Selective Inference in Propensity Score Analysis

Yoshiyuki Ninomiya
Department of Statistical Inference and Mathematics
The Institute of Statistical Mathematics

Yuta Umezu
School of Information and Data Science, Nagasaki University

Ichiro Takeuchi
Department of Computer Science, Nagoya Institute of Technology
RIKEN Center for Advanced Intelligence Project

Abstract

Selective inference (post-selection inference) is a methodology that has attracted much attention in recent years in the fields of statistics and machine learning. Naive inference based on data that are also used for model selection tends to show an overestimation, and so the selective inference conditions the event that the model was selected. In this paper, we develop selective inference in propensity score analysis with a semiparametric approach, which has become a standard tool in causal inference. Specifically, for the most basic causal inference model in which the causal effect can be written as a linear sum of confounding variables, we conduct Lasso-type variable selection by adding an $\ell_1$ penalty term to the loss function that gives a semiparametric estimator. Confidence intervals are then given for the coefficients of the selected confounding variables, conditional on the event of variable selection, with asymptotic guarantees. An important property of this method is that it does not require modeling of nonparametric regression functions for the outcome variables, as is usually the case with semiparametric propensity score analysis.

Keywords: causal inference; confidence interval; inverse-probability-weighted estimation; Lasso; pivot statistic; post-selection inference; variable selection
1 Introduction

It has often been pointed out that generating a hypothesis of interest from data and then using the same data to construct a statistical test or a confidence interval for that hypothesis is inappropriate. Breiman (2001) called it a “quiet scandal of statistics.” The so-called “file drawer problem” can be regarded as this kind of problem. A typical example in regression is naive inference on the coefficients of the selected variables after variable selection. To deal with this variable selection problem, “simultaneous inference” was proposed by Berk et al. (2013). Simultaneous inference gives a reasonable inference no matter what model is selected, but as one can imagine from the methods of multiple testing, it is rather conservative. In these situations, a “selective inference” approach, which is based on the work of Lee and Taylor (2014), Lee et al. (2016), and Tibshirani et al. (2016), has recently gained attention in the field of statistics and machine learning. Selective inference conditions on that the model has been selected. At about the same time as these papers were published, Taylor and Tibshirani (2015) explained the great potential of selective inference. It has been extensively developed since then.

On the other hand, causal inference has been a hot topic in biostatistics and econometrics and has recently been of importance in machine learning (see, e.g., Peters et al. 2017; Hernán and Robins 2020). In particular, propensity score analysis with a semiparametric approach has become a standard tool, as it is a theoretically guaranteed method that avoids the bias introduced by the existence of confounding. To illustrate the motivation for this paper, let us examine one of the simplest models in causal inference, where the causal effect can be written as a linear sum of confounding variables, as follows:

\[
Y_i = T_i X'_i \beta^{(1)} + (1 - T_i) X'_i \beta^{(0)} + \xi_i \quad (1 \leq i \leq n).
\]

The subscript \(i\) \((\in \{1, \ldots, n\})\) represents the \(i\)-th sample, \(Y_i\) \((\in \mathbb{R})\) is the outcome variable, \(T_i\) \((\in \{0, 1\})\) is the assignment variable, and \(X_i = (X_{i1}, \ldots, X_{ip})'\) is the confounding variable vector for \(Y_i\) and \(T_i\). In addition, \(\xi_i\) \((\in \mathbb{R})\) is an unobserved random variable that depends on \(X_i\), and the conditional distribution given \(X_i\) is supposed to be normal. We assume that \(T_i \perp Y_i \mid X_i\), which is called the ignorable treatment assignment condition. In this model, the parameter of the causal effect is \(\beta^{(1)} - \beta^{(0)}\), and the basic semiparametric method for consistent estimation is the inverse-probability-weighted estimation using the propensity score \(e(X_i) \equiv P(T_i = 1 \mid X_i)\) (Rubin 1985; Robins et al. 1994). It has the advantage of not requiring an estimation of \(E(\xi_i \mid X_i)\), which is difficult to model, and we rely upon it in this paper.
Now, supposing that the dimension $p$ of the confounding variable vector is high, let us try to estimate the causal effect by selecting variables to improve the estimation accuracy. Specifically, letting $\| \cdot \|_1$ be the $\ell_1$ norm, we consider

$$\sum_{i=1}^{n} \left( \frac{T_i}{e(X_i)} - \frac{1 - T_i}{1 - e(X_i)} \right) Y_i - X'_i \beta \right)^2 + \lambda \| \beta \|_1$$

and give its minimizer with respect to $\beta$ as a Lasso-type estimator (Tibshirani 1996). As mentioned at the beginning, this type of variable selection affects the subsequent inference of the causal effect, for example, in constructing a confidence interval for $\beta^{(1)} - \beta^{(0)}$. Therefore, we will attempt to combine causal inference with selective inference. Specifically, on the basis of the method of Lee et al. (2016), we can give a confidence interval with a confidence coefficient of $\alpha$ conditional on the event of variable selection for each component of $\beta^{(1)} - \beta^{(0)}$ with the guarantee of asymptotics. The same problem has been addressed by Zhao et al. (2017), who give a nonparametric estimation of $E(\xi_i \mid X_i)$ without using propensity scores. One of the reasons why propensity score analysis has become a hot topic is that it does not necessarily require this nonparametric estimation; we put much value on it in this paper.

As a preparation, Section 2 introduces propensity score analysis in causal inference models and selective inference with a focus on Lee et al. (2016). Section 3 deals with selective inference in propensity score analysis and builds a foundation for selective inference by conditioning on the assignment variables before the outcome variables, an operation that is not usually considered in propensity score analysis. Then, asymptotically guaranteed confidence intervals are given by using higher-order asymptotic theory. Sections 4 and 5 describe numerical experiments and real data analyses comparing the proposed method with a naive method that ignores the influence of model selection. We find that there is a significant difference between the two methods and that the naive method lacks validity to a large extent. In Section 6, we extend the proposed method to handle especially developed versions of propensity score analysis and conclude in Section 7.

## 2 Preparation

### 2.1 Causal inference model

Let us assume that there are $H$ types of treatments, and let $Y^{(h)} (\in \mathbb{R})$ be the potential outcome variable when the $h$-th treatment is received. In addition, let $T^{(h)}$ be the assignment variable that is 1 when $Y^{(h)}$ is observed and 0 when it is not ($h \in \{1, \ldots, H\}$, $\sum_{h=1}^{H} T^{(h)} = 1$). Moreover, let $X (\in \mathbb{R}^p)$ be the confounding variable vector, i.e., $p$ kinds of confounding variables
are observed; we consider the model,

\[ Y \equiv \sum_{h=1}^{H} T^{(h)} Y^{(h)} = \sum_{h=1}^{H} T^{(h)} \{\mu^{(h)}(X) + f(X) + \epsilon^{(h)}\}. \]  

(1)

Here, \( f : \mathbb{R}^p \to \mathbb{R} \) is a nonlinear function and \( \epsilon^{(h)} \) is the error variable that follows a normal distribution \( N(0, \sigma^2) \) independently of \( X \). Note that \( Y \) is the observed outcome variable. Letting \((c^{(1)}, \ldots, c^{(H)})' \in \mathbb{R}^H\) be a contrast satisfying \( \sum_{h=1}^{H} c^{(h)} = 0 \), we suppose that the object of interest is the conditional average treatment effect \( \sum_{h=1}^{H} c^{(h)} \mu^{(h)}(x) \) for \( X = x \). If \( H = 2 \), and \( h = 1 \) and \( h = 2 \) represent the control and treatment groups, respectively, and if \( \mu^{(1)}(x) \) and \( \mu^{(2)}(x) \) do not depend on \( x \), then this model is the simplest Rubin’s causal inference model, and the causal effect \( \mu^{(2)} - \mu^{(1)} \) defined by \((c^{(1)}, c^{(2)})' = (-1, 1)' \) becomes the simple one estimated in Rosenbaum and Rubin (1983) or Rubin (1985).

Suppose there are \( n \) samples that follow this model and the variables in the \( i \)-th sample are identified by the subscript \( i \). Letting \( \tilde{Y} = (Y_1, \ldots, Y_n)' \), \( T^{(h)} = \text{diag}(T_1^{(h)}, \ldots, T_n^{(h)}) \), \( \tilde{X} = (X_1, \ldots, X_n)' \), \( \tilde{\mu}^{(h)}(\tilde{X}) = (\mu^{(h)}(X_1), \ldots, \mu^{(h)}(X_n))' \), \( \tilde{f}(\tilde{X}) = (f(X_1), \ldots, f(X_n))' \), and \( \tilde{\epsilon}^{(h)} = (\epsilon_1^{(h)}, \ldots, \epsilon_n^{(h)})' \), we express the model as

\[ \tilde{Y} = \sum_{h=1}^{H} \tilde{T}^{(h)} \tilde{Y}^{(h)} = \sum_{h=1}^{H} \tilde{T}^{(h)} \{\tilde{\mu}^{(h)}(\tilde{X}) + \tilde{f}(\tilde{X}) + \tilde{\epsilon}^{(h)}\}. \]

(2)

We will assume the usual conditions for this setting. First, we assume the weakly ignorable treatment assignment condition (Imbens 2000),

\[ Y_i^{(h)} \perp T_i^{(h)} \mid X_i \quad (i \in \{1, \ldots, n\}, h \in \{1, \ldots, H\}). \]

(3)

Note that \( Y_i^{(h)} \) in this condition can be changed to \( \epsilon_i^{(h)} \). Second, we assume the positivity condition \( 0 < P(T_i^{(h)} = 1 \mid X_i) < 1 \) \((i \in \{1, \ldots, n\})\). Third, we assume that the samples are independent, that is,

\[ (T_i^{(h)}, X_i, \epsilon_i^{(h)}) \perp (T_l^{(h)}, X_l, \epsilon_l^{(h)}) \quad (i \neq l, i, l \in \{1, \ldots, n\}, h \in \{1, \ldots, H\}). \]

As a natural consequence, this means that \( Y_i \perp Y_l \) \((i \neq l, i, l \in \{1, \ldots, n\})\).

In this paper, we will suppose that the dimension \( p \) of the confounding variable vector is high and try to estimate the conditional average treatment effect with higher accuracy by simultaneously conducting variable selection and estimation. Specifically, we will use a Lasso-type method (Tibshirani 1996) in Section 3. Note that a model selection method for causal inference was recently proposed by Baba et al. (2017), but it deals with the selection of marginal structures and is not used here.
In (2), the $H - 1$ potential outcome variables $Y_i^{(h)}$ with $T_i^{(h)} = 0$ are considered to be missing. Since $E(Y_i^{(h)}) \neq E(Y_i^{(h)} \mid T_i^{(h)} = 1)$ in general, simply estimating the causal effect $\sum_{h=1}^{H} c^{(h)}(x)$ by using the least-squares method based on minimization of $\|\sum_{h=1}^{H} c^{(h)} T^{(h)} Y - \sum_{h=1}^{H} c^{(h)} \mu^{(h)}(X)\|_2^2$ would result in a large bias. Here, $\| \cdot \|_2$ is the $\ell_2$ norm. As described in Section 1, this paper deals with inverse-probability-weighted estimation using the propensity score $c^{(h)}(X_i) \equiv P(T_i^{(h)} = 1 \mid X_i)$ to avoid the bias.

In this estimation method, the missing values are pseudo-recovered by multiplying the observed values by the inverse of the propensity score as weights; then, the usual estimation is conducted. Specifically, letting $\tilde{X}$ served values by the inverse of the propensity score as weights; then, the usual estimation is $\sum_{h=1}^{H} c^{(h)} \tilde{T}^{(h)} \tilde{Y} - \sum_{h=1}^{H} c^{(h)} \tilde{\mu}^{(h)}(\tilde{X})\|_2^2$ would result in a large bias. Here, $\| \cdot \|_2$ is the $\ell_2$ norm. As described in Section 1, this paper deals with inverse-probability-weighted estimation using the propensity score $c^{(h)}(X_i) \equiv P(T_i^{(h)} = 1 \mid X_i)$ to avoid the bias.

In this estimation method, the missing values are pseudo-recovered by multiplying the observed values by the inverse of the propensity score as weights; then, the usual estimation is conducted. Specifically, letting $\tilde{W}^{(h)}(\tilde{T}^{(h)}, \tilde{X}) \equiv \text{diag}(\tilde{T}_1^{(h)} / c^{(h)}(X_1), \ldots, \tilde{T}_n^{(h)} / c^{(h)}(X_n))$ and $\tilde{T} = (\tilde{T}^{(1)}, \ldots, \tilde{T}^{(H)})$ and defining $\tilde{W}(\tilde{T}, \tilde{X}) \equiv \sum_{h=1}^{H} c^{(h)} \tilde{W}^{(h)}(\tilde{T}^{(h)}, \tilde{X})$ as the weight matrix, the inverse-probability-weighted estimator can be found by minimizing the squared loss function,

$$
\left\| \tilde{W}(\tilde{T}, \tilde{X}) \tilde{Y} - \sum_{h=1}^{H} c^{(h)} \tilde{\mu}^{(h)}(\tilde{X}) \right\|_2^2.
$$

When the propensity scores are unknown, we assume some parametric function for $c^{(h)}(X_i)$, maximize the likelihood function $\prod_{i=1}^{n} \prod_{h=1}^{H} c^{(h)}(X_i) T_i^{(h)}$ to obtain the estimator $\hat{c}^{(h)}(X_i)$, and use it instead. The inverse-probability-weighted estimator is consistent under the weakly ignorable treatment assignment condition in (3).

### 2.2 Selective inference

In Section 2.1, we suggested the use of variable selection. After the variable selection, in order to measure the extent to which the selected variables have an impact on the causal effects, tests are performed or confidence intervals are constructed. However, the problem is that the $p$-values used in this process are no longer reliable, because the selected variables are likely to be significant, or in other words, the model using the selected variables likely overfits the data.

To explain the inference after variable selection in detail, we will omit the superscript $(h)$ as $H = 1$ in (2); i.e., $\tilde{T}$ is an $n \times n$ identity matrix $I_n$. Let $\tilde{X}$ be a non-random matrix $\tilde{x}$, and let $\tilde{f}(\tilde{x})$ be an $n$-dimensional zero vector $0_n$. Moreover, we will consider a linear function of $\tilde{x}$ as a model for $\tilde{\mu}(\tilde{x})$. Then, for the subset $M \subset \{1, \ldots, p\}$, we define an estimand as

$$
\beta^M \equiv \arg \min_{b^M} \text{E}(\| \tilde{Y} - \tilde{x}_M b^M \|_2) = (\tilde{x}_M \tilde{x}_M)^{-1} \tilde{x}_M \tilde{\mu}(\tilde{x}),
$$

which minimizes the expected squared error for the linear sum of the confounding variables belonging to $M$, where $\tilde{x}_M = (\tilde{x}_{ij})_{i \in \{1, \ldots, n\}, j \in M}$. If the models $M$ and $M^*$ are different, then
\( \beta_{j}^{\hat{M}} \) and \( \beta_{j}^{\hat{M}^*} \) are generally different, i.e., the target of inference is different for each model selected. Consequently, we can say that inference after selection has some ambiguity.

In terms of confidence interval construction, the simultaneous inference mentioned in Section 1 is a method of creating an interval in which all the regression coefficients regardless of which model is selected are included with probability \( 1 - \alpha \) or greater. When \( \hat{M} \) is the selected model and the regression coefficients in that model are denoted as \( \beta_{j}^{\hat{M}} (j \in \hat{M}) \), this method finds \( C_{j}^{\hat{M}} (j \in \hat{M}) \) such that

\[
P(\forall j \in \hat{M}, \ \beta_{j}^{\hat{M}} \in C_{j}^{\hat{M}}) \geq 1 - \alpha.
\]

Since the interval is guaranteed for all \( j \), a problem arises in that the interval becomes considerably wide when the size \( |\hat{M}| \) is large.

On the other hand, selective inference, in terms of confidence interval construction, is a method of creating an interval in which the regression coefficients in the selected model are included with probability \( 1 - \alpha \) or greater under the condition that the model was selected. The method is to find \( C_{j}^{\hat{M}} (j \in M) \) such that

\[
P(\beta_{j}^{\hat{M}} \in C_{j}^{\hat{M}} | \hat{M} = M) \geq 1 - \alpha.
\]

Accordingly, the false coverage rate is controlled to be

\[
E\left( \frac{|\{j \in \hat{M} : \beta_{j}^{\hat{M}} \notin C_{j}^{\hat{M}}\}|}{|\hat{M}|} ; |\hat{M}| > 0 \right) \leq \alpha. \tag{5}
\]

In the cornerstone papers, beautiful inferences were presented for marginal screening and Lasso model selection in the setting of normal distributions. Then, their validity was theoretically demonstrated in more generalized settings. For example, Fithian et al. (2014) showed the optimality of the method in terms of uniformly most powerful unbiasedness in the setting of exponential families of distributions, while Tian and Taylor (2017), Tian and Taylor (2018), and Tibshirani et al. (2018) showed that the distributions of the pivot statistics for getting \( p \)-values or confidence intervals converge uniformly to the actual distribution when a normal distribution is used, even when the noise has a non-normal distribution. Furthermore, Liu et al. (2018) proposed to increase the power of the original method by reducing the over-conditioning. In this paper, we develop selective inference for causal inference on the basis of Lee et al. (2016).

We will explain the method of Lee et al. (2016). Let us denote the usual Lasso estimator by \( \hat{\beta} = (\hat{\beta}_{1}, \ldots, \hat{\beta}_{p})' \), the collection of non-zero estimators by \( \hat{\beta}^{\hat{M}} \), and its sign by \( s^{\hat{M}} = \)
sign($\hat{\beta}^M$). In this case, from the Karuch-Kuhn-Tucker conditions, for any $s \in \{-1, 1\}^{|M|}$, there exists an $n \times n$ matrix $A(M, s)$ and an $n$-dimensional vector $b(M, s)$ such that

$$\{\hat{M}^+, s^\perp = s\} = \{A(M, s)Y \leq b(M, s)\}.$$ 

Now, using an appropriate unit vector $e_j (\in \mathbb{R}^{|M|})$, we define $\hat{\eta}_j^\perp \equiv \hat{x}_M(\hat{x}_M'\hat{x}_M)^{-1}e_j$. Since the target parameter can be written as $\beta_j^M = \eta_j^\perp \hat{\mu}(\hat{x})$, we use the statistic $\hat{\eta}_j^\perp Y$ to create its confidence interval. Then, by using the strong properties of a normal distribution, we obtain

$$F_{\beta_j^M, \sigma^2_j \hat{\eta}_j^\perp \hat{\eta}_j^\perp}((\hat{\eta}_j^\perp Y) \mid \{A(M, s)Y \leq b(M, s)\}) \sim \text{Unif}(0, 1).$$

Here, $F_{\mu, \sigma^2}()$ denotes the cumulative distribution function of $\text{TN}(\mu, \sigma^2, a, b)$, which is $\text{N}(\mu, \sigma^2)$ truncated into the interval $[a, b]$, $\hat{Z} = \{I_n - \hat{\eta}_j^\perp (\hat{\eta}_j^\perp \hat{\eta}_j^\perp)^{-1} \hat{\eta}_j^\perp \}\hat{Y}$,

$$V_{s, j}^{-}(\hat{Z}) = \max_{k: \{A(M, s)\hat{\eta}_j^\perp (\hat{\eta}_j^\perp \hat{\eta}_j^\perp)^{-1}\} < 0} \{b(M, s)k - (A(M, s)\hat{Z})_k\}/(A(M, s)\hat{\eta}_j^\perp (\hat{\eta}_j^\perp \hat{\eta}_j^\perp)^{-1})_k,$$

$$V_{s, j}^{+}(\hat{Z}) = \min_{k: \{A(M, s)\hat{\eta}_j^\perp (\hat{\eta}_j^\perp \hat{\eta}_j^\perp)^{-1}\} > 0} \{b(M, s)k - (A(M, s)\hat{Z})_k\}/(A(M, s)\hat{\eta}_j^\perp (\hat{\eta}_j^\perp \hat{\eta}_j^\perp)^{-1})_k,$$

and $\text{Unif}(0, 1)$ denotes a continuous uniform distribution in $[0, 1]$. We have obtained a pivot statistic that is unique to a normal distribution, and from it we can construct a statistical test or a confidence interval for $\beta_j^M$, with the significance level or coverage probability controlled in a conditional sense.

### 3 Main Result

Consider selective inference for the causal inference model in (2). This problem has been addressed by Zhao et al. (2017), who do not use propensity scores, but perform nonparametric estimation of $\hat{f}(\hat{X})$. One of the reasons why causal inference has become a hot topic is the rapid development of propensity score analysis, which does not necessarily require this nonparametric estimation, and so, in this section, we will avoid the nonparametric estimation as well. This avoidance would require an essential generalization of selective inference, unlike Zhao et al. (2017).

Now, for $\hat{\mu}^{(h)}(\hat{X})$ in (2), let us choose a linear function of $\hat{X}$ as a model. For the subset $M \subset \{1, \ldots, p\}$, because the causal effect is $\sum_{h=1}^{H} c^{(h)} \hat{\mu}^{(h)}(X)$, we define an estimand as

$$\beta^M \equiv \arg \min_{b^M} E\left(\left\| \sum_{h=1}^{H} c^{(h)} \hat{Y}^{(h)} - X_M b^M \right\|_2^2 \mid \hat{X} \right) = (X_M'X_M)^{-1}X_M' \sum_{h=1}^{H} c^{(h)} \hat{\mu}^{(h)}(X),$$
which minimizes the expected squared error for the linear sum of the confounding variables belonging to \( M \), where \( \hat{X}_M = (\hat{X}_{ij})_{i \in \{1, \ldots, n\}, j \in M} \). Note that, in this model, the causal effect is captured by \( x'_M \beta^M \). Then, denoting the selected model as \( \hat{M} \), we try to find \( C^M_j \) \((j \in M)\), which gives the conditional coverage,

\[
P(\beta^M_j \in C^M_j \mid \hat{M} = M) \geq 1 - \alpha
\]  

(6)

for the regression coefficient \( \beta^M_j \) \((j \in M)\).

This allows us to construct a statistical test or a confidence interval for \( \beta^M_j \). Using an appropriate unit vector \( e_j \) \((j \in \mathbb{R}^{|M|})\), we have \( \beta^M_j = e'_j(\hat{X}'_M \hat{X}_M)^{-1} \hat{X}'_M \sum_{h=1}^H c^{(h)}(\hat{X}) \).

Then, defining \( \tilde{\eta}_j = \hat{X}_M(\hat{X}'_M \hat{X}_M)^{-1}e_j \), later we consider the distribution of \( \tilde{\eta}'_j \hat{W}(\hat{X}, \hat{X})' \hat{Y} \) conditional on \( \hat{M} = M \). Since it is not possible to perfectly capture the true model by \( \hat{M} \) in finite data, and the true structure may not be included in the model, \( \hat{X}_M \beta^M \) is not itself the conditional average treatment effect \( \sum_{h=1}^H c^{(h)}(\mu^{(h)}(\hat{X})) \). However, it is a problem with selective inference itself, and it is not addressed in this paper.

As a model selection method, by letting \( \beta = \sum_{h=1}^H c^{(h)}(\beta^{(h)}) \) after using \( \tilde{\mu}^{(h)}(\hat{X}) = \hat{X}_M \beta^{(h)} \) in (4), we propose a Lasso-type one that estimates \( \beta \) by

\[
\hat{\beta} = \arg\min_{\beta} \left\{ \frac{1}{2} \| \hat{W}(\hat{T}, \hat{X})' \hat{Y} - \hat{X}_M \beta \|^2_2 + \lambda \| \beta \|_1 \right\}.
\]  

(7)

Note that \( \hat{Y} \) has a nonlinear component \( \hat{f}(\hat{X}) \), however, this nonlinear component is canceled out in \( \hat{W}(\hat{T}, \hat{X})' \hat{Y} \) and there is no problem in using Lasso for linear models. Since Lasso gives a sparse solution, the selected model can be expressed as

\[
\hat{M} = \{ j : \hat{\beta}_j \neq 0 \}.
\]

Here, we write \( \hat{\beta}^M = (\hat{\beta}_j)_{j \in M} \) for the collection of non-zero estimators and \( \hat{s}^M = \text{sign}(\hat{\beta}^M) \) for their sign.

The fact that \( \hat{\beta} \) is the solution of (7) is equivalent to the existence of \( \hat{s} = (\hat{s}_1, \ldots, \hat{s}_p)' \) satisfying the Karush-Kuhn-Tucker conditions,

\[
\hat{\beta}_j \neq 0 \Rightarrow \hat{s}_j = \text{sign}(\hat{\beta}_j),
\]

\[
\hat{\beta}_j = 0 \Rightarrow \hat{s}_j \in [-1,1]
\]

and

\[
\hat{X}' (\hat{X} \hat{\beta} - \hat{W}(\hat{T}, \hat{X})' \hat{Y}) + \lambda \hat{s} = 0.
\]
For the model $M$, if the candidate for the sign of its regression coefficients is $s \in \{-1, 1\}^{\lvert M \rvert}$, then we can use this equivalence conditions to show

$$\{\mathcal{M} = M, \ s^M = s\} = \{A(M, s)\hat{W}(\hat{T}, \hat{X})\hat{Y} \leq b(M, s)\}. \quad (8)$$

Here, we use the definitions,

$$A(M, s) = \frac{1}{\lambda} \begin{pmatrix} \hat{X}_M'\hat{X}_M^{-1} & -
\hat{X}_M'\hat{X}_M^{-1} \hat{X}_M' \\
\hat{X}_M'\hat{X}_M^{-1} & -\diag(s)\hat{X}_M'\hat{X}_M^{-1} \end{pmatrix},$$

and

$$b(M, s) = \begin{pmatrix} 1_{\lvert M^c \rvert} - \hat{X}_M'\hat{X}_M^{-1}s \\
1_{\lvert M^c \rvert} + \hat{X}_M'\hat{X}_M^{-1}s \\
\diag(s)\hat{X}_M'\hat{X}_M^{-1}s \end{pmatrix},$$

where $M^c = \{1, \ldots, p\} \setminus M$, $\hat{X}_{M^c} = (\hat{x}_{ij})_{i\in\{1, \ldots, n\}, j\in M^c}$ and $1_{\lvert M^c \rvert}$ is a $\lvert M^c \rvert$-dimensional one-vector. Its derivation method is the same as the one used by Lee et al. (2016). Hence, we also have

$$\{\mathcal{M} = M\} = \bigcup_{s \in \{-1, 1\}^{\lvert M \rvert}} \{A(M, s)\hat{W}(\hat{T}, \hat{X})\hat{Y} \leq b(M, s)\}. \quad (9)$$

Although we mentioned that we would consider the conditional coverage in (6), conditioning also on the sign $s^M$ leads to an easier problem; hence, we will consider

$$P(\beta_j^M \in C_j^M \mid \mathcal{M} = M, \ s^M = s) = P\{\beta_j^M \in C_j^M \mid A(M, s)\hat{W}(\hat{T}, \hat{X})\hat{Y} \leq b(M, s)\} \geq 1 - \alpha$$

as in Lee et al. (2016). This interval $C_j^M$ also satisfies (6), but if one wants tighter coverage, one can condition on (9). To find $C_j^M$, we consider the distribution of $\hat{\beta}_j^M \hat{W}(\hat{T}, \hat{X})\hat{Y}$ conditional on (8). Let $\hat{D}(\hat{T}, \hat{X}) \equiv \hat{W}(\hat{T}, \hat{X})^2\hat{\eta}_j\hat{W}(\hat{T}, \hat{X})^2\hat{\eta}_j$ conditional on (8). Using the same derivation method as in Lee et al. (2016), we can rewrite the conditioned polyhedral region by using $\hat{\eta}_j\hat{W}(\hat{T}, \hat{X})\hat{Y}$ and

$$\hat{Z} \equiv \{I_n - \hat{D}(\hat{T}, \hat{X})\hat{\eta}_j\hat{W}(\hat{T}, \hat{X})\hat{Y},$$

from which we get

$$\{A(M, s)\hat{W}(\hat{T}, \hat{X})\hat{Y} \leq b(M, s)\} = \{\forall_{s,j}(\hat{Z}, \hat{T}, \hat{X}) \leq \hat{\eta}_j\hat{W}(\hat{T}, \hat{X})\hat{Y} \leq \forall_{s,j}^+(\hat{Z}, \hat{T}, \hat{X}), \ \forall_{s,j}^0(\hat{Z}, \hat{T}, \hat{X}) \geq 0\},$$

as in Lee et al. (2016).
where
\[
\begin{align*}
\mathcal{V}^-_{s,j}(\tilde{Z}, \tilde{T}, \tilde{X}) &\equiv \max_{k: (A(M, s)D(T, \tilde{X}))_k < 0} \frac{b(M, s)_k - (A(M, s)\tilde{Z})_k}{(A(M, s)D(T, \tilde{X}))_k}, \\
\mathcal{V}^+_{s,j}(\tilde{Z}, \tilde{T}, \tilde{X}) &\equiv \min_{k: (A(M, s)D(T, \tilde{X}))_k > 0} \frac{b(M, s)_k - (A(M, s)\tilde{Z})_k}{(A(M, s)D(T, \tilde{X}))_k}
\end{align*}
\]
and
\[
\mathcal{V}^0_{s,j}(\tilde{Z}, \tilde{T}, \tilde{X}) \equiv \min_{k: (A(M, s)D(T, \tilde{X}))_k = 0} \{b(M, s)_k - (A(M, s)\tilde{Z})_k\}.
\]

Given this setup, we want to construct a statistical test or a confidence interval for the target parameter \(\beta^M_j = e_j'(\tilde{X}'_M\tilde{X}_M)^{-1}\tilde{X}'_M \sum_{h=1}^H c^{(h)}(\tilde{\mu}^{(h)}(\tilde{X}))\). However, unlike in Section 2.2, \(\tilde{Y}\) is not normally distributed and \(\tilde{f}(\tilde{X})\) exists. Also, \(\tilde{X}\) is random, although it is a trivial difference. Therefore, we consider using the realizations \(\tilde{t}\) and \(\tilde{x}\) to further condition on that \(\tilde{T} = \tilde{t}\) and \(\tilde{X} = \tilde{x}\). This is a difficult operation to understand because, \(\tilde{T}\) is not conditioned first in the usual propensity score analysis, but it is essentially necessary to extract the properties of a normal distribution. It is also difficult to intuitively understand how such conditioning guarantees valid inferences, but as we will see later, it is easy to show that it does. In general, under arbitrary conditioning, inference about the target parameter becomes impossible, but in this setting, under the weakly ignorable treatment assignment condition, inference based on asymptotics is possible. Also, as in the usual propensity score analysis, it is independent of \(\tilde{f}(\tilde{X})\).

Under this new conditioning, \(\tilde{\eta}_j^\prime \tilde{W}(\tilde{T}, \tilde{X})\tilde{Y}\) and \(\tilde{Z}\) become independent; thus, letting \(\tilde{z}\) be the realization of \(\tilde{Z}\), we have
\[
[\tilde{\eta}_j^\prime \tilde{W}(\tilde{T}, \tilde{X})\tilde{Y} \mid A(M, s)\tilde{W}(\tilde{t}, \tilde{x})\tilde{Y} \leq b(M, s), \tilde{Z} = \tilde{z}, \tilde{T} = \tilde{t}, \tilde{X} = \tilde{x}] = [\tilde{\eta}_j^\prime \tilde{W}(\tilde{t}, \tilde{x})\tilde{Y} \leq \tilde{\eta}_j^\prime \tilde{W}(\tilde{t}, \tilde{x})\tilde{Y} \leq V^0_{s,j}(\tilde{z}, \tilde{t}, \tilde{x}), \tilde{T} = \tilde{t}, \tilde{X} = \tilde{x}].
\] (12)

In addition, because the ignorable treatment assignment condition is satisfied, \(\tilde{e}^{(h)}\) follows \(N(0_n, \sigma^2 I_n)\) under this conditioning. Hence, by letting
\[
\kappa^M_j(\tilde{t}, \tilde{x}) \equiv e_j'(\tilde{x}'_M\tilde{x}_M)^{-1}\tilde{x}'_M \sum_{h=1}^H c^{(h)}(\tilde{\mu}^{(h)}(\tilde{x}))\} \tilde{e}(\tilde{x})\}
\] (13)
and
\[
\zeta^M_j(\tilde{t}, \tilde{x}) \equiv \sigma^2 e_j'(\tilde{x}'_M\tilde{x}_M)^{-1}\tilde{x}'_M \tilde{W}(\tilde{t}, \tilde{x})\tilde{W}(\tilde{t}, \tilde{x})\tilde{W}(\tilde{t}, \tilde{x})\tilde{W}(\tilde{t}, \tilde{x})\tilde{e},
\] (14)
it can be seen that the conditional distribution is a truncated normal distribution,
\[
TN(\kappa^M_j(\tilde{t}, \tilde{x}), \zeta^M_j(\tilde{t}, \tilde{x}), V^-_{s,j}(\tilde{z}, \tilde{t}, \tilde{x}), V^+_{s,j}(\tilde{z}, \tilde{t}, \tilde{x})).
\]
Thus, letting $F_{\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j}}(\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j}) \equiv F_{\nu^{\alpha}_{j} \nu^{\alpha}_{j}}(\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j})$, we obtain

$$[F_{\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j}}(\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j}) \mid A(M, s) \hat{W}(\hat{T}, \hat{X}) \hat{Y} \leq b(M, s), \hat{Z} = \hat{z}, \hat{T} = \hat{t}, \hat{X} = \hat{x}]$$

$$= [F_{\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j}}(\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j}) \mid A(M, s) \hat{W}(\hat{t}, \hat{x}) \hat{Y} \leq b(M, s), \hat{Z} = \hat{z}, \hat{T} = \hat{t}, \hat{X} = \hat{x}]$$

$$\sim \text{Unif}(0, 1).$$

Denoting the conditional density function of $V$ by $p_V$, we have shown that

$$p_F(\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j} \hat{W}(\hat{T}, \hat{X}) \hat{Y} \mid \hat{Z}, \hat{T}, \hat{X}(v \mid \hat{z}, \hat{t}, \hat{x}) = 1_{[0,1]}(v),$$

from which it can be seen that

$$p_F(\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j} \hat{W}(\hat{T}, \hat{X}) \hat{Y}(v)$$

$$= \int \sum \left\{ \int p_F(\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j} \hat{W}(\hat{T}, \hat{X}) \hat{Y} \mid \hat{Z}, \hat{T}, \hat{X}(v \mid \hat{z}, \hat{t}, \hat{x}) p_{\hat{Z}, \hat{T}, \hat{X}}(\hat{z} \mid \hat{t}, \hat{x}) d\hat{z} \right\} p_{\hat{Z}, \hat{T}, \hat{X}}(\hat{t}, \hat{x}) d\hat{x}$$

$$= 1_{[0,1]}(v).$$

This leads us to the following theorem.

**Theorem 1.** Let $F^{(a,b)}(\cdot)$ be the cumulative distribution functions of $TN(\mu, \sigma^2, a, b)$, which is $N(\mu, \sigma^2)$ truncated into the interval $[a, b]$, and let $\nu^{\alpha}_{s,j}, \nu^{\alpha}_{j} \nu^{\alpha}_{j}$ and $\zeta^{M}_{j}$ be the functions defined in (10), (11), (13) and (14), respectively. Then,

$$[F_{\nu^{\alpha}_{j} \nu^{\alpha}_{j}}(\hat{\nu}^{\alpha}_{j} \nu^{\alpha}_{j}) \mid A(M, s) \hat{W}(\hat{T}, \hat{X}) \hat{Y} \leq b(M, s)] \sim \text{Unif}(0, 1)$$

holds.

The key point here is that, like the conditioning of $\hat{Z}$, the conditioning of $\hat{T}$ and $\hat{X}$ does not interfere with the inference.

This theorem cannot be used for the inference about $\beta^{M}_{j}$ without further consideration, because unlike the selective inference in Lee et al. (2016), $\beta^{M}_{j}$ does not appear explicitly in this pivot statistic. Therefore, we consider higher-order asymptotic theory and extract $\beta^{M}_{j}$ from $\kappa^{M}_{j}(\hat{T}, \hat{X})$, which appears as the mean parameter of the truncated normal distribution. Then, we construct an inference guaranteed asymptotically by evaluating the asymptotic behavior of $\kappa^{M}_{j}(\hat{T}, \hat{X})$. Note that this operation is peculiar to the selective inference for propensity score analysis.

In (13) of $(\hat{t}, \hat{x})$ as $(\hat{T}, \hat{X})$, we decompose the parts other than $c'(\hat{X}'_{M} \hat{X}_{M})^{-1}$ and evaluate their expectations. Then, we obtain

$$E \left[ X_{M,i} \sum_{h=1}^{H} c(h) \frac{T^{(h)}_{i}}{e^{(h)}(X_{i})} (\mu^{(h)}(X_{i}) + f(X_{i})) \mid \hat{X} \right] = X_{M,i} \sum_{h=1}^{H} c(h) \{\mu^{(h)}(X_{i}) + f(X_{i})\}$$

11
since the sum of contrasts is 0, where $X_{M,i} = (X_{ij})_{j \in M}$.

Thus, we have

$$
\kappa_j^M(\hat{T}, \hat{X}) - \beta_j^M
= e_j'(\hat{X}^*_M \hat{X}_M)^{-1} \hat{X}_M^* \sum_{h=1}^H c(h) \{ W(h)(\hat{T}^*(h), \hat{X}) - I_n \} \{ \mu(h)(\hat{X}) + \hat{f}(\hat{X}) \} = o_p(1)
$$

from the law of large numbers. We prepare an estimator $Y_i^{(h)}$ for $\mu^{(h)}(x_i) + f(x_i)$, and let $\hat{Y}^{(h)} = (Y_1^{(h)}, \ldots, Y_n^{(h)})'$ and $\hat{Y} = (\hat{Y}^{(1)}\top, \ldots, \hat{Y}^{(H)}\top)'$. Accordingly, we define

$$
\tau_j^M(\hat{Y}^\top, \hat{T}, \hat{X}) \equiv e_j'(\hat{X}^*_M \hat{X}_M)^{-1} \hat{X}_M^* \sum_{h=1}^H c(h) \{ W(h)(\hat{T}^*(h), \hat{X}) - I_n \} \hat{Y}^{(h)} = o(1).
$$

(15)

This leads us to the following lemma.

**Lemma 1.** Let $F_{\mu, \sigma^2}^{[a,b]}(\cdot)$ be the cumulative distribution functions of $TN(\mu, \sigma^2)$ truncated into the interval $[a, b]$, and let $\nu_{a,j}^-, \nu_{a,j}^+, \tau_j^M$ and $\sigma_j^M$ be the functions defined in (10), (11), (15) and (14), respectively. Then, if $Y_i^{(h)} = \{ \mu^{(h)}(x_i) + f(x_i) \} = o_p(1)$ under the condition,

$$
[F_{\beta_j^M + \gamma_j^M (Y^\top, T, X)}^M(\hat{T}, \hat{X}) \mid \mu^{(h)}(x_i) + f(x_i) \leq b(M, s)] \sim Unif(0, 1)
$$

holds as $n \to \infty$.

Note that although this lemma makes use of asymptotics unlike Theorem 1, it is essentially different from those of Tian and Taylor (2017) and Tibshirani et al. (2018). If we try to use the results of these papers simply because the error in the context of causal inference, $f(X) + \epsilon$, has a non-Gaussian distribution, the estimation will have a large bias due to the existence of confounding. On the other hand, in this paper, we assume a Gaussian distribution for $\epsilon$, and this lemma does not mean that our result is valid even if we make it non-Gaussian.

Similarly to Lee et al. (2016), if we condition on (9), we only have to use the normal distribution $N(\mu, \sigma^2)$ truncated into the union set $S$ of intervals. Writing its cumulative distribution function as $F_{\mu, \sigma^2}^{S} (\cdot)$, we get

$$
\left\{ F_{\mu, \sigma^2}^{S} (\hat{T}, \hat{X}) \mid \mu^{(h)}(x_i) + f(x_i) \leq b(M, s) \right\} \sim Unif(0, 1).
$$
In addition, because \( F_{\beta_j^M+\gamma_j^M}(y_1,\tilde{T},\tilde{X}) \), \( \zeta_j^M(\tilde{T},\tilde{X}) \), (\( \tilde{y}_j^f(\tilde{T},\tilde{X})Y \)) is a monotonically decreasing function with respect to \( \beta_j^M \), we can see that if we set \( L \) and \( U \) to satisfy

\[
F_{\beta_j^M+\gamma_j^M}(y_1,\tilde{T},\tilde{X}), \zeta_j^M(\tilde{T},\tilde{X}) \) \( \tilde{y}_j^f(\tilde{T},\tilde{X})Y \) = 1 - \frac{\alpha}{2}
\]

and

\[
F_{u+\gamma_j^M}(y_1,\tilde{T},\tilde{X}), \zeta_j^M(\tilde{T},\tilde{X}) \) \( \tilde{y}_j^f(\tilde{T},\tilde{X})Y \) = \frac{\alpha}{2},
\]

we get

\[
P(\beta_j^M \in [L, U] \mid \tilde{M} = M) \to 1 - \alpha,
\]

which gives the asymptotically guaranteed conditional coverage.

Next, we develop a theory using a concrete and easily conceivable estimator for \( \mu^{(h)}(X_i) + f(X_i) \). Suppose we have a sequence of real numbers \( \{\delta_n\} \) that converges to 0, and let \( \mathcal{N}^{(h)} \equiv \{l \neq i : \|X_l - X_i\|_2 \leq \delta_n \}, T_i^{(h)} = 1 \). We take \( \delta_n \) such that \( O(1) \neq \mathcal{N}_{i}^{(h)} = o(n) \). For example, if the set of possible values of \( X_i \) is a closed set of bounded open sets in \( \mathbb{R}^p \), then we only have to take \( \delta_n \) such that \( O(n^{-1}) \neq \delta_n = o(1) \). Letting \( \beta^{(h)} \equiv \arg\min_{\beta} \|\tilde{T}^{(h)}(\tilde{Y} - \tilde{X})\|_2^2 \) and \( \tilde{Y}^{(h)*} \equiv (X_l^{(h)} + \sum_{l \in \mathcal{N}_{i}^{(h)}(Y_l - X_l^{(h)})/|\mathcal{N}_{i}^{(h)}|, \ldots, X_l^{(h)} + \sum_{l \in \mathcal{N}_{i}^{(h)}(Y_l - X_l^{(h)})/|\mathcal{N}_{i}^{(h)}|}) \), if \( f(\cdot) \) is Lipschitz continuous, then it is obvious that \( \kappa_j^M(\tilde{T}, \tilde{X}) - \{\beta_j^M + \tau_j^M(\tilde{Y}^{(h)}*, \tilde{T}, \tilde{X})\} = \text{op}(n^{-1/2}) \), where \( \tau_j^M \) is the function defined in \( (15) \) and \( \tilde{Y}^{(h)*} = (\tilde{Y}^{(1)*'}, \ldots, \tilde{Y}^{(H)*'})' \). Thus, we obtain

\[
E[\tilde{y}_j^f(\tilde{T}, \tilde{X})Y - \tau_j^M(\tilde{Y}^{(h)*}, \tilde{T}, \tilde{X}) \mid \tilde{T}, \tilde{X}] = \beta_j^M + \text{op}(n^{-1/2}).
\]  

For the evaluation of the variance,

\[
V[\tilde{y}_j^f(\tilde{T}, \tilde{X})Y - \tau_j^M(\tilde{Y}^{(h)*}, \tilde{T}, \tilde{X}) \mid \tilde{T}, \tilde{X}] = E([\tilde{y}_j^f(\tilde{T}, \tilde{X})Y - \kappa_j^M(\tilde{T}, \tilde{X})]^2 \mid \tilde{T}, \tilde{X}),
\]

the conditional expectation of the square of the first term is \( \zeta_j^M(\tilde{T}, \tilde{X}) \), the conditional expectation of the square of the second term is

\[
\tilde{y}_j^f \sum_{h=1}^{H} \sum_{k=1}^{H} c^{(h)}c^{(k)}E[\{\tilde{W}^{(h)}(\tilde{T}^{(h)}, \tilde{X}) - I_n\}c^{(h)}c^{(k)} \mid \tilde{T}, \tilde{X}] \tilde{y}_j^f,
\]  

and the conditional expectation of the product of the first and second terms is

\[
\tilde{y}_j^f \sum_{h=1}^{H} \sum_{k=1}^{H} c^{(h)}c^{(k)}E[\tilde{W}^{(h)}(\tilde{T}^{(h)}, \tilde{X})c^{(h)}c^{(k)} \mid \tilde{T}, \tilde{X}] \tilde{y}_j^f,
\]  

13
where \( \epsilon^* = \bar{Y}^* - E(\bar{Y}^* \mid \bar{T}, \bar{X}) \).

In (17), if \( h \neq k \), then \( \epsilon^{(h)*} \) and \( \epsilon^{(k)*} \) are independent, and if \( h = k \), then \( \epsilon^{(h)*} \epsilon^{(k)*} \) has at most \( \sum_{i=1}^n |\Lambda_i^{(h)}|^2 \) non-zero components. Given the expectation with respect to \( \tilde{T} \) of the conditional expectation, the off-diagonal components of this matrix are 0 due to the weakly ignorable treatment assignment and the independence of \( \{T_i^{(h)}\} \); moreover, the diagonal components are \( (\sigma^2/|\Lambda_i^{(h)}|)\{1 - e^{(h)}(X_i)\}/e^{(h)}(X_i) \) from the assignment condition. Hence, we can see that (17) becomes

\[
\sigma^2 \sum_{h=1}^H c^{(h)}(\tilde{\eta}_j^{(h)} \tilde{K}^{(h)}(\tilde{X}) \tilde{\eta}_j + o_p(n^{-3/2}),
\]

where \( \tilde{K}^{(h)}(\tilde{X}) = \text{diag}\{1/e^{(h)}(X_1), \ldots, 1/e^{(h)}(X_n)\}/|\Lambda_n^{(h)}| \}. \) In (18), if \( h \neq k \), then \( \epsilon^{(h)} \) and \( \epsilon^{(k)*} \) are independent, and if \( h = k \), then \( \epsilon^{(h)} \epsilon^{(k)*} \) has at most \( \sum_{i=1}^n |\Lambda_i^{(h)}| \) non-zero components. Given the expectation with respect to \( \tilde{T} \) of the conditional expectation, the off-diagonal components of this matrix are 0 due to the weakly ignorable treatment assignment condition and the independence of \( \{T_i^{(h)}\} \). Moreover, the diagonal components are also 0 because of the assignment condition and the independence of \( \epsilon_i^{(h)} \) and \( \epsilon_i^{(h)*} \). Hence, we can see that (18) becomes \( o_p(n^{-3/2}) \).

Thus, by defining

\[
\rho_j^M(\tilde{T}, \tilde{X}) = \epsilon_j^M(\tilde{T}, \tilde{X}) + \sigma^2 \sum_{h=1}^H c^{(h)}(\tilde{\eta}_j^{(h)} \tilde{K}^{(h)}(\tilde{X}) \tilde{\eta}_j,
\]

it can be seen that

\[
V(\tilde{\eta}_j^{(h)} \tilde{W}(\tilde{T}, \tilde{X}) \bar{Y} - \tau_j^M(\bar{Y}^*, \tilde{T}, \tilde{X}) \mid \tilde{T}, \tilde{X}) = \rho_j^M(\tilde{T}, \tilde{X}) + o_p(n^{-3/2}).
\]

From (16) and (21), we obtain the following theorem, which is an asymptotic version of (5).

**Theorem 2.** Let \( F_{\mu, \sigma^2}(\cdot) \) be the cumulative distribution functions of \( T \in N(\mu, \sigma^2) \) truncated into the interval \( [a, b] \), and let \( V_{s,j}, V_{s,j}^+, \gamma_j^M \) and \( \rho_j^M \) be the functions defined in (10), (11), (15) and (20), respectively. Then, if we define \( L_j^M \) and \( U_j^M \) such that

\[
\int_{U_j^M} \int_{\bar{Y}^*, \tilde{T}, \tilde{X}, \rho_j^M(\tilde{T}, \tilde{X})} (\tilde{\eta}_j^{(h)} \tilde{W}(\tilde{T}, \tilde{X}) \bar{Y} = 1 - \frac{\alpha}{2}
\]

and

\[
\int_{L_j^M} \int_{\bar{Y}^*, \tilde{T}, \tilde{X}, \rho_j^M(\tilde{T}, \tilde{X})} (\tilde{\eta}_j^{(h)} \tilde{W}(\tilde{T}, \tilde{X}) \bar{Y} = \frac{\alpha}{2}
\]

then

\[
E\left( \left\{ j \in \hat{M} : \beta_j^M \notin [L_j^M, U_j^M] \right\} \mid |\hat{M}| > 0 \right) \leq \alpha
\]

holds as \( n \to \infty \).
The key point here is that the pivot statistic does not depend on the nonparametric function \( \hat{f}(\hat{X}) \).

**Remark.** Regarding \( \hat{Y}^* \), for example, when \( H = 2 \) and \((c^{(1)}, c^{(2)}) = (-1, 1)\), if we define \( \hat{Y}^{(1)*}= (X'_1\hat{\beta}^{(1)}) + \{\sum_{l \in \mathcal{N}_1^{(1)}} (Y_l - X'_1\hat{\beta}^{(1)} - X'_0\hat{\beta}^{(1)})\} / (|\mathcal{N}_1^{(1)}| + |\mathcal{N}_1^{(2)}|), \ldots, X'_n\hat{\beta}^{(1)} + \{\sum_{l \in \mathcal{N}_n^{(1)}} (Y_l - X'_1\hat{\beta}^{(1)}) + \sum_{l \in \mathcal{N}_n^{(2)}} (Y_l - X'_1\hat{\beta}^{(1)} - X'_0\hat{\beta}^{(1)})\} / (|\mathcal{N}_n^{(1)}| + |\mathcal{N}_n^{(2)}|)\}' \) and \( \hat{Y}^{(2)*}= (X'_1\hat{\beta}^{(2)}) + \{\sum_{l \in \mathcal{N}_1^{(1)}} (Y_l + X'_1\hat{\beta}^{(2)} - X'_0\hat{\beta}^{(2)}) + \sum_{l \in \mathcal{N}_1^{(2)}} (Y_l - X'_1\hat{\beta}^{(2)})\} / (|\mathcal{N}_1^{(1)}| + |\mathcal{N}_1^{(2)}|), \ldots, X'_n\hat{\beta}^{(2)} + \{\sum_{l \in \mathcal{N}_n^{(1)}} (Y_l + X'_1\hat{\beta}^{(2)} - X'_0\hat{\beta}^{(2)}) + \sum_{l \in \mathcal{N}_n^{(2)}} (Y_l - X'_1\hat{\beta}^{(2)})\} / (|\mathcal{N}_n^{(1)}| + |\mathcal{N}_n^{(2)}|)\)' \), the accuracy tends to increase. Note that in the definition of \( \hat{K}^{(h)}(\hat{X}) \), \( |\mathcal{N}_1^{(h)}| \) becomes \( |\mathcal{N}_1^{(1)}| + |\mathcal{N}_1^{(2)}| \).

4 Numerical experiment

We numerically verified the usefulness of the proposed method. Letting \( H = 2 \), for the confounding variable \( X \) (\( \in \mathbb{R}^p \)), the outcome variable \( Y \) (\( \in \mathbb{R} \)) was observed through the model,

\[
Y = T\mu(X) + f(X) + \epsilon.
\]

That is, the treatment group was represented by \( T = 1 \) and the control group by \( T = 0 \). The error variable \( \epsilon \) was assumed to follow a normal distribution \( \mathcal{N}(0, \sigma^2) \) independently of \( X \), and the variance was assumed to be known. This model is a special case of (1), where the causal effect is \( \mu(x) \).

The setup of the numerical experiment is as follows. In each run, we choose one of the following

\[\text{(F1)} \quad f(x) = 0\]

\[\text{(F2)} \quad f(x) = 3x_1 + x_2 + x_3 + x_4 + x_5 - 3.5\]

\[\text{(F3)} \quad f(x) = x_1 + 0.5x_2 + 0.5x_3 + \pi \sin(\pi x_4)/32 + \pi \sin(\pi x_5)/32 - 1.125\]

as an infinite-dimensional nuisance parameter \( f(\cdot) \). In addition, we chose one of the following

\[\text{(E1)} \quad e(x) = 0.5\]

\[\text{(E2)} \quad e(x) = 1/\{1 + \exp(-x_1 - 0.5x_2 - 0.5x_3 - 0.5x_4 - 0.5x_5 + 1.5)\}\]

as the propensity score \( e(\cdot) \) and one of the following

\[\text{(M1)} \quad \mu(x) = 0\]

\[\text{(M2)} \quad \mu(x) = 3x_1 + x_2 + x_3 + x_4 + x_5 - 3.5\]
as the causal effect $\mu(\cdot)$. Note that (E1) indicates a completely random assignment. In the case of (M1), all variables are inactive for $\mu(x)$, while in the case of (M2), the variables $\{x_1, x_2, x_3, x_4, x_5\}$ are active for $\mu(x)$. Each component of the confounding variable vector independently follows a continuous uniform distribution $U(0, 1)$ or Bernoulli distribution $Ber(1/2)$, and the data are generated from the above model with a sample size of $n = 1000$. Here, we set $p$ to 5 or 25 and the error variance $\sigma^2$ to $0.25^2$.

We compared the proposed method (SI) with a method that ignores the influence of model selection (Naive). Model selection in Naive is conducted by using Lasso and then confidence intervals are constructed using a normal distribution instead of a truncated normal distribution. Because Naive ignores the fact that the selected variables are likely to be significant, it is supposed that it will have a large number of false positives and the validity of its inference will be impaired.

We evaluated the size $|\hat{M}|$ of the model selected by Lasso and the false coverage rate (FCR) by conducting 1,000 Monte Carlo simulations. We also evaluated the probability that $\beta_1, \ldots, \beta_5$ is significant, i.e., the probability that $\beta_1 = 0, \ldots, \beta_5 = 0$ is not covered in the confidence interval when $p = 5$, and numbers of true positive (TP) and false positive (FP) when $p = 25$. In all cases, the significance level was set to 0.05; i.e., the confidence coefficient was set to 0.95. Therefore, the closer the FCR is to 0.05, the more valid the method is. Note that with $\hat{M}$ as the selected model and $C^\hat{M}_j$ as the confidence interval for $\hat{\beta}_j^M$, TP is the number of intervals that did not cover zero when $\hat{\beta}_j^M$ is non-zero, $|\{j \in \hat{M} : \hat{\beta}_j^M \neq 0, 0 \notin C^\hat{M}_j\}|$, and FP is the number intervals that did not cover zero when $\beta_j^M$ is zero, $|\{j \in \hat{M} : \hat{\beta}_j^M = 0, 0 \notin C^\hat{M}_j\}|$.

4.1 Case of $p = 5$

First, let us consider (M1), where all variables are inactive. The confounding variables followed a Bernoulli distribution, and the sequence $\delta_n$ used to define the neighborhood $N_i^{(h)}$ was set to 0. The tuning parameter of Lasso was $\lambda = \sigma n^{-1/2}(\log p)^{1/2}$ when the infinite-dimensional nuisance parameter was (F1), and $\lambda = 2\sigma n^{-1/2}(\log p)^{1/2}$ when it was (F2) or (F3). Table 1 shows $|\hat{M}|$, the non-coverage probability, and the FCR. Since $\beta_j = 0$ is correct for all variables, the closer the non-coverage probability is to 0.05, the better. Although the non-coverage probability of SI seems to be somewhat large when the propensity score and the nuisance parameter are (E2) and (F2), it is approximately 0.05 in most other cases. On the other hand, if we look at Naive, we can see that when the nuisance parameter is (F2) or (F3), 0 is not covered for all the
Table 1: Case in which \( p = 5 \), \( X_{ij} \sim \text{Ber}(1/2) \) and (M1) is correct. The figures in parentheses are standard deviations. In all settings, SI controls the FCR appropriately, but the inference of Naive is invalid.

| e(\(x\)) | f(\(x\)) | \(|\hat{M}|\) | \(\beta_1\) | \(\beta_2\) | \(\beta_3\) | \(\beta_4\) | \(\beta_5\) | FCR |
|---------|----------|-----------|---------|---------|---------|---------|---------|-----|
| SI      | 0.039    | 0.031     | 0.026   | 0.045   | 0.047   | 0.035   |
|         | (0.194)  | (0.175)   | (0.159) | (0.208) | (0.212) | (0.124) |
| Naive   | 0.080    | 0.085     | 0.070   | 0.089   | 0.088   | 0.079   |
|         | (0.272)  | (0.279)   | (0.255) | (0.285) | (0.284) | (0.178) |
|        | (F1)     | 2.593     | 1.164   | 0.194   | 0.175   | 0.124   |
| SI      | 0.049    | 0.055     | 0.038   | 0.050   | 0.059   | 0.051   |
|         | (0.215)  | (0.227)   | (0.190) | (0.218) | (0.235) | (0.123) |
| Naive   | 1.000    | 1.000     | 1.000   | 1.000   | 1.000   | 0.999   |
|         | (0.000)  | (0.000)   | (0.000) | (0.000) | (0.000) | (0.032) |
|        | (E1)     | 3.641     | 1.012   | 0.215   | 0.217   | 0.156   |
| SI      | 0.040    | 0.051     | 0.042   | 0.043   | 0.048   | 0.039   |
|         | (0.196)  | (0.219)   | (0.201) | (0.204) | (0.213) | (0.156) |
| Naive   | 1.000    | 1.000     | 1.000   | 1.000   | 1.000   | 0.842   |
|         | (0.000)  | (0.000)   | (0.000) | (0.000) | (0.000) | (0.365) |
|        | (F3)     | 1.675     | 1.141   | 0.196   | 0.217   | 0.134   |
| SI      | 0.043    | 0.049     | 0.040   | 0.050   | 0.044   | 0.042   |
|         | (0.202)  | (0.216)   | (0.196) | (0.217) | (0.204) | (0.134) |
| Naive   | 0.081    | 0.076     | 0.080   | 0.064   | 0.070   | 0.068   |
|         | (0.274)  | (0.265)   | (0.272) | (0.246) | (0.255) | (0.157) |
|        | (F1)     | 2.734     | 1.154   | 0.246   | 0.218   | 0.153   |
| SI      | 0.065    | 0.067     | 0.063   | 0.050   | 0.079   | 0.062   |
|         | (0.246)  | (0.250)   | (0.244) | (0.218) | (0.270) | (0.153) |
| Naive   | 1.000    | 1.000     | 1.000   | 1.000   | 1.000   | 0.999   |
|         | (0.000)  | (0.000)   | (0.000) | (0.000) | (0.000) | (0.032) |
|        | (E2)     | 3.744     | 1.004   | 0.246   | 0.218   | 0.145   |
| SI      | 0.045    | 0.066     | 0.056   | 0.053   | 0.050   | 0.042   |
|         | (0.207)  | (0.248)   | (0.230) | (0.224) | (0.219) | (0.145) |
| Naive   | 1.000    | 1.000     | 1.000   | 1.000   | 1.000   | 0.886   |
|         | (0.000)  | (0.000)   | (0.000) | (0.000) | (0.000) | (0.318) |

selected variables, and the validity of the inference is impaired. Note that when the nuisance parameter is (F1), the non-coverage probability is greater than 0.05, although this value is not as extreme as when the nuisance parameter is (F2) or (F3). The FCR shows a similar trend, where SI keeps the FCR at almost 0.05, but Naive results in a very large value.
Table 2 shows results for when (M2) is the causal effect. Here, the confounding variables followed a continuous uniform distribution, $\delta_n = (p/6)^{1/2}$ and $\lambda = 5\sigma n^{-1/2}(\log p)^{1/2}$. In this case, all variables are active, so the closer the non-coverage probability is to 1, the better. The non-coverage probability of both methods is close to 1, but Naive does not keep the FCR at 0.05. On the other hand, SI keeps it at almost 0.05.

Figure 1 compares the confidence intervals obtained by SI and Naive. In all panels, the nuisance parameter is (F3). Figures 1(a) and 1(b) respectively use (E1) and (E2) as the propensity scores in the setting of Table 1. On the other hand, Figures 1(c) and 1(d) respectively use (E1) and (E2) as the propensity scores in the setting of Table 2. Note that the horizontal axis represents the variables selected by Lasso, and in Figure 1(a), the variables $x_2, x_3, x_4, x_5$ are selected, while in the remaining figures, all variables are selected. Since (M1) is correct in Figures 1(a) and 1(b), the confidence intervals should contain 0. In both panels, it can be seen that most of Naive’s results do not contain 0, but those of SI do contain 0. In Figure 1(a), the two confidence intervals for SI do not include the estimates, which may seem somewhat strange, but a similar phenomenon has been reported in Liu et al. (2018). Figure 1(c) shows that the confidence intervals for SI appropriately include the true values of the parameters except for $x_2$, but the confidence intervals for Naive do not include the true values for any variable.

4.2 Case of $p = 25$

Table 3 shows the results for when (M2) is the causal effect; i.e., the first 5 variables are active and the remaining 20 variables are inactive. In all settings, the tuning parameter of Lasso was set to $\lambda = 2\sigma n^{-1/2}(\log p)^{1/2}$. The sequence $\delta_n$ used to define the neighborhood $\mathcal{N}_i^{(h)}$ was set to $(p/2)^{1/2}$ for $X_{ij} \sim \text{Ber}(1/2)$ and $(p/6)^{1/2}$ for $X_{ij} \sim \text{U}(0,1)$. As in the case of $p = 5$, we can see that SI appropriately controls the FCR, keeping it at almost 0.05 in all settings. In addition, since TP is close to 5, the confidence intervals for the active variables are appropriately away from 0, and FP is sufficiently small. On the other hand, the FCR for Naive is well above 0.05, suggesting that reasonable confidence intervals were not obtained. The values of TP, FP, and $|\hat{M}|$ indicate that almost all of the confidence intervals for the variables selected by Lasso are away from 0.

Figure 2 compares the confidence intervals for SI and Naive with a confidence coefficient of 0.95 where the nuisance parameters and propensity scores are (F3) and (E2). As explained in Table 3, Naive has extremely short confidence intervals due to loss of the validity of the
Table 2: Case in which $p = 5$, $X_{ij} \sim U(0,1)$ and (M2) is correct. The figures in parentheses are standard deviations. In all settings, SI removes a high percentage of 0’s from the confidence intervals while appropriately controlling the FCR.

| $e(x)$ | $f(x)$ | $|\hat{M}|$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | FCR     |
|--------|--------|-----------|-----------|-----------|-----------|-----------|-----------|--------|
| SI     |        |           | 1.000     | 0.998     | 0.998     | 0.999     | 0.999     | 0.054   |
| (F1)   | 4.987 (0.113) |           |           |           |           |           |           |         |
|        |        |           | (0.000)   | (0.045)   | (0.045)   | (0.032)   | (0.032)   | (0.099) |
| Naive  |        |           | 1.000     | 1.000     | 1.000     | 1.000     | 1.000     | 0.417   |
|        |        |           | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.233) |
| SI     |        |           | 1.000     | 0.994     | 0.995     | 0.992     | 0.990     | 0.067   |
| (E1)   | 4.460 (0.754) |           |           |           |           |           |           |         |
|        |        |           | (0.000)   | (0.076)   | (0.068)   | (0.089)   | (0.101)   | (0.127) |
| Naive  |        |           | 1.000     | 1.000     | 1.000     | 1.000     | 1.000     | 0.737   |
|        |        |           | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.215) |
| SI     |        |           | 1.000     | 0.998     | 0.991     | 0.999     | 0.997     | 0.064   |
| (F3)   | 4.888 (0.334) |           |           |           |           |           |           |         |
|        |        |           | (0.000)   | (0.045)   | (0.096)   | (0.032)   | (0.055)   | (0.114) |
| Naive  |        |           | 1.000     | 1.000     | 1.000     | 1.000     | 1.000     | 0.591   |
|        |        |           | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.230) |
| SI     |        |           | 1.000     | 1.000     | 0.998     | 0.999     | 0.999     | 0.999   |
| (F1)   | 4.980 (0.140) |           |           |           |           |           |           | 0.054   |
|        |        |           | (0.000)   | (0.000)   | (0.045)   | (0.032)   | (0.032)   | (0.101) |
| Naive  |        |           | 1.000     | 1.000     | 1.000     | 1.000     | 1.000     | 0.458   |
|        |        |           | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.227) |
| SI     |        |           | 1.000     | 0.992     | 0.987     | 0.993     | 0.991     | 0.066   |
| (E2)   | 4.445 (0.778) |           |           |           |           |           |           |         |
|        |        |           | (0.000)   | (0.089)   | (0.112)   | (0.083)   | (0.097)   | (0.125) |
| Naive  |        |           | 1.000     | 1.000     | 1.000     | 1.000     | 1.000     | 0.749   |
|        |        |           | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.221) |
| SI     |        |           | 1.000     | 0.998     | 0.998     | 0.997     | 0.996     | 0.057   |
| (F3)   | 4.864 (0.384) |           |           |           |           |           |           |         |
|        |        |           | (0.000)   | (0.045)   | (0.046)   | (0.056)   | (0.064)   | (0.108) |
| Naive  |        |           | 1.000     | 1.000     | 1.000     | 1.000     | 1.000     | 0.614   |
|        |        |           | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.000)   | (0.228) |

inference, and the confidence interval does not include 0 even when the true value is 0. On the other hand, the confidence intervals for SI contain a high percentage of true values.
5 Data analysis

In practice, the error variance $\sigma^2$ and propensity score $e(x)$ must be estimated in some way. As described in Sections 6.1 and 6.3, appropriate estimators can be substituted for them. Thereafter, the error variance estimate $\hat{\sigma}^2$ can be calculated using the method in Section 6.1 as $\hat{\delta}_n = (p/2)^{1/2}$, and the propensity score estimate $\hat{e}(x)$ can be calculated from a logistic regression model. Using the estimates, the tuning parameter of Lasso was set to $\lambda = \hat{\sigma}n^{-1/2}(\log p)^{1/2}$. The following subsections describe the application of the proposed method to two real datasets.
Table 3: Cases in which $p = 25$ and $(M2)$ is correct. The figures in parentheses are standard deviations. In all settings, SI controls the FCR appropriately, but the inference of Naive is invalid.

| $e(x)$ | $f(x)$ | $X_{ij} \sim \text{Ber}(1/2)$ | $X_{ij} \sim \text{U}(0,1)$ |
|--------|--------|-------------------------------|-------------------------------|
|        |        | $|\hat{M}|$ | TP | FP | FCR | $|\hat{M}|$ | TP | FP | FCR |
| SI     | (F1)   | 4.991   | 0.323 | 0.049 |      | 5.000 | 0.002 | 0.051 |
|        |        | (0.114) | (0.592) | (0.068) |      | (0.000) | (0.045) | (0.098) |
| Naive  | (E1)   | 11.775  | (2.154) | 5.000 | 6.775 | 0.830 | (0.380) | 5.000 | 0.143 | 0.436 |
|        | (F2)   | (0.000) | (1.066) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| SI     | (F3)   | 4.972   | 0.879 | 0.058 |      | 4.827 | 0.392 | 0.058 |
|        |        | (0.213) | (1.213) | (0.074) |      | (0.423) | (0.647) | (0.079) |
| Naive  | (E2)   | 4.982   | 14.822 | 0.965 |      | 4.852 | 6.547 | 0.893 |
|        | (F2)   | (0.133) | (1.979) | (0.040) |      | (0.385) | (1.997) | (0.093) |
| SI     | (F3)   | 4.990   | 0.573 | 0.053 |      | 4.992 | 0.076 | 0.053 |
|        |        | (0.148) | (0.847) | (0.065) |      | (0.126) | (0.273) | (0.089) |
| Naive  | (E3)   | 5.000   | 10.960 | 0.921 |      | 4.998 | 1.678 | 0.690 |
|        |        | (2.198) | (2.198) | (0.066) |      | (0.045) | (1.230) | (0.187) |
| SI     | (F4)   | 4.979   | 0.405 | 0.048 |      | 4.994 | 0.013 | 0.053 |
|        |        | (0.211) | (0.729) | (0.080) |      | (0.089) | (0.113) | (0.098) |
| Naive  | (E4)   | 5.000   | 8.841 | 0.874 |      | 5.000 | 0.287 | 0.491 |
|        |        | (2.322) | (2.322) | (0.089) |      | (0.000) | (0.539) | (0.229) |
| SI     | (F3)   | 4.938   | 0.834 | 0.053 |      | 4.780 | 0.391 | 0.054 |
|        |        | (0.280) | (0.935) | (0.056) |      | (0.498) | (0.673) | (0.077) |
| Naive  | (E3)   | 4.954   | 15.425 | 0.968 |      | 4.818 | 7.026 | 0.904 |
|        |        | (1.906) | (1.887) | (0.038) |      | (2.121) | (2.085) | (0.084) |
| SI     | (F3)   | 4.963   | 0.647 | 0.052 |      | 4.974 | 0.101 | 0.054 |
|        |        | (0.309) | (1.028) | (0.075) |      | (0.222) | (0.359) | (0.099) |
| Naive  | (E3)   | 5.000   | 12.238 | 0.933 |      | 4.997 | 2.053 | 0.724 |
|        |        | (2.244) | (2.244) | (0.059) |      | (0.055) | (1.359) | (0.174) |

5.1 Application to the lalonde dataset

The lalonde dataset is treated in LaLonde (1986) and is included in the R package Matching. We set the group that took the U.S. job training program in 1976 as $t = 1$ and the group that
Figure 2: Comparison of confidence intervals for SI and Naive for cases using (F3), (E2), and (M2) as models. The black (red) dots are the estimates for SI (Naive), and the solid lines are the confidence intervals with a confidence coefficient of 0.95.

did not take the program as $t = 0$. We predicted the difference in annual income in 1978 with training as a causal effect. The confounding variables were age (age), years of education (educ), black or not (black), Hispanic or not (hisp), married or not (married), high school graduate or higher (nodegr), income in 1974 (re74), income in 1975 (re75), zero income in 1974 or not (u74), zero income in 1975 or not (u75), i.e., $p = 10$, and the outcome variable was income in 1978 (re78). The sample size was $n = 445$.

When Lasso was used, 5 variables were selected: age, educ, re74, re75 and u74. For these coefficients, we compared the confidence intervals with a confidence coefficient of 0.95 given by the proposed method (SI) and a method ignoring the influence of model selection (Naive). As shown in Table 4, for all variables, the confidence interval for Naive is shorter than that for SI. This is because, as mentioned in the previous section, Naive ignores that the selected variable is likely to be significant. In both methods, the confidence intervals for educ and u74 do not contain 0, indicating that they are significant, while the confidence intervals for age and re75 contain 0, indicating that they are not significant. On the other hand, SI and Naive give different results for re74. Since u74 is a variable that is 0 if there is income in 1974 and 1 if there is not, if u74 is 1, the corresponding re74 is 0. Therefore, the SI result can be interpreted as that the causal effect is affected by whether there was income or not in 1974, rather than by the size of the income in 1974 itself.
Table 4: Confidence intervals with a confidence coefficient of 0.95 for the 5 variables obtained by applying Lasso to the Lalonde dataset.

| Selected variable | SI Estimates | lower | upper | Naive Estimates | lower | upper |
|-------------------|--------------|-------|-------|----------------|-------|-------|
| age               | 49.353       | -164.745 | 165.970 | 51.345          | -66.302 | 168.991 |
| educ              | 721.188      | 228.418 | 1196.124 | 862.309         | 390.500 | 1334.117 |
| re74              | 0.240        | -0.173 | 0.479  | 0.236           | 0.019  | 0.454 |
| re75              | 0.275        | -0.341 | 0.645  | 0.293           | -0.040 | 0.625 |
| u74               | 6459.088     | 3709.181 | 8956.314 | 6480.235        | 3997.591 | 8962.880 |

5.2 Application to E-commerce dataset

The MineThatData dataset is published by Hillstrom (2008) and contains 12 variables such as purchase amount and district classification code for 64,000 customers. The objective is to predict the purchase price of a product from the type of email delivered (male, female, or unsent). The sample size of 64,000 was too large computational load for our resources, so we set the assignment variable to that of emails delivered to males ($t = 1$) and unsent emails ($t = 0$). Moreover, we set the sample size to $n = 6,382$ and the district code (zip_code) to Rural. The confounding variables were the number of months since the last purchase (recency), the amount of purchases made in the last year (history), whether the user purchased men’s products in the last year (mens), whether the user purchased women’s products in the last year (womens), whether the user became a new user in the last 12 months (newbie), whether the user received the email within two weeks of the delivery (newbie), whether the user visited the site within two weeks after the mail was delivered (visit), and whether the user made a purchase within two weeks after the mail was delivered (conversion), i.e., $p = 7$. The outcome variable was the purchase amount when the user made a purchase (spend).

After the preprocessing described above, the error variance and propensity scores were estimated, and Lasso selected four variables: recency, history, newbie and conversion. Table 5 compares the confidence intervals with a confidence coefficient of 0.95 for these coefficients. In the case of Naive, all four variables are significant, while in the case of SI, only history and conversion are significant. A natural interpretation of the SI result is that it is important for a customer to make a large number of purchases within two weeks of receiving an email in order to predict the purchase amount. In the case of Naive, history is negative and significant, indicating
Table 5: Confidence intervals with a confidence coefficient of 0.95 for the 4 variables obtained by applying Lasso to the E-commerce dataset.

| Selected variable | SI | Estimates | lower | upper | Naive | Estimates | lower | upper |
|-------------------|----|-----------|-------|-------|-------|-----------|-------|-------|
| recency           |    | −0.094    | −0.059| 0.225 | −0.143| −0.274    | −0.011|
| history           |    | −0.002    | 0.000 | 0.004 | −0.003| −0.005    | −0.001|
| newbie            |    | 0.704     | −0.640| 1.631 | 1.419 | 0.494     | 2.343 |
| conversion        |    | 13.378    | 8.661 | 17.857| 20.191| 15.730    | 24.651|

that a large purchase amount over the preceding year has a negative effect on the prediction of the purchase amount, which is rather unnatural. On the other hand, in SI, although its estimate is negative, its confidence interval is positive; i.e., it is a natural interval estimation.

6 Extension

6.1 Case of unknown variance

Up to this point, we have treated the variance \( \sigma^2 \) of \( \epsilon(h) \) in (1) as known. However, it is usually unknown in practical applications. As such, it would seem that we only have to substitute a consistent estimator, but since the model contains \( f(\cdot) \) and we are trying to develop a method that avoids its identification, its determination is more difficult than usual. Here, let us choose an appropriate sequence of real numbers \( \{\delta_n\} \) that converges to 0, define \( N_i^\dagger = \{l \neq i: \|X_l - X_i\|_2 < \delta_n, T_l = T_i\} \) using \( T_i = (T_i^{(1)},\ldots,T_i^{(H)}) \), and use

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{|N_i^\dagger|}{1 + |N_i^\dagger|} \left( Y_i - \frac{1}{|N_i^\dagger|} \sum_{l \in N_i^\dagger} Y_l \right)^2.
\]

If the set of possible values of \( X_i \) is finite, or if the set is compact and \( f(\cdot) \) is continuous, this estimator is obviously a consistent estimator of \( \sigma^2 \).

6.2 Generalization of error structure, model selection method, and causal effect

In Section 3, we developed selective inference for basic causal inference models. There is no difficulty in extending it to the case of certain error structures, model selection methods, and causal effects. Here, we assume that, in (1), the variance of \( \epsilon(h) (\in \mathbb{R}^r) \) is \( \Sigma \) in the setup of multivariate regression, conduct a model selection by using an elastic net (Zou and Hastie 2005)
as is done in Lee et al. (2016), and consider the causal effect  \( E(\sum_{h=1}^{H} e^{(h)}Y^{(h)} \mid T^{(1)} = 1) \) in the group receiving the first treatment as an estimand. Note that this causal effect is the average treatment effect on the treated or the average treatment effect on the untreated when  \( H = 2 \) (Imbens 2004). Below, we assume the strongly ignorable treatment assignment condition that assures a valid estimation. The variance of  \( e^{(h)} \) (\( e^{(h)} \in \mathbb{R}^{nr} \)) in (2) is  \( \tilde{\Sigma} \otimes I_n \), which we will denote as  \( \tilde{\Sigma} \). Let  \( \lambda_1 \) and  \( \lambda_2 \) be the tuning parameters. We will use

\[
\tilde{\beta} = \arg \min_{\beta} \left[ \frac{1}{2} \{ \tilde{W}(\tilde{T}, \tilde{X}) \tilde{Y} - \tilde{X} \beta \}^{\top} \tilde{\Sigma}^{-1} \{ \tilde{W}(\tilde{T}, \tilde{X}) \tilde{Y} - \tilde{X} \beta \} + \lambda_1 \| \beta \|_1 + \frac{\lambda_2}{2} \| \beta \|_2^2 \right]
\]

instead of (7). Here,  \( \tilde{W}(\tilde{T}, \tilde{X}) = \sum_{h=1}^{H} e^{(h)}W^{(h)}(\tilde{T}^{(h)}, \tilde{X}) \), and  \( \tilde{W}(h)(\tilde{T}^{(h)}, \tilde{X}) \equiv \text{diag}\{T_1^{(h)} e^{(1)}(X_1)/\{e^{(h)}(X_1)E(T_1^{(h)})\}, \ldots, T_n^{(h)} e^{(1)}(X_n)/\{e^{(h)}(X_n)E(T_n^{(1)})\}\} \) is the weight matrix for estimating the causal effect described above.

In this setup, the coefficients in (9) are given by

\[
A(M, s) = \frac{1}{\lambda_1} \left( \begin{array}{c}
\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \{ I_n - \tilde{\Sigma}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M + \lambda_2 I_M \}^{-1} \tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \{ I_n - \tilde{\Sigma}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M + \lambda_2 I_M \}^{-1} \\
\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \{ I_n - \tilde{\Sigma}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M + \lambda_2 I_M \}^{-1} \tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \{ I_n - \tilde{\Sigma}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M + \lambda_2 I_M \}^{-1} \\
-\text{diag}(s)(\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M + \lambda_2 I_M)^{-1} \tilde{X}_M^{\top} \tilde{\Sigma}^{-1}
\end{array} \right)
\]

and

\[
b(M, s) = \left( \begin{array}{c}
1_{|M|} - \tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M (\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M + \lambda_2 I_M)^{-1} s \\
1_{|M|} + \tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M (\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M + \lambda_2 I_M)^{-1} s \\
-\text{diag}(s)(\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M + \lambda_2 I_M)^{-1} s
\end{array} \right).
\]

Then, by letting  \( \tilde{E}(\tilde{X}) \equiv \text{diag}\{e^{(1)}(X_1), \ldots, e^{(1)}(X_n)\}, \tilde{K}^{(h)}(\tilde{X}) \equiv \text{diag}\{e^{(1)}(X_1)/\{e^{(h)}(X_1) - 1\}\}, \ldots, e^{(1)}(X_n)/\{e^{(h)}(X_n) - 1\}\}/|\mathcal{N}_1^{(h)}| \),

\[
\tilde{\eta}_j = \tilde{\Sigma}^{-1} \tilde{X}_M (\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M)^{-1} e_j,
\]

\[
\kappa_j^M(\tilde{T}, \tilde{X}) = e_j^{(h)}(\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M)^{-1} \tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \sum_{h=1}^{H} e^{(h)}W^{(h)}(\tilde{T}^{(h)}, \tilde{X}) \{ \tilde{\mu}^{(h)}(\tilde{X}) + \tilde{f}(\tilde{X}) \},
\]

\[
\zeta_j^M(\tilde{T}, \tilde{X}) = e_j^{(h)}(\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M)^{-1} \tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{W}(\tilde{T}, \tilde{X}) \tilde{\Sigma} \tilde{W}(\tilde{T}, \tilde{X}) \tilde{\Sigma}^{-1} \tilde{X}_M (\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M)^{-1} e_j,
\]

\[
\tau_j^M(\tilde{Y}^{\top}, \tilde{T}, \tilde{X}) = e_j^{(h)}(\tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \tilde{X}_M)^{-1} \tilde{X}_M^{\top} \tilde{\Sigma}^{-1} \sum_{h=1}^{H} c^{(h)}\{ \tilde{W}(h)(\tilde{T}^{(h)}, \tilde{X}) - \tilde{E}(\tilde{X}) \otimes I_r \} \tilde{Y}^{(h)^\top}
\]

and

\[
\rho_j^M(\tilde{T}, \tilde{X}) = \kappa_j^M(\tilde{T}, \tilde{X}) + \tilde{\eta}_j \sum_{h=1}^{H} c^{(h)^2} \{ \tilde{K}^{(h)}(\tilde{X}) \otimes \tilde{\Sigma} \} \tilde{\eta}_j,
\]

it can be shown that Theorem 1, Lemma 1 and Theorem 2 hold.
6.3 Case of unknown propensity score

In Section 3, we treated the propensity scores as known, but in reality, they are often unknown and are usually estimated by performing some kind of discriminant analysis. We can formally use the results of Section 3 as if the estimated propensity scores were known, but for the sake of a more rigorous methodology, we will take the effect of the estimation into account. We will regress \( \tilde{x} \) on \( \tilde{t} \) with some function and write \( \hat{e}(h)(X_i) \) as the estimator of the propensity score \( e(h)(X_i) = P(T_i(h) = 1 \mid X_i) \). Here, we assume that there is no misspecification in the regression model of the propensity scores and that \( \hat{e}(h)(X_i) - e(h)(X_i) = O_p(n^{-1/2}) \). Note that the results in this section do not depend on the representation of this regression model.

In \( \tilde{W}(\tilde{T}, \tilde{X}) \) which appears in Section 3, we replace \( e(h)(X_i) \) with \( \hat{e}(h)(X_i) \) and write it as \( \hat{W}(\tilde{T}, \tilde{X}) \). Note that under the conditioning of (8), \( \hat{e}(h)(X_i) \) is a random variable, but under the conditioning of (12), which is unique to this paper, \( \hat{e}(h)(X_i) \) is a non-random variable. This does not require using the asymptotic variance of the estimator when the propensity score is unknown, which has an interesting paradox of being smaller than when it is known (Hirano et al. 2003). Then, Theorem 1 and Lemma 1 still hold. The different consideration is needed in the derivation of Theorem 2, where we evaluate the variance of \( \hat{\eta}'j\hat{W}(\tilde{T}, \tilde{X})\hat{Y} - \tau^M_j(\hat{Y}^*, \tilde{T}, \tilde{X}) \). Because of the fluctuations in \( \hat{e}(h)(X_i) \), the evaluation of the matrix in (17) is modified to include \( E(\hat{e}(h)\hat{e}(k)*) \times O_p(n^{-1}) \). As a result, the conditional expectation of the second terms still has the form (19). As well, the evaluation of the matrix in (18) is modified to include \( E(\hat{e}(h)\hat{e}(k)*) \times O_p(n^{-1/2}) \). As a result, \( o_p(n^{-3/2}) \) changes to \( O_p(n^{-3/2}) \) in the evaluation of (21). Despite this change, Theorem 2 still holds.

6.4 Case of doubly robust estimation

Doubly robust estimation is often used as a potentially more efficient alternative to inverse-probability-weighted estimation (Robins and Rotnitzky 1995). In this estimation, we directly model the regression of the outcome variable on the confounding variable, i.e., \( \mu(h)(X) + f(X) \), and we write its regression function as \( g(h)(X) \). If either the modeling or the propensity score modeling is correct, this method can give a consistent estimator of the causal effect, which is why it is called doubly robust (Scharfstein et al. 1999). Note that unlike in the previous section, in doubly robust estimation, it is not necessarily true that \( \hat{e}(h)(X_i) - e(h)(X_i) = O_p(n^{-1/2}) \).

For simplicity, we will use \( \{A(M, s)\hat{W}(\tilde{T}, \tilde{X})\hat{Y} \leq b(M, s)\} \) for the conditioned event. Note that, although we will not consider it here, there is no difficulty in using double robust estimating
In line with this, we will let \( \hat{\rho} \) holds. where \( 0 \) valid and we use \( \hat{\rho} \) statistic. If this goodness-of-fit statistic is positive, the propensity score model is regarded as 

\[
e^*_j(X_M \hat{X}_M)^{-1} \hat{X}_M \sum_{h=1}^H e^{(h)} \{ \hat{W}^{(h)}(T^{(h)}, \hat{X}) \hat{Y}^{(h)} \} + \{ I_n - \hat{W}^{(h)}(T^{(h)}, \hat{X}) \} \hat{g}^{(h)}(\hat{X}).
\]

(22)

Here, \( \hat{g}^{(h)}(\hat{X}) = (\hat{g}^{(h)}(X_1), \ldots, \hat{g}^{(h)}(X_n))^\top \), and we assume that when this modeling is correct, the differences between it and \( \hat{\mu}^{(h)}(\hat{X}) + \hat{f}(\hat{X}) \) and between it and \( E\{ \hat{g}^{(h)}(\hat{X}) \mid \hat{X} \} \) are each \( \mathrm{O}_p(n^{-1/2}) \). Then, by defining

\[
\tau^M_\cdot(Y_t, T, X) \equiv e^*(X_M \hat{X}_M)^{-1} \hat{X}_M \sum_{h=1}^H e^{(h)} \{ \hat{W}^{(h)}(T^{(h)}, \hat{X}) - I_n \} \{ \hat{Y}^{(h)} - E\{ \hat{g}^{(h)}(\hat{X}) \mid \hat{X} \} \},
\]

and replacing the estimator \( \hat{\eta}_j \hat{W}(T, \hat{X}) \hat{Y} \) with (22), we can see that Lemma 1 holds.

Next, as \( \hat{Y}^{(h)} \) above, we consider using its natural estimator like \( \hat{Y}^{(h)} \) in Section 3. However, \( \hat{Y}^{(h)} \) converges slowly, so if the modeling of the propensity score is incorrect and the expectation of \( \hat{W}^{(h)}(T^{(h)}, \hat{X}) - I_n \) does not converge to the zero matrix, we cannot expect quick convergence in evaluations like those of (16) and (21). As a naive solution, we propose to use a goodness-of-fit statistic for \( \hat{Y}^{(h)} - \hat{g}^{(h)}(\hat{X}) \) or \( 0_n \) for \( \hat{Y}^{(h)} - E\{ \hat{g}^{(h)}(\hat{X}) \mid \hat{X} \} \) by case separation. For example, suppose that the possible values of \( \hat{e}^{(h)}(X) \) estimated using a \( q \)-dimensional discriminant model are discrete, and the estimate is \( \hat{e}^{(h)} \) when \( X \in X_r (\subset \mathbb{R}^p) \), where \( \bigcup_{r=1}^R X_r = \mathbb{R}^p \). Then, by letting \( \hat{p}^{(h)}_i \equiv \sum_{X_j \in X_r} T^{(h)}_i / \{ i : X_i \in X_r \} \), we only have to consider \( d(T, X) \equiv -2 \sum_{r=1}^R \sum_{X_j \in X_r} \sum_{h=1}^H T^{(h)}_i \log(\hat{p}^{(h)}_i / \hat{e}^{(h)}_i) + (LH - q) \log n \), which is equivalent to a difference between Bayesian information criteria (BIC), as the goodness-of-fit statistic. If this goodness-of-fit statistic is positive, the propensity score model is regarded as valid and we use \( \hat{Y}^{(h)} - \hat{g}^{(h)}(\hat{X}) \). If it is negative, the propensity score model is regarded as not valid and we use \( 0_n \). Then, from the consistency of BIC, we can ensure the convergence as in Section 3. Specifically, we define

\[
\tau^M_\cdot(Y^*, T, X)
\]

\[
e^*(X_M \hat{X}_M)^{-1} \hat{X}_M \{ d(T, \hat{X}) > 0 \} \sum_{h=1}^H e^{(h)} \{ \hat{W}^{(h)}(T^{(h)}, \hat{X}) - I_n \} \{ Y^{(h)*} - \hat{g}^{(h)}(\hat{X}) \}.
\]

In line with this, we will let \( \hat{K}^{(h)}(\hat{X}) \equiv \text{diag}[1/e^{(h)}(X_1) - 1] / |N_1|, \ldots, 1/e^{(h)}(X_n) - 1] / |N_n| \}. Moreover, we define

\[
\hat{\rho}^M_j(T, \hat{X}) \equiv \zeta^M_j(T, \hat{X}) + \sigma^2 I \{ d(T, \hat{X}) > 0 \} \sum_{h=1}^H e^{(h)} \{ \hat{\eta}_j \hat{K}^{(h)}(\hat{X}) \hat{\eta}_j \},
\]

where \( I \{ \cdot \} \) is the indicator function. Using these definitions, it can be seen that Theorem 2 holds.
6.5 Case of time-varying confounding

Dealing with time-varying confounding is a fundamental topic in causal inference (Bang and Robins 2005; Daniel et al. 2013). Here, suppose that there are \( m \) time points, indexed by \( j \) (\( j = 1, \ldots, m \)). Let \( T_j \) be the assignment variable representing the binary time-varying treatment and \( X_j \) be the time-varying confounding variable at time \( j \). The collections \( T = (T_1, \ldots, T_m)' \) and \( X = (X_1', \ldots, X_m')' \) are called the treatment history and the confounding history, respectively. Let us denote the realization of \( T \) as \( t = (t_1, \ldots, t_m)' \) and the potential outcome variable for \( t \) as \( Y^{(t)} \). We assume the following structure,

\[
Y = \sum_{t \in \{0,1\}^m} I(T = t)Y^{(t)} = \sum_{t \in \{0,1\}^m} I(T = t)\{\mu^{(t)} + f(X) + \epsilon\}
\]
as in Daniel et al. (2013). Then, for this \( \mu^{(t)} \), we consider the model \( (t - 1_m/2)'\beta \). Note that \( \beta_j \) is a parameter that represents the treatment effect at time \( j \). As in the preceding sections, the variables for the \( i \)-th sample are denoted by the subscript \( i \) and their collection is denoted by the tilde symbol \( \tilde{\cdot} \). In this setting, the ignorable treatment assignment condition defined by (3) does not give a consistent estimation, so

\[
Y^{(t)}_i \perp T_{j,i} \mid X_{1,i}, \ldots, X_{j,i}, T_{1,i}, \ldots, T_{j-1,i} \quad (t \in \{0,1\}^m; \ i = 1, \ldots, n; \ j = 1, \ldots, m)
\]
is assumed instead. Here, when \( j = 1 \), the assignment variable is assumed to be unconditioned.

The propensity score in this setting is \( e^{(t)}(X) = P(T_1 = t_1 \mid X_1)\prod_{j=2}^m P(T_j = t_j \mid X_1, \ldots, X_j, T_1 = t_1, \ldots, T_{j-1} = t_{j-1}) \), and the weight matrix is \( W^{(t)}(\tilde{T}, \tilde{X}) = \text{diag}\{I(T_1 = t)/e^{(t)}(X_1), \ldots, I(T_n = t)/e^{(t)}(X_n)\} \). We conduct model selection by minimizing \( \sum_{t \in \{0,1\}^m} \|W^{(t)}(\tilde{T}, \tilde{X})\tilde{Y} - 1_n(t - 1_m/2)'\beta\|^2/2 + \lambda\|\beta\|_1 \). Once a model \( M \) is selected, we use the inverse-probability-weighted estimator,

\[
\frac{1}{n^2|M|^{-2}} e'_j \sum_{t \in \{0,1\}^m} \left(t^M - \frac{1|M|}{2}\right) 1_n'W^{(t)}(\tilde{T}, \tilde{X})\tilde{Y},
\]
as a consistent estimator of \( \beta_M^j \). Although the model selection and estimation have different expressions from those in Section 3, it makes no difference that the squared losses are based on a linear function of \( \tilde{Y} \) minus a linear function of \( \beta \). Therefore, for example, an event in which model \( M \) with sign \( s \) is selected is expressed as \( A(M, s, \tilde{T}, \tilde{X})\tilde{Y} \leq b(M, s) \), and \( V^-_{s,j} \) and \( V^+_{s,j} \) are easily determined, i.e., the argument can be developed in the same way as in Section 3. Specifically, by letting \( \tilde{\kappa}^{(t)}_j(\tilde{T}, \tilde{X}) \equiv \text{diag}\{1/e^{(t)}(X_1) - 1\}/|V^-_1|, \ldots, \{1/e^{(t)}(X_n) - 1\}/|V^-_n|\},

\[
\kappa^M_j(\tilde{T}, \tilde{X}) = \frac{1}{n^2|M|^{-2}} e'_j \sum_{t \in \{0,1\}^m} \left(t^M - \frac{1|M|}{2}\right) 1_n'W^{(t)}(\tilde{T}, \tilde{X})\{\tilde{\mu}^{(t)} + \tilde{f}(X)\},
\]

28
\[ \zeta_j^M(\tilde{T}, \tilde{X}) = \frac{\sigma^2}{n^22^{2|\mathcal{M}|-4}} \sum_{t \in \{0,1\}^m} e_j^t \left( t^M - \frac{1}{2\mathcal{M}} \right) 1_n'(\tilde{W}^{(t)}(\tilde{T}, \tilde{X})^21_n(t^M - \frac{1}{2\mathcal{M}}) \right) e_j, \]

\[ \tau_j^M(\tilde{Y}^\dagger, \tilde{T}, \tilde{X}) = \frac{1}{n^22^{2|\mathcal{M}|-2}} \sum_{t \in \{0,1\}^m} \left( t^M - \frac{1}{2\mathcal{M}} \right) 1_n'\{\tilde{W}^{(t)}(\tilde{T}, \tilde{X}) - I_n\} \tilde{Y}(t)^\dagger \]

and

\[ \rho_j^M(\tilde{T}, \tilde{X}) = \zeta_j^M(\tilde{T}, \tilde{X}) + \frac{\sigma^2}{n^22^{2|\mathcal{M}|-4}} \sum_{t \in \{0,1\}^m} e_j^t \left( t^M - \frac{1}{2\mathcal{M}} \right) 1_n'\tilde{K}^{(t)}(\tilde{X})1_n(t^M - \frac{1}{2\mathcal{M}}) \right) e_j, \]

it can be seen that Theorem 1, Lemma 1 and Theorem 2 hold.

7 Conclusion

We developed selective inference in a basic causal inference model where the least-squares method is inappropriate for when inverse-probability-weighted estimation is used as the standard for propensity score analysis. Conditioning on the assignment variables first is usually not done in propensity score analysis, but doing so, we were able to construct a pivot statistic that follows a uniform distribution. Moreover, by dealing with higher-order asymptotics, we gave asymptotically guaranteed confidence intervals in the context of selective inference. Numerical experiments showed that a method that ignores the quiet scandal of statistics results in significant deviations from the preset coverage of the confidence intervals, whereas our method maintains the coverage. By comparing the two methods on benchmark data, we showed that there is a significant difference between them.

To carry on the beautiful methodology described by Lee et al. (2016), we considered in this paper model selection with basic sparse estimation in a setting that follows a normal distribution if there is no assignment. However, it is also important to consider model selection with an information criterion in a setting that includes a non-normal distribution. The information criterion for this case was recently developed by Baba et al. (2017), and it is expected that there will be no difficulty in implementing it in the method of Charkhi and Claeskens (2018). On the other hand, in order to treat the error distribution non-parametrically and to construct an elaborate theory based on the uniform convergence of pivot statistics, it is necessary to develop the works of Tian and Taylor (2017) and Tibshirani et al. (2018).

The selective inference is developing rapidly, and there are ongoing discussions on increasing the power of test. In fact, Kivaranovic and Leeb (2021) has shown that the standard method of selective inference can give too wide confidence intervals due to \( \text{E}(U_j^M - L_j^M) = \infty \). Then, for example, Tian and Taylor (2018) proposed a modern version of sample splitting, specifically
adding an appropriate randomization to the data, which is further discussed in Rasines and Young (2021). Depending on how the randomization is realized, these methods may yield different results, and so they cannot be accepted in all the fields of application, however, a considerable improvement in the power has been observed. In addition, Liu et al. (2018) pointed out that it is over-conditioning to condition on \( \hat{M} = M \) when inferring each \( \beta_j \), and increased the power by conditioning on a wider region. Since the selective inference in propensity score analysis involves many non-trivial arguments in constructing valid inferences themselves, and since the above-mentioned improvements are expected to be realized by simply combining them, we have not dealt with them in this paper. However, they are significant and will be an important subject in the future.

ACKNOWLEDGEMENT

Yoshiyuki Ninomiya was supported by JSPS KAKENHI (16K00050), Yuta Umezu was supported by JSPS KAKENHI (18K18010), and Ichiro Takeuchi was supported by JSPS KAKENHI (17H00758, 20H00601), JST CREST (JPMJCR1502) and RIKEN Center for Advanced Intelligence Project.

REFERENCES

Baba, T., Kanemori, T., and Ninomiya, Y. (2017). A Cp criterion for semiparametric causal inference, Biometrika, 104, 845–861.

Bang, H. and Robins, J. M. (2005). Doubly robust estimation in missing data and causal inference models, Biometrics, 61, 962–973.

Berk, R., Brown, L., Buja, A., Zhang, K., and Zhao, L. (2013). Valid post-selection inference, The Annals of Statistics, 41, 802–837.

Breiman, L. (2001). Statistical modeling: The two cultures, Statistical Science, 16, 199–231.

Charkhi, A. and Claeskens, G. (2018). Asymptotic post-selection inference for the Akaike information criterion, Biometrika, 105, 645–664.

Daniel, R. M., Cousens, S. N., De Stavola, B. L., Kenward, M. G., and Sterne, J. A. C. (2013). Methods for dealing with time-dependent confounding, Statistics in Medicine, 32, 1584–1618.
Fithian, W., Sun, D., and Taylor, J. (2014). Optimal inference after model selection, *arXiv preprint:1410.2597*.

Hernán, M. A. and Robins, J. M. (2020). *Causal inference: what if*: Boca Raton: Chapman & Hall/CRC.

Hillstrom, K. (2008). The MineThatData e-mail analytics and data mining challenge, *MineThatData blog*, http://blog.minethatdata.com/2008/03/minethatdata-e-mail-analytics-and-data.html.

Hirano, K., Imbens, G. W., and Ridder, G. (2003). Efficient estimation of average treatment effects using the estimated propensity score, *Econometrica*, 71, 1161–1189.

Imbens, G. W. (2000). The role of the propensity score in estimating dose-response functions, *Biometrika*, 87, 706–710.

——— (2004). Nonparametric estimation of average treatment effects under exogeneity: A review, *Review of Economics and Statistics*, 86, 4–29.

Kivaranovic, D. and Leeb, H. (2021). On the length of post-model-selection confidence intervals conditional on polyhedral constraints, *Journal of the American Statistical Association*, 116, 845–857.

LaLonde, R. J. (1986). Evaluating the econometric evaluations of training programs with experimental data, *The American Economic Review*, 604–620.

Lee, J. D., Sun, D. L., Sun, Y., and Taylor, J. E. (2016). Exact post-selection inference, with application to the lasso, *The Annals of Statistics*, 44, 907–927.

Lee, J. D. and Taylor, J. E. (2014). Exact post model selection inference for marginal screening, In *Advances in Neural Information Processing Systems*, 27, 136–144.

Liu, K., Markovic, J., and Tibshirani, R. (2018). More powerful post-selecion inference, with application to the lasso, *arXiv preprint:1801.09037*.

Peters, J., Janzing, D., and Schölkopf, B. (2017). *Elements of causal inference: foundations and learning algorithms*: The MIT Press.

Rasines, D. G. and Young, G. A. (2021). Splitting strategies for post-selection inference, *arXiv preprint arXiv:2102.02159*.
Robins, J. M., Rotnitzky, A., and Zhao, L. P. (1994). Estimation of regression coefficients when some regressors are not always observed, *Journal of the American Statistical Association*, 89, 846–866.

Robins, J. M. and Rotnitzky, A. (1995). Semiparametric efficiency in multivariate regression models with missing data, *Journal of the American Statistical Association*, 90, 122–129.

Rosenbaum, P. R. and Rubin, D. B. (1983). The central role of the propensity score in observational studies for causal effects, *Biometrika*, 70, 41–55.

Rubin, D. B. (1985). The use of propensity scores in applied Bayesian inference, *Bayesian Statistics*, 2, 463–472.

Scharfstein, D. O., Rotnitzky, A., and Robins, J. M. (1999). Adjusting for nonignorable drop-out using semiparametric nonresponse models, *Journal of the American Statistical Association*, 94, 1096–1120.

Taylor, J. and Tibshirani, R. J. (2015). Statistical learning and selective inference, *Proceedings of the National Academy of Sciences*, 112, 7629–7634.

Tian, X. and Taylor, J. (2017). Asymptotics of selective inference, *Scandinavian Journal of Statistics*, 44, 480–499.

——— (2018). Selective inference with a randomized response, *The Annals of Statistics*, 46, 679–710.

Tibshirani, R. (1996). Regression shrinkage and selection via the lasso, *Journal of the Royal Statistical Society: Series B*, 58, 267–288.

Tibshirani, R. J., Taylor, J., Lockhart, R., and Tibshirani, R. (2016). Exact post-selection inference for sequential regression procedures, *Journal of the American Statistical Association*, 111, 600–620.

Tibshirani, R. J., Rinaldo, A., Tibshirani, R., and Wasserman, L. (2018). Uniform asymptotic inference and the bootstrap after model selection, *Annals of Statistics*, 46, 1255–1287.

Zhao, Q., Small, D. S., and Ertefaie, A. (2017). Selective inference for effect modification via the lasso, *arXiv preprint:1705.08020*.

Zou, H. and Hastie, T. (2005). Regularization and variable selection via the elastic net, *Journal of the Royal Statistical Society: Series B*, 67, 301–320.

32