Constant mean curvature foliation properties in the extended Reissner-Nordström spacetimes

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Abstract

We solve the spacelike, spherically symmetric, constant mean curvature hypersurfaces in the maximally extended Reissner-Nordström spacetime with the charge smaller than the mass. Based on these results, we construct constant mean curvature foliations with fixed or varied mean curvature in each slice in this spacetime.

1 Introduction

In general relativity, spacelike constant mean curvature (CMC) hypersurfaces in spacetimes are important geometric objects. Not only CMC hypersurfaces are used in the analysis on Einstein constraint equations \cite{1, 6} and in the gauge condition in the Cauchy problem of the Einstein equations \cite{2, 3}, but also the CMC foliation in cosmological spacetime is identified as the absolute time function \cite{14}. Some introduction to the CMC foliation theory can be found in \cite{8} and references therein.

In this article we are interested in studying spacelike, spherically symmetric, constant mean curvature (SS-CMC) hypersurfaces in the maximally extended Reissner-Nordström spacetime with the charge smaller than the mass and finding CMC foliations in this spacetime. The Reissner-Nordström spacetime is a static solution of the Einstein-Maxwell field equations. The Reissner-Nordström metric reduces to the Schwarzschild metric if there is no charge. Since we have known many results on the SS-CMC hypersurfaces and CMC foliation properties in the extended Schwarzschild spacetime (Kruskal extension) \cite{7, 8, 9, 10, 11, 12}, they are very helpful to explore CMC related questions in the Reissner-Nordström spacetime. Remark that Tuite and Ó Murchadha also studied spacelike CMC hypersurfaces in the Reissner-Nordström spacetime \cite{13}. Here we provide different points of view and show more properties in this topic.

To sum up, we answer some CMC hypersurfaces related questions in the Reissner-Nordström spacetime. These results are summarized as the following main theorem.
Main Theorem. Consider the maximally extended Reissner-Nordström spacetime with the charge smaller than the mass.

(a) The initial value problem for the spacelike, spherically symmetric, constant mean curvature hypersurface equation in the maximally extended Reissner-Nordström spacetime is solvable and the solution is unique.

(b) The Dirichlet problem for the spacelike, spherically symmetric, constant mean curvature hypersurface equation in the maximally extended Reissner-Nordström spacetime is solvable and the solution is unique.

(c) Given $H \in \mathbb{R}$, we can construct a CMC foliation in the maximally extended Reissner-Nordström spacetime so that each slice has the constant mean curvature $H$.

(d) There is a CMC foliation with mean curvature in each slice increasing along the future time direction in the maximally extended Reissner-Nordström spacetime.

More precise statements of these results can be seen in Theorem 1-3 and Theorem 6.

The idea for proving these results comes from the analyses when dealing with CMC hypersurfaces in the Schwarzschild spacetime [7, 8, 9, 10]. Although this main theorem can be viewed as a generalization of theorems in the Schwarzschild spacetime, we should mention the essential differences and difficulties between the Reissner-Nordström spacetime and the Schwarzschild spacetime. One main difference is that the maximally extended Reissner-Nordström spacetime is formed by infinitely many Reissner-Nordström spacetimes. Compared to the extended Schwarzschild spacetime (Kruskal extension), it is formed by gluing only two Schwarzschild spacetimes. Because we focus on the Reissner-Nordström spacetime with the charge smaller than the mass, every Reissner-Nordström spacetime in standard coordinates is divided into three regions, $0 < r < r_-$, $r_- < r < r_+$, and $r > r_+$. The mechanism to glue infinitely many Reissner-Nordström spacetimes is through the region $0 < r < r_-$, but no such region is in the Schwarzschild spacetime.

In order to construct a CMC foliation with fixed mean curvature in each slice, once we get a CMC foliation in a pair of Reissner-Nordström spacetimes, we can copy the same CMC foliation to other pairs and get the whole CMC foliation in the maximally extended Reissner-Nordström spacetimes. However, if we want to find a CMC foliation with varied mean curvature in each slice, we have to select different slices on different pairs of spacetimes so that the mean curvature is increasing along the future time direction. This will be the essential difficulties in this topic. Since different Reissner-Nordström spacetime regions cause some different situations, we separately estimate each term of the derivative of a function to show the positivity or negativity, which is equivalent to the
increasing mean curvature property in different regions. Fortunately, we can construct a CMC foliation with required properties in the maximally extended Reissner-Nordström spacetime.

The organization of this paper is as follows. Section 2 is a brief introduction to the Reissner-Nordström spacetime and its Penrose diagram. We describe all spacelike, spherically symmetric, constant mean curvature hypersurfaces in the extended Reissner-Nordström spacetime and solve the initial value problem for the constant mean curvature equation in section 3. Dirichlet problem for the constant mean curvature equation is discussed in section 4. In section 5, we will construct two types of constant mean curvature foliations in the extended Reissner-Nordström spacetime. Finally, we summarize results of this paper in section 6 and put some detailed computations in Appendix.

2 The Reissner-Nordström spacetime

In this section, we will briefly introduce the Reissner-Nordström spacetime and its Penrose diagram. The Reissner-Nordström spacetime is a 4-dimensional time-oriented Lorentzian manifold equipped with the metric

\[ ds^2 = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]

where \( M > 0 \) represents the gravitational mass, and \( e \) is the electric charge. When \( e = 0 \), the Reissner-Nordström spacetime reduces to the Schwarzschild spacetime. Here we denote \( h(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2} \) in convenience.

In this article, we will focus on the case \( e^2 < M^2 \). It implies \( h(r) = 0 \) has two real roots \( r_\pm = M \pm \sqrt{M^2 - e^2} \), and the metric is regular in region I (\( r > r_+ \)), region II (\( r_- < r < r_+ \)), and region III (\( 0 < r < r_- \)). When \( r = r_\pm \), they are coordinates singularities. The metric can be analytically extended at \( r = r_\pm \). The maximally extended Reissner-Nordström spacetime can be constructed by connecting infinitely many Reissner-Nordström spacetimes. Its Penrose diagram is illustrated in Figure 1, where regions I’, II’, and III’ come from another Reissner-Nordström spacetime and we upside down of the spacetime to combine them. We refer to Hawking and Ellis’ book [5, pages 156–159] for more discussions on the construction of the maximally extended Reissner-Nordström spacetime.

Here we set \( X \)-axis and \( T \)-axis in this Penrose diagram and choose \( \partial_T \) as the future directed timelike vector field in the maximally extended Reissner-Nordström spacetime.
3 Initial value problem for the SS-CMC equation in the Reissner-Nordström spacetime

Now we are interested in the spacelike, spherically symmetric, constant mean curvature (SS-CMC) hypersurfaces in the maximally extended Reissner-Nordström spacetime. All SS-CMC hypersurfaces are solved and these constructions are similar to the constructions in the Schwarzschild spacetime [9]. First, we solve the SS-CMC equation in each Reissner-Nordström spacetime region, and then we determine the correct parameters to smoothly glue two SS-CMC hypersurfaces at every coordinate singularity so that the joined hypersurface is the SS-CMC hypersurface defined in the extended Reissner-Nordström spacetime. Compared to the Schwarzschild spacetime case, we still need to consider SS-CMC hypersurfaces in region III and look at the behavior of SS-CMC hypersurfaces at $r = r_-$. This case is in fact similar to the case in region I, and it does not cause any problem.

Here we only state the key ingredients for the construction of SS-CMC solutions, and these materials will be used to describe CMC foliations in the maximally extended

Figure 1: Penrose diagram for the maximally extended Reissner-Nordström spacetime with $e^2 < M^2$. 
Reissner-Nordström spacetime in section 5. Suppose that \( \Sigma : (t = f(r), r, \theta, \phi) \) is an SS-CMC hypersurface in the Reissner-Nordström spacetime. The SS-CMC equation in the Reissner-Nordström spacetime is

\[
    f'' + \left( \frac{1}{h} - (f')^2 h \right) \left( \frac{2h}{r} + \frac{h'}{2} + \frac{f'}{h} \right) f' + 3H \left( \frac{1}{h} - (f')^2 h \right) \frac{3}{2} = 0,
\]

where \( h(r) = 1 - \frac{2M}{r} + \frac{\ell^2}{r^2} \) and it implies \( h'(r) = \frac{2M}{r^2} - \frac{2\ell^2}{r^4} \), and \( H \) is the mean curvature. The spacelike condition is equivalent to \( \frac{1}{h} - (f')^2 h > 0 \). Notice that the minus sign (or plus sign) of the term \( \mp 3H \left( \frac{1}{h} - (f')^2 h \right) \frac{3}{2} \) is applied in regions I, II, III (or I', II', III').

The reason for different signs of this term in SS-CMC equations is that we choose \( \partial_T \) as the future timelike direction in the Penrose diagram, and regions I', II', III' come from the upside down of some Reissner-Nordström spacetime. It implies that \( \partial_T \) points to opposite direction between regions I and I', II and II', III and III', respectively.

Taking region II for example, suppose that SS-CMC hypersurfaces are piecewisely divided into graphs of functions with positive slope or negative slope (allowing \( f'(r) = +\infty \) or \(-\infty \) at some point). Denote \( l(r) = \frac{1}{\sqrt{-h(r)}} \left( -Hr + \frac{\ell}{r^2} \right), \)

\[
f'(r) = \begin{cases} 
    \frac{1}{-h(r)} \sqrt{\frac{\ell^2(r)}{l^2(r)-1}} & \text{if } f'(r) > 0 \\
    \frac{1}{h(r)} \sqrt{\frac{\ell^2(r)}{l^2(r)-1}} & \text{if } f'(r) < 0,
\end{cases}
\]

and we require \( l(r) > 1 \).

Similarly, in region II', denote \( l(r) = \frac{1}{\sqrt{-h(r)}} \left( Hr - \frac{\ell}{r^2} \right), \)

\[
f'(r) = \begin{cases} 
    \frac{1}{-h(r)} \sqrt{\frac{\ell^2(r)}{l^2(r)-1}} & \text{if } f'(r) > 0 \\
    \frac{1}{h(r)} \sqrt{\frac{\ell^2(r)}{l^2(r)-1}} & \text{if } f'(r) < 0,
\end{cases}
\]

and we require \( l(r) > 1 \).

Since the SS-CMC equation is a second order ordinary differential equation, the solution of \( f'(r) \) gives a freedom \( c \), and the solution of \( f(r) \) gives another freedom, called \( \bar{c} \) for instance.

From this explicit formula, we know that the domain of the function \( f(r) \) depends on the parameter \( c \). In region II, we get the condition \( c > Hr^3 + r(-r^2 + 2Mr - e^2)\frac{1}{2} \) denote \( F(H, r) \), so the domain of \( f(r) \) is

\[
    \{ r \in (r_-, r_+) | c > F(H, r) \} \cup \{ r \in (r_-, r_+) | c = F(H, r) \text{ and } f(r) \text{ is finite} \}.
\]

\(^1\)Compared to the paper \(^3\) page 13, line −8, or page 15, line −8, here we change the sign of the parameter, that is, let \( c = -c_2 \). The reason for changing the sign of this parameter is that it will be easier to describe the CMC foliation properties in section 5.
In region $\Pi'$, we get the condition $c < Hr^3 - r(-r^2 + 2Mr - e^2)^{\frac{1}{2}} \bar{G}(H, r)$, and the domain of $f(r)$ is

$$\{r \in (r_-, r_+) | c < G(H, r)\} \cup \{r \in (r_-, r_+) | c = G(H, r) \text{ and } f(r) \text{ is finite}\}.$$  

Let $C_H = \max_{r \in [r_-, r_+]} F(H, r) = F(H, R_H)$ and $c_H = \min_{r \in [r_-, r_+]} G(H, r) = G(H, r_H)$. Now we are ready to describe SS-CMC hypersurfaces in the extended Reissner-Nordström spacetime.

(A) If $c \in (c_H, C_H)$, two graphs of $f(r)$ with positive slope and negative slope can be smoothly glued at the point which satisfies $f'(r) = +\infty$ or $-\infty$ by choosing correct $\bar{c}$, so the union of two graphs of $f(r)$ forms an SS-CMC hypersurface in region $\Pi$ or $\Pi'$. This joined SS-CMC hypersurface will touch the coordinates singularities $r = r_-$ or $r = r_+$ in the extended Reissner-Nordström spacetime sense, and we can find an SS-CMC hypersurface in different region with the same $c$ and correct $\bar{c}$ to glue them smoothly and finally we get the whole SS-CMC hypersurface. In this case, the SS-CMC hypersurface will range $\Pi'$, $\Pi$, $\Pi$ (see Figure 2 (A)), or $\Pi'$, $\Pi$, $\Pi'$, or $\Pi'$, $\Pi'$, $\Pi$, $\Pi'$, $\Pi'$.

(B) If $c = C_H$ (or $c = c_H$), $f(r)$ is defined on $(r_-, R_H)$ or $(R_H, r_+)$. By calculating the order of $f'(r)$ near $r = R_H$ (or $r = r_+$), we know that $f'(r) \in O(|r - R_H|^{-1})$ (or $f'(r) \in O(|r - r_+|^{-1})$) so that $\lim_{r \to R_H} |f(r)| = \infty$ (or $\lim_{r \to r_+} |f(r)| = \infty$). This SS-CMC hypersurface will touch coordinate singularities $r = r_-$ or $r = r_+$, and we can find an SS-CMC hypersurface in region $\Pi'$, $\Pi'$, $\Pi'$, or $\Pi'$, $\Pi'$, $\Pi'$, or $\Pi'$, $\Pi'$, $\Pi'$, $\Pi'$, $\Pi'$.

(C) If $c > C_H$ (or $c < c_H$), $f(r)$ is defined on $(r_-, r_+)$ in region $\Pi'$ (or $\Pi'$), Such SS-CMC hypersurface will touch coordinate singularities $r = r_-$ or $r = r_+$, and we can find an SS-CMC hypersurface in region $\Pi'$, $\Pi'$, $\Pi'$, or $\Pi'$, $\Pi'$, $\Pi'$, or $\Pi'$, $\Pi'$, $\Pi'$, $\Pi'$, $\Pi'$, $\Pi'$, $\Pi'$.

(D) Besides SS-CMC hypersurfaces range over different regions, there are another types of SS-CMC hypersurfaces, called cylindrical hypersurfaces. These hypersurfaces are of the form $(t, r = r_0, \theta, \phi)$, where $r_0 \in (r_-, r_+)$ in region $\Pi$ (see Figure 2 (D)) or region $\Pi'$ with constant mean curvature

$$H(r) = \frac{\pm 1}{3\sqrt{-h(r)}} \left( \frac{2h(r)}{r} + \frac{h'(r)}{2} \right) = \frac{\pm(2r^2 - 3Mr + e^2)}{3r^2\sqrt{-r^2 + 2Mr - e^2}}.$$ 

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From the above discussion, we can state and prove the theorem about the existence and uniqueness of the initial value problem for the SS-CMC equation in the maximally extended Reissner-Nordström spacetime.

**Theorem 1.** Given \( H \in \mathbb{R} \), a point \((T_0, X_0)\), and a value \( V \) with \( 1 - V^2 > 0 \) in the Penrose diagram of the Reissner-Nordström spacetime, there is a unique function \( T = T(X) \) such that \( T(X_0) = T_0 \), \( T'(X_0) = V \), and the hypersurface \( \Sigma : (T = T(X), X, \theta, \phi) \) is spacelike \((1 - (T'(X))^2) > 0\) with constant mean curvature \( H \).

**Proof.** The analyses (A)–(D) indicate that the SS-CMC equation in the standard coordinates in each region is solvable, and the initial values \( T(X_0) = T_0 \) and \( T'(X_0) = V \) will uniquely determine \( c \) and \( \bar{c} \) at the SS-CMC hypersurface containing \((T_0, X_0)\). For SS-CMC hypersurfaces in different regions, in order to smoothly glue together, their \( c \) and \( \bar{c} \) are also uniquely determined. Finally, since there is a one-to-one correspondence between the standard coordinates in each region and the Penrose diagram, the theorem is described in the maximally extended Reissner-Nordström spacetime sense and is proved. 

4 Dirichlet problem for the SS-CMC equation in the Reissner-Nordström spacetime

One application of Theorem 1 is to answer the Dirichlet problem for the SS-CMC equation in the extended Reissner-Nordström spacetime. Here we only focus on the Dirichlet
problem with symmetric boundary data because our goal is to construct $T$-axisymmetric SS-CMC foliations in next section.

Before describing the Dirichlet problem, let us go back to the construction (A) and (D) in section\(^3\) to collect all $T$-axisymmetric SS-CMC hypersurfaces. An SS-CMC hypersurface $\Sigma$ is determined by two parameters $c$ and $\bar{c}$. The effect of the parameter $\bar{c}$ in the standard coordinates $(t, r, \theta, \phi)$ means the translation of an SS-CMC hypersurface $\Sigma$ along the $t$-direction, and it presents the Lorentzian isometry in the Penrose diagram. In case (A), given $H \in \mathbb{R}$ and $c \in (c_H, C_H)$, among the family of SS-CMC hypersurfaces $\{\Sigma_{H, c, \bar{c}}\}_{\bar{c} \in \mathbb{R}}$, there is a unique hypersurface with $T$-axisymmetric. We call it TSS-CMC hypersurface and use the notation $\Sigma_{H, c}$ to represent it.

In the construction (D), if $c = C_H$, the only TSS-CMC hypersurface is the cylindrical hypersurface $(t, r = R_H, \theta, \phi)$ in region $\Pi$. Similarly, if $c = c_H$, the only TSS-CMC hypersurface is the cylindrical hypersurface $(t, r = r_H, \theta, \phi)$ in region $\Pi'$.

Thus, we know all TSS-CMC hypersurfaces in the extended Reissner-Nordström spacetime. Conversely, given two symmetric boundary data in the extended Reissner-Nordström spacetime, Dirichlet problem will ask that whether there is a unique SS-CMC hypersurface satisfying the boundary data. Precisely, we have the following theorem.

**Theorem 2.** Given $H \in \mathbb{R}$ and boundary data $(T_0, X_0, \theta, \phi)$, $(T_0, -X_0, \theta, \phi)$ in the maximally extended Reissner-Nordström spacetime, there exists a unique TSS-CMC hypersurface $\Sigma : (T = T(X), X, \theta, \phi)$ with mean curvature $H$ satisfying the boundary value conditions $T(X_0) = T(-X_0) = T_0$.

**Proof.** The idea to prove Theorem 2 is similar to the Schwarzschild spacetime\[^7\] pages 9–15. First of all, we use the shooting method to get the existence of the Dirichlet problem. Given $H \in \mathbb{R}$, $V$ with $1 - V^2 > 0$, and $(T_0, X_0, \theta, \phi)$, consider the family of hypersurfaces $\{\Sigma_{H, c(V), \bar{c}(T_0, X_0)}\}_{1 - V^2 > 0}$. This family collects SS-CMC hypersurfaces with mean curvature $H$, $T(X_0) = T_0$, and $T'(X_0) = V$. In this notation $\{\Sigma_{H, c(V), \bar{c}(T_0, X_0)}\}_{1 - V^2 > 0}$, we just remark that the parameter $c$ is determined by $V$, $\bar{c}$ is determined by $T_0$ and $X_0$, and $\{\Sigma_{H, c(V), \bar{c}(T_0, X_0)}\}_{1 - V^2 > 0}$ is continuously varied when $V$ changes.

When $V$ tends to 1, $\Sigma_{H, c(V), \bar{c}(T_0, X_0)}$ tends to some future null cone. When $V$ goes to $-1$, $\Sigma_{H, c(V), \bar{c}(T_0, X_0)}$ approaches to some past null cone. Since the other boundary data $(T_0, -X_0, \theta, \phi)$ lies between these future null cone and past null cone, by the Intermediate Value Theorem, there is an SS-CMC hypersurface $\Sigma$ in the family $\{\Sigma_{H, c(V), \bar{c}(T_0, X_0)}\}_{1 - V^2 > 0}$ satisfying $T(-X_0) = T_0$.

This SS-CMC hypersurface $\Sigma$ is in fact symmetric with respect to the $T$-axis. We can argue it by contradiction. Suppose that $\Sigma$ is not $T$-axisymmetric. Consider its $T$-axisymmetric reflection hypersurface called $\tilde{\Sigma}$. Two hypersurfaces $\Sigma$ and $\tilde{\Sigma}$ have different
parameters $c$ because they have different slopes at the boundary point. On the other hand, the $T$-axisymmetric reflection does not change the radius of the “throat”, that is, the extreme value of the $r$-component function restricted on an SS-CMC hypersurface.

From the construction of SS-CMC hypersurfaces (A) and (D) in section 3, we know that the radius of the throat is determined by $c$, so $\Sigma$ and $\bar{\Sigma}$ must share the same parameter $c$, and it leads to the contradiction.

Now we will give a proof of the uniqueness part. First, consider the family of hypersurfaces $\{\Sigma_{H,c,\varepsilon}|T(X_0) = T(-X_0) = T_0\}_{H \in \mathbb{R}}$. This family collects all TSS-CMC hypersurfaces passing through $(T_0, X_0, \theta, \phi)$ and $(T_0, -X_0, \theta, \phi)$. Hypersurfaces in this family are continuously changed with respect to $H$. Next, we argue the uniqueness of the Dirichlet problem by contradiction. Suppose not, that is, given $H \in \mathbb{R}$, there are two TSS-CMC hypersurfaces $\Sigma^1 : (T = T^1(X), X, \theta, \phi)$ and $\Sigma^2 : (T = T^2(X), X, \theta, \phi)$ satisfying $T^1(X_0) = T^2(X_0) = T_0$, $T^1(-X_0) = T^2(-X_0) = T_0$, and $T^1(X) \neq T^2(X)$ in $(-X_0, X_0)$. Without loss of generality, assume that two hypersurfaces have causality relation $\Sigma^1 \ll \Sigma^2$ in $(-X_0, X_0)$. We slightly perturb $\Sigma^2$ in the family $\{\Sigma_{H,c,\varepsilon}|T(X_0) = T(-X_0) = T_0\}_{H \in \mathbb{R}}$ to $\Sigma$ so that $\Sigma^1 \ll \Sigma$ in $(-X_0, X_0)$, and $\Sigma$ has constant mean curvature $H + \varepsilon$ for some $\varepsilon > 0$.

Consider $d_{\Sigma^1}|_{\Sigma} : \Sigma \to [0, \infty]$ the Lorentzian distance function with respect to $\Sigma^1$ restricted on $\Sigma$. We refer [1] or [2] page 7 for more discussions on this distance function. One property is that the Laplacian of $d_{\Sigma^1}|_{\Sigma}$, called $\Delta_{\Sigma}(d_{\Sigma^1}|_{\Sigma})$, at each point $q \in \Sigma$ is related to the Laplacian of $d_{\Sigma^1}$ with respect to the ambient spacetime, the mean curvature of $\Sigma$, and the Hessian of $d_{\Sigma^1}$ applying on the unit normal part $\nu$. More precisely, we have the following equality:

$$\Delta_{\Sigma}(d_{\Sigma^1}|_{\Sigma})(q) = \Delta_{\Sigma^1}(q) + \text{Hess}(d_{\Sigma^1})(q; \nu, \nu) + 3H_{\Sigma}(q) \sqrt{1 + |\nabla(d_{\Sigma^1}|_{\Sigma})(q)|^2}.$$

Since $\Sigma^1 \neq \Sigma$, we know that the maximum value of $d_{\Sigma^1}|_{\Sigma}$ is positive and will achieve at some point $q \in \Sigma$. Since the Reissner-Nordström spacetime satisfies the timelike convergence condition, which implies $\Delta_{\Sigma^1}(q) \geq -3H_{\Sigma^1}(p)$ (see [1] or [2] page 7), where $p$ is the orthogonal projection of $q$ on $\Sigma^1$, we get at the maximum point $q$,

$$0 \geq \Delta_{\Sigma}(d_{\Sigma^1}|_{\Sigma})(q) \geq \text{Hess}(d_{\Sigma^1})(q; \nu, \nu) + 3H_{\Sigma}(q) \sqrt{1 + |\nabla(d_{\Sigma^1}|_{\Sigma})(q)|^2} - 3H_{\Sigma^1}(p) = 3H_{\Sigma}(q) - 3H_{\Sigma^1}(p) = 3\varepsilon > 0.$$

It leads to a contradiction. Remark that we use the result $\text{Hess}(d_{\Sigma^1})(q; \nu, \nu) = 0$ in the above discussion because of the perpendicular property of maximum distance on two submanifolds [7 Proposition 2]. Therefore, the uniqueness of the Dirichlet problem is proved. \[\square\]
5 Constant mean curvature foliations in the Reissner-Nordström spacetime

In this section, we will construct two types of $T$-axisymmetric, spacelike, spherically symmetric, constant mean curvature (TSS-CMC) foliations in the maximally extended Reissner-Nordström spacetime. One foliation has the same constant mean curvature for every hypersurface, and the other foliation has increasing constant mean curvatures when observing hypersurfaces along the $T$-direction.

Recall that a TSS-CMC foliation in the extended Reissner-Nordström spacetime is a family of hypersurfaces $\{\Sigma_s\}$ satisfying (i) every $\Sigma_s$ is a TSS-CMC hypersurface, (ii) for any $s_1 \neq s_2$, $\Sigma_{s_1}$ and $\Sigma_{s_2}$ are disjoint, and (iii) $\bigcup_s \Sigma_s$ covers the whole spacetime.

**Theorem 3.** Given $H \in \mathbb{R}$, there is a TSS-CMC foliation in the maximally extended Reissner-Nordström spacetime so that every hypersurface has the mean curvature $H$.

**Proof.** Given $H \in \mathbb{R}$, consider the closed loop formed by the graphs of $(r, F(H, r))$ and $(r, G(H, r))$ in the $rc$-plane, where $F(H, r) = Hr^3 + r(-r^2 + 2Mr - e^2)^{\frac{1}{2}}$ and $G(H, r) = Hr^3 - r(-r^2 + 2Mr - e^2)^{\frac{1}{2}}$ are defined in section 3. Figure 3 illustrates the closed loop when $H > 0$ in the $rc$-plane. We parameterize this closed loop by $\gamma(s)$, where $s \in \mathbb{R}$, $\gamma(0) = (r, F(H, r)) = (r, G(H, r))$, and when $s$ increases, the loop $\gamma(s)$ goes counterclockwise. Since $\gamma(s)$ is a closed loop, we can further set the parameter $s$ with period $T > 0$ so that $\gamma(s + kT) = \gamma(s)$ for all $k \in \mathbb{Z}$.

First, we choose a pair of Reissner-Nordström spacetimes. One Reissner-Nordström spacetime consists of regions $I$, $II$, $III$, and another adjacent Reissner-Nordström spacetime consists of regions $I'$, $II'$, $III'$, and they contain $(T, X, \theta, \phi) = (0, 0, \theta, \phi)$. We will say that for $0 \leq s \leq T$, there is a one-to-one correspondence between every point of $\gamma(s)$ and a TSS-CMC hypersurface $\Sigma_s$ in this spacelike extension of a pair of Reissner-Nordström spacetimes. This is because $r$-component of $\gamma(s)$ tells us the radius of the “throat” of the TSS-CMC hypersurface, that is, the $T$-intercept of the TSS-CMC hypersurface in the Penrose diagram. From the theorem of the initial value problem for the SS-CMC equation, an SS-CMC hypersurface is determined by the position of the throat on $T$-axis, and the uniqueness also implies that this SS-CMC hypersurface is $T$-axisymmetric. Hence we establish the one-to-one correspondence relation.

See Figure 3. Here we point out some special points on the loop and their correspondences. The point $\gamma(0)$ corresponds to a TSS-CMC hypersurface $\Sigma_0$ at the bottom of the pair of Reissner-Nordström spacetimes. When $s$ increases, $\gamma(s)$ goes the loop counterclockwise, and the $T$-intercept of the TSS-CMC hypersurface in Penrose diagram is increasing. The point $\gamma(s_1)$ achieves the minimum value of the function $G(H, r)$, so it corresponds to
Figure 3: Two graphs of functions \((r, F(H, r))\) and \((r, G(H, r))\) form a closed loop in the case \(H > 0\). This closed loop is parameterized by \(\gamma(s), s \in \mathbb{R}\) with period \(T > 0\). Every \(\gamma(s)\) corresponds to a TSS-CMC hypersurface in the extended Reissner-Nordström spacetime so that we construct a TSS-CMC foliation.

the cylindrical hypersurface \(r = r_H\) in region \(\Pi'\), called \(\Sigma_{s_1}\). The \(r\)-component of \(\gamma(s_2)\) is \(r = r_+\) so that \(T\)-intercept of the corresponding TSS-CMC hypersurface \(\Sigma_{s_2}\) is \(T = 0\). The point \(\gamma(s_3)\) achieves the maximum value of the function \(F(\mathcal{H}, r)\), so it corresponds to the cylindrical hypersurface \(r = R_H\) in region \(\Pi\), called \(\Sigma_{s_3}\). The point \(\gamma(T)\) corresponds to a TSS-CMC hypersurface \(\Sigma_T\) at the top of the pair of Reissner-Nordström spacetimes.

Notice that \(\Sigma_0\) and \(\Sigma_T\) have the same shape but with different \(T\)-intercepts in the Penrose diagram.

Consider \(\cup_{0 \leq s \leq T} \Sigma_s\), where \(\Sigma_s\) is the TSS-CMC hypersurface which corresponds to the point \(\gamma(s)\), then \(\cup_{0 \leq s \leq T} \Sigma_s\) will foliate the Reissner-Nordström spacetimes between \(\Sigma_0\) and \(\Sigma_T\). The argument is as follows: (i) Suppose that \((T_0, X_0, \theta, \phi)\) and \((T_0, -X_0, \theta, \phi)\) lie between \(\Sigma_0\) and \(\Sigma_T\). By the existence of the Dirichlet problem for the SS-CMC equation, there exists a TSS-CMC hypersurface \(\Sigma\) passing through them. Next, \(\Sigma\) must have the causality relation that \(\Sigma_0 \ll \Sigma \ll \Sigma_T\). Suppose not, say \(\Sigma_0 \not\ll \Sigma\) for example, then
they intersect at some symmetric boundary points or at \( X = 0 \). The former case will contradict to the uniqueness of the Dirichlet problem for the SS-CMC equation and the later case will contradict to the uniqueness of the initial value problem for the SS-CMC equation. (ii) Furthermore, this causality argument are also applied to prove that any two hypersurfaces \( \Sigma_{s_1} \) and \( \Sigma_{s_2} \) in the family are disjoint.

Finally, we copy the family of hypersurfaces \( \{ \Sigma_s \}_{0 \leq s \leq T} \) and paste it to other pairs of the Reissner-Nordström spacetimes. In other words, consider \( \{ \Sigma_s \}_{(k-1)T \leq s \leq kT} \), where \( k \in \mathbb{Z} \), then \( \cup_{(k-1)T \leq s \leq kT} \Sigma_t \) will foliate the Reissner-Nordström spacetimes between \( \Sigma_{(k-1)T} \) and \( \Sigma_{kT} \). Therefore, \( \cup_{s \in \mathbb{R}} \Sigma_s \) is a TSS-CMC foliation in the maximally extended Reissner-Nordström spacetime.

If we want to construct a TSS-CMC foliation with varied constant mean curvature in each slice, the copy and paste method in Theorem 3 does not work. However, one-to-one correspondence between a TSS-CMC hypersurface with mean curvature \( H \) and a point on the graph of \( F(H, r) \) or \( G(H, r) \) still holds and this is a crucial observation. Here we will view \( F(H, r) \) and \( G(H, r) \) as two variables functions, and we will piecewisely analyze this CMC foliation problem.

Look at the function \( F(H, r) \) first. When \( H \) goes over all real numbers, the range of \( F(H, r) \) will cover the strip \( r_− \leq r \leq r_+ \) in the \( rc \)-plane. In other words, given a point \( (r_0, c_0) \) in the strip \( r_− \leq r \leq r_+ \), there is a unique mean curvature \( H_0 \in \mathbb{R} \) such that \( c_0 = F(H_0, r_0) \). Notice that \( c_0 = F(H_0, r_0) \) will determine the \( T \)-intercept of a TSS-CMC hypersurface with mean curvature \( H_0 \) in region \( \Pi \).

In order to construct a TSS-CMC foliation with varied \( H \) in the spacelike extension of the region \( \Pi \), we have to find a curve \( \gamma(c) \) in the strip \( r_− \leq r \leq r_+ \) satisfying the following properties:

(a) For fixed \( H \), \( \gamma(c) \) intersects the graph of \( F(H, r) \) only once. This property will imply that any two points on the curves correspond to two TSS-CMC hypersurfaces with different constant mean curvatures.

(b) The curve \( \gamma(c) \) is a graph of some monotonic function in the \( rc \)-plane. The monotonicity property will imply that the constant mean curvature \( H(c) \) of corresponding TSS-CMC hypersurface is monotonic along the \( T \)-direction (with respect to \( c \)).

(c) In order to get a TSS-CMC foliation on the whole spacetime, we need to continuously glue the curve \( \gamma(c) \) to another curves corresponding to TSS-CMC hypersurfaces in different regions, so we require some behavior of the curve when \( r \) tends to \( r_− \) or \( r_+ \).
Based on these observations, we derive Proposition 4. We put the proof of Proposition 4 in Appendix, and we label each equation according to the required properties listed above.

**Proposition 4.** For the function $F(H, r) = H^3 + r(-r^2 + 2Mr - e^2)\frac{1}{2}$ where $r \in [r_-, r_+]$ and $H \geq 0$, there exists a function $y(r)$ defined on $[r_-, r_+]$ satisfying

\[
\begin{align*}
(a) \quad & \frac{dy}{dr} \neq \frac{3y}{r} - \frac{(-r^2 + 3Mr - 2e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} \\
(b) \quad & \frac{dy}{dr} < 0 \text{ for all } r \in (r_-, r_+) \\
(c) \quad & \lim_{r \to r_-} y(r) \text{ and } \lim_{r \to r_+} y(r) = y(r_+) \geq 0 \text{ are finite.}
\end{align*}
\]

Next, we look at the foliation in region $\Pi'$. When $H$ goes over all real numbers, the range of $G(H, r)$ is also the strip $r_- \leq r \leq r_+$ in the $rc$-plane. That is, given a point $(r_1, c_1)$ in the strip $r_- \leq r \leq r_+$, there is $H_1 \in \mathbb{R}$ such that $c_1 = G(H_1, r_1)$, and $G(H_1, r_1)$ will determine the $T$-intercept of a TSS-CMC hypersurface with mean curvature $H_1$ in region $\Pi'$.

In order to construct a TSS-CMC foliation with varied $H$ in the spacelike extension of the region $\Pi'$, we have to find a curve $\gamma(c)$ in the strip $r_- \leq r \leq r_+$ satisfying the properties (a) (change $F(H, r)$ to $G(H, r)$), (b), and (c). We formulate these properties as Proposition 5 and the proof can be found in Appendix.

**Proposition 5.** For the function $G(H, r) = H^3 - r(-r^2 + 2Mr - e^2)\frac{1}{2}$ where $r \in [r_-, r_+]$ and $H \geq 0$, there exists a function $y(r)$ defined on $[r_-, r_+]$ satisfying

\[
\begin{align*}
(a) \quad & \frac{dy}{dr} \neq \frac{3y}{r} + \frac{(-r^2 + 3Mr - 2e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} \\
(b) \quad & \frac{dy}{dr} > 0 \text{ for all } r \in (r_-, r_+) \\
(c) \quad & \lim_{r \to r_-} y(r) = y(r_-) > 0 \text{ and } \lim_{r \to r_+} y(r) \text{ are finite.}
\end{align*}
\]

Now we are ready to construct a TSS-CMC foliation with varied mean curvature in such slice.

**Theorem 6.** There is a TSS-CMC foliation $\{\Sigma_e : (T = T_e(X), X, \theta, \phi)\}_{c \in \mathbb{R}}$ in the maximally extended Reissner-Nordström spacetime such that constant mean curvature $H(c)$ is increasing along the $T$-direction (with respect to $c$). Moreover, this family of TSS-CMC hypersurfaces is symmetric with respect to the $X$-axis, that is, $T_{-e}(X) = -T_e(X)$, and the mean curvature ranges from $-\infty$ to $\infty$.

**Proof.** It suffices to find a family of TSS-CMC hypersurfaces $\{\Sigma_e\}_{c \geq 0}$ with nonnegative mean curvature, and $H(c)$ is increasing from 0 to $\infty$ with respect to $c$, and $\{\Sigma_e\}_{c \geq 0}$
foliates the maximally extended Reissner-Nordström spacetime on the region $T \geq 0$. By the $X$-axis reflection, that is, consider $\{\Sigma_{-c} : (T = T_{-c}(X) \overset{\text{def}}{=} -T_c(X), X, \theta, \phi)\}_{c \geq 0}$, then $\{\Sigma_c\}_{c \geq 0} \cup \{\Sigma_{-c}\}_{c \geq 0}$ will foliate the maximally extended Reissner-Nordström spacetime.

Taking $\Sigma_0 : (T = T_0(X) \equiv 0, X, \theta, \phi)$, then $\Sigma_0$ is a TSS-CMC hypersurface with mean curvature $H = 0$, and $\Sigma_0$ will correspond to $\gamma(0) = (r_+, c = F(0, r_+) = 0)$ in the $rc$-plane. Starting from $c = F(0, r_+) = 0$, by Proposition 4, there exists a function $y_1(r)$ satisfying (1) and $\lim_{r \to r_+} y_1(r) = F(0, r_+) = 0$. Let $c_1 = \lim_{r \to r_-} y_1(r)$, then there exists $H = H(c_1)$ such that $c_1 = F(H(c_1), r_-) = G(H(c_1), r_-)$. The graph of $y_1(r)$ is denoted by $\gamma_1(c) = (r, c = y_1(r))$.

Next, from $c_1 = G(H(c_1), r_-)$, by Proposition 5, there exists a function $y_2(r)$ satisfying (2) and $\lim_{r \to r_-} y_2(r) = c_1$. Let $c_2 = \lim_{r \to r_+} y_2(r)$, then there exists $H = H(c_2)$ such that $c_2 = G(H(c_2), r_+) = F(H(c_2), r_+)$. The graph of $y_2(r)$ is denoted by $\gamma_2(c) = (r, c = y_2(r))$.

We continue this process. For $k \in \mathbb{N}$, from the value $c_{2k}$, by Proposition 4, there exists a function $y_{2k+1}(r)$ satisfying (1) and $\lim_{r \to r_+} y_{2k+1}(r) = c_{2k}$. The graph of $y_{2k+1}(r)$ is denoted by $\gamma_{2k+1}(c) = (r, c = y_{2k+1}(r))$. Let $c_{2k+1} = \lim_{r \to r_-} y_{2k+1}(r)$. From the value $c_{2k+1}$, by Proposition 5, there exists a function $y_{2k+2}(r)$ satisfying (2) and $\lim_{r \to r_-} y_{2k+2}(r) = c_{2k+1}$. The graph of $y_{2k+2}(r)$ is denoted by $\gamma_{2k+2}(c) = (r, c = y_{2k+2}(r))$, and let $c_{2k+2} = \lim_{r \to r_+} y_{2k+2}(r)$.

Let $\gamma(c) = \bigcup_{i=1}^{\infty} \gamma_i(c)$, then the corresponding TSS-CMC hypersurfaces $\{\Sigma_c\}_{c \geq 0}$ will foliate the Reissner-Nordström spacetime with $T \geq 0$.

![Figure 4](image)

Figure 4: We first find a curve $\gamma_1(c) = (r, c = y_1(r))$ satisfying Proposition 4 and $y_1(r_+) = 0$, then find another curve $\gamma_2(c) = (r, c = y_2(r))$ satisfying Proposition 5 and $y_2(r_-) = y_1(r_-)$. We continue the process to get $\bigcup_{i=1}^{\infty} \gamma_i(c)$ and the corresponding TSS-CMC hypersurfaces will foliate the extended Reissner-Nordström spacetime on $T \geq 0$. By $X$-axis reflection, we will get a TSS-CMC foliation on the whole spacetime.
6 Conclusions

The Reissner-Nordstr"om spacetime and the Schwarzschild spacetime share the same metric form $ds^2 = -h(r) dt^2 + \frac{1}{h(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$, where $h(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}$ and $h(r) = 1 - \frac{2M}{r}$, respectively. When the charge smaller than the mass, it is natural to ask CMC related questions in the Reissner-Nordstr"om spacetime metric when we know fruitful results on the CMC properties in the Schwarzschild spacetime. This paper will discuss the initial value problem, Dirichlet problem for the CMC equation, and CMC foliations with fixed mean curvature or varied mean curvature in each slice in the Reissner-Nordstr"om spacetime.

In Schwarzschild case [9], SS-CMC equations and their solutions in standard coordinates can be written in terms of $h(r)$, so solutions of the SS-CMC equation in the Reissner-Nordstr"om spacetime are similarly derived by changing $h(r)$ to $1 - \frac{2M}{r} + \frac{e^2}{r^2}$. Two SS-CMC hypersurfaces can be smoothly glued at every coordinate singularity by checking the blow up order and the next order of the function $f(r)$, the solution of the SS-CMC equations, which is similar to the Schwarzschild case. The only difference in the Reissner-Nordstr"om spacetime case is that we have to take care of SS-CMC hypersurfaces in regions III and III', and our analyses also work in these regions. Thus, the initial value problem is solved in both existence and uniqueness.

Dirichlet problem for the SS-CMC equation in the Reissner-Nordstr"om spacetime is also solvable after proving the initial value problem. Shooting method also works to prove the existence part. For uniqueness part, geometric analysis method provides a powerful tool. We can argue it by contradiction: For different TSS-CMC hypersurfaces satisfying the Dirichlet problem, consider the maximal timelike geodesic between two hypersurfaces, then the Laplacian of the Lorentzian distance function of this geodesic gives the relation between two mean curvatures of hypersurfaces. We can find the inequality is not consistent. This argument can be applied to the spacetime with timelike convergence condition.

To show the TSS-CMC foliation properties in the Reissner-Nordstr"om spacetime with fixed mean curvature, they are highly related to the existence and uniqueness of the Dirichlet problem for SS-CMC equations with symmetric boundary data. This is because the existence part is equivalent that hypersurfaces cover the spacetime. The uniqueness part is equivalent that any two hypersurfaces are disjoint. This equivalent statement is also established in the Schwarzschild spacetime case [7, Theorem 8].

Since the extended Reissner-Nordstr"om spacetime consists of infinity many pairs of Reissner-Nordstr"om spacetimes, for TSS-CMC foliation with fixed mean curvature, we can choose the TSS-CMC hypersurfaces with periodicity. However, we have to be careful to
choose special TSS-CMC hypersurfaces with varied mean curvature to foliate the extended Reissner-Nordström spacetime. This TSS-CMC foliation problem is solved because we can find a one-to-one correspondence for every TSS-CMC hypersurface to the point in the rc-plane with $r_- \leq r \leq r_+$, which detects the position of the TSS-CMC hypersurface on the T-axis. So we can change every required TSS-CMC foliation properties to equivalent conditions (a)–(c) in Proposition 4 and Proposition 5 and show the existence of a function satisfying (a)–(c). Thus, TSS-CMC foliation with varied $H$ is established.

Appendix

We will give proofs of Proposition 4 and Proposition 5 here.

**Proposition 4.** For the function $F(H, r) = H r^3 + r(-r^2 + 2Mr - e^2)^{\frac{1}{2}}$ where $r \in [r_-, r_+]$ and $H \geq 0$, there exists a function $y(r)$ defined on $[r_-, r_+]$ satisfying

\[
\begin{align*}
(a) & \quad \frac{dy}{dr} \neq \frac{3y}{r} - \frac{(-r^2 + 3Mr - 2e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} \\
(b) & \quad \frac{dy}{dr} < 0 \text{ for all } r \in (r_-, r_+) \\
(c) & \quad \lim_{r \to r_-} y(r) \text{ and } \lim_{r \to r_+} y(r) = y(r_+) \geq 0 \text{ are finite.}
\end{align*}
\]

**Proof.** First of all, we compute

\[
\frac{\partial F}{\partial r}(H, r) = 3H r^2 + \frac{(-2r^2 + 3Mr - e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} = \frac{3y}{r} - \frac{(-r^2 + 3Mr - 2e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}}.
\]

Here we replace $H$ with $y$ and $r$ by the relation $y = H r^3 + r(-r^2 + 2Mr - e^2)^{\frac{1}{2}}$ in the last equality, and this will be the right hand side of (a). In order to find a function $y(r)$ satisfying (a), it suffices to find a function $p(r) > 0$ such that

\[
\frac{dy}{dr} \neq \frac{3y}{r} - \frac{(-r^2 + 3Mr - 2e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} - p(r).
\]

When multiplying the integrating factor $e^{\int \frac{-3}{r} dr} = r^{-3}$ on both sides of the above differential equation, it becomes

\[
\frac{d}{dr} \left( r^{-3} y(r) \right) = \frac{(-r^2 + 3Mr - 2e^2)}{r^{3}(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} - \frac{p(r)}{r^3}.
\]

The function $y(r)$ is solved by choosing the initial value of the integration $y(r_+) \geq 0$:

\[
y(r) = \frac{y(r_+)}{r_+^3} r^3 + r(-r^2 + 2Mr - e^2)^{\frac{1}{2}} + r^3 \int_r^{r_+} \frac{p(x)}{x^3} dx.
\]
Next, we hope that the function \( y(r) \) has property (b), so we compute

\[
y'(r) = \frac{3y(r_+)}{r_+^3}r^2 + \frac{(-2r^2 + 3Mr - e^2)}{(-r^2 + 2Mr - e^2)^{\frac{3}{2}}} + 3r^2 \int_r^{r_+} \frac{p(x)}{x^3} \, dx - p(r).
\]

Notice that the root of \( q(r) \) \( q(r) \defeq \frac{(-2r^2 + 3Mr - e^2)}{(-r^2 + 2Mr - e^2)^{\frac{3}{2}}} = 0 \) is \( r_* = \frac{3M + \sqrt{9M^2 - 8e^2}}{4} \) and it implies that \( q(r) > 0 \) on \((r_-, r_*)\) and \( q(r) < 0 \) on \((r_*, r_+)\). Consider

\[
p(r) = q(r) \cdot \chi_{(r_-, r_*)}(r) + Cr^\alpha = \frac{(-2r^2 + 3Mr - e^2)}{(-r^2 + 2Mr - e^2)^{\frac{3}{2}}} \cdot \chi_{(r_-, r_*)}(r) + Cr^\alpha,
\]

where \( \chi_{(r_-, r_*)}(r) \) is the characteristic function (indicator function) of the set \((r_-, r_*)\), and \( \alpha < -1 \) is a fixed constant, and \( C \) is a constant to be determined later. Since

\[
3r^2 \int_r^{r_+} \frac{p(x)}{x^3} \, dx = 3r^2 \int_r^{r_+} \frac{-2x^2 + 3Mx - e^2}{x^3(-x^2 + 2Mx - e^2)^{\frac{3}{2}}} \cdot \chi_{(r_-, r_*)}(x) \, dx + 3Cr^2 \int_r^{r_+} x^{\alpha - 3} \, dx
\]

\[
\leq 3C_1 r^2 - \frac{3Cr^2}{2 - \alpha} \left( \frac{1}{r_+^{2-\alpha}} - \frac{1}{r^2-\alpha} \right),
\]

where \( C_1 = \int_r^{r_+} \frac{-2x^2 + 3Mx - e^2}{x^3(-x^2 + 2Mx - e^2)^{\frac{3}{2}}} \cdot \chi_{(r_-, r_*)}(x) \, dx \) is a finite value, we have

\[
y'(r) \leq \frac{3y(r_+)}{r_+^3}r^2 + 3C_1 r^2 - \frac{3Cr^2}{2 - \alpha} \left( \frac{1}{r_+^{2-\alpha}} - \frac{1}{r^{2-\alpha}} \right) - Cr^\alpha + (1 - \chi_{(r_-, r_*)}(r))q(r)
\]

\[
= \left( \frac{3y(r_+)}{r_+^3} + 3C_1 - \frac{3Cr^2}{2 - \alpha r^{2-\alpha}} \right) r^2 + \left( \frac{3}{2 - \alpha} - 1 \right) Cr^\alpha + (1 - \chi_{(r_-, r_*)}(r))q(r).
\]

Since \( \alpha < -1 \), we can choose \( C \) large enough so that \( y'(r) < 0 \) for all \( r \in (r_-, r_+) \).

Finally, since \( p(r) \) is of order \( \mathcal{O}(r - r_-)^{-\frac{1}{2}} \), the improper integral in \( (3) \) is finite, so \( \lim_{r \to r_-} y(r) \) is finite. The other limit \( \lim_{r \to r_+} y(r) = y(r_+) \) is finite because of the initial value setting in \( (3) \). \( \square \)

**Proposition 5.** For the function \( G(H, r) = H \frac{r^3}{3} - r(-r^2 + 2Mr - e^2)^{\frac{1}{2}} \) where \( r \in [r_-, r_+] \) and \( H \geq 0 \), there exists a function \( y(r) \) defined on \([r_-, r_+]\) satisfying

\[
\left\{ \begin{array}{l}
(a) \quad \frac{dy}{dr} \neq \frac{3y}{r} + \left( \frac{-r^2 + 3Mr - 2e^2}{(-r^2 + 2Mr - e^2)^{\frac{3}{2}}} \right) \\
(b) \quad \frac{dy}{dr} > 0 \text{ for all } r \in (r_-, r_+) \\
(c) \quad \lim_{r \to r_-} y(r) = y(r_-) > 0 \text{ and } \lim_{r \to r_+} y(r) \text{ are finite.}
\end{array} \right.
\]

**Proof.** First of all, we compute

\[
\frac{\partial G}{\partial r} (H, r) = 3H r^2 - \frac{(-r^2 + 3Mr - e^2)}{(-r^2 + 2Mr - e^2)^{\frac{3}{2}}};
\]

\[
= \frac{3y}{r} + \left( \frac{-r^2 + 3Mr - 2e^2}{(-r^2 + 2Mr - e^2)^{\frac{3}{2}}} \right).
\]

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Here we replace $H$ with $y$ and $r$ by the relation $y = Hr^3 - r(-r^2 + 2Mr - e^2)^{\frac{1}{2}}$ in the last equality, and this will be the right hand side of (a). In order to find a function $y(r)$ satisfying (a), it suffices to find a function $p(r) > 0$ such that
\[
\frac{dy}{dr} - \frac{3y}{r} = \frac{(-r^2 + 3Mr - 2e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} + p(r).
\]
When multiplying the integrating factor $e^{\int -\frac{3}{r}dr} = r^{-3}$ on both sides of the differential equation, it becomes
\[
\frac{d}{dr} (r^{-3}y(r)) = \frac{(-r^2 + 3Mr - 2e^2)}{r^3(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} + \frac{p(r)}{r^3}.
\]
The function $y(r)$ is solved by choosing the initial value of the integration $y(r_-) > 0$:
\[
y(r) = \frac{y(r_-)}{r_-^3} r^3 - r(-r^2 + 2Mr - e^2)^{\frac{1}{2}} + r^3 \int_{r_-}^r \frac{p(x)}{x^3} dx.
\tag{4}
\]
Next, we hope that the function $y(r)$ has property (b), so we compute
\[
y'(r) = \frac{3y(r_-)}{r_-^3} r^2 - \frac{(-2r^2 + 3Mr - e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} + 3r^2 \int_{r_-}^r \frac{p(x)}{x^3} dx + p(r).
\]
Notice that the root of $q(r) \overset{\text{def}}{=} \frac{(-2r^2 + 3Mr - e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}}$ is $r_* = \frac{3M + \sqrt{9M^2 - 8e^2}}{4}$ and it implies that $q(r) > 0$ on $(r_-, r_*)$ and $q(r) < 0$ on $(r_*, r_+)$. Consider
\[
p(r) = q(r) \cdot \chi(r_-, r_*)(r) = \frac{(-2r^2 + 3Mr - e^2)}{(-r^2 + 2Mr - e^2)^{\frac{1}{2}}} \cdot \chi(r_-, r_*)(r) \geq 0,
\]
where $\chi(r_-, r_*)(r)$ is the characteristic function (indicator function) of the set $(r_-, r_*)$. Then we have
\[
y'(r) = \frac{3y(r_-)}{r_-^3} r^2 + 3r^2 \int_{r_-}^r \frac{p(x)}{x^3} dx + (\chi(r_-, r_*)(r) - 1)q(r) > 0.
\]
Finally, since $p(r)$ is of order $O(|r - r_+|^{-\frac{1}{2}})$, the improper integral in (4) is finite, so \[\lim_{r \to r_+} y(r) \text{ is finite.}\] The other limit \[\lim_{r \to r_-} y(r) = y(r_-) \text{ is finite because of the initial value setting in (3).}\]

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