STABLE INVARIANTS FOR MULTIDIMENSIONAL PERSISTENCE

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Abstract. In this paper we explain how to convert discrete invariants into stable ones via what we call hierarchical stabilization. We illustrate this process by constructing stable invariants for multi-parameter persistence modules with respect to so called simple noise systems. For one parameter, we recover the standard barcode information. For more than one parameter we prove that the constructed invariants are in general NP-hard to calculate.

1. Introduction

Data analysis aims to understand the mechanism of the source that generates the data. A dataset is typically given in the form of a finite set $U$ together with a sequence of $r$ maps out of $U$ into metric spaces called measurements. One way to analyze such data is to use these measurements to produce summaries or signatures describing the structure of the data with a hope of gaining some insight into the source. To reduce the dependency of the signatures on the accuracy of the measurements, which is necessary in the presence of heterogeneity, noise, variability, missing information etc., one can utilize homological tools from algebraic topology. Constructing and studying homology based stable summaries is the aim of topological data analysis (TDA). For $r = 1$ (only one measurement), this approach has been applied very successfully to for example data coming from medicine [3, 16, 19], biology [9, 13], robotics [6, 18] and image processing [5, 2, 1].

For $r > 1$, mathematical foundations still need to be developed to construct and understand homology based stable signatures. This is important as studying correlations between different measurements is essential in data analysis. The potential and value of the multi-parameter topological data analysis can be seen for example in [3] where information extracted from a 2-parameter rank invariant yields a better classification rate than the 1-parameter signatures given by barcodes. In section 3 we present a simple procedure inspired by hierarchical clustering which in our mind is the essence of persistence. This procedure, which we call hierarchical stabilization, is about converting discrete invariants into stable ones and we use it to define continuous multi-parameter invariants such as the stable rank.

Computational challenges are other road blocks for making multi-parameter homological stable signatures applicable. Describing these challenges is the focus of this paper. Our main result (Theorem 11.6) is proving NP-hardness of computing the stable rank invariant (introduced in [20] where it was called the feature counting function) with respect to a certain class of noise systems when $r \geq 2$. Since for $r = 1$, the stable rank invariant is similar to the usual barcode (Proposition 10.3),

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our main result illustrates computational difference between 1 and multi-parameter situations and why understanding correlations is a much harder problem. The entire paper is written with implementation in mind. For all the introduced or recalled concepts, such as Betti diagrams and numbers, considered noise systems (called simple), related shifts, and the stable rank, we indicate how they can be calculated or implemented.

The first step in TDA is to transform a data system (a finite set $\mathcal{U}$ with $r$ measurements on it), via for example the Cech, Vietoris-Rips, or their witness version constructions, into a multi-parameter simplicial complex. The result of this step is a functor $F: \mathbb{Q}^r \to \text{Spaces}$ indexed by the poset $\mathbb{Q}^r$ of $r$ tuples of non-negative rational numbers where $(v_1, \ldots, v_r) \geq (w_1, \ldots, w_r)$ if and only if $v_i \geq w_i$ for all $i$. By applying homology with coefficients in a field $K$ we get a so called multi-parameter persistence module $H_i(F, K): \mathbb{Q}^r \to \text{Vect}_K$ (a functor indexed by $\mathbb{Q}^r$ with values in the category of $K$ vector spaces). This step is important as it relaxes the dependency on the accuracy of the measurements. Since $\mathcal{U}$ is a finite set, the obtained functors are quite special: they are finitely generated and tame [20]. The category of tame functors with values in a category of vector spaces has properties similar to the category of graded modules over the polynomial ring in $r$ variables ([20]). In particular all tame functors have free resolutions of length no greater than $r$. It follows that for $r = 1$, similarly to finitely generated modules over a PID, finitely generated and tame functors can be classified. Barcoding is a particularly useful form of this classification for data analysis purposes since barcodes are not only computable in polynomial time in terms of the data system but more importantly barcodes are also stable (small changes in the data lead to small changes in its barcode). Since for $r > 1$ the moduli of tame and finitely generated functors is a very complicated algebraic variety ([8]), it is not possible to classify all such functors by easily visualizable, computable, and stable invariants. For $r > 1$, instead of a complete classification (as it is for $r = 1$), we need other methods of defining and extracting stable information about the data system out of the associated multi-parameter persistence module. It is exactly for that purpose we have developed the hierarchical stabilization. The need for a stabilisation procedure comes from the fact that algebraic invariants of multi-parameter persistence modules such as minimal number of generators, Betti tables, Hilbert polynomials etc. tend to change drastically when the initial data is altered even slightly. That is why the classical commutative algebra invariants are not useful for data analysis. For data analysis we need stable invariants. Illustrating the hierarchical stabilisation methods for the rank invariant and the so called simple noise systems is another aim of this paper.

2. Notation

2.1. We use bold letters and black-board bold letters to denote small categories and their sets of objects respectively. For example $\mathbf{I}$ denotes the small category whose set of objects is denoted by $\mathbbm{I}$.

Let $\mathbf{I}$ be a poset, $S \subseteq \mathbbm{I}$ a subset of its objects, and $i$ an object in $\mathbf{I}$. We write $S \leq i$ if $j \leq i$ for any $j$ in $S$.

For a poset $\mathbf{I}$, we use the symbol $\mathbf{I}_\infty$ to denote the poset obtained from $\mathbf{I}$ by adding an extra object $\infty$ such that $i \leq \infty$ for any $i$ in $\mathbbm{I}$. 


Similarly, numbers, while to be 0-1 functions of the form $f$ and $f^t$ with the set of morphisms from $v$ to $w$ being empty if $v \not\leq w$ and having a unique morphism $v \leq w$ otherwise. According to our convention, the symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of natural, non-negative rational and real numbers respectively. Similarly, $\mathbb{N}'$ and $\mathbb{Q}'$ denote the sets of $r$-tuples of natural and non-negative rational numbers, while $\mathbb{N}^r$ and $\mathbb{Q}^r$ denote the posets of $r$-tuples of natural and non-negative rational numbers respectively, with $(v_1, \ldots, v_r) \leq (w_1, \ldots, w_r)$ if $v_i \leq w_i$, for all $i$. According to this notation, $f: \mathbb{Q}^r \to \mathbb{N}^r$ denotes a function between sets while $g: \mathbb{Q}^r \to \mathbb{N}^r$ denotes a functor between categories, which in this case is a function with the property $g(v) \leq g(w)$ for any $v \leq w$ in $\mathbb{Q}^r$.

The symbol $e_n$ denotes the element in $\mathbb{N}^r$ whose coordinates are 0 except the $n$-th coordinate which is 1. For a subset $S \subset \{1, \ldots, r\}$, we set $e_S := \sum_{n \in S} e_n$. Elements of $S \subset \{1, \ldots, r\}$ are naturally ordered by their size $S = \{n_1 < \cdots < n_{|S|}\}$ and we call the index $i$ of the element $n_i$ its order in $S$.

2.2. Let $T$ be a set. A multi-subset of $T$ is by definition a function $f: T \to \mathbb{N}$. For such a multi-subset, its value $f(t)$ is called the multiplicity of $t$. The set $\{t \in T \mid f(t) \neq 0\}$ is called the support of $f$ and is denoted by supp$(f)$. A multi-subset is called finite if its support is finite, in which case the sum $\sum_{t \in T} f(t)$ is called the rank of $f$ and denoted by rank$(f)$. We use symbols Mult($T$) to denote the set of all multi-subsets of $T$.

2.3. An extended pseudometric on a set $T$ is a function $d: T \times T \to \mathbb{R} \cup \{\infty\}$ such that: $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y$, and $z$ in $T$ (here we of course take $r \leq \infty$ and $\infty + r = \infty$ for any $r$ in $\mathbb{R} \cup \{\infty\}$).

3. Hierarchical stabilization

The aim of clustering is to partition a data set into parts that aggregate elements sharing similar characteristics (whatever we choose them to be) and separate elements with different characteristics. One way to make a decision about which partition to choose is to assemble possible partitions into a dendrogram and study how different partitions are related to each other. Producing dendrograms, not just partitions, is what a hierarchical clustering is about. Dendrograms have an important advantage over partitions. They form a metric space (see for example [7]), which means that one can measure how close or how far apart dendrograms can be.

In this way one can study stability of a particular hierarchical clustering method, which is essential in data analysis. In this section we use the same hierarchical idea to show how to turn discrete invariants into stable and continuous ones.

Consider the set $\text{Mult}(\mathbb{Q})$ of multi-subsets of $\mathbb{Q}$ (see 2.2) which are simply functions of the form $f: \mathbb{Q} \to \mathbb{N}$. For $\epsilon$ in $\mathbb{Q}$, two multi-sets $f, g: \mathbb{Q} \to \mathbb{N}$ are defined to be $\epsilon$-close if, for any $\tau$ in $\mathbb{Q}$, the following inequalities hold: $g(\tau) \geq f(\tau + \epsilon)$ and $f(\tau) \geq g(\tau + \epsilon)$. This leads to an extended pseudometric (see 2.3) on the set $\text{Mult}(\mathbb{Q})$ called the interleaving distance:

$$d(f, g) := \begin{cases} 
\infty & \text{if } f \text{ and } g \text{ are not } \epsilon\text{-close for any } \epsilon \\
\inf\{\epsilon \mid f \text{ and } g \text{ are } \epsilon\text{-close}\} & \text{otherwise}
\end{cases}$$
From now on the symbol Mult(\(\mathbb{Q}\)) is used to denote the metric space of multi-subsets of \(\mathbb{Q}\) with the interleaving distance as a metric. In the context of persistence, interleaving distances have been introduced in [10] and have since then been extensively studied (see for example [15]).

Let \(T\) be a set. A discrete invariant on \(T\) is simply a function \(f: T \to \mathbb{N}\). The aim of this section is to explain how to stabilize \(f\) to obtain a continuous invariant. The first step is to choose an extended pseudometric \(d: T \times T \to \mathbb{R} \cup \{\infty\}\) on \(T\) (we can not talk about stability and continuity without being able to measure distances). For any such choice, we construct a function \(\hat{f}: T \to \text{Mult}(\mathbb{Q})\) called the \textit{(hierarchical) stabilization of} \(f\). For \(x\) in \(T\) and \(\tau\) in \(\mathbb{Q}\) define:

\[
\hat{f}(x)(\tau) := \min\{f(y) \mid d(x, y) \leq \tau\}
\]

Thus the value of the multi-set \(\hat{f}(x)\) at \(\tau\) is the smallest value \(f\) takes on the disc \(B(x, \tau) := \{y \in T \mid d(x, y) \leq \tau\}\). We stress again that the stabilization \(\hat{f}\) of \(f\) depends on the choice of an extended pseudometric on \(T\). Note that if \(\epsilon \geq \tau\), then \(B(x, \tau) \subset B(x, \epsilon)\) and consequently \(\hat{f}(x)(\tau) \geq \hat{f}(x)(\epsilon)\).

3.1. \textbf{Proposition.} For any choice of an extended pseudometric on \(T\), the function \(\hat{f}: T \to \text{Mult}(\mathbb{Q})\) is 1-Lipschitz, i.e. for any \(x\) and \(y\) in \(T\), \(d(x, y) \geq d(\hat{f}(x), \hat{f}(y))\).

\textbf{Proof.} If \(d(x, y) = \infty\) there is nothing to prove. Assume \(d(x, y) < \infty\). By the triangle inequality, for any \(\tau\) and \(\epsilon\) in \(\mathbb{Q}\) such that \(\epsilon \geq d(x, y)\), we have inclusions \(B(y, \tau) \subset B(x, \tau + \epsilon)\) and \(B(x, \tau) \subset B(y, \tau + \epsilon)\). It then follows that \(\hat{f}(y)(\tau) \geq \hat{f}(x)(\tau + \epsilon)\) and \(\hat{f}(x)(\tau) \geq \hat{f}(y)(\tau + \epsilon)\). That means that \(\hat{f}(x)\) and \(\hat{f}(y)\) are \(\epsilon\)-close. As this happens for all \(\epsilon \geq d(x, y)\), we can conclude \(d(x, y) \geq d(\hat{f}(x), \hat{f}(y))\). \(\Box\)

The input for the hierarchical stabilization has three ingredients: (i) a set \(T\), (ii) a discrete invariant \(f: T \to \mathbb{N}\), and (iii) a choice of an extended pseudometric on \(T\). The outcome is a 1-Lipschitz function \(\hat{f}: T \to \text{Mult}(\mathbb{Q})\). In the next few sections we illustrate how to apply this hierarchical stabilization process when (i) \(T\) is the set of finitely generated tame functors (see 6.2), (ii) \(f: T \to \mathbb{N}\) is a classical homological invariant such as the \(i\)-th Betti number (see 6.2), and (iii) the pseudometric on \(T\) is induced by a noise system (see Section 7). Our aim for this article has been to show that, for \(r \geq 2\), calculating the stabilization of the 0-th Betti number is in general an NP-hard problem, which we prove in Section 11.

4. Homological invariants of vector space valued functors

Let \(K\) be a field and \(I\) a poset. In this section we recall how homological invariants of certain functors of the form \(F: I \to \text{Vect}_K\) are defined and constructed.

4.1. \textbf{Freeness.} For \(i\) in \(I\), \(K_I(i, -): I \to \text{Vect}_K\) denotes the composition of the representable functor \(\text{mor}_I(i, -): I \to \text{Sets}\) with the linear span functor \(K: \text{Sets} \to \text{Vect}_K\). We often omit the subscript \(I\) and write \(K(i, -)\). Since \(I\) is a poset, \(K(i, j) = K\) if \(i \leq j\) and \(K(i, j) = 0\) if \(i \not\leq j\). The functor \(K(i, -)\) has the following Yoneda property explaining why it is called \textbf{free on one generator in degree} \(i\). For any \(F: I \to \text{Vect}_K\) the homomorphism \(\text{Nat}(K(i, -), F) \to F(i)\) that assigns to a natural transformation \(\phi: K(i, -) \to F\) the element \(\phi_i(id_i)\) in \(F(i)\) is an isomorphism. In this way any element \(g\) in \(F(i)\) (an element with coordinate \(i\), see 2.1) determines a unique natural transformation, denoted by the same symbol \(g: K(i, -) \to F\), which maps the element \(id_i\) in \(K(i, i)\) to \(g\) in \(F(i)\). In particular
there is a non-zero natural transformation $K(i, -) \rightarrow K(j, -)$ if and only if $j \leq i$. Moreover, any such non-zero natural transformation is a monomorphism. It follows that $K(i, -)$ and $K(j, -)$ are isomorphic if and only if $i = j$.

Functors of the form $\oplus_{i \in I}(K(i, -) \otimes V_i)$ are called free. By the Yoneda property, for any $F: I \rightarrow \text{Vect}_K$, a sequence of elements $\{g_s \in F(i_s)\}_{s \in S}$ determines a unique natural transformation denoted by $[g_s]_{s \in S}: \oplus_{s \in S} K(i_s, -) \rightarrow F$. This Yoneda property and the fact that $I$ is a poset imply that two free functors $\oplus_{i \in I}(K(i, -) \otimes V_i)$ and $\oplus_{i \in I}(K(i, -) \otimes W_i)$ are isomorphic if and only if, for any $i$ in $I$, the vector spaces $V_i$ and $W_i$ are isomorphic. Thus a free functor is up to an isomorphism determined by the sequence of vector spaces $\{V_i\}_{i \in I}$. Based on the properties of this sequence, we use the following dictionary about a free functor $P = \oplus_{i \in I}(K(i, -) \otimes V_i)$:

1. The vector space $V_i$ is called the 0-th homology of $P$ at $i$ and is denoted by $H_0P(i)$. With this notation $P$ is isomorphic to $\oplus_{i \in I}(K(i, -) \otimes H_0P(i))$.
2. The set $\{i \in I \mid H_0P(i) \neq 0\}$ is called the support of $P$ and is denoted by $\text{supp}(P)$.
3. $P$ is of finite type if $H_0P(i)$ is finite-dimensional for all $i$.
4. If $P$ is of finite type, the multi-subset $\beta_0P: I \rightarrow \mathbb{N}$ of $I$, defined as $\beta_0P(i) := \dim H_0P(i)$, is called the Betti diagram of $P$. Two finite type free functors are isomorphic if and only if they have the same Betti diagrams.
5. $P$ is of finite rank if it is of finite type and its Betti diagram $\beta_0P$ is a finite multi-set (see 2.2).
6. Let $P$ be of finite rank. The number $\text{rank}(\beta_0P) := \sum_{i \in I} \dim H_0P(i)$ is also called the rank or the Betti number of $P$ and is denoted by $\text{rank}(P)$.

4.2. Minimality. Recall that a morphism $\phi: X \rightarrow Y$ in a category is called minimal if any morphism $f: X \rightarrow X$ satisfying $\phi = \phi f$ is an isomorphism (see [4]). A natural transformation $\phi: P \rightarrow F$ of functors indexed by $I$ with values in $\text{Vect}_K$ is called a minimal cover of $F$ if $P$ is free and $\phi$ is both minimal and an epimorphism. Minimal covers are unique up to an isomorphism: if $\phi: P \rightarrow F$ and $\phi': P' \rightarrow F$ are minimal covers of $F$, then there is an isomorphism (not necessarily unique) $f: P \rightarrow P'$ such that $\phi = \phi'f$. Furthermore any $g: P \rightarrow P'$, for which $\phi = \phi'g$, is an isomorphism (a consequence of minimality). Note however that in this generality a minimal cover may not exist.

A set $\{g_s \in F(i_s)\}_{s \in S}$ generates a functor $F: I \rightarrow \text{Vect}_K$ if the induced natural transformation $[g_s]_{s \in S}: \oplus_{s \in S} K(i_s, -) \rightarrow F$ is an epimorphism. A functor is finitely generated if it is generated by a finite set. It is called cyclic if it is generated by one element. A free functor is finitely generated if and only if it is of finite rank. For a finitely generated $F$, a set $\{g_s \in F(i_s)\}_{s \in S}$ is called a minimal set of generators if it generates $F$ and no sequence with fewer elements can generate $F$.

Assume $F$ is finitely generated and admits a minimal cover $P \rightarrow F$. Then $P$ is free (by definition) and also finitely generated. For such a functor admitting a minimal cover, $\{g_s \in F(i_s)\}_{1 \leq s \leq n}$ is its minimal set of generators if and only if the natural transformation $[g_s]_{1 \leq s \leq n}: \oplus_{1 \leq s \leq n} K(i_s, -) \rightarrow F$ is a minimal cover. It follows that all minimal sets of generators of $F$ have the following common property: the number of generators in such a set that belong to $F(i)$ is equal to the value of the Betti diagram $\beta_0P(i)$. In particular, the number of elements in a minimal set of generators of $F$ is equal to the rank of $P$ and the set of coordinates of elements in a minimal set of generators of $F$ coincides with $\text{supp}(P)$.
4.3. **Resolutions.** An exact sequence \( \cdots \rightarrow P_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} F \xrightarrow{\delta_{-1}} 0 \) of functors indexed by \( I \) with values in \( \text{Vect}_K \) is called a **minimal free resolution** of \( F \) if \( P_n \rightarrow \text{Ker}(\delta_{n-1}) \) is a minimal cover for any \( n \geq 0 \) (see 4.2). Any two minimal resolutions of \( F \) are isomorphic. Assume \( F : I \rightarrow \text{Vect}_K \) admits a minimal resolution as above. For such a functor we are going to use the following dictionary:

1. The vector space \( H_0 P_n(i) \) (see 4.1) is called the **\( n \)-th homology** of \( F \) at \( i \) and is denoted by \( H_n F(i) \). With this notation, \( P_n \) is isomorphic to \( \bigoplus_{i \in I}(K(i, -) \otimes H_n F(i)) \).
2. The set \( \{ i \in I \mid H_0 F(i) \neq 0 \} \) is called the **support** of \( F \) and is denoted by \( \text{supp}(F) \).
3. \( F \) is of **finite type** if \( P_n \) is of finite type for any \( n \geq 0 \) (see 4.1).
4. If \( F \) is of finite type, the multi-subset \( \beta_n F : I \rightarrow \mathbb{N} \) of \( I \), defined as \( \beta_n F(i) := \dim H_n F(i) \), is called the **\( n \)-th Betti diagram** of \( F \).
5. \( F \) is of **finite rank** if \( P_n \) is of finite rank for any \( n \geq 0 \) (see 4.1). Thus \( F \) is of finite rank if and only if it is of finite type and all its Betti diagrams \( \beta_n F \) are finite for all \( n \geq 0 \).
6. Let \( F \) be of finite rank. For \( n \geq 0 \), the number \( \text{rank}(\beta_n F) = \sum_{i \in I} \beta_n F(i) = \sum_{i \in I} \dim H_n F(i) \) is called the **\( n \)-th Betti number** of \( F \) and is denoted by \( \text{rank}_n F \). We also use the term **rank** of \( F \) to denote its \( 0 \)-th Betti number, \( \text{rank}_0 F \).

Note that all the notions defined in this paragraph do not depend on the choice of a minimal free resolution of \( F \).

4.4. **Bars.** Let \( a \leq b \) be comparable objects in the poset \( I \). Consider the natural transformation \( K(b, -) \rightarrow K(a, -) \) induced by the element \( a \leq b \) in \( \text{mor}_I(a, b) \). The cokernel of this natural transformation is denoted by \( [a, b) \). The functor \( [a, b) \) is called the **bar** starting in \( a \) and ending in \( b \). The exact sequence \( 0 \rightarrow K(b, -) \rightarrow K(a, -) \rightarrow [a, b) \rightarrow 0 \) is a minimal free resolution. Consequently:

\[
H_n[a, b)(i) = \begin{cases} 
K & \text{if } n = 0, i = a \\
K & \text{if } n = 1, i = b \\
0 & \text{otherwise}
\end{cases}
\beta_n[a, b)(i) = \begin{cases} 
1 & \text{if } n = 0, i = a \\
1 & \text{if } n = 1, i = b \\
0 & \text{otherwise}
\end{cases}
\]

\[
\text{rank}_n[a, b) = \begin{cases} 
1 & \text{if } n = 0 \text{ or } n = 1 \\
0 & \text{otherwise}
\end{cases}
\]

5. **Functors indexed by \( \mathbb{N}^r \)**

The aim of this section is to recall how to effectively calculate the homology, the Betti diagrams, and Betti numbers of functors of the form \( F : \mathbb{N}^r \rightarrow \text{Vect}_K \). We call such functors **frames**. All the material presented here is standard as the category of frames is equivalent to the category of \( r \)-graded modules over the polynomial ring with \( r \) variables and coefficients in the field \( K \).

5.1. **Semisimplicity.** A functor \( F : \mathbb{N}^r \rightarrow \text{Vect}_K \) is called **semisimple** if \( F(v < u) \) is the zero homomorphism for any non-identity relation \( v < u \). For example, the unique functor \( U_w : \mathbb{N}^r \rightarrow \text{Vect}_K \) such that \( U_w(w) = K \) and \( U_w(v) = 0 \) if \( v \neq w \) is semisimple. A semisimple functor \( F : \mathbb{N}^r \rightarrow \text{Vect}_K \) is isomorphic to the direct sum \( \bigoplus_{v \in \mathbb{N}^r}(U_v \otimes F(v)) \) and thus two such functors are isomorphic if and only if they have isomorphic values.
5.2. **Semisimplifications.** Consider a frame $F : \mathbb{N}^r \to \text{Vect}_K$. For an object $v$ in $\mathbb{N}^r$ and $i \in \mathbb{N}$, define $\delta_i : \Delta F(v)_i \to \Delta F(v)_{i-1}$ to be the homomorphism:

$$
\delta_i : \bigoplus_{S \subset \{1, \ldots, r\} \mid |S| = i} F(v - e_S) \to \bigoplus_{T \subset \{1, \ldots, r\} \mid |T| = i - 1} F(v - e_T)
$$

given by the matrix with the following coordinates (see 2.1 for the notation):

$$(\delta_i)_{T,S} := \begin{cases} 0 & \text{if } T \not\subset S \\ (-1)^a F((v - e_S) < (v - e_T)) & \text{if } T \subset S \text{ and where } a \text{ is the order in } S \text{ of the only element in } S \setminus T \end{cases}$$

Note that $\Delta F(v)_i = 0$ if $i > r$. A consequence of the fact that $F$ is a functor is the equality $\delta_i \delta_{i-1} = 0$. Thus, for a given $v$ and all $i$ in $\mathbb{N}$, these homomorphisms define a chain complex denoted by $\Delta F(v)$ and called the **Koszul complex** of $F$ at $v$. Note further that if $v \leq w$, then the homomorphism $\Delta F(v) \to \Delta F(w)$, induced by $F((v - e_S) \leq (w - e_S))$ for any $S \subset \{1, \ldots, r\}$, is a map of chain complexes. These maps in fact define a functor $\Delta F: \mathbb{N}^r \to \text{Ch}_>(K)$ with values in the category of non-negative chain complexes $\text{Ch}_>(K)$.

By taking the $i$-th homology we obtain a frame $H_i(\Delta F): \mathbb{N}^r \to \text{Vect}_K$. Note that if $v < w$, then $\Delta F(v < w)$ maps any summand $F(v - e_S)$ in $\Delta F(v)_i$ into the image of $\delta_{i+1}: \Delta F(w)_{i+1} \to \Delta F(w)_i$. This means that $H_i(\Delta F)(v < w)$ is the trivial homomorphism and consequently the homology functors $H_i(\Delta F): \mathbb{N}^r \to \text{Vect}_K$ are all semisimple. We therefore also call them **semisimplifications** of $F$. For example:

$$
H_i(\Delta U_w) = \bigoplus_{S \subset \{1, \ldots, r\} \mid |S| = i} U_{w + e_S} \quad \text{and} \quad H_i(\Delta K(w, -)) = \begin{cases} U_w & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}
$$

$$
H_i(\Delta \oplus_{v \in \mathbb{N}^r} (K(v, -) \otimes V_v)) = \begin{cases} \oplus_{v \in \mathbb{N}^r} U_v \otimes V_v & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}
$$

Thus if $F$ is free, then $F$ is isomorphic to $\oplus_{v \in \mathbb{N}^r} K(v, -) \otimes H_0(\Delta F(v))$ and we see that in this case $H_0(\Delta F(v))$ is isomorphic to $H_0 F(v)$ as defined in 4.1.

If $v$ is a minimal element in the poset $\mathbb{N}^r$ for which $F(v) \neq 0$, then $\Delta F(v) = F(v)$ and hence $H_0(\Delta F)(v) = F(v)$. It follows that $F \neq 0$ if and only if $H_0(\Delta F) \neq 0$. Thus the 0-th semisimplification detects non triviality of a frame.

Let $0 \to F_0 \to F_1 \to F_2 \to 0$ be an exact sequence of frames. As the Koszul complex is formed by taking direct sums and direct sums preserve exactness, we get an exact sequence of chain complexes $0 \to \Delta F_0 \to \Delta F_1 \to \Delta F_2 \to 0$. By taking the homology we then obtain a long exact sequence of semisimple frames:

$$
\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
\end{array}
$$
For example, let \( v < w \) in \( \mathbb{N}^r \) and consider the exact sequence of frames \( 0 \to K(w, -) \to K(v, -) \to [v, w] \to 0 \) (see 4.4). This leads to an exact sequence of semisimple frames \( 0 \to H_1(\Delta[v, w]) \to H_0(\Delta K(w, -)) \to H_0(\Delta K(v, -)) \to H_0(\Delta[v, w]) \to 0 \). Thus:

\[
H_i(\Delta[v, w]) = \begin{cases} 
U_v & \text{if } i = 0 \\
U_w & \text{if } i = 1 \\
0 & \text{if } i > 1
\end{cases}
\]

Since \( H_0 \) detects non triviality, it also detects epimorphisms: a natural transformation \( \phi: F_1 \to F_2 \) is an epimorphism if and only if \( H_0(\Delta \phi): H_0(\Delta F_1) \to H_0(\Delta F_2) \) is an epimorphism. Detection of epimorphisms and non triviality combine with the above long exact sequence can be used to show:

- \([g_s]_{s \in S}: \bigoplus_{s \in S} K(w_s, -) \to F\) is an epimorphism if and only if \( H_0(\Delta [g_s]_{s \in S}) \) is an epimorphism.
- \( \phi: P_0 \to F \) is a minimal cover (see 4.2) if and only if \( P_0 \) is free and \( H_0(\Delta \phi) \) is an isomorphism.
- Any frame \( F \) admits a minimal resolution \( \cdots \to P_1 \to P_0 \to F \to 0 \).
- If \( \cdots \to P_1 \to P_0 \to F \to 0 \) is a minimal resolution, then (i) \( H_n F(v) \) and \( H_n(\Delta F(v)) \) are isomorphic, (ii) \( P_n \) is isomorphic to \( \oplus_{v \in \mathbb{N}^r} K(v, -) \otimes H_n(\Delta F(v)) \), (iii) \( P_n = 0 \) for \( n > r \).

The main point of this paragraph is: the Koszul complex \( \Delta F \) is a very effective tool for calculating the homology of a frame as defined in 4.3.

5.3. Betti diagrams, numbers, and Euler characteristic. Since polynomial rings over fields are Noetherian, a subfunctor of a finitely generated frame is also finitely generated. It follows that if \( F: \mathbb{N}^r \to \text{Vect}_K \) is finitely generated, then so are all the functors in its minimal resolution \( \cdots \to P_1 \to P_0 \to F \to 0 \) and all its semisimplifications \( H_n(\Delta F) \). It follows that \( F \) is finitely generated if and only if it is of finite rank as defined in 4.3. For such an \( F \) we can use the Koszul complex to calculate its \( n \)-th Betti diagram and Betti number as follows:

\[
\beta_n F(v) = \dim H_n(\Delta F(v)) \quad \text{rank}_n F = \sum_{v \in \mathbb{N}^r} \dim H_n(\Delta F)(v)
\]

In particular, \( \sum_{v \in \mathbb{N}^r} \dim H_0(\Delta F)(v) \) is the minimal number of generators of \( F \).

Since for \( n > r \), \( \text{rank}_n(F) = 0 \), we can define the Euler characteristic for finitely generated frames as follows:

\[
\chi(F) := \sum_{n=0}^r (-1)^n \text{rank}_n F
\]

For example:

\[
\beta_n K(w, -)(v) = \begin{cases} 
1 & \text{if } n = 0 \text{ and } v = w \\
0 & \text{otherwise}
\end{cases} \quad \text{rank}_n K(w, -) = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{if } n > 0
\end{cases} \quad \chi(K(w, -)) = 1
\]

\[
\beta_n [u, w](v) = \begin{cases} 
1 & \text{if } n = 0 \text{ and } v = u \\
1 & \text{if } n = 1 \text{ and } v = w \\
0 & \text{otherwise}
\end{cases} \quad \text{rank}_n [u, w] = \begin{cases} 
1 & \text{if } n \leq 1 \\
0 & \text{if } n > 1
\end{cases}
\]
\[
\chi((u, w)) = 0
\]

\[
\beta_n U_w(v) = \begin{cases}
1 & \text{if } v = w + e_S \text{ where } S \subset \{1, \ldots, r\} \text{ and } |S| = n \\
0 & \text{otherwise}
\end{cases}
\]

\[
\text{rank}_n U_w = \binom{r}{n} \quad \chi(U_w) = 0
\]

6. Tame functors

In this section we recall the definition and basic properties of tame functors introduced in [20]. We also explain how to calculate homological invariants such as Betti diagrams, Betti numbers, and the minimal number of generators of tame functors.

6.1. Tameness. Choose a positive rational number \(\alpha\) called a resolution and consider two functors: \(\alpha: \mathbb{N}^r \rightarrow \mathbb{Q}^r\) which is the multiplication by \(\alpha\) that maps \((n_1, \ldots, n_r)\) to \((\alpha n_1, \ldots, \alpha n_r)\) and \([\alpha^{-1}]: \mathbb{Q}^r \rightarrow \mathbb{N}^r\) that maps \(v = (v_1, \ldots, v_r)\) to \([\alpha^{-1}v] := \left(\left\lfloor \frac{v_1}{\alpha} \right\rfloor, \ldots, \left\lfloor \frac{v_r}{\alpha} \right\rfloor\right)\) where for a non-negative rational number \(t\), the symbol \([t]\) denotes the biggest natural number smaller or equal than \(t\). Note that the composition \([\alpha^{-1}]\alpha: \mathbb{N}^r \rightarrow \mathbb{N}^r\) is the identity and the functor \([\alpha^{-1}]\) is constant on all sub-posets of the form:

\[
[\alpha n_1, \alpha(n_1 + 1)] \times [\alpha n_2, \alpha(n_2 + 1)] \times \cdots \times [\alpha n_r, \alpha(n_r + 1)] \subset \mathbb{Q}^r
\]

for any \(r\)-tuple of natural numbers \((n_1, \ldots, n_r)\). We call these sub-posets right open \(\alpha\)-cubes.

A functor \(G: \mathbb{Q}^r \rightarrow \text{Vect}_K\) is called \(\alpha\)-tame if it is isomorphic to the composition \(F[\alpha^{-1}]\) for some \(F: \mathbb{N}^r \rightarrow \text{Vect}_K\) (called an \(\alpha\)-frame of \(G\)). Since \([\alpha^{-1}]\alpha\) is the identity, the frame \(F\) has to be necessarily isomorphic to \(G\alpha: \mathbb{N}^r \rightarrow \text{Vect}_K\). It is then clear that \(G: \mathbb{Q}^r \rightarrow \text{Vect}_K\) is \(\alpha\)-tame if and only if its restriction to any right open \(\alpha\)-cube is isomorphic to a constant functor. This happens if and only if \(G(\alpha[\alpha^{-1}v] \leq v)\) is an isomorphism for any \(v\) in \(\mathbb{Q}^r\).

A functor \(G: \mathbb{Q}^r \rightarrow \text{Vect}_K\) is called tame, if it is \(\alpha\)-tame for some \(\alpha\) (called a resolution of \(G\)). For example the free functor \(K(w, -): \mathbb{Q}^r \rightarrow \text{Vect}_K\) on one generator in degree \(w = (w_1, \ldots, w_r)\) (see 4.1) is tame. Let \(n_i = 0\) and \(d_i = 1\) if \(w_i = 0\) and, in the case \(w_i \neq 0\), let \(n_i\) and \(d_i\) to be coprime natural numbers such that \(w_i = \frac{n_i}{d_i}\). Let \(d\) be a common multiple of \(d_1, \ldots, d_r\). Then \(K(w, -)\) is constant on any right open \(1/d\)-cube in \(\mathbb{Q}^r\) and hence it is \(1/d\)-tame.

Tameness is preserved by finite direct sums (see [20, Corollary 5.3]). More generally, for an exact sequence \(0 \rightarrow G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow 0\) of functors indexed by \(\mathbb{Q}^r\), if two out of \(G_0, G_1, G_2\) are tame, then so is the third one. For example, for any \(v \leq w\) in \(\mathbb{Q}^r\), the functor \([v, w]\) (see 4.4) is tame. Tameness however is not preserved in general by subfunctors or quotients. For example consider \(K((0, 0), -): \mathbb{Q}^2 \rightarrow \text{Vect}_K\). Its subfunctor \(G_0 \subset K((0, 0), -)\) given by:

\[
G_0(v_1, v_2) = \begin{cases}
0 & \text{if } v_1 + v_2 < 1 \\
K & \text{if } v_1 + v_2 \geq 1
\end{cases}
\]

is not tame. Neither is the quotient \(K((0, 0), -)/G_0\). Note that in this example \(G_0\) is not finitely generated. If \(\phi: G_0 \rightarrow G_1\) is a natural transformation of tame functors, then according to [20, Proposition 5.2] \(\ker(\phi), \text{coker}(\phi), \text{and im}(\phi)\) are also
tame. It follows that a finitely generated subfunctor of a tame functor is always tame. Moreover a subfunctor of a finitely generated and tame functor is tame if and only if this subfunctor is finitely generated.

### 6.2. Betti diagrams, numbers, and Euler characteristic

Consider an \( \alpha \)-tame functor \( G : Q^r \to \text{Vect}_K \). Let \( \cdots \to P_n \to \cdots \to P_0 \to G\alpha \to 0 \) be a minimal free resolution of its frame \( G\alpha : N^r \to \text{Vect}_K \). According to 5.2, the functor \( P_n \) is isomorphic to \( \bigoplus_{v \in N^r} (K_{Q^r}(\alpha v, -) \otimes H_n(\Delta(G\alpha))(v)) \). Precomposing with \( [\alpha^{-1}] : Q^r \to N^r \) is a faithful and exact operation. Moreover \( K_{Q^r}(\alpha v, -)[\alpha^{-1}] \) is isomorphic to \( K_{Q^r}(\alpha v, -) \). These facts imply that:

\[
\cdots \to P_n[\alpha^{-1}] \to \cdots \to P_0[\alpha^{-1}] \to G\alpha[\alpha^{-1}] = G \to 0
\]

is a minimal resolution of \( G \) and \( P_n[\alpha^{-1}] \) is isomorphic to:

\[
\bigoplus_{v \in N^r} (K_{Q^r}(\alpha v, -) \otimes H_n(\Delta(G\alpha))(v))
\]

We can use these observations to conclude:

- Any \( \alpha \)-tame functor has a minimal free resolution by \( \alpha \)-tame functors. The length of such a resolution does not exceed \( r \).
- \( H_n G(u) = \begin{cases} H_n \Delta(G\alpha)(v) & \text{if } u = \alpha v \\ 0 & \text{otherwise} \end{cases} \)
- \( G \) is finely generated if and only if \( G\alpha \) is finitely generated.
- \( G \) is finitely generated if and only if it is of finite rank (see 4.3).
- If \( \phi : G_0 \to G_1 \) is a natural transformation between finitely generated tame functors, then \( \ker(\phi) \), \( \text{im}(\phi) \), and \( \text{coker}(\phi) \) are also finitely generated tame functors.
- If \( G \) is finitely generated, then:

\[
\beta_n G(u) = \begin{cases} \beta_n (G\alpha)(v) = \dim H_n \Delta(G\alpha)(v) & \text{if } u = \alpha v \\ 0 & \text{otherwise} \end{cases}
\]

\[
\text{rank}_n G = \sum_{v \in N^r} \dim H_n(\Delta G\alpha)(v)
\]

- Let \( G \) be finitely generated and let \( \{g_s \in G(v_s)\}_{s=1}^n \) be a minimal set of generators for \( G \). Then the number of generators in this set whose coordinate is \( v \) (i.e. generators that belong to \( G(v) \)) is given by \( \beta_0 G(v) \). In particular \( n = \text{rank}_0 G \). Moreover the set of coordinates of the generators coincides with \( \text{supp}(G) \).

A finitely generated tame functor is of finite rank and the length of its minimal free resolution is not exceeding \( r \). For such a functor \( G \), we can define its Euler characteristic as:

\[
\chi(G) := \sum_{n=1}^r (-1)^n \text{rank}_n G
\]

Let the symbol \( T(Q^r, \text{Vect}_K) \) denote the set of finitely generated tame functors. The functions \( \text{rank}_n, |\cdot| : T(Q^r, \text{Vect}_K) \to \mathbb{N} \) (the \( n \)-th rank and the absolute value of the Euler characteristic) are the discrete invariants we plan stabilize (see section 3). The last ingredient needed for the hierarchical stabilization is a choice of a pseudometric on \( T(Q^r, \text{Vect}_K) \). Our next step is to recall how to construct such metrics using so called noise systems [20], which is the subject of the next section.
7. Noise systems

Recall from [20, Definition 6.1] that a noise system is a sequence \( C = \{C_\epsilon\}_{\epsilon \in \mathbb{Q}} \) indexed by non-negative rational numbers, called components, such that:

1. the zero functor belongs to \( C_\epsilon \) for any \( \epsilon \) in \( \mathbb{Q} \),
2. if \( \tau \leq \epsilon \), then \( C_\tau \subseteq C_\epsilon \),
3. if \( 0 \rightarrow G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow 0 \) is an exact sequence of finitely generated tame functors, then:
   a. if \( G_1 \) is in \( C_\epsilon \), then so are \( G_0 \) and \( G_2 \),
   b. if \( G_0 \) is in \( C_\tau \) and \( G_2 \) is in \( C_\epsilon \), then \( G_1 \) is in \( C_{\epsilon+\tau} \).

A noise system \( \{C_\epsilon\}_{\epsilon \in \mathbb{Q}} \) is closed under direct sums if, for any \( \epsilon \), the set \( C_\epsilon \) is closed under direct sums: \( G_0 \) and \( G_1 \) belong to \( C_\epsilon \) if and only if their direct sum \( G_0 \oplus G_1 \) belongs to \( C_\epsilon \). Noise systems closed under direct sums are quite special. They are determined by cyclic functors (see 4.2):

7.1. Proposition. Let \( C = \{C_\epsilon\}_{\epsilon \in \mathbb{Q}} \) and \( D = \{D_\epsilon\}_{\epsilon \in \mathbb{Q}} \) be noise systems closed under direct sums. Then \( C_\epsilon = D_\epsilon \) if and only if they contain the same cyclic functors.

Proof. Assume \( C_\epsilon \) and \( D_\epsilon \) contain the same cyclic functors. Let \( G \) be in \( C_\epsilon \) and choose its minimal set of generators \( \{g_s \in G(v_s)\}_{s=1}^n \). Then, for any \( s \), the cyclic subfunctor of \( G \) generated by \( g_s \) also belongs to \( C_\epsilon \). By assumption, these cyclic subfunctors belong to \( D_\epsilon \) and so does their direct sum. As a quotient of this direct sum, the functor \( G \) is therefore a member of \( D_\epsilon \). This shows the inclusion \( C_\epsilon \subseteq D_\epsilon \). By symmetry we then get an equality \( C_\epsilon = D_\epsilon \). □

We think about the component \( C_\epsilon \) as the closed disc of radius \( \epsilon \) around the zero functor and regard its members as functors which are \( \epsilon \) small. This notion of smallness can be extended to a notion of proximity among arbitrary finitely generated tame functors \( G_0 \) and \( G_1 \) as follows. First, recall [20, Section 8] that a natural transformation \( \psi \) between functors in \( \mathcal{T}(Q', \text{Vec}_K) \) is called an \( \epsilon \)-equivalence if \( \ker(\psi) \) is in \( C_\epsilon \), \( \coker(\psi) \) is in \( C_0 \), and \( a+b \leq \epsilon \). Second, define \( G_0 \) and \( G_1 \) to be \( \epsilon \)-close if there exists \( G_2 \) in \( \mathcal{T}(Q', \text{Vec}_K) \) and natural transformations \( G_0 \leftarrow G_2 : \phi \) and \( \psi : G_2 \rightarrow G_1 \) such that \( \phi \) is a \( \tau \)-equivalence, \( \psi \) is a \( \mu \)-equivalence, and \( \tau + \mu \leq \epsilon \).

Finally set:

\[
d(G_0, G_1) := \begin{cases} 
\inf \{ \epsilon \mid G_0 \text{ and } G_1 \text{ are } \epsilon\text{-close} \} & \text{if } G_0 \text{ and } G_1 \text{ are close for some } \epsilon \\
\infty & \text{otherwise}
\end{cases}
\]

A key result of [20] states that the above defined \( d \) is a pseudometric on the set \( \mathcal{T}(Q', \text{Vec}_K) \). For \( F \) in \( \mathcal{T}(Q', \text{Vec}_K) \), denote the set of all functors that are \( \epsilon \)-close to \( F \) by \( B(F, \epsilon) \). This is the closed disc around \( F \) of radius \( \epsilon \) with respect to the metric \( d \). For example \( B(0, \epsilon) = C_\epsilon \).

We now have all three ingredients needed for hierarchical stabilization (see Section 3):

(i) a set \( \mathcal{T}(Q', \text{Vec}_K) \),
(ii) discrete invariants \( \text{rank}_n, |\chi| : \mathcal{T}(Q', \text{Vec}_K) \rightarrow \mathbb{N} \),
(iii) a pseudometric (induced by a noise system) on \( \mathcal{T}(Q', \text{Vec}_K) \).
For any choice of noise system \( \{ C_r \}_{r \in Q} \), by minimizing over the discs \( B(G, \tau) \), we obtain 1-Lipschitz functions (see Section 3):
\[
\hat{\text{rank}}_0, [\chi] : \mathbb{T}(Q^r, \text{Vect}_K) \to \text{Mult}(Q)
\]
For example, let \( v \leq w \) be in \( Q^r \). Then:
\[
\hat{\text{rank}}_0 K(v, -)(\tau) = \begin{cases} 1 & \text{if } K(v, -) \not\in S, \\ 0 & \text{if } K(v, -) \in S. \end{cases}
\]
\[
\hat{\text{rank}}_0[v, w](\tau) = \begin{cases} 1 & \text{if } [v, w] \not\in S, \\ 0 & \text{if } [v, w] \in S. \end{cases}
\]

The rest of the paper is devoted to explaining a strategy of how to calculate the invariant \( \hat{\text{rank}}_0 \) with respect to so called simple noise systems.

8. Simple noise systems

Recall that the 0-th Betti number \( \hat{\text{rank}}_0 G \) (also called the rank) of a finitely generated tame functor \( G \) is equal to the size of any minimal set of generators of \( G \). Let \( C = \{ C_r \}_{r \in Q} \) be a noise system and \( G : Q^r \to \text{Vect}_K \) a finitely generated tame functor. By definition, to calculate the value of \( \hat{\text{rank}}_0 G : Q \to \mathbb{N} \) at \( \tau \), we need to find the smallest 0-th Betti number among the functors in the disc \( B(G, \tau) \).

In general, the set \( B(G, \tau) \) might be infinite even in the case the underlying field is finite. Our strategy is to understand under what circumstances we only need to calculate the 0-th Betti numbers of finitely many functors in order to identify \( \hat{\text{rank}}_0 G(\tau) \). For that purpose consider the following collection of subfunctors of \( G \):
\[
B_{\leq}(G, \tau) := \{ G_0 \mid G_0 \text{ is a tame subfunctor of } G \text{ such that } G/G_0 \text{ belongs to } C_r \}
\]
This collection may or may not have any minimal element with respect to the inclusion relation. However, in the case that \( C_r \) is closed under direct sums, if some minimal element in \( B_{\leq}(G, \tau) \) exists, than it has to be unique:

8.1. Proposition. Assume \( C_r \) is closed under direct sums. Let \( F \) and \( G \) be finitely generated tame functors.

1. If \( G_0 \) and \( G_1 \) are minimal elements in \( B_{\leq}(G, \tau) \), then \( G_0 = G_1 \).
2. Let \( G[\tau] \) and \( F[\tau] \) be minimal elements in \( B_{\leq}(G, \tau) \) and \( B_{\leq}(F, \tau) \) respectively. Then any natural transformation \( \phi : G \to F \) maps \( G[\tau] \) into \( F[\tau] \). Moreover, if \( \phi \) is an epimorphism, then so is its restriction \( \phi : G[\tau] \to F[\tau] \).
3. If \( F[\tau], G[\tau] \) and \( (F \oplus G)[\tau] \) exist, then \( (F \oplus G)[\tau] = F[\tau] \oplus G[\tau] \).

Proof. (1): The functor \( G/G_0 \oplus G/G_1 \) belongs to \( C_r \) as the noise is closed under direct sums. The image of the homomorphism \( \pi : G \to G/G_0 \oplus G/G_1 \), given by the projections, is therefore also a member of \( C_r \) and consequently \( \ker(\pi) = G_0 \cap G_1 \) belongs to \( B_{\leq}(G, \tau) \). Minimality of \( G_0 \) and \( G_1 \) implies \( G_0 = G_0 \cap G_1 = G_1 \).

(2): Since \( F/(F[\tau]) \) belongs to \( C_r \), then so does the image of the composition of \( \phi : G \to F \) and the quotient \( F \to F/(F[\tau]) \). The kernel of this composition therefore belongs to \( B_{\leq}(G, \tau) \). By construction \( \phi \) maps this kernel into \( F[\tau] \). To finish the argument just notice that, by minimality, \( G[\tau] \) is a subfunctor of this kernel.

If \( \phi \) is an epimorphism, then \( F/\phi(G[\tau]) \) is a quotient of \( G/(G[\tau]) \). Consequently \( F/\phi(G[\tau]) \) belongs to \( C_r \) and hence we have an inclusion \( F[\tau] \subset \phi(G[\tau]) \). This together with what we already have shown implies equality \( F[\tau] = \phi(G[\tau]) \).

(3): Since \( C_r \) is closed under direct sums, we have \( (F \oplus G)[\tau] \subset F[\tau] \oplus G[\tau] \). By part (2), the inclusions \( F \subset F \oplus G \supset G \) induce inclusions \( F[\tau] \subset (F \oplus G)[\tau] \supset G[\tau] \). That gives \( F[\tau] \oplus G[\tau] \subset (F \oplus G)[\tau] \). \(\square\)
In the case $C_\tau$ is closed under direct sums, if it exists, we denote this minimal element in $B_{C}(G, \tau)$ by $G_{C}[\tau] \subset G$, or simply by $G[\tau] \subset G$ if the noise system is fixed. The subfunctor $G[\tau] \subset G$ is also called the $\tau$-shift of $G$. Note that in the case $G[\tau] \subset G$ exists, since $C_\tau$ is closed under taking quotients, a tame subfunctor $G_0 \subseteq G$ has the property that $G/G_0$ belongs to $C_\tau$ if and only if $G[\tau] \subseteq G_0$. Thus in this case $B_\tau(G, \tau)$ can be identified with the set:

$$\{G_0 \mid G_0 \text{ is a tame subfunctor of } G \text{ such that } G[\tau] \subseteq G_0 \subseteq G\}$$

We are now ready to define what we call a simple noise system.

**8.2. Definition.** A noise system $C = \{C_\epsilon\}_{\epsilon \in \mathbb{Q}}$ is called simple if:

- it is closed under direct sums,
- for any finitely generated tame functor $G : \mathbb{Q}^r \to \text{Vect}_K$ and any $\tau$ in $\mathbb{Q}$, the set $B_\tau(G, \tau)$ contains the minimal element $G[\tau] \subset G$,
- $\text{rank}_0 G[\tau] \leq \text{rank}_0 G$ for any $\tau$ in $\mathbb{Q}$.

**8.3. Example.** Let $C = \{C_\epsilon\}_{\epsilon \in \mathbb{Q}}$ be a noise system. For $\epsilon$ in $\mathbb{Q}$, set:

$$X_\epsilon := \{v \in \mathbb{Q}^r \mid K(v, -) \text{ belongs to } C_\epsilon\}$$

If $\tau \leq \epsilon$, then since $C_\tau \subset C_\epsilon$, we have $X_\tau \subset X_\epsilon$. We can therefore consider the domain noise system with respect to the sequence $\{X_\epsilon\}_{\epsilon \in \mathbb{Q}}$ (see [20, Section 6.5]) whose components are given by:

$$DC_\epsilon := \{F \in T(\mathbb{Q}^r, \text{Vect}_K) \mid F(v) = 0 \text{ if } v \notin X_\epsilon\}$$

For example, let $w$ be in $X_\epsilon$. Then $K(w, -)$ belongs to $C_\epsilon$. For any $w \leq v$, as a subfunctor of $K(w, -)$, the functor $K(v, -)$ also belongs to $C_\epsilon$, which means $v$ is in $X_\epsilon$. As $K(w, -)(v) \neq 0$ if and only if $w \leq v$, we can conclude that $K(w, -)$ is a member of $DC_\epsilon$. Consequently, any tame functor generated by finitely many elements whose coordinates belong to $X_\epsilon$ is a member of $DC_\epsilon$.

We claim $DC$ is a simple noise system. Being closed under direct sums is clear. Let $G$ be a finitely generated tame functor and $\{g_s \in G(v_s)\}_{s=1}^n$ be its minimal set of generators. For $\tau$ in $\mathbb{Q}$, define $G' \subset G$ to be the subfunctor generated by $\{g_s \in G(v_s) \mid v_s \notin X_\tau\}$. Thus $G/G'$ is generated by elements whose coordinates belong to $X_\tau$ and therefore $G/G'$ belongs to $DC_\tau$. Let $G_0 \subseteq G$ be a tame subfunctor for which the quotient $G/G_0$ is a member of $DC_\tau$. Then $(G/G_0)(v) = 0$ if $v \notin X_\tau$. In particular all the generators of $G'$ are mapped to 0 via the quotient $G \to G/G_0$. This means $G' \subset G_0$ and we can conclude that $G' \subset G$ is the minimal subfunctor whose quotient belongs to $DC_\tau$. Moreover, by construction, $\text{rank}_0 G' \leq \text{rank}_0 G$. The noise system $DC$ is therefore simple.

Note that $G' \subset G$ defined above can also be described as the subfunctor generated by all the elements $g \in G(v)$ whose coordinate $v$ does not belong to $X_\tau$.

Assume $C$ is closed under direct sums. Consider an exact sequence of finitely generated tame functors $0 \to G_0 \to G_1 \to G_2 \to 0$ where $G_0$ is in $C_\tau$ and $G_2$ is in $DC_\tau$. We claim that $G_1$ belongs to $C_{\max\{\epsilon, \tau\}}$. To see this, let $\text{supp}(G_2) = \{v_s\}_{s=1}^n$ and notice that since $G_2$ is in $DC_\tau$, it is a quotient of $P = \oplus_{s=1}^n K(v_s, -)$ where, for any $s$, $K(v_s, -)$ belongs to $C_\tau$. Because $P$ is free, we can lift the quotient natural transformation $P \to G_2$ to $P \to G_1$ and so $G_1$ is a quotient of $G_0 \oplus P$. As $G_0 \oplus P$ belongs to $C_{\max\{\epsilon, \tau\}}$, then so does $G_1$.

Assume further that $C$ is simple. Any functor in $DC_\tau$ is a quotient of a finite direct sum of free functors $K(v, -)$ where $v$ belongs to $X_\tau$. Any such functor is therefore
a member of \( C_r \) as \( C \) is closed under direct sums. It follows that \( DC_r \subset C_r \), and hence, for any finitely generated tame functor \( G \), there is a sequence of inclusions \( G_C[\tau] \subset G_{DC}[\tau] \subset G \).

Recall that for a subset \( X \subset \mathbb{Q}' \) and an element \( v \in \mathbb{Q}' \), we write \( X \leq v \) if \( w \leq v \) for any \( w \) in \( X \) (see 2.1).

8.4. **Theorem.** Let \( \{ C_r \}_{r \in \mathbb{Q}} \) be a simple noise system and \( G: \mathbb{Q}' \to \text{Vect}_K \) a finitely generated tame functor. Let \( v \in \mathbb{Q}' \) such that \( \text{supp}(G[\tau]) \leq v \). Then:

\[
\text{rank}_0 G(\tau) = \min \left\{ \text{rank}_0 F \left| \begin{array}{l}
F \text{ is a tame subfunctor of } G \text{ such that } \\
G[\tau] \subset F \subset G \text{ and } \text{supp}(F) \leq v
\end{array} \right. \right\}
\]

**Proof.** The proof is very similar to the proof of [20, Proposition 10.1]. Let \( n := \text{rank}_0 G(\tau) \) and \( G \xleftarrow{G_1 : \phi} \to \phi \xrightarrow{G_1} G_2 \) be natural transformations between finitely generated tame functors such that \( \text{rank}_0 G_2 = n \) and:

\[
\ker(\phi) \in C_a, \quad \coker(\phi) \in C_b, \quad \ker(\psi) \in C_c, \quad \coker(\psi) \in C_d, \quad a + b + c + d \leq \tau
\]

Consider the subfunctor \( G_2[d] \subset G_2 \) and let \( \{ g_s \in G_2[d](v_s) \}_{s=1}^k \) be a minimal set of generators. Since the considered noise system is simple, \( k \leq n \). By the minimality of \( G_2[d] \subset G_2 \), we have \( G_2[d] \subset \text{im}(\psi) \). Thus we can choose elements \( \{ g'_s \in G_1(v_s) \}_{s=1}^k \) that are mapped via \( \psi \) to the chosen minimal generators of \( G_2[d] \).

Let \( G'_1 \subset G_1 \) be the subfunctor generated by these elements and \( G' \subset G \) be its image \( \phi(G'_1) \). All these functors are finitely generated and tame and they can be arranged into a commutative diagram where the indicated natural transformations are monomorphisms and epimorphisms:

\[
\begin{array}{ccc}
G'_2 & \xrightarrow{G'_1} & G_2[d] \\
\downarrow \phi & & \downarrow \psi \\
G & \xrightarrow{G_1} & G_2
\end{array}
\]

We claim that \( G/G' \) belongs to \( C_{b+c+d} \subset C_r \). This is a consequence of the additivity property of noise systems applied to the following exact sequences:

\[
0 \to \ker(\phi)/(\ker(\psi) \cap G'_1) \to G_1/G'_1 \to G_2/G_2[d] \to 0
\]

The above claim implies \( G[\tau] \subset G' \). Let \( \{ x_s \in G'(w_s) \}_{s=1}^l \) be a minimal set of generators of \( G' \). Define \( G'' \subset G' \) to be generated by \( \{ x_s \mid 1 \leq s \leq l \text{ and } w_s \leq v \} \). Since \( \text{supp}(G[\tau]) \leq v \), for degree reasons \( G[\tau] \) is included in \( G'' \). By construction \( \text{rank}_0 G'' \leq \text{rank}_0 G' \leq k \) and \( k \leq n \), which gives:

\[
\text{rank}_0 G(\tau) \geq \min \left\{ \text{rank}_0 F \left| \begin{array}{l}
F \text{ is a tame subfunctor of } G \text{ such that } \\
G[\tau] \subset F \subset G \text{ and } \text{supp}(F) \leq v
\end{array} \right. \right\}
\]

As the reverse inequality \( \leq \) is obvious, the equality of the theorem is proven.

The set of tame subfunctors of \( G \) considered in Theorem 8.4 can still be infinite. To reduce the calculation of \( \text{rank}_0 G(\tau) \) to a finite set we need an additional step.
8.5. Corollary. Let \( \{ C_v \}_{v \in \mathbb{Q}} \) be a simple noise system and \( G : \mathbb{Q}^r \rightarrow \text{Vect}_K \) a finitely generated tame functor. Choose a rational number \( \alpha \) such that both \( G \) and its shift \( G[\tau] \) are \( \alpha \)-tame. Choose \( v \) in \( \mathbb{N}^r \) so that \( \text{supp}(G[\tau] \alpha) \leq v \). Then:

\[
\hat{\text{rank}}_0 G(\tau) = \min \left\{ \text{rank}_0 F \mid F : \mathbb{N}^r \rightarrow \text{Vect}_K \text{ is a subfunctor of } G \alpha \text{ s.t.} \right. \\
G[\tau] \alpha \subset F \subset G \alpha \text{ and } \text{supp}(F) \leq v \}
\]

Proof. Since \( \text{supp}(G[\tau] \alpha) \leq v \), then \( \text{supp}(G[\tau]) \leq \alpha v \). We can then use Theorem 8.4 to obtain a finitely generated tame functor \( F' : \mathbb{Q}^r \rightarrow \text{Vect}_K \) such that \( G[\tau] \subset F' \subset G \), \( \text{supp}(F') \leq \alpha v \), and \( \text{rank}_0 F' = \hat{\text{rank}}_0 G(\tau) \). By restricting along \( \alpha : \mathbb{N}^r \rightarrow \mathbb{Q}^r \) we get a sequence of frames \( G[\tau] \alpha \subset F' \alpha \subset G \alpha \). Since \( \text{supp}(F') \leq \alpha v \), we also have \( \text{supp}(F' \alpha) \leq v \). Finally, as the restriction preserve the rank, \( \text{rank}_0 F' \alpha = \hat{\text{rank}}_0 G(\tau) \). That shows that the left side of the equality in the statement of the corollary can not be smaller than the right side. As the opposite inequality is clear, the corollary is proven. \( \square \)

A consequence of Corollary 8.5 is that to determine the value \( \hat{\text{rank}}_0 G(\tau) \) for a finite field \( K \) we need to perform only finitely many operations as the following set is finite:

\[
\{ F : \mathbb{N}^r \rightarrow \text{Vect}_K \mid F \text{ is a subfunctor of } G \alpha \text{ such that } \text{supp}(F) \leq v \}
\]

Definition 8.2 describes a simple noise system in terms of certain global properties of its components. For implementation and calculation purposes a constructive description of simple noise systems is needed. A description that would also lead to an explicit construction of the subfunctor \( G[\tau] \subset G \), so we can use Theorem 8.4 and Corollary 8.5 to calculate the stabilization of the rank.

9. Persistence contours

Recall (see 2.1) that \( \mathbb{Q}^r_\infty \) is the poset obtained from \( \mathbb{Q}^r \) by adding an extra object \( \infty \) such that \( v \leq \infty \) for any \( v \) in \( \mathbb{Q}^r \). For any \( v \) in \( \mathbb{Q}^r_\infty \), we set \( v + \infty := \infty \).

9.1. Notation. Let \( \{ C_v \}_{v \in \mathbb{Q}} \) be a simple noise system. For \( (v, \epsilon) \) in \( \mathbb{Q}^r_\infty \times \mathbb{Q} \), define \( C(v, \epsilon) \) in \( \mathbb{Q}^r_\infty \) as follows:

- \( C(\infty, \epsilon) := \infty \).
- Consider the functors \( K(v, -)[\epsilon] \subset K(v, -) \). Because the noise system is simple, \( \text{rank}_0 K(v, -)[\epsilon] \leq \text{rank}_0 K(v, -) = 1 \). Thus either \( K(v, -)[\epsilon] \) is trivial or it is free on one generator. If \( K(v, -)[\epsilon] = 0 \), then we set \( C(v, \epsilon) := \infty \). If \( K(v, -)[\epsilon] \) is free on one generator, define \( C(v, \epsilon) \) to be the object in \( \mathbb{Q}^r \) for which \( K(v, -)[\epsilon] \) is isomorphic to \( K(C(v, \epsilon), -) \).

We claim that if \( w \leq v \) in \( \mathbb{Q}^r_\infty \) and \( \tau \leq \epsilon \) in \( \mathbb{Q} \), then \( C(w, \tau) \leq C(v, \epsilon) \). This means that the association \( (v, \epsilon) \mapsto C(v, \epsilon) \) is in fact a functor \( C : \mathbb{Q}^r_\infty \times \mathbb{Q} \rightarrow \mathbb{Q}^r_\infty \). The claim is obvious if \( v = \infty \) or \( v \) is in \( \mathbb{Q}^r \) and \( K(v, -)[\epsilon] = 0 \). In the remaining case of \( v \) being in \( \mathbb{Q}^r \) and \( K(v, -)[\epsilon] \) being non-trivial, since \( C_r \subset C_\epsilon \), we have monomorphisms:

\[
K(C(v, \epsilon), -) = K(v, -)[\epsilon] \rightarrow K(v, -)[\tau] \rightarrow K(v, -)
\]

\[
K(w, -)[\tau] \rightarrow K(w, -)
\]
The functor $K(w, -)[\tau]$ is therefore non-trivial. The relation $C(w, \tau) \leq C(v, \epsilon)$ is then a consequence of the fact that $K(C(v, \epsilon), -)$ is a subfunctor of $K(C(w, \tau), -)$.

The functor $C: \mathbf{Q}_\mathcal{K}^r \times \mathcal{Q} \to \mathbf{Q}_\mathcal{K}^r$ is not arbitrary. It satisfies the following two additional properties for any $v$ in $\mathbf{Q}_\mathcal{K}^r$ and any $\epsilon$ and $\tau$ in $\mathcal{Q}$:

$$v \leq C(v, \epsilon) \quad C(C(v, \epsilon), \tau) \leq C(v, \epsilon + \tau)$$

The first property is clear when $C(v, \epsilon) = \infty$. If $C(v, \epsilon) \neq \infty$, then $v \neq \infty$. In this case the relation $v \leq C(v, \epsilon)$ is a consequence of $K(C(v, \epsilon), -)$ being a subfunctor of $K(v, -)$. The second property is also clear if $C(v, \epsilon + \tau) < \infty$. If $C(v, \epsilon + \tau) < \infty$, then we have a sequence of monomorphisms:

$$K(C(v, \epsilon + \tau), -) = K(v, -)[\epsilon + \tau] \subset K(v, -)[\epsilon] \subset K(v, -)[\epsilon] \subset K(v, -)$$

Thus $K(v, -)[\epsilon][\tau]$ is non trivial, and according to our notation, isomorphic to $K(C(v, \epsilon), \tau, -)$. As this functor contains $K(C(v, \epsilon + \tau), -)$, we get the claimed relation $C(C(v, \epsilon), \tau) \leq C(v, \epsilon + \tau)$.

We formalize these properties in:

9.2. **Definition.** A **persistence contour** is a functor $C: \mathbf{Q}_\mathcal{K}^r \times \mathcal{Q} \to \mathbf{Q}_\mathcal{K}^r$, such that, for any $v$ in $\mathbf{Q}_\mathcal{K}^r$ and any $\epsilon$ and $\tau$ in $\mathcal{Q}$:

1. $v \leq C(v, \epsilon)$,
2. $C(C(v, \epsilon), \tau) \leq C(v, \epsilon + \tau)$.

We have seen how a simple noise system leads to a persistence contour. It turns out that this procedure can be reversed.

9.3. **Notation.** Let $D: \mathbf{Q}_\mathcal{K}^r \times \mathcal{Q} \to \mathbf{Q}_\mathcal{K}^r$ be a persistence contour.

- For $\epsilon$ in $\mathcal{Q}$ define $D_\epsilon \subset \mathcal{T}(\mathbf{Q}^r, \text{Vect}_K)$ to be the collection of the finitely generated tame functors $G: \mathbf{Q}^r \to \text{Vect}_K$ for which $G(v \leq D(v, \epsilon))$ is the zero homomorphism whenever $D(v, \epsilon) \neq \infty$.

- Let $G$ be a finitely generated tame functor. Choose its minimal set of generators $\{g_s \in G(v_s)\}_{s=1}^n$ and $\tau$ in $\mathcal{Q}$. For any $s$ such that $D(v_s, \tau) \neq \infty$, set $b_s := \mu(g_s \in G(D(v_s, \tau)))$. Define $G[\tau] \subset G$ to be the subfunctor generated by $\{b_s \mid 1 \leq s \leq n \text{ and } D(v_s, \tau) \neq \infty\}$.

If $D(v, \tau) = \infty$, then since $D$ is a functor, $D(w, \tau) = \infty$ for any $v \leq w$. It follows that $K(v, -)$ belongs to $D_\tau$ if and only if $D(v, \tau) = \infty$. Furthermore:

$$K(v, -)[\tau] = \begin{cases} 0 & \text{if } D(v, \tau) = \infty \\ K(D(v, \tau), -) & \text{if } D(v, \tau) \neq \infty \end{cases}$$

Note also that if $C, D: \mathbf{Q}_\mathcal{K}^r \times \mathcal{Q} \to \mathbf{Q}_\mathcal{K}^r$ are persistence contours such that $C(v, \epsilon) \leq D(v, \epsilon)$ for any $(v, \epsilon)$ in $\mathbf{Q}_\mathcal{K}^r \times \mathcal{Q}$, then $C \subset D_\epsilon$ for any $\epsilon$ in $\mathcal{Q}$.

A priori the subfunctor $G[\tau] \subset G$, defined in 9.3, depends on the choice of the generators of $G$. In Proposition 9.4 we are going to show that $G[\tau] \subset G$ is the minimal subfunctor, with respect to inclusion, for which $G/(G[\tau])$ belongs to $D_\tau$.

9.4. **Proposition.** The sequence $D = \{D_\epsilon\}_{\epsilon \in \mathcal{Q}}$ is a simple noise system.

**Proof. Noise conditions.** It is clear that the zero functor belongs to $D_\epsilon$ for any $\epsilon$. Let $\tau \leq \epsilon$, $G$ be in $D_\tau$, and $v$ be an element in $\mathcal{Q}^r$ for which $D(v, \epsilon) \neq \infty$. Since $D(v, \tau) \leq D(v, \epsilon)$, we also have $D(v, \tau) \neq \infty$. The homomorphism $G(v \leq D(v, \tau))$ is therefore trivial, and hence so is the composition $G(v \leq D(v, \epsilon)) = G(D(v, \tau) \leq$
$D(v, e) \circ G(v \leq D(v, \tau))$. Thus $G$ is in $D_\tau$. This shows $D_\tau \subset D_\varepsilon$. The first two requirements for a noise system are therefore satisfied (see Section 7).

Let $0 \to G_0 \to G_1 \to G_2 \to 0$ be an exact sequence. Assume $G_1$ belongs to $D_\varepsilon$. By definition $G_1(v \leq D(v, e))$ is the zero homomorphism for any $v$ for which $D(v, e) \neq \infty$. Since $G_1(v) \to G_2(v)$ is an epimorphism and $G_0(D(v, e)) \to G_1(D(v, e))$ is a monomorphism, the homomorphisms $G_2(v \leq D(v, e))$ and $G_0(v \leq D(v, e))$ have to be also trivial. Thus both $G_0$ and $G_2$ belong to $D_\varepsilon$.

Assume $G_0 \in D_\tau$ and $G_2 \in D_\varepsilon$. We need to show $G_1$ belongs to $D_{\varepsilon + \tau}$. Let $v$ be in $\mathbb{Q}'$ such that $D(v, \tau + \varepsilon) \neq \infty$. Since $D(v, e) \leq D(v, \tau + \varepsilon) \geq D(D(v, e), \tau)$, we have $D(v, e) \neq \infty \neq D(D(v, e), \tau)$ and thus we can form the following commutative diagram where the indicated homomorphisms are trivial:

\[
\begin{array}{ccccccccc}
0 & \to & G_0(v) & \to & G_1(v) & \to & G_2(v) & \to & 0 \\
& & & \downarrow{G_1(v \leq D(v, e))} & & \downarrow{G_2(v \leq D(v, e), \tau)} & & \downarrow{G_1(D(v, e), \tau)} & & \downarrow{G_2(D(v, e), \tau)} & & 0 \\
0 & \to & G_0(D(v, e)) & \to & G_1(D(v, e)) & \to & G_2(D(v, e)) & \to & 0 \\
& & \downarrow{G_2(D(v, e) \leq D(v, e), \tau)} & & \downarrow{G_1(D(v, e))} & & \downarrow{G_2(D(v, e), \tau)} & & \downarrow{G_1(D(v, e), \tau)} & & 0 \\
0 & \to & G_0(D(v, e), \tau) & \to & G_1(D(v, e), \tau) & \to & G_2(D(v, e), \tau) & \to & 0
\end{array}
\]

Commutativity and exactness implies that the middle vertical composition:

$G_1(D(v, e) \leq D(D(v, e), \tau)) \circ G_1(v \leq D(v, e)) = G_1(v \leq D(D(v, e), \tau))$

is also the zero homomorphism. Consequently so is the composition:

$G_1(v \leq D(v, e + \tau)) = G(D(D(v, e), \tau) \leq D(v, e + \tau)) \circ G(v \leq D(D(v, e), \tau))$

This means $G_1$ belongs to $D_{\varepsilon + \tau}$.

**Simplicity.** A direct sum of zero homomorphisms is a zero homomorphism. The noise system $D$ is therefore closed under direct sums. Let $G$ be a finitely generated tame functor. We are going to show that $G[\varepsilon] \subset G$, defined in 9.3, is the minimal subfunctor for which the quotient $G/G[\varepsilon]$ belongs to $D_\tau$. Since by construction $\text{rank}_0 G[\varepsilon] \leq \text{rank}_0 G$, we could thus conclude $D$ to be simple.

Let $\{g_s \in G(v_s)\}_{s=1}^n$ be a minimal set of generators of $G$. By definition $G[\varepsilon] \subset G$ is the subfunctor generated by $\{h_s \mid 1 \leq s \leq n \text{ and } D(v_s, \tau) \neq \infty\}$. First we show $G/(G[\varepsilon])$ belongs to $D_\tau$. Let $v$ be an object in $\mathbb{Q}'$ such that $D(v, \tau) \neq \infty$. Any element $x$ in $G(v)$ can be written as $x = \sum_{v_s \leq v} \lambda_s G(v_s \leq v)(g_s)$ and thus:

\[
G(v \leq D(v, \tau))(x) = \sum_{v_s \leq v} \lambda_s G(D(v_s, \tau) \leq D(v, \tau)) \circ G(v \leq D(v_s, \tau))(g_s) = \\
= \sum_{v_s \leq v} \lambda_s G(D(v_s, \tau) \leq D(v, \tau))(h_s)
\]

This implies that $G(v \leq D(v, \tau))$ maps any $x$ in $G(v)$ into $G[\varepsilon]$. Consequently the homomorphism $G/(G[\varepsilon])(v \leq D(v, \tau))$ is trivial.

Let $G_0 \subset G$ be a tame functor for which $G/G_0$ belongs to $D_\tau$. Thus, for any $s$ such that $D(v_s, \tau) \neq \infty$, the homomorphism $G(v_s \leq D(v_s, \tau))$ maps the generator $g_s$ to an element in $G_0$, and so $h_s = G(v_s \leq D(v_s, \tau))(g_s)$ belongs to $G_0$. Consequently the functor $G[\varepsilon]$, generated by these $h_s$, is a subfunctor of $G_0$. \[\square\]
9.5. Example. Choose an object \( w \) in \( \mathbb{Q}_\tau \). For any \( (v, \epsilon) \) in \( \mathbb{Q}_\tau^r \times \mathbb{Q} \), define:

\[
S_w(v, \epsilon) := v + \epsilon w
\]

In particular, according to our convention \( \infty + \epsilon w = \infty \), we have \( S_w(\infty, \epsilon) = \infty \). It is straightforward to verify that \( S_w \) is a persistence contour. For this contour the following equality holds \( S_w(S_w(v, \epsilon), \tau) = v + \epsilon w + \tau w = S_w(v, \epsilon + \tau) \). The functor \( S_w \) is called the standard persistence contour in the direction \( w \). The associated simple noise system is given by:

\[
(S_w)_\epsilon = \left\{ G: \mathbb{Q}_\tau \to \text{Vect} \mid G \text{ is a finitely generated tame functor such that } G(v \leq v + \epsilon w) \text{ is trivial for any } v \right\}
\]

In [20] this noise system is called the standard noise in the direction of \( w \). The \( \tau \)-shift \( G_{S_w}[\tau] \subset G \) is given by the subfunctor generated by all the elements of the form \( G(v_s \leq v_s + \tau w)(g_s) \) where \( \{g_s \in G(v_s)\}^s_{s=1} \) is a minimal set of generators of \( G \). For example, consider \( v \leq u \) in \( \mathbb{Q}_\tau \). The functor \( [v, u] \) (see 4.4) belongs to \( (S_w)_\epsilon \) if and only if \( u \leq v + \epsilon w \).

We are now ready to state and prove a constructive characterization of simple noise systems. Any such noise is associated with a unique persistence contour.

9.6. Theorem. The function that assigns to a persistence contour the noise system defined in 9.3 is a bijection between the set of persistence contours and the set of simple noise systems.

Proof. We are going to prove that the construction 9.1 of a persistence contour associated to a simple noise system is the inverse to the function claimed to be a bijection in the theorem.

Let \( D: \mathbb{Q}_\infty^r \times \mathbb{Q} \to \mathbb{Q}_\infty^r \) be a persistence contour and \( D' = \{D'_\epsilon\}_{\epsilon \in \mathbb{Q}} \) be the associated simple noise system as defined in 9.3. In 9.4 it was shown that, for a finitely generated tame functor \( G \), the subfunctor \( G[\tau] \subset G \) constructed in 9.3 coincides with \( G_{D'}[\tau] \), i.e., it is the smallest subfunctor for which \( G/G[\tau] \) belongs to \( D'_\tau \). According to this construction:

\[
K(v, -)[\tau] = \begin{cases} 0 & \text{if } D(v, \tau) = \infty \\ K(D(v, \tau) -) & \text{if } D(v, \tau) \neq \infty \end{cases}
\]

Thus the persistence contour, constructed in 9.1 for the noise system \( D' \), is equal to \( D: \mathbb{Q}_\infty^r \times \mathbb{Q} \to \mathbb{Q}_\infty^r \).

Let \( D = \{D_\epsilon\}_{\epsilon \in \mathbb{Q}} \) be a simple noise system. Consider the associated persistence contour \( D: \mathbb{Q}_\infty^r \times \mathbb{Q} \to \mathbb{Q}_\infty^r \) as defined in 9.1. Let \( D' = \{D'_\epsilon\}_{\epsilon \in \mathbb{Q}} \) be the noise system associated to the contour \( D \) as define in 9.3. We need to show \( D = D' \). Since both of these noise systems are closed under direct sums, it is enough to prove \( D \) and \( D' \) contain the same cyclic functors (see 7.1).

Let \( G \) be a cyclic functor. Choose an epimorphism \( \pi: K(-, -) \to G \). Assume \( D(v, \tau) = \infty \). This has two consequences. First, \( K(v, -)[\tau] = 0 \) and so \( K(v, -) \) is a member of \( D_\tau \). Second, \( K(v, -) \) belongs to \( D'_\tau \) (see the paragraph after 9.3). As its quotient, \( G \) then belongs to both \( D_\tau \) and \( D'_\tau \).
Assume $D(v, \tau) \neq \infty$. Then $K(v, -)_D[\tau] = K(D(v, \tau), -)$ and we have a commutative square, where the indicated arrows are epimorphisms (see 8.1):

\[
\begin{array}{c}
K(D(v, \tau), -) \downarrow \\
G_D[\tau] \longleftarrow G
\end{array}
\]

Commutativity of this diagram implies that the homomorphism $G(v \leq D(v, \tau))$ is trivial if and only the homomorphism $K(D(v, \tau), -) \rightarrow G_D[\tau]$ is trivial. As this is an epimorphism we get an equivalence: $G_D[\tau] = 0$ if and only if $G(v \leq D(v, \tau))$ is trivial. The statement $G_D[\tau] = 0$ is equivalent to $G$ belonging to $D_{\tau}$ and the homomorphism $G(v \leq D(v, \tau))$ being trivial is equivalent to $G$ belonging to $D'_{\tau}$.

We can conclude that $D_{\tau}$ and $D'_{\tau}$ contain the same cyclic functors. □

9.7. **Truncations.** Let $C : \mathbb{Q}_\infty^r \times \mathbb{Q} \rightarrow \mathbb{Q}_\infty^r$ be a persistence contour and $C = \{C_i\}_{i \in \mathbb{Q}}$ be the associated simple noise system (see 9.3). Choose an element $u$ in $\mathbb{Q}^r$. For $(v, \epsilon) \in \mathbb{Q}_\infty^r \times \mathbb{Q}$ define:

\[
(C/u)(v, \epsilon) := \begin{cases} 
C(v, \epsilon) & \text{if } u \not\leq C(v, \epsilon) \\
\infty & \text{if } u \leq C(v, \epsilon)
\end{cases}
\]

The fact that $C/u$ is a functor and the relation $v \leq (C/u)(v, \epsilon)$ are clear. Thus to show $C/u$ is a persistence contour, it remains to prove $(: (C/u)((C/u)(v, \epsilon), \tau) \leq (C/u)(v, \epsilon + \tau)$. This inequality is obvious if $u \leq C(v, \epsilon)$ as in this case both sides are equal to $\infty$. Assume $u \not\leq C(v, \epsilon)$. Since $v \leq C(v, \epsilon)$, then also $u \not\leq v$. Thus in this case $(C/u)((C/u)(v, \epsilon), \tau) = C(C(v, \epsilon), \tau)$ and $(C/u)(v, \epsilon + \tau) = C(v, \epsilon + \tau)$ and the required inequality also holds in this case as $C$ is a persistence contour.

The contour $C/u$ is called the **truncation** of $C$ at $u$.

For example, let $S_w : \mathbb{Q}_\infty^r \times \mathbb{Q} \rightarrow \mathbb{Q}_\infty^r$ be the standard persistence contour in the direction of $w$ (see 9.5). Then

\[
(S_w/u)(v, \epsilon) := \begin{cases} 
v + \epsilon w & \text{if } u \not\leq v + \epsilon w \\
\infty & \text{if } u \leq v + \epsilon w
\end{cases}
\]

The noise system $C/u$, associated with the truncated contour $C/u$, is called the **truncation** of $C$ at $u$. This simple noise system $C/u$ is the smallest noise system such that $K(u, -)$ belongs to $(C/u)_0$ and $C_{\epsilon} \subset (C/u)_\epsilon$ for any $\epsilon$ in $\mathbb{Q}$.

Let $G : \mathbb{Q}^r \rightarrow \text{Vect}_K$ be a finitely generated tame functor and $\{g_s \in G(v_s)\}_{s=1}^n$ be its a minimal set of generators. The translation $G_{C/u}[\tau] \subset G$ is the subfunctor generated by the following set:

\[
\{G(v_s \leq v_s + \tau w)(g_s) \mid \text{for } s \text{ such that } C(v_s, \tau) \neq \infty \text{ and } u \not\leq C(v_s, \tau)\}
\]

10. **The case** $r = 1$

Throughout this section $r = 1$. Let us also fix a simple noise system $C = \{C_i\}_{i \in \mathbb{Q}}$ in $\mathbb{Q}(\mathbb{Q}, \text{Vect}_K)$. Let $C : \mathbb{Q}_\infty \times \mathbb{Q} \rightarrow \mathbb{Q}_\infty$ be the associated persistence contour.

For $v \leq w$ in $\mathbb{Q}$, functors of the form $[v, w)$ (see 4.4) and $K(v, -)$ are called **bars** of length $w - v$ and $\infty$ respectively. The bars are the indecomposable elements in $\mathbb{Q}(\mathbb{Q}, \text{Vect}_K)$, i.e. any finitely generated tame functor indexed by $\mathbb{Q}$ is isomorphic to a finite direct sum of bars and the isomorphism types of these summands.
are uniquely determined by the isomorphism type of the functor (see for example [11], [12], [20, Proposition 5.6]). Such a direct sum decomposition is called a bar decomposition. The rank $\operatorname{rank}_0 G$ is equal to the number of bars in a bar decomposition of $G$.

A subfunctor of a finitely generated free functor is also free and of smaller or equal rank. Consequently, for a tame subfunctor $F \subset G$, there is an inequality $\operatorname{rank}_0 F \leq \operatorname{rank}_0 G$. This together with 8.4 implies:

10.1. Proposition. For any finitely generated tame functor $G \colon Q \to \text{Vect}_K$ and $\tau$ in $Q$:

$$\wedge \operatorname{rank}_0 G(\tau) = \operatorname{rank}_0 G[\tau]$$

Since taking the shift commutes with direct sums (see 8.1.(3)), according to the above proposition, so does the stabilization of the rank: if $F, G \colon Q \to \text{Vect}_K$ are finitely generated and tame, then for any $\tau$ in $Q$

$$\operatorname{rank}_0 (F \oplus G)(\tau) = \operatorname{rank}_0 (F \oplus G)[\tau] = \operatorname{rank}_0 (F[\tau] \oplus G[\tau]) = \operatorname{rank}_0 F[\tau] + \operatorname{rank}_0 G[\tau] = \wedge \operatorname{rank}_0 F(\tau) + \wedge \operatorname{rank}_0 G(\tau)$$

Thus if we decompose $G$ as a direct sum of bars $\oplus_{s=1}^n B_s$, then:

$$\wedge \operatorname{rank}_0 G(\tau) = \wedge \operatorname{rank}_0 (\oplus_{s=1}^n B_s)(\tau) = \sum_{s=1}^n \wedge \operatorname{rank}_0 B_s(\tau) = \sum_{s=1}^n \operatorname{rank}_0 B_s[\tau]$$

Recall:

$$\operatorname{rank}_0 K(v, -)[\tau] = \begin{cases} 1 & \text{if } C(v, \tau) \neq \infty \text{ or equivalently if } K(v, -) \not\in C_\tau \\ 0 & \text{if } C(v, \tau) = \infty \text{ or equivalently if } K(v, -) \in C_\tau \end{cases}$$

$$\operatorname{rank}_0 [v, w]\tau = \begin{cases} 1 & \text{if } C(v, \tau) < w \text{ or equivalently if } [v, w] \not\in C_\tau \\ 0 & \text{if } C(v, \tau) \geq w \text{ or equivalently if } [v, w] \in C_\tau \end{cases}$$

We can conclude that, for a finitely generated tame functor $G \colon Q \to \text{Vect}_K$, $\operatorname{rank}_0 G(\tau)$ is equal to the number of bars, in a bar decomposition of $G$, that do not belong to $C_\tau$ (a more general statement can be found in [20, Section 10]).

10.2. Example. Let $S$ be the simple noise system in $T(Q, \text{Vect}_K)$ associated with the standard contour $S \colon Q_\infty \times Q \to Q_\infty$ given by $S(v, \epsilon) = v + \epsilon$ (see 9.5). For this noise system, the $\tau$-shift of $G = \left( \bigoplus_{1 \leq s \leq n} [v_s, w_s] \right) \oplus \left( \bigoplus_{1 \leq t \leq m} K(u_t, -) \right)$ has the following bar decomposition:

$$G_S[\tau] = \bigoplus_{1 \leq s \leq n} [v_s + \tau, w_s] \oplus \bigoplus_{1 \leq t \leq m} K(u_t + \tau, -)$$

The value $\wedge \operatorname{rank}_0 G(\tau) = \operatorname{rank}_0 G_S[\tau]$ is therefore given by the number of bars, in a bar decomposition of $G$, of length strictly bigger than $\tau$. Thus the invariant $G \mapsto \wedge \operatorname{rank}_0 G$ encodes information about the length of bars in a bar decomposition of $G$ but not where the bars start and end. This loss of information can be recovered however by considering truncations of the standard noise (see 9.7). Choose $u$ in $Q$ and consider the truncated noise system $S / u$ (see 9.7). Recall that $S / u$ is associated with the contour given by:

$$(S / u)(v, \epsilon) = \begin{cases} v + \epsilon & \text{if } v + \epsilon < u \\ \infty & \text{if } u \leq v + \epsilon \end{cases}$$
For this truncated noise system the $\tau$-shift of the functor $G$ has the following bar decomposition:

$$G_{S/u}[\tau] = \bigoplus_{1 \leq s \leq n, \tau < v_s - v_s, v_s < u - \tau} (v_s + \tau, w_s) \oplus \bigoplus_{1 \leq t \leq m, u_t < u - \tau} K(u_t + \tau, -)$$

The value $\text{rank}_0 G(\tau) = \text{rank}_0 G_{S/u}[\tau]$ is therefore given by the number of bars, in a bar decomposition of $G$, of length strictly bigger than $\tau$ and whose starting points are strictly smaller than $u - \tau$.

10.3. Proposition. Let $S$ be the standard noise system considered in 10.2. Then two finitely generated tame functors $F, G : Q \to \text{Vect}_K$ are isomorphic if and only if $\text{rank}_0 F_{S/u}[\tau] = \text{rank}_0 G_{S/u}[\tau]$ for any $\tau$ and $u$ in $Q$.

Proof. The if implication is obvious. Assume $\text{rank}_0 F_{S/u}[\tau] = \text{rank}_0 G_{S/u}[\tau]$ for any $\tau$ and $u$ in $Q$. Fix bar decompositions of $F$ and $G$. If we choose $u$ strictly bigger than the beginnings of the bars in the decompositions of $F$ and $G$, then the equalities $\text{rank}_0 F = \text{rank}_0 F_{S/u}[0] = \text{rank}_0 G_{S/u}[0] = \text{rank}_0 G$ imply then decompositions of $F$ and $G$ have the same number of bars. To show $F$ and $G$ are isomorphic we proceed by induction on the number of different lengths of bars in the decomposition of $F$.

Assume first all the bars in the decomposition of $F$ have the same length $l_F$. If there is a bar in $G$ of length $l$ different than $l_F$, then the numbers $\text{rank}_0 F_{S/u}[\tau]$ and $\text{rank}_0 G_{S/u}[\tau]$ would be different if we choose $\max\{l_F, l\} > \tau > \min\{l_F, l\}$ and $u$ to be strictly larger than $\tau$ plus the beginnings of all the bars in the decompositions of $F$ and $G$. Thus all the bars in the decomposition of $G$ have also the same length $l_F$. By setting $\tau = 0$ and varying $u$, we can then see that the number of bars in $F$ and $G$, which start at a particular point, are the same. Consequently $F$ and $G$ are isomorphic.

Assume $F$ has bars of different length in its bar decomposition. Let $l_F$ and $l_G$ be the biggest length and $l'_F$ and $l'_G$ be the next biggest length of any bar in the decomposition of $F$ and $G$ respectively. Choose $\tau$ satisfying $l_F > \tau > l'_F$ and $u$ to be bigger than $\tau$ plus the beginnings of all the bars in the decomposition of $F$. Since $\text{rank}_0 G_{S/u}[\tau] = \text{rank}_0 F_{S/u}[\tau] \neq 0$, then $G$ contains bars of length bigger than $\tau$. As this holds for any $\tau$ satisfying $l_F > \tau > l'_F$, we can conclude $l_G \geq l_F$. If $l_G > l_F$, then for $l_G > \tau > l'_F$ and $u$ bigger than $\tau$ plus the beginnings of all the bars in the decomposition of $G$, the numbers $\text{rank}_0 F_{S/u}[\tau]$ and $\text{rank}_0 G_{S/u}[\tau]$ would be different. We therefore have $l_G = l_F$. Furthermore the number of bars of length $l_F$ in the decompositions of both $F$ and $G$ is the same. Let $F'$ and $G'$ be the direct sums of all the bars of length $l_F$ in the decompositions of respectively $F$ and $G$. Note that for $\tau \geq \max\{l'_F, l'_G\}$ and any $u$:

$$\text{rank}_0 F'_{S/u}[\tau] = \text{rank}_0 F_{S/u}[\tau] = \text{rank}_0 G_{S/u}[\tau] = \text{rank}_0 G'_{S/u}[\tau]$$

Furthermore for any $\tau' \leq \tau < l_F$ and any $u$:

$$\text{rank}_0 F'_{S/u}[\tau'] = \text{rank}_0 F_{S/u}[\tau'] = \text{rank}_0 G'_{S/u}[\tau'] = \text{rank}_0 G'_{S/u}[\tau']$$

It follows $\text{rank}_0 F'_{S/u}[\tau] = \text{rank}_0 G'_{S/u}[\tau]$ for any $\tau$ and $u$. According to the first step of the induction, the functors $F'$ and $G'$ are therefore isomorphic. Let $F''$ and $G''$ be the direct sums of all the bars of length different than $l_F$ in the decompositions of $F$ and $G$ respectively. Then $F$ is isomorphic to $F' \oplus F''$ and $G$ is isomorphic to
G' ⊕ G''. By additivity of the shift and the rank, for any τ and u:
\[ \text{rank}_0 F''_{S/u}[\tau] = \text{rank}_0 F_{S/u}[\tau] - \text{rank}_0 F'_0[\tau] = \]
\[ = \text{rank}_0 G_{S/u}[\tau] - \text{rank}_0 G'_{S/u}[\tau] = \text{rank}_0 G''_{S/u}[\tau] \]
By the inductive hypothesis \( F'' \) and \( G'' \) are isomorphic and so are \( F \) and \( G \). \( \square \)

11. The case \( r \geq 2 \).

Let \( K \) be a finite field. In 8.5 we proved that, for a simple noise system, calculating \( \text{rank}_0 G(\tau) \) requires only finitely many operations. For \( r \geq 2 \), one should not however expect to find a fast algorithm to do that. The aim of this section is to show that calculating \( \text{rank}_0 G(\tau) \) is in general an NP-hard problem if \( r \geq 2 \). This is in contrast with the case of \( r = 1 \) where the complexity is closely related to the complexity of Gaussian elimination (this is the complexity of finding bar decompositions of such functors, see for example [21], [14]).

To show this NP-hardness we are going to compare calculating \( \text{rank}_0 G(\tau) \) with the RANK-3 problem introduced in [17]. The RANK-3 problem is an example of a more general problem which we now describe.

11.1. Input: a sequence \( x_1, \ldots, x_k \) of vectors in \( K^n \) and a sequence \( L_1, \ldots, L_k \) of vector subspaces of \( K^n \) each given by a system of linear equations.

11.2. Output: \( \min \{ \dim(L) \mid L \text{ is a subspace of } K^n \text{ s.t. } x_s \in L + L_t \text{ for } 1 \leq s \leq k \} \).

The above problem can be reformulated in terms of finding the smallest rank of a matrix in a certain set of matrices. An \( n \times k \) matrix \( A = [c_1 \cdots c_k] \) is said to belong to the input above if, for any \( 1 \leq s \leq k \), the vector \( c_s - x_s \) is in \( L_s \). Note that if \( A \) belongs to the input, then \( x_s \in \text{span}(c_1, \ldots, c_k) + L_s \) for any \( s \). On the other hand if, for all \( s \), \( x_s = c_s + y_s \) where \( c_s \) is in \( L \) and \( y_s \) is in \( L_s \), then the matrix \( [c_1 \cdots c_k] \) belongs to the input. It is then clear that the number in the output above coincides with:
\[ \min \{ \text{rank}(A) \mid A \text{ belongs to the input} \} \]

The following two sources of inputs are of primary interest to us:

11.3. Example. Let \( X = (V, E) \) be a finite graph with \( V = \{1, 2, \ldots, n\} \). Consider the following collection of \( n \times n \) matrices \( A \) with coefficients in \( K \):
\[ \mathcal{A}(X) := \{ A \mid A_{ss} = 1 \text{ for any } s \in V \text{ and } A_{st} = 0 \text{ for any } s, t \in E \} \]
For example if \( X \) is colored by \( \chi(X) \) (the chromatic number of \( X \)) colors, then the \( n \times n \) matrix \( M \), for which \( M_{xt} = 1 \) if \( s \) and \( t \) have the same color and \( M_{xt} = 0 \) otherwise, belongs to the collection \( \mathcal{A}(X) \). Since this matrix is of rank \( \chi(X) \) (see [17, Section 1]), the collection \( \mathcal{A}(X) \) satisfies the assumptions of [17, Theorem 3.1]. According to this theorem, for an arbitrary graph \( X \), the RANK-3 problem of deciding if \( \mathcal{A}(X) \) contains a matrix of rank 3 is NP-complete.

Note that \( \mathcal{A}(X) \) consists of all the matrices that belong to the input given by the sequence \( e_1, \ldots, e_n \) of vectors in the standard basis for \( K^n \) and the sequence \( L_1, \ldots, L_n \) of vector subspaces of \( K^n \) where \( L_s \) is defined as \( L_s = \{ y \in K^n \mid y_t = 0 \text{ if } \{ s, t \} \in E \text{ or } t = s \} \). We can therefore conclude that deciding:
\[ \min \{ \text{rank}(A) \mid A \text{ belongs to the input} \} = \min \{ \text{rank}(A) \mid A \in \mathcal{A}(X) \} \leq 3 \]
is an NP complete problem. Thus in general the problem of calculating \(11.2\) is as hard as the RANK-3 problem.

11.4. Example. Let \(C: Q_{\infty}^r \times Q \rightarrow Q_{\infty}^r\) be a persistence contour and \(C\) the associated noise system. The initial data consists of:

- a tame functor \(G: Q^r \rightarrow \text{Vect}_K\),
- its minimal set of generators \(\{g_s \in G(v_s)\}_{s=1}^n\),
- an element \(\tau\) in \(Q\),
- an element \(u\) in \(Q^r\) such that \(v_s \leq u \leq C(v_s, \tau)\) for any \(1 \leq s \leq n\).

With this initial data we can form the following input:

- the vector space \(G(u)\),
- the sequence \(G(v_1 \leq u)(g_1), \ldots, G(v_n \leq u)(g_n)\) of vectors in \(G(u)\),
- the sequence \(L_1, \ldots, L_n\) of vector subspaces of \(G(u)\) defined as follows:

\[
L_s = \begin{cases} 
\text{Ker } G(u \leq C(v_s, \tau)) & \text{if } C(v_s, \tau) \neq \infty \\
G(u) & \text{if } C(v_s, \tau) = \infty 
\end{cases}
\]

We claim that the output for the input above coincides with \(\widehat{\text{rank}}_0 G(\tau)\):

**Output \(\geq \widehat{\text{rank}}_0 G(\tau)\):** Let \(L\) be the subspace of \(G(u)\) of the smallest dimension such that \(G(v_s \leq u)(g_s) \in L + L_s\) for \(1 \leq s \leq n\). Choose a basis \(\{f_1, \ldots, f_m\}\) for \(L\) and consider the subfunctor \(F \subset G\) generated by the elements \(\{f_s \in G(u)\}_{s=1}^m\). Note that \(\text{rank}(F) = m\). Moreover, since the \(\tau\)-shift \(G[\tau]\) is generated by the elements \(\{G(v_s \leq C(v_s, \tau))(g_s) \mid 1 \leq s \leq n\text{ and } C(v_s, \tau) \neq \infty\}\), the functor \(F\) contains \(G[\tau]\). Thus according to Theorem 8.4 \(m \geq \text{rank}_0 G(\tau)\).

**Output \(\leq \widehat{\text{rank}}_0 G(\tau)\):** Let \(H\) be a finitely generated subfunctor of \(G\) of the smallest rank such that \(G[\tau] \subset H \subset G\). Choose a minimal set of generators \(\{h_i \in H(w_i)\}_{i=1}^m\). Since \(H\) is a subfunctor of \(G\) we can express its generators as linear combinations:

\[
h_i = \sum_{v_s \leq w_i} a_{is} G(v_s \leq w_i)(g_s)
\]

We use the coefficients \(a_{is}\) to define the following vectors in \(G(u)\) for \(1 \leq i \leq m\):

\[
f_i = \sum_{v_s \leq w_i} a_{is} G(v_s \leq u)(g_s)
\]

Define \(L\) to be the subspace of \(G(u)\) generated by these \(f_1, \ldots, f_m\). Choose \(s\) such that \(1 \leq s \leq n\) and \(C(v_s, \tau) \neq \infty\). The fact that \(H\) contains the \(\tau\)-shift \(G[\tau]\) implies that the vector \(G(v_s \leq C(v_s, \tau))(g_s)\) can be expressed as a linear combination of \(\{G(u \leq C(v_s, \tau))(f_i) \mid w_i \leq v_s\}\). Since \(G\) is a functor, the element \(G(v_s \leq C(v_s, \tau))(g_s)\) can then also be expressed as a linear combination of \(\{G(u \leq C(v_s, \tau))(f_i)\}_{i=1}^m\). This means that the vector \(G(v_s \leq u)(g_s)\) belongs to \(L + \text{Ker } G(u \leq C(v_s, \tau))\). Since \(\dim(L) \leq m\) the claimed inequality follows.

It turns out that Example 11.3 is a special case of 11.4 and our next step is to explain how the input given by a graph in 11.3 can be described as an input induced by a tame functor as explained in 11.4. We do that with a help of so called band functors:

11.5. Band Functors. Let \(n \geq 0\) be a natural number and \(L_0, \ldots, L_n\) be a sequence of subspaces of \(K^{n+1}\). A band functor \(B(L_0, \ldots, L_n): Q^2 \rightarrow \text{Vect}_K\) is by definition a 1-tame functor whose 1-frame (see 6.1) is constructed as follows:
1-Frame: Let $P: \mathbb{N}^2 \to \text{Vect}_K$ be the free functor given by the direct sum $P := \bigoplus_{s=0}^{n} K((n-s,s),-)$ Note that $(2n-t,n+t) \geq (n-s,s)$ for any $0 \leq s,t \leq n$. Thus $P(2n-t,n+t) = \bigoplus_{s=0}^{n} K((n-s,s),(2n-t,n+t)) = K^{n+1}$. For $0 \leq t \leq n$, we regard $L_t$ to be the subspace of $K^{n+1} = P(2n-t,n+t)$. Define the frame $B: \mathbb{N}^2 \to \text{Vect}_K$ to be the unique functor for which the following homomorphisms form a surjective natural transformation $\pi: P \to B$:

$$\pi_{(a,b)}: P(a,b) \to B(a,b) \text{ is given by }$$

$$\left\{ \begin{array}{ll}
id & \text{if } a \leq 2n, b \leq 2n, a + b < 3n \\
\text{the quotient } K^{n+1} \to K^{n+1}/L_{b-n} & \text{if } a \leq 2n, n \leq b \leq 2n, a + b = 3n \\
P(a,b) \to 0 & \text{otherwise} \\
\end{array} \right.$$

The following tables give the non-zero values of $B$ for $n = 1, 2,$ and 3:

| $n$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|----|----|----|----|----|----|----|----|----|
| 3  | K | K | K | K | K | K | K | K |
| 2  | K | K | K | K | K | K | K | K |
| 1  | K | K | K | K | K | K | K | K |
| 0  | 1 | 2 | 3 | 1 | 2 | 3 | 4 | 5 |

Fix the standard persistence contour $S: Q_\infty^2 \times Q \to Q_\infty^2$ given by $S(v,\tau) = v + \tau(1,1)$ (see 9.5). Consider the following initial data (see 11.4):

- the band functor $B(L_0, \ldots, L_n): \mathbb{N}^2 \to \text{Vect}_K$,
- its minimal set of generators given by $\{1 \in B(n-s,s) = K\}_{s=0}^{n}$,
- the element $\tau = n$ in $\mathbb{Q}$,
- the element $u = (n,n)$ in $\mathbb{Q}^2$.

Note that $(n-s,s) \leq (n,n) \leq (n-s,s) + n(1,1)$ for any $0 \leq s \leq n$. Thus this initial data satisfies the required assumption (see 11.4). We can therefore conclude that $\text{rank}_0 B(\tau)$ coincides with the output associated with the input given by the sequence of vectors $e_0, \ldots, e_n$ in $K^{n+1}$ and the sequence of subspaces $L_0, \ldots, L_n$ of $K^{n+1}$. Since no condition was put on these subspaces we could choose them to coincide with the subspaces associated to an arbitrary graph as explained in Example 11.3. This shows that computing $\text{rank}_0 B(\tau)$ is at least as hard as deciding the RANK-3 problem. We can therefore conclude:

11.6. Theorem. Let $S: Q_\infty^2 \times Q \to Q_\infty^2$ be the standard persistence contour given by $S(v,\tau) = v + \tau(1,1)$ and $G: \mathbb{Q}^2 \to \text{Vect}_K$ be an arbitrary finitely generated
tame functor. Deciding $\hat{\text{rank}}_0 G(\tau) \leq 3$ is an NP-complete problem. Consequently computing $\hat{\text{rank}}_0 G(\tau)$ is NP-hard.

11.7. **Corollary.** Let $r \geq 2$. Consider an arbitrary finitely generated tame functor $G: Q^r \rightarrow \text{Vect}_K$ and an arbitrary simple noise system in $\mathcal{T}(Q^r, \text{Vect}_K)$. Then deciding if $\hat{\text{rank}}_0 G(\tau) \leq 3$ is an NP-complete problem. Consequently computing $\hat{\text{rank}}_0 G(\tau)$ is NP-hard.

**Proof.** Note that any 2-parameter persistence module can be viewed as an $r$-parameter persistence module for $r > 2$ where all the maps in the added dimensions are identities. Hence the above statement contains Theorem 11.6 as a special case, by which the result follows.$\square$

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