Light-Cone Gauge Quantization of 2D Sigma Models

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ABSTRACT

This work describes the formulation of the manifestly ghost-free (spacetime) light-cone gauge for bosonic string theory with non-trivial spacetime metric, antisymmetric tensor, dilaton and tachyon fields. The action is a general two-dimensional sigma model, corresponding to a closed string theory with a second order action in the Polyakov picture. The spacetime fields must have a symmetry generated by a null, covariantly constant spacetime vector in order for the light-cone gauge to be accessible. Also, the theory must be Weyl invariant. The conditions for Weyl invariance are computed within the light-cone gauge, reproducing the usual beta functions. The calculation of the dilaton beta function and the critical dimension is somewhat novel in this ghost-free theory. Some exactly solvable light-cone theories are discussed.

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1 Introduction

The light-cone gauge played an important role in the early development of string theory. It provided a way to consistently quantize the Nambu-Goto string from first principles, giving a manifestly unitary theory \([1, 2]\). The bosonic string was found to suffer from an anomaly in the Lorentz algebra unless the number of spacetime dimensions is 26, and the Regge intercept is 1. This was the first definite problem encountered with string theory in less than the critical dimension. The operator formalism had required \(D \leq 26\) to be ghost-free \([3]\), and it had favored the critical dimension in order to have proper factorization of open string amplitudes \([4]\), but it was not clear that the theory was inconsistent in fewer than 26 dimensions.

String interactions were incorporated into the light-cone theory \([5]\), and the three string vertex was found to be Lorentz invariant exactly in the critical dimension \([6]\). Scattering amplitudes were calculated using Neumann function techniques. The theory was then reformulated as the first string field theory \([7]\). The Mandelstam diagrams of the first quantized theory were cut into propagators and vertices, and the theory was second quantized. The Feynman rules for both the open, and the closed bosonic string were derived. Light-cone gauge was also instrumental in the development of fermionic string theory \([8]\). The supersymmetric versions of light-cone gauge quantum mechanics, first quantized field theory and second quantized field theory have been developed as well \([9]\).

Then the light-cone gauge was largely abandoned in favor of formulations in which the target space is treated covariantly. The advent of the Polyakov picture revolutionized string theory \([10]\). The critical dimension was identified with the vanishing of the Weyl anomaly. Ghosts were introduced to fix the intrinsic geometry of the worldsheet \([10]\). The covariant formulations that followed—conformal field theory \([11]\), the operator formalism \([12]\) and covariant string field theory \([13]\)—have the advantage of not singling out any direction in spacetime. Also, there is a gauge symmetry, conformal invariance, that is very powerful in analyzing the theory. Despite the fact that the covariant formulations retain more unphysical degrees of freedom, scattering amplitudes at the zero- and one-loop order are calculated more easily than in light-cone gauge. This is because there is no complete operator formalism for the light-cone gauge. The tree level amplitudes are easy when \(p^+ = 0\) for all but two of the strings; otherwise, operator techniques cannot be used since \(e^{ip^+X^-}\) is not well-defined \([3]\).

The Nambu-Goto action had allowed for string propagation in curved spacetime. A non-trivial spacetime metric could enter the action in the Polyakov picture, too, but other spacetime fields could be added to the action, as well. The dilaton, in particular, was found to
be vital to renormalization on a curved worldsheet [14]. Sigma model perturbation theory and background field techniques were developed to determine the conditions for Weyl invariance [14]–[25]. The resulting beta functionals generalized the concept of the critical dimension, requiring the spacetime background fields to satisfy differential equations to assure Weyl invariance. When these equations are violated, the worldsheet theory is not scale invariant, so the Liouville mode must be quantized as well, leading to non-critical strings.

The development of the covariant formulations has been independent of the light-cone gauge for the most part, after the invention of conformal field theory. The one striking exception is the covariant closed string field theory. The required non-polynomial action was elusive, and a great deal of effort went into studying the relatively simple light-cone gauge closed string field theory. It only has three string vertices and no contact terms. It still provides one of the few known triangulations of moduli space [26].

The goal of this article is to quantize the general two-dimensional bosonic sigma model in the light-cone gauge. This amounts to using the light-cone gauge to quantize closed string theory in the Polyakov picture [1]. The worldsheet geometry plays a fundamental role, since the light-cone gauge fixes the worldsheet reparameterization invariance. This is done without first going to the conformal gauge (as advocated by [27]). The light-cone gauge avoids propagating reparameterization ghosts. It eliminates the negative norm modes, at the expense of manifest worldsheet covariance.

The light-cone gauge is found to be accessible and non-singular provided that all the spacetime fields (the metric, antisymmetric tensor field, dilaton and tachyon backgrounds) have a symmetry generated by a null covariantly constant spacetime vector. Also, the spacetime fields must satisfy the usual Weyl invariance conditions (vanishing beta functions) as in the conformal gauge—i.e. it must be a critical string theory. The way these conditions arise is somewhat novel, especially the \((D - 26)\) term of the dilaton beta function in this ghost-free theory. These conditions guarantee a consistent string quantum mechanics. They also seem to be enough to allow interactions and a consistent string field theory, but we will not go through a complete analysis.

Having recounted a few of the many wonderful successes of the covariant formulations, one might well ask why the light-cone gauge should be revived in the more general Polyakov framework. Will it only provide a cumbersome check of results from the covariant techniques? There are two applications of current interest which could benefit greatly from a light-cone

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1 This is not to be confused with Polyakov’s light-cone gauge quantization of two-dimensional quantum gravity.
gauge analysis. In both cases it is the fact that the light-cone gauge explicitly retains only physical degrees of freedom that makes it useful. All of the unphysical degrees of freedom, like the Lorentz and reparameterization ghosts, are removed by gauge fixing.

The first application is the plane-fronted gravitational waves, and their stringy generalizations [28, 29]. Sigma models with these target spaces have been shown to be Weyl invariant to all orders in sigma model perturbation theory. Perhaps they are exactly solvable. Covariant techniques could be used for the analysis. They are not intractable since the sigma model perturbation theory stops at one loop in any calculation. Even so, the bookkeeping is unwieldy. Light-cone gauge simplifies the calculations enormously, and the generalized plane-fronted wave background fields meet the requirements for light-cone gauge quantization exactly. In fact, Horowitz and Steif have already used light-cone gauge quantum mechanics to study a string propagating through a plane-fronted wave to lowest order in $\alpha'$ [30]. Our work puts their analysis on a sound footing, proving unitarity, for instance. It also prepares the way for a full string field theory treatment using the relatively simple light-cone closed string field theory.

The second application is the class of sigma models with two-dimensional target spaces coming from Liouville theory and non-critical strings [31]. These models of $c \leq 1$ matter coupled to two-dimensional quantum gravity were first solved by matrix models [32]. The continuum solutions are less complete, and many mysteries remain. One of their most striking features is the presence of physical states at discrete values of the momentum. The naive light-cone Hilbert space would be trivial—nothing but the tachyon—because there are no transverse dimensions. It is interesting to see why the usual light-cone gauge fails, and how it might be generalized to include the special states. This turns out to be a subtle problem [34], and its solution will be presented elsewhere [35].

Having cited some possible applications, we are in a position to state what we should demand of a light-cone quantization, since we might be willing to forgo some of the usual properties of the light-cone gauge. In order of increasing stringency, these properties are the elimination of as many unphysical degrees of freedom as possible, the elimination of Lorentz ghosts, and the absence of all ghosts so that the theory is manifestly unitary. In addition, we might demand that the theory have a string field theory representation, perhaps even a simple one. We should also demand that we be able to check the consistency of the theory from within the light-cone gauge. Traditionally this is done by computing the Lorentz anomaly, but it is a big restriction to demand that spacetime have a Lorentz isometry between $X^+$ and two transverse dimensions. The formulation that we will develop has all of the usual
light-cone gauge properties (except the Lorentz isometry), but we will point out where some of these restrictions could be eased in future applications.

2 Gauge Fixing

The generating functional $Z$ for bosonic closed string theory in the Polyakov picture is (see \[27\] for a review)

$$Z = \sum_{\text{topologies}} \int [\mathcal{D}g_{ab}] \mathcal{D}X^\mu e^{iS[g_{ab}, X]}$$

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left\{ g^{ab}G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \frac{\epsilon^{ab}}{\sqrt{g}} B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu - \alpha' R^{(2)} \Phi + T \right\}.$$

(2.1)

The action $S[g_{ab}, X]$ is a functional of the worldsheet metric $g_{ab}$ and the $D$ spacetime coordinate fields $X^\mu$. These fields live on a worldsheet parameterized by the coordinates $\sigma^a$ where $\sigma^0 = \tau$, $\sigma^1 = \sigma$, and $\sigma$ ranges from 0 to $2\pi$. Both the spacetime and the worldsheet have the Lorentzian signature $(-, +, \cdots, +)$. The spacetime fields consist of the metric $G_{\mu\nu}(X)$, an antisymmetric tensor field $B_{\mu\nu}(X)$, the dilaton $\Phi(X)$ and the tachyon $T(X)$. They are functionals of the fields $X^\mu$. The tachyon would be absent from the corresponding supersymmetric model, and it has been included primarily due to the important role it plays in two dimensions where it is massless. $\epsilon^{ab}$ is the Levi-Civita antisymmetric tensor (density) such that $\epsilon^{01} = 1$. Note that spacetime indices are Greek, worldsheet indices are Roman, and repeated indices are summed. The action is usually Wick rotated to a Euclidean worldsheet where the sum over topologies becomes a sum over the genera of compact Riemann surfaces. We will perform the Wick rotation before quantization in section 3.

The integral over $g_{ab}$ denotes an integral over worldsheet metrics modulo worldsheet diffeomorphisms. The action is invariant under reparameterizations of the world-sheet coordinates where the fields transform as

$$\delta X^\mu = \epsilon^a \partial_a X^\mu + \cdots$$

$$\delta g^{ab} = -\left( \nabla_a \epsilon^b + \nabla_b \epsilon^a \right) = \epsilon^c \partial_c g^{ab} - g^{ac} \partial_c \epsilon^b - g^{bc} \partial_c \epsilon^a$$

$$\delta \sqrt{g} = -\frac{1}{2} \sqrt{g} g_{ab} \delta g^{ab} = \sqrt{g} \nabla_a \epsilon^a = \partial_a (\epsilon^a \sqrt{g})$$

(2.2)

under the diffeomorphism $\sigma^a \rightarrow \sigma^a + \epsilon^a(\sigma, \tau)$. The $X^\mu$ transformation is typically very complicated and not exactly calculable (with non-trivial background fields), but it may be calculated to low orders in sigma model perturbation theory. For our purposes it is enough
to know that the correction term transforms as a spacetime vector depending on $\Phi$ and $T$ classically and on all the background fields in the quantum theory.

For certain configurations of the space-time metric, antisymmetric tensor, dilaton and tachyon backgrounds, the theory is also invariant under Weyl scalings

$$\delta g^{ab} = \Lambda(\sigma^c) g^{ab}. \quad (2.3)$$

The background fields will be chosen such that the Weyl anomaly cancels and the Liouville mode decouples in the quantum theory. The Weyl mode does not decouple for generic background configurations. In fact, the trace of the stress tensor (which generates Weyl transformations) takes the form

$$\sqrt{g} T^a = \beta^T (X) \sqrt{g} + \beta^\Phi (X) \sqrt{g} \partial_\mu X^\mu \partial^\nu X^\nu + \beta^B_{\mu \nu} (X) \epsilon^{ab} \partial_\mu X^\mu \partial_\nu X^\nu \quad (2.4)$$

in conformal gauge. The beta functionals $\beta^G_{\mu \nu}$, $\beta^B_{\mu \nu}$, $\beta^\Phi$ and $\beta^T$ must vanish for the action to be Weyl invariant. Actually, the beta functions are equivalent to the equations of motion for the massless fields entering the low-energy effective action, so they must vanish if the theory is to be sensible. This is not a requirement put in by hand. It is difficult to calculate the beta functionals exactly for general actions, but they may be found using a perturbation expansion in $\alpha'$. The well-known result is

$$\beta^G_{\mu \nu} = R_{\mu \nu} + 2 \nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\lambda \mu \sigma} H^{\lambda \mu \sigma} - \partial_\mu T \partial_\nu T + \cdots$$

$$\beta^\Phi = -\frac{\alpha'}{16\pi^2} \left[ \frac{26 - D}{3\alpha'} + R - 4 \partial_\mu \Phi \partial^\mu \Phi + 4 \nabla^2 \Phi - \frac{1}{12} H^2 - \partial_\mu T \partial^\mu T + 2 T^2 + \cdots \right]$$

$$\beta^B_{\mu \nu} = \nabla_\lambda H^\lambda_{\mu \nu} - 2 (\nabla_\lambda \Phi) H^\lambda_{\mu \nu} + \cdots$$

$$\beta^T = -2 \nabla^2 T + 4 G^{\mu \nu} \partial_\mu \Phi \partial_\nu T - 4 T + \cdots \quad (2.5)$$

where the field strength for the antisymmetric tensor field is given by $H_{\alpha \beta \gamma} = \partial_\alpha B_{\beta \gamma} + \partial_\beta B_{\gamma \alpha} + \partial_\gamma B_{\alpha \beta}$ [21]. We will consider sigma models for which these beta functionals are zero, for now, in order to formulate the general light-cone gauge quantization. Once the formalism is established, we will return to the question of the Weyl anomaly within the light-cone gauge. If other (non-critical) backgrounds are used as the starting point for quantization, additional degrees of freedom (such as the Liouville mode) enter during quantization such that the equations are solved in the end. The Liouville mode couples through the Weyl anomaly, for instance. This is somewhat awkward to treat in light-cone gauge, so we will postpone this discussion. For now, we assume that we are dealing with a general critical string theory.
The basic idea with the light-cone gauge is to use reparameterization invariance to gauge away the oscillator contribution to one of the $X^\mu$ coordinates—in particular a null (light-cone) coordinate. In fact, the reparameterization group is large enough to do this and at the same time to fix a conformally flat world-sheet metric. The conformal factor may then be set to unity in a Weyl invariant theory. The oscillators may be eliminated from any one of the coordinates, but when a null coordinate is chosen, the other null coordinate becomes an auxiliary field. It may be expressed in terms of the transverse coordinates through the constraints, so that only the physical degrees of freedom enter the gauge fixed theory. There are no ghosts.

This section will develop the formalism classically. The approach we will take is somewhat unconventional, but it is appropriate for quantizing the Polyakov theory in the second order formalism. It only works for critical string theories, where the beta functionals vanish. The standard light-cone gauge approaches to quantizing the $D=26$ bosonic string include quantizing the Nambu-Goto action \[1, 5\], quantizing the second order action by choosing the light-cone gauge subsequent to choosing the conformal gauge \[27\] and quantizing using hybrid light-cone/conformal gauges \[36\]. These approaches are either inappropriate for the quantization of an action with a dilaton term (such as Nambu-Goto) or they have reparameterization ghosts. Another option is to quantize the first order Polyakov action \[37\], but this is inconvenient for the discussion of general sigma model backgrounds. So we will quantize the Polyakov action in the second order formalism.

Consider the variation of the action (2.1) with respect to the fields $X^\mu$ and $g^{\mu\nu}$. This yields the classical field equations

$$0 = -4\pi\alpha' \frac{\delta S}{\delta X^\mu} = -2G_{\mu\nu} \left( \Delta X^\nu + \Gamma^\nu_{\alpha\beta} g^{ab} \partial_a X^\alpha \partial_b X^\beta \right)$$

$$+ \frac{e^{ab}}{\sqrt{g}} H_{\mu\alpha\beta} \partial_a X^\alpha \partial_b X^\beta - \alpha' R^{(2)} \partial_\mu \Phi + \partial_\mu T$$

$$0 = \frac{-4\pi\alpha'}{\sqrt{g}} \frac{\delta S}{\delta g^{ab}} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - \alpha' R^{(2)} \partial_\mu \Phi + \alpha' (\nabla_a \nabla_b X^\mu) \partial_\mu \Phi + \alpha' (\partial_a X^\mu \partial_b X^\nu) \partial_\mu \partial_\nu \Phi$$

$$- \frac{1}{2} g^{ab} \left\{ g^{cd} G_{\mu\nu} \partial_c X^\mu \partial_d X^\nu - \alpha' R^{(2)} \Phi + 2\alpha' \nabla^c \nabla_c \Phi + T \right\}$$

(2.6)

where the Laplacian is given by $\Delta = \frac{1}{\sqrt{g}} \partial_a \sqrt{g} g^{ab} \partial_b$. The variation with respect to the worldsheet metric is the stress-energy tensor, $T_{ab}$. Because string theory is Weyl invariant, it is useful to decompose the metric into a Weyl part, $e^{ab}$, and a unit determinant part, $\gamma_{ab}$. That is,

$$g^{ab} = e^{-\phi(\sigma)} \gamma^{ab} \quad \text{with} \quad \det \gamma^{ab} = -1.$$  

(2.7)
Note that $\gamma^{ab} = \sqrt{g}g^{ab}$ has two degrees of freedom. The covariant metric on the space of $\gamma$’s is given by

$$|\delta \gamma^{ab}|^2 = \int d^2 \sigma \gamma_{ac} \gamma_{bd} \delta \gamma^{ab} \delta \gamma^{cd}.$$  

(2.8)

Rewriting the action with this metric decomposition, we find

$$Z = \sum_{\text{topologies}} \int [D\phi] D\gamma_{ab} D\sigma e^{iS[g_{ab}, X]}$$

$$S = -\frac{1}{4\pi\alpha'} \int d^2 \sigma \left\{ \gamma^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - \alpha'(-\Delta^{(\gamma)} \phi + R^{(\gamma)} \Phi + e^{\phi} T) \right\}$$  

(2.9)

where the integral over $\phi$ is trivial in the full quantum theory due to the required Weyl invariance, and the $\gamma$ integral runs over the two independent components. The variation of the action with respect to the metric (i.e. the stress tensor) may be reexpressed as

$$0 = -4\pi\alpha' \frac{\delta S}{\delta \gamma_{ab}} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - \alpha' R^{(2)}_{\mu\nu} \Phi + \alpha' (\partial_a X^\mu \partial_b X^\nu) \partial_\mu \partial_\nu \Phi - \frac{1}{2} \gamma_{ab} \left[ (G_{\mu\nu} + \alpha' \partial_\mu \partial_\nu \Phi) \gamma^{cd} \partial_c X^\mu \partial_d X^\nu - \alpha' R^{(2)}_{\mu\nu} \Phi + \alpha' (\nabla^2 X^\mu) \partial_\mu \Phi \right]$$

$$0 = -4\pi\alpha' \frac{\delta S}{\delta \phi}_{\text{classical}} = \alpha' \Delta^{(\gamma)} \Phi + e^{\phi} T$$  

(2.10)

The Weyl anomaly cancels the classical variation of the action with respect to $\phi$ in the quantum theory. It will be convenient to define the $\gamma$ stress tensor,

$$T^{(\gamma)}_{ab} = -4\pi\alpha' \frac{\delta S}{\delta \gamma_{ab}} = T_{ab} - \frac{1}{2} \gamma_{ab} (\alpha' \Delta^{(\gamma)} \Phi + e^{\phi} T).$$  

(2.11)

It is traceless since $\text{det} \gamma_{ab} = -1$. Note that $T_{01} = T^{(\gamma)}_{01}$ in gauges with $\gamma^{01} = 0$.

The key to the light-cone gauge is that when the two fields $X^+$ and $\gamma^{00}$ are fixed, the resulting constraints insure that two other fields, $X^-$ and $\gamma^{01}$, are auxiliary. In fact, these fields enter the constraints linearly, so the constraints may be used to solve for them in terms of the transverse fields. This eliminates the maximum number of degrees of freedom explicitly. In the process the worldsheet metric is fixed to be flat, as required ultimately in order to express scattering amplitudes in terms of the usual Mandelstam diagrams for light-cone gauge. We will examine how this works in detail.

The main point of this section is to show under what conditions the light-cone gauge is non-singular. The gauge is not accessible with generic target space fields. First, the gauge conditions must solve the classical equations of motion. Then, the resulting constraints must be solvable. This is what prohibits the light-cone gauge for a flat $D < 26$ target space (upon
quantization). Finally, it should be possible to solve certain constraints as operator equations for the fields \( X^- \) and \( \gamma^{01} \). The gauge fixing does not require Faddeev-Popov ghosts, so this reduces the theory down to the (off-shell) physical degrees of freedom. These criteria are met for a large class of target space fields, as shown in section 4.

2.1 Fixing \( X^+ \) and \( \gamma^{00} \)

Consider the following gauge choice:

\[
\begin{align*}
X^+ &= p^+ \tau + x_0^+ \\
\gamma^{00} &= -1
\end{align*}
\]  

where

\[
X^\pm = \frac{1}{\sqrt{2}}(X^1 \pm X^0).
\]  

According to equation (2.2), infinitesimal reparameterizations about this gauge choice give

\[
\begin{align*}
\delta X^+ &= \epsilon^a \partial_a (p^+ \tau + x_0^+) + \cdots = p^+ \epsilon^0 + \cdots \\
\delta \gamma^{00} &= \partial_c [\epsilon^c (-1)] - 2 \gamma^{0c} \partial_c \epsilon^0 = \partial_0 \epsilon^0 - \partial_1 \epsilon^1
\end{align*}
\]  

when \( \sigma^a \rightarrow \sigma^a + \epsilon^a (\sigma, \tau) \). The additional terms in \( \delta X^+ \) generically lead to ghosts. They transform as the + component of a spacetime vector depending on \( \Phi \) or \( T \), classically. After quantization and renormalization, these terms also depend on \( G_{\mu\nu} \) and \( B_{\mu\nu} \). A condition sufficient to insure the absence of Faddeev-Popov ghosts is that all such vectors vanish; that is, \( \partial_- \Phi = \partial_- T = \partial_- G_{\mu\nu} = \partial_- B_{\mu\nu} = 0 \), and \( G_{--} = G_{-i} = B_{-\nu} = 0 \). This condition arises from other considerations as well, as we will see below. Setting these terms to zero, the Faddeev-Popov determinant is

\[
\Delta_{FP} = \begin{vmatrix}
p^+ & 0 \\
\partial_0 & -\partial_1
\end{vmatrix}.
\]  

We have used the fact that \( \gamma^{01} \) will be fixed to zero. Since the relevant part (the diagonal) of the determinant does not depend on time derivatives, it just contributes a constant factor to the measure. The corresponding ghosts do not propagate. At the level of quantum mechanics, the determinant could be absorbed into the overall normalization of the path integral, except for the dependence on \( p^+ \). It is odd that the physical parameter \( p^+ \) should enter the normalization, which is independent of the dynamics. It turns out that this \( p^+ \) dependence cancels that coming from the eventual elimination of \( X^- \). In fact, we will see that rescaling the worldsheet by \( p^+ (\sigma \rightarrow \sigma/p^+, \tau \rightarrow \tau/p^+) \) makes both determinants equal to unity. This is especially important, since \( p^+ \) is not a Lorentz scalar. When the spacetime
metric has a global Lorentz isometry, it may be used to check scattering amplitudes and to detect the occurrence of anomalies. Lorentz invariance would be spoiled by $p^+$ in the normalization of the path integral.

The gauge choice (2.12) does not completely fix the reparameterization invariance, since the transformation $\sigma_1 \rightarrow \sigma_1 + \epsilon^1(\tau)$ leaves the gauge condition unchanged. The residual gauge invariance will be used to eliminate $\gamma^{01}$. Then the gauge (2.12) is accessible and unique on any Mandelstam worldsheet with at least one vertex (The two point amplitude at tree level with its cylindrical worldsheet has the residual invariance $\sigma \rightarrow \sigma + \text{const.}$ and $\tau \rightarrow \tau + \text{const.}$).

2.2 The Auxiliary Fields $X^-$ and $\gamma^{01}$

The gauge choice should also constrain the other null coordinate for it to be a useful light-cone gauge. Both $X^-$ and $\gamma^{01}$ turn out to be auxiliary fields. Consider the variation of the gauge fixed action with respect to $\gamma^{01}$:

$$0 = -4\pi\alpha' \frac{\delta S}{\delta \gamma^{01}} = G_{\mu
u} \partial_0 X^\mu \partial_1 X^\nu + \alpha' (\nabla_0 \nabla_1 X^\mu) \partial_\mu \Phi + \alpha' (\partial_0 X^\mu \partial_1 X^\nu) \partial_\mu \partial_\nu \Phi. \quad (2.16)$$

This is a constraint since $\gamma^{01}$ is auxiliary. We would like to solve it for $X^-$, the other light-cone coordinate, expressing it in terms of the transverse coordinates and $p^+$. Of course, $X^-$ is an operator, and the only known way to have a sensible algebra is if it is expressed as a (differential) polynomial in the other fields. It cannot be realized as the square-root of the transverse stress-tensor, for example. So equation (2.16) must be linear in $X^-$. Then $G_{-\mu} = 0$, $\partial_- G_{-\mu} = 0$, $\partial_- \partial_- G_{\mu\nu} = 0$ and $\partial_- \partial_- \Phi = 0$. Also, $G_{-i} = 0$, since $\partial_{a} X^i$ cannot appear in the coefficient of $X^-$ if it is to be inverted. Similarly, $G_{+-}$ must be constant, which will be chosen to be 1 by rescaling $X^-$. Now equation (2.16) becomes

$$0 = p^+ \partial_1 X^- + G_{i+} p^+ \partial_1 X^i + G_{ij} \partial_0 X^i \partial_1 X^j + \alpha' \left\{ (\nabla_0 \nabla_1 X^-) \partial_- \Phi + (\partial_0 X^- \partial_1 X^i) \partial_i \partial_- \Phi + (p^+ \partial_1 X^-) \partial_+ \partial_- \Phi \right\} \quad (2.17)$$

Note that only a few terms coming from the dilaton contain $\tau$ derivatives of $X^-$. If we require additionally that $\partial_- \Phi = 0$, then the terms in the braces in (2.17) vanish, and $X^-$ is

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The only caveat is that the equation might factorize. If the dilaton is constant, then the spacetime metric could be rescaled by an $X^-$ dependent conformal factor, $G_{\mu\nu} = \Lambda(X^-) \tilde{G}_{\mu\nu}$. It would factor out of (2.16). This possibility is ruled out by requiring $\gamma^{01} = 0$. 

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an auxiliary field. $X^-$ is expressed in terms of the transverse stress tensor as

$$X^-= -\int d\sigma' \left\{ \frac{1}{p^+} T^{\mu\nu}_\sigma (\sigma', \tau) + G_{++} \partial_1 X^i + \alpha' \partial_1 \partial_+ \Phi \right\} + x^-(\tau)$$

(2.18)

where the $\sigma$-independent piece $x^-(\tau)$ is undetermined. It cannot be eliminated without using constraints involving time derivatives. The trivial integral over the dilaton term has not been done as a reminder that the $\sigma$ independent part is missing. Eliminating $X^-$ produces the determinant $(\det p^+ \partial_1)^{-1}$. The $p^+$ dependence cancels that in the Faddeev-Popov determinant, and the remainder of the determinant is a constant which may be absorbed into the normalization of the path integral.

Thus, we arrive at a simple expression if we impose two sets of conditions on the background fields. Since it may be possible to relax these conditions, we will restate them. The first set of conditions insures that $X^-$ has an explicit representation on the transverse Fock space. They are

$$G_{-\mu} = 0 \quad \text{except} \quad G_{-+} = 1$$

$$\partial_2 G_{\mu\nu} = 0$$

$$\partial_2 \Phi = 0.$$  

(2.19)

(In terms of the inverse metric the conditions are $G^{+\mu} = 0$ except $G^{+-} = 1$.) The second set of conditions guarantee that $X^-$ is an auxiliary field, so that it may be eliminated by its equations of motion without producing a non-trivial Jacobian. These conditions are

$$G_{--} = 0$$

$$G_{-i} = 0$$

$$\partial_- \Phi = 0.$$  

(2.20)

Once the first set of restrictions is imposed, the second set only forbids a dilaton linear in $X^-$. The complete set of restrictions is sufficient to prohibit the Faddeev-Popov ghosts discussed deriving (2.15).

As an example of how these restrictions may be circumvented, consider choosing the light-cone gauge subsequent to fixing the conformal gauge in the usual D=26 bosonic string, as has been advocated by many authors [27]. The 26 dimensional spacetime with $G_{\mu\nu} = \eta_{\mu\nu}$ and $B_{\mu\nu} = \Phi = T = 0$ certainly meets the requirements for a light-cone quantization given above, but the conformal gauge approach is not ghost-free. Gauge fixing produces the non-trivial Faddeev-Popov determinants and ghosts familiar from the usual conformal gauge treatment, 

\[\text{The linear dilaton is important for } c \leq 1 \text{ Liouville theory. Evidently, the usual light-cone gauge will not suffice, but a generalized light-cone gauge is possible.} \]

\[\text{[3]}\]
along with additional determinants and Jacobians. All of these factors cancel in the end to give a trivial measure, as required. There is no complete discussion of this cancellation in the literature, but it must occur since the gauge fixed action is identical to the one we find, up to the measure. It would be interesting to see how this works. Most of the no-ghost statements in light-cone gauge rely on the implicit equivalence with the Nambu-Goto light-cone string, where the no-ghost theorem is more straightforward and well established. While this example does not violate the restrictions placed on the background fields above, it does show how there may be cancellations among the various measure factors which we have not considered. Such cancellations can be very complicated, as they are in the conformal gauge example.

It remains to show that the other $\gamma^{01}$ degree of freedom may be eliminated using the constraints. Since $X^-$ has been eliminated as an auxiliary field, varying the action with respect to it yields a constraint,

$$0 = -4\pi\alpha' \frac{\delta S}{\delta X^-} = -2G_{-\nu} \left( \Delta X^\nu + \Gamma_{\alpha\beta}^\nu g^{ab} \partial_a X^\alpha \partial_b X^\beta \right)$$

$$+ \epsilon_{ab} \frac{H_{-\alpha\beta} \partial_a X^\alpha \partial_b X^\beta - \alpha' R^{(2)} \partial_- \Phi + \partial_- T.}$$  \hfill (2.21)

In order to solve this constraint for $\gamma^{01}$, all of the dependence on the $X^i$ fields must vanish. There is no cancellation among the terms for general field configurations, so

$$\Gamma_{-\alpha\beta} = H_{-\alpha\beta} = \partial_- \Phi = \partial_- T = 0. \hfill (2.22)$$

Note that $\partial_- \Phi = 0$, as required above for $X^-$ to be an auxiliary field. Also, using the previous restrictions on the metric, the vanishing Christoffel symbol becomes

$$0 = \Gamma_{-\alpha\beta} = -\frac{1}{2} \partial_- G_{\alpha\beta}. \hfill (2.23)$$

This also rules out the $X^-$ dependent conformal factor mentioned in footnote 2. The only field that can depend on $X^-$ is $B_{\mu\nu}$. In fact, the antisymmetric tensor possesses a gauge invariance that may be used to eliminate the $X^-$ dependence from it, too. The theory is invariant under

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \hfill (2.24)$$

where $\Lambda_\mu(X)$ is any functional of $X^\mu$. Consider the gauge transformation given by

$$\Lambda_\nu = - \int^{X^-} dX^- B_{-\alpha}(X^+, X^-, X^i)$$

$$\Lambda_- = 0 \hfill (2.25)$$
Then $B_{-\mu} \rightarrow B_{-\mu} + \partial_- \Lambda_{\mu} - \partial_\mu \Lambda_- = 0$. The fields still satisfy (2.22), so $0 = H_{\alpha\beta} = \partial_- B_{\alpha\beta} + \partial_\alpha B_{\beta-} + \partial_\beta B_{-\alpha} = \partial_- B_{\alpha\beta}$. Evidently, all of the spacetime fields are independent of $X^-$ in this gauge.

### 2.3 The Geometry of Light-Cone Sigma Models

Horowitz and Steif [30] have investigated the conditions necessary to impose the light-cone gauge subsequent to the conformal gauge in a theory essentially with $\Phi = T = 0$ (at tree level in sigma model perturbation theory). They point out that the conditions may be phrased geometrically: the spacetime metric must admit a covariantly constant null vector. A generalization of this condition is necessary to get all the benefits of a light-cone gauge theory in the general sigma models we have considered, as well. The field $K_\mu \equiv \partial_\mu X^+$ is a covariantly constant null vector. It trivially satisfies the Killing equation $\nabla_{(\alpha} K_{\beta)} = 0$. In fact, $K_\mu$ not only generates an isometry in $G_{\mu\nu}$, but also symmetries in the other fields

$$L_K G_{\mu\nu} = L_K B_{\mu\nu} = L_K \Phi = L_K T = 0, \quad (2.26)$$

where $L_K$ is the Lie derivative in the $K_\mu$ direction. Note that this is an invariant statement, not restricted to the special coordinates suited for the light-cone gauge. Thus, to have a full light-cone quantization, all of the spacetime fields must have a symmetry generated by a single, covariantly constant null vector (in the $X^-$ free gauge for $B_{\mu\nu}$).

There are spacetime field configurations which satisfy both the background field equations for Weyl invariance (2.5) and the light-cone gauge quantization conditions (2.26). This is exactly the kind of configuration studied in “Compactification Propagation” [29]. For example, consider the metric and dilaton given by

$$ds^2 = dX^+ dX^- + \sum_i [2\pi R_i(X^+)]^2 (dx^i)^2$$

$$\Phi(X^+) = \frac{1}{2} \int X^+ \int \sum_i \frac{R''_i}{R_i}$$

where $R_i(X^+)$ for $i = 1, \cdots, 24$ is an arbitrary function of $X^+$, and the primes denote differentiation with respect to $X^+$. These fields certainly satisfy the light-cone gauge requirements, since they are independent of $X^-$. They also satisfy the background field equations to all orders in $\alpha'$, as shown in the reference [25]. Since the fields meet both requirements, the corresponding sigma model may be quantized in the light-cone gauge. This is just one of a large class of light-cone sigma models. We will consider more examples below.
2.4 Additional Constraints and Weyl Invariance

We now consider only those backgrounds meeting the light-cone gauge requirements. Then the equation (2.21) reduces to

\[ \Delta X^+ = e^\phi \partial_1 \gamma^{10} p^+ = 0 \]  

so \( \gamma^{10} \) must be independent of \( \sigma \). Under an infinitesimal reparameterization about \( \gamma^{10} = 0, \gamma^{00} = -1 \) and \( \gamma^{11} = 1 \), the variation of \( \gamma^{10} \) is

\[ \delta \gamma^{10} = \partial_0 \left[ c(\gamma^{10}) \right] - \gamma^{0c} \partial_0 c - \gamma^{1c} \partial_1 c = \partial_0 c - \partial_1 c. \]  

We can now use the residual reparameterization invariance \( \sigma^1 \rightarrow \sigma^1 + \epsilon^1(\tau) \) to set \( \gamma^{10} = 0 \) at one value of \( \sigma \), say \( \sigma_0 \). Then the equation of motion guarantees that it vanishes everywhere, giving the Jacobian \([p^+ \det \partial_1]^{-1}\). This fixes two degrees of freedom of the worldsheet metric, leaving only the Weyl mode. It decouples from the quantum theory according to our ansatz.

It should be noted that while \( \gamma^{10} \) is an auxiliary field, it is partially gauge fixed using (2.29) which contains a time derivative. Fortunately, the resulting determinant is trivial and does not require ghosts. The would-be ghost action is given by

\[ S_{\gamma^{10}}^{\text{ghost}} = \int d\tau b \partial_0 c \]  

where \( b \) and \( c \) are the ghosts. They only depend on \( \tau \), not \( \sigma \). The ghosts do not interact, and they do not depend on the worldsheet geometry, so they just contribute to the overall normalization of the path integral. They are irrelevant. This is important, because flat Minkowski space is a special case of the general backgrounds studied here, and the measure is known to be flat in that case.

The variation of the action with respect to the gauge fixed field \( \gamma^{00} \) gives the Virasoro constraint \( T_{00}^{(\gamma)} = 0 \). The zero mode (\( \sigma \)-independent) part of this constraint is just the mass-shell condition. The non-zero mode part is less obvious. Consider

\[ 0 = \left. \frac{\delta S}{\delta \gamma^{00}} \right|_{nzm} = \left\{ T_{00} - \frac{1}{2} g_{00} \text{tr} T^{(g)} \right\}_{nzm} \]
\[ = \left\{ T_{11} - \frac{1}{2} \sqrt{g} \text{tr} T^{(g)} \right\}_{nzm} \]
\[ = \left\{ \int \partial_0 T_{01} - \frac{1}{2} \sqrt{g} \text{tr} T^{(g)} \right\}_{nzm} \]
\[ = -\frac{1}{2} \sqrt{g} \text{tr} T^{(g)} \left|_{nzm} \right. \]  

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where $\text{tr}T^{(g)} = g^{ab}T_{ab}^{(g)}$, and we have used the stress tensor conservation equation $\nabla^a T_{ab} \sim 0$. Classically the stress tensor conservation equation reads

$$\nabla^a T_{ab} = 2\pi \alpha' \partial_b X^\mu \frac{\delta S}{\delta X^\mu}$$

which vanishes due to the classical equations of motion and the $X^+$ and $X^-$ constraints. The key point in (2.31) is that since $T_{01}$ and $\gamma_{01}$ vanish, $T_{11}$ must be independent of sigma. The auxiliary field equation $T_{01} = T_{++} - T_{--} = 0$ gives the full set of Virasoro constraints except for the mass-shell condition. Evidently the non-zero mode part of the $T_{00}^{(\gamma)}$ constraint only requires that the trace of the stress tensor vanish. This is the reason the theory must be Weyl invariant. Otherwise, the constraint is violated, and the light-cone gauge fails. Another way to put this is that the light-cone gauge constraints $T_{ab}^{(\gamma)} = 0$ are incompatible with worldsheet general covariance $0 = \nabla^a T_{ab}^{(g)} = \frac{1}{2} \partial_b \text{tr}T^{(g)} + \nabla^a T_{ab}^{(\gamma)}$ unless the trace vanishes. This problem is even more fundamental than the fact that we cannot solve the constraints for $X^{-}$ if the trace in non-zero.

At the classical level the trace of the stress tensor is $\alpha' \Delta^{(\gamma)} \Phi + e^{\phi} T$ (2.10). This gets renormalized upon quantization. The classical trace may be cancelled by the contribution from the Weyl anomaly. It is the trace of the renormalized stress tensor that must vanish in the quantum theory. We will see how this works in section 3.

The zero mode part of the constraint gives the mass-shell condition. The equation $T_{00}^{(\gamma)} = 0$ may be solved for $\dot{x}^- \equiv \partial_\tau x^-$ in terms of the transverse $X^i$ fields. The result is

$$\dot{x}^-(\tau) = -\frac{1}{2\pi p^+} \int_0^{2\pi} d\sigma \left\{ T_{00}^{tr}(\gamma) + p^+ G_{++} \partial_0 X^i - 2a \right\}$$

(2.33)

where the transverse stress tensor is

$$T_{00}^{tr}(\gamma) = \frac{1}{2} G_{ij} (\partial_0 X^i \partial_0 X^j + \partial_1 X^i \partial_1 X^j) + \frac{1}{2} \alpha' \left(- \frac{1}{p^+} \partial_+^2 + \partial_0^2 + \partial_1^2 \right) \Phi.$$  

(2.34)

We have introduced a normal ordering constant, $a$, which vanishes classically, but is fixed to a non-zero value in an anomaly free quantum theory. Its value is $(D-2)/24$, as determined by Lorentz invariance (if present classically) or modular invariance.

It is interesting to see how the usual result for the $D = 26$ critical string arises. It has the spacetime fields $B_{\mu\nu} = \Phi = T = 0$ and $G_{\mu\nu} = \eta_{\mu\nu}$. The trace of the stress tensor vanishes classically (2.31) and quantum mechanically (as is well known in the conformal gauge, and will be checked within the light-cone gauge in the next section). Equation (2.33) reduces to the usual mass-shell condition

$$\dot{x}^-(\tau) = -\frac{1}{p^+} (L_0^{tr} \bar{L}_0^{tr} - 2).$$

(2.35)
where $L^0_T$ ($\bar{L}^0_T$) is the transverse part of the zeroth left-moving (right-moving) Virasoro generator. This gives the mass $M^2 = p^+ p^- + p^i p^i = 2(N - 1)$ where $N$ is the combined right and left oscillator number. This is the correct mass-shell relation for the critical string. Equation (2.33) generalizes the mass-shell for arbitrary target spaces.

The final constraint comes from varying the gauge fixed $X^+$. This variation is one of the terms in the stress tensor conservation equation (2.32), which provides a convenient expression in terms of the other fields

$$0 = \frac{\delta S}{\delta X^+} = \frac{1}{2\pi \alpha' p^+} \nabla^a T_{a0} - \frac{1}{p^+} \partial_0 X^- \frac{\delta S}{\delta X^-} - \frac{1}{p^+} \partial_0 X^i \frac{\delta S}{\delta X^i}.$$ (2.36)

The first term is the divergence of the stress tensor which vanishes if the theory is Weyl invariant in the light-cone gauge, as explained above. The second term is the $X^-$ constraint, and the third term is proportional to the classical equations of motion for the $X^i$ fields. Thus, the $X^+$ constraint is satisfied classically, given Weyl invariance. This agrees with the expectation that exactly four fields, $X^+, X^-, \gamma^{00}$ and $\gamma^{01}$, should be eliminated in light-cone gauge fixing. The constraint $\frac{\delta S}{\delta X^+}$ is redundant. It is interesting to note that the $\gamma^{00}$ constraint is almost trivial, too. Historically these constraints have not been emphasized because they vanish classically when $\Phi = T = 0$, as in the $D = 26$ bosonic string.

Energy-momentum conservation carries over to the quantum theory. So if the theory is Weyl invariant, the conservation equations continue to hold. In the conformal gauge it is possible to preserve energy-momentum conservation even when the trace of the stress tensor is non-zero. The resulting Ward identities greatly ease the calculation of the Weyl anomaly [38]. This is not possible in the light-cone gauge, since the gauge is only consistent when the anomaly cancels. There is no light-cone gauge for non-critical strings, although it is possible to treat the corresponding dilatonic critical strings.

### 2.5 The Light-Cone Action

This completes gauge fixing to the light-cone gauge classically. It remains to quantize the system and introduce string interactions. The gauge fixed form of the generating functional is

$$Z = \sum_{\text{topologies}} \int d\tau d\sigma \int D X^i e^{iS[X^i]}$$

$$S_{g.f.} = \int d\tau \left\{ p^+ \dot{x}^- + \frac{1}{4\pi \alpha'} \int_0^{2\pi} d\sigma \left[ G_{ij} \partial_a X^i \partial^a X^j - 2p^+ G_{+i} \partial_0 X^i - (p^+)^2 G_{++} \right.$$

$$+ B_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j + 2 p^+ B_{pi} \partial_0 X^i - \alpha' R^{(c)} \Phi + T \right\}. (2.37)$$
\( \tau \) and \( \sigma \) are the worldsheet moduli (the interaction times and locations) and \( R^{(\gamma)} \) has delta function support at the vertices. We have used Weyl invariance to set \( \phi = 0 \). The measure for the \( X^i \) fields is flat, as we have checked at each step of gauge fixing.

The Hamiltonian is also derived easily. It is constructed as a Legendre transform of the Lagrangian, as usual. The canonical momentum densities are given by

\[
P_i(\sigma) = 2\pi \alpha' \frac{\partial L}{\partial \dot{X}^i} = G_{ij} \dot{X}^j + p^+ G_{+i} - B_{ij} \partial_1 X^j. \quad (2.38)
\]

Then the Hamiltonian is

\[
H = \frac{1}{2\pi \alpha'} \int_0^{2\pi} d\sigma \left( \dot{X}^i P_i - \mathcal{L} \right) = \frac{1}{4\pi \alpha'} \int_0^{2\pi} d\sigma \left[ G^{ij} P_i P_j + G_{ij} \dot{X}^i \dot{X}^j' - (p^+)^2 G^{-} + 2p^+ P_i G^{i-} + 2p^+ B_{ij} \dot{X}^i \dot{X}^j + G^{ij} B_{ik} B_{jl} \dot{X}^k \dot{X}^l + \alpha' R^{(\gamma)} \Phi + T \right] \quad (2.39)
\]

where \( \mathcal{L} \) is the Lagrangian density (i.e. half the \( \sigma \) integrand in (2.37)). The primes denote \( \sigma \) derivatives. Since \( P^-(\sigma) \) is conjugate to \( X^+ = p^+ \tau \), the Hamiltonian is equal to \( P^-(\sigma)/p^+ \).

### 3 Light-Cone Quantum Mechanics

The quantization of a general sigma model may be accomplished using standard techniques extended to the light-cone gauge. It is most natural to use functional methods, since the light-cone operator formalism is problematic. There is a class of models which may be solved exactly in the light-cone gauge, but most actions with non-trivial space-time fields require the use of perturbation theory on the worldsheet. The resulting string quantum mechanics is very complicated, due to the worldsheet interactions. That is, even without string interactions in which the string branches and joins, the two-dimensional field theory is interacting. As with most interacting field theories, it is not integrable, but at each order in perturbation theory we are able to investigate whether the actions studied in section 2 may be quantized consistently.

The usual technique for analyzing quantum sigma models is to employ sigma model perturbation theory. Then quantities such as the effective action may be computed as power expansions in the coupling constant \( \sqrt{\alpha'}/r \), where \( r \) is some relevant length scale (such as the compactification radius). We will not investigate whether sigma model perturbation theory is compatible with the usual light-cone techniques for calculating scattering amplitudes. There is no reason to expect the generalization of Neumann functions and string field propagators not to make sense as series expansions, but such considerations are beyond the scope of
this paper. Many of the questions about the consistency of quantization may be addressed within the context of string quantum mechanics (without string interactions). Sigma model perturbation theory is perfectly natural in this milieu.

There are many approaches to sigma model perturbation theory that have been developed in the conformal gauge. Most of these also work in the light-cone gauge with minor alterations. A few techniques do not carry over, so we will discuss the problems. Our goal is to develop a framework to test the consistency of the light-cone quantization, not to explicitly calculate beta functionals or effective actions using light-cone gauge. To that end, we will not be concerned with selecting the best formulation for doing computations, rather we will describe how to use the covariant techniques within the light-cone gauge. The essential point is that the Weyl anomaly can be calculated, and it must vanish for the light-cone gauge to be non-singular.

In this section we will quantize the general light-cone actions found in the previous section. The fields are quantized within sigma model perturbation theory, either canonically or functionally. The physical Hilbert space of states is represented in terms of the transverse oscillators. It consists of all the mass eigenstates in the transverse Fock space. Finally, the question of anomalies is addressed. The conformal anomaly manifests itself as an anomaly in the Lorentz algebra when the target space has a Lorentz isometry. In general, the anomaly violates a gauge fixing constraint, so it must vanish.

3.1 Sigma Model Perturbation Theory

The quantization of the gauge-fixed action using sigma model perturbation theory begins with the division of the $X^\mu$ fields into classical and quantum pieces, $X^\mu_0$ and $\tilde{X}^\mu$, respectively.

$$X^\mu(\sigma^a) = X^\mu_0(\sigma^a) + \tilde{X}^\mu(\sigma^a).$$

(3.1)

Of course, in the light-cone gauge $X^+$ has no quantum part, and $X^-$ is largely irrelevant. The counterterms necessary to renormalize the theory are covariant in terms of the classical fields $X^\mu_0$, so it greatly simplifies the analysis to write the action in an explicitly covariant form. Covariant background field techniques are thoroughly explained in the literature (in the conformal gauge) [15]. The main obstacle to a covariant action is that the naive quantum fields are not covariant in spacetime. They need to be expressed in terms of a spacetime vector. Then the fluctuations of the metric and the other background fields are given by a covariant expansion about their classical values. The quantum fields may be expressed
covariantly in terms of geodesic coordinates

\[ \tilde{X}^\mu = x^\mu - \frac{1}{2} \Gamma^\mu_{\sigma_1 \sigma_2} x^{\sigma_1} x^{\sigma_2} + \ldots \]  

where \( x^\mu \) is tangent to the geodesic \( \lambda^\mu(t) \) running from \( X_0^\mu \) at \( t = 0 \) to \( X_0^\mu + \tilde{X}^\mu \) at \( t = 1 \). At arbitrary \( t \), \( \lambda^\mu(t) = X_0^\mu + x^\mu t - \frac{1}{2} \Gamma^\mu_{\sigma_1 \sigma_2} x^{\sigma_1} x^{\sigma_2} t^2 + \cdots \), so \( x^\mu = \partial_t \lambda^\mu(0) \). Note that the coordinates are defined separately at each \( \sigma^a \), so \( x^\mu = x^\mu(\sigma^a) \).

Since the quantum field \( x^\mu \) transforms covariantly, all of the spacetime backgrounds are explicitly covariant in terms of the classical field \( X_0^\mu \). The metric is a particularly simple example:

\[ G_{\mu \nu}(X) = G_{\mu \nu}(X_0) - \frac{1}{3} R_{\mu \lambda \nu \rho}(X_0) x^\lambda x^\rho - \frac{1}{6} \nabla_\kappa R_{\mu \lambda \nu \rho}(X_0) x^\kappa x^\lambda x^\rho + \cdots \]  

where \( x^+ = 0 \). The light-cone gauge requirements are that \( R_{-\mu \rho} = B_{-\mu} = 0 \) and \( D_- \) annihilates any tensor. These conditions are true everywhere, including \( X_0^\mu \), so \( x^- \) does not appear. The expansion of \( \partial_a X^\mu \) is

\[ \partial_a X^\mu = \partial_a X_0^\mu + \nabla_a x^\mu + \frac{1}{3} R_{\lambda \nu \mu \rho}(X_0) \partial_a x^\lambda x^\nu x^\rho + \cdots \]

with \( \nabla_a x^\mu = \partial_a x^\mu + \Gamma_{\lambda \nu}^\mu(x_0) \partial_a x^\lambda x^\nu \). These expansions may be derived using Riemann normal coordinates, but there are other tricks as well (cf. [24]).

The kinetic term for the \( x^i \) fields turns out to be somewhat complicated due to the classical backgrounds. The spacetime metric enters the quadratic term

\[ G_{\mu \nu}(X_0) \nabla_a x^\mu \nabla^a x^\nu \]  

which is messy because the metric \( G_{\mu \nu}(X_0) \) is not constant. The propagator is simplified by introducing a vielbein \( e^i_\mu(X_0) \) such that

\[ e^i_\mu(X_0) e^j_\nu(X_0) \eta_{ij} = G_{\mu \nu}(X_0). \]

Then \( G_{\mu \nu}(X_0) \nabla_a x^\mu \nabla_b x^\nu = (\nabla_a x)^i (\nabla_b x)^j \) with \( (\nabla_a x)^i = \partial_a x^i + \omega^i_{\mu} \partial_a X_0^\mu x^\rho \) and \( x^i = e^i_\mu(X_0) x^\mu \). The spacetime spin connection one-form is given by \( \omega^i_{\mu} = e^j_\mu \nabla_{\lambda} e^{ij} \). The kinetic term is \( \partial_a x^i \partial^a x^i \) up to spin connection terms, so it gives a simple propagator.

We may now change variables in the path integral (2.1) from \( X^\mu \) to \( x^i \). This involves a shift and a spacetime coordinate change. This change of variables is certainly possible before light-cone gauge fixing, since the measure may be assumed to be invariant under spacetime diffeomorphisms and there is no Jacobian. After the field redefinition gauge fixing may proceed as described in section 2. Once the gauge has been fixed, a field redefinition of
$X^+$ could change the form of the light-cone gauge conditions, so it is simplest to make the field redefinition before fixing the gauge.

We are now ready to rewrite the action (2.37) in a background field expansion. The expansions for $G_{\mu\nu}, B_{\mu\nu}, \Phi, T$ and $\partial_a X^\mu$ may be inserted in the action. The $X^+$ field is gauge fixed, so it only appears in the classical background field $X^0_0$. $X^-$ has been eliminated as an auxiliary field, as described in section 2. We will drop the tachyon and anti-symmetric tensor backgrounds in order to simplify renormalization. These backgrounds may be included using techniques well-developed in the literature [14, 21], but it simplifies our discussion to omit them. Since $X^0_0$ satisfies the classical equations of motion, the generating functional (without string interactions) becomes

$$Z_{X^0_0} = \int Dx^I e^{-S[X^0_0,x^I]} = \int Dx^I e^{-S[X_0] - S_{X_0}^I[x^I]}$$

$$S_{X^0_0} = \int d\tau p^+ + \frac{1}{4\pi\alpha'} \int d^2\sigma \left\{ (\nabla_a x)^I (\nabla^a x)_I + R_{\mu\nu\rho\sigma} (X_0) \partial_\nu X^\mu_0 \partial^\rho X^\nu_0 x^I x^J + \frac{4}{3} R_{\mu\nu\rho\sigma} (X_0) (\nabla_a x)^\mu (\nabla^a x)^\nu x^I x^J - \frac{1}{3} \alpha' (-\partial_\nu \partial^\rho \phi) [\Phi (X_0) + \nabla_\nu \Phi (X_0) x^I + \nabla_\nu \nabla_\rho \Phi (X_0) x^I x^J] + \cdots \right\}$$

where the action has been Wick rotated ($\tau \rightarrow i\tau$) to a Euclidean worldsheet. The ellipsis represents the terms with derivatives of the Riemann tensor or higher derivatives of the dilaton. These higher order terms are given in reference [17], along with an algorithm to generate them. Any worldsheet curvature is due to the the non-trivial Weyl mode. The term linear in $x^I$ in the metric expansion vanishes since $X^0_0$ is chosen to satisfy the classical equations of motion with $\Phi = 0$. Since the dilaton breaks Weyl invariance classically, it is put in the quantum part of the action. Note that there are no $X^+$ fluctuations.

The action (3.7) is easily quantized using path integral techniques. Collecting the quadratic terms gives the full kinetic term

$$\frac{1}{4\pi\alpha'} \left[ \partial_\nu x^I \partial^\nu x_I + 2\omega_{\mu IJ} \partial_\mu X^\nu_0 x^J \partial^\nu x^I + \omega_{\mu IJ} \omega_{\nu JK} \partial_\mu X^\nu_0 \partial_\sigma X^\sigma_0 x^J x^K + \alpha' \partial_\nu \partial^\sigma \phi \nabla_\nu \nabla_\sigma \Phi (X_0) x^I x^J \right],$$

Actually, $\frac{1}{4\pi\alpha'} \partial_\nu x^I \partial^\nu x_I$ is used as the kinetic term, with the other terms considered as ‘mass’ terms. The spin connection is not covariant, so it does not enter unless it is differentiated, and the Liouville mode $\phi$ may be taken to be small and smooth. Both types of terms may be treated perturbatively. The propagator is

$$\langle x^I(\sigma, \tau) x^J(\sigma', \tau') \rangle = \alpha' \delta^{IJ} \sum_{k^1 = -\infty}^{\infty} \sum_{k^1 \neq 0} \int dk^0 e^{ik_0 (\sigma^0 - \sigma'^0)}$$
which is the kernel for the kinetic term on the cylinder. At short distances the propagator goes like
\[
\langle x'(\sigma, \tau) x'(\sigma', \tau') \rangle \sim -2\alpha' \delta^{ij} \log |\rho - \rho'| \quad \text{as } \rho \to \rho' \tag{3.10}
\]
This is the usual free field propagator for a flat worldsheet. The worldsheet curvature is treated perturbatively.

### 3.2 The Hilbert Space of States

The fields have mode expansions given by
\[
x^I(\sigma, \tau) = 2\alpha' p^I \tau + i\sqrt{2\alpha'} \left( \sum_{n=-\infty}^{\infty} \alpha_n^I e^{n\rho} + \sum_{n=-\infty}^{\infty} \overline{\alpha}_n^I e^{n\rho} \right) \tag{3.11}
\]
where \(\rho = \tau + i\sigma\). The oscillators satisfy the commutation relations
\[
[\alpha_m^I, \alpha_n^J] = m \delta^{ij} \delta_{m,-n} \\
[\overline{\alpha}_m^I, \overline{\alpha}_n^J] = m \delta^{ij} \delta_{m,-n} \\
[\alpha_m^I, \overline{\alpha}_n^J] = 0 \tag{3.12}
\]
All of the gauge symmetry has been fixed by the light-cone gauge, so the physical Hilbert space of states is represented by the whole Fock space of transverse oscillators acting on the vacuum labeled by \(p^\mu\)
\[
\mathcal{H}_\text{phys} = \text{span} \left\{ \prod_{I=1}^{D-2} (\alpha^I_N)^{m_{I,N}} \cdots (\alpha^I_{-1})^{m_{I,1}} \left( (\overline{\alpha}_m^I)^{\overline{m}_{I,N}} \cdots (\overline{\alpha}_m^I)^{\overline{m}_{I,1}} \right) |p^\mu\rangle + \mathcal{O}(\alpha') \right\}
\]
such that \(\sum_{I,n} n m_{I,n} = \sum_{I,n} n \overline{m}_{I,n} = N\) for \(N = 0, 1, 2, \ldots\)\n\[
\tag{3.13}
\]
such that \(p^\mu\) satisfies the mass-shell condition (2.33). The physical states must be \(L^r_0\) and \(T^r_0\) eigenstates for the mass-shell equation to have a solution. The monomials in the oscillators shown in equation (3.13) are eigenstates at \(\alpha' = 0\), but must be corrected for \(\alpha' \neq 0\). These corrections may change the mass spectrum (the eigenvalues) as well. Note that the left- and right-moving oscillator numbers are equal, as required by \(\sigma\) translation invariance for the closed string. The vacuum, \(|p^\mu\rangle\), is annihilated by the positive frequency modes of the string, \(\alpha_n^I |p^\mu\rangle = \overline{\alpha}_n^I |p^\mu\rangle = 0\) for \(n > 0\), for all \(\alpha'\). The Hilbert space is also endowed with an algebraic structure given by the three-point functions of the states. Even though the cardinality of the Hilbert space is determined by the number of transverse dimensions, not all actions with the same number of dimensions have isomorphic Hilbert spaces. Both the mass spectrum and the three-point functions may be computed in sigma model perturbation theory.
3.3 Renormalization and the Beta Functions

The action (3.7) suffers from both quadratic and logarithmic divergences in the UV. The propagator diverges logarithmically at short distances, so any contraction of two of the quantum fields in the action is potentially problematic. The quadratic divergences reflect the presence of the tachyon. They renormalize the cosmological constant of the two dimensional quantum gravity. It is the logarithmic divergences that can destabilize the light-cone gauge. The action must be renormalized in a way consistent with worldsheet reparameterization invariance, such as dimensional regularization with minimal subtraction. In the process the spacetime fields get renormalized. The dimension two metric operator undergoes an additive renormalization, and the dimension zero dilaton receives both multiplicative and additive renormalizations due to the worldsheet curvature. Even though the action may be Weyl invariant classically, the renormalized worldsheet couplings (i.e. the physical spacetime metric, antisymmetric tensor field, etc.) may vary with the Weyl scale, since there is no regulator that is both reparameterization and Weyl invariant. Unless the Weyl anomaly vanishes, the light-cone gauge is not consistent.

The light-cone quantum mechanics that is emerging is remarkably similar to the sigma model perturbation theory in the conformal gauge. The action is identical, except for the absence of the kinetic term \( \partial_{\alpha}x^{+}(\nabla^{a}x)^{I} \) and the interaction terms with \( x^{+} \). Due to the light-cone restrictions on the backgrounds, \( x^{-} \) would not appear in the interaction terms. Since \( x^{+} \) would only contract with \( x^{-} \), the sole divergent term missing from the light-cone action is the kinetic term itself (which contributes to the dilaton beta function). Except for this possible difference in \( \beta^{\Phi} \) at leading order, the beta functions must be identical. The only caveat is that not all of the calculational methods used in the conformal gauge are appropriate for the light-cone gauge.

The standard renormalization of the divergent sigma model actions uses dimensional regularization with minimal subtraction. This is how we will renormalize the light-cone action (3.7). In \( 2 + \epsilon \) dimensions the action is given by

\[
S_{X_{0}} = \frac{1}{4\pi\alpha'} \int d^{2+\epsilon} \sigma \, e^{\frac{1}{4}e\phi} \, 2p^{+} \partial_{0} X^{-}[X_{0}^{\mu}, x^{I}] \\
+ \frac{1}{4\pi\alpha'} \int d^{2+\epsilon} \sigma \left\{ e^{\frac{1}{4}e\phi} \left[ (\nabla_{a}x)^{I}(\nabla^{a}x)^{J} + R_{\mu I, \nu J}(X_{0}) \partial_{a}X_{0}^{\mu} \partial^{a}X_{0}^{\nu} x^{I} x^{J} \\
+ \frac{4}{3} R_{\mu I K, \nu J L}(X_{0}) \partial_{a}X_{0}^{\mu} (\nabla_{a}x)^{K} x^{I} x^{J} + \frac{1}{3} R_{K I L, J}(X_{0}) (\nabla_{a}x)^{K} (\nabla^{a}x)^{L} x^{I} x^{J} \\
- \alpha' (-\partial_{a} \partial^{a} \phi - \frac{\epsilon}{4} \partial_{a} \partial^{a} \phi) \left[ \Phi(X_{0}) + \nabla_{I} \Phi(X_{0}) x^{I} + \nabla_{I} \nabla_{J} \Phi(X_{0}) x^{I} x^{J} \right] + \cdots \right\} \tag{3.14}
\]
where the worldsheet curvature to which the dilaton couples is taken to be the dimensional
continuation of $\sqrt{g}R^{(2)}/(d - 1)$, for convenience. It is now clear that the regulated form of
the action is not Weyl invariant. The Weyl scale $(g_{ab} = e^{\phi}\delta_{ab})$ enters through the worldsheet
volume element $\sqrt{g}$ as well as the curvature.

The action (3.14) has been split into light-cone and transverse parts, because the trans-
verse part is almost identical to the corresponding part of the action in the conformal gauge.
The light-cone part is more unusual. Because of the Weyl mode $\phi$, the $X^-$ oscillators reenter
the action in $2 + \epsilon$ dimensions
\[ \int d\tau\ 2p^\pm \dot{x}^- \rightarrow \int d\tau\ 2p^\pm \dot{x}^- + \frac{1}{4\pi\alpha'} \int d^{2+\epsilon}\sigma\ e\phi p^\pm \partial_\sigma X^- + \cdots. \] (3.15)
The classical solution for $X^-$ in terms of the transverse fields is singular, with poles as $\epsilon \rightarrow 0$,
so this term may contribute to the Weyl dependence of the renormalized action.

The form of the trace of the stress tensor is given in (2.4) for the transverse action which
is covariant on the worldsheet
\[ \sqrt{g}T^a_a = \beta^T(X) \sqrt{g} + \beta^\Phi(X) \sqrt{g} R^{(2)} + \beta^G_{\mu\nu}(X) \sqrt{g} \partial_\mu X^\mu \partial_\nu X^\nu + \beta^B_{\mu\nu}(X) \epsilon^{ab} \partial_\mu X^\mu \partial_\nu X^\nu. \] (3.16)
This may be integrated to get the $\phi$-dependent part of the effective action,
\[ S_{\text{eff}}[\phi] = \int d^2\sigma \left\{ \beta^T(X) e^{\phi} + \frac{1}{2} \beta^\Phi(X) \partial_\sigma \partial^\sigma \phi^2 + \beta^G_{\mu\nu}(X) \phi \delta^{ab} \partial_\mu X^\mu \partial_\nu X^\nu + \beta^B_{\mu\nu}(X) \phi \epsilon^{ab} \partial_\mu X^\mu \partial_\nu X^\nu \right\}, \] (3.17)
where we have dropped a term proportional to the scalar curvature that is related to $\beta^\Phi$.
The corresponding terms are easily extracted from the action (3.14). The exponential $e^{\frac{1}{4\epsilon}\phi}$ is
expanded, and a field redefinition $x^I = (1 - \frac{1}{4\epsilon}\phi)y^I$ is performed to simplify the propagator [27].
This is related to a required wave function normalization. The poles in $\epsilon$ in the action
are discarded through minimal subtraction, leaving finite terms which contribute to the Weyl
anomaly. The Weyl anomaly coefficients are read off using
\[ \langle x^I x^J \rangle \sim -\frac{\alpha'}{2\epsilon} \delta^{IJ}, \] (3.18)
which is the singular part of the propagator.

Consider first this transverse part of the action. It is identical to the corresponding
part of the action in conformal gauge, as explained above, so using the methods we have
just outlined the anomaly coefficients may be computed identically in both cases. The only
difference is the leading order part of the dilaton beta function, which counts the number
of dimensions minus the ghost contribution. In the conformal gauge it is proportional to
(26 – D), whereas in the light-cone gauge it is (2 – D) due to the absence of the propagating ghosts and the light-cone coordinates. The resulting beta functions are

\[
(\beta_{\mu \nu}^G)_{\text{tr}} = R_{\mu \nu} + 2\nabla_{\mu} \nabla_{\nu} \Phi + \cdots \\
(\beta^\Phi)_{\text{tr}} = -\frac{\alpha'}{16\pi^2} \left[ \frac{2 - D}{3\alpha'} + R - 4\partial_\mu \Phi \partial^\mu \Phi + 4\nabla^2 \Phi + \cdots \right].
\]

(3.19)

At this point we have neglected the two contributions to the Weyl anomaly that appear in the light-cone gauge differently from the conformal gauge. The first contribution comes from the dimensional continuation of the \( \int 2p^+ \dot{x}^- \) term, and the second comes from the measure. Together they eliminate the discrepancy in the dilaton beta function.

First consider the \( X^- \) term. Expanded out in terms of the transverse fields, it is given by

\[
\frac{-1}{4\pi \alpha'} \int d^{2+\epsilon} \sigma \left\{ (e^{\frac{\epsilon}{2}\phi} - 1) \int_\sigma d\sigma' \partial_0 \left[ (\nabla_0 x)^I (\nabla_1 x)_I + R_{\mu \nu, IJ}(X_0) \partial_0 X_0^\mu \partial_1 X_0^\nu x^I x^J \right. \\
+ \frac{2}{3} R_{\mu KJ}(X_0) \left[ \partial_0 X_0^K (\nabla_1 x)^K + \partial_1 X_0^K (\nabla_0 x)^K \right] x^I x^J + \frac{1}{3} R_{K I L J}(X_0) (\nabla_0 x)^K (\nabla_1 x)^L x^I x^J \] \\
+ \alpha' (\nabla_0 \nabla_0 + \partial_0 \partial_0 \phi) \left[ \Phi(X_0) + \nabla_I \Phi(X_0) x^I + \nabla_J \nabla_I \Phi(X_0) x^I x^J \right] + \cdots \right\}.
\]

(3.20)

The terms contributing to the Weyl anomaly may be organized in much the same way as with the transverse part of the action. After all, \( X^- \) is proportional to \( T_{01}^{tr} \), which is the variation of the action with respect to \( \gamma^{01} \). The Weyl anomaly takes the form

\[
(S_{eff}[\phi])^{X^-} = \int d^2 \sigma \left\{ \frac{1}{2} [\beta^\Phi(X)]_{\text{tr}} \partial_0 \partial_0 \phi^2 + \phi \partial_0 \int_\sigma \left[ [\beta_{\mu \nu}^G(X)]_{\text{tr}} \partial_0 X_0^\mu \partial_1 X_0^\nu \right] \right\},
\]

(3.21)

up to terms that vanish because \( X_0^\mu \) satisfies the classical equations of motion. The transverse part is actually the whole metric beta functional, so it will be required to vanish. The dilaton beta functional is incomplete, and the \( X^- \) contribution merely cancels the part of the Weyl anomaly that depends on \( \tau \) derivatives of \( \phi \).

The remaining part of the anomaly is cancelled by the measure. There are two contributions to the measure: one from the Faddeev-Popov determinant (2.13) and one from the Jacobians and from solving for \( X^- \) (2.18) and \( \gamma^{01} \) (2.28). Dropping the factors of \( p^+ \) which cancel, the two contributions are

\[
\Delta_{FP} = \left| \begin{array}{cc} 1 & 0 \\ \partial_0 & -\partial_1 \end{array} \right| = \left[ \det(\partial_1^{\text{vector}}) \right]^{1/2}
\]

(3.22)

\[
\mathcal{J} = \det(\partial_1) = \left[ \det(\partial_1^{\text{scalar}}) \right]^{-1/2}
\]

where the subscripts \text{vector} and \text{scalar} refer to the fields on which the operators act. Specifically, \( (\partial_1^{\text{vector}}) \) acts on the \( \sigma \) component of the reparameterization vector. The expressions
involving det(\partial^2_1) are derived using the covariant measure on the space of metrics (2.8) and the measure on the space of fields \(X^i\). These determinants may be evaluated in sigma model perturbation theory, giving the contribution

\[
\Delta_{FPJ}(\phi) = -\frac{1}{2\pi^2} \int d^2\sigma \partial_1 \partial_1 \phi^2 + \mu e^0
\]

(3.23)

This combines with the contributions from the transverse action and the \(X^-\) term to produce the complete beta functionals

\[
\beta^G_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi + \cdots \]

\[
(\beta^\Phi)^{tr} = -\frac{\alpha'}{16\pi^2} \left[ \frac{26 - D}{3\alpha'} + R - 4\partial_\mu \Phi \partial^\mu \Phi + 4\nabla^2 \Phi + \cdots \right].
\]

(3.24)

These are the usual beta functionals. They correspond to the light-cone form of the Weyl anomaly

\[
S_{eff}[\phi] = \frac{1}{4\pi} \int d^2\sigma \left\{ \beta^T(X) e^0 + \beta^\Phi(X) \phi \partial^2_1 \phi + f(X) \partial_1 \partial_1 \phi \\
+ \beta^G_{\mu\nu}(X) \phi \partial^a \partial^b X_\mu \partial^c X_\nu + \beta^B_{\mu\nu}(X) \phi \epsilon^{abc} \partial^a X_\mu \partial^b X_\nu \right\},
\]

(3.25)

where we have reinstated the tachyon and anti-symmetric tensor for completeness. This differs from the usual effective action for the Liouville mode. \(\phi\) does not propagate, so (3.25) cannot be quantized in any naive fashion to yield a non-critical light-cone string.

This calculation of the Weyl anomaly has identified the critical dimension for light-cone string theory without resorting to computing the Lorentz anomaly. The Weyl anomaly is the more fundamental of the two, since it can exist even when the sigma model does not have a global Lorentz isometry. The light-cone gauge cannot be fixed if the anomaly does not vanish, as we found in section 2. One might ask what happens if the theory is formulated violating this constraint. After all, the light-cone action (2.37) still exists. The first problem is that the Weyl mode does not decouple, so it must be quantized, too. If one sets \(\phi = 0\) by fiat, the action is not renormalizable. Also, the underlying gauge invariance that insures proper factorization of loop graphs would be spoiled, and the theory would be inconsistent once interactions were included. The measure at the vertices would be wrong. The Weyl anomaly must vanish for the light-cone gauge to be consistent.

Before we proceed to introduce interactions, we will point out one technique for computing the Weyl anomaly that does not work in light-cone gauge. It is the trick of using conformal Ward identities to compute the dilaton beta functional on a flat worldsheet [8]. Since the \(\beta^\Phi\) term in the effective action is quadratic in \(\phi\), the classical two-point function

\[
\left\langle \frac{\delta}{\delta \phi(\rho)} \frac{\delta}{\delta \phi(\rho')} \right\rangle = \left\langle T^a_\phi(\rho) T^a_\phi(\rho') \right\rangle,
\]

(3.26)
contains the $\beta^\Phi$ information. In conformal field theory, this is related to the two-point function of the holomorphic part of the stress tensor via conformal Ward identities, $\langle \nabla^a T_{ab} \cdots \rangle = 0$, allowing an extremely simple computation of the leading term of the dilaton beta functional in terms of the central charge. Unfortunately, these Ward identities are violated in the non-critical light-cone gauge systems. The holomorphic part of the stress tensor is being set to zero when we solve for $X^-$, but the trace does not vanish. The Ward identities do not hold, and without the use of the Ward identities, we must resort to the calculation of the critical dimension using a curved worldsheet, as we have done above. Once the leading term is calculated, the higher order terms may be found using the elegant method of Curci and Paffuti [17]. It uses beta function consistency conditions to compute the dilaton beta function on a flat worldsheet.

4 Interactions and Exactly Solvable Models

Once we have a light-cone gauge theory that is consistent at the level of quantum mechanics, the next step is to see whether we can add string interactions. An interacting string theory allows the calculation of N-string scattering amplitudes in a string loop perturbation theory. In the first quantized formalism, the amplitude at a fixed order in the string coupling is expressed as an integral over the moduli of a Riemann surface representing the geometry of the N-string scattering process. Each Riemann surface has a definite geometry in which strings split and join at vertices. Since we are considering closed string theory, the basic string vertex is one in which two strings join to form a single string, or the time-reversed vertex in which one string splits to form two. This is the type of string interaction that must be added (possibly with contact terms as well).

4.1 String Interactions

Since the worldsheet is no longer just a cylinder, we must reconsider one aspect of gauge fixing. The equations of motion for $X^+$ may not be satisfied by the previous gauge choice. For simplicity we will consider tachyon scattering. The backgrounds are taken to satisfy the Weyl invariance conditions and the light-cone gauge requirements discussed above. The
N-tachyon scattering amplitude before gauge fixing is given by

\[
\langle X_1^\mu \cdots X_N^\mu \rangle = \sum_{\text{loops}} \int d\mu \int [DG_{ab}] D\mu e^{-S[g_{ab},X]} \delta(X^\mu(\sigma^a_1) - X^\mu_1) \cdots \delta(X^\mu(\sigma^a_N) - X^\mu_N)
\]

\[
\langle p_1^\mu \cdots p_N^\mu \rangle = \sum_{\text{loops}} \int d\mu \int [DG_{ab}] D\mu e^{-S[g_{ab},X]} e^{ip_1^\mu X_1^\mu(\sigma^a_1)} \cdots e^{ip_N^\mu X_N^\mu(\sigma^a_N)}
\]

\[
S = \frac{1}{4\pi \alpha'} \int d^2\sigma \sqrt{g} \left\{ g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \frac{\epsilon^{ab}}{\sqrt{g}} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - \alpha' R^{(2)} \Phi + T \right\}
\]

(4.1)

where the sum over loops is a sum over the genera of the Euclidean worldsheet. The integral over \(\mu\) sums over the moduli space of N-punctured surfaces at fixed genus, including the integrals \(\int d^2\sigma \sqrt{g}\) over the location of the punctures. We will describe the modular integral more precisely below. The N-tachyon momentum amplitude is just the Fourier transform of the position amplitude. The exponentials must be normal ordered in the quantum theory. Scattering of the higher string states may be treated similarly using well-known tricks.

Classically, we may pull the \(X^+\) and \(X^-\) parts of the exponentials into the action, giving

\[
S' = \frac{1}{4\pi \alpha'} \int d^2\sigma \sqrt{g} \left\{ g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \frac{\epsilon^{ab}}{\sqrt{g}} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - \alpha' R^{(2)} \Phi + T + i \sum_{r=1}^N (p_+^r X^- + p_-^r X^+) \right\}
\]

(4.2)

Since all the background fields are independent of \(X^-\), the equation of motion for \(X^+\) is

\[
\Delta X^+ = i \sum_{r=1}^N p_+^r (1/\sqrt{g}) \delta^{(2)}(\sigma^a - \sigma^a_r).
\]

(4.3)

This has the simple solution \(X^+ = \tau\), having let \(\tau_r \rightarrow \pm \infty\). The factor of \(p^+\) in the previous gauge condition is dropped since different string legs have different values of \(p^+\), and \(X^+\) must be consistent around loops. Each leg is rescaled compared to the cylindrical worldsheet by its value of \(p^+\) \((\sigma \rightarrow \sigma/p^+, \tau \rightarrow \tau/p^+)\). Since \(\sigma\) runs from \(-2\pi|p^+|\) to \(2\pi|p^+|\), the canonical momentum is still \(p^+\). Figure 1 shows the worldsheet after the rest of the gauge fixing has been carried out. This is called a Mandelstam diagram. Note that the width is constant due to \(p^+\) momentum conservation. The essential property of the Mandelstam diagram is that it puts the \(X^-\) vertex operators in the infinite past or the infinite future, so that the ill-defined object \(e^{ip^+} X^-\) does not contribute except through the width of the external legs.

The geometry of the Mandelstam diagram is that of cylinders joined by three string vertices. There is a delta function singularity in the curvature at the vertex which contributes \(-1\) to the Euler character of the worldsheet. This curvature would enter the \(X^+\) equation
of motion if the dilaton depended on $X^-$. Then $X^+ = \tau$ would no longer be a solution. For this reason, as well as to prevent ghosts, we have required $\partial_- \Phi = 0$. This point must be considered further in the treatment of $c = 1$ [35].

The moduli of the Mandelstam diagram are shown in Figure 1. For an $N$-string scattering diagram with $g$ loops (genus $g$), they consist of the $g$ internal $p^+$ momenta which determine the radii of the internal legs, the $N - 2 + 2g$ interaction times and the $N - 2 + 3g$ angles, modulo the global time-translation and $\sigma$-rotation invariance. Thus, there are $2(N - 3 + 3g)$ real moduli, as required.

The connection between the Mandelstam diagrams and the non-singular parameterizations of Riemann surfaces may seem obscure at this point. From the point of view of physics, the connection is very simple, especially at tree level. Consider gauge fixing on the complex
plane with coordinates $z$ and $\overline{z}$, where the interaction points $z_r$ are at a finite distance from the origin. The solution to the classical equation of motion (4.3) is given by

$$X^+ = \sum_{r=1}^{N} 2p^+_r \log |z - z_r|$$

(4.4)

Since $X^+$ satisfies Laplace’s equation everywhere except the points $z_r$, we may consider the conformal transformation, $z \to \rho$, where

$$\rho(z) = \sum_{r=1}^{N} 2p^+_r \log(z - z_r).$$

(4.5)

The dilaton and tachyon are independent of $X^-$, so $X^+$ transforms essentially as a weight zero field under conformal transformations, and $X^+ = \Re e(\rho)$ in the new coordinates. We may set $\rho = \tau + i\sigma$, so that the light-cone gauge condition is reproduced. The transformation (4.3) is called a Schwarz-Christoffel transformation, and it maps the N-punctured plane onto the Mandelstam diagram exactly once. The turning points of the Schwarz-Christoffel map occur at the vertices of the Mandelstam surface, so the solutions of $\partial_\tau \rho = 0$ relate the light-cone moduli to the Koba-Nielson variables $z_r$.

Thus, we have shown that the light-cone gauge is well-defined on Mandelstam diagrams provided the backgrounds meet the requirements for a consistent quantum mechanics. We could proceed to formulate scattering amplitudes for these backgrounds using a double perturbation series in both $\alpha'$ and the string coupling, but instead we consider backgrounds which are exactly solvable at the worldsheet level and do not have contributions at higher order in $\alpha'$.

### 4.2 Exactly Solvable Models

Consider the light-cone action (2.37)

$$S_{g.f.} = \int d\tau \left\{ p^+ \dot{x}^- + \frac{1}{4\pi \alpha'} \int d\sigma \left[ G_{ij} \partial_0 X^i \partial^0 X^j - 2p^+ G_{++i} \partial_0 X^i - (p^+)^2 G_{++} 
+ B_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j + 2p^+ B_{++i} \partial_1 X^i - \alpha' R(\gamma) \Phi + T \right] \right\}.$$  

(4.6)

This action is exactly solvable in terms of the classical equations of motion provided it is at most quadratic in the fields $X^i$. The backgrounds must satisfy the requirements

$$\partial_k G_{ij} = \partial_k B_{ij} = 0$$
$$\partial_k \partial_0 G_{++} = \partial_k \partial_0 B_{++} = 0$$
$$\partial_k \partial_0 \partial_1 G_{++} = \partial_k \partial_0 \partial_1 T = 0$$

(4.7)
since this guarantees that the transverse coordinates enter at most quadratically. The dilaton only contributes at the vertices (where the curvature is located), so it does not affect the solvability.

Since the $X^+$ dependence of the spacetime fields is not constrained by solvability, there is potentially a large class of backgrounds for which the sigma models are exactly solvable in the light-cone gauge. Many of these sigma models would be non-trivial in the conformal gauge. There are certainly some configurations of the metric and dilaton for which this is true, since the dilatonic gravitational waves studied in “Compactification Propagation” fall into this class.

In fact, the techniques of that paper are easily extended to find all solutions of the background equations with $T = 0$ that meet the solvability conditions. The metric beta function with $T = 0$ forces $\Phi$ to be independent of the transverse coordinates, up to a linear term, $\mathcal{Q} X^i$. All the beta functions except $\beta_{++}$ vanish automatically in the critical dimension. $\beta_{++} = 0$ may be solved for $\Phi(X^+)$:

$$\Phi = \frac{1}{8} \int^{X^+} \int \left\{ G^{ik} G^{jl} H_{ij} H_{kl} - 4 R_{++} \right\} + Q^i X^i$$

because the integrand only depends on $X^+$. Since the $X^+$ dependence of the metric and the antisymmetric tensor field is unconstrained, there is indeed a large class of solutions. Each of these yields an exactly solvable sigma model in the light-cone gauge.

The spacetime fields only depend on $\sigma$ through the coordinates $X^i$, giving a diagonal action in the oscillator basis. It simplifies the field theory to consider fields that are asymptotically constant functions of $X^+$, so that the potentials are flat at infinity. The worldsheet has been rescaled, so the oscillator expansion of $X^i$ in (3.11) becomes

$$X^i(\sigma, \tau) = x^i(\tau) + i \frac{\sqrt{2\alpha'}}{2} \left( \sum_{n=-\infty}^{\infty} \alpha^i_n(\tau) e^{in\sigma/(2p^+)} + \sum_{n=-\infty}^{\infty} \bar{\alpha}^i_n(\tau) e^{in\sigma/(2p^+)} \right)$$

(4.9)

The spacetime fields may be decomposed explicitly in terms of the transverse coordinates

$$G_{+i} = G_{+i}^{(0)}(\tau) + [G_{+i}^{(1)}(\tau)]_j X^j$$
$$B_{+i} = B_{+i}^{(0)}(\tau) + [B_{+i}^{(1)}(\tau)]_j X^j$$
$$G_{++} = G_{++}^{(0)}(\tau) + [G_{++}^{(1)}(\tau)]_j X^j + [G_{++}^{(2)}(\tau)]_{jk} X^j X^k$$
$$T = T^{(0)}(\tau) + [T^{(1)}(\tau)]_j X^j + [T^{(2)}(\tau)]_{jk} X^j X^k$$

(4.10)

where we have set $X^+ = \tau$. Because the spacetime fields only depend on $\sigma$ through the $X^i$ fields, the terms in the action linear in the transverse fields drop out. The resulting form of the action is

$$S_{int} = \frac{p^+}{2\alpha'} \int d\tau A^i_n T \cdot M_{ij}(\tau) \cdot A^j_{-n}$$

(4.11)
with

\[
A^i_n(\tau) = \begin{pmatrix}
\dot{\alpha}^i_n \\
\ddot{\alpha}^i_n \\
\alpha^i_n
\end{pmatrix}
\quad \text{and} \quad
M_{ij}(\tau) = G_{ij}(\tau) \begin{pmatrix}
-1 & 0 \\
0 & n^2
\end{pmatrix} + \cdots.
\tag{4.12}
\]

This is a slightly unusual form for a quadratic action, because the kinetic term is multiplied by a time-dependent function. It is still exactly solvable, however. The propagator is entirely determined by the classical equations of motion for the \(X^i\) fields. We will not display the propagator explicitly, since it is a messy expression whose exact form does not affect what follows. Horowitz and Steif [30] have used an approximate form of the propagator to examine the excitation of a string passing through a plane-fronted wave at lowest order in the string coupling. The Bogoliubov transformation is easily calculated once the propagator is known.

Once the string propagator is determined in the oscillator basis, all that remains is to determine the vertex in that basis. The only vertex that is necessary for the \(D = 26\) closed bosonic string field theory is the three string vertex. No contact terms are necessary. This vertex is simply an overlap delta functional in the position representation, giving the decomposition in the oscillator basis

\[
|V\rangle = \exp \left\{ -\tau_0 \sum_{r=1}^3 1/(2p^+_r) + \frac{1}{2} \sum_{r,s} \sum_{m,n=1}^\infty \overline{N}^{rs}_{mn} \alpha_{-m} \cdot \alpha_{-n}\right.
\right.
\]

\[
\left. + \sum_{r,s} \sum_{m=1}^\infty \overline{N}^{rs}_{mn} \alpha_{-m} \cdot \mathcal{P} - \frac{\tau_0}{16p^+_1 p^+_2 p^+_3} \mathcal{P}^2 \right\} |0\rangle \delta^{(D-2)}(\sum p_r)
\tag{4.13}
\]

where \(\tau_0\) is the interaction time, \(|0\rangle\) is the oscillator vacuum, \(\mathcal{P}^i = 2(p^+_1 p^i_1 - p^+_2 p^i_2)\), \(\overline{N}^{rs}_{mn} = N^{rs}_{mn} e^{\nu \tau_0/(2p^+_1)} e^{\nu \tau_0/(2p^+_2)}\) and \(\overline{N}^r_m = N^r_m e^{\nu \tau_0/(2p^+_1)}\). \(N^{rs}_{mn}\) and \(N^r_m\) are the standard Neumann function coefficients. Since the conformal map from a smooth Riemann surface to the Mandelstam diagram is non-singular except at the vertices, we expect the usual three string vertex to work for the general light-cone sigma models, as well. The one difference is the zero modes. The oscillators have time-dependent frequencies due to the \(X^+\) dependence in the spacetime fields, so the vertex depends on \(\tau_0\) in a more complicated fashion. Also, the zero mode integral that produces the momentum conservation delta function may be altered by the backgrounds. The oscillator terms represented by the Neumann coefficients should not change, however, and the zero mode modifications may be determined explicitly in many cases. We will not check for contact terms. This completes the requirements for the first quantized string field theory. Scattering amplitudes may be computed using the usual LSZ techniques. The wavefunctionals may be second quantized to obtain the full closed string field theory. We leave the details for future work.


5 Conclusion

This paper has described the formulation of the manifestly ghost-free light-cone gauge for the second order action in the Polyakov picture. The action is that of a two-dimensional sigma model, giving a bosonic string theory with spacetime metric, antisymmetric tensor, dilaton and tachyon fields. These fields must have a symmetry generated by a null, covariantly constant spacetime vector in order for the light-cone gauge to be fixed. Also, the theory must be Weyl invariant. These two conditions are satisfied by a large class of non-trivial critical string theories, including time-dependent wave-like backgrounds. The conditions for Weyl invariance have been computed within the light-cone gauge, reproducing the usual beta functions. The calculation of the dilaton beta function and the critical dimension is somewhat unusual because of the absence of propagating ghosts.

These results confirm the notion that the light-cone gauge is only sensible in critical string theories; i.e. those that could have been quantized in the conformal gauge. The absence of a Lorentz anomaly, as in two and three dimensions, does not guarantee a consistent light-cone theory. Still there are many interesting sigma models which may be quantized in the light-cone gauge. Each has a consistent, unitary string quantum mechanics and a relatively simple string field theory. These models include the exactly solvable models discussed in section four. They also include more complex, wave-like backgrounds.

The requirement of a null symmetry is not extremely restrictive, but it may seem strange and superfluous considering the many excluded models that can be quantized in conformal gauge. It is interesting to note, however, that models with a flat direction possess an extra $N = 2$ supersymmetry involving the ghosts \[40\]. The light-cone theories may just be the unitary, $N = 2$ models in the conformal gauge. It is certainly an interesting class of string models which is largely unexplored.

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References
[1] P. Goddard, J. Goldstone, C. Rebbi and C.B. Thorn, *Nucl. Phys.* **B56** (1973) 109

[2] Y. Nambu, Lectures at the Copenhagen Symposium (1970) unpublished;
   T. Goto, *Prog. Theo. Phys.* **46** (1971) 1560

[3] E. Del Giudice, P. Di Vecchia and S. Fubini *Ann. Phys.* **70** (1972) 378;
   R.C. Brower, *Phys. Rev.* **D6** (1972) 1655;
   P. Goddard and C.B. Thorn, *Phys. Lett.* **40B** (1972) 235

[4] C. Lovelace, *Phys. Lett.* **34B** (1971) 500

[5] S. Mandelstam, *Nucl. Phys.* **B64** (1973) 205; *Phys. Rep.* **C13** (1974) 259; “The
   Interacting-String Picture and Functional Integration,” in *Unified String Theories*, eds
   M.B. Green and D.J. Gross, (World Scientific, Singapore, 1986)

[6] S. Mandelstam, *Nucl. Phys.* **B83** (1974) 413

[7] M. Kaku and K. Kikkawa, *Phys. Rev.* **D10** (1974) 1110, 1823; E. Cremmer and J.L.
   Gervais, *Nucl. Phys.* **B76** (1974) 209; *Nucl. Phys.* **B90** (1975) 410

[8] S. Mandelstam, *Phys. Lett.* **46B** (1973) 447; *Nucl. Phys.* **B69** (1974) 77

[9] M.B. Green and J.H. Schwarz, *Nucl. Phys.* **B181** (1981) 502; *Nucl. Phys.* **B258** (1982)
   252; *Phys. Lett.* **109B** (1982) 444

[10] A.M. Polyakov, *Phys. Lett.* **103B** (1981) 207

[11] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333;
    D. Friedan, Z. Qiu and S. Shenker, *Phys. Rev. Lett.* **52** (1984) 1575; *Phys. Lett.* **151B**
    (1985) 55;
    D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B271** (1986) 93

[12] C. Vafa, *Phys. Lett.* **170B** (1987) 47;
    L. Alvarez-Gaumé, C. Gomez, G. Moore and C. Vafa, *Nucl. Phys.* **B303** (1988) 455

[13] E. Witten, *Nucl. Phys.* **B268** (1986) 253;
    D.J. Gross and A. Jevicki, *Nucl. Phys.* **B283** (1987) 1;
    T. Kugo, H. Kunitomo and K. Suehiro, *Phys. Lett.* **226B** (1989) 48;
    M. Kaku in *Functional Integration, Geometry and Strings*, 25th Karpacz Winter School
    1989, Z. Haba and J. Sobcyk, eds., (Birkhaeuser, Basel, 1989);
M. Saadi and B. Zwiebach, *Ann. Phys.* **192** (1989) 213;
B. Zwiebach, *Nucl. Phys.* **B390** (1993) 33

[14] E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys.* **B261** (1985) 1

[15] D. Friedan, *Ann. Phys.* **163** (1985) 318; *Phys. Rev. Lett.* **45** (1980) 1057

[16] L. Alvarez-Gaumé, D.Z. Freedman and S. Mukhi, *Ann. Phys.* **134** (1981) 85

[17] G. Curci and G. Paffuti, *Nucl. Phys.* **B286** (1987) 399

[18] S.J. Hathrell, *Ann. Phys.* **142** (1982) 34; *Ann. Phys.* **139** (1982) 136

[19] C.M. Hull and P.K. Townsend, *Nucl. Phys.* **B274** (1986) 349

[20] R.R. Metsaev and A.A. Tseytlin, *Nucl. Phys.* **B293** (1987) 385

[21] C.G. Callan, D. Friedan, E.J. Martinec, and M.J. Perry, *Nucl. Phys.* **B262** (1985) 593

[22] C.G. Callan, I.R. Klebanov, and M.J. Perry, *Nucl. Phys.* **B278** (1986) 78

[23] C.G. Callan and L. Thorlacius, “Sigma Models and String Theory,” in *TASI-88*, (World Scientific, Singapore, 1989) p. 795

[24] A.M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Pub., New York, 1987) Ch. 10

[25] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, V.1 (Cambridge U. Press, London, 1987) Ch. 3

[26] S.B. Giddings and S.A. Wolpert, *Commun. Math. Phys.* **109** (1987) 177;
S.B. Giddings, “The Equivalence of Polyakov and Light-Cone Bosonic String Theory,” *Proc. 13th Annual Mtg. of APS Div. of Particles and Fields* (1987) 411;
S.B. Giddings and E. D’Hoker, *Nucl. Phys.* **B291** (1987) 90; K. Aoki, E. D’Hoker and D.H. Phong, *Nucl. Phys.* **B342** (1990) 149

[27] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, V.1,2 (Cambridge U. Press, London, 1987) Ch. 2, 13

[28] G.T. Horowitz and A.R. Steif, *Phys. Rev. Lett.* **64** (1990) 260;
D. Amati and C. Klimčík, *Phys. Lett.* **B219** (1989) 443;
A.A. Tseytlin, *Phys. Lett.* **B288** (1992) 279

33
[29] R.E. Rudd, *Nucl. Phys.* **B352** (1991) 489

[30] G.T. Horowitz and A.R. Steif, *Phys. Rev.* **D42** (1990) 1950

[31] V. Knizhnik, A.M. Polyakov, A. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 819; T.L. Curtright and C.B. Thorn, *Phys. Rev. Lett.* **48** (1982) 1309; E. Braaten, T.L. Curtright and C.B. Thorn, *Phys. Lett.* **B118** (1982) 115; *Ann. Phys.* **147** (1983) 365; E. Braaten, T.L. Curtright, G. Ghandour and C.B. Thorn, *Phys. Rev. Lett.* **51** (1983) 19; *Ann. Phys.* **153** (1984) 147; F. David, *Mod. Phys. Lett.* **A3** (1988) 1651; J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989) 509

[32] D.J. Gross and A.A. Migdal, *Phys. Rev. Lett.* **64** (1990) 717; M. Douglas and S. Shenker, *Nucl. Phys.* **B335** (1990) 635; E. Brézin and V.A. Kazakov, *Phys. Lett.* **B236** (1990) 144

[33] D.J. Gross and N. Miljković, *Phys. Lett.* **B238** (1990) 217; E. Brézin, V.A. Kazakov and A.B. Zamolodchikov, *Nucl. Phys.* **B338** (1990) 673; P. Ginsparg and J. Zinn-Justin, *Phys. Lett.* **B240** (1990) 333

[34] E. Smith, *Nucl. Phys.* **B382** (1992) 229; *Phys. Lett.* **B305** (1993) 344

[35] R.E. Rudd, Princeton Ph.D. Thesis, 1992, unpublished.

[36] R. Tzani, *Phys. Rev.* **D38** (1988) 3112; *Phys. Rev.* **D39** (1989) 3745; *Phys. Rev.* **D43** (1991) 1254

[37] W. Siegel, *Introduction to String Field Theory* (World Scientific, Singapore, 1988)

[38] L. Alvarez-Gaumé and E. Witten, *Nucl. Phys.* **B234** (1984) 269

[39] J. Polchinski, *Nucl. Phys.* **B289** (1987) 465

[40] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, *Nucl. Phys.* **B401** (1993)