A COUPLE OF REAL HYPERBOLIC DISC BUNDLES OVER SURFACES

SASHA ANAN’IN, PHILIPY V. CHIOVETTO

ABSTRACT. Applying the techniques developed in [AGG], we construct new real hyperbolic manifolds whose underlying topology is that of a disc bundle over a closed orientable surface. By the Gromov-Lawson-Thurston conjecture [GLT], such bundles \( M \to S \) should satisfy the inequality \( |e_M/\chi_S| \leq 1 \), where \( e_M \) stands for the Euler number of the bundle and \( \chi_S \) for the Euler characteristic of the surface. In this paper, we construct new examples that provide a maximal value of \( |e_M/\chi_S| = \frac{3}{5} \) among all known examples. The former maximum, belonging to Feng Luo [Luo], was \( |e_M/\chi_S| = \frac{1}{2} \).

Dedicated to Krolik

1. Introduction

Topologically or differentially, every open disc bundle \( M \to S \) over a closed connected orientable surface \( S \) can be completely characterized by two numbers: the Euler characteristic \( \chi_S \) of the surface and the Euler number \( e_M \) of the bundle, i.e., the number of self-intersections of a section of the bundle. Note that, taking an unramified finite cover of \( S \) and pullbacking the bundle, one gets the same value of \( |e_M/\chi_S| \).

The conjecture of Gromov, Lawson, and Thurston [GLT, p. 28] suggests a numerical criterion deciding whether a bundle can be equipped with a complete real hyperbolic geometry.

1.1. GLT-conjecture. A disc bundle \( M \to S \) over a closed connected orientable surface \( S \) of genus \( g \geq 2 \) admits a complete real hyperbolic structure iff \( |e_M/\chi_S| \leq 1 \).

In [AGG], Conjecture 1.1 was extended (with the same bound) to the complex hyperbolic case.

1.2. Known results. The best proven upper bound belongs to Misha Kapovich [Kap] who showed that \( |e_M| \leq \exp \left( \exp (10^8|\chi_S|) \right) \) for any complete real hyperbolic 4-manifold homotopically equivalent to a closed orientable surface; so, without actually using the fact that it admits a disc bundle structure. (In this case, \( e_M \) stands for the self-intersection of the generator of \( H_2 M \) represented by a homotopy equivalence \( S \to M \).) It is worthwhile mentioning that, in such settings, Nicolaas H. Kuiper [Kui2] constructed examples with \( |e_M/\chi_S| > \frac{2}{\sqrt{g}} > 1 \).

In the other direction, the best results belong to N. H. Kuiper [Kui1] and to Feng Luo [Luo]. In [Kui1, Theorem 6, p. 68], it was constructed a series of disc bundles admitting complete real hyperbolic geometry with any rational value of \( |e_M/\chi_S| \) in the interval \([0, \frac{1}{2}]\). (Though, we are not sure that this result is literally correct as there are a few miscalculations in the exposition.) F. Luo constructed an example with a maximal known (before our paper) value \( |e_M/\chi_S| = \frac{1}{2} \). Since the surface \( S \) in F. Luo’s example has genus 2, taking a finite unramified cover of \( S \), one gets examples satisfying the relation \( |e_M/\chi_S| = \frac{1}{2} \) with \( S \) of an arbitrary genus \( g \geq 2 \).

2000 Mathematics Subject Classification. 57N16 (57M50, 57S30).
Key words and phrases. Real hyperbolic disc bundles, right-angled polyhedra, GLT-conjecture.

The second author is supported by the grant Iniciação Científica FAPESP 2014/26282-5
1.3. Main result. Using the ideas of [AGG] and a coordinate free approach to hyperbolic geometry exposed in [AGr2, AGr3], we construct 3 new examples of disc bundles with $|eM/\chi S| = \frac{3}{2}$.

Two of them have $eM = 12$ with $S$ of genus 11. The third one has $eM = 24$ with $S$ of genus 21. The first two come from right-angled necklace polyhedra with 84 codimension 1 faces and 21 cycles of codimension 2 faces. The third one has a similar flavor with 164 codimension 1 faces and 41 cycles of codimension 2 faces.

In fact, we construct 3 orbifolds whose 4-sheeted covers provide the above manifolds. Each orbifold comes from a right-angled necklace polyhedron $P$ symmetric with respect to a regular elliptic isometry $r$ of order 24 for the first 2 examples and of order 44 for the third one. The face-pairing of these polyhedra are reflections in their totally geodesic codimension 1 faces.

For the other examples of disc bundles of a similar type, see Theorem 3.10.

2. Construction

We start the construction by fixing a regular elliptic isometry $r$ of indicated order $n$. Denote by $P_1$ and $P_2$ its totally geodesic $r$-stable planes. The planes intersect orthogonally at the unique $r$-fixed point $b$.

Next, we pick a generic totally geodesic hyperplane $H_0 \subset \mathbb{H}^3$. Geometrically, the pair $r, H_0$ is given by the distances from $b$ to the intersections $H_0 \cap P_1$ and $H_0 \cap P_2$. So, it can be described by means of 2 convenient real parameters $x_1, x_2$ responsible for these distances.

Then we copy the hyperplane, $H_i := r^i H_0$. The conditions that $C_i := H_i \cap H_{i+1} \not\subset \partial \mathbb{H}^4$ and that the other pairs of hyperplanes $H_i$ and $H_j$, $i - j \not\equiv n \pm 1$, are ultraparallel, i.e., $H_i \cap H_j = \emptyset$, is equivalent by Lemma 3.1 to a finite number of inequalities linear in $x_1, x_2$. Thus, we arrive at a convex region $R \subset \mathbb{R}^2(x_1, x_2)$. In what follows, we assume $(x_1, x_2) \in R$.

Intersecting those closed half-spaces limited by the $H_i$’s that contain the point $b$, we get a convex polyhedron $P$ bounded by closed solid cylinders $B_i \subset H_i$ and by a piece $\partial_0 P \subset \partial \mathbb{H}^4$ of the absolute bounded by a torus $T \subset \partial \mathbb{H}^4$. In turn, the solid cylinder $B_i$ is bounded inside $H_i$ by the ultraparallel totally geodesic planes $C_{i-1}, C_i$ and by a cylinder inside $T$; such cylinders form the torus $T$. Denoting $\partial_0 P := \bigcup_{i=1}^n B_i$, we see that $\partial_0 P$ is a solid torus bounded by $T$ and that $\partial P = \partial_0 P \sqcup_T \partial_1 P$.

Every solid cylinder $B_i$ is fibred by totally geodesic planes called slices of $B_i$. Indeed, the geodesic segment $\Gamma_i$, that joins the closest points in $C_{i-1}$ and in $C_i$ lists the fibres in question: through any $p \in \Gamma_i$, we have, inside $H_i$, a totally geodesic plane orthogonal to $\Gamma_i$. Note that $C_{i-1}$ and $C_i$ are among the slices; they are the initial and the final slices. Denote by $M_i$ the middle slice, i.e., the one passing through the middle point of $\Gamma_i$.

Every solid cylinder $B_i$ is fibred as well by closed geodesic segments called strings of $B_i$. They can be described as follows. Take any totally geodesic plane $F$ such that $\Gamma_i \subset F \subset H_i$. The intersection $F \cap B_i$ is bounded in $F$ by the geodesics $F \cap C_{i-1}$ and $F \cap C_i$, both orthogonal to $\Gamma_i$, and by two arcs on the absolute. A string of $B_i$ is the segment of a line, inside some $F$, equidistant from the geodesic containing the segment $\Gamma_i$. The segment $\Gamma_i$ and the mentioned two arcs on the absolute are among the strings of $B_i$. Clearly, the reflection $\sigma_i$ in the middle slice $M_i$ of $B_i$ stabilizes any string of $B_i$ and interchanges the endpoints of the string.

Pick a point $q_0 \in \partial \mathbb{H}^4 \cap C_0$ and let $q_0 \in s_n \subset B_n$ be the string of $B_n$ that contains $q_0$. Then $q_{n-1} := \sigma_n q_0 \in \partial \mathbb{H}^4 \cap C_{n-1}$ is the other endpoint of $s_n$. Next, we take the string $s_{n-1}$ of $B_{n-1}$ such that $q_{n-1} \in s_{n-1} \subset B_{n-1}$, and so on. Finally, we get a simple curve $s := s_1 \cup s_2 \cup \cdots \cup s_n$ with the endpoints $q_0 \in \partial \mathbb{H}^4 \cap C_0$ and $q_n := q_0$ $\sigma_1 \sigma_2 \cdots q_0 \in \partial \mathbb{H}^4 \cap C_0$, where $\partial \mathbb{H}^4 \cap B_i \supset s_i$ is a string of $B_i$ for all $1 \leq i \leq n$. We call such a curve $s \subset T$ the string of $P$ generated by $q_0 \in \partial \mathbb{H}^4 \cap C_0$. 
2.1. Lemma (cf. [AGG, Lemma 2.25, p. 4317]). If $H_0$ is orthogonal to $P_1$ (in terms of the parameters, this means that $x_2 = 0$), then any string of $P$ is closed and contractible in $\partial P_1$. So, $\partial P_1$ is a solid torus and the slice bundle of $\partial P_0$ is extendable to $P$ in this case.

Proof. When $H_0$ is orthogonal to $P_1$, the intersection $Q := P \cap P_1$ is a regular $n$-gon centred at $b$ in the hyperbolic plane $P_1$. The polyhedron $P$ is simply the union of all those totally geodesic planes orthogonal to $P_1$ that pass through a point of $Q$. The slices of $\partial P_0$ are built over the points of the boundary $\partial Q$. Thus, we get the slice bundle of $\partial P_0$ extended to $P$.

The isometry $\sigma_i$ is the trivial extension of the reflection in the middle point of the corresponding side of $Q$. Hence, the isometry $\sigma_1 \ldots \sigma_n$ is a trivial extension of an elliptic isometry of $P_1$ with the fixed point $C_0 \cap P_1$. In other words, the restriction $\sigma_1 \ldots \sigma_n|_{C_0}$ is the identity, implying that any string of $P$ is closed.

In order to visualize a contraction of a closed string $s \subset T$ in $\partial P_1$, one can simply shrink the $n$-gon $Q$ (say, keeping its $r$-rotational symmetry about $b$).

The fact that $\partial P_1$ is a solid torus follows from the Dehn lemma.

If any string of the polyhedron $P$ is closed, we say that $P$ is fibred. As we saw, this is equivalent to $\sigma_1 \ldots \sigma_n|_{C_0} = 1_{C_0}$. Denote $\sigma := \sigma_0$. Then $\sigma_i = r^i \sigma^{-1}$ and $\sigma_1 \ldots \sigma_n = (r\sigma)^n$ because $r^n = 1$. Since $C_0$ is $r\sigma$-stable, we conclude that $P$ is fibred iff $r\sigma|_{C_0}$ is an elliptic isometry of $C_0$ whose order divides $n$.

It follows from the connectedness of the region $R$ and from Lemmas 2.1 and 3.1 that $\partial P_0$ is a solid torus. By [AGG, Lemma 2.19, p. 4312], $P$ is topologically a closed 4-ball and the slice bundle of $\partial P_0$ is extendable to $P$.

Suppose that $P$ is fibred. Take a couple of disjoint strings $s, s' \subset T$ of $P$. Since $s$ and $s'$ are contractible inside $P$ with 2-discs $D, D' \subset P$, $s = \partial D$ and $s' = \partial D'$, one can calculate the algebraic number of intersections of $D$ and $D'$. This number $eP$ (clearly independent of the choice of $s, s'$) is the Euler number of the fibred polyhedron $P$.

Suppose that a fibred polyhedron $P$ is equipped with face-pairing for the codimension 1 faces $B_i$ and that the conditions of Poincaré’s polyhedron theorem (PPT) are satisfied (see, for example, [AGr1]). Then we obtain a disc bundle $M$ over a 2-dimensional orbifold $S$ and $eP$ is the Euler number of this bundle because the (face-pairing) isometries preserve the slice bundles and the string bundles of the $B_i$’s. (See, for instance, [BoS] for a treatment of bundles over orbifolds. Another option is to glue a few copies of $P$ forming a fundamental polyhedron for a manifold and to note that $eP$ and $\chi S$ get multiplied by the number of copies.)

A closed curve $c \subset T$ that generates the group $H_1(\partial P_0, \mathbb{Z})$ is said to be trivializing if $[c] = 0$ in $H_1(\partial P_0, \mathbb{Z})$. In the case considered in Lemma 2.1, any string of $P$ is trivializing.

2.2. Lemma [AGG, Remark 2.22, p. 4314]. Let $P$ be a fibred polyhedron, let $T \supset c$ be a string of $P$, and let $T \supset c$ be a trivializing curve of $P$. Then $eP = \#s \cap c$. In other words, $[s] = eP \cdot [g]$ in the group $H_1(\partial P_1, \mathbb{Z})$, where $g := \partial \Pi_2 \cap C_0$.

Now we get a tool to measure the Euler number $eM$. Given fibred polyhedron satisfying the conditions of PPT, one can deform it into a ‘plane’ one (dealt with in Lemma 2.1) because the region $R$ is convex. At the beginning of the deformation, the Euler number $eP$ of a ‘plane’ polyhedron equals 0 by Lemmas 2.1 and 2.2. During the deformation, we keep track of how many times the polyhedron becomes fibred, i.e., how many times a string of the polyhedron becomes closed. Of course, counting these events, we should take care of the signs. So, it is better to say that the Euler number of the fibred polyhedron at the end of the deformation equals the algebraic number of times it was fibred during the deformation, including the last moment and not including the initial ‘plane’ moment.

We apply this method at the end of the proof of Theorem 3.9. The count is simple there because the chosen deformation provides a monotonic evolution of a string.
3. Calculation

Let $b_1, b_2, b_3, b_4, b$ be an orthonormal basis of signature $-\ldots - +$ in an $\mathbb{R}$-linear space $V$ equipped with a symmetric bilinear form $\langle - , - \rangle$. Then the real hyperbolic space $\mathbb{H}^4_{\mathbb{R}}$, its absolute $\partial \mathbb{H}^4_{\mathbb{R}}$, and $\mathbb{H}^4 := \mathbb{H}^4_{\mathbb{R}} \cup \partial \mathbb{H}^4_{\mathbb{R}}$ are known to be identified respectively with

$$\mathbb{H}^4 := \{ p \in \mathbb{P}_{\mathbb{R}} V \mid \langle p, p \rangle > 0 \}, \quad \partial \mathbb{H}^4 := \{ p \in \mathbb{P}_{\mathbb{R}} V \mid \langle p, p \rangle = 0 \}, \quad \mathbb{H} := \{ p \in \mathbb{P}_{\mathbb{R}} V \mid \langle p, p \rangle \geq 0 \}.$$ 

Pick some numbers $k, m, n \in \mathbb{N}$ such that $n$ is even and $1 < k < m < \frac{n}{2}$. Denote by

$$r := \begin{bmatrix} c_1 & -s_1 & 0 & 0 & 0 \\ s_1 & c_1 & 0 & 0 & 0 \\ 0 & 0 & c_m & -s_m & 0 \\ 0 & 0 & s_m & c_m & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad p_0 := \begin{bmatrix} \sqrt{s_2} \\ 0 \\ \sqrt{s_2} \\ 0 \\ \sqrt{s_1+s_2} \end{bmatrix}, \quad 0 < x_1, \quad 0 \leq x_2, \quad 1 < x_1 + x_2,$$

the regular elliptic isometry of $\mathbb{H}^4$ with the unique fixed point $b \in \mathbb{H}^4$ and a point $p_0 \in \mathbb{P}_{\mathbb{R}} V \setminus \mathbb{H}^4$, both written in the above basis, where $x_1, x_2$ are some real parameters subject to the displayed inequalities, $c_i := \cos \frac{2\pi}{n}$, and $s_i := \sin \frac{2\pi}{n}$. Clearly, $r^n = 1$, $r \in SO$, and $\langle p_0, p_0 \rangle = -1$. For any $i \in \mathbb{Z}$, denote

$$p_i := r^i p_0 = \begin{bmatrix} c_i & -s_i & 0 & 0 & 0 \\ s_i & c_i & 0 & 0 & 0 \\ 0 & 0 & c_{mi} & -s_{mi} & 0 \\ 0 & 0 & s_{mi} & c_{mi} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{s_2} \\ 0 \\ \sqrt{s_2} \\ 0 \\ \sqrt{s_1+s_2} \end{bmatrix} = \begin{bmatrix} c_i \sqrt{s_2} \\ s_i \sqrt{s_2} \\ c_{mi} \sqrt{s_2} \\ s_{mi} \sqrt{s_2} \\ \sqrt{s_1+s_2} \end{bmatrix}.$$

Obviously,

$$g_i := \langle p_0, p_i \rangle = (1 - c_i) x_1 + (1 - c_{mi}) x_2 - 1.$$

Denote by $H_i \subset \mathbb{H}^4$ the totally geodesic hyperplane corresponding to the $\mathbb{R}$-linear subspace $p_i^{\perp} \subset V$. Let $C_i := H_i \cap H_{i+1}$.

**3.1. Lemma.** The conditions that $C_i \cap \partial \mathbb{H}^4$ and that $H_i \cap H_j = \emptyset$ for all $i, j$ such that $i - j \equiv \pm 1 \pmod{n}$ are equivalent to the inequalities

$$\begin{equation}
0 < x_1, \quad 0 \leq x_2, \quad 1 < x_1 + x_2, \quad (1 - c_i) x_1 + (1 - c_{mi}) x_2 < 2 < (1 - c_i) x_1 + (1 - c_{mi}) x_2,
\end{equation}$$

for $2 \leq i \leq \frac{n}{2}$. The convex region $R \subset \mathbb{R}^2(x_1, x_2)$ given by the inequalities (3.2) is nonempty because it contains the segment $S := \left( \frac{2}{1-c_2}, \frac{2}{1-c_1} \right) \times 0 \subset R$.

**Proof.** The condition that $C_i \cap \partial \mathbb{H}^4$ is known to be equivalent to $-1 < \langle p_i, p_{i+1} \rangle < 1$ and the condition that $H_i \cap H_j = \emptyset$ is known to be equivalent to $\langle p_i, p_j \rangle \notin [-1, 1]$, taking into account that $\langle p_k, p_i \rangle = -1$ for all $k$. Since $0 \leq g_i + 1$ and $0 < g_i + 1$, due to the symmetry related to the action of $r$, we arrive at the inequalities (3.2). For $x_2 = 0$, the inequalities take the form $1 < x_1$ and $(1 - c_1) x_1 < 2 < (1 - c_2) x_1$, i.e., the form $1 < \frac{2}{1-c_2} < x_1 < \frac{2}{1-c_1}$ with $-1 < c_2 < c_1$.

In the sequel, we assume $(x_1, x_2) \in R$.

Obviously, $C_i \cap C_j = \emptyset$ unless $i \equiv n \cdot j$. Therefore, the ultraparallel planes $C_{i-1}, C_i \subset H_i \simeq \mathbb{H}^3_{\mathbb{R}}$ limit in $H_i$ a solid cylinder $B_i$, and these cylinders form a solid torus $\partial P := \bigcup_{i=1}^{n} B_i$ bounded by a torus $T \subset \partial \mathbb{H}^4$ as claimed in Section 2.

All the $C_i$’s are in a same closed half-space of $\mathbb{H}^4$ limited by a hyperplane $H_j$ as, otherwise, there would exist some $C_{i-1}$ and $C_i$, both disjoint with $H_j$, in different half-spaces, which would cause an impossible intersection $H_i \cap H_j \neq \emptyset$. 

Let $P$ stand for the intersection of those closed half-spaces limited by the $H_i$’s that contain the point $b$; the $B_i$’s are codimension 1 faces of $P$ and the $C_i$’s are codimension 2 faces of $P$. By Lemma 2.1, $\partial P$ is a solid torus, $P$ is a closed 4-ball, and $\partial P = \partial_0 P \cup \partial_1 P$.

Denote by $\sigma$ the reflection in the middle slice $M_0$ of $B_0$ and by $\tau$, the reflection in $H_0$. Clearly, $\sigma_i := r^i \sigma r^{-i}$ is the reflection in the middle slice $M_i$ of $B_i$ and $\tau_i := r^i \tau r^{-i}$ is the reflection in $H_i$.

3.3. Lemma. Suppose that $g_1 = 0$, i.e.,

$$ (1 - c_1)x_1 + (1 - c_n)x_2 = 1. $$

Then $(\tau_{i+1})^2 = 1$ for all $i$ and the polyhedron $\mathbb{H}^4_\mathbb{R} \cap P$ endowed with the face-pairing isometries $\tau_i$ (identifying every codimension 1 face $B_i$ with itself) satisfies the conditions of Poincaré’s polyhedron theorem.

Proof. The hyperplanes $H_i$ and $H_{i+1}$ are orthogonal along $C_i$ because $\langle p_i, p_{i+1} \rangle = g_1 = 0$. As the reflection $\tau_i$ is given by the rule $\tau_i : v \mapsto v + 2\langle v, p_i \rangle p_i$, the equality $(\tau_{i+1})^2 = 1$ follows straightforwardly from $\langle p_i, p_{i+1} \rangle = 0$ and $\langle p_i, p_i \rangle = \langle p_{i+1}, p_{i+1} \rangle = 1$.

Since $\tau_i$ sends the interior of $P$ into the exterior of $P$ and every geometric cycle of codimension 2 faces of $P$ has length 4 and, therefore, total angle $2\pi$, the conditions of PPT are satisfied (see [AGr1, Theorem 3.2, p. 303] and [AGr1, Proposition 2.1, p. 300]) $\blacksquare$.

3.5. Lemma. Let $U := \mathbb{R}p_0 + \mathbb{R}p_1$ and $W := \mathbb{R}p_0 + \mathbb{R}(p_1 - p_{n-1})$. Then $C_0$ and $M_0$ correspond respectively to the $\mathbb{R}$-linear subspaces $U^\perp$ and $W^\perp$. The isometries $\sigma$ and $r\sigma$ are given by the rules

$$ v \mapsto v + 2\langle v, p_0 \rangle p_0 + \langle v, p_1 - p_{n-1} \rangle (p_1 - p_{n-1}), $$

$$ r\sigma v = rv + 2\langle v, p_0 \rangle p_1 + \langle v, p_1 - p_{n-1} \rangle (p_2 - p_0). $$

The $\mathbb{R}$-linear subspace $U$ is $r\sigma$-stable and $\begin{bmatrix} 0 & 1 \\ -1 & 2g_1 \end{bmatrix}$ is the matrix of $r\sigma|_U$ in the basis $p_0, p_1$. The point $f_0 := (1 - g_1)b + \langle b, p_0 \rangle (p_0 + p_1) \in \mathbb{H}^4_\mathbb{R} \cap C_0$ is a fixed point of $r\sigma$.

Proof. By definition, $C_0$ corresponds to $U^\perp$. In other words, $C_0$ corresponds to $p_0^\perp \cap (p_1 + g_1 p_0)^\perp$ and, similarly, $C_{n-1}$ corresponds to $p_0^\perp \cap (p_{n-1} + g_1 p_0)^\perp$, where $p_1 + g_1 p_0, p_{n-1} + g_1 p_0 \in p_0^\perp$. Since

$$ \langle p_1 + g_1 p_0, p_1 + g_1 p_0 \rangle = \langle p_1 + g_1 p_0, p_1 \rangle = -1 + g_1^2 < 0, $$

$$ \langle p_{n-1} + g_1 p_0, p_{n-1} + g_1 p_0 \rangle = \langle p_{n-1} + g_1 p_0, p_{n-1} \rangle = -1 + g_1^2 < 0, $$

$$ \langle p_1 + g_1 p_0, p_{n-1} + g_1 p_0 \rangle = \langle p_1 + g_1 p_0, p_{n-1} \rangle = g_2 + g_1^2 > 0, $$

the middle point of $\Gamma$ equals $m_0 := (p_1 + g_1 p_0) + (p_{n-1} + g_1 p_0) = p_1 + p_{n-1} + 2g_1 p_0$ and $M_0$ corresponds to $p_0^\perp \cap ((p_1 + g_1 p_0) - (p_{n-1} + g_1 p_0))^\perp$.

Clearly, $p_0$ and $p_1 - p_{n-1}$ are orthogonal. Hence, the rule (3.6) acting as $v \mapsto v$ for any $v \in W^\perp$ and as $v \mapsto -v$ for any $v \in W$ in view of $\langle p_1 - p_{n-1}, p_1 - p_{n-1} \rangle = -2g_2 - 2$ defines the isometry $\sigma$. The formula (3.7) is now immediate. It implies the equalities $r\sigma p_0 = -p_1$ and $r\sigma p_1 = p_0 + 2g_1 p_1$ providing the indicated matrix.

Taking $rb = b$ into account, we see that $\langle b, p_i \rangle$ is independent of $i$, hence, $\langle f_0, p_0 \rangle = \langle f_0, p_1 \rangle = 0$. Therefore, $g_1 < 1$ implies $(f_0, f_0) = (1 - g_1)(f_0, b) = (1 - g_1)^2 + 2(1 - g_1)(b, p_0)^2 > 0$, i.e., $f_0 \in \mathbb{H}^4_\mathbb{R} \cap C_0$. Finally,

$$ r\sigma f_0 = r f_0 - \frac{(f_0, p_{n-1})}{g_2 + 1}(p_2 - p_0) = (1 - g_1)b + \langle b, p_0 \rangle(p_1 + p_2) - \ldots $$
\[ -(1 - g_1)(b, p_0) + \langle b, p_0 \rangle(g_1 + g_2)/(p_2 - p_0) = (1 - g_1)b + \langle b, p_0 \rangle(p_0 + p_1) = f_0 \]

3.8. Lemma. The isometry \( r\sigma|_{C_0} \) of \( C_0 \) is a rotation by \( a \) about \( f_0 \), where

\[ \cos a = \frac{(1 - c_1^2)c_m x_1 + c_1(1 - c_m^2)x_2}{(1 - c_1^2)x_1 + (1 - c_m^2)x_2} . \]

Proof. The isometry \( r\sigma \) preserves orientation. By Lemma 3.5, the isometry \( r\sigma|_{U} \) preserves orientation. Consequently, the isometry \( r\sigma|_{C_0} \) also preserves orientation. By Lemma 3.5, it has to be a rotation by some angle \( a \) about \( f_0 \). Since \( \text{tr}(r\sigma|_{U}) = 2g_1 \) by Lemma 3.5 and

\[ \text{tr}(r\sigma) = \text{tr} r + 2g_1 + \frac{\langle p_2 - p_0, p_1 - p_1 \rangle}{g_2 + 1} = 1 + 2c_1 + 2c_m + 2g_1 + \frac{g_1 - g_3}{g_2 + 1} \]

by (3.7), we obtain

\[ \cos a = c_1 + c_m + \frac{g_1 - g_3}{2(g_2 + 1)} = c_1 + c_m + \frac{(c_3 - c_1)x_1 + (c_3m - c_m)x_2}{2(1 - c_2)x_1 + 2(1 - c_2m)x_2} . \]

Taking \( c_2 = 2c_1^2 - 1 \), \( c_3 = 4c_1^3 - 3c_1 \), \( c_{2m} = 2c_{m}^2 - 1 \), and \( c_{3m} = 4c_m^3 - 3c_m \) into account, we get

\[ \cos a = c_1 + c_m + \frac{(c_1^3 - c_1)x_1 + (c_m^3 - c_m)x_2}{(1 - c_1^2)x_1 + (1 - c_m^2)x_2} = \frac{(1 - c_1^2)c_m x_1 + c_1(1 - c_m^2)x_2}{(1 - c_1^2)x_1 + (1 - c_m^2)x_2} . \]

3.9. Theorem. Suppose that the solution of the system

\[
\begin{cases}
(1 - c_1)x_1 + (1 - c_m)x_2 = 1 \\
(1 - c_1^2)(c_k - c_m)x_1 = (1 - c_m^2)(c_1 - c_k)x_2
\end{cases}
\]

satisfies the inequalities \( 2 < (1 - c_i)x_1 + (1 - c_{mi})x_2 \) for all \( 2 \leq i \leq \frac{n}{2} \), where \( k, m, n \in \mathbb{N} \), \( n \) is even, \( 1 < k < m < \frac{n}{2} \), and \( c_i := \cos \frac{2\pi i}{n} \). Then there exists a disc bundle \( M \to S \) over a closed connected orientable surface \( S \) admitting a complete real hyperbolic geometry such that \( |\epsilon M/\chi S| = \frac{4m - 4k}{n - 4} \).

Proof. First, we observe that the solution clearly satisfies the inequalities \( 0 < x_1, 0 < x_2 \), and \( (1 - c_1)x_1 + (1 - c_m)x_2 < 2 \). The inequality \( 1 < x_1 + x_2 \) follows from the inequality \( 2 < (1 - c_i)x_1 + (1 - c_{mi})x_2 \) with \( i = \frac{n}{2} \) because \( c_i = -1 \) and \( -c_{mi} < 1 \). In other words, we get a point \((x_1, x_2) \in R\) in the region \( R \).

Let \( p(t) := (x_1(t), x_2(t)) \), \( t \in [0, 1] \), be a linearly parameterized path in \( R \) that joins a point in the segment \( S \subset R \) (see Lemma 3.1) with the point \( p(1) = (x_1, x_2) \). The function \( a(t) \) is continuous and, by Lemma 3.8, the function \( \cos a(t) \) has a form \( \cos a(t) = \frac{a_1t + a_2}{a_3t + a_4} \) for some \( a_1, a_2, a_3, a_4 \in \mathbb{R} \).

By Lemma 3.8, \( \cos a(0) = \frac{(1 - c_1^2)c_m x_1(0)}{(1 - c_1^2)x_1(0)} = c_m \) and \( \cos a(1) = \frac{(1 - c_1^2)c_m x_1 + c_1(1 - c_m^2)x_2}{(1 - c_1^2)x_1 + (1 - c_m^2)x_2} = c_k \) due to the second equation of the system. Hence, the function \( \cos a(t) \) is not constant and is therefore monotonic. Consequently, the function \( a(t) \) is monotonic.

It was understood in Section 2 that the polyhedron \( P(t) \) is fibred if \( r\sigma|_{C_0} \) is a periodic isometry whose order divides \( n \), i.e., if \( \cos a(t) = c_j \) for some \( j \in \mathbb{Z} \). Since \( a(t) \) is monotonic, \( \cos a(0) = c_m \), and \( \cos a(1) = c_k \), we conclude that \( eP(1) = m - k \).

By Lemma 3.3, the polyhedron \( P(1) \) satisfies the conditions of PPT due to the first equation of the system. It remains to observe that the Euler characteristic of the corresponding orbifold \( S \) equals \( \chi S = \frac{n}{4} - \frac{n}{2} + 1 = -\frac{n - 4}{4} \).
3.10. Calculation. Taking $(k, m, n) := (2, 5, 24)$ or $(k, m, n) := (3, 6, 24)$ or $(k, m, n) := (5, 11, 44))$, one can check the $11$ (or $11$ or $21$) inequalities of Theorem 3.9 thus arriving at $|eM/\chi S| = \frac{2}{3}$. ■

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Departamento de Matemática, ICMC, Universidade de São Paulo, Caixa Postal 668, 13560-970–São Carlos–SP, Brasil

E-mail address: sasha@icmc.usp.br

Departamento de Matemática, ICMC, Universidade de São Paulo, Caixa Postal 668, 13560-970–São Carlos–SP, Brasil

E-mail address: philipy.chioveto2@usp.br