Dyon degeneracies from Mathieu moonshine

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Abstract: We construct the Siegel modular forms associated with the theta lift of twisted elliptic genera of $K3$ orbifolded with $g'$ corresponding to the conjugacy classes of the Mathieu group $M_{24}$. We complete the construction for all the classes which belong to $M_{23} \subset M_{24}$ and two other classes outside the subgroup $M_{23}$. For this purpose we provide the explicit expressions for all the twisted elliptic genera in all the sectors of these classes. We show that the Siegel modular forms satisfy the required properties for them to be generating functions of 1/4 BPS dyons of type II string theories compactified on $K3 \times T^2$ and orbifolded by $g'$ which acts as a $\mathbb{Z}_N$ automorphism on $K3$ together with a $1/N$ shift on a circle of $T^2$. In particular the inverse of these Siegel modular forms admit a Fourier expansion with integer coefficients together with the right sign as predicted from black hole physics. Our analysis completes the construction of the partition function for dyons as well as the twisted elliptic genera for all the 7 CHL compactifications.
1 Introduction

The partition function of 1/4 BPS dyons for $\mathcal{N} = 4$ string compactifications have been studied extensively. Starting from the original proposal [1] for the degeneracy of dyons in heterotic string theory on $T^6$ and the study of its asymptotic property [2], it has been generalized to certain CHL compactifications [3]. The degeneracy of dyons can be obtained from the Fourier coefficients of the inverse of an appropriate $Sp(2, \mathbb{Z})$ Siegel modular forms or its subgroup. For the case of the heterotic string on $T^6$, it is the Igusa cusp form of weight 10 which is the theta lift or the multiplicative lift of the elliptic genus of $K3$. The elliptic genus of $K3$ plays a role in the degeneracy since the counting of these 1/4 BPS states is done in the type II picture which is compactified on $K3 \times T^2$ [4, 5]. For the case of CHL compactifications [6] considered it turns out that the Siegel modular forms are theta lifts of the twisted elliptic genus of $K3$ [7, 8]. This is because the CHL compactifications are dual to $(K3 \times T^2)/\mathbb{Z}_N$ where the orbifold acts as an order $\mathbb{Z}_N$ Nikulin’s automorphism [9] on $K3$ together with a $1/N$ shift on one of the circles of $T^2$ [10, 11]. The construction so far has been for the case of $N = 2, 3, 5, 7$ CHL models, that is 4 out of the 7 CHL models.

With the discovery of Mathieu moonshine in $K3$ [12], it has been seen $K3$ admits 26 twining elliptic genera corresponding to the 26 conjugacy classes of the Mathieu
group $M_{24}$. Before we proceed let us define the twisted elliptic genus of $K3$ by an automorphism $g'$ of order $\mathbb{Z}_N$, given by

$$F^{(r,s)}(\tau, z) = \frac{1}{N} \text{Tr}_{RR} g'^{r} e^{2\pi i z F_{K3} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}}.$$  \hspace{1cm} (1.1)

Here the trace is taken over the Ramond-Ramond sector of the $\mathcal{N} = (4, 4, 4)$ superconformal field theory of $K3$ with central charge $(6, 6)$ and $F$ is the fermion number. The $K3$ CFT is orbifolded by the action of $g'$, a $\mathbb{Z}_N$ automorphism. The values $(r, s)$ run from 0 to $N - 1$. For $g'$ belonging to the 26 conjugacy classes of $M_{24}$ only the twining character $F^{(0,1)}$ has been constructed in [13–15]. The names of these classes and the corresponding cycle and the cycle shape are listed in tables 1 and 2. The order of the corresponding automorphism in $K3$ is also listed. For the $M_{24}$ conjugacy classes $pA, p = 2, 3, 5, 7$, the twisted elliptic genera in all the sectors was given earlier in [7]. These genera are obtained by orbifolding the $K3$ by $g'$ which is an order $N = 2, 3, 5, 7$ automorphism. The Siegel modular forms which capture the degeneracy of 1/4 BPS states in the $\mathcal{N} = 4$ theories obtained by type II compactified on the orbifold $(K3 \times T^2)/\mathbb{Z}_N$ have also been constructed in [7]. The most direct method of constructing these Siegel modular forms is thorough the theta lift of the corresponding twisted elliptic genus of $K3$. For this purpose it is necessary to know the Fourier expansion of the twisted elliptic genus in all its sectors. In this paper we extend this construction of Siegel modular forms to the other conjugacy classes of $M_{24}$. The construction is carried out for all classes in table 1 and for the first two classes in table 2. We also demonstrate that the inverse of these Siegel modular forms have the required properties to be generating functions of 1/4 BPS states of type II string compactified on orbifolds of $K3 \times T^2$ by $g'$ on $K3$ corresponding to these conjugacy classes together with a shift of $1/N$ on one of the circles of $T^2$. See [16, 17] for reviews. Our main objective is to study the Fourier expansion of the resulting Siegel modular forms and observe that their coefficients are integers as well as positive in accordance with the conjecture of [18].

As we have remarked the first step towards constructing the Siegel modular form obtained as a theta lift of the twisted elliptic genus of $K3$ is the knowledge of the Fourier expansion in all the sectors $F^{(r,s)}$. Other than the conjugacy classes $pA, p = 2, 3, 5, 7$ only the twining character $F^{(0,1)}$ is known. To obtain the other sectors we use the following transformation property of the twisted elliptic genus under modular transformation

$$F^{(r,s)} \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = \exp \left( 2\pi i \frac{cz^2}{c\tau + d} \right) F^{(cs + ar + ds + br)}(\tau, z),$$ \hspace{1cm} (1.2)

with

$$a, b, c, d \in \mathbb{Z}, \hspace{0.5cm} ad - bc = 1.$$ \hspace{1cm} (1.3)

In (1.2) the indices $cs + ar$ and $ds + br$ belong to $\mathbb{Z}$ mod $N$. For example the $(0,1)$ sector on the LHS of (1.2) is related to the $(1,0)$ sector its arguments is evaluated
at \((-1/\tau, z/\tau)\). However this is not sufficient, since we require a Fourier expansion of the \((1,0)\) sector to construct the theta lift, in fact we need further relations to express the the expansion in terms of \(e^{-2\pi i/\tau}\) in terms ordinary \(q = e^{2\pi i \tau}\) expansions. We find several identities involving modular forms of \(\Gamma_0(N)\) which will enable us to perform this explicitly in this paper. For \(N\) prime this procedure is enough to determine all the sectors of the twisted elliptic genus. But, when \(N\) is composite it is not possible to relate all the sectors to the \((0,1)\) sector by modular transformation. The various sectors of the twisted elliptic genus break up into sub-orbits under the action of modular transformations. For example for the class \(4B\) with \(N = 4\) in table 1, the sectors \(F^{(0,2)}, F^{(2,0)}, F^{(2,2)}\) form a sub-orbit and cannot be related to \(F^{(0,1)}\). We determine the twisted elliptic genus in these sub-orbits using its correspondence with the cycle shape of \(M_{24}\). This correspondence is sufficient to determine the complete twisted elliptic genus for all the classes given in table 1.

Among the classes in table 2, we construct the twisted elliptic genus for classes \(2B\) and \(3B\). The cycle shape of conjugacy classes belonging to this table is such that one cannot use it to determine the twisted elliptic genus in the sub-orbits. For example squaring the cycle in the \(2B\) class leads to the identity. In [19], an explicit rational CFT consisting of 6, \(SU(2)\) WZW models at level 1 in which the \(2B\) orbifold can be performed was introduced. This construction enables the evaluation of the twisted elliptic genus in all the sectors. The twisted elliptic genus exhibits the following property which is known as ‘quantum symmetry’. This essentially means that the sum of the twisted elliptic genus in all its sectors vanishes. For example for the case of \(2B\) which is an order 4 action in \(K3\) quantum symmetry implies the equality

\[
\sum_{r,s=0}^{3} F^{(r,s)}(\tau, z) = 0. \tag{1.4}
\]

Using this symmetry we obtain the twisted elliptic genus for the \(3B\) in all the sectors.

At this point it is important to mention the the [20] also constructs the twisted elliptic genus for all the orbifolds considered in this paper as well as orbifolds by non-cyclic groups. A detailed comparison of our work with [20] will made in section 2.4. But we remark here that the explicit expressions that we derive here are not found in the main body of [20].

We then construct and determine the weights \(k\) of the Siegel modular form \(\tilde{\Phi}_k(\rho, \sigma, v)\) obtained from the theta lift of the twisted elliptic genus corresponding to the conjugacy classes in table (1) and the first two classes in table (2). We study the factorization property on divisors as \(v \to 0\). This enables us to obtain the asymptotic degeneracies of 1/4 BPS black holes of large charges in type II string theory compactified on the orbifold \((K3 \times T^2)/\mathbb{Z}_N\) including the sub-leading corrections. Using the analysis in [21], we see the sub-leading corrections agree precisely with that obtained using the entropy function method including the Gauss-Bonnet term.
in these theories. We also obtain the degeneracies of 1/2 BPS states of these theories which are either purely electric or magnetic. Finally we explicitly evaluate the degeneracies of the low charge dyons in these states using these Siegel modular forms by extracting out the respective Fourier coefficients. This is given by the expression

\[ -B_6 = -(-1)^{Q \cdot P} \int_C d\rho d\sigma dv \, e^{-\pi i (N_\rho Q^2 + \sigma/N P^2 + 2v \cdot Q \cdot P)} \frac{1}{\Phi(\rho, \sigma, v)}, \]

where \( C \) is a contour in the complex 3-plane which we will define. \( Q, P \) refer to the electric and magnetic charge of the dyons in the heterotic frame. We emphasize that the evaluation of the degeneracy \(-B_6\) for low charge dyons is possible only due to the explicit knowledge of the twisted elliptic genus in all its sectors. A subset of these Fourier coefficients represent single centered black holes. From the fact that the single centered black holes carry zero angular momentum, it is conjectured that the sign of \(-B_6\) is positive [18]. We verify this prediction for low charge dyons. All these properties of \((\Phi_k)^{-1}\) indicate that they capture the degeneracy of dyons in \( \mathcal{N} = 4 \) theories compactified on orbifolds \((K3 \times T^2)/\mathbb{Z}_N\) where \( \mathbb{Z}_N \) acts as \( g' \) an order \( N \) automorphism in \( K3 \) together with a \( 1/N \) shift on one of the circles of \( T^2 \).

The construction of Siegel modular forms for the cases of composite order 4B, 6A, 8A in table 1 together with the earlier construction of the Siegel modular forms for the classes \( pA, p = 2, 3, 5, 7 \) completes the study of the spectrum of 1/4 BPS dyons in all the 7 CHL compactifications introduced in [6, 10].

Again here we remark the construction of the modular forms \( \tilde{\Phi}_k \) given the twisted elliptic genus is quite straightforward and our method is the extension of the method first introduced for cyclic orbifolds in [21]. Recently this construction has been extended for non-cyclic orbifolds in [22–24]. However to our knowledge that the observation of positivity of the Fourier coefficients of the inverse of \( \tilde{\Phi}_k \) which is in agreement with the conjecture of [22–24] for all the orbifolds considered in this paper is new. This observation has been made possible by the careful construction of all the twisted elliptic genera performed in section 2. It is also important to note that a proof for the positivity conjecture of [18] has been made only for the case of a class of Fourier coefficients of the partition function \( \Phi_{10} \) by [25]. A general proof of the positivity conjecture for \( \Phi_{10} \) as well as all the cyclic orbifolds considered in this paper is an open question.

The organization of the paper is as follows: In section 2, we construct the twisted elliptic genus for different orbifolds of \( K3 \) in each sector, We first discuss the orbifolds of \( K3 \) corresponding to the classes in table 1 and then move on to the classes 2B and 3B of table 2. In section 3, we use the twisted elliptic genus to construct the Siegel modular forms that capture degeneracies of 1/4 BPS dyons of type II theories compactified on \((K3 \times T^2)/\mathbb{Z}_N\) where \( \mathbb{Z}_N \) acts as a order \( N \) automorphism on \( K3 \) together with a \( 1/N \) shift on one of the circles of \( T^2 \). We show that low lying coefficients of the 1/4 BPS index are positive as expected from black hole
considerations in section 3.1. Appendix A lists various identities relating modular forms involving expansions in $e^{-2\pi i \tau}$ to $e^{2\pi i \tau}$. Finally appendix B lists the twisted elliptic genus for the $14A$ and the $15A$ conjugacy class.

| Conjugacy Class | Order | Cycle shape | Cycle |
|-----------------|-------|-------------|-------|
| 1A              | 1     | $1^{24}$    | $(1)$ |
| 2A              | 2     | $1^8 \cdot 2^6$ | (1, 8)(2, 12)(4, 15)(5, 7)(9, 22)(11, 18)(14, 19)(23, 24) |
| 3A              | 3     | $1^6 \cdot 3^6$ | (3, 18, 20)(4, 22, 24)(5, 19, 17)(6, 11)(8, 7, 15, 10)(9, 12, 14) |
| 5A              | 4     | $1^4 \cdot 5^4$ | (2, 21, 13, 16, 23)(3, 5, 15, 22, 14)(4, 12, 20, 17, 7)(9, 18, 19, 10, 24) |
| 7A              | 7     | $1^3 \cdot 7^3$ | (1, 17, 5, 21, 24, 10, 6)(2, 12, 13, 9, 4, 23, 20)(3, 8, 22, 7, 18, 14, 19) |
| 7A              | 7     | $1^3 \cdot 7^3$ | (1, 21, 6, 5, 10, 17, 24)(2, 9, 20, 13, 23, 12, 4)(3, 7, 19, 22, 14, 8, 18) |
| 11A             | 11    | $1^2 \cdot 11^2$ | (1, 3, 10, 4, 14, 15, 5, 24, 13, 17, 18)(2, 21, 23, 9, 20, 19, 6, 12, 16, 11, 22) |
| 23A             | 23    | $1^1 \cdot 23^1$ | (1, 7, 6, 24, 14, 4, 16, 12, 20, 9, 11, 5, 15, 10, 19, 18, 23, 17, 3, 2, 8, 22, 21) |
| 23B             | 23    | $1^1 \cdot 23^1$ | (1, 4, 11, 8, 6, 12, 15, 17, 21, 14, 9, 19, 2, 7, 16, 5, 23, 22, 24, 20, 10, 3) |

Table 1: Conjugacy classes of $M_{23} \subset M_{24}$ (Type 1)

| Conjugacy Class | Order | Cycle shape | Cycle |
|-----------------|-------|-------------|-------|
| 4B              | 4     | $1^4 \cdot 2^4 \cdot 3^4$ | (1, 17, 21, 9)(2, 13, 24, 15)(3, 14, 5, 8)(6, 16, 12, 18, 20, 22) |
| 6A              | 6     | $1^2 \cdot 2^2 \cdot 3^2 \cdot 3^2$ | (1, 8)(2, 24, 11, 12, 13, 23)(8, 20, 10)(4, 15)(5, 19, 9, 7, 14, 22)(6, 16, 13) |
| 8A              | 8     | $1^2 \cdot 2 \cdot 4 \cdot 4 \cdot 8^2$ | (1, 13, 17, 24, 21, 15, 9, 2)(3, 16, 23, 6)(4, 22, 14, 12, 5, 18, 8, 20)(7, 11) |
| 14A             | 14    | $1^4 \cdot 2^2 \cdot 7^1 \cdot 14^1$ | (1, 12, 17, 13, 5, 9, 21, 4, 24, 23, 10, 20, 6, 2)(3, 18, 8, 14, 22, 19, 7)(11, 15) |
| 14B             | 14    | $1^4 \cdot 2^2 \cdot 7^1 \cdot 14^1$ | (1, 13, 21, 23, 6, 12, 5, 4, 10, 2, 17, 9, 24, 20)(3, 14, 7, 8, 19, 18, 22)(11, 15) |
| 15A             | 15    | $1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$ | (2, 13, 21, 23, 16)(3, 7, 9, 5, 4, 18, 15, 12, 19, 22, 20, 10, 14, 17, 24)(6, 8, 11) |
| 15B             | 15    | $1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$ | (2, 23, 16, 13, 21)(3, 12, 24, 15, 17, 18, 14, 4, 10, 5, 20, 9, 22, 7, 19)(6, 8, 11) |

Table 2: Conjugacy classes of $M_{24} \not\subset M_{23}$ (Type 2)

2 Twisted Elliptic Genus

In this section we construct the twisted elliptic genus of the conjugacy classes in table 1 and then for the classes of $2B$ and $3B$ from table 2. Among the classes in table 1, the complete elliptic genus for the classes $pA$ with $p = 2, 3, 5, 7$ were given in [7]. To quote the result we first define

$$A = \frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2},$$

$$B(\tau, z) = \frac{\theta_4(\tau, z)^2}{\eta(\tau)^6}.$$
and
\[ E_N(\tau) = \frac{12i}{\pi(N - 1)} \partial_\tau [\ln \eta(\tau) - \ln \eta(N\tau)]. \tag{2.2} \]

Under $SL(2, \mathbb{Z})$ transformation $A(\tau, z)$ transforms as a weak Jacobi form of weight 0 and index 1 and $B(\tau, z)$ transforms as a weak Jacobi form of weight $-2$ and index 1. Now $E_N(\tau)$ transforms as a modular form of weight 2 under the group $\Gamma_0(N)$. Its transformations under $T$ and $S$ transformations of $SL(2, \mathbb{Z})$ are given by
\[ E_N(\tau + 1) = E_N(\tau), \quad E_N(-1/\tau) = -\frac{\tau^2}{N} E_N(\tau/N). \tag{2.3} \]

Then the twisted elliptic genera in all the sectors for the classes $pA$ with $p = 2, 3, 5, 7$ are given by
\[ F^{(0,0)}(\tau, z) = \frac{8}{N} A(\tau, z), \]
\[ F^{(0,s)}(\tau, z) = \frac{8}{N(N + 1)} A(\tau, z) - \frac{2}{N + 1} B(\tau, z) E_N(\tau), \quad \text{for } 1 \leq s \leq (N - 1), \]
\[ F^{(r,rk)} = \frac{8}{N(N + 1)} A(\tau, z) + \frac{2}{N(N + 1)} E_N(\frac{\tau + k}{N}) B(\tau, z), \]
for $1 \leq r \leq (N - 1), 1 \leq k \leq (N - 1)$.

Note that $rk$ is defined up to mod $N$. Here $N = 2, 3, 5, 7$ corresponding to the classes $pA$ respectively. Let us discuss the low lying coefficients in the expansion of $F^{(0,s)}$ which is given by
\[ F^{(0,s)}(\tau, z) = \sum_{j \in \mathbb{Z}, n=0}^{\infty} c^{(0,s)}(4n - j^2)e^{2\pi i n\tau} e^{2\pi ijz}. \tag{2.5} \]

Then it is easy to see from (2.4) that the low lying coefficients satisfy the following property
\[ \sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2. \tag{2.6} \]

The above set of equations corresponds to the number of $(0,0), (0,2), (2,0), (2,2)$ forms of the $pA$ orbifold of $K3$. As expected, these are the same as $K3$, since the orbifold preserves these forms [21]. The $(2,0)$ and $(0,2)$ forms are holomorphic forms which are required to be preserved if Type II theory compactified on the $pA$ orbifold $(K3 \times T^2)/\mathbb{Z}_N$ needs to be a $\mathcal{N} = 4$ theory. The orbifold preserves the 0-form as well as the top-form of $K3$. In fact the twisted elliptic genus satisfies the stronger property
\[ c^{(0,s)}(\pm 1) = \frac{2}{N}, \quad s = 0, \ldots, N - 1. \tag{2.7} \]

Now we can also see that
\[ \sum_{s=0}^{N-1} c^{(0,0)}(0) = 2 \left( \frac{24}{N + 1} - 2 \right). \tag{2.8} \]
The last equation corresponds to the number of the (1, 1) forms which are reduced from the $K_3$ value of 20 to 12, 8, 4, 2 for $N = 2, 3, 5, 7$ respectively. Finally the orbifold action for all these classes on $K_3$ produces another $K_3$. Therefore, the elliptic genus of $K^3/Z_N$ should be the same as that of $K_3$. This implies that we should obtain

$$\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z).$$  \hspace{1cm} (2.9)

Substituting the expressions for the twisted elliptic genus given in (2.4), we see that this is ensured by the following identity satisfied by $E_N(\tau)$ for $N$ prime.

$$\sum_{s=0}^{N-1} E_N\left(\frac{\tau + s}{N}\right) - NE_N(\tau) = 0.$$  \hspace{1cm} (2.10)

We will first focus on the classes with prime orders, 11A and 23A and obtain the twisted elliptic genera for these cases. We then move to the classes with composite order 4A, 6A, 8A, 14A, 14B, 15A, 15B. All cases with composite orders involve sub-orbits in the sectors of the twisted elliptic genus. To determine the twisted elliptic genus in these sub-orbits we use its correspondence with the cycle structure in $M_{24}$. Finally we discuss the cases of 2B, 3B which are automorphisms of order 4, 6 respectively. For these cases we use quantum symmetry to determine the twisted elliptic genus in the sub-orbits.

2.1 The conjugacy class 11A and 23A

11A class

The twining character for this class was determined in [13–15], it is given by

$$F^{(0,0)} = \frac{8}{11} A(\tau, z),$$  \hspace{1cm} (2.11)

$$F^{(0,1)} = \frac{2}{33} A(\tau, z) - B(\tau, z) \left(\frac{1}{6} \mathcal{E}_{11}(\tau) - \frac{2}{5} \eta^2(\tau)\eta^2(11\tau)\right).$$

To determine the twisted elliptic genus in all the sectors we will use the transformation law given in (1.2). Since $A(\tau, z), B(\tau, z)$ are weak Jacobi forms under $SL(2, \mathbb{Z})$ it is the transformation property of the $\Gamma_0(11)$ forms $\mathcal{E}_{11}(\tau)$ and $\eta^2(\tau)\eta^2(11\tau)$ in (2.3) under $SL(2, \mathbb{Z})$ which allows us to move to the other sectors.

It is first useful to show that $F^{(0,1)} = F^{(0,s)}$ for all $s = 2, \cdots, N-1$. It is important to point out if this fact is assumed for prime $N$ then the construction of all the sectors proceeds very straightforwardly using modular transformations. However as we will see we can prove the equality $F^{(0,1)} = F^{(0,s)}$, this will involve several steps. For this

\footnote{We have multiplied the twining character in [13–15] by 1/11. The reason for this normalization will be explained subsequently.}
we will need the following identities obeyed by the Dedekind $\eta$-function and $E_N$ for $N$ odd.

$$\eta(\tau + \frac{1}{2}) = e^{\pi i/24} \frac{\eta^3(2\tau)}{\eta(\tau)\eta(4\tau)}, \quad (2.12)$$

$$E_N(\tau + \frac{1}{2}) = -E_N(\tau) - 4E_N(4\tau) + 6E_N(2\tau). \quad (2.13)$$

Note that the first equation is a simple consequence of the definition of the $\eta$ function in its product form. The second equation can be obtained using the first equation and the definition (2.2).

First let us show $F^{(0,1)} = F^{(0,2)}$. Since we will have to keep track of how the factor $E_{11}(\tau)$ and $\eta^2(\tau)\eta^2(11\tau)$ transforms under $SL(2,\mathbb{Z})$, let us label these coefficients in the twisted elliptic genus in terms of the sector it occurs. Let

$$f^{(0,1)} = E_{11}(\tau), \quad g^{(0,1)} = \eta^2(\tau)\eta^2(11\tau). \quad (2.14)$$

Now under an $S$ transformation we know that from (1.2) we obtain the $F^{(1,0)}$ sector or the $F^{(10,0)}$ sector. Therefore using the transformation in (2.3), we obtain

$$F^{(1,0)}(\tau, z) = F^{(10,0)} = \frac{2}{33} A(\tau, z) + B(\tau, z) \left( \frac{1}{66} E_{11}(\tau) - \frac{2}{55} \eta^2(\tau) \eta^2(11\tau) \right). \quad (2.15)$$

We label the coefficients

$$f^{(1,0)}(\tau) = f^{(10,0)} = \frac{1}{\tau^2} E_{11}(\tau), \quad g^{(1,0)}(\tau) = g^{(10,0)} = \frac{1}{\tau^2} \eta^2(\tau) \eta^2(11\tau). \quad (2.16)$$

Now from the $F^{(1,0)}$ sector we can obtain the $F^{(1,s)}$ sector by a $(T)^s$ transformation and it results in

$$F^{(1,s)}(\tau, z) = \frac{2}{33} A(\tau, z) + B(\tau, z) \left( \frac{1}{66} E_{11}(\tau + s/11) - \frac{2}{55} \eta^2(\tau + s/11) \eta^2(\tau + s) \right). \quad (2.17)$$

An equation identical to the above exists for the sectors $F^{(10,s)}$. Thus the coefficients

$$f^{(1,s)}(\tau) = f^{(10,10s)}(\tau) = E_{11}(\tau + s/11), \quad g^{(1,s)}(\tau) = g^{(10,10s)}(\tau) = \eta^2(\tau + s/11) \eta^2(\tau + s) \quad (2.18)$$

Let us now move to the $F^{(2,s)}$ sectors. For this we begin with $F^{(1,2)}$ and perform the $S$ transformation. From (1.2) we see that we can obtain $F^{(2,10)}$ sector. The $S$ transformation also relates $F^{(1,2)} = F^{(9,1)}$ and the conclusions we obtain for the $F^{(2,10)}$ sector also holds for the $F^{(9,1)}$ sector. This sector will have the coefficients of $B$ given by

$$f^{(2,10)}(\tau) = \frac{1}{\tau^2} E_{11}(-\frac{1}{11\tau} + \frac{2}{11}), \quad (2.19)$$

$$g^{(2,10)}(\tau) = \frac{1}{\tau^2} \eta^2(-\frac{1}{11\tau} + \frac{2}{11}) \eta^2(-\frac{1}{\tau} + 2).$$
However, this is not useful, since we would like to obtain a $q$ expansion for these twisted sectors. We will establish the identity

\[
f^{(2,10)}(\tau) = \frac{1}{\tau^2} E_{11}(\frac{\tau + 5}{11}) = f^{(1,5)}(\tau),
\]

\[
g^{(2,10)}(\tau) = \frac{1}{\tau^2} \eta^5(\frac{\tau + 5}{11}) \eta^2(\frac{\tau + 5}{11}) = g^{(1,5)}(\tau).
\]

This enables us to perform the $q$ expansion in these sectors. To begin, we see that using the transformations under $S$ and $T$ (2.3) we obtain

\[
E_{11}(\frac{\tau}{11} + 2) = \frac{1}{(2 - 1/\tau)^2} E_{11}(\frac{1}{2} - \frac{1}{2(2\tau - 1)}).
\]

Using the $S$ transformation on the LHS of (2.22) leads us to

\[
E_{11}(\frac{\tau}{11} + 2) = \frac{(2\tau - 1)^2}{11} \left[ 4E_{11}(\frac{2(2\tau - 1)}{11}) + E_{11}(\frac{2\tau - 1}{22}) - 6E_{11}(\frac{2\tau - 1}{11}) \right].
\]

Substituting the above equation into (2.21) we obtain

\[
E_{11}(\frac{\tau}{11} + 2) = \tau^2 \left[ -4E_{11} \left( \frac{2\tau - 1}{22} \right) - E_{11} \left( \frac{2\tau - 1}{22} \right) + 6E_{11}(\frac{2\tau - 1}{11}) \right],
\]

\[
= \tau^2 E_{11}(\frac{\tau + 5}{11}).
\]

In the last line we have again used the identity (2.13) with $\tau$ replaced by $(2\tau - 1)/22$. Therefore using (2.24) we see that the first identity in equation (2.20) is true. Exactly a similar manipulation involving $g^{(2,10)}$ but using (2.12) to remove shifts by $1/2$ will enable us to prove the second equation in (2.20). Thus we obtain the relation

\[
F^{(2,10)} = F^{(1,5)}
\]

Now using the $T$ transformation and the fact that $N = 11$ is prime we can obtain all the twisted sectors $F^{(2,s)}$ as well as $F^{(9,s)}$.

To show that $F^{(0,1)} = F^{(0,2)}$, we start with (2.25) and perform a $(T)^6$ transformation to both sides of the equation using (1.2). Then we obtain $F^{(2,0)} = F^{(1,0)}$ and thus by a $S$ transformation we see that $F^{(0,1)} = F^{(0,2)}$.

Similar manipulations allow us to obtain all the other sectors. Let us briefly go over one more case. We start with $F^{(2,4)}$ which can be obtained by performing a $T^{-5}$ transformation. Thus the coefficient of the $B$ term is given by

\[
f^{(2,4)}(\tau) = \frac{1}{\tau^2} E_{11}(\frac{\tau + 2}{11})
\]
Lets do the $S$ transformation on $F^{(2,4)}$ with will take us to either $F^{(4,9)}$ or $F^{(7,2)}$ and the coefficient of the $B$ term in this sector contains

$$f^{(4,9)}(\tau) = \frac{1}{\tau^2} E_{11} \left( -\frac{1}{11\tau} + \frac{2}{11} \right). \quad (2.27)$$

Therefore using the same set of manipulations we see that $F^{(4,9)} = F^{(7,2)} = F^{(1,5)}$. Therefore by a first $T^{-(5)}$ and then an $S$ action we can show $F^{(0,4)} = F^{(0,7)} = F^{(0,1)}$. Using these steps we obtain the relations among different sectors as given in the following chart.

| $S$          | $T^{-(5)}$          | $S$          |
|--------------|---------------------|--------------|
| $F^{(1,2)}$  | $F^{(2,10)} = F^{(9,1)} = F^{(1,5)}$ | $F^{(2,0)} = F^{(9,0)} = F^{(1,0)} = F^{(10,0)}$ | $F^{(0,2)} = F^{(0,9)} = F^{(0,1)} = F^{(0,10)}$ |
| $F^{(2,4)}$  | $F^{(4,9)} = F^{(7,2)} = F^{(1,5)}$ | $F^{(4,0)} = F^{(7,0)} = F^{(2,0)}$ | $F^{(0,4)} = F^{(0,7)} = F^{(0,2)}$ |
| $F^{(4,8)}$  | $F^{(8,7)} = F^{(3,4)} = F^{(1,5)}$ | $F^{(8,0)} = F^{(3,0)} = F^{(4,0)}$ | $F^{(0,8)} = F^{(0,3)} = F^{(0,4)}$ |
| $F^{(3,6)}$  | $F^{(6,8)} = F^{(5,3)} = F^{(1,5)}$ | $F^{(6,0)} = F^{(5,0)} = F^{(3,0)}$ | $F^{(0,6)} = F^{(0,5)} = F^{(0,3)}$ |

Table 3: Table showing the $S$ and $T$ transformation starting from the twisted sector given in the leftmost column.

Going through these steps we obtain the following formula for the twisted elliptic genus for 11A.

$$F^{(0,0)} = \frac{8}{11} A(\tau, z), \quad (2.28)$$

$$F^{(0,s)} = \frac{2}{33} A(\tau, z) - B(\tau, z) \left( \frac{1}{6} E_{11}(\tau) - \frac{2}{5} \eta^2(\tau) \eta^2(11\tau) \right),$$

$$F^{(r,rs)} = \frac{2}{33} A(\tau, z) + B(\tau, z) \left( \frac{1}{66} E_{11}(\tau + s) - \frac{2}{55} \eta^2(\tau + s) \eta^2\left( \frac{\tau + s}{11} \right) \right).$$

The low lying values of the twisted elliptic genus for the 11A case satisfies

$$c^{(0,s)}(\pm 1) = \frac{2}{11}, \quad s = 0, \cdots N - 1, \quad (2.29)$$

$$\sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2.$$

Thus type II compactifications on such the orbifold $(K3 \times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as the 11A automorphism on $K3$ together with a 1/11 shift on one of the circles of $T^2$ preserves $N = 4$ supersymmetry. Note that the normalization of multiplying the twining character given in [13–15] by 1/11 is to ensure the condition given in (2.29). However the number of (1, 1) forms preserved by the orbifold vanishes as

$$\sum_{s=0}^{N-1} c^{(0,s)}(0) = 0. \quad (2.30)$$
Thus even the Kähler form of $K3$ is projected out, which implies this orbifold is not geometric. We can evaluate the elliptic genus of the $11A$ orbifold of $K3$, the result is given by

$$\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z). \quad (2.31)$$

We have verified this equation by performing a $q$-expansion of both the left hand side as well as the right hand side. The fact that the elliptic genus of the $11A$ orbifold of $K3$ satisfies this identity implies that the $11A$ orbifold of $K3$ is $K3$ itself.

The structure of all the twisted sectors is similar to that seen in the $pA$ cases in (2.4). In fact for the cases in $pA$, if one is given just the twining elliptic genus we can obtain the complete twisted elliptic genus using the same manipulations discussed for the $11A$.

23A/B class

The twining characters for the conjugacy classes 23A and 23B are identical and was determined in [13–15]. It is given by

$$F^{(0,0)} = \frac{8}{23} A(\tau, z), \quad \quad F^{(0,1)} = \frac{1}{69} A - B \left( \frac{1}{12} E_{23} - \frac{1}{22} f_{23,1}(\tau) - \frac{7}{22} \eta^2(\tau) \eta^2(23\tau) \right). \quad (2.32)$$

We can use the same procedure as discussed for the class $11A$ in the previous section to determine the twisted elliptic genus in all the sectors. Essentially we use the transformation law given in (1.2) to move to twisted elliptic genus in the other sectors from the $(0,1)$ sector. As discussed in the previous section we need identities satisfied by the modular forms $E_{23}(\tau)$, $\eta^2(\tau)\eta^2(23\tau)$ and $f_{23,1}(\tau)$ to express the expansion in terms of $e^{-2\pi i/\tau}$ in terms of a the usual $q$ expansion. Note that all these transform as modular forms under $\Gamma_0(23)$. The identities analogous to equation (2.24) can be found by similar manipulations. The new form $f_{23,1}(\tau)$ under $\Gamma_0(23)$ has been constructed in [26, 27] which involves Hecke eigenforms. A closed formula for $f_{23,1}(\tau)$ in terms of $\eta$ functions is provided in the ancillary files associated with [20]. This is given by

$$f_{23,1}(\tau) = 2 \frac{\eta^3(2\tau)\eta^3(23\tau)}{\eta(2\tau)\eta(46\tau)} + 8\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 8\eta^2(2\tau)\eta^2(46\tau) + 5\eta^2(\tau)\eta^2(23\tau) \quad (2.33)$$

It can be seen that from (2.33) that the $S$ transformation of $f_{23,1}(\tau)$ is given by

$$f_{23,1}(\frac{-1}{\tau}) = \frac{\tau^2}{23} f_{23,1}(\frac{\tau}{23}). \quad (2.34)$$
The low lying coefficients of this twisted elliptic genus satisfy

\[ f_{23,1}( -\frac{1}{23\tau} + \frac{1}{23} ) = \tau^2 f_{23,1}(\tau + \frac{22}{23}), \]
\[ f_{23,1}( -\frac{1}{23\tau} + \frac{2}{23} ) = \tau^2 f_{23,1}(\tau + \frac{11}{23}), \]
\[ f_{23,1}( -\frac{1}{23\tau} + \frac{3}{23} ) = \tau^2 f_{23,1}(\tau + \frac{15}{23}). \]

(2.35)
(2.36)
(2.37)

Therefore combining the modular transformation obeyed by the twisted elliptic genus \( E \) of the conjugacy class 23A as well as the identities in (2.35) and (2.38) we obtain the twisted elliptic genus \( f_{23,1}(\tau) \) given in the previous section we have shown that the modular form \( E_{23}(\tau) \) as well as \( \eta^2(\tau)\eta^2(23\tau) \) obey the identities (2.35) and (2.38).

A similar analysis given in the previous section can be used to prove these identities. We have not done this, but have numerically verified these identities. Again following a similar analysis to \( E_{11}(\tau) \) given in the previous section we have shown that the modular form \( E_{23}(\tau) \) and \( \eta^2(\tau)\eta^2(23\tau) \) obey the identities (2.39) and (2.40).

Therefore combining the modular transformation obeyed by the twisted elliptic genus (1.2) as well as the identities in (2.35) and (2.38) we obtain the twisted elliptic genus of the conjugacy class 23A. The result is given by

\[ F^{(0,k)}(\tau, z) = \frac{1}{23} \left[ \frac{1}{3} A - B \left( \frac{23}{12} E_{23}(\tau) - \frac{23}{22} f_{23,1}(\tau) - \frac{161}{22} \eta^2(\tau)\eta^2(23\tau) \right) \right], \]
\[ F^{(r,\ell)}(\tau, z) = \frac{1}{23} \left[ \frac{1}{3} A + B \left( \frac{12}{12} E_{23}(\tau + k) - \frac{1}{22} f_{23,1}(\tau + k) - \frac{7}{22} \eta^2(\tau + k)\eta^2(\frac{\tau + k}{23}) \right) \right]. \]

(2.39)

The low lying coefficients of this twisted elliptic genus satisfy

\[ c^{(0,s)}(\pm 1) = \frac{2}{23}, \quad s = 0, \ldots, N - 1, \]
\[ \sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2 \]

(2.40)

As we have discussed earlier, this implies that type II compactifications on the orbifold \( K3 \times T^2/\mathbb{Z}_N \) preserves \( \mathcal{N} = 4 \) supersymmetry. We also have

\[ \sum_{s=0}^{N-1} c^{(0,s)}(0) = -2. \]

(2.41)

When the RHS side of this equation is a positive integer, it corresponds to the number of \((1,1)\) forms preserved by the orbifold. Here, we obtain a result which is
a negative integer, the orbifold is therefore not geometric. Just as in the case of the
11A orbifold, the elliptic genus of 23A/23B orbifold reduces to that of K3. This can
be seen by showing the twisted elliptic genera of the 23A/23B orbifold satisfies
\[ \sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z). \] (2.42)

We have verified this identity by substituting the twisted elliptic genus from (2.39)
and performing the q expansion on both sides of the above equation.

2.2 Automorphisms $g'$ with composite order and $g' \in M_{23}$

Let us consider automorphisms $g'$ with composite order and those which belong to
$M_{23} \subset M_{24}$. Examples of these are the classes 4B, 6A, 8A, 14A, 15A given in table
1. When the order of the automorphism $g'$ is composite, we cannot use the $SL(2, \mathbb{Z})$
modular transformation in (1.2) to arrive at all the sectors of the twisted elliptic
genus started from the twining character. For example for the case of 4B which is of
the order 4 we cannot reach the sectors $(0, 2), (2, 0), (2, 2)$ starting from the twining
character $(0, 1)$. We call these sectors sub-orbits. In general if the order $N$ admits a
factorization
\[ N = \prod_i n_i \] (2.43)
then there is a sub-orbit for each divisor. Since the sub-orbits are not accessible
by modular transformations from the twining character $(0, 1)$ one needs to make a
choice of a particular character in these sectors. To be more specific, consider the
sub-orbit corresponding to the divisor $n_i$ we need to make a choice for the character
\[ F^{(0,n_i)}(\tau_i) = \frac{1}{N} \text{Tr}_{RR}[(1)^{F_{K3} + F_{K3}} g^{n_i} e^{2\pi i z F_{K3}} q^{L_0 - c/24} \bar{q}^{L_0 - \bar{c}/24}]. \] (2.44)

We will see in all the cases for composite orders with $g' \in M_{23} \subset M_{24}$, we will see that
from the cycle shape of $g'^{n_i}$ corresponds to a conjugacy class of order $N/n_i'$. Therefore
by appealing to Mathieu moonshine symmetry we can choose for $F^{(0,n_i)}(\tau, z)$, the
twining character corresponding to the conjugacy class with the cycle structure of
$g^{n_i}$. We show that with these choices we can complete the construction of the twisted
elliptic genera for the remaining conjugacy classes in table 1.

4B class

The twining character for the 4B conjugacy class is given by
\[ F^{(0,1)}(\tau, z) = \frac{A(\tau, z)}{3} - \frac{B(\tau, z)}{4} \left( -\frac{1}{2} \mathcal{E}_2(\tau) + 2\mathcal{E}_4(\tau) \right). \] (2.45)

Since the modular forms involved in the twining character is in $\Gamma_0(N)$, the order
of the automorphism corresponding to the 4B class is 4. Therefore the sectors
(0, 2), (2, 0), (2, 2) are not accessible using $SL(2, \mathbb{Z})$ modular transformations. Now the cycle shape of $g'$ in this class is given by $1^4 \cdot 2^2 \cdot 4^4$ and the cycle shape of $g^2$ is given by $1^8 \cdot 2^8$. From table 1 we see that this cycle shape coincides with the conjugacy class 2A. Therefore we choose for the twisted elliptic genus in the sector (0, 2) to be identical to be the 1/2 the twisting character (0, 1) of the 2A conjugacy class. The choice of normalization is because we are in an order 4 conjugacy class. We will also show that this normalization results in the expected values for the low lying coefficients of the elliptic genus. Similarly sectors (2, 0) and the (2, 2) of the 4B conjugacy class coincide with 1/2 the twisted sectors (1, 0) and (1, 1) of the 2A class. The rest of the sectors can be determined by using the relation (1.2) and identities relating expansions in $e^{-2\pi i/\tau}$ to $e^{2\pi i \tau}$. For this we need the following identities

$$E_2(\tau + 1/2) = -E_2(\tau) + 2E_2(2\tau),$$

$$E_4(\tau + 1/2) = \frac{1}{3}(-E_2(\tau) + 4E_2(2\tau)).$$

One can prove these identities using the definition of $E_N(\tau)$ in (2.2) together with the first equation of (2.12). The identities in (2.46) allow us to obtain the (2, 1) or the (2, 3) sector from the (1, 2) using a similar analysis followed in section 2.1. The result for the twisted elliptic genus using these inputs is given by

$$F^{(0,0)}(\tau, z) = 2A(\tau, z),$$

$$F^{(0,1)}(\tau, z) = F^{(0,3)}(\tau, z) = \frac{1}{4} \left[ \frac{4A}{3} - B \left( -\frac{1}{3}E_2(\tau) + 2E_4(\tau) \right) \right],$$

$$F^{(1,s)}(\tau, z) = F^{(3,3s)} = \frac{1}{4} \left[ \frac{4A}{3} + B \left( -\frac{1}{6}E_2(\frac{\tau + s}{2}) + \frac{1}{2}E_4(\frac{\tau + s}{4}) \right) \right],$$

$$F^{(2,1)}(\tau, z) = F^{(2,3)} = \frac{1}{4} \left( \frac{4A}{3} - \frac{B}{3} \left( 3E_2(\tau) - 4E_2(2\tau) \right) \right),$$

$$F^{(0,2)}(\tau, z) = \frac{1}{4} \left( \frac{8A}{3} - \frac{4B}{3} E_2(\tau) \right),$$

$$F^{(2,2s)}(\tau, z) = \frac{1}{4} \left( \frac{8A}{3} + \frac{2B}{3} E_2(\frac{\tau + s}{2}) \right).$$

Note that sector (0, 1) is the twining character given by [13–15] for the 4B conjugacy class. Using this, the modular transformation property (1.2) and the relations in (2.46) we obtain the sectors (2, 1), (2, 3). Finally the sectors (0, 2), (2, 2s) belong to the sub-orbit which can be identified with the 2A class. Note the twisted elliptic genus for this sub-orbit is 1/2 of that twisted elliptic genus for the 2A class. It is interesting to note that our result in (2.47) for the twisted elliptic genus coincides with that obtained in [28]. This was obtained prior to the discovery of the $M_{24}$ symmetry. The approach followed in [28] involved writing down the possible $\Gamma_0(4)$ and $\Gamma_0(2)$ forms allowed in the (0, s) sectors and constraining the coefficients using topological data.
Let us now evaluate the low lying coefficients of the elliptic genus. We have
\[ c^{(0,s)}(\pm 1) = \frac{1}{2}, \quad s = 0, \ldots, N - 1, \tag{2.48} \]
and
\[ \sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2 \]
and
\[ \sum_{s=0}^{N-1} c^{(0,s)}(0) = 6. \tag{2.49} \]
This equation implies that the number of $(1, 1)$ forms due to the orbifolding is down to 6 from 20 of the $K3$. This agrees with the analysis of [10] which studies the orbifold of $K3$ dual to the $N = 4$ CHL compactification. We can therefore identify the compactification of type II on $(K3 \times T^2)/\mathbb{Z}_4$ where $\mathbb{Z}_4$ is the $4A$ automorphism to be dual to the $N = 4$ heterotic CHL compactification. Let us now evaluate the full elliptic genus of $K3$ orbifolded by the $4A$ automorphism. This is given by
\[ \sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z). \tag{2.50} \]
To show this we substitute the twisted elliptic genus given in (2.47) along with the identity in (2.10) and finally use the relation
\[ \frac{1}{4} \sum_{s=0}^{3} \mathcal{E}_4(\frac{\tau + s}{4}) = \mathcal{E}_2(\tau). \tag{2.51} \]

**6A class**

The twining character for the $6A$ conjugacy class is given by
\[ F^{(0,1)} = \frac{1}{6} \left( \frac{2A}{3} - B \left( -\frac{1}{6} \mathcal{E}_2(\tau) - \frac{1}{2} \mathcal{E}_3(\tau) + \frac{5}{2} \mathcal{E}_6(\tau) \right) \right) \tag{2.52} \]
From this, it is easy to see that that $6A$ automorphism is of the order 6, which admits 2 and 3 as non-trivial divisors. Therefore there are 2 independent sub-orbits of orders 3 and 2 respectively. These sub-orbits cannot be accessed using $SL(2,\mathbb{Z})$ modular transformations from the $(0,1)$ sector. The sub-orbits are the following twisted sectors
\[ a : (0, 2), (0, 4); (2, 0), (2, 2), (2, 4); (4, 0), (4, 2), (4, 4), \tag{2.53} \]
\[ b : (0, 3), (3, 0), (3, 3). \]
Now to determine the twisted elliptic genus in the sub-orbit $a$, first examine the $(0, 2)$ sector. The cycle shape of $g'$ for the $6A$ conjugacy class can be read out from
the table 1 and is given by $1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$. The cycle shape of $g'^2$ for 6A is given by $1^6 \cdot 3^6$ which is identical to that cycle shape of the conjugacy class 3A. Therefore we take the twisted elliptic genus of for the sub-orbit (a) to be 1/2 of the twisted elliptic genera of the 3A class. Similarly for the sub-orbit (b) the cycle shape is obtained by looking at $g'^3$ which is $1^8 \cdot 2^8$. This coincides with the cycle structure of the 2A conjugacy class. Therefore for the twisted elliptic genera of the sub-orbit (b) we can take 1/3 the twisted elliptic genera of the 2A conjugacy class.

The sectors other than the sub-orbits (a) and (b) can be reached using the $SL(2,\mathbb{Z})$ transformation given in (1.2). Again to convert expansions in $e^{-2\pi i/\tau}$ to expansions in $e^{2\pi i\tau}$ we need the following identity obtained by using (2.2) and the first equation of (2.12).

$$E_6(\nu + 1/2) = \frac{1}{5} (-E_2(\nu) + 2E_2(2\nu) + 4E_3(2\nu)). \quad (2.54)$$

Let’s illustrate this in obtaining the (3, 1) or the (3, 5) sectors from the (0, 1) sector. First using the $S$ transformation on the twining character in (2.52) we obtain the (1, 0) sector which is given by

$$F^{(1, 0)}(\tau, z) = F^{(5, 0)}(\tau, z) = \frac{1}{6} \left[ \frac{2A}{3} + B \left( -\frac{1}{12}E_2(\tau) - \frac{1}{6}E_3(\tau) + \frac{5}{12}E_6(\tau) \right) \right]. \quad (2.55)$$

Then using $T^3$ transformation we can reach the (1, 3) sector, which is given by

$$F^{(1, 3)} = F^{(5, 3)} = \frac{1}{6} \left[ \frac{2A}{3} + B \left( -\frac{1}{12}E_2(\frac{\tau + 3}{2}) - \frac{1}{6}E_3(\frac{\tau + 3}{3}) + \frac{5}{12}E_6(\frac{\tau + 3}{6}) \right) \right]. \quad (2.56)$$

We can now use the $S$ transformation, to obtain the (3, 1) or the (3, 5) sectors. It is easy to see from the argument of $E_6$ in (2.56) we will require the identity in (2.54) to perform the $S$ transformation. The relations we need are

$$E_6(\frac{\tau}{6} + \frac{1}{2}) = \frac{1}{5} \left( -E_2(\tau/6) + 2E_2(\tau/3) + 4E_3(\tau/3) \right), \quad (2.57)$$

$$E_6(\frac{-1/6}{\tau} + \frac{1}{2}) = \frac{1}{5} \left( -E_2(-1/6\tau) + 2E_2(-1/3\tau) + 4E_3(-1/3\tau) \right),$$

$$= \frac{\tau^2}{5} \left( 18E_2(3\tau) - 12E_3(\tau) - 9E_2(\frac{3\tau}{2}) \right).$$

This results in the following expression for the (3, 1) or the (3, 5) sector

$$F^{(3, 1)}(\tau, z) = F^{(3, 5)}(\tau, z) = A \frac{9}{9} - B \frac{12}{12}E_3(\tau) - B \frac{12}{72}E_2(\frac{\tau + 1}{2}) + B \frac{8}{8}E_2(\frac{3\tau + 1}{2}). \quad (2.58)$$

From the (3, 1) sector by performing the $T$ and then the $S$ transformation we can reach the (2, 3) sector and again we will require the use of the identity (2.54) as well as

$$E_2(-\frac{1}{2\tau} + \frac{1}{2}) = \tau^2E_2(\frac{\tau + 1}{2}). \quad (2.59)$$
Finally from the $$(2, 3)$$ sector by $T$ transformations we can reach the $$(2, 1)$$ sector as well as the $$(2, 5)$$ sector.

Using all these inputs the sectors of the twisted elliptic genus for $6A$ are given by

$$F^{(0,0)} = \frac{4}{3} A; \quad F^{(0,1)} = F^{(0,5)}; \quad F^{(0,2)} = F^{(0,4)};$$

$$F^{(0,1)} = \frac{1}{6} \left[ 2A - \frac{3}{2} B \mathcal{E}_3(\tau) \right],$$

$$F^{(0,2)} = \frac{1}{6} \left[ 8A - 4B \mathcal{E}_3(\tau) \right],$$

$$F^{(0,3)} = \frac{1}{6} \left[ 8A - \frac{4}{3} B \mathcal{E}_2(\tau) \right].$$

$$F^{(1,k)} = F^{(5,5k)} = \frac{1}{6} \left[ \frac{2A}{3} + B \left( -\frac{1}{12} \mathcal{E}_3\left(\frac{\tau + k}{2}\right) + \frac{1}{6} \mathcal{E}_3\left(\frac{\tau + 3k}{6}\right) \right) \right],$$

$$F^{(2,2k+1)} = \frac{A}{9} + \frac{B}{36} \mathcal{E}_3\left(\frac{\tau + 2 + k}{3}\right),$$

$$F^{(4,4k+1)} = \frac{A}{9} + \frac{B}{36} \mathcal{E}_3\left(\frac{\tau + 1 + k}{3}\right),$$

$$F^{(3,1)} = F^{(3,5)} = \frac{A}{9} - \frac{B}{12} \mathcal{E}_3(\tau) - \frac{B}{72} \mathcal{E}_2\left(\frac{\tau + 1}{2}\right) + \frac{B}{8} \mathcal{E}_2\left(\frac{3\tau + 1}{2}\right),$$

$$F^{(3,2)} = F^{(3,4)} = \frac{A}{9} - \frac{B}{12} \mathcal{E}_3(\tau) - \frac{B}{72} \mathcal{E}_2\left(\frac{\tau}{2}\right) + \frac{B}{8} \mathcal{E}_2\left(\frac{3\tau}{2}\right),$$

$$F^{(2r,2rk)} = \frac{1}{6} \left[ 2A + \frac{1}{2} B \mathcal{E}_3\left(\frac{\tau + k}{3}\right) \right],$$

$$F^{(3,3k)} = \frac{1}{6} \left[ \frac{8A}{3} + \frac{2}{3} B \mathcal{E}_2\left(\frac{\tau + k}{2}\right) \right].$$

The low lying coefficients of the $6A$ twisted elliptic genus is given by

$$c^{(0,s)}(\pm 1) = \frac{1}{3}, \quad s = 0, \ldots, 5,$$

$$\sum_{s=0}^{5} c^{(0,s)}(\pm 1) = 2.$$
Therefore the number of (1, 1) forms is 4. This agrees with [10] which studies the orbifold of $K3$ dual to the $N = 6$ CHL compactification. We therefore identify the compactification of type II on $(K3 \times T^2)/\mathbb{Z}_6$ where $\mathbb{Z}_6$ is the 6A automorphism to be dual to the $N = 4$ heterotic CHL compactification. The full elliptic genus of $K3$ orbifolded by the 6A automorphism is given by

$$
\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z). \quad (2.66)
$$

Thus the result of the 6A orbifold of $K3$ is $K3$ itself.

### 8A Class

The twining character for the 8A conjugacy class is given by

$$
F^{(0,1)} = \frac{1}{8} \left[ \frac{2A}{3} - B \left( -\frac{1}{2} E_4(\tau) + \frac{7}{3} E_8(\tau) \right) \right]. \quad (2.67)
$$

Therefore the 8A automorphism in $K3$ is of the order 8, this admits 2 and 4 as non-trivial divisors. The independent sub-orbits are of the length 4 and 2 respectively. From table 1, the cycle shape of the conjugacy class for 8A is given by $1^4 \cdot 2^2 \cdot 4^4$. The cycle shape of $g^2$ is $1^4 \cdot 2^2 \cdot 4^4$ which is identical to the 4B conjugacy class. The sectors of the twisted elliptic genus belonging to this sub-orbit of order 4

$$(0, 2), (0, 4), (0, 6);$$

$$(2, 0), (2, 2), (2, 4), (2, 6);$$

$$(4, 0), (4, 2), (4, 4), (4, 6);$$

$$(6, 0), (6, 2), (6, 4), (6, 6).$$

Since the 4A already has a sub-orbit of order 2 which coincides with the sub-orbit of the 8A conjugacy class we do not need to consider this sub-orbit independently. Thus the twisted sectors in this sub-orbit is taken to be $1/2$ that of the 4B conjugacy class given in (2.47). The rest of the sectors can be determined by using the modular transformation (1.2) and the identity

$$
E_8(\tau + 1/2) = \frac{1}{7} (-E_2(\tau) + 2E_2(2\tau) + 6E_4(2\tau)). \quad (2.69)
$$

Going through a similar analysis as in the case of 6A we obtain

$$
F^{(0,0)}(\tau, z) = A(\tau, z),\quad F^{(0,1)} = F^{(0,3)} = F^{(0,5)} = F^{(0,7)},
$$

$$
= \frac{1}{8} \left[ \frac{2A}{3} - B \left( -\frac{1}{2} E_4(\tau) + \frac{7}{3} E_8(\tau) \right) \right]. \quad (2.70)
$$

- 18 -
\[ F^{(r, r)}(\tau, z) = \frac{1}{8} \left[ \frac{2A}{3} + \frac{B}{8} \left( -E_4(\tau + k) + \frac{7}{3} E_8(\tau + k) \right) \right] . \]  

(2.71)

where \( r = 1, 3, 5, 7. \)

\[ F^{(2, 1)} = F^{(6, 3)} = F^{(2, 5)} = F^{(6, 7)}, \]

(2.72)

\[ = \frac{1}{8} \left[ \frac{2A}{3} + \frac{B}{3} \left( -E_2(2\tau) + \frac{3}{2} E_4(\frac{2\tau + 1}{4}) \right) \right] ; \]

\[ F^{(2, 3)} = F^{(6, 5)} = F^{(2, 7)} = F^{(6, 1)}, \]

(2.73)

\[ = \frac{1}{8} \left[ \frac{2A}{3} + \frac{B}{3} \left( -E_2(2\tau) + \frac{3}{2} E_4(\frac{2\tau + 3}{4}) \right) \right] . \]

Finally the low lying coefficients of this orbifold satisfy

\[ c^{(0, s)}(\pm 1) = \frac{1}{4}, \quad s = 0, \ldots 7, \]  

(2.74)

\[ \sum_{s=0}^{7} c^{(0, s)}(\pm 1) = 2. \]

and

\[ \sum_{s=0}^{7} c^{(0, s)}(0) = 2. \]  

(2.75)

The above equation implies that the number of \( (1, 1) \) forms is 2 which agrees with the \( K3 \) orbifold dual to the \( N = 8 \) CHL compactification [10]. The full elliptic genus of \( K3 \) orbifolded by the \( 8A \) automorphism is given by

\[ \sum_{r, s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z) \]  

(2.76)
Thus the elliptic genus of the 8A orbifold of K3 is K3 itself.

We have seen that the K3 orbifold by the 4A, 6A and the 8A conjugacy class can be identified with type II on \((K3 \times T^2)/\mathbb{Z}_N\) orbifolds dual to \(N = 4, 6, 8\) CHL compactifications of the heterotic string respectively. Therefore with the result for the twisted elliptic genus for these cases along with the twisted elliptic genus for the \(pA\) cases with \(p = 2, 3, 5, 7\) completes the analysis of the twisted elliptic genus for all the 7 CHL compactifications discussed in [10].

There are 2 remaining conjugacy classes in table (1). These are the 14A and the 15A conjugacy classes. The construction for the twisted elliptic genus for these classes proceeds along similar lines as that discussed for the composite orders. The result is given in appendix B.

2.3 Automorphisms \(g'\) with composite order and \(g' \notin M_{23}\)

The conjugacy classes listed in table 2 are all of composite orders. Therefore they admit sub-orbits under the action of \(SL(2, \mathbb{Z})\). However the cycle shape in the sub-orbit is not unique enough to determine the twisted elliptic genus. For instance consider the conjugacy class 2B in table 2, squaring \(g'\) leads to a cycle shape of the identity class. Recently [19] constructed an explicit rational conformal field theory consisting of 6 \(SU(2)\) WZW models at level 1 which realizes K3. The action of the orbifold by \(g'\) belonging to the conjugacy class 2B is explicitly realized in this CFT. It was observed that this orbifold satisfied the property called ‘quantum symmetry’

\[
\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 0
\]

In this section starting from the twining characters for the 2B and 3B conjugacy class given in [19] we determine all the sectors of the twisted elliptic genus. This is done by assuming quantum symmetry together with the following condition on the low lying coefficients of the twisted elliptic genus

\[
\sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2
\]

where order \(N\) is 4, 9 for the 2B and 3B conjugacy class respectively. As we have discussed earlier, the above condition on the low lying coefficients of the twisted elliptic genus ensures that the type II theory compactified on \((K3 \times T^2)/\mathbb{Z}_N\) preserves \(\mathcal{N} = 4\) supersymmetry.

**Twisted elliptic genus of 2B**

An explicit realization of the 2B orbifold of K3 was given in [19] in which K3 is realized a rational CFT consisting of 6 \(SU(2)\) WZW models at level 1. Rather than
use this realization, we will start from the twining character given in [13–15] for the 2B conjugacy class

\[ F^{(0,1)}(\tau, z) = \frac{B(\tau, z)}{2}(E_2(\tau) - E_4(\tau)) \]  

(2.79)

Note that this is distinct from the classes belonging to table 1 in that it does not have any component of the weak Jacobi form \( A(\tau, z) \). It is clear from the structure of the twining character, the 2B automorphism is of the order 4. Using the modular transformations (1.2) together with the identities in (2.46), we can determine the elliptic genus in the following sectors to be given by

\[ F^{(0,1)}(\tau, z) = F^{(0,3)}(\tau, z), \]  

(2.80)

\[ F^{(1,s)}(\tau, z) = F^{(3,3s)}(\tau, z) = -\frac{B(\tau, z)}{4}\left(E_2\left(\frac{\tau + s}{2}\right) - E_4\left(\frac{\tau + s}{4}\right)\right), \]

\[ F^{(2,1)}(\tau, z) = F^{(2,3)}(\tau, z) = \frac{B(\tau, z)}{2}\left(-\frac{1}{6}E_2(\tau) + \frac{2}{3}E_2(2\tau)\right). \]

The remaining sectors (0, 2), (2, 0), (2, 2) belong to a sub-orbit. To determine the structure of the elliptic genus in this sub-orbit let us first focus on the (0, 2) sector. We assume that is a \( \Gamma_0(2) \) weak Jacobi form. Thus it can be written as

\[ F^{(0,2)}(\tau, z) = \alpha A(\tau, z) + \beta B(\tau, z)E_2(\tau). \]  

(2.81)

where \( \alpha, \beta \) are undetermined constants. Now the sectors (2, 0) and (2, 2) can be determined using the modular transformations (1.2) to be

\[ F^{(2,2s)}(\tau, z) = \alpha A(\tau, z) - \frac{\beta}{2}E_2\left(\frac{\tau + s}{2}\right). \]  

(2.82)

Imposing the equations (2.77) and (2.78) we obtain

\[ \alpha = \beta = -\frac{2}{3}. \]  

(2.83)

To summarize the twisted elliptic genus for the 2B conjugacy class is given by

\[ F^{(0,0)}(\tau, z) = 2A; \quad F^{(0,1)}(\tau, z) = F^{(0,3)}(\tau, z), \]  

(2.84)

\[ F^{(0,1)}(\tau, z) = \frac{B(\tau, z)}{2}(E_2(\tau) - E_4(\tau)), \]

\[ F^{(0,2)}(\tau, z) = -\frac{2A(\tau, z)}{3} - \frac{2B(\tau, z)}{3}E_2(\tau), \]

\[ F^{(1,s)}(\tau, z) = F^{(3,3s)}(\tau, z) = -\frac{B(\tau, z)}{4}\left(E_2\left(\frac{\tau + s}{2}\right) - E_4\left(\frac{\tau + s}{4}\right)\right), \]

\[ F^{(2,1)}(\tau, z) = F^{(2,3)}(\tau, z) = \frac{B(\tau, z)}{2}\left(-\frac{1}{6}E_2(\tau) + \frac{2}{3}E_2(2\tau)\right), \]

\[ F^{(2,2s)}(\tau, z) = -\frac{2A(\tau, z)}{3} + \frac{B(\tau, z)}{3}E_2\left(\frac{\tau + s}{2}\right). \]
We have also evaluated the complete twisted elliptic genus using the explicit rational CFT realization of this orbifold in [19] and have verified that it agrees with that given in (2.84). Evaluating the low lying coefficient corresponding to the invariant (1, 1) forms of $K3$ we obtain

$$\sum_{s=0}^{3} c^{(0,s)}(0) = 0. \quad (2.85)$$

Therefore, as expected the orbifold corresponding to the $2B$ conjugacy class is non-geometric.

**Twisted elliptic genus of 3B**

Among the conjugacy classes in table 2) the $3B$ class can also be completely determined using the quantum symmetry (2.77) and the supersymmetry condition (2.78). The twining character in this class is given by

$$F^{(0,1)}(\tau, z) = -\frac{2B(\tau, z)}{9} \frac{\eta^6(\tau)}{\eta^2(3\tau)} \quad (2.86)$$

From the modular properties of the $\eta$ function it is clear that the $3B$ automorphism is of order 9. The following sectors

$$(0,3), (0,6); \quad (2.87)$$

$$(3,0), (3,3), (3,6);$$

$$(6,0), (6,3), (6,6)$$

forms a sub-orbit under $SL(2, \mathbb{Z})$ modular transformations. The remaining sectors can be obtained from the twining character in (2.86) by using the transformation (1.2), together with the modular properties of the $\eta$ function. Once that is obtained we assume the following Jacobi weak form of $\Gamma_0(3)$ for the $(0,3)$ sector of the (2.87).

$$F^{(0,3)}(\tau, z) = \alpha A(\tau, z) + \beta B(\tau, z) E_3(\tau). \quad (2.88)$$

Here $\alpha, \beta$ are undetermined constants. Then using modular transformation (1.2) and the identities in (A.8) for $N = 3$ we can obtain the twisted elliptic genus in the sub-orbit. Finally imposing the conditions (2.77) and (2.78) we determine the constants $\alpha, \beta$ as

$$\alpha = -\frac{1}{9}, \quad \beta = \frac{1}{4}. \quad (2.89)$$
Using all these steps we obtain the twisted elliptic genus for the 3B conjugacy class to be given by

\[
F^{(0,0)}(\tau, z) = \frac{8A(\tau, z)}{9};
\]

\[
F^{(0,1)}(\tau, z) = F^{(0,2)} = F^{(0,4)} = F^{(0,5)} = F^{(0,7)} = F^{(0,8)};
\]

\[
F^{(0,1)}(\tau, z) = -\frac{2B(\tau, z)}{9} \frac{\eta^6(\tau)}{\eta^2(3\tau)};
\]

\[
F^{(0,3)}(\tau, z) = -\frac{A(\tau, z)}{9} - \frac{B(\tau, z)}{4} \mathcal{E}_3(\tau),
\]

\[
F^{(r,rs)}(\tau, z) = \frac{2B(\tau, z) \eta^6(\tau + s)}{3 \eta^2(\frac{\tau+s}{3})}, \quad r = 1, 2, 4, 5, 7, 8
\]

\[
F^{(3,1)}(\tau, z) = -\frac{2B(\tau, z) e^{2\pi i/3} \eta^6(\tau)}{9 \eta^2(3\tau)};
\]

\[
F^{(3,2)}(\tau, z) = -\frac{2B(\tau, z) e^{-2\pi i/3} \eta^6(\tau)}{9 \eta^2(3\tau)};
\]

\[
F^{(3r,3rk)}(\tau, z) = -\frac{A(\tau, z)}{9} + \frac{B(\tau, z)}{12} \mathcal{E}_3(\tau + k/3).
\]

The number of (1,1) forms is given by consider the following low lying coefficients of the twisted elliptic genus

\[
\sum_{s=0}^{8} c^{(0,s)}(0) = -2
\]

Again the orbifold of K3 by the 3B conjugacy class is non-geometric.

The rest of the conjugacy classes in table (2) have more than one sub-orbits. quantum symmetry given in (2.77) and the supersymmetry condition (2.78) is not enough to determine the unknown constants in these sub-orbits. It will be interesting to determine the twisted elliptic genera for all the remaining conjugacy classes of table 2.

2.4 Comparison with literature

As remarked earlier the work of [20] provides the mathematical justification for the construction of the twisted elliptic genus over for all the cyclic orbifolds considered in this paper. To compare with the results of this paper, let us briefly review their construction. Let \( g' \) be the cyclic orbifold corresponding to the conjugacy class of \( M_{24} \) with order \( N \). Then the twisted elliptic genus admits the following decomposition in terms of the characters of the \( N = 4 \) superconformal algebra with central charge
\[ c = 6 \]

\[ F^{(r,s)}(\tau, z) = \sum_{k=n+\frac{r}{2}}^{\infty} \text{Tr}_{H_{g^{r,s},k}}(\rho_{g^{r,s},k}(g'^{s})) \text{ch}_{h=\frac{1}{2}+k,l}(\tau, z) \]  

(2.92)

Note that \( l = \frac{1}{2} \) except when \( h = 1/4 \) for which both \( l = \frac{1}{2} \) and \( l = 0 \) are understood to be present in the sum. The vector space \( H_{g^{r,s},k} \) is finite dimensional and is the projective representation of the centralizer \( C_{M_{24}}(g') \) which satisfies properties detailed in [20]. Thus the problem of determining the twisted twining elliptic genera reduces to determining characters of the projective representations. Though not easy to extract from the ancillary files provided along with [20], a careful examination of the files lists out some of the twisted twining elliptic genera for the orbifolds considered in the paper. Notably it is only the \( F^{(1,s)} \) sector which is listed out in the ancillary files. The files also enable the evaluation of the characters of the projective representations and a verification of the expansion of the twisted elliptic genera as given in (2.92). However explicit expressions such as that given in equations (2.4), (2.28), (2.60) - (2.63) are not listed in the main body of the paper \(^2\). To arrive at these reasonably compact expressions we had to perform modular transformations and use identities such as (2.24), (2.35), (2.54). These identities were demonstrated for all the cases except for the orbifold by the class 23A/B. Let us also emphasize that the \( F^{(1,s)} \) sector which are provided in the ancillary files associated with [20] are the simplest to obtain by modular transformations. We have also not made any assumption that the \( F^{(0,1)} \) sector is the same as the \( F^{(0,s)} \) sector for \( s \neq 1 \) and \( N \) prime, but have arrived at it as a consequence of the identities derived in this paper as can be seen from the detailed discussion of the case of 11A class.

Note that explicit formulae for the twisted elliptic genera for \( pA \) orbifolds with \( p = 2, 3, 5, 7 \) were known even before the discovery of moonshine symmetry in [7] and before the work of [20]. Since the latter paper as well the present work uses modular transformations (1.2) to obtain the twisted elliptic genera we are assured that both the constructions agree. In our discussion we have also explicitly compared the low lying coefficients of the twisted elliptic genera for the case of 4B, 6A, 8A orbifolds with the Hodge numbers of the CHL compactifications discussed in [6, 10] and found agreement. The check that sum over all the sectors of the orbifolds when \( g' \in M_{23} \) yields back the elliptic genus of \( K3 \) was also performed in [20] as can be read from the discussion in the text \(^3\). In this work this is assured by the identities of the kind given (2.10).

As we have discussed earlier, when the order of the orbifold is composite, there are sub-orbits in the twisted sectors which cannot be reached by modular transformations from the twining character. We have used moonshine symmetry to determine the

\(^2\) We have been informed by Mathias Gaberdiel that these explicit expressions were known to the authors of [20] however they did not write them out in the body of their paper.

\(^3\)See discussion in the beginning of section 4. of [20].
twisted elliptic genus in these sectors. The treatment of such situations in [20] is more
general. Their discussion also encompasses orbifolds by non-cyclic groups. Though
not treated explicitly, the case of the cyclic orbifolds is implicit in their discussion 4.

3 1/4 BPS dyon partition functions

Given the twisted elliptic genus one can construct a Siegel modular form as follows
[21]. The twisted elliptic genus can be expanded as

\[ F^{(r,s)}(\tau, z) = \sum_{b=0}^{1} \sum_{j \in 2 + b, n \in \mathbb{Z}/N} c_b^{(r,s)}(4n - j^2)e^{2\pi i n \tau + 2\pi i jz}. \]

(3.1)

Then a Siegel modular form associated with the twisted elliptic genus is given by

\[ \tilde{\Phi}(\rho, \sigma, v) = e^{2\pi i (\tilde{\alpha}\rho + \tilde{\beta}\sigma + v)} \prod_{b=0,1} \prod_{r=0}^{N-1} \prod_{k',l' \geq 0, j < 0, k'=l'=0} (1 - e^{2\pi i (k'\sigma + l\rho + jv)}) \sum_{s=0}^{N-1} e^{2\pi is/N} c_b^{r,s}(4k'l' - j^2). \]

(3.2)

where

\[ \tilde{\beta} = \frac{1}{24N} \chi(M), \]

(3.3)

\[ \tilde{\alpha} = \frac{1}{24N} \chi(M) - \frac{1}{2N} \sum_{s=0}^{N-1} Q_{0,s} e^{-2\pi is/N}. \]

Q_{r,s} = N(c_{0,r,s}(0) + 2c_{1,r,s}(-1)).

Evaluating \( \tilde{\alpha}, \tilde{\beta} \) for the twisted elliptic genus corresponding to all the conjugacy
classes considered in the previous section as well as the \( pA \) classes with \( p = 1, 2, 3, 5, 7 \)
we obtain

\[ \tilde{\alpha} = 1, \quad \tilde{\beta} = \frac{1}{N}. \]

(3.4)

Here \( N \) is the order of the orbifold action. This Siegel modular form in (3.2) transforms as a weight \( k \) form under appropriate sub-groups of \( Sp(2, \mathbb{Z}) \). The weight \( k \) is
related to the low lying coefficients of the twisted elliptic genus and is given by

\[ k = \frac{1}{2} \sum_{0}^{N-1} c_{0,s}^{r,s}(-1). \]

(3.5)

The weights of the Siegel modular forms corresponding to the twisted elliptic genera
constructed in this paper is listed in table 4 and 5.

\[ ^{4} \text{We thank Mathias Gaberdiel for correspondence which enabled us to compare our work with} \]

[20].
| Type 1 | pA | 4B | 6A | 8A | 14A | 15A |
|--------|----|----|----|----|-----|-----|
| Weight | $\frac{24}{p+1} - 2$ | 3  | 2  | 1  | 0   | 0   |

Table 4: Weight of Siegel modular forms corresponding to classes in $M_{23}$

| Type 2 | 2B | 3B |
|--------|----|----|
| Weight | 0  | -1 |

Table 5: Weight of Siegel modular forms corresponding to the classes $\not\in M_{23}$

Now consider type II theory compactified on $(K^3 \times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as the automorphism $g'$ belonging to any of the conjugacy classes together with a $1/N$ shift along one of the circles of $T^2$, $S^1$. Then by the analysis in [21], the generating function of the index of 1/4 BPS states in this theory is given by $1/\tilde{\Phi}(\rho, \sigma, v)$. Let us work in the dual heterotic frame in which the orbifolded heterotic theory is compactified in general on $T^6$. For example the cases of the $pA$ orbifolds of $K^3 \times T^2$ with $p = 2, 3, 4, 5, 6, 7, 8$ corresponds to the $N = 2, 3, 4, 5, 6, 7, 8$ CHL compactifications on the heterotic side. Let us label the charges of the 1/4 BPS state by $(Q, P)$ corresponding to the electric and magnetic charge of the dyon. Let $Q^2, P^2$ and $Q \cdot P$ denote the continuous T-duality invariants in this duality frame. Then the 1/4 BPS index in this frame is given by

$$-B_6(Q, P) = \frac{1}{N} \left(-1\right)^{Q-P+1} \int_{\mathcal{C}} d\rho d\sigma dv \ e^{-\pi i(N\rho Q^2 \sigma P^2 / N + 2 \nu Q \cdot P)} \frac{1}{\tilde{\Phi}(\rho, \sigma, v)}. \quad (3.6)$$

The contour $\mathcal{C}$ is defined over a 3 dimensional subspace of the 3 complex dimensional space ($\rho = \rho_1 + i \rho_2, \sigma = \sigma_1 + i \sigma_2, v = v_1 + iv_2$).

$$\rho_2 = M_1, \quad \sigma_2 = M_2, \quad v_2 = -M_3, \quad 0 \leq \rho_1 \leq 1, \quad 0 \leq \sigma_1 \leq N, \quad 0 \leq v_1 \leq 1. \quad (3.7)$$

The choice of $(M_1, M_2, M_3)$ is determined by the domain in which one needs to evaluate the index $-B_6$ [29, 30]. We pick up the Fourier coefficients by expanding $1/\tilde{\Phi}$ in powers of $e^{2\pi i \rho}, e^{2\pi i \sigma}$ and $e^{-2\pi i v}$. For this expansion to make sense we must have [21, 29]

$$M_1, M_2 >> 0, \quad M_3 << 0, \quad |M_3| << M_1, M_2. \quad (3.8)$$

Since this is an index, the Fourier coefficient $-B_6$ must be an integer. Let us now focus on 1/4 BPS states which are single centered black holes. Then from the fact that the single centered black holes carry zero angular momentum, it is predicted that the index $-B_6$ for these black holes is positive [18]. The argument for this goes
as follows. Given the domain (3.8), these 1/4 BPS black states have regular event horizons and are single centered only if the charges satisfy the condition [18]

\[ Q \cdot P \geq 0, \quad (Q \cdot P)^2 < Q^2 P^2, \quad Q^2, P^2 > 0. \]  

(3.9)

Thus if we can show that the index \(-B_6\) is positive for states satisfying the condition (3.9), then it will imply the \(-B_6\) is positive for single centered 1/4 BPS dyons as predicted from black hole considerations. In the next section we show that for low lying charges satisfying (3.9), \(-B_6\) is indeed positive for all the Siegel modular forms associated with the twisted elliptic genera constructed in this paper. This is the generalization of the observation seen first in [18] for the \(pA\) conjugacy classes with \(p = 1, 2, 3, 5, 7\). For \(p = 1\) and for a special class of charges it was proved that the coefficient \(-B_6\) is positive [25].

Before we proceed we will study 2 properties of the Siegel modular forms which are theta lifts of the twisted elliptic genera constructed in this paper. First the Siegel modular forms factorize in the \(v \to 0\) limit as

\[ \lim_{v \to 0} \tilde{\Phi}_k(\rho, \sigma, v) \sim v^2 f^{(k+2)}(\rho) g^{(k+2)}(\sigma). \]  

(3.10)

where \(f^{(k+2)}, g^{(k+2)}\) are weight \(k + 2\) modular forms transforming under \(\Gamma_0(N)\). The explicit modular forms on which the \(\Phi_k's\) factorize are given in table 6. The function \(1/f^{(k+2)}(\rho)\) is the partition function of purely electric states while \(1/g^{(k+2)}(\sigma)\) is the partition function of purely magnetic states. In fact \(f^{(k+2)}\) and \(g^{(k+2)}\) are related to each other by a \(S\) transformation.
The second property we discuss is the asymptotic property of the index in (3.6) when the charges \(Q, P\) are equally large. The procedure to obtain the asymptotic behaviour has been developed in [4, 21, 31], which we summarize briefly. Consider another Siegel modular form \(\hat{\Phi}(\rho, \sigma, v)\) of weight \(k\) associated with the twisted elliptic genus defined by

\[
\hat{\Phi}(\rho, \sigma, v) = e^{2\pi i (\hat{\alpha} \rho + \hat{\beta} \sigma + v)}
\]

\[
\prod_{b=0,1}^{N-1} \prod_{r=0}^{L_b} (1 - e^{2\pi i r/N} e^{2\pi i (k' \sigma + l \rho + j v)}) e^{-2\pi i \sigma r/N} c_{b,0}^{N,0,s} (4k' - j)^2.
\]

Here we have,

\[
\hat{\beta} = \hat{\alpha} = \hat{\gamma} = \frac{1}{24} \chi(M) = 1
\]

Under \(v \to 0\), this modular form factorizes symmetrically in \(\rho\) and \(\sigma\) as

\[
\lim_{v \to 0} \hat{\Phi}(\rho, \sigma, v) \sim v^2 h^{(k+2)}(\rho) h^{(k+2)}(\sigma)
\]

Then the leading behaviour of the index \(-B_6\) is given by

\[
-B_6(Q, P) \sim \exp(-S(Q, P))
\]

---

\(^5\text{We follow the discussion in [21].}\)
where $S(Q, P)$ is obtained by minimizing the function
\[
-S(Q, P) = \frac{\pi}{2\tau_2} |Q^2 + \tau P^2|^2 - \ln(h^{(k+2)}(\tau)) - \ln(h^{(k+2)}(-\tau)) - (k + 2) \ln(2\tau_2). \tag{3.15}
\]
with respect to $\tau_1, \tau_2$. The minimum lies at
\[
\tau_1 = \frac{Q \cdot P}{P^2}, \quad \tau_2 = \frac{1}{P^2} \sqrt{Q^2 P^2 - (Q \cdot P)^2}. \tag{3.16}
\]
Substituting the above values for $\tau_1, \tau_2$ results in the asymptotic behaviour of the index $-B_6$. The list of the $\Gamma_0(N)$ modular forms for the models constructed in this paper is provided in table 7.

| Conjugacy Class | $h^{(k+2)}(\rho)$ |
|-----------------|--------------------|
| pA              | $\eta^{k+2}(\rho)\eta^{k+2}(pp)$ |
| 4B              | $\eta^4(4\rho)\eta^2(2\rho)\eta^4(\rho)$ |
| 6A              | $\eta^2(\rho)\eta^2(2\rho)\eta^2(3\rho)\eta^2(6\rho)$ |
| 8A              | $\eta^2(\rho)\eta(2\rho)\eta(4\rho)\eta^2(8\rho)$ |
| 14A             | $\eta(\rho)\eta(2\rho)\eta(7\rho)\eta(14\rho)$ |
| 15A             | $\eta(\rho)\eta(3\rho)\eta(5\rho)\eta(15\rho)$ |
| 2B              | $\frac{\eta^8(4\rho)}{\eta^4(2\rho)}$ |
| 3B              | $\frac{\eta^3(9\rho)}{\eta(3\rho)}$ |

Table 7: Factorization of $\Phi_k(\rho, \sigma, v)$ as $\lim v \to 0$ as shown in (3.13), $p \in \{1, 2, 3, 5, 7, 11, 23\}$

Let us now compare this to the behaviour of the entropy of single centered large charge $1/4$ BPS dyons in these $\mathcal{N} = 4$ theories obtained compactifying type II theory on $(K3 \times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as the automorphisms on $K3$ together with a $1/N$ shift on one of the circles of $T^2$. Apart from the usual 2 derivative terms in the effective action, a one loop computation shows that the coefficient of the Gauss-Bonnet term is given by
\[
\Delta L = \phi(a, S)(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2). \tag{3.17}
\]
where $a, S$ is the axion and dilaton moduli in the heterotic frame. The function $\phi(a, S)$ is given by
\[
\phi(a, S) = -\frac{1}{64\pi^2}((k + 2) \ln S + \ln h^{(k+2)}(a + iS) + \ln h^{(k+2)}(-a + iS)). \tag{3.18}
\]
It is important to note that the $\Gamma_0(N)$ modular form $h^{(k+2)}(\tau)$ for each of the compactifications is identical to the $\Gamma_0(N)$ form that occurs in the factorization (3.13) [21]. Now evaluating the Hawking-Bekenstein-Wald entropy including the correction due to the Gauss-Bonnet term using the entropy function formalism leads to the following minimizing problem. The entropy is given by minimizing the function

$$E(Q, P) = \frac{\pi}{2\tau_2} |Q^2 + \tau P^2|^2 - \ln h^{(k+2)}(\tau) - \ln h^{(k+2)}(-\tau) - (k+2) \ln(2\tau_2). \quad (3.19)$$

Here $\tau = a + iS$. The entropy function is identical to the statistical entropy function (3.15) which occurred while obtaining the asymptotic behaviour of $-B_6$. Thus the partition function $1/\tilde{\Phi}(\rho, \sigma, v)$ captures the degeneracy of large charge single centered 1/4 BPS black holes in these class of $\mathcal{N} = 4$ compactifications including the correction from the Gauss-Bonnet term.

The construction of the Siegel modular form given the coefficient of the twisted elliptic genus of $K3$ is quite straightforward and for cyclic orbifolds, this was first given in [21]. Recently the references [22–24] extend it for non-cyclic orbifolds. It is important to emphasize the there are 2 modular forms associated with the twisted elliptic genus of $K3$. The $\tilde{\Phi}_k$ and the $\hat{\Phi}_k$ are constructed in equations (3.2) and (3.11) respectively. The Fourier expansion of the inverse of $\tilde{\Phi}_k$ capture the degeneracy of the 1/4 BPS dyon and its zeros at $v \to 0$ are associated with the walls of marginal stability of the dyon. The zero’s of $\hat{\Phi}_k$ are however associated with the asymptotic growth of the degeneracies for large charges. We mention that $\hat{\Phi}_k$ has not been constructed for the orbifolds listed in this paper in the references [22–24]. We emphasize that our objective in constructing the Siegel modular form $\tilde{\Phi}_k$ in particular is to verify that the Fourier expansions of the inverse of these forms are integers and positive as predicted by the conjecture of [18]. This was verified earlier by [18] for the Siegel modular forms $\tilde{\Phi}_k$, associated with the $pA, p = 1, 2, 3, 5, 7$ orbifolds. In this next section we extend this observation for all the orbifolds discussed in this paper. To our knowledge, this observation has not been seen in the works of [22–24]. We also emphasize that to obtain this observation the explicit construction of the twisted elliptic genus in all its sectors together with the normalizations as discussed earlier is important.

### 3.1 Positivity and integrality of the 1/4 BPS index

In this subsection we provide the list for the index $-B_6$ for low lying charges for all the Siegel modular forms $\tilde{\Phi}_k$ associated with the twisted elliptic genera constructed. From the expansion of $\tilde{\Phi}_k$ in Fourier coefficients in the domain (3.8) together with the expression for $-B_6$ in (3.6) we see that the electric charge $Q^2$ is quantized in units of $2/\mathbb{Z}_N$, while the magnetic charge $P^2$ is quantized in units of $2\mathbb{Z}$ and the angular momentum $Q \cdot P$ is an integer. We see that the index $-B_6$ for the low lying charges examined is always an integer. Furthermore for charges satisfying the
condition (3.9) it is positive. This property is a sufficient condition which ensures that single centered black holes carry zero angular momentum. One important point to emphasize is that it is possible to obtain the Fourier expansion of the Siegel modular forms for low lying charges only after the explicit construction of the twisted elliptic genus. As a check on our Mathematica routines to obtain these Fourier coefficients, we have verified that our routine reproduces all the tables given in [18] for the $pA$ orbifold of $K3$ with $p = 1, 2, 3, 5, 7$.

| $(Q^2, P^2)$ \ $Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|-------------------------------|----|---|---|---|---|
| $(1/2, 2)$                    | -512 | 176 | 8 | 0 | 0 |
| $(1/2, 4)$                    | -1536 | 896 | 80 | 0 | 0 |
| $(1/2, 6)$                    | -4544 | 3616 | 480 | 0 | 0 |
| $(1/2, 8)$                    | 11752 | 12848 | 2176 | 24 | 0 |
| $(1, 4)$                      | -4592 | 5024 | 832 | 16 | 0 |
| $(1, 6)$                      | -13408 | 22464 | 36786 | 224 | 0 |
| $(1, 8)$                      | -33568 | 88320 | 26176 | 1760 | 0 |
| $(3/2, 6)$                    | -37330 | 112316 | 36786 | 2998 | 38 |
| $(3/2, 8)$                    | -80896 | 491920 | 196960 | 23616 | 592 |

**Table 8:** Some results for the index $-B_6$ for the $4B$ orbifold of $K3$ for different values of $Q^2$, $P^2$ and $Q \cdot P$

| $(Q^2, P^2)$ \ $Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|-------------------------------|----|---|---|---|---|
| $(1/3, 2)$                    | -98 | 40 | 1 | 0 | 0 |
| $(1/3, 4)$                    | -224 | 148 | 12 | 0 | 0 |
| $(1/3, 6)$                    | -546 | 478 | 49 | 0 | 0 |
| $(1/3, 8)$                    | -1120 | 1352 | 186 | 0 | 0 |
| $(2/3, 4)$                    | -512 | 592 | 92 | 0 | 0 |
| $(2/3, 6)$                    | -1240 | 2080 | 436 | 8 | 0 |
| $(2/3, 8)$                    | -2504 | 6416 | 1676 | 0 | 0 |
| $(1, 6)$                      | -2926 | 7880 | 2172 | 116 | 0 |
| $(1, 10)$                     | -2450 | 81380 | 32300 | 3494 | 49 |
| $(1, 12)$                     | -4696 | 234900 | 104176 | 13856 | 316 |

**Table 9:** Some results for the index $-B_6$ for the $6A$ orbifold of $K3$ for different values of $Q^2$, $P^2$ and $Q \cdot P$
| $(Q^2, P^2)$ | $Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|---------------|-------------|-----|---|---|---|---|
| $(1/4, 2)$    |             | -60 | 20| 0 | 0 | 0 |
| $(1/4, 4)$    |             | -120| 68| 2 | 0 | 0 |
| $(1/4, 6)$    |             | -280| 196| 10| 0 | 0 |
| $(1/4, 8)$    |             | -520| 504| 40| 0 | 0 |
| $(1/2, 6)$    |             | -560| 724| 96| 0 | 0 |
| $(1/2, 8)$    |             | -1038| 1998| 352| 2 | 0 |
| $(3/4, 6)$    |             | -1114| 2280| 450| 6 | 0 |
| $(3/4, 8)$    |             | -2024| 6704| 1728| 56 | 0 |
| $(3/4, 10)$   |             | -3860| 18256| 5564| 300 | 0 |
| $(3/4, 12)$   |             | -6168| 46456| 16296| 1192 | 4 |

**Table 10:** Some results for the index $-B_6$ for the 8A orbifold of $K3$ for different values of $Q^2$, $P^2$ and $Q \cdot P$

| $(Q^2, P^2)$ | $Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|---------------|-------------|-----|---|---|---|---|
| $(2/11, 2)$   |             | -50 | 10| 0 | 0 | 0 |
| $(2/11, 4)$   |             | -100| 30| 0 | 0 | 0 |
| $(2/11, 6)$   |             | -200| 82| 1 | 0 | 0 |
| $(4/11, 6)$   |             | -400| 276| 18| 0 | 0 |
| $(6/11, 6)$   |             | -800| 806| 83| 0 | 0 |
| $(6/11, 8)$   |             | -1438| 2064| 314| 2 | 0 |
| $(6/11, 10)$  |             | -2584| 4962| 937| 16 | 0 |
| $(6/11, 12)$  |             | -4328| 11132| 2558| 72 | 0 |
| $(6/11, 22)$  |             | -34000| 366378| 139955| 12760| 114 |

**Table 11:** Some results for the index $-B_6$ for the 11A orbifold of $K3$ for different values of $Q^2$, $P^2$ and $Q \cdot P$

| $(Q^2, P^2)$ | $Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|---------------|-------------|-----|---|---|---|---|
| $(1/7, 2)$    |             | -18 | 4 | 0 | 0 | 0 |
| $(1/7, 4)$    |             | -24 | 10| 0 | 0 | 0 |
| $(1/7, 6)$    |             | -54 | 24| 0 | 0 | 0 |
| $(2/7, 6)$    |             | -72 | 70| 5 | 0 | 0 |
| $(2/7, 8)$    |             | -96 | 156| 16| 0 | 0 |
| $(3/7, 8)$    |             | -216| 406| 65| 0 | 0 |
| $(3/7, 10)$   |             | -412| 890| 165| 2 | 0 |
| $(4/7, 12)$   |             | -710| 4682| 1443| 58 | 0 |
| $(5/7, 12)$   |             | -1180| 11512| 4156| 292 | 0 |
| $(5/7, 14)$   |             | -1622| 24744| 9816| 908 | 5 |

**Table 12:** Some results the index $-B_6$ for the 14A orbifold of $K3$ for different values of $Q^2$, $P^2$ and $Q \cdot P
| $(Q^2, P^2) \backslash Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $(2/15, 2)$ | -8 | 4 | 0 | 0 | 0 |
| $(2/15, 4)$ | -16 | 8 | 0 | 0 | 0 |
| $(2/15, 6)$ | -24 | 20 | 0 | 0 | 0 |
| $(2/5, 8)$ | -120 | 274 | 45 | 0 | 0 |
| $(2/5, 10)$ | -203 | 578 | 113 | 1 | 0 |
| $(4/15, 6)$ | -48 | 50 | 4 | 0 | 0 |
| $(4/5, 8)$ | -80 | 102 | 13 | 0 | 0 |
| $(8/15, 12)$ | -440 | 2844 | 898 | 40 | 0 |
| $(2/3, 12)$ | -638 | 6818 | 2498 | 178 | 0 |
| $(4/5, 18)$ | 8236 | 141252 | 73651 | 12124 | 419 |

**Table 13:** Some results for the index $-B_6$ for the $15A$ orbifold of $K3$ for different values of $Q^2$, $P^2$ and $Q \cdot P$.

| $(Q^2, P^2) \backslash Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $(2/23, 2)$ | -8 | 1 | 0 | 0 | 0 |
| $(2/23, 4)$ | -12 | 3 | 0 | 0 | 0 |
| $(2/23, 6)$ | -20 | 7 | 1 | 0 | 0 |
| $(4/23, 6)$ | -30 | 53 | 6 | 0 | 0 |
| $(4/23, 8)$ | -42 | 91 | 11 | 0 | 0 |
| $(6/23, 6)$ | -48 | 103 | 23 | 2 | 0 |
| $(6/23, 8)$ | -66 | 190 | 47 | 4 | 0 |
| $(6/23, 10)$ | -104 | 312 | 74 | 6 | 0 |

**Table 14:** Some results for the index $-B_6$ for the $23A$ orbifold of $K3$ for different values of $Q^2$, $P^2$ and $Q \cdot P$.

| $(Q^2, P^2) \backslash Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $(1/2, 2)$ | 320 | 288 | 24 | 0 | 0 |
| $(1/2, 4)$ | 0 | 512 | 256 | 0 | 0 |
| $(1/2, 6)$ | -752 | 1120 | 888 | 48 | 0 |
| $(1/2, 8)$ | 384 | 3328 | 2048 | 384 | 0 |
| $(1, 4)$ | 32 | 4416 | 2240 | 32 | 0 |
| $(1, 6)$ | -2304 | 22464 | 13248 | 224 | 0 |
| $(1, 8)$ | 5920 | 42944 | 27328 | 5920 | 64 |
| $(3/2, 6)$ | -2008 | 102380 | 66172 | 9032 | 28 |
| $(3/2, 8)$ | 59392 | 372736 | 243712 | 59392 | 2048 |

**Table 15:** Some results for the index $B_6$ for the $2B$ orbifold of $K3$ for different values of $Q^2$, $P^2$ and $Q \cdot P$. 


| $(Q^2, P^2)$ | $Q \cdot P$ | -2 | 0 | 1 | 2 | 3 |
|-----------|------------|----|----|----|----|----|
| $(2/9, 2)$ | 0          | 18 | 0  | 0  | 0  | 0  |
| $(2/9, 4)$ | 18         | 27 | 0  | 0  | 0  | 0  |
| $(2/9, 6)$ | 0          | 78 | 21 | 0  | 0  | 0  |
| $(4/9, 4)$ | 42         | 150| 33 | 0  | 0  | 0  |
| $(4/9, 6)$ | 0          | 270| 81 | 0  | 0  | 0  |
| $(4/9, 8)$ | 0          | 378| 162| 0  | 0  | 0  |
| $(2/3, 6)$ | 0          | 918| 297| 0  | 0  | 0  |
| $(2/3, 8)$ | 0          | 2460|1239| 93 | 0  | 0  |

Table 16: Some results for the index $-B_6$ for the 3B orbifold of K3 for different values of $Q^2$, $P^2$ and $Q \cdot P$.

It is interesting to note that the non-geometric orbifolds $11A, 23A, 23B, 2B, 3B$ also satisfy the positivity constraints conjectured by [18]. We have attached the mathematica files which generate the Fourier coefficients for the $11A$ and $3B$ orbifolds as ancillary files.

4 Conclusions

We have constructed the twisted elliptic genera for K3 orbifolded by automorphisms corresponding to all the conjugacy classes which lie $M_{23} \subset M_{24}$ as well as 2 conjugacy classes which does not lie in $M_{23}$. Our method involves the use of the modular transformation property of the twisted elliptic genus and discovering identities satisfied by $\Gamma_0(N)$ modular forms which relate expansions in $e^{-2\pi i/\tau}$ to expansions in $e^{2\pi i \tau}$. We also used inputs from $M_{23}$ symmetry to determine the twisted elliptic genus in sectors which form sub-orbits under $SL(2, \mathbb{Z})$.

We then constructed Siegel modular forms associated with the twisted elliptic genera that capture the degeneracy of 1/4 BPS states in $\mathcal{N} = 4$ theories obtained by compactifying type II theory on $(K3 \times T^2)/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as a order $N$ automorphism associated with the conjugacy class of $M_{24}$ on K3 together with a $1/N$ shift on one of the circles of $T^2$. We show that the dyon partition function satisfied the required properties expected from black hole physics. In particular the Fourier coefficients of the 1/4 BPS index are integers and certain low lying charges are positive in agreement with the conjecture of [18]. This is a sufficient condition predicted from the fact that single centered black holes carry zero angular momentum. The construction of the twisted elliptic genus as well as the dyon partition function associated with the 4A, 6A, 8A classes done in this paper, along with the earlier studied cases of $pA$ with $p = 2, 3, 5, 7$ completes this analysis for all the CHL models.

It is worthwhile to complete this analysis of this paper for the remaining 9 conjugacy classes of table 2. The construction for the twisted elliptic genera corresponding to these classes would required new ingredients. One possible direction is to use pos-
itivity and integrality of the low lying coefficients in the associated Siegel modular form to determine the twisted elliptic genera in the sectors which form sub-orbits under $SL(2,\mathbb{Z})$. These conjugacy classes have more than one sub-orbits. One can also verify if the Siegel modular forms constructed from the twisted elliptic genera for these classes provided in the ancillary files associated with [20] is in agreement with the positivity conjecture of [18].

The references [22–24] has studied more general non-cyclic twisted twining elliptic genera of $K3$ than considered in this paper. It is important to check if the more general twining elliptic genera considered in these references admit a $1/4$ BPS dyon partition function with integral Fourier coefficients and obey the positivity constraints as expected from black hole physics. Recently multiplicative lifts of more general weak Jacobi forms $^6$ as well as the the Siegel modular forms of $Sp(2,\mathbb{Z})$ of weight 35 and 12 were studied and were shown to have properties which make them candidates for partition of black holes [32]. It will be interesting to check if the Fourier coefficients of these Siegel modular forms also satisfy the positivity constraints required from black hole physics.

The discovery of the Mathieu moonshine symmetry has provided useful insights in string compactifications [33–35] as well as provided new examples where precision microscopic counting of black holes is possible as seen in this paper. It is certainly worthwhile to explore the implication of this symmetry further.

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A S-transformations for the $\eta$ function and $\mathcal{E}_N$

In this appendix we derive identities which for $\eta$ functions and $\mathcal{E}_N$. These identities relate expansions in $e^{-2\pi i/\tau}$ on one side to expansions in $e^{2\pi i\tau}$. These identities are used in the explicit construction of the twisted elliptic genus in all the sectors.

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$^6$These were Jacobi forms of weight 0 but index $> 1$. 
We begin with the relation between $\eta(-1/\tau + 1/N)$ to $\eta((\tau - N)/N^2)$

\[
\eta \left( \frac{-1}{\tau} + \frac{1}{N} \right) = \eta \left( \frac{\tau - N}{N\tau} \right) = \eta \left( \frac{-N\tau}{\tau - N} \right) \left( \frac{(iN\tau)}{\tau - N} \right)^{1/2} = \eta \left( -N - \frac{N^2}{\tau - N} \right) \left( \frac{(iN\tau)}{\tau - N} \right)^{1/2} = e^{-i\pi N/12} \eta \left( \frac{\tau - N}{N^2} \right) \left( \frac{(iN\tau)}{\tau - N} \right)^{1/2} \left( \frac{-i(\tau - N)}{N^2} \right)^{1/2} = e^{-i\pi N/12} \sqrt{\frac{\tau}{N}} \eta \left( \frac{\tau - N}{N^2} \right).
\]

(A.1)

Now for odd $N$ we find an identity for $\eta(-1/\tau + 2/N)$

\[
\eta \left( \frac{-1}{\tau} + \frac{2}{N} \right) = \eta \left( \frac{2\tau - N}{N\tau} \right) = \eta \left( \frac{-N\tau}{2\tau - N} \right) \left( \frac{(iN\tau)}{2\tau - N} \right)^{1/2} = \eta \left( -N - \frac{N^2}{4\tau - 2N} \right) \left( \frac{(iN\tau)}{2\tau - N} \right)^{1/2}.
\]

(A.2)

Let $N = 2m - 1$.

\[
\eta \left( \frac{-N}{2} - \frac{N^2}{4\tau - 2N} \right) \left( \frac{(iN\tau)}{2\tau - N} \right)^{1/2} = \eta \left( \frac{1}{2} - m - \frac{N^2}{4\tau - 2N} \right) \left( \frac{(iN\tau)}{2\tau - N} \right)^{1/2} = e^{\pi i/24 - \pi i m/12} \eta^3 \left( \frac{-N^2}{2\tau - N} \right) \left( \frac{(iN\tau)}{2\tau - N} \right)^{1/2} = e^{-\pi m/12} \eta \left( \frac{2\tau - N}{2N^2} + 1/2 \right) \sqrt{\tau/N}.
\]

(A.3)

In the second line of the above equation we have used the identity in (2.12) to relate $\eta$ functions at 1/2 shifts. Then using (A.1) and (A.3) we obtain

\[
\eta \left( \frac{-1}{\tau} + \frac{1}{N} \right) = e^{-i\pi m/12} \eta \left( \frac{2\tau - N}{2N^2} + 1/2 \right) \sqrt{\tau/N}, \quad N = 2m + 1.
\]

(A.4)

Let us now proceed to obtain an identity involving the shift $\eta(-1/\tau - 2/N)$ for odd
\[ \eta \left( \frac{-1}{\tau} - \frac{2}{N} \right) = \eta \left( \frac{-2\tau - N}{N\tau} \right) \] (A.5)

\[ = \eta \left( \frac{N\tau}{2\tau + N} \right) \left( \frac{-iN\tau}{2\tau + N} \right)^{1/2} \]

\[ = \eta \left( \frac{N}{2} - \frac{N^2}{4\tau + 2N} \right) \left( \frac{-iN\tau}{2\tau + N} \right)^{1/2}. \]

Let \( N = 2m + 1 \), then we obtain

\[ \eta \left( \frac{N}{2} - \frac{N^2}{4\tau + 2N} \right) \left( \frac{-iN\tau}{2\tau + N} \right)^{1/2} = \eta \left( \frac{1}{2} + m - \frac{N^2}{4\tau + 2N} \right) \left( \frac{-iN\tau}{2\tau + N} \right)^{1/2} \] (A.6)

\[ = e^{\pi i/24 - \pi i m/12} \eta \left( \frac{-N^2}{2\tau + N} \right) \eta \left( \frac{-2N^2}{2\tau + N} \right) \left( \frac{-iN\tau}{2\tau + N} \right)^{1/2} \]

\[ = -ie^{\pi i m/12} \eta \left( \frac{2\tau + N}{2N^2 + 1/2} \right) \sqrt{\tau/N}. \]

Here again we have used the identity in n (2.12) to relate \( \eta \) functions at 1/2 shifts. Combining (A.5) and (A.6) we obtain

\[ \eta \left( \frac{-1}{\tau} - \frac{2}{N} \right) = -ie^{\pi i m/12} \eta \left( \frac{2\tau + N}{2N^2 + 1/2} \right) \sqrt{\tau/N}, \quad N = 2m + 1. \] (A.7)

Using A.1, A.4, A.7 and the definition of \( \mathcal{E}_N(\tau) \) we obtain the relations

\[ \mathcal{E}_N \left( \frac{-1}{N\tau} + \frac{1}{N} \right) = \tau^2 \mathcal{E}_N \left( \frac{\tau + N - 1}{N} \right); \quad (A.8) \]

\[ \mathcal{E}_N \left( \frac{-1}{N\tau} + \frac{2}{N} \right) = \tau^2 \mathcal{E}_N \left( \frac{2\tau + N - 1}{2N} \right), \quad N = 2m + 1; \quad (A.9) \]

\[ \mathcal{E}_N \left( \frac{-1}{N\tau} - \frac{2}{N} \right) = \tau^2 \mathcal{E}_N \left( \frac{2\tau + N + 1}{2N} \right) \quad N = 2m + 1. \quad (A.10) \]

The first relation is true for all \( N \) and the last two for \( N \) being odd. Therefore we can use the last two equations for \( N = 3, 5, 7 \). We use these relations repeatedly for obtaining different sectors of the twisted elliptic genus for the 14A and 15A conjugacy class.
Finally using (2.12) to relate $\eta$ functions at $1/2$ shifts we obtain

\begin{align}
\mathcal{E}_2(\tau + 1/2) &= -\mathcal{E}_2(\tau) + 2\mathcal{E}_2(2\tau), \\
\mathcal{E}_4(\tau + 1/2) &= \frac{1}{3}(-\mathcal{E}_2(\tau) + 4\mathcal{E}_2(2\tau)), \\
\mathcal{E}_6(\tau + 1/2) &= \frac{2}{5}\mathcal{E}_2(2\tau) + \frac{4}{5}\mathcal{E}_3(2\tau) - \frac{1}{5}\mathcal{E}_2(\tau), \\
\mathcal{E}_8(\tau + 1/2) &= \frac{1}{7}(-\mathcal{E}_2(\tau) + 2\mathcal{E}_2(2\tau) + 6\mathcal{E}_4(2\tau)), \\
\mathcal{E}_{14}(\tau + 1/2) &= \frac{1}{13}(-\mathcal{E}_2(\tau) + 2\mathcal{E}_2(2\tau) + 12\mathcal{E}_7(2\tau)), \\
\mathcal{E}_{15}(\tau + 1/2) &= -\mathcal{E}_{15}(\tau) + 6\mathcal{E}_{15}(2\tau) - 4\mathcal{E}_{15}(4\tau).
\end{align}

From the definition of $\mathcal{E}_N$ in terms of the weight 2 Eisenstein series

\[ \mathcal{E}_N(\tau) = \frac{1}{N-1}(NE_2(N\tau) - E_2(\tau)). \]

we obtain the relations

\begin{align}
\mathcal{E}_{15}(\tau + k/3) &= \frac{1}{7}\mathcal{E}_3(\tau + \frac{k}{3}) + \frac{6}{7}\mathcal{E}_5(3\tau), \quad k \in \mathbb{Z}, \\
\mathcal{E}_{15}(\tau + k/5) &= \frac{2}{7}\mathcal{E}_5(\tau + \frac{k}{3}) + \frac{5}{7}\mathcal{E}_3(5\tau), \quad k \in \mathbb{Z}.
\end{align}

B Conjugacy class 14A and 15A

In this appendix we construct the twisted elliptic genera of $K3$ orbifolded by automorphisms corresponding to the conjugacy class 14A and 15A.

Conjugacy class 14A

\begin{align}
F^{(0,1)}(\tau, z) &= F^{(0,3)} = F^{(0,5)} = F^{(0,9)} = F^{(0,11)} = F^{(0,13)}, \\
&= \frac{1}{14} \left[ \frac{A}{3} - B \left( -\frac{1}{36}\mathcal{E}_2(\tau) - \frac{7}{12}\mathcal{E}_7(\tau) + \frac{91}{36}\mathcal{E}_{14}(\tau) \\
-\frac{14}{3}\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) \right) \right]; \\
F^{(r, rk)} &= \frac{1}{14} \left[ \frac{A}{3} + B \left( -\frac{1}{72}\mathcal{E}_2(\tau + \frac{k}{2}) - \frac{1}{12}\mathcal{E}_7(\tau + \frac{k}{7}) + \frac{13}{72}\mathcal{E}_{14}(\tau + \frac{k}{14}) \\
-\frac{1}{3}\eta(\tau + k)\eta(\frac{\tau + k}{2})\eta(\frac{\tau + k}{7})\eta(\frac{\tau + k}{14}) \right) \right];
\end{align}

where $r=1,3,5,9,11,13$ and $rk$ is Mod 14.
The even twisted sectors with odd twining characters can be found by similar manipulations as discussed in detail for the case of the 11A conjugacy class. This leads to the following equalities.

\[ F^{(2,13)} = F^{(12,1)} = F^{(6,11)} = F^{(8,3)} = F^{(4,5)} = F^{(10,9)}. \] (B.3)

Combining all these results into a single formula we obtain

\[
F^{(2r,2rk+7)} = \frac{1}{14} \left[ A + B \left( -\frac{1}{6} E_2(\tau) - \frac{1}{12} E_7(\frac{\tau + k}{7}) + \frac{1}{3} E_7(\frac{2\tau + 2k}{7}) \right) \right. \\
\left. - \frac{2}{3} \eta(\tau + k) \eta(2\tau + 2k) \eta(\frac{\tau + k}{7}) \eta(\frac{2\tau + 2k}{7}) \eta(\frac{\tau}{7}) \right];
\] (B.4)

where \( k \) runs from 0 to 6 and except 3 and \( r \) from 1 to 6. Next the following sectors are given by

\[
F^{(7,2k+1)} = \frac{1}{14} \left[ A + B \left( -\frac{7}{12} E_7(\tau) + \frac{49}{72} E_2(\frac{7\tau + 1}{2}) - \frac{1}{7} E_2(\frac{\tau + 1}{2}) \right) \right. \\
\left. + \frac{7}{3} \eta(\tau) \eta(7\tau) \eta(\frac{\tau + 1}{2}) - \frac{1}{7} \eta(\tau) \right];
\] (B.5)

\[
F^{(7,2k)} = \frac{1}{14} \left[ A + B \left( -\frac{7}{12} E_7(\tau) + \frac{49}{72} E_2(\frac{7\tau}{2}) - \frac{1}{7} E_2(\frac{\tau}{2}) \right) \right. \\
\left. + \frac{7}{3} \eta(\tau) \eta(7\tau) \eta(\frac{\tau}{2}) \right].
\]

Finally the sectors belonging to the 2A and 7A sub-orbits are given by

\[
F^{(0,0)} = \frac{4}{7} A. \] (B.6)

\[
F^{(0,2k)} = \frac{1}{14} \left[ A - \frac{7}{4} B E_7(\tau) \right] \quad k \text{ runs from 1 to 6,} \] (B.7)

\[
F^{(2r,2rk)} = \frac{1}{14} \left[ A + \frac{1}{4} B E_7(\frac{\tau + k}{7}) \right]; \quad k \text{ runs from 0 to 6.} \] (B.8)

\[
F^{(0,7)} = \frac{1}{14} \left[ \frac{8}{3} A - \frac{2}{3} B E_2(\tau) \right], \quad \text{(B.9)}
\]

\[
F^{(7,7k)} = \frac{1}{14} \left[ \frac{8}{3} A + \frac{2}{3} B E_2(\frac{\tau + k}{2}) \right] \quad k \text{ runs from 0 to 1.}
\]

**Conjugacy class 15A**

\[
F^{(0,1)}(\tau, z) = F^{(0,2)} = F^{(0,4)} = F^{(0,7)} = F^{(0,8)} = F^{(0,11)} = F^{(0,13)} = F^{(0,14)}; \] (B.9)

\[
= \frac{1}{15} \left[ A - B \left( -\frac{1}{16} E_3(\tau) - \frac{5}{24} E_5(\tau) + \frac{35}{16} E_{15}(\tau) - \frac{15}{4} \eta(\tau) \eta(3\tau) \eta(5\tau) \eta(15\tau) \right) \right].
\]
\[ F^{(r,rk)} = \frac{1}{15} \left[ \frac{A}{3} + B \left( -\frac{1}{48} \varepsilon_3(\frac{\tau+k}{3}) - \frac{1}{24} \varepsilon_5(\frac{\tau+k}{5}) + \frac{7}{48} \varepsilon_{15}(\frac{\tau+k}{15}) \right) \right] \]

where \( r = 1, 2, 4, 7, 8, 11, 13, 14 \) and \( rk \) is mod 15. The sectors belonging to the 5A and 3A sub-orbits are given by

\[ F^{(0,0)} = \frac{8}{15} A, \quad (B.11) \]
\[ F^{(0,3k)} = \frac{1}{15} \left( \frac{4}{3} A - \frac{5}{3} B \varepsilon_5(\tau) \right) \quad k \text{ runs from 1 to 4}; \quad (B.12) \]
\[ F^{(3r,3rk)} = \frac{1}{15} \left( \frac{4}{3} A + \frac{1}{3} B \varepsilon_5(\frac{\tau+k}{5}) \right); \quad k \text{ runs from 0 to 4.} \]
\[ F^{(0,5k)} = \frac{1}{15} \left( 2A - \frac{3}{2} B \varepsilon_3(\tau) \right); \quad (B.13) \]
\[ F^{(5r,5rk)} = \frac{1}{15} \left( 2A + \frac{1}{2} B \varepsilon_3(\frac{\tau+k}{3}) \right) \quad k \text{ runs from 0 to 2.} \]

Finally the remaining sectors are given by

\[ F^{(3r,5s+3rk)} = \frac{1}{15} \left( \frac{A}{3} + B \left( -\frac{1}{4} \varepsilon_3(\tau) - \frac{1}{24} \varepsilon_5(\frac{\tau+k}{5}) + \frac{3}{8} \varepsilon_5(\frac{3\tau+3k}{5}) \right) \quad (B.14) \]

where \( k \) runs from 0 to 4 and \( s = 1 \) to 4.

\[ F^{(5r,3s+5rk)} = \frac{1}{15} \left( \frac{A}{3} + B \left( \frac{5}{24} \varepsilon_5(\tau) + \frac{1}{12} \varepsilon_3(\frac{3\tau}{3}) - \frac{5}{24} \varepsilon_5(\frac{\tau+k}{3}) \right) \quad (B.15) \]

where \( k \) runs from 0 to 2 and \( s = 1 \) to 2.

The low lying coefficients of the twisted elliptic genus in conjugacy classes 14A as well as 15A satisfy

\[ c^{(0,s)}(\pm 1) = \frac{2}{N}, \quad \sum_{s=0}^{N-1} c^{(0,s)}(\pm 1) = 2, \quad (B.16) \]
\[ \sum_{s=0}^{N-1} c^{(0,s)}(0) = 0, \quad \sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = 8A(\tau, z) \]

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