Matriz de rigidez y vector de carga de una viga de Timoshenko de dos capas.
Stiffness matrix and loading vector of a two-layer Timoshenko composite beam

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Deseo dedicar este trabajo a mi amada familia...
Agradecimientos

I would like to thank all those people who in one way or another helped the development of this work. A special thanks to my mentor and thesis director José Darío Aristizabal Ochoa, who started me in this great job. To my wife, mother, father and friends who supported me.
Resumen

Este trabajo presenta un resumen de los resultados obtenidos de la investigación realizada durante los estudios de doctorado. Inicialmente la propuesta del trabajo de grado consistía en la obtención de la "Matriz de rigidez y vector de carga de una viga de Timoshenko de dos capas" (ver Capítulo 5), sin embargo se ha adjuntado a este documento otros capítulos que se encuentran íntimamente relacionados y que fueron también fruto del trabajo de investigación.

Los capítulos 1 y 2 presentan la formulación teórica y la verificación con ejemplos, respectivamente, de la matriz de rigidez y el vector de carga de una viga pretensada incluyendo los efectos de largo plazo. El capítulo 3 presenta el análisis de segundo orden de una viga columna sobre fundación elástica con deflexión inicial y conexiones semirrígidas. Los capítulos 4, 5 y 6 presentan el análisis de una viga de Timoshenko de dos capas. En el capítulo 4 se presenta la formulación para un sólo elemento, en el capítulo 5 se presenta la derivación de la matriz de rigidez y se hace la verificación con aplicaciones al diseño de vigas mixtas de acero y concreto. Finalmente en el capítulo 6 se usa la formulación desarrollada en el capítulo 5 para realizar el análisis de nudos adhesivos.

Los capítulos 3 al 6 cuentan con el identificador único y permanente para las publicaciones electrónicas (DOI) en el encabezado de cada capítulo para una fácil referencia.

This paper presents a summary of the results obtained from the research carried out during the doctoral studies.

Initially, the proposal of the degree work consisted of obtaining the "Stiffness matrix and loading vector of a two-layer Timoshenko beam" (see Chapter 5 and 6), however it has been attached to this document other chapters that are closely related and that were also the result of the research work of these years.

Chapters 1 and 2 present the theoretical formulation and verification with examples, respectively, of the stiffness matrix and load vector of a prestressed beam including long-term effects. Chapter 3 presents the second order analysis of a column beam on elastic foundation with initial deflection and semi-rigid connections. Chapters 4, 5 and 6 present the analysis of a two-layer...
Tymoshenko beam. In chapter 4 the formulation for a single element is presented, in chapter 5 the bypass of the stiffness matrix is presented and verification is made with applications to the design of mixed steel and concrete beams. Finally in chapter 6 the formulation developed in chapter 5 is used to perform the analysis of adhesive joints.

Chapters 3 through 6 have the unique and permanent Digital Object Identifier (DOI) in the heading of each chapter for easy reference.
Contents

List of figures xviii

List of tables xix

1 Stiffness matrix and loading vector of a prestressed concrete beam including long
term effects: i) Theory 1
1.1 Abstract .......................................................... 1
1.2 Introduction ....................................................... 1
1.3 Proposed model and equilibrium equations ......................... 3
  1.3.1 Strain Compatibility between concrete and reinforcing steels ..... 3
  1.3.2 Constitutive laws of each material ............................ 4
  1.3.3 Equilibrium at the beam section level ......................... 5
1.4 Profile of the prestressed steel and losses in the initial tension forces ..... 7
1.5 Governing differential equations and their solutions ................. 9
  1.5.1 Solution of the differential equation of axial deformations ....... 9
  1.5.2 Solution of governing differential equation of flexural deformations . 11
1.6 Stiffness matrix and loading vector in local coordinates ............. 14
1.7 Stiffness matrix and loading vector in global ........................ 14
1.8 Net axial force, shear and bending moment diagrams .................. 15
1.9 Stress Diagrams ...................................................... 15
1.10 Summary and conclusions ........................................... 15
  1.10.1 Apendice I ..................................................... 16
  1.10.2 Apendice II: Notation ......................................... 18

2 Stiffness matrix and loading vector of a prestressed concrete beam including long
term effects: ii) verification and examples. 21
2.1 Abstract: ................................................................. 21
2.2 Introduction ........................................................ 21
2.3 Verification and examples and comprehensive examples ........... 22
  2.3.1 Example 1-. Analysis of a two-span prestressed beam ........... 22
  2.3.2 EXAMPLE 2-. Four-span post-tensioned concrete girder ....... 28
  2.3.3 EXAMPLE 3-.Two-span prestressed concrete beam ............... 32
2.4 Summary and conclusions ........................................... 38

3 Second-order analysis of a beam-column on elastic foundation partially restrained
  axially with initial deflections and semirigid connections. 39
  3.1 Abstract: ............................................................. 39
  3.2 Introduction ........................................................ 40
  3.3 Structural model ................................................... 41
    3.3.1 Governing equations ......................................... 42
    3.3.2 Compatibility Conditions (Displacements and Rotations at Ends A and B) 46
    3.3.3 Second-order Axial Stiffness ................................ 47
  3.4 Verification and comprehensive examples .......................... 48
  3.5 Summary and conclusions ......................................... 59
  3.6 APPENDIX .......................................................... 59
    3.6.1 APPENDIX I .................................................... 59
    3.6.2 APPENDIX II .................................................. 60
    3.6.3 APPENDIX III ................................................ 61

4 Elastic analysis of composite beams and beams retrofitted with frp laminates with
  generalized end conditions. 63
  4.1 Abstract .......................................................... 63
  4.2 Introduction ........................................................ 64
  4.3 Structural model ................................................... 65
  4.4 Governing equations ................................................. 65
    4.4.1 Translational and Rotational Equilibrium ....................... 66
    4.4.2 Conditions of compatibility and constitutive laws ............ 66
    4.4.3 Boundary Conditions .......................................... 72
  4.5 Comprehensive examples ............................................ 73
    4.5.1 Example 1: Simply supported concrete-steel composite beam .... 73
    4.5.2 Example 2: Simply supported steel-concrete composite beam .... 76
# Contents

4.5.3 Example 3: Simple supported R/C rectangular beam retrofitted with CFRP laminate .......................... 79
4.6 Conclusions .................................................................................. 80

5 Stiffness matrix and loading vector of a two-layer Timoshenko beam. 83
  5.1 Abstract: ...................................................................................... 83
  5.2 Introduction ................................................................................. 84
  5.3 Structural model .......................................................................... 85
  5.4 Governing equations .................................................................... 85
  5.5 Translational and Rotational Equilibrium. ....................... 86
  5.6 Compatibility conditions and constitutive laws of all materials. .. 87
  5.7 Proposed solution. ....................................................................... 88
  5.8 Index notation. ............................................................................. 91
  5.9 Forces acting along the two layers. .............................................. 92
  5.10 Displacements along the two layers. ........................................... 93
  5.11 Stiffness Matrix. ....................................................................... 95
  5.12 Comprehensive examples and verification. .................... 97
     5.12.1 Example 1. Simply supported concrete-steel composite beam ... 97
     5.12.2 Example 2. Two-span continuous steel-concrete composite beam . 102
  5.13 Summary and conclusions ......................................................... 105
  5.14 Acknowledgements ................................................................. 105
  5.15 Appendix I. List of matrices ..................................................... 105

6 A novel linear matrix method to analyze adhesive joints 109
  6.1 Abstract: ...................................................................................... 109
  6.2 Introduction ................................................................................. 110
  6.3 Structural model. ......................................................................... 112
     6.3.1 Governing equations. ............................................................. 112
     6.3.2 Stiffness Matrix and Loading Vector. ................................. 113
  6.4 COMPREHENSIVE EXAMPLES AND VERIFICATION. .......... 115
     6.4.1 EXAMPLE 1. Adhesive stresses of four unbalanced steel-aluminum joints. ................................................................. 115
     6.4.2 EXAMPLE 2. Adhesive stresses of four unbalanced steel-aluminum joints. ................................................................. 119
  6.5 SUMMARY AND CONCLUSIONS ............................................. 123
6.6 NOMENCLATURE ........................................... 123

References .................................................. 125
List of figures

1.1 Symmetric cross-section: Strain and Stress distributions, and linear elastic materials .................................................. 4
1.2 Assumed parabolic profile of the prestressed steel tendon ............... 8
1.3 The 6-DOF and corresponding local coordinates of the prestressed beam. 9
1.4 Variables used in the prestressed cable parabolic profile. ............... 18

2.1 Example 1: Two-span prestressed concrete beam with parabolic profile of steel cable along each span (after T. Y. Lin [51], Example 11.2, p. 400). ....... 23
2.2 Two-span rectangular beam with 9 Degrees-of-freedom .................... 23
2.3 EXAMPLE 1: Axial Force Diagram ........................................... 26
2.4 EXAMPLE 1: Shear Force Diagram ............................................. 26
2.5 EXAMPLE 1: Bending Moment Diagram ...................................... 27
2.6 EXAMPLE 1: Cross section top and bottom stresses (σt and σb) along the two-span beam. ...................................................... 27
2.7 Four span posttensioned frame. ................................................. 28
2.8 EXAMPLE 2: Degrees of Freedom, numbering of 12- beam segments, and cable profile along the prestressed four-span girder (dimensions are in mm). .... 28
2.9 EXAMPLE 2: Axial Force Distribution along the four-span girder ......... 32
2.10 EXAMPLE 2: Shear Force Distribution along the four-span girder ........ 33
2.11 EXAMPLE 2: Bending Moment Diagram along the four-span girder ...... 33
2.12 EXAMPLE 2: Cross section top and bottom stresses (σt, σb) along the four-span girder. ......................................................... 34
2.13 Modelo Estructural viga pretensada. ......................................... 34
2.14 EXAMPLE 3: Axial Force Distribution along the two-span beam. ......... 36
2.15 EXAMPLE 3: Shear diagram along the two-span beam. ................... 36
2.16 EXAMPLE 3: Bending Moment diagram along the two-span beam. ....... 37
2.17 EXAMPLE 3: Cross section top and bottom stresses ($\sigma_t$, $\sigma_b$) along the two-span beam. ................................................................. 37

3.1 Structural model of Beam-Column under arbitrary transverse loading. .... 42

3.2 Free Body Diagram of Differential Element. ........................................... 43

3.3 Sign convention for the bending moments, shears, deflections, and rotations at the ends. ............................................................... 46

3.4 Mid-span vertical deflection of a perfectly hinged-hinged beam subject to a uniformly distributed load. Curves (i), (ii), (iii), (iv) and (v) correspond to $m=$ 1, 1/2, ¼, 9.2 and 14.1, respectively. .................................................. 49

3.5 Mid-span vertical deflection of a perfectly hinged-hinged beam subject to a concentrated load at $x=L/2$. Curves (i), (ii), (iii), (iv) and (v) correspond to $m=$ 1, 1/2, ¼, 9.2 and 14.1 respectively. .................................................... 50

3.6 Mid-span vertical deflection of a perfectly hinged-hinged beam under a concentrated load at $x=L/3$. Curves (i), (ii), (iii), (iv) and (v) correspond to $m=$ 1, 1/2, ¼, 9.2 and 14.1 respectively. .................................................... 50

3.7 Mid-span vertical deflection of a perfectly hinged-hinged beam subject to a uniformly distributed load $q$ and $\rho_a=\rho_b=0$. .................. 52

3.8 Critical axial stiffness $S_{\Delta,cr}$-versus-$m$ for the distributed load $q$: A) both ends hinged ($\rho_a=\rho_b=0$); B) both ends clamped ($\rho_a=\rho_b=1$) .......... 53

3.9 Critical axial stiffness $S_{\Delta,cr}$-versus-$u$ for a hinged-hinged beam ($\rho_a=\rho_b=0$) under distributed load $q$. .................................................. 54

3.10 Axial Load $P$-vs-Det(A) ................................................................. 54

3.11 $P$-vs-lateral deflection $\delta_t$ for $u=$ -6, -4, -2, 0, 2, 4, and 6 ............... 56

3.12 (a) $(y/d_1)$-vs-$u$; (b) $P$-vs-$(y/d_1)$; and (c) $P$-vs-$u$. ....................... 57

3.13 Variation of $y/d_1$-vs-$u$ for the post-buckling behavior of modes 1-to-5 .... 58

4.1 Differential element and free body diagram of both components and connecting interface. ................................................................. 65

4.2 Applied Transverse Loads along the beam and Forces and Moments at ends $x=0$ and $x=L$. ................................................................. 70

4.3 Concrete-steel composite beam of Example 1. .................................. 74

4.4 Transverse Load on a concrete-steel composite beam. ....................... 74

4.5 Shear connection relationships. .......................................................... 75
List of figures

4.6 Slip diagrams for different models, loads and rigidities (theoretical and experimental results) ........................................ 76
4.7 Simply Supported Steel-Concrete Composite Beam ........................................ 77
4.8 Normal and shear stresses along the composite beam ........................................ 78
4.9 Simple supported R/C beam retrofitted with a CFRP laminate ........................................ 79
4.10 Normal and shear stresses along a beam retrofitted with a FRP laminate .......................... 80

5.1 Sign convention for the bending moments, shears, deflections, and rotations at the ends .............................................................. 86
5.2 Degrees of freedom of the two-layer beam ................................................................. 92
5.3 Applied transverse loads along the two-layer beam and forces and moments at ends $x = 0$ and $L$ ................................................................. 94
5.4 Geometrical and mechanical properties, structural model and degrees of freedom of composite Steel-Concrete beam ........................................ 98
5.5 Shear connection relationship (after Ollgaard, 1971) ...................................................... 99
5.6 Example 1: a) Stresses along the interface, b) axial forces, c) shear forces and d) bending moments along the layers 1 and 2 ................................................ 100
5.7 Example 1: Slip diagrams for two different loads (Theoretical-vs-experimental results) .......................................................... 101
5.8 Example 1: Load-deflection curve (Theoretical-vs-experimental results) ......................... 101
5.9 Example 2: Geometrical and loading conditions of beam CTB6 ................................. 102
5.10 Example 2: Structural model and degrees of freedom for composite beam ........................ 103
5.11 Example 2: Midspan Deflection of beam CTB6 (Theoretical-vs-experimental results) .................................................. 104
5.12 Example 2: a) Stresses along the interface, b) axial forces, c) shear forces and d) bending moments along layers 1 and 2 ........................................... 104

6.1 Structural model and free body diagrams ................................................................. 112
6.2 Degrees of freedom of the system ................................................................. 114
6.3 Applied transverse loads along the beam and forces and moments at ends $x = 0$ and $x = L$ .............................................................................................................................................. 115
6.4 Example 1: Unbalanced adhesively bonded steel-aluminum joints and degrees of freedom of each joint system .................................................. 116
6.5 EXAMPLE 1: Normalized stresses along the interface for stiffened plate joint, case (a) .............. 118
6.6 EXAMPLE 1: Normalized stresses along the interface for stiffened plate joint, case (b) ................................. 118
6.7 EXAMPLE 1: Normalized stresses along the interface for a single-lap joint (c) without and (c’) with transverse load. ................................. 119
6.8 EXAMPLE 2: Geometry and degrees of freedom of the single-lap joint tested by Tsai and Morton (1995) ........................................ 120
6.9 EXAMPLE 2: (a) Normal and shear strain distributions and (b) longitudinal strain distribution for the laminated composite single-lap joint. ................................. 121
6.10 EXAMPLE 2: Longitudinal, normal and shear (a) strain and (b) stresses distributions ................................. 122
List of tables

2.1  Cable Eccentricities (in mm) ......................................................... 23
2.2  Cable end and mid-span eccentricities of the prestressed steel cables in each
     beam element (mm) ...................................................................... 29
2.3  Vertical Deflections and reactions in global coordinates ....................... 31
Chapter 1

Stiffness matrix and loading vector of a prestressed concrete beam including long term effects: i) Theory

by Mauricio Areiza-Hurtado, J. Darío Aristizábal-Ochoa.

1.1 Abstract

The stiffness matrix and load vector of a prismatic prestressed-concrete beam with symmetrical cross section about its major axis and subject to transverse static loads along its span including long term effects and prestress losses are presented. The proposed formulation is based on equilibrium conditions, strain compatibility, and constitutive laws of all materials involved. It also includes thermal strains and deformations of both steel and concrete, as well as the long-term effects caused by creep and shrinkage of the concrete and relaxation of the prestressed steel. In the development of the proposed formulation, the equivalent transversal load method is shown as a corollary. Three comparative and verification examples are presented in a companion paper that shows the accuracy and simplicity of the proposed method and corresponding equations.

1.2 Introduction

Prestressed concrete beam structures are currently analyzed using either the basic method, the pressure-line method, or the equivalent transverse load method. The benefits and difficulties
of each of these three methods are well known in the technical literature [15, 51, 40]. For statically determinate structures, the application of these three methods presents the same level of difficulty. On the other hand, for statically indeterminate structures, the rotations and displacements of the ends of each member are evaluated first using the virtual work method in order to determine the bending moment, shear and axial forces diagrams along the span of each element using the flexibility method [39]. On the other hand, in the structural analysis of indeterminate structures the redundant degrees of freedom are first released so that the structure as a whole becomes statically determinate, then compatibility conditions are applied at the locations of the released degrees of freedom to determine the magnitude of the redundant forces and moments. Superposition of all moments including those caused by the tensioning process (primary moments) along with the moments caused by the redundant forces (secondary moments) results in the moments diagrams. Finally, once the moment diagram for each beam has been found, the method of the pressure line can be used to find the stresses across in the beam section along its span. Another method used in the analysis of statically indeterminate prestressed structures is the equivalent load method [39, 2]. In this method the tensioning forces are replaced by equivalent transverse loads along each member and it is commonly used along with the moment distribution method (i.e., the Cross method) to determine the moment diagrams of each member. Losses in the tensioning forces of the prestressed cables along each member must be taken into account in the analysis and design of prestressed concrete members due to the fact that they can become significant in their actual behavior. Generally, the losses caused by friction between the cables and the duct and those caused by curvature of the cable affect the behavior and load capacity of prestressed members. In the technical literature, it is common to assume that the total loss is the sum of these two components. The losses which are time-dependent, such as those due to creep and shrinkage of concrete and relaxation of prestressing steel, can be considered in the proposed matrix method simply by modifying the modulus of elasticity of the materials involved. The combined effects of losses caused by friction, shrinkage and temperature in statically indeterminate structures such as continuous beams and frames have been generally treated in the technical literature using a compact formulation. This is the main reason why the work presented herein deals with the stiffness matrix and load vector (i.e., fixed end moments and forces) of a prestressed beam element. It is assumed that the profile of the prestressed steel cable can be defined by second-order polynomials or parabolas along each member span. The proposed stiffness matrix and load vector can be used together in a classical manner in the analysis of two-dimensional indeterminate prestressed concrete structures. The proposed method is based on the basic laws of compatibility of deformations, constitutive laws...
of materials and static equilibrium to find the global stiffness matrix and load vector of complex framed structures. Unlike the flexibility method mentioned above, when the proposed method is used, it is not necessary to find the primary- and secondary-moment diagrams caused by the prestressed steel cables. Similar to the method of the equivalent transversal load, the moment diagrams obtained are directly those produced by the tensioning of the cables. However, it is possible to include the combined effects of losses by friction, shrinkage of the concrete, and thermal expansion of all materials in a rational and efficient manner as a routine in the matrix software. The load vector presented herein includes a set of equivalent loads of the equivalent transverse load method, that is, uniform or concentrated transverse loads along the beam span and the loads and moments at both ends of the beam.

1.3 Proposed model and equilibrium equations

The variations of the curvature and the axial deformation of the centroidal axis of a prestressed beam with the imposed loads can be obtained by applying equilibrium, compatibility of deformations and constitutive laws of the materials along the geometric characteristics of the beam cross section [15].

1.3.1 Strain Compatibility between concrete and reinforcing steels

As shown in Fig. 1.1, it is assumed that the beam cross sections remain plane, therefore the strains of the concrete, steel bars, and prestressed steel (i.e. $\varepsilon_c$, $\varepsilon_s$ and $\varepsilon_p$, respectively) can be expressed in terms of the beam curvature $\phi$ and the concrete strain at the centroid of the cross section as follow:

$$\varepsilon_c = \varepsilon_{cen} - \phi y$$  \hspace{1cm} (1.1)

$$\varepsilon_s = \varepsilon_{cen} - \phi y$$  \hspace{1cm} (1.2)

$$\varepsilon_p = \varepsilon_{cen} - \phi y + \Delta_{ep}$$  \hspace{1cm} (1.3)
1.3.2 Constitutive laws of each material

The concrete, steel rebars and prestressed cables are assumed to be linear elastic as indicated by Eqs. 1.4-1.6 and Fig. 1.1.

**Concrete**

\[ f_c = E_c \varepsilon_{cf}; \quad \varepsilon_{cf} = \varepsilon_c - \varepsilon_{co} \]  
\[ (1.4) \]

**Steel bars**

\[ f_s = E_s \varepsilon_{sf}; \quad \varepsilon_{sf} = \varepsilon_s - \varepsilon_{so} \]  
\[ (1.5) \]

**Prestressed steel**

\[ f_p = E_p \varepsilon_{pf}; \quad \varepsilon_{pf} = \varepsilon_p - \varepsilon_{po} \]  
\[ (1.6) \]
1.3 Proposed model and equilibrium equations

The effects of creep on the concrete elastic modulus $E_c$ are based on experimental data [15] according to the following expression:

$$E_{c,\text{eff}} = \frac{E_{ci}}{1 + \phi(t, t_i)}$$

where:

$$\phi(t, t_i) = 3.5 k_c k_f \left( 1.58 - \frac{H}{120} \right) t_i^{-0.118} \left( \frac{t-t_i}{10 + (t-t_i)^{0.6}} \right)$$

$$k_f = \frac{1}{0.67 + (f^c_{c}/9000)} \text{ psi}$$

$$k_f = \frac{1}{0.67 + (f^c_{c}/62)} \text{ MPa}$$

$k_c$: volume-to-surface ratio factor and $H$ is the relative humidity in percent. [15]

The effect of relaxation of the prestressed steel on its elastic modulus $E_p$ is assumed to be as follows, [15]:

$$E_{p,\text{eff}} = \frac{f_p}{f_{pi}} E_p$$

where:

$$\frac{f_p}{f_{pi}} = 1 - \frac{\log(t)}{45} \left( \frac{f_{pi}}{f_{py}} - 0.55 \right)$$

1.3.3 Equilibrium at the beam section level

The equilibrium of horizontal forces and bending moments at the beam section level is presented. The axial force $N_o$ and the moment of decompression $M_o$ are defined below. Note that the subscript $s$ used in this formulation refers to the conventional steel rebars, but it could also be used for any type of linear elastic reinforcement such as bars or sheets made of fiber reinforced polymers (GFRP, CFRP), considering using the micromechanics theory and the general theory of lamination for the calculation of its mechanical properties (Modules and resistances) [28, 11]; which will depend on the fiber volume fraction, orientation angle, material and position of each sheet within the laminate, among others.

Longitudinal equilibrium

Using Eqs. 1.1-1.6, the net axial force $N$ and the corresponding axial strain $\varepsilon_{cen}$ at the centroid of the beam can be found as follows:

$$N = \int_{A_c} f_c dA_c + \int_{A_s} f_s dA_s + \int_{A_p} f_p dA_p = E_c \varepsilon_{cen} A_{tr} + N_o \quad (1.7)$$
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects:

i) Theory

\[ \varepsilon_{cen} = \frac{du}{dx} = \frac{N - N_o}{E_c A_{tr}} \]  \hspace{1cm} (1.8)

where:

\[ A_{tr} = \int_{A_c} dA_c + \int_{A_s} E_s dA_s + \int_{A_p} E_p dA_p. \]

\[ N_o = \int_{A_p} E_p \Delta_{ep} dA_p - \int_{A_c} E_c \varepsilon_{co} dA_c - \int_{A_s} E_s \varepsilon_{so} dA_s - \int_{A_p} E_p \varepsilon_{po} dA_p. \]

\( A_{tr} \) and \( N_o \) are generally referred in the technical literature \[15\] as the transformed area of the beam cross-section and the decompression axial force, respectively.

Rotational equilibrium

Likewise, using Eqs. 1.1 - 1.6, the bending moment \( M \) and the curvature \( \phi \) of the beam are:

\[ -M = \int_{A_c} f_c y dA_c + \int_{A_s} f_s y dA_s + \int_{A_p} f_p y dA_p = -E_c \phi I_{tr} - M_o \] \hspace{1cm} (1.9)

\[ \phi = \frac{d^2 y}{dx^2} = \frac{M - M_o}{E_c I_{tr}} \] \hspace{1cm} (1.10)

where:

\[ I_{tr} = \int_{A_c} y^2 dA_c + \int_{A_s} \frac{E_s}{E_c} y^2 dA_s + \int_{A_p} \frac{E_p}{E_c} y^2 dA_p \]

\[ M_o = -\int_{A_p} E_p \Delta_{ep} y dA_p + \int_{A_c} E_c \varepsilon_{co} y dA_c + \int_{A_s} E_s \varepsilon_{so} y dA_s + \int_{A_p} E_p \varepsilon_{po} y dA_p \]

\( I_{tr} \) and \( M_o \) are generally referred in the technical literature \[15\] as the second moment of inertia of the beam transformed cross-section and moment of decompression, respectively. Eqs. (1.9) and (1.10) relate the axial strain and the curvature of the beam in terms of the applied axial load \( N \) and bending moment \( M \) and along the beam longitudinal centroidal axis. The solution
of these two equations allows to establish the behavior of the beam at the sectional level. To determine the behavior of the beam, both at the sectional level and for any section along its span, the following equations of equilibrium of the differential element must be satisfied:

\[ \frac{dN}{dx} = 0 \quad (1.11) \]

\[ \frac{dV}{dx} = -q \quad (1.12) \]

\[ \frac{dM}{dx} = V \quad (1.13) \]

### 1.4 Profile of the prestressed steel and losses in the initial tension forces

It is assumed that the eccentricity of the prestressed steel tendon (i.e., its profile) measured with respect to the neutral axis of the transformed section is assumed to be parabolic as follows:

\[ e(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad (1.14) \]

The coefficients \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) in Eq. (1.8) can be determined by the eccentricities at both ends and at \( x = \lambda L \) \((0 < \lambda < 1)\) of the beam as shown in Fig. 1.2. Appendix I presents useful relationships for the determination of cable eccentricities used in multi-span beams. According to Fig. 1.2:

Evaluating Eq.(1.14) at the ends points and at an intermediate point, we obtain: \( e(0) = \alpha_1 = e_0; \ e(\lambda L) = \alpha_1 + \alpha_2 (\lambda L) + \alpha_3 (\lambda L)^2 \) and \( e(L) = \alpha_1 + \alpha_2 (L) + \alpha_3 (L)^2 \).

Therefore:
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects:

**i) Theory**

Fig. 1.2 Assumed parabolic profile of the prestressed steel tendon

\[
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} = \frac{1}{L^2 \lambda - L^2 \lambda^2} \begin{bmatrix}
L^2 \lambda - L^2 \lambda^2 & 0 & 0 \\
L^2 \lambda^2 - L^2 & L^2 - L^2 \lambda^2 & 0 \\
L - L \lambda & -L & L \lambda
\end{bmatrix} \begin{bmatrix}
e_0 \\
e_m \\
e_L
\end{bmatrix}
\]

The losses in the initial tension forces along the prestressed tendon caused by the combined effects of friction and curvature [15] along the beam are generally calculated using the following expression:

\[P(x) = P_0 \exp(-\mu \alpha + kx)\]

where:

- \(P_0\): Axial force in the steel tendon at \(x = 0\).
- \(\mu\): Friction coefficient (0.2 ≤ \(\mu\) ≤ 0.5). See Table 2-2 of Ref. [15].

\[\alpha(x) = e'(x) - e'(0) = \alpha_2 + 2 \alpha_3 x - \alpha_2 = 2 \alpha_3 x\]

angle made by the tangent of the steel cable profile at \(x\) with respect to that the left support of the beam (see Fig. 1.2)

- \(k\): Wobble coefficient per unit length of steel tendon. See Table 2-2 of Ref. [15].
Therefore:

\[ P(x) = P_0 \exp^{fx} \]  

(1.15)

where:

\[ f = -(2\mu \alpha_3 + k) \]

### 1.5 Governing differential equations and their solutions

Fig. 1.3 shows the 6-DOF of the prestressed beam and the local coordinates \((x, y, z)\). The governing differential equations for the axial degrees of freedom (1 and 4), transverse and rotational degrees of freedom (2, 3, 5 and 6) at the ends of the beam, their respective solutions and corresponding forces and bending moments are presented and discussed in the next two sections.

![Fig. 1.3 The 6-DOF and corresponding local coordinates of the prestressed beam.](image)

**1.5.1 Solution of the differential equation of axial deformations**

Assuming that \(\varepsilon_{co}, \varepsilon_{so}, y \varepsilon_{po}\) are zero, the decompression axial force \(N_o\) becomes:  

\[ N_o = \int_A p \Delta_{ep} dA_p = E_p \Delta_{ep} \int_A dA_p = P(x) = P_0 \exp^{fx} \]  

and substituting this expression into Eq. (1.8), the axial strain along the beam span becomes:
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects:

i) Theory

\[
\frac{du}{dx} = \frac{N - P_0 \exp^{fx}}{E_cA_{tr}} = \frac{N_{net}}{E_cA_{tr}} \tag{1.16}
\]

Where the beam net axial force is: \(N_{net} = N - P_0 \exp^{fx}\)

Deriving Eq. (1.16) with respect to \(x\), using Eq. (1.11) and making \(A_{tr} = A, I_{tr} = I\) and \(E_c = E\):

\[
\frac{d^2u}{dx^2} = -\frac{\exp^{fx}fP_0}{EA} \tag{1.17}
\]

Eq. (1.17) is a second-order differential equation that governs the elastic axial behavior of a beam subject to an axial force from a prestressed cable along its span including the cable force losses caused by friction and wobble effects. The solution to Eq. (1.17) is as follows:

\[
u = d_1 + d_2x - \frac{\exp^{fx}P_0}{EAf} \tag{1.18}\]

Substituting (1.18) into Eq. (1.16):

\[
N = EAd_2 \tag{1.19}\]

Using Eqs. (1.18) and (1.19), the boundary conditions at \(x = 0\) and \(x = L\) become:

\[
N(0) = N_a = -EAd_2, \quad N(L) = N_b = EAd_2
\]

\[
u(0) = u_a = d_1 - \frac{P_0}{EAf}, \quad u(L) = u_b = d_1 + d_2L - \frac{\exp^{fx}P_0}{EAf}
\]

These four end conditions can be expressed in matrix form as shown by Eqs. (1.20) and (1.21):

\[
\begin{bmatrix}
N_a \\
N_a
\end{bmatrix} =
\begin{bmatrix}
0 & -EA \\
0 & EA
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} \tag{1.20}
\]
1.5 Governing differential equations and their solutions

\[
\begin{bmatrix}
    u_a \\
    u_b
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    1 & L
\end{bmatrix} \begin{bmatrix}
    d_1 \\
    d_2
\end{bmatrix} - \frac{P_0}{EA} \begin{bmatrix}
    1 \\
    \exp^{fL}
\end{bmatrix}
\]

(1.21)

Combining Eqs. (1.20) and (1.21), Eq. (1.22) can be obtained:

\[
\begin{bmatrix}
    N_a \\
    N_a
\end{bmatrix} = \frac{EA}{L} \begin{bmatrix}
    1 & -1 \\
    -1 & 1
\end{bmatrix} \begin{bmatrix}
    u_a \\
    u_b
\end{bmatrix} + P_0 \begin{bmatrix}
    \frac{1}{fL}(1 - \exp^{fL}) \\
    -\frac{1}{fL}(1 - \exp^{fL})
\end{bmatrix}
\]

(1.22)

1.5.2 Solution of governing differential equation of flexural deformations

Assuming that \(\varepsilon_{co}, \varepsilon_{so}\) and \(\varepsilon_{po}\) are zero, the moment of decompression \(M_o\) becomes:

\[M_o = - \int A_p E_p \Delta_{ep} y dA_p = -E_p \Delta_{ep} A_p = -E_p \Delta_{ep} A_p e = -P(x) e(x)\]

Using Eq. (1.10):

\[EI \frac{d^2 y}{dx^2} = M - M_o = M_{net}\]

(1.23)

Differentiating Eq. (1.23) with respect to \(x\) once and twice and taking into account the static equilibrium of the differential element according to Eqs. (1.12) and (1.13), the following expressions for \(V_{net}\) and \(q_{net}\) can be obtained:

\[EI \frac{d^3 y}{dx^3} = V - \frac{dM_o}{dx} = V_{net}\]

(1.24)

\[EI \frac{d^4 y}{dx^4} = -q - \frac{d^2 M_o}{dx^2} = q_{net}\]

(1.25)

where:

\[M_o = -P_0 \exp^{fL} (\alpha_1 + \alpha_2 x + \alpha_3 x^2)\]
Eqs. (1.23) and (1.24) define the net bending moment and the net transverse shear force along the beam span. Whereas Eq. (1.25) shows that the net load on the structure is the sum of two components: the applied transverse load \( q \) and that related to the tensioning load of the cables. It is interesting to note that Eqs. (1.16), (1.23)-(1.25) when used in the analysis of prestressed concrete beams gives identical results to those obtained by the equivalent transverse load method proposed by T.Y. Lin [38]. Finally, Eq. (1.25) can be expressed as follows:

\[
EI \frac{d^4 y}{dx^4} = -q + P_0 \frac{\exp f x}{EI} \left( (f^2 \alpha_1 + 2f \alpha_2 + 2 \alpha_3) + (f^2 \alpha_2 + 4f \alpha_3) x + f^2 \alpha_3 x^2 \right) \quad (1.26)
\]

Eq. (1.26) is the governing differential equation of the transverse deflection \( y \) of a R/C beam of symmetrical cross section (Fig. 1.1) subject to uniform transversal load \( q \) as well as to a prestressed force applied by bonded steel cables with a parabolic profile including the prestressed force losses along its span. Note that the assumed cable profile can be used to model horizontal straight cables, inclined straight cables and parabolic cables. Eq. (1.26) is a linear non-homogeneous 4th-order differential equation with constant coefficients whose full solution is as follows:

\[
y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 - \frac{q x^4}{24 E I} + \frac{\exp f x P_0}{E I f^4} \left( (f^2 \alpha_1 - 2f \alpha_2 + 6 \alpha_3) + (f^2 \alpha_2 - 4f \alpha_3) x + f^2 \alpha_3 x^2 \right) \quad (1.27)
\]

Using Eqs. (1.23), (1.24) and (1.27) the following expressions for the shear \( V \) and the bending moment \( M \) can be obtained: \( V = 6EIc_4 - qx \) and \( M = 2EIc_3 + 6EIc_4 x - \frac{qx^2}{2} \). Therefore, the shear force and bending moment at \( x = 0 \) and \( x = L \) become: \( V_a = 6EIc_4; \quad M_a = -2EIc_3; \quad V_b = -6EIc_4 + qL; \) and \( M_b = 2EIc_3 + 6EIc_4L - \frac{qL^2}{2} \). These values can be expressed in matrix form as follow:

\[
\begin{bmatrix}
V_a \\
M_a \\
V_b \\
M_b
\end{bmatrix} = EI
\begin{bmatrix}
0 & 0 & 6 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & -6 \\
0 & 0 & 2 & 6L
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
qL \\
-\frac{qL^2}{2}
\end{bmatrix}
\]
1.5 Governing differential equations and their solutions

Or:

\[
\{ M \} = [S] \{ c \} + \{ J \}
\]  
(1.28)

Using Eq. (1.27) the displacements and rotations at \( x = 0 \) and \( x = L \) become:

\[
y_a = c_1 + \frac{P_0}{EI} f^2 \alpha_1 - 2f \alpha_2 + 6\alpha_3
\]
\[
\theta_a = c_2 + \frac{P_0}{EI} f^2 \alpha_1 - f \alpha_2 + 2\alpha_3
\]
\[
y_b = c_1 + c_2 L + c_3 L^2 + c_4 L^3 - \frac{L^4 q}{24EI} + \frac{P_0}{EI} f^2 \alpha_1 + f (fL - 2) \alpha_2 + (6 - 4fL + f^2 L^2) \alpha_3
\]
\[
\theta_b = c_2 + 2c_3 L + 3c_4 L^2 - \frac{L^3 q}{6EI} + \frac{P_0}{EI} f^2 \alpha_1 + f (-1 + fL) \alpha_2 + (2 - 2fL + f^2 L^2) \alpha_3
\]

These displacements and rotations can be expressed in matrix form as follows:

\[
\begin{bmatrix}
y_a \\
\theta_a \\
y_b \\
\theta_b
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & L & L^2 & L^3 \\
0 & 1 & 2L & 3L^2
\end{bmatrix} \begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix} +
\begin{bmatrix}
\frac{P_0}{EI} f^2 \alpha_1 - 2f \alpha_2 + 6\alpha_3 \\
\frac{P_0}{EI} f^2 \alpha_1 - f \alpha_2 + 2\alpha_3 \\
- \frac{L^4 q}{24EI} + \frac{P_0}{EI} f^2 \alpha_1 + f (fL - 2) \alpha_2 + (6 - 4fL + f^2 L^2) \alpha_3 \\
- \frac{L^3 q}{6EI} + \frac{P_0}{EI} f^2 \alpha_1 + f (-1 + fL) \alpha_2 + (2 - 2fL + f^2 L^2) \alpha_3
\end{bmatrix}
\]

Or:

\[
\{ \delta \} = [R] \{ c \} + \{ F \}
\]  
(1.29)

Substituting Eq. (1.29) into Eq. (1.28), the following matrix equation is obtained:

\[
\{ M \} = [K_m] \{ \delta \} + \{ M_m \}
\]  
(1.30)

Where:

\[
[K_m] = [SR^{-1}], \{ M_m \} = \{ J \} - [K_m] \{ F \}
\]
1.6 Stiffness matrix and loading vector in local coordinates

Combining Eqs. (1.22) and (1.30) into a single matrix expression (see Appendix I) the following matrix equation can be obtained:

\[
\begin{bmatrix}
  P_a \\
  V_a \\
  M_a \\
  P_b \\
  V_b \\
  M_b 
\end{bmatrix}
= 
\begin{bmatrix}
  k_p & 0 & 0 & -k_p & 0 & 0 \\
  0 & k_{11} & k_{12} & 0 & k_{13} & k_{14} \\
  0 & k_{21} & k_{22} & 0 & k_{23} & k_{24} \\
  -k_p & 0 & 0 & k_p & 0 & 0 \\
  0 & k_{31} & k_{32} & 0 & k_{33} & k_{34} \\
  0 & k_{41} & k_{42} & 0 & k_{43} & k_{44}
\end{bmatrix}
\begin{bmatrix}
  x_a \\
  y_a \\
  \theta_a \\
  x_b \\
  y_b \\
  \theta_b
\end{bmatrix}
+ 
\begin{bmatrix}
  m_{ep1} \\
  m_{ep2} \\
  m_{ep3} \\
  m_{ep4} \\
  m_{ep5} \\
  m_{ep6}
\end{bmatrix}
\]

Using the local coordinate system of the beam, this last equation can be expressed in condensed form as follows:

\[
\{F\} = [\bar{K}] \{\bar{X}\} + \{\bar{F}_0\}
\]

(1.31)

Where the matrix \([\bar{K}]\) and the vector \(\{\bar{F}_0\}\) represent the stiffness matrix and the load vector (fixed end actions) respectively in local coordinates.

1.7 Stiffness matrix and loading vector in global

Eq. (1.31) can be transformed from the local to the global coordinate system by means of the appropriate transformation matrix as follows:

\[
\{F\} = [K] \{X\} + \{F_0\}
\]

(1.32)

Where:

\[
[K] = [T][K][T]^{-1}, \{F_0\} = [T]\{F_0\}
\]

\[
[T] = \begin{bmatrix}
  \lambda & 0 \\
  0 & \lambda
\end{bmatrix}, \lambda = \begin{bmatrix}
  c & -s \\
  s & c
\end{bmatrix}, c = \cos \theta, s = \sin \theta
\]
1.8 Net axial force, shear and bending moment diagrams

Once the integration constants of each element are known, the axial force, shear and net bending moment can be obtained from Eqs. (1.16), (1.23) and (1.24) as follow:

\[ N_{net} = EAd_2 - \exp^{fx}P_0 \]  \hspace{1cm} (1.33)

\[ V_{net} = 6EIc_4 - qx + \exp^{fx}P_0 (f \alpha_1 + (1 + fx) \alpha_2 + x(2 + fx) \alpha_3) \]  \hspace{1cm} (1.34)

\[ M_{net} = 2c_3EI + 6c_4EIx - \frac{q}{2}x^2 + \exp^{fx}P_0 (\alpha_1 + x\alpha_2 + x^2\alpha_3) \]  \hspace{1cm} (1.35)

1.9 Stress Diagrams

The stress diagrams of the top and bottom fibers of the beam cross-section can be determined from Eqs. (1.36) and (1.37):

\[ \sigma_b = \frac{N_{net}}{A} + \frac{M_{net}}{Z_b} \]  \hspace{1cm} (1.36)

\[ \sigma_t = \frac{N_{net}}{A} - \frac{M_{net}}{Z_t} \]  \hspace{1cm} (1.37)

Where: \( Z_t, Z_b \) are the top fiber and bottom fiber modules of the beam cross section, respectively.

1.10 Summary and conclusions

The stiffness matrix, the load vector and the transfer functions of a prestressed concrete beam with a tendon that follows a parabola of second order subject to a uniform transverse load \( q \) are presented in detail. The proposed method and corresponding equations can be used for the analysis indeterminate structures made of prestressed continuous beams with prestressed or post-
tensioned tendons with horizontal, inclined straight lines and parabolic profiles. The proposed method is based on the constitutive laws of the materials, compatibility of deformations and static equilibrium. The effects caused by force losses in the tendons, thermal deformation and shrinkage have been included in the proposed method. It is shown that as \( f \) tends to zero (i.e., when the effects of the force losses in the tendons are negligible) the stiffness matrix and the load vector correspond to that of a regular reinforced concrete beam while effects of concrete creep and relaxation of the tensioning steel are taken into account rationally by affecting their modulus of elasticity. Equations are presented to calculate the diagrams of the axial force, shear, net bending moment and the stresses in the extreme fibers of the beam cross section along its span. Three comprehensive examples are presented in detail in a companion paper that show the effectiveness and accuracy the proposed method and corresponding equations. It is concluded that the effects of force losses caused by friction and curvature along the prestressed tendons are of great importance in the analysis of prestressed structures and should be taken into account in their analysis and design. The proposed stiffness matrix and load vector can be incorporated into conventional matrix structural analysis software. The equivalent load method for prestressed and post-tensioned beam analysis proposed by T. Y Lin [5] has been demonstrated to be valid. Therefore, the effects of post-tensioning can be replaced by a set of external equivalent loads.

### 1.10.1 Apéndice I

**Stiffness Matrix of a prismatic prestressed beam:**

\[
K = \begin{bmatrix}
  k_p & 0 & 0 & k_p & 0 & 0 \\
  0 & k_{11} & k_{12} & 0 & -k_{11} & k_{12} \\
  0 & k_{12} & k_{22} & 0 & -k_{12} & k_{24} \\
  k_p & 0 & 0 & k_p & 0 & 0 \\
  0 & -k_{11} & -k_{12} & 0 & k_{11} & -k_{12} \\
  0 & k_{12} & k_{24} & 0 & -k_{12} & k_{22}
\end{bmatrix}
\]

where:

\[
k_{11} = \frac{12EI}{L^3}, \quad k_{12} = \frac{6EI}{L^2}, \quad k_{22} = \frac{4EI}{L}, \quad k_{24} = \frac{2EI}{L}, \quad k_p = \frac{EA}{L}
\]

**Loading Vector including force losses in a prismatic prestressed beam:**
1.10 Summary and conclusions

\[ M_{ep} = \frac{q}{L^2} \begin{pmatrix} 0 & 0 \\ 6L & L^2 \\ 0 & 0 \\ 6L & -L^2 \end{pmatrix} + \frac{P_0}{L^3} \begin{pmatrix} A_1 \\ -6L A_2 - 12A_3 + 12A_5 + 6L A_6 \\ -4L^2 A_2 - 6L A_3 + 6L A_5 + 2L^2 A_6 \\ 6L A_2 + 12A_3 + 12A_5 - 6L A_6 \\ -2L^2 A_2 - 6L A_3 + 6L A_5 - 4L^2 A_6 \end{pmatrix} \]

Where:

\[
A_1 = \frac{L^2}{f} (1 - \exp(f L)) \quad A_2 = \frac{(f^2 \alpha_2 - f \alpha_2 + 2 \alpha_3)}{f^3} \quad A_3 = \frac{(f^2 \alpha_1 - 2 f \alpha_2 + 6 \alpha_3)}{f^4} \quad A_4 = -\frac{L^2}{f} (1 - \exp(f L)) \\
A_5 = \frac{\exp(f L) (f^2 \alpha_1 + f (-2 + f L) \alpha_2 + (6 - 4 f L + f^2 L^2) \alpha_3)}{f^4} \quad A_6 = \frac{\exp(f L) (f^2 \alpha_1 + f (-1 + f L) \alpha_2 + (2 - 2 f L + f^2 L^2) \alpha_3)}{f^3}
\]

**Loading Vector excluding prestressed force losses of a prismatic prestressed beam:**

The fixed end moments and forces of a prestressed beam neglecting the losses of the applied prestressed force (i.e., \( f \to 0 \)) are as follow:

\[
M_{ep} = \begin{pmatrix}
-L_0 & L_0 \alpha_2 - L_0 \alpha_3 \\
L_0 q - P_0 \left( \alpha_2 + L \alpha_3 \right) & \frac{L_0 q}{L} + P_0 \left( \alpha_1 - \frac{L^2 \alpha_1}{6} \right) \\
L^2 q L_2 + P_0 \left( \alpha_1 - \frac{L^2 \alpha_1}{6} \right) & P_0 \\
L_0 q - P_0 \left( -\alpha_1 - L \alpha_2 - \frac{5L^2 \alpha_3}{6} \right) & -\frac{L^2 q L_2 + P_0 \left( -\alpha_1 - L \alpha_2 - \frac{5L^2 \alpha_3}{6} \right)}{L^2} \end{pmatrix}
\]

**Variables used in the parabolic profile of a prestressed cable**

Input data: : \( e_1, e_3, L_1 \) y \( L_2 \). (See Fig. 1.4)

Resultados: \( e_{1m} = \frac{(3e_1+e_3)L_1+4e_1 L_2}{4(L_1+L_2)} \), \( e_{2m} = \frac{(e_1+3e_3)L_2+4e_1 \alpha_1}{4(L_1+L_2)} \)

\( e_2 = \text{eccentricity at the inflection point of the cable profile} = e_2 = \frac{e_1 L_1 + e_3 L_2}{(L_1+L_2)^2} \),
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects:

i) Theory

1.10.2 Apéndice II: Notación

$L$: Bar length.

$\varepsilon_{cen}$: Strain along the beam centroidal axis.

$\phi$: Curvature along the beam centroidal axis.

$\Delta_{ep}$: Deformación de pretensado.

$c, s, p$: Sub-indexes referring to concrete, steel bars and prestressed steel, respectively.

$E_c, E_s, E_p$: Elastic moduli of concrete, steel rebars and prestressed steel, respectively.

$\varepsilon_c, \varepsilon_s, \varepsilon_p$: Net strains of concrete, steel rebars and prestressed steel, respectively.

$\varepsilon_{cf}, \varepsilon_{sf}, \varepsilon_{pf}$: Strains caused by the applied forces, respectively.

$\varepsilon_{co} = \varepsilon_{csh} + \varepsilon_{cth}$: Concrete Strains caused by shrinkage and temperature, respectively.

$\varepsilon_{so} = \varepsilon_{sth}$: Strains in the steel bars caused by temperature. $\varepsilon_{po} = \varepsilon_{pth}$: Strains in the prestressed steel caused by temperature.

$k_c$: Volume-to-surface ratio factor and $H$ is the relative humidity in percent, [15].

$A_{tr}$: Transformed area of the beam cross-section.

$N$: Centroidal axial force.

$N_o$: Decompression axial force.

$M$: Bending Moment.
$I_{tr}$: Second moment of inertia of the beam transformed cross-section.

$M_0$: Moment of decompression.

$V$: Shear Force.

$q$: Transversal distributed load.

$e$: Eccentricity respect to the neutral axes of the Steel tendon.

$\mu$: Friction coefficient.

$P_0$: Axial force in the steel tendon at $x = 0$.

$k$: Wobble coefficient per unit length of steel tendon.

$u$: Axial displacement.

$y$: Transversal displacement.

$\{F\}$: Global forces at the nodes.

$[K]$: Global stiffness matrix.

$\{X\}$: Global displacements vector.

$\sigma_b$: Stress at the top of the beam cross-section.

$\sigma_t$: Stress at the bottom of the beam cross-section.
Chapter 2

Stiffness matrix and loading vector of a prestressed concrete beam including long term effects: ii) verification and examples.

by Mauricio Areiza-Hurtado, J. Darío Aristizábal-Ochoa.

2.1 Abstract:

The stiffness matrix and load vector of a prismatic prestressed-concrete beam with symmetrical cross section about its major axis and subject to transverse loads along its span including long term effects and prestress losses are presented in a companion paper. The proposed formulation is based on equilibrium conditions, strain compatibility, and constitutive laws of all materials involved including the thermal strains and deformations of both steel and concrete, as well as the long-term effects caused by creep and shrinkage of the concrete and relaxation of the prestressed steel. Three comparative and verification examples are presented in this paper that shows the accuracy and simplicity of the proposed method.

2.2 Introduction

The structural analysis of indeterminate prestressed concrete beam structures using the classical stiffness matrix and load vector is presented in a companion paper. Superposition of all moments including those caused by the tensioning process (primary moments) along with the moments
caused by the redundant forces (secondary moments) and losses in the tensioning forces of the prestressed cables along each member are taken into account. It is shown that the losses caused by friction between the cables and the duct and those caused by curvature of the cable affect the behavior and load capacity of prestressed members. In the technical literature, it is common to assume that the total loss is the sum of these two components. The time-dependent losses, such as those due to creep and shrinkage of concrete and relaxation of prestressing steel, are also considered in the proposed matrix method simply by modifying the modulus of elasticity of the materials involved. The combined effects of losses caused by friction, shrinkage and temperature in statically indeterminate structures such as continuous beams and frames have been generally treated in the technical literature using a compact formulation. It is assumed that the profile of the prestressed steel cable can be defined by second-order polynomials or parabolas along each member span. The proposed stiffness matrix and load vector can be used together in a classical manner in the analysis of two-dimensional indeterminate prestressed concrete structures. The proposed method is based on the basic laws of compatibility of deformations, constitutive laws of materials and static equilibrium to find the global stiffness matrix and load vector of complex framed structures. Three comprehensive examples of indeterminate multi-span prestressed concrete structure are presented in detail that show the effectiveness and accuracy the proposed method and corresponding equations developed in the companion paper. The diagrams of the axial force, shear, net bending moment and the stresses in the extreme fibers of each beam cross section along a multi-span prestressed concrete structure are presented.

2.3 Verification and examples and comprehensive examples

2.3.1 Example 1-. Analysis of a two-span prestressed beam

A two-span rectangular beam 304.8 mm x 762 mm (12"x30") shown by Fig. 2.1 is prestressed with a parabolic cable with an initial load $P = 1423.4$ kN (320 kips). Assume that the cable is continuous at the intermediate support. Calculate the net shear, the net bending moment and top and bottom stress diagrams along the two-span beam when subjected to transverse load $DD + DL = 23.35$ N/mm (1.6 kips/ft). Compare the calculated results with those reported by T. Y. Lin, [51], example 11-2, page 400. Use $E=27579$ N/mm² (4000ksi).

Solution:

Fig. 2.2 shows the numbering of the nine degrees of freedom of the two-span continuous beam. The unrestrained degrees of freedom are the rotations at the supports (1, 2, 3) and the
2.3 Verification and examples and comprehensive examples

Fig. 2.1 Example 1: Two-span prestressed concrete beam with parabolic profile of steel cable along each span (after T. Y. Lin [51], Example 11.2, p. 400).

Table 2.1 Cable Eccentricities (in mm)

| Beam Element | $e_0$ | $e_m$ | $e_L$ |
|--------------|-------|-------|-------|
| 1            | 0     | -12   | 0     |
| 2            | 0     | -12   | 0     |

horizontal displacement of the central and right supports (4, 5). The degrees of freedom 6, 7, 8 and 9 are fully restrained.

Table 2.1 shows the eccentricities of the cable at the ends and center for each beam span. The relationships presented in Appendix I of the companion paper were used to calculate the coefficients of the second-order polynomial that defines the profile of each cable.

Therefore, the coefficients $\alpha_1$, $\alpha_2$ and $\alpha_3$ of the parabola in Eq. (1.14) presented in the companion paper are as follow:

\[
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3
\end{bmatrix} = 
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & \lambda L & \lambda^2 L^2 \\
  1 & L & L^2
\end{bmatrix}^{-1}
\begin{bmatrix}
  e_0 \\
  e_m \\
  e_L
\end{bmatrix} = 
\begin{bmatrix}
  0 \\
  -0.08 \\
  1.3 e - 4
\end{bmatrix}
\]
The stiffness matrix and load vector of beams 1 and 2 obtained using Eq. (1.32) presented in the companion paper are as follow:

\[
[K] = \begin{bmatrix}
2400 & 0 & 0 & -2400 & 0 & 0 \\
0 & 6 & 1800 & 0 & -6 & 1800 \\
0 & 1800 & 7.2e+5 & 0 & -1800 & 3.6e+005 \\
-2400 & 0 & 0 & 2400 & 0 & 0 \\
0 & -6 & -1800 & 0 & 6 & -1800 \\
0 & 1800 & 3.6e+005 & 0 & -1800 & 7.2e+5
\end{bmatrix}
\]

\[
\{M_{PE,1}\} = \begin{bmatrix}
-310.3 \\
39.8 \\
1471.7 \\
310.3 \\
40.2 \\
-1564.1
\end{bmatrix}, \quad \{M_{PE,2}\} = \begin{bmatrix}
-291.6 \\
39.9 \\
1623.7 \\
291.6 \\
40.1 \\
-1710.5
\end{bmatrix}
\]

Using the stiffness matrix and load vector of spans 1 and 2 just shown above, it is now possible to assemble the global 9x9 stiffness matrix and 9x1 load vector of the two-beam system by the traditional matrix stiffness method. Solving the system of equations, the rotations \(\theta_1, \theta_2\) and \(\theta_3\) horizontal displacements \(u_4\) and \(u_4\) were obtained:

\[
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
u_4 \\
u_5
\end{bmatrix} = \begin{bmatrix}
-1.961E - 03 \\
-1.658E - 04 \\
2.459E - 03 \\
-1.293E - 01 \\
-2.508E - 01
\end{bmatrix}
\]

And the support reactions corresponding to the degrees of freedom 6, 7, 8 and 9 are:

\[
\begin{bmatrix}
F_6 \\
F_7 \\
F_8 \\
F_9
\end{bmatrix} = \begin{bmatrix}
0 \\
36.018 \\
87.965 \\
36.018
\end{bmatrix}
\]
With these last two sets of values (nodal rotations, displacements and reaction forces), the constants of integration of each beam determined using Eqs. (1.21) and (1.29) presented in the companion paper are:

Beam 1:

\[
\begin{align*}
\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= \begin{pmatrix} -2.1505 \\ 0 \end{pmatrix}, \\
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} &= \begin{pmatrix} -2.036e+007 \\ 693.9 \\ 0 \\ 5.5583e-008 \end{pmatrix}
\end{align*}
\]

Beam 2:

\[
\begin{align*}
\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= \begin{pmatrix} -2.1505 \\ 0 \end{pmatrix}, \\
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} &= \begin{pmatrix} -1.9136e+007 \\ 652.19 \\ -1.1062e-005 \\ 6.7874e-008 \end{pmatrix}
\end{align*}
\]

Using Eqs. (1.33)-(1.35) presented in the companion paper the net axial force, the net shear force and net bending moment diagrams along the two-span beam including the combined effects of the applied transverse load and tensioning cable forces can be calculated. Figs. 2.3, 2.4 and 2.5 show these diagrams including and excluding the tensioning cable losses and compared to those calculated using Sap2000 computer program.

Fig. 2.6 shows the stress diagrams along the top and bottom fibers ($\sigma_t$ and $\sigma_b$) of the two-span beam determined using Eqs. (1.36) and (1.37) presented in the companion paper.

The results calculated using the proposed method are in good agreement with those reported by T. Y. Lin as well with those obtained using Sap2000. The results show that the diagrams of the shear force, bending moment and stresses in the extreme fibers are slightly affected by the tension force losses in the cables. However, the axial load diagrams show that tension force losses affect their behavior.
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects:

ii) verification and examples.

Fig. 2.3 EXAMPLE 1: Axial Force Diagram

Fig. 2.4 EXAMPLE 1: Shear Force Diagram
2.3 Verification and examples and comprehensive examples

Fig. 2.5 EXAMPLE 1: Bending Moment Diagram

Fig. 2.6 EXAMPLE 1: Cross section top and bottom stresses ($\sigma_t$ and $\sigma_b$) along the two-span beam.
2.3.2 EXAMPLE 2-. Four-span post-tensioned concrete girder

Consider the four-span bridge girder of rectangular section 1.524 m width and 1.98 m height (60 in by 78 in), shown in Fig. 2.7 is post-tensioned with tendons consisting of twenty 15 mm (0.6 in.) of \( f_{pu} = 1860 \text{ MPa (270 ksi)} \). The symmetrical tendons are simultaneously stressed to 0.75\( f_{pu} \), that is 3870 kN (871 kips) from the left end and then anchored. The beam is subject to a uniform transverse load \( q = 23.35 \text{ N/mm (1.6 kips/ft)} \). The Calculate: the shear stress and bending moment diagrams and the stresses on the extreme fibers. This example is presented by Collins and Mitchell [15], p. 51 along with the friction loss calculations. Assume \( E_c = 27579 \text{ N/mm}^2 (4,000 \text{ ksi}) \).

Solution:

Fig. 2.8 shows the profile of the post-tensioned cable. In order to use the proposed method, the continuous four-span girder is divided into 12 beam elements each with its own cable parabolic profile with the nodes located at the inflection points of the cable and at the simple supports A, B, C, D and E. The degrees-of-freedom along the girder have been numbered from left to right in the order horizontal \( x \), vertical \( y \), and rotational \( \theta \).
Table 2.2 Cable end and mid-span eccentricities of the prestressed steel cables in each beam element (mm)

| Element | $e_0$ | $e_m$ | $e_L$ |
|---------|-------|-------|-------|
| 1       | 0     | -571.5 | -762.0 |
| 2       | -762.0 | -481.3 | 360.2 |
| 3       | 360.2 | 547.4 | 609.6 |
| 4       | 609.6 | 543.3 | 344.2 |
| 5       | 344.2 | -762.0 | 344.2 |
| 6       | 344.2 | 543.3 | 609.6 |
| 7       | 609.6 | 543.3 | 344.2 |
| 8       | 344.2 | -762.0 | 344.2 |
| 9       | 344.2 | 543.3 | 609.6 |
| 10      | 609.6 | 547.4 | 360.2 |
| 11      | 360.2 | -481.3 | -762.0 |
| 12      | -762.0 | -571.5 | 0 |

Table 2.2 shows the eccentricity at the center and at the ends of each cable for each of the elements that make up the bridge girder. The relationships presented in Appendix I have been used for purpose.

The stiffness matrix and the loading vector of each beam element which are shown below are calculated in local coordinates (units in Newtons and mm) using the equations presented in Appendix I of the companion paper.

**Stiffness matrix of beam elements: 1, 2, 11 and 12:**

$$[K] = \begin{bmatrix}
6.07E+06 & 0 & 0 & -6.07E+06 & 0 & 0 \\
0 & 1.27E+05 & 8.69E+08 & 0 & -1.27E+05 & 8.69E+08 \\
0 & 8.69E+08 & 7.94E+12 & 0 & -8.69E+08 & 3.97E+12 \\
-6.07E+06 & 0 & 0 & 6.07E+06 & 0 & 0 \\
0 & -1.27E+05 & -8.69E+08 & 0 & 1.27E+05 & -8.69E+08 \\
0 & 8.69E+08 & 3.97E+12 & 0 & -8.69E+08 & 7.94E+12
\end{bmatrix}$$

**Stiffness matrix of beam elements 3 and 10:**
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects:

\[
[K] = \begin{bmatrix}
2.73E+07 & 0 & 0 & -2.73E+07 & 0 & 0 \\
0 & 1.15E+07 & 1.76E+10 & 0 & -1.15E+07 & 1.76E+10 \\
0 & 1.76E+10 & 3.57E+13 & 0 & -1.76E+10 & 1.79E+13 \\
-2.73E+07 & 0 & 0 & 2.73E+07 & 0 & 0 \\
0 & -1.15E+07 & -1.76E+10 & 0 & 1.15E+07 & -1.76E+10 \\
0 & 1.76E+10 & 1.79E+13 & 0 & -1.76E+10 & 3.57E+13
\end{bmatrix}
\]

Stiffness matrix of beam elements 4, 6, 7 and 9:

\[
[K] = \begin{bmatrix}
2.28E+07 & 0 & 0 & -2.28E+07 & 0 & 0 \\
0 & 6.68E+06 & 1.22E+10 & 0 & -6.68E+06 & 1.22E+10 \\
0 & 1.22E+10 & 2.98E+13 & 0 & -1.22E+10 & 1.49E+13 \\
-2.28E+07 & 0 & 0 & 2.28E+07 & 0 & 0 \\
0 & -6.68E+06 & -1.22E+10 & 0 & 6.68E+06 & -1.22E+10 \\
0 & 1.22E+10 & 1.49E+13 & 0 & -1.22E+10 & 2.98E+13
\end{bmatrix}
\]

Stiffness matrix of beam elements 5 and 8:

\[
[K] = \begin{bmatrix}
2.73E+06 & 0 & 0 & -2.73E+06 & 0 & 0 \\
0 & 11543 & 1.76E+08 & 0 & -11543 & 1.76E+08 \\
0 & 1.76E+08 & 3.57E+12 & 0 & -1.76E+08 & 1.79E+12 \\
-2.73E+06 & 0 & 0 & 2.73E+06 & 0 & 0 \\
0 & -11543 & -1.76E+08 & 0 & 11543 & -1.76E+08 \\
0 & 1.76E+08 & 1.79E+12 & 0 & -1.76E+08 & 3.57E+12
\end{bmatrix}
\]

Loading vectors (units in kNewtons and m):

\[
\{F_{0,1}\} = \begin{bmatrix} -3780.6 \\ 364.0 \\ -144.7 \\ 3780.6 \\ -43.7 \\ 2940.6 \end{bmatrix} , \quad \{F_{0,2}\} = \begin{bmatrix} -3580.4 \\ -137.7 \\ -3085.1 \\ 3580.4 \\ 458.0 \\ -1000.0 \end{bmatrix} , \quad \{F_{0,3}\} = \begin{bmatrix} -3408.0 \\ -221.2 \\ 1418.4 \\ 3408.0 \\ 292.4 \\ -2201.1 \end{bmatrix} , \quad \{F_{0,4}\} = \begin{bmatrix} -3282.4 \\ 297.3 \\ 2205.0 \\ 3282.4 \\ -211.9 \\ -1273.7 \end{bmatrix}
\]
2.3 Verification and examples and comprehensive examples

Table 2.3 Vertical Deflections and reactions in global coordinates

| Degrees of Freedom | Deflections (mm) and Rotations* (rad) | Reactions | Degrees of Freedom | Force (kN) |
|--------------------|---------------------------------------|-----------|--------------------|------------|
| $X_1$              | 0.5232                                | $F_{40}$  | -163.8             |
| $X_2$              | -0.10575                              | $F_{41}$  | 2.4353             |
| $X_3$              | 0.0037                                | $F_{42}$  | 18866.0            |
| $X_{10}$           | 0.2355                                | $F_{43}$  | -85.7              |
| $X_{11}$           | -0.0690                               | $F_{44}$  | -12.9              |
| $X_{12}$           | -0.0005                               | $F_{45}$  | 13653.0            |
| $X_{19}$           | -0.0352                               | $F_{46}$  | 22.9               |
| $X_{20}$           | -0.0690                               | $F_{47}$  | 17.1               |
| $X_{21}$           | -0.0001                               | $F_{48}$  | -2968.5            |

\[
\{ F_{0.5} \} = \begin{bmatrix} -3040.3 \\ 595.0 \\ 3040.3 \\ 357.0 \\ -630.2 \end{bmatrix}, \quad \{ F_{0.6} \} = \begin{bmatrix} -2813.0 \\ 2813.0 \\ 232.8 \\ -1837.5 \end{bmatrix}, \quad \{ F_{0.7} \} = \begin{bmatrix} -2712.9 \\ 2712.9 \\ -167.7 \\ -1057.2 \end{bmatrix}, \quad \{ F_{0.8} \} = \begin{bmatrix} -2512.7 \\ 2512.7 \\ 356.8 \\ -834.5 \end{bmatrix},
\]

\[
\{ F_{0.9} \} = \begin{bmatrix} -2324.9 \\ -114.4 \\ 948.5 \\ 2324.9 \\ 199.8 \\ -1523.2 \end{bmatrix}, \quad \{ F_{0.10} \} = \begin{bmatrix} -2239.3 \\ 233.4 \\ 2239.3 \\ -162.2 \\ -897.5 \end{bmatrix}, \quad \{ F_{0.11} \} = \begin{bmatrix} -2131.9 \\ 331.7 \\ 2131.9 \\ -11.4 \\ 1625.2 \end{bmatrix}, \quad \{ F_{0.12} \} = \begin{bmatrix} -2018.8 \\ 44.7 \\ 727.5 \\ 2020.0 \\ -126.0 \end{bmatrix},
\]

The stiffness matrix and loading vector of the entire structure are assembled in global coordinates. Then, the system of equations is solved for the nodal displacements and the reactions at the column supports. Once the displacements of each element in local coordinates are known, the integration constants are calculated using Eqs (1.21) and (1.29) presented in the
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects: ii) verification and examples.

companion paper. The axial force, shear force and the net bending moment diagrams shown Figs. 2.9 to 2.11 are determined using Eqs. (1.33 - 1.35) presented in the companion paper. Finally, the stress diagrams of the top and bottom fibers of the beam cross-section can be determined from Eqs. (1.36) and (1.37) also presented in the companion paper.

Fig. 2.9 EXAMPLE 2: Axial Force Distribution along the four-span girder

The results show that the shear forces, moments and stresses found using the proposed method are similar to those presented by Collins and Mitchell [15] on p. 51. In this example, the friction losses completely change the behavior of the element, particularly at the support areas, even with sign changes in the moments.

2.3.3 EXAMPLE 3-.Two-span prestressed concrete beam

The following example has been treated by Naaman [5] p. 589. Consider a continuous two-beam prestressed beam. The profile of the post-tensioned tendon is shown in Fig. 2.13. The post-tensioning force is 2,668.9 kN (600 kips). Assume $b = 914.4$ mm (36 in.), and $h = 1,219.2$ mm (48 in.). Calculate: diagrams of axial force, shear force, bending moment and stresses on the extreme fibers. Assume $E_c = 27,579$ N/mm2 (4,000 ksi).

Solution:
2.3 Verification and examples and comprehensive examples

Fig. 2.10 EXAMPLE 2: Shear Force Distribution along the four-span girder

Fig. 2.11 EXAMPLE 2: Bending Moment Diagram along the four-span girder
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects:

ii) verification and examples.

Fig. 2.12 EXAMPLE 2: Cross section top and bottom stresses \((\sigma_t, \sigma_b)\) along the four-span girder.

Similar to the previous examples, the structure is divided into several beam segments. Note that each beam segment is made with inclined straight-line cables. The numbering of the degrees of freedom (DOFs) is shown in Fig. 2.13. Notice that the restrained DOFs are numbered last (12, 13, 14, and 15).

Fig. 2.13 Modelo Estructural viga pretensada.

The stiffness matrix and the load vector of each beam element shown below are calculated in the local coordinates (see Appendix I in the companion paper). The global stiffness matrix
2.3 Verification and examples and comprehensive examples

and load vector of the entire structure is then assembled and the system of linear equations are solved using the classic methods of matrix analysis.

Stiffness matrix of beam elements 1-4 (units in Newtons and mm) are as follows:

\[
\begin{bmatrix}
2.52E + 06 & 0 & 0 & -2.52E + 06 & 0 & 0 \\
0 & 25218 & 1.54E + 08 & 0 & -25218 & 1.54E + 08 \\
0 & 1.54E + 08 & 1.25E + 12 & 0 & -1.54E + 08 & 6.25E + 11 \\
-2.52E + 06 & 0 & 0 & 2.52E + 06 & 0 & 0 \\
0 & -25218 & -1.54E + 08 & 0 & 25218 & -1.54E + 08 \\
0 & 1.54E + 08 & 6.25E + 11 & 0 & -1.54E + 08 & 1.25E + 12 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.68E + 06 & 0 & 0 & -1.68E + 06 & 0 & 0 \\
0 & 7472.1 & 6.83E + 07 & 0 & -7472.1 & 6.83E + 07 \\
0 & 6.83E + 07 & 8.33E + 11 & 0 & -6.83E + 07 & 4.17E + 11 \\
-1.68E + 06 & 0 & 0 & 1.68E + 06 & 0 & 0 \\
0 & -7472.1 & -6.83E + 07 & 0 & 7472.1 & -6.83E + 07 \\
0 & 6.83E + 07 & 4.17E + 11 & 0 & -6.83E + 07 & 8.33E + 11 \\
\end{bmatrix}
\]

Load vectors of beam elements 1-4 (units in kNewtons and m) are as follows:

\[
\begin{bmatrix}
-2637.2 \\
98.5 \\
401.9 \\
2637.2 \\
-98.5 \\
799.0 \\
\end{bmatrix},
\begin{bmatrix}
-2559.3 \\
-106.3 \\
-782.5 \\
2559.3 \\
106.3 \\
-1160.8 \\
\end{bmatrix},
\begin{bmatrix}
-2468.8 \\
103.2 \\
1137.8 \\
2468.8 \\
-103.2 \\
750.3 \\
\end{bmatrix},
\begin{bmatrix}
-2395.8 \\
-90.2 \\
-734.6 \\
2395.8 \\
90.2 \\
-365.1 \\
\end{bmatrix}
\]

Figs. 2.14-2.17 show the diagrams of axial force, shear force, bending moment and stresses along the beam top and bottom fibers of the prestressed beam.

This example shows that the calculated force losses have little effect on the behavior of the two-span beam when subject to post-tensioning load.
Stiffness matrix and loading vector of a prestressed concrete beam including long term effects:

ii) verification and examples.

Fig. 2.14 EXAMPLE 3: Axial Force Distribution along the two-span beam.

Fig. 2.15 EXAMPLE 3: Shear diagram along the two-span beam.
2.3 Verification and examples and comprehensive examples

Fig. 2.16 EXAMPLE 3: Bending Moment diagram along the two-span beam.

Fig. 2.17 EXAMPLE 3: Cross section top and bottom stresses ($\sigma_t$, $\sigma_b$) along the two-span beam.
2.4 Summary and conclusions

Three comprehensive examples of indeterminate multi-span prestressed concrete structure are presented in detail that show the effectiveness and accuracy the proposed method and corresponding equations developed in the companion paper. The diagrams of the axial force, shear, net bending moment and the stresses in the extreme fibers ($\sigma_t, \sigma_b$) of each beam cross section along a multi-span prestressed concrete structure are presented.

It is concluded that the effects of force losses caused by friction and curvature along the prestressed tendons are of great importance in the analysis of prestressed structures and must be taken into account in their analysis and design. The proposed stiffness matrix and load vector can be incorporated into conventional matrix structural analysis software. The equivalent load method for prestressed and post-tensioned beam analysis proposed by T. Y [38] has been demonstrated to be valid. Therefore, the effects of post-tensioning can be replaced by a set of external equivalent loads.
Chapter 3

Second-order analysis of a beam-column on elastic foundation partially restrained axially with initial deflections and semirigid connections.

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3.1 Abstract:

The second-order analysis of a prismatic beam-column on elastic foundation partially restrained axially with initial imperfections and semi-rigid end-connections subject to transverse load is presented. The simultaneous effects of bending and shear deformations, and concentric axial forces at both ends are also included. The transverse loads and the initial transverse deformations are modeled using Fourier series allowing great variety of cases of loads and imperfections. The proposed structural model is also capable of capturing the phenomenon of snap-through, snap-back (i.e, the buckling reversals), buckling modes and the corresponding elastic critical loads of an imperfect Timoshenko beam-column with initial transverse deformations and its post-buckling behavior.

KEYWORDS: beam-columns; elastic foundation; frames; imperfections; second-order analysis; stability; stress reversals.
3.2 Introduction

The elastic stability and second-order analyses of beams-columns with initial imperfections of sinusoidal shape have been studied by several researchers [19], [16], [41], [8] and [52] using classical methods of equilibrium, energy principles or the finite element method. In general, these methods involve the calculation of the buckling loads and the corresponding buckling shapes. The elastic stability of framed structures made of beam-columns subject to static axial loads are generally carried out using matrix methods and linear algebra, where the characteristic values and vectors are the critical loads and modes of buckling of the structure as a whole, respectively.

Beam-columns with initial lateral deformations with supports that restrain the transverse and longitudinal displacements can present the phenomena of snap-through, snap-back and deflection-reversals; result of the nonlinearity of the problem [53], [42], [13], [47] and [1]. It is common to present the results of these analyzes using the well-known Cross-Load-vs-Vertical displacement diagrams of some reference point (generally the midpoint in beams). These diagrams allow an explanation of the above-mentioned phenomena (snap-through, etc.) and in general a description of the path of the midpoint as a function of the transversal load applied [55]. New applications have been found by other researchers using this type of beams with initial deformations and restrained supports, such as in the fabrication of stiff elastic composite materials and in the vibration damping systems with negative stiffness phase and nonlinear bi-stable behavior [33] and [31].

A beam with initial transverse deformations when is axially restrained at its ends and subject to transverse loads, an axial load is induced from the horizontal reactions of the supports which can lead to lateral instability causing the phenomenon of snap-through [12]. Similarly, snap-through may happen in arch beams subject to compressive transverse loads. These second-order effects, as well as those caused by an elastic foundation, shear deformations and forces along the member, and semi-rigid connections and lateral bracings at the ends have been studied in detail by the senior author [19], [16], [41], [52], [42], [7] and [6].

The main objective of this paper is to present a general method for the second-order analysis of prismatic beam-columns with initial imperfections and discuss in detail using three comprehensive examples their buckling and post-buckling behavior caused by the loss of horizontal stiffness of the end supports. The formulation and the corresponding governing equations and solutions proposed herein include the combined effects of bending, shear and axial deformations, elastic foundation, semi-rigid connections with rotation, transverse and
axial displacements of the supports. The transverse load and the initial transverse deflections are modeled using Fourier series allowing to represent any applied transverse load and any initial transverse deflections (see Appendix II). In addition, the incorporation of semi-rigid connections at both supports allows the complete analysis of beam-columns with any support conditions. This paper is organized as follows: the first two parts present the proposed structural model, the corresponding governing equations, their solutions along the compatibility conditions (displacements and rotations at both ends), and the second-order axial stiffness. The third part presents the verification of the proposed solutions using three comprehensive examples. Example 1 presents the buckling reversals of a hinged-hinged beam with initial imperfections subject to symmetric transverse loads. The loss of lateral stability (snap-through) due to a loss of axial stiffness of the support is presented in Example 2. The critical loads, post-buckling behavior and modes of configuration of a hinged-hinged beam axially restrained with initial imperfections subject to a concentrated transverse load at L/3 are presented in Example 3.

3.3 Structural model

Consider the 2-D beam-column elastically connected at both ends A and B as shown by Fig. 1. The member AB consists of the beam-column itself A'B' connected to the ends A and B by elastic springs AA' and BB' (with flexural stiffness $\kappa_a$ and $\kappa_b$), and linear transverse springs (with shear stiffness $S_a$ and $S_b$), respectively. It is assumed that the beam-column A'B' is: 1) prismatic with cross sectional properties $A$ and $I$ (about the bending axis) and span $L$; 2) made of a linear elastic material homogeneous and isotropic material with elastic moduli $E$ and $G$; 3) uniformly supported on an elastic “Winkler” foundation with a ballast modulus $k_s$ along its span $L$; and 4) subjected to axial $P$ (tension or compression) applied at its ends and to transverse load $q(x)$ as shown by Fig.3.1.

Note that any transverse sways or rotations of the member ends are partially restrained by the translational springs ($S_a$ and $S_b$) and flexural springs ($\kappa_a$ and $\kappa_b$), respectively. The flexural connectors of $\kappa_a$ and $\kappa_b$, have units of force $\times$ distance/radian and vary between zero (perfectly hinged or “pinned”) to infinite (perfectly clamped or rigid). The relationships $R_a = \kappa_a/(EI/L)$ and $R_b = \kappa_b/(EI/L)$ are denoted as the stiffness indices of the end flexural connections. These indices vary from zero for perfectly hinged connections to infinite for “clamped” or perfectly rigid connections (where: $I$= moment of inertia of the beam-column cross-section; $L$= beam-column span; $y E$= elastic modulus). For convenience the following parameters used by Aristizabal-Ochoa [7], $ho_a = \frac{1}{1+3/R_a}$; and $\rho_b = \frac{1}{1+3/R_b}$ $\rho_a$ and $\rho_b$ are denoted as the fixity factors
Second-order analysis of a beam-column on elastic foundation partially restrained axially with initial deflections and semirigid connections.

![Structural model of Beam-Column under arbitrary transverse loading.](image)

Fig. 3.1 Structural model of Beam-Column under arbitrary transverse loading.

at the top and bottom ends of the member, respectively. These factors vary from zero for perfectly hinged connections to 1 for “clamped” or perfectly rigid connections.

### 3.3.1 Governing equations

The total deflection \( y \) of the beam-column is made of two components \( y_o \) and \( y_e \) (i.e., the initial deflection and that caused by the applied transverse loads), as follows:

\[
y = y_o + y_e \tag{3.1}
\]

The deflection caused by the applied transverse loads \( y_e \) is made of two components \( y_b \) and \( y_s \) caused by the bending moment and transverse shear along the member span, respectively.

\[
y_e = y_b + y_s \tag{3.2}
\]

The tangent to the center line at any point \( x \) along the beam-column span consists two components, \( \theta = dy_b/dx \) (caused by the bending moment), and \( \gamma = dy_s/dx \) (caused by the total transverse shear force as shown by [52], [8] and [6]).

Assuming small shear distortions and deflections as well as linear elastic conditions:

\[
tan \gamma \cong \gamma = \frac{\Delta y_s}{\Delta x} = \frac{Q}{A_sG} \tag{3.3}
\]

Taking into account that:

\[
\frac{d\theta}{dx} = -\frac{M}{EI} \tag{3.4}
\]
Applying the second derivative with respect to \( x \) to both sides of Eq. (3.2):

\[
\frac{d^2 y_e}{dx^2} = \frac{d\theta}{dx} + \frac{dy}{dx}
\]  

(3.5)

Applying equilibrium to the differential element shown in Fig. 3.2 and neglecting high-order differentials, Eqs. (3.6) and (3.7) can be obtained:

\[
\frac{dV}{dx} = k_s y - q
\]  

(3.6)

\[
\frac{dM}{dx} = V + P \frac{dy}{dx}
\]  

(3.7)

As shown by Hetenyi [32], on page 127 and using Fig. 3.2, it is obtained that \( Q = P \sin \theta + V \cos \theta \) as \( \theta \to 0 \), then

\[
Q = P\theta + V
\]  

(3.8)

Notice that the slope of the center line caused by shear includes the contribution of the transverse component of the axial load \( (dy_b/dx) \) as well as the contribution of \( V \) as shown by Eq. (3.8).

Fig. 3.2 Free Body Diagram of Differential Element.

Substituting expression (3.8) into Eq. (3.3) and deriving once and using Eqs. (3.4) and (3.6), Eq. (3.9) can be obtained:

\[
\frac{dy}{dx} = \frac{1}{A_s G} \left( -P \frac{M}{EI} + k_s y - q \right)
\]  

(3.9)
Substituting Eqs. (3.4) and (3.9) into Eq. (3.5), Eq. (3.10) can be obtained:

\[
\frac{d^2 y_e}{dx^2} = \frac{1}{A_s G} (k_s y - q) - \frac{M}{EI} \left( 1 + \frac{P}{A_s G} \right)
\]

(3.10)

Deriving Eq. (3.10) once with respect to \(x\) and taking into account Eq. (3.7), Eq. (3.11) can be obtained:

\[
\frac{d^3 y_e}{dx^3} = k_s \frac{dy}{dx} - \frac{1}{A_s G} \frac{dq}{dx} + \frac{M}{EI} \left( 1 + \frac{P}{A_s G} \right) \left( V + P \frac{dy}{dx} \right)
\]

(3.11)

Deriving Eq. (3.11) again with respect to \(x\) and taking into account Eq. (3.6), \(V\) is eliminated and Eq. (3.12) can be obtained.

\[
\frac{d^4 y_e}{dx^4} = -\frac{P}{EI} \left( 1 + \frac{P}{A_s G} \right) \frac{d^2 y}{dx^2} - k_s \frac{d^2 y}{dx^2} + q \frac{d^2 q}{dx^2}
\]

(3.12)

Substituting Eq. (3.1) into (3.12) and grouping common terms, Eq. (3.13) can be obtained.

\[
\frac{d^4 y_e}{dx^4} + 2 \alpha \frac{d^2 y_e}{dx^2} + \beta y_e = -2 \alpha \frac{d^2 y_o}{dx^2} - \beta y_o + \frac{\varphi}{\beta} q - \frac{1}{A_s G} \frac{d^2 q}{dx^2}
\]

(3.13)

where: \( \alpha = \frac{P}{2EI} \left( \frac{n P}{A_s G} + 1 \right) - \frac{n k_s}{2A_s G} \), \( \beta = \frac{k_s}{EI} \left( \frac{n P}{A_s G} + 1 \right) \); and \( \varphi = \frac{k_s}{\beta} \).

Eq. (3.13) is a fourth-order non-homogeneous linear differential equation with constant coefficients that governs the elastic behavior of a beam-column with initial imperfections, supported laterally on a uniform Winkler elastic foundation with a coefficient of ballast \( k_s \), and subject to an axial load \( P \) (tension or compression) at both ends. Notice that the governing Eq. (3.13) includes the coupling between shear and bending deformations along the beam-column. In addition, it captures particular cases of the governing differential equation of beams and beam-columns developed by other researchers as indicated next when \( y_o = 0 \). Similar differential equations that describe the second-order and post-buckling behavior of beam-columns have been presented elsewhere [8], [6], [32], [23], [9] and [42].

The arbitrary transverse load \( q(x) \) and the initial imperfection shape \( y_o \) shown in Fig. 3.1 can be expressed using Fourier complex series (for \( 0 < x < L \)) as: \( q(x) = \sum_{n=-\infty}^{\infty} A_n e^{i \omega_n x} \) and \( y_o(x) = \sum_{n=-\infty}^{\infty} B_n e^{i \omega_n x} \) with

\[
A_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \omega_n x} dx; \quad B_n = \frac{1}{2L} \int_{-L}^{L} g(x) e^{-i \omega_n x} dx; \quad \text{and} \quad \omega_n = \frac{n \pi}{L}
\]

The complete solution to Eq. (3.13) consists of two parts: the homogeneous and the particular solution. The homogeneous solution can be expressed using the Euler complex exponential
form, which captures the exact solution when $P$ is tension or compression (avoiding the problem of exchanging from trigonometric to hyperbolic function when $P$ varies from compression to tension) as shown by [32]. The particular solution is obtained using Fourier complex series, allowing the exact evaluation of the load vector for any initial imperfect shape and transverse loading applied to the beam-column.

The solution to (3.13) can be expressed as:

$$y_e = c_1 e^{m_1 x} + c_2 e^{-m_1 x} + c_3 e^{m_2 x} + c_4 e^{-m_2 x} + \sum_{n=-\infty}^{\infty} a_n e^{i\omega_n x} \tag{3.14}$$

where: $m_1 = \sqrt{-\alpha + \sqrt{\alpha^2 - \beta}}$; $m_2 = \sqrt{-\alpha - \sqrt{\alpha^2 - \beta}}$; and $a_n = \frac{B_n (2\alpha \omega_n^2 - \beta) + \left(\frac{\omega_n^2}{\alpha \omega_n^2} + \frac{1}{\beta}\right) \Lambda_n}{\omega_n^4 - 2\alpha \omega_n^2 + \beta}$ (see Appendix II)

Introducing Eq. (3.14) into Eq. (3.1):

$$y(x) = y_e + y_o = y_x = \sum_{i=1}^{4} c_i f_{i,x} + f_{5,x} \tag{3.15}$$

where: $f_{5,x} = \sum_{m=-\infty}^{\infty} D_n e^{i\omega_m x}$; and $D_n = a_n + B_n$

The $k$-derivative of $y_x$ is:

$$y_x^k = \sum_{i=1}^{4} c_i f_{i,x}^k + f_{5,x}^k \tag{3.16}$$

Note that the sub-index $x$ indicates the abscissa where the function is evaluated ($0 \leq x \leq L$), $f_i$ refers to the function $i$ with its corresponding constant $c_i$, and $k$ its derivative, i.e. $f_{i,x}^k = \frac{d^k f_i}{dx^k} = m_i^k e^{m_i x} \text{ and } f_{5,x}^k = \sum_{n=-\infty}^{\infty} \omega_n^k D_n e^{i\omega_n x}$.

Fig.3.3 shows the right-hand sign convention (+ directions) utilized in this publication for the end moments and shears ($M_a$, $M_b$, $V_a$, and $V_b$), as well as for the end rotations and displacements ($\theta_a$, $\theta_b$, $\Delta_a$, and $\Delta_b$).

Using Eqs. (3.10) and (3.11), $M$ and $V$ can be expressed as follows:

$$V = (P_s - P) \frac{dy}{dx} - \phi \frac{d^3 y}{dx^3} + \phi \frac{d^3 y_o}{dx^3} - \omega \frac{dq}{dx} \tag{3.17}$$

$$M = -\phi \frac{d^2 y}{dx^2} + \phi \frac{d^2 y_o}{dx^2} + P_s y - \omega q \tag{3.18}$$

where: $P_s = \frac{\phi_k}{\Lambda_s G}$; and $\omega = \frac{P_s}{k_s}$. 
Second-order analysis of a beam-column on elastic foundation partially restrained axially with initial deflections and semirigid connections.

Fig. 3.3 Sign convention for the bending moments, shears, deflections, and rotations at the ends.

3.3.2 Compatibility Conditions (Displacements and Rotations at Ends A and B)

Taking into account Eqs. (3.1) and (3.2), Eq. (3.19) can be obtained:

\[
\frac{dy_b}{dx} - \frac{dy_s}{dx} - \frac{dy_o}{dx} = 0 \quad (3.19)
\]

Substituting Eq. (3.8) into Eq (3.3) and taking into account Eqs. (3.15), (3.17) and (3.19), the following differential equation is obtained:

\[
\frac{dy_b}{dx} \left( 1 - \psi P_s \right) y_x' + \psi \phi_y'' + \left[ (\psi P - 1) y_o' - \psi \phi y_o'' + \psi \omega q_x' \right] = 0 \quad (3.20)
\]

where: \( \psi = \frac{1}{A_s G + p} \)

Applying compatibility to the ends connections \( a \) and \( b \), respectively:

\[
V_a = S_a y \bigg|_{x=0} \quad (3.21a)
\]

\[
V_b = -S_b y \bigg|_{x=L} \quad (3.21b)
\]

\[
M_a = -k_a \frac{dy_b}{dx} \bigg|_{x=0} \quad (3.21c)
\]

\[
M_b = k_b \frac{dy_b}{dx} \bigg|_{x=L} \quad (3.21d)
\]

Substituting Eqs. (3.16)-(3.18) and (3.20) into Eqs. (3.21):
3.3 Structural model

\[ \sum_{i=1}^{4} c_i [(P_s - P) f_i,0 - \varphi f_i,,, - S_a f_i,0] = S_a f_5,0 - (P_s - P) f_5,0 + \varphi f_5,,, - \varphi y_o,0 + \omega q_0 \] (3.22a)

\[ \sum_{i=1}^{4} c_i [(P_s - P) f_i,L - \varphi f_i,,, + S_b f_i,L] = -S_b f_5,L - (P_s - P) f_5,L + \varphi f_5,,, - \varphi y_o,L + \omega q_L \] (3.22b)

\[ \sum_{i=1}^{4} c_i \left[ P_s f_i,0 + k_a (1 - \psi P_s) f_i,0 - \varphi f_i,,, + k_a \psi \varphi f_i,,, \right] = \\
- P_s f_5,0 - k_a (1 - \psi P_s) f_5,0 + \varphi f_5,,, - k_a (\psi P - 1) y_o,0 - \varphi y_o,0 + k_a \psi \varphi y_o,0 \\
+ \omega q_0 - k_a \psi \omega q_0 - k_a \psi \varphi f_5,,, \] (3.22c)

\[ \sum_{i=1}^{4} c_i \left[ P_s f_i,L - k_b (1 - \psi P_s) f_i,L - \varphi f_i,,, - k_b \psi \varphi f_i,,, \right] = \\
- P_s f_5,L - k_b (1 - \psi P_s) f_5,L + \varphi f_5,,, + k_b \psi \varphi f_5,,, + k_b (\psi P - 1) y_o,L \\
- \varphi y_o,L - k_b \psi \varphi y_o,L + \omega q_L + k_b \psi \omega q_L \] (3.22d)

Eqs.(3.22) can be expressed in matrix form as \([A] \{C\} = \{b\}\), (See appendix I), therefore:

\[ \{C\} = [A]^{-1} \{b\} \] (3.23)

Eqs.(3.22) constitute a system of 4 equations with four unknowns whose solution is expressed in matrix form by Eq. (3.23). Introducing Eq. (3.23) into (3.16) it is possible to find the deflection along a beam-column with initial imperfection of any shape subject to a transverse load of any shape taking into account the effects of shear, elastic foundation, and semirigid connections at both ends.

3.3.3 Second-order Axial Stiffness

The total shortening of a column beam is made up of three components caused by the external transverse loads; the initial deflections; and the applied axial load as follow:

\[ \delta_t = \frac{1}{2} \int_0^L \left( \frac{dy}{dx} \right)^2 dx - \frac{1}{2} \int_0^L \left( \frac{dy_o}{dx} \right)^2 dx + \frac{PL}{AE} \frac{P}{S_A} \] (3.24)
The relationship between the axial load and the total axial deformation can be expressed using Eq. (3.25):

\[ P = S_1 \frac{EA}{L} \delta_t \]  

(3.25)

where: \( S_1 = \frac{1}{1 + \frac{4EA\omega}{S_4L^2}} \); and \( \Omega = \int_0^L \left( \frac{dy}{dx} \right)^2 - \int_0^L \left( \frac{dy_2}{dx} \right)^2 \)

Or

\[
\Omega = \sum_{i=1}^{4} \sum_{j=1}^{4} c_i c_j m_i m_j \int_0^L e^{(m_i + m_j)x} dx + 2i \sum_{i=1}^{4} \sum_{j=-\infty}^{\infty} c_i D_j m_i \omega_j \int_0^L e^{(m_i + i\omega) x} dx
\]

\[ - \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} D_i D_j \omega_i \omega_j \int_0^L e^{i(\omega_i + \omega_j) x} dx + \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} B_i B_j \omega_i \omega_j \int_0^L e^{i(\omega_i - \omega_j) x} dx \]

\( \Omega \) can be reduced to:

\[
\Omega = \sum_{i=1}^{4} \sum_{j=1}^{4} c_i c_j m_i m_j \eta (m_i, m_j) + 2i \sum_{i=1}^{4} \sum_{j=-\infty}^{\infty} c_i D_j m_i \omega_j \eta (m_i, i\omega_j)
\]

\[ - \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (D_i D_j - B_i B_j) \omega_i \omega_j \eta (i\omega_i, i\omega_j) \]

Where: \( \eta (a, b) = \begin{cases} 
\int_0^L e^{(a+b)x} dx = \frac{e^{(a+b)L} - 1}{a+b} & \text{when } a + b \neq 0 \\
\int_0^L dx = L & \text{when } a + b = 0 
\end{cases} \)

### 3.4 Verification and comprehensive examples

**EXAMPLE 1: Simple supported beam axially restrained under symmetrical transverse loads**

A beam with an initial imperfection given by \( y_0 = d_1 \sin (\pi x/L) \) with both ends perfectly hinged and axially restrained against horizontal displacements is subjected to the following transverse loads cases: (1) a uniformly distributed load along its entire span; and (2) a concentrated load at midspan. Determine the variation of the midspan deflection for each of the applied transverse loads.

**Solution.-** This problem has been previously solved and discussed by Timoshenko and Gere [52] and Smith-Pardo et al [42]. To find the zero crossing of the beam deflection with the applied transverse load \( y_0/d_1 \) versus \( u \) using the proposed formulation, it is necessary to determine the zero crossing of \( \delta_t \) against \( P \) for each value of \( u \), then the zero crossing of \( \delta_t \) against \( P \) curve are the axial loads that satisfy the condition that \( \delta_t = 0 \) in Eq. (3.24). Once these axial loads are known, the deflection of the element at its midpoint is determined for each configuration can be
found. For the calculation of $\delta_t$ the integration constants using Eq. (3.23) and then $y$ and $y_o$ can be expressed using Fourier series (see Appendix II).

To use the proposed formulation, the effects of shear deformation ($G \rightarrow \infty$) and of the elastic foundation ($k_s = 0$) must be neglected and fixity factors ($\rho_a = \rho_b = 0$) and lateral stiffnesses at the ends of the member are assumed $S_a = S_b = \infty$.

Figs.3.4, 3.5 and 3.6 show the curves applied transversal load versus the deflection $t$ for different values of $m = \frac{4I}{Ad_1}$ and for normalized loads as defined by Timoshenko and Gere [8] with $u = \frac{5}{384} \frac{qL^4}{EId_1}$ for distributed load and $u = \frac{1}{48} \frac{QL^3}{EId_1}$ for concentrated load. Fig.3.4 corresponds to a distributed load $q$, Fig.3.5 corresponds to a concentrated load $Q$ at $x = L/2$. Fig.3.6 corresponds to a concentrated load at $x = L/3$.

Fig. 3.4 Mid-span vertical deflection of a perfectly hinged-hinged beam subject to a uniformly distributed load. Curves (i), (ii), (iii), (iv) and (v) correspond to $m = 1, 1/2, 1/4, 9.2$ and $14.1$, respectively.

Results presented by Timoshenko and Gere [52] as well as those by Smith-Pardo and Aristizabal-Ochoa [42] compare well with those calculated using the proposed formulation. Appendix II shows the Fourier series expansion for a trapezoidal load. Note that the proposed Fourier series for the applied load allow the modeling of uniform, triangular, and trapezoidal distributed transverse loads, as well as concentrated forces and moments as shown previously by Areiza-Hurtado et al [6].
Second-order analysis of a beam-column on elastic foundation partially restrained axially with initial deflections and semirigid connections.

Fig. 3.5 Mid-span vertical deflection of a perfectly hinged-hinged beam subject to a concentrated load at $x = L/2$. Curves (i), (ii), (iii), (iv) and (v) correspond to $m = 1, 1/2, 1/4, 9.2$ and $14.1$ respectively.

Fig. 3.6 Mid-span vertical deflection of a perfectly hinged-hinged beam under a concentrated load at $x = L/3$. Curves (i), (ii), (iii), (iv) and (v) correspond to $m = 1, 1/2, 1/4, 9.2$ and $14.1$ respectively.
EXAMPLE 2: Effects of the Axial Restraint Stiffness

Consider the case of distributed load of Example 1 again and study the influence of the stiffness of the axial restraint $S_\Delta = 0, 0.1, 1$ and $\infty$ for the following two cases of rotational end conditions: A) hinged-hinged ends ($\rho_a = \rho_b = 0$) and B) clamped-clamped ends ($\rho_a = \rho_b = 1$)

Solution-.

**Case A:** Figs. 3.7(a)-(e) show the vertical displacement at mid-span of the beam against the transverse load, appropriately normalized ($y/d_1$-vs-$u$) for $\rho_a = \rho_b = 0$ for different values of the parameters $m = 1, 1/2, 1/4, 1/9.1$ and $1/14.2$ (studied by [42]) and $S_\Delta = 0, 0.1, 1$ and $\infty$ (cases i, ii, iii, iv), respectively. Figs. 3.7(a)-(e) also how a linear load-deformation behavior when $S_\Delta = 0$, corresponding to the case of a simply supported beam. Note that, as expected, the slope of this line is the same for any value of $m$.

Fig. 3.7(a) shows that as the axial stiffness $S_\Delta$ increases, the beam deforms into well-known curved shapes. Note that for the particular case of $m= 1$ even when $S_\Delta = \infty$ it is not possible for the curve to have negative rigidity, making unfeasible the snap-through phenomenon to take place. Fig. 3.7(b)-(e) show that there is a critical value of axial stiffness ($S_{\Delta,cr}$) at which the load-displacement curves have negative stiffness, with three possible configuration values for the same transverse load $q$ at which the snap-through phenomenon may occur. Note that one of the three possible configurations corresponds to the negative stiffening zone making the beam unstable. The smaller the parameter $m$, the smaller the axial stiffness $S_{\Delta,cr}$ required to achieve negative stiffness in the load-displacement diagram and vice versa.

Fig. 3.8 shows that $S_{\Delta,cr}$ increases exponentially for cases A and B as the parameter $m$ increases. The curve also shows a vertical asymptote at $m = 0.955$, indicating that for values of $m \geq 0.955$ there are not zones of negative stiffness, and consequently the phenomenon of snap through is not feasible. However, when $m < 0.955$ the snap-through phenomenon occurs requiring that $S_\Delta = \infty$ for $m = 0.955$. That is, to avoid the snap-through phenomenon $m$ must be greater or equal to 0.955 regardless of the stiffness of the axial restraint.

**Case B:** Similar to the beam with hinged-ends (case A), Fig. 3.8 shows the $m$-vs-$S_{\Delta,cr}$ relationship for the beam with clamped-ends (case B). Fig. 3.8 shows that the value of ($S_{\Delta,cr}$) necessary to avoid the phenomenon of snap-through for case of a beam with both ends clamped can be obtained with a much smaller value of the parameter $m$ when is compared to that of case A.

Consider Fig. 3.9, where only curves iii) and iv) of case A are shown. Assume that the beam is subject to a load corresponding to $u = 2.21$ and $S_\Delta = \infty$ is in a static equilibrium, i.e., case iv) point $P$ in Fig. 3.9. However, if the transverse load remains constant at $u = 2.21$, but the axial
Second-order analysis of a beam-column on elastic foundation partially restrained axially with initial deflections and semirigid connections.

Fig. 3.7 Mid-span vertical deflection of a perfectly hinged-hinged beam subject to a uniformly distributed load $q$ and $\rho_a=\rho_b=0$. Figures (a), (b), (c), (d) and (e) correspond to $m= 1, 1/2, 1/4, 1/9.2$ and $1/14.1$, respectively. Cases $i$, $ii$, $iii$ and $iv$ correspond to $S_\Delta= 0, 0.01, 1$, and $\infty$ respectively.
3.4 Verification and comprehensive examples

Fig. 3.8 Critical axial stiffness $S_{\Delta,cr}$-versus-$m$ for the distributed load $q$: A) both ends hinged ($\rho_a = \rho_b = 0$); (B) both ends clamped ($\rho_a = \rho_b = 1$)

Fig. 3.8 shows the critical axial stiffness $S_{\Delta,cr}$ versus $m$ for the distributed load $q$: A) both ends hinged ($\rho_a = \rho_b = 0$); (B) both ends clamped ($\rho_a = \rho_b = 1$). The restraint changes from $S_{\Delta} = \infty$ to $S_{\Delta} = 1$ (i.e., case iii, point P* in Fig 3.9) the structure would not be able to maintain its static equilibrium and suffer from the snap-through phenomenon and would suddenly go from point P to point P**, causing a sudden change in the configuration of the structure, going from tension to compression, which could be considered as state Structural service limit. Then, there is a value of $S_{\Delta} = S_{\Delta,adm}$ which it would be the admissible axial stiffness that would guarantee that the beam-column with initial imperfection will not suffer from snap-through and its configuration will not change suddenly.

**EXAMPLE 3: Critical Buckling loads and post buckling configuration modes**

Analyze the beam from the previous case with a concentrated load at $L/3$ of the support. Consider that both ends of the member are perfectly hinged but fully restrained to move horizontally and vertically. Assume $m = 1/20$. Determine the critical buckling loads and the post-buckling behavior of the beam.

**Solution-.** Eq. (3.23) is used to determine the critical loads of the beam-column corresponding to the first five modes of buckling. Fig. 3.10 shows the axial load curve $P/\left(\frac{EI}{L^2}\right)$ against the value of determinant of matrix $[A]$. From Fig. 3.10, the critical loads $P_{cr}/\left(\frac{EI}{L^2}\right)$ of the beam-column are: 9.867, 39.456, 88.732, 157.616, and 246.014, respectively.
Second-order analysis of a beam-column on elastic foundation partially restrained axially with initial deflections and semirigid connections.

Fig. 3.9 Critical axial stiffness $S_{\Delta,cr}$-versus-$m$ for a hinged-hinged beam ($\rho_a = \rho_b = 0$) under distributed load $q$.

Fig. 3.10 Axial Load $P$-vs-Det($A$)
Note that all curves in Figs. 3.11(a)-(g) become asymptotic as the axial load $P/\left(\frac{EI}{L^2}\right)$ approach the corresponding $P_{cr}/\left(\frac{EI}{L^2}\right)$ (i.e., 9.867, 39.456, 88.732, 157.616, 246.014, etc.) according to the proposed method of analysis. Note also that depending on the value of the parameter $u$, the curves have 1, 2, 3, 4, or 7 crossings with the horizontal axis at $\delta_t=0$ due to the relative "movement" of the curves "as a whole" with respect to the horizontal axis. Note that as the $u$-parameter varies from -6 to +6, each curve descends and then ascends in the vertical direction, which will suggest the appearance or disappearance of roots in Eq. (3.13) see Fig. 3.11. According to the proposed method, the curves are made of "branches" as shown in Fig. 3.11 for $u= -6$ or $u = 2$, with the five "branches" numbered. Branch 1 has only a root when it intersects the horizontal axis and generates open curves in the diagrams $y/d$-vs- $u$, $P/\left(\frac{EI}{L^2}\right)$ -vs-$y/d$ and $P/\left(\frac{EI}{L^2}\right)$ -vs-$u$ shown in Figs. 3.12(a)-(c). Branches 2, 3, 4, . . . intersect the horizontal axis once or twice depending on the value of the $u$ parameter and generate closed curves in the $y/d$-vs-$u$, $P/\left(\frac{EI}{L^2}\right)$ -vs-$y/d$ and $P/\left(\frac{EI}{L^2}\right)$ -vs-$u$ diagrams.

Figs. 3.12(a)-(c) are the superposition in the same graph of different modes of post-buckling of the beam-column. Analyzing Fig. 3.12(a) it can be observed that this curve is composed of five curves (see Fig. 3.13), each built from the solutions of each of the 5 branches of Fig. 3.11 as they intercept the horizontal axis. Note that the curves that come from branches 2, 3, 4 and 5 generate closed curves, which do not connect with each other due to the asymptotic nature of the curves presented in Fig. 3.11. Figs. 3.12(a)-(c) show that the post-buckling modes of the beam-column are separated by the critical loads, which in order to pass from one mode to another requires that the element suffer from infinitely large displacements. Since the beam-column cannot naturally jump from one mode to another due to the impossibility for the beam-column to deform infinitely, it must follow the path of minimum work, that is, the first mode shape. Consequently, the beam-column cannot deflect naturally into the higher modes 2, 3, 4, etc.

Figs. 3.13(a)-(c) show the post-buckling behavior corresponding to each of the first-five buckling modes, respectively. Note that the first-mode corresponds to branch 1 of the curves of Figs. 3.11(a)-(g), and it is the only mode that generates open curves in Fig. 3.12 of positive stiffness. Whereas in the buckling modes 2, 3 and 4, the post-buckling behavior are from branches 2, 3, and 4 and their intersections of the curves shown in Figs. 3.11(a)-(g) with the horizontal axis at $\delta_t=0$ are the corresponding critical load, respectively. Note that first-buckling mode is located between zero load and the first critical load (9.869 $\frac{EI}{L^2}$), the second buckling mode is between 9.869 $\frac{EI}{L^2}$ and 39.456 $\frac{EI}{L^2}$ and so on. Fig. 12(a) shows the location of modes 1, 2, 3, 4 and 5 on the same ($y/d_1$)-vs-$u$ plot.
Second-order analysis of a beam-column on elastic foundation partially restrained axially with initial deflections and semirigid connections.

Fig. 3.11 $P$-vs-lateral deflection $\delta_t$ for $u = -6, -4, -2, 0, 2, 4,$ and $6$
Fig. 3.12 (a) \( \frac{y}{d_1} \)-vs-\( u \); (b) \( P \)-vs-\( \frac{y}{d_1} \); and (c) \( P \)-vs-\( u \).
Fig. 3.13 Variation of $y/d_1$-vs-$u$ for the post-buckling behavior of modes 1-to-5
3.5 Summary and conclusions

A general method for the second-order analysis of prismatic beam-columns with initial imperfections is presented and discussed in detail. The proposed formulation and the corresponding governing equations and solutions include the combined effects of bending, shear and axial deformations, elastic foundation, semi-rigid connections with rotation, transverse and axial displacements of the supports. The transverse load and the initial transverse deflection are modeled using Fourier series allowing to represent any applied transverse load and any initial transverse deflections. In addition, the incorporation of semi-rigid connections at both supports allows the complete analysis of beam-columns with any support conditions. The buckling and post-buckling behaviors caused by the loss of horizontal stiffness of the end supports of the member are presented and validated using three comprehensive examples. Example 1 presents the buckling reversals of a hinged-hinged beam with initial imperfections subject to symmetric transverse loads. The loss of lateral stability (snap-through) due to a loss of axial stiffness of the support is presented in Example 2. The critical loads, post-buckling behavior and modes of configuration of a hinged-hinged beam axially retrained with initial imperfections subject to a concentrated transverse load at L/3 are presented in Example 3.

Based on the obtained results it is concluded that: 1) there is a minimum axial bracing that guarantees the non-occurrence of the snap-through phenomenon; 2) it is possible that a loss of axial rigidity causes the snap-through phenomenon to occur in beam-columns; 3) buckling modes with disconnected paths exist; 4) a curved element cannot pass from one mode to another, since the structure would be required to take infinite deformed values; 5) A curved element must follow the minimum work curve, according to Fig. 12c, mode 1 requires the lowest load values and will be the path that the structure naturally follows; and 6) as the \( m \) parameter decreases it will be possible to obtain more buckling modes for reasonable values of axial load.

3.6 APPENDIX

3.6.1 APPENDIX I

\[ \{C\} = \{c_1 \ c_2 \ c_3 \ c_4\}^T \]

La columna \( i \) de la matriz \([A]\), \([A]_i\) se expresa de la siguiente manera:
Second-order analysis of a beam-column on elastic foundation partially restrained axially with initial deflections and semirigid connections.

\[ [A]_i = \begin{bmatrix}
(P_s - P) f'_{i,0} - \varphi f'''_{i,0} - S_a f_{i,0} \\
(P_s - P) f'_{i,L} - \varphi f'''_{i,L} + S_b f_{i,L} \\
P_s f_{i,0} + k_a (1 - \psi P_s) f'_{i,0} - \varphi f'''_{i,0} + k_a \psi \varphi f'''_{i,0} \\
P_s f_{i,L} - k_b (1 - \psi P_s) f'_{i,L} - \varphi f'''_{i,L} - k_b \psi \varphi f'''_{i,L}
\end{bmatrix} \]

El vector \( \{ b \} \) se expresa de la siguiente manera:

\[ \{ b \} = \begin{bmatrix}
S_a f_{5,0} - (P_s - P) f'_{5,0} + \varphi f'''_{5,0} - \varphi y'_{0,0} + \omega q' \\
- S_b f_{5,L} - (P_s - P) f'_{5,L} + \varphi f'''_{5,L} - \varphi y'_{0,L} + \omega q' \\
-P_s f_{5,0} - k_a (1 - \psi P_s) f'_{5,0} + \varphi f'''_{5,0} - k_a \psi \varphi f'''_{5,0} + \omega q_0 - k_a \psi \omega q'_0 - k_a \psi \varphi f'''_{5,0} \\
-P_s f_{5,L} + k_b (1 - \psi P_s) f'_{5,L} + \varphi f'''_{5,L} + k_b \psi \varphi f'''_{5,L} - k_b \psi \varphi f'''_{5,L} + \omega q_L + k_b \psi \omega q'_L
\end{bmatrix} \]

### 3.6.2 APPENDIX II

For the particular case of trapezoidal loading (shown in Fig. 4a) can be defined as follows:

\[
f(x) = \begin{cases} 
0 & ; \ 0 \leq x < a \\
q_a + \frac{q_b - q_a}{b-a} (x-a) & ; \ a \leq x \leq b \\
0 & ; \ b < x \leq L
\end{cases}
\]

It is obtained that: \( A_0 = \frac{1}{2L} \left\{ C_s (b-a) + \frac{m_s}{2} \left( b^2 - a^2 \right) \right\} \)

\[
A_n = \frac{i}{2} \left\{ e^{-i\omega_n b} \left( C_s + m_s b - \frac{im_s}{\omega_n} \right) - e^{-i\omega_n a} \left( C_s + m_s a - \frac{im_s}{\omega_n} \right) \right\}
\]

where: \( m_s = \frac{q_b - q_a}{b-a} ; C_s = q_a - m_s a ; n = 0, \pm 1, \pm 2, \pm 3, \ldots \)

These expressions can be used to model the following types of transverse load:

- Uniformly distributed load: \( q_a = q_b \);
- Triangular distributed load: \( q_a = 0, q_b \);
- Concentrated load at a: \( b = a + \varepsilon \); \( q_a = q_b = \frac{Q}{\varepsilon} \); \( \varepsilon \to 0 \);
- Concentrated moment at a: \( b = a + \varepsilon \) and \( q_a = -q_b = \frac{6M}{\varepsilon^2} \) as \( \varepsilon \to 0 \)

For the particular case of initial deflection of sinusoidal shape:
3.6 APPENDIX

\[ y_0(x) = d \sin \left( \frac{\pi x}{L} \right) = d \left( \frac{1}{2i} e^{i \frac{\pi x}{L}} - \frac{1}{2i} e^{-i \frac{\pi x}{L}} \right) = \sum_{n=-\infty}^{\infty} B_n e^{i \omega_n x} \]

By comparing the terms of the summation, it can be found that:

\[ B_n = \begin{cases} 
\frac{d}{2i} & \text{for } n = 1 \\
-\frac{d}{2i} & \text{for } n = -1 \\
0 & \text{for } \text{other } n 
\end{cases} \]

3.6.3 APPENDIX III

The following symbols are used in this publication:

- **A**: Cross sectional area of the beam-column;
- **A_s**: Cross sectional shear area of the beam-column;
- **E**: Elastic Modulus of the material;
- **G**: Elastic Shear Modulus of the material;
- **I**: principal moment of inertia of the beam-column
- **I_A'**: about the bending axis;
- **k_s**: ballast coefficient of the supporting soil;
- **L**: beam-column span;
- **m = \frac{4I}{Ad_1^2}**: normalized load factor defined by Timoshenko;
- **n**: shear-form factor of the beam-column cross section;
- **y, y_0, y_e, y_b and y_s**: deflection total, initial, caused by external loads, bending and shear, respectively.
- **M_A and M_B**: applied bending moments at ends A and B, respectively;
- **P**: applied axial load and the ends of the beam-column;
- **Q**: applied concentrated transverse load;
- **q(x)**: applied transverse load;
- **R_A and R_B**: stiffness indices of the flexural connections at ends A and B, respectively;
- **S_A and S_B**: stiffness indices of the transverse bracings at ends A and B, respectively;
- **S_\Delta**: stiffness indices of the horizontal bracings at ends B;
- **u = \frac{5}{384} \frac{qL^4}{EI_d}**: for uniformly distributed load q, or **u = \frac{1}{48} \frac{QL^3}{EI_d}** for concentrated load Q. **V_A and V_B**: applied shears at ends A and B, respectively;
- **k_a and k_b**: stiffness of the flexural connections at ends A and B, respectively;
- **\delta_t**: total horizontal displacement at end B.
- **i = \sqrt{-1}**
Chapter 4

Elastic analysis of composite beams and beams retrofitted with FRP laminates with generalized end conditions.

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4.1 Abstract

The elastic analysis of composite concrete-steel beams and beams retrofitted with FRP laminates is presented in detail. The proposed method and corresponding equations allow the analysis of these beams with generalized end conditions subjected to any transversal loads (concentrated and distributed forces and moments) including the simultaneous effects of shear, bending and axial deformations of each component and connecting material along its span. Three comprehensive examples are presented and the calculated results are compared with those obtained and reported by other researchers showing the efficiency and accuracy of the proposed method and corresponding equations.

KEYWORDS: Composite beams, retrofitted beams, Steel beams, structural analysis
4.2 Introduction

Composite beams and beams retrofitted with FRP laminates are generally made of two structural components inter-connected mechanically or with a layer of an adhesive material extending along the beam span. The main purpose of connecting two structural components along the span is to improve the strength and stiffness of the whole beam and also to optimize both its cost and ultimate performance. These beams are commonly used in Civil Engineering structures like buildings and bridges.

Knowledge of the distribution and magnitude of stresses and deformations along the two components and the interconnection (adhesive or mechanical) are of critical importance in the proper design of composite beams and beams retrofitted with FRP laminates. The distribution and magnitude of stresses and deformations along the beam depend not only on the mechanical properties and stiffness of the components and interconnection but also on the support conditions, applied loads and length of the interconnection [17]. To properly determine the distribution and magnitude of stresses and deformations along the interconnection is quite cumbersome involving the solution of a system of differential equations with two variables and the application of a relative large number of boundary conditions [17, 20, 29, 43, 45]. Currently to find these stresses there are three methods available in the technical literature: analytical methods [20], [29], [44, 50, 20, 36]; finite differences [48]; and the finite element method [45], [37, 56, 26, 49, 52]. Some of these methods are rather complex and difficult to implement, and others are generally over-simplified involving few variables in their development leading many times to unreliable solutions.

An analytical method that takes into account the interaction between shear and normal stress in the interface, the axial as well as the bending and shear effects and deformations of each component in composite beams with generalized end conditions subject to distributed and concentrated loads and moments is proposed herein. The governing equation derived in this paper includes both shear and bending deformations on the composite beam. Similar equation was presented by Cosenza [17] including uniform transverse load but neglecting shear effects. Liu et al [54] presented a similar one including the shear effects but neglecting transverse loads, while Quang-Huy Nguyen et al [44] presented an equation but neglecting the interaction between the normal and shear stresses at the connection. Sua and Gao [50] presented an analysis that includes shear deformations and the interaction between the normal and shear stresses at the connection but neglected the effects of transverse loads on this interaction.
4.3 Structural model

Consider a beam made up of two perfectly straight prismatic structural components (i= 1 and 2) inter-connected by a uniformly distributed linear elastic interface with vertical and horizontal stiffness $k_v$ and $k_h$, respectively. Both structural components are assumed to be perfectly linear elastic, homogeneous and isotropic with elastic modulus $E_i$, shear modulus $G_i$, and cross sectional area and moment of inertia $A_i$ and $I_i$, respectively. The top face of component 1 and the bottom face of component 2 are subject to distributed external loads $q_1$ and $q_2$ along the beam span, respectively. Fig. 4.1 shows a differential beam element as well as the loads on the interface and on components 1 and 2.

Fig. 4.1 Differential element and free body diagram of both components and connecting interface.

4.4 Governing equations

The governing equations of the system derived in this section are based on equilibrium, strain compatibility and constitutive laws of all materials involved. It is assumed that plane sections
remain plane and that the transverse deflections of the member are relatively small compared its dimensions so that the principle of superposition can be applied. Axial, bending and shear deformations along the member and its components are included in the analysis described below.

4.4.1 Translational and Rotational Equilibrium

Fig. 4.1 shows the infinitesimal element of the composite beam as well as the free body diagram of each component. The translational and rotational equilibrium equations of components 1 and 2 are as follow:

\[
\frac{dV_1}{dx} = -q_1 + \sigma b
\]  (4.1)

\[
\frac{dV_2}{dx} = -q_2 - \sigma b
\]  (4.2)

\[
\frac{dP_1}{dx} = -\tau b
\]  (4.3)

\[
\frac{dP_2}{dx} = \tau b
\]  (4.4)

\[
\frac{dM_1}{dx} = V_1 + \tau bc_1
\]  (4.5)

\[
\frac{dM_2}{dx} = V_2 + \tau bd_2
\]  (4.6)

4.4.2 Conditions of compatibility and constitutive laws

The tangent of the centerline at any point along the x-axis consists of two components (caused by the bending moment) and (caused by the transverse shear force):
4.4 Governing equations

\[ y''_1 = y''_{b,1} + y''_{s,1} \] \hspace{1cm} (4.7)

\[ y''_2 = y''_{b,2} + y''_{s,2} \] \hspace{1cm} (4.8)

The axial deformation of the bottom fiber of component 1 and that of the top fiber of component 2 caused by the axial force and the bending moment combined are as follow:

\[ u''_{1,B} = -\frac{P_1}{(EA)_1} + \frac{M_{1c_1}}{(EI)_1} \] \hspace{1cm} (4.9)

\[ u''_{2,T} = -\frac{P_2}{(EA)_2} - \frac{M_{2d_2}}{(EI)_2} \] \hspace{1cm} (4.10)

The angular deformations caused by bending and transverse shear are given by Eqs. (4.11), (4.12), (4.13), (4.14), respectively:

\[ y''_{b,1} = -\frac{M_1}{(EI)_1} \] \hspace{1cm} (4.11)

\[ y''_{b,2} = -\frac{M_2}{(EI)_2} \] \hspace{1cm} (4.12)

\[ y'_{s,1} = \frac{V_1}{(A_sG)_1} \] \hspace{1cm} (4.13)

\[ y'_{s,2} = \frac{V_2}{(A_sG)_2} \] \hspace{1cm} (4.14)
Elastic analysis of composite beams and beams retrofitted with FRP laminates with generalized end conditions.

Notice that second-order shear force induced by the axial load as the beam deflects along its span (as shown by Timoshenko and Gere [52] on page 132) have been neglected. Otherwise it would lead to a more complex non-linear analysis.

It is assumed that the interface is also linear elastic with the following constitutive laws:

\[ \tau = k_h (u_{1,B} - u_{2,T}) \]  \hspace{1cm} (4.15)

\[ \sigma = k_v (y_1 - y_2) \]  \hspace{1cm} (4.16)

Where: \( k_h \) and \( k_v \) are the horizontal and vertical stiffness of the interface, respectively; \( u_{1,B} \), \( u_{2,T} \) and \( y_1, y_2 \) are the displacements along the X and Y directions of components 1 and 2, respectively.

Substituting Eqs. (4.11) and (4.12) into (4.7) and (4.8):

\[ y_1'' = -\frac{M_1}{(EI)_1} + \frac{\sigma b - q_1}{(A_s G)_1} \]  \hspace{1cm} (4.17)

\[ y_2'' = -\frac{M_2}{(EI)_2} - \frac{\sigma b + q_2}{(A_s G)_2} \]  \hspace{1cm} (4.18)

Substituting Eq. (4.9) and (4.10) into (4.15) and successively deriving:

\[ \tau' = k_p \left( -\frac{P_1}{(EA)_1} + \frac{M_1 c_1}{(EI)_1} + \frac{P_2}{(EA)_2} + \frac{M_2 d_2}{(EI)_2} \right) \]  \hspace{1cm} (4.19)

\[ \tau'' - \alpha^2 \tau = k_h \left( \frac{c_1 V_1}{(EI)_1} + \frac{d_2 V_2}{(EI)_2} \right) \]  \hspace{1cm} (4.20)

\[ \tau''' - \alpha^2 \tau' - c k_h \sigma = -k_h \left( \frac{c_1 q_1}{(EI)_1} + \frac{d_2 q_2}{(EI)_2} \right) \]  \hspace{1cm} (4.21)
4.4 Governing equations

Substituting Eq. (4.17) and (4.18) into the second derivative of (4.16) and deriving consecu-
tively twice, Eqs. (4.22), (4.23) and (4.24) can be obtained:

\[
\sigma'' - 4n^4 \sigma = k_v \left( \frac{M_2}{(EI)_2} - \frac{M_1}{(EI)_1} \right) + k_v \left( \frac{q_2}{(A_sG)_2} - \frac{q_1}{(A_sG)_1} \right) \quad (4.22)
\]

\[
\sigma''' - 4n^4 \sigma' + c k_v \tau = k_v \left( \frac{V_2}{(EI)_2} - \frac{V_1}{(EI)_1} \right) + k_v \left( \frac{q_2'}{(A_sG)_2} - \frac{q_1'}{(A_sG)_1} \right) \quad (4.23)
\]

\[
\sigma'''' - 4n^4 \sigma'' + c k_v' \tau' + 4 \beta^4 \sigma = k_v \left( \frac{q_1}{(EI)_1} - \frac{q_2}{(EI)_2} \right) + k_v \left( \frac{q_2''}{(A_sG)_2} - \frac{q_1''}{(A_sG)_1} \right) \quad (4.24)
\]

Where:

\[
4 \beta^4 = k_v \left( \frac{1}{(EI)_1} + \frac{1}{(EI)_2} \right) ; \quad 4n^4 = k_v \left( \frac{1}{(A_sG)_1} + \frac{1}{(A_sG)_2} \right) ; \quad c = b \left( \frac{c_1}{(EI)_1} - \frac{d_2}{(EI)_2} \right) \quad \text{and} \quad \alpha^2 = bk_h \left( \frac{1}{(EA)_1} + \frac{c_1^2}{(EI)_1} + \frac{c_1}{(EA)_2} + \frac{d_2^2}{(EI)_2} \right)
\]

Substituting Eq. (4.21) into (4.24) and integrating once and after some algebra the following
sixth-order differential equation on \( \tau \) can be obtained:

\[
\tau'''''' - \left( \alpha^2 + 4n^4 \right) \tau''''' + 4 \left( n^4 \alpha^2 + \beta^4 \right) \tau'' - 4 \beta^4 \alpha^2 (1 - j) \tau = \frac{k_hk_vb(c_1 + d_2)}{(EI)_1 (EI)_2} T(x)
\]

\[-k_hk_vb \left( \frac{c_1}{(EI)_1 (A_sG)_2} + \frac{d_2}{(EI)_2 (A_sG)_1} \right) T''(x) - k_h \left( \frac{c_1q_1'''}{(EI)_1} + \frac{d_2q_2'''}{(EI)_2} \right) \quad (4.25)
\]

Where:

\[
j = \frac{k_hk_vc_1}{4\beta^4\alpha^2} \quad \text{and} \quad T(x) = - \int_0^x (q_1 + q_2) \, dx \quad (= \text{total shear force along } x)
\]

Eq. (4.25) is the governing equation that includes shear and bending deformations on
the composite beam of Fig.4.2 subjected to transverse loads of any shape. Note that it is a
non-homogeneous sixth-order differential equation of constant coefficients. As previously
explained in the introduction, a similar equation was presented by Cosenza [17] including
uniform transverse load but neglecting shear effects. Liu et al [54] presented a similar one
including the shear effects but neglecting transverse loads, while Quang-Huy Nguyen et al [44]
presented an equation but neglecting the interaction between the normal and shear stresses at the connection. Sua and Gao [50] presented an analysis that includes shear deformations and the interaction between the normal and shear stresses at the connection but neglected the effects of transverse loads on this interaction.

![Diagram of applied transverse loads along the beam and forces and moments at ends x = 0 and x = L.]

The solution to Eq. (4.25) is made of the homogeneous ($\tau_h$) and non-homogeneous parts ($\tau_p$) that depend on the applied transversal loads.

$$\tau = \tau_h + \tau_p$$  \hspace{1cm} (4.26)

Where:

\[ \tau_h = \sum_{i=1}^{6} C_i \exp(m_i x) \]

Note that $m_i$ are the roots of the characteristic polynomial of Eq. (4.25) given by Eq. (4.27):

$$m^6 - (\alpha^2 + 4n^4) m^4 + 4\left(n^4 \alpha^2 + \beta^4 \right) m^2 - 4\beta^4 \alpha^2 (1 - j) = 0$$  \hspace{1cm} (4.27)

To find the particular solution $\tau_p$, the arbitrary external loads are expressed using Fourier series ($0 \leq x \leq L$) as follows:

\[ q_i = a_{i,0} + \sum_{n=1}^{\infty} \left\{ a_{i,n} \cos(\omega_n x) + b_{i,n} \sin(\omega_n x) \right\} \]

Where:

\[ \omega_n = \frac{2\pi}{L} n; \quad a_{i,0} = \frac{1}{L} \int_0^L q_i (x) \, dx, \quad a_{i,n} = \frac{2}{L} \int_0^L q_i (x) \cos(\omega_n x) \, dx, \quad \text{and} \quad b_{i,n} = \frac{2}{L} \int_0^L q_i (x) \sin(\omega_n x) \, dx, \]

for $i = 1, 2$. 
The particular solution is given by Eq. (4.28).

\[ \tau_p = A_0 + D_0 x + \sum_{n=1}^{\infty} \{ A_n \cos(\omega_n x) + B_n \sin(\omega_n x) \} \] (4.28)

Where:

\[ A_0 = -\frac{\delta_1 (V_{1,0} + V_{2,0})}{\alpha_{13}}, \quad D_0 = \frac{\delta_1 (b_{1,0} + b_{2,0})}{\alpha_{13}}, \]

\[ A_n = -\frac{(b_{1,n} + b_{2,n}) \delta_1 + (b_{1,n} + b_{2,n}) \alpha_{1}^2 \delta_2 + (b_{1,n} \delta_3 + b_{2,n} \delta_4) \omega_n^2}{\omega_n^2 + \alpha_{11} \omega_n^4 + \alpha_{12} \omega_n^6 + \alpha_{13} \omega_n^8}, \]

\[ B_n = \frac{(a_{1,n} + a_{2,n}) \delta_1 + (a_{1,n} + a_{2,n}) \alpha_{1}^2 \delta_2 + (b_{1,n} \delta_3 + b_{2,n} \delta_4) \omega_n^2}{\omega_n^2 + \alpha_{11} \omega_n^4 + \alpha_{12} \omega_n^6 + \alpha_{13} \omega_n^8}, \]

\[ \alpha_{11} = (\alpha^2 + 4n^4), \quad \alpha_{12} = 4 \left( n^4 \alpha^2 + \beta^4 \right), \quad \alpha_{13} = 4 \beta^4 \alpha^2 (1 - j), \]

\[ \delta_1 = \frac{k_h k_v (c_1 + d_2)}{(EI)_1 (EI)_2}, \quad \delta_2 = k_h k_v b \left( \frac{c_1}{(EI)_1 (A_s G)_1} + \frac{d_2}{(EI)_2 (A_s G)_1} \right), \quad \delta_3 = \frac{k_h c_1}{(EI)_1} \text{ and } \delta_4 = \frac{k_h d_2}{(EI)_2}. \]

This solution is valid for any transverse loads including concentrated loads and moments, uniform and non-uniform loads. For the particular case of uniformly distributed transverse loads loads \( q_1 \) and \( q_2 \) along the beam span the solution is given by Eq. (4.29).

\[ \tau_p = \gamma_1 T(x) \] (4.29)

Where:

\[ \gamma_1 = \frac{k_h k_v b (c_1 + d_2)}{(EI)_1 (EI)_2 \alpha^2}; \quad \text{and } T(x) = -\int_0^x (q_1 + q_1) dx \]

By replacing Eq. (4.26) into (4.21) the normal stress \( \sigma(x) \) can be determined as a linear combination of the integration constants using Eq. (4.30).

\[ \sigma = \frac{1}{ck_h} \left\{ \sum_{i=1}^{6} C_i m_i (m_i^2 - \alpha^2) \exp(m_i x) + \tau''_{p} - \alpha^2 \tau'_{p} + k_h \left( \frac{c_1 q_1}{(EI)_1} + \frac{d_2 q_2}{(EI)_2} \right) \right\} \] (4.30)
Elastic analysis of composite beams and beams retrofitted with FRP laminates with generalized end conditions.

To calculate the axial force and shear force diagrams as well as that of the bending moment along components 1 and 2, Eqs. (4.1) to (4.6) must be integrated taking account the boundary conditions. Note that Eqs. (4.7)-(4.14) can be used to calculate deformations and rotations along the beam.

### 4.4.3 Boundary Conditions

The approach proposed herein is a generalized formulation capable of analyzing the composite beam subject not only to different loadings but with different boundary conditions without having to determine all the constants of integration for each particular case [17].

Fig. (4.2) shows the composite beam with its components (i= 1, 2) subject to axial and shear forces (\(P_i\) and \(V_i\), respectively) as well as to bending moments (\(M_i\)) at both ends (\(j = 0 \text{ and } L\)) as follow:

At \(x = 0\):
- \(M_1 = M_{1,0}\), \(V_1 = V_{1,0}\), \(P_1 = P_{1,0}\), \(M_2 = M_{2,0}\), \(V_2 = V_{2,0}\), and \(P_2 = P_{2,0}\)

At \(x = L\):
- \(M_1 = M_{1,L}\), \(V_1 = V_{1,L}\), \(P_1 = P_{1,L}\), \(M_2 = M_{2,L}\), \(V_2 = V_{2,L}\), and \(P_2 = P_{2,L}\)

Eq. (4.31) can be obtained combining Eqs. (4.21) and (4.22):

\[
\tau''' - (\alpha^2 + 4n^4) \tau'' + 4n^4 \alpha^2 \tau' = ck'h^q \left( \frac{M_2}{(EI)_2} - \frac{M_1}{(EI)_1} \right) - kh \left( \frac{c_1q''}{(EI)_1} + \frac{d_2q''}{(EI)_2} \right) + \delta_2 (q_1 + q_2) - (\tau'''' - (\alpha^2 + 4n^4) \tau'' + 4n^4 \alpha^2 \tau')
\]

Applying boundary conditions (at \(x = 0\) and \(x = L\)) to Eqs. (4.19, 4.20) and (4.31) and substituting Eq. (4.26), the following six equations can be obtained:

\[
\sum_{i=1}^{6} c_i m_i = k_h \left( -\frac{P_{1,0}}{(EA)_1} + c_1 M_{1,0} + \frac{P_{2,0}}{(EA)_2} + \frac{d_2 M_{2,0}}{(EI)_2} \right) - \tau'_p(0) \tag{4.32}
\]

\[
\sum_{i=1}^{6} c_i M_i^2 = k_h \left( \frac{c_1 V_{1,0}}{(EI)_1} + \frac{d_2 V_{2,0}}{(EI)_2} \right) - (\tau''(0) - \alpha^2 \tau_p(0)) \tag{4.33}
\]
\[
\sum_{i=1}^{6} c_i \left\{ m_i^5 - (\alpha^2 + 4n^4) m_i^3 + 4n^4 \alpha^2 m_i \right\} = ck_hk_v \left( \frac{M_{2.0}}{(EI)_2} - \frac{M_{1.0}}{(EI)_1} \right) - k_h \left( \frac{c_1 q''_{1.0}}{(EI)_1} + \frac{d_2 q''_{2.0}}{(EI)_2} \right) + \\
\delta_2 (q_{1,0} + q_{2,0}) - (\tau''''_{p}) (0) - (\alpha^2 + 4n^4) \tau''_{p} (0) + 4n^4 \alpha^2 \tau'_{p} (0) 
\]

(4.34)

\[
\sum_{i=1}^{6} c_i m_i \exp (m_i L) = k_h \left( - \frac{P_{1.L}}{(EA)_1} + \frac{c_1 M_{1.L}}{(EI)_1} + \frac{P_{2.L}}{(EA)_2} + \frac{d_2 M_{2.L}}{(EI)_2} \right) - \tau'_{p} (L) 
\]

(4.35)

\[
\sum_{i=1}^{6} c_i (m_i^2 - \alpha^2) \exp (m_i L) = k_h \left( \frac{c_1 V_{1.L}}{(EI)_1} + \frac{d_2 V_{2.L}}{(EI)_2} \right) - (\tau''_{p} (L) - \alpha^2 \tau_{p} (L)) 
\]

(4.36)

\[
\sum_{i=1}^{6} c_i \left\{ m_i^5 - (\alpha^2 + 4n^4) m_i^3 + 4n^4 \alpha^2 m_i \right\} \exp (m_i L) = ck_hk_v \left( \frac{M_{2.0}}{(EI)_2} - \frac{M_{1.0}}{(EI)_1} \right) - \\
k_h \left( \frac{c_1 q''_{1.0}}{(EI)_1} + \frac{d_2 q''_{2.0}}{(EI)_2} \right) + \delta_2 (q_{1,L} + q_{2,L}) - (\tau''''_{p}) (L) - (\alpha^2 + 4n^4) \tau''_{p} (L) + 4n^4 \alpha^2 \tau'_{p} (L) 
\]

(4.37)

Eqs. (4.32) to (4.37) can be expressed in matrix form as \([A] \{c\} = \{b\}\), where \([A]\) is the coefficient matrix, \(\{b\}\) is vector of external loads, and \(\{c\}\) is vector containing the six unknown constants of integration. Note that the shear and normal bonding stresses (\(\sigma\) and \(\tau\)) can be obtained introducing \(\{c\} = [A]^{-1} \{b\}\) into Eqs. (4.26) and (4.30), respectively.

### 4.5 Comprehensive examples

#### 4.5.1 Example 1: Simply supported concrete-steel composite beam

Fig. 4.3 shows a concrete-steel composite beam experimentally tested by Abdel Aziz [10] and the test results discussed later by Fabbrocino [25]. The geometrical and mechanical properties of the beam are included in Fig. 4.3. The shear area correction factors are designated by \(k_{s,1}\) and \(k_{s,2}\).
Elastic analysis of composite beams and beams retrofitted with FRP laminates with generalized end conditions.

_solution_. To compare the experimental results with those obtained with the proposed model, the concrete component is subjected to a distributed transverse load shown in Fig. 4.4 and defined as follows:

\[
\eta(a, b) = \begin{cases} 
0 & x < a \\
\frac{bq_a - aq_b}{b-a} + \frac{q_b - q_a}{b-a}x & a \leq x \leq b \\
0 & x > b 
\end{cases}
\]

Fig. 4.3 Concrete-steel composite beam of Example 1.

Fig. 4.4 Transverse Load on a concrete-steel composite beam.
4.5 Comprehensive examples

Eq. (4.28) allows the expansion of the transversal load $q(x)$ of element in Fourier series and it can be used to model trapezoidal, triangular, uniform, concentrated loads and moments. Fig. 4.4 shows that for concentrated loads $e$ tends to zero. The experimental program does not report the vertical stiffness of the connection. Assume that $k_v = k_h/5$ as suggested by Cosenza [17].

![Diagram](image)

Fig. 4.5 Shear connection relationships.

Fig. 4.5 shows the slip/shear-force diagram of the connection (for a single bolt) between the concrete element and the steel element along the length of the beam introduced by J. G. Ollgaard [34]. The values of the parameters $\alpha$, $\beta$, $Q_{50\%}$, $k_{50\%}$, and the maximum shear force, $Q_{\text{max}}$, are also shown in Fig. 4.5. To determine the horizontal shear stiffness connection $k_h$, it is recommended to use the secant stiffness $k_{50\%}$.

Fig. 4.6 show the slip distribution over the span of the beam subjected to two different loads (257 and 344 kN) calculated using the proposed model as well as the theoretical results reported by Fabbrocino [25] and those obtained experimentally by Abdel Aziz [10]. Both figures show that the results from the linear proposed model using a single member compare well with the experimental results and those by Fabbrocino who used a nonlinear analysis. In addition, Fig. 4.6(b) shows the variation of the shear stresses along the beam for three different values of the vertical stiffness $k_v$ ($= k_h/20$, $k_h/5$ and $k_h/0.001$). It must be emphasized that in
Elastic analysis of composite beams and beams retrofitted with FRP laminates with generalized end conditions.

The linear proposed model a single member was used since the applied transverse load was modeled using Fourier series [45], [50], [6] and [46].

4.5.2 Example 2: Simply supported steel-concrete composite beam

Analyze the composite beam shown in Fig. 4.7 made of a concrete slab and a steel beam connected by steel bolts. Assume that: \( L = 5 \text{m} \); a steel profile HEB200; concrete slab wide and thick; the steel bolts have a diameter of and spacing with stiffness \( P/s = 0.12E3 \text{ kN/mm} \) and \( T/u = 0.024E3 \text{ kN/mm} \). Consider two loading conditions: 1) load applied on the concrete slab \( (q_1 = q \text{ and } q_2 = 0) \) and; 2) a load applied to the steel beam \( (q_1 = 0 \text{ and } q_2 = q) \). Compare the calculated results using the proposed method with those presented by Cosenza [17].

**Solution**. The boundary conditions are:

\[
P_{1.0} = 0; \ V_{1.0} = 0; \ M_{1.0} = 0; \ P_{2.0} = 0; \ V_{2.0} = V_0 \text{ and } M_{2.0} = 0 \\
P_{1,L} = 0; \ V_{1,L} = 0; \ M_{1,L} = 0; \ P_{2.0} = 0; \ V_{2.L} = -V_0 \text{ and } M_{2,L} = 0
\]

In this case \( q_1 \) and \( q_2 \) are uniformly distributed loads applied the entire beam span and according to Eq (1.16):

\[
\tau_p (x) = \gamma_1 T (x) = \gamma_1 \left[ (q_1 + q_2) \left( \frac{L}{2} - x \right) \right] \text{ and } V_0 = \frac{(q_1 + q_2)L}{2}
\]

Fig. 4.6 Slip diagrams for different models, loads and rigidities (theoretical and experimental results).
The normal and shear stiffness of the connecting steel bolts according to Cosenza [17] are:

\[ k_h = \frac{E}{b p} \] \quad \text{and} \quad \[ k_v = \frac{T}{b p} \]

where \( b \) = the width of the steel top flange and \( p \) = the horizontal spacing between bolts.

Fig. 4.8 shows the results obtained using the proposed method and those reported by Cosenza [17] showing good agreements. Note that: 1) when the load is applied along the bottom of the steel beam (i.e., component 2) normal tensile stress are induced along the connectors over the entire length of the beam; however, when the load is applied on top of the concrete slab (i.e., component 1) normal tension stresses are induced near to the two extremes of the beam (about \( L/8 \) from the supports) and normal compression along the rest of the beam span; and 2) shear stresses along at the interface are identical for both cases.

As shown in Examples 1 and 2, the distribution of the local stresses of shear connectors calculated from the proposed linear model using a single member compare well with the experimental results and those by Fabbrocino [25] who used a nonlinear analysis. However, it is important to emphasize that in the case of ductile shear connectors such as shear studs, such local behavior is not important. In actual design, it is assumed that load-redistribution is acceptable along the shear studs. In the case of brittle connectors, such local behavior could be important.
Elastic analysis of composite beams and beams retrofitted with FRP laminates with generalized end conditions.

Fig. 4.8 Normal and shear stresses along the composite beam.
### Example 3: Simple supported R/C rectangular beam retrofitted with CFRP laminate

Analyze the simple supported R/C concrete beam retrofitted with CFRP laminate shown in Fig. 4.9. The beam has a length $L_1 = 2.4\text{m}$ with $E_1 = 30,000\text{ MPa}$, wide and thick. The carbon fiber sheet adheres over the entire width of the beam and is cut off at a distance $r = 300\text{mm}$. The module of the CFRP laminate is $E_2 = 230,000\text{ MPa}$ and $t_2 = 3\text{mm}$ thick. The adhesive used has thickness $t_a = 2\text{mm}$, elastic modulus $E_a = 1500\text{ MPa}$ and shear modulus $G_a = 580\text{ MPa}$.

Fig. 4.9 Simple supported R/C beam retrofitted with a CFRP laminate.

**Solution.** According to Cosenza [17] the normal and shear stiffness of the connection are given by: $k_h = \frac{G_a t_a}{t_a}$ and $k_v = \frac{E_a t_a}{t_a}$ where $t_a$ is the thickness of the adhesive. The boundary conditions on components 1 and 2 are as follow:

- $P_{1,0} = 0$ ; $V_{1,0} = V_0$ ; $M_{1,0} = M_0$ ; $P_{2,0} = 0$ ; $V_{2,0} = 0$ and $M_{2,0} = 0$
- $P_{1,L} = 0$ ; $V_{1,L} = -V_0$ ; $M_{1,L} = M_0$ ; $P_{2,L} = 0$ ; $V_{2,L} = 0$ and $M_{2,L} = 0$

In this particular case $q_1 = q$, (uniformly distributed load on the beam span) and $q_2 = 0$, so according to Eq. (4.29):

$$\tau_p(x) = \gamma_1 T(x) = \gamma_1 \left[ (q_1 + q_2) \left( \frac{L_1}{2} - r - x \right) \right] , \quad V_0 = (q_1 + q_2) \left( \frac{L_1}{2} - r \right) \quad \text{and} \quad M_0 = (q_1 + q_2) \left( L_1 - r - x \right) / 2$$
Fig. 4.10 shows the distribution of both normal and shear stresses along the FRP laminate. Note that: 1) the results obtained by the proposed model and reported by Cosenza [17] are in good agreement; 2) as expected, the shear diagram at the interface obtained with the proposed model is anti-symmetric about the beam mid-span (i.e., $x = L/2$) and crossing the origin, thus satisfying the symmetry of the composite beam and the applied transversal load.

![Graph showing normal and shear stresses along a beam retrofitted with a FRP laminate.](image)

**Fig. 4.10 Normal and shear stresses along a beam retrofitted with a FRP laminate.**

### 4.6 Conclusions

The elastic static analysis of composite beams made of two structural components interconnected mechanically with bolts (such as composite concrete-steel beams) or with a layer of an adhesive material (such as beams retrofitted with FRP laminates) extending along the beam span is presented in detail in this paper. The top face of component 1 and the bottom face of component 2 are subject to two different distributed loads $q_1$ and $q_2$ along the beam
span both modeled using Fourier series, respectively. The proposed analytical method and corresponding equations take into account the interaction between shear and normal stresses in the interconnection, as well as the bending, axial and shear effects of each structural component. It is shown that the distribution and magnitude of stresses and deformations along the beam depend on several variables including the mechanical properties, stiffness of the components and interconnection, support conditions, applied loads, and length of the interconnection. To verify the proposed method and corresponding equations three comprehensive examples were carried out and the calculated results were compared with analytical and experimental results available in the technical literature presented by other researches.

The governing equation of the system is a non-homogeneous sixth-order differential equation with constant coefficients. To find its solution, a total of twelve boundary conditions must be applied (axial forces, shear and bending moments). The solution proposed herein involves a total of twelve actions at the ends of the beam (x = 0 and L) including axial and shear forces, and bending moments at the ends of components 1 and 2. The proposed method is capable to analyze composite beams and beams retrofitted with FRP laminates with any case of end conditions and loadings using a single beam element.
Chapter 5

Stiffness matrix and loading vector of a two-layer Timoshenko beam.

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5.1 Abstract:

The stiffness matrix, transfer functions and the corresponding loading vector of a two-layer prismatic Timoshenko beam that includes the transversal and longitudinal relative displacements along the interface, the interaction between the normal and shear stresses and the effects of the coupling between the bending and shear deformations are derived in a classical manner. The proposed method can be used in the structural analysis and design of coupled elements and frame systems of composite elements such as simply and continuous steel-concrete composite beams. The proposed expressions for the load vector are general for any type or combination of transverse loads that fit a second-order polynomial curve, including uniformly, trapezoidal and parabolic distributed transverse loads. The transfer functions necessary to determine the adhesive stresses, axial and shear forces, bending moments, deflections and rotations along the member are also derived. Two comprehensive examples are presented to show the effectiveness and validity of the proposed method and corresponding equations.

KEYWORDS: Stiffness matrix; loading vector; two-layer Timoshenko beam; coupled systems.
5.2 Introduction

The analysis of framed structures made of beams, columns and composite members is of importance in structural engineering. Numerous researchers have developed expressions for the first- and second-order stiffness matrices of single-layer Timoshenko’s beams and beam-columns subjected to transverse and axial forces as well as to bending and torsional moments [Arboleda-Monsalve et al. [5]; Areiza-Hurtado et al. [6]; Aydogan [9]; Cheng and Pantelides [14]; Aristizabal-Ochoa [7]; among many others]. These expressions have been implemented in the analysis of beams and beam-columns of mixed steel-concrete elements by homogenizing the cross section and neglecting the partial interaction at the interface of the coupled elements. Several expressions have been developed to include this effect by using approaches such as empirical, classic, finite element and finite difference methods [Ansourian [4], Dall’Asta and Zona [18], Fabbrocino et al. [21], Faella et al. [22], Foraboschi [24], Gara et al. [27], He and Yang [30], and Ranzi et al. [26]].

Ecsedi and Baksa [20] presented an analytical solution for a two-layered beam with interlayer slip. They derived an analytical solution for the deflection, cross-sectional rotation and internal forces neglecting effects of the transverse relative movement. Nguyen et al. [44] and Keo et al. [43] derived the “exact” stiffness matrix for the analysis of partially connected shear deformable two- and multi-layered Timoshenko beams, respectively, subjected to external forces and moments at the ends and to a uniformly distributed transverse load along the span. They neglected the transversal separation at the interface of the coupled elements. Jiang et al. [48] used the finite element method to develop a two-node composite beam-element that includes the interaction between the normal and shear stresses along the interface as well as the shear effects on the beam deformations. Ranzi et al. [26] derived an analytical formulation for the analysis of two-layered composite beams using the principle of virtual work. They presented the finite element solution in both its weak and strong forms and includes the longitudinal and vertical partial interaction. However, the authors did not present an expression for the load vector.

The main objective of this publication is to present the stiffness matrix, as well as the loading vector and transfer functions of a two-layer prismatic Timoshenko beam including the effects of the transversal and longitudinal relative displacements along the interface, the interaction between the normal and shear stresses, and the coupling between the bending and shear deformations. The expression developed for the load vector considers any type or combination of transverse loads that fits a second-order polynomial, including uniformly
distributed, trapezoidal and parabolic transverse loads. The derived matrices and load vector for coupled elements are limited to first-order linear analyses. However, the proposed 12x12 stiffness matrix can be coupled with 6x6 first- and second-order matrices available in the technical literature for single beam elements, so framed systems composed of single and composite beam elements can be analyzed.

In addition to the aforementioned capabilities, the proposed method can be applied to the analysis of structural elements retrofitted or strengthened with fiber reinforced polymer materials, and coupled shear walls (Cosenza [17]). The expressions presented for the load vector can also be extended to include different types of transversal loads by using a function that fits the transversal load (Areiza-Hurtado [6]). For the sake of simplicity, the expressions derived herein are used only for the analysis of single and continuous mixed steel-concrete beams. Finally, the validity of the proposed method and matrices is verified against available solutions. Two comprehensive examples are presented to validate the proposed method and equations.

5.3 Structural model

Fig. 4.1 shows the free-body diagram of the differential element of the coupled system, the external forces and moments applied at the ends of each layer and the normal and shear stresses along the interface connection. The top and bottom layers are loaded transversally along the span with an applied external load. The layers are made of isotropic linear elastic and homogenous materials with elastic moduli $E_i$ and $G_i$ (with i= 1, 2). The cross-sectional properties of each layer about the bending axis are the cross-sectional area $A_i$, moment of inertia $I_i$, and effective area for shear $A_{si}$. The two layers are connected by a continuous interface with a uniform vertical and horizontal stiffness $k_v$ and $k_h$, respectively.

5.4 Governing equations

The stiffness matrix and loading vector of the two-layer Timoshenko beam just described above are derived by applying the basic concepts of transverse and rotational equilibrium on the differential element shown in Fig. 5.1, the compatibility conditions at the ends of the member, and the constitutive laws of all materials involved. It is assumed small deflections and that plane sections remain plane after deformations along each layer.
Fig. 5.1 Sign convention for the bending moments, shears, deflections, and rotations at the ends.

5.5 Translational and Rotational Equilibrium.

The translational and rotational equilibrium equations are (Fig. 4.1):

\[
\frac{dV_1}{dx} = -q_1 + \sigma b 
\]  \hspace{1cm} (5.1)

\[
\frac{dV_2}{dx} = -q_2 - \sigma b  
\]  \hspace{1cm} (5.2)

\[
\frac{dP_1}{dx} = -\tau b  
\]  \hspace{1cm} (5.3)

\[
\frac{dP_2}{dx} = \tau b  
\]  \hspace{1cm} (5.4)

\[
\frac{dM_1}{dx} = V_1 + \tau bc_1 
\]  \hspace{1cm} (5.5)
5.6 Compatibility conditions and constitutive laws of all materials.

As shown by Timoshenko and Gere [52], the tangent to the center line at any point \( x \) along the beam span is caused by the bending moment \( \theta = dy_b/dx \), and by the transverse shear force \( \gamma = dy_s/dx \), that is: \( y/dx = dy_b/dx + dy_s/dx \). The subscripts \( b \) and \( s \) indicate bending and shear, respectively. Therefore:

\[
y''_1 = y''_{b,1} + y''_{s,1}
\]

\[
y''_2 = y''_{b,2} + y''_{s,2}
\]

The horizontal deformations caused by the applied axial load and the end bending moment at the bottom of layer 1 and the top of layer 2 are given by Eqs. (5.9) and (5.10), respectively. The subscript \( B \) indicates bottom and likewise \( T \) indicates top.

\[
u''_{1,B} = -\frac{P_1}{(EA)_1} + \frac{M_1c_1}{(EI)_1}
\]  

(5.9)

\[
u''_{2,T} = -\frac{P_2}{(EA)_2} - \frac{M_2d_2}{(EI)_2}
\]

(5.10)

The curvature and rotation caused by the bending moment and shear force at the center line of elements 1 and 2 are given by Eqs. by Eqs. (5.11), (5.12), (5.13), (5.14), as follows:

\[
y''_{b,1} = \frac{M_1}{(EI)_1}
\]

(5.11)
\[ y''_{b,2} = -\frac{M_2}{(EI)_2} \]  
(5.12)

\[ y'_{s,1} = \frac{V_1}{(A_sG)_1} \]  
(5.13)

\[ y'_{s,2} = \frac{V_2}{(A_sG)_2} \]  
(5.14)

Notice that the shear deformation induced by the axial load component is not considered in Eqs. (5.13), (5.14). Therefore, the proposed method is limited to the first-order analyses of coupled systems.

Assuming a linear-elastic isotropic material, the constitutive laws relating the stresses and displacements along the interface are as follows:

\[ \tau = k_h (u_{1,B} - u_{2,T}) \]  
(5.15)

\[ \sigma = k_v (y_1 - y_2) \]  
(5.16)

Where: \( k_h \) and \( k_v \) are the horizontal and vertical stiffness of the interface, respectively; \( u_{1,B} \), \( u_{2,T} \) and \( y_1 \), \( y_2 \) are the displacements along the X and Y directions of components 1 and 2, respectively.

### 5.7 Proposed solution.

Substituting Eqs. (5.11), (5.12) and the first derivative of Eqs.(5.13), (5.14) into Eqs. (5.7), (5.8), then Eqs. (5.17), (5.18) can be obtained:

\[ y''_1 = -\frac{M_1}{(EI)_1} + \frac{\sigma_b - q_1}{(A_sG)_1} \]  
(5.17)
5.7 Proposed solution.

\[ y''_2 = -\frac{M_2}{(EI)_2} - \frac{\sigma b + q_2}{(A_sG)_2} \]  \hspace{1cm} (5.18)

Eqs. (5.19) to (5.21) can be obtained by substituting Eqs. (5.9), (5.10) into the first derivative of Eq. (5.15) and then differentiating three times:

\[ \tau' = k_h \left( -\frac{P_1}{(EA)_1} + \frac{M_1c_1}{(EI)_1} + \frac{P_2}{(EA)_2} + \frac{M_2d_2}{(EI)_2} \right) \]  \hspace{1cm} (5.19)

\[ \tau'' - \alpha^2 \tau = k_h \left( \frac{c_1V_1}{(EI)_1} + \frac{d_2V_2}{(EI)_2} \right) \]  \hspace{1cm} (5.20)

\[ \tau''' - \alpha^2 \tau' - ck_h \sigma = -k_h \left( \frac{c_1q_1}{(EI)_1} + \frac{d_2q_2}{(EI)_2} \right) \]  \hspace{1cm} (5.21)

Substituting Eqs. (5.17), (5.18) into the second derivative of Eq. (5.16) and differentiating three times, Eqs. (5.22) to (5.24) can be obtained:

\[ \sigma'' - 4n^4 \sigma = k_v \left( \frac{M_2}{(EI)_2} - \frac{M_1}{(EI)_1} \right) + k_v \left( \frac{q_2}{(A_sG)_2} - \frac{q_1}{(A_sG)_1} \right) \]  \hspace{1cm} (5.22)

\[ \sigma''' - 4n^4 \sigma' + ck_v \tau = k_v \left( \frac{V_2}{(EI)_2} - \frac{V_1}{(EI)_1} \right) + k_v \left( \frac{q_2'}{(A_sG)_2} - \frac{q_1'}{(A_sG)_1} \right) \]  \hspace{1cm} (5.23)

\[ \sigma'''' - 4n^4 \sigma'' + ck_v \tau' + 4\beta^4 \sigma = k_v \left( \frac{q_1}{(EI)_1} - \frac{q_2'}{(EI)_2} \right) + k_v \left( \frac{q_2''}{(A_sG)_2} - \frac{q_1''}{(A_sG)_1} \right) \]  \hspace{1cm} (5.24)

Where:

\[ 4\beta^4 = k_v b \left( \frac{1}{(EI)_1} + \frac{1}{(EI)_2} \right) \] ; \[ 4n^4 = k_v b \left( \frac{1}{(A_sG)_1} + \frac{1}{(A_sG)_2} \right) \] ; \[ c = b \left( \frac{c_1}{(EI)_1} - \frac{d_2}{(EI)_2} \right) \] and \[ \alpha^2 = bk_h \left( \frac{1}{(EA)_1} + \frac{c_1^2}{(EI)_1} + \frac{d_2^2}{(EI)_2} \right) \]

Eq. (5.25) can be obtained in terms of the shear stress by substituting from Eq. (5.21) into Eq. (5.24):
\[ \tau^{'''''} - \delta_1 \tau^{''''} + \delta_2 \tau^{'''} - \delta_3 \tau' = \gamma_1 (q_1 + q_2) + \gamma_2 (q_1'' + q_2'') + \gamma_3 q_1''' + \gamma_4 q_2''' \]  

(5.25)

Where:

\[ \delta_1 = (\alpha^2 + 4\eta^4), \quad \delta_2 = 4 (\eta^4 \alpha^2 + \beta^4), \quad \delta_3 = 4\beta^4 \alpha^2 (1 - j), \]

\[ \gamma_1 = -\frac{k_h k_m (c_1 + d_2)}{(EI_1 (A_s G_2)} \left( \frac{c_1}{(EI_1 (A_s G_2)} + \frac{d_2}{(EI_2 (A_s G_1)} \right), \]

\[ \gamma_2 = b k_h k_v \left( \frac{c_1}{(EI_1 (A_s G_2)} + \frac{d_2}{(EI_2 (A_s G_1)} \right), \]

\[ \gamma_3 = -\frac{k_h c_1}{(EI_1)}, \quad \gamma_4 = -\frac{k_h d_2}{(EI_2)}, \quad j = \frac{k_h c_2}{4\beta^4 \alpha^2} \]

Expression (5.25) is a seventh-order non-homogeneous linear differential equation with constant coefficients that governs the elastic behavior of a two-layer Timoshenko beam subject to an external transversal load. Eq. (5.25) includes the coupling between the transverse and the relative longitudinal displacements along the interface and that between bending and shear deformations, and captures particular cases presented by other authors. For instance, when the shear deformations are neglected, it yields to the governing differential equation developed by Cosenza and Pecce [17], and when the transversal load is zero, it yields to the equation presented by Liu et al. [54].

The complete solution to Eq. (5.25) consists of two parts: the homogenous (\( \tau_p \)) and the particular solution (\( \tau_h \)) as follows:

\[ \tau = \tau_h + \tau_p \]  

(5.26)

The particular solution (\( \tau_p \)) depends on the transverse load applied to the two-layer beam, and \( \tau_h = \sum_{i=1}^{6} C_i \exp(m_i x) + R_1 \). The coefficients \( m_i \) of the homogenous solution can be found from the solution of the characteristic equation associated to Eq. (5.25) as follows:

\[ m^6 - (\alpha^2 + 4\eta^4) m^4 + 4 (\eta^4 \alpha^2 + \beta^4) m^2 - 4\beta^4 \alpha^2 (1 - j) = 0 \]  

(5.27)

The external loads \( q_1 \) and \( q_2 \) (see Fig. 4.1) are assumed to be second-order polynomials of the form \( q_1 = r_1 s_1 x + t_1 x^2 \) and \( q_2 = r_2 s_2 x + t_2 x^2 \), respectively. Then, the non-homogeneous solution of the differential equation becomes: \( \tau_p = j_1 x + j_2 x^2 + j_3 x^3 \).
Where:

\[
  j_1 = -\frac{2(\gamma_1 \delta_2 + \gamma_2 \delta_1)(t_1 + t_2) + \gamma_2 \delta_3 (r_1 + r_2)}{\delta_1^2}, \quad j_2 = -\frac{\gamma_1 (s_1 + s_2)}{2\delta_3}, \quad j_3 = -\frac{\gamma_1 (t_1 + t_2)}{3\delta_3}
\]

By substituting Eq. (5.26) into (5.21), the normal stress can be found as a linear combination of the integration constants, as follows:

\[
  \sigma = \frac{1}{ckh} \sum_{i=1}^{6} C_i m_i \left( m_i^2 - \alpha^2 \right) \exp(m_i x) + \frac{1}{ckh} \left( \tau_{p}''' - \alpha^2 \tau_{p}' - \gamma_3 q_1 - \gamma_4 q_2 \right)
\]  

5.8 Index notation.

For convenience, the following definitions with repeated index notation (known as Einstein's convention) are introduced:

\[
  \tau = \sum_{i=1}^{6} C_i f_{i,x} + R_1 + \tau_{p,x} = C_i f_{i,x} + R_1 + \tau_{p,x}
\]

\[
  \sigma = \sum_{i=1}^{6} C_i g_{i,x} + \sigma_{p,x} = C_i g_{i,x} + \sigma_{p,x}
\]

Where:

\[
  f_{i,x} = \exp(m_i x), \quad g_{i,x} = \frac{1}{ckh} m_i \left( m_i^2 - \alpha^2 \right) \exp(m_i x),
\]

\[
  \tau_{p,x} = j_1 x + j_2 x^2 + j_3 x^3, \quad \sigma_{p,x} = w_1 + w_2 x + w_3 x^2,
\]

\[
  w_1 = \left( \frac{6j_3 - \alpha^2 j_1 - \gamma r_1 - \gamma r_2}{ckh} \right), \quad w_2 = -\left( \frac{2\alpha^2 j_2 + \gamma s_1 + \gamma s_2}{ckh} \right), \quad w_3 = -\left( \frac{3\alpha^2 j_3 + \gamma t_1 + \gamma t_2}{ckh} \right)
\]

The following notation is introduced to represent the derivative and integral of the functions \( f_{i,x} \) and \( g_{i,x} \), respectively:

\[
  f_{i,x}^{(n)} = m_i^n \exp(m_i x); \quad \text{the } n^{th} \text{ derivative of the function } f_{i,x}; \text{ and}
\]

\[
  (n)f_{i,x} = m_i^{-n} \exp(m_i x); \quad \text{the } n^{th} \text{ integral of the function } f_{i,x}
\]
Similarly, the \( n^{th} \) derivative and integral of the function \( g_{i,x} \) can be expressed as follows:

\[
g^{(n)}_{i,x} = m_i^n \frac{1}{ck_h} m_i \left( m_i^2 - \alpha^2 \right) \exp(m_i x)
\]

\[
^{(n)}g_{i,x} = m_i^{-n} \frac{1}{ck_h} m_i \left( m_i^2 - \alpha^2 \right) \exp(m_i x)
\]

Note that the apostrophe on the right side represents a derivative and the one on the left side an integral.

5.9 Forces acting along the two layers.

The Timoshenko’s conventional sign convention shown in Fig. 4.1 and utilized in the derivation of the governing equation of the two-layer Timoshenko beam is not suitable for matrix analysis. Therefore, the following sign convention (i.e., positive directions) for the end forces and moments, as well as for the end rotations and transverse deflections are utilized in this publication to determine the stiffness matrix and load vector of the two-layer beam (Fig. 5.2).

Substituting Eqs. (5.26) and (5.28) into Eqs. (5.1) to (5.6) and carrying out the corresponding integrals, the forces and moments acting at the end of the elements can be expressed as follows:

\[
P_1 = -b \left( C_i f_{i,x} + R_{1,x} + \tau_{p,x} \right) + R_2
\]  

(5.29)

\[
V_1 = - q_{1,x} + b \left( C_i g_{i,x} + \sigma_{p,x} \right) + R_3
\]  

(5.30)
5.10 Displacements along the two layers.

The displacements and rotations along each layer can be determined by substituting Eqs. (5.26) and (5.28) into Eqs. (5.7) to (5.10), and carrying out the integrals resulting in the following expressions:

\[ M_1 = -''q_{1,x} + b(C_i''g_{i,x} + ''\sigma_{p,x}) + R_3x + bc_1(C_i'f_{i,x} + R_1x + '\tau_{p,x}) + R_4 \] (5.31)

\[ P_2 = b(C_i'f_{i,x} + R_1x + '\tau_{p,x}) + R_5 \] (5.32)

\[ V_2 = -'q_{2,x} - b(C_i'g_{i,x} + '\sigma_{p,x}) + R_6 \] (5.33)

\[ M_2 = -''q_{2,x} - b(C_i''g_{i,x} + ''\sigma_{p,x}) + R_6x + bd_2(C_i'f_{i,x} + R_1x + '\tau_{p,x}) + R_7 \] (5.34)

5.10 Displacements along the two layers.

The displacements and rotations along each layer can be determined by substituting Eqs. (5.26) and (5.28) into Eqs. (5.7) to (5.10), and carrying out the integrals resulting in the following expressions:

\[ u_1 = -\frac{1}{(EA)_1} \left[ -b(C_i''f_{i,x} + R_1\frac{x^2}{2} + ''\tau_{p,x}) + R_2x \right] + R_8 \] (5.35)

\[ y_1 = -\frac{1}{(EI)_1} \left[ -'''q_{1,x} + b(C_i'''g_{i,x} + '''\sigma_{p,x}) + R_3\frac{x^3}{6} + bc_1(C_i'''f_{i,x} + R_1\frac{x^3}{6} + '''\tau_{p,x}) + \right. \]
\[ \left. R_4\frac{x^2}{2} \right] + R_9x + \frac{1}{(A_G)_1} \left[ -''q_{1,x} + b(C_i''g_{i,x} + ''\sigma_{p,x}) + R_3x \right] + R_{10} \] (5.36)

\[ \theta_1 = -\frac{1}{(EI)_1} \left[ -'''q_{1,x} + b(C_i'''g_{i,x} + '''\sigma_{p,x}) + R_3\frac{x^2}{2} + bc_1(C_i''f_{i,x} + R_1\frac{x^2}{2} + ''\tau_{p,x}) + \right. \]
\[ \left. R_4x \right] + R_9 \] (5.37)

\[ u_2 = -\frac{1}{(EA)_2} \left[ b(C_i''f_{i,x} + R_1\frac{x^2}{2} + ''\tau_{p,x}) + R_5x \right] + R_{11} \] (5.38)
Stiffness matrix and loading vector of a two-layer Timoshenko beam.

\[ y_2 = -\frac{1}{(EI)^2} \left[ -^{'''}q_{2,x} - b \left( C_i^{'''}g_{i,x} + ^{'''}\sigma_{p,x} \right) + R_6 \frac{x^3}{6} + bd_2 \left( C_i^{'''}f_{i,x} + R_1 \frac{x^3}{6} + ^{'''}\tau_{p,x} \right) + R_7 \frac{x^2}{2} \right] + R_{12}x + \frac{1}{(A_sG)^2} \left[ -^{''}q_{2,x} - b \left( C_i^{''}g_{i,x} + ^{''}\sigma_{p,x} \right) + R_6x \right] + R_{13} \quad (5.39) \]

\[ \theta_2 = -\frac{1}{(EI)^2} \left[ -^{'''}q_{2,x} - b \left( C_i^{'''}g_{i,x} + ^{'''}\sigma_{p,x} \right) + R_6 \frac{x^2}{2} + bd_2 \left( C_i^{'''}f_{i,x} + R_1 \frac{x^2}{2} + ^{'''}\tau_{p,x} \right) + R_7x \right] + R_{12} \quad (5.40) \]

Fig. 5.3 shows the composite beam with its layers (\(i = 1, 2\)) subject to axial and shear forces (\(P_{i,j}\) and \(V_{i,j}\), respectively) as well as to bending moments (\(M_{i,j}\)) at both ends (\(j = 0\) and \(L\)) as follows:

Fig. 5.3 Applied transverse loads along the two-layer beam and forces and moments at ends \(x = 0\) and \(L\).

At \(X = 0\):
\[ M_1 = M_{1,0} ; P_1 = P_{1,0} ; V_1 = V_{1,0} ; M_2 = M_{2,0} ; P_2 = P_{2,0} ; V_2 = V_{2,0} \]

At \(X = L\):
\[ M_1 = M_{1,L} ; P_1 = P_{1,L} ; V_1 = V_{1,L} ; M_2 = M_{2,L} ; P_2 = P_{2,L} ; V_2 = V_{2,L} \]
5.11 Stiffness Matrix.

In order to determine the shear and moments as well as the displacements and rotations described in Eqs (5.29) to (5.34) and (5.35) to (5.40), respectively, it is necessary to determine 19 constants of integration (i.e. \( C_i \), \( R_k \) with \( i = 1, 2, \ldots, 6 \) and \( k = 1, 2, 3, \ldots, 13 \)). Since only twelve boundary conditions exist (i.e., twelve linear equations), it is necessary to find seven additional linear equations relating the integration constants. This system of linear equations is found by substituting Eq. (5.35) to (5.40) into Eqs. (5.15) and (5.16), and is presented in Eq. (5.41) relating only the \( R_k \) constants.

\[
\{ R^1 \} = [H] \{ R^2 \} + \{ E \} \tag{5.41}
\]

Where:

\[
\{ R^1 \} = \{ R_1, R_2, R_3, R_4, R_5, R_6, R_8, R_{10} \}^T \quad \text{and} \quad \{ R^2 \} = \{ R_5, R_7, R_9, R_{11}, R_{12}, R_{13} \}^T
\]

Matrices \([H]\), \([R]\) and \([E]\) are listed in Appendix I. The following expressions are obtained by evaluating Eqs (5.29) to (5.34) and Eqs (5.35) to (5.40) at \( x = 0 \) and \( x = L \), and considering the sign convention presented in Fig. 5.2:

\[
\{ F \} = [M_1] \{ C \} + [M_2] \{ R^1 \} + [M_3] \{ R^2 \} + \{ J \} \tag{5.42}
\]

\[
\{ \delta \} = [M_4] \{ C \} + [M_5] \{ R^1 \} + [M_6] \{ R^2 \} + \{ L \} \tag{5.43}
\]

Where:

\[
\{ F \} = \{ P_1^0, V_1^0, M_1^0, P_2^0, V_2^0, M_2^0, P_1^L, V_1^L, M_1^L, P_2^L, V_2^L, M_2^L \}^T
\]

\[
\{ \delta \} = \{ u_1^0, y_1^0, \theta_1^0, u_2^0, y_2^0, \theta_2^0, u_1^L, y_1^L, \theta_1^L, u_2^L, y_2^L, \theta_2^L \}^T
\]

\[
\{ R^1 \} = \{ R_1, R_2, R_3, R_4, R_5, R_6, R_{10} \}^T
\]

\[
\{ R^2 \} = \{ R_5, R_7, R_9, R_{11}, R_{12}, R_{13} \}^T
\]
For convenience and quick reference, matrices $[M_1]$, $[M_2]$, $[M_3]$, $[M_4]$, $[M_5]$, $[M_6]$ and $[M_1]$ are also listed in Appendix I.

Now, by substituting Eq (5.41) into Eqs (5.42) and (5.43), the following matrices are obtained:

\[
\{F\} = \begin{bmatrix} M_1 & M_2H + M_3 \end{bmatrix} \begin{bmatrix} C \\ R^2 \end{bmatrix} + \{M_2E + J\} 
\]

(5.44)

\[
\{\delta\} = \begin{bmatrix} M_4 & M_5H + M_6 \end{bmatrix} \begin{bmatrix} C \\ R^2 \end{bmatrix} + \{M_5E + L\} 
\]

(5.45)

Finally, by replacing Eq. (5.45) into Eq. (5.44), the following matrix is found:

\[
\{F\} = [K] \{\delta\} + \{M_{EP}\} 
\]

(5.46)

Where:

\[
[K] = \begin{bmatrix} M_1 & M_2H + M_3 \\ M_4 & M_5H + M_6 \end{bmatrix}^{-1} 
\]

(5.47)

\[
M_{EP} = \{M_2E + J\} - [K] \{M_5E + L\} 
\]

(5.48)

Expression $\{F\} = [K] \{\delta\}$ is obtained from Eq. (5.46) when the transverse loads are zero (i.e., $q_1 = q_2 = 0$). Note that Eq. (5.47) represents the stiffness matrix of a two-layer Timoshenko beam relating the vector of end moments and shears $\{F\}$ with the vector of displacements and rotations $\{\delta\}$. $[K]$ is 12x12 matrix corresponding to three degrees of freedom at each layer end. The stiffness matrix takes into account the coupling effects between shear and bending deformations of each layer as well as those between the normal and shear stresses along the interface. The degree of interaction between the shear and normal stresses is governed by the parameter $j$ according to Cosenza and Pecce [17]. When $j = 0$, Eqs (5.21) and (5.24) become
5.12 Comprehensive examples and verification.

5.12.1 Example 1. Simply supported concrete-steel composite beam

Fig. 5.4 shows the concrete-steel composite beam experimentally tested by Abdel Aziz [10]. The geometrical and mechanical properties of the composite beam as well as the structural model and degrees of freedom are included in Fig. 5.4. The shear area correction factor are designated by $k_{s,1}$ and $k_{s,2}$ and are taken as 2/3 and 5/6, respectively.

Solution. The beam was divided into two two-layer Timoshenko beam segments with each segment consisted of twelve degrees of freedom. The stiffness matrix of each segment was determined using the proposed method and it is as follows:

$$
\begin{bmatrix}
1.11E+06 & 9.21E+03 & -1.87E+06 & -2.18E+05 & 1.74E+04 & -1.95E+07 & -7.88E+05 & -3.80E+03 & 1.26E+06 & -1.02E+05 & -2.28E+04 & 6.64E+06 \\
3.49E+04 & 1.01E+07 & -9.21E+03 & -3.19E+04 & -8.62E+06 & 3.80E+03 & 1.00E+03 & -3.12E+05 & -3.80E+03 & -4.02E+03 & 3.12E+06 \\
6.14E+09 & 1.67E+06 & -7.96E+06 & -2.16E+09 & 1.26E+06 & 3.02E+05 & 3.62E+07 & -1.26E+06 & -2.41E+06 & 1.54E+09 \\
9.28E+05 & -1.74E+04 & 1.95E+07 & -1.02E+05 & 3.80E+03 & -1.26E+06 & -6.08E+05 & 2.28E+04 & -6.64E+06 \\
6.68E+04 & 4.72E+07 & 2.28E+04 & -4.02E+03 & 2.41E+06 & -2.28E+04 & -3.08E+04 & 3.54E+07 \\
7.49E+10 & 6.44E+06 & -3.12E+06 & 1.54E+09 & -6.64E+06 & -3.54E+07 & 2.56E+10 \\
1.11E+06 & -9.21E+03 & -1.67E+06 & -2.16E+09 & -1.74E+04 & -1.95E+07 \\
9.28E+05 & -1.74E+04 & 1.95E+07 & -1.02E+05 & 3.80E+03 & -1.26E+06 & -6.08E+05 & 2.28E+04 & -6.64E+06 \\
6.68E+04 & 4.72E+07 & 2.28E+04 & -4.02E+03 & 2.41E+06 & -2.28E+04 & -3.08E+04 & 3.54E+07 \\
7.49E+10 & 6.44E+06 & -3.12E+06 & 1.54E+09 & -6.64E+06 & -3.54E+07 & 2.56E+10 \\
1.11E+06 & -9.21E+03 & -1.67E+06 & -2.16E+09 & -1.74E+04 & -1.95E+07 \\
9.28E+05 & -1.74E+04 & 1.95E+07 & -1.02E+05 & 3.80E+03 & -1.26E+06 & -6.08E+05 & 2.28E+04 & -6.64E+06 \\
6.68E+04 & 4.72E+07 & 2.28E+04 & -4.02E+03 & 2.41E+06 & -2.28E+04 & -3.08E+04 & 3.54E+07 \\
7.49E+10 & 6.44E+06 & -3.12E+06 & 1.54E+09 & -6.64E+06 & -3.54E+07 & 2.56E+10 \\
\end{bmatrix}
$$

Fig. 5.5 shows the slip/shear-force diagram of the connection for a single stud between the concrete and steel. The maximum shear strength of the stud connector is taken after Ollgaard et
Stiffness matrix and loading vector of a two-layer Timoshenko beam.

Fig. 5.4 Geometrical and mechanical properties, structural model and degrees of freedom of composite Steel-Concrete beam.

al. [34] as $Q_{max} = \frac{1}{2}A_s\sqrt{f'_cE_c} \leq 65A_s (ksi)$ [where $f'_c$ is the concrete compressive strength (ksi), $E_c$ the modulus of elasticity (ksi), and $A_s$, the cross sectional area of the stud shear connector (in$^2$)]. The two independent parameters describing the mechanical behavior of a type-C headed stud are $\alpha = 0.8$ and $\beta = 0.7mm^{-1}$ (Fabbrocino et al. [25]; Johnson and Molenstra [35]). To determine the horizontal shear stiffness connection $k_h$ is recommended to use the secant stiffness $k_{50\%}$. Then:

$$k_h = \frac{k_{50\%} \#bolts}{Lb} = \frac{83409 \times 18}{5000 \times 180} = 1.67 \frac{N}{mm^3}$$

Fig. 5.6 shows the normalized peel and shear stresses along the interface, the axial and shear forces and the bending moments at the top and bottom layers of the simply supported beam of Example 1. Fig. 5.6a shows that the compressive normal stresses acting along the interface are greater near the supports and under the applied concentrated load and then they become almost zero or tension normal stresses on the rest of the span. On the other hand, the shear stresses are maxima at the ends of the element and zero at the center of the span. The shear stresses follow a stress distribution similar to that predicted by the Jourawsky’s theory for a simply supported beam, where due to the symmetry of the problem, the shear stress is zero at the point of application of the concentrated load. Fig. 5.6b shows the internal axial force acting at the top and bottom elements. This force variation is the result of the shear flow developed along the interface. Since the external axial force is zero, satisfaction of the axial equilibrium implies that $P_1 = -P_2$, as shown in Fig. 5.6b. Fig. 5.6c shows the shear load diagram. Because the concentrated load is applied directly on top of element 1, the shear force distribution acting
5.12 Comprehensive examples and verification.

Fig. 5.5 Shear connection relationship (after Ollgaard, 1971)

on element 1 at midspan is discontinuous (abrupt jump). On the other side, and because the support reactions are acting on element 2, there is a smooth transition of the shear force at midspan and it is maximum at the supports. Note that $V_1 + V_2 = V_{ext}$, which implies that shear forces equilibrium is met. Fig. 5.6d presents the bending moments acting on the two elements. From static equilibrium, it is known that total external moment ($M_{ext}$) is balanced by the internal couple formed by the axial load and the moments acting on the two elements. Therefore, from equilibrium $M_1 + M_2 + P_1 (d_1 + d_2) = M_{ext}$.

Fig. 5.7 shows the slip distribution over the span of the beam when subjected to two different loads (257 and 344 kN) and calculated using the proposed model as well as the theoretical results reported by Fabbrocino et al. [25] and those obtained experimentally by Abdel Aziz [10]. Both figures show that the results from the linear matrix analysis proposed herein compared well with the experimental results and those obtained with the nonlinear analysis method proposed by Fabbrocino.

Fig. 5.8 shows the load-deflection curve response measured at midspan of the beam (bottom of layer 2). As shown in this figure, there is an excellent agreement between the results obtained experimentally and those calculated using the proposed method.
Fig. 5.6 Example 1: a) Stresses along the interface, b) axial forces, c) shear forces and d) bending moments along the layers 1 and 2
Fig. 5.7 Example 1: Slip diagrams for two different loads (Theoretical-vs-experimental results)

Fig. 5.8 Example 1: Load-deflection curve (Theoretical-vs-experimental results)
5.12.2 Example 2. Two-span continuous steel-concrete composite beam

For the two-span continuous steel-concrete composite beam shown in Fig. 5.9 determine: a) the peel and shear stresses along the interface and b) the axial force, shear force and bending moments along the top and bottom elements. Beam (CTB6) was part of an experimental program conducted by Ansourian [3]. The beam has two spans of 4.50 m each subject to its self-weight of 3.3 kN/m and to a concentrated load of 160 kN applied at the center of each span. The dimensions and loading conditions are shown in Fig. 5.9. The elastic modulus $E$ and shear modulus $G$ used for the analysis are: $E_1 = 34000$ MPa and $G_1 = 14167$ MPa for the top element, and $E_2 = 210000$ MPa and $G_2 = 80769$ MPa for the bottom element. In this work, shear stiffness factors of 2/3 and 5/6 were adopted for the steel beam and concrete slab sections, respectively. The horizontal shear stiffness is computed as $k_h = \frac{k_{sag} \times \text{#bolts}}{L_b} = \frac{190.8 \times 60}{9000 \times 120} = 10.6 \frac{N}{mm^2}$. Assume that $k_v = k_h/5$ as suggested by Cosenza and Pecce [17].

Solution: Fig. 5.10 shows the structural model and degrees of freedom and their numbering used for the steel-concrete composite beam. The continuous beam was divided into four two-layer Timoshenko beam segments. The stiffness matrix and load vector of each segment are:

![Diagram showing the structural model and degrees of freedom](image-url)
the axial load to the moment equilibrium is significant. and 2 are relatively low when compared to the total moment, indicating that the contribution of

5.12 Comprehensive examples and verification.

Fig. 5.10 Example 2: Structural model and degrees of freedom for composite beam

\[
[K] = \begin{bmatrix}
2.55E + 06 & 1.76E + 04 & -1.12E + 07 & -5.85E + 05 & 3.31E + 04 & -3.12E + 07 & -1.73E + 06 & -1.89E + 04 & 6.91E + 06 & -2.34E + 05 & -3.19E + 04 & 1.03E + 07 \\
1.06E + 05 & 2.43E + 07 & -1.76E + 04 & -9.87E + 04 & -1.88E + 04 & -2.05E + 03 & 1.41E + 06 & -1.89E + 04 & -5.47E + 03 & 3.85E + 06 \\
1.33E + 10 & 1.12E + 07 & -1.93E + 07 & -9.14E + 09 & 8.91E + 06 & -1.41E + 06 & 9.23E + 08 & -6.91E + 06 & -3.66E + 06 & 1.89E + 09 \\
9.60E + 05 & -3.31E + 04 & 3.12E + 07 & -2.34E + 05 & 1.89E + 04 & -6.91E + 06 & -3.19E + 05 & -3.19E + 04 & 4.93E + 07 \\
1.11E + 05 & 2.79E + 07 & 3.19E + 04 & -5.47E + 03 & 3.66E + 06 & -3.19E + 04 & -5.47E + 03 & 3.85E + 06 & 5.24E + 06 \\
2.33E + 10 & 1.03E + 07 & -3.85E + 06 & -3.51E + 04 & 3.12E + 07 & -2.45E + 07 & 1.76E + 04 & -5.85E + 05 & -3.12E + 07 \\
2.55E + 06 & -1.76E + 04 & 1.12E + 07 & -5.85E + 06 & -3.51E + 04 & 3.12E + 07 & -2.45E + 07 & 1.76E + 04 & -5.85E + 05 & -3.12E + 07 \\
1.06E + 05 & -2.45E + 07 & 1.76E + 04 & -5.85E + 05 & -3.12E + 07 & -2.45E + 07 & 1.76E + 04 & -5.85E + 05 & -3.12E + 07 \\
2.33E + 10 & 1.03E + 07 & -3.85E + 06 & -3.51E + 04 & 3.12E + 07 & -2.45E + 07 & 1.76E + 04 & -5.85E + 05 & -3.12E + 07 \\
1.33E + 10 & 1.12E + 07 & -3.85E + 06 & -3.51E + 04 & 3.12E + 07 & -2.45E + 07 & 1.76E + 04 & -5.85E + 05 & -3.12E + 07 \\
9.35E + 05 & 3.31E + 04 & 3.12E + 07 & -2.79E + 07 & 1.11E + 05 & -2.79E + 07 & 1.11E + 05 & -2.79E + 07 & 1.11E + 05 & -2.79E + 07 \\
9.35E + 05 & 3.31E + 04 & 3.12E + 07 & -2.79E + 07 & 1.11E + 05 & -2.79E + 07 & 1.11E + 05 & -2.79E + 07 & 1.11E + 05 & -2.79E + 07
\end{bmatrix}

And:

\[
[M_E] = \begin{bmatrix}
1338.6 & 2338 & 659410 & -1338.6 & 1374.53 & 505220 & -1338.6 & 2338 & -659410 & 1338.6 & 1374.5 & -505220 \\
\end{bmatrix}^T
\]

Fig. 5.11 shows the load-deflection curve response of the linear portion of beam CTB6, measured at the beam midspan (bottom of layer 2, i.e. degrees of freedom 19 and 23) and compared to those obtained using the proposed model. As shown in the figure, there is a good agreement between the results obtained experimentally by Ansourian [4] and those calculated using the proposed method.

Fig. 5.12 shows the interface stresses and internal forces and moments for the first span of the continuous composite beam of Example 2. Figs. 5.12a and 5.12b show the normal and shear stresses and axial load diagrams, respectively. The results are similar to those presented and discussed in Example 1. From Fig. 5.12c, it is interesting to notice that the shear force acting on element 1 (\(V_1\)) at the point of application of \(P\) has a magnitude greater than the value of the applied external shear force (\(V_{ext}\)) and the static equilibrium at that point still holds. Fig. 5.12d shows that for the continuous composite beam the bending moments acting on layers 1 and 2 are relatively low when compared to the total moment, indicating that the contribution of the axial load to the moment equilibrium is significant.
Fig. 5.11 Example 2: Midspan Deflection of beam CTB6 (Theoretical-vs-experimental results).

Fig. 5.12 Example 2: a) Stresses along the interface, b) axial forces, c) shear forces and d) bending moments along layers 1 and 2.
5.13 Summary and conclusions

A matrix method suitable for the analysis of frame systems made of two-layer prismatic Timoshenko beams is derived in a classical manner. The proposed method includes the effects of the interaction between the normal and shear stresses, the coupling between the bending and shear deformations and the longitudinal and transverse partial interaction along the interface of the coupled layers. The 12x12 stiffness matrix and the corresponding load vector are presented. The expression developed for the load vector considers any type or combination of transverse loads that fit a second-order polynomial function. One of the main advantages of the proposed method is that the 12x12 stiffness matrix can be coupled with 6x6 first- and second-order matrices so frame systems composed of single and coupled elements can be analyzed. The transfer functions necessary to determine the interface stresses, axial and shear forces, bending moments, deflections and rotations along the members are also presented. The expressions derived herein can be implemented in the analysis of structural elements retrofitted or strengthened with fiber reinforced polymer materials. Finally, two comprehensive examples are presented in detail to demonstrate the effectiveness and validity of the proposed method and corresponding matrices and the calculated values verified against available experimental results.

5.14 Acknowledgements

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5.15 Appendix I-. List of matrices.

\[ [H] = - [H_1]^{-1} [H_2], \quad \{E\} = - [H_1]^{-1} \{S\} \]

\[ \{S\} = \left\{ 0, -\frac{j_1}{k_h}, -\frac{j_2}{k_h}, -\frac{w_1}{k_v}, -\frac{w_2}{k_v}, -\frac{2w_3}{k_v}, \frac{b w_1 - r_1}{(A_s G)_1}, \frac{b w_1 + r_2}{(A_s G)_2}, \frac{b w_2 - s_1}{(A_s G)_1} \right\}^T \]
Stiffness matrix and loading vector of a two-layer Timoshenko beam.

\[
[H_1] = \begin{bmatrix}
\frac{-1}{k_h} & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{-1}{(EA)_1} & 0 & \frac{c_1}{(EI)_1} & 0 & 0 & 0 \\
\frac{\alpha^2}{k_h} & 0 & \frac{c_1}{(EI)_1} & 0 & \frac{d_2}{(EI)_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{1}{(A_s G)_1} & 0 & \frac{-1}{(EI)_1} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{(EI)_1} & \frac{1}{(EI)_1} & 0 & 0 \\
-c & 0 & \frac{-1}{(EI)_1} & 0 & \frac{1}{(EI)_1} & 0 & 0 \\
0 & 0 & c_1 & -1 & -d_2 & 0 & 0 \\
\frac{1}{(EA)_2} & \frac{d_2}{(EI)_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & \frac{1}{(EI)_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
[H_2] = \begin{bmatrix}
-\frac{f_i,0}{g_i,0} \\
\frac{f_i,0}{g_i,0} \\
-\frac{g_i,0 - c_1 f_i,0}{g_i,0} \\
\frac{f_i,0}{f_i,0} \\
-\frac{g_i,0}{g_i,0} \\
\frac{g_i,0 - d_2 f_i,0}{g_i,0} \\
\frac{f_i,L}{f_i,L} \\
-\frac{g_i,L}{g_i,L} \\
\frac{g_i,L + c_1 f_i,L}{g_i,L} \\
-\frac{f_i,L}{f_i,L} \\
\frac{f_i,L}{g_i,L} \\
-\frac{g_i,L + d_2 f_i,L}{g_i,L}
\end{bmatrix}, \quad [M_2] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
bL & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
bLc_1 & 0 & L & 1 & 0 & 0 & 0 \\
-bL & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
bd_2L & 0 & 0 & 0 & L & 0 & 0
\end{bmatrix}
\]

\[[M_1] = b\]
\[ [M_3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ [M_4] = b \]

\[ [M_5] = \begin{bmatrix} \frac{b t^2}{2EA_1} & -\frac{L}{EA_1} & 0 & 0 & 0 & 0 \\ \frac{b c_1 l^3}{6EI_1} & 0 & \frac{L}{(A_s G_1)} & -\frac{L^3}{6(EI_1)} & -\frac{L^2}{2(EL_1)} & 0 \\ -\frac{b c_1 l^2}{2(EL_1)} & 0 & 0 & \frac{L}{(A_s G_1)} & -\frac{L^3}{6(EI_1)} & 0 \\ \frac{b l^2}{2(EL_1)} & 0 & 0 & 0 & \frac{L}{(A_s G_1)} & -\frac{L^3}{6(EI_1)} \\ -\frac{bd L^2}{2(EL_1)} & 0 & 0 & 0 & 0 & \frac{L}{(A_s G_1)} \\ \frac{b d L^2}{2(EL_1)} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
Note that $[M_6]$ and $[L]$ are 12x6 matrices expressed in terms of the functions $f_{i,x}$ and $g_{i,x}$ as described in the notation section.
6.1 Abstract:

A novel linear matrix method to analyze balanced and unbalanced adhesive joints is presented in this paper. The stiffness matrix, loading vector and transfer functions for the analysis of adhesive joints such as stiffened plate, single-strap and single-lap joints are derived in a classic manner. A matrix formulation is derived for a two-layer beam composed of two elements (adherends) and an adhesive. The proposed method takes into account the following effects: a) longitudinal and transversal relative displacements along the interface; b) the interaction between the normal and shear stresses; c) the coupling between the axial, bending and shear deformations in the adherends; and d) the transverse load applied on the two adherends. The expressions developed for the load vector are general for any type, or combination, of transverse load that fits a second-order polynomial curve, including uniformly distributed transverse load, trapezoidal and parabolic loads. The transfer functions necessary to determine the adhesive stresses, axial and shear forces, bending moments, deflections and rotations along the members are also presented in detail. Two comprehensive examples are presented to show the effectiveness and validity of the proposed method and corresponding equations.

KEYWORDS: Stiffness matrix; loading vector; two-layer beam; adhesive joints.
6.2 Introduction

The use and construction of adhesive joints is used commonly for certain engineering applications. These joints consist of three main components, two adherends and an adhesive material, with similar (balanced joints) or different (unbalanced joints) geometrical and mechanical properties. Adhesively bonded joints are the most commonly and widely used joints in several industries such as aerospace, astronautic, energy, civil and mechanical, among others—This is due to their superior physical and mechanical properties in comparison with other available materials. Stiffened plate, single-strap and single-lap joints are among the most common types of adhesive joints.

Numerous experimental and theoretical studies have been conducted to determine the distribution and magnitude of the stresses developed along the interface of adhesive joints. Several expressions have been proposed for the stress analysis using empirical, classical, finite element and finite difference methods (Andruet et al. 2001; Budhe, S. 2017; Chen and Qiao 2012; He 2011; Her 1999; Kumar and Tampi 2016; Neto et al. 2012; Panigrahi 2013; Rodríguez et al. 2011; Romilly and Clark 2008; Rudawska 2010; Weißgraeber et al. 2014; Zhao et al. 2011 among others).

Her (1999) presented an analytical solution based on the classical elastic theory to analyze single-lap and double-lap joints. The expressions developed by Her did not include the effects of the peeling stresses on the adhesive and the effects of the bending and shear deformations on the adherends. Zou et al. (2004) used the classical laminate theory and constitutive equations of the adhesive to develop an expression for the analysis of balanced stiffened plate, single-lap and single-strap joints. Zou’s model takes into account the peeling and shear stresses along the adhesive interface and the axial and bending deformations of the adherends but it neglects the shear effects on the member deformation. Zhao et al. (2011) derived 2D closed-form analytical expressions for the elastic analysis of stresses of unbalanced adhesive single-lap joints and compared the calculated results with previously published results from analytical and nonlinear finite element methods. Zhao et al. (2011) also presented the complete stress-strain and stress-displacement equations for the adherends and the adhesive. More recently, Liu et al. (2014) presented an analytical solution for the linear elastic analysis of balanced and unbalanced adhesively bonded joints with different displacement boundary conditions subject to external forces. Their model includes the effects of the normal and shear stresses along the adhesive interface and the deformations induced by the axial and shear forces as well as the bending
moments. In their approach, it is necessary to solve a set of linear equations of each type of joint to determine the integration constants required to solve the proposed expressions.

On the other hand, several researchers have conducted linear and non-linear finite element analysis, implementing very efficient elements and advanced methods for the analysis of adhesively bonded joints. Among these works, Tsai and Morton (1995) conducted an experimental and numerical study on laminated composite single-lap joints to assess the effect of a spew fillet on the adhesive stresses. The results showed good agreements between the in-plane surface deformations obtained experimentally on some specimens and those obtained numerically by implementing a nonlinear finite element analysis. Andruet et al. (2001) developed two- and three-dimensional finite elements to analyze adhesively bonded joints of arbitrary geometry of the adhesive layer including the effects of a crack on the element. However, his model required more than 250 elements to obtain a good estimate of the joint behavior. Castagnetti and Dragoni (2006) presented a finite element method to estimate the post-elastic response of complex bonded structures. In their FE method, the authors used a computational technique named Tied Mesh and modeled the adherends as plates or shell elements and the adhesive as a single cohesive element.

The main objective of this paper is to present a novel classical approach for the analysis of various types of adhesive bonded joints (single-lap, single-strap, stiffened plate, L- and T-shape joints, etc.) based on the well-known concepts of the stiffness matrix theory for structural analysis of framed structures. The derivation of the 12x12 stiffness matrix, the 12x6 loading vector and the transfer functions of a two-layer beam element is presented and discussed in detail. The proposed methodology includes the effects of (a) the transversal and longitudinal relative displacements along the interface, (b) the interaction between the normal and shear stresses, (c) the coupling between axial, flexural and shear deformations and (d) any transverse load applied on the adherends that fits a second-order polynomial function (uniformly distributed transverse load, trapezoidal and parabolic loads). The proposed 12x12 matrix can be coupled with 6x6 matrices developed for single beam elements (Areiza-Hurtado et al. 2005), so joints such as the T- and L-shape joints, where the adherends extend beyond the end of the adhesive, can be analyzed. The transfer functions necessary to determine the adhesive stresses, axial and shear forces, bending moments, deflections and rotations along the members are also derived.

Finally, the validity of the proposed method and the corresponding matrices is verified against available solutions. Two comprehensive examples are presented to validate the proposed method and equations. Example 1 presents the adhesive stresses results for three types of unbalanced adhesively bonded steel-aluminum joints (stiffened plat, single-strap and single-lap
A novel linear matrix method to analyze adhesive joints. The solution to the single-strap joint presents a case that involves the assembling of two-layer beam elements. The effect of a uniformly distributed transverse load on the adhesive stress distribution of a single-lap joint was also analyzed. Example 2 compares the results of an experimental program conducted on a balanced single-lap joint with those obtained with the proposed method. The two examples presented herein show the effectiveness and validity of the proposed matrix method.

### 6.3 Structural model.

Fig. 6.1 shows the free-body diagram of the differential elements comprising the adhesively bonded joint, the external forces and moments applied at the element end and the normal and shear stresses along the interface of the adhesive. The elastic bending and shear moduli of the adherends are $E_i$ and $G_i$, respectively. The cross-sectional properties of each member, about the bending axis, are the cross-sectional area $A_i$, moment of inertia $I_i$, and effective area for shear $A_{si}$. The subscript $i$ indicates the top $i = 1$ and bottom $i = 2$ elements. The uniform vertical and horizontal stiffnesses of the adhesive are $k_v$ and $k_h$, respectively.

![Fig. 6.1 Structural model and free body diagrams.](image)

### 6.3.1 Governing equations.

The proposed stiffness matrix and loading vector of the adhesively joint described above are derived in a classical manner by applying the basic concepts of transversal and rotational
equilibrium on the differential element shown in Fig. 6.1, the compatibility conditions at the ends of the member and the constitutive laws of all materials involved. It is assumed small deflections and that plane sections remain plane after deformations along each layer. The linear elastic response of the two-layer beam described above and subject to an external transversal load is described by Eq. 6.1. This matrix model was previously developed by Areiza-Hurtado et al (2019) to study two-layer Timoshenko composite beams. In this paper, however, the matrix model and equations are used to analyze adhesive joints. Derivation of Eq. 6.1 is presented in section 5.5 of the previous chapter.

\[
\tau^{VII} - \delta_1 \tau^V + \delta_2 \tau^{III} - \delta_3 \tau^I = \gamma_1 (q_1 + q_2) + \gamma_2 (q_1^H + q_2^H) + \gamma_3 q_1^V + \gamma_4 q_2^V \tag{6.1}
\]

Where: 
\[
\delta_1 = (\alpha^2 + 4\eta^4); \quad \delta_2 = 4 (\eta^4 \alpha^2 + \beta^4); \quad \delta_3 = 4 \beta^4 \alpha^2 (1 - j); \quad \gamma_1 = -\frac{k_h k_v (c_1 + d_2)}{(ET)_1^2 (ET)_2}; \quad \gamma_2 = b k_h k_v \left(\frac{c_1}{(ET)_1 (\alpha, G)} + \frac{d_2}{(ET)_2 (\alpha, G)}\right); \quad \gamma_3 = -k_h \frac{c_1}{(ET)_1^2}; \quad \gamma_4 = -k_h \frac{d_2}{(ET)_2^2}; \quad \text{and} \quad j = \frac{k_h k_v c_2^2}{4 \beta^4 \alpha^2}
\]

Eq. 6.1 is a seventh-order nonhomogeneous linear differential equation with constant coefficients. The complete solution to Eq. 6.1 consists of a homogenous solution \( \tau_h \) and the particular solution \( \tau_p \), which are also presented in Chapter 5.

### 6.3.2 Stiffness Matrix and Loading Vector.

The stiffness matrix of a two-layer Timoshenko beam relating the vectors of end moments and shears \( \{F\} \) with the vectors of displacements and rotations \( \{\delta\} \) is giving by Eq. (6.2)

\[
\{F\} = [K] \{\delta\} + \{M_{EP}\} \tag{6.2}
\]

Where:

\[
[K] = [M_1 \quad M_2 H + M_3] [M_4 \quad M_5 H + M_6]^{-1} \tag{6.3}
\]

\[
\{M_{EP}\} = \{M_2 E + J\} - [K] \{M_5 E + L\} \tag{6.4}
\]

Matrices and vectors \([M_1], [M_2], [M_3], [M_4], [M_5], [M_6], [H], [E], [J]\) and \([L]\) are listed in Appendix I of Chapter 5.
\( \{F\} = [K] \{\delta\} \) can be obtained from Eq. 6.2 when the transverse loads are zero (i.e., \( q_1 = q_2 = 0 \)). \([K]\) is 12x12 matrix corresponding to three degrees of freedom at each layer end, see Fig. 6.2. The stiffness matrix takes into account the coupling effects between shear and bending deformations of each layer as well as those between the normal and shear stresses along the interface. The degree of interaction between the shear and normal stresses is governed by the parameter \( j \) according to Cosenza and Pecce (2001). When \( j = 0 \), Eqs (5.21) and (5.24) become uncoupled and can be solved independently. In this case, the two layer beam is balanced (Liu et al. 2014).

![Fig. 6.2 Degrees of freedom of the system.](image)

The load vector of the two-layer Timoshenko beam can be obtained from Eq. 6.2 making \( \{\delta\} = 0 \), then \( \{F\} = \{M_{EP}\} \). The load vector consists of equivalent forces and bending moments at the ends of each layer of the member as the end displacements and rotations become zero, see Fig. 6.3. This is a 12x1 column vector corresponding to the three degrees of freedom of each end of the two layers. Using the above formulation, the structural response of adhesive joints (stiffened plate, single-strap and single-lap joints) can be investigated by means of the well-known stiffness matrix method, analogous to that in the analysis of framed structures, or by replacing the existing matrix and loading vector with the proposed ones in available or customized structural engineering computer programs.
6.4 COMPREHENSIVE EXAMPLES AND VERIFICATION.

6.4.1 EXAMPLE 1. Adhesive stresses of four unbalanced steel-aluminum joints.

Determine the adhesive stresses of the four cases of unbalanced adhesively bonded steel-aluminum joints shown in Fig. 6.4 for different values of the adherends thickness ratio \((H_1/H_2)\) where \(H_1\) and \(H_2\) are the thicknesses of the upper and lower adherends, respectively. Cases (a) and (b) represent a stiffened plate/joint and a single-strap joint, respectively. Cases (c) and (c’) show a single-lap joint without and with transverse load, respectively. The structural models and degrees of freedom are shown in Fig. 6.4. The elastic modulus and Poisson’s ratio for the upper and lower adherends are assumed to be \(E_1=200\) GPa, \(\nu_1=0.29\) (steel), and \(E_2=70\) GPa, \(\nu_2=0.34\) (aluminum), respectively, and for the adhesive, \(E_a=2.5\) GPa and \(\nu_a=0.25\). \(h_a=0.25\) mm is the thickness of the adhesive layer and \(L=50\) mm is the length of the upper element. For case (c’) and when \(H_1/H_2=1.0\), assume a uniformly distributed transverse load equal to \(q = \chi \left( \frac{P}{L} \right)\), where \(P=P_o\) and \(\chi = -1.5 : -0.5 : 0 : 0.5 : 1.5\). Compare the results with those reported by Liu et al. (2014).

**Solution:** Figs. 6.5 and 6.6 shows the normalized peeling and shear stresses along the interface of the unbalanced stiffened plate/joint and single-strap joint for cases (a) and (b). As shown in Fig. 6.4, the stiffened plate/joint was analyzed by using a single two-layer beam element and
Fig. 6.4 Example 1: Unbalanced adhesively bonded steel-aluminum joints and degrees of freedom of each joint system.
the single-strap joint by assembling two beam elements. The local stiffness matrices of a single element (see Fig. 2), for cases (a) and (b) calculated using the proposed method are as follow:

\[
[K] = \begin{bmatrix}
4.53E04 & 4.15E02 & -3.86E04 & 2.69E04 & 5.04E04 & 6.2E05 & 3.14E04 & 1.76E05 & 2.49E04 & -1.39E04 & 1.56E03 & 1.16E04 & -7.95E03 & -6.81E02 & 3.67E03 \\
-1.95E02 & -6.43E04 & 2.64E03 & 2.32E04 & 3.90E04 & 3.90E04 & 2.49E04 & -7.95E03 & 1.56E03 & 1.16E04 & -7.95E03 & -6.81E02 & 3.67E03 \\
-1.76E03 & -3.43E04 & -5.34E03 & -7.34E03 & -1.56E03 & -1.56E03 & 4.14E03 & 6.81E03 & 1.16E04 & 1.16E04 & -7.95E03 & -6.81E02 & 3.67E03 \\
2.67E05 & 2.67E05 & 3.67E03 & -1.44E03 & 3.14E03 & 3.14E03 & 4.31E04 & 4.31E04 & 4.31E04 & 4.31E04 & -7.95E03 & -6.81E02 & 3.67E03 \\
\end{bmatrix}
\]

The results shown in Fig. 6.5 and 6.6 are in excellent agreement with the general analytical solution and finite element analysis presented by Liu et al. (2014) for the unbalanced adhesively joints. One of the main limitations of Liu’s approach is that depending on the boundary conditions of the joint type and the symmetry of the problem (i.e., geometric and mechanical properties) it becomes necessary to formulate and solve its particular system of linear equation to determine the constants of integration. While with the proposed matrix method any type of joint can be analyzed.

Fig. 6.7 shows the normalized peeling and shear stresses along the interface of the single-lap joint with and without transverse load (cases c’ and c), respectively. Note that additional bending moments \(M_1\) and \(M_2\) must be applied to the elements to satisfy the static equilibrium condition (Zhao et al. 2010). The single-lap joint with transverse load was analyzed using the stiffness matrix a single two-layer beam element and its corresponding load vector for the values of the distributed transverse load shown in Fig. 6.4. It can be observed that depending on the magnitude and direction of the transverse load, the adhesive stresses can significantly increase or decrease and, in some cases, even change the direction of the stresses. Again, the results for the case without transverse load are in excellent agreement with the results reported by Liu et al. (2014).
Fig. 6.5 EXAMPLE 1: Normalized stresses along the interface for stiffened plate joint, case (a).

Fig. 6.6 EXAMPLE 1: Normalized stresses along the interface for stiffened plate joint, case (b).
6.4 COMPREHENSIVE EXAMPLES AND VERIFICATION.

6.4.2 EXAMPLE 2. Adhesive stresses of four unbalanced steel-aluminum joints.

Determine the normal and shear strains as well as the peeling and shear stresses of the balanced single-lap joint shown in Fig. 6.8. The geometry of the joint and the structural model and degrees of freedom are also shown in Fig. 6.8. This joint was part of an experimental program carried out by Tsai and Morton (1995) in which Moiré interferometry was used to measure the deformation of the laminated adherends and adhesive layers. The geometric parameters are as follow: length of the outer adherend \( l' = 101.6 \, \text{mm} \), thickness of the adherend \( H = 2 \, \text{mm} \), length of the adhesive \( 2c = 25.4 \, \text{mm} \) and thickness of the adhesive layer \( h_a = 0.13 \, \text{mm} \). The width of the adherend and adhesive is 25.4 mm. The adherend was made up of graphite/epoxy (XAS/914C) laminae and layed-up with the following fiber orientation: \([0/45/-45/0]_2s\). The material properties of an orthotropic unidirectional lamina are \( E_l = 138 \, \text{GPa} \) (longitudinal elastic modulus), \( E_t = 9.4 \, \text{GPa} \) (transverse elastic modulus) and \( G_{lt} = 6.7 \, \text{GPa} \) (in-plane shear) and \( \nu_{lt} = 0.32 \) (Poisson’s ratio). The subscripts \( l \) and \( t \) represent the fiber and transversal directions of a single lamina. For the adhesive, \( E_a = 2.5 \, \text{GPa} \) and \( \nu_a = 0.25 \).

Solution: The mechanical properties of the laminae were determined using the classic lamination theory (Gay 2003; Tsai 1992). From this theory, the elastic properties of the laminae in the \( x \) and \( y \) global directions are as follow: \( E_x = 81 \, \text{GPa} \); \( E_y = 24.08 \, \text{GPa} \); \( G_{xy} = 21.15 \, \text{GPa} \); \( \nu_{yx} = 0.638 \) and \( \nu_{xy} = 0.19 \). Then, the following parameters were used to compute the stiffness
A novel linear matrix method to analyze adhesive joints

matrix and its corresponding load vector coefficients: $E_1 = E_2 = 81$ GPa and $G_1 = G_2 = G_{xy} = 21.15$ GPa.

Fig. 6.9a shows the adhesive normal and shear strain distributions measured from the Moiré experiment for a load of 4448 N and those obtained from the proposed method. Noticed that in the linear range, about 3% for the $\nu_{xy}$ and 1.5 for the $\nu_y$ (Tsai and Morton, 1995), there is a good agreement between the results obtained experimentally and those calculated using the proposed linear matrix analysis. Fig 6.9b shows the longitudinal strain distribution measured in the experiment at points 1 and 2 (Fig. 6.8) which compare well with those obtained using the nonlinear finite element method (NFEM) reported by Tsai and Morton (1995) and those obtained using the proposed model. Again, there is a good agreement in the linear range between the values calculated using the proposed method and the experimental results.

Figs 6.10a and 6.10b show the adhesive longitudinal, normal and shear strain distributions as well as their corresponding stresses obtained from the linear matrix formulation proposed herein for a load of 7619 N and those obtained by the NFEM presented by Tsai and Morton (1995). For the NFEM analysis, Tsai and Morton (1995) used over 300 elements to model the joint. Notice that the excellent agreements between the results of these two methods.
Fig. 6.9 EXAMPLE 2: (a) Normal and shear strain distributions and (b) longitudinal strain distribution for the laminated composite single-lap joint.
Fig. 6.10 EXAMPLE 2: Longitudinal, normal and shear (a) strain and (b) stresses distributions.
6.5 SUMMARY AND CONCLUSIONS

A novel linear matrix method for the analysis of various types of adhesively bonded balanced and unbalanced joints (i.e., single-lap, single-strap, stiffened plate joints, and L- and T- shaped joints) is presented and discussed in detail. The proposed matrix formulation is capable of determining the adhesive stresses along the interface of a two-layer prismatic beam element made up of two adherends and an adhesive. One of the novelties of this approach is that it allows the analysis of many types of joints simply by assembling the stiffness matrix and load vector similar to the matrix analysis of plane framed structures. The proposed 12x12 stiffness matrix and its corresponding load vector can be coupled with 6x6 matrices available in the technical literature and developed for single beam elements so joints where the adherends extend beyond the end of the adhesive can be analyzed. The proposed method takes into account the effects of the interaction between the peeling and shear stresses, as well as the coupling between the axial, bending and shear deformations and the transverse and longitudinal partial deformation along the interface of the adhesive. The effect of any type or combination of transverse loads that fits a second-order polynomial is also considered. Finally, two comprehensive examples are presented to validate the proposed method against available experimental, analytical and finite element solutions.

6.6 NOMENCLATURE

\( i \) = index of the elements of the two-layer beam. Top element \((i = 1)\) and bottom element \((i = 2)\);
\((EA)_i\) = Axial stiffness of element \(i\);
\((EI)_i\) = Bending stiffness of element \(i\);
\((A_sG)_i\) = Effective shear stiffness of element \(i\);
\(P_i\) = Compression or tension axial load applied at the centroid and ends of element \(i\);
\(V_i\) = Shear force applied at the centroid and ends of element \(i\);
\(M_i\) = Bending moment applied at the centroid and ends of element \(i\);
\(q_1\) and \(q_2\) = Applied transverse load at the top of element 1 and bottom of element 2, respectively.
\(\sigma\) = Normal stress along the interface of elements 1 and 2;
\(\tau\) = Shear stress acting along the interface of elements 1 and 2;
\(b\) = Width of the contact between elements 1 and 2;
\(d_i\) = Distance from the neutral axis of element \(i\) to the top of element \(i\);
$c_i =$ Distance from the neutral axis of element $i$ to the bottom of element $i$;

$\tau_h, \tau_p =$ Homogeneous and particular solutions of the governing differential equation, respectively;

$\theta =$ Slope due to bending moment of the centroidal line;

$\gamma =$ Slope due to transverse shear force of the centroidal line;

$k_h, k_v =$ Horizontal and vertical shear stiffness, respectively;

$k_s =$ Shear stiffness factor;

$r_i, s_i, t_i =$ Second-order polynomial coefficients of $q_i$;

$j_1, j_2, j_3 =$ Coefficients of the non-homogeneous solution to the applied transverse load;

$x =$ Coordinate along the centroidal axis of the coupled elements;

$y =$ Total vertical deflection of the centroidal axis of the coupled elements;

$u_i, y_i =$ Horizontal and vertical displacements along element $i$, respectively;

$\theta_i =$ Rotation along element $i$;

$\{F\} =$ Vector of end moments and shears;

$\{\delta\} =$ Vector of displacements and rotations;

$[K], \{M_{EP}\} =$ Stiffness matrix and vector of fixed-end forces and moments, respectively;

$\{R^1\}, \{R^2\} =$ Vectors of constants of integration according to the boundary conditions, $R_k (k =$ 1 to 13);

$\{C\} =$ Vector of constants of integration according to the boundary conditions, $C_i (i =$ 1 to 6);

$L_i =$ Span of element $i$. 
References

[1] A., L. and J., L. (2016). Non-linear buckling of elliptical curved beams. *Int. J. Non. Linear. Mech.*, 82:132–143.

[2] Aalami, B. O. (2000). Structural modeling of post-tensioned members. *Struct. Engineering, ASCE*, 126:157–162.

[3] Ansourian, P. (1981). Experiments on continuous composite beams. *Proceedings Institute of Civil Engineers. part 2.*, 71:25–51.

[4] Ansourian, P. (2011). Behaviour of stiffened composite beams with partial shear interaction accounting for time effects. *Procedia Engineering*, 14:402–409.

[5] Arboleda-Monsalve, L. G., Z.-M. D. G. and Aristizabal-Ochoa, J. D. (2008). Timoshenko beam-column with generalized end conditions on elastic foundation: Dynamic-stiffness matrix and load vector. *Journal of Sound and Vibration*, 310:1057–1079.

[6] Areiza-Hurtado M., V.-P. C. and Dario., A.-O. J. (2005). Second-order stiffness matrix and loading vector of a beam-column with semirigid connections on an elastic foundation. *ASCE J. Eng. Mech.*, 131:752–762.

[7] Aristizabal-Ochoa, J. D. (1997). First- and second-order stiffness matrices and load vector of beam-columns with semirigid connections. *ASCE J. Struct. Eng.*, 123:669–678.

[8] Aristizabal-Ochoa, J. D. (2010). Second-order slope deflection equations for imperfect beamcolumn structures with semi-rigid connections. *Eng. Struct.*, 32:2440–2454.

[9] Aydoğan, M. (1995). Stiffness-matrix formulation of beams with shear effect on elastic foundation. *J. Struct. Engrg., ASCE*, 121:1265–1270.

[10] Aziz, K. A. (1986). Modelisation et etude experimentale de poutres mixtes acier-beton a connexion partielle ou espacee, doktorska disertacija. *Institut National des Sciences Appliques des Rennes*.

[11] Barbero, E. J. (1999). *Introduction to Composite Materials Design*. Tylor and Francis Group, ISBN 1-56032-701-4.

[12] Bazzucchi F., M. A. and A., C. (2017). Interaction between snap-through and eulerian instability in shallow structures. *Int. J. Non. Linear. Mech.*, 88:11–20.
[13] Chen J. and Hung, S. (2012). Exact snapping loads of a buckled beam under a midpoint force. *Appl. Math. Model.*, 36:1776–1782.

[14] Cheng, F. Y. and Pantelides, C. P. (1988). Static timoshenko beam-columns on elastic media. *Journal of Structural Engineering*, 114:1152–1172.

[15] Collins, M. P. and Mitchell, D. (1997). *Prestressed Concrete Structures*. Prentice Hall College, ISBN 13: 9780136916352.

[16] Colorado-Urrea G. J., a. A.-O. J. D. (2014). Second-order stiffness matrix and load vector of an imperfect beam-column with generalized end conditions on a two-parameter elastic foundation. *Eng. Struct.*, 70:260–270.

[17] Cosenza, E. (2001). Shear and normal stresses interaction in coupled structural systems. *J Struct Eng*, 127:84–88.

[18] Dall’Asta, A. and Zona, A. (2002). Non-linear analysis of composite beams by a displacement approach. *Computers and Structures*, 80:2217–2228.

[19] Darío, A.-O. J. (2000). Second-order axial deflections of imperfect 3-d beam-column. *ASCE J. Eng. Mech.*, 126(11):1201–1208.

[20] Ecsedi, I. and Baksa, A. (2016). Analytical solution for layered composite beams with partial shear interaction based on timoshenko beam theory. *Eng Struct*, 115:107–117.

[21] Fabbrocino, G., M. G. and Cosenza, E. (2002). Modelling of continuous steel–concrete composite beams: computational aspects. *Computers and Structures*, 80:2241–2251.

[22] Faella, C., M. E. and Nigro, E. (2010). Steel–concrete composite beams in partial interaction: Closed-form ‘exact’ expression of the stiffness matrix and the vector of equivalent nodal forces. *Engineering Structures*, 32:2744–2754.

[23] Feodosiev, V. I. (1980). *Resistencia de Materiales*. Editorial Mir Moscu, URS.

[24] Foraboschi, P. (2009). Analytical solution of two-layer beam taking into account nonlinear interlayer slip. *Journal of Engineering Mechanics*, 135:1129–1146.

[25] G. Fabbrocino, G. M. and Cosenza, E. (1999). Non-linear analysis of composite beams under positive bending. *Comput Struct*, 70:77–89.

[26] G. Ranzi, F. G. and Ansourian, P. (2006). General method of analysis for composite beams with longitudinal and transverse partial interaction. *Comput Struct*, 84:2373–2384.

[27] Gara, F., C. S. L. G. and Dezi, L. (2014). A higher order steel–concrete composite beam model. *Engineering Structures*, 80:260–273.

[28] Gay, D. and Hoa., S. V. (2007). *Composite Materials. Design and Applications*. CRC Press, ISBN 13:978-1-4200-4519-2.
[29] H. Yuan, H. Deng, Y. Y.-Y. W. and Zhenggeng, Z. (2016). Element-based effective width for deflection calculation of steel-concrete composite beams. *J Constr Steel Res*, 121:163–172.

[30] He, G. and Yang, X. (2015). Dynamic analysis of two-layer composite beams with partial interaction using a higher order beam theory. *International Journal of Mechanical Sciences*, 90:102–112.

[31] Hečzko J., D. Z. and C., R. H. (2014). Negative stiffness materials for vibration damping?: a material realization of a nonlinear bistable element. *International Journal of Innovations in Materials Science and Engineering (IMSE)*, 1.

[32] Hetényi, M. (1979). *Beams On Elastic Fundation*. The University Of Michigan Press.

[33] J., L. R. S. D. W. (2002). Dramatically stiffer elastic composite materials due to a negative stiffness phase. *J. Mech. Phys. Solids*, 50:979–1009.

[34] J.G. Ollgaard, R. S. and Fisher, J. (1971). Shear strength of stud connectors in lightweight and normal-weight concrete. *AISC Eng. J.*, 71:55–64.

[35] Johnson, R. P. and Molenstra, N. (1991). Partial shear connection in composite beams for buildings. *Proceedings Institute of Civil Engineers. part 2.*, 91:679–704.

[36] K. Belakhdar, A. Tounsi, S. B.-E. A. A. B. and Hassar, S. M. E. (2010). On the reduction of the interfacial stresses in a repaired beam with an adhesively bonded frp plate. *Instrumentation Science and Technology*, 17:1–14.

[37] Lezgy-Nazargah, M. (2014). An isogeometric approach for the analysis of composite steel–concrete beams. *Thin Walled Struct*, 84:406–415.

[38] LIN., T. Y. (1963). Load-balancing method for desingn and analysis of prestressed concrete structures. *J. Am. Concr. institute.*, 60:719–742.

[39] Lin, T. Y. and Thornton, K. (1972). Secondary moment and moment redistribution in continuous prestressed concrete beams. *PCI J.*, 18:8–20.

[40] Navy, E. G. (2003). *Prestressed Concrete A Fundamental Approach*. Pearson Education, Inc., New Jersey 07458.

[41] Nistor M., W. R. and I., S. (2017). Relationship between euler buckling and unstable equilibria of buckled beams. *Int. J. Non. Linear. Mech.*, 95:151–161.

[42] P. S.-P. J. and Dario, A.-O. J. (1999). Buckling reversals of axially restrained imperfect beam-column. *ASCE J. Eng. Mech.*, 125.

[43] P. Keo, Q.-H. Nguyen, H. S. and Hijajj, M. (2016). Derivation of the exact stiffness matrix of shear-deformable multi-layered beam element in partial interaction. *Finite Elem Anal Des*, 112:40–49.
[44] Q.-H. Nguyen, E. M. and Hjiaj, M. (2011). Derivation of the exact stiffness matrix for a
two-layer timoshenko beam element with partial interaction. *Eng Struct*, 33:298–307.

[45] Q.-H. Nguyen, M. H. and Guezouli, S. (2011). Exact finite element model for shear-deformable two-layer beams with discrete shear connection. *Finite Elem Anal Des*, 47:718–727.

[46] Q.-H. Nguyen, M. H. and Lai, V.-A. (2014). Force-based f.e. for large displacement inelastic analysis of two-layer timoshenko beams with interlayer slips. *Finite Elem Anal Des*, 85:1–10.

[47] S., H. and J., C. (2012). Snapping of a buckled beam on elastic foundation under a midpoint force. *Eur. J. Mech. –A/Solids*, 31:90–100.

[48] S.-F. Jiang, X. Z. and Zhou, D. (2014). Novel two-node linear composite beam element with both interface slip and shear deformation into consideration: formulation and validation. *Int J Mech Sci*, 85:110–119.

[49] Santos, H. and Silberschmidt, V. (2014). Hybrid equilibrium finite element formulation for composite beams with partial interaction. *Compos Struct*, 108:646–656.

[50] Sua, Y.-Y. and Gao, X.-L. (2014). Analytical model for adhesively bonded composite panel-flange joints based on the timoshenko beam theory. *Compos Struct*, 107:112–118.

[51] T. Y. LIN., N. H. (1981). *Design Of Prestressed Concrete Structures*. John Willey and Sons, ISBN-13: 978-9812531179.

[52] Timoshenko S. P, G. J. M. (1961.). *Theory of Elastic Stability*. McGraw-Hill Book Company, 2 ed.

[53] Šnirc L’. and J., R. (2017). Statics and dynamics of snap-through effect. *Procedia Engineering*, 190:540–546.

[54] Z. Liu, Y. Huang, Z. Y. S. B. and Valvo, P. (2014). A general solution for the two-dimensional stress analysis of balanced and unbalanced adhesively bonded joints. *Int J Adhes Adhes*, 54:112–123.

[55] Zhou, Y., S. I. E. T. and M., S. (2015). Nonlinear elastic buckling and postbuckling analysis of cylindrical panels. *Finite Elem. Anal. Des.*, 96:41–50.

[56] Zona, A. and Ranzi, G. (2011). Finite element models for nonlinear analysis of steel–concrete composite beams with partial interaction in combined bending and shear. *Finite Elem Anal Des*, 47:98–118.