Energy–momentum density in small regions: the classical pseudotensors

Lau Loi So¹,4, James M Nester¹,²,³ and Hsin Chen¹

¹ Department of Physics, National Central University, Chungli 320, Taiwan
² Institute of Astronomy, National Central University, Chungli 320, Taiwan
³ Center for Mathematics and Theoretical Physics, National Central University, Chungli 320, Taiwan

E-mail: s0242010@webmail.tku.edu.tw and nester@phy.ncu.edu.tw

Received 12 January 2009, in final form 2 March 2009
Published 1 April 2009
Online at stacks.iop.org/CQG/26/085004

Abstract

The values for the gravitational energy–momentum density, given by the famous classical pseudotensors: Einstein, Papapetrou, Landau–Lifshitz, Bergmann–Thompson, Goldberg, Møller and Weinberg, in the small region limit are found to the lowest non-vanishing order in normal coordinates. All except Møller’s have the zeroth-order material limit required by the equivalence principle. However for small vacuum regions, we find that none of these classical holonomic pseudotensors satisfies the criterion of being proportional to the Bel–Robinson tensor. Generalizing an earlier work which had identified one case, we found another independent linear combination satisfying this requirement—and hence a one-parameter set of linear combinations of the classical pseudotensors with this desirable property.

PACS numbers: 04.20.—q, 04.20.Cv, 04.20.Fy

1. Introduction

The localization of energy for gravitating systems remains an outstanding problem. Unlike all other source and interaction fields, the standard techniques for identifying an energy–momentum density for the gravitational field have only yielded various expressions which are inherently reference frame dependent; these non-covariant expressions are often referred to as pseudotensors. This non-covariant feature can be understood as an inevitable consequence of the equivalence principle, which precludes the detection of the gravitational field at a point—so one cannot have a pointwise well-defined energy–momentum density for gravitating systems (for a good discussion of this point, see chapter 20 in [1]).

⁴ Present address: Department of Physics, Tamkang University, Tamsui 251, Taiwan.
The energy–momentum pseudotensor approach has largely been displaced by the more modern perspective of quasi-local: energy–momentum is to be associated with a closed 2-surface (for a review of the quasi-local idea see [2]). One quasi-local formulation is in terms of the Hamiltonian. The Hamiltonian for evolving a (generally finite) spacetime region includes a boundary term. Quasi-local quantities are associated with this Hamiltonian boundary term. There are many possible quasi-local expressions simply because there are many possible boundary terms. They are all physically meaningful, for each distinct boundary term is associated with a distinct physical boundary condition (which is given by what must be held fixed in the Hamiltonian variation). We note that this Hamiltonian quasi-local approach includes all the traditional pseudotensors, since each is generated by a superpotential which serves as a special type of Hamiltonian boundary term [3–7]. Via the Hamiltonian formulation the ambiguities of the traditional pseudotensors (which expression? which coordinate system?) are clarified. Hence, from the perspective of this Hamiltonian boundary term approach to quasi-local energy–momentum the traditional pseudotensors are still of interest.

For pseudotensor expressions the physical energy–momentum of the gravitational field is inextricably bound up with the choice of reference frame. This problem has long been recognized. Indeed soon after Einstein had proposed his expression for gravitational energy it was noted that with certain choices of coordinates it could give both a nonzero energy for Minkowski space and a vanishing energy for the Schwarzschild solution [8, 9]. Notwithstanding this, in many cases the reference frame ambiguity is not a problem—simply because there is an obvious natural choice of reference frame.

In particular in evaluating the energy–momentum for an asymptotically flat gravitating system, the asymptotic Minkowski space provides a natural and unambiguous reference for the coordinate system. For this case almost all the classic pseudotensors (except for Möller’s 1958 expression [10]) give the standard value for the total energy–momentum. Now there is another case where there is a natural and unambiguous reference: namely in a small region, where one can use the flat tangent space at some interior point to determine a Minkowski coordinate system. Here we shall test the classical pseudotensors in this limit.

Specifically we will be concerned with the famous pseudotensors due to Einstein [11], Papapetrou [12–14], Landau–Lifshitz [15], Bergmann–Thompson [16], Goldberg [17], Möller [10] and Weinberg [1, 18]. Although in some interesting cases many of the pseudotensors give identical answers (see [19]), in the small vacuum region limit considered here that will not be the case.

Note that a good energy–momentum expression for gravitating systems should satisfy a variety of requirements (see, e.g., [2, 20]), including giving the standard values for the total quantities for asymptotically flat space and reducing to the material energy–momentum in the appropriate limit (equivalence principle). No entirely satisfactory expression has yet been identified. One of the most restrictive requirements is positivity.

It is generally accepted that gravitational energy should be positive; indeed positive energy proofs have been heralded (e.g., [20–22]). Positivity is difficult to prove in general. One can regard positivity as an important test for quasi-local energy expressions. One limit that is not so difficult, and which has not been systematically investigated for all the classical pseudotensor expressions, is the small region limit. The small region requirements have not yet been applied to many energy–momentum expressions. We found that they afford both interesting restrictions and unexpected freedom.

For a small region within matter the equivalence principle requires that the energy–momentum expression should be dominated by the material energy–momentum tensor. On the other hand, the positivity of gravitational energy in a small vacuum region is assured if
its Taylor series expansion in Riemann normal coordinates is at the second order a positive multiple of the Bel–Robinson tensor.

Some time ago Deser et al [23] presented a discussion of pseudotensors in a small region, which has been a major source of inspiration for us. In that work the main techniques that we will use here were developed. They examined the Taylor expansion of a pseudotensor around a preselected (vacuum) point using Riemann normal coordinates (RNC). Considering pseudotensors derived from superpotentials, it was noted that the leading order non-vanishing vacuum expansion was at the second order and would be quadratic in the Riemann tensor. They then identified four basis expressions for such terms and related them to the famous Bel–Robinson tensor.

They argued that it is desired that one get the Bel–Robinson tensor, \( B_{\alpha\beta\mu\nu} \). They found exactly one such expression from a certain linear combination, \( B_{\alpha\beta\mu\nu} = \partial_{\mu}^2 \left(L_{\alpha\beta} + \frac{1}{2} E_{\alpha\beta}\right) \), of the Landau–Lifshitz and Einstein pseudotensors, a combination which they argued was unique.

We have reexamined the issue, considering all the aforementioned pseudotensors. Here we report in detail on our results, which were first obtained in [24] and briefly announced in [25]. We found that no classical holonomic pseudotensor gives the required second-order vacuum result (surely this contributed to the difficulty in finding a positive gravitational energy proof). However, using similar methods to the work just mentioned, we found another independent combination of the Bergmann–Thompson, Papapetrou and Weinberg pseudotensors, and thus a one-parameter set of pseudotensors, with the same desired Bel–Robinson property. (This was overlooked in the earlier work because they had not allowed for pseudotensors which require the explicit use of the Minkowski metric as a reference).

2. The classical pseudotensors

The classical pseudotensors can be obtained by suitably rearranging Einstein’s equation:

\[
G_{\mu\nu} = \kappa T_{\mu\nu},
\]

(1)

(where \( \kappa := 8\pi G/c^4 \)). One specific way is to choose a suitable superpotential \( U^{\mu\lambda}_{\nu} \equiv U^{[\mu\lambda]}_{\nu} \) and define the associated gravitational energy–momentum density pseudotensor by

\[
2\kappa t_{\mu\nu} := \partial_{\lambda} U^{\mu\lambda}_{\nu} - 2|g|^{1/2} G_{\mu\nu}.
\]

(2)

Einstein’s equation now takes the form

\[
2\kappa T_{\mu\nu} := 2\kappa \left(|g|^{1/2} T_{\mu\nu} + t_{\mu\nu}\right) = \partial_{\lambda} U^{\mu\lambda}_{\nu}.
\]

(3)

Because of the antisymmetry of the superpotential the total energy–momentum density complex is automatically conserved: \( \partial_{\mu} T_{\mu\nu} \equiv 0 \).

There are some variations on the idea, the classical pseudotensorial total energy–momentum density complexes all follow from associated superpotentials according to one of the patterns

\[
2\kappa T_{\mu\nu} = \partial_{\lambda} U^{\mu\lambda}_{\nu}, \quad 2\kappa T^{\mu\nu} = \partial_{\mu} U^{\mu\lambda}_{\nu}, \quad 2\kappa T^{\mu\nu} = \partial_{\alpha} H^{\alpha\mu\beta\nu},
\]

(4)

where the superpotentials have certain symmetries which automatically guarantee conservation: specifically \( U^{\mu\lambda}_{\nu} \equiv U^{[\mu\lambda]}_{\nu} \), \( U^{\mu\lambda}_{\nu} \equiv U^{[\mu\lambda]}_{\nu} \), while \( H^{\alpha\mu\beta\nu} \) has the symmetries of the Riemann tensor. In particular, the Einstein total energy–momentum density follows from the Freud superpotential [26]

\[
U^{\mu\lambda}_{\nu} := -|g|^{1/2} g^{\beta\gamma} \Gamma^\mu_{\beta\gamma} \delta^{\mu\lambda}_{\alpha\nu},
\]

(5)
while the Bergmann–Thompson [16], Landau–Lifshitz [15], Papapetrou [12], Weinberg [18] and Møller [10] expressions can be obtained from the respective superpotentials

\[ U_{\mu \nu}^{\text{BT}} := g^{\alpha \delta} U_{\alpha \delta}^{\mu \nu}, \]

equivalently

\[ H_{\mu \nu}^{\text{LL}} := |g|^{1/2} g_{\mu \nu}, \]

\[ H_{\mu \nu}^{\text{P}} := \delta_{\mu i} \delta_{\nu j} g^{ij}, \]

\[ H_{\mu \nu}^{\text{W}} := \delta_{\mu i} \delta_{\nu j} g^{ij} \left( -\frac{1}{2} \right) g_{cd} \]

\[ U_{\mu \nu}^{\text{M}} := -|g|^{1/2} g^{\alpha \beta} \Gamma_{\alpha \beta \gamma}^{\mu} \]

Regarding the detailed form of these expressions, the significance of \( \bar{g}_{ab} \) appearing in some of them will be explained shortly, and it should be noted that all indices here and throughout this work refer to spacetime and range from 0 to 3, otherwise our conventions follow [1].

These expressions are all non-covariant. As they depend on the coordinates in a non-tensorial way, they can at best be expected to give sensible energy–momentum values only in certain coordinates—which are in some suitable sense nearly Minkowski coordinates. Given that we have such coordinates we also have an underlying Minkowski space reference structure. Some of the superpotentials explicitly include this reference metric, which here has the Minkowski values \( \bar{g}_{ij} = \text{diag}(-1, +1, +1, +1) \). Our basic philosophy is that energy–momentum is properly a covector, and hence properly the energy–momentum density should be a weight one density with the index positions \( t_{\mu \nu} \). For all of the classical pseudotensors this can be achieved by introducing suitable factors of the Minkowski metric and its determinant. According to this philosophy only the Einstein and Møller expressions have proper expressions that do not explicitly need the Minkowski metric associated with the chosen coordinates.

It should be noted that, while in general identifying physically meaningful Minkowski coordinates is problematical, in the small region case of interest to us here at any chosen point there is a natural local Minkowski structure, as we discuss in the following section.

3. Some technical background

3.1. Riemann normal coordinates

As Riemann first argued (see, e.g., [27]), at any preselected point one can choose coordinates such that at the point \( x^\alpha = 0 \), the metric coefficients have the standard flat values, the first derivatives of the metric vanish, and the second derivatives have the minimum number (20 for \( n = 4 \)) of independent values. Specifically

\[ g_{\alpha \beta}|_0 = \bar{g}_{\alpha \beta}, \quad \partial_\mu g_{\alpha \beta}|_0 = 0, \quad 3 \partial_{\alpha \mu} g_{\nu \rho}|_0 = -(R_{\alpha \mu \nu \rho} + R_{\alpha \nu \beta \rho})|_0, \]

where \( R_{\alpha \beta \mu \nu} \) is the Riemannian curvature tensor and, in our case, \( \bar{g}_{\alpha \beta} = \eta_{\alpha \beta} = \text{diag}(-1, +1, +1, +1) \) is the Minkowski spacetime metric. The corresponding Lévi-Civitá connection values are

\[ \Gamma_{\beta \gamma}^{\alpha}|_0 = 0, \quad 3 \partial_\mu \Gamma_{\beta \gamma}^{\alpha}|_0 = -(R_{\alpha \mu \nu} + R_{\alpha \nu \beta})|_0. \]

3.2. Quadratic curvature basis

It turns out that when expanded in RNC the lowest non-vanishing vacuum energy–momentum expressions are of the second order and are quadratic in the curvature tensor:
$t_{\mu\nu} \sim (R \cdot R \cdot)_{\mu\nu\rho\sigma} x^i x^j$. An investigation [23] of all such possible terms (taking into account the Weyl = vacuum Riemann tensor symmetries) shows that they can be written in terms of

$$Q_{\mu\nu\rho\sigma} := R_{\mu\rho\sigma\alpha} R_{\nu}^{\alpha} \equiv Q_{\rho\sigma\mu\nu} \equiv Q_{\mu\rho\nu\sigma}. \quad (13)$$

where all the symmetries have been indicated. The cited work defined the three basis combinations

$$X_{\mu\nu\alpha\beta} := 2Q_(\mu\nu)_{\alpha\beta}, \quad Y_{\mu\nu\alpha\beta} := 2Q_{\alpha\beta(\mu\nu)}, \quad Z_{\mu\nu\alpha\beta} := Q_{\alpha\rho\nu\sigma} + Q_{\alpha\nu\rho\sigma}, \quad (14)$$

along with the trace tensor

$$T_{\mu\nu\rho\sigma} = -\frac{1}{6}g_{\mu\nu}Q_{\rho\sigma\alpha\beta}. \quad (15)$$

One could write all our vacuum expansions to the second order as linear combinations of $X, Y, Z, T$. However, it is more suitable for physical purposes to use the Bel–Robinson tensor.

3.3. The Bel–Robinson and two other tensors

The Bel–Robinson tensor

$$B_{\mu\nu\alpha\beta} := R_{\mu\rho\sigma\alpha} R_{\nu}^{\rho} \sigma + R_{\mu\rho\sigma\beta} R_{\nu}^{\rho} \sigma - \frac{1}{8}g_{\mu\nu}R_{\rho\sigma\tau\rho\sigma\tau} \quad (16)$$

has many well-known remarkable properties, see e.g., [23, 28]. For our considerations we are interested in it only in the vacuum, where the Riemann tensor reduces to the Weyl tensor. In this case the Bel–Robinson tensor is completely symmetric and traceless, and the last term admits an alternate form using

$$4R_{\rho\sigma\tau\rho\sigma\tau} = g_{\alpha\beta}R_{\kappa\lambda\gamma\delta}R_{\kappa\lambda\gamma\delta}. \quad (17)$$

In vacuum the Bel–Robinson tensor $B$ and two other convenient tensors $S$ and $K$ are given by

$$B_{\mu\nu\alpha\beta} := R_{\mu\rho\sigma\alpha} R_{\nu}^{\rho} \sigma + R_{\mu\rho\sigma\beta} R_{\nu}^{\rho} \sigma - \frac{1}{8}g_{\mu\nu}R_{\rho\sigma\tau\rho\sigma\tau},$$

$$S_{\mu\nu\alpha\beta} := R_{\mu\rho\sigma\alpha} R_{\nu}^{\rho} \sigma + R_{\mu\rho\sigma\beta} R_{\nu}^{\rho} \sigma + \frac{1}{8}g_{\mu\nu}R_{\rho\sigma\tau\rho\sigma\tau}, \quad (18)$$

$$K_{\mu\nu\alpha\beta} := R_{\mu\rho\sigma\alpha} R_{\nu}^{\rho} \sigma + R_{\mu\rho\sigma\beta} R_{\nu}^{\rho} \sigma - \frac{3}{8}g_{\mu\nu}R_{\rho\sigma\tau\rho\sigma\tau}. \quad (19)$$

In terms of the aforementioned basis for quadratic terms

$$B = Z + 3T, \quad S = -2X + 2Z - 6T, \quad K = Y + 9T. \quad (20)$$

(Here and below we have suppressed some obvious indices.)

Our leading order non-vanishing vacuum expressions will be given as linear combinations of $B, S$ and $K$. As can be directly verified, these three combinations (by virtue of (17)) satisfy the divergence free condition,

$$\partial_\alpha (x^i x^j t_{ij}^{\alpha\beta}) \equiv 2x^i t_{ij}^{\alpha\beta} \equiv 0, \quad (22)$$

and all such tensors are some linear combination of these three. While the tensor $S$ has been known for a long time, we here draw attention to the tensor $K$ which also enjoys this divergence-free property.

4. The small region limit

Here the small region values for the classical pseudotensors are presented.
4.1. Einstein

The Einstein total energy–momentum complex can be obtained from the Freud superpotential (5):

\[ 2\kappa T^\mu_\nu_E := \partial_\lambda(U^{\mu\lambda}_F)^\nu := \partial_\lambda(-|g|^{\frac{1}{2}} g^{\beta\sigma} \Gamma^\alpha_{\beta\rho} \delta^{\mu\lambda\gamma} \nu) \]

\[ = -|g|^{\frac{1}{2}} g^{\beta\sigma} \left( \frac{1}{2} R^\alpha_{\beta\lambda\nu} - \Gamma^\alpha_{\lambda\rho} \Gamma^\rho_{\beta\nu} \right) \delta^{\mu\lambda\gamma} \nu - \partial_\lambda \left( |g|^{\frac{1}{2}} g^{\beta\sigma} \Gamma^\alpha_{\beta\rho} \delta^{\mu\lambda\gamma} \nu \right) \]

\[ = 2|g|^{\frac{1}{2}} G^\mu_\nu - |g|^{\frac{1}{2}} \left[ \left( \Gamma^\delta_{\lambda\rho} \Gamma^{\alpha\sigma\gamma} - \Gamma^{\sigma\beta\gamma} \Gamma^\alpha_{\beta\rho} \right) g^{\mu\lambda\nu} \right]. \]

Using the Einstein field equation (1) this relation takes the form

\[ T^\mu_\nu_E = |g|^{\frac{1}{2}} T^\nu_\mu + \rho^\mu_\nu, \]

where the Einstein energy–momentum pseudotensor density is

\[ t^\mu_\nu := -(2\kappa)^{-1} |g|^{\frac{1}{2}} \left[ \left( \Gamma^\delta_{\lambda\rho} \Gamma^{\alpha\sigma\gamma} - \Gamma^{\sigma\beta\gamma} \Gamma^\alpha_{\beta\rho} \right) g^{\mu\lambda\nu} \right]. \]

Expanding in normal coordinates, to zeroth order the pseudotensor vanishes, so (26) reduces to

\[ T^\mu_\nu_E = |g|^{\frac{1}{2}} T^\nu_\mu + \rho^\mu_\nu, \]

which is the matter interior limit expected from the equivalence principle. In vacuum the complex (29) reduces to the pseudotensor, which has its first non-vanishing contribution at the second order:

\[ 2\kappa T^\mu_\nu_E = 2\kappa \rho^\mu_\nu = |g|^{\frac{1}{2}} \left( -2\Gamma^\mu_{\rho\sigma} \Gamma^{\rho\sigma\nu} + \delta^\mu_\nu \Gamma^{\rho\sigma\nu} \right) \simeq \frac{x_i x_j}{2 \cdot 3 \cdot 3} (4B - S)_{ij} \rho^\mu_\nu, \]

a result known for some time [1, 23, 28, 29].

4.2. Bergmann–Thompson, Landau–Lifshitz and Goldberg

The total energy–momentum complex proposed by Bergmann and Thompson [16] can be obtained from a transveected version of the Freud superpotential (6):

\[ 2\kappa T^{\mu\nu}_{BT} := \partial_\lambda \left( g^{\lambda\nu} U^{\mu\lambda}_F \right) = 2\kappa \left( \partial_\lambda g^{\lambda\nu} \right) g^{\nu\lambda} \]

\[ = -|g|^{\frac{1}{2}} g^{\rho\sigma} \left( \frac{1}{2} R^{\mu\rho\sigma\nu} - \Gamma^{\mu\rho\sigma} \Gamma^{\nu\rho\sigma} \right) g^{\lambda\nu} \rho \gamma - \partial_\lambda \left( |g|^{\frac{1}{2}} g^{\rho\sigma} \Gamma^{\mu\rho\sigma} \right) g^{\lambda\nu} \rho \gamma. \]

Consequently the Bergmann–Thompson gravitational energy–momentum pseudotensor density, found from a decomposition of the form \( T_{BT} = |g|^{\frac{1}{2}} T + \rho_{BT} \), is

\[ t^{\mu\nu}_{BT} = t^\mu_\nu + g^{\lambda\nu} \rho \gamma - |g|^{\frac{1}{2}} g^{\rho\sigma} \Gamma^{\mu\rho\sigma} g^{\lambda\nu} \rho \gamma. \]

In normal coordinates to zeroth order the pseudotensor \( \rho_{BT} \) vanishes so \( T_{BT} = |g|^{\frac{1}{2}} T \), which is the desired limit inside of matter. In vacuum the complex (29) reduces to the pseudotensor, which has its first non-vanishing contribution at the second order:

\[ 2\kappa t^{\mu\nu}_{BT} = |g|^{\frac{1}{2}} \left[ 2\Gamma^{\nu\lambda\rho}_{\mu\sigma\rho} - 2\Gamma^{\nu\lambda\rho}_{\mu\sigma\rho} \Gamma^{\gamma\rho\sigma\lambda} + g^{\mu\nu} \Gamma^{\nu\lambda\gamma} \Gamma^{\rho\sigma\lambda} \right] \simeq \frac{x_i x_j}{2 \cdot 3 \cdot 3} (7B + S)_{ij}. \]

The better known Landau–Lifshitz expression [15] can be obtained from this transveected superpotential with an additional improper density weight factor (7):

\[ 2\kappa T^{\mu\nu}_{LL} := \partial_\lambda \left( |g|^{\frac{1}{2}} g^{\lambda\nu} U^{\mu\lambda}_F \right) = 2\kappa |g|^{\frac{1}{2}} T^{\mu\nu}_E - |g|^{\frac{1}{2}} g^{\rho\sigma} \Gamma^{\mu\rho\sigma} g^{\lambda\nu} \rho \gamma. \]

A short calculation shows that this extra weight factor makes no contribution to the zeroth-order matter-interior limit nor to the second-order result in vacuum. Thus the Landau–Lifshitz vacuum result is given by (31), as was noted some time ago [23].
By the way, Goldberg [17] has proposed an infinite class of arbitrary density weight energy–momentum expressions obtained from those of Einstein and Landau–Lifshitz in a manner similar to that just considered. For essentially the same reasons, Goldberg’s weighted density factors will not yield any modified values in the small region limits we are considering here.

4.3. Papapetrou

For our analysis of the Papapetrou [12] and Weinberg [18] expressions we use the Einstein tensor expansion
\[ 2G_{\mu\nu} \equiv \delta_{\mu\alpha} \delta_{\nu\beta} [ \frac{1}{2} g^{\alpha\gamma} g_{\gamma\delta} - g^{\mu\nu} g^{\delta\gamma} ] \partial_{\delta\rho} g_{\gamma\sigma} + (2g^{\rho\mu} g^{\nu\gamma} - g^{\mu\nu} g^{\rho\gamma}) \times (\partial_{\rho} g^{\sigma\delta} \Gamma_{\delta\beta\gamma} - \partial_{\beta} g^{\sigma\delta} \Gamma_{\delta\rho\gamma} + \Gamma^{\sigma}_{\delta\alpha} \Gamma^{\delta}_{\beta\gamma} - \Gamma^{\sigma}_{\delta\beta} \Gamma^{\delta}_{\rho\alpha}), \]

For the Papapetrou energy–momentum complex [12–14] we find, using (33) and the appropriate superpotential (8),
\[ 2\kappa T_{\mu\nu}^{P} \equiv \delta_{\mu\alpha} \delta_{\nu\beta} [ (|g|^{-1/2} g_{\gamma\delta}) \partial_{\delta\rho} g_{\gamma\sigma} ] \partial_{\rho} g_{\gamma\sigma} \]
\[ \equiv \delta_{\mu\alpha} \delta_{\nu\beta} [ (|g|^{-1/2} g_{\gamma\delta}) \partial_{\delta\rho} g_{\gamma\sigma} ] \partial_{\rho} g_{\gamma\sigma} \]
\[ \approx 2|g|^{-1/2} G_{\alpha\beta} + |g|^{-1} (2g^{\rho\mu} g^{\nu\gamma} - g^{\mu\nu} g^{\rho\gamma}) \left[ \partial_{\gamma} g^{\alpha\delta} \Gamma_{\delta\rho\gamma} + \Gamma^{\alpha}_{\delta\rho} \Gamma^{\delta}_{\gamma\rho} \right] \]
\[ + \delta_{\mu\alpha} \delta_{\nu\beta} \Delta_{\gamma\delta} g_{\gamma\delta} \left[ |g|^{-1/2} g_{\gamma\delta} g^{\rho\sigma} \partial_{\sigma} g_{\gamma\delta} \right] \]
\[ \approx 2|g|^{-1/2} G_{\alpha\beta} + |g|^{-1} (2g^{\rho\mu} g^{\nu\gamma} - g^{\mu\nu} g^{\rho\gamma}) \left[ \partial_{\gamma} g^{\alpha\delta} \Gamma_{\delta\rho\gamma} + \Gamma^{\alpha}_{\delta\rho} \Gamma^{\delta}_{\gamma\rho} \right] \]
\[ + \delta_{\mu\alpha} \delta_{\nu\beta} \Delta_{\gamma\delta} g_{\gamma\delta} \left[ |g|^{-1/2} g_{\gamma\delta} g^{\rho\sigma} \partial_{\sigma} g_{\gamma\delta} \right], \]

where we use for any metric expression \( \Delta F \equiv F - \bar{F} \). Here we have indicated one way to arrange things so that we can get the terms of the desired orders. In normal coordinates to zeroth order within matter expansion (36) clearly reduces to the desired result. In vacuum to second order a lengthy calculation yields
\[ 2\kappa T_{\mu\nu}^{P} \approx \frac{x^i j}{3} \left[ 4B - S - K \right]_{ij} \]

4.4. Weinberg

With the help of (33) and the appropriate superpotential (9) for the Weinberg [18] expression (note: the same expression is considered in [1] section 20.2) we find
\[ 2\kappa T_{\mu\nu}^{W} \equiv \delta_{\mu\alpha} \delta_{\nu\beta} [ (|g|^{-1/2} g_{\gamma\delta}) \partial_{\delta\rho} g_{\gamma\sigma} ] \partial_{\rho} g_{\gamma\sigma} \]
\[ \equiv \delta_{\mu\alpha} \delta_{\nu\beta} [ (|g|^{-1/2} g_{\gamma\delta}) \partial_{\delta\rho} g_{\gamma\sigma} ] \partial_{\rho} g_{\gamma\sigma} \]
\[ \approx 2|g|^{-1/2} G_{\alpha\beta} + |g|^{-1} (2g^{\rho\mu} g^{\nu\gamma} - g^{\mu\nu} g^{\rho\gamma}) \left[ \partial_{\gamma} g^{\alpha\delta} \Gamma_{\delta\rho\gamma} + \Gamma^{\alpha}_{\delta\rho} \Gamma^{\delta}_{\gamma\rho} \right] \]
\[ - \delta_{\mu\alpha} \delta_{\nu\beta} \Delta_{\gamma\delta} g_{\gamma\delta} \left[ |g|^{-1/2} g_{\gamma\delta} g^{\rho\sigma} \partial_{\sigma} g_{\gamma\delta} \right] \]
\[ \approx 2|g|^{-1/2} G_{\alpha\beta} + |g|^{-1} (2g^{\rho\mu} g^{\nu\gamma} - g^{\mu\nu} g^{\rho\gamma}) \left[ \partial_{\gamma} g^{\alpha\delta} \Gamma_{\delta\rho\gamma} + \Gamma^{\alpha}_{\delta\rho} \Gamma^{\delta}_{\gamma\rho} \right] \]
\[ - \delta_{\mu\alpha} \delta_{\nu\beta} \Delta_{\gamma\delta} g_{\gamma\delta} \left[ |g|^{-1/2} g_{\gamma\delta} g^{\rho\sigma} \partial_{\sigma} g_{\gamma\delta} \right], \]

Here we have indicated one way to get the desired orders. In normal coordinates, as expected, to zeroth order it gives the desired material limit; in vacuum to second order after a long calculation we found
\[ 2\kappa T_{\mu\nu}^{W} \approx \frac{x^i j}{3} \left[ -B - 2S - 3K \right]_{ij} \]

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4.5. Møller

Møller’s holonomic energy–momentum complex [10] follows from the superpotential (10). We find

\[ 2\kappa T^\mu_{\text{Møller}} := -\partial_\nu (|g|^{1/2} g^{\mu\nu} \Gamma^\rho_{\nu\rho}) \tilde{g}^{\rho\lambda}_{\text{Møller}} = \partial_\lambda \left[ |g|^{1/2} g^{\rho\mu} g^{\lambda\nu} (\partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\rho\nu}) \right] \]

\[ \equiv |g|^{1/2} g^{\rho\mu} g^{\lambda\nu} (\partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\rho\nu}) + \partial_\lambda \left[ |g|^{1/2} g^{\rho\mu} g^{\lambda\nu} \right] (\partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\rho\nu}) \]

\[ \simeq |g|^{1/2} R^\mu_{\nu\lambda\sigma} - \frac{1}{2} g^{\mu\rho} g^{\nu\lambda} (\partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\rho\nu} + \partial_{\nu\lambda} g_{\rho\sigma} - \partial_{\rho\sigma} g_{\nu\lambda}) + \mathcal{O}(\Gamma^2) \]

\[ \simeq |g|^{1/2} R^\mu_{\nu\lambda\sigma} + \mathcal{O}(\lambda^2). \]

Note that the Møller energy–momentum complex fails to give the correct small region material limit (by the way, it also fails to agree with the proper asymptotic limit presented in [1], section 20.2). Using Einstein’s equation and (2) we have to zeroth order

\[ 2\kappa T^\mu_{\text{Møller}} = -(T^\mu_{\nu} + \frac{1}{2} \delta^\mu_{\nu} T^I_I), \]

which is non-vanishing inside of matter. Therefore the Møller expression does not satisfy the equivalence principle. From our perspective Møller’s holonomic expression is thus disqualified as a satisfactory description of energy–momentum.

Although it is consequently only of academic interest, nevertheless, for completeness, we briefly report here on the small region vacuum limit of Møller’s pseudotensor. From a long calculation, which requires the fourth-order Riemann normal coordinate expansion for the metric [30],

\[ g_{\alpha\beta,\sigma\lambda\mu\nu} = P \left( -\frac{1}{32} R^\alpha_{\sigma\beta\lambda\mu} + \frac{3}{16} R^I_{\sigma\beta,\lambda\mu\nu} \right), \]

where \( P \) is the \( \sigma\lambda\mu\nu \) symmetrization projection operator, we found to second order

\[ 2\kappa T^\mu_{\text{Møller}} \simeq \frac{\chi^I_I}{3} \left[ 2B - \frac{1}{2} S - K \right]_{\mu\nu} \]

5. Combinations of the classical pseudotensors

The Møller expression is ruled out of our further considerations by its material and asymptotic limit. The other four expressions are satisfactory asymptotically and in the material limit. However none of them give a good small vacuum value. Following the example of [23] we shall consider linear combinations of the four satisfactory classical pseudotensors. In the aforementioned work, it was noted that in vacuum a certain combination of the Einstein and the Landau–Lifshitz (or equivalently the Bergmann–Thompson) expressions is to second order proportional to the Bel–Robinson tensor; specifically they found

\[ B_{\rho\mu\nu} = \tilde{g}^2_{\mu\nu} \left( t_\ast + \frac{1}{2} t_{1L} \right)_{\rho\beta}. \]

This raises a couple of issues not addressed in that earlier work. In particular, one should pay attention to the overall normalization in order to satisfy the small region matter interior limit (the same normalization will also give the correct magnitude for the total energy–momentum of an asymptotically flat space). One also must cope with the mixed index positions which naturally occur for the Einstein pseudotensor in contrast with the two contravariant indices of the other three pseudotensors. Algebraically at the RNC origin it matters not whether one raises the index on the Einstein expression with \( g_{\alpha\beta} \) or \( \bar{g}_{\alpha\beta} \). However only the latter choice is allowed if we wish to preserve the conservation property. Thus we see another reason why we
are paralyzed unless we allow for the explicit appearance of the Minkowski metric in RNC. Hence we define

\[ t_{\mu \nu}^E := \bar{g}^{\nu \gamma} t_{E \gamma}^\mu. \]  

(50)

With this convention we can consider the linear combinations (suppressing indices for simplicity)

\[ 2 \kappa t := 2\kappa [et_E + bt_{BT} + pt_P + wt_W] \simeq (e + b + p + w)2G \]

\[ + \frac{1}{3}xx [B(2e + \frac{2}{3}b + 4p - w) - S(\frac{1}{2}e - \frac{1}{2}b + p + 2w) - K(p + 3w)], \]  

(51)

where we use \( \simeq \) to mean that only the zeroth-order and second-order vacuum values have been indicated. In order to have the correct asymptotic and material limits one must require \( e + b + p + w = 1 \), and for a good small vacuum limit to Bel–Robinson the coefficients of \( S \) and \( K \) should vanish—thus three restrictions on the four parameters. A convenient way to parametrize the set of acceptable coefficients is

\[ e = 1 - \lambda, \quad b = 2/3, \quad p = 3\lambda/2 - 1, \quad w = -\lambda/2 + 1/3, \]  

(52)

which yields

\[ t(\lambda) = (1 - \lambda)t_E + \frac{2}{3}t_{BT} + \left( \frac{\lambda}{2} - \frac{1}{3} \right)(3t_P - t_W) \]

\[ = \left[ t_E + \frac{2}{3}t_{BT} - \frac{1}{3}(3t_P - t_W) \right] + \lambda \left[ -t_E + \frac{1}{2}(3t_P - t_W) \right] \]  

(54)

\[ \simeq (2\kappa)^{-1} [2G + \lambda B_{xx}/2] = T + \lambda B_{xx}/4\kappa. \]  

(55)

The combination in the first bracket of (54) gives both the desired asymptotic and small region material result, but vanishes to second order in the small vacuum region limit; the combination in the second bracket vanishes both asymptotically and in the material limit, while in the small vacuum limit it gives the desired second-order Bel–Robinson contribution. Formally we can choose any value for \( \lambda \). Physically we want \( \lambda > 0 \). We know of no principle to fix the magnitude of \( \lambda \). Some values stand out: \( \lambda = 2/3 \) gives the (here properly normalized) case found earlier [23]:

\[ t(2/3) = (1/3)t_E + (2/3)t_{BT} \simeq T + B_{xx}/6\kappa. \]  

(56)

The choice \( \lambda = 1 \) gives another simple case:

\[ t(1) = (2/3)t_{BT} + (1/6)(3t_P - t_W) \simeq T + B_{xx}/4\kappa. \]  

(57)

Note that the earlier work found just one expression. The restricted form of the expressions considered there (i.e., not explicitly containing \( \bar{g} \)) are not in our opinion justified. To take linear combinations we need to get the indices on the Einstein pseudotensor at the same level, that requires \( \bar{g}_{\mu \nu} \). Moreover, the Landau–Lifshitz expression is actually of the wrong density weight (this can be adjusted by including a suitable power of \( |\bar{g}| \)). To the order considered in that work one can equally well use the Bergmann–Thompson expression, which we have used here.

Comparing with the earlier work, we note that both the Weinberg and Papapetrou expressions cannot be constructed without explicitly using a reference. We certainly want to allow for these expressions—especially the latter which has been generalized to define the total energy for an asymptotically anti-de Sitter space [31].
6. Conclusion

In this work, we found the values for the gravitational energy–momentum density given by the famous classical pseudotensors: Einstein, Papapetrou, Landau–Lifshitz, Bergmann–Thompson, Goldberg, Möller and Weinberg, in the small region limit to the lowest non-vanishing order in normal coordinates. All except Möller’s were found to have the zeroth-order material limit required by the equivalence principle. For small vacuum regions we found that none of these classical holonomic pseudotensors satisfies the second-order vacuum criterion of being proportional to the Bel–Robinson tensor.

However, certain linear combinations of these classical holonomic pseudotensors do give the desired second-order vacuum result. Generalizing an earlier work [23] which had identified one case, we found another independent linear combination satisfying this requirement—and hence a one-parameter set of linear combinations of the Einstein, Bergmann–Thompson, Papapetrou and Weinberg pseudotensors with this same desired property. (This had been overlooked because the earlier work had not allowed for the explicit use of the Minkowski metric as a reference in the pseudotensor.)

Regarding the physical meaning of these combinations, although that has not been worked out in detail, there is a straightforward way to find it. Each viable parameter choice fixes a specific Hamiltonian boundary term—and thereby not just the value of the Hamiltonian but also an associated boundary condition. Nevertheless these combinations may still appear rather artificial, as being mathematically and physically contrived. This situation may be contrasted with the case of another (nonholonomic) energy–momentum expression: elsewhere we have shown that the tetrad–teleparallel energy–momentum gauge current expression [32] naturally has the desired Bel–Robinson property [33].

Although it still allows for a certain amount of freedom, the small vacuum region Bel–Robinson positivity requirement provides a strong restriction which excludes many otherwise satisfactory energy–momentum expressions.

Acknowledgments

We would like to thank the Taiwan NSC for their financial support under the grant nos NSC 93-2112-M-008-001, 94-2112-M008-038, 95-2119-M008-027, 96-2112-M-008-005 and 97-2112-M-008-001. JMN was also supported in part by the National Center of Theoretical Sciences.

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