Relativistic BCS-BEC Crossover at Finite Temperature and Its Application to Color Superconductivity

Lianyi He and Pengfei Zhuang

Physics Department, Tsinghua University, Beijing 100084, China

(Dated: February 1, 2008)

The non-relativistic $G_0G$ formalism of BCS-BEC crossover at finite temperature is extended to relativistic fermion systems. The uncondensed pairs contribute a pseudogap to the fermion excitations. The theory recovers the BCS mean field approximation at zero temperature and the non-relativistic results in a proper limit. For massive fermions, when the coupling strength increases, there exist two crossovers from the weak coupling BCS superfluid to the non-relativistic BEC state and then to the relativistic BEC state. For color superconductivity at moderate baryon density, the matter is in the BCS-BEC crossover region, and the behavior of the pseudogap is quite similar to that found in high temperature superconductors.

PACS numbers: 11.30.Qc, 12.38.Lg, 11.10.Wx, 25.75.Nq

I. INTRODUCTION

It is well-known that, by adjusting the attractive coupling strength among the constituents, a fermion system may undergo a smooth crossover from the Bardeen-Cooper-Schrieffer (BCS) superfluidity/superconductivity in degenerate fermion gas to the Bose-Einstein condensation (BEC) of composite molecules. Such a BCS-BEC crossover is theoretically due to the fact that the wave functions of BCS and BEC ground states are essentially the same. The BCS-BEC crossover is expected to be realized in high temperature superconductor and atomic fermion gas via using an external magnetic field to change the s-wave scattering length.

The superconductivity in Quantum Chromodynamics (QCD), i.e., the color superconductivity, is naturally considered as a system to study the relativistic BCS-BEC crossover. Due to the asymptotic property of QCD, there may exist a crossover from the BCS superconductivity with weakly bound quark pairs at high baryon density to the BEC state of composite hadrons at low baryon density. Such a BCS-BEC crossover in QCD may also be realized in chiral condensed matter and in pion superfluid. At moderate baryon density, while a diquark BEC state may not be realized due to the chiral symmetry restoration, the attractive coupling strength is obviously not located in the weak coupling region. It is shown in many effective QCD models that, the quark energy gap at moderate baryon density is about 100 MeV which is already of the order of the Fermi energy. The strong coupling in this case may induce a so-called pseudogap effect, which has been investigated in two flavor color superconductivity above the critical temperature. A natural question is how the pseudogap modifies the critical temperature and thermodynamics of the color superconductor. To answer this question, one needs to construct a relativistic theory at finite temperature which can describe the pseudogap and possible BCS-BEC crossover.

The BCS-BEC crossover in relativistic fermion systems are recently investigated in the Nozieres–Schmitt-Rink (NSR) theory above the critical temperature, the boson-fermion model, and the BCS-Leggett mean field theory at zero temperature. It is shown that, not only the BCS superfluidity and the non-relativistic BEC (NBEC) of heavy molecules but also the NBEC and the relativistic BEC (RBEC) of nearly massless molecules can be smoothly connected. In the RBEC state, anti-fermion pairs (anti-bosons) are excited and become nearly degenerate with fermion pairs (bosons). From the NSR theory at $T \geq T_c$, where $T_c$ is the critical temperature, the difference between the NBEC and RBEC states is significant.

It is widely known that, at zero temperature the mean field theory is a good approximation to describe the BCS-BEC crossover, and the pair fluctuations can be safely neglected even at strong coupling. Only around the unitary limit, i.e., the infinite scattering length limit, the pair fluctuations are somewhat important to obtain a proper value of the universal constant. In our previous paper, we investigated the generalization from non-relativistic to relativistic BCS-BEC crossover at zero temperature in the BCS-Leggett mean field theory. At finite temperature, however, the condensed pairs with zero momentum can be thermally excited, and one should go beyond the mean field approximation to treat properly the uncondensed pairs.

There exist many methods to treat pair fluctuations at finite temperature. In the NSR theory, which is also called $G_0G_0$ theory, the pair fluctuations enter only the number equation, and the fermion loops which appear in the pair propagator are constructed by bare Green function $G_0$. As a consequence, such a theory is in principle not self-consistent and is valid only at $T \geq T_c$. For the study of BCS-BEC crossover, one needs a theory which is valid not only above the critical temperature but also in the symmetry breaking phase. While such a strict theory is not yet reached so far, some $T$-matrix approaches are recently developed, see for instance. Among them, the asymmetric pair approximation or the so-called $G_0G$ scheme is a competitive one. The effect of the pair fluctuations in the $G_0G$ method is treated as a fermion
pesudogap which has been widely discussed in high temperature superconductivity. In contrast to the NSR theory ($G_0G$ scheme), the $G_0G$ scheme is self-consistent and keeps the Ward identity \[ \delta S = 0. \]

In the study of color superconductivity at moderate density, the color condensed phase is of great interest. The NSR theory \[ [16, 17], \] which seems valid in the normal phase, can only predict the transition temperature of color superconductivity. A necessary task in this field of research is to develop a relativistic BCS-BEC crossover theory in the symmetry breaking phase. In this paper, we will generalize the $G_0G$ scheme to relativistic fermion systems. A necessary requirement for such a generalization is to recover the non-relativistic limit \[ [4] \] and mean field limit \[ [3] \] properly. With this theory, we can calculate the critical temperature $T_c$ for arbitrary coupling and describe the BCS-NBEC-RBEC crossover at finite temperature. It, as an application, can be used to study the pseudogap effect on color superconductivity.

The paper is organized as follows. In section II we review the BCS mean field theory for relativistic superfluidity/superconductivity. In the framework of the $G_0G$ scheme, we include in Section III the contribution from the uncondensed pairs and construct coupled equations for the superfluid order parameter and pseudogap. In section IV we apply the theory to massive fermions and study the BCS-NBEC-RBEC crossover at finite temperature. In section V we apply the theory to color superconducting quark matter. We will calculate the transition temperature and the quark pseudogap and show the significance of the fluctuations at moderate baryon density. We summarize in section VI.

II. BCS MEAN FIELD THEORY

We consider a model with only fermions as elementary blocks. The Lagrangian density can be written as

\[ \mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + \mathcal{L}_I, \]

where $\psi, \bar{\psi}$ denote the Dirac fermion fields with mass $m$, and $\mathcal{L}_I$ indicates the attractive interaction among fermions. Since the dominant interaction is the $J^P = 0^+$ scalar channel, the interaction for the pairing between different spins can take the form\[ [16, 19] \]

\[ \mathcal{L}_I = \frac{g}{4} (\bar{\psi} i\gamma^5 C \psi^T) (\psi^T C i\gamma_5 \psi), \]

where $g$ is the attractive coupling constant, and $C = i\gamma_0 \gamma_2$ is the charge conjugation matrix. Generally, by adjusting the coupling strength, the crossover from condensation of spin-zero Cooper pairs with large size at weak coupling to the Bose-Einstein condensation of deeply bound bosons at strong coupling can be realized. In our model, only fermions are elementary particles. Another type of model which is used to discuss the BCS-BEC crossover in high temperature superconductors and atomic Fermi gases is the so-called boson-fermion model where both fermions and bosons are considered as elementary blocks. Such a model is recently generalized to study the relativistic BCS-BEC crossover \[ [18] \].

In order to develop a finite temperature theory including pair fluctuations in the symmetry breaking phase, we first review in this section the BCS mean field theory in the functional integral approach and $G_0G$ formalism.

A. Functional Integral Approach

In the functional integral approach, we start the calculation from the partition function in imaginary time formalism,

\[ Z = \int D\bar{\psi} D\psi e^{i\beta \int d^3x (\mathcal{L} + \mu \bar{\psi} \psi)} \]

with the effective boson action

\[ S_{\text{eff}} = \int_0^\beta d\tau \int d^3x \left[ \frac{|\Delta(x)|^2}{g} - \frac{1}{2\beta} \text{Tr} \ln[\beta \mathbf{G}^{-1}] \right] \]

in terms of the inverse Nambu-Gorkov propagator

\[ \mathbf{G}^{-1} = i\gamma^\mu \partial_\mu - m + \mu \gamma_0 \sigma_3 + i\gamma_5 \Delta \sigma_+ + i\gamma_5 \Delta^* \sigma_-, \]

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ are defined in the Nambu-Gorkov space with $\sigma_i(i = 1, 2, 3)$ being the Pauli matrices.

The mean field theory is a good approximation to describe the BCS-BEC crossover at low enough temperature, namely $T \ll T_c$, since the dominant contribution of fluctuations to the effective potential is from the Goldstone mode and is proportional to $T^3 \mathbb{R}$. In the mean field approximation, we consider a uniform static saddle point $\Delta(x) = \Delta_\text{sc}$ which satisfies the stationary condition $\delta S_{\text{eff}}[\Delta_\text{sc}] / \delta \Delta_\text{sc} = 0$. The thermodynamic potential $\Omega_{\text{mf}} = S_{\text{eff}}[\Delta_\text{sc}] / (\beta V)$ at the saddle point can be evaluated as

\[ \Omega_{\text{mf}} = \frac{\Delta^2}{g} - \int \frac{d^3k}{(2\pi)^3} \left[ (E^+ \Delta^+ \Delta^- \Delta^+ \Delta^- - \xi^+_k - \xi^-_k) \right. \\
\left. - \frac{1}{\beta} \left( \ln(1 + e^{-\beta E^+_k}) + \ln(1 + e^{-\beta E^-_k}) \right) \right], \]

where we have defined the quasi-particle energies $E^+_k = \sqrt{(\xi^+_k)^2 + \Delta^2}$ with $\xi^+_k = \epsilon_k + \mu$ and $\epsilon_k = \sqrt{k^2 + m^2}$.
Minimizing $\Omega_n$, we get the gap equation to determine the order parameter $\Delta_{sc}$ in the symmetry breaking phase,

$$
\frac{1}{g} = \int \frac{d^3 k}{(2\pi)^3} \left[ 1 - 2f(E_k^-) + \frac{1 - 2f(E_k^+)}{2E_k^-} \right],
$$

(8)

where $f(x) = 1/(e^{x} + 1)$ is the Fermi-Dirac distribution function. In the study of BCS-BEC crossover, people often consider the thermodynamics in canonical ensemble with fixed fermion density $n$ by fixing the Fermi momentum $k_f$ through the relation $n = k_f^3/(3\pi^2)$ at zero temperature. At finite temperature, the density can be obtained from the first order derivative of the thermodynamic potential with respect to the chemical potential,

$$
n = \int \frac{d^3 k}{(2\pi)^3} \left[ 1 - \frac{\xi_k^+}{E_k^+} (1 - 2f(E_k^-)) \right. \\
- \left. \left( 1 - \frac{\xi_k^-}{E_k^-} (1 - 2f(E_k^+)) \right) \right],
$$

(9)

The first and second terms in the square bracket on the right hand side of equations (8) and (9) correspond respectively to fermion and anti-fermion degrees of freedom.

### B. $GoG$ Formalism

Now we reexpress the BCS mean field theory in the $GoG$ formalism[2, 3, 23]. Such a formalism is convenient for us to go beyond the BCS and include uncondensed pairs at finite temperature. Let us start from the fermion propagator $S$ in the symmetry breaking phase. The inverse propagator reads

$$
S^{-1}(k) = \begin{pmatrix} G_0^{-1}(k, \mu) & i\gamma_5 \Delta_{sc} \\ i\gamma_5 \Delta_{sc} & G_0^{-1}(k, -\mu) \end{pmatrix}
$$

(10)

with the inverse free propagator

$$
G_0^{-1}(k, \mu) = (i\omega_n + \mu)\gamma_0 - \gamma \cdot k - m,
$$

(11)

where $k = (i\omega_n, k)$ is the fermion four momentum at finite temperature with $\omega_n$ being the fermion frequency $\omega_n = (2n + 1)\pi T$ ($n = 0, \pm 1, \pm 2, \cdots$). The propagator can be formally expressed as

$$
S(k) = \begin{pmatrix} G(k, \mu) & F(k, \mu) \\ F(k, -\mu) & G(k, -\mu) \end{pmatrix}
$$

(12)

with the diagonal and off-diagonal elements

$$
G(k, \mu) = \left[ G_0^{-1}(k, \mu) - \Sigma_{sc}(k) \right]^{-1}, \\
F(k, \mu) = -iG(k, \mu)i\gamma_5 \Delta_{sc} G_0(k, -\mu),
$$

(13)

where the fermion self-energy $\Sigma_{sc}$ is defined as

$$
\Sigma_{sc}(k) = i\gamma_5 \Delta_{sc} G_0(k, -\mu)i\gamma_5 \Delta_{sc},
$$

(14)

with the help of the energy projectors

$$
\Lambda_{\pm}(k) = \frac{1}{2} \left[ 1 \pm \frac{\gamma \cdot (k + m)}{E_k} \right],
$$

(15)

the propagator elements can be explicitly evaluated as

$$
G(k, \mu) = \left( i\omega_n + \xi_k^- \right) \Lambda_{+\mu} + \left( i\omega_n - \xi_k^+ \right) \Lambda_{-\mu}, \\
F(k, \mu) = \frac{i}{2} \Delta_{sc} \Lambda_{+\mu} + \frac{i}{2} \Delta_{sc} \Lambda_{-\mu},
$$

(16)

The gap equation for the order parameter $\Delta_{sc}$ is related to the off-diagonal element,

$$
\Delta_{sc} = -i\frac{g}{2} \sum_k \text{Tr} [i\gamma_5 F(k, \mu)] \\
= -i\frac{g}{2} \Delta_{sc} \sum_k \text{Tr} \left[ \Sigma_0(k, \mu) \Sigma_0(\mu, -k) \right],
$$

(17)

and the fermion number is controlled by the diagonal element,

$$
n = -i \sum_k \text{Tr} [\gamma_0 G(k, \mu)]
$$

(18)

with the four momentum integration $\sum_k = iT \sum_n \int d^3 k / (2\pi)^3$ at finite temperature. Completing the Matsubara frequency summation, we can reobtain the gap equation (8) and number equation (9).

In the BCS mean field theory, fermion–fermion pairs and anti-fermion–anti-fermion pairs explicitly enter the system below $T_c$ only through the condensate $\Delta_{sc}$. In the $GoG$ formalism, the fermion self-energy can equivalently be expressed as

$$
\Sigma_{sc}(k) = \sum_q t_{sc}(q) G_0(q - k, \mu)
$$

(19)

associated with a condensed-pair propagator given by

$$
t_{sc}(q) = \frac{\Delta_{sc}^2}{T} \delta(q),
$$

(20)

where $q = (i\nu_n, q)$ is the boson four momentum with boson frequency $\nu_n = 2n\pi T$.

The BCS theory can be related to a specific pair susceptibility $\chi$ defined by

$$
\chi_{BCS}(q) = -i \sum_k \text{Tr} [G(k, \mu) G_0(q - k, \mu)],
$$

(21)

with which, the gap equation for the condensate $\Delta_{sc}$ can be written as

$$
1 - g\chi_{BCS}(0) = 0.
$$

(22)

This implies that the uncondensed pair propagator should be of the form

$$
t(q) = \frac{ig}{1 - g\chi_{BCS}(q)},
$$

(23)
and $t^{-1}(q = 0)$ is proportional to the pair chemical potential $\mu_{\text{pair}}$. Therefore, the fact that in the symmetry breaking phase the pair chemical potential is zero leads to the BEC-like condition

$$t^{-1}(q = 0) = 0. \quad (24)$$

While the uncondensed pairs do not play any real role in the BCS mean field theory, such a specific choice of the pair susceptibility and the BEC-like condition tell us a way how to go beyond the BCS mean field theory and include the effect of uncondensed pairs.

**III. BEYOND MEAN FIELD THEORY**

While the uncondensed pairs can be safely neglected at weak coupling, they should be included for a self-consistent theory at arbitrary coupling and at finite temperature. We now go beyond the BCS mean field approximation and include the uncondensed pairs in the $G_0G$ formalism. It is clear that, in the BCS mean field approximation the fermion self-energy $\Sigma_{\text{sc}}$ includes contribution only from the condensed pairs. At finite temperature, the condensed pairs with zero total momentum can be thermally excited, and the total propagator should contain both the condensed (sc) and uncondensed or “pseudogap”-associated (pg) contributions,

$$t(q) = t_{\text{pg}}(q) + t_{\text{sc}}(q),$$

$$t_{\text{pg}}(q) = \frac{iq}{1 - g\chi(q)}, \quad q \neq 0,$$

$$t_{\text{sc}}(q) = \frac{\Delta_{\text{sc}}^2}{T}\delta(q). \quad (25)$$

Now the total fermion self-energy becomes

$$\Sigma(k) = \sum_q t(q)G_0(q - k, \mu) = \Sigma_{\text{sc}}(k) + \Sigma_{\text{pg}}(k), \quad (26)$$

with the mean field part

$$\Sigma_{\text{sc}}(k) = \sum_q t_{\text{sc}}(q)G_0(q - k, \mu) \quad (27)$$

and the pseudogap related part

$$\Sigma_{\text{pg}}(k) = \sum_q t_{\text{pg}}(q)G_0(q - k, \mu). \quad (28)$$

With the full propagator

$$G(k, \mu) = \left[ G_0^{-1}(k, \mu) - \Sigma(k) \right]^{-1} \quad (29)$$

in terms of the total self-energy, the pair susceptibility is still given by

$$\chi(q) = -\frac{i}{2} \sum_k \text{Tr} G(k, \mu)G_0(q - k, \mu). \quad (30)$$

The $G_0G$ formalism used here is diagrammatically illustrated in Fig. 1.

Note that, the feedback of the pair fluctuations on the order parameter $\Delta_{\text{sc}}$ is included, and it and the chemical potential $\mu$ are in principle determined by the BEC condition $t_{\text{pg}}^{-1}(0) = 0$ and the number equation $n = -i \sum_k \text{Tr} [\gamma_0 G(k, \mu)]$.

The above equations are hard to handle analytically. In the symmetry breaking phase with $T \leq T_c$, the BEC condition $t_{\text{pg}}^{-1}(0) = 0$ implies that $t_{\text{pg}}(q)$ is peaked at $q = 0$. This allows us to approximate

$$\Sigma(k) \simeq -\Delta_{\text{sc}}^2 G_0(-k, \mu), \quad (31)$$

where $\Delta^2$ contains contributions from the condensed and uncondensed pairs,

$$\Delta^2 = \Delta_{\text{sc}}^2 + \Delta_{\text{pg}}^2 \quad (32)$$

with the pseudogap $\Delta_{\text{pg}}$ defined as

$$\Delta_{\text{pg}}^2 = -\sum_{q \neq 0} t_{\text{pg}}(q). \quad (33)$$

It is necessary to point out that, above the critical temperature $T_c$ such an approximation is no longer good, since the BEC condition is not valid in normal phase.

Since the pseudogap $\Delta_{\text{pg}}$ looks similar to the condensate $\Delta_{\text{sc}}$, a natural question is whether a finite $\Delta_{\text{pg}}$ breaks the symmetry of the system. If yes, $\Delta_{\text{sc}}$ will no longer be considered as the order parameter of the phase transition. By omitting a term of the order of $O \left( \Delta_{\text{sc}}^2 / \Lambda^2 \right)$, where $\Lambda$ is a momentum cutoff, the inverse fermion propagator including the feedback of the pair fluctuations can be written as

$$S^{-1}(k) = \left( \begin{array}{cc} G_0^{-1}(k, \mu) - \Sigma_{\text{pg}}(k) & i\gamma_5\Delta_{\text{sc}} \\ ik_5\Delta_{\text{sc}} & G_0^{-1}(k, -\mu) - \Sigma_{\text{pg}}'(k) \end{array} \right) \quad (34)$$

where $\Sigma_{\text{pg}}' = \Sigma_{\text{pg}}(\mu \rightarrow -\mu)$. It is now clear that, the pseudogap appears in the diagonal elements of the Nambu-Gorkov propagator and does not break the symmetry of the system. On the other hand, parallel to the discussion in non-relativistic theory\cite{6, 7, 24}, we can show

\[ \text{FIG. 1: Diagrammatic representation of the propagator } t_{\text{pg}} \text{ for the uncondensed pairs and the fermion self-energy. The total fermion self-energy contains contributions from condensed } (\Sigma_{\text{sc}}) \text{ and uncondensed } (\Sigma_{\text{pg}}) \text{ pairs. The dashed, thin solid and thick solid lines in } t_{\text{pg}} \text{ represent, respectively, the coupling constant } g/2, \text{ bare propagator } G_0 \text{ and full propagator } G. \text{ This diagram is taken from } [29]. \]
that $\Delta_{pg}^2$ is just the fluctuation of the order parameter field $\Delta(x)$,
\[ \Delta_{pg}^2 = \langle |\Delta|^2 \rangle - \langle |\Delta| \rangle^2, \] (35)
and hence it does not break the symmetry.

Under the approximation (31), all the equations in the mean field theory are still valid, the only change is the replacement of $E^\pm_k = \sqrt{\langle \xi^2_k \rangle^2 + \Delta_{sc}^2}$ by $E^\pm_k = \sqrt{\langle \xi^2_k \rangle^2 + \Delta^2}$. For instance, the diagonal element $G$ of the full propagator, the fermion number $n$ and the gap equation for $\Delta$ take exactly their mean field forms [14], [9] and [8]. The equations [8], [9] and [33] determine self-consistently the order parameter $\Delta_{sc}$, the pseudogap $\Delta_{pg}$ and the chemical potential $\mu$ as functions of temperature $T$. Note that the pair fluctuation effect is self-consistently included in the coupled equations through the pseudogap $\Delta_{pg}$. It is necessary to point out that, the $G_0G$ approach we used is quite different from the NSR theory. In the NSR theory, the pair fluctuations enter only the number equation via adding a molecule number term [16, 17].

However, solving such a coupled set of equations is still rather complicated. Fortunately, the BEC condition allows us to do further approximations for the pair propagator $t_{pg}(q)$. Using the BEC condition $1 - g\chi(0) = 0$, the T-matrix can be written as
\[ t_{pg}(q) = -\frac{i}{\chi(q) - \chi(0)}. \] (36)
Since the pseudogap is dominated by the gapless pair dispersion in long wavelength limit, we can expand the susceptibility around $q = 0$ in this limit,
\[ t_{pg}(q) \simeq -\frac{i}{Z_1 q_0 + Z_2 q_0^2 - \xi^2 q_0^2 + i\Gamma(q)}, \] (37)
where the coefficients $Z_1, Z_2$ and $\xi^2$ are defined as
\[ Z_1 = \left. \frac{\partial \chi}{\partial q_0} \right|_{q=0}, \quad Z_2 = \left. \frac{1}{2} \frac{\partial^2 \chi}{\partial q_0^2} \right|_{q=0}, \]
\[ \xi^2 = \left. -\frac{1}{2} \frac{\partial^2 \chi}{\partial q_0^2} \right|_{q=0}, \] (38)
and we have considered the fact that the susceptibility depends only on $q_0^2$. The explicit expressions for $Z_1, Z_2$ and $\xi^2$ are listed in Appendix A.

In the symmetry breaking phase where the temperature is low, it is believed that the pairs are long-lived and we can neglect their width $\Gamma$. With the expansion for the pair propagator, the equation (33) now takes a simple form
\[ \Delta_{pg}^2 = \frac{1}{Z_2} \int \frac{d^3q}{(2\pi)^3} \frac{1 + b(\omega_q - \nu) + b(\omega_q + \nu)}{2\omega_q}, \] (39)
where $b(x) = 1/(e^{bx} - 1)$ is the Bose-Einstein distribution function and $\omega_q$ and $\nu$ are defined as
\[ \omega_q = \sqrt{\nu^2 + c^2 q^2}, \quad \nu = \frac{Z_1}{2Z_2}, \quad c^2 = \frac{\xi^2}{Z_2}, \] (40)

The first term on the righthand side of (39) suffers ultraviolet divergence, but it can be dropped out via renormalization [6, 7].

Let us first discuss some conclusions from the above equations without detailed numerical calculations.
1) At zero temperature, the pseudogap $\Delta_{pg}$ vanishes automatically and the theory is reduced to the BCS mean field approximation [19].
2) If the coupling is not so strong that the molecule binding energy $E_b$ satisfies $E_b \ll 2m$, the theory is reduced to its non-relativistic version [6] for systems with $k_f \ll m^{-1}$. 3) If $Z_1$ dominates the propagator $t_{pg}$, the pair dispersion is quadratic in $|q|$, and the pseudogap $\Delta_{pg}$ becomes proportional to $T$ at low temperature. On the other hand, if $Z_2$ is the dominant term, the pair dispersion is linear in $|q|$ and $\Delta_{pg}$ becomes proportional to $T$ at low temperature. In the next section, we will show that the first case happens in the NBEC region and the second case occurs in the RBEC region.

4) From the explicit expression of $Z_1$ shown in Appendix A
\[ Z_1 = \frac{1}{2\pi} \left[ \frac{n}{2} - \int \frac{d^3k}{(2\pi)^3} \left( f(\xi_k^-) - f(\xi_k^+) \right) \right], \] (41)
the quantity in the square brackets is just the total number density $n_B$ of the bound pairs (bosons),
\[ n_B = Z_1 \Delta^2. \] (42)
From the relation $\Delta^2 = \Delta_{sc}^2 + \Delta_{pg}^2$, $n_B$ can be decomposed into the condensed pair number $n_{sc}$ and the uncondensed pair number $n_{pg}$,
\[ n_{sc} = Z_1 \Delta_{sc}^2, \quad n_{pg} = Z_1 \Delta_{pg}^2. \] (43)
The fraction of the condensed pairs can be defined by
\[ P_c = \frac{n_{sc}}{n} = \frac{2Z_1 \Delta_{sc}^2}{n}. \] (44)

5) In the weak coupling BCS region, we expect the fermion number density
\[ n \simeq 2 \int \frac{d^3k}{(2\pi)^3} \left( f(\xi_k^-) - f(\xi_k^+) \right) \] (45)
which leads to $n_B = 0$ in this region. In the deep BEC region, however, almost all the fermions form two body bound states which results in $n_B \approx n/2$. At zero temperature, we have $\Delta_{pg} = 0, n_B = n_{sc}$ and $P_c = 1$, while at the critical temperature $T_c$ the order parameter $\Delta_{sc}$ disappears, and the uncondensed pair number $n_{pg}$ becomes dominant and is approximately equal to $n/2$.

Numerically, the transition temperature $T_c$ can be calculated from (39) and the generalized equations [8] and [9] by setting $\Delta_{sc} = 0$. Usually, at and above $T_c$ where the order parameter $\Delta_{sc}$ disappears, the pseudogap $\Delta_{pg}$
increase with temperature, we expect that \( \Delta_{pg} \) where \( \Omega_{\text{cond}} \) contains both the fermion and boson contributions, \( \Omega \). We will do such a generalization, but the numerical results in the following will be presented mainly at \( T \leq T_c \).

Above the critical temperature \( T_c \), the order parameter \( \Delta_{nc} \) vanishes, and the BEC condition is no longer valid, \( 1 - g\chi(0) \neq 0 \). As a consequence, the propagator of the pair takes the form

\[
t_{pg}(q) = \frac{-i}{\chi(q) - \chi(0) - Z_0}
\]

with \( Z_0 = 1/g - \chi(0) \). As an estimation of \( \Delta_{pg} \) and \( T^* \), we still perform the expansion for the susceptibility around \( q = 0 \),

\[
t_{pg}(q) \approx \frac{-i}{Z_1q_0 + Z_2q_0^2 - \xi^2|q|^2 - Z_0 + i\Gamma(q)}.
\]

Now the pseudogap equation becomes

\[
\Delta_{pg}^2 = \frac{1}{Z_2} \int \frac{d^3q}{(2\pi)^3} \frac{b(\omega'_q - \nu) + b(\omega'_q + \nu)}{2\omega'_q}
\]

with the definition

\[
\omega'_q = \sqrt{\nu^2 + \lambda^2 + c^2q^2}, \quad \lambda^2 = Z_0/Z_2.
\]

The equation (48) together with the number equation (49) determines the pseudogap \( \Delta_{pg} \) and chemical potential \( \mu \) above \( T_c \). Since the pair dispersion is now no longer gapless in the long-wavelength limit, and \( Z_0 \) will generally increase with temperature, we expect that \( \Delta_{pg} \) will drop down and approach zero at the dissociation temperature \( T^* \).

In the end of this section, we discuss the thermodynamics of the system. The naive BCS mean field theory does not include the contribution from the uncondensed bosons which, however, dominate the thermodynamics at strong coupling. In the present theory, considering the uncondensed pairs, the total thermodynamic potential \( \Omega \) contains both the fermion and boson contributions,

\[
\Omega = \Omega_{\text{cond}} + \Omega_{\text{fermion}} + \Omega_{\text{boson}},
\]

where \( \Omega_{\text{cond}} \) is from the condensed pairs,

\[
\Omega_{\text{cond}} = \frac{\Delta_{pg}^2}{g},
\]

\( \Omega_{\text{fermion}} \) from the fermion excitations,

\[
\Omega_{\text{fermion}} = \int \frac{d^3k}{(2\pi)^3} \left[ \left( \xi^+_k + \xi^-_k - E^+_k - E^-_k \right) \right]
\]

\[\left( \frac{1}{\epsilon_k - m} + \frac{1}{\epsilon_k + m} \right).\]

Under the approximation (37) for the pair propagator, the boson part in the symmetry breaking phase can be evaluated as

\[
\Omega_{\text{boson}} = \sum_q \ln |1 - g\chi(q)|.
\]

As we will show below, the former and the latter correspond to the NBEC and RBEC region, respectively. The bosons and fermions behave very differently in thermodynamics. As is well known, the specific heat \( C \) of an ideal boson gas is proportional to \( T^2 \) with \( \alpha = 3/2 \) and \( 3 \) corresponding to non-relativistic and ultra-relativistic systems, but the naive BCS mean field theory predicts an exponential law \( C \propto e^{-\Delta_0/T} \), where \( \Delta_0 \) is the gap at zero temperature, \( \Delta_0 = \Delta(T = 0) \).

**IV. BCS-NBEC-RBEC CROSSOVER WITH MASSIVE FERMIONS**

In this section, we study the BCS- BEC crossover when the coupling constant \( g \) increases. Since our model is non-renormalizable, a proper regularization is needed. In the case with massive fermions we employ the often used non-relativistic regularization, namely, to replace the bare coupling \( g \) by a renormalized coupling \( \frac{U_{\text{fr}}}{g_{\text{fr}}} \),

\[
\frac{1}{U} = \frac{1}{g} - \frac{1}{2} \int_{|k| \leq \Lambda} \frac{d^3k}{(2\pi)^3} \left( \frac{1}{\epsilon_k - m} + \frac{1}{\epsilon_k + m} \right).
\]

The effective s-wave scattering length \( a_s \) can be related to \( U \) by \( U = 4\pi a_s/m \). While this is a natural extension of the non-relativistic regularization to relativistic systems, the ultraviolet divergence can not be completely removed, and a cutoff \( \Lambda \) still exists in the theory. In this regularization, the solution of the coupled equations depends on three dimensionless parameters: the effective...
coupling constant $\eta = 1/(k_f a_c)$, the quantity $\zeta = k_f / m$ which reflects the fermion number density, and the cutoff $\Lambda / m$.

We assume in this section that the fermion density $n$ is not very high and satisfies the relation $n < m^3$ or $\zeta < 1$. In this case the system is not ultra-relativistic and can even be treated non-relativistically in some parameter region. From the study in NSR theory above $T_c$ and in the BCS-Leggett theory at $T = 0$, if the dimensionless coupling $\eta$ varies from $-\infty$ to $+\infty$, the system will undergo two crossovers\[16, 17, 18, 19\], the crossover from the BCS state to the NBEC state around $\eta \sim 0$ and the crossover from the NBEC state to the RBEC state around $\eta \sim \zeta^{-1}$. The NBEC state and the RBEC state are characterized by the molecule binding energy $E_b$. We have $E_b \ll 2m$ in the NBEC state and $E_b \sim 2m$ in the RBEC state.

1) BCS region. In weak coupling BCS region, there exist no bound pairs in the system. In this case, $Z_1$ is sufficiently small and $Z_2$ dominates the pair dispersion\[7\], and we have $\Delta_{pg}^2 \propto 1/(Z_2 e^3)$ after a simple algebra. Since $\Delta$ should be small in the weak coupling region, and $c$ can be proven to be approximately equal to the Fermi velocity\[7\], the pseudogap $\Delta_{pg}$ is very small and can be safely neglected in this region, as we expected. Therefore, the BCS mean field approximation is good enough at any temperature, and the critical temperature satisfies the well known relation $T_c \approx 0.57 \Delta_0$. For example, in the non-relativistic limit with $k_f \ll m$, the anti-fermion degrees of freedom can be ignored and the pair susceptibility recovers its non-relativistic version\[7\], see the result in Appendix A. The critical temperature can be expressed as\[6\]

$$T_c = \frac{8 \epsilon^{-2}}{\pi} \epsilon_f e^{2 \eta / \pi},$$  \hspace{1cm} (58)

where $\gamma$ is the Euler constant and $\epsilon_f = k_f^2/(2m)$ is the Fermi kinetic energy. In this region, even though $Z_2$ dominates the pair dispersion, we can show that $c \propto \Delta$ is vanishingly small due to the weak coupling. Since the boson contribution to thermodynamics can be neglected, the specific heat at low temperature takes the well known form $C \propto e^{-\Delta_0 / T}$.

2) NBEC region. In the non-relativistic BEC region with $\eta > 1$ but $\eta \ll \zeta^{-1}$\[15\], the molecule binding energy $E_b$ is much less than $2m$, namely $|\mu - m| \ll m$, the boson mass is approximately $2m$, and the system can be regarded as a non-relativistic boson gas, if $k_f / m$ is small enough. Assuming $k_f \ll m$, the anti-fermion degrees of freedom can be neglected, and we can recover the non-relativistic result\[6\]. In this region, the gap $\Delta$ becomes as large as the Fermi kinetic energy $\epsilon_f$. From $Z_1 \propto 1/\Delta^2$ and $Z_2 \propto 1/\Delta^4$, $Z_1$ is the dominant one and the pair dispersion becomes quadratic in $|q|$. In this case, the propagator of the uncondensed pairs can be approximated by

$$t_{pg}(q) \simeq \frac{-i Z_1^{-1}}{q_0 - |q|^2 / (2m_B)},$$  \hspace{1cm} (59)

where the pair mass $m_B$ is defined by $m_B = Z_1 / 2 \zeta^2$, and we have the simple relation

$$Z_1 \Delta_{pg}^2 = \int \frac{d^3 q}{(2\pi)^3} b \left( \frac{|q|^2}{2m_B} \right) = \left( \frac{m_B T}{2\pi} \right)^{3/2} \zeta \left( \frac{3}{2} \right).$$  \hspace{1cm} (60)

Since $Z_1 \Delta_{pg}^2$ is equal to the total boson density $n_B$ at $T = T_c$, we arrive at the standard critical temperature for Bose-Einstein condensation in non-relativistic boson gas\[21\],

$$T_c = \frac{2\pi}{m_B} \left( \frac{n_B}{\zeta(\frac{3}{2})} \right)^{2/3}.$$  \hspace{1cm} (61)

The boson mass $m_B$ is generally expected to be equal to the boson chemical potential $\mu_B = 2\mu$. In the non-relativistic limit $k_f \ll m$, we can show $m_B \simeq 2m$ and $Z_1 \Delta_{pg}^2 \simeq n/2$ at $T = T_c$, the critical temperature becomes $T_c = 0.218 \epsilon_f$. Since $Z_1$ dominates the pair dispersion, the pseudogap is proportional to $T^{3/4}$ and the specific heat is proportional to $T^{3/2}$ at low temperature.

3) RBEC region. In this region we have the molecule binding energy $E_b = 2m$ and chemical potential $\mu \to 0$. In this case, non-relativistic limit can not be reached even for $k_f \ll m$\[19\]. Since the bosons with mass $m_B = 2\mu$ become nearly massless in this region, the anti-bosons and anti-fermions can be easily excited, and the system contains both bosons and anti-bosons. From the relation

$$n_B = n_b - n_c = Z_1 \Delta_{pg}^2$$  \hspace{1cm} (62)

at $T = T_c$, where $n_b$ and $n_c$ are the boson and antiboson numbers, while $n_b$ and $n_c$ are both very large, their difference produces a small pure boson density $n_B \simeq n/2$. On the other hand, for $\mu \to 0$ we can expand $Z_1$ in powers of chemical potential $\mu$,

$$Z_1 \approx R \mu + O(\mu^3) = \frac{R}{2} m_B + O(\mu^3)$$  \hspace{1cm} (63)

and hence $Z_2$ dominates the pair dispersion, which means that the pseudogap is proportional to $T$ at low temperature. In this case, the propagator of the uncondensed pairs can be approximated by

$$t_{pg}(q) \simeq \frac{-i Z_2^{-1}}{q_0 - c^2 |q|^2},$$  \hspace{1cm} (64)

which leads to the relation

$$Z_2 \Delta_{pg}^2 \simeq \int \frac{d^3 q}{(2\pi)^3} \frac{b c |q|}{c |q|} = \frac{T^2}{12c^3}.$$  \hspace{1cm} (65)

Combining the above equations, we find

$$T_c = \left( \frac{24 \epsilon^3 Z_2 n_B}{R m_B} \right)^{1/2}.$$  \hspace{1cm} (66)

In the RBEC limit $\mu \to 0$, we can approach to the standard critical temperature for ultra-relativistic Bose-Einstein condensation\[22, 23\],

$$T_c = \left( \frac{3m_B}{m_B} \right)^{1/2}.$$  \hspace{1cm} (67)
Since \( n_B \) is almost fixed and \( m_B \to 0, T_c \) would approach to infinity in the RBEC limit. In the ultra-relativistic boson gas, the specific heat at low temperature is proportional to \( T^3 \).

We now turn to numerical calculations. From the coupled equations \[8\], \[9\] and \[39\], we can solve the critical temperature \( T_c \), chemical potential \( \mu(T_c) \) and pseudogap \( \Delta_{pg}(T_c) \) at \( T_c \) as functions of the coupling \( \eta \) at fixed \( k_f/m \). In Fig.\[4\] we show the numerical results with the parameters \( \Lambda/m = 10 \) and \( k_f/m = 0.5 \). The BCS-NBEC-RBEC crossover can be seen directly from the behavior of the chemical potential \( \mu \). In the BCS region with \( -\infty < \eta < 0.5 \), \( \mu \) is larger than the fermion mass and approaches to the Fermi energy \( E_f \) in the weak coupling limit \( \eta \to -\infty \). The NBEC region is located around \(-0.5 < \eta < 4 \) and the RBEC region is at about \( \eta > 4 \).

The critical coupling \( \eta \approx 4 \) for the RBEC state is consistent with our analytical result

\[
\eta_c = \frac{2}{\pi k_f} \ln \left( \frac{\Lambda}{m} + \sqrt{(\Lambda/m)^2 + 1} \right)
\]

(68)
derived in \[19\]. The difference between NBEC and RBEC states is that the chemical potential \( \mu \) is of the order of \( m \) in the NBEC region but approaches zero in the RBEC region.

The critical temperature, plotted as the solid line in Fig.\[2\], shows significant change from the weak to strong coupling. To compare it with the standard critical temperature for the ideal boson gas, we solve the equation \[22\]

\[
\int \frac{d^3q}{(2\pi)^3} \left[ b \left( \epsilon_q^B - \mu_B \right) - b \left( \epsilon_q^B + \mu_B \right) \right] \bigg| \mu_B = m_B = n_B
\]

(69)
with \( \epsilon_q^B = \sqrt{q^2 + m_B^2} \), boson mass \( m_B = 2\mu \) and boson number \( n_B = n/2 \), and show the obtained critical temperature as dashed line in Fig.\[2\]. In the weak coupling region \( T_c \) is very small and agrees with the BCS theory. In the NBEC region \( T_c \) changes smoothly and there is no remarkable difference between the solid and dashed lines. Around the coupling \( \eta_c = 4 \), \( T_c \) increases rapidly and then varies smoothly again. In the RBEC region, the critical temperature from our calculation deviates significantly from the standard critical temperature for ideal boson gas. Note that, \( T_c \) is of the order of the Fermi kinetic energy \( \epsilon_f \simeq k_f^2/(2m) \) in the NBEC region but becomes as large as the Fermi energy \( E_f \) in the RBEC region. The pseudogap \( \Delta_{pg} \) at \( T = T_c \), shown in Fig.\[2\], behaves similarly as the critical temperature. To see clearly the pseudogap region, we present in Fig.\[2\]a the limit temperature \( T^* \) as a dotted line. The pseudogap exists between the solid and dotted lines and begin to vanish on the dotted line.

To explain why the critical temperature in the RBEC region deviates remarkably from the standard one for ideal boson gas, we calculate the boson number fraction \( n_B = n_B/(n/2) \) and the fermion number fraction \( n_F = 1 - n_B \) and show them as functions of the coupling \( \eta \) in Fig.\[3\]. While there are only bosons at \( T_c \) in the NBEC region, \( n_B \) is obviously less than 1 in the RBEC region. This conclusion is consistent with the results from the NSR theory \[16\], \[17\]. In the NBEC region, the binding energy of the molecules is \( E_b \simeq 1/ma^2 = 2\eta^2\epsilon_f \), which is much larger than the critical temperature \( T_c \), and the molecules can be safely regarded as point bosons at temperature near \( T_c \). However, the critical temperature in the RBEC region is as large as the Fermi energy \( E_f \), which is of the order of the molecule binding energy \( E_b \simeq 2m \). Due to the competition between the condensation and dissociation of composite bosons in hot medium, the molecules can not be regarded as point bosons and the critical temperature should deviates from the result.
FIG. 3: The boson number fraction $r_B$ and the fermion number fraction $r_F$ at the critical temperature $T_c$, as functions of the coupling $\eta$ at $\Lambda/m = 10$ and $k_f/m = 0.5$.

for ideal boson gas. This may be a general characteristic of a composite boson system, when the condensation temperature $T_c$ is of the order of the molecule binding energy. The phenomenon can be explained by the competition between free energy and entropy [17]: In terms of entropy a two-fermion state is more favorable than a one-fermion state, but in terms of free energy it is less favorable. Since the condensation temperature $T_c$ in the RBEC region is of the order of $(n_B/m_B)^{1/2} \sim (n/\mu)^{1/2}$, we conclude that only for a system with sufficiently small value of $k_f/m$, the standard RBEC critical temperature can be reached and is much smaller than $2m$.

V. APPLICATION TO MASSLESS FERMIONS: COLOR SUPERCONDUCTIVITY

As a natural application of the relativistic $G_0G$ formalism, we calculate in this section the transition temperature and pseudogap in two flavor color superconductivity at moderate baryon density. The two flavor color superconducting quark matter corresponds to the ultra-relativistic case with $n \gg m^3$, where $m$ is the current quark mass. At moderate baryon density, the quark energy gap due to color superconductivity is of the order of 100 MeV, which is not located in the weak coupling region. As a result, the pseudogap effect is expected to be significantly important near the critical temperature. To apply the present theory directly, we employ the generalized Nambu–Jona-Lasinio(NJL) model with scalar diquark channel, which has been widely used to study color superconductivity at moderate baryon density. The Lagrangian density is defined as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + G_a \left[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\tau\psi)^2\right] + G_d \sum_{a=2,5,7} (\bar{\psi}\gamma^5 \tau_2 \lambda_a C\bar{\psi})^T (\psi^T C\bar{\psi})^5 \tau_2 \lambda_a \psi,$$

(70)

where $\psi$ and $\bar{\psi}$ denote the quark fields with two flavors ($N_f = 2$) and three colors ($N_c = 3$), $\tau_i (i = 1, 2, 3)$ are the Pauli matrices in flavor space and $\lambda_a (a = 1, 2, \ldots, 8)$ are the Gell-Mann matrices in color space, and $G_a$ and $G_d$ are coupling constants in meson and diquark channels.

At moderate baryon density, the chiral symmetry has already been restored and we need not consider the chiral condensate $\langle \bar{\psi}\psi \rangle$. Since the current quark mass $m$ is about 5 MeV, the quarks are nearly massless. The order parameter field for color superconductivity is defined as

$$\Phi_a = -2G_d \psi^T C\bar{\psi}\gamma^5 \tau_2 \lambda_a \psi.$$

(71)

To simplify the calculation, one usually considers a spontaneous color breaking from the SU(3) symmetry to a SU(2) subgroup. Due to the residual color SU(2) symmetry, the effective potential in mean field approximation should depend only on the combination $\Delta_2 + \Delta_3^2 + \Delta_5^2$ with $\Delta_a = \langle \Phi_a \rangle$, and we can choose a specific gauge $\Delta_3 = \Delta_5 = 0$ without loss of generality. In this gauge, the red and green quarks participate in the condensation, but the blue one does not.

The detailed formalism of the $G_0G$ theory in the NJL model is similar to what we shown in sections II and III but becomes somewhat complicated due to the presence of color and flavor degrees of freedom. Comparing the quark propagator in the NJL model with the one shown in above sections, the relativistic $G_0G$ scheme can be directly applied to the study of color superconductivity, provided that we consider carefully the difference between the pairing including a blue quark and the pairing with only red and green quarks. The dispersion for red and green quarks is identical with the one obtained in the toy model, their excitation gap is $\Delta = (\Delta_2^2 + \Delta_3^2)^{1/2}$, and the pair susceptibility $\chi(q)$ should be multiplied by a factor $N_f(N_c - 1)$ where $N_f$ and $N_c$ are flavor and color numbers of quarks. The new feature is that a gapless blue quark in the naive BCS mean field theory obtains a gap $\Delta_{pg}$ in the $G_0G$ scheme. This can be understood by the fact that, the color symmetry is controlled only by the order parameters themselves, and fluctuations of any order parameter field $\Phi_a$ do not change it. At and above the critical temperature, $\Delta_{sc} = 0$, the color symmetry is restored, all colors become degenerate, and their gaps are just the pseudogap.

The two flavor quark matter may exist in the region of $\mu = 350 - 500$ MeV, where the strange quarks are not yet excited. Unlike the study in above sections in the canonical ensemble with fixed fermion number, people usually investigate color superconductivity in the grand canonical ensemble with fixed quark chemical potential. In this case, the quark number is not directly coupled to the calculation of the order parameter $\Delta_a$ and pseudogap $\Delta_{pg}$. For numerical calculations, we take the current quark mass $m = 5$ MeV, the often used quark momentum cutoff $\Lambda = 650$ MeV, and a fixed quark chemical potential $\mu = 400$ MeV. We have checked that a reasonable change in the value of $\mu$ does not bring qualitative difference. As is conventionally considered in the literatures, we use the pairing gap $\Delta_0$ at zero temperature to reflect the strength of the diquark coupling constant $G_d$. 
In Fig. 4 we show the critical temperature $T_c$ as a function of $\Delta_0$ in the $G_0G$ theory and in the BCS mean field theory. While the critical temperature is not strongly modified by the diquark fluctuations in a wide range of $\Delta_0$, the difference between the two is up to 20% in the strong coupling region with $\Delta_0 \simeq 200$ MeV. In Fig. 5 we show the pseudogap $\Delta_{pg}$ at the critical temperature $T_c$. In a wide range of the coupling, the pseudogap is of the order of 100 MeV, which is as large as the diquark condensate $\Delta_{sc}$ at zero temperature. Such a behavior means that the two flavor color superconductivity at moderate density is in the BCS-BEC crossover region and quite like the high temperature superconductivity in cuprates[6, 7]. Since $\Delta_{sc}$ vanishes at $T = T_c$, the large pseudogap will bring significant effect at and above $T_c$, such as the non-Fermi liquid behavior. In Fig. 6 we show the temperature dependence of the diquark condensate $\Delta_{sc}$ and pseudogap $\Delta_{pg}$ at two values of $\Delta_0$. With increasing temperature, while the diquark condensate decreases, the pseudogap increases from zero. At low temperature, especially at zero temperature, we can safely neglect the pseudogap.

While the pseudogap is small at low temperature and dominates the system only near and above $T_c$, the diquark fluctuations bring significant contribution to thermodynamics at any temperature. In the low temperature region, the temperature behavior of the pseudogap is significantly important, since it can tell us whether the coefficient $Z_1$ or $Z_2$ dominates the pair fluctuations. In Fig. 7 we show the pseudogap at low temperature. In the region of $T/T_c \leq 0.1$, it obeys a perfect power law $\Delta_{pg} \propto T^{3/4}$, which means that $Z_1$ is the dominant one for the pair susceptibility.

Considering the uncondensed diquarks, the total ther-
Since the coefficient $Z_{1}$ controls the pair fluctuations at low temperature, the specific heat satisfies the power law $C \propto T^{3/2}$. As we mentioned above, the diquark contribution can be neglected only at sufficiently weak coupling. While the color superconductor at moderate baryon density may not reach the BEC condition, the effect of diquark fluctuations on the thermodynamics may be remarkable, and it may bring significant astrophysical consequences, such as the cooling process in compact stars.

VI. SUMMARY

We have generalized the non-relativistic $G_{a}G$ formalism of BCS-BEC crossover to relativistic fermion systems. The theory can describe the superfluidity/superconductivity with arbitrary strength of attractive interaction, both in the symmetric phase and symmetry breaking phase. The beyond-BCS effect at strong coupling brings in thermally excited bosons and contributes a pseudogap to fermion excitations. In such a formalism, we confirmed that there exists a BCS-NBEC-BEC crossover in relativistic fermion systems.

For color superconductivity at moderate baryon density, while the BEC state can not be reached, the effect of diquark fluctuations is still remarkable and the naive BCS mean field theory breaks down when the temperature is close to the critical value. We investigate the two flavor color superconductivity at quark chemical potential $\mu = 350 - 500$ MeV where the gap at zero temperature is of the order of 100 MeV. We found that the beyond-BCS effect strongly suppresses the transition temperature, and the pseudogap is very large near the critical temperature. This may strongly modify the thermodynamics of quark matter and bring significant astrophysical consequences in the study of compact stars.

Such a theory can be applied to not only diquark condensate $\langle \bar{q}q \rangle$ at finite baryon density but also chiral condensate $\langle \bar{q}q \rangle$ at finite temperature and pion superfluidity at finite isospin density. The observation of $\bar{q}q$ bound states in strongly coupled quark-gluon plasma and a large thermal quark mass above the chiral phase transition temperature in lattice QCD indicate strongly the significance of the $\bar{q}q$ Bose-Einstein condensation and the quark pseudogap effect. The study in this direction is under progress.

Acknowledgement: We thank Dr. Mei Huang for drawing our attention to [6] and [7] and Gaofeng Sun for a careful check on the calculations in the Appendix. This work is supported by the grants No. NSFC10428510, No.10575058 and the major state basic research developing program 2007CB815000.

APPENDIX A: PAIR SUSCEPTIBILITY AND ITS EXPANSION COEFFICIENTS

In this appendix, we evaluate the explicit expression of the pair susceptibility and its momentum expansion. Completing the trace in Dirac space and the Matsubara summation over the fermion frequencies, we obtain from the equations (30), (16) and (11)

\[
\chi(q) = \int \frac{d^3k}{(2\pi)^3} \left[ \left( 1 - f(E_k^-) - f(\xi_{q-k}^-) \right) \frac{E_k^- + \xi_{q-k}^-}{2E_k^-} - \left( 1 - f(E_k^-) - f(\xi_{q-k}^-) \right) \frac{E_k^- - \xi_{q-k}^-}{2E_k^-} \right] \left( \frac{1}{2} - \frac{k \cdot (q-k) - m^2}{2\epsilon_{kq-k}} \right)
\]

\[
+ \left( 1 - f(E_k^-) - f(\xi_{q-k}^-) \right) \frac{E_k^- - \xi_{q-k}^-}{2E_k^-} \left( \frac{1}{2} + \frac{k \cdot (q-k) - m^2}{2\epsilon_{kq-k}} \right)
\]

\[
+ \left( E_{k,\xi_k}^\pm, q_0 \rightarrow E_{k,\xi_k}^\mp, -q_0 \right).
\]

Taking its first and second order derivatives with respect to $q_0$, we have

\[
Z_1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left[ \left( 1 - f(E_k^-) - f(\xi_k^-) \right) \frac{E_k^- + \xi_k^-}{E_k^- - \xi_k^-} + f(E_k^-) - f(\xi_k^-) \right] \left( E_{k,\xi_k}^\pm, \xi_k \rightarrow E_{k,\xi_k}^\mp, \xi_k \right),
\]

\[
Z_2 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left[ \left( 1 - f(E_k^-) - f(\xi_k^-) \right) \frac{E_k^- + \xi_k^-}{(E_k^- + \xi_k^-)^2} - \left( E_{k,\xi_k}^\pm, \xi_k \rightarrow E_{k,\xi_k}^\mp, \xi_k \right) + \left( E_{k,\xi_k}^\pm, \xi_k \rightarrow E_{k,\xi_k}^\mp, \xi_k \right). \right. \]
Using the relation \((E_k^\pm)^2 - (\xi_k^\pm)^2 = \Delta^2\), the coefficients can be rewritten as

\[
Z_1 = \frac{1}{2\Delta^2} \left[ n - 2 \int \frac{d^3k}{(2\pi)^3} \left( f(\xi_k^+) - f(\xi_k^-) \right) \right], \\
Z_2 = \frac{1}{2\Delta^4} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{(E_k^-)^2 + (\xi_k^-)^2}{E_k^-} \left( 1 - 2f(\xi_k^-) \right) - 2\xi_k^- \left( 1 - 2f(\xi_k^-) \right) \right] + \left( E_k^\pm, \xi_k^\pm \rightarrow E_k^\mp, \xi_k^\mp \right).
\]

Taking the second order derivative of the susceptibility \(\chi\) with respect to \(q\), we obtain another coefficient

\[
\xi^2 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2E_k} \left( 1 - f(E_k^-) - f(\xi_k^-) \right) + \frac{f(E_k^-) - f(\xi_k^-)}{E_k^- - \xi_k^-} \right] \xi_k^2 - \frac{k^2x^2}{\xi_k^2} \\
- \left( \frac{1}{E_k^2} \left( 1 - f(E_k^-) - f(\xi_k^-) \right) \frac{f(E_k^-) - f(\xi_k^-)}{(E_k^- - \xi_k^-)^2} + \frac{2f'(\xi_k^-)}{\Delta^2} \right) \frac{k^2x^2}{\xi_k^2} \\
- \left( \frac{1}{E_k^2} \left( 1 - f(E_k^-) - f(\xi_k^-) \right) \frac{f(E_k^-) - f(\xi_k^-)}{(E_k^- - \xi_k^-)^2} - \frac{2f(E_k^-)}{2E_k^-} \right) \frac{\xi_k^2 - k^2x^2}{2\xi_k^2} \\
+ \left( E_k^\pm, \xi_k^\pm \rightarrow E_k^\mp, \xi_k^\mp \right)
\]

with \(x = \cos \theta\) and \(f'(x)\) being the first order derivative of the Fermi-Dirac distribution.

In the non-relativistic limit with \(k_f \ll m, |\mu - m|, \Delta \ll m\), all the terms including anti-fermion dispersions can be safely neglected, and the relativistic dispersions are reduced to \(\xi_k = k^2/(2m) - (\mu - m)\) and \(E_k = \sqrt{\xi_k^2 + \Delta^2}\). Taking into account \(|q| \ll m\), we have

\[
\chi_{NR}(q) = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1 - f(E_k^-) - f(\xi_{q-k})}{E_k^- + \xi_{q-k} - q_0} \frac{E_k + \xi_k}{2E_k} - \frac{f(E_k^-) - f(\xi_{q-k})}{E_k^- - \xi_{q-k} + q_0} \frac{E_k - \xi_k}{2E_k} \right],
\]

which is just the same as the one given in [6, 7], and the expansion coefficients \(Z_1, Z_2\) and \(\xi^2\) are reduced to [6, 7]

\[
Z_1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left[ 1 - f(E_k^-) - f(\xi_k) \right] \left( \frac{E_k + \xi_k}{E_k^- + \xi_k} + \frac{f(E_k^-) - f(\xi_k^-)}{E_k^- - \xi_k^-} \right) \\
= \frac{1}{2\Delta^2} \left[ n - 2 \int \frac{d^3k}{(2\pi)^3} f(\xi_k) \right], \\
Z_2 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left[ 1 - f(E_k^-) - f(\xi_k^-) \right] \left( \frac{E_k + \xi_k^-}{E_k^- + \xi_k^-} + \frac{f(E_k^-) - f(\xi_k^-)}{E_k^- - \xi_k^-} \right) \\
= \frac{1}{2\Delta^4} \int \frac{d^3k}{(2\pi)^3} \frac{E_k^2 + \xi_k^2}{E_k^-} \left( 1 - 2f(E_k^-) \right) - 2\xi_k \left( 1 - 2f(\xi_k^-) \right), \\
\xi^2 = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{4mE_k} \left( 1 - f(E_k^-) - f(\xi_k) \right) \left( \frac{E_k + \xi_k}{E_k^- + \xi_k} + \frac{f(E_k^-) - f(\xi_k^-)}{E_k^- - \xi_k^-} \right) \\
- \frac{k^2}{6m^2} \left( \frac{1}{E_k} \left( 1 - f(E_k^-) - f(\xi_k) \right) \frac{f(E_k^-) - f(\xi_k^-)}{(E_k^- - \xi_k^-)^2} + \frac{2f'(\xi_k)}{\Delta^2} \right) \right].
\]

In the RBEC limit with \(\mu \rightarrow 0\), we can expand \(Z_1\) in powers of \(\mu\), \(Z_1 \simeq R\mu + O(\mu^3)\), with the expansion coefficient \(R\) given by

\[
R = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1 - 2f(E_k^-)}{E_k^3} - 2 \frac{E_k^2 f'(E_k^-)}{E_k^3} + 2 \frac{f'(\xi_k)}{\Delta^2} \right].
\]
where $E_k = \sqrt{\epsilon_k^2 + \Delta^2}$ is the dispersion at $\mu = 0$, and $Z_2$ and $\xi^2$ can be simplified as

$$Z_2 = \frac{1}{4\Delta^2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{E_k^2 + \epsilon_k^2}{E_k} (1 - 2f(E_k)) - 2\epsilon_k (1 - 2f(\epsilon_k)) \right],$$

$$\xi^2 = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2\Delta^2} \left( (1 - 2f(\epsilon_k)) - \frac{\epsilon_k}{E_k} (1 - 2f(E_k)) \right) \frac{\epsilon_k^2 - k^2x^2}{\epsilon_k^2} \right. $$

$$\left. - \left( \frac{1}{2\Delta^2} \left( \frac{E_k^2 + \epsilon_k^2}{E_k} (1 - 2f(E_k)) - 2\epsilon_k (1 - 2f(\epsilon_k)) \right) + \frac{2f'(\epsilon_k)}{\Delta^2} \right) \frac{k^2x^2}{\epsilon_k^2} \right] .$$

(A8)

[1] D.M.Eagles, Phys.Rev.186, 456(1969)
[2] A.J.Leggett, in Modern Trends in the Theory of Condensed Matter, edited by A.Pekalski and J.Przystawa (Springer-Verlag, Berlin, 1980)
[3] P.Nozieres and S.Schmitt-Rink, J.Low.Temp.Phys 59, 195(1985)
[4] C.A.RSa de Melo, M.Randeria and J.R.Engelbrecht, Phys.Rev.Lett.71, 3202(1993)
[5] J.R.Engelbrecht, M.Randeria and C.A.RSa de Melo, Phys.Rev.B55, 15153(1997)
[6] Q.Chen, J.Stajic, S.Tan and K.Levin, Phys.Rept.412, 1(2005)
[7] Q.Chen, Generalization of the BCS Theory to Short Coherence Length Superconductors: A BCS–Bose–Einstein Crossover Scenario(PhD Thesis, the University of Chicago, 2000)
[8] V.M.Loktev, R.M.Quick and S.G.Sharapov, Phys.Rept.349, 1(2001)
[9] M.Greiner, C.A.Regal and D.S.Jin, Nature426, 537(2003); M.W.Zwierlein, et.al., Nature435, 1047(2005)
[10] K.Rajagopal and F.Wilczek, hep-ph/0011333

D.K.Hong, Acta Phys.Pol.B32, 1253(2001); M. Alford, Ann.Nucl.Part.Sci.51, 131(2001); T.Schäfer, hep-ph/0304281

D.H.Rischke, Prog.Part.Nucl.Phys.52, 197(2004); M.Buballa, Phys.Rept.407, 205(2005); H.-C.Ren, hep-ph/0404074

M.Huang, Int.J.Mod.Phys.E14, 675(2005); I.A.Shovkovy, Found.Phys.35, 1309(2005).
[11] G.Sun, L.He and P.Zhuang, Phys.Rev.D75, 096004(2007)
[12] P.Cstorina, G.Nardulli and D.Zappala, Phys.Rev.D72, 076006(2005)
[13] E.Babaev, Phys.Rev.D62, 074020(2000); Int.J.Mod.Phys.A16, 1175(2001)
[14] R.Rapp, T.Schäfer, E.V.Shuryak and M.Velkovsky, Phys.Rev.Lett.81, 53(1998); M.Alford, K.Rajagopal and F.Wilczek, Phys.Lett.B422, 247(1998); D.T.Son, Phys.Rev.D59, 094019(1999)
[15] M.Kitazawa, T.Koide, T.Kunihiro and Y.Nemoto, Phys.Rev.D65, 091504(R)(2002); ibid 70, 056003(2004)
[16] Y.Nishida and H.Abuki, Phys.Rev. D72, 096004(2005)
[17] H.Abuki, Nucl.Phys.A791, 117(2007)
[18] J.Deng, A.Schmitt and Q.Wang, Phys.Rev.D76, 034013(2007)
[19] L.He and P.Zhuang, Phys.Rev.D75, 096003(2007)
[20] P.Pieri, L.Pisani and G.C.Strinati, Phys.Rev.B70, 094508(2004)
[21] Kerson Huang, Statistical Mechanics(Wiley, New York, 1987)
[22] J.I.Kapusta, Finite Temperature Field Theory (Cambridge University Press, Cambridge, 1989)
[23] H.E.Haber and H.A.Weldon, Phys.Rev.Lett.46,1497(1981); J.I.Kapusta, Phys.Rev.D24,426(1981)
[24] I.Kosztin, Q.J.Chen, B.Janko and K.Levin, Phys.Rev.B58, R939(1998); Q.J.Chen, I.Kosztin, B.Janko and K.Levin, Phys.Rev.Lett.81, 4708(1998); I.Kosztin, Q.J.Chen, Y.J.Kao and K.Levin, Phys.Rev.B61, 11662(2000)
[25] L.P.Kadanoff and P.C.Martin, Phys.Rev.124, 670(1961)
[26] E.V.Shuryak, Nucl.Phys.A774, 387(2006); hep-ph/0608177
[27] P.Petreczky, F.Karsch, E.Laermann, S.Stickan and I.Wetzorke, Nucl.Phys.Proc.Suppl.106, 513(2002); hep-lat/0110006
[28] L.He and P.Zhuang, in progress
[29] Q.J.Chen, Y.He, C.C.Chien and K.Levin, Phys.Rev.B75, 014521(2007); K.Levin and Q.J.Chen, cond-mat/0610006