CORRELATION FUNCTIONS OF STRICT PARTITIONS AND TWISTED FOCK SPACES

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Abstract. Using twisted Fock spaces, we formulate and study two twisted versions of the $n$-point correlation functions of Bloch-Okounkov [BO], and then identify them with $q$-expectation values of certain functions on the set of (odd) strict partitions. We find closed formulas for the 1-point functions in both cases in terms of Jacobi $\theta$-functions. These correlation functions afford several distinct interpretations.

Introduction

Bloch-Okounkov [BO] studied certain correlation functions on the infinite wedge representation (i.e. the Fock space of a pair of free fermions), which is closely related to the correlation functions of vertex operators [Zhu]. Recently it has gradually become clear that this correlation function (and its variant) affords several distinct interpretations:

1. It can be regarded as a certain character of representations of the $\mathcal{W}_{1+\infty}$ algebra of level one.
2. It can be regarded as the $q$-expectation value of a certain function on the set of partitions.
3. It can be interpreted as a generating function of the stationary Gromov-Witten invariants of an elliptic curve.
4. A variant of this correlation function can be interpreted as a generating function of equivariant intersection numbers on Hilbert schemes of points on the affine plane.
5. It can be interpreted as a generating function of certain structure constants of the class algebras of the symmetric groups.

The items (1) and (2) were points of view taken in [BO] (also see [Ok]). More on representation theory of $\mathcal{W}_{1+\infty}$ (which is a central extension of the Lie algebra of differential operators on the circle) of positive integral levels using Fock spaces can be found in [FKRW, AFMO]. Item (3) is due to the recent remarkable work of Okounkov-Pandharipande [OP]. Item (4) is due to Li, Qin, and the author [LQW], and (5) is essentially equivalent to (4) based on the results of Vasserot, Lascoux-Thibon and ours (cf. [LQW]; also see [LT, Wa]). We refer to [LT, Wa] for a closely related study of the class algebras of the symmetric groups and more generally of wreath products and their connections to the $\mathcal{W}_{1+\infty}$ algebra.

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The purpose of this paper is to present (two) twisted versions of the above correlation functions. This is also partly motivated by the representation theory of distinguished Lie subalgebras \( \hat{D}^\pm \) of \( W_{1+\infty} \) [KWy] via the twisted fermionic Fock spaces of Neveu-Schwarz type and of Ramond type, which are intimately related to Lie subalgebras of \( \hat{gl}_\infty \) [DJKM]. One can interpret our correlation functions in a way analogous to (1), (2), and (5) above, although the connections with the counterpart of (5) will not be presented in this paper. We believe that there will be a geometric interpretation of our correlation functions.

The first construction uses the twisted Fock space \( \mathcal{F} \) of a fermion with integer indices (i.e. the Ramond sector). The \( n \)-point correlation functions of certain distinguished operators (quadratic in fermion generators) in \( \mathcal{F} \) are further identified as the \( q \)-expectation value of certain functions on the set of strict partitions. The second construction uses the twisted Fock space \( \mathcal{F} \) of a fermion with half-odd-integer indices (i.e. the Neveu-Schwarz sector). The \( n \)-point correlation functions of certain operators in \( \mathcal{F} \) are then interpreted as the \( q \)-expectation value of functions on the set of odd strict partitions. Using combinatorial methods including an identity of Euler on partitions, we derive clean closed formulas of the the 1-point functions in terms of Jacobi \( \theta \)-functions in both cases. From the combinatorial viewpoint, the representation theory serves as a motivation to suggest “right” combinatorial questions (which are justified by nice answers). Our results should have implications on random partition theory (compare [Ok]).

In both cases, we obtain the \( q \)-difference equations for the \( n \)-point correlation functions and our formulation also indicates clearly the nature of poles of the \( n \)-point correlation functions. These results have their counterparts in [BO] and our method is analogous to [OK]. Indeed, such properties play a key role in [BO] to establish closed formulas for their \( n \)-point correlation functions. We expect these properties will also play an important role in eventually finding closed formulas for the \( n \)-point functions studied in the paper. Our correlation function also admits a natural super counterpart (cf. Remark 3.3).

The paper is organized as follows. In Sect. 1 we study a distinguished operator on \( \mathcal{F} \) and formulate the correlation functions. In Sect. 2 we use a combinatorial method to derive the 1-point function. In Sect. 3 we establish a \( q \)-difference equation for the \( n \)-point functions. Sect. 4 is a counterpart of Sect. 1 where the setup is switched to the Fock space \( \mathcal{F} \). In Sect. 5 we formulate and establish the counterparts in \( \mathcal{F} \) of Sections 2 and 3. We end in Sect. 6 with discussions.

**Notations.** We denote by \( \mathbb{Z} \) the set of integers, \( \mathbb{Z}_+ \) the set of non-negative integers, and \( \frac{1}{2} + \mathbb{Z} \) the set of half-odd-integers.

**Acknowledgment.** After we finished this work, a very interesting paper of Milas [Mi] appeared in [arXiv:math.QA/0303154] whose results and approach are independent of and complementary to ours. He systematically used vertex operator techniques to study closely related correlation functions and established precise relations between [BO] and [Zhu], while the methods in this paper are mainly combinatorial and Lie theoretic.
1. Correlation functions on the Fock space of Ramond type

Introduce a fermionic field of Ramond type

\[ \phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n} \]

with the following commutation relations:

\[ [\phi_m, \phi_n]_+ = \delta_{m,-n}, \quad m, n \in \mathbb{Z}. \]

Note that \( \phi_0^2 = 1/2 \), and \( \phi_0 \) anticommutes with all \( \phi_n, n \neq 0 \).

Denote by \( \mathcal{F} \) the Fock space of \( \phi(z) \) with the highest weight vector \( |0\rangle \) annihilated by \( \phi_n, n > 0 \). Recall a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) is strict if \( \lambda_1 > \lambda_2 > \ldots > \lambda_\ell > 0 \). The number \( \ell \) is often referred to as the length of \( \lambda \) and denoted by \( \ell(\lambda) \).

Introduce the notations

\[ \phi_\lambda = \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_\ell} |0\rangle \]

\[ \phi_\lambda \phi_0 = \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_\ell} \phi_0 |0\rangle . \]

Then \( \mathcal{F} \) has a linear basis given by \( \phi_\lambda \) and \( \phi_\lambda \phi_0 \), where \( \lambda \) runs over the set \( SP \) of all strict partitions (including the empty partition). The following graded operator \( L_0 \) (which can be identified with the zero-mode of a Virasoro algebra) defines a natural \( \mathbb{Z}_+ \)-grading on \( \mathcal{F} \):

\[ L_0 \phi_\lambda = |\lambda| \cdot \phi_\lambda, \quad L_0 \phi_\lambda \phi_0 = |\lambda| \cdot \phi_\lambda \phi_0 \]

for all \( \lambda \in SP \). A subalgebra \( b_\infty \) of \( \widehat{gl}_\infty \) acts on \( \mathcal{F} \) as follows [DJKM] (also cf. [KWy]). The algebra \( b_\infty \) is spanned by \( E_{i,j} - E_{-j,-i} \), where \( i, j \in \mathbb{Z} \), and a central element \( C \), and the action is given by

\[ E_{i,j} - E_{-j,-i} = : \phi_{-i} \phi_j : \quad (1.1) \]

and \( C \) acts as the identity operator \( I \). Recall that the normal ordered product \( : \phi_{-i} \phi_j : \) equals \( \phi_{-i} \phi_j \) unless \( i = j < 0 \); if \( i = j < 0 \), it equals \( \phi_{-i} \phi_j - 1 \).

It is convenient to introduce the generating field

\[ B(z, w) = \sum_{i,j \in \mathbb{Z}} (E_{i,j} - E_{-j,-i})z^iw^{-j}. \]

Then the representation of \( b_\infty \) on \( \mathcal{F} \) can be concisely formulated by

\[ B(z, w) = : \phi(z) \phi(w) : . \]

The Fock space \( \mathcal{F} \) decomposes into a direct sum of two irreducible representations of \( b_\infty \):

\[ \mathcal{F} = \mathcal{F}_0 \bigoplus \mathcal{F}_1. \]

Here \( \mathcal{F}_0 \) (resp. \( \mathcal{F}_1 \)) are spanned by \( \phi_\lambda \) with \( \ell(\lambda) \) even (resp. odd) and \( \phi_\lambda \phi_0 \) with \( \ell(\lambda) \) odd (resp. even), where \( \lambda \in SP \).

Introduce the following operator in \( b_\infty \) (where \( t \) is some variable):

\[ B_0(t) = \sum_{k>0} (t^k - t^{-k}) (E_{k,k} - E_{-k,-k}) + \frac{t + 1}{2(t - 1)} I \quad (1.2) \]
Note that $\frac{t+1}{2(t-1)}$ can be rewritten in a more symmetric form as $\frac{t^{1/2}+t^{-1/2}}{2(t^{1/2}-t^{-1/2})}$.

The operator $B_0(t)$ can be formally understood as

$$B_0(t) = \sum_{k \in \mathbb{Z}} t^k \phi_{-k} \phi_k$$

which is the zero-mode of $\phi(tz)\phi(z)$ without normal ordering. Note that for $|t| > 1$, we have

$$B_0(t) = :B_0(t): + \left(1/2 + \sum_{k < 0} t^k \right) I$$

where

$$:B_0(t): = \sum_{k > 0} (t^k - t^{-k}) : \phi_{-k} \phi_k : .$$

We introduce the $n$-point correlation functions

$$R(t_1, \ldots, t_n) = \text{Tr} |_{\mathcal{F}_0} q^{L_0} B_0(t_1) \cdots B_0(t_n).$$

Similarly we define

$$:R : (t_1, \ldots, t_n) = \text{Tr} |_{\mathcal{F}_0} q^{L_0} : B_0(t_1) : \cdots : B_0(t_n) : .$$

It is clear that a multiple of 2 shows up if we have used $\text{Tr} |_{\mathcal{F}}$ instead of $\text{Tr} |_{\mathcal{F}_0}$ above.

Apparently, there is a simple identity to express $R(t_1, \ldots, t_n)$ in terms of $R : (t_1, \ldots, t_n)$, and vice versa. Further, for $\lambda \in SP$,

$$:B_0(t) : (\phi_{\lambda}) = \sum_{k \geq 1} (t^{\lambda_k} - t^{-\lambda_k}) \cdot \phi_{\lambda}$$

$$:B_0(t) : (\phi_{\lambda} \phi_0) = \sum_{k \geq 1} (t^{\lambda_k} - t^{-\lambda_k}) \cdot \phi_{\lambda} \phi_0. \quad (1.3)$$

2. The correlation function via strict partitions

Denote by $SP_n$ the set of strict partitions of $n$, and thus $SP = \bigcup_{n=0}^{\infty} SP_n$. Denote by

$$(a; q)_r = (1-a)(1-aq) \cdots (1-aq^{r-1}), \quad (a; q)_{\infty} = \prod_{i=0}^{\infty} (1-aq^i).$$

It is well known that

$$\sum_{\lambda \in SP} q^{||\lambda||} = (-q; q)_{\infty} = (q; q^2)_{\infty}^{-1}.$$ 

Given a function $f$ on the set $SP$, we denote by

$$(f|_{SP}^S) = (-q; q)_{\infty}^{-1} \sum_{\lambda \in SP} f(\lambda) q^{||\lambda||}.$$
Since $\langle f \rangle_{q}^{SP} = 1$, we may think of $\langle f \rangle_{q}^{SP}$ as the $q$-expectation value of $f$. We regard $t^{\lambda k}$ ($k \geq 1$) as a function on $SP$ (which by our convention takes value 1 on $\lambda$ with $\ell(\lambda) < k$).

Lemma 2.1. We have

$$\sum_{\lambda \in SP} t^{\lambda_{1}q|\lambda|} = 1 + \sum_{n=1}^{\infty} (1 + q)(1 + q^{2}) \cdots (1 + q^{n-1})q^{n}t^{n}.$$

Proof. Follows from the fact that the $n$-th term on the right-hand side of the identity equals the sum of $t^{\lambda_{1}q|\lambda|}$ for $\lambda_{1} = n$. □

The following lemma is a key step in our approach.

Lemma 2.2. For $k \geq 1$, we have

$$\sum_{\lambda \in SP} t^{\lambda_{k+1}q|\lambda|} = \frac{q^{k(k+1)/2}}{(1 - q)(1 - q^{2}) \cdots (1 - q^{k})} \cdot \sum_{\lambda \in SP} (q^{k}t)^{\lambda_{1}}q|\lambda|.$$

Proof. By writing $\lambda = (a_{1}, \ldots, a_{k}, \mu)$ with $\mu = (\mu_{1}, \mu_{2}, \ldots) \in SP$ and $a_{1} > \ldots > a_{k} > \mu_{1}$, we obtain that

$$\sum_{\lambda \in SP} t^{\lambda_{k+1}q|\lambda|} = \sum_{a_{1} > \ldots > a_{k} > \mu_{1}} \sum_{\mu \in SP} t^{\mu_{1}}q^{a_{1} + \ldots + a_{k} + |\mu|}.$$

Note also that (for a fixed $\mu_{1}$)

$$\sum_{a_{1} > \ldots > a_{k} > \mu_{1}} q^{a_{1} + \ldots + a_{k}} = \sum_{a_{1} > \ldots > a_{k} > 0} q^{a_{1} + \ldots + a_{k}}q^{k\mu_{1}}.$$

Thus the proof of the Lemma reduces to the following identity

$$\sum_{a_{1} > \ldots > a_{k} > 0} q^{a_{1} + \ldots + a_{k}} = \frac{q^{k(k+1)/2}}{(1 - q)(1 - q^{2}) \cdots (1 - q^{k})}.$$

This identity can be proved by a direct computation (as done in an earlier version), or as the referee suggests, it is equivalent to the simple fact that a strict partition of length $k$ can be obtained from a partition of length $\leq k$ by adding row by row $(k, k-1, \ldots, 1)$. □

Theorem 2.3. We have

$$\langle \sum_{k \geq 1} t^{\lambda k} \rangle_{q}^{SP} = 1 + \sum_{n=1}^{\infty} \frac{q^{n}t^{n}}{1 + q^{n}}.$$

Proof. We recall a well-known identity of Euler (cf. e.g. Corollary 2.2, [An]):

$$\sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}z^{k}}{(1 - q)(1 - q^{2}) \cdots (1 - q^{k})} = \prod_{r=0}^{\infty} (1 + q^{r}z).$$
Here and below it is understood that the term for $k = 0$ equals 1. We shall use an equivalent form of the Euler identity by setting $z$ above to be $qz$:

$$\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}z^k}{(1-q)(1-q^2) \cdots (1-q^k)} = \prod_{r=0}^{\infty} (1 + q^{r+1}z). \quad (2.1)$$

By Lemmas 2.1 and 2.2, we have

$$\sum_{\lambda \in SP} \left( \sum_{k \geq 0} t^{\lambda_{k+1}} \right) q^{\lambda} = \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(1-q)(1-q^2) \cdots (1-q^k)} \cdot \sum_{\lambda \in SP} (q^k t)^{\lambda} q^{\lambda}$$

$$= \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(1-q)(1-q^2) \cdots (1-q^k)} + \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(1-q)(1-q^2) \cdots (1-q^k)} \cdot \sum_{n=1}^{\infty} \frac{q^k t}{(1-q)(1-q^2) \cdots (1-q^k)} (1 + q)(1 + q^2) \cdots (1 + q^{n-1}) q^n (q^k t)^n$$

$$= \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}}{(1-q)(1-q^2) \cdots (1-q^k)}$$

$$+ \sum_{n=1}^{\infty} (1 + q)(1 + q^2) \cdots (1 + q^{n-1})(qt)^n \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}(q^n)^k}{(1-q)(1-q^2) \cdots (1-q^k)}. \quad (2.2)$$

Applying (2.1) to the r.h.s. of (2.2) with $z = 1$ and $z = q^n$, we have

$$\sum_{\lambda \in SP} \left( \sum_{k \geq 0} t^{\lambda_{k+1}} \right) q^{\lambda} z^{\ell(\lambda)}$$

$$= \prod_{r=0}^{\infty} (1 + q^{r+1}) + \sum_{n=1}^{\infty} (1 + q)(1 + q^2) \cdots (1 + q^{n-1})(qt)^n \cdot \prod_{r=0}^{\infty} (1 + q^{n+r+1})$$

$$= (-q; q)_{\infty} + (-q; q)_{\infty} \sum_{n=1}^{\infty} \frac{q^n t^n}{1 + q^n}.$$

This finishes the proof. \(\square\)

**Remark 2.4.** The same type of argument can be used to establish

$$\sum_{\lambda \in SP} \left( \sum_{k \geq 0} t^{\lambda_{k+1}} \right) q^{\lambda} z^{\ell(\lambda)} = (-qz; q)_{\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{q^n t^n z}{1 + q^n z} \right). \quad (2.3)$$

When $z = 1$, it specializes to Theorem 2.3. Another distinguished specialization is setting $z = -1$.

Our argument can also be adapted to give an alternative argument for Theorem 6.5 of [BO], which is the counterpart of our Theorem 2.3 above. The argument
therein does not seem to apply directly to our case, as the conjugation symmetry of partitions used there is not available for strict partitions.

By (1.3), we have

\[ R : (t_1, \ldots, t_n) = (-q; q)_{\infty} \left( \prod_{i=1}^{n} \sum_{k \geq 1} (t_i^k - t_i^{-k}) \right) q. \]  

(2.4)

Therefore,

\[ R(t_1, \ldots, t_i, \ldots, t_n) = -R(t_1, \ldots, t_i^{-1}, \ldots, t_n) \]

\[ : R : (t_1, \ldots, t_i, \ldots, t_n) = -: R : (t_1, \ldots, t_i^{-1}, \ldots, t_n). \]

Recall the Jacobi \( \theta \)-functions:

\[ \theta_{j,1}(q, t) = \sum_{n \in \mathbb{Z}} q^{n^2} t^n, \quad j = 0, 1. \]

Define

\[ \mathbb{B}(q, t) = \frac{\theta_{1,1}(q, -t)}{\theta_{0,1}(q, -t)} = (-q^{-1/2}t) \frac{(t; q^2)_{\infty} (q^2t^{-1}; q^2)_{\infty}}{(qt; q^2)_{\infty} (qt^{-1}; q^2)_{\infty}}. \]

(2.5)

The second equality above can be easily derived by using twice the celebrated Jacobi triple product identity.

**Theorem 2.5.** For \( |q| < 1, |q| < |t| < |q|^{-1} \),

1. the 1-point correlation function : \( R : (t) \) is given by

\[ : R : (t) = (-q; q)_{\infty} \sum_{n=1}^{\infty} \frac{q^n (t^n - t^{-n})}{1 + q^n} \]

\[ = (-q; q)_{\infty} \sum_{r=0}^{\infty} \left( \frac{(-1)^r q^{r+1} t}{1 - q^{r+1} t} - \frac{(-1)^r q^{r+1} t^{-1}}{1 - q^{r+1} t^{-1}} \right). \]

2. the one-point function \( R(t) \) is given by

\[ R(t) = (-q; q)_{\infty} \cdot t \frac{d}{dt} \ln \mathbb{B}(q, t). \]

**Proof.** Part (1) follows from (2.4), Theorem 2.3 and the following simple identity:

\[ \sum_{n=1}^{\infty} \frac{q^n t^n}{1 + q^n} = \sum_{r=0}^{\infty} \frac{(-1)^r q^{r+1} t}{1 - q^{r+1} t}. \]

By definition of \( R(t) \), we have

\[ R(t) =: R : (t) + \frac{t^{1/2} + t^{-1/2}}{2(t^{1/2} - t^{-1/2})} \cdot (-q; q)_{\infty}. \]
Now part (2) is a consequence of part (1) and the following simple identities (and their counterparts with \( t \) replaced by \( t^{-1} \)):

\[
\frac{d}{dt} \ln \left( t^{1/2} (1 - t) \right) = \frac{t^{1/2} + t^{-1/2}}{2(t^{1/2} - t^{-1/2})}
\]

\[
\frac{d}{dt} \ln \left( (q^2 t; q^2)_{\infty} (qt; q^2)_{\infty}^{-1} \right) = \sum_{r=0}^{\infty} \frac{(-1)^r q^{r+1} t}{1 - q^{r+1} t}.
\]

This finishes the proof. \( \square \)

It is straightforward to check that

\[
\mathbb{B}(q, qt) = -\mathbb{B}(q, t)^{-1}
\]

and thus by Theorem 2.5 that

\[
R(qt) = -R(t).
\]

**Remark 2.6.** The modular transformation properties of \( \mathbb{B}(q, t) \) (by letting \( q = e^{2\pi i \tau} \) and \( t = e^{2\pi i z} \)) and thus of \( R(t) \) (which actually depends on \( q \) as well) can be derived from the well-known modular properties of the \( \theta \)-functions and \( (q^{-1}; q)_{\infty} \).

### 3. Difference equations for the correlation functions

In this section we derive a \( q \)-difference equation satisfied by the correlation function \( R(t_1, \ldots, t_n) \).

**Theorem 3.1.** The function \( R(t_1, \ldots, t_n) \) satisfies the following \( q \)-difference equation:

\[
R(qt_1, t_2, \ldots, t_n) = \sum_{s=0}^{n-1} \sum_{1 < i_1 < \cdots < i_s \leq n} \sum_{\varepsilon_{i_a} = \pm 1} (-1)^{1+\#\varepsilon} R(t_1^{\varepsilon_{i_1}} \cdots t_s^{\varepsilon_{i_s}}, t_{i_1}, \ldots, \hat{t}_{i_1}, \ldots, \hat{t}_{i_s}, \ldots)
\]

where \( \#\varepsilon \) denotes the number of \(-1\)'s among \( \varepsilon_{i_1}, \ldots, \varepsilon_{i_s} \), and \( \hat{t}_i \) means that the very term is removed.

**Proof.** By (1.1) and (1.2), we have

\[
[B_0(t), \phi_k] = -(t^k - t^{-k}) \phi_k, \quad k \in \mathbb{Z}.
\]

Therefore,

\[
B_0(t_2) \cdots B_0(t_n) \phi_k = \sum_{S \subset \{2, \ldots, n\}} (-1)^{|S|} \prod_{i \in S} (t^k - t^{-k}) \cdot \phi_k \prod_{i \not\in S} B_0(t_i).
\]

This further implies that

\[
\text{Tr} \left| F_0 q^{L_0} \phi_{-k} B_0(t_2) \cdots B_0(t_n) \phi_k \right| = \sum_{S \subset \{2, \ldots, n\}} (-1)^{|S|} \prod_{i \in S} (t^k - t^{-k}) \cdot \text{Tr} \left| F_0 q^{L_0} \phi_{-k} \phi_k \prod_{i \not\in S} B_0(t_i) \right|.
\]
On the other hand, by using \( \phi_k q^{L_0} = q^k q^{L_0} \phi_k \) and the cyclic property of a trace, we have
\[
\text{Tr} \left| \mathcal{F}_0 q^{L_0} \phi_k \mathcal{B}_0(t_2) \cdots \mathcal{B}_0(t_n) \phi_k \right| = \text{Tr} \left| \mathcal{F}_0 q^{L_0} \phi_{-k} \mathcal{B}_0(t_2) \cdots \mathcal{B}_0(t_n) \right| = q^k \text{Tr} \left| \mathcal{F}_0 q^{L_0} \phi_k \phi_{-k} \mathcal{B}_0(t_2) \cdots \mathcal{B}_0(t_n) \right|.
\]

Define
\[
\widehat{\mathcal{B}}_0(t) = \sum_{k \in \mathbb{Z}} t^k \phi_k \phi_{-k}.
\]

By multiplying both sides of (3.4) with \( t_1^{\lambda} \) and then summing over \( k \), we obtain with the help of (3.4) that
\[
\text{Tr} \left| \mathcal{F}_0 q^{L_0} \widehat{\mathcal{B}}_0(qt_1) \mathcal{B}_0(t_2) \cdots \mathcal{B}_0(t_n) \right|
\]
\[
= \sum_{s=0}^{n-1} \sum_{1 < i_1 < \cdots < i_s \leq n} \sum_{\varepsilon_{i_a} = \pm 1} (-1)^{s+\# \varepsilon} R(t_1 t_{i_1}^{\varepsilon_{i_1}} \cdots t_{i_s}^{\varepsilon_{i_s}}, \ldots, \widehat{t}_1, \ldots, \widehat{t}_i, \ldots).
\]

Noting that
\[
\mathcal{B}_0(t) = \mathcal{B}_0(t) : + \frac{t^{1/2} + t^{-1/2}}{2(t^{1/2} - t^{-1/2})} I, \quad |t| > 1,
\]
\[
\widehat{\mathcal{B}}_0(t) = - \mathcal{B}_0(t) : - \frac{t^{1/2} + t^{-1/2}}{2(t^{1/2} - t^{-1/2})} I, \quad |t| < 1,
\]
we have
\[
\text{Tr} \left| \mathcal{F}_0 q^{L_0} \widehat{\mathcal{B}}_0(qt_1) \mathcal{B}_0(t_2) \cdots \mathcal{B}_0(t_n) \right| = - R(q t_1, t_2, \ldots, t_n).
\]

This finishes the proof. \( \square \)

**Remark 3.2.** Our formulation indicates clearly the nature of poles when regarding \( R(t_1, \ldots, t_n) \) as a meromorphic function. For example, (3.5) implies that
\[
R(t_1, t_2, \ldots, t_n) = \frac{t_1 + 1}{2(t_1 - 1)} R(t_2, \ldots, t_n) + \text{regular terms on } (t_1 = 1).
\]

As in [BO, OK], the singularities of \( R(t_1, \ldots, t_n) \) and the \( q \)-difference functions determine \( R(t_1, \ldots, t_n) \). Indeed this is the key strategy used in loc. cit. to establish a compact closed formula for their \( n \)-point correlation functions. It is expected the singularities of \( R(t_1, \ldots, t_n) \) and the \( q \)-difference equation can be very useful eventually in finding a closed formula for \( R(t_1, \ldots, t_n) \). Similar remarks apply to the function \( S(t_1, \ldots, t_n) \) studied below.

**Remark 3.3.** In light of (2.3), we can introduce a generalization of the \( n \)-point function \( R(t_1, \ldots, t_n) \) by considering
\[
R(t_1, \ldots, t_n; z) = \text{Tr} \left| \mathcal{F}_0 q^{L_0} z^{\alpha_0} \mathcal{B}_0(t_1) \cdots \mathcal{B}_0(t_n) \right|,
\]
where \( \alpha_0 \in \sum_{k>0} \phi_k \phi_{-k} \). The specialization to \( z = -1 \), denoted by \( R^-(t_1, \ldots, t_n) \), might be regarded as a super \( n \)-point function, which is no less natural to be
considered than the original $n$-point functions. It can be shown in a way similar to Theorem 3.1 to satisfy a $q$-difference equation:

$$R^-(q t_1, t_2, \ldots, t_n) = -R^-(t_2, \ldots, t_n) + \sum_{s=0}^{n-1} \sum_{1<i_1<\ldots<i_s\leq n} \sum_{\varepsilon_{i_1}=\pm 1} (-1)^{s+\#\varepsilon} R^-(t_{i_1}^{\varepsilon_{i_1}}, t_{i_1}^{\varepsilon_{i_1}}, \ldots, \hat{t}_{i_1}, \ldots, \hat{t}_{i_s}, \ldots).$$

Noting

$$[\alpha_0, \phi_{\pm k}] = \mp \phi_{\pm k}, \quad k > 0,$$

we see by (2.3) that (for $n=1$)

$$R(t; z) = \sum_{\lambda \in SP} \left( \sum_{k \geq 1} (t^{\lambda_k} - t^{-\lambda_k}) \right) q^{|\lambda|} z^{(\lambda)} + (-qz; q)_\infty \frac{t^{1/2} + t^{-1/2}}{2(t^{1/2} - t^{-1/2})}.$$

In particular, by using (3.6) we can show as in the proof of Theorem 2.5 that

$$R^-(t) = (q; q)_\infty \cdot t \frac{d}{dt} \ln \left( t^{-\frac{1}{2}} (t; q)_\infty (qt^{-1}; q)_\infty \right).$$

Similar remarks apply to the function $S(t_1, \ldots, t_n)$ studied below.

4. Correlation functions on the twisted Fock space of Neveu-Schwarz type

Introduce a fermionic field of Neveu-Schwarz type

$$\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{-n}$$

with the following commutation relations:

$$[\varphi_m, \varphi_n]_+ = \delta_{m,-n}, \quad m, n \in \frac{1}{2} + \mathbb{Z}.$$

Denote by $\mathbb{F}$ the Fock space of $\varphi(z)$ with the highest weight vector $|0\rangle$ annihilated by $\varphi_n$, $n > 0$. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is called *odd strict* if $\lambda_1 > \lambda_2 > \ldots > \lambda_\ell > 0$ and all $\lambda_i$ are odd integers. Introduce the notations

$$\varphi_\lambda = \varphi_{-\lambda_1/2} \varphi_{-\lambda_2/2} \ldots \varphi_{-\lambda_\ell/2} |0\rangle.$$

Then $\mathbb{F}$ has a linear basis given by $\varphi_\lambda$, where $\lambda$ runs over the set $OSP$ of all odd strict partitions (including the empty partition). The following grading operator $L_0$ defines a natural $\mathbb{Z}_+\mathbb{Z}$-grading on $\mathbb{F}$:

$$L_0(\varphi_\lambda) = (|\lambda|/2) \cdot \varphi_\lambda, \quad \lambda \in OSP.$$
A subalgebra $d_\infty$ of $\hat{gl}_\infty$ acts on $F$ as follows [DJKM] (also cf. [KWY]). The algebra $d_\infty$ is spanned by $E_{i,j} - E_{1-j,1-i}$, where $i, j \in \mathbb{Z}$, and a central element $C$, and the action is given concisely in terms of the generating field

$$\sum_{i,j \in \mathbb{Z}} (E_{i,j} - E_{1-j,1-i}) z^{i+\frac{1}{2}} w^{-j+\frac{1}{2}} =: \varphi(z) \varphi(w) :$$

and $C = I$. The Fock space $F$ decomposes into a direct sum of two irreducible representations of $d_\infty$:

$$F = F_0 \bigoplus F_1.$$ 

Here $F_0$ (resp. $F_1$) are spanned by $\varphi_\lambda$ with $\ell(\lambda)$ even (resp. odd).

Introduce the following operator $D_0(t)$ in $d_\infty$:

$$D_0(t) = \sum_{i \in \frac{1}{2} + \mathbb{Z}^+} \left( t^{i+\frac{1}{2}} - t^{\frac{1}{2} - i} \right) (E_{i,i} - E_{1-i,1-i}) + \frac{t^{1/2}}{t-1} I.$$ 

The operator $D_0(t)$ can be formally understood as

$$D_0(t) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} t^k \varphi_{-k} \varphi_k$$

which is the zero-mode of $\varphi(tz) \varphi(z)$ (without normal ordering). Indeed, for $|t| > 1$, we have

$$D_0(t) = :D_0(t): + \left( \sum_{k \in \frac{1}{2} + \mathbb{Z}, k < 0} t^k \right) I = :D_0(t): + \frac{t^{1/2}}{t-1} I$$ 

where

$$:D_0(t): = \sum_{k \in \frac{1}{2} + \mathbb{Z}} t^k \varphi_{-k} \varphi_k :.$$

We introduce the $n$-point correlation functions

$$S(t_1, \ldots, t_n) = \text{Tr} |_F q^{L_0} D_0(t_1) \cdots D_0(t_n)$$

and similarly define

$$:S:(t_1, \ldots, t_n) = \text{Tr} |_F q^{L_0} :D_0(t_1): \cdots :D_0(t_n):$$

By definition, there is a simple identity of relating $S(t_1, \ldots, t_n)$ to $:S:(t_1, \ldots, t_n)$. It is clear for $\lambda \in OSP$ that

$$:D_0(t):(\varphi_\lambda) = \sum_{k \geq 1} (t^{\lambda_k/2} - t^{-\lambda_k/2}) \cdot \varphi_\lambda.$$ (4.1)
5. The correlation function via odd strict partitions

In this section, we develop the counterpart of Sections 2 and 3. We omit the proofs which are similar to the earlier cases.

Note that
\[ \sum_{\lambda \in OSP} q^{\lambda|/2} = (-q^{1/2}; q)_{\infty}. \]

Given a function \( f \) on the set \( OSP \) of odd strict partitions, we denote by
\[ \langle f \rangle_{OSP}^{OSP} = (-q^{1/2}; q)_{\infty}^{-1} \sum_{\lambda \in OSP} f(\lambda)q^{\lambda|/2}. \]

Since \( \langle I \rangle_{OSP}^{OSP} = 1 \), we may think of \( \langle f \rangle_{OSP}^{OSP} \) the \( q \)-expectation value of \( f \). We regard \( t_{\lambda k} \) (\( k \geq 1 \)) as a function on \( OSP \) which in particular takes value 1 on \( \lambda \) with \( \ell(\lambda) < k \).

Lemma 5.1. We have
\[ \sum_{\lambda \in OSP} t_{\lambda 1/2}q^{\lambda|/2} = 1 + \sum_{n=1}^{\infty} \frac{(1 + q^{1/2})(1 + q^{3/2}) \cdots (1 + q^{n-1/2})q^{n-1/2}t^{n-1/2}}{(1 - q)(1 - q^2) \cdots (1 - q^k)}. \]

Proof. Follows from the fact that the \( n \)-th term on the right-hand side of the identity equals the sum of \( t_{\lambda 1/2}q^{\lambda|/2} \) for \( \lambda_1 = 2n - 1 \).

Lemma 5.2. For \( k \geq 1 \), we have
\[ \sum_{\lambda \in OSP} t_{\lambda k+1/2}q^{\lambda|/2} = \frac{q^{k(k+1)/2}}{(1 - q)(1 - q^2) \cdots (1 - q^k)} \cdot \sum_{\lambda \in OSP} (q^k t_{\lambda k/2})q^{\lambda|/2}. \]

Proof. Follows from the same type of argument as in the proof of Lemma 2.2.

Theorem 5.3. We have
\[ \langle \sum_{k \geq 1} t_{\lambda k/2} \rangle_{OSP}^{OSP} = 1 + \sum_{n=1}^{\infty} \frac{q^{n-1/2}t^{n-1/2}}{1 + q^{n-1/2}}. \]

Recall \( B(q, t) \) was defined in (2.5).

Theorem 5.4. For \(|q| < 1, |q| < |t| < |q|^{-1} \),
(1) the 1-point correlation function \( S : (t) \) is given by
\[
S : (t) = (-q^{1/2}; q)_{\infty} \cdot \sum_{n=1}^{\infty} \frac{q^{n-1/2} (t^{n-1/2} - t^{-n+1/2})}{1 + q^{n-1/2}}
\]
\[
= (-q^{1/2}; q)_{\infty} \cdot \sum_{r=0}^{\infty} \left( \frac{(-1)^r (q^{r+1} t^2)}{1 - q^{r+1} t^2} - \frac{(-1)^r (q^{r+1} t^{-2})}{1 - q^{r+1} t^{-2}} \right)
\]

(2) the 1-point correlation function \( S(t) \) is given by
\[
S(t) = (-q^{1/2}; q)_{\infty} \cdot t \frac{d}{dt} \left( \frac{\ln \mathcal{B}(q^{1/2}, t^{1/2})}{\mathcal{B}(q^{1/2}, -t^{1/2})} \right).
\]

Proof. Part (1) follows from Theorem 5.3 and the following simple identity:
\[
\sum_{n=1}^{\infty} \frac{q^{n-1/2} t^{n-1/2}}{1 + q^{n-1/2}} = \sum_{r=0}^{\infty} \frac{(-1)^r (q^{r+1} t^2)}{1 - q^{r+1} t^2}.
\]

By definition of \( S(t) \), we have
\[
S(t) =: S : (t) + \frac{t^2}{t - 1} \cdot (-q^{1/2}; q)_{\infty}.
\]

We can rewrite by the definition (2.5) of \( \mathcal{B} \) that
\[
\ln \mathcal{B}(q^{1/2}, t^{1/2}) - \ln \mathcal{B}(q^{1/2}, -t^{1/2})
= \frac{1 - t^{1/2}}{1 + t^{1/2}} \prod_{k \geq 1} (1 - q^{k/2} t^{1/2})^{-1} (1 + q^{k/2} t^{1/2})^{(-1)^{k+1}} \cdot \prod_{k \geq 1} (1 - q^{k/2} t^{-1/2})^{-1} (1 + q^{k/2} t^{-1/2})^{(-1)^{k+1}}.
\]

(5.1)

Part (2) can now be derived from (5.1) and part (1) of the Theorem.

It follows from (2.6) and Theorem 5.3 that \( S(qt) = -S(t) \). The modular transformation properties of \( S(t) \) can be derived from the well-known modular properties of the \( \theta \)-functions and \((-q^{1/2}; q)_{\infty}\).

The same proof as in Theorem 3.3 also leads to the following \( q \)-difference equation for \( S \). It is interesting to note that the two distinct functions \( R(t_1, \ldots, t_n) \) and \( S(t_1, \ldots, t_n) \) satisfy the same difference equations but are distinguished by their singularities.

Theorem 5.5. The meromorphic function \( S(t_1, \ldots, t_n) \) satisfies the following difference equation:
\[
S(qt_1, t_2, \ldots, t_n)
= \sum_{s=0}^{n-1} \sum_{1 \leq i_1 < \cdots < i_s \leq n} \sum_{\varepsilon_{i_a} = \pm 1} (-1)^{1 + s + \# \varepsilon} S(t_1 t_{i_1}^{\varepsilon_{i_1}} \cdots t_{i_a}^{\varepsilon_{i_a}}, \ldots, \hat{t}_{i_1}, \ldots, \hat{t}_{i_a}, \ldots)
\]

where \# \( \varepsilon \) denotes the number of \(-1\)'s among \( \varepsilon_{i_1}, \ldots, \varepsilon_{i_a} \).
6. Discussions

In this paper we have studied certain distinguished operators on twisted Fock spaces and the corresponding correlation functions. These functions satisfy certain $q$-difference equations and they can be interpreted as $q$-expectation values of functions on the set of (odd) strict partitions. We found closed formulas for the 1-point functions in terms of Jacobi $\theta$-functions etc, which implies nice modular transformation properties.

The Lie algebra $\mathcal{W}_{1+\infty}$ of differential operators on the circle has two distinguished subalgebras $\widehat{D}^\pm$ induced from anti-involutions [KWY]. There exists canonical Lie algebra homomorphisms from $\widehat{D}^\pm$ to the Lie algebras $b_\infty$, $d_\infty$. In this way, the Fock spaces $\mathcal{F}$ and $\mathcal{F}$ can be regarded as representations of these Lie algebras $\widehat{D}^\pm$, and they decompose into a direct sum of two irreducibles. The study of correlation functions in this paper can thus be interpreted as character formulas of these $\widehat{D}^\pm$-modules. This is similar to the case of [BO], where the correlation functions of the infinite wedge representations were also interpreted as characters of $\mathcal{W}_{1+\infty}$.

In [FKRW] (also cf. [AFMO] and the references therein), irreducible quasi-finite modules of $\mathcal{W}_{1+\infty}$ of higher levels using Fock space were studied in detail, and a similar study has been carried out [KWY] for irreducible quasi-finite modules of $\widehat{D}^\pm$ of higher levels using untwisted and twisted Fock spaces. It will be interesting to study the correlation functions on these modules of higher levels.

In is an interesting question to find closed formulas for the correlation functions $R(t_1, \ldots, t_n)$ and $S(t_1, \ldots, t_n)$ when $n > 1$. The $q$-difference equations and singularities of the functions $R$ and $S$ are expected to play an important role. While the vertex operators are not essentially used in the paper, the correlation functions here can be related to the correlation functions in vertex algebras (cf. [Zhu, DLM, Mi]), and this connection could be useful in understanding these functions further. We refer to Milas [Mi] for very interesting results in this direction.

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