Abstract

In this article we give the realization of the Klein’s Program for geometrical structures (Riemannian spaces and fiber bundles with connection) with arbitrary variable curvature within the framework of infinite deformed groups. These groups generalize gauge groups to the case of nontrivial action on the base space of bundles with use of idea of groups deformations.

We also show that infinite deformed groups give a group-theoretic description of gauge fields (gravitational field with its metric or vierbein part similarly to gauge fields of internal symmetry) which is alternative for their geometrical interpretation.
Introduction

An infinite (local gauge) symmetry lays in a basis of modern theories of fundamental interactions. Theory of gravitation (general relativity) is based on the idea of covariance with respect to the group of space-time diffeomorphisms. Theories of strong and electroweak interactions are gauge theories of internal symmetry. Moreover, the existence of these interactions is considered to be necessary for ensuring the local gauge symmetries.

But any physical theory can be written in covariant form without introduction of a gravitational field. Similarly, as first had been emphasized in [1], for any theory with global internal symmetry \( G \) corresponding gauge symmetry \( G^g \) can be ensured without introduction of nontrivial gauge fields with help of pure gauging. Presence of the gravitational or the gauge fields of internal symmetry is being manifested in presence of deformation - curvature of Riemannian space, or fiber bundles with connection accordingly.

Formally nontrivial gauge fields are being entered by continuation of derivatives up to covariant derivatives \( \partial_\mu \rightarrow \nabla_\mu \). Their commutators characterize strength of a field, which is considered, and from the geometrical point of view - curvature of corresponding space. On the other hand, covariant derivatives set infinitesimal space-time translations in gauge fields (curved spaces). That is why it is possible to suppose, that for introduction of nontrivial arbitrary gauge fields it is necessary to consider groups (wider than gauge groups \( G^g \)) which would generalize gauge groups \( G^g \) to a case of nontrivial action on space-time manifold and contain the information about arbitrary gauge fields in which occur a motion. Consequently, such groups must contain the information about appropriate geometrical structures with arbitrary variable curvature and set these geometrical structures on manifolds where they act. Hence from the mathematical point of view such groups should realize the Klein’s Erlanger Program [2] for these geometrical structures.

For a long time it was considered that such groups do not exist. E.Cartan [3] has named the situation in question as Riemann-Klein’s antagonism - antagonism between Riemann’s and Klein’s approach to geometry. There are attempts of modifying of the Klein’s Program for geometrical structures with arbitrary variable curvature by means of refusal of group structure of used transformations with usage of categories [4], quasigroups [5] and so on. One can encounter widespread opinion that nonassociativity is an algebraic equivalent of the geometrical notion of a curvature [5].

In our article we shall show that realization of the Klein’s Program for the geometrical structures with arbitrary variable curvature (Riemannian space and fiber bundles with connection) can be executed within the framework of the so called infinite deformed groups which generalize of gauge groups to the case of nontrivial action on the base space of bundles with use of idea of groups deformations.

Such groups have been constructed in [6]. Klein’s Erlanger Program was realized for fiber bundles with connection in [7] and for Riemannian space in [8].

This is important for physics because the widely known gauge approaches to gravity (see, for example, [9]) in fact gives gauge interpretation neither to metric fields nor to vierbein ones. An interpretation of these as connections in appropriate fibrings has been achieved in a way of introduction (explicitly or implicitly) of the assumption about existence of the background flat space (see, for example, [10]). That is unnatural for gravity. The reason for these difficulties lies in the fact that the fiber bundles formalism is appropriate only for the internal symmetry Lie group, which do not act on the space-time manifold. But for the gravity this restriction is obviously meaningless because it does not permit consider gravity as the gauge theory of the translation group.

In this article we also shall show that infinite deformed groups give a group-theoretic description
of gauge fields (gravitational field with its metric or vierbein part similarly to gauge fields of internal symmetry) which is alternative for their geometrical interpretation [11].

This approach allow to overcome the well known Coleman-Mandula no-go theorem within the framework of infinite deformed groups and gives new possibilities to unification gravity with gauge theories of internal symmetry [12].

1. Generalized Gauge Groups

Gauge groups of internal symmetry $G^g$ are a special case of infinite groups and have simple group structure - the infinite direct product of the finite-parameter Lie groups $G^g = \prod_{x \in M} G$ where product takes on all points $x$ of the space-time manifold $M$. Groups $G^g$ act on $M$ trivially:

$$x'^\mu = x^\mu.$$ 

For the aim of a generalization of groups $G^g$ to the case of nontrivial action on the space-time manifold $M$, let’s now consider a Lie group $G_M$ with coordinates $\tilde{g}^\alpha$ (indexes $\alpha, \beta, \gamma, \delta$) and the multiplication law:

$$(\tilde{g} \cdot \tilde{g}')^\alpha = \tilde{\varphi}^\alpha(\tilde{g}, \tilde{g}'),$$

which act (perhaps inefficiency) on the space-time manifold $M$ with coordinates $x^\mu$ (indexes $\mu, \nu, \pi, \rho, \sigma$) according to the formula:

$$x'^\mu = \tilde{f}^\mu(x, \tilde{g}).$$

The infinite Lie group $G_M^g$ is parameterized by smooth functions $\tilde{g}^\alpha(x)$ which meet the condition:

$$\det\{d_\nu \tilde{f}^\mu(x, \tilde{g}(x))\} \neq 0, \forall x \in M,$$

where $d_\nu := d/dx^\nu$. The multiplication law in $G_M^g$ is determined by the formulae:

$$(\tilde{g} \times \tilde{g}')^\alpha = \tilde{\varphi}^\alpha(\tilde{g}(x), \tilde{g}'(x')),$$  \hspace{1cm} (1)

$$x'^\mu = \tilde{f}^\mu(x, \tilde{g}(x)).$$  \hspace{1cm} (2)

It is a simple matter to check that these operations truly make $G_M^g$ a group. Formula (2) sets the action of $G_M^g$ on $M$. In the case of trivial action of the group $G_M$ on $M$, $G_M^g$ becomes the ordinary gauge group $G^g = \prod_{x \in M} G$. The groups $G_M^g$ we call generalized gauge groups.

For the clearing of the groups deformations idea we shall consider spheres of different radius $R$. All of them have isomorphic isometry groups - groups of rotations $O(3)$. The information about radius of the spheres is in structural constants of groups $O(3)$, which in the certain coordinates may be written as: $F_{12}^3 = 1/R^2, F_{13}^3 = -1, F_{23}^3 = 1$. Isomorphisms of groups $O(3)$, which change $R$, correspond to deformations of the sphere.

For gauge groups $G_M^g$ some isomorphisms also play a role of deformations of space of groups representations, but as against isomorphisms of finite-parameter Lie groups such isomorphisms are more substantial, as these allow to independently deform space in its different points.

Let us pass from the group $G_M^g$ to the group $G_M^{gH}$ isomorphic to it by the formula $g^\alpha(x) = H^\alpha(x, \tilde{g}(x))$ (Latin indexes assume the same values as the corresponding Greek ones). The smooth functions $H^\alpha(x, \tilde{g})$ have the properties:
1) \( H^a(x, 0) = 0 \quad \forall x \in M; \)

2) \( \exists K^a(x, g): \quad K^a(x, H(x, \tilde{g})) = \tilde{g}^a \quad \forall \tilde{g} \in G, \ x \in M. \)

The group \( G^{qH}_M \) multiplication law is determined by its isomorphism to the group \( G^{q}_M \) and formulae (1), (2):

\[
(g * g')^a(x) = \varphi^a(x, g(x), g'(x')) := H^a(x, \varphi(K(x, g(x)), K(x', g'(x')))),
\]

\[
x'^\mu = f^\mu(x, g(x)) := \tilde{f}^\mu(x, K(x, g(x))).
\]

Formula (4) sets the action of \( G^{qH}_M \) on \( M \).

Such transformations between the groups \( G^q_M \) and \( G^{qH}_M \) we call deformations of infinite Lie groups, since (together with changing of the multiplication law) the corresponding deformations of geometric structures of manifolds subjected to group action are directly associated with them. The functions \( H^a(x, \tilde{g}) \) we call deformation functions, functions

\[
h(x)^a_\alpha := \partial H^a(x, \tilde{g})/\partial \tilde{g}^\alpha|_{\tilde{g}=0}
\]

- deformation coefficients, and the groups \( G^{qH}_M \) - infinite (generalized gauge) deformed groups.

Let us consider expansion:

\[
\varphi^a(x, g, g') = g^a + g'^a + \gamma(x)^a_{bc} g^b g'^c + \frac{1}{2} \rho(x)^a_{bcd} g^b g^d g'^c + \ldots
\]

The functions \( \varphi^a \), setting the multiplication law (3) in the group \( G^{qH}_M \), are explicitly \( x \)-dependent, so the coefficients of expansion (5) are \( x \)-dependent as well. So, \( x \)-dependent became structure coefficients of group \( G^{qH}_M \) (structure functions versus structure constants for ordinary Lie groups)

\[
F(x)^a_{bc} := \gamma(x)^a_{bc} - \gamma(x)^a_{cd}
\]

The coefficients

\[
R(x)^a_{dab} := \rho(x)^a_{dab} - \rho(x)^a_{db},
\]

which we call curvature coefficients of the deformed group \( G^{qH}_M \).

Since

\[
\xi(x)^a_{\mu} := \partial_\mu f^\mu_H(x, g)|_{g=0} = h(x)^a_\alpha \xi(x)^\mu_\alpha,
\]

where \( \partial_\mu := \partial/\partial g^\mu \) and \( h(x)^a_\alpha \) is reciprocal to the \( h(x)^a_\alpha \) matrix, the generators \( X_\alpha := \xi(x)^a_{\mu} \partial_\mu \) (\( \partial_\mu := \partial/\partial x^\mu \)) of the deformed group \( G^{qH}_M \) are expressed through the generators \( \tilde{X}_\alpha := \tilde{\xi}(x)^a_{\mu} \partial_\mu \) of the group \( G^{q}_M \) with the help of deformation coefficients:

\[
X_\alpha = h(x)^a_\alpha \tilde{X}_\alpha.
\]

So, in infinitesimal (algebraic) level, deformation is reduced to independent in every point \( x \in M \) nondegenerate liner transformations of generators of the initial Lie group.

**Theorem 1.** Commutators of generators of the infinite (generalized gauge) deformed group are lliner combinations of generators with structure functions as coefficients [6]:

3
\[ [X_a, X_b] = F(x)^c_{ab} X_c. \]  

(8)

For the generalized gauge nondeformed group \( G_M^g \) we have:

\[ [\tilde{X}_\alpha, \tilde{X}_\beta] = \tilde{F}_{\alpha\beta}^{\gamma} \tilde{X}_\gamma, \]

where \( \tilde{F}_{\alpha\beta}^{\gamma} \) - structure constants of the initial Lie group \( G_M^g \).

2. Group-Theoretic Description of Connections in Fiber Bundles and Gauge Fields of Internal Symmetry

Let \( P = M \times V \) be a principal bundle with the base \( M \) (space-time) and a structure group \( V \) with coordinates \( \tilde{\upsilon}^i \) (indexes \( i, j, k \)) and the multiplication law \( (\tilde{\upsilon} \cdot \tilde{\upsilon}')^i = \tilde{\varphi}^i(\tilde{\upsilon}, \tilde{\upsilon}') \). As usually, we define the left \( l_5 : P = M \times V \rightarrow P' = M \times \tilde{\upsilon}^{-1} \cdot V \) and the right \( r_5 : P = M \times V \rightarrow P' = M \times V \cdot \tilde{\upsilon} \) action \( V \) on \( P \).

Let’s consider a group \( G_M = T_M \otimes V \) where \( T_M \) is the group of space-time translations. The group \( G_M \) is parameterized by the pair \( \tilde{t}^\mu, \tilde{\upsilon}^i \), has the multiplication law:

\[ (\tilde{g} \cdot \tilde{g}')^\mu = \tilde{l}^\mu, \quad (\tilde{g} \cdot \tilde{g}')^i = \tilde{\varphi}^i(\tilde{\upsilon}, \tilde{\upsilon}') \]

and act on the \( M \) inefficiently: \( x'^\mu = x^\mu + \tilde{l}^\mu, \quad \upsilon'^i = l^i_{\upsilon}(\upsilon) \).

The group \( G_M^g \) is parameterized by the functions \( \tilde{l}^\mu(x), \tilde{\upsilon}^i(x) \) which meet the condition:

\[ \det\{\delta^\mu_\nu + \partial_\nu \tilde{l}^\mu(x)\} \neq 0, \quad \forall x \in M. \]

The multiplication law in \( G_M^g \) is:

\[ (\tilde{g} \times \tilde{g}')^\mu(x) = \tilde{l}^\mu(x) + \tilde{l}'^\mu(x'), \quad (\tilde{g} \times \tilde{g}')^i(x) = \tilde{\varphi}^i(\tilde{\upsilon}(x), \tilde{\upsilon}'(x')), \]

(9)

\[ x'^\mu = x^\mu + \tilde{l}^\mu(x), \]

(10)

where (10) determines the inefficient action of \( G_M^g \) on \( M \) with the kern of inefficiency - gauge group \( V^g \). The group \( G_M^g \) has the structure \( Diff M \times V^g \), act on \( P \) as:

\[ x'^\mu = x^\mu + \tilde{l}^\mu(x), \quad \upsilon'^i = l^i_{\upsilon(x)}(\upsilon) \]

and is the group \( aut P \) of automorphisms of the principal bundle \( P \).

Let us deform the group \( G_M^g \rightarrow G_M^{gH} \) with help of deformation functions with additional properties:

3) \( H^\mu(x, \tilde{t}, \tilde{\upsilon}) = \tilde{l}^\mu \quad \forall \tilde{t} \in T, \ \tilde{\upsilon} \in V, \ x \in M; \)

4) \( H^i(x, 0, \tilde{\upsilon}) = \tilde{\upsilon}^i \quad \forall \tilde{\upsilon} \in V, \ x \in M. \)
The deformed group $G^g_M$ is parameterized by the functions:

$$t^\mu(x) = \tilde{t}^\mu(x), \quad v^i(x) = H^i(x, \tilde{t}(x), \tilde{v}(x)).$$

Obviously, the group $G^g_M$, as well as the group $G^g_P$, has the structure $Diff M \times V^g$ and act on $P$ as:

$$x'^\mu = x^\mu + t^\mu(x), \quad v'^i = t^i K(x, t(x), v(x))(v),$$

where functions $K^i(x, t(x), v(x))$ are determined by equation: $K^i(x, t(x), v(x)) = \tilde{v}^i(x)$. Properties 3), 4) result in the fact that among deformation coefficients of the group $G^g_M$, $x$-dependent is only $h(x)^i_\mu = \partial_\mu H^i(x, \tilde{t}, 0)|_{\tilde{t}} = -A(x)^i_\mu$ (where $\partial_\mu := \partial / \partial \tilde{t}^\mu$).

Generators of the $G^g_M$-action on $P$ (11) are split in the pair:

$$X_\mu = \partial_\mu + A(x)^i_\mu \tilde{X}_i, \quad X_i = \tilde{X}_i$$

where $\tilde{X}_i$ are generators of the left action of the group $V$ on $P$. This results in the natural splitting of tangent spaces $T_u$ in any point $u \in P$ into the direct sum $T_u = T\tau_u \oplus T\nu_u$ subspaces:

$$T\tau_u = \{t^\mu X_\mu\}, \quad T\nu_u = \{v^i X_i\}.$$ The distribution $T\tau_u$ is invariant with respect to the right action of the group $V$ on $P$, and $T\nu_u$ is tangent to the fiber. So $T\tau_u$ one can treated as horizontal subspaces of $T_u$ and generators $X_\mu$ - as covariant derivatives. This set in the principal bundle $P$ connection and deformation coefficients $A(x)^i_\mu$ are the coordinates of the connection form, which on submanifold $M \subset P$ may be written as $\omega^i = A(x)^i_\mu dx^\mu$. Necessary condition of existence of group $G^g_M$ (8) for generators $X_\mu$ yield:

$$[X_\mu, X_\nu] = F(x)^i_{\mu\nu} X_i,$$

where

$$F(x)^i_{\mu\nu} X_i = \tilde{F}^i_{jk} A(x)^j_\mu A(x)^k_\nu + \partial_\mu A(x)^i_\nu - \partial_\nu A(x)^i_\mu$$

- structure functions of the group $G^g_M$ and $\tilde{F}^i_{jk}$ - structure constants of the Lie group $V$. Relationship (12) one can write as:

$$d\omega^i = -\frac{1}{2} \tilde{F}^i_{jk} \omega^j \wedge \omega^k + \Omega^i,$$

where form

$$\Omega^i = \frac{1}{2} F(x)^i_{\mu\nu} dx^\mu \wedge dx^\nu$$

play the role of the curvature form on submanifold $M$. So, equation (14) is a structure equation for connection, which has be set on the principal bundle $P$ by action of group $G^g_M$.

**Theorem 2.** Acting on the principal bundle $P = M \times V$ deformed group $G^g_M = Diff M \times V^g$ sets on $P$ structure of connection. Any connection on the principal bundle $P = M \times V$ may be set thus [7].

This theorem realizes Klein’s Erlanger Program for fiber bundles $P = M \times V$ with connection.
We should emphasize that for the setting of a geometrical structure on $P$ it is enough to consider the infinitesimal action (11) of the group $G^g_M$.

The potentials of gauge fields of internal symmetry are identified with deformation coefficients $A(x)_\mu^i$, a strength tensor - with structure functions $F(x)_{ij\mu}^i$ of the group $G^g_M$. All groups $G^g_M$ obtained one from another by internal automorphisms, which are generated by the elements $v(x) \in V^g$, describe the same gauge field. These automorphisms lead to gauge transformations for fields $A(x)_\mu^i$ and for infinitesimal $v^i(x)$ yield:

$$A'(x)_\mu^i = A(x)_\mu^i - \tilde{F}_{jk}^i A(x)_k^j v^k(x) - \partial_\mu v^i(x). \quad (15)$$

3. Group-Theoretic Description of Riemannian Spaces and Gravitational Fields

The structure of Riemannian space is a special case of structure of affine connection in vierbein bundle and consequently it can be set by the way described above with the application of the deformed group $DiffM \times SO(n)^g$. If we force the coordination of connection with a metric and vanishing of torsion, generators of translations $X_\mu = \partial_\mu + \Gamma(x)_\mu^{mn} \tilde{S}_{mn}$, where $\tilde{S}_{mn}$ is generators of group $SO(n)$, become covariant derivatives in Riemannian space. For the setting of Riemannian structure by such means, it is enough to consider the group $DiffM \times SO(n)^g$ on algebraic level - on level its generators.

Potentials of a gravitational field in the given approach are represented by the connection coefficients $\Gamma(x)_\mu^{mn}$ instead of the metrics or vierbein fields that would correspond to sense of a gravitational field as a gauge field of translation group which is born by an energy - momentum tensor, instead of spin.

Now we shall show that the Riemannian structure on $M$ is naturally set also by a narrower group than $DiffM \times SO(n)^g$, namely, the deformed group of diffeomorphisms $T^g_M$, though it demands consideration of its action on $M$ up to the second order on parameters.

Let $G_M = T_M$ where $T_M$ is the group of space-time translations. In this case $(\tilde{t} \times \tilde{t'})^\mu = \tilde{t}^\mu + \tilde{t'}^\mu$ and $x'^\mu = x^\mu + \tilde{t'}$. The group is parameterized by the functions $\tilde{t}^\mu(x)$, which meet the condition:

$$\det\{\delta^\mu_\nu + \partial_\nu \tilde{t}^\mu(x)\} \neq 0, \quad \forall x \in M.$$  

The multiplication law in $T^g_M$ is:

$$(\tilde{t} \times \tilde{t'})^\mu(x) = \tilde{t}^\mu(x) + \tilde{t'}^\mu(x'), \quad (16)$$

$$x'^\mu = x^\mu + \tilde{t}'(x), \quad (17)$$

where (17) determines the action of $T^g_M$ on $M$. The multiplication law indicates that $T^g_M$ is the group of space-time diffeomorphisms $DiffM$ in additive parameterization. Thus, in the approach considered, the group $T^g_M = DiffM$ becomes the gauge group of local translations. The generators of the $T^g_M$-action on $M$ (17) are simply derivatives $\tilde{X}_\mu = \partial_\mu$ and this fact corresponds to the case of the flat space $M$ and the absence of gravitational field.

Let us deform the group $T^g_M \rightarrow T^g_M\cdot t^m(x) = H^m(x, \tilde{t}(x))$. The multiplication law in $T^g_M$ is determined by the formulae:
yield: So, formulae (6), (7) for structure coefficients and curvature coefficients of deformed group $X$ of space $M$.

In this formulae deformation coefficients $h$.

Formula (19) sets the action of $T^g_H$ on $M$.

Let us consider expansion:

$$H^m(x, t) = h(x)^m_\mu [\tilde{\mu} + \frac{1}{2} \Gamma(x)^\mu_\nu_\rho \tilde{\nu} \tilde{\rho} + \frac{1}{6} \Delta(x)^m_\mu_\nu_\rho_\pi \tilde{\nu} \tilde{\rho} \tilde{\pi}].$$

(20)

With usage of the formula (18), for coefficients of expansion (5) we can obtain:

$$\gamma^m_{\nu \pi} = h^m_\mu (\Gamma^\mu_\nu_\pi + h^\nu_\rho \partial_\pi h^\mu_\rho),$$

(21)

$$\rho^m_{\nu \pi \rho} = h^m_\mu (\Delta^\mu_\nu_\pi_\rho - \Gamma^\mu_\nu_\pi_\rho_\mu \Gamma^\pi_\sigma_\rho_\sigma - \Gamma^\mu_\nu_\pi_\rho_\mu_\gamma \Gamma^\sigma_\gamma_\rho_\sigma).$$

(22)

So, formulae (6), (7) for structure coefficients and curvature coefficients of deformed group $T^g_H$ yield:

$$F^m_{\mu \nu} = -(\partial_\mu h^\nu_\rho - \partial_\nu h^\mu_\rho),$$

(23)

$$R^\mu_{\nu \pi \rho} = \partial_\pi \Gamma^\mu_\nu_\rho - \partial_\nu \Gamma^\mu_\pi_\rho + \Gamma^\mu_\pi_\sigma \Gamma^\sigma_\nu_\rho - \Gamma^\mu_\nu_\sigma \Gamma^\sigma_\pi_\rho.$$  

(24)

In this formulae deformation coefficients $h(x)^m_\mu$ and $h(x)^{\mu}_m$ we use for changing Greek index to Latin (and vice versa).

Formulae (23) and (24) say that groups $T^g_H$ contain information about geometrical structure of space $M$ where they act. The generators $X_n = h^\mu_\nu \partial_\nu$ of the $T^g_H$-action (19) on $M$ can be treated as vierbeins. Structure functions $F^m_{\mu \nu}$ differ from non-holonomy coefficients only by a factor -1/2.

Let us write the multiplication law of the group $T^g_H$ (18) for infinitesimal second factor:

$$(t * \tau)^m(x) = t^m(x) + \lambda(x, t(x))^{m\tau}_n(x'),$$

(25)

where $\lambda(x, t(x))^m_n := \partial_\nu \varphi^m(x, t, t')|_{t=0}$. Formula (25) gives the rule for the addition of vectors, which set in different points $x$ and $x'$ or a rule of the parallel transport of a vector field $\tau$ from point $x'$ to point $x$:

$$\tau^{m}_{\parallel}(x) = \lambda(x, t(x))^{m}_{\tau^m}(x'),$$

or in coordinate basis:

$$\tau^{\mu}_{\parallel}(x) = \partial_\nu H^\mu(x, t)\tau^\nu(x + \tilde{t}).$$

This formula determines the covariant derivative:

$$\nabla_\nu \tau^\mu(x) = \partial_\nu \tau^\mu(x) + \Gamma(x)^\mu_\nu_\rho \tau^\rho(x),$$

(26)

where functions $\Gamma(x)^\mu_\nu_\rho$ set the second order of expansion (20) and play the role of coefficients of an affine connection. They are symmetric on the bottom indexes, so torsion equals zero. If $h_{mn}$ is a metric of a flat space, in the manifold $M$ we can determine metrics $g_{\mu \nu} = h^m_\mu h^\rho_\nu h_{mn}$. If we force
\[ \gamma_{mns} + \gamma_{nms} = 0 \] (lowering indexes we fulfill with help of metric \( \eta_{mn} \)), we can show that coefficients \( \Gamma^p_{\mu\nu} \) of expansion (20) may be written as:

\[
\Gamma^p_{\mu\nu} = \frac{1}{2} g^{pq} (\partial_\mu g_{q\nu} + \partial_\nu g_{q\mu} - \partial_q g_{\mu\nu}).
\] (27)

So, these coefficients coincide with Christoffel symbols \( \{^p_{\mu\nu} \} \) and curvature coefficients \( R^\mu_{\lambda\kappa\nu} \) of the group \( T^gH_M \) coincide with the Riemann curvature tensor.

**Theorem 3.** Acting on the manifold \( M \) the deformed group \( T^gH_M \) sets on \( M \) structure of a Riemannian space. Any Riemannian structure on the manifold \( M \) may be set thus [8].

This theorem realizes Klein’s Erlanger Program for Riemannian space \( M \).

Information about Christoffel symbols is contained in the second order of expansion (20) of deformation functions and about curvature - in functions \( \rho^m_{pnr} \), which determine the third order of expansion (5) of the multiplication law of the group \( T^gH_M \). So, in this approach we need consider not only infinitesimal - algebraic - level in the group \( T^gH_M \) (as in previous section), but higher levels, too.

The gravitational field potentials are identified with deformation coefficients \( h^m(x)_\mu \), a strength tensor of the gravitational field - with structure functions \( F^p(x)_m^m \) of the group \( T^gH_M \). All groups \( T^gH_M \) obtained one from another by internal automorphisms describe the same gravitational field. These automorphisms, which can always be connected with the coordinate transformations on \( M \), lead to a general covariance transformation law for fields \( h^m(x)_\mu \) and for infinitesimal \( t^m(x) \) yield:

\[
h'(x)_\mu^m = h(x)_\mu^m - F(x)_m^m h(x)_\mu^p t^p(x) - \partial_\mu t^m(x).
\] (28)

The transformation law (28) is similar to the transformation law (15) for potentials of gauge fields of internal symmetry and the only difference consists in the replacement of structure constants of the finite-parameter Lie group \( V \) by structure functions of the infinite deformed group \( T^gH_M \). This fact permits us to interpret the group \( T^gH_M \) as the gauge translation group and the fields \( h^m(x)_\mu \) as the gauge fields of the translation group.

So, we show that infinite deformed (generalized gauge) groups:

1. set on manifolds where they act geometrical structures of fiber bundles with connection or Riemannian spaces with arbitrary variable curvature;
2. give a group-theoretic description of gauge fields of internal symmetry as well as gravitational fields.

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