Universal Metric Spaces According to
W. Holsztynski

Sergei Ovchinnikov
Mathematics Department
San Francisco State University
San Francisco, CA 94132
sergei@sfsu.edu

October 27, 2018

Abstract
In this note we show, following W. Holsztynski, that there is a continuous
metric $d$ on $\mathbb{R}$ such that any finite metric space is isometrically embeddable
into $\mathbb{R}$.

Let $\mathcal{M}$ be a family of metric spaces. A metric space $U$ is said to be a
universal space for $\mathcal{M}$ if any space from $\mathcal{M}$ is (isometrically) embeddable in
$U$.

Fréchet [2] proved that $\ell^\infty$ (the space of all bounded sequences of real num-
bers endowed with the sup norm) is a universal space for the family $\mathcal{M}$ of all
separable metric spaces. Later, Uryson [5] constructed an example of a separable
universal space for this $\mathcal{M}$ ($\ell^\infty$ is not separable).

In this note we establish the following result which is Theorem 5 in [3].

Theorem 1. There exists a metric $d$ in $\mathbb{R}$, inducing the usual topology, such
that every finite metric space embeds in $(\mathbb{R}, d)$.

Our proof essentially follows the original Holsztynski’s approach [4].

We say that a metric space $(X, d)$ is $\varepsilon$–dispersed if $d(x, y) \geq \varepsilon$ for all $x \neq y$
in $X$ ($\varepsilon > 0$). Clearly, any $\varepsilon$– metric space is also $\varepsilon'$–dispersed for any positive
$\varepsilon' < \varepsilon$. The following proposition will be used to construct universal spaces for
particular families $\mathcal{M}$.

Proposition 1. Let $f : X \to Y$ be a continuous surjection from a metric space
$(X, d)$ onto a metric space $(Y, D)$. Then $(X, d_\varepsilon)$ where

$$d_\varepsilon(x, y) = \max\{\min\{d(x, y), \varepsilon\}, D(f(x), f(y))\}$$

is a universal space for the family of $\varepsilon$–dispersed subspaces of $(Y, D)$ and metrics
$d$ and $d_\varepsilon$ are equivalent on $X$. 

\[\]
Proof. \( d_\varepsilon \) is a distance function. Indeed, \( d_\varepsilon \) is symmetric and \( d_\varepsilon(x, y) = 0 \) if and only if \( x = y \). We have
\[
\max\{ \min\{d(x, y), \varepsilon\}, D(f(x), f(y))\} + \max\{ \min\{d(y, z), \varepsilon\}, D(f(y), f(z))\} \geq D(f(x), f(y)) + D(f(y), f(z)) \geq D(f(x), f(z))
\]
and
\[
\max\{ \min\{d(x, y), \varepsilon\}, D(f(x), f(y))\} + \max\{ \min\{d(y, z), \varepsilon\}, D(f(y), f(z))\} \geq \min\{d(x, y), \varepsilon\} + \min\{d(y, z), \varepsilon\} = \min\{d(x, y) + d(y, z), \varepsilon\} \geq \min\{d(x, z), \varepsilon\}
\]
Hence, \( d_\varepsilon(x, y) + d_\varepsilon(y, z) \geq d_\varepsilon(x, z) \).

Let \( Z \) be an \( \varepsilon \)-dispersed subspace of \( Y \). Since \( f \) is surjective, for any \( z \in Z \), there is \( x_z \in X \) such that \( f(x_z) = z \). Let \( X' = \{x_z : z \in Z\} \). By (\( I \)), \( d_\varepsilon(x, y) = D(f(x), f(y)) \) for all \( x, y \in X' \). Thus \( f \) establishes an isometry between \((Z, D)\) and \((X', d_\varepsilon)\). \( \square \)

In what follows, \( \mathcal{M} \) is the family of all finite metric spaces.

We define
\[
I^n = \{ \bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq n, \ 1 \leq i \leq n \}
\]
and \( J_n = [n - 1, n] \) for \( n \geq 1 \). \( I^n \) is a metric space with the distance function
\[
D(\bar{x}, \bar{y}) = \max\{|x_i - y_i| : 1 \leq i \leq n\},
\]
and \( J_n \) is a metric space with the usual distance.

Let \((X, d)\) be a finite metric space. We define
\[
p = |X|, \ q = [\text{Diam}(X)], \ r = [\varepsilon^{-1}],
\]
where \( \varepsilon = \min\{d(x, y) : x \neq y\}, \) and \( n = \max\{p, q, r\} \). Clearly, \((X, d)\) is \( \frac{1}{n} \)-dispersed.

**Proposition 2.** (The Kuratowski embedding) \((X, d)\) is embeddable into \( I^n \).

Proof. Let \( X = \{x_1, \ldots, x_p\} \). We define \( f : X \to I^n \) by
\[
f(x_i) = (d(x_i, x_1), \ldots, d(x_i, x_p), 0, \ldots, 0),
\]
for \( 1 \leq i \leq p \). (Since \( n \geq \text{Diam}(X) \), \( f(x_i) \in I^n \).) We have, by the triangle inequality,
\[
D(f(x_k), f(x_m)) = \max_j \{|d(x_k, x_j) - d(x_m, x_j)|\} \leq d(x_k, x_m).
\]
On the other hand, \( |d(x_k, x_j) - d(x_m, x_j)| = d(x_k, x_m) \) for \( j = m \). Therefore, \( D(f(x_k), f(x_m)) = d(x_k, x_m) \) for all \( 1 \leq k, m \leq p \). \( \square \)
Let $f_n$ be a continuous surjection from $J_n$ onto $I^n$ (a “Peano curve” IV(4)) and $d_n(x, y)$ be the distance function on $J_n$ defined by (1) for $\varepsilon = \frac{1}{n}$. Note, that $d_n$ is equivalent to the usual distance on $J_n$. By Proposition 1, $J_n$ is a universal space for any $\frac{1}{n}$-dispersed subspace of $I^n$. By Proposition 2, any finite metric space is embeddable in $(J_n, d_n)$ for some $n$.

It is easy to show that there is a continuous distance function on $\mathbb{R}$ that coincides with $d_n(x, y)$ on $J_n$ for all $n$. Indeed, let $d_1(x, y)$ and $d_2(x, y)$ be two continuous distance functions on intervals $[a, b]$ and $[b, c]$, respectively. Then $d(x, y)$ defined by

$$d(x, y) = \begin{cases} d_1(x, y), & \text{if } x, y \in [a, b], \\ d_2(x, y), & \text{if } x, y \in [b, c], \\ d_1(x, b) + d_2(b, y), & \text{if } x \in [a, b] \text{ and } y \in [c, d]. \end{cases}$$

is a continuous distance function on $[a, c]$. In fact, thus defined $d$ is equivalent to the usual metric on $[a, c]$.

By applying this process consecutively, we obtain a required distance function on $\mathbb{R}$.

References

[1] Dugundji, J. *Topology* (Wm. C. Brown Publishers, Dubuque, Iowa, 1989).

[2] Fréchet, M., Les dimensions d’un ensemble abstrait, *Math. Ann.* 68 (1910), 145–168.

[3] Holsztynski, W., $\mathbb{R}^n$ as universal metric space, *Notices Amer. Math. Soc.* 25 A–367, 1978.

[4] Holsztynski, W., Personal communication, 2000.

[5] Uryson, P.S., Sur un espace métrique universel, *Bull. de Sciences Math.* 5 (1927), 1–38.