Complexity and Behind the Horizon Cut Off

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Abstract

Motivated by $T\bar{T}$ deformation of a conformal field theory we compute holographic complexity for a black brane solution with a cut off using “complexity=action” proposal. In order to have a late time behavior consistent with Lloyd’s bound one is forced to have a cut off behind the horizon whose value is fixed by the boundary cut off. Using this result we compute holographic complexity for two dimensional AdS solutions where we get expected late times linear growth. It is in contrast with the naively computation which is done without assuming the cut off where the complexity approaches a constant at the late time.
1 Introduction

According to the “complexity=action” proposal (CA) the quantum computational complexity of a holographic state is given by the on-shell action evaluated on a bulk region known as the ‘Wheeler-De Witt’ (WDW) patch \[1,2\]

\[ \mathcal{C}(\Sigma) = \frac{I_{\text{WDW}}}{\pi \hbar}. \] (1.1)

Here the WDW patch is defined as the domain of dependence of any Cauchy surface in the bulk whose intersection with the asymptotic boundary is the time slice \( \Sigma \).

An interesting feature of the complexity is that it grows linearly with time at the late time with slope given by Lloyd’s bound \[3\] that is twice of the energy of the state. Holographic complexity for two-sided black holes has been calculated in \[4\] where it was shown that although at the late time the growth rate approaches a constant value that is twice of the mass of the black hole, the constant is approached from above, violating the Lloyd’s bound \[3\].

Another recent interesting development in the literature of theoretical higher energy is to study a conformal theory deformed by an irrelevant operator such as the one which is quadratic in the stress energy tensor known as \( T T \) deformation. Although typically deforming a conformal field theory by an irrelevant operator would remove UV fixed point and makes it non-local at high energies, it was shown that for the mentioned deformation the resultant theory is still exactly solvable \[6,7\].

To be concrete let us consider a two dimensional conformal field theory deformed by the corresponding operator as follows

\[ I_{\text{QFT}} = I_{\text{CFT}} + \mu \int d^2 x T \bar{T}. \] (1.2)

There are some interesting features of the resultant quantum field theory. First of all it is UV complete. Moreover the spectrum of the deformed theory can be determined non-perturbatively and rather in a compact form. More precisely for a conformal field theory on a cylinder with the circumference \( L \) the energy level \( E_n(\mu, L) \) for a state denoted by conformal dimensions \((\Delta_n, \bar{\Delta}_n)\) is given by \[6,7\]

\[ E_n(\mu, L) = \frac{2L}{\mu} \left( 1 - \sqrt{1 - \frac{2\pi \mu}{L^2} \left( M_n + \frac{2\pi \mu}{L^2} J_n^2 \right) } \right), \] (1.3)

where \( M_n = \Delta_n + \bar{\Delta}_n - \frac{c}{12} \), and \( J_n = \Delta_n - \bar{\Delta}_n \).

In the context of AdS/CFT correspondence it was proposed that the above deformation has a holographic dual. The corresponding dual gravitational theory may be described by an AdS_3 metric with a finite radial cut off \[8\]. The radial cut off \( r_c \) is given in terms of the deformed parameter \( \mu \), by \( r_c^2 = \frac{16 \pi \mathcal{G}}{\mu} \).

Using AdS/CFT correspondence the generalization of \( T \bar{T} \) deformation to higher dimensional conformal field theories has also been studied in \[9,10\]. Following \[8\] one would also expect that a
$d+2$ dimensional AdS black brane solution with a radial cut off could provide a holographic dual for a $d+1$ dimensional $T\bar{T}$ deformed conformal field theory. Given the corresponding geometry by

$$ds^2 = \frac{\ell^2}{r^2} \left( -f(r) dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^{d} dx_i^2 \right), \quad f(r) = 1 - \left( \frac{r}{r_h} \right)^{d+1}, \quad (1.4)$$

where $r_h$ and $\ell$ are radius of horizon the AdS radius, respectively, the spectrum of energy of the deformed theory is $[9,10]$

$$E = \frac{V_d \ell^{d+1}}{8\pi G} \frac{1}{r_c^{d+1}} \left( 1 - \sqrt{1 - \frac{r_c^{d+1}}{r_h^{d+1}}} \right), \quad (1.5)$$

with $V_d$ being the volume of $d$-dimensional internal space of the metric parametrized by $x_i$, $i = 1, \cdots, d$.

Motivated by $T\bar{T}$ deformation and its holographic dual in the present paper we would like to compute the complexity growth of a black brane at a finite cut off using CA proposal. We observe that requiring to reach the Lloyd’s bound at the late time enforces us to have a cut off behind the horizon whose value is fixed by boundary cut off. More precisely for black brane solutions denoting cut off radius inside the horizon by $r_0$ one finds (at leading order)

$$r_0 r_c^2 = 2 \pi^2 r_h^3. \quad (1.6)$$

To explore the significance of our result we will then study holographic complexity for AdS$_2$ vacuum solutions of certain two dimensional Maxwell-Dilaton gravities. One observes that if we naively compute the complexity without taking into account the behind horizon cut off the rate of growth vanishes at the late time. On the other hand if one assumes that the UV cut off would set a cut off behind the horizon given by (1.6) the complexity exhibits late time linear growth, as expected.

The paper is organized as follows. In the next section we will compute holographic complexity for back brane solutions in the present of a cut off where we show how the inside cut off would emerge. In section three we will study complexity for AdS$_2$ taking into account the enforced behind the horizon cut off. The last section is devoted to conclusions.

## 2 CA complexity for cut off geometries

In this section we would like to compute holographic complexity for a black brane solution with a radial cut off. To do so, following CA proposal we will need to compute on shell action on the WDW patch associated with a boundary state given at $\tau = t_L + t_R$. Here $t_L(t_R)$, is time coordinate of left (right) boundary on the eternal black brane (see figure 1). Of course since we are interested in the late time behavior of the complexity it is sufficient to compute on shell action over the intersection of the WDW patch with the future interior shown by dark blue color in the figure.
The theory is defined at a radial finite cut off \( r_c \) that fixes a cut off behind the horizon denoted by \( r_0 \).

To proceed we note that the action consists of several parts including bulk, boundary and joint points as follows \[ 12, 14 \]

\[
I^{(0)} = \frac{1}{16\pi G_N} \int d^{d+2}x \sqrt{-g}(R - 2\Lambda) + \frac{1}{8\pi G_N} \int_{\Sigma_{d+1}} K_t d\Sigma_t \\
\pm \frac{1}{8\pi G_N} \int_{\Sigma_{d+1}}^{} K_s d\Sigma_s \pm \frac{1}{8\pi G_N} \int_{\Sigma_{d+1}}^{} K_n dS d\lambda \pm \frac{1}{8\pi G_N} \int_{J^d} a dS. \quad (2.1)
\]

Here the timelike, spacelike, and null boundaries and also joint points are denoted by \( \Sigma_{d+1}^t, \Sigma_{d+1}^s, \Sigma_{d+1}^n \) and \( J^d \), respectively. The extrinsic curvature of the corresponding boundaries are given by \( K_t, K_s \) and \( K_n \). The function \( a \) at the intersection of the boundaries is given by the logarithm of the inner product of the corresponding normal vectors and \( \lambda \) is the null coordinate defined on the null segments. The sign of different terms depends on the relative position of the boundaries and the bulk region of interest (see \[ 14 \] for more details).

The null boundaries \( B_1 \) and \( B_2 \) of the future interior are

\[
B_1 : \ t = t_R + r^*(r_c) - r^*(r), \quad B_2 : \ t = -t_L - r^*(r_c) + r^*(r), \quad (2.2)
\]

where \( r^*(r) \) is the tortoise coordinate. The null vectors associated with these null boundaries are also given by

\[
k_1 = \alpha \left( \partial_t + \frac{1}{f(r)} \partial_r \right), \quad k_2 = \beta \left( \partial_t - \frac{1}{f(r)} \partial_r \right), \quad (2.3)
\]

Here \( \alpha \) and \( \beta \) are two free constant parameters appearing due to the ambiguity of the normalization.
of null vectors.

Using this notation the bulk part of the on shell action is

$$ I_{\text{FI}}^{\text{bulk}} = -\frac{V_d \ell^d}{4\pi G_N} (d+1) \int_{r_h}^{r_0} \frac{dr}{r^{d+2}} \left( \frac{\tau}{2} + r^*(r_c) - r^*(r) \right) $$

$$ = -\frac{V_d \ell^d}{8\pi G_N} \left( \frac{2}{dr_h^d} - \frac{2}{d r_0^d} \right) - \frac{V_d \ell^d}{8\pi G_N} \left( \frac{1}{r_{h}^{d+1}} - \frac{1}{r_{0}^{d+1}} \right) (\tau + \tau_c). \quad (2.4) $$

where $\tau_c = 2(r^*(r_c) - r^*(r_0))$. Here to find the last expression we have performed an integration by parts.

There are five boundaries four of which are null that have zero contribution if one uses the Affine parameter to parametrize the null directions. Therefore we are left with a space like boundary at future singularity whose contribution is given by

$$ I_{\text{FI}}^{\text{surf}_1} = -\frac{1}{8\pi G_N} \int d^d x \int_{-t_L - r^*(r_c) + r^*(r)}^{t_R + r^*(r_c) - r^*(r)} dt \sqrt{|h|} K_s \bigg|_{r=r_0}, \quad (2.5) $$

where $K_s$ is the the trace of extrinsic curvature of the boundary at $r = r_0$ and $h$ is the determinant of the induced metric on it. To compute this term it is useful to note that for a constant $r$ surface using the metric (1.4) one has

$$ \sqrt{|h|} K = -\sqrt{g^{rr}} \partial_r \sqrt{|h|} = -\frac{1}{2} \frac{\ell^d}{r^d} \left( \partial_r f(r) - \frac{2(d+1)}{r} f(r) \right), \quad (2.6) $$

therefore the boundary term (2.5) reads

$$ I_{\text{FI}}^{\text{surf}_1} = \frac{V_d \ell^d}{8\pi G_N} (d+1) \left( \frac{1}{2r_h^{d+1}} - \frac{1}{r_0^{d+1}} \right) (\tau + \tau_c). \quad (2.7) $$

Note that there is also another boundary term to be evaluated at the surface cut off behind the horizon that is given by

$$ I_{\text{FI}}^{\text{surf}_2} = \frac{1}{8\pi G_N} \int d^d x \int_{-t_L - r^*(r_c) + r^*(r)}^{t_R + r^*(r_c) - r^*(r)} dt \sqrt{|h|} \bigg| \frac{d}{\ell} \bigg|_{r=r_0} = \frac{V_d \ell^d}{8\pi G_N} \frac{d}{r_0^{d+1}} \sqrt{\frac{r_h^{d+1}}{r_0^{d+1}}} - 1 (\tau + \tau_c). \quad (2.8) $$

There are also five joint points, two points at $r_0$ and three at the horizon $r = r_h$. Of course those at the horizon are not at the same point, though the coordinate system $r$ cannot make any distinction between them. To label these points it is convenient to use the following coordinate system (15),

$$ u = -e^{-\frac{1}{2} f'(r_0)}(r^*(r) - t), \quad v = -e^{-\frac{1}{2} f'(r_0)}(r^*(r) + t), \quad (2.9) $$
by which the points are located at \((\epsilon_u, v_0), (u_0, \epsilon_v)\) and \((\epsilon_u, \epsilon_v)\) as depicted in figure 1. Here in order to regularize quantities like \(r^{*}(r)\) at \(r = r_h\) we have put the horizon at \(v = \epsilon_v\) and \(u = \epsilon_u\) for small \(\epsilon_v\) and \(\epsilon_u\). In what follows the radial coordinate associated with these three points are also labeled by \(r_{v_0}, r_{u_0}\) and \(r_{\epsilon}\), respectively. Using this notation the contribution of joint points is \([11]\)

\[
I_{\text{joint}}^{\text{FI}} = \frac{V_{d}\ell^{d}}{8\pi G_N} \left( \frac{\log \frac{\alpha \beta r^2_0}{r_0^d}}{r_0^d} + \frac{\log \frac{\alpha \beta r^2_0}{r^d}(r_{u_0})}{r^d} - \frac{\log \frac{\alpha \beta r^2_0}{r^d}(r_{v_0})}{r^d} \right) (2.10)
\]

\[
= -\frac{V_{d}\ell^{d}}{8\pi G_N} \left( \frac{\log |f(r_{\epsilon})| - \log |f(r_{u_0})| - \log |f(r_{v_0})|}{r_h^d} \right) + \frac{\log \frac{\alpha \beta r^2_0}{r^d}}{r^d} + \frac{\log \frac{\alpha \beta r^2_0}{r^d}}{r_0^d} \right).
\]

Here we have used the fact that \(\{r_{u_0}, r_{v_0}, r_{\epsilon}\} \approx r_h\). On the other hand by making use of the fact that \([15]\)

\[
\log |f(r_{u,v})| = \log |uv| + c_0 + O(uv) \quad \text{for } uv \to 0,
\]

one arrives at

\[
I_{\text{joint}}^{\text{FI}} = \frac{V_{d}\ell^{d}}{8\pi G_N} \left( \frac{\log |u_0 v_0| + c_0}{r_h^d} - \frac{\log |f(r_0)|}{r_h^d} \right) - \frac{V_{d}\ell^{d}}{8\pi G_N} \left( \frac{\log \frac{\alpha \beta r^2_0}{r^d}}{r_0^d} \right) \). (2.12)
\]

The only remaining part of the action to be considered is a term needed to remove the ambiguity associated with the normalization of null vectors \([14, 16, 17]\)

\[
I_{\text{amb}} = \frac{1}{8\pi G_N} \int d\lambda d^d x \sqrt{\gamma} \Theta \log \frac{\tilde{\ell} \Theta}{d},
\]

where \(\tilde{\ell}\) is an undetermined length scale and \(\gamma\) is the determinant of the induced metric on the joint point where two null segments intersect, and

\[
\Theta = \frac{1}{\sqrt{\gamma}} \frac{\partial \sqrt{\gamma}}{\partial \lambda}.
\]

In the the present case the contribution of this term is (for more details see \([11]\))

\[
I_{\text{amb}}^{\text{FI}} = \frac{V_{d}\ell^{d}}{8\pi G_N} \left( \frac{\log \frac{\alpha \beta r^2_0}{r^d}}{r_h^d} - \frac{\log \frac{\alpha \beta r^2_0}{r^d}}{r_0^d} \right) + \frac{V_{d}\ell^{d}}{8\pi G_N} \left( \frac{2}{d r_h^d} - \frac{2}{d r_0^d} \right). (2.15)
\]

Taking all parts contributing to the on shell action into account one arrives at

\[
I_{\text{FI}} = \frac{V_{d}\ell^{d}}{8\pi G_N} \left[ \left( \frac{d}{r_h^{d+1}} - \frac{d}{r_0^{d+1}} + \frac{d}{r_0^{d+1}} \sqrt{r_0^{d+1} - 1} \right) \left( \tau + \tau_{\epsilon} \right) + \frac{(d + 1)r^{*}(r_0) + c_0 r_h}{r_h^{d+1}} - \frac{\log |f(r_0)|}{r_0^d} \right]
\]
leading to the following rate of growth

\[
\frac{dI_{FI}}{d\tau} = V_d \ell^d \frac{1}{8\pi G_N} \left( \frac{1}{r_h^{d+1}} - \frac{1}{r_0^{d+1}} + \frac{1}{r_0^{d+1}} \sqrt{\frac{r_0^{d+1}}{r_h^{d+1}} - 1} \right),
\]

that is indeed the late time expression for the holographic complexity of the corresponding black brane solution [11]. Now the aim is to compare the above result with Lloyd’s bound that in our case should be read from the energy spectrum (1.5) that can be recast into the following inspiring form

\[
E = V_d \ell^d \frac{1}{16\pi G_N} r_h^{d+1} + V_d \ell^d d \frac{1}{16\pi G_N} r_c^{d+1} \left( 1 - \sqrt{1 - \frac{r_c^{d+1}}{r_h^{d+1}}} \right)^2.
\]

Therefore if one assumes that at the late time the growth rate of complexity saturates the Lloyd’s bound, \(2E\), one may conclude that

\[
\frac{1}{r_0^{d+1}} \left( \sqrt{\frac{r_0^{d+1}}{r_h^{d+1}}} - 1 - 1 \right) = \frac{1}{r_c^{d+1}} \left( \sqrt{1 - \frac{r_c^{d+1}}{r_h^{d+1}}} - 1 \right)^2,
\]

which at leading order reduces to

\[
r_0^2 r_c^2 = 2\pi^2 r_h^3,
\]

This means that the cut off at the singularity is fixed by the UV cut off at the boundary. In other words this leads to a conclusion that as soon as we fixed the UV cut off we are not allowed to consider another independent cut off inside the horizon (let say near the singularity) and the UV cut off will automatically regularize the modes inside the horizon. This is, indeed, the main result of the present paper.

To explore the importance of the above conclusion in what follows we will study holographic complexity for AdS vacuum solutions of certain two dimensional gravities.

3 Complexity for AdS\(_2\) geometry

In this section we shall study holographic complexity for certain two dimensional Maxwell-Dilaton gravities that admit AdS vacuum solutions. The first model we will consider has the following action\(^1\)

\[
I = \frac{1}{8G} \int d^2x \sqrt{-g} \left( e^\phi \left( R + \frac{2}{\ell^2} \right) - F^2 \right).
\]

---

\(^1\)This is indeed one of the simplest example of two dimensional gravity having non-trivial vacuum. One could, as well, consider rather more complicated actions (see e.g. [18, 20]).
Using the entropy function formalism \[21\] one can show that the above action admits constant dilaton AdS$_2$ vacuum solution as follows \[22\]

\[ds^2 = \ell^2 \left(-\left(r^2 - r_h^2\right)dt^2 + \frac{dr^2}{r^2 - r_h^2}\right), \quad e^\phi = 4G^2 Q^2 \ell^2, \quad F_{rt} = 2GQ \ell,\] (3.2)

whose entropy is

\[S_{BH} = 2\pi G Q^2 \ell^2,\] (3.3)

that it is independent of $r_h$. Let us compute holographic complexity for a state given at $\tau = t_L + t_R$. The corresponding WDW patch is depicted in the figure 2.

One may naively compute on shell action in the WDW patch shown in the figure 2 with two joint points denoted by $r_m$ and $r_{m'}$ (the later point is drown by dashed lines). Positions of the corresponding points are obtained from the following equations

\[\tau = -2\left(r^*(r_c) - r^*(r_{m'})\right) = 2\left(r^*(r_c) - r^*(r_m)\right),\] (3.4)

where $r_c$ is a UV cut off.

Following CA proposal the idea is to evaluate on shell action on the corresponding WDW with a UV cut off but no, a priori, restriction on modes behind the horizon. This means that there is no cut off behind the horizon and both corners denoted by $m$ and $m'$ should be taken into account.

Figure 2: Penrose diagram of AdS$_2$ geometry. The green part is covered by AdS global coordinates, while the Rindler coordinates cover a portion shown in the figure. The actual WDW patch is shown by blue color.
With this assumption the bulk part of the on shell action reads

\begin{align*}
I_{\text{bulk}} &= GQ^2 \ell^2 \left( \int_{r_{m'}}^{r_h} dr \left( \tau + 2(r^*(r_c) - r^*(r)) \right) + 2 \int_{r_h}^{r_c} dr \ 2(r^*(r_c) - r^*(r)) \right) \\
&= GQ^2 \ell^2 \left( (r_m - r_{m'}) \frac{\tau}{2} + \right. \\
&\left. + \int_{r_m}^{r_c} dr \left( \tau + 2(r^*(r_c) - r^*(r)) \right) \right). \quad (3.5)
\end{align*}

By making use of an integration by parts one finds

\begin{align*}
I_{\text{bulk}} &= GQ^2 \ell^2 \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_{m'})| \right), \quad (3.6)
\end{align*}

where \( f(r) = r^2 - r_h^2 \).

On the other hand using Affine parameter to parametrize the null direction one gets zero contribution from null boundaries. Therefore the only part one needs to further consider is the contribution of join points. Denoting the null vectors by

\begin{align*}
k_1 &= \alpha \left( \partial_t - \frac{1}{f(r)} \partial_r \right), \\
k_2 &= \beta \left( \partial_t + \frac{1}{f(r)} \partial_r \right), \quad (3.7)
\end{align*}

one gets

\begin{align*}
I_{\text{joint}} &= \frac{e^\phi}{4G} \left( \log \left| \frac{\alpha \beta}{\ell^2 f(r_m)} \right| + \log \left| \frac{\alpha \beta}{\ell^2 f(r_{m'})} \right| - 2 \log \left| \frac{\alpha \beta}{\ell^2 f(r_c)} \right| \right) \\
&= GQ^2 \ell^2 \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_{m'})| \right). \quad (3.8)
\end{align*}

Interestingly enough the free parameters \( \alpha \) and \( \beta \) drop from the final result which means that there is no ambiguity associated with the normalization of null vectors. Therefore we do not need any further counter terms, except possibly the one that could cancel the most divergent term of the on shell action, \( \log f(r_c) \). Of course since we are interested in the time derivative of the action this term does not play any role.

Taking all terms contributing to the on shell action one arrives at

\begin{align*}
I &= I_{\text{bulk}} + I_{\text{joint}} = 2GQ^2 \ell^2 \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_{m'})| \right), \quad (3.9)
\end{align*}

whose time derivative is

\begin{align*}
\frac{dI}{d\tau} &= 2GQ^2 \ell^2 (r_m - r_{m'}). \quad (3.10)
\end{align*}

It is then notable that at the late time when \( \\{r_m, r_{m'}\} \rightarrow r_h \) the rate of growth vanishes, leading
to a constant late time complexity that is counter intuitive. Indeed we would expect to get linear growth at the late time.

Of course in light of our result in the previous section this conclusion is, indeed, misleading. In fact, as we have already demonstrated in the previous section, setting a UV cut off at the boundary would enforce us to have a cut off inside the horizon that prevents us to have access to all regions on WDW located behind the horizon.

In other words, as soon as we set the UV cut off, $r_c$, at the boundary we will also have to consider a cut off behind the horizon given by $r_0 \sim \frac{r^3}{r_c}$ at leading order. Actually having this cut off will remove the joint point $r_m'$ from the WDW patch and instead we would have a space like boundary at $r = r_0$. Therefore one should redo our computations for on shell action for a new WDW patch that has no joint point $m'$, as shown with blue color in the figure 2.

To proceed let us again start with the bulk action. In this case one gets

$$I_{\text{bulk}} = GQ^2 \ell^2 \left( \int_{r_0}^{r_h} dr \left( \tau + 2(r^*(r_c) - r^*(r)) \right) + 2 \int_{r_h}^{r_c} dr \left( -\tau + 2(r^*(r_c) - r^*(r)) \right) \right),$$

(3.11)

that can be recast to the following form after making use of an integration by parts

$$I_{\text{bulk}} = GQ^2 \ell^2 \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_0)| - r_0 (\tau + 2(r^*(r_c) - r^*(r_0))) \right).$$

(3.12)

The boundary contributions associated with null boundaries are still zero when Affine parametrization is used. Of course in the present case we have a space like boundary whose contribution is

$$I_{\text{surf}} = -\frac{1}{4G} \int dte^\phi \sqrt{-h} \left( K_s - \frac{1}{\ell} \right) \bigg|_{r_0} = GQ^2 \ell^2 (r_0 + r_h) (\tau + 2(r^*(r_c) - r^*(r_0))).$$

(3.13)

As for joint points we have

$$I_{\text{joint}} = \frac{e^\phi}{4G} \left( \log \left| \frac{\alpha \beta}{\ell^2 f(r_m)} \right| + \log \left| \frac{\alpha}{\ell \sqrt{f(r_0)}} \right| + \log \left| \frac{\beta}{\ell \sqrt{f(r_c)}} \right| - 2 \log \left| \frac{\alpha \beta}{\ell^2 f(r_c)} \right| \right)$$

$$= GQ^2 \ell^2 \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_0)| \right).$$

(3.14)

Now putting all terms together one arrives at

$$I = 2GQ^2 \ell^2 \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_0)| \right) + GQ^2 \ell^2 r_h (\tau + 2(r^*(r_c) - r^*(r_0))).$$

(3.15)
It is easy to show
\[ \frac{dI}{dt} = GQ^2 \ell^2 (r_h + 2r_m), \]  
which approaches a constant at the late time
\[ \frac{dI}{dt} = 3GQ^2 \ell^2 r_h = 3(2\pi GQ^2 \ell^2) \left( \frac{r_h}{2\pi} \right) = 3S_{BH} T. \]

Here \( T \) is the Hawking temperature associated with the geometry. This is in agreement with what is expected; namely one has late time linear growth with slope given by entropy times temperature. Of course the actual numerical factor does not look universal.

To further explore the above picture better it is also constructive to consider another two dimensional model admitting AdS\(_2\) vacuum solutions as follows
\[ S_2 = \frac{1}{8G} \int d^2x \sqrt{-g} e^\phi \left( R + \frac{2}{\ell^2} - \frac{\ell^2}{4} e^{2\phi} F^2 \right). \]

Using the entropy function formalism \[21\] one can show that the above action admits the AdS\(_2\) vacuum solution as follows \[22\]
\[ ds^2 = \frac{\ell^2}{4} \left( -(r^2 - r_h^2) dt^2 + \frac{dr^2}{r^2 - r_h^2} \right), \quad e^\phi = \sqrt{4GQ\ell^2}, \quad F_{tr} = \sqrt{\frac{1}{16GQ\ell^2}} \]

with the entropy,
\[ S_{BH} = 2\pi \sqrt{\frac{Q\ell^2}{4G}}. \]

Now the aim is to compute the holographic complexity for this model. Of course the procedure is the same as that we considered in the previous case and the only difference is the numerical factors. More precisely for the bulk term one finds
\[ I_{\text{bulk}} = -\frac{\ell}{4} \sqrt{\frac{Q}{G}} \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_0)| - r_0 (\tau + 2(r^*(r_c) - r^*(r_0))) \right). \]

As for joint points one gets
\[ I_{\text{joint}} = \frac{\ell}{2} \sqrt{\frac{Q}{G}} \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_0)| \right), \]

while for the surface term one has
\[ I_{\text{surf}} = \frac{\ell}{2} \sqrt{\frac{Q}{G}} (r_0 + r_h) \left( \tau + 2(r^*(r_c) - r^*(r_0)) \right). \]

\[2\] See \[23\] for non-constant dilation solution of the model.
Therefore the total action is found

\[
I = \frac{\ell}{4 \sqrt{\frac{Q}{G}}} \left( 2 \log |f(r_c)| - \log |f(r_m)| - \log |f(r_0)| \right) + \frac{\ell}{2 \sqrt{\frac{Q}{G}}} r_h (\tau + 2(r^*(r_c) - r^*(r_0))) \quad (3.24)
\]

resulting to the following rate of growth for the on shell action

\[
\frac{dI}{dt} = \frac{\ell}{4 \sqrt{\frac{Q}{G}}} (r_m + 2r_h) \quad (3.25)
\]

which approaches a constant at late time

\[
\frac{dI}{dt} = 3 \frac{\ell}{4 \sqrt{\frac{Q}{G}}} r_h = \frac{3}{2} S_{\text{BH}} T \quad (3.26)
\]

Note that the same as previous one, had not we considered the inside surface cut off, the complexity growth would have been zero at the late time.

### 4 Conclusions

In this paper we have studied holographic complexity for an AdS black brane geometry with a radial cut off using CA proposal. Within this explicit example we have found that as soon as one sets a UV cut off at the boundary the model enforces us to have a cut off behind the horizon whose value is fixed by the UV cut off. Indeed in the present case one has

\[
1 = \frac{1}{r_0^{d+1}} \left( \sqrt{\frac{r_0^{d+1}}{r_h^{d+1}}} - 1 - 1 \right) = \frac{1}{r_c^{d+1}} \left( \sqrt{1 - \frac{r_c^{d+1}}{r_h^{d+1}}} - 1 \right)^2 \quad (4.1)
\]

It is worth mentioning that in order to get a consistent result fulfilling the Lloyd’s bound it was crucial to consider the contribution of certain counter term on the cut off surface behind the horizon.

In this paper we have only considered uncharged black hole with flat boundary. It would be interesting to find an expression for behind the horizon cut off in terms of the UV cut off for a general charged black hole. In general the cut off \( r_0 \) is a function of UV cut off; \( r_0 = r_0(r_h, r_c) \), though it might not have such a simple expression as above. Actually this relation should be intuitively understood from the fact that the energy is a charge defined at the boundary while the late time behavior of complexity is evaluated from the action behind the horizon.

If our result works for a generic black hole, it means that near singularity modes may be regularized through a UV cut off. It is, however, important to note that our conclusion will not affect the results people have found so far in the literature, though it might shed light on some new problem such as how to deal with Riemann tensor squared.
Actually in order to explore the importance of our result we have studied holographic complexity for AdS$_2$ vacuum solutions in certain two different Maxwell-Dilaton gravities. We have found that the complexity is finite at late times if one does not consider the cut off enforced by the UV cut off, that seems counter intuitive. Indeed one would expect that the complexity exhibits linear growth at the late time. On the other hand if one considers behind the horizon cut off fixed by the UV cut off, indeed one gets the corresponding linear growth.

Two dimensional AdS solutions we have considered were supported by a constant Dilaton, though it would be interesting to consider the case where the Dilaton is not constant. This might be more interesting as it could provide a holographic dual for SYK model [24, 25] (see for example [26–28]).

Note added: While we were in the final stage of our work, the paper [29] appeared in the arXiv where the complexity of two dimensional gravity has also been studied. In this paper the authors resolved the undesired late time behavior by adding a new charge to the model. This in fact could be naturally accommodated if one considers the model as a dimensionally reduced four dimensional RN black hole.

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