Quasi-projective relation algebras and directed cylindric algebras of any dimension are categorically equivalent

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Abstract. We show that the class of relations with quasi projections $QRA$ and Nemeti’s directed cylindric algebras $CA^\uparrow$ are categorically equivalent. There exists a functor from the former to the latter that is strongly invertible. We also prove that such algebras enjoy the superamalgamation property. Using pairing functions, stimulated by quasi-projection, we formulate and prove a Gödels second incompleteness theorem for finite variable fragments, and we discuss Maddux’s-like representations for $QRA$, extended to $CA^\uparrow$ by Sagi, in connection to forcing in set theory.

1 Quasi-projective relation algebras

The pairing technique due to Alfred Tarski, and substantially generalized by Istvan Nemeti, consists of defining a pair of quasi-projections $p_0$ and $p_1$ so that in a model $\mathcal{M}$ say of a certain sentence $\pi$, where $\pi$ is built out of these quasi-projections, $p_0$ and $p_1$ are functions and for any element $a, b \in \mathcal{M}$, there is a $c$ such that $p_0$ and $p_1$ map $c$ to $a$ and $b$, respectively. We can think of $c$ as representing the ordered pair $(a, b)$ and $p_0$ and $p_1$ are the functions that project the ordered pair onto its first and second coordinates.

Such a technique, ever since introduced by Tarski, to formalize, and indeed successfully so, set theory, in the calculas of relations manifested itself in several re-incarnations in the literature some of which are quite subtle and sophisticated. One is Simon’s proof of the representability of quasi-relation algebras $QRA$ (relation algebras with quasi projections) using a neat embedding theorem for cylindric algebras $[?]$. The proof consists of stimulating a neat embedding theorem via the quasi-projections, in short it is actually a a completeness proof. The idea implemented is that quasi-projections, on the

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one hand, generate extra dimensions, and on the other it has control over such a stretching. The latter property does not come across very much in Simon’s proof, but below we will give an exact rigorous meaning to such property. This method can is used by Simon to apply a Henkin completeness construction. We shall use Simon’s technique to further show that QRA has the superamalgamation property; this is utterly unsurprising because Henkin constructions also prove interpolation theorems. This is the case, e.g. for first order logics and several of its non-trivial extensions arising from the process of algebraising first order logic, by dropping the condition of local finiteness reflecting the fact that formulas contain only finitely many (free) variables. A striking example in this connection is the algebras studied by Sain and Sayed Ahmed [5], [8].

This last condition is unwarranted from the algebraic point of view, because it presents an equational formalism of first order logic.

The view, of capturing extra dimensions, using also quasi-projections comes along also very much so, in Németis directed cylindric algebras (introduced as a CA counterpart of QRA). In those, Sági defined quasi-projections also to achieve a completeness theorem for higher order logics. The technique used is similar to Maddux’s proof of representation of QRA, which further emphasizes the correlation. We start with making the notion of extra dimensions explicit. We formulate its dual notion, that of compressing dimensions, known as taking neat reducts. The definition of neat reducts in the standard definition adopted by Henkin, Monk and Tarski in their monograph, deals only with the latter case, but it proves useful to stretch the definition a little allowing arbitrary subsets of \( \alpha \) not just initial segments.

**Definition 1.1.** Let \( C \in \mathbf{CA}_\alpha \) and \( I \subseteq \alpha \), and let \( \beta \) be the order type of \( I \). Then

\[
Nr_I C = \{ x \in C : c_i x = x \text{ for all } i \in \alpha \sim I \},
\]

\[
\mathfrak{N}r_I C = (Nr_I C, +, \cdot, 0, 1, c, d)_{i,j<\beta},
\]

where \( \beta \) is the unique order preserving one-to-one map from \( \beta \) onto \( I \), and all the operations are the restrictions of the corresponding operations on \( C \). When \( I = \{i_0, \ldots, i_{k-1}\} \) we write \( \mathfrak{N}r_{i_0, \ldots, i_{k-1}} C \). If \( I \) is an initial segment of \( \alpha \), \( \beta \) say, we write \( \mathfrak{N}r_\beta C \).

Similar to taking the \( n \) neat reduct of a CA, \( \mathfrak{A} \) in a higher dimension, is taking its RA reduct, its relation algebra reduct. This has universe consisting of the 2 dimensional elements of \( \mathfrak{A} \), and composition and converse are defined using one spare dimension. A slight generalization, modulo a reshuffling of the indices:

**Definition 1.2.** For \( n \geq 3 \), the relation algebra reduct of \( C \in \mathbf{CA}_n \) is the algebra

\[
\mathfrak{R}aC = (Nr_{n-2,n-1} C, +, \cdot, 1, \cdot, 1').
\]
where \( 1' = d_{n-2,n-1}, \bar{x} = s_{n-1}^{0}s_{n-2}^{n-1}x \) and \( x;y = c_0(s_0^{n-1}.s_0^{n-2}y) \). Here \( s_i^j(x) = c_i(x \cdot d_{ij}) \) when \( i \neq q \) and \( s_i^j(x) = x \).

But what is not obvious at all is that an RA has a CA_\( n \) reduct for \( n \geq 3 \). But Simon showed that certain relations algebras do; namely the QRAs.

**Definition 1.3.** A relation algebra \( \mathfrak{B} \) is a QRA if there are elements \( p,q \) in \( \mathfrak{B} \) satisfying the following equations:

1. \( \bar{p};p \leq 1',q; q \leq 1 \);
2. \( \bar{p}; q = 1 \).

In this case we say that \( \mathfrak{B} \) is a QRA with quasi-projections \( p \) and \( q \). To construct cylindric algebras of higher dimensions 'sitting' in a QRA, we need to define certain terms. seemingly rather complicated, their intuitive meaning is not so hard to grasp.

**Definition 1.4.** Let \( x \in \mathfrak{B} \in RA \), then , we need \( \text{dom}(x) = 1'; (x;\bar{x}) \) and \( \text{ran}(x) = 1'; (\bar{x}; x), x^0 = 1', x^{n+1} = x^n; x, x \)

is a functional element if \( x; \bar{x} \leq 1' \).

Given a QRA, which we denote by \( Q \), we have quasi-projections \( p \) and \( q \) as mentioned above. Next we define certain terms in \( Q \), cf. [7]:

\[
\epsilon^n = \text{dom} q^{n-1},
\]
\[
\pi_i^n = \epsilon^n; q^i; p, i < n - 1, \pi_{n-1}^{(n)} = q^{n-1},
\]
\[
\xi^{(n)} = \pi_i^{(n)}; \pi_j^{(n)},
\]
\[
t_i^{(n)} = \prod_{i \neq j < n} \xi_j^{(n)}, t^{(n)} = \prod_{j < n} \xi_j^{(n)},
\]
\[
c_i^{(n)} x = x; t_i^{(n)},
\]
\[
d_{ij}^{(n)} = 1; (\pi_i^{(n)} \cdot \pi_j^{(n)}),
\]
\[
1^{(n)} = 1; \epsilon^{(n)}.
\]

and let

\[
\mathfrak{B}_n = (B_n, +, \cdot, -, 0, 1^{(n)}, c_i^{(n)}, d_{ij}^{(n)})_{i,j < n},
\]

where \( B_n = \{ x \in B : x = 1; x; t^{(n)} \} \). The intuitive meaning of those terms is explained in [7], right after their definition on p. 271.

**Theorem 1.5.** Let \( n > 1 \)

1. Then \( \mathfrak{B}_n \) is closed under the operations.
2. \( \mathcal{B}_n \) is a CA\(_n\).

**Proof.** (1) is proved in [7] lemma 3.4 p.273-275 where the terms are definable in a QRA. That it is a CA\(_n\) can be proved as [7] theorem 3.9.

**Definition 1.6.** Consider the following terms.

\[
suc(x) = 1; (\bar{p}; x; \bar{q})
\]

and

\[
pred(x) = \bar{p}; \text{ran} x; q.
\]

It is proved in [7] that \( \mathcal{B}_n \) neatly embeds into \( \mathcal{B}_{n+1} \) via \( \text{succ} \). The successor function thus codes extra dimensions. The thing to observe here is that we will see that \( \text{pred} \); its inverse; guarantees a condition of commutativity of two operations: forming neat reducts and forming subalgebras; it does not make a difference which operation we implement first, as long as we implement both one after the other. So the function \( \text{succ} \) captures the extra dimensions added.

From the point of view of definability it says that terms definable in extra dimensions add nothing, they are already term definable. And this indeed is a definability condition, that will eventually lead to strong interpolation property we want.

**Theorem 1.7.** Let \( n \geq 3 \). Then \( \text{succ} : \mathcal{B}_n \to \{a \in \mathcal{B}_{n+1} : c_0 a = a\} \) is an isomorphism into a generalized neat reduct of \( \mathcal{B}_{n+1} \). Strengthening the condition of surjectivity, for all \( X \subseteq \mathcal{B}_n, n \geq 3 \), we have (*)

\[
\text{succ}(\mathcal{G} \mathcal{B}_n X) \cong \mathcal{N}_r 1, 2, ..., n \mathcal{G} \mathcal{B}_{n+1} \text{succ}(X).
\]

**Proof.** The operations are respected by [7] theorem 5.1. The last condition follows because of the presence of the functional element \( \text{pred} \), since we have \( \text{suc}(\text{pred} x) = x \) and \( \text{pred}(\text{suc} x) = x \), when \( c_0 x = x \), [7] lemmas 4.6-4.10.

**Theorem 1.8.** Let \( n \geq 3 \). Let \( \mathcal{C}_n \) be the algebra obtained from \( \mathcal{B}_n \) by reshuffling the indices as follows; set \( c_0^n = c_0^n \mathcal{B}_n \) and \( c^n_n = c_0^n \mathcal{B}_n \). Then \( \mathcal{C}_n \) is a cylindric algebra, and \( \text{suc} : \mathcal{C}_n \to \mathcal{N}_r \mathcal{C}_n, \mathcal{C}_{n+1} \) is an isomorphism for all \( n \). Furthermore, for all \( X \subseteq \mathcal{C}_n \) we have

\[
\text{suc}(\mathcal{G} \mathcal{C}_n X) \cong \mathcal{N}_r \mathcal{G} \mathcal{C}_{n+1} \text{suc}(X).
\]

**Proof.** immediate from [LM]

**Theorem 1.9.** Let \( \mathcal{C}_n \) be as above. Then \( \text{suc}^m : \mathcal{C}_n \to \mathcal{N}_r \mathcal{C}_m \) is an isomorphism, such that for all \( X \subseteq A \), we have

\[
\text{suc}^m(\mathcal{G} \mathcal{C}_n X) = \mathcal{N}_r \mathcal{G} \mathcal{C}_m \text{suc}^{n-1}(X).
\]
Proof. By induction on $n$. \hfill \Box

Now we want to neatly embed our $QRA$ in $\omega$ extra dimensions. At the same we do not want to lose, our control over the stretching; we still need the commutativity of taking, now $Ra$ reducts with forming subalgebras; we call this property the $RaS$ property. To construct the big $\omega$ dimensional algebra, we use a standard ultraproduct construction. So here we go. For $n \geq 3$, let $C^+_n$ be an algebra obtained by adding $c_i$ and $d_{ij}$'s for $\omega > i, j \geq n$ arbitrariness and with $Rd^+_n C^+_n = B_n$. Let $C = \prod_{n \geq 3} C^+_n / G$, where $G$ is a non-principal ultrafilter on $\omega$. In our next theorem, we show that the algebra $A$ can be neatly embedded in a locally finite algebra $\omega$ dimensional algebra and we retain our $RaS$ property.

Theorem 1.10. Let

$$i : A \rightarrow RaC$$

be defined by

$$x \mapsto (x, \text{suc}(x), \ldots \text{suc}^{n-1}(x), \ldots n \geq 3, x \in B_n)/G.$$ 

Then $i$ is an embedding, and for any $X \subseteq A$, we have

$$i(\text{Sg}^n X) = Ra\text{Sg}^n i(X).$$

Proof. The idea is that if this does not happen, then it will not happen in a finite reduct, and this impossible [8]. \hfill \Box

Theorem 1.11. Let $Q \in RA$. Then for all $n \geq 4$, there exists a unique $A \in S\text{Mr}_3 C A_n$ such that $Q = RaA$, such that for all $X \subseteq A$, $\text{Sg}^Q X = Ra\text{Sg}^n X$.

Proof. This follows from the previous theorem together with $RaS$ property. \hfill \Box

Corollary 1.12. Assume that $Q = RaA \cong RaB$ then this lifts to an isomorphism from $A$ to $B$.

The previous theorem says that $Ra$ as a functor establishes an equivalence between $QRA$ and a reflective subcategory of $Lf_\omega$. We say that $A$ is the $\omega$ dilation of $Q$. Now we are ready for:

Theorem 1.13. $QRA$ has $SUPAP$.

Proof. We form the unique dilatons of the given algebras required to be superamalgamated. These are locally finite so we can find a superamalgam $D$. Then $RaD$ will be required superamalgam; it contains quasiports because the base algebras does. Let $A, B \in QRA$. Let $f : C \rightarrow A$ and
Let $g : \mathcal{C} \to \mathcal{B}$ be injective homomorphisms. Then there exist $\mathfrak{A}^+, \mathfrak{B}^+, \mathcal{C}^+ \in \text{CA}_{\omega}^+, e_A : \mathfrak{A} \to \text{RaA}^+, e_B : \mathfrak{B} \to \text{RaB}^+$ and $e_C : \mathcal{C} \to \text{RaC}^+$. We can assume, without loss, that $\mathcal{G}_\mathfrak{A}^+ e_A(A) = \mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathcal{C}^+$. Let $f(C)^+ = \mathcal{G}_\mathcal{C}^+ e_C(f(C))$ and $g(C)^+ = \mathcal{G}_\mathcal{B}^+ e_B(g(C))$. Since $\mathcal{C}$ has $\text{UNEP}$, there exist $\tilde{f} : \mathcal{C}^+ \to f(C)^+$ and $\tilde{g} : \mathcal{C}^+ \to g(C)^+$ such that $(e_A \restriction f(C)) \circ f = \tilde{f} \circ e_C$ and $(e_B \restriction g(C)) \circ g = \tilde{g} \circ e_C$. Both $\tilde{f}$ and $\tilde{g}$ are monomorphisms. Now $Lf_\omega$ has $\text{SUPAP}$, hence there is an $\mathfrak{D}^+$ in $K$ and $k : \mathfrak{A}^+ \to \mathfrak{D}^+$ and $h : \mathfrak{B}^+ \to \mathfrak{D}^+$ such that $k \circ \tilde{f} = h \circ \tilde{g}$. $k$ and $h$ are also monomorphisms. Then $k \circ e_A : \mathfrak{A} \to \text{RaD}^+$ and $h \circ e_B : \mathfrak{B} \to \text{RaD}^+$ are one to one and $k \circ e_A \circ f = h \circ e_B \circ g$. Let $\mathfrak{D} = \text{RaD}^+$. Then we obtained $\mathfrak{D} \in \text{QRA}$ and $m : \mathfrak{A} \to \mathfrak{D}$, $n : \mathfrak{B} \to \mathfrak{D}$ such that $m \circ f = n \circ g$. Hence, it follows that $\mathfrak{A}^+ = \mathcal{G}_\mathfrak{A}^+ e_C(C) = e_C(C)$. So, there exists $t \in C$ with $z' = e_C(t)$. Then we get $e_A(a) \leq \tilde{f}(e_C(t))$ and $\tilde{g}(e_C(t)) \leq e_B(b)$. It follows that $e_A(a) \leq \tilde{f}(z')$ and $\tilde{g}(z') \leq e_B(b)$. Now by hypothesis

One can prove the theorem using the dimension restricted free algebra $B = \mathcal{F}_\mathfrak{A}^+ \text{CA}_{\omega}$, where $\rho(0) = 2$. This corresponds to a countable first order language with a sequence of variables of order type $\omega$ and one binary relation. The idea is that $\mathcal{F}_\mathfrak{A}^+ \text{QRA} \cong \text{RaA}^+ \text{CA}_{\omega}$. So let $a, b \in \mathcal{F}_\mathfrak{A}^+ \text{QRA}$ be such that $a \leq b$. Then there exists $y \in \mathcal{G}_\mathcal{B}^+ \{x\}$ were $x$ is the free generator of both, such that $a \leq y \leq b$.

But we need to show that pairing functions can be defined in $\text{RaA}^+ \mathcal{F}_\mathfrak{A}^+ \text{CA}_{\omega}$. We have one binary relation $E$ in our language; for convenience, we write $x \in y$ instead of $E(x, y)$, to remind ourselves that we are actually working in the language of set theory. We define certain formulas culminating in formulating the axioms of a finitely undecidable theory, better known as Robinson’s arithmetic in our language. These formulas are taken from Németi [??]. (This is not the only way to define quasi-projections) We need to define, the quasi projections. Quoting Andréka and Németi in [??], we do this by ’brute force’.

$$x = \{y\} =: y \in x \land (\forall z)(z \in x \implies z = y)$$

$$\{x\} \in y =: \exists z(z = \{x\} \land z \in y)$$
Now we define the pairing functions:

\[ p_0(x, y) =: \text{pair}(x) \land \{y\} \in x \]

\[ p_1(x, y) =: \text{pair}(x) \land [x = \{y\} \lor (\{y\} \notin x \land y \in \cup x \lor \exists y(y \in z)] \]

\( p_0(x, y) \) and \( p_1(x, y) \) are defined.

## 2 Pairing functions in Németis directed CAs

We recall the definition of what is called weakly higher order cylindric algebras, or directed cylindric algebras invented by Németi and further studied by Sági and Simon. Weakly higher order cylindric algebras are natural expansions of cylindric algebras. They have extra operations that correspond to a certain kind of bounded existential quantification along a binary relation \( R \). The relation \( R \) is best thought of as the ‘element of relation’ in a model of some set theory. It is an abstraction of the membership relation. These cylindric-like algebras are the cylindric counterpart of quasi-projective relation algebras, introduced by Tarski. These algebras were studied by many authors including Andréka, Givant, Németi, Maddux, Sági, Simon, and others. The reference [7] is recommended for other references in the topic. It also has reincarnations in Computer Science literature under the name of Fork algebras. We start by recalling the concrete versions of directed cylindric algebras:

**Definition 2.1.** (P–structures and extensional structures.)

Let \( U \) be a set and let \( R \) be a binary relation on \( U \). The structure \( \langle U; R \rangle \) is defined to be a \( P \)-structure \(^2\) iff for every elements \( a, b \in U \) there exists an element \( c \in U \) such that \( R(d, c) \) is equivalent with \( d = a \) or \( d = b \) (where \( d \in U \) is arbitrary), that is,

\[ \langle U; R \rangle \models (\forall x, y)(\exists z)(\forall w)(R(w, z) \iff (w = x \lor w = y)). \]

The structure \( \langle U; R \rangle \) is defined to be a weak \( P \)-structure iff

\[ \langle U; R \rangle \models (\forall x, y)(\exists z)(R(x, z) \land R(y, z)). \]

\(^2\)"P" stands for “pairing” or “pairable".
The structure \( \langle U; R \rangle \) is defined to be extensional iff every two points \( a, b \in U \) coincide whenever they have the same “\( R \)-children”, that is,

\[
\langle U; R \rangle \models (\forall x, y)(((\forall z)R(z, x) \iff R(z, y)) \Rightarrow x = y).
\]

We will see that if \( \langle U; R \rangle \) is a P–structure then one can “code” pairs of elements of \( U \) by a single element of \( U \) and whenever \( \langle U; R \rangle \) is extensional then this coding is “unique”. In fact, in \( \text{RCA}_3^\uparrow \) (see the definition below) one can define terms similar to quasi–projections and, as with the class of QRA’s, one can equivalently formalize many theories of first order logic as equationa l theories of certain \( \text{RCA}_3^\uparrow \)’s. Therefore \( \text{RCA}_3^\uparrow \) is in our main interest. \( \text{RCA}_\alpha^\uparrow \) for bigger \( \alpha \)’s behave in the same way, an explanation of this can be found in [6] and can be deduced from our proof, which shows that \( \text{RCA}_3^\uparrow \) has implicitly \( \omega \) extra dimensions.

**Definition 2.2.** (\( \text{Cs}_\alpha^\uparrow, \text{RCA}_\alpha^\uparrow \))

Let \( \alpha \) be an ordinal. Let \( U \) be a set and let \( R \) be a binary relation on \( U \) such that \( \langle U; R \rangle \) is a weak P–structure. Then the full w–directed cylindric set algebra of dimension \( \alpha \) with base structure \( \langle U; R \rangle \) is the algebra:

\[
\langle \mathcal{P}(^\alpha U); \cap, -, C_i^{\uparrow(R)}, C_i^{\downarrow(R)}, D_{i,j}^U \rangle_{i,j \in \alpha},
\]

where \( \cap \) and \( - \) are set theoretical intersection and complementation (w.r.t. \( ^\alpha U \)), respectively, \( D_{i,j}^U = \{ s \in ^\alpha U : s_i = s_j \} \) and \( C_i^{\uparrow(R)}, C_i^{\downarrow(R)} \) are defined as follows. For every \( X \in \mathcal{P}(^\alpha U) \):

\[
C_i^{\uparrow(R)}(X) = \{ s \in ^\alpha U : (\exists z \in X)(R(z_i, s_i) \land (\forall j \in \alpha)(j \neq i \Rightarrow s_j = z_j)) \},
\]

\[
C_i^{\downarrow(R)}(X) = \{ s \in ^\alpha U : (\exists z \in X)(R(s_i, z_i) \land (\forall j \in \alpha)(j \neq i \Rightarrow s_j = z_j)) \}.
\]

The class of w–directed cylindric set algebras of dimension \( \alpha \) and the class of directed cylindric set algebras of dimension \( \alpha \) are defined as follows.

\[
w - \text{Cs}_\alpha^\uparrow = S\{ \mathcal{A} : \mathcal{A} \text{ is a full w–directed cylindric set algebra of dimension } \alpha \text{ with base structure } \langle U; R \rangle, \text{ for some weak P–structure } \langle U; R \rangle \}\}
\]

\[
\text{Cs}_\alpha^\uparrow = S\{ \mathcal{A} : \mathcal{A} \text{ is a full w–directed cylindric set algebra of dimension } \alpha \text{ with base structure } \langle U; R \rangle, \text{ for some extensional P–structure } \langle U; R \rangle \}\}
\]

The class \( \text{RCA}_\alpha^\uparrow \) of representable directed cylindric algebras of dimension \( \alpha \) is defined to be \( \text{RCA}_\alpha^\uparrow = \text{SPCs}_\alpha^\uparrow \).

The main result of Sagi in [6] is a direct proof for the following:
**Theorem 2.3.** \( \text{RCA}_\alpha^\uparrow \) is a finitely axiomatizable variety whenever \( \alpha \geq 3 \) and \( \alpha \) is finite

\( \text{CA}_3^\uparrow \) denotes the variety of directed cylindric algebras of dimension 3 as defined in \([6]\) definition 3.9. In \([6]\), it is proved that \( \text{CA}_3^\uparrow = \text{RCA}_3^\uparrow \). A set of axioms is formulated on p. 868 in \([6]\). Let \( \mathfrak{A} \in \text{CA}_3^\uparrow \). Then we have quasi-projections \( p, q \) defined on \( \mathfrak{A} \) as defined in \([6]\) p. 878, 879. We recall their definition, which is a little bit complicated because they are defined as formulas in the corresponding second order logic. Let \( \mathcal{L} \) denote the untyped logic corresponding to directed \( \text{CA}_3 \)'s as defined p. 876-877 in \([6]\). It has only 3 variables. There is a correspondence between formulas (or formal schemes) in this language and \( \text{CA}_3^\uparrow \) terms. This is completely analogous to the correspondence between \( \text{RCA}_n \) terms and first order formulas containing only \( n \) variables. For example \( v_i = v_j \) corresponds to \( d_{ij}, \exists^i v_i (v_i = v_j) \) correspond to \( c_i \delta_{ij} \). In \([6]\) the following formulas (terms) are defined:

**Definition 2.4.** Let \( i, j, k \in 3 \) distinct elements. We define variable-free \( \text{RCA}_3^\uparrow \) terms as follows:

\[
\begin{align*}
v_i \in_R v_j & \quad \text{is} \quad \exists^i v_j (v_i = v_j), \\
v_i = \{v_j\}_R & \quad \text{is} \quad \forall v_k (v_k \in_R v_j \iff v_k = v_j), \\
\{v_i\}_R \in_R v_j & \quad \text{is} \quad \exists v_k (v_k \in_R v_j \land v_k = \{v_i\}_R), \\
v_i = \{\{v_j\}_R\}_R & \quad \text{is} \quad \exists v_k (v_k = \{v_j\}_R \land v_i = \{v_k\}_R), \\
v_i \in_R \cup v_j & \quad \text{is} \quad \exists v_k (v_i \in_R v_k \land v_k \in_R v_j).
\end{align*}
\]

Therefore pair \( (a \text{ pairing function}) \) can be defined as follows:

\[
\begin{align*}
\exists v_j \forall v_k (\{v_k\}_R \in_R v_i \iff v_j = v_k) & \land \\
\forall v_j \exists v_k (v_j \in_R v_i \Rightarrow v_k \in_R v_j) & \land \\
\forall v_j \forall v_k (v_j \in R \cup v_i \land \{v_j\} \not\subseteq R v_i \land v_k \in R \cup v_i \land \{v_k\} \not\subseteq R v_i \Rightarrow v_j = v_k).
\end{align*}
\]

It is clear that this is a term built up of diagonal elements and directed cylindrifications. The first quasi-projection \( v_i = P(v_j) \) can be chosen as:

\[
\text{pair}_j \land \forall^i v_j \exists^i v_j (v_i = v_j).
\]

and the second quasiprojection \( v_i = Q(v_j) \) can be chosen as:

\[
\text{pair}_j \land ((\forall v_i \forall v_k (v_i \in_R v_j \land v_k \in_R v_j \Rightarrow v_i = v_k)) \Rightarrow v_i = P(v_j)) \land \\
(\exists v_i \exists v_k (v_i \in_R v_j \land v_k \in_R v_j \land v_i \neq v_k) \Rightarrow (v_i \neq P(v_j) \land \exists^i v_j \exists^i v_j (v_i = v_j))).
\]

**Theorem 2.5.** Let \( \mathfrak{B} \) be the relation algebra reduct of \( \mathfrak{A} \); then \( \mathfrak{B} \) is a relation algebra, and the variable free terms corresponding to the formulas \( v_i = P(v_j) \) and \( v_j = Q(v_j) \) call them \( p \) and \( q \), respectively, are quasi-projections.
Proof. One proof is very tedious, though routine. One translates the functions as variable free terms in the language of $\text{CA}_3$ and use the definition of composition and converse in the $\text{RA}$ reduct, to verify that they are quasi-projections. Else one can look at their meanings on set algebras, which we recall from Sagi \[6\]. Given a cylindric set algebra $\mathcal{A}$ with base $U$ and accessibility relation $R$

\[(v_i = P(v_j)^{\mathcal{A}} = \{s \in 3^U : (\exists a, b \in U)(s_j = (a, b)_R, s_i = a)\}\]

\[(v_i = Q(v_j)^{\mathcal{A}} = \{s \in 3^U : (\exists a, b \in U)(s_j = (a, b)_R, s_i = b)\}.\]

First $P$ and $Q$ are functions, so they are functional elements. Then it is clear that in this set algebras that $P$ and $Q$ are quasi-projections. Since $\text{RCA}_3^\uparrow$ is the variety generated by set algebras, they have the same meaning in the class $\text{CA}_3^\uparrow$.

Now we can turn the class around. Given a $\text{QRA}$ one can define a directed $\text{CA}_n$, for every finite $n \geq 2$. This definition is given by Németi and Simon in \[4\]. It is very similar to Simon’s definition above (defining $\text{CA}$ reducts in a $\text{QRA}$, except that directed cylindrifiers along a relation $R$ are implemented.

**Theorem 2.6.** The concrete category $\text{QRA}$ with morphisms injective homomorphisms, and that of $\text{CA}^\uparrow$ with morphisms also injective homomorphisms are equivalent. in particular $\text{CA}^\uparrow$ of dimension 3 is equivalent to $\text{CA}^\uparrow$ for $n \geq 3$.

**Proof.** Given $\mathfrak{A}$ in $\text{QRA}$ we can associate a directed $\text{CA}_3$, homomorphism are restrictions and vice versa; these are inverse Functors. However, when we pass from an $\text{QRA}$ to a $\text{CA}^\uparrow$ and then take the $\text{QRA}$ reduct, we may not get back exactly to the $\text{QRA}$ we started off with, but the new quasi projections are definable from the old ones. Via this equivalence, we readily conclude that $\text{RCA}_3 \rightarrow \text{RCA}_n$ are also equivalent.

**Corollary 2.7.** The class $\text{CA}^\uparrow$ has the super amalgamation property.

**Proof.** The functor from $\text{QRA}$ to $\text{CA}^\uparrow$ preserves order.

**3 Godel’s first for finite variable fragments**

This section is a summary of work of Németi \[3\], reported in \[1\]. There has been some debate over the impact of Gödel’s incompleteness theorems on Hilbert’s Program, and whether it was the first or the second incompleteness theorem that delivered the coup de grace.

Undoubtedly the opinion of those most directly involved in the developments were convinced that the theorems did have a decisive impact.
Gödel announced the second incompleteness theorem in an abstract published in October 1930: no consistency proof of systems such as Principia, Zermelo-Fraenkel set theory, or the systems investigated by Ackermann and von Neumann is possible by methods which can be formulated in these systems.

Gödel’s theorems have a profound impact Hilbert’s program. Through a careful Gödel coding of sequences of symbols (formulas, proofs), Gödel showed that in theories $T$ which contain a sufficient amount of arithmetic, it is possible to produce a formula $Pr(x, y)$ which "says" that $x$ is (the code of) a proof of (the formula with code) $y$. Specifically, if $0 = 1$ is the code of the formula $0 = 1$, then $ConT = \forall(x\neg Pr(x, 0 = 1))$ may be taken to "say" that $T$ is consistent (no number is the code of a derivation in $T$ of $0 = 1$). The second incompleteness theorem ($G_2$) says that under certain assumptions about $T$ and the coding apparatus, $T$ does not prove $ConT$.

This shattered Hilbert’s hopes of proving that set theory is consistent, by finitary means, presumably formalizable in set theory (it is hard to visualize 'finitary means" that is not formalizable in set theory, or even Peano arithmetic). This means that mathematicians will be always threatened that one day, some mathematician, or rather set-theoretician, will find an inconsistency. Nevertheless, with the amount of research done in set theory, in the last decades, deems this possibility as far fetched, and some mathematicians go as far as to say impossible. This is a fair view, if there were a consistency we would have probably stumbled upon it by now.

In the above cited results, the ideas are not too difficult, but implementing the details is highly technical and complicated. Németi generalized Gödel’s first theorem as follows:

**Theorem 3.1.**  
(1) There is a computable, structural translation $tr : L_\omega \rightarrow L_3(E, 2)$ such that $tr$ has a recursive image and the following are true for all sets of sentences $Th \cup \{\phi\}$ in $L_\omega$

- $(a) Th \models \phi \iff tr(Th) \vdash_n tr(\phi)$.
- $(b) Th \models \phi \iff tr(Th) \models tr(\phi)$.

(2) There is a computable, structural translation function $tr : L_\omega(E, 2) \rightarrow L(E, 2)$ such that $tr$ has a recursive range and the following (c) and (d) are true

- $(c)$ Statements (a) and (b) above hold and $Th \models \neg tr(\bot)$. Furthermore, $ZF \models \neg tr(\bot)$.
- $(d)$ $\neg tr(\bot) \models \phi \iff tr(\phi)$

Using this translation map he proves:

**Theorem 3.2.** There is a formula $\psi \in L_3$ such that no consistent recursive extension $T$ of $\psi$ is complete, and moreover, no recursive extension of $\psi$ separates the $\vdash$ consequences of $\psi$ from the $\psi$ refutable sentences.
Proof. We give a sketch of proof for $L_4$. This is implicit in the Tarski Givant approach, when they interpreted $ZF$ in $RA$. $L_4$ is very close to $RA$ but not quite $RA$, it is a little bit stronger. The technique is called the *pairing* technique, which uses quasi projections to code extra variable, establishing the completeness theorem above for $\vdash_n$.

We have one binary relation $E$ in our language; for convenience, we write $x \in y$ instead of $E(x, y)$, to remind ourselves that we are actually working in the language of set theory. We define certain formulas culminating in formulating the axioms of a finite undecidable theory, better known as Robinson’s arithmetic in our language. These formulas are taken from Németi. We need to define the quasi projections. Quoting Andréka and Németi, we do this by ‘brute force’. We now formulate the desired $\lambda$.

Having defined the pairs, we go on as follows:

$x \in Ord =: "x$ is an ordinal, i.e. $x$ is transitive and $\in$ is a total ordering on $x$,

\[ x \in Ford =: x \in Ord \land "every element of $x$ is a successor ordinal " \]

i.e. $x$ is a finite ordinal .

\[ x = 0 =: "x$ has no element " \]

\[ sx = z =: z = x \cup \{x\}, \]

\[ x \leq y =: x \subseteq y, \]

\[ x < y =: x \leq y \land x \neq y, \]

\[ x + y = z =: \exists v(z = x \cup v \land x \cap v = 0 \land "there exists a bijection between $v$ and $y" ") \]

\[ x \cdot y = z =: "there is a bijection between $z$ and $x \times y" ") \]

\[ xexp_y = z : \text{ there is a bijection between } z \text{ and the set of all functions from } y \text{ to } x" \]

Now $\lambda$ is the formula saying that: 0, $s$, $+$, $\cdot$, $exp$ are functions of arities 0, 1, 2, 2, 2 on $Ford$ and

\[ (\forall xy \in Ford)(sx \neq 0 \land sx = sy \rightarrow x = y) \land (x < sy \iff x \leq y) \land \]

\[-(x < 0) \land (x < y \lor x = y \lor y < x) \land (x + 0 = x) \land (x + sy = s(x + y)) \land (x.0 = 0) \land (x \cdot sy = x \cdot y + x) \land (xexp0 = s0) \land (xexp sy = xexp y \cdot x). \]

Now the existence of the desired incompletable $\lambda$ readily follows: $\lambda \in Fm_0^\omega$. Let $p = r(p_0(x, y))$ and $q = r(p_1(x, y))$ be the pairing functions as defined
above, where \( r \) be the recursive function mapping \( Fm_3^2 \) into \( RAT \). (It is not hard to construct such an function, that also preserves meaning).

\[
\pi_{RA} = (p^\lor; p \to \text{Id}) \cdot (q^\lor; q \to \text{Id}) \cdot (p^\lor; q).
\]

Then \( \pi_{RA} \in RAT \) since \( p_i(x, y) \in Fm_3^2 \). Let \( \lambda \in Fm_\omega^0 \) be inseparable and let \( \eta = (r(\text{tr}(\lambda))) \cdot \pi_{RA} \). From the definition of \( r \) and \( f \) we have \( \eta \in RAT_1 \). Let \( \mathfrak{f}_m \) be the algebra of restricted formulas using 4 variables. Let \( G = Fr_1\text{SimRA} \).

Let \( h : G \to \mathfrak{Ra}\mathfrak{f}_m \) be the homomorphism that takes the free generator of \( G \) to \( x \in y \). Let \( \psi = h(\eta) \). Then \( \psi \in Fm^{13} \). \( \psi \) is the desired formula. (Here we use that the \( \mathfrak{Ra} \) reduct of a \( CA_4 \) is a relation algebra.

The generalization of Gödel’s first theorem, has a very natural algebraic counterpart; the least that can be said for his second. The following is slightly new and it depends only on Gödel’s incompleteness theorem for \( L_1 \). The free algebras addressed in the next theorem are called dimension restricted free algebras.

**Corollary 3.3.**

(i) Let \( \omega \geq m > 3 \). Let \( \beta \) be a cardinal \( < \omega \) and \( \rho : \beta \to \varphi(3) \) such that \( \rho(i) \geq 2 \) for some \( i \in \beta \). Then \( Fr_\beta^m S\mathfrak{Nr}_3 CA_m \) is not atomic.

(ii) Let \( m \geq n > 3 \) and \( \rho : \beta \to \varphi(n) \) where \( \beta < \omega \) and \( \rho(i) \geq 2 \) for some \( i \in \beta \). Then \( Fr_\beta^m S\mathfrak{Nr}_n CA_m \) is not atomic. In particular, \( Fr_\beta CA_4 \) and \( Fr_\beta RCA_4 \) are not atomic.

**Corollary 3.4.** (Maddux) For each finite \( n \geq 3 \), The equational theories of \( Df_n \) and \( CA_n \) are undecidable

Maddux’s proof followed an entirely different route, using the undecidability of the word problem for semigroups.

### 4 Gödel’s second for finite variable fragments

Our work here is inspired by work of Andreka Madaras and Nemtii, on working out a Gödel’s second incompleteness theorem for certain strong enough axiomatizations of special relativity. Having a periodic object in their model, the succeed to code \( N \), and then the rest follows like the classical case.

We work with \( n = 3 \), and we assume that we have equality. All the results extend to the case when we do not have equality but we have a tenary relation symbol, instead of a binary one. (This follows from theorem 3.1).

Gödel’s second theorem follows from the first by formalizing the meta mathematical proof of it into the formal system whose consistency is at stake. So
such theories should be strong enough to encode the proof of the first incompleteness theorem. Roughly the provability relation \( p(x, y) \) (\( x \) proves \( y \)) not only proves, when it does it can prove that it proves. Given a theory \( T \) containing arithmetic, let \( Prb_T(\sigma) \) denotes \( \exists x p(x, \sigma) \). Formally:

**Definition 4.1.** A theory \( T \) is strong enough if when \( T \) proves \( \phi \) then \( T \) proves that \( T \) proves \( \phi \) In more detail,

1. \( T \) contains Robinson’s arithmetic

2. for any sentence \( \sigma \), \( T \vdash \sigma \), then \( T \vdash Prb_T(\sigma) \)

3. for any sentence \( \sigma \), \( T \vdash (Prb_T(\sigma) \rightarrow Prb_T Prb_T(\sigma)) \)

4. For any sentences \( \rho \) and \( \sigma \), \( T \vdash Prb_T(\rho \rightarrow \sigma) \rightarrow (Prb_T \rho \rightarrow Prb_T \sigma) \).

Strong theories are strong enough not to prove their consistency, if they are consistent. Robinsons arithmetic is not strong enough but \( PA \) and \( ZF \) are. So we need to capture at least \( PA \) in \( L_n \). This will be done in a minute. In fact, we can capture the whole of \( ZF \), but we will be content only with \( PA \), which is sufficient for our process.

Clearly \( \psi \) is consistent (we are in \( ZF \) set theory). Now, we can interpret Robinson’s arithmetic \( Q \) in our theory \( \psi \), and this way we can prove all those parts of Gödel’s incompleteness theorems (together with the related theorems like Rosser’s) which hold for \( Q \).

However, we want to establish stronger incompleteness results which hold for Peano’s Arithmetic \( PA \), like for example that \( PA \) does not prove \( Con(PA) \). So far what we have is not enough, to render this form of Godel’s second incompleteness theorem.

\( PA \) is stronger than \( Q \); because it has the induction schema. So what strikes one as the obvious thing to do, is to introduce an axiom schema \( Ax(ind) \) which postulates a natural induction principle for the theory of \( \psi \).

We note that Németi defined \( \psi \) in a language with only one binary relation, but the operation symbols of Peano arithmetic are definable in \( Th(\psi) \) (See above). In particular, the successor function \( succ \) is definable (This analogous to the the interpretability of Peano arithmetic in set theory).

Our work in what follows is inspired and is in fact very close to the work of Andréka et all, when they formalized Godel’s second, in strong enough first order fragments of special relativity.

Now the induction schema has the form \( ind(\psi, x) \) is defined as follows.

\[
\forall x ((\psi(0) \land \psi(x) \rightarrow \psi(suc(x))) \Rightarrow (\forall x)\psi(x)).
\]

Now,
\[ \text{Ax(ind)} := \{ \text{ind}(\psi, x) : \psi(x) \text{ is a formula using 3 variables} \} \]

And we define \( T^+ \) as follows:
\[
T^+ := \psi + \text{Ax(ind)}
\]

By definition, \( T^+ \) is an extension of \( \psi \) by a finite schema of axioms, it is consistent and it is valid in the standard models of \( \psi \).

**Theorem 4.2.** There is a formula \( \text{Con}(T^+) \) using only 3 variables, such that in each model \( \mathfrak{M} \models T^+ \) this formula expresses the consistency of \( T^+ \). Furthermore,
\[ T^+ \not\vdash \text{Con}(T^+) \]

and
\[ T^+ \not\vdash \neg\text{Con}(T^+) \]

**Proof.** Firstly, \( PA \) can be interpreted in \( T^+ \) because the axioms of \( T^+ \) were chosen in such a way as to make this true. The axiom system \( T^+ \) is given by a finite schema, completely analogous with the axiom system of \( PA \). Therefore, the axiom theory \( T^+ \) can also be formalized in \( PA \). Hence in \( T^+ \), like \( PA \), there is formula \( \text{pr}(x, y) \) expressing that \( x \) is the Gödel number of a proof from \( T^+ \) of a formula \( \varphi \) of whose Gödel number is \( y \). Now, \( \exists \text{pr}(x, y) \) is a provability formula \( \pi(y) \) which in \( T^+ \) expresses that \( y \) is the Gödel number of an \( L_3 \) formula provable in \( T^+ \). Furthermore, one can easily check that the Löb conditions (as presented, e.g., in [10. Def.2.16. p.163]) are satisfied by \( \pi(y) \) and by \( T^+ \). Now, we choose \( \text{Con}(T^+) \) to be \( \neg \pi(\text{False}) \). The rest follows the standard proof. Also, the generalization for (consistent) extensions of \( T^+ \) with finitely many new axioms can be proved like the classical case; if we have a \( \sigma_1 \) definition of the Gödel numbers of the axioms of \( T^+ \) then we can extend this \( \sigma_1 \)-definition to “\( T^+ \) an extra (concrete) axiom, say \( \varphi \)”, since \( \varphi \) has a concrete Gödel number \([\varphi]\). 

Our next theorem says that truth in our theory is independent of \( ZF \):

**Theorem 4.3.** There is a formula \( \varphi \) using 3 variables and an extension \( T^{++} \) of \( T^+ \) in \( L_3 \) such that truth of statement (i) below is independent of \( ZFC \).

(i)
\[ T^{++} \models \varphi \]

**Proof.** Choose \( T^{++} \) such that \( \text{Th}(\omega) \) of full first-order arithmetic can be interpreted in \( T^{++} \). In \( \text{Th}(\omega) \) there exist a formula, \( \psi \), such that the statement “\( \omega \models \psi \)” is independent of \( ZFC \) (assuming \( ZFC \) is consistent). Such a \( \psi \) is the Gödelian formula \( \text{Con}(ZF) \), then “\( \text{tr}(\psi) \in T^{++} \)” or equivalently “\( T^{++} \models \text{tr}(\psi) \)” is a statement about \( T^{++} \) whose truth is independent from \( ZFC \). 

\[ \square \]
5 Forcing in relation and cylindric algebras

Tarski used the theory of relation algebras to express Zermelo-Fraenkel set theory as a system of equations without variables. Representations of relation algebras will take us back to set-theoretic relational systems.

On the other hand, Cohen’s method of forcing provides us a way to build new models of set theory and to establish the independence of many set-theoretic statements. In [9] a way of building the missing link to connect relation algebras and the method of forcing is presented. Let $\text{QRA}$ stand for the class of quasi relation algebras. Maddux proved using a technique which we call a Maddux style representation, that every $\text{QRA}$ is representable.

Now, see [9] p.55, theorem 13,

\textbf{Theorem 5.1.}  
(1) Let $\mathfrak{A}$ be a simple countable $\text{QRA}$ that is based on a model $(M, \in)$ of set theory. Let $h$ be a Maddux style representation of $\mathfrak{A}$. If $d \in A$ is well founded relation on $M$, then $h(d)$ is well founded.

(2) Let $\mathfrak{A}$ be a simple countable $\text{CA}_3$ that is based on a model $(M, \in)$ of set theory. Let $h$ the Sagi representation. If $R \in A$ is well founded then so $h(d)$.

So Maddux’s and Sagi’s style representations, in fact preverses well foundeness of relations, which is not an elementary fact. In Theorem 14, p. 61 of [9], a characteriszation of simple $\text{QRA}$’s with a distinguished element that are isomorphic to an algebra of relations arising from a countable transitive model of enough set theory is given.

So let $h$ be the Maddux style representation of such an $\mathfrak{A}$, on a set algebra with base $U$. Then $U$ is countable, and $h(e)$ ”set like”. By Mostowski Collapsing theorem, there is a transitive $M$ and a one to one map $g$ from $U$ onto $M$, such that $g$ is an isomorphism between $(U, h(e))$ and $(M, \in)$, where $\in$ is the real membership. $(M, \in)$ is also, a model of enough set theory. Let $M[G]$ is generic extension of $M$, formed by the methods of forcing, and take the $\text{QRA}$, call it $\mathfrak{A}[G]$ corresponding to $(M[G], \in)$. Assume for example that $\mathfrak{A}$ models the translation of the continuum hypothesis, while $M[G]$ models its negation. Then we can conclude that $\mathfrak{A}$ and $\mathfrak{A}[G]$ are simple countable relation algebras that are equationaly distinct. similary for the corresponding directed $\text{CA}s$.

One can carry similar investigations in the context of directed cylindric algebras instead of $\text{QRA}$, by noting that representations of such algebras defined by Sagi also preserves well foundness.

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