Non-Hermitian Chern insulator

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A non-Hermitian extension of the Chern insulator and its bulk-boundary correspondence are investigated. It is shown that in addition to the robust chiral edge states that reflect the nontrivial topology of the bulk (nonzero Chern number), the anomalous helical edge states localized only at one edge can appear, which are unique to the non-Hermitian Chern insulator.

I. INTRODUCTION

Over the past decade, remarkable progress has been achieved in phases of matter characterized by the topology of its wavefunction [1–5]. Such topological phases were explored in solid-state systems including insulators [6–13], superconductors [14–15], and semimetals [16–21], as well as in photonic [22–26] and atomic [27–33] systems, all of which are classified according to spatial dimension and symmetry [34–39]. A hallmark of these topological phases is the bulk-boundary correspondence: the topologically protected gapless states appear as a consequence of the nontrivial topology of the gapped bulk. Examples include chiral edge states in Chern insulators [7], helical edge states in quantum spin Hall insulators [11–13], and Majorana zero modes in topological superconducting wires [15].

Recently, there has been growing interest in non-Hermitian topological phases of matter both in theory [40–69] and experiment [70–77]. In general, non-Hermiticity originates from the exchange of energy and/or particles with the environment [78–81], and several phenomena unique to the nonconservative systems have been theoretically proposed [82–103] and experimentally observed [104–121]. One of the unique features of non-Hermitian systems is the presence of a level degeneracy called exceptional point [122–124], at which the eigenstates coalesce to render the Hamiltonian nondiagonalizable. The exceptional points bring about novel functionalities with no Hermitian counterparts such as unidirectional invisibility [92–111] and enhanced sensitivity [95–97, 120–121]. Recent studies have also revealed that non-Hermiticity alters the nature of the bulk-boundary correspondence in topological systems [11–14, 34–41, 72, 73, 76]. Non-Hermiticity was shown to amplify the topologically protected edge states [14], which were experimentally observed in one dimension [72, 73] and two dimensions [76]. Furthermore, the presence of exceptional points makes edge states anomalous, where the edge states can be localized only at one edge in a non-Hermitian extension of the Su-Schrieffer-Heeger model (non-Hermitian system in one dimension that respects chiral symmetry) [47]. Despite these recent studies, the bulk-boundary correspondence in non-Hermitian systems has yet to be fully understood, especially in two dimensions.

In this work, we investigate a non-Hermitian extension of the Chern insulator and its bulk-boundary correspondence. In Sec. II, we give the topological invariants for a general non-Hermitian system in two dimensions without symmetry (2D class A). In Sec. III, we consider a typical non-Hermitian Chern insulator and provide its phase diagram under the periodic boundary condition. It consists of the gapped phases characterized by the Chern number and the gapless phases characterized by the topological charges accompanied by exceptional points. In Sec. IV, we investigate the topologically protected edge states. The chiral edge states are shown to be robust even in the non-Hermitian systems; they can also be amplified (lasing) due to the presence of non-Hermiticity. Moreover, we find the emergence of the helical edge states localized only at one edge in some phases. These anomalous helical edge states do not have Hermitian counterparts and hence they are unique to the non-Hermitian Chern insulator. In Sec. V, we conclude the paper with outlooks.

II. TOPOLOGICAL INVARIANTS

We first provide the topological invariants of a general non-Hermitian system in two dimensions without symmetry (2D class A). If the Hamiltonian $H (k)$ is diagonalizable, it can be expressed as

$$H (k) = \sum_n E_n (k) |\varphi_n (k)\rangle \langle \chi_n (k)| .$$  \hspace{1cm} (1)

Here $E_n (k)$ is a complex eigenenergy and $|\varphi_n (k)\rangle$ ($|\chi_n (k)\rangle$) is the corresponding right (left) eigenstate [123–126], which satisfy

$$H (k) |\varphi_n (k)\rangle = E_n (k) |\varphi_n (k)\rangle ,$$

$$H^\dagger (k) |\chi_n (k)\rangle = E^*_n (k) |\chi_n (k)\rangle ,$$  \hspace{1cm} (2)

and

$$\langle \chi_m (k) |\varphi_n (k)\rangle = \delta_{mn} ,$$

$$\sum_n |\varphi_n (k)\rangle \langle \chi_n (k)| = I.$$  \hspace{1cm} (3)
where $I$ is the identity matrix. Here $|\varphi_n(k)\rangle$ and $|\chi_n(k)\rangle$ are equal to each other in the presence of Hermiticity, but they are not for non-Hermitian Hamiltonians. If the complex band $n$ is separated from the others (i.e., $\forall k, m \ E_n(k) \neq E_n(k)$), we can define the Berry connection $A_n^i(k)$ and the Berry curvature $F_n(k)$ of the complex band $n$ as

\begin{align}
A_n^i(k) &:= i \langle \chi_n(k) | \partial_{k_i} \varphi_n(k) \rangle, \\
F_n(k) &:= \partial_{k_i} A_n^i(k) - \partial_{k_j} A_n^j(k),
\end{align}

and hence the (first) Chern number as

$$C_n := \int_{\text{BZ}} \frac{d^2k}{2\pi} F_n \in \mathbb{Z}. \quad (6)$$

The Chern number remains unchanged as long as the complex band is separated and the Hamiltonian is diagonalizable. Due to the difference between the left and right eigenstates, there is some arbitrariness concerning the definition of the Berry connection. For instance, it can also be defined as $A_n^i(k) := i \langle \varphi_n(k) | \partial_{k_i} \chi_n(k) \rangle$. However, the Chern number does not depend on such choices of the eigenstates and can be uniquely defined $[60]$. On the other hand, the Chern number is not well-defined if the Hamiltonian is nondiagonalizable at a wavenumber $k = k_{\text{EP}}$ (i.e., some eigenstates coalesce and linearly depend on each other) $[122, 123]$. Instead, in the presence of such a defective point (exceptional point), we can define another topological invariant given by $[19, 50, 60, 73, 116, 117]$

$$\nu(k_{\text{EP}}) := \int_{\Gamma(k_{\text{EP}})} \frac{dk}{2\pi i} \nabla_k \log (E_m - E_n), \quad (7)$$

where the two bands $E_m(k)$ and $E_n(k)$ coalesce at $k = k_{\text{EP}}$, and $\Gamma(k_{\text{EP}})$ is a closed loop in momentum space that encircles $k_{\text{EP}}$. Here fractional $\nu(k_{\text{EP}})$ implies that $E_m(k) - E_n(k)$ is multi-valued and $k_{\text{EP}}$ becomes a branch point in momentum space. The defined charge $\nu(k_{\text{EP}})$ is topological in that it cannot be changed unless the exceptional point is annihilated.

III. PHASES AND COMPLEX SPECTRA

In the following, we consider the non-Hermitian Chern insulator given by

$$H(k) = (m + t \cos k_x + t \cos k_y) \sigma_x + (i\gamma + t \sin k_x) \sigma_y + (t \sin k_y) \sigma_z, \quad (8)$$

where $t$, $m$, and $\gamma$ are real parameters and we assume $t > 0$. In the case of $\gamma = 0$, the model reduces to the well-known Hermitian Chern insulator that is characterized by the Chern number $[113]$. The eigenstates form two bands and their complex energy dispersions are obtained as

$$E_{\pm}(k) = \pm \sqrt{(m + t \cos k_x + t \cos k_y)^2 + (i\gamma + t \sin k_x)^2 + (t \sin k_y)^2}. \quad (9)$$

The Hamiltonian becomes defective if and only if

$$\exists \ k_{\text{EP}} \in [-\pi, \pi]^2 \quad \text{s.t.} \quad E_{\pm}(k_{\text{EP}}) = 0. \quad (10)$$

This requires $\sin(k_{\text{EP}})_x = 0$ and reduces to the following conditions:

1. Exceptional points appear on $(k_{\text{EP}})_x = 0$, and then Eq. $\text{(10)}$ reduces to

$$\gamma^2 = (m + t)^2 + t^2 + 2t (m + t) \cos(k_{\text{EP}})_y, \quad (11)$$

where $(k_{\text{EP}})_y \in [-\pi, \pi]$ exists if and only if

$$\begin{cases} |m| \leq |\gamma| \leq |m + 2t| & \text{for } m \geq -t; \\ |m + 2t| \leq |\gamma| \leq |m| & \text{for } m \leq -t. \quad (12) \end{cases}$$

2. Exceptional points appear on $(k_{\text{EP}})_x = \pm \pi$, and then Eq. $\text{(10)}$ reduces to

$$\gamma^2 = (m - t)^2 + t^2 + 2t (m - t) \cos(k_{\text{EP}})_y, \quad (13)$$

where $(k_{\text{EP}})_y \in [-\pi, \pi]$ exists if and only if

$$\begin{cases} |m - 2t| \leq |\gamma| \leq |m| & \text{for } m \geq t; \\ |m| \leq |\gamma| \leq |m - 2t| & \text{for } m \leq t. \quad (14) \end{cases}$$

Hence we obtain the phase diagram (Fig. 1) and the corresponding complex band structures (Fig. 2).
broad gapless phases that accompany pairs of exceptional points (see Fig. 2(e-h)), whereas they are separated by a point in the Hermitian counterpart (γ = 0). The appearance of such large gapless phases arises from the fact that a level degeneracy requires fine-tuning 2 parameters for a general 2 × 2 non-Hermitian matrix, although it requires fine-tuning 3 parameters for a general 2 × 2 Hermitian matrix [120, 121, 122]; non-Hermitian systems in two dimensions can be gapless without fine-tuning parameters or the presence of certain symmetry. As explained in the last section, the topological invariants can be defined for each exceptional point as Eq. (7). For instance, if there appears a pair of exceptional points at \( k_{EP} = (0, \pm \kappa) \) (\( \kappa > 0 \)) as in Fig. 2(e-f), the energy dispersions near the exceptional points are given by

\[
E_\pm (k) \simeq \sqrt{2t' \gamma k_x \pm t^2 (\sin 2\kappa)} (k_y \mp \kappa).
\]

Here the square-root singularity implies that the exceptional points \( k = k_{EP} \) become branch points and that the topological charges defined by Eq. (7) are \( \nu = \pm 1/2 \) for \( k_{EP} = (0, \pm \kappa) \). Although the Chern number cannot be defined, we find the emergence of the anomalous helical edge states in these gapless phases, as discussed in the next section.

IV. EDGE STATES

The hallmark of the Hermitian Chern insulator is the emergence of the chiral edge states that correspond to the nontrivial topology of the bulk [115]. It is shown that the bulk-edge correspondence in the presence of non-Hermiticity is sensitive to the boundary conditions [127]. We first investigate the system with open boundaries in the \( y \) direction, and then the system with open boundaries in the \( x \) direction.

A. Open boundaries in the \( y \) direction

We first consider the following non-Hermitian Chern insulator with periodic boundaries in the \( x \) direction and open boundaries in the \( y \) direction:

\[
\hat{H} = \sum_{k_x} \sum_y \left\{ \hat{c}_{k_x,y}^\dagger \left( t (\sigma_x + i \sigma_y) \frac{2}{2} \hat{c}_{k_x,y+1}^\dagger + \text{H.c.} \right) \right. \\
+ \left. \hat{c}_{k_x,y} \left( (m + t \cos k_x) \sigma_x + (\gamma + t \sin k_x) \sigma_y \right) \hat{c}_{k_x,y}^\dagger \right\},
\]

where \( \hat{c}_{k_x,y} \) (\( \hat{c}_{k_x,y}^\dagger \)) annihilates (creates) a fermion with two internal degrees of freedom on site \( y \) and with momentum \( k_x \).

In the gapped phase with the topologically trivial bulk (Fig. 2(a-b)), no gapless states appear between the gapped complex bands (Fig. 3(a-b)). In the gapped

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**FIG. 2.** Complex band structures of the non-Hermitian Chern insulator \( E_\pm = E_\pm (k_x, k_y) \) given by Eq. (4). The orange (blue) band represents \( E_+ (E_-) \). (a-b) Real and imaginary parts of the gapped and topologically trivial bands with zero Chern number \( C = 0 \) \((t = 1.0, m = -3.0, \gamma = 0.5)\). (c-d) Real and imaginary parts of the gapped and topologically trivial bands with nonzero Chern number \( C = -1 \) \((t = 1.0, m = -1.0, \gamma = 0.5)\). (e-f) Real and imaginary parts of the gapless bands with a pair of exceptional points on \( k_x = 0 \) \((t = 1.0, m = -2.0, \gamma = 1.0)\). (g-h) Real and imaginary parts of the gapless bands with two pairs of exceptional points on both \( k_x = 0 \) and \( k_x = \pm \pi \) \((t = 1.0, m = -0.2, \gamma = 1.0)\).
and the corresponding edge states are given by

\[
\hat{\Psi}_{\text{left}}(k_x) \propto \sum_{y=1}^{L_y} \left( -\cos k_x - \frac{m}{t} \right)^{y-1} \hat{c}_{x,y}(1+i),
\]

\[
\hat{\Psi}_{\text{right}}(k_x) \propto \sum_{y=1}^{L_y} \left( -\cos k_x - \frac{m}{t} \right)^{y-1} \hat{c}_{x,y}(1-i),
\]

which satisfy \([\hat{H}, \hat{\Psi}_{\text{left/right}}] = E_{\text{left/right}} \hat{\Psi}_{\text{left/right}}]\) in the thermodynamic limit \(L_y \to \infty\) (see Appendix A for the detailed derivation [128–131]). Here \(|\cos k_x + m/t| < 1\) is required for the localization (normalization) of these edge states. The obtained edge states are immune to disorder and hence topologically protected (see Appendix B for the details). Remarkably, the left edge state has the largest imaginary part for \(\gamma > 0\), whereas the right edge state has the smallest imaginary part. Physically, this results in the amplification (lasing) of the left edge state and the attenuation of the right edge state, just like a topological insulator laser recently proposed and realized [70].

In the gapless phases, there appear exceptional points on \(k_x = 0\) and/or \(k_x = \pm \pi\) (Fig. 3(e-h)). Whereas no edge states appear in Fig. 3(e-f), a pair of chiral edge states emerges in both edges as shown in Fig. 3(g-h), which are also described by Eqs. (17) and (18). The latter case corresponds to the gapless phase with \(\nu_0, \nu_\pi \neq 0\) between the gapped \(C = +1\) and \(C = -1\) phases (see the green regions in Fig. 1). Such a gapless phase connects the gapped and topologically nontrivial phases provided that \(C_4\) rotational symmetry is maintained. Despite the absence of the energy gap, these chiral edge states are also stable against weak perturbations (see Appendix B for the details), which implies their topological origin. We remark that Ref. [132] discusses a similar gapless topological phase in Hermitian topological superconductors in two dimensions.

### B. Open boundaries in the \(x\) direction

We next consider the following non-Hermitian Chern insulator with open boundaries in the \(x\) direction and periodic boundaries in the \(y\) direction:

\[
\hat{H} = \sum_x \sum_{k_y} \left[ \hat{c}_{x+1,k_y}^\dagger \left( t \sigma_z \frac{\sigma_x + i \sigma_y}{2} \hat{c}_{x,k_y} + \text{H.c.} \right) \right. \\
+ \hat{c}_{x,k_y}^\dagger \left( m + t \cos k_y \right) \sigma_x + i \gamma \sigma_y + (t \sin k_y) \sigma_z \hat{c}_{x,k_y} \right],
\]

where \(\hat{c}_{x,k_y}^\dagger (\hat{c}_{x,k_y})\) annihilates (creates) a fermion with two internal degrees of freedom on site \(x\) and with momentum \(k_y\).

As in the case with the results obtained in the last subsection, the chiral edge states appear in the gapped phase with nonzero Chern number (Fig. 4(c-d)), whereas no
edge states appear in the gapped phase with zero Chern number (Fig. 4(a-b)). However, the energy dispersions are drastically different from those with periodic boundaries (Fig. 2). In particular, the spectra are entirely real in some cases (see Fig. 4(b, f)), which are to be contrasted with the corresponding periodic cases (Fig. 2(b, f)). The eigenstates also differ from the periodic case: all the eigenstates are localized at one edge in Fig. 4. The energy dispersions of the chiral edge states are obtained as

\[ E_{\text{left}}(k_y) = -t \sin k_y, \quad E_{\text{right}}(k_y) = t \sin k_y, \]

and the corresponding edge states are given by

\[
\hat{\Psi}_{\text{left}}(k_y) \propto \sum_{x=1}^{L_x} \left( -\cos k_y - \frac{m + \gamma}{t} \right)^{-1} \hat{c}_{x,k_y}(0 1), \\
\hat{\Psi}_{\text{right}}(k_y) \propto \sum_{x=1}^{L_x} \left( -\cos k_y - \frac{m - \gamma}{t} \right)^{-1} \hat{c}_{x,k_y}(1 0).
\]

Here the imaginary parts of the edge states vanish in contrast to Eq. (17). In addition, the localization lengths depend on the non-Hermiticity \( \gamma \) in contrast to Eq. (18), and they are different according to which edge they are localized at; the left edge state is present for \( |m + \gamma| < 2t \), while the right edge state is present for \( |m - \gamma| < 2t \).

Remarkably, the energy gap can be open (Fig. 4(e-h)) even in the gapless phases for the corresponding periodic systems (Fig. 2(e-h)), and there appears a pair of helical edge states that are localized only at the left edge (Fig. 4(e-f)). The emergence of such anomalous edge states localized only at one edge reminds us of those in a one-dimensional system with chiral symmetry [47]. In another phase, two pairs of helical edge states appear, one localized at the right edge and the other localized at the left edge (Fig. 4(g-h)). These anomalous helical edge states are immune to disorder that does not close the energy gap, and hence they are topologically protected (see Appendix B for the details). It is also notable that finite-size effects are strong in this system (see Appendix C for the details). In Fig. 4(g-h), the energy gap closes around \( k_y = 0 \) for a larger system with \( L_x = 200 \) and the right edge states disappear. Nevertheless, the energy gaps are still open and the edge states survive even in larger systems for the other cases.

The emergence of the anomalous helical edge states is qualitatively understood as follows. Let us consider the topological phase transition from the gapped phase with \( C = -1 \) to that with \( C = 0 \), through the \( \nu_0 \neq 0 \) phase (see Fig. 1). In the \( C = -1 \) phase, the chiral edge states appear at both edges given as Eq. (21), and all the eigenstates except for the right chiral edge state are localized at the left edge. If we approach the boundary between the \( C = -1 \) and \( \nu_0 \neq 0 \) phases, the right chiral edge state gradually becomes delocalized, whereas the left chiral edge state does not. At the boundary between the \( C = -1 \) and \( \nu_0 \neq 0 \) phases, the bulk gap closes (even under the open boundary condition along the \( x \) direction), and the right chiral edge state is absorbed into the bulk eigenstates and shifted to the left edge. In fact, the right chiral edge state vanishes at the phase boundary \( |m - \gamma| = 2t \), as described above. Inside the \( \nu_0 \neq 0 \) phase, the bulk eigenstates acquire an energy gap, but the helical edge states remain gapless. At the boundary between the \( \nu_0 \neq 0 \) and \( C = 0 \) phases, the
helical edge states are absorbed into the bulk eigenstates and disappear inside the $C = 0$ phase.

Since the helical edge states emerge in the $\nu_0 \neq 0$ phase, they may be related to the topological charges of exceptional points given by Eq. (7). In addition, we point out that the helical edge states in Fig. 3(e-f) seem to bridge the two exceptional points that appear in the periodic systems (see Fig. 3(e-f)).

V. CONCLUSION AND OUTLOOK

In this work, we have investigated the non-Hermitian Chern insulator and its bulk-boundary correspondence. We have provided the phase diagram of the system with periodic boundaries; the gapped phases characterized by the Chern number are separated by the gapless phases characterized by the topological charges accompanied by exceptional points. We have also investigated topologically protected edge states for the system with open periodic systems (see Fig. 2(e-f)).

An important outstanding issue is to find the topological invariants that characterize the anomalous helical edge states and to establish their bulk-boundary correspondence. Moreover, counterparts of the anomalous edge states may appear even in different dimensions (including three dimensions) and symmetry classes, which we leave for future work.

Note added. — After completion of this work, we became aware of a recent related work [134], which also investigates a non-Hermitian Chern insulator and its edge states.

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Appendix A: Exact forms of the chiral edge states

We analytically provide the energy dispersions and wavefunctions of the chiral edge states in the non-Hermitian Chern insulator [128]. We first consider the chiral edge state at the left edge of the Hamiltonian given by Eq. (10). If the edge state is expressed as

$$\hat{\Psi}_{\text{left}} \propto \sum_{y=1}^{L_y} \lambda^{y-1} (\hat{c}_{k_x,y} \hat{v}),$$

where $\lambda$ denotes a parameter that determines the localization length (given by $-\log |\lambda|^{-1}$) and $\hat{v}$ is a two-component vector. Then the Schrödinger equation $[\hat{H}, \hat{\Psi}_{\text{left}}] = E_{\text{left}} \hat{\Psi}_{\text{left}}$ reduces to

$$(\lambda^{-1} T^\dagger + M + \lambda T) \hat{v} = E_{\text{left}} \hat{v}$$

in the bulk and

$$(M + \lambda T) \hat{v} = E_{\text{left}} \hat{v}$$

at the left edge. Here $T$ and $M$ are defined as $T := t (\sigma_x + i \sigma_y) / 2$ and $M := (m + t \cos k_x) \sigma_x + (i \gamma + t \sin k_x) \sigma_y$, and we take the semi-infinite limit $L_y \to \infty$ and neglect the effect of the right edge. Combining Eqs. (A2) and (A3), we have $T^\dagger \hat{v} = 0$, which leads to

$$(\lambda^{-1} T^\dagger + M + \lambda T) \hat{v} = E_{\text{left}} \hat{v}$$

FIG. 5. Stability of the chiral and helical edge states. (a-d) Complex spectrum of the disordered non-Hermitian Chern insulator given by Eq. (11) as a function of the wavenumber $k_x$. The periodic boundary condition is imposed in the $x$ direction, while the open boundary condition is imposed in the $y$ direction ($L_y = 30$). The chiral edge states are robust against disorder for both (a-b) $t = 1.0$, $m_x = -1.0 + 0.5 \epsilon_y$, and $\gamma_y = 0.5 + 0.5 \epsilon'_y$, and (c-d) $t = 1.0$, $m_x = -0.2 + 0.2 \epsilon_y$, and $\gamma_y = 1.0 + 0.2 \epsilon'_y$, where $\epsilon_y$ and $\epsilon'_y$ are random variables uniformly distributed over $[-0.5, 0.5]$. (e-f) Complex spectrum of the disordered non-Hermitian Chern insulator given by Eq. (B2) as a function of the wavenumber $k_y$. The open boundary condition is imposed in the $x$ direction ($L_x = 30$), while the periodic boundary condition is imposed in the $y$ direction. The helical edge states are robust against disorder for $t = 1.0$, $m_x = -2.0 + 0.5 \epsilon_x$, and $\gamma_x = 1.0 + 0.5 \epsilon'_x$.
is expressed as of the Hamiltonian given by Eq. (19). If the edge state also obtained in a similar manner.

Whereas the chiral or helical edge states survive in (c-f), the y direction. The same sets of parameters are used as in Fig. 4. Whereas the chiral or helical edge states survive in (c-f), the band gap closes near \( k_y = 0 \) and hence the helical edge states at the right edge vanish in (g-h).

to \( \vec{v} = (1 \ i)^T \). Hence with Eq. (A3) we have

\[
E_{\text{left}} = (t \sin k_x + i \gamma) (1 \ i)^T = [\lambda t + (m + t \cos k_x + \gamma)] (1 \ 0)^T, \tag{A4}
\]

which indicates Eqs. (17) and (18) in the main text. For the presence of this edge state, there exists a wavenumber \( k_x \) that satisfies \( \cos k_x + m/t < 1 \), which reduces to \( |m| < 2t \). The other chiral edge state at the right edge is also obtained in a similar manner.

We next consider the chiral edge state at the left edge of the Hamiltonian given by Eq. (19). If the edge state is expressed as

\[
\hat{\Psi}_{\text{left}} \propto \sum_{x=1}^{L_x} \lambda_{\text{left}}^{x-1} \hat{c}_{x,k_y} \hat{v}, \tag{A5}
\]

the Schrödinger equation \( \hat{H} \hat{\Psi}_{\text{left}} = E_{\text{left}} \hat{\Psi}_{\text{left}} \) reduces to Eqs. (A2) and (A3) with \( T := t (\sigma_x + i \sigma_y)/2 \) and \( M := (m + t \cos k_y) \sigma_x + i \gamma \sigma_y + (t \sin k_y) \sigma_z \). We again have \( T^\dagger \hat{v} = 0 \) and hence \( \hat{v} = (0 \ 1)^T \). With Eq. (A3), we have

\[
E_{\text{left}} + t \sin k_y (0 \ 1)^T = [\lambda_{\text{left}} t + (m + t \cos k_x + \gamma)] (1 \ 0)^T, \tag{A6}
\]

which indicates Eqs. (20) and (21) in the main text. While the other chiral edge state at the right edge is also obtained in a similar manner, their localizations depend differently on the non-Hermiticity \( \gamma \); the left edge state appears for \( |m + \gamma| < 2t \), while the right one appears for \( |m - \gamma| < 2t \).

**Appendix B: Robustness against disorder**

The chiral and helical edge states revealed in the main text are immune to disorder. To confirm this, we first consider the following disordered non-Hermitian Chern insulator with periodic boundaries in the \( x \) direction and open boundaries in the \( y \) direction:

\[
\hat{H} = \sum_k \sum_y \left\{ \hat{c}_{x+1,k_y}^\dagger \frac{t (\sigma_x + i \sigma_y)}{2} \hat{c}_{x,k_y} + \text{H.c.} \right\} + \hat{c}_{x,k_y}^\dagger \left( m_x + t \cos k_y \right) \sigma_x + i \gamma_x \sigma_y + (t \sin k_y) \sigma_z \hat{c}_{x,k_y}. \tag{B1}
\]

where \( m_y \) and \( \gamma_y \) denote the disordered parameters on site \( y \). The chiral edge states appear even in the presence of disorder and hence are topologically protected (see Fig. 5(u-d)). We also consider the following disordered non-Hermitian Chern insulator with open boundaries in the \( x \) direction and periodic boundaries in the \( y \) direction:

\[
\hat{H} = \sum_x \sum_{k_y} \left\{ \hat{c}_{x+1,k_y}^\dagger \frac{t (\sigma_x + i \sigma_y)}{2} \hat{c}_{x,k_y} + \text{H.c.} \right\} + \hat{c}_{x,k_y}^\dagger \left( m_x + t \cos k_y \right) \sigma_x + i \gamma_x \sigma_y + (t \sin k_y) \sigma_z \hat{c}_{x,k_y}. \tag{B2}
\]

where \( m_x \) and \( \gamma_x \) denote the disordered parameters on site \( x \). As well as the chiral edge states, the helical edge states are topologically protected (see Fig. 6(e-f)).

**Appendix C: Finite-size effect**

Figure 6 represents the complex spectra of the non-Hermitian Chern insulator with open boundaries in the \( x \) direction and periodic boundaries in the \( y \) direction. In contrast to the results of Fig. 4 with \( L_x = 30 \), the results of Fig. 6 are on larger systems with \( L_x = 200 \). Remarkably, in Fig. 6(g-h), the band gap closes near \( k_y = 0 \) and hence the helical edge states at the right edge disappear, which are to be contrasted with Fig. 4(g-h).
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