ON POLARIZATION TYPES OF LAGRANGIAN FIBRATIONS

BENJAMIN WIENECK

Abstract. The generic fiber of a Lagrangian fibration is an abelian variety. This fact is used to associate to each Lagrangian fibration a polarization type which comes from a polarization on a generic fiber. It follows that this polarization type stays constant in a family of Lagrangian fibrations i.e. it is a deformation invariant. Conjecturally the polarization type should only depend on the deformation type of the total space. Indeed, we prove that the polarization type of $K3^{[n]}$–type fibrations is always principal.

Contents

1. Introduction 1
2. Irreducible symplectic manifolds and their fibrations 3
3. Moduli space of Lagrangian fibrations 5
4. Polarization types of Lagrangian fibrations 11
5. Beauville–Mukai systems 16
6. Polarization types of $K3^{[n]}$–type fibrations 18
References 22

1. INTRODUCTION

The geometry of irreducible holomorphic symplectic manifolds or compact hyperkähler manifolds seems to be quite rigid since only four deformation types are known. The only possible nontrivial fibrations such manifolds can admit are Lagrangian fibrations as D. Matsushita showed, see Theorem 2.4. Lagrangian fibrations help us to understand the geometry of irreducible holomorphic symplectic manifolds. It is hoped that Lagrangian fibrations will be useful for the classification of irreducible holomorphic symplectic manifolds, see [Saw03].

Let $f : X \to B$ be a Lagrangian fibration. It is well known that all smooth fibers are abelian varieties even if $X$ is not projective. Given a smooth fiber $F$ an immediate question is to ask for natural polarizations on it which is by definition the first Chern class $H = c_1(L)$ of an ample line bundle $L$ of $F$.

By F. Campana, see Proposition 4.3, it is known that for each smooth fiber $F$ one can find a Kähler class $\omega$ on $X$ such that the restriction $\omega|_F$ is integral and
primitive and hence defines a polarization on $F$. An ad–hoc definition of the polarization type of a Lagrangian fibration would be to set $d(f) := d(\omega|_F)$ where the latter one is the polarization type of the polarization on $F$ given by $\omega|_F$. Indeed this does not depend on the chosen smooth fiber $F$ and the chosen $\omega$, see Proposition 4.7, Theorem 4.9 and Proposition 4.10.

The main result is the following.

**Theorem 1.1** (Section 4) Let $f : X \to B$ be a Lagrangian fibration with $\dim X = 2n$. Then we can associate to $f$ a tuple $d(f) \in \mathbb{Z}^n$ of integers called the polarization type such that the following holds.

(i) Theorem 4.9 The polarization type is a deformation invariant of the fibration i.e. if $f' : X' \to B'$ is a Lagrangian fibration deformation equivalent to $f$ then $d(f) = d(f')$.

(ii) Theorem 6.1 If $X$ is of $K3^{[n]}$–type then $d(f)$ is principal i.e. $d(f) = (1, \ldots, 1)$.

Ultimately we want the polarization type $d(f)$ only to depend on the deformation type of the total space of $f : X \to B$ for every deformation type of irreducible holomorphic symplectic manifolds. But this is possibly a too optimistic conjecture.

The proof of Theorem 1.1 (ii) involves moduli theory of Lagrangian fibrations of $K3^{[n]}$–type or $K3^{[n]}$–type fibrations i.e. Lagrangian fibrations $f : X \to \mathbb{P}^n$ such that $X$ is of $K3^{[n]}$–type, this means $X$ is deformation equivalent to the Hilbert scheme $S^{[n]}$ of $n$ points of some $K3$ surface $S$.

In section 3 the moduli theory of Lagrangian fibrations is explained which relies on methods developed by E. Markman in [Mar11] and [Mar14]. Beside that a result of D. Matsushita, see Theorem 3.2, plays an important role which says that every Lagrangian fibration can be considered as a member of a family of Lagrangian fibrations parametrized by the deformation space $\text{Def}(X, L)$ of the pair $(X, L)$ where $L$ is the pull–back of an ample line bundle on the base space. We will describe how to obtain a connected component of the moduli space of $K3^{[n]}$–type fibrations.

Section 5 recalls the notion of Beauville–Mukai systems which are examples of $K3^{[n]}$–type fibrations and it is shown that their polarization type is principal i.e. given by $(1, \ldots, 1) \in \mathbb{Z}^n$ which is needed for the proof of Theorem 6.1 (ii).

---

This means we have an $S$–morphism $\phi : \mathcal{X} \to P$ such that $S$ is a connected complex space with finitely many irreducible components, $\mathcal{X} \to S$ is a family of irreducible holomorphic symplectic manifolds and $P \to S$ is a family of projective varieties such that $\phi_t := \phi|_{\mathcal{X}_t} : \mathcal{X}_t \to P_t$ is a Lagrangian fibration for all $t \in S$ and there are points $t_i \in S$, $i = 1, 2$, such that $f = \phi_{t_1}$ and $f' = \phi_{t_2}$.
The rough idea of the proof of Theorem 1.1 (ii) is the following. Every connected component of the moduli of $K3^{[n]}$-type fibrations contains a Beauville–Mukai system. Then each two fibrations in such a component are deformation equivalent as fibrations hence the polarization type must be principal by Theorem 1.1 (i) since the polarization types of Beauville–Mukai systems are principal.

Acknowledgements: I thank my advisor Klaus Hulek, Eyal Markman, Christian Lehn and Malek Joumaah for helpful discussions. In particular Christian Lehn for also introducing me to this problem. Especially I want to thank Eyal Markman for hospitality at the University of Massachusetts in Amherst and explaining me so much about the moduli of Lagrangian fibrations.

2. Irreducible symplectic manifolds and their fibrations

In this section we recall the basic facts about irreducible holomorphic symplectic manifolds and their fibrations which are Lagrangian.

Definition 2.1 A compact Kähler manifold $X$ is called hyperkähler or irreducible holomorphic symplectic if $X$ is simply connected and $H^0(X, \Omega^2_X)$ is generated by a nowhere degenerate holomorphic two–form $\sigma$.

Note that $\sigma$ is automatically symplectic since every holomorphic form on a compact Kähler manifold is closed.

The most basic example is provided by the Douady space $S^{[n]}$ of $n$ points for a K3 surface $S$ which parametrizes zero–dimensional subspaces of $S$ of length $n$. A. Beauville [Bea84] showed that $S^{[n]}$ is an irreducible holomorphic symplectic manifold of dimension $2n$. A complex manifold is called of $K3^{[n]}$–type if it is deformation equivalent to $S^{[n]}$ for a K3 surface $S$.

The second cohomology $H^2(X, \mathbb{Z})$ of any irreducible holomorphic symplectic manifold $X$ admits the well known Beauville–Bogomolov–Fujiki quadratic form $q_X$ which is non–degenerate and of signature $(3, b_2(X)−3)$, see [MGJ03, 23.3]. The associated bilinear form is denoted by $(\cdot, \cdot)$. On an abstract lattice we also denote the bilinear form by $(\cdot, \cdot)$. The lattice $H^2(X, \mathbb{Z})$ with the Beauville–Bogomolov–Fujiki form is a deformation invariant of the manifold $X$. For manifolds of $K3^{[n]}$–type this lattice is isometric to the $K3^{[n]}$–type lattice

$$\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} \oplus \langle 2 − 2n \rangle,$$

see [Bea84, Prop. 6] where $\langle 2 − 2n \rangle$ denotes the lattice of rank one with generator $l$ such that $(l, l) = 2 − 2n$, $E_8(-1)$ the negative definite root lattice and $U$ the unimodular rank two hyperbolic lattice.

A marking on an irreducible holomorphic manifold $X$ is the choice of an isometry $\eta : H^2(X, \mathbb{Z}) \to \Lambda$. The tuple $(X, \eta)$ is then called a marked pair or a marked irreducible holomorphic symplectic manifold.
If $X$ is a fixed irreducible holomorphic symplectic manifold set $\Lambda := H^2(X, \mathbb{Z})$ and consider the Kuranishi family $\pi : \mathcal{X} \to \text{Def}(X)$ with $\mathcal{X}_0 := \pi^{-1}(0) = X$. We will view the base $\text{Def}(X)$ sometimes as a germ but also as a representative which we choose small enough i.e. simply connected. Then by Ehresmann’s theorem we can choose trivialization $\Sigma : R^2\pi_* \mathbb{Z} \to \Lambda_{\text{Def}(X)}$ also called a marking and define the local period map by

$$\mathcal{P} : \text{Def}(X) \to \mathbb{P}(\Lambda_C), \quad t \mapsto [\Sigma_t(H^{2,0}(\mathcal{X}_t))]$$

where $\Lambda_C := \Lambda \otimes \mathbb{C}$. It takes values in the period domain of type $\Lambda$ \cite[22.3]{MGJ03} namely

$$\Omega_\Lambda := \{ p \in \mathbb{P}(\Lambda_C) \mid (p, p) = 0 \text{ and } (p, \bar{p}) > 0 \}$$

which is connected since the signature of $q_X$ is $(3, \text{rk } \Lambda - 3)$.

**Theorem 2.2** (Local Torelli, \cite{Bea84}, 8.) The period map $\mathcal{P} : \text{Def}(X) \to \Omega_\Lambda$ is an open embedding.

Two marked pairs $(X_i, \eta_i)$, $i = 1, 2$, are called isomorphic if there is an isomorphism $f : X_1 \to X_2$ such that $\eta_2 = \eta_1 \circ f^*$. There exists a moduli space of marked pairs $\mathcal{M}_\Lambda := \{(X, \eta) \text{ marked pair } \} / \cong$ which can be constructed by gluing all deformation spaces $\text{Def}(X)$ of irreducible holomorphic symplectic manifolds $X$ with $H^2(X, \mathbb{Z})$ isometric to $\Lambda$. This gives a non–Hausdorff complex manifold of dimension $\text{rk } \Lambda - 2$. The global period mapping

$$\mathcal{P} : \mathcal{M}_\Lambda \to \Omega_\Lambda, \quad (X, \eta) \mapsto [\eta(H^{2,0}(X))]$$

is locally given by $\mathcal{P} : \text{Def}(X) \to \Omega_\Lambda$ and hence is again a local biholomorphism by the Local Torelli. If one takes an arbitrary connected component $\mathcal{M}_\Lambda^0$ of $\mathcal{M}_\Lambda$ then by a result of D. Huybrechts \cite[Prop. 25.12]{MGJ03} the restriction $\mathcal{P} : \mathcal{M}_\Lambda^0 \to \Omega_\Lambda$ is surjective.

If $L$ denotes a line bundle on $X$ by abuse of notation we also denote the universal family of the pair $(X, L)$ by $\pi : \mathcal{X}_L \to \text{Def}(X, L)$ which comes with an universal line bundle $\mathcal{L}$ on $\mathcal{X}_L$ such that $(\mathcal{X}_L)_0 = X$ and $\mathcal{L}_0 = L$, see \cite[Cor. 1]{Bea84}. This family is the restriction of the Kuranishi family $\pi : \mathcal{X} \to \text{Def}(X)$ to $\text{Def}(X, L)$. We consider again $\text{Def}(X, L)$ as a germ but as well as a proper space. A representative is locally given by $(e_1(L), \cdot) = 0$ in $\Omega_\Lambda$ hence it is a smooth hypersurface in $\text{Def}(X)$, see \cite[26.1]{MGJ03} and one defines $\mathcal{X}_L$ as the preimage of it.

For completeness we give the following definitions.

**Definition 2.3** Let $X_i$, $i = 1, 2$, be two irreducible holomorphic symplectic manifolds. An isometry $P : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is called a parallel transport operator if there exists a family $\pi : \mathcal{X} \to S$ of irreducible holomorphic symplectic manifolds, points $t_i$ such that $\mathcal{X}_{t_i} = X_i$ and a continuous path $\gamma$ such that the parallel transport $P_\gamma$ along $\gamma$ in the local system $R^2\pi_* \mathbb{Z}$ coincides with $P$. For $X := X_1 = X_2$ it is also called a monodromy operator and the subgroup $\text{Mon}^2(X)$ of $O(H^2(X, \mathbb{Z}))$ generated by monodromy operators is called the monodromy group.
Due to D. Matsushita much is known about nontrivial fiber structures on irreducible holomorphic symplectic manifolds.

**Theorem 2.4 (Matsushita, [Mat00])** Let $f : X \to B$ be a surjective holomorphic map with connected fibers from an irreducible holomorphic symplectic manifold $X$ of dimension $2n$ to a normal complex space $B$ such that $0 < \dim B < 2n$. Then the following statements hold.

(i) $B$ is projective and of dimension $n$.
(ii) For all $t \in B$ the fiber $X_t := f^{-1}(t)$ is Lagrangian subspace i.e. $\sigma|_{X_t} = 0$.
(iii) If $X_t$ is smooth then it is a projective complex torus i.e. an abelian variety.

Such a fibration $f : X \to B$ as in the Theorem is called a *Lagrangian fibration*.

If the base of the Lagrangian fibration is smooth even more is known due to a deep result of J.-M. Hwang which was recently slightly generalized by C. Lehn and D. Greb to the non–projective case.

**Theorem 2.5 (Hwang, [Hwa08], [LG13])** Let $f : X \to B$ be a Lagrangian fibration such that $B$ is smooth and $\dim X = 2n$. Then $B \cong \mathbb{P}^n$.

The basic example of a Lagrangian fibrations is provided by the Douady space of $n$ points $S^{[n]}$ parametrizing zero–dimensional subschemes $Z$ of length $l(Z) := \dim \mathcal{O}_Z(Z) = n$ of an elliptic K3 surface $f : S \to \mathbb{P}^1$. Then one uses the Douady–Barlet map

$$\rho : S^{[n]} \longrightarrow S^{(n)}, \quad Z \longmapsto \sum_{z \in Z} (\dim \mathcal{O}_{Z,z}) z$$

which is a resolution of singularities of the $n$–th symmetric product $S^{(n)} := (S \times \cdots \times S)/\Sigma_n$ to obtain a morphism

$$S^{[n]} \xrightarrow{\rho} S^{(n)} \xrightarrow{f \times \cdots \times f} (\mathbb{P}^1)^{(n)} \cong \mathbb{P}^n.$$

which is a Lagrangian fibration and the smooth fibers are given by products of elliptic curves which are the fibers of $f : S \to \mathbb{P}^1$. Note that two–dimensional Lagrangian fibrations are exactly the elliptic K3 surfaces.

### 3. Moduli space of Lagrangian fibrations

This section has the purpose to explain what we mean by the *moduli space of Lagrangian fibrations*. Most of the constructions and explanations can be found in [Mar11] and [Mar14].

D. Matsushita [Mat09] constructed a local *moduli space of Lagrangian fibrations*. 
Definition 3.1  
(i) A family of Lagrangian fibrations over a connected complex space $S$ with finitely many irreducible components is an $S$–morphism

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & P \\
S & \xrightarrow{} & P
\end{array}
\]

where $X \to S$ is a family of irreducible holomorphic symplectic manifolds and $P \to S$ is a family of projective varieties such that for every $s \in S$ the restriction $\phi|_{X_s} : X_s \to P_s$ to the irreducible homorphic symplectic manifold $X_s$ is a Lagrangian fibration.

(ii) Two Lagrangian fibrations $f_1$ and $f_2$ are deformation equivalent if there is a family of Lagrangian fibrations over a connected complex space $S$ containing $f_1$ and $f_2$.

Let $\pi : \mathfrak{X} \to \text{Def}(X)$ denote the Kuranishi family of an irreducible holomorphic symplectic manifold $X = \pi^{-1}(0)$. For a line bundle $L$ on $X$ let $\pi : \mathfrak{X}_L \to \text{Def}(X, L)$ denote the universal family of the pair $(X, L)$. Further denote by $\mathcal{L}$ the universal line bundle on $\mathfrak{X}_L$ and set $\mathcal{L}_t := \mathcal{L}|_{\mathfrak{X}_{L,t}}$.

Theorem 3.2 [Mat09, Cor. 1.1] Let $f : X \to B$ be a Lagrangian fibration and $L$ be the pullback of a very ample line bundle on $B$. Then $\pi_*\mathcal{L}$ is locally free and there exists a family of Lagrangian fibrations

\[
\begin{array}{ccc}
\mathfrak{X}_L & \xrightarrow{\zeta} & \mathbb{P}(\pi_*\mathcal{L}) \\
\pi & \xrightarrow{} & \text{Def}(X, L)
\end{array}
\]

over $\text{Def}(X, L)$ containing $f$.

Note that $\dim \text{Def}(X) = h^1(X, \mathcal{T}_X) = b_2(X) - 2$. For a fixed deformation type of irreducible holomorphic symplectic manifolds with lattice $\Lambda$ one can glue all total spaces $\text{Def}(X, L)$ of such families $\mathfrak{X}_L \to \text{Def}(X, L)$ for $f : X \to \mathbb{P}^n$ a Lagrangian fibration and $L$ a line bundle on $X$ as in the theorem to obtain a moduli space of Lagrangian fibrations which is a non–Hausdorff smooth hypersurface $H_{\Lambda}$ of dimension $b_2(X) - 3$ in the moduli space of marked irreducible holomorphic symplectic manifolds $\mathcal{M}_\Lambda$.

In the $K3^{[n]}$–type case E. Markman [Mar14] provides methods to describe a connected component of the moduli space of Lagrangian fibrations lattice–theoretically.

From now on let $X$ always denote a manifold of $K3^{[n]}$–type and let $\Lambda$ be the $K3^{[n]}$–type lattice.
For a period \( p \in \Omega_\Lambda \) let \( \Lambda(p) \) denote the integral Hodge structure of weight two of \( \Lambda \) determined by the period \( p \) that is
\[
\Lambda^{2,0}(p) = p, \quad \Lambda^{0,2}(p) = \bar{p} \quad \text{and} \quad \Lambda^{1,1}(p) = \{ m \in \Lambda_\mathbb{C} \mid (m, p) = (m, \bar{p}) = 0 \}.
\]
We also set \( \Lambda^{1,1}(p, R) := \{ m \in \Lambda_R \mid (m, p) = 0 \} \) for \( R \in \{ \mathbb{Z}, \mathbb{R} \} \). Further consider the cones
\[
\mathcal{C}_p := \{ m \in \Lambda^{1,1}(p, \mathbb{R}) \mid (m, m) > 0 \} \quad \text{and} \quad \tilde{\mathcal{C}}_\Lambda := \{ m \in \Lambda_\mathbb{R} \mid (m, m) > 0 \}.
\]
The latter one \( \tilde{\mathcal{C}}_\Lambda \) is called the positive cone. It is connected and \( H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \cong \mathbb{Z} \), see [Mar11, Lem. 4.1].

**Lemma 3.3** [Mar14, 4.3]

(i) The cone \( \mathcal{C}_p \) has two connected components.

(ii) A choice of an orientation of \( \tilde{\mathcal{C}}_\Lambda \) determines a connected component of \( \mathcal{C}_p \) of all \( p \in \Omega_\Lambda \).

Choose a primitive isotropic class \( \lambda \in \Lambda \) and set
\[
\Omega_{\lambda\perp} := \{ p \in \Omega_\Lambda \mid (p, \lambda) = 0 \}.
\]
Further choose a generator of \( H^2(\tilde{\mathcal{C}}_\Lambda, \mathbb{Z}) \) i.e. an orientation of \( \tilde{\mathcal{C}}_\Lambda \) so that by Lemma 3.3 we get a connected component \( \mathcal{C}_p^+ \) for every period \( p \). If additionally the period \( p \) is contained in \( \Omega_{\lambda\perp} \) then by definition \( \lambda \) belongs to \( \Lambda^{1,1}(p, \mathbb{R}) \) and \( \lambda \) is contained in the boundary of one of the connected components of \( \mathcal{C}_p \) since \( \lambda \) is isotropic. Then consider only such periods which belong to the closure of the distinguished connected component \( \mathcal{C}_p^+ \) of \( \mathcal{C}_p \) of Lemma 3.3 i.e.
\[
\Omega_{\lambda\perp}^+ := \{ p \in \Omega_{\lambda\perp} \mid \lambda \in \Lambda^{1,1}(p, \mathbb{Z}) \quad \text{and} \quad \lambda \in \partial \mathcal{C}_p^+ \}
\]
which is a connected component of \( \Omega_{\lambda\perp} \) by construction.

Let \( \mathfrak{M}_\Lambda \) denote the moduli space of isomorphism classes of marked pairs \( (X, \eta) \) i.e. \( X \) is an irreducible holomorphic symplectic manifold of \( K3^{[n]} \)–type and \( \eta : H^2(X, \mathbb{Z}) \to \Lambda \) is marking i.e. an isometry. Choose a connected component \( \mathfrak{M}_\Lambda^0 \) of \( \mathfrak{M}_\Lambda \) and consider the period mapping
\[
\mathcal{P} : \mathfrak{M}_\Lambda^0 \to \Omega_\Lambda, \quad (X, \eta) \mapsto [\eta(H^{2,0}(X))].
\]

Assume that \( \mathfrak{M}_\Lambda^0 \) is compatible with the orientation of \( \tilde{\mathcal{C}}_\Lambda \). This means the following. If \( (X, \eta) \in \mathfrak{M}_\Lambda^0 \) then there is a canonical choice for the connected component of \( \tilde{\mathcal{C}}_X = \{ x \in H^2(X, \mathbb{R}) \mid (x, x) > 0 \} \) namely the positive cone that is the connected component \( \mathcal{C}_X^+ \) which contains the Kähler cone \( \mathcal{K}_X \) of \( X \). Set \( p := \mathcal{P}(X, \eta) \) then \( \eta(H^{1,1}(X, \mathbb{R})) = \Lambda^{1,1}(p, \mathbb{R}) \) and compatibility means that \( \eta(\mathcal{C}_X^+) = \mathcal{C}_p^+ \). Now set
\[
\mathfrak{M}_\Lambda^0_{\lambda\perp} := \mathcal{P}^{-1}(\Omega_{\lambda\perp}^+) = \{(X, \eta) \in \mathfrak{M}_\Lambda^0 \mid \eta^{-1}(\lambda) \text{ is of type } (1,1) \text{ and in } \partial \mathcal{C}_X^+ \}
\]
which is a connected hypersurface of \( \mathfrak{M}_\Lambda^0 \) by [Mar14, Lem. 4.4]. Set
\[
\mathfrak{U}_\Lambda^0_{\lambda\perp} := \{(X, \eta) \in \mathfrak{M}_\Lambda^0_{\lambda\perp} \mid \eta^{-1}(\lambda) \text{ is nef}\}.\]
We claim that this space is a connected component of the moduli space $\mathcal{H}_3^{[n]}$-type fibrations $\mathcal{H}_3$, see Theorem 3.7 below. In particular it is connected and open in $\mathcal{H}_3^0$.

**Remark 3.4** Recall that a holomorphic line bundle $L \in \text{Pic}(X) \cong \text{NS}(X)$ on an irreducible holomorphic symplectic manifold $X$ is called nef if $c_1(L)$ belongs to closure of the Kähler cone $\tilde{K}_X$ in $H^{1,1}(X, \mathbb{R})$.

**Lemma 3.5** Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration with $X$ not necessarily of $K3^{[n]}$-type and let $L := f^*A$ be the pullback of a nontrivial line bundle $A$ on $\mathbb{P}^n$.

(i) $L$ is isotropic with respect to the Beauville–Bogomolov quadratic form $q_X$.

(ii) If $A = \mathcal{O}_{\mathbb{P}^n}(k)$, $k \geq 0$, i.e. it admits nontrivial sections then $L$ is nef.

(iii) Assume $A = \mathcal{O}_{\mathbb{P}^n}(k)$, $k \geq 1$, i.e. it is very ample and let $\mathcal{L}$ be the universal bundle of the Kuranishi family of the pair $(X, L)$. Then nefness of $\mathcal{L}_t$ is an open condition in $\text{Def}(X, L)$. Furthermore the non–nef locus in $\text{Def}(X, L)$ is the union of finitely many analytic subsets.

**Proof:**

(i) By Fujiki’s relation [MGJ03, Prop. 3.9]

\[ c_X \cdot q_X (L)^n = \int_X c_1(L)^{2n} = \int_X f^*(c_1(A))^{2n} = \int_{\mathbb{P}^n} c_1(A)^{2n} = 0 \]

since $f : X \to \mathbb{P}^n$ is holomorphic (hence orientable) and $c_1(A)^{2n} = 0$. As $c_X \neq 0$ we have $q_X(L) = 0$.

(ii) If $C \subset X$ is a rational curve then $L.C = \deg(f^*A|_C) \geq 0$ since $A$ admits nontrivial sections by assumption and by [Huy03, Prop. 3.2] this implies that $c_1(L)$ is contained in the closure of the Kähler cone i.e. it is nef.

(iii) For an primitive and isotropic line bundle nefness is equivalent to basepoint freeness by [Mar14, Thm. 1.3] (note that there you can drop the non–speciality assumption by [Mar14, Rmk. 1.8]). Let $\zeta : \mathcal{X}_L \to P$ denote the family of Lagrangian fibration containing $f$ by Theorem 3.2. Let $H$ denote a representative of $\text{Def}(X, L)$. Locally around $0 \in H$ the line bundle $\mathcal{L}_t$ is given by $\zeta^*\mathcal{O}_P(1)$ and so it is nef and isotropic by (ii) and (i). In particular $d := h^0(\mathcal{X}_t, \mathcal{L}_t)$ is constant. Choose global sections $s_1, \ldots, s_d$ of $\mathcal{L}$ such that $s_1|_{\mathcal{X}_t}, \ldots, s_d|_{\mathcal{X}_t}$ is a frame of $\mathcal{L}_t$ for every $t \in H$. Then $\mathcal{L}_t$ is nef iff $s_i(t) = 0$ for every $i$ i.e.

\[ V = \{ t \in H \mid s_1(t) = \ldots = s_d(t) = 0 \} \]

is a union of finitely many closed analytic subsets and $H \setminus V$ is open in $H$ and consists precisely of all points $t \in H$ with $\mathcal{L}_t$ nef.

\[ \square \]

The space $\mathcal{H}_3^0$ compares to Matsushita’s local moduli space in the following way.

**Proposition 3.6** Let $(X, \eta)$ be a marked irreducible holomorphic symplectic manifold, $f : X \to \mathbb{P}^n$ be a Lagrangian fibration on $X$ and $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$. Set
\[ \lambda := \eta(c_1(L)). \] Then \( \Omega^0_{\lambda, \perp} \) and \( \text{Def}(X, L) \) are locally isomorphic around \((X, \eta)\) and 0 respectively.

**Proof:** Note that \( \lambda \) is isotropic by Lemma 3.5. Under the assumption that \( \text{Def}(X) \) is chosen sufficiently small there exists an unique extension \( \Sigma : R^2 \pi_* \mathbb{Z} \to A_{\text{Def}(X)} \) of \( \eta \) i.e. \( \Sigma_0 = \eta \) and we have a local isomorphism \( F : \text{Def}(X) \to \mathcal{M}_\Lambda \) by the construction of the moduli of marked pairs. More precisely it is given by

\[ F : \text{Def}(X) \longrightarrow \mathcal{M}_\Lambda, \quad t \longmapsto (X_t, \Sigma_t) \]

and we have the following diagram

\[
\begin{array}{ccc}
\text{Def}(X) & \longrightarrow & \text{Def}(X, L) \\
\downarrow F & & \downarrow F \\
\mathcal{M}_\Lambda & \overset{\cong}{\longleftarrow} & \mathcal{M}^0_{\Lambda, \perp}
\end{array}
\]

By Lemma 3.5 (iii) we can choose \( \text{Def}(X, L) \) small such that \( \mathcal{L}_t \) is nef for every \( t \in \text{Def}(X, L) \). If we restrict \( F \) to \( \text{Def}(X, L) \) it takes values in \( \Omega^0_{\lambda, \perp} \): For \( t \in \text{Def}(X, L) \) the mapping \( t \mapsto c_1(\mathcal{L}_t) \) is a section of \( R^2 \pi_* \mathbb{Z}_{|\text{Def}(X, L)} \) so in particular \( t \mapsto \Sigma_t(c_1(\mathcal{L}_t)) \in \Lambda \) is continuous. Hence it is constant as \( \Lambda \) is discrete and \( \text{Def}(X, L) \) sufficiently small. This implies that \( \Sigma_t(c_1(\mathcal{L}_t)) = \Sigma_t(L) = \lambda \) so \( \Sigma_t^{-1}(\lambda) = c_1(\mathcal{L}_t) \) is nef i.e. \( F(t) \in \Omega^0_{\lambda, \perp} \). Since \( \text{Def}(X, L) \) and \( \mathcal{M}^0_{\Lambda, \perp} \) are hypersurfaces in \( \text{Def}(X) \) and \( \mathcal{M}_\Lambda \) respectively \( F(\text{Def}(X, L)) \) is an open set in \( \mathcal{M}^0_{\Lambda, \perp} \) and contained in \( \Omega^0_{\lambda, \perp} \). Hence \( \text{Def}(X, L) \) and \( \Omega^0_{\lambda, \perp} \) are locally isomorphic.

\[ \square \]

**Theorem 3.7** Let \( \lambda \) be a primitive and isotropic element in the \( K_3^{[n]} \)-lattice \( \Lambda \). The space \( \Omega^0_{\lambda, \perp} \) has the following properties.

(i) It parametrizes isomorphism classes of marked pairs \((X, \eta)\) of \( \mathcal{M}^0_{\Lambda} \) with \( X \) of \( K_3^{[n]} \)-type admitting a Lagrangian fibration \( f : X \to \mathbb{P}^n \) such that \( \eta \left(c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1))\right) = \lambda \).

(ii) It is open in \( \mathcal{M}^0_{\Lambda, \perp} \) and connected.

(iii) It is a non-Hausdorff smooth hypersurface in the moduli space of marked pairs \( \mathcal{M}_\Lambda \).

**Proof:**

(i) Let \((X, \eta) \in \Omega^0_{\lambda, \perp} \). As \( H^1(X, \mathcal{O}_X) = 0 \) the exponential sequence on \( X \) is

\[ \cdots \longrightarrow \text{Pic}(X) \overset{c_1}{\longrightarrow} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow \cdots. \]

Since \( \eta^{-1}(\lambda) \) is of type \((1, 1)\) it is in the kernel of \( H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \) so we can find an unique line bundle \( L \) on \( X \) such that \( c_1(L) = \eta^{-1}(\lambda) \) then by [Mar14, Thm. 1.3] \( L \) induces a Lagrangian fibration \( f : X \to |L^*| = \mathbb{P}^n \) since \( \eta^{-1}(\lambda) \) is nef by assumption. Note that you can drop the non-speciality assumption in [Mar14, Thm. 1.3] by [Mar14, Rmk. 1.8].
Conversely let \((X, \eta)\) be a marked pair admitting a Lagrangian fibration \(f : X \to \mathbb{P}^n\). Then \(\lambda := \eta(c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1)))\) is primitive and isotropic by Lemma 3.5. We then have \(\mathcal{P}(X, \eta) \in \Omega_{\lambda^+}^+\) i.e. \((X, \eta)\) is in \(\mathfrak{M}_{\lambda^+}\). In particular \(\eta^{-1}(\lambda) = c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1))\) is nef also by Lemma 3.5. This implies that \((X, \eta)\) is in \(\mathfrak{U}_{\lambda^+}\).

(ii) By Proposition 3.6 and Lemma 3.5 (iii) \(\mathfrak{U}_{\lambda^+}\) is open. We know that \(\mathfrak{M}_{\lambda^+}\) is connected and by Lemma 3.5 (iii) the complement \(\mathfrak{M}_{\lambda^+} \setminus \mathfrak{U}_{\lambda^+}\) is the union of countably many analytic subsets i.e. \(\mathfrak{U}_{\lambda^+}\) is the complement of a real codimension two subset in the connected space \(\mathfrak{M}_{\lambda^+}\). Hence \(\mathfrak{U}_{\lambda^+}\) must be connected by [Ver11, Lem. 4.10].

(iii) This was shown in Proposition 3.6.

\[\square\]

When do two Lagrangian fibrations on \(K3^{[n]}\)-type manifolds lie in the same connected component \(\mathfrak{U}_{\lambda^+}\)?

**Definition 3.8** [Mar11, 5.2] Let \(X_i, i = 1, 2\), denote denote two irreducible holomorphic symplectic manifolds, \(L_i\) line bundles on \(X_i\) and \(e_i\) classes in \(H^2(X_i, \mathbb{Z})\). The pairs \((X_1, L_1)\) and \((X_2, L_2)\) are called deformation equivalent if there exist a family \(\pi : \mathcal{X} \to S\) of irreducible holomorphic symplectic manifolds over a connected complex space \(S\) with finitely many irreducible components, a section \(e\) of \(R^2\pi_*\mathbb{Z}\), points \(t_i\) in \(S\) such that \(\mathcal{X}_{t_i} = X_i\) and \(e_{t_i} = c_1(L_i)\).

**Proposition 3.9** Let \(f_i : X_i \to \mathbb{P}^n, i = 1, 2\), denote two Lagrangian fibrations with \(X_i\) of \(K3^{[n]}\)-type and set \(L_i := f_i^*\mathcal{O}_{\mathbb{P}^n}(1)\). Then the following statements are equivalent.

(i) The Lagrangian fibrations \(f_i\) are deformation equivalent.

(ii) The pairs \((X_i, L_i)\) are deformation equivalent.

(iii) There exist markings \(\eta_i : H^2(X_i, \mathbb{Z}) \to \Lambda\) such that

\[\eta_1(c_1(L_1)) = \eta_2(c_1(L_2))\]

and \(\eta_2^{-1} \circ \eta_1\) is a parallel transport operator.

(iv) There exist markings \(\eta_i : H^2(X_i, \mathbb{Z}) \to \Lambda\) such that the marked pairs \((X_i, \eta_i)\) are contained in the same connected component \(\mathfrak{U}_{\lambda^+}\) for a primitive isotropic class \(\lambda\) in the \(K3^{[n]}\)-type lattice.

**Proof:** (i) \(\Rightarrow\) (ii) Consider a family of Lagrangian fibrations \(\phi : \mathcal{X} \to P\) over a complex space \(S\) with points \(t_i\) such that \(\phi_{t_i} = f_i\) where we can assume that \(P\) is a projective bundle as the \(f_i\) are fibered over \(\mathbb{P}^n\). Let \(\pi : \mathcal{X} \to S\) denote the family of irreducible holomorphic symplectic manifolds belonging to the family \(\phi\). Let \(\mathcal{L} := \phi^*\mathcal{O}_P(1)\) then \(e_t := c_1(\mathcal{L}|_{X_t})\) defines a section of \(R^2\pi_*\mathbb{Z}\) such that \(e_{t_i} = L_i\) hence the pairs \((X_i, L_i)\) are deformation equivalent.

(ii) \(\Rightarrow\) (iii) Let \(\pi : \mathcal{X} \to S\) be a family of irreducible holomorphic symplectic manifolds with \(S\) connected, \(t_i\) points such that \(\mathcal{X}_{t_i} = X_i\) and \(e\) a section of \(R^2\pi_*\mathbb{Z}\)
with \( e_t = c_1(L_t) \). As \( R^2\pi_*\mathcal{Z} \) is a local system we can find a neighbourhood \( U \) of \( t_2 \) and a marking \( \Sigma : R^2\pi_*\mathcal{Z}|_U \to \Lambda_U \). As \( S \) is connected we can choose a path \( \gamma \) connecting \( t_1 \) with \( t_2 \). Then \( \gamma \) is parallel along \( e \) i.e. \( \gamma^*e \) is a flat section of \( \gamma^*R^2\pi_*\mathcal{Z} \). Consider the parallel transport \( P_\gamma : H^2(X_1, \mathcal{Z}) \to H^2(X_2, \mathcal{Z}) \) along \( \gamma \). Note that \( P_\gamma(e_{t_1}) = e_{t_2} \). Define \( \eta_2 := \Sigma t_2 \) and \( \eta_1 := \eta_2 \circ P_\gamma \). Hence we have \( \eta_2^{-1} \circ \eta_1 = P_\gamma \) and 

\[
\eta_1(c_1(L_1)) = \eta_1(e_{t_1}) = \eta_2(P_\gamma(e_{t_1})) = \eta_2(e_{t_2}) = \eta_2(c_1(L_2)).
\]

(iii) \( \Rightarrow \) (iv) As \( \eta_2^{-1} \circ \eta_1 \) is a parallel transport operator the manifolds \( X_1 \) belong to a family of irreducible holomorphic symplectic manifolds. Hence the marked pairs \( (X_i, \eta_i) \) belong to a connected component \( \mathcal{M}_k^0 \) of the moduli of marked pairs. The condition that \( X : \eta_1(c_1(L_1)) = \eta_2(c_1(L_2)) \) then implies that the pairs \( (X_i, \eta_i) \) are contained in \( \mathcal{M}_k \), see also the proof of Theorem 3.7 (ii).

(iv) \( \Rightarrow \) (i) Choose a path \( \gamma \) in \( \mathcal{M}_k \) connecting the marked pairs \( (X_i, \eta_i) \). By Theorem 2.4 we can choose finitely many points \( x_1, \ldots, x_N \) with \( (X_1, \eta_1) = x_1 \) and \( (X_2, \eta_2) = x_2 \) of \( \gamma \) such that every \( x_k \) admits a neighbourhood \( \text{Def}_k \) and a family of Lagrangian fibrations \( \zeta_k : \mathcal{X}_k \to P_k \) over \( \text{Def}_k \) containing the Lagrangian fibration associated to the point \( \gamma \), and such that \( \text{Def}_k \cap \text{Def}_{k+1} \neq \emptyset \) for \( k = 1, \ldots, N-1 \) as subsets of \( \mathcal{M}_k \). Choose points in \( z_k \) in \( \text{Def}_k \cap \text{Def}_{k+1} \) for \( k = 1, \ldots, N-1 \).

- Set \( S := \bigsqcup_{k=1}^N \text{Def}_k / \sim \) where \( \sim \) glues \( \text{Def}_k \) and \( \text{Def}_{k+1} \) at the point \( z_k \) for \( k = 1, \ldots, N-1 \).
- Further set \( \mathcal{X} := \bigsqcup_{k=1}^N \mathcal{X}_k / \sim \) where \( \sim \) glues \( \mathcal{X}_k \) and \( \mathcal{X}_{k+1} \) at \( (\mathcal{X}_k)_z \) and \( (\mathcal{X}_{k+1})_{z_k} \). Note that those fibers are isomorphic.
- Denote by \( \pi_k : \mathcal{X}_k \to \text{Def}_k \) the family of irreducible holomorphic symplectic manifolds belonging to the family \( \phi_k \). Then the map \( \pi : \mathcal{X} \to S \) defined by \( \pi|_{\mathcal{X}_k} := \pi_k \) is well defined and is a family of irreducible holomorphic symplectic manifolds.
- Set \( P := \bigsqcup_{k=1}^N P_k / \sim \) where \( \sim \) glues \( P_k \) and \( P_{k+1} \) at the projective spaces \( (P_k)_z \) and \( (P_{k+1})_{z_k} \) for \( k = 1, \ldots, N-1 \). We get a morphism \( P \to S \) which is induced by the morphisms \( P_k \to \text{Def}_k, k = 1, \ldots, N \). This map is a family of projective spaces hence a projective bundle.

Putting everything together we can define a map \( \phi : \mathcal{X} \to P \) locally given by the \( \zeta_k : \mathcal{X}_k \to P_k, k = 1, \ldots, N \). This defines by construction a family of Lagrangian fibrations over \( S \) containing \( f_1 = \phi_{t_1} \) and \( f_2 = \phi_{t_2} \).

### 4. Polarization types of Lagrangian fibrations

Let \( X \) be an irreducible holomorphic symplectic manifold of dimension \( 2n \) and \( f : X \to B \) a Lagrangian fibration. For a general point \( t \in B \) the associated fiber \( F := f^{-1}(t) \) is an abelian variety [Cam05, Prop 2.1] even when \( X \) is not projective, see Proposition 4.3. In this section it is explained how to associate to \( f \) a tuple \( d(f) \in \mathbb{Z}^n \) which is called the polarization type of the fibration.
Definition 4.1 Let $F$ denote a smooth fiber. We say that a Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ is special with respect to $F$ if the restriction $\omega|_F$ is integral i.e. contained in $H^2(F, \mathbb{Z})$ and primitive i.e. indivisible. We call such an $\omega$ just special if there is no confusion with the fiber $F$.

Example 4.2 Of course every ample line bundle $\mathcal{L} \in \text{Pic}(X)$ defines a Kähler class $c_1(\mathcal{L}) \in H^{1,1}(X, \mathbb{Z})$ which is integral on all fibers and for each smooth fiber $F$ we can find a natural number $k$ such that $\omega := \frac{1}{k}c_1(\mathcal{L})$ is special with respect to $F$.

Proposition 4.3 [Cam05, Prop 2.1] For every smooth fiber $F$ there is a Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ which is special with respect to $F$.

Proof: We have a surjective projection $p : H^2(X, \mathbb{R}) \to H^{1,1}(X, \mathbb{R})$ which is induced by the Hodge decomposition. As $H^2(X, \mathbb{Q})$ is dense in $H^2(X, \mathbb{R})$ also $p(H^2(X, \mathbb{Q}))$ is dense in $H^{1,1}(X, \mathbb{R})$. Since the Kähler cone $\mathcal{K}_X$ is open in $H^{1,1}(X, \mathbb{R})$ we have $p(H^2(X, \mathbb{Q})) \cap \mathcal{K}_X \neq \emptyset$ so that we can find a class $\alpha \in H^2(X, \mathbb{Q})$ with $p(\alpha) \in \mathcal{K}_X$. As $F$ is Lagrangian and $H^{2,0}(X)$ is generated by the holomorphic symplectic form the restriction $r : H^{2,0}(X) \to H^{2,0}(F)$ is zero, hence the non–(1,1) parts of $\alpha$ are in the kernel of $r$ so we have $r(\alpha) = (r(p(\alpha))$. Then take a positive number $m > 0$ such that $m\alpha \in H^2(X, \mathbb{Z})$ and $mr(\alpha)$ is primitive. Consequently $\omega := m\alpha$ is a special Kähler class with respect to $F$ since $r(\omega) = r(m\alpha) \in H^2(F, \mathbb{Z})$.

A polarization on an abelian variety $A$ is by definition the first Chern class $c_1(L) \in H^2(A, \mathbb{Z})$ of an ample line bundle $L$ on $A$. The restriction $\omega|_F$ of a Kähler class which is special with respect to the smooth fiber $F$ defines a primitive polarization on the abelian variety $F$.

Lemma 4.4 Let $\mathcal{K}_X$ be the Kähler cone of $X$, $F$ a smooth fiber and $r : H^2(X, \mathbb{C}) \to H^2(F, \mathbb{C})$ the restriction.

(i) We have $\text{rk } r = 1$.

(ii) $G := r(\mathcal{K}_X) \subset H^{1,1}(F, \mathbb{R})$ is a ray and contains integral points.

Proof: (i) Consider the space

$$D_F := \{ t \in \text{Def}(X) \mid \text{there exists a deformation } \mathcal{F}_t \subset \mathfrak{X}_t \text{ of } F \} .$$

Let $L$ be the pullback of a very ample line bundle on $B$ by $f$. Then $\text{Def}(X, L) \subset D_F$: By Theorem 3.2 we have a family $\zeta : \mathfrak{X}_L \to P$ of Lagrangian fibrations over $\text{Def}(X, L)$ such that $\zeta_0 = f$. Write $F = X_{t_0}$. Then we can choose a neighbourhood $U$ of 0 and a local holomorphic section $s : U \to P$ of the $\mathbb{P}^n$–fibration $\mathfrak{X}_L \to P$ such that $s(0) = t_0$. Then the fiber product $\mathcal{F} := U \times_P \mathfrak{X}_L$ gives a deformation $pr_U : \mathcal{F} \to U$ of $\mathcal{F}_0 = X_{t_0} = F$ hence $U \subset \text{Def}(X, L)$ i.e. $\text{Def}(X, L) \subset D_F$ as germs. By [Voi92, Prop 1.2, Lemma 1.5] $\text{Def}(X, L) \subset D_F$ is a complex submanifold and for
its codimension in $\text{Def}(X)$ one has $\text{codim} D_F = \text{rk} r$. As $\text{Def}(X, L) \subset D_F$ we have in particular

$$1 = \text{codim} \text{Def}(X, L) \geq \text{codim} D_F = \text{rk} r \geq 1$$

as a Kähler class on $X$ restricts to a nontrivial element in $H^2(F, \mathbb{C})$. We conclude that $\text{rk} r = 1$.

(ii) By (i) we have $\text{rk} r = 1$. In particular one has $\text{rk}(r : H^{1,1}(X, \mathbb{R}) \to H^{1,1}(F, \mathbb{R})) = 1$. As $\mathcal{K}_X$ is open in $H^{1,1}(X, \mathbb{R})$ it follows that $\dim G = 1$. Since $G$ is the restriction of the Kähler cone of $X$ to $F$ it is in particular a ray. By Proposition 4.3 $G$ contains integral points.

□

Remark 4.5  
(i) If $\pi : \mathfrak{X} \to \text{Def}(X)$ denotes the Kuranishi family then the local system $\mathbb{R}^2 \pi_* \mathbb{C}_X$ is trivial as we assume $\text{Def}(X)$ to be simply connected. By Ehresmann’s theorem we can choose a differentiable trivialization

$$\mathfrak{X} \leftarrow \rho \longrightarrow X \times \text{Def}(X) \leftarrow \text{Def}(X)$$

where we denote by $\rho_t := \rho|_X : X \to \mathfrak{X}_t$ the associated fiber diffeomorphism. Further we can choose a relative holomorphic form $\sigma$ i.e. a section of $\Omega^2_{\mathfrak{X}/\text{Def}(X)}$ such that the restriction $\sigma_t := \sigma|_{\mathfrak{X}_t}$ is a holomorphic symplectic form on $\mathfrak{X}_t$. Then the space $D_F$ in the proof of Lemma 4.4 can also be defined as

$$D_F = \left\{ t \in \text{Def}(X) \mid r(\rho_t(\sigma_t)) = 0 \in H^2(F, \mathbb{C}) \right\},$$

see [Voi92, Thm 0.1]. Hence the space $D_F$ parametrizes deformations $F_t \subset \mathfrak{X}_t$ of $F$ which stay Lagrangian with respect to the Lagrangian fibration $\zeta_t : \mathfrak{X}_t \to P_t$.

(ii) From the proof it also follows that $D_F = \text{Def}(X, L)$ as germs as $D_F$ is contained in $\text{Def}(X, L)$ but both have codimension one in $\text{Def}(X)$.

Let $\Delta \subset B$ denote the discriminant locus of the Lagrangian fibration $f : X \to B$ that is set parametrizing the singular fibers. Note that $\Delta$ is an in general reducible hypersurface in $B$, see [HO09, Prop 3.1]. Then $B^\circ := B - \Delta$ is a connected open subset and the restriction $g := f|_{f^{-1}(B^\circ)} : f^{-1}(B^\circ) \to B^\circ$ is a proper holomorphic submersion. Let $\mathcal{C}^\infty$ denote the sheaf of smooth real functions. By Ehresmann’s theorem $\mathcal{H} := \mathbb{R}^2 g_* \mathbb{R} \otimes \mathcal{C}^\infty_{B^\circ}$ is a differentiable real vector bundle on $B^\circ$ which comes with a canonical flat connection $\nabla$ called the Gauss–Manin connection, see [Voi02, 9.2.1].

For each $t \in B^\circ$ consider the restriction $r_t : H^2(X, \mathbb{R}) \to H^2(X_t, \mathbb{R})$. And set $\mathcal{G}_t := r_t(\mathcal{K}_X) \cap H^2(X_t, \mathbb{Z})$. By Lemma 4.4 $\mathcal{G}_t$ is a non–empty semigroup of rank
one. We can define a mapping

(1) \[ \alpha : B^\circ \to H \]

such that \( \alpha(t) \in G_t \subset H^2(X_t, \mathbb{R}) \) is the unique integral and primitive element in \( G_t \) for all \( t \in B^\circ \).

**Proposition 4.6** The \( G_t \) form a local system \( G \) of semigroups on \( B^\circ \). The map \( \alpha : B^\circ \to H \) can be considered as a section of \( G \) and is in particular continuous.

**Proof:** Consider the family of sections

(2) \[ \varphi : \mathcal{K}_X \times B^\circ \to H, \quad (\omega, t) \mapsto \omega|_{X_t}. \]

Then the image of \( \varphi \) is the union of rays in each \( H^2(X_t, \mathbb{R}) \) considered in Lemma 4.4 containing integral points. Note that the family \( \varphi \) is differentiable as for each \( \omega \in \mathcal{K}_X \) the corresponding section \( \varphi(\omega, \cdot) \) is differentiable. It is in particular flat i.e. \( \nabla \varphi(\omega, \cdot) = 0 \) for each \( \omega \in \mathcal{K}_X \) which follows by the Cartan–Lie formular, see [Voi02, 9.2.2].

Let \( \mathcal{H}^\mathfrak{N} \) be the sheaf of flat sections of \( \mathcal{H} = R^2g_*\mathbb{R} \otimes C_\mathbb{R}^\infty \). As \( \varphi \) is a flat family the image \( \text{im} \varphi \) is a local system of semigroups which is contained in \( \mathcal{H}^\mathfrak{N} \). Then define \( \mathcal{G} := \text{im} \varphi \cap R^2g_*\mathbb{Z} \) which is in a canonical way a local system whose stalks are given precisely by \( G_t \).

Take a open covering \( B^\circ = \bigcup_i U_i \) such that \( \mathcal{G} \) is trivial on each \( U_i \) say \( \mathcal{G}(U_i) = G \) for all \( i \) where \( G := G_t \) for a fixed \( t \). For each \( i \) the restriction \( \alpha|_{U_i} \) is the unique primitive element in \( G \). They glue to an unique global section of \( G \) which is precisely \( \alpha \). Hence \( \alpha \) is continuous as a section of the local system \( G \) and in particular as a map \( B^\circ \to H \).

Clearly \( \alpha(t) \in H^2(X_t, \mathbb{Z}) \) defines a polarization on the abelian variety \( X_t \) for every \( t \in B^\circ \). Any polarization on an abelian variety one can associate a tuple of integers which is called the polarization type in the following way.

Since \( X_t \) is an abelian variety we have an identification \( H^2(X_t, \mathbb{Z}) = \wedge^2 H_1(X_t, \mathbb{Z})^\vee \), see [BL03, Cor. 1.3.2]. We then can see \( \alpha(t) : \Lambda_t \otimes \Lambda_t \to \mathbb{Z} \) as an alternating integral form on the lattice \( \Lambda_t := H_1(X_t, \mathbb{Z}) \). By the elementary divisor theorem we can find a basis of \( \Lambda_t \) such in this basis we have

\[
\alpha(t) = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}
\]

where \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) is an integral diagonal matrix with \( \lambda_i > 0 \) and \( \lambda_i|\lambda_{i+1} \).

The tuple

\[ d(f, t) := (\lambda_1, \ldots, \lambda_n) \]

is called the polarization type of \( \alpha(t) \) and it does a priori depend on \( t \in B^\circ \).

**Proposition 4.7** The polarization type \( d(f, \cdot) : B^\circ \to \mathbb{Z}^n \) is constant.
Proof: By construction for \( t \in B^o \) the associated tuple \( d(f,t) \) is the diagonal of the representation matrix of \( \alpha(t) : \Lambda_t \times \Lambda_t \to \mathbb{Z} \) with respect to a chosen basis \( b_1(t), \ldots, b_{2n}(t) \) of the lattice \( H_1(X_t, \mathbb{Z}) \). This correspondence is continuous and since \( d(f,\cdot) \) is integer valued in each component it is locally constant hence constant as \( B^o \) is connected. \( \square \)

**Definition 4.8** For each Lagrangian fibration \( f : X \to B \) the associated tuple \( d(f) \) in \( \mathbb{Z}^n \) is called the polarization type of \( f \).

**Theorem 4.9** The polarization type stays constant in a family of Lagrangian fibrations. In particular two Lagrangian fibrations which are deformation equivalent have the same polarization type.

**Proof:** The proof is similar to the one of Proposition 4.6. Let \( \phi : \mathcal{X} \to P \) be a family of Lagrangian fibrations parametrized by a complex space \( S \). Setting \( \mathcal{B} := \cup_{s \in S} B^o_s \) where as before \( B^o_s := P_s - \Delta_s \) is the base of the Lagrangian fibration \( \phi_s := \phi|_{\mathcal{X}_s} : \mathcal{X}_s \to P_s \) without the discriminant locus. Note that \( \mathcal{B} \) is connected as it is \( P \) without a real codimension two subset. Set \( \psi := \phi|_{\pi^{-1}(\mathcal{B})} : \phi^{-1}(\mathcal{B}) \to \mathcal{B} \) which is a holomorphic submersion.

With same argument as in Proposition 4.6 we get a section \( \alpha : \mathcal{B} \to R^2\psi_\ast\mathbb{Z} \) such that for \( t \in B_s \) the value \( \alpha(t) \) coincides with \( \alpha_s(t) \) where \( \alpha_s \) is the continuous mapping \( \alpha_s : B^o_s \to \mathcal{H}_s \) as in 1 which is associated to the Lagrangian fibration \( \phi_s \). Let \( d(\alpha(t)) \) denote the polarization type of the polarization \( \alpha(t) \) on the abelian variety \( (\mathcal{X}_s)_t \) for \( t \in B^o_s \). As \( \mathcal{B} \) is connected the continuous map \( d(\alpha(\cdot)) : \mathcal{B} \to \mathbb{Z}^n \) must be constant. Since \( d(\phi_s) = d(\alpha_s(t)) = d(\alpha(t)) \) for \( t \in B^o_s \) we see that \( d(\phi_s) \) is constant on \( S \). \( \square \)

**Proposition 4.10** Let \( f : X \to B \) be a Lagrangian fibration and \( \omega \) a special Kähler class with respect to a smooth fiber \( F \). Then \( d(f) \) is given by the polarization type of \( \omega|_F \).

**Proof:** As \( \omega|_F \) is the restriction of a Kähler class it is contained in the ray \( G \) of Lemma 4.4. Since \( \omega|_F \) is primitive it is in the image of \( \alpha : B^o \to \mathcal{H} \) of 1 i.e. \( \alpha(t) = \omega|_F \) for \( F = X_t \). \( \square \)

**Example 4.11** Let \( f : S \to \mathbb{P}^1 \) be an elliptic K3 surface and \( \omega \) a special Kähler class on \( S \) with respect to a smooth fiber \( F \). As \( F \) is an elliptic curve we have \( H^2(F, \mathbb{Z}) \cong \mathbb{Z} \). Since \( \alpha := \omega|_F \) is primitive it is the generator of \( H^2(F, \mathbb{Z}) \) and so \( \alpha = c_1(L) \) for an ample line bundle of degree \( \deg(L) = 1 \). We then have

\[
    d(f) = \int_F c_1(L) = \deg(L) = 1
\]

as one can identify degree map with integration of the first Chern class.
Theorem 4.12  The associated Lagrangian fibrations of two marked pairs which are defining points in the same connected component $\Lambda_{X_0}^\perp$ of the moduli of Lagrangian fibrations for a primitive isotropic class $\lambda$ in the $K3^{[n]}$–type lattice $\Lambda$ have the same polarization type.

Proof: By Proposition 3.9 the associated Lagrangian fibrations are deformation equivalent and the claim follows by Proposition 4.9.  

A very optimistic conjecture is the following.

Conjecture 4.13  Let $f_i : X_i \to B_i$, $i = 1, 2$, be two Lagrangian fibrations such that $X_1$ and $X_2$ are deformation equivalent. Then $d(f_1) = d(f_2)$.

As every known irreducible holomorphic symplectic manifold can be deformed to one which admits a Lagrangian fibration this conjecture would basically give a new deformation invariant. In section 6 we verify this conjecture for manifolds of $K3^{[n]}$–type.

5. Beauville–Mukai systems

This section has the purpose to recall the notion of Beauville–Mukai systems which are important examples of $K3^{[n]}$–type fibrations and to determine their polarization type.

Let $S$ be a projective K3 surface and let $H^\bullet(S)$ denote the Mukai lattice i.e.

$$H^\bullet(S) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

together with the bilinear form defined by $(v, w) := (v_2, w_2) - \int_S v_0 \wedge w_0 + v_4 \wedge w_4$ where $(v_2, w_2) = \int_S v_2 \wedge w_2$ denotes the intersection form on $H^2(S, \mathbb{Z})$ and $v = v_0 + v_2 + v_4$ with $v_i \in H^i(S, \mathbb{Z})$ the decomposition in $H^\bullet(S)$ and similarly for $w$.

This lattice is even, unimodular, of rank 24 and isometric to

$$\tilde{\Lambda} := \Lambda_{K3} \oplus U_{\oplus 2} = E_8(-1)^{\oplus 2} \oplus U_{\oplus 4}$$

where $\Lambda_{K3} \cong H^2(S, \mathbb{Z})$ is the the K3 lattice, $E_8(-1)$ the negative definite root lattice and $U$ the unimodular rank two hyperbolic lattice. We identify $H^4(S, \mathbb{Z}) = \mathbb{Z}$ where we use the Poincare dual to a point as a generator and similarly $H^0(S, \mathbb{Z}) = \mathbb{Z}$ by taking the Poincare dual of $S$.

A Mukai vector is a tuple $v = (r, l, s) \in H^0(S, \mathbb{Z}) \oplus H^{1,1}(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ such that $r \geq 0$ and $l$ is effective if $r = 0$. By [HL10, 4C.] for each Mukai vector $v$ there is a distinguished countable system of hyperplanes called $v$–walls in the the real ample cone $A(S) \otimes \mathbb{R}$ which is locally finite. An ample divisor $H$ on $S$ is called $v$–generic if it lies outside the minimal union of such $v$–walls. For a coherent sheaf
Choose a Mukai vector \( v \) together with an \( v \)-generic ample class \( H \) and consider the moduli space \( M_H(v) \) of \( H \)-stable pure coherent sheaves \( \mathcal{F} \) on \( S \) with \( v(\mathcal{F}) = v \). Then general results of many authors, see for instance [Muk84] and [Yos01], imply that \( M_H(v) \) is a holomorphic symplectic manifold of dimension \( 2n = (v, v) + 2 \). It is not necessarily compact. To compactify one needs to add semistable sheaves but under the assumption that \( v \) is primitive any \( H \)-semistable sheaf \( \mathcal{F} \) with \( v(\mathcal{F}) = v \) is automatically \( H \)-stable and we denote the moduli space in this case just by \( M_H(v) \) which is a K3\(^{[n]}\)-type manifold, see [Yos01, Prop. 4.12].

Let \( v \) be a primitive Mukai vector on \( S \) of the form \( v = (0, c_1(D), s) \) where \( D \) is a nef divisor on \( S \). Note that we have \( h^0(S, D) = \frac{1}{2}(D, D) + 2 = n + 1 \). Choose an \( v \)-generic ample class \( H \) on \( S \) hence \( M_H(v) \) is an irreducible holomorphic symplectic manifold. It comes with a natural Lagrangian fibration in the following way.

**Remark 5.1** Let \( \mathcal{F} \) be a sheaf on \( S \) such that \( C := \text{supp}(\mathcal{F}) \) is a smooth irreducible curve of genus \( g \). Then \( \text{rk}(\mathcal{F}) = 0, c_1(\mathcal{F}) = c_1(\mathcal{O}_S(C)) \) and by Riemann–Roch

\[
v(\mathcal{F}) = (0, c_1(\mathcal{F}), \frac{1}{2} c_1^2(\mathcal{F}) - c_2(\mathcal{F})) = (0, C, 1 - g + d)
\]

where \( d \) denotes the degree of the restriction of \( \mathcal{F} \) to \( C \).

Considering \( M_H(v) \) we only deal with sheaves \( \mathcal{F} \) on \( S \) with \( v(\mathcal{F}) = v \) i.e. \( \mathcal{F} \) is of rank zero with Chern class \( c_1(\mathcal{F}) = c_1(D) \). In particular \( \mathcal{F} \) is supported on a divisor which is an element of \( |D| \cong \mathbb{P}^n \). Then we can define

\[
\pi : M_H(v) \longrightarrow |D|^*, \quad \mathcal{F} \longmapsto \text{supp} \mathcal{F}
\]

to obtain a holomorphic map \( M_H(v) \to |D|^* \). Since \( M_H(v) \) is irreducible holomorphic symplectic \( \pi \) is a Lagrangian fibration by Matsushita’s Theorem 2.4. If \( C \) is a smooth curve and an element of \( |D| \) then the fiber \( \pi^{-1}(C) \) is the Jacobian of the curve \( C \) by construction.

**Definition 5.2** The Lagrangian fibration \( \pi : M_H(v) \to |D|^* \) with the primitive Mukai vector \( v \) and polarization \( H \) on \( S \) as above is called a Beauville–Mukai system.

**Remark 5.3** More classically Beauville–Mukai systems arise from linear systems induced by a smooth curve in \( S \) in the following way. Let \( C \subset S \) denote an irreducible smooth curve of genus \( n \). Under the assumption that \( \text{Pic}(S) \) is generated by \( \mathcal{O}_S(C) \) all curves in the linear system \( |C| \) are reduced and irreducible. By Riemann–Roch it follows that \( |C| = \mathbb{P}^n \). Let \( \mathcal{C} \to \mathbb{P}^n \) denote the associated family of curves. The relative compactified Jacobian \( \pi : X := \overline{\text{Pic}^d(C/\mathbb{P}^n)} \to \mathbb{P}^n \) exists, see [D’S79, II, 1-4] or [AK80, Thm. 6.6].

\[\text{Note that } v(\mathcal{F}) = (\text{rk}(\mathcal{F}), c_1(\mathcal{F}), \chi(\mathcal{F}) - \text{rk}(\mathcal{F})) = (\text{rk}(\mathcal{F}), c_1(\mathcal{F}), \frac{1}{2} c_1(\mathcal{F})^2 - c_2(\mathcal{F}) + \text{rk}(\mathcal{F})) \text{ as } \sqrt{\text{Td}(S)} = 1 + \omega \text{ where } \omega \text{ denote the Poincare dual of a point since we consider sheaves on a K3 surface.} \]
By setting $v := (0, c_1(C), d + 1 - n)$ the identification of $\overline{\text{Pic}}^d(C/\mathbb{P}^n)$ with $M_H(v)$ is given by the following map. For a pair $(\mathcal{C}, \mathcal{F})$ as above consider the inclusion $\iota : \mathcal{C}_t \hookrightarrow S$. Then associate to it the element $\iota_\star \mathcal{F} \in M_H(v)$.

In particular one can see $M_H(v)$ as a generalization of the classical definition of a Beauville–Mukai system since the construction with the compactified Picard scheme only works if the linear system contains only reduced and irreducible curves.

**Lemma 5.4** Let $A$ be an abelian variety.

(i) If $\text{End}(A) = \mathbb{Z}$ then one has for the Picard number $\rho(A) = 1$.

(ii) If $A = \text{Jac}(C)$ is a Jacobian of a curve $C$ and $\rho(A) = 1$ then the primitive polarization on $A$ is principal.

**Proof:**

(i) By [BL03, Prop. 5.2.1] there is an isomorphism $\text{NS}(A) \otimes \mathbb{Q} \cong V$ where $V$ is a $\mathbb{Q}$–subspace of $\text{End}(A) \otimes \mathbb{Q}$. The latter has dimension 1 by assumption hence $\rho(A) = \dim_\mathbb{Q} \text{NS}(A) \otimes \mathbb{Q} = 1$.

(ii) It is well known that on every Jacobian of a curve there exists a primitive principal polarization, see [BL03, Prop 11.1.2]. Since $\rho(A) = 1$ it must be unique. \hfill \Box

**Theorem 5.5** [CvdG92, Thm 1.1, Cor. 1.2] Let $S$ be a K3 surface and $V$ a linear system on it. If $C$ is a general element of $V$ and $A = \text{Jac}(C)$ the Jacobian of $C$ then $\text{End}(A) = \mathbb{Z}$.

**Proof:** This follows directly from [CvdG92, Thm 1.1, Cor. 1.2] since K3 surfaces satisfy $H^1(S, \mathcal{O}_S) = 0$. \hfill \Box

**Corollary 5.6** The Picard number of the generic smooth fiber of a Beauville–Mukai system $\pi : X \to |D|^*$ equals one. In particular $d(\pi) = (1, \ldots, 1)$.

**Proof:** The first statement follows immediately from Theorem 5.5 and Lemma 5.4 (i) since the generic smooth fiber is a Jacobian of a smooth curve. A special Kähler class $\omega$ with respect to a fiber $F$ which is a Jacobian of a curve restricts to the unique primitive polarization on $F$ since $\rho(F) = 1$. Then by Lemma 5.4 (ii) above this polarization is principle hence $d(\pi) = d(\omega|_F) = (1, \ldots, 1)$ by Proposition 4.10. \hfill \Box

## 6. Polarization types of $K3^{[n]}$–type fibrations

In this section we verify Conjecture 4.13 for $K3^{[n]}$–type manifolds which follows mainly by methods developed by E. Markman [Mar14].

**Theorem 6.1** Let $f : X \to \mathbb{P}^n$ be a Lagrangian fibration with $X$ of $K3^{[n]}$–type. Then the polarization type $d(f)$ is principal i.e. it is given by $d(f) = (1, \ldots, 1)$.

For the following see also section 2. of [Mar14].
Definition 6.2 Let \( \lambda \) be an element in an arbitrary lattice \( \Lambda \). Then define the divisibility of \( \lambda \) as

\[
\text{Div}(\lambda) := \max \{ k \in \mathbb{N} \mid (\lambda, \cdot)/k \text{ is an integral class of } \Lambda^* \}.
\]

Now let \( \Lambda \) denote the \( K3^{[n]} \)-type lattice and let \( X \) be a \( K3^{[n]} \)-type manifold. Fix a positive integer \( d \) such that \( d^2 \) divides \( n - 1 \). Next we associate to an isotropic and primitive class \( \lambda \) a lattice in the following way.

Let \( \tilde{\Lambda} \) denote the Mukai lattice, see equation 3. Then by [Mar10, Thm. 1.10] \( X \) comes with a natural \( O(\tilde{\Lambda}) \)-orbit \( \iota_X \) of primitive isometric embeddings \( \iota : H^2(X, \mathbb{Z}) \to \tilde{\Lambda} \). Choose a primitive isometric embedding \( \iota : H^2(X, \mathbb{Z}) \to \tilde{\Lambda} \) in \( \iota_X \). Since \( \iota(H^2(X, \mathbb{Z})) \) is of rank 23 and the Mukai lattice is of rank 24 the orthogonal complement \( \iota(H^2(X, \mathbb{Z}))^\perp \) is of rank 1. Choose a generator \( v \) of \( \iota(H^2(X, \mathbb{Z}))^\perp \). Note that \( (v, v) = 2n - 2 \). Then define the lattice \( H(\lambda) \) to be the saturation of \( \langle v, \iota(\lambda) \rangle \subset \tilde{\Lambda} \) i.e. \( H(\lambda) \) is the maximal sublattice of \( \tilde{\Lambda} \) such that \( H(\lambda) \) is of rank two and contains \( \langle v, \iota(\lambda) \rangle \). Two pairs \( (H_1, v_1) \) and \( (H_2, v_2) \) are called isometric if there is an isometry \( g : H_1 \to H_2 \) such that \( g(v_1) = v_2 \). The isometry class of \( (H(\lambda), v) \) depends only on \( \lambda \) since the \( O(\tilde{\Lambda}) \)-orbit \( \iota_X \) is natural. Denote by \( H_{n,d} \) the lattice \( \mathbb{Z}^2 \) together with the pairing

\[
\frac{2n - 2}{d^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

and denote by \( I_{n,d} \) the set of isometry classes of pairs \( (H, w) \) such that \( H \) is isometric to \( H_{n,d} \) and \( w \) is a primitive class in \( H \) with \( (w, w) = 2n - 2 \). Given an positive integer \( d \) let \( I_d(X) \subset H^2(X, \mathbb{Z}) \) be the subset of primitive isotropic classes \( \lambda \) with respect to the Beauville–Bogomolov form such that \( \text{Div}(\lambda) = d \). Note that this set is clearly monodromy–invariant i.e. \( \text{Mon}^2(X) \cdot I_d(X) \subset I_d(X) \). Note that if \( X' \) is deformation equivalent to \( X \) and \( P : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z}) \) a parallel transport operator then \( I_d(X') = P(I_d(X)) \).

Lemma 6.3 [Mar14, Lem. 2.5] Let \( \lambda \) denote a primitive isotropic class in \( H^2(X, \mathbb{Z}), \) let \( \iota \) and \( v \) as above and set \( d := \text{Div}(\lambda) \).

(i) The square \( d^2 \) divides \( n - 1 \) and the lattice \( H(\lambda) \) is isometric to \( H_{n,d} \).

(ii) The map defined by

\[
h : I_d(X) \to I_{n,d}, \quad \lambda' \mapsto [(H(\lambda'), v)]
\]

induces a bijection \( I_d(X)/\text{Mon}^2(X) \to I_{n,d} \).

(iii) For the pair \( (H(\lambda), v) \) there exists an integer \( b \) such that \( (\iota(\lambda) - bv)/\text{Div}(\lambda) \) is an integral class of \( H(\lambda) \) and the isometry class \( h(\lambda) \) of \( (H(\lambda), v) \) is represented by \( (H_{n,d}, (d, b)) \) for any such integer \( b \).

Remark 6.4 The integer \( b \) in Lemma 6.3 (iii) satisfies \( \gcd(d, b) = 1 \). Indeed, if they would have a common divisor say \( r > 1 \), then we could write the equation in Lemma 6.3 (iii) as

\[
\iota(\lambda) = dm + bv = r(d'm + b'v)
\]
for integers $m, d'$ and $b'$ which is a contradiction to the fact that $\iota(\lambda)$ is primitive.

The map $h : I_d(X) \to I_{n,d}$ is also called a monodromy–invariant, see [Mar11, 5.4] for the general notion.

**Lemma 6.5** [Mar14, Ex. 3.1] Let $d$ be a positive integer such that $d^2$ divides $n - 1$ and let $b$ an integer satisfying $\gcd(d, b) = 1$. Then there exists a Beauville–Mukai system $\pi : M_H(v) \to \mathbb{P}^n$ and a primitive isotropic class $\alpha \in H^2(M_H(v), \mathbb{Z})$ such that the following holds.

(i) $\text{Div}(\alpha) = d$,

(ii) the monodromy–invariant $h(\alpha)$ is represented by $(H_{n,d}, (d, b))$,

(iii) $c_1(\pi^*\mathcal{O}_{\mathbb{P}^n}(1)) = \alpha$.

**Proof:** This is a recap of [Mar14, Ex. 3.1]. Let $S$ be a K3 surface together with a nef and primitive line bundle $L$ on $S$ of Bogomolov degree $(2n - 2)/d^2$ e.g. take a $(2n - 2)/d^2$–polarized K3 surface. Set $\beta := c_1(L)$ and let $s$ be an integer such that $sb \equiv 1 \mod d$. Then $v := (0, b\beta, s)$ is a Mukai vector. In particular $v$ is primitive since $\beta$ is primitive and $\text{gcd}(d, s) = 1$. Choose an $v$–generic ample class $H$. We have $(v, v) = d^2(\beta, \beta) = 2n - 2$ hence $M_H(v)$ is irreducible holomorphic symplectic of dimension $2n$ and we obtain a Beauville–Mukai system $\pi : M_H(v) \to |L^d|$ as described in section 5. We have Mukai’s Hodge isometry

$$\Theta : v^\perp \to H^2(M_H(v), \mathbb{Z})$$

which can be defined as follows. Choose a quasi–universal family of sheaves $\mathcal{E}$ on $S$ of simplitude $\rho \in \mathbb{N}$. That is $\mathcal{E} \in \text{Coh}(S \times M_H(v))$ such that $\mathcal{E}$ is flat over $M_H(v)$ and for every class $\mathcal{F} \in M_H(v)$ one has $\mathcal{E}|_{S \times \{\mathcal{F}\}} \cong \mathcal{F}^\oplus \rho$. Then set

$$\Theta(x) := \frac{1}{\rho} \left[ (\text{pr}_{M_H(v)})! \left( (\text{ch}(\mathcal{E})(\text{pr}_{M_H(v)})^* \left( \sqrt{\text{Td}(S)x^\vee} \right) \right) \right]_2$$

where $x^\vee = -x_0 + x_2 + x_4$ for $x = x_0 + x_2 + x_4$ and $[.]_2$ denotes the part in $H^2(S \times M_H(v), \mathbb{Z})$. For the details see [Yos01, 1.2]. Set $\alpha := \Theta(0, 0, 1)$ which is clearly isotropic and define $\iota : H^2(M_H(v), \mathbb{Z}) \to H^*(S, \mathbb{Z})$ to be $\Theta^{-1}$ composed with the inclusion $v^\perp \hookrightarrow H^*(S, \mathbb{Z})$.

(i) An element $(r, c, t)$ belongs to $v^\perp$ iff

$$0 = ((0, d\beta, s), (r, c, t)) = d(\beta, c) - rs \iff rs = d(\beta, c).$$

Hence $d$ divides $r$ since $\text{gcd}(d, s) = 1$. Further we have $((0, 0, 1), (r, c, t)) = r$ for all $(r, c, t) \in v^\perp$ hence $\text{Div}((0, 0, 1)) \geq d$. As the K3–lattice is unimodular we have $\text{Div}_{H^2(S, \mathbb{Z})}(\beta) = 1$ in $H^2(S, \mathbb{Z})$. This implies that $\text{Div}(\beta) = 1$ in $v^\perp$ hence we can find an element $c \in H^2(S, \mathbb{Z})$ such that $s = (c, \beta)$.

Then $(d, c, 0)$ is contained in $v^\perp$ and $((0, 0, 1), (d, c, 0)) = d$ hence

$$\text{Div}(\alpha) = \text{Div}(0, 0, 1) = d.$$ 

(ii) We have $\iota(\alpha) - bv = (0, 0, 1) - (0, bd\beta, bs) = (0, bd\beta, 1 - bs)$ which is divisible by $d$ since $sb \equiv 1 \mod d$ by assumption. By Lemma 6.3 (iii) the monodromy–invariant $h(\alpha)$ is represented by $(H_{n,d}, (d, b))$. 


(iii) Let $\omega = [p] \in H^4(S, \mathbb{Z})$ denote Poincare dual of a point $p \in S$. By our notation we have $\omega = (0, 0, 1) = \omega^\vee \in H^\bullet(S)$. Since $S$ is a K3 surface one has $\sqrt{\text{Td}(S)} = 1 + \omega$ hence $\sqrt{\text{Td}(S)}^\vee = \omega$. Note that $\mathcal{E}$ is a sheaf of rank zero hence $\text{ch}(\mathcal{E}) = \rho c_1(\mathcal{E}) + \xi = \rho[D] + \xi$ for some divisor $D$ in $S \times M_H(v)$ and for some terms $\xi$ of higher degree. Further $(\text{pr}_x)^*\omega = [p \times M_H(v)] \in H^1(S \times M_H(v), \mathbb{Z})$ and $[(\text{pr}_M(v))((\xi \cdot [p \times M_H(v)]))]_2 = 0$ due to degree reasons. Then we have

$$
\Theta(0, 0, 1) = (\text{pr}_{M_H(v)}((D \cdot [p \times M_H(v)]))
= \{\mathcal{F} \in M_H(v) \mid p \in \text{supp}(\mathcal{F})\}
= \pi^*[C \in |L^d| \mid p \in C]$

since $V := \{C \in |L^d| \mid p \in C\}$ is an hyperplane in a projective space, hence $[V] = c_1(\mathcal{O}_{|L^d|}(1))$.

\[\square\]

For a fixed K3$^{[n]}$–type manifold $X$ we have constructed the monodromy–invariant $h : I_d(X) \to I_{n,d}$. Is $X'$ is another K3$^{[n]}$–type manifold then we denote the monodromy–invariant also by $h : I_d(X') \to I_{n,d}$.

Lemma 6.6 [Mar11, Lem. 5.17] Let $X_i$, $i = 1, 2$, be two K3$^{[n]}$–type manifolds and $e_i \in I_d(X_i)$. Assume that $h(e_1) = h(e_2)$, $e_i = c_1(L_i)$ for holomorphic line bundles $L_i$ and there are Kähler classes $\kappa_i$ such that $(\kappa_i, e_i) > 0$. Then the pairs $(X_i, L_i)$ are deformation equivalent in the sense of Definition 3.8.

Lemma 6.7 Let $\lambda$ be a nontrivial isotropic class in $H^{1,1}(X, \mathbb{R})$ with $X$ an arbitrary irreducible holomorphic symplectic manifold. Then the Beauville–Bogomolov quadratic form satisfies $(x, \lambda) > 0$ for every class $x$ in the positive cone $\mathcal{C}_X^+$.

**Proof:** Let $x \in \mathcal{C}_X^+$. As $\mathcal{C}_X^+$ is self–dual the cone coincide with its dual i.e. $\mathcal{C}_X^+ = (\mathcal{C}_X^+)^\vee$. This means $(x, y) > 0$ for all $y \in \mathcal{C}_X^+$. The condition $(\lambda, \lambda) = 0$ means nothing as that $\lambda$ is contained in the closure $\mathcal{C}_X^+$ of the positive cone hence $(x, \lambda) \geq 0$. As $(x, x) > 0$ and the signature of the Beauville–Bogomolov form on $H^{1,1}(X, \mathbb{R})$ is $(1, b_2(X) - 3)$ the orthogonal complement $x^\perp$ in $H^{1,1}(X, \mathbb{R})$ is a negative definite subspace. Considering the decomposition of $\lambda$ with respect to $H^{1,1}(X, \mathbb{R}) = \langle x \rangle \oplus x^\perp$ i.e.

$$0 \neq \lambda = \frac{(x, \lambda)}{(x, x)} x + \left(\lambda - \frac{(\lambda, x)}{(x, x)} x\right)$$

we see that $(x, \lambda) \neq 0$. Otherwise $(\lambda, \lambda) < 0$ which would be a contradiction. Hence $(x, \lambda) > 0$. \[\square\]
Proof of Theorem 6.1: Let $f : X \to \mathbb{P}^n$ a K3$^n$-type fibration and set $L := f^* \mathcal{O}_{\mathbb{P}^n}(1)$. Then $\lambda := c_1(L)$ is isotropic with respect to the Beauville–Bogomolov quadratic form by Lemma 3.5. Let $d := \text{Div}(\lambda)$ denote the divisibility of $\lambda$. Consider the monodromy–invariant $h : I_d(X) \to I_{n,d}$. By Lemma 6.3 (iii) there exists an integer $b$ such that $h(\lambda)$ is represented by $(H_{n,d}, (d, b))$. Then $\gcd(b, d) = 1$ by Remark 6.4 hence Lemma 6.5 gives a Beauville–Mukai system $\pi : X' \to \mathbb{P}^n$ together with a primitive isotropic class $\alpha \in H^2(X', \mathbb{Z})$ such that $\text{Div}(\alpha) = d$, $L' := \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$ satisfies $c_1(L') = \alpha$ and $h(\alpha)$ is represented also by $(H_{n,d}, (d, b))$ i.e. $h(\alpha) = h(\lambda)$.

Further by Lemma 6.7 we have $(\omega, L) > 0$ and $(\omega', L') > 0$ for ever Kähler classes $\omega$ on $X$ and $\omega'$ on $X'$ as $L$ and $L'$ are isotropic. Hence we can apply Lemma 6.6 which says that the pairs $(X, L)$ and $(X', L')$ are deformation equivalent. By Proposition 3.9 there exist markings $\eta$ and $\eta'$ on $X$ and $X'$ respectively such that the pairs $(X, \eta)$ and $(X', \eta')$ are contained in the same connected component of the moduli of Lagrangian fibrations $\mathcal{U}_{\lambda'}^\text{iso}$, for a primitive isotropic class $\lambda' \in \Lambda$. By Proposition 4.12 and Corollary 5.6 and we have

$$d(f) = d(\pi) = (1, \ldots, 1)$$

which concludes the proof. □

References

[AK80] Allen Altman and Steven Kleiman. Compactifying the Picard scheme. Adv. in Math., 35:50–112, 1980. 17

[Bea84] Arnaud Beauville. Variétés kähleriennes dont la premièr classe de chern est nulle. J. Differential Geom., 18:755–782, 1984. 3, 4

[BL03] Christina Birkenhake and Herbert Lange. Complex Abelian Varieties, volume 302 Second Edition of Grundlehren der mathematischen Wissenschaft. Springer, 2003. 14, 18

[Cam05] Frédéric Campana. Isotrivialité de certaines familles kählériennes de variété non projectives. arXiv:0408148v2, 2005. 11, 12

[CvdG92] Ciro Ciliberto and Gerard van der Geer. On the Jacobian of a hyperplane section of a surface. Classification of irregular varieties (Trento, 1990) Lecture Notes in Math., Springer, Berlin, 1515:33–40, 1992. 18

[D’S79] Cyril D’Souza. Compactification of generalized Jacobians. Proc. Indian Acad. Sci., 88:419–457, 1979. 17

[HL10] Daniel Huybrechts and Manfred Lehn. The geometry of moduli spaces of sheaves. Cambridge University Press., Cambridge, second edition edition, 2010. 16

[HO09] Jun-Muk Hwang and Keiji Oguiso. Characteristic foliation on the discriminantal hypersurface of a holomorphic Lagrangian fibration. Amer. Journal of Math., 131:981–1007, 2009. 13

[Huy03] Daniel Huybrechts. The kähler cone of a compact hyperkähler manifold. Math. Ann, pages 499–513, 2003. 8

[Hwa08] Jun-Muk Hwang. Base manifolds for fibrations of projective irreducible symplectic manifolds. Invent. Math. 174, 3:625–644, 2008. 5

[LG13] Christian Lehn and Daniel Greb. Base manifolds for lagrangian fibrations on hyper-kähler manifolds. arXiv:1303.3919, 2013. 5
[Mar10] Eyal Markman. Integral constraints on the monodromy group of the hyperkähler resolution of a symmetric product of a k3 surface. Internat. J. of Math. 21, 2:169–223, 2010.

[Mar11] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In Wolfgang Ebeling et. al., editor, Complex and Differential Geometry, volume 8, pages 257–323. Springer Proceedings in Math., 2011.

[Mar14] Eyal Markman. Lagrangian fibrations of holomorphic-symplectic varieties of K3[^n]_type. In Anne Frühbis-Krüger et. al., editor, Algebraic and Complex Geometry, volume 71. Springer Proceedings in Math., 2014.

[Mat00] Daisuke Matsushita. Equidimensionality of Lagrangian fibrations on holomorphic symplectic manifolds. Math. Res. Lett., 7:389–391, 2000.

[Mat09] Daisuke Matsushita. On deformation of deformations of Lagrangian fibrations. 2009.

[MGJ03] Daniel Huybrechts Mark Gross and Dominic Joyce. Calabi-Yau Manifolds and Related Geometries. Springer, 2003.

[Muk84] Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. Invent. math, 77:101–116, 1984.

[Saw03] Justin Sawon. Abelian fibred holomorphic symplectic manifolds. Jour. Math. 2, 1:97–230, 2003.

[Ver11] Misha Verbitsky. A global torelli theorem for hyperkähler manifolds. arXiv:0908.4121, 2011.

[Voi92] Claire Voisin. Sur la stabilité des sous-variétés lagrangiennes des variétés symplectique holomorphes. Complex projective geometry Cambridge Univ. Press, 1992.

[Voi02] Claire Voisin. Hodge Theory and Complex Algebraic Geometry I, volume 76 of Cambridge studies in advanced mathematics. Cambridge University Press, 2002.

[Yos01] Kota Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. Math. Ann. 321, 4:817–884, 2001.

Benjamin Wieneck, Institut für Algebraische Geometrie and Graduiertenkolleg 1463 "Analysis, Geometry and String Theory", Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
E-mail address: wieneck@math.uni-hannover.de