Bound State Solutions of the Dirac Equation in the Extreme Kerr Geometry

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Abstract

In this paper we consider bound state solutions, i.e., normalizable time-periodic solutions of the Dirac equation in the exterior region of an extreme Kerr black hole with mass $M$ and angular momentum $J$. It is shown that for each azimuthal quantum number $k$ and for particular values of $J$ the Dirac equation has a bound state solution, and that the energy of this Dirac particle is uniquely determined by $\omega = -\frac{kM}{\sqrt{J}}$. Moreover, we prove a necessary and sufficient condition for the existence of bound states in the extreme Kerr-Newman geometry, and we give an explicit expression for the radial eigenfunctions in terms of Laguerre polynomials.

1 Introduction

In Boyer-Lindquist coordinates $(t, r, \theta, \phi)$ with $r \in (0, \infty)$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$ the metric of a Kerr-Newman black hole of mass $M$, angular momentum $J$, and charge $Q$ is given by (compare [11, Section 12.3])

$$ds^2 = \frac{\Delta}{U} (dt - a \sin^2 \theta \, d\phi)^2 - U \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - \frac{\sin^2 \theta}{U} \left( a \, dt - (r^2 + a^2) \, d\phi \right)^2,$$

where $a := \frac{J}{M}$ is the Kerr parameter and

$$U(r, \theta) := r^2 + a^2 \cos^2 \theta, \quad \Delta(r) := r^2 - 2Mr + a^2 + Q^2.$$

In the following we consider the extreme case $M^2 = a^2 + Q^2$, where the function $\Delta$ has only one zero $\rho := M = \sqrt{a^2 + Q^2}$, i.e., $\Delta(r) = (r - \rho)^2$. This means, in particular, that the Cauchy horizon and the event horizon coincide. On such an extreme Kerr-Newman manifold we study the Dirac equation for a particle with rest mass $m$ and charge $e$ in the exterior region $r \in (\rho, \infty)$. The Dirac equation has the form

$$(\mathcal{R} + \mathcal{A}) \Psi = 0 \quad (1)$$
with

\[ R := \begin{pmatrix} imr & 0 & \sqrt{\Delta} D_+ & 0 \\ 0 & -imr & 0 & \sqrt{\Delta} D_- \\ \sqrt{\Delta} D_- & 0 & -imr & 0 \\ 0 & \sqrt{\Delta} D_+ & 0 & imr \end{pmatrix}, \]

\[ A := \begin{pmatrix} -am \cos \theta & 0 & 0 & L_+ \\ 0 & am \cos \theta & L_- & 0 \\ 0 & L_+ & -am \cos \theta & 0 \\ -L_- & 0 & 0 & am \cos \theta \end{pmatrix} \]

and the differential operators

\[ D_\pm := \frac{\partial}{\partial r} \mp \frac{1}{\Delta} \left[ (r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \phi} - ieQr \right], \]

\[ L_\pm := \frac{\partial}{\partial \theta} + \cot \theta \pm i \left[ a \sin \theta \frac{\partial}{\partial t} \mp \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right]. \]

Moreover, by rearranging (1), we can write the Dirac equation in Hamiltonian form

\[ i \frac{\partial}{\partial t} \Psi = H \Psi, \tag{2} \]

where \( H \) is a first order \((4 \times 4)\) matrix differential operator acting on spinors \( \Psi \) on hypersurfaces \( t = \text{const} \). A simple scalar product on such a hypersurface in the exterior region \( r \in (\rho, \infty) \) is given by

\[ (\Psi, \Phi) := \int_\rho^\infty \int_0^\pi \int_0^{2\pi} \overline{\Psi}(t, r, \theta, \phi) \Phi(t, r, \theta, \phi) \sin \theta \frac{r^2 + a^2}{\Delta(r)} \, d\phi \, d\theta \, dr, \]

where \( \overline{\Psi} \) denotes the complex conjugated, transposed spinor. Note that the Hamiltonian \( H \) is in general not symmetric with respect to this scalar product. However, there exists a scalar product \([\cdot, \cdot]\) on the spinors on hypersurfaces \( t = \text{const} \) which is equivalent to \((\cdot, \cdot)\) such that \( H \) is symmetric with respect to \([\cdot, \cdot]\) (see [6] for the details).

In this paper we are looking for time-periodic solutions

\[ \Psi(t, r, \theta, \phi) = e^{-i\omega t} \Psi_0(r, \theta, \phi), \quad \Psi_0 \neq 0, \tag{3} \]

of the Dirac equation (1), where \( \omega \in \mathbb{R} \) and \( \Psi_0 \) is normalizable, i.e., \((\Psi_0, \Psi_0) = (\Psi, \Psi) < \infty \). If such a solution exists, then \( \omega \) is an eigenvalue of \( H \) for the eigenspinor \( \Psi_0 \), and \( \omega \) represents the one-particle energy of the bound state \( \Psi \). It is well known (see [5] and [6]) that normalizable time-periodic solutions do not arise in the non-extreme case \( M^2 > a^2 + Q^2 \) and in the Reissner-Nordstrøm geometry \( a = 0 \). Here we consider the Dirac equation on an extreme Kerr-Newman manifold and we prove – at least for the extreme Kerr case \( Q = 0 \),
$a \neq 0$ – that bound state solutions exist for particular values of $a$. To this end, we employ the ansatz (3) with

$$\Psi_0(r, \theta, \phi) = e^{-ik\phi} \begin{pmatrix} f_1(r)g_1(\theta) \\ f_2(r)g_2(\theta) \\ f_2(r)g_1(\theta) \\ f_1(r)g_2(\theta) \end{pmatrix},$$

where $k \in \{\pm \frac{1}{2}, \pm \frac{3}{2}, \ldots\}$ is a half-integer, and instead of (1) we investigate the equations

$$R \Psi = \lambda \Psi, \quad A \Psi = -\lambda \Psi$$

with some separation parameter $\lambda \in \mathbb{R}$. Now, if we define

$$f(r) := \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}, \quad r \in (\rho, \infty), \quad g(\theta) := \begin{pmatrix} g_1(\theta) \\ g_2(\theta) \end{pmatrix}, \quad \theta \in [0, \pi],$$

then the Dirac equation can be separated into a radial part

$$\left( \begin{array}{cc} (r - \rho) \frac{\partial}{\partial r} + \frac{iV(r)}{r-\rho} & imr - \lambda \\ -imr - \lambda & (r - \rho) \frac{\partial}{\partial r} - \frac{iV(r)}{r-\rho} \end{array} \right) f(r) = 0,$$

where $V(r) := \omega \left( r^2 + a^2 \right) + ka + eQr$, and an angular part

$$\left( \begin{array}{cc} \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \theta} - W(\theta) & -am \cos \theta + \lambda \\ am \cos \theta + \lambda & -\frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \theta} - W(\theta) \end{array} \right) g(\theta) = 0,$$

where $W(\theta) := a\omega \sin \theta + \frac{k}{a} \sin \theta$ (see [2], [10]). In the following, a point $\omega \in \mathbb{R}$ is called energy eigenvalue of (1), if there exist some $\lambda \in \mathbb{R}$ and nontrivial solutions $f$ of (5), $g$ of (6) satisfying the normalization conditions

$$\int_{\rho}^{\infty} |f(r)|^2 \frac{r^2 + a^2}{\Delta(r)} dr < \infty, \quad \int_{0}^{\pi} |g(\theta)|^2 \sin \theta d\theta < \infty.$$

Then, for the spinor $\Psi$ given by (3) and (4), it follows that

$$(\Psi, \Psi) = 2\pi \left( \int_{\rho}^{\infty} |f(r)|^2 \frac{r^2 + a^2}{\Delta(r)} dr \right) \left( \int_{0}^{\pi} |g(\theta)|^2 \sin \theta d\theta \right),$$

and the conditions (7) imply $(\Psi, \Psi) < \infty$. This way, the eigenvalue equation $H \Psi_0 = \omega \Psi_0$ and the normalization condition $(\Psi_0, \Psi_0) < \infty$ have been reduced to a pair of boundary value problems for $(2 \times 2)$ systems of ordinary differential equations which are coupled by the energy eigenvalue $\omega$ and the separation parameter $\lambda$. At first, we try to get some information about the angular eigenvalues $\lambda$ in dependence of $\omega$. For this purpose, we rewrite in Section 2 the angular Dirac equation (6) and the second condition in (4) as an eigenvalue
equation for some self-adjoint differential operator $A$. From oscillation theory it follows that $A$ has purely discrete spectrum, and perturbation theory yields that the eigenvalues $\lambda_j$, $j \in \mathbb{Z}$, of $A$ depend analytically on $am$ and $a\omega$. Subsequently, in Section 3, we investigate the radial Dirac equation (5) for a fixed azimuthal quantum number $k \in \{\pm \frac{1}{2}, \pm \frac{3}{2}, \ldots\}$. The first condition in (7) and the asymptotic behavior of the solutions of (5) at $r = \rho$ and $r = \infty$ imply

$$\omega = -\frac{ka + eQ \rho}{a^2 + \rho^2}.$$ 

In this case the system (5) can be reduced either to a Bessel or a Whittaker equation, which allows a more detailed analysis of the solutions of (5). By this means, we obtain a necessary and sufficient condition for $\omega$ being an energy eigenvalue of (1). In particular, it turns out that $\omega$ is an energy eigenvalue if

$$m^2 - \omega^2 > 0, \quad \lambda_j^2 + \rho^2 m^2 - \mu^2 > \frac{1}{4}, \quad n + \frac{pm^2 - \omega \mu}{\sqrt{m^2 - \omega^2}} + \kappa = 0,$$

where $n$ is a positive integer, $j \in \mathbb{Z} \setminus \{0\}$, and

$$\mu := 2\rho\omega + eQ, \quad \kappa := \sqrt{\lambda_j^2 + \rho^2 m^2 - \mu^2}.$$ 

Since $\lambda_j$ depends on $\omega$, it is not obvious that these (in)equalities can be satisfied. In order to prove that bound state solutions of the Dirac equation actually exist, we restrict our attention in Section 4 to the Kerr case $Q = 0$ and $a \neq 0$, where the energy eigenvalue is uniquely determined by $\omega = -\frac{k}{2a}$. Using the estimates from Section 2, we can show that bound states appear for countably many values of $a$.

## 2 The Angular Dirac Equation

In this section we study the angular part (6) of the separated Dirac equation for some fixed half-integer $k$. For this reason, we write (6) in the form

$$\frac{1}{2 \sin \theta} \left[ 2 \begin{pmatrix} 0 & \sin \theta \\ -\sin \theta & 0 \end{pmatrix} u'(\theta) + \begin{pmatrix} 0 & \cos \theta \\ -\cos \theta & 0 \end{pmatrix} u(\theta) \right. + \left. \begin{pmatrix} -2L \sin \theta \cos \theta & 2k + 2\Omega \sin^2 \theta \\ 2k + 2\Omega \sin^2 \theta & 2L \sin \theta \cos \theta \end{pmatrix} u(\theta) \right] = \lambda u(\theta), \quad (8)$$

where $L := am$ and $\Omega := a\omega$. For fixed values of $L$ and $\Omega$, the differential operator $A$ generated by the left hand side of (8) is a self-adjoint operator acting on the Hilbert space $L^2((0, \pi), 2 \sin \theta)^2$ of square-integrable vector functions with respect to the weight function $2 \sin \theta$. From oscillation theory for Dirac systems (see [12, Section 16]) it follows that the spectrum of $A$ consists of discrete eigenvalues $\lambda_j = \lambda_j(L, \Omega)$, $j \in \mathbb{Z}$, where $|\lambda_j(L, \Omega)| \to \infty$ as $|j| \to \infty$ pointwise.
on \( \mathbb{R}^2 \). Moreover, \( A = A(L, \Omega) \) depends analytically on \( L \) and \( \Omega \), and the partial derivatives with respect to \( L \) and \( \Omega \), respectively, are given by

\[
\frac{\partial A}{\partial L} = \begin{pmatrix}
-\cos \theta & 0 \\
0 & \cos \theta
\end{pmatrix}, \quad \frac{\partial A}{\partial \Omega} = \begin{pmatrix}
0 & \sin \theta \\
\sin \theta & 0
\end{pmatrix}.
\]

Hence, by perturbation theory (see [5, Chap. VII, §3, Sec. 4]), also the eigenvalues \( \lambda_j = \lambda_j(L, \Omega) \), \( j \in \mathbb{Z} \), depend analytically on \( L \) and \( \Omega \), and we obtain the estimates

\[
\left| \frac{\partial \lambda_j}{\partial L} \right| \leq \left\| \frac{\partial A}{\partial L} \right\| \leq 1, \quad \left| \frac{\partial \lambda_j}{\partial \Omega} \right| \leq \left\| \frac{\partial A}{\partial \Omega} \right\| \leq 1,
\]

where \( \| \cdot \| \) denotes the operator norm of a \((2 \times 2)\) matrix.

**Lemma 1** The eigenvalues \( \lambda_j = \lambda_j(L, \Omega) \) of the angular Dirac operator (8) depend analytically on \( L \) and \( \Omega \). Moreover, \( |\lambda_j| \to \infty \) as \( |j| \to \infty \) locally uniformly on \( \mathbb{R}^2 \).

**Proof.** For fixed \((L, \Omega) \in \mathbb{R}^2\), we have \( |\lambda_j(L, \Omega)| \to \infty \) as \( |j| \to \infty \), and from (9) it follows that \( |\frac{\partial \lambda_j}{\partial L}|, |\frac{\partial \lambda_j}{\partial \Omega}| \) are uniformly bounded on \( \mathbb{R}^2 \). \( \square \)

### 3 The Radial Dirac Equation

In the following we consider the radial part (5) of the separated Dirac equation in the extreme case \( \Delta(r) = (r - \rho)^2 \). First, we introduce a new variable \( x := r - \rho \) (the coordinate distance from the event horizon) and we write (5) in the form

\[
f'(x) = \begin{pmatrix}
-\frac{i \tau}{x^2} - \frac{i \mu}{x} - i \omega & \frac{\lambda - im \rho}{x} - im \\
\frac{\lambda + im \rho}{x} + im & \frac{\tau}{x^2} + \frac{\mu}{x} + i \omega
\end{pmatrix} f(x), \quad x \in (0, \infty),
\]

where

\[
\tau := \omega (\rho^2 + a^2) + ka + eQ \rho, \quad \mu := 2\rho \omega + eQ.
\]

Now, let \( S \) be the unitary matrix

\[
S := \frac{1}{\sqrt{2}} \begin{pmatrix}
-1 & i \sigma \\
\sigma & -i
\end{pmatrix}
\]

with \( \sigma := \text{sign} \omega \) (and \( \text{sign} 0 := 1 \)). By means of the transformation \( f(x) = Sw(x) \), the differential equation (10) is equivalent to the system

\[
w'(x) = \begin{pmatrix}
\frac{\sigma \lambda}{x^2} & -\frac{\sigma \tau}{x^2} + \frac{\rho m + \sigma \mu}{x} + m - |\omega| \\
\frac{\rho m + \sigma \mu}{x} + m + |\omega| & \frac{\sigma \lambda}{x^2}
\end{pmatrix} w(x)
\]

on the interval \((0, \infty)\), and since \( |f(x)| = |w(x)| \), a point \( \omega \in \mathbb{R} \) is an energy eigenvalue of (11) if and only if (12) has a nontrivial solution \( w \) satisfying

\[
\int_0^\infty |w(x)|^2 \frac{(x + \rho)^2 + a^2}{x^2} \, dx < \infty
\]

(13)
provided that $\lambda$ is an eigenvalue of the angular Dirac operator $A$. In the following
we present some necessary conditions for $\omega \in \mathbb{R}$ being an energy eigenvalue of the
Dirac equation (11), and we start with some results on the solutions of singular
systems more general than (12).

Lemma 2 Let $y$ be a solution of the differential equation
\[ y'(z) = (C + R(z)) y(z), \quad z \in [1, \infty), \tag{14} \]
where $C$ and $R(z)$ are $(2 \times 2)$ matrices, $\det C > 0$, $R(z) \to 0$ as $z \to \infty$, $R'$ is
integrable on $[1, \infty)$, and $\text{tr} (C + R) \equiv 0$. If $y \neq 0$, then there exists a constant
$\delta > 0$ such that $|y(z)| \geq \delta$ for all $z \in [1, \infty)$.

Proof. Let
\[ \Lambda := \begin{pmatrix} -i \sqrt{\det C} & 0 \\ 0 & i \sqrt{\det C} \end{pmatrix}. \]
Since $\text{tr} C = 0$, $\pm i \sqrt{\det C}$ are the eigenvalues of the constant matrix $C$, and
there exists an invertible matrix $T$ such that $T^{-1} CT = \Lambda$. Furthermore, we can fix
some point $z_0 \in [1, \infty)$ such that $\det (C + R(z)) > 0$ for all $z \in [z_0, \infty)$. Now,
$\pm i \sqrt{\det (C + R(z))}$ are the eigenvalues of the matrix $C + R(z)$, and Eastham’s
Theorem [3, Theorem 1.6.1] implies that the system (14) has a fundamental
matrix $Y(z) = TH(z)e^{iD(z)}$, where $H(z) \to I$ as $z \to \infty$ ($I$ is the $(2 \times 2)$ unit
matrix) and $D$ denotes the diagonal matrix function
\[ D(z) := \text{diag} \left( -\int_{z_0}^z \sqrt{\det (C + R(t))} \, dt, \int_{z_0}^z \sqrt{\det (C + R(t))} \, dt \right). \]
If $y$ is a nontrivial solution of (14), then there exists some vector $c \in \mathbb{C}^2 \setminus \{0\}$
such that $y(z) = Y(z)c$, and we obtain
\[ c = Y(z)^{-1}y(z) = e^{-iD(z)}H(z)^{-1}T^{-1}y(z), \quad z \in [z_0, \infty). \]
Since $e^{-iD(z)}$ is a unitary matrix for all $z \in [z_0, \infty)$, it follows that
\[ |c| = |H(z)^{-1}T^{-1}y(z)| \leq \|H(z)^{-1}T^{-1}\| |y(z)|, \quad z \in [z_0, \infty). \]
In addition, $\lim_{z \to \infty} H(z) = I$ implies $\liminf_{z \to \infty} |y(z)| \geq c \|T^{-1}\|^{-1} > 0$.
Finally, as $y$ is continuous on $[1, \infty)$ and $|y(z)| \neq 0$ for all $z \in [1, \infty)$ by the existence
and uniqueness theorem, we have $|y(z)| \geq \delta$ for all $z \in [1, \infty)$ with some constant $\delta > 0$. \hfill \Box

Corollary 1 If $\omega \in \mathbb{R}$ is an energy eigenvalue of (11), then $\tau = 0$. This means,
\[ \omega = -\frac{ka + eQ\rho}{a^2 + \rho^2}. \tag{15} \]
Proof. Suppose that $\tau \neq 0$, and let $w$ be a nontrivial solution of \((12)\). By means of the transformation $y(z) = w \left( \frac{1}{z} \right)$, the differential equation \((12)\) on the interval \((0, 1]\) is equivalent to the asymptotically constant system

$$y'(z) = (C + R(z)) y(z), \quad z \in [1, \infty),$$

where

$$C := \begin{pmatrix} 0 & \sigma \tau \\ -\sigma \tau & 0 \end{pmatrix}, \quad R(z) := \begin{pmatrix} -\sigma \lambda & \rho \lambda - \sigma \mu \rho - m - |\omega| \\ -\frac{\rho m + \sigma \mu}{z} & \frac{\sigma \lambda}{z} \end{pmatrix}. $$

As $\text{tr} \ C = 0$ and $\det \ C = \tau^2 > 0$, Lemma 2 implies that there exists a constant $\delta > 0$ such that $|y(z)| \geq \delta$ on $[1, \infty)$ and therefore $|w(x)| \geq \delta$ on $(0, 1]$. Hence, the normalization condition is not satisfied, and it follows that $\omega$ is not an energy eigenvalue of \((1)\). $\Box$

Since we intend to find energy eigenvalues of the Dirac equation, we assume in what follows that $\tau = 0$ holds. Then the differential equation \((12)\) becomes

$$xw'(x) = \begin{pmatrix} \sigma \lambda & \rho m - \sigma \mu + (m - |\omega|) \rho \\ \rho m + \sigma \mu + (m + |\omega|) \rho & -\sigma \lambda \end{pmatrix} w(x). \quad (17)$$

Lemma 3 Let $y$ be a nontrivial solution of the differential equation

$$xy'(x) = (A + xB) y(x), \quad x \in (0, 1], \quad (18)$$

where $A, B$ are $2 \times 2$ matrices and $\text{tr} \ A = 0$, $\det \ A \geq -\frac{1}{4}$. If $y \neq 0$, then there exists a constant $\varepsilon > 0$ such that $|y(x)| \geq \varepsilon \sqrt{x}$ for all $x \in (0, 1]$.

Proof. Let $J$ be the canonical form of $A$. Since $\text{tr} \ A = 0$, $\pm \sqrt{-\det A}$ are the eigenvalues of $A$. Therefore,

$$J = \begin{pmatrix} -\sqrt{-\det A} & 0 \\ 0 & \sqrt{-\det A} \end{pmatrix} \quad (19)$$

if $\det A \neq 0$, and we have

$$J = \begin{pmatrix} 0 & \nu \\ 0 & 0 \end{pmatrix} \quad (20)$$

with some $\nu \in \{0, 1\}$ if $\det A = 0$. Moreover, let $T$ be an invertible matrix such that $T^{-1} A T = J$. By means of the transformation $T \tilde{y}(z) = y \left( \frac{1}{z} \right)$, \((18)\) is equivalent to the system

$$\tilde{y}'(z) = \begin{pmatrix} -\frac{1}{z} J - \frac{1}{z^2} T^{-1} B T \end{pmatrix} \tilde{y}(z), \quad z \in [1, \infty). \quad (21)$$

From [4, Theorem 1.8.1] and the Levinson Theorem (see [4, Theorem 1.3.1]) it follows that \((21)\) has a fundamental matrix $\tilde{Y}(z) = H(z) z^{-J}$, where $H$ is a
continuous \((2 \times 2)\) matrix function which satisfies \(\lim_{z \to \infty} H(z) = I\). Hence, 
\[ Y(x) = T H \left( \frac{1}{x} \right) x^J \]
is a fundamental matrix of the differential equation (18). Now, if \(y\) is a nontrivial solution of (18), then there exists some vector \(c \in \mathbb{C}^2 \setminus \{0\}\) such that \(y(x) = Y(x)c\), and we obtain the estimate 
\[ |c| = \left| x^{-J} H \left( \frac{1}{x} \right)^{-1} T^{-1} y(x) \right| \leq b \|x^{-J}\| \|y(x)\|, \quad x \in (0, 1], \quad (22) \]
where \(b := \max_{x \in (0, 1]} \|H \left( \frac{1}{x} \right)^{-1} T^{-1}\|\). If \(\det A > 0\), then \(x^{-J} = e^{-\log x \cdot J}\) is a unitary matrix (since \(J\) is a diagonal matrix with purely imaginary entries), and it follows that \(\|x^{-J}\| = 1\). In the case that \(J\) is the Jordan matrix (20), 
\[ x^{-J} = I - \log x \cdot J \]
is a fundamental matrix of the differential equation (18). Moreover, we set 
\[ \kappa := \sqrt{\lambda^2 + \rho^2 m^2 - \mu^2} > \frac{1}{2}. \]

**Lemma 4** If \(\omega \in \mathbb{R}\) is an energy eigenvalue of (11), then 
\[ m^2 - \omega^2 > 0. \quad (24) \]

**Proof.** According to Corollary 2 \(\omega\) is not an energy eigenvalue of (11) if \(m^2 - \omega^2 < 0\). Now, we suppose that \(m = |\omega|\) and we will prove that \(\omega\) is not an energy eigenvalue even in this case. Introducing 
\[ w(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \]
\(\omega\) is an energy eigenvalue of (11) if and only if the system 
\[ \begin{align*}
x u'(x) &= \sigma \lambda u(x) + (\rho m - \sigma \mu) v(x), \\
x v'(x) &= [(\rho m + \sigma \mu) + 2mx] u(x) - \sigma \lambda v(x) \end{align*} \quad (25) \]
and 
\[ \begin{align*}
x u'(x) &= \sigma \lambda u(x) + (\rho m - \sigma \mu) v(x), \\
x v'(x) &= [(\rho m + \sigma \mu) + 2mx] u(x) - \sigma \lambda v(x) \end{align*} \quad (26) \]
has a nontrivial solution \((u, v)\) satisfying
\[
\int_0^\infty (|u(x)|^2 + |v(x)|^2) \frac{(x + \rho)^2 + a^2}{x^2} \, dx < \infty, \tag{27}
\]
where \(\lambda\) is an eigenvalue of the angular Dirac operator \(A\).

If \(\rho \mu - \sigma \mu = 0\), then equation (26) implies \(u(x) = c_1 x^\sigma \) with some constant \(c_1 \in \mathbb{C}\), and from (27) it follows that \(c_1 = 0\). Further, from (26) and \(u \equiv 0\) we obtain \(v(x) = c_2 x^{-\sigma} \) with some constant \(c_2 \in \mathbb{C}\), and (26) gives \(c_2 = 0\), i.e., \(v \equiv 0\). Hence, \(\omega\) is not an energy eigenvalue of (1) in the case \(\rho \mu - \sigma \mu = 0\).

Next, we assume that \(\rho \mu - \sigma \mu \neq 0\). From (26) it follows that
\[
v(x) = \frac{x u'(x) - \sigma \lambda u(x)}{\rho \mu - \sigma \mu}, \tag{28}
\]
and replacing \(v\) in (26) with (28) gives
\[
x^2 u''(x) + x u'(x) - \left[\lambda^2 + \rho^2 m^2 - \mu^2 + 2m(\rho \mu - \sigma \mu)\right] u(x) = 0. \tag{29}
\]

We first investigate the case \(\rho \mu - \sigma \mu < 0\). By means of the transformation
\[
u(x) = \tilde{u} \left(\sqrt{8m(\sigma \mu - \rho \mu)} x\right)
\]
(29) is on the interval \((0, \infty)\) equivalent to Bessel’s differential equation
\[
z^2 \tilde{u}''(z) + z \tilde{u}'(z) + \left[z^2 - 4\kappa^2\right] \tilde{u}(z) = 0. \tag{30}
\]
The Bessel function \(J_\nu\) and the Neumann function \(Y_\nu\) of order \(\nu = 2\kappa \geq 1\) form a fundamental system of solutions of (30). These functions have the asymptotic behavior \(J_\nu(z) \sim \frac{\Gamma(\nu)}{\sqrt{\pi z}} / \Gamma(\nu + 1)\), \(Y_\nu(z) \sim -\frac{\Gamma(\nu)}{\sqrt{\pi z}} (\frac{z}{2})^{-\nu}\) as \(z \to 0\) (the properties of the special functions used in this proof and in the following text can be found, for example, in [1] or [9]). Hence, if \(u\) is a solution of (29) which satisfies (27), then there exists a constant \(c \in \mathbb{C}\) such that \(u(x) = c J_\nu(\sqrt{8m(\sigma \mu - \rho \mu)} x)\), and since \(J_\nu(z) \sim \sqrt{2/(\pi z)} \cos(z - \frac{\nu + 1}{2})\) as \(z \to \infty\), we obtain \(c = 0\) according to the normalization condition (27). This means, \(u \equiv 0\) on \((0, \infty)\), and from (26) it follows that \(v \equiv 0\) on \((0, \infty)\). Therefore, \(\omega\) is not an energy eigenvalue of (1) if \(\rho \mu - \sigma \mu < 0\).

Finally, let us consider the case \(\rho \mu - \sigma \mu > 0\). By means of the transformation
\[
u(x) = \tilde{u} \left(\sqrt{8m(\rho \mu - \sigma \mu)} x\right),
\]
(29) is on the interval \((0, \infty)\) equivalent to the differential equation
\[
z^2 \tilde{u}''(z) + z \tilde{u}'(z) - \left[z^2 + 4\kappa^2\right] \tilde{u}(z) = 0. \tag{31}
\]
The modified Bessel functions \(I_\nu\) and \(K_\nu\) of order \(\nu = 2\kappa \geq 1\), which form a fundamental system of solutions of (31), asymptotically behave like \(I_\nu(z) \sim (\frac{z}{2})^\nu / \Gamma(\nu + 1)\) and \(K_\nu(z) \sim \frac{\Gamma(\nu)}{\frac{z}{2}} (\frac{z}{2})^{-\nu}\) as \(z \to 0\). Hence, if \(u\) is a solution of
then the integrability condition \[ u = c I, \] implies \[ u = c I \] with some constant \( c \in \mathbb{C} \), and since \( I, z \sim e^{\nu} / \sqrt{2\pi z} \) as \( z \to \infty \), it follows that \( c = 0 \). Hence, \( u \equiv 0 \) on \((0, \infty)\), and gives \( v \equiv 0 \) on \((0, \infty)\). This proves that \( \omega \) is not an energy eigenvalue of \((1)\) if \( \rho m - \sigma \mu > 0 \).

In the following we suppose that the conditions \((15), (23)\) and \((24)\) are satisfied. Further, let \( T \) be the invertible matrix

\[
T := \begin{pmatrix}
-\sqrt{m - |\omega|} & \sqrt{m - |\omega|} \\
\sqrt{m + |\omega|} & \sqrt{m + |\omega|}
\end{pmatrix}.
\]

By means of the transformation \( w(x) = Ty(x) \), \((12)\) is on the interval \((0, \infty)\) equivalent to the system

\[
xy'(x) = \begin{pmatrix}
-\alpha - \gamma x & -\beta - \sigma \lambda \\
\beta - \sigma \lambda & \alpha + \gamma x
\end{pmatrix} y(x),
\]

where

\[
\alpha := \frac{\rho m^2 - \omega \mu}{\sqrt{m^2 - \omega^2}}, \quad \beta := \frac{(\rho|\omega| - \sigma \mu)m}{\sqrt{m^2 - \omega^2}}, \quad \gamma := \sqrt{m^2 - \omega^2},
\]

and since \( \|T^{-1}||^{-1} |y(x)| \leq |w(x)| \leq ||T|| |y(x)| \), a point \( \omega \in \mathbb{R} \) is an energy eigenvalue of \((1)\) if and only if the differential equation \((33)\) has a nontrivial solution \( y \) satisfying

\[
\int_0^\infty |y(x)|^2 \frac{(x + \rho)^2 + a^2}{x^2} \, dx < \infty.
\]

**Theorem 1** A point \( \omega \in \mathbb{R} \) is an energy eigenvalue of \((1)\) if and only if there exists an eigenvalue \( \lambda \in \mathbb{R} \) of the angular Dirac equation \((6)\) such that

\[
\omega = -\frac{ka + eQ \rho}{a^2 + \rho^2}, \quad m^2 - \omega^2 > 0, \quad \lambda^2 + \rho^2 m^2 - \mu^2 > \frac{1}{4},
\]

and either \( \beta - \sigma \lambda = 0 \) or \( 1 + n + \alpha + \kappa = 0 \) holds with some non-negative integer \( n \), where

\[
\kappa := \sqrt{\lambda^2 + \rho^2 m^2 - \mu^2}, \quad \mu := 2\rho \omega + eQ, \quad \sigma := \text{sign} \omega,
\]

and \( \alpha, \beta, \gamma \) are given by \((34)\).

If \( \beta - \sigma \lambda = 0 \) and \( \alpha + \kappa = 0 \), then the radial eigenfunctions are constant multiples of

\[
f(x) = x^\kappa e^{-\gamma x} ST \begin{pmatrix}
1 \\
0
\end{pmatrix}, \quad x \in (0, \infty),
\]

where the matrices \( S \) and \( T \) are given by \((11)\) and \((32)\), respectively.

If \( 1 + n + \alpha + \kappa = 0 \), then the radial eigenfunctions are constant multiples of

\[
f(x) = x^\kappa e^{-\gamma x} ST \begin{pmatrix}
(n + 1) L_{n+1}^{(2\kappa)}(2\gamma x) \\
(\beta - \sigma \lambda)L_n^{(2\kappa)}(2\gamma x)
\end{pmatrix}, \quad x \in (0, \infty),
\]
where \( L_n^{(2\kappa)} \) denotes the generalized Laguerre polynomial of degree \( n \) and order \( 2\kappa \).

**Proof.** Let \( \lambda \in \mathbb{R} \) be an eigenvalue of the angular Dirac equation (3). Introducing

\[
y(x) =: \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},
\]

\( \omega \) is an energy eigenvalue of (1) if and only if the system

\[
xu'(x) = -(\alpha + \gamma x)u(x) - (\beta + \sigma \lambda)v(x), \tag{36}
\]

\[
xv'(x) = (\beta - \sigma \lambda)u(x) + (\alpha + \gamma x)v(x) \tag{37}
\]

has a nontrivial solution \((u, v)\) which satisfies

\[
\int_0^\infty (|u(x)|^2 + |v(x)|^2) \frac{(x + \rho)^2 + a^2}{x^2} \, dx < \infty. \tag{38}
\]

We first assume that \( \beta - \sigma \lambda = 0 \). In this case, equation (37) implies \( v(x) = c_1 x^\alpha e^{\gamma x} \) with some constant \( c_1 \in \mathbb{C} \), and since \( \gamma > 0 \), (38) gives \( c_1 = 0 \). Further, from (36) and \( v \equiv 0 \) we obtain \( u(x) = c_2 x^{-\alpha} e^{-\gamma x} \) with some constant \( c_2 \in \mathbb{C} \), and (38) implies \( \alpha < -\frac{1}{2} \) in the case \( c_2 \neq 0 \). A short calculation shows that \( \alpha^2 - \beta^2 = \rho^2 m^2 - \mu^2 = \kappa^2 - \lambda^2 \), and therefore we have \( 0 = \beta^2 - \lambda^2 = \kappa^2 - \alpha^2 \).

Since \( \alpha < 0 < \kappa \), we obtain \( \alpha + \kappa = 0 \).

Now, let \( \beta - \sigma \lambda \neq 0 \). From (37) it follows that

\[
u(x) = \frac{\alpha + \gamma x}{\beta - \sigma \lambda}.
\]

Replacing \( u \) in (36) with (39) gives

\[
x^2v''(x) + xv'(x) - \left[ \lambda^2 + \alpha^2 - \beta^2 + (1 + 2\alpha)\gamma x + \gamma^2 x^2 \right] v(x) = 0. \tag{40}
\]

Since \( \lambda^2 - \beta^2 = \rho^2 m^2 - \mu^2 \) and \( \kappa^2 = \lambda^2 + \rho^2 m^2 - \mu^2 \), (40) takes the form

\[
x^2v''(x) + xv'(x) - \left[ \kappa^2 + (1 + 2\alpha)\gamma x + \gamma^2 x^2 \right] v(x) = 0. \tag{41}
\]

Further, by means of the transformation

\[
v(x) = \frac{1}{\sqrt{x}} \tilde{v}(2\gamma x),
\]

(41) is equivalent to Whittaker’s differential equation

\[
\tilde{v}''(z) + \left[ -\frac{1}{4} - \frac{1 + \alpha}{z} + \frac{1 - \kappa^2}{z^2} \right] \tilde{v}(z) = 0. \tag{42}
\]

A solution of (42) is the Whittaker function

\[
M_{-\frac{1}{2} - \alpha, \kappa}(z) = z^{\frac{1}{2} + \alpha} e^{-\frac{1}{2} z} M(1 + \alpha + \kappa, 1 + 2\alpha, z)
\]
where $M(p, q, z)$ denotes the Kummer function

$$M(p, q, z) := \sum_{n=0}^{\infty} \frac{(p)_n z^n}{(q)_n n!}$$

(the Pochhammer symbol is defined by $(p)_0 := 1$ and $(p)_n := p(p + 1) \cdots (p + n - 1)$ if $n \geq 1$). Thus, for some constant $c \in \mathbb{C} \setminus \{0\}$,

$$v(x) = c x^\kappa e^{-\gamma x} M(1 + \alpha + \kappa, 1 + 2\kappa, 2\gamma x)$$

is a nontrivial solution of (41), and the function $\frac{1}{x}v(x)$ is square integrable in a neighborhood of $x = 0$. Note that $\kappa$ and $-\kappa$ are the characteristic exponents of the differential equation (41). Hence, a solution of (41) which is linearly independent of (43) has an asymptotic behavior like $c 0 + o(1)$ as $x \to 0$ with some constant $c_0 \neq 0$, and since $\kappa > \frac{1}{2}$, such a solution cannot satisfy the normalization condition (38).

Now, $M(p, q, z) = \frac{\Gamma(q)}{\Gamma(p)} z^{p-q} e^z [1 + O(1/z)]$ as $z \to \infty$ if $p \neq 0, -1, -2, \ldots$, and if $n := -p$ is a non-negative integer, then $M(-n, q, z)$ reduces to a polynomial of degree $n$. In particular,

$$M(-n, q, z) = \frac{n!}{(q)_n} L_{n+1}(2\gamma x)$$

where $L_{n+1}(2\gamma x)$ denotes the generalized Laguerre polynomial of degree $n$ and order $q-1$. Consequently, if $-(1 + \alpha + \kappa) \notin \mathbb{N}$, then the solution (43) has the property that

$$v(x) = c x^\alpha e^{\gamma x} \left[1 + O \left(\frac{1}{x}\right)\right], \quad x \to \infty,$$

with some constant $c \neq 0$, and since $\gamma > 0$, $v$ does not match the normalization condition (38).

In the following we suppose that there exists a non-negative integer $n$ such that $n + 1 + \alpha + \kappa = 0$. In this case,

$$v(x) = (\beta - \sigma \lambda) \frac{(1 + 2\kappa)_n}{n!} x^\kappa e^{-\gamma x} M(-n, 1 + 2\kappa, 2\gamma x)$$

is a nontrivial solution of (41). Applying the differential relation

$$z M'(p, q, z) + (q - p - z) M(p, q, z) = (q - p) M(p - 1, q, z),$$

we can evaluate and simplify the expression in (39):

$$u(x) = \frac{x v'(x) - (\alpha + \gamma x) v(x)}{\beta - \sigma \lambda}$$

$$= (\kappa - \alpha) \frac{(1 + 2\kappa)_n}{n!} x^\kappa e^{-\gamma x} M(-1 - n, 1 + 2\kappa, 2\gamma x)$$

$$= (n + 1) x^\kappa e^{-\gamma x} L_{n+1}(2\gamma x).$$
Since \((u, v)\) satisfy the normalization condition (38), \(\omega\) is an energy eigenvalue of the Dirac equation (1), and the radial eigenfunctions are constant multiples of
\[
f(x) = x^\kappa e^{-\gamma x} ST\left(\frac{(n+1)L_{n+1}^{(2\kappa)}(2\gamma x)}{\beta - \sigma \lambda L_{n}^{(2\kappa)}(2\gamma x)}\right).
\]
□

As an immediate consequence of Theorem 1, we obtain the following well known result (see [5, Section V]).

**Corollary 3** In the extreme Reissner-Nordstrøm geometry, bound state solutions of the Dirac equation do not exist.

**Proof.** In the case \(a = 0\) we have \(\rho = |Q|\), and if a point \(\omega \in \mathbb{R}\) is an energy eigenvalue of (1), then \(\omega = -e|Q|\) according to Theorem 1. The condition \(m^2 - \omega^2 > 0\) yields \(m^2 - e^2 > 0\), and since \(\mu = 2\rho \omega + eQ = -eQ\), we obtain \(\alpha = |Q|\sqrt{m^2 - e^2} > 0\). Therefore \(\alpha + \kappa > \frac{1}{2}\), and the condition \(\alpha + \kappa = -n\) is not satisfied for any non-negative integer \(n\). This implies that (1) has no bound state solutions of the form (4). □

**Remark 1** We can also expect that the Dirac equation has no bound state solutions in an extreme Kerr-Newman black hole background if the angular momentum \(J\) is sufficiently small compared to the charge \(Q\).

We conclude this section with the following observation. Applying the unitary transformation
\[
f(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} F(x) \\ G(x) \end{pmatrix}
\]
and taking into account \(\rho = M\), the radial Dirac equation (10) with \(\tau = 0\) is equivalent to
\[
F'(x) = \frac{\lambda}{x} F(x) - \left[\omega - m - \frac{Mm - \mu}{x}\right] G(x),
\]
\[
G'(x) = -\frac{\lambda}{x} G(x) + \left[\omega + m + \frac{Mm + \mu}{x}\right] F(x),
\]
where \(\mu = 2M\omega + eQ\). In Planck units \(\hbar = c = G = 1\), this system is formally the same as that for the radial Dirac equation in Minkowski space-time with Coulomb potential \(-\frac{\omega}{x}\), Newtonian (scalar) potential \(\frac{Mm}{x}\), and spin-orbit coupling parameter \(\lambda\) (see [7, Example 9.8]).

### 4 The Kerr Case

In this section we consider the Dirac equation on the extreme Kerr manifold, i.e., we assume \(Q = 0\) and \(M = |a| > 0\).
Lemma 5 If, for some half-integer \( k \), the Dirac equation for a particle with rest mass \( m \) has a normalizable time-periodic solution with azimuthal quantum number \( k \) in an extreme Kerr geometry with mass \( M \) and angular momentum \( J \), then \( \frac{|k|}{2} < Mm < \frac{|k|}{\sqrt{2}} \), and the energy of this particle is given by \( \omega = -\frac{kM}{2} \).

Proof. Let \( L := am \). In the case \( Q = 0 \), we have \( \rho = M = |a| \) and \( \mu = 2|a|\omega \). Hence, Theorem \ref{thm:dirac} implies that a point \( \omega \in \mathbb{R} \) is an energy eigenvalue of \( (\ref{eq:dirac}) \) if and only if for some \( j \in \mathbb{Z} \)

\[
\alpha(L) := \frac{2L^2 - k^2}{\sqrt{4L^2 - k^2}}, \quad \beta(L) := -\frac{|kL|}{\sqrt{4L^2 - k^2}}, \quad \lambda_j(L) := \frac{\lambda_j(L)^2 + L^2 - k^2}{\sqrt{2}}, \quad \kappa_j(L) := \sqrt{\lambda_j(L)^2 + L^2 - k^2}.
\]

A necessary condition for \( \omega = -\frac{k}{2} = -\frac{kM}{2} \) being an energy eigenvalue of the Dirac equation \( (\ref{eq:dirac}) \) is \( \alpha(L) < 0 \), i.e., \( \frac{k^2}{4} < L^2 < \frac{k^2}{2} \), where \( L^2 = a^2m^2 = M^2m^2 \).

Since \( k \) is a half-integer, we have \( |k| \geq \frac{1}{2} \), and Lemma \ref{lem:halfinteger} immediately yields the following result.

Corollary 4 The inequality

\[
Mm > \frac{1}{4}
\]

is a necessary condition for the existence of bound states in the extreme Kerr geometry. Here, \( M \) is the mass of the extreme Kerr black hole and \( m \) is the rest mass of the Dirac particle.

Theorem 2 For a fixed half-integer \( k \), there exist two sequences \( (a^-_n)_{n \in \mathbb{N}} \) and \( (a^+_n)_{n \in \mathbb{N}} \) with the properties

\[
a^-_n < 0 < a^+_n, \quad |a^+_n| \in \left( \frac{|k|}{2m}, \frac{|k|}{\sqrt{2}m} \right), \quad \lim_{n \to \infty} |a^+_n| = \frac{|k|}{2m},
\]

such that the Dirac equation has a normalizable time-periodic solution with azimuthal quantum number \( k \) in the extreme Kerr black hole background with mass \( M = |a^+_n| \) and angular momentum \( J = a^+_nM \), and the one-particle energy of this bound state is given by \( \omega = -\frac{k}{2a^+_n} \).
Proof. Again, let \( L := am \). From Lemma 5 it follows that energy eigenvalues of the Dirac equation appear at most in the case \( |L| \in \left( \frac{|k|}{2}, \frac{|k|}{\sqrt{2}} \right) \). Now, by Lemma 1, the function \( \lambda_j \) depends continuously on \( L \), and \( |\lambda_j| \to \infty \) uniformly on the compact set \( K := \left[ -\frac{|k|}{\sqrt{2}}, -\frac{|k|}{2} \right] \cup \left[ \frac{|k|}{2}, \frac{|k|}{\sqrt{2}} \right] \) as \( |j| \to \infty \). Hence, there exists an integer \( j = j(k) \) such that the last inequality in (44) holds for all \( L \in K \). Since \( \alpha(L) \to -\infty \) as \( |L| \to \frac{|k|}{2} \), \( \alpha(L) \to 0 \) as \( |L| \to \frac{|k|}{\sqrt{2}} \), \( \alpha(L) < 0 \) if \( \frac{|k|}{2} < |L| < \frac{|k|}{\sqrt{2}} \), and \( \alpha, \lambda_j \) are continuous functions on \( K \), the intersection theorem yields that for fixed \( n \in \mathbb{N} \) the equation \( \alpha(L) = -n - \kappa_j(L) \) has at least one solution \( L_n^\pm = a_n^\pm m \) in the interval \( \left( \frac{|k|}{2}, \frac{|k|}{\sqrt{2}} \right) \). In addition, \( \alpha(L_n^\pm) = -1 - n - \kappa_j(L_n^\pm) \to -\infty \) as \( n \to \infty \), which implies \( \lim_{n \to \infty} |a_n^\pm| = \frac{|k|}{2m} \). \( \square \)

Remark 2 We can expect that a similar result holds for extreme Kerr-Newman black holes if the charge \( Q \) is sufficiently small compared to the angular momentum \( J \).

Remark 3 Since the functions \( \alpha \) and \( -n - \kappa_j \) both depend analytically on \( L \), but are not identical, there can exist at most finitely many solutions of the equation \( \alpha(L) = -n - \kappa_j(L) \) for each half-integer \( k, n \in \mathbb{N}_0 \), and \( j \in \mathbb{Z} \). In particular, bound states for the Dirac equation exist on at most countably many extreme Kerr black holes.

The figures below are the results of numerical computations and give some examples for the radial and angular density functions \(|f|^2, |g|^2\) of the bound state solutions for different values of \( k \) and \( a \). In each example, the half-integer \( k \) and the corresponding values of \( am \) and \( \frac{\omega}{m} \) in Planck units \( h = c = G = 1 \) are specified.

Figure 1: \( k = \frac{5}{2}, am = -1.264065, \frac{\omega}{m} = 0.988873 \)
Figure 2: $k = \frac{5}{2}, am = -1.266630, \frac{\omega_m}{\omega} = 0.986871$

Figure 3: $k = \frac{17}{2}, am = -4.594167, \frac{\omega_m}{\omega} = 0.925086$

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References

[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*, Dover Publications, Inc., New York, 1966.

[2] S. Chandrasekhar, *The solution of Dirac's equation in Kerr geometry*, Proc. Roy. Soc. Lond. A 349 (1976), no. 1659, 571–575.
[3] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill Company, Inc., New York – Toronto – London, 1955.

[4] M. S. P. Eastham, The Asymptotic Solution of Linear Differential Systems. Applications of the Levinson Theorem, London Mathematical Society Monographs (New Series) 4, Oxford University Press, New York, 1989.

[5] F. Finster, J. Smoller, and S.-T. Yau, Non-existence of time-periodic solutions of the Dirac equation in a Reissner-Nordstrøm black hole background, J. Math. Phys. 41 (2000), no. 4, 2173–2194.

[6] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau, Nonexistence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry, Commun. Pure Appl. Math. 53 (2000), no. 7, 902–929.

[7] W. Greiner, Relativistic Quantum Mechanics. Wave Equations, Theoretical Physics: Text and Exercise Books, 3, Springer, Berlin, 1990.

[8] T. Kato, Perturbation theory for linear operators, Springer, Berlin – Heidelberg – New York, 1966.

[9] W. Magnus, F. Oberhettinger, R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer, Berlin – Heidelberg – New York, 1966.

[10] D. Page, Dirac equation around a charged, rotating black hole, Phys. Rev. D 14 (1976), pp. 1509.

[11] R. M. Wald, General Relativity, The University of Chicago Press, Chicago – London, 1984.

[12] J. Weidmann, Spectral Theory of Ordinary Differential Operators, Lecture Notes in Mathematics 1258, Springer, Berlin – New York, 1987.