THE WEIGHT FILTRATION ON THE CONSTANT SHEAF ON A PARAMETERIZED SPACE

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ABSTRACT. On an \( n \)-dimensional locally reduced complex analytic space \( X \) on which the shifted constant sheaf \( \mathbb{Q}^X[n] \) is perverse, it is well-known that, locally, \( \mathbb{Q}^X[n] \) underlies a mixed Hodge module of weight \( \leq n \) on \( X \), with weight \( n \) graded piece isomorphic to the intersection cohomology complex \( I^X \), with constant \( \mathbb{Q} \) coefficients. In this short paper we identify the weight \( n-1 \) graded piece \( \text{Gr}^W_{n-1} \mathbb{Q}^X[n] \) in the case where \( X \) is a “parameterized space”, using the comparison complex, a perverse sheaf naturally defined on any space for which the shifted constant sheaf \( \mathbb{Q}^X[n] \) is perverse. We give concrete computations of the perverse sheaf \( W_0Q^V(f)[2] \) in the case where \( X = V(f) \) is a surface in \( \mathbb{C}^3 \).

1. Introduction

Let \( W \) be an open neighborhood of the origin in \( \mathbb{C}^N \), let \( X \subseteq W \) be a (reduced) complex analytic space containing \( 0 \) of pure dimension \( n \), on which the (shifted) constant sheaf \( \mathbb{Q}^X[n] \) is perverse (e.g., if \( X \) is a local complete intersection), and let \( \pi : Y \to X \) be the normalization of \( X \).

There is then a surjection of perverse sheaves \( \mathbb{Q}^X[n] \to I^X \to 0 \), where \( I^X \) is the intersection cohomology complex on \( X \) with constant \( \mathbb{Q} \) coefficients. Since the category of perverse sheaves is Abelian, we obtain a short exact sequence

\[
0 \to N^X \to \mathbb{Q}^X[n] \to I^X \to 0.
\]

The perverse sheaf \( N^X \) is called the comparison complex on \( X \), and was first defined by the author and David Massey in [7] and subsequently studied in several papers by the author [5], [6] and Massey [9].

By shrinking \( W \) if necessary, the perverse sheaf \( \mathbb{Q}^X[n] \) underlies a graded-polarizable mixed Hodge module (Prop 2.19, Prop 2.20, [11]) of weight \( \leq n \). Moreover, by Saito’s theory of (graded polarizable) mixed Hodge modules in the local complex analytic context, the perverse cohomology objects of the usual sheaf functors naturally lift to cohomology functors in the context of (graded polarizable) mixed Hodge modules (but not on their derived category level as in the algebraic context as in Section 4 of [11]). Moreover, by (4.5.9) [11], the quotient morphism \( \mathbb{Q}^X[n] \to I^X \) induces an isomorphism

\[
\text{Gr}^W_n \mathbb{Q}^X[n] \xrightarrow{\sim} I^X;
\]

consequently, the short exact sequence (1) identifies the comparison complex \( N^X \) with \( W_{n-1} \mathbb{Q}^X[n] \). This then endows \( N^X \) with the structure of a mixed Hodge module of weight \( \leq n-1 \) with weight filtration \( W_kN^X = W_k\mathbb{Q}^X[n] \) for \( k \leq n-1 \). In this short paper, we explicitly identify the graded piece \( \text{Gr}^W_{n-1} N^X = \text{Gr}^W_{n-1} \mathbb{Q}^X[n] \) in the case where the normalization of \( X \) is a rational homology manifold, and give...
concrete computations of $W_{n-2}\mathbb{Q}^*_X[n]$ in the case where $X = V(f)$ is a surface in $\mathbb{C}^3$.

Before we state our main result, we first recall a theorem of Borho and MacPherson [1] giving us several equivalent characterizations of rational homology manifolds:

**Theorem 1.1.** ([B-M]) The following are equivalent:

1. $X$ is a rational homology manifold, i.e., for all $p \in X$, for all $k$, $H^k(X, X \setminus \{p\}; \mathbb{Q}) = 0$ unless $k = 2n$, and $H^{2n}(X, X \setminus \{p\}; \mathbb{Q}) \cong \mathbb{Q}$.
2. The natural morphism $\mathbb{Q}^*_X[n] \to \mathbb{I}^*_X$ is an isomorphism.
3. $D(\mathbb{Q}^*_X[n]) \cong \mathbb{Q}^*_X[n]$, where $D$ is the Verdier duality functor.

Let $\pi : (Y, S) \to (X, 0)$ be the normalization of $X$, where $S := \pi^{-1}(0)$. The normalization map is a small map in the sense of Goresky and MacPherson [4], and so there is an isomorphism $\pi_*\mathbb{I}^*_Y \cong \mathbb{I}^*_Y$, where $\mathbb{I}^*_Y$ is intersection cohomology on $Y$ with constant $\mathbb{Q}$ coefficients. Thus, when the normalization is a rational homology manifold, $\mathbb{I}^*_X \cong \pi_*\mathbb{Q}^*_Y[n]$. In this case, by taking the long exact sequence in stalk cohomology of (1), we then find that $\mathbb{N}^*_{\mathbb{X}}$ has cohomology concentrated in degree $-n + 1$, and in that degree, we have $\dim H^{-n+1}(\mathbb{N}^*_{\mathbb{X}})_p = |\pi^{-1}(p)| - 1$.

From this, it follows that

$$D := \text{supp}\mathbb{N}^*_X = \{p \in X \mid |\pi^{-1}(p)| > 1\}$$

is a purely $(n-1)$-dimensional set (it is the support of a perverse sheaf concentrated in degree $-n + 1$), and $D \subseteq \Sigma X$. Throughout this paper, we will assume the normalization of $X$ is a rational homology manifold; additionally, we will assume that $D = \Sigma X$, so that $\Sigma X$ will always be purely $(n-1)$-dimensional.

We will also use the following result throughout this paper, in which the vanishing of the cohomology sheaves of the comparison complex $\mathbb{N}^*_X$ places strong constraints on the topology of the normalization $Y$.

**Theorem 1.2.** ([H., [6]].) The normalization $Y$ of $X$ is a rational homology manifold if and only if $\mathbb{N}^*_X$ has cohomology sheaves concentrated in degree $-n + 1$; i.e., for all $p \in X$, $H^k(\mathbb{N}^*_X)_p$ is non-zero only possibly when $k = -n + 1$.

Letting $\Sigma X$ denote the singular locus of $X$, and let $i : \Sigma X \hookrightarrow X$. We can then find a smooth, Zariski open dense subset $U \subseteq \Sigma X$ over which the normalization map restricts to a covering projection $\tilde{\pi} : \pi^{-1}(U) \to U \subseteq \Sigma X$ (see Section 6.2, [4]). Let $l : U \hookrightarrow \Sigma X$ and $m : \Sigma X \setminus U \hookrightarrow \Sigma X$ denote the respective open and closed inclusion maps. Let $\tilde{m} := i \circ m$, $\tilde{l} := i \circ l$. Note that $\dim_0 \Sigma X \setminus U \leq n - 2$, as it is the complement of a Zariski open set (we will need this later in Proposition 2.2).

**Example 1.3.** Consider the Whitney umbrella $V(f) \subseteq \mathbb{C}^3$ with $f(x, y, z) = y^2 - x^3 - z^2$. Then, the normalization of $V(f)$ is smooth, and given by the map $\pi(u, t) = (u^2 - t, u(u^2 - t), t)$.

The critical locus of $f$ is $\Sigma f = V(x, y)$, and it is easy to see that over $\Sigma f \setminus \{0\}$, $\pi$ is a 2-to-1 covering map; thus, we set $U = \Sigma f \setminus \{0\}$.

**Example 1.4.** Suppose $V(f) \subseteq \mathbb{C}^3$ is a (reduced) surface with $\dim_0 \Sigma f = 1$ whose normalization is a rational homology manifold. Then, it is easy to see that $U = \Sigma f \setminus \{0\}$; this follows from the fact that $\mathbb{I}^*_{V(f)}$ is constructible with respect to the Whitney stratification $\{\Sigma f \setminus \{0\}, \{0\}\}$ of $\Sigma f$, along with the description of the stalk cohomology of $\mathbb{I}^*_{V(f)}$ given by the isomorphism $\mathbb{I}^*_{V(f)} \cong \pi_*\mathbb{Q}^*_Y[2]$.  


We will examine this setting in more detail in Section 3.

Our main result is the following.

Main Theorem 1 (Theorem 2.6). Suppose the normalization of $X$ is a rational homology manifold. Then, there is an isomorphism $\text{Gr}_{n-1}^W i^* N^*_X \cong \mathbb{I}_{\Sigma X}(i^* N^*_X)$, so that the short exact sequence of perverse sheaves on $X$

$$0 \to m_* p H^0 (m^! i^* N^*_X) \to i^* N^*_X \to \mathbb{I}_{\Sigma X}(i^* N^*_X) \to 0$$

identifies $W_{n-2} i^* N^*_X \cong m_* p H^0 (m^! i^* N^*_X)$. Here, $\mathbb{I}_{\Sigma X}(i^* N^*_X)$ denotes the intermediate extension of the perverse sheaf $i^* N^*_X$ to all of $\Sigma X$, and $p H^0 (-)$ denotes the 0-th perverse cohomology functor.

Since the map $i : \Sigma X \hookrightarrow X$ is a closed inclusion, it preserves weights. Moreover, the support of $N^*_X$ is contained in the singular locus $\Sigma X$, and so $i_* i^* N^*_X \cong N^*_X$. Consequently, we have the following.

Corollary 1.5 (Corollary 2.7, Theorem 2.8). Suppose the normalization of $X$ is a rational homology manifold. Then, there are isomorphisms

$$\text{Gr}_{n-1}^W \mathbb{Q}^*_X[n] \cong \text{Gr}_{n-1}^W i^* N^*_X \cong i_* \mathbb{I}_{\Sigma X}(i^* N^*_X),$$

and

$$W_{n-2} \mathbb{Q}^*_X[n] \cong W_{n-2} i^* N^*_X \cong m_* p H^0 (m^! i^* N^*_X) \cong m_* \ker \{ \phi_g [-1] i^* N^*_X \xrightarrow{\text{var}} \psi_g [-1] i^* N^*_X \},$$

where is any $g$ complex analytic function on $\Sigma X$ such that $V(g)$ contains $\Sigma X \setminus \mathcal{U}$, but does not contain any irreducible component of $\Sigma X$, and $m' : V(g) \hookrightarrow \Sigma X$ is the closed inclusion.

In the case where $X = V(f)$ is a surface in $\mathbb{C}^3$, we explicitly compute $W_0 \mathbb{Q}^*_V(f)[2]$; the vanishing of this perverse sheaf places strong constraints on the topology of the singular set $\Sigma f$ of $V(f)$.

Main Theorem 2 (Theorem 3.1). If $V(f)$ is a surface in $\mathbb{C}^3$ whose normalization is a rational homology manifold, and $\dim_0 \Sigma f = 1$, then

$$W_0 \mathbb{Q}^*_V(f)[2] \cong V_0$$

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional $\mathbb{Q}$-vector space, of dimension

$$\dim_\mathbb{Q} V = 1 - |\pi^{-1}(0)| + \sum_C \dim_\mathbb{Q} \ker \{ \text{id} - h_C \},$$

where $C$ is the collection of irreducible components of $\Sigma f$ at 0, and for each $C$, $h_C$ is the (internal) monodromy operator on the local system $H^{-1}(N^*_V(f))_{|C \setminus \{0\}}$. Note that $|\pi^{-1}(0)|$ is, of course, equal to the number of irreducible components of $V(f)$ at 0.
2. Main Result

In this section, we first prove a general result, Lemma 2.1, about perverse sheaves that will allow us to construct the short exact sequence mentioned in Theorem 2.6, and that $N^*_X$ satisfies the hypotheses of this lemma. Then, we examine the weight filtration on $I^*_X(l^*N^*_X)$ and show that it underlies a polarizable Hodge module of weight $n - 1$ in Proposition 2.5. With all this, we can state and prove Theorem 2.6 and Corollary 2.7.

Recall the category of perverse sheaves $Perv(X)$ is the Abelian subcategory of the bounded derived category of $\mathbb{C}$-constructible sheaves $D^b(X)$ given by the heart of the perverse $t$-structure, $Perv(X) = \mathcal{P}D^{\leq 0}(X) \cap \mathcal{P}D^{\geq 0}(X)$. Here,

- $\mathcal{P} \in \mathcal{P}D^{\leq 0}(X)$ if $\mathcal{P}$ satisfies the support condition: for all $k \in \mathbb{Z}$,
  \[ \dim_{\mathbb{C}} \text{supp} H^k(\mathcal{P}) \leq -k. \]

- $\mathcal{P} \in \mathcal{P}D^{\geq 0}(X)$ if $\mathcal{P}$ satisfies the support condition, where again $\mathcal{P}$ denotes the Verdier duality functor.

The following lemma is due to the author and David Massey.

**Lemma 2.1.** Suppose $X$ is a complex analytic space, $\mathcal{P}$ a perverse sheaf on $X$, $l : \mathcal{U} \hookrightarrow X$ a Zariski open subset and $m : Z = X \setminus \mathcal{U} \hookrightarrow X$ its closed analytic complement. Then, if $m^*[-1]\mathcal{P} \in \mathcal{P}D^{\leq 0}(Z)$, there is a short exact sequence

\[ 0 \to m_*\mathcal{P}^{\leq 0}(m^!\mathcal{P}) \to \mathcal{P} \to I^*_X(l^*\mathcal{P}) \to 0 \]

of perverse sheaves on $X$, where $I^*_X(l^*\mathcal{P}) := \text{im} \mathcal{P}^{\leq 0}(l_!l^*\mathcal{P}) \to \mathcal{P}$ denotes the intermediate extension of $l^*\mathcal{P}$ to all of $X$.

**Proof.** The natural morphism $\mathcal{P}^{\leq 0}(l_!l^*\mathcal{P}) \to \mathcal{P}^{\leq 0}(l_*l^*\mathcal{P})$ factors as

\[ \mathcal{P}^{\leq 0}(l_!l^*\mathcal{P}) \to \mathcal{P} \to \mathcal{P}^{\leq 0}(l_*l^*\mathcal{P}). \]

From the other natural distinguished triangle associated to this pair of subsets,

\[ l_!l^*\mathcal{P} \to \mathcal{P} \to m_*m^*\mathcal{P} \to l, \]

we see that surjectivity of $\alpha$ follows from the vanishing of

\[ \mathcal{P}H^0(m_*m^*\mathcal{P}) \cong m_*\mathcal{P}H^0(m^*\mathcal{P}). \]

By assumption, $m^*[-1]\mathcal{P} \in \mathcal{P}D^{\leq 0}(Z)$, so that $\mathcal{P}H^k(m^*[-1]\mathcal{P}) = 0$ for all $k > 0$. Thus,

\[ \mathcal{P}H^0(m_*m^*\mathcal{P}) \cong m_*\mathcal{P}H^1(m^*[-1]\mathcal{P}) = 0; \]

hence, $\alpha$ is surjective, and we have $\text{im} \beta = \text{im}(\beta \circ \alpha) \cong I^*_X(l^*\mathcal{P})$. We then obtain the isomorphism $I^*_X(l^*\mathcal{P}) \cong \text{im}\{\mathcal{P} \to \mathcal{P}H^0(l_*l^*\mathcal{P})\}$.

Finally, the result follows from the long exact sequence in perverse cohomology associated to the distinguished triangle

\[ m_*m^!\mathcal{P} \to \mathcal{P} \to l_*l^*\mathcal{P} \to l, \]

since $m_*m^!\mathcal{P} \in \mathcal{P}D^{\geq 0}(X)$ and $l_*l^*\mathcal{P} \in \mathcal{P}D^{\geq 0}(X)$, (see, e.g., Proposition 10.3.3 of [8], or Theorem 5.2.4 of [2]).
From the introduction, let $\Sigma X$ denote the singular locus of $X$, and let $i : \Sigma X \hookrightarrow X$. We can then find a smooth, Zariski open dense subset $U \subseteq \Sigma X$ over which the normalization map restricts to a covering projection $\tilde{\pi} : \tilde{\pi}^{-1}(U) \to U \subseteq \Sigma X$ (see Section 6.2, [4]). Let $i : U \hookrightarrow \Sigma X$ and $m : \Sigma X \setminus U \to \Sigma X$ denote the respective open and closed inclusion maps. Let $\tilde{m} := i \circ m$, $\tilde{l} := i \circ l$. Note that $\dim_0 \Sigma X \setminus U \leq n - 2$, as it is the complement of a Zariski open set.

**Proposition 2.2.** If the normalization of $X$ is a rational homology manifold, then $\tilde{m}^*[-1]|N^*_X \in \mathcal{L}(\Sigma X \setminus U)$.

**Proof.** We wish to show that for all $k \in \mathbb{Z}$,

$$\dim \text{supp } H^k(\tilde{m}^*[-1]|N^*_X) \leq -k.$$ 

However, $\text{supp } H^k(\tilde{m}^*[-1]|N^*_X)$ is non-empty only for $k - 1 = -n + 1$, i.e., when $k = -n + 2$. In this degree, the support is equal to $\Sigma X \setminus U$. Since this set is the complement of a Zariski open dense subset of $\Sigma X$,

$$\dim \text{supp } H^{-n+2}(\tilde{m}^*N^*_X) \leq n - 2,$$

as desired. \qed

**Remark 2.3.** For surfaces $X = V(f)$ with curve singularities, $\tilde{m}^*[-1]|N^*_X(f) \in \mathcal{L}(\Sigma f \setminus U)$ if and only if the normalization is a rational homology manifold (see Section 3).

In general, $\tilde{m}^*[-1]|N^*_X \in \mathcal{L}(\Sigma X \setminus U)$ places strict constraints on the possible cohomology groups of the real link of $X$ at different points $p \in \Sigma f$, denoted $K_{X,p}$, i.e., the intersection of $X$ with a sphere of sufficiently small radius at $p$.

**Remark 2.4.** Generically along an irreducible component $C$ of $\Sigma X$, $N^*_X$ is isomorphic to a local system $\tilde{l}^*(N^*_{X|C})$ in degree $-n + 1$, and in that degree, we have

$$H^{-n+1}(N^*_X)_p \cong \tilde{H}^{-n}(K_{X,p}; \mathbb{Q}),$$

where $\tilde{H}$ denotes reduced hypercohomology. This description follows immediately from short exact sequence (1). Since $\mathbb{I}_X^* \cong \pi_1 \mathbb{Q}^*_Y[n]$, this reduced hypercohomology is actually just

$$\tilde{H}^{-n}(K_{X,p}; \mathbb{I}_X^*) \cong \tilde{H}^{0}(K_{Y,\pi_1^{-1}(p)}; \mathbb{Q}),$$

where

$$K_{Y,\pi_1^{-1}(p)} = \bigcup_{q \in \pi_1^{-1}(p)} K_{Y,q}.$$ 

Since $Y$ is normal (and thus locally irreducible) it is clear that one has $\tilde{H}^{0}(K_{Y,q}; \mathbb{Q}) \cong \mathbb{Q}$ for all $q \in Y$. After noting that $H^{-n}(\mathbb{I}_X^*)_p = \tilde{H}^{0}(K_{X,p})$ (that is, intersection cohomology of $K_{X,p}$ with topological indexing), $H^{-n}(\mathbb{I}_X^*)_p$ has a pure Hodge structure of weight 0 (see, e.g., A. Durfee and M. Saito [3]).

**Proposition 2.5.** Let $C$ be an irreducible component of $\Sigma X$ at $0$. Then, $\tilde{l}^*(N^*_{X|C})$ underlies a polarizable variation of Hodge structure of weight 0.

Consequently, $\mathbb{I}_X^*(\tilde{l}^*N^*_X)$ underlies a polarizable Hodge module of weight $n - 1$ on $\Sigma X$. 

5 The Weight Filtration for Parameterized Hypersurfaces
Proof. Since $\hat{l}^*\mathbf{N}_X^\bullet$ underlies a mixed Hodge module whose underlying perverse sheaf is a local system (up to a shift) on the complex manifold $U$, this local system underlies an admissible graded polarizable variation of mixed Hodge structures on $U$ by Theorem 3.27 [11].

To show that this mixed Hodge structure is pure of weight zero, we can check on stalks at points $p \in U$. Let $i_p : \{p\} \hookrightarrow U$; then, the stalk cohomology $H^k(\cdot)_p$ agrees with perverse cohomology $^pH^k(i^*_p)$. So, applying $H^k(i^*_p)$ on the category of mixed Hodge modules to the short exact sequence (1), we get by Proposition 2.19, Proposition 2.20, and Theorem 3.9 of [11] a short exact sequence in the category of graded polarizable mixed Hodge structures, whose underlying sequence of vector spaces is

$$0 \to \mathbb{Q}_{\{p\}} \to H^{-n}(\mathbf{I}_X^*)_p \to H^{-n+1}(\mathbf{N}_X^*)_p \to 0. \quad (2)$$

However, $\pi : Y \to X$ is a finite map, and therefore exact for the perverse $t$-structure (and mixed Hodge modules), with

$$H^{-n}(\mathbf{I}_X^*)_p \cong H^{-n}(\pi_*\mathbb{Q}_X^*[n])_p \cong \bigoplus_{y \in \pi^{-1}(p)} \mathbb{Q}_{\{y\}}.$$

Since this stalk is pure of weight zero, the surjection in (2) implies $H^{-n}(\mathbf{N}_X^*)_p$ is also pure of weight zero.

□

From the introduction, we have the inclusions $\hat{m} : \Sigma X \setminus U \to X$ and $\hat{l} : U \to X$, which give the distinguished triangle

$$m_*m^!i^*\mathbf{N}_X^\bullet \to i^*\mathbf{N}_X^\bullet \to l_*\hat{l}^*\mathbf{N}_X^\bullet \to.$$  

By Lemma 2.1, Proposition 2.2, and Proposition 2.5 we now have a short exact sequence of perverse sheaves coming from a short exact sequence of mixed Hodge modules (Corollary 2.20 [11])

$$0 \to m_*m^!H^0(m^!i^*\mathbf{N}_X^\bullet) \to i^*\mathbf{N}_X^\bullet \to \mathbf{I}_{\Sigma X}^\bullet(\hat{l}^*\mathbf{N}_X^\bullet) \to 0 \quad (3)$$

where $i^*\mathbf{N}_X^\bullet$ has weight $\leq n - 1$ (recall $\mathbf{N}_X^\bullet$ has weight $\leq n - 1$, and $i^*$ does not increase weights [10] pg. 340), and $\mathbf{I}_{\Sigma X}^\bullet(\hat{l}^*\mathbf{N}_X^\bullet)$ has weight $n - 1$. Since a short exact sequence of mixed Hodge modules is strictly compatible with the weight filtration, and the functor $Gr_{n-1}^W$ is exact on the Abelian category of polarizable mixed Hodge modules, we have the short exact sequence of mixed Hodge modules and their underlying perverse sheaves

$$0 \to Gr_{n-1}^W m_*m^!H^0(m^!i^*\mathbf{N}_X^\bullet) \to Gr_{n-1}^W i^*\mathbf{N}_X^\bullet \to \mathbf{I}_{\Sigma X}^\bullet(\hat{l}^*\mathbf{N}_X^\bullet) \to 0.$$

We can now state and prove our main result. We would like to express our thanks to Jörg Schürmann for suggesting a simplified version of our original proof.

**Theorem 2.6.** Suppose the normalization of $X$ is a rational homology manifold. Then, there is an isomorphism $Gr_{n-1}^W i^*\mathbf{N}_X^\bullet \cong \mathbf{I}_{\Sigma X}^\bullet(\hat{l}^*\mathbf{N}_X^\bullet)$, so that the short exact sequence of perverse sheaves on $X$

$$0 \to m_*m^!H^0(m^!i^*\mathbf{N}_X^\bullet) \to i^*\mathbf{N}_X^\bullet \to \mathbf{I}_{\Sigma X}^\bullet(\hat{l}^*\mathbf{N}_X^\bullet) \to 0$$

identifies $W_{n-2}i^*\mathbf{N}_X^\bullet \cong m_*m^!H^0(m^!i^*\mathbf{N}_X^\bullet).$
Proof. Since \( \text{Gr}_{n-1}^W i^* N_X^* \) underlies a pure Hodge module, it is by definition semi-simple as a perverse sheaf, i.e., a direct sum of simple intersection cohomology sheaves with irreducible support. Hence, we can write \( \text{Gr}_{n-1}^W i^* N_X^* \) as direct sum of a semi-simple perverse sheaf \( M^* \) with support in \( \Sigma X \backslash \mathcal{U} \) and a semi-simple perverse sheaf whose summands are all not supported on \( \Sigma X \backslash \mathcal{U} \). This second semi-simple perverse sheaf has to be \( I_{\Sigma X}^\bullet ( \hat{l}^* N_X^* ) \), by pulling back the short exact sequence (3) by \( \hat{l}^* \).

Finally, we claim \( M^* = 0 \). Since \( M^* \) is a direct summand of \( \text{Gr}_{n-1}^W i^* N_X^* \), we have a surjection of perverse sheaves
\[
i^* N_X^* \to \text{Gr}_{n-1}^W i^* N_X^* \to M^*.
\]

But \( H^0(m^*) \) is right exact for the perverse t-structure (since \( m^* \) is a closed inclusion), so we also get a surjection
\[
0 = H^0(m^*) \to H^0(m^* M^*) = M^* \to 0,
\]
where the last equality follows from the fact that \( M^* \) is supported on \( \Sigma X \backslash \mathcal{U} \).

\[\square\]

Corollary 2.7. There are isomorphisms
\[
\text{Gr}_{n-1}^W \mathbb{Q}_X[n] \cong \text{Gr}_{n-1}^W N_X^* \cong \mathcal{I}_{\Sigma X}^\bullet ( \hat{l}^* N_X^* ),
\]
and
\[
W_{n-2} \mathbb{Q}_X[n] \cong W_{n-2} N_X^* \cong \mathcal{I}_{\Sigma X}^\bullet ( \hat{l}^* N_X^* ) .
\]

As mentioned in the introduction, this trivially follows from the fact that \( i_* \) preserves weights ([10], pg. 339), is exact for the perverse t-structure, and from the fact that \( i_* i^* N_X^* \cong N_X^* \), since the support of \( N_X^* \) is contained in \( \Sigma X \).

At first glance, the formula for \( W_{n-2} i^* N_X^* \) appears quite abstruse. We now give a much more geometric interpretation of this perverse sheaf.

Theorem 2.8. Let \( g \) be a complex analytic function on \( \Sigma X \) such that \( V(g) \) contains \( \Sigma X \backslash \mathcal{U} \), but does not contain any irreducible component of \( \Sigma X \). Then,
\[
W_{n-2} i^* N_X^* \cong m' \ker \{ \phi_g[-1] i^* N_X^* \xrightarrow{\text{var}} \psi_g[-1] i^* N_X^* \},
\]
where the kernel is taken in the category of perverse sheaves on \( \Sigma X \), var is the variation morphism, and \( m' : V(g) \to \Sigma X \) is the closed inclusion.

Proof. We first note that such a function \( g \) exists locally by the prime avoidance lemma. Then, \( \Sigma X \backslash V(g) \subseteq \mathcal{U} \), and we have as perverse sheaves
\[
\mathcal{I}_{\Sigma X}^\bullet ( i^* N_X^* |_{\Sigma X \backslash V(g)} ) \cong \mathcal{I}_{\Sigma X}^\bullet ( \hat{l}^* N_X^* ),
\]
since the normalization is still a covering projection away from \( V(g) \) in \( \Sigma X \). One notes then that the proofs of Proposition 2.2, Proposition 2.5, and Theorem 2.6 remain unchanged with these new choices of complementary subspaces \( V(g) \hookrightarrow \Sigma X \) and \( \Sigma X \backslash V(g) \xrightarrow{l} \Sigma X \), so that
\[
\text{Gr}_{n-1}^W i^* N_X^* \cong \mathcal{I}_{\Sigma X}^\bullet ( i^* N_X^* |_{\Sigma X \backslash V(g)} )
\]
and
\[
W_{n-2} i^* N_X^* \cong m' \text{ } H^0(m' l^* i^* N_X^* ).
\]
The claim then follows by taking the long exact sequence in perverse cohomology of the variation distinguished triangle

$$\phi_g[-1]i^*N_X^* \xrightarrow{\text{var}} \psi_g[-1]i^*N_X^* \to m'^i[1]i^*N_X^* \xrightarrow{\text{var}}$$

yielding

$$0 \to pH^0(m'^i i^*N_X^*) \to \phi_g[-1]i^*N_X^* \xrightarrow{\text{var}} \psi_g[-1]i^*N_X^* \to pH^1(m'^i i^*N_X^*) \to 0.$$ 

\[ \square \]

3. The Surface Case

Suppose \( X = V(f) \) is a surface in \( \mathbb{C}^3 \); we want to compute \( W_0\mathbb{Q}_V(f)[2] \) using the isomorphism

\[ W_0\mathbb{Q}_V(f)[2] = W_0N_V^*(f) \cong \tilde{m}_*pH^0(m^i i^*N_V^*(f)). \]

The main tool we use is the following: if \( \dim_0 \Sigma f = 1 \), then \( \Sigma f|U \) is zero dimensional (or empty), and perverse cohomology on a zero-dimensional space is just ordinary cohomology. Recall that \( Y \xrightarrow{\sim} V(f) \) is the normalization map.

**Theorem 3.1.** Suppose the normalization of \( V(f) \) is a rational homology manifold. Then,

\[ W_0\mathbb{Q}_V(f)[2] \cong V_{\{0\}}^* \]

is a perverse sheaf concentrated on a single point, i.e., a finite-dimensional \( \mathbb{Q} \)-vector space, of dimension

\[ \dim V = 1 - |\pi^{-1}(0)| + \sum_C \dim \ker \{ \text{id} - h_C \}, \]

where \( \{C\} \) is the collection of irreducible components of \( \Sigma f \) at \( 0 \), and for each component \( C, h_C \) is the (internal) monodromy operator on the local system \( H^{-1}(N_V^*(f))|_{C\backslash\{0\}} \).

Note that \( |\pi^{-1}(0)| \) is, of course, equal to the number of irreducible components of \( V(f) \) at \( 0 \).

**Proof.** First, note that we have \( \Sigma f\backslash\mathcal{U} = \{0\} \), and \( \mathcal{U} = \bigcup_{C}(C\backslash\{0\}) \), where each \( C\backslash\{0\} \) is homeomorphic to a punctured complex disk. Then, we find

\[ pH^0(m^i i^*N_V^*(f)) \cong H^0(m^i i^*N_V^*(f)) \cong H^0(\Sigma f, \Sigma f\backslash\{0\}; i^*N_V^*(f)). \]

We can compute this last term from the long exact sequence in relative hypercohomology with coefficients in \( N_V^*(f) \):

\[ 0 \to H^{-1}(\Sigma f, \Sigma f\backslash\{0\}; i^*N_V^*(f)) \to H^{-1}(N_V^*(f))_0 \to H^{-1}(\Sigma f\backslash\{0\}; i^*N_V^*(f)) \to \]

\[ H^0(\Sigma f, \Sigma f\backslash\{0\}; i^*N_V^*(f)) \to H^0(N_V^*(f))_0 \to H^0(\Sigma f\backslash\{0\}; i^*N_V^*(f)) \to 0 \]

The cosupport condition on \( i^*N_V^*(f) \) implies \( H^{-1}(\Sigma f, \Sigma f\backslash\{0\}; i^*N_V^*(f)) = 0 \). Additionally, since \( H^0(N_V^*(f))_0 \) is only supported on \( \{0\} \), it follows that \( H^0(\Sigma f\backslash\{0\}; i^*N_V^*(f)) = 0 \) as well. Since the normalization of \( V(f) \) is rational homology manifold, \( H^0(N_V^*(f))_0 = 0 \) by Theorem 1.2, and \( \dim H^{-1}(N_V^*(f))_0 = |\pi^{-1}(0)| - 1 \).
Finally, 
\[
\mathbb{H}^{-1}(\Sigma f \setminus \{0\}; i^*N_{V(f)}^\bullet) \cong \bigoplus_C \mathbb{H}^{-1}(C \setminus \{0\}; i^*N_{V(f)}^\bullet).
\]
This last term is easily seen to be (the sum of) global sections of the local system \(H^{-1}(N_{V(f)}^\bullet|_{C \setminus \{0\}})\), which is just \(\ker \{\text{id} - h_C\}\). Taking the alternating sums of the dimensions of the terms in the resulting short exact sequence 
\[
0 \to H^{-1}(N_{V(f)}^\bullet|_0) \to \bigoplus_C \ker \{\text{id} - h_C\} \to \mathbb{H}^0(\Sigma f \setminus \{0\}; i^*N_{V(f)}^\bullet) \to 0
\]
yields the desired result. 

**Example 3.2.** Let \((x, y, z) = y^2 - x^3 - zx^2\), so that \(V(f)\) is the Whitney umbrella. Then, \(\Sigma f = V(x, y)\), and \(V(f)\) has (smooth) normalization given by \(F(u, t) = (u^3 - t, u(u^2 - t), t)\). Then, it is easy to see that the internal monodromy operator \(h_C\) along the component \(V(x, y)\) is multiplication by 2, so \(\ker \{\text{id} - h_C\} = 0\). Hence, 
\[
W_0Q_{V(f)}[2] = 0.
\]

**Example 3.3.** Let \((x, y, z) = xz^2 - y^3\), so that \(\Sigma f = V(y, z)\). Then, the normalization \(Y\) is equal to 
\[
Y = V(u^3 - xy, uy - xz, uz - y^2) \subseteq \mathbb{C}^4,
\]
(i.e., the affine cone over the twisted cubic) and the normalization map \(\pi\) is induced by the projection \((u, x, y, z) \mapsto (x, y, z)\). By Section 4, \(Y\) is a rational homology manifold. The internal monodromy operator \(h_C\) on \(H^{-1}(N_{V(f)}^\bullet|_{V(y, z) \setminus \{0\}})\) is trivial, so \(\ker \{\text{id} - h_C\} \cong \mathbb{Q}\). Thus, 
\[
W_0Q_{V(f)}[2] \cong \mathbb{Q}\{0\}.
\]

**Example 3.4.** \((x, y, z) = xyz\), so \(\Sigma f = V(x, y) \cup V(y, z) \cup V(x, z)\). Then, \(|\pi^{-1}(0)| = 3\), and the internal monodromy operators \(h_C\) are all the identity. It then follows that 
\[
W_0Q_{V(f)}[2] \cong \mathbb{Q}\{0\}.
\]

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