The conformable fractional grey system model

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Abstract

The fractional order grey models (FGM) have appealed considerable interest of research in recent years due to its higher effectiveness and flexibility than the conventional grey models and other prediction models. However, the definitions of the fractional order accumulation (FOA) and difference (FOD) is computationally complex, which leads to difficulties for the theoretical analysis and applications. In this paper, the new definition of the FOA are proposed based on the definitions of Conformable Fractional Derivative, which is called the Conformable Fractional Accumulation (CFA), along with its inverse operation, the Conformable Fractional Difference (CFD). Then the new Conformable Fractional Grey Model (CFGM) based on CFA and CFD is introduced with detailed modelling procedures. The feasibility and simplicity and the CFGM are shown in the numerical example. And the at last the comprehensive real-world case studies of natural gas production forecasting in 11 countries are presented, and results show that the CFGM is much more effective than the existing FGM model in the 165 subcases.

Key words: Grey System Model; Conformable Fractional Calculus; Conformable Fractional Accumulation; Fractional Grey Model; CFGM Model; Natural Gas Production

1 Introduction

The grey models play a key role in the Grey System Theory pioneered by Deng in 1982. The main idea of grey forecasting was initially proposed by Deng in 1983, and the first grey model was developed in 1984 \([1]\). The commonly used statistical prediction models, such as the empirical models \([2]\), semi-parametric models \([3]\), artificial neural networks (ANN) \([4]\), and other hybrid models \([5, 6]\), often need large amount of sample data to make convinced predictions. But the idea of grey models can often built on very simple samples. In the early report by Deng \([7]\), it was shown that the basic GM(1,1) model can be built with only 4 points to make acceptable predictions. And recent theoretical and empirical studies have also shown that the grey prediction models can be quite efficient in time series forecasting with very few data \([8]\), and can even be more efficient than the machine learning models \([9]\). With high effectiveness in time series forecasting with small samples, the grey models have recently been used in a wide variety of application fields, such as manufacturing industries \([10]\), energy marketing \([11]\), energy economics \([12]\), policies \([13]\), etc.

Significant efforts of researchers have been made to improve the grey model family. In our opinion, the main goal of these researches is aiming at improving the prediction accuracy of the grey models. The first kind of researches to improve the grey models is to optimize the parameters of the model. One of the most important method is the reconstruction of the back ground values proposed by Tan \([14]\), which was mainly used for univariate grey models. Zeng et.al \([15]\) have developed the method.
of dynamic background value to improve the multivariate grey model, which can be regarded as an extension of the method by Tan. On the other hand, some researchers have noticed that the parameters of most grey models are obtained using the least squares method, which limited the flexibility of the grey models. Pei et al. employed the nonlinear least squares for the parameter estimation of the nonlinear Bernoulli models, which has been presented to be more efficient than the existing models. And it has been pointed out by Ma and Liu that the inconsistency of the solution and the discrete form of the grey model is the essential reason which lead large errors to the GMC(1, n) model, and the optimized parameter transforms have been proposed. Then the GMCO(1, n) model with optimal parameters has been proved to unbiased to arbitrary linear dynamical grey systems [16]. The second kind of methods mainly focus on the modelling mechanism of the grey models. The discrete grey modelling technique (DGMT) is one of the most important new mechanisms, which was initially introduced by Xie et al. in 2009 [17]. The DGMT has been applied to build several effective grey models, such as the NDGM model [18]. And recently it has also been extended to build the multivariate grey models, such as the DGM(1, N) [19], RRGDM(1, n) [20], TDVGM(1, N) [21], etc. Thirdly, some other methods, such as the intelligent optimizers [22], kernel machine learning [23], data grouping [24], mega-trend-diffusion [25], have also been introduced to build the grey models.

However, almost all the main stream grey models are essentially linear models. Actually, early in 2002, Deng [26] has pointed out in his Chinese book five that nonlinearity widely existed in the real-world applications, so the linear grey models are not sufficient for more general cases, such as the processes of the underground fluid flow [27, 28], the management of the unconventional petroleum reservoirs [29], heating effect of the buildings in plateaus [30], etc. In Deng’s main idea, it was implied that the nonlinear grey models should be developed on nonlinear differential equations [26], thus he firstly introduced the Bernoulli equation, but leaving no discussions on the computational details. Chen et al. have initially presented the detailed computational steps of the nonlinear grey Bernoulli model [31], and soon some improved works have been illustrated by other researchers [32, 33, 34]. Wang et al. generalized such idea and developed several multivariate grey models by using the power terms [35, 36].

In addition to the above method, the fractional order accumulation (FOA) is also one of the most important innovations for the grey models. The FOA was firstly introduced by Wu et al. in 2013 to build a fractional grey model (FGM) [37]. Being different to the use of nonlinear structures, the FOA was used as an nonlinear data preprocessing method to describe the nonlinear series for the grey models. With high performance of improving the grey models and innovative methodology, the FOA and FGM soon appealed considerable interest of research in nearly 5 years, and have been widely used in the real-world applications, such as the weapon system cost [38], gas emission [39], etc. According to Wu’s results, the FOA can perform as an error reducer to the grey models [40], and the FGM is also effective in time series forecasting with small samples. It was noticed that the basic structure of the FGM models would not be changed when introducing the FOA, thus it can also be used for building other commonly used grey models. For instance, Wu et al. have rebuilt the nonhomogenous discrete grey model using the FOA [41]. The FOA has also been proved to be efficient to revise the relational analysis, thus the fractional order grey relational analysis has also been proosed [42]. With the enlightening of the FOA, some researchers started to broaden the usage of the fractional calculus. One of the typical works is the grey model with fractional derivatives by Yang and Xue [43], which introduced the fractional order calculus to the continuous grey models framework, and it has also been applied to develop a novel interval model [44]. Essentially, the FOA can be regarded as a preprocessing method for the grey models, and it is also a general form of the well-known 1-AGO in the traditional Grey System Theory. With the success of the FOA, some researchers also mentioned that the variation of the preprocessing methods can be effective to improve the grey models, thus some new operators have also been proposed in recent years, such as the weakening buffer operator (WBO) [45, 46], opposite-direction accumulation (ODA) [45] and also the inverse fractional order accumulation (IFOA) [47]. Above all, the introduction of the FOA has made significant contributions to the development of Grey System Theory, thus it has also been listed as one of the most achievements in the new millennium [48].

However, the definition introduced by Wu is just one case of the fractional order calculus and differencing. In the point of view of computational complexity, the commonly used definitions of the
fractional accumulation and its corresponding fractional differencing in the existing grey models are not easy to implement, and it leads to high difficulties to the deeper theoretical analysis. As mentioned above, the fractional order calculus has also been introduced to the grey models by Yang and Xue [43], but the analytic solution of such fractional models contains infinite series, this is obviously not easy to use and analyze. Moreover, such difficulties would also hinder the development of the fractional order grey models and even the Grey System Theory.

In our investigations, a new definition of the fractional order derivative has been proposed by Khalil et al. in 2014 [49], which is called the Conformable Fractional Derivative. The definition of the CFD is much simpler than the “old” definitions of the fractional order derivatives (FOD), such as the Riemann-Liouville definition and Caputo definition. It was proved by Khalil et al. the CFD has very good properties, and often can solve many problems which are difficult or impossible when using the “old” definitions. With such significant improvements, the CFD has soon appealed significant interest by the researchers in recent 4 years, and many valuable new findings have been presented. Hammad and Khalil [50] proposed the Fractional Fourier Series (FFS) based on the CFD, and it was shown that the FFS is quite efficient to solve the partial differential equations. With the new formulation of CFD, many important differential equations have been redefined [51, 52, 53], and it is also important to see that the comprehensive analysis can be easily carried out with the CFD in these works.

According to the reviews above, we are motivated to put forward the idea of use of the CFD by Khalil et al. to fix the issues of computational complexity for the existing fractional grey models. Thus, we firstly introduce the new definitions of the fractional order difference and accumulation based on the CFD by Khalil, and name them as the Conformable Fractional Difference and Accumulation, and then use them to build the novel Conformable Fractional Grey Model (CFGM). Comprehensive numerical example and real-world case studies would also presented in order to compare the properties of the CFGM and the existing FGM.

The rest of this paper is organized as follows: the Section 2 gives a brief overview of the definition of the Conformable Fractional Derivative by Khalil et al., and presents the definitions of the Conformable Fractional Difference and Accumulation; Section 3 presented the modelling procedures of the new CFGM in a very brief way; a numerical example with detailed steps of the CFGM is presented in Section 4; the cases studies of predicting the natural gas production in the 11 countries are shown in Section 5, along with comprehensive discussions on the properties of CFGM comparing to the FGM, and the conclusions and perspectives are shown in Section 6.

2 The conformable fractional accumulation

In this section, we firstly give a brief overview of the conformable fractional derivative and some important properties. Then the conformable fractional difference can be define in a very natural way, and the conformable fractional accumulation is just the inverse operator of it.

2.1 Definition of the conformable fractional derivative

The conformable fractional derivative defined by Khalil, et al can be represented as follows:

Definition 1. (see [49]) Given a differentiable function $f : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of $f$ with $\alpha$ order is defined as

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1]$.

Within the definition given above, we can easily obtain the following property of the the conformable derivative as

Theorem 1. (see [49]) If the function $f$ is differentiable, then we have

$$T_\alpha(f)(t) = t^{1-\alpha} \frac{df(t)}{dt}.$$
for all \( t > 0, \alpha \in (0, 1] \).

**Proof.** Let \( h = \varepsilon t^{1-\alpha} \), then \( \varepsilon = ht^{\alpha-1} \). Therefore,

\[
T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} = \frac{t^{1-\alpha}}{h^{\alpha-1}} \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = t^{1-\alpha} \frac{df(t)}{dt}.
\]

The Theorem 1 is very important as it describes the relationship between the conformable fractional derivative and the common derivative. And in the next subsection it will be shown in the next subsection that this property would be quite useful for us to define the conformable fractional difference and accumulation.

Without loss of generality, we can also define the higher order derivative and the similar relationship to the commonly defined derivatives. Firstly we notice that the Definition 1 is actually in the following form

\[
T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{[\alpha]-\alpha}) - f(t)}{\varepsilon} = t^{[\alpha]-\alpha} \frac{df(t)}{dt},
\]

where \([\cdot]\) is the ceil function, i.e. the \([\alpha]\) is the smallest integer not smaller than \( \alpha \). Thus we can still use the similar formulation to define the higher order conformable derivative.

**Remark 1.** (see \([49]\)) The \( \alpha \) order (coin\((n, n+1]) \) conformable derivative is defined as follows

\[
T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{[\alpha]-\alpha}) - f(t)}{\varepsilon} = t^{[\alpha]-\alpha} \frac{df(t)}{dt}.
\]

for all \( t > 0, \alpha \in (n, n+1], n \in N^+ \), and \( f \) is \((n+1)\) differentiable. And also, set \( h = \varepsilon t^{1-\alpha} \), then \( \varepsilon = ht^{\alpha-1} \), we have

\[
T_\alpha(f)(t) = \frac{t^{[\alpha]-\alpha}}{h^{\alpha-1}} \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = t^{[\alpha]-\alpha} \frac{df(t)}{dt}.
\]

Then when \( \alpha = n + 1 \), there is \([\alpha] - \alpha = 0 \), thus we have

\[
T_{n+1}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{(n+1)-\alpha}) - f(t)}{\varepsilon} = \frac{df^{n+1}(t)}{dt^{n+1}}.
\]

### 2.2 Definition of the conformable fractional accumulation and difference

It is well known that the first order difference can be easily defined by using the approximation of the first order derivative as \( \Delta f(k) \approx \frac{f(k+1) - f(k)}{h} = \lim_{h \to 0} \frac{f(k+h) - f(k)}{h} = f(k+1) - f(k) \). Considering the Theorem 1 it is very natural to give the definition of the conformable fractional difference as follows:

**Definition 2.** The conformable fractional difference (CFD) of \( f \) with \( \alpha \) order is defined as

\[
\Delta^\alpha f(k) = k^{1-\alpha} \Delta f(k) = k^{1-\alpha} [f(k) - f(k - 1)].
\]

for all \( k \in N^+, \alpha \in (0, 1] \).

Within this definition, we are going to define the conformable fractional accumulation. At first, let’s recall the definition of the first order accumulation as

\[
\nabla f(k) = \sum_{j=1}^{k} f(j).
\]

It is known that the first order accumulation is the inverse operator of the first order difference, because

\[
\Delta \nabla f(k) = \Delta \left( \sum_{j=1}^{k} f(j) \right) = \sum_{j=1}^{k} f(j) - \sum_{j=1}^{k-1} f(j) = f(k).
\]
Without loss of generality, we denote the conformable fractional accumulation as $\nabla^\alpha$, and it should also satisfy the following relationship:

$$\Delta^\alpha \nabla^\alpha f(k) = f(k). \quad (10)$$

Considering the Definition 2 of CFD, we can rewrite the Eq. (10) as

$$\Delta^\alpha \nabla^\alpha f(k) = k^{1-\alpha} \Delta (\nabla^\alpha f(k)) = f(k). \quad (11)$$

Dividing Eq. (11) by $k^{1-\alpha}$, we have

$$\Delta (\nabla^\alpha f(k)) = f(k) k^{1-\alpha}. \quad (12)$$

Taking the first order accumulation of Eq. (12), we can obtain the definition of conformable fractional accumulation as

**Definition 3.** The conformable fractional accumulation (CFA) of $f$ with $\alpha$ order is defined as

$$\nabla^\alpha f(k) = \nabla \left( \frac{f(k)}{k^{1-\alpha}} \right) = \sum_{j=1}^{k} \frac{f(j)}{j^{1-\alpha}}. \quad (13)$$

for all $k \in \mathbb{N}^+$, $\alpha \in (0, 1]$.

Comparing to the higher order conformable derivative defined in Remark 1, we can easily define the higher order CFD.

**Definition 4.** The $\alpha$ order ($\alpha \in (n, n+1]$) CFD is defined as

$$\Delta^\alpha f(k) = k^{[\alpha]-\alpha} \Delta^n f(k) \quad (14)$$

for all $n \in \mathbb{N}$.

Obviously, when $\alpha = 1$ it yields the $n+1$ order difference $\Delta^{n+1}$. And the Definition 4 is a uniform definition of the CFD as it holds for all nonnegative $\alpha$ including $\alpha = 0$.

Notice that the higher order CFA is still the inverse operator of the higher order CFD. i.e.

$$\Delta^\alpha \nabla^\alpha f(k) = f(k)$$

for $\alpha \in (n, n+1]$.

Recalling the Definition 4, we have

$$k^{[\alpha]-\alpha} \Delta^\alpha \nabla^\alpha f(k) = f(k). \quad (15)$$

Similarly, we can also obtain the definition of the $\alpha$ order CFA by dividing (15) by $k^{[\alpha]-\alpha}$ and using the relationship $\nabla^n \Delta^\alpha f(k) = f(k)$.

**Definition 5.** The $\alpha$ order ($\alpha \in (n, n+1]$) CFA is defined as

$$\nabla^\alpha f(k) = \nabla^n \left( \frac{f(k)}{k^{[\alpha]-\alpha}} \right). \quad (16)$$

When $\alpha = n + 1$ the CFA yields the $(n+1)$ order accumulation $\nabla^{n+1}$. And also the Definition 5 is a uniform definition for the CFA as it holds for all nonnegative $\alpha$ including $\alpha = 0$. 
2.3 Comparison to the existing fractional order accumulation and difference

The fractional order accumulation (FOA) introduced by Wu [37], which is often used in the existing fractional grey models is usually defined as

$$\nabla_\alpha^W x(k) = \sum_{j=1}^{k} \left( \frac{k-j+\alpha-1}{k-j} \right) x(j).$$  \hspace{1cm} (17)

And its inverse operation, the fractional order difference (FOD), is defined as

$$\Delta_\alpha^W x(k) = \sum_{j=1}^{k} \left( \frac{k-j-\alpha-1}{k-j} \right) x(j).$$  \hspace{1cm} (18)

The coefficients in Eqs. (17) and (18) can be uniformly defined as

$$\left( \frac{k-j+\alpha-1}{k-j} \right) = \frac{(k-j+\alpha-1)(k-j+\alpha-1)\cdots(\alpha+1)\alpha}{(k-j)!}.$$  \hspace{1cm} (19)

The fractional order $\alpha$ can be arbitrary fractional numbers (actually it can be any real numbers with such formulations).

It can be clear to see that the definitions of the FOA and FOD above are more complex. And it would be shown in the numerical example that the implementations of the CFA and CFD are quite simple.

3 The Conformable Fractional Grey Model

Within the definitions of the CFA and CFD we can rebuild the classical grey system model GM(1, 1), and these procedures are presented in this section.

3.1 Formulation of the conformable fractional grey model

With the original series $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n))$, we firstly denote the $\alpha$ order CFA as

$$X^{(\alpha)} = (x^{(\alpha)}(1), x^{(\alpha)}(2), ..., x^{(\alpha)}(n)),$$  \hspace{1cm} (20)

where

$$x^{(\alpha)}(k) = \nabla_\alpha^{\alpha} x^{(0)}(k) = \sum_{j=1}^{k} \frac{x^{(0)}(j)}{k^{\alpha-\alpha}}, \hspace{0.5cm} (\alpha > 0).$$  \hspace{1cm} (21)

Then the Conformable Fractional Grey Model is represented as

$$\frac{d x^{(\alpha)}(t)}{dt} + ax^{(\alpha)}(t) = b.$$  \hspace{1cm} (22)

In the rest of this paper, we abbreviate it as CFGM for convenience. And the differential equation (22) is called its whitening equation. When $\alpha = 1$, it yields the conventional GM(1, 1) in Liu’s book [54]. And if the fractional order accumulated series (20) is computed by the existing FOA in (17), the CFGM model can be translated to the existing FGM model by Wu [37].

The discrete form of the whitening equation can be obtained using the trapezoid formula as

$$\left( x^{(\alpha)}(k) - x^{(\alpha)}(k-1) \right) + \frac{\alpha}{2} \left( x^{(\alpha)}(k) + x^{(\alpha)}(k-1) \right) = b.$$  \hspace{1cm} (23)

Similar procedures can be found in recent researches.
3.2 Parameters estimation

Within the discrete form (23) we can easily obtain the parameter estimation of the CFGM model using the least squares method with given samples and $\alpha$ as

$$[\hat{a}, \hat{b}]^T = (B^T B)^{-1} B^T Y,$$  \hspace{1cm} (24)

where

$$B = \begin{bmatrix}
-\frac{1}{4} \left( x^{(\alpha)}(2) + x^{(\alpha)}(1) \right) & 1 \\
-\frac{1}{4} \left( x^{(\alpha)}(3) + x^{(\alpha)}(2) \right) & 1 \\
\vdots & \vdots \\
-\frac{1}{4} \left( x^{(\alpha)}(n) + x^{(\alpha)}(n-1) \right) & 1
\end{bmatrix}, \quad Y = \begin{bmatrix}
x^{(\alpha)}(2) - x^{(\alpha)}(1) \\
x^{(\alpha)}(3) - x^{(\alpha)}(2) \\
\vdots \\
x^{(\alpha)}(n) - x^{(\alpha)}(n-1)
\end{bmatrix}.$$  \hspace{1cm} (25)

3.3 Response function and restored values

Notice that the initial point $x^{(\alpha)}(1) = \nabla x^{(0)}(1) = x^{(0)}(1)$, the response function can be obtained by solving the whitening equation (22) as

$$x^{(\alpha)}(t) = (x^{(0)}(1) - \frac{\hat{b}}{\hat{a}}) e^{-\hat{a}(t-1)} + \frac{\hat{b}}{\hat{a}}.$$  \hspace{1cm} (26)

Thus the predicted values of the CFA series can be computed using the estimated parameters and the discrete form of the response function as

$$\hat{x}^{(\alpha)}(k) = (x^{(0)}(1) - \frac{\hat{b}}{\hat{a}}) e^{-\hat{a}(k-1)} + \frac{\hat{b}}{\hat{a}}.$$  \hspace{1cm} (27)

Then restored values can be obtained using the CFD as

$$\hat{x}^{(0)}(k) = \Delta^\alpha \hat{x}^{(\alpha)}(k) = k^{\alpha-\alpha} \left( \hat{x}^{(\alpha)}(k) - \hat{x}^{(\alpha)}(k-1) \right).$$  \hspace{1cm} (28)

3.4 Computation steps

The computation steps of CFGM model with given sample and $\alpha$ can be summarized as follows:

**Step 1:** Compute the $\alpha$ order CFA series $\left( x^{(\alpha)}(1), x^{(\alpha)}(2), ..., x^{(\alpha)}(n) \right)$ of the given sample $\left( x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n) \right);$  

**Step 2:** Compute the parameters $\hat{a}$ and $\hat{b}$ using Eq.(24);  

**Step 3:** Compute the predicted values of CFA series $\hat{x}^{(\alpha)}(k)$ and the restored values using the response function (27) and the CFD (28) by $k$ from 1 to $n + p$, respectively. Where $n$ is the number of samples and $p$ is the number of prediction steps.

3.5 Brute force method for selecting the optimal $\alpha$

It should be noticed that the above procedures are given with the assumption that the value $\alpha$ is given. Thus we can build a very simple optimization problem for the optimal $\alpha$ as

$$\min_{\alpha} \text{MAPE} = \sum_{j=1}^{n} \left| \frac{x^{(0)}(j) - x^{(0)}(j)}{x^{(0)}(j)} \right| \times 100\%$$
In this section we present a numerical example to show the computational steps of the CFGM model. The optimization problem is essentially a nonlinear programming with nonlinear objective function and nonlinear constraints. There are a lot of nonlinear optimizers used for the grey system models in recent publications. However, in this paper we do not use the cutting-edge optimizer, and on the other hand we only use the Brute Force (BF) method. The BF method is a basic method for solving the optimization problems, it can solve almost all the problems but with quite low efficiency (with large number of iterations and low accuracy). Thus if the BF is available to select an appropriate \(\alpha\) for the CFGM model, it is sufficient to prove its simplicity and applicability.

In this paper we enumerate all the values in the interval \([0, 2]\) with step 0.01, then use the computational steps presented in subsection 3.4 and select the \(\alpha\) corresponds to the minimum MAPE as the optimal value.

4  Numerical example

In this section we present a numerical example to show the computational steps of the CFGM model with the raw data \(X(0) = (55.70, 59.01, 62.10, 62.27, 60.81, 58.39, 55.43, 52.18, 48.80, 45.41)\). In the following contexts, we use the first 5 points to build the CFGM model, leaving last 5 points for testing. The decimals listed in this section are often truncated in order to make them easy to display. And all the numbers are set to be “Real number” in Matlab R2015a in all computational steps in this paper.

4.1  Computing the CFA of the original series

Table 1: Computational details for the CFA with \(\alpha = 1.1\)

| \(k\) | \(x(0)(k)\) | \(k^{[\alpha]−\alpha}\) | \(x(\alpha−1)(k)\) | \(\sum_{j=1}^{k} x(0)(j)\) | \(\sum_{j=1}^{k} x(\alpha−1)(j)\) |
|-----|-------------|----------------|-----------------|---------------------|---------------------|
| 1   | 55.7        | 1.00           | 55.70           | 55.70               | 55.70               |
| 2   | 59          | 1.87           | 31.62           | 87.32               | 143.02              |
| 3   | 62.7        | 2.69           | 23.33           | 110.64              | 253.66              |
| 4   | 61.3        | 3.48           | 17.60           | 128.25              | 381.91              |
| 5   | 61.4        | 4.26           | 14.42           | 142.67              | 524.58              |

Computation of the CFA of the original series is the first step to build the CFGM model. For convenience, we firstly set \(\alpha = 1.1\) to illustrate the computational steps of CFA, and the detailed steps are listed in Table 1.

In the first row of Table 1 we present the formulations in the computational steps, and the corresponding values are listed in the following rows.

We firstly list the values of \(x(0)(k)\) in the second column, and the values of \(k^{[\alpha]−\alpha}\) in the third column, respectively.
Recalling the uniform Definition 5 of the higher order CFA, we need firstly to compute the values of \( x^{(0)}(k) \), which are listed in the fourth column in Table 1.

Noticing that \( \alpha = 1.1 \in (1, 2] \), then \([\alpha] - \alpha = 0.1\), the computational formulation for the CFA should be

\[
x^{(1.1)}(k) = \nabla^2 \left( \frac{x^{(0)}(k)}{k^{0.1}} \right).
\]

This implies that we need to computing the second order accumulation of \( \frac{x^{(0)}(k)}{k^{0.1}} \). Noticing that the \( k^{0.1} \) is actually

\[
x^{(0.1)}(k) = \nabla \left( \frac{x^{(0)}(k)}{k^{0.1}} \right)
\]

, then we can write \( x^{(\alpha)}(k) = \sum_{j=1}^{k} x^{(\alpha-1)}(j) \), thus we list the values of \( x^{(\alpha-1)}(k) \) before the values of \( x^{(\alpha)}(k) \) in the fifth and sixth column, respectively.

The above descriptions indicate that the CFA can be easily implemented. Actually, the Table 1 is implemented in the software Microsoft Excel 2010.

Using the similar computational steps, we can easily obtain the values of CFA with

\[
\alpha = 0.1, 0.2, \ldots, 2
\]

. The subfigures in Fig. 1 presents the plots of CFA with \( \alpha = 0.1, 0.2, \ldots, 1 \) and \( \alpha = 1.1, 1.2, \ldots, 2 \). It can be seen that the CFA series approaches to the 1-AGO series when \( \alpha \) approaches to 1, and it approaches to 2-AGO series when \( \alpha \) approaches to 2. It can also be seen that for each point the CFA value \( x^{(\alpha)}(k) \) becomes larger with larger \( \alpha \), and the growing speed also increases with larger \( \alpha \).

![Figure 1: The CFA series with different \( \alpha \).](image)

### 4.2 Modelling the CFGM

In this step we choose the order as \( \alpha = 0.59 \), thus the CFA series can be obtained using the similar steps presented in Subsection 4.1 as \( X^{(0.59)} = \left(55.70, 100.10, 140.07, 174.79, 206.53\right) \).

The matrices \( B \) and \( Y \) in (24) can be constructed as

\[
B = \begin{bmatrix}
-77.9024 & 1 \\
-120.0858 & 1 \\
-157.4283 & 1 \\
-190.6591 & 1 \\
\end{bmatrix}, \quad
Y = \begin{bmatrix}
44.4048 \\
39.9621 \\
34.7229 \\
31.7388 \\
\end{bmatrix}
\]
Then we obtain the parameters using the least squares solution (24) as

\[
[a, b]^T = (B^T B)^{-1} B^T Y = [0.1152, 53.4382]^T.
\]  

By substituting the parameters \(a, b\) into the response function (27) we have

\[
x^{(\alpha)}(k) = -408.17e^{-0.1152(k-1)} + 463.87.
\]  

Then the restored can be obtained using (32) by \(k\) from 1 to 10 as

\[
X^{(0.59)}(0.59) = (55.7, 100.11, 139.69, 174.96, 206.39, 234.4, 259.37, 281.61, 301.43, 319.1).
\]  

Then the restored values can be obtained using the CFD in (28) as

\[
X^{(0)} = (55.7, 59.01, 62.10, 62.27, 60.81, 58.39, 55.43, 52.18, 48.80, 45.41).
\]  

The predicted values of \(X^{(0.59)}\) and \(X^{(0)}\) are plotted in Fig. 2.

![Figure 2: Predicted values by CFGM model with \(\alpha = 0.59\). (a) CFA series; (b) The restored values.](image)

### 4.3 Selecting the optimal \(\alpha\)

The Brute Force strategy used in this paper is quite simple, which is actually repeating the computational steps in the above subsections, the fitting MAPEs with \(\alpha\) in the interval \([0, 2]\) with step 0.01 are plotted in Fig. 3. It can be seen that the values of MAPE often jump when the \(\alpha\) is near an integer, thus the subfigures in Fig. 3 are also presented in order to provide a clearer picture. And in this example, the optimal \(\alpha\) is obtained at \(\alpha = 0.59\), which is the value we used in the above subsection. And it is shown that the optimal \(\alpha\) is quite easy to obtain, which indicates that Brute Force strategy is available.

### 5 Applications and analysis

In this section, we carry out the case study of predicting the annual natural gas production of 11 countries to show the performance in real world applications comparing to the existing fractional grey model FGM [37].
Figure 3: MAPEs of CFGM model with $\alpha$ in $[0, 2]$ by step 0.01. (a) The overall picture of MAPEs with $\alpha$ in $[0, 2]$; (b) The picture of MAPEs with $\alpha$ in $[0, 1]$; (c) The picture of MAPEs with $\alpha$ in $[1, 0.1, 2]$.

Table 2: Annual natural gas (NG) production ($10^9$ m³) of the 11 countries from 2008 to 2016

| No. | YEAR | 2008  | 2009  | 2010  | 2011  | 2012  | 2013  | 2014  | 2015  | 2016  |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1   | UAE   | 50.2  | 48.8  | 51.3  | 52.3  | 54.3  | 54.6  | 54.2  | 60.2  | 61.9  |
| 2   | Brazil| 14.3  | 11.9  | 14.6  | 16.7  | 19.3  | 21.3  | 22.7  | 23.1  | 23.5  |
| 3   | Bolivia| 10.8  | 8.4   | 8.2   | 6.6   | 5.7   | 4.8   | 4.6   | 4.5   | 4.5   |
| 4   | Denmark| 66.5  | 62.7  | 70.5  | 64.1  | 63.8  | 68.6  | 57.9  | 43.3  | 40.2  |
| 5   | Nigeria| 77.1  | 89.3  | 131.2 | 145.3 | 157   | 177.6 | 174.1 | 178.5 | 181.2 |
| 6   | Qatar  | 36.2  | 26    | 37.3  | 40.6  | 43.3  | 36.2  | 45    | 50.1  | 44.9  |
| 7   | Turkmenistan| 66.1 | 64.4 | 42.4 | 59.5 | 62.3 | 62.3 | 67.1 | 69.6 | 66.8 |
| 8   | Brunei | 12.2  | 11.4  | 12.3  | 12.8  | 12.6  | 12.2  | 11.9  | 11.6  | 11.2  |
| 9   | Italy  | 8.4   | 7.3   | 7.6   | 7.7   | 7.8   | 7    | 6.5   | 6.2   | 5.3   |
| 10  | India  | 30.5  | 37.6  | 49.3  | 44.5  | 38.9  | 32.1  | 30.5  | 29.3  | 27.6  |

5.1 Background and data collection

Clean energy is one of most important resources, which will be vital to the global economics and natural environment in the future. Natural gas (NG) is one kind of clean energy with low price and high efficiency. And now many countries are using NG as one of the most important fuels.

However, most recent researches only focus on the future trend of the largest gas producers such as
the USA, China, Russia, etc [55]. In this paper, we selected 11 countries as mid-sized gas producers. The annual production data are collected from 2008 to 2016, which are available from the BP Statistical Review of World Energy 1.

![Initial point](image)

Data not used  Data for modelling  Data for testing

Figure 4: The time series cross validation

### 5.2 Overall performance in comparison to the existing fractional grey model

Table 3: Modelling accuracy evaluation criteria used for TSCV

| AE  | MAE  | STD |
|-----|------|-----|
| $\varepsilon_{i,j}^A = |\varepsilon_{i,j}|$ | $\overline{\varepsilon}_{i,j}^A = \frac{1}{N} \sum_{i,j} \varepsilon_{i,j}^A$ | $\sqrt{\frac{1}{N} \sum_{i,j} \left( \varepsilon_{i,j}^A - \overline{\varepsilon}_{i,j}^A \right)^2}$ |

| SE  | MSE  | STD |
|-----|------|-----|
| $\varepsilon_{i,j}^S = \varepsilon_{i,j}$ | $\overline{\varepsilon}_{i,j}^S = \frac{1}{N} \sum_{i,j} \varepsilon_{i,j}^S$ | $\sqrt{\frac{1}{N} \sum_{i,j} \left( \varepsilon_{i,j}^S - \overline{\varepsilon}_{i,j}^S \right)^2}$ |

| APE | MAPE | STD |
|-----|------|-----|
| $\varepsilon_{i,j}^P = \left| \frac{\varepsilon_{i,j}}{\varepsilon_{i,j}^{(0)}} \right| \times 100$ | $\overline{\varepsilon}_{i,j}^P = \frac{1}{N} \sum_{i,j} \varepsilon_{i,j}^P$ | $\sqrt{\frac{1}{N} \sum_{i,j} \left( \varepsilon_{i,j}^P - \overline{\varepsilon}_{i,j}^P \right)^2}$ |

In order to provide a comprehensive comparison of CFGM to the existing FGM model, we use the time series cross validation (TSCV) in this section, which has been quite efficient to evaluate the overall performance and robustness of the times series models in [56].

The main idea of TSCV is illustrated in Fig. 4. For each case listed in Table 2 the prediction models would be built on different subcases shown in 4 with different initial points and different numbers of sample data. The Brute Force method is also used to select the optimal $\alpha$ for the CFGM and FGM in this subsection. The $\alpha$ for CFGM is searched in the [0, 2] by step 0.01, and that for FGM is searched in a wider range $[-2, 2]$ by step 0.01.

1Available at the website of British Petroleum Company [https://www.bp.com/en/global/corporate/energy-economics/statistical-review-of-world-energy.html](https://www.bp.com/en/global/corporate/energy-economics/statistical-review-of-world-energy.html)
According to the TSCV shown in Fig. 4, the CFGM and FGM model would be built on 15 subcases in each case\(^2\), and this means we need to build these models for \(15 \times 11 = 165\) times in total.

The modelling accuracy criteria we used in this paper are the mean squares error (MSE), mean absolute error (MAE) and mean absolute percentage error (MAPE) with standard deviation (STD) for all subcases.

Firstly we define the error for each point as

\[
\varepsilon_{i,j} = \hat{x}_{i}^{(0)}(j) - x_{i}^{(0)}(j),
\]

where \(i\) represents the number of each subcase (also the number of original series used for initial point), and \(j\) represents the number of points in each subcase. The \(x_{i}^{(0)}(j)\) represents the \(j^{th}\) point when \(i^{th}\) point is the initial point, and \(\hat{x}_{i}^{(0)}(j)\) represents the corresponding restored value.

Thus the MSE, MAE and MAPE with the corresponding STD are defined in Table 3.

The overall evaluation criteria for fitting accuracy are listed in Table 4. The MSE, MAE and MAPE of CFGM are smaller than FGM model in 8, 7 and 7 cases, respectively, which indicates that the overall fitting accuracy of CFGM model is better than FGM. Meanwhile, it can be seen that most STDs of CFGM model are also smaller than FGM model, which indicates that the stability of CFGM model is also better than FGM model. On the other hand, it should also be noticed that the fitting accuracy of CFGM is quite close to the FGM model although it has better accuracy.

| Case | MSE  | MAE  | MAPE(%) |
|------|------|------|---------|
| 1    | 0.6762 ± 1.5977 0.4706 ± 1.2755 | 0.4503 ± 1.5696 0.3531 ± 1.6242 | 0.8151 ± 3.0833 0.6404 ± 3.1979 |
| 2    | 0.0313 ± 0.0724 0.1037 ± 0.2520 | 0.1028 ± 2.3606 0.2563 ± 1.9661 | 0.5340 ± 17.5909 1.4204 ± 15.5630 |
| 3    | 0.1323 ± 0.3300 0.1817 ± 0.3947 | 0.2017 ± 1.9584 0.2563 ± 1.9661 | 1.0867 ± 14.2824 1.6036 ± 17.2361 |
| 4    | 0.0485 ± 0.0811 0.0355 ± 0.0855 | 0.1552 ± 1.0819 0.1012 ± 1.1046 | 2.6744 ± 12.4777 3.2386 ± 12.7874 |
| 5    | 8.4423 ± 14.7093 9.4880 ± 18.2279 | 2.0447 ± 4.0094 2.0254 ± 4.0964 | 3.3530 ± 6.1291 3.2386 ± 6.9701 |
| 6    | 12.8673 ± 20.5661 23.5152 ± 53.7708 | 2.4822 ± 22.2533 2.9278 ± 22.1466 | 1.6616 ± 26.8327 2.2180 ± 26.8304 |
| 7    | 6.6457 ± 12.2297 7.4043 ± 12.3512 | 1.6339 ± 4.3122 1.8317 ± 4.4426 | 4.2040 ± 15.7513 4.9719 ± 16.5651 |
| 8    | 4.0174 ± 6.7740 8.7518 ± 15.9248 | 1.3110 ± 9.3218 1.8508 ± 9.1328 | 2.3761 ± 20.7557 3.5410 ± 20.5943 |
| 9    | 0.0031 ± 0.0062 0.0014 ± 0.0032 | 0.0336 ± 0.0401 0.0672 ± 0.3734 | 0.2724 ± 3.4182 0.5487 ± 3.1928 |
| 10   | 0.0128 ± 0.0262 0.0220 ± 0.0494 | 0.0703 ± 0.4215 0.0834 ± 0.4057 | 0.9519 ± 5.3180 1.1255 ± 5.1085 |
| 11   | 3.4843 ± 6.9808 4.1196 ± 12.0940 | 1.7151 ± 5.1944 0.9921 ± 5.3509 | 3.1228 ± 14.3985 2.4508 ± 14.6638 |

Total: 8:3 7:4 7:4

\(^2\)As there are 9 points in each case, the subcases should be with 4 to 8 points for modelling when the initial point is the first point in the original series, and then there are 5 subcases. Similarly, we could know that there are 4, 3, 2, 1 subcases when the 2\(^{nd}\), 3\(^{rd}\), 4\(^{th}\), 5\(^{th}\) point are used as initial point, respectively. So we have 5+4+3+2+1=15 subcases in each case.
Table 5: Prediction performance of CFGM and FGM in Cross validation

| Case  | CFGM (MSE) | FGM (MSE) | CFGM (MAE) | FGM (MAE) | CFGM (MAPE) | FGM (MAPE) |
|-------|------------|-----------|------------|-----------|-------------|-------------|
| Case1 | 30.4939 ± 39.3554 | 34.5486 ± 39.5743 | 4.0966 ± 3.3726 | 4.8863 ± 3.3146 | 7.3025 ± 5.4229 | 8.0961 ± 5.3269 |
| Case2 | 8.3676 ± 18.3188 | 22.6661 ± 42.6136 | 2.0370 ± 2.0838 | 3.3110 ± 3.4710 | 14.2821 ± 14.7859 | 16.7184 ± 14.2824 |
| Case3 | 18.8652 ± 26.8104 | 25.9883 ± 39.2119 | 3.3413 ± 2.8156 | 3.8781 ± 3.3572 | 16.7184 ± 14.2824 | 19.4072 ± 17.0292 |
| Case4 | 1.6338 ± 1.8054 | 6.2599 ± 18.7491 | 1.1026 ± 0.6560 | 1.4047 ± 2.1007 | 24.1863 ± 14.5933 | 30.7946 ± 46.4865 |
| Case5 | 196.2533 ± 280.8344 | 524.1852 ± 721.9064 | 11.1167 ± 8.6493 | 13.8827 ± 10.8847 | 24.6414 ± 21.2041 |
| Case6 | 425.8787 ± 767.7857 | 1371.6504 ± 2805.3065 | 24.7774 ± 27.9288 | 26.9264 ± 49.4552 | 42.7472 ± 34.8749 | 59.1887 ± 108.8847 |
| Case7 | 153.6544 ± 342.6568 | 4992.1914 ± 19111.8027 | 9.1558 ± 8.4781 | 32.7378 ± 66.9320 | 20.2662 ± 18.7234 | 59.8024 ± 108.8847 |
| Case8 | 2443.2359 ± 8753.3487 | 4992.1914 ± 19111.8027 | 21.6147 ± 45.1016 | 48.7507 ± 94.2844 | 32.2738 ± 66.9320 | 48.7507 ± 94.2844 |
| Case9 | 0.4373 ± 0.7388 | 0.7558 ± 1.7631 | 0.4786 ± 0.4630 | 0.5634 ± 0.6700 | 4.1491 ± 4.0524 | 4.8841 ± 5.8270 |
| Case10 | 1.0815 ± 1.7394 | 1.2261 ± 1.9004 | 0.7954 ± 0.6797 | 0.8540 ± 0.7151 | 13.5638 ± 12.3565 | 14.7381 ± 13.0126 |
| Case11 | 34.2671 ± 38.8337 | 1041.3573 ± 4094.3538 | 5.0070 ± 3.0770 | 15.7837 ± 28.5575 | 17.0717 ± 10.3767 | 54.8024 ± 102.3495 |

The overall evaluation criteria for prediction accuracy are listed in Table 5. The MSE, MAE and MAPE of CFGM are smaller than FGM model in almost all cases, only except the MAE in case 1, in which the CFGM performances quite close to FGM. Thus it is very clear that the CFGM model is quite competitive to the FGM model in prediction accuracy. Meanwhile, also almost all STDs of CFGM are smaller than FGM, except the STD of AE in case 1 and STD of APE in case 1. Thus it is also shown that the stability of CFGM model is better than FGM. On the other hand, it can be seen that the upper bounds of the criteria of CFGM are quite smaller than that of FGM. e.g. the maximum MAPE of CFGM is 32.2738%, and that of FGM is as large as 59.8024%. Thus, it is sufficient to show that the CFGM model is more effective in prediction than CFGM with more acceptable robustness.

And it should also be noticed that the α for FGM model is searched in a wider range than CFGM model, thus it is implied that it would be easier for the CFGM to select the optimal α than FGM in the applications.

### 5.3 About the parameter α

The α is one partial deterministic term to the CFGM model, and in the numerical example 4.3 it has been shown that the value of α would effect the modelling accuracy of the CFGM model. In this subsection we will analyze the ranges which contains the optimal α for most cases.

The proportion of optimal α for CFGM and FGM are shown in the subfigureg in Fig. 5. It is shown that 84% optimal α are obtained in (0, 1) for CFGM, and 15% optimal α are obtained at 0. In total, we can see that there are 99% values are obtained in [0, 1). And only a few optimal α are obtained in (1, 2], with just 1.22%.

As for the FGM model, most optimal α are obtained in (0, 1), which take 72% in all the cases. But there still exist 25% points obtained in (1, 2]. According to the results shown above, it is clear to see that the FGM model needs a wider range to select the optimal α than CFGM. And it is shown that we
almost only need to search the optimal $\alpha$ in the range of $[0, 1)$ for the FGM model in the applications. This finding indicates that it is easier to optimize the $\alpha$ for CFGM. Moreover, it is also very useful for us to design more precise algorithms or try to use other optimizers, as the available searching range has been shown in this finding.

5.4 Some typical cases and analysis

For better explanation, we choose some typical cases to compare the features of CFGM model to the FGM model. The three cases chosen in this subsection are cases 7-9, where the maximum errors of CFGM and FGM appear in case 8 and 7, respectively, and they perform closely in case 9, as shown in Table 5.

5.4.1 The non-smooth series with one inflexion point

The CFGM model has the maximum errors in case 8, with MAPE as 32.2738%. We firstly plotted the raw data of case 8, the NG production of Turkmenistan, in subfigure (a) in Fig. 6. It can be shown that the series started to change its direction at the second point. Intuitively, we call such point as the inflexion point in the rest of this paper. And in this case, the second point is its main inflexion point.

With such an inflexion point, we can see that the CFGM and FGM model all perform poorly in subfigure (b) in Fig. 6, with quite large MAPE, which implies that it is such large errors which lead to the poor overall performance in this case. In subfigure (c), one non-inflexion point added for modelling, and then it can be seen that the accuracy of these models is improved significantly. It is very interesting to see that when two non-inflexion points added for modelling, the performance of CFGM becomes much better immediately. However, the performance of FGM model is still not well at all.

With this case, we can see that the inflexion point is very important to the CFGM model, which would make it inefficient in the applications. However, the inference of the inflexion point can be weakened when more non-inflexion points added.

Moreover, it can be seen that the CFGM model is more sensitive to the newly added non-inflexion point than the FGM model, which makes it more effective in this case.
The main inflexion point

Modelling with one inflexion point

One non-inflexion point added for modelling

Two non-inflexion point added for modelling

Figure 6: The prediction results for case 8 with 1 main inflexion point. (a) The raw data of NG production of Turkmenistan. (b) 4 points for modelling with 1 inflexion points. (c) 5 points for modelling with 1 new non-inflexion point. (d) 5 points for modelling with 2 new non-inflexion points.

5.4.2 The non-smooth series with two inflexion points

In case 9, the CFGM has similar performance to the FGM, although it has better accuracy than FGM. The original series of case 9 has two inflexion points as shown in subfigure (a) in Fig. 7. Thus we can see that if we use only four points to build the models, they only reflect the overall trend of the direction from the second to fourth points, as shown in subfigure (b) in Fig. 7. However, it is clear to see that the CFGM correctly catches the overall trend when only one non-inflexion point added for modelling, while the change of FGM is not significant, as shown in subfigure (c). This case clearly indicates that the CFGM is much more sensitive to the new information than FGM model.
5.4.3 The non-smooth series with four inflexion points

The case 7 is the most special case as it has four inflexion points out of the total 9 points, as shown in subfigure (a) in Fig. 8. In another point of view, it can be seen that the direction is changed with every three or less points, thus it approaches to the oscillatory series.

With only one inflexion point for modelling, the CFGM and FGM all have poor performance, and the FGM overestimate the trend of the original series, as shown in subfigure (b) in Fig. 8. Their performances become better when a new inflexion point added, because the last four points for modelling roughly reflect the overall trend of the last series, shown in subfigure (c).

However, the situations shown in subfigures (d) and (e) indicate that the performance of CFGM become worse with newly added points, whatever the inflexion or non-inflexion point. And we can see in these subfigures that the effects of the newly added points to the CFGM is quite significant. But for the FGM model, it always follows the overall increasing trend of the original series, even when the last point added shown in the subfigure (f).

This case presents quite a clear picture of the sensitivities of the CFGM and FGM, which indicates the CFGM is much more sensitive to the new information than FGM model. However, it also indicates that such sensitivity may also lead to worse performance to the CFGM model.
Figure 8: The prediction results for case 7 with 4 inflexion points. (a) The raw data of NG production of Nigeria; (b) 4 points for modelling with 1 inflexion point; (c) The results of CFGM and FGM with 5 points for modelling, including 2 inflexion points; (d) 6 points for modelling with 3 inflexion points; (e) 7 points for modelling with 1 new non-inflexion point; (f) 8 points for modelling with 4 inflexion points.

6 Conclusions

In this paper we proposed a novel definition of conformable fractional difference and accumulation, and then built the novel conformable fractional grey model (CFGM). The computational formulations of CFA and CFD are quite simpler than the conventional FOA and FOD used in the existing fractional grey models, which also lead to the simplicity of the CFGM model, as shown in the numerical example.

The real world applications in predicting the natural gas consumptions in 11 countries are carried
out, and the CFGM model has been compared to the existing FGM model. Priority of the CFGM over the FGM model can be clearly summarized from the results. Firstly, the CFGM model performs quite better than FGM in most cases, especially the prediction accuracy of CFGM model is higher than FGM model in all cases (except only one indicator for one case). Secondly, the fractional parameter $\alpha$ needs narrower range (often in $[0,1]$) to tune, which implies that the CFGM would be easier to tune than FGM in the applications, and it has been shown in this paper that even the simple and rough Brute Force method can be efficient in the numerical example and the case studies. Thirdly, the CFGM is more sensitive to the new information, which is quite useful to avoid the interference of the inflexion points. However, the sensitivity of CFGM model may also be its disadvantage in the applications, especially when the original series is not stable.

In summary, it can be seen that the new definition of CFA and CFD is eligible to build the new grey model, and it can be more efficient than the conventional fractional grey models. With its simplicity and effectiveness, the new definition of the CFA and CFD can be expected to rebuild the family of the existing fractional order grey models, and this would made considerable contributions to the development of the Grey System Theory. On the other hand, this paper is also the initial work on the conformable fractional differencing theory, which also provides a new direction of the difference equations.

7 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

8 Acknowledgment

This research was supported by the National Natural Science Foundation of China (Grant No. 71771033), the Doctoral Research Foundation of Southwest University of Science and Technology (no. 16zx7140,15zx7141), the Open Fund (PLN201710) of State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation (Southwest Petroleum University).

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