ON FREE LOOP SPACES OF TORIC SPACES

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This paper is dedicated to Samuel Gitler Hammer who brought us much joy and interest in Mathematics.

1. Introduction

Let

\[ Z_K = Z(K; (D^2, S^1)) \]

be a moment-angle complex also known as a polyhedral product where \( K \) is a finite simplicial complex with \( m \) vertices. In the special cases for which \( K \) is a polytopal sphere, \( Z_K \) is a manifold with orbit space given by a simple convex polytope

\[ P^m(K) = Z_K/T^m \]

where the torus of rank \( m, T^m \), acts naturally on \( Z_K \). The topology/geometry of the free loop space of the Davis-Januszkiewicz space \( DJ(K) = ET^m \times T^m Z_K \) and related spaces here is tightly tied to the geometry of \( P^m(K) \). The purpose of this note is to develop the dichotomy in the next Theorem arising from \( LX \) the free loop space of a space \( X \) together with the connections to \( P^m(K) \).

**Theorem 1.1.** If \( X = Z(K; (D^2, S^1)) \), then the Hilbert-Poincaré series for the rational homology of

\[ LZ(K; (D^2, S^1)) \]

has exponential growth if and only if \( Z(K; (D^2, S^1)) \) contains a wedge of two spheres as a retract, and is thus hyperbolic. Thus the following are equivalent:

1. The Hilbert-Poincaré series for the rational homology of \( LZ(K; (D^2, S^1)) \) is a rational function.
2. The space \( Z(K; (D^2, S^1)) \) has totally finite rational homotopy groups, in other words \( Z(K; (D^2, S^1)) \) is elliptic.
The previous theorem follows at once by combining a theorem of Pascal Lambrechts [11] together with Theorem 1.3 of [2] which illustrates this dichotomy in the case $X = Z_K$. The growth of free loop spaces has also been developed in [6].

Gurvich in his thesis [9] showed that in case $K$ is polytopal sphere, then $Z_K$ is elliptic if and only if $P^m(K)$ is a product of simplices. The next corollary follows from Gurvich’s result together with Theorem 1.1.

**Corollary 1.2.** Let $K$ be a polytopal sphere, and $X = Z(K; (D^2, S^1))$. Then following are equivalent:

1. The Hilbert-Poincaré series for the rational homology of $LZ(K; (D^2, S^1))$ is a rational function.
2. The space $Z(K; (D^2, S^1))$ has totally finite rational homotopy groups, that is, $Z(K; (D^2, S^1))$ is elliptic.
3. The simple polytope $P^m(K)$ is a product of simplices.

In what follows, an analogous theorem is correct in case $Z(K; (D^2, S^1))$ is replaced by either $DJ(K)$ the associated Davis-Januszkiewicz space, a Davis-Januszkiewicz manifold given by

$$M = Z_K/T^q,$$

or mildly more general spaces. However, this seemingly direct variation of Theorem 1.1 seems to require additional work when using the methods below.

Remarks addressing earlier work on irrational Hilbert-Poincaré series follow next. J. E. Roos first proved that the Hilbert-Poincaré series for the free loop space of $S^3 \vee S^3$ is irrational [13] following Serre’s method showing that the Hilbert-Poincaré series for $\Omega^2(S^3 \vee S^3)$ is irrational [16]. One common theme here is the application of the Lech-Mahler-Skolem theorem which identifies whether certain infinite series are given by rational functions [16, 13]. However, it is unclear whether these methods extend directly to many of the cases in this paper.

A result due to Pascal Lambrechts is described next [11]. Lambrechts proves that if $X$ is a coformal, 1-connected CW complex of finite type, and is hyperbolic, then the rational Betti numbers of the free loop space have exponential growth. Examples are wedges of two spheres each of dimension greater than 1. (Aside: Let $X$ be a simply connected CW complex with
rational cohomology of finite type. Let $\Lambda(V;d)$ denote the Sullivan minimal model for $X$. Then $\Lambda(V;d)$ is said to be coformal provided $d^2(V) \subset \Lambda^2 V$.

By Theorem 1.3 in [2], either $Z_K$ is rationally homotopy equivalent to a finite product of odd spheres in which case $Z_K$ is elliptic, or $Z_K$ has a wedge of two spheres both of dimension greater than one as a retract in which case, it is hyperbolic. The structure of the minimal non-faces determines whether the moment-angle complex is elliptic or hyperbolic.

It is natural to try to extend the proof of Theorem 1.1 from $Z_K$ to the Davis-Januskiewicz space

$$DJ(K) = ET^m \times_{T^m} Z_K,$$

or a quotient of a $Z_K$ by a free action of some product of circles

$$M = Z_K/T^q.$$

The natural proof of Theorem 1.1 does not immediately extend to $DJ(K)$ for the following reasons: Panov, and Ray prove that DJ(K) is coformal precisely in the cases where $K$ is a flag complex [12]. Hence, the natural extension of Lambrecht’s theorem does not apply directly to $DJ(K)$ in the cases where $K$ fails to be a flag complex. However, the proof works in case $K$ is a flag complex. Namely, it follows from [2], [11], and [12], that if $K$ is a flag complex, the free loop space of $DJ(K)$ has irrational Hilbert-Poincare series if and only if $Z(K;(D^2,S^1))$ is hyperbolic. However, a technical variation gives the next result.

**Theorem 1.3.** If $X = DJ(K)$ or $X = ET^m \times_{T^q} Z_K$ where $T^q \subset T^m$, then the following are equivalent:

1. The Hilbert-Poincaré series for the rational homology of $LX$ has sub-exponential growth.
2. The space $Z(K;(D^2,S^1))$ has totally finite rational homotopy groups, that is, $Z(K;(D^2,S^1))$ is elliptic.

Since the Hochschild homology of the cohomology ring for $DJ(K)$ is the cohomology of the free loop space of $DJ(K)$ as a special case of [7], the next result follows.

**Corollary 1.4.** The Hochschild cohomology of the Stanley-Reisner ring (or face ring of $K$) has Hilbert-Poincaré series which has sub-exponential growth if and only if $Z(K;(D^2,S^1))$ has totally finite rational homotopy groups (i.e. is elliptic). Furthermore, if $K$ is a polytopal
sphere, the Hochschild cohomology of the Stanley-Reisner ring has Hilbert-Poincaré series which is a rational function if and only if the simple polytope $P^m(K)$ is a product of simplices.

If $T^q$ is a torus of maximal rank in $T^m$ which acts freely on $Z_K$ where $K$ is a polytopal sphere, then the results above together with [11, 8], give the next corollary.

**Corollary 1.5.** If

1. $T^q$ is a torus of maximal rank in $T^m$ which acts freely on $Z_K$ where $K$ is a polytopal sphere, and
2. the simple polytope $P^m(K)$ is not a product of simplices,

then there exist infinitely many geometrically distinct periodic geodesics on the standard toric manifold $Z_K/T^q$ (a simply-connected, closed, smooth manifold).

A related question is to work out the precise cohomology of $LX$, a problem considered by N. Seeliger [15]. In the special case for which $Z(K; (D^2, S^1))$ is rationally elliptic, the homology of the free loop space is just that of a product of odd dimensional spheres with a product of pointed loop spaces of odd dimensional spheres. To work out the homology of $LDJ(K)$ in the rationally elliptic case, it suffices to work out the differentials in the spectral sequence for $L(Z(K; (D^2, S^1))) \to L(DJ(K)) \to L(\mathbb{CP}(\infty))^m$ where there is a homotopy equivalence

$$L(\mathbb{CP}(\infty))^m \to \mathbb{CP}(\infty)^m \times (S^1)^m.$$ 

One example arises from Ganea’s fibration

$$S^3 \to \mathbb{CP}^\infty \vee \mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty.$$ 

In this case $K$ has two vertices without an edge between the vertices, $Z_K = S^3$, and $DJ(K) = \mathbb{CP}^\infty \vee \mathbb{CP}^\infty$. The upshot is that Hilbert-Poincaré series for $L(\mathbb{CP}^\infty \vee \mathbb{CP}^\infty)$ is rational function by a computation which is omitted. The interested reader is invited to do this computation.

2. **Proof of Theorem 1.1**

Note that $Z(K; (D^2, S^1))$ is either rationally hyperbolic, or rationally elliptic. Furthermore, $Z(K; (D^2, S^1))$ is rationally hyperbolic if and only if it has a rational wedge of two simply-connected spheres as a retract by [2]. Similarly, $Z(K; (D^2, S^1))$ is rationally elliptic if
and only if it is homotopy equivalent to a finite product of simply-connected odd dimensional spheres by \[2\].

Assume that \(Z(K; (D^2, S^1))\) is rationally hyperbolic. Thus \(Z(K; (D^2, S^1))\) has a rational wedge of two simply-connected spheres as a retract by \[2\]. Appealing to Lambrecht’s theorem \[11\], the Hilbert-Poincaré series for the rational homology of the free loop space of \(Z(K; (D^2, S^1))\) has exponential growth as the Hilbert-Poincaré series for the free loop space of a wedge of two simply-connected spheres has exponential growth. Thus the rational homology of \(LZ(K; (D^2, S^1))\) has irrational Hilbert-Poincaré series.

Note that \(Z(K; (D^2, S^1))\) is rationally elliptic if and only if it is rationally homotopy equivalent to a product of odd spheres \[2\]. The free loop space of a product of odd spheres is rationally (or indeed after inverting 2) homotopy equivalent to the product of odd spheres with the pointed loop space of the finite product of odd spheres. So rationally elliptic implies that the Hilbert-Poincaré series for the free loops is rational.

These remarks imply Theorem 1.1 as any space of the homotopy type of a finite, 1-connected CW-complex is either elliptic, or hyperbolic.

3. Extensions

Let

\[F \to E \to B\]

be a fibration of path-connected spaces where the base \(B\) may have a non-trivial fundamental group, but is assumed to act trivially on the homology of the fibre \(F\). One example addressed below is given by

\[B = L((\mathbb{CP}^\infty)^m),\]

or equivalently

\[B = (\mathbb{CP}^\infty)^m \times (S^1)^m\]

with

\[E = L(DJ(K)).\]
Theorem 3.1. Let
\[ F \to E \to B \]
be a fibration of path-connected spaces where the base $B$ may have a non-trivial fundamental group, but which acts trivially on the homology of the fibre $F$ (namely, trivial local coefficients). Further, assume that there is a free resolution of $\mathbb{Q}$ as a $H^*(B;\mathbb{Q})$-module which has sub-exponential growth. If the Hilbert-Poincaré series for the rational homology of $F$ has exponential growth, then so does the Hilbert-Poincaré series for the rational homology of $E$.

Example 3.2. If $B = L((\mathbb{C}P^\infty)^m)$, the rational cohomology ring is a tensor product of exterior algebras and polynomial algebras. An application of standard versions of the Koszul resolution gives that this case of $B$ satisfies the sub-exponential growth condition in Theorem 3.2. If the local coefficient system is trivial, the Hilbert-Poincaré series for the rational homology of $F$ has exponential growth, then so does the Hilbert-Poincaré series for the rational homology of $E$.

The elegant proof below is due to Ran Levi and Kathryn Hess [10].

Proof. The proof is by contradiction in which (1) the Hilbert-Poincaré series for the rational homology of $F$ has exponential growth, and (2) the Hilbert-Poincaré series for the rational homology of $E$ has sub-exponential growth.

Consider the fibration
\[ F \longrightarrow E \longrightarrow B \]
for which the local coefficient system in homology is assumed to be trivial.

By Dwyer [5], the Eilenberg-Moore spectral sequence abuts to the cohomology of $F$.

By assumption, there is a free resolution of $\mathbb{Q}$ as a $H^*(B;\mathbb{Q})$-module which has sub-exponential growth. If the rational cohomology of $E$ has sub-exponential growth, then both the $E_2$-term $\text{Tor}_{H^*(B;\mathbb{Q})}(\mathbb{Q}, H^*(E;\mathbb{Q}))$ and $E_\infty$ of the Eilenberg-Moore spectral sequence have sub-exponential growth. Thus $F$ will have sub-exponential growth which is a contradiction.

Thus the rational homology of $E$ must have exponential growth. □
4. Proof of Theorem 1.3

A theorem which is required in the proof of Theorem 1.3 is stated next, but proven in section 5.

**Theorem 4.1.** The local coefficient system for the fibration

\[ L(Z_K) \to L(ET^m \times T^q Z_K) \to L(BT^q) \]

is trivial.

Consider the fibration

\[ L(Z_K) \to L(ET^m \times T^q Z_K) \to L(BT^q). \]

By Theorem 3.1, and Theorem 4.1 it follows that if \( L(Z_K) \) is hyperbolic, then so is \( L(ET^m \times T^q Z_K) \). Similarly, if \( L(Z_K) \) is elliptic, then the rational homology of \( L(ET^m \times T^q Z_K) \) has sub-exponential growth.

The Hilbert-Poincaré series for \( L(Z_K) \) has exponential growth if and only if \( Z_K \) is hyperbolic by the above remarks. Appealing to Theorem 3.1, \( L(ET^m \times T^q Z_K) \) has Hilbert-Poincaré series which has exponential growth if and only if \( Z_K \) is hyperbolic.

As an extra feature in the case where \( K \) is a polytopal sphere, the Hilbert-Poincaré series for \( L(ET^m \times T^m Z_K) \) is irrational if and only if \( Z_K \) is hyperbolic by Theorems 3.1 and Corollary 1.2.

5. Local systems and the free loop space of \( Z_K/T \)

The proof of the following theorem is given next.

**Theorem 5.1.** The local coefficient system for the fibration

\[ L(Z_K) \to L(ET^m \times T^q Z_K) \to L(BT^q) \]

is trivial.
Proof. The definition of local coefficients in homology is given first. Let $F \to E \to B$ be a fibration of path-connected spaces. Let $\alpha : (S^1, \bar{p}) \to (B, \ast)$ represent an element in the fundamental group of the base $B$ where $\bar{p}$ denotes a base-point in $S^1$, and $\ast$ denotes the base-point in $B$.

Then there is a lift $H$ giving a commutative diagram

$$
\begin{array}{c}
[0, 1] \times F \\ \downarrow \\
S^1 \\
\end{array}
\xrightarrow{H}
\begin{array}{c}
E \\
\downarrow \\
B \\
\end{array}
$$

where $H(0, f) = f$ in the fibre. There is an induced map

$$\tilde{\alpha} : F \to F$$

defined by $\tilde{\alpha}(f) = H(1, f)$ for all $f \in F$. The local system is said to be trivial in case $\tilde{\alpha}_\ast$ is the identity map on the level of homology of the fibre for all $\alpha$.

The local coefficient system for the fibration

$$L(Z_K) \to L(ET^m \times_{T^n} Z_K) \to L(BT^n)$$

is considered next.

The proof is in several steps with the first case arising from the free loop space of $S^2$. Recall the fibration with cross-section

$$\Omega X \to LX \to X$$

for spaces $X$. If

$$X = S^2,$$

then the fundamental group of $L(S^2)$ is $\mathbb{Z}$. One choice of map realizing a generator is given by the composite

$$S^1 \xrightarrow{E} \Omega S^2 \xrightarrow{} L(S^2)$$

where $E : X \to \Omega \Sigma(X)$ is the Freudenthal suspension. The purpose of considering this example is that it provides the method for analyzing the setting provided by Theorem ??.
One way to construct the map $S^1 \to \Omega S^2$ is by using the natural action of $SO(2)$: The group $SO(2) = S^1$ acts on the unit 2-sphere in $\mathbb{R}^3$ with center at the origin by rotation orthogonal to the $z$-axis fixing both the north pole, and south pole

$$\rho : SO(2) \times S^2 \to S^2.$$  

The loop arises by spinning one half of a great circle through the north pole and south pole replacing the other half by a constant arc from the south pole to the north pole.

Let $\vec{n}$ denote the north pole in $S^2$. Notice that the choice of action of $SO(2)$ fixes $\vec{n}$ as well as the south pole.

Thus this action extends to actions on both the pointed loop space by which spins a path from the north pole to the south pole around the equator while fixing a path from the south pole to the north pole as well as free loop space of $S^2$ together with the following commutative diagram which gives a map of $S^1 \to LS^2$ realizing a generator of the fundamental group.

$$
\begin{array}{ccc}
SO(2) \times S^1 & \xrightarrow{\rho} & S^1 \\
1 \times E & \downarrow & \downarrow E \\
SO(2) \times \Omega S^2 & \longrightarrow & \Omega(S^2) \\
\downarrow & & \downarrow \\
SO(2) \times L(S^2) & \longrightarrow & L(S^2).
\end{array}
$$

Consider the restriction of the maps in the diagram to

$$SO(2) \times \Omega S^2 \longrightarrow \Omega(S^2)$$

to the loop which is an equatorial circle containing the north and south poles realizes a generator on the level of the fundamental groups. (This loop spins a 1/2 of a great circle fixing the other half.)

Now extend the action of $SO(2)$ to an action on $DJ(K)$ as well as the free loops of $DJ(K)$ as follows. First define an action of $SO(2)$ on $S^1$ by multiplication. Let $T^q \subset T^m$ and set

$$ET^m = (ES^1)^m.$$ 

Fix a coordinate $1 \leq j \leq m$. The group $S^1$ acts on $(ES^1)^m$ by left translation in the natural way by restriction of the action to the fixed coordinate $j$:

$$\rho_j : SO(2) \times (ES^1)^m \to (ES^1)^m.$$
The formula for this action on the level of the Borel construction is as follows:

\[ \rho_j(z, (e_1, \cdots, e_m), x) = ((e_1, \cdots, z e_j, \cdots, e_m), x) \]

where \( z \in SO(2), (e_1, \cdots, e_m) \in (ES^1)^m \) and \( x \in Z(K) \). This action on \((ES^1)^m \times_T Z_K\) descends to an action on the Borel construction \((ET)^m \times_{T^q} Z_K\) since \( T^m \) is abelian.

The next lemma is an inspection with details omitted: the lemma follows at once from the definitions and the fact that \( S^1 \) is abelian.

**Lemma 5.2.** The action \( \rho_j : SO(2) \times (ES^1)^m \to (ES^1)^m \) descends to an action

\[ \hat{\rho}_j : SO(2) \times L((ES^1)^m \times_{T^q} Z_K) \to L((ES^1)^m \times_{T^q} Z_K). \]

Furthermore, there is a commutative diagram

\[
\begin{array}{ccc}
SO(2) \times L(Z_K) & \longrightarrow & L(Z_K) \\
\downarrow & & \downarrow \\
SO(2) \times L((ES^1)^m \times_{T^q} Z_K) & \longrightarrow & L((ES^1)^m \times_{T^q} Z_K) \\
\downarrow & & \downarrow \\
SO(2) \times L((ES^1)^m / T^q) & \longrightarrow & L((ES^1)^m / T^q). \\
\end{array}
\]

Thus the local coefficient system for the fibration

\[ L(Z_K) \to L(ET^m \times_{T^q} Z_K) \to L(BT^m) \]

is trivial.

\[ \square \]

6. Free loop spaces in the elliptic case

Assume that \( Z_K = Z(K; (D^2, S^1)) \) is rationally elliptic, then it is a finite product of odd dimensional spheres by [2]. The free loop space \( L(S^{2n+1}) \) is homotopy equivalent to

\[ S^{2n+1} \times \Omega(S^{2n+1}) \]

as long as the prime 2 has been inverted. In this case of \( L(Z_K) \), the free loop space is a product of free loop spaces of odd dimensional spheres.

One remark is that the natural spectral sequence for

\[ L(DJ(K)) \to L(\mathbb{C}P^\infty)^m \]
frequently supports a non-trivial differential as in the case of the free loops of Ganea’s fibration
\[ L(S^3) \to L(\mathbb{CP}^\infty \vee \mathbb{CP}^\infty) \to L(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \]
for which \( K \) is two points, and \( DJ(K) = \mathbb{CP}^\infty \vee \mathbb{CP}^\infty \). This differential propagates to several related cases.

It is natural to conjecture that if \( Z_K \) is rationally elliptic, then the Hilbert-Poincaré series for the free loop space of \( L(ET^m \times_{T^q} Z_K) \) is a rational function.

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