Computations with reachable elements in simple Lie algebras

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Abstract

We report on some computations with reachable elements in simple Lie algebras of exceptional type within the SLA package of GAP4. These computations confirm the classification of such elements by Elashvili and Grélaud. Secondly they answer a question from Panyushev. Thirdly they show in what way a recent result of Yakimova for the Lie algebras of classical type extends to the exceptional types.

1 Introduction

Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \) (or over an algebraically closed field of characteristic 0). For \( e \in \mathfrak{g} \) we denote its centraliser in \( \mathfrak{g} \) by \( \mathfrak{g}_e \). In [9] an \( e \) in \( \mathfrak{g} \) is defined to be reachable if \( e \in [\mathfrak{g}_e, \mathfrak{g}_e] \). Such an element has to be nilpotent. In [3], Elashvili and Grélaud gave a classification of reachable elements in \( \mathfrak{g} \) (in this paper such elements are called compact, in analogy with [1]).

By the Jacobson-Morozov theorem a nilpotent \( e \in \mathfrak{g} \) lies in an \( \mathfrak{sl}_2 \)-triple \( (h, e, f) \) (where \( [e, f] = h, [h, e] = 2e, [h, f] = -2f \)). By the adjoint representation the subalgebra spanned by such a triple acts on \( \mathfrak{g} \). Since the eigenvalues of \( \text{ad} h \) are integers, we get a grading

\[
\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(k)
\]

where \( \mathfrak{g}(k) = \{ x \in \mathfrak{g} \mid [h, x] = kx \} \). Now set \( \mathfrak{g}(k)_e = \mathfrak{g}(k) \cap \mathfrak{g}_e \), and let \( \mathfrak{g}(\geq 1)_e \) denote the subalgebra spanned by all \( \mathfrak{g}(k)_e \), \( k \geq 1 \).

Panyushev ([9]) showed that, for \( \mathfrak{g} \) of type \( A_n \), \( e \) is reachable if and only if \( \mathfrak{g}(\geq 1)_e \) is generated as Lie algebra by \( \mathfrak{g}(1)_e \). Here we call this the Panyushev property of \( \mathfrak{g} \). In [9] it is stated that this property also holds for the other classical types and the question is posed whether it holds for the exceptional types. In [10] a proof is given that the Panyushev property holds in types \( B_n, C_n, D_n \). Our computations confirm that the Panyushev property holds also for the Lie algebras of exceptional type.
Yakimova ([10]) studied the stronger condition $g_e = [g_e, g_e]$. For the purposes of this paper we call elements $e$ satisfying this condition strongly reachable. She showed that for $g$ of classical type, $e$ is strongly reachable if and only if the nilpotent orbit of $e$ is rigid. (This means that it is not induced, cf. [8], [2], [7].) Furthermore, this is shown to fail for $g$ of exceptional type. As a result of our calculations we find all rigid nilpotent orbits whose representatives are not strongly reachable. From this we conclude that $e$ is strongly reachable if and only if $e$ is both reachable and rigid. We note that one direction of this statement can be shown in a uniform way for all $g$: if $e$ is strongly reachable then it is reachable, but also rigid by [10], Proposition 11. The converse for exceptional types follows from our calculations in two ways. Firstly we compute the list of all strongly reachable orbits and the list of all nilpotent orbits that are reachable and rigid, and find that they are the same. Second, the Panyushev property, which we checked by computation for the exceptional types, also implies the statement. For the classical types we have, of course, the stronger theorem from [10].

The SLA package ([5]), written in the language of the computer algebra system GAP4 ([4]), has functionality for working with the nilpotent orbits in simple Lie algebras. In particular the package contains the classification of such orbits. Using this it is straightforward to approach the above questions by computational means. Indeed, for a nilpotent orbit the system easily computes a representative $e$, and a corresponding $\mathfrak{sl}_2$-triple. Then using functions present in GAP4 we can compute the centralizer, $g_e$, and its derived subalgebra, and check whether $e$ lies in it. This gives us the list of reachable nilpotent orbits. Secondly, a similar procedure yields the list of strongly reachable orbits. Thirdly, SLA has a function for computing the grading corresponding to an $\mathfrak{sl}_2$-triple. With that it is straightforward to check whether $g(\geq 1)e$ is generated by $g(1)e$. The appendix contains the code for the functions implementing these procedures.

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2 Reachable nilpotent elements in the Lie algebras of exceptional type

Tables 1, 2, 3, 4, and 5 contain the nilpotent orbits that by our calculations are reachable. The content of the tables is as follows. The first column has the label of the orbit, and the second column the weighted Dynkin diagram. The third and fourth columns contain a × if the orbit is, respectively, strongly reachable and rigid. We note that the classification of rigid nilpotent orbits is known (see [2], [7]).

Table 1: Reachable nilpotent orbits in $E_6$.

| label | characteristic | Strong | Rigid |
|-------|----------------|--------|-------|
| $A_1$ | $0\ 0\ 0\ 0\ 0\ 0$ | ×      | ×     |
Reachable nilpotent orbits in $E_6$.

| label        | characteristic | Strong | Rigid |
|--------------|---------------|--------|-------|
| $2A_1$       | 1 0 0 0 1     | ×      | ×     |
| $3A_1$       | 0 0 1 0 0     | ×      | ×     |
| $A_2 + A_1$  | 1 0 0 0 1     |        |       |
| $A_2 + 2A_1$ | 0 1 0 1 0     |        |       |
| $2A_2 + A_1$ | 1 0 1 0 1     | ×      | ×     |

Table 2: Reachable nilpotent orbits in $E_7$.

| label        | characteristic | Strong | Rigid |
|--------------|---------------|--------|-------|
| $A_1$        | 1 0 0 0 0 0   | ×      | ×     |
| $2A_1$       | 0 0 0 0 1 0   | ×      | ×     |
| $(3A_1)'$    | 0 1 0 0 0 0   | ×      | ×     |
| $4A_1$       | 0 0 0 0 0 1   | ×      | ×     |
| $A_2 + A_1$  | 1 0 0 0 1 0   |        |       |
| $A_2 + 2A_1$ | 0 0 1 0 0 0   | ×      | ×     |
| $2A_2 + A_1$ | 0 1 0 0 1 0   | ×      | ×     |
| $A_4 + A_1$  | 1 0 1 0 1 0   |        |       |

Table 3: Reachable nilpotent orbits in $E_8$.

| label        | characteristic | Strong | Rigid |
|--------------|---------------|--------|-------|
| $A_1$        | 0 0 0 0 0 0 1 | ×      | ×     |
| $2A_1$       | 1 0 0 0 0 0   | ×      | ×     |
| $3A_1$       | 0 0 0 0 0 1 0 | ×      | ×     |
| $4A_1$       | 0 0 0 0 0 0   | ×      | ×     |
| $A_2 + A_1$  | 1 0 0 0 0 1   | ×      | ×     |
| $A_2 + 2A_1$ | 0 0 0 0 1 0   | ×      | ×     |
| $A_2 + 3A_1$ | 0 1 0 0 0 0   | ×      | ×     |
Reachable nilpotent orbits in $E_8$.

| label             | characteristic | Strong | Rigid |
|-------------------|----------------|--------|-------|
| $2A_2 + A_1$      | 1 0 0 0 0 0 1 0 | ×      | ×     |
| $A_4 + A_1$       | 1 0 0 0 0 1 0 1 |        |       |
| $2A_2 + 2A_1$     | 0 0 0 1 0 0 0 0 | ×      | ×     |
| $(A_3 + 2A_1)''$  | 0 1 0 0 0 0 1 0 | ×      | ×     |
| $D_4(a_1) + A_1$  | 0 0 0 0 0 1 0 1 | ×      | ×     |
| $A_3 + A_2 + A_1$ | 0 0 1 0 0 0 0 0 | ×      | ×     |
| $2A_3$            | 1 0 0 1 0 0 0 0 | ×      | ×     |
| $A_4 + 2A_1$      | 0 0 1 0 0 0 0 1 |        |       |
| $A_4 + A_3$       | 0 0 1 0 0 1 0 0 | ×      | ×     |

Table 4: Reachable nilpotent orbits in $F_4$.

| label             | characteristic | Strong | Rigid |
|-------------------|----------------|--------|-------|
| $A_1$             | 1 0 0 0 0 0 1 0| ×      | ×     |
| $\tilde{A}_1$     | 0 0 0 1 0 0 0 0| ×      | ×     |
| $A_1 + \tilde{A}_1$| 0 1 0 0 0 0 0 0| ×      | ×     |
| $A_2 + \tilde{A}_1$| 0 0 1 0 0 0 0 0| ×      | ×     |

Table 5: Reachable nilpotent orbits in $G_2$.

| label | characteristic | Strong | Rigid |
|-------|----------------|--------|-------|
| $A_1$ | 1 0 0 0 0 0 1 0| ×      | ×     |

We make the following comments.

- Here the reachable elements are exactly the same as in the paper of Elashvili and Grélaud. Therefore our calculations confirm their result.

- The rigid nilpotent orbits that are not strongly reachable are
  - in type $E_7$: $(A_3 + A_1)' (41,40)$,
  - in type $E_8$: $A_3 + A_1 (84,83)$, $D_5(a_1) + A_2 (46,45)$, $A_5 + A_1 (46,45)$,
  - in type $F_4$: $\tilde{A}_2 + A_1 (16,15)$,
– in type $G_2$: $A_1 (6,5)$.

Here the pair of integers in brackets is $(\dim \mathfrak{g}_e, \dim [\mathfrak{g}_e, \mathfrak{g}_e])$.

- In type $E_6$ all rigid orbits are strongly reachable. Hence in this type the situation is the same as for the classical types: $e$ is strongly reachable if and only if the orbit of $e$ is rigid.

- The last two columns of all tables are equal. This shows that for the exceptional types the following theorem holds: $e$ is strongly reachable if and only if $e$ is both reachable and rigid.

- This last statement also follows from the Panyushev property. Indeed, $e$ rigid implies that $\mathfrak{g}(0)_e$ is semisimple, so $[\mathfrak{g}(0)_e, \mathfrak{g}(0)_e] = \mathfrak{g}(0)_e$. Furthermore, $[\mathfrak{g}(0)_e, \mathfrak{g}(1)_e] = \mathfrak{g}(1)_e$ by [10], Lemma 8 (where this is shown to hold for all nilpotent $e$). By the Panyushev property this implies that $[\mathfrak{g}_e, \mathfrak{g}_e] = \mathfrak{g}_e$.

- We see that for all nilpotent orbits that are rigid but not strongly reachable the codimension of $[\mathfrak{g}_e, \mathfrak{g}_e]$ in $\mathfrak{g}_e$ is 1. Since a rigid orbit is reachable if and only if it is strongly reachable, we get that in all those cases $e$ spans the quotient $\mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$.

Example 1 Let us consider the nilpotent orbit in the Lie algebra of type $E_7$ with label $A_3 + A_2$. This orbit is not reachable. It has a representative with diagram

```
\begin{verbatim}
 29 32 31 27 30
\end{verbatim}
```

This means that the representative is $e = x_{29} + x_{32} + x_{31} + x_{27} + x_{30}$, where $x_i$ denotes the root vector corresponding to the $i$-th positive root (enumeration as in GAP4, cf. [6]). Furthermore, the Dynkin diagram of these roots is as shown above. This representative is stored in the package SLA.

Now, if the orbit were reachable then $e \in [\mathfrak{g}_e, \mathfrak{g}_e] \cap \mathfrak{g}(2)$. Using the SLA package we can easily compute the latter space:

```gap
gap> L := SimpleLieAlgebra("E",7,Rationals);
gap> o := NilpotentOrbits(L);
gap> sl2 := SL2Triple( o[19] );
[ (2)*v.90+(3)*v.92+(2)*v.93+(3)*v.94+(4)*v.95, (6)*v.127+(9)*v.128+(12)*v.129+
+(18)*v.130+(14)*v.131+(10)*v.132+(5)*v.133, v.27+v.29+v.30+v.31+v.32 ]
gap> g := SL2Grading( L, sl2[2] );
gap> g2 := Subspace( L, g[1][2] );
gap> der := LieDerivedSubalgebra(LieCentralizer(L,Subalgebra(L,[sl2[3]])));<Lie algebra of dimension 33 over Rationals>
gap> BasisVectors( Basis( Intersection( g2, der ) ) );
[ v.18, v.23+(-1)*v.24+v.28, v.24+(-1)*v.25+(-1)*v.28,
v.27+(-1)*v.29+v.30+(-1)*v.31+(-1)*v.32, v.33+(-1)*v.36+v.37,
v.34+(-1)*v.36+v.37, v.39 ]
```
First we make some comments on the above computation. The $\mathfrak{sl}_2$-triple comes ordered as $(f, h, e)$. So the second element is the neutral element, and the third element is the nil-positive element, i.e., the representative, which is as indicated above. So the second element defines the grading, which we compute with \texttt{SL2Grading}. In the subsequent line the subspace $\mathfrak{g}(2)$ is defined, followed by $[\mathfrak{g}_e, \mathfrak{g}_e]$. Finally a basis of the intersection is computed.

We see that one of the basis vectors of the intersection is
\[
v.27 + (-1)v.29 + v.30 + (-1)v.31 + (-1)v.32.
\]

So we see that $e = (x_{29} + x_{32} + x_{31}) + (x_{27} + x_{30})$ does not lie in $[\mathfrak{g}_e, \mathfrak{g}_e]$ but $(x_{29} + x_{32} + x_{31}) - (x_{27} + x_{30})$ does!

**Appendix: the code**

\begin{verbatim}
ReachableOrbits:= function( L )
    # this returns the nilpotent orbits of L that are reachable.
    local o, reachables, i, sl2, e, K;
    o:= NilpotentOrbits(L);
    reachables:= [ ];
    for i in [1..Length(o)] do
        sl2:= SL2Triple( o[i] );
        e:= sl2[3];
        K:= LieCentralizer( L, Subalgebra(L,[e]) );
        if e in LieDerivedSubalgebra(K) then
            Add( reachables, o[i] );
        fi;
    od;
    return reachables;
end;

PanyushevProperty:= function( L )
    # this function returns true if the Panyushev property
    # holds for L, otherwise false is returned.
    local reachables, sl2, K, prop, r, c, M, g;
end;
\end{verbatim}
reachables := ReachableOrbits(L);
prop := true;
for r in reachables do
  sl2 := SL2Triple( r );
  g := SL2Grading( L, sl2[2] );
  K := LieCentralizer( L, Subalgebra(L,[sl2[3]]) );
  c := List( g[1], u -> BasisVectors( Basis( Intersection( K, Subspace(L,u) ) ) ) );
  M := Subalgebra( L, Flat( c ) );
  if Dimension( Subalgebra( L, c[1] ) ) <> Dimension(M) then
    Print("Property not verified for ",r[1][3],"\n");
    prop := false;
  fi;
od;
return prop;
end;

StronglyReachableOrbits := function( L )
  # returns the strongly reachable nilpotent orbits of L.
  local o, good, i, sl2, e, K;

  o := NilpotentOrbits(L);
  good := [ ];
  for i in [1..Length(o)] do
    sl2 := SL2Triple( o[i] );
    e := sl2[3];
    K := LieCentralizer( L, Subalgebra(L,[e]) );
    if LieDerivedSubalgebra(K)=K then
      Add( good, o[i] );
    fi;
  od;

  return good;
end;

As a small illustration we give a sample session for the Lie algebra of type E_6.

gap> RequirePackage("sla");
gap> L:= SimpleLieAlgebra("E",6,Rationals);;
gap> r:= ReachableOrbits( L );;
gap> List( r, WeightedDynkinDiagram );
[ [ 0, 1, 0, 0, 0, 0 ], [ 1, 0, 0, 0, 0, 1 ], [ 0, 0, 0, 1, 0, 0 ],
  [ 1, 1, 0, 0, 0, 1 ], [ 0, 0, 1, 0, 1, 0 ], [ 1, 0, 0, 1, 0, 1 ] ]
gap> r:= StronglyReachableOrbits( L );;
gap> List( r, WeightedDynkinDiagram );
[ [ 0, 1, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 1, 0 ], [ 1, 0, 0, 1, 0, 1 ] ]
gap> PanyushevProperty(L);
true

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