Abstract
Through the Curry-Howard isomorphism between logics and calculi, necessity modality in logic is interpreted as types representing program code. Particularly, $\lambda\Box$, which was proposed in influential work by Davies, and its successors have been widely used as a logical foundation for syntactic meta-programming. However, it is less known how to extend calculi based on modal type theory to handle more practical operations including manipulation of variable binding structures.

This paper constructs such a modal type theory in two steps. First, we reconstruct contextual modal type theory by Nanevski, et al. as a Fitch-style system, which introduces hypothetical judgment with hierarchical context. The resulting type theory, Fitch-style contextual modal type theory $\lambda[]$, is generalized to accommodate not only S4 but also K, T, and K4 modalities, and proven to enjoy many desired properties. Second, we extend $\lambda[]$ with polymorphic context, which is an internalization of contextual weakening, to obtain a novel modal type theory $\lambda\forall[]$. Despite the fact that it came from observation in logic, polymorphic context allows both binding manipulation and hygienic code generation. We claim this by showing a sound translation from $\lambda\Box$ to $\lambda\forall[]$. 

1 Introduction
Syntactic metaprogramming enables programs to manipulate code fragments by generating, embedding or evaluating them. It can be seen in many applications such as macros, staged computation and proof assistants. Type safety of syntactic metaprogramming is extensively researched especially for staged computation,
and many type systems including MetaML \cite{13,22} have been developed. Under the Curry-Howard isomorphism \cite{20}, those type systems correspond to constructive modal logic including S4 modal logic \cite{7,17} and linear temporal logic \cite{6,23}. While those modal type theories give logical foundations of various aspects of staged computation, the logical counterpart of manipulation of variable binding structures has been unclear. MacroML\cite{8} and SND\cite{21}, which are surface type systems on MetaML, tackle this problem, but they only allow a restricted form of binding manipulation.

Nanevski’s contextual modal type theory (CMTT) \cite{15} partly solves this problem. CMTT extends S4 modal types to have context, where $\Gamma T$ stands for an open code of type $T$ under the environment $\Gamma$. Contextual modal types allow programs access to free variables in open code, and hence enables to express binding manipulation. However, contextual modal types are less flexible than those of linear temporal types in some cases. For example, a linear temporal type $\Diamond (S \rightarrow T) \rightarrow \Diamond S \rightarrow \Diamond T$ stands for a function that takes open code under arbitrary environment. CMTT cannot express this type because contextual modal type $[\Gamma]T$ only accepts specific context $\Gamma$. Nanevski and Pfenning\cite{14} proposed support polymorphism for this purpose in their former work, but its relation to neither contextual modality nor $\lambda\Diamond$ is not formalized.

We aim to give a logical foundation for syntactic metaprogramming with flexible binding manipulation. To this end, we introduce two novel type theories. First one is $\lambda_\Box$, which is a Fitch-style reconstruction of Nanevski’s contextual modal type theory. This reconstruction generalizes contextual modality to accommodate K, T, K4 and S4 modalities and gives $\lambda_\Box$ like quasiquotation syntax, which many macro systems adopt \cite{11,16}. The second one is $\lambda\forall\Box$, which is a generalization of $\lambda_\Box$ with contextual polymorphism. The notion of contextual polymorphism is obtained by internalizing context weakening of hypothetical judgment. Despite the logical background of contextual polymorphism, it also endows excellent expressibility of code generation. Finally, we formally show that $\lambda\Diamond$ terms are embeddable to $\lambda\forall\Box$ terms, through context extraction.

**Organization.** Section 2 presents former results of logical foundations of syntactic meta-programming and compare our work with them. Section 3 shows how to reconstruct contextual modal type theory based on Fitch-style judgment. Section 4 extends Fitch-style contextual modal type theory by introducing contextual polymorphism. Section 5 proves that $\lambda\Diamond$ can be embedded into $\lambda\forall\Box$. Section 6 concludes the paper.

## 2 Related Work

### 2.1 Modal Type Theory and Staged Computation

Through the research on the Curry-Howard isomorphism between logics and calculi, it is considered that constructive modal logic corresponds to multi-stage calculi.

In staged computation, two kinds of modalities have been mainly used. Davies’ $\lambda\Diamond$ corresponds to linear temporal logic. The type $\Diamond T$ can be interpreted as “open code of type $T$”, and $\lambda\Diamond$ provides logical basis for MetaML \cite{13} and its variant MetaOCaml \cite{22}. $\lambda\Diamond$ is not capable of run-time code evaluation because evaluating code at a different stage will cause “undefined variable error”. Taha
and Nielsen’s $\lambda^\alpha$ [22] solves this problem by labeling stages with environment classifiers. Tsukada and Igarashi’s $\lambda^\alpha$ [23] provides a logical foundation of $\lambda^\alpha$, where environment classifiers are interpreted as polymorphism over stage transitions. Polymorphic contexts in $\lambda^\forall[1]$ and environment classifiers are similar notions in the sense that both are polymorphism related to environments. However, they are essentially different notions: environment classifiers abstract stage transition while polymorphic contexts abstract context itself.

Another is S4 modality, which we simply call “box modality”. A box modal type $\square T$ can be interpreted as “closed code of type T”. S4 modal types are capable of code evaluation both at run-time and through multiple stages, and hence they are used as a logical foundation for code evaluation [7,13,24]. We refer to two formulations for box modalities. Dual-context formulation [7,17] uses two-level hypothetical judgment to describe S4 modality and introduces meta variables as terms. Recent work of Kavvos [9] generalizes dual-context formulation to K, T, K4 and GL modality. On the other hand, Fitch-style formulation [4,7,12,18] uses hypothetical judgment with context stack and introduces lisp-like quasiquotation syntax. Martini and Masini [12] point out that this formulation introduces K, T, K4 and S4 modalities. These formulations are compared by Davies and Pfenning [7] for the case of S4 modality, and they prove both formulations are logically equivalent.

2.2 Contextual Modal Type Theory

There is some work that generalizes box modality with environment [10,14,19]. CMTT by Nanevski et al. [15] provides logical foundations for those calculi. CMTT generalizes S4 modality of dual-context into contextual modality. $\lambda_{\forall[1]}^\forall$ is a Fitch-style reconstruction of CMTT, and hence it allows K, T, K4 and S4 modalities.

Fitch-style formulation makes it easier to reason calculi with quasiquotation from the viewpoint of contextual modality. $\lambda_{\forall[1]}^{\text{open}}$ and $\lambda_{\forall[1]}^{\text{poly}}$ by Kim et al. [16] are multi-stage calculi with quasiquotation syntax, and $\lambda_{\forall[1]}^{\text{poly}}$ can be regarded as their logical foundations. Particularly, $\lambda_{\forall[1]}^{\text{open}}$ introduces let-polymorphism for type environment, which is the special case of polymorphic contexts in $\lambda_{\forall[1]}^\forall$. $\lambda_{\forall[1]}^{\text{open}}$ and $\lambda_{\forall[1]}^{\text{poly}}$ by Rhiger [19] are also multi-staged calculi similar to both $\lambda_{\forall[1]}^\forall$ and $\lambda_{\forall[1]}^{\forall}$.

Contextual modal types and linear temporal types are similar in that they handle open code. In fact, the existence of embeddings from $\lambda_{\forall[1]}^\forall$ into $\lambda_{\forall[1]}^{\text{open}}$ and $\lambda_{\forall[1]}^{\text{poly}}$ was claimed. However, the former embedding is claimed to be unsound by Davies [6]. The latter embedding to Rhiger’s $\lambda_{\forall[1]}^{\text{open}}$ is also unsound. One of the counterexamples is the $\lambda_{\forall[1]}^\forall$ type judgment $\vdash \lambda x:\langle \square S \to \square T \rangle. \langle \lambda y:S,xy\rangle:(\square S \to \square T) \to \square(S \to T)$, which is translated into an ill-typed judgment. Therefore our translation into K $\lambda_{\forall[1]}^\forall$ is the first work that formalizes the relationship between contextual modal types and linear temporal types.

Contextual modality is also applied in proof assistants. Nanevski et al. provide contextual extension of dependent types in their work of CMTT. Cave and Pientka [22] develop it as the basis of Beluga, a programming language for theorem proving. Beluga can type object-level LF terms with contextual types, and it uses first-class contexts as meta-level values. First-class contexts of Beluga are similar to polymorphic contexts in $\lambda_{\forall[1]}^\forall$; we consider there may be close a relationship between them.
3 Fitch-Style Contextual Modal Type Theory

In this section, we introduce Fitch-style contextual modal type theory $\lambda[\Box]$. First, we informally explain quasiquotation syntax and Fitch-style hypothetical judgment. Then we give a formal description of the calculus and its properties. Finally, we show examples.

3.1 Quasiquotation with Explicit Environment

Quasiquotation plays an essential role in syntactic metaprogramming, especially in macros [11, 16]. However, a naive implementation of quasiquotation causes unintended variable capture. The bind macro in Common Lisp reveals the problem.

\begin{verbatim}
(defmacro bind (x) '(lambda (y) ,x))
\end{verbatim}

This macro takes a code fragment $x$ and embed it into a lambda expression. Therefore (bind (+ x y)) will expand to (lambda (y) (+ x y)), where $y$ is captured by the introduced lambda expression. The problem is that this bind macro captures free variables no matter whether the programmer intends. In practice, Common Lisp provides gensym which generates fresh identifier which does not conflict with existing identifiers. Our solution to this problem is to annotate quasiquotation with explicit environment. In this approach, quoted codes come with their environments like '<x y>(+ x y). Here, the part <x y> represents free variables of code. Unquotation comes with definition of free variables like ,<z (+ z w)>x. Therefore, ,<z (+ z w)>'<x y>(+ x y) is equivalent to (+ z (+ z w)). With this idea, the bind macro can be rewritten as follows.

\begin{verbatim}
(defmacro bind1 (x) '<y>(lambda (z) ,<y z>x))
(defmacro bind2 (x) '<y z>(lambda (w) ,<y z>x))
\end{verbatim}

Both macros assume that $x$ is a code with two free variables and unquote $x$ with instantiating its free variables with $y$ and $z$. bind1 binds the second free variable in $x$ while bind2 introduces a lambda abstraction which binds nothing.

Still, those macros produce broken codes given unintended inputs e.g. with free variables not listed in the annotations, or given inconsistent terms, e.g., with quotations and unquotations with unmatched numbers of variables in the annotations. To avoid this, we propose adequate type systems by exploiting Fitch-style hypothetical judgments.

3.2 Fitch-style Formulation of Contextual Modality

In the long history of modal logic, many formulations of natural deduction were proposed. Some work extends hypothetical judgment to handle multiple meta-levels [1, 7, 12, 18] although they differ in notation. We call this idea Fitch-style following Clouston [4].

In natural deduction for propositional logic, a hypothetical judgment $\Gamma \vdash A$ states that “$A$ holds assuming $\Gamma$”. Fitch-style formulation generalizes this to have a context stack as following.

\begin{verbatim}
\Gamma_0; \ldots; \Gamma_2; \Gamma_1 \vdash A
\end{verbatim}

4
In this judgment, each context in the context stack is an assumption for each meta-level: \( \Gamma_1 \) for object level, \( \Gamma_2 \) for meta level, \( \Gamma_3 \) for meta-meta level and so on.

Using Fitch-style hypothetical judgments, contextual modality is defined by following rules. Here we write \( \Psi \) for a context stack. \( \llbracket \cdot \rrbracket \) -introduction states that object-level hypothetical judgment \( \Gamma \vdash A \) corresponds to contextual modality \( \llbracket \Gamma \rrbracket A \). \( \llbracket \cdot \rrbracket \) -elimination is defined so that they make proof-theoretic harmony.

\[
\begin{align*}
\frac{\Psi, \Gamma \vdash A}{\Psi \vdash \llbracket \Gamma \rrbracket A} \quad & \frac{\Psi \vdash \llbracket B_1, \ldots, B_n \rrbracket A}{\Psi, \Gamma \vdash A} \quad \text{for } 1 \leq i \leq n
\end{align*}
\]

It is worth noting that this rules corresponds to K modal logic when \( \Gamma \) is always empty. We can also obtain T, K4 and S4 contextual modality slightly changing the elimination rule ins the same way as Martini and Masini’s method [12].

3.3 Syntax

The syntax of \( \lambda_{\llbracket \cdot \rrbracket} \) is given as follows. We write \( x,y \) for variables and \( n,m \in \mathbb{N} \).

| Types     | \( S,T \) ::= \( \tau \mid S \to T \mid \llbracket \vec{S} \rrbracket T \) |
|-----------|-------------------------------------------------|
| Terms     | \( M,N \) ::= \( x \mid \lambda x : T.M \mid M N \mid \iota(\Gamma)M \mid \eta(\vec{N})M \) |
| Context   | \( \Gamma \) ::= \( \cdot \mid \Gamma, x : T \) |
| Type Sequence | \( \vec{S}, \vec{T} \) ::= \( \cdot \mid \vec{S}, T \) |
| Term Sequence | \( \vec{M}, \vec{N} \) ::= \( \cdot \mid \vec{M}, N \) |
| Context Stack | \( \Psi \) ::= \( \cdot \mid \Psi ; \Gamma \) |

For a context \( \Gamma \), we write \( \text{dom}(\Gamma) \) for the domain (i.e., the list of variables) of \( \Gamma \), and \( \text{rg}(\Gamma) \) for the range (i.e., the list of types) of \( \Gamma \). Any context is required that its domain is distinct. Note that it is acceptable that the domains of contexts in a context stack overlap, like \( x : S ; x : T, y : U \). In \( \lambda_{\llbracket \cdot \rrbracket} \), free variables have their levels: \( \text{FV}_l(M) \) is the set of level-\( l \) free variables in \( M \), which corresponds to the \( l \)-th context in the context stack.

A type is either a base type, a function type or a contextual modal type. In addition to the standard \( \lambda \)-terms, two terms are added. A quotation \( \iota(\Gamma)M \) stands for code, where \( \Gamma \) binds \( M \). An unquotation \( \eta(\vec{N})M \) stands for code evaluation. Term/type sequence are also added to the syntax. We assume that \( \alpha \)-renaming is implicitly performed whenever it is necessary to avoid variable capture.

3.4 Type System

Figure [4] shows the typing rules in \( \lambda_{\llbracket \cdot \rrbracket} \). The judgment \( \Psi \vdash M : T \) denotes that \( M \) has type \( T \) under the context stack \( \Psi \), and the judgment \( \Psi \vdash \vec{M} : \vec{T} \) denotes that each term of \( \vec{M} \) has each type of \( \vec{T} \). In the rules (Var), (Abs), and (App), variables in the object-level context are only concerned. The rules (Quo) and (Unq) correspond to introduction and elimination of contextual modal types.

By slightly changing the rule, We can think of four variations of (Unq). \( K \) is the most basic variant where \( n \) is always 1. T variant allows \( n = 0 \), K4 variant allows \( n > 0 \), and S4 variant allows both. We write \( \Psi \vdash_{\llbracket \cdot \rrbracket} M : T \) when \( \Psi \vdash M : T \) holds especially in \( \lambda_{\llbracket \cdot \rrbracket} \) of variant \( A \) for each \( A \in \{ K, T, K4, S4 \} \).
There are two sorts of meta operations in \( \lambda I \): substitution and level-substitution. We omit the definitions, which is obtained by extending those of Fitch-style modal calculus \([7,12]\). For \( l \geq 1 \), a substitution \([N_1/x_1, \ldots, N_m/x_m]\) is a meta operation that maps a term to a term. It substitutes level-\( l \) free variables \( x_1, \ldots, x_m \) in a term with terms \( N_1, \ldots, N_m \), respectively. We also write \([\vec{N}/\Gamma]\) for point-wise composition when \( \vec{N} \) and \( \Gamma \) are the same length.

The following substitution lemma formally states the property of substitution.

**Lemma 3.1 (Substitution Lemma).** If \( \Psi; \Gamma_i; \Gamma_{i-1}; \ldots; \Gamma_1 \vdash M : T \) and \( \Psi; \Gamma_i'; \Gamma_{i-1}; \ldots; \Gamma_1 \vdash \vec{N} : \vec{T} \) where \( \{ n = 1 \text{ for } K, \ n = 0, 1 \text{ for } T \} \), then \( \Psi; \Gamma_i; \Gamma_{i-1}; \ldots; \Gamma_1 \vdash M[\vec{N}/\Gamma] : T \).

For \( l \geq 1 \) and \( n \geq 0 \), a level substitution \( \uparrow^n I \) is a meta operation that maps a term to a term. Proof theoretically, a level substitution manipulates the structure of the context-stack. The following lemma formally states this idea. Note that this lemma varies among four variations K, T, K4, and S4.

**Lemma 3.2 (Level Substitution Lemma).** If \( \Psi; \Gamma_0; \Delta_i; \ldots; \Delta_t \vdash M : T \) holds, then \( \Psi; \Gamma_0; \Delta_i; \ldots; \Delta_n, \Delta_1; \ldots; \Delta_t \vdash M \uparrow^n I : T \) also holds for any \( \Gamma_1 \ldots \Gamma_n \) if \( \text{dom}(\Gamma_n) \cap \text{dom}(\Delta_i) = \emptyset \) and \( n = 1 \) in \( K \), \( n = 0, 1 \) in \( T \), \( n \geq 1 \) in \( K4 \), and \( n \geq 0 \) in \( S4 \).

### 3.5 Reduction and Expansion

Now we are ready to define \( \beta \)-reduction and \( \eta \)-expansion rules.

**Definition 3.3.** \( \beta \)-reduction \( \rightarrow_\beta \) and \( \eta \)-expansion \( \rightarrow_\eta \) are the relations on terms which are closed under the following rules and congruence rules, which are
We show some examples of $\lambda$-terms. We omit type annotation of $\lambda$-abstraction due to the limitation of space. (1-3) correspond to the contextual variant of modal axioms K, T and 4. Contextual modality may be viewed as a metatheoretical framework of propositional logic. (4-6) represent weakening, exchange, and contraction of hypothetical judgment.

(1) $\vdash \lambda x. \lambda y. (\langle z: S \rangle (, 1(z)x, 1(z)y) : [U][S \rightarrow T] \rightarrow [U]S \rightarrow [U]T$

(2) $z : S \vdash T. \lambda x., \eta(z)x : [S]T \rightarrow T$

(3) $\vdash_{K4} \lambda x. (\langle z: U \rangle (, 2(y)x) : [S]T \rightarrow [U][S]T$

(4) $\vdash \lambda x. (\langle z: U, y:S \rangle (, 1(y)x) : [S]T \rightarrow [U,S]T$

(5) $\vdash \lambda x. (\langle y: S, z:T \rangle (, 1(z)y) : [S,T]U \rightarrow [T,S]U$

(6) $\vdash \lambda x. (\langle y: S \rangle (, 1(y)x) : [S,S]T \rightarrow [S]T$

From the viewpoint of syntactic metaprogramming, $\lambda_1$ can handle open code. Suppose $\lambda_1$ has the `if` statement and the boolean type. The `$\text{or}$` macro, which takes two code fragments $x$ and $y$ and produces `if $x$ then true else $y$`, can be represented as $\text{or} = \lambda x. \lambda y. (\langle z: T \rangle (\langle i(z)x \text{ then true else } i(z)y)$. This `$\text{or}$` macro has the type $[T]\text{bool} \rightarrow [T]\text{bool} \rightarrow [T]\text{bool}$, where $x$ and $y$ are code fragments of type `$\text{bool}$` under the context $[T]$. We can also represent the `$\text{bind}$` macro, which was introduced in the beginning of this section, as the following $\lambda_1$ terms:

$$\text{bind1} = \lambda x. (\langle y: S \rangle \lambda z., 1(y,z)x) \quad \text{bind2} = \lambda x. (\langle y: S, z:T \rangle \lambda w., 1(y,z)x)$$

$\text{bind1}$ has type $[S,T]U \rightarrow [S][T \rightarrow U]$, and $\text{bind2}$ has type $[S,T]U \rightarrow [S,T][W \rightarrow U]$. Given a code fragment $x$ of type $[S,T]U$, $\text{bind1}$ binds the free variable in $x$ of type $S$ while $\text{bind2}$ binds nothing.
4 Polymorphic Context

While contextual modal types enable us to handle open code with explicit context, the type system is too strict to be used in practice. Let us review the example of the or macro. Its type \[ [T]boo → [T]boo → [T]boo \] only accepts code with context \([T]\), so we have to define different functions for all possible contexts. This inflexibility is critical when we attempt to apply contextual modality in real-world code generation.

To solve this problem, we introduce polymorphic context to contextual modality. Using polymorphic context, the type of the generalized or macro is generalized as \[ \forall \gamma.[\gamma]boo → [\gamma]boo → [\gamma]boo \]. In this type, \(\gamma\) is a context variable which stands for an arbitrary context and \(\forall\gamma\) quantifies occurrences of \(\gamma\). This or macro accepts code with arbitrary context by substituting \(\gamma\).

From the viewpoint of the Curry-Howard isomorphism, polymorphic context is interpreted as the internalization of context weakening. In natural deduction for propositional logic, weakening can be written as the following statement.

**Weakening** If \(\Gamma \vdash T\) holds then \(\Delta,\Gamma \vdash T\) also holds for any context \(\Delta\).

Here, a meta variable \(\Delta\) stands for arbitrary context irrelevant to the proof. We gain polymorphic context by embedding this meta variable into the object system of natural deduction. As a result, context weakening is written as \(\delta,\Gamma \vdash T\) where \(\delta\) is now an object variable.

When we assign terms, we obtain another sort of variables, weakening variables which we denote by \(i\) or \(j\). In a type system, contexts come with variables which label each assumption. This principle is also applied to context variables. For example, assigning terms to a judgment \(\gamma, S, T \vdash U\), we obtain the following type judgment.

\[ i; \gamma, x; S, y; T \vdash M; U \]

A weakening variable \(i\) stands for a sequence of variables distinct from the rest of the context \((x\) and \(y\) in this case), and therefore \(i; \gamma\) stands for a context. When we substitute \(\gamma\) with some context, \(i\) is also replaced by a sequence of variables which are fresh to \(x\) and \(y\). Therefore we gain the following judgment substituting \(\gamma\) with \(P, Q\).

\[ v; P, w; Q, x; S, y; T \vdash M; U \]

This is the basic concept of polymorphic context. In the rest of this section, we give formal definition and properties of \(\lambda_W[\_]\) as an extension of \(\lambda_\_\) with polymorphic context. \(\lambda_W[\_]\) subsumes hypothetical judgment as quotation and unquotation, and therefore the discussion on context variables in hypothetical judgment is generalized to terms and types.

4.1 Syntax

Let \(CVar\) be the set of context variables and \(WVar\) the set of weakening variables. We use meta variables \(\gamma, \delta \in CVar\), and \(i, j \in WVar\). \(\lambda_W[\_]\) terms and types are defined as follows in addition to \(\lambda_\_\).

The type \(\forall \gamma.T\) is polymorphic context type which abstracts a context variable while a context abstraction \(\Lambda \gamma.M\) and a context application \(M[\delta]\) correspond
Types ∶∶
\[ S, T, U \vdash \ldots \mid \forall \gamma.T \]
Terms ∶∶
\[ M, N, L \vdash \ldots \mid \Lambda \gamma.M \mid M[\bar{T}] \]
Type Sequence ∶∶
\[ \bar{S}, \bar{T}, \bar{U} \vdash \ldots \mid \bar{S}, \gamma \]
Term Sequence ∶∶
\[ \bar{M}, \bar{N}, \bar{L} \vdash \ldots \mid \bar{M}, i \]
Context ∶∶
\[ \Gamma, \Delta \vdash \ldots \mid \Gamma, i; \gamma \]

The additional typing rules of \( \lambda\forall \) are shown in Figure 2. For the sequence judgment, \((\text{SeqC})\) is added for weakening variables and context variables. This addition indirectly changes the \((\text{Quo})\) and \((\text{Unq})\) rules. For the term judgment, \((\text{Poly})\) and \((\text{Inst})\) rules are added which corresponds to the introduction and elimination rules for polymorphic context type. Those rules are similar to the rules for polymorphic type in System F [20]: a context variable \( \gamma \) can be quantified when there is no free occurrence in the context stack, and a quantified context variable can be replaced by any type sequence.

4.2 Substitution

In \( \lambda\forall \), there are three kinds of substitution. \textit{Level substitution} is almost the same as that of \( \lambda_1 \). \textit{Term substitution} is obtained by generalizing substitution in \( \lambda_1 \). In \( \lambda\forall \), substitution content \( \sigma \) is defined inductively as follows.

\[ \sigma = \cdot \mid \sigma, M/x \mid \sigma, \bar{M}/i \]

Given a substitution content \( \sigma \) and a level \( l \), a term substitution \([ \sigma ]_l \) is a meta-operation on terms. We omit the definition because it is almost the same as substitution in \( \lambda_1 \).
Given a type sequence $\vec{S}$ and a context variable $\gamma$, context substitution $[\vec{S}/\gamma]$ is a meta operation inductively defined on $\lambda_{\forall[]}_{\vec{S}}$ objects (see Figure 3). Although most of the cases are straightforward, the definition for the case of a quotation is quite complicated. The basic idea is that to replace free occurrences of $\gamma$ in $\Gamma$ is to \textit{weaken} the context $\Gamma$. $\textit{weaken}_{\vec{S},\gamma}$ performs weakening inductively and returns a pair of a context and a substitution content $(\Delta, \sigma)$ where $\Delta$ stands for the replaced context, and $\sigma$ for how weakening variables corresponding to $\gamma$ are to be substituted with fresh variables.

Given a type sequence $\vec{S}$ and a context variable $\gamma$, $\textit{weaken}_{\vec{S},\gamma}$ is defined using some auxiliary functions. $\textit{weaken}_{\vec{S},\gamma}$ is the main component of $\lambda_{\forall[]}_{\vec{S}}$. Given a finite set of variables and fresh variables $V$, $\textit{genSym}_{V}$ generates a sequence of variables which are fresh with respect to $V$.

We also define context substitution on $\lambda_{\forall[]}_{\vec{S}}$ type judgment.

$$(\Gamma_1; \ldots; \Gamma_1 \vdash M : T)[\vec{S}/\gamma] = \Delta_1; \ldots; \Delta_1 \vdash M[\vec{S}/\gamma][\sigma_1]; \ldots; [\sigma_1]; \gamma : T[\vec{S}/\gamma]$$

where $(\Delta_i, \sigma_i) = \textit{weaken}_{\vec{S}_{\gamma}}(\Gamma_i)$

As a result, the following context substitution lemma holds.

**Lemma 4.1** (Context Substitution Lemma). If $\Psi \vdash M : T$ holds, $(\Psi \vdash M : T)[\vec{S}/\gamma]$ holds for any $\vec{S}$ and $\gamma$.

### 4.3 Reduction/Expansion Rules

Given a context $\Gamma$ and a term sequence $\vec{M}$, we define a substitution $\vec{M}/\Gamma$ where $\cdot / \cdot = \cdot$, $(\vec{M}, N)/\Gamma = \vec{M}/\Gamma, N/x$ and $(\vec{M}, i)/\Gamma = \vec{M}/\Gamma, i/j$.

**Definition 4.2.** $\beta$-reduction $\rightarrow_{\beta}$ and $\eta$-reduction $\rightarrow_{\eta}$ are the relations closed under the following rules and congruence rules, which we omit here.

$$(\lambda x : T. M)N \rightarrow_{\beta} M[N/x]_1$$

$$(\cdot)

\text{when $\vec{N}/\Gamma$ is defined}

\text{when $\Psi \vdash M : T \rightarrow S$}

\text{when $\Psi \vdash M : [\vec{T}]S$ and $rg(\Gamma) = \vec{T}$}

\text{when $\Psi \vdash M : \forall \delta S$.}$$

\begin{align*}
\psi \vdash & \vec{M} : \vec{T} \\
\psi \vdash & M : T
\end{align*}
As in Section 3, λ

Type
(Subject Reduction/Expansion)

\(\tau[\tilde{S}/\gamma] = \tau\)

\((U \rightarrow T)[\tilde{S}/\gamma] = U[\tilde{S}/\gamma] \rightarrow U[\tilde{S}/\gamma]\)

\(((\tilde{U}T))[\tilde{S}/\gamma] = \tilde{U}[\tilde{S}/\gamma]T[\tilde{S}/\gamma]\)

\((\forall \tilde{\delta}.T)[\tilde{S}/\gamma] = \begin{cases} 
\forall \tilde{\delta}.T \quad & \text{when } \gamma = \delta \\
\forall \tilde{\delta}.(T[\tilde{S}/\gamma]) \quad & \text{otherwise}
\end{cases}\)

\(\text{Type Sequence, Term Sequence (omitted: defined point-wise)}\)

\((\Delta \delta.M)[\tilde{S}/\gamma] = \begin{cases} 
\Delta \delta.M \quad & \text{when } \gamma = \delta \\
\Delta \delta.(M[\tilde{S}/\gamma]) \quad & \text{otherwise}
\end{cases}\)

\(\text{weaken}_{\gamma,\tilde{S}}(\cdot, \Delta, \sigma) = (\Delta, \sigma)\)

\(\text{weaken}_{\gamma,\tilde{S}}((\Gamma, i; \delta), \Delta, \sigma) = \begin{cases} 
\text{weaken}_{\gamma,\tilde{S}}((\Gamma, \Delta'), (\sigma, \text{dom}(\Delta')/i)) \quad & \text{if } \gamma = \delta \\
\text{where } \Delta' = \text{genSym}_{(\text{dom}(\Gamma') \cup \text{dom}(\Delta))}(\tilde{S}) \\
\text{weaken}_{\gamma,\tilde{S}}((\Gamma, \Delta, i; \delta), \sigma) \quad & \text{otherwise}
\end{cases}\)

\(\text{genSym}_{\mathcal{V}}(\cdot) = \cdot\)

\(\text{genSym}_{\mathcal{V}}((\tilde{S}, T)) = \text{genSym}_{\mathcal{V} \cup \{i\}}(\tilde{S}), x:T \text{ for some } x \notin V\)

\(\text{genSym}_{\mathcal{V}}((\tilde{S}, \gamma)) = \text{genSym}_{\mathcal{V} \cup \{i\}}(\tilde{S}), i: \gamma \text{ for some } i \notin V\)

\(\text{weaken}_{\gamma,\tilde{S}}(\Gamma) = \text{weaken}_{\gamma,\tilde{S}}(\Gamma, \cdot, \cdot)\)

Figure 3: Context Substitution

**Theorem 4.3** (Subject Reduction/Expansion).  
(i) If \(\Psi \vdash M : T \text{ and } M \rightarrow_{\beta} N\), then \(\Psi \vdash N : T\).

(ii) If \(\Psi \vdash M : T \text{ and } M \rightarrow_{\eta} N\), then \(\Psi \vdash N : T\).

**4.4 Examples**

As in Section 3, \(\lambda_{\mathcal{V}[]}\) admits generalized modal axioms and metatheorems of hypothetical judgments. For example, context weakening is represented as \(\vdash \Lambda \gamma.\Lambda \delta.\lambda x.\cdot(i; \delta; \cdot; \cdot), 1(i); x) : \forall \gamma. \forall \delta. ((\gamma)T \rightarrow [\delta, \gamma]T)\).

From the viewpoint of syntactic metaprogramming, polymorphic context allows flexible manipulation of open code. As discussed in the beginning of this section, the or macro is generalized as the term \(\Lambda \gamma.\lambda x.\lambda y.\cdot(i; \gamma)\text{if } 1(i); x \text{ then true else } i(i); y\).

It is typed as \(\forall \gamma. [(\gamma)\text{bool} \rightarrow [\gamma)\text{bool} \rightarrow [\gamma)\text{bool}\text{ and thus accepts code with any environment by virtue of polymorphic context. The bind macros can also be represented as the following }\lambda_{\mathcal{V}[]}\text{ term:}\)

\(\text{bind1} = \Lambda \gamma.\Lambda \delta.\lambda x.\cdot(i; \gamma; j; \delta)\lambda y.\cdot(i; y; \cdot; \cdot)\)
\n\(\text{bind2} = \Lambda \gamma.\lambda x.\cdot(i; \gamma)\lambda y.\cdot(\iota; x)\)
\textbf{bind1} has type \( \forall \gamma. \forall \delta [\gamma, S, \delta] T \rightarrow [\gamma, \delta] (S \rightarrow T) \), and \textbf{bind2} has type \( \forall \gamma. [\gamma] T \rightarrow [\gamma] (S \rightarrow T) \). Clearly, \textbf{bind1} macro binds free variable of type \( S \) among the context \([\gamma, S, \delta]\) while \textbf{bind2} binds no variable in \([\gamma]\). The point is that weakening variables \( i, j \) will not be replaced with bound variables such as \( y \) due to capture avoiding substitution. This systematically prevents macros from unintended variable capture. In other words, polymorphic context preserves lexical scoping and achieves hygienic code generation.

5 Context Extraction for Linear Temporal Type Theory

As we have seen, \( \lambda_{\forall[]} \) allows flexible manipulation of open code. In this section, we formalize the relationship between \( K \lambda_{\forall[]} \) and \( \lambda_{\bigcirc} \) by giving a sound translation from \( \lambda_{\bigcirc} \) to \( K \lambda_{\forall[]} \), which extracts implicit contexts of \( \lambda_{\bigcirc} \). We call this translation context extraction. The key observation is the following construction:

\[ \vdash \lambda x. \langle (i : \delta) \lambda y, i, (i, y) (x \delta, S) \rangle : \langle \forall \gamma. [\gamma] S \rightarrow [\gamma] T \rangle \rightarrow [\delta] (S \rightarrow T) \]

This suggests \( \lambda_{\forall[]} \) has some relation to \( \lambda_{\bigcirc} \), which is characterized by the axiom \( \bigcirc A \rightarrow \bigcirc B \rightarrow \bigcirc (A \rightarrow B) \). We further analyze this relation in the following.

5.1 \( \lambda_{\bigcirc} \) as Fitch-style Linear Temporal Type Theory

Figure 4 shows an overview of the type theory \( \lambda_{\bigcirc} \). Although this definition is arranged in accordance with \( K \lambda_{\forall[]} \) and \( \lambda_{\bigcirc} \) by giving a sound translation from \( \lambda_{\bigcirc} \) to \( K \lambda_{\forall[]} \), which extracts implicit contexts of \( \lambda_{\bigcirc} \). We call this translation context extraction. The key observation is the following construction: with K-S4 modality, it is known that the proposition \( (\Box A \rightarrow \Box B) \rightarrow \Box (A \rightarrow B) \) is not provable. However, a very similar proposition is, in fact, provable with polymorphic context.

\[ \vdash \lambda x. \langle (i : \delta) \lambda y, i, (i, y) (x \delta, S) \rangle : \langle \forall \gamma. [\gamma] S \rightarrow [\gamma] T \rangle \rightarrow [\delta] (S \rightarrow T) \]

This suggests \( \lambda_{\forall[]} \) has some relation to \( \lambda_{\bigcirc} \), which is characterized by the axiom \( (\bigcirc A \rightarrow \bigcirc B) \rightarrow \bigcirc (A \rightarrow B) \). We further analyze this relation in the following.

5.2 Context Extraction

First we introduce the notion of context/type sequence allocator. We fix \( Ctx \) for the set of \( \lambda_{\forall[]} \) context, and \( Typ \) for the set of \( \lambda_{\forall[]} \) type sequence. We also write \( \Gamma \uplus \Delta \) and \( \vec{S} \uplus \vec{T} \) for the concatenation of \( \Gamma \) and \( \Delta \), \( \vec{S} \) and \( \vec{T} \).

\textbf{Definition 5.1} (Context / Type Sequence Allocator). (1) A context allocator is a function \( \mathbb{N} \rightarrow Ctx \). For context allocators \( P \) and \( Q \), we define the following operations on context allocators.

(a) \( P \uparrow \) is a context allocator where \( P \uparrow (x) = P(x + 1) \)
Typing Rules

Definition 5.2. (1) For a \( \lambda_{\Diamond} \) type \( T_s \), the context depth of \( T_s \), written as \( D(T_s) \), is inductively defined as follows:

\[
D(\tau) = 0 \quad D(S_a \rightarrow T_s) = D(T_s) \quad D(\Diamond T_s) = D(T_s) + 1
\]

(2) For \( n \in \mathbb{N} \), \( GenCa(n) \) generates a pair of a context allocator and a sequence of context variables:

\[
GenCa(n) = (P, (\gamma_1, \ldots, \gamma_n)) \quad where \quad P(x) = \begin{cases} \gamma_x & \text{if } 1 \leq x \leq n \\ \epsilon(x) & \text{otherwise} \end{cases}
\]

for fresh variables \( \gamma_1, \ldots, \gamma_n \) and \( i_1, \ldots, i_n \).

Now we are ready to define a translation from \( \lambda_{\mathcal{F}[1]} \) types to \( \lambda_{\Diamond} \) types. We use the following abbreviations for context abstraction/application. Given a
sequence of context variables $\vec{\gamma} = \gamma_1, \ldots, \gamma_n$, we write $\forall \vec{\gamma}. T$ for $\forall \gamma_1, \ldots, \forall \gamma_n. T$ and $\Lambda \vec{\gamma}. M$ for $\Lambda \gamma_1, \ldots, \Lambda \gamma_n. M$. Given a type sequence allocator $\tilde{P}$, we write $M[\tilde{P}]_n$ for $M[P(0)] \ldots [P(n-1)]$.

**Definition 5.3** (Context Extraction for Linear Temporal Types). Given a type sequence allocator $\tilde{P}$, we define a translation $\langle \gamma \rangle_{\tilde{P}}$ from $\lambda_\circ$ types to $\lambda_{\forall[1]}$ types:

$$
\langle S_n \rightarrow T_n \rangle_{\tilde{P}} = \forall \vec{\gamma}. \langle S_n \rangle_{rg(Q)} \rightarrow \langle T_n \rangle_{\tilde{P}} \cdot \langle Q \cdot \vec{\gamma} \rangle = GenCa(D(S_n)) \quad \langle \forall \vec{\gamma}. T_n \rangle_{\tilde{P}} = \tau
$$

where $(Q, \vec{\gamma}) = GenCa(D(S_n))$ and we also use this notation to define context extraction for contexts and context stacks. $GenCa$ may return different results because it generates “some fresh variables”, and therefore generative translation is useful to keep track of what is generated.

**Definition 5.4** (Context Extraction for Contexts). (1) A generative translation $\Gamma \leftarrow \langle \Gamma_s \rangle^P$ states that a $\lambda_\circ$ context $\Gamma_s$ is translated to a $\lambda_{\forall[1]}$ context $\Gamma$ with generating context an allocator $P$. This judgment is derived by the following rules:

$$
\frac{\cdot \leftarrow \langle \gamma \rangle^P}{\Gamma \leftarrow \langle \Gamma_s \rangle^P \quad T \leftarrow \langle T_n \rangle_{\tilde{P}}^{Q, \vec{\gamma}} \quad x \notin dom(\Gamma)}
$$

(2) A generative translation $\Psi \leftarrow \langle \Psi_s \rangle^P$ states that a $\lambda_\circ$ past context stack $\Psi_s$ is translated to a $\lambda_{\forall[1]}$ context stack $\Psi$ with generating context allocator $P$. This judgment is derived by the following rules:

$$
\frac{\cdot \leftarrow \langle \gamma \rangle^P}{\Psi \leftarrow \langle \Psi_s \rangle^P \quad \Gamma \leftarrow \langle \Gamma_s \rangle^Q \quad \Psi; P(0), \Gamma \leftarrow \langle \Psi_s; \Gamma_s \rangle^{P \oplus Q}}
$$

(3) A generative translation $P \leftarrow \emptyset$ states that a $\lambda_\circ$ future context stack $\Theta_s$ is translated to a context allocator $P$. This judgment is derived by the following rules:

$$
\frac{\cdot \leftarrow \emptyset}{\Gamma \leftarrow \langle \Gamma_s \rangle^P \quad Q \leftarrow \emptyset \quad \Gamma \leftarrow \emptyset \cdot \emptyset = \emptyset}
$$

Then we define a term translation which corresponds to the type translation. We use auxiliary $\lambda_{\forall[1]}$ terms $weak^T_{P,Q}, contr^T_{P,Q}$, $exchg^T_{P,Q,P',Q'}$, which satisfy the following properties. We omit the definition due to space limitation.
Lemma 5.5. (1) If $P(x) \subseteq Q(x)$ for all $x \in \mathbb{N}$, $\vdash \text{weak}^T_{P,Q} \langle T_\gamma \rangle_{r\gamma(P)} \rightarrow \langle T_\gamma \rangle_{r\gamma(Q)}$.

(2) $\vdash \text{cont}^T_{P,Q} \langle T_\gamma \rangle_{r\gamma(P\otimes P\otimes Q)} \rightarrow \langle T_\gamma \rangle_{r\gamma(P\otimes Q)}$.

(3) $\vdash \text{exch}^T_{P,Q,P',Q'} \langle T_\gamma \rangle_{r\gamma(P\otimes P\otimes Q\otimes Q')} \rightarrow \langle T_\gamma \rangle_{r\gamma(P\otimes P\otimes Q\otimes Q')}$

To simplify the definition of translation, we annotate a $\lambda_\square$ term $M_\square$ with its type $T_\square$, like $M_{\square,T_{\square}}$. We can safely introduce this modification because we can infer annotations from a derivation tree.

Definition 5.6 (Context Extraction for Terms and Judgments). (1) Given a generative translations $\Psi \triangleq \Psi_x^P$ and $Q \triangleq |\Theta_x|$, we define a translation

\[
\text{exch}_{P,Q}^T \Psi \triangleq \Psi_x^P
\]

from type-annotated $\lambda_\square$ terms to $\lambda_\square$ terms:

\[
\Psi(\cdot) \triangleq \text{weak}^T_{R,\Phi \otimes Q} \quad \text{where } (R \otimes Q) \triangleq \Gamma \triangleq |\Theta_x|, \quad Q \triangleq |\Theta_x|.
\]

\[
\langle (\lambda x : S_x.M_x)^T \rangle_{Q = |\Theta_x|} = \text{weak}^T_{R,\Phi \otimes Q} \quad \text{where } (R \otimes Q) \triangleq \Gamma \triangleq |\Theta_x|, \quad Q \triangleq |\Theta_x|.
\]

\[
\langle (\lambda x : S_x.M_x)^T \rangle_{Q = |\Theta_x|} = \text{weak}^T_{R,\Phi \otimes Q} \quad \text{where } (R \otimes Q) \triangleq \Gamma \triangleq |\Theta_x|, \quad Q \triangleq |\Theta_x|.
\]

\[
\langle (\lambda x : S_x.M_x)^T \rangle_{Q = |\Theta_x|} = \text{weak}^T_{R,\Phi \otimes Q} \quad \text{where } (R \otimes Q) \triangleq \Gamma \triangleq |\Theta_x|, \quad Q \triangleq |\Theta_x|.
\]

We can confirm the soundness of context extraction by induction on derivation trees in $\lambda_\square$. As a result, the following theorem holds.

Theorem 5.7. If $\Gamma_\square \mid \Theta_\square \vdash M_\square : T_\square$ holds in $\lambda_\square$ then $\langle \Gamma_\square \mid \Theta_\square \vdash M_\square : T_\square \rangle$ also holds in $K \lambda_\square$.

6 Conclusion

In this paper, we have proposed a novel type theory $\lambda_\square$ and its extension $\lambda_\square$. $\lambda_\square$ is a Fitch-style reconstruction of contextual modal type theory, which was introduced by Nanevski et al. as a logical foundation for explicit substitution and meta-variables. Thanks to the reconstruction, the contextual modality is easily generalized to weaker modality such as $K$. This enables us to model computation in syntactic meta-programming including macro systems using contextual
modality since macro systems do not involve eval-like cross-level computation in
general. To obtain more expressivity, we introduced polymorphism over contexts
in modality into $\lambda[]$. We finally proved that there is a sound translation from
$\lambda_\Box$ to $\lambda_V[]$, which we think asserts $\lambda_V[]$ is indeed useful as a logical foundation
for syntactic meta-programming.

There are some topics we have left for future work. In this paper, we have
proved strong normalization and confluency for $\lambda[]$ but not for $\lambda_V[]$. We think
this can be proved through a translation to System F [20]. Also, it is unclear
whether the translation from $\lambda_\Box$ to $\lambda_V[]$ preserves reductions. To identify the
expressive power of $\lambda_V[]$, we need to establish a method to compare $\lambda_V[]$ with
$\lambda^\bowtie$, which is a type theory extending $\lambda_\Box$ and is also known to soundly embed
$\lambda_\Box$ into itself. We also expect that it would be possible to define translations
between the S4 fragment of $\lambda[]$ and the original CMTT by Nanevski et al. using
the technique used to prove the equivalence between the Fitch-style S4 and
dual-context S4 calculi.

Acknowledgements.

We would like to thank Yoshihiko Kakutani and Masami Hagiya for valuable
discussions and comments. This work is based on our work-in-progress paper
“Kripke-style Contextual Modal Type Theory” presented at LFMTP 2017.

References

[1] Tijn Borghuis. Modal Pure Type Systems. Journal of Logic, Language and
Information, 7(3):265–296, 1998.

[2] Andrew Cave and Brigitte Pientka. Programming with binders and indexed
data-types. ACM SIGPLAN Notices, 47:413, 2012.

[3] Andrew Cave and Brigitte Pientka. First-class substitutions in contextual
type theory. the Eighth ACM SIGPLAN international workshop, page 15,
2013.

[4] Ranald Clouston. Fitch-Style Modal Lambda Calculi. 2017.

[5] Rowan Davies. A temporal-logic approach to binding-time analysis. Pro-
ceedings 11th Annual IEEE Symposium on Logic in Computer Science,
(October):184–195, 1996.

[6] Rowan Davies. A Temporal Logic Approach to Binding-Time Analysis.
Journal of the ACM, 64(1):1–45, 2017.

[7] Rowan Davies and Frank Pfenning. A Modal analysis of Staged Computation.
Journal of the ACM, 48(3):555–604, 2001.

[8] Steven E. Ganz, Amr Sabry, and Walid Taha. Macros as multi-stage computa-
tions: Type-safe, generative, binding macros in macroml. In Proceedings
of the Sixth ACM SIGPLAN International Conference on Functional Pro-
gramming, ICFP ’01, pages 74–85, New York, NY, USA, 2001. ACM.
[9] G. A. Kavvos. Dual-Context Calculi for Modal Logic. In Logic in Computer Science, Reykjavik, Feb 2017.

[10] Ik-Soon Kim, Kwangkeun Yi, and Cristiano Calcagno. A Polymorphic Modal Type System for Lisp-Like Multi-Staged Languages. ACM SIGPLAN Notices, 41(1):257–268, 2006.

[11] Geoffrey B Mainland. Why It’s Nice to be Quoted: Quasiquoting for Haskell. Applied Sciences, (August):73–81, 2007.

[12] Simone Martini and Andrea Masini. A Computational Interpretation of Modal Proofs. In Proof Theory of Modal Logic, pages 213–241. 1996.

[13] Eugenio Moggi, Walid Taha, Zine El-abidine Abidine Benaissa, and Tim Sheard. An idealized metaML: Simpler, and more expressive. Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), 1576:193–207, 1999.

[14] Aleksandar Nanevski and Frank Pfenning. Staged computation with names and necessity. Journal of Functional Programming, 15(6):893–939, 2005.

[15] Aleksandar Nanevski, Frank Pfenning, and Brigitte Pientka. Contextual modal type theory. ACM Transactions on Computational Logic, 9(3):1–49, 2008.

[16] Lionel Parreaux, Antoine Voizard, Amir Shaikhha, and Christoph Koch. Unifying Analytic and Staticnally-Typed Quasiquotes. Proceedings of the ACM on Programming Languages, 2(13):1–33, 2018.

[17] Frank Pfenning and Rowan Davies. A Judgmental Reconstruction of Modal Logic. Mathematical Structures in Computer Science, 11(4):511–540, 2001.

[18] Frank Pfenning and H. C. Wong. On a Modal λ-Calculus for S4. Electronic Notes in Theoretical Computer Science, 1(C):515–534, 1995.

[19] Morten Rhiger. Staged computation with staged lexical scope. Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), 7211 LNCS:559–578, 2012.

[20] Morten Heine Sørensen and Pawel Urzyczyn. Lectures on the Curry-Howard Isomorphism, Volume 149 (Studies in Logic and the Foundations of Mathematics). Elsevier Science Inc., New York, NY, USA, 2006.

[21] Walid Taha and Patricia Johann. Staged notational definitions. In Proceedings of the 2Nd International Conference on Generative Programming and Component Engineering, GPCE ’03, pages 97–116, New York, NY, USA, 2003. Springer-Verlag New York, Inc.

[22] Walid Taha and Michael Florentin Nielsen. Environment classifiers. ACM SIGPLAN Notices, 38:26–37, 2003.

[23] Takeshi Tsukada and Atsushi Igarashi. A Logical Foundation for Environment Classifiers. Logical Methods in Computer Science, 6(4):1–43, 2010.

17
[24] Yosihiro Yuse and Atsushi Igarashi. A modal type system for multi-level generating extensions with persistent code. Proceedings of the 8th ACM SIGPLAN symposium on Principles and practice of declarative programming - PPDP ’06, page 201, 2006.
A Omitted definitions in $\lambda[]$

A.1 Free Variables

For $l \geq 1$ and a term $M$, $FV_l(M)$ is the set of level-$l$ free variables in $M$, which is defined as follows. We also define free variables of a term sequence.

Term

$$FV_l(x) = \begin{cases} \{x\} & \text{when } l = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$FV_l(MN) = FV_l(M) \cup FV_l(N)$$

$$FV_l(\lambda x : T.M) = \begin{cases} FV_l(M) - \{x\} & \text{if } l = 1 \\ FV_l(M) & \text{otherwise} \end{cases}$$

$$FV_l(\cdot) = \emptyset \quad FV_l(M, M) = FV_l(M) \cup FV_l(N)$$

Term Sequence

$$FV_l(\cdot) = \emptyset \quad FV_l(M, N) = FV_l(M) \cup FV_l(N)$$

A.2 Meta Operations

For $l \geq 1$ and a substitution content $\sigma$, substitution application $\cdot[\sigma]_l$ is defined as follows:

Term

$$x[\sigma]_l = \begin{cases} N & \text{if } l = 1 \text{ and } N/x \in \sigma \\ x & \text{else} \end{cases}$$

$$(MN)[\sigma]_l = (M[\sigma])_l(N[\sigma])_l$$

$$(\lambda x : A.M)[\sigma]_l = \begin{cases} \lambda x : A.(M[\sigma])_l & \text{if } l = 1, \ x \notin \text{dom}(\sigma) \\ \lambda x : A.(M[\sigma])_l & \text{if } l > 1 \text{ and } x \notin FV_l(\cdot) \end{cases}$$

$$(\cdot, k(\tilde{N})M)[\sigma]_l = \begin{cases} (\cdot, k(\tilde{N}[\sigma])_l)M[\sigma]_{l-k} & \text{if } l \leq k \\ (\cdot, (\cdot, k(\tilde{N}[\sigma])_l)M[\sigma]_{l-k})_l & \text{else} \end{cases}$$

$$('\langle \Gamma \rangle M)[\sigma]_l = '\langle \Gamma \rangle (M[\sigma])_{l+1}$$

Term Sequence

$$\cdot[\sigma]_l = \cdot \quad (\tilde{M}, N)[\sigma]_l = \tilde{M}[\sigma]_l, N[\sigma]_l$$

Note that this substitution is capture-avoiding: it renames bindings of $\lambda$ abstraction when there is a name conflict. Although there is apparently no collision check for quotation, it works because the substitution and the bindings of the quotation are at different levels.

For $l \geq 1$ and $n \geq 0$, a level substitution application $\cdot \uparrow^n_l$ is defined as follows:

Term

$$x \uparrow^n_l = x \quad ('\langle \Gamma \rangle M) \uparrow^n_l = '\langle \Gamma \rangle (M \uparrow^n_l)$$
\[(MN) \downarrow^k_l = (M \downarrow^k_l)(N \downarrow^k_l)\]

\[(\lambda x:A.M) \downarrow^k_l = \lambda x:A.(M \downarrow^k_l) \quad (s^{k+(\lambda^nM)}M) \downarrow^k_l = \begin{cases} s^{k+(\lambda^nM)}M \downarrow^k_{l-k^l} & \text{when } l \leq k^l \\ s^{k+(\lambda^nM)}M \downarrow^k_l & \text{otherwise} \end{cases}\]

**Term Sequence**

\[
\cdot \downarrow^k_l = \cdot \\
(M, N) \downarrow^k_l = \tilde{M} \downarrow^k_l, N \downarrow^k_l
\]

### A.3 Congruence Rules

**Definition A.1.** Let \(\sim\) be a binary relation on terms. \(\sim\) is congruent iff it satisfies the following conditions.

\[
M_1 \sim M_2 \Rightarrow \lambda x:T.M_1 \sim \lambda x:T.M_2 \\
M_1 \sim M_2 \Rightarrow (M_1N) \sim (M_2N) \\
M_1 \sim M_2 \Rightarrow (NM) \sim (NM) \\
M_1 \sim M_2 \Rightarrow \iota(l,M_1,\hat{\emptyset})N \sim \iota(l,M_2,\hat{\emptyset})N
\]

## B Strong Normalization and Confluency of \(\lambda[\_]\)

Let \(\lambda_\_\) be the simply typed lambda calculus with implication. It is known that \(\lambda_\_\) enjoys strong normalization [20], and we define a translation from \(\lambda[\_]\) to \(\lambda_\_\) in order to prove strong normalization of \(\lambda[\_]\). We write \(\Gamma \vdash \lambda M : T\) for a type judgment of \(\lambda_\_\).

**Definition B.1.** We define \([\cdot]\), a translation from \(\lambda[\_]\) objects to those of \(\lambda_\_\):  

**Terms**  
\[
[x] = x \\
[\lambda x:T.M] = \lambda x: [T] [M] \\
[M.N] = [M][N]
\]

**Types**  
\[
[\tau] = \tau \\
[T \rightarrow S] = [T] \rightarrow [S] \\
[[T_1, \ldots, T_n]S] = [T_1] \rightarrow \cdots \rightarrow [T_n] \rightarrow [S]
\]

**Context**  
\[
[\Gamma_1, \ldots, \Gamma_n] = [[\Gamma_1], \ldots, [\Gamma_n]]
\]

where dom(\(\Gamma_i\)) \(\cap\) dom(\(\Gamma_j\)) = \(\emptyset\) for any \(1 \leq i < j \leq n\)

**Lemma B.2.** If \(\Psi \vdash M : T\) holds and \([\Psi]\) is defined, then \([\Psi]\) \(\vdash_\lambda [M] : [T]\) holds.

**Lemma B.3.** If \(\Psi \vdash M : T\) and \(M \rightarrow_\beta M'\) and \([\Psi]\) is defined, then \([M] \rightarrow_\beta^* [M']\).
Theorem B.4 (Strong Normalization). \( \rightarrow_\beta \) is strongly normalizing.

Proof. Assume that there is an infinite reduction sequence in Fitch-style. There is no infinite sequence of reduction \( \rightarrow_\beta \) from some point because this reduction strictly reduces the size of the term. Then there also exists an infinite reduction sequence in \( \lambda_\gamma \) by Lemma B.3 and this result contradicts the strong normalization of \( \lambda_\gamma \). \( \square \)

We prove confluency using Newman’s lemma \([20]\).

Lemma B.5 (Weak Confluence). If \( M \rightarrow_\beta N_1 \) and \( M \rightarrow_\beta N_2 \), there exists \( L \) such that \( N_1 \rightarrow_\beta^* L \) and \( N_2 \rightarrow_\beta^* L \).

Theorem B.6 (Confluence). If \( M \rightarrow^* N_1 \) and \( M \rightarrow^* N_2 \), there exists \( L \) such that \( N_1 \rightarrow^* L \) and \( N_2 \rightarrow^* L \).

Proof. By Theorem B.4 Lemma B.5 and Newman’s lemma \([20]\). \( \square \)

C Definition of \( weak^T_{P,Q}, contr^T_{P,Q}, exchg^T_{P,Q,P',Q'} \)

Definition C.1 (Weakening / Contraction / Exchange of Context Allocators).

(i) Given a \( \lambda_\gamma \) type \( T_x \) and context allocators \( P \) and \( Q \), \( weak^T_{P,Q} \) is a \( \lambda_\gamma[] \) term inductively defined as follows.

\[
weak^T_{P,Q} = \lambda x: \tau.x
\]

\[
weak^S_{T_x} = \lambda x: \forall \gamma.\langle S_x \rangle^{R,\gamma} \rightarrow \langle T_x \rangle^{rg(P)\equiv R,\gamma,\lambda y: \langle S_y \rangle^{rg(R)\equiv R,\gamma}}
\]

\[
weak^O_{P,Q} = \lambda x: [rg(P(0))\equiv P(0)\equiv Q(0)]\langle T_x \rangle^{rg(P\equiv P',Q\equiv Q')}\langle T_x \rangle^{rg(P\equiv P',Q\equiv Q')}
\]

(ii) Given a \( \lambda_\gamma \) type \( T \) and context allocators \( P \) and \( Q \), \( contr^T_{P,Q} \) is a \( \lambda_\gamma[] \) term inductively defined as follows.

\[
contr^T_{P,Q} = \lambda x: \tau.x
\]

\[
contr^S_{T_x} = \lambda x: \forall \gamma.\langle S_x \rangle^{R,\gamma} \rightarrow \langle T_x \rangle^{rg(P\equiv P\equiv Q\equiv Q',\gamma)}
\]

\[
contr^O_{P,Q} = \lambda x: [rg(P(0)\equiv P(0)\equiv Q(0))]\langle T_x \rangle^{rg(P\equiv P',Q\equiv Q')}\langle T_x \rangle^{rg(P\equiv P',Q\equiv Q')}
\]

(iii) Given a \( \lambda_\gamma \) type \( T \) and context allocators \( P, Q, P' \) and \( Q' \), \( exchg^T_{P,Q,P',Q'} \) is a \( \lambda_\gamma[] \) term inductively defined as follows.

\[
exchg^T_{P,Q,P',Q'} = \lambda x: \tau.x
\]

\[
exchg^S_{P,Q,P',Q'} = \lambda x: \forall \gamma.\langle S_x \rangle^{R,\gamma} \rightarrow \langle T_x \rangle^{rg(P\equiv Q\equiv Q',\gamma)}
\]

\[
exchg^O_{P,Q,P',Q'} = \lambda x: [rg(P(0)\equiv Q(0)\equiv P'(0)\equiv Q'(0))]\langle T_x \rangle^{rg(P\equiv P',Q\equiv Q')}\langle T_x \rangle^{rg(P\equiv P',Q\equiv Q')}
\]