Background independent holographic description:
From matrix field theory to quantum gravity

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Abstract

We propose a local renormalization group procedure where length scale is changed in spacetime dependent way. Combining this scheme with an earlier observation that high energy modes in renormalization group play the role of dynamical sources for low energy modes at each scale, we provide a prescription to derive background independent holographic duals for field theories. From a first principle construction, it is shown that the holographic theory dual to a $D$-dimensional matrix field theory is a $(D+1)$-dimensional quantum theory of gravity coupled with matter fields of various spins. The gravitational theory has $(D+1)$ first-class constraints which generate local spacetime transformations in the bulk. The $(D+1)$-dimensional diffeomorphism invariance is a consequence of the freedom to choose different local RG schemes.
I. INTRODUCTION

Can one prove AdS/CFT correspondence[1–3]? If so, it will not only give us more insight into the precise content of the duality but also open the door to construct holographic duals for general quantum field theory. There have been efforts to derive holographic duals directly from boundary field theories[4–16]. One approach is to build up a bulk spacetime, by introducing dynamical sources and their conjugate fields in exchange of decimating high energy modes at each step of renormalization group (RG)[8, 14, 16]. This procedure amounts to an exact change of variables from the original $D$-dimensional fields into the $(D + 1)$-dimensional variables in functional integration, where the length scale becomes the extra dimension in the bulk[17, 18].

The first principle construction provides microscopic justification for the dictionary of the AdS/CFT correspondence. If the prescription is applied to a $D$-dimensional gauge theory, one obtains a $(D + 1)$-dimensional field theory of closed loops which are coupled with a two-form gauge field in the bulk[16]. The two-form gauge field is an emergent gauge field in the sense that its dynamics is solely generated by other loop fields. Because the gauge group is compact, there is a topological defect (NS-brane) for the gauge field. Proliferation of the defects describes quantum tunnelings between different topological sectors. For a sufficiently large $N$, the topological defects are dynamically suppressed, leading to the deconfinement of the two-form gauge field in the bulk. It has been emphasized that those phases that admit ‘classical’ holographic description possess a non-trivial quantum order associated with the spontaneous suppression of the tunneling between different topological sectors[16]. This is analogous to the quantum order that is present in exotic phases of condensed matter systems with an emergent one-form gauge field[19], which often requires a large number of flavors.

Despite some progresses, the construction[8, 14, 16] has an important drawback: it is not background independent. As a result, it has not been easy to see the emergence of gravitational theory in the holographic description. In this paper, we provide a prescription to construct holographic duals in a background independent manner. Using the prescription, we show that a $D$-dimensional matrix field theory can be mapped into a $(D + 1)$-dimensional quantum gravity coupled with matter fields of various spins. This allows one to identify the boundary field theory as a quantum theory of gravity in the bulk.

Here is an outline of the paper. In Sec. II, we start by defining a concrete matrix field theory whose holographic dual will be constructed in the remaining of the paper. A theory is defined by
specifying sources for all operators allowed by symmetry. A local operator is constructed from traces of the fundamental matrix field and its derivatives. Although the field theory is nominally defined on the flat $D$-dimensional spacetime, one can view the theory with spacetime dependent sources as a theory defined on a curved background spacetime. In Sec. III, we eliminate multi-trace operators by introducing a dynamical source and its conjugate field for each single-trace operator, where the conjugate fields represent the operators themselves. In particular, the $D$-dimensional metric and its conjugate field that represents the energy-momentum tensor become dynamical. In Sec. IV, a local coarse graining is performed where the length scale is increased at a rate that depends on spacetime. As the high energy modes are integrated out, non-trivial actions are generated for the dynamical sources and the conjugate fields. One contribution is a source dependent determinant for the quadratic action of the high energy mode that is integrated out. This includes the $D$-dimensional curvature term of the dynamical metric generated a la Sakharov’s induced gravity [20]. The other contribution represents double-trace operators generated from quantum correction, which become a quadratic action for the conjugate field as the double-trace operators are removed by another set of auxiliary fields. In Sec. V, we take the advantage of the $D$-dimensional diffeomorphism invariance, which is ensured by the fact that the $D$-dimensional metric is fully dynamical, to shift the $D$-dimensional coordinates of the low energy modes relative to the coordinates of the high energy field. In Sec. VI, it is shown that one can construct a $(D + 1)$-dimensional theory as the coarse graining procedure is repeatedly applied to the low energy mode. In Sec. VII, we show that the theory in the bulk takes the form of a $(D + 1)$-dimensional canonical quantum gravity if one interprets the extra dimension associated with the scale as a time. In particular, there are $(D + 1)$ local constraints which originate from the fact that the partition function is independent of the local RG scheme: the partition function is invariant under the changes in the speed of local coarse graining and the $D$-dimensional shift. From this, it can be shown that those constraints are first-class, which generate $(D + 1)$-dimensional local spacetime transformations. In Sec. VIII, we apply the prescription to a simple toy model ($0$-dimensional matrix model) to illustrate the main idea in the simplest setting. In Sec. IX, the difference between the present holographic description and the conventional RG is contrasted. In particular, we emphasize the fact that the beta function is promoted to a ‘Heiserberg’ equation for quantum operators in the holographic description.
II. MODEL

A. Matrix field theory

Consider a matrix quantum field theory defined on a \( D \)-dimensional flat spacetime. To be concrete, we consider a theory of \( N \times N \) real traceless symmetric matrix field \( \Phi(x) \) with the global \( O(N) \) symmetry under which the matrix field transforms as an adjoint field. The ‘partition function’ is

\[
Z[J] = \int D\Phi \exp \left[ iN^2 \int d^Dx \left( -\mathcal{J}^m O_m + V[O_m; \mathcal{J}^{\{m_i\};\{\nu_j\}}] \right) \right].
\]

(1)

Here \( O_m \)'s denote single-trace operators constructed from \( \Phi \) and its derivatives. In general, one can take \( \{O_m\} \) to be a complete set of primary single-trace operators. Here we use the basis where \( O_m \) takes the form of

\[
O_{[q+1;\{\mu_j^i\}]} = \frac{1}{N^{\mathrm{tr}}} \left[ \Phi \left( \partial_{\mu_1^1} \partial_{\mu_2^2} \ldots \partial_{\mu_{p_1}^{p_1}} \Phi \right) \left( \partial_{\mu_2^2} \partial_{\mu_3^3} \ldots \partial_{\mu_{p_2}^{p_2}} \Phi \right) \ldots \left( \partial_{\mu_{p_1}^{p_1}} \partial_{\mu_{p_2}^{p_2}} \ldots \partial_{\mu_{p_q}^{p_q}} \Phi \right) \right],
\]

(2)

where \( q + 1 \) is the order in the matrix field, and \( \{\mu_j^i\} \) specifies the spacetime indices. General single-trace operators can be written as linear combinations of these operators and their derivatives. For simplicity, we assume that there is no boundary in spacetime. Any operator that has overall derivatives is removed by integration by part in Eq. (1). Throughout the paper, we will use the compressed label, say \( m \) to denote the full indices, \( [q, \{\mu_j^i\}] \) of a single-trace operator. Explicit indices will be used only when it is needed. \( \mathcal{J}^m(x) \) is the spacetime dependent sources for the corresponding operator \( O_m \). The information on the signature of the background metric is solely encoded in the sources. We assume that the spacetime has the Minkowskian metric with the signature \((-1, 1, 1, \ldots, 1)\) for \( x^\mu \) with \( \mu = 0, 1, \ldots, (D - 1) \). \( V \) represents a multi-trace deformation,

\[
V[O_m; \mathcal{J}^{\{m_i\};\{\nu_j\}}] = \sum_{q=1}^{\infty} \mathcal{J}^{\{m_i\};\{\nu_j\}} O_{m_1} \left( \partial_{\nu_1^1} \ldots \partial_{\nu_{p_1}^{p_1}} O_{m_2} \right) \left( \partial_{\nu_{p_1}^{p_1}} \ldots \partial_{\nu_{p_2}^{p_2}} O_{m_3} \right) \ldots \left( \partial_{\nu_{p_q}^{p_q}} \ldots \partial_{\nu_{p_q}^{p_q}} O_{m_{q+1}} \right),
\]

(3)

where \( \mathcal{J}^{\{m_i\};\{\nu_j\}}(x) \)'s are sources for multi-trace operators. All repeated indices are summed over.

To make sense of the partition function, the theory should be regularized. Here we use the Pauli-Villar regularization. Namely, the sources for high derivative terms are turned on in the quadratic action for the matrix field to suppress UV divergence in loop integrals. For example, one can use a regularized kinetic term, \(-\mathrm{tr}[\Phi \Box e^{-\frac{M^2}{\Box}} \Phi] \), where \( \Box = \partial_\mu \partial^\mu \). The mass scale \( M \)
in the higher derivative terms plays the role of a UV cut-off. It is noted that the divergence in the determinant of the quadratic action is not regularized by the higher derivative terms. In this sense, the partition function itself is not well defined. What is well defined is the ratio between two partition functions with two different sets of sources where the divergences from the determinants cancel. For example, the divergence in the determinant is canceled in correlation functions of local operators.

B. From flat to curved background spacetimes

Suppose the manifold is endowed with a background metric $G_{\mu\nu}$. One can define covariant operators that transform as tensor density of weight one under coordinate transformations,

$$O_n^G = \frac{1}{N} \sqrt{|G|} \text{tr} \left[ \Phi \left( \nabla_{\mu_1} G \nabla_{\mu_2} G \ldots \nabla_{\mu_p} G \Phi \right) \left( \nabla_{\mu_1} G \nabla_{\mu_2} G \ldots \nabla_{\mu_p} G \Phi \right) \ldots \right],$$

where $\sqrt{|G|} \equiv \sqrt{\det G_{\alpha\beta}}$ and $\nabla_{\mu}^G$ is the covariant derivative associated with the background metric. Any operator $O_m$ defined on the flat spacetime can be expressed as a linear combination of the covariant operators,

$$O_m(x) = c_m^G(x)O_n^G(x),$$

where $c_m^G(x)$ is the transformation matrix. It is a function of the metric $G_{\mu\nu}$ and its derivative at the position $x$. For example,

$$O_{[2,\mu\nu]} = \frac{1}{\sqrt{|G|}} \left[ O_n^G + \Gamma_{\mu\nu}^\lambda O_\lambda^G \right],$$

where $O_{[2,\mu\nu]} = \frac{1}{N} \text{tr}(\Phi \partial_{\mu} \partial_{\nu} \Phi)$, $O_n^G = \frac{1}{N} \sqrt{|G|} \text{tr}(\Phi \nabla_{\mu}^G \nabla_{\nu}^G \Phi)$, $O_{[2,\lambda]} = \frac{1}{N} \sqrt{|G|} \text{tr}(\Phi \nabla_{\lambda} \Phi)$, and $\Gamma_{\mu\nu}^\lambda$ is the Christoffel symbol for the metric $G_{\mu\nu}$. Therefore, the same Lagrangian in Eq. (1) can be written in terms of these ‘covariant operators’,

$$\mathcal{L} = N^2 \left\{ -\mathcal{J}^{G;m} O_m^G + V(O_m^G; \mathcal{J}^{G;\{m\},\{\nu\}}) \right\},$$

where

$$\mathcal{J}^{G;m}(x) = \mathcal{J}^n(x)c_n^m(G).$$

The inverse of the transformation is given by

$$O_m^G(x) = d_m^G(x)O_n(x),$$

$$\mathcal{J}^m(x) = J^{G;n}(x)d_n^m(G),$$

where $d_n^m(G)$ and $J^{G;m}(x)$ are the transformation matrix and the inverse transformation matrix, respectively.
with \( c_m^b(G) c_{m^b}(G) = \delta_a^b \). The multi-trace operators can be also expressed in terms of the covariant operators and the covariant derivatives of them,

\[
V G m; \mathcal{J} G m; \{v_j^i\} = \sum_{q=1}^{\infty} \frac{\mathcal{J} G m; \{v_j^i\}}{|G|^{\frac{2}{q}}} O_m^G \left( \nabla G v_1^i \cdots \nabla G v_{m_1}^i O_{m_2} \right) \left( \nabla G v_1^p \cdots \nabla G v_{m_q}^p O_{m_{q+1}} \right),
\]

(10)

where \( \mathcal{J} G m; \{v_j^i\} \) can be similarly expressed as a linear combination of \( \mathcal{J} \{v_i^j\} \) so that Eq. (10) coincides with Eq. (3). The explicit form of the transformation is not important. The factor of \(|G|^{-\frac{2}{q}}\) is introduced to make the whole expression to have weight one when \( \mathcal{J} G m; \{v_j^i\} \) has weight zero.

This means that the original theory defined on the flat spacetime can be viewed as a theory defined on a curved spacetime with any background metric. The theory does not depend on the background metric because different choices of metric can be compensated by metric dependent sources. However, there is a natural choice of metric. We choose the metric \( G(0)_{\alpha\beta} \), such that the two derivative kinetic term takes the canonical form, that is, \( J(0)[2, \mu\nu] = G(0)_{\mu\nu} \), where \( J(0)[2, \mu\nu] \) denotes the source for \( \sqrt{|G(0)|} tr \left[ \Phi \nabla G(0)_{\mu} \nabla G(0)^{\mu} \Phi \right] \). This is always possible because one can make \( J(0)[2, \mu\nu] \) symmetric in \( \mu \) and \( \nu \) without loss of generality. In \( D > 2 \), there is a unique metric that satisfies the canonical condition, \( J(0)[2, \mu\nu] = G(0)_{\mu\nu} \) for a given set of sources (For a proof of this, see Appendix A). Such choice of metric is not unique at \( D = 2 \) where one needs an extra condition to fix the freedom associated with the dilatation. Here we assume that \( D > 2 \). In this choice of the background metric, the kinetic term takes the canonical form

\[
\mathcal{L} = -N \sqrt{|G(0)|} G(0)_{\mu\nu} tr \left[ \Phi \nabla G(0)_{\mu} \nabla G(0)^{\mu} \Phi \right] + \ldots
\]

(11)

where the same metric is used for the covariant derivative in each tensorial operator and the source for the kinetic term with two derivatives. In this sense, an action on a flat spacetime with space-time dependent sources defines a natural curved background spacetime. Physically, this amounts to measuring the distance on the manifold based on the cost of the action in the limit that the amplitude of field is small and the field changes slowly in spacetime. For example, one can set the distance between two points to be 1 when the quadratic action that is needed to twist fields between the two points is \( N \) per unit twist and per unit square modulus for each field. In this choice of metric, we denote

\[
\mathcal{J}^{(0)m}(x) \equiv \mathcal{J}^n(x) c^n_m(G(0)),
\]

\[
O^{(0)}_m(x) \equiv d^n_m(G(0)) O_n(x)
\]

(12)
to write

\[ \mathcal{L} = N^2 \left\{ -\mathcal{J}^{(0)m} O_m^{(0)} + V[O_m^{(0)}, \mathcal{J}^{(0);\{m\}}, \{\nu\}] \right\}. \quad (13) \]

Here the background geometry is in general curved, but it is non-dynamical.

### III. AUXILIARY FIELDS AND GAUGE FIXING

In the holographic construction\[^8, 14, 16\], sources become dynamical in the bulk. Therefore one needs to introduce a dynamical field in the bulk for each independent operator. Since multi-trace operators can be written as products of single-trace operators, it is convenient to remove the multi-trace operators at the expense of making the sources for the single trace operators dynamical.

We introduce a pair of auxiliary fields for every single-trace operator\[^16\],

\[ Z = \int Dj^{(1)n} Dp^{(1)n} D\Phi \ e^{i \int d^D x \ \mathcal{L}_1}, \quad (14) \]

where

\[ \mathcal{L}_1 = N^2 \left\{ j^{(1)m}(p^{(1)m} - O_m^{(0)}) - \mathcal{J}^{(0)m} f_m^{n}(G^{(0)}, g)p^{(1)n} + V[f_m^{n}(G^{(0)}, g)p^{(1)n}; \mathcal{J}^{(0);\{m\}}, \{\nu\}] \right\} \]

\[ = N^2 \left\{ (j^{(1)n} - \mathcal{J}^{(0)n} f_m^{n}(G^{(0)}, g)) p^{(1)n} - j^{(1)m} O_m^{(0)} + V[f_m^{n}(G^{(0)}, g)p^{(1)n}; \mathcal{J}^{(0);\{m\}}, \{\nu\}] \right\}. \quad (15) \]

Here \( j^{(1)n}(x) \) and \( p^{(1)n}(x) \) are \( D \)-dimensional auxiliary fields which in general carry \( D \)-dimensional spacetime indices. \( g_{\mu\nu} \) is an arbitrary \( D \)-dimensional metric that is used to define a new set of tensorial operators \( O_m^{\mu} \). It is noted that \( g \) is in general different from \( G^{(0)} \). The matrix that transforms fields defined with different background metrics is given by

\[ f_m^{n}(G^{(0)}, g) = d_m^{\ a}(G^{(0)}) c_a^{ n}(g), \quad (16) \]

which satisfies \( f(g, g) = I, f(g^{(1)}, g^{(1)}) f(g^{(2)}, g^{(2)}) f(g^{(3)}, g^{(3)}) = f(g^{(1)}, g^{(2)}) f(g^{(2)}, g^{(1)}) = I \). \( j^{(1)n} \)'s are dynamical sources for single-trace operators, and \( p^{(1)n} \)'s represent the operators themselves\[^16\]. In Eq. (15), \( j^{(1)n} \) is a Lagrangian multiplier that enforces the constraint \( p^{(1)n} = O_m^{\mu} \).

Once \( j^{(1)n} \) and \( p^{(1)n} \) are integrated out, Eq. (15) becomes Eq. (13).

Since the partition function is independent of the metric \( g_{\mu\nu} \), we can formally integrate over different choices of \( g_{\mu\nu} \) and divide by the volume of the space of metric (gauge volume),

\[ Z = \int \frac{Dg Dj^{(1)n} Dp^{(1)n} D\Phi}{V_{\text{gauge}}} e^{i \int d^D x \ \mathcal{L}_1}. \quad (17) \]
The resulting theory has a gauge symmetry associated with different choices of the metric $g_{\mu\nu}$. It is emphasized that this gauge redundancy is different from the coordinate redundancy generated by the $D$-dimensional diffeomorphism. Rather it is associated with different choices of background metric in a fixed coordinate system. Under the gauge transformation, the fields transform as

\[
\begin{align*}
  g_{\mu\nu} &\to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}, \\
  j^{(1)m} &\to j'^{(1)m} = j^{(1)n} f^m_n (g, g + \delta g), \\
  p^{(1)}_m &\to p'^{(1)}_m = f^m_n (g + \delta g, g) p^{(1)}_n, \\
  O^g_m &\to O'^{g+\delta g} = f^m_n (g + \delta g, g) O^g_n.
\end{align*}
\]  

Again we fix the gauge by requiring that the quadratic kinetic term has the canonical form. Namely, we will choose the gauge where $j^{(1)[2,\mu\nu]} = g^{\mu\nu}$. Here we have to take into account a non-trivial determinant in gauge fixing because the sources are dynamical unlike the case in Sec. II. The determinant associated with the gauge fixing can be obtained from the standard Fadeev-Popov method. We first define $\Delta(j, g)$ such that

\[
\int D\delta g \delta \left( j^n f^{[2,\mu\nu]}_n (g, g + \delta g) - (g + \delta g)^{\mu\nu} \right) \Delta(j, g) = 1. \tag{19}
\]

This identity is inserted into the partition function,

\[
\begin{align*}
  Z &= \int Dg D\bar{J}^{(1)n} Dp^{(1)}_n D\Phi D\delta g \\
  &\quad \times \frac{V_{\text{gauge}}}{\Delta(j^{(1)}, g)\delta \left( j^{(1)n} f^{[2,\mu\nu]}_n (g, g + \delta g) - (g + \delta g)^{\mu\nu} \right)} e^{i \int d^{D}x L_1[j^{(1)}, p^{(1)}, g, \Phi]}. \tag{20}
\end{align*}
\]

By changing the variables,

\[
\begin{align*}
  G^{(1)\mu\nu} &= g^{\mu\nu} + \delta g^{\mu\nu}, \\
  J^{(1)m} &= j^{(1)n} f^m_n (g, g + \delta g), \\
  P^{(1)}_m &= f^m_n (g + \delta g, g) p^{(1)}_n,
\end{align*} \tag{21}
\]

and by using the gauge invariance of $L_1$ and the facts that

\[
\begin{align*}
  \begin{vmatrix}
  \frac{\partial j^{(1)}}{\partial J^{(1)}} \\
  \frac{\partial p^{(1)}}{\partial P^{(1)}} \\
  \frac{\partial \delta g}{\partial G^{(1)}}
\end{vmatrix} &= 1, \\
  \Delta(j^{(1)n} f^m_n (g, g + \delta g), g + \delta g) &= \Delta(j^{(1)m}, g). \tag{22}
\end{align*}
\]
we obtain
\[
Z = \int \frac{DG^{(1)} D J^{(1)n} D P^{(1)} D \Phi D \delta g}{V_{\text{gauge}}} \Delta(J^{(1)}, G^{(1)}) \delta \left( J^{(1)} [2, \mu \nu] - G^{(1)[\mu \nu]} \right) e^{i \int d^D x \mathcal{L}'_1[J^{(1)}, P^{(1)}, G^{(1)[\mu \nu]}, \Phi]}
\]
\[
= \int D J^{(1)n} D P^{(1)} D \Phi \Delta(J^{(1)}) e^{i \int d^D x \mathcal{L}'[J^{(1)}, P^{(1)}, J^{(1)[2, \mu \nu]}, \Phi]},
\]
where
\[
\mathcal{L}'_1 = N^2 \left\{(J^{(1)n} - J^{(0)m} f_m^{(1)} (0, 1)) P_n^{(1)} - J^{(1)m} O_m^{(1)} + V[f_m^{(1)} (0, 1) P_n^{(1)} ; J^{(0)}; \{m\}, \{\nu\}] \right\}.
\]
Here \( f_m^{(1)} (0, 1) = f_m^{(1)} (G^{(0)}, G^{(1)}) \) and \( O_m^{(1)} \) refers to covariant operators written in the background metric \( G^{(1)[\mu \nu]} = J^{(1)[2, \mu \nu]} \) which is also the source for the kinetic term with two derivatives. In the following, we will use \( G^{\mu \nu} \) and \( J^{[2, \mu \nu]} \) interchangeably. The determinant becomes
\[
\Delta(J) \equiv \Delta(J, J^{[2, \mu \nu]})
\]
\[
= \left| \int D \delta G \delta \left( J^n f_n^{[2, \mu \nu]} (G, G + \delta G) - (G + \delta G)^{\mu \nu} \right) \right|_{G^{\mu \nu} = J^{[2, \mu \nu]}}^{-1}
\]
\[
= \left| \int D \delta G \delta \left( - \int d y J^n(x) \frac{\delta f_n^{[2, \mu \nu]} (x)}{\delta G_{\alpha \beta} (y)} \delta G^{\alpha \beta} (y) + \delta G^{\mu \nu} (x) \right) \right|_{G^{\mu \nu} = J^{[2, \mu \nu]}}^{-1}
\]
\[
= \det \left[ \delta^{(\mu \nu)}_{(\alpha \beta)} \delta(x - y) + J^n(x) \frac{\delta f_n^{[2, \mu \nu]} (x)}{\delta G_{\alpha \beta} (y)} \right]_{G^{\mu \nu} = J^{[2, \mu \nu]}},
\]
where \( \frac{\delta f_n^{[2, \mu \nu]} (x)}{\delta G_{\alpha \beta} (y)} \big|_{G' = G} \) and we used the fact that \( \frac{\delta f_n^{[2, \mu \nu]} (G, G')}{\delta G^{\mu \nu} (y)} \big|_{G' = G} = - \frac{\delta f_n^{[2, \mu \nu]} (G, G')}{\delta G^{\mu \nu} (y)} \big|_{G' = G}. \)
\( \delta^{(\mu \nu)}_{(\alpha \beta)} \) is the Kronecker delta function for symmetrized indices, and \( \delta(x - y) \) is the \( D \)-dimensional Dirac delta function. To obtain the expression in the third line from the second line in Eq. (25), we use the fact that there is one and only one solution for the gauge fixing condition (see Appendix A).

IV. COARSE GRANING

Now we perform a coarse graining by integrating out high energy modes of the matrix field \( \Phi \). Although the sources \( J^{(1)m} \) are also dynamical fields, one can treat them as background fields when one integrates out high energy modes of \( \Phi \). We focus on the functional integration of the original dynamical field \( \Phi \) which is coupled to the sources \( J^{(1)m} \),
\[
Z_\Phi[J^{(1)}] \equiv \int D \Phi e^{i \int d^D x \left[ \operatorname{tr}(\Phi M \Phi) + U_{J^{(1)}}[\Phi] \right]},
\]
where $M_{J(1)}$ is the kernel for the quadratic action that includes the two and higher derivative terms,

$$M_{J(1)} = -\sqrt{|G(1)|} \left[ J^{(1)[2]} + G^{(1) \mu \nu} \nabla_\mu \nabla_\nu + \sum_{n=3}^{\infty} J^{(1)[2, \mu_1 \ldots \mu_n]} \nabla_{\mu_1} \ldots \nabla_{\mu_n} \right]$$

(27)

and $U_{J(1)}[\Phi]$ includes all other single-trace operators, which are at least cubic in $\Phi$. There is no operator linear in $\Phi$ because $\Phi$ is traceless. The sources for the higher derivative terms in Eq. (27) have the engineering scaling dimension $[J^{[2, \mu_1 \ldots \mu_n]}] = -(n-2)$, and the mass scales associated with the sources play the role of UV cut-offs. It is interesting to note that there are in general many scales. Moreover, the cut-off scales are fluctuating because the sources are dynamical. Each configuration of $J^{[2, \mu_1 \ldots \mu_n]}$ describes a theory of $\Phi$ with a different set of UV cut-off scales. We perform a real space RG transformation\cite{21,22} by lowering some of these energy scales. For this, an auxiliary traceless real symmetric matrix field $\tilde{\Phi}$ is added to the original theory,

$$Z_\Phi[J^{(1)}] = \left[ \det \tilde{M} \right]^{\frac{(N+2)(N-1)}{4}} \int D\Phi D\tilde{\Phi} e^{i \int d^Dx [N\text{tr}(\Phi M_{J(1)} \Phi) + N\text{tr}(\tilde{\Phi} S \tilde{\Phi}) + U_{J(1)}[\Phi]]},$$

(28)

where $\tilde{M}$ is an arbitrary kernel for the auxiliary field. Here $\frac{(N+2)(N-1)}{2}$ is the number of independent components of a real traceless symmetric matrix. We go into a new basis $\phi$ and $\tilde{\phi}$,

$$\Phi(x) = \phi(x) + \tilde{\phi}(x),$$

$$\tilde{\Phi}(x) = \int dy \left( A(x,y) \phi(y) + B(x,y) \tilde{\phi}(y) \right),$$

(29)

where the functions $A$ and $B$ are uniquely chosen from the conditions that the low energy field $\phi$ and the high energy field $\tilde{\phi}$ do not mix at the quadratic level, and that the low energy field has a set of UV-cut off scales which are smaller than those for the original field $\Phi$. Then the partition function takes the form of

$$Z_\Phi[J^{(1)}] = \left[ \det S \right] \left[ \det M_{J(1)'} \right]^{\frac{(N+2)(N-1)}{4}} \int D\phi D\tilde{\phi} e^{i \int d^Dx [N\text{tr}(\phi M_{J(1)'} \phi) + N\text{tr}(\tilde{\phi} S \tilde{\phi}) + U_{J(1)}[\phi + \tilde{\phi}]]},$$

(30)

where the quadratic action for the low energy mode is given by

$$M_{J(1)'} = -\sqrt{|G(1)|} \left[ J^{(1)[2]} + G^{(1) \mu \nu} \nabla_\mu \nabla_\nu + \sum_{n=3}^{\infty} J^{(1)'[2, \mu_1 \ldots \mu_n]} \nabla_{\mu_1} \ldots \nabla_{\mu_n} \right].$$

(31)

Here the rescaled sources are

$$J^{(1)'}[2, \mu_1 \ldots \mu_n](x) = e^{c_n \alpha^{(1)}(x) dz} J^{(1)[2, \mu_1 \ldots \mu_n]}(x),$$

(32)
where \( c_n \) is a set of constants which determine how we rescale the set of UV cut-off scales. Changing the sources for high derivative terms of \( \phi \) in this way is equivalent to lowering the UV cut-off scale associated with the source \( J^{(1)[2,\mu_1,...,\mu_n]} \) by a factor of \( e^{-c_n \alpha^{(1)} dz} \), where \( \alpha^{(1)} \) is the rate at which the UV cut-off is lowered, and \( dz \) is an infinitesimal constant. It is noted that one can choose different speeds of coarse graining at different points in spacetime, and \( \alpha^{(1)}(x) \) is in general position dependent. This is a local RG procedure where the speed of coarse graining is spacetime dependent. Specifying \( \{ c_n \} \) corresponds to choosing a particular RG scheme. One natural choice would be \( c_n = (n-2) \) which reflects the fact that \( J^{(1)[2,\mu_1,...,\mu_n]} \) has dimension \(- (n-2)\). This amounts to lowering all UV cut-off scales in the same way. However, this choice is not ideal for our purpose: the ratio between the determinants of the original field and the low energy field,

\[
\ln[\det \mathcal{M}_{J^{(1)'}} \det \mathcal{M}_{J^{(1)}^{-1}}] \sim \alpha^{(1)} dz \int d^D k \frac{\sum_n c_n J^{(1)[2,\mu_1,...,\mu_n]}_k k_{\mu_1}...k_{\mu_n}}{\sum_n i^n J^{(1)[2,\mu_1,...,\mu_n]} k_{\mu_1}...k_{\mu_n}},
\]

which is divergent if \( c_n = (n-2) \). In the conventional RG procedure, the determinants do not play an important role, and one can ignore the divergent determinant in computing beta functions. In our case, the determinants are important because they provide a non-trivial action for the dynamical sources. This difference comes from the fact that sources are dynamical fields in our approach, instead of constants. In order to avoid the UV divergence, we choose, among many other choices, the following prescription,

\[
c_n = (n-2) \quad \text{for} \quad n \leq n_c, \\
= 0 \quad \text{for} \quad n > n_c,
\]

where \( n_c \) is a large but fixed number. For sufficiently large momenta, Eq. (33) becomes

\[
\ln[\det \mathcal{M}_{J^{(1)'}} \det \mathcal{M}_{J^{(1)}^{-1}}] \sim \alpha^{(1)} dz \int d^D k \frac{c_n J^{(1)[2,\mu_1,...,\mu_n]}_k k_{\mu_1}...k_{\mu_n}}{\sum_n i^n J^{(1)[2,\mu_1,...,\mu_n]} k_{\mu_1}...k_{\mu_n}},
\]

which is finite. This is a well defined coarse graining procedure, where we are rescaling the sources for higher derivative terms up to the \( n_c \)-th order to eliminate high energy mode while avoiding the divergence in the ratio of the determinants. In a sense, we are performing a coarse graining with two sets of scales. The first set of scales associated with the high derivative terms up to the \( n_c \)-th order plays the role of the usual UV cut-off that is rescaled to thin out high energy modes. The
second set of scales associated with the high derivative terms with more than $n_c$ derivatives cuts off the UV divergence in the ratio of the determinants. The conventional scheme is reproduced when $n_c$ is taken to be infinite. It is emphasized that the specific form of rescaling in Eq. (34) is not important. There exist many other schemes that regularize the divergences in the determinants. What follows below is independent of the specific choice. The propagator of the high energy mode is given by the difference between the propagators of the original field and the low energy field

$$ S^{-1} = M^{-1}_{J(1)} - M^{-1}_{J(1)'} . $$

(36)

Therefore the propagator of the high energy mode is $O(dz)$.

![Diagram](image)

FIG. 1: Two ways of generating quantum corrections to the linear order in $dz$. Each circle denotes trace of a chain of matrix fields. Solid lines represent chains of low energy fields and each dashed line represents a high energy field. (a) Contraction of a pair of high energy fields within a single-trace operator generates two singe-trace operators (the first and the third) and one double-trace operator (the second). In the large $N$ limit, only the second term is $O(N^2)$. (b) At the quadratic order, one can fuse two single-trace operators each of which contains one high energy mode. This leads to one double-trace operator and two single-trace operators, all of which are $O(N^2)$.

Integrating out the high energy mode, we obtain an effective theory for the low energy mode,

$$ Z_{\Phi}[J^{(1)}] = \left[ \det M_{J(1)'} \det M_{J(1)}^{-1} \right] \frac{(N+2)(N-1)}{4} \int D\phi e^{i \int d^D x \left[ N \text{tr} (\phi M_{J(1)'} \delta J^{(1)} \phi) + U_{J(1)} + \delta W^{(1)} \right]} , $$

(37)

where $\delta J^{(1)}$ is the quantum correction to the sources for the single-trace operators, and $\delta W^{(1)}$ denotes double-trace operators generated from quantum corrections. Because the propagator of the high energy mode $\tilde{\phi}$ is of the order of $dz$, only two diagrams contribute to the quantum corrections to the linear order in $dz$. The first contribution comes from contracting two high energy fields.
within one single-trace operator as is shown in Fig. 1(a),

$$\delta L = -N \sqrt{|G^{(1)}| J^{(1)[q+1, \mu_j]}} \left\langle \text{tr} \left[ \phi \left( \nabla_{\mu_1} \phi \right) \ldots \left( \nabla_{\mu_{p-1}} \phi \right) \left( \nabla_{\mu_p} \phi \right) \right] \right\rangle,$$

(38)

where \( \nabla_{\mu_i} \phi \equiv \left( \nabla_{\mu_1} \phi \right) \nabla_{\mu_2} \phi \ldots \nabla_{\mu_p} \phi \) and

$$< O >_\phi = i \frac{S^{-1}(x, y) + S^{-1}(y, x)}{8N} \left[ (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) - \frac{2}{N} \delta_{ab} \delta_{cd} \right],$$

(40)

one obtains

$$\delta L = -i \frac{\sqrt{|G^{(1)}|}}{8} J^{(1)[q+1, \mu_j]} \nabla_{\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_p} \left[ S^{-1}(x, y) + S^{-1}(y, x) \right]_{y=x} \left\langle \text{tr} \left[ \phi \left( \nabla_{\mu_1} \phi \right) \ldots \left( \nabla_{\mu_{p-1}} \phi \right) \left( \nabla_{\mu_p} \phi \right) \right] \right\rangle \text{tr} \left[ \nabla_{\mu_1} \phi \ldots \nabla_{\mu_p} \phi \right].$$

(41)

Here we drop the contributions that are sub-leading in \( 1/N \). Integrating by part, if necessary, one obtains operators of the form,

$$N^2 \sum_{p,q=0}^{\infty} \frac{\delta_{q(1)} J^{(1)[mn \mu_1 \ldots \mu_p \mu_{p+1} \ldots \mu_q]}}{\sqrt{|G^{(1)}|}} \left( \nabla_{\mu_1} \ldots \nabla_{\mu_p} O_{m}^{(1)} \right) \left( \nabla_{\nu_1} \ldots \nabla_{\nu_q} O_{n}^{(1)} \right).$$

(42)

This expression represents double-trace operators when both \( O_{m}^{(1)} \) and \( O_{n}^{(1)} \) are non-trivial. In the special case where \( O_{m}^{(1)} = 1 \) with \( p = 0 \) (or \( O_{n}^{(1)} = 1 \) with \( q = 0 \)), it reduces to single-trace operators. The special case occurs when two high energy fields are adjacent to each other in a chain of matrix fields within a single-trace operator, as in \( J^{(1)[4, \mu]} \frac{1}{N} \text{tr} \left[ \phi \phi \phi \phi \nabla_{\mu} \nabla_{\nu} \phi \right] \). The second contribution comes from fusing two single-trace operators as is shown in Fig. 1(b),

$$\delta L' = i \frac{N^2}{2} \int d^D y \sqrt{|G^{(1)}(x)|} \sqrt{|G^{(1)}(y)|} J^{(1)[q+1, \mu_j]}(x) J^{(1)[p+1, \nu_j]}(y) \left\langle \text{tr} \left[ \phi \left( \nabla_{\mu_1} \phi \right) \ldots \left( \nabla_{\mu_{p-1}} \phi \right) \left( \nabla_{\mu_p} \phi \right) \right] \text{tr} \left[ \phi \left( \nabla_{\nu_1} \phi \right) \ldots \left( \nabla_{\nu_{q-1}} \phi \right) \left( \nabla_{\nu_q} \phi \right) \right] \right\rangle.$$

(43)
Contracting the high energy modes, one obtains both single-trace and double-trace operators, 
\[
\delta \mathcal{L}' = -\frac{1}{16} \int d^Dy \sqrt{|G^{(1)}(x)|} \sqrt{|G^{(1)}(y)|} [J^{(1)}[\mu, \nu, \alpha](x), J^{(1)}[\mu, \nu, \alpha](y)] \times
\]
\[
\nabla^\nu_{\{\mu\}} \nabla^\mu_{\{\nu\}} [S^{-1}(x, y) + S^{-1}(y, x)] \left\{ -2 \text{tr} \left[ \phi \left( \nabla^\nu_{\{\mu\}} \phi \right) \ldots \left( \nabla^\nu_{\{\mu\}} \phi \right) \right] \right. 
\]
\[
\left. \times \text{tr} \left[ \phi \left( \nabla^\nu_{\{\mu\}} \phi \right) \ldots \left( \nabla^\nu_{\{\mu\}} \phi \right) \right] \right.
\]
\[
+ N \text{tr} \left[ \phi \left( \nabla^\nu_{\{\mu\}} \phi \right) \ldots \left( \nabla^\nu_{\{\mu\}} \phi \right) \right] \left( \nabla^\nu_{\{\mu\}} \phi \right) \ldots \left( \nabla^\nu_{\{\mu\}} \phi \right)
\]
\[
+ N \text{tr} \left[ \phi \left( \nabla^\nu_{\{\mu\}} \phi \right) \ldots \left( \nabla^\nu_{\{\mu\}} \phi \right) \right] \left( \nabla^\nu_{\{\mu\}} \phi \right) \ldots \left( \nabla^\nu_{\{\mu\}} \phi \right)
\}
\]
\[
(44)
\]

In this expression, it is understood that \( \phi = \phi(x) \) and \( \phi' = \phi(y) \). Although the quantum corrections appear to be non-local, the propagator for the high energy mode decays exponentially in real space, allowing one to do a gradient expansion. The scale that controls the gradient expansion is the UV cut-off which is set by the dynamical sources. This results in local double-trace operators of the form in Eq. (42) and quantum corrections to single-trace operators. The determinant \[\det \mathcal{M}_{\alpha(1)}, \det \mathcal{M}_{\alpha(1)}^{-1} \] (N+2)(N-1) gives rise to a Casimir energy that depends on the source \( J^{(1)} \). The Casimir energy provides a ‘potential’ energy for the dynamical sources while the double-trace operators becomes a quadratic ‘kinetic’ term for the conjugate fields as will be shown in Sec. VI and VII.

Note that even though the action for \( \Phi \) in Eq. (24) has only single-trace operators, double-trace operators are generated in the renormalized action \[10, 11\]. Triple or higher trace operators are at least order of \( dz^2 \) and can be ignored in the small \( dz \) limit. After exponentiating the determinant into a quantum action for the dynamical sources, the total Lagrangian for the low energy field and dynamical sources can be written in the following form,

\[
\mathcal{L}_2 = N^2 \left\{ V[f_m^n(0, 1) P^{(1)}_n; J^{(0)}; \mu, \nu] \right. 
\]
\[
\left. + \left( J^{(1)m} - J^{(0)m} f_m^n(0, 1) \right) P^{(1)}_n \right. 
\]
\[
\left. + \delta_{\alpha(1)} \mathcal{L}[J^{(1)m}] \right. 
\]
\[
\left. - \left( J^{(1)m} + \delta_{\alpha(1)} J^{(1)m} \mu \right) \nabla^{(1)}_\mu O^{(1)}_m \right. 
\]
\[
\left. + \delta_{\alpha(1)} \frac{J^{(1)m} \mu \nu}{\sqrt{|G^{(1)}|}} \left( \nabla^{(1)}_\mu O^{(1)}_m \right) \left( \nabla^{(1)}_\nu O^{(1)}_n \right) \right\}. (45)
\]
the sources for double-trace operators generated from quantum corrections in Eqs. (41) and (44). In the Casimir energy and the quantum corrections to the sources, it is enough to keep only those contributions to the order of $dz$. Therefore, $\delta_{\alpha(1)} L[J^{(1)m}]$, $\delta_{\alpha(1)} J^{(1)m\{\mu\}}$ and $\delta_{\alpha(1)} J^{(1)mn\{\mu\}\{\nu\}}$ are linear in $\alpha^{(1)} dz$. In general, all terms that respect the $D$-dimensional diffeomorphism invariance are allowed in the Casimir energy and the quantum corrections to the sources.

\[
\delta_{\alpha(1)} L[J^{(1)m}] = dz \alpha^{(1)}(x) \sqrt{|G^{(1)}|} \left\{ C_0[J^{(1)}] + C_1[J^{(1)}] \mathcal{R} + \ldots \right\}, \tag{46}
\]

\[
\delta_{\alpha(1)} J^{(1)m\{\mu\}} = dz \alpha^{(1)}(x) A^{m\{\mu\}}[J^{(1)}], \tag{47}
\]

\[
\delta_{\alpha(1)} J^{(1)mn\{\mu\}\{\nu\}} = dz \alpha^{(1)}(x) B^{mn\{\mu\}\{\nu\}}[J^{(1)}]. \tag{48}
\]

Here $\delta_{\alpha(1)} L[J^{(1)m}]$ is the Casimir energy which includes the cosmological constant $C_0$ and the $D$-dimensional Ricci scalar $\mathcal{R}$. It also includes higher order terms in the curvature tensor and derivative action for other sources, $J^{[2,\mu_1,\ldots,\mu_n]}$. Since the Casimir energy comes from the determinant of the quadratic operator of $\Phi$, it depends only on the sources for the quadratic operators. On the other hand, in Eqs. (47) and (48), cubic or higher order operators are also renormalized by quantum corrections. $C_0$, $C_1$, $A^{m\{\mu\}}$ and $B^{mn\{\mu\}\{\nu\}}$ are finite functions of the sources $J^{(1)}$ and their covariant derivatives. Note that one needs to include descendant operators for quantum corrections in Eq. (45) to express $\delta_{\alpha(1)} J^{(1)m\{\mu\}}$ and $\delta_{\alpha(1)} J^{(1)mn\{\mu\}\{\nu\}}$ as linear functions of $\alpha^{(1)}$ without derivative as in Eqs. (47) and (48).

It is noted that the partition function is independent of $\alpha^{(1)}(x)$. This is because $\alpha^{(1)}(x)$ is an arbitrary function introduced to change the length scale in the coarse graining procedure. One can choose any speed of RG without affecting the partition function. If one modifies the parameter $\alpha^{(1)}$ to $\alpha^{(1)} + \delta \alpha$, the quantum corrections will be modified accordingly, exactly undoing the changes caused by $\delta \alpha$. In other words, one has to add quantum corrections (counter terms) so that the partition function computed from the low energy effective theory with a lower UV cut-off is equal to the one computed from the bare theory with the original cut-off. This is nothing but the cut-off independence of the partition function in the Wilsonian RG. The fact that the partition function is independent of the choice of $\alpha^{(1)}(x)$ will become important later to obtain the $(D+1)$-dimensional diffeomorphism invariance in the holographic description as will be discussed in Sec. VII.
FIG. 2: The $D$-dimensional coordinate of the low energy mode is shifted infinitesimally relative to the coordinate of the field defined at high energy.

V. SHIFT

One key difference of the present construction from the conventional RG procedure[21, 22] is that the sources (and the conjugate fields) that are coupled with the low energy field are dynamical. In particular, the low energy field is covariantly coupled with the fully dynamical $D$-dimensional metric. Therefore, there is a freedom to choose the coordinate system for the low energy mode without modifying the form of the Lagrangian. To take advantage of this extra freedom, we will choose a new coordinate system for the low energy mode which is infinitesimally shifted along a $D$-dimensional direction compared to the original spacetime[12]. The point is that one does not have to use the same $D$-dimensional coordinate for the low energy field as the one for the high energy field as is illustrated in Fig. 2. For this, we single out the action for the low energy mode,

$$S_l = N^2 \int d^D x \left[ - (J^{(1)} m + \delta_{\alpha(1)} J^{(1)} m \mu \nabla^{(1)} \mu \right) O^{(1)} m + \delta_{\alpha(1)} J^{(1)} mn \mu \nu \nabla^{(1)} \mu \nabla^{(1)} \nu \right] \sqrt{|G^{(1)}|} \left( \nabla^{(1)} \mu \right) \left( \nabla^{(1)} \nu \right) \right],$$

and change the variable,

$$\tilde{\phi}(y) = \phi(x)$$

with $y^{\mu} = x^{\mu} + N^{(1)} \mu(x) dz$. In the new coordinate system, the metric and the sources are transformed as tensors. The operators transform as tensor densities of weight one.

To the linear order of $dz$, we have

$$S_l = N^2 \int d^D y \left[ - \tilde{J}^{(1)} m \tilde{O}^{(1)} m - \delta_{\alpha(1)} \tilde{J}^{(1)} m \mu \nu \nabla^{(1)} \mu \nabla^{(1)} \nu \right] \sqrt{|\tilde{G}^{(1)}|} \left( \nabla^{(1)} \mu \right) \left( \nabla^{(1)} \nu \right) \right],$$

(51)
where $\tilde{O}_{m}^{(1)}$s are the covariant operators constructed with the covariant derivative $\tilde{\nabla}^{(1)}$ with the new metric

$$\tilde{G}_{\mu\nu}^{(1)}(y) = \frac{\partial x^\lambda}{\partial y^\mu} \partial x^\sigma \partial y^\nu G_{\lambda\sigma}^{(1)}(x).$$

These operators are coupled with the new sources given by

$$\tilde{J}^{(1)[q,\{\mu_i\}]}(y) = \prod_{i,j} \left[ \frac{\partial y^\mu_i}{\partial x^\nu_i} \frac{\partial y^\nu_j}{\partial x^\nu_j} \right] J^{(1)[q,\{\nu^i\}]}(x).$$

We can ignore the shift in $\delta^{(1)}_{\alpha} J^{(1)m\{\mu}\{\nu\}}$ and $\delta^{(1)}_{\alpha} J^{(1)mn\{\mu\}\{\nu\}}$ because they are already order of $dz$.

The Lagrangian in the shifted coordinate becomes

$$\mathcal{L}_1 = N^2 \left\{ - \tilde{J}^{(1)m} f_m^n (G^{(1)} + \delta^{(1)}_{N(1)} G^{(1)} , G^{(1)}) O_n^1 - \delta^{(1)}_{\alpha} J^{(1)m\{\mu\}} \nabla^{(1)}_{\{\mu\}} O_m^1 \right. $n

$$+ \frac{\delta^{(1)}_{\alpha} J^{(1)mn\{\mu\}\{\nu\}}}{\sqrt{|G^{(1)}|}} (\nabla^{(1)}_{\{\mu\}} O_m^1)(\nabla^{(1)}_{\{\nu\}} O_n^1) \right\}$$

$$= N^2 \left\{ - \left( J^{(1)m} + \delta^{(1)}_{\alpha} J^{(1)m\{\mu\}} \nabla^{(1)}_{\{\mu\}} + \delta^{(1)}_{N(1)} J^{(1)m} \right) O_m^1 \right.$n

$$+ \frac{\delta^{(1)}_{\alpha} J^{(1)mn\{\mu\}\{\nu\}}}{\sqrt{|G^{(1)}|}} (\nabla^{(1)}_{\{\mu\}} O_m^1)(\nabla^{(1)}_{\{\nu\}} O_n^1) \right\},$$

where $\delta^{(1)}_{N(1)} G^{(1)\mu\nu} = \tilde{G}^{(1)\mu\nu} - G^{(1)\mu\nu}$ and

$$\delta^{(1)}_{N(1)} J^{(1)m} = \left( \tilde{J}^{(1)m} - J^{(1)m} \right) + J^{(1),n} \left( f_m^n (G^{(1)} + \delta^{(1)}_{N(1)} G^{(1)}, G^{(1)} - \delta^{(1)}_{\alpha} J^{(1)m}) \right).$$

As was the case for $\alpha^{(1)}$, the partition function is clearly independent of $N^{(1)\mu}$, because different choices of shift merely corresponds to different choices of coordinate system for the low energy mode. This completes one cycle of the RG procedure. We have a theory of the low energy field coupled with dynamical sources whose fluctuations are controlled by the action generated from the high energy mode,
VI. CONSTRUCTION OF BULK THEORY

Now we repeat the procedures in Secs. III-V. Another set of auxiliary fields are introduced to remove the double-trace operators for the low energy fields followed by the gauge fixing to obtain

\[ Z = \int D J^{(1)n} D P_n^{(1)} D J^{(2)n} D P_n^{(2)} D \Phi \Delta(J^{(1)}) \Delta(J^{(2)}) e^{i \int d^D x \mathcal{L}_3}, \]  

where we use the notation $\Phi$ for the low energy mode to avoid introducing a new notation for the low energy mode at each step of RG, and the Lagrangian is given by

\[ \mathcal{L}_3 = N^2 \left\{ V[f_m^n(0,1) P_n^{(1)}; J^{(0)}; \{ n_i \}, \{ \nu^j \}] + (J^{(1)n} - J^{(0)n} f_m^n(0,1)) P_n^{(1)} + \delta_{\alpha(1)} L [J^{(1)m}] \\
+ J^{(2)n} (P_n^{(2)} - O_n^{(2)}) - (J^{(1)m} + \delta_{\alpha(1)} J^{(1)m} \nabla^{(1)}_{\{ \mu \}} + \delta_{N(1)} J^{(1)m}) f_m^n(1,2) P_n^{(2)} \\
+ \frac{\delta_{\alpha(1)} J^{(1)mn\{ \mu \}\{ \nu \}}}{\sqrt{|G^{(1)}|}} (\nabla^{(1)}_{\{ \mu \}} f_m^{k(1,2)} P_k^{(2)})(\nabla^{(1)}_{\{ \nu \}} f_m^{k'(1,2)} P_{k'}^{(2)}) \right\}. \]  

(58)

Here $O_n^{(2)}$'s represent covariant operators constructed with the metric $G^{(2)\mu\nu} = J^{(2)[2,\mu\nu]}$. $\Delta(J^{(2)})$ in Eq. (57) is the Jacobian generated from the gauge fixing. As was done in Sec. IV, high energy modes are integrated out with a spacetime dependent coarse graining rate $\alpha^{(2)}(x)$ to generate the Casimir energy for $J^{(2)n}$ and another set of quantum corrections to single-trace operators and double-trace operators, which are proportional to $\alpha^{(2)}(x)dz$. This is followed by another infinitesimal shift along the $D$-dimensional direction $N^{(2)\mu}(x)$ as in Sec. V. Note that the $\alpha^{(2)}(x)$ and $N^{(2)\mu}(x)$ are independent of $\alpha^{(1)}(x)$ and $N^{(1)\mu}(x)$. Namely, we can choose different rate of coarse graining and different shift at each scale. The double trace operators are again removed by introducing a third set of auxiliary fields $J^{(3)m}$ and $P_n^{(3)}$.

If we repeat these steps $L$ times, the partition function can be written as

\[ Z = \int \prod_{l=1}^{L} \left[ D J^{(l)n} D P_n^{(l)} \Delta(J^{(l)}) \right] D \Phi e^{i \int d^D x \mathcal{L}_4}, \]  

(59)
where

\[
\mathcal{L}_4 = N^2 V \left[ f^m_n (0, 1) P^{(1)}_n \right] \mathcal{J}^{(0)}; \{ m_i \}, \{ \nu_j \} \\
+ N^2 \sum_{l=0}^{L} \left\{ (J^{(l+1)n} - J^{(l)m} f^m_n (l, l + 1)) P^{(l+1)}_n \right\} \\
+ N^2 \sum_{l=1}^{L} \left\{ \delta_{\alpha(l)} \mathcal{L}[J^{(l)m}] - \left( \delta_{\alpha(l)} J^{(l)m(\mu)} \nabla^{(l)}_{\mu} + \delta_{N^{(l)} J^{(l)m}} \right) f^{m}_n (l, l + 1) P^{(l+1)}_n \right\} \\
+ \frac{\delta_{\alpha(l)} J^{(l+1)m(\mu)} (\nu_j)}{\sqrt{|G(l)|}} \left( \nabla^{(l)}_{\mu} f^m_k (l, l + 1) P^{(l+1)}_{k} \right) \left( \nabla^{(l)}_{\nu} f^m_n (l, l + 1) P^{(l+1)}_{n} \right) \\
- N^2 J^{(L+1)n} O^{(L+1)}_n. \tag{60}
\]

Here it is understood that \( J^{(0)n} = \mathcal{J}^{(0)n} \).

Now we first take the limit with \( dz \to 0 \) and \( L \to \infty \) with fixed \( z_L = Ldz \), where \( z = ldz \) becomes a continuous coordinate that labels the length scale in the range \( 0 \leq z \leq z_L \). \( J^{(l)n}(x) \), \( P^{(l)}_n(x) \), \( \alpha^{(l)}(x) \) and \( N^{(l)}(x) \) become \( D + 1 \)-dimensional fields \( J^n(x, z) \), \( P_n(x, z) \), \( \alpha(x, z) \) and \( N^\mu(x, z) \), respectively. Then, we take the \( z_L \to \infty \) limit, which amounts to taking the low energy limit where one push the RG procedure to the IR limit. In this limit, the partition function becomes

\[
Z[\mathcal{J}] = \int \mathcal{D}J(x, z) \mathcal{D}P(x, z) \mathcal{M}(J) e^{i \left( S_{UV}[P(x, 0)] + S[J(x, z), P(x, z)] + S_{IR}[J(x, \infty)] \right)} \left|_{J(x, 0) = \mathcal{J}(x)} \right., \tag{61}
\]

where

\[
\mathcal{D}J(x, z) \mathcal{D}P(x, z) \equiv \prod_{l=1}^{\infty} \left[ D J^{(l)}(x) \right] D P^{(l)}(x) \\
\mathcal{M}(J) \equiv \prod_{l=1}^{\infty} \Delta(J^{(l)}) \\
S_{UV} = N^2 \int d^Dx \ V[P_m(x, 0); \mathcal{J}^{(0)}; \{ m_i \}, \{ \nu_j \}] \\
S = N^2 \int d^Dx \int dz \ \left[ \left( \mathcal{D}_z J^n \right) P_n - \alpha(x, z) \mathcal{H} - N^\mu(x, z) \mathcal{H}_\mu \right] \\
S_{IR} = -i \ln \int \mathcal{D}\Phi \ e^{-i N^2 \int d^Dx J_m(x, \infty) G^m(x, \infty)}. \tag{62}
\]

Here \( S_{UV} \) and \( S_{IR} \) are the actions defined at the UV (\( z = 0 \)) and the IR (\( z = \infty \)) boundaries respectively. \( S \) is the bulk action. \( \mathcal{D}_z J^n \equiv (\partial_z J^n + J^m \mathcal{A}_m^n) \) is a ‘covariant derivative’ for a vector.
$J^n$ defined in the space of operators with the connection,
\[
A^n_m(x, z) = \partial_z f^n_m(G(x, z), G(x, z')) \bigg|_{z' = z}
\]
\[
= \int dy \partial_z G^{\mu\nu}(y) \frac{\delta f^n_m(x)}{\delta G^{\mu\nu}(y)}.
\]
(63)

Note that $D_z$ is not related to the covariant derivative in the $D$-dimensional spacetime $\nabla_\mu$. $H$ and $H_\mu$ are given by
\[
H = A^{(m)}[J(x)](\nabla_{\{\mu\}}P_m) - \frac{B^{mn\{\mu\}\{\nu\}}[J(x)]}{\sqrt{|G|}}(\nabla_{\{\mu\}}P_m)(\nabla_{\{\nu\}}P_n)
\]
\[
- \sqrt{|G|} \left\{ C_0[J(x)] + C_1[J(x)]R + \ldots \right\},
\]
(64)
\[
H_\mu = -2\sqrt{|G|}\nabla^\nu \left[ \frac{1}{\sqrt{|G|}} \left( P_{[2,\mu\nu]}(x) + \int dy J^n(y) \frac{\delta f^n_m(y)}{\delta G^{\mu\nu}(x)} P_m(y) \right) \right]
\]
\[
- \sum_{[q,\{\mu_i\}]
eq [2,\mu\nu]} \left[ \sqrt{|G|} \sum_{a,b} \nabla_\nu \left( \frac{1}{\sqrt{|G|}} J^{[q,\{\mu_1\mu_2\ldots \mu_{b-1}\nu \mu_{b+1}\ldots \}]} P_{[q,\{\mu_1\mu_2\ldots \mu_{b-1}\nu \mu_{b+1}\ldots \}]} \right) \right.
\]
\[
+ \left( \nabla_\mu J^{[q,\{\mu_i\}]} \right) P_{[q,\{\mu_i\}]} \right]\right],
\]
(65)

with $\mu = 0, 1, \ldots, (D - 1)$.

Starting from the $D$-dimensional matrix field theory, we obtained a $(D+1)$-dimensional theory for dynamical sources and operators. The extra dimension parameterized by $z$ represents the length scale in the RG. One can choose different speed $\alpha(x, z)$ of RG at different points in spacetime and scale. Therefore $z$ is not a gauge invariant quantity. What is gauge invariant is the length scale whose infinitesimal increment is given by $d\tau = \alpha dz$. In this sense, the physical length scale can be viewed as a proper length along the direction of the extra dimension. This is illustrated in Fig. 3.

The theory in the bulk has the dynamical metric and its conjugate momentum as dynamical degrees of freedom. Therefore it is natural to expect that the bulk theory is a gravitational theory.

In the next section, we will see that the theory in the bulk indeed respects the $(D + 1)$-dimensional diffeomorphism invariance.

VII. HAMILTONIAN GRAVITY

If the RG scale $z$ is interpreted as ‘time’, the theory can be viewed as a Hamiltonian system where the sources $J^m$’s play the role of ‘coordinates’ and the operators $P_m$’s are the ‘momenta’.
FIG. 3: (a) Bulk spacetime made of the $D$-dimensional boundary spacetime and the semi-infinite line that represents the length scale in the RG procedure. Each step of coarse graining, say the $l$-th step, generates a set of $D$-dimensional fields $(J^{(l)}(x), P^{(l)}_{nl}(x))$ that represent dynamical sources and operators at that scale. These fields are combined into $(D + 1)$-dimensional fields $(J^{n}(x, z), P^{n}(x, z))$ in the bulk, where the extra coordinate is given by $z = l dz$. Each ‘vertical’ line traces the positions of the bulk fields which are generated from the original field variable $\Phi(x)$ at each $x$ in the boundary spacetime. The spacetime dependent shift $N^\mu(x, z)$ causes the bulk fields to have different $D$-dimensional coordinates from that of $\Phi(x)$. Each ‘horizontal’ line represents the manifold in the bulk spacetime with an equal $z$ coordinate. Because the speeds of coarse graining are in general different at different points in spacetime, two points within the manifold with an equal $z$ do not in general have the same proper length along the extra dimension, where the proper length is the scale in the RG. (b) The same bulk spacetime where the coordinate $z$ is used instead of the proper length along the extra dimension. The vertical lines have the same meaning as in (a). Each horizontal line represents the manifold with an equal proper length, that is, the set of points with the same length scale in RG. Note that an horizontal line that is concave upward in (a) is concave downward in (b).

The sources and operators are conjugate to each other as expected. The ‘Hamiltonian’ is given by

$$H = \sum_{M=0}^{D} \int d^D x \ N^M \mathcal{H}_M,$$

(66)

where $N^D(x, z) \equiv \alpha(x, z)$ and $\mathcal{H}_D \equiv \mathcal{H}$. Note that the ‘time’ $x^D = z$ is different from the real time $x^0$ in the boundary field theory. The Hamiltonian in Eq. (66) generates the evolution along the time $x^D$ associated with increasing length scale of the system, not along the real time $x^0$. In this sense, one can regard the Hamiltonian as a generator for a quantum beta function. One difference
from the usual Hamiltonian system is that the ‘covariant’ derivative $D_z$ is used in the action. The non-trivial connection originates from the fact the fields $J^a(x, z)$, $P_n(x, z)$ defined at different length scales in general have different metric. Because the definition of the covariant operators and their sources is tied with the metric, a change in metric effectively induces changes in all sources. Physically, the momentum canonically conjugate to the metric is the energy-momentum tensor given by $\pi_{[2, \mu \nu]} = \frac{1}{N^2} \frac{\delta S}{\delta G^{\mu \nu}}$. There are many other contributions to the energy momentum tensor besides $P_{[2, \mu \nu]} = \sqrt{|G|} \text{tr}[\Phi \nabla_\mu \nabla_\nu \Phi]$ because metric enters not only in $\sqrt{|G|} \text{tr}[\Phi \nabla_\mu \nabla_\nu \Phi]$ but also in the definition of all other covariant operators. This suggests that $P_{[2, \mu \nu]}$ is not the canonical momentum of the metric, which is also reflected in the non-trivial measure $\Delta(J)$ in the functional integration, and the unconventional form of the shift for the metric in Eq. \((65)\). In order to go to the canonical basis, we define a new momentum for the metric and keep the same conjugate momenta for all other variables,

$$\pi_{[2, \mu \nu]}(x) = P_{[2, \mu \nu]}(x) + \int dy \ J^a(y) \frac{\delta f_{n, m}(y)}{\delta G^{\mu \nu}(x)} P_m(y), \quad \pi_m = P_m, \quad \text{for } m \neq [2, \mu \nu]. \quad (67)$$

The last term in Eq. \((67)\) takes into account the metric dependence in general operators. The Jacobian from the change of variable,

$$\left| \frac{\delta P_{[2, \mu \nu]}(y)}{\delta \pi_{[2, \alpha \beta]}(x)} \right| = \det \left[ \delta_{(\mu \nu)}^{(\alpha \beta)} \delta(x - y) + J^a(y) \frac{\delta f_{n, [2, \mu \nu]}(y)}{\delta G^{\alpha \beta}(x)} \right]^{-1} \quad \Delta(J)^{-1} \quad (69)$$

exactly cancels $\Delta(J)$ in the measure. The partition function and the action takes the canonical form\((23)\) in the new variables,

$$Z[J] = \int D\pi(x, z) D\pi(x, z) \ e^{i \left( S_{UV}[\pi(x, 0)] + S[J(x, z), \pi(x, z)] + S_{1\mu}[J(x, \infty)] \right)} \bigg|_{J(x, 0) = J(x)}, \quad (70)$$

where

$$S = N^2 \int d^D x dz \left[ (\partial_z J^a) \pi_n - \alpha(x, z) \mathcal{H} - N^a(x, z) \mathcal{H}_\mu \right]. \quad (71)$$

Note that $D_z$ is replaced by the usual derivative in the canonical variables. Moreover, the ‘momentum constraint’ $\mathcal{H}_\mu$ that generates the $D$-dimensional shift takes the standard form,

$$\mathcal{H}_\mu = -2 \nabla^\nu \pi_{[2, \mu \nu]} - \sum_{[q, \{\mu_i\}] \neq [2, \mu \nu]} \sum_{a, b} \nabla_\nu \left( J^{[\mu_i]} \pi_{[q, \{\mu_i\}]} \right) + (\nabla_\mu J^{[q, \{\mu_i\}]}) \pi_{[q, \{\mu_i\}]}. \quad (72)$$
It is noted that $J^m$’s and $\pi_m$’s are $D$-dimensional contra-variant tensors with weight zero and covariant tensor density with weight one, respectively. To obtain the ‘Hamiltonian constraint’ $\mathcal{H}$, one has to convert Eq. (67) to express $P_{[2,\mu\nu]}$ as a linear combination of $\pi_m$’s and plug in the expression to Eq. (64). Since the full expression is complicated, we focus on the metric and its conjugate momentum. Among many other terms, $\mathcal{H}$ includes the linear and quadratic terms for the conjugate momentum, the cosmological constant and the $D$-dimensional curvature, 

$$\mathcal{H} = \tilde{A}^{\mu\nu}[J(x)]\pi_{[2,\mu\nu]} \frac{B^{\mu\nu\lambda\sigma}[J(x)]}{\sqrt{|G|}} \pi_{[2,\lambda\sigma]} - \sqrt{|G|}\left\{C_0[J(x)] + C_1[J(x)]\mathcal{R}\right\} + \ldots,$$

(73)

where ... represents the higher dimensional terms that involve covariant derivatives of $\pi$ and the curvature. Cubic or higher order terms in $\pi_{[2,\mu\nu]}$ are not allowed because at most double-trace operators are generated out of single-trace operators at each step of RG. The linear term in the conjugate momentum arises because the operators that are quartic in $\Phi$, such as $\frac{1}{N}\text{tr} [\Phi^3 \nabla_\mu \nabla_\nu \Phi]$, renormalizes the metric through the quantum correction in Eq. (41). It is interesting to note that the kinetic term for the conjugate momentum originates from the beta function under the RG, while the potential term for the metric originates from the Casimir energy. Besides the dynamical gravitational mode, the theory also includes other degrees of freedom, including the higher spin fields for $\frac{1}{N}\text{tr} [\Phi \nabla_{\mu_1} \nabla_{\mu_2} \ldots \nabla_{\mu_n} \Phi]$ and the fields associated with the single-trace operators that are cubic or higher order in $\Phi$. As was noted in Sec. IV, the latter fields do not have the bare ‘potential energy’ because the Casimir energy is independent of those fields. However, they do have the quadratic kinetic term in general because double-trace terms are generated for those operators under the RG. Although the bare action for those higher order sources are ultra-local along the $D$-dimensional space, potential terms that involve derivatives along the $D$-dimensional space will be generated dynamically, as other heavier fields are integrated out in the bulk [16].

In the large $N$ limit, the bulk fields become classical. In particular, non-perturbative fluctuations of the bulk fields are dynamically suppressed [16]. The on-shell action in the bulk computes the partition function of the original matrix field theory in the large $N$ limit. The classical equation of motion is given by 

$$\partial_z J^n = \{J^n, H\}, \quad \partial_z \pi_n = \{\pi_n, H\},$$

(74)

where the Poisson bracket is defined by

$$\{A, B\} = \int d^Dx \left[ \frac{\delta A}{\delta J^n} \frac{\delta B}{\delta \pi_n} - \frac{\delta A}{\delta \pi_n} \frac{\delta B}{\delta J^n} \right].$$

(75)
To solve the equation of motion, we need another set of boundary conditions besides $J^n(x, 0) = \mathcal{J}^n(x)$. The second set of boundary condition is dynamically imposed by the regularity condition in the IR limit [8]. At the saddle point, we have

$$Z[J(x)] = e^{i(S_{UV} + \tilde{S})} Z[J(x, \infty)],$$

(76)

where the bulk action is evaluated at the saddle point configuration, and $J(x, \infty)$ is determined from the condition that the bulk action is finite in the IR limit. At the first glance, this expression does not seem meaningful because both $Z[J(x)]$ and $Z[J(x, \infty)]$ are not well defined due to the divergent determinants. However, correlation functions are well defined because the divergences in the determinants cancel. Correlation functions of local operators are given by

$$\langle O_{n_1}(x_1) O_{n_2}(x_2) \ldots O_{n_k}(x_{nk}) \rangle = \frac{i^n}{N^{2n}} \left. \frac{\partial^k Z[J'(x)]}{\partial w^{n_1} \partial w^{n_2} \ldots \partial w^{n_k}} \right|_{w=0}$$

(77)

where

$$J'(n) = J^n(x) + \sum_{i=1}^k w^{n_i} \delta_{n_i} \delta(x - x_i).$$

(78)

If non-local sources are turned on, both the saddle point solution in the bulk and $J(x, \infty)$ are modified. For perturbations that are localized both in space and time in $D$-dimensions as the one in Eq. (78), it is expected that $J(x, \infty)$ does not depend on the perturbation, $w^{n_i}$. This is because local perturbations are always irrelevant and die out in the IR limit in unitary theories. Therefore, the contribution from $Z[J(x, \infty)]$ will drop out in Eq. (77).

It is noted that the $D$-dimensional general covariance does not allow mass term for the metric. This does not preclude the possibility that metric fluctuations become massive as other fields are condensed, breaking the $D$-dimensional Lorentz symmetry. Here we consider the simple case where the sources $J^m(x)$ at the boundary respect the $D$-dimensional Lorentz symmetry, and the vacuum does not break the symmetry spontaneously. In this case, the saddle point configuration of the bulk fields will also respect the $D$-dimensional Lorentz symmetry. Mathematically, this means that the only tensor that has a non-zero spin and has a non-zero expectation in the bulk is the metric. Then, the tensors for the conjugate momentum in Eq. (73) take the form,

$$\tilde{A}^{\mu\nu}[J(x)] = A[J(x)] G^{\mu\nu},$$

$$\tilde{B}^{\mu\nu\lambda\sigma}[J(x)] = B_1[J(x)] G^{\mu\nu} G^{\lambda\sigma} + B_2[J(x)] G^{\mu\lambda} G^{\nu\sigma}$$

(79)

at the saddle point, where $A[J(x)]$, $B_1[J(x)]$ and $B_2[J(x)]$ are scalar functions of the sources and their covariant derivatives.
Although the theory in the bulk is a quantum theory of dynamical metric in $(D+1)$-dimensional space, it is not clear whether this theory has the diffeomorphism invariance in the bulk, which is the key property of gravitational theories. In the canonical formalism, the $(D + 1)$-dimensional diffeomorphism invariance would show up as $(D + 1)$ first-class constraints. If $A[J(x)], B_1[J(x)]$ and $B_2[J(x)]$ were just constants, they have to satisfy specific conditions in order for the Hamiltonian constraint to be first-class. For generic values of $A, B_1$ and $B_2$, the Hamiltonian constraint $\mathcal{H}$ is not first-class, in which case the theory does not have the full $(D + 1)$-dimensional diffeomorphism invariance. Given that the coefficients are dynamically determined, it seems highly unlikely that they have the saddle point values of the fixed ratio at all points in the bulk independent of $J^n$. However, we have to be more careful here because the present theory is not a pure gravitational theory. As a result, $A, B_1$ and $B_2$ depend on other dynamical fields which themselves have non-trivial Poisson bracket with their own conjugate momenta. Namely, we can not just replace $A, B_1$ and $B_2$ with the saddle point values when we determine the nature of the constraint. In other words, one should compute the Poisson bracket among the constraints, treating all dynamical fields on the equal footing. Instead of computing the Poisson bracket explicitly, here we use a simple argument to show that all $(D + 1)$-constraints are first-class.

As was emphasized in Secs. IV and V, the partition function does not depend on the choice of the lapse $N^D(x, z) = \alpha(x, z)$ and the shift $N^\mu(x, z)$. From the fact that the partition function is independent of $N^M(x, z)$, we obtain

$$< \mathcal{H}_M(x, z) > = \frac{1}{Z} \frac{\delta Z}{\delta N^M(x, z)} = 0.$$ (80)

Therefore the lapse and the shift play the role of Lagrangian multipliers which impose the local constraints,

$$\mathcal{H} = 0, \quad \mathcal{H}_\mu = 0$$ (81)

inside the bulk spacetime. Since the above equality holds at any time $z$, we have

$$\frac{\partial}{\partial z} < \mathcal{H}_M(x, z) > = \int d^Dy \ N^M(y, z) \langle \{\mathcal{H}_M(x, z), \mathcal{H}_M'(y, z)\} \rangle = 0.$$ (82)

In order for this to be true for any choices of $N^M(x, z)$, we have

$$\{\mathcal{H}_M(x, z), \mathcal{H}_M'(y, z)\} = 0$$ (83)

at the saddle point. This implies that the $(D + 1)$ constraints are first-class classically. These constraints generate local spacetime transformations in the bulk. The Hamiltonian constraint $\mathcal{H}$
generates the transformation, 
\[ x^\mu \rightarrow x^\mu, \quad z \rightarrow z + dlN^D(x, z), \]  
(84)

whereas the momentum constraint \( \mathcal{H}_\mu \) generates
\[ x^\mu \rightarrow x^\mu + dlN^\mu(x, z), \quad z \rightarrow z. \]  
(85)

A general \((D+1)\)-dimensional diffeomorphism generated by a combination of the two corresponds to choosing a different prescription for the local RG procedure.

VIII. A SIMPLE EXAMPLE

Because the construction is rather complicated, it will be useful to apply the prescription to a simple toy model to illustrate the backbone idea. Here we provide an explicit construction for the simplest possible matrix model: 0-dimensional matrix theory. The partition function is given by
\[ Z[J] = \int d\Phi \exp \left[ i \left( -N \sum_{n=2}^{\infty} \mathcal{J}^n \text{tr}(\Phi^n) + N^2 V[\text{tr}(\Phi^n)/N] \right) \right], \]  
(86)

where \( \Phi \) is a real traceless symmetric matrix and \( V[\text{tr}(\Phi^n)/N] \) is a general non-linear function of single-trace operators which may be expanded as
\[ V = \sum_{q=2}^{\infty} \sum_{m_1, m_2, \ldots, m_q} N^{-q} \mathcal{J}^{m_1, m_2, \ldots, m_q} \text{tr}(\Phi^{m_1}) \text{tr}(\Phi^{m_2}) \ldots \text{tr}(\Phi^{m_q}). \]  
(87)

Because there is no spacetime, the partition function is given by a single matrix integration. \( \mathcal{J}^n \)'s (\( \mathcal{J}^{m_1, m_2, \ldots, m_q} \)'s) represent the sources for the single-trace (multi-trace) operators. We assume that the sources have small imaginary components so that the integration is well defined, e.g., \( \text{Im} J^n = -\epsilon \) for even \( n \); \( \text{Im} J^n = 0 \) for odd \( n \).

To remove the multi-trace operators in \( V \), we introduce a pair of auxiliary fields, \( J^{(1), n} \), \( P^{(1)}_n \) for each single-trace operator,
\[ Z = \int dJ^{(1), n} dP^{(1)}_n d\Phi \ e^{iS_1}, \]  
(88)

where
\[ S_1 = N^2 \left\{ J^{(1), n} \left( P^{(1)}_n - \frac{\text{tr}(\Phi^n)}{N} \right) - \mathcal{J}^n P^{(1)}_n + V[P^{(1)}_m] \right\}. \]  
(89)
with \( V[P_m^{(1)}] = \sum_{q=2}^{\infty} \sum_{m_1, m_2, \ldots, m_q} J_{m_1, m_2, \ldots, m_q} P_{m_1}^{(1)} P_{m_2}^{(1)} \cdots P_{m_q}^{(1)}. \) The contours of \( P_n^{(1)} \)'s are along the real axis, but the contours of \( J^{(1), n} \) are chosen slightly off the real-axis as \( I_m J^{(1), n} = I_m J^n \), which guarantees that integration for \( \Phi \) is well defined. Now we have only single-trace operators for \( \Phi \) which are coupled to the dynamical sources \( J^{(1), n} \). \( P_n^{(1)} \) is the conjugate variable which corresponds to the single-trace operator, \( 1/N \text{tr}(\Phi^n) \). This can be seen from Eq. (89) where \( J^{(1), n} \) plays the role of a Lagrangian multiplier which enforces the constraint, \( P_n^{(1)} = 1/N \text{tr}(\Phi^n) \).

There is only one operator which is quadratic in \( \Phi \). We use its source \( J^{(1), 2} \) as a scale to generate a renormalization group transformation. Because \( <\Phi^2> \sim 1/J^{(1), 2} \), we can regard \( 1/J^{(1), 2} \) as a UV cut-off, and generate RG flow by lowering \( 1/J^{(1), 2} \) [8]. The fact that \( 1/J^{(1), 2} \) plays the role of a UV cut-off can be understood from the observation that with a smaller \( 1/J^{(1), 2} \) the fluctuations of \( \Phi^2 \) decreases. Using the method described in Eq. (29), the original matrix field \( \Phi \) can be written as a sum of the low energy field \( \phi \) and the high energy field \( \tilde{\Phi} \),

\[
Z = \left[ \frac{\tilde{m}^2 J^{(1), 2'}}{J^{(1), 2}} \right]^{(N+2)(N-1)/4} \int dJ^{(1), n} dP_n^{(1)} d\phi d\tilde{\Phi} e^{iS_2},
\]

where

\[
S_2 = N^2 \left\{ V[P_m^{(1)}] + (J^{(1), n} - J^n) P_n^{(1)} \right\} - N J^{(1), 2'} \text{tr}(\phi^2) - N \tilde{m}^2 \text{tr}(\tilde{\Phi}^2) - N \sum_{n=3}^{\infty} J^{(1), n} \text{tr}(\phi + \tilde{\Phi})^n
\]

with \( J^{(1), 2'} = J^{(1), 2} e^{2\alpha^{(1)} dz} \) and \( \tilde{m}^2 = \frac{J^{(1), 2}}{2\alpha^{(1)} dz} \). Here \( dz \) is an infinitesimal parameter and \( \alpha^{(1)} \) is the rate at which the UV cut-off is lowered in the first step of RG. Integrating out the high energy mode, one obtains the effective action which includes a Casimir energy, quantum corrections to the single-trace operators, and double-trace operators,

\[
S_3 = N^2 \left\{ V[P_m^{(1)}] + (J^{(1), n} - J^n) P_n^{(1)} \right\} - i\alpha^{(1)} dz \frac{(N + 2)(N - 1)}{2} - N \left( J^{(1), n} + \alpha^{(1)} dz A^n[J^{(1)}] \right) \text{tr}(\phi^n) + \alpha^{(1)} dz B^{nl}[J^{(1)}] \text{tr}(\phi^n) \text{tr}(\phi^l),
\]

where

\[
A^n[J^{(1)}] = 2 J^{(1), 2} \delta_{n, 2} + \frac{1}{2 J^{(1), 2}} \left[ \sum_{k+l=2+n} l k J^{(1), k} J^{(1), l} \frac{i(n + 1)(n + 2)}{2N^2} \left( 1 - \frac{2}{N} \right) J^{(1), n+2} \right],
\]

\[
B^{nl}[J^{(1)}] = \frac{1}{2 J^{(1), 2}} \left[ (l + 1)(n + 1) J^{(1), l+1} J^{(1), n+1} - \frac{i(n + l + 2)}{2} J^{(1), n+l+2} \right].
\]

In this 0-dimensional matrix model, the Casimir energy is a constant independent of the sources. However, in higher dimensions, the Casimir energy is in general a function of dynamical sources,
including metric as we saw in Sec. IV. Now we introduce another set of auxiliary fields \( J^{(2),n} \), \( P^{(2)}_n \) to remove the double-trace operators that are generated from quantum corrections. Then \( \phi \) is again divided into the low energy mode and the high energy mode, by rescaling \( J^{(2),2} \) by \( e^{2\alpha^{(2)} dz} \). Integrating out the high energy mode generates double-trace operators, which are removed by another set of auxiliary fields. Repeating these steps, one can write down the original partition function in terms of the integration of the dynamical sources \( J^{(k),n} \) and the conjugate variables \( P^{(k)}_n \) introduced at each step of RG. As \( dz \to 0 \), the discrete RG step becomes a continuous dimension, and \( J^{(k),n}, P^{(k)}_n, \alpha^{(k)} \) become functions of \( z : J^n(z), P_n(z), \alpha(z) \). The original 0-dimensional theory in Eq. (86) is now mapped into an one-dimensional theory,

\[
Z[J] = \int D J^n(z) D P_n(z) \ e^{i \left( S_{UV}[P(0)] + S[J(z), P(z)] + S_{IR}[J(T)] \right) \bigg |_{J^n(0) = J^n}},
\]

(94)

where

\[
S_{UV} = N^2 V[P_m(0)], \quad S = N^2 \int_0^T dz \ [ (\partial_z J^n) P_n - \alpha(z) \mathcal{H}] , \quad S_{IR} = -i \ln \int d\phi \ e^{-iN \sum_n J^n(T) \text{tr}(\phi^n)}.
\]

(95)

Here \( T \) is the RG ‘time’ at which we stop the coarse graining procedure. This creates a boundary at \( z = T \) and a boundary action \( S_{IR} \). The partition function is independent of \( T \), and one can take \( T \to \infty \) to push the ‘IR boundary’ to infinity. The ‘Hamiltonian’ in the bulk is given by

\[
\mathcal{H} = i \frac{(N+2)(N-1)}{2N^2} + 2J^2 P_2
\]

\[
+ \frac{1}{2J^2} \sum_{n \geq 2} \left[ \sum_{k,l \geq 3; k+l=2+n} lk J^k J^l + \frac{i(n+1)(n+2)}{2N} \left( 1 - \frac{2}{N} \right) J^{n+2} \right] P_n
\]

\[
- \frac{1}{2J^2} \left[ \sum_{l,n \geq 2} (l+1)(n+1)J^{l+1} J^{n+1} P_n P_l - \sum_{l+n \geq 2} \frac{i(n+l+2)}{2} J^{n+l+2} P_n P_l \right].
\]

(96)

For \( P_n \) with \( n < 2 \) we use the convention, \( P_0 = 1 \) and \( P_1 = 0 \), which reflect the fact that \( \frac{1}{N} \text{tr}(I) = 1 \) and \( \frac{1}{N} \text{tr}(\phi) = 0 \). This is the one-dimensional holographic description for the 0-dimensional matrix theory.

From the expression in Eq. (94), one immediately realizes that the partition function can be viewed as a transition amplitude of a quantum mechanical system if \( z \) is identified as time and \( \mathcal{H} \) as
Hamiltonian. In the Hamiltonian interpretation, the sources and the conjugate fields are promoted to quantum operators, and satisfy the commutation relation,

\[
[\hat{J}^n, \hat{P}_m] = i\frac{1}{N^2} \delta^n_m,
\]

(97)

where \([\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}\) is the usual commutator and \(\frac{1}{N^2}\) plays the role of the Planck constant.

The partition function can be written as

\[
Z = < \Psi_f | e^{-iN^2 \int_T^0 dz \alpha(z) \mathcal{H}} | \Psi_i >,
\]

(98)

where the initial and the final wavefunctions are given by

\[
< P_n | \Psi_i > = e^{-iN^2 \int_0^T d\alpha(z) \mathcal{H}} | P_n >,
\]

\[
< J^n | \Psi_f > = e^{iS_{IR}[J^n]}.
\]

(99)

Since Hamiltonian is not Hermitian, the evolution is not unitary, which is consistent with the fact RG flow is irreversible. Note that \(\alpha(z)\) becomes the lapse function along the time direction. Moreover, the partition function is independent of the choice of \(\alpha(z)\). Choosing a different \(\alpha(z)\) amounts to using a different parameterization along the RG flow. The independence of the partition function under the reparameterizaion of RG flow is nothing but the diffeomorphism invariance of the bulk theory. This one-dimensional diffeomorphism invariance is expressed in terms of the constraint,

\[
\frac{\delta Z}{\delta \alpha(z)} = -iN^2 < \Psi_f | e^{-iN^2 \int_0^T d\alpha(w) \mathcal{H} dw} \mathcal{H} e^{-iN^2 \int_0^T d\alpha(w) \mathcal{H} dw} | \Psi_i > = 0.
\]

(100)

However, \(\mathcal{H} | \Psi_i >\) does not identically vanish because the boundary at UV explicitly breaks the diffeomorphism invariance.

**IX. HOLOGRAPHIC (QUANTUM) RG VS. CONVENTIONAL (CLASSICAL) RG**

Finally, we compare the holographic description with the conventional RG. In the present construction, the effective action contains only single-trace operators at all energy scales. This greatly simplifies the RG procedure. The price one has to pay is that one has to promote the sources of the single-trace operators to dynamical variables. In other words, the couplings are not mere constants any more, but they have non-trivial quantum fluctuations. Accordingly, the beta functions that govern the change of couplings under the RG flow are *quantum* operator equations not
FIG. 4: (a) In the conventional RG, multi-trace operators are generated at low energies although only single-trace operators are turned on at UV. Once the initial condition is given, there is a unique RG trajectory determined by the classical beta function. (b) In the holographic description, one only needs to keep track of single-trace operators under the RG flow at the expense of making the sources for the single-trace operators dynamical variables. The partition function is given by sum over all possible RG trajectories for the single-trace operators. One has the freedom to employ different RG schemes, namely, one can choose different ‘speed’ of RG flow at different scales. This freedom corresponds to the diffeomorphism invariance in the bulk.

Classical equations,

$$\frac{i}{N^2} \frac{\partial j_n}{\partial z} = [\hat{j}_n, \hat{H}],$$  \hspace{1cm} (101)

where we have chosen the gauge $\alpha(z) = 1$. The equation for the sources has to be supplemented by the equation for the conjugate operators, $\frac{i}{N^2} \frac{\partial \hat{P}_n}{\partial z} = [\hat{P}_n, \hat{H}]$. In the large $N$ limit, the quantum beta function reduces to the classical equation in Eq. (74) with the identification $[A, B] \rightarrow \frac{i}{N^2} \{A, B\}$ where $\{A, B\}$ is the Poisson bracket. The ‘Hamiltonian’ that governs the quantum RG flow is dynamically generated from integrating out high energy modes at each step of RG. This is in contrast to the conventional RG where the RG trajectory is uniquely determined once an initial condition is given. In conventional RG, there is no quantum fluctuations for coupling constants, but one has to keep all multi-trace operators along the RG flow. This is illustrated in Fig. 4.
X. SUMMARY AND DISCUSSIONS

From a first-principle construction, it is shown that a $D$-dimensional matrix field theory is mapped into a $(D + 1)$-dimensional quantum theory of gravity, where the metric in the bulk spacetime is fully dynamical. The construction starts from the observation that one can identify high energy modes as fluctuating sources for the low energy modes in RG\[8\]. For matrix field theories, this is implemented by introducing a dynamical source and its conjugate momentum for each primary single-trace operator to remove multi-trace operators at each step of RG\[16\]. In particular, there is a spin two source and its conjugate momentum that represent the dynamical metric and the energy-momentum tensor, respectively. While the dynamical sources and momenta are initially introduced as auxiliary fields, they acquire non-trivial dynamics as high energy modes are integrated out. On the one hand, the double-trace operators that are generated from single-trace operators through quantum correction provides the quadratic kinetic term for the conjugate momenta. On the other hand, the potential terms, including the curvature term for the $D$-dimensional dynamical metric, are generated from the source dependent determinant for the high energy mode that is integrated out at each step of RG. The kinetic and potential terms together can be viewed as a $(D + 1)$-dimensional action written in the canonical formalism, once the extra dimension corresponding to the length scale of RG is interpreted as a time. The bulk theory takes the form of quantum theory of gravity coupled with matter fields of various spins. Because of the freedom to choose different local RG scheme without modifying the partition function, one has $(D + 1)$-dimensional diffeomorphism in the bulk, which in turn leads to $(D + 1)$ local constraints. The Hamiltonian constraint originates from the gauge freedom in choosing the spacetime dependent speed of coarse graining in the local RG procedure, while the $D$ momentum constraints are associated with relabeling the $D$-dimensional coordinates of low energy modes relative to the coordinates of the high energy modes. Because different choices of local RG scheme merely correspond to choosing different gauge, the $(D + 1)$ local constraints are first-class.

The holographic dual for the matrix model includes dynamical gravity and other fields. Generically, the cosmological constant in the bulk is expected to be order of the UV cut-off of the boundary field theory. Then the saddle point geometry in the bulk will have a curvature that is comparable to the scale that controls the gradient expansion for the action in the bulk. In this case, there will be no sense of locality within the distance scale over which the bulk spacetime is flat, although the geometry is classical due to the suppressed quantum fluctuations for a sufficiently
large $N$. It would be of great interest to find boundary field theories whose gravity dual have a weakly curved bulk spacetime through the explicit construction. This would require stabilizing the theory at a strong coupling.

For a general field theory, it is not easy to derive the dual theory in a closed form because one has to keep a large number of fields in the bulk. However, we have a concrete prescription to identify gravitational duals starting from boundary field theories. Using this prescription, one can try to examine the properties of the field theories which have simple gravity duals. For example, it will be interesting to see if one can identify the field theory whose holographic dual is the pure gravity.

XI. ACKNOWLEDGMENT

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XII. APPENDIX A : EXISTENCE AND UNIQUENESS OF CANONICAL METRIC

We constructively prove the statement that there is one and only one metric in which the quadratic kinetic term has the canonical form as in Eq. (11) for a given set of sources. Under a change of the metric used in covariant derivatives, only those operators that are quadratic in $\Phi$ mix with the kinetic term. So we focus on the quadratic Lagrangian,

$$\mathcal{L}^{(2)} = -\sum_{n=0}^{\infty} j^{\mu_1,..,\mu_n} \text{tr} \left[ \Phi \partial_{\mu_1} .. \partial_{\mu_n} \Phi \right].$$

(102)

The term with $n = 1$ can be absorbed into the term with $n = 0$ via an integration by part, but it is more convenient to keep it for now. The goal is to find the metric in which the same Lagrangian is written in the canonical form,

$$\mathcal{L}^{(2)} = -\sum_{n=0}^{\infty} \sqrt{|G|} J^{\mu_1,..,\mu_n} \text{tr} \left[ \Phi \nabla_{\mu_1} .. \nabla_{\mu_n} \Phi \right],$$

(103)
where $\nabla_\mu$ is the covariant derivative associated with the canonical metric that satisfies the condition, $G^{\mu\nu} = J^{\mu\nu}$.

The canonical metric will be a local functional of the sources $j^{\mu_1\ldots\mu_n}(x)$. Our strategy is to compute the canonical metric using a gradient expansion of the sources. Suppose that $G_v^{\mu\nu}$ is the metric that coincides with the canonical metric up to the $v$-th order in derivative. Namely, $G_v^{\mu\nu}$ is made of the terms that have $v$ or less derivatives of the sources in the canonical metric. The exact canonical metric is $G^{\mu\nu}_\infty$. The Lagrangian in Eq. (102) can be expressed in terms of the operators constructed with the covariant derivative $\nabla^v_\mu$ with the metric $G_v^{\mu\nu}$,

$$L^{(2)} = -\sum_{n=0}^\infty j^{\mu_1\ldots\mu_n} \text{tr} \left[ \Phi \nabla^{\mu_1}_\mu \cdots \nabla^{\mu_n}_\mu \Phi \right].$$

To the zeroth order in derivative, the canonical metric is completely determined from $j^{\mu\nu}$,

$$\sqrt{G_0} G_0^{\mu\nu} = j^{\mu\nu}.$$  \hspace{1cm} (105)

It is noted that $G_0^{\mu\nu}$ itself is completely fixed in $D > 2$. In this metric, the source for the two derivative operator becomes

$$j_0^{\mu\nu} = \sqrt{|G_0|} G_0^{\mu\nu} + j^{\mu\alpha\beta} \Gamma^{\nu}_{0;\alpha\beta} + j^{\alpha\beta\nu} \Gamma^{\mu}_{0;\alpha\beta} + j^{\alpha\beta\nu} \Gamma^{\mu}_{0;\alpha\beta} + \ldots,$$  \hspace{1cm} (106)

where $\Gamma^{\nu}_{0;\alpha\beta}$ is the Christoffel symbol for the metric $G_0^{\mu\nu}$, and ... represents the terms that have at least two derivatives of the source $j^{\mu\nu}$, such as $j^{\mu_1\nu_1 \Gamma^{\delta}_{0;\alpha\beta} \Gamma^{\nu_2}_{0;\alpha\beta}}$ and $j^{\mu_1\nu_1 \nabla^{\mu_2}_{0;\alpha\beta} \Gamma^{\nu_2}_{0;\alpha\beta}}$. To the first order in derivative, the canonical metric is given by

$$\sqrt{|G_1|} G_1^{\mu\nu} = \sqrt{G_0} G_0^{\mu\nu} + j^{\mu_1\nu_1} \Gamma^{\nu}_{0;\alpha\beta} + j^{\alpha\beta\nu} \Gamma^{\mu}_{0;\alpha\beta} + j^{\alpha\beta\nu} \Gamma^{\mu}_{0;\alpha\beta}.$$  \hspace{1cm} (107)

Note that the difference between $G_1^{\mu\nu}$ and $G_0^{\mu\nu}$ has one derivative in the source. If we rewrite the Lagrangian using the covariant operators associated with $G_1^{\mu\nu}$, the source for the two derivative operator $\text{tr} \left[ \Phi \nabla^{\mu}_\mu \nabla^{\nu}_\nu \Phi \right]$ differs from $\sqrt{G_1} G_1^{\mu\nu}$ by terms that have at least two derivatives of the sources. If one repeats this procedure, one can uniquely determine $G_v^{\mu\nu}$ from $G_{v-1}^{\mu\nu}$ by adding terms that have $v$ derivatives of the sources. This proves that there is a unique canonical metric for a given set of sources.

There is a useful corollary. Suppose there are two theories which have canonical kinetic terms defined on two different curved backgrounds,

$$L_a = -N \sqrt{|G_a|} G_a^{\mu\nu} \text{tr} \left[ \Phi \nabla^{\mu}_\mu \nabla^{\nu}_\nu \Phi \right] + \ldots,$$  \hspace{1cm} (108)
with \( a = 1, 2 \). If \( G_{1}^{\mu\nu} \neq G_{2}^{\mu\nu} \), the two theories are distinct in the following sense. If the Lagrangians are re-expressed in terms of the operators defined on the flat manifold in Eq. (2), the two theories should have different sets of sources.

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