WEILPOSEDNESS AND DECAYING PROPERTY OF VISCOS SURFACE WAVE

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Abstract. In this paper, we consider an incompressible viscous flow without surface tension in a finite-depth domain of three dimensions, with free top boundary and fixed bottom boundary. This system is governed by a Navier-Stokes equation in above moving domain and a transport equation for the top boundary. Traditionally, we consider this problem in Lagrangian coordinates with perturbed linear form. In the series of papers [1], [2] and [3], I. Tice and Y. Guo introduced a new framework using geometric structure in Eulerian coordinates to study both local and global wellposedness of this system. Following this path, we extend their result in local wellposedness from small data case to arbitrary data case. Also, we give a simpler proof for global wellposedness in infinite domain. Other than the geometric energy estimates, time-dependent Galerkin method, and interpolation estimate with Riesz potential and minimum count, which are introduced in these papers, we utilize three new techniques: (1) using e-Poisson integral to construct a diffeomorphism between fixed domain and moving domain; (2) using bootstrapping argument to prove a comparison result for steady Navier-Stokes equation for arbitrary data of free surface; (3) redefining the energy and dissipation to replace the original complicated bootstrapping argument to show interpolation estimate.

Keywords: free surface, geometric structure, interpolation

1. INTRODUCTION

1.1. Problem Presentation. We consider a viscous incompressible flow in the moving domain.

\( \Omega(t) = \{ y \in \Sigma \times R \mid -b(y_1, y_2) < y_3 < \eta(y_1, y_2, t) \} \)

Here we can take either \( \Sigma = R^2 \) or \( \Sigma = (L_1T) \times (L_2T) \) for which \( T \) denotes the 1-torus and \( L^1, L^2 > 0 \) the periodicity lengths. The lower boundary \( b \) is fixed and given satisfying

\( 0 < b(y_1, y_2) < \bar{b} = \text{constant} \quad \text{and} \quad b \in C^\infty(\Sigma) \)

When \( \Sigma = R^2 \), we further require that \( b(y_1, y_2) \to b_0 = \text{positive constant} \) as \( |y_1| + |y_2| \) \( \to \) \( \infty \) and \( b - b_0 \in H^s(\Sigma) \) for any \( s \geq 0 \). When \( \Sigma = (L_1T) \times (L_2T) \), we just denote \( b_0 = 1/2(\max\{b(y_1, y_2)\} + \min\{b(y_1, y_2)\}) \).

It is easy to see this also implies \( b - b_0 \in H^s(\Sigma) \) for any \( s \geq 0 \) since \( \Sigma \) is bounded. We denote the initial domain \( \Omega(0) = \Omega_0 \). For each \( t \), the flow is described by velocity and pressure \((u, p) : \Omega(t) \mapsto R^3 \times R \) which satisfies the incompressible Navier-Stokes equation

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p & = \mu \Delta u \quad \text{in} \quad \Omega(t) \\
\nabla \cdot u & = 0 \quad \text{in} \quad \Omega(t) \\
(pI - \mu \mathbb{D}(u))\nu & = g\eta \nu \quad \text{on} \quad \{ y_3 = \eta(y_1, y_2, t) \} \\
u & = 0 \quad \text{on} \quad \{ y_3 = -b(y_1, y_2) \} \\
\partial_t \eta & = u_3 - u_1 \partial_y \eta - u_2 \partial_y \eta \quad \text{on} \quad \{ y_3 = \eta(y_1, y_2, t) \} \\
u(t = 0) & = u_0 \quad \text{in} \quad \Omega_0 \\
\eta(t = 0) & = \eta_0 \quad \text{on} \quad \Sigma
\end{align*}
\]

for \( \nu \) the outward-pointing unit normal vector on \( \{ y_3 = \eta \} \), \( I \) the 3x3 identity matrix, \((\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i \) the symmetric gradient of \( u \), \( g \) the gravitational constant and \( \mu > 0 \) the viscosity. As described in [1], the fifth equation in (1.1.3) implies that the free surface is convected with the fluid. Note that in (1.1.3), we have make the shift of actual pressure \( \bar{p} \) by constant atmosphere pressure \( p_{atm} \) according to \( p = \bar{p} + gy_3 - p_{atm} \).

We will always assume the natural condition that there exists a positive number \( \rho \) such that \( \eta_0 + b \geq \rho > 0 \) on \( \Sigma \), which means that the initial free surface is always strictly separated from the bottom. Also without loss of generality, we may assume that \( \mu = g = 1 \), which in fact will not infect our proof. In the following, we will use the term “infinite case” when \( \Sigma = R^2 \) and “periodic case” when \( \Sigma = (L_1T) \times (L_2T) \).
1.2. Previous Results. Historically, this problem is studied in several different settings according to whether we consider the viscosity and surface tension, and the different choices of domains.

For the inviscid case, traditionally, we replace the no-slip condition $u = 0$ with no penetration condition $u \cdot n = 0$ on $\Sigma_b$. It is often assumed that the initial fluid is irrotational and then this curl-free condition will be preserved in the later time. This allows to reformulate the problem to one only on the free surface. The local wellposedness in this framework was proved by Wu [12, 13] and Lannes [14]. Local wellposedness without irrotational assumption was proved by Zhang-Zhang [15], Christodoulou-Lindblad [16], Lindblad [17], Coutand-Shkoller [18] and Shatah-Zheng [19]. In the viscous case, vorticity will be naturally introduced at the free surface, so the surface formulation will not work. Also, only in the irrotational case the global wellposedness was shown as in Wu [22] and Germain-Masmoudi-Shatah [23].

In the viscous case without surface tension, the local wellposedness of equation (1.1.3) was proved by Solonnikov and Beale. Solonnikov [11] employed the framework of Holder spaces to study the problem in a bounded domain, all of whose boundary is free. Beale [4] utilized the framework of $L^2$ spaces to study the domain like ours. Both of them took the equation as a perturbation of the parabolic equation and make use of the regularity results for linear equations. Abel [20] extended the result to the framework of $L^p$ spaces. Also, Hataya [24] proved the global wellposedness in periodic case.

Many authors have also considered the effect of surface tension, which can help stabilize the problem and gain regularity. However, most of these results are related to global wellposedness for small data, e.g. in Bae [21].

Almost all of above results are built in the Lagrangian coordinate and ignore the natural energy structure of the problem. In [1], [2] and [3], Y. Guo and I. Tice introduced the geometric energy framework and prove both the local and global wellposedness in small data. The main idea in their proof is that instead of considering perturbation of the parabolic equation, we restart to prove the regularity under time-dependent basis. In our paper, we will follow this path and employ a similar argument.

In [1], it is always assumed that the initial data $u_0$ and $\eta_0$ is sufficiently small to guarantee that: (1) the transform between fixed domain and moving domain is a diffeomorphism; (2) the elliptic estimate in linear Navier-Stokes problem achieves the best regularity; (3) in proving boundedness and contraction of iteration sequence, the constant is small enough to get required estimate. In Section 2 of our paper, we will drop this assumption and prove the local wellposedness for arbitrary initial data. Hence, we have to use different techniques to recover the above three results.

Also, in proving global wellposedness and decaying property in [2], the author introduced a quite complicated interpolation argument to increase the decaying rate of energy with minimum count. In Section 3 of our paper, through redefining the energy and dissipation, we greatly simplify the proof in that: (1) avoiding the complicated bootstrapping argument for interpolation results; (2) avoiding the bootstrapping argument in comparison theorem; (3) handling the energy and dissipation 1-minimum count and 2-minimum simultaneously instead of separately.

1.3. Geometric Formulation. In order to work in a fixed domain, we want to flatten the free surface via a coordinate transformation. Beale introduced a flatten transform in [4] and we will use a slightly modified version as the geometric transform. We define a fixed domain

$$\Omega = \{x \in \Sigma \times R \mid -b_0 < x_3 < 0\}$$

for which we will write the coordinate $x \in \Omega$. In this slab, we take $\Sigma : \{x_3 = 0\}$ as the upper boundary and $\Sigma_b : \{x_3 = -b_0\}$ as the lower boundary. Simply denote $x' = (x_1, x_2)$ and $x = (x_1, x_2, x_3)$. In order to introduce the mapping between $\Omega$ and $\Omega(t)$, we need to utilize slightly different techniques to construct the extension of free surface.

1.3.1. Geometric Formulation for Local Wellposedness. We will prove the local wellposedness for arbitrary initial data and arbitrary smooth bottom, so the mapping should be related to the data itself. Hence, we define the $\epsilon$-Poisson integral for $\eta$:

$$\tilde{\eta}^\epsilon = \mathcal{P}_\epsilon \eta = \begin{cases} \int_{R^2} \hat{\eta}(\xi) e^{i|x|\xi} e^{-2\pi i \epsilon \xi} \, d\xi & \text{for infinite case} \\ \sum_{n \in (L_1^{-1}Z) \times (L_2^{-1}Z)} e^{2\pi i n \cdot x'} e^{i|n|x_3} \tilde{\eta}(n) & \text{for periodic case} \end{cases}$$

where $\hat{\eta}(\xi)$ and $\tilde{\eta}(n)$ denotes the Fourier transform of $\eta(x')$ in either continuous or discrete form and $0 < \epsilon < 1$ is the parameter. The detailed definitions are shown in (2.1.1) and (2.1.10).

Consider the geometric transform from $\Omega$ to $\Omega(t)$:

$$\Phi^\epsilon : (x_1, x_2, x_3) \mapsto (x_1, x_2, \frac{b}{b_0} (x_3 + \tilde{\eta}^\epsilon x_3) + \tilde{\eta}^\epsilon (1 + \frac{x_3}{b_0})) = (y_1, y_2, y_3)$$
This transform maps $\Omega$ into $\Omega(t)$ and its Jacobian matrix

\begin{equation}
\nabla \Phi^\epsilon = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
A^\epsilon & B^\epsilon & J^\epsilon
\end{pmatrix}
\end{equation}

and the transform matrix

\begin{equation}
A^\epsilon = ((\nabla \Phi^\epsilon)^{-1})^T = \begin{pmatrix}
1 & 0 & -A^\epsilon K^\epsilon \\
0 & 1 & -B^\epsilon K^\epsilon \\
0 & 0 & K^\epsilon
\end{pmatrix}
\end{equation}

where

\begin{equation}
\tilde{b} = 1 + \frac{x_3}{b_0}
\end{equation}

Note that this transform is not necessarily a homomorphism for arbitrary $\epsilon$. By our assumption on initial data that $\eta_0(x') + b_0(x') \geq \rho > 0$ and theorem 2.7, we can always choose a sufficiently small $\epsilon$ depending on $\|\eta_0\|_{H^2}$ such that there exists a $\delta > 0$ satisfying $J^\epsilon(0) > \delta > 0$, then this nonzero Jacobi implies the transform $\Phi^\epsilon$ is a homomorphism at $t = 0$. If we further assume $\eta_0 \in H^{7/2}(\Sigma)$, we can conclude $\Phi^\epsilon$ is a $C^1$ diffeomorphism from $\Omega$ to $\Omega_0$. In the following, we will just write $\bar{\eta}$ instead of $\eta^\epsilon$ for simplicity, and the same fashion applies to $A, \Phi, A, B, J$ and $K$.

Define some transformed operators as follows.

\begin{align}
(\nabla_A f)_i &= A_{ij} \partial_j f \\
\nabla_A \cdot \bar{g} &= A_{ij} \partial_j g_i \\
\Delta_A f &= \nabla_A \cdot \nabla_A f \\
N &= (-\partial_1 \eta, -\partial_2 \eta, 1) \\
(\mathbb{D}_A u)_{ij} &= A_{ik} \partial_k u_j + A_{jk} \partial_k u_i \\
S_A(p, u) &= pI - \mathbb{D}_A u
\end{align}

where the summation should be understood in the Einstein convention. If we extend the divergence $\nabla_A \cdot$ to act on symmetric tensor in the natural way, then a straightforward computation reveals that $\nabla_A \cdot S_A(p, u) = \nabla_A p - \Delta_A u$ for vector fields satisfying $\nabla_A \cdot u = 0$.

In our new coordinate, the original equation system (1.1.3) becomes

\begin{align}
\begin{cases}
\partial_t u - \partial_t \bar{\eta} \bar{b} K \partial_3 u + u \cdot \nabla_A u - \Delta_A u + \nabla_A p = 0 \quad &\text{in } \Omega \\
\nabla_A \cdot u = 0 \quad &\text{in } \Omega \\
S_A(p, u) N = \eta N \quad &\text{on } \Sigma \\
u = 0 \quad &\text{on } \Sigma_b \\
u(x, 0) = u_0(x) \quad &\text{in } \Omega \\
\eta(x', 0) = \eta_0(x') \quad &\text{on } \Sigma \\
\partial_t \eta + u_3 \partial_3 \eta + u_3 \partial_2 \eta = u_3 \quad &\text{on } \Sigma
\end{cases}
\end{align}

where we can split the system into a Naiver-Stokes equation and a transport equation.

Since $A$ is determined by $\eta$ through the transform, so all the quantities above is related to $\eta$, i.e. the geometric structure of the free surface. This is the central idea of our proof. It is noticeable that in proving local wellposedness of above equation system, we must verify $\Phi(t)$ is a $C^1$ diffeomorphism for any $t \in [0, T]$, where the theorem holds.

### 1.3.2. Geometric Formulation for Global Wellposedness

In the global part, we need to consider the infinite case when domain $\Omega(t)$ always possess a flat bottom, which means $b = \text{constant}$. Also our theorem only holds for sufficiently small data, so we may directly define $\bar{\eta}$ as the Poisson integral of $\eta$ which is shown in (A.3.1), i.e. $\bar{\eta} = \bar{P} \eta$. Then the map can be taken as

\begin{align}
\Phi : (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + \bar{\eta}(1 + \frac{x_3}{b})) = (y_1, y_2, y_3)
\end{align}
which also successfully transform $\Omega$ to $\Omega(t)$. It is obvious that the Jacobi matrix and transform matrix are identical to those in the local case except that we need to redefine the quantities
\[
\tilde{b} = 1 + x_3/b \\
A = \partial_1 \tilde{y}b \\
B = \partial_2 \tilde{y}b \\
J = 1 + \tilde{y}/b + \partial_1 \tilde{y}b \\
K = 1/J
\]
(1.3.10)

Naturally, the definition of weighted operators in (1.3.7) and the transformed equation (1.3.8) are the same as those in local part with replacing the related quantities with above definition. We can easily see that as long as $\|\eta\|_{H^{5/2}(\Sigma)}$ is sufficiently small, $\Phi$ is a diffeomorphism. Hence, we do not need the discussion as local part.

1.4. Main Theorem. In this paper, we will prove both the local wellposedness and global wellposedness for higher order regularity.

**Theorem 1.1.** Let $N \geq 3$ be an integer. Assume the initial data $\eta_0 + b \geq 2\bar{b}\delta > 0$ for some $\delta > 0$. Suppose that $u_0$ and $\eta_0$ satisfy the estimate $K_0 = \|u_0\|^2_{H^2 \Omega_0} + \|\eta_0\|^2_{H^{2N+1/2}(\Sigma)_0} < \infty$ as well as the $N^{th}$ compatible condition (2.6.3). Then there exists $0 < T_0 < 1$ such that for $0 < T < T_0$, there exists a unique solution $(u, p, \eta)$ to system (1.3.8) on the interval $[0, T]$ that achieves the initial data. Furthermore, the solution obeys the estimate
\[
\begin{align*}
\sum_{j=0}^{N} \sup_{0 \leq t \leq T} \|\partial_t^j u\|^2_{H^{2N-2j}} + \sum_{j=0}^{N} \int_0^T \|\partial_t^j u\|^2_{H^{2N-2j+1}} + \|\partial_t^{N+1} u\|^2_{Y^*} \\
+ \left( \sum_{j=0}^{N-1} \sup_{0 \leq t \leq T} \|\partial_t^j p\|^2_{H^{2N-2j-1}} + \sum_{j=0}^{N} \int_0^T \|\partial_t^j p\|^2_{H^{2N-2j}} \right) \\
+ \left( \sum_{j=1}^{N} \sup_{0 \leq t \leq T} \|\eta\|^2_{H^{2N+1/2}(\Sigma)} + \sum_{j=0}^{N} \int_0^T \|\partial_t^j \eta\|^2_{H^{2N-2j+3/2}(\Sigma)} \right) \\
\leq C(\Omega_0, \delta) P(K_0)
\end{align*}
\]
where $C(\Omega_0, \delta)$ is a positive constant depending on the initial domain $\Omega_0$ and separation quantity $\delta$ and $P(\cdot)$ is a single variable polynomial satisfying $P(0) = 0$. The solution is unique among functions that achieve the initial data and for which the left hand side of the estimate is finite. Moreover, $\eta$ is such that the mapping $\Phi(t)$ defined by (1.3.3) is a $C^{2N-2}$ diffeomorphism for each $t \in [0, T]$.

**Remark 1.2.** In above theorem, $H^k$ denotes the usual Sobolev space in $\Omega$ and $H^k(\Sigma)$ denotes the usual Sobolev space on $\Sigma$, while we will state the definition of $\|\cdot\|_{Y^*}$ later. Because the map $\Phi(t)$ is a $C^{2N-2}$ diffeomorphism, then we may change variable to $y \in \Omega(t)$ to produce solution of (1.1.3).

**Theorem 1.3.** Let $N \geq 4$ be an integer. Suppose the initial data $(u_0, \eta_0)$ satisfy the appropriate compatible conditions stated in local wellposedness theorem. Then there exists a $\kappa > 0$ such that if
\[
\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \leq \kappa
\]
(1.4.2)
then there exists a unique solution $(u, p, \eta)$ on the interval $[0, \infty)$ that achieves the initial data. Also, the solution obeys the estimate
\[
\mathcal{G}_N(\infty) \leq C(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) \leq C \kappa
\]
(1.4.3)
where $C > 0$ is a universal constant.

**Remark 1.4.** The detailed definition for $\mathcal{E}_{2N}$, $\mathcal{F}_{2N}$ and $\mathcal{G}_N$ is presented in (3.2.1) to (3.2.12). This theorem easily implies the algebraic decaying of energy $\mathcal{E}_{N+2,2}$ and $\mathcal{E}_{N+2,1}$.

1.5. Convention and Terminology. We now mention some of the definitions, bits of notations and conventions we will use throughout the paper.

(1) We will employ Einstein summation convention to sum up repeated indices for vector and tensor operations.
Throughout the paper $C > 0$ will denote a constant only depend on the parameter of the problem, $\mathcal{N}$ and $\Omega$, but does not depend on the data. They are referred as universal and can change from one inequality to another one. When we write $C(z)$, it means a certain positive constant depending on quantity $z$.

There are two exceptions to above rules in the local part. The first one is that in the elliptic estimates and Korn’s inequality, there are constants depending on the initial domain $\Omega_0$. Although this should be understood as depending on the initial free surface $y_0$, since its dependent relation is given implicitly and cannot be simplified further, we will also call them universal. The second one is that we will also call the constant $C(\delta)$ universal, though $\delta$ is determined by initial domain $\Omega_0$. To note that apart from these, all the other constants related to initial data $\Omega_0$, $u_0$ and $y_0$ should be specified in detail.

(3) We will employ notation $a \lesssim b$ to denote $a \leq Cb$, where $C$ is a universal constant as defined above.

(4) We will write $P(\cdot)$ to denote the single variable polynomial. This type of polynomial always satisfies $P(0) = 0$ unless it is specified as an exception. This notation is used to denote some very complicated polynomial expressions, however whose details we do not really care. This kind of polynomial may change from line to line.

(5) For convenience, we will typically write $H^0 = L^2$, except for notation $L^2([0, T]; H^k)$. We write $H^k(\Omega)$ with $k \geq 0$ and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for standard Sobolev space. Same style of notations also holds for $L^2([0, T]; H^k)$ and $L^\infty([0, T]; H^k)$. When we write $\|\cdot\|_{H^k}$, this always means the Sobolev norm in $\Omega$, otherwise, we will point out the exact space it stands for, e.g. $\|\cdot\|_{H^k(\Sigma)}$. A similar fashion is adopted for $\|\cdot\|_{L^2 H^k}$ and $\|\cdot\|_{L^\infty H^k}$.

(6) We write the multi-indices $\mathbb{N}^{1+m} = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)\}$ to emphasize that the 0-index term is related to temporal derivatives. For just spatial derivatives, we write $\mathbb{N}^m$ without the 0-index. We define the parabolic counting of such multi-indices as $|\alpha| = 2\alpha_0 + \alpha_1 + \ldots + \alpha_m$. We always write $Df$ to denote the horizontal derivative of $f$ and $\nabla f$ for the full derivative. For a given norm $\|\cdot\|$ and integers $k, m \geq 0$, we introduce the following notation for the sums of spatial derivatives.

$$
\|D^m f\| = \sum_{\alpha \in \mathbb{N}^m} \|\partial^{\alpha} f\|, \quad \|\nabla^m f\| = \sum_{\alpha \in \mathbb{N}^m} \|\partial^{\alpha} f\|
$$

$$
\|D_k f\| = \sum_{\alpha \in \mathbb{N}^{k+1}} \|\partial^{\alpha} f\|, \quad \|\nabla_k f\| = \sum_{\alpha \in \mathbb{N}^{k+1}} \|\partial^{\alpha} f\|
$$

where $D$ and $\nabla$ means horizontal derivatives and $\bar{D}$ and $\bar{\nabla}$ means full derivatives. Also we define

$$
\|D^k f\| = \|D_k f\|, \quad \|\nabla^k f\| = \|\nabla_k f\|
$$

$$
\|\bar{D} f\| = \|\bar{D} f\|, \quad \|\bar{\nabla} f\| = \|\bar{\nabla} f\|
$$

1.6. Structure of This Paper. In Section 2, we will discuss the local behavior for arbitrary initial data, which includes three main parts: estimates for linear Navier-Stokes equation, estimates for transport equation and the construction of iteration. Combining all these, the local wellposedness easily follows. In Section 3, we will prove the global wellposedness for small data in horizontally infinite domain. Since many lemmas have been proved in [2], here we only present the main part of our simplification in interpolation theorem and comparison theorem, with merely stating the result for other parts.

2. Local Wellposedness for Large Data

2.1. $\epsilon$-Poisson Integral. In this section, we will define the $\epsilon$-Poisson integral, which is utilized to construct geometric transform as (1.3.3), and discuss the basic properties of it.

2.1.1. Poisson Integral in Infinite Case. For a function $f$ defined on $\Sigma = \mathbb{R}^2$, the $\epsilon$-Poisson integral is defined by

$$
\mathcal{P}^\epsilon f(x, x_3) = \int_{\mathbb{R}^2} \hat{f}(\xi)e^{\epsilon|\xi|x_3}e^{2\pi ix_3} \xi d\xi
$$

where $\hat{f}(\xi)$ denotes the Fourier transform of $f(x')$ in $\mathbb{R}^2$ and $0 < \epsilon < 1$ is the parameter. Although $\mathcal{P}^\epsilon f$ is defined on $\mathbb{R}^2 \times (-\infty, 0)$, we will only consider the part $\mathbb{R}^2 \times (-b_0, 0)$ here.
Lemma 2.1. Let $\mathcal{P}\epsilon f$ be the $\epsilon$-Poisson integral of function $f$ which is in homogeneous Sobolev space $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$. Then we have

\begin{equation}
\|\nabla^q \mathcal{P}\epsilon f\|_{H^0}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q-1/2}}^2
\end{equation}

where $C > 0$ is a constant independent of $\epsilon$. In particular, we have

\begin{equation}
\|\mathcal{P}\epsilon f\|_{H^0}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q-1/2}(\Sigma)}^2
\end{equation}

Proof. By definition of Fourier transform, Fubini theorem and Parseval identity, we may bound

\begin{align*}
\|\nabla^q \mathcal{P}\epsilon f\|_{H^0}^2 &\leq (2\pi)^2 0 \leq \frac{\pi\epsilon}{\epsilon} \|f\|_{H^{q-1/2}(\Sigma)}^2
\end{align*}

We can simply take $C = \pi$ to achieve the estimate. Furthermore, considering

\begin{equation}
1 - e^{-2\epsilon b_0 |\xi|} \leq \min \left\{ \frac{1}{2\epsilon |\xi|^3}, b_0 \right\}
\end{equation}

and $0 < \epsilon < 1$, we have

\begin{equation}
\|\mathcal{P}\epsilon f\|_{H^0}^2 \leq \frac{\pi b_0}{\epsilon} \|f\|_{H^0(\Sigma)}^2
\end{equation}

Hence, (2.1.3) easily follows.

Lemma 2.2. Let $\mathcal{P}\epsilon f$ be the $\epsilon$-Poisson integral of function $f$. Then for $q \in \mathbb{N}$ and $s > 1$, we have

\begin{equation}
\|\nabla^q \mathcal{P}\epsilon f\|_{L^\infty}^2 \leq C \|f\|_{H^{q+s}(\Sigma)}^2
\end{equation}

Proof. A simple application of Sobolev embedding and lemma 2.1 reveals

\begin{equation}
\|\nabla^q \mathcal{P}\epsilon f\|_{L^\infty}^2 \leq C \|\nabla^q \mathcal{P}\epsilon f\|_{H^{q+1/2}}^2 \leq C \|f\|_{H^{q+s}(\Sigma)}^2
\end{equation}

The following lemma illustrate the specialty of derivative in vertical direction.

Lemma 2.3. Let $\mathcal{P}\epsilon f$ be the $\epsilon$-Poisson integral of function $f$. Then we have

\begin{equation}
\|\partial_3 \mathcal{P}\epsilon f\|_{L^\infty}^2 \leq C \epsilon \|f\|_{H^{3/2}(\Sigma)}^2
\end{equation}

where $C > 0$ is a constant independent of $\epsilon$.

Proof. We can simply bound as follows.

\begin{equation}
\|\partial_3 \mathcal{P}\epsilon f\|_{L^\infty}^2 \leq C \|\partial_3 \mathcal{P}\epsilon f\|_{H^{1/2}}^2 \leq C \epsilon^2 \|\mathcal{P}\epsilon f\|_{H^1}^2 \leq C \epsilon \|f\|_{H^{3/2}(\Sigma)}^2
\end{equation}

Note that the first inequality is based on Sobolev embedding theorem and the third one is a straightforward application of lemma 2.1, so we only need to verify the second inequality in detail. By definition, it is easy to see

\begin{equation}
\partial_3 \mathcal{P}\epsilon f = \epsilon \int_{R^2} |\xi| \hat{f}(\xi)e^{\epsilon |\xi| \cdot x_3} e^{2\pi i x_3 \cdot \xi} d\xi
\end{equation}

Hence, we have

\begin{align*}
\|\partial_3 \mathcal{P}\epsilon f\|_{H^0}^2 &= \epsilon^2 \int_{R^2} \int_{-b_0}^0 \|\xi\|^2 |\hat{f}(\xi)|^2 e^{2\epsilon |\xi| x_3} d\xi dx_3 = \epsilon^2 (2\pi)^2 \int_{R^2} \int_{-b_0}^0 \|\xi\|^2 |\hat{f}(\xi)|^2 e^{2\epsilon |\xi| x_3} d\xi dx_3 \\
&\leq \frac{\epsilon^2}{(2\pi)^2} \|\partial_3 \mathcal{P}\epsilon f\|_{H^0}^2
\end{align*}

Similarly, we can show that for $i, j = 1, 2, 3$

\begin{equation}
\|\partial_i \mathcal{P}\epsilon f\|_{H^0}^2 \leq \frac{\epsilon^2}{(2\pi)^2} \|\partial_{i1} \mathcal{P}\epsilon f\|_{H^0}^2
\end{equation}
\[ \|\partial_{3j}\mathcal{P}\epsilon f\|_{H^0}^2 \leq \frac{\epsilon^2}{(2\pi)^{3q}} \|\partial_{111}\mathcal{P}\epsilon f\|_{H^0}^2 \]

So we have
\[ (2.1.9) \quad \|\partial_{3j}\mathcal{P}\epsilon f\|_{H^2}^2 \leq C\epsilon^2 \|\mathcal{P}\epsilon f\|_{H^3}^2 \square \]

### 2.1.2. Poisson Integral in Periodic Case.
Suppose \( \Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T}) \), we define the \( \epsilon \)-Poisson integral as follows.
\[ (2.1.10) \quad \mathcal{P}\epsilon f(x',x_3) = \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} e^{2\pi i n \cdot x'} e^{i n \cdot x_3 \epsilon} \hat{f}(n) \]

where
\[ (2.1.11) \quad \hat{f}(n) = \int_{\Sigma} f(x') \frac{e^{-2\pi i n \cdot x'}}{L_1 L_2} dx' \]

**Lemma 2.4.** Let \( \mathcal{P}\epsilon f \) be the \( \epsilon \)-Poisson integral of function \( f \) which is in homogeneous Sobolev space \( H^{q-1/2}(\Sigma) \) for \( q \in \mathbb{N} \). Then we have
\[ (2.1.12) \quad \|\nabla^q \mathcal{P}\epsilon f\|_{H^0}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q-1/2}}^2 \]

where \( C > 0 \) is a constant independent of \( \epsilon \). In particular, we have
\[ (2.1.13) \quad \|\mathcal{P}\epsilon f\|_{H^0}^2 \leq \frac{C}{\epsilon} \|f\|_{H^{q-1/2}(\Sigma)}^2 \]

**Proof.** By Fubini theorem and Parseval identity, we may bound
\[
\|\nabla^q \mathcal{P}\epsilon f\|_{H^0}^2 \leq \left(2\pi\right)^{2q} \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} \int_{-b_0}^0 |n|^{2q} e^{2\epsilon |n| \pi} \left|\hat{f}(n)\right|^2 \, dx_3
\]
\[ = \left(2\pi\right)^{2q} \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} |n|^{2q} \left|\hat{f}(n)\right|^2 \left(1 - \frac{1 - e^{-2\epsilon |n|}}{2\epsilon |n|}\right) \]
\[ \leq \frac{(2\pi)^{2q}}{2\epsilon} \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} |n|^{2q-1} \left|\hat{f}(n)\right|^2 \leq \frac{\pi}{\epsilon} \|f\|_{H^{q-1/2}}^2 \]

We can simply take \( C = \pi \) to achieve the estimate. Furthermore, considering
\[ \frac{1 - e^{-2\epsilon |n|}}{2\epsilon |n|} \leq \min \left\{ \frac{1}{2\epsilon |n|}, b_0 \right\} \]
and \( 0 < \epsilon < 1 \), we have
\[ (2.1.14) \quad \|\mathcal{P}\epsilon f\|_{H^0}^2 \leq \frac{\pi b_0}{\epsilon} \|f\|_{H^0(\Sigma)}^2 \]

Hence, (2.1.13) easily follows. \( \square \)

Similar to infinite case, we give an \( L^\infty \) estimate for Poisson integral.

**Lemma 2.5.** Let \( \mathcal{P}\epsilon f \) be the \( \epsilon \)-Poisson integral of function \( f \). Then for \( q \in \mathbb{N} \) and \( s > 1 \), we have
\[ \|\nabla^q \mathcal{P}\epsilon f\|_{L^\infty}^2 \leq C \|f\|_{H^{q+s}(\Sigma)}^2 \]

**Proof.** A simple application of Sobolev embedding and lemma (2.4) reveals
\[ (2.1.15) \quad \|\nabla^q \mathcal{P}\epsilon f\|_{L^\infty}^2 \leq C \|\nabla^q \mathcal{P}\epsilon f\|_{H^{q+s}}^2 \leq C \|f\|_{H^{q+s}(\Sigma)}^2 \]

We still need the following lemma to illustrate the specialty of derivative in vertical direction.

**Lemma 2.6.** Let \( \mathcal{P}\epsilon f \) be the \( \epsilon \)-Poisson integral of function \( f \). Then we have
\[ (2.1.16) \quad \|\partial_3 \mathcal{P}\epsilon f\|_{L^\infty}^2 \leq C\epsilon \|f\|_{H^{q/2}(\Sigma)}^2 \]

where \( C > 0 \) is a constant independent of \( \epsilon \).
Proof. We can simply bound as follows.

\begin{equation}
\|\partial_3 P^e f\|_{L^\infty}^2 \leq C \|\partial_3 P^e f\|_{H^2}^2 \leq C \epsilon^2 \|P^e f\|_{H^2}^2 \leq C \epsilon \|f\|_{H^{5/2}(\Sigma)}^2
\end{equation}

Note that the first inequality is based on Sobolev embedding theorem and the third one is a straightforward application of lemma (2.1), so we only need to verify the second inequality in details.

By definition, it is easy to see

\begin{equation}
\partial_3 P^e f = \epsilon \sum_{n \in (L_1^{-1} Z) \times (L_2^{-1} Z)} |n| \hat{f}(\xi) e^{in|x_3|e^{2\pi x'}}
\end{equation}

Hence, we have

\[
\|\partial_3 P^e f\|_{H^0}^2 = \epsilon^2 \sum_{n \in (L_1^{-1} Z) \times (L_2^{-1} Z)} \int_{-b_0}^0 |n|^2 \left| \hat{f}(n) \right|^2 e^{2|n|x_3} \, dx_3
\]

\[
= \frac{\epsilon^2}{(2\pi)^2} \left( \frac{2\pi}{(2\pi)^2} \sum_{n \in (L_1^{-1} Z) \times (L_2^{-1} Z)} \int_{-b_0}^0 |n|^2 \left| \hat{f}(n) \right|^2 e^{2|n|x_3} \, dx_3 \right)
\]

\[
\leq \frac{\epsilon^2}{(2\pi)^2} \|\partial_1 P^e f\|_{H^0}^2
\]

Similarly, we can show that for \(i, j = 1, 2, 3\)

\[
\|\partial_3 P^e f\|_{H^0}^2 \leq \frac{\epsilon^2}{(2\pi)^2} \|\partial_{1i} P^e f\|_{H^0}^2
\]

\[
\|\partial_3 P^e f\|_{H^0}^2 \leq \frac{\epsilon^2}{(2\pi)^2} \|\partial_{1j} P^e f\|_{H^0}^2
\]

So we have

\begin{equation}
\|\partial_3 \tilde{\eta}\|_{H^2}^2 \leq C \epsilon^2 \|\tilde{\eta}\|_{H^3}^2
\end{equation}

\[\square\]

2.1.3. Homomorphism of the Geometric Transform.

**Theorem 2.7.** If the initial data satisfies \(\eta_0 + b_0 > \rho > 0\) for all \(x' \in \Sigma\), then we can choose a \(\epsilon > 0\) such that there exists a \(\delta > 0\) satisfying \(J^e(0) \geq \delta > 0\).

**Proof.** Naturally there always exists \(\delta > 0\), such that \(\eta_0(x') + b_0(x') \geq 2b_0\delta > 0\) for arbitrary \(x' \in \Sigma\). Based on lemma (2.3) and lemma (2.6), if we take \(\epsilon \leq \delta^2/(4C^2 \|\eta_0\|_{H^{3/2}})\), we have

\begin{equation}
\|\partial_3 \tilde{\eta}_0\|_{L^\infty} \leq \delta/2
\end{equation}

where \(\tilde{\eta}_0\) denotes \(\tilde{\eta}\) at \(t = 0\). Furthermore, by fundamental theorem of calculus, we can estimate for any \(x_3 \in [-b_0, 0]\) and \(x' \in \Sigma\)

\begin{equation}
|\tilde{\eta}_0(x', x_3) - \tilde{\eta}_0(x', 0)| \leq \int_{-b_0}^0 |\partial_3 \tilde{\eta}_0(x', z)| \, dz \leq b_0 \|\partial_3 \tilde{\eta}_0\|_{L^\infty} \leq b_0 \delta/2
\end{equation}

Since \(\tilde{\eta}_0(x', 0) = \eta_0(x')\), we have

\[
J^e(0) = \frac{b}{b_0} + \frac{\tilde{\eta}_0}{b_0} + \partial_3 \tilde{\eta}_0 \bar{\tilde{\eta}} = \frac{b + \eta_0}{b_0} + \frac{\tilde{\eta}_0 - \eta_0}{b_0} + \partial_3 \tilde{\eta}_0 \bar{\tilde{\eta}}
\]

\[
\geq 2\delta - \delta/2 - \delta/2 = \delta > 0
\]

This means for given initial data \(\eta_0\), if we take \(\epsilon = \delta^2/(4C^2 \|\eta_0\|_{H^{3/2}})\), then \(J^e \geq \delta > 0\). \(\square\)

2.2. Preliminaries.
2.2.1. Transform Estimates. In order to study the linear problem in the slab domain, we will employ the idea that transforming the variable-coefficient problem into a constant-coefficient problem through diffeomorphism. So before estimating, we need to confirm the mapping $\Phi$ is an isomorphism from $\Omega$ to $\Omega(t)$ and determine the relation of corresponding norms between these two spaces.

Lemma 2.8. Let $\Psi: \Omega \rightarrow \Omega'$ be a $C^1$ diffeomorphism satisfying $\Psi \in H_{loc}^{k+1}$, the Jacobi $J = \det(\nabla \Psi) > \delta > 0$ a.e. in $\Omega$ and $\nabla \Psi - I \in H^k(\Omega)$ for an integer $k \geq 3$. If $v \in H^m(\Omega')$, then $v \circ \Psi \in H^m(\Omega)$ for $m = 0, 1, \ldots, k+1$, and

$$\|v \circ \Psi\|_{H^m(\Omega)} \lesssim P(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|v\|_{H^m(\Omega')},$$

for a polynomial $P(\cdot)$. Similarly, for $u \in H^m(\Omega)$, $u \circ \Psi^{-1} \in H^m(\Omega')$ for $m = 0, 1, \ldots, k+1$ and

$$\|u \circ \Psi^{-1}\|_{H^m(\Omega')} \lesssim P(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|u\|_{H^m(\Omega)}.$$ 

Let $\Sigma' = \Psi(\Sigma)$ denote the upper boundary of $\Omega'$. If $v \in H^{m-1/2}(\Sigma')$ for $m = 1, \ldots, k-1$, then $v \circ \Psi \in H^{m-1/2}(\Sigma)$, and

$$\|v \circ \Psi\|_{H^{m-1/2}(\Sigma')} \lesssim C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|v\|_{H^{m-1/2}(\Sigma')}.$$ 

If $u \in H^{m-1/2}(\Sigma)$ for $m = 1, \ldots, k-1$, then $u \circ \Psi^{-1} \in H^{m-1/2}(\Sigma')$ and

$$\|u \circ \Psi^{-1}\|_{H^{m-1/2}(\Sigma')} \lesssim C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|u\|_{H^{m-1/2}(\Sigma)}.$$

where $C(\|\nabla \Psi - I\|_{H^k(\Omega)})$ denote a constant depending on $\|\nabla \Psi - I\|_{H^k(\Omega)}$.

Proof. The proof is almost identical to that of lemma 2.3.1 in [1]. It is easy to see from the original proof, in (2.2.1) and (2.2.2), the constant $C$ is actually a polynomial of $\|\nabla \Psi - I\|_{H^k(\Omega)}$. All the other part of the proof is exactly the same. 

Remark 2.9. Based on our assumptions on $\bar{\eta}$ and $b$, it is easy to see $\Phi$ defined in 1.3.3 is a $C^1$ diffeomorphism satisfying the hypothesis in lemma 2.8. Moreover, the universal constant $C$ now can be taken in a succinct form

$$C = C(\|\eta\|_{H^{k+1/2}(\Sigma)})$$

If we only consider the estimate in the domain $\Omega$ and need to specify the constant, we can take the explicit form

$$C = P(\|\eta\|_{H^{k+1/2}(\Sigma)}).$$

2.2.2. Functional Spaces. Now we introduce several functional spaces. We write $H^k(\Omega)$ and $H^k(\Sigma)$ for usual Sobolev space of either scalar or vector functions. Define

$$W(t) = \{u(t) \in H^1(\Omega) : u(t)|_{\Sigma_b} = 0\}$$

$$X(t) = \{u(t) \in H^1(\Omega) : u(t)|_{\Sigma_b} = 0, \nabla \cdot u(t) = 0\}$$

Moreover, let $L^2([0, T]; H^k(\Omega))$ and $L^2([0, T]; H^k(\Sigma))$ denotes the usual time-involved Sobolev space. Define

$$W = \{u \in L^2([0, T]; H^1(\Omega)) : u|_{\Sigma_b} = 0\}$$

$$X = \{u \in L^2([0, T]; H^1(\Omega)) : u|_{\Sigma_b} = 0, \nabla \cdot u = 0\}$$

It is easy to see $W$, $X$, $W$ and $X$ are all Hilbert spaces with the inner product defined exactly the same as in $H^1(\Omega)$ or $L^2([0, T]; H^1(\Omega))$. Hence, in the subsequence, we will take the norms of these spaces the same as $\|\cdot\|_{H^1}$ or $\|\cdot\|_{L^2H^1}$.

Furthermore, we define the functional space for zero upper boundary.

$$V(t) = \{u(t) \in H^1(\Omega) : u(t)|_{\Sigma_b} = 0\}$$

Also we will need an orthogonal decomposition $H^0(\Omega) = Y(t) \oplus Y^\perp(t)$, where

$$Y^\perp(t) = \{\nabla \cdot \varphi(t) : \varphi(t) \in V(t)\}$$

In our use of these spaces, we will often drop the $(t)$ when there is no potential of confusion. Finally we will define the regular divergence-free space in $H^1(\Omega)$,

$$X_0 = \{u \in H^1(\Omega) : u|_{\Sigma_b} = 0, \nabla \cdot u = 0\}$$

$$X_0 = \{u \in L^2([0, T]; H^1(\Omega)) : u|_{\Sigma_b} = 0, \nabla \cdot u = 0\}$$

Sometimes, we will consider the functional space with the similar properties as defined above in transformed space $\Omega'$. If that is the case, we will employ the notation $W(\Omega')$ or $X(\Omega')$ to specify the space.
they live in. The same fashion can be applied to other functional spaces. We will use \( \langle \cdot, \cdot \rangle_{H^0} \) to denote the normalized inner product in \( H^0 \), i.e.

\[
(2.2.12) \quad \langle u, v \rangle_{H^0} = \int_\Omega J u \cdot v
\]

and \( \langle \cdot, \cdot \rangle_{L^2H^0} \) to denote the normalized inner product in \( L^2([0, T]; H^0) \), i.e.

\[
(2.2.13) \quad \langle u, v \rangle_{L^2H^0} = \int_0^t \int_\Omega J u \cdot v
\]

Also we will use \( \langle \cdot, \cdot \rangle_W \) and \( \langle \cdot, \cdot \rangle_{W} \) to denote the regular inner product in \( H^1 \) if both arguments are in \( W \) or \( W \). Moreover, we define the regular inner product on \( \Sigma \) as follows.

\[
(2.2.14) \quad \langle u, v \rangle_{H^0(\Sigma)} = \int_{\Sigma} u \cdot v
\]

\[
(2.2.15) \quad \langle u, v \rangle_{L^2H^0(\Sigma)} = \int_0^t \int_\Sigma u \cdot v
\]

The following result gives a version of Korn’s type inequality for initial domain \( \Omega_0 \). Here we employ Beale’s idea in [4].

**Lemma 2.10.** Suppose \( \Omega_0 \) is the initial domain and \( \eta_0 \in H^{5/2}(\Sigma) \). Then

\[
(2.2.16) \quad \|v\|^2_{H^1(\Omega_0)} \lesssim \|Dv\|^2_{H^0(\Omega_0)}
\]

for any \( v \in H^1(\Omega_0) \) and \( v = 0 \) on \( \{y_3 = -b\} \).

**Proof.** In the periodic case, \( \Sigma \) is bounded and \( v|_{\Sigma} = 0 \), so naturally Korn’s inequality is valid (see proof of lemma 2.7 in [4]). Then we consider the infinite case. First we prove the decaying property of initial surface \( \eta_0 \), i.e. for \( |\alpha| \leq 1 \), \( |\partial^\alpha \eta_0| \to 0 \) as \( |x| \to \infty \) in the slab. For horizontal derivative \( \partial^\alpha \),

\[
\partial^\alpha \tilde{\eta}_0 = \int_{R^2} (2\pi i \xi)^\alpha e^{\xi x} |\xi| \tilde{\eta}_0(\xi) d\xi
\]

where \( \tilde{\eta}_0 \) is the Fourier transform of \( \eta_0 \) in \( R^2 \). Then

\[
\int_{R^2} \left| (2\pi i \xi)^\alpha e^{\xi x} |\xi| \tilde{\eta}_0(\xi) \right| d\xi \lesssim \int_{R^2} |\xi|^{2\alpha} e^{\xi x} |\xi| \tilde{\eta}_0(\xi) \left( 1 + |\xi|^{5/4} \right) \left( \frac{1}{1 + |\xi|^{5/4}} \right) d\xi
\]

\[
\lesssim \left( \int_{R^2} |\xi|^{2\alpha} \tilde{\eta}_0(\xi) \right)^2 \left( 1 + |\xi|^{5/4} \right)^2 \left( \int_{R^2} \frac{e^{2\xi x} |\xi| d\xi}{(1 + |\xi|^{5/4})^2} \right)^{1/2} \lesssim \|\eta_0\|_{H^{\alpha+5/4}} \left( \int_{R^2} \frac{e^{2\xi x} |\xi| d\xi}{(1 + |\xi|^{5/4})^2} \right)^{1/2} < \infty
\]

The last inequality is valid since \( x_3 < 0 \). Hence, \( (2\pi i \xi)^\alpha e^{\xi x} |\xi| \tilde{\eta}_0(\xi) \in L^1(R^2) \). A similar proof can justify the result if \( \partial^\alpha = \partial_x \). Since \( |x| \to \infty \) naturally implies \( |x'| \to \infty \) in the slab, the Riemann-Lebesgue lemma implies \( |\partial^\alpha \tilde{\eta}_0| \to 0 \) as \( |x| \to \infty \).

We construct a mapping \( \sigma = \Phi(0) \) as defined in (1.3.3), which maps \( \Omega = \{R^3 : -b_0 < x_3 < 0\} \) to \( \Omega_0 \). Denote \( \sigma(x) = x + \sigma(x) \). The above decaying property and our assumption on \( b \) leads to \( \partial^\alpha \sigma(x) \to 0 \) as \( |x| \to \infty \) for \( |\alpha| \leq 1 \).

We partition the slab \( \Omega \) into cubes

\[
Q_{j,k} = \{x : j < x_1 < j + 1, k < x_2 < k + 1, -b_0 < x_3 < 0\}
\]

In each cube, we have Korn’s inequality

\[
\|v\|^2_{H^1(Q)} \leq C(\|Dv\|^2_{H^0(Q)} + \|v\|^2_{H^0(Q)})
\]

Employing the compactness argument as Beale did, under the condition \( v = 0 \) on \( \{x_3 = -b_0\} \), we can strengthen this result to

\[
\|v\|^2_{H^1(Q)} \leq C \|Dv\|^2_{H^0(Q)}
\]

This argument relies on the fact that for such \( v \)

\[
\|v\|^2_{H^0(Q)} \leq C \|Dv\|^2_{H^0(Q)}
\]
Now suppose \( D \subseteq \Omega_0 \) is the image of union of such cubes contained in \( \{|x| \geq R\} \) for some \( R \). Applying last estimate to \( \nu = u \circ \sigma \) and then transform to \( \Omega_0 \), we have
\[
\|u\|_{H^1(D)} \leq C \|\nabla u\|_{H^1(D)} + \epsilon \|u\|_{H^1(D)}
\]
where \( \epsilon < 1 \) when \( R \) is large enough based on the decaying property. Then we use Korn’s inequality in the bounded domain \( \Omega_0 - D \) and combine with above estimate to obtain the estimate required. \( \Box \)

Next, we will show the equivalence of certain quantities. Although the result is quite obvious for small initial data (see lemma 2.1 of [1]), now we need to utilize above Korn’s inequality to show this for arbitrary data.

**Lemma 2.11.** There exists a \( 0 < \epsilon_0 < 1 \), such that if \( \|\eta - \eta_0\|_{H^{5/2}} < \epsilon_0 \), then the following relation
\[
(2.2.17) \quad \|u\|_{H^0}^2 \lesssim \int_{\Omega} J|u|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)}) \|u\|_{H^1}^2
\]
\[
(2.2.18) \quad \frac{1}{(1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3} \|u\|_{H^1}^2 \lesssim \int_{\Omega} J|\nabla u|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3 \|u\|_{H^1}^2
\]
holds for all \( u \in W \).

**Proof.** (2.2.17) is just a natural corollary of relation \( \delta \lesssim \|J\|_{L^\infty} \lesssim (1 + \|\nabla \eta\|_{L^\infty}) \lesssim (1 + \|\eta\|_{H^{5/2}}) \lesssim (1 + \|\eta_0\|_{H^{5/2}}) \) based on our assumption and Sobolev embedding.

To derive (2.2.18), notice that
\[
(2.2.19) \quad \int_{\Omega} J|\nabla u|^2 = \int_{\Omega} J|\nabla \eta u|^2 + \int_{\Omega} J(\nabla \eta u + \nabla \eta_0 u) : (\nabla \eta u - \nabla \eta_0 u)
\]
For the first term on the right-handed side, based on the integral substitution and Korn’s inequality we proved above, we have
\[
\int_{\Omega} J|\nabla \eta u|^2 \lesssim \frac{1}{1 + \|\eta_0\|_{H^{5/2}(\Sigma)}} \int_{\Omega} J_0 |\nabla \eta_0 u|^2 \lesssim \frac{1}{1 + \|\eta_0\|_{H^{5/2}(\Sigma)}} \int_{\Omega} |\nabla \eta|^2 \lesssim \frac{1}{1 + \|\eta_0\|_{H^{5/2}(\Sigma)}} \|\eta\|_{H^1}^2
\]
For the second term on the right-handed side, using Korn’s inequality in slab \( \Omega \), we can naturally estimate
\[
\left| \int_{\Omega} J(\nabla \eta u + \nabla \eta_0 u) : (\nabla \eta u - \nabla \eta_0 u) \right| \lesssim \|J\|_{L^\infty} \|A + A_0\|_{L^\infty} \|A - A_0\|_{L^\infty} \int_{\Omega} |\nabla \eta|^2 \lesssim \epsilon_0 (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3 \int_{\Omega} |\nabla u|^2 \lesssim \epsilon_0 (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3 \|u\|_{H^1}^2
\]
For given initial data, we can always take \( \epsilon_0 \) sufficiently small to absorb the second term into the first one. Then we have
\[
\int_{\Omega} J|\nabla u|^2 \lesssim \frac{1}{(1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3} \|u\|_{H^1}^2
\]
The first part of (2.2.18) is verified. For the second part, notice that
\[
\int_{\Omega} J|\nabla u|^2 \lesssim (1 + \|\eta\|_{H^{5/2}}) \int_{\Omega} |\nabla u|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}}) \max\{1, \|AK\|_{L^\infty}, \|BK\|_{L^\infty}, \|K\|_{L^\infty}\} \|u\|_{H^1}^2
\]
Since it is easy to see
\[
\max\{1, \|AK\|_{L^\infty}, \|BK\|_{L^\infty}, \|K\|_{L^\infty}\} \lesssim 1 + (1 + \|\nabla \eta\|_{L^\infty}) \|K\|_{L^\infty} \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^2
\]
then the second part of (2.2.18) follows. \( \Box \)

**Remark 2.12.** Throughout this section, we can always assume the restriction of \( \eta \) depending on \( \epsilon_0 \) is justified, and finally we will verify this condition for \( t \in [0, T] \) in the nonlinear part.

Now we present a lemma about the differentiability of norms in time-dependent space.

**Lemma 2.13.** Suppose that \( u \in W \) and \( \partial_t u \in W^* \). Then the mapping \( t \mapsto \|u(t)\|_{H^1}^2 \) is absolute continuous, and
\[
(2.2.20) \quad \frac{d}{dt} \|u(t)\|_{H^1}^2 = 2(\partial_t u(t), u(t)) + \int_{\Omega} |u(t)|^2 \partial_t J(t)
\]
Moreover, \( u \in C^0([0, T]; H^0(\Omega)) \). If \( v \in W \) and \( \partial_t v \in W^* \) as well, we have

\[
\frac{d}{dt} \int_{\Omega} u(t) \cdot v(t) = (\partial_t u(t), v(t)) + (\partial_t v(t), u(t)) + \int_{\Omega} u(t) \cdot v(t) \partial_t J(t)
\]

(2.2.21)

**Proof.** This is exactly the same result as lemma 2.4 in [1], so we omit the proof here. \( \square \)

Next we test the estimate for \( H^{-1/2} \) boundary functions.

**Lemma 2.14.** If \( v \in H^0 \) and \( \nabla A \cdot v \in H^0 \), then \( v \cdot N \in H^{-1/2}(\Sigma) \), \( v \cdot v \in H^{-1/2}(\Sigma_b) \) (\( \nu \) is the outer normal vector on \( \Sigma_b \)) and satisfies the estimate

\[
\| v \cdot N \|_{H^{-1/2}(\Sigma)} + \| v \cdot v \|_{H^{-1/2}(\Sigma_b)} \lesssim (1 + \| \eta \|_{H^{3/2}})^2 \left( \| v \|_{H^0} + \| \nabla A \cdot v \|_{H^0} \right)
\]

(2.2.22)

**Proof.** We only need to prove the result on \( \Sigma \); the result on \( \Sigma_b \) can be derived in a similar fashion considering \( J \mathcal{A} \vec{e}_3 = \nu \) on \( \Sigma_b \).

Let \( \phi \in H^{1/2}(\Sigma) \) be a scalar function and let \( \tilde{\phi} \in V \) be a bounded extension. We have

\[
\int_{\Sigma} \phi v \cdot N = \int_{\Sigma} J \mathcal{A}_{ij} v_i \phi (e_j \cdot e_3) = \int_{\Omega} \nabla A \cdot (v \tilde{\phi}) J = \int_{\Omega} \tilde{\partial}_A \cdot v J + v \cdot \nabla A \tilde{\phi} J
\]

\[
\lesssim (1 + \| \eta \|_{H^{3/2}}) \left( \| \tilde{\phi} \|_{H^0} \| \nabla A \cdot v \|_{H^0} + \| v \|_{H^0} \| \nabla A \tilde{\phi} \|_{H^0} \right)
\]

\[
\lesssim (1 + \| \eta \|_{H^{3/2}})^2 (\| v \|_{H^0} + \| \nabla A \cdot v \|_{H^0}) \| \tilde{\phi} \|_{H^{1/2}(\Sigma)}
\]

Since \( \phi \) is arbitrary, our result naturally follows. \( \square \)

**Remark 2.15.** Recall the space \( \mathcal{Y}(t) \subset H^0 \). It can be shown that if \( v \in \mathcal{Y}(t) \), then \( \nabla A \cdot v = 0 \) in the weak sense, such that lemma 2.14 implies that \( v \cdot N \in H^{-1/2}(\Sigma) \) and \( v \cdot v \in H^{-1/2}(\Sigma_b) \). Moreover, since the elements of \( \mathcal{Y}(t) \) are orthogonal to each \( \nabla A \phi \) for \( \phi \in V(t) \), we find that \( v \cdot v = 0 \) on \( \Sigma_b \).

We want to connect the divergence-free space and divergence-\( A \)-free space, so we define

\[
M = M(t) = K \nabla \Phi = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ AK & BK & 1 \end{pmatrix}
\]

Note that \( M \) is invertible and \( M^{-1} = J \mathcal{A}^T \). Since \( \partial_j (J \mathcal{A}_{ij}) = 0 \) for \( i = 1, 2, 3 \), we have the following relation

\[
p = \nabla A \cdot v \Leftrightarrow Jp = J \nabla A \cdot v = J \mathcal{A}_{ij} v_i = \partial_j (J \mathcal{A}_{ij} v_i) = \partial_j (J \mathcal{A}^T v_j) = \partial_j (M^{-1} v)_j = \nabla \cdot (M^{-1} v)
\]

Then if \( \nabla A \cdot v = 0 \), we have \( \nabla \cdot (M^{-1} v) = 0 \). \( M \) induces a linear operator \( \mathcal{M}_t : u \rightarrow \mathcal{M}_t(u) = M(t) u \).

It has the following property.

**Lemma 2.16.** For each \( t \in [0, T] \), \( k = 0, 1, 2 \), \( \mathcal{M}_t \) is a bounded linear isomorphism from \( H^k(\Omega) \) to \( H^k(\Omega) \) and from \( X_0 \) to \( X \). The bounding constant is given by \( (1 + ||\eta(t)||_{H^{3/2}(\Sigma)})^2 \). If we further define \( \mathcal{M} \) defined by \( \mathcal{M}u(t) = \mathcal{M}_t u(t) \), then it is a bounded linear isomorphism from \( L^2([0, T]; H^k(\Omega)) \) to \( L^2([0, T]; H^k(\Omega)) \) and from \( X_0 \) to \( X \). The bounding constant is given by \( (1 + \sup_{0 \leq t \leq T} ||\eta(t)||_{H^{3/2}(\Sigma)})^2 \).

**Proof.** It is easy to see for each \( t \in [0, T] \),

\[
||\mathcal{M}_t u||_{H^k} \lesssim ||\mathcal{M}_t e||_{C^2} \leq ||u||_{H^k} \lesssim (1 + ||\eta(t)||_{H^{3/2}(\Sigma)})^2 ||u||_{H^k}
\]

for \( k = 0, 1, 2 \), which implies \( \mathcal{M}_t \) is a bounded operator from \( H^k \) to \( H^k \). Since \( \mathcal{M}_t \) is invertible, we can estimate \( ||\mathcal{M}^{-1} e||_{H^k} \lesssim (1 + ||\eta(t)||_{H^{3/2}(\Sigma)})^2 ||e||_{H^k} \). Hence \( \mathcal{M}_t \) is an isomorphism between \( H^k \). Above analysis implies \( \mathcal{M}_t \) maps divergence-free function to divergence-\( A \)-free function. Then it is also an isomorphism from \( X_0 \) to \( X(t) \).

A similar argument can justify the case for \( L^2([0, T]; H^k) \) and \( X \). \( \square \)

2.2.3. **Pressure as a Lagrangian Multiplier.** It is well-known that in usual Navier-Stokes equation, pressure can be taken as a Lagrangian multiplier. In our new settings now, proposition 2.2.9 in [1] gives construction of pressure from transformed equation 1.3.8, which is valid for small free surface. So our result in arbitrary initial data is not completely ready, which we will present here.

For \( p \in H^0 \), we define the functional \( S_t \in W^* \) by \( S_t(v) = \langle p, \nabla A \cdot v \rangle_{H^0} \). By the Riesz representation
theorem, there exists a unique $Q_t(p) \in W$ such that $S_t(v) = \langle Q_t(p), v \rangle_W$ for all $v \in W$. This defines a linear operator $Q_t : H^0 \rightarrow W$, which is bounded since we may take $v = Q_t(p)$ to see
$$
\|Q_t(p)\|^2_W = S_t(v) = \langle p, \nabla_A \cdot v \rangle_{(0,T)} \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)}) \|p\|_{H^0} \|\nabla_A \cdot v\|_{H^0}
$$
so we have $\|Q_t(p)\|_W \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)}) \|p\|_{H^0}$. Similarly, for $S \in W^*$, we may also define a bounded linear operator $Q : L^2([0,T]; H^0) \rightarrow W$ via the relation $\langle p, \nabla_A \cdot v \rangle_{L^2H^0} = \langle Q(p), v \rangle_W = S(v)$ for all $v \in W$. Similar argument shows $\|Q(p)\|_W \lesssim (1 + \sup_{0 \leq t \leq T} \|\eta(t)\|_{H^{3/2}(\Sigma)})^3 \|p\|_{L^2H^0}$.

**Lemma 2.17.** Let $p \in H^0$, then there exists a $v \in W$ such that $\nabla_A \cdot v = p$ and $\|v\|_W \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^6 \|p\|_{H^0}$. If instead $p \in L^2([0,T]; H^0)$, then there exists a $v \in W$ such that $\nabla_A \cdot v = p$ for a.e. $t$ and $\|v\|_W \lesssim (1 + \sup_{0 \leq t \leq T} \|\eta(t)\|_{H^{3/2}(\Sigma)}) \|p\|_{L^2H^0}$.

**Proof.** In the proof of lemma 3.3 of [4], it is established that for any $q \in L^2(\Omega)$, the problem $\nabla \cdot u = q$ admits a solution $u \in W$ such that $\|u\|_{H^1} \lesssim \|q\|_{H^0}$. A simple modification of this proof in infinite case can be applied to periodic case. Let define $g = Jp$, then
$$
\|q\|^2_{H^0} = \int_\Omega |q|^2 = \int \|p\|^2 J \lesssim \|J\|^2_{\mathbb{Z}} \|p\|_{H^0}^2 \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^2 \|p\|_{H^0}^2
$$
Hence, we know $v = M(t)u \in W$ satisfies $\nabla_A v = p$ and
$$
\|v\|^2_W \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^6 \|u\|^2_W \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^6 \|q\|^2_{H^0} 
$$
A similar argument may justify the case for $W$. \hfill \Box

With this lemma in hand, we can show the range of operator $Q_t$ and $Q$ is closed in $W$ and $W'$.

**Lemma 2.18.** $R(Q_t)$ is closed in $W$, and $R(Q)$ is closed in $W$.

**Proof.** For $p \in H^0$, let $v \in W$ be the solution of $\nabla_{A_0} \cdot v = p$ provided by lemma 2.17. Then
$$
\|p\|_{H^0} \lesssim \langle p, \nabla_A \cdot v \rangle_{(0,T)} = \langle Q_t(p), v \rangle_W \lesssim \|Q_t(p)\|_W \|v\|_W \lesssim \|Q_t(p)\|_W (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^6 \|p\|_{H^0}
$$
such that
$$
\frac{1}{(1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^3} \|Q_t(p)\|_W \lesssim \|p\|_{H^0} \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^6 \|Q_t(p)\|_W
$$
Hence $R(Q_t)$ is closed in $W$. A similar analysis shows that $R(Q)$ is closed in $W$. \hfill \Box

Now we can perform a decomposition of $W$ and $W'$.

**Lemma 2.19.** We have that $W = X \oplus R(Q_t)$, i.e. $X^\perp = R(Q_t)$. Also, $W = X \oplus R(Q)$, i.e. $X^\perp = R(Q)$.

**Proof.** By lemma 2.18, $R(Q_t)$ is closed subspace of $W$, so it suffices to show $R(Q_t)^\perp = X$. Let $v \in R(Q_t)^\perp$, then for all $p \in H^0$, we have
$$
\langle p, \nabla_A \cdot v \rangle_{(0,T)} = \langle Q_t(p), v \rangle_W = 0
$$
and hence $\nabla_A \cdot v = 0$, which implies $R(Q_t)^\perp \subseteq X$. On the other hand, suppose $v \in X$, then $\nabla_A \cdot v = 0$ implies
$$
\langle Q_t(p), v \rangle_W = \langle p, \nabla_A \cdot v \rangle_{(0,T)} = 0
$$
for all $p \in H^0$. Hence $V \in R(Q_t)^\perp$ and we see $X \subseteq R(Q_t)^\perp$. So we finish the proof. A similar argument can show $W = X \oplus R(Q)$. \hfill \Box

**Proposition 2.20.** If $\lambda \in W^*$ such that $\lambda(v) = 0$ for all $v \in X$, then there exists a unique $p(t) \in H^0$ such that
$$
\langle p(t), \nabla_A \cdot v \rangle_{H^0} = \lambda(v) \quad \text{for all } v \in W
$$
and $\|p(t)\|_{H^0} \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^6 \|\lambda\|_{W^*}$.

If $\Lambda \in W^*$ such that $\Lambda(v) = 0$ for all $v \in X$, then there exists a unique $p \in L^2([0,T]; H^0)$ such that
$$
\langle p, \nabla_A \cdot v \rangle_{L^2H^0} = \Lambda(v) \quad \text{for all } v \in W
$$
and $\|p\|_{L^2H^0} \lesssim (1 + \sup_{0 \leq t \leq T} \|\eta(t)\|_{H^{3/2}(\Sigma)})^3 \|\Lambda\|_{W^*}$. 
Proof. If \( \lambda(v) = 0 \) for all \( v \in X \), then the Riesz representation theorem yields the existence of a unique \( u \in X^\perp \) such that \( \lambda(v) = \langle u, v \rangle_W \) for all \( v \in W \). By lemma 2.19, \( u = Q(p) \) for some \( p \in H^0 \). Then 
\[
\lambda(v) = \langle Q(p), v \rangle_W = \langle p(t), \nabla_A \cdot v \rangle_{H^0} \quad \text{for all } v \in W.
\]
For the sake of simplicity, we will temporarily ignore the time dependence of \( \lambda(t) \).

As for the estimate, by lemma 2.17, we may find \( v \in W \) such that \( \nabla_A \cdot v = p \) and \( \|v\|_W \lesssim (1 + \|\eta(t)\|_{H^{k/2}(\Sigma)})^6 \|p\|_{H^0} \).

So the desired estimate holds and a similar argument can show the result for \( \Lambda \). \( \square \)

2.3. Elliptic Estimates. In this section, we will study two types of elliptic problems, which will be employed in deriving the linear estimates. For both equations, we will present wellposedness theorems to the higher regularity.

2.3.1. \( A \)-Stokes Equation. Let us consider the stationary Navier-Stokes problem.

\[
\begin{aligned}
&\nabla_A \cdot S_A(p, u) = F \quad \text{in } \Omega \\
&\nabla_A \cdot u = G \quad \text{in } \Omega \\
&S_A(p, u) N = H \quad \text{on } \Sigma \\
&u = 0 \quad \text{on } \Sigma_b
\end{aligned}
\]  

(2.3.1)

Since this problem is stationary, we will temporarily ignore the time dependence of \( \eta, A \), etc.

Before discussing the higher regularity result for this equation, we should first define the weak formulation. Our method is quite standard; we will introduce \( p \) after first solving a pressureless problem. Suppose \( F \in W^*, \; G \in H^0 \) and \( H \in H^{-1/2}(\Sigma) \). We say \( (u, p) \in W \times H^0 \) is a weak solution to (2.3.1) if \( \nabla_A \cdot u = G \) a.e. in \( \Omega \), and

\[
\frac{1}{2} \langle [\nabla_A u, \nabla_A v]_{H^0} - \langle p, \nabla_A \cdot v \rangle_{H^0} = \langle F, v \rangle_{H^0} - \langle H, v \rangle_{H^0(\Sigma)}
\]

(2.3.2)

for all \( v \in W \).

Lemma 2.21. Suppose \( F \in W^*, \; G \in H^0 \) and \( H \in H^{-1/2}(\Sigma) \), then there exists a unique weak solution \( (u, p) \in W \times H^0 \) to (2.3.1).

Proof. By lemma 2.17, there exists a \( \bar{u} \in W \) such that \( \nabla_A \cdot \bar{u} = F^2 \). Naturally we can switch the unknowns to \( w = u - \bar{u} \) such that in the weak formulation \( w \) is such that \( \nabla_A \cdot w = 0 \) and satisfies

\[
\frac{1}{2} \langle [\nabla_A w, \nabla_A v]_{H^0} - \langle p, \nabla_A \cdot v \rangle_{H^0} = -\frac{1}{2} \langle [\nabla_A \bar{u}, \nabla_A v]_{H^0} + \langle F, v \rangle_{H^0} - \langle H, v \rangle_{H^0(\Sigma)}
\]

(2.3.3)

for all \( v \in W \).

To solve this problem, we may restrict our test function to \( v \in X \) such that the pressure term vanishes. A direct application of Riesz representation theorem to the Hilbert space whose inner product is defined as \( \langle u, v \rangle = \langle [\nabla_A u, \nabla_A v]_{H^0} \rangle \) provides a unique \( w \in X \) such that

\[
\frac{1}{2} \langle [\nabla_A w, \nabla_A v]_{H^0} = -\frac{1}{2} \langle [\nabla_A \bar{u}, \nabla_A v]_{H^0} + \langle F, v \rangle_{H^0} - \langle H, v \rangle_{H^0(\Sigma)}
\]

(2.3.4)

for all \( v \in X \).

In order to introduce the pressure, we can define \( \lambda \in W^* \) as the difference of the left and right hand sides in (2.3.4). So \( \lambda(v) = 0 \) for all \( v \in X \). Then by proposition 2.20, there exists a unique \( p \in H^0 \) satisfying

\[
\langle p, \nabla_A \cdot v \rangle_{H^0} = \lambda(v) \quad \text{for all } v \in W,
\]

which is equivalent to (2.3.3). \( \square \)

The regularity gain available for solution to (2.3.1) is limited by the regularity of the coefficients of the operator \( \nabla_A, \nabla_{\perp}, \nabla_{\perp} \), and hence by the regularity of \( \eta \). In the next lemma, we will present some preliminary elliptic estimates.

Lemma 2.22. Suppose that \( \eta \in H^{k+1/2}(\Sigma) \) for \( k \geq 3 \) such that the map \( \Phi \) defined in (1.3.3) is a \( C^1 \) diffeomorphism of \( \Omega \) to \( \Omega' = \Phi(\Omega) \). If \( F \in H^0(\Omega), \; G \in H^1(\Omega) \) and \( H \in H^{1/2}(\Sigma) \), then the equation

(2.3.1)

admits a unique strong solution \( (u, p) \in H^2 \times H^1 \), i.e. \( (u, p) \) satisfies (2.3.1) in strong sense. Moreover, for \( r = 2, \ldots, k - 1 \) we have the estimate

\[
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\]

(2.3.5)

whenever the right hand side is finite, where \( C(\eta) \) is a constant depending on \( \|\eta\|_{H^{k+1/2}(\Sigma)} \).

Proof. This lemma is exactly the same as lemma 3.6 in [1], so we omit the proof here. \( \square \)
Notice that the estimate (2.3.5) can only go up to $k - 1$ order, which does not fully satisfy our requirement. Hence, in the following we will employ approximating argument to improve this estimate. For clarity, we divide it into two steps. In the next lemma, we first prove that the constant can actually only depend on the initial free surface.

**Lemma 2.23.** Let $k \geq 6$ be an integer and suppose that $\eta \in H^{k+1/2}(\Sigma)$ and $\eta_0 \in H^{k+1/2}(\Sigma)$. Then there exists $\epsilon_0 > 0$ such that if $\|\eta - \eta_0\|_{H^{-3/2}(\Sigma)} \leq \epsilon_0$, solution to 2.3.1 satisfies

\[
\|\eta\|_{H^{r-1}} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\]

for $r = 2, \ldots, k - 1$, whenever the right hand side is finite, where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$.

**Proof.** Based on lemma 2.22, we have the estimate

\[
\|u\|_{H^{r-1}} + \|p\|_{H^{r-1}} \lesssim C(\eta) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\]

for $r = 2, \ldots, k - 1$, whenever the right hand side is finite, where $C(\eta)$ is a constant depending on $\|\eta\|_{H^{k+1/2}(\Sigma)}$. Define $\xi = \eta - \eta_0$, then $\xi \in H^{k+1/2}(\Sigma)$.

Let us denote $A_0$ and $N_0$ of quantities in terms of $\eta_0$. We rewrite the equation 2.3.1 as a perturbation of initial status

\[
\begin{align*}
\nabla A_0 \cdot S_{A_0}(p, u) & = F^0 + F^0 & \text{in } \Omega \\
\nabla A_0 \cdot u & = G + G^0 & \text{in } \Omega \\
S_{A_0}(p, u)N_0 & = H + H^0 & \text{on } \Sigma \\
u & = 0 & \text{on } \Sigma_b
\end{align*}
\]

where

\[
F^0 = \nabla A_0 \cdot S_A(p, u) + \nabla A_0 \cdot S_{A_0-A}(p, u)
\]

\[
G^0 = \nabla A_0 \cdot u
\]

\[
H^0 = S_{A_0}(p, u)(N_0 - N) + S_{A_0-A}(p, u)
\]

Suppose that $\|\xi\|_{H^{k-3/2}(\Sigma)} \leq 1$, which implies $\|\xi\|_{H^{k-3/2}(\Sigma)} \leq \|\xi\|_{H^{k-3/2}(\Sigma)} < 1$ for any $l > 1$. A straightforward calculation reveals that

\[
\|F^0\|_{H^{r-2}} \leq C(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^4 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^{r-1}} + \|p\|_{H^{r-1}})
\]

\[
\|G^0\|_{H^{r-1}} \leq C(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^2 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^{r-1}})
\]

\[
\|H\|_{H^{r-3/2}(\Sigma)} \leq C(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^2 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^{r-1}} + \|p\|_{H^{r-1}})
\]

for $r = 2, \ldots, k - 1$.

Since the initial surface function $\eta_0$ satisfies all the requirement of lemma 2.22, we arrive at the estimate that for $r = 2, \ldots, k - 1$

\[
\|u\|_{H^{r-1}} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F + F^0\|_{H^{r-2}} + \|G + G^0\|_{H^{r-1}} + \|H + H^0\|_{H^{r-3/2}(\Sigma)} \right)
\]

where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$. Combining all above, we will have

\[
\|u\|_{H^{r-1}} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} + C(\eta_0)(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^4 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^{r-1}} + \|p\|_{H^{r-1}}) \right)
\]

So if

\[
\|\xi\|_{H^{k-3/2}(\Sigma)} \leq \min \left\{ \frac{1}{2}, \frac{1}{4C(\eta_0)(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^4} \right\}
\]

we can absorb the extra term in right hand side into left hand side and get a succinct form

\[
\|u\|_{H^{r-1}} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\]

for $r = 2, \ldots, k - 1$. \qed
Note that the above lemma only concerns about regularity up to \( k - 1 \) and we actually need two more order. Then the next result allows us to achieve this with a bootstrapping argument.

**Proposition 2.24.** Let \( k \geq 6 \) be an integer, and suppose that \( \eta \in H^{k+1/2}(\Sigma) \) as well as \( \eta_0 \in H^{k+1/2}(\Sigma) \) satisfying \( \| \eta - \eta_0 \|_{H^{k+1/2}(\Sigma)} \leq \epsilon_0 \). Then solution to 2.3.1 satisfies

\[
(2.3.15) \quad \| u \|_{H^r} + \| p \|_{H^{r-1}} \lesssim C(\eta_0) \left( \| F \|_{H^{r-2}} + \| G \|_{H^{r-1}} + \| H \|_{H^{r-3/2}(\Sigma)} \right)
\]

for \( r = 2, \ldots, k + 1 \), whenever the right hand side is finite, where \( C(\eta_0) \) is a constant depending on \( \| \eta_0 \|_{H^{k+1/2}(\Sigma)} \).

**Proof.** If \( r \leq k - 1 \), then this is just the conclusion of lemma 2.23, so our main aim is to gain two more regularity here for \( r = k \) and \( r = k + 1 \). In the following, we first define an approximate sequence for \( \eta \). In the case that \( \Sigma = R^2 \), we let \( \rho \in C_0^\infty(\Sigma) \) be such that \( \text{supp}(\rho) \subset B(0, 2) \) and \( \rho = 1 \) for \( B(0, 1) \). For \( m \in \mathbb{N} \), define \( \eta^m \) by \( \hat{\eta}^m(\xi) = \rho(\xi/m)\hat{\eta}(\xi) \) where \( \hat{\eta} \) denotes the Fourier transform. For each \( m \), \( \eta^m \in H^1(\Sigma) \) for arbitrary \( j \geq 0 \) and also \( \eta^m \to \eta \) in \( H^{k+1/2}(\Sigma) \) as \( m \to \infty \). In the periodic case, we define \( \hat{\eta}^m \) by throwing away the higher frequencies: \( \hat{\eta}^m(n) = 0 \) for \( |n| \geq m \). Then \( \eta^m \) has the same convergence property as above. Let \( A^m = N^m \) be defined in terms of \( \eta^m \).

Consider the problem (2.3.1) with \( A \) and \( N \) replaced by \( A^m \) and \( N^m \). Since \( \eta^m \in H^{k+5/2} \), we can apply lemma (2.22) to deduce the existence of \( (u^m, p^m) \) that solves

\[
(2.3.16) \quad \begin{cases}
\nabla A^m \cdot \mathcal{S}_{A^m}(p^m, u^m) = F & \text{in } \Omega \\
\nabla A^m \cdot v^m = G & \text{in } \Omega \\
\mathcal{S}_{A^m}(p^m, u^m)N_0 = H & \text{on } \Sigma \\
u^m = 0 & \text{on } \Sigma_b
\end{cases}
\]

and such that

\[
(2.3.17) \quad \| u^m \|_{H^r} + \| p^m \|_{H^{r-1}} \lesssim C(\| \eta^m \|_{H^{k+5/2}(\Sigma)}) \left( \| F \|_{H^{r-2}} + \| G \|_{H^{r-1}} + \| H \|_{H^{r-3/2}(\Sigma)} \right)
\]

for \( r = 2, \ldots, k + 1 \). We can rewrite above equation in the following shape, as long as we split the \( D_{A^m}u^m \) term.

\[
(2.3.18) \quad \begin{cases}
-\Delta A^m u^m + \nabla A^m p^m = F + \nabla A^m G & \text{in } \Omega \\
\nabla A^m \cdot u^m = G & \text{in } \Omega \\
(p^m I - D_{A^m}u^m)N_m = H & \text{on } \Sigma \\
u^m = 0 & \text{on } \Sigma_b
\end{cases}
\]

In the following, we will prove an improved estimate for \( (u^m, p^m) \) in terms of \( \| \eta^m \|_{H^{k+1/2}(\Sigma)} \). We divide the proof into several steps.

**Step 1: Preliminaries**

To abuse the notation, within this bootstrapping procedure, we always use \( (u, p, \eta) \) instead of \( (u^m, p^m, \eta^m) \) to make the expression succinct, but in fact they should be understood as the approximate sequence. Also it is easy to see the term \( \nabla A^m G \) will not affect the shape of estimate because \( \| \nabla A^m G \|_{H^{r-1}} \lesssim \| \eta^m \|_{H^{k+1/2}(\Sigma)} \| G \|_{H^k} \). Hence, we still write \( F \) here to indicate the forcing term in first equation.

We write explicitly each terms in above equations, which will be hired in the following.

\[
(2.3.19) \quad \begin{aligned}
\partial_t u_1 + \partial_{x_1} u_1 &+ (1 + A^2 + B^2)K^2 \partial_{x_3} u_1 - 2AK \partial_{x_1} u_1 - 2BK \partial_{x_3} u_1 \\
&+ (AK \partial_3(AK) + BK \partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K \partial_3 K) \partial_3 u_1 + \partial p - AK \partial_3 p = F_1
\end{aligned}
\]

\[
(2.3.20) \quad \begin{aligned}
\partial_t u_2 + \partial_{x_2} u_2 &+ (1 + A^2 + B^2)K^2 \partial_{x_3} u_2 - 2AK \partial_{x_1} u_2 - 2BK \partial_{x_3} u_2 \\
&+ (AK \partial_3(AK) + BK \partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K \partial_3 K) \partial_3 u_2 + \partial p - BK \partial_3 p = F_2
\end{aligned}
\]

\[
(2.3.21) \quad \begin{aligned}
\partial_t u_3 + \partial_{x_3} u_3 &+ (1 + A^2 + B^2)K^2 \partial_{x_3} u_3 - 2AK \partial_{x_1} u_3 - 2BK \partial_{x_3} u_3 \\
&+ (AK \partial_3(AK) + BK \partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K \partial_3 K) \partial_3 u_3 + \partial p = F_3
\end{aligned}
\]

\[
(2.3.22) \quad \begin{aligned}
\partial_t u_1 - AK \partial_3 u_1 + \partial_2 u_2 - BK \partial_3 u_2 + K \partial_3 u_3 = G
\end{aligned}
\]

where for all above, \( A, B \) and \( K \) should be understood in terms of \( \eta^m \). For convenience, we define

\[
(2.3.23) \quad Z = C(\eta_0)P(\eta) \left( \| F \|_{H^{r-1}}^2 + \| G \|_{H^k}^2 + \| H \|_{H^{r-3/2}(\Sigma)}^2 \right)
\]
where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$ and $P(\eta)$ is the polynomial of $\|\eta\|_{H^{k+1/2}(\Sigma)}$.

Step 2: $r = k$ case

For $k - 1$ order elliptic estimate, we have

\[(2.3.24)\]

$$\|u\|_{H^{k-1}}^2 + \|p\|_{H^{k-2}}^2 \leq C(\eta_0) \left( \|F\|_{H^{k-3}}^2 + \|G\|_{H^{k-2}}^2 + \|H\|_{H^{k-5/2}(\Sigma)}^2 \right) \lesssim Z$$

By lemma 2.23, the bounding constant $C(\eta_0)$ only depend on $\|\eta_0\|_{H^{k+1/2}}$.

For $i = 1, 2$, since $(\partial_i u, \partial_i p)$ satisfies the equation

\[(2.3.25)\]

$$\begin{cases}
-\Delta A(\partial_i u) + \nabla A(\partial_i p) = \bar{F} & \text{in } \Omega \\
\nabla A \cdot (\partial_i u) = \bar{G} & \text{in } \Omega \\
((\partial_i p)I - D_{A}(\partial_i u))N = \bar{H} & \text{on } \Sigma \\
\partial_i u = 0 & \text{on } \Sigma_b
\end{cases}$$

where

$$\bar{F} = \partial_i F + \nabla A \cdot \nabla A u + \nabla A \cdot \nabla A \partial_i u - \nabla A \partial_i u \partial_i A u$$

$$\bar{G} = \partial_i G - \nabla A \cdot u$$

$$\bar{H} = \partial_i H - (pI - D_{A}A)\partial_i N + D_{A\partial_i u}N$$

Employing $k - 1$ order elliptic estimate, we have

\[(2.3.26)\]

$$\|\partial_i u\|_{H^{k-1}}^2 + \|\partial_i p\|_{H^{k-2}}^2 \lesssim C(\eta_0) \left( \|F\|_{H^{k-3}}^2 + \|G\|_{H^{k-2}}^2 + \|H\|_{H^{k-5/2}(\Sigma)}^2 \right)$$

Since except for the derivatives of $F$, $G$, and $H$, all the other terms on the right hand side has the form $\|A \cdot B\|_{H^k}$, in which $A = \partial^2 \eta$ and $B = \partial^3 u$ or $\partial^3 p$. These kinds of estimates can be achieved by lemma A.1. Because this lemma will be repeated used in the following estimates, we will not mention it every time and all of the following estimate can be derived in the same fashion. Hence, we have the forcing estimate

\[(2.3.27)\]

$$\|\bar{F}\|_{H^{k-3}}^2 + \|\bar{G}\|_{H^{k-2}}^2 + \|\bar{H}\|_{H^{k-5/2}}^2 \lesssim \|F\|_{H^{k-3}}^2 + \|G\|_{H^{k-2}}^2 + \|H\|_{H^{k-3/2}(\Sigma)}^2 + C(\eta_0)P(\eta) \left( \|u\|_{H^{k-1}}^2 + \|p\|_{H^{k-2}}^2 \right) \lesssim Z$$

In detail, this means

\[(2.3.28)\]

$$\|\partial_1 u\|_{H^{k-1}}^2 + \|\partial_1 p\|_{H^{k-2}}^2 + \|\partial_2 u\|_{H^{k-1}}^2 + \|\partial_2 p\|_{H^{k-2}}^2 \lesssim Z$$

Hence, most parts in $\|u\|_{H^k} + \|p\|_{H^{k-1}}$ has been covered by this estimate, except those with highest order derivative of $\partial_1$.

Multiplying $A$ to (2.3.21) and adding it to (2.3.19) will eliminate the $\partial_3 p$ term and get

$$(\partial_1 u_1 + A\partial_1 u_3) + (\partial_{22} u_1 + A\partial_{22} u_3) + (1 + A^2 + B^2)K^2(\partial_{33} u_1 + A\partial_{33} u_3) - 2AK(\partial_{33} u_1 + A\partial_{33} u_3) - 2BK(\partial_{33} u_1 + A\partial_{33} u_3) + (AK\partial_3(AK) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3K)(\partial_1 u_1 + A\partial_1 u_3) + \partial_3 p = F_1 + AF_3$$

Then taking derivative $\partial_3^{k-2}$ on both sides and focus in the term $(1 + A^2 + B^2)K^2(\partial_3^{k-2} u_1 + A\partial_3^{k-2} u_3)$, the estimate of all the other terms in $H^0$ norm implies that

\[(2.3.29)\]

$$\|\partial_3^{k-2} u_1 + A\partial_3^{k-2} u_3\|_{H^0}^2 \lesssim Z$$

Similarly, we have

\[(2.3.30)\]

$$\|\partial_3^{k-2} u_2 + B\partial_3^{k-2} u_3\|_{H^0}^2 \lesssim Z$$

Rearrange the terms in (2.3.22), we get

\[(2.3.31)\]

$$K(1 + A^2 + B^2)\partial_3 u_3 = G - \partial_1 u_1 - \partial_2 u_2 + AK(\partial_3 u_1 + A\partial_3 u_3) + BK(\partial_3 u_2 + B\partial_3 u_3)$$

Taking derivative $\partial_3^{k-1}$ on both sides, focusing in the term $K(1 + A^2 + B^2)\partial_3^{k-1} u_3$ employing all of the estimate we have known, we can show

\[(2.3.32)\]

$$\|\partial_3^{k-1} u_3\|_{H^0}^2 \lesssim Z$$

Combining (2.3.29), (2.3.30) and (2.3.32), it is easy to see we can get

\[(2.3.33)\]

$$\|\partial_3^{k-1} u_1\|_{H^0}^2 + \|\partial_3^{k-1} u_2\|_{H^0}^2 \lesssim Z$$
Plugging this to (2.3.21) and taking derivative $\partial_k^{k-2}$ on both sides, we get

$$
(2.3.34) \quad \|\partial_k^{k-1}p\|^2_{H^{k'}} \lesssim \mathcal{Z}
$$

Combining this with all above estimate, we have proved

$$
(2.3.35) \quad \|u\|^2_{H^{k+1}} + \|p\|^2_{H^{k-1}} \lesssim \mathcal{Z}
$$

Therefore, we have proved the case $r = k$.

Step 3: $r = k + 1$ case: first loop

For $i, j = 1, 2$, since $(\partial_{ij}u, \partial_{ij}p)$ satisfies the equation

$$
(2.3.36) \quad \begin{cases}
-\Delta_A(\partial_{ij}u) + \nabla_A(\partial_{ij}p) = \tilde{F} & \text{in } \Omega \\
\nabla_A \cdot (\partial_{ij}u) = \tilde{G} & \text{in } \Omega \\
((\partial_{ij}p)I - D_A(\partial_{ij}u))N = \tilde{H} & \text{on } \Sigma \\
(\partial_{ij}u) = 0 & \text{on } \Sigma_b
\end{cases}
$$

where

\begin{align*}
\tilde{F} &= \partial_{ij}F + \nabla_{\partial_{ij}A} \cdot \nabla_A u + \nabla_A \cdot \nabla_{\partial_{ij}A} u + \nabla_{\partial_{ij}A} \cdot \nabla_A u + \nabla_{\partial_{ij}A} \cdot \nabla_A u + \nabla_{\partial_{ij}A} \cdot \nabla_A (\partial_{ij}u) \\
&\quad + \nabla_{\partial_{ij}A} \cdot \nabla_A (\partial_{ij}u) + \nabla_A \cdot \nabla_{\partial_{ij}A} (\partial_{ij}u) + \nabla_A \cdot \nabla_{\partial_{ij}A} (\partial_{ij}u) - \nabla_{\partial_{ij}A} (\partial_{ij}p) - \nabla_{\partial_{ij}A} (\partial_{ij}p) \\
\tilde{G} &= \partial_{ij}G - \nabla_{\partial_{ij}A} \cdot u - \nabla_{\partial_{ij}A} \cdot (\partial_{ij}u) - \nabla_{\partial_{ij}A} \cdot (\partial_{ij}u) \\
\tilde{H} &= \partial_{ij}H - (pI - D_A u) \partial_{ij}N - ((\partial_{ij}p)I - D_A (\partial_{ij}u)) \partial_{ij}N - ((\partial_{ij}p)I - D_A (\partial_{ij}u)) \partial_{ij}N \\
&\quad + (D_{\partial_{ij}A} u + D_{\partial_{ij}A} (\partial_{ij}u) + D_{\partial_{ij}A} (\partial_{ij}u)) \partial_{ij}N
\end{align*}

Employing $k - 1$ order elliptic estimate, we have

$$
(2.3.37) \quad \|\partial_{ij}u\|^2_{H^{k+1}} + \|\partial_{ij}p\|^2_{H^{k-2}} \lesssim \|\tilde{F}\|^2_{H^{k-3}} + \|\tilde{G}\|^2_{H^{k-2}} + \|\tilde{H}\|^2_{H^{k-3/2}(\Sigma)} \lesssim \mathcal{Z}
$$

where the forcing estimate can be taken in a similar argument as in $r = k$ case.

In detail, this is actually

$$
(2.3.38) \quad \|\partial_{11}u\|^2_{H^{k+1}} + \|\partial_{12}u\|^2_{H^{k+1}} + \|\partial_{22}u\|^2_{H^{k+1}} + \|\partial_{11}p\|^2_{H^{k-1}} + \|\partial_{12}p\|^2_{H^{k-1}} + \|\partial_{22}p\|^2_{H^{k-1}} \lesssim \mathcal{Z}
$$

Step 4: $r = k + 1$ case: second loop

Similar to $r = k$ case argument, for $i = 1, 2$, taking derivative $\partial_k^{k-2} \partial_i$ on both sides of (2.3.29) and focus in the term $\partial_k^{k} \partial_i u_1 + A \partial_k^{k} \partial_i u_3$, we get

$$
(2.3.39) \quad \|\partial_k^{k} \partial_i u_1 + A \partial_k^{k} \partial_i u_3\|^2_{H^{k'}} \lesssim \mathcal{Z}
$$

Similarly, we have

$$
(2.3.40) \quad \|\partial_k^{k} \partial_i u_2 + B \partial_k^{k} \partial_i u_3\|^2_{H^{k'}} \lesssim \mathcal{Z}
$$

Plugging in this result to (2.3.22) and taking derivative $\partial_k^{k-2} \partial_i$ on both sides, we will get

$$
(2.3.41) \quad \|\partial_k^{k} \partial_i u_3\|^2_{H^{k'}} \lesssim \mathcal{Z}
$$

An easy estimate for these three terms implies

$$
(2.3.42) \quad \|\partial_k^{k} \partial_i u\|^2_{H^{k'}} \lesssim \mathcal{Z}
$$

Plugging this to (2.3.21) and taking derivative $\partial_k^{k-2} \partial_i$ on both sides, we get

$$
(2.3.43) \quad \|\partial_k^{k-1} \partial_i p\|^2_{H^{k'}} \lesssim \mathcal{Z}
$$

Combining all above estimate with estimate of previous steps, we have shown

$$
(2.3.44) \quad \|\partial_{ij}u\|^2_{H^{k+1}} + \|\partial_{ij}p\|^2_{H^{k-1}} \lesssim \mathcal{Z}
$$

In detail, this is actually

$$
(2.3.45) \quad \|\partial_{13}u\|^2_{H^{k+1}} + \|\partial_{23}u\|^2_{H^{k+1}} + \|\partial_{13}p\|^2_{H^{k-1}} + \|\partial_{23}p\|^2_{H^{k-1}} \lesssim \mathcal{Z}
$$
Step 5: $r = k + 1$ case: third loop
Again we use the same trick as above. Taking derivative $\partial^{k+1}_3 u$ on both sides of (2.3.29) and focusing in the term $\partial^{k+1}_3 u_1 + A\partial^{k+1}_3 u_3$, we can bound $\partial^{k+1}_3 u_1 - A\partial^{k+1}_3 u_3$. Similarly, we control $\partial^{k+1}_3 u_2 - B\partial^{k+1}_3 u_3$. Plugging in this result to (2.3.22) and taking derivative $\partial^{k+1}_3 u$ on both sides, we can estimate $\partial^{k+1}_3 u_3$. Then we have

$$\|\partial^{k+1}_3 u\|_{H^p}^2 \lesssim Z$$

Plugging this to (2.3.21) and taking derivative $\partial^{k+1}_3 p$ on both sides, we get

$$\|\partial^{k+1}_3 p\|_{H^p}^2 \lesssim Z$$

Combining this will all above estimate, we get

$$\|\partial^{k+1}_{33} u\|_{H^{k-1}}^2 + \|\partial^{k+1}_{33} p\|_{H^{k-2}}^2 \lesssim Z$$

Step 6: $r = k + 1$ case: conclusion
To synthesize, (2.3.38), (2.3.45) and (2.3.48) imply that all the second order derivative of $u$ in $k - 1$ norm and $p$ in $k - 2$ norm is controlled, so we naturally have the estimate

$$\|u\|_{H^{k+1}} + \|p\|_{H^{k+2}} \lesssim C(\eta_0)P(\eta) \left( \|F\|_{H^{k+1}} + \|G\|_{H^k} + \|H\|_{H^{k-1/2}}^2 \right)$$

This is just what we desired.

Now let us go back to original notation, since $P(\cdot)$ is a fixed polynomial, this gives an estimate

$$\|u^m\|_{H^{k+1}} + \|p^m\|_{H^k} \lesssim C(\eta_0)P(u^m) \left( \|F\|_{H^{k+1}} + \|G\|_{H^k} + \|H\|_{H^{k-1/2}}^2 \right)$$

where $C(\eta_0)$ depends on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$.

This bound implies that the sequence $(u^m, p^m)$ is uniformly bounded in $H^{k+1} \times H^k$, so we can extract weakly convergent subsequence $u^m \rightharpoonup u^0$ and $p^m \rightharpoonup p^0$.

In the second equation of (2.3.16), we multiplied both sides by $J^m w$ for $w \in C^\infty_0$ to see that

$$\int_\Omega GwJ^m = \int_\Omega (\nabla_A \cdot u^m)wJ^m = -\int_\Omega u^m \cdot (\nabla_A w)J^m - \int_\Omega u^0 \cdot (\nabla_A w)J = \int_\Omega (\nabla_A \cdot u^0)wJ$$

which implies $\nabla_A \cdot u^0 = G$. Then we multiply the first equation by $wJ^m$ for $w \in W$ and integrate by parts to see that

$$\frac{1}{2} \int_\Omega D_{A=} u^m : D_{A=} wJ^m - \int_\Omega p^m \nabla_{A=} \cdot wJ^m = \int_\Omega F \cdot wJ^m - \int_{\Sigma} H \cdot w$$

Passing to the limit $m \to \infty$, we deduce that

$$\frac{1}{2} \int_\Omega D_{A=} u^0 : D_{A=} wJ - \int_\Omega p^0 \nabla_{A=} \cdot wJ = \int_\Omega F \cdot wJ - \int_{\Sigma} H \cdot w$$

which reveals, upon integrating by parts again, $(u^0, p^0)$ satisfies (2.3.1). Since $(u, p)$ is the unique strong solution to (2.3.1), we have $u^0 = u$ and $p^0 = p$. This, weakly lower semi-continuity and estimate (2.3.50) imply (2.3.54).

\begin{remark}
The key part of this proposition is that as long as we can deduce $\eta_0 \in H^{k+1/2}(\Sigma)$ and $\eta \in H^{k+1/2}(\Sigma)$, we can achieve $u \in H^{k+1}$ and $p \in H^k$, which is the highest possible regularity we can expect.
\end{remark}
Remark 2.26. By our notation rule, since \( C(\eta_0) \) depends on \( \Omega_0 \) and is given implicitly, we can take it as a universal constant, so the estimate may be rewritten as follows.

\[
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)}
\]

for \( r = 2, \ldots, k + 1 \)

2.3.2. A-Poisson Equation. We consider the elliptic problem

\[
\begin{aligned}
\Delta_p f &= f & \text{in} & \Omega \\
p &= g & \text{on} & \Sigma \\
\nabla_p \cdot \nu &= h & \text{on} & \Sigma_b
\end{aligned}
\]

(2.3.55)

For the weak formulation, we suppose \( f \in V^*, \ g \in H^{1/2}(\Sigma) \) and \( h \in H^{-1/2}(\Sigma_b) \). Let \( \tilde{p} \in H^1 \) be an extension of \( g \) such that \( \text{supp}(\tilde{p}) \subset \{ -\inf (b) / 2 < x_3 < 0 \} \). Then we can switch the unknowns to \( q = p - \tilde{p} \). Hence, the weak formulation of (2.3.55) is

\[
\langle \nabla \tilde{p} \cdot \tilde{\nabla} \phi \rangle_{H^0} = -\langle \nabla \tilde{p} \cdot \tilde{\nabla} \phi \rangle_{H^0} - \langle f, \phi \rangle_{V^*} + \langle h, \phi \rangle_{-1/2}
\]

(2.3.56)

for all \( \phi \in V, \) where \( \langle \cdot, \cdot \rangle_{V^*} \) denotes the dual pairing with \( V \) and \( \langle \cdot, \cdot \rangle_{-1/2} \) denotes the dual pairing with \( H^{-1/2}(\Sigma_b) \). The existence and uniqueness of a solution to (2.3.56) is given by standard argument for elliptic equation.

If \( f \) has a more specific fashion, we can rewrite the equation to accommodate the structure of \( f \). Suppose the action of \( f \) on an element \( \phi \in V \) is given by

\[
\langle f, \phi \rangle_{V^*} = \langle f_0, \phi \rangle_{H^0} + \langle f_0, \tilde{\nabla} \phi \rangle_{H^0}
\]

where \( \langle f_0, f_0 \rangle_{H^0} = H^0 \times H^0 \) with \( \|f_0\|_{H^0}^2 + \|f_0\|_{H^0}^2 = \|f\|_{V^*}^2 \). Then we rewrite (2.3.56) into

\[
\langle \tilde{\nabla} \tilde{p} + f_0, \tilde{\nabla} \phi \rangle_{H^0} = -\langle f_0, \phi \rangle_{H^0} + \langle h, \phi \rangle_{-1/2}
\]

(2.3.58)

Hence, it is possible to say \( p \) is a weak solution to equation

\[
\begin{aligned}
\nabla \cdot (\nabla p + f_0) &= f_0 \\
p &= g \\
(\nabla p + f_0) \cdot \nu &= h
\end{aligned}
\]

(2.3.59)

This formulation will be used to construct the higher order initial conditions in later sections.

Lemma 2.27. Suppose that \( \eta \in H^{k+1/2}(\Sigma) \) for \( k \geq 3 \) such that the map \( \Phi \) is a \( C^1 \) diffeomorphism of \( \Omega \) to \( \Omega' = \Phi(\Omega) \). If \( f \in H^0, g \in H^{1/2} \) and \( h \in H^{1/2} \), then the equation (2.3.55) admits a unique strong solution \( p \in H^2 \). Moreover, for \( r = 2, \ldots, k - 1 \), we have the estimate

\[
\|p\|_{H^r} \lesssim C(\eta) \left( \|f\|_{H^{r-2}} + \|g\|_{H^{r-1/2}(\Sigma)} + \|h\|_{H^{r-3/2}(\Sigma_b)} \right)
\]

(2.3.60)

whenever the right hand side is finite, where \( C(\eta) \) is a constant depending on \( \|\eta\|_{H^{k+1/2}(\Sigma)} \).

Proof. This is exactly the same as lemma 3.8 in [1], so we omit the proof here. \( \square \)

Next, we will prove the boundning posture for the estimate can actually only depend on the initial surface. Since we do not need optimal regularity result for this equation, we do not need the bootstrapping argument now.

Proposition 2.28. Let \( k \geq 6 \) be an integer suppose that \( \eta \in H^{k+1/2}(\Sigma) \) and \( \eta_0 \in H^{k+1/2}(\Sigma) \). There exists \( \epsilon_0 > 0 \) such that if \( \|\eta - \eta_0\|_{H^{k-3/2}} \leq \epsilon_0 \), then solution to (2.3.55) satisfies

\[
\|p\|_{H^r} \lesssim C(\eta_0) \left( \|f\|_{H^{r-2}} + \|g\|_{H^{r-1/2}(\Sigma)} + \|h\|_{H^{r-3/2}(\Sigma_b)} \right)
\]

for \( r = 2, \ldots, k - 1 \), whenever the right hand side is finite, where \( C(\eta_0) \) is a constant depending on \( \|\eta_0\|_{H^{k+1/2}(\Sigma)} \).

Proof. The proof is similar to lemma 2.23. We rewrite the problem as a perturbation of the initial status.

\[
\begin{aligned}
\Delta_\tilde{p} \tilde{p}^m &= f + f^m & \text{in} & \Omega \\
\tilde{p}^m &= g + g^m & \text{on} & \Sigma \\
\nabla_\tilde{p} \cdot \tilde{\nabla} \tilde{p}^m &= h + h^m & \text{on} & \Sigma_b
\end{aligned}
\]

(2.3.62)

The constant in this elliptic estimate only depends on the initial free surface. We may estimate \( f^m, g^m \) and \( h^m \) in terms of \( p^m, \eta_0 \) and \( \xi^m = \eta^m - \eta_0 \). The smallness of \( \xi^m \) implies that we can absorb these terms into left hand side, which is actually (2.3.63). \( \square \)
Remark 2.29. Similarly, we can also take this constant \( C(\eta_{0}) \) as a universal constant and rewrite the estimate as follows.

\[
\|p\|_{H^{r}} \lesssim \|f\|_{H^{r-2}} + \|g\|_{H^{r-1/2}(\Sigma)} + \|h\|_{H^{r-3/2}(\Sigma_{b})}
\]

for \( r = 2, \ldots, k - 1 \).

2.4. Linear Navier-Stokes Estimates.

\[
\begin{aligned}
\partial_t u - \Delta_A u + \nabla_A u &= F & \text{in } \Omega \\
\nabla_A \cdot u &= 0 & \text{in } \Omega \\
(pI - D_A u)N &= H & \text{on } \Sigma \\
u &= 0 & \text{on } \Sigma_b
\end{aligned}
\]

In this section, we will study the linear Navier-Stokes problem (2.4.1). Following the path of I. Tice and Y. Guo in [1]. We will employ two notions of solution: weak and strong. Then we prove the wellposedness theorem from lower regularity to higher regularity. To note that, most parts of the results and proofs here are identical to section 4 of [1], so we won’t give all the details. The only difference is the constants related to free surface since our surface has less restrictions.

2.4.1. Weak Solution and Strong Solution. The weak formulation of linear problem (2.4.1) is as follows.

\[
\langle \partial_t u, v \rangle_{L^2 H^0} + \frac{1}{2} \langle D_A u, D_A v \rangle_{L^2 H^0} = \langle p, \nabla_A \cdot v \rangle_{L^2 H^0} = \langle F, v \rangle_{L^2 H^0} - \langle H, v \rangle_{L^2 H^0(\Sigma)}
\]

Definition 2.30. Suppose that \( u_0 \in H^0 \), \( F \in W^* \) and \( H \in L^2([0, T]; H^{-1/2}(\Sigma)) \). If there exists a pair \((u, p)\) achieving the initial data \( u_0 \) and satisfies \( u \in W \), \( p \in L^2([0, T]; H^0) \) and \( \partial_t u \in W^* \), such that the (2.4.2) holds for any \( v \in W \), we call it a weak solution.

If further restricting the test function \( v \in \mathcal{X} \), we have a pressureless weak formulation.

\[
\langle \partial_t u, v \rangle_{L^2 H^0} + \frac{1}{2} \langle D_A u, D_A v \rangle_{L^2 H^0} = \langle F, v \rangle_{L^2 H^0} - \langle H, v \rangle_{L^2 H^0(\Sigma)}
\]

Definition 2.31. Suppose that \( u_0 \in \mathcal{Y}(0) \), \( F \in \mathcal{X}^* \) and \( H \in L^2([0, T]; H^{-1/2}(\Sigma)) \). If there exists a function \( u \) achieving the initial data \( u_0 \) and satisfies \( u \in \mathcal{X} \), \( \partial_t u \in \mathcal{X}^* \), such that the (2.4.3) holds for any \( v \in \mathcal{X} \), we call it a pressureless weak solution.

Remark 2.32. It is noticeable that in \( \Omega \), \( W^* \subset \mathcal{X}^* \) and \( \|u\|_{\mathcal{X}^*} \leq \|u\|_{W^*} \); on \( \Sigma \), for \( u \in \mathcal{X} \), we have \( u|_{\Sigma} \in L^2([0, T]; H^{1/2}(\Sigma)) \).

Since our main aim is to prove higher regularity of the linear problem, so the weak solution is a natural byproduct of the proof for strong solution in next subsection. Hence, we will not give the existence proof here and only present the uniqueness.

Lemma 2.33. Suppose \( u \) is a pressureless weak solution of (2.4.1). Then for a.e. \( t \in [0, T] \),

\[
\frac{1}{2} \int_{\Omega} |J(u(t))|^2 + \frac{1}{2} \int_{t}^{t} \int_{\Omega} J|\mathbb{D}_A u|^2 = \frac{1}{2} \int_{\Omega} J(0)|u_0|^2 + \frac{1}{2} \int_{t}^{t} \int_{\Omega} |u|^2 \partial_t J + \int_{t}^{t} \int_{\Omega} J(F \cdot u) - \int_{t}^{t} \int_{\Sigma} H \cdot u
\]

Also

\[
\|u\|_{L^\infty H^0}^2 + \|u\|_{L^2 H^1}^2 \lesssim C_0(\eta) \left( \|u_0\|^2_{H^0} + \|F\|^2_{\mathcal{X}^*} + \|H\|_{L^2 H^{-1/2}(\Sigma)}^2 \right)
\]

where

\[
C_0(\eta) = P(\|\eta\|_{H^{1/2}(\Sigma)}) + \|\partial_\eta \|_{H^{1/2}(\Sigma)} \exp \left( TP(\|\partial_\eta \|_{H^{1/2}(\Sigma)}) \right)
\]

Hence, if a pressureless weak solution exists, then it is unique.

Proof. The proof is almost the same as that of lemma 4.1 and proposition 4.2 in [1]. The only difference is that we should replace the lemma 2.1 in [1] used in that proof by our lemma 2.11 now.

Next, we define the strong solution.

Definition 2.34. Suppose that

\[
\begin{align*}
u_0 &\in H^2 \cap \mathcal{X}(0) \\
F &\in L^2([0, T]; H^1) \cap C^0([0, T]; H^0) \\
H &\in L^2([0, T]; H^{3/2}(\Sigma)) \cap C^0([0, T]; H^{1/2}(\Sigma)) \\
\partial_t F &\in \mathcal{X}^* \\
\partial_t H &\in L^2([0, T]; H^{-1/2}(\Sigma))
\end{align*}
\]
If there exists a pair \((u, p)\) achieving the initial data \(u_0\) and satisfies
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &\in X^* \\
\frac{\partial u}{\partial t} &\in L^2(0, T; H^1) \cap C^0([0, T]; H^0) \\
p &\in L^2(0, T; H^2) \cap C^0([0, T]; H^1)
\end{align*}
\]
such that they satisfy (2.4.1) in the strong sense, we call it a strong solution.

2.4.2. Lower Regularity Theorem.

**Theorem 2.35.** Assume the initial data and forcing terms satisfy the condition (2.4.7). Define the divergence-A preserving operator \(D_t u\) by
\[
D_t u = \frac{\partial u}{\partial t} - Ru \quad \text{for} \quad R = \frac{\partial}{\partial t} MM^{-1}
\]
where \(M\) is defined in (2.2.23). Furthermore, define the orthogonal projection onto the tangent space of the surface \(\{x_3 = \eta_0\}\) according to
\[
\Pi_0(v) = v - (v \cdot N_0)N_0|N_0|^{-2} \quad \text{for} \quad N_0 = (-\partial_1 \eta_0, -\partial_2 \eta_0, 1)
\]
Suppose the initial data satisfy the compatible condition
\[
\Pi_0(H(0) + D_{A_0} u_0 N_0) = 0
\]
Then the equation (2.4.1) admits a unique strong solution \((u, p)\) which satisfies the estimate
\[
\begin{align*}
\|u\|_{L^2 H^2} + \|\frac{\partial u}{\partial t}\|_{L^2 H^1} + \|\frac{\partial^2 u}{\partial t^2}\|_{L^2 X^*} + \|u\|_{L^\infty H^2} + \|\frac{\partial u}{\partial t}\|_{L^\infty H^2} + \|p\|_{L^2 H^2} + \|\frac{\partial p}{\partial t}\|_{L^2 H^1} \\
< L(\eta) \left( \|u_0\|_{L^2 H^2} + \|F(0)\|_{H^0} + \|H(0)\|_{H^{1/2}} + \|F\|_{L^2 H^2} + \|\frac{\partial p}{\partial t}\|_{X^*} + \|H\|_{L^2 H^{1/2}} + \|\frac{\partial u}{\partial t}\|_{L^2 H^{-1/2}} \right)
\end{align*}
\]
where
\[
L(\eta) = P(K(\eta)) \exp \left( TP(K(\eta)) \right)
\]
for
\[
K(\eta) = \sup_{0 \leq t \leq T} \left( \|\partial t^2(\eta(t))\|_{H^{3/2}(\Sigma)} + \|\partial t(\eta(t))\|_{H^{3/2}(\Sigma)} + \|\partial t^2(\eta(t))\|_{H^{3/2}(\Sigma)} \right)
\]
The initial pressure \(p(0)\) in \(H^1\), is determined in terms of \(u_0, \eta_0, F(0)\) and \(H(0)\) as the weak solution to
\[
\begin{align*}
\nabla_{A_0} \cdot (\nabla_{A_0} p(0) - F(0)) &= -\nabla_{A_0} (R(0) u_0) \quad \text{in} \ \Omega \\
p(0) &= (H(0) + D_{A_0} u_0 N_0) \cdot N_0 |N_0|^{-2} \quad \text{on} \ \Sigma \\
(\nabla_{A_0} p(0) - F(0)) \cdot \nu &= \Delta_{A_0} u_0 \cdot \nu \quad \text{on} \ \Sigma_b
\end{align*}
\]
in the sense of (2.3.59).

Also, \(D_t u(0) = \frac{\partial}{\partial t} u(0) - R(0) u_0\) satisfies
\[
D_t u(0) = \Delta_{A_0} u_0 - \nabla_{A_0} p(0) + F(0) - R(0) u_0 \in X(0).
\]
Moreover, \((D_t u, \partial_t p)\) satisfies
\[
\begin{align*}
\partial_t (D_t u) - \Delta_A (D_t u) + \nabla_A (\partial_t p) &= D_t F + G^F \quad \text{in} \ \Omega \\
\nabla_A \cdot (D_t u) &= 0 \quad \text{in} \ \Omega \\
S_A (\partial_t p, D_t u) N &= \partial_t H + G^H \quad \text{on} \ \Sigma \\
D_t u &= 0 \quad \text{on} \ \Sigma_b
\end{align*}
\]
in the weak sense of (2.4.3), where
\[
\begin{align*}
G^F &= -(R + \partial_t J K) \Delta_{A_0} u - \partial_t Ru + (\partial_t J K + R + R^T) \nabla_A p + \nabla_A \cdot (D_A (Ru) - RD_{A_0} u + D_{A_0} A_{u}) \\
G^H &= D_A (Ru) N - (p l - D_{A_0} u) \partial_t N + D_{A_0} A_{u} N
\end{align*}
\]
**Proof.** Since this theorem is almost identical to theorem 4.3 in [1], we will only point out the differences here without giving details. In order for contrast, we use the same notation and statement as [1] here.

1. The only difference in the assumption part is that in theorem 4.3 of [1], \(K(\eta)\) is sufficiently small, but our \(K(\eta)\) can be arbitrary. Hence, in [1], we can always bound \(P(K(\eta)) \leq 1 + K(\eta)\), but now we have to keep this term as a polynomial (this is the only difference of \(L(\eta)\) between these two theorems).

2. It is noticeable that each lemma used in proving theorem 4.3 of [1] has been recovered or slightly modified in our preliminary section. To note that these kind of modification merely involves the bounded constant due to the arbitrary \(K(\eta)\), but does not change the conclusion and the shape of estimate at all. Therefore, all these modification will only contribute to \(P(K(\eta))\) term.
(3) In elliptic estimate, we now use proposition 2.24 to replace proposition 3.7 in [1], which gives exactly the same estimates.

Based on all above, we can easily repeat the proving process in [1] with no more work, so our result naturally follows. \hfill \square

2.4.3. Initial Data and Compatible Condition. We first define some quantities related to the initial data:

\begin{align}
\mathcal{H}_0(u, p) &= \sum_{j=0}^{N} \left\| \partial_t^j u(0) \right\|_{L^2(\Omega)}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j p(0) \right\|_{L^2(\Omega)}^2 \\
\mathcal{K}_0(u_0) &= \left\| u_0 \right\|_{L^2(\Omega)}^2 \\
\mathcal{H}_0(\eta) &= \left\| \eta(0) \right\|_{L^2(\Omega)}^2 + \sum_{j=1}^{N} \left\| \partial_t^j \eta(0) \right\|_{L^2(\Omega)}^2 \\
\mathcal{K}_0(\eta) &= \left\| \eta \right\|_{L^2(\Omega)}^2 \\
\mathcal{H}_0(F, H) &= \sum_{j=0}^{N-1} \left\| \partial_t^j F(0) \right\|_{L^2(\Omega)}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H(0) \right\|_{L^2(\Omega)}^2
\end{align}

Furthermore, we need to define mappings \( G_1 \) on \( \Omega \) and \( G_2 \) on \( \Sigma \):

\begin{align}
G^1(v, q) &= -(R + \partial_t JK) \Delta_A v - \partial_t Rv + (\partial_t JK + R + R^T) \nabla_A q + \nabla_A \cdot (\mathbb{D}_A(\mathbf{R}v) - R\mathbb{D}_A v + \mathbb{D}_{\partial_t A} v) \\
G^2(v, q) &= \mathbb{D}_A(\mathbf{R}v)N - (qI - \mathbb{D}_A v) \partial_t N + \mathbb{D}_{\partial_t A} v N
\end{align}

The mappings above allow us to define the arbitrary order forcing terms. For \( j = 1, \ldots, N \), we have

\begin{align}
F^0 &= F \\
H^0 &= H \\
F^j &= D_t F^{j-1} + G^1(D_t^{j-1} u, \partial_t^{j-1} p) = D_t^j F + \sum_{l=0}^{j-1} D_t^l G_1(D_t^{j-l-1} u, \partial_t^{j-l-1} p) \\
H^j &= D_t H^{j-1} + G^2(D_t^{j-1} u, \partial_t^{j-1} p) = D_t^j H + \sum_{l=0}^{j-1} D_t^l G_2(D_t^{j-l-1} u, \partial_t^{j-l-1} p)
\end{align}

Finally, we define the general quantities we need to estimate as follows.

\begin{align}
\mathcal{E}(u, p) &= \sum_{j=0}^{N} \left\| \partial_t^j u \right\|_{L^2(\Omega)}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j p \right\|_{L^2(\Omega)}^2 \\
\mathcal{D}(u, p) &= \sum_{j=0}^{N} \left\| \partial_t^j u \right\|_{L^2(\Omega)}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j p \right\|_{L^2(\Omega)}^2 \\
\mathcal{K}(u, p) &= \mathcal{E}(u, p) + \mathcal{D}(u, p) \\
\mathcal{E}(\eta) &= \left\| \eta \right\|_{L^2(\Omega)}^2 + \sum_{j=1}^{N} \left\| \partial_t^j \eta \right\|_{L^2(\Omega)}^2 \\
\mathcal{D}(\eta) &= \left\| \eta \right\|_{L^2(\Omega)}^2 + \sum_{j=2}^{N+1} \left\| \partial_t^j \eta \right\|_{L^2(\Omega)}^2 \\
\mathcal{K}(\eta) &= \mathcal{E}(\eta) + \mathcal{D}(\eta) \\
\mathcal{K}(F, H) &= \sum_{j=0}^{N-1} \left\| \partial_t^j F \right\|_{L^2(\Omega)}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H \right\|_{L^2(\Omega)}^2
\end{align}
Before discussing the higher regularity, we first define the compatible condition of initial data. For \( j = 0, \ldots, N - 1 \), we say that the \( j^{th} \) compatible condition is satisfied if

\[
\begin{align*}
&D_l^j u(0) \in H^2 \cap X(0) \\
&\Pi_0(D_l^j u(0)) + \partial_{\mathcal{A}} D_l^j u(0) \mathcal{N}_0 = 0
\end{align*}
\]

(2.4.32)

Then we may construct the initial condition of higher order satisfying the compatible condition. Since this section is almost identical to section 4.3 in Y. Guo and I. Tice’s paper (see pp 34-35 of [1]), we omit the details here. The core idea is to construct the initial data iteratively from lower to higher order. Then we have the estimate.

\[
\mathcal{H}_0(u,p) \leq P(\mathcal{H}_0(\eta)) \left( \|u_0\|_{H^{2N}}^2 + \mathcal{H}_0(F,H) \right)
\]

(2.4.33)

2.4.4. Higher Regularity Theorem.

**Theorem 2.36.** Suppose the initial data \( u_0 \in H^{2N}, \eta_0 \in H^{2N+1/2}(\Sigma) \) and the initial condition is constructed as above to satisfy the \( j^{th} \) compatible condition for \( j = 0, 1, \ldots, N - 1 \), with the forcing function \( K(F,H) + K_0(F,H) < \infty \). Then there exists a universal constant \( T_0(\eta) > 0 \) depending on \( L(\eta) \) as defined in the following, such that if \( 0 < T < T_0(\eta) \), then there exists a unique strong solution \((u,p)\) to the equation (2.4.1) on \([0,T]\) such that

\[
\begin{align*}
&\partial_t^j u \in L^2([0,T]; H^{2N-2j+1}) \cap C^0([0,T]; H^{2N-2j}) \quad \text{for} \quad j = 0, \ldots, N \\
&\partial_t^j p \in L^2([0,T]; H^{2N-2j-1}) \cap C^0([0,T]; H^{2N-2j-1}) \quad \text{for} \quad j = 0, \ldots, N - 1 \\
&\partial_t^{N+1} u \in H^s
\end{align*}
\]

Also the pair \((u,p)\) satisfies the estimate

\[
K(u,p) \leq L(\eta) \left( \|u_0\|_{H^{2N}}^2 + \mathcal{H}_0(F,H) + K(F,H) \right)
\]

(2.4.35)

where

\[
L(\eta) = P(K(\eta)) \exp \left( T P(K(\eta)) \right)
\]

(2.4.36)

Furthermore, the pair \((D_l^j u, \partial_{\mathcal{A}}^j p)\) satisfies the equation

\[
\begin{align*}
&\partial_t (D_l^j u) - \Delta_{\mathcal{A}} (D_l^j u) + \nabla_{\mathcal{A}} (\partial_{\mathcal{A}}^j p) = F^j \quad \text{in} \quad \Omega \\
&\nabla_{\mathcal{A}} \cdot (D_l^j u) = 0 \quad \text{in} \quad \Omega \\
&S_{\mathcal{A}} (\partial_{\mathcal{A}}^j p, D_l^j u) \mathcal{N} = H^j \quad \text{on} \quad \Sigma \\
&D_l^j u = 0 \quad \text{on} \quad \Sigma_b
\end{align*}
\]

(2.4.37)

in the strong sense for \( j = 0, \ldots, N - 1 \) and in the weak sense for \( j = N \).

**Proof.** Similar to lower regularity case, this theorem is almost identical to theorem 4.8 in [1]. Due to the same reason that our free surface \( \eta \) can be arbitrary, the only difference here is that we keep the polynomial of \( K(\eta) \) and \( \mathcal{H}_0(\eta) \) in the final estimate which cannot be further simplified. Hence, our result easily follows. \( \square \)

2.5. Transport Equation. In this section, we will prove the wellposedness of the transport equation

\[
\begin{align*}
&\partial_t \eta + u_1 \partial_{\eta} \eta + u_2 \partial_{\eta} \eta = u_3 \\
&\eta(t = 0) = \eta_0
\end{align*}
\]

(2.5.1)

Throughout this section, we can always assume \( u \) is known and satisfies some bounded properties, which will be specified in the following.

2.5.1. Wellposedness of Transport Equation. Define the quantity

\[
Q(u) = \sum_{j=0}^{N} \left( \|\partial_{\eta}^j u\|_{L^2 H^{2N-2j+1}}^2 + \sum_{j=0}^{N-1} \|\partial_{\eta}^j u\|_{L^\infty H^{2N-2j}}^2 \right)
\]

(2.5.2)

Obviously, we have the relation \( Q(u) \leq K(u,p) \).

**Theorem 2.37.** Suppose that \( K_0(\eta_0) < \infty \) and \( Q(u) < \infty \). Then the problem (2.5.1) admits a unique solution \( \eta \) satisfying \( K(\eta) < \infty \) and achieving the initial data \( \partial_{\mathcal{A}}^j \eta(0) \) for \( j = 0, \ldots, N \). Moreover, there exists a \( 0 < \bar{T} < \infty \), depending on \( Q(u) \), such that if \( 0 < T < \bar{T} \), then we have the estimate

\[
K(\eta) \leq P(K_0(\eta_0)) + P(Q(u))
\]

where \( P(\cdot) \) is a polynomial satisfying \( P(0) = 0 \).
Proof. We divide the proof into several steps.

Step 1: Solvability of the equation:
Similar to proof of theorem 5.4 in [1], based on proposition 2.1 in [5], since \( u \in L^2([0, T]; H^{2N+1/2}(\Sigma)) \), then the equation (2.5.1) admits a unique solution \( \eta \in L^\infty([0, T]; H^{2N+1/2}(\Sigma)) \) achieving the initial data \( \eta(0) = \eta_0 \).

Step 2: Estimate of \( E(\eta) \):
By lemma B.1, we have
\[
\|\eta\|_{L^\infty H^{2N+1/2}(\Sigma)} \leq \exp \left( C \int_0^T \|u(t)\|_{H^{2N+1/2}(\Sigma)} dt \right) \left( \sqrt{K(\eta_0)} + \int_0^T \|u_3(t)\|_{H^{2N+1/2}(\Sigma)} dt \right)
\]
for \( C > 0 \). The Cauchy-Schwarz inequality implies
\[
C \int_0^T \|u(t)\|_{H^{2N+1/2}(\Sigma)} dt \lesssim C \int_0^T \|u(t)\|_{H^{2N+1/2}(\Sigma)} dt \lesssim \sqrt{T} \sqrt{Q(u)}
\]
So if \( T \leq 1/(2Q(u)) \), then
\[
\exp \left( C \int_0^T \|u(t)\|_{H^{2N+1/2}(\Sigma)} dt \right) \leq 2
\]
Hence, we have
\[
\|\eta\|_{L^\infty H^{2N+1/2}(\Sigma)} \lesssim K_0(\eta_0) + T Q(u) \lesssim K_0(\eta_0) + 1
\]
Based on the equation, we further have
\[
\|\partial_t \eta\|_{L^\infty H^{2N-1/2}(\Sigma)} \lesssim \|u_2\|_{L^\infty H^{2N-1/2}(\Sigma)} + \|u\|_{L^\infty H^{2N-1/2}(\Sigma)} \|\eta\|_{L^\infty H^{2N+1/2}(\Sigma)} \lesssim K_0(\eta_0) Q(u) + Q(u) + T Q(u)^2 \lesssim (K_0(\eta_0) + 1) Q(u)
\]
The last inequality is based on the choice of \( T \) as above. The solution \( \eta \) is temporally differentiable to get the equation
\[
\partial_t (\partial_t \eta) + u_1 \partial_1 (\partial_t \eta) + u_2 \partial_2 (\partial_t \eta) = \partial_t u_3 - \partial_t u_1 \partial_1 \eta - \partial_t u_2 \partial_2 \eta
\]
So we have the estimate
\[
\|\partial_t^2 \eta\|_{L^\infty H^{2N-5/2}(\Sigma)} \lesssim \|u_2\|_{L^\infty H^{2N-5/2}(\Sigma)} \|\partial_t \eta\|_{L^\infty H^{2N-3/2}(\Sigma)} + \|\partial_t u_3\|_{L^\infty H^{2N-3/2}(\Sigma)} \|\partial_t \eta\|_{L^\infty H^{2N-3/2}(\Sigma)} + \|\partial_t u\|_{L^\infty H^{2N-3/2}(\Sigma)} \|\eta\|_{L^\infty H^{2N-3/2}(\Sigma)} \lesssim \left( K_0(\eta_0) + 1 \right) \left( Q(u)^2 + Q(u) \right)
\]
In a similar fashion, we can achieve the estimate for \( \partial_t^j \eta \) as \( j = 1, \ldots, N \).
\[
\|\partial_t^j \eta\|_{L^\infty H^{2N-2j+3/2}(\Sigma)} \lesssim \left( K_0(\eta_0) + 1 \right) \sum_{i=1}^j Q(u)^i
\]
To sum up, we have
\[
E(\eta) \lesssim (1 + K_0(\eta_0)) P(Q(u))
\]

Step 3: Estimate of \( D(\eta) \):
Similar to above argument, by differentiating on both sides and utilize the transport equation iteratively, we can show
\[
D(\eta) \lesssim (1 + K_0(\eta_0)) P(Q(u))
\]
Naturally, summarizing all above, we have
\[
K(\eta) = E(\eta) + D(\eta) \lesssim (1 + K_0(\eta_0)) P(Q(u))
\]
Then our result easily follows. □

Next, we introduce a lemma to describe the different between \( \eta \) and \( \eta_0 \) in a small time period.
Lemma 2.38. If $Q(u) + K_0(\eta_0) < \infty$, then for any $\epsilon > 0$, there exists a $\bar{T}(\epsilon) > 0$ depending on $Q(u)$ and $K_0(\eta_0)$, such that for any $0 < T < \bar{T}$, we have the estimate
\[
\|\eta - \eta_0\|_{L^\infty H^{2N+1/2}(\Sigma)} \leq \epsilon
\]

Proof. $\xi = \eta - \eta_0$ satisfies the transport equation
\[
\begin{cases}
\partial_t \xi + u_1 \partial_1 \xi + u_2 \partial_2 \xi = u_3 - u_1 \partial_1 \eta_0 - u_2 \partial_2 \eta_0 & \text{on } \Sigma \\
\xi(0) = 0
\end{cases}
\]
By the transport estimate stated above, we have
\[
\begin{align*}
\|\xi\|_{L^\infty H^{2N+1/2}(\Sigma)} & \leq \exp \left( C \int_0^T \|u(t)\|_{H^{2N+1/2}(\Sigma)} dt \right) \left( \int_0^T \|u_3(t) - u_1(t) \partial_1 \eta_0 - u_2(t) \partial_2 \eta_0\|_{H^{2N+1/2}(\Sigma)} dt \right) \\
& \lesssim T Q(u) K_0(\eta_0)
\end{align*}
\]
Hence, when $T$ is small enough, our result naturally follows. \qed

2.5.2. Forcing Estimates. Now we need to estimate the forcing terms which appears on the right-hand side of the linear Navier-Stokes equation. The forcing terms is as follows.
\[
F = \partial_2 \eta \tilde{b} K \partial_1 u - u \cdot \nabla_A u
\]
\[
H = \eta N
\]
Recall that we define the forcing quantities as follows.
\[
\begin{align*}
\mathcal{K}(F, H) &= \sum_{j=0}^{N-1} \|\partial_j^2 F\|^2_{L^2 H^{2N-2j-1}} + \|\partial_j^N F\|^2_{X^0} + \sum_{j=0}^{N} \|\partial_j^2 H\|^2_{L^2 H^{2N-2j-1/2}(\Sigma)} \\
& \quad + \sum_{j=0}^{N-1} \|\partial_j^2 F\|^2_{L^\infty H^{2N-2j-2}} + \sum_{j=0}^{N-1} \|\partial_j^2 H\|^2_{L^2 H^{2N-2j-3/2}(\Sigma)}
\end{align*}
\]
\[
\begin{align*}
\mathcal{H}_0(F, H) &= \sum_{j=0}^{N-1} \|\partial_j^2 F(0)\|^2_{H^{2N-2j-2}} + \sum_{j=0}^{N-1} \|\partial_j^2 H(0)\|^2_{H^{2N-2j-3/2}(\Sigma)}
\end{align*}
\]
In the estimate of Navier-Stokes-transport system, we also need some other forcing quantities.
\[
\begin{align*}
\mathcal{F}(F, H) &= \sum_{j=0}^{N-1} \|\partial_j^2 F\|^2_{L^2 H^{2N-2j-1}} + \|\partial_j^N F\|^2_{L^2 H^0} + \sum_{j=0}^{N} \|\partial_j^2 H\|^2_{L^\infty H^{2N-2j-1/2}(\Sigma)}
\end{align*}
\]
\[
\begin{align*}
\mathcal{H}(F, H) &= \sum_{j=0}^{N-1} \|\partial_j^2 F\|^2_{L^2 H^{2N-2j-1}} + \sum_{j=0}^{N-1} \|\partial_j^2 H\|^2_{L^2 H^{2N-2j-1/2}(\Sigma)}
\end{align*}
\]

Theorem 2.39. The forcing terms satisfies the estimate
\[
\begin{align*}
\mathcal{K}(F, H) &\lesssim P(\mathcal{K}(\eta)) + P(Q(u)) \\
\mathcal{H}_0(F, H) &\lesssim P(K_0(\eta_0)) + P(K_0(\eta_0)) \\
\mathcal{F}(F, H) &\lesssim P(\mathcal{K}(\eta)) + P(Q(u)) \\
\mathcal{H}(F, H) &\lesssim T \left( P(\mathcal{K}(\eta)) + P(Q(u)) \right)
\end{align*}
\]
where $P(\cdot)$ is a polynomial with $P(0) = 0$.

Proof. The estimates follow from simple but lengthy computation, involving standard argument, so we will only present a sketch of the proof here.
After we expand all the terms with Leibniz rule and plug in the definition of $F^j$ and $H^j$, it is easy to see that the basic form of estimate is $\|X Y\|^2_{L_t^\infty H^k}$ or $\|X Y\|^2_{L^\infty H^k}$ where $X$ includes the terms only involving $\tilde{\eta}$ and $Y$ includes the terms involving $u, p, F$ or $H$. Employing lemma A.1, we can always bound $\|X Y\|^2_{L_t^\infty H^k} \lesssim \|X\|^2_{L_t^\infty H^{k_1}} \|Y\|^2_{L_t^\infty H^{k_2}}$ or $\|X Y\|^2_{L^\infty H^k} \lesssim \|X\|^2_{L^\infty H^{k_1}} \|Y\|^2_{L^\infty H^{k_2}}$, and also $\|X Y\|^2_{L^\infty H^k} \lesssim \|X\|^2_{L_t^\infty H^{k_1}} \|Y\|^2_{L_t^\infty H^{k_2}}$. The principle of choice in these two forms for $L^2 H^k$ is not to exceed the order of derivative in the definition of right-hand sides in the estimates. A detailed estimate can show this goal can always be achieved. In (2.5.24), note the trivial bound
\[
\|\partial_1^N F\|^2_{X^0} \lesssim \|\partial_1^N F\|^2_{L^2 H^0}
\]
Also in (2.5.27), the key part is the appearance of bounding constant $T$. We first trivially bound it as follows.

$$
\sum_{j=0}^{N-1} \left\| \partial_t^j F \right\|_{L^2 H^{2N-2j+1}(\Omega)}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H \right\|_{L^2 H^{2N-2j+1/2}(\Sigma)}^2 \\
\leq T \left( \sum_{j=0}^{N-1} \left\| \partial_t^j F \right\|_{L^2 H^{2N-2j-1}(\Omega)}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H \right\|_{L^2 H^{2N-2j+1/2}(\Sigma)}^2 \right)
$$

Finally, an application of inequality $2ab \leq a^2 + b^2$ will imply the required estimate.

**Remark 2.40.** The reason why we can get the constant $T$ lies in that, in the nonlinear Navier-Stokes equation, the $u$ in nonlinear terms actually has one less derivative than the linear part, which makes it available to use lemma A.15. The appearance of $T$ in (2.5.27) will play a key role in the following nonlinear iteration argument.

### 2.6. Navier-Stokes-Transport System

In this section, we will study the nonlinear system

\[
\begin{aligned}
\partial_t u - \partial_t \eta b_k \partial_k u + u \cdot \nabla_A u - \Delta Au + \Delta p = 0 & \quad \text{in } \Omega \\
\nabla_A \cdot u = 0 & \quad \text{in } \Omega \\
S_A(p,u) = \eta \mathcal{N} & \quad \text{on } \Sigma \\
u = 0 & \quad \text{on } \Sigma_b \\
u(x,0) = u_0(x) & \\
\partial_\eta \eta + u_1 \partial_\eta \eta + u_3 \partial_\eta = u_3 & \quad \text{on } \Sigma \\
\eta(x',0) = \eta_0(x') & 
\end{aligned}
\]

(2.6.1)

We will first show a revised version of construction of the initial data and then employ an iteration argument to show the wellposedness of this system.

#### 2.6.1. Initial Data and Compatible Condition

Before we turn into the nonlinear equation, we first need to determine the higher order initial condition.

Define

\[
K_0 = K_0(u_0) + K_0(\eta_0)
\]

(2.6.2)

We say $(u_0, \eta_0)$ satisfies the $N^{th}$ order compatible condition if

\[
\begin{aligned}
\nabla_{A_0} \cdot (D^j_1 u(0)) = 0 & \quad \text{in } \Omega \\
D^j_1 u(0) = 0 & \quad \text{on } \Sigma_b \\
\Pi_0 H^j(0) + D_A \Pi_0 D^j_1 u(0)\mathcal{N}(0) = 0 & \quad \text{on } \Sigma
\end{aligned}
\]

(2.6.3)

for $j = 0, \ldots, N - 1$. Then we can employ an iteration argument to construct the initial condition of higher order. We will not repeat the detailed process here (see section 5.2 of [1]) and only give an estimate for the constructed data.

**Theorem 2.41.** Suppose $(u_0, \eta_0)$ satisfies $K_0 < \infty$. Let $\partial_1^j u(0)$, $\partial_1^j \eta(0)$ for $j = 0, \ldots, N$ and $\partial_1^j p(0)$ for $j = 0, \ldots, N - 1$ defined as we stated. Then

\[
K_0 \leq H_0(u, p) + H_0(\eta) \lesssim P(K_0)
\]

(2.6.4)

for a given polynomial $P(\cdot)$ with $P(0) = 0$.

**Proof.** This is a natural corollary of our construction, so we omit the proof here.

#### 2.6.2. Construction of Iteration

For given initial data $(u_0, \eta_0)$, by extension lemma A.16, there exists $u^0$ defined in $\Omega \times [0, \infty)$ such that $Q(u^0) \lesssim K_0(u_0)$ achieving the initial data to $N^{th}$ order. By solving transport equation with respect to $u^0$, theorem 2.37 implies that there exists $\eta^0$ defined in $\Omega \times [0, T_0)$ such that $K(\eta^0) \lesssim (1 + K_0(\eta_0))P(Q(u^0)) \lesssim P(K_0) < \infty$. This is our start point.

For any integer $m \geq 1$, define the approximate solution $(u^m, p^m, \eta^m)$ on the existence interval $[0, T_m]$ by the following iteration.

\[
\begin{aligned}
\partial_t u^m - \Delta u^{m-1} + \nabla_A u^{m-1}p^m + \Delta \eta^{m-1}bK^{m-1}\partial_3 u^{m-1} & \quad \text{in } \Omega \\
\Sigma_A \cdot u^m = 0 & \quad \text{in } \Omega \\
(p^m I - D_{A-1} u^m)\mathcal{N}^{m-1} = \eta^{m-1}N^{m-1} & \quad \text{on } \Sigma \\
u^m = 0 & \quad \text{on } \Sigma_b \\
\partial_\eta \eta^m + u_1^m \partial_\eta \eta^m + u_2^m \partial_\eta \eta^m = u_3^m & \quad \text{on } \Sigma
\end{aligned}
\]

(2.6.5)
where \((u^m, \eta^m)\) achieves the same initial data \((u_0, \eta_0)\), and \(A^m, N^m, K^m\) are given in terms of \(\eta^m\). This is only a formal definition of iteration. In the following theorems, we will finally prove that this approximate sequence can be defined for any \(m \in \mathbb{N}\) and the existence interval \(T_m\) will not shrink to 0 as \(m \to \infty\).

2.6.3. Boundedness Theorem.

Remark 2.42. Before we start to prove the boundedness result, it is useful to notice that, based on the linear estimate (2.4.35) and forcing estimate in lemma 2.39, this sequence can always be constructed and we can directly derive an estimate

\[
K(u^{m+1}, p^{m+1}) + K(\eta^{m+1}) \lesssim P(Q(u^m) + K(\eta^m) + K_0)
\]

when \(T\) is sufficiently small, where \(P(\cdot)\) is a polynomial satisfying \(P(0) = 0\). Since the initial data can be arbitrarily large, this estimate cannot meet our requirement. Hence, we have to go back to the energy structure and derive a stronger estimate. However, this result naturally implies a lemma, which will be used in the following:

If

\[
Q(u^{m-1}) + K(\eta^{m-1}) \leq Z
\]

then

\[
K(u^m, p^m) + K(\eta^m) \leq CP(K_0 + Z)
\]

Theorem 2.43. Assume \(J^0 > \delta > 0\) and the initial data \((u_0, \eta_0)\) satisfies the \(N^{th}\) compatible condition. Then there exists a constant 0 < \(Z < \infty\) and 0 < \(\bar{T} < 1\) depending on \(K_0\), such that if 0 < \(T < \bar{T}\) and \(K_0 < \infty\), then there exists an infinite sequence \((u^m, p^m, \eta^m)_{m=0}^{\infty}\) satisfying the iteration equation (2.6.5) within the existence interval \([0, T]\) and the following properties.

1. The iteration sequence satisfies the estimate

\[
Q(u^m) + K(\eta^m) \leq Z
\]

for arbitrary \(m\), where the temporal norm is taken with respect to \(T\).

2. For any \(m, J^m(t) > \delta/2\) for 0 ≤ \(t \leq T\).

Proof. Let’s denote the above two assertions related to \(m\) as statement \(P_m\). We use induction to prove this theorem. To note that in the following, we will extensively use the notation \(P(\cdot)\), however, these polynomials should be understood as explicitly given, but not necessarily written out here. Also they can change from line to line.

Step 1: \(P_0\) case:
This is only related to the initial data. Obviously, the construction of \(u^0\) leads to \(Q(u^0) \leq K_0\). By transport estimate (2.37), we have \(K(\eta^0) \lesssim P(K_0)\) for \(P(0) = 0\). We can choose \(Z \geq P(K_0)\), so the first assertion is verified.

Define \(\xi^0 = \eta^0 - \eta_0\) the difference between free surface at later time and its initial data. Then the estimate in lemma 2.38 implies \(\|\xi^0\|_{L^\infty H^{5/2}} \lesssim T \bar{Z}\). Naturally, \(\sup_{t \in [0, T]} J^0(t) - J^0(0) \lesssim \|\xi^0\|_{L^\infty H^{5/2}} \lesssim T \bar{Z}\). Thus if we take \(T \leq \delta/(2\bar{Z})\), then \(J^0(t) \geq \delta/2\). So the second assertion is also verified. In a similar fashion, we can verify the closeness assumption we made in linear Navier-Stokes equation, i.e. \(\eta\) and \(\eta_0\) is close enough within \([0, T]\) can also be verified. Hence, \(P_0\) is true.

In the following, we will assume \(P_{m-1}\) is true for \(m \geq 1\) and prove \(P_m\) is also true. As long as we can show this, by induction, \(P_n\) is valid for arbitrary \(n \in \mathbb{N}\). Certainly, the induction hypothesis and above remark implies \(Q(u^{m-1}) + K(\eta^{m-1}) \leq Z\) and \(K(u^m, p^m) + K(\eta^m) \leq CP(K_0 + Z)\).

Step 2: \(P_m\) case: estimate of \(u^m\) via energy structure
By theorem 2.36, the pair \((D^t_N u^m, D^t_N \eta^m)\) satisfies the equation (2.4.37) in the weak sense, i.e.

\[
\begin{aligned}
\partial_t (D^N_t u^m) - \Delta A^{m-1} (D^N_t u^m) + \nabla A^{m-1} (D^N_t p^m) &= F^N & \text{in } \Omega \\
\nabla A^{m-1} \cdot (D^N_t u^m) &= 0 & \text{in } \Omega \\
S_{A^{m-1}} (D^N_t u^m, D^N_t p^m) N &= H^N & \text{on } \Sigma \\
D^N_t u^m &= 0 & \text{on } \Sigma_b
\end{aligned}
\]
where $F^N$ and $H^N$ is given in terms of $u^m$ and $\eta^{m-1}$. Hence, we have the standard weak formulation, i.e. for any test function $\psi \in \mathcal{X}^{m-1}$, the following holds

$$\langle \partial_t D_t^N u^m, \psi \rangle_{L^2 H^0} + \frac{1}{2} \langle D_{A \cdot m-1} D_t^N u^m, D_{A \cdot m-1} \psi \rangle_{L^2 H^0} = \langle D_t^N u^m, F^N \rangle_{L^2 H^0} - \langle D_t^N u^m, H^N \rangle_{L^2 H^0(\Sigma)}$$

(2.6.10)

Therefore, when we plug in the test function $\psi = D_t^N u^m$, we have the natural energy structure

$$2 \int_{\Omega} J(0) \left| D_t^N u^m(0) \right|^2 + \frac{1}{2} \int_{\Omega} \partial_J \left| D_t^N u^m \right|^2 + \int_{\Omega} \int J F^N \cdot D_t^N u^m - \int_{\Omega} \int H^N \cdot D_t^N u^m$$

A preliminary estimate is as follows.

**LHS** $\gtrsim \left\| D_t^N u^m \right\|_{L^2 H^0}^2 + \left\| D_t^N u^m \right\|_{L^2 H^1}^2$

**RHS** $\lesssim P(K_0) + TZ \left\| D_t^N u^m \right\|_{L^2 H^0}^2$

$$+ \sqrt{TZ} \left| F^N \right|_{L^2 H^0} \left\| D_t^N u^m \right\|_{L^2 H^0} + \sqrt{T} \left\| H^N \right|_{L^2 H^{-1/2}(\Sigma)} \left\| D_t^N u^m \right\|_{L^2 H^{1/2}(\Sigma)}$$

$$\lesssim P(K_0) + TZ \left\| D_t^N u^m \right\|_{L^2 H^0}^2 + \sqrt{T} \left| F^N \right|_{L^2 H^0}^2 + \sqrt{T} \left| H^N \right|_{L^2 H^{-1/2}(\Sigma)}^2$$

for a polynomial $P(0) = 0$. Taking $T \leq 1/(16L^2)$ and absorbing the extra term on RHS into LHS implies

$$\left\| D_t^N u^m \right\|_{L^2 H^0}^2 + \left\| D_t^N u^m \right\|_{L^2 H^1}^2 \lesssim P(K_0) + \sqrt{T} \left| F^N \right|_{L^2 H^0}^2 + \sqrt{T} \left| H^N \right|_{L^2 H^{-1/2}(\Sigma)}^2$$

(2.6.12)

Drop the $\left\| D_t^N u^m \right\|_{L^2 H^0}^2$ term and we derive further the estimate for $\partial_t^N u^m$

$$\left\| \partial_t^N u^m \right\|_{L^2 H^1}^2 \lesssim P(K_0) + \sqrt{T} \left| F^N \right|_{L^2 H^0}^2 + \sqrt{T} \left| H^N \right|_{L^2 H^{-1/2}(\Sigma)}^2 + \left\| D_t^N u^m - \partial_t^N u^m \right\|_{L^2 H^1}^2$$

(2.6.13)

Then we need to estimate each term on RHS. For the middle two terms, it suffices to show it is bounded, however, for the last term, we need a temporal constant $T$ within the estimate, which can be done by lemma A.15. Since these estimates are similar to the proof of lemma ??, we will not give the details here.

$$\left\| F^N \right|_{L^2 H^0}^2 \lesssim P(K_0(\eta^{m-1})) \left( \sum_{j=0}^{N-1} \left\| \partial_t^j u^m \right\|_{L^2 H^2}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j p^m \right\|_{L^2 H^1}^2 \right) + \mathcal{F}(F, H)$$

$$\lesssim P(K_0 + Z) + \mathcal{F}(F, H)$$

$$\left\| H^N \right|_{L^2 H^{-1/2}(\Sigma)}^2 \lesssim P(K_0(\eta^{m-1})) \left( \sum_{j=0}^{N-1} \left\| \partial_t^j u^m \right\|_{L^2 H^2}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j p^m \right\|_{L^2 H^1}^2 \right) + \mathcal{F}(F, H)$$

$$\lesssim P(K_0 + Z) + \mathcal{F}(F, H)$$

$$\left\| D_t^N u^m - \partial_t^N u^m \right\|_{L^2 H^1} \lesssim T \left\| D_t^N u^m - \partial_t^N u^m \right\|_{L^2 H^1}$$

$$\lesssim T P(K_0(\eta^{m-1})) \left( \sum_{j=0}^{N-1} \left\| \partial_t^j u^m \right\|_{L^2 H^1}^2 \right) + \mathcal{F}(F, H)$$

Therefore, to sum up, we have

$$\left\| \partial_t^N u^m \right\|_{L^2 H^1} \lesssim P(K_0) + \sqrt{T} P(K_0 + Z) + \sqrt{T} \mathcal{F}(F, H)$$

(2.6.14)

Step 3: $P_m$ case: estimate of $u^m$ via elliptic estimate

For $0 \leq n \leq N - 1$, the $n^{th}$ order Navier-Stokes equation is as follows.

$$\begin{align*}
\partial_t (D_t^n u^m) - \Delta_A (D_t^n u^m) + \nabla_A (\partial_t^p p^m) &= F^n & \text{in} \quad \Omega \\
\nabla_A \cdot (D_t^n u^m) &= 0 & \text{in} \quad \Omega \\
S_A(D_t^n u^m, \partial_t^p p^m)N &= H^n & \text{on} \quad \Sigma \\
D_t^n u^m &= 0 & \text{on} \quad \Sigma_b
\end{align*}$$

(2.6.15)
where $F^n$ and $H^n$ is given in terms of $u^m$ and $\eta^{m-1}$.

A straightforward application of elliptic estimate reveals the fact.

(2.6.16) \[ \|D_t^n u^m\|^2_{L^2 H^{2N-2n+1}} \lesssim \|\nabla^2 D_t^n u^m\|^2_{L^2 H^{2N-2n-1}} + \|F^n\|^2_{L^2 H^{2N-2n-1}} + \|H^n\|^2_{L^2 H^{2N-2n-1/2}} \]

A more reasonable form is as follows.

(2.6.17) \[ \|\nabla^2 D_t^n u^m\|^2_{L^2 H^{2N-2n+1}} \lesssim \|F^n\|^2_{L^2 H^{2N-2n-1}} + \|H^n\|^2_{L^2 H^{2N-2n-1/2}} + \|\nabla^2 D_t^n u^m\|^2_{L^2 H^{2N-2n-1}} + \|\nabla^2(D^n_t u^m - \nabla^2 u^m)\|^2_{L^2 H^{2N-2n-1}} + \|D^n_t u^m - \nabla^2 u^m\|^2_{L^2 H^{2N+1}} \]

Then we give a detailed estimate for each term on RHS. These estimates can be easily obtained as what we did before, so we omit the details here. It is noticeable that the appearance of $T$ is due to lemma A.15.

\[ \|F^n\|^2_{L^2 H^{2N-2n-1}} \lesssim TP(K(\eta^{m-1})) \left( \sum_{j=0}^{N-2} \left\| \partial_j^2 u^m \right\|^2_{L^\infty H^{2N-2j-1}} + \sum_{j=0}^{N-2} \left\| \partial_j^2 u^m \right\|^2_{L^\infty H^{2N-2j-2}} \right) + \mathcal{H}(F, H) \]

\[ \|H^n\|^2_{L^2 H^{2N-2n-1/2}(\Sigma)} \lesssim TP(K(\eta^{m-1})) \left( \sum_{j=0}^{N-2} \left\| \partial_j^2 u^m \right\|^2_{L^\infty H^{2N-2j-1}} + \sum_{j=0}^{N-2} \left\| \partial_j^2 u^m \right\|^2_{L^\infty H^{2N-2j-2}} \right) + \mathcal{H}(F, H) \]

\[ \|\partial_t(D^n_t u^m - \partial^n_t u^m)\|^2_{L^2 H^{2N-2n+1}} \lesssim TP(K(\eta^{m-1})) \left( \sum_{j=0}^{N-1} \left\| \partial_j^2 u^m \right\|^2_{L^\infty H^{2N-2j-1}} \right) \]

\[ \|D^n_t u^m - \partial^n_t u^m\|^2_{L^2 H^{2N-2n+1}} \lesssim TP(K(\eta^{m-1})) \left( \sum_{j=0}^{N-2} \left\| \partial_j^2 u^m \right\|^2_{L^\infty H^{2N-2j+1}} \right) \]

Therefore, to sum up, we have

(2.6.18) \[ \sum_{n=0}^{N-1} \left\| \partial^n_t u^m \right\|^2_{L^2 H^{2N-2n+1}} \lesssim TP(K_0 + Z) + \mathcal{H}(F, H) \]

**Step 4: $F_m$ case: synthesis of estimate for $u^m$**

Combining (2.6.14) and (2.6.18) with lemma A.14 implies

(2.6.19) \[ Q(u^m) \lesssim P(K_0) + \sqrt{TP(1 + K_0 + Z)} + \sqrt{TF}(F, H) + \mathcal{H}(F, H) \]

Then, by forcing estimate (2.39), we have

\[ F(F^0, H^0) \lesssim P(K(\eta^{m-1})) + P(Q(u^{m-1})) \lesssim P(Z) \]

\[ \mathcal{H}(F^0, H^0) \lesssim T \left( P(K(\eta^{m-1})) + P(Q(u^{m-1})) \right) \lesssim TP(Z) \]

Hence, to sum up, we achieve the estimate

(2.6.20) \[ Q(u^m) \lesssim P(K_0) + \sqrt{TP}(K_0 + Z) \]

This is actually

(2.6.21) \[ Q(u^m) \lesssim CP(K_0) + \sqrt{TCP}(K_0 + Z) \]

for some universal constant $C > 0$. So we can take $Z \geq 2CP(K_0)$. If $T$ is sufficiently small depending on $Z$, we can bound $Q(u^m) \lesssim 2CP(K_0) \lesssim Z$. 
Step 5: $\mathbb{P}_m$ case: estimate of $\eta^m$ via transport estimate

Employing transport estimate in lemma 2.37

(2.6.22) \[ \mathcal{K}(\eta^m) \lesssim P(K_0) + P(Q(u^m)) \]

for $P(0) = 0$. Hence, we can take $Z \geq P(K_0) + P(2CP(K_0))$, then we have

(2.6.23) \[ \mathcal{K}(\eta^m) \lesssim Z \]

Step 6: $\mathbb{P}_m$ case: estimate of $J^m(t)$

Define $\xi^m = \eta^m - \eta_0$, then transport estimate in lemma 2.38 implies $\| \xi^m \|_{L^\infty H^{3/2}} \lesssim T Z$. Naturally, $\sup_{t \in [0,T]} |J^m(t) - J^m(0)| \lesssim \| \xi^m \|_{L^\infty H^{3/2}} \lesssim T Z$. Thus if we take $T \leq \delta/(2Z)$, then $J^m(t) \geq \delta/2$. A similar argument can justify the closeness assumption of $\eta^m$ and $\eta_0$ in studying linear Navier-Stokes equation.

Synthesis:

Above estimates reveals that if we take $Z = CP(K_0)$ where the polynomial can be given explicitly by summarizing all above and $T$ small enough depending on $Z$, we have

(2.6.24) \[ Q(u^m) + \mathcal{K}(\eta^m) \leq Z \]

and

(2.6.25) \[ J^m(t) \geq \delta/2 \quad \text{for} \quad t \in [0,T] \]

Therefore, $\mathbb{P}_m$ is verified. By induction, we conclude that $\mathbb{P}_n$ is valid for any $n \in \mathbb{N}$. It is noticeable that $Z = P(K_0)$ where $P(\cdot)$ actually satisfies $P(0) = 0$.

Theorem 2.44. Assume exactly the same condition as Theorem (2.43), then actually we have the estimate

(2.6.26) \[ \mathcal{K}(u^m, p^m) + \mathcal{K}(\eta^m) \lesssim P(K_0) \]

for a polynomial satisfying $P(0) = 0$.

Proof. This is a natural corollary of above theorem. Since we know $Q(u^{m-1}) \lesssim Z$, by remark 2.42, it implies $\mathcal{K}(u^m, p^m)$ is bounded for any $m \in \mathbb{N}$. For convenience, in the following we also call this bound $Z$.

2.6.4. Contraction Theorem. Define

(2.6.27) \[ \mathfrak{M}(v, q; T) = \| v \|_{L^2 H^3}^2 + \| v \|_{L^2 H^3}^2 + \| \partial_t v \|_{L^2 H^3}^2 + \| \partial_t v \|_{L^2 H^3}^2 + \| q \|_{L^2 H^3}^2 + \| q \|_{L^2 H^3}^2 \]

(2.6.28) \[ \mathfrak{M}(\zeta; T) = \| \zeta \|_{L^2 H^{3/2}(\Sigma)}^2 + \| \partial_t \zeta \|_{L^2 H^{3/2}(\Sigma)}^2 + \| \partial_t \zeta \|_{L^2 H^{3/2}(\Sigma)}^2 \]

Theorem 2.45. For $j = 1, 2$, Suppose that $v^j$, $q^j$, $w^j$ and $\zeta^j$ achieve the same initial condition for different $j$ and satisfy

\[
\begin{align*}
\partial_t v^j - \Delta A^j v^j + \nabla A^j q^j &= \partial_t \zeta^j \bar{h} K^j \partial_\zeta w^j - w^j \cdot \nabla A^j v^j \quad \text{in} \quad \Omega \\
\nabla A^j \cdot v^j &= 0 \quad \text{in} \quad \Omega \\
S_{A^j}(q^j, v^j) N^j &= \zeta^j N^j \quad \text{on} \quad \Sigma \\
v^j &= 0 \quad \text{on} \quad \Sigma_b \\
\partial_t \zeta^j &= w^j \cdot N^j \quad \text{in} \quad \Omega 
\end{align*}
\]

where $A^j$, $N^j$ and $K^j$ are in terms of $\zeta^j$. Suppose $\mathcal{K}(w^j, 0)$, $\mathcal{K}(v^j, q^j)$ and $\mathcal{K}(\zeta^j)$ is bounded by $Z$. Then there exists $0 < \tilde{T} < 1$ such that for any $0 < T < \tilde{T}$, we have the following contraction relation

(2.6.29) \[ \mathfrak{M}(v^1 - v^2, q^1 - q^2; T) \leq \frac{1}{2} \mathfrak{M}(w^1 - w^2, 0; T) \]

(2.6.30) \[ \mathfrak{M}(\zeta^1 - \zeta^2; T) \lesssim \mathfrak{M}(w^1 - w^2, 0; T) \]

Proof. We divide this proof into several steps.
Step 1: Lower order equations
define \( v = v^1 - v^2, q = q^1 - q^2, w = w^1 - w^2 \) and \( \zeta = \zeta^1 - \zeta^2 \), which has trivial initial condition. Then they satisfy the equation as follows.

\[
\begin{aligned}
\partial_v v + \nabla_{A^1} \cdot S_{A^1}(q, v) &= H^1 + \nabla_{A^1} \cdot (\mathbb{D}(A^1 - A^2)v^2) \quad \text{in } \Omega \\
\nabla_{A^1} \cdot v &= H^2 \\
S_{A^1}(q, v)N^1 &= H^3 + \mathbb{D}(A^1 - A^2)v^2N^1 \quad \text{on } \Sigma \\
v &= 0 \quad \text{on } \Sigma_b
\end{aligned}
\]  

(2.6.31)

where

\[
\begin{aligned}
H^1 &= \nabla_{(A^1 - A^2)}(\mathbb{D}_{A^1}v^2) - (A^1 - A^2)\nabla p^2 + \partial_t \zeta^1 \bar{b}K^1 \partial_tA w + \partial_t \zeta^2 \bar{b}K^2 \partial_tA w^2 \\
+ \bar{\partial}_t \zeta^1 \bar{b}(K^1 - K^2) \partial_tA w^2 - w \cdot \nabla_{A^1} w^1 - w^2 \cdot \nabla_{A^1} w - w^2 \cdot \nabla_{(A^1 - A^2)}w^2 \\
H^2 &= -\nabla_{(A^1 - A^2)} \cdot v^2 \\
H^3 &= -q^2(N^1 - N^2) + \mathbb{D}_{A^1}v^2(N^1 - N^2) - \mathbb{D}(A^1 - A^2)v^2(N^1 - N^2) + \zeta^1 N^1 + \zeta^2(N^1 - N^2)
\end{aligned}
\]

The solutions are sufficiently regular for us to differentiate in time, which is the following equation.

\[
\begin{aligned}
\partial_t(\partial_v v) + \nabla_{A^1} \cdot S_{A^1}(\partial_t q, \partial_v v) &= \tilde{H}^1 + \nabla_{A^1} \cdot (\mathbb{D}(\partial_t A^1 - \partial_t A^2)v^2) \quad \text{in } \Omega \\
\nabla_{A^1} \cdot \partial_v v &= \tilde{H}^2 \\
S_{A^1}(\partial_t q, \partial_v v)N^1 &= \tilde{H}^3 + \mathbb{D}(\partial_t A^1 - \partial_t A^2)v^2N^1 \quad \text{on } \Sigma \\
\partial_t v &= 0 \quad \text{on } \Sigma_b
\end{aligned}
\]  

(2.6.33)

where

\[
\begin{aligned}
\tilde{H}^1 &= \partial_t H^1 + \nabla_{\partial_t A^1} \cdot (\mathbb{D}(A^1 - A^2)v^2) + \nabla_{A^1} \cdot (\mathbb{D}(A^1 - A^2)v^2) + \nabla_{\partial_t A^1} \cdot (\mathbb{D}_{A^1}v^2) + \nabla_{A^1} \cdot (\mathbb{D}_{\partial_t A^1}v^2) - \nabla_{\partial_t A^1} q \\
\tilde{H}^2 &= \partial_t H^2 - \nabla_{\partial_t A^1} \cdot v \\
\tilde{H}^3 &= \partial_t H^3 + \mathbb{D}(A^1 - A^2) \partial_t v^2N^1 + \mathbb{D}(A^1 - A^2)v^2 \partial_t N^1 - S_{A^1}(q, v)\partial_t \bar{N} + \mathbb{D}_{\partial_t A^1}v^1 N^1
\end{aligned}
\]

Step 2: Energy evolution for \( \partial_t v \)

Multiply \( J^1 \partial_t \bar{v} \) on both sides to get the natural energy structure:

\[
\frac{1}{2} \int_{\Omega} |\partial_t v|^2 J^1 + \frac{1}{2} \int_{0}^{t} \int_{\Omega} |\partial_t v|^2 J^1 = \int_{0}^{t} \int_{\Omega} J^1 \tilde{H}^1 \cdot \partial_t v + \int_{0}^{t} \int_{\Omega} J^1 \tilde{H}^2 \partial_t q - \int_{0}^{t} \int_{\Sigma} \tilde{H}^3 \cdot \partial_t v \\
+ \frac{1}{2} \int_{0}^{t} \int_{\Omega} |\partial_t v|^2 \partial_t J^1 - \frac{1}{2} \int_{0}^{t} \int_{\Omega} J^1 \mathbb{D}(\partial_t A^1 - \partial_t A^2)v^2 : \mathbb{D}_{A^1} \partial_t v
\]

LHS is simply the energy and dissipation term, so we focus on estimate of RHS.

\[
\begin{aligned}
\int_{0}^{t} \int_{\Omega} J^1 \tilde{H}^1 \cdot \partial_t v &\lesssim \int_{\Sigma} \int \tilde{H}^1_{\mathbb{H}^0} \|\partial_t v\|_{\mathbb{H}^0} \lesssim \int \|\partial_t v\|_{L^\infty \mathbb{H}^0} \sqrt{T} \|\tilde{H}^1\|_{L^2 \mathbb{H}^0} \\
&\lesssim \sqrt{T} Z \sqrt{\mathbb{N}^v} \|\tilde{H}^1\|_{L^2 \mathbb{H}^0}
\end{aligned}
\]

\[
\begin{aligned}
\int_{0}^{t} \int_{\Omega} J^1 \tilde{H}^2 \partial_t q &= \int_{0}^{t} \int_{\Omega} J^1 q \tilde{H}^2 - \int_{0}^{t} \int_{\Omega} (\partial_t J^1 q \tilde{H}^2 + J^1 q \partial_t \tilde{H}^2) \\
&\lesssim \int \|q\|_{L^\infty \mathbb{H}^0} \sqrt{T} Z \|\tilde{H}^2\|_{L^2 \mathbb{H}^0} + \|q\|_{L^\infty \mathbb{H}^0} \sqrt{T} Z \|\partial_t \tilde{H}^2\|_{L^2 \mathbb{H}^0} \\
&\lesssim \int \|q\|_{L^\infty \mathbb{H}^0} \sqrt{T} Z \sqrt{\mathbb{N}^v} \left( \|\tilde{H}^2\|_{L^2 \mathbb{H}^0} + \|\partial_t \tilde{H}^2\|_{L^2 \mathbb{H}^0} \right)
\end{aligned}
\]

\[
\begin{aligned}
\int_{0}^{t} \int_{\Sigma} \tilde{H}^3 \cdot \partial_t v &\lesssim \int_{\Omega} \|\tilde{H}^3\|_{H^{-1/2}(\Sigma)} \|\partial_t v\|_{H^{1/2}(\Sigma)} \lesssim \sqrt{T} \|\tilde{H}^3\|_{L^\infty H^{-1/2}(\Sigma)} \|\partial_t v\|_{L^2 H^{1/2}(\Sigma)} \\
&\lesssim \sqrt{T} \sqrt{\mathbb{N}^v} \|\tilde{H}^3\|_{L^\infty H^{-1/2}(\Sigma)} \\
\int_{0}^{t} \int_{\Omega} |\partial_t v|^2 \partial_t J^1 &\lesssim \|\partial_t J^1\|_{L^\infty \mathbb{H}^2} T \|\partial_t v\|_{L^\infty \mathbb{H}^0} \lesssim T Z \mathbb{N}^v
\end{aligned}
\]
So we need several estimate on $\tilde{H}^i$. For the following terms, it suffices to show they are bounded. We use the usual way to estimate these quadratic terms. Since we have repeatedly used this method, we will not give the details here.

$$\left\| \tilde{H}^1 \right\|_{L^2 H^0} \lesssim Z \left( \left\| \zeta \right\|_{L^2 H^{3/2}} + \left\| \partial_t \zeta \right\|_{L^2 H^{1/2}} + \left\| \partial_t^2 \zeta \right\|_{L^2 H^{-1/2}} + \left\| w \right\|_{L^2 H^1} + \left\| \partial_t w \right\|_{L^2 H^1} + \left\| v \right\|_{L^2 H^1} \right)$$

$$\lesssim Z \left( \sqrt{\Theta^v} + \sqrt{\Theta^w} + \sqrt{\Theta^t} \right)$$

$$\left\| \tilde{H}^2 \right\|_{L^2 H^0} \lesssim Z \left( \left\| \partial_t \zeta \right\|_{L^2 H^{1/2}} + \left\| \zeta \right\|_{L^2 H^{1/2}} + \left\| v \right\|_{L^2 H^1} \right) \lesssim Z \left( \sqrt{\Theta^v} + \sqrt{\Theta^w} \right)$$

$$\left\| \partial_t \tilde{H}^2 \right\|_{L^2 H^0} \lesssim Z \left( \left\| \partial_t^2 \zeta \right\|_{L^2 H^{1/2}} + \left\| \partial_t \zeta \right\|_{L^2 H^{1/2}} + \left\| \zeta \right\|_{L^2 H^{1/2}} + \left\| \partial_t v \right\|_{L^2 H^1} + \left\| v \right\|_{L^2 H^1} \right)$$

$$\lesssim Z \left( \sqrt{\Theta^v} + \sqrt{\Theta^w} \right)$$

$$\left\| \tilde{H}^3 \right\|_{L^\infty H^{-1/2}(\Sigma)} \lesssim Z \left( \left\| \zeta \right\|_{L^\infty H^{1/2}} + \left\| \partial_t \zeta \right\|_{L^\infty H^{1/2}} + \left\| v \right\|_{L^\infty H^1} \right)$$

$$\lesssim Z \left( \sqrt{\Theta^v} + \sqrt{\Theta^w} \right)$$

Also we have the following estimate.

$$\int_{\Omega} J^1 \cdot \tilde{H}^2 \lesssim Z \left\| \zeta \right\|_{L^\infty H^0} \left\| \tilde{H}^2 \right\|_{L^\infty H^0}$$

$$\lesssim Z \left\| \zeta \right\|_{L^\infty H^0} \left( \left\| \zeta \right\|_{L^\infty H^{1/2}} + \left\| \partial_t \zeta \right\|_{L^\infty H^{1/2}} + \left\| v \right\|_{L^\infty H^1} \right)$$

$$\lesssim Z \sqrt{\Theta^v} \left( \left\| \zeta \right\|_{L^\infty H^{1/2}} + \left\| \partial_t \zeta \right\|_{L^\infty H^{1/2}} + \left\| v \right\|_{L^\infty H^1} \right)$$

$\zeta$ satisfies the equation

$$\begin{cases} \partial_t \zeta + w_1 \partial_1 \zeta + w_2 \partial_2 \zeta = -N^2 \cdot w \\
\zeta(0) = 0 \end{cases}$$

Employing transport estimate and the boundedness of higher order norms, we have

$$\left\| \zeta \right\|_{L^\infty H^{5/2}} \lesssim \sqrt{T} Z \left\| \zeta \right\|_{L^\infty H^{1/2}} \lesssim \sqrt{T} Z \sqrt{\Theta^v}$$

$\partial_t \zeta$ satisfies the equation

$$\begin{cases} \partial_t (\partial_t \zeta) + w_1 \partial_1 (\partial_t \zeta) + w_2 \partial_2 (\partial_t \zeta) = -N^2 \cdot \partial_t w - \partial_t N w - \partial_t w_1 \partial_1 \zeta - \partial_t w_2 \partial_2 \zeta \\
\partial_t \zeta(0) = 0 \end{cases}$$

Similar argument as above shows that

$$\left\| \partial_t \zeta \right\|_{L^\infty H^{1/2}} \lesssim \sqrt{T} Z \left( \left\| \partial_t w \right\|_{L^2 H^{1/2}} + \left\| w \right\|_{L^2 H^{1/2}} \right) \lesssim \sqrt{T} Z \sqrt{\Theta^v}$$

Combining above transport estimate and lemma A.14, we have the final version

$$\int_{\Omega} J^1 \cdot \tilde{H}^2 \lesssim \sqrt{T} Z \sqrt{\Theta^v} \sqrt{\Theta^w} + Z \sqrt{\Theta^v} \left\| v \right\|_{L^\infty H^2}$$

$$\lesssim \sqrt{T} Z \sqrt{\Theta^v} \sqrt{\Theta^w} + Z \sqrt{\Theta^v} \left( \left\| v \right\|_{L^2 H^2} + \left\| \partial_t v \right\|_{L^2 H^0} \right)$$

$$\lesssim \sqrt{T} Z \sqrt{\Theta^v} \sqrt{\Theta^w} + \sqrt{T} Z \sqrt{\Theta^w} \left( \left\| v \right\|_{L^\infty H^2} + \left\| \partial_t v \right\|_{L^\infty H^0} \right)$$

$$\lesssim \sqrt{T} Z \left( \sqrt{\Theta^v} \sqrt{\Theta^w} + \Theta^v \right)$$

Then we consider simplify the last term in RHS of energy structure.

$$\int_0^t \int_{\Omega} J^1 D(\partial_t A_1 - \partial_t A_2) v^2 : D A_1 \partial_t v \lesssim \frac{1}{4\epsilon} \left\| J^1 D(\partial_t A_1 - \partial_t A_2) v^2 \right\|_{L^2 H^0}^2 + \epsilon \left\| D A_1 \partial_t v \right\|_{L^2 H^0}^2$$
where $\epsilon$ should be determined in order for the second term in RHS to be absorbed in LHS.

$$\|J^2 D(\overline{\partial}_A \cdot - \partial_0 \cdot A^0) v^2\|_{L^2 H^0} \lesssim \sqrt{T} \|J^2 R(\overline{\partial}_A \cdot - \partial_0 \cdot A^0) v^2\|_{L^\infty H^0} \lesssim \sqrt{T} \mathcal{Z} \mathcal{W}$$

To summarize, using Cauchy inequality

(2.6.35) \[\|v_1\|_{L^2 H^0} + \|v_2\|_{L^2 H^1} \lesssim \sqrt{T} (\mathcal{W} + \mathcal{W} + \mathcal{M})\]

Step 3: Elliptic estimate for $v$

Based on standard elliptic regularity theory, we have the estimate

$$\|v\|_{H^{r+2}} + \|q\|_{H^{r+1}} \lesssim \|\partial_t v\|_{H^r} + \|H^1\|_{H^r} + \|H^2\|_{H^{r+1}} + \|H^3\|_{H^{r+1}} + \|\nabla A^1 \cdot (D_n A^{-2} v^2)\|_{H^r} + \|D_n A^{-2} v^2 N^1\|_{H^{r+1}}$$

Set $r = 0$ and take $L^\infty$ on both sides

$$\|v\|_{L^\infty H^2} + \|q\|_{L^\infty H^1} \lesssim \|\partial_t v\|_{L^\infty H^0} + \|H^1\|_{L^\infty H^0} + \|H^2\|_{L^\infty H^1} + \|H^3\|_{L^\infty H^1} + \|\nabla A^1 \cdot (D_n A^{-2} v^2)\|_{L^\infty H^0} + \|D_n A^{-2} v^2 N^1\|_{L^\infty H^1}$$

Set $r = 1$ and take $L^2$ on both sides

$$\|v\|_{L^2 H^2} + \|q\|_{L^2 H^1} \lesssim \|\partial_t v\|_{L^2 H^1} + \|H^1\|_{L^2 H^1} + \|H^2\|_{L^2 H^2} + \|H^3\|_{L^2 H^2} + \|\nabla A^1 \cdot (D_n A^{-2} v^2)\|_{L^2 H^1} + \|D_n A^{-2} v^2 N^1\|_{L^2 H^2}$$

We need to estimate all the RHS terms. We can employ the usual way to estimate quadratic terms in $L^2 H^k$ norm, however, for $L^\infty H^k$ norm, we use lemma A.4. Then we have the estimate

$$\|v\|_{L^\infty H^2} + \|q\|_{L^\infty H^1} + T \mathcal{Z} (\mathcal{W} + \mathcal{M})$$

To summarize, we have

(2.6.36) \[\|v\|_{L^\infty H^2} + \|q\|_{L^\infty H^1} + \|v\|_{L^2 H^3} + \|q\|_{L^2 H^2} \lesssim \|\partial_t v\|_{L^2 H^1} + \|\partial_t v\|_{L^\infty H^0} + T \mathcal{Z} (\mathcal{W} + \mathcal{M})\]

In total of (2.6.35) and (2.6.36), we get the succinct form of estimate

(2.6.37) \[\mathcal{W} \lesssim \sqrt{T} (\mathcal{W} + \mathcal{W} + \mathcal{M})\]

Step 4: Transport estimate for $\zeta$

$\zeta$ satisfies the equation

$$\begin{cases}
\partial_t \zeta + w_1 \partial_1 \zeta + w_2 \partial_2 \zeta = -N^2 \cdot w \\
\zeta(0) = 0
\end{cases}$$

A straightforward application of transport estimate can show

$$\|\zeta\|_{L^\infty H^{1/2} (\Sigma)} \lesssim \exp \left( C \int_0^T \|w_1\|_{H^3} \right) \int_0^T \|N^2 \cdot w\|_{H^{1/2} (\Sigma)}$$

$$\lesssim \sqrt{T} \mathcal{Z} \|w\|_{L^2 H^{1/2} (\Sigma)} \lesssim \sqrt{T} \mathcal{Z} \mathcal{W}$$

Similar to the proof of transport estimate in theorem 2.37, we can use the transport equation to estimate the higher order derivatives.

$$\|\partial_t \zeta\|_{L^\infty H^{1/2} (\Sigma)} \lesssim \|\zeta\|_{L^\infty H^{1/2} (\Sigma)} \|w\|_{L^\infty H^{1/2} (\Sigma)} + \|\zeta\|_{L^\infty H^{1/2} (\Sigma)} \|w_1\|_{L^\infty H^{1/2} (\Sigma)} \lesssim \mathcal{Z} \mathcal{W}$$

In the same fashion, we can easily show

$$\|\partial_t^2 \zeta\|_{L^2 H^{1/2}} \lesssim \mathcal{Z} \mathcal{W}$$

To sum up, we have the estimate

(2.6.38) \[\mathcal{W} \lesssim \mathcal{W}\]
In (2.6.37), for $T$ sufficiently small, we can easily absorb all $\mathfrak{N}^v$ term from RHS to LHS and replace all $\mathfrak{M}$ with $\mathfrak{N}$ to achieve

$$\mathfrak{N}^v \lesssim \sqrt{T \mathfrak{Z} \mathfrak{N}^w}$$

Certainly, the smallness of $T$ can guarantee the contraction.

2.6.5. Local Wellposedness Theorem. Now we can combine theorem 2.44 and 2.45 to produce a unique strong solution to the equation system (1.3.8).

**Theorem 2.46.** Assume that $(u^0, \eta^0)$ satisfies $K_0 < \infty$ and that the initial data are constructed to satisfy the $N$th compatible condition. Then there exist $0 < T_0 < 1$ such that if $0 < T < T_0$, then there exists a solution triple $(u, p, \eta)$ to the Navier-Stokes equation on the time interval $[0, T]$ that achieves the initial data and satisfies

$$K(u, p) + K(\eta) \leq CP(K_0)$$

for a universal constant $C > 0$ and a polynomial $P(\cdot)$ satisfying $P(0) = 0$. The solution is unique among functions that achieve the initial data and satisfy $K(u, p) + K(\eta) < \infty$. Moreover, $\eta$ is such that the mapping $\Phi$ is a $C^{2N-2}$ diffeomorphism for each $t \in [0, T]$.

**Proof.** Since we have proved the boundedness and contraction of the iteration sequence as [1] did, then a similar argument as in theorem 6.3 in [1] can justify our result. Hence, we will omit the details here. The basic idea is that using boundedness to derive weak convergence and using the interpolation between lower order and higher order and the contraction to derive strong convergence such that we could pass to limit in the iteration sequence.

3. Global Wellposedness for Horizontally Infinite Domain

3.1. Preliminaries. In this section, we will present two types of formulation for the system and describe the corresponding energy structure. Since they are almost identical to that in section 2 of [2], we will not give detailed proofs.

3.1.1. Geometric Structure Form. Suppose that $\eta, u$ are known and $A, N$, etc. are given in terms of $\eta$ as usual. Then we consider the linear equation for $(v, q, \xi)$

$$\begin{align*}
\partial_t v - \partial_\eta \tilde{b} K \partial_3 v + u \cdot \nabla A v + \nabla A \cdot S_A(q, v) &= F_1 \quad \text{in } \Omega \\
\nabla A \cdot v &= F_2 \quad \text{in } \Omega \\
S_A(q, v)N &= \xi N + F_3 \quad \text{on } \Sigma \\
\partial_\xi \xi - v \cdot N &= F_4 \quad \text{on } \Sigma \\
v &= 0 \quad \text{on } \Sigma_b
\end{align*}$$

We have the following energy structure.

**Lemma 3.1.** Suppose $(u, p, \eta)$ satisfies system (1.3.8) and $(v, q, \xi)$ is the solution of system (3.1.1), then

$$\begin{align*}
\partial_t \left( \frac{1}{2} \int_\Omega |v|^2 + \frac{1}{2} \int_\Sigma |\xi|^2 \right) + \frac{1}{2} \int_\Omega J |\nabla A v|^2 &= \int_\Omega J (v \cdot F_2 + q F_2) + \int_\Sigma -v \cdot F_3 + \xi F_4
\end{align*}$$

The geometric structure form will be utilized to estimate the temporal derivatives. Hence, we may apply differential operator $\partial^\alpha = \partial_t^{\alpha_0}$ to system (1.3.8) with the resulting equation being (3.1.1) for $v = \partial^\alpha u$, $q = \partial^\alpha p$ and $\xi = \partial^\alpha \eta$, where
Compare to the original equation (1.3.8), we can write the detailed form for the nonlinear terms.

\[(3.1.3)\quad F^1 = F^{1,1} + F^{1,2} + F^{1,3} + F^{1,4} + F^{1,5} + F^{1,6}\]

\[(3.1.4)\quad F_{i}^{1,1} = \sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta (\partial_t \tilde{\eta} K) \partial^{\alpha-\beta} \partial_i u_i + \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} \partial_i \tilde{\eta} \partial^\beta (\tilde{b} K) \partial_3 u_i\]

\[(3.1.5)\quad F_{i}^{1,2} = -\sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \left( \partial^\beta (u_j A_{jk}) \partial^{\alpha-\beta} \partial_k u_i + \partial^\beta A_{jk} \partial^{\alpha-\beta} \partial_k p \right)\]

\[(3.1.6)\quad F_{i}^{1,3} = \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} A_{i\beta} \partial^\beta \partial_h (A_{jm} \partial_m u_j + A_{jm} \partial_m u_i)\]

\[(3.1.7)\quad F_{i}^{1,4} = \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} A_{i\beta} \partial_h (\partial^\beta A_{il} \partial^{\alpha-\beta} \partial_j u_j + \partial^\beta A_{il} \partial^{\alpha-\beta} \partial_j u_i)\]

\[(3.1.8)\quad F_{i}^{1,5} = \partial^\beta \partial_h \tilde{K} \partial_3 u_i\]

\[(3.1.9)\quad F_{i}^{1,6} = A_{i\beta} \partial_h (\partial^\beta A_{i\beta} \partial_j u_j + \partial^\beta A_{i\beta} \partial_j u_i)\]

\[(3.1.10)\quad F^2 = F^{2,1} + F^{2,2}\]

\[(3.1.11)\quad F^{2,1} = -\sum_{0<\beta<\alpha} C_{\alpha,\beta} \partial^\beta A_{i\beta} \partial^{\alpha-\beta} \partial_j u_i\]

\[(3.1.12)\quad F^{2,2} = -\partial^\alpha A_{i\beta} \partial_j u_i\]

\[(3.1.13)\quad F^3 = F^{3,1} + F^{3,2}\]

\[(3.1.14)\quad F^{3,1} = \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \partial^\beta D\eta (\partial^{\alpha-\beta} \eta - \partial^{\alpha-\beta} p)\]

\[(3.1.15)\quad F^{3,2} = \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \left( \partial^\beta (N_j A_{jm}) \partial^{\alpha-\beta} \partial_m u_j + \partial^\beta (N_j A_{jm}) \partial^{\alpha-\beta} \partial_m u_i \right)\]

\[(3.1.16)\quad F^4 = \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} \partial^\beta D\eta \cdot \partial^{\alpha-\beta} u\]

In all above, $C_{\alpha,\beta}$ represents constants depending on $\alpha$ and $\beta$.

### 3.1.2. Perturbed Linear Form

We may also take the system (1.3.8) as the perturbation of the linear system with constant coefficients.

\[
\begin{aligned}
\partial_t u - \Delta u + \nabla p &= G^1 & \text{in} & \Omega \\
\nabla \cdot u &= G^2 & \text{in} & \Omega \\
(p I - \nabla u - \eta I) e_3 &= G^3 & \text{on} & \Sigma \\
\partial_t \eta - u_3 &= G^4 & \text{on} & \Sigma \\
u &= 0 & \text{on} & \Sigma_b
\end{aligned}
\]

This form satisfies the following energy structure.

**Lemma 3.2.** Suppose $(u, p, \eta)$ satisfies system (3.1.17), then

\[(3.1.18)\quad \partial_t \left( \frac{1}{2} \int_{\Omega} |u|^2 + \frac{1}{2} \int_{\Sigma} |\eta|^2 \right) + \frac{1}{2} \int_{\Omega} J |Du|^2 = \int_{\Omega} u \cdot G^2 + p G^2 + \int_{\Sigma} -u \cdot G^3 + \eta G^4\]

Compare to the original equation (1.3.8), we can write the detailed form for the nonlinear terms.
Lemma 3.3. There exists a universal preliminary estimates.

3.1.3. Definition of Energy and Dissipation. In this section we will give the definition of energy and dissipation with or without minimum count. Compared to those in [2] (2.46-2.55), the main difference here lies in the full dissipation part, where we consider at most two vertical derivatives for velocity and one vertical derivative for pressure. It is this key improvement that largely simplifies the whole proof.

\begin{align}
G^1 &= G^{1,1} + G^{1,2} + G^{1,3} + G^{1,4} + G^{1,5} \\
G^{1,1}_i &= (\delta_{ij} - A_{ij}) \partial_i p \\
G^{1,2}_i &= \rho_j A_{jk} \partial_k u_i \\
G^{1,3}_i &= [K^2(1 + A^2 + B^2) - 1] \partial_{33} u_i - 2AK \partial_{33} u_i - 2BK \partial_{33} u_i \\
G^{1,4}_i &= [-K^3(1 + A^2 + B^2) \partial_3 J + AK^2(\partial_i J + \partial_3 A) \\
&\quad + BK^2(\partial_j J + \partial_3 B) - K(\partial_1 A + \partial_2 B)] \partial_3 u_i \\
G^{1,5}_i &= \partial_i \eta(1 + x_3/b)K \partial_3 u_3
\end{align}

\begin{align}
G^2 &= (1 - J)(\partial_1 u_1 + \partial_2 u_2) + A \partial_2 u_1 + B \partial_3 u_2
\end{align}

\begin{align}
G^3 &= \partial_1 \eta \left(\begin{array}{c}
p - \eta - 2(\partial_1 u_1 - AK \partial_3 u_1) \\
- \partial_2 u_1 - \partial_1 u_2 - BK \partial_3 u_1 + AK \partial_3 u_2 \\
- \partial_3 u_1 - K \partial_3 u_1 + AK \partial_3 u_2
\end{array}\right) + \partial_2 \eta \left(\begin{array}{c}
- \partial_1 u_1 - \partial_3 u_2 + BK \partial_3 u_1 + AK \partial_3 u_2 \\
p - \eta - 2(\partial_2 u_2 - BK \partial_3 u_2) \\
- \partial_2 u_3 - K \partial_3 u_2 + BK \partial_3 u_3
\end{array}\right) + \partial_3 \eta \left(\begin{array}{c}
(K - 1) \partial_3 u_1 + AK \partial_3 u_3 \\
(K - 1) \partial_3 u_2 + BK \partial_3 u_3 \\
2(K - 1) \partial_3 u_3
\end{array}\right)
\end{align}

where based on the equations we have that

\begin{align}
p - \eta &= \partial_1 \eta (-\partial_1 u_3 - K \partial_3 u_1 + AK \partial_3 u_3) + \partial_2 \eta (-\partial_2 u_3 - K \partial_3 u_2 + BK \partial_3 u_3) + 2K \partial_3 u_3
\end{align}

\begin{align}
G^4 &= -D \eta \cdot u
\end{align}

3.1.3. Preliminary Estimates. Here we record several preliminary lemmas in estimating.

Lemma 3.3. There exists a universal $0 < \delta < 1$ such that if $\|\eta\|_{H^{3/2}(\Sigma)}^2 \leq \delta$, then

\begin{align}
\|J - 1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \leq \frac{1}{2}
\end{align}

Proof. The same as lemma 2.4 in [2].

Lemma 3.4. For $i = 1, 2$, define $U_i : \Sigma \rightarrow \mathbb{R}$ by

\begin{align}
U_i(x') &= \int_{x_3}^{0} J(x', x_3) u_i(x', x_3) dx_3
\end{align}

Then $\partial_i \eta = -\partial_1 U_1 - \partial_2 U_2$ on $\Sigma$.

Proof. The same as lemma 2.5 in [2].

3.2. Definition of Energy and Dissipation. In this section we will give the definition of energy and dissipation with or without minimum count. Compared to those in [2] (2.46-2.55), the main difference here lies in the full dissipation part, where we consider at most two vertical derivatives for velocity and one vertical derivative for pressure. It is this key improvement that largely simplifies the whole proof.
3.2.1. Energy and Dissipation. We define the energy and dissipation with 2-minimum count as follows.

\[
\mathcal{E}_{n,2} = \|D_2^{2n-1}\|_{H^0}^2 + \|D_2D_2^{2n-1}\|_{H^0}^2 + \|\nabla \partial_t \|_{H^0}^2 + \|D_2^{2n}\|_{H^n(\Sigma)}^2
\]

\[
\mathcal{D}_{n,2} = \|D_2^{2n}\|_{H^0}^2
\]

We define the energy and dissipation without minimum count as follows, where \(I_\lambda u\) and \(I_\lambda \eta\) denote the Riesz potential of \(u\) and \(\eta\) as defined in (A.4.1) and (A.4.2).

\[
\mathcal{E}_n = \|I_\lambda u\|_{H^0}^2 + \|I_\lambda \eta\|_{H^0(\Sigma)} + \|D_0^{2n}u\|_{H^0}^2 + \|D_0^{2n}\eta\|_{H^0(\Sigma)}^2
\]

\[
\mathcal{D}_n = \|\mathbb{D}I_\lambda u\|_{H^0}^2 + \|\mathbb{D}D_0^{2n}u\|_{H^0}^2
\]

\[
\mathcal{E}_n = \|I_\lambda u\|_{H^0}^2 + \|I_\lambda \eta\|_{H^0(\Sigma)} + \sum_{j=0}^n \|\partial_t^j u\|_{H^{2n-2j}}^2 + \sum_{j=0}^{n-1} \|\partial_t^j p\|_{H^{2n-2j-1}}^2
\]

\[
\mathcal{D}_n = \|I_\lambda u\|_{H^1}^2 + \|\partial_t D_0^{2n-1}u\|_{H^2}^2 + \|\nabla \partial_t D_0^{2n-1} p\|_{H^0}^2
\]

\[
\mathcal{K} = \|\nabla u\|_{L^\infty}^2 + \|\nabla^2 u\|_{L^\infty}^2 + \sum_{i=1}^2 \|D_0u_i\|_{H^2(\Sigma)}^2
\]

\[
\mathcal{F}_{2N} = \|\eta\|_{H^{N+1/2}(\Sigma)}^2
\]

\[
\mathcal{G}_N(t) = \sup_{0 \leq r \leq t} \mathcal{E}_{2N}(r) + \sum_{m=1}^2 \frac{1}{1+r} \mathcal{E}_{N+1,2m}(r) + \sup_{0 \leq r \leq t} \frac{\mathcal{F}_{2N}(r)}{\eta(1+r)}
\]

Finally, we define the energy with 1-minimum count as follows.

\[
\mathcal{E}_{n,1} = \|Du\|_{H^{2n-1}}^2 + \sum_{j=1}^n \|\partial_t^j u\|_{H^{2n-2j}}^2 + \|Dp\|_{H^{2n-2}}^2 + \sum_{j=1}^{n-1} \|\partial_t^j p\|_{H^{2n-2j-1}}^2
\]

Since we only consider the decaying for energy and utilize the interpolation relation between \(\mathcal{E}_{n,2}\) and \(\mathcal{E}_{n,1}\) to estimate, it is not necessary to define the dissipation with 1-minimum count and the horizontal quantities.

3.2.2. Other Necessary Quantities. We define some useful quantities in the estimate as follows.

\[
\mathcal{E}_{2N} = \|\eta\|_{H^{N+1/2}(\Sigma)}^2
\]
3.2.3. Some Initial Estimates. Based on remark 2.6-2.8 in [2], we naturally have the following lemmas.

**Lemma 3.5.** \( \dot{\varepsilon}_{n,2} \) satisfies that

\[
\frac{1}{2} \left( |D^2 u|_H^2 + |D^2 \eta|_H^2 \right) \leq \dot{\varepsilon}_{n,2} \leq \frac{3}{2} \left( |D^2 u|_H^2 + |D^2 \eta|_H^2 \right)
\]

**Lemma 3.6.** For \( N \geq 4 \), we have \( \dot{\varepsilon}_{N+2,2} \lesssim \dot{\varepsilon}_{2N} \) and \( \mathcal{D}_{N+2,2} \lesssim \dot{\varepsilon}_{2N} \).

**Lemma 3.7.** We have the estimate \( \|I_3 \partial \eta\|_{H_N(\Sigma)}^2 \lesssim \|u\|_{H^r}^2 \lesssim \dot{\varepsilon}_{2N} \).

3.3. Interpolation Estimates.

3.3.1. Lower Interpolation Estimates. In the following, we will provide several interpolation estimates in the form of

\[
\begin{align*}
\|X\|^2 & \lesssim (\varepsilon_{N+2,2})^\theta (\varepsilon_{2N})^{1-\theta} \\
\|X\|^2 & \lesssim (\mathcal{D}_{N+2,2})^\theta (\varepsilon_{2N})^{1-\theta}
\end{align*}
\]

where \( \theta \in [0,1] \), \( X \) is some quantity, and \( ||:|| \) is some norm.

For brevity, we will write the interpolation power in the tables below. For example, the following part of the table

\[
\begin{array}{ccc}
L^2 & \dot{\varepsilon}_{N+2,2} & \mathcal{D}_{N+2,2} \\
X & \theta & \kappa
\end{array}
\]

should be understood as

\[
\begin{align*}
\|X\|^2 & \lesssim (\varepsilon_{N+2,2})^\theta (\varepsilon_{2N})^{1-\theta} \\
\|X\|^2 & \lesssim (\mathcal{D}_{N+2,2})^\theta (\varepsilon_{2N})^{1-\theta}
\end{align*}
\]

When we write \( \varepsilon_{N+2,2} \sim \mathcal{D}_{N+2,2} \) in a table, it means that \( \theta \) is the same when interpolating between \( \varepsilon_{N+2,2} \) and \( \varepsilon_{2N} \) and between \( \mathcal{D}_{N+2,2} \) and \( \varepsilon_{2N} \). When we write multiple entries, it means the same interpolation estimates hold for each item listed. Sometimes, an interpolation index \( r \) appears, we can always take arbitrary \( r \in (0,1) \).

Although most of the estimates are straightforward, some of them have more complicated structures and need further computation. The most common cases are as follows.

1. For \( 0 \leq \theta \leq \kappa \leq 1 \)

\[
\begin{equation}
\varepsilon_{N+2,2} \varepsilon_{2N}^{1-\theta} + \varepsilon_{N+2,2}^{1-\kappa} \leq \varepsilon_{N+2,2} \varepsilon_{2N}^{1-\theta} + \varepsilon_{N+2,2}^{1-\kappa} \\
\lesssim \varepsilon_{N+2,2} \varepsilon_{2N}^{1-\theta} + \varepsilon_{N+2,2}^{1-\kappa} \lesssim \varepsilon_{N+2,2} \varepsilon_{2N}^{1-\theta}
\end{equation}
\]

We may apply the same fashion for \( \varepsilon_{N+2,2} \) replaced by \( \mathcal{D}_{N+2,2} \).

2. For \( \theta_1 + \theta_2 \geq 1 \)

\[
\begin{equation}
\|X\|^2 \lesssim \varepsilon_{N+2,2} \varepsilon_{2N}^{1-\theta_1} + \varepsilon_{N+2,2}^{1-\theta_2} \lesssim \varepsilon_{N+2,2} \varepsilon_{2N}^{1-\theta_1} + \varepsilon_{2N}^{1-\theta_2} \lesssim \varepsilon_{N+2,2} \varepsilon_{2N}^{1-\theta_1} \lesssim \varepsilon_{N+2,2} \varepsilon_{2N}^{1-\theta_2}
\end{equation}
\]

where we utilize the bound \( \varepsilon_{2N} \leq 1 \).

With this notation explained, we can state some basic interpolation estimates now.

**Lemma 3.8.** The following tables encode the power in the \( L^\infty \) or \( L^2 \) interpolation estimates for \( u \) with its derivatives either in \( \Omega \) or \( \Sigma \).

\[
\begin{array}{ccc}
L^\infty & \varepsilon_{N+2,2} \sim \mathcal{D}_{N+2,2} & \lambda/(\lambda + 2) \\
u & \lambda/(\lambda + 2) & \lambda/(\lambda + 2) \\
Du & 2/(2 + r) & \lambda/(\lambda + 2) \\
\nabla u & 1/2 & \lambda/(\lambda + 2) \\
\nabla \nabla u & 2/(2 + r) & \lambda/(\lambda + 2) \\
\nabla^2 u & \lambda/(\lambda + 2) & \lambda/(\lambda + 2)
\end{array}
\]

\[
\begin{array}{ccc}
L^2 & \varepsilon_{N+2,2} \sim \mathcal{D}_{N+2,2} & \lambda/(\lambda + 2) \\
u & \lambda/(\lambda + 2) & \lambda/(\lambda + 2) \\
Du & 2/(2 + r) & \lambda/(\lambda + 2) \\
\nabla u & 1/2 & \lambda/(\lambda + 2) \\
\nabla \nabla u & 2/(2 + r) & \lambda/(\lambda + 2) \\
\nabla^2 u & \lambda/(\lambda + 2) & \lambda/(\lambda + 2)
\end{array}
\]
The following tables encode the power in the $L^\infty$ or $L^2$ interpolation estimates for $\eta$ and $\bar{\eta}$ with their derivatives either in $\Omega$ or $\Sigma$.

\[
\begin{array}{|c|c|c|}
\hline
L^\infty & \mathcal{E}_{N+2,2} & D_{N+2,2} \\
\hline
\eta, \bar{\eta} & (\lambda + 1)/(\lambda + 2) & (\lambda + 1)/(\lambda + 3) \\
D\eta, \nabla \eta & (\lambda + 2)/(\lambda + 2 + r) & (\lambda + 2)/(\lambda + 3) \\
D^2\eta, \nabla^2 \eta & 1 & (\lambda + 3)/(\lambda + 3 + r) \\
\partial_\eta \eta, \partial \bar{\eta} & 1 & 2/(2 + r) \\
\hline
\end{array}
\]

(3.3.9)

The following tables encode the power in the $L^\infty$ or $L^2$ interpolation estimates for $p$ with its derivatives either in $\Omega$ or $\Sigma$.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
L^\infty & \mathcal{E}_{N+2,2} & D_{N+2,2} & L^2 & \mathcal{E}_{N+2,2} & D_{N+2,2} \\
\hline
\nabla p & 1/2 & 1/2 & \nabla p & 0 & 0 \\
D p & 2/(2 + r) & 2/3 & D p & 1/2 & 1/3 \\
D \nabla p & 2/(2 + r) & 2/(2 + r) & D \nabla p & 1/2 & 1/2 \\
D^2 p & 1 & 2/(2 + r) & D^2 p & 1 & 2/3 \\
\hline
\end{array}
\]

(3.3.10)

The following tables encode the power in the $L^\infty$ or $L^2$ interpolation estimates for $G^i$.

\[
\begin{array}{|c|c|c|}
\hline
L^\infty & \mathcal{E}_{N+2,2} & D_{N+2,2} \\
\hline
G^1 & 1 & (3\lambda + 5)/(2\lambda + 6) \\
DG^1 & 1 & 1 \\
G^2 & 1 & 1 \\
DG^2 & 1 & 1 \\
\nabla G^2 & 1 & 1 \\
G^3 & 1 & (3\lambda + 5)/(2\lambda + 6) \\
DG^3 & 1 & 1 \\
G^4 & 1 & 1 \\
\hline
\end{array}
\]

(3.3.11)

for

\[
m_1 = \min\{(2\lambda^2 + 6\lambda + 2)/(\lambda^2 + 5\lambda + 6), (3\lambda + 3)/(2\lambda + 6)\} \\
m_2 = \min\{(2\lambda^2 + \lambda + 4)/(\lambda^2 + 5\lambda + 6), (3\lambda + 5)/(2\lambda + 6)\}
\]

Proof. The estimate follows directly from the Sobolev embedding and lemma A.11-A.13, using the bounds $\|I_\lambda \eta\|_{H^0}^2 \lesssim \mathcal{E}_{2N}$ and $\|I_\lambda \partial_\eta \eta\|_{H^0}^2 \lesssim \mathcal{E}_{2N}$ which is a natural corollary of lemma 3.7. For the terms of $G^i$, we may utilize the estimating method for complicated structures stated at the beginning of this section and the natural product rule

(3.3.12) $\|XY\|_{L^\infty}^2 \lesssim \|X\|_{L^\infty}^2 \|Y\|_{L^\infty}^2$

(3.3.13) $\|XY\|_{L^2}^2 \lesssim \|X\|_{L^2}^2 \|Y\|_{L^2}^2$

(3.3.14) $\|XY\|_{L^2}^2 \lesssim \|X\|_{L^\infty}^2 \|Y\|_{L^\infty}^2$

where in $L^2$ bound, we may take the larger value of $\theta$ produced by the two options. □
Then we can show several important lemmas which greatly improve the estimates above and will play the key role in the a priori estimates.

**Lemma 3.9.** We have the estimate

\[ \mathcal{K} \lesssim \mathcal{E}_{2N}^{r/(2+r)} \mathcal{E}_{N+2}^{2/(2+r)} \]

*Proof.*

\[ \mathcal{K} = \| \nabla u \|_{L^\infty}^2 + \| \nabla^2 u \|_{L^\infty}^2 + \sum_{i=1}^2 \| D u_i \|_{H^2(\Sigma)}^2 \]

We need to estimate each term above.

1. \( \| \nabla u \|_{L^\infty}^2 \):

By the definition, we have

\[ \| \nabla u \|_{L^\infty}^2 \lesssim \| D u \|_{L^\infty}^2 + \| \partial_3 u \|_{L^\infty}^2 \]

By the divergence equation, we have

\[ \| \partial_3 u_i \|_{L^\infty(\Sigma)}^2 \lesssim \| D u_i \|_{L^\infty(\Sigma)}^2 + \| G^3 \|_{L^\infty(\Sigma)}^2 \]

For \( i = 1, 2 \), by Poincare, we can naturally estimate

\[ \| \partial_3 u_i \|_{L^\infty(\Sigma)}^2 \lesssim \| \partial_3^2 u_i \|_{L^\infty(\Sigma)}^2 + \| \partial_3 u_i \|_{L^\infty(\Sigma)}^2 \]

The upper boundary condition implies that

\[ \| \partial_3 u_i \|_{L^\infty(\Sigma)}^2 \lesssim \| D u_i \|_{L^\infty(\Sigma)}^2 + \| G^3 \|_{L^\infty(\Sigma)}^2 \lesssim \| D \nabla u_i \|_{L^\infty(\Sigma)}^2 + \| G^3 \|_{L^\infty(\Sigma)}^2 \]

Considering the Navier-Stokes equation, we have

\[ \| \partial_3^2 u_i \|_{L^\infty}^2 \lesssim \| \partial_3 u_i \|_{L^\infty}^2 + \| D^2 u_i \|_{L^\infty}^2 + \| D p \|_{L^\infty}^2 \]

Therefore, to summarize all above

\[ \| \nabla u \|_{L^\infty}^2 \lesssim \| D u \|_{L^\infty}^2 + \| G^2 \|_{L^\infty}^2 + \| D \nabla u_i \|_{L^\infty(\Sigma)}^2 + \| G^3 \|_{L^\infty(\Sigma)}^2 + \| \partial_3 u_i \|_{L^\infty(\Sigma)}^2 + \| D u_i \|_{L^\infty}^2 + \| D p \|_{L^\infty}^2 \]

The estimates of each term above with lemma 3.8 satisfy our requirement

2. \( \| \nabla^2 u \|_{L^\infty}^2 \):

By the definition, we have

\[ \| \nabla^2 u \|_{L^\infty}^2 \lesssim \| D \nabla u \|_{L^\infty}^2 + \| \partial_3^2 u_i \|_{L^\infty}^2 \]

The first term naturally satisfies our requirement while the second term has been successfully estimated in the first step above. Hence, we only need to estimate the third one. Taking \( \partial_3 \) in the divergence equation reveals

\[ \| \partial_3^2 u_i \|_{L^\infty}^2 \lesssim \| D \nabla u_i \|_{L^\infty}^2 + \| \nabla G^2 \|_{L^\infty}^2 \]

where all the terms can be easily estimated via lemma 3.8.

3. \( \sum_{i=1}^2 \| D u_i \|_{H^2(\Sigma)}^2 \):

By trace theorem and Poincare lemma, we have

\[ \sum_{i=1}^2 \| D u_i \|_{H^2(\Sigma)}^2 \lesssim \| D^3 \nabla u_i \|_{H^0}^2 + \| D^2 \nabla u_i \|_{H^0}^2 + \| D u_i \|_{H^1}^2 \]

The first and second term are naturally estimated, so we only need to focus on the last one. By Poincare, we have

\[ \| D u_i \|_{H^1} \lesssim \| D \partial_3 u_i \|_{H^0}^2 + \| D^2 u_i \|_{H^0}^2 \]

and also

\[ \| D \partial_3 u_i \|_{H^0}^2 \lesssim \| D \partial_3^2 u_i \|_{H^0}^2 + \| D \partial_3 u_i \|_{H^1}^2 \]
The boundary condition shows that
\[(3.3.28) \quad ||D\partial_3 u_i||^2_{H^p(\Sigma)} \lesssim ||D^2 u_3||^2_{H^p(\Sigma)}^2 + ||DG^3||^2_{H^p(\Sigma)} \lesssim ||D^2 \partial_3 u_3||^2_{H^p(\Sigma)} + ||DG^3||^2_{H^p(\Sigma)}\]

By Navier-Stokes equation, we have
\[(3.3.29) \quad ||D\partial_3^2 u_i||^2_{H^p(\Sigma)} \lesssim ||D\partial_3 u_i||^2_{H^p(\Sigma)} + ||D^3 u_i||^2_{H^p(\Sigma)} + ||D^2 p||^2_{H^p(\Sigma)} + ||DG^1||^2_{H^p(\Sigma)}\]

Hence, in total, we have
\[(3.3.30) \quad \sum_{i=1}^2 ||Du_i||^2_{H^2} \lesssim ||D^2 \nabla u_i||^2_{H^p(\Sigma)} + ||Du_i||^2_{H^p(\Sigma)} + ||D^3 \partial_3 u_3||^2_{H^p(\Sigma)} + ||DG^3||^2_{H^p(\Sigma)} + ||D\partial_3 u_i||^2_{H^p(\Sigma)} + ||D^2 p||^2_{H^p(\Sigma)} + ||DG^1||^2_{H^p(\Sigma)}\]

The estimate of each term above with lemma 3.8 satisfies our requirement, so we finish this proof. □

**Lemma 3.10.** We have the estimate
\[(3.3.31) \quad \tilde{\mathcal{E}}_{N+2,2} \lesssim \mathcal{E}_{2N}^{1/(\lambda+3)} D_{N+2,2}^{(\lambda+2)/(\lambda+3)}\]

**Proof.** The dissipation \(\mathcal{D}_{N+2,2}\) can control all the terms in \(\tilde{\mathcal{E}}_{N+2,2}\) except that \(\partial_3 \eta, D^2 \eta\) and \(D^{2(N+2)} \eta\). A direct application of lemma A.10 reveals that
\[(3.3.32) \quad ||D^{2(N+2)} \eta||^2_{H^p(\Sigma)} \lesssim \mathcal{E}_{2N}^{1/(4N-\delta)} D_{N+2,2}^{(4N-\delta)/(4N-\delta)}\]

Since \(N \geq 3\), above estimate satisfies our requirement. Also lemma 3.8 reveals that the estimate of \(D^2 \eta\) will not be an obstacle. Hence, the dominating term is \(\partial_3 \eta\). By transport equation, we have
\[(3.3.33) \quad ||\partial_3 \eta||^2_{H^p(\Sigma)} \lesssim ||u_3||^2_{H^p(\Sigma)} + ||G^4||^2_{H^p(\Sigma)}\]

By Poincare, we can estimate
\[(3.3.34) \quad ||u_3||^2_{H^p(\Sigma)} \lesssim ||\partial_3 u_3||^2_{H^p(\Sigma)}\]

while the divergence equation and Poincare imply that
\[(3.3.35) \quad ||\partial_3 u_3||^2_{H^p(\Sigma)} \lesssim ||Du_3||^2_{H^p(\Sigma)} + ||G^2||^2_{H^p(\Sigma)} \lesssim ||D\partial_3 u_3||^2_{H^p(\Sigma)} + ||G^2||^2_{H^p(\Sigma)}\]

Similar to the third step in the proof of lemma 3.9, by Poincare, we have
\[(3.3.36) \quad ||D\partial_3 u_3||^2_{H^p(\Sigma)} \lesssim ||D\partial_3^2 u_3||^2_{H^p(\Sigma)} + ||D\partial_3 u_3||^2_{H^p(\Sigma)}\]

The boundary condition shows that
\[(3.3.37) \quad ||D\partial_3 u_3||^2_{H^p(\Sigma)} \lesssim ||D^2 u_3||^2_{H^p(\Sigma)} + ||DG^3||^2_{H^p(\Sigma)} \lesssim ||D^2 \partial_3 u_3||^2_{H^p(\Sigma)} + ||DG^3||^2_{H^p(\Sigma)}\]

By Navier-Stokes equation, we have
\[(3.3.38) \quad ||D\partial_3^2 u_i||^2_{H^p(\Sigma)} \lesssim ||D\partial_3 u_i||^2_{H^p(\Sigma)} + ||D^3 u_i||^2_{H^p(\Sigma)} + ||D^2 p||^2_{H^p(\Sigma)} + ||DG^1||^2_{H^p(\Sigma)}\]

where \(D^2 p\) dominates. Utilizing Poincare, we have
\[(3.3.39) \quad ||D^2 p||^2_{H^p(\Sigma)} \lesssim ||D^2 \partial_3 p||^2_{H^p(\Sigma)} + ||D^2 p||^2_{H^p(\Sigma)}\]

in which we may again use the boundary condition to estimate
\[(3.3.40) \quad ||D^2 p||^2_{H^p(\Sigma)} \lesssim ||D^2 \eta||^2_{H^p(\Sigma)} + ||D^2 \partial_3 u_3||^2_{H^p(\Sigma)} + ||D^2 G^1||^2_{H^p(\Sigma)}\]

Hence, in total, we can estimate that
\[(3.3.41) \quad ||\partial_3 \eta||^2_{H^p(\Sigma)} \lesssim \quad \text{[Expressions not shown due to page layout issues]}\]

We may easily check via lemma 3.8 that each term above satisfies our requirement, so we finish this proof. □
Lemma 3.11. We have the estimate
\[ \mathcal{E}_{N+2,1} \lesssim \mathcal{E}_{N+2,2}^{(\lambda+1)/(\lambda+2)} \mathcal{E}_{2N}^{1/(\lambda+2)} \]

Proof. We only need to estimate the lower order terms which merely appears in \( \mathcal{E}_{N+2,1} \), which are \( \partial_t \mathcal{P} \) and \( \partial_t \mathcal{Q} \). Based on the lemma 3.8, the estimates of \( \partial_t \mathcal{P} \) and \( \partial_t \mathcal{Q} \) have satisfied our requirement, so we focus on \( \partial_t \mathcal{P} \). By a similar argument as lemma 3.10, we have that
\[ \|D\mathcal{P}\|_{H^p}^2 \lesssim \|D\partial_t \mathcal{P}\|_{H^p}^2 + \|D\partial_t \mathcal{Q}\|_{H^p}^2 \]
\[ \lesssim \|D\mathcal{Q}\|_{H^p}^2 + \|D\partial_t \mathcal{Q}\|_{H^p}^2 + \|\partial_t^2 \mathcal{Q}\|_{H^p}^2 + \|D\Delta \mathcal{Q}\|_{H^p}^2 + \|D\mathcal{Q}^2\|_{H^p}^2 \]
where each term can be estimated directly through lemma 3.8. Then our result easily follows. \( \square \)

3.3.2. Higher Interpolation Estimates. Then we need to give several estimates for the higher order terms.

Lemma 3.12. We have the estimate
\[ \|D^{2N+4}\|_{H^{1/2}(\Sigma)}^2 + \|\nabla^{2N+5}\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^{2/(4N-7)} \mathcal{D}_{N+2}^{(4N-9)/(4N-7)} \]

Proof. See lemma 3.18 in [2]. \( \square \)

Lemma 3.13. We have the estimate
\[ \|u\|_{H^{1/2}(\Sigma)}^2 + \|\nabla u\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{D}_{N+2}^{1/3} \]

Proof. Trace theorem implies that
\[ \|u\|_{H^{1/2}(\Sigma)}^2 + \|\nabla u\|_{H^{1/2}(\Sigma)}^2 \lesssim \|\nabla u\|_{H^p}^2 + \|\nabla^2 u\|_{H^p}^2 + \|D^2 \nabla u\|_{H^p}^2 + \|D^2 \nabla^2 u\|_{H^p}^2 \]
The last two terms naturally satisfy our requirement, so we concentrate on the first two. Similar to the argument in lemma 3.9, we have
\[ \|\nabla u\|_{H^p}^2 + \|\nabla^2 u\|_{H^p}^2 \lesssim \|D\mathcal{P}\|_{H^p}^2 + \|\mathcal{Q}\|_{H^p}^2 + \|D\partial_t \mathcal{Q}\|_{H^p}^2 + \|\partial_t^2 \mathcal{Q}\|_{H^p}^2 \]
\[ + \|\mathcal{Q}\|_{H^p}^2 + \|D\partial_t \mathcal{Q}\|_{H^p}^2 + \|D\partial_t \mathcal{P}\|_{H^p}^2 + \|D\partial_t \mathcal{Q}\|_{H^p}^2 \]
The estimate of each term on the right hand side easily lead to our result. \( \square \)

Lemma 3.14. Let \( P = P(K, D\eta) \) be a polynomial of \( K \) and \( D\eta \). Then there exists a \( \theta > 0 \) such that
\[ \| (D^{2N+4}) u \|_{H^{1/2}(\Sigma)}^2 + \| (D^{2N+4}) \nabla u \|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2} \]
Let \( Q = Q(K, \theta, \nabla \eta) \) be a polynomial. Then there exists a \( \theta > 0 \), such that
\[ \| (\nabla^{2N+5}) Q \nabla u \|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,2} \]

Proof. This lemma is identical to lemma 3.19 in [2]. The above two lemmas implies that all the results used in proving lemma 3.19 have been recovered now, so this lemma is still valid. \( \square \)

3.4. Nonlinear Estimates.

3.4.1. Estimates of Perturbation Terms. Now we will give several estimates for the nonlinear terms in perturbed linear form. Similar to theorem 4.1 and 4.2 in [2], the estimating methods for these terms are quite standard, so we will merely give the basic rules in estimating:

1. Since all the terms are quadratic or of higher order, then we may expand the differential operators with Leibniz rule and estimate resulting quadratic terms by the basic interpolation estimate in lemma 3.8 either in \( L^\infty \) or \( L^2 \) norm, combining with the definition of energy and dissipation, trace theorem and Sobolev embedding. For the choices of form (3.3.13) or (3.3.14), we should always take the higher interpolation index for \( \mathcal{E}_{N+2,2} \) and \( \mathcal{D}_{N+2,2} \).

2. For the \( 2N \) level, there is no minimum count for the derivatives, hence we only need to refer to the definition of energy and dissipation; however, in \( N + 2 \) level, we have to resort to lemma 3.8 for terms that cannot be estimated directly. Also, lemma 3.14 can be utilized for more complicated terms.

3. Note that the most important difference between our proof and that of theorem 4.1 and 4.2 in [2] is that we concentrate on the horizontal derivatives instead of the full derivatives. Thereafter, in the estimates related to \( \mathcal{D}_{N+2,2} \), we should try to avoid the introduction of vertical derivatives via Sobolev embedding, especially for pressure \( p \).
For the term in form of $\|X\|_{H^2(\Sigma)}$, instead of using trace theorem directly, we should first write it into $\|X\|_{H^0(\Sigma)} + \|DX\|_{H^0(\Sigma)} + \|D^2X\|_{H^0(\Sigma)}$ and then apply trace theorem, which will only produce one vertical derivatives rather than three.

Considering that the proofs for the following estimates are lengthy but not difficult based on the above rules, and also most of them have been achieved in that of theorem 4.1 and 4.2 of [2], we will omit the details here and only present the theorems.

**Lemma 3.15.** There exists a $\theta > 0$ such that

\[
\left\| D^2 \nabla_0^{2(N+2)-4} G^1 \right\|^2_{H^0} + \left\| D^2 \nabla_0^{2(N+2)-4} G^2 \right\|^2_{H^0} + \left\| D^2 \nabla_0^{2(N+2)-4} G^3 \right\|^2_{H^{1/2}(\Sigma)} \lesssim \mathcal{E}^\theta_{2N} \mathcal{E}_{N+2,2}
\]

and

\[
\left\| D_2^{2(N+2)-1} G^1 \right\|^2_{H^0} + \left\| D_2^{2(N+2)-1} G^2 \right\|^2_{H^0} + \left\| D_2^{2(N+2)-1} G^3 \right\|^2_{H^{1/2}(\Sigma)} \lesssim \mathcal{E}^\theta_{2N} \mathcal{D}_{N+2,2}
\]

**Lemma 3.16.** There exists a $\theta > 0$ such that

\[
\left\| \nabla_0^{4N-2} G^1 \right\|^2_{H^0} + \left\| \nabla_0^{4N-1} G^2 \right\|^2_{H^0} + \left\| \nabla_0^{4N-2} G^3 \right\|^2_{H^{1/2}(\Sigma)} \lesssim \mathcal{E}^{1+\theta}_{2N}
\]

and

\[
\left\| D_0^{4N-1} G^1 \right\|^2_{H^0} + \left\| D_0^{4N-1} G^2 \right\|^2_{H^0} + \left\| D_0^{4N-1} G^3 \right\|^2_{H^{1/2}(\Sigma)} \lesssim \mathcal{E}^\theta_{2N} \mathcal{D}_{2N} + K \mathcal{F}_{2N}
\]

### 3.4.2. Estimates of Other Nonlinearities

Here is a few lemmas for some other nonlinear terms which have been proved in proposition 4.3-4.5 of [2]. Although our definition of energy and dissipation is a little bit different, we do not even need to revise the proof here.

**Lemma 3.17.** We have that

\[
\left\| I_\lambda G^1 \right\|^2_{H^1} + \left\| I_\lambda G^2 \right\|^2_{H^2} + \left\| I_\lambda \partial_t G^2 \right\|^2_{H^0} \lesssim \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}
\]

and

\[
\left\| I_\lambda G^3 \right\|^2_{H^1(\Sigma)} + \left\| I_\lambda G^4 \right\|^2_{H^1(\Sigma)} \lesssim \mathcal{E}_{2N} \min\{\mathcal{E}_{2N}, \mathcal{D}_{2N}\}
\]

Also

\[
\left\| I_\lambda G^4 \right\|^2_{H^0(\Sigma)} \lesssim \mathcal{D}^2_{2N}
\]

**Lemma 3.18.** It holds that

\[
\left\| I_\lambda [(AK) \partial_3 u_1 + (BK) \partial_3 u_2] \right\|^2_{H^0} + \sum_{i=1}^2 \left\| I_\lambda [u \partial_i K] \right\|^2_{H^0} \lesssim \mathcal{D}^2_{2N}
\]

and

\[
\left\| I_\lambda [(1 - K) u] \right\|^2_{H^0} \lesssim \mathcal{E}^{1/(1+\lambda)}_{2N} \mathcal{D}^{(1+2\lambda)/(1+\lambda)}_{2N}
\]

Also

\[
\left\| I_\lambda [(1 - K) G^2] \right\|^2_{H^0} \lesssim \mathcal{E}_{2N} \mathcal{D}^2_{2N}
\]

**Lemma 3.19.** We have that

\[
\left\| \partial_t^{2N+1} A \right\|^2_{H^0} \lesssim \mathcal{D}_{2N}
\]

and

\[
\left\| \partial_t^{N+3} A \right\|^2_{H^0} \lesssim \mathcal{D}_{N+2,2}
\]
3.5. Energy Estimates. In this section, we will present the energy estimate for horizontal derivatives. Since this part has been carefully completed in section 5 of [2] and is not the main improvement of our new proof, it is not necessary for us to show all the details. Hence, we will not give all the details of the proofs and only comment when we need different technique to achieve the estimates due to the changing of definition of energy and dissipation.

3.5.1. Estimates of Highest and Lowest Temporal Derivatives.

Lemma 3.20. Let $\partial^0 = \partial_t^2 N$ and let $F^i$ be defined as in geometric structure form. Then

$$\| F^1 \|_{H^0}^2 + \| \partial_t(F^2) \|_{H^0}^2 + \| F^3 \|_{H^0(\Sigma)}^2 + \| F^4 \|_{H^0(\Sigma)}^2 \lessgtr \mathcal{E}_{2N} \mathcal{D}_{2N}$$

Proof. This is identical to theorem 5.1 in [2], so we omit the proof here. \qed

Lemma 3.21. Let $\partial^0 = \partial_t^{N+2}$ and let $F^i$ be defined as in geometric structure form. Then

$$\| F^1 \|_{H^0}^2 + \| \partial_t(F^2) \|_{H^0}^2 + \| F^3 \|_{H^0(\Sigma)}^2 + \| F^4 \|_{H^0(\Sigma)}^2 \lessgtr \mathcal{E}_{2N} \mathcal{D}_{N+2}$$

Also, if $N \geq 3$, then there exists a $\theta > 0$ such that

$$\| F^2 \|_{H^0}^2 \lessgtr \mathcal{E}_N^\theta \mathcal{E}_{N+2}$$

Proof. This is a revised version of theorem 5.2 in [2]. The first part is exactly the same as the original theorem, so it is naturally valid. However, we will utilize different techniques in the second part. By the definition, $F^2 = F^{2,1} + F^{2,2}$. By the same argument as (5.5) in [2], we have

$$(3.5.4) \quad \| F^{2,1} \|_{H^0}^2 \lessgtr \mathcal{E}_{2N}^\theta \mathcal{E}_{N+2}$$

For $F^{2,2}$, a calculation reveals that

$$(3.5.5) \quad F^{2,2} = \partial_t^{N+2}(\partial_t \eta \tilde{b} K) \partial_3 u_1 + \partial_t^{N+2}(\partial_2 \eta \tilde{b} K) \partial_3 u_2 - \partial_t^{N+2} K \partial_3 u_3$$

We can estimate

$$(3.5.6) \quad \| \partial_t^{N+2}(\partial_t \eta \tilde{b} K) \partial_3 u_1 \|_{H^0}^2 \lessgtr \| \partial_t^{N+2}(\partial_t \eta \tilde{b} K) \|_{H^0}^2 \| \partial_3 u_1 \|_{L^\infty}^2$$

Lemma 3.9 implies that

$$(3.5.7) \quad \| \partial_3 u_1 \|_{L^\infty} \lesssim \| \nabla u \|_{L^\infty} \lesssim \mathcal{E}_N^{r/(2+r)} \mathcal{E}_N^{2/(2+r)}$$

Also, we have that for $0 \leq |\alpha| \leq N + 2$, there exists a $\theta > 0$ such that

$$(3.5.8) \quad \| \partial_3^\alpha \eta \|_{H^{1/2}(\Sigma)} \lesssim \| \partial_3^\alpha \eta \|_{H^0(\Sigma)} \| \partial_3^\alpha \eta \|_{H^1(\Sigma)} \lesssim \mathcal{E}_N^{1-\theta} \mathcal{E}_N^{\theta}$$

Hence we have

$$(3.5.9) \quad \| \partial_t^{N+2}(\partial_t \eta \tilde{b} K) \|_{H^0}^2 \lesssim \| \partial_3^\alpha \partial_t \eta \|_{H^0}^2 \| \partial_t^{N+2-\alpha}(\tilde{b} K) \|_{L^\infty} \lesssim \mathcal{E}_N^{1-\theta} \mathcal{E}_N^{\theta}$$

Then when $r$ is sufficiently small, we can naturally estimate

$$(3.5.10) \quad \| \partial_t^{N+2}(\partial_t \eta \tilde{b} K) \partial_3 u_1 \|_{H^0}^2 \lesssim \mathcal{E}_N^{\theta} \mathcal{E}_N^{N+2,2}$$

Similarly, we have

$$(3.5.11) \quad \| \partial_t^{N+2}(\partial_2 \eta \tilde{b} K) \partial_3 u_2 \|_{H^0}^2 \lesssim \mathcal{E}_N^{\theta} \mathcal{E}_N^{N+2,2}$$

The same argument as (5.11) in [2] shows that

$$(3.5.12) \quad \| \partial_t^{N+2} K \partial_3 u_3 \|_{H^0}^2 \lesssim \mathcal{E}_N^{\theta} \mathcal{E}_N^{N+2,2}$$

Therefore, our result easily follows. \qed

The following three lemmas is just the restatement of proposition 5.3-5.5 in [2], then we will only present the lemmas without the proofs.

Lemma 3.22. There exists a $\theta > 0$ such that

$$(3.5.13) \quad \| \partial_t^{2N} u(t) \|_{H^0}^2 + \| \partial_t^{2N} \eta(t) \|_{H^0(\Sigma)}^2 + \int_0^t \| \partial_t^{2N} u(t) \|_{H^0}^2 \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t \mathcal{E}_{2N}^\theta \mathcal{D}_{2N}$$
Lemma 3.23. Let $F^2$ be given with $\partial_\alpha^p = \partial_\alpha^{N+2}$. Then it holds that
\[
(3.5.14) \quad \partial_t \left( \| \sqrt{J} \partial_\alpha^{N+2} u \|^2_{H^0} + \| \partial_\alpha^{N+2} \eta \|^2_{H^0(\Sigma)} - 2 \int_\Omega J \partial_\alpha^{N+2} p F^2 \right) + \| \nabla \partial_\alpha^{N+2} u \|^2_{H^0} \lesssim \sqrt{e_2 N} \mathcal{D} N + 2.2
\]

Lemma 3.24. It holds that
\[
(3.5.15) \quad \| u(t) \|^2_{H^0} + \| \eta(t) \|^2_{H^0(\Sigma)} + \int_0^t \| \nabla u \|^2_{H^0} \lesssim \epsilon_2 N(0) + \int_0^t \sqrt{e_2 N} \mathcal{D} N_2 N
\]

3.5.2. Estimates of Mixed Derivatives.

Lemma 3.25. Let $\alpha \in \mathbb{N}^2$ be such that $|\alpha| = 4N$. Then
\[
(3.5.16) \quad \left\| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \right\| \lesssim \sqrt{e_2 N} \mathcal{D} N_2 N + \sqrt{D_2 N} \mathcal{K} F_2 N
\]

Proof. Essentially, this lemma is completely identical to lemma 6.1 in [2]. We only need to notice that although our definition of energy and dissipation is different from that in [2], the terms related to $\eta$ is exactly the same. Also we should refer to the general rules for estimating in the section of nonlinear estimates to deal with $\| X \|_{H^0(\Sigma)}$ form. Hence, this lemma is still valid now. \hfill \qed

Lemma 3.26. Suppose that $\alpha \in \mathbb{N}^{1+2}$ is such that $\alpha_0 \leq 2N - 1$ and $1 \leq |\alpha| \leq 4N$. Then there exists a $\theta > 0$ such that
\[
(3.5.17) \quad \| D^{1+1}_1 u \|^2_{H^0} + \| D^2 u \|^2_{H^0(\Sigma)} + \| D^2 u \|^2_{H^0(\Sigma)} + \| D^2 u \|^2_{H^0(\Sigma)} + \| D^2 \eta \|^2_{H^0(\Sigma)}
\]

\begin{align*}
+ \int_0^t \| D^{1+1}_1 \nabla u \|^2_{H^0} + \| D^2 \nabla u \|^2_{H^0} & \lesssim \epsilon_2 N(0) + \int_0^t \epsilon_2 N \mathcal{D} N_2 N + \sqrt{D_2 N} \mathcal{K} F_2 N\n\end{align*}

Proof. This lemma is identical to proposition 6.2 in [2], and the proof is almost the same. We only need to notice that now we should utilize lemma 3.16 to estimate the nonlinear terms to replace lemma 4.2 in [2]. Hence, the result still holds. \hfill \qed

Lemma 3.27. We have the estimate
\[
(3.5.18) \quad \| D^{N+3} G^4 \|^2_{H^{1+2}(\Sigma)} \lesssim \mathcal{D}^{1+2/(4N-7)} N_{+2,2}
\]

Also, there exists a $\theta > 0$ such that
\[
(3.5.19) \quad \| D^2 G^4 \|^2_{H^0(\Sigma)} \lesssim \epsilon_2 N \mathcal{D}^{1+1/(\lambda+3)}_{N+2,2}
\]

and
\[
(3.5.20) \quad \| D^2 G^4 \|^2_{H^1} \lesssim \epsilon_2 N \mathcal{D}^{1+1/(\lambda+3)}_{N+2,2}
\]

Proof. This is basically lemma 6.3 in [2], the only difference is that we need to estimate $\| \nabla u \|_{H^0}$ directly now. By a similar argument as in lemma 3.9, we have
\[
(3.5.21) \quad \| \nabla u \|^2_{H^0} \lesssim \| D u \|^2_{H^0} + \| G \|^2_{H^0} + \| D \nabla u \|^2_{H^0} + \| \partial_\alpha u \|^2_{H^0} + \| D^2 u \|^2_{H^0}
\]

Hence, an application of lemma 3.8 implies that $\| \nabla u \|^2_{H^0} \lesssim \mathcal{D}^{(\lambda+1)/(\lambda+3)} N_{+2,2}$. Using exactly the same argument for the other part of the proof, we can see the validity of this lemma. \hfill \qed

Lemma 3.28. Suppose that $\alpha \in \mathbb{N}^{1+2}$ is such that $\alpha_0 \leq N + 1$ and $2 \leq |\alpha| \leq 2(N + 2)$. Then there exists a $\theta > 0$ such that
\[
(3.5.22) \quad \partial_t \left( \| D^{2+3} u \|^2_{H^0} + \| D^2 u \|^2_{H^0} + \| D^2 \nabla u \|^2_{H^0} + \| \partial_\alpha u \|^2_{H^0} + \| D^2 u \|^2_{H^0} \right)
\]

\begin{align*}
+ \| \nabla u \|^2_{H^0} + \| D^2 u \|^2_{H^0} & \lesssim \epsilon_2 N \mathcal{D} N_{+2,2}\n\end{align*}

Proof. This is almost the same as proposition 6.4 in [2], so we omit the detailed proof here. We only need to note that for higher derivatives, we should use lemma 3.15 to replace lemma 4.1 in [2] to estimate nonlinear terms. Then for lower derivatives, we should utilize lemma 3.8, 3.10 and 3.27 to replace lemma 3.1, 3.16 and 6.3 in [2], which share exactly the same results. Therefore, the conclusion naturally holds. \hfill \qed
3.5.3. Estimates of Riesz Potentials. In the following, we will record several lemmas related to Riesz potentials, which have been proved in lemma 6.5-6.7 in [2]. Note that all the preliminary lemmas have been recovered in lemma 3.17 and 3.18, so we do not even need to revise the proofs.

Lemma 3.29. It holds that
\begin{align}
\|I_N \rho\|_{H^0}^2 & \lesssim \mathcal{E}_2 N \\
\|I_N \partial_t \rho\|_{H^0}^2 & \lesssim \mathcal{E}_2 N^{1/(1+\lambda)}
\end{align}

Lemma 3.30. It holds that
\begin{align}
\int_{\Omega} I_N \partial_t \rho G^2 \lesssim \sqrt{\mathcal{E}_2 N} D_{2N}
\end{align}

Lemma 3.31. It holds that
\begin{align}
\|I_N u\|_{H^0}^2 + \|I_N \eta\|_{H^0(S)}^2 + \int_0^t \|\mathbb{D} I_N u\|_{H^0}^2 \leq \mathcal{E}_2 N(0) + \int_0^t \sqrt{\mathcal{E}_2 N} D_{2N}
\end{align}

3.5.4. Synthesis of Energy Estimates. Assembling lemma 3.22, 3.23, 3.24, 3.26, 3.28 and 3.31, we have the following energy estimate.

Proposition 3.32. Let \( F^2 \) be given with \( \partial_t^{N+2} \). Then there exists a \( \theta > 0 \) such that
\begin{align}
\partial_t \left( \mathcal{E}_{N+2,2} - 2 \int_{\Omega} \mathcal{D}_{N+2,2} \leq \mathcal{E}_2 N \right) + \mathcal{D}_{N+2,2} \lesssim \mathcal{E}_2 N D_{N+2,2}
\end{align}

Proposition 3.33. There exists a \( \theta > 0 \) such that
\begin{align}
\mathcal{E}_2 N(t) + \int_0^t \mathcal{D}_{2N}(r) dr \leq \mathcal{E}_2 N(0) + \mathcal{E}_2 N(t)^{3/2} + \int_0^t \mathcal{E}_2 N(r)^{\theta} D_{2N} dr \\
+ \int_0^t \sqrt{\mathcal{D}_{2N}(r) K(r) \mathcal{F}_{2N}(r)} dr
\end{align}

3.6. Comparison Estimates. In this section, we will prove several comparison results between horizontal derivatives and full derivatives, both in energy and dissipation.

3.6.1. Dissipation Comparison Estimates.

Proposition 3.34. There exists a \( \theta > 0 \) such that
\begin{align}
\mathcal{D}_{N+2,2} \lesssim \mathcal{D}_{N+2,2} + \mathcal{E}_2 N \mathcal{D}_{N+2,2}
\end{align}

Proof. We define
\begin{align}
\mathcal{Y}_{N+2,2} = \left\| \mathcal{D}_2^{2(N+2)-1} G \right\|_{H^0}^2 + \left\| \mathcal{D}_2^{2(N+2)-1} G \right\|_{H^1}^2 + \left\| \mathcal{D}_2^{2(N+2)-1} G \right\|_{H^{1/2}(\Sigma)}^2 + 2 \left\| \mathcal{D}_1^{2(N+2)-1} G \right\|_{H^{1/2}(\Sigma)}^2
\end{align}

and
\begin{align}
\mathcal{Z} = \mathcal{D}_{N+2,2} + \mathcal{Y}_{N+2,2}
\end{align}

Then the proof is divided into several steps.

Step 1: Application of Korn’s inequality.
Since any horizontal derivatives of velocity vanishes on the lower boundary, we can directly apply Korn’s inequality (see lemma A.5) to achieve that
\begin{align}
\|D_2^{2(N+2)} u\|_{H^0}^2 \leq \|D_2^{2(N+2)} \mathbb{D} u\|_{H^0}^2 = \mathcal{D}_{N+2,2} \leq \mathcal{Z}
\end{align}

Step 2: Initial estimates of velocity and pressure.
Let \( 0 \leq j \leq (N + 2) - 1 \) and \( \alpha \in \mathbb{N}^2 \) be such that \( 2 \leq 2j + |\alpha| \leq 2(N + 2) - 1 \). Hence, we naturally have
\begin{align}
\|\partial^\alpha_\nu \partial_t^{j+1} u\|_{H^0}^2 \leq \|D_2^{2(N+2)} u\|_{H^1}^2 \leq \mathcal{Z}
\end{align}

We apply the operator \( \partial^\alpha \partial_t \partial_3 \) to the divergence equation in (3.1.17) and rearrange the terms
\begin{align}
\partial^\alpha \partial_t \partial_3 (\partial_3 u_3) = \partial^\alpha \partial_t \partial_3 (-\partial_1 u_1 - \partial_2 u_2) + \partial^\alpha \partial_t \partial_3 G^2
\end{align}
Thus we have
\[ (3.6.7) \quad \left\| \partial^o \partial^i \partial^2 u_3 \right\|_{H^0}^2 \lesssim \left\| D_2^{(N+2)} u \right\|_{H^1}^2 + \left\| \partial^o \partial^i G^2 \right\|_{H^1}^2 \lesssim Z \]
Hence, it is easily implied that
\[ (3.6.8) \quad \left\| \Delta \partial^o \partial^i u_3 \right\|_{H^0}^2 \lesssim Z \]
Apply the operator \( \partial^o \partial^i \) to the third equation in (3.1.17), then we have
\[ (3.6.9) \quad \left\| \partial^o \partial^i \partial_3 p \right\|_{H^0}^2 \lesssim \left\| \partial^o \partial^i \Delta u \right\|_{H^0}^2 + \left\| \partial^o \partial^i G^1 \right\|_{H^0}^2 \lesssim Z \]
So we only need to estimate the terms related to \( \partial^2 u_i \) and \( \partial_3 p \) for \( i = 1, 2 \).

Step 3: Estimates of velocity and pressure in the upper domain.
In order to achieve above estimate, we need to employ a localization argument. Define a cutoff function \( \chi_1 = \chi_1(x_3) \) given by \( \chi_1(x_3) = 1 \) for \( x_3 \in \Omega_1 = [-2b/3,0] \) and \( \chi_1(x_3) = 0 \) for \( x_3 \notin (-3b/4,1/2) \).
Define the curl of the velocity \( w = \nabla \times u \), which means \( w_1 = \partial_2 u_3 - \partial_3 u_2 \) and \( w_2 = \partial_3 u_1 - \partial_1 u_3 \).
Therefore, \( \chi_1 w_i \) for \( i = 1, 2 \) satisfy the equation
\[ (3.6.10) \quad \Delta \partial^o \partial^i (\chi_1 w_i) = \chi_1 \partial^o \partial^i \partial^2 w_i + 2(\partial_3 \chi_1)(\partial^o \partial^i \partial_3 w_i) + (\partial^3_3 \chi_1)(\partial^o \partial^i w_i) - \chi_1 \nabla \times (\partial^o \partial^i G^1) \]
in \( \Omega \) as well as the boundary condition
\[ (3.6.11) \quad \left\{ \begin{aligned}
\partial^o \partial^i (\chi_1 w_1) &= 2\partial_3 \partial^o \partial^i u_3 + \partial^o \partial^i G^3 \cdot e_2 \quad \text{on } \Sigma \\
\partial^o \partial^i (\chi_1 w_2) &= -2\partial_3 \partial^o \partial^i u_3 - \partial^o \partial^i G^3 \cdot e_1 \quad \text{on } \Sigma \\
\partial^o \partial^i (\chi_1 w_3) &= \partial^o \partial^i (\chi_1 w_2) = 0 \quad \text{on } \Sigma_b
\end{aligned} \right. \]
Based on the standard elliptic estimate for Poisson equation, we have
\[ (3.6.12) \quad \left\| \partial^o \partial^i (\chi_1 w_1) \right\|_{H^1}^2 \lesssim \left\| \Delta \partial^o \partial^i (\chi_1 w_1) \right\|_{H^-1}^2 + \left\| \partial^o \partial^i (\chi_1 w_1) \right\|_{H^{1/2}(\Sigma)}^2 \]
Hence, we have to estimate each term on the right hand side.
Let \( \phi \in H^1_0(\Omega) \). For \( \alpha \neq 0 \) we may write \( \alpha = \beta + (\alpha - \beta) \) with \( |\beta| = 1 \). Then an integration by parts reveals that
\[ (3.6.13) \quad \left| \int_{\Omega} \phi \chi_1 \partial^o \partial^i \partial^2 w_i \right| = \left| \int_{\Omega} \partial^\beta \phi \chi_1 \partial^o \partial^i \partial^2 w_i \right| \leq \|\phi\|_{H^1} \left\| \chi_1 D_2^{(N+2)} w_i \right\|_{H^0}
\]
Since \( 2(j + 1) + |\alpha - \beta| \in [3,2(N + 2)] \), we naturally have
\[ (3.6.14) \quad \left\| \chi_1 D_2^{(N+2)} w_i \right\|_{H^-1}^2 \lesssim \left\| D_2^{(N+2)} u \right\|_{H^1}^2 \lesssim Z \]
Then if we take the supreme over all \( \phi \) such that \( \|\phi\|_{H^1} \leq 1 \), then we may get
\[ (3.6.15) \quad \left\| \chi_1 \partial^o \partial^i \partial^2 w_i \right\|_{H^-1}^2 \lesssim Z \]
A similar argument without integration by parts shows that the above result is also true for \( \alpha = 0 \) since in this case \( j \leq (N + 2) - 1 \) implies \( 4 \leq 2(j + 1) \leq 2(N + 2) \). In a similar fashion, we may apply integration by parts for \( \partial_3 \) for the other terms in \( H^{-1} \) norm and achieve that
\[ (3.6.16) \quad \left\| 2(\partial_3 \chi_1)(\partial^o \partial^i \partial_3 w_i) \right\|_{H^{-1}}^2 \lesssim \left( \|\partial_3 \chi_1\|_{L^\infty}^2 + \|\partial^3_3 \chi_1\|_{L^\infty}^2 \right) \left\| D_2^{(N+2)} w_i \right\|_{H^0}^2 \lesssim \left\| D_2^{(N+2)} u \right\|_{H^1}^2 \lesssim Z \]
\[ (3.6.17) \quad \left\| (\partial^3_3 \chi_1)(\partial^o \partial^i w_i) \right\|_{H^{-1}}^2 \lesssim \left\| \partial^3_3 \chi_1 \right\|_{L^\infty}^2 \left\| D_2^{(N+2)} w_i \right\|_{H^0}^2 \lesssim \left\| D_2^{(N+2)} u \right\|_{H^1}^2 \lesssim Z \]
\[ (3.6.18) \quad \left\| \chi_1 \nabla \times (\partial^o \partial^i G^1) \right\|_{H^{-1}}^2 \lesssim \left( \|\chi_1\|_{L^\infty}^2 + \|\partial_3 \chi_1\|_{L^\infty}^2 \right) \left\| D_2^{(N+2)-1} G^1 \right\|_{H^0}^2 \lesssim Z \]
Above four estimates together yield
\[ (3.6.19) \quad \left\| \Delta \partial^o \partial^i (\chi_1 w_i) \right\|_{H^-1}^2 \lesssim Z \]
Then for the boundary terms, a direct application of trace theorem shows that

\[(3.6.20) \quad \|\partial^\alpha \partial_3^j u_3\|_{H^{1/2}(\Sigma)}^2 + \|\partial^\alpha \partial_2^j u_3\|_{H^{1/2}(\Sigma)}^2 \leq \|D_2^{(N+2)} u\|_{H^1}^2 \lesssim Z\]

\[(3.6.21) \quad \|\partial^\alpha \partial_1^j G^3\|_{H^{1/2}(\Sigma)}^2 \leq \|D_2^{(N+2)-1} G^3\|_{H^{1/2}(\Sigma)}^2 \lesssim Z\]

Then above implies that

\[(3.6.22) \quad \|\partial^\alpha \partial_1^j (\chi_1 u_1)\|_{H^{1/2}(\Sigma)}^2 \lesssim Z\]

Hence, the elliptic estimate gives us the final form

\[(3.6.23) \quad \|\partial^\alpha \partial_1^j w_1\|_{H^1(\Omega_1)}^2 \leq \|\partial^\alpha \partial_1^j (\chi_1 u_1)\|_{H^1}^2 \lesssim Z\]

Since \(\partial^\alpha_2 \partial^\alpha_1^j u_1 = \partial_3 \partial^\alpha \partial_1^j (w_2 + \partial_1 u_3)\) and \(\partial^\alpha_2 \partial^\alpha_1^j u_2 = \partial_3 \partial^\alpha \partial_1^j (\partial_2 u_3 - w_1)\), we can deduce from above that for \(i = 1, 2\),

\[(3.6.24) \quad \|\partial^\alpha_2 \partial^\alpha_1^j u_i\|_{H^0(\Omega_1)}^2 \lesssim Z\]

which, by considering the Navier Stokes equation, further implies

\[(3.6.25) \quad \|\partial_3 \partial^\alpha \partial_1^j p\|_{H^0(\Omega_1)}^2 \lesssim Z\]

Combining all above, we have achieved that

\[(3.6.26) \quad \|D_2^{(N+2)-1} u\|_{H^2(\Omega_1)}^2 + \|D_2^{(N+2)-1} \nabla p\|_{H^0(\Omega_1)}^2 \lesssim Z\]

Step 4: Estimates of velocity and pressure in the lower domain.

Now we extend above estimate to the lower domain \(\Omega_2 = [-b, -b/3]\). Define a cutoff function \(\chi_2 \in C_c^\infty(\mathbb{R})\) such that \(\chi_2(x_3) = 1\) for \(x_3 \in \Omega_2\) and \(\chi_2(x_3) = 0\) for \(x_3 \notin (-2b, -b/6)\). Notice that this cutoff function satisfies that \(\text{supp}(\nabla \chi_2) \in \Omega_1\), which will play a key role in the following.

Then the localized variables satisfy the equation

\[(3.6.27) \quad \begin{cases}
-\Delta(\chi_2 u) + \nabla(\chi_2 p) = -\partial_t (\chi_2 u) + \chi_2 G^1 + H^1 & \text{in } \Omega \\
\nabla \cdot (\chi_2 u) = \chi_2 G^2 + H^2 & \text{in } \Omega \\
\chi_2 u = 0 & \text{on } \Sigma_1 \\
\chi_2 u = 0 & \text{on } \Sigma_2
\end{cases}\]

where

\[(3.6.28) \quad H^1 = \partial_3 \chi_2 (\partial_3 u - 2 \partial_3 u) - (\partial_3^2 \chi_2) u \quad H^2 = \partial_3 \chi_2 u_3\]

Let \(0 \leq j \leq (N + 2) - 1\) and \(\alpha \in \mathbb{N}^2\) such that \(2 \leq 2j + |\alpha| \leq 2(N + 2) - 1\). Then we may apply the differential operator \(\partial^\alpha \partial_1^j\) on (3.6.27) and consider the elliptic estimate (B.2.3) for Navier Stokes equation, which implies that

\[(3.6.29) \quad \|\partial^\alpha \partial_1^j (\chi_2 u)\|_{H^1}^2 + \|\partial^\alpha \partial_1^j (\chi_2 p)\|_{H^0}^2 \lesssim \|\partial^\alpha \partial_1^{j+1} (\chi_2 u)\|_{H^{-1}}^2 + \|\partial^\alpha \partial_1^j (\chi_2 G^1 + H^1)\|_{H^{-1}}^2 + \|\partial^\alpha \partial_1^j (\chi_2 G^2 + H^3)\|_{H^0}^2\]

Similar to the argument in the upper domain, via integration by parts, it is easy to deduce the \(H^{-1}\) estimates that

\[(3.6.30) \quad \|\partial^\alpha \partial_1^{j+1} (\chi_2 u)\|_{H^{-1}}^2 + \|\partial^\alpha \partial_1^j (\chi_2 G^1)\|_{H^{-1}}^2 + \|\partial^\alpha \partial_1^j (\chi_2 G^2)\|_{H^0}^2 \lesssim Z\]

For the remaining part, we may directly estimate

\[(3.6.31) \quad \|\partial^\alpha \partial_1^j (H^1)\|_{H^{-1}}^2 + \|\partial^\alpha \partial_1^j H^2\|_{H^0}^2 \lesssim \|\partial^\alpha \partial_1^j H^1\|_{H^0}^2 + \|\partial^\alpha \partial_1^j H^2\|_{H^0}^2 \lesssim \|D_2^{(N+2)-1} u\|_{H^2(\Omega_1)}^2 + \|D_2^{(N+2)-1} \nabla p\|_{H^0(\Omega_1)}^2 \lesssim Z\]
where we utilize the property for $\nabla \chi_2$ stated above. Thus we have

$$\left\| \partial_t \partial_\alpha \partial_{\chi_2} p \right\|_{H^0(\Omega_2)}^2 \lesssim \left\| \partial_\alpha \partial_t (\chi_2 p) \right\|_{H^0}^2 \lesssim Z \tag{3.6.32}$$

which, based on the Navier-Stokes equation (3.1.17), further implies

$$\left\| \partial_t^2 \partial_\alpha \partial_{\chi_2} u \right\|_{H^0(\Omega_2)}^2 \lesssim Z \tag{3.6.33}$$

Combining all above, we have

$$\left\| D_2^{(N+2)-1} u \right\|_{H^2(\Omega_2)}^2 + \left\| D_2^{(N+2)-1} \nabla p \right\|_{H^0(\Omega_2)}^2 \lesssim Z \tag{3.6.34}$$

Step 5: Estimates of free surface.

With above estimates in hand, we may derive the estimates for $\eta$ by employing the boundary condition of (3.1.17), i.e.

$$\eta = p - 3 \partial_3 u_3 - G_3^3 \tag{3.6.35}$$

We may take horizontal derivative on both sides of above equation, i.e. for $\alpha \in \mathbb{N}^2$ and $3 \leq |\alpha| \leq 2(N + 2) - 1$, we have

$$\left\| \partial_\alpha \eta \right\|_{H^{1/2}(\Sigma)}^2 \lesssim \left\| \partial_\alpha p \right\|_{H^{1/2}(\Sigma)}^2 + \left\| \partial_\alpha \partial_3 u_3 \right\|_{H^{1/2}(\Sigma)}^2 + \left\| \partial_\alpha G_3^3 \right\|_{H^{1/2}(\Sigma)}^2$$

$$\lesssim \left\| \nabla D_2^{(N+2)-1} p \right\|_{H^0}^2 + \left\| D_2^{(N+2)-1} u \right\|_{H^2}^2 + \left\| \partial_\alpha G_3^3 \right\|_{H^{1/2}(\Sigma)}^2 \tag{3.6.36}$$

For the temporal derivative of $\eta$, we need to resort to the transport equation on the upper boundary, i.e.

$$\partial_t \eta = u_3 + G^4 \tag{3.6.37}$$

For $\alpha \in \mathbb{N}^2$ and $1 \leq |\alpha| \leq 2(N + 2) - 1$, by trace theorem, Poincare theorem and the divergence equation $\nabla \cdot u = G^2$, we have

$$\left\| \partial_\alpha \partial_t \eta \right\|_{H^{1/2}(\Sigma)}^2 \lesssim \left\| \partial_\alpha u_3 \right\|_{H^{1/2}(\Sigma)}^2 + \left\| \partial_\alpha G^4 \right\|_{H^{1/2}(\Sigma)}^2$$

$$\lesssim \left\| \partial_\alpha u_3 \right\|_{H^1}^2 + Z$$

$$\lesssim \left\| \partial_\alpha u_3 \right\|_{H^0}^2 + \left\| \partial_\alpha Du_3 \right\|_{H^2}^2 + \left\| \partial_\alpha \partial_3 u_3 \right\|_{H^0}^2 + Z$$

$$\lesssim \left\| \partial_\alpha Du_3 \right\|_{H^0}^2 + \left\| \partial_\alpha G^2 \right\|_{H^2}^2 + Z$$

Similarly, for $j = 2, \ldots, (N + 2) + 1$, we may apply $\partial_t^{j-1}$ to transport equation and for $\alpha \in \mathbb{N}^2$ and $0 \leq |\alpha| \leq 2(N + 2) - 2j + 2$

$$\left\| \partial_\alpha \partial_t^{j-1} \eta \right\|_{H^{1/2}(\Sigma)}^2 \lesssim \left\| \partial_\alpha \partial_t^{j-1} u_3 \right\|_{H^{1/2}(\Sigma)}^2 + \left\| \partial_\alpha \partial_t^{j-1} G^4 \right\|_{H^{1/2}(\Sigma)}^2$$

$$\lesssim \left\| \partial_\alpha \partial_t^{j-1} u_3 \right\|_{H^1}^2 + Z$$

Step 6: Synthesis.

To summarize all above, since $\Omega = \Omega_1 \cup \Omega_2$ we have the estimate

$$\mathcal{D}_{N+2,2} \lesssim \mathcal{D}_{N+2,2} + \mathcal{Y}_{N+2,2} \tag{3.6.40}$$

By lemma 3.15, it is obvious that there exists a $\theta > 0$ such that

$$\mathcal{Y}_{N+2,2} \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2,2} \tag{3.6.41}$$

Therefore, our result naturally follows. \( \square \)

**Proposition 3.35.** There exists a $\theta > 0$ such that

$$\mathcal{D}_{2N} \lesssim \mathcal{D}_{2N} + \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}_F_{2N} \tag{3.6.42}$$
Proof. It is easy to see that most part of the proof is identical to that of proposition 3.34, except that we do not need the minimum count and interpolation now. Also we will utilize lemma 3.16 instead to estimate the nonlinear terms in perturbed linear form. Finally, a direct estimate for Riesz potential term (3.6.43)

\[ \| I_{\lambda} u \|^2_{H^1, \Sigma} \lesssim \| \mathcal{D} I_{\lambda} u \|^2_{H^0} \]
can close the proof.

3.6.2. Energy Comparison Estimates.

**Proposition 3.36.** There exists a \( \theta > 0 \) such that

\[ \mathcal{E}_{N+2,2} \lesssim \mathcal{E}_{N+2,2}' + \mathcal{E}_{2N}^\theta \mathcal{E}_{N+2,2} \]

**Proof.** We define

\[ W_{N+2,2} = \left\| D^2 \nabla_{0}^{2(N+2)-4} G \right\|^2_{H^0} + \left\| D^2 \nabla_{0}^{2(N+2)-3} G \right\|^2_{H^0} + \left\| D^2 \nabla_{0}^{2(N+2)-4} G \right\|^2_{H^0(\Sigma)} \]

and

\[ Z = \mathcal{E}_{N+2,2} + W_{N+2,2} \]

Then we can directly apply elliptic estimate (B.2.10) in perturbed linear form. For \( \alpha \in \mathbb{N}^2 \) and \( |\alpha| = 2 \), we have

\[ \| \partial^\alpha u \|^2_{H^2(\Sigma)} + \| \partial^\alpha p \|^2_{H^2(\Sigma)} \lesssim \| \partial^\alpha \partial_t u \|^2_{H^2(\Sigma)} + \| \partial^\alpha G \|^2_{H^2(\Sigma)} \]

and

\[ \| \partial^\alpha G \|^2_{H^2(\Sigma)} + \| \partial^\alpha G \|^2_{H^2(\Sigma)} + \| \partial^\alpha \partial_t \|^2_{H^2(\Sigma)} \lesssim W_{N+2,2} \lesssim Z \]

Naturally, we have

\[ \| \partial^\alpha G \|^2_{H^2(\Sigma)} + \| \partial^\alpha G \|^2_{H^2(\Sigma)} + \| \partial^\alpha \partial_t \|^2_{H^2(\Sigma)} \lesssim \mathcal{E}_{N+2,2} \lesssim Z \]

So the only remaining term is

\[ \| \partial^\alpha \partial_t u \|^2_{H^2(\Sigma)} \lesssim \| \partial_t u \|^2_{H^2(\Sigma)} \]

which should be estimated with a finite induction in the following. For \( j = 1, \ldots, (N + 2) - 1 \), we have the elliptic estimate

\[ \left| \partial_{t}^{j} u \right|_{H^{2}(\Sigma)} + \left| \partial_{t}^{j} p \right|_{H^{2}(\Sigma)} \lesssim \left| \partial_{t}^{j+1} u \right|_{H^{2}(\Sigma)} + \left| \partial_{t}^{j} G \right|_{H^{2}(\Sigma)} \]

\[ + \left| \partial_{t}^{j} G \right|_{H^{2}(\Sigma)} \]

By the definition, it is easy to see

\[ \left| \partial_{t}^{j} G \right|_{H^{2}(\Sigma)} + \left| \partial_{t}^{j} G \right|_{H^{2}(\Sigma)} \lesssim \mathcal{E}_{N+2,2} \lesssim Z \]

and

\[ \left| \partial_{t}^{j} \eta \right|_{H^{2}(\Sigma)} \lesssim \mathcal{E}_{N+2,2} \lesssim Z \]

When \( j = (N + 2) - 1 \), we have

\[ \left| \partial_{t}^{j+1} u \right|_{H^{2}(\Sigma)} = \left| \partial_{t}^{N+2} u \right|_{H^{0}} \lesssim \mathcal{E}_{N+2,2} \lesssim Z \]

Hence, the elliptic estimate shows

\[ \left| \partial_{t}^{N+1} u \right|_{H^{2}(\Sigma)} + \left| \partial_{t}^{N+1} p \right|_{H^{1}} \lesssim Z \]

Inductively, we can estimate all the temporal derivatives of velocity and pressure for \( j = 1, \ldots, N \), which will finally close the proof.

Therefore, we have shown that

\[ \mathcal{E}_{N+2,2} \lesssim \mathcal{E}_{N+2,2} + W_{N+2,2} \]
Based on lemma 3.15, there exists a $\theta > 0$ such that

\[(3.6.57) \quad W_{N+2.2} \lesssim \mathcal{E}_{2N+2}^\theta \mathcal{E}_{N+2.2} \]

Hence, our result naturally follows. \hfill \qed

**Proposition 3.37.** There exists a $\theta > 0$ such that

\[(3.6.58) \quad \mathcal{E}_{2N} \lesssim \mathcal{E}_{2N} + \mathcal{E}_{2N}^{1+\theta} \]

*Proof.* It is easy to see that the proof is almost the same as that of proposition 3.36 except that there is no minimum count now and we should utilize lemma 3.16 instead. So we omit the proof here. \hfill \qed

### 3.7. A Priori Estimates

First, we concentrate on the $2N$ level a priori estimate.

**Lemma 3.38.** There exists $C > 0$ such that

\[(3.7.1) \quad \sup_{0 \leq r \leq t} F_{2N}(r) \lesssim \exp\left(C \int_0^t \sqrt{K(r)dr}\right) \left[ F_{2N}(0) + t \int_0^t (1 + \mathcal{E}_{2N}(r)) D_{2N}(r) dr + \left( \int_0^t \sqrt{K(r)F_{2N}(r)dr} \right)^2 \right] \]

*Proof.* This lemma is almost identical to lemma 9.1 in [2], so we omit the detailed proof here and only outline some differences here. Based on trace theorem, we have that for $0 \leq |\beta| \leq 4N$ and $\beta \in \mathbb{N}^2$

\[(3.7.2) \quad \|\partial_\beta u\|_{H^{1/2}(\Sigma)}^2 \lesssim \|\partial_\beta u\|_{H^1}^2 \lesssim D_{2N} \]

Hence, although our new definition of dissipation is slightly different from the original paper, we can still estimate and achieve the same form. \hfill \qed

**Proposition 3.39.** There exists a universal constant $0 < \delta < 1$ such that if $\mathcal{G}_N(T) \lesssim \delta$, then

\[(3.7.3) \quad \sup_{0 \leq r \leq t} F_{2N}(r) \lesssim F_{2N}(0) + t \int_0^t D_{2N}(r) dr \]

and

\[(3.7.4) \quad \int_0^t K(r) F_{2N}(r) dr \lesssim \delta^{(8+2\lambda)/(8+4\lambda)} F_{2N}(0) + \delta^{(8+2\lambda)/(8+4\lambda)} \int_0^t D_{2N}(r) dr \]

\[(3.7.5) \quad \int_0^t \sqrt{D_{2N}(r)K(r)F_{2N}(r)} dr \lesssim F_{2N}(0) + \delta^{(8+2\lambda)/(16+6\lambda)} \int_0^t D_{2N}(r) dr \]

for all $0 \leq t \leq T$. \hfill \qed

*Proof.* By lemma 3.9, we have that $K \lesssim \mathcal{E}_{N+2.2}^{2/3+r/2} \mathcal{E}_{2N}^{r/2}$ and $\mathcal{E}_{2N} \leq 1$, we may take $r = 2\lambda/(4 + \lambda)$ to achieve that

\[(3.7.6) \quad K \lesssim \mathcal{E}_{N+2.2}^{8+2\lambda)/(8+4\lambda)} \]

Then utilizing the decaying of $\mathcal{E}_{N+2.2}$, the other part of the proof is identical to proposition 9.2 and corollary 9.3 in [2], so we will omit it here. \hfill \qed

**Theorem 3.40.** There exists a universal constant $0 < \delta < 1$ such that if $\mathcal{G}_N(T) \lesssim \delta$, then

\[(3.7.7) \quad \sup_{0 \leq r \leq t} \mathcal{E}_{2N}(r) + \int_0^t D_{2N} + \sup_{0 \leq r \leq t} \frac{F_{2N}(r)}{1+r} \lesssim \mathcal{E}_{2N}(0) + F_{2N}(0) \]

for all $0 \leq t \leq T$. \hfill \qed

*Proof.* This is identical to theorem 9.4 of [2]. Notice that in the section of energy estimates and comparison estimates, we have recovered all the necessary results for $2N$ level a priori estimate here. \hfill \qed

Then we consider the a priori estimate for $N + 2$ level.

**Lemma 3.41.** Let $F^2$ be defined with $\partial_1^{N+2}$. There exists a universal constant $0 < \delta < 1$ such that if $\mathcal{G}_N(T) \lesssim \delta$, then

\[(3.7.8) \quad \frac{2}{3} \mathcal{E}_{N+2.2}^2(t) \leq \mathcal{E}_{N+2.2} - 2 \int_0^t J(t)\partial_1^{N+1} p(t)F^2(t) \leq \frac{4}{3} \mathcal{E}_{N+2.2}(t) \]

for all $0 \leq t \leq T$. \hfill \qed

*Proof.* It is the same as lemma 9.6 of [2]. \hfill \qed
Theorem 3.42. There exists a universal constant $0 < \delta < 1$ such that if $G_N(T) \lesssim \delta$, then
\begin{equation}
(3.7.9)
\sup_{0 \leq r \leq t} (1 + r)^{2+\lambda} E_{N+2,2}(r) \lesssim E_{2N}(0)
\end{equation}
for all $0 \leq t \leq T$.

Proof. Utilizing Lemma 3.10, we may employ the same argument as theorem 9.7 in [2] to show this. Also notice that all the necessary lemmas have been recovered in the section of energy estimates and comparison estimates.

Next, we will give an interpolation argument to achieve the decaying results for energy with different minimum count.

Theorem 3.43. There exists a universal constant $0 < \delta < 1$ such that if $G_N(T) \lesssim \delta$, then
\begin{equation}
(3.7.10)
\sup_{0 \leq r \leq t} (1 + r)^{1+\lambda} E_{N+2,1}(r) \lesssim E_{2N}(0)
\end{equation}
for all $0 \leq t \leq T$.

Proof. By Lemma 3.11, we have that
\begin{equation}
(3.7.11)
E_{N+2,1} \lesssim E_{2N}^{(1+\lambda)/(2+\lambda)} E_{2N}^{1/(2+\lambda)}
\end{equation}
Combining with Theorem 3.42 and Theorem 3.40, we can easily see that
\begin{equation}
(3.7.12)
(1 + r)^{1+\lambda} E_{N+2,1}(r) \lesssim \left( (1 + r)^{2+\lambda} E_{N+2,2}(r) \right)^{(1+\lambda)/(2+\lambda)} \left( E_{2N}(r) \right)^{1/(2+\lambda)}
\end{equation}
Hence, our result easily follows.

Finally, we can sum up all above to achieve the a priori estimate.

Theorem 3.44. There exists a universal constant $0 < \delta < 1$ such that if $G_N(T) \lesssim \delta$, then
\begin{equation}
(3.7.13)
G_N(t) \lesssim E_{2N}(0) + F_{2N}(0)
\end{equation}
for all $0 \leq t \leq T$.

Proof. To sum up the conclusion of Theorem 3.40, 3.42 and 3.43 easily implies this.

3.8. Global Wellposedness. Since we introduce several terms related to Riesz potential in the a priori estimates, we have to revise our local wellposedness result here to adopt this, which is perfectly completed in Theorem 10.7 of [2].

Theorem 3.45. Suppose that initial data are given satisfying the $2N$th compatible condition and $\|u_0\|_{H^{2N}}^2 + \|\eta_0\|_{H^{2N+1/2}(\Sigma)}^2 + \|I_{\lambda}u_0\|_{H^0}^2 + \|I_{\lambda}\eta_0\|_{H^0(\Sigma)}^2 < \infty$. Let $\epsilon > 0$. There exists a $\delta_0 = \delta_0(\epsilon)$ and a $\delta_0 > 0$ such that if $0 < T < T_0$ and $\|u_0\|_{H^{2N}}^2 + \|\eta_0\|_{H^{2N}(\Sigma)}^2 \leq \delta_0$, then there exists a unique solution $(u, p, \eta)$ to system (1.3.8) on the interval $[0, T]$ that achieves the initial data. The solution obeys the estimates
\begin{equation}
(3.8.1)
T_0 = C(\epsilon) \min \left\{ 1, \frac{1}{\|\eta_0\|_{H^{2N+1/2}(\Sigma)}^2} \right\} > 0
\end{equation}
where $C(\epsilon) > 0$ is a constant depending on $\epsilon$, such that if $0 < T < T_0$ and $\|u_0\|_{H^{2N}}^2 + \|\eta_0\|_{H^{2N}(\Sigma)}^2 \leq \delta_0$, then there exists a unique solution $(u, p, \eta)$ to system (1.3.8) on the interval $[0, T]$ that achieves the initial data. The solution obeys the estimates
\begin{equation}
(3.8.2)
\sup_{0 \leq t \leq T} \mathcal{E}_{2N}(t) + \sup_{0 \leq t \leq T} \|I_{\lambda} p(t)\|_{H^0}^2 + \int_0^T D_{2N}(t)dt + \int_0^T \left( \|\partial_t^{2N+1} u(t)\|_{H^{1}^*}^2 + \|\partial_t^{2N} p(t)\|_{H^0}^2 \right) dt \lesssim \epsilon + \|I_{\lambda} u_0\|_{H^0}^2 + \|I_{\lambda} \eta_0\|_{H^0(\Sigma)}^2
\end{equation}
and
\begin{equation}
(3.8.3)
\sup_{0 \leq t \leq T} \left( \mathcal{E}_{2N}(t) - \|I_{\lambda} u\|_{H^0}^2 - \|I_{\lambda} \eta\|_{H^0(\Sigma)}^2 \right) \leq \epsilon
\end{equation}
\begin{equation}
(3.8.4)
\sup_{0 \leq t \leq T} \mathcal{F}_{2N}(t) \leq C \mathcal{F}_{2N}(0) + \epsilon
\end{equation}
Finally, we come to the conclusion for global wellposedness. Since we have recovered the a priori estimate and do not improve the argument for this part, the reader may directly refer to theorem 11.1 and 11.2 in [2], which completes the whole proof.

**Theorem 3.46.** Suppose that the initial data \( (u_0, \eta_0) \) satisfy the \( 2N \)th compatible condition. There exists a \( \kappa > 0 \) such that if \( \|u_0\|^2_{H^{s_N}} + \|\eta_0\|^2_{H^{s_{N+1/2}}(\Sigma)} \leq \kappa \), then there exists a unique solution \( (u, p, \eta) \) on the interval \([0, \infty)\) that achieves the initial data. The solution obeys the estimate

\[
G_N(\infty) \leq C(\|u_0\|^2_{H^{s_N}} + \|\eta_0\|^2_{H^{s_{N+1/2}}(\Sigma)}) < C\kappa
\]

**Appendix A. Analytic Tools**

**A.1. Products in Sobolev Space.** We will need some estimates of the products of functions in Sobolev spaces. Since these results have been proved in lemma A.1 and lemma A.2 of [1], we will present the statement of the lemmas here without proof.

**Lemma A.1.** Let \( U \) denote either \( \Sigma \) or \( \Omega \).

1. Let \( 0 \leq r \leq s_1 \leq s_2 \) be such that \( s_1 > n/2 \). Let \( f \in H^{s_1}(U) \), \( g \in H^{s_2}(U) \). Then \( fg \in H^r(U) \) and

\[
\|fg\|_{H^r(U)} \lesssim \|f\|_{H^{s_1}(U)} \|g\|_{H^{s_2}(U)}
\]

2. Let \( 0 \leq r \leq s_1 \leq s_2 \) be such that \( s_2 > r + n/2 \). Let \( f \in H^{s_1}(U) \), \( g \in H^{s_2}(U) \). Then \( fg \in H^r(U) \) and

\[
\|fg\|_{H^r(U)} \lesssim \|f\|_{H^{s_1}(U)} \|g\|_{H^{s_2}(U)}
\]

3. Let \( 0 \leq r \leq s_1 \leq s_2 \) be such that \( s_2 > r + n/2 \). Let \( f \in H^{-r}(\Sigma) \), \( g \in H^{s_2}(\Sigma) \). Then \( fg \in H^{-s_1}(\Sigma) \) and

\[
\|fg\|_{H^{-s_1}(\Sigma)} \lesssim \|f\|_{H^{-r}(\Sigma)} \|g\|_{H^{s_2}(\Sigma)}
\]

**Lemma A.2.** Suppose that \( f \in C^1(\Sigma) \) and \( g \in H^{1/2}(\Sigma) \). Then \( fg \in H^{1/2}(\Sigma) \) and

\[
\|fg\|_{H^{1/2}(\Sigma)} \lesssim \|f\|_{C^1(\Sigma)} \|g\|_{H^{1/2}(\Sigma)}
\]

**A.2. Poincare-Type Inequality.** We need several Poincare-type inequality in \( \Omega \). Since all these lemmas have been proved in lemma A.10-A.13 in [2], we will only give the statement here without proof.

**Lemma A.3.** It holds that

\[
\|f\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^2(\Sigma)}^2 + \|\partial_\nu f\|_{L^2(\Omega)}^2
\]

for all \( f \in H^1(\Omega) \). Also, if \( f \in W^{1,\infty}(\Omega) \), then

\[
\|f\|_{L^\infty(\Omega)}^2 \lesssim \|f\|_{L^2(\Sigma)}^2 + \|\partial_\nu f\|_{L^\infty(\Omega)}^2
\]

**Lemma A.4.** It holds that \( \|f\|_{H^p(\Sigma)} \lesssim \|\partial_\nu f\|_{H^p} \) for \( f \in H^1(\Omega) \) such that \( f = 0 \) on \( \Sigma_b \). It also holds that \( \|f\|_{L^\infty(\Sigma)} \lesssim \|\partial_\nu f\|_{L^\infty(\Omega)} \) for \( f \in W^{1,\infty}(\Omega) \) such that \( f = 0 \) on \( \Sigma_b \).

**Lemma A.5.** It holds that \( \|u\|_{H^1} \lesssim \|\nabla u\|_{H^0} \) for all \( u \in H^1(\Omega; \mathbb{R}^3) \) such that \( f = 0 \) on \( \Sigma_b \).

**Lemma A.6.** It holds that \( \|\nabla f\|_{H^0} \lesssim \|\nabla f\|_{H^0} \) for all \( f \in H^1(\Omega) \) such that \( f = 0 \) on \( \Sigma_b \). Also, \( \|f\|_{W^{1,\infty}(\Omega)} \lesssim \|\nabla f\|_{L^\infty(\Omega)} \) for all \( f \in W^{1,\infty}(\Omega) \) such that \( f = 0 \) on \( \Sigma_b \).

**A.3. Poisson Integral.** For a function \( f \) defined on \( \Sigma = \mathbb{R}^2 \), the Poisson integral \( \mathcal{P}f \) in \( \mathbb{R}^2 \times (-\infty, 0) \) is defined by

\[
\mathcal{P}f(x', x_3) = \int_{\mathbb{R}^2} \hat{f}(\xi)e^{2\pi i x_3 e^{2\pi i x' \xi}} d\xi
\]

where \( \hat{f}(\xi) \) is the Fourier transform of \( f(x') \) on \( \mathbb{R}^2 \). Then within the slab \( \mathbb{R}^2 \times (-b, 0) \), we have the following estimate based on lemma A.5 in [2].

**Lemma A.7.** The Poisson integral satisfies that for \( q \in \mathbb{N} \),

\[
\|\nabla^q \mathcal{P}f\|_{H^0} \lesssim \|f\|_{H^{q-1/2}(\Sigma)}
\]

\[
\|\nabla^q \mathcal{P}f\|_{L^\infty(\Omega)} \lesssim \|f\|_{H^q(\Sigma)}
\]

where \( \dot{H}^q \) denotes the usual homogeneous Sobolev space.
A.4. Riesz Potential. For a function $f$ defined in $\Omega$, we define the Riesz potential

\begin{equation}
I_\lambda f(x', x_3) = \int_{-b}^{0} \int_{R^2} \hat{f}(\xi, x_3) |\xi|^{-\lambda} e^{2\pi i x' \cdot \xi} d\xi dx_3
\end{equation}

Similarly, for $f$ defined on $\Sigma$, we set

\begin{equation}
I_\lambda f(x') = \int_{R^2} \hat{f}(\xi) |\xi|^{-\lambda} e^{2\pi i x' \cdot \xi} d\xi
\end{equation}

We have the following lemmas to describe the product of Riesz potential and its interaction with the horizontal derivatives as lemma A.3 and A.4 in [2].

Lemma A.8. Let $\lambda \in (0, 1)$. If $f \in H^0(\Omega)$ and $g, Dg \in H^1(\Omega)$, then

\begin{equation}
\|I_\lambda (fg)\|_{H^0} \lesssim \|f\|_{H^0} \|g\|_{H^1}^{1-\lambda} \|Dg\|_{H^1}^{1-\lambda}
\end{equation}

Lemma A.9. Let $\lambda \in (0, 1)$. If $f \in H^k(\Omega)$ for $k \geq 1$ an integer, then

\begin{equation}
\|I_\lambda D^k f\|_{H^0} \lesssim \|D^{k-1} f\|_{H^0} \|D^k f\|_{H^0}^{1-\lambda}
\end{equation}

A.5. Interpolation Estimates. Here we record several interpolation estimate utilized in our proof. Since they have been proved in lemma A.6-A.8 and lemma 3.18 in [2], we will omit the proof now.

Lemma A.10. For $s, q > 0$ and $0 \leq r \leq s$, we have the estimate

\begin{equation}
\|f\|_{H^r(\Sigma)} \lesssim \|f\|_{H^s(\Sigma)}^{q/(r+q)} \|f\|_{H^{s+r}(\Sigma)}^{r/(r+q)}
\end{equation}

whenever the right hand side is finite.

Lemma A.11. Let $\mathcal{P} f$ be the Poisson integral of $f$, defined on $\Sigma$. Let $\lambda \geq 0$, $q, s \in \mathbb{N}$, and $r \geq 0$. Then the following estimates hold.

1. Let

\begin{equation}
\theta = \frac{s}{q+s+\lambda}, \quad 1 - \theta = \frac{q+\lambda}{q+s+\lambda}
\end{equation}

Then

\begin{equation}
\|\nabla^q \mathcal{P} f\|_{L^\infty}^2 \lesssim \left(\|I_\lambda f\|_{H^0}^2\right)^\theta \left(\|D^{q+s} f\|_{H^0}^2\right)^{1-\theta}
\end{equation}

2. Let $r+s > 1$,

\begin{equation}
\theta = \frac{r+s-1}{q+s+r+\lambda}, \quad 1 - \theta = \frac{q+\lambda+1}{q+s+r+\lambda}
\end{equation}

Then

\begin{equation}
\|\nabla^q \mathcal{P} f\|_{H^r}^2 \lesssim \left(\|I_\lambda f\|_{H^0}^2\right)^\theta \left(\|D^{q+s} f\|_{H^r}^2\right)^{1-\theta}
\end{equation}

3. Let $s > 1$. Then

\begin{equation}
\|\nabla^q \mathcal{P} f\|_{L^\infty}^2 \lesssim \|D^s f\|_{H^r}^2.
\end{equation}

Lemma A.12. Let $f$ be defined on $\Sigma$. Let $\lambda \geq 0$. Then we have the following estimates.

1. Let $q, s \in (0, \infty)$ and

\begin{equation}
\theta = \frac{s}{q+s+\lambda}, \quad 1 - \theta = \frac{q+\lambda}{q+s+\lambda}
\end{equation}

Then

\begin{equation}
\|D^q f\|_{H^s}^2 \lesssim \left(\|I_\lambda f\|_{H^0}^2\right)^\theta \left(\|D^{q+s} f\|_{H^s}^2\right)^{1-\theta}
\end{equation}
(2) Let $q, s \in \mathbb{N}$, $r \geq 0$, $r + s > 1$, 

(A.5.9) \[ \theta = \frac{r + s - 1}{q + s + r} \quad 1 - \theta = \frac{q + \lambda + 1}{q + s + r + \lambda} \]

Then 

(A.5.10) \[ \|D^q f\|_{L^2}^2 \lesssim \left( \|I_\lambda f\|_{H^0}^2 \right)^\theta \left( \|D^{q+s} f\|_{H^r}^2 \right)^{1-\theta} \]

**Lemma A.13.** Let $f$ be defined in $\Omega$. Let $\lambda \geq 0$, $q, s \in \mathbb{N}$, and $r \geq 0$. Then we have the following estimates.

(1) Let 

(A.5.11) \[ \theta = \frac{s}{q + s + \lambda} \quad 1 - \theta = \frac{q + \lambda + 1}{q + s + r + \lambda} \]

Then 

(A.5.12) \[ \|D^q f\|_{H^r}^2 \lesssim \left( \|I_\lambda f\|_{H^1}^2 \right)^\theta \left( \|D^{q+s} f\|_{H^{r+1}}^2 \right)^{1-\theta} \]

(2) Let $r + s > 1$

(A.5.13) \[ \theta = \frac{r + s - 1}{q + s + r + \lambda} \quad 1 - \theta = \frac{q + \lambda + 1}{q + s + r + \lambda} \]

Then 

(A.5.14) \[ \|D^q f\|_{L^\infty(\Omega)}^2 \lesssim \left( \|I_\lambda f\|_{H^1}^2 \right)^\theta \left( \|D^{q+s} f\|_{H^{r+1}}^2 \right)^{1-\theta} \]

and 

(A.5.15) \[ \|D^q f\|_{L^\infty(\Omega)}^2 \lesssim \left( \|I_\lambda f\|_{H^1}^2 \right)^\theta \left( \|D^{q+s} f\|_{H^{r+1}}^2 \right)^{1-\theta} \]

**A.6. Continuity and Temporal Derivative.** In the following, we give two important lemmas to connect $L^2 H^k$ norm and $L^\infty H^k$ norm.

**Lemma A.14.** Suppose that $u \in L^2([0, T]; H^{s_1}(\Omega))$ and $\partial_t u \in L^2([0, T]; H^{s_2}(\Omega))$ for $s_1 \geq s_2 \geq 0$ and $s = (s_1 + s_2)/2$. Then $u \in C^0([0, T]; H^s(\Omega))$ and satisfies the estimate 

(A.6.1) \[ \|u\|_{L^\infty H^s} \leq \|u(0)\|_{H^s}^2 + \|u\|_{L^2 H^{s_2}}^2 + \|\partial_t u\|_{L^2 H^{s_2}}^2 \]

where the $L^2 H^k$ norm and $L^\infty H^k$ norm are evaluated in $[0, T]$.

**Proof.** Considering the extension theorem in Sobolev space, we only need to prove this result in the $R^n$ case. The periodic case can be derived in a similar fashion. Using Fourier transform, 

\[ \partial_t^r u(t) \|_{H^r}^2 = 2\mathfrak{R} \left( \int_{R^n} \langle \xi \rangle^{2s} \hat{u}(\xi, t) \overline{\partial_t^r \hat{u}(\xi, t)} d\xi \right) \leq 2 \int_{R^n} \langle \xi \rangle^{2s} |\hat{u}(\xi, t)|^2 |\partial_t^r \hat{u}(\xi, t)|^2 d\xi \]

\[ = 2 \int_{R^n} \langle \xi \rangle^{s_1} |\hat{u}(\xi, t)|^2 |\partial_t \hat{u}(\xi, t)|^2 d\xi \leq \int_{R^n} \langle \xi \rangle^{2s_1} |\hat{u}(\xi, t)|^2 d\xi + \int_{R^n} \langle \xi \rangle^{2s_2} |\partial_t \hat{u}(\xi, t)|^2 d\xi \]

So integrate with respect to time on $[0, t]$ 

\[ \|u(t)\|_{H^r}^2 \leq \|u(0)\|_{H^s}^2 + \|u\|_{L^2 H^{s_2}}^2 \quad \square \]

The following lemma shows the estimate in another direction.

**Lemma A.15.** For any $u \in L^\infty([0, T]; H^k)$ within $[0, T]$, we must have $u \in L^2([0, T]; H^k)$ and satisfies the estimate 

(A.6.2) \[ \|u\|_{L^2 H^k}^2 \leq T \|u\|_{L^\infty H^k}^2 \]

**Proof.** The result is simply based on the definition of these two norms. \square
A.7. Extension Theorem. The following are two extension theorems which will be used to construct start point of iteration from initial data in proving wellposedness of Naiver-Stokes-transport system. Since it is identical as lemma A.5 and A.6 in [1], we omit the proof here.

Lemma A.16. Suppose that \( \partial_t^j u(0) \in H^{2N-2j}(\Omega) \) for \( j = 0, \ldots, N \), then there exists a extension \( u \) achieving the initial data, such that

\[
\partial_t^j u \in L^2([0, \infty); H^{2N-2j+1}(\Omega)) \cap L^\infty([0, \infty); H^{2N-2j}(\Omega))
\]

for \( j = 0, \ldots, N \). Moreover,

\[
\sum_{j=0}^N \left\| \partial_t^j u \right\|^2_{L^2 H^{2N-2j+1}} + \left\| \partial_t^j u \right\|^2_{L^\infty H^{2N-2j}} \leq \sum_{j=0}^N \left\| \partial_t^j u(0) \right\|^2_{H^{2N-2j}}
\]

Lemma A.17. Suppose that \( \partial_t^j p(0) \in H^{2N-2j-1}(\Omega) \) for \( j = 0, \ldots, N - 1 \), then there exists a extension \( p \) achieving the initial data, such that

\[
\partial_t^j p \in L^2([0, \infty); H^{2N-2j-1}(\Omega)) \cap L^\infty([0, \infty); H^{2N-2j-1}(\Omega))
\]

for \( j = 0, \ldots, N - 1 \). Moreover,

\[
\sum_{j=0}^{N-1} \left\| \partial_t^j p \right\|^2_{L^2 H^{2N-2j-1}} + \left\| \partial_t^j p \right\|^2_{L^\infty H^{2N-2j-1}} \leq \sum_{j=0}^{N-1} \left\| \partial_t^j p(0) \right\|^2_{H^{2N-2j-1}}
\]

APPENDIX B. ESTIMATES FOR FUNDAMENTAL EQUATIONS

B.1. Transport Estimates. Let \( \Sigma \) be either infinite or periodic. Consider the equation

\[
\begin{cases}
\partial_t \eta + u \cdot \nabla \eta = g & \text{in } \Sigma \times (0, T) \\
\eta(t=0) = \eta_0 & \text{on } \Sigma \times (0, T)
\end{cases}
\]

We have the following estimate of the regularity solution to this equation, which is a particular case of a more general result proved in proposition 2.1 of [5]. Note that the result in [5] is stated for \( \Sigma = \mathbb{R}^2 \), but the same result holds in the periodic case as described in [9].

Lemma B.1. Let \( \eta \) be a solution to equation (B.1.1). Then there exists a universal constant \( C > 0 \) such that for any \( s \geq 3 \)

\[
\| \eta \|_{L^\infty H^s} \leq \exp \left( C \int_0^t \| Du(r) \|_{H^{3/2}} \, dr \right) \left( \| \eta_0 \|_{H^s} + \int_0^t \| g(r) \|_{H^s} \, dr \right)
\]

Proof. Use \( p = p_2 = 2 \), \( N = 2 \) and \( \Sigma = s \) in proposition 2.1 of [9], along with the embedding \( H^{3/2} \hookrightarrow B_{2, \infty}^s \cap L^\infty \).

Lemma B.2. Let \( \eta \) be a solution to (B.1.1). Then there is a universal constant \( C > 0 \) such that for any \( 0 \leq s < 2 \)

\[
\sup_{0 \leq r \leq t} \| \eta(r) \|_{H^s(\Sigma)} \leq \exp \left( C \int_0^t \| Du(r) \|_{H^{3/2}(\Sigma)} \, dr \right) \left( \| \eta_0 \|_{H^s(\Sigma)} + \int_0^t \| g(r) \|_{H^s(\Sigma)} \, dr \right)
\]

Proof. The same as lemma A.9 in [2].

B.2. Elliptic Estimates. We need two different type of elliptic estimates in our proof either in weak sense or strong sense.

Lemma B.3. (weak solution) We call \( (u, p) \in H^1_0(\Omega) \times H^0(\Omega) \) a weak solution to the elliptic equation

\[
\begin{cases}
-\Delta u + \nabla p = \phi \in H^{-1}(\Omega) & \text{in } \Omega \\
\nabla \cdot u = \psi \in H^0(\Omega) & \text{in } \Omega \\
u = 0 & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma \setminus \Omega
\end{cases}
\]

if \( \nabla \cdot u = \psi \) in \( \Omega \) and \( (u, p) \) satisfy the weak formulation that for arbitrary \( f \in H^1_0(\Omega) \), it holds that

\[
\langle Du : Df \rangle_{H^0} + \langle p, \nabla \cdot f \rangle_{H^0} = \langle \phi, f \rangle_{H^{-1}}
\]

where \( \langle \cdot, \cdot \rangle_{H^0} \) represents the inner product in \( H^0(\Omega) \) and \( \langle \cdot, \cdot \rangle_{H^{-1}} \) denotes the dual pairing between \( H^1_0(\Omega) \) and \( H^{-1}(\Omega) \). Then we have the estimate

\[
\| u \|^2_{H^1} + \| p \|^2_{H^0} \lesssim \| \phi \|^2_{H^{-1}} + \| \psi \|^2_{H^0}
\]
Then we have the new weak formulation that for arbitrary $f \in H^0_0(\Omega)$, it is valid that
\begin{equation}
\langle Dw : Df \rangle_{H^0} + \langle p, \nabla \cdot f \rangle_{H^0} = \langle \phi, f \rangle_{H^{-1}} - \langle Dw : Df \rangle_{H^0}
\end{equation}

We may further restrict that $f$ satisfies $\nabla \cdot f = 0$ as the pressureless weak formulation that
\begin{equation}
\langle Dw : Df \rangle_{H^0} = \langle \phi, f \rangle_{H^{-1}} - \langle Dw : Df \rangle_{H^0}
\end{equation}

Riesz theorem implies that there exists a pressureless weak solution $w \in H^1_0(\Omega)$ satisfying $\nabla \cdot w = 0$ and the estimate
\begin{equation}
\|w\|^2_{H^1} \lesssim \|\phi\|^2_{H^{-1}} + \|\psi\|^2_{H^0} \lesssim \|\phi\|^2_{H^{-1}} + \|\psi\|^2_{H^0}
\end{equation}

Then it is well-known that pressure can be taken as the Lagrangian multiplier for the Navier-Stokes system. Similar to proposition 2.9 in [1], we have that there exists a $p \in H^1(\Omega)$ such that the weak formulation (B.2.5) holds and it satisfies
\begin{equation}
\|p\|^2_{H^1} \lesssim \|\phi\|^2_{H^{-1}} + \|\psi\|^2_{H^0} + \|w\|^2_{H^1} \lesssim \|\phi\|^2_{H^{-1}} + \|\psi\|^2_{H^0}
\end{equation}

Therefore, $(u, p)$ satisfy the weak formulation (B.2.2) and our result easily follows. \hfill \Box

**Lemma B.4. (strong solution)** Suppose $(u, p)$ solve the equation
\begin{equation}
\begin{cases}
-\Delta u + \nabla p = \phi \in H^{r-2}(\Omega) & \text{in } \Omega \\
\nabla \cdot u = \psi \in H^{r-1}(\Omega) & \text{in } \Omega \\
(pI - D_u)e_3 = \varphi \in H^{r-3/2}(\Sigma) & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
\end{equation}
in the strong sense. Then for $r \geq 2$,
\begin{equation}
\|u\|^2_{H^r} + \|p\|^2_{H^{r-1}} \lesssim \|\phi\|^2_{H^{r-2}} + \|\psi\|^2_{H^{r-1}} + \|\varphi\|^2_{H^{r-3/2}}
\end{equation}
whenever the right hand side is finite.

**Proof.** The same as lemma A.15 in [1]. \hfill \Box

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