Mass Matrix Models

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Abstract

It is possible that the hierarchy in the masses and mixing of quarks is a result of a horizontal symmetry. The smallness of various parameters is related to their suppression by high powers of a scale of new physics. We analyze in detail the structure of such symmetries in view of new experimental data and present explicit models consistent with all phenomenological constraints. We show that it is possible that the flavor dynamics can be at accessible energies – as low as a $TeV$. 
1. Introduction

The experimental values of the quark sector parameters, even if consistent with each other, may provide important hints for new physics beyond the standard model. In recent years, much progress has been achieved in the determination of these parameters: heavy quark symmetry and additional experimental measurements allowed a more accurate determination of $|V_{cb}| = 0.040 \pm 0.007$; measurements in CLEO and ARGUS of the leptonic spectrum near the endpoint in charmless $B$ decays provided a first determination of $|V_{ub}/V_{cb}| = 0.10 \pm 0.03$; and the combination of direct searches in CDF with precision electroweak measurements in LEP gave stringent bounds on $m_t \approx 135 \pm 45 \, GeV$. Consequently, all six quark masses and three mixing angles became known to within a factor of two or so. This calls for a re-examination of various frameworks that try to explain the hierarchy in these parameters.

Most theoretical frameworks that provide such an explanation suggest that the new physics responsible for the hierarchy takes place at a very high energy scale, typically the GUT scale (see, for example, refs. [1 – 9]). If this is indeed the case, it would be very difficult to test these ideas beyond the numerical estimates that they provide for the quark masses and mixings. If, on the other hand, the physics of flavor lies at the $TeV$ scale (see, for example, refs. [10 – 12]) then it may be accompanied with rich phenomenology that may be accessible to future experiments.

In this work, we study the possibility that the hierarchy in the quark sector parameters is a result of a horizontal discrete symmetry. We analyze in detail the structure of such symmetries in view of the new experimental data; and we carefully examine the possibility that the symmetry is broken at low enough energies to be tested in experiment.

We have in mind supersymmetric models. Most of our discussion is in this context and we ignore the soft breaking terms. However, our ideas are more general and apply also to non-supersymmetric models.
In sections 2–4 we discuss general features of horizontal symmetries and their breaking mechanism: in section 2 we prove that horizontal symmetries cannot be exact and require that the scalar sector is extended beyond the single Higgs doublet of the standard model; in section 3 we discuss the phenomenological consequences of the symmetry breaking at various scales and analyze the implications of natural flavor conservation; and in section 4 we show that only Abelian horizontal symmetries are relevant in a large class of models. In sections 5–6 we introduce a more specific framework, where mixing and hierarchy of masses arise from non-renormalizable terms: in section 5 we discuss the theoretical framework and in section 6 we present the requirements that arise from the measured values of the quark sector parameters. In section 7 we present several explicit models that explain the observed hierarchy in the quark sector parameters, and in section 8 we study the possibility that the new physics related to these models takes place at a low enough scale to be observed in experiment. Finally, we give our conclusions in section 9. In an appendix we extend our ideas to the lepton sector.

2. General Features of Horizontal Symmetries

Our basic assumption in this work is that there is a horizontal symmetry $H$ that gives a certain structure to the quark mass matrices. Throughout our discussion, we assume that $H$ is a discrete symmetry which might or might not be gauged. We note that spontaneously broken discrete symmetries may cause cosmological problems by creating domain walls. We have nothing to add in this respect.

In this section, we prove two well known facts about horizontal symmetries: that they cannot be exact and that they require extending the scalar sector beyond the single Higgs doublet of the standard model. We present the proofs in a way which is useful for our argumentation later.

We denote quark fields by $Q$, $\bar{d}$ and $\bar{u}$ for, respectively, $(3, 2)_{1/6}$, $(\bar{3}, 1)_{1/3}$ and $(\bar{3}, 1)_{-2/3}$ representations of $SU(3)_C \times SU(2)_L \times U(1)_Y$. Scalar fields are generically denoted by $\phi$. (Later we will distinguish between $\phi_d$, $\phi_u$ and $S$ which denote,
respectively, $\frac{1}{2}$, $\frac{3}{2}$ and $0$ representations of $SU(2)_L \times U(1)_Y$. Yukawa terms are then of the general form

$$\mathcal{L}_{\text{Yukawa}} = \lambda_{ij}^{da} Q_i \phi^a d_j + \lambda_{ij}^{ub} Q_i \phi^b \bar{u}_j + \text{h.c.}. \quad (2.1)$$

The mass matrices $M_u = \lambda^{ub} \langle \phi^b \rangle$ and $M_d = \lambda^{da} \langle \phi^a \rangle$ are diagonalized by bi-unitary transformations,

$$U_L M_u U_R^\dagger = D_u \equiv \text{diag}(m_u, m_c, m_t),$$
$$D_L M_d D_R^\dagger = D_d \equiv \text{diag}(m_d, m_s, m_b), \quad (2.2)$$

with the CKM mixing matrix

$$V = U_L D_L^\dagger. \quad (2.3)$$

Under $H$, the fields transform as

$$Q \rightarrow LQ, \quad \bar{d} \rightarrow R_d \bar{d}, \quad \bar{u} \rightarrow R_u \bar{u}, \quad \phi \rightarrow P\phi. \quad (2.4)$$

In the mass basis, left handed quarks transform as

$$d \rightarrow L_d d, \quad u \rightarrow L_u u, \quad (2.5)$$

where

$$L_u = U_L L U_L^\dagger, \quad L_d = D_L L D_L^\dagger. \quad (2.6)$$

Note that $L_u$ and $L_d$ are related through

$$L_d = V^\dagger L_u V. \quad (2.7)$$

First, we prove that if $H$ is not broken, there are either degenerate quarks or vanishing mixing angles. If $H$ is unbroken, then the mass matrices are invariant
under the symmetry operation, namely

\[ LM_d R_d = M_d, \quad LM_u R_u = M_u, \]  \hspace{2cm} (2.8)

and consequently

\[ LM_d M_d^\dagger L = M_d M_d^\dagger, \quad LM_u M_u^\dagger L = M_u M_u^\dagger. \]  \hspace{2cm} (2.9)

In the mass basis, the conditions (2.9) translate into

\[ [L_d, D^2_d] = [L_u, D^2_u] = 0. \]  \hspace{2cm} (2.10)

Then the fact that there is no degeneracy in either quark sector forces \( L_d \) and \( L_u \) to be diagonal. However, eq. (2.7) implies that \( L_d \) and \( L_u \) have the same eigenvalues.

If the three eigenvalues are different from each other, then (2.7) requires that all mixing angles vanish (\( V \) is a permutation \( \times \) phase matrix). If two eigenvalues are equal but different from the third one, then one mixing angle vanishes (\( V \) is block diagonal).

The fact that none of the mixing angles vanishes forces \( L_d \) and \( L_u \) to be proportional to the unit matrix and \( H \) is just an overall phase transformation.

More intuitively, if the horizontal symmetry is not broken, then quarks in the same representation remain degenerate. Since there is no degeneracy in the spectrum, the quarks should be in three one dimensional representations. If these representations are different, there cannot be mixing between them and if they are all isomorphic representations, the horizontal symmetry is trivial. We conclude: as there is no degeneracy among quarks and as all three quark generations mix, there can be no unbroken horizontal symmetry.

Second, we prove that if \( H \) is broken by the VEV of a single Higgs doublet, there are either degenerate quarks or vanishing mixing angles. (The situation in supersymmetric theories with two Higgs fields will be discussed in the next section.)
which is just a phase transformation. Therefore, we can always find a symmetry $H' \subset H \times U(1)_Y$ isomorphic to $H$ under which the Higgs field is neutral. Note that $H'$ is an exact horizontal symmetry of the full theory and is not spontaneously broken. Then, as in the case of unbroken $H$, it implies either degenerate quarks or trivial mixing angles.

We conclude: as there is no degeneracy among quarks and as all three quark generations mix, a horizontal symmetry requires extending the scalar sector beyond a single Higgs doublet.

3. Spontaneous Breaking of Horizontal Symmetries

From the phenomenological point of view, the scale $\Lambda_H$ at which the spontaneous breakdown of the horizontal symmetry takes place can reside in one of three ranges:

(i) High $\Lambda_H$: above $10^3 - 10^4$ TeV. In this range there are no obvious phenomenological signatures to the existence of $H$, except, of course, for the numerical estimates of the quark sector parameters that it provides.

(ii) Intermediate $\Lambda_H$: above a few TeV and below $10^3 - 10^4$ TeV. In this range, the new physics related to $H$ may affect rare processes such as $K - \bar{K}$ mixing.

(iii) Low $\Lambda_H$: a few TeV or less. In this range, the new physics related to $H$ may be directly accessible in future experiments.

Models with a high $\Lambda_H$ are phenomenologically “safe”: scalar doublets and singlets have their masses at the scale $\Lambda_H$ except for one Higgs (two in the SUSY framework). The light scalar sector is similar to the minimal (SUSY) standard model. But it is somewhat disappointing that all “flavor physics” takes place at a high energy scale and cannot be further tested in experiments.

If, on the other hand, $H$ is broken at an intermediate or maybe even at the electroweak scale, then the implications for phenomenology of the scalar sector are very interesting. (For a recent related discussion, see ref. [13].) As we showed in
the previous section, the scalar sector should be extended beyond a single Higgs doublet. However, a generic multi Higgs model leads to large flavor changing neutral currents (FCNC). For nondiagonal Yukawa couplings of order one, the bounds from neutral meson mixing,

\[
M(\phi_i) \gtrsim \begin{cases} 
4 \times 10^3 \text{ TeV} \sqrt{\lambda^i_{ds} \lambda^i_{sd}} & (K - \bar{K} \text{ mixing}), \\
6 \times 10^2 \text{ TeV} \sqrt{\lambda^i_{uc} \lambda^i_{cu}} & (D - \bar{D} \text{ mixing}), \\
6 \times 10^2 \text{ TeV} \sqrt{\lambda^i_{db} \lambda^i_{bd}} & (B - \bar{B} \text{ mixing}),
\end{cases}
\tag{3.1}
\]

do not allow intermediate or low \( \Lambda_H \).

A well known mechanism to avoid FCNC is that of Natural Flavor Conservation (NFC) [14], which requires that there is only one scalar doublet coupled to each quark sector:

\[
\mathcal{L}_{\text{Yukawa}} = \lambda^d_{ij} Q_i \phi_d \bar{d}_j + \lambda^u_{ij} Q_i \phi_u \bar{u}_j + \text{h.c.}
\tag{3.2}
\]

This is the situation in supersymmetric models. An important characteristic feature of these couplings is the existence of a global \( U(1)_X \) symmetry:

\[
\bar{d} \to e^{i\alpha} \bar{d}, \quad \phi_d \to e^{-i\alpha} \phi_d,
\tag{3.3}
\]

and all other light fields carry \( X = 0 \). Note that \( U(1)_X \) is not necessarily a symmetry of the full Lagrangian but only of the terms in equation (3.2). Such a situation with an extended symmetry of the Yukawa terms is natural either in supersymmetric models or with suitably chosen discrete symmetries. The Yukawa Lagrangian (3.2) is, however, unacceptable in the framework of horizontal symmetry: The combination of spontaneously broken horizontal symmetry and NFC leads to either degenerate quarks or trivial mixing angles [15 – 17].

To prove this statement, note that as there is only one scalar doublet coupled to each quark sector, each of them is in a one dimensional representation of \( H \).
The transformation law (2.4) can be rewritten for the scalar fields:

\[ \phi_d \to P_d \phi_d, \quad \phi_u \to P_u \phi_u, \quad (3.4) \]

where \( P_q \) is a phase transformation. Then we can define a symmetry \( H' \) of the Yukawa terms (3.2), isomorphic to \( H \), that operates on the various fields as

\[ Q \to LQ, \quad \bar{d} \to R_d P_d \bar{d}, \quad \bar{u} \to R_u P_u \bar{u}, \quad (3.5) \]
\[ \phi_d \to \phi_d, \quad \phi_u \to \phi_u. \]

\( H' \) is a subgroup of \( H \times U(1)_X \times U(1)_Y \times U(1)_B \) (\( U(1)_B \) is baryon number symmetry). Such redefinitions of the horizontal symmetry will be used extensively below. Since both \( \phi_u \) and \( \phi_d \) transform trivially under \( H' \), \( H' \) is not broken. As proved in the previous section, this implies either degenerate quarks or trivial mixing angles.

This conclusion is not modified even if there are additional scalars that transform nontrivially under \( H \) but do not couple to quarks: we define all of them to be singlets of \( H' \). Note that even though \( U(1)_X \) is not necessarily a symmetry of the full Lagrangian, \( H' \) is.

Thus, to allow for an intermediate or low \( \Lambda_H \), we would like to find a framework where NFC is broken, but by a small amount. The no-go proof above points at a way around it. One could violate NFC by non-renormalizable terms. They are naturally small because they are suppressed by inverse powers of some high scale \( M \). If there is only one Higgs, the result in the previous section shows that it is impossible to break the horizontal symmetry. With only two Higgs fields, \( \phi_u \) and \( \phi_d \), the horizontal symmetry cannot be broken, even when non-renormalizable operators are taken into account, without violating the \( U(1)_X \) symmetry. We are, therefore, led to consider terms like \( Q \phi_d (\phi_u \phi_d)^n \bar{d} \) and \( Q \phi_u (\phi_u \phi_d)^n \bar{u} \) which violate this symmetry and lead to flavor changing processes. Alternatively, we can preserve the \( U(1)_X \) symmetry in the Yukawa couplings and the non-renormalizable terms but introduce more scalars, \( S \), which are invariant under \( SU(2) \times U(1) \) with \( U(1)_X \) invariant couplings like \( Q \phi_d S^n \bar{d} \) and \( Q \phi_u S^n \bar{u} \).
The framework of this work is then defined as follows: We assume that there is a horizontal symmetry $H$ which is spontaneously broken, preferably at a low scale $\Lambda_H$. In addition, there is Natural Flavor Conservation broken by non-renormalizable terms and suppressed, therefore, by a high energy scale $M > \Lambda_H$.

4. Abelian or nonabelian?

Within our framework, the tree level renormalizable Yukawa terms are of the form (3.2). Then $\phi_d$ and $\phi_u$ are necessarily in one dimensional representations of $H$. If all the scalars that couple to quarks are in one dimensional representations of $H$, then all quarks are also in one dimensional representations.

To prove this statement, note that as the scalars have only phase transformations under $H$, we must have

$$[L, \lambda^u \lambda^{u\dagger}] = [L, \lambda^d \lambda^{d\dagger}] = 0. \quad (4.1)$$

For three quark generations, there are three possibilities:

(i) $L$ is an irreducible representation of $H$. Then eq. (4.1) implies that both $\lambda^u \lambda^{u\dagger}$ and $\lambda^d \lambda^{d\dagger}$ are proportional to the unit matrix, leading to degeneracy among all three quarks. Such a degeneracy cannot be lifted enough by small corrections.

(ii) $L$ is reducible to a two dimensional and a one dimensional representations. Then (4.1) implies that the two generations in the two dimensional representation are degenerate. But more important, the single generation in one dimensional representation does not mix with the other two generations. This holds to all orders since any combination of scalars is still in one dimensional representation of $H$.

A way out of this scenario is to add an $SU(2)$-singlet scalar in a two dimensional representation of $H$. Then nonrenormalizable terms may induce the required mixing.

* A similar point was recently made in a different context in ref. [18].
(iii) $L$ consists of three one dimensional representations. This implies that $R_d$ and $R_u$ also reduce to one dimensional representations.

The conclusion is that if all scalars reside in one dimensional representations of a horizontal symmetry, then only an Abelian (sub-)group $H$ is relevant.

5. Mass and Mixing Hierarchy from Nonrenormalizable Terms

The mechanism that we employ to create the hierarchy in the masses and mixing of quarks is similar to that suggested by Froggatt and Nielsen [1]. To explain the general idea, it is simplest to consider the spontaneous breaking of the horizontal symmetry by a VEV of a scalar $S$ which is a singlet of the standard model. Then the breaking scale is simply

$$\Lambda_H \sim \langle S \rangle .$$

(5.1)

We further assume that at some higher energy scale, $M$, natural flavor conservation is broken. Specifically, we will introduce additional pairs of mirror fermions with masses of order $M$.

Consider first the possibility that the $U(1)_X$ symmetry of (3.3) can act also on the massive fermions such that all Yukawa terms in the Lagrangian are invariant. When we integrate out the massive fermions the effective Yukawa couplings below $M$ are of the form

$$\frac{\lambda_{ij}^d}{M^{m_{ij}}} Q_i \phi_d S^{m_{ij}} \bar{d}_j + \frac{\lambda_{ij}^u}{M^{n_{ij}}} Q_i \phi_u S^{n_{ij}} \bar{u}_j + \text{h.c.}$$

(5.2)

In non-supersymmetric models we should also allow powers of $S^\dag$. They can be incorporated in the following discussion by allowing negative $m$ and $n$ but we will ignore this possibility. Without loss of generality, we can use $U(1)_Y \times U(1)_X$ rotations and rescaling to redefine the horizontal charges (which we also denote by
$H$) of the scalar fields to

$$H(\phi_d) = H(\phi_d) = 0, \quad H(S) = -1. \quad (5.3)$$

Then the power $m_{ij}(n_{ij})$ in eq. (5.2) is simply the horizontal charge difference between $Q_i$ and $d_j(u_j)$:

$$m_{ij} = H(Q_i) + H(\bar{d}_j), \quad n_{ij} = H(Q_i) + H(\bar{u}_j). \quad (5.4)$$

(We assume for simplicity that the horizontal symmetry is $U(1)_H$ rather than a discrete subgroup of it. Suitable terms in the scalar potential explicitly break $U(1)_H$ to a discrete subgroup. As mentioned above, such a situation can be natural.) In the quark mass matrix,

$$\mathcal{L}_{\text{mass}} = \lambda_{ij}^d \langle \phi_d \rangle \left( \frac{\Lambda_H}{M} \right)^{m_{ij}} Q_i \bar{d}_j + \lambda_{ij}^u \langle \phi_u \rangle \left( \frac{\Lambda_H}{M} \right)^{n_{ij}} Q_i \bar{u}_j + \text{h.c.}, \quad (5.5)$$

a small parameter

$$\epsilon = \frac{\Lambda_H}{M}. \quad (5.6)$$

appears (we absorb a Yukawa coupling of $S$ to massive fields into the definition of $\Lambda_H$). The hierarchy in the quark sector parameters appears because various mixing angles and mass ratios depend on different powers $\epsilon^m$. Even if $\epsilon$ by itself is not a very small number, the physical parameters may be very small if they depend on high powers $m$, namely if there are large $H$-charge differences between quarks.

The mixing angles are determined by the $H$-charge differences between the quark doublets:

$$|V_{us}| \sim e^{[H(Q_1) - H(Q_2)]}, \quad |V_{cb}| \sim e^{[H(Q_2) - H(Q_3)]}, \quad |V_{ub}| \sim e^{[H(Q_1) - H(Q_3)]}. \quad (5.7)$$

Eq. (5.7) implies that

$$|V_{ub}| \sim |V_{us}V_{cb}| \quad (5.8)$$

always holds in our framework. As for the mass ratios, they depend on the charges
\[
\frac{m_{d_i}}{m_{d_j}} \sim \epsilon [H(Q_i) - H(Q_j) + H(\bar{d}_i) - H(\bar{d}_j)], \quad \frac{m_{u_i}}{m_{u_j}} \sim \epsilon [H(Q_i) - H(Q_j) + H(\bar{u}_i) - H(\bar{u}_j)].
\] (5.9)

Note that with the Yukawa Lagrangian (5.2), there are no flavor changing neutral currents mediated by the scalar doublet fields. The effective Yukawa couplings of \( \phi_d, (\lambda_{\text{eff}}^d)_{ij} = \lambda_{ij}^d \left( \frac{S}{M} \right)^{m_{ij}}, \) and of \( \phi_u, (\lambda_{\text{eff}}^u)_{ij} = \lambda_{ij}^u \left( \frac{S}{M} \right)^{n_{ij}}, \) can be diagonalized simultaneously with the corresponding mass matrices. Only the singlet scalar \( S \) mediates FCNC. This is rather fortunate: lower bounds on the doublet masses from FCNC may be difficult to satisfy because they may be in conflict with upper bounds from triviality or unitarity. In particular, in a supersymmetric framework, there is in general a strong upper bound on the mass of the lightest neutral scalar \( \star \), of order 150 GeV (see e.g. refs. [19–21]). On the other hand, the mass of a singlet scalar is not related to the electroweak breaking scale and could be easily set to satisfy constraints from FCNC. We will study these constraints in section 8.

Next, we consider the possibility that \( U(1)_X \) of eq. (3.3) is not a symmetry of the full Lagrangian and this leads to Yukawa interactions of the form

\[
\frac{\lambda_{ij}^d}{M^{m_{ij}+2k_{ij}}} Q_i \phi_d (\phi_d \phi_u)^{k_{ij}} S^{m_{ij}} \bar{d}_j + \frac{\lambda_{ij}^u}{M^{n_{ij}+2l_{ij}}} Q_i \phi_u (\phi_d \phi_d)^{l_{ij}} S^{n_{ij}} \bar{u}_j + \text{h.c.}
\] (5.10)

with non-zero \( k_{ij} \) and \( l_{ij} \). This has two important implications:

a. There is an additional small parameter,

\[
\eta^2 = \frac{\langle \phi_d \rangle \langle \phi_u \rangle}{M^2}. \quad (5.11)
\]

Together with \( \epsilon \), a more detailed explanation of the hierarchy in parameters may become possible.

\* We thank T. Banks for stressing the relevance of these bounds to our problem.
b. There are FCNC mediated by the doublet scalar fields (and not only by the singlet). This may lead, as discussed above, to phenomenological problems in some cases.

From (5.10) it can be seen that we can do with no singlets at all, with just $\eta^2$ as our small parameter. This scenario, however, would turn out to lead to considerable difficulties.

Three comments are in order.

(i) Another generalization (which we will study in detail below) is based on adding more singlets. For example, with two singlets, $S_1$ and $S_2$, there are two small parameters (even when $U(1)_X$ holds at low energy), $\epsilon_1$ and $\epsilon_2$. This, again, would allow more structure to the mass matrices though some of the predictivity is lost.

(ii) In general $U(1)_H \times U(1)_X$ is explicitly broken by the scalar potential and the horizontal symmetry of the full Lagrangian is a discrete subgroup $H \subset U(1)_H \times U(1)_X$. It is easy to construct models such that $H$ invariance guarantees that the Yukawa couplings are invariant under $U(1)_H \times U(1)_X$.

(iii) The expressions for the mixing angles (5.7) and for the masses (5.9) imply that, if $H(d_i) - H(d_j) = H(Q_i) - H(Q_j)$, then the corresponding $2 \times 2$ mass matrix has the form

$$M \sim \begin{pmatrix} \epsilon^2 m & \epsilon m \\ \epsilon^2 m & 1 \end{pmatrix} \implies |V_{ij}| \sim \sqrt{\frac{m_i}{m_j}}.$$

(5.12)

It is well known that such a relation holds to within a few percent for the two light generations, $|V_{us}| \approx \sqrt{\frac{m_u}{m_s}}$. We emphasize that while the order of magnitude relation (5.12) is easy to derive in our framework, an actual equality as the experimental values may suggest (see next section) is difficult to obtain. The reason is that the natural way to obtain an equality [3],

$$M = \begin{pmatrix} 0 & a \epsilon^m \\ a \epsilon^m & b \end{pmatrix} \implies |V_{ij}| = \sqrt{\frac{m_i}{m_j}},$$

(5.13)

requires not only $M_{11} = 0$ but also $|M_{12}| = |M_{21}|$. The latter equality is not natural.
without left-right symmetry, but we were unable to find left-right symmetric models consistent with all our requirements.

6. Numerology

In this section we introduce the experimental values of the quark sector parameters and analyze the hierarchy required in the mass matrices to produce these values.

For the mixing angles, we take (see ref. [22] and references therein)

\[ |V_{us}| = 0.2205 \pm 0.0018, \quad |V_{cb}| = 0.040 \pm 0.007, \quad |V_{ub}/V_{cb}| = 0.10 \pm 0.03. \quad (6.1) \]

(The complex KM phase in the mixing matrix is not a small parameter, \( \sin \delta = \mathcal{O}(1) \). We do not discuss CP violation in this work.) Note that the relation (5.8), \( |V_{ub}/V_{cb}| \sim |V_{us}| \) holds to within a factor of 2–3. This is encouraging, because the relation is a very general feature of our framework.

For quark masses at 1 GeV, we take [23]

\[
\begin{align*}
    m_u &= 5.1 \pm 1.5 \text{ MeV}, \quad m_c = 1.35 \pm 0.05 \text{ GeV}, \quad m_t \sim 225 \pm 75 \text{ GeV}, \\
    m_d &= 8.9 \pm 2.6 \text{ MeV}, \quad m_s = 175 \pm 55 \text{ MeV}, \quad m_b = 5.6 \pm 0.4 \text{ GeV},
\end{align*} \quad (6.2)
\]

(we used \( m_t^{\text{phys}} \approx 0.6 m_t(1 \text{ GeV}) \)) leading to the mass ratios:

\[
\begin{align*}
    \frac{m_d}{m_s} &= 0.051 \pm 0.004, \quad \frac{m_s}{m_b} = 0.032 \pm 0.012, \\
    \frac{m_u}{m_c} &= 0.0038 \pm 0.0012, \quad \frac{m_c}{m_t} \sim 0.006^{+0.003}_{-0.002}, \\
    \frac{m_b}{m_t} &\sim 0.025^{+0.015}_{-0.008}.
\end{align*} \quad (6.3)
\]

The largest of the small parameters is the Cabibbo angle, \( |V_{us}| \sim 0.22 \). If we
set our small expansion parameter to equal $|V_{us}|$, then we should roughly aim at

$$
\epsilon \sim |V_{us}|, \\
\epsilon^2 \sim |V_{cb}|, \frac{m_d}{m_s}, \frac{m_s}{m_b}, \frac{m_b}{m_t}, \\
\epsilon^3 \sim |V_{ub}|, \frac{m_u}{m_c}, \frac{m_c}{m_t}.
$$

With this order of magnitude estimate, we have

$$
\det M_d \sim \langle \phi_d \rangle^3 \epsilon^3, \quad \det M_u \sim \langle \phi_u \rangle^3 \epsilon^9
$$

(6.5)

where $\langle \phi_d \rangle$ is of the same order as $\langle \phi_u \rangle$. As we will see below, the very high powers of $\epsilon$ mean that we would need a large number of fields in the high energy theory to produce this hierarchy.

If we allow $m_u = 0$ (a possibility which is still controversial [24 -- 27]), the determinant of the block of massive $u$ quarks is $m_t m_c \sim \langle \phi_u \rangle^2 \epsilon^3$. If we explain the small ratio $m_b/m_t$ dynamically, namely $m_b/m_t \sim \langle \phi_d \rangle / \langle \phi_u \rangle$ as in [28], rather than obtaining the hierarchy from the Yukawa couplings, then $\det M_d \sim \langle \phi_d \rangle^3 \epsilon^6$. In either of these possibilities the required high energy model could have fewer massive fields.

As we will later see, it is difficult to construct a viable low energy model with $\epsilon$ as large as 0.22. We may need two small parameters, $\epsilon_1 \sim 0.04$ and $\epsilon_2 \sim 0.008$:

$$
\epsilon_1 \sim |V_{cb}|, \frac{m_s}{m_b}, \frac{m_b}{m_t}, \\
\epsilon_2 \sim |V_{ub}|, \frac{m_u}{m_c}, \frac{m_c}{m_t}, \\
\frac{\epsilon_2}{\epsilon_1} \sim |V_{us}|, \sqrt{\frac{m_d}{m_s}}.
$$

(6.6)

With this order of magnitude estimate, we have

$$
\det M_d \sim \langle \phi_d \rangle^3 \epsilon_1^3 \epsilon_2^2, \quad \det M_u \sim \langle \phi_u \rangle^3 \epsilon_2^3.
$$

(6.7)

If $m_b/m_t \sim \langle \phi_d \rangle / \langle \phi_u \rangle$ then $\det M_d \sim \langle \phi_d \rangle^3 \epsilon_2^2$. If $m_u = 0$ then $m_t m_c \sim \langle \phi_u \rangle^2 \epsilon_2$. If $m_b/m_t \sim \langle \phi_u \rangle$ then $\det M_d \sim \langle \phi_u \rangle^3 \epsilon_2^2$.
Finally, we mention the possibility that there is only one small parameter, \( \epsilon \sim 0.04 \) and a much cruder estimate of the various parameters can be achieved:

\[
|V_{us}| \sim 1 \text{ or } \epsilon, \\
|V_{cb}|, \frac{m_d}{m_s}, \frac{m_b}{m_t} \sim \epsilon, \\
|V_{ub}|, \frac{m_u}{m_c}, \frac{m_c}{m_t} \sim \epsilon \text{ or } \epsilon^2.
\] (6.8)

7. Explicit Examples

We now turn to some explicit examples of the general ideas discussed above. We first construct a low energy model with two small parameters \( \epsilon_1 \) and \( \epsilon_2 \). We assume that they arise from the VEV’s of two singlet fields \( S_1 \) and \( S_2 \):

\[
\epsilon_1 = \frac{\langle S_1 \rangle}{M} \sim 0.04, \quad \epsilon_2 = \frac{\langle S_2 \rangle}{M} \sim 0.008.
\] (7.1)

These small numbers are not extremely small and are perfectly natural. As we will see, we absorb a typical Yukawa coupling into the definition of \( \langle S_i \rangle \). This makes \( \epsilon \) smaller than the ratio of scales. Furthermore, the ratio between the two VEV’s,

\[
\frac{\epsilon_2}{\epsilon_1} \sim 0.2,
\] (7.2)

is a number of order one. It can arise dynamically by minimizing the scalar potential. We do not address this issue here.

Let us consider the following set of matrices, where the various entries give the corresponding orders of magnitude (and not exact numbers):

\[
M_u = \langle \phi_u \rangle \begin{pmatrix}
\epsilon_2^2 & 0 & \epsilon_2 \\
\epsilon_1 \epsilon_2 & \epsilon_2 & \epsilon_1 \\
\epsilon_2 & 0 & 1
\end{pmatrix}, \quad M_d = \langle \phi_d \rangle \begin{pmatrix}
\epsilon_2^2 & \epsilon_1 \epsilon_2 & \epsilon_1 \epsilon_2 \\
\epsilon_1 \epsilon_2 & \epsilon_1^2 & \epsilon_1^2 \\
\epsilon_2 & \epsilon_1 & \epsilon_1
\end{pmatrix}.
\] (7.3)

For \( \langle \phi_u \rangle \sim \langle \phi_d \rangle \) they lead to exactly the required relations of eq. (6.6).
Such mass matrices can arise from non-renormalizable terms in the low energy Lagrangian with two fermions and a number of scalars. We take the horizontal symmetry to be $U(1)_{H_1} \times U(1)_{H_2}$ and also impose the $U(1)_X$ symmetry on the Yukawa couplings. We actually have in mind an anomaly free discrete subgroup of $U(1)_X \times U(1)_{H_1} \times U(1)_{H_2}$. For example, we can consider the symmetry $Z_3 \times Z_3 \times Z_5 \subset U(1)_X \times U(1)_{H_1} \times U(1)_{H_2}$. Then the models presented below are free of QCD anomalies. Anomalies with the $SU(2)$ gauge group cannot be discussed without assigning charges to the leptons and without deciding whether the horizontal symmetry is an R-symmetry or not. All the results concerning the quark mass matrices are unchanged, so we use the simpler presentation with $U(1)$ charges.

Using $U(1)_Y \times U(1)_X$ we can set the horizontal charges of the scalars to be

\begin{align*}
H_1(\phi_u) = H_1(\phi_d) = H_1(S_2) = 0, \quad H_1(S_1) = -1, \\
H_2(\phi_u) = H_2(\phi_d) = H_2(S_1) = 0, \quad H_2(S_2) = -1.
\end{align*}

(7.4)

Using the baryon number symmetry $U(1)_B$ we can set $H_1(Q_3) = H_2(Q_3) = 0$. Then the mass matrices (7.3) are consistent with the $H_1$ and $H_2$ charges for quarks:

| field | $H_1$ | $H_2$ |
|-------|-------|-------|
| $Q_1$ | 0     | 1     |
| $Q_2$ | 1     | 0     |
| $Q_3$ | 0     | 0     |
| $\bar{u}_1$ | 0     | 1     |
| $\bar{u}_2$ | -1    | 1     |
| $\bar{u}_3$ | 0     | 0     |
| $\bar{d}_1$ | 0     | 1     |
| $\bar{d}_2$ | 1     | 0     |
| $\bar{d}_3$ | 1     | 0     |

(7.5)

We can replace the two fields $S_1$ and $S_2$ with a single scalar $S$ such that $\epsilon = \frac{(S)}{M} \sim 0.2$ satisfies $\epsilon^3 = \epsilon_2$ and $\epsilon^2 = \epsilon_1$ (thus explaining $\epsilon_2 = \epsilon_1$). The
$U(1)_{H_1} \times U(1)_{H_2}$ symmetry is replaced then by a single $U(1)_H$ generated by $H = 2H_1 + 3H_2$, with $H(S) = -1$. This allows also non-zero couplings at the $(1,2)$ and $(3,2)$ entries of $M_u$. Clearly, in such a scheme there is no need to explain the ratio $\frac{e_2}{e_1} = 0.2$.

We now turn to the high energy theory. As in the mechanism of [1], we would like to introduce massive fields such that when they are integrated out we derive the non-renormalizable terms needed in the low energy theory. There are many ways to do that. For example we can add $SU(2)$ singlet color triplet fields $U_i$ with charge +2/3 and $D_i$ with charge −1/3:

| field | $H_1$ | $H_2$ |
|-------|-------|-------|
| $U_1$ | 1     | 0     |
| $U_2$ | 0     | 0     |
| $U_3$ | 0     | 1     |
| $D_1$ | 0     | 0     |
| $D_2$ | 0     | 0     |
| $D_3$ | 1     | 0     |
| $D_4$ | 0     | 1     |
| $D_5$ | −1    | 1     |

(7.6)

and fields with conjugate horizontal and gauge quantum numbers $\bar{U}_i$ and $\bar{D}_i$.

For the up sector we need a $6 \times 6$ matrix whose rows correspond to \{Q$_1$, Q$_2$, Q$_3$, U$_1$, U$_2$, U$_3$\} and its columns to \{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{U}_1, \bar{U}_2, \bar{U}_3\}. For the down sector we need an $8 \times 8$ matrix whose rows correspond to \{Q$_1$, Q$_2$, Q$_3$, D$_1$, D$_2$, D$_3$, D$_4$, D$_5$\} and columns to \{\bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{D}_1, \bar{D}_2, \bar{D}_3, \bar{D}_4, \bar{D}_5\}. When there are two or more fields with the same quantum numbers, we use the freedom to mix them to set some of the explicit mass terms in the matrices below to zero. Using the quantum numbers
we find the orders of magnitude for the various entries ($M$ is the high scale):

\[
M_{\text{full}}^u = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \langle \phi_u \rangle \\
0 & 0 & 0 & \langle \phi_u \rangle & 0 & 0 \\
0 & 0 & \langle \phi_u \rangle & 0 & \langle \phi_u \rangle & 0 \\
0 & \langle S_2 \rangle & \langle S_1 \rangle & M & \langle S_1 \rangle & 0 \\
\langle S_2 \rangle & 0 & 0 & 0 & M & 0 \\
0 & 0 & \langle S_2 \rangle & 0 & \langle S_2 \rangle & M
\end{pmatrix}, \quad (7.7)
\]

\[
M_{\text{full}}^d = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \langle \phi_d \rangle & 0 \\
0 & 0 & 0 & 0 & 0 & \langle \phi_d \rangle & 0 \\
0 & 0 & 0 & \langle \phi_d \rangle & \langle \phi_d \rangle & 0 & 0 \\
\langle S_2 \rangle & \langle S_1 \rangle & \langle S_1 \rangle & M & 0 & 0 & 0 \\
\langle S_2 \rangle & \langle S_1 \rangle & \langle S_1 \rangle & 0 & M & 0 & 0 \\
0 & 0 & 0 & \langle S_1 \rangle & \langle S_1 \rangle & M & 0 \\
0 & 0 & 0 & \langle S_2 \rangle & \langle S_2 \rangle & 0 & M & \langle S_1 \rangle \\
0 & \langle S_2 \rangle & \langle S_2 \rangle & 0 & 0 & 0 & M
\end{pmatrix}. \quad (7.8)
\]

(The two fields $D_1$ and $D_2$ with identical quantum numbers are needed to ensure $\det M_{\text{full}}^d \neq 0$.) In finding eqs. (7.7) and (7.8) we used the fact that the theory is supersymmetric and there are no terms proportional to $S_i^\dagger$. After integrating out the massive fields we find for the light three quark generations precisely the mass matrices (7.3).

Although we can have different sets of fields in the high energy theory leading to an acceptable mass matrix for the light fields, a simple argument shows that we need at least three $U$ and at least five $D$ quarks (we can also replace a pair of $SU(2)$ singlets $U$ and $D$ with a single colored doublet). The mass matrices (7.3) have

\[
\det M_u = \langle \phi_u \rangle^3 \epsilon_2^3 = \langle \phi_u \rangle^3 \langle S_2 \rangle^3 M^{-3},
\]

\[
\det M_d = \langle \phi_d \rangle^3 \epsilon_1^3 \epsilon_2 = \langle \phi_d \rangle^3 \langle S_1 \rangle^3 \langle S_2 \rangle^2 M^{-5}. \quad (7.9)
\]

Since the determinants of $M_{\text{full}}^u$ and $M_{\text{full}}^d$ are polynomials in $\langle \phi_q \rangle$, $\langle S_i \rangle$ and $M$ and they differ from the determinants of $M_u$ and $M_d$ by powers of $M$, we need at least
three $U$ quarks and at least five $D$ quarks. This argument is very general. It is independent of the gauge and horizontal quantum numbers of the massive fermions. It also counts correctly different fermions with identical quantum numbers which are needed to make the determinant non-zero.

If $m_u = 0$, we can set some of the entries in $M_u$ to zero. The determinant of the block of massive light particles is then $\langle \phi_u \rangle^2 \epsilon_2$. By the previous argument such a matrix can be obtained with only one $U$ quark in the high energy theory. An example of such a theory is the previously discussed theory but without $U_2$ and $U_3$.

If the smallness of $m_b/m_t$ arises from a small ratio of VEV’s $\langle \phi_d \rangle / \langle \phi_u \rangle$, we can have $m_b \sim \langle \phi_d \rangle$ without a small $\epsilon$ parameter. In this case $\det M_d = \langle \phi_d \rangle^3 \epsilon_2^2$. Such a matrix can be obtained with only two $D$ quarks in the massive fermion sector. An example of such a model is derived by modifying the $H_1$ and $H_2$ charges of the singlet down quarks of the model above to

| field  | $H_1$ | $H_2$ |
|--------|-------|-------|
| $\bar{d}_1$ | $-1$ | $1$  |
| $\bar{d}_2$ | $0$  | $0$  |
| $\bar{d}_3$ | $0$  | $0$  |

and by dropping the massive fermions $D_1$, $D_2$ and $D_5$. After integrating out the two heavy fields we find

$$M_d = \langle \phi_d \rangle \begin{pmatrix} 0 & \epsilon_2 & \epsilon_2 \\ \epsilon_2 & \epsilon_1 & \epsilon_1 \\ 0 & 1 & 1 \end{pmatrix}. \quad (7.11)$$
8. The Scales in the Problem

Within our theoretical framework, there are three energy scales: the electro-
weak breaking scale \( \sim \langle \phi_u \rangle \), the horizontal symmetry breaking scale \( \sim \Lambda_H \sim \langle S \rangle \),
and the mass scale of the extra massive fermions \( \sim M \). In the explicit models
presented in the previous section, neither \( \langle S \rangle \) nor \( M \) are fixed, but their ratio
\( \langle S \rangle / M \) is constrained to be of order 0.04. It is then possible that the new physics
related to the scales \( \langle S \rangle \) and \( M \) may be at energies accessible to future experiments.

On the other hand, various constraints imply that the scales \( M \) and \( \langle S \rangle \) cannot
be arbitrarily low. At the scale \( M \), new colored multiplets appear which affect the
running of \( \alpha_s \). Requiring that no Landau pole appears up to some high energy scale
gives a lower bound on \( M \). At the scale \( \langle S \rangle \) there are scalars which mediate flavor
changing neutral currents. Requiring that these contributions would not exceed
the experimental values gives a lower bound on \( \langle S \rangle \). In this section we study these
bounds.

The general formula for the running of \( \alpha_s \) in our framework is

\[
[\alpha_s(M_P)]^{-1} = [\alpha_s(m_Z)]^{-1} + \frac{7}{2\pi} \ln \frac{M_{\text{SUSY}}}{m_Z} + \frac{3}{2\pi} \ln \frac{M}{M_{\text{SUSY}}} + \frac{3 - N_T}{2\pi} \ln \frac{M_P}{M} \tag{8.1}
\]

where \( M_{\text{SUSY}} \) is the scale of the various sparticles and \( N_T \) is the number of pairs
of extra color triplets and color anti-triplets. The requirement that there is no
Landau pole below the Planck mass \( M_P \) can be translated into a lower bound on
\( M \) of the form

\[
M_{\text{min}} = M_P \exp(-Z/N_T), \tag{8.2}
\]

where \( Z \) depends on \( M_{\text{SUSY}} \):

\[
Z = \begin{cases} 
174 & (i) \ M_{\text{SUSY}} = m_Z, \\
183 & (ii) \ M_{\text{SUSY}} = 10m_Z. 
\end{cases} \tag{8.3}
\]
This leads to

\[
\begin{array}{cccc}
N_T & M_{\min}^{(i)}[TeV] & M_{\min}^{(ii)}[TeV] \\
5  & 7               & 1.3           \\
6  & 3 \cdot 10^3   & 6 \cdot 10^2  \\
7  & 2 \cdot 10^5   & 4 \cdot 10^4  \\
8  & 3 \cdot 10^6   & 1 \cdot 10^6  \\
9  & 4 \cdot 10^7   & 1 \cdot 10^7  \\
\end{array}
\]

(8.4)

We conclude that the new physics related to the scale \( M \) may be directly accessible to future experiments if \( N_T \leq 5 \). If \( \langle S \rangle \sim 0.04M \), then the new physics related to the scale \( \langle S \rangle \) may be directly accessible if \( N_T \leq 5 \) and could affect rare processes if \( N_T \leq 7 \). Clearly, if the gauge group changes between \( M \) and \( M_P \) (e.g. \( SU(3) \) is embedded in \( SU(4) \)) these bounds can be significantly weaker.

In our theoretical framework, there are necessarily scalars that mediate FCNC at tree level. It is often thought that the constraints on such scalars force them to be heavier than a thousand \( TeV \). This conclusion assumes that the relevant Yukawa couplings of these scalars are of order one. Clearly, if the couplings are smaller, the bounds are weaker. If the couplings of these additional scalars are similar in their magnitude to those of the Higgs which leads to masses, they must be small [29 – 32]. In particular, it was pointed out in ref. [31] that if the couplings of these additional scalars are of the Fritzsch type they can be as light as a \( TeV \). A more complete analysis related to flavor symmetries was given in ref. [13]. There it is pointed out that the additional scalars can be even lighter than a \( TeV \).

The most stringent bounds in our models arise from mixing of neutral mesons. For all models discussed in the previous section, only singlet fields \( S_a \) contribute at tree level to meson mixing. The effective nondiagonal Yukawa couplings of these fields are typically of order

\[
\lambda_i^a \lambda_j^a \sim \frac{m_i m_j}{\langle S_a \rangle^2}.
\]

(8.5)

The mass of the field \( S_a \) is also at the scale \( \langle S_a \rangle \). Then the bounds of eq. (3.1)
can be translated into lower bounds on the scale $\langle S \rangle$ of the horizontal symmetry breaking:

$$\langle S \rangle \gtrsim \begin{cases} 
0.4 \text{ TeV} & (K - \bar{K} \text{ mixing}), \\
0.2 \text{ TeV} & (D - \bar{D} \text{ mixing}), \\
0.4 \text{ TeV} & (B - \bar{B} \text{ mixing}). 
\end{cases} \quad (8.6)$$

Note that the bounds are very weak because the singlet mass increases with $\langle S \rangle$ while its Yukawa couplings decrease with $\langle S \rangle$.

Additional bounds can be derived from rare decays such as $B \to X_{\mu^+\mu^-}$, but they depend on details of the leptonic sector, and are typically weaker than (8.6).

We mentioned the possibility that the $U(1)_X$ symmetry is broken, leading to non-renormalizable terms of the form (5.10). In this case, the situation with FCNC is entirely different as the scalar doublets contribute as well. The nondiagonal Yukawa couplings of the doublets in this class of models are typically of order

$$\lambda_{ij}^q \lambda_{ji}^q \sim \frac{m_i m_j}{\langle \phi_q \rangle^2}. \quad (8.7)$$

Taking $\langle \phi_q \rangle \sim 0.2 \text{ TeV}$, we find from eq. (3.1):

$$M(\phi_q) \gtrsim \begin{cases} 
0.8 \text{ TeV} & (K - \bar{K} \text{ mixing}), \\
0.3 \text{ TeV} & (D - \bar{D} \text{ mixing}), \\
0.7 \text{ TeV} & (B - \bar{B} \text{ mixing}). 
\end{cases} \quad (8.8)$$

The strongest bound, coming from $K - \bar{K}$ mixing, is in conflict with the unitarity bound on the Higgs mass of 0.75 TeV (for a review see ref. [19]). Strictly speaking, this bound applies to the single Higgs of the minimal standard model, but more generally it applies approximately to the doublet scalar that carries the VEV. It is even more difficult to accommodate (8.8) in a supersymmetric framework. There the upper bound on the mass of the lightest neutral scalar is of order 0.15 TeV. Again, strictly speaking the bound applies only in the minimal supersymmetric standard model, but if we require that perturbativity holds to high energy scales, this bound becomes very general [20, 21].
A second problem in models with no $U(1)_X$ symmetry arises if some of the quark sector parameters depend on the ratio defined in eq. (5.11), \( \eta^2 = \frac{(\langle \phi_d \rangle)(\langle \phi_u \rangle)}{M^2} \).

The experimental value of the parameters will then set the scale $M$. For example, if $\eta^2$ takes the role of our small parameter $\epsilon_2$, namely $\eta^2 \sim 0.008$, then $M \lesssim 2 \, \text{TeV}$ which may be in conflict with the bounds from Landau poles, or force the scale $\langle S \rangle$ to be too low for FCNC bounds. Furthermore, the light fermions would have exotic components of order $\langle \phi \rangle / M \sim 0.1$ which is inconsistent with various electroweak precision measurements [33]. If $\eta^2$ just explains the first generation masses, namely $\eta^2 \sim \frac{m_d}{m_u}$ or $\frac{m_b}{m_t}$, then it could be useful in explaining these parameters without forcing $M$ to be too small.

A third problem arises in models of broken $U(1)_X$ where the hierarchy $m_b \ll m_t$ results from $\langle \phi_d \rangle \ll \langle \phi_u \rangle$. If, for example, $\langle \phi_d \rangle \sim 0.1 \langle \phi_u \rangle$, then the bounds on $M(\phi_d)$ are about ten times stronger than those in (8.8). This is not necessarily in conflict with the stringent supersymmetric upper bound mentioned above, because in this case the lightest neutral scalar is dominantly $\phi_u$. But in supersymmetric models it is in conflict with perturbativity, while in non-supersymmetric models, it sets a lower bound on the scale $\langle S \rangle$ beyond the direct reach of future experiments.

We should mention that even in models which are $U(1)_X$-symmetric, the light scalars may mediate FCNC because they have small components of the singlet fields. However, as these components are of order $\langle \phi \rangle^2 / \langle S \rangle^2$, the contributions to neutral meson mixing are usually smaller than those from the heavy scalars which are dominantly singlets [32].

To summarize, models with $U(1)_X$ symmetry and a small number of massive colored supermultiplets allow for rich phenomenology at energies accessible to future experiments - a $\text{TeV}$ or even lower. Models with $U(1)_X$ symmetry and a large number of massive colored supermultiplets can explain all the details of the observed hierarchy in the quark sector parameters, but have no directly accessible phenomenology (similarly to the original Froggatt-Nielsen models [1]). Supersymmetric models with no $U(1)_X$ symmetry are probably not viable.
9. Conclusions

The new data on $m_t$, $V_{cb}$ and $V_{ub}$ motivated us to reexamine the old problem of the quark mass matrix. Perhaps the most puzzling aspect of the quark mass matrix is the large hierarchy between the entries. Therefore, as a first approximation we are not interested in explaining the precise values of the parameters in the mass matrix but focus on the order of magnitudes.

We parametrize the mass matrices as

$$
M_u = \langle \phi \rangle \begin{pmatrix}
\epsilon_1^2 & 0 & \epsilon_2 \\
\epsilon_1 \epsilon_2 & \epsilon_2 & \epsilon_1 \\
\epsilon_2 & 0 & 1
\end{pmatrix}, \\
M_d = \langle \phi \rangle \begin{pmatrix}
\epsilon_1^2 & \epsilon_1 \epsilon_2 & \epsilon_1 \epsilon_2 \\
\epsilon_1 \epsilon_2 & \epsilon_1^2 & \epsilon_1^2 \\
\epsilon_2 & \epsilon_1 & \epsilon_1
\end{pmatrix},
$$

leading to

$$
\epsilon_1 \sim |V_{cb}|, \frac{m_s}{m_b}, \frac{m_b}{m_t} \\
\epsilon_2 \sim |V_{ub}|, \frac{m_c}{m_t}, \frac{m_t}{m_c} \\
\frac{\epsilon_2}{\epsilon_1} \sim |V_{us}|, \sqrt{\frac{m_d}{m_s}}.
$$

Experimentally, these relations are satisfied quite well with $\epsilon_1 \sim 0.04$ and $\epsilon_2 \sim 0.008$. These are small but not extremely small numbers. The large hierarchy occurs by raising these numbers to large powers.

Following [1] and [5] we attempt to explain the hierarchy in (9.1) in terms of tree level exchanges of massive particles with mass of order $M$ and the small parameters $\epsilon_1 = \frac{\langle S_1 \rangle}{M}$ and $\epsilon_2 = \frac{\langle S_2 \rangle}{M}$ are related to the expectation values of two massive scalars (we absorb a Yukawa coupling of $S$ to massive fields into its VEV). This is most easily achieved if we assume that $m_u = 0$ and there are two Higgs fields with $\langle \phi_d \rangle \sim \epsilon_1 \langle \phi_u \rangle$. Then the matrices in (9.1) can be replaced with

$$
M_u = \langle \phi_u \rangle \begin{pmatrix}
0 & 0 & 0 \\
0 & \epsilon_2 & \epsilon_1 \\
0 & 0 & 1
\end{pmatrix}, \\
M_d = \langle \phi_d \rangle \begin{pmatrix}
0 & \epsilon_2 & \epsilon_2 \\
\epsilon_2 & \epsilon_1 & \epsilon_1 \\
0 & 1 & 1
\end{pmatrix}.
$$

A high energy theory containing a single $U$-like quark and a couple of $D$-like quarks
can lead to (9.3).

A more detailed description with a non-zero \(m_u\) and an explanation of the hierarchy between \(\langle \phi_u \rangle\) and \(\langle \phi_d \rangle\) needs two more \(U\)-like quarks and three more \(D\)-like quarks in the high energy theory. Such a theory reproduces all the relations in (9.2).

The parametrization (9.1) is not unique. Other parametrizations are possible and they lead one to consider different high energy theories. Our models should therefore be viewed merely as an existence proof to the approach presented here. In constructing a complete model one needs to find the discrete horizontal symmetry (rather than pretend that it is continuous) and make sure that it does not suffer from anomalies.

Previous attempts to explain the hierarchy in the quark mass matrices in terms of tree level exchanges of massive fields [1, 5] assumed that the relevant flavor dynamics occurs at super high energies. Both \(\langle S_i \rangle\) and \(M\) are very large \((10^{10} - 10^{19}\text{GeV})\) with a small (but not extremely small) ratio between them \(\epsilon_i\). Unlike these authors, we suggest that the entire flavor dynamics can take place at experimentally accessible energies in the TeV range.

It is amusing to note that the set of particles that we need to add to the standard model are common in string inspired models. There we typically have a number of generations and anti-generations containing pairs of mirror fields with the quantum numbers of the standard quarks and leptons. Furthermore, the \(E_6\) generations contain also pairs of mirror singlet \(D\) quarks and pairs of mirror lepton doublets as well as fields like our \(S\). Also, string models often have large discrete symmetries which could be used as horizontal symmetries. It is fortunate that unlike grand unified theories these discrete symmetries can act differently on different fields in the same generation (in fact it is not clear how to group the fields into generations). The particle content of a string inspired model with four generations and a single anti-generation is large enough to produce all the hierarchies we need except that \(m_u = 0\). Although with more generations it is possible to induce a
non-zero mass for the up quark we would like to remind the reader that $m_u = 0$ leads to a possible solution to the strong CP problem and may also be consistent [24-26].

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**Appendix: Lepton Masses**

The observed hierarchy in the charged lepton masses can be explained by a similar mechanism to the quark sector parameters. The experimental values of the masses are

\[ m_e = 0.51 \text{ MeV}, \quad m_\mu = 105.7 \text{ MeV}, \quad m_\tau = 1777 \text{ MeV}, \]  

(A.1)

leading to

\[ \frac{m_e}{m_\mu} = 0.005, \quad \frac{m_\mu}{m_\tau} = 0.06, \quad \frac{m_\tau}{m_t} = 0.008^{+0.004}_{-0.002}. \]  

(A.2)

(In this work we take the neutrinos to be massless.)

With the two small parameters defined in section 6, $\epsilon_1 \sim 0.04$ and $\epsilon_2 \sim 0.008$, we should aim at

\[ \epsilon_1 \sim \frac{m_\mu}{m_\tau}, \quad \epsilon_2 \sim \frac{m_e}{m_\mu}, \quad \frac{m_\tau}{m_t}. \]  

(A.3)

We should point out, however, that other possibilities exist. With this order of
magnitude estimate, we have

$$\det M_\ell \sim \langle \phi_d \rangle ^3 \epsilon_1^2 \epsilon_2^4,$$

(A.4)

requiring at least six heavy charged leptons.

We denote leptons doublets by $L_i$ and charged lepton singlets by $\ell_i^+$ and $E_i^\pm$ for light and heavy fields, respectively. We use the same set of scalar fields as in eq. (7.4). To produce the mass ratios (A.3), we take

$$\begin{array}{ccccccc}
\text{field} & H_1 & H_2 \\
L_1 & 1 & 1 \\
L_2 & 1 & 0 \\
L_3 & 0 & 0 \\
\ell_{1,2,3}^+ & 0 & 1 \\
E_{1,2,3}^- & 0 & 0 \\
E_{4,5}^- & 1 & 0 \\
E_6^- & 1 & 1
\end{array}$$

(A.5)

and $E_i^+$ fields with conjugate horizontal and gauge numbers to those of $E_i^-$. In the full theory we have a $9 \times 9$ mass matrix whose rows correspond to

$\{L_1, L_2, L_3, E_1^-, E_2^-, E_3^-, E_4^-, E_5^-, E_6^-\}$ and its columns to $\{\ell_1^+, \ell_2^+, \ell_3^+, E_1^+, E_2^+, E_3^+, E_4^+, E_5^+, E_6^+\}$. Using the quantum numbers in (A.5) we find the orders of magnitude for the various entries:

$$M_\ell^{\text{full}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \langle \phi_d \rangle \\
0 & 0 & 0 & 0 & 0 & 0 & \langle \phi_d \rangle & \langle \phi_d \rangle & 0 \\
0 & 0 & 0 & \langle \phi_d \rangle & \langle \phi_d \rangle & \langle \phi_d \rangle & 0 & 0 & 0 \\
\langle S_2 \rangle & \langle S_2 \rangle & \langle S_2 \rangle & M & 0 & 0 & 0 & 0 & 0 \\
\langle S_2 \rangle & \langle S_2 \rangle & \langle S_2 \rangle & 0 & M & 0 & 0 & 0 & 0 \\
\langle S_2 \rangle & \langle S_2 \rangle & \langle S_2 \rangle & 0 & 0 & M & 0 & 0 & 0 \\
0 & 0 & 0 & \langle S_1 \rangle & \langle S_1 \rangle & \langle S_1 \rangle & M & 0 & 0 \\
0 & 0 & 0 & \langle S_1 \rangle & \langle S_1 \rangle & \langle S_1 \rangle & 0 & M & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \langle S_2 \rangle & \langle S_2 \rangle & M
\end{pmatrix}.$$ 

(A.6)
After integrating out the heavy fields, we get for the three light generations

\[
M_\ell = \langle \phi_d \rangle \begin{pmatrix}
\epsilon_2^2 \epsilon_1 & \epsilon_2^2 \epsilon_1 & \epsilon_2^2 \epsilon_1 \\
\epsilon_2 \epsilon_1 & \epsilon_2 \epsilon_1 & \epsilon_2 \epsilon_1 \\
1 & 1 & 1
\end{pmatrix},
\]

(A.7)

leading to the mass ratios of (A.3). Clearly, some of the six massive \( E \)'s can be replaced with massive \( SU(2) \) doublets with appropriate horizontal charges without affecting the low energy hierarchy.

If we explain the small ratio \( m_\tau/m_t \) dynamically, namely \( m_\tau/m_t \sim \langle \phi_d \rangle / \langle \phi_u \rangle \), then \( \det M_\ell \sim \langle \phi_d \rangle^3 \epsilon_1^2 \epsilon_2 \). This can be derived from a model with three heavy charged leptons only. An example of such a model is obtained by modifying the \( H_2 \) charges of the \( \ell_i^+ \) fields of the model above to

\[
H_2(\ell_i^+) = 0,
\]

(A.8)

and by dropping the massive leptons \( E_1, E_2 \) and \( E_3 \). After integrating out the heavy fields we find

\[
M_\ell = \langle \phi_d \rangle \begin{pmatrix}
\epsilon_2 \epsilon_1 & \epsilon_2 \epsilon_1 & \epsilon_2 \epsilon_1 \\
1 & 1 & 1
\end{pmatrix}.
\]

(A.9)

Finally, let us mention that the bounds on the masses of scalars with non-diagonal couplings to charged lepton mass eigenstates are

\[
M(\phi_i) \gtrsim \begin{cases}
100 \text{ TeV} \sqrt{\lambda_{e\mu}^i \lambda_{ee}^i} & (\mu \rightarrow eee), \\
1 \text{ TeV} \sqrt{\lambda_{r\ell}^i \lambda_{\ell\ell}^i} & (\tau \rightarrow \ell\ell\ell). 
\end{cases}
\]

(A.10)
REFERENCES

1. C.D. Froggatt and H.B. Nielsen, *Nucl. Phys.* B147 (1979) 277.

2. A. De Rujula, H. Georgi and S.L. Glashow, *Ann. Phys.* 109 (1977) 258.

3. H. Fritzsch, *Phys. Lett.* 70B (1977) 437.

4. S. Dimopoulos and C. Jarlskog, *Phys. Lett.* 86B (1979) 297.

5. S. Dimopoulos, *Phys. Lett.* B129B (1983) 417; J. Bagger, S. Dimopoulos, E. Masso and M. Reno, *Nucl. Phys.* B258 (1985) 565; J. Bagger, S. Dimopoulos, H. Georgi and S. Raby, In: *Proc. Fifth Workshop on Grand Unification*. Eds. Kang, K., Fried, H. and Frampton, P., Singapore, World Scientific (1984).

6. R. Cahn and H. Harari, *Nucl. Phys.* B176 (1980) 135.

7. B.R. Greene, K.H. Kirklin, P.J. Miron and G.G. Ross, *Nucl. Phys.* B292 (1987) 606.

8. A. Davidson, S. Ranfone and K.C. Wali, *Phys. Rev.* D41 (1990) 208.

9. S. Dimopoulos, L.J. Hall and S. Raby, *Phys. Rev. Lett.* 68 (1992) 1984; *Phys. Rev.* D45 (1992) 4192.

10. B.S. Balakrishna and R.N. Mohapatra, *Phys. Lett.* B216 (1989) 349.

11. D. Kaplan, *Nucl. Phys.* B365 (1991) 259.

12. Z.G. Berezhiani and R. Rattazi, LBL-32889 (1992).

13. A. Antaramian, L.J. Hall and A. Rasin, *Phys. Rev. Lett.* 69 (1992) 1871.

14. S.L. Glashow and S. Weinberg, *Phys. Rev.* D15 (1977) 1958.

15. R. Gatto, G. Morchio and F. Strocchi, *Phys. Lett.* 83B (1979) 348.

16. R. Gatto, G. Morchio, G. Sartori and F. Strocchi, *Nucl. Phys.* B163 (1980) 221.

17. G. Segre and H.A. Weldon, *Phys. Lett.* 86B (1979) 291; *Ann. Phys.* 124 (1980) 37.
18. M.Y. Wang and E.D. Carlson, HUTP-92/A062 (1992).

19. J.F. Gunion, H.E. Haber, G. Kane and S. Dawson, *The Higgs Hunter’s Guide*, (Addison-Wesley, 1990).

20. H.E. Haber and M. Sher, *Phys. Rev.* D35 (1987) 2206.

21. G.L. Kane, C. Kolda and J.D. Wells, UM-TH-92-24 (1992).

22. Y. Nir and U. Sarid, WIS-92/52/Jun-PH (1992), *Phys. Rev.* D, in press.

23. J. Gasser and H. Leutwyler, *Phys. Rep.* 87 (1982) 77.

24. D. Kaplan and A. Manohar, *Phys. Rev. Lett.* 56 (1986) 2004.

25. J. Donoghue and D. Wyler, *Phys. Rev.* D45 (1992) 892.

26. K. Choi, *Nucl. Phys.* B383 (1992) 58.

27. H. Leutwyler, *Nucl. Phys.* B337 (1990) 108.

28. T. Banks, *Nucl. Phys.* B303 (1988) 172.

29. B. McWilliams and L-F. Li, *Nucl. Phys.* B179 (1981) 62.

30. O. Shanker, *Nucl. Phys.* B206 (1982) 253.

31. T.P. Cheng and M. Sher, *Phys. Rev.* D35 (1987) 3484.

32. H.E. Haber and Y. Nir, *Nucl. Phys.* B335 (1990) 363.

33. E. Nardi, E. Roulet and D. Tommasini, *Phys. Rev.* D46 (1992) 3040.