2.53) differs slightly from our results, i.e. the branes’ position labeled by $s$ is purely imaginary. This is not unexpected since the same already happened for spherical branes in $H_3^+$ model [25].

5 Relation to boundary super-Liouville theory

In the previous section, we have computed one point functions of bulk operators with two different boundary conditions by utilizing the crossing symmetry. In this section, we reproduce the disk amplitudes for super $\text{AdS}_2$ branes by making use of the relation to super-Liouville field theory with boundary. In fact, the structure constants of bulk OSP(1|2) WZNW model have been computed by utilizing the relation to super-Liouville field theory in [11]. The relation is a generalization of the one by Ribault and Teschner in [9], which relates sphere amplitudes of the SL(2) WZNW model (or the $H_3^+$ model) and those of Liouville field theory. The extension of Ribault-Teschner relation to disk amplitudes has been done in [40], see also [41].

5.1 Boundary super-Liouville theory

Let us first review $\mathcal{N} = 1$ super-Liouville field theory with boundary by closely following [14]. The theory consists of a bosonic field $\varphi$ and its super partner $\psi$, whose action is given by

$$S_{\mathcal{L}} = \frac{1}{2\pi} \int d^2z \left[ \partial \varphi \bar{\partial} \varphi + \frac{Q_\varphi}{4} \sqrt{g} \mathcal{R} \varphi + \bar{\psi} \partial \psi + \psi \partial \bar{\psi} \right] + 2i \mu_L b^2 \int d^2z \psi \bar{\psi} e^{b \varphi} , \quad (5.1)$$

where $\mathcal{R}$ represents the curvature of the worldsheet. Here the background charge is related to the parameter $b$ by $Q_\varphi = b + 1/b$, and the central charge is $c = 3/2 + 3Q_\varphi^2$. In the NSNS-sector, primary fields are $V_a = e^a \varphi$ with conformal weight $\Delta_a = \alpha (Q - \alpha)/2$. In the RR-sector, they are defined with chiral spin fields $\sigma^\pm$ and $\bar{\sigma}^\pm$, which have operator products with the fermions as

$$\psi(z) \sigma^\pm(0) \sim \frac{\sigma^\mp(0)}{\sqrt{2z^{\mp}}} , \quad \bar{\sigma}^\pm(\bar{z}) \bar{\psi}(0) \sim \frac{i \bar{\sigma}^\mp(0)}{\sqrt{2\bar{z}^{\mp}}} . \quad (5.2)$$

With non-chiral products $\sigma^{a\bar{a}} = \sigma^a \bar{\sigma}^{\bar{a}}$ ($a, \bar{a} = \pm$), we define primary fields in the RR-sector as $\Theta_a = \sigma^{a\bar{a}} e^{a\varphi}$. Their conformal weights are $\tilde{\Delta}_a = \alpha (Q - \alpha)/2 + 1/16$.

In the presence of boundary we can add boundary terms as

$$S_{\mathcal{L}}^B = \int dt \left[ \frac{1}{4\pi} Q_\varphi \sqrt{g} \mathcal{K} \varphi + \frac{1}{4} \Theta \partial_t \Theta + \mu_B b \Theta \psi e^{b \varphi}/2 \right] . \quad (5.3)$$
Here we denote the curvature of boundary as $K$, and $\Theta$ is a Grassmann odd field living only at the boundary. In particular, free field correlator is given by $\langle \Theta(t_1)\Theta(t_2) \rangle = \text{sgn}(t_1 - t_2)$. As boundary conditions we assign

$$T(z) = \bar{T}(\bar{z}), \quad T_F(z) = \zeta \bar{T}(\bar{z})$$

(5.4)

at $z = \bar{z}$, where $T$ is the energy momentum tensor and $T_F$ is its superpartner. The boundary operators are defined as

$$B_\beta(t) = e^{\beta \varphi/2}(t) = e^{\beta \varphi_L(t)}, \quad \Theta^\pm_\beta(t) = \sigma^\pm e^{\beta \varphi/2}(t) = \sigma^\pm e^{\beta \varphi_L(t)},$$

(5.5)

which are inserted at the boundary of worldsheet. Here $\varphi_L$ is the holomorphic part of the original field $\varphi$.

As we will see below, bulk one point functions on a disk in the OSP(1|2) WZNW model are related to those in super-Liouville field theory. These one point functions in super-Liouville field theory are of the form

$$\langle V_\alpha(z) \rangle_{u^-} = \frac{U_+(\alpha;u_-)}{|z-\bar{z}|^{2\Delta_L}}, \quad \langle \Theta^{\alpha}_{\alpha}(z) \rangle_{u^-} = \frac{U_-(\alpha,a;u_-)}{|z-\bar{z}|^{2\Delta_L}}.$$ 

(5.6)

Here we have assigned the boundary condition (5.4) with parameter $\zeta$. The subscript $u_\pm$ represents the boundary condition and is related to the parameter $\mu_B$ in (5.3) as

$$\mu_B = \left( \frac{2\mu_L}{\cos(\pi \gamma/2)} \right)^{1/2} \sinh(\pi u_+ b), \quad \mu_B = \left( \frac{2\mu_L}{\cos(\pi \gamma/2)} \right)^{1/2} \cosh(\pi u_- b).$$

(5.7)

The coefficients are obtained in [14] as

$$U_+ = -2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu_L \pi \gamma(bQ_\phi/2))^{2bQ_\phi/2\pi\alpha} \Gamma(b(\alpha - Q_\phi)) \Gamma(\frac{1}{b}(\alpha - Q_\phi)) U_+, \quad \hat{U}_+(\alpha;u_+) = \cosh(\pi(2\alpha - Q_\phi)u_+),$$

(5.8)

for the NSNS-sector and

$$U_- = 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu_L \pi \gamma(bQ_\phi/2))^{2bQ_\phi/2\pi\alpha} \Gamma(\frac{1}{b} + b(\alpha - Q_\phi)) \Gamma(\frac{1}{b} + \frac{1}{b}(\alpha - Q_\phi)) \hat{U}_-, \quad \hat{U}_-(\alpha,a;u_+) = a \sinh(\pi(2\alpha - Q_\phi)u_+), \quad \hat{U}_-(\alpha,a;u_-) = \cosh(\pi(2\alpha - Q_\phi)u_-)$$

(5.9)

for the RR-sector.

### 5.2 Boundary OSP(1|2) model

The action of WZNW model associated with supergroup can be obtained as in the case with bosonic group. First adopt a specific parametrization of the (super) group element,
and then insert it in the general expression of WZNW action. Introducing auxiliary fields, we may express the action of the OSP(1|2) WZNW model as \[ S^O = \frac{1}{\pi} \int d^2 z \left[ \frac{1}{2} \partial \phi \partial \bar{\phi} + \frac{b}{8} \sqrt{g} R \phi + \bar{\beta} \partial \gamma + \bar{\bar{\beta}} \partial \bar{\gamma} + p \bar{\partial} \theta + \bar{p} \partial \bar{\theta} \right] \] (5.10)

\[ + i \lambda \int d^2 z (p + \beta \theta) (\bar{p} - \bar{\beta} \bar{\theta}) e^{b \phi}. \]

Namely, we have a free boson \( \phi \) with background charge \( Q_\phi = b = 1/\sqrt{2k-3} \), and \( (\beta, \gamma) \)-system with the conformal weights \((1, 0)\). In addition to these bosonic fields, there are free fermions \((p, \theta)\) with the conformal weights \((1, 0)\). The central charge of the system is \( c = 1 + 3b^2 = \frac{2k}{2k-3} \) as desired. The interaction terms can be treated perturbatively.

In order to define vertex operators, it is convenient to bosonize the fermionic fields as

\[ \theta = \exp(iY_L), \quad p = \exp(-iY_L), \quad \bar{\theta} = \exp(iY_R), \quad \bar{p} = \exp(-iY_R), \]

and then define \( Y = Y_L + Y_R \). Using the new field the vertex operators are

\[ V^s_h(\mu|z) = |\mu|^{-2h+1+s} e^{i4Y} e^{\mu \gamma - \bar{\mu} \bar{\gamma}} e^{-2b(h-\frac{1}{2})\phi} \] (5.12)

with \( s = 0, 1 \) in the RR-sector and \( s = 1/2 \) in the NSNS-sector. In the RR-sector and the NSNS-sector, the conformal weights are \( \Delta_h = -2b^2 h(h - 1/2) \) and \( \hat{\Delta}_h = -2b^2 h(h - 1/2) - 1/8 \), respectively. For the vertex operators with \( s = 0, 1 \), we can change the basis by the Fourier transform as

\[ V^\pm_h(\mu|z) = \frac{1}{\pi} |\mu|^{-2h+2} \int d^2 x \int d\xi d\bar{\xi} e^{\mu \xi - \bar{\mu} \bar{\xi}} (1 \pm |\mu|^{-1} \xi \bar{\xi}) \Phi_h(x, \xi|z). \] (5.13)

Here

\[ V^\pm_h(\mu|z) = |\mu|^{-2h+1} (|\mu| \bar{\partial} \theta \pm 1) e^{\mu \gamma - \bar{\mu} \bar{\gamma}} e^{-2b(h-\frac{1}{2})\phi} \] (5.14)

are the orthogonal basis as discussed in [11] and \( \Phi_h(x, \xi|z) \) is defined in (3.7).

Now we would like to include a boundary to the worldsheet, and hence to find out proper boundary terms. In the \( H_3^+ \) model, the corresponding boundary terms are constructed in [11], and in our case they are discussed in appendix C (see also appendix D). For super AdS\(_2\) branes they are given by

\[ S^O_B = \int du \left[ \frac{1}{2\pi} \frac{b}{8} \sqrt{g} K \phi + \frac{1}{4} \Theta \partial_u \Theta + \lambda_B e^{b \phi/2} \Theta (\beta \theta + p) \right] \]

with \( \beta = -\bar{\beta} \) and \( p = \zeta \bar{p} \) at the boundary. In terms of bosonization, the boundary condition for fermions are mapped to \( Y_L = Y_R + i \ln \zeta \). The parameter \( \lambda_B \) plays the same role as \( \mu_B \) in super-Liouville field theory. The boundary operators may be defined as

\[ B^\pm_I (v|u) = |v|^{-(l+\frac{1}{2})+\frac{1}{2} s} e^{\frac{i}{4} 2Y} e^{\frac{1}{2} (v \gamma - \bar{v} \bar{\gamma})} e^{-b(l-\frac{1}{2})\phi} \] (5.15)
with $\tau = 0, 1/2, 1$. Next task is to rewrite the following generic correlation function

$$\Omega = \left\langle \prod_{i=1}^{n} V_{h_i}^{s_i}(\mu_i|z_i) \prod_{a=1}^{m} B_{\tau_a}^{s_a}(v_a|u_a) \right\rangle$$

(5.16)
on a disk in terms of that of super-Liouville field theory.

### 5.3 Relation between the two theories

In [10] it was shown that the relation between $H^+_3$ model and Liouville theory in [9] can be rederived in the path integral formulation. Utilizing this method the relation has been extended to that for sphere amplitudes of OSP(1|2) model and super-Liouville theory in [11]. In this subsection we extend the result to the boundary case. In the path integral formulation the correlation function (5.16) is written as

$$\Omega = \int d\phi d^2\gamma d^2\beta d^2\theta e^{-S_{O} - S_{B}} \prod_{i=1}^{n} V_{h_i}^{s_i}(\mu_i|z_i) \prod_{a=1}^{m} B_{\tau_a}^{s_a}(v_a|u_a) .$$

(5.17)

Following [11] we first integrate over the fields $\beta, \gamma$, and then perform field redefinitions to reduce to super-Liouville field theory. The analysis is quite analogous to the bulk case, so here we explain it only briefly.

We start from the observation that integration over the zero mode of $\gamma - \bar{\gamma}$ leads to

$$\sum_{i=1}^{n} (\mu_i + \bar{\mu}_i) + \sum_{a=1}^{m} v_a = 0 .$$

(5.18)

Then we move to the non-zero modes. Since $\gamma$ appears only linearly in the exponent of the path integral expression (5.17), we can integrate them out, which yields delta-functionals of $\beta$. After integration over $\beta$, the field $\beta(z)$ is replaced by a function $B(z)$ defined by

$$B(z) = - \sum_{i=1}^{n} \frac{\mu_i}{z - z_i} - \sum_{i=1}^{n} \frac{\bar{\mu}_i}{z - \bar{z}_i} - \sum_{a=1}^{m} \frac{v_a}{z - u_a} .$$

(5.19)

In the same way, $\bar{\beta}(z)$ is replaced by $-\bar{B}(\bar{z})$. As in [14] it is essential to define $y_{\nu}, \bar{y}_{\nu}, t_{\alpha'}$ by

$$B(z) = u \prod_{i=1}^{n'} (z - y_{\nu})(z - \bar{y}_{\nu}) \prod_{a=1}^{m'} (z - t_{\alpha'}) \prod_{i=1}^{n'} (z - z_i)(z - \bar{z}_i) \prod_{a=1}^{m'} (z - u_a) .$$

(5.20)

with $2n' + m' = 2n + m - 2$. Up to the permutation of $y$'s or $t$'s, the above equation defines a map from the old variables $\mu, \bar{\mu}, \nu$ to the new variables $u, y, \bar{y}, t$.

Since the field $\beta(z)$ is now replaced by $B(z)$, the coefficients in the action are some functions depending on $z$. In order to remove the coordinate dependence, we perform the shifts as

$$\varphi := \phi + \frac{1}{2b} |B|^2 , \quad Y'_L := Y'_L - \frac{i}{2} \ln B , \quad Y'_R := Y'_R - \frac{i}{2} \ln \bar{B} .$$

(5.21)
If we fermionize the new boson $Y' = Y'_L + Y'_R$ by
\[ \psi = \sqrt{2} \exp(\pm iY'_L), \quad \tilde{\psi} = \sqrt{2} \exp(\pm iY'_R), \] (5.22)
then we find that the action becomes the one of super-Liouville field theory plus the one of free fermion $(\chi, \bar{\chi})$
\[ S^F = \frac{1}{2\pi} \int d^2 z \left[ \chi \partial \chi + \bar{\chi} \partial \bar{\chi} \right]. \] (5.23)
The boundary conditions for fermions are mapped to
\[ \psi = \zeta \text{sgn} B \tilde{\psi}, \quad \chi = \zeta \text{sgn} B \bar{\chi}. \] (5.24)
This is because $Y'_L$ and $Y'_R$ are defined in (5.21), and $B = \bar{B} = |B|$ or $e^{-\pi i}B = e^{\pi i} \bar{B} = |B|$ at the boundary. The relation between parameters of boundary term are given by
\[ \sqrt{2} \text{sgn} B \lambda_B = \mu_B b. \] (5.25)

During the change of variables, we receive extra contributions from kinetic terms. One is the insertion of extra fields $V^{1/2}_{-1/2b}(y'_r)$ and $B^{1/2}_{-1/2b}(t'_a)$ with
\[ V^s_a = e^{i\phi Y'/2} e^{\alpha \varphi}, \quad B^r_{\beta} = e^{i\pi Y'/2} e^{\beta \varphi/2}, \] (5.26)
and another is an overall factor $|u|\Xi^{1/\beta + 1/2}$ with
\[ \Xi = \prod_{i<j} |z_i - z_j|^2 \prod_{i,j} |z_i - u_a|^2 \prod_{a<b} |y_{i'j'}|^2 \prod_{i'j'} |y_{i'j'} - \bar{y}_{i'j'}|^2 \prod_{i'j'} |y_{i'j'} - t_{a'}|^2 \prod_{a'<b'} |t_{a'} - t_{b'}|^2 \prod_{a'<b'} |z_{i'a'} - u_{a'}|^2 \prod_{a',a'} |u_{a'} - y_{a'}|^2 \prod_{a,a'} (u_{a'} - t_{a'})^{-1}. \] (5.27)
In the end we arrive at the expression as
\[ \Omega = \delta \left( \sum_i (\mu_i + \bar{\mu}_i) + \sum_a \nu_a \right) |u| |\Xi|^{-1/\beta - 1/2} \times \] (5.28)
\[ \times \left\langle \prod_{i=1}^n V^{s_i-\frac{1}{2}} \left( z_i \right) \prod_{a=1}^m B^{r_a-\frac{1}{2}} \left( u_a \right) \prod_{i'=1}^{n'} V^{\frac{1}{4}}_{-\delta} \left( y_{i'} \right) \prod_{a'=1}^{m'} B^{\frac{1}{4}}_{-\delta} \left( t_{a'} \right) \right\rangle, \]
where
\[ \alpha_i = -2b \left( \frac{1}{2} - \frac{1}{2\pi} \right) + \frac{1}{2\pi}, \quad \beta_a = -2b \left( \frac{1}{2} - \frac{1}{2\pi} \right) + \frac{1}{2\pi}. \] (5.29)
The momenta receive extra contributions as in (5.29), and the background charge is shifted from $Q_\phi = b$ to $Q_\phi = b + 1/b$. The right hand side should be computed in super-Liouville field theory with a pair of free fermions $(\chi, \bar{\chi})$. Thanks to the condition
2n' + m' = 2n + m - 2, we do not need to include extra fields for one point functions of bulk operator on a disk and two point functions of boundary operators. In the next subsection we compute one point functions of bulk operator on a disk in the OSP(1|2) WZNW model utilizing the relation [9].

5.4 One point function of bulk operators

According to (5.28), one point function of bulk operator on a disk can be written as

$$\langle V^s_h(\mu | z) \rangle = \delta(\mu + \bar{\mu})||\mu||z - \bar{z}|^{1 + \frac{1}{8\Delta_h}}^{\frac{1}{2} + \frac{1}{4}} \left\langle V^s_{\frac{3}{2}}(z) \right\rangle,$$  (5.30)

where the right hand side should be computed in the super Liouville field theory with a pair of free fermions. Let us first restrict the label to \( s = 0, 1 \). In these cases, the right hand side involves one point function of \( e^{\pm iY'/2} \), which should be rewritten in terms of fermions \( \psi, \chi \). If we define order and disorder operators of Ising model by \( \Sigma^\pm \), then we have a map as

$$\sqrt{2} \cos Y'/2, \sqrt{2} \sin Y'/2 \leftrightarrow (\Sigma^+)^2, (\Sigma^-)^2.$$  (5.31)

With the description of Ising model in terms of free fermion we moreover have [10]

$$\Sigma^+ = \frac{1}{\sqrt{2}}(\sigma^{++} \mp \sigma^{+-}), \quad \Sigma^- = \frac{1}{\sqrt{2i}}(\sigma^{+-} \mp \sigma^{--}).$$  (5.32)

With these maps we can rewrite the expression of (5.30) in terms of one point function in the RR-sector of Liouville field theory. Defining \( \Sigma^\pm_\alpha = \Sigma^\pm e^{i\alpha} \) and \( \Sigma^\pm_\chi \), we therefore have

$$\langle V^\pm_h(\mu | z) \rangle \propto \delta(\mu + \bar{\mu})||\mu||z - \bar{z}|^{1 + \frac{1}{8\Delta_h}}^{\frac{1}{2} + \frac{1}{4}} \left\langle \Sigma^\pm_{\chi}(z) \right\rangle,$$  (5.33)

where \( V^\pm_h(\mu | z) \) was defined in (5.14). Notice that one point functions with \( \Sigma^- \) vanish. Furthermore, the one point function of free fermion \( \chi \) is of the form as \( \langle \Sigma^+_\chi(z) \rangle = U^+_\chi |z - \bar{z}|^{-1/8} \) with some constant \( U^+_\chi \). The explicit expression of (5.9) then leads to

$$\langle V^\pm_h(\mu | z) \rangle = \frac{\delta(\mu + \bar{\mu})||\mu||U^+_h}{|z - \bar{z}|^{2\Delta_h}}.$$  (5.34)

---

9 If we want to study more complicated cases, then we need to develop the map between correlators with free boson \( Y' \) and with free fermions \( \chi, \psi \) on a worldsheet with boundary. These cases will not be treated in this paper, but it might be useful to consult with the work [42], which deals with similar problems.

10 The choice of \( \mp \) may be related to the definition of energy operator \( E = \pm i\psi \bar{\psi} \). Moreover, we have a duality transformation as \( E \leftrightarrow -E \) and \( \Sigma^+ \leftrightarrow \Sigma^- \). For the bulk case as in [11], it is merely convention. However, in the presence of boundary, it should matter and the choice should be related to the boundary condition for fermion \( \psi \).

11 Once we fix the bosonization rule, only one sign of \( \langle V^\pm_h(\mu | z) \rangle \) is non-vanishing.
with

\[ U_h = A_0(\mu_L \pi \gamma \left( \frac{b^2 + 1}{2} \right))^{-2h + \frac{1}{2}} \Gamma \left( \frac{1}{2} - b^2 (2h - \frac{1}{2}) \right) \Gamma (-2h + 1) \sinh (2\pi bu_+(2h - \frac{1}{2})) , \]

\[ U_h = A_0(\mu_L \pi \gamma \left( \frac{b^2 + 1}{2} \right))^{-2h + \frac{1}{2}} \Gamma \left( \frac{1}{2} - b^2 (2h - \frac{1}{2}) \right) \Gamma (-2h + 1) \cosh (2\pi bu_- (2h - \frac{1}{2})) \]

for \( \zeta \, \text{sgn} \, B = +1 \) and \( \zeta \, \text{sgn} \, B = -1 \), respectively. Here \( A_0 \) is some constant independent of \( h \). The above results are consistent with the previous ones (4.23) and (4.24) with the identifications of \( \nu = \mu_L \pi \gamma \left( \frac{b^2 + 1}{2} \right) \) and \( \zeta = -\epsilon \). Moreover, the parameters for boundary condition should be related as \( 2\pi bu_+ = -r - \pi i/2 \, \text{sgn} \, \text{Im} \mu \) and \( 2\pi bu_- = -r - \pi i/2 \, \text{sgn} \, \text{Im} \mu \), which correspond to (5.25) through (5.7).

One advantage of this approach is that it is easy to extend to the NSNS-sector contrary to the method with the crossing symmetry. Using the relation (5.28) (or (5.30) with \( s = 1/2 \)) one point function of this operator can be written in terms of super-Liouville field theory as

\[ \langle V_h^{\frac{1}{2}}(\mu|z) \rangle = \delta(\mu + \bar{\mu})|\mu| |z - \bar{z}|^{\frac{1}{2} + \frac{b^2}{4} - \frac{1}{4}} \left\langle V_0^{0}(-2b(h - \frac{1}{2}) + \frac{1}{2}) \right\rangle. \]  

(5.35)

The above correlator does not include spin operators, and hence we just need to use the expression of (5.8) in super-Liouville field theory. The result is given by

\[ \langle V_h^{\frac{1}{2}}(\mu|z) \rangle = \frac{\delta(\mu + \bar{\mu})|\mu| \tilde{U}_h}{|z - \bar{z}|^{2\Delta_h}} \]  

(5.36)

with

\[ \tilde{U}_h = \tilde{A}_0(\mu_L \pi \gamma \left( \frac{1+b^2}{2} \right))^{-2h + \frac{1}{2}} \Gamma (1 - b^2 (2h - \frac{1}{2})) \Gamma (-2h + \frac{1}{2}) \cosh (2\pi bu_+(2h - \frac{1}{2})) . \]  

(5.37)

Here \( \tilde{A}_0 \) is a constant independent of \( h \), and the identification is used as \( 2\pi bu_\pm = -r - \pi i/2 \, \text{sgn} \, \text{Im} \mu \).

6 Annulus amplitude and Cardy condition

In section 4 we have computed one point functions of bulk operator by solving the constraint coming from the channel duality. Addition to this constraint, there is another constraint for one point function, which arises from the open-closed duality. From the information of bulk one point functions, we can construct boundary states describing branes. Then, we can compute the overlap between boundary states by considering the exchange of closed strings. On the other hand, we can obtain the same amplitude in the language of open string as a partition function with proper boundary conditions. They are related by modular transformation of the annulus amplitude, which yields a strong consistency condition to the one point functions of closed strings. This condition is called
the Cardy condition. In rational conformal field theories the Cardy condition almost
fixes boundary states, but in non-rational theories like our model it is not that strong.
Nonetheless this condition still gives useful information to boundary states as we will see
in this section.

6.1 Boundary states for super AdS brane in the RR-sector

Boundary states have information of how closed strings couple to branes. For a while
we focus on super AdS branes and closed strings in the RR-sector. In particular, the
coupling to primary states is given by

$$B \langle r; \epsilon | h; x, \xi \rangle \equiv \langle \Phi_h(x, \xi | \frac{1}{2} \rangle_{r, \epsilon} .$$ \hspace{1cm} (6.1)

Here $r$ and $\epsilon$ represent the boundary conditions as before. The coupling to higher modes is
fixed by the boundary condition $[4,1]$. Later we change the basis from $x$-basis to $m$-basis by utilizing a Fourier transformation

$$\Phi_{h,a,\bar{a}}(z) = \int \frac{d^2x}{|x|^2} d\xi d\bar{\xi} x^{h-m+\frac{a}{2}} \bar{x}^{h-\bar{m}+\frac{\bar{a}}{2}} \xi^{1-a} \bar{\xi}^{1-a} \Phi_h(x, \xi | z) .$$ \hspace{1cm} (6.2)

Here we assume that $m = (n + ip)/2$, $\bar{m} = (n - ip)/2$ with $n \in \mathbb{Z}$ and $p \in \mathbb{R}$. Following
[25], the overlaps are computed as

$$B \langle r; \epsilon | h; n, p, a, \bar{a} \rangle = 2\pi \delta(p) \delta_{a, \bar{a}} \cdot 2\pi A(h, n, s)_{r, \epsilon} ,$$ \hspace{1cm} (6.3)

with

$$A(h, n, 0|r, +1) = d_n^h(\pi_n^0 \cosh(2h - \frac{1}{2})r - \pi_n^1 \sinh(2h - \frac{1}{2})r) ,$$ \hspace{1cm} (6.4)

$$A(h, n, 1|r, +1) = -2hd_n^{h+\frac{1}{2}}(\pi_n^0 \cosh(2h - \frac{1}{2})r - \pi_n^1 \sinh(2h - \frac{1}{2})r) ,$$ \hspace{1cm} (6.5)

$$A(h, n, 0|r, -1) = d_n^h(\pi_n^1 \cosh(2h - \frac{1}{2})r - \pi_n^0 \sinh(2h - \frac{1}{2})r) ,$$ \hspace{1cm} (6.6)

$$A(h, n, 1|r, -1) = 2hd_n^{h+\frac{1}{2}}(\pi_n^1 \cosh(2h - \frac{1}{2})r - \pi_n^0 \sinh(2h - \frac{1}{2})r) .$$ \hspace{1cm} (6.7)

The functions $d_n^h$ and $\pi_n^\delta$ ($\delta = 0, 1$) are defined as

$$d_n^h = \frac{\Gamma(-2h + 1)}{\Gamma(1 - h + \frac{n}{2})\Gamma(1 - h - \frac{n}{2})} , \quad \pi_{2m}^\delta = 1 - \delta , \quad \pi_{2m+1}^\delta = \delta$$ \hspace{1cm} (6.8)

for $m \in \mathbb{Z}$.

12Orthogonal basis consists of linear combinations, such as $\Phi_{m, \bar{m}} = \frac{1}{\sqrt{2}}(\Phi_{m, \bar{m}}^{h,0,0} \pm \Phi_{m, \bar{m}}^{h,1,1})$. 
Using the boundary states for super AdS branes, the annulus amplitude can be written as
\[
Z^R_{r,s} = B\langle r; \epsilon | (-1)^F \tilde{q}^H | r; \epsilon \rangle_B = \int_{\mathbb{S}} \frac{dh}{2\pi} \int_{\mathbb{C}} d^2x \int d\xi d\tilde{\xi} \chi^h_{R,-}(\tilde{q})_B\langle r; \epsilon | h; x, \xi \rangle\langle h; x, \xi | r; \epsilon \rangle_B.
\]
(6.9)

Here the Hamiltonian is \( H^R = L_0 + \tilde{L}_0 - c/12 \), and only the states in the RR-sector are summed over. The worldsheet modulus is given by \( \tilde{\tau} = -1/\tau \) and \( \tilde{q} = \exp(2\pi i \tilde{\tau}) \). We define \( F = \frac{1}{2}(F_c + \tilde{F}_c) \), where the numbers of Grassmann odd fields are counted by \( F_c \) and \( \tilde{F}_c \) for holomorphic and anti-holomorphic parts, respectively. The integral domain \( \mathbb{S} \) is restricted to the physical line as \( h = 1/4 + iP \) with \( P \in \mathbb{R} \). Since there are three bosons and two fermions with periodic boundary condition and without any singular vectors, the character is given by \( \chi^h_{R,-}(\tilde{q}) = \tilde{q}^{2\tilde{q}^2 P^2} \eta^{-1}(\tilde{q}) \).

The overlap is actually a divergent quantity basically due to the infinite volume of super AdS branes, so we have to adopt a regularization scheme. In order to see this it is convenient to change the basis as
\[
Z^R_{r,s} = \int_{\mathbb{S}} \frac{dh}{2\pi} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{dp}{2\pi} \chi^h_{R,-}(\tilde{q})_B\langle r; \epsilon | h; n, p, 1, 1 \rangle\langle h; n, p, 0, 0 | r; \epsilon \rangle_B
\]
(6.10)

First of all we replace a delta-function as \( 2\pi \delta_T(p) = \frac{2}{P} \sin TP \) as in [25]. A difficult part is on the summation over \( n \). Because we have to sum over bosonic and fermionic contributions, naively they are canceled with each other. Therefore, we pick up effectively one bosonic contribution among them. With this regularization, the overlap can be computed as
\[
Z^R_{r,s} = \epsilon 4\pi T |A_b|^2 \int_{-\infty}^{\infty} \frac{dP}{4\pi} \frac{\cos 4Pr}{\cosh 2\pi P \cosh 2\beta P}.
\]
(6.11)

Here we have used the fact that \( \cos^2 \pi h + \cos^2 \pi (\frac{1}{2} - h) = 1 \).

Performing the open-closed duality, we may map the overlap between boundary states into the partition function of open string. Utilizing the formula of modular transformation
\[
\chi^h_{R,-}(\tilde{q}) = 2b \int_{-\infty}^{\infty} dP' \chi^{h'}_{R,-}(q) \exp(8\pi ib^2 P' P) \sim
\]
(6.12)
we can rewrite the overlap (6.11) as
\[
Z^R_{r,s} = T \int_{0}^{\infty} dP' N(P'; r; \epsilon) \chi^{h'}_{R,-}(q).
\]
(6.13)

\footnote{There should be a better regularization than this ad hoc way, but here we adopt this anyway. This regularization may be justified by the character of degenerate representation with \( (r, s) = (2J + 1, 0) \). The representation is of \( 2J + 1 \)-dimensional and Grassmann odd and even states appear alternatively. Therefore, the character is given by \( \pm \chi^h_{R,-}(\tilde{q}) = \tilde{q}^{2\tilde{q}^2 P^2} \eta^{-1}(\tilde{q}) \) with \( P = -ir/4 \), and an analytic continuation would lead to our character.}
For the following analysis it might be convenient to express the density of state as

\[
N(P'|r; \epsilon) = \frac{\epsilon}{2\pi} \frac{\partial}{\partial P'} \int_0^\infty dt \left( \frac{\cosh \frac{Q \epsilon}{4} - \cosh \left( \frac{b-1}{4} \right) t}{\sinh t/2 \sinh t/2b} \right) \frac{\partial}{\partial P'} \ln \left( \frac{tb(P' + \frac{r}{2\pi b}) + \sin b(P' - \frac{r}{2\pi b})}{\sinh t/2 \sinh t/2b} \right)
\]

(6.14)

where we set \(|A_b|^2 = 2b|\).

6.2 Annulus amplitude from open strings

Before going into the detail of the Cardy condition, we first study the partition function of open string stretched between the super AdS\(_2\) brane with the label \((r, \epsilon)\). For the calculation of the partition function, we sum up the spectrum of open strings with the density of states \(\rho(P|r; \epsilon)\).

The spectral density is related to the reflection amplitude \(R(P|r; \epsilon)\) as

\[
\rho(P|r; \epsilon) \sim \frac{L}{\pi} + \frac{1}{2\pi i} \frac{\partial}{\partial P'} \ln R(P|r; \epsilon).
\]

(6.15)

Here \(L\) is a cut-off scale, and the wave-function is assumed not to go into the region \(L < \phi\) due to the exponential potential. For the detail, see, for example, appendix B in [25].

The reflection relation can be read off from the two point functions of boundary operators, which are computed in appendix D. Boundary operators may be constructed by \(\Psi_\rho^\rho'(t, \eta|u)\) as in the bulk case. The representation of zero modes of OSP\((1|2)\) current algebra is parameterized by \(l, t, \eta,\) and the position on the boundary of worldsheet is denoted by \(u\). The extra labels \(\rho, \rho'\) represent boundary conditions across the inserted point \(u\), which are related to \(r, r'\) by \(\rho = r/(4\pi b^2) - i/(8b^2)\). Let us denote \(V_\rho^\rho'(t|u) = \Psi_\rho^\rho'(t, 0|u)\), then the two point functions

\[
\left\langle V_\rho^\rho'(1|1) V_\rho'^\rho'(0|0) \right\rangle = \delta(l_1 - l_2) d(l_1; \rho, \rho'),
\]

(6.16)

\[
\left\langle p V_\rho^\rho'(1|1) p V_\rho'^\rho'(0|0) \right\rangle = \delta(l_1 - l_2) d'(l_1; \rho, \rho'),
\]

(6.17)

are computed with the function \(d(l_1; \rho, \rho')\) in (D.32) or (D.33) and \(d'(l_1; \rho, \rho')\) in (D.49) or (D.50).

Following the above general argument we can compute the spectral density from the two point functions. Let us start from the primary \(V_\rho^\rho'(t|u)\). Since the identification is \(\epsilon = -\zeta\) as discussed in section 3, we have

\[
\rho(P|r) \sim -\frac{1}{2\pi i} \frac{\partial}{\partial P'} \ln S_{\text{NS}}(2b(l + 2i\rho)) S_{\text{NS}}(2b(l - 2i\rho))
\]

(6.18)

for \(\epsilon = +1\) and

\[
\tilde{\rho}(P|r) \sim -\frac{1}{2\pi i} \frac{\partial}{\partial P'} \ln S_{\text{R}}(2b(l + 2i\rho)) S_{\text{R}}(2b(l - 2i\rho))
\]

(6.19)
for $\epsilon = -1$. Here we picked up the terms depending on the parameter $r$ (or $\rho$). We also set $l = 1/4 + iP$. The spectral density for the descendant $p\mathcal{V}^{p,\rho'}_l$ can be obtained in the same way, and given by $\rho(P|r)$ in (6.19) for $\epsilon = +1$ and $\rho(P|r)$ in (6.18) for $\epsilon = -1$.

Here we remark that the spectral densities are different for $\mathcal{V}^{p,\rho'}_l$ and $p\mathcal{V}^{p,\rho'}_l$, even though in principle they could be related with each other by OSP$(1|2)$ current algebra. A similar situation arises in $\mathcal{N} = 1$ super-Liouville theory, where the reflection relation and worldsheet supersymmetry do not commute [14]. Since the first type operator is Grassmann even and the second type operator is Grassmann odd, we may construct the following partition function as

$$Z^0_R(r; \epsilon) = \text{Tr}_R \frac{1+(-1)^F}{2} q^{H_o} + \text{Tr}_R \frac{1-(-1)^F}{2} q^{H_o}.$$  (6.20)

Here $H_o = L_0 - c/24$ is the Hamiltonian for the open strings and we sum over states in the R-sector. The fermion number is counted by $F$. For the first term we use the spectral density of the primary $\mathcal{V}^{p,\rho'}_l$ and for the second term we use the one of the descendant $p\mathcal{V}^{p,\rho'}_l$.

The above partition function is a divergent quantity, so we need to regularize it. There are two types of divergence. One type comes from the limit of $L \to \infty$ in (6.15), and it is useful to consider the relative partition function $Z^0_R(r; \epsilon) - Z^0_R(r_*; \epsilon)$ with reference boundary condition $r_*$. Another is due to the sum over the eigen-value $m$ of $J^3_0$. For the contribution from $\text{Tr}_R(-1)^F q^{H_o}$, naively bosonic and fermionic contributions cancel out. Thus we pick up one type of contribution as before. The character is given by

$$\chi^h_{R,\epsilon}(q) = \theta(q)^{2\bar{p}^2 P^2} \eta^{-1}(q)$$  (6.21)

with

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{(n-\frac{1}{2})^2/2} z^{n-\frac{1}{2}}$$  (6.22)

$$= 2e^{\pi i \tau/4} \cos \pi y \prod_{m=1}^{\infty} (1 - q^m)(1 + zq^m)(1 + z^{-1}q^m).$$

Here we set $z = \exp(2\pi iy)$.

---

14 Similar quantity can be defined in the NS-sector, but we do not deal with it in this paper.
6.3 Boundary states for super AdS branes in the NSNS-sector

It is easy to observe that the overlap computed in subsection [6.1] cannot be mapped to the one in subsection [6.2] by the modular transformation. The reason is that the character \( \chi_{h}^{R, -}(q) \) is related to the same character by the transformation \( \tilde{\tau} = -1/\tau \rightarrow \tau \) as in (6.12).

In order to resolve this problem, we include the NSNS-sector in the spectrum of closed strings exchanged between the branes. The character for the sector is given by

\[
\chi_{NS, -}^{h}(\tilde{q}) = q^{2b^{2}P^{2}}\vartheta_{4}(0, \tilde{\tau}) \quad \text{(6.23)}
\]

Here we have used theta function defined as

\[
\vartheta_{4}(y|\tau) = \sum_{n=-\infty}^{\infty} (-1)^{n}q^{n^{2}/2}z^{n} = \prod_{m=1}^{\infty}(1-q^{m})(1-zq^{m-\frac{1}{2}})(1-z^{-1}q^{m-\frac{1}{2}}) \quad \text{(6.24)}
\]

whose modular transformation is

\[
\vartheta_{4}(y/\tau, -1/\tau) = (-i\tau)^{1/2}\exp(\pi iy^{2}/\tau)\vartheta_{2}(y, \tau) \quad \text{(6.25)}
\]

This character is related to the other character \( \chi_{h}^{R, +}(q) \) in the partition function of open string as

\[
\chi_{NS, -}^{h}(\tilde{q}) = 4b^{2}i \int_{-\infty}^{\infty} dP' \chi_{R, +}^{h'}(q) \exp(8\pi ib^{2}PP') \quad \text{(6.26)}
\]

With the help of the relation between OSP(1|2) WZNW model and \( \mathcal{N} = 1 \) super-Liouville field theory, we have obtained the one point function of closed string in the NSNS-sector as in (5.36). We can map from the \( \mu \)-basis to \( x \)-basis as

\[
V_{\mu}^{\frac{1}{2}}(\mu|z) = \frac{1}{\pi} |\mu|^{-2h+2} \int_{C} d^{2}xe^{\mu x-\bar{\mu} \bar{x}} \Phi_{h}^{\frac{1}{2}}(x|z) \quad \text{(6.27)}
\]

and in the \( x \)-basis the one point function can be written as

\[
\langle \Phi_{h}^{\frac{1}{2}}(x|z) \rangle = \frac{\tilde{U}(h; r)}{|x + \bar{x}|^{2h+\frac{1}{2}}|z - \bar{z}|^{2\Delta_{h}}} \quad \text{(6.28)}
\]

where

\[
\tilde{U}(h; r) = \tilde{A}_{b}b^{-1}e^{-2h+\frac{1}{2}\Gamma(1 - b^{2}(2h - \frac{1}{2}))e^{-\sigma(2h-\frac{1}{2})r}} \quad \text{(6.29)}
\]

Notice that the coefficient depends on the sign \( \sigma = \text{sgn}(x + \bar{x}) \). The overall factor \( \tilde{A}_{b} \) will be fixed later by using the Cardy condition. As before we define the boundary state as

\[
B(r; \epsilon|h; x, \frac{1}{2}) \equiv \langle \Phi_{h}^{\frac{1}{2}}(x|\frac{1}{2}) \rangle_{r, \epsilon} \quad \text{(6.30)}
\]
We again change the basis from $x$-basis to $m$-basis by utilizing the formula
\[
\Phi_{m,m}^{\frac{1}{2}}(z) = \int \frac{d^2x}{|x|^2} x^{\frac{h}{2} - m + \frac{1}{2}} x^{\frac{h}{2} - m + \frac{1}{2}} \Phi_i^m(x|z),
\]
then we have
\[
B\langle r; |h; n, p, \frac{1}{2}\rangle = 2\pi\delta(p) \cdot 2\pi\hat{A}_b^{-1}\nu^{-2h+\frac{1}{2}}\Gamma(1 - b^2(2h - \frac{1}{2}))A(h, n|r) \quad (6.32)
\]
with
\[
A(h, n|r) = d_n^{h-\frac{1}{2}}(\pi_n^0 \cosh(2h - \frac{1}{2})r - \pi_n^1 \sinh(2h - \frac{1}{2})r).
\]

Let us compute the following annulus amplitude as
\[
Z_r^{NS} = B\langle \epsilon|(-1)^F q^{\hat{H}^{NS}}|r; \epsilon\rangle_B = \int_{\mathbb{S}} \frac{dh}{2\pi} \int_\mathbb{C} d^2x \chi_{NS,-}^{h} \langle \hat{q}| r\; \epsilon; n, p, \frac{1}{2}\rangle \langle \hat{h}\; x; n, p, \frac{1}{2}\rangle \langle \hat{h}\; x; \frac{1}{2}\rangle |r; \epsilon\rangle_B,
\]
where we consider the Hamiltonian for the NSNS-sector. The overlap with closed strings in the NSNS-sector diverges but the type of divergence is the same as the one for SL(2) WZNW model. Therefore we can adopt the same regularization as in [25]. In the $m$-basis the overlap can be regularized as
\[
Z_r^{NS} = \int_{\mathbb{S}} \frac{dP}{\pi} \frac{1}{2\pi} \sum_{n=-\lambda+1}^{\lambda} \int_{\mathbb{R}} \frac{dh}{2\pi} \chi_{NS,-}^{h} \langle \hat{q}| r\; \epsilon; n, p, \frac{1}{2}\rangle \langle \hat{h}\; x; n, p, \frac{1}{2}\rangle \langle \hat{h}\; x; \frac{1}{2}\rangle |r; \epsilon\rangle_B = \lambda 4\pi T |\hat{A}_b|^2 \int_{\delta}^\infty \frac{dP}{\pi} \cosh^2 2P - \cos^2 2Pr \sinh^2 \pi P \sinh 2\pi P \sinh 2\pi b \sinh 2\pi P \chi_{NS,-}^{h}(\hat{q}). \quad (6.35)
\]
Here the range of $n$ is set as $-\lambda < n < \lambda$ and the delta-function is replace as $2\pi\delta_T(p) = \frac{2}{P} \sin Tp$. We also set the cut-off $\delta$ for the integration over $P$, as the integral diverges in the limit of $\delta \to 0$. In order to obtain the finite part we may consider a difference as
\[
Z_r^{NS} - Z_{r,=}^{NS} = \lambda 4\pi T |\hat{A}_b|^2 \int_{\delta}^\infty \frac{dP}{\pi} \cosh^2 2P - \cos^2 2Pr \sinh^2 \pi P \sinh 2\pi P \sinh 2\pi P \chi_{NS,-}^{h}(\hat{q}) \quad (6.36)
\]
with a reference boundary parameter $r_*$. After the modular transformation with (6.26) we find
\[
Z_r^{NS} - Z_{r,=}^{NS} = 4\kappa T \int_0^\infty dP' (\tilde{N}(P'|r) - \tilde{N}(P'|r*)) \chi_{R,+}^{h'}(q) \quad (6.37)
\]
with
\[
\tilde{N}(P'|r) = \frac{1}{2\pi} \frac{\partial}{\partial P'} \int_0^\infty dt \left( \frac{\cosh \frac{Q t}{4} + \cosh \left(\frac{b-b^{-1}}{4}P'\right)}{\sinh \frac{tb}{2} \sinh t/2b} \right) \sinh tb/2 \sinh t/2b \quad (6.38)
\]
Here we have set $|\hat{A}_b|^2 = 4b$. The relation between $\lambda$ and $\kappa$ is given by $-i\tau \kappa = \lambda$ as discussed in [25].
6.4 Open-closed duality

As pointed out in the previous subsection, we should sum over the NSNS-sector of closed strings as well as the RR-sector. Now the amplitude is given by

$$Z_{r; \epsilon} = B \langle r'; \epsilon | (-1)^F \tilde{q}^{1/2}_{HR} | r; \epsilon \rangle_B + B \langle r; \epsilon | (-1)^F \tilde{q}^{1/2}_{NS} | r; \epsilon \rangle_B = Z^R_{r; \epsilon} + Z^NS_{r; \epsilon}.$$  \hspace{1cm} (6.39)

With the application of modular transformation, the above amplitude can be written in terms of open strings in the R-sector. In fact, using (6.13) and (6.37), we find

$$Z_{r; \epsilon} - Z_{r^*; \epsilon} = 2T \int_{-\infty}^{\infty} dP (\rho(P|r) - \rho(P|r^*)) (4\kappa \chi_{h^R, an}^+(q) + \epsilon \chi_{h^R,-}(q))$$

$$+ (\rho'(P|r) - \rho'(P|r^*)) (4\kappa \chi_{h^R, an}^+(q) - \epsilon \chi_{h^R,-}(q)),$$ \hspace{1cm} (6.40)

where the spectral densities are given by (6.18) and (6.19). From this we can conclude that the amplitude (6.39) for super AdS branes is consistent with the partition function of open strings in the R-sector.

6.5 Fuzzy supersphere brane

The couplings of closed strings in the NSNS-sector to fuzzy supersphere branes are obtained in (4.31) and (4.32) by utilizing the factorization constraint. Here we would like to check the Cardy condition for this type of branes. We define the boundary states by

$$B \langle s; \epsilon|h; x, \xi \rangle \equiv \langle \Phi_h(x, \xi|\frac{1}{2}) \rangle_{s, \epsilon},$$ \hspace{1cm} (6.41)

then the annulus amplitude can be written as

$$B \langle s'; \epsilon'|(-1)^F \tilde{q}^{1/2}_{HR} | s; \epsilon \rangle_B = \int_{\mathbb{S}} \frac{dh}{2\pi} \int_{\mathbb{C}} d^2 x \int d\xi d\chi_{h^R,-}(\tilde{q})_B \langle s'; \epsilon'|h; x, \xi \rangle \langle h; x, \xi| s; \epsilon \rangle_B.$$ \hspace{1cm} (6.42)

First we study the case with $\epsilon = \epsilon' = +1$. Using the explicit form of one point function (4.31), the overlap can be evaluated as

$$B \langle s'; +1 |(-1)^F \tilde{q}^{1/2}_{HR} | s; +1 \rangle_B \propto \int dP \frac{\sinh 2s'P \sinh 2sP}{\cosh 2\pi b^2 P} \chi_{h^R,-}(\tilde{q}),$$ \hspace{1cm} (6.43)

where we have set $s = 2\pi b^2 n$, $s' = 2\pi b^2 m$ with positive integer $n, m$. Using the relation

$$\frac{\sinh 4\pi b^2 m P \sinh 4\pi b^2 n P}{\cosh 2\pi b^2 P} = \sum_{l=0}^{2\min(n, m)-1} (-1)^l \cosh 2\pi b^2 (2n + 2m - 2l - 1)P,$$ \hspace{1cm} (6.44)

we may find

$$Z(q|s, s'; +1) = \sum_{J=|n-m|}^{n+m-1} (-1)^{n+m-J-1} \chi_{h^R,-}(q).$$ \hspace{1cm} (6.45)
Here the representation with $h = -J/2$ is of $2J + 1$-dimensional, and the character is given by $\pm \chi_{R,-}^{-J/2}(q)$. Next we choose $\epsilon = \epsilon' = -1$. Then the overlap becomes

$$B\langle s'; -1|(-1)^F q^\frac{1}{2} H^R |s; -1\rangle_B \propto \int dP \frac{\cosh 2s' P \cosh 2s P}{\cosh 2\pi b^2 P} \chi_{R,-}^h(\tilde{q}) \ ,$$

(6.46)

where we have set $s = \pi b^2(2n + 1)$, $s' = \pi b^2(2m + 1)$ with non-negative integer $n, m$. Now the relation

$$\cosh 2\pi b^2(2m + 1) P \cosh 2\pi b^2(2n + 1) P \cosh 2\pi b^2 P = \sum_{l=0}^{2 \min(n,m)} (-1)^l \cosh 2\pi b^2(2n + 2m - 2l + 1) P$$

(6.47)

leads to

$$Z(q|s, s'; -1) = \sum_{J = |n - m|}^{n+m} (-1)^{n+m-J} \chi_{R,-}^{-J/2}(q) \ .$$

(6.48)

In this way, we have shown that the couplings of RR-states to the fuzzy supersphere branes are consistent with the partition function of open strings.

Let us consider the couplings of closed strings in the NSNS-sector as well. For the super AdS branes, we obtained the coupling of NSNS-states with the help of the relation to $N = 1$ super Liouville field theory as in (5.36). However, this approach is not applicable to the super spherical branes as far as we know. Fortunately, the Cardy condition is actually strong enough to guess the coupling of NSNS-states almost uniquely at least in this case. Suppose that the one point function is given by

$$\langle \Phi_{\tilde{h}}^\dagger (x)|z \rangle_{s\epsilon} = \frac{\tilde{U}(h; s, \epsilon)}{|x + \bar{x}|^{2h + 1} |z - \bar{z}|^{2\Delta_{\tilde{h}}}}$$

(6.49)

with

$$\tilde{U}(h; s, \epsilon) \propto \nu^{-2h+1} \Gamma(1 - b^2(2h - \frac{1}{2})) \sin(2h - \frac{1}{2}))$$

(6.50)

up to an overall factor. Moreover we set $s = \pi b^2 n$, $s' = \pi b^2 m$ with non-negative integer $n, m$. With this ansatz the overlap is

$$B\langle s'; \epsilon|(-1)^F q^\frac{1}{2} H^{NS} |s; \epsilon\rangle_B \propto \int dP P \frac{\sinh 2s' P \sinh 2s P}{\sinh 2\pi b^2 P} \chi_{NS,-}^h(\tilde{q}) \ ,$$

(6.51)

which becomes after the modular transformation

$$Z(q|s, s'; \epsilon) = \sum_{J = |n - m|/2}^{(n+m)/2-1} \chi_{R,+}^{-J/2}(q)$$

(6.52)
if we choose a proper normalization. Here we have used the relation
\[
\frac{\sinh 2\pi b^2 m P \sinh 2\pi b^2 n P}{\sinh 2\pi b^2 P} = \sum_{l=0}^{\min(n,m)-1} \sinh 2\pi b^2 (n + m - 2l - 1) P .
\] (6.53)

The expression (6.52) has a meaning as the partition function of open strings only if \(m + n \in 2\mathbb{Z}\). Referring to the coupling of RR-states, it is natural to guess that \(m, n \in 2\mathbb{Z}\) for \(\epsilon = +1\) and \(m, n \in 2\mathbb{Z} + 1\) for \(\epsilon = -1\).

7 Conclusion and discussions

In this article, we have analyzed branes in the OSP(1|2) WZNW model. We have used two different methods. The first and common method is to determine correlation functions in the boundary theory via factorization constraints. We used this to determine bulk one-point functions and boundary two-point functions. The results can be checked by strong consistency conditions. These are the Cardy condition and agreement with the semiclassical limit. For non-compact branes worldsheet duality is subtle and involves a spectral density. We were able to treat this issue as in the case of the bosonic \(H_3^+\) model [25]. The second method is to establish a correspondence to boundary \(\mathcal{N} = 1\) super-Liouville theory and using the known results from this model, e.g. [14]. This correspondence is a generalization of bulk correspondence between the OSP(1|2) model and super-Liouville theory in [11], see also the bosonic counterpart [9].

A boundary CFT is completely specified by bulk one-point functions, bulk-boundary two-point functions and boundary three-point functions. Thus some work is still left in solving the boundary OSP(1|2) WZNW model. For the super AdS\(_2\) branes the correspondence to \(\mathcal{N} = 1\) super-Liouville theory seems to be a good way to address this issue. Another open problem is the treatment of NS-sectors. The fermions in WZNW models naturally have Ramond-boundary conditions. So far, we have only studied the NS-sector through the correspondence to super-Liouville theory. A more direct way might be relevant. In [43] GL(N|N) models with NS-boundary conditions for the fermions are discussed.

There are several interesting applications and generalizations of our work. First of all, it would be interesting to generalize our analysis to WZNW models on other supergroups. A candidate is the OSP(2|2) WZNW model. This model has several types of branes; spherical ones and AdS-type branes with either Dirichlet or Neumann boundary conditions in the U(1) direction. For the AdS-type branes it seems possible to establish a correspondence to boundary \(\mathcal{N} = 2\) super-Liouville theory. The appropriate boundary action of the OSP(2|2) WZNW model is suggested using a relation between supergroup WZNW models and \(\mathcal{N} = (2, 2)\) superconformal theories [43]. Such a correspondence would provide bulk one-point functions from [19].
The correspondence between the $H^+_3$ model and bosonic Liouville theory has been used to show a strong-weak duality (the Fateev-Zamolodchikov-Zamolodchikov duality [44]) between the cigar CFT and sine-Liouville theory [45]. Using the boundary action of [41] we could extend this duality to worldsheets with boundary. Furthermore, we would like to understand correspondences and dualities involving supergroups, such as of the OSP-type and their cosets. There are also $\mathcal{N} = (2, 2)$ super conformal field theories on these spaces [3, 46]. To these cases mirror symmetry should be applied similarly to the case of the fermionic Euclidean black hole [47].

Acknowledgement
We are very grateful to Vincent Bouchard, Gaston Giribet, David Ridout, Peter Roenne, and Volker Schomerus. The work of YH is supported in part by JSPS Research Fellowship.

A Some super analysis

We list some useful formulae. There are two ways to extend complex conjugation to the ring of Grassmann numbers. We denote them by a bar and by the super star $\sharp$. The ordinary bar-operation is defined by

$$
\overline{c\theta} = \bar{c}\bar{\theta}, \quad \overline{\theta} = \theta, \quad \overline{\theta_1\theta_2} = \theta_2\theta_1 \tag{A.1}
$$

for a complex number $c$ and Grassmann odd elements $\theta, \theta_1$ and $\theta_2$. The superstar is

$$(c\theta)^\sharp = \bar{c}\theta^\sharp, \quad \theta^\sharp = -\theta, \quad (\theta_1\theta_2)^\sharp = \theta_1^\sharp\theta_2^\sharp. \tag{A.2}$$

They are related as follows; define the map $\alpha$ as $\alpha(x) = 1$ if $x$ is Grassmann even and $\alpha(\theta) = i$ if $\theta$ is Grassmann odd. Then it is straightforward to check, that

$$X^\sharp = \alpha(\overline{X}) \tag{A.3}$$

for any Grassmann number $X$.

The analog of transposition for super matrices is the super transpose $st$

$$
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}. \tag{A.4}
$$

Hermitean conjugation is defined as

$$\dagger = st \circ \sharp. \tag{A.5}$$

It is important that this map is of order two (while the supertranspose and the superstar are each of order four).
Four point function with a degenerate operator

Generically it is quite difficult to obtain explicit expressions of four point function. The situation might change if one of the vertex operators belongs to a degenerate representation. In this appendix we study the case with \( h_1 = k/2 \).

B.1 Null equation from the degenerate operator

As discussed in subsection 3.4 the state \(|k/2\rangle\) is degenerate since a descendant \(|\theta\rangle = J^-_{-1}|k/2\rangle\) is null. Therefore, we can set the operator corresponding to \(|\theta\rangle\) to be zero, and this yields differential equation which correlation functions with the degenerate operator should satisfy. In the language of operator the null condition can be written as

\[
J^-_{-1}(x,\xi)\Phi_{k/2}(x,\xi|z) = 0 . \tag{B.1}
\]

Here \( J^-_{-1}(x,\xi) \) is given by the mode expansion of the following operator as

\[
J^-(x,\xi|z) = e^{-2\xi j_0^+} e^{-xJ_0^+} J^-(z)e^{xJ_0^+} e^{2\xi j_0^+} \tag{B.2}
\]

\[
= J^-(z) + 2xJ^3(z) + x^2J^+(z) - 2\xi J^{-}(z) + 2x\xi J^+(z) .
\]

We consider correlation functions with the insertion of \((B.1)\). With the help of Ward identity

\[
\left\langle J^-(x,\xi|z) \prod_{i=1}^{N} \Phi_{h_i}(x_i,\xi_i|z_i) \right\rangle \tag{B.3}
\]

\[
= -\sum_{i=1}^{N} \left\{ (x - x_i + \frac{1}{2}\xi \xi_i)^2 \partial_{x_i} + (x - x_i)(\xi - \xi_i)\partial_{\xi_i} - 2(x - x_i + \xi \xi_i)h_i \right\} \left\langle \prod_{i=1}^{N} \Phi_{h_i}(x_i,\xi_i|z_i) \right\rangle
\]

we can see that the correlation function with \( \Phi_{k/2} \) should satisfy the following differential equations

\[
\sum_{i=1}^{N} \frac{1}{z - z_i} \left\{ (x - x_i + \frac{1}{2}\xi \xi_i)^2 \partial_{x_i} + (x - x_i)(\xi - \xi_i)\partial_{\xi_i} - 2(x - x_i + \xi \xi_i)h_i \right\}
\]

\[
\times \left\langle \Phi_{k/2}(x,\xi|z) \prod_{i=1}^{N} \Phi_{h_i}(x_i,\xi_i|z_i) \right\rangle = 0 , \tag{B.4}
\]

which is called as the null equation.

Let us focus on four point function. In this case we can set as \((z_1, z_2, z_3, z_4) = (z_\infty, 1, z, 0)\), \((x_1, x_2, x_3, x_4) = (x_\infty, 1, x, 0)\), and \((\xi_1, \xi_2, \xi_3, \xi_4) = (x_\infty\eta, 1, \xi, 0)\), and finally
take the limit of \( z_\infty, x_\infty \to \infty \). Picking up the term proportional to \( z_\infty^{-1} x_\infty^2 \), we obtain a differential equation

\[
\mathcal{D}_0 \left( \Phi_\frac{1}{2}(x_1, \xi_1|z_1) \prod_{i=2}^{4} \Phi_{h_i}(x_i, \xi_i|z_i) \right) = 0 \quad \text{(B.5)}
\]

with

\[
\mathcal{D}_0 \equiv z \partial_{x_2} + (1 + \eta \xi) \partial_{x_3} + (z + 1) \partial_{x_4} + \eta(z \partial_{\xi_2} + \partial_{\xi_3} + (z + 1) \partial_{\xi_4}) . \quad \text{(B.6)}
\]

This equation largely restricts the form of correlator. Recall that the four point function is of the form \( \Omega = \sum_{i} \gamma_{i} \). Expanding by the fermionic parameters, the holomorphic part may be written as

\[
\Omega = X_{12}^{-\gamma_{12}} X_{13}^{-\gamma_{13}} X_{14}^{-\gamma_{14}} X_{24}^{-\gamma_{24}} (A(X, z) + B(X, z)\chi_3\chi_2 + C(X, z)\chi_3 + D(X, z)\chi_2) . \quad \text{(B.7)}
\]

Here \( X_{ij} \) are defined in \( \text{(3.11)} \) and the cross ratios \( X, \chi_3, \chi_2 \) are in \( \text{(3.32)} \) and \( \text{(3.33)} \). Replacing the correlator by its holomorphic part \( \Omega \quad \text{(B.7)} \) and setting \( (x_i, \xi_i, z_i) \) as the fixed values at the final point, we find

\[
(x - z)A' + \gamma_{24} A + \eta \xi ((x - z)B' + (\gamma_{24} + 1)B - xA') \quad \text{(B.8)}
\]

\[
+ (x - z)\eta C' + (\gamma_{24} + \frac{1}{2})\eta C + (x - z)(\xi - x\eta)D' + (\gamma_{24} + \frac{1}{2})(\xi - x\eta)D = 0 .
\]

Here \( ' \) represents the derivative with respect to \( x \). Expanding this equation by \( \eta, \xi \) the above equation yields four independent differential equations. Solutions can be find as

\[
A(X, z) = a(z)(X - z)^{-\gamma_{24}} , \quad B(X, z) = b(z)(X - z)^{-\gamma_{24} - 1} , \quad C(X, z) = c(z)(X - z)^{-\gamma_{24} - \frac{1}{2}} , \quad D(X, z) = d(z)(X - z)^{-\gamma_{24} - \frac{1}{2}} . \quad \text{(B.9)}
\]

Here we should notice that \( A(X) \to A(x) - x\eta\xi A'(x) \). In this way, the null equation fixes the \( (X, \chi_3, \chi_2) \)-dependence completely, and now the problem is to obtain the explicit form of the functions \( a(z), b(z), c(z), d(z) \) of \( z \).

### B.2 Solutions to the KZ equation

It is known that correlators of WZNW models satisfy the KZ equation, and in our case it is written as \( \text{(3.15)} \). Here we fix the functions \( a(z), b(z), c(z), d(z) \) by utilizing the KZ equation. Since the KZ operator itself is bosonic, it does not mix the bosonic and fermionic parts of conformal blocks. First we focus on the bosonic part with \( a(z), b(z) \) and later we study the fermionic part with \( c(z), d(z) \). Notice that only the bosonic part appears in the two point function on a disk as seen in section 4.
Differential equations for $a(z)$ and $b(z)$ can be obtained by inserting the holomorphic part of correlator (B.7) into the KZ equation (3.15) with $N = 4$. After a tedious but straightforward calculation we find

$$\kappa z(z - 1) \partial_z a(z) = [\alpha_1(z - 1) + \beta_1 z] a(z) - \frac{1}{2} b(z), \quad (B.10)$$

$$\kappa z(z - 1) \partial_z b(z) = z \gamma_{24} \left[ \frac{1}{2} (\gamma_{12} - 1) + \frac{24}{z} \right] a(z) + [\alpha_1'(z - 1) + \beta_1' z] b(z), \quad (B.11)$$

where we have compared the terms with $(x - z)^{-24}$ and $\eta \xi (x - z)^{-24}$. The coefficients are given by

$$\alpha_1 = \frac{1}{2} \gamma_{24} (\gamma_{24} + 1) - \gamma_{24} (h_3 + h_4 + \frac{1}{2}) + h_3 h_4, \quad (B.12)$$

$$\beta_1 = \frac{1}{2} \gamma_{24} (\gamma_{24} + 1) - \gamma_{24} (h_3 + h_2) + h_3 h_2, \quad (B.13)$$

$$\alpha_1' = \frac{1}{2} (\gamma_{24} + 1) (\gamma_{24} + 2) - (\gamma_{24} + 1) (h_3 + h_4 + \frac{1}{2}) + \frac{h_3}{2} + \frac{h_4}{2} + h_3 h_4, \quad (B.14)$$

$$\beta_1' = \frac{1}{2} (\gamma_{24} + 1) (\gamma_{24} + 2) - (\gamma_{24} + 1) (h_3 + h_2 + \frac{3}{2}) + \frac{h_3}{2} + \frac{h_2}{2} + h_3 h_2 - \frac{1}{4} \gamma_{24}. \quad (B.15)$$

Since they are two independent first order differential equations, we have two independent solutions to them.

With the two solutions, we can write down the conformal block as a linear combination of two independent functions. We set the form as

$$F_i^S = \frac{1}{4 \pi} \left[ (h_1 - h_2)(h_1 - h_2 - \frac{1}{2}) - h_3 (h_3 - \frac{1}{2}) - h_4 (h_4 - \frac{1}{2}) \right] \left( 1 - \frac{1}{\eta \xi} \right) \left( 1 - \frac{1}{24} \right),$$

$$\times \left( f_{a,1}^S(z)(x(1 - \eta \xi) - z)^{-24} + f_{b,1}^S(z) \eta \xi (x - z)^{-24-1} \right) \quad (B.16)$$

with $i = 1, 2$, which behave as

$$F_1^S = \frac{1}{4 \pi} (h_1 - h_2)(h_1 - h_2 - \frac{1}{2}) - h_3 (h_3 - \frac{1}{2}) - h_4 (h_4 - \frac{1}{2}) \eta \xi \xi^{-24} + \ldots, \quad (B.17)$$

$$F_2^S = \frac{1}{4 \pi} (h_1 - h_2 - \frac{1}{2})(h_1 - h_2 - 1) - h_3 (h_3 - \frac{1}{2}) - h_4 (h_4 - \frac{1}{2}) \eta \xi \eta^{-24-1} + \ldots. \quad (B.18)$$

The overall normalization is our convention. The functions $f_{a,1}^S(z)$ and $f_{b,1}^S(z)$ are found to be

$$f_{a,1}^S = F \left( 1 + \frac{-1 + h_1 - h_2 + h_3 + h_4}{4 \pi} - h_3 (h_3 - \frac{1}{2}) - h_4 (h_4 - \frac{1}{2}) - 1 + \frac{-2h_1 + 2h_2}{4 \pi}; z \right), \quad (B.19)$$

$$f_{b,1}^S = \left( \frac{-1 + h_1 - h_2 + h_3 - h_4}{4 \pi} - h_3 (h_3 - \frac{1}{2}) - h_4 (h_4 - \frac{1}{2}) \right) \times \left( 1 + \frac{-1 + h_1 - h_2 + h_3 + h_4}{4 \pi} - 1 - \frac{1 + h_1 - h_2 - h_3 - h_4}{4 \pi}; z \right),$$

$$f_{a,2}^S = \left( \frac{-1 + h_1 + h_2 - h_3 - h_4}{4 \pi} - h_3 (h_3 - \frac{1}{2}) - h_4 (h_4 - \frac{1}{2}) \right) \left( 1 + \frac{-1 + h_1 - h_2 + h_3 + h_4}{4 \pi} - 2 + \frac{-2h_1 + 2h_2}{4 \pi}; z \right),$$

$$f_{b,2}^S = \frac{-1 + h_1 + h_2 - h_3 - h_4}{4 \pi} \times \left( \frac{-2h_1 + 2h_2}{4 \pi} - 1 + \frac{-2h_1 + 2h_2}{4 \pi}; z \right),$$

$$f_{b,1}^S = \frac{-1 + h_1 + h_2 - h_3 - h_4}{4 \pi} \times \left( \frac{-2h_1 + 2h_2}{4 \pi} + 1 + \frac{-2h_1 + 2h_2}{4 \pi}; z \right).$$
Here \( F(a, b, c; z) \) denotes the hypergeometric function.

Above expressions are given in terms of \( z \), thus they are suitable for the region with \( z \sim 0 \) or \( z_3 \sim z_4 \). We may say that they are in the \( s \)-channel expression. In the \( t \)-channel expression with \( z \sim 1 \) or \( z_3 \sim z_2 \), it is suitable to express in terms of \( 1 - z \) as well as \( 1 - x \). These expressions are simply given with \( F_i^S \) by exchanging \( z, x \) and \( 1 - z, 1 - x \), and moreover \( h_2 \) and \( h_4 \). We denote them by \( F_i^T \) with \( i = 1, 2 \). These two expression are related by

\[
F_1^S = e^{-\pi i \gamma_24} (F_{11}^{ST} F_1^T + F_{12}^{ST} F_2^T) , \quad F_2^S = e^{-\pi i \gamma_24} (F_{21}^{ST} F_1^T + F_{22}^{ST} F_2^T)
\]

(B.20)

with

\[
F_{11}^{ST} = \frac{\Gamma(1 - \frac{1}{4} - \frac{h_1 + h_2}{4\kappa}) \Gamma(\frac{1}{4} - \frac{h_1 + h_2}{4\kappa})}{\Gamma(1 - \frac{1}{4} - \frac{h_1 - h_2 - h_3 - h_4}{4\kappa}) \Gamma(\frac{1}{4} - \frac{h_1 - h_2 - h_3 - h_4}{4\kappa})},
\]

(B.21)

\[
F_{12}^{ST} = -\frac{4\kappa \Gamma(1 - \frac{1}{4} - \frac{h_1 + h_2}{4\kappa}) \Gamma(1 + \frac{1}{4} + \frac{h_1 - h_2}{4\kappa})}{\Gamma(1 - \frac{1}{4} - \frac{h_1 - h_2 - h_3 - h_4}{4\kappa}) \Gamma(1 + \frac{1}{4} + \frac{h_1 - h_2 - h_3 - h_4}{4\kappa})},
\]

(B.22)

\[
F_{21}^{ST} = -\frac{\Gamma(\frac{1}{4} + \frac{h_1 + h_2}{4\kappa}) \Gamma(\frac{1}{4} - \frac{h_1 + h_2}{4\kappa})}{\Gamma(1 + \frac{1}{4} + \frac{h_1 + h_2 - h_3 - h_4}{4\kappa}) \Gamma(1 - \frac{1}{4} - \frac{h_1 + h_2 - h_3 - h_4}{4\kappa})},
\]

(B.23)

\[
F_{22}^{ST} = \frac{\Gamma(\frac{1}{4} - \frac{h_1 + h_2}{4\kappa}) \Gamma(\frac{1}{4} + \frac{h_1 - h_2}{4\kappa})}{\Gamma(1 + \frac{1}{4} + \frac{h_1 - h_2 - h_3 - h_4}{4\kappa}) \Gamma(1 - \frac{1}{4} - \frac{h_1 - h_2 - h_3 - h_4}{4\kappa})}.
\]

(B.24)

Let us move to the fermionic part with \( c(z), d(z) \). As mentioned above, there are no such contributions to the disk amplitude, so we study them here only for completeness. We hope to report on more systematic analysis of closed strings in the OSP(1|2) WZNW model somewhere else. In the same way as before, we can show that the KZ equation leads to two independent first order differential equations for \( c(z) \) and \( d(z) \) as

\[
\kappa z(z - 1) \partial_z c(z) = [\alpha_2(z - 1) + \beta_2 z] c(z) + \frac{1}{4} (\gamma_{12} - \frac{1}{2}) z d(z) ,
\]

(B.25)

\[
\kappa z(z - 1) \partial_z d(z) = -\frac{1}{4} (\gamma_{24} + \frac{1}{2}) - \frac{h_4}{2} c(z) + [\alpha'_2(z - 1) + \beta'_2 z] d(z) .
\]

(B.26)

Here the coefficients are given by

\[
\alpha_2 = \frac{1}{2} (\gamma_{24} + \frac{1}{2}) (\gamma_{24} + \frac{3}{2}) - (\gamma_{24} + \frac{1}{2}) (h_3 + h_4 + \frac{1}{2}) + h_3 h_4 ,
\]

(B.27)

\[
\beta_2 = \frac{1}{2} (\gamma_{24} + \frac{1}{2}) (\gamma_{24} + \frac{3}{2}) - (\gamma_{24} + \frac{1}{2}) (h_3 + h_2 + \frac{1}{2}) + \frac{h_3}{2} + h_3 h_2 ,
\]

(B.28)

\[
\alpha'_2 = \frac{1}{2} (\gamma_{24} + \frac{1}{2}) (\gamma_{24} + \frac{3}{2}) - (\gamma_{24} + \frac{1}{2}) (h_3 + h_4 + \frac{3}{4}) + \frac{h_3}{2} + \frac{h_3}{2} + h_3 h_4 ,
\]

(B.29)

\[
\beta'_2 = \frac{1}{2} (\gamma_{24} + \frac{1}{2}) (\gamma_{24} + \frac{3}{2}) - (\gamma_{24} + \frac{1}{2}) (h_3 + h_2 + \frac{1}{2}) + \frac{h_3}{2} + h_3 h_2 .
\]

(B.30)
With two independent solutions to the above equations, we can write down the conformal block as

\[
G_i^S = z^{\frac{1}{24}[(h_1-h_2)(h_1-h_2-\frac{1}{2})-h_3(h_3-\frac{1}{2})-h_4(h_4-\frac{1}{2})]}(1-z)^{\frac{1}{24}[(h_1-h_4)(h_1-h_4-\frac{1}{2})-h_2(h_2-\frac{1}{2})-h_3(h_3-\frac{1}{2})]} \times \left(f_{c,i}^S(z)\eta(x-z)^{-\gamma_{24}-\frac{1}{4}} + f_{d,i}^S(z)(\xi - x\eta)(x-z)^{-\gamma_{24}-\frac{1}{4}}\right)
\]

with \(i = 1, 2\), which behave like

\[
G_1^S = z^{\frac{1}{24}[(h_1-h_2-\frac{1}{2})(h_1-h_2-1)-h_3(h_3-\frac{1}{2})-h_4(h_4-\frac{1}{2})]}\eta x^{-\gamma_{24}-\frac{1}{2}} + \ldots ,
\]

\[
G_2^S = z^{\frac{1}{24}[(h_1-h_2)(h_1-h_2-\frac{1}{2})-h_3(h_3-\frac{1}{2})-h_4(h_4-\frac{1}{2})]}(\xi - x\eta)x^{-\gamma_{24}-\frac{1}{2}} + \ldots .
\]

The functions are

\[
f_{c,1}^S = z^{\frac{1-2h_1+2h_4}{4\kappa}} F\left(\frac{\frac{1}{2}-h_1+h_2-h_3-h_4}{4\kappa}, \frac{-\frac{1}{2}+h_1+h_2-h_3-h_4}{4\kappa}, \frac{1-2h_1+2h_2}{4\kappa}; z\right),
\]

\[
f_{d,1}^S = \frac{\frac{1}{2}-h_1+h_2-h_3+h_4}{1-2h_1+2h_2} z^{\frac{1-2h_1+2h_2}{4\kappa}} F\left(1 - \frac{\frac{1}{2}-h_1-h_2+h_3+h_4}{4\kappa}, \frac{\frac{1}{2}-h_1+h_2-h_3-h_4}{4\kappa}, 1 + \frac{1-2h_1+2h_2}{4\kappa}; z\right),
\]

\[
f_{c,2}^S = -\frac{\frac{1}{2}+h_1-h_2-h_3-h_4}{4\kappa} F\left(1 - \frac{\frac{1}{2}-h_1-h_2-h_3+h_4}{4\kappa}, 1 - \frac{3}{2}-3h_1+h_2+h_3+h_4}{4\kappa}, 2 - \frac{1-2h_1+2h_2}{4\kappa}; z\right),
\]

\[
f_{d,1}^S = F\left(-\frac{\frac{1}{2}-h_1+h_2-h_3+h_4}{4\kappa}, 1 - \frac{\frac{3}{2}-3h_1+h_2+h_3+h_4}{4\kappa}, 1 - \frac{1-2h_1+2h_2}{4\kappa}; z\right).
\]

In the \(t\)-channel expression, functions are given by replacing \(h_2 \leftrightarrow h_4\) and \((z, x) \leftrightarrow (1 - z, 1 - x)\). The relation between two expressions is

\[
G_1^S = e^{-\pi i(\gamma_{24}+\frac{1}{2})}(G_{11}^{ST} G_1^T + G_{12}^{ST} G_2^T), \quad G_2^S = e^{-\pi i(\gamma_{24}+\frac{1}{2})}(G_{21}^{ST} G_1^T + G_{22}^{ST} G_2^T),
\]

with

\[
G_{11}^{ST} = -\frac{\Gamma(\frac{1-2h_1+2h_2}{4\kappa})\Gamma(1 + \frac{1-2h_1-2h_4}{4\kappa})}{\Gamma(1 - \frac{\frac{1}{2}-h_1-h_2+h_3+h_4}{4\kappa})\Gamma(\frac{\frac{1}{2}-h_1+h_2-h_3-h_4}{4\kappa})},
\]

\[
G_{12}^{ST} = \frac{\Gamma(\frac{1-2h_1+2h_2}{4\kappa})\Gamma(\frac{1-2h_1+2h_4}{4\kappa})}{\Gamma(\frac{\frac{3}{2}-3h_1+h_2+h_3+h_4}{4\kappa})\Gamma(\frac{\frac{1}{2}-h_1+h_2-h_3+h_4}{4\kappa})},
\]

\[
G_{21}^{ST} = \frac{\Gamma(1 - \frac{1-2h_1+2h_2}{4\kappa})\Gamma(1 + \frac{1-2h_1-2h_4}{4\kappa})}{\Gamma(1 - \frac{\frac{1}{2}-h_1+h_2-h_3+h_4}{4\kappa})\Gamma(1 - \frac{3}{2}-3h_1+h_2+h_3+h_4}{4\kappa})},
\]

\[
G_{22}^{ST} = \frac{\Gamma(1 - \frac{1-2h_1+2h_2}{4\kappa})\Gamma(\frac{1-2h_1+2h_4}{4\kappa})}{\Gamma(\frac{\frac{1}{2}-h_1+h_2+h_3+h_4}{4\kappa})\Gamma(\frac{1-2h_1-2h_4}{4\kappa})}.
\]
Now that we have the complete set of conformal blocks, the four point function of bulk operators can be written down explicitly. Recall that the operator product expansion involving $\Phi_{k/2}$ has the simple form as in (4.12). Therefore, the four point function $g_4(z, x, \eta, \xi)$ can be expanded with the above conformal blocks as

$$
g_4(z, x, \eta, \xi) = C(h_2)C(h_2 - h_3, h_4)|F_4^S|^2 - \tilde{C}(h_2)\tilde{C}(h_2 - h_3, h_4)|\tilde{F}_4^S|^2 \quad (B.40)
$$

$$
+ \tilde{C}(h_2)C(h_2 - h_2 - \frac{1}{2}, h_3, h_4)|G_4^S|^2 + C(h_2)\tilde{C}(h_2 - h_2, h_3, h_4)|G_2^S|^2 .
$$

After the modular transformation, the above expression is transferred into the $t$-channel. The expression is single-valued around $z \sim 1$ only if the three point functions satisfy the constraints

$$
\frac{C(h_2)C(h_2 - h_3, h_4)}{\tilde{C}(h_2)C(h_2 - h_2 - \frac{1}{2}, h_3, h_4)} = \frac{F_{21}^{ST}F_{21}^{ST}}{F_{11}^{ST}F_{12}^{ST}} , \quad \frac{\tilde{C}(h_2)\tilde{C}(h_2 - h_3, h_4)}{C(h_2)\tilde{C}(h_2 - h_2, h_3, h_4)} = -\frac{G_{21}^{ST}G_{22}^{ST}}{G_{11}^{ST}G_{12}^{ST}} .
$$

These conditions are not enough to fix uniquely the three point functions as (B.33), and we need to make use of another four point function with a different degenerate operator, such as, $\Phi_{k-1}$. In this case the operator product expansion is of the form as $\Phi_h \Phi_{k-1} \sim C[\Phi_{k-1-h}]_{oo} + C[\Phi_h]_{oo} + C[\Phi_{k-1-h}]_{oo}$.

### B.3 Basic three point functions of bulk operator

The coefficients $C(h)$ and $\tilde{C}(h)$ in (4.12) are actually important quantities to compute the one point function of bulk operator on a disk. Here we obtain them with the information of (B.41). These can be computed with the explicit form of three point function (3.23), but it might be instructive to find them in a different route. It would be also important if we want to obtain the three point functions without referring super-Liouville field theory.

Before using (B.41) we would like to obtain constraint equations for $C(h)$ and $\tilde{C}(h)$ from three point functions as analyzed in [48] for the SL(2) WZNW model. We consider the correlator

$$
\langle \Phi^\frac{k}{2}(x, \xi)\Phi_h(y_1, \eta_1)\Phi^\frac{k}{2-h}(y_2, \eta_2) \rangle . \quad (B.42)
$$

Utilizing the OPE formula (4.12) we can expand around $x \sim y_1$ and $x \sim y_2$. Comparing the two expressions we can find the relation

$$
C(\frac{k}{2}, h, \frac{k}{2} - h) = C(h)D(\frac{k}{2} - h) = C(\frac{k}{2} - h)D(h) . \quad (B.43)
$$

Here $D(h)$ is given in (3.19), which appears in the two point function (3.18). In the same way from

$$
\langle \Phi^\frac{k}{2}(x, \xi)\Phi_h(y_1, \eta_1)\Phi^\frac{k}{2-h-\frac{1}{2}}(y_2, \eta_2) \rangle , \quad (B.44)
$$

46
we would obtain
\[ \tilde{C}(\frac{k}{2}, h, \frac{k}{2} - h - \frac{1}{2}) = \tilde{C}(h) D(\frac{k}{2} - h - \frac{1}{2}) = \tilde{C}(\frac{k}{2} - h - \frac{1}{2}) D(h). \]  
(B.45)

Now is the time to use (B.41). Setting \( h_2 = h_3 = h, h_1 = h_4 = k/2 \), we have
\[ \frac{C(h)C(\frac{k}{2} - h, h, \frac{k}{2})}{C(h)C(\frac{k}{2} - h - \frac{1}{2}, h, \frac{k}{2})} = b^4 \left( \gamma(\sqrt{b^2(k - 2h - 1)}) \right)^2 \gamma(2b^2h) \gamma(2b^2(h - \frac{1}{2})) . \]  
(B.46)

With the first equations of (B.43) and (B.45), this equation leads to
\[ \frac{C(h)^2}{C(h)^2} = b^4 \nu^2 \left( \gamma(\sqrt{b^2(k - 2h - 1)}) \gamma(2b^2h) \right)^2. \]  
(B.47)

Combining with the second equations of (B.43) and (B.45), we may have
\[ C(h) = D(h) , \quad \tilde{C}(h) = \frac{D(h)}{b^2\nu \gamma(b^2(k - 2h - 1)) \gamma(2b^2h)} \]  
(B.48)
up to a common constant factor. We can show that the explicit form of three point functions (3.23) also leads to the same expression.

C Super AdS\(_2\) branes

In this appendix we find the boundary action of the AdS\(_2\)-like branes in the OSP(1|2) WZNW model.

C.1 The bulk action

Let us start by stating the bulk action. We follow [11] with notation as
\[ S_{\text{bulk}} = \frac{k}{\pi} \int d^2z \left[ \partial\phi \bar{\partial}\phi + e^{-2\phi}(\partial\bar{\gamma} - \bar{\partial}\theta)(\bar{\partial}\gamma - \theta\partial\bar{\theta}) + 2e^{-\phi}\bar{\partial}\theta\partial\bar{\theta} \right]. \]  
(C.1)

In the free field realization this becomes
\[ S_{\text{bulk}} = S_0 + S_{\text{int}}, \]
\[ S_0 = \frac{1}{\pi} \int d^2z \left[ \frac{1}{2} \partial\phi \bar{\partial}\phi + \frac{b}{8}\sqrt{g}R\phi + \beta \bar{\partial}\gamma + \bar{\beta}\partial\gamma + p\bar{\partial}\theta + \bar{p}\partial\bar{\theta} \right], \]  
(C.2)
\[ S_{\text{int}} = -\frac{1}{\pi} \int d^2z \left[ \frac{1}{k} \beta \bar{\beta} e^{2b\phi} + \frac{1}{2k}(p + \beta\theta)(\bar{p} + \bar{\beta}\bar{\theta})e^{b\phi} \right]. \]

The bulk equation of motion for the auxiliary fields \( p \) and \( \beta \) are
\[ \beta = ke^{-2b\phi}(\partial\bar{\gamma} - \bar{\partial}\theta), \quad p + \beta\theta = -2ke^{-b\phi}\partial\bar{\theta}, \]  
(C.3)
\[ \bar{\beta} = ke^{-2b\phi}(\bar{\partial}\gamma - \theta\partial\bar{\theta}), \quad \bar{p} + \bar{\beta}\bar{\theta} = 2ke^{-b\phi}\bar{\partial}\theta . \]
C.2 The boundary action

We use the same parametrization as [11]. Further let $t^a$ be the generators of OSP(1|2), and denote the current by

$$\tilde{J}(z) = ky^{-1}\partial y = \sum_a t^a \tilde{J}^a. \quad (C.4)$$

Then we get

$$\tilde{J}^H(z) = 2k\tilde{\phi} + 2ke^{-2\phi}\tilde{\gamma} - 2k\tilde{\theta}e^{-2\phi}\tilde{\theta} + \tilde{\theta}e^{-\phi}\tilde{\theta};$$

$$\tilde{J}^{E+}(z) = ke^{-2\phi}\tilde{\gamma} - ke^{-2\phi}\theta\tilde{\theta};$$

$$\tilde{J}^{E-}(z) = k\tilde{\theta}\tilde{\theta} - ke^{-2\phi}\tilde{\gamma} - 2k\tilde{\phi} + k\tilde{\gamma} + ke^{-2\phi}\tilde{\theta}\tilde{\theta} + 2ke^{-\phi}\tilde{\gamma}\tilde{\theta}, \quad (C.5)$$

$$\tilde{J}^{F+}(z) = -2ke^{-2\phi}\tilde{\gamma}\tilde{\theta} - 2ke^{-2\phi}\tilde{\theta}\tilde{\theta} + 2ke^{-\phi}\tilde{\theta},$$

$$\tilde{J}^{F-}(z) = 2k\tilde{\theta}\tilde{\theta} - 2k\tilde{\phi}\tilde{\phi} - 2ke^{-2\phi}\tilde{\gamma}\tilde{\gamma} + 2ke^{-2\phi}\tilde{\theta}\tilde{\gamma} + 2ke^{-\phi}\tilde{\gamma}\tilde{\theta},$$

and similarly for $J$. We introduce a short-hand notation

$$\beta' = ke^{-2b\phi}(\partial\tilde{\gamma} - \tilde{\theta}\tilde{\theta}), \quad p' + \beta'\theta = -2ke^{-b\phi}\partial\tilde{\theta}, \quad (C.6)$$

$$\bar{\beta}' = ke^{-2b\phi}(\partial\tilde{\gamma} - \tilde{\theta}\tilde{\theta}), \quad \bar{p}' + \bar{\beta}'\bar{\theta} = 2ke^{-b\phi}\partial\tilde{\theta}.$$ 

Then the gluing conditions for the gluing map given by conjugation with $X^\epsilon$ read

$$\tilde{\gamma} - \gamma = ce^{\phi} - e\tilde{\theta}, \quad \beta' = \bar{\beta}', \quad p' = \epsilon\bar{p'},$$

$$4k(\partial - \bar{\partial})\phi - 4\beta'ce^{\phi} = (\bar{\theta} + e\tilde{\theta})(p' - \epsilon\bar{p}' + \beta'(\theta - e\tilde{\theta})), \quad (C.7)$$

$$e4k\tilde{\theta}\tilde{\theta} + 4k\partial\tilde{\theta} = 2k(\theta + e\tilde{\theta})(\partial + \bar{\partial})\phi + ce^{\phi}(p' - \epsilon\bar{p}' + \beta'(\theta - e\tilde{\theta})).$$

Using the bulk equations of motion (C.3) the gluing conditions have the above form in the free field formalism just with $\beta', \bar{\beta}', p', \bar{p}'$ replaced by $\beta, \bar{\beta}, p, \bar{p}$.

The position of the brane is parameterized by the number $c$. If $c \neq 2$, then we propose the following action

$$S = S_{\text{bulk}} + \frac{k}{2\pi} \frac{1}{2 - c} \int du e^{-\phi}(\theta + e\tilde{\theta})(\partial + \bar{\partial})(\theta + e\tilde{\theta}), \quad (C.8)$$

where $S_{\text{bulk}}$ is (C.1). Variation of the action under the Dirichlet constraint $\tilde{\gamma} - \gamma = ce^{\phi} - e\tilde{\theta}$ at the boundary vanishes, if the above gluing conditions hold and the usual bulk EOMs.
In the free field realization this becomes

\[
S = S_{\text{bulk}} + S_{\text{bdy}},
\]

where the interaction term of the bulk action is as before, but the free term gets partially integrated

\[
S_0 = \frac{1}{2\pi} \int d^2z \left[ \frac{1}{2} \partial\phi \bar{\partial}\phi + \frac{b}{8} \sqrt{g} R\phi - \gamma \partial\beta - \gamma \bar{\partial}\bar{\beta} + \theta \partial\bar{\theta} + \bar{\theta} \partial\theta \right].
\]

The boundary term is

\[
S_{\text{bdy}} = S_{0,\text{bdy}} + S_{\text{int, bdy}},
\]

\[
S_{0,\text{bdy}} = \frac{1}{2\pi} \int du \left[ \Theta(\partial + \bar{\partial}) \Theta \right] + \frac{k}{2 - c} \int du \frac{b}{8} \sqrt{g} K\phi,
\]

\[
S_{\text{int, bdy}} = \frac{1}{2\pi} \int du \left[ c e^{b\phi} \beta + e^{b\phi/2} \Theta(\beta \frac{1}{2}(\theta - \bar{\epsilon}\bar{\theta}) + p) \right].
\]

Here, we introduced the additional fermionic boundary degree of freedom \(\Theta\). Furthermore, we imposed the Dirichlet conditions

\[
\beta = \bar{\beta} \quad \text{and} \quad p = \epsilon\bar{\theta}
\]

at the boundary. Integrating the auxiliary fields \(\beta, \bar{\beta}, p, \bar{p}\) and \(\Theta\) gives the old action. Moreover, the variation of the boundary action vanishes provided our choice of gluing conditions for the currents holds. Note also the extra boundary linear dilaton term coming with the geodesic curvature \(K\) normal to the boundary.

Finding a similar formalism for the fuzzy-spherical branes seems to be very difficult. One of the reasons is that in the first order form one has boundary conditions of the form \(p = \bar{\partial}\bar{\theta}\) which were considered in [49].

\section{Two point functions of boundary operator}

In the presence of boundary, we can insert boundary operators and consider their correlation functions. We assign boundary conditions corresponding to the super AdS\(_2\) branes, then the boundary operators can be defined by \(\Psi_{l,\rho,\rho'}(t, \eta|u) = \mathcal{V}_{l,\rho,\rho'}(t|u) + \eta\mathcal{W}_{l,\rho,\rho'}(t|u)\) as explained in subsection 6.2. In this appendix, we compute the following two types of two point functions. The first type is the two point function of primary operators

\[
\langle \mathcal{V}_{l_1,\rho}^{\rho',\rho}(1|1) \mathcal{V}_{l_2,\rho'}^{\rho',\rho}(0|0) \rangle = \delta(l_1 - l_2) d(l_1; \rho, \rho'),
\]

\[\text{(D.1)}\]
where we set \( \eta_i = 0 \) just for simplicity. The second type is the two point function of descendants, and its form is assumed to be

\[
\left\langle pV_{l_1}^{\rho,\rho'}(1|1)pV_{l_2}^{\rho,\rho}(0|0) \right\rangle = \delta(l_1 - l_2)d'(l_1; \rho, \rho') .
\] (D.2)

The function \( d'(l_1; \rho, \rho') \) could be different from the one for primaries. In order to compute \( d(l_1; \rho, \rho') \) and \( d'(l_1; \rho, \rho') \), we make use of a degenerate operator \( V_{-1/2}^{\rho,\rho'}(t|u) \). Contrary to the case of bulk operator, the boundary operator with \( l = -1/2 \) is not always degenerate and the situation depends on the boundary conditions \( \rho, \rho' \). Here we assume that the operator is degenerate when \( \rho' = \rho \pm i/2 \), and we will see it is indeed the case. See \([50, 14]\) for more details.

### D.1 First type of two point function

First we study the two point function of primary operators \((D.1)\). We obtain a constraint equation for the coefficient \( d(l_1; \rho, \rho') \) by utilizing the three point function with the insertion of a degenerate boundary operator \( V_{-1/2}^{\rho,\rho'} \). The operator product expansion involving the operator is given as

\[
V_{-1/2}^{\rho'',\rho'}(t_1|u_1)V_{1/2}^{\rho',\rho}(t_2|u_2) = |u_{12}|^{2\beta^2}c_+(l)[V_{-1/2}^{\rho'',\rho}(t_2|u_2)]_e
\] (D.3)

\[
+ |u_{12}|^{2\beta^2}c_0(l)[V_{1/2}^{\rho'',\rho}(t_2|u_2)]_o + |u_{12}|^{2\beta^2}c_-(l)t_{12}[V_{1/2}^{\rho'',\rho}(t_2|u_2)]_e.
\]

Taking different limits of the three point function \( \langle V_{-1/2}^{\rho'',\rho'}V_{1/2}^{\rho',\rho}V_{1/2}^{\rho,\rho''} \rangle \), we have a constraint equation

\[
c_-(l)d(l + \frac{1}{2}; \rho'', \rho) = c_+(l + \frac{1}{2})d(l; \rho', \rho) .
\] (D.4)

We can obtain the explicit form of \( d(l; \rho', \rho) \) by solving this equation, however in order to do so we need to know \( c_-(l) \) and \( c_+(l) \) in the above operator product expansion.

In the following we compute \( c_-(l) \), \( c_+(l) \) by utilizing a free field realization of \( \text{OSP}(1|2) \) model developed in appendix \([\square]\). In the free field formulation the interaction terms may be given by\(^{15}\)

\[
S_{\text{int}} = i\lambda \int d^2z \left[ (p + \beta \theta)(\bar{p} - \bar{\beta} \bar{\theta})e^{2\phi} \right] + \lambda_B \int du \left[ \bar{\phi}e^{2\phi/2} \Theta(\beta \theta + p) \right] .
\] (D.5)

Here we omit the terms with \( \beta \bar{\beta} e^{2\phi} \) and \( \beta e^{\phi} \) since they can be generated from the other terms. Treating the interaction terms perturbatively, the boundary conditions are set as

\[
\beta = -\bar{\beta} , \quad \gamma = -\bar{\gamma} , \quad p = \zeta \bar{p} , \quad \theta = \zeta \bar{\theta} .
\] (D.6)

\(^{15}\)Here we change the notation as \( \bar{\gamma} \rightarrow -\bar{\gamma} \) and \( \bar{\beta} \rightarrow -\bar{\beta} \) from the previous one.
With this type of free field realization, vertex operators are usually given in the $m$-basis. However, in our case, the expression in the $x$-basis is suitable, which was actually discussed in [51] for the SL(2) WZNW model. Following their arguments we use the leading part as
\[ \mathcal{V}_l(t|u) = |\gamma - t|^{-2l}e^{b\phi} \]  \hspace{1cm} (D.7)

The overall factor is just our convention. This convention is equivalent to set $c_+(l) = 1$ since we have
\[ \mathcal{V}_{-\frac{1}{2}}(t_1|u_1)\mathcal{V}_{l}(t_2|u_2) \sim |u_{12}|^{2b^2l}|\gamma - t_1|^{-2l+1}e^{b(l-1/2)\phi}(u_2) + \cdots \]  \hspace{1cm} (D.8)

for $t_1 \to t_2, u_1 \to u_2$. See section 4 of [48] for similar calculations.

Similarly we compute $c_-(l)$ using the operator product expansions of free fields
\[ \beta(z)\gamma(w) \sim -\frac{1}{z-w}, \quad p(z)\theta(w) \sim \frac{1}{z-w}, \quad \phi(z, \bar{z})\phi(w, \bar{w}) \sim -\ln |z-w|^2 . \]  \hspace{1cm} (D.9)

Anti-holomorphic part is treated by utilizing the mirror trick and operator product expansions are given in a similar manner. In particular, the scalar field $\phi(u)$ inserted at the boundary has the OPE relation as $\phi(u_1)\phi(u_2) \sim -4\ln |u_{12}|$. For the calculation of $c_-(l)$ we have to include the effect of the interaction terms. Since there are both bulk and boundary interaction terms, we separate the function as $c_-(l) = c_{(v)}(l) + c_{(b)}(l)$, where $c_{(v)}(l)$ and $c_{(b)}(l)$ denote the contributions with bulk and boundary interactions, respectively.

When the bulk interaction term contributes, we have to evaluate
\[ -i\lambda \int d^2z(z + \beta\theta)(\bar{\beta} - \bar{\beta}\theta)e^{b\phi}(z)|\gamma - t_1|e^{-\frac{1}{2}b\phi}(u_1)|\gamma - t_2|^{-2l}e^{b\phi}(u_2) , \]  \hspace{1cm} (D.10)

where the integration is over the upper half plane $\text{Im } z \geq 0$. Applying the Wick contraction of free fields, it reduces to
\[ 2l\lambda\zeta|u_{12}|^{2b^2l}t_{12} \int d^2z \frac{|u_1 - z|^{2b^2}}{|z - u_2|^{4b^2l}|z - \bar{z}|^{b^2+1}} \left( \frac{1}{z - u_2} + \frac{1}{\bar{z} - u_2} \right) |\gamma - t_2|^{-2l-1}e^{b(l+\frac{1}{2})\phi} , \]  \hspace{1cm} (D.11)

if we pick up the terms proportional to $t_{12}$. Non-trivial contribution arises from the $(\beta, \gamma)$-system, which can be computed as
\[ \beta(z)|\gamma(u_1) - t_2 - t_{12}||\gamma(u_2) - t_2|^{-2l} \sim \frac{-2l}{z - u_2}t_{12}|\gamma(u_2) - t_2|^{-2l-1} \]  \hspace{1cm} (D.12)

for the term we want. In the following we may set $u_1 = 1, u_2 = 0$ for simplicity. From the above expression we can see that the bulk contribution $c_{(v)}$ is written in a integral form as
\[ c_{(v)} = 2l\lambda\zeta \int d^2z \frac{|1 - z|^{2b^2}}{|z|^{4b^2l}|z - \bar{z}|^{b^2+1}} \left( \frac{1}{z} + \frac{1}{\bar{z}} \right) . \]  \hspace{1cm} (D.13)
When the boundary interaction terms contribute, we have to compute
\[
\sum_{i,j=1}^{3} \frac{\lambda_{i}^{B} \lambda_{j}^{B}}{2} \int_{C_{i}} dx_{1} \int_{C_{j}} dx_{2} \Theta(p + \beta \theta) e^{b \phi/2}(x_{1}) \Theta(p + \beta \theta) e^{b \phi/2}(x_{2}) \tag{D.14}
\]
\[\times |\gamma - t_{1}| e^{-\frac{b \phi}{2}(1)} |\gamma - t_{2}|^{-2} e^{b \phi(0)}.\]

Now that we inserted boundary operators at \(u_{1} = 1\) and \(u_{2} = 0\), the boundary conditions for the regions \(C_{1} = [-\infty, 0], C_{2} = [0, 1], C_{3} = [0, \infty]\) are different. Boundary conditions are related to the parameter \(\lambda_{B}\) in the boundary interaction term of (D.5), and we use \(\lambda_{i}^{B}\) with \(i = 1, 2, 3\) for the parameter in each region. As in the same way as \(c_{\gamma}^{(v)}\) we obtain
\[
c_{\gamma}^{(b)} = 2l \sum_{i,j=1}^{3} \frac{\lambda_{i}^{B} \lambda_{j}^{B}}{2} \int_{C_{i}} dx_{1} \int_{C_{j}} dx_{2} \frac{|(1 - x_{1})(1 - x_{2})|^{b^{2}}}{|x_{1}x_{2}|^{2b^{2}}|x_{1} - x_{2}|^{b^{2}+1}} \left(\frac{1}{x_{1}} + \frac{1}{x_{2}}\right). \tag{D.15}
\]

As seen above, the coefficient \(c_{\gamma}(l)\) can be written in terms of integral, and fortunately the integrals can be performed explicitly as in [50, 14].

Now the problem is to compute the following form of the integral as
\[
J = \int dx_{1} \int dx_{2} \frac{|(1 - x_{1})(1 - x_{2})|^{b^{2}}}{|x_{1}x_{2}|^{2b^{2}}|x_{1} - x_{2}|^{b^{2}+1}} \left(\frac{1}{x_{1}} + \frac{1}{x_{2}}\right). \tag{D.16}
\]
For the purpose it is convenient to rewrite
\[
J = \frac{1}{2b^{2}l} \partial_{u} I(u)|_{u=0}, \quad I(u) = \int dx_{1} \int dx_{2} \frac{|(1 - x_{1})(1 - x_{2})|^{b^{2}}}{|(x_{1} - u)(x_{2} - u)|^{2b^{2}}|x_{1} - x_{2}|^{b^{2}+1}}. \tag{D.17}
\]

Since the integral \(I(u)\) can be reduced to
\[
I(u) = (1 - u)^{b^{2} - b^{2}l+1} I(0), \tag{D.18}
\]
we obtain a simple formula as
\[
J = \frac{4b^{2}l - b^{2} - 1}{2b^{2}l} I(0). \tag{D.19}
\]

In this way we have observed that the integral \(J\) is related to another integral \(I(0)\) in a simple way. This fact is quite nice since \(I(0)\) have been computed in the last part of section 3 in [14], therefore we can borrow their results. It is also possible to reproduce their results by explicit calculations.

With the above integral formula, we can obtain the explicit expression of \(c_{\gamma}(l)\). The contribution with bulk interaction term is
\[
c_{\gamma}^{(v)} = -\lambda \zeta I_{0} \sin(\pi b^{2}) \sin^{2}(2\pi b^{2}l) \tag{D.20}
\]

52
with
\[ I_0 = 2l \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} - b^2 \right)}{\pi \sin \pi b^2} \Gamma(-2b^2l) \Gamma(2b^2l) \Gamma(\frac{1}{2} - \frac{b^2}{2} + 2b^2l) \Gamma(\frac{1}{2} - \frac{b^2}{2} - 2b^2l). \] (D.21)

The contribution with boundary interaction terms is
\[ c_-(b) = I_0 \left( - (\lambda_B^1)^2 \sin \pi b^2 \frac{\sin \pi b^2}{2} - (\lambda_B^2)^2 \sin \pi b^2 \cos \pi (2b^2l - \frac{b^2}{2}) \right) \]
\[ + (\lambda_B^3)^2 \sin \pi b^2 \cos \pi (2b^2l + \frac{b^2}{2}) \]
\[ + \lambda_B^1 \lambda_B^2 \sin \pi b^2 \cos \pi (2b^2l - \frac{b^2}{2}) \]
\[ - \lambda_B^1 \lambda_B^2 \sin \pi b^2 \cos \pi (2b^2l + \frac{b^2}{2}) \sin \pi b^2 \frac{\sin \pi b^2}{2} \cos \pi (2b^2l - \frac{b^2}{2}) \cos \pi (2b^2l + \frac{b^2}{2}) \).
\] (D.22)

We relate the parameters of boundary condition \( \lambda_B \) and \( \rho \) by the following formula as
\[ (\lambda_B^1)^2 \cos \frac{\pi b^2}{2} = \lambda \sinh \pi b^2 \rho \]
\[ (\lambda_B^3)^2 \cos \frac{\pi b^2}{2} = \lambda \cosh \pi b^2 \rho \] (D.23)
for \( \zeta = +1 \) and \( \zeta = -1 \), respectively. Here we set \( \rho_1 = \rho, \rho_2 = \rho', \rho_3 = \rho'' = \rho' - i/2 \). For \( \zeta = +1 \), we then find
\[ c_-(l) = - 4l \ln \pi b^2 \cos \pi b^2 \pi (i \rho' + i \rho + l) \sin b^2 \pi (i \rho' - i \rho + l) \]
\[ \times \sin b^2 \pi (i \rho' + i \rho + l + \frac{1}{2}) \cos b^2 \pi (i \rho' - i \rho + l + \frac{1}{2}) \],
and for \( \zeta = -1 \)
\[ c_-(l) = - 4l \ln \pi b^2 \cos \pi b^2 \sin b^2 \pi (i \rho' + i \rho + l) \sin b^2 \pi (i \rho' - i \rho + l) \]
\[ \times \cos b^2 \pi (i \rho' + i \rho + l + \frac{1}{2}) \cos b^2 \pi (i \rho' - i \rho + l + \frac{1}{2}) \].

When we set \( \rho'' = \rho' + i/2 \), we again obtain \( c_-(l) \) as above but with \( \rho \rightarrow - \rho, \rho' \rightarrow - \rho' \).

Now that we know \( c_+(l) \) and \( c_-(l) \), we have prepared for solving the constraint equation \[D.4\]. A solution for the coefficient \( d(l; \rho, \rho') \) to the constraint can be written in terms of special functions \( G(x) \) and \( S(x) \). With \( Q = b + 1/b \), they are defined by \[50\]
\[ \ln G(x) = \int_0^\infty \frac{dt}{t} \frac{e^{-Qt/2} - e^{-xt}}{(1 - e^{-bt})(1 - e^{-b/t})} \left[ \frac{(Q/2 - x)^2}{2} e^{-t} + \frac{(Q/2 - x)}{t} \right] , \]
(D.26)
\[ \ln S(x) = \int_0^\infty \frac{dt}{t} \frac{\sinh(Q/2 - x)t}{2 \sinh bt/2 \sinh t/2b} - \frac{(Q - 2x)}{t} \],
(D.27)
and behaves under shifts as
\[ G(x + b) = \frac{b^{1/2-bx}}{\sqrt{2\pi}} \Gamma(bx) G(x) \]
\[ G(x + 1/b) = \frac{b^{x/b-1/2}}{\sqrt{2\pi}} \Gamma(x/b) G(x) \]
(D.28)
\[ S(x + b) = 2 \sin(\pi bx) S(x) \]
\[ S(x + 1/b) = 2 \sin(\pi x/b) S(x) \].
(D.29)

53
It is also useful to define as in [14]

\[
G_{NS}(x) = G(\frac{d}{2})G(\frac{d+Q}{2}) , \quad G_R(x) = G(\frac{d+b}{2})G(\frac{d+b-1}{2}) , \quad \text{(D.30)}
\]

\[
S_{NS}(x) = S(\frac{d}{2})S(\frac{d+Q}{2}) , \quad S_R(x) = S(\frac{d+b}{2})S(\frac{d+b-1}{2}) . \quad \text{(D.31)}
\]

For \( \zeta = +1 \) a solution for the coefficient \( d(l; \rho', \rho) \) is

\[
d(l; \rho', \rho) = \Gamma(1-2l)(\pi \lambda \gamma(\frac{d}{2}(1+b^2))b^{1-b^2})^{-2l(1-\frac{4}{d})}G_{NS}(b-4bl)G_{R}(4bl-b)^{-1} \quad \text{(D.32)}
\]

\[
\times [S_{NS}(2b(l+i\rho+i\rho'))S_{NS}(2b(l-i\rho+i\rho'))S_{NS}(2b(l+i\rho-i\rho'))S_{NS}(2b(l+i\rho-i\rho'))]^{-1} .
\]

For \( \zeta = -1 \) it is given by

\[
d(l; \rho', \rho) = \Gamma(1-2l)(\pi \lambda \gamma(\frac{d}{2}(1+b^2))b^{1-b^2})^{-2l(1-\frac{4}{d})}G_{NS}(b-4bl)G_{R}(4bl-b)^{-1} \quad \text{(D.33)}
\]

\[
\times [S_{NS}(2b(l+i\rho+i\rho'))S_{NS}(2b(l-i\rho+i\rho'))S_{NS}(2b(l-i\rho-i\rho'))S_{NS}(2b(l+i\rho-i\rho'))]^{-1} .
\]

In section 6.2 we compute the spectral density of open string by using the two point functions. In particular, the dependence of boundary condition would be important.

### D.2 Second type of two point function

In order to compute the spectral density of open string, we also need the coefficient \( d'(l; \rho', \rho) \) in (D.32). The function can be computed in a similar way as before. In this case the relevant OPE is

\[
\cal{V}_{-\frac{1}{2}}^{\rho'', \rho'}(t_1|u_1)p\cal{V}_{\frac{1}{2}}^{\rho', \rho}(t_2|u_2) = |u_{12}|^{2b^2} \bar{c}_+(l)[p\cal{V}_{\frac{1}{2}}^{\rho'', \rho}(t_2|u_2)]_o \quad \text{(D.34)}
\]

\[
+ |u_{12}|^{b^2-1}c_0(l)[\cal{V}_{\frac{1}{2}}^{\rho'', \rho}(t_2|u_2)]_e + |u_{12}|^{b^2-2b^2} \bar{c}_-(l)t_{12}[p\cal{V}_{\frac{1}{2}}^{\rho'', \rho}(t_2|u_2)]_o
\]

with \( \rho'' = \rho' - i/2 \). Taking different limits of three point function \( \langle \cal{V}_{-\frac{1}{2}}^{\rho'', \rho'}p\cal{V}_{\frac{1}{2}}^{\rho', \rho}p\cal{V}_{\frac{1}{2}}^{\rho'', \rho''} \rangle \), we have

\[
\bar{c}_-(l)d'(l + \frac{1}{2}; \rho'', \rho) = \bar{c}_+(l + \frac{1}{2})d'(l; \rho', \rho) . \quad \text{(D.35)}
\]

First problem is therefore to find out the explicit forms of \( \bar{c}_+(l) \) and \( \bar{c}_-(l) \). After that we solve the constraint equation in order to obtain \( d'(l; \rho', \rho) \).

As before we express the vertex operator in terms of free fields, and now we adopt

\[
p\cal{V}_{\frac{1}{2}}(t|u) = |\gamma - t|^{-2l}pe^{bl\phi} . \quad \text{(D.36)}
\]

This overall normalization is the same as the requirement \( \bar{c}_+(l) = 1 \) since

\[
\cal{V}_{-\frac{1}{2}}(t_1|u_1)p\cal{V}_{\frac{1}{2}}(t_2|u_2) \sim |u_{12}|^{2b^2}|\gamma - t_2|^{-2l+1}pe^{b(t-1/2)\phi}(u_2) + \ldots \quad \text{(D.37)}
\]
for $t_1 \to t_2$, $u_1 \to u_2$. A difficult part is the computation of $\tilde{c}_-(l)$, which we divide as $\tilde{c}_-(l) = \tilde{c}_-^{(v)}(l) + \tilde{c}_-^{(b)}(l)$ as before. The contribution with the bulk interaction is given by

$$
-i\lambda \int d^2z \rho p + \beta \theta)(\bar{\rho} - \bar{\beta} \theta) e^{b\phi}(z) |\gamma - t_1| e^{-\frac{1}{2}b\phi}(1)p|\gamma - t_2|^{-2l} e^{b\phi}(0),
$$

thus we have

$$
\tilde{c}_-^{(v)} = 2l\lambda \int d^2z \frac{|1 - z|^{2b^2}}{|z|^{[4b^2]}|z - \bar{z}|^{b^2 + 1}} \left( 1 + \frac{(z - \bar{z})^2}{z\bar{z}} \right) \left( \frac{1}{z} + \frac{1}{\bar{z}} \right).
$$

For the boundary contribution we have to calculate

$$
\sum_{i,j=1}^3 \frac{\lambda_i^j \lambda_j^i}{2} \int_{\mathcal{C}_i} \int_{\mathcal{C}_j} dx_1 \int dx_2 \Theta(p + \beta \theta)e^{b/2\phi}(x_1) \Theta(p + \beta \theta)e^{b/2\phi}(x_2) dx_1 dx_2
$$

$$
\times |\gamma - t_1| e^{-\frac{1}{2}b\phi}(1)p|\gamma - t_2|^{-2l} e^{b\phi}(0),
$$

which leads to

$$
\tilde{c}_-^{(b)} = 2l \sum_{i,j=1}^3 \frac{\lambda_i^j \lambda_j^i}{2} \int_{\mathcal{C}_i} \int dx_1 \int dx_2 \frac{|(1 - x_1)(1 - x_2)|^{b^2}}{|x_1 x_2|^{[2b^2]}|x_1 - x_2|^{b^2 + 1}} \left( 1 + \frac{(x_1 - x_2)^2}{x_1 x_2} \right) \left( \frac{1}{x_1} + \frac{1}{x_2} \right).
$$

The integrals can be divided into two parts, and the first terms are the same as before. Therefore we just need to examine the second terms.

Notice that the second terms are of the form

$$
\tilde{J} = \int dx_1 \int dx_2 \frac{|(1 - x_1)(1 - x_2)|^{b^2}}{|x_1 x_2|^{[2b^2]}|x_1 - x_2|^{b^2 + 1}} \left( \frac{1}{x_1} + \frac{1}{x_2} \right).
$$

Therefore we can reduce them into more simpler integrals as

$$
\tilde{J} = \frac{4b^2 l - b^2 - 1}{2b^2 l + 1} \tilde{I}(0), \quad \tilde{I}(0) = \int dx_1 \int dx_2 \frac{|(1 - x_1)(1 - x_2)|^{b^2}}{|x_1 x_2|^{[2b^2]}|x_1 - x_2|^{b^2 + 1}}.
$$

The integrals of $\tilde{I}(0)$ have not been calculated in [14], so we have to do it by ourselves. By combining the first term and the second term in the integral, we obtain the explicit form of the bulk contribution as

$$
\tilde{c}_-^{(v)} = -\lambda \tilde{c}_0 \sin(\pi b^2) \sin^2(2\pi b^2 l)
$$

with

$$
\tilde{c}_0 = 2l \left( 1 - \frac{b^2(1 + b^2)}{(2b^2 l)^2 - 1} \right) \frac{\gamma(\frac{1}{2}(1 + b^2))}{\pi \sin \pi b^2} \Gamma(-2b^2 l) \Gamma(2b^2 l) \Gamma(\frac{1}{2} - \frac{b^2}{2}) \Gamma(\frac{1}{2} - \frac{b^2}{2} - 2b^2 l).
$$
The contribution with the boundary interaction terms is
\[
c_{(b)} = \tilde{I}_0 \left( -\left(\lambda_B^1 \right)^2 \sin \pi b^2 \cos \frac{b^2}{2} - \left(\lambda_B^2 \right)^2 \sin \pi b^2 \cos \pi \left(2b^2 l - \frac{b^2}{2}\right) \right) + \left(\lambda_B^3 \right)^2 \sin \pi b^2 \cos \pi \left(2b^2 l + \frac{b^2}{2}\right) - \lambda_B^1 \lambda_B^2 \sin \pi b^2 \cos \pi \left(2b^2 l - \frac{b^2}{2}\right) + \lambda_B^1 \lambda_B^3 \sin \pi b^2 \cos \pi \left(2b^2 l + \frac{b^2}{2}\right) \right) .
\]

Compared to the previous result, we have opposite signs as \(\lambda_B^1 \lambda_B^2 \rightarrow -\lambda_B^1 \lambda_B^2\) and \(\lambda_B^1 \lambda_B^3 \rightarrow -\lambda_B^1 \lambda_B^3\) along with the different factor in \(\tilde{I}_0\). For \(\zeta = +1\), we then find
\[
c_{-}(l) = - 4\lambda \tilde{I}_0 \sin \pi b^2 \cos b^2 \pi (i\rho' + i\rho + l) \cos b^2 \pi (i\rho' - i\rho + l) \quad (D.46)
\]
and for \(\zeta = -1\)
\[
c_{-}(l) = - 4\lambda \tilde{I}_0 \sin \pi b^2 \cos b^2 \pi (i\rho' + i\rho + l) \cos b^2 \pi (i\rho' - i\rho + l) \quad (D.47)
\]
\[
\times \sin b^2 \pi (i\rho' + i\rho + l + \frac{1}{2}) \sin b^2 \pi (i\rho' - i\rho + l + \frac{1}{2}) .
\]

When we set \(\rho'' = \rho' + i/2\), we again obtain \(c_{-}(l)\) as above but with \(\rho \rightarrow -\rho\), \(\rho' \rightarrow -\rho'\).

Using the explicit form of \(\tilde{c}_{+}(l)\) and \(\tilde{c}_{-}(l)\) we solve the constraint equation \((D.35)\) for \(d''(l; \rho', \rho)\). A solution may be given by
\[
d''(l; \rho', \rho) = g(l) \Gamma(1 - 2l)(\pi \lambda \gamma(\frac{1}{2}(1 + b^2)) b^{-1 - b^2})^{-2(1 - \frac{1}{2})} G_R(b - 4bl) G_R(4bl - b)^{-1}
\]
\[
\times [S_{NS}(2b(l + i\rho + i\rho')) S_R(2b(l - i\rho + i\rho')) S_{NS}(2b(l - i\rho - i\rho')) S_R(2b(l + i\rho - i\rho'))]^{-1}
\]
\[
(D.49)
\]
for \(\zeta = +1\) and
\[
d''(l; \rho', \rho) = g(l) \Gamma(1 - 2l)(\pi \lambda \gamma(\frac{1}{2}(1 + b^2)) b^{-1 - b^2})^{-2(1 - \frac{1}{2})} G_R(b - 4bl) G_R(4bl - b)^{-1}
\]
\[
\times [S_{R}(2b(l + i\rho + i\rho')) S_R(2b(l - i\rho + i\rho')) S_R(2b(l - i\rho - i\rho')) S_R(2b(l + i\rho - i\rho'))]^{-1}
\]
\[
(D.50)
\]
for \(\zeta = -1\). The factor \(g(l)\) depends on the notation, and in the present case it is given by a solution to
\[
\frac{g(l)}{g(l + 1/2)} = 1 - \frac{b^2(1 + b^2)}{(2b^2 l)^2 - 1} .
\]
\[
(D.51)
\]
Thus we may use
\[
g(l) = \frac{\Gamma(2l - b^{-2}) \Gamma(2l + b^{-2})}{\Gamma(2l + \sqrt{1 + b^{-2} + b^{-4}}) \Gamma(2l - \sqrt{1 + b^{-2} + b^{-4}})} .
\]
\[
(D.52)
\]
\(^{16}\)If we compute the boundary contributions naively, then we have the same signs as before. We change the signs according to the prescription for \(N = 1\) super-Liouville field theory in [14, 52].
References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] arXiv:hep-th/9711200.

[2] K. B. Efetov, “Supersymmetry and theory of disordered metals,” Adv. Phys. 32, 53 (1983); “Supersymmetry in disorder and chaos,” Cambridge University Press (1997).

[3] G. Giribet, Y. Hikida and T. Takayanagi, “Topological string on OSP(1|2)/U(1),” arXiv:0907.3832 [hep-th].

[4] T. Takayanagi and N. Toumbas, “A matrix model dual of type 0B string theory in two dimensions,” JHEP 0307, 064 (2003) [arXiv:hep-th/0307083].

[5] M. R. Douglas, I. R. Klebanov, D. Kutasov, J. M. Maldacena, E. J. Martinec and N. Seiberg, “A new hat for the $c = 1$ matrix model,” arXiv:hep-th/0307195.

[6] T. Quella and V. Schomerus, “Free fermion resolution of supergroup WZNW models,” JHEP 0709 (2007) 085 [arXiv:0706.0744 [hep-th]].

[7] J. Teschner, “On structure constants and fusion rules in the SL(2,C)/SU(2) WZNW model,” Nucl. Phys. B 546, 390 (1999) [arXiv:hep-th/9712256].

[8] J. Teschner, “On the Liouville three point function,” Phys. Lett. B 363, 65 (1995) [arXiv:hep-th/9507109].

[9] S. Ribault and J. Teschner, “$H^+_3$ WZNW correlators from Liouville theory,” JHEP 0506, 014 (2005) [arXiv:hep-th/0502048].

[10] Y. Hikida and V. Schomerus, “$H^+_3$ WZNW model from Liouville field theory,” JHEP 0710, 064 (2007) [arXiv:0706.1030 [hep-th]].

[11] Y. Hikida and V. Schomerus, “Structure constants of the OSP(1|2) WZNW model,” JHEP 0712, 100 (2007) [arXiv:0711.0338 [hep-th]].

[12] R. C. Rashkov and M. Stanishkov, “Three-point correlation functions in $\mathcal{N} = 1$ super Liouville theory,” Phys. Lett. B 380, 49 (1996) [arXiv:hep-th/9602148].

[13] R. H. Poghosian, “Structure constants in the $\mathcal{N} = 1$ super-Liouville field theory,” Nucl. Phys. B 496, 451 (1997) [arXiv:hep-th/9607120].

[14] T. Fukuda and K. Hosomichi, “Super Liouville theory with boundary,” Nucl. Phys. B 635, 215 (2002) [arXiv:hep-th/0202032].
[15] T. Creutzig, T. Quella and V. Schomerus, “Branes in the GL(1|1) WZNW-model,” Nucl. Phys. B 792, 257 (2008) [arXiv:0708.0583 [hep-th]].

[16] T. Creutzig and P. B. Ronne, “The GL(1|1)-symplectic fermion correspondence,” Nucl. Phys. B 815 (2009) 95 [arXiv:0812.2835 [hep-th]].

[17] T. Creutzig and V. Schomerus, “Boundary correlators in supergroup WZNW models,” Nucl. Phys. B 807 (2009) 471 [arXiv:0804.3469 [hep-th]].

[18] T. Creutzig, “Branes in supergroups,” arXiv:0908.1816 [hep-th].

[19] K. Hosomichi, “N = 2 Liouville theory with boundary,” JHEP 0612 (2006) 061 [arXiv:hep-th/0408172].

[20] C.-r. Ahn and M. Yamamoto, “Boundary action of N = 2 super-Liouville theory,” Phys. Rev. D 69 (2004) 026007 [arXiv:hep-th/0310046].

[21] N. P. Warner, “Supersymmetry in boundary integrable models,” Nucl. Phys. B 450 (1995) 663 [arXiv:hep-th/9506064].

[22] T. Quella, V. Schomerus and T. Creutzig, “Boundary spectra in superspace sigma-models,” JHEP 0810 (2008) 024 [arXiv:0712.3549 [hep-th]].

[23] T. Creutzig, “Geometry of branes on supergroups,” Nucl. Phys. B 812 (2009) 301 [arXiv:0809.0468 [hep-th]].

[24] J. L. Cardy, “Boundary conditions, fusion rules and the Verlinde formula,” Nucl. Phys. B 324 (1989) 581.

[25] B. Ponsot, V. Schomerus and J. Teschner, “Branes in the Euclidean AdS3,” JHEP 0202, 016 (2002) [arXiv:hep-th/0112198].

[26] A. Y. Alekseev and V. Schomerus, “D-branes in the WZW model,” Phys. Rev. D 60 (1999) 061901 [arXiv:hep-th/9812193].

[27] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, “String theory on AdS3,” JHEP 9812, 026 (1998) [arXiv:hep-th/9812046].

[28] D. Kutasov and N. Seiberg, “More comments on string theory on AdS3,” JHEP 9904, 008 (1999) [arXiv:hep-th/9903219].

[29] L. Alvarez-Gaume and P. Zaugg, “Structure constants in the N = 1 superoperator algebra,” Annals Phys. 215 (1992) 171 [arXiv:hep-th/9109050].

[30] H. Dorn and H. J. Otto, “Two and three point functions in Liouville theory,” Nucl. Phys. B 429, 375 (1994) [arXiv:hep-th/9403141].

58
[31] A. B. Zamolodchikov and A. B. Zamolodchikov, “Structure constants and conformal bootstrap in Liouville field theory,” Nucl. Phys. B 477, 577 (1996) [arXiv:hep-th/9506136].

[32] V. G. Kac and M. Wakimoto, “Modular invariant representations of infinite dimensional Lie algebras and superalgebras,” Proc. Nat. Acad. Sci. 85 (1988) 4956.

[33] I. P. Ennes, A. V. Ramallo and J. M. Sanchez de Santos, “osp(1|2) conformal field theory,” arXiv:hep-th/9708094.

[34] A. Belavin, V. Belavin, A. Neveu and A. Zamolodchikov, “Bootstrap in supersymmetric Liouville field theory. I: NS sector,” Nucl. Phys. B 784, 202 (2007) [arXiv:hep-th/0703084].

[35] V. A. Belavin, “On the $\mathcal{N} = 1$ super Liouville four-point functions,” Nucl. Phys. B 798, 423 (2008) [arXiv:0705.1983 [hep-th]].

[36] A. B. Zamolodchikov and V. A. Fateev, “Operator algebra and correlation functions in the two-dimensional Wess-Zumino SU(2)×SU(2) chiral model,” Sov. J. Nucl. Phys. 43 (1986) 657 [Yad. Fiz. 43 (1986) 1031].

[37] A. Parnachev and D. A. Sahakyan, “Some remarks on D-branes in AdS$_3$,” JHEP 0110, 022 (2001) [arXiv:hep-th/0109150].

[38] P. Lee, H. Ooguri and J. w. Park, “Boundary states for AdS$_2$ branes in AdS$_3$,” Nucl. Phys. B 632, 283 (2002) [arXiv:hep-th/0112188].

[39] B. Ponsot, “Monodromy of solutions of the Knizhnik-Zamolodchikov equation: SL(2,\mathbb{C})/SU(2) WZNW model,” Nucl. Phys. B 642, 114 (2002) [arXiv:hep-th/0204085].

[40] K. Hosomichi and S. Ribault, “Solution of the $H^+_3$ model on a disc,” JHEP 0701, 057 (2007) [arXiv:hep-th/0610117].

[41] V. Fateev and S. Ribault, “Boundary action of the $H^+_3$ model,” JHEP 0802, 024 (2008) [arXiv:0710.2093 [hep-th]].

[42] K. Hori, “Notes on bosonization with boundary,” Prog. Theor. Phys. Suppl. 177, 42 (2009).

[43] T. Creutzig and P. B. Ronne, “From world-sheet supersymmetry to super target spaces,” in preparation.

[44] V. Fateev, A. B. Zamolodchikov and A. B. Zamolodchikov, unpublished.
[45] Y. Hikida and V. Schomerus, “The FZZ-duality conjecture - A proof,” JHEP 0903 (2009) 095 [arXiv:0805.3931 [hep-th]].

[46] T. Creutzig, P. B. Ronne and V. Schomerus, “$\mathcal{N} = 2$ superconformal symmetry in super coset models,” Phys. Rev. D 80 (2009) 066010 [arXiv:0907.3902 [hep-th]].

[47] K. Hori and A. Kapustin, “Duality of the fermionic 2d black hole and $\mathcal{N} = 2$ Liouville theory as mirror symmetry,” JHEP 0108 (2001) 045 [arXiv:hep-th/0104202].

[48] A. Giveon and D. Kutasov, “Notes on AdS$_3$,” Nucl. Phys. B 621, 303 (2002) [arXiv:hep-th/0106004].

[49] T. Creutzig, T. Quella and V. Schomerus, “New boundary conditions for the $c = -2$ ghost system,” Phys. Rev. D 77, 026003 (2008) [arXiv:hep-th/0612040].

[50] V. Fateev, A. B. Zamolodchikov and A. B. Zamolodchikov, “Boundary Liouville field theory. I: Boundary state and boundary two-point function,” [arXiv:hep-th/0001012].

[51] K. Hosomichi, K. Okuyama and Y. Satoh, “Free field approach to string theory on AdS$_3$,” Nucl. Phys. B 598, 451 (2001) [arXiv:hep-th/0009107].

[52] T. Fukuda, “How to approach $\mathcal{N} = 1$ super-Liouville theory,” Ph.D. thesis (in Japanese).