A Counterexample to Las Vergnas’ Strong Map Conjecture on Realizable Oriented Matroids

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Abstract

The Las Vergnas strong map conjecture asserts that any strong map of oriented matroids \( f : \mathcal{M}_1 \to \mathcal{M}_2 \) can be factored into extensions and contractions. This conjecture is known to be false due to a construction by Richter-Gebert, who finds a strong map that is not factorizable; however, in his example, \( \mathcal{M}_1 \) is not realizable. The question of whether there exists a non-factorizable strong map between realizable oriented matroids remains open. In this paper, we provide a counterexample to the strong map conjecture on realizable oriented matroids, which is a strong map \( f : \mathcal{M}_1 \to \mathcal{M}_2 \), where \( \mathcal{M}_1 \) is an alternating oriented matroid of rank 4 and \( f \) has corank 2. We prove that the map is not factorizable by showing that there is no uniform oriented matroid \( \mathcal{M}' \) of rank 3 such that \( \mathcal{M}_1 \to \mathcal{M}' \to \mathcal{M}_2 \).

Keywords Las Vergnas’ strong map conjecture · Strong map · Oriented matroid · Point configurations

Mathematics Subject Classification 52C40 · 52B40 · 55R40

1 Background

The strong map conjecture, first posed by Las Vergnas, asserts that any strong map of oriented matroids \( f : \mathcal{M}_1 \to \mathcal{M}_2 \) can be factored into extensions and contractions.
The conjecture holds for ordinary matroids [5], but for oriented matroids, a counterexample has been constructed by Richter-Gebert [10]. However, $\mathcal{M}_1$ is not realizable in Richter-Gebert’s construction. The question of whether Las Vergnas’ conjecture holds when $\mathcal{M}_1$ is realizable remains open. In this paper, we present a counterexample disproving this conjecture.

**Theorem 1.1** There is a strong map $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$, where $\mathcal{M}_1$ is a realizable oriented matroid of rank 4 on eight elements and $f$ is corank 2, that is not factorizable into extensions and contractions.

Las Vergnas’ conjecture on realizable oriented matroids has its own significance as a part of the “combinatorial Grassmannian” program [9]. The program is stimulated by pioneering works of Gelfand and MacPherson [4,7], who in [4] proposed a formula that calculates rational Pontrjagin classes of a differentiable manifold from combinatorial data. In their proof, they make use of a modified formulation of Chern–Weil theory, so it is not possible to calculate any $\mathbb{Z}/p\mathbb{Z}$-characteristic classes following the same argument. A possible way to remedy this deficit is to adopt the definition of characteristic classes via Grassmannians. Let us recall some standard facts about characteristic classes (see [2] or [8] for a comprehensive treatment). Let $p : E \rightarrow B$ be a real vector bundle on a manifold $B$, a characteristic class of the bundle is an invariant taking value in the cohomology ring $H^*(B)$ of certain coefficients. If $p$ is an $\mathbb{R}^k$-bundle, there is a canonical map (up to isotopy) from $B$ to an infinite real Grassmannian $G^\infty_k$, coined the Gauss map, and characteristic classes are pull-backs of certain cohomology classes on the infinite real Grassmannian. This definition is purely topological, so one would expect that we are able to rewrite the definition using combinatorial data with less effort. MacPherson [9] suggests the following object as a substitute of $G_k(\mathbb{R}^n)$: the (chain complex of) poset of all oriented matroids of rank $k$ on $n$ elements, ordered with respect to weak maps, called MacPhersonian and denoted as MacP$(n,k)$. Let $\mathcal{F}^n$ be the free oriented matroid of rank $n$; MacP$(n,k)$ is the poset of rank $k$ strong image of $\mathcal{F}^n$. One can obtain a more general object by substituting $\mathcal{F}^n$ with an arbitrary rank $n$ oriented matroid $\mathcal{M}$ (one can assume $\mathcal{M}$ is realizable for our purpose), called the OM-Grassmannian and denoted as $\mathcal{G}_k(\mathcal{M})$. The combinatorial Grassmannian program is the study of the homotopy type of $\mathcal{G}_k(\mathcal{M})$. The conjecture that $\mathcal{G}_k(\mathcal{M})$ and $G_k(\mathbb{R}^n)$ are homotopy equivalent has been disproved by Gaku Liu [6], and whether MacP$(n,k)$ and $G_k(\mathbb{R}^n)$ are homotopy equivalent remains an open question.

The Las Vergnas strong map conjecture is related to the combinatorial Grassmannian program in the following way: the (non-compact) Stiefel manifold $V_k(\mathbb{R}^n)$ is the set of all $k$-tuples of linearly independent vectors, and there is a surjective mapping $p : V_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n)$ obtained by sending the $k$-tuples to the linear space they span. For every pt $p \in G_k(\mathbb{R}^n)$, $p^{-1}(\{pt\})$ is isomorphic to $GL(k,\mathbb{R})$, so $V_k(\mathbb{R}^n)$ can be viewed as a principal $GL(k,\mathbb{R})$-bundle over $G_k(\mathbb{R}^n)$. The oriented matroid counterpart of the Stiefel manifold is defined as follows: let $\mathcal{M}$ be an oriented matroid of rank $n$; the OM-Stiefel space $V_k(\mathcal{M})$ is defined as all “non-degenerate” $n-k$ extensions, i.e., if the set of new elements is $S$, the contraction $\mathcal{M}/S$ should have rank $k$. Thus, there is a poset mapping $\tilde{p} : V_k(\mathcal{M}) \rightarrow \mathcal{G}_k(\mathcal{M})$ defined by contracting $S$. A natural question is whether the preimage of every point is homotopic to $GL(k,\mathbb{R})$. Note that the

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Las Vergnas strong map conjecture implies the subjectivity of $\tilde{p}$. Our counterexample indicates that there is a point with an empty preimage.

2 Oriented Matroids

For completeness, we include a brief introduction to the theory of oriented matroids, in which we attempt to cover most conventions and facts we use in this paper. One could refer to [1] for a detailed treatment.

Datum of oriented matroids can be encoded by circuits, vectors, cocircuits, covectors, topes or chirotopes. Let $E$ be the ground set. Circuits, vectors, cocircuits, covectors, and topes are all signed vectors on $E$. A signed vector $X$ is a mapping $X: E \to \{-1, 0, 1\}$. $X^{-1}(1)$, and $X^{-1}(-1)$ are denoted as $X^+$ and $X^-$, respectively. We use two ways to write the signed vectors; for example, when $E = \{1, \ldots, 5\}$, $X^+ = \{1, 3\}$ and $X^- = \{2, 4\}$, $X = (+ + + - 0)$ or $X = 1234$. $\emptyset$ is the signed vector $X$ with $X^+ = X^- = \emptyset$; $1$ is the signed vector $X$ with $X^+ = X$. If $X$ is a signed vector, define $-X$ to be the signed vector with $(-X)(i) = -X(i)$, which is called the opposite of $X$. Given a set of signed vectors $X$, reorientation of an element $e \in E$ is the operation of reversing the values of $X(e)$ for all signed vectors $X \in X$. The support of a signed vector is defined as $X^+ \cup X^-$, denoted as $X$, and the size of $X$ is defined as the size of support. Signed vector $X$ has full support iff $X = E$. Two signed vectors $Y$ and $Z$ are perpendicular iff in their component-wise products $X^+ \subset Z^-$ and $X^- \subset Z^+$. A natural partial ordering on signed vectors exists: $X \preceq X'$ iff $X^+ \subseteq X'^+$ and $X^- \subseteq X'^-$. If $E' \subseteq E$, restriction of $X$ on $E'$ is a signed vector on $E'$, defined as $X|_{E'}(i) = X(i)$ for $i \in E'$. The chirotope is an anti-symmetric mapping $\chi : E' \to \{1, 0, -1\}$, in which $r = r(M)$ is the rank of the oriented matroid. An oriented matroid can be encoded by a set of circuits, or cocircuits, etc., satisfying certain sets of axioms [1, Chap. 3]. For completeness, we include the covector axiomatization of oriented matroids here.

Definition 2.1 An oriented matroid is a pair $M = (E, L)$, where covector $L$ is a set of signed vectors on $E$ such that:

1. $0 \in L$,
2. $X \in L \Rightarrow -X \in L$,
3. $X, Y \in L \Rightarrow X \circ Y \in L$,
4. (covector elimination) $X, Y \in L, e \in S(X, Y) \Rightarrow$ there exists $Z \in L$ such that $Z(e) = 0$ and $Z(f) = (X \circ Y)(f)$ for $f \notin S(X, Y)$,

$$S(X, Y) := \{e \in E \mid X(e) = -Y(e) \neq 0\},$$

and $X \circ Y$ is the signed vector defined as

$$(X \circ Y)(e) = \begin{cases} X(e), & \text{if } X(e) \neq 0, \\ Y(e), & \text{otherwise}. \end{cases}$$

Point configurations are important models of oriented matroids. A finite set of points $E = \{v_1, \ldots, v_n\}$ in affine space $\mathbb{R}^{r-1}$ is a point configuration if their affine closure
is $\mathbb{R}^{r-1}$. We can associate it with an oriented matroid $M$. Each affine dependency $\sum \lambda_i v_i = 0$, $\sum \lambda_i = 0$ defines a vector $X$ of $M$ by $X^+ = \{v_i \mid \lambda_i > 0\}$ and $X^- = \{v_i \mid \lambda_i < 0\}$. Geometrically, this implies the convex hulls of $X^+$ and $X^-$ are intersecting at interior points. Each $w \in \mathbb{R}^*$, $a \in \mathbb{R}$, defines a covector $X$ of $M$ such that $X^+ = \{v_i \mid \langle v_i, w \rangle > a\}$, $X^- = \{v_i \mid \langle v_i, w \rangle < a\}$. That is, $X^+$ and $X^-$ lie in two half-spaces cut by the hyperplane $\{v_i \mid \langle v_i, w \rangle = a\}$. Circuits are non-zero $\preceq$-minimal vectors, cocircuits are non-zero $\preceq$-minimal covectors, and topes are $\preceq$-maximal covectors. A chirotope is an alternating function on $2^{[n]}$. An oriented matroid is realizable iff it arises in this way for some $\{v_e \mid e \in E\}$, up to reorientation of elements. One could verify that every vector is perpendicular to every covector, which is a property that also holds for non-realizable oriented matroids.

An oriented matroid is acyclic iff $1$ is a covector. And an oriented matroid is uniform iff $\chi(i_1, \ldots, i_r) \neq 0$ for all $i_1, \ldots, i_r$. In a uniform oriented matroid, the size of the circuits is always $r + 1$ and the size of cocircuits is always $n - r$. Define $\Phi_r(n) := \sum_{i=0}^r \binom{n}{i}$, the number of topes is $2\Phi_{r-1}(n - 1)$. It is worthwhile to point out that the converse is also true by [3], which provides an alternative axiomatization of a uniform oriented matroid.

**Theorem 2.2** Let $T$ be a set of full support signed vectors on $[n]$; $T$ is the set of topes of a rank $r$ uniform oriented matroid iff:

1. $\#T = 2\Phi_{r-1}(n - 1)$,
2. $X \in \mathcal{L} \Rightarrow -X \in \mathcal{L}$,
3. (VC-dimension) for any $Q \in \binom{[n]}{r+1}$, there exists a signed vector $c_Q$ (together with its opposite) supported on $Q$ such that $T \perp c_Q$ (or equivalently, $T|_Q \neq c_Q$) for every $T \in T$.

We further define several operations on oriented matroids for stating the strong map conjecture of Las Vergnas. If $M_1$ is on ground set $E_1$ and $M_2$ is on ground set $E_1 \cup \{u\}$, they are of the same rank and their chirotopes coincide on $E_1$, we say $M_2$ is a single extension of $M_1$ and $M_1$ is a single deletion of $M_2$ obtained by deleting $u$, written as $M_1 = M_2 \backslash u$ or $M_2 \hookrightarrow M_1$. An extension is a composite of single extensions, and a deletion is a composite of single deletions. If $E' \subseteq E_1$, the restriction of $M_1$ on $E'$ is the oriented matroid that deletes all elements not in $E'$. Contractions are defined as follows: if $M_1$ is on ground set $E$ and $u \in E$, the contraction of $u$ is defined as an oriented matroid $M_2$ on $E \setminus \{u\}$ with chirotope $\chi_2(\chi_1, \ldots, x_{r-1}) = \chi_1(u, x_1, \ldots, x_{r-1})$, written as $M_1 = M_2 / u$ or $M_1 \rightarrow M_2$. The contraction of a subset $U \subseteq E$ is composite of contracting all elements in $U$.

There is a strong map from $M_1$ to $M_2$ iff they are on the same ground set and every covector of $M_1$ is a covector of $M_2$. In this case, we write the strong map $f: M_1 \rightarrow M_2$ (for a general discussion, see [1, p. 319]). The corank of a strong map is defined as $r(M_1) - r(M_2)$. A composition of extensions and contractions on the same set of elements is always a strong map (we say that such a strong map is factorizable for short). The strong map conjecture asks whether the converse is true. The conjecture holds if the corank is $1$, the rank of $M_2$ is $1$ or the rank of $M_1$ is only
Proposition 2.3 Let $T_1, T_2$ be topes of oriented matroids $M_1, M_2$, respectively. We further assume that $M_2$ is uniform. Then, there exists a strong map $M_1 \to M_2$ iff $T_2 \subseteq T_1$.

Proof The “only if” part is trivial since every tope is a covector; we will prove the “if” part. Let $C_1, C_2$ be covectors of oriented matroids $M_1, M_2$, respectively. Suppose $X \in C_2$; then, since $M_2$ is uniform, any $X' \geq X$ is a covector of $M_2$. Let $P(X) = \{X' \mid X' \geq X, X' \text{ has full support}\}$; we then have $P(X) \subseteq T_2 \subseteq T_1 \subseteq C_1$.

Then, we prove that $X$ is a covector of $M_1$. Observe that if $X_1, X_2$ are two covectors that differ in only one index (i.e., there exists $i \in E$ s.t. $X_1(e) = X_2(e)$ for $e \neq i$ and $X_1(i) = -X_2(i)$), then by covector elimination $X'$ with $X'(e) = X_2(e)$ for $e \neq i$ with $X'(i) = 0$ being a covector, applying this property on $P(X)$ iteratively yields $X \in C_1$. Thus, $M_1 \to M_2$. \qed

2.1 Alternating Oriented Matroid

Alternating oriented matroids represent an important family of oriented matroids with many nice properties. An alternating oriented matroid is an oriented matroid on $[n]$ with rank $r$ and chirotope $\chi(e_1, \ldots, e_r) = 1$ if $1 \leq e_1 \leq \cdots \leq e_r \leq n$. The main fact we need is that the topes of a rank $k$ alternating oriented matroid are all signed vectors with at most $k - 1$ sign changes. For example, if $k = 4$, the topes are signed vectors of the form $(+ \cdots + - - \cdots -), (\cdots + + + \cdots -)$ or $(\cdots + - - \cdots + - \cdots +)$ or their opposite. Alternating oriented matroids are always realizable by momentum curve $t \mapsto (t, \ldots, t^{r-1})$, $t \in [n]$.

3 Construction and Verification of a Counterexample

In this section, we identify oppositely signed vectors and choose signed vectors with the non-zero first component being positive as representatives. The counterexample is the following strong map $f : M_1 \to M_2$. $M_1$ is a rank 4 alternating oriented matroid on ground set $E = [n] := \{1, \ldots, n\}$ with $n$ being even. $M_2$ is a rank 2 oriented matroid defined as follows. Let $\sigma$ be the permutation $(1 \, 2 \, 3 \, 4 \ldots (n - 1 \, n)$; the chirotope of $M_2$ is defined as $\chi(i, j) = 1$ iff $\sigma(i) \geq \sigma(j)$. The topes of $M_2$ are all in the form of $(+ \cdots + - - \cdots -)$ or $(+ \cdots + - + \cdots -)$. Thus, by Proposition 2.3, $f : M_1 \to M_2$ is a strong map.

We first give an intuitive (and invalid) explanation of why $f$ is not factorizable when $n$ is sufficiently large. Note that $M_1$ can be realized by moment curve $h : t \mapsto (t, t^2, t^3)$. If $f$ is a factorizable strong map, it can be realized as a projective transformation. Let the transformation be $g$; then, $g(h(t))$ is a rational function in the form of $p_1(t)/p_2(t)$, in which $p_1, p_2$ are polynomials that are at most cubic. Therefore, the number of $t$ with $(g \circ h)'(t) = 0$ is at most 4, and the number of poles is at most 3. We can thus realize $M_2$ as $[n] \to \mathbb{R}$, which is the restriction of $g \circ h$ on $[n]$. Then, for every $i = 2, \ldots, n-1$, either $g(h(i)) < g(h(i + 1))$, $g(h(i)) < g(h(i - 1))$
or \( g(h(i)) > g(h(i + 1)), g(h(i)) > g(h(i - 1)) \) holds, hence there is a critical point or pole of \( g \circ h \) near \( i \), but the number of points satisfying this condition is at least \( n - 2 \), which leads to a contradiction.

This argument is not valid for two reasons. First, the realization is not necessarily the moment curve; second, the extension may be not realizable. We give a strict proof that when \( n = 8 \), \( f \) is not factorizable.

Observe that if \( f \) is a factorizable strong map, then since deletions and contractions commute, there exists an oriented matroid \( \mathcal{M}' \) of rank 3 such that \( f = f_1 \circ f_2 \), where \( f_1: \mathcal{M}_1 \to \mathcal{M}', f_2: \mathcal{M}' \to \mathcal{M}_2 \) are both strong maps. We can further assume that \( \mathcal{M}' \) is uniform by perturbing the extension element (Proposition 7.2.2(2) in [1]).

To show that no uniform oriented matroid \( \mathcal{M}' \) makes \( f_1 \) and \( f_2 \) both strong maps, we start by considering the case of \( n = 6 \).

**Lemma 3.1** Let \( \mathcal{M}_1, \mathcal{M}_2 \) be two oriented matroids defined above with \( n = 6 \). If \( \mathcal{M}' \) is a uniform oriented matroid of rank 3 such that \( f_1: \mathcal{M}_1 \to \mathcal{M}', f_2: \mathcal{M}' \to \mathcal{M}_2 \) are both strong maps. Then,

1. \((-+---), (+---), \notin \mathcal{T}(\mathcal{M}').
2. \((-+00), (+00++) \) are circuits of \( \mathcal{M}' \).

**Proof**

1. We prove this result by enumerating all possible \( \mathcal{T}(\mathcal{M}') \). By Theorem 2.2 and properties of strong maps, \( \mathcal{T}(\mathcal{M}') \) should satisfy \( \# \mathcal{T}(\mathcal{M}') = 16, \mathcal{T}(\mathcal{M}_2) \subset \mathcal{T}(\mathcal{M}') \subset \mathcal{T}(\mathcal{M}_1) \) and the VC-dimensional property. Note that \( \# \mathcal{T}(\mathcal{M}_1) = 26 \) and \( \# \mathcal{T}(\mathcal{M}_2) = 6 \), so there are \( \binom{26}{6} = 184,756 \) cases to check. By brute force search, we find 20 possible \( \mathcal{T}(\mathcal{M}') \). For all of them, \((-+---), (+---), \notin \mathcal{T}(\mathcal{M}'). \) (See Supplementary File, or https://github.com/PeterWu-Biomath/OM-Strong-Map for an implementation and detailed explanation of the proposed algorithm.)

2. Note that \((-+---) \) is the only signed vector in \( \mathcal{T}(\mathcal{M}_1) \) perpendicular to \((+-00)\); hence, for all \( T \in \mathcal{T}(\mathcal{M}'), T \perp (+--00) \). Thus, \((+-000) \) is a circuit of \( \mathcal{M}' \). Following the same argument, \((+00+++) \) is a circuit of \( \mathcal{M}' \). \( \square \)

Finally, the nonfactorizability in the case of \( n = 8 \) follows immediately from Lemma 3.1. Assume \( \mathcal{M}' \) is a uniform oriented matroid such that \( f_1 \) and \( f_2 \) are both strong maps. First, restricting on \( S = \{1, 2, 3, 4, 5, 6\} \), \( \mathcal{M}_1 \mid S \) is an alternating oriented matroid and \( \mathcal{M}_2 \mid S \) isomorphic to \( \mathcal{M}_2 \) in the case of \( n = 6 \). Hence, the circuit of \( \mathcal{M}' \) supported on \( \{1, 2, 3, 4, 5, 6\} \) is \((+-00-+00) \). Following the same argument, by restricting on \( \{1, 2, 5, 6, 7, 8\} \), the circuit on \( \{1, 2, 5, 6\} \) should be \((+-00+-00) \), which leads to a contradiction.

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