Ternary numbers and algebras
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Abstract

The Calabi-Yau spaces with $SU(n)$ holonomy can be studied by the algebraic way through the integer lattice where one can construct the Newton reflexive polyhedra or the Berger graphs. Our conjecture is that the Berger graphs can be directly related with the $n$-ary algebras. To find such algebras we study the $n$-ary generalization of the well-known binary norm division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, which helped to discover the most important "minimal" binary simple Lie groups, $U(1)$, $SU(2)$ and $G(2)$. As the most important example, we consider the case $n=3$, which gives the ternary generalization of quaternions (octonions), $3^n$, $n=2,3$, respectively. The ternary generalization of quaternions is directly related to the new ternary algebra (group) which are related to the natural extensions of the binary $su(3)$ algebra ($SU(3)$ group). Using this ternary algebra we found the solution for the Berger graph: a tetrahedron.
1 Introduction

Our interest in ternary algebras and symmetries started from the study of the geometry based on the holonomy principle, discovered by Berger [1]. The $CY_n$ spaces with $SU(n)$ holonomy [2, 3] have a special interest for us. Our conjecture [5, 6, 7] is that $CY_3$ ($CY_n$) spaces are related to the ternary ($n$-ary) symmetries, which are natural generalization of the binary Cartan–Killing–Lie symmetries.

There are some motivations in modern quantum physics and cosmology why we are searching for new symmetries beyond Lie algebras/groups. One of the main reasons is related with the conjecture that the Standard Model of quarks and leptons cannot be solved in terms of binary Lie groups. This problem can be formulated as incompleteness of the Standard Model in terms of binary Lie groups [5, 6, 7, 8]. The theory of superstrings is also based on binary Lie groups, in particular on the D-dimensional Lorentz group, and therefore the description of the Standard Model in the superstrings approach did not bring us to success. In our opinion, the main problem with the superstring approaches (also GUTs, SUGRA) is the inadequate external symmetry at the string scale, $M_{\text{str}} \gg M_{\text{SM}}$: we believe that the D-Lorentz symmetry must be generalized. This is valid also for GUTs or SUGRA approaches. So far the construction of the quantum theory of superstrings has not been finished and it might be helpful to know the point limit of superstring theories in any dimension $4 \leq D \leq 10$. This limit is known only for $D = 4$, where there exist the renormalizable quantum field theories based on the $D = 1 + 3$ Lorentz group symmetry. The extra uncompactified dimensions make quantum field theories with Lorentz symmetry much less credible, since the power counting is worse. A possible way out is to suppose that the propagator is more convergent than $1/p^2$, such a behaviour can be obtained if we consider, instead of binary symmetry algebra, algebras with higher order relations (That is, instead of binary operations such as addition or product of 2 elements, we start with composition laws that involve at least $n$ elements of the considered algebra, $n$-ary algebras). For instance, a ternary symmetry could be related with membrane dynamics. To solve the Standard Model problems we suggested to generalize their external and internal binary symmetries by addition of ternary symmetries based on the ternary algebras [6, 7, 8]. For example, ternary symmetries seem to give very good possibilities to overcome the above-mentioned problems, i.e. to make the next progress in understanding of the space-time geometry of our Universe. We suppose that the new symmetries beyond the well-known binary Lie algebras/superalgebras could allow us to build the renormalizable theories for space-time geometry with dimension $D > 4$. It seems very plausible that using such ternary symmetries will offer a real possibility to overcome the problems of quantization of membranes and could represent a further progress beyond string theories.

2 Geometry and algebra of reflexive numbers

All modern theories based on the binary Lie algebras have a common property since the algebras/symmetries are related with some invariant quadratic forms. In all these approaches, a wide class of simple classical Lie algebras is used; the ones whose Cartan–
Killing classification contains four infinite series $A_r = sl(r + 1)$, $B_r = so(2n + 1)$, $C_r = sp(2r)$, $D_r = so(2r)$, and five exceptional algebras $G_2$, $F_4$, $E_6$, $E_7$, $E_8$. There are several ways of studying such classification [14, 15, 16, 17, 18, 19, 20, 21]. We can recall two of them, one is through the Cartan matrices and Dynkin graphs, the second through the theory of numbers and Clifford algebras. Our interest in the new $n$-ary algebras and their classification started from a study of infinite series of $CY_n$ spaces characterized by holonomy groups [1]. More exactly, the $CY_n$ space can be defined as the quadruple $(M, J, g, \Omega)$, where $(M, J)$ is a complex compact $n$-dimensional manifold of complex structure $J$, $g$ is a Kähler metrics with $SU(n)$ holonomy group, and $\Omega_n = (n, 0)$ and $\Omega_n = (0, n)$ are non-zero parallel tensors which are the holomorphic volume forms.

A $CY_{n-2}$ space can be realized as an algebraic manifold $M$ in a weighted projective space $CP^{n-1}(\vec{k})$ where the weight vector reads $\vec{k} = (k_1, \ldots, k_n)$. This manifold is defined by

$$M \equiv \{(x_1, \ldots, x_n) \in CP^{n-1}(\vec{k}) : \mathcal{P}(x_1, \ldots, x_n) \equiv \sum \limits_{\vec{m}} c_{\vec{m}} x^{\vec{m}} = 0\},$$

i.e. as the zero locus of a quasi-homogeneous polynomial of degree $d_k = \sum_{i=1}^{n} k_i$, with the monomials $x^{\vec{m}} \equiv x_1^{m_1} \cdots x_n^{m_n}$. The points in $CP^{n-1}$ satisfy the property of projective invariance $\{x_1, \ldots, x_n\} \approx \{\lambda^{k_i} x_1, \ldots, \lambda^{k_n} x_n\}$ leading to the constraint $\vec{m} \cdot \vec{k} = d_k$.

The classification of $CY_n$ can be done through the reflexivity of the weight vectors $\vec{k}$ (reflexive numbers), which can be defined in terms of the Newton reflexive polyhedra [9] or Berger graphs [6]. The Newton reflexive polyhedra are determined by the exponents of the monomials participating in the $CY_n$ equation [9]. The term “reflexive” is related with the mirror duality of Calabi–Yau spaces and the corresponding Newton polyhedra [2]. The Berger graphs can be constructed directly through the reflexive weight numbers $\vec{k} = (k_1, \ldots, k_{n+2})[d_k]$ by the procedure shown in [6, 7]. According to the universal algebraic approach [11] (see also [10, 11]) one can find a section in the reflexive polyhedron and, according to the $n$-arity of this algebraic approach, the reflexive polyhedron can be constructed from 2-, 3-,... Berger graphs. It was conjectured that the Berger graphs might correspond to $n$-ary Lie algebras [6, 7]. In these articles we tried to decode those Berger graphs by using the method of the ”simple roots” (for illustration, see Table 1). In this table some general properties of the Berger graphs are presented, which correspond to the subclass of the ”simply -laced” reflexive numbers (Egyptian numbers). A simply–laced number $\vec{k} = (k_1, \ldots, k_n)$ with degree $d_k = \sum_{i=1}^{n} k_i$, is defined such that

$$\frac{d}{k_i} \in \mathbb{Z}^+ \text{ and } d > k_i.$$  \hfill (2)

For these numbers there is a simple way of constructing the corresponding affine Berger graphs together with their Coxeter labels [6, 7]. The corresponding Cartan and Berger matrices of these graphs are symmetric. In the Cartan case they correspond to the $ADE$ series of simply–laced algebras. In dimensions $n = 1, 2, 3$, the Egyptian numbers are $(1), (1, 1), (1, 1, 1), (1, 1, 2), (1, 2, 3)$. For $n = 4$, from all 95 reflexive numbers, 14 are simply–laced Egyptian numbers for which some general properties can be illustrated (see Table 1) [7].
Table 1: Rank, Coxeter number $h$, Casimir depending on $B_{ii}$ and determinants for the non-affine exceptional Berger graphs. The maximal Coxeter labels coincide with the degree of the corresponding reflexive simply–laced vector. The determinants in the last column for the infinite series (0,0,1,1,1)[3], (0,0,1,1,2)[4] and (0,0,1,2,3)[6] are independent from the number $l$ of internal binary $B_{ii} = 2$ nodes. The numbers $1_3$ and $2_3$ denote the number of nodes with $B_{ii} = 3$.

3 Division algebras and Lie algebras

Now we should briefly recall the four norm division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [12, 13, 14, 15, 18, 19]. An algebra $A$ will be a vector space that is equipped with a bilinear map $f : A \times A \rightarrow A$ called by multiplication and a nonzero element $1 \in A$ called the unit, such that $f(1, a) = f(a, 1) = a$. These algebras admit an anti-involution (or conjugation) $(a^\ast)^\ast = a$ and $(ab)^\ast = b^\ast a^\ast$. A norm division algebra is an algebra $A$ that is also a normed vector space with $N(ab) = N(a)N(b)$. Such algebras exist only for $n = 1, 2, 4, 8$ dimensions where the following identities can be obtained:

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = (z_1^2 + \cdots + z_n^2)$$

(3)
The doubling process, which is known as the Cayley-Dickson process, forms the sequence of division algebras
\[ \mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O}. \] (4)

Note that next algebra is not a division algebra. So \( n = 1 \mathbb{R} \) and \( n = 2 \mathbb{C} \) these algebras are the commutative associative normed division algebras. The quaternions, \( \mathbb{H}, n = 4 \) form the non-commutative and associative norm division algebra. The octonion algebra \( n = 8, \mathbb{O} \) is an non-associative alternative algebra. If the discovery of complex numbers took a long period about some centuries years, the discovery of quaternions and octonions was made in a short time, in the middle of the XIX century by W. Hamilton [12], and by J. Graves and A.Cayley [13]. The complex numbers, quaternions and octonions can be presented in the general form:

\[ \hat{q} = x_0 e_0 + x_p e_p, \quad \{x_0, x_p\} \in \mathbb{R}, \] (5)

where \( p = 1 \) and \( e_1 \equiv i \) for complex numbers \( \mathbb{C} \), \( p = 1, 2, 3 \) for quaternions \( \mathbb{H} \), and \( p = 1, 2, ..., 7 \) for \( \mathbb{O} \). The \( e_0 \) is as unit and all \( e_p \) are imaginary units with conjugation \( \overline{e_p} = -e_0 \). For quaternions we have the main relation

\[ e_m e_p = -\delta_{mp} + f_{mpl} e_l, \] (6)

where \( \delta_{mp} \) and \( f_{mpl} \equiv \epsilon_{mpl} \) are the well-known Kronecker and Levi-Cevita tensors, respectively. For octonions the completely antisymmetric tensor \( f_{mpl} = 1 \) for the following seven triple associate cycles:

\[ \{mpl\} = \{123\}, \{145\}, \{176\}, \{246\}, \{257\}, \{347\}, \{365\}. \] (7)

There are also 28 non-associate cycles. Each triple associate cycle corresponds to a quaternionic subalgebra. These algebras have a very close link with geometry. For example, the unit elements \( x^2 = 1, x \in \mathbb{R}, |\hat{q}| = x_0^2 + x_1^2 = 1 \) in \( \mathbb{C}_1 \), \( |\hat{q}| = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \) in \( \mathbb{H}_1 \), \( |\hat{q}| = x_0^2 + x_1^2 + ... + x_7^2 = 1 \) in \( \mathbb{O}_1 \), define the spheres, \( S^0, S^1, S^3, S^7 \), respectively. The complex and quaternionic unit elements have the \( U(1) \) and \( SU(2) \) group properties, respectively. The \( A \) series contains the complex rotations in the unit circle, \( S^1 \), and \( S^3 \) is a Lie group. The \( B \) and \( C \) groups both contain the quaternion rotations on the unit sphere \( S^3 \), and \( S^7 \) is a Lie group. The \( D \) series contain the Lorentz group in \( D = 3 + 1 \), which consists of two copies of \( S^3 \)-3-rotations and 3-boosts. The exceptional groups do not include \( S^7 \) as a Lie group. Thus \( S^7 \) is the only unit sphere in a division algebra that is not a Lie group. The reason is that the octonions are not associative; their associator is

\[ \{e_m, e_p, e_l\} = (e_m e_p) e_l - e_m (e_p e_l) = f_{mplk} e_k, \] (8)

where the tensor \( f_{mplk} \) is completely antisymmetric and it is non-zero for the following seven 4-cycles:
\{mplk\} = \{4567\}, \{2367\}, \{2345\}, \{1357\}, \{1346\}, \{1256\}, \{1247\}.

(9)

It is also the case when three elements \(\{e_a, e_b, e_c\}\) are not in the same three associate cycles, for example, \((e_1e_5)e_7 - e_1(e_5e_7) = 2e_3\). Note that the octonions form the alternative algebra since the alternative condition, \(\{a, b, c\} + \{c, b, a\} = 0\), is always valid. The octonions are directly linked to the five exceptional groups \(G(2), F(4), E(6), E(7), E(8)\) [17, 18]. The automorphism of octonions is \(G(2)\).

4 Nambu–Filippov ternary algebras

The new n-ary symmetries can be related to algebras that are based on the generalization of the Lie binary commutation relation

\[ [x, y] = xy - yx \tag{10} \]

by the ternary commutations relations

\[ [x_1, x_2, \ldots, x_n] = (-1)^{\tau(\sigma)}[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}], \tag{11} \]

where \(\sigma\) runs over the symmetric group \(S_n\) and the number \(\tau(\sigma)\) equals 0 or 1, depending on the parity of the permutation \(\sigma\) (see [10] [22] [23] [24] [25] [32] [27] [28] [29] [30] [31] [34] [35] [36] [11]).

More exactly, a ternary Lie algebra is defined by a ternary antisymmetric operation \(A \times A \times A \rightarrow A\) with the Jacobi-like identity [23]. Fillipov considered the n-ary generalizations of Lie algebras \((n > 2)\). The most simple example of this Lie algebra can be the n-vector product of the \((n + 1)\)-dimensional vectors \(\vec{x}_1, \ldots, \vec{x}_n\) which is equal to the following determinant:

\[
[\vec{x}_1, \ldots, \vec{x}_n]_n = \vec{x}_1 \times \vec{x}_2 \times \ldots \times \vec{x}_n = \begin{vmatrix}
  x_{11} & x_{12} & \ldots & x_{1n} & \vec{e}_1 \\
  x_{21} & x_{22} & \ldots & x_{2n} & \vec{e}_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_{(n+1)1} & x_{(n+1)2} & \ldots & x_{(n+1)n} & \vec{e}_{n+1}
\end{vmatrix}, \tag{12}
\]

where \((x_{1l}, \ldots x_{(n+1)l})\) are the coordinates of the vector \(\vec{x}_l\) and \(\{\vec{e}_l\}\) is the orthonormal basis.

The other simple example is the algebra of polynomials of \(n\)-variables \(x_1, \ldots x_n\) where n-ary operation is the functional Jacobian:

\[
[f_1, \ldots, f_n]_n = \det \left| \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} \right|. \tag{13}
\]

The n-ary Poisson-like structure of the determinant have been used for the generalization of the classical Hamiltonian mechanics in which the binary Poisson bracket can be replaced by ternary [22] or by n-ary brackets [23] [25], respectively.
There exist several examples of multi-Hamiltonian systems possessing dynamical or hidden symmetries, which can be realized within the generalized Nambu–Hamiltonian mechanics using the $n$-ary Nambu-Poisson brackets ($n > 2$). Among such systems one can consider the $SO(4)$ Kepler problem [26]. The well-known Kepler Hamiltonian is

$$H = \frac{\vec{p}^2}{2} - \frac{1}{r},$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Such a Hamiltonian has the $SO(3)$ rotational symmetry giving the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ as an integral of motion. This integral of motion with a Hamiltonian implies that the orbit lies in 2-dimensional plane, but cannot explain why it is closed. Laplace discovered the other hidden $SO(3)$ symmetry, which gives the additional integral of motion $\vec{A} = \vec{p} \times \vec{L} - \vec{r}/r$. The Kepler problem was naturally solved in Nambu Hamiltonian mechanics, in which the equations of motion are given by $n$-Poisson–Nambu bracket:

$$\frac{df}{dt} = [H_1, \ldots, H_5, f]. \quad (14)$$

Other $n$-ary analogues of Lie algebras have also been considered by Bremner [35, 36]. Bremner found the minimal (Jacobi) identity in the totally associative case, which has degree 7. One calls algebra $A$ totally associative if it satisfies the polynomial identities

$$t_i(a_1, a_2, \ldots, a_{2n-1}) = a_1 \cdots (a_i \cdots a_{i+n-1}) \cdots a_{2n-1} - a_1 \cdots (a_{i+1} \cdots a_{i+n}) \cdots a_{2n-1} = 0, \quad (15)$$

for $1 \leq i \leq n - 1$. The minimal identity in the partially associative case, which has degree 5, was found by Gnedbaye [32]: $A$ is called partially associative if it satisfies the polynomial identity

$$p(a_1, \ldots, a_{2n-1}) = \sum_{i=1}^{i=n} (-1)^{n-1} a_1 \cdots a_{i-1} (a_i \cdots a_{i+n-1}) a_{i+n} \cdots a_{2n-1} = 0. \quad (16)$$

Also the ternary symmetries have been intensively discussed in quantum physics [30, 31], in conformal field theories [21].

One of the best ways of studying the $n$-ary algebras/symmetries is through the $n$-ary generalizations of Clifford algebras. The binary Clifford algebra is an associative algebra, which contains and is generated by a vector space having a quadratic form: $e_k e_l + e_l e_k = 2 g_{kl}$; $\{e_a\}$ is the basis of the vector space and the signature of this space is determined by the metric $g_{kl} = diag(-1, \ldots, -1_t, +1, \ldots, 1_s)$.

The binary Clifford algebras are closely related with the study of the rotation groups of multidimensional spaces. The $1_t + 3_s$ Minkowski space can be described by the quadratic form $x^2 + y^2 + z^2 - c^2 t^2$, which remains invariant under a general Lorentz group. The general Lorentz group $O(3, 1)$ consists of a proper orthochronous Lorentz group $O_0(3, 1)$ and three reflections (discrete transformations) $P, T, PT$, where $P$ and $T$ are space and time reversal. In the general case of the real space $R^{n,t}$, the orthogonal group $O(s, t)$ is represented by the semidirect product of a connected component $O(s, t)_0$ and a discrete subgroup $\{1, P, T, PT\}$. The double covering of the orthogonal group $O(s, t)$ is a
Clifford group $Pin(s,t)$, which can be completely constructed within a Clifford algebra, i.e. $Pin(s,t) \subset Cl_{s,t}$ \cite{[37]}. The discrete symmetries are represented by fundamental automorphisms of the Clifford algebras, i.e. $\{1, P, T, PT\} \approx \text{Aut}(\mathbb{r}mCl)$. In contrast with the transformations $P, T, PT$, the operation $C$ of charge conjugation is represented by a pseudoautomorphism $A \rightarrow \bar{A}$. An extended automorphism group $Ext(Cl)$ is isomorphic to a CPT group $\{1, P, T, PT, C, CP, CT, CPT\}$. The n-ary Clifford algebras are related to the generalizations of orthogonal groups.

5 The ternary generalization of quaternions and the tripling Cayley-Dickson method

We would now like to discuss the doubling Cayley–Dickson method and generalize its to construct the ternary form division algebras. The complex numbers are 2-dimensional algebra with basis $e_0$ and $e_1 \equiv i$,

$$C = \mathbb{R} \oplus \mathbb{R}e_1,$$  \hfill (17)

where $e_0^2 = e_0$, $e_1e_0 = e_0e_1 = e_1$ and $e_1$ is the imaginary unit, $i^2 = -e_0$. Considering one additional basis imaginary unit element $e_2 \equiv j$ in the Cayley–Dickson doubling process, we can obtain the quaternions:

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j.$$  \hfill (18)

By the Cayley–Dickson method, if $a, b, c, d \in X$, when $(a, b), (c, d) \in X^2$. And the product is defined in $X^2$ through

$$(a, b)(c, d) = (ad - c^*b, bc^* + da) \hfill (19)$$

This means that quaternions can be considered as a pair of complex numbers:

$$q = (a + ib) + j(c + id), \hfill (20)$$

where

$$j(c + id) = (c + \bar{id})j = (c - id)j.$$  \hfill (21)

so, that we can see that $ij = -ji = k$.

The quaternions

$$q = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3, \quad q \in \mathbb{H}, \hfill (22)$$

produce over $\mathbb{R}$ a 4-dimensional norm division algebra, where the fourth imaginary unit $e_3 = e_1e_2 \equiv k$ appears. The main multiplication rules of all these four elements are the following:

$$i^2 = j^2 = k^2 = -1 \hfill (23)$$

$$ij = k \quad ji = -k.$$
All other identities can be obtained from cyclic permutations of \( i, j, k \). The imaginary quaternions \( i, j, k \) produce the \( \text{su}(2) \) algebra. The matrix realization of quaternions has been done through the Pauli matrices:

\[
\sigma_0, \ i\sigma_1, i\sigma_2, i\sigma_3.
\]

(24)

The unit quaternions \( q = a1 + bi + cj + dk \in H_1, \ q\bar{q} = 1 \), produce the \( SU(2) \) group:

\[
qq = a^2 + b^2 + c^2 + d^2 = 1, \ \{a, b, c, d\} \in S^3, \ S^3 \approx SU(2).
\]

(25)

Similarily, continuing the Cayley–Dickson doubling process, we can build the octo-

nions:

\[
\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l,
\]

(26)

where we introduced the new basis element \( l \equiv e_4 \). As a result of this process, the basis \( \{1, i, j, k\} \) of \( \mathbb{H} \) is complemented to a basis \( \{1 = e_0, i = e_1, j = e_2, k = e_3 = e_1e_2, l = e_4, il = e_5 = e_1e_4, jl = e_6 = e_2e_4, kl = e_7 = e_3e_4\} \) of \( \mathbb{O} \):

\[
o = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7,
\]

(27)

where we can see the following seven associative cycle triples:

\[
\{123 : e_1e_2 = e_3\}, \ \{145 : e_1e_4 = e_5\}, \ \{176 : e_1e_7 = e_6\}, \ \{246 : e_2e_4 = e_6\}, \ \{257 : e_2e_5 = e_7\}, \ \{347 : e_3e_4 = e_7\}, \ \{365 : e_3e_6 = e_5\}.
\]

(28)

The doubling process can consequently produce new algebras:

\[
\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S} \to \ldots,
\]

(29)

but just the first four from this list are the norm division alternative algebras.

In order find new division algebras, we can just relax some constraints. To do this, let us consider two basic elements, \( q_0, q_1 \), with the following constraints:

\[
q_1 \cdot q_0 = q_0 \cdot q_1 = q_1, \ \ \ q_1^3 = q_0.
\]

(30)

In this case a new element can be introduced, as \( q_2 = q_1^2 = q_1^{(-1)} \), i.e. \( q_2q_1 = q_1q_2 = q_0 \).

From these three elements a new field \( \mathbb{T}C \) can be built:

\[
\mathbb{T}C = \mathbb{R} \oplus \mathbb{R}q_1 \oplus \mathbb{R}q_1^2,
\]

(31)

with the new numbers

\[
\hat{z} = x_0q_0 + x_1q_1 + x_2q_2, \ \ \ \ x_i \in \mathbb{R}, \ \ \ i = 0, 1, 2,
\]

(32)
which are the ternary generalization of the complex numbers. Note that the multicomplex numbers, $\mathbb{MC}_n = \{ x = \sum_{i=0}^{i=n-1} x_i q^i, q^n = -q_0, x_i \in \mathbb{R} \}$ have been suggested in articles [38, 39, 21].

Let us define consequent two operations of the $\mathbb{Z}_3$-conjugation:

\[ \bar{q}_1 = jq_1, \quad \bar{q}_1 = j^2 q_1, \]

(33)

where $j = \exp{(2i\pi)/3}$. Since $q_2 = q_1^2$ we can easily obtain

\[ q_2 = j^2 q_2, \quad \bar{q}_2 = j q_2. \]

(34)

One can also define a $\mathbb{Z}_2$-conjugation which transforms $q_1 \rightarrow q_1^* = q_2$ and $q_2 \rightarrow q_2^* = q_1$, i.e. $q_1^* = q_2$ and $q_2^* = q_1$. Two $\mathbb{Z}_3$-conjugation operations can be applied, respectively:

\[ \bar{z} = x_0 q_0 + x_1 j q_1 + x_2 j^2 q_2, \]

\[ \overline{\bar{z}} = x_0 q_0 + x_1 j^2 q_1 + x_2 j q_2. \]

(35)

We now introduce the cubic form:

\[ \langle \hat{z} \rangle = \hat{z} \overline{\bar{z}} \hat{z} = x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_3, \]

(36)

And also easily check the following relation:

\[ \langle \hat{z}_1 \hat{z}_2 \rangle = \langle \hat{z}_1 \rangle \langle \hat{z}_2 \rangle, \]

(37)

which indicates the group properties of the $\mathbb{T}_C$ numbers. More exactly, the unit $\mathbb{T}_C$ numbers produce the Abelian ternary group. According to the ternary analogue of the Euler formula, the following ternary complex functions [38, 39, 21] can be constructed:

\[ \Psi = \exp{(q_1 \phi_1 + q_2 \phi_2)}, \quad \psi_1 = \exp{(q_1 \phi_1)}, \quad \psi_2 = \exp{(q_2 \phi_2)}, \]

(38)

where $\phi_i$ are the group parameters. For the functions $\psi_i$, $i = 0, 1, 2$, i.e. we have the following analogue of Euler, formula:

\[ \Psi = \exp{(q_1 \phi + q_2 \phi_2)} = f q_0 + g q_1 + h q_2, \]

\[ \psi_1 = \exp{(q_1 \phi)} = f_1 q_0 + g_1 q_1 + h_1 q_2, \]

\[ \psi_2 = \exp{(q_2 \phi)} = f_2 q_0 + h_2 q_1 + g_2 q_2, \]

(39)

Consequently, e can now introduce the conjugation operations for these functions. For example, for $\psi_1$ we can get:

\[ \bar{\psi}_1 = \exp{(\bar{q}_1 \phi)} = \exp{(j \cdot q_1 \phi)} = f q_0 + j g q_1 + j^2 h q_2, \]

\[ \overline{\bar{\psi}}_1 = \exp{\left(\overline{\bar{q}_1} \phi\right)} = \exp{(j^2 \cdot q_1 \phi)} = f_1 q_0 + j^2 g_1 q_1 + j h_1 q_2, \]

(40)
with the following constraints:

\[
\psi_1 \bar{\psi}_1 \bar{\psi}_1 = \exp (q_1 \phi) \exp (j \cdot q_1 \phi) \exp (j^2 \cdot q_1 \phi) = q_0,
\]

which gives us the following link between the functions, \(f, g, h\):

\[
f_1^3 + g_1^3 + h_1^3 - 3f_1 g_1 h_1 = 1.
\]

This surface (see figure 1, appendix: \(x^3 + y^3 + u^3 - 3xu = 1\) and [33]) is a ternary analogue of the \(S^1\) circle and it is related with the ternary Abelian group, \(TU(1)\).

![Figure 1: The surface for \(f_1^3 + g_1^3 + h_1^3 - 3f_1 g_1 h_1 = 1\)]

At the next step, we can construct the analogue of binary quaternions \(\mathbb{H}\), based on the three elements \(q_0, q_1, q_7\). Our new constraint \(q_1^3 = q_7^3 = q_0\), we can introduce the following six new products from \(q_1\) and \(q_7\): \(q_1^3, q_1^2 q_7, q_1 q_7^2, q_1^2 q_7 q_1, q_1^3 q_7^2\) and \(q_1^2 q_7^2\). More exactly, to get the new \(3^2\) ternary \(\mathbb{H}\) numbers, we suggest the tripling Cayley-Dickson process starting from \(\mathcal{T}C^1\).
\[ H = TC \oplus TCq_7 \oplus TCq_7^2, \]  
\[ (43) \]

or more exactly

\[
Q = (x_0q_0 + x_1q_1 + x_2q_1^2) + q_7(y_0q_0 + y_1q_1 + y_2q_1^2) + q_7^2(z_0q_0 + z_1q_1 + z_2q_1^2) \\
= (x_0q_0 + x_1q_1 + x_2q_4) + (y_0q_7 + jy_1q_2 + jy_2q_6) + (z_0q_8 + z_1q_3 + j^2z_2q_5) \\
= (x_0q_0 + y_0q_7 + z_0q_8) + (x_1q_1 + y_1q_2 + z_1q_3) + (x_2q_4 + jy_2q_6 + j^2z_2q_5),
\]
\[ (44) \]

where we defined the new unit elements \( q_1^2 = q_4, q_7q_1 = q_2, q_7^2 = q_8, q_7^2q_1 = q_3, q_7q_1^2 = jq_6, \) \( q_7^2q_1^2 = j^2q_5 \) with \( q_3^3 = q_0. \)

To find the table of multiplication (see table 2) of all \( q_a \) basis elements we can recall the identity between three unit imaginary elements, \( e_1, e_2, e_3, \) in binary quaternion algebra:

\[
e_1e_2e_3 = e_2e_3e_1 = e_3e_1e_2 = -e_0, \\
e_3e_2e_1 = e_2e_1e_3 = e_1e_3e_2 = +e_0. \]
\[ (45) \]

Since we suppose that the \( H \) numbers are the ternary generalizations of quaternions we can start from the following triple identities for \( q_1, q_2, q_3: \)

\[
q_1q_2q_3 = q_2q_3q_1 = q_3q_1q_2 = j^2q_0, \\
q_3q_2q_1 = q_2q_1q_3 = q_1q_3q_2 = jq_0.
\]
\[ (46) \]

where \( j = \exp(2\pi i/3). \)

Introducing the new elements

\[
q_1^2 = q_4, \quad q_2^2 = q_5, \quad q_3^2 = q_6, \]
\[ (47) \]

we can immediately get the following triple identities:

\[
q_4q_5q_6 = q_5q_6q_4 = q_6q_4q_5 = j^2q_0, \\
q_6q_5q_4 = q_5q_4q_6 = q_4q_6q_5 = jq_0.
\]
\[ (48) \]

We suggest the following 24 commutation relations:

\[
q_1q_2 = jq_2q_1, \quad q_2q_3 = jq_3q_2, \quad q_3q_1 = jq_1q_3, \\
q_4q_5 = jq_5q_4, \quad q_5q_6 = jq_6q_5, \quad q_6q_4 = jq_4q_6, \\
q_5q_1 = jq_1q_5, \quad q_6q_2 = jq_2q_6, \quad q_4q_3 = jq_3q_4, \\
q_6q_1 = j^2q_1q_6, \quad q_5q_3 = j^2q_4q_2, \quad q_4q_2 = j^2q_5q_3, \\
q_1q_7 = jq_7q_1, \quad q_2q_7 = jq_7q_2, \quad q_3q_7 = jq_7q_3, \\
q_1q_8 = j^2q_8q_1, \quad q_2q_8 = j^2q_8q_2, \quad q_3q_8 = j^2q_8q_3, \\
q_7q_4 = jq_4q_7, \quad q_7q_5 = jq_5q_7, \quad q_7q_6 = jq_6q_7, \\
q_8q_4 = j^2q_4q_8, \quad q_8q_5 = j^2q_5q_8, \quad q_8q_6 = j^2q_6q_8
\]
\[ (49) \]
and 4 commuting pairs:

\[ q_1 q_4 = q_4 q_1, q_2 q_5 = q_5 q_2, q_3 q_6 = q_6 q_3, q_7 q_8 = q_8 q_7. \]  

(50)

The \( q_k \) elements that satisfy the ternary algebra are:

\[ \{ A, B, C \} \rightarrow S_3 = ABC + BCA + CAB - BAC - ACB - CBA. \]  

(51)

Here \( j = \exp(2i\pi/3) \) and \( S_3 \) is the permutation group of three elements.

We can consider the \( 3 \times 3 \) matrix realization of \( q \)-algebra:

\[
\begin{align*}
q_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & q_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, & q_3 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix} \\
q_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & q_5 &= \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, & q_6 &= \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix} \\
q_7 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & j^2 & 0 \end{pmatrix}, & q_8 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, & q_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]  

(52)

which satisfy the ternary algebra:

\[ \{ q_k, q_l, q_m \} \rightarrow S_3 = f_{klm}^n q_n. \]  

(53)

We can check that each triple commutator \( \{ q_k, q_l, q_m \} \) is defined by triple numbers, \( \{ klm \} \), with \( k, l, m = 0, 1, 2, ..., 8 \), that it gives just one matrix \( q_n \) with the corresponding coefficient \( f_{klm}^n \) given in Table 3.
\[ q_a^3 = q_0 \quad a = 1, 2, \ldots, 8 \quad j = \exp(\frac{\sqrt{3}}{2} \pi i) \]

\[ q_7 q_1 = q_2 \]
\[ q_7 q_2 = q_3 \]
\[ q_7 q_3 = q_1 \]

\[ q_{1} q_{7} = j q_{2} \]
\[ q_{2} q_{7} = j q_{3} \]
\[ q_{3} q_{7} = j q_{1} \]

\[ q_1^2 = q_4 \]
\[ q_2^2 = q_5 \]
\[ q_3^2 = q_6 \]
\[ q_7^2 = q_8 \]

\[ q_1 q_2 q_3 = j^2 q_0 \]
\[ q_4 q_5 q_6 = j^2 q_0 \]
\[ q_2 q_1 q_3 = j q_0 \]
\[ q_5 q_4 q_6 = j q_0 \]
Table 3: The ternary commutation relations

| $N$ | $\{klm\} \rightarrow \{n\}$ | $f_{klm}^n$ | $N$ | $\{klm\} \rightarrow \{n\}$ | $f_{klm}^n$ | $N$ | $\{klm\} \rightarrow \{n\}$ | $f_{klm}^n$ |
|-----|----------------|----------|-----|----------------|----------|-----|----------------|----------|
| 1   | $\{123\} \rightarrow \{0\}$ | $3(j^2 - j)$ | 2   | $\{124\} \rightarrow \{2\}$ | $j(1 - j)$ | 3   | $\{126\} \rightarrow \{3\}$ | $2(j^2 - j)$ |
| 4   | $\{125\} \rightarrow \{1\}$ | $j(1 - j)$ | 5   | $\{127\} \rightarrow \{5\}$ | $2(1 - j)$ | 6   | $\{128\} \rightarrow \{4\}$ | $2(j^2 - 1)$ |
| 7   | $\{120\} \rightarrow \{6\}$ | $(j^2 - j)$ | 8   | $\{134\} \rightarrow \{3\}$ | $(j^2 - j)$ | 9   | $\{136\} \rightarrow \{1\}$ | $(j^2 - j)$ |
| 10  | $\{135\} \rightarrow \{2\}$ | $2(j^2 - j)$ | 11  | $\{137\} \rightarrow \{4\}$ | $2(j - 1)$ | 12  | $\{138\} \rightarrow \{6\}$ | $2(1 - j^2)$ |
| 13  | $\{130\} \rightarrow \{5\}$ | $j(1 - j)$ | 14  | $\{146\} \rightarrow \{6\}$ | $(j^2 - j)$ | 15  | $\{145\} \rightarrow \{5\}$ | $(j - j^2)$ |
| 16  | $\{147\} \rightarrow \{7\}$ | $(j^2 - j)$ | 17  | $\{148\} \rightarrow \{8\}$ | $(j - j^2)$ | 18  | $\{140\} \rightarrow O$ | 0 |
| 19  | $\{165\} \rightarrow \{4\}$ | $2j(1 - j)$ | 20  | $\{167\} \rightarrow \{8\}$ | $2(1 - j^2)$ | 21  | $\{168\} \rightarrow \{0\}$ | $3(1 - j^2)$ |
| 22  | $\{160\} \rightarrow \{7\}$ | $(1 - j^2)$ | 23  | $\{157\} \rightarrow \{0\}$ | $3(1 - j)$ | 24  | $\{158\} \rightarrow \{7\}$ | $(1 - j)$ |
| 25  | $\{150\} \rightarrow \{8\}$ | $(1 - j)$ | 26  | $\{178\} \rightarrow \{1\}$ | $(j - j^2)$ | 27  | $\{170\} \rightarrow \{2\}$ | $(j - 1)$ |
| 28  | $\{180\} \rightarrow \{3\}$ | $(j^2 - 1)$ | 29  | $\{234\} \rightarrow \{1\}$ | $2(j^2 - j)$ | 30  | $\{236\} \rightarrow \{2\}$ | $(j - j^2)$ |
| 31  | $\{235\} \rightarrow \{3\}$ | $(j - j^2)$ | 32  | $\{237\} \rightarrow \{6\}$ | $2(1 - j)$ | 33  | $\{238\} \rightarrow \{5\}$ | $2(j^2 - 1)$ |
| 34  | $\{230\} \rightarrow \{4\}$ | $(j^2 - j)$ | 35  | $\{246\} \rightarrow \{5\}$ | $2(j - j^2)$ | 36  | $\{245\} \rightarrow \{4\}$ | $(j - j^2)$ |
| 37  | $\{247\} \rightarrow \{8\}$ | $2(1 - j^2)$ | 38  | $\{248\} \rightarrow \{0\}$ | $3(1 - j^2)$ | 39  | $\{240\} \rightarrow \{7\}$ | $(1 - j^2)$ |
| 40  | $\{265\} \rightarrow \{6\}$ | $(j^2 - j)$ | 41  | $\{267\} \rightarrow \{0\}$ | $3(1 - j)$ | 42  | $\{268\} \rightarrow \{7\}$ | $2(1 - j)$ |
| 43  | $\{260\} \rightarrow \{8\}$ | $1 - j$ | 44  | $\{257\} \rightarrow \{7\}$ | $(j^2 - j)$ | 45  | $\{258\} \rightarrow \{8\}$ | $(j - j^2)$ |
| 46  | $\{250\} \rightarrow O$ | 0 | 47  | $\{278\} \rightarrow \{2\}$ | $(j - j^2)$ | 48  | $\{270\} \rightarrow \{3\}$ | $(j - 1)$ |
| 49  | $\{280\} \rightarrow \{1\}$ | $(j^2 - 1)$ | 50  | $\{346\} \rightarrow \{4\}$ | $(j^2 - j)$ | 51  | $\{345\} \rightarrow \{6\}$ | $2(j^2 - j)$ |
| 52  | $\{347\} \rightarrow \{0\}$ | $3(1 - j)$ | 53  | $\{348\} \rightarrow \{7\}$ | $2(1 - j)$ | 54  | $\{340\} \rightarrow \{8\}$ | $(1 - j)$ |
| 55  | $\{365\} \rightarrow \{5\}$ | $j^2 - j$ | 56  | $\{367\} \rightarrow \{7\}$ | $(j^2 - j)$ | 57  | $\{368\} \rightarrow \{8\}$ | $(j^2 - j)$ |
| 58  | $\{360\} \rightarrow O$ | 0 | 59  | $\{357\} \rightarrow \{8\}$ | $2(1 - j^2)$ | 60  | $\{358\} \rightarrow \{0\}$ | $3(1 - j^2)$ |
| 61  | $\{350\} \rightarrow \{7\}$ | $(1 - j^2)$ | 62  | $\{378\} \rightarrow \{3\}$ | $(j - j^2)$ | 63  | $\{370\} \rightarrow \{1\}$ | $(j - 1)$ |
| 64  | $\{380\} \rightarrow \{2\}$ | $(j^2 - 1)$ | 65  | $\{465\} \rightarrow \{0\}$ | $3(j - j^2)$ | 66  | $\{467\} \rightarrow \{3\}$ | $2(j - 1)$ |
| 67  | $\{468\} \rightarrow \{1\}$ | $2(1 - j^2)$ | 68  | $\{460\} \rightarrow \{2\}$ | $(j - j^2)$ | 69  | $\{457\} \rightarrow \{1\}$ | $2(1 - j)$ |
| 70  | $\{458\} \rightarrow \{2\}$ | $2(j^2 - 1)$ | 71  | $\{450\} \rightarrow \{3\}$ | $(j^2 - j)$ | 72  | $\{478\} \rightarrow \{4\}$ | $(j^2 - j)$ |
| 73  | $\{470\} \rightarrow \{6\}$ | $(1 - j)$ | 74  | $\{480\} \rightarrow \{5\}$ | $(1 - j^2)$ | 75  | $\{467\} \rightarrow \{2\}$ | $2(j - 1)$ |
| 76  | $\{658\} \rightarrow \{3\}$ | $2(1 - j^2)$ | 77  | $\{650\} \rightarrow \{1\}$ | $(j - j^2)$ | 78  | $\{678\} \rightarrow \{6\}$ | $(j^2 - j)$ |
| 79  | $\{670\} \rightarrow \{5\}$ | $(1 - j)$ | 80  | $\{680\} \rightarrow \{4\}$ | $(1 - j^2)$ | 81  | $\{578\} \rightarrow \{5\}$ | $(j^2 - j)$ |
| 82  | $\{570\} \rightarrow \{4\}$ | $(1 - j)$ | 83  | $\{580\} \rightarrow \{6\}$ | $(1 - j^2)$ | 84  | $\{780\} \rightarrow O$ | 0 |
There are $C_3^2 = 84$ ternary commutation relations, but there are also $C_8^2 = 28$ commutation relations which correspond to the $su(3)$ algebra! Therefore, it is natural to represent the $q$-numbers as a ternary generalization of quaternions. The $S_3$ commutation relations naturally go to the binary, $S_2$, Lie commutation relations:

\[ \{q_a, q_b, q_0\}_{S_3} = q_a q_b q_0 + q_b q_0 q_a + q_0 q_a q_b - q_b q_a q_0 - q_a q_0 q_b - q_0 q_b q_a = q_a q_b - q_b q_a, \]  
(54)

where $a \neq b \neq 0$. On table for those 28 cases, one can see that we have $\{kl0\}$.

Let’s review ternary-quaternions in matrix form:

\[ Q = q_a x_a = \begin{pmatrix} z_0 & z_1 & \tilde{z}_2 \\ \tilde{z}_1 & \tilde{z}_2 & z_0 \end{pmatrix}, \]  
(55)

where

\[ \begin{align*}
z_0 &= x_0 + x_7 + x_8, \quad \tilde{z}_0 = x_0 + j x_7 + j^2 x_8, \\
z_1 &= x_1 + x_2 + x_3, \quad \tilde{z}_1 = x_1 + j x_2 + j^2 x_3, \\
z_2 &= x_4 + x_5 + x_6, \quad \tilde{z}_2 = x_4 + j x_5 + j^2 x_6.
\end{align*} \]  
(56)

According to the permutation groups $Z_2$ and $Z_3$, one can consider the $Z_2$ and the $Z_3$ transposition operations:

\[ \begin{align*}
t(Z_2) &: \quad 1 \to 2; \quad 2 \to 1 \\
T_+(Z_3) &: \quad 1 \to 2; \quad 2 \to 3; \quad 3 \to 1 \\
T_-(Z_3) &: \quad 3 \to 2; \quad 2 \to 1; \quad 1 \to 3.
\end{align*} \]  
(57)

For example, in the $Z_3$ transposition, we can obtain the following transformation (conjugations) for the $q_i \ (i = 1, 2, 3)$ elements:

\[ \begin{align*}
q_1^{T+} &= q_1, & q_2^{T+} &= j^2 q_2, & q_3^{T+} &= j q_3, \\
q_4^{T+} &= q_4, & q_5^{T+} &= j^2 q_5, & q_6^{T+} &= j q_6, \\
q_0^{T+} &= q_0, & q_7^{T+} &= j^2 q_7, & q_8^{T+} &= j q_8.
\end{align*} \]  
(58)

and

\[ \begin{align*}
q_1^{T-} &= q_1, & q_2^{T-} &= j q_2, & q_3^{T-} &= j^2 q_3, \\
q_4^{T-} &= q_4, & q_5^{T-} &= j q_5, & q_6^{T-} &= j^2 q_6, \\
q_0^{T-} &= q_0, & q_7^{T-} &= j q_7, & q_8^{T-} &= j^2 q_8.
\end{align*} \]  
(59)

respectively.

In the two subsequent $Z_3^+$ transpositions, the $Q$-matrices are transformed in the following way:
respectively.

\[ Q_{TT}^+ = \begin{pmatrix} x_0 + j^2x_7 + jx_8 & x_1 + j^2x_2 + jx_3 & x_4 + jx_5 + j^2x_6 \\ x_4 + j^2x_5 + jx_6 & x_0 + x_7 + x_8 & x_1 + x_2 + x_3 \\ x_1 + jx_2 + j^2x_3 & x_4 + x_5 + x_6 & x_0 + jx_7 + j^2x_8 \end{pmatrix}, \quad (61) \]

and

\[ Q_{TT}^+ = \begin{pmatrix} x_0 + jx_7 + j^2x_8 & x_1 + jx_2 + j^2x_3 & x_4 + x_5 + x_6 \\ x_4 + j^2x_5 + jx_6 & x_0 + j^2x_7 + jx_8 & x_1 + j^2x_2 + jx_3 \\ x_1 + x_2 + x_3 & x_4 + j^2x_5 + jx_6 & x_0 + x_7 + x_8 \end{pmatrix}. \]

In this ternary conjugation, the cubic form \( QQ^T Q_{TT} \),

\[
(x_0^3 + x_7^3 + x_8^3 - 3x_0x_7x_8) + (x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3) + (x_4^3 + x_5^3 + x_6^3 - 3x_4x_5x_6) \\
-3x_0(x_1x_4 + x_2x_5 + x_3x_6) - 3j^2x_7(x_1x_5 + x_2x_6 + x_3x_4) - 3jx_8(x_1x_6 + x_2x_4 + x_3x_5),
\]

is "binary" complex. This form is also symmetric under the complex conjugation and the following transformations: \( x_1 \leftrightarrow x_4, x_2 \leftrightarrow x_5, x_3 \leftrightarrow x_6, x_7 \leftrightarrow x_8 \). One can take in account the binary complex conjugation to define the following norm:

\[
< Q > = (QQ^T Q_{TT})(QQ^T Q_{TT})^t \quad (63)
\]

, where the \( T \) is \( T_+ \) or \( T_- \). This norm can result in an appearance of group properties of ternary quaternions \( < Q > = 1 \).

If the binary alternative division algebras (real numbers, complex numbers, quaternions, octonions) over the real numbers have the dimensions \( 2^n \), \( n = 0, 1, 2, 3, 4, \ldots \), the ternary algebras have the following dimensions \( 3^n \), \( n = 0, 1, 2, 3, 4, \ldots \), respectively:

| \( \mathbb{R} \) | \( 2^0 = 1 \) | \( \mathbb{R} \) | \( 3^0 = 1 \) |
|---|---|---|---|
| \( \mathbb{C} \) | \( 2^1 = 1 + 1 \) | \( \mathbb{T} \) | \( 3^1 = 1 + 1 + 1 \) |
| \( \mathbb{H} \) | \( 2^2 = 1 + 2 + 1 \) | \( \mathbb{TH} \) | \( 3^2 = 1 + 2 + 3 + 2 + 1 \) |
| \( \mathbb{O} \) | \( 2^3 = 1 + 3 + 3 + 1 \) | \( \mathbb{TO} \) | \( 3^3 = 1 + 3 + 6 + 7 + 6 + 3 + 1 \) |
| \( \mathbb{S} \) | \( 2^4 = 1 + 4 + 6 + 4 + 1 \) | \( \mathbb{TS} \) | \( 3^4 = 1 + 4 + 10 + 16 + 19 + 16 + 10 + 4 + 1 \) |

In the last line the sedenions do not produce the division algebra. For both cases we have the unit elements \( e_0 \) and \( q_0 \), and the \( (n - 1) \) basis elements

\[
\{ e_a : e_a^2 = -e_0 \}, \\
\{ q_a : q_a^3 = q_0 \},
\]

respectively.

\[
\mathbb{R} \rightarrow \mathbb{T} \mathbb{C} \rightarrow \mathbb{T} \mathbb{H} \rightarrow \mathbb{T} \mathbb{O} \rightarrow \mathbb{T} \mathbb{S} \rightarrow \ldots
\]

(66)
To build the ternary "octonions" from ternary "quaternion" one needs the additional basis element. For illustration we take the following three basis elements: the previous two, \( q_1, q_7 \) and new third element \( q_{21} \). Then, applying the generalized Cayley–Dickson method, we can get the 27-dimensional algebra with \( q_0 = q_0, a = 1, 2, ..., 26 \) (see figure [6]):

\[
TO = (x_0q_0 + x_1q_1 + x_2q_2 + x_3q_3 + x_4q_4 + x_5q_5 + x_6q_6 + x_7q_7 + x_8q_8)
+ (y_0q_0 + y_1q_1 + y_2q_2 + y_3q_3 + y_4q_4 + y_5q_5 + y_6q_6 + y_7q_7 + y_8q_8)q_{21}
+ (z_0q_0 + z_1q_1 + z_2q_2 + z_3q_3 + z_4q_4 + z_5q_5 + z_6q_6 + z_7q_7 + z_8q_8)q_{21}^2.
\]  

(67)

6 Appendix: \( x^3 + y^3 + u^3 - 3xyu = 1 \).

\[
\begin{align*}
\rho &= x + y + u \\
z &= x + jy + j^2u \\
z &= x + j^2y + ju
\end{align*}
\]

(68)

(We use the sign \( \tilde{z} \) to define binary conjugation of ordinary complex numbers, and sign \( \bar{z} \) ternary complex conjugation.) Using these new variables one can rewrite the cubic equation

\[
x^3 + y^3 + u^3 - 3xyu = 1
\]

in the following form

\[
\rho |z|^2 = 1.
\]

(70)

Let substitute the \( \rho \) in this equation:

\[
\begin{align*}
z\bar{z} &= (x + jy + j^2u)(x^2 + j^2y + ju) = (x^2 + y^2 + u^2) - (xy + xu + yu) \\
&= (x - \frac{y + u}{2})^2 + \frac{3}{4}(y^2 + u^2) - \frac{3}{2}yu = \\
&= (\rho - \frac{3}{2}(y + u))^2 + \frac{3}{4}(y - u)^2 \\
&= \frac{1}{\rho},
\end{align*}
\]

(71)

or

\[
(\rho - \frac{3}{2}(y + u))^2 + (\frac{\sqrt{3}}{2}(y - u))^2 = (\frac{1}{\sqrt{\rho}})^2.
\]

(72)
Take new variables

\[
\frac{3}{2}(y + u) - \rho = \frac{1}{\sqrt{\rho}} \cos \phi \\
\frac{\sqrt{3}}{2}(y - u) = \frac{1}{\sqrt{\rho}} \sin \phi.
\] (73)

In result we can get the next parametrization

\[
x = \frac{\rho}{3} - \frac{2}{3\sqrt{\rho}} \cos \phi \\
y = \frac{\rho}{3} + \frac{1}{3\sqrt{\rho}}(\cos \phi + \sqrt{3} \sin \phi) \\
u = \frac{\rho}{3} + \frac{1}{3\sqrt{\rho}}(\cos \phi - \sqrt{3} \sin \phi)
\] (74)

or

\[
x = \frac{\rho}{3} - \frac{2}{3\sqrt{\rho}} \cos \phi \\
y = \frac{\rho}{3} + \frac{2}{3\sqrt{\rho}} \sin(\phi + \pi/6) \\
u = \frac{\rho}{3} + \frac{2}{3\sqrt{\rho}} \sin(\phi - \pi/6)
\] (75)

Note, that

\[
\rho > 0, \quad 0 \leq \phi \leq 2\pi.
\] (76)

7 Ternary algebras of higher rank

Now we would like to create the ternary algebra of rank 3 from the simple ternary algebra, which can be described by a tetrahedron Berger graph [6]. This is similar to the way of creating the \(su(3)\) \((su(r + 1))\) Lie algebra from \(su(2)\)-algebras. The \(su(r + 1)\) algebra is the algebra of rank \(r\) having \(r\)-simple roots \(\alpha_a\) \((a = 1, 2, ..., r)\), each of them corresponding to the \(su(2)/u(1)\) algebra. This is a way to construct the \(su(r + 1)\) algebra from \(r\)-\(su(2)\) algebras! We can try to do same to construct the Berg, algebras from the minimal simple Berg-algebra.

Let us take three ternary algebras based on the \(q_A(m, n, i), q_B(n, p, j), q_C(m, p, k)\) matrices, where \(A(m, n, p), B(n, p, j), C(m, p, k) = 0, 1, 2, ..., 8\). According to our conception
Figure 2: The rank-3 Berger graph
(Figure 2) we would like to obtain the ternary algebra of higher rank defined by generators \( q_D(i, j, k) \). For this we should consider all ternary commutation relations: \([Q_a, Q'_a, Q''_a]_{S_3}\), where \( Q_a \) is the complete set of all considered elements \( Q_a = \{ q_A, q_B, q_C \} \). For example, we can consider the following:

\[
q_A = \begin{pmatrix} 0 & mn & 0 & 0 & 0 & 0 \\ 0 & 0 & ni & 0 & 0 & 0 \\ im & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad q_{TA} = \begin{pmatrix} 0 & 0 & 0 & mi & 0 & 0 \\ nm & 0 & 0 & 0 & 0 & 0 \\ 0 & in & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad q_A^0 = \begin{pmatrix} mn & 0 & 0 & 0 & 0 & 0 \\ 0 & nn & 0 & 0 & 0 & 0 \\ 0 & 0 & ii & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[(77)\]

\[
q_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & np & -0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_j \\ 0 & jn & 0 & 0 & 0 & 0 \end{pmatrix} \quad q_{TB} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & nj \\ 0 & pn & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad q_B^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pp & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & jj \end{pmatrix}
\]

\[(78)\]

\[
q_C = \begin{pmatrix} 0 & 0 & 0 & mp & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pk & 0 \\ km & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad q_{TC} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & mk \\ 0 & 0 & 0 & 0 & 0 & 0 \\ pm & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & kp & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad q_C^0 = \begin{pmatrix} mm & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pp & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & kk \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[(79)\]

\[
q_D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ik & 0 \\ 0 & 0 & 0 & 0 & 0 & kj \\ 0 & 0 & ji & 0 & 0 & 0 \end{pmatrix} \quad q_{TD} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ki & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad q_D^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & kk \\ 0 & 0 & 0 & 0 & 0 & jj \end{pmatrix}
\]

\[(80)\]

\[
t_{su(2)} = \begin{pmatrix} mm & 0 & 0 & 0 & 0 & m_j \\ 0 & mn & 0 & 0 & nk & 0 \\ 0 & 0 & ii & pi & 0 & 0 \\ 0 & 0 & ip & pp & 0 & 0 \\ 0 & kn & 0 & 0 & kk & kj \\ jm & 0 & 0 & 0 & 0 & jj \end{pmatrix}
\]
8 Appendix: Quaternary algebra

Now we would like to illustrate the quaternary algebra.

\[ q_m^4 = 1, \quad m = 0, 1, 2, 3, \ldots, 14, 15. \]  

\[ q_{m+1} = q_{13}^m q_1, \quad q_{4-m} = q_{15}^m q_4, \quad m = 0, 1, 2, 3 \]
\[ q_{5+m} = q_{13}^m q_5, \quad q_{8-m} = q_{15}^m q_8, \quad m = 0, 1, 2, 3, \]
\[ q_{9+m} = q_{13}^m q_9, \quad q_{12-m} = q_{15}^m q_{12}, \quad m = 0, 1, 2, 3, \]

\[ q_{13}^2 = q_{14}, \quad q_{13}^3 = q_{15}, \]
\[ q_{14}^2 = q_0, \]
\[ q_{15}^2 = q_{14}, \quad q_{15}^3 = q_{13} \]

\[ q_1^2 = q_5, \quad q_2^2 = j q_7, \quad q_3^2 = j^2 q_5, \quad q_4^2 = j^3 q_7, \]
\[ q_1^3 = q_9, \quad q_2^3 = j^3 q_{12}, \quad q_3^3 = j^2 q_{11}, \quad q_4^3 = j q_{10} \]

\[ q_5^2 = q_0, \quad q_6^2 = j^2 q_{14}, \quad q_7^2 = q_0, \quad q_8^2 = j^2 q_{14}. \]

\[ q_5^2 = q_5, \quad q_{10}^2 = j^3 q_7, \quad q_{11}^2 = j^2 q_5, \quad q_{12}^2 = j q_7, \]
\[ q_9^3 = q_1, \quad q_{10}^3 = j q_4, \quad q_{11}^3 = j^2 q_3, \quad q_{12}^3 = j^3 q_2 \]

We can consider the 4 × 4 matrix realization of \( q \)-algebra:

\[
q_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad q_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & j & 0 \\
0 & 0 & 0 & j^2 \\
j^3 & 0 & 0 & 0
\end{pmatrix}, \quad q_3 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & j^2 & 0 \\
0 & 0 & 0 & 1 \\
j^2 & 0 & 0 & 0
\end{pmatrix}, \quad q_4 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & j^3 & 0 \\
0 & 0 & 0 & j^2 \\
j & 0 & 0 & 0
\end{pmatrix}
\]
\[ q_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad q_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & j \\ j^2 & 0 & 0 & 0 \\ 0 & j^3 & 0 & 0 \end{pmatrix}, \quad q_7 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & j^2 \\ 1 & 0 & 0 & 0 \\ 0 & j^2 & 0 & 0 \end{pmatrix}, \quad q_8 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & j^3 \\ j^2 & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{pmatrix}, \]
\[ q_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad q_{10} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ j & 0 & 0 & 0 \\ 0 & j^2 & 0 & 0 \\ 0 & 0 & j^3 & 0 \end{pmatrix}, \quad q_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ j^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & j^2 & 0 \end{pmatrix}, \quad q_{12} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ j^3 & 0 & 0 & 0 \\ 0 & j^2 & 0 & 0 \\ 0 & 0 & j & 0 \end{pmatrix}, \]
\[ q_{13} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & j^2 & 0 \\ 0 & 0 & 0 & j^3 \end{pmatrix}, \quad q_{14} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & j^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & j^2 \end{pmatrix}, \quad q_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & j^3 & 0 & 0 \\ 0 & 0 & j^2 & 0 \\ 0 & 0 & 0 & j \end{pmatrix}, \quad q_{16} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

where \( j = \exp 2i\pi/4 \).
9 Conclusions and Acknowledgements

Our interest in the search for the \( n \)-ary algebras has a long history and started in 1998 when we have read the pioneer article of P. Candelas and M. Font [40]. We found the ”minimal” non-Abelian ternary algebra which we intend using for solving the set of Berger graphs. This could help us to find the ternary generalization of Cartan–Killing–Lie classification. Also it is very important to understand the ternary ”octonions” and find a link between binary and ternary exceptional graphs and algebras. Examples of the ternary algebras, which are presented here, could help us to find some physical applications, for example, in solving of the tetrahedron Baxter equation. The other physical application is related to searching for a ternary generalization of Lorentz group. In the end we hope that the ternary symmetries could help us to make the quantum theory of membranes.

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