MIRROR SYMMETRY FOR DOUBLE COVER CALABI-YAU VARIETIES

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Abstract. The presented paper is a continuation of the series of papers [17, 18]. In this paper, utilizing Batyrev and Borisov’s duality construction on nef-partitions, we generalize the recipe in [17, 18] to construct a pair of singular double cover Calabi–Yau varieties \((Y, Y^\lor)\) over toric manifolds and compute their topological Euler characteristics and Hodge numbers. In the 3-dimensional cases, we show that \((Y, Y^\lor)\) forms a topological mirror pair, i.e., \(h^{p,q}(Y) = h^{3-p,q}(Y^\lor)\) for all \(p, q\).

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0. Introduction

0.1. Motivations. Mirror symmetry from physics has successfully made numerous non-trivial predictions in algebraic geometry and has been investigated intensively in the last decades. Roughly speaking, a mirror pair is a pair of Calabi–Yau varieties \((M, M^\lor)\) such that under certain identification, which is called the mirror map, the A-model correlation function of \(M\) is identified with the B-model correlation function of \(M^\lor\) and vice versa.

The first mirror pair was written down by Greene and Plesser [13], the quintic and the (orbifold) Fermat quintic threefold. Utilizing reflexive polytopes, Batyrev gave a recipe for constructing mirror pairs for Calabi–Yau hypersurfaces in Gorenstein toric varieties [5]. Soon later Batyrev and Borisov generalized the construction to Calabi–Yau complete intersections in Gorenstein toric varieties via nef-partitions [1].

During the last two decades, to test mirror symmetry, many techniques had been developed and numerous numerical quantities had been calculated explicitly. The first convincing evidence was the successful prediction of the numbers of rational curves on quintic threefolds in \(\mathbb{P}^4\) by Candelas et. al. [6] in a vicinity of the so-called maximal

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unipotent monodromy point in the moduli. Hosono, Klemm, Theisen, and Yau calculated the $B$-model correlation functions as well as the mirror maps to test the mirror symmetry for Calabi–Yau hypersurfaces in 4-dimensional Gorenstein toric Fano varieties [14]. It was observed by Hosono, Lian, and Yau in [15] that the Gröbner basis for the toric ideal determines a finite set of differential operators for the local solutions to the $A$-hypergeometric system, one of the most important tools to study the $B$-model correlation function introduced by Gel’fand, Kapranov and Zelevinskii [12]. They also proved the existence of rank one points of the $A$-hypergeometric system for the family of Calabi–Yau hypersurfaces in certain toric varieties, where mirror symmetry is expected [16].

The family of $K3$ surfaces arising from double covers branched along six lines in $\mathbb{P}^2$ in general positions were studied by Matsumoto, Sasaki, and Yoshida [24,25] as a higher dimensional analogue of the Legendre family. The parameter space $P(3,6)$ of this $K3$ family admits various compactifications – (a) a GIT compactification (a.k.a. the Baily–Borel–Satake compactification) [9,26] and (b) a toroidal compactification constructed by Reuvers [28]. However, Hosono, Lian, Takagi, and Yau observed in [18] that none of these compactifications admits a priori the so-called large complex structure limit points (LCSL points for short hereafter). In order to study mirror symmetry, they constructed a new compactification of $P(3,6)$ and found LCSL points on it by relating it to (a) and (b). We briefly explain their idea. The $GL_3(\mathbb{C})$-action on $\mathbb{P}^2$ allows us to rearrange the hyperplanes to the coordinate axes so that the $K3$ family is in fact parameterized by three lines in $\mathbb{P}^2$. This procedure is called the partial gauge fixing in [18]. After the partial gauge fixing, it turns out that the period integrals of our $K3$ family satisfy certain $A$-hypergeometric system with $A \in \text{Mat}_{5 \times 9}(\mathbb{Z})$, a fractional exponent $\beta \in \mathbb{Q}^5$. The matrix $A$ can be recognized as the integral matrix associated to certain nef-partition on the base $\mathbb{P}^2$ and the torus $(\mathbb{C}^*)^5$ can be identified with $L \otimes \mathbb{C}^*$, where $L$ is the lattice relation of $A$. Consequently, $P(3,6)$ admits a toroidal compactification via the associated secondary fan. Standard techniques for Calabi–Yau hypersurfaces or complete intersections in toric varieties are still applicable and results in [14,16] can be straightened into this situation. Because of this striking similarity with the classical complete intersections, we shall call such a double cover a fractional complete intersection. Based on numerical evidences, it is conjectured that the mirror of the said $K3$ family is given by certain double covers over a del Pezzo surface of degree 6, which is a blow-up of three torus invariant points on $\mathbb{P}^2$ ([17, Conjecture 6.3]). Note that such a del Pezzo surface can be obtained from Batyrev–Borisov’s duality construction for the associated nef-partition on $\mathbb{P}^2$.

0.2. Statements of main results. The aim of this paper is to study the conjecture [17, Conjecture 6.3] and its further generalization. Consider a nef-partition $(\Delta, \{\Delta_i\}_{i=1}^r)$ and its dual nef-partition $(\nabla, \{\nabla_i\}_{i=1}^r)$ in the sense of Batyrev and Borisov (the precise definitions of nef-partition and the dual nef-partition will be given in §1.2). Let $P_\Delta$ and $P_\nabla$ be the toric varieties defined by $\Delta$ and $\nabla$. Let $X \to P_\Delta$ and $X^\vee \to P_\nabla$ be maximal projective crepant partial resolutions (MPCP resolutions for short hereafter) of $P_\Delta$ and $P_\nabla$. The nef-partitions on $P_\Delta$ and $P_\nabla$ determine nef-partitions on $X$ and $X^\vee$. Let
$E_1, \ldots, E_r$ and $F_1, \ldots, F_r$ be the sum of toric divisors representing nef-partitions on $X$ and $X^\vee$, respectively. In this article, we will assume that

$$X \text{ and } X^\vee \text{ are both smooth.}$$

Said differently, both $\Delta$ and $\nabla$ admit a regular triangulation.

Let $s_j \in H^0(X, 2E_j)$ be a smooth section and $Y$ be the double cover over $X$ branched along $s_1 \cdot \cdots \cdot s_r$. Deforming the sections $s_j$ yields a family of Calabi–Yau double covers over $X$, which is parameterized by a suitable open set in the product of $H^0(X, E_j)$. We now elaborate how to define a partial gauge fixing for such a family (see §2.1 for details), which turns out to be crucial in this paper.

A partial gauge fixing is a decomposition of the section $s_j$ into a product of a canonical section of $E_j$ and a smooth section of $E_j$. In other words, $s_j = s_{j,1}s_{j,2}$ with $s_{j,k} \in H^0(X, E_j)$ such that $\text{div}(s_{j,1}) \equiv E_j$ and $\text{div}(s_{j,2})$ is smooth. The original double cover family will restrict to a subfamily parametrized by

$$V \subset H^0(X, E_1) \times \cdots \times H^0(X, E_r).$$

A parallel construction can be applied on the dual side. Let $Y \to V$ and $Y^\vee \to V^\vee$ be partial gauge fixings for those families. Let $Y$ and $Y^\vee$ be the fiber of these families.

We observe that $Y$ and $Y^\vee$ form a topological mirror pair.

**Theorem 0.1 (Theorem 2.2).** We have $\chi_{\text{top}}(Y) = (-1)^n \chi_{\text{top}}(Y^\vee)$, where $n = \dim Y$ and $\chi_{\text{top}}(\cdot)$ denotes the topological Euler characteristic.

Since $Y$ and $Y^\vee$ are orbifolds, the Hodge numbers $h^{p,q}(Y)$ are well-defined. Moreover, by construction, $X \setminus B$ is affine, where $B$ is the branched locus of the cover $Y \to X$. It follows that $h^{p,q}(Y) = h^{p,q}(X)$ for all $p, q$ with $p + q \neq n$. In particular, when $n = 3$, we can prove

**Theorem 0.2 (Theorem 2.3).** We have $h^{p,q}(Y) = h^{3-p,q}(Y^\vee)$ for all $p, q$.

The calculation of the Euler characteristics boils down to computation of intersection numbers on toric varieties, which turns out to be a consequence of a combinatorial formula by Danilov and Khovanskii [8].

Based on these results, we propose the following conjecture, which can be served as a generation of [17, Conjecture 6.3].

**Conjecture.** $Y$ is mirror to $Y^\vee$.

We shall emphasize that none of $Y$ and $Y^\vee$ is smooth. The conjecture is served as an extension of the classical mirror correspondence to singular Calabi–Yau varieties.

**Remark 0.1.** The quantum test, i.e., the correspondence between enumerative geometry (more precisely, the Chen–Ruan orbifold Gromov–Witten invariants) and complex geometry (deformation of complex structures), for the conjecture will be treated in our forthcoming paper.

We work over $\mathbb{C}$, the field of complex numbers.
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1. Preliminaries

1.1. Cyclic covers. In this paragraph, let \( X \) be a smooth projective variety, \( L \) be a line bundle over \( X \) and \( \mathcal{L} \) be the sheaf of sections of \( L \). For \( s \in \Gamma(X, L^r) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, L^r) \), the dual \( s^\vee : \mathcal{L}^{-r} \to \mathcal{O}_X \) determines an \( \mathcal{O}_X \)-algebra structure on \( A_s' := \bigoplus_{i=0}^{r-1} \mathcal{L}^{-i} \). In fact, we have an identification

\[
A_s' = \bigoplus_{i=0}^{\infty} \mathcal{L}^{-i}/\mathcal{I},
\]

where \( \mathcal{I} \) is the sheaf of \((\bigoplus_{i=0}^{\infty} \mathcal{L}^{-i})\)-module generated by

\[
\{s^\vee(\ell) - \ell : \ell \text{ is a local section of } \mathcal{L}^{-r}\}.
\]

Note that the multiplication on \( A_s' \) is given by the usual multiplication on sections \( \mathcal{L}^{-i} \times \mathcal{L}^{-j} \to \mathcal{L}^{-i-j} \) and further composed with \( s^\vee \) if \( i+j \geq r \). Let \( Y'_s := \text{Spec}_{\mathcal{O}_X}(A'_s) \) and \( Y_s \to Y'_s \) be the normalization. We denote by \( D_s \) the scheme-theoretic zero of \( s \).

**Definition 1.1.** The scheme \( Y_s \) is called the \( r \)-fold cyclic cover over \( X \) branched over \( D_s \) or simply the \( r \)-fold cover if the context is clear.

We are mainly interested in the situation that \( \text{codim}_X \text{Sing}(D_s) \geq 2 \), which implies that \( Y'_s \) is already normal and consequently \( Y_s = Y'_s \). We denote by \( \omega_X \) and \( \omega_Y \) the dualizing sheaf of \( X \) and \( Y \). We can summarize these results in the next proposition.

**Proposition 1.1.** \( Y'_s \) is Cohen–Macaulay. Furthermore, if \( \text{codim}_X \text{Sing}(D_s) \geq 2 \), then \( Y'_s \) is a normal variety, \( \omega_Y \cong \mathcal{O}_Y \) if and only if \( \omega_X \otimes \mathcal{L}^{r-1} \cong \mathcal{O}_X \).

**Proof.** See Proposition \[A.3\] for the proof. \(\square\)

**Example 1.2.** Let \( X = \mathbb{P}^n \) and \( \mathcal{L} = \mathcal{O}_X(d) \). We list some \( r \)-fold cyclic covers over \( X \) which satisfy \( \omega_Y \cong \mathcal{O}_Y \). In this case, the criterion in Proposition \[1.1\] boils down to the numerical constraint \( n+1 = d(r-1) \).

- \( n = 1 \).
  - (1a) \( d = r = 2 \). The cyclic cover \( Y \) is an elliptic curve and the attached family is known as the Legendre family. The general fiber (branched over four distinct points) has non-zero \( j \)-invariant.
  - (1b) \( d = 1 \) and \( r = 3 \). The cyclic cover \( Y \) is also an elliptic curve, whose \( j \)-invariant is zero.
- \( n = 2 \).
  - (2a) \( d = 3 \) and \( r = 2 \).
  - (2b) \( d = 1 \) and \( r = 4 \).
- \( n = 3 \).
(3a) $d = 4$ and $r = 2$.
(3b) $d = 2$ and $r = 3$.
(3c) $d = 1$ and $r = 5$.

We can also compute the Euler characteristic for the cyclic covers. Let us recall that for an $n$-dimensional complex analytic variety $W$, the Euler characteristic is defined to be

$$
\chi(W) := \sum_{k=0}^{2n} (-1)^k \dim H^k(W) = \sum_{k=0}^{2n} (-1)^k \dim H^k_c(W).
$$

If $U \to W$ is a finite étale cover of degree $r$, then we have $\chi(U) = r \cdot \chi(W)$.

Let $\pi : Y \to X$ be an $r$-fold cyclic cover and $D$ be the ramification locus. Then $Y \setminus \pi^{-1}(D) \to X \setminus D$ is a finite étale cover of degree $r$. We then have

$$
\chi(Y) = \chi(\pi^{-1}(D)) + \chi(Y \setminus \pi^{-1}(D)) = \chi(D) + r \cdot \chi(X \setminus D) = \chi(D) + r(\chi(X) - \chi(D)).
$$

### 1.2. Toric varieties and Batyrev–Borisov’s duality construction

To elaborate the singular mirror duality in this paper, we review the construction of classical mirror duality pair of Calabi–Yau complete intersections in toric varieties introduced by Batyrev and Borisov \[1\]. Let us begin with the following data.

- Let $N = \mathbb{Z}^n$ be a lattice of rank $n$ and $M := \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ be the dual lattice. We denote by $N_\mathbb{R}$ and $M_\mathbb{R}$ the tensor products $N \otimes_\mathbb{Z} \mathbb{R}$ and $M \otimes_\mathbb{Z} \mathbb{R}$.
- For a complete fan $\Sigma$ in $N_\mathbb{R}$, we denote by $\Sigma(k)$ the set of all $k$-dimensional cones in $\Sigma$. For convenience, we write $\Sigma(1) = \{ \rho_1, \ldots, \rho_p \}$. The same notation $\rho_i$ is used to denote the primitive generator of the corresponding 1-cone. The support of $\Sigma$ is denoted by $|\Sigma|$.
- The toric variety defined by $\Sigma$ is denoted by $X_\Sigma$ or simply by $X$ if the context is clear. Let $T = (\mathbb{C}^*)^n$ be its maximal torus. Each $\rho \in \Sigma(1)$ determines a Weil divisor $D_\rho$ on $X$.
- Let $D = \sum_\rho a_\rho D_\rho$ be a torus invariant divisor. The divisor polytope $\Delta_D$ is defined by

$$
\Delta_D := \{ m \in M_\mathbb{R} : \langle m, \rho \rangle \geq -a_\rho, \forall \rho \in \Sigma(1) \}.
$$

- A polytope in $M_\mathbb{R}$ is called lattice polytope if its vertices belong to $M$. For a lattice polytope $\Delta$ in $M_\mathbb{R}$, we denote by $\Sigma_\Delta$ the normal fan of $\Delta$. The toric variety determined by $\Delta$ is denoted by $\mathbf{P}_\Delta$, i.e., $\mathbf{P}_\Delta = X_{\Sigma_\Delta}$.
- A reflexive polytope $\Delta \subset M_\mathbb{R}$ is a lattice polytope containing the origin $0 \in M_\mathbb{R}$ in its interior and such that the polar dual $\Delta^\vee$ is again a lattice polytope. If $\Delta$ is a reflexive polytope, then $\Delta^\vee$ is also a lattice polytope and satisfies $(\Delta^\vee)^\vee = \Delta$.

The normal fan of $\Delta$ is the face fan of $\Delta^\vee$ and vice versa.

Let $I_1, \ldots, I_r$ be a nef-partition on $\mathbf{P}_\Delta$, that is, $\Sigma_\Delta(1) = \sqcup_{s=1}^{r} I_s$ and $E_s := \sum_{\rho \in I_s} D_\rho$ is numerical effective for each $s$. This gives rise to a Minkowski sum decomposition...
$\Delta = \Delta_1 + \cdots + \Delta_r$, where $\Delta_i = \Delta_{E_i}$ is the section polytope of $E_i$. The Batyrev–Borisov duality construction goes in the following way.

Let $\nabla_k$ be the convex hull of $\{0\} \cup I_k$ and $\nabla = \nabla_1 + \cdots + \nabla_r$ be their Minkowski sum. It turns out that $\nabla$ is a reflexive polytope in $N_\mathbb{R}$ whose polar polytope is given by $\nabla^\vee = \text{Conv}(\Delta_1, \ldots, \Delta_r)$ and $\nabla_1 + \cdots + \nabla_r$ corresponds to a nef-partition on $P_\nabla$, called the dual nef-partition. The corresponding nef toric divisors are denoted by $F_1, \ldots, F_r$.

Then the section polytope of $F_j$ is $\Delta_j = \Delta$.

Let $X \to P_\Delta$ and $X^\vee \to P_\nabla$ be MPCP resolutions for $P_\Delta$ and $P_\nabla$. Via pullback, the nef-partitions on $P_\Delta$ and $P_\nabla$ determine nef-partitions on $X$ and $X^\vee$ and they determine the families of Calabi–Yau complete intersections inside $X$ and $X^\vee$ respectively.

Recall that the section polytopes $\Delta_i$ and $\nabla_j$ correspond to $E_i$ on $P_\Delta$ and $F_j$ on $P_\nabla$, respectively. To save the notation, the corresponding nef-partitions and toric divisors on $X$ and $X^\vee$ will be still denoted by $\Delta_i$, $\nabla_j$ and $E_i$, $F_j$ respectively.

There is another point of view which is useful for us. Given a nef-partition on $X$ as above, corresponding to $\Delta = \Delta_1 + \cdots + \Delta_r$, one constructs a cone in $\mathbb{R}^r \times M_\mathbb{R}$ by

$$\sigma_\Delta := \left\{ \left( \lambda_1, \ldots, \lambda_r, \sum_{i=1}^r \lambda_i w_i \right) : w_i \in \Delta_i \text{ and } \lambda_i \geq 0 \right\}.$$ 

Then the dual cone $\sigma_\Delta^\vee \subset \mathbb{R}^r \times N_\mathbb{R}$ can be identified with the cone $\sigma_\nabla \subset \mathbb{R}^r \times N_\mathbb{R}$ constructed from the dual nef-partition $\nabla_1 + \cdots + \nabla_r$. $\sigma_\Delta$ and $\sigma_\nabla$ arising in this way give a pair $(\sigma_\Delta, \sigma_\nabla)$ of the so-called reflexive Gorenstein cones with index $r$. See [3] for further discussions.

The following proposition may be known to experts.

**Proposition 1.2.** Assume that $X$ and $X^\vee$ are both smooth. Let $\{e_i\}_{i=1}^r$ be the standard basis of $\mathbb{R}^r$. We denote by $S$ the convex hull of $0$ and $e_i \times (\Delta_i \cap M)$, $i = 1, \ldots, r$, in $\mathbb{R}^r \times M_\mathbb{R}$. Then the normalized volume of $S$ in $\mathbb{R}^r \times M_\mathbb{R}$ is equal to the normalized volume of $\nabla^\vee = \text{Conv}(\Delta_1, \ldots, \Delta_r)$ in $M_\mathbb{R}$.

**Proof.** Let $W$ be the total space of the vector bundle $\mathcal{O}(F_1) \oplus \cdots \oplus \mathcal{O}(F_r)$ over $X^\vee$. Since $X^\vee$ is assumed to be smooth, $W$ is a smooth toric variety having the same Euler characteristic with $X^\vee$. The normalized volume of $S$ is equal to the number of maximal cones in the toric variety $W$ and therefore it is equal to the Euler characteristic of $X^\vee$, that is, the normalized volume of $\nabla^\vee$. \hfill \square

Let $Z_1, \ldots, Z_k$ be nef torus invariant divisors on $X$ and $\Delta_{Z_i}$ be the section polytope of $Z_i$. Let $\Delta_{Z_1} \ast \cdots \ast \Delta_{Z_k}$ be the Cayley polytope of $\Delta_{Z_i}$, i.e., the convex hull of the polyhedra $e_1 \times \Delta_{Z_1}, \ldots, e_k \times \Delta_{Z_k}$ in the space $\mathbb{R}^k \times M_\mathbb{R}$. Similarly for each nonempty subset $J \subset \{1, \ldots, k\}$, we define $\Delta_{*J} := \ast_{j \in J} \Delta_{Z_j} \subset \mathbb{R}^{\mid J\mid} \times M_\mathbb{R}$. Let $\Lambda$ and $\Lambda_J$ be the pyramids with vertex $0$ and base $\Delta_{1} \ast \cdots \ast \Delta_r$ and $\Delta_{*J}$ in $\mathbb{R}^r \times M_\mathbb{R}$ and $\mathbb{R}^{\mid J\mid} \times M_\mathbb{R}$ respectively. Now we can state a result due to Danilov and Khovanskii.

**Theorem 1.3** (cf. [3], §6). For general $D_i$ in the linear system $|Z_i|$, we have

$$\chi(D_1 \cap \cdots \cap D_k \cap T) = - \sum_{J} (-1)^{n+\mid J\mid-1} \text{vol}_{n+\mid J\mid}(\Lambda_J),$$
where the summation runs over all nonempty subsets $J \subset \{1, \ldots, k\}$ and $\text{vol}_k$ is the normalized volume in $k$-dimensional spaces.

**Example 1.3** (Double covers over $\mathbb{P}^2$ branched along six lines). Let $X = \mathbb{P}^2$ and $\Delta = \text{Conv}\{(2,-1),(-1,2),(-1,-1)\}$ be the section polytope of $-K_X$. We denote by $\rho_1$, $\rho_2$, and $\rho_3$ the primitive vectors $(1,0)$, $(0,1)$, and $(-1,-1)$ respectively generating the 1-cones of the normal fan of $\Delta$, i.e., the standard $\mathbb{P}^2$ fan. Then the divisors $E_i := D_{\rho_i}$, $i = 1, 2, 3$, define a nef-partition on $X = \mathbb{P}_\Delta$. Correspondingly we have the decomposition $\Delta = \Delta_1 + \Delta_2 + \Delta_3$. Batyrev–Borisov duality applies to $\Delta$ and $\nabla$ such that $\nabla^\vee = \text{Conv}(\Delta_1, \Delta_2, \Delta_3)$. Namely if we define $\nabla_i := \text{Conv}(0, \rho_i)$, then we obtain the decomposition $\nabla = \nabla_1 + \nabla_2 + \nabla_3$ which corresponds to the dual nef-partition $F_1 + F_2 + F_3$ on $X^\vee = \mathbb{P}_{\nabla}$. Having $\nabla = \text{Conv}\{ (\pm 1,0), (0,\pm 1), (1,1), (-1,-1)\}$, or from the face fan of $\nabla^\vee$, we determine the normal fan of $\nabla$, which is described by the following primitive generators of 1-dimensional cones

$$
\begin{align*}
\nu_1 &= (-1,1), \quad \nu_2 = (-1,0), \quad \nu_3 = (0,-1), \\
\nu_4 &= (1,-1), \quad \nu_5 = (1,0), \quad \text{and} \quad \nu_6 = (0,1).
\end{align*}
$$

From these data, we see that $\mathbb{P}_{\nabla}$ is isomorphic to $\mathbb{P}^2$ blown up at three points. Also the dual nef-partition is given by

$$
F_k = D_{\nu_{2k-1}} + D_{\nu_{2k}}, \quad k = 1, 2, 3.
$$

2. **Mirror symmetry for singular Calabi–Yau double covers over toric manifolds**

The mirror duality between singular $K3$ surfaces (see Example 2.2 below) was discovered in [17, 18]. In this paragraph, we put the mirror duality into a more general framework: we formulate the mirror duality for the pair of singular Calabi–Yau varieties, which are double covers over certain pair of dual toric manifolds.

Let us keep the notation in [1.2]. Starting with a reflexive polytope $\Delta$ in $M_\mathbb{R}$ and a decomposition $\Delta_1 + \cdots + \Delta_r$, representing a nef-partition $E_1 + \cdots + E_r$ of $-K_{\mathbb{P}_\Delta}$, we have the corresponding dual polytope $\nabla$ in $N_\mathbb{R}$ and the dual decomposition $\nabla_1 + \cdots + \nabla_r$ representing the dual nef-partition $F_1 + \cdots + F_r$ of $-K_{\mathbb{P}_{\nabla}}$. Let $X$ and $X^\vee$ be the MIPC resolution of $\mathbb{P}_\Delta$ and $\mathbb{P}_{\nabla}$ respectively. Hereafter, we will simply call the decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ a nef-partition on $X$ for short with understanding the nef-partition $E_1 + \cdots + E_r$. Likewise for the decomposition $\nabla = \nabla_1 + \cdots + \nabla_r$. Also, unless otherwise stated, we assume that

$$
X \text{ and } X^\vee \text{ are both smooth.}
$$

Equivalently, we assume that both $\Delta$ and $\nabla$ admit uni-modular triangulations. From the duality, we have

$$
H^0(X^\vee, F_i) \cong \bigoplus_{\rho \in \nabla_i \cap N} \mathbb{C} \cdot t^\rho \quad \text{and} \quad H^0(X, E_i) \cong \bigoplus_{m \in \Delta_i \cap M} \mathbb{C} \cdot t^m.
$$

Here we use the same notation $t = (t_1, \ldots, t_n)$ to denote the coordinates on the maximal torus of $X$ and $X^\vee$. 

From Proposition 1.1, a double cover $Y$ has trivial canonical bundle if and only if $L \cong \omega_X^{-1}$. The branched locus of $Y \rightarrow X$ is linearly equivalent to $-2K_X$.

**Definition 2.1.** Given a decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ representing a nef-partition $E_1 + \cdots + E_r$ on $X$, the double covers branched along the nef-partition over $X$ is the double cover $Y \rightarrow X$ constructed from the section $s = s_1 \cdots s_r$ with

$$(s_1, \ldots, s_r) \in H^0(X, 2E_1) \times \cdots \times H^0(X, 2E_r),$$

where $E_i$ is the corresponding toric divisor to $\Delta_i$.

**Example 2.2** (Families of singular $K3$ surfaces). Let us retain the notation in Example 1.3. Let $Y \rightarrow X := \mathbb{P}^2$ be the double cover branched along six lines in general positions. $Y$ is a singular $K3$ surface with $15 A_1$-singularities.

The indicial ring for the Picard–Fuchs equation of this family was calculated in [17, Proposition 4.4]. It was proved that the intersection pairing $\langle \theta_i, \theta_j \rangle$ is identical to the intersection matrix of the divisors $\tilde{L}_i$ in $Y^\vee$, $i = 1, \ldots, 4$. Here $Y^\vee$ is a double cover over $X^\vee$ and $\tilde{L}_i$ is the pullback of $L_i = F_i$ for $i = 1, 2, 3$ and $L_4 = H$ on $X^\vee$ (the pullback of the hyperplane class on $X^\vee \rightarrow \mathbb{P}^2$). For notation and details, see [17, §4 and §6].

In order to generalize the duality construction to double covers over toric varieties, we need the concept of “partial gauge fixings”.

2.1. **Partial gauge fixings.** In the $K3$ example, the gauge fixed family over $\mathbb{P}^2$ is the subfamily when the “half” of the branched divisors are fixed to be the toric divisors. Inspired by this, we are led to consider the case when $s_i \in H^0(X, 2E_i)$ is of the form $s_i = s_{i,1} s_{i,2}$ with $s_{i,1}, s_{i,2} \in H^0(X, E_i)$. We further assume that $s_{i,1}$ is the section corresponding to the lattice point $0 \in \Delta_i \cap M$, i.e., the scheme-theoretic zero of $s_{i,1}$ is $E_i$, and that the scheme-theoretic zero of $s_{i,2}$ is non-singular. In this manner, we obtain a subfamily of double covers branched along the nef-partition over $X$ parameterized by an open subset

$$V \subset H^0(X, E_1)^\vee \times \cdots \times H^0(X, E_r)^\vee.$$

**Definition 2.3.** Given a decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ representing a nef-partition $E_1 + \cdots + E_r$ on $X$, the subfamily $\mathcal{V} \rightarrow V$ constructed above is called the gauge fixed double cover branched along the nef-partition over $X$ or simply the gauge fixed double cover if no confuse occurs.

Given a decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ representing a nef-partition $E_1 + \cdots + E_r$ on $X$ as above, we denote by $\mathcal{V} \rightarrow V$ the gauge fixed double cover family. A parallel construction is applied for the dual decomposition $\nabla = \nabla_1 + \cdots + \nabla_r$ representing the dual nef-partition $F_1 + \cdots + F_r$ over $X^\vee$ and this yields another family $\mathcal{V}^\vee \rightarrow W$, where $W$ is an open subset in

$$H^0(X^\vee, F_1)^\vee \times \cdots \times H^0(X^\vee, F_r)^\vee.$$

This construction generalizes our previous example on double covers over $\mathbb{P}^2$. 
Example 2.4 (Families of singular $K3$ surfaces continued). Let $Y^\vee$ be the gauged fixed double cover branched along the nef-partition $F_1 + F_2 + F_3$ over $X^\vee$. Let us write down the period integral for the family $\mathcal{Y}^\vee \to W$.

Let $w_1, \ldots, w_6$ be the homogeneous coordinates corresponding to divisors $D_{\nu_1}, \ldots, D_{\nu_6}$ for $X^\vee$. Let $t_1, t_2$ be the coordinates on the maximal torus of $X^\vee$. These are related by $t_i = \prod_j w_j^{\nu_i,j}$ ($i = 1, 2$), which gives

$$t_1 = w_1^{-1}w_2^{-1}w_5, \quad t_2 = w_1w_3^{-1}w_4^{-1}w_6.$$  \hspace{1cm} (2.1)

In terms of homogeneous coordinates, we have $s_{1,1} = w_1w_2$, $s_{2,1} = w_3w_4$, and $s_{3,1} = w_5w_6$ for each $0 \in \Delta_i$ representing $H^0(X^\vee, F_i)$. For the other half of sections $s_{i,2} \in H^0(X^\vee, F_i)$, we write them with parameters $(a_1, b_1, a_2, b_2, a_3, b_3)$ as follows:

$$s_{1,2} = a_1w_1w_2 + b_1w_4w_5,$$

$$s_{2,2} = a_2w_3w_4 + b_2w_1w_6,$$

$$s_{3,2} = a_3w_5w_6 + b_3w_2w_3.$$  

Then we can write the period integral as a function on $W$

$$\int \frac{\Omega_{X^\vee}}{\sqrt{s_{1,1}s_{2,1}s_{3,1}s_{1,2}s_{2,2}s_{3,2}}} = \int \frac{\Omega_{X^\vee}}{w_1w_2w_3w_4w_5w_6} \frac{1}{\sqrt{h_1h_2h_3}} = \int \frac{dt_1 \wedge dt_2}{t_1t_2} \frac{1}{\sqrt{h_1h_2h_3}},$$

where

$$h_1 = w_1^{-1}w_2^{-1}s_{1,2} = a_1 + b_1t_1,$$

$$h_2 = w_3^{-1}w_4^{-1}s_{2,2} = a_2 + b_2t_2,$$

$$h_3 = w_5^{-1}w_6^{-1}s_{3,2} = a_3 + b_3t_1^{-1}t_2^{-1}$$

and $\Omega_{X^\vee}$ is a generator in $H^0(X^\vee, \Omega^3_{X^\vee}(-K_{X^\vee}))$. It is straightforward to prove that the period integrals are governed by the GKZ $A$-hypergeometric equations with

$$A = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}, \quad \beta = \begin{bmatrix}
-1/2 \\
-1/2 \\
-1/2 \\
0 \\
0 \\
0
\end{bmatrix}.$$  

2.2. Topological mirror duality. Let $Y$ and $Y^\vee$ be the general fiber in $\mathcal{Y} \to V$ and $\mathcal{Y}^\vee \to W$. Note that by construction, $Y$ and $Y^\vee$ have trivial canonical bundles with at worst quotient singularities.

**Theorem 2.1.** The Hodge numbers $h^{p,q}(Y)$ and $h^{p,q}(Y^\vee)$ are well-defined and they are equal to $h^{p,q}(X)$ and $h^{p,q}(X^\vee)$ respectively for $p + q \neq n$

**Proof.** We defer a proof in Appendix [A] (cf. Proposition [A.4]) where we also provided some generalities about cyclic covers. \hfill $\square$

First of all, under our hypothesis on $X$ and $X^\vee$, we have

**Theorem 2.2.** $\chi_{\text{top}}(Y) = (-1)^n\chi_{\text{top}}(Y^\vee)$.  

Theorem 2.3. When \( s_{i,1} \) and \( s_{i,2} \) respectively. Note that \( E_{i,1} = E_i \) and \( \cup_{i=1}^{r} E_{i,1} \) equals to the union of all toric divisors on \( X \). Under our gauge fixing, the Euler characteristic of the branched locus \( D \) for \( Y \to X \) is
\[
\chi(D) = \chi(\bigcup_{i=1}^{r} E_{i,1}) + \chi(T \cap (E_{1,2} \cup \cdots \cup E_{r,2})) \\
= \chi(X) + \chi(T \cap (E_{1,2} \cup \cdots \cup E_{r,2})).
\]
Therefore, from (1.3), we can compute
\[
\chi(Y) = 2\chi(X) - \chi(D) \\
= \chi(X) - \chi(T \cap (E_{1,2} \cup \cdots \cup E_{r,2})).
\]
By inclusion-exclusion principle, Theorem 1.3, and Proposition 1.2
\[
-\chi(T \cap (E_{1,2} \cup \cdots \cup E_{r,2})) = (-1)^{r-1} \cdot (-1)^{n+r-1} \text{vol}_{n+r}(\Lambda) \\
= (-1)^{n} \chi(Y). \tag{2.2}
\]
Hence we have
\[
\chi(Y) = \chi(X) + (-1)^{n} \chi(Y) \\
= (-1)^{n} \chi(Y) + (-1)^{n} \chi(X) \\
= (-1)^{n} \chi(Y).
\]

In the case of Calabi–Yau threefolds, having Euler characteristic and all the Hodge numbers \( h^{p,q} \) with \( p + q \neq 3 \) in hand, we can completely determine the Hodge diamond. In fact, we have

**Theorem 2.3.** When \( n = 3 \), we have \( h^{p,q}(Y) = h^{3-p,q}(Y) \) for all \( p, q \).

**Proof.** A priori we have \( h^{p,q}(Y) = h^{p,q}(X) = 0 \) for all \( p + q \neq 3 \) and \( p \neq q \) since \( X \) is a toric manifold. Note that \( \chi(X) = 2(1 + h^{1,1}(X)) \) by Serre duality and \( \chi(Y) = \chi(X) - \chi(Y) = 2(h^{1,1}(X) - h^{1,1}(Y)) \). Therefore, we have
\[
h^{2,1}(Y) = h^{1,1}(Y) - \frac{\chi(Y)}{2} = h^{1,1}(X) = h^{1,1}(Y),
\]
where the last equality follows from Proposition A.4.

Based on the numerical results, we propose that

**Conjecture.** \( Y \to V \) is mirror to \( Y \to W \).

Note that \( Y \) and \( Y \) are families of singular Calabi–Yau threefolds. The above conjecture is a generalization of the symmetry observed for singular K3 surfaces [17, Conjecture 6.3].

**Example 2.5.** We retain the notation in Example 1.3 and 2.2. Let \( Y \to V \) be the gauge fixed double cover family over \( X \) along the nef-partition \( \{\rho_1, \rho_2\} \cup \{\rho_3\} \). Equivalently, \( Y \to V \) is the family of double covers over \( X \) branched along 4 lines and 1 quadric and 3 of the lines are coordinate axises. In the present case, \( V \) is an open subset of
\( H^0(X, \mathcal{O}(2))^\vee \times H^0(X, \mathcal{O}(1))^\vee \). We denote by \([x : y : z]\) the homogeneous coordinates on \(X\). Then period integrals for \(y \rightarrow V\) is then of the form
\[
(2.3) \quad \int \frac{d\mu}{\sqrt{xyz(c_1 x + c_2 y + c_3 z)(d_1 x^2 + d_2 y^2 + d_3 z^2 + d_4 xy + d_5 xz + d_6 yz)}}
\]
with \(d\mu = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy\) and \(c_i, d_j \in \mathbb{C}\). Mimicking the argument in Example 2.4, we see that the period integrals are governed by a GKZ \(A\)-hypergeometric system given by the line bundles
\[
(2.4) \quad A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}.
\]

It is natural to consider the period integrals before the gauge fixing. Consider the family of double covers over \(X\) branched along 4 lines and 1 quadric in general positions. Such a family can be parameterized by an open subset
\[
U \subset \text{Mat}_{3 \times 4}(\mathbb{C}) \times \text{Mat}_{6 \times 1}(\mathbb{C}) = (H^0(X, \mathcal{O}(1))^\vee)^4 \times H^0(X, \mathcal{O}(2))^\vee.
\]
Precisely, the element
\[
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \times \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \\ b_{51} \\ b_{61} \end{bmatrix} \in U
\]
determines a double cover branched along the lines \(a_{1i} x + a_{2i} y + a_{3i} z\) for \(i = 1, \ldots, 4\) and a quadric \(b_{11} x^2 + b_{21} y^2 + b_{31} z^2 + b_{41} xy + b_{51} xz + b_{61} yz\). Moving around inside \(U\) yields a family of double covers over \(X\). The period integrals are of the form
\[
\omega(a, b) := \int \frac{d\mu}{\prod_{i=1}^{4}(a_{1i} x + a_{2i} y + a_{3i} z)(b_{11} x^2 + b_{21} y^2 + b_{31} z^2 + b_{41} xy + b_{51} xz + b_{61} yz)}.
\]

It is straightforward to check that \(\omega = \omega(a, b)\) satisfies the system of PDEs consisting of three sets of equations (See Appendix C for details), which can be thought as a generalized Aomoto–Gelfand systems on \(\text{Mat}_{3 \times 4}(\mathbb{C}) \times \text{Mat}_{6 \times 1}(\mathbb{C})\).

**Remark 2.6.** The system of the equations in \(\text{(C.1)}, \text{(C.2)}, \text{and (C.3)}\) can be identified with the *tautological systems* defined in [23] with a fractional exponent \(\beta\). Indeed, \(\text{(C.1)}\) is the Euler operator, \(\text{(C.2)}\) is the symmetry operator generated by the \(\text{GL}(3, \mathbb{C})\)-action on \(\mathbb{P}^2\), and \(\text{(C.3)}\) is the polynomial operators determined by the embedding
\[
(2.5) \quad \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^5
\]
given by the line bundles \(\mathcal{O}(1), \mathcal{O}(1), \mathcal{O}(1), \mathcal{O}(1), \text{and } \mathcal{O}(2)\).

Conversely, starting with the tautological system as above, we can perform a gauge fixing to reduce the system to a GKZ system given by the data in \(\text{(2.4)}\). One can then
explicitly write down the secondary fan compactification, the unique holomorphic period near the LCSL point and the mirror map.

We also remark that even in the case of (classical) Calabi–Yau complete intersections in a projective homogeneous manifold \( X \) endowed with a semi-simple Lie group \( G \)-action, it is not clear how to write down a holomorphic series solution to the corresponding tautological system at the rank one point where the existence was proven in [19, 20, 22].

**Appendix A. A proof of Theorem 2.1 and generalities on cyclic covers**

In this paragraph, we recall the construction of cyclic covers over a smooth projective variety and investigate the Calabi–Yau condition. We also recall the Hodge theory needed for our cyclic covers. Let us fix the following notation throughout this section.

- Let \( X \) be an \( n \)-dimensional smooth projective variety.
- Let \( L \) be a line bundle on \( X \) and \( \mathcal{L} \) be the sheaf of sections of \( L \). As an algebraic variety, \( L = \text{Spec} \mathcal{O}_X(\oplus_{i=0}^{\infty} \mathcal{L}^{-i}) \), where \( \mathcal{L}^{-1} \) is the dual of \( \mathcal{L} \) and \( \mathcal{L}^0 := \mathcal{O}_X \).
- Fix an integer \( r \geq 0 \). For \( s \in H^0(X, L^r) = \text{Hom}_X(\mathcal{O}_X, \mathcal{L}^r) \) a non-zero section, we denote by \( D_s \) the scheme-theoretic zero locus of \( s \).
- Let \( \Omega_X^k(\log \mathcal{D}_{\text{red}}) := \Omega_X^k(\log D_{\text{red}}) \) be the sheaf of logarithmic differential \( k \)-forms with poles along \( D \).
- For a normal projective variety \( Z \), we denote by \( \omega_Z \) the sheaf of top exterior product of the Kähler differential on \( Z \) and by \( K_Z \) the canonical divisor on \( Z \). Note that under the normality hypothesis on \( Z \), \( \omega_Z \) is isomorphic to the dualizing sheaf of \( Z \).

**A.1. Basic properties.** For \( s \in H^0(X, L^r) \), let \( Y'_s := \text{Spec} \mathcal{O}_X(\mathcal{L}'_s) \) and \( Y_s \) be the cyclic cover over \( X \) defined in §1.1.

**Proposition A.1.** \( Y'_s \) is smooth if and only if \( D_s \) is smooth.

**Proof.** Put \( Y' := Y'_1 \) for simplicity. The question is local. Let \( \pi' : Y' \to X \) be the structure morphism. Fix \( y \in Y' \) and put \( x = \pi'(y) \). We take an affine open neighborhood \( U \subset X \) of \( x \) over which \( L \) is trivial. Then

\[
Y'|_{\pi'^{-1}(U)} \simeq \mathcal{O}_U[y]/(y^r - f) \subset \mathbb{A}^1 \times X,
\]

where \( f \) is the local function representing \( s \). If \( x_1, \ldots, x_n \) be a system of local coordinates around \( x \), \( Y'|_{\pi'^{-1}(U)} \) is singular at \( q = (y, x) \in Y'|_{\pi'^{-1}(U)} \) if and only if

\[
y^r - f(x) = 0, \quad ry^{r-1} = 0, \quad \text{and} \quad \partial f / \partial x_i = 0, \quad i = 1, \ldots, n.
\]

(A.1)

For \( r \geq 2 \), this is equivalent to \( y = 0, \ f(x) = 0 \) and \( \partial f / \partial x_i = 0, \ i = 1, \ldots, n \), which means \( D_s \) is singular at \( x \in X \). \( \square \)

For simplicity we put \( Y = Y_s \) and denote by \( \pi : Y_s \to X \) the structure morphism. The morphism \( \pi \) is étale over \( X \setminus D_s \), with degree \( r \). If \( D_s \) is non-singular, the pull-back section \( \pi^*(s) \in H^0(Y, \pi^*L^r) \) defines a smooth subvariety \( (D_{\pi^*(s)})_{\text{red}} \). According to [21, Lemma 4.2.4], \( \pi^*\Omega_X^k(\log D_s) = \Omega_Y^k(\log D_{\pi^*(s)}) \). In particular, for \( k = \dim X \), we have
Proposition A.2. With the same notation, we have
\begin{equation}
\pi^*(\omega_X \otimes L^r) \simeq \omega_Y \otimes \pi^*L.
\end{equation}
Consequently, \( \omega_Y \simeq \mathcal{O}_Y \) if and only if \( \pi^*(\omega_X \otimes L^r) \otimes \pi^*L^{-1} \simeq \mathcal{O}_Y \).

We are mainly interested in the case when \( Y'_s \) or \( Y_s \) are singular. To handle this situation, by compactifying the total space of \( L \), we regard \( Y'_s \) as a hypersurface in a certain projective space bundle over \( X \). To explain this in more detail, let us recall the construction of projective bundle spaces over \( X \).

Let \( \mathcal{E} \) be a locally free sheaf of rank \( (m+1) \) over \( X \) and \( Z := \text{Proj}_{\mathcal{E}_X}(\text{Sym}^*(\mathcal{E})) \) be the associated projective space bundle. We denote by \( \eta : Z \to X \) the structure morphism. We have the relative Euler sequence
\begin{equation}
0 \to \Omega_{Z/X} \to \eta^*\mathcal{E}(-1) \to \mathcal{O}_Z \to 0.
\end{equation}
Here \( \eta^*\mathcal{E}(-1) = \eta^*\mathcal{E} \otimes \mathcal{O}_{Z/X}(1)^Y \) and \( \mathcal{O}_{Z/X}(1) \) is the relative ample sheaf. Taking exterior products yields \( \omega_{Z/X} \simeq \eta^*(\wedge^{m+1}\mathcal{E})(-m-1) \).

Given \( X, L, s, r \) as above, let \( Y' = Y'_s \) and \( Y = Y_s \) as before. We consider the rank two bundle \( \mathcal{E} := \mathcal{O}_X \oplus L^{-1} \) and the associated projective space bundle \( \eta : Z \to X \). Note that \( \wedge^2\mathcal{E} \simeq L^{-1} \) and therefore \( \omega_Z \simeq \omega_{Z/X} \otimes \eta^*\omega_X \simeq \eta^*L^{-1} \otimes \eta^*\omega_X \otimes \mathcal{O}_{Z/X}(-2) \).

\( Y' \) can be regarded as a hypersurface in \( Z \). Since \( \mathcal{O}_{Z/X}(-2) \) is trivial over \( Y' \), the dualizing sheaf \( \omega_{Y'} \) is trivial if and only if \( L^{r} \simeq L \otimes \omega_X^{-1} \).

Remark A.1. \( Y' \) may not be an anti-canonical hypersurface in \( Z \). (Indeed, it is never the case unless \( r = 2 \)).

From the viewpoint of hyperplane sections, we obtain

Proposition A.3. \( Y' \) is Cohen–Macaulay. Furthermore, if \( \text{codim}_X \text{Sing}(D_s) \geq 2 \), then \( Y' \) is normal and \( Y = Y' \). In this case \( \omega_Y \simeq \mathcal{O}_Y \) if and only if \( \omega_X \otimes L^{r-1} \simeq \mathcal{O}_X \).

Proof. Since \( Z \) is smooth, it is Cohen–Macaulay. The first statement is clear. Now suppose \( \text{codim}_X \text{Sing}(D_s) \geq 2 \). Then \( Y' \) is regular in codimension one and hence, by Serre’s criterion, \( Y' \) is normal and \( Y = Y' \). It then follows that the canonical sheaf of \( Y \) (the top exterior power of the Kähler differential) is isomorphic to the dualizing sheaf \( \omega_Y \), and the later one is locally free by adjunction formula. In particular, the canonical sheaf of \( Y \) is Cartier. We compute
\begin{equation}
\omega_Y \simeq \omega_Z \otimes \mathcal{O}_Z(Y)\big|_Y
\simeq \mathcal{O}_Z(r - 2) \otimes \eta^*\omega_X \otimes \eta^*L^{r-1} \big|_Y
\simeq \eta^*(\omega_X \otimes L^{r-1})\big|_Y,
\end{equation}
where \( \pi = \eta|_Y \). If \( \omega_Y \simeq \mathcal{O}_Y \), then, by projection formula,
\( (\omega_X \otimes L^{r-1}) \otimes \pi_*\mathcal{O}_Y \simeq \pi_*\mathcal{O}_Y \).
Since the isomorphism respects the $\mathbb{Z}/r\mathbb{Z}$-action, from the eigenspace decompositions, it follows $\omega_X \otimes \mathcal{L}^{r-1} \simeq \mathcal{O}_X$. □

**Remark A.2.** This interpretation allows us to reduce the general smooth cyclic covers to the case of classical hypersurfaces. We will discuss it in Appendix B.

A.2. **Hodge numbers.** We review some basic facts about the Hodge theory for orbifolds proved in [3][29] and [2] §1, which are applicable to the case of cyclic covers over a smooth manifold.

Let $s \in H^0(X, L^r)$ with $D := D_s$ being a simple normal crossing divisor. In particular, $\text{codim}_X \text{Sing}(D) \geq 2$. Let $\pi : Y \rightarrow X$ be the cyclic cover. $Y$ is smooth outside $\pi^{-1}(\text{Sing}(D))$. Denote by $Y^{\text{reg}}$ the non-singular part of $Y$ and $j : Y^{\text{reg}} \rightarrow Y$. In [29], Steenbrink defined $\tilde{\Omega}^k_Y := j_*\Omega^k_{Y^{\text{reg}}}$ and proved that

(a) There is a canonical, purely weight $k$ Hodge structure on $H^k(Y, \mathbb{Q})$. (cf. [29 Corollary 1.5]).

(b) There is a spectral sequence for hypercohomology groups

$$H^q(Y, \tilde{\Omega}^p_Y) \Rightarrow H^{p+q}(Y, \tilde{\Omega}^*_{Y}) = H^{p+q}(Y, \mathbb{C}).$$

(cf. [29 Theorem 1.12]).

(c) The hard Lefschetz theorem holds for $Y$. (cf. [29 Theorem 1.13]).

In addition, as observed by Arapura in [2], we have

(d) There is an isomorphism

$$H^k(Y - E, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(Y, \tilde{\Omega}^p_Y(\log E)),$$

where $\tilde{\Omega}^p_Y(\log E) := j_*\Omega^p_{Y^{\text{reg}}}(\log E \cap Y^{\text{reg}})$ and $E := \pi^{-1}(D)$.

Since $\pi$ is finite,

$$H^q(Y, \tilde{\Omega}^p_Y) \simeq H^q(X, \pi_*\tilde{\Omega}^p_Y).$$

Since $Y$ is normal, we have $(\Omega^p_Y)^{\vee \vee} \simeq j_*\Omega^p_{Y^{\text{reg}}} = \tilde{\Omega}^p_Y$. Most statements in [11 Lemma 3.16] can be extended to our case.

**Proposition A.4** (See also [2 Lemma 1.5]). Let $L$ be an ample line bundle and $s \in H^0(X, L^r)$. Assume that $D := D_s$ is a simple normal crossing divisor. Then we have

$$\pi_*\tilde{\Omega}^p_Y(\log E) \simeq \bigoplus_{i=0}^{r-1} \Omega^p_X(\log D^{(i)}) \otimes \mathcal{L}^{-i}$$

and

$$\pi_*\Omega^p_Y \simeq \bigoplus_{i=0}^{r-1} \Omega^p_X(\log D^{(i)}) \otimes \mathcal{L}^{-i},$$

where $D^{(i)} = D$ for $i \neq 0$ and $D^{(0)} = 0$. Furthermore, if $p + q \neq n$, then we have

$$H^q(\Omega^p_X(\log D) \otimes \mathcal{L}^{-i}) = 0$$

for all $i \neq 0$. Consequently,

$$h^{p,q}(X, \mathbb{C}) = h^{p,q}(Y, \mathbb{C}), \text{ for } p + q \neq n.$$

**Proof.** Look at the fibred diagram

$$\begin{array}{ccc}
Y^{\text{reg}} & \xrightarrow{j} & Y \\
\downarrow & & \downarrow \pi \\
X \setminus D_{\text{sing}} & \longrightarrow & X.
\end{array}$$
The corresponding pushforward formulae hold for $Y^{\text{reg}} \to X \setminus D_{\text{sing}}$ by Lemma 3.16(a)(d). Pushing forward the equality via $j$, we obtain (i) and (ii) since both sides of them are reflexive sheaves.

For the second part, we observe that, for $E := \pi^{-1}(D)$,
\[ H^q(\Omega^p_X(\log D) \otimes \mathcal{L}^{-i}) \subset H^{p+q}(Y - E, \mathbb{C}) = 0, \text{ as } p + q > n, \]
by the affine vanishing theorem [10 Corollary 1.5]. The statement for $p + q < n$ follows from the hard Lefschetz on $Y$ and a duality argument. \qed

**Remark A.3.** For non-ample $L$, these statements still hold if $D$ is a simple normal crossing divisor such that $X \setminus D$ is affine.

**APPENDIX B. A MIRROR CONSTRUCTION OF SMOOTH CYCLIC COVERS**

In this paragraph, we explain how to construct the (topological) mirror Calabi–Yau family when $Y = Y'$ is a smooth cyclic cover $X$ (for notation, see §A.1).

Let us outline the procedure. As we noted in §A.1, we can compactify the total space of $L \to X$ by $Z := \text{Proj}_{\mathcal{O}_X}(\operatorname{Sym}^* \mathcal{E})$ and $Y$ can be realized as a hypersurface in $Z$. Note that $Z$ is a smooth semi-Fano toric variety. However, $Y$ may not be an anti-canonical hypersurface in $Z$. To remedy this defect, we contract the infinite divisor, which is disjoint from $Y$ since $Y$ is a cyclic cover, and obtain a morphism $\phi : Z \to Z'$. Now one can easily prove that $Z'$ is semi-Fano and $\phi(Y')$ is an anti-canonical hypersurface in $Z'$. The Batyrev’s duality construction applies. We keep the notation in §1.2.

**B.1. Projective bundle spaces and its toric contraction.** Let $X$ be a smooth semi-Fano toric variety and $Y \to X$ be a smooth cyclic $r$-fold cover over $X$. Let $L$ be a big and nef divisor on $X$. We fix a torus invariant divisor $D = \sum_{j=1}^p a_j D_j$ with $a_j \geq 0$ such that $L \simeq \mathcal{O}_X(D)$. Put $Z = \text{Proj}_X(\operatorname{Sym}^* \mathcal{E})$, where $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{L}^r$. This is a toric variety and we now describe its toric data.

Let $e = (0,1) \in \tilde{N} := N \times \mathbb{Z}$. Consider
\[ S_1 := \{ \tilde{\rho}_j = \rho_j - a_j e : j = 1, \ldots, p \}, \text{ and } S_2 := \{ e, -e \}. \]

Any maximal cone $\tau \in \Sigma(n) := \Sigma_X(n)$ determines two maximal cones in $\tilde{N}$:
\[ \tau_0 = \text{Cone}(\{ \tilde{\rho}_j : \rho_j \in \tau(1) \} \cup \{ e \}), \text{ and } \]
\[ \tau_\infty = \text{Cone}(\{ \tilde{\rho}_j : \rho_j \in \tau(1) \} \cup \{ -e \}). \]

**Definition B.1.** Let $\Sigma_Z$ be the collection of $\tau_0$ and $\tau_\infty$ as well as all their faces for all $\tau \in \Sigma(n)$. The following proposition is straightforward.

**Proposition B.1.** $\Sigma_Z$ is a fan and defines the toric variety $Z$. Furthermore, from the construction, the infinite divisor is given by the 1-cone $\mathbb{R}_{\geq 0} \cdot (-e)$.

Next, we construct another toric variety $Z'$. We remove the 1-cone $\mathbb{R}_{\geq 0} \cdot (-e)$ from $\Sigma_Z$ and glue all the maximal cones $\tau_\infty$ together. Let $\Sigma_{Z'}$ be the resulting fan and $Z'$ be the toric variety defined by $\Sigma_{Z'}$. The map $\Sigma_Z \to \Sigma_{Z'}$ determines a divisorial contraction.
\( \phi : Z \to Z' \), which only contracts the infinite divisor to a point. \( Z' \) has at most one singular point, which corresponds to the glued cone \( \cup_{\tau \in \Sigma(n)} \tau_\infty \).

The canonical bundle \( \omega_Z \cong \mathcal{O}(2) \otimes L' \otimes \omega_X \), where \( \eta : Z \to X \) is the structure morphism and \( \mathcal{O}(1) \) is the relative ample sheaf. It is easy to check that \( O_{Z/X}(1) \cong \mathcal{O}(D - e) \).

\[ (B.2) \quad D_e \sim \sum_{j=1}^p a_j D_{\rho_j} + D_{-e} \text{ on } Z. \]

**Proposition B.2.** The divisor \( H := D_{-e} + \sum_{j=1}^p a_j D_{\rho_j} \) is base point free.

*Proof.* We only have to show that the divisor \( H \) is numerically effective, which implies base point free in toric cases [7, Theorem 6.3.12]. We can prove this by using the notion of primitive collections and corresponding curves on toric varieties. We leave the details to the reader. \( \square \)

**Remark B.2.** The morphism \( \phi : Z \to Z' \) does not affect \( Y \). Hence \( Y \cong \phi(Y) \) is also a Calabi–Yau hypersurface in \( Z' \).

Under a stronger hypothesis on \( D \), we can show that \( Z' \) is Fano.

**Proposition B.3.** If \( D = \sum_{j=1}^p a_j D_{\rho_j} \) is ample, then the morphism \( \phi_H \) determined by \( |H| \) defined in Proposition (B.2) is exactly the contraction map \( \phi : Z \to Z' \).

*Proof.* It is straightforward to check that the polytope \( \Delta_H \) has a positive volume. Since \( H \) is already nef, it follows that \( H \) is big. By the argument in [27, Proposition 1.2], the ampleness of \( D \) implies that the fan of the image toric variety is \( \Sigma_{Z'} \). \( \square \)

**B.2. The Calabi–Yau condition.** Now we impose the Calabi–Yau condition for the cyclic cover \( Y \). Because of Proposition A.3, this automatically implies that \( L \) is big and nef. The first consequence is that \( Z' \) can not be too singular.

**Lemma B.4.** \( K_{Z'} \) is Cartier.

*Proof.* The Calabi–Yau condition is equivalent to

\[ (B.3) \quad \sum_{j=1}^p (r - 1)a_j D_j \sim \sum_{j=1}^p D_j, \]

which, in toric language, amounts to that there exists an \( m \in M \) such that

\[ (B.4) \quad \langle m, \rho_j \rangle = (r - 1)a_j - 1, \; \forall j = 1, \ldots, p. \]

Namely,

\[ (B.5) \quad \langle -m, \rho_j \rangle + (r - 1)a_j = 1, \; \forall j = 1, \ldots, p. \]

This is equivalent to saying that \( -(m - (r - 1)) \in \hat{M} \) defines \( K_{Z'} \) on the glued cone \( \cup_{\tau \in \Sigma(n)} \tau_\infty \) (defined in §B.1). Since all the other cones in \( \Sigma_{Z'} \) are regular, this implies that \( K_{Z'} \) is Cartier. \( \square \)

**Proposition B.5.** \(-K_{Z'} \) is big and nef. Consequently, \( Z' \) is semi-Fano.
Proof. From the previous lemma, we know that
\[(B.6) \quad K_Z = \phi^* K_{Z'} + (r - 2)D_e.\]
Hence \(\phi^* K_{Z'} = K_Z - (r - 2)D_e = -rH,\) where \(H\) is defined in Proposition \[B.2.\] It follows that \(\phi^*(-K_{Z'})\) is nef, which implies \(-K_{Z'}\) is nef since \(\phi: Z \rightarrow Z'\) is surjective. Note that from the projection formula we also have \(\mathcal{O}_{Z'}(-K_{Z'}) \simeq \mathcal{O}_Z(rH)\) and
\[(B.7) \quad H^0(Z', \mathcal{O}_{Z'}(-K_{Z'})) = H^0(Z, \mathcal{O}_Z(rH)).\]

The fact that \(\phi\) is birational implies that \(-K_{Z'}\) is big.

\[\square\]

Corollary B.6. The image hypersurface \(\phi(Y)\) is in the anti-canonical class.

Let \(X\) be a semi-Fano proper toric variety and \(L\) be a line bundle such that \(L^{r-1} \otimes \omega_X \simeq \mathcal{O}_X\). Let \(\varphi_{-K_X}\) be the support function of \(-K_X\). Then \(\varphi_{-K_X}\) is convex. On each \(\tau \in \Sigma(n)\) there exists an \(m_\tau \in M\) so that
\[(B.8) \quad \varphi_{-K_X} |_\tau(v) = \langle m_\tau, v \rangle, \quad v \in |\tau|.
\]
We glue together those maximal cones \(\tau\) having the same \(m_\tau\). The resulting cone is still strongly convex since \(-K_X\) is big and nef. The set of these strongly convex cones and all its faces form a new complete fan \(\Sigma_X'.\) Let \(X'\) be the toric variety associated to \(\Sigma_X'.\) We have (cf. \[27\] Proposition 1.2])

Proposition B.7. There exists a birational map \(\psi: X \rightarrow X'\) such that \(\Sigma_X\) is a subdivision of \(\Sigma_X',\) the divisor class \(\psi([-K_X])\) is ample and \(\psi^*(-K_X) = [-K_X].\) Moreover, \(\Sigma_X'\) is the normal fan of \(\Delta_{-K_X}\).

Here \(\psi([-K_X])\) is the cycle-theoretic pushforward and \(\psi^*\) is the usual pullback of line bundles. One notices that, by its very definition, \(\psi([-K_X])\) is isomorphic to the sum of all the torus invariant Weil divisors, which is \(-K_X'.\) Therefore, \(\psi^* \omega_X' \simeq \omega_X'\) and \(X'\) is Gorenstein and Fano. In particular, if \(X\) is smooth, we see that \(\psi: X \rightarrow X'\) is a crepant resolution.

Proposition B.8. Let \(X\) be a complete, smooth, and semi-Fano toric variety. Then \(\psi: X \rightarrow X'\) is a MPCP resolution.

Proof. For simplicity, let \(\Delta := \Delta_{-K_X}, \Delta'\) be its dual polytope. We only have to prove that \(\Sigma_X(1) = \Delta' \cap N \setminus \{0\}.\) From
\[(B.9) \quad K_X = \psi^* K_{X'} + \sum_{\rho \in \Sigma(1) \setminus \Sigma'(1)} (\varphi_{K_X'}(\rho) - 1)D_{\rho},\]

\(\psi\) is crepant implies \(\varphi_{K_X'}(\rho) = 1\) for all \(\rho \in \Sigma(1) \setminus \Sigma'(1)\) and that \(\Sigma_X(1) \subset \Delta' \cap N \setminus \{0\}.\) For the opposite direction, one observes that \(X\) is already smooth. Therefore we must have \(\Sigma(1) \supset \Delta' \cap N \setminus \{0\}.\)

\[\square\]

Back to our situation, \(Z := \text{Proj}_{\mathcal{O}_X}(\text{Sym}^* (\mathcal{E})),\) where \(\mathcal{E} = \mathcal{O}_X \oplus \mathcal{L}'\), is a smooth toric variety and let \(Z'\) be the variety obtained by contracting the infinite divisor as before. By Proposition \[B.5\] \(Z'\) is semi-Fano. By Proposition \[B.7\] there exists a contraction \(Z' \rightarrow Z''\) such that \(Z''\) is Fano.

Put \(\Delta := \Delta_{-K_{Z'}}(= \Delta_{-K_{Z''}})\) and let \(\Delta'\) be its dual polytope.
Proposition B.9. For $r \geq 3$, we have $\Sigma_{Z'}(1) = \Delta^N \cap N \setminus \{0\}$. Consequently, any simplicialization of $\Sigma_{Z'}$ gives an MPCP resolution of $Z''$.

Proof. The equation (B.9) holds for the contraction $Z' \to Z''$. We have $\bar{\Sigma}'(1) \subset \Delta^N \cap N \setminus \{0\}$ as before.

Note that $Z'$ has only one singular point. It then suffices to show that the facet $F$ spanned by $\{\bar{\rho}_j : j = 1, \ldots, p\}$ does not contain any relative interior integral points. The support function of that facet is

\begin{equation}
\langle (m, (r - 1)), \bar{\rho}_j \rangle = -1.
\end{equation}

In particular, it intersects with the $(n + 1)^{th}$-axis at $(0, -1/(r - 1)) \in \bar{M}$. If $r \geq 3$, then $F$ contains no relative interior integral points. Otherwise, some maximal cone $\tau_\infty$ in $\Sigma_{Z'}$ contains an integral point in the convex hull of $\tau(1) \cup \{0\}$ and therefore $Z$ is not smooth. This gives a contradiction. \qed

The case $r = 2$ is much simpler. The Calabi–Yau condition implies that $Z$ is a maximal projective crepant partial resolution of $Z''$. Indeed, $Z'$ is not an MPCP resolution due to the presence of the integral point $(0, -1)$ on the facet $F$. However, we can resolve the singularity by simply adding the corresponding 1-cone. This is just a reverse construction of our map $Z \to Z'$.

B.3. The mirror construction. Now we begin with a tuple $(X, L, s, r)$ satisfying the Calabi–Yau condition for some $r \geq 2$ and $Y$ is the $r$-fold cyclic cover as before. We have constructed $Z$, $Z'$ and $\phi : Z \to Z'$. $Z'$ is a Gorenstein semi-Fano toric variety and $Y \simeq \phi(Y)$ is an anti-canonical hypersurface in $Z'$. Let $Z' \to Z''$ be the toric morphism constructed in Proposition B.7 and $\bar{\Sigma}_{Z'}$ be any simplicialization of $\Sigma_{Z'}$. Now $Z''$ is Fano and $Y$ can be regarded as (family) of hypersurfaces in a MPCP resolution of $Z''$ or $Z$ depending on $r \geq 3$ or $r = 2$). Manipulating the Batyrev’s toric mirror construction to $Z''$ yields the desired mirror family.

Appendix C. Picard–Fuchs equations for double covers

In this paragraph, we list the equations in the PDE systems which govern the period integrals (2.3).

\begin{equation}
\sum_{i=1}^6 b_{1i} \frac{\partial}{\partial b_{1i}} \omega = -\frac{1}{2} \omega, \quad \sum_{i=1}^3 a_{ij} \frac{\partial}{\partial a_{ij}} \omega = -\frac{1}{2} \omega, \quad j = 1, \ldots, 4.
\end{equation}
\[
\left( \sum_{k=1}^{4} a_{1k} \frac{\partial}{\partial a_{2k}} + 2b_{11} \frac{\partial}{\partial b_{41}} + b_{41} \frac{\partial}{\partial b_{21}} + b_{51} \frac{\partial}{\partial b_{61}} \right) \omega = 0,
\]

\[
\left( \sum_{k=1}^{4} a_{1k} \frac{\partial}{\partial a_{3k}} + 2b_{11} \frac{\partial}{\partial b_{51}} + b_{41} \frac{\partial}{\partial b_{21}} + b_{51} \frac{\partial}{\partial b_{31}} \right) \omega = 0,
\]

\[
\left( \sum_{k=1}^{4} a_{2k} \frac{\partial}{\partial a_{3k}} + 2b_{21} \frac{\partial}{\partial b_{61}} + b_{41} \frac{\partial}{\partial b_{11}} + b_{61} \frac{\partial}{\partial b_{31}} \right) \omega = 0,
\]

\[
\left( \sum_{k=1}^{4} a_{2k} \frac{\partial}{\partial a_{1k}} + 2b_{21} \frac{\partial}{\partial b_{51}} + b_{41} \frac{\partial}{\partial b_{11}} + b_{61} \frac{\partial}{\partial b_{41}} \right) \omega = 0,
\]

\[
\left( \sum_{k=1}^{4} a_{3k} \frac{\partial}{\partial a_{1k}} + 2b_{31} \frac{\partial}{\partial b_{41}} + b_{51} \frac{\partial}{\partial b_{11}} + b_{61} \frac{\partial}{\partial b_{41}} \right) \omega = 0,
\]

\[
\left( \sum_{k=1}^{4} a_{3k} \frac{\partial}{\partial a_{2k}} + 2b_{31} \frac{\partial}{\partial b_{61}} + b_{51} \frac{\partial}{\partial b_{11}} + b_{61} \frac{\partial}{\partial b_{21}} \right) \omega = 0,
\]

\[
\left( \sum_{k=1}^{4} a_{1k} \frac{\partial}{\partial a_{1k}} + 2b_{11} \frac{\partial}{\partial b_{11}} + b_{41} \frac{\partial}{\partial b_{41}} + b_{51} \frac{\partial}{\partial b_{51}} + 1 \right) \omega = 0,
\]

\[
\left( \sum_{k=1}^{4} a_{2k} \frac{\partial}{\partial a_{2k}} + 2b_{21} \frac{\partial}{\partial b_{21}} + b_{41} \frac{\partial}{\partial b_{41}} + b_{61} \frac{\partial}{\partial b_{61}} + 1 \right) \omega = 0,
\]

\[
\left( \sum_{k=1}^{4} a_{3k} \frac{\partial}{\partial a_{3k}} + 2b_{31} \frac{\partial}{\partial b_{31}} + b_{51} \frac{\partial}{\partial b_{51}} + b_{61} \frac{\partial}{\partial b_{61}} + 1 \right) \omega = 0.
\]
\[
\begin{align*}
\left( \frac{\partial^2}{\partial a_{ij} \partial a_{kl}} - \frac{\partial^2}{\partial a_{il} \partial a_{kj}} \right) \omega &= 0, \quad 1 \leq i, k \leq 3, \quad 1 \leq j, l \leq 4. \\
\left( \frac{\partial^2}{\partial b_{11} \partial b_{21}} - \frac{\partial^2}{\partial b_{41} \partial b_{61}} \right) \omega &= 0, \\
\left( \frac{\partial^2}{\partial b_{21} \partial b_{31}} - \frac{\partial^2}{\partial b_{61} \partial b_{51}} \right) \omega &= 0, \\
\left( \frac{\partial^2}{\partial a_{11} \partial b_{21}} - \frac{\partial^2}{\partial a_{21} \partial b_{41}} \right) \omega &= 0, \\
\left( \frac{\partial^2}{\partial a_{11} \partial b_{41}} - \frac{\partial^2}{\partial a_{21} \partial b_{11}} \right) \omega &= 0, \\
\left( \frac{\partial^2}{\partial a_{11} \partial b_{51}} - \frac{\partial^2}{\partial a_{21} \partial b_{61}} \right) \omega &= 0, \\
\left( \frac{\partial^2}{\partial a_{11} \partial b_{31}} - \frac{\partial^2}{\partial a_{21} \partial b_{61}} \right) \omega &= 0, \\
\left( \frac{\partial^2}{\partial a_{11} \partial b_{61}} - \frac{\partial^2}{\partial a_{21} \partial b_{51}} \right) \omega &= 0.
\end{align*}
\]

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