ATIYAH’S $L^2$-INDEX THEOREM

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1. Introduction

The $L^2$-Index Theorem of Atiyah [1] expresses the index of an elliptic operator on a closed manifold $M$ in terms of the $G$-equivariant index of some regular covering $\tilde{M}$ of $M$, with $G$ the group of covering transformations. Atiyah’s proof is analytic in nature. Our proof is algebraic and involves an embedding of a given group into an acyclic one, together with naturality properties of the indices.

2. Review of the $L^2$-Index Theorem

The main reference for this section is Atiyah’s paper [1]. All manifolds considered are smooth Riemannian, without boundary. Covering spaces of manifolds carry the induced smooth and Riemannian structure. Let $M$ be a closed manifold and let $E$, $F$ denote two complex (Hermitian) vector bundles over $M$. Consider an elliptic pseudo-differential operator

$$D : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

acting on the smooth sections of the vector bundles. One defines its space of solutions

$$S_D = \{ s \in C^\infty(M, E) \mid Ds = 0 \}.$$

The complex vector space $S_D$ has finite dimension (see [13]), and so has $S_{D^*}$ the space of solutions of the adjoint $D^*$ of $D$ where

$$D^* : C^\infty(M, F) \rightarrow C^\infty(M, E)$$

is the unique continuous linear map satisfying

$$\langle Ds, s' \rangle = \int_M \langle Ds(m), s'(m) \rangle_E \, dm = \langle s, D^*s' \rangle = \int_M \langle s(m), D^*s'(m) \rangle_E \, dm$$

for all $s \in C^\infty(M, E)$, $s' \in C^\infty(M, F)$. One now defines the index of $D$ as follows:

$$\text{Index}(D) = \dim_C(S_D) - \dim_C(S_{D^*}) \in \mathbb{Z}.$$
An explicit formula for $\text{Index}(D)$ is given by the famous Atiyah-Singer Theorem (cf. [2]). Consider a not necessarily connected, regular covering $\pi: \tilde{M} \to M$ with countable covering transformation group $G$. The projection $\pi$ can be used to define an elliptic operator

$$\tilde{D} := \pi^*(D) : C^\infty_c(\tilde{M}, \pi^*E) \to C^\infty_c(\tilde{M}, \pi^*F).$$

Denote by $S_{\tilde{D}}$ the closure of \{ $s \in C^\infty_c(\tilde{M}, \pi^*E)$ | $\tilde{D}s = 0$ \} in $L^2(\tilde{M}, \pi^*E)$. Let $\tilde{D}^*$ denote the adjoint of $\tilde{D}$. The space $S_{\tilde{D}}$ is not necessarily finite dimensional, but being a closed $G$-invariant subspace of the $L^2$-completion $L^2(\tilde{M}, \pi^*E)$ of the space of smooth sections with compact supports $C^\infty_c(\tilde{M}, \pi^*E)$, its von Neumann dimension is therefore defined as follows. Write

$$\mathcal{N}(G) = \{ P : \ell^2(G) \to \ell^2(G) \text{ bounded and } G\text{-invariant} \}$$

for the group von Neumann algebra of $G$, where $G$ acts on $\ell^2(G)$ via the right regular representation. Then $S_{\tilde{D}}$ is a finitely generated Hilbert $G$-module and hence can be represented by an idempotent matrix $P = (p_{ij}) \in M_n(\mathcal{N}(G))$ (recall that a finitely generated Hilbert $G$-module is isometrically $G$-isomorphic to a Hilbert $G$-subspace of the Hilbert space $\ell^2(G)^n$ for some $n \geq 1$, see [9]). One then sets

$$\dim_G(S_{\tilde{D}}) = \sum_{i=1}^n \langle p_{ii}(e), e \rangle = \kappa(P) \in \mathbb{R},$$

where by abuse of notation $e$ denotes the element in $\ell^2(G)$ taking value 1 on the neutral element $e \in G$ and 0 elsewhere (see Eckmann’s survey [9] on $L^2$-cohomology for more on von Neumann dimensions). The map $\kappa : M_n(\mathcal{N}(G)) \to \mathbb{C}$ is the Kaplansky trace. One defines the $L^2$-index of $\tilde{D}$ by

$$\text{Index}_G(\tilde{D}) = \dim_G(S_{\tilde{D}}) - \dim_G(S_{\tilde{D}}^*).$$

We can now state Atiyah’s $L^2$-Index Theorem.

**Theorem 2.1 (Atiyah [1]).** For $D$ an elliptic pseudo-differential operator on a closed Riemannian manifold $M$

$$\text{Index}(D) = \text{Index}_G(\tilde{D})$$

for any countable group $G$ and any lift $\tilde{D}$ of $D$ to a regular $G$-cover $\tilde{M}$ of $M$.

In particular, the $L^2$-index of $\tilde{D}$ is always an integer, even though it is a priori given in terms of real numbers. The following serves as an illustration of the $L^2$-Index Theorem.

**Example 2.2 (Atiyah’s formula [1]).** Let $\Omega^*$ be the de Rham complex of complex valued differential forms on the closed connected manifold $M$ and consider the de Rham differential $D = d + d^* : \Omega^{ev} \to \Omega^{odd}$. Let $\pi : \tilde{M} \to M$ be the universal cover of $M$ so that $G = \pi_1(M)$. Then
• Index(D) = \chi(M), the ordinary Euler characteristic of M.
• Index_G(D) = \sum_j (-1)^j \beta^j(M), the L^2-Euler characteristic of M.

The \beta^j(M)'s denote the L^2-Betti numbers of M. Thus the L^2-Index Theorem translates into Atiyah's formula
\chi(M) = \sum_j (-1)^j \beta^j(M).

We recall that the L^2-Betti numbers \beta^j(M) are in general not integers. For instance, if \pi_1(M) is a finite group, one checks that
\beta^j(M) = \dfrac{1}{|\pi_1(M)|} b^j(\widehat{M}),
where b^j(\widehat{M}) stands for the ordinary j'th Betti number of the universal cover \widehat{M} of M. In particular, for 1 < |\pi_1(M)| < \infty, \beta^0(M) = 1/|\pi_1(M)| is not an integer and the L^2-Index Theorem reduces to the well-known fact that
\chi(M) = \dfrac{\chi(\widehat{M})}{|\pi_1(M)|}.

It is a conjecture (Atiyah Conjecture) that for a general closed connected manifold M the L^2-Betti numbers \beta^j(M) are always rational numbers, and even integers in case that \pi_1(M) is torsion-free. For some interesting examples, which might lead to counterexamples, see Dicks and Schick [8].

3. Hilbert modules

Recall that for H < G and X an H-space, the induced G-space is
G \times_H X = (G \times X)/H
where H acts on G \times X via h \cdot (g, x) = (gh^{-1}, hx) and the left G-action on G \times_H X is given by g \cdot [k, x] = [gk, x] (where [k, x] denotes the class of the pair (k, x) \in G \times X in G \times_H X). For A \subseteq \ell^2(H)^n a Hilbert H-module one defines Ind^G_H(A) the induced Hilbert G-module as follows:
Ind^G_H(A) = \{f : G \to A, f(gh) = h^{-1}f(g), \sum_{\gamma \in G/H} \|f(\gamma)\|^2 < \infty\}.

On Ind^G_H(A) the action of G is given as follows:
(\gamma \cdot f)(\mu) = f(\gamma^{-1}\mu), \quad \gamma, \mu \in G and f \in Ind^G_H(A).

For \widehat{M} an H-free, cocompact Riemannian manifold and \widehat{D} an H-equivariant pseudo-differential operator on \widehat{M}, one can express the lift \widehat{D} of \widehat{D} to \overline{\widehat{M}} = G \times_H \widehat{M} as follows. Fix a set R of representatives for G/H and write \pi : \overline{\widehat{M}} \to \widehat{M} for the projection; a section \overline{\mathfrak{s}} \in C_c(\overline{\widehat{M}}, \pi^*E) is a collection
\overline{\mathfrak{s}} = \{\overline{s}_r\}_{r \in R},
where $\tilde{s}_r \in C^\infty_c(\tilde{M}, E)$ is the zero section for all but finitely many $r$’s, and $\pi([g, \tilde{m}]) = \tilde{s}_r(h\tilde{m})$, if $[r, h\tilde{m}] = [g, \tilde{m}] \in G \times H \tilde{M}$. Now the lift $\tilde{D}$ of $\overline{D}$ to $\overline{M} = G \times H \tilde{M}$ satisfies

$$\overline{D\tilde{s}_r} = \{\tilde{D}\tilde{s}_r\}_{r \in R}.$$ 

### Lemma 3.1

Let $M$ be a closed Riemannian manifold, $D$ a pseudo-differential operator on $M$ and $\tilde{M}$ a regular cover of $M$ with countable transformation group $H$. Consider an inclusion $H < G$ and form the regular cover $\overline{M} = G \times H \tilde{M}$ of $M$. Then for the lifts $\tilde{D}$ of $D$ to $\tilde{M}$ and $\overline{D}$ of $D$ to $\overline{M}$,

$$\text{Index}_H(\tilde{D}) = \text{Index}_G(\overline{D}).$$

**Proof.** It is enough to see that $S_{\overline{D}} \cong \text{Ind}_H^G(S_{\tilde{D}})$. Indeed, it is well-known (see [9]) that for a Hilbert $H$-module $A$ one has

$$\dim_H(A) = \dim_G(\text{Ind}_H^G(A)).$$

For $R$ a fixed set of representatives for $G/H$, the map

$$\varphi_R : \text{Ind}_H^G(S_{\tilde{D}}) \to S_{\overline{D}}
$$

$$f \mapsto \{f(r)\}_{r \in R}$$

is well-defined by $H$-equivariance of the elements of $S_{\tilde{D}}$ and one checks that it defines a $G$-equivariant isometric bijection. Similarly for the adjoint operators. \hfill \Box

The following example is a particular case of the previous lemma.

### Example 3.2

Let us look at the case $\tilde{M} = M \times G$. A section $\tilde{s} \in C^\infty_c(\tilde{M}, \pi^*E)$ is an element $\tilde{s} = \{s_g\}_{g \in G}$ where $s_g \in C^\infty(M, E)$ and $s_g = 0$ for all but finitely many $g$’s. Note that $L^2(\tilde{M}, \pi^*E)$ can be identified with $\ell^2(G) \otimes L^2(M, E)$. Now

$$\tilde{D}\tilde{s} = \{Ds_g\}_{g \in G} \in C^\infty_c(\tilde{M}, \pi^*F)$$

and hence $S_{\tilde{D}}$ may be identified with $\ell^2(G) \otimes S_D \cong \ell^2(G)^d$, where $d = \dim_C(S_D)$. In this identification the projection $P$ onto $S_{\tilde{D}}$ becomes the identity in $M_d(N(G))$ and thus

$$\dim_C(S_{\tilde{D}}) = \sum_{i=1}^d \langle e, e \rangle = d = \dim_C(S_D).$$

A similar argument for $D^*$ shows that in this case not only the $L^2$-Index of $\tilde{D}$ coincides with the Index of $D$, but also the individual terms of the difference correspond to each other. This is not the case in general, see Example 2.2.
4. On $K$-homology

Many ideas of this section go back to the seminal article by Baum and Connes [3], which has been circulating for many years and has only recently been published.

An elliptic pseudo-differential operator $D$ on the closed manifold $M$ can also be used to define an element $[D] \in K_0(M)$, the $K$-homology of $M$, and according to Baum and Douglas [4], all elements of $K_0(M)$ are of the form $[D]$. The index defined in Section 2 extends to a well-defined homomorphism (cf. [4])

$$\text{Index} : K_0(M) \to \mathbb{Z},$$

such that $\text{Index}([D]) = \text{Index}(D)$. On the other hand, the projection $\text{pr} : M \to \{pt\}$ induces, after identifying $K_0(\{pt\})$ with $\mathbb{Z}$, a homomorphism

$$\text{pr}_*: K_0(M) \to \mathbb{Z},$$

which, as explained in [4], satisfies

$$\text{pr}_*[D] = \text{Index}([D]).$$

More generally (cf. [4]), for a not necessarily finite CW-complex $X$, every $x \in K_0(X)$ is of the form $f_*[D]$ for some $f : M \to X$, and $K_0(X)$ is obtained as a colimit over $K_0(M_\alpha)$, where the $M_\alpha$ form a directed system consisting of closed Riemannian manifolds (these homology groups $K_0(X)$ are naturally isomorphic to the ones defined using the Bott spectrum; sometimes, they are referred to as $K$-homology groups with compact supports). The index map from above extends to a homomorphism

$$\text{Index} : K_0(X) \to \mathbb{Z},$$

such that $\text{Index}(x) = \text{Index}([D])$ if $x = f_*[D]$, with $f : M \to X$.

We now consider the case of $X = BG$, the classifying space of the discrete group $G$, and obtain thus for any $f : M \to BG$ a commutative diagram

$$
\begin{array}{ccc}
K_0(M) & \xrightarrow{\text{Index}} & \mathbb{Z} \\
\downarrow f_* & & \\
K_0(BG) & \xrightarrow{\text{Index}} & \mathbb{Z}.
\end{array}
$$

Note that (*) from above implies the following naturality property for the index homomorphism.

**Lemma 4.1.** For any homomorphism $\varphi : H \to G$ one has a commutative diagram

$$
\begin{array}{ccc}
K_0(BH) & \xrightarrow{\text{Index}} & \mathbb{Z} \\
\downarrow (B\varphi)_* & & \\
K_0(BG) & \xrightarrow{\text{Index}} & \mathbb{Z}.
\end{array}
$$
We now turn to the $L^2$-index of Section 2. It extends to a homomorphism

$$\text{Index}_G : K_0(BG) \to \mathbb{R}$$

as follows. Each $x \in K_0(BG)$ is of the form $f_*([y])$ for some $y = [D] \in K_0(M)$, $f: M \to BG$, $M$ a closed smooth manifold and $D$ an elliptic operator on $M$. Let $\tilde{D}$ be the lifted operator to $\tilde{M}$, the $G$-covering space induced by $f: M \to BG$. Then put

$$\text{Index}_G(x) := \text{Index}_G(\tilde{D}).$$

One checks that $\text{Index}_G(x)$ is indeed well-defined, either by direct computation, or by identifying it with $\tau(x)$, where $\tau$ denotes the composite of the assembly map $K_0(BG) \to K_0(C^*_rG)$ with the natural trace $K_0(C^*_rG) \to \mathbb{R}$ (for this latter point of view, see Higson-Roe [10]; for a discussion of the assembly map see e.g. Kasparov [12], or Valette [14]). The following naturality property of this index map is a consequence of Lemma 3.1.

**Lemma 4.2.** For $H < G$ the following diagram commutes

$$
\begin{array}{ccc}
K_0(BH) & \xrightarrow{\text{Index}_H} & \mathbb{R} \\
\downarrow & & \uparrow \\
K_0(BG) & \xrightarrow{\text{Index}_G} & \mathbb{R}.
\end{array}
$$

Atiyah’s $L^2$-Index Theorem [2.1] for a given $G$ can now be expressed as the statement (as already observed in [10])

$$\text{Index}_G = \text{Index} : K_0(BG) \to \mathbb{R}.$$ 

5. **Algebraic proof of Atiyah’s $L^2$-index theorem**

Recall that a group $A$ is said to be *acyclic* if $H_*(BA,\mathbb{Z}) = 0$ for $* > 0$. For $G$ a countable group, there exists an embedding $G \to A_G$ into a countable acyclic group $A_G$. There are many constructions of such a group $A_G$ available in the literature, see for instance Kan-Thurston [11, Proposition 3.5], Berrick-Varadarajan [5] or Berrick-Chatterji-Mislin [6]; these different constructions are to be compared in Berrick’s forthcoming work [7]. It follows that the suspension $\Sigma BA_G$ is contractible, and therefore the inclusion $\{e\} \to A_G$ induces an isomorphism

$$K_0(B\{e\}) \cong K_0(BA_G).$$

Our strategy is as follows. We show that the Atiyah $L^2$-Index Theorem holds in the special case of acyclic groups, and finish the proof combining the above embedding of a group into an acyclic group.
Proof of Theorem 2.1. If a group $A$ is acyclic, the equation $\text{Index}_A = \text{Index}$ follows from the diagram

$$
\begin{array}{cccccc}
K_0(BA) & \xrightarrow{\text{Index}_A} & \mathbb{R} & \xleftarrow{\text{Index}} & K_0(BA) \\
\cong & & \uparrow & & \cong \\
K_0(B\{e\}) & \xrightarrow{\text{Index}_{\{e\}}} & \mathbb{Z} & \xleftarrow{\text{Index}} & K_0(B\{e\})
\end{array}
$$

because $\text{Index}_{\{e\}} = \text{Index}$ on the bottom line. For a general group $G$, consider an embedding into an acyclic group $A_G$ and complete the proof by using Lemma 3.1 together with Lemmas 4.1 and 4.2. □

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