Exotic $E_{11}$ branes as composite gravitational solutions

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Abstract
A two-parameter group element is presented that interpolates between M-brane solutions. The group element is used to interpret a number of exotic branes related to the generators of the adjoint representation of $E_{11}$ as non-marginal half-BPS bound states of M-branes. It is conjectured that the adjoint representation of $E_{11}$ contains only generators related to bound states of fundamental M-branes which, in the limit, may be understood as membrane molecules.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The fundamental solutions of M-theory, the KK-wave [1], the membrane [2], the fivebrane [3] and the KK-monopole [4, 5] were known as solutions of 11-dimensional supergravity or, even, general relativity, prior to the first investigations of M-theory. Solutions particular to M-theory are thin on the ground. More intricate solutions of M-theory can be constructed from the fundamental solutions either as marginal brane intersections or as non-marginal bound states. These solutions are valid beyond the weak-coupling limit of M-theory and are solutions that probe the nature of M-theory’s extension of supergravity.

Of these two classes of solution the M-brane intersections [6] have been classified (for reviews see [7, 8] and the references therein), as having the brane intersection rules of their counterparts with Euclidean worldvolumes, the S-branes [9, 10]. The basic brane intersections are $M2 \perp M2(0)$, $M2 \perp M5(1)$, $M5 \perp M5(3)$ and intersections involving the KK-wave and KK-monopole are also known [11–13]. A marginal brane intersection involving $N$ different branes is typically described by $N$ harmonic functions and preserves $\frac{1}{N}$ of the background.

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1 This notation indicates the pair of branes involved in the intersection, while the number in brackets indicates the number of worldvolume directions along the intersections, so that $M2 \perp M2(0)$ indicates two $M2$-branes intersecting at a point.
supersymmetry. Although classified, the exact forms of the harmonic functions encoding localized intersecting solutions are not known in a closed form. In IIA supergravity, there is an integral form of a localized solution of D2-branes localized in the worldvolume of D6-branes [14] and this construction technique, relying on the special property that D6-branes uplift to a pure gravitational solution, remains the state of the art method.

However, marginal solutions are easy to construct from two branes under less general conditions:

- when both branes are smeared in their relative transverse directions [11];
- when one of the branes is smeared (e.g. for an M2 (012) brane intersecting a second M2 (034) brane at a point, where the harmonic function encoding one of the brane solutions is a function of only the mutual transverse coordinates of both branes); and
- in the near-core limit, where an exact solution to the harmonic functions can be computed in the near horizon limit (or equivalently, a large charge limit) of one of the branes.

In contrast the non-marginal, bound states of M-theory have not been catalogued, but a number of such solutions have been found. The first discovered of these was named the dyonic membrane [15] and was lifted from $N = 2D = 8$ supergravity to an 11-dimensional setting. It consists of an M2-brane delocalized within an M5-brane, the mass of the solution is proportional to $\sqrt{Q^2 + P^2}$, where $Q$ is the (electric) charge of the M2-brane and $P$ the (magnetic) charge of the M5-brane. The mass being proportional to the square root (or sum of square roots) of the charges squared is a characteristic feature of non-marginal, bound states and indicative of the mass energy of the individual states contributing to a binding energy. The dyonic membrane preserves half of the supersymmetries of the background and indeed half-BPS, non-marginal states in general, are characterized by one harmonic function and a number of parameters which interpolate between the constituent marginal brane solutions. In the dyonic membrane solution there is a parameter which interpolates between the M2- and the M5-brane solutions. In the type II supergravities, other half-supersymmetric bound states include the $SL(2, Z)$ multiplet of dyonic bound states of the F1 string and the D1-brane [16] and bound states of strings and Dp-branes [17].

There are a number of techniques used to construct bound states of M-branes. The most common of these makes use of U-duality, but solutions have also been constructed in great generality by an analysis of zero modes [18–20]. Such solutions are interesting for a number of reasons beyond cataloguing the non-perturbative, strongly coupled regime of M-theory; however, we will study their existence in this paper for their own intrinsic value. There is no systematic method for constructing non-marginal solutions from marginal solutions in 11 dimensions. It will be shown in this paper that M-theory bound state solutions can be extracted from the Kac–Moody algebra $E_{11}$ using a straightforward ansatz to relate the bound states to a number of marginal solutions.

The Kac–Moody algebra $E_{11}$ is conjectured to be a symmetry algebra of M-theory [21] and an ansatz has been found, in the form of a solution-generating group element, which encodes the half BPS M-brane solutions [22]. To each positive root of the adjoint representation of $E_{11}$ the method related a half BPS M-brane solution and a harmonic function. Since there are an infinite number of such roots the construction suggested that there existed an infinite set of unknown, exotic branes beyond the four basic M-branes. These exotic branes would also be described in terms of one harmonic function and would preserve, supposedly, one half of the supersymmetry. Upon dimensional reduction, the mixed symmetry tensors associated with

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2 The KK-monopole was not included in the solutions of [22], although one can see from [23] where the solution is presented in the context of $E(11)$ that it also fits into the solution-generating group element ansatz. We will review this solution later in this paper.
exotic brane solutions are of direct use for deriving maximal gauged supergravity theories [24, 25] in lower dimensions, as ten-forms used to complete the IIB superalgebra [26, 27] or as gauge potentials for Dp-branes in ten dimensions [28]. However, there is some doubt about the role played by the full \( E_{11} \) symmetry in M-theory and especially whether the high level roots associated with exotic branes have a physical significance directly in 11 dimensions. Work on U-duality multiplets of brane charges in lower dimensions [29–33] revealed the existence of exotic brane charges and also their need for a higher dimensional origin. In [34, 35], it was shown that the exotic brane charges could be derived from the fundamental, or \( l_1 \), representation of \( E_{11} \). Furthermore the tensions of these exotic solutions were derived and it was seen that with the exception of the KK-wave, the M2-brane and the M5-brane, all the branes related to positive roots within the \( E_{11} \) adjoint representation had divergent tensions in a non-compact background, indicative of solutions in the strongly coupled regime. This observation is a clue to interpret such branes as non-marginal bound state solutions which is the purpose of this paper. A second clue lies in the observation that the tensions of the exotic states indicate objects which sweep out more dimensions with their worldvolume than there are dimensions in the background spacetime. A simple resolution to this problem occurs if the exotic states are actually composite solutions.

In section 2, we demonstrate our preferred method for visualizing the generators of the \( E_{11} \) algebra as tensor representations of \( SL(11) \). By dint of the choice of vector space basis and inner product, a very rapid, back-of-the-envelope generation of the Young tableaux appearing at the general level, \( l \), in the decomposition is achieved. In section 3, the half-BPS M-branes and their relation to roots of \( E_{11} \) is recalled, with particular focus on the KK-monopole. In section 4, the dyonic membrane, the prototype bound state of M-branes, is studied and the relationship between the roots of \( E_{11} \) and non-marginal bound state solutions is given. The central thesis of this paper is outlined: that to each positive root of \( E_{11} \) a non-marginal, bound state solution may be associated. In particular, the two-parameter group element relating three fundamental brane solutions associated with roots: \( \beta_1, \beta_2 \) and \( \beta_3 \equiv \beta_1 + \beta_2 \), is

\[
g_{\beta_3} = \exp \left( -\sum_{i} \frac{1}{\beta_i^2} \ln N_i (\beta_i \cdot H) \right) \exp \left( (1 - N_1)N_2 \beta_1 \beta_2 \sin \xi E_{\beta_1} \right)
+ \left( i(1 - N_2)N_1 \beta_1 \beta_2 \tan \xi E_{\beta_2} + (1 - N_1)N_1 \beta_1 \beta_2 \cos \xi E_{\beta_3} \right),
\]

(1)

where \( N_1 \) and \( N_2 \) are harmonic functions on the overall transverse space coordinates for the brane solutions associated with the roots \( \beta_1 \) and \( \beta_2 \). They are related by a continuous parameter, \( \xi \), defined by

\[
\cos^2 \xi = \frac{1 - N_2}{1 - N_1}.
\]

(2)

We generate a number of solutions to indicate the general method including the transversely boosted M2- and M5-branes. In section 5, we consider some solutions which interpolate between three brane charges and develop the ansatz for the group element, finally using the group element to construct a bound state solution interpolating between the membrane, two fivebranes and the KK-monopole described by one harmonic function and two angle parameters.

2. The \( E_{11} \) algebra as Young tableaux

Our aim in this section is to give the definitions of the Kac–Moody algebra \( E_{11} \) and to derive the algebraic content of its adjoint representation in a simple way as products of \( SL(11) \) Young tableaux using the Littlewood–Richardson rule.
The Kac–Moody algebra of $E_{11}$ is defined from its Cartan matrix, $A_{ab}$, which may be read off from its Dynkin diagram and is shown in figure 1. There is a node on the Dynkin diagram for each positive, simple root, $\alpha_a$, and the Cartan matrix is defined in terms of these as

$$A_{ab} = \frac{2 \langle \alpha_a, \alpha_b \rangle}{\langle \alpha_a, \alpha_a \rangle}.$$  

(3)

The Kac–Moody algebra is defined in terms of its Cartan matrix as

$$[H_a, E_b] = A_{ab} E_b \quad [H_a, F_b] = -A_{ab} F_b \quad [E_a, F_b] = \delta_{ab} H_a$$

(4)

$$[E_a, [E_a, \ldots [E_a, E_b, \ldots]]] = 0 \quad [F_a, [F_a, \ldots [F_a, F_b, \ldots]]] = 0,$$

(5)

where $H_a$, $E_a$, and $F_a$ are the Chevalley generators of the Cartan subalgebra, the positive roots and the negative roots, respectively. In each of the relations of the the second line, (5), there are $(1 - A_{ab})$ commutators—these relations are the Serre relations. The relations of the first line are easily understood as $r$ copies of the $SU(2)$ algebra, where $r$ is the rank of the Cartan matrix. In the Dynkin diagram there are $r$ nodes and each node indicates an $SU(2)$ algebra. The number of lines connecting the nodes $a$ and $b$ in the Dynkin diagram is given by the negative off-diagonal entries of the Cartan matrix, namely $-A_{ab}$. When two positive root generators do not commute, there exists a third positive root generator whose associated root is the sum of the two roots associated with the two generators in the commutator

$$[E_a, E_b] = E_{a+b} \neq 0 \iff \alpha_a + \alpha_b \in \Pi^+,$$

(6)

where $\Pi^+$ is the set of positive roots of the algebra. The Serre relations indicate the termination of a root string in the algebra, but being a set of nested commutators they are very difficult to work with. Some of the information encoded in the Serre relations can be expressed as a condition on the root length squared, that, if met, indicates the existence of that root in the root system and an associated generator in the algebra. Some information is lost; we will not recover the root multiplicity in this fashion—however, this can be calculated recursively using, for example, the Peterson formula. Let us derive the condition on the root length squared from the Serre relations for $E_{11}$.

The algebra $E_{11}$ is a simply laced algebra, meaning that all its simple roots have the same length. We will normalize our roots so that their length squared is 2. This simplifies the definition of the Cartan matrix

$$\alpha_a^2 = \langle \alpha_a, \alpha_a \rangle = 2 \iff A_{ab} = \langle \alpha_a, \alpha_b \rangle.$$ 

(7)

For a simply laced algebra, the entries of the Cartan matrix can take only three values: 2 for $A_{aa}$, $-1$ for $A_{ab}$ if node $a$ is connected to node $b$ in the Dynkin diagram and 0 for $A_{ab}$ if nodes $a$ and $b$ are not directly connected by a line on the Dynkin diagram. This gives three possible expressions for the Serre relations on the simple positive root generators$^3$ summarized in

$^3$ It is sufficient to focus on the positive root generators, since the negative root generators will have an associated root which is simply the negative of its positive counterpart. If the positive root exists, then so does the negative root.
Table 1. The Serre relations on the generators of the positive simple roots of a simply laced algebra.

| $A_{ab}$ | $1 - A_{ab}$ | Serre relation |
|----------|--------------|----------------|
| 2        | $-1$         | $[E_a, E_b] = 0$ |
| 0        | $1$          | $[E_a, E_b] = 0$ |
| $-1$     | 2            | $[E_a, [E_a, E_b]] = 0$ |

Table 1. The first two cases are not interesting, but in the third case, the commutator $[E_a, E_b]$ may be non-zero—this is the case when the nodes $a$ and $b$ are connected by a line on the Dynkin diagram. In the case of $E_{11}$, if we commence with the root $\alpha_{11}$ we may construct a new root $\alpha_{11} + \alpha_8$, since there is a connection between nodes 11 and 8 on the Dynkin diagram. We could not, for example, find a root $\alpha_{11} + \alpha_{10}$ since nodes 11 and 10 are not connected on the Dynkin diagram. The root system of a simply laced algebra consists of sums of simple roots which are all connected when superposed on the Dynkin diagram. In this way, the Serre relation imposes irreducibility on any representation of the generators of the positive roots. Conversely if $\alpha_{11} + \alpha_{10}$ had been a root, then the representation of the generators would have been reducible since there would be a subrepresentation of $SU(2)$ associated with node 10 of the Dynkin diagram. We have sketched a direct relation between $A_{ab} = \langle \alpha_a, \alpha_b \rangle$ being negative and the Serre relations giving a non-zero commutator for the positive root generators. Consequently we can now relate the Serre relations to a condition on the root length.

Consider the root length for the case we have just considered of adding two simple roots, $\alpha_a$ and $\alpha_b$, to find a new root $\alpha_a + \alpha_b$. In this case the new root length is

$$(\alpha_a + \alpha_b)^2 = \alpha_a^2 + \alpha_b^2 + 2\langle \alpha_a, \alpha_b \rangle$$

$$= 2 + 2 + 2A_{ab}$$

$$= 2,$$

where we have used the observation that if $\alpha_a + \alpha_b$ is a root then $A_{ab} = -1$. Now suppose we have a third root, $\alpha_c + \alpha_b + \alpha_a$, where $\alpha_c$ is another simple root. Since we have asserted it is a root, we see that

$$[E_c, [E_a, E_b]] = [E_c, [E_a, E_b]] = E_{a+b+c} \neq 0.$$  

(8)

Let us expand the nested commutator expression using the Jacobi identity

$$[E_c, [E_a, E_b]] = [[E_c, E_a], E_b] + [E_a, [E_c, E_b]] \neq 0.$$  

(9)

Both the left-hand side and the right-hand side are non-zero, so we have three possible conditions from the Serre relations:

1. $A_{ab} = -1$, $A_{ac} = -1$ and $A_{bc} = 0,$
2. $A_{ab} = -1$, $A_{bc} = 0$ and $A_{bc} = -1,$
3. $A_{ab} = -1$, $A_{ac} = -1$ and $A_{bc} = -1.$

Let us consider the root length squared for the different cases

$$(\alpha_a + \alpha_b + \alpha_c)^2 = \alpha_a^2 + \alpha_b^2 + \alpha_c^2 + 2\langle \alpha_a, \alpha_b \rangle + 2\langle \alpha_a, \alpha_c \rangle + 2\langle \alpha_b, \alpha_c \rangle$$

$$= 2 + 2 + 2A_{ab} + 2A_{ac} + 2A_{bc}$$

$$= \begin{cases} 2 & \text{cases 1 and 2} \\ 0 & \text{case 3.} \end{cases}$$

Cases 1 and 2 are the minimal examples for the existence of a new root according to the Serre relations; in these cases, node $c$ is connected to only one of nodes $a$ and $b$ and the root length
squared is equal to 2. In case 3, node c is connected to both nodes a and b and the root length squared decreases by 2. One may continue this process and build up the root string, it is clear that if one does this that the Serre relations provide a bound on the root length squared. Explicitly, a root, β, in a simply laced algebra will always satisfy

$$\beta^2 = 2, 0, -2, -4, \ldots.$$  \hfill (10)

Our intention now is to apply this condition to the roots associated with generators with the correct index structure to appear in the algebra.

By the level decomposition of the $E_{11}$ algebra into $SL(11)$ representations, the generators of $E_{11}$ may be represented by Young tableaux. Recall that upon deletion of the 11th node of the $E_{11}$ Dynkin diagram the remaining diagram is that of $A_{10}$, or $SL(11)$. A generic root associated with the $E_{11}$ algebra requires multiple deletions of $\alpha_{11}$ before a root of $SL(11)$ remains, the number of deletions of $\alpha_{11}$ needed for this to occur is called the level of the decomposition. The deleted root can be written as a sum of two parts, the first part, $x$, being orthogonal to all the simple roots of $A_{10}$ and the second part, $-\lambda_8$, being in the $A_{10}$ weight lattice:

$$\alpha_{11} = x - \lambda_8,$$  \hfill (11)

where $\lambda_8$ is a fundamental weight of $A_{10}$, having the defining property that $\langle \lambda_1, \alpha_j \rangle = \delta_{ij}$. Consequently the inner products between the roots of $A_{10}$ and $\alpha_{11}$ are those of the $E_{11}$ Cartan matrix. The deletion of $\alpha_{11}$ corresponds to a representation of $A_{10}$ whose highest weight is $\lambda_8$ which is a rank 3 antisymmetric $SL(11)$ tensor.

At this point, it will be useful to express the roots of $E_{11}$ in a vector space basis, $\{e_1, e_2, \ldots, e_{11}\}$, instead of a simple root basis, $\{\alpha_1, \alpha_2, \ldots, \alpha_{11}\}$. One can rewrite the simple roots as

$$\alpha_i = e_i - e_{i+1} \quad i \leq 10 \quad \alpha_{11} = e_9 + e_{10} + e_{11},$$  \hfill (12)

on a vector space endowed with the inner product

$$\langle a, b \rangle = \sum_{i=1}^{11} a_i b_i - \frac{1}{9} \sum_{i=1}^{11} a_i \sum_{j=1}^{11} b_j$$  \hfill (13)

where

$$a = \sum_i a_i e_i, \quad b = \sum_i b_i e_i.$$  \hfill (14)

One can check that the roots expressed in this basis with this inner product obey the inner products of the $E_{11}$ Cartan matrix. Now we are in a position to notice a curious thing, namely that in this basis the index structure of the associated generators may be read off immediately from the root. The rule to follow is simply that in this basis the index structure of the associated generators may be read off immediately. For example, consider the roots $\{\alpha_1, \ldots, \alpha_{10}\}$ which are the simple positive roots of $A_{10}$. Following these rules, we can write down the simple root generators of $A_{10}$ ($\alpha_i = e_i - e_{i+1}$) as $K_i^{e_i}$ by simply reading off the coefficients of the roots in the $e_i$ basis. For $\alpha_{11} = e_9 + e_{10} + e_{11}$, we write down the generator $R^{(10)11}$ and for $\beta_{M5} = e_5 + e_7 + \cdots + e_{11}$, we find the generator $R^{67 \cdots 11}$. Since the full $E_{11}$ algebra is constructed from multiple commutators of these generators, the rule for reading off the index structure of the $SL(11)$ tensor generators from the root vector in the $\{e_i\}$ basis is true for the full algebra. As an example of a mixed symmetry generator, consider the KK-monopole whose gauge field is associated with the generator $R^{4 \cdots 11, 11}$—one readily deduces that the associated root is $e_4 + \cdots + e_{10} + 2e_{11}$. For a general root $\beta \equiv \sum_i e_i e_i$, the Young tableau for
the associated generator has $c_{11}$ boxes in its top row, $c_{10}$ boxes in the next row and so on down to $c_1$ boxes in the bottom row. We are able to rapidly associate Young tableaux with roots of $E_{11}$ expressed in the $e_i$ basis or vice versa.

Returning to the decomposition and having observed that the deleted root, $\alpha_{11} = e_9 + e_{10} + e_{11}$, is associated with an antisymmetric three tensor, we can construct the $E_{11}$ algebra. At level 1 we draw the Young tableau for an antisymmetric 3-tensor:

![Young tableau for an antisymmetric 3-tensor](image)

(15)

To be explicit we would write the coordinates 11, 10, 9 descending from top to bottom in the Young tableau; this would correspond to a highest weight $SL(11)$ tensor. In the following, we will assume that an empty Young tableau indicates the highest weight $SL(11)$ tensor: each column will be associated with coordinates starting with 11 in the top box and decreasing in units until the base of each column is reached. We may now check the root length of the Young tableau (to confirm that it is not projected out by the Serre relations) using (13), and find it is 2, as expected and it is in the root system. At the next level, we use the Littlewood–Richardson rule to construct the products of the Young tableau with the Young tableaux associated with roots at the previous level. At level 2, we find the highest weight Young tableaux:

![Young tableaux at level 2](image)

(16)

These roots have lengths 2, 4, 6 and 8, respectively. Hence, only the first Young tableau, corresponding to an antisymmetric 6-tensor, appears at level 2. At level 3 we have

![Young tableaux at level 3](image)

(17)

The Young tableaux shown have roots with length squared 0 and 2, and the rest of the Young tableaux do not appear in the algebra because their associated root length squared is greater than 2. It is straightforward to check that the consequence of moving a box from one column to an adjacent one directly to its right, while still having a valid Young tableau (i.e. the columns have equal or descending length from left to right) is to add 2 to the root length squared. Consequently the root length condition means that one constructs only the tallest and thinnest Young tableaux at each level. One can give a general rule for constructing the highest weight Young tableaux at an arbitrary level, $l$, associated with the $E_{11}$ algebra: take $3l$
boxes and arrange them into the tallest and thinnest Young tableaux, with the constraint that the column height is at most 11. Those that have length squared less than or equal to 2 are, up to multiplicity considerations, associated with generators present in the algebra. At level 3, considered above, the completely antisymmetric 9-tensor is not present in the algebra due to multiplicity considerations. It appears that there is at least one Young tableau per level, after level 2, which has outer multiplicity of zero and hence is not present in the algebra [36]. It would be interesting to understand the relatively infrequent occurrence of roots with outer multiplicity 0. For the purpose of this paper, we will ignore multiplicity considerations.

3. $E_{11}$ and the standard M-theory solutions

A solution-generating group element encoding $\frac{1}{2}$-BPS solutions of 11-dimensional supergravity was found in [22] and its generalization in [37] was shown to reconstruct all the $\frac{1}{2}$-BPS solutions to the maximally oxidized supergravities associated with a $G^{+++}$ symmetry. The group element takes the following form:

$$g_\beta = \exp \left( -\frac{1}{\beta^2} \ln (NH \cdot \beta) \right) \exp ((1-N)E_\beta),$$

where $\beta$ is a root in the adjoint representation of $E_{11}$, $E_\beta$ is its associated generator in the algebra, $H$ are the elements of the Cartan sub-algebra and $N$ is a harmonic function. The harmonic function $N$ appears as a solution to the equations of motion found by varying the generic gravity action

$$\mathcal{L} = \int d^{11}x \sqrt{-g} \left[ R - \frac{1}{n!} (F_{a_1...a_n})^2 \right],$$

where $F_{a_1...a_n}$ is the field strength derived from the potential $A_{a_1...a_n}$. This gauge field appears as the coefficient of the generator $E_\beta$ in the group element of equation (18). As such the group element gives an on-shell description of the $\frac{1}{2}$-BPS brane solutions. Note that for these solutions described by a single active potential, the Chern–Simons term vanishes; for more complicated potentials, this will no longer be the case.

3.1. Fundamental solutions of M-theory

Let us familiarize ourselves with the group element by reviewing the examples of the KK-wave, the M2-brane and the M5-brane solutions as given in [22]. The KK-monopole, which we will refer to herein as the KK6-brane or KK6-monopole as upon dimensional reduction it gives rise to the D6-brane, has been related to a group element of the form of (18) [23]. Here we will review how it is encoded in the group element but in a more complicated way than the other half-BPS solutions.

The KK-wave solution is associated with the roots of the $SL(11)$ sub-algebra which appear at level 0 in the decomposition of the $E_{11}$ algebra. For example, consider the case when $\beta_{KK} = \alpha_{10}$ in (18), so that $E_\beta = K_{10}^{11}$. In this case,

$$H \cdot \beta_{KK} = K_{10}^{10} - K_{11}^{11}.$$

We use (18) to read off the line element

$$ds^2 = dx_1^2 + \cdots + dx_9^2 - N^{-1} dt_{10}^2 + N(dx_{11} - (N^{-1} - 1) dt_{10})^2,$$

where we have chosen the tenth coordinate to be timelike. Note that the gauge field in this case gives an off-diagonal vielbein component premultiplying the generator $K_{10}^{11}$; in tangent

4 The gauge field is also known as the dual gravity field and is a purely gravitational solution.
space indices, this is just $(1 - N)$ whereas in worldvolume indices it becomes $(N^{-1} - 1)$, the factor rotating $dx_{11}$ into $dt_{10}$. The ansatz for solution generation is to take the function $N(x_1, \ldots, x_9)$ to be a harmonic function, specifically $N = 1 + \frac{Q}{r^3}$.

The M2-brane solution is associated with the simple root $\beta_{M2} = e_9 + e_{10} + e_{11}$ of $E_{11}$ appearing at level 1 in the algebra. This root is associated with the highest weight of the generator $R^{91011}$, the component $R^{91011}$ of $E_6$. Consequently,

$$H \cdot \beta_{M2} = -\frac{1}{3}(K^{1} + \cdots + K^{8}) + \frac{2}{3}(K^{9} + \cdots + K^{11})_1. \tag{20}$$

From equation (18), one can read off the vielbein components to find the line element of the M2-brane

$$ds^2_{M2} = N^\frac{1}{2}(dx_1^2 + \cdots + dx_8^2) + N^{-\frac{1}{2}}(-dt_6^2 + dx_{10}^2 + dx_{11}^2), \tag{21}$$

where $N(x_1, \ldots, x_8)$ is a harmonic function of the transverse coordinates:

$$N = 1 + \frac{Q}{r^3}. \tag{22}$$

It satisfies $d \ast F = 0$, where $F = dN^{-1}$, which can be rewritten as the curved space Laplace equation

$$\partial_{\mu}(\sqrt{|g|}g^{\mu\nu}g^{99}g^{1010}g^{1111}F_{\nu1011}) = 0. \tag{23}$$

The M5-brane is derived from the level 2 root $\beta_{M5} = e_6 + \cdots + e_{11}$, so that,

$$H \cdot \beta_{M5} = -\frac{2}{3}(K^{1} + \cdots + K^{5}) + \frac{1}{3}(K^{6} + \cdots + K^{11}) \tag{24}$$

and the method gives the line element

$$ds^2_{M5} = M^\frac{1}{2}(dx_1^2 + \cdots + dx_5^2) + M^{-\frac{1}{2}}(-dt_6^2 + \cdots + dx_{10}^2 + dx_{11}^2), \tag{25}$$

where $M(x_1, \ldots, x_5)$ is harmonic, so that

$$M = 1 + \frac{Q}{r^3}. \tag{26}$$

The root at level 4 has $\beta_{K6} = e_4 + \cdots + e_{11} + 2e_{11}$ and is associated with the KK6-brane. The relation between the harmonic functions encoding the KK6 monopole solution and the root $\beta_{K6}$ was given in [23]. The solution has not been understood in terms of the solution-generating group element (18) previously and so we discuss its derivation in detail here.

We compute

$$H \cdot \beta_{K6} = -(K^{1} + \cdots + K^{3}) + K^{11}, \tag{27}$$

which gives a diagonalized metric

$$ds^2_{K6} = N(dx_1^2 + \cdots + dx_3^2) + N^{-1}(dx_{11}^2)^2 + d\Omega^2_{(1,6)}. \tag{28}$$

Unlike the membrane and fivebrane solutions, the KK6-brane is a pure gravity solution. Its gauge field $A^{4,11,11}$ is dual to the vielbein field and as with the KK-wave adds an off-diagonal contribution to the metric. The field strength associated with the dual gravity field is dualized as

$$\frac{1}{3!}e_{a_1 \cdots a_4}F_{a_1 \cdots a_3, b} = F^{a_1 a_2 a_3}b = 2\xi[a_1 \xi^{a_2} b]_b. \tag{29}$$

In particular, for the highest weight with component $A_{4,11,11}$, whose field strength is related to the harmonic function $N(x_1, x_2, x_3)$ as

$$F_{411,11} = \partial_k N^{-1} = -N^{-2}\partial_k N \quad \Rightarrow \quad F^{411,11} = -N\partial^k N = -\partial_k N, \tag{30}$$
where \( i, j, k \in \{x_1, x_2, x_3\} \) and by raising the indices prior to taking the Hodge dual we ensure that the dual field has a set of covariant indices. Consequently,
\[
E_{ij}^{11} = 2 \delta_{ij} h_{3j}^{11} = -\epsilon_{ijk} \partial_k N = \nabla \wedge A.
\]
(31)
From the solution-generating group element, we conclude that it has line element
\[
d s_{KK6}^2 = N(dx_1^2 + \cdots + dx_5^2) + N^{-1}(dx_{11}^2 - A_i \cdot \dot{x}_i)^2 + d\Omega_{(1,6)}^2.
\]
(32)
where \( N(x_1, x_2, x_3) = 1 + \hat{\partial} \hat{r}^2 \) is a harmonic function and \( i = \{1, 2, 3\} \). This is a Wick rotation of the Gross–Sorkin–Perry KK-monopole. Consider as a particular example that case when only the off-diagonal vierbein component \( h_3^{11} \) is non-zero, which corresponds to limiting \( N \) to be harmonic in only \( x_1 \) and \( x_2 \) [23], then the line element is a smeared gravitational monopole:
\[
d s_{KK6}^2 = N(dx_1^2 + \cdots + dx_5^2) + N^{-1}(dx_{11}^2 + (1 - N) dx_3^2 - dt^2 + \cdots + dx_{10}^2
\]
\[
N(x_1, x_2) = Q \ln(r),
\]
(33)
where we have used equation (31) to identify the components \( A_i \) with the field \( h_3^{11} = 1 - N \).

Returning to the general KK6-brane solution of equation (32) and rewriting the metric in spherical coordinates, \( \theta, \phi \) and a shifted radial coordinate, \( \hat{r} = r + K \), we arrive at a form derived from the Euclidean Taub-NUT metric embedded in 11-dimensional spacetime:
\[
d s_{NUT}^2 = N(\hat{r}^2 - K^2)(d\theta^2 + \sin^2 \theta d\phi^2) + N^{-1}(dx_{11}^2 - 2K \cos \theta d\phi)^2 + d\Sigma_{1,6}^2
\]
\[
N = \frac{\hat{r} + K}{\hat{r} - K}.
\]
(34)
This is the extremal version of the Euclidean Taub-NUT metric in 11 dimensions.

The new technique apparent in deriving this solution from the roots of equation (2) is the use of dual gauge potentials. As we see above, choosing to work with a dual field can cast a solution in a recognizable form. In the case of the KK6-brane, the \((9, 1)\) field strength that one expects to be associated with the \((8, 1)\) gauge field is recast as the field strength of the dual graviton and is absorbed into the volume element.

3.1.1. Hodge duality. The fundamental solutions are paired up by Hodge duality: the M2(S2)-brane is related to the S5(M5)-brane and the KK wave to the KK6 monopole. In terms of the roots that the solutions are derived from the dual solutions are mapped to each other by
\[
\beta_{\pm M_p} = \beta_0 - \beta_{M_p},
\]
(35)
where
\[
\beta_0 \equiv e_3 + e_4 + \cdots + e_{11}
\]
(36)
This is a root at level 3 in the decomposition of \( E_{11} \) into representations of \( SL(11) \) which does not appear in the algebra due to multiplicity considerations. It is associated with a tensor with nine antisymmetrized indices. Hodge duality in 11 dimensions relates gauge fields with \( p + 1 \) antisymmetric indices to those with \( 9 - (p + 1) \) antisymmetric indices, hence the role of \( \beta_0 \) in relating dual solutions. We propose the following group element for dual solutions:
\[
g_{\beta} = \exp \left( -\frac{1}{\beta_{\pm M_p}^2} \ln N(H \cdot \beta_0) + \frac{1}{\beta_{\pm M_p}^2} \ln N(H \cdot \beta_{M_p}) \right)
\]
\[
\times \exp \left( (1 - N) N^{-1} \beta_{\pm M_p}^2 \hat{E}_{\beta_{M_p}} \right).
\]
(37)
Consider the example of the $S_5$ derived from the $M_2$ group element. Taking 
\[ \beta_{M_2} = e_9 + e_{10} + e_{11}, \quad \beta_{S_5} = \beta_9 - \beta_{M_2} = e_3 + \cdots + e_8 \]
\[ \Rightarrow \beta_{S_5} \cdot \beta_{M_2} = -2. \quad (38) \]
Substituting this into (37) gives 
\[ g_\beta = \exp\left( -\frac{1}{2} \ln N (H \cdot \beta_{S_5}) \right) \exp \left( (1 - N) N R^{21011} \right). \]

Reading off the metric gives 
\[ ds^2 = \frac{N}{2} (d\chi_3^2 + \cdots + d\chi_8^2) + d\chi_1^2 + d\chi_2^2 + d\chi_9^2 + \cdots + d\chi_{11}^2. \quad (39) \]
Using the vielbein to convert the tangent space indices of the gauge field to worldvolume indices, we find the dual field strength 
\[ F_4 = -dN. \quad (40) \]

Similarly we can directly find the KK6 monopole metric (up to a Wick rotation) from the KK-wave group element. The relevant roots and inner products are 
\[ \beta_{KK} = e_3 - e_{11}, \quad \beta_{KK6} = \beta_9 - \beta_{KK} \]
\[ \Rightarrow \beta_{KK} \cdot \beta_{KK6} = -2. \quad (41) \]

The dualized group element for the KK6 monopole is 
\[ g_\beta = \exp\left( -\frac{1}{2} \ln N (H \cdot \beta_{KK}) \right) \exp \left( (1 - N) N K^{311} \right). \quad (42) \]

After converting the gauge field to worldvolume indices, one reproduces the KK6 monopole volume element, and one can see clearly that the solution is purely gravitational in this dual form.

### 3.2. Marginal intersecting M-brane solutions

The intersecting brane solutions were first understood in [38, 39] to arise for branes associated with roots, $\beta_1$ and $\beta_2$, such that $\beta_1 \cdot \beta_2 = 0$. The combination of two solution-generating group elements (18) one finds is [22] 
\[ g_1 g_2 = \exp \left( -\frac{1}{2} \ln N_1 (H \cdot \beta_1) - \frac{1}{2} \ln N_2 (H \cdot \beta_2) \right) \]
\[ \times \exp \left( (1 - N_1) N_2 F_{\beta_1} + (1 - N_2) F_{\beta_2} \right). \quad (43) \]

This group element does not possess a manifest symmetry between the harmonic functions of the two brane solutions, except in the case when $\beta_1 \cdot \beta_2 = 0$. The cases when this occurs correspond to the basic M-brane intersections. We show examples of the roots and the associated marginal solutions in table 2.

In table 2, the number in brackets indicates the number of overlapping spacelike dimensions and we indicate that the Taub-NUT direction of the KK6-brane is included amongst the intersection directions by a prime, so that KK6 $\perp M_2(1')$ indicates a KK-monopole intersecting an M2-brane over a string in the Taub-NUT direction. These solutions, and multiple combinations of them, were originally found using properties of supersymmetry, dimensional reduction, the method of harmonic superposition (subsequently identified as a no-force condition between the constituent branes) and analysis of the equations of motion [6, 9–13, 40–45]. The solutions in table 2 are described by two harmonic functions and preserve $\frac{1}{4}$ of the supersymmetry of the background. The solutions may be treated as building blocks and one may combine the intersection rules to find solutions with $N$ harmonic functions.
and preserving \( \frac{1}{3} \) of the supersymmetries. For example one can construct a \( \frac{1}{8} \)-BPS solution involving three M2-branes each of which intersects the other M2-branes over a point, i.e. \( \text{M2}(012) \perp \text{M2}(034) \perp \text{M2}(056) \). The limiting factor in constructing these intersections is the dimension of the background spacetime, or, equivalently, the amount of supersymmetry that may be broken.

For any of the solutions of table 2, the volume element may be reconstructed using the harmonic superposition rule \([11]\). The two metrics of the contributing brane solutions are superposed and their harmonic functions are restricted to be harmonic in only the overall transverse directions. This is equivalent to smearing a constituent brane over the worldvolume directions of the other constituent branes. For example considering the \( \text{M2} \perp \text{M2}(0) \) solution, from the solution-generating group element for \( \beta_1 \) and \( \beta_2 \) (which we can read from table 2), the two membranes have the following worldvolumes:

\[
\begin{align*}
\text{dx}_1^2 &= N_1^{-1} \left( N_1^{-1} (dx_{10}^2 + dx_{11}^2 - dt_{11}^2) + dy_j dy^j \right) \quad i \in \{1, \ldots, 8\} \tag{44} \\
\text{dx}_2^2 &= N_2^{-1} \left( dx_{10}^2 + dx_{11}^2 - dt_{11}^2 \right) + dy_j dy^j \quad j \in \{1, \ldots, 6, 9, 10\}.
\end{align*}
\]

Superposing the solutions gives

\[
\begin{align*}
\text{dx}^2_{\perp 2} &= N_1^{-1} N_2^{-1} \left( dx_{10}^2 + dx_{11}^2 - dt_{11}^2 \right) - N_1^{-1} N_2^{-1} dt_{11}^2 \\
&\quad + dy_j dy^j \quad k \in \{1, \ldots, 6\}.
\end{align*}
\tag{45}
\]

The solutions are smeared in their relative transverse directions so that

\[
N_i = 1 + \frac{Q_i}{r}, \quad \text{where} \quad r^2 = y_k y^k.
\tag{46}
\]

The field strength for the intersecting solution is a sum of the field strengths for the individual solutions:

\[
\mathcal{F}_{4(\perp 12)} = -dt^{11} \wedge \left( dN_1^{-1} \wedge dx^9 \wedge dx^{10} + dN_2^{-1} \wedge dx^7 \wedge dx^8 \right).
\tag{47}
\]

This may be read from the group element \(43\) after the tangent space indices of the gauge fields are converted into wordspace indices under the action of the vielbein of the individual brane

### Table 2. Roots satisfying \( \beta_1 \cdot \beta_2 = 0 \) associated with basic marginal brane intersections.

| \( \beta_1 \) | \( \beta_2 \) | Marginal solution |
|-----------------|----------------|------------------|
| \( e_{10} - e_{11} \) | \( e_7 + e_{10} + e_{11} \) | Boosted M2 |
| \( e_9 + e_{10} + e_{11} \) | \( e_7 + e_9 + e_{11} \) | M2 \perp M2(0) |
| \( e_{10} - e_{11} \) | \( e_9 + e_{10} + e_{11} \) | Boosted M5 |
| \( e_9 + \cdots + e_{11} \) | \( e_6 + \cdots + e_{11} \) | M2 \perp M5(1) |
| \( e_9 + \cdots + e_{11} \) | \( e_6 + \cdots + e_{11} \) | M5 \perp M5(3) |
| \( e_9 - e_{10} \) | \( e_{11} + \cdots + e_{10} + 2e_{11} \) | Boosted KK6 |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_3 + e_4 + e_{11} \) | KK6 \perp M2(1') |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_4 + e_6 + e_{11} \) | M2 \in KK6 |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_5 + \cdots + e_{10} + 2e_{11} \) | M5 \in KK6 |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_6 + \cdots + e_{10} + 2e_{11} \) | KK6 \perp KK6(5') |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_7 + \cdots + e_{10} + 2e_{11} \) | \( e_7 + e_9 + e_{11} \) |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_8 + \cdots + e_{10} + 2e_{11} \) | \( e_8 + e_9 + e_{11} \) |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_9 + e_{10} + e_{11} \) |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_9 + \cdots + e_{10} + 2e_{11} \) |
| \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_9 + \cdots + e_{10} + 2e_{11} \) | \( e_9 + \cdots + e_{10} + 2e_{11} \) |
Finally in combining the final two exponentials of 43 to read off the gauge field components, we note that $[E_{\beta_1}, E_{\beta_2}] = 0$ which will be in contrast to the non-marginal bound states that we consider next.

4. Composite gravitational solutions

In this section, we outline the procedure that will allow us to associate roots with non-marginal, bound states of marginal solutions. The first example we will reproduce is the dyonic membrane which we will associate with the level 2 root in the decomposition of $E_{11}$.

The solution-generating group element is formed of two parts: the first exponential contains the gravitational solution and the second the gauge fields. The root at a given level, $l$, may always be re-expressed as a sum of roots from lower levels, $l_i$, such that $l = \sum_{i=0}^{n} l_i$. The level 0 roots are some element of the root system of $SU(11)$. The lowest level roots, at $l = 1, 2, 3$, correspond to well-known solutions of supergravity derived in the previous section. Given an arbitrary high-level (greater than level 2) root, we may always partition it into a sum of roots from levels $l = 0, 1, 2, 3$ whose solutions are well understood. In this paper, we will consider the decomposition of real roots but the interpretation of roots as composite solutions presented here will extend also to the null and imaginary roots; indeed it has previously been proposed \[46\] that the null and imaginary roots of $E_{10}$ correspond to Minkowski branes, and furthermore that certain imaginary roots may be decomposed into sums of two roots in order to better investigate their physical nature in broad agreement with the central thesis of this paper.

As we will see, this will result in introducing multiple parameters to describe a single solution with one harmonic function. The construction of solution-generating group elements will vary from solution to solution; in this section, we will propose group elements for two parameter half-BPS bound state solutions. The construction in general may be complicated by the large number of parameters involved but the steps taken here for the two parameter examples are reproducible in the more general examples which we consider in section 5. The prototype solution that we will derive using the solution-generating group element is the dyonic membrane \[15\].

4.1. The dyonic membrane

The M5-brane solution is associated with a root of $E_{11}$ that appears at level 2 in the decomposition. It has a unique expression as a sum of roots associated with membrane solutions

$$\beta_{M5} = 2\beta_{M2} + \beta_{pp},$$

where

$$\beta_{M2} = e_9 + e_{10} + e_{11},$$

$$\beta_{pp} = e_6 + e_7 + e_8 - e_9 - e_{10} - e_{11}.$$

The root $\beta_{pp}$ is a sum of roots in the $A_{10}$ sub-algebra of the decomposition; in fact, here $\beta_{pp} = \alpha_6 + 2\alpha_7 + 3\alpha_8 + 2\alpha_6 + \alpha_{10}$, and have the effect of rotating indices on the gauge fields. There are two gauge fields associated with the two M2 roots, $\beta_{M2}$: $R_{(1)}^{91011}$ and $R_{(2)}^{91011}$. By allowing the $SL(11)$ generators to act on the $R_{(1)}^{91011}$ generators and lower one set of the

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\[5\] We thank Axel Kleinschmidt for drawing our attention to this paper.
membrane indices, we are left with an M2 (678) brane root, with the corresponding generator $R^{(1)}_{678}$, and an S2 (91 011) brane root, with generator $R^{(78)}_{91011}$, where we have indicated the worldvolume directions in brackets:

$$\beta_{M2} = e_6 + e_7 + e_8, \quad \beta_{S2} = e_9 + e_{10} + e_{11}. \quad (51)$$

Note that the time coordinate is rotated by the $SL(11)$ adjoint action [47]. Consequently we take the $x^6$ direction to be temporal in this pair of solutions. We also observe that $\beta_{M2} \cdot \beta_{S2} = -1$ which, as outlined in section 1, indicates that one can add to the root string and find a generator associated with the root $\beta_{M2} + \beta_{S2}$; this is to be contrasted with the situation for marginal brane intersections for which $\beta_1 \cdot \beta_2 = 0$. Using the superposition procedure described earlier in this paper, we find the volume element

$$\begin{align*}
dx^2_{M2,S2} &= (N_1 N_2)^{\frac{1}{2}} \left( dx_1^2 + \cdots + dx_5^2 + N_1^{-1}(-d\xi_6^2 + dx_7^2 + dx_8^2) \\
&\quad + N_2^{-1}(dx_9^2 + dx_{10}^2 + dx_{11}^2) \right). \quad (52)
\end{align*}$$

We note that we recover the dyonic membrane solution [15] if we take as our ansatz for $N_1$ and $N_2$:

$$N_1 = 1 + \frac{Q}{r^3}, \quad N_2 = 1 + \frac{Q \cos^2 \xi}{r^3}, \quad (53)$$

where $\xi$ is a free parameter and when $\xi = 0$ ($N_2 = N_1$), we have the magnetic fivebrane solution, while when $\xi = \frac{\pi}{2}$ ($N_2 = 1$) we have the electric membrane solution. This suggests that the ansatz for the solution-generating group element should be augmented with an angle variable that measures the difference between the membrane charges $(1 - N_1)$ and $(1 - N_2)$ as

$$\frac{1 - N_2}{1 - N_1} = \cos^2 \xi. \quad (54)$$

To find the interpolating group element given in equation (1), we need the following inner products:

$$\beta_{M2} \cdot \beta_{S2} = -1 \quad \beta_{M2} \cdot \beta_{M5} = 1 \quad \beta_{S2} \cdot \beta_{M5} = 1. \quad (55)$$

The group element (1) for the dyonic membrane is

$$g_\mu = \exp \left( -\frac{1}{\beta^2} \ln N_1 (H \cdot \beta_{M2}) - \frac{1}{\beta^2} \ln N_2 (H \cdot \beta_{S2}) \right) \times \exp \left( (1 - N_1)^\frac{1}{2} \left( 1 - \frac{N_1}{N_2} \right)^\frac{1}{2} R^{078} + (1 - N_2)^\frac{1}{2} \left( 1 - \frac{N_2}{N_1} \right)^\frac{1}{2} R^{91011} \right) \times \exp \left[ (1 - N_1)^\frac{1}{2} \left( 1 - N_2 \right)^\frac{1}{2} \left( \frac{N_1}{N_2} \right)^\frac{1}{2} R^{6-11} \right] \right). \quad (56)$$

The solution-generating group element is now dependent upon two parameters, $r$ and $\xi$. We note that when $N_2 \to 1$, it approaches the group element for a membrane solution, when $N_1 \to 1$ it approaches the solution-generating group element for a spacelike membrane solution, and when $N_1 \to N_2$ it approaches the group element that encodes the fivebrane solution. Also note that under interchange of $N_1$ and $N_2$, the gauge field components for the M2-brane and the S2-brane are interchanged while the expression for the fivebrane gauge field component is mapped to itself.

We can use equation (54) to write a line element in terms of $N_1(r)$ and $\xi$ giving

$$\begin{align*}
dx^2_{M2\oplus S2} &= N_1^\frac{1}{2} \left( \sin^2 \xi + N_1 \cos^2 \xi \right)^\frac{1}{2} \left( dx_1^2 + \cdots + dx_5^2 + N_1^{-1}(-d\xi_6^2 + dx_7^2 + dx_8^2) \\
&\quad + (\sin^2 \xi + N_1 \cos^2 \xi)^{-1}(dx_9^2 + dx_{10}^2 + dx_{11}^2) \right). \quad (57)
\end{align*}$$
The group element in equation (56) encodes the four-form field strength of the dyonic membrane solution. It is helpful to write the gauge field for the fivebrane in a dual form; this amounts to the insertion of $N_2^{-\frac{(5 - 77)}{2}} = N_2$ in front of the $R^{[67891011]}$ generator. With this insertion when $N_2 = N_1$, we find the solution-generating group element for an M5-brane written in a dual form. The gauge part of the group element is modified to read

$$A_{{678}}^T R^{[678]} + A_{{91011}}^T R^{[91011]} + A_{{67891011}}^T R^{[67891011]}$$

$$\equiv \left((1 - N_1) N_2^{-\frac{1}{2}} \sin \xi\right) R^{[678]} + i(1 - N_2) N_1^{-\frac{1}{2}} \tan \xi) R^{[91011]}$$

$$+ ((1 - N_1) (N_1 N_2)^{\frac{i}{2}} \cos \xi) R^{[6,11]},$$

where $T$ indicates that the gauge field is given in tangent space coordinates. Note that in this form when $\xi = 0$, the interpolating group element reduces to the dual form of the group element for a single M5-brane.

To extract the field strength, we premultiply each active gauge field component $A^T$ by the appropriate product of vielbeins $e_a^a$, which we read off from the volume element (52) to find the gauge field component in worldvolume indices. Explicitly,

$$A_{{678}}^W = e_6^a e_7^b e_8^c A_{{678}}^T = (N_1^{\frac{1}{2}} - 1) \sin \xi$$

$$A_{{91011}}^W = e_9^a e_{10}^b e_{11}^c A_{{91011}}^T = i(N_2^{\frac{1}{2}} - 1) \tan \xi$$

$$A_{{6,11}}^W = e_6^a \cdots e_{11}^c A_{{6,11}}^T = (1 - N_1) \cos \xi.$$  

These potentials are the components of two three-forms and a six-form. The six-index gauge potential $A_{{6,11}}^W$ sources a dual four-form field strength. To find the field strength, we take the exterior derivative of the relevant forms to find

$$F_{4(M2\oplus M5)} = d(N_1^{\frac{1}{2}}) \sin \xi \wedge dx_6 \wedge dx_7 \wedge dx_8 \wedge * d(N_1) \cos \xi$$

$$- \frac{i}{2 N_2} d(N_1) \sin 2\xi \wedge dx_9 \wedge dx_{10} \wedge dx_{11}. \quad (62)$$

The exterior derivative and the Hodge dual ($*$) act in the five-dimensional subspace with coordinates $[x_1, \ldots, x_5]$, which is mutually transverse to the worldvolumes of the M2 and S2 solution worldvolumes. The solution may be described as a bound state of an M2(678) and an S2(91011) brane, or in the more usual language as the bound state of an M2(678) and an M5(67891011) brane. Due to the non-zero commutator of $[R^{678}, R^{91011}]$ we find that the field strength is no longer a simple sum of the field strengths for the constituent M2- and S2-branes; instead we find additional terms in the field strength that give rise to a binding energy. The harmonic functions of the solution are derived from the harmonic function of the membrane smeared in the directions longitudinal to the constituent brane worldvolumes, which is the prescription for harmonic superposition [11]. This was implicit in the ansatz (54) where $N_1$ and $N_2$ are harmonic functions of $\{x_1, x_2, x_3, x_4, x_5\}$.

The dyonic membrane illustrates and guides our principle observation that one may extract marginal bound state solutions from the roots of the adjoint algebra of $E_{11}$. Let us state the basic steps.

- Roots are decomposed into sums of roots associated with marginal solutions, in the limit into sums of M2- and S2-brane roots.
- Each solution is smeared along the longitudinal directions of all branes, allowing all harmonic functions to be related by one of the harmonic functions by angle parameters.
- The constituent branes are superposed.
- The gauge field is determined by the conditions that
(a) in the limits of the angle parameters it reduces to lower level marginal or non-marginal solutions, and in particular to individual brane solutions;
(b) interchange of constituent gauge fields and the corresponding generators, which do not effect the commutators required to obtain the solution, leave the overall gauge field unaltered;
(c) upon conversion to worldvolume indices using the veilbein of the background of the bound state, the gauge fields may be written using the harmonic function and angle parameters which interpolate between the constituent electric brane states. Magnetic S-brane states will have imaginary gauge fields and a singular limit in the angle parameters.

Using this set-up we can find other non-marginal states from the $E_{11}$ algebra.

### 4.2. The transversely boosted $M2$- and $M5$-branes

Amongst the marginal M-brane solutions are the longitudinally boosted $M2$- and $M5$-branes, but amongst the non-marginal solutions we can find the transversely boosted branes [48].

Consider the following root, corresponding to a boosted $M2$-brane:

\[
\beta_{M2+} = e_8 + e_9 + e_{10} = \beta_{M2} + \beta_{pp}, \quad \text{where} \quad \beta_{KK} = e_8 - e_{11}.
\]

Following the prescription, we smear the KK-wave along the $M2$-brane directions so that the harmonic functions of the solution are

\[
N_1 = 1 + \frac{Q}{r^6}, \quad N_2 = 1 + \frac{Q \cos^2 \xi}{r^6}, \quad \text{where} \quad r^2 = x_1^2 + \cdots + x_7^2 + x_{11}^2.
\]

The composite roots $\beta_{pp}$ and $\beta_{M2}$ give the diagonal components of the metric

\[
d^2_{M2} = -N_1^{-1} \, dx_8^2 + N_1 \, dx_{11}^2 + d\Omega_5^2
\]

where we indicate by $d\Omega$ that this is not the full line element for the KK-wave as only the diagonal parts of the metric are indicated. The superposed diagonal solutions give the crucial vielbein components that will be used shortly to reconstruct the off-diagonal part of the metric from the group element in this case

\[
d^2_{M2+} = N_2^{-1} \left( N_2^{-1} \left( dx_8^2 + dx_{10}^2 + dx_{11}^2 \right) - N_1^{-1} \, dx_{11}^2 + dy^2 \right).
\]

To find the group element (1) we calculate the following inner products:

\[
\beta_{KK} \cdot \beta_{M2_1} = -1 \quad \beta_{KK} \cdot \beta_{M2_2} = 1 \quad \beta_{M2_1} \cdot \beta_{M2_2} = 1.
\]

The group element for this solution has the particular form:

\[
g_\beta = \exp \left( -\frac{1}{\beta^2} \ln N_1 (H \cdot \beta_{pp}) - \frac{1}{\beta^2} \ln N_2 (H \cdot \beta_{M2}) \right)
\]

\[
\times \exp \left( (1 - N_1)^\frac{1}{2} \left( 1 - \frac{N_1}{N_2} \right)^\frac{1}{2} K_{811} + (1 - N_2)^\frac{1}{2} \left( 1 - \frac{N_2}{N_1} \right)^\frac{1}{2} R_{8910} + (1 - N_1)^\frac{1}{2} (1 - N_2)^\frac{1}{2} \left( \frac{N_1}{N_2} \right)^\frac{1}{2} R_{8910} \right).
\]

Using the veilbein of (66) to write the gauge components in terms of worldvolume coordinates, we find an off-diagonal component of the metric and a three-form gauge field respectively:
Finally including the off-diagonal metric component (69) gives the volume element of the transversely boosted M2-brane [48]:

\[ ds^2_{M2^+} = N_1^3 N_2^{-1} \left( dx_9^2 + dx_{10}^2 + N_1 \left( dx_{11} - (N_1^{-1} - 1) \sin \xi dt_8 \right)^2 \right) - N_1^{-1} dt_8^2 + dy_i dy^i. \] (71)

By writing \( N_1 = 1 + W \) and \( \xi = \theta + \frac{\pi}{2} \), we find the transversely boosted M2-brane in the variables of [48]. We recall the observation here of [48] that this solution may be written more simply, and more obviously, as a boosted membrane by a Lorentz transformation of the coordinates, where the boost parameter is \( \xi \). The solution interpolates between the KK-wave (\( \xi = \frac{\pi}{2} \)) and an M2-brane solution (\( \xi = 0 \)) but \( \xi \) now has the physical interpretation of a boost parameter with the Lorentz transformation given by

\[ x_{11} \rightarrow \frac{1}{\cos \xi} (\bar{x}_{11} - \sin \xi \bar{t}_8), \quad t_8 \rightarrow \frac{1}{\cos \xi} (\bar{t}_8 - \sin \xi \bar{x}_{11}). \] (72)

The transversely boosted M5 solution is recovered in a similar way from a group element analogous to (68).

5. The \( E_{11} \) guide to exotic solutions

5.1. The KK-branes of M-theory

In this section we derive the M-theory KK-brane solutions of [49] from the associated roots of the adjoint representation of \( E_{11} \). The existence of two KK-branes in M-theory, the \( M_{26} \) and the \( M_{53} \), has been argued for by oxidizing ten-dimensional KK-branes of ten-dimensional string theory (which were derived using S-duality) [49]. They also arise from the U-duality transformations of M-theory [35] which correspond to Weyl reflections of \( E_{11} \).

A useful tool for identifying the roots of \( E_{11} \) associated with the KK-branes of M-theory is the mass formula\(^6\) identified in [35] from which the masses of the \( M_{26} \) and the \( M_{53} \) may be associated with particular roots appearing in the \( l_1 \), or charge, representation of \( E_{11} \). This formula, after division by the brane volume, correctly reproduced all the brane tensions of 11-dimensional supergravity, as well as the tensions of the Dp-branes of the IIA and IIB string theories together with the correct powers of the string coupling constant, \( g_s \), and \( \alpha' \). Furthermore, it gave agreement with the tensions of exotic objects which had been predicted as a consequence of the U-duality symmetry applied to known brane charges [30, 32, 33]. The mass of the exotic objects is a monomial function of the radii of compactification with at least one of the radii appearing nonlinearly—we will use this as our definition of a KK-brane. For example, the mass of the KK6 brane, which can be seen in the particle/flux multiplet of [30, 32, 33], is

\[ M_{KK6} = \frac{R_4 R_5 R_6 R_7 R_9 R_{10} R_{11}^2}{f_p}. \] (73)

The radius \( R_{11} \) is squared in this mass, so we define it to be a KK-brane. The mass formula of [35] was given as a map from the charge algebra, or \( l_1 \) representation, of \( E_{11} \). Previously [50], an injective map was constructed which took roots of the adjoint of \( E_{11} \), whose generators are

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\(^6\) A similar mass formula for \( E_{10} \) was used in [46].
associated with gauge fields in the physical theory, into the weights of the $l_1$ representation, whose generators are associated with brane charges in the physical theory. This injective map removed a single index from the generators of the $E_{11}$ adjoint in a systematic fashion. Thus, the three-form $R^{\alpha_1 \alpha_2 \alpha_3}$ was mapped to a two-form charge $Z_{\alpha_1 \alpha_2}$, the six-form generator to a five-form charge $Z_{\alpha_1 \ldots \alpha_5}$ and so on. It is therefore a straightforward step to give a mass formula that acts directly upon the adjoint of $E_{11}$.\footnote{However, we realize that while this is a useful shortcut we should not lose track of the physical notion that the mass is derived from the the charge algebra, the $l_1$ representation.} Given a root, $\beta$, of $E_{11}$ expressed in the $e_i$ basis such that

$$\beta = \sum_{i=1}^{11} b_i e_i,$$

the mass formula in the compact setting is

$$M'_{\beta} = \prod_{i=1}^{11} \left( \frac{R_i}{l_p} \right)^{b_i},$$

where $R_i$ are the radii of the compact directions. The prime is used here to indicate that all longitudinal directions have been compactified. When a direction, $x^i$, is not compactified, the constant $R_i$ is set to 1 [35]. For example, the M2-brane is associated with the root $\alpha_{11} = e_9 + e_{10} + e_{11}$ and its mass derived from the root is

$$M'_{M2} = \frac{R_9 R_{10} R_{11}}{l_p^3}.$$  

However, at least one of the longitudinal directions of the M2-brane is timelike and it is natural not to compactify this direction; if we associate the direction $x^{11}$ to time and decompactify, then we set $R_{11} = 1$ to obtain

$$M_{M2} = \frac{R_9 R_{10}}{l_p^5}.$$  

In general, we will not consider the compactification of the timelike direction, so let us rewrite our mass formula to take into account a non-compact timelike direction

$$M_{\beta} = \left( \frac{1}{l_p^5} \right)^{b_i} \prod_{i=1}^{10} \left( \frac{R_i}{l_p} \right)^{b_i},$$

where we have used $b_i$ to indicate the coefficient of the timelike coordinate of $e_i$ in the root. The $E_{11}$ adjoint algebra is expressed in terms of representations of $SL(11)$ by the deletion of the root $\alpha_{11}$. The number of times $\alpha_{11}$ must be deleted from a root in $E_{11}$ to arrive at a highest weight of $SL(11)$ is called the level and denoted as $m_{11}$ here. As a consequence of the formulae of [35] one can express the mass formula in terms of the level as

$$M_{\beta} = \prod_{i=1}^{10} \frac{R_i^b}{l_p^{2m_{11} b_i}}.$$  

This formula is very useful; for example, we can use it to immediately locate the root associated with the M2$_6$ solution and the M5$_3$ solution, by searching for their masses, which are [49]

$$M_{M2_6} = \frac{R_3 R_5 (R_6 \ldots R_{11})^2}{l_p^{15}}$$

$$M_{M5_3} = \frac{R_3 \ldots R_6 (R_9 \ldots R_{11})^2}{l_p^{12}}.$$
Referring to equation (79), we find that the corresponding roots, both appearing in the adjoint of $E_{11}$, are

$$\beta_{M2}\epsilon = (0, 0, 1, 1, 1, 2, 2, 2, 2, 2) = (0, 0, 1, 2, 3, 5, 7, 9, 6, 3, 5)\alpha_i,$$

$$\beta_{M5}\epsilon = (0, 0, 1, 1, 1, 1, 1, 2, 2) = (0, 0, 1, 2, 3, 4, 5, 6, 4, 2, 4)\alpha_i,$$

(82)

(83)

where we have expressed the roots in both the $e_i$ coordinate basis and the $\alpha_i$ simple root basis. For reference, we also draw the Young tableaux of the associated generators:

There are a number of tables of adjoint roots of $E_{11}$ in the literature, the first appearing in [36], where one can verify that the outer multiplicity of these roots is not zero. However, for our purposes it is most useful to see the roots expressed in the $e_i$ basis, and for this reason the reader is also referred to the tables of [51], to confirm these roots do appear in the root tables of the adjoint of $E_{11}$. Alternatively the reader may prefer to use the simple methods outlined in section 1.

These roots may be expressed as sums of lower level roots in a number of ways. In both the case of the $M_{26}$ and the $M_{53}$ the root may be written as a sum of roots $(\beta_{pp}, \beta_{M2}, \beta_{M5})$.

Our aim is to construct the Young tableaux of equation (84) as a combination of the Young tableaux of the $M2$- and the $M5$-brane,

$$\begin{align*}
\text{and } & \quad \begin{array}{c}
\end{align*}$$

(85)

with coordinate rotations corresponding to roots in the local sub-algebra.

5.1.1. The $M_{26}$ brane. Let us begin with the $M_{26}$ solution. We observe that

$$\beta_{M26} = \beta_{M2} + 2\beta_{M5} + \beta_{pp},$$

(86)

where $\beta_{pp}$ indicates a sum of roots from the local sub-algebra and we deduce that

$$\beta_{pp} = (0, 0, 1, 1, 1, 0, 0, -1, -1, -1)\alpha_i = (0, 0, 1, 2, 3, 3, 3, 3, 2, 1)\alpha_i.$$

(87)

The root for the $M_{26}$-brane comprises two copies of the root for the $M5$-brane and one copy of the root associated with the $M2$-brane together with the application of the local generators associated with $\beta_{pp}$, which rotate the coordinates $\{x^9, x^{10}, x^{11}\}$ to $\{x^3, x^4, x^5\}$. We recall that the action of the local sub-algebra is to rotate the time coordinate [47], meaning in this case
that this composite solution may be interpreted as either $M_2 \oplus S_5 \oplus S_5$ or $M_5 \oplus S_2 \oplus S_5$. In terms of representations of $A_{10}$, the $M_2e$ corresponds to the irreducible representation with highest weight $\lambda_2 + \lambda_5$, by the identification above we are stating that this representation occurs within the representation whose highest weight is $\lambda_5 + 2\lambda_3$. We will consider the $M_2 \oplus S_5 \oplus S_5$ interpretation here.

Following the procedure described in section 3, we smear the constituent brane solutions over their relative transverse directions to obtain three harmonic functions. However, as the transverse space is two-dimensional, it is impossible to find three real harmonic functions such that $1 - M_1M_2 = 1 - N$; instead we use the following harmonic, holomorphic and anti-holomorphic functions:

$$N = 1 + Q \ln r, \quad M_1 = 1 + Q \ln \omega \cos^2 \xi, \quad M_2 = 1 + Q \ln \bar{\omega} \cos^2 \xi$$

$$\Rightarrow \cos \xi = \sqrt{\frac{1 - \frac{1}{2}(M_1 + M_2)}{1 - N}}.$$

where $r^2 = x_1^2 + x_2^2 = \omega \bar{\omega}$, with $\omega = x_1 + ix_2$. The solution interpolates between the membrane ($\xi = \frac{\pi}{2}$) and the exotic $M_2e$ ($\xi = 0$) brane. The function $N$ is harmonic in the overall two-dimensional transverse space, but the functions $M_1$ and $M_2$ are holomorphic and anti-holomorphic, respectively [49]. The $M_2e$ solution has also been derived from a Kac–Moody symmetry following a different method in [23]. The metric for the full solution is constructed from just the diagonal elements of the metric and may now be read off using the roots for the constituent solutions, $\beta_{M_2}$ and $\beta_{S5}$:

$$\beta_{M_2} = e_3 + e_4 + e_5, \quad \beta_{S5} = e_6 + \cdots + e_{11}. \quad (89)$$

Superposing the three component solutions, we obtain

$$\begin{align*}
ds^2_{M_2e} &= N^\frac{1}{2}(M_1M_2)^\frac{1}{2}(dx_1^2 + dx_2^2 + N^{-1}(-dr^2 + dx_4^2 + dx_5^2) \\
&\quad + (M_1M_2)^{-1}(dx_6^2 + \cdots + dx_{11}^2)). \quad (90)
\end{align*}$$

The group element for this solution is short, principally because $[R^{345}, R^{6..11}] = 0$ and $[R^{6..11}, R^{6..11}] = 0$. This solution is in a different class to those we have considered hitherto. In the earlier examples we considered sets of generators whose commutators gave the generator of interest. More precisely if we had followed the same procedure here we would have considered three different generators such as $R^{6..11}$, $R^{5611}$ and $R^{3910}$ whose commutators generate $R^{3..11,91011}$. The group element would have been parameterized by three variables: $r$ and two angle variables. With this example, we see that it is possible to find alternative group elements whose form differs from that of equation (1) but recreates M-brane solutions. We propose the group element

$$g_B = \exp \left( -\frac{1}{\beta^2} \ln N(H \cdot \beta_{M_2}) - \frac{1}{\beta^2} \ln (M_1M_2)(H \cdot \beta_{S5}) \right) \exp \left( \left( 1 - \frac{N}{M_1M_2} \right) R^{345} \right)$$

$$\quad + \left( 1 - \frac{\sqrt{M_1M_2}}{N} \right) \left( 1 + \frac{\sqrt{M_1M_2}}{N} \right) R^{67891011}. \quad (91)$$

Once the tangent space indices of the gauge fields are transformed into worldvolume indices, we find

$$A_{345}^w = \frac{M_1M_2}{N} - 1, \quad A_{6..11}^w = -\frac{B}{M_1M_2}, \quad (92)$$

where we use $B \equiv \frac{1}{2}(M_1 - M_2)$ and the following identity:
The solution interpolates between the \( M_2 \) and the \( M_2^6 \)-brane. In reproducing the line element of the \( M_2^6 \) from the \( E_{11} \) fields, we have gained an understanding of the \( M_2^6 \) as a composite solution. The line element of the \( M_2^6 \) consists of two \( S^5 \)-branes and an \( M_2 \) solution. One could also consider the Wick rotation of the solution which would consist of two \( M_5 \)-branes and an \( S^2 \)-brane, or indeed attempt to understand this solution in terms of the group element of (1).

5.1.2. The \( M_5^3 \) brane. Let us repeat the process of finding the line element for the \( M_5^3 \) KK-brane from the \( E_{11} \) group element, using the candidate root of \( E_{11} \) associated with the \( M_5^3 \) given in equation (83). Proceeding as before, we note that

\[
\beta_{M_5^3} = 2\beta_{M_2^3} + \beta_{pp}.
\]

Proceeding as before, we associate the coordinate rotation associated with \( \beta_{pp} \) to the gauge fields of the \( M_5 \)-brane or to one of the gauge fields of the pair of \( M_2 \)-branes, giving \( M_5 \oplus S^2 \oplus S^2 \) and \( M_2 \oplus S^5 \oplus S^5 \), respectively. As before we focus on the former case and identify it with the \( M_5^3 \) brane of [49]. By writing the individual metrics and harmonically superposing the solutions, we arrive at the line element

\[
d s_{M_5^3}^2 = M_2^3 (N_1 N_2) \left( dx_1^2 + dx_2^2 \right) + \left( \frac{M}{N_1 N_2} \right)^{\frac{3}{4}} \left( -dt_2^2 + dx_3^2 + \cdots + dx_8^2 \right) + \left( \frac{M}{N_1 N_2} \right)^\frac{3}{2} \left( dx_9^2 + \cdots + dx_{11}^2 \right).
\]

The \( M_5^3 \) solution is reproduced from the group element analogous to that used for the \( M_2^6 \) solution (91). The functions \( M, N_1 \) and \( N_2 \) are identified as in (88) under an interchange of the labels \( N \) and \( M \).

5.1.3. An exotic pure gravity solution: \( WM_7 \). One other exotic M-theory solution is derived in [49] in addition to the \( M_2^6 \) and \( M_5^3 \) branes, denoted by the \( WM_7 \) which is a deformation of the KK-wave. Its mass is

\[
M_{WM_7} = \frac{(R_4 \cdots R_{10})^2 R_{11}}{l_{11}^p}.
\]

Hence it is expected to be derived from the level 6 root:

\[
\beta_{WM_7} = e_1 + 2(e_4 + \cdots + e_{10}) + 3e_{11} = (0, 0, 1, 3, 5, 7, 9, 11, 7, 3, 6)_{a_1}.
\]

This root is associated with the highest weight Young tableau:

\[
\begin{array}{ccccccc}

 & & & A & A & A & A \\
 & & & A & A & A & A \\
 & & & A & A & A & A \\
 & & & A & A & A & A \\
 & & & A & A & A & A \\
 & & & A & A & A & A \\
 & & & A & A & A & A \\

\end{array}
\]
We observe that
\[
\beta_{WM}^7 = 2 \beta_{KK}^6 + \beta_{KK}^9
\]
where \( \beta_{KK}^9 = e_3 - e_{11} \). (100)

For this solution, we may use a short-cut via Hodge duality to find the solution; hence, the group element encoding the solution is
\[
g_\beta = \exp \left( -\frac{1}{\beta^2} \ln N(H \cdot \beta_{KK}) - \frac{1}{\beta^2} \ln (M_1 M_2) (H \cdot \beta_{KK}) \right) \times \exp \left( \left( 1 - \frac{N}{M_1 M_2} \right) K^3_{11} \right) .
\]
(101)

Following our method, the functions \( M_1, M_2 \) and \( N \) are smeared so that they are harmonic in the directions \( x_1 \) and \( x_2 \), as in the previous examples. The functions are identical to those given in equation (88). In this example, we demonstrate how it is possible to take advantage of Hodge dualization to remove antisymmetric blocks of nine boxes from the Young tableau. The generator has been ‘dualized’ twice using the method outlined in section 3, that is, we have used the observation that
\[
2 \beta_9 - \beta_{WM}^7 = \beta_{KK}.
\]
(102)

The volume element encoded in the group element (101) is
\[
\text{d} s^2_{WM} = (M_1 M_2) (\text{d} x_1^2 + \text{d} x_2^2) - \left( \frac{N}{M_1 M_2} \right)^{-1} \text{d} t_3^2 \\
+ \frac{N}{M_1 M_2} \left( \text{d} x_{11} - \left( \frac{M_1 M_2}{N} - 1 \right) \text{d} t_3 \right)^2 + \text{d} \Omega^2_7.
\]
(103)

Recalling that,
\[
M_1 M_2 = (1 - (1 - N) \cos^2 \xi)^2 + (B \cos^2 \xi)^2.
\]
(104)

We see that it interpolates between the KK-wave and the WM\(_7\) brane. There is a tower of similar roots in the \( E_{11} \) root system, the so-called dual roots [52], for which we can readily write down a group element and metric. Consider the set of dual roots made from dressing the KK6 root up with multiple copies of \( \beta_9 \), i.e.
\[
\beta_{KK6^n} = \beta_{KK6} + n \beta_9,
\]
(105)

such that the root is dualized to
\[
n \beta_9 - \beta_{KK6^n} = \beta_{KK}.
\]
(106)

We predict a group element
\[
g_\beta = \exp \left( -\frac{1}{\beta^2} \ln N(H \cdot \beta_{KK}) - \frac{1}{\beta^2} \ln (M_1 \ldots M_{n+1}) (H \cdot \beta_{KK}) \right) \times \exp \left( \left( 1 - \frac{N}{M_1 \ldots M_{n+1}} \right) K^3_{11} \right)
\]
(107)

that encodes the metric
\[
\text{d} s^2_{KK6^n} = (M_1 \ldots M_{n+1}) (\text{d} x_1^2 + \text{d} x_2^2) - \left( \frac{N}{M_1 \ldots M_{n+1}} \right)^{-1} \text{d} t_3^2 \\
+ \frac{N}{M_1 \ldots M_{n+1}} \left( \text{d} x_{11} - \left( \frac{M_1 \ldots M_{n+1}}{N} - 1 \right) \text{d} t_3 \right)^2 + \text{d} \Omega^2_7.
\]
(108)
The metric corresponds to a deformation of the KK6-brane. However, the exact format of the harmonic functions and the interpolating angles remains to be expressed.

5.2. The KK6-brane revisited

In the previous sections, we have investigated a solution-generating technique using simple group elements to reproduce known composite solutions from higher level roots associated with the Borel sub-algebra of the adjoint representation of $E_{11}$. Through this method we have given a consistent interpretation of exotic branes as bound states of the fundamental M-branes. However, the method of partitioning a root into sums of lower level roots, which are in turn associated with brane solutions, may give more than one interpretation of the constituent states. To highlight this point we examine again the level 3 root associated with the KK6 brane. In the first instance we will interpret it as a bound states of two M-branes before suggesting a group element consisting of three membrane charges and two interpolation angles.

The level 4 root $\beta_{KK6} = e_4 + \cdots + e_{10} + 2e_{11}$, which we saw earlier was associated with the KK6 monopole and the embedded Taub-NUT solution is associated with the generator $R^{456...11,11}$ and may be partitioned into roots from lower levels in two different ways:

$$\beta_{KK6} = 3\beta_{M2} + \beta_{pp}^{(1)}$$
$$\beta_{KK6} = \beta_{M5} + \beta_{M2} + \beta_{pp}^{(2)}.$$  (109)

where

$$\beta_{pp}^{(1)} = e_4 + e_5 + e_6 + e_7 + e_8 - 2e_9 - e_{10} - e_{11}$$
$$\beta_{pp}^{(2)} = e_4 + e_5 - e_9 - e_{10}.$$  (111)

The roots $\beta_{pp}^{(i)}$ correspond to multiple applications of $SL(11)$ generators $K^a_{ab}$ and have the effect of rotating the active coordinates of the gauge field. For example $\beta_{pp}^{(2)}$ rotates the coordinates $\{x_9, x_{10}\}$ into $\{x_4, x_5\}$ and $\beta_{pp}^{(1)}$ rotates the coordinates $\{x_9, x_9, x_{10}, x_{10}, x_{11}\}$ into $\{x_4, x_5, x_6, x_7, x_8\}$.

Consider the partition of $\beta_{KK6}$ given in (109), the rotation associated with $\beta_{pp}^{(2)}$ could be applied to either the M5 solution or the M2 solution. The rotation has the effect of transforming one of the brane solutions while ensuring that it remains an electric solution [47], while the unrotated solutions will be spacelike, or magnetic, versions of the brane solutions. We have two choices from (110):

$$\beta_{KK6} = \beta_{M5} + \beta_{S2} = \beta_{S5} + \beta_{M2}.$$  (112)

Let us consider the partition into an M5 and an S2 root. Taking the M5 to be aligned along $\{t_4, x_5, \ldots, x_{11}\}$ and the S2 along $\{x_9, x_{10}, x_{11}\}$, the superposed volume element is

$$ds^2_{S2@M5} = M^2N^4(dx_1^2 + \cdots + dx_3^2 + M^{-1}(dt_4^2 + \cdots + dx_9^2)$$
$$+ N^{-1}(dx_9^2 + dx_{10}^2 + M^{-1}N^{-1}dx_{11}^2)).$$  (113)

We can interpret the solution as an M5-brane intersecting an S2-brane at a point, noting that when $N = 1$ the solution is that of an electric M5-brane, while when $M = 1$ the solution is that of a magnetic spacelike S2-brane solution, as expected by our construction. Furthermore, when $N = M$, the solution becomes that of equation (42). The interpolating, two-parameter
group element (1) for these roots is

\[ g_\beta = \exp \left( -\frac{1}{\beta^2} \ln M(H \cdot \beta_{M5}) - \frac{1}{\beta^2} \ln N(H \cdot \beta_{S2}) \right) \]

\[ \times \exp \left( (1 - M)^{\frac{1}{2}} \left( 1 - \frac{M}{N} \right)^{\frac{1}{2}} R^{4567811} + (1 - N)^{\frac{1}{2}} \left( 1 - \frac{N}{M} \right)^{\frac{1}{2}} R^{91011} \right) \]

\[ + (1 - M)^{\frac{1}{2}} (1 - N)^{\frac{1}{2}} \left( \frac{M}{N} \right)^{\frac{1}{2}} R^{11111} \].

(114)

As before we have introduced an angle variable relating the ratio of the charges of the solutions:

\[ \frac{1 - N}{1 - M} = \cos^2 \xi. \]

The field strength for this interpolating solution according to the two-parameter group element ansatz is

\[ F_t = (\sin \xi) \star \text{d}M^{-1} \wedge \text{d}x_2 \wedge \cdots \wedge \text{d}x_8 \wedge \text{d}x_{11} \]

\[ = \frac{1}{2N^2}(\sin 2\xi) \text{d}M \wedge \text{d}x_9 \wedge \text{d}x_{10} \wedge \text{d}x_{11}. \]

(115)

We may write the metric for this solution in terms of one harmonic function, \( M \), which interpolates between an M5-brane background and the pure gravitational solution of the KK6 as

\[ ds^2_{M5\rightarrow KK6} = (M^2(\sin^2 \xi + M \cos^2 \xi))^2(dx_5^2 + \cdots + dx_7^2 + M^{-1}(-dx_8^2 + \cdots + dx_9^2)) \]

\[ + (\sin^2 \xi + M \cos^2 \xi)^{-1}((dx_6^2 + dx_{10}^2) \]

\[ + M^{-1}(dx_{11}^2 + (1 - M) \cos^2 \xi dx_5^2)) \].

(116)

Given the interpretation of the dyonic membrane as a bound state of an M2- and an M5-brane, this presentation suggests that the KK6 may be understood as a bound state of two M5-branes intersecting along a four-brane.

Let us turn to the second partition of \( \beta_{KK6} \) given in equation (109), which, once we include the rotation due to \( \beta_{KK6}^{(2)} \), interprets the solution as an object composed of one M2 solution and two S2 solutions. The volume elements for these solutions are

\[ ds^2_{M2} = N_1^{\frac{1}{2}}(dx_5^2 + \cdots + dx_7^2) + N_1^{-\frac{1}{2}}(-dx_8^2 + \cdots + dx_9^2) + N_1^{\frac{1}{2}}(dx_6^2 + \cdots + dx_{11}^2) \]

\[ ds^2_{S2_1} = N_2^{\frac{1}{2}}(dx_5^2 + \cdots + dx_7^2 + dx_8^2 + dx_9^2 + dx_{10}^2) + N_2^{-\frac{1}{2}}(dx_6^2 + dx_7^2 + dx_{11}^2) \]

\[ ds^2_{S2_2} = N_3^{\frac{1}{2}}(dx_5^2 + \cdots + dx_7^2) + N_3^{-\frac{1}{2}}(dx_6^2 + dx_{10}^2 + dx_{11}^2). \]

(117)

Superposing these solutions gives

\[ ds^2_{M2} = (N_1N_2N_3)^{\frac{1}{2}}[(dx_5^2 + \cdots + dx_7^2) + N_1^{-1}(-dx_8^2 + \cdots + dx_9^2) \]

\[ + N_2^{-1}(dx_6^2 + dx_7^2) + N_3^{-1}(dx_6^2 + dx_{10}^2) + (N_2N_3)^{-1}dx_{11}^2] \]

(118)

which one can interpret as one membrane and two spacelike membranes which intersect over a point. When either \( N_2 = 1 \) or \( N_3 = 1 \), the solution becomes the dyonic membrane.

To encode this solution in a group element we will introduce two angle parameters, \( \xi \) and \( \upsilon \), defined by

\[ \frac{1 - N_2}{1 - N_1} = \cos^2 \xi \quad \frac{1 - N_3}{1 - N_1} = \cos^2 \upsilon. \]

(119)

To derive the gauge part of the three-parameter solution-generating group element, we are guided by the symmetry under interchange of the two component magnetic branes and by the
The gauge field components when transformed into worldvolume indices are
\[
A^w_{456} = (N_1^{-1} - 1) \sin \xi \sin \upsilon \\
A^w_{7811} = i(N_2^{-1} - 1) \tan \xi \sin \upsilon \\
A^w_{9101} = i(N_3^{-1} - 1) \sin \xi \tan \upsilon \\
A^w_{4567811} = (1 - N_1)N_2^{-1} \cos \xi \sin \upsilon \\
A^w_{45691011} = (1 - N_1)N_3^{-1} \cos \upsilon \sin \xi \\
A^w_{4567891011.11} = (1 - N_1)N_3^{-1} \cos \xi \cos \upsilon.
\]

The solution is invariant under the interchange of the harmonic functions associated with the S2-branes, \(N_2\) and \(N_3\), and the gauge field components are permuted accordingly. The ansatz for writing down the field strength requires us to recognize \textit{a priori} which components are dualized and which are not. In the list above, the first three fields are taken as components of a three-form whose exterior derivative contributes to the four-form field strength. The remaining fields are taken as dualized components: for the two six-forms the Hodge dual is taken in the five coordinates transverse to the field; however, for the mixed-symmetry tensor, the Hodge dual in the transverse coordinates gives a symmetric rank 2 tensor which contributes
to the gravity sector and modifies the volume element of the solution. We find a four-form field strength and an off-diagonal vielbein component:

\[
F_4 = d(N_1^{-1}) \sin \xi \sin \nu \wedge dt \wedge dx_5 \wedge dx_6
- \frac{i}{2N_2^{2}} d(N_1 \sin 2\xi \sin \nu \wedge dx_7 \wedge dx_8 \wedge dx_{11}
- \frac{i}{2N_3^{2}} d(N_1 \sin 2\nu \sin \xi \wedge dx_9 \wedge dx_{10} \wedge dx_{11}
+ \star dN_1 \cos \xi \sin \nu + \star dN_1 \cos \nu \sin \xi
\]

\[
\partial_i (e_{jk}^{11} = \epsilon_{ijk} \partial_k (1 - N_1) \cos \xi \cos \nu \equiv \nabla \wedge A_i) \tag{123}
\]

The metric is corrected to read

\[
ds^2 = (N_1 N_2 N_3)^{\frac{1}{2}} (dx_1^2 + \cdots + dx_3^2 + N_1^{-1} (-dx_4^2 + \cdots + dx_6^2))
+ N_2^{-1} (dx_7^2 + dx_8^2)
+ N_3^{-1} (dx_9^2 + dx_{10}^2)
+ (N_2 N_3)^{-1} (dx_{11} + (1 - N_1) \cos \xi \cos \nu (dx_1 + dx_2 + dx_3)^2). \tag{124}
\]

When \(\nu = \frac{\pi}{2}\) (respectively \(\xi = \frac{\pi}{2}\)), such that \(N_3 = 1 (N_2 = 1)\), the solution reduces to the dyonic membrane. In the limit \(\nu = \xi = \frac{\pi}{2}\), so that \(N_2 = N_3 = 1\), we recover the membrane solution. There are similar limits which give the KK6 solution \((\nu = \xi = 0 N_1 = N_2 = N_3)\) and two different M5-brane limits \((\xi = \frac{\pi}{2}, \nu = 0) N_3 = 1, N_1 = N_2\), and \(\xi = 0, \nu = \frac{\pi}{2} N_3 = 1, N_1 = N_3\). We indicate these limits pictorially in figure 2.

There is also a solution which we have not seen before that interpolates between the KK6-brane and an M5-brane, which occurs when either \(\xi = 0\) or \(\nu = 0\). Let us consider the example where \(\nu = 0\) and hence \(N_3 = N_1\). The volume element is

\[
ds^2 = N_1 \frac{1}{2} N_2 \frac{1}{2} (dx_1^2 + \cdots + dx_3^2) + N_1^{-1} (-dx_4^2 + \cdots + dx_6^2)
+ N_2^{-1} (dx_7^2 + dx_8^2)
+ N_3^{-1} (dx_9^2 + dx_{10}^2)
+ (N_1 N_2)^{-1} (dx_{11} + (1 - N_1) \cos \xi (dx_1 + dx_2 + dx_3)^2). \tag{125}
\]

The field strength becomes

\[
F_4 = \star d(N_1) \sin \xi. \tag{126}
\]
There are another two simple limits of the decomposed KK6 solution that we have not considered hitherto; these are indicated by the diagonals in figure 2. Upon setting \( \nu = \xi \), so that \( N_2 = N_3 \), we find that the volume element of the solution becomes

\[
dx^2 = N_1^2 \left( \frac{1}{2} \sin^2 \xi (1 - N_1) \right)^{\frac{1}{2}} \left[ \left( dx_1^2 + \cdots + dx_5^2 \right) + N_1^{-1} \left( -dx_1^2 + \cdots + dx_5^2 \right) \right. \\
\left. + \left( N_1 + (1 - N_1) \sin^2 \xi \right)^{-1} \left( dx_6^2 + dx_7^2 + dx_8^2 + dx_{11}^2 \right) \right]
\]

(127)

The field strength becomes

\[
F_4 = d(N_1^{-1}) \sin^2 \xi \wedge dx_1 \wedge dx_5 \wedge dx_6 - \frac{i}{N_2^2} d(N_1) \sin^2 \xi \cos \xi \wedge dx_7 \wedge dx_8 \wedge dx_{11}
\]

\[
- \frac{i}{N_3^2} d(N_1) \sin^2 \xi \cos \xi \wedge dx_9 \wedge dx_{10} \wedge dx_{11} + \star_2 dN_1 \frac{1}{2} \sin 2\xi + \star_3 dN_1 \frac{1}{2} \sin 2\xi.
\]

(128)

The second limit worthwhile considering relates two M5-branes and appears when we set \( \nu = \frac{\pi}{2} - \xi \). In this limit, \( \sin \nu = \cos \xi \), \( \cos \nu = \sin \xi \) so that \( 1 - N_3 = (1 - N_1 \sin^2 \xi) \) and

\[
1 - N_3 = N_2 - N_1.
\]

(129)

Our construction prohibits us from taking the limit \( N_1 \to 1 \), corresponding to the vanishing of the M2 charge. However, as \( N_1 \) approaches 1, the above relation indicates that the charges of the two S2-branes \((1 - N_2)\) and \((1 - N_3)\) are approaching opposite values. The parameter \( \xi \) interpolates between two M5-branes whose longitudinal directions are \( \{x_4, x_5, x_6, x_7, x_8, x_{11}\} \) and \( \{x_4, x_5, x_6, x_9, x_{10}, x_{11}\} \); hence, its variation in this limit has the effect of rotating a two-dimensional subspace of an M5-brane: \( \{x_7, x_8\} \to \{x_9, x_{10}\} \) as \( \xi = 0 \to \frac{\pi}{2} \). The volume element is

\[
dx^2 = N_1^4 \left( N_1 + \frac{1}{4} \sin^2 2\xi (1 - N_1) \right) \left[ \left( dx_1^2 + \cdots + dx_5^2 \right) + N_1^{-1} \left( -dx_1^2 + \cdots + dx_5^2 \right) \right. \\
\left. + \left( N_1 + (1 - N_1) \sin^2 \xi \right)^{-1} \left( dx_6^2 + dx_7^2 + dx_8^2 + dx_{11}^2 \right) \right] \\
+ \left( N_1 + \frac{1}{4} \sin^2 2\xi (1 - N_1) \right)^{-1} \left( dx_{11} - \frac{1}{2} (1 - N_1) \sin 2\xi \left( \sum_{i=1}^{3} dx_i \right) \right)^2
\]

(130)

6. Discussion

In the decomposition of \( E_{11} \) into tensors of \( SL(11) \), the gauge fields of the M2-brane and M5-brane appear without discrimination together with an infinite set of mixed symmetry tensors whose direct role in 11 dimensions is unclear\(^8\). In this paper, we have provided evidence that the mixed symmetry tensors of \( E_{11} \) may be interpreted as bound states of M-branes in M-theory in the form of a group element (1) encoding these solutions. The group element was paramaterized by two continuous variables: the radial distance \( r \) from the origin of the solution in spacetime and an angle variable that interpolated between brane solutions. The angle variable ranging from 0 to \( \frac{\pi}{2} \) moves along a path connecting the generators of \( E_{11} \).

\(^8\) Although an \( E_6 \) multiplet [23] (corresponding to a subset of the \( E_{11} \) Young tableaux with a maximum height of nine boxes) has been shown to solve the supergravity equations of motion.
The transversely boosted M2- and M5-branes as well as the dyonic membrane fell into a class of solutions encoded in the group element (1). The gauge fields appearing in these solutions mirrored the generators used to move along a path in the adjoint representation of $E_{11}$ under the adjoint action of the algebra. However, we also gave examples of the M2_{6}, M5_{3} and W5\text{M7} solutions whose interpolating group element deviated from (1). In these examples the orientation of the three component branes was chosen so that the generators associated with their gauge fields formed an Abelian sub-algebra. The M2_{6} and the M5_{3} were derived using a two-parameter group element whose gauge parameter was modified from the form of (1). Interestingly in these examples it was made manifest that given a mixed symmetry Young tableau there is more than one way to decompose it into sub-tableaux before finding a composite solution. For two-parameter group elements this point was examined again by reinterpreting the KK6 monopole solution as a bound state of an M5-brane and an S2-brane, although without choosing generators which formed an Abelian sub-algebra. Finally, a further decomposition of the KK6 monopole into three membrane solutions gave a suggested three-parameter group element. In general at level $l$, we expect to use an $l$-parameter \(^9\) interpolating group element to describe the solution as a superposition of membrane states. Diagrammatically one may imagine the interpolating solutions as an \(l-1\)-dimensional hypercube, having \(l\) vertices corresponding to M2- or S2-branes, the transition along the edges of the hypercube correspond to varying one angle parameter. In this way, one may imagine all the states of the adjoint representation as bound states of membranes, or membrane molecules.

We have concentrated, in this paper, on investigating M-theory bound state solutions about which little is known. A strong test of the form of the group element given in (1) is that upon dimensional reduction it gives rise to other well-known bound state solutions. In particular, in a forthcoming work [53], we intend to reproduce the bound states of string theory by a similar analysis. There are some interesting questions associated with this reduction, in particular the question of whether we can use the group element formulation to produce localized bound state brane solutions. Specifically bound states which include D6-branes are known to give localized brane solutions [14] as opposed to the smeared ones that we have considered in the present work, since they arise as a dimensional reduction of a pure gravity solution (as a reduction of the KK6 monopole). In this paper and elsewhere [23], an infinite tower of gravitational solutions have been derived and it is interesting to wonder whether other localized brane solutions may be derived by dimensionally reducing the bound states of this tower of solutions.

Amongst the examples of composite solutions considered in section 4 of this paper were the transversely boosted M2- and M5-branes where the interpolating solution parameterised the path from $K^{a}_{4}$ to $R^{abcd}$. In the example shown in section 4.2, the action $[K^{8}_{11}, R^{0101}] = R^{0010}$ was encoded in the group element and a rather complicated metric was derived which included the angle parameter, $\xi$. In [48], this angle parameter was interpreted as the boost parameter of the Lorentz transformation associated with the rotation generator $K^{8}_{11}$ in $SO(1,10)$ and by observing that the resulting field strength should simplify to an M2-brane field strength written in Lorentz boosted coordinates, it was possible to read off the Lorentz transformations on the coordinates. In an identical approach, one would expect to be able to interpret the action of the $R^{ab}$ generator on the coordinates by considering the field strength of the dyonic membrane. In this case, the problem is not so straightforward since the transformation should map an involved four-form field strength to a simple seven-form field strength of the fivebrane.

\(^9\) Where there is one radial parameter, $r$, and $l-1$ interpolating angle variables.
One can postulate that the map acts on the coordinate three-forms as
\[ dx_6 \wedge dx_7 \wedge dx_8 \rightarrow \frac{1}{\sin \xi} \sqrt{\frac{N_1}{N_2}} \wedge d\Omega, \]
\[ dx_9 \wedge dx_{10} \wedge dx_{11} \rightarrow -i \frac{1}{\tan \xi} \sqrt{\frac{N_2}{N_1}} \wedge d\Omega, \]
so that the four-form field strength is transformed into a seven-form field strength
\[ F_4(M_2 \oplus M_5) \rightarrow d \frac{1}{\sqrt{N_1 N_2}} \wedge d\Omega, \]
where \( d\Omega = -d\bar{t}_6 \wedge d\bar{x}_7 \wedge \cdots \wedge d\bar{x}_{11} \) is a six-form and \( \bar{x}_i \) are the transformed coordinates whose transformation bring the volume element to
\[ d\bar{s}^2_{M_2 \oplus S_2} = (N_1 N_2)^{-\frac{1}{2}} (dx_1^2 + \cdots + dx_5^2 + (\sqrt{N_1 N_2})^{-1} (-d\bar{t}_2^2 + d\bar{x}_7^2 + \cdots + d\bar{x}_{11}^2)). \]
However, finding the precise nature of the coordinate transformations as was possible with the Lorentz transformations, corresponding to the action of \( R^{abc} \) remains an open problem.

By relating the high-level roots of \( E_{11} \) to composite bound state solutions, we hope to gain an understanding of the field equations to be satisfied by exotic brane solutions. For the low-level single membrane field strength, the equation of motion does not depend upon the Chern–Simons term of the bosonic part of the supergravity action. The dyonic fivebrane equations of motion in \( D = 8 \) are dependent upon the dimensionally reduced Chern–Simons term; the curved space Laplace equations in \( D = 8 \) are sourced. In 11 dimensions, the fivebrane equations of motion are not generically of the same form as the membrane equations of motion; there is a contribution to the equations of motion from the Chern–Simons term. Understanding the transformation of \( A_{[3]} \) and \( F_{[4]} \) under the action of \( R^{abc} \) ought to determine the transformed Lagrangian including the transformation of the Chern–Simons term and the equations of motion. The corresponding Lagrangian will deviate from a sum of Lagrangians associated with the constituent branes by additional terms coming from both the kinetic terms and the Chern–Simons terms and the difference corresponds to the binding energy of the system.

The dyonic membrane is a solution to the equations of motion for arbitrary angle parameter and not just 0 or \( \frac{\pi}{2} \). That it falls into the class of bound state solutions encoded in the group element of (1) suggests that one must consider the full and continuous \( E_{11} \) symmetry as a symmetry of M-theory and not only the discrete subgroup generated by U-duality transformations.

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