Quark confinement and color transparency in a gauge-invariant formulation of QCD

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Abstract

We examine a nonlocal interaction that results from expressing the QCD Hamiltonian entirely in terms of gauge-invariant quark and gluon fields. The interaction couples one quark color-charge density to another, much as electric charge densities are coupled to each other by the Coulomb interaction in QED. In QCD, this nonlocal interaction also couples quark color-charge densities to gluonic color. We show how the leading part of the interaction between quark color-charge densities vanishes when the participating quarks are in a color singlet configuration, and that, for singlet configurations, the residual interaction weakens as the size of a packet of quarks shrinks. Because of this effect, color-singlet packets of quarks should experience final state interactions that increase in strength as these packets expand in size. For the case of an SU(2) model of QCD based on the ansatz that the gauge-invariant gauge field is a hedgehog configuration, we show how the infinite series that represents the nonlocal interaction between quark color-charge densities can be evaluated nonperturbatively, without expanding it term-by-term. We discuss the implications of this model for QCD with SU(3) color and a gauge-invariant gauge field determined by QCD dynamics.

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I. INTRODUCTION

In previous work, [1–3], we developed a gauge-invariant formulation of QCD in which the QCD Hamiltonian is described as a functional of gauge-invariant operator-valued quantities—spinor (quark) fields, gauge fields and their canonical momenta. One outgrowth of this work was the demonstration of an important similarity between QED and QCD. In a formulation of QED in which the charged spinor field represents a gauge-invariant quantity, a nonlocal interaction between charge densities—the Coulomb interaction—appears as part of the Hamiltonian. On the other hand, as is well-known, there is no explicit nonlocal interaction in standard versions of QED in covariant gauges in which the charged spinor, \( \psi \), represents a gauge-dependent field. However, when covariant-gauge QED is transformed to a representation in which Gauss’ law is implemented and \( \psi \) becomes a gauge-invariant charged field, the same nonlocal Coulomb interaction that is seen in the Coulomb gauge appears explicitly in that case as well, even though the covariant-gauge condition continues to apply to it. [4] These observations illustrate that, when only gauge-invariant fields are used in constructing the Hamiltonian—in a variety of gauges, including but not limited to the Coulomb gauge—the interactions between charged fields and pure-gauge parts of gauge fields vanish, and a nonlocal interaction between charge densities appears in their stead. The fact that the Coulomb interaction is, by far, the most important electromagnetic force in the low-energy regime, provides strong incentive for formulating QCD in terms of gauge-invariant fields, to explore the implications of these observations for our understanding of QCD.

The organization of this paper is as follows: In the section following this Introduction, we review and expand on relevant material from our previous work, which provides a foundation for describing QCD in terms of gauge invariant fields. This part of our discussion is exact, and requires no approximations. In a later section, we will use these results to suggest how quark confinement and color transparency could be understood as a consequence of the gauge-invariant formulation of the QCD Hamiltonian. And finally, we will invoke an ansatz about the form of the gauge-invariant gauge field to illustrate a technique for evaluating the nonlocal interaction nonperturbatively.

II. THE QCD HAMILTONIAN AS A FUNCTIONAL OF GAUGE-INARIANT FIELDS

We will review here some technical developments that we discussed previously, [1–3] and that are essential for the investigation reported in this work. One of these developments is the construction of a set of gauge-invariant quark and gluon operator-valued fields. Another is the transformation of the QCD Hamiltonian to a representation in which it is expressed in terms of these gauge-invariant fields. In our work, the QCD Hamiltonian, \( \tilde{H} \), is expressed in terms of gauge-invariant operator-valued fields in a representation in which \( \psi \) designates the gauge-invariant quark field. It is given by
\[ \tilde{H} = \int d\mathbf{r} \left[ \frac{1}{2} \Pi_i^a(r) \Pi_i^a(r) + \frac{1}{4} F_{ij}^a(r) F_{ij}^a(r) + \psi^\dagger(r) (\beta m - i\alpha_i \partial_i) \psi(r) \right] + \tilde{H}'. \] (2.1)

\( \tilde{H}' \) describes interactions involving the gauge-invariant quark field, and can be expressed as

\[ \tilde{H}' = \tilde{H}_{j-A} + \tilde{H}_G + \tilde{H}_{LR}. \] (2.2)

As shown in Ref. [3], \( \tilde{H}_G \) vanishes in the representation (the so-called \( \mathcal{N} \) representation) in which this Hamiltonian is described, and will not be given any further consideration in this work. \( \tilde{H}_{j-A} \) describes the interaction of the gauge-invariant gauge field with the transverse gauge-invariant quark color-current density, and is given by

\[ \tilde{H}_{j-A} = -g \int d\mathbf{r} \psi^\dagger(r) \alpha_i \frac{\lambda^i}{2} \psi(r) A_{GI}^h(r); \] (2.3)

\( \tilde{H}_{LR} \) is the nonlocal interaction

\[ \tilde{H}_{LR} = H_{g-Q} + H_{Q-Q} \] (2.4)

with 1

\[ H_{g-Q} = \int d\mathbf{r} \left\{ + \text{Tr} \left[ \sum_{r=0}^\infty g^{r+1} (-1)^r f_{(r)}^{\tilde{d}h} f_{(r')}^{dore} \Pi_i^a(r) \frac{\lambda^a}{2} A_{GI}^h(r) V_C^{-1}(r) \frac{\lambda^b}{2} V_C(r) \frac{1}{\partial^2} \left( \mathcal{T}^\tilde{\gamma}_{(r)}(r) j_0^h(r) \right) \right] \right\} \] (2.5)

and

\[ H_{Q-Q} = \left\{ \frac{1}{2} \sum_{r=0}^\infty \sum_{r'=0}^\infty g^{r+r'} (-1)^{r+r'} f_{(r)}^{\tilde{d}h} f_{(r')}^{dore} \frac{\lambda^b}{2} \frac{\partial}{\partial^2} \left( \mathcal{T}^\tilde{\gamma}_{(r)}(r) j_0^h(r) \right) \left( \mathcal{T}^\tilde{\gamma}_{(r')}(r) j_0^h(r) \right) \right\}; \] (2.6)

here \( A_{GI}^h(r) \) is the transverse, gauge-invariant gauge field constructed in Ref. [2] and \( j_0^h \) is the gauge-invariant quark color-charge density \( j_0^h = g \psi^\dagger \lambda^h / 2 \psi \); \( f_{(\eta)}^{\tilde{\alpha}\beta\gamma} \) is the chain of structure constants

\[ f_{(\eta)}^{\tilde{\alpha}\beta\gamma} = f^{\alpha[1]} b^{[1]} [ f^{\beta[1]} \alpha^{[2]} b^{[2]} b^{[2]} \alpha^{[3]} b^{[3]} \ldots f^{b^{[\eta-2]} \alpha^{[\eta-1]} b^{[\eta-1]} f^{b^{[\eta-1]} \alpha^{[\eta-1]} \gamma} \] (2.7)

summed over repeated Lie group indices; \( V_C^{-1}(r) \frac{\lambda^b}{2} V_C(r) \) is a quantity that transforms like an SU(N) vector (where \( N = 3 \) for QCD and \( N = 2 \) for Yang-Mills theory); and \( \left( \mathcal{T}^\tilde{\gamma}_{(r)}(r) j_0^h(r) \right) \) is given by

\[ \mathcal{T}^\tilde{\gamma}_{(r)}(r) j_0^h(r) = A_{GI}^{\delta(1)}(r) \frac{\partial j_0^h(r)}{\partial^2} \left( A_{GI}^{\delta(2)}(r) \frac{\partial j_0^h(r)}{\partial^2} \left( \ldots \left( A_{GI}^{\delta(r)}(r) \frac{\partial j_0^h(r)}{\partial^2} (j_0^h(r)) \right) \right) \right). \] (2.8)

Eq. (2.5) can be understood as the non-Abelian analog of the Coulomb interaction in QED, and to have a structure similar to that of the Coulomb interaction, when the quantity

1Eq. (2.5) corrects a typographical error in Eq. (23) in Ref. [3].
\( \mathcal{K}_d^0(r) = \sum_{r=0} g f^{dgh} (-g)^r \left( \mathcal{T}_r(r) j_0^0(r) \right) \) is substituted for the Abelian \( j_0(r) \) that represents the electric charge density in QED. The same \( \mathcal{K}_d^0(r) \) also participates in the nonlocal interaction described in Eq. (2.3), where it couples to “glue”-color, \( \mathcal{K}_g^d \) given by

\[
\mathcal{K}_g^d(r) = g f^{dare} \text{Tr} \left[ V^{-1}(r) \frac{\lambda^c}{2} V_C(r) \frac{\lambda^b}{2} \right] A_{gi}^e(r) \Pi_{bi}(r).
\]

Before proceeding with an application of our earlier work to a discussion of quark color confinement and color transparency, we need to clarify some questions about the formalism we have constructed. Our first remark addresses the fact that gauge-invariant fields are not generally unique, either in Abelian theories, such as QED, or in QCD. Since the Gauss’ law operator is the generator of infinitesimal gauge transformations, any unitary operator that commutes with the Gauss’ law operator can be applied to a gauge-invariant state, or can be used to unitarily transform gauge-invariant operators, without interfering with their gauge invariance. A number of such unitary operators that commute with the Gauss’ law operator for QED were displayed in Ref. [4], and further operators, that transform gauge-invariant operators in other Abelian gauge theories without interfering with their gauge invariance, have also been constructed. [5] Constructing such operators for QCD would be more difficult, but there is little doubt that it would be possible to do so. We therefore need to address the non-uniqueness of the gauge-invariant operators used in our formulation.

In previous work, [4,5], we have used the following criterion for accepting a gauge-invariant field as useful for a gauge-invariant formulation of a gauge theory; we propose to apply that same criterion in this case as well. We require that when the interactions between the pure gauge degrees of freedom and gauge-invariant matter fields have been eliminated in favor of nonlocal interactions involving gauge-invariant matter fields — \( H_{Q-Q} \) and \( H_{g-Q} \) in the present case — the remaining interactions are restricted to interactions of the gauge-invariant gauge field with transverse current densities only. The unavailability of a longitudinal component of the gauge-invariant gauge field in \( H_{j-A} \) precludes the formation, through the operation of current conservation, of a coupling between charge density and a longitudinal gauge field. No further interactions between matter-field charge densities can therefore be transmitted through virtual loops of gauge-invariant gauge field components, making the nonlocal interactions the dominant features in a description of low-energy dynamics.

In most gauge-invariant formulations of QED, such as those in the Coulomb gauge, the Coulomb interaction, \(-\frac{1}{2} \int dr j_0(r) \nabla^{-2} j_0(r)\), and the interaction of the transverse (gauge-invariant) part of the gauge field with the current density, constitute the interaction Hamiltonian. That formulation satisfies the criterion we have proposed. In axial \((A_3 = 0)\) gauge QED, however, the nonlocal interaction is given by \(-\frac{1}{2} \int dr j_0(r) \partial_3^{-2} j_0(r)\) and is accompanied by further interactions of the gauge field with charge as well as current densities. Further unitary transformations are then necessary to eliminate contributions from these additional interactions, through virtual photon loops, to the formation of forces between static charges. Appropriately chosen unitary transformations not only eliminate these interactions among static charges through virtual loops, but also transform \(-\frac{1}{2} \int dr j_0(r) \nabla^{-2} j_0(r)\) to the Coulomb interaction \(-\frac{1}{2} \int dr j_0(r) \nabla^{-2} j_0(r)\) and thereby restore rotational symmetry to the nonlocal interaction, and still maintain the axial gauge condition. [4] It should also be noted that,
in QED, our proposed criterion requires that gauge-invariance be imposed on matter fields through unitary transformations that do not dress charged fields with transverse propagating photons that have no role in implementing Gauss’ law. A similar requirement, that charged particles not be dressed spuriously with transverse, propagating photons, was proposed by Haagensen and Johnson as necessary for avoiding “false confinement” in Wilson loop calculation for QED.

When applied to QCD, we note that the nonlocal interactions in $\tilde{H}'$ are $H_{Q-Q}$ and $H_{g-Q}$, and that $\tilde{H}_{j-A}$ describes the interaction of the transverse, gauge-invariant gluon field $A^a_{\text{Gl}}(r)$ with quark color transverse current densities only. $H_g$ is a term in the transformed Hamiltonian in which $\mathcal{G}^a$ appears either on the extreme left or right. $\mathcal{G}^a$ represents the Gauss’s law operator in the representation in which the transformed Hamiltonian $\tilde{H}$ is expressed. $\mathcal{G}^a$ always annihilates either the “bra” or “ket” state vector of matrix elements taken between allowed states. And, since $\mathcal{G}^a$ commutes with $\tilde{H}$, state vectors that implement Gauss’s law initially, will continue to do so as they time-evolve under the influence of $\tilde{H}$. The nonlocal $H_{Q-Q}$ and $H_{g-Q}$ therefore describe all the interactions between static quarks. No interactions between quark color charge densities can be transmitted through virtual loops generated by the gauge-invariant gluon field $A^a_{\text{Gl}}(r)$ without involving $H_{Q-Q}$ or $H_{g-Q}$; and, therefore, $\tilde{H}_{j-A}$ can be expected to make only relatively unimportant contributions to low-energy processes. This fact supports our choice of gauge-invariant quark and gluon fields as appropriate for formulating a gauge-invariant description of QCD dynamics.

We will also make some clarifying remarks about the relation between two representations used in this and earlier work: [2,3] the $\mathcal{C}$ and the $\mathcal{N}$ representations. As we pointed out in earlier work, [2,3] the Gauss’s law operator $\tilde{\mathcal{G}}^a(r) = \partial_i \Pi_i^a(r) + g f^{abc} A_i^b(r) \Pi_i^c(r) + j_i^a(r)$ with $j_i^a(r) = \partial_i \psi^\dagger(r) \frac{e}{2} \psi(r)$, and the “pure glue” Gauss’s law operator $\mathcal{G}^a(r) = \partial_i \Pi_i^a(r) + g f^{abc} A_i^b(r) \Pi_i^c(r)$ are unitarily equivalent, so that $\mathcal{G}^a$ may be taken to represent $\tilde{\mathcal{G}}^a(r)$ in a different, unitarily equivalent representation. We refer to the representation in which $\tilde{\mathcal{G}}^a(r)$ is the Gauss’s law operator, the $\mathcal{C}$ representation; and the representation in which $\mathcal{G}$ represents the entire Gauss’s law operator, with the color-charge density $j_i^a(r)$ included (though implicitly only) the $\mathcal{N}$ representation. The unitary equivalence is expressed as

$$\tilde{\mathcal{G}}^a(r) = \mathcal{U}_C \mathcal{G}^a(r) \mathcal{U}_C^{-1}. \quad (2.10)$$

The Gauss’s law operator is the generator of infinitesimal gauge transformations; and the criterion for the gauge invariance of an operator $\xi$ is that it commute with the Gauss’s law operator. In this work, when we use the Gauss’s law operator to determine gauge invariance, it is important to distinguish between the Gauss’s law operators in the $\mathcal{C}$ and the $\mathcal{N}$ representations. An operator $\xi$ represents two different quantities in the two representations. The quantity represented by $\xi$ in the $\mathcal{C}$ representation has the form $\mathcal{U}_C^{-1} \xi \mathcal{U}_C$ in the $\mathcal{N}$ representation. But $\xi$, when appearing in the $\mathcal{N}$ representation, refers to a different quantity, whose form in the $\mathcal{C}$ representation would be $\mathcal{U}_C \xi \mathcal{U}_C^{-1}$. Since the quark field $\psi$ trivially commutes with $\mathcal{G}^a(r)$, $\psi$ is manifestly gauge-invariant in the $\mathcal{N}$ representation. The unitary operator $\mathcal{U}_C$ transforms the quark field $\psi$ so that

$$\psi_{\text{G}}(r) = \mathcal{U}_C \psi(r) \mathcal{U}_C^{-1} = V_C(r) \psi(r) \quad (2.11)$$
is the form that the gauge-invariant gauge field takes in the more usual \( \mathcal{C} \) representation. The unitary transformation given in Eq. \((2.11)\) has no effect on the gauge field. But the transformation it effects on the quark field is significant, because it allows us to use the quark field \( \psi(\mathbf{r}) \) to represent the gauge-invariant quark field in the \( \mathcal{N} \) representation. The unitary operator \( V_C(\mathbf{r}) \) is a functional of gauge fields and of the Gell-Mann matrices \( \lambda^h \); its structure was discussed extensively in Ref. [3]. Since, in the \( \mathcal{N} \) representation, \( \psi(\mathbf{r}) \) is the expression for that same gauge-invariant field that is described by \( \psi_{\mathcal{G}I}(\mathbf{r}) \) in the \( \mathcal{C} \) representation, \( V_C(\mathbf{r}) \) is implicitly included in the quark field \( \psi(\mathbf{r}) \) in the \( \mathcal{N} \) representation. In the \( \mathcal{N} \) representation, \( \psi(\mathbf{r}) \) therefore implicitly consists of glue as well as quark field components. When the quark field is gauge-transformed within the \( \mathcal{C} \) representation, \( \psi(\mathbf{r}) \rightarrow \exp[ig(\lambda^h/2)\chi^h(\mathbf{r})]\psi(\mathbf{r}) \), where \( \chi^h(\mathbf{r}) \) is an arbitrary time-independent c-number field in the adjoint representation of \( SU(3) \); that same gauge transformation transforms \( V_C(\mathbf{r}) \) to \( V_C(\mathbf{r})\exp[-ig(\lambda^h/2)\chi^h(\mathbf{r})] \), so that \( \psi_{\mathcal{G}I}(\mathbf{r}) \) is gauge-invariant. In the \( \mathcal{N} \) representation, the gauge invariance of \( \psi(\mathbf{r}) \) is a trivial consequence of the structure of the Gauss’s law operator.

These considerations have important consequences for the behavior of charge and current densities under gauge transformations. As a simple illustrative example, we consider the electron field operator \( \psi \) in QED in the temporal gauge. Under a gauge transformation, \( A_\mu \rightarrow A_\mu - \partial_\mu \chi \), and the corresponding change in \( \psi \) is \( \psi \rightarrow \exp(i\chi)\psi \). There are a number of ways of constructing gauge-invariant electron field operators in QED, [4] but one that is very useful for this discussion was provided by Dirac, [5] who defined the gauge-invariant electron field \( \psi_{\mathcal{G}I} = \exp[-ie(1/\partial^2)\partial_iA_i]\psi \). Under a gauge transformation, compensating changes occur in \( \psi \) and in \( (1/\partial^2)\partial_iA_i \), so that \( \psi_{\mathcal{G}I} \) remains gauge-invariant. The electric charge density operator can be represented either as \( e\psi_{\mathcal{G}I}^\dagger \psi_{\mathcal{G}I} \) or as \( e\psi^\dagger \psi \). Since the longitudinal photons (they are zero-norm ghosts) used to dress the electron field to make it gauge-invariant are electrically neutral, they do not affect the electric charge density, and the two expressions, \( e\psi_{\mathcal{G}I}^\dagger \psi_{\mathcal{G}I} \) and \( e\psi^\dagger \psi \), are identical.

To discuss the non-Abelian case, we will use the \( SU(2) \) version of QCD, because the difference between the Abelian and the non-Abelian cases can be illustrated very graphically in that system. We therefore use the \( SU(2) \) version of \( V_C(\mathbf{r}) \) in Eq. \((2.11)\), which is given by

\[
V_C(\mathbf{r})_{SU(2)} = \exp[-ig(\vec{\tau}/2)\cdot\vec{Z}(\mathbf{r})],
\]

where \( \vec{Z}(\mathbf{r}) = (1/\partial^2)\partial_i\vec{B}_i \), and where \( \vec{B}_i \) is a complicated functional of gauge fields, all of which commute with each other and with the spinor field \( \psi \). [2] We can identify the gauge-invariant color charge density in the \( \mathcal{C} \) representation of this \( SU(2) \) model of QCD as \( \vec{j}_0_{\mathcal{G}I} = g\psi_{\mathcal{G}I}^\dagger(\vec{\tau}/2)\psi_{\mathcal{G}I} \). In this non-Abelian case, \( \vec{j}_0_{\mathcal{G}I} \) is no longer identical to \( \vec{j}_0 \). When the substitution \( \psi_{\mathcal{G}I} = \exp[-ig(\vec{\tau}/2)\cdot\vec{Z}(\mathbf{r})] \psi \) is made, the gauge-invariant \( \vec{j}_0_{\mathcal{G}I} \) can be expressed in terms of \( \vec{j}_0 = g\psi^\dagger(\vec{\tau}/2)\psi \) by

\[
\vec{j}_0_{\mathcal{G}I} = (\hat{Z}\times\vec{j}_0)(\hat{Z} + \hat{Z}\times(\vec{j}_0\times\hat{Z})\cos(Z) + (\vec{j}_0\times\hat{Z})\sin(Z)),
\]

where \( \hat{Z} \) is the unit vector \( \hat{Z}(\mathbf{r}) = \vec{Z}(\mathbf{r})/Z(\mathbf{r}) \). Eq. \((2.12)\) shows that in this case, and in its \( SU(3) \) version, dressing the gauge-dependent quark field with the gluons required to make it gauge-invariant, does affect the color charge density, since the gluons themselves carry color. Under the infinitesimal gauge transformation \( \delta\vec{A}_i = \partial_i\delta\vec{\chi} + g\vec{A}_i\times\delta\vec{\chi} \), the change in \( \vec{j}_0 \) is exactly compensated by the change in \( \hat{Z} \), [2] so that \( \vec{j}_0_{\mathcal{G}I} \) remains untransformed, and maintains its orientation in the — in this instance \( SU(2) \) — color space. The fact that a
A quantity is gauge-invariant does not mean that it has no preferred orientation in color space; it does mean that, whatever orientation it has, will not be altered by a gauge transformation. The same remark applies to the gauge-invariant gauge field $A^a_{\text{GI}}(\mathbf{r})$ as well.

In view of the relation between $[\tilde{\mathbf{j}}_0^a]_{\text{GI}}$ and $\tilde{\mathbf{j}}_0$, it is particularly important to realize that $\tilde{\mathbf{j}}_0$ is the form that $[\tilde{\mathbf{j}}_0]_{\text{GI}}$ takes in the $\mathcal{N}$ representation. The color-charge density $j_0^i(\mathbf{r})$ as well as the color current density $j^a_i(\mathbf{r})$ therefore are gauge-invariant operators in the $\mathcal{N}$ representation, and implicitly include glue as well as quark ingredients to constitute gauge-invariant color-charge and color-current densities. It is important to emphasize this point. The quantity that is represented by $j_0^a(\mathbf{r})$ in the $\mathcal{C}$ representation is different from the quantity that is represented by $j_0^a(\mathbf{r})$ in the $\mathcal{N}$ representation. $j_0^a(\mathbf{r})$, when it appears in the $\mathcal{C}$ representation, consists of quark field components only, and transforms gauge-covariantly (not gauge-invariantly) under an infinitesimal gauge transformation, in the form

$$\delta [j_0^a(\mathbf{r})]_{\mathcal{C}-\text{rep}} = gf^{abc} [j_0^b(\mathbf{r})]_{\mathcal{C}-\text{rep}} \delta \chi^c .$$  \hspace{1cm} (2.13)$$

But $j_0^a(\mathbf{r})$, when it appears in the $\mathcal{N}$ representation, is gauge invariant! A gauge-invariant field tensor can also be defined as $\mathbf{F}_{\text{GI}}^a(\mathbf{r})$

$$F_{\text{GI}ij}^a(\mathbf{r}) = \partial_j A_{\text{GI}i}^a(\mathbf{r}) - \partial_i A_{\text{GI}j}^a(\mathbf{r}) - gf^{abc} A_{\text{GI}i}^b(\mathbf{r}) A_{\text{GI}j}^c(\mathbf{r})$$  \hspace{1cm} (2.14)$$

and then $F_{\text{GI}ij}^a(\mathbf{r}) \Delta_2^a = F_{ij}^a(\mathbf{r}) V_\mathcal{C}(\mathbf{r}) \Delta_2^a V_\mathcal{C}^{-1}(\mathbf{r})$ and $F_{\text{GI}ij}^a(\mathbf{r}) K_{ij}^a(\mathbf{r}) = F_{ij}^a(\mathbf{r}) F_{ij}^a(\mathbf{r})$. Similarly, $\mathbf{K}^a_i(\mathbf{r})$ and $\Pi_i^a(\mathbf{r})$ are gauge-invariant quantities, so that all the terms appearing in $\tilde{H}$ are gauge-invariant. In Ref. [2], we verified that the expressions we obtained for the gauge-invariant fields — gauge as well as quark fields — by expanding $V_\mathcal{C}$ to arbitrary orders, agree with the expressions arrived at perturbatively by Lavelle and McMullan.

The fact that, in the $\mathcal{N}$ representation, $j_0^a(\mathbf{r})$ and $j_i^a(\mathbf{r})$ represent gauge-invariant charge and current densities respectively, enables us to use them to represent physical observables. This makes it very convenient to formulate this work in the $\mathcal{N}$ representation. It would, in principle, have been possible to carry out the investigation we are presenting here in either the $\mathcal{C}$ or the $\mathcal{N}$ representation. But the use of the $\mathcal{N}$ representation is a very powerful tool for expressing the Hamiltonian in this gauge-invariant formulation. We will therefore proceed with this discussion, by using the $\mathcal{N}$ representation to examine how the nonlocal interactions $H_{Q-Q}$ and $H_{g-Q}$ could provide a mechanism for the confinement of quarks when these are not in color-singlet configurations. A number of authors have discussed the importance of implementing Gauss’s law in the quantization of QCD and Yang-Mills theory. And other authors have also speculated that Gauss’s law might be responsible for color confinement in QCD, and that, therefore, gauge-invariant degrees of freedom might be necessary to make color confinement manifest. A somewhat similar expression to our Eq. (2.6) — later expressed as Eq. (3.2) — has been given by T. D. Lee.

\[\text{In this non-relativistic notation, } V_i \text{ refers to a contravariant vector and } \partial_i \text{ refers to the covariant derivative. This convention is used extensively throughout this work.}\]
by Christ and Lee, \[24\] but not with the same gauge-invariant gauge field $A^0_{GI}$ between the inverse Laplacians that appears in our work. The gauge-invariant gauge field $A^0_{GI}$, positioned between the inverse Laplacians, is instrumental in providing for gauge invariance of the interaction Hamiltonian. In Ref. \[24\], unitary operators ($u$ and $U$) transform the spinor (quark) fields so as to bring them into compliance with Gauss’s law in a formulation in which Gauss’s law is assumed to already be implemented in the gauge sector. Our work, reported in Ref. \[24\], is based on the initial implementation of Gauss’s law for the gauge sector, so that the functionals we construct, that correspond to the angle variables in $u$ and $U$ in Ref. \[24\], are explicitly given in terms of operator-valued gauge fields. These explicit expressions, which we obtained by implementing Gauss’s law for the gauge sector, are required to establish gauge invariance of the quark and gluon fields and of the color charge and current densities. In spite of the important differences between the non-Abelian analogs to the Coulomb interaction that appear in Refs. \[23,24\] and the one given in our Eqs. (2.6), (3.2) and (3.3), the fact that nonlocal interactions of the same general structure appear in these two treatments, which are quite different in their objectives and in the technical procedures used, supports the idea that such nonlocal interactions can have an important role in QCD dynamics.

We do not give a proof, in this work, that the long-range interactions $H_{Q-Q}$ and $H_{g-Q}$ confine quarks. We do present arguments, however, that these nonlocal interactions, which arise naturally when the QCD Hamiltonian is represented in terms of gauge-invariant fields, are interesting candidates for describing low-energy QCD dynamics, and that these nonlocal forces might have an important role in the confinement of quarks and the non-confinement of color-singlet configuration of quarks.

### III. QUARK CONFINEMENT, COLOR TRANSPARENCY, AND THE NONLOCAL INTERACTION

We will here examine to what extent the structure of $\mathcal{K}_0^b(r)$ can be responsible for the confinement of quarks, or of wave packets composed of quarks. To facilitate this discussion, we will display $\mathcal{K}_0^b(r)$ in the expanded form

\[
\mathcal{K}_0^b(r) = -j_0^b(r) + (-g) \int \frac{dy}{4\pi|y-r|} A^{\delta_{Gi}}_{Gl}(r) \partial_i \int \frac{dx}{4\pi|x-r|} j_0^a(x) + \\
g^2 \int \frac{dy}{4\pi|y-r|} A^{\delta_{Gi}}_{Gl}(r) \partial_i \int \frac{dx}{4\pi|x-y|} j_0^a(x) + \cdots \\
+ (-g)^n \int \frac{dy}{4\pi|y-r|} \cdots \int \frac{dx}{4\pi|x-y|} j_0^a(x) + \cdots \\
A^{\delta_{Gl}}_{Gl}(y) = \int \frac{dy}{4\pi|y-r|} A^{\delta_{Gi}}_{Gl}(r) \partial_i \int \frac{dx}{4\pi|x-y|} j_0^a(x) + \cdots.
\]
Eq. (2.8) can be substituted into Eq. (2.6), and a partial integration carried out, so that
\( (1/2) \int d\mathbf{r} \frac{\partial}{\partial r} \mathcal{K}^b_0(\mathbf{r}) \frac{\partial}{\partial \mathbf{r}} \mathcal{K}^b_0(\mathbf{r}) \) is transformed into \( -(1/2) \int d\mathbf{r} \mathcal{K}^b_0(\mathbf{r}) (1/\partial^2) \mathcal{K}^b_0(\mathbf{r}) \).

The first question to consider is how the nonlocal interaction described in Eqs. (2.6) and (3.1) would depend on the color of the participating quark configurations. We turn our attention to Eq. (2.6) and represent it as

\[
H_{Q-Q} = -\frac{1}{2} \int d\mathbf{r} \mathcal{K}^b_0(\mathbf{r}) \left( \frac{1}{\partial^2} \right) \mathcal{K}^b_0(\mathbf{r}) = \frac{1}{2} \int d\mathbf{r} \mathcal{J}^b(\mathbf{r}) \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) \mathcal{J}^b(\mathbf{x}) .
\]

(3.2)

\( \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) \) is a Green function that is the non-Abelian analog of \( 1/(4\pi|\mathbf{r}-\mathbf{x}|) \), but which differs from its QED analog in that it does not only refer to spatial points, but also depends on the gauge-invariant gauge field \( A^{\delta}_{GI}(\mathbf{r}) \). \( \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) \) can easily be read from Eq. (3.1), and shown to be

\[
\mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) = \frac{\delta_{ab}}{4\pi|\mathbf{r}-\mathbf{x}|} + 2g f^{b_{(1)}b_{a}} \int \frac{dy}{4\pi|\mathbf{r}-\mathbf{y}|} A^{\delta}_{GL(1)}(\mathbf{y}) \partial_i \frac{1}{4\pi|\mathbf{y}-\mathbf{x}|} + \cdots
\]

\(-3g^2 f^{b_{(1)}b_{(1)}c} f^{c_{(1)}\delta_{(2)a}} \int \frac{dy_1}{4\pi|\mathbf{r}-\mathbf{y}_1|} A^{\delta}_{GI}(\mathbf{y}_1) \partial_i \int \frac{dy_2}{4\pi|\mathbf{y}_1-\mathbf{y}_2|} A^{\delta}_{GI}(\mathbf{y}_2) \partial_j \frac{1}{4\pi|\mathbf{y}_2-\mathbf{x}|} + \cdots
\]

\((-1)^{(n-1)(n+1)} g^n f^{b_{(1)}b_{(1)}c} f^{c_{(1)}\delta_{(2)}a} \cdots f^{c_{(n-1)}\delta_{(n)a}} \int \frac{dy_1}{4\pi|\mathbf{r}-\mathbf{y}_1|} A^{\delta}_{GI}(\mathbf{y}_1) \partial_i \int \frac{dy_2}{4\pi|\mathbf{y}_1-\mathbf{y}_2|} \cdots \int \frac{dy_n}{4\pi|\mathbf{y}_{n-1}-\mathbf{y}_n|} A^{\delta}_{GI}(\mathbf{y}_n) \partial_x \frac{1}{4\pi|\mathbf{y}_n-\mathbf{x}|} + \cdots.
\]

(3.3)

\( \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) \) can be seen to be symmetric under the combined interchange \( \mathbf{r} \leftrightarrow \mathbf{x} \) and \( a \leftrightarrow b \), when we assume that \( A^{\delta}_{GI}(\mathbf{y}) \to 0 \) when \( |\mathbf{y}| \to \infty \). Eqs. (3.2) and (3.3) define a nonlocal interaction between gauge-invariant quarks or packets of quarks that is an analog of the Coulomb interaction in QED. \( \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) \) is determined not only by the inverse Laplacians, but also by the matrix elements of the gauge-invariant gauge fields \( A^{\delta}_{GI}(\mathbf{y}) \) in the gluonic medium in which the quarks are immersed. Once the spatial dependence of \( \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) \) in a particular state of quark-gluon matter is fixed, it describes a force that acts on quark color charges at points \( \mathbf{x} \) and \( \mathbf{y} \) respectively. Particularly in cases in which quark-antiquark creation is supressed and in which the quarks are at rest, or nearly at rest — as in the case of heavy, static quarks — \( \mathcal{F}^{ba}(\mathbf{r}, \mathbf{x}) \) can play a very similar role in QCD to the Coulomb interaction in QED.

We will apply Eq. (3.2) to a state consisting of two packets of quarks that are immersed in glue, but are well-separated from each other in the sense that the quarks in each of the two packets can be represented using a complete set of orbitals, and that the two sets of orbitals occupied by the quarks in the two packets have a negligible overlap. We will not attempt to make any quantitative models for how such separated packets arise, but will only explore the interactions of such separated packets once they have arisen. We will furthermore assume in this discussion that quark-antiquark pair creation and annihilation can be neglected or “quenched”. We consider the expectation value

\[
\langle qQ | H_{Q-Q} | Qq \rangle = \langle q_h \cdots q_i Q_n \cdots Q_1 | H_{Q-Q} | q_i \cdots q_h Q_1 \cdots Q_n \rangle ,
\]

(3.4)
where \( q_1 \cdots q_k \) represents a set of quarks in orbitals \( u_j(r) \) and \( Q_1 \cdots Q_n \) a set of quarks in orbitals \( U_i(r) \), so that both sets of orbitals are localized, and the overlap of \( u_j(r) \) and \( U_i(r) \) is negligible. In \( |q_1 \cdots q_k Q_1 \cdots Q_n \rangle \), a set of creation operators for quarks \( q_1, \cdots q_k, Q_1, \cdots Q_n \), are applied to a state \( |g \rangle \), which represents a gluonic medium that obeys the Gauss’s law (in the \( \mathcal{N} \) representation)

\[
\mathcal{G}^a |g \rangle = 0. \tag{3.5}
\]

In Ref. [2], a procedure was developed for constructing states such as \( |g \rangle \). Since, in the \( \mathcal{N} \) representation, quark creation and annihilation operators commute with the Gauss’s law operator \( \mathcal{G}^a \), multiquark Fock states can be constructed by applying quark creation operators to \( |g \rangle \) without invalidating Gauss’s law applied to the resulting multiquark Fock state.

Because quark and glue field operators commute, and because of the negligible overlap of of the quark orbitals \( u_j(r) \) and \( U_i(r) \), \( \langle qQ | H_{Q-Q} | Qq \rangle \) simplifies, through cluster decomposition, to

\[
\langle qQ | H_{Q-Q} | Qq \rangle = \int dx \bar{r}_{\mathcal{F}^{ba}(r,x)}(q_k \cdots q_1 | j_0^b(r) | q_1 \cdots q_k \rangle \langle Q_n \cdots Q_1 | j_0^a(x) | Q_1 \cdots Q_n \rangle, \tag{3.6}
\]

where \( \mathcal{F}^{ba}(r,x) \) represents the expectation value \( \mathcal{F}^{ba}(r,x) = \langle g | \mathcal{F}^{ba}(r,x) | g \rangle \).

In evaluating \( \int dx \bar{r}_{\mathcal{F}^{ba}(r,x)} j_0^a(x) \), — one of the integrals required for Eq. (3.2) — we can expand about a point \( x_0 \) located where the orbitals \( U_i(r) \) are large in magnitude. The integral then can be expressed as the Taylor’s series, in a kind of “color multipole” expansion,

\[
\int dx \left\{ \mathcal{F}^{ba}(r,x_0) + X_i \partial_i \mathcal{F}^{ba}(r,x_0) + \frac{1}{2} X_i X_j \partial_i \partial_j \mathcal{F}^{ba}(r,x_0) + \cdots \right\} j_0^a(x) \tag{3.7}
\]

where \( X_i = (x-x_0)_i \) and \( \partial_i = \partial / \partial x_i \). When we perform the integration in Eq. (3.7), the first term contributes \( \mathcal{F}^{ba}(r,x_0) Q^a \), where \( Q^a = \int dx j_0^a(x) \) — the integrated “color charge”. Since the color charge is the generator of infinitesimal rotations in SU(3) space, it will annihilate any multiquark state vector in a singlet color configuration. Multiquark packets in a singlet color configuration therefore are immune to the leading term of the nonlocal \( H_{Q-Q} \). Color-singlet configurations of quarks are only subject to the color multipole terms, which act as color analogs to the Van der Waals interaction.

The scenario that this model suggests is that the leading term in \( H_{Q-Q} \), namely \( \mathcal{Q}^b \mathcal{F}^{ba}(r_0,x_0) Q^a \), for a quark color charge \( \mathcal{Q}^a \) at \( r_0 \) and another quark color charge \( \mathcal{Q}^b \) at \( x_0 \), as well as \( H_{g-Q} \), which describes the coupling between quark color charge and color-bearing gluonic matter, are responsible for the confinement of quarks and packets of quarks that are not in color-singlet configurations. In this scenario, the multipole terms, which are the only parts of \( H_{Q-Q} \) that affect the dynamics of color-singlet quark configurations, do not confine packets, but result in final-state interactions that act on color-singlets as these move through, or emerge from a gluonic medium. To confirm this scenario, it would be necessary to evaluate \( \mathcal{F}^{ba}(r,x) \) and its spatial derivatives. This requires knowledge of the spatial dependence of \( A^\delta_{GI}(r) \). In Refs. [2,3], relationships of \( A^\delta_{GI}(r) \) to other gauge-invariant (and gauge-covariant) quantities were established that would, in principle, allow
specification of \( A_\delta g_{i1}(r) \) (or, rather, the specification of the set of possible values of \( A_\delta g_{i1}(r) \), since the nonlinearity of the equations that determine those relationships may well signify that \( A_\delta g_{i1}(r) \) does not have a single unique value). But the explicit construction of \( A_\delta g_{i1}(r) \) is not within the scope of this current work. We will, rather, explore some features of the behavior of color-singlet quark packets as these move through a gluonic medium, assuming that \( \mathcal{F}_{ba}(r, x) \) is a reasonably well-behaved function of \( r \) and \( x \).

In order to examine the dynamics of color-singlet quark configurations as these move through a medium consisting of gauge-invariant glue and quark matter, we will explore the effect of the multipole terms on “small” color-singlet packets. Small packets occupy a limited region of space in the interior of gluonic matter, so that \( \mathcal{F}_{ba}(r, x) \) does not vary significantly over the spatial domain in which the packet functions \( u_j(r, x) \) and \( U_i(x) \) make sizable contributions. Given the assumption that \( \mathcal{F}_{ba}(r, x_0) \) varies only gradually within a volume occupied by quark packets, the effect of these color multipole forces on a packet of quarks in a color-singlet configuration becomes more significant as the packet increases in size. As small quark packets move through gluonic matter, they will experience only insignificant effects from the multipole contributions to \( H_{Q-Q} \), since, as can be seen from Eq. (3.7), the factors \( \mathcal{X}_i, \mathcal{X}_i \mathcal{X}_j, \ldots, \mathcal{X}_{i(1)} \ldots \mathcal{X}_{i(n)} \), keep the higher order multipole terms from making significant contributions to \( \int dxd\mathcal{F}_{ba}(r, x)\mathcal{J}_0(x) \) when they are integrated over small packets of quarks. As the size of the quark packets increases, the regions over which the multipoles are integrated also increases, and the effect of the multipole interactions on the color-singlet packets can become larger. This dependence on packet size of the final-state interactions experienced by color-singlet states — i.e. the increasing importance of final-state interactions as color-singlet packets grow in size — is in qualitative agreement with the characterizations of color transparency and color coherence given by Miller and by Jain, Pire and Ralston. Eqs. (3.2), (3.3) and (3.7) generalize the multipole expansion of a time-independent nonlocal interaction from the electromagnetic case, in which the leading term vanishes for a neutral object and leaves a residue of higher order multipoles, to QCD, in which color neutrality — an attribute of color-singlet states — corresponds to electrical neutrality in electrodynamics. In the language of Jain, Pire and Ralston, our analysis provides a model for how color neutrality protects a quark packet from the color-monopole force, with the result that such color-singlet packets can survive — when the region over which the multipole interaction is integrated is small enough and the effect of \( \mathcal{F}_{ba}(r, x) \) on it coherent enough — to become asymptotic states observed in studies of exclusive processes.

As previously mentioned, quark and gluon color is also coupled directly, in the form

\[
H_{g-Q} = -\int dr d\mathbf{x} \left[ K_g^{b}(\mathbf{r}) (4\pi |(\mathbf{r - x})|)^{-1} K_0^{b}(\mathbf{x}) + K_0^{b}(\mathbf{x}) (4\pi |(\mathbf{r - x})|)^{-1} K_g^{b}(\mathbf{r}) \right]. \tag{3.8}
\]

Whether our observations about the behavior of quark packets coupled to each other by \( H_{Q-Q} \) also apply to quark packets coupled to gluonic matter directly by \( H_{g-Q} \), depends on the distribution of glue — whether, when the integration in Eq. (3.3) ranges over regions in which \( \mathbf{x} \approx \mathbf{r} \), the expansion in Eq. (3.7) is valid. We have to defer more detailed discussion of this question until the dynamical equations that determine the distribution of glue in this gauge-invariant formulation of the theory have been solved.
Another point of interest about Eq. (3.6) is the fact that the spatial dependence of the long-range nonlocal interaction described by \( H_{Q-Q} \) is largely determined by \( \bar{F}^{ba}(r, x) \), which represents the expectation value \( \bar{F}^{ba}(r, x) = \langle g | F^{ba}(r, x) | g \rangle \). And \( |g\rangle \) implements the Gauss’s law given in Eq. (3.5), in which the Gauss’s law operator includes only the gluon color charge density. We observe, therefore, that the gluonic medium has a far more important role in determining the long-range behavior of the nonlocal interaction \( H_{Q-Q} \) than the quarks, which are acted on by \( H_{Q-Q} \) (and by \( H_{g-Q} \)), but which have no role in transmitting these interactions. This result can therefore connect our work to other models for understanding color confinement. If it turns out that quarks, and not only gluons, are essential in transmitting confining forces from one group of quarks to another, then \( H_{Q-Q} \) and \( H_{g-Q} \) cannot be the only mechanisms for quark confinement. However, if other methods of analysis corroborate that the gluonic medium has the primary role in effecting quark confinement, than \( H_{Q-Q} \) and \( H_{g-Q} \) become more interesting candidates as descriptions of confining forces.

IV. NON-PERTURBATIVE APPROACH TO THE NONLOCAL INTERACTION

IN AN SU(2) MODEL OF QCD

In order to pursue the analysis undertaken in Section III, and to demonstrate a procedure for evaluating \( H_{Q-Q} \) nonperturbatively, we will make use of Yang-Mills theory — the SU(2) version of this model — for which the structure constants \( f^{\delta \beta \gamma}(\eta) \) are \( \delta^{\beta \gamma} \). We will also ignore correlations among gauge-invariant gauge fields, and replace all \( A^{\delta}_{GI}(r) \) in Eq. (3.3) with the corresponding \( \langle A^{\delta}_{GI}(r) \rangle = \langle g | A^{\delta}_{GI}(r) | g \rangle \). Furthermore, we will model \( \langle A^{\delta}_{GI}(r) \rangle \), which are transverse fields in the adjoint representation of SU(2), as the manifestly transverse “hedgehog” configuration

\[
\langle A^{\delta}_{GI}(r) \rangle = \epsilon^{\delta ij} r_j \phi(r). \tag{4.1}
\]

Although there is no reason to believe that the ansatz given in Eq. (4.1) follows from the dynamical equations that determine \( A^{\delta}_{GI}(r) \), it is a convenient choice for examining to what extent the structure of \( F^{ba}(r, x) \) — as distinct from the precise form of \( A^{\delta}_{GI}(r) \) — enables us to nonperturbatively evaluate the infinite series given in Eq. (3.3). This simplified SU(2) model can also help us to identify the features of the gauge-invariant gauge field that might be significant for the confinement of quarks or color-bearing quark packets.

When we substitute \( \epsilon^{\tilde{\alpha} \tilde{\beta} \tilde{\gamma}}(\eta) \) for \( f^{\alpha \beta \gamma}(\eta) \), and use Eq. (4.1) in Eq. (3.1), we find that the integral in Eq. (3.1) that is linear in the gauge-invariant gauge field, can be expressed as

\[
(-g) \epsilon^{\delta \alpha \beta \gamma}(\eta) \langle A^{\delta\alpha\beta\gamma}_{GI}(r) \rangle \partial_i \int \frac{d\mathbf{x}_1}{4\pi |\mathbf{r} - \mathbf{x}_1|} \bar{j}_i^{\alpha}(\mathbf{x}_1) = ig \epsilon^{\delta \gamma ba}(r) L^\delta \phi(r) \int \frac{d\mathbf{x}_1}{4\pi |\mathbf{r} - \mathbf{x}_1|} \bar{j}_i^{\alpha}(\mathbf{x}_1), \tag{4.2}
\]

where \( L^\delta \) represents the \( \delta \)-component of the orbital angular momentum. In the case of the term quadratic in the gauge-invariant gauge field, we observe that
\[ g^2 \epsilon^\delta_{(1)^b_1} \epsilon^{\delta_{(1)^a}} \langle A_{\Omega(1)}(r) \rangle \partial_i \int \frac{dx_2}{4\pi |r - x_2|} \left( A_{\Omega_{ij}^{(2)}}(x_2) \right) \partial_j \int \frac{dx_1}{4\pi |x_2 - x_1|} J_0^a(x_1) = -g^2 \epsilon^\delta_{(1)^b_1} e^{s_1^\delta_{(2)^a}} \phi(r) L^\delta_{(1)} \int \frac{dx_2}{4\pi |r - x_2|} \phi(x_2) L^\delta_{(2)} \int \frac{dx_1}{4\pi |x_2 - x_1|} J_0^a(x_1). \tag{4.3} \]

\( L^\delta_{(2)} \) can be shifted to the left of the \( x_2 \) integration, by noting that for any reasonably well-behaved \( \psi(x_2) \),

\[ \int \frac{dx_2}{4\pi |r - x_2|} \phi(x_2) e^{\delta_{(2)^j}\delta_{ij}}(x_2) \frac{\partial}{\partial(x_2)_j} \psi(x_2) = -\epsilon^{\delta_{(2)^j}\delta_{ij}} \int \frac{dx_2}{4\pi |r - x_2|} \phi(x_2) \left( (x_2)_i \frac{\partial}{\partial(x_2)_j} \frac{1}{4\pi |r - x_2|} \right) \psi(x_2) = -\epsilon^{\delta_{(2)^j}\delta_{ij}} \frac{\partial}{\partial r_i} \int dx_2 \phi(x_2) \left( \frac{1}{4\pi |r - x_2|} \right) \psi(x_2) = -i L^\delta \int dx_2 \phi(x_2) \left( \frac{1}{4\pi |r - x_2|} \right) \psi(x_2), \tag{4.4} \]

and that the identical procedure can be carried out on every term in the series given in Eq. (3.1), to yield an expression in which all orbital angular momentum operators are on the extreme left-hand-side of the expression, and are functions of \( r_\ell \) and \( \partial/\partial r_\ell \). We thus obtain

\[ K^b_{0}(r) = \sum_{n=0}^{\infty} n^a \epsilon^{\delta_{(1)^b_0}} L^\delta_{(n)}(r) \Phi^a_{(n)}(r) = -i L^\delta \sum_{n=0}^{\infty} n^a \epsilon^{\delta_{(1)^b_0}} L^\delta_{(n)}(r) \Phi^a_{(n)}(r) \tag{4.5} \]

where \( L^\delta_{(n)} = \prod_{i=1}^{n} L^\delta_{i} \) and \( \epsilon^{\delta_{(1)^b_0}} \) is given by the SU(2) version of Eq. (2.7) (with the convention that \( \epsilon^{\delta_{(1)^b_0}} = \epsilon^{\delta_{ba}}, \epsilon^{\delta_{(0)^b_0}} = -\delta_{ab}, \) and \( L^\delta_{(0)} = 1 \)), and

\[ \Phi^a_{(n)}(r) = \phi(r) \int \frac{dy_{(1)}}{4\pi |r - y_{(1)}|} \phi(y_{(1)}) \int \frac{dy_{(2)}}{4\pi |y_{(1)} - y_{(2)}|} \phi(y_{(2)}) \cdots \int \frac{dy_{(\ell)}}{4\pi |y_{(\ell-1)} - y_{(\ell)}|} \phi(y_{(\ell)}) \cdots \int \frac{dy_{(n)}}{4\pi |y_{(n-1)} - y_{(n)}|} \phi(y_{(n-1)}) \int \frac{dy_{(n)}}{4\pi |y_{(n-1)} - y_{(n)}|} \phi(y_{(n-1)}) \int \frac{dy_{(n)}}{4\pi |y_{(n)} - y_{(n)}|} \phi(y_{(n)}) j_0^a(y_{(n)}), \tag{4.6} \]

with the convention that \( \Phi^a_{(0)}(r) = j_0^a(r) \).

Because of the simplicity of the SU(2) structure constants — the Kronecker delta is the only symmetric structure constant in SU(2) — it is possible to significantly simplify \( i^n \epsilon^{\delta_{ba}} L^\delta_{(n)} \). It was previously pointed out that the SU(2) chain of structure constants \( \epsilon^{\delta_{ba}_{\gamma\gamma}} \) can be represented as

\[ \epsilon^{\delta_{ba}_{\gamma\gamma}} = (-1)^{\frac{n}{2}} \delta_{a_1 a_2} \delta_{a_3 a_4} \cdots \delta_{a_{n-3} a_{n-2}} \epsilon^{a_{n-1} b_1} \epsilon^{b_0 a_{n}} \tag{4.7} \]

and

\[ \epsilon^{\delta_{ba}_{\gamma\gamma}} = (-1)^{\frac{n-1}{2}} \delta_{a_1 a_2} \delta_{a_3 a_4} \cdots \delta_{a_{n-2} a_{n-1}} \epsilon^{a_{n} b_1} \epsilon^{b_0 a_{n}} \tag{4.8} \]

for even and odd \( n \) respectively. \[ It \] is a trivial consequence of Eqs. (1.7) and (1.8) that \( i^n \epsilon^{\delta_{ba}} L^\delta_{(n)} \) can be represented as

\[ i^n \epsilon^{\delta_{ba}} L^\delta_{(n)} = L^b_{(n)} = A_n \left( L^a L^b + L^b L^a \right) + B_n \delta_{ab} + i C_n \epsilon_{ab} L^2, \tag{4.9} \]

where \( A_n, B_n, \) and \( C_n \) only depend on numerical constants and on the Casimir operator \( L^2 \). From simple recursion relations, \( i.e. D_n + D_{(n-1)} = D_{(n-2)} L^2 \) where \( D_n = A_n + C_n = \]

\[ \]

13
\[ B_{n+1}/L^2, \text{ and } F_n + F_{n+1} + D_n + D_{n-1}L^2 = 0 \text{ where } F_n = A_n - C_n, \] we can obtain the following explicit expressions for \( A_n, B_n, \) and \( C_n. \)

\[
A_n = (-1)^n \frac{1}{2} \sum_{s=0}^{[n/2]} \frac{(s - 1)!}{(n - 2s - 2)! (s + 1)!} L^{2s} \quad \text{for } n \geq 2, \tag{4.10}
\]

\[
B_n = (-1)^{(n-1)} \sum_{s=0}^{[n/2]} \frac{(s - 2)!}{(n - 2s - 2)! (s)!} L^{2(s+1)} \quad \text{for } n \geq 2, \tag{4.11}
\]

and

\[
C_n = (-1)^{(n-1)} \sum_{s=0}^{[(n-1)/2]} \frac{(s - 1)!}{(n - 2s - 1)! (s + 1)!} L^{2s} \quad \text{for } n \geq 2, \tag{4.12}
\]

where \([ (n - 2)/2 ] = (n - 2)/2 \) when \( n \) is even, and \([ (n - 2)/2 ] = (n - 3)/2 \) when \( n \) is odd.

for \( n < 2, A_0 = A_1 = B_1 = C_0 = 0, \) and \( B_0 = C_1 = -1. \) Because the algebra of the elements \((L^aL^b + L^bL^a), \delta_{ab}, \) and \( i\epsilon_{abi}L^i \) is closed under multiplication, it is easily shown — using the notation introduced in Eq. (4.9) — that

\[
\mathcal{L}_{(n+1)}^{ba} = -\mathcal{L}_{(n)}^{bs}\mathcal{L}_{(k)}^{sa} \tag{4.13}
\]

where the repeated index \( s \) is summed, and where

\[
A_{n+k} = \left( 2L^2 - \frac{5}{2} \right) A_n A_k + (A_n B_k + B_n A_k) - \frac{3}{2} (A_n C_k + C_n A_k) - \frac{1}{2} C_n C_k, \]

\[
B_{n+k} = A_n A_k L^2 + (A_n C_k + C_n A_k) L^2 + B_n B_k + C_n C_k L^2, \quad \text{and}
\]

\[
C_{n+k} = - \left( 2L^2 - \frac{3}{2} \right) A_n A_k + (B_n C_k + C_n B_k) + \frac{1}{2} (A_n C_k + C_n A_k) - \frac{1}{2} C_n C_k. \tag{4.14}
\]

Since there are only three independent coefficients in any \( \mathcal{L}_{(n)}^{ba} \) — viz. \( A_n, B_n, \) and \( C_n \) — the following equations must always have a unique solution for the constants \( x_i \) for \( i = 1 \rightarrow 3: \)

\[-A_0 = x_1 A_1 + x_2 A_2 + x_3 A_3,\]

\[-B_0 = x_1 B_1 + x_2 B_2 + x_3 B_3, \quad \text{and} \]

\[-C_0 = x_1 C_1 + x_2 C_2 + x_3 C_3, \tag{4.15}\]

so that the linear combination

\[
\mathcal{L}_{(0)}^{ba} + x_1 \mathcal{L}_{(1)}^{ba} + x_2 \mathcal{L}_{(2)}^{ba} + x_3 \mathcal{L}_{(3)}^{ba} = 0. \tag{4.16}
\]

Moreover, Eq. (4.13) has the effect of generalizing the validity of Eqs. (4.15) and (4.11) so that they apply to coefficients \( A_{n+i}, B_{n+i}, \) and \( C_{n+i} \) for any fixed \( n \) and \( i = 0 \rightarrow 3. \) The effect of these identities and of the explicit values of the three \( x_i \) is that

\[
\mathcal{L}_{(n+3)}^{ba} + 2\mathcal{L}_{(n+2)}^{ba} + (1 - L^2) \mathcal{L}_{(n+1)}^{ba} - L^2 \mathcal{L}_{(n)}^{ba} = 0 \tag{4.17}
\]
for any $n$. The Casimir operator, $L^2$, is treated like any constant in this analysis.

Eq. (4.17) enables us to obtain a differential equation for $K^b_0(r)$ whose features — including its order — reflect the SU(2) algebra as expressed in Eq. (4.9). We observe that, for $n \geq 1$,

$$
\nabla^2 \frac{1}{\phi(r)} \Phi^b_{(n)} = - \Phi^b_{(n-1)},
$$

which, with the use of Eq. (4.17), leads to:

$$
\nabla^2 \frac{1}{\phi(r)} \left[ K^b_0(r) + j^b_0(r) \right] = L^b_{(1)} K^a_0(r),
$$

$$
\nabla^2 \frac{1}{\phi(r)} \left( \nabla^2 \frac{1}{\phi(r)} \left[ K^b_0(r) + j^b_0(r) \right] \right) + L^b_{(1)} \nabla^2 \frac{1}{\phi(r)} j^b_0(r) = -L^b_{(2)} K^a_0(r) \quad \text{and} \quad (4.20)
$$

$$
\left\{ \nabla^2 \frac{1}{\phi(r)} \left[ \nabla^2 \frac{1}{\phi(r)} \left[ K^b_0(r) + j^b_0(r) \right] \right] \right\} + L^b_{(1)} \nabla^2 \frac{1}{\phi(r)} j^b_0(r) - L^b_{(2)} \nabla^2 \frac{1}{\phi(r)} j^b_0(r) = L^b_{(3)} K^a_0(r). \quad (4.21)
$$

We can use Eq. (4.17) to combine Eqs. (4.19)-(4.21) in such a manner that the right-hand sides vanish, to generate a sixth-order equation for $K^a_0(r)$, given below:

$$
\nabla^2 \frac{1}{\phi(r)} \left[ \nabla^2 \frac{1}{\phi(r)} \left( \nabla^2 \frac{1}{\phi(r)} K^b_0(r) \right) \right] - 2 \nabla^2 \frac{1}{\phi(r)} \left( \nabla^2 \frac{1}{\phi(r)} K^b_0(r) \right) + L^2 K^b_0(r) + \\
+ \left( 1 - L^2 \right) \nabla^2 \frac{1}{\phi(r)} K^b_0(r) = - \left( 1 - L^2 \right) \delta_{ba} + \left( 2 L^b_{(1)} + L^b_{(2)} \right) \nabla^2 \frac{1}{\phi(r)} j^b_0(r) + \\
+ \left( 2 \delta_{ba} - L^b_{(1)} \right) \nabla^2 \frac{1}{\phi(r)} \left( \nabla^2 \frac{1}{\phi(r)} j^a_0(r) \right) - \nabla^2 \frac{1}{\phi(r)} \left[ \nabla^2 \frac{1}{\phi(r)} \left( \nabla^2 \frac{1}{\phi(r)} j^b_0(r) \right) \right]. \quad (4.22)
$$

Although Eqs. (4.3) and (4.6) represent $K^b_0(r)$ as an infinite series, which we might expect to have to evaluate perturbatively, Eq. (4.22) is a differential equation that $K^b_0(r)$ obeys as a whole. The derivation of Eq. (4.22) therefore has enabled us to bypass the perturbative representation of $K^b_0(r)$.

We will make the further ansatz that $\phi(r)$ is a complex constant, $\kappa$. Then, we can write Eq. (4.22) in the form

$$
\left( \frac{\nabla^6}{\kappa^2} - 2 \frac{\nabla^4}{\kappa} + (1 - L^2) \nabla^2 + \kappa L^2 \right) K^b_0(r) = s^b(r)
$$

where $s^b(r)$ is the source term

$$
s^b(r) = \left\{ \left[ - \left( 1 - L^2 \right) \delta_{ba} + \left( 2 L^b_{(1)} + L^b_{(2)} \right) \right] \nabla^2 + \left( 2 \delta_{ba} - L^b_{(1)} \right) \frac{\nabla^4}{\kappa} - \frac{\nabla^6}{\kappa^2} \right\} j^b_0(r). \quad (4.24)
$$
In Eq. (4.24), $s^b(r)$ has been expanded in inverse powers of $\kappa$, and the leading term has been kept $\kappa$-independent. Eq. (4.23) is a differential equation in which $K^b_0(r)$ is related to a source. The spatial dependence of $K^b_0(r)$ can be determined by finding a Green function, $G(r, x)$, for which

$$\left(\frac{\nabla^6}{\kappa^2} - 2\frac{\nabla^4}{\kappa} + (1 - L^2)\nabla^2 + \kappa L^2\right) G(r, x) = -\delta(r - x). \quad (4.25)$$

Because the differential operator in Eq. (4.25) is explicitly dependent on the orbital angular momentum $L^2$, it is desirable to expand $G(r, x)$ in terms of partial waves. We express $G(r, x)$ as

$$G(r, x) = \sum_{\ell, m} g_\ell(r, x) Y_{\ell, m}(\theta_x, \phi_x) Y_{\ell, m}(\theta_r, \phi_r) \quad (4.26)$$

and find that $g_\ell(r, x)$ obeys

$$D_r g_\ell(r, x) = -\frac{1}{r^2} \delta(r - x) \quad (4.27)$$

where $D_r$ is the differential operator

$$D_r = \left[ \frac{1}{\kappa^2} \left( \nabla_r^2 - \frac{\ell(\ell + 1)}{r^2} \right)^3 - \frac{2}{\kappa} \left( \nabla_r^2 - \frac{\ell(\ell + 1)}{r^2} \right)^2 + \left[ 1 - \ell(\ell + 1) \right] \left( \nabla_r^2 - \frac{\ell(\ell + 1)}{r^2} \right) + \kappa \ell (\ell + 1) \right]. \quad (4.28)$$

where $\nabla_r^2 = \partial_r^2 + (2/r)\partial_r$. We expand $g_\ell(r, x)$ as

$$g_\ell(r, x) = \frac{2}{\pi} \int_0^\infty g_\ell(\alpha) j_\ell(\alpha r) j_\ell(\alpha x) \alpha^2 d\alpha \quad (4.29)$$

and adjust $g_\ell(\alpha)$ so that

$$D_r g_\ell(r, x) = -\frac{1}{r^2} \delta(r - x), \quad (4.30)$$

which leads to

$$g_\ell(\alpha) = \frac{1}{\kappa} \left[ \left( \frac{\alpha^2}{\kappa} + 1 \right) \left( \frac{\alpha^4}{\kappa^2} + \frac{\alpha^2}{\kappa} - \ell(\ell + 1) \right) \right]^{-1} \quad (4.31)$$

Since $\kappa$ is a complex constant, we can parameterize it as $\kappa = |\kappa| \exp[i\beta]$ and, for $\ell > 0$, represent $g_\ell(r, x)$ as

$$g_\ell(r, x) = \frac{\kappa^2}{\pi} \int_{-\infty}^{\infty} \frac{j_\ell(\alpha r) j_\ell(\alpha x) \alpha^2 d\alpha}{(\alpha - i\sqrt{|\kappa e^{i\beta/2}|})(\alpha + i\sqrt{|\kappa e^{i\beta/2}|})} \times \left[ \left( \alpha - ae^{i\beta/2} \right) \left( \alpha + ae^{i\beta/2} \right) \left( \alpha - ibe^{i\beta/2} \right) \left( \alpha + ibe^{i\beta/2} \right) \right]^{-1} \quad (4.32)$$

where
\[ a = \sqrt{|\kappa|} \left( \frac{\sqrt{4\ell(\ell + 1) + 1 - 1}}{2} \right) \quad \text{and} \quad b = \sqrt{|\kappa|} \left( \frac{\sqrt{4\ell(\ell + 1) + 1 + 1}}{2} \right). \] (4.33)

For \( \ell = 0 \), the corresponding expression is

\[ g_0(r, x) = \frac{\kappa^2}{\pi r x} \int_{-\infty}^{\infty} \frac{\sin(\alpha r) \sin(\alpha x) d\alpha}{\alpha^2 (\alpha - i \sqrt{\kappa |e^{i\beta/2}})^2 (\alpha + i \sqrt{\kappa |e^{i\beta/2}})^2}. \] (4.34)

The integrals in Eqs. (4.32) and (4.34) are simple to evaluate, but only some of the features of the resulting expressions are relevant to this discussion. For \( \ell > 0 \), the dominant behavior of all the \( g_\ell(r, x) \) as \( r \) and \( x \) increase in size is to decay exponentially in \( r \) and \( x \). For \( \ell = 0 \) the poles at \( \alpha = \pm i \sqrt{\kappa |e^{i\beta/2}} \) result in contributions that similarly decay exponentially in \( r \) and \( x \) unless \( e^{i\beta/2} = \pm i \); but the doubly degenerate pole at \( \alpha = 0 \) produces the contribution

\[ [g_0(r, x)](\alpha=0) = \frac{1}{2r x} ((r + x) - |r - x|). \] (4.35)

The discussion presented in this section leads us to make the following observations: The series representation, in Eq. (4.5), of \( K_0^b(r) \) — a quantity that encapsulates the nonlocal interactions between quark color-charge densities \( j_0^b(r) \) with each other and with gluonic color-charge — together with the SU(2) identities given in Eq. (4.17), lead to a differential equation for \( K_0^b(r) \) with source terms that are functionals of \( j_0^b(r) \). The derivation of this differential equation — Eq. (4.22) — enables us to eliminate the need for a term-by-term iterative expansion of \( K_0^b(r) \). We have arrived at Eq. (4.22) by replacing the SU(3) structure constants that apply to QCD with their corresponding SU(2) equivalents, and we have simplified Eq. (4.22) to the form given in Eq. (4.23) by imposing an ad hoc ansatz that fixes the functional dependence of the gauge-invariant gauge field on spatial variables and SU(2) indices as shown in Eq. (4.1). In spite of the special assumptions that apply to this nonperturbative evaluation of \( K_0^b(r) \), it can serve as a useful model for a similar approach applicable to a more realistic treatment of QCD with SU(3) structure constants and with gauge-invariant gauge fields that reflect more of the the dynamics of this theory.

The solutions we obtained for our simplified SU(2) model — Eqs. (4.29)–(4.35) — indicate that our simplifying assumptions lead to a form for the gauge-invariant gauge field that does not make \( H_{Q-Q} \) a confining nonlocal interaction. The fact that Eq. (4.23) is a sixth-order equation might have led us to anticipate that \( F^{ba}(r, x) \) would be a confining interaction. Even fourth-order equations can lead to Green functions with linear potentials that confine. But, in this present case, the SU(2) structure constants lead to the pole structure for the \( \ell \)-th partial wave shown in Eq. (4.32), with \( \sqrt{\kappa \ell (\ell + 1)} \) acting like a mass term in the differential equation that defines the Green function. Green functions that confine with linear potentials, or even with more rapidly rising ones, require higher order degenerate poles. Moreover, if the Green function’s exponential decrease with distance is to be avoided, the degenerate poles must be on the real-\( \alpha \) axis; and, if oscillatory behavior of the Green function is also to be avoided, the degenerate poles must be at the origin in the \( \alpha \) plane. In our model, only \( g_0(r, x) \) has a degenerate pole at the origin, and the pole is not degenerate to a sufficiently high order to support confinement. We could expect that the richer algebra...
of the SU(3) structure constants would lead to a higher order differential equation, and that the SU(3) Green functions therefore could lead to confinement — perhaps through a more degenerate pole structure. But in order to determine whether that is the case, it will be necessary to find an expression for the gauge-invariant gauge field that adheres more closely to the dynamics of the theory, and is not based on an ad hoc ansatz.

To illustrate this discussion with a specific example, we consider a hypothetical case in which both $L^2$ and $1 - L^2$ in Eq. (4.23) vanish — an obvious impossibility, since the eigenvalues of $L^2$ are quantized to a set of possible eigenvalues that forbid this. Nevertheless, the consequences of this assumption serve a useful illustrative purpose. For this hypothetical case, we obtain the following expression for $\bar{g}_0(r, x)$, the $\ell = 0$ component of the Green function:

$$\bar{g}_0(r, x) = \kappa^2 \pi r x \int_{-\infty}^{\infty} \frac{\sin(\alpha r)\sin(\alpha x) d\alpha}{\alpha^4 (\alpha^2 + 2\kappa)},$$

(4.36)

which includes a contribution from the quadruply degenerate pole at $\alpha = 0$,

$$[\bar{g}_0(r, x)](\alpha = 0) = -\frac{\kappa}{24 r x} \left( (r + x)^3 - |r - x|^3 \right).$$

(4.37)

The spatial dependence of $[\bar{g}_0(r, x)](\alpha = 0)$ is consistent with a confining potential for color-bearing quark packets. Although the dynamics of the model we are exploring does not lead to Eq. (4.37), the result may nevertheless serve a useful illustrative purpose. The model we are investigating in this section, which includes SU(2) structure constants and the “hedgehog” representation of the gauge-invariant gauge field, with the spatial function $\phi(r)$ represented as the constant $\phi(r) = \kappa$, is itself a toy model used to represent the less tractable theory that has SU(3) structure constants and a gauge-invariant gauge field obtained from the dynamical equations presented in Refs. [2,3]. It is relevant to inquire what changes can occur in the behavior of $K_b^0(r)$, and therefore also of the coupling term for quark color-charge densities, $\mathcal{F}^{ba}(r, x)$, when changes are made in the equations that determine the Green function $G(r, x)$, which might well be duplicated in the full SU(3) version of QCD.

The model proposed in this section, consisting of SU(2) structure constants and the hedgehog ansatz for the gauge-invariant gauge field, does not provide us with much information about $H_{g-Q}$ — the direct coupling between quark and glue color. When the gauge-invariant gauge field appears as part of $\mathcal{F}^{ba}(r, x)$, it is evaluated in expressions that do not contain the operator $\Pi^b_i(r)$ conjugate to the gauge-dependent gauge field. Ignoring field correlations, and replacing $A^b_{Gi}(r)$ with $\langle A^b_{Gi}(r) \rangle$ in $\mathcal{F}^{ba}(r, x)$, therefore is a legitimate and useful approximation. The model is not, however, as applicable to the representation of $K_b^0(r) = gf^{cde} \text{Tr} \left[ V^{-1}_{C}(r) \frac{\lambda^d}{2} V_{C}(r) \frac{\lambda^b}{2} \right] A^c_{Gi}(r) \Pi^b_i(r)$, in which $A^c_{Gi}(r)$ and $\Pi^b_i(r)$ appear together. Our model does not include an expression for $\Pi^b_i(r)$ that would be consistent with the hedgehog ansatz, or even respect the necessary commutation rules between the gauge field and its canonical adjoint. The effect of $H_{g-Q}$ on color confinement therefore remains to be addressed until the dynamics of the gauge-invariant gauge field have been more fully explored.
V. DISCUSSION

In this work, we have analyzed the nonlocal interaction that results when QCD is formulated in terms of gauge-invariant quark and gluon operator-valued fields. We have shown that this nonlocal interaction involves the quark color-charge density in a way that is roughly analogous to the role of the electric charge density in the Coulomb interaction in a gauge-invariant formulation of QED; but the functional form of this nonlocal interaction — the QCD analog of \(1/(4\pi|\mathbf{r} - \mathbf{x}|)\) in QED — is \(\mathcal{F}^{ba}(\mathbf{r}, \mathbf{x})\), which is a nonlocal functional that depends not only on spatial variables, but also involves the gauge-invariant gauge field. \(\mathcal{F}^{ba}(\mathbf{r}, \mathbf{x})\) is an infinite series, in which the \(n\)-th order term contains the gauge-invariant gauge field to the \(n\)-th power. But the series has such a regular structure, that an explicit form for the \(n\)-th order term can easily be written, without requiring knowledge of the lower order terms in the series. A nonperturbative treatment of this nonlocal interaction term is therefore not nearly as inaccessible as would be the case for S-matrix elements in perturbative QCD.

One feature of the nonlocal interaction between quark color-charge densities is that the monopole color charge is its leading term, and that higher order multipole interactions — Van der Waals type forces — succeed it in a series expansion. A color-charge monopole force as the leading term for quarks or color-bearing ensembles of quarks, and the consequence of that idea — that color singlets, the natural QCD analogs of electrically neutral ensembles of charges, would not feel that force — is not new. This idea — and the term “color neutrality” to designate it — have, for example, been suggested by Jain, Pire and Ralston. These authors also have suggested that there is a connection between color transparency and the lesser importance of higher order Van der Waals-like multipoles for small-sized color singlet configurations. What is new in our work is that these features of a nonlocal QCD interaction are no longer a conjecture motivated by phenomenology only, but are inherent in the representation of the QCD Hamiltonian in terms of gauge-invariant quark and gluon fields.

The gauge-invariant form of the QCD Hamiltonian contains the kinetic energy terms for the gauge-invariant quark and gluon fields, the nonlocal interaction discussed in this paper, and the additional interaction term \(\hat{H}_{j-A} = -g \int \bar{\psi}(\mathbf{r})\alpha_i \frac{\lambda_a}{2} \psi(\mathbf{r}) A_{GIi}(\mathbf{r}) d\mathbf{r}\), which is the QCD analog of the QED interaction term \(-e \int \bar{\psi}(\mathbf{r})\alpha_i \psi(\mathbf{r}) A_{GIi}(\mathbf{r}) d\mathbf{r}\). In QED, this latter term describes the interaction of the electron current with the gauge-invariant gauge field (in QED, \(A_{GIi}(\mathbf{r})\) is just the transverse component of the gauge field). In QED, this interaction couples electrons to the two helicity modes of the photons — the propagating, observable quantized modes of the transverse electromagnetic vector potential. As is well known, the current appearing in this interaction has a \(v/c\) dependence, that makes it relatively unimportant in the low-energy regime, in which the Coulomb interaction is, by far, the most important interaction between charged particles. Since the corresponding term in the QCD Hamiltonian similarly involves a current density — the transverse color-current density in this case — it is reasonable to expect that its \(v/c\) dependence will also make it much less important than the nonlocal interaction between quark color-charge densities in the low-energy regime. The interaction term \(H_{g-Q} + H_{Q-g}\) therefore is a very interesting candidate low-energy limit of of the QCD interaction — a nonlocal interaction term between
quark color-charge densities with each other and with gluonic color, that, in analogy with
the Coulomb interaction in QED, would describe the most important features of QCD in
the low-energy regime.

It is important to make the following distinction between different parts of this work:
On the one hand, there are relations like Eqs. (3.1)-(3.3), which are exact. These relations
are useful to the extent to which we have identified serviceable gauge-invariant fields that
lead to a gauge-invariant Hamiltonian in which the nonlocal interactions that result from
the implementation of gauge invariance are dominant in describing static quarks. These
relations are not dependent on any approximations. They are direct and inevitable conse-
quences of transforming the QCD Hamiltonian to a representation in which it is expressed
in terms of the gauge-invariant fields we have constructed. On the other hand, there are the
contents of Section IV, which serve an important illustrative purpose, but are dependent on
a number of simplifying assumptions. Our discussion, in Section III, of the implications of
the nonlocal interaction for color confinement and for color transparency, are not dependent
on the substitution of the SU(2) for the SU(3) algebra, or the ansatz that the gauge-invariant
gauge fields are uncorrelated or that they have a hedgehog configuration, which we made
in Section IV. This part of the discussion in Section III is dependent only on assumptions
about the functional dependence of $F^{ba}(r, x)$ on the spatial variables $r$ and $x$, and about
the sizes of the quark packets coupled by this nonlocal interaction. We have developed
this model far enough so that we can explore the consequences of our model, given these
assumptions. We cannot, at this point, establish that these assumed quark configurations
will necessarily arise in the low-energy regime.

The differential equation, Eq. (4.22) and its special form Eq. (4.23), and the Green
function solution of Eq. (4.23) given in Section IV, depend on two simplifying assumptions
— the SU(2) algebra and the hedgehog configuration of the gauge-invariant gauge field.
These results are important for establishing a pattern for nonperturbative treatments of
the nonlocal interaction in QCD and for demonstrating the role of the SU(N) algebra in
generating higher order differential equations for the Green function that connects quarks
to each other. But the specific solutions — Eqs. (4.32)-(4.35) — reported in Section III only
apply to the “toy model” based on SU(2) structure constants and the hedgehog ansatz for
the gauge-invariant gauge field.

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