Infrared structure of $e^+e^- \rightarrow 3$ jets at NNLO - the $C_F^2$ contribution

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We discuss the infrared structure of $e^+e^- \rightarrow 2$ and 3 jets at NNLO in QCD perturbation theory and describe subtraction terms that render the separate parton-level contributions finite. As a first result, we find that the NNLO $C_F^2$ contribution to the first moment of the Thrust distribution $\langle 1 - T \rangle = -20.4 \pm 4$.

1. Introduction

Hadronic event shapes and jet production observables can be measured very accurately at LEP and future high energy electron positron colliders. By confronting these data with theoretical calculations, one can determine the strong coupling constant. Analyzing the different sources of error on these determinations, it becomes clear that the largest source of uncertainty is theoretical and mainly due to the truncation of the perturbation series at next-to-leading order (NLO).

To improve this situation, the calculation of next-to-next-to-leading order (NNLO) corrections to jet observables is mandatory. For an $n$-jet observable, several ingredients are required: the two-loop $n$-parton matrix elements, the one-loop $(n+1)$-parton matrix elements and the tree level $(n+2)$-parton matrix elements.

In the specific case of three-jet final states in $e^+e^-$ annihilation the primary process is $\gamma^* \rightarrow q\bar{q}g$, the decay of a virtual photon into a quark–antiquark pair accompanied by a gluon. The individual partonic channels that contribute through to NNLO are shown in Table 1. In the recent past, enormous progress has been made in the calculation of two-loop QCD matrix elements, which are now known for $\gamma^* \rightarrow q\bar{q}g$ [12]. The one-loop matrix elements with one additional parton [3] and the tree-level matrix elements with two more partons are also known and form part of NLO programs for four-jet production [4].

2. The infrared problem

Now that all the pieces are available, all that remains is to combine them in a way that produces numerically stable results for physical observables. To achieve this, the contributions of the processes shown in Table 1 are weighted by jet functions which select three-jet final states from the partonic final state momenta. At a given order, all partonic multiplicity channels contributing to this order have to be summed. However, each partonic channel contains infrared singularities which, after summation, cancel among each other. While infrared singularities from purely
virtual corrections are obtained immediately after integration over the loop momenta, their extraction is more involved for real emission (or mixed real-virtual) contributions. Here, the infrared singularities become only explicit after integrating the real radiation matrix elements over the phase space appropriate to the jet observable under consideration.

Exactly how to accomplish this task is presently an open question - see Refs. [6,7] and the problem of integrating out double real emission contributions has so far only been addressed in specific calculations [9,10,11,12], each of which requires a subset of the ingredients needed for generic jet observables at NNLO.

The infrared singularities of the real radiation contributions can be extracted using infrared subtraction terms. These subtraction terms are constructed such that they approximate the full real radiation matrix elements in all singular limits while still being sufficiently simple to be integrated analytically over a section of phase space that encompasses all regions corresponding to singular configurations. In all cases, the subtraction terms must be local in phase space. However, there are two distinct approaches to derive the subtraction terms. In the first, one uses phase space remappings together with the iterated sector decomposition [13,14] to extract the singularities from individual terms in the matrix elements in terms of plus prescriptions [15,16,17,18]. In the second [5], one identifies one- and two-particle subtraction functions that approximate the full matrix elements in all of the singular limits and which are sufficiently simple to be integrated analytically over the unresolved phase space. It is this latter approach that we follow here.

One-particle subtraction at tree level is well understood from NLO calculations and general algorithms are available for one-particle subtraction at one-loop [19,20,21,22], in a form that has recently been integrated analytically [21,22].

Similarly, tree-level two-particle subtraction terms have been extensively studied in the literature [23,24,25]. However their integration over the unresolved phase space remains an outstanding issue.

To specify the notation, we define the tree level $m$-parton contribution to the $J$-jet cross section in $d$-dimensions by,

$$d\sigma^B \sim d\Phi_m |M_m|^2 F^{(m)}_J.$$  

where $\sim$ hides all QCD-independent factors, the sum over all configurations with $m$ partons and symmetry factors for identical partons in the final state. $d\Phi_m$ is the phase space for $m$ partons and $|M_m|$ is the tree level $m$-parton matrix element. The jet function $F^{(m)}_J$ defines the procedure for building $J$-jets out of $m$ partons. The main property of $F^{(m)}_J$ is that the jet observable defined above is collinear and infrared safe.

### 3. NLO infrared subtraction terms

At NLO, we consider the following $m$-jet cross section,

$$d\sigma^{mNLO} = \int d\Phi_m \left( d\sigma^{NLO}_R - d\sigma^{NLO}_S \right)$$

where $\sim$ hides all QCD-independent factors, the sum over all configurations with $m$ partons and symmetry factors for identical partons in the final state. $d\Phi_m$ is the phase space for $m$ partons and $|M_m|$ is the tree level $m$-parton matrix element. The jet function $F^{(m)}_J$ defines the procedure for building $J$-jets out of $m$ partons. The main property of $F^{(m)}_J$ is that the jet observable defined above is collinear and infrared safe.

The cross section $d\sigma^{R}_{NLO}$ has the same expression as the Born cross section $d\sigma^{R}_{NLO}$ above except that $m \rightarrow m+1$, while $d\sigma^{S}_{NLO}$ is the one-loop virtual correction to the $m$-parton Born cross section $d\sigma^{B}$. The cross section $d\sigma^{S}_{NLO}$ is a local counter-term for $d\sigma^{R}_{NLO}$. It has the same unintegrated singular behavior as $d\sigma^{R}_{NLO}$ in all appropriate limits. Their difference is free of divergences and can be integrated over the $(m+1)$-parton phase space numerically. The subtraction term $d\sigma^{R}_{NLO}$ has to be integrated analytically over all singular regions of the $m+1$-parton phase space. The resulting cross section added to the virtual contribution yields an infrared finite result.

A systematic procedure for finding NLO infrared subtraction terms in the second method is the dipole formalism derived by Catani and Seymour [26]. Their subtraction terms are obtained as sum of dipoles $\sum D_{ijk}$ (where each dipole corresponds to a single infrared singular configuration) such that,

$$d\sigma^{R}_{NLO} - d\sigma^{S}_{NLO} = d\Phi_{m+1} \left[ |M_{m+1}|^2 F^{(m+1)}_J \right].$$
\[- \sum_{\text{pairs } i,j \neq i,j} D_{ijk} \mathcal{M}_m((\tilde{p}_{ij}, \tilde{p}_k))^2 F_j^{(m)}(\tilde{p}_{ij}, \tilde{p}_k) \]\n
The dipole contribution \( D_{ijk} \) involves the \( m \)-parton amplitude depending on the redefined on-shell momenta \( \tilde{p}_{ij}, \tilde{p}_k \) and the dipole \( D_{ijk} \) which depends only on \( p_i, p_j, p_k \). The momenta \( p_i, p_j \) and \( p_k \) are respectively the emitter, unresolved parton and the spectator momenta corresponding to a single dipole term. The redefined on-shell momenta \( \tilde{p}_{ij}, \tilde{p}_k \) are linear combinations of them.

### 4. NNLO infrared subtraction terms

At NNLO, \( m \)-jet production is induced by final states containing up to \( (m+2) \) partons, including the one-loop virtual corrections to \( (m+1) \)-parton final states. As at NLO, one has to introduce subtraction terms for the \( (m+1) \)- and \( (m+2) \)-parton contributions. Schematically the NNLO \( m \)-jet cross section reads,

\[
d\sigma_{NNLO} = \int d\Phi_{m+2} (d\sigma_{NNLO}^R - d\sigma_{NNLO}^S) + \int d\Phi_{m+1} (d\sigma_{NNLO}^{V,1} - d\sigma_{NNLO}^{V,S,1}) + \int d\Phi_{m+2} d\sigma_{NNLO}^S + \int d\Phi_{m+1} d\sigma_{NNLO}^{V,S,1} + \int d\Phi_m d\sigma_{NNLO}^{V,2}
\]

where \( d\sigma_{NNLO}^S \) denotes the real radiation subtraction term coinciding with the \( (m+2) \)-parton tree level cross section \( d\sigma_{NNLO}^R \) in all singular limits. Likewise, \( d\sigma_{NNLO}^{V,S,1} \) is the one-loop virtual subtraction term coinciding with the one-loop \( (m+1) \)-parton cross section \( d\sigma_{NNLO}^{V,1} \) in all singular limits. Finally, the two-loop correction to the \( m \)-parton cross section is denoted by \( d\sigma_{NNLO}^{V,2} \).

In the simple case of two-jet production, the subtraction terms have been fully identified [24].

The four-particle contribution is

\[
d\sigma_{NNLO}^R = \Phi_4 |M_4|^2 F_2^{(4)}.
\]

Motivated by the fact that for three-jet production, the sum over dipoles is essentially equivalent to the three-parton matrix element, the four-parton subtraction term is given by,

\[
d\sigma_{NNLO}^S = \Phi_4 |M_4|^2 F_2^{(2)} + \sum_{ijk} |M_3|^2 D_{ijk} \left( F_2^{(3)} - F_2^{(2)} \right). (5)
\]

The first term precisely cancels the real radiation when the four-parton configuration is perceived as a two-jet event. The second removes configurations when only one parton is unresolved, but the jet algorithm still sees only two jets. The final term eliminates cases when we double count. Taken together, Eqs. 5 and 6 yield a finite result.

The one-loop three-parton contribution is,

\[
d\sigma_{NNLO}^{V,1} = \Phi_3 |M_3^{V,1}|^2 F_2^{(3)} (6)
\]

and the subtraction term is

\[
d\sigma_{NNLO}^{V,S,1} = \Phi_3 |M_3^{V,S,1}|^2 F_2^{(2)} - \sum_{ijk} |M_3|^2 D_{ijk} \left( F_2^{(3)} - F_2^{(2)} \right). (7)
\]

As before, we subtract the full virtual matrix elements so that the first term precisely cancels \( d\sigma_{NNLO}^{V,1} \) when the three-parton configuration is perceived as a two-jet event. The second and third terms are merely the reappearance of the four-parton subtraction terms when a single parton is unresolved. Taken together, Eqs. 7 and 8 yield a finite result. Finally, the two-parton contribution is made up of the two-loop contribution

\[
d\sigma_{NNLO}^{V,2} = \Phi_2 |M_2|^2 |M_2^{V,2}|^2 F_2^{(2)}
\]

and the analytically integrated subtractions terms (the first term on the RHS of Eqs. 6 and 8),

\[
\int d\Phi_{m+2} d\sigma_{NNLO}^S + \int d\Phi_{m+1} d\sigma_{NNLO}^{V,S,1} = \Phi_2 |M_2|^2 \left[ \int d\Phi_T |M_4|^2 + \int d\Phi_D |M_3^{V,1}|^2 \right] F_2^{(2)} (9)
\]

where, the matrix elements are normalized to the two-parton matrix element such that

\[
|M_j|^2 = \frac{1}{|M_2|^2} |M_j|^2.
\]
and
\[ d\Phi_3 = d\Phi_2 \, d\Phi_D, \quad d\Phi_4 = d\Phi_2 \, d\Phi_T, \]
(11)
defines the dipole and tripole phase space, \( d\Phi_D \) and \( d\Phi_T \) that the subtraction terms must be integrated over. All of the master integrals necessary to perform the integrations in Eq. 10 have been evaluated in Ref. [16]. The result is an analytic expression in \( d \)-dimensions such that when taken together, Eqs. 9 and 10 yield a finite result. For the inclusive hadronic \( R \)-ratio, \( F_j^{(m)} = 1 \) and the only remaining contribution is the two-parton piece which agrees with that found in the literature [28]. Similar results for distributions have been found using the sector decomposition method [18].

5. Three-jet event shapes at NNLO

The subtraction terms based on subtracting the full matrix element described above are specific to the two-jet configurations. Subtracting the full tree-level five-parton matrix elements and integrating it analytically is neither sensible nor feasible. On the other hand, one might expect that the singular behaviour of the five-parton matrix elements can be represented by simpler building blocks that depend on only four of the five parton momenta multiplied by three-parton matrix elements that depend on momenta built from the original parton momenta. In this case, we can repeat the same steps as for the two-jet subtraction terms and use Ref. [16] to analytically integrate the subtraction term.

At NNLO, the three-jet cross section contains seven colour structures,
\[ \frac{1}{\sigma_0} d\sigma^{NLO}_3 = C_F \left[ A C_A^2 + B C_A C_F + C C_F^2 \right. \]
\[ \left. + D C_A N_F + E C_F N_F + F N_F^2 + G N_F \chi \left( \frac{4}{N} - N \right) \right] . \]
(12)
Because of the QED-like behaviour of the \( C_F^2 \) colour factor contribution, the subtraction terms (for this colour factor) are very similar to the two-jet case. Accordingly, we have implemented these terms in the NLO four-jet program EERAD2 [3] and find that the five- and four-parton contributions are numerically finite. At the same time, the analytic integration of the subtraction terms precisely cancels the infrared poles found in the two-loop matrix elements [4].

5.1. The average value of 1-Thrust

One of the classic event shape variables is Thrust. The first moment of the Thrust distribution \( \langle 1 - T \rangle \) is safe from large infrared logarithms and is therefore a theoretically clean and experimentally relevant observable. The perturbative expansion of \( \langle 1 - T \rangle \) is given by
\[ \langle 1 - T \rangle = \int (1 - T) \frac{d\sigma}{\sigma_0} dT \]
\[ = C_F \left[ \left( \frac{\alpha_s}{2\pi} \right) A + \left( \frac{\alpha_s}{2\pi} \right)^2 B + \left( \frac{\alpha_s}{2\pi} \right)^3 C + \ldots \right] \]
where \( A = 1.57, B = 32.3 \). The NNLO coefficient \( C \) receives contributions from all seven colour structures. We find that the \( C_F^2 \) colour factor has the value,
\[ C|_{C_F^2} = -20.4 \pm 4. \]

6. Summary

We have discussed the infrared singularity structure of jets in electron-positron annihilation at NNLO. For 2-jet events the subtraction terms necessary to render the individual partonic contributions finite has been worked through in detail [27]. In the case of 3-jet events, the subtraction terms for the \( C_F^2 \) colour factor have been identified and analytically integrated. We have implemented these in a numerical program and produced first NNLO results for a 3-jet event shape. Analogous results for the remaining six colour structures are in progress.

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