QNMnonreciprocal_resonators: an openly available toolbox for computing the QuasiNormal Modes of nonreciprocal resonators

Tong Wu, Philippe Lalanne
LP2N, Institut d’Optique d’Aquitaine, IOGS, Univ. Bordeaux, CNRS

wutong1121@sina.com
philippe.lalanne@institutoptique.fr

Last revision: June, 2021

QNMnonreciprocal_resonators is an openly available COMSOL application; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation, either version 3 of the License, or (at your option) any later version. It is composed of
- the present user guide document,
- QNM_Dolmen_multipole.mph(.m), a COMSOL model provided under. mph or .m extensions.

The COMSOL application is distributed in the hope that it will be useful, but WITHOUT ANY WARRANTY; without even the implied warranty of MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the GNU General Public License for more details http://www.gnu.org/license/.
Table of contents
1. QNMnonreciprocal_resonators.................................................................................................................................................. 3
   1.1 How to acknowledge and cite ........................................................................................................................................ 3
   1.2 Units and conventions of input/output data for QNMtoolbox_multipole ..................................................................... 3
   1.3 Outline of the theoretical background ......................................................................................................................... 3
2. The permittivity and permeability of Yttrium iron garnet ............................................................................................... 5
3. A Tutorial: Computing QNMs for a YIG wire in AIR ........................................................................................................ 5
   2.1 Modeling Instructions ...................................................................................................................................................... 5
4. References .......................................................................................................................................................................... 16
1. QNMnonreciprocal_resonators

QNMnonreciprocal_resonators is an extension (posted in 2021) of the QNMEig solver of the freeware package MAN. It provides a comprehensive presentation of the computation and normalization of electromagnetic quasinormal modes (QNMs) of resonators composed of nonreciprocal materials. It features a theoretical background on the topic and a COMSOL model that illustrates how to put into practice the theory on the example of a Yttrium iron garnet wire in a homogenous background. This document provides the necessary details on how the model is built so that the interested readers may easily modify it for computing QNMs of other nonreciprocal resonators.

The present model has been successfully tested with another COMSOL model operating with QNMpole, the second QNM solver of MAN that performs a pole search gradient descent algorithm to compute and normalize QNMs [1]. The corresponding COMSOL model and Matlab programs used in the QNMpole are also included in the version 7.2 of MAN and following ones.

1.1 How to acknowledge and cite

We kindly ask that you reference the MAN package from IOGS-CNRS and its authors in any publication/report for which you used it. The preferred citation for QNMnonreciprocal_resonators is the following paper:
Wu, T.; Gurioli, M.; Lalanne, P. “Nanoscale Light Confinement: the Q’s and V’s”, ACS Photonics 2021, (https://doi.org/10.1021/acsphotonics.1c00336).

1.2 Units and conventions of input/output data for QNMtoolbox_multipole

Unit. All the input information is required to be in the SI unit. Accordingly, the output information is given in the SI unit as well.

Convention. The time-dependent terms \( \exp(i\omega t) \) are used in the COMSOL model provided in this toolbox. We use \( \exp(-i\omega t) \) convention for the formulas in Sections 1.3 and 2.

1.3 Outline of the theoretical background

The Section is organized as follows. It starts with the classical derivation of a general form of the Lorentz reciprocity theorem. The theorem is then used to normalize the modes and introduce the definition of the mode volume. Unlike our previous work [2,3], we do not assume reciprocity, namely, the formulas are valid for systems with \( \mathbf{\tilde{\mu}}^T \neq \mathbf{\tilde{\mu}} \) or \( \mathbf{\tilde{\varepsilon}} \neq \mathbf{\tilde{\varepsilon}} \) (the superscript \( T \) being the transpose operator). The theory presented here can be viewed as an extension of Supplementary Material 1 of ref. [2]. Related works may also be found in refs. [4,5].

1.3.1 Unconjugated form of Lorentz reciprocity theorem for nonreciprocal systems

We consider systems satisfying the following Maxwell’s equations:
\[
\nabla \times \mathbf{E}_1 = i\omega_1 \tilde{\mu}_1 \mathbf{H}_1, \quad \nabla \times \mathbf{H}_1 = -i\omega_1 \tilde{\varepsilon}_1 \mathbf{E}_1 - i\omega_1 \mathbf{p}_1, \tag{1}
\]
which features a source term \(-i\omega_1 \mathbf{p}_1\).

Lorentz reciprocity theorem relates two different solutions of Maxwell’s equations, \((\mathbf{E}_1, \mathbf{H}_1, \omega_1, \mathbf{p}_1, \tilde{\varepsilon}_1, \tilde{\mu}_1)\) and \((\mathbf{E}_2, \mathbf{H}_2, \omega_2, \mathbf{p}_2, \tilde{\varepsilon}_2, \tilde{\mu}_2)\) labeled by indices 1 and 2. It is derived by applying the divergence theorem to the vector \(\mathbf{E}_2 \times \mathbf{H}_1 - \mathbf{E}_1 \times \mathbf{H}_2\) and by using Eq. (1).
\[
\int_{\Sigma} (\mathbf{E}_2 \times \mathbf{H}_1 - \mathbf{E}_1 \times \mathbf{H}_2) \cdot d\mathbf{S} = i \int_{\Omega} \mathbf{H}_1 \cdot [(\omega_2 \tilde{\mu}_2 - \omega_1 \tilde{\mu}_1) \cdot \mathbf{H}_2 - \mathbf{E}_1 \cdot (\omega_2 \tilde{\varepsilon}_2 - \omega_1 \tilde{\varepsilon}_1) \cdot \mathbf{E}_2 + (\omega_1 \mathbf{p}_1 \cdot \mathbf{E}_2 - \omega_2 \mathbf{p}_2 \cdot \mathbf{E}_1)] d\mathbf{V}, \tag{2}
\]
where \(\Sigma\) is an arbitrarily closed surface defining a volume \(\Omega\).

1.3.2 QNM orthogonality

Lorentz reciprocity theorem can also be used to show that the QNMs satisfy an unconjugated orthogonality relation.

For that purpose, we set the second solution as the \(m^{th}\) QNM of a system with a permittivity \(\tilde{\varepsilon}\) and permeability \(\tilde{\mu}\), that is \((\mathbf{E}_2, \mathbf{H}_2, \omega_2, \mathbf{p}_2, \tilde{\varepsilon}_2, \tilde{\mu}_2) = (\mathbf{\bar{E}}^{(R)}_n, \mathbf{\bar{H}}^{(R)}_n, \tilde{\omega}_n^{(R)}, \mathbf{\bar{\varepsilon}}, \mathbf{\bar{\mu}})\), whereas the first solution corresponds to the \(m^{th}\) QNM of a system with a permittivity \(\tilde{\varepsilon}^T\) and permeability \(\tilde{\mu}^T\), that is \((\mathbf{E}_1, \mathbf{H}_1, \omega_1, \mathbf{p}_1, \tilde{\varepsilon}_1, \tilde{\mu}_1) = (\mathbf{\bar{E}}^{(L)}_m, \mathbf{\bar{H}}^{(L)}_m, \tilde{\omega}_m^{(L)}, \mathbf{\bar{\varepsilon}}, \mathbf{\bar{\mu}}^T)\). QNMs have divergent fields far away from the resonator and their treatment in a “generalized” Hilbert space requires regularization. We chose to regularize QNM with perfectly matched layers (PMLs) [2]. Thus, we choose the integral domain \(\Omega\) to be
the whole space including the PML regions so that $\Sigma$ represents the outer surfaces of PMLs, which are made of perfect electric/magnetic conductors in practice. Perfect electric/magnetic conductors impose zero tangential components of electric/magnetic fields on $\Sigma$. As a result, the surface integral on the left-hand-side of Eq. (2) vanishes [3], and this leads to a simple volume integral relation
\[ \int_{V} \bar{H}^{(L)}_{m} \cdot \left[ \bar{\omega}_{m} \bar{\mu}(\bar{\omega}_{m}) - \bar{\omega}_{m} \bar{\mu}(\omega_{m}) \right] \cdot \bar{E}^{(R)}_{m} - \int_{V} \bar{E}^{(L)}_{m} \cdot \left[ \bar{\omega}_{m} \bar{\varepsilon}(\bar{\omega}_{m}) - \bar{\omega}_{m} \bar{\varepsilon}(\omega_{m}) \right] \cdot \bar{E}^{(R)}_{m} dV = 0. \] (3)

Note that, for nonreciprocal systems, $\bar{H}^{(L)}_{m} \neq \bar{H}^{(R)}_{m}$ and $\bar{E}^{(L)}_{m} \neq \bar{E}^{(R)}_{m}$, but they share the same eigenvalues, that is $\bar{\omega}^{(L)}_{m} = \bar{\omega}^{(R)}_{m} = \bar{\omega}_{m}$ [5]. In the following part of the document, we refer to $(\bar{H}^{(L)}_{m}, \bar{E}^{(L)}_{m})$ as the left QNM and $(\bar{H}^{(R)}_{m}, \bar{E}^{(R)}_{m})$ as the right QNM.

Equation (3) is the orthogonality relation. Note that the permittivity and permeability are those of the actual (real) system without transposition. However, the orthogonality requires two fully independent computations, one for the real system QNMs and a second one for the transposed system. This is not the case for reciprocal materials.

1.3.3 The modal excitation coefficients and mode volumes

With the knowledge of the orthogonality of QNMs, we now discuss how to compute the QNM excitation coefficients. We consider a single electric dipole $p$ located at $r_{n}$ in the vicinity of a resonator. The dipole radiates a field $(\bar{E}, \bar{H})$ at the real frequency $\omega$. Owning to the completeness [3], we are able to expand the field everywhere in the PMLized space
\[ \bar{E}(r, \omega) = \sum_{m} \alpha_{m}(\omega) \bar{E}^{(R)}_{m}(r), \] (4)
where $\alpha_{m}$ are complex coefficients to be determined.

By applying Eq. (2) to the total field $(\bar{E}, \bar{H})$ created by the dipole at the frequency $\omega$ and to the $n^{th}$ left QNM $(\bar{E}^{(L)}_{n}, \bar{H}^{(L)}_{n})$, and using the modal expansion of Eq. (4), we obtain a linear system of equations
\[ \sum_{m} B_{nm}(\omega) \alpha_{m}(\omega) = -\omega p \cdot \bar{E}^{(L)}_{n}(r_{n}), \] where the frequency-dependent coefficient $B_{nm}$ is given by
\[ B_{nm} = \int_{V} \bar{H}^{(L)}_{m} \cdot \left[ \bar{\omega}_{m} \bar{\mu}(\bar{\omega}_{m}) - \bar{\omega}_{m} \bar{\mu}(\omega_{m}) \right] \cdot \bar{H}^{(L)}_{n} - \int_{V} \bar{E}^{(L)}_{m} \cdot \left[ \bar{\omega}_{m} \bar{\varepsilon}(\bar{\omega}_{m}) - \bar{\omega}_{m} \bar{\varepsilon}(\omega_{m}) \right] \cdot \bar{E}^{(L)}_{n} dV. \] (5)

In the absence of dispersion, all the non-diagonal terms are vanished according to Eq. (3) and the expansion coefficient is analytically given by
\[ \alpha_{n}(\omega) = \frac{-\omega p \cdot \bar{E}^{(L)}_{n}(r_{n})}{(\omega - \bar{\omega}_{n}) \int_{V} \bar{E}^{(L)}_{m} \cdot \bar{\mu}(\omega_{m}) \cdot \bar{E}^{(L)}_{m} \cdot \bar{H}^{(L)}_{m} dV}. \] (6)

In the general case of dispersive media, the off-diagonal coefficients $B_{nm}$ are not equal to zero. One thus has to solve a linear system of equations to obtain $\alpha_{m}$. To guarantee a safe and accurate numerical implementation, we note that $B_{nm}$ is null for $\omega = \bar{\omega}_{m}$, and we introduce $A_{nm} = (\omega - \bar{\omega}_{m})^{-1} B_{nm}$, so that the linear system of equations can be rewritten as
\[ \sum_{m} A_{nm}(\omega) \alpha_{m}(\omega) = -\omega p \cdot \bar{E}^{(L)}_{n}, \] (7)
where the unknowns are now $\alpha_{m}(\omega) = (\omega - \bar{\omega}_{m})^{-1} \alpha_{m}(\omega)$.

The linear system Eq. (7) is not diagonal, and it is not possible in general to derive a closed-form expression. However, we can still show that $\alpha_{n}$ has a pole and use this property to derive an approximate analytical expression. For $\omega \approx \bar{\omega}_{n}$, since $A_{nm}(\bar{\omega}_{n}) = 0$ for $n \neq m$ according to Eq. (3), we therefore find
\[ \alpha_{n}(\omega) \approx \frac{-\omega p \cdot \bar{E}^{(L)}_{n}(r_{n})}{(\omega - \bar{\omega}_{n}) \int_{V} \bar{E}^{(L)}_{m} \cdot \bar{\mu}(\omega_{m}) \cdot \bar{E}^{(L)}_{m} \cdot \bar{H}^{(L)}_{m} dV}. \] (8)

We thus obtain an approximate closed-form expression for $\alpha_{n}$ valid in the vicinity of $\bar{\omega}_{n}$. To derive Eqs. (6-8), we have strictly followed the approach in [2]. The approximate Eq. (8) has been successfully used in many earlier works starting with [2]. It is convenient, but it could also be set into a rigorous format. Using auxiliary fields to linearize the nonlinear eigenvalue problem of Eq. (1) (which is nonlinear in $\omega$ for dispersive material), we may derive an exact analytical expression for the excitation coefficients. This machinery that is described in [3] is not provided here for the sake of simplicity.

1.3.4 Mode volume

The Purcell factor is the modification of the spontaneous decay rate due to the resonance. Following [2], it is easily shown that the non-Hermitian Purcell factor $F_{n}$ of the resonance $\bar{E}^{(R)}_{n}$ (i.e. the) is
\[ F_{n} = \frac{3}{4\pi^{2} L^{3}_{n}} \text{Re} \left( \frac{\bar{\omega}_{n}}{\bar{\varepsilon}_{n}} \frac{\bar{\omega}^{3}_{n}}{\omega^{2}} \frac{\bar{\varepsilon}_{n}}{\omega^{2} \bar{\varepsilon}_{n}^{2} + \bar{\varepsilon}_{n}^{2}} \left[ 1 + 2 Q_{n} \omega \bar{\omega}_{n} \text{Im}(\bar{\varepsilon}_{n}) \right] \right), \] (9)
and that the mode volume is given
\[
\tilde{\mathcal{V}}_n = \int_\Omega \mathcal{E}_n^{(L)} \frac{\partial (\mathcal{E}_n^{(R)})}{\partial \omega} - \mathcal{H}_n^{(L)} \frac{\partial (\mathcal{H}_n^{(R)})}{\partial \omega} \, dV. 
\]

(10)

The Purcell factor takes a form similar to that of the classical expression derived in a Hermitian framework, except for the last bracketed term. \(Q_n = \Omega_n/\Gamma_n\) is the quality factor of the mode with complex eigenfrequency \(\tilde{\omega}_n = \Omega_n - i \Gamma_n/2\). \(n_b\) is the background refractive index at the dipole position. \(\lambda_n = 2\pi c/\Omega_n\) is the resonance wavelength.

We, therefore, can define the normalization factor as

\[
QN = \int_\Omega \mathcal{E}_n^{(L)} \cdot \mathcal{E}_n^{(R)} - \mathcal{H}_n^{(L)} \cdot \mathcal{H}_n^{(R)} \, dV. 
\]

(11)

Equation (10) is very similar to that derived for reciprocal resonators in [2]. Note that the expression is unchanged by scaling either \([\mathcal{E}_n^{(L)}, \mathcal{H}_n^{(L)}]\) or \([\mathcal{E}_n^{(R)}, \mathcal{H}_n^{(R)}]\) by any multiplicative factor, as expected.

Note that Eq. (10) is identical to the expression given at the end of the ‘Summary and Perspectives’ Section in [6]: the present document thus provides a demonstration of the expression that was not demonstrated in [6] for compactness.

2. THE PERMITTIVITY AND PERMEABILITY OF YTTRIUM IRON GARNET

In this toolbox, we consider a simple case of a 2D Yttrium iron garnet (YIG) wire in a homogeneous background. The users may easily extend it for studying resonators with other materials, dimensions, and geometries following the steps given in the next Tutorial section.

Under an external dc magnetic field along the \(z\) direction, the YIG exhibits a strong gyromagnetic anisotropy, with the relative permeability tensor taking the form

\[
\bar{\mu} = \begin{bmatrix} \mu_r & -i \kappa_r & 0 \\ i \kappa_r & \mu_r & 0 \\ 0 & 0 & \mu_{\infty} \end{bmatrix} \mu_0, 
\]

(12)

where the matrix elements \(\mu_r\) and \(\kappa_r\) are determined by the Landau-Lifshitz-Gilbert equation and can be expressed as [7]

\[
\mu_r = \mu_{\infty} \left[ 1 + \frac{(\omega_H - i \omega) \omega_M}{(\omega_H - i \omega)^2 - \omega^2} \right], 
\]

(13)

and

\[
\kappa_r = \frac{\mu_{\infty} \omega_M}{(\omega_H - i \omega)^2 - \omega^2}, 
\]

(14)

where \(\mu_{\infty} = 1\), \(\alpha\) is the damping parameter, \(\omega_H = \gamma H_b\), \(\omega_M = \gamma M_s\) with \(\gamma\) and \(M_s\) being the gyromagnetic ratio and saturated magnetization, respectively, \(H_b\) being the external static magnetic field which leads to broken time-reversal symmetry. On the other hand, the YIG permittivity is taken as a constant scalar \(\tilde{\varepsilon} = \varepsilon_{z} E_0 \, \text{diag}(1,1,1)\).

At last, note the simple relation [8]

\[
\tilde{\mu}^{-1} = \begin{bmatrix} d & -ib & 0 \\ ib & d & 0 \\ 0 & 0 & \mu_{\infty}^{-1} \end{bmatrix} \mu_0^{-1}. 
\]

(15)

where \(d = \mu_r/(\mu_r^2 - \kappa_r^2)\) and \(b = -\kappa_r/(\mu_r^2 - \kappa_r^2)\), which may help to derive the weak form expression later.

3. A TUTORIAL: COMPUTING QNMs FOR A YIG WIRE IN AIR

In this part, we provide details on how to compute the QNMs of a classical nonreciprocal resonator, a YIG wire in a homogenous background. Note that in QNMEig, there are already several well-documented examples for computing modes for dispersive resonators. However, they are all limited to systems with reciprocal materials, and QNMnonreciprocal_resonators can be considered as an extension of the QNMEig for non-reciprocal systems.

2.1 Modeling Instructions

Open COMSOL Multiphysics. From its File menu, choose New.
NEW
In the New window, click Model Wizard.

MODEL WIZARD
Since we need to compute both the left and right QNMs, unlike the other models in QNMEig, here we need two emws and two weak form PDEs.

1. In the Model Wizard window, click 2D.
2. In the Select physics tree, select Radio Frequency->Electromagnetic Waves, Frequency Domain (emw).
3. Click Add.
4. In the Select physics tree, select Mathematics->PDE Interfaces, Weak Form PDE (w).
5. Click Add.
6. In the Select physics tree, select Radio Frequency->Electromagnetic Waves, Frequency Domain (emw).
7. Click Add.
8. In the Select physics tree, select Mathematics->PDE Interfaces, Weak Form PDE (w).
9. Click Add.
10. Click Study
11. In the Select Study tree, select Preset Studies for Selected Physics Interfaces>Eigenfrequency
12. Click Done

Here the ‘emw2’ and ‘w2’ are built for computing the left QNM.

GLOBAL DEFINITIONS
Define the model geometric and material parameters.

Parameters 1
1. In the Model Builder window, under Global Definitions click Parameters 1.
2. In the Settings window for the Parameters, locate the Parameters section.
3. In the table, enter the following settings:

| Name      | Expression | Description                                      |
|-----------|------------|--------------------------------------------------|
| Lair      | a*1.5      | Geom: air background width                       |
| r         | 0.35*a     | Geom: wire radius                                |
| Lpml      | a/4        | Geom: PML thickness                              |
| a         | 26[mm]     | Geom: length of the air domain                   |
| epsrinf  | 15         | Material: permittivity of YIG                    |
| murinf    | 1          | Material: mu_inf given in section 2              |
| epsilonb  | 1          | Material: permittivity of the background medium  |
| lambda_pml| c_const/freqg | Material: typical absorbing wavelength of PMLs   |
| freqg     | 8.8466 [GHz] | Freq: frequency to search for QNMs              |
| Omega     | (freqg*2*pi)| Freq: normalization factor for the weak form     |
The Nomega can be set as an arbitrary value but should be of the same order of freq*2*pi. It is used in the auxiliary-field equation of the Weak Form PDE module (see later) to make the two equations of the quadratic polynomial eigenproblem [3] have similar magnitudes, thereby increasing the numerical stabilities.

Parameters 2
4. In the Model Builder window, right-click Global Definitions and select Parameters.
5. Type YIG materials in the Label text field.
6. In the Settings window for Parameters 2, locate the Parameters section.
7. In the table, enter the following settings:

| Name     | Expression          | Description                                |
|----------|---------------------|--------------------------------------------|
| omegam1  | 175[mT]*gamma1      | Material: omega_m given in section 2        |
| omega01  | gamma1*Hs1*mu0_const| Material: omega_0 given in section 2        |
| Hs1      | 900[Oe]             | Material: H0 given in section 2             |
| gamma1   | 2*pi*28[GHz/T]      | Material: gamma given in section 2          |
| alpha1   | 3e-4                | Material: alpha given in section 2          |

This table gives the parameters which are used to compute the \( \tilde{\mu} \) of YIG.

GEOMETRY
The geometry consists of a YIG wire in air background, surrounded by a PML.

Circle 1
1. In the Geometry toolbar, click Circle.
2. In the Settings window for Circle, locate the Size and Shape section.
3. In the Radius text field, type r.

Rectangle 1
4. In the Geometry toolbar, click Rectangle.
5. In the Settings window for Rectangle, locate the Size and Shape section.
6. In the Width text field, type Lair+Lpml*2.
7. In the Height text field, type Lair+Lpml*2.
8. Locate the Position section. From the Base list, select Center.
9. Locate the Layers section. In the Layer 1 text field, type Lpml.
10. Choose Layers to the left, Layers to the right, Layers on bottom, and Layers on top checkboxes.
11. Click the Build All Objects.

The geometry of the system: a YIG wire in air background, surrounded by a PML.
DEFINITIONS
Define PML domains and PML types. Define the resonator and background. Define different variables used for normalizing the QNMs.

Explicit 1
1. In the Definitions toolbar, click Explicit.
2. In the Settings window for Explicit, type PML in the Label text field.
3. Set Domains 1, 2, 3, 4, 6, 7, 8, 9.

Explicit 2
4. In the Definitions toolbar, click Explicit.
5. In the Settings window for Explicit, type sca in the Label text field.
6. Set Domain 10 only.

Explicit 3
7. In the Definitions toolbar, click Explicit.
8. In the Settings window for Explicit, type Air background and its attached PML in the Label text field.
9. Set Domains 1, 2, 3, 4, 5, 6, 7, 8, 9.

Perfectly matched layers
We model the PML as a non-dispersive material.

10. In the Definitions toolbar, click Perfectly Matched Layer.
11. In the Settings window for Perfectly Matched Layer, locate the Domain Selection section.
12. From the Selection list, choose PML.
13. Locate the Geometry section. From the Type list, select Cartesian.
14. Locate the Scaling section. From the Typical wavelength from list, choose User defined.
15. In Typical wavelength text field, type lambda_pml.

Variables
Here we define the complex eigenfrequencies that will be used in the weak form later.

16. On the Definitions toolbar, click Variables.
17. In the Settings window for Variables, locate the Geometric Entity Selection section.
18. Locate the Variables section. In the table, enter the following settings:

| Name          | Expression     | Description                  |
|---------------|----------------|------------------------------|
| QNM_omega     | emw.iomega/i   | Complex eigenfrequency of E^L|
| QNM_omega2    | emw2.iomega/i  | Complex eigenfrequency of E^R|

19. In the Definitions toolbar, click Variables.
20. In the Settings window for Variables, locate the Geometric Entity Selection section.
21. From the Geometric entity level list, choose Domain.
22. From the Selection list, choose Air background and its attached PML.
23. Locate the Variables section. In the table, enter the following settings:
Variables 3

Define the analytical expression of $\frac{\partial(\omega \mu^\prime)}{\partial \omega \vec{H}_n^{(r)}}$. Its three components are given by $\text{dwudwH}_x$, $\text{dwudwH}_y$, and $\text{dwudwH}_z$. These values will be used later to compute $QN$.

24. In the Definitions toolbar, click Variables.
25. In the Settings window for Variables, locate the Geometric Entity Selection section.
26. From the Geometric entity level list, choose Domain.
27. From the Selection list, choose sca.
28. Locate the Variables section. In the table, enter the following settings:

| Name     | Expression                                      | Description                                      |
|----------|-------------------------------------------------|-------------------------------------------------|
| fac1     | $i^*\alpha_1^*w^2+\omega_0^1+\omega_m^1$       |                                                  |
| fac3     | $i^*w^2^*\omega_m^1$                           |                                                  |
| fac2     | $i^*\alpha_1^*w^2^+\omega_0^1$                 |                                                  |
| lowduinvu| $(w^2^2-fac1^2)^*(w^2^2-fac2^2^2)$               |                                                  |
| fac4     | $(2^*\omega_0^1+\omega_m^1)^w^2^*(1+\alpha_1^1^2)$|                                                  |
| fac5     | $i^*\alpha_1^*((1+\alpha_1^1^2)^w^2^2-\omega_0^1^*(\omega_0^1^*+\omega_m^1))$|      |
| upduinvu11| $\omega_m^1^*((fac4+fac5)$                         |                                                  |
| upduinvu12| $i^*\omega_m^1^*((1+\omega_m^1^*^2)^w^2^2+\omega_0^1^*(\omega_0^1^*+\omega_m^1)$|      |
| dwudw11  | $1/w^2^+upduinvu11/lowduinvu$                    |                                                  |
| dwudw12  | upduinvu12/lowduinvu                            |                                                  |
| dwudw21  | upduinvu12/lowduinvu                            |                                                  |
| dwudw22  | $1/w^2^+upduinvu11/lowduinvu$                    |                                                  |
| dwudwH_x | $dwudw11^*cER_x^+dwudw12^*cER_y^/(-i)/\mu_0^{const}$|      |
| dwudwH_y | $dwudw21^*cER_x^+dwudw22^*cER_y^/(-i)/\mu_0^{const}$|      |
| dwudwH_z | $cER_z/w^2^/(-i)/\mu_0^{const}$                  |                                                  |

Variables 4

Define the analytical expression of $\frac{\partial(\omega \mu^\prime)}{\partial \omega \vec{H}_n^{(r)}}$ outside the resonator.

29. In the Definitions toolbar, click Variables.
30. In the Settings window for Variables, locate the Geometric Entity Selection section.
31. From the Geometric entity level list, choose Domain.
32. From the Selection list, choose Air background and its attached PML.
33. Locate the Variables section. In the table, enter the following settings:

| Name     | Expression | Description |
|----------|------------|-------------|
| dwudwH_x | HR_x       |             |
| dwudwH_y | HR_y       |             |
| dwudwH_z | HR_z       |             |

Variables 5
Since we have changed the weak form, \( \text{emw}(2).H_x, \text{emw}(2).H_y, \) and \( \text{emw}(2).H_z \) no longer give the correct magnetic fields. \( H_{Lx}, H_{Ly}, \) and \( H_{Lz} \) are the correct magnetic field for the left QNMs. \( H_{Rx}, H_{Ry}, \) and \( H_{Rz} \) are the correct magnetic field for the right QNMs. We also give the curl of \( \vec{E}_{n}^{(R)} \), \( cER_x, cER_y, \) and \( cER_z \).

34. On the **Definitions** toolbar, click **Variables**.
35. In the **Settings** window for Variables, locate the **Geometric Entity Selection** section.
36. Locate the **Variables** section. In the table, enter the following settings:

| Name     | Expression                                                                 | Description |
|----------|---------------------------------------------------------------------------|-------------|
| w        | \( \text{emw}.\omega_0/\text{i} \)                                   |             |
| w2       | \( \text{emw2}.\omega_0/\text{i} \)                                   |             |
| HLx      | \( \frac{1}{\text{i} \omega_0 \mu_0} (M_{1x} + \text{emw}\text{curlEx} \text{invmuinf}) \) |             |
| HLy      | \( \frac{1}{\text{i} \omega_0 \mu_0} (M_{1y} + \text{emw}\text{curlEy} \text{invmuinf}) \) |             |
| HLz      | \( \frac{1}{\text{i} \omega_0 \mu_0} (\text{emw}\text{curlEz} \text{invmuinf}) \) |             |
| invmuinf | \( (\mu_{\infty})^{-1} \)                                            |             |
| cERx     | \( \text{emw2}\text{curlEx} \)                                      |             |
| cERY     | \( \text{emw2}\text{curlEy} \)                                      |             |
| cERz     | \( \text{emw2}\text{curlEz} \)                                      |             |
| HRx      | \( \frac{1}{\text{i} \omega_0 \mu_0} (N_{1x} + \text{emw2}\text{curlEx} \text{invmuinf}) \) |             |
| HRy      | \( \frac{1}{\text{i} \omega_0 \mu_0} (N_{1y} + \text{emw2}\text{curlEy} \text{invmuinf}) \) |             |
| HRz      | \( \frac{1}{\text{i} \omega_0 \mu_0} (\text{emw2}\text{curlEz} \text{invmuinf}) \) |             |

Variables 6
Define the numerical expression of \( \partial (\omega \vec{H}) / \partial \omega \vec{H}_{n}^{(R)} \). In contrast to Variables 3, here these values are computed numerically.

37. In the **Definitions** toolbar, click **Variables**.
38. In the **Settings** window for Variables, locate the **Geometric Entity Selection** section.
39. From the **Geometric entity level** list, choose **Domain**.
40. From the **Selection list**, choose **sca**.
41. Locate the **Variables** section. In the table, enter the following settings:

| Name       | Expression                                                                 | Description |
|------------|---------------------------------------------------------------------------|-------------|
| mur\_YIG   | \( \mu_{\infty} + \mu_{\infty} \omega_0 (\omega_0 + \text{i} \alpha_1 \omega_2) / ((\omega_0 + \text{i} \alpha_1 \omega_2)^2 - \omega_2^2) \) |             |
| kappa\_YIG | \( \omega_0 (\omega_0 + \text{i} \alpha_1 \omega_2) / ((\omega_0 + \text{i} \alpha_1 \omega_2)^2 - \omega_2^2) \) \mu_{\infty} |             |
| dwkappa\_YIG | \( d(\text{mur}_\text{YIG}^2,\omega_2) \) |             |
| dmur\_YIG  | \( d(\text{mur}_\text{YIG}^2,\omega_2) \) |             |
| dmu11      | \( \text{dmu}_1 \) \text{YIG} |             |
| dmu12      | \( \text{dmu}_2 \) \text{YIG}_i |             |
| dmu21      | \( -\text{dmu}_2 \text{YIG}_i \) |             |
| dmu22      | \( \text{dmu}_2 \text{YIG} \) |             |
| duwHx      | \( \text{dmu}_1 \text{H}_{Rx} + \text{dmu}_2 \text{H}_{Ry} \) |             |
| duwHy      | \( \text{dmu}_1 \text{H}_{Rx} + \text{dmu}_2 \text{H}_{Ry} \) |             |
| duwHz      | \( \text{H}_{Rz} \) |             |

Variables 7
Define the numerical expression of \( \partial (\omega \vec{H}) / \partial \omega \vec{H}_{n}^{(R)} \) outside the resonator.
42. In the Definitions toolbar, click Variables.
43. In the Settings window for Variables, locate the Geometric Entity Selection section.
44. From the Geometric entity level list, choose Domain.
45. From the Selection list, choose Air background and its attached PML.
46. Locate the Variables section. In the table, enter the following settings:

| Name   | Expression | Description |
|--------|------------|-------------|
| duwHx  | HRx        |             |
| duwHy  | HRy        |             |
| duwHz  | HRz        |             |

ELECTROMAGNETIC WAVES, FREQUENCY DOMAIN (emw)
emw and w are used to solve for the left QNMs. Here we only consider the case where the electric field component of the QNM is polarized along z axis.

1. In the Settings window for Electromagnetic Waves, Frequency Domain, locate the Components section.
2. From the Electric field components solved for list, choose Out-of-plane vector.
3. On the top of the Modal Builder window, click Show option (the one with an eye icon), select at least Advanced Physics and Discretization options.
4. In the Physics toolbar, select Electromagnetic Waves, Frequency Domain (emw).
5. In the Settings window for Electromagnetic Waves, Frequency Domain (emw), locate the Domain Selection section.
6. From the Selection list, choose All Domains.
7. In the Physics toolbar, from the Domains section list, choose Weak Contribution.
8. From the Selection list, choose sca.
9. In Weak expression text field, type -M1x*test(emw.curlEx)-M1y*test(emw.curlEy).

Wave Equation, Electric 1
The weak expression for the resonator domain.

10. In the Settings window for Wave Equation, Electric 1, locate the Electric Displacement Field section.
11. From the Electric displacement field model list, choose Relative permittivity.
12. From Relative permittivity list, choose User defined. In the text field, enter epsrinf.
13. In the Settings window for Variables, locate the Magnetic Field section.
14. From the Constitutive relation, choose Relative permeability.
15. From Relative permeability list, choose User defined. In the text field, enter murinf.
16. In the Settings window for Variables, locate the Conduction Current section.
17. From the Electric conductivity, choose User defined. In the text field, enter 0.

Wave Equation, Electric 2
The weak expression for the background domain.
18. In the **Physics** toolbar, from the **Domains** section list, choose **Wave Equation, Electric**.
19. In the **Settings** window for **Wave Equation, Electric 2**, locate **Domain Selection** section.
20. From the **Selection** list, choose **Air background and its attached PML**.
21. From the **Relative permittivity** list, choose **User defined**. In the text field, enter epsilonb.
22. From the **Relative permittivity** list, choose **User defined**. In the text field, enter 1.
23. In the **Settings** window for **Variables**, locate the **Conduction Current** section.
24. From the **Electric conductivity** list, choose **User defined**. In the text field, enter 0.

Weak Form PDE (w)
The weak expression for the auxiliary fields M1x and M1y.

\[ \mathbf{M}(\mathbf{r}) \cdot \mathbf{F} = \left( (\varepsilon_0^{-1} - \varepsilon_\infty^{-1}) \nabla \times \mathbf{E}_m^{(L)}(\mathbf{r}) \right) \cdot \mathbf{F}. \]

1. In the **Physics** toolbar, select **Weak Form PDE**.
2. In the **Settings** window for **Weak Form PDE**, locate the **Domain Selection** section.
3. From the **Selection** list, choose: sca.
4. Locate the **Discretization** section. From the **Shape function type** list, choose **Lagrange**.
5. Locate the **Dependent Variables** section. In the **Field name** text field, enter M1. In the **Number of dependent** variables text field, enter 2. In the **Dependent variables** text field, enter M1x, M1y.

Weak Form PDE 1
6. In the **Modal Builder** window, under the **Weak Form PDE (w)** module, click **Weak Form PDE 1**.
7. In the **Settings** window for **Weak Form PDE 1**, locate the **Weak Expressions** section, enter the following expression:

\[
(test(M1x)*emw.curlEx+test(M1y)*emw.curlEy)*(omegam1*(i*alpha1*QNM_omega+omega01+omega_m1))/nomega^2+(test(M1x)*emw.curlEy-test(M1y)*emw.curlEx)*(-i)*(QNM_omega*omegam1)/nomega^2-(test(M1x)*M1x+test(M1y)*M1y)*(1+alpha1^2)*QNM_omega^2-2*i*alpha1*QNM_omega*(omega01+omega_m1)-(omega01+omega_m1)^2)*murinf/nomega^2
\]

ELECTROMAGNETIC WAVES, FREQUENCY DOMAIN (emw2)
emw and w are used to solve for the right QNMs.

Repeat what has been done in ELECTROMAGNETIC WAVES, FREQUENCY DOMAIN (emw1) but replace all the M1x and M1y with N1x and N1y. Note that \( \mathbf{N}(\mathbf{r}) = (\varepsilon_0^{-1} - \varepsilon_\infty^{-1}) \nabla \times \mathbf{E}_m^{(L)}(\mathbf{r}) \), which is different from \( \mathbf{M}(\mathbf{r}) \).

Weak Form PDE (w2)
Repeat what has been done in Weak Form PDE (w) but replace all the M1x and M1y with N1x and N1y and in the **Settings** window for **Weak Form PDE 1**, locate the **Weak Expressions** section, enter the following expression:

\[
(test(N1x)*emw2.curlEx+test(N1y)*emw2.curlEy)*(omegam1*(i*alpha1*QNM_omega2+omega01+omega_m1))/nomega^2+(test(N1x)*emw2.curlEy-test(N1y)*emw2.curlEx)*(-i)*(QNM_omega2*omegam1)/nomega^2-(test(N1x)*N1x+test(N1y)*N1y)*(1+alpha1^2)*QNM_omega^2-2*i*alpha1*QNM_omega*(omega01+omega_m1)-(omega01+omega_m1)^2)*murinf/nomega^2
\]

Mesh 1
Free Triangular 1
1. In the Model Builder window, right-click Mesh 1 and choose Free Triangular.
2. In the Settings window for Free Triangular locate the Domain Selection section.
3. From the Geometric entity level list, choose Domain.
4. From the Selection list, choose sca.
5. Right-click Free Triangular 1 and choose Size.
6. In the Settings window for Size 1, locate the Element Size section and choose Custom.
7. From the Geometric entity level list, choose Domain.
8. From the Selection list, choose sca.
9. Locate the Element Size section, in the Maximum element size text field, type r/10.

Free Triangular 2
10. In the Model Builder window, right-click Mesh 1 and choose Free Triangular.
11. In the Settings window for Free Triangular locate the Domain Selection section.
12. From the Geometric entity level list, choose Domain.
13. Select Domain 5 only.
14. Right-click Free Triangular 1 and choose Size.
15. In the Settings window for Size 1, locate the Element Size section and choose Custom.
16. From the Geometric entity level list, choose Domain.
17. Select Domain 5 only.
18. Locate the Element Size section, in the Maximum element size text field, type r/2.

Mapped 1
19. In the Model Builder window, right-click Mesh 1 and choose Mapped.
20. In the Settings window for Mapped locate the Domain Selection section.
21. From the Geometric entity level list, choose Remaining.
22. Right-click Mapped 1 and choose Distribution.
23. In the Settings window for Distribution 1, locate the Distribution section.
24. From the Distribution type list, choose Fixed number of elements.
25. In the Number of elements field, type 5.
26. Click Build All.

The mesh of the system.

Study 1
Search for the left QNMs.

1. In the Model Builder window, click Study 1>>Step 1: Eigenfrequency.
2. Activate the Desired number of eigenfrequencies check box. Type the number of modes.
3. From the Unit list, choose GHz.
4. In the Search for eigenfrequencies around, type freqg.
5. Locate the Physics and Variables Selection section, clear the ELECTROMAGNETIC
6. Click Compute.

Study 2
Search for the right QNMs.

7. In the Study toolbar, select Add study.
8. In the Add study window, click General Studies>> Eigenfrequency.
9. In the Model Builder window, click Study 2>>Step 1: Eigenfrequency.
10. Activate the Desired number of eigenfrequencies check box. Type the number of modes and make sure it is equal to that in Study 1.
11. From the Unit list, choose GHz.
12. In the Search for eigenfrequencies around, type freq.
13. Locate the Physics and Variables Selection section, clear the ELECTROMAGNETIC WAVES, FREQUENCY DOMAIN (emw) and Weak Form PDE (w) checkboxes.
14. Click Compute.

Data Sets
Join 1
Because we need both the left QNMs computed with Study 1 and the right QNM computed with Study 2, we need to join two data sets.

1. In the Study toolbar, select More Data Sets>>Base Data Det>>Join.
2. In the Model Builder window, click Join 1.
3. In the Settings window for Join 1, locate the Data 1 section.
4. From the Data list, choose Study 1/Solution 1 (sol1).
5. From the Solutions list, choose One.
6. From the Eigenfrequency list, choose a mode which we are concerned about.
7. In the Settings window for Join 1, locate the Data 2 section.
8. From the Data list, choose Study 2/Solution 2 (sol2).
9. From the Solutions list, choose One.
10. From the Eigenfrequency list, choose the mode which has the same frequency as that in Data 1.
11. In the Settings window for Join 1, locate the Combination section.
12. From the Combination list, choose Explicit.
Derived Values

Surface integral

Evaluate the $QN$ defined in Eq. (11).

1. In the Study toolbar, select More Derived Values>>Integration>>Surface integration.
2. In the Model Builder window, click Surface integration 1.
3. From the Data set list, choose Join 1.
4. Locate the Expressions section. In the table, enter the following settings:

| Expression | Description         |
|------------|---------------------|
| $(data1(\text{emw}.\text{Ex})*data2(\text{emw2}.\text{Dx})+data1(\text{emw}.\text{Ey})*data2(\text{emw2}.\text{Dy})$ $+$ $data1(\text{emw}.\text{Ez})*data2(\text{emw2}.\text{Dz}))$*$data1(\text{pml1}.\text{detInvT})$ $-$ $(data1(\text{HLx})*data2(\text{dwudwH}.\text{x})+data1(\text{HLy})*data2(\text{dwudwH}.\text{y})$ $+$ $data1(\text{HLz})*data2(\text{dwudwH}.\text{z}))$*$data1(\text{pml1}.\text{detInvT})$*$\mu_0_{\text{const}}$ $(data1(\text{emw}.\text{Ex})*data2(\text{emw2}.\text{Dx})+data1(\text{emw}.\text{Ey})*data2(\text{emw2}.\text{Dy})$ $+$ $data1(\text{emw}.\text{Ez})*data2(\text{emw2}.\text{Dz}))$*$data1(\text{pml1}.\text{detInvT})$$-$ $$(data1(\text{HLx})*data2(\text{dwudwH}.\text{x})+data1(\text{HLy})*data2(\text{dwudwH}.\text{y})$ $+$ $data1(\text{HLz})*data2(\text{dwudwH}.\text{z}))$*$data1(\text{pml1}.\text{detInvT})$*$\mu_0_{\text{const}}$ $(data1(\text{HLx})*data2(\text{duwHx})+data1(\text{HLy})*data2(\text{duwHy})$ $+$ $data1(\text{HLz})*data2(\text{duwHz}))$*$\mu_0_{\text{const}}$*$data1(\text{pml1}.\text{detInvT})$ | $QN_E$ $QN_H$ $QN=QN_E+QN_H$ $QN_H$ (Method 2) |

5. Click Evaluate>>New table.

2D Plot Group

Plot the map for $\text{Re}(\vec{E}_m^{(R)} \cdot \vec{E}_m^{(L)}/QN)$.

1. In the Study toolbar, select 2D Plot Group.
2. From the Data set list, choose Join 1.
3. In the Model Builder window, right-click 2D Plot Group 2, select Surface.
4. In the **Model Builder** window, click **Surface**.

5. Locate the **Expressions** section. In the table, enter the following settings: \(\text{real(data1(emw.Ez)*data2(emw2.Ez)/X)}\) with \(X\) being the value of \(QN=QN_E+QN_H\) computed in the previous step.

![Image](image.png)

\[\text{Re}(\mathbf{E}_m^{(R)} / \mathbf{E}_m^{(L)})/QN)\]

### 4. REFERENCES

1. Bai, Q.; Perrin, M.; Sauvan, C.; Hugonin, J. P.; Lalanne, P. “Efficient and intuitive method for the analysis of light scattering by a resonant nanostructure”, *Opt. Express* **2013**, 21, 27371.

2. Sauvan, C.; Hugonin, J. P.; Maksymov, I. S.; Lalanne, P. “Theory of the spontaneous optical emission of nanosize photonic and plasmon resonators”, *Phys. Rev. Lett.* **2013**, 110, 237401.

3. Yan, W.; Faggiani, R.; Lalanne, P. “Rigorous modal analysis of plasmonic nanoresonators”, *Phys. Rev. B* **2018**, 97, 205402.

4. Vial, B.; Nicolet, A.; Zolla, F.; Commandré, M. “Quasimodal expansion of electromagnetic fields in open two-dimensional structures”, *Phys. Rev. A* **2014**, 89, 023829.

5. Pick, A.; Zhen, B.; Miller O.D.; Hsu C.W.; Hernandez F.; Rodriguez A.W.; Soljačić M.; Johnson S.G. “General Theory of Spontaneous Emission Near Exceptional Points”, *Opt. Express* **2017**, 25, 12325.

6. Wu, T.; Gurioli, M.; Lalanne, P. “Nanoscale Light Confinement: the Q's and V's”, *ACS Photonics* **2021**, (https://doi.org/10.1021/acsphotonics.1c00336).

7. Rameshti, B. Z.; Bauer, G. E. “Indirect coupling of magnons by cavity photons”, *Phys. Rev. B* **2018**, 97, 014419.

8. Yang, B.; Wu, T.; Zhang, X. “Topological properties of nearly flat bands in two-dimensional photonic crystals”, *J. Opt. Soc. Am. B* **2017**, 34, 831.