Extension of Chern-Simons Forms

Spyros Konitopoulos and George Savvidy

+ Demokritos National Research Center, Ag. Paraskevi, Athens, Greece

Abstract

We investigate metric independent, gauge invariant and closed forms in the generalized YM theory. These forms are polynomial on the corresponding fields strength tensors - curvature forms and are analogous to the Pontryagin-Chern densities in the YM gauge theory. The corresponding secondary characteristic classes have been expressed in integral form in analogy with the Chern-Simons form. Because they are not unique, the secondary forms can be dramatically simplified by the addition of properly chosen differentials of one-step-lower-order forms. Their gauge variation can also be found yielding the potential anomalies in the gauge field theory.
1 Introduction

The chiral anomalies, Abelian and non-Abelian \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18\], can be derived by a differential geometric method without having to evaluate Feynman diagrams. Indeed, the non-Abelian anomaly in \((2n - 2)\)-dimensional space-time may be obtained from the Abelian anomaly in \(2n\) dimensions by a series of reduction (transgression) steps \[6, 7, 8, 9, 10, 11, 12, 13, 18\]. The \(U_A(1)\) gauge anomaly is given by the Pontryagin-Chern-Simons \(2n\)-form \[6, 7, 8, 9, 10, 11, 12, 13, 18\]:

\[
d * J^A \propto \mathcal{P}_{2n} = Tr(G^n) = d \omega_{2n-1},
\]

(1.1)

where \(\omega_{2n-1}\) is the Chern-Simons form in \(2n - 1\) dimensions \[6, 7, 12\]:

\[
\omega_{2n-1}(A) = n \int_0^1 dt \, Str(A, G_t^{n-1}),
\]

(1.2)

\(G = dA + A^2\) is the 2-form Yang-Mills (YM) field-strength tensor of the 1-form vector field \(A = -ig A^a_L L^a dx^\mu\) and \(G_t = tG + (t^2 - t)A^2\). The non-Abelian anomaly \[1, 2, 3, 4, 5\] can be obtained by the gauge variation of \(\omega_{2n-1}\) \[6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18\]:

\[
\delta \omega_{2n-1} = d\omega_{2n-2},
\]

(1.3)

where the \((2n - 2)\)-form has the following integral representation \[6, 7\]:

\[
\omega_{2n-2}(\xi, A) = n(n - 1) \int_0^1 dt (1 - t) \, Str \left( d\xi, A, g_t^{n-2} \right).
\]

(1.4)

Here \(\xi = \xi^a L_a\) is a scalar gauge parameter and \(Str\) denotes a symmetrized trace. The covariant divergence of the non-Abelian left and right handed currents is given by this \((2n - 2)\)-form.

In recent articles \[19, 20, 21\] the authors found closed invariant forms similar to the Pontryagin-Chern-Simons forms in non-Abelian tensor gauge field theory \[22, 23, 24\]. The first series of closed invariant forms are defined in \(D = 2n + 4\) dimensions and are given by the expression

\[
\Phi_{2n+4} = tr(G_4 G^n) = Str(G_4, G^n) = d\psi_{2n+3},
\]

(1.6)

where the corresponding secondary \((2n + 3)\)-form \(\psi_{2n+3}\) is in \(D = 2n + 3\) dimensions

\[
\psi_{2n+3} = Str(A_3, G^n)
\]

(1.7)

\(^1L^a\) are the generators of the Lie algebra.

\(^2\)In this article we shall use the symmetrized trace

\[
Str(A_1, A_2, ..., A_n) = \frac{1}{n!} \sum_{(i_1, ..., i_n)} (A_{i_1} A_{i_2} ... A_{i_n}),
\]

(1.5)

where the sum is over all permutations. Its properties are described in the Appendix B of the article \[10\].
and \( G_4 = dA_3 + \{A, A_3\} \). It turns out that the introduction of \( \text{Str} \) in the above equations leads to very crucial simplifications in all our subsequent derivations. For compact notation, when some of the entries of \( \text{Str} \) are the same, we write them in power form. The second series of forms is defined in \( D = 2n + 6 \) dimensions [21]:

\[
\Xi_{2n+6} = \text{Str}(G_6, G^n) + n\text{Str}(G_4^2, G^{n-1}) = d\phi_{2n+5}.
\]

(1.8)

The general expression for the secondary \((2n+5)\)-form \( \phi_{2n+5} \) will be constructed in this article. The third series of invariant closed forms found in this article \( \Upsilon_{2n+8} \) in \( D = 2n + 8 \) dimensions is

\[
\Upsilon_{2n+8} = \text{Str}(G_8, G^n) + 3n\text{Str}(G_4, G_6, G^{n-1}) + n(n-1)\text{Str}(G_4^3, G^{n-2}) = d\rho_{2n+7}.
\]

(1.9)

Its secondary form \( \rho_{2n+7} \) will be presented in the next sections.

All forms \( \Phi_{2n+4}, \Xi_{2n+6} \) and \( \Upsilon_{2n+8} \) are analogous to the Pontryagin-Chern-Simons densities \( \mathcal{P}_{2n} \) in the YM gauge theory (1.1) in the sense that they are gauge invariant, closed and metric independent.

Our aim is to investigate this rich class of topological invariants of extended gauge theory as well as to find out potential gauge anomalies performing transgressions analogous to (1.1) and (1.3):

\[
\mathcal{P}_{2n} \Rightarrow \omega_{2n-1} \Rightarrow \omega^1_{2n-2}.
\]

(1.10)

Therefore we shall perform the following transgressions:

\[
\Phi_{2n+4} \Rightarrow \psi_{2n+3} \Rightarrow \psi^1_{2n+2},
\]

\[
\Xi_{2n+6} \Rightarrow \phi_{2n+5} \Rightarrow \phi^1_{2n+4},
\]

\[
\Upsilon_{2n+8} \Rightarrow \rho_{2n+7} \Rightarrow \rho^1_{2n+6}.
\]

(1.11)

We shall find explicit expressions for these primary invariants in terms of higher order polynomials of the curvature forms on a vector bundle. The most difficult challenge will be the evaluation and differentiation of the very complicated noncommutative polynomial expressions as well as the search of the most simple expressions for the secondary forms. The secondary forms are not uniquely defined. Indeed, the secondary form \( \psi_{2n+3} \) is defined modulo the exterior derivative of an arbitrary \((2n+2)\)-form \( \psi_{2n+3} \sim \psi_{2n+3} + d\alpha_{2n+2} \), the form \( \phi_{2n+5} \) modulo the exterior derivative of a \((2n+4)\)-form \( \phi_{2n+5} \sim \phi_{2n+5} + d\beta_{2n+4} \) and the form \( \rho_{2n+7} \) modulo the exterior derivative of a \((2n+6)\)-form \( \rho_{2n+7} \sim \rho_{2n+7} + d\gamma_{2n+6} \). When the difference of two closed forms is an exact form, they are said to be cohomologous to each other. Therefore the problem is to find out the most simple representatives in the set of equivalence classes. Conveniently chosen exact forms will dramatically simplify the

\footnote{In the Appendix one can find the definition of tensor gauge fields and the corresponding curvature forms.}
expressions. These problems will be solved by using properties of symmetrized traces (1.5) defined in [10].

In Section 2 we shall present a general construction and analysis of the primary forms $\Phi_{2n+4}$ and $\Xi_{2n+6}$, their secondary forms $\psi_{2n+3}$ and $\phi_{2n+5}$ and the corresponding anomalies represented by $\psi_{2n+2}^1$ and $\phi_{2n+4}^1$. The material of this section is not completely new, but the alternative derivation in terms of symmetrized traces will allow to extend the results to the higher-dimensional forms $\Upsilon_{2n+8}$. In Section 3 we shall derive the explicit expressions for the primary form $\Upsilon_{2n+8}$, written in terms of the symmetrized traces (1.5), starting from the low-dimensional forms listed in [21]. Next, we shall find the secondary forms $\rho_{2n+7}$ and the corresponding gauge anomalies $\rho_{2n+6}^1$ associated with each of the independent gauge transformations $\delta_\xi, \delta_\zeta_2, \delta_\zeta_4, \delta_\zeta_6$. In the conclusion we summarize the primary and secondary invariant forms constructed in the article. In the Appendix we present useful formulas for the gauge transformations of the fields, the corresponding Bianchi identities and a one-parameter deformation of fields generalizing deformation of [6, 7].

2 Gauge and Metric Independent Forms

We shall start by deriving the already known results for the form $\Phi_{2n+4}$ using the properties of the symmetrized traces. This approach will allow to extend the derivation to more complicated cases. Indeed, the form can be represented in terms of a symmetrized trace as [20, 21]

$$\Phi_{2n+4} = Str\left(G_4, G^n\right)$$

so that its gauge invariance with respect to the standard-scalar gauge transformations $\delta_\xi$ and the tensor gauge transformations $\delta_\zeta_2$ can be easily checked:

$$\delta_\xi \Phi_{2n+4} = Str\left(\delta_\xi G_4, G^n\right) + n Str\left(G_4, \delta_\xi G, G^{n-1}\right) =$$

$$= Str\left([G_4, \xi], G^n\right) + n Str\left(G_4, [G, \xi], G^{n-1}\right) = 0,$$

$$\delta_\zeta_2 \Phi_{2n+4} = Str\left(\delta_\zeta_2 G_4, G^n\right) = Str\left([G, \zeta_2], G^n\right) =$$

$$= \frac{1}{n + 1} \left[Str\left([G, \zeta_2], G^n\right) + ... + Str\left(G^n, [G, \zeta_2]\right)\right] = 0. \quad (2.2)$$

On the last steps we used the identity (B.10) of the Appendix B of the article [10]. Our next step is to check that $\Phi_{2n+4}$ is a closed form. Indeed,

$$d \Phi_{2n+4}(A, A_3) = Str\left(DG_4, G^n\right) + n Str\left(G_4, DG, G^{n-1}\right) = Str\left([G, A_3], G^n\right) =$$

$$= \frac{1}{n + 1} \left[Str\left([G, A_3], G^n\right) + Str\left(G, [G, A_3], G^{n-1}\right) + ... + Str\left(G^n, [G, A_3]\right)\right] = 0,$$
where we used the Bianchi identity \( DG = 0 \). To find out the secondary form we shall use a one-parameter deformation of the gauge potentials: \( \[6, 20, 21\] 

\[
A_t = tA, \quad A_{3t} = tA_3, \quad A_{5t} = tA_5, \quad A_{7t} = tA_7, \quad A_{9t} = tA_9, ....,
\]
defined in the Appendix \((5.6), (5.7)\), take the derivative and employ \((B.13)\) of \([10]\):

\[
\frac{d}{dt}\Phi_{2n+4}(A_t, A_{3t}) = Str\left(\frac{dG_{4t}}{dt}, G^n_t\right) + nStr\left(G_{4t}, \frac{dG_t}{dt}, G^{n-1}_t\right) =
\]

\[
= Str\left(D_tA_3 + t\{A, A_3\}, G^n_t\right) + nStr\left(G_{4t}, D_tA, G^{n-1}_t\right) =
\]

\[
= dStr\left(A_3, G^n_t\right) + ndStr\left(G_{4t}, A, G^{n-1}_t\right) +
\]

\[
+ Str\left\{\{A, A_3\}, G^n_t\right\} - nStr\left\{G_t, A_{3t}, A, G^{n-1}_t\right\} =
\]

\[
= d\left\{Str\left(A_3, G^n_t\right) + nStr\left(G_{4t}, A, G^{n-1}_t\right)\right\}.
\quad (2.3)
\]

Integrating the above equation over the parameter \( t \) in the interval \([0,1]\) we get the integral representation of the secondary form

\[
\psi_{2n+3} = \int_0^1 dt \left[ Str\left(A_3, G^n_t\right) + nStr\left(G_{4t}, A, G^{n-1}_t\right)\right].
\quad (2.4)
\]

This secondary form is not unique. It can be modified by the addition of the differential of a one-step-lower-order form \( d\alpha_{2n+2} \). Two closed forms which differ by an exact form are said to be cohomologous to each other. The integral on the right hand side of the equation looks complicated, but if we add a properly chosen exact form \( d\alpha_{2n+2} \), then it will be dramatically simplified. Let us take it in the following form:

\[
\alpha_{2n+2} = -n \int_0^1 dt Str\left(A_{3t}, A, G^{n-1}_t\right).
\]

Then we get:

\[
\psi_{2n+3} \sim \psi_{2n+3} + d\alpha_{2n+2} = \psi_{2n+3} - n \int_0^1 dt Str\left(D_tA_{3t}, A, G^{n-1}_t\right) +
\]

\[
+ n \int_0^1 dt Str\left(A_{3t}, D_tA, G^{n-1}_t\right) - n(n-1) \int_0^1 dt Str\left(A_{3t}, D_tG_t, G^{n-2}_t\right),
\]

where \( D_tG_t = 0 \). Using the relations \((5.2), (5.7)\) and the integral representation \((2.4)\), we have:

\[
\psi_{2n+3} \sim \psi_{2n+3} - n \int_0^1 dt Str\left(G_{4t}, A, G^{n-1}_t\right) + n \int_0^1 dt Str\left(A_{3t}, \frac{\partial G_t}{\partial t}, G^{n-1}_t\right) =
\]

\[
= \int_0^1 dt \left\{Str\left(A_3, G^n_t\right) + nStr\left(A_{3t}, \frac{\partial G_t}{\partial t}, G^{n-1}_t\right)\right\} = \int_0^1 dt \frac{\partial}{\partial t}Str\left(A_{3t}, G^n_t\right) = Str\left(A_3, G^n\right).
\quad (2.5)
\]

\(^4\)The symbol "\( \sim \)" denotes the cohomology relation between the two forms.
Thus, the secondary form gets the following compact form
\[ \psi_{2n+3} = \text{Str} \left( A_3, G^n \right). \] (2.6)

The secondary forms (2.4) and (2.6) are representatives of the same cohomology class, because their difference is an exact form \( d\alpha_{2n+2} \), but, as one can see (2.6), has a much more simple expression. Using (5.2), (5.5) we can verify that
\[ d\psi_{2n+3} = \Phi_{2n+4}: \]

\[ d\text{Str} \left( A_3, G^n \right) = \text{Str} \left( DA_3, G^n \right) - n\text{Str} \left( A_3, DG, G^{n-1} \right) = \text{Str} \left( G_4, G^n \right). \]

The form (2.6) allows to find the potential anomalies of the theory, by the following transgression steps. The gauge invariance of the primary form \( \Phi_{2n+4} \) means that \( \delta \Phi_{2n+4} = d(\delta \psi_{2n+3}) = 0 \). By employing the Poincare’s lemma it follows that \( \delta \psi_{2n+3} = d\psi_{2n+2} \), where \( \psi_{2n+2} \) is the potential anomaly. Thus, in order to proceed we have to calculate the gauge variation of the secondary form with respect to the gauge transformations \( \delta \xi \) and \( \delta \zeta \). We have
\[ \delta \xi \psi_{2n+3} = \text{Str} \left( \delta \xi A_3, G^n \right) + n\text{Str} \left( A_3, \delta \xi G, G^{n-1} \right) = \text{Str} \left( [A_3, \xi], G^n \right) + n\text{Str} \left( A_3, [G, \xi], G^{n-1} \right) = 0, \] (2.7)

that is, the secondary form is gauge invariant with respect to standard-scalar gauge transformations \( \delta \xi \) and therefore there are no gauge anomalies associated with the scalar gauge transformations. But with respect to the tensor gauge transformations \( \delta \zeta \) there are anomalies
\[ \delta \zeta \psi_{2n+3} = \text{Str} \left( \delta \zeta A_3, G^{n-1} \right) = \text{Str} \left( D\zeta, G^n \right) = d\text{Str} \left( \zeta, G^n \right). \] (2.8)

Therefore the anomaly is
\[ \psi_{2n+2}^{(1)}(\zeta, A) = \text{Str} \left( \zeta, G^n \right). \] (2.9)

In summary, we have the expressions (2.1) for primary form \( \Phi_{2n+4} \), the expression (2.6) for the secondary form \( \psi_{2n+3} \) and (2.9) for the anomaly.

We shall now move to the next primary form \( \Xi_{2n+6} \), which can be written in terms of symmetrized traces as [21]:
\[ \Xi_{2n+6} = \text{Str} \left( G_6, G^n \right) + n\text{Str} \left( G_4^2, G^{n-1} \right). \] (2.10)

Each term of this expression is independently gauge invariant. Indeed,
\[ \delta \xi \text{Str} \left( G_6, G^n \right) = \text{Str} \left( \delta \xi G_6, G^n \right) + n\text{Str} \left( G_6, \delta \xi G, G^{n-1} \right) = \text{Str} \left( [G_6, \xi], G^n \right) + n\text{Str} \left( G_6, [G, \xi], G^{n-1} \right) = 0, \] (2.11)

\(^5\)The identity (B.10) of the Appendix B [10] should be used.
and for the second term we shall get

\[ \delta t \text{Str} \left( G^2_G, G^{n-1} \right) = 2 \text{Str} \left( \delta t G, G^{n-1} \right) + (n - 1) \text{Str} \left( G^2_{G}, \delta G, G^{n-2} \right) \]

\[ = 2 \text{Str} \left( [G, \xi], G^{n-1} \right) + (n - 1) \text{Str} \left( G^2_{[G, \xi]}, G^{n-2} \right) = 0, \]

where in the last two equations we again used (B.10) of [10]. However only the sum of these terms is a closed form. We can check the closeness of the form \( \Xi_{2n+6} \) by taking the exterior derivative:

\[ d\Xi_{2n+6} = \text{Str} \left( DG_6, G^n \right) + n \text{Str} \left( G_6, DG, G^{n-1} \right) + \]

\[ + 2n \text{Str} \left( DG_4, G^{n-1} \right) + n(n - 1) \text{Str} \left( G^2_{DG}, G^{n-2} \right) = \]

\[ = 2 \text{Str} \left( [G, A], G^n \right) + 2n \text{Str} \left( [G, A], G^{n-1} \right) + 2n \text{Str} \left( [G, A], G^{n-1} \right) = (2.12) \]

\[ = 2 \left\{ \text{Str} \left( [G, A], G^n \right) + n \text{Str} \left( G, [G, A], G^{n-1} \right) \right\} + \text{Str} \left( [G, A], G^n \right) = 0. \]

On the first step we used (B.13) of [10] and on the second - the Bianchi identities \( DG = 0 \). On the last step, the terms in the big brace as well as the last term are zero because of (B.10) of [10]. Again, according to Poincaré’s lemma, this equation implies that \( \Xi_{2n+6} \) can be locally written as an exterior derivative of a certain \((2n + 5)\)-form. In order to find that form we need to differentiate \( \Xi_{2n+6} \) over the deformation parameter \( t \). We have:

\[ \frac{d}{dt} \Xi_{2n+6}(A_t, A_{3t}, A_{5t}) = \text{Str} \left( \frac{dG_6}{dt}, G^n \right) + n \text{Str} \left( G_6, \frac{dG}{dt}, G^{n-1} \right) + 2n \text{Str} \left( \frac{dG_4}{dt}, G^{n-2} \right) + \]

\[ + n(n - 1) \text{Str} \left( G^2_{G}, \frac{dG}{dt}, G^{n-2} \right) = \text{Str} \left( D_t A_5, G^n \right) + n \text{Str} \left( \left\{ A, A_{3t} \right\}, G^n \right) + 2n \text{Str} \left( \left\{ A, A_{3t} \right\}, G^{n-1} \right) + \]

\[ + n(n - 1) \text{Str} \left( G^2_{DG}, D_t A, G^{n-1} \right) + 2n \text{Str} \left( D_t A_3, G^{n-1} \right) + 2n \text{Str} \left( \left\{ A, A_{3t} \right\}, G^{n-1} \right) + \]

\[ + n(n - 1) \text{Str} \left( G^2_{G}, D_t A, G^{n-1} \right) + n \text{Str} \left( D_t G_6, A, G^{n-1} \right) + \]

\[ + 2n \text{Str} \left( A_3, G^{n-1} \right) + 2n \text{Str} \left( A_3, D_t G_4, G^{n-1} \right) + 2n \text{Str} \left( \left\{ A, A_{3t} \right\}, G^{n-1} \right) + \]

\[ n(n - 1) \text{Str} \left( G^2_{DG}, A, G^{n-2} \right) - 2n(n - 1) \text{Str} \left( D_t G_6, G_4, A, G^{n-2} \right) = \]
\[ = d \left\{ n \text{Str} \left( G_{6t}, A, G_t^{n-1} \right) + n(n - 1) \text{Str} \left( G_{4t}^2, A, G_t^{n-2} \right) + 2n \text{Str} \left( G_{4t}, A_3, G_t^{n-1} \right) + \text{Str} \left( A_5, G_t^n \right) \right\} + \]

\[ + 2 \left[ \text{Str} \left( \{ A_3, A_{3t} \}, G_t^n \right) + n \text{Str} \left( A_3, [G_t, A_{3t}], G_t^{n-1} \right) \right] + \]

\[ + 2n \left[ \text{Str} \left( \{ A, A_{3t} \}, G_{4t}, G_t^{n-1} \right) - \text{Str} \left( [G_{4t}, A_{3t}], A, G_t^{n-1} \right) \right] + \]

\[ + 2n \left[ \text{Str} \left( \{ A, A_{3t} \}, G_{4t}, G_t^{n-1} \right) + \text{Str} \left( A, [G_{4t}, A_{3t}], G_t^{n-1} \right) - (n - 1) \text{Str} \left( [G_t, A_{3t}], G_{4t}, A, G_t^{n-2} \right) \right] + \]

\[ + \text{Str} \left( \{ A, A_{5t} \}, G_t^n \right) - n \text{Str} \left( [G_t, A_{5t}], A, G_t^{n-1} \right) = d\phi_{2n+5}. \] (2.13)

On the second, third and last steps we used the relations (5.7) and (B.13), (B.10) of [10] respectively. Integrating the above equation over the parameter \( t \) in the interval \([0, 1]\) we shall get the following integral representation of the secondary form:

\[ \phi_{2n+5} = \int_0^1 dt \left\{ n \text{Str} \left( G_{6t}, A, G_t^{n-1} \right) + n(n - 1) \text{Str} \left( G_{4t}^2, A, G_t^{n-2} \right) + \right. \]

\[ + 2n \text{Str} \left( G_{4t}, A_3, G_t^{n-1} \right) + \text{Str} \left( A_5, G_t^n \right) \right\}. \] (2.14)

As we already mentioned above, the secondary form is not unique and it can be modified by the addition of the differential of a one-step-lower-order form \( d\beta_{2n+4} \) (1.11). As in the case of \( \psi_{2n+3} \), we will use this freedom in order to simplify our result. Adding \( d\beta_{2n+4} \), where

\[ \beta_{2n+4} = - \int_0^1 dt \left[ n \text{Str} \left( A_{5t}, A, G_t^{n-1} \right) + n(n - 1) \text{Str} \left( G_{4t}, A_{3t}, A, G_t^{n-2} \right) \right], \]

we get:

\[ \phi_{2n+5} \leftarrow \phi_{2n+5} + d\beta_{2n+4} = \]

\[ = \phi_{2n+5} + \int_0^1 dt \left[ - n \text{Str} \left( D_t A_{5t}, A, G_t^{n-1} \right) + n \text{Str} \left( A_{5t}, D_t A, G_t^{n-1} \right) - \right. \]

\[ - n(n - 1) \text{Str} \left( A_{5t}, A, D_t G_{4t}, G_t^{n-2} \right) - n(n - 1) \text{Str} \left( D_t G_{4t}, A_{3t}, A, G_t^{n-2} \right) - \]

\[ - n(n - 1) \text{Str} \left( G_{4t}, D_t A_{3t}, A, G_t^{n-2} \right) + n(n - 1) \text{Str} \left( G_{4t}, A_{3t}, D_t A, G_t^{n-2} \right) - \]

\[ - n(n - 1)(n - 2) \text{Str} \left( G_{4t}, A_{3t}, A, D_t G_t, G_t^{n-3} \right) \],

where one should use the Bianchi identities \( D_t G_t = 0 \) and (B.13) of [10]. Next, with the aid of (5.2) and (5.7) we get,

\[ \phi_{2n+5} \leftarrow \phi_{2n+5} + \int_0^1 dt \left[ - n \text{Str} \left( G_{6t}, A, G_t^{n-1} \right) + n \text{Str} \left( \{ A_{3t}, A_{3t} \}, A, G_t^{n-1} \right) + \right. \]

\[ + n \text{Str} \left( A_{5t}, \frac{\partial G_t^1}{\partial t}, G_t^{n-1} \right) - n(n - 1) \text{Str} \left( [G_t, A_{3t}], A_{3t}, A, G_t^{n-2} \right) - \]

\[ - n(n - 1) \text{Str} \left( G_{4t}^2, A, G_t^{n-2} \right) + n(n - 1) \text{Str} \left( G_{4t}, A_3, \frac{\partial G_t^1}{\partial t}, G_t^{n-2} \right) \].
One can see that the first, the third and fifth terms cancel with the first two terms of $\phi_{2n+5}$ (2.14). With the aid of (B.9) of [10], the second and the forth terms combine to give $Str\left(A_3, \{A, A_3t\}, G_{n-1}^n\right)$. Finally using the equation $tD_1A_3 = G_{4t}$ and (5.2) we shall get the following expression for $\phi_{2n+5}$:

$$
\int_0^1 dt \left[ 2nStr\left(G_{4t}, A_3, G_{n-1}^n\right) + Str\left(A_5, G_n^n\right) + nStr\left(\{A, A_3t\}, A_{3t}, G_{n-1}^n\right) + 
+nStr\left(A_{3t}, \frac{\partial G_t}{\partial t}, G_{n-1}^n\right) + n(n - 1)Str\left(G_{4t}, A_{3t}, \frac{\partial G_t}{\partial t}, G_{n-2}^n\right) \right] =
$$

$$
= \int_0^1 dt \left[ Str\left(A_5, G_{n}^n\right) + nStr\left(A_{3t}, \frac{\partial G_t}{\partial t}, G_{n-1}^n\right) + 
+nStr\left(D_tA_3 + t\{A, A_3\}, A_{3t}, G_{n-1}^n\right) + nStr\left(G_{4t}, A_3, G_{n-1}^n\right) + 
+n(n - 1)Str\left(G_{4t}, A_{3t}, \frac{\partial G_t}{\partial t}, G_{n-2}^n\right) \right] =
$$

$$
= \int_0^1 dt \frac{\partial}{\partial t} \left[ Str\left(A_{5t}, G_t^n\right) + nStr\left(G_{4t}, A_3, G_{n-1}^n\right) \right] .
$$

Hence, after the integration we get

$$
\phi_{2n+5} = Str\left(A_5, G_n^n\right) + nStr\left(A_3, G_4, G_{n-1}^n\right) .
$$

By comparing the representations of the secondary form $\phi_{2n+5}$ in (2.14) and in (2.15) it becomes clear that the last expression is much more simple and transparent. Let us verify that the exterior derivative of the above form leads us back to $\Xi_{2n+6}$.

$$
d\phi_{2n+5} = Str\left(DA_5, G_n^n\right) - nStr\left(A_5, DG, G_{n-1}^n\right) + nStr\left(DG_4, A_3, G_{n-1}^n\right) + 
+nStr\left(G_4, DA_3, G_{n-1}^n\right) - n(n - 1)Str\left(G_4, A_3, DG, G_{n-2}^n\right) =
$$

$$
= Str\left(G_6, G_n^n\right) - Str\left(\{A_3, A_3\}, G_n^n\right) + nStr\left([G, A_3], A_3, G_{n-1}^n\right) + 
+nStr\left(G_4^2, G_{n-1}^n\right) = Str\left(G_6, G_n^n\right) + nStr\left(G_4^2, G_{n-1}^n\right) = \Xi_{2n+6},
$$

where on the second step the second and third terms cancel because of (B.10) of [10].

In order to find out the potential anomalies we have to calculate the gauge variation of the
secondary form $\phi_{2n+5}$ with respect to the scalar, rank-2 and rank-4 gauge parameters. We have

$$
\delta_\xi \phi_{2n+5} = \delta_\xi \left[ \text{Str}(A_5, G^n) + n \text{Str}(G_4, A_3, G^{n-1}) \right] = \\
= \text{Str}(\delta_\xi A_5, G^n) + n \text{Str}(\delta_\xi G_4, A_3, G^{n-1}) + n \text{Str}(\delta_\xi G_4, A_3, G^{n-1}) + \\
+ n \text{Str}(G_4, \delta_\xi A_3, G^{n-1}) + n(n-1) \text{Str}(G_4, A_3, \delta_\xi G, G^{n-2}) = \\
= \text{Str}([A_5, \xi], G^n) + n \text{Str}(A_5, [G, \xi], G^{n-1}) + n \text{Str}([G_4, \xi], A_3, G^{n-1}) + \\
+ n \text{Str}(G_4, [A_3, \xi], G^{n-1}) + n(n-1) \text{Str}(G_4, A_3, [G, \xi], G^{n-2}) = 0.
$$

Thus, there are no anomalies in the standard gauge symmetry. But there are potential anomalies in the higher-rank gauge symmetries. Indeed,

$$
\delta_\zeta \phi_{2n+5} = \text{Str}(\delta_\zeta A_5, G^n) + n \text{Str}(\delta_\zeta G_4, A_3, G^{n-1}) + n \text{Str}(\delta_\zeta G_4, A_3, G^{n-1}) = \\
= 2 \text{Str}([A_3, \zeta], G^n) + n \text{Str}([G, \zeta], A_3, G^{n-1}) + n \text{Str}(G_4, D\zeta_2, G^{n-1}) = \\
= \text{Str}([A_3, \zeta], G^n) + n \text{Str}(G_4, D\zeta_2, G^{n-1}) = \\
= \text{Str}([A_3, \zeta], G^n) + n d \text{Str}(G_4, \zeta_2, G^{n-1}) - n \text{Str}([G, A_3], \zeta_2, G^{n-1}) = \\
= n d \text{Str}([\zeta_2, G_4, G^{n-1}]) = (2.16)
$$

and

$$
\delta_\zeta \phi_{2n+5} = \text{Str}(D\zeta_4, G^n) = d \text{Str}(\zeta_4, G^n). (2.17)
$$

Hence the anomalies are:

$$
\phi^{(1)}_{2n+4}(\zeta_4, A) = \text{Str}(\zeta_4, G^n) \\
\phi^{(1)}_{2n+4}(\zeta_2, A, A_3) = n \text{Str}(\zeta_2, G_4, G^{n-1}) (2.18)
$$

In summary, we have the expressions (2.10) for primary form $\Xi_{2n+6}$, the expression (2.15) for the secondary form $\phi_{2n+5}$ and (2.18) for the anomalies.
3 The Form $\Upsilon_{2n+8}$

In the recent article [21] the authors found the following exact, metric independent forms, linear in $G_8$:

$$
\Upsilon_{10} = Tr(GG_8 + 3G_4G_6) = Str(G_8, G) + 3Str(G_4, G_6),
$$

$$
\Upsilon_{12} = Tr(G^2G_8 + 3GG_4G_6 + 3GG_6G_4 + 2G_4^3) = Str(G_8, G^2) + 6Str(G_4, G_6, G) + 2Str(G_4^3). \quad (3.1)
$$

In order to find the general expression for the forms linear in $G_8$ let us first find the next form $\Upsilon_{14}$. For that let us consider the linear combination of all possible rank-14 Lorentz invariant traces which can be constructed in terms of field strength tensors:

$$
\Upsilon_{14} = Tr \left( G^3G_8 + c_1GG_6^2 + c_2G_4^4G_6 + c_3G^2G_4G_6 + c_4GG_4GG_6 + c_5G^2G_6G_4 + +c_6G^4G_6 + c_7GG_4^3 + c_8G^3G_4^2 + c_9GG_4GG_4G + c_{10}G^5G_4 + c_{11}G^7 \right). \quad (3.2)
$$

The two terms with the coefficients $c_6 = c_8$ can be dropped since they compose the form $\Xi_{14} (2.10)$. The terms with coefficients $c_{10}$ and $c_{11}$ can also be dropped since they represent the forms $\Phi_{14} (2.1)$ and $P_{14} (1.1)$ respectively. Using the Bianchi identities (5.5) one can calculate the derivatives of the following terms:

$$
d Tr(G^3G_8) = 3Tr(G^3(G_6A_3 - A_2G_6 + G_4A_5 - A_5G_4)),
$$

$$
d Tr(G^2G_4G_6) = Tr(G^3(A_3G_6 - G_4A_5) + 2G^2(C_4A_3 - G_4A_3G_4) + G^2G_4G_6A_5 - G^2A_3GG_6),
$$

$$
d Tr(GG_4GG_6) = Tr(GA_3(GG_6G - G^2G_6) + 2GG_4G(G_4A_3 - A_3G_4) + (GG_4G^2 - G^2G_4)A_5),
$$

$$
d Tr(G^2G_6G_4) = Tr(2G^2(G_4A_3G_4 - A_3G_4^2) + G^2(G_6GA_3 - GG_6A_3) + G^2(GA_5 - A_5G_4)G_4),
$$

$$
d Tr(GG_4^3) = Tr(G^2(A_3G_4^2 - G_4^2A_3) + GG_4G(A_3G_4 - G_4^2A_3)),
$$

and see that the following combination is a closed form:

$$
d Tr(G^3G_8 + 3G^2G_4G_6 + 3GG_4GG_6 + 3G^2G_6G_4 + 6GG_4^3) = 0. \quad (3.3)
$$

Hence,

$$
\Upsilon_{14} = Tr(G^3G_8 + 3G^2G_4G_6 + 3GG_4GG_6 + 3G^2G_6G_4 + 6GG_4^3) =
= Str(G_8, G^3) + 9Str(G_4, G_6, G^2) + 6Str(G_4^3) \quad (3.4)
$$

and it can be written in terms of symmetric trace. The rest of the terms with the coefficients $c_1, c_2, c_9$ do not comprise any closed form.
One can check that the $\Upsilon_{14}$ is gauge invariant. In terms of the standard gauge parameter we get:

$$
\delta_\xi \Upsilon_{14} = Tr \left[ G^3, [G, \xi] G_8 + G^3 \left( [G_8, \xi] + 3[G_6, \xi] + 3[G_4, \xi] + [G, \xi] \right) + 3[G^2, \xi] G_4 G_6 + 
\right.
$$

$$
+ 3G^2 \left( [G_4, \xi] + [G, \xi] \right) G_6 + 3G^2 G_4 \left( [G_6, \xi] + 2[G_4, \xi] + [G, \xi] \right) +
$$

$$
+ 3[G^2, \xi] G_6 G_4 + 3G^2 \left( [G_6, \xi] + 2[G_4, \xi] + [G, \xi] \right) G_4 + 3G^2 G_6 \left( [G_4, \xi] + [G, \xi] \right) +
$$

$$
+ 3[G, \xi] G_4 G G_6 + 3G \left( [G_4, \xi] + [G, \xi] \right) G G_6 + 3G G_4 [G, \xi] G_6 +
$$

$$
+ 3G G_4 G \left( [G_6, \xi] + 2[G_4, \xi] + [G, \xi] \right) + 6[G, \xi] G_4^3 + 6G [G^2, \xi] \right] = 0.
$$

(3.5)

In an analogous way one can easily prove that $\Upsilon_{14}$ is invariant under the transformations of the higher-tensor gauge parameters $\zeta$, $\zeta_4$, $\zeta_6$.

Having in hand the series of forms $\Upsilon_{10,12,14}$ one can guess a general expression for $\Upsilon_{2n+8}$ and check that it fulfills all the required properties. We suggest the following general form for $\Upsilon_{2n+8}$:

$$
\Upsilon_{2n+8} = Str \left( G_8, G^n \right) + 3nStr \left( G_4, G_6, G^{n-1} \right) + n(n-1)Str \left( G^3_4, G^{n-2} \right).
$$

(3.6)

As one can see, each term of the $\Upsilon_{2n+8}$ is separately gauge invariant. Variating over the standard gauge parameter we get:

$$
\delta_\xi Str \left( G_8, G^n \right) = Str \left( \delta_\xi G_8, G^n \right) + nStr \left( G_6, \delta_\xi G, G^{n-1} \right) =
$$

$$
= Str \left( [G_8, \xi], G^n \right) + nStr \left( G_8, [G, \xi], G^{n-1} \right) = 0,
$$

$$
\delta_\xi Str \left( G_4, G_6, G^{n-1} \right) = Str \left( [G_4, \xi], G_6, G^{n-1} \right) + Str \left( G_4, [G_6, \xi], G^{n-1} \right) +
$$

$$
+ (n-1)Str \left( G_4, G_6, [G, \xi], G^{n-2} \right) = 0,
$$

$$
\delta_\xi Str \left( G^3_4, G^{n-1} \right) = 3Str \left( \delta_\xi G_4, G^2_4, G^{n-1} \right) + (n-1)Str \left( G_4^3, \delta_\xi G, G^{n-2} \right) =
$$

$$
= 3Str \left( [G_4, \xi], G^2_4, G^{n-1} \right) + (n-1)Str \left( G^3_4, [G, \xi], G^{n-2} \right) = 0
$$

(3.7)

Analogously, one can check that each term of $\Upsilon_{2n+8}$ is separately invariant under the variation over the higher-tensor gauge parameters. The last two calculations clearly demonstrate the power of the use of the symmetrized traces and of their properties.
Taking the exterior derivative of $\Upsilon_{2n+8}$ one can become convinced that it is a closed form:

$$d\Upsilon_{2n+8} = \text{Str} \left( DG_8, G^n \right) + n \text{Str} \left( G_8, DG, G^{n-1} \right) + 3n \text{Str} \left( DG_4, G_6, G^{n-1} \right) + 3n \text{Str} \left( G_4, DG_6, G^{n-1} \right) + 3n(n-1) \text{Str} \left( G_4, G_6, DG, G^{n-2} \right) + 3n(n-1) \text{Str} \left( DG_4, G_4^2, G^{n-2} \right) + n(n-1)(n-2) \text{Str} \left( G_4^3, DG, G^{n-3} \right)$$

$$= 3 \text{Str} \left( [G_6, A_3], G^n \right) + 3 \text{Str} \left( [G_4, A_5], G^n \right) + \text{Str} \left( [G, A_7], G^n \right) + 3 \text{Str} \left( [G_4, [G, A_3]], G^n \right) + 6n \text{Str} \left( G_4, [G_4, A_3], G^{n-1} \right) + 3n(n-1) \text{Str} \left( [G, A_3], G_4^2, G^{n-2} \right) = 3 \left\{ \text{Str} \left( [G_6, A_3], G^n \right) + n \text{Str} \left( G_6, [G, A_3], G^{n-1} \right) \right\} + 3 \left\{ \text{Str} \left( [G_4, A_3], G^n \right) + (n-1) \text{Str} \left( G_4, G_4, [G, A_3], G^{n-2} \right) \right\} + 3 \left\{ \text{Str} \left( [G_4, A_5], G^n \right) + n \text{Str} \left( G_4, [G, A_5], G^{n-1} \right) \right\} + \frac{1}{n+1} (n+1) \text{Str} \left( [G, A_7], G^n \right) = 0.$$ (3.8)

Again, according to Poincaré’s lemma, this equation implies that $\Upsilon_{2n+8}$ can be locally written as an exterior derivative of a certain $(2n+7)$-form $\rho_{2n+7}$. In order to find that form we need to differentiate $\Upsilon_{2n+8}$ over the deformation parameter $t$, as we did in the previous section:

$$\frac{d}{dt} \Upsilon_{2n+8} = \text{Str} \left( \frac{dG_8}{dt}, G^n_t \right) + n \text{Str} \left( G_8, \frac{dG_t}{dt}, G^{n-1}_t \right) + 3n \text{Str} \left( \frac{dG_4}{dt}, G_6, G^{n-1}_t \right) + 3n \text{Str} \left( \frac{dG_4}{dt}, G_6, G^{n-1}_t \right) + 3n(n-1) \text{Str} \left( \frac{dG_4}{dt}, G_6, G^{n-2}_t \right) + 3n(n-1)(n-2) \text{Str} \left( \frac{dG_4}{dt}, G_6, G^{n-3}_t \right)$$

$$= \text{Str} \left( D_t A_7, G^n_t \right) + 6 \text{Str} \left( \{ A_3, A_5 \}, G^n_t \right) + \text{Str} \left( \{ A, A_7 \}, G^n_t \right) + n \text{Str} \left( \{ A_3, A_5 \}, G^n_t \right) + 3n \text{Str} \left( D_t A_3, G_6, G^{n-1}_t \right) + 3n \text{Str} \left( D_t A_3, G_6, G^{n-1}_t \right) + 6n \text{Str} \left( D_t A_3, G_6, G^{n-1}_t \right) + 3n(n-1) \text{Str} \left( D_t A_3, G_6, G^{n-2}_t \right) + 3n(n-1) \text{Str} \left( D_t A_3, G_6, G^{n-3}_t \right) + n(n-1)(n-2) \text{Str} \left( D_t A_3, G_6, G^{n-3}_t \right).$$
In some of the terms we can extract the covariant exterior derivatives outside the symmetrized traces:

\[
\frac{d}{dt} Y_{2n+8} = d \text{Str} \left( A_7, G_t^n \right) + 6 \text{Str} \left( \{ A_3, A_5 t \}, G_t^n \right) + \text{Str} \left( \{ A, A_7 t \}, G_t^n \right) + \\
+ n \text{Str} \left( D_t G_8 t, A, G_t^{n-1} \right) + 3 n \text{Str} \left( A_3, G_6 t, G_t^{n-1} \right) + \\
+ 3 n \text{Str} \left( A_3, D_t G_6 t, G_t^{n-1} \right) + 3 n \text{Str} \left( A_3, A_3 t, G_t^{n-1} \right) + 3 n \text{Str} \left( G_4 t, A, G_t^{n-1} \right) - \\
- 3 n \text{Str} \left( D_t G_4 t, A, G_t^{n-1} \right) + 3 n \text{Str} \left( A_3, A_3 t, G_t^{n-1} \right) + \\
+ 6 n (n-1) \text{Str} \left( G_4 t, G_6 t, G_t^{n-2} \right) - 3 n(n-1) \text{Str} \left( D_t G_4 t, G_6 t, A, G_t^{n-2} \right) - \\
- 3 n(n-1) \text{Str} \left( G_4 t, D_t G_6 t, A, G_t^{n-2} \right) + 3 n(n-1) \text{Str} \left( A_3, G_4 t, G_4 t, G_t^{n-2} \right) + \\
+ 6 n(n-1) \text{Str} \left( A_3, D_t G_4 t, G_4 t, G_t^{n-2} \right) + 3 n(n-1) \text{Str} \left( A_3, A_3 t, G_4 t, G_4 t, G_t^{n-2} \right) + \\
+ n(n-1)(n-2) \text{Str} \left( G_3^2 t, A, G_t^{n-3} \right) - 3 n(n-1)(n-2) \text{Str} \left( D_t G_4 t, G_4 t^2, A, G_t^{n-3} \right).
\]

As one can see, some of the terms are written as exterior derivatives. We shall collect them in the formula below and then combine the rest of the terms in square brackets:

\[
\frac{d}{dt} Y_{2n+8} = d\left\{ \text{Str} \left( A_7, G_t^n \right) + n \text{Str} \left( G_8 t, A, G_t^{n-1} \right) + 3 n \text{Str} \left( A_3, G_6 t, G_t^{n-1} \right) + \\
+ 3 n \left( G_4 t, A, G_t^{n-1} \right) + 3 n(n-1) \text{Str} \left( G_4 t, G_6 t, A, G_t^{n-2} \right) + \\
+ 3 n(n-1) \text{Str} \left( A_3, G_4 t, G_t^{n-2} \right) + n(n-1)(n-2) \text{Str} \left( G_3^3 t, A, G_t^{n-3} \right) \right\} + \\
+ 3 n \left[ \text{Str} \left( \{ A, A_3 t \}, G_t^n \right) + n \text{Str} \left( A_3, [G_t, A_5 t], G_t^{n-1} \right) \right] + \\
+ 3 n \left[ \text{Str} \left( \{ A_3 t, A_5 t \}, G_t^n \right) - n \text{Str} \left( [G_t, A_3 t], A_5 t, G_t^{n-1} \right) \right] + \\
+ 3 n \left[ \text{Str} \left( \{ A, A_7 t \}, G_t^n \right) - n \text{Str} \left( [G_t, A_7 t], A, G_t^{n-1} \right) \right] + \\
+ 3 n \left[ \text{Str} \left( \{ A_3 t, G_4 t, A_3 t \}, G_t^{n-1} \right) - \text{Str} \left( [G_6 t, A_3 t], A, G_t^{n-1} \right) - (n-1) \text{Str} \left( [G_4 t, A_3 t], G_6 t, A, G_t^{n-1} \right) \right] + \\
+ 3 n \left[ \text{Str} \left( Go t, \{ A, A_5 t \}, G_t^{n-1} \right) - \text{Str} \left( [G_4 t, A_5 t], A, G_t^{n-1} \right) - (n-1) \text{Str} \left( [G_4 t, [G_t, A_5 t]], A, G_t^{n-2} \right) \right] + \\
+ 6 n \left[ \text{Str} \left( A_3, [G_4 t, A_3 t], G_t^{n-1} \right) + \text{Str} \left( G_4 t, \{ A_3, A_3 t \}, G_t^{n-1} \right) + (n-1) \text{Str} \left( A_3, [G_t, A_3 t], G_4 t, G_t^{n-2} \right) \right] + \\
+ 3 n(n-1) \left[ \text{Str} \left( \{ A, A_3 t \}, G_4 t, G_4 t, G_t^{n-2} \right) - 2 n \text{Str} \left( [G_4 t, G_4 t, A_3 t], A, G_t^{n-2} \right) \right] - \\
- (n-2) \text{Str} \left( [G_t, A_3 t], G_4 t^2, A, G_t^{n-3} \right) \right].
\]

The terms in the square brackets vanish thanks to the identity (B.10). Therefore we have the
following integral representation for the secondary form:

$$\rho_{2n+7} = \int_0^1 dt \left\{ n\text{Str} \left( G_{8t}, A, G_t^{n-1} \right) + 3n(n-1)\text{Str} \left( G_{4t}, G_{6t}, A, G_t^{n-2} \right) + 
+n(n-1)(n-2)\text{Str} \left( G_{4t}^3, A, G_t^{n-3} \right) + 3n\text{Str} \left( G_{6t}, A_3, G_t^{n-1} \right) + 
+3n(n-1)\text{Str} \left( G_{4t}^2, A_3, G_t^{n-2} \right) + 3n\text{Str} \left( G_{4t}, A_5, G_t^{n-1} \right) + \text{Str} \left( A_7, G_t^n \right) \right\}. \quad (3.9)$$

As we already discussed in the introduction and in the previous section, the secondary forms are defined modulo exact forms and in the given case up to \((2n+7)\)-form \(\rho_{2n+7} \sim \rho_{2n+7} + d\gamma_{2n+6} \). Therefore, we have to choose an appropriate candidate for \(\gamma_{2n+6} \). It appears that to simplify the result the exterior derivative of the following form should be subtracted:

$$\gamma_{2n+6} = \int_0^1 dt \left[ n\text{Str} \left( A_7t, A, G_t^{n-1} \right) + n(n-1)(n-2)\text{Str} \left( G_{4t}^2, A_3t, A, G_t^{n-3} \right) + 
+n(n-1)\text{Str} \left( G_{6t}, A_3t, A, G_t^{n-2} \right) + 2n(n-1)\text{Str} \left( G_{4t}, A_5t, A, G_t^{n-2} \right) - n\text{Str} \left( A_3t, A_5, G_t^{n-1} \right) \right].$$

Subtracting the first two terms of the \(d\gamma_{2n+6} \) from \(\rho_{2n+7} \) we will get:

$$\text{Str} \left( A_7, G^n \right) + n(n-1)\text{Str} \left( G_{4t}^2, A_3, G_t^{n-2} \right) +$$

$$+ \int_0^1 \left\{ 3n(n-1)\text{Str} \left( G_{4t}, G_{6t}, A, G_t^{n-2} \right) + 3n\text{Str} \left( G_{6t}, A_3, G_t^{n-1} \right) + 3n\text{Str} \left( G_{4t}, A_5, G_t^{n-1} \right) +$$

$$+3n\text{Str} \left( \{A_{3t}, A_{5t}\}, A, G_t^{n-1} \right) - 2n(n-1)\text{Str} \left( \{A, A_{3t}\}, G_{4t}, A_{3t}, G_t^{n-2} \right) -$$

$$-2n(n-1)(n-2)\text{Str} \left( \{G_t, A_{3t}\}, G_{4t}, A_{3t}, A, G_t^{n-3} \right) \right\}.$$  

Subtracting now the last three terms of the \(d\gamma_{2n+6} \), we will get:

$$\rho_{2n+7} = \text{Str} \left( A_7, G_t^n \right) + n(n-1)\text{Str} \left( G_{4t}, G_{4t}, A_3, G_t^{n-2} \right) +$$

$$+n\text{Str} \left( G_{6t}, A_3, G_t^{n-1} \right) + 2n\text{Str} \left( G_{4t}, A_5, G_t^{n-1} \right). \quad (3.10)$$

Let us check that the exterior derivative of the simplified secondary form gives us back the primary
form $\Upsilon_{2n+8}$. We have,

$$d\rho_{2n+7} = \text{Str} \left( DA_7, G^n \right) + n\text{Str} \left( DG_6, A_3, G^{n-1} \right) + n\text{Str} \left( G_6, DA_3, G^{n-1} \right) +$$

$$+ 2n\text{Str} \left( DG_4, A_5, G^{n-1} \right) + 2n\text{Str} \left( G_4, DA_5, G^{n-1} \right) +$$

$$+ 2n(n-1)\text{Str} \left( DG_4, G_4, A_3, G^{n-2} \right) + n(n-1)\text{Str} \left( G_4^2, DA_3, G^{n-2} \right) =$$

$$= \text{Str} \left( G_8, G^n \right) - (2 + 1)\text{Str} \left( \{A_3, A_5\}, G^n \right) + 2n\text{Str} \left( [G_4, A_3], A_3, G^{n-1} \right) +$$

$$+ n\text{Str} \left( [G, A_5], A_3, G^{n-1} \right) - n\text{Str} \left( G_6, G_4, G^{n-1} \right) + 2n\text{Str} \left( [G, A_3], A_5, G^{n-1} \right) +$$

$$+ 2n\text{Str} \left( G_4, G_6, G^{n-1} \right) - 2n\text{Str} \left( G_4, \{A_3, A_3\}, G^{n-1} \right) +$$

$$+ 2n(n-1)\text{Str} \left( [G, A_3], G_4, A_3, G^{n-2} \right) + n(n-1)\text{Str} \left( G_4^3, G^{n-2} \right) = \Upsilon_{2n+8}.$$  

Due to (B.10) of [10] the first part of the second term cancels with the sixth one, the second part of the second term cancels with the forth one, and the third term cancels with the ninth one. The secondary form allows to find the potential anomalies of the theory by performing the transgression steps. Thus in order to find out potential anomalies we have to calculate the gauge variation of the secondary form $\rho_{2n+7}$ with respect to the scalar, rank-2, rank-4 and rank-6 gauge parameters:

$$\delta_\xi \rho_{2n+7} = \text{Str} \left( [A_7, \xi], G^n \right) + n\text{Str} \left( A_7, [G, \xi], G^{n-1} \right) +$$

$$+ n\text{Str} \left( [G_6, \xi], A_3, G^{n-1} \right) + n\text{Str} \left( G_6, [A_3, \xi], G^{n-1} \right) +$$

$$+ n(n-1)\text{Str} \left( G_6, A_3, [G, \xi], G^{n-2} \right) + 2n\text{Str} \left( [G_4, \xi], A_5, G^{n-1} \right) +$$

$$+ 2n\text{Str} \left( G_4, [A_5, \xi], G^{n-1} \right) + 2n(n-1)\text{Str} \left( G_4, A_5, [G, \xi], G^{n-2} \right) +$$

$$+ 2n(n-1)\text{Str} \left( [G_4, \xi], G_4, A_3, G^{n-2} \right) + n(n-1)\text{Str} \left( G_4^2, [A_3, \xi], G^{n-2} \right) +$$

$$+ n(n-1)(n-2)\text{Str} \left( G_4^2, A_3, [G, \xi], G^{n-3} \right) = 0,$$  

(3.11)

where the identity (B.13) of [10] was used. There are no anomalies in the standard gauge symmetry.
The variation over the rank-2 gauge parameter gives:

\[
\delta_{\zeta_2} \rho_{2n+7} = 3 \text{Str} \left( [A_5, \zeta_2], G^n \right) + 2n \text{Str} \left( [G_4, \zeta_2], A_3, G^{n-1} \right) + n \text{Str} \left( G_6, D\zeta_2, G^{n-1} \right) +
+2n \text{Str} \left( [G, \zeta_2], A_5, G^{n-1} \right) + 4n \text{Str} \left( G_4, [A_3, \zeta_2], G^{n-1} \right) +
+2(n - 1) \text{Str} \left( [G_2, \zeta_2], G_4, A_3, G^{n-2} \right) + n(n - 1) \text{Str} \left( G_4^2, D\zeta_2, G^{n-2} \right) =
= \text{Str} \left( [A_5, \zeta_2], G^n \right) + nd \text{Str} \left( G_6, \zeta_2, G^{n-1} \right) - n \text{Str} \left( D G_6, \zeta_2, G^{n-1} \right) +
+2n \text{Str} \left( [G_4, [A_3, \zeta_2], G^{n-1} \right) + n(n - 1) d \text{Str} \left( G_4^2, \zeta_2, G^{n-2} \right) -
- 2n(n - 1) \text{Str} \left( [G, A_3], [G_4, \zeta_2, G^{n-2} \right),
\] (3.12)

where the sum of the third and the fourth terms and the sum of the last three terms vanish due to (B.10). The variation over the rank-4 gauge parameter gives:

\[
\delta_{\zeta_4} \rho_{2n+7} = 3 \text{Str} \left( [A_3, \zeta_4], G^n \right) + n \text{Str} \left( [G, \zeta_4], A_3, G^{n-1} \right) + 2n \text{Str} \left( G_4, D\zeta_4, G^{n-1} \right) =
= 2 \text{Str} \left( [A_3, \zeta_4], G^n \right) + 2nd \text{Str} \left( G_4, \zeta_4, G^{n-1} \right) - n \text{Str} \left( D G_4, \zeta_4, G^{n-1} \right) =
= 2 \text{Str} \left( [A_3, \zeta_4], G^n \right) + 2n \text{Str} \left( [A_3, G], \zeta_4, G^{n-1} \right) + 2nd \text{Str} \left( \zeta_4, G_4, G^{n-1} \right)
= 2nd \text{Str} \left( \zeta_4, G_4, G^{n-1} \right)
\] (3.13)

and the variation over rank-6 gauge parameter is:

\[
\delta_{\zeta_6} \rho_{2n+7} = \text{Str} \left( D \zeta_6, G^n \right) = d \text{Str} \left( \zeta_6, G^n \right).
\] (3.14)

Hence the corresponding anomalies are:

\[
\rho_{2n+6}^{(1)} (\zeta_6, A) = \text{Str} \left( \zeta_6, G^n \right),
\]

\[
\rho_{2n+6}^{(1)} (\zeta_4, A, A_3) = 2n \text{Str} \left( \zeta_4, G_4, G^{n-1} \right),
\]

\[
\rho_{2n+6}^{(1)} (\zeta_2, A, A_3, A_5) = n \text{Str} \left( \zeta_2, G_6, G^{n-1} \right) + n(n - 1) \text{Str} \left( \zeta_2, G_4^2, G^{n-2} \right)
\] (3.15)

and there are no anomalies with respect to the standard gauge transformations.

In summary, we have the expressions (3.6) for primary form \( \Upsilon_{2n+8} \), the expression (3.10) for the secondary form \( \rho_{2n+6} \) and (3.15) for the anomalies.
4 Conclusion

In this article we are interested in enumerating and classifying metric independent, gauge invariant and closed forms in generalized YM theory. The forms that we constructed are defined in various dimensions, are based on non-Abelian tensor gauge fields and are polynomial on the corresponding fields strength tensors - curvature forms. All these forms $\Phi_{2n+4}$, $\Xi_{2n+6}$ and $T_{2n+8}$ are analogous to the Pontryagin-Chern-Simons densities $\mathcal{P}_{2n}$ in YM gauge theory (1.1). They are closed forms, but not globally exact.

The secondary characteristic classes $\psi_{2n+3}$, $\phi_{2n+5}$ and $\rho_{2n+7}$ have been expressed in integral form (2.4), (2.14) and (3.9) in analogy with the Chern-Simons form (1.4). The secondary forms are not unique, because they can be modified by the addition of the differential of a one-step-lower-order forms. By adding the properly chosen exact forms (2.6), (2.15) and (3.10) respectively to the secondary forms $\psi_{2n+3}$, $\phi_{2n+5}$ and $\rho_{2n+7}$, we are led to much more simple expressions. The gauge variation of the secondary forms can also be found: (2.9), (2.18) and (3.15) yielding the potential anomalies in gauge field theory. The above general considerations should be supplemented by an explicit calculation of loop diagrams involving chiral fermions. The argument in favor of the existence of these potential anomalies is based on the fact that they fulfill Wess-Zumino consistency conditions [4, 6, 7, 8, 9, 18]. The integrals of these forms over the corresponding space-time coordinates provide us with new topological Lagrangians [21] and with a generalization of the Chern-Simons quantum field theory [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39].

At the same time, these densities constructed on a high-dimensional manifold have their own value. Their integrals represent global geometric invariants suggesting the existence of new topological characterization of the manifolds [6, 7, 10, 16, 17].

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The gauge transformations of non-Abelian tensor gauge fields were defined in \[22, 23, 24\]:

\[
\begin{align*}
\delta A & = D\xi, \\
\delta A_3 & = D\zeta_2 + [A_3, \xi] \\
\delta A_5 & = D\zeta_4 + 2[A_3, \zeta_2] + [A_5, \xi], \\
\delta A_7 & = D\zeta_6 + 3[A_3, \zeta_4] + 3[A_5, \zeta_2] + [A_7, \xi], \\
\delta A_9 & = D\zeta_8 + 4[A_3, \zeta_6] + 6[A_5, \zeta_4] + 4[A_7, \zeta_2] + [A_9, \xi], \\
\end{align*}
\]

\[
\begin{align*}
& \text{........... ...................................}, \\
\end{align*}
\]

where \(DA_{2n+1} = dA_{2n+1} + \{A, A_{2n+1}\}\) and the corresponding field-strength tensors are

\[
\begin{align*}
G & = dA + A^2 = DA - A^2 \\
G_4 & = dA_3 + \{A, A_3\} = DA_3, \\
G_6 & = dA_5 + \{A, A_5\} + \{A_3, A_3\} = DA_5 + \{A_3, A_3\}, \\
G_8 & = dA_7 + \{A, A_7\} + 3\{A_3, A_5\} = DA_7 + 3\{A_3, A_5\}, \\
G_{10} & = dA_9 + \{A, A_9\} + 4\{A_3, A_7\} + 3\{A_5, A_5\}, \\
& \text{........... ...................................}
\end{align*}
\]

The general variations of the field-strength tensors are:

\[
\begin{align*}
\delta G & = D(\delta A), \\
\delta G_4 & = D(\delta A_3) + \{A_3, \delta A\}, \\
\delta G_6 & = D(\delta A_5) + \{A_5, \delta A\} + 2\{A_3, \delta A_3\}, \\
\delta G_8 & = D(\delta A_7) + \{A_7, \delta A\} + 3\{A_5, \delta A_3\} + 3\{A_3, \delta A_5\}, \\
\delta G_{10} & = D(\delta A_9) + \{A_9, \delta A\} + 4\{A_7, \delta A_3\} + 6\{A_5, \delta A_5\} + 4\{A_3, \delta A_7\}, \\
& \text{........... ...................................}
\end{align*}
\]
The gauge transformations of the field-strength tensors follow from (5.3) and (5.1). They are homogeneous:

\[ \delta G = [G, \xi], \]
\[ \delta G_4 = [G_4, \xi] + [G, \zeta_2], \]
\[ \delta G_6 = [G_6, \xi] + 2[G_4, \zeta_2] + [G, \zeta_4], \]
\[ \delta G_8 = [G_8, \xi] + 3[G_6, \zeta_2] + 3[G_4, \zeta_4] + [G, \zeta_6], \]
\[ \delta G_{10} = [G_{10}, \xi] + 4[G_8, \zeta_2] + 6[G_6, \zeta_4] + 4[G_4, \zeta_6] + [G, \zeta_8], \]

The Bianchi identities are given by

\[ DG = 0, \]
\[ DG_4 + [A_3, G] = 0, \]
\[ DG_6 + 2[A_3, G_4] + [A_5, G] = 0, \]
\[ DG_8 + 3[A_3, G_6] + 3[A_5, G_4] + [A_7, G] = 0, \]
\[ DG_{10} + 4[A_3, G_8] + 6[A_5, G_6] + 4[A_7, G_4] + [A_9, G] = 0, \]

where \( DG_{2n} = dG_{2n} + [A, G_{2n}] \). Generalizing Zumino’s construction [9], we introduce a one-parameter family of potentials and field-strengths as:

\[ A_t = tA, \quad A_{3t} = tA_3, \quad A_{5t} = tA_5, \quad A_{7t} = tA_7, \quad A_{9t} = tA_9, \]
\[ G_t = tG + (t^2 - t)A^2, \]
\[ G_{4t} = tG_4 + (t^2 - t)[A, A_3], \]
\[ G_{6t} = tG_6 + (t^2 - t)(\{A, A_5\} + \{A_3, A_3\}), \]
\[ G_{8t} = tG_8 + (t^2 - t)(\{A, A_7\} + 3\{A_3, A_5\}), \]
\[ G_{10t} = tG_{10} + (t^2 - t)(\{A, A_9\} + 4\{A_3, A_7\} + 3\{A_5, A_5\}), \]

The Bianchi identities hold for the deformed fields as well.

\[ \frac{\partial G_t}{\partial t} = dA + 2tA^2 = D_tA, \]
\[ \frac{\partial G_{4t}}{\partial t} = dA_3 + 2t\{A, A_3\} = D_tA_3 + t\{A, A_3\}, \]
\[ \frac{\partial G_{6t}}{\partial t} = dA_5 + 2t\left(\{A, A_5\} + \{A_3, A_3\}\right) = D_tA_5 + t\{A, A_5\} + 2t\{A_3, A_3\}, \]
\[ \frac{\partial G_{8t}}{\partial t} = dA_7 + 2t\left(\{A, A_7\} + 3\{A_3, A_5\}\right) = D_tA_7 + t\{A, A_7\} + 6t\{A_3, A_5\}, \]
where $D_t A_{2n+1} = dA_{2n+1} + \{A_t, A_{2n+1}\}$. Because we used the properties (B.10) of the symmetrized traces which are defined in [10] we shall present them in the form convenient for our purposes:

$$
\sum_{i=1}^{n} (-1)^{(d_1+\cdots+d_i-1)d_0} Str(\Lambda_1, \ldots, [\Theta, \Lambda_i], \ldots \Lambda_n) = 0, \quad (B.10)
$$

where $d_i$ is the rank of the form $\Lambda_i$ and $\Theta$ is an even form. But if both $\Theta$ and $\Lambda_i$ are odd forms, then the commutator should be replaced by the anticommutator. For the exterior derivative we use [10]:

$$
dStr(\Lambda_1, \ldots, \Lambda_i, \ldots \Lambda_n) = \sum_{i=1}^{n} (-1)^{d_1+\cdots+d_i-1} Str(\Lambda_1, \ldots, D\Lambda_i, \ldots \Lambda_n). \quad (B.13)
$$

References

[1] S. L. Adler, *Axial vector vertex in spinor electrodynamics*, Phys. Rev. 177 (1969) 2426.

[2] J. S. Bell and R. Jackiw, *A PCAC puzzle: $\pi_0$ to $\gamma\gamma$ in the sigma model*, Nuovo Cim. A 60 (1969) 47.

[3] W. A. Bardeen, *Anomalous Ward identities in spinor field theories*, Phys. Rev. 184 (1969) 1848.

[4] J. Wess and B. Zumino, *Consequences of anomalous Ward identities*, Phys. Lett. B 37 (1971) 95.

[5] P. H. Frampton and T. W. Kephart, *The Analysis Of Anomalies In Higher Space-time Dimensions*, Phys. Rev. D 28 (1983) 1010.

[6] B. Zumino, *Chiral Anomalies And Differential Geometry*, Lectures Given At Les Houches, August 1983, LBL-16747, UCB-PTH-83/16

[7] R. Stora, *Algebraic Structure And Topological Origin Of Anomalies*, LAPP-TH-94, Nov 1983. 20pp. Seminar given at Cargese Summer Inst.: Progress in Gauge Field Theory, Cargese, France, Sep 1-15, 1983. Published in Cargese Summer Inst.1983:0543

[8] L. D. Faddeev, *Operator Anomaly For The Gauss Law*, Phys. Lett. B 145 (1984) 81.

[9] L. D. Faddeev and S. L. Shatashvili, *Algebraic and Hamiltonian Methods in the Theory of Nonabelian Anomalies*, Theor. Math. Phys. 60 (1985) 770 [Teor. Mat. Fiz. 60 (1984) 206].

[10] B. Zumino, Y. -S. Wu and A. Zee, *Chiral Anomalies, Higher Dimensions, and Differential Geometry*, Nucl. Phys. B 239, 477 (1984).
[11] J. Manes, R. Stora and B. Zumino, *Algebraic Study Of Chiral Anomalies*, Commun. Math. Phys. **102** (1985) 157.

[12] S. B. Treiman, E. Witten, R. Jackiw and B. Zumino, *Current Algebra And Anomalies, Singapore*, Singapore: World Scientific (1985) 537p

[13] L. D. Faddeev, *Hamiltonian approach to the theory of anomalies*, in *Schladming 1987, Proceedings, Recent Developments In Mathematical Physics* 137-159.

[14] L. D. Faddeev and S. L. Shatashvili, *Realization of the Schwinger Term in the Gauss Law and the Possibility of Correct Quantization of a Theory with Anomalies*, Phys. Lett. B **167** (1986) 225.

[15] D. K. Faddeev, *The operation $\delta$ is the coboundary operator in homological algebra*. Dokl. Akad. Nauk SSSR, **8** (1947) 361.

[16] L. Alvarez-Gaume and P. H. Ginsparg, *The Topological Meaning of Nonabelian Anomalies*, Nucl. Phys. B **243** (1984) 449.

[17] L. Alvarez-Gaume and P. H. Ginsparg, *Geometry Anomalies*, Nucl. Phys. B **262** (1985) 439.

[18] L. Alvarez-Gaume, *An Introduction To Anomalies*, HUTP-85/A092. Lectures given at the International School of Mathematical Physics on *Fundamental Problems of Gauge Field Theory*, Erice, Italy, 1-14 July 1985.

[19] G. Savvidy, *Topological mass generation in four-dimensional gauge theory*, Phys. Lett. B **694**, 65 (2010) [arXiv:1001.2808 [hep-th]].

[20] I. Antoniadis and G. Savvidy, *New gauge anomalies and topological invariants in various dimensions*, Eur. Phys. J. C **72** (2012) 2140 [arXiv:1205.0027 [hep-th]].

[21] I. Antoniadis and G. Savvidy, *Extension of Chern-Simons forms and new gauge anomalies*, Int. J. Mod. Phys. A **29** (2014) 401 [arXiv:1304.4398 [hep-th]].

[22] G. Savvidy, *Non-Abelian tensor gauge fields: Generalization of Yang-Mills theory*, Phys. Lett. B **625** (2005) 341 [arXiv:hep-th/0509049]

[23] G. Savvidy, *Non-abelian tensor gauge fields. I*, Int. J. Mod. Phys. A **21** (2006) 4931.

[24] G. Savvidy, *Non-abelian tensor gauge fields. II*, Int. J. Mod. Phys. A **21** (2006) 4959.
[25] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Commun. Math. Phys. **121** (1989) 351.

[26] A. S. Schwarz, *On Quantum Fluctuations of Instantons*, Lett. Math. Phys. **2** (1978) 201.

[27] A. S. Schwarz, *The Partition Function of Degenerate Quadratic Functional and Ray-Singer Invariants*, Lett. Math. Phys. **2** (1978) 247.

[28] J. F. Schonfeld, *A Mass Term For Three-Dimensional Gauge Fields*, Nucl. Phys. B **185** (1981) 157.

[29] S. Deser, R. Jackiw and S. Templeton, *Three-Dimensional Massive Gauge Theories*, Phys. Rev. Lett. **48** (1982) 975.

[30] S. Deser, R. Jackiw and S. Templeton, *Topologically massive gauge theories*, Annals Phys. **140** (1982) 372

[31] E. Witten, *Chern-Simons gauge theory as a string theory*, Prog. Math. **133** (1995) 637 [hep-th/9207094].

[32] C. Beasley and E. Witten, *Non-Abelian localization for Chern-Simons theory*, J. Diff. Geom. **70** (2005) 183 [hep-th/0503126].

[33] E. Witten, *Quantization Of Chern-simons Gauge Theory With Complex Gauge Group*, Commun. Math. Phys. **137** (1991) 29.

[34] S. Axelrod, S. Della Pietra and E. Witten, *Geometric Quantization Of Chern-simons Gauge Theory*, J. Diff. Geom. **33** (1991) 787.

[35] R. Dijkgraaf and E. Witten, *Topological Gauge Theories and Group Cohomology*, Commun. Math. Phys. **129** (1990) 393.

[36] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Topological field theory*, Phys. Rept. **209** (1991) 129.

[37] M. Blau and G. Thompson, *Topological Gauge Theories of Antisymmetric Tensor Fields*, Annals Phys. **205** (1991) 130.

[38] M. Blau and G. Thompson, *A New Class Of Topological Field Theories And The Ray-singer Torsion*, Phys. Lett. B **228** (1989) 64.
[39] G. Thompson, *1992 Trieste lectures on topological gauge theory and Yang-Mills theory*, In *Trieste 1992, Proceedings, High energy physics and cosmology* 1-75 and Trieste Int. Cent. Theor. Phys. - IC-93-112 (93/05,rec.Jul.) 75 p. (310578) (see Conference Index) [hep-th/9305120].