Large mode-2 internal solitary waves in three-layer flows

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Abstract

In this paper, we investigate mode-2 solitary waves in a three-layer stratified flow model. Travelling wave solutions to both the fully nonlinear problem (Euler equations), and a three-layer extension of the strongly nonlinear two-layer Miyata-Choi-Camassa equations are found numerically. Mode-2 solitary waves with speeds slower than the linear mode-1 long-wave speed are typically expected to be generalised solitary waves: that is, their tails consist of a resonant mode-one periodic wave train. Herein we evidence the existence of mode-2 embedded solitary waves, that is, we show that for specific values of the parameters, the amplitude of the oscillations in the tail are zero. For sufficiently thick middle layers, we also find branches of mode-2 solitary waves with speeds that exceed the mode-1 linear long wave speed. Further, we show how some forms of the embedded solitary waves are intimately linked to the conjugate states of the problem.

1 Introduction

Waves propagating in density stratified fluids are responsible for mass, momentum, temperature, and biomass transfer in the world’s oceans. The ocean is a continuously stratified fluid, for which linear theory predicts infinitely many modes of propagation. Most previous literature, both observational and theoretical, concerns waves of the first baroclinic mode (mode-1), which have the simplest vertical structure. In this paper, second baroclinic mode (mode-2) solitary waves are considered in detail within both the Euler and model equations for a three-layer system. The relative lack of literature on mode-2 waves is likely due to both the complexity of the problem and that observations of mode-2 internal waves are far more recent than those of mode-1.

The simplest description of internal waves in density stratified fluids is given by the study of waves propagating at the interface of two immiscible inviscid fluids of constant but different density, bounded above and below by solid walls (the rigid lid approximation). Although in the context of the ocean the salted water column is bounded above by a free-surface, Evans and Ford [1996] demonstrated that for weak stratification, the surface being ‘free’ has minimal effect on mode-1 solutions. This is not necessarily the case when the stratification across the two-layers is strong [Barros and Gavrilyuk, 2007; Kodaira et al., 2016]. The two layer model is particularly relevant for stratification where there exists two regions of approximately constant density with a thin intermediate mixing region, and has been the subject of many studies. Amick and Turner [1986] proved the existence of branches of solitary waves that have a limiting amplitude, where the solutions become so-called ‘tabletop solitary waves’. As the solitary wave indefinitely broadens, the solutions limit to a wavefront (i.e. an internal bore). Such a solution is a connection between conjugate states, named as such by Benjamin [1966]. Numerical computations confirming this can be seen in a variety of papers, such as in the work of Turner and Vanden-Broeck [1988], Sha and Vanden-Broeck [1993], and Dias and Vanden-Broeck [2003]. Wavefronts are not the only limiting configuration for two-layer solitary waves. Large amplitude overhanging waves have been found, typically in settings with weak stratification [Pullin and Grimshaw 1988; Maklakov and Sharipov 2018; Guan et al. 2021].

Recent observations suggest that mode-2 waves are more common than previously thought [Duda et al., 2004; Yang et al., 2009; Shroyer et al., 2010; Ramp et al., 2012; Khimchenko and Serebryany, 2016]. Although observations by Shroyer et al. [2010] found mode-2 waves off the coast of New Jersey were 10 to 100 times less energetic than their mode-1 counterparts, they are capable of travelling

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large distances with so-called ‘trapped-cores’, inducing mass transport [Brandt and Shipley, 2014; Deepwell and Stastna, 2016]. Mode-2 waves have also been the subject of experimental studies (for example, Kao and Pao, 1980; Stamp and Jacka, 1995; Honji et al., 1995; Carr et al., 2015).

The rigid-lid two-layer model admits only waves of the first baroclinic mode. To capture the behaviour of higher modes, one must consider a model with two or more interfaces between layers, or alternatively continuously stratified fluids. In this paper, we explore mode-2 waves in a three-layer rigid-lid configuration. Linear and weakly nonlinear theories predict that both interface displacements have the same polarity for mode-1, while mode-2 waves have interfaces with opposite polarities. If the lower interface is of depression (and hence the upper interface is of elevation), then the wave is called convex. The opposite case, that is of a lower interface of elevation, is known as concave. Whether mode-2 solitary waves are convex or concave can be predicted by the Korteweg-de Vries (KdV) equation, where it is found that thicker middle layers result in concave waves.

We present a numerical scheme similar to the one used by Rusas and Grue [2002] to compute travelling wave solutions to the full Euler equations. These solutions will be compared and contrasted with travelling wave solutions to a long wave strongly nonlinear approximation of the Euler equations that we will refer to as the three-layer Miyata-Choi-Camassa equations (denoted MCC3). Typically, mode-2 solitary waves are within the linear spectrum of the system, insofar as their speed is expected to lie within the range of speeds of mode-1 waves. Hence, they are often associated with non-decaying oscillatory mode-1 tails, due to resonance with a mode-1 short wave travelling with the same speed. Such solutions are called generalised solitary waves (GSW). However, it could happen that solitary wave solutions within the linear spectrum have a flat far-field (i.e. no resonant tail). These were coined embedded solitary waves (ESW) by Yang et al. [1999]. One may find these solutions as particular cases of GSW, for which the amplitude of the oscillatory tail goes to zero, such as the ESWs discovered for the modified fifth-order KdV equation by Champneys et al. [2002]. Recent work by Barros et al. [2020] found numerically examples of ESWs to the MCC3 system.

The goal of this paper is to present numerical evidence that ESWs also exist for the full Euler system, together with elucidating the parameters for which they occur, their limiting configurations, and comparing solutions of the Euler equations to those of MCC3. To the authors knowledge, the ESWs in this paper are the first nontrivial ones to be computed for the full Euler equations. With the exception of the case when the domain and density stratification are symmetric about the mid-channel, and under the Boussinesq approximation (see [Davis and Acrivos, 1967; Tung et al., 1982]), all mode-2 waves found in literature are GSWs. Mode-2 GSWs for the full Euler system have been found for both layered models [Rusas and Grue, 2002; Wang et al., 2014], and continuous density stratified models [Vanden-Broeck and Turner, 1992; Olsthoorn et al., 2013]. Furthermore, previous attempts to find ESWs in a single layer flow for the full Euler equations for gravity-capillary [Champneys et al., 2002] and gravity-flexural [Gao, 2016] waves have given negative results. Model equations which decouple the modes, such as the KdV and Gardner equations, do not exhibit mode-2 solutions with resonant mode-1 tails.

For concave mode-2 solitary waves, it was found recently for the MCC3 equations that the speed of the wave could exceed the mode-1 long wave speed [Barros et al., 2020]. We show herein that concave mode-2 solitary wave branches outside the linear spectrum are also a feature of the full Euler system. The bifurcation structure of these solution branches is not the same as that of mode-1 solitary waves, since they emerge from branches of GSWs as the speed exceeds $c^+_0$, and hence their bifurcation is not described by weakly nonlinear theory.

As with two-layer solitary waves, a feature of the mode-2 ESW branches found in this paper is that the solutions broaden with increasing amplitude and can, in some cases, limit to a tabletop solitary wave with speeds and amplitudes given by the conjugate state equations. This was also found as limiting behaviour for three-layer mode-1 solitary waves [Rusas and Grue, 2002]. The conjugate state equations for a three-layer fluid have been explored graphically by Lambi [2000] and Rusas and Grue [2002]. It should be noted that the conjugate state equations only inform us as to which two states conserve momentum and mass, and do not confirm whether a connection (i.e. heteroclinic orbit) exists.

A favourable feature of the MCC3 equations is that they share the same solutions to the conjugate state equations as the full Euler system. Barros et al. [2020] showed that travelling-wave solutions of the MCC3 equations are governed by a Hamiltonian system with two degrees of freedom. The conjugate state equations for the full Euler system can be shown to be equivalent to finding critical points of the MCC3 potential at which the potential is also zero. The results of this paper demonstrate the nature of the critical points allows educated guesses about the types of homoclinic orbits found.
In particular, if the conjugate state is a maximum of the potential, then one may expect solution branches of solitary waves which limit to a wavefront. If the conjugate state is a saddle of the potential, then more exotic solutions can be found, where the homoclinic orbit includes one or more oscillations near the stable manifold of the saddle. These solutions are related to the curious ‘multi-hump’ solutions found in Barros et al. [2020], which are solitary waves characterised by multiple oscillations on one or both interfaces. Multi-hump waves have also been witnessed in experimental conditions by Liapidevskii and Gavrilov [2018], who also found numerical solutions to a modified MCC3 system, which includes the additional assumption that the pressure in the middle layer is purely hydrostatic.

The insight gained by exploring critical points of the potential of the MCC3 system extend to the full Euler system: it is found that, although the MCC3 solutions are at times quantitatively poor, the qualitative similarities are remarkable. We believe the poor quantitative comparison is associated with the invalidation of the long wave approximation. Throughout the paper, comparison between the Euler and MCC3 systems is made, with results in line with the findings for the two-layer MCC (MCC2) equations Camassa et al. [2006]. The impressive qualitative comparison motivates more analytical studies into the global bifurcation structure of the MCC3 equations.

Due to the weak stratification of the world’s oceans, many oceanic models of internal waves make use of the Boussinesq approximation, in which the stratification is only included in the gravitation potential term (and ignored for inertial terms). The use of the Boussinesq approximation is a topic of mild controversy, as it has been shown by Long [1965] and Benjamin [1966] that the Boussinesq approximation can effect the existence of internal solitary waves, and have implications on the hydrodynamic stability of the flow Barros and Choi [2011] Boonkasame and Milewski [2012]. In this paper, we compare the solution spaces both with and without the Boussinesq approximation. In particular, inclusion of the Boussinesq approximation, along with symmetric stratification and equal bottom and top layer depths, results in an invariant subspace of mode-2 waves within the system with branches of solutions which are equivalent to two-layer mode-1 waves reflected about the midline. Gavrilov and Liapidevskii [2009] Barros et al. [2020]. Breaking the symmetry, either by varying the densities, layer depths, or removing the Boussinesq approximation, is typically expected to result in a GSW.

The paper is organised as follows. In section two, we formulate the problem to be solved, and introduce the steady MCC3 equations for comparison. In section three, we explore the conjugate state equations for MCC3. In section four, we present and describe the numerical results. Concluding remarks are given in section five. In the Appendix, we present the numerical scheme used to compute fully nonlinear solutions.

2 Formulation

Consider three immiscible fluids stably stratified in a two-dimensional channel of height $H$. The fluids are assumed to be incompressible and of constant density, and we assume that the resulting flow is irrotational. The density of each fluid is taken to be $\rho_i, i = 1, 2, 3$ (from top to bottom), and we require $\rho_3 > \rho_2 > \rho_1$ for a stable stratification. Subscripts will be used in this way for other quantities throughout the paper. We consider a periodic domain of length $\lambda$. Although this paper concerns

![Figure 1: A sketch of the flow configuration. Three layers of constant density fluid are bounded between walls at $y = 0$ and $y = 1$. The layers are separated by interfaces $y = \eta_i$.](image)
embedded solitary waves with a flat far-field, the ESW found in this paper are particular cases of
generalised solitary waves, which we approximate with long periodic waves. The period is verified
to be sufficiently long to not have appreciable effects on the ESWs. We take Cartesian coordinates
\((x, y)\), where \(y\) is perpendicular to the solid boundaries. The wave is assumed to be symmetric about
\(x = 0\), and we choose \(y = 0\) at the bottom wall. We denote by \(H\) the depth of fluid layer \(i\) at
the end of the wavelength, such that \(H = H_1 + H_2 + H_3\). Throughout the paper, we choose \(H\) and \(\rho_2\) as
the characteristic length and density respectively. We scale velocities such that the acceleration due
to gravity \(g\) is also taken to be unity. The flow configuration is shown in figure [1]

2.1 Euler equations

Let the velocity field in each layer be given by \(\mathbf{u}_i = (u_i, v_i)\). There exists a velocity potential \(\phi_i\) and
streamfunction \(\psi_i\) in each layer, such that

\[
\begin{align*}
    u_i &= \frac{\partial \phi_i}{\partial x}, \\
    v_i &= \frac{\partial \phi_i}{\partial y} - \frac{\partial \psi_i}{\partial x},
\end{align*}
\]

Hence in each layer, the flow is governed by

\[
\nabla^2 \phi_i = 0.
\]

We consider travelling wave solutions moving to the right with constant speed \(c\). We take a frame
of reference travelling with the wave, such that the quiescent flow far from a solitary wave has a
horizontal velocity of \(c\) in each layer. We parameterise the lower and upper interfaces as \(y = \eta_i(x)\)
and \(y = \eta_2(x)\) respectively. We also denote the interface displacements as \(\zeta_i\), which reads

\[
\begin{align*}
    \zeta_2 &= \eta_2 - H_3, \\
    \zeta_1 &= \eta_1 - H_2 - H_3.
\end{align*}
\]

As well as kinematic boundary conditions, that is

\[
\mathbf{u} \cdot \mathbf{n} = 0,
\]

we enforce continuity of pressure on the interfaces, which (making use of Bernoulli’s equation) gives

\[
\frac{1}{2} [\rho_{i+1} |\nabla \phi_{i+1}|^2 - \rho_i |\nabla \phi_i|^2] + \Delta_i \eta_i = B_i, \quad \text{for } y = \eta_i(x) \quad (i = 1, 2).
\]

In the above, \(\mathbf{n}\) refers to the unit normal of the curve, \(\Delta_i = \rho_{i+1} - \rho_i\), and \(B_i\) are unknown constants
to be found as part of the solution. The Boussinesq approximation, where differences in density only
effect the buoyancy term, is commonly applied to scenarios where the density stratification is weak
((\(\rho_3 - \rho_1\))/\(\rho_2 \ll 1\)). This modifies the above boundary condition to give

\[
\frac{1}{2} (|\nabla \phi_{i+1}|^2 - |\nabla \phi_i|^2) + \Delta_i \eta_i = B_i, \quad \text{for } y = \eta_i(x) \quad (i = 1, 2).
\]

All the solutions we seek are assumed to be symmetric about \(x = 0\). Furthermore, we enforce

\[
\begin{align*}
    u_3 &= u_2 = -c, \quad \text{at } x = \pm \frac{\lambda}{2}, \quad y = \eta_2, \\
    u_2 &= u_1 = -c, \quad \text{at } x = \pm \frac{\lambda}{2}, \quad y = \eta_1.
\end{align*}
\]

This boundary condition ensures that (assuming the solution has no oscillatory tail) the wave prop-
agates with constant speed \(c\) to the right, in a frame of reference to which the wave is steady.

For the numerical code, which is designed to compute periodic solutions, we introduce the variable
\(\bar{c}\), which is the wavelength averaged horizontal velocity in the bottom fluid for a fixed value of \(y\):

\[
\bar{c} = -\frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} u_3(x, y) \, dx, \quad y = \text{constant}.
\]
c \cdot k \cdot c^+ (k) + c^- (k)

Figure 2: A typical dispersion relation, showing the fast mode \((c^+)\) and slow mode \((c^-)\) as functions of \(k\). The value \(k^*\) is such that \(c^- (0) = c^+ (k^*)\).

The irrotationality of the flow ensures that the above value is independent of the value of \(\hat{y}\) chosen. We will approximate solitary waves with long periodic waves. It is the case that, given the solitary waves are not generalised (i.e. the solution is flat in the far-field), as \(\lambda \to \infty\), one finds \(\tilde{c} \to c\). Before describing model equations which approximate the above system, we first note that the above system has a linear dispersion relation \([\text{Rus}}} {\text{as and Grue, 2002}\) given by

\[
a_4 c^4 k^2 - a_2 c^2 k + a_0 = 0,
\]

where \(k\) is the wavenumber, and

\[
a_4 = 1 + \rho_3 C_2 C_3 + \rho_4 \rho_1 C_3 C_1 + \rho_1 C_1 C_2,
\]

\[
a_2 = \Delta_2 (C_2 + \rho_1 C_1) + \Delta_1 (\rho_2 C_3 + C_2),
\]

\[
a_0 = \Delta_1 \Delta_2,
\]

and \(C_i = \coth(H_i k)\). There exists two modes: a faster mode (mode-1), denoted \(c^+\), and a slower mode (mode-2), denoted by \(c^-\). Both modes monotonically decrease from their maximum at \(k = 0\), denoted \(c_{\pm 0}\). A typical dispersion relation is shown in figure 2. The resonant tail of a mode-2 GSW is associated with a mode-1 periodic wave which propagates at the same speed. Indeed, it can be seen in the figure that for mode-2 solitary waves bifurcating at \(c^- 0\), there exists a finite value of \(k = k^*\) such that \(c^- 0 = c^+ (k^*)\).

We note that, under the Boussinesq approximation, the above system is invariant under the transformation \(g \to -g\), and \(\Delta_i \to -\Delta_i\). Furthermore, when the stratification and channel heights are symmetric \((H_1 = H_3\) and \(\Delta_1 = \Delta_2)\), we recover the so-called symmetric Boussinesq configuration. A feature of this parameter choice is the existence of families of mode-2 solutions with the property that \(\zeta_1 = -\zeta_2\). The curve \(y = 0.5\) is a streamline, and the flow can be viewed as a two-layer flow with layer heights \(H_3\) and \(H_2/2\). Surprisingly, not all mode-2 solutions for the symmetric Boussinesq configuration have this additional symmetry, such as the large amplitude concave waves shown in section 4.2.

### 2.2 Travelling waves for the MCC3 equations

The depth averaged equations for a single homogeneous fluid with a free-surface were derived by Serre [1953] and Green and Naghdi [1976]. Miyata [1988] and Choi and Camassa [1999] independently derived the equivalent equations for two fluids separated by an interface under. The system is known as the two-layer Miyata-Choi-Camassa equations (denoted in this paper as MCC2). Camassa et al. [2006] performed an extensive analysis on the validity of the MCC2 equation for mode-1 solitary waves, with a comparison to both full Euler solutions and experimental observations. These equations have since been generalised to three layers [Jo and Choi [2014], Barros et al. [2020], N-layers with variable bottom topography [Choi [2000], and to include higher order dispersion [Ostrovsky and Grue [2003], Debsarma et al. [2010]]. We denote the shallow water parameter in each layer as \(\epsilon_i = H_i / \lambda\), and the amplitude parameter as \(\zeta_i = \zeta_i / H_i\). The three-layer model (MCC3) we will consider assumes \(O(\epsilon_i) = O(\epsilon_j)\), has errors of \(O(\epsilon_i^2)\), and has no smallness assumption on the parameter \(\delta_i\). Hence, it is possible that the model can remain asymptotically valid for large amplitude solutions. In a frame
of reference travelling with a wave of constant speed $c$, and under the assumption that the flow in the far-field is unperturbed, one finds the Euler-Lagrange equations (see Barros et al. [2020])

$$\frac{\partial L}{\partial \zeta_i} - \frac{d}{dx} \left[ \frac{\partial L}{\partial \zeta_i'} \right] = 0, \quad i = 1, 2,$$

(14)

where the Lagrangian is given by

$$L = \frac{1}{6} c^2 \left[ \left( \rho_1 \frac{H_1^2}{h_1} + \frac{H_2^2}{h_2} \right) \zeta_1'^2 + \frac{H_2^2}{h_2} \zeta_1' \zeta_2' + \left( \frac{H_2^2}{h_2} + \rho_3 \frac{H_3^2}{h_3} \right) \zeta_2'^2 \right] - V,$$

(15)

with a potential $V(\zeta_1, \zeta_2)$

$$V = \frac{1}{2} \left\{ \left( \Delta_1 \zeta_1^2 + \Delta_2 \zeta_2^2 \right) - c^2 \left[ \left( \Delta_1 + \rho_1 \frac{H_1}{h_1} - \frac{H_2}{h_2} \right) \zeta_1 + \left( \Delta_2 - \rho_3 \frac{H_3}{h_3} + \frac{H_2}{h_2} \right) \zeta_2 \right] \right\}$$

(16)

In the above, $\zeta_i' = d\zeta_i/dx$. The system is reduced, under the Boussinesq approximation, by removing every instance of $\rho_1$ and $\rho_3$, except where they appear in $\Delta_1$ and $\Delta_2$.

Travelling wave solutions to (14) were the object of study in Barros et al. [2020]. Although GSWs were found to be prevalent, they numerically computed ESWs for special values of parameters. Due to the additional condition imposed that the tails have no oscillations, the ESWs computed exist in a parameter space with one less dimension than that of GSWs. The asymptotic limit $H_2 \to 0$ was used to find ‘compacton’ multi-hump solutions. These solutions, characterised by $p$ humps on the upper interface and $q$ on the lower, can be numerically extended into finite values of $H_2$ via the method of continuation. By doing so, mode-2 multi-hump GSWs were also exhibited, and ESWs were found along these GSW branches. Interestingly, it was suggested that not all truly localised mode-2 solitary waves are embedded, since concave solitary waves were found outside the linear spectrum. These features will also be further explored in the following sections.

3 Conjugate states and limiting solitary waves of the MCC3 model

As with other internal wave models, large amplitude solitary waves for the system considered in this paper are found to be related to the conjugate states of the system. In particular, many solitary wave branches have been found to broaden as the amplitude increases, approaching a tabletop solitary wave, and limiting to a wavefront solution (i.e. a heteroclinic orbit between conjugate states). We demonstrate in this section that further insight into the rich tapestry of behaviours for large amplitude solutions can be gained by exploring the critical points of the MCC3 potential $V$ (equation (16)).

In this section, we present the conjugate state equations for the three-layer configuration, and use these to discuss large amplitude broad solutions to the MCC3 system, found computationally using a pseudospectral collocation method. Large amplitude solutions to the full Euler system are explored further in section 4 where many qualitative similarities with the MCC3 system are found.

Figure 3: Configuration when solving the conjugate state equations.
Consider a uniform flow from \( x = +\infty \) with velocity \(-c\). We consider \( c > 0 \) and therefore upstream refers to the limiting (trivial) state to right of the wavefront, and downstream refers to the limiting state at \( x = -\infty \). The depth of each upstream layer is given by \( H_i \). The flow downstream has the lower and upper interface perturbed by \( \hat{\zeta}_2 \) and \( \hat{\zeta}_1 \) respectively, and we denote the new layer depths as \( \hat{H}_i \). The flow configuration is shown in figure 3. In this frame, the problem is assumed steady.

The speed in each layer for the downstream state is given by \( \hat{u}_i \), which can be expressed in terms of \( c \) and \( \zeta_i \) by conservation of mass

\[
\hat{u}_i = -\frac{c}{H_i}, \quad i = 1, 2, 3, \tag{17}
\]

where \( \hat{H}_3 = H_3 + \hat{\zeta}_2 \), \( \hat{H}_2 = H_2 - \hat{\zeta}_2 + \hat{\zeta}_1 \), and \( \hat{H}_1 = H_1 - \hat{\zeta}_1 \). There are three unknowns: \( \hat{\zeta}_1 \), \( \hat{\zeta}_2 \), and \( c \). One equation comes from enforcing conservation of momentum, given by

\[
\int_{x \to -\infty} P + \rho u^2 \, dy = \int_{x \to \infty} P + \rho u^2 \, dy. \tag{18}
\]

Making use of Bernoulli’s equation to find the pressure \( P \), Lamb [2000] showed this condition is equivalent to

\[
c^2 \left\{ (1 + \Delta_2) \frac{\hat{\zeta}_2^3}{H_3^2} + \frac{(\hat{\zeta}_1 - \hat{\zeta}_2)^3}{H_2^2} - (1 - \Delta_1) \frac{\hat{\zeta}_1^3}{H_1^2} \right\} = 0. \tag{19}
\]

The two equations which close the system are continuity of pressure across both interfaces. This is found to be

\[
\frac{c^2}{2} \left\{ (1 + \Delta_2) \left( 1 - \frac{H_3^2}{H_3^2} \right) - \left( 1 - \frac{H_2^2}{H_2^2} \right) \right\} - \Delta_2 \hat{\zeta}_2 = 0, \tag{20}
\]

\[
\frac{c^2}{2} \left\{ 1 - \frac{H_2^2}{H_2^2} - (1 - \Delta_1) \left( 1 - \frac{H_1^2}{H_1^2} \right) \right\} - \Delta_1 \hat{\zeta}_1 = 0. \tag{21}
\]

These equations are explored by Lamb [2000] and Rusas and Grue [2002]. Lamb [2000] finds solutions by graphically observing the intersection of two plane algebraic curves, while Rusas and Grue [2002] use an iterative Newton-Raphson’s method. For any non-trivial solution \( (\hat{\zeta}_1, \hat{\zeta}_2) \neq (0, 0) \), the flow downstream is called a conjugate state to the upstream flow. Note that solving these algebraic equations does not imply a continuous connection between conjugate states within the fluid equations.

The conjugate state equations described above are recovered using the full Euler system. However, as was found for a two-layer system with a free-surface by Barros [2016], the conjugate states of the full Euler equations are the same as those for the MCC3 equations. Certainly, it can be seen immediately that equations (20) and (21) are equivalent to \( V_{\hat{\zeta}_1} = 0 \) and \( V_{\hat{\zeta}_2} = 0 \) respectively. Furthermore, it can be shown that equation (19) is equivalent to

\[
2V - V_{\hat{\zeta}_1} - V_{\hat{\zeta}_2} = 0. \tag{22}
\]

Therefore, conjugate states (to Euler and MCC3) can be viewed as non-trivial critical points of the potential of the MCC3 equations for which \( V = 0 \). At the origin \( (\hat{\zeta}_1, \hat{\zeta}_2) = (0, 0) \), the potential \( V \) is equal to zero, and is a local minimum when \( c \in (0, c_0^+) \), a saddle when \( c \in (c_0^-, c_0^+) \), and a local maximum for \( c > c_0^+ \). For fixed \( H_i \) and \( \rho_i \), one can plot the critical points of \( V \) for a given \( c \), and observe how the critical points evolve as \( c \) varies. As an illustration, consider figure 4, where we choose parameters \( \Delta_1 = \Delta_2 = 0.01 \), \( H_2 = 0.03 \), and \( H_1 = 1.2H_3 \). We restrict our attention to mode-2 conjugate states by considering the lower-right quadrant \( (\hat{\zeta}_1 > 0, \hat{\zeta}_2 < 0) \). For the given parameter values, \( c_0^- = 0.012 \) and \( c_0^+ = 0.069 \). Therefore, \( c \in (c_0^-, c_0^+) \) for all the contour plots in the figure, and hence the origin is a saddle of \( V \). The black curves are level sets of \( V \), where the bold curve corresponds to \( V = 0 \). The blue curves, which are independent of \( c \), are found by rearranging equations (20)-(21) to eliminate \( c \), and are given by

\[
\Delta_1 \hat{\zeta}_1 \left[ \rho_3 \left( 1 - \frac{H_2^2}{H_3^2} \right) - \left( 1 - \frac{H_2^2}{H_2^2} \right) \right] - \Delta_2 \hat{\zeta}_2 \left[ \left( 1 - \frac{H_2^2}{H_2^2} \right) - \rho_1 \left( 1 - \frac{H_2^2}{H_1^2} \right) \right] = 0. \tag{23}
\]
Figure 4: Contour plot of the potential $V$ in the lower-right quadrant for given values of the speed $c \in (c^0_-, c^0_+)$. The thick black curve is the contour $V = 0$. The critical points are given by the crosses, which are restricted to moving along the blue curves. The parameters are $H_2 = 0.03$, $H_1 = 1.2H_3$, and $\Delta_1 = \Delta_2 = 0.01$. Excluding the origin, there are four critical points in the lower right quadrant: a local minimum, a local maximum, and two saddles, given by green, black, and blue crosses respectively. The same colour scheme will be used throughout this section.

The crosses correspond to critical points which must lie on the blue curves, and it can be seen there are four non-trivial such critical points in the lower-right quadrant. The critical points are a local minimum, two saddles, and a local maximum. These correspond with green, blue, and black crosses respectively. There are two further maxima, but these are outside the domain window. In panel (b), the $V = 0$ curve intersects one of the saddles, and the other saddle in panel (c). Hence, the speeds $c$, and the values of $\hat{\zeta}_i$ at these saddles are conjugate states of the system. In panel (d), the small contour enclosing the maximum will eventually (as $c$ increases) collapse to that point, yielding a third mode-2 conjugate state in the lower-right quadrant.

In figure 5, we plot branches of conjugate states for the parameter regime $\Delta_1 = \Delta_2 = 0.01$ and $H_1 = 1.2H_3$. The figure shows how the number of conjugate states varies as $H_2$ varies. The black dotted curves are $c^0_-$ and $c^0_+$. Blue curves are mode-1 conjugate states, while red curves are mode-2. If the curve is solid, the conjugate state is a maximum of $V$, while dashed curves imply a saddle.

Let us first discuss mode-1 conjugate states. It can be seen that for smaller values of $H_2$, there is one maximum conjugate state with speeds $c > c^+_1$, while for larger values of $H_2$ there are two. In both cases, our numerical tests show that heteroclinic orbits from the origin to a maximum can be found, provided $c > c^+_1$. Hence, when $H_2$ is sufficiently large, two heteroclinic orbits are found. Furthermore, they exhibit different polarities, as illustrated in figure 6. There is also a saddle mode-1 conjugate state. However, no heteroclinic orbits or large amplitude solutions related to this critical point were found numerically. For mode-2 conjugate states, there is one maximum and two saddles. The maximum is restricted to small values of $H_2$, and it lies within the linear spectrum of the system. The origin is a saddle of $V$, and solitary wave solutions computed in this region typically have mode-1 oscillatory tails about the origin (GSWs). However, for special parameters one can find solutions where these oscillations are of zero amplitude, resulting in truly localised ESWs. The process by which these special parameters are found is described in section 4.1.2. Figure 7 shows a mode-2 tabletop ESW for $\Delta_1 = \Delta_2 = 0.01$ and $H_2 = 0.03$. The particular value of $H_1$ for which the oscillatory tail vanishes is found as part of the solution. For these specific parameters $H_1 = 0.9891H_3$.

Unlike the maximum conjugate state, we found no heteroclinic orbits to the saddles inside the linear spectrum. Despite this, large amplitude waves are found, but are characterised by oscillations...
Figure 5: Solutions to the conjugate state equations for $\Delta_1 = \Delta_2 = 0.01$, $H_1 = 1.2H_3$. Varying $H_2$, the figure shows how the number of branches changes, and their corresponding speed $c$. The black dotted curves are the linear long wave speeds. Red and blue curves are mode-2 and mode-1 conjugate states respectively. If the curve is dashed, then the conjugate state is a saddle of $V$, while solid curves are local maximums. Panels (b) and (c) are blow ups of different regions of the figure in panel (a).

Figure 6: Two mode-1 MCC3 elevation solitary waves with $\Delta_1 = \Delta_2 = 0.01$, $H_2 = 0.85$, $H_1 = 1.2H_3$. The solution in panels (a1) and (a2) is an elevation tabletop solitary wave with speed $c = 0.0501$, while (b1) and (b2) show a depression tabletop solitary wave with speed $c = 0.0503$. In panel (a1) and (b1), we show the projection of the solutions (in red) over the $(\zeta_1, \zeta_2)$-plane. Panel (a2) and (b2) show the solutions (a1) and (b1) in the physical space.

Figure 7: A mode-2 MCC3 convex ESW with $\Delta_1 = \Delta_2 = 0.01$, $H_2 = 0.03$, $H_1 = 0.9891H_3$, and $c = 0.0354$. In panel (a), we show the projection of the solution (in red) over the $(\zeta_1, \zeta_2)$-plane. Panel (b) shows the solutions in the physical space.
Figure 8: A mode-2 MCC3 convex ESW wave with \( \Delta_1 = \Delta_2 = 0.01 \), \( H_2 = 0.03 \), \( H_1 = 1.2H_3 \), and \( c = 0.03372 \). In panel (a), we show the projection of the solution (in red) over the \((\zeta_1, \zeta_2)\)-plane. Panel (b) shows the solutions in the physical space. Solutions of this type necessarily require speeds greater than the speed such that the saddle is a conjugate state of the system, since we require the value of \( V \) at the saddle to be less than zero.

Figure 9: A mode-2 MCC3 concave solitary wave with \( \Delta_1 = \Delta_2 = 0.01 \), \( H_2 = 0.85 \), \( H_1 = 1.2H_3 \), and \( c = 0.0359 \). In panel (a), we show the projection of the solution (in red) over the \((\zeta_1, \zeta_2)\)-plane. Panel (b) shows the solutions in the physical space. Solutions of this type necessarily require speeds greater than the speed of the saddle conjugate state of the system, since we require the value of \( V \) at the saddle about which the solution oscillates to be less than zero.

Figure 10: Solutions to the conjugate state equations for \( \Delta_1 = \Delta_2 = \Delta \), \( H_1 = H_3 \), with the Boussinesq approximation. Panels (b) is a blow up of a region in figure (a).
along the broadened section of the wave and with speeds exceeding that of the conjugate state, as in figure 8. These solutions are so-called multi-hump ESWs, found both asymptotically in the limit as $H_2 \to 0$ and numerically by Barros et al. [2020]. As we will see in section 4, these solutions are also present in the fully nonlinear theory.

In figure 5, there is a special value of $H_2 = H_2^*$ at which the curve for the mode-2 conjugate state is tangent to the curve $c = c_0$. It can be shown that this corresponds to the criticality condition for the KdV equation for which the quadratic nonlinearity coefficient vanishes. The KdV theory predicts convex (concave) mode-2 waves for values of $H_2$ less (greater) than $H_2^*$. This characterisation for the polarity of solutions seems to be in agreement with numerical solutions found in this paper.

As the value of $H_2$ increases beyond $H_2^*$, it may be that a mode-2 saddle conjugate state exceeds $c_0^+$. In this case, one can find mode-2 solitary wave solutions which are no longer embedded. Although we are able to compute large amplitude solitary waves, not all have a wavefront as the limiting solution. They are typically characterised by oscillations on the broadened section, as in figure 9, although special parameter values can be found such that the oscillations about the broadened section vanish.

We stress that the considerations made above on the existence of multiple mode-2 conjugate states for small values of $H_2$ do not hold in general. For example, if in figure 5 we took $H_1 = 2H_3$, a similar picture would be obtained, but without the two mode-2 conjugate states in panel (b). Therefore, across all values of $H_2$, there would be only one saddle mode-2 conjugate state.

We note that when the Boussinesq approximation is considered, similar considerations apply. However, when $H_1 = H_3$ and $\Delta_1 = \Delta_2$, the criticality condition of the KdV equation for mode-1 is met. As a consequence, the mode-1 KdV equation has no solitary wave solutions. When higher order nonlinearities are accounted for, mode-1 solutions are expected to exist provided $H_2$ is sufficiently large. This is confirmed for the MCC3 model, and it can be shown that the curve for the mode-1 conjugate state starts precisely at $H_2 = 4/13$, in complete agreement with Gardner theory [Talipova et al., 1999; Lamb, 2005]. Figure 10 shows the conjugate states for this configuration. While the figure appears to only show one mode-1 conjugate state, the blue curve corresponds to two different conjugate states, under the symmetry $\Delta_i \to -\Delta_i$, $g \to -g$.

Finally, to unveil the regions in parameter space for which the mode-2 conjugate state exits the linear spectrum, we adopt the Boussinesq assumption with $\Delta_1 = \Delta_2 = \Delta$. The results are shown in figure 11(a) and, as expected, such a feature only occurs for thick intermediate layers. When $H_1 = H_3$, the value of $H_2$ for which this occurs is $H_2 = 6H_1$ (resulting in $H_2 = 0.75$) in agreement with figure 10. In the case when $H_2$ is small, we can use asymptotics to predict for which parameters there is one (or three) mode-2 conjugate states, as shown in figure 11(b). In agreement with the previous figures, the three conjugate states exist when $H_1/H_3$ is close to unity.

### 4 Results for the Euler equations and comparison with MCC3

In this section, we discuss numerically computed mode-2 solitary waves for the three-layer configuration. Fully nonlinear waves in the Euler equations are computed using the numerical method given...
Figure 12: Symmetric configuration solitary wave solution branches. The black and blue curves correspond to $H_2 = 0.06$ and $H_2 = 0.2$ respectively. The dashed, dotted, and solid curves correspond to the KdV, MCC2, and full Euler models respectively. The solutions corresponding to points (a)-(c) are shown in figure 14.

in Appendix A. Comparisons are made with the MCC3 model, and the conjugate states discussed in section 3 are referred to when describing large amplitude solutions.

4.1 Convex waves

We first consider the case where parameters are chosen such that the waves are convex. From section 3, this occurs for values of $H_2$ between zero and a value $H_2^*$ given by the criticality condition. As such, the conjugate states are attained at speeds $c < c_0^+$. Hence, we expect mode-2 solitary waves to be within the linear spectrum, typically characterised by oscillatory tails. Along these branches of GSWs, special values of the parameters are found such that the tails have zero amplitude.

4.1.1 Symmetric Boussinesq configuration

Along branches of GSWs, the parameters for which the solutions have no oscillations in the far-field are typically not known a priori, and must be found as part of the solution. This is not the case, however, if one makes the Boussinesq approximation with $H_1 = H_3$ and $\Delta_1 = \Delta_2$. With this symmetric configuration, there exists for all $H_2 \neq 0.5$ a branch of Boussinesq mode-2 solitary waves bifurcating from zero amplitude at $c_0$ and ending in a heteroclinic connection between the origin and a conjugate state. In other words, there exists a Boussinesq branch with $\Delta_1 = \Delta_2$ for which the ‘special’ parameters yielding solutions with non-oscillatory tails is precisely $H_1 = H_3$. The interface displacements of these solutions are related via $\zeta_1 = -\zeta_2$ and are, in fact, equivalent to a mode-1 solitary wave travelling in a two-layer fluid with layer depths $H_3$ and $H_2/2$. We briefly discuss solutions in this configuration, before seeking solutions in which the symmetry is broken.

Figure 12 shows symmetric Boussinesq solitary wave branches in the $(c/c_0, \zeta_1(0))$-plane for $H_2 = 0.06$ and $H_2 = 0.2$. We have set $\Delta_1 = \Delta_2$, where the value chosen does not matter, since for the
Boussinesq system, any change can be expressed as a rescaling of $c$. In the figure, we plot the branches for the KdV model (dotted curves), the MCC2 model (dashed curves), and full Euler (solid curves). The black curves correspond to $H_2 = 0.06$, while the blue curves are for $H_2 = 0.2$. The solution branches bifurcate from $c_0$. Since the KdV equation admits soliton solutions of arbitrary amplitude, the solution branch does not terminate until the interfaces collide with the wall. Meanwhile, the MCC2 and Euler solutions broaden, until eventually approaching wavefronts, the amplitude and speed of which can be recovered from the conjugate state equations. This limiting behaviour for two-layer internal solitary waves has been known since the work of Amick and Turner [1986]. It can be seen from the figure that the MCC2 performs better for the branch with $H_2 = 0.02$ when compared with $H_2 = 0.06$. This is because as the intermediate layer becomes shallower, the waves become short relative to the depth of the bottom layer. While technically of infinite wavelength, one can still measure a horizontal length scale. Here, we make use of the so-called effective wavelength, defined by

$$\lambda_e = \frac{1}{\zeta_1(0)} \int_0^\infty \zeta_1(x) \, dx.$$  \hfill (24)

Koop and Butler [1981] used this parameter to compare the two-layer KDV equation to observations, while Camassa et al. [2006] used it when testing the validity of the MCC2 equation. We demonstrate the shortening of the wave relative to the depth of the layers as $H_2$ decreases in figure 12. We have plotted the same branches as in figure 12 but this time showing the shallow water parameter for the lower ($\epsilon_3 = H_3/\lambda_e$) and middle ($\epsilon_2 = H_2/\lambda_e$) layer. It can be seen for the branch with $H_2 = 0.2$ that both $\epsilon_i$ remain relatively small, getting largest for moderate amplitude. Meanwhile, for $H_2 = 0.06$, the wave is not shallow relative to the bottom layer for almost the whole branch. This explains the greater deviation between the MCC2 and Euler models for this choice of $H_2$. Despite this, it can be seen from figures 12 and 13 that the $H_2 = 0.06$ MCC2 and Euler branches agree very well for large amplitude. This is related to the fact that the MCC2 and Euler models share the same conjugate states. We plot three solutions along the $H_2 = 0.06$ branch in figure 14. While there are discrepancies between the MCC2 and Euler solutions in figures 14(a) and 14(b), the agreement in figure 14(c) is excellent. The speeds of the MCC2 and Euler solutions in figure 14(c) are the same (that of the conjugate state), within numerical errors. These solutions can be computed with arbitrarily large areas in the broadened section of the wave, only limited by computational resources. It can also be seen that by point (a) along the branch, the KdV equation fails to capture the behaviour of the full Euler system. We also note that, while in some regions of the solution branch the quantitative agreement between the MCC2 and Euler system is poor, they share many qualitative features. The results above are in broad agreement with the findings of Camassa et al. [2006], and allows us to clearly demonstrate two points that will recur through the rest of the results. First, the MCC3 system can become a poor approximation of the Euler system for mode-2 waves due to the invalidation of the shallow water approximation. Second, since the MCC3 and Euler system have the same conjugate states, tabletop solitary waves are well described by strongly nonlinear theory.

Above, we explored convex mode-2 solutions to the Boussinesq symmetric configuration. Below, we discuss the solution space for convex mode-2 solitary waves to the MCC3 and full Euler models when this symmetry is broken.

Figure 14: The solid curves in panels (a)-(c) show the corresponding Euler solutions (a)-(c) from the $H_2 = 0.06$ branch in figure 12. Meanwhile, the dashed and dotted curves correspond to the MCC2 and KdV solutions respectively with the same amplitude. Only the lower interface is shown. The upper interface is a reflection of this profile about $y = 0.5$. The speed of the MCC2 solutions in figure 14(c) are the same (that of the conjugate state), within numerical errors. These solutions can be computed with arbitrarily large areas in the broadened section of the wave, only limited by computational resources. It can also be seen that by point (a) along the branch, the KdV equation fails to capture the behaviour of the full Euler system. We also note that, while in some regions of the solution branch the quantitative agreement between the MCC2 and Euler system is poor, they share many qualitative features. The results above are in broad agreement with the findings of Camassa et al. [2006], and allows us to clearly demonstrate two points that will recur through the rest of the results. First, the MCC3 system can become a poor approximation of the Euler system for mode-2 waves due to the invalidation of the shallow water approximation. Second, since the MCC3 and Euler system have the same conjugate states, tabletop solitary waves are well described by strongly nonlinear theory.

Above, we explored convex mode-2 solutions to the Boussinesq symmetric configuration. Below, we discuss the solution space for convex mode-2 solitary waves to the MCC3 and full Euler models when this symmetry is broken.
Figure 15: Panel (a) shows a branch of GSW with $\Delta_1 = \Delta_2 = 0.01$, $H_3 = 0.06$, and $Y(0) = -0.03$, with varying values of $H_2$ (and $H_1$). The dotted curve is $K = 0$. The red, black, and blue profiles in panel (b) correspond to the solutions (1), (2), and (3) respectively.

4.1.2 Non-symmetric configuration

It is known that the oscillations in the tail of a GSW can occur to orders smaller than the accuracy of a particular numerical scheme [Sun and Shen, 1993, Grimshaw and Joshi, 1995, Sun, 1999]. To overcome this difficulty, we use the criterion proposed by Champneys et al. [2002], which allows for the observation of an ESW while measuring parameters to orders of accuracy that can be achieved numerically. More precisely, denote the curvature of the lower interface at the last meshpoint in the tail as $K$, where $K$ is counted positive if the radius of curvature lies inside the bottom fluid. We state that an ESW is found along branches of GSW when it passes through $K = 0$. The justification is as follows: the domain is periodic, and hence the solution ends on a wave trough if $K > 0$, and a wave crest if $K < 0$. The solution which occurs at $K = 0$ must have waves with zero amplitude in the tail. This technique has been used to explore the existence of ESW for fully nonlinear gravity-capillary waves [Champneys et al., 2002] and gravity-flexural waves [Gao, 2016]. In all the cases above, no ESWs were found. In fact, positive results for the existence of ESWs are only available for reduced long wave models (see e.g. Champneys et al. [2002] for ESWs in a fifth-order modified KdV equation; Barros et al. [2020] for ESWs in the MCC3 model). The results to follow are the first positive result in the context of the full Euler equations.

Using a solution to the symmetric Boussinesq configuration with $H_3 = 0.47$, $H_2 = 0.06$, and $\Delta_1 = \Delta_2 = 0.01$ as an initial guess, we break the symmetry by removing the Boussinesq approximation. The numerical scheme described in Appendix A then converges to a GSW. We fix $Y(0)$ and vary the value of $H_3$ to obtain a branch of GSW, shown in figure 15(a), where $Y$ is the perturbation of the lower interface translated to have mean zero (see appendix, equation (45)). We plot the solution branch with $K$ on the vertical axis. We see that the solution given by (2) is an ESW. When plotting solution branches in the figures that follow, a cross shall denote a GSW, and a diamond an ESW. Figure 15(b) shows the perturbation of the lower interface $Y$ for the solutions (1)-(3). It demonstrates how $K$ going from positive to negative transitions through a localised solution.

We now modify the procedure to enforce the condition that $K = 0$. As suggested by figure 15(a), this requires removing a degree of freedom from the problem: for example, instead of fixing the amplitude and varying the speed (or vice versa), we must allow both to vary. Alternatively, one can allow an additional parameter to vary (for example, fix the amplitude and allow $c$ and $H_1$ to vary). The above procedure allows the computation of branches of ESWs. In the discussion that follows, we compute branches of ESWs by fixing $H_2$, $\Delta_1$, and $\Delta_2$, and varying the value of $H_3$.

Let $A$ be defined by

$$A = \max (|\xi_1(0)|, |\xi_2(0)|).$$

(25)

Figure 16 shows the three branches of Boussinesq ESWs plotted in the $(H_3, A)$-plane with $H_2 = 0.25$, and $\Delta_2 = 1.1\Delta_1$ (red curves), $\Delta_1 = 1.1\Delta_2$ (blue curves), and $\Delta_1 = \Delta_2$ (black curves). The dashed curves are MCC3 solutions, while the solid curves are Euler solutions. It can be seen the black curves
correspond to the symmetric Boussinesq configuration with $H_1 = H_3$. A break in the stratification symmetry results in a solution branch either side of the $\Delta_1 = \Delta_2$ branch. As $A$ increases along the branch, the waves get broader, becoming a tabletop solitary wave, and limiting to a wavefront. This limiting solution is a heteroclinic orbit from a saddle origin to a maximum conjugate state. The solution denoted by (4) is shown in figure 17, where a solution of the conjugate state equations (which is a maximum of the MCC3 potential $V$) is given by the red curves.

It is not clear from the numerical scheme how the solution branches terminate at the other end. The Euler and MCC3 numerical solvers struggle to converge beyond the points plotted in the figure. The amplitude of the waves by this point is very small. Due to the Boussinesq approximation and the values of $\Delta_1$, it is the case that the blue branch is a reflection of the red branch across the curve $H_1 = H_3$. The inset in figure 16 shows the speed-amplitude bifurcation diagram for $H_3 = 0.35$, $\Delta_1 = 0.01$, and $\Delta_2 = 0.011$. For the MCC3 and Euler solutions, we recover only an isolated point, corresponding to the red diamonds. Meanwhile KdV would predict (incorrectly) a complete branch of solutions, given by the dashed curve. Figure 18 shows a comparison of the MCC3 and Euler profiles relating to the diamonds, as well as a KdV solution matching the amplitude of the Euler solution. It can be seen that the KdV solution is not as broad as the MCC3 or Euler solution.

Keeping other parameters the same, but removing the Boussinesq approximation, one finds a similar solution space. The key difference is that, unlike the Boussinesq case, the branch with symmetric density stratification does not lie on the curve $H_1 = H_3$, and the solutions are not symmetric about $y = 0.5$. This is demonstrated in figure 16 where Euler ESW branches are shown with and without the Boussinesq approximation, with the same parameter values used in figure 16. This comparison

![Figure 16: Boussinesq ESW branches for the MCC3 (dashed curves) and Euler (solid curves) with $H_2 = 0.25$, $\Delta_1 = 0.01$, $\Delta_2 = 0.011$ (red), and $\Delta_1 = 0.011$, $\Delta_2 = 0.01$ (blue). The black curves are the symmetric configuration solitary wave branch ($\Delta_1 = \Delta_2$). The inset shows the speed-amplitude bifurcation diagram for $H_3 = 0.35$, $\Delta_1 = 0.01$, and $\Delta_2 = 0.011$. The MCC3 and Euler branches are an isolated point (given by the diamonds), while a full branch of solitary waves exists for the KdV model (dashed curve). The solution (4) is shown in figure 16.](image1.png)

![Figure 17: Solution (4) from figure 16. The MCC3 and Euler solutions are plotted in black solid and dashed curves, and are almost indistinguishable. The red lines show a solution to the conjugate state equations given the incoming flow from the left. To four significant figures, both the Euler and MCC solution have the parameter values $H_3 = 0.3561$, $H_1 = 0.3939$, and $c = 0.03618$.](image2.png)
Figure 18: Boussinesq ESW solutions with $H_3 = 0.35$, $H_2 = 0.25$, $\Delta_1 = 0.01$, and $\Delta_2 = 0.011$ for the Euler model (solid curve) and the MCC3 model (dashed curve). The dotted curve is a KdV solitary wave, with an amplitude matching the Euler solution.

Figure 19: Euler ESW branches without and with the Boussinesq approximation for the solid and dashed curves respectively. The middle layer has depth $H_2 = 0.25$, while the stratifications are $\Delta_1 = 0.01$, $\Delta_2 = 0.011$ (red), $\Delta_1 = 0.011$, $\Delta_3 = 0.01$ (blue) and $\Delta_1 = \Delta_2 = 0.01$ (black). Demonstrates the sensitivity the Boussinesq approximation can have on the solution space.

The Euler solution space for $H_2 = 0.06$ has similar features seen for the $H_2 = 0.25$ solution space. However, we found that the code failed to converge for very large amplitudes, and hence we were not able to approach a limiting wavefront solution. Figure 20 shows Euler branches (computed as far as the numerical scheme allows) without the Boussinesq approximation with $H_2 = 0.06$. While reasonable agreement is found between the Euler and MCC3 ESW branches when $H_2 = 0.25$, for $H_2 = 0.06$ stark differences appear. We suspect the cause of the increased discrepancy is the invalidation of the shallow water approximation. For example, in the same spirit as figure 2, compare the linear dispersion relation for two sets of parameter values. We fix $\Delta_1 = \Delta_2 = 0.01$ and $H_1 = H_3$, and vary $H_2$. For $H_2 = 0.06$, the value of $k = k^*$ for which $c^+ (k^*) = c^- (0)$ is approximately $k^* = 22.3$, while for $H_2 = 0.25$, it is $k^* = 6.4$. Using these values to determine the horizontal length scale, the shallow water parameter relative to the bottom layer is $\epsilon_3 \approx 1.7$ and $\epsilon_3 \approx 0.38$ for $H_2 = 0.06$ and $H_2 = 0.25$ respectively. It can be seen that the shallow water assumption is invalidated by the relatively short wavelength of the periodic tail for smaller $H_2$.

Figures 21 and 22 show branches of ESWs for the MCC3 equation with and without the Boussinesq approximation respectively. We fix $\Delta_1 = 0.01$, $H_2 = 0.06$, and vary $\Delta_2$. For the Boussinesq case,
when $\Delta_2 = \Delta_1$, the ESW branch with $H_1 = H_3$ is recovered, as above. However, along this branch there is a bifurcation point occurring about $A \approx 0.1$. The bifurcating branch no longer satisfies $H_1 = H_3$. Numerical computations show that this bifurcation point occurs for smaller values of $A$ as $H_2$ is increased, and no longer exists for $H_2 \approx 0.22$. Breaking the density symmetry results in the red and blue curves, which lie in domains bounded by the $\Delta_1 = \Delta_2$ curves. The branches for large $A$ ultimately end in a wavefront, while again it is unclear from the numerical results how the branches ends with small values of $A$ terminate. For the non-Boussinesq model, we no longer have the symmetric Boussinesq solution branch. Figure 22(a), shows branches for four choices of $\Delta_2$. Like for $H_2 = 0.25$ (see figure 19), the $\Delta_2 = \Delta_1$ branch without the Boussinesq approximation no longer lies on $H_3 = H_1$. The two branches for $\Delta_2 = 0.010575$ and $\Delta_2 = 0.01016$ are indistinguishable from figure 22(a). To see the difference, figure 22(b) shows a blow up of the region of the solution space where the solution branches almost touch. It can be seen that the endpoints of the branches switch in between these values. We conjecture that for some $\Delta_2$ in between these two values, there exists a solution branch with a bifurcation point, bearing resemblance to the $\Delta_1 = \Delta_2$ branches for the MCC3 Boussinesq model.

### 4.1.3 Multi-hump solitary waves

Thus far, the ESWs we have seen are the typical ‘bulge-shaped waves’ observed in laboratory experiments (for example, Carr et al. [2015], Barros et al. [2020] found compacton solutions comprised of $p$ and $q$ humps on the lower and upper interface respectively (referred to as a $(p, q)$ compacton). They were recovered in the asymptotic limit $H_2 \to 0$, but were found numerically to persist for finite values of $H_2$. In this section, we show evidence of their existence in the context of the Euler equations.

For the full Euler system, one cannot get onto these new ESW branches by remaining on the single-humped branches discussed in the previous section. Furthermore, using the MCC3 solutions as an initial guess proved difficult, due to the Euler solver requiring exceptionally good initial guesses to converge. Instead, we recovered new ESW solutions by venturing onto branches of GSWs, moving through parameter space to seek a new solution with $K = 0$. We describe one such path as follows.
We begin with the solution labelled (2) in figure 20. We fix $\zeta_2(0)$ and vary the parameter $\Delta_1$. The resulting branch of GSW is shown in figure 23(a). We take the solution (5) (with $\Delta_1 = 0.021$), and then compute the branch shown in figure 23(b) by varying the speed. Finally, fixing $\Delta_1$ and $\zeta_2(0)$ but allowing $H_2$ to vary, we obtain the branch in figure 23(c). These branches are shown in three-dimensional parameter space in figure 24. Branches of GSW and ESW are plotted with dashed and solid curves respectively. Solutions (5)–(7) are shown in figure 25 and it can be seen that the solution (7) is an ESW. By fixing $K = 0$ at the last mesh point, we are able to find branches of ESWs characterised by two humps on the upper interface and one on the lower interface, resembling solution (7). One such branch is shown in figure 26 with $H_2 = 0.15$ and $\Delta_1 = \Delta_2 = 0.01$ for both the full Euler and MCC3 equations, where the parameter $Q$ on the $y$-axis measures the volume of the perturbation, given by

$$Q = \int_{-\infty}^{0} |\zeta_1| + |\zeta_2| \, dx. \quad (26)$$

The solutions (8)–(11) from figure 26 are shown in physical space in figure 27. It can be seen that at with increasing values of $Q$ the waves broaden. However, unlike the single-humped ESWs seen in section 4.1.2, the broadened section is not flat. To better understand this, we consider the projection of the MCC3 solutions onto the $(\zeta_1, \zeta_2)$-plane, as shown in figure 28. As the amplitude of the solution increases, the trajectory gets closer to the maximum critical point, shown by the black cross. The single-humped ESW branch with the same values of $H_2$ and $\Delta_1$ has a limiting solution which is a heteroclinic orbit from the origin to the maximum critical point. Remarkably, the multi-hump branch from figure 26 appears to approach a solution (see (11)) which consists of a quasi-heteroclinic orbit combined with a quasi-homoclinic orbit originating at the maximum critical point: in other words a solitary wave riding on a conjugate state.

One can smoothly, via continuation, get from any MCC3 solution along this branch to a $(1, 2)$ compacton in panels (a) and (b) respectively. One can also reduce the value of $H_2$ for the Euler solutions, but we found computing solutions with $H_2 < 0.05$ difficult due to stiffness in the numerical method.

More multi-hump ESW branches with $H_2 = 0.15$ and $\Delta_1 = \Delta_2$ can be found, such as the MCC3 branches shown in figure 29. All the branches shown have a vertical asymptote at $H_3 \approx 0.4276$. The solutions (9)–(10) from figure 26 lie on the lowest branch which is to the left of the asymptote. As discussed in the previous paragraph, these solutions approach a $(1, 2)$ compacton as $H_2 \to 0$. The branch directly above approaches a $(1, 3)$ compacton as $H_2 \to 0$, and the one above that a $(1, 4)$ compacton, and so on. Meanwhile, to the right of the asymptote ($H_3 > 0.4276$), the branches approach an $(n, 1)$ compacton as $H_2 \to 0$, where $n = 1$ is the for the lowest most branch, and $n = 2$ for the second lowest branch, and so on. The lowest branch is a single-humped ESW branch. The solutions given by the green, blue, and red diamond are shown in figure 31 by the green, red and blue curves respectively. The green single-humped profile is akin to those seen in section 4.1.2. The blue and red curves are multi-humped ESWs. The solutions are characterised by a broadened section with multiple humps, and our numerical results suggest there are solutions with an arbitrary number of such oscillations. Note that the upper interface also has oscillations, albeit of smaller amplitude, as shown by the blow up in panel (b). Only when $H_2 = 0$ do the oscillations on the top interface disappear, resulting in an $(n, 1)$ compacton. As with the single-humped ESWs, we found that the
Figure 24: The branches from figure 23 plotted in \((c, H_3, K)\)-space. The shaded region is given by \(K = 0\), and any solution lying on this plane is an ESW. The black branch along which the solution (2) lies corresponds with the black branch of ESWs seen in figure 20.

Figure 25: Solutions (5)–(7) from figure 23. All solutions have \(H_2 = 0.06\), \(\Delta_2 = 0.1\), and \(\Delta_1 = 0.021\). Solution (7) is an ESW.

Figure 26: A multi-hump ESW branch with \(\Delta_1 = \Delta_2 = 0.01\) and \(H_2 = 0.15\) for the MCC3 model (red) and Euler model (black). The solutions (8)-(11) along the MCC3 branch are shown in figure 27 by the red curves, while the Euler solution with the same \(H_3\) is shown in black.
Figure 27: Solutions (8)-(11) from figure 26. The red curves are the MCC3 solutions, while the black curves are the corresponding Euler solution with the same value of $H_3$. The speeds of the solutions (8)-(11) are $c = 0.02396$, $c = 0.02772$, $c = 0.03498$, and $c = 0.03536$ for the MCC3 system and $c = 0.02374$, $c = 0.02764$, $c = 0.03507$, and $c = 0.03536$ for the Euler equations.

Figure 28: MCC3 solutions (9)–(11) from figure 27, projected onto the $(\zeta_1, \zeta_2)$-plane. Black curves are level sets of $V$, where the bold black curve is $V = 0$. The red curve is the trajectory of the solution, originating at the origin and bounded by $V = 0$. The green, blue, and black crosses are minimum, saddle, and maximum critical points of $V$ respectively.

Figure 29: MCC3 solutions with $\Delta_1 = \Delta_2 = 0.01$, $H_3 = 0.13$, and $H_2 = 0.06$ and $H_2 = 0$ for panels (a) and (b) respectively.
Figure 30: Branches of MCC3 ESWs with $H_2 = 0.15$ and $\Delta_1 = \Delta_2 = 0.01$. The green, blue, and red diamonds lie on $H_3 = 0.55$, and these solutions are shown in figure 31. We also annotate solutions (9)-(10) from figure 26 given by the black diamonds.

Figure 31: MCC3 ESWs with $H_3 = 0.55$, $H_2 = 0.15$, and $\Delta_1 = \Delta_2 = 0.01$. The green, blue, and red solutions correspond to the green, blue, and red diamonds shown in figure 30 respectively. The black solution has a speed $c = 0.02916$, while the blue and red solutions have speeds approaching that of the conjugate state ($c \approx 0.03126$). Panel (b) is a blow up of the upper interface where oscillations cannot be seen in panel (a).

Figure 32: ESWs for the MCC3 (red) and Euler (black) equations with $H_3 = 0.3$ and $\Delta_1 = \Delta_2 = 0.01$. The solutions in panel (a) have $H_2 = 0.15$, where the MCC3 and Euler speeds are $c = 0.03112$ and $c = 0.03109$ respectively. Meanwhile, panel (b) has $H_2 = 0.06$ with MCC3 and Euler speeds of $c = 0.02895$ and $c = 0.02888$ respectively.
MCC3 system becomes a poorer approximation for the Euler equations as $H_2$ decreases. For example, in figure 12 we compare MCC3 and Euler multi-hump profiles for $H_2 = 0.15$ and $H_2 = 0.06$ in panel (a) and (b) respectively. It can be seen that the wavelength and amplitude of oscillations at the center of the wave have worse agreement for the solutions in panel (b).

4.2 Concave waves outside the linear spectrum

All of the mode-2 solitary waves considered so far have speeds less than $c_0^+$, and are hence ESWs. However, it is not always the case that mode-2 waves must be within the linear spectrum. This is predicted by the fact that mode-2 conjugate states may have speeds exceeding $c_0^+$, provided that the value of $H_2$ is sufficiently large (see figure 11(a)). The mode-2 conjugate state with $c > c_0^+$ is a saddle of the MCC3 potential. We have seen already in the previous section how large amplitude ESWs can have a broadened section with periodic-like orbits in the vicinity of a non-trivial saddle point, resulting in multi-hump solitary waves. Such solutions are also found outside the linear spectrum, and indeed one has been shown already for the MCC3 system in figure 9. For simplicity, we will restrict our attention to the Boussinesq system.

We begin by exploring solutions with $H_2 = 0.8$, $H_1 = H_3$, and $\Delta_1 = \Delta_2$. A bifurcation diagram is shown in figures 33 and 53 for the Euler and MCC3 systems respectively. The horizontal axis is the volume of the perturbation of the lower interface over half the domain. The agreement, like for convex waves, is qualitatively good but quantitatively poor (we note that the figures have different scalings on the horizontal axis). The two blue curves correspond to two mode-1 solitary wave branches which bifurcate from $c = c_0^+$. Since this is the symmetric Boussinesq configuration, one branch can be recovered from the other by the mapping $g \rightarrow -g$, $\zeta_1 \rightarrow -\zeta_2$, $\zeta_2 \rightarrow -\zeta_1$. The red branch is mode-2 solitary waves with $\zeta_1 = -\zeta_2$, which bifurcate from $c = c_0^-$ and approach a tabletop solitary wave. As expected, the limiting solution is a heteroclinic orbit from the origin to the mode-2 conjugate state of the system. Along the branch there are bifurcation points represented by triangles. The $\zeta_1 = -\zeta_2$ symmetry is broken along the bifurcating branches, which are shown by the black curves. In fact, two new branches bifurcate from each bifurcation point, but they are related to each other by the aforementioned mapping for the two mode-1 branches.

We compare solutions with the same speed for the MCC3 and Euler system. Solutions (11), along the first bifurcating branch and with $c = 0.338$, are plotted in figure 35 for the Euler and MCC3 system in black and red respectively. In both cases, the upper interface no longer monotonically decreases to a minimum value from its crest. Following the bifurcating branch, the solutions decay slowly at $x \rightarrow \pm \infty$ as the speed approaches $c_0^+$, resulting in numerical difficulties, especially for the full Euler code due to the increased complexity of the solver. A possible explanation for this phenomena is that the solution is experiencing nonlinear resonance with a mode-1 solitary wave travelling at the same speed. As the speed of the wave approaches $c_0^+$, the mode-1 solutions become broader with slower decay. Hence, as the bifurcating mode-2 solitary waves resonate with mode-1 waves of decreasing amplitude (i.e. as $c \rightarrow c_0^+$), the decay of the wave becomes weak.

Solutions from the remaining bifurcating branches are multi-humped solitary waves. Figure 36 shows the solutions (12)-(15) from figures 33 and 34 where the Euler and MCC3 profiles are shown in black and red respectively. For comparison, the speeds are kept the same across all the solutions. It can be seen that the interfaces have additional oscillations as one considers the sequence of bifurcating branches. Our numerical exploration suggests infinitely many of these bifurcating branches exist. We can have a broadened section with periodic-like orbits in the vicinity of a non-trivial saddle point, provided that the speed of the wave have worse agreement for the solutions in panel (b).

Next, we consider a solution branch with non-symmetric stratification, by setting $H_2 = 0.8$, $H_1 = H_3$, $\Delta_1 = 0.01$ and $\Delta_2 = 0.011$. For simplicity, we shall consider the MCC3 system only. The solution space for mode-2 solitary waves is shown in figure 37 where we plot each branch in a different colour. There are two key differences with the solution space for the symmetric Boussinesq configuration. First, there is no longer a branch satisfying $\zeta_1 = -\zeta_2$, and hence none of the branches with $c > c_0^+$ smoothly enter the linear spectrum while remaining truly localized. In the linear
Figure 33: Boussinesq Euler solution branches with $H_1 = H_3 = 0.1$, $\Delta_1 = \Delta_2 = 0.01$. The two blue curves are mode-1 solitary wave branches. The red curve is the symmetric Boussinesq mode-2 branch, while the black curves are mode-2 branches which bifurcate from the symmetry branch. The triangles are bifurcation points. The solutions (11)–(15) corresponding to the crosses are plotted in figures 35–36.

Figure 34: This figure is the MCC3 comparison with figure 33. Note that the horizontal axis has a different scale than in figure 33. The solutions (11)–(15) corresponding to the crosses are plotted in figures 35–36.

Figure 35: Solution (11) from figures 33–34. The black curves are the Euler solution, while the red curves are MCC3. The solutions have the speed $c = 0.0388$. 

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Figure 36: Solutions (12)–(15) from figures 33–34. The black curves are the Euler solution, while the red curves are MCC3. The solutions all have the same speed $c = 0.03552$.

Figure 37: A single MCC3 branch from figure 34 showing the bifurcation point, and solutions along the branch.
Figure 38: Boussinesq MCC3 mode-2 solitary wave solution branches with parameter values $\Delta_1 = 0.011$, $\Delta_2 = 0.01$, $H_2 = 0.8$, and $H_1 = H_3 = 0.1$. Different colours are used to distinguish separate branches. Panel (b) shows a blow-up of the red, blue, and green branch from panel (a).

Figure 39: The green solution branch from figure 38 with solutions (a)–(d) shown in their respective panels.
spectrum, one may still find ESWs, but they are isolated points along branches of GSW, as seen for convex waves in section 4.1.2. The second difference is that there is no sequence of bifurcation points. Instead, the branches of solutions are isolated: where they cross no longer corresponds to a bifurcation point. This being said, numerics still suggest there is an infinite sequence of multi-humped solution branches with an increasing number of oscillations on the broadened section. We plot in figure 39 the green branch from figure 38 along with a number of solutions along the branch. It can be seen that, at either end of the branch, the solution is multi-humped solitary wave with a mode-1 solitary wave either side of the mode-2 solution.

5 Conclusion

In this paper, we have presented a plethora of steady three-layer mode-2 internal solitary waves. Using the criterion proposed by Champneys et al. [2001], we establish numerically that truly localised mode-2 solitary waves can be embedded in the linear spectrum. In addition to single-humped profiles, we find branches of exotic multi-hump solutions for both the fully nonlinear Euler equations and the reduced MCC3 equations. For MCC, this extends the work of Liapidevskii and Gavrilov [2018] and Barros et al. [2020], who previously computed multi-humped waves within the strongly nonlinear regime, but limited to small depths of the intermediate layer. In fact, such solutions reveal to be prolific throughout the parameter space.

The MCC3, being a long wave strongly nonlinear model, is found to be in good agreement with the Euler system as long as the shallow water assumption is not invalidated. We have found that, even in cases where the long wave approximation is not applicable, qualitative agreement still holds, but significant quantitative differences can be found between the solutions of the two models.

The mode-2 solutions found in this paper tend to broaden with increasing values of amplitude. In some cases, the broadened section is flat (i.e. tabletop solitary waves), while in others, numerous oscillations (humps) develop in the broadened section. The study of the conjugate states, which are equivalent to critical points of the MCC3 system with a zero level energy, allows us to distinguish the two different behaviours.

Since all computations in this paper concern steady waves, no information is obtained regarding their stability. Exploration of the stability would raise interesting questions, particularly when there is multiplicity of solutions, such as in figure 30. If more than one solution were stable, as it is the case for the Gardner mode-1 solutions with a thick intermediate layer [Kalisch and Nguyen, 2010, Kurkina et al., 2015], it would be of interest to see how they could all be generated in a laboratory.

This paper has focussed primarily on solutions in which the stratification is weak ($\Delta_1 \approx \Delta_2 \ll 1$). However, owing to the large number of parameters in the problem, a complete description of the solution space for steady waves is still to be found. Previous computations by Rusas and Grue [2002] for stronger stratification have found existence of overhanging solutions for three-layer mode-1 waves. Whether such behaviour could be found for mode-2 waves is as of yet unknown.

Declaration of interests. The authors report no conflict of interest.

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A Numerical method for Euler equations

To solve the system of equations described in section 2, we reformulate the problem such that $x$ and $y$ are functions of the independent variables $\phi_3$ and $\psi_3$. The desire to solve this system of equations in the potential space is motivated by the fact that the unknown interfaces are isolines of the streamfunctions $\psi_i$. The unknowns are expressed in terms of integrals concerning boundary values via Cauchy’s integral formula. This method has been adopted by many authors for the two-layer model (for example, Turner and Vanden-Broeck [1986]). We note that this method is very similar to that used by Rusás and Grue [2002] for the three-layer model. However, they remain in the physical space and resolve boundary integrals seeking the unknown velocities in terms of $x$ and $y$. The advantage of working in the potential space is the reduction in the number of unknowns, as demonstrated below.

Consider first the bottom fluid layer. Denoting $\phi = \phi_3$, we can write the complex potential $f_3$ in the lower fluid layer as $f_3 = \phi + i\psi_3$. Without loss of generality, choose $\psi_3 = 0$ on the bottom wall, and $\phi = 0$ at $x = 0$. The potential space of the bottom fluid layer, and it’s reflection across the wall $\psi_3 = 0$, is given by

$$\Omega_3^f = \{(\phi, \psi_3) : \phi \in [-\tilde{c}\lambda/2, \tilde{c}\lambda/2], \psi_3 \in (-2Q_3, 0)\},$$

where we denote the flux of fluid in the bottom fluid layer as $Q_3$. We conformally map $\Omega_3^f$ to an annulus in the $t$-plane via the mapping

$$t = \exp\left(-\frac{kif_3}{\tilde{c}}\right).$$

We parameterise the perturbation to the uniform stream of the lower interface as $(X(\phi), Y(\phi))$ and the upper interface as $(\tilde{X}(\phi), \tilde{Y}(\phi))$, where

$$\int_0^{c\lambda/2} Y(\phi) \, d\phi = 0, \quad \int_0^{c\lambda/2} \tilde{Y}(\phi) \, d\phi = 0.$$

The function $F = x_{\phi\phi} + 1/\tilde{c} + iy_{\phi\phi}$, where subscripts denote partial differentiation, is an analytic function of $f_3$, and hence an analytic function of $t$. Applying Cauchy’s integral formula to the function $F$ over
the annulus in the $t$-plane, where we denote the boundary of the annulus traversed counterclockwise as $C$, one finds that

$$F(t_0) = \frac{1}{\pi i} \oint_C \frac{F(t)}{t - t_0} \, dt,$$

(30)

where $t_0$ is taken on the boundary of the annulus. Making use of (28), and taking real parts, the above integral gives

$$\frac{\partial \lambda}{2} \left( X_\phi(\phi_0) + \frac{1}{\tilde{c}} \right) = \frac{1}{2} \int_{-\tilde{\epsilon} \lambda/2}^{\tilde{\epsilon} \lambda/2} Y_\phi \cot \left( \frac{\Theta_-}{2} \right) \, d\phi$$

$$- \int_{-\tilde{\epsilon} \lambda/2}^{\tilde{\epsilon} \lambda/2} \frac{(X_\phi + \tilde{c}) (1 - R_3 \cos \Theta_-)}{1 + R_3^2 - 2R_3 \cos \Theta_-} \, d\phi$$

$$+ \int_{-\tilde{\epsilon} \lambda/2}^{\tilde{\epsilon} \lambda/2} \frac{Y_\phi \sin \Theta_-}{1 + R_3^2 - 2R_3 \cos \Theta_-} \, d\phi.$$  

(31)

Here, $\Theta_\pm = k(\phi_0 \pm \phi)/c$ and $R_3 = e^{2Q_3 k/\tilde{c}}$. The above integral expresses $X_\phi(\phi_0)$ at some point $\phi_0$ on the interface in terms of integrals concerning boundary values of $X_\phi$ and $Y_\phi$.

Next, we consider the intermediate fluid layer. We write $\phi_2 = h(\phi)$, and denote the flux in the layer as $Q_2$. Without loss of generality, let $\psi_2 = 0$ on the lower interface $y = \eta_2$. Hence, the potential space of the intermediate layer $f_2 = h(\phi) + \psi_2$ is given by

$$\Omega_f^2 = \{(h(\phi), \psi_2) : \phi \in [-\tilde{c} \lambda/2, \tilde{c} \lambda/2], \psi_2 \in (0, Q_2)\}.$$  

(32)

The far-field condition (7) requires that $h$ satisfies

$$h \left( \frac{\pm \tilde{c} \lambda}{2} \right) = 1.$$  

(33)

One can map the above space to an annulus in the $s$-plane via the mapping

$$s = \exp \left( \frac{ki f_2}{\tilde{c} h_0} \right).$$  

(34)

where

$$h_0 = \frac{h(\tilde{c} \lambda/2) - h(0)}{\tilde{c} \lambda/2}.$$  

(35)

The parameter $h_0$ is a measure of the wavelength averaged shear between the bottom and intermediate layer. When $h_0 = 1$, then $\phi_3$ and $\phi_2$ vary the same over one wavelength, and hence the average shear is zero. Typically, $h_0$ is different from unity. Again solving Cauchy’s integral formula on the region bounded by the annulus in the $s$-plane, one can derive two boundary integral equations. First we find that

$$\frac{\partial h_0 \lambda}{2} \left( \frac{X_\phi(\phi_0)}{h_\phi(\phi_0)} + \frac{1}{\tilde{c} h_0} \right) = \frac{1}{2} \int_{-\tilde{\epsilon} \lambda/2}^{\tilde{\epsilon} \lambda/2} Y_\phi \cot \left( \frac{\alpha_-}{2} \right) \, d\phi$$

$$- \int_{-\tilde{\epsilon} \lambda/2}^{\tilde{\epsilon} \lambda/2} \frac{(\tilde{X}_\phi + \frac{h_\phi}{\tilde{c} h_0}) (1 - R_2 \cos \alpha_-)}{1 + R_2^2 - 2R_2 \cos \alpha_-} \, d\phi$$

$$+ \int_{-\tilde{\epsilon} \lambda/2}^{\tilde{\epsilon} \lambda/2} \frac{\tilde{Y}_\phi \sin \alpha_-}{1 + R_2^2 - 2R_2 \cos \alpha_-} \, d\phi.$$  

(36)
Here, $R_2 = e^{Q_2 k^2}$ and $\alpha_{\pm} = k(h(\phi_0) \pm h(\phi))/(\tilde{c} h_0)$. Second, we get

$$
\frac{\tilde{c} h_0 \lambda}{2} \left( \frac{\bar{x}_\phi(\phi_0)}{\bar{g}_\phi(\phi_0)} + \frac{1}{\tilde{c} g_0} \right) = -\frac{1}{2} \int_{-\tilde{c} \lambda/2}^{\tilde{c} \lambda/2} \bar{y}_\phi \cot \left( \frac{\alpha_-}{2} \right) \, d\phi
$$

$$
- \int_{-\tilde{c} \lambda/2}^{\tilde{c} \lambda/2} \left( \frac{\bar{x}_\phi + \frac{b_0}{\tilde{g} h_0}}{1 + R_2^2 - 2 R_2 \cos \alpha_-} \right) (1 - R_2 \cos \alpha_-) \, d\phi
$$

$$
- \int_{-\tilde{c} \lambda/2}^{\tilde{c} \lambda/2} \frac{Y_\phi \sin \alpha_-}{1 + R_2^2 - 2 R_2 \cos \alpha_-} \, d\phi.
$$

(37)

Finally, denoting $\phi_1 = g(\phi)$ and the flux in the upper layer as $Q_1$, we map the upper interface and its reflection across the top boundary in the $(g, \psi_1)$-space to an annulus. Following the same methodology as above, and upon assuming $\psi_3 = 0$ on the upper interface $y = \eta_1$, we get

$$
\frac{\tilde{c} g_0 \lambda}{2} \left( \frac{\bar{x}_\phi(\phi_0)}{g_\phi(\phi_0)} + \frac{1}{\tilde{c} g_0} \right) = -\frac{1}{2} \int_{-\tilde{c} \lambda/2}^{\tilde{c} \lambda/2} \bar{y}_\phi \cot \left( \frac{\beta_-}{2} \right) \, d\phi
$$

$$
- \int_{-\tilde{c} \lambda/2}^{\tilde{c} \lambda/2} \left( \frac{\bar{x}_\phi + \frac{b_0}{\tilde{g} h_0}}{1 + R_1^2 - 2 R_1 \cos \beta_-} \right) (1 - R_1 \cos \beta_-) \, d\phi
$$

$$
- \int_{-\tilde{c} \lambda/2}^{\tilde{c} \lambda/2} \frac{Y_\phi \sin \beta_-}{1 + R_1^2 - 2 R_1 \cos \beta_-} \, d\phi.
$$

(38)

Here, $R_1 = \exp(2kQ_1/(\tilde{c} g_0))$, $\beta_{\pm} = k(g(\phi_0) \pm g(\phi))/(\tilde{c} g_0)$, and

$$
g_0 = \frac{g(\tilde{c} \lambda/2) - g(0)}{\tilde{c} \lambda/2}.
$$

(39)

The far-field condition [8] requires that

$$
g_\phi \left( \frac{\pm \tilde{c} \lambda}{2} \right) = 1.
$$

(40)

It is left to satisfy the continuity of pressure condition [5]. We can rewrite these equations as

$$
h_\phi(\phi) = \left[ \frac{\rho_1 + \frac{2}{q^2} (\Delta_2 Y - B_2)}{1} \right]^{1/2},
$$

(41)

$$
g_\phi(\phi) = \left[ \frac{1}{\rho_1} \frac{\tilde{h}_\phi^2 + \frac{2}{\tilde{q}^2} (\Delta_1 \tilde{Y} - B_1)}{1} \right]^{1/2},
$$

(42)

where

$$
q = (X_\phi^2 + Y_\phi^2)^{-1/2}, \quad \tilde{q} = (\tilde{X}_\phi^2 + \tilde{Y}_\phi^2)^{-1/2}.
$$

(43)

Similar expressions are found when using the Boussinesq approximation.

We wish to solve the reformulated problem numerically. We use the assumed symmetry about $x = 0$ such that we only have to solve the problem over half a wavelength. We discretise $\phi \in [-\tilde{c} \lambda/2, 0]$ into $N + 1$ mesh points $\phi_I$ as follows

$$
\phi_I = -\frac{\tilde{c} \lambda}{2} \left\{ \frac{N + 1 - I}{N} \right\}, \quad I = 1, \cdots, N + 1.
$$

(44)

The midpoints are give by $\phi^m_I = (\phi_I + \phi_{I+1})/2$. We express some of the unknowns as a Fourier series.
Then we write $C$ counterclockwise as $F$ integrals with the singularity removed. For example, consider the integral of some analytic function seeking values inside the domains, such as in the evaluation of occurring in the integrals are handled by evaluating the integrals at midpoints. However, when in the numerical scheme used by Rus˚ as and Grue [2002], and hence justifies our decision to solve the which we solve via Newton’s method. This is a much smaller system than the 8

...corresponds to a uniform stream with layers of depth $X$, Given $satisfying the boundary integral equation (31). The integrals are evaluated at midpoints, and ap-

We differentiate (45) and (47) with respect to $\phi$ We take

We then compute the solution, such as some measure of the amplitude (for example, $Y$). This results in a $2N + 6$ unknowns. We differentiate (45) and (47) with respect to $\phi$ to find $Y_\phi$ and $\tilde{Y}_\phi$. We then find $b_n$ by numerically satisfying the boundary integral equation (41). The integrals are evaluated at midpoints, and approximated using the trapezoidal rule. The discretised system is then solved via Newton’s method. Given $X_\phi$, $Y_\phi$, and $Y$, one can recover $h_\phi$ explicitly using equation (42). Expressing $h_\phi$ explicitly in other unknown variables (and hence reducing the number of unknowns in the nonlinear solver) was inspired by the work of Wang et al. [2014], who deployed such a trick for two-layer hydroelastic interfacial waves with a free-surface. We integrate $h_\phi$ numerically to find $h$, and then compute $X_\phi$ at midpoints explicitly by re-arranging equation (37). A four-point interpolation formula is used to approximate values of $X_\phi$ at the meshpoints $\phi_i$. One can then compute $g_\phi$ explicitly using (42), and then integrate for values of $g$ at each meshpoint. It is left then to solve the integral equations (36) and (38). Satisfying these equations at midpoints results in $2N$ equations. Another two equations are obtained by satisfying the far-field equations (33) and (40). These equations ensure that any ESW computed with this numerical scheme had a far-field given by a uniform stream with speeds $-c$ in each layer. Another three equations are obtained by fixing $H_i$ for each layer. We do this by discretising $\psi_i$ in each layer, and then evaluating values of $x_\phi$ at the first midpoint $\phi_1^M$. We then compute $H_i$ using the identities

$$H_3 = \int_0^{Q_3} x_\phi(\phi_1^M, \psi_3) d\psi_3, \quad H_2 = \int_0^{Q_2} x_\phi(\phi_1^M, \psi_2) \frac{h_\phi(\phi_1^M)}{g_\phi(\phi_1^M)} d\psi_2, \quad H_1 = \int_0^{Q_1} x_\phi(\phi_1^M, \psi_1) \frac{1}{g_\phi(\phi_1^M)} d\psi_1.$$

(48)

Here, values of $x_\phi$ inside the flow domains (rather than on the boundaries) needs to be recovered using Cauchy’s integral equation, resulting in equations similar to (41). For an ESW, the far-field corresponds to a uniform stream with layers of depth $H_i$. A final equation is obtained by fixing something about the solution, such as some measure of the amplitude (for example, $Y(0)$), or the speed of the wave $\tilde{c}$ [9]. The system is solved via Newton’s method, and we say a solution has converged when the $L_\infty$ norm of the residuals are of the order $O(10^{-10})$.

As mentioned before, the above method results in a $2N + 6$ system of unknowns and equations, which we solve via Newton’s method. This is a much smaller system than the $8N + 1$ unknowns found in the numerical scheme used by Rusás and Gruc [2002], and hence justifies our decision to solve the integrals in the potential space.

We briefly mention here that when evaluating the integrals on the boundary, the singularity occurring in the integrals is handled by evaluating the integrals at midpoints. However, when seeking values inside the domains, such as in the evaluation of $H_i$, it is found that the singular nature of the integrals becomes problematic for points close to the boundary. To solve this, we rewrite the integrals with the singularity removed. For example, consider the integral of some analytic function $F$ on the annulus in the $t$-plane, given by (28). Denoting the boundary of the annulus traversed counterclockwise as $C$, and let $t_0$ be some point inside the annulus which maps to $(\phi_0, \psi_0)$ in $\Omega_f$. Then we write

$$F(t_0) = \frac{1}{2\pi i} \int_C \frac{F(t) - F(t_1)}{t - t_0} dt + \frac{1}{2\pi i} \int_C \frac{F(t_1)}{t - t_0} dt,$$

(49)

where $t_1 = e^{-ki\phi_0/e}$. The first integral is evaluated numerically, where the singularity as $t_0 \to t_1$ is removed by the new numerator. The second integral is evaluated analytically, which using Cauchy’s integral formula gives simply $F(t_1)$. Hence, the computation is the same as in (30), except with a new correction term, given by the term in the square brackets below

$$F(t_0) = \frac{1}{2\pi i} \int_C \frac{F(t)}{t - t_0} dt + \left[ F(t_1) - \frac{F(t_1)}{2\pi i} \int_C \frac{1}{t - t_0} dt \right].$$

(50)
The correction term measures inaccuracies in evaluating the integral of a simple pole numerically when using the trapezoidal rule. The errors only become large near the boundary (that is, $t_0$ near $t_1$). It is found the above method improves the convergence of the evaluation of the integral as the number of meshpoints $N$ is increased.