Quality factor of a matter-wave beam.

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Imperfections in dilute atomic beams propagating in the paraxial regime and in potentials of cylindrical symmetry have been characterized experimentally through the measurement of a parameter analogous to a beam quality factor [Riou et al., Phys. Rev. Lett. 96, 070404 (2006)]. We propose a generalization of this parameter, which is suitable to describe dilute matter waves propagating beyond the paraxial regime and in fully general linear atom-optical systems. The presented quality factor shows that the atomic beam symmetry can be traded for a better transverse collimation.

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I. INTRODUCTION

There is a strong analogy between the propagation of light and matter waves. It is manifest in the paraxial wave equation for the mode \( U \) of an electromagnetic field \( E(x, y, z) = U(x, y, z)e^{i(kz-\omega t)} \hat{e} \) which takes the form of a bidimensional Schrödinger equation

\[
2ik \frac{\partial U(x, y, z)}{\partial z} = -\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) U(x, y, z) + F(x, y)U(x, y, z),
\]

with a function \( F(x, y) \) depending on the local refraction index. A spectacular achievement proving the similarity of the atomic and light fields was the realization of a quasicontinuous matter beam analog to a laser [2–9]. It has been shown recently that the extraction of a quasicontinuous atom laser from a Bose-Einstein condensate by a weak RF [10–13] or Raman [14] outcoupling involves the propagation through a discontinuous potential which degrades the beam collimation. To quantify this effect, Riou et al. [10] introduced a parameter analogous to a quality factor. It allows one to estimate the transverse width of a cylindrical atomic beam falling in a uniform gravitational field and in the paraxial regime, i.e., with atoms strongly accelerated along one privileged direction.

However, an important difference between light and matter waves is that the latter generally do not propagate in this paraxial regime. This is typically the case for a condensate expanding after a sudden trap shut down, which can be viewed as a pulsed atom laser. Even for a weakly outcoupled atom laser, the paraxial approximation is often valid only after a certain propagation time [10] during which the beam must be treated in a more general framework. A comparison between the usual Schrödinger equation and the paraxial wave equation [11] shows that for these matter waves, time plays the role devoted to the axis of propagation \( O\z \) in paraxial optics, and that the transverse space is three dimensional and spanned by \( \{ \hat{e}_x, \hat{e}_y, \hat{e}_z \} \) instead of \( \{ e_x, e_y \} \).

To extend the concept of matter beam quality beyond the paraxial regime, it is necessary to reconsider its physical significance in optics. For cylindrical light beams, a quality factor compares the divergence of a non-ideal beam to a standard set by a perfect Gaussian mode. It conveys the idea of degradation in the collimation or, equivalently, that of diminution in phase-space density. A relevant quality parameter should thus be left invariant during the propagation in perfect linear optical systems which preserve the beam collimation.

Such systems are modelled by a finite set of \( ABCD \) matrices which describe linear input-output relations in the light beam phase-space [15]. Their atomic counterpart is the quantum evolution under an Hamiltonian quadratic in position and impulsion, which involves similar relations with \( 3 \times 3 \) \( ABCD \) matrices [16, 17]. The phase space evolution of the atomic beam then also amounts to a time-dependent map in the arguments of the Wigner distribution [1].

\[
W(r, p, t) = W \left( \tilde{D}(r - \xi) - \frac{1}{m} \tilde{P}(p - m\phi), \right.
\]

\[\left. -m\tilde{C}(r - \xi) + \tilde{A}(p - m\phi), t_0 \right) \right).
\]

The \( \tilde{\cdot} \) stands for the transposition, the vectors \( r, p \) are the position and impulsion, the matrices \( A, B, C, D \) and vectors \( \xi, \phi \) are time-dependent parameters determined in Ref. [15]. Thanks to the unimodularity of the \( ABCD \) matrices, the propagation under those quadratic Hamiltonians preserves the phase-space density and as such do not alter the atomic beam collimation. In this view, these Hamiltonians can be considered as perfect aberrationless atom-optical systems and should thus leave invariant a well-defined quality factor.

This is not the case for the parameter used so far to characterize the quality of matter beams [10, 14]. This is disturbing, since its current definition would sometimes lead to ignoring possible improvements of the atom laser collimation through non cylindrical atom-optical elements such as astigmatic atomic lenses [15]. It seems thus useful to propose a new quality factor for atom lasers which respects this invariance requirement. This is the main purpose of this paper.
II. QUALITY FACTOR OF A CYLINDRICAL BEAM

Let us first reconsider the definition of a quality factor for a cylindrical light beam, before extending this concept to atom optics. Partially coherent light beams can be described by means of a first-order field correlation function \( \Gamma \) relating the field amplitude at different points of planes transverse to the direction of propagation \( O_z \).

For a cylindrical beam, only a single transverse direction \( O_x \) needs to be considered in the correlation function

\[
\Gamma(x_1, x_2, z) = \langle E(x_1, z) E^*(x_2, z) \rangle .
\]

The Wigner transform of this function provides a phase space picture of the beam

\[
W(x, k_x, z) = \int dx' \Gamma(x + \frac{x'}{2}, x - \frac{x'}{2}, z) e^{-ik_xx'},
\]

which can be used to define moments in the transverse position and wave vectors

\[
\langle m(x, k_x, z) \rangle = \int dx \, dk_x \, m(x, k_x, z) W(x, k_x, z) .
\]

Because the beam is cylindrical and transverse \( \langle x \rangle = \langle k_x \rangle = 0 \). The moments \( \Delta x = \sqrt{\langle x^2 \rangle} \) and \( \Delta k_x = \sqrt{\langle k_x^2 \rangle} \) are, respectively, the transverse squared width and wave-vector dispersions of the beam.

Optimal collimation is achieved with a perfectly coherent Gaussian mode, for which the position and direction moments verify at the waist

\[
\Delta x|_w,\text{Gaussian} \Delta k_x|_w,\text{Gaussian} = \frac{1}{2} .
\]

Partially coherent cylindrical light beams have a lesser collimation quantitatively estimated thanks to a quality factor \( M_z^2 \) \[19\]. This parameter can be expressed as a combination of moments left invariant during the propagation in aberrationless cylindrical optical systems

\[
\frac{M_z^2}{2} = \sqrt{(\langle x^2 \rangle)_z (\langle k_x^2 \rangle)_z - \langle x k_x \rangle_z^2} . \tag{2}
\]

At the beam waist, this expression reduces to the product of transverse squared width and divergence which has a clear interpretation in terms of phase-space dispersion

\[
\frac{M_z^2}{2} = \Delta x|_w \Delta k_x|_w . \tag{3}
\]

This is indeed the definition which has been adopted to express the quality of a matter beam \[10\, [14\] , once taken into account De Broglie relation between momentum and wave vector \( p = h k \)

\[
M_z^2 = \frac{2}{\hbar} \Delta x|_w \Delta p_x|_w . \tag{4}
\]

This definition is consistent if one considers only matter-waves and atom-optical systems with cylindrical symmetry, and a propagation in the paraxial regime: the parameter \( M_z^2 \) is then unaltered by the propagation in aberrationless systems.

However, these conditions are far too restrictive to characterize the matter beams in most experiments, which generally involve potentials without any symmetry and a matter-wave propagation outside the paraxial regime. A three-dimensional quality factor \( M_3 \), invariant under the whole set of possible \( ABCD \) matrices, is then required. Following previous treatments of optical and quantum mechanical invariants \[21\, [22\], we now examine how to properly define this parameter.

III. QUADRATIC PROPAGATION INVARIANT FOR MATTER BEAMS

Our goal is to replace the parameter \( M_z \) by an \( ABCD \) invariant combination of moments which still provides an insight on the matter beam divergence. In the unidimensional case, the price to pay to express the product \( \Delta x|_w \Delta k_x|_w \) \[5\] as an invariant expression \[2\] was to introduce an additional moment \( \langle x k_x \rangle \) mixed in wave vector and position. This term can be viewed as an element of a variance matrix. Such matrices have been introduced in optics \[20\], and they can also characterize a partially coherent atomic beam \[22\]:

\[
V = \begin{pmatrix}
\Delta x_{rr} & \Delta x_{rv} \\
\Delta x_{vr} & \Delta x_{vv}
\end{pmatrix} .
\]

The matrices \( \Delta_{ab} \) are

\[
\Delta_{ab} = \begin{pmatrix}
\langle a_x b_x \rangle & \langle a_x b_y \rangle & \langle a_x b_z \rangle \\
\langle a_y b_x \rangle & \langle a_y b_y \rangle & \langle a_y b_z \rangle \\
\langle a_z b_x \rangle & \langle a_z b_y \rangle & \langle a_z b_z \rangle
\end{pmatrix}
\]

with the vectors \( a, b = r, p/m \). We now derive the variance evolution when the atomic field propagates under an Hamiltonian which is quadratic in position and impulse

\[
H(\hat{r}, \hat{p}) = \frac{\hat{p} \beta(t) \hat{p}}{2m} - \frac{\hat{r} \alpha(t) \hat{p}}{m} - \frac{m \hat{r}}{2} \Gamma(t) \hat{r} - m g(t) \hat{r} + f(t) \cdot \hat{p} . \tag{5}
\]

\( \alpha(t), \beta(t) \) and \( \gamma(t) \) are \( 3 \times 3 \) matrices; \( f(t) \) and \( g(t) \) are three dimensional vectors. It follows from the equations of motion for the position and impulse operators in the Heisenberg picture that the variance matrix \( V \) satisfies

\[
\frac{dV}{dt} = \Gamma V + V \Gamma \quad \Gamma(t) = \begin{pmatrix}
\alpha(t) & \beta(t) \\
\gamma(t) & \alpha(t)
\end{pmatrix} .
\]

The integration of this last equation involves the time-ordering operator \( T \) and the atom-optical \( ABCD \) ma-
The most straightforward attempt to generalize the uni-
Eq. (2) instead of single coordinates
laws in optics
The evolution of the variance matrix in the quadratic
M
 cyclic property of the trace, the definition (8) can be
simple and identical to optics [20]. Combining relation
\[ \frac{d}{dt} \text{tr} \left[ \begin{pmatrix} A(t,t_0) & B(t,t_0) \\ C(t,t_0) & D(t,t_0) \end{pmatrix} \right] = T \exp \left[ - \int_{t_0}^t dt' \left( \begin{pmatrix} \alpha(t') & \beta(t') \\ \gamma(t') & \alpha(t') \end{pmatrix} \right) \right]. \]
The evolution of the variance matrix in the quadratic Hamiltonian [3] is then similar to the transformation laws in optics
\[ V(t) = M(t,t_0) V(t_0) \tilde{M}(t,t_0). \] (6)
The most straightforward attempt to generalize the uni-
dimensional quality factor \( M_1^2 \) to three dimensions would consist in considering norms and scalar products in Eq. (3) instead of single coordinates
\[ \sqrt{(x^2 + y^2 + z^2)(p_x^2 + p_y^2 + p_z^2)} - (\mathbf{r} \cdot \mathbf{p})^2. \]
At an instant satisfying \( \langle \mathbf{r} \cdot \mathbf{p} \rangle = 0 \), the multidimensional equivalent of a “waist”, this expression would express the greater phase space occupied by the matter beam. Unfortunately, this quantity is not left invariant in the propagation under a general quadratic Hamiltonian.

Nonetheless, it is possible to define a quality factor which respects the invariance requirement and still provides a meaningful insight into the beam phase space. We use the fact that the \( ABCD \) matrices associated with matter wave propagation are symplectic, i.e., they verify at all times the following relation
\[ M(t,t_0)^{-1} = L M(t,t_0) \tilde{L} \iff L = \tilde{M}(t,t_0) L M(t,t_0), \] (7)
with the matrix
\[ L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]
This symplectic structure can indeed be used to generate a family of invariants [20, 22]. The lowest order of this invariant family indeed generalizes the definition of the current matter beam quality factor
\[ M_3^2(t) = \frac{4m^2}{3h^2} \text{Tr} \left[ V(t) L V(t) L \right]. \] (8)
The constant is adjusted to yield \( M_3 = 1 \) for a Gaussian matter wave. Proving the invariance of this parameter is simple and identical to optics [20]. Combining relation [3] describing the variance matrix propagation with the cyclic property of the trace, the definition (8) can be recast as
\[ M_3^4(t) = \frac{4m^2}{3h^2} \text{Tr} \left[ V(t_0) M(t,t_0) \tilde{M}(t,t_0) V(t_0) \tilde{M}(t,t_0) L M(t,t_0) \right]. \]
The symplectic relation [4] then gives the desired invariance of \( M_3^4(t) \):
\[ M_3^4(t) = \frac{4m^2}{3h^2} \text{Tr} \left[ V(t_0) L V(t_0) L \right] = M_3^4(t_0). \] (9)
Its expression in terms of position and impulsion momenta is
\[ M_3^4 = \frac{4}{3h} \left( (x^2)(y^2) - (x p_x)^2 + (y^2)(p_x^2) - (y p_y)^2 \\
+ (z^2)(p_z^2) - (z p_z)^2 + 2(xy)(p_x p_y) - (xp_y)(yp_x) \\
+ 2((xz)(p_x p_z) - (xp_z)(zp_x)) + 2((yz)(p_y p_z) - (yp_z)(zp_y)) \right). \] (10)
This expression can be rewritten in a more compact form as a sum of moments along single and multiple directions:
\[ M_3^4 = \frac{1}{3} [Q_x + Q_y + Q_z - A_{xx} - A_{yy} - A_{zz}] \]
\[ Q_\eta = \frac{4}{h^2} \langle \eta^2 \rangle |\eta|^2 \]
\[ A_{\eta \eta'} = \frac{8}{h^2} (\langle \eta \eta' \rangle \langle \eta' \eta \rangle - \langle \eta \eta' \rangle \langle \eta \eta' \rangle) \] (11)
The quantities \( Q_\eta \) correspond to the fourth power of the previous matter-wave quality factor \( M_1 \) considered along the three spatial directions, while the terms \( A_{\eta \eta'} \) reflect correlations between different directions. If one chooses the coordinate system along the beam principal axis, the position momenta satisfy \( \langle \eta \rangle = 0 \) for \( \eta \neq \eta' \), yielding a simpler expression for \( A_{\eta \eta'} \):
\[ A_{\eta \eta'} = \frac{8}{h^2} \langle \eta \eta' \rangle \langle \eta' \eta \rangle \]
These two families of parameters have different physical significance and obey different constraints. The parameters \( Q_\eta \) are a manifestation of the beam divergence and are bounded below by the Heisenberg principle: \( Q_\eta \geq 1 \). The terms \( A_{\eta \eta'} \) reflect the beam asymmetry and can be of either sign, they cancel for spherical clouds. As previously announced, a Gaussian matter wave satisfies \( M_3 = 1 \).

This atomic beam quality factor thus reflects the departure from a fundamental Gaussian mode: a beam with a quality factor \( M_3 \gg 1 \) needs to be expanded onto several modes, which is likely to degrade the fringe pattern in an atomic interference experiment or to add additional noise in the atomic beam [25]. In principle, if these modes were put into a fully coherent and accurately controlled superposition, a high value of the quality factor would merely induce an additional complexity in the fringe pattern. This ideal situation is, however, not encountered in practice, since the coefficients of the mode decomposition are generally not accessible. For most precision interferometric measurement, a simple TEM\(_{00}\) mode is indeed highly preferable. In the LIGO experiment, the beam quality factor of the laser involved is maintained at a value \( M \approx 1 \) thanks to a mode-cleaning step which filters out high order modes [24, 27]. This quality factor requirement, reflecting the control on the beam structure, should also apply to accurate interferometers involving atom lasers.
It seems in general more advantageous to minimize the quantities $Q$ which are directly related to the beam collimation. In this respect, the fact that the invariant quantity be $M_3$ instead of the coefficients $Q$, has a practical interest. Considering for instance a cylindrical beam with given $Q_x = Q_y = Q_z = M_1^2$ and $Q_z$, it is possible to reduce $Q_z$ while keeping the quality factor $M_3$ constant by increasing $Q_z$ or by generating negative asymmetrical parameters $A_{xy}$ [29]. One can thus trade some longitudinal collimation or cylindrical symmetry for a better transverse collimation. The corresponding $ABCD$ transformation could be implemented by astigmatic atom-optical lenses such as electromagnetic waves with an asymmetric wave-front [13]. This possibility of improvement is ignored in Ref. [10, 14], which assume that the transverse beam quality factor $Q$ [Eq. (10)], which addresses the general propagation of an atom laser. Higher order invariants proportional to $k = \left( \frac{\pi}{\sigma} \right)^{2k} \text{Tr} [V(t) L V(t) L]_k$ could also be considered to describe the atomic beam, but they do not provide the same insight into the phase-space density.

Note: Since the publication of this paper, a different nonlinear matter wave quality factor has been proposed [29], which is preserved in presence of mean-field interactions. However, this nonlinear parameter is suitable only for uniform and paraxial atomic beams propagating in transverse potentials of cylindrical symmetry. The generalizations of the matter wave quality factor presented in this paper and in Ref. [29] are indeed complementary: they enable one to apply the concept of beam quality beyond the paraxial and beyond the linear regime of propagation respectively. The non-paraxial and the non-linear matter wave quality factors can thus be fruitfully applied in different contexts.

IV. CONCLUSION

A quality factor has been defined for matter waves $[M_3$, Eq. (10)], which addresses the general propagation of an atom laser. It generalizes the currently adopted beam quality factor $[M_1$, Eq. (1)], indeed only appropriate to describe the propagation of matter-waves in the paraxial regime and in cylindrical potentials, and which can lead to underestimate the optimal collimation of an atom laser. The author is greatly indebted to Christian J. Bordé for enlightening discussions on atom optics. He thanks Emeric De Clercq for manuscript reading. This work was supported by DGA.

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