EXISTENCE, CONVERGENCE AND LIMIT MAP OF THE LAPLACIAN FLOW

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Abstract. We prove short time existence and uniqueness of the Laplacian flow starting at an arbitrary closed $G_2$-structure. We establish long time existence and convergence of the Laplacian flow starting near a torsion-free $G_2$-structure. We analyze the limit map of the Laplacian flow in relation to the moduli space of torsion-free $G_2$-structures. We also present a number of results which constitute a fairly complete algebraic and analytic basis for studying the Laplacian flow.

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1. INTRODUCTION

The Riemannian holonomy group of a 7-dimensional manifold $M$ equipped with a torsion-free $G_2$-structure is contained in the Lie group $G_2$. As a consequence, $M$ is Ricci flat. If the fundamental group of $M$ is finite, then the holonomy group of $M$ actually equals $G_2$ and the spinor bundle of $M$ splits off a parallel $\mathbb{R}$ summand w.r.t. the Levi-Civita connection. These properties are the main reason for the importance of torsion-free $G_2$-structures and in general, $G_2$-structures, in differential geometry. In particular, $G_2$ holonomy appears as an important case of the Berger classification of holonomy groups of Riemannian manifolds. Note that manifolds with $G_2$ holonomy play an important role in $M$-theory. Namely the compactification of $M$-theory on a manifold with $G_2$ holonomy leads to an $\mathcal{N} = 1$ $(3+1)$-dimensional quantum field theory, which is similar to the compactification of heterotic string theory on Calabi-Yau manifolds. (The parallel $\mathbb{R}$ summand of the spinor bundle provides the $\mathcal{N} = 1$ supersymmetry.)

A fundamental problem here is how to deform a given $G_2$-structure on a manifold to a torsion-free $G_2$-structure. R. Bryant proposed the following Laplacian flow for closed $G_2$-structures
\[
\frac{\partial \sigma}{\partial t} = \Delta_\sigma \sigma, \tag{1.1}
\]
where $\Delta_{\sigma}$ denotes the Hodge Laplacian of the Riemannian metric induced by the $G_2$-structure $\sigma$, cf. [B2]. In [BX], this flow is interpreted as the gradient flow of Hitchin’s volume functional w.r.t. an unusual metric.

It turns out that the structures of the Laplacian flow are rather complicated. On the other hand, the Laplacian flow shares some features with the Ricci flow, which is worth noting. Under the Laplacian flow, the induced metric $g = g(t)$ evolves as follows [B2]
\[
\frac{\partial g}{\partial t} = -2Ric + \frac{8}{21}|\tau|^2g + \frac{1}{4}j(\ast(\tau \wedge \tau)), \tag{1.2}
\]
where $\tau$ denotes the adjoint torsion of $\sigma$ (cf. Section 2), $\ast$ the Hodge star, and $j$ a certain linear operator associated with $\sigma$. Thus we see the leading part $-2Ric$ which appears in the Ricci flow. The perturbation part is given by the adjoint torsion $\tau$, which is a key quantity because its vanishing is equivalent to the
torsion-free condition. One may wonder whether it is possible to use the Ricci flow to deform $G_2$-structures. However, the induced metric of a $G_2$-structure $\sigma$ does not determine $\sigma$ completely, and in general a metric may not be induced from a $G_2$-structure. Hence a suitable additional coupled equation would be needed in order to use the Ricci flow to deform $G_2$-structures. Based on some calculations we are convinced that such coupled equations do not exist in general.

There are four main parts in this paper. First, we prove the short time existence and uniqueness of solutions of the Laplacian flow with given initial data. Second, we establish the long time existence and convergence of the Laplacian flow starting near torsion-free $G_2$-structures, and determine the limit projection of the Laplacian flow into the moduli space of torsion-free $G_2$-structures. This reveals the deep relation of the Laplacian flow with the moduli space of torsion-free $G_2$-structures. Fourth, we present a number of results which constitute a fairly complete algebraic and analytic basis for studying the Laplacian flow. These include algebraic formulas, differential identities, a linear parabolic theory, and a detailed analysis of the basic analytic structure of the Laplacian flow and the gauge fixed Laplacian flow.

The highlights of our main results in the first three parts are formulated in the following three main theorems. (We refer to the subsequent sections for the results in the last part.) A closed solution means a solution given by closed $G_2$-structures.

**Theorem 1.1.** Let $M$ be a compact 7-dimensional manifold. Let $\sigma_1$ be a closed $G_2$ structure of class $C^{4+\mu}$ on $M$ for some $0 < \mu < 1$. Then there is a closed $C^{2+\mu, (3+\mu)/2}$ solution $\sigma = \sigma(t)$ of the Laplacian flow on a time interval $[0, T]$ with $T > 0$, such that $\sigma(0) = \sigma_1$. This solution is unique among all $C^{2+\mu, (2+\mu)/2}$ functions (with closed $G_2$-structures as values) with the initial value $\sigma_1$. For each $0 < \epsilon < T$, there is a family of diffeomorphisms $\phi(\cdot, t)$ of class $C^{3+\mu, (4+\mu)/2}$ on $[\epsilon, T]$, such that $\sigma(t) = \phi(\cdot, t)^* \sigma(\cdot)$ is a $C^\infty$ solution of the Laplacian flow. Moreover, there holds $\sigma \in C^{l-2, (l-1)/2}$ on $M \times [0, T]$, provided that $\sigma_1 \in C^l$ for a noninteger $l > 4 + \mu$. In particular, $\sigma$ is smooth if $\sigma_1$ is smooth.

For an estimate for $T$ from below and other a priori estimates, we refer to Theorem 6.4 and the proof of Theorem 1.1. The definitions of the involved function spaces are given in Appendix. In particular, the space $C^{4+\mu}$ means the Hölder space $C^{4+\mu}$ in the conventional notation. The parabolic Hölder spaces, i.e. the $C^{1/2}$ spaces, and their generalizations $C^{1/2}$ spaces, involve spatial and time derivatives in patterns which are particularly suitable for handling second order partial differential equations of parabolic type or related types. Several statements of this theorem actually hold true under more general or weaker assumptions. On the other hand, short time existence and uniqueness of solutions of Sobolev classes can also be obtained for Sobolev initial data.

**Theorem 1.2.** Let $\sigma_0$ be a smooth torsion-free $G_2$-structure on a compact manifold $M$ of dimension 7. Let $0 < \mu < 1$. Then there exists a strong $C^{2+\mu}$-neighborhood $U_{\sigma_0}$ of $\sigma_0$ in the space of closed $G_2$-structures on $M$ such that whenever $\sigma_1 \in U_{\sigma_0}$, the Laplacian flow (1.1) starting at $\sigma_1$ has a unique closed smooth solution $\sigma = \sigma(t)$ on $M \times [0, \infty)$ which converges exponentially to a smooth torsion free $G_2$-structure $\sigma_\infty$ as $t \to \infty$. Thus torsion-free $G_2$-structures are stable in the space of closed $G_2$-structures with respect to the Laplacian flow.

**Theorem 1.3.** Let $\mathcal{F}$ denote the limit map of the Laplacian flow in the situation of Theorem 1.1, i.e. $\mathcal{F}(\sigma_1) = \sigma_\infty$. Then $\mathcal{F} : U_{\sigma_0} \to T$ is a smooth map, where $T$ denotes the space of smooth torsion-free $G_2$-structures on $M$. Moreover, there holds

$$\pi \circ \mathcal{F} = \Pi,$$

(1.3)

where $\Pi$ is a canonical projection into the moduli space $T / \text{Diff}_0(M)$ of smooth torsion-free $G_2$-structures on $M$ and $\pi$ is the quotient projection from $T$ onto $T / \text{Diff}_0(M)$.

For relevant definitions (such as strong $C^{2+\mu}$ neighborhood and the projection $\Pi$) we refer to Sections 9 and 10. Next we explain the backgrounds and main ideas of the above results.

**Existence and uniqueness of short time solutions**

As it turns out, existence and uniqueness of short time solutions of the Laplacian flow are a rather delicate problem. Indeed, the Laplacian flow is not a parabolic equation, and there seems to be no way to restore full parabolicity for it by a transformation such as gauge fixing as employed in the DeTurck trick for the Ricci flow. Previously, it was proved in [BX] via rather complicated computations that a partial parabolicity, namely the parabolicity in the direction of closed forms, can be restored for the Laplacian flow by a certain gauge fixing, i.e. the gauge fixed Laplacian flow is parabolic in the said direction. However, the gauge fixed Laplacian flow fails to be parabolic in the complementary directions. To cope with this situation of lack of
full parabolicity, the set-up of Fréchet space of smooth forms and Nash-Moser implicit function theorem were employed in [BX]. This way, short time existence and uniqueness of closed smooth solutions of the Laplacian flow starting at a smooth closed $G_2$-structure were obtained in [BX].

In this paper, we first introduce a new gauge fixing for the Laplacian flow, which restores the partial parabolicity, i.e. the parabolicity in the direction of closed forms. This new gauge fixing is simpler and more transparent than the one used in [BX], and is based on a new identity for the Hodge Laplacian, which in turn is based on some delicate differential identities involving splittings of the exterior differential via irreducible $G_2$-representations. Next we develop a new linear parabolic theory for closed forms which is tailored to handle operators which are only parabolic in the direction of closed forms. Using this theory we are then able to establish the short time existence and uniqueness of the Laplacian flow starting at $C^{4+\mu}$ initial data and obtain estimates depending only on the $C^{4+\mu}$ properties of the initial data. (Note that we avoid using the Nash-Moser implicit function theorem.)

The improvement to $C^4$ initial data with $l \geq 4 + \mu$ provided by Theorem 1.1 in comparison with the result in [BX] is an obvious analytic aspect of it. More important is the complete understanding and resolution of the problem of short time solutions of the Laplacian flow. The existence for $C^4$ initial data, the associated estimates, as well as the basic $C^{4\mu}$ set-up also play an important role for establishing the long time existence and convergence of the Laplacian flow and the smoothness of its limit map as presented in Theorem 1.2 and Theorem 1.3 as will further be explained below. Moreover, the framework and strategy for Theorem 1.1 also allow to handle e.g. the Laplacian flow on complete noncompact manifolds. This will be presented in a subsequent paper.

**Long time existence and convergence**

The second main theorem of this paper, Theorem 1.2, provides the first result on long time behavior of the Laplacian flow. From the dynamical point of view, this result can be viewed as stability of torsion-free $G_2$-structures in regard to the Laplacian flow and the Hitchin volume functional. As the Laplacian flow is very natural geometrically, this dynamical stability is also very natural from a geometric point of view. We also believe that it is significant for the $M$-theory. Previously, the dynamical stability of Einstein metrics w.r.t. the volume-normalized Ricci flow and that of Ricci flat metrics w.r.t. the Ricci flow were proved by the second named author under the condition of positive first eigenvalue of the Lichnerowicz Laplacian, as a consequence of a general convergence result for the Ricci flow [Y1]. We would like to mention that we have also obtained a general long time convergence result for the Laplacian flow under the assumption of small torsion of the initial $G_2$-structure [XY1].

The basic scheme of the proof for Theorem 1.2 is to derive exponential decay estimates for the solution under the assumption of certain smallness and boundedness. The said smallness and boundedness on a small time interval follow from our results on short time solutions, but are not known a priori for all time. Hence it is crucial to obtain strong feedback via exponential decay, such that they can be shown to always hold true. The key starting point of the exponential decay is the exponential $L^2$-decay, which is based on the spectral property of the Hodge Laplacian. Such an exponential $L^2$-decay scheme was first implemented successfully in [Y1] for proving long time convergence of the Ricci flow. The situation in this paper is more delicate for the following reason. The involved PDE has a second order perturbation term besides the leading Laplacian term, which makes it more difficult to apply the maximum principle to convert $L^2$ estimates into $C^0$ estimates. Moreover, for the purpose of establishing the smoothness of the limit map of the Laplacian flow, we need to derive linear power decay estimates rather than estimates with fractional powers. Here the conventional $L^2$ version of Moser type maximum principle is not suitable. We derive an $L^1$ version instead and apply it to overcome the trouble.

Another tool employed here is a result on the local smooth structure of the moduli space of torsion-free $G_2$-structures. It is used to locate the target torsion-free $G_2$-structure for the Laplacian flow to converge to. (The actual limit differs from this target by a diffeomorphism.) The said result is a refinement of D. Joyce’s well-known result [J] on the same topic, and its proof is presented in [XY2]. Note that this result can be viewed as the stationary version of Theorem 1.1. Indeed, it is in part based on our understanding of some features of the Laplacian flow, see [XY2] for details.

Note that the analytic set-up for the above scheme of exponential decay has to be carefully chosen. Indeed, the $C^{4\mu}$ spaces and the parabolic estimates in Section 5 play a crucial role here. The main reason for this is that the estimates in these spaces require minimal amount of bounds while providing strong control directly, in contrast to e.g. Sobolev space estimates which leave a large gap because of Sobolev embeddings.

The limit map and its projection into the moduli space
Our third main theorem, Theorem 1.3 (and additional results in Section 10), is the first result of its kind regarding the limit map of a nonlinear geometric evolution equation, the space of its stationary solutions, and the corresponding moduli space. Besides individual torsion-free $G_2$-structures, their moduli space is an important geometric object. In particular, it plays an important role in M-theory. It is therefore very desirable to understand the relation between the Laplacian flow and the space of torsion-free $G_2$-structures and the associated moduli space. (The application of the result in [XY2] mentioned above is only one aspect of this relation.) Theorem 1.3 provides a complete understanding of this deep relation.

The smoothness of the limit map of the Laplacian flow is rather intricate. Indeed, one encounters analytic troubles if one deals with the Laplacian flow directly. Indeed, the equation satisfied by the difference of two solutions of the Laplacian flow (with two different initial values) fails to be parabolic, and hence it is not clear how to derive estimates for this difference directly. (This goes back to the lack of parabolicity of the Laplacian flow itself. We are able to handle it in the context of short time solutions by a suitable gauge fixing as explained before.) Our basic strategy for proving the said smoothness is to go through the gauge fixed Laplacian flow. The proof requires a number of additional ingredients, and involves various exponential decay estimates. Indeed, the linear theory in Section 5, Theorem 1.2 and the techniques in its proof have to be applied in various fashions. In particular, as mentioned above, the linear power nature of the decay estimates is crucial here.

Finally, the identification of the limit projection of the Laplacian flow in terms of a canonical projection is achieved via the detailed convergence analysis of the Laplacian flow.

Now some additional brief descriptions of the main content of the subsequent sections. In Section 2, we present a short introduction to the basics of $G_2$-structures. We explain the basic concepts, present some useful facts and algebraic formulas, such as the important $G_2$-irreducible decompositions of forms and associated formulas, and also provide some basic set-ups of this paper. In Section 3, we derive the new identity for the Hodge Laplacian mentioned above. Along the way, we present a detailed treatment of a typical one of Bryant’s differential identities, and also derive a new one. In Section 4, we present some additional differential identities. Note that the differential identities in these two sections are tied to the irreducible decompositions of forms and are only available when a $G_2$-structure is present. Obviously, the applications of these differential identities in the study of the Laplacian flow as presented in [BX] and this paper offer a unique new perspective in geometric analysis and nonlinear analysis. In Section 5, we develop the new linear parabolic theory for closed forms described above. A subtle point here is that the corresponding linear parabolic problem for exact forms is ill-behaved due to the lack of completeness of some involved function spaces. (This phenomenon is uncovered for the first time in this paper.) In Section 6, we construct our new gauge, which is motivated by the Hodge Laplacian identity in Section 3, and apply the theory in Section 5 and the classic inverse function theorem to prove existence and uniqueness of short time solutions of the gauge fixed Laplacian flow. Note that the inverse function theorem immediately implies a local uniqueness. To obtain global uniqueness, we utilize the special quadratic structure in the equation to handle its nonlinear second order perturbation part, and appeal to the Bochner-Weitzenb"ock formula for the Hodge Laplacian. In Section 7, we apply the results of Section 6 to derive existence and uniqueness of short time solutions of the Laplacian flow. Here the differential identities in Section 4 play an important role.

In Section 8, we prove long time convergence at exponential rate of the gauge fixed Laplacian flow starting near a torsion-free $G_2$-structure. In Section 9 we combine the result in the previous section and results on the local smooth structure of the moduli space of torsion-free $G_2$-structures to derive long time existence and convergence of the Laplacian flow. In the last section, we prove the smoothness of the limit map of the Laplacian flow and identify its projection into the moduli space of torsion-free $G_2$-structures.

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2. $G_2$-Structures

In Subsection 2.1 we present some basics of $G_2$-structures. In Subsection 2.2, we describe the decompositions of forms into irreducible components, which play a crucial role for various computations in this paper.

2.1. Basics.
Lemma 2.1. Let $e_i, i = 1, 2, \ldots, 7$ denote the standard orthonormal basis of $\mathbb{R}^7$ and $e^i = dx^i$ its dual basis. The standard $G_2$-structure on $\mathbb{R}^7$ is

$$
\sigma_{\mathbb{R}^7} = e^1 \wedge (e^2 \wedge e^3 + e^4 \wedge e^5 + e^6 \wedge e^7) + e^2 \wedge (e^4 \wedge e^6 - e^5 \wedge e^7)
$$

$$
- e^3 \wedge (e^4 \wedge e^7 + e^5 \wedge e^6)
$$

where $\omega_{\mathbb{R}^7} = e^2 \wedge e^3 + e^4 \wedge e^5 + e^6 \wedge e^7$ is the standard symplectic form on $\mathbb{R}^6$ and $\Omega_{\mathbb{R}^7} = dz^1 \wedge dz^2 \wedge dz^3$ is the standard holomorphic volume form on $\mathbb{C}^3 = \mathbb{R}^6$, w.r.t. the decomposition $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{R}^6$. (Thus $z^1 = x^2 + \sqrt{-1} x^3, z^2 = x^4 + \sqrt{-1} x^5$ and $z^3 = x^6 + \sqrt{-1} x^7$.) The group $G_2$ can be defined as follows

$$
G_2 = \{ A \in GL(7, \mathbb{R}) : A^* \sigma_{\mathbb{R}^7} = \sigma_{\mathbb{R}^7} \}.
$$

(2.1)

It is a 14-dimensional compact, connected, simply-connected and simple Lie subgroup of $SO(7)$, cf. [B1][B2].

Set $\Lambda^2_+((\mathbb{R}^7)^*) = \{ L^* \sigma_{\mathbb{R}^7} : L \in GL(\mathbb{R}, 7) \}$. This is the set of constant $G_2$-structures on $\mathbb{R}^7$. It is open in $\Lambda^2((\mathbb{R}^7)^*)^*$, cf. [B2]. Let $M$ be a smooth 7-dimensional manifold. For each $p \in M$, set $\Lambda^2_+T_p^*M = \{ \sigma \in \Lambda^2T_p^*M : \sigma = L^* \sigma_{\mathbb{R}^7}, \text{ for an isomorphism } L : T_p M \to \mathbb{R}^7 \}$. Then $\Lambda^2_+(T^*M) = \cup_{p \in M} \Lambda^2_+T_p^*M$ is an open subbundle of $\Lambda^3T^*M$ as a fiber bundle. This is the bundle of positive 3-forms. An $A$ in $\sigma = A^* \sigma_{\mathbb{R}^7}$ is called an inducing map of $\sigma$. An induced orthonormal basis for $\sigma$ is $A^{-1}e_1, \ldots, A^{-1}e_7$, where $A$ is an inducing map of $\sigma$.

**Definition 2.1** Let $l \geq 0$. $G_2$-structures of class $C^l$, or $C^l$ $G_2$-structures are defined to be 3-forms of class $C^l$ with values in $\Lambda^2_+T^*M$. In other words, they are $C^l$ sections of $\Lambda^2_+T^*M$. (It is easy to show that they are in one-to-one correspondence with $C^l$ principal $G_2$ subbundles of the principal frame bundle of $M$.)

Note that the existence of a $G_2$-structure (of class $C^l, l \geq 0$) is equivalent to the vanishing of the first two Stiefel-Whitney classes, i.e. equivalent to $M$ being orientable and spinnable, cf. [B2].

Since $G_2 \subset SO(7)$, a $C^l$ $G_2$-structure $\sigma$ induces a $C^l$ Riemannian metric $g_\sigma$ on $M$ and an $C^l$ orientation of $M$, namely $g_\sigma = L^* g_{\mathbb{R}^7}$ and $\text{dvol}_{\sigma} = L^* (e^1 \wedge \cdot \cdot \cdot \wedge e^7)$, if $\sigma = L^* \sigma_{\mathbb{R}^7}$. All quantities associated with $g_\sigma$ will often be indicated by the subscript $\sigma$. For example, $*_\sigma$ denotes the Hodge $\ast$ of $g_\sigma$. Note that $g_\sigma$ can be given by an explicit algebraic formula in terms of $\sigma$. Indeed there holds, as is easy to verify

$$
g_\sigma(u, v) = \frac{\langle u, \sigma \rangle \wedge \langle v, \sigma \rangle \wedge \sigma}{6 \text{dvol}_\sigma}.
$$

(2.3)

at each $p \in M$ and for all $u, v \in T_pM$. Moreover, there holds

$$
\text{dvol}_\sigma = 6^{-\frac{7}{2}} (\text{det}_\Omega \sigma)^{\frac{3}{2}} \Omega,
$$

(2.4)

where $\Omega$ denotes an arbitrary volume form at any given $p$ (i.e. a nonzero element of $\Lambda^7(T^*M)$, and the determinant $\text{det}_\Omega \sigma$ is defined to be the determinant of the quadratic form $\langle u, \sigma \rangle \wedge \langle v, \sigma \rangle \wedge \sigma / \Omega$ on a basis $u_1, \ldots, u_7$ such that $\Omega(u_1, \ldots, u_7) = 1$. Hence the formula (2.3) gives the metric $g_\sigma$ explicitly in terms of $\sigma$.

Next we note the following simple, but important fact.

**Lemma 2.1.** There are universal positive numbers $\epsilon_0 \leq 1, \mu_0$ and $C_0$ with the following property. Let $p \in M$. If $\sigma = L^* \sigma_{\mathbb{R}^7}$, $\gamma \in \Lambda^3T^*M$ and $| \gamma - \sigma |_{\sigma} \leq \epsilon_0$, then $\gamma \in \Lambda^3_+T^*M$. Moreover, there holds $| \gamma |_{\sigma} \leq C_0$ and the eigenvalues of $g_\sigma$, w.r.t. $g_\sigma$ are bounded below by $\mu_0$.

**Proof.** Since $\Lambda^3((\mathbb{R}^7)^*)^*$ is open in $\Lambda^3((\mathbb{R}^7)^*)$, there is a positive number $\epsilon_0 \leq 1$ such that $\gamma \in \Lambda^3((\mathbb{R}^7)^*)^*$ whenever $\gamma \in \Lambda^3((\mathbb{R}^7)^*)$ and $| \gamma - \sigma |_{\sigma} \leq \epsilon_0$. By continuity and compactness, there is a positive number $\mu_0$ such that the eigenvalues of $g_\sigma$, w.r.t. the Euclidean metric are bounded from below by $\mu_0$. The claims of the lemma then follow from the induced nature of $g_\sigma$. \qed

**Definition 2.2** The total torsion of a $G_2$-structure $\sigma$ is defined to be $\nabla_\sigma \sigma$. Its adjoint torsion $\tau = \tau_\sigma$ is defined to be

$$
\tau = d^* \sigma = - *_\sigma d *_\sigma \sigma.
$$

(2.5)

Note that $d\tau = \Delta_\sigma \sigma$, if $\sigma$ is a closed $G_2$-structure, i.e. a $G_2$-structure which is a closed form. A $G_2$-structure is said to be torsion-free, provided that its total torsion vanishes everywhere. (If we do not specify the $C^l$ class of $\sigma$ in a discussion, then $\sigma$ is assumed to be in $C^l$ for the minimal $l$ as required in the discussion.)

A fundamental fact [B1][B2][FG][S] is that a $G_2$-structure is torsion-free precisely when the induced metric has a subgroup of $G_2$ as its holonomy group and hence is Ricci-flat. On the other hand, it is well-known [B2][FG][S] that a $G_2$-structure $\sigma$ is torsion-free precisely when it is closed and its adjoint torsion vanishes,
i.e. when $\sigma$ is a harmonic form (w.r.t. $g_\sigma$) in the case of a compact $M$. Indeed, the full torsion $\nabla_\sigma \sigma$ can be expressed in terms of $\sigma \sigma$ and $\tau_\sigma$, which follows from the arguments in [Proof of Proposition 2.2, B2], see also [Theorem 2.27, K] for an explicit formula. This explicit formula leads to the following lemma regarding closed $G_2$-structures.

**Lemma 2.2.** Let $\sigma$ be a closed $G_2$-structure. Then there holds at each $p \in M$

$$\nabla_\sigma \sigma = -\frac{1}{3} < \sigma_\sigma, *_\sigma \sigma >_{2,1}, \tag{2.6}$$

where $< \cdot, \cdot >_{2,1}$ denotes the contraction $< \cdot, \cdot >_{2,1} : \otimes^2 T^*_p M \times (T^*_p M \otimes \Lambda^3 T^*_p M) \to T^*_p M \otimes \Lambda^3 T^*_p M$ given by

$$< \alpha_1 \otimes \alpha_2, \alpha_3 \otimes \gamma >_{2,1} = (\alpha_2 \otimes \alpha_3) \alpha_1 \otimes \gamma \tag{2.7}$$

for $\alpha_1, \alpha_2, \alpha_3 \in T^*_p M$ and $\gamma \in \Lambda^3 T^*_p M$. (Note that $\Lambda^4 T^*_p M \subset T^*_p M \otimes \Lambda^3 T^*_p M$. For relevant discussions of a similar contraction see Lemma 2.27 below.)

**Proof.** This is a reformulation of [Theorem 2.27, K] in the special case of a closed $G_2$-structure. \qed

### 2.2. Irreducible Decomposition of Forms.

Let $\sigma$ be a $G_2$-structure on $M$. For each $p \in M$ and $1 \leq j \leq 7$, the exterior space $\Lambda^j T^*_p M$ decomposes orthogonally into irreducible representations of $G_2$, which then leads to the corresponding decompositions of the bundles $\Lambda^j T^* M$, and hence of differential $j$-forms. We have [B2]

$$\Lambda^2 T^* M = \Lambda_1^2(T^* M) \oplus \Lambda_2^2(T^* M),$$

$$\Lambda^3 T^* M = \Lambda_3^2(T^* M) \oplus \Lambda_4^2(T^* M),$$

and the corresponding ones $\Omega^2(M) = \Omega_1^2(M) \oplus \Omega_2^2(M)$ etc. (as well as for forms of various $C^l$ classes), where the subscript indicates the dimension of representation. We have the characterizations

$$\Lambda_1^2(T^* M) = \{c \sigma_\sigma : c \in \mathbb{R}, p \in M\}, \Lambda_2^2(T^* M) = \{*_\sigma (\alpha \wedge \sigma) : \alpha \in T^* M\},$$

$$\Lambda_3^2(T^* M) = \{\gamma \in \Lambda^3(T^* M) : \gamma \wedge \sigma = 0, \gamma \wedge *_\sigma \sigma = 0\},$$

$$\Lambda_4^2(T^* M) = \{*_\sigma (\alpha \wedge *_\sigma \sigma) : \alpha \in \Lambda^2 T^* M = \{\alpha \in \Lambda^2 T^* M : \alpha \wedge \sigma = 2 *_\sigma \alpha\},$$

$$\Lambda_5^2(T^* M) = \{\alpha \in \Lambda^2 T^* M : \alpha \wedge \sigma = -*_\sigma \alpha\}, \tag{2.9}$$

cf. [B2]. (Obviously, e.g. $\gamma \wedge \sigma$ means $\gamma \wedge \sigma_\sigma$ for $\gamma \in \Lambda^3(T^*_p M)$.) It follows that

$$\pi_2^2 \alpha = \frac{1}{3} \alpha + \frac{1}{3} *_\sigma (\alpha \wedge \sigma), \quad \pi_4^2 \alpha = \frac{2}{3} \alpha - \frac{1}{3} *_\sigma (\alpha \wedge \sigma), \tag{2.10}$$

where $\pi_j$ denotes the orthogonal projection from $\Lambda^j T^*_p M$ to $\Lambda_j^2(T^*_p M), p \in M$. On the other hand, by (2.9), the formula for the decomposition of $\gamma \in \Lambda^3 T^*_p M$ for $p \in M$ can be written as follows

$$\gamma = f^0 \sigma + *_\sigma (f^1 \wedge \sigma) + f^3, \tag{2.11}$$

with $f^0 \in \mathbb{R}, f^1 \in T^*_p (M)$ and $f^3 = \pi_2^3 \gamma$. ($\sigma$ stands for $\sigma_\sigma$.) We present a formula for computing $f^1$, which will be needed later. For this purpose, we first present two lemmas, which will also be useful for other purposes.

**Lemma 2.3.** Let $p \in M$ and $\alpha_1, \alpha_2 \in T^*_p M$. Then there hold

$$(\alpha_1 \wedge \sigma) \cdot (\alpha_2 \wedge \sigma) = 4 \alpha_1 \cdot \alpha_2 \tag{2.12}$$

and

$$(\alpha_1 \wedge *_\sigma \sigma) \cdot (\alpha_2 \wedge *_\sigma \sigma) = 3 \alpha_1 \cdot \alpha_2, \tag{2.13}$$

where $\sigma$ means $\sigma_\sigma$.

**Proof.** By the induced nature of the metric $g_\sigma$, it suffices to consider the Eulidean space. So we can assume $\sigma_\sigma = \sigma_{\mathbb{R}}$. By linearity, it suffices to verify (2.12) and (2.13) for $\alpha_1 = e_i$ and $\alpha_2 = e_j$. Since $G_2 \subset SO(7)$ and it acts transitively on unit vectors and on orthonormal pairs [B1][B2], we can assume $(i, j) = (1, 1)$ or $(1, 2)$. Now it is straightforward to verify

$$(e^1 \wedge *_{\mathbb{R}} \sigma) \cdot (e^1 \wedge *_{\mathbb{R}} \sigma) = 4, \quad (e^1 \wedge *_{\mathbb{R}} \sigma) \cdot (e^2 \wedge *_{\mathbb{R}} \sigma) = 0. \tag{2.14}$$

On the other hand, using the formula

$$*_\sigma \sigma = e^1 \wedge e^5 \wedge e^6 \wedge e^7 + e^2 \wedge e^3 \wedge e^6 \wedge e^7 + e^3 \wedge e^4 \wedge e^5 \wedge e^7 + e^1 \wedge e^3 \wedge e^5 \wedge e^7$$

$$- e^1 \wedge e^3 \wedge e^6 \wedge e^7 - e^1 \wedge e^2 \wedge e^5 \wedge e^6 - e^1 \wedge e^2 \wedge e^5 \wedge e^7$$

(2.15)
it is also straightforward to verify
\[(e^1 \wedge *\sigma_{R^7}) \cdot (e^1 \wedge *\sigma_{R^7}) = 3, (e^1 \wedge *\sigma_{R^7}) \cdot (e^2 \wedge *\sigma_{R^7}) = 0.\] (2.16)

\[\square\]

**Lemma 2.4.** Let \( p \in M \). Consider the linear map \( \sigma_p \wedge : T_p M^* \to \Lambda^4 T^*_p M \) and its adjoint \((\sigma_p \wedge)^* : \Lambda^3 T^*_p M \to T^*_p M \). We have the following formula
\[(\sigma_p \wedge)^* = \sigma_p \gamma_{\sigma} |_{\Lambda^4 T^*_p M},\] (2.17)

where \( \gamma_{\sigma} : \Lambda^3 T^*_p M \times (\Lambda^3 T^*_p M \otimes T^*_p M) \to T^*_p M \) denotes the front contraction w.r.t. \( g_\sigma \), i.e.
\[\gamma_{\sigma} = (\gamma_1 \cdot \gamma_2)\alpha\] (2.18)

for \( \gamma_1, \gamma_2 \in \Lambda^3 T^*_p M \) and \( \alpha \in T^*_p M \). (Note that it equals \( \frac{1}{6} \) times the restriction of the front contraction between \( \otimes^3 T^*_p M \) and \( \otimes^4 T^*_p M \) which is given by
\[(\gamma_1, \gamma_2 \otimes \alpha) \mapsto (\gamma_1 \cdot \gamma_2)\alpha\] (2.19)

for \( \gamma_1, \gamma_2 \in \otimes^3 T^*_p M \) and \( \alpha \in T^*_p M^* \). The factor \( \frac{1}{6} \) is due to the fact that the inner product between \( \gamma_1, \gamma_2 \in \Lambda^3 T^*_p M \) equals \( \frac{1}{6} \) times their inner product as elements of \( \otimes^3 T^*_p M \).

**Proof.** It suffices to consider the Euclidean space. We need to verify
\[(\sigma_{R^7} \wedge \alpha) \cdot \gamma = \alpha \cdot (\sigma_{R^7} \gamma)\] (2.20)

for all \( \alpha \in (R^7)^* \) and \( \gamma \in \Lambda^4 (R^7)^* \). Since \( G_2 \subset SO(7) \) and it acts transitively on unit vectors, we can assume \( \alpha = e_1 \). There holds
\[\sigma_{R^7} \wedge e^1 = -e^1 \wedge e^2 \wedge e^4 \wedge e^6 + e^1 \wedge e^2 \wedge e^5 \wedge e^7 + e^1 \wedge e^3 \wedge e^4 \wedge e^7 + e^1 \wedge e^3 \wedge e^5 \wedge e^6.\] (2.21)

Hence we have for \( \gamma = \sum_{i<j<k<l} a_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l \)
\[(\sigma_{R^7} \wedge e^1) \cdot \gamma = \alpha_{1246} + \alpha_{1257} + \alpha_{1347} + \alpha_{1356}.\] (2.22)

On the other hand, there holds \( e^1 \wedge e^2 \wedge e^3 \gamma = \alpha_{1234} e^4 + \alpha_{1235} e^5 + \alpha_{1236} e^6 + \alpha_{1237} e^7 \) etc. and hence a straightforward calculation yields
\[e^1 \cdot (\sigma_{R^7} \gamma) = -\alpha_{1246} + \alpha_{1257} + \alpha_{1347} + \alpha_{1356}.\] (2.23)

\[\square\]

Now we present the formulas for computing \( f^0, f^1 \) and \( f^3 \) in (2.11).

**Lemma 2.5.** The forms \( f^0, f^1 \) and \( f^3 \) in (2.11) can be computed from \( \gamma \) as follows
\[f^0 = \frac{1}{7} \gamma \cdot \sigma, f^1 = -\frac{1}{4} \sigma_\gamma (\sigma \gamma), f^3 = \gamma - f^0 - \sigma (f^1 \wedge \sigma).\] (2.24)

In other words, we have
\[\pi^3_{\gamma} = \frac{1}{7} (\gamma \cdot \sigma) \sigma, \pi^3_{\gamma} = \frac{1}{4} \sigma ((\sigma \gamma (\sigma \gamma)) \wedge \sigma).\] (2.25)

**Proof.** Taking the inner product of (2.11) with an arbitrary element \( \sigma (f \wedge \sigma) \) of \( \Lambda^3 T^*_p M \) (with \( f \in T^*_p M \)) we deduce, on account of Lemma 2.4.3,
\[\gamma \cdot \sigma (f \wedge \sigma) = \sigma (f^1 \wedge \sigma) \cdot \sigma (f \wedge \sigma) = 4f^1 \cdot f.\] (2.26)

Since \( \gamma \cdot \sigma (f \wedge \sigma) = -\sigma \gamma \cdot (\sigma \wedge f) \), we can apply Lemma 2.4 to arrive at the formula for \( f^1 \) in (2.24). The formula for \( f^0 \) in (2.24) is obtained by taking the inner product of (2.11) with \( \sigma \).

\[\square\]

Note that a general \( G_2 \) structure \( \sigma \) has four torsion forms \( \tau_0, \tau_1, \tau_2 \) and \( \tau_3 \), with e.g. \( \tau_2 \) having values in \( \Lambda^3 T^*_p M \), see [B2]. If \( \sigma \) is closed, then its adjoint torsion \( \tau \) is precisely \( \tau_2 \). Indeed, in that case, we have by [Proposition 1, B2] the equation \( \tau_2 \wedge \sigma = d \star_\sigma \sigma \). Hence we have by the above characterizations \( \tau_2 = -\star_\sigma (\tau_2 \wedge \sigma) = -\star_\sigma d \star_\sigma \sigma = \tau \).
3. An identity for the Hodge Laplacian on 3-forms

The main purpose of this section is to present a new identity for the Hodge Laplacian, which will play a crucial role in Section 6 for constructing a suitable gauge fixing for the Laplacian flow.

3.1. A New Differential Identity of First Order.

In [B2] Bryant introduced differentials (exterior derivatives) which are adapted to the above decompositions of forms. These adapted differentials are very natural and indeed unique up to zeroth order perturbations. He also found remarkable (but very natural) identities for those differentials [B2]. Here we present a typical one of them and obtain a new one.

As before, a $G_2$-structure $\sigma$ on $M$ is given.

**Definition 3.1** The differential $d^2_7 : \Omega^1(M) \to \Omega^2_7(M)$ is defined to be

$$d^2_7 \alpha = \ast_\sigma d(\alpha \wedge \ast_\sigma) = \ast_\sigma (d\alpha \wedge \ast_\sigma - \ast\tau).$$  \hspace{1cm} (3.1)

The differential $d^2_{14} : \Omega^1(M) \to \Omega^2_{14}(M)$ is defined to be

$$d^2_{14} \alpha = \pi^2_1 d\alpha.$$ \hspace{1cm} (3.2)

(We can also define $d^2_7$ on $\Omega^2(M)$ and $\Omega^2_{14}(M)$. But the formulas for $d^2_7$ on different spaces are different. The situations with $d^2_{14}$ and other adapted differentials are similar.)

The differential identity (3.3) below without the lower order term can be found in [B2] for the special case of a torsion-free $\sigma$. For the purpose of computations in dealing with the Laplacian flow, we need to understand the precise nature of the additional lower order term which appears in the identity in the general case.

**Lemma 3.1.** We drop the subscript $\sigma$ in the notations. There holds for all $\alpha \in \Omega^1(M)$ (or $\alpha \in C^1(T^* M)$)

$$d\alpha = \frac{1}{3} \ast (d_7^2 \alpha \wedge \ast) + d_{14}^2 \alpha + \frac{1}{3} \ast (\ast \sigma \wedge \ast (\alpha \wedge \ast \tau)).$$ \hspace{1cm} (3.3)

**Proof.** The identity (3.3) is equivalent to the identity

$$\pi^2_1 d\alpha = \frac{1}{3} \ast (d_7^2 \alpha \wedge \ast) + \frac{1}{3} \ast (\ast \sigma \wedge \ast (\alpha \wedge \ast \tau)).$$ \hspace{1cm} (3.4)

To prove (3.4) we first observe

$$\pi^2_1 d\alpha = \pi^2_1 \left( \sum_i e^i \wedge (e_i \lrcorner \nabla \alpha) \right) = F(\nabla \alpha),$$ \hspace{1cm} (3.5)

where $e_i$ denotes a local orthonormal basis and

$$F_p(\Theta) = \pi^2_1 \left( \sum_i e^i \wedge (e_i \lrcorner \Theta) \right)$$ \hspace{1cm} (3.6)

for all $p \in M$ and $\Theta \in T^*_p M \otimes T^*_p M$. On the other hand, we have

$$\ast (d_7^2 \alpha \wedge \ast) + \ast (\ast \sigma \wedge \ast (\alpha \wedge \ast \tau)) = \ast (\ast \sigma \wedge (d_7^2 \alpha + \ast (\alpha \wedge \ast \tau)))$$

$$= \ast (\ast \sigma \wedge \ast (d_7^2 \alpha + \ast \tau))$$

$$= \check{F}(\nabla \alpha),$$ \hspace{1cm} (3.7)

where

$$\check{F}_p(\Theta) = \sum_i \ast (\ast \sigma \wedge \ast ((e^i \wedge e_i \Theta) \wedge \ast \sigma)).$$ \hspace{1cm} (3.8)

By (3.8), $\check{F}_p$ has values in $\Lambda^2_7(T^*_p M)$.

It is easy to verify that $F$ and $\check{F}$ are independent of the choice of the basis. Let $F$ and $\check{F}$ stand for $F_p$ and $\check{F}_p$ respectively for an arbitrary $p \in M$. They are linear maps from $T^*_p M \otimes T^*_p M$ to $\Lambda^2(T^*_p M)$ and $\Lambda^2_7(T^*_p M)$. One readily verifies that they are $G_2$ equivariant. Now we have the orthogonal decomposition into irreducible $G_2$ representations

$$T^*_p M \otimes T^*_p M = \text{span}(g_{\sigma}|_p) \oplus S^0_2(T^*_p M) \oplus \Lambda^2_7(T^*_p M) \oplus \Lambda^2_{14}(T^*_p M),$$ \hspace{1cm} (3.9)

where $S^2(T^*_p M)$ consists of traceless symmetric 2-tensors. The dimensions of these representations are obviously different from each other. By Schur lemma, the restrictions of $F$ and $\check{F}$ to the complement of $\Lambda^2_7(T^*_p M)$ are trivial. On the other hand, it is easy to see that their restrictions $F_{\Lambda^2_7}$ and $\check{F}_{\Lambda^2_7}$ to $\Lambda^2_7(T^*_p M)$ are
nontrivial. Indeed, we can choose the basis $e_1$ to be induced from the standard basis on $\mathbb{R}^7$ via an inducing map of $\sigma$. Then the formula (2.1) holds true for $\sigma$. Using it we easily deduce for $\Theta = e^1 \otimes e^2$

$$F(e^1 \otimes e^2) = \pi^2(e^1 \wedge e^2) = \frac{1}{3} e^1 \wedge e^2 + \frac{1}{3} (e^1 \wedge e^2 \wedge \sigma)$$

$$= \frac{1}{3} (e^1 \wedge e^2 - e^4 \wedge e^7 - e^5 \wedge e^6)$$

(3.10)

and

$$\tilde{F}(e^1 \otimes e^2) = *(\sigma \wedge *(e^1 \wedge e^2 \wedge \sigma)) = *(\sigma \wedge e^3)$$

$$= e^1 \wedge e^2 - e^4 \wedge e^7 - e^5 \wedge e^6.$$  

(3.11)

By Schur’s lemma, $F_{\Lambda^2}$ and $\tilde{F}_{\Lambda^2}$ are isomorphisms. Since $\Lambda^1(T^*_\star M)$ is odd dimensional, the isomorphism $F_{\Lambda^2} \tilde{F}^{-1}_{\Lambda^2}$ has at least one nontrivial eigenspace. By the irreducibility we then conclude that it is a scalar multiple of the identity. By (3.10) and (3.11) the scalar is $\frac{1}{2}$. Hence we conclude that $F = \frac{1}{2} \tilde{F}$, which leads to (3.4).

An alternative proof of (3.5) is in terms of the characterization (2.9), the orthogonality relations and integration by parts, analogous to the proof of Lemma 4.2 below. □

Next we present the said new differential identity.

**Lemma 3.2.** There holds for all $\alpha \in \Omega^1(M)$

$$d\alpha = *(d^2\alpha \wedge *\sigma) - *(d\alpha \wedge \sigma) + \frac{1}{3} \xi (\{(*(\sigma \wedge *(\alpha \wedge *\tau))\})$$

(3.12)

where $\xi = \xi_\sigma$ is defined as follows

$$\xi(\gamma) = \gamma + *(\sigma \wedge \gamma).$$

(3.13)

**Proof.** By the identity (3.3) and the formulas in (2.9) we deduce

$$d\alpha \wedge \sigma = \frac{2}{3} d^2\alpha \wedge *\sigma - *d^2_{14}\alpha + \frac{1}{3} \sigma \wedge *(\sigma \wedge *(\alpha \wedge *\tau)),$$

(3.14)

which leads to

$$*(d\alpha \wedge \sigma) = \frac{2}{3} *(d^2\alpha \wedge *\sigma) - d^2_{14}\alpha + \frac{1}{3} [\sigma \wedge *(\sigma \wedge *(\alpha \wedge *\tau))].$$

(3.15)

Adding (3.3) and (3.15) we then arrive at (3.12). □

### 3.2. A New Identity for the Hodge Laplacian on 3-Forms.

Let $\sigma$ be a given closed $G_2$-structure on $M$. In the ensuing computations in this subsection, we’ll drop the subscript $\sigma$. Thus $* = *_{\sigma}, \Delta = \Delta_{\sigma}$ and $\tau = \tau_{\sigma}$. For a closed form $\theta \in C^2(\Lambda^3T^*_\star M)$ we apply the decomposition (2.11) and compute

$$-\Delta \theta = -*d*d\theta + d*d*\theta = d*d*\theta,$$

(3.16)

$$*d*d\theta = *d*(f^0\sigma + *(f^1 \wedge \sigma) + f^3)$$

$$=*\{df^0 \wedge *\sigma + df^1 \wedge \sigma + d*f^3\} - f^0\tau.$$  

(3.17)

Next we consider the differential operator $H = H_{\sigma}$:

$$H(\theta) = *d*(\frac{4}{3}f^0\sigma + *(f^1 \wedge \sigma) - f^3)$$

$$=*\{df^0 \wedge *\sigma + df^1 \wedge \sigma - d*f^3\} - \frac{4}{3}f^0\tau.$$  

(3.18)

This is an important operator because of its role in the linearization of the Laplacian flow, as will be shown in Section 6 below. We would like to compute the difference $d \circ H - \Delta = d(H + *d*)$. By (3.18) and (3.17) there holds

$$H + *d*)\theta = \frac{7}{3} *(df^0 \wedge *\sigma) + 2*(df^1 \wedge \sigma) - \frac{7}{3}f^0\tau.$$  

(3.19)

We would like to convert the term $2*(df^1 \wedge \sigma)$ involving the 2-form $df^1$ into an expression involving a 1-form. This is achieved by the following lemma. The 2-form $df^1$ still appears in the new formula (3.20), but is separated from other quantities. Hence it disappears in (3.21) because of differentiation.
Lemma 3.3. There holds
\[
(H + d^* d)\theta = \frac{7}{3} (d^0 f^* \sigma) + 2 (d^1 f^1 \wedge \sigma) - 2 d f^1 \\
+ \frac{2}{3} \xi (\ast(\sigma \wedge (f^1 \wedge \tau))) - \frac{7}{3} \xi \tau
\]
\[
= \frac{7}{3} (d^0 f^* \sigma) + 2 (d^1 f^1 \wedge \sigma) - 2 d f^1 \\
+ \frac{2}{3} \xi (\ast(\sigma \wedge (f^1 \wedge \tau))) - \frac{7}{3} \xi \tau.
\]

\[(3.20)\]

Consequently,
\[
\Delta \theta = d(H(\theta)) - d\left(\frac{7}{3} (d^0 f^* \sigma) + 2 (d^1 f^1 \wedge \sigma) \right. \\
+ \frac{2}{3} \xi (\ast(\sigma \wedge (f^1 \wedge \tau))) - \frac{7}{3} \xi \tau \biggr) \\
\]
\[
\quad + d \left[ \frac{7}{3} \xi \tau - \frac{2}{3} \xi (\ast(\sigma \wedge (f^1 \wedge \tau))) \biggr].
\]

\[(3.21)\]

Proof. Applying (3.12) with \(\alpha = f^1\) we obtain
\[
\ast (d^1 \wedge \sigma) = -df^1 + \ast (d^1 f^1 \wedge \sigma) + \frac{1}{3} \xi (\ast(\sigma \wedge (\alpha \wedge \tau)\)).
\]

Combining this with (3.19) we then arrive at (3.20). \(\square\)

4. Additional differential identities

In this section we establish several differential identities which will be used in Section 7 for proving the uniqueness of the solution of the Laplacian flow with given initial data. As in the last section, the proofs of these identities determine the precise forms of the additional lower order terms which arise in the situation of a general closed \(G_2\)-structure in comparison with a torsion-free \(G_2\)-structure.

Let a closed \(G_2\)-structure \(\sigma\) on \(M\) be given. As in the last section, we drop the subscript \(\sigma\) in the notations.

4.1. Two First Order Identities.

Definition 4.1 The differential \(d^1_7 : \Omega^0(M) \rightarrow \Omega^1(M)\) is defined to be
\[
d^1_7 f = df.
\]

The differential \(d^1_7 : \Omega^1(M) \rightarrow \Omega^0(M)\) is defined to be the former \(L^2\)-adjoint of \(d^1_7\), thus
\[
d^1_7 \alpha = d^* \alpha = -\ast d^* \alpha.
\]

The differential \(d^1_{27} : \Omega^1(M) \rightarrow \Omega^3_{27}(M)\) is defined to be
\[
d^1_{27} \alpha = \pi^1_{27} d^* (\alpha \wedge \sigma).
\]

The differential \(d^1_{14} : \Omega^0_{14}(M) \rightarrow \Omega^1(M)\) is defined to be the formal \(L^2\)-adjoint of \(d^1_{14}\), whose definition is given in the last section. Thus \(d^1_{14} = (d^1_{14})^*\). Finally, we define \(d^1_{27} : \Omega^0_{14}(M) \rightarrow \Omega^3_{27}(M)\) by the formula \(d^1_{27} \beta = \pi^1_{27} d^* \beta\).

First we present a new differential identity. It has the remarkable feature of expressing the co-differential of a special kind of 2-form in terms of its differential. This is impossible for general 2-forms.

Lemma 4.1. There holds for all \(\beta \in \Omega^2(M)\)
\[
\ast (d^* d) \beta = \frac{1}{2} \sigma \wedge \ast d \beta - \frac{1}{2} \sigma \wedge (e^1 \wedge \ast (\alpha \wedge \nabla_{\epsilon^1} \sigma)) - \ast (\alpha \wedge \ast \tau),
\]

where \(\alpha \in \Omega^1(M)\) is uniquely determined by the equation \(\beta = \ast(\alpha \wedge \sigma)\) (according to (2.4)). In other words, there holds
\[
d^1_7 \alpha \equiv \star d (\alpha \wedge \ast \sigma) = \frac{1}{2} \sigma \wedge d (\alpha \wedge \ast \sigma) - \frac{1}{2} \sigma \wedge (e^1 \wedge \ast (\alpha \wedge \nabla_{\epsilon^1} \sigma)) - \ast (\alpha \wedge \ast \tau)
\]

for all 1-forms \(\alpha\).
Proof. There holds
\[ *d(\alpha \wedge *\sigma) = *(d\alpha \wedge *\sigma) - *(\alpha \wedge *\tau) = \Phi(\nabla \alpha) - *(\alpha \wedge *\tau), \]
where for \( p \in M \) and \( \Theta \in T^*_p M \otimes T^*_p M \)
\[ \Phi_p(\Theta) = *(e^i \wedge (e_i \wedge \Theta) \wedge *\sigma). \]
On the other hand, there holds
\[ \sigma - \star d * (\alpha \wedge *\sigma) = \sigma - \star (e^i \wedge *(\nabla_{e_i} \alpha \wedge *\sigma + \alpha \wedge *\nabla_{e_i} \sigma)) = \Psi(\nabla \alpha) + \sigma - \star (e^i \wedge *(\alpha \wedge *\nabla_{e_i} \sigma)), \]
where
\[ \Psi_p(\Theta) = \sigma - \star (e^i \wedge *(e_i \wedge \Theta \wedge *\sigma)) \]
with the above \( \Theta \). For a fixed \( p \), \( \Phi_p \) and \( \Psi_p \) are \( G_2 \)-equivariant linear maps from \( T^*_p M \otimes T^*_p M \) into \( T^*_p M \).
By the arguments in the proof of Lemma, \( \Phi_p = \lambda \Psi_p \) for a scalar \( \lambda \). Consider an induced orthonormal basis \( e_i \) and its dual \( e^i \). There hold
\[ \Phi_p(e^1 \wedge e^2) = *(e^1 \wedge e^2 \wedge *\sigma) = *e^1 \wedge e^2 \wedge e^4 \wedge e^5 \wedge e^7 = e^3 \]
and
\[ \Psi_p(e^1 \wedge e^2) = \sigma - \star (e^1 \wedge *(e^2 \wedge *\sigma)) \]
\[ = \sigma - \star (e^1 \wedge *e^2 \wedge e^4 \wedge e^5 \wedge e^7 - e^1 \wedge e^2 \wedge e^3 \wedge e^5 \wedge e^7 + e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^6)) \]
\[ = \sigma - \star e^2 \wedge e^3 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^5 \wedge e^7) \]
\[ = 2e^3. \]
It follows that \( \Phi_p = \frac{1}{2} \Psi_p \), which leads to (4.3). \( \square \)

The differential identity in the next lemma without the lower order terms can be found in [B2] for the special case of a torsion-free \( G_2 \)-structure.

Lemma 4.2. There holds for all \( \alpha \in \Omega^1(M) \)
\[ d * (\alpha \wedge *\sigma) = -\frac{3}{7} (d_1^\alpha) \sigma - \frac{1}{2} * (d_2^\alpha \wedge \sigma) + d_\tau \alpha + \zeta(\alpha) \]
with
\[ \zeta(\alpha) = \zeta_\sigma(\alpha) = -\frac{1}{7} ((\alpha \wedge *\sigma) \wedge *\tau) \sigma - \frac{1}{4} \sigma - \star (e^i \wedge *(\alpha \wedge *\nabla_{e_i} \sigma)) - \frac{1}{2} * (\alpha \wedge *\tau). \]
Proof. First we decompose \( d * (\alpha \wedge *\sigma) \) into irreducible parts
\[ d * (\alpha \wedge *\sigma) = \pi_1^3 d * (\alpha \wedge *\sigma) + \pi_2^3 d * (\alpha \wedge *\sigma) + \pi_2^3 d * (\alpha \wedge *\sigma). \]
The first part can be determined by employing orthogonal relations and integration by parts. (One can also argue as in the proof of Lemma 3.1 and Lemma. By 2.3, it can be written as \( f \sigma \) for a scalar function \( f \). Taking the \( L^2 \) inner product of (4.14) with \( \tilde{f} \sigma \) for an arbitrary scalar function \( \tilde{f} \) we infer
\[ 7 \int_M f \tilde{f} = -\int_M d * (\alpha \wedge *\sigma) \cdot \tilde{f} \sigma \]
\[ = -\int_M (\alpha \wedge *\sigma) \cdot (d \tilde{f} * \sigma) \]
\[ = -\int_M (\alpha \wedge *\sigma) \cdot (d(\tilde{f} \sigma)) - \int_M \tilde{f} \sigma (\alpha \wedge *\sigma) \cdot *\tau. \]
Appealing to Lemma 2.3 we then deduce
\[ 7 \int_M f \tilde{f} = -3 \int_M \alpha \cdot \tilde{f} - \int_M \tilde{f} (\alpha \wedge *\sigma) \cdot *\tau \]
\[ = -3 \int_M \tilde{f} d_1^\alpha - \int_M \tilde{f} (\alpha \wedge *\sigma) \cdot *\tau. \]
We conclude that \( f = -\frac{3}{7} d_1^\alpha - \frac{1}{2} (\alpha \wedge *\sigma) \cdot *\tau \) and hence
\[ \pi_1^3 d * (\alpha \wedge *\sigma) = -\frac{3}{7} (d_1^\alpha) \sigma - \frac{1}{7} ((\alpha \wedge *\sigma) \wedge *\tau) \sigma. \]
Next we determine the second term in the decomposition (4.14). By Lemma 2.5 and Lemma we deduce
\[
\pi^3 \partial^* (\alpha \wedge \sigma) = -\frac{1}{4} * \left( (\sigma \wedge \partial^* (\alpha \wedge \sigma) \wedge \sigma) \right) \\
= -\frac{1}{2} (d^2 \alpha \wedge \sigma) - \frac{1}{4} \sigma \wedge * (\epsilon^3 \wedge \sigma (\alpha \wedge \nabla_{\xi} \sigma)) - \frac{1}{2} * (\alpha \wedge \tau). 
\]
(4.18)

Combining (4.17) and (4.18) we arrive at (4.12).

4.2. An Identity for the Hodge Laplacian on 1-Forms.

The following second order differential identity without the lower order term can be found in [B2] for the special case of a torsion-free $G_2$-structure.

**Lemma 4.3.** There holds for all $\alpha \in \Omega^1(M)$
\[
\Delta \alpha = (d^2 d_1^2 + d^2 d_2^2) \alpha + \frac{1}{3} * d * \xi * (\star (\alpha \wedge \tau)), 
\]
where $\xi$ is given in (3.13).

**Proof.** There holds $\Delta \alpha = dd^* \alpha + d^* d\alpha = dd^* \alpha + * d \alpha$. By the definitions of $d_1^2$ and $d_2^2$ we have
\[
* d * d\alpha = * d(d_2^2 \alpha \wedge \sigma) - * d(d\alpha \wedge \sigma) + \frac{1}{3} * d * \xi * (\star (\alpha \wedge \tau)). 
\]
(4.21)

By the definition of $d_2^2$ the first term on the above right hand side is precisely $d_2^2 d_2^2 \alpha$. The second term vanishes because $\sigma$ is closed. Combining the above calculations we arrive at (4.19). □

5. Linear Parabolic Theory

As mentioned in Introduction, the gauge fixed Laplacian flow is parabolic only in the direction of closed forms. Hence there are troubles with applying the conventional theory of parabolic equations. For this reason, an approach in terms of Nash-Moser implicit function theorem was adopted in [BX]. In this section, we develop a new linear parabolic theory for closed forms, which will enable us to construct short time solutions of the gauge fixed Laplacian flow via the classical implicit function theorem.

For the sake of completeness, we also include the corresponding theory for exact forms. (We also have the corresponding theories for co-closed and co-exact forms.) There is a subtlety here as mentioned in Introduction. The structure of the Laplacian equation or the gauge-fixed Laplacian equation for closed forms allows one to treat them as equations for exact forms, namely one can assume that $\sigma - \sigma_0$ is exact with $\sigma_0$ denoting the initial $G_2$-structure. However, the linear parabolic theory for exact forms is not suitable for treating the issue of short time solutions due to the lack of completeness of the involved function spaces $C^{4,1/2}_d (\pi^* (\Lambda^3 T^* M))$, cf. the discussions below.

The parabolic Hölder spaces, namely the $C^{4,1/2}$ spaces, play an important role in this paper both for handling short time solutions and the convergence of the Laplacian flow. These function spaces are used e.g. in the classical text [LSU]. They were first introduced in a geometric set-up in [Y3]. Alternatively, we can also use parabolic Sobolev spaces to handle short time solutions. But the use of the $C^{4,1/2}$ spaces is crucial for proving convergence, see Section 5. The definition of the $C^{4,1/2}$ spaces is given in Appendix.

In this section, $M$ stands for a compact smooth manifold of dimension $n \geq 2$. Fix a background metric $g_\ast$ on $M$, which is used to make various measurements. It is required to have enough smoothness in each context. Throughout this section, all norms are measured w.r.t. to $g_\ast$, unless otherwise indicated. Note that we can choose $g_\ast$ according to our needs in each situation. For example, we can choose $g_\ast$ to be the induced metric of a given torsion-free $G_2$-structure in the context of Theorem 1.2. We can translate easily the measurements w.r.t. one background metric into measurements w.r.t. another background metric.
5.1. Linear parabolic theory for general forms.

Consider the vector bundle $E = \Lambda^j T^* M, 1 \leq j \leq 7$. We’ll fix $j$ in the discussions below. Let $C^l(E)$ denote the space of $C^l$ sections of $E$, equipped with the $C^l$-norm, which is defined w.r.t. $g_*$. Fix $T > 0$, let $\pi = \pi_{[0, T]} : M \times [0, T] \to M$ denote the projection $\pi_{[0, T]}(p, t) = p$. Let $C^l_{(l), 2}(\pi^* E)$ denote the $C^l_{(l), 2}$ sections of $\pi^* E$, equipped with the $C^l_{(l), 2}$-norm, which is defined w.r.t. $g_*$. Note that a section $\gamma$ of $\pi^* E$ has arguments $(p, t) \in M \times [0, T]$ and satisfies $\gamma(p, t) \in E_p$.

Let $l > 2$ and $U$ an open subset of $C^l_{(l), 2}(\pi^* E)$. To each operator $F : U \to C^{l-2, (l-2)/2}(\pi^* E)$ we associate its $P$-operator

$$P_F : U \to C^{l-2, (l-2)/2}(\pi^* E)$$

(5.1)

given by $P_F = \frac{\partial}{\partial t} + F$ and its $P$-map

$$P_F : U \to C^{l-2, (l-2)/2}(\pi^* E) \times C^l(E)$$

(5.2)

given by $P_F(\gamma) = (\frac{\partial \gamma}{\partial t} + F(\gamma), \gamma(0))$.

**Theorem 5.1.** Let $g_0$ be a $C^l$ metric on $M$ for a given noninteger $l > 2$, and $\Delta$ its Hodge Laplacian. Let $\pi = \pi_{[0, T]}$ for a given $T > 0$. There is a positive constant $\delta_0 = \delta_0(\|g_0\|_{C^0}, \|g_0^{-1}\|_{C^0}, l, g_*)$ with the following properties. Let $\Phi_0 \in C^{l-1, (l-1)/2}(\text{Hom}(\pi^*(\Lambda^j T^* M, \Lambda^{j-1} T^* M))$ and

$$\Phi_1 \in C^{l-1, (l-1)/2}(T^* M \otimes \Lambda^j T^* M, \Lambda^{j-1} T^* M).$$

Set $\Phi(\gamma) = \Phi_0(\gamma) + \Phi_1(\nabla \gamma)$. Assume

$$\|\Phi_1\|_{C^0} \leq \delta_0.$$  

(5.3)

Then the $P$-map of the operator $\Delta + d \circ \Phi$

$$P_{\Delta + d \circ \Phi} : C^{l_{(l), 2}}(\pi^* \Lambda^j T^* M) \to C^{l-2, (l-2)/2}(\pi^* \Lambda^j T^* M) \times C^l(\Lambda^j T^* M)$$

(5.4)

is an isomorphism. Moreover, there hold

$$\|P_{\Delta + d \circ \Phi}\| \leq C \text{ and } \|P_{\Delta + d \circ \Phi}^{-1}\| \leq C$$

(5.5)

for a positive constant $C = C(n, l, T, \|g_0\|_{C^{l-1}}, \|g_0^{-1}\|_{C^0}, \|\Phi_0\|_{C^{l-1, (l-1)/2}}, \|\Phi_1\|_{C^{l-1, (l-1)/2}}, g_*)$.

The number $\delta_0$ depends on each involved scalar quantity decreasingly, while the number $C$ has increasing dependences. The dependences of $\delta_0$ and $C$ on $g_*$ are in terms of its Riemannian norm $\|g_*\|_{C^{l-1}}$ (see [Y3] for the definition of this norm). The dependences of constants on $g_*$ below are all of the same nature.

**Proof of Theorem 5.1.** We have the following Bochner-Weitzenböck formula

$$\Delta = \nabla^* \nabla + \mathcal{R},$$

(5.6)

where $\mathcal{R} = \mathcal{R}_j$ is a linear action of the curvature operator of $g_0$ on $j$-forms. In a local chart, the leading term of the operator $\nabla^* \nabla$ takes the form $- \sum_{ij} g^{ij} \partial_i \partial_j$. Hence the parabolic theory in [LSU] can be applied, and the desired isomorphism property and estimates follow, see [Y2] or [Y3] for details. Note that the smallness condition (5.8) is for the purpose of obtaining uniform strong ellipticity of the operator $-\Delta - d \circ \Phi$. 

Theorem 5.1 will be applied below to establish a linear parabolic theory for closed forms. On the other hand, we have the following time-interior version of Theorem 5.1 which will be used in Section 7 for handling long time existence and convergence of the Laplacian flow.

**Lemma 5.2.** Assume the same set-up as in Theorem 5.1. Moreover, assume (5.5). Let $\gamma \in C^{l_{(l), 2}}(\pi^* E)$ and $\alpha \in C^{l-2, (l-2)/2}(\pi^* E)$ satisfy

$$\frac{\partial \gamma}{\partial t} + \Delta \gamma + d(\Phi(\gamma)) = \alpha$$

(5.7)

on $M \times [0, T]$. Let $0 < \epsilon_1 < \epsilon_2 < T$. Then there is a positive constant $C = C(l, T, (\epsilon_2 - \epsilon_1)^{-1})$ depending only on $l, T$ and $(\epsilon_2 - \epsilon_1)^{-1}$ such that

$$\|\gamma\|_{C^{l_{(l), 2}}(M \times [\epsilon_2, T])} \leq C \cdot C(l, T, (\epsilon_2 - \epsilon_1)^{-1})[\|\alpha\|_{C^{l-2, (l-2)/2}(M \times [\epsilon_1, T])} + (\epsilon_2 - \epsilon_1)^{-1}\|\gamma\|_{C^{l-2, (l-2)/2}(M \times [\epsilon_1, T])}],$$

(5.8)

where $C$ is the constant in (5.7).
Proof. Fix a nonnegative smooth function $\eta$ on $\mathbb{R}$ such that $\eta(t) = 0$ for $t \leq 0$ and $\eta(t) = 1$ for $t \geq 1$. Then we set $\eta_{\epsilon_1, \epsilon_2}(t) = \eta((\epsilon_2 - \epsilon_1)^{-1}(t - \epsilon_1))$ and $\tilde{\gamma} = \eta_{\epsilon_1, \epsilon_2}(t)\gamma$. There holds
\[
\frac{\partial\tilde{\gamma}}{\partial t} + \Delta\tilde{\gamma} + d(\Phi(\tilde{\gamma})) = \eta_{\epsilon_1, \epsilon_2}\alpha - \eta_{\epsilon_1, \epsilon_2}\gamma
\]  
(5.9)
on $M \times [0, T]$. Let $\delta = l - |t|$. Then there hold
\[
\|\eta_{\epsilon_1, \epsilon_2}\|_{C^{l-2,(l-2)/2}(M \times [0, T])} \leq C(l, T, (\epsilon_2 - \epsilon_1)^{-1})
\]  
(5.10)and
\[
\|\eta_{\epsilon_1, \epsilon_2}\|_{C^{l-2,(l-2)/2}(M \times [0, T])} \leq (\epsilon_2 - \epsilon_1)^{-1}C(l, T, (\epsilon_2 - \epsilon_1)^{-1}),
\]  
(5.11)where
\[
C(l, T, x) = C(0)(\max\{T^{1-\delta}, T^{1-\delta/2}, T^{(1-\delta)/2}\}x^{\frac{|l|+1}{2}} + \max_{0 \leq j \leq \frac{|l|}{2}} x^j)
\]  
(5.12)for a positive constant $C(0)$ depending only on $|l|$. Then it follows that
\[
\|\eta_{\epsilon_1, \epsilon_2}\alpha\|_{C^{l-2,(l-2)/2}(M \times [0, T])} \leq C(l, T, (\epsilon_2 - \epsilon_1)^{-1})\|\alpha\|_{C^{l-2,(l-2)/2}(M \times [\epsilon_1, T])}
\]  
(5.13)and
\[
\|\eta_{\epsilon_1, \epsilon_2}\gamma\|_{C^{l-2,(l-2)/2}(M \times [0, T])} \leq (\epsilon_2 - \epsilon_1)^{-1}C(l, T, (\epsilon_2 - \epsilon_1)^{-1})\|\gamma\|_{C^{l-2,(l-2)/2}(M \times [\epsilon_1, T])}.
\]  
(5.14)Applying Theorem 5.1 we then arrive at
\[
\|\tilde{\gamma}\|_{C^{l/2}(M \times [0, T])} \leq C \cdot C(l, T, (\epsilon_2 - \epsilon_1)^{-1})\|\alpha\|_{C^{l-2,(l-2)/2}(M \times [\epsilon_1, T])} + (\epsilon_2 - \epsilon_1)^{-1}\|\gamma\|_{C^{l-2,(l-2)/2}(M \times [\epsilon_1, T])},
\]  
(5.15)which implies (5.8). \qed

Theorem 5.3. Assume the same set-up as in Theorem 5.1. Moreover, assume (5.3). Let $\gamma \in C^{l,2,(l-2)/2}(\pi^* E)$ and $\alpha \in C^{l-2,(l-2)/2}(\pi^* E)$ satisfy (5.7) on $M \times [0, T]$. Let $0 < \epsilon < T$. Then there is a positive constant $C(l, T, \epsilon^{-1}, C)$ depending only on $l, T, \epsilon^{-1}$ and the $C$ in (5.7) such that
\[
\|\gamma\|_{C^{l/2}(M \times [\epsilon, T])} \leq C(l, T, \epsilon^{-1}, C)(\|\alpha\|_{C^{l-2,(l-2)/2}(M \times [\epsilon_1, T])} + \|\gamma\|_{C^{m-2,(m-2)/2}(M \times [0, T])}),
\]  
(5.16)where $m = l - 2k \geq 0$ for the largest nonnegative integer $k$. (If $l = 2k + \mu$ for $0 < \mu < 1$, then $m = \mu$. If $l = 2k + 1 + \mu$ for $0 < \mu < 1$, then $m = 1 + \mu$.)

Proof. Apply Lemma 5.2 successively to the sequence of pairs $(\epsilon/2, \epsilon), (\epsilon/4, \epsilon/2), \ldots$ (playing the role of $(\epsilon_1, \epsilon_2)$), with a sequence of decreasing $l$, i.e., $l, l-2, \ldots$. After finitely many steps we then arrive at (5.16). \qed

5.2. Linear parabolic theory for closed forms.

We set for $l \geq 0$
\[
C^l_o(\Lambda^j T^* M) = \{ \gamma \in C^l(\Lambda^j T^* M) : d\gamma = 0 \}.
\]  
(5.17)
Here the equation $d\gamma = 0$ is in the sense of distribution in the case $0 \leq l < 1$, i.e.
\[
\langle \gamma, d^*_y, \theta \rangle_{L^2_y^*} = 0
\]  
(5.18)for all $\theta \in \Omega^{j+1}(M)$, where $d^*_y$ is the co-differential associated with $g_*$, and $\langle \cdot, \cdot \rangle_{L^2_y^*}$ denotes the $L^2$ inner product w.r.t. $g_*$. (We can also replace $g_*$ by a given $g_0$ as in Theorem 5.1.) Obviously, $C^l_o(\Lambda^j T^* M)$ is a closed subspace, and hence a Banach subspace of $C^l(\Lambda^j T^* M)$. For a noninteger $l > 0$ we set
\[
C^{l,1/2}(\pi^* \Lambda^j T^* M) = \{ \gamma \in C^{l,1/2}(\pi^* \Lambda^j T^* M) : d\gamma(\cdot, t) = 0 \text{ for each } t \in [0, T] \}
\]  
(5.19)which is obviously a closed subspace, and hence a Banach subspace of $C^{l,1/2}(\pi^* \Lambda^j T^* M)$. (Again, the equation $d\gamma(\cdot, t) = 0$ is in the sense of distribution in the case $0 < l < 1$.)
Theorem 5.4. Let $l > 2$ be a noninteger. Let $g_0, \Delta, \Phi_0, \Phi_1, \Phi$ and $\delta_0$ be the same as in Theorem 5.1. Assume $T$. Then the P-map of the operator $\Delta + d \circ \Phi$

$$P_{\Delta + d \circ \Phi} : C_{o}^{l \setminus /2}(\pi^* \Lambda^{j}T^* M) \rightarrow C_{o}^{l - 2, (l - 2) / 2}(\pi^* \Lambda^{j}T^* M)$$

is an isomorphism. Moreover, there hold

$$\|P_{\Delta + d \circ \Phi}\| \leq C$$

and $$\|P_{\Delta + d \circ \Phi}^{-1}\| \leq C$$

for a positive constant $C = C(n, l, T, \|g_0\|_{C^{1,1}}, \|g_0^{-1}\|_{C^{0}}, \|\Phi_0\|_{C^{1,1}(\Lambda^{j}T^* M)}, \|\Phi_1\|_{C^{1,1}(\Lambda^{j}T^* M)}).$ Thus, for each $\beta \in C_{o}^{l}(\Lambda^{j}T^* M)$ and $\gamma \in C_{o}^{l - 2, (l - 2) / 2}(\pi^* \Lambda^{j}T^* M)$ there is a unique solution $\gamma \in C_{o}^{l \setminus /2}(\pi^* \Lambda^{j}T^* M)$ of the initial value problem

$$\frac{d\gamma}{dt} + \Delta \gamma + d(\Phi(\gamma)) = \alpha \quad \text{(on } [0, T]),$$

such that

$$\|\gamma\|_{C^{l,1 / 2}} \leq C(\|\alpha\|_{C^{l - 2, (l - 2) / 2}} + \|\beta\|_{C^{l,1 / 2}}).$$

Proof. We have

$$d \circ \Phi(\gamma) = \sum_i e^i \wedge [(\nabla e_i, \Phi_0)(\gamma) + \Phi_0(\nabla e_i, \gamma) + (\nabla e_i, \Phi_1)(\nabla \gamma) + \Phi_1(\nabla e_i, \nabla \gamma)].$$

Applying Theorem 5.1 we infer that the extended P-map

$$P_{\Delta + d \circ \Phi} : C_{o}^{l \setminus /2}(\pi^* \Lambda^{j}T^* M) \rightarrow C_{o}^{l - 2, (l - 2) / 2}(\pi^* \Lambda^{j}M) \times C^{1}(\Lambda^{j}T^* M)$$

is an isomorphism and satisfies the estimate (5.5). Hence it suffices to show

$$C_{o}^{l \setminus /2}(\pi^* \Lambda^{j}T^* M) = P_{\Delta + d \circ \Phi}(C_{o}^{l - 2, (l - 2) / 2}(\pi^* \Lambda^{j}M)).$$

First we show that the LHS of (5.27) is contained in the RHS of (5.27). It suffices to show that

$$\frac{d\gamma}{dt} + \Delta \gamma + d(\Phi(\gamma)) = \alpha \quad \text{(on } [0, T]).$$

To show the opposite inclusion, consider $\gamma = P_{\Delta + d \circ \Phi}(\alpha, \beta)$ for some $\beta \in C_{o}^{l}(\Lambda^{j}T^* M)$ and $\alpha \in C_{o}^{l - 2, (l - 2) / 2}(\pi^* \Lambda^{j}M)$. Thus $\gamma, \alpha$ and $\beta$ satisfy the equation (5.22). Assume $l > 3$. Taking the differential in the equation we deduce

$$\begin{aligned}
\frac{d\gamma}{dt} + d* d\gamma &= 0,
\frac{d\gamma}{dt}(\gamma, 0) &= 0.
\end{aligned}$$

But $dd^* \gamma = \Delta d\gamma$. Hence we infer $d\gamma \equiv 0$. Indeed, this can be shown directly as follows. Multiplying the above equation by $d\gamma$ and then integrating (first in space, then in time) lead to

$$\|d\gamma(\cdot, t)\|_{L^2}^2 + \int_0^t \|d^* d\gamma(\cdot, t)\|_{L^2}^2 = \|d\gamma(\cdot, 0)\|_{L^2} = 0,$$

where the $L^2$-norms are w.r.t. $g_0$. It follows that $\gamma \in C_{o}^{l \setminus /2}(\pi^* \Lambda^{j}T^* M)$. The case $2 < l < 3$ requires a different argument which also applies to the case $l > 3$. Choose a complete set of $L^2$-orthonormal eigenforms $\gamma_k$ of degree $j$ for $\Delta$, such that each $\gamma_k$ is either harmonic, exact, or coexact. This is possible for the following reason. Let $\phi$ be an eigenform with nonzero eigenvalue $\lambda$. We write $\phi = h + d\psi + d^* \chi$, where $h$ is harmonic. There holds

$$dd^* \psi + d^* dd\chi = \lambda h + \lambda d\psi + \lambda d^* \chi.$$
Multiplying the equation (5.22) with $d^*\phi_i$ and integrating lead to
\[
\frac{da_i}{dt} = \frac{d}{dt} <d\gamma, \phi_i> = <\frac{\partial}{\partial t} d^*\phi_i> = -<\Delta \gamma + d\Phi(\gamma), d^*\phi_i> = -<d^*d\gamma, d^*\phi_i> = -<d\gamma, dd^*\phi_i> = -\lambda_i a_i.
\] (5.34)
Since $\lambda_i > 0$ and $a_i(t, 0) = 0$, we infer $a_i = 0$. Consequently, $\gamma \in C^{l,1/2}_a(\pi^*\Lambda^j T^*M)$.

\section{Linear parabolic theory for exact forms.}

We set for $l \geq 1$
\[
C^l_d(\Lambda^j T^*M) = d(C^{l+1}(\Lambda^{j-1} T^* M)).
\] (5.35)
For a noninteger $l \geq 1$ we set
\[
C^{l,1/2}_d(\pi^*\Lambda^j T^*M) = d(C^{l+1,(l+1)/2}(\pi^*\Lambda^j T^* M)).
\] (5.36)

Employing basic linear elliptic estimates one can easily show that $C^l_d(\Lambda^j T^*M)$ is a closed and hence Banach subspace of $C^l(\Lambda^j T^*M)$. However, as it turns out, $C^{l,1/2}_d(\pi^*\Lambda^j T^*M)$ is not a closed subspace of $C^{l,1/2}(\pi^*\Lambda^j T^*M)$, and hence it is not a Banach space. The analytic reason for this is the lack of involvement of the time derivative in its definition.

As a consequence of Theorem 5.4 we obtain the following result for exact forms.

\begin{theorem}
Let $l > 2$ be a noninteger. Let $g_0, \Delta, \Phi_0, \Phi_1$ and $\delta_0$ be the same as in Theorem 5.1. Assume (5.3). Then the parabolic map
\[ P = P_{\Delta+\delta\Phi} : C^{l,1/2}_d(\pi^*\Lambda^j T^*M) \to C^{l-2,(l-2)/2}_d(\pi^*\Lambda^j M) \times C^l_d(\Lambda^j T^* M) \] (5.37)
is an isomorphism. Moreover, the estimate (5.21) holds true with the same $C$.
\end{theorem}

\begin{proof}
As in the proof of Theorem 5.4 it suffices to show
\[ C^{l,1/2}_d(\pi^*\Lambda^j T^*M) = P_{\Delta+\delta\Phi}^{-1}(C^{l-2,(l-2)/2}_d(\pi^*\Lambda^j M) \times C^l_d(\Lambda^j T^* M)) \] (5.38)
for the extended $P$-map. It is easy to see that the LHS of (5.38) is contained in its RHS. On the other hand, if $\alpha$ and $\beta$ are exact, integrating the equation (5.22) in time shows that $\gamma$ is also exact. Hence the RHS of (5.38) is also contained in the LHS of (5.38).
\end{proof}

\begin{remark}
Obviously, the analog of Theorem 5.4 for co-closed forms and the analog of Theorem 5.5 for co-exact forms hold true if $d \circ \Phi$ is replaced by $d^* \circ \Phi$.
\end{remark}

\section{Short time solutions of the gauge fixed Laplacian flow}

From now on $M$ stands for a compact 7-dimensional manifold which admits closed $G_2$ structures. As in the last section, we fix a background metric $g_0$ on $M$. All function norms in this section are associated with $g_0$. But pointwise norms and other geometric quantities are associated with an initial $G_2$-structure $\sigma_0$ in some situations. This will be made clear in the discussions below.

\subsection{Gauge fixing.}

To construct short time smooth solutions of the Laplacian flow, we employ as in [BX] the following DeTurck type gauge fixing of the Laplacian flow
\[
\frac{\partial \sigma}{\partial t} = \Delta_\sigma \sigma + L_{X(\sigma)} \sigma,
\] (6.1)
where $X(\sigma)$ is a vector field associated with $\sigma$ and $L_{X(\sigma)}$ denotes the Lie derivative. The game of this gauge fixing is to find a suitable $X(\sigma)$ such that the operator $\Delta_\sigma \sigma + L_{X(\sigma)} \sigma$ has maximal (strong) ellipticity. In [BX], a vector field is constructed from the induced metric and its Levi-Civita connection. Based on the new differential identities in Section 3, we introduce a new vector field which has a more transparent structure.

Let a reference closed $G_2$-structure $\sigma_0$ be given. We set $\theta = \sigma - \sigma_0$ for a $G_2$-structure $\sigma$ and write
\[
\theta = f^0 \sigma_0 + *_{\sigma_0} (f^1 \wedge \sigma_0) + f^3
\] (6.2)
as in (2.11). We define
\[ X_{\sigma_0}(\theta) = (\frac{7}{3} df^0 + 2 (d_f^1)_{\sigma_0} f^1)_{\sigma_0}.
\] (6.3)
It is motivated by the identity (3.21). Obviously, $X_{\sigma}(\theta)$ is defined for an arbitrary 3-form $\theta$ given by (3.2).

**Definition 6.1.** The $\sigma_0$-gauged Laplacian flow is defined to be

$$\frac{\partial \sigma}{\partial t} = \Delta_\sigma \sigma + L_{X_{\sigma_0}(\sigma - \sigma_0)} \sigma.$$  

(6.4)

For closed $\sigma$ we have $L_{X_{\sigma_0}(\sigma - \sigma_0)} \sigma = d(X_{\sigma_0}(\sigma - \sigma_0), \omega)$. Hence the $\sigma_0$-gauged Laplacian flow for closed $\sigma$ can be written as follows

$$\frac{\partial \sigma}{\partial t} = \Delta_\sigma \sigma + d(X_{\sigma_0}(\sigma - \sigma_0), \omega).$$  

(6.5)

Next we relate the operator $\Delta_\sigma \sigma + d(X_{\sigma_0}(\sigma - \sigma_0), \omega)$ to the Hodge Laplacian of $\sigma_0$. We shall adopt the following notations: we use $D$ to denote the linearization, i.e. the directional derivative, of an operator, and write it as $D$ in the case of a pointwise operator without involving partial derivatives (thus a finite dimensional operator). For example, $(D_\sigma \Delta_\sigma)(\theta) = \frac{d}{ds} \Delta_{\sigma + s\theta}(\sigma + s\theta)|_{s=0}$ and $(D_\sigma \ast_\sigma \sigma)(\theta) = \frac{d}{ds} \ast_{\sigma + s\theta}(\sigma + s\theta)|_{s=0}$.

**Lemma 6.1.** Let a closed $G_2$ structure $\sigma_0$ be given. There holds for an arbitrary closed $G_2$-structure $\sigma$

$$\Delta_\sigma \sigma + d(X_{\sigma_0}(\theta), \omega) = -\Delta_{\sigma_0} \theta - d(\Phi_{\sigma_0}(\theta))$$  

(6.6)

with

$$\Phi_{\sigma_0}(\theta) = A(\sigma_0, \sigma_0 + \theta, \theta, \nabla_\sigma \theta) + B(\sigma_0, \sigma_0 + \theta, \tau_0, \theta) - \tau_0,$$

(6.7)

where $\theta = \sigma - \sigma_0$ as above, and $A$ and $B$ are smooth in their first two arguments and linear in the other two arguments. The functions $A$ and $B$ are pointwise functions, e.g. $B(\sigma_0, \sigma, \tau_0, \theta)(p) = B(\sigma_0(p), \sigma(p), \tau_0(p), \theta(p))$. They are also universal, i.e. their formulas are independent of the point and the manifold. In other words, these formulas are induced from the case of the Euclidean space in terms of an inducing map.

**Proof.** Because $\sigma$ and $\sigma_0$ are closed, we have $\Delta_\sigma \sigma = -d \ast_\sigma d \ast_\sigma \sigma$ and $\Delta_{\sigma_0} \sigma_0 = -d \ast_{\sigma_0} d \ast_{\sigma_0} \sigma_0$. Hence we obtain

$$\Delta_\sigma \sigma - \Delta_{\sigma_0} \sigma_0 = -d(\ast_\sigma d \ast_\sigma \sigma - \ast_{\sigma_0} d \ast_{\sigma_0} \sigma_0).$$

(6.8)

There holds

$$\ast_\sigma d \ast_\sigma \sigma - \ast_{\sigma_0} d \ast_{\sigma_0} \sigma_0 = \ast_\sigma d(\ast_\sigma \sigma - \ast_{\sigma_0} \sigma_0) + (\ast_\sigma - \ast_{\sigma_0}) d(\ast_\sigma \sigma - \ast_{\sigma_0} \sigma_0) \ast_\sigma - \ast_{\sigma_0} d \ast_{\sigma_0} \sigma_0.$$

(6.9)

We have $D_{\sigma_0}(\ast_\sigma \sigma)(\theta) = \ast_{\sigma_0}(\frac{4}{3}f^0 \sigma_0 + \ast_{\sigma_0}(f^1 \wedge \sigma_0) - f^3)$, cf. (3). It follows that

$$\ast_\sigma \sigma - \ast_{\sigma_0} \sigma_0 = \ast_{\sigma_0}(\frac{4}{3}f^0 \sigma_0 + \ast_{\sigma_0}(f^1 \wedge \sigma_0) - f^3) \ast_\sigma - \ast_{\sigma_0} d \ast_{\sigma_0} \sigma_0.$$

(6.10)

where $q$ is given by

$$q(\sigma_0, \sigma, \theta, \theta) = \int_0^1 \int_0^1 tD^2(\ast_\sigma \sigma)|_{\sigma_0 + st(\sigma - \sigma_0)}(\theta, \theta) ds dt.$$  

(6.11)

with $D^2$ denoting the second derivative operator. Thus $q$ is smooth in its first two arguments and linear in the other two arguments. Note that $q$ is a universal pointwise function and involves no derivative of $\sigma_0$ or $\sigma$.

Now we infer from the above formulas

$$\Delta_\sigma \sigma = \Delta_{\sigma_0} \sigma_0 - d(H_{\sigma_0} \theta) + d((\ast_\sigma - \ast_{\sigma_0}) d(\ast_\sigma \sigma - \ast_{\sigma_0} \sigma_0) + (\ast_\sigma - \ast_{\sigma_0}) d \ast_{\sigma_0} \sigma_0) - d \ast_{\sigma_0} d(q(\sigma_0, \sigma, \theta, \theta)).$$

(6.12)

where $H_{\sigma_0} \theta = \ast_{\sigma_0} d \ast_{\sigma_0} (\frac{4}{3}f^0 \sigma_0 + \ast_{\sigma_0}(f^1 \wedge \sigma_0) - f^3)$ is the operator introduced in (3.18). Consequently,

$$\Delta_\sigma \sigma + d(X_{\sigma_0}(\theta), \omega) = \Delta_{\sigma_0} \sigma_0 - d(H_{\sigma_0} \theta) + \frac{7}{3}(df^0)_{\sigma_0} \omega + 2(df^1 f^1 \sigma_0) \omega + d((\ast_\sigma - \ast_{\sigma_0}) d(\ast_\sigma \sigma - \ast_{\sigma_0} \sigma_0) + (\ast_\sigma - \ast_{\sigma_0}) d \ast_{\sigma_0} \sigma_0) - d \ast_{\sigma_0} d(q(\sigma_0, \sigma, \theta, \theta)).$$

(6.13)

Applying (3.21) with $\sigma_0$ playing the role of $\sigma$ we then deduce (6.6) with

$$\Phi_{\sigma_0}(\theta) = -\tau_0 + (\ast_\sigma - \ast_{\sigma_0}) d(\ast_\sigma \sigma - \ast_{\sigma_0} \sigma_0) + (\ast_\sigma - \ast_{\sigma_0}) \ast_{\sigma_0} \tau_0$$

$$+ \ast_{\sigma_0} d(q(\sigma_0, \sigma, \theta, \theta)) - \frac{7}{3}(df^0)_{\sigma_0} \theta + 2(df^1 f^1)_{\sigma_0} \theta$$

$$+ \frac{2}{3} \xi_{\sigma_0} \ast_{\sigma_0} \ast_{\sigma_0} \sigma_0 \wedge \ast_{\sigma_0}(f^1 \wedge \ast_{\sigma_0} \tau_0)).$$

(6.14)
It is easy to see that \((\sigma_\theta - \sigma_0) d(\sigma_\theta - \sigma_0) = -\frac{2}{3} (df^0)_{\tau_{\sigma_0}} \theta - 2(df^1)_{\tau_{\sigma_0}} \theta\) can be written in the form
\[A_1(\sigma_0, \sigma, \theta, \nabla_{\sigma_0} \theta),\]
the expression \((\sigma_\theta - \sigma_0)^* \sigma_\theta \tau_0 + \frac{\delta}{\pi} \xi_{\sigma_0} (\sigma_\theta \sigma_0 \wedge \sigma_0 \wedge \tau_0)\) can be written in the form \(B_1(\sigma_0, \sigma, \nabla_{\sigma_0} \theta, \theta)\) and \((\sigma_\theta - \sigma_0)^* \sigma_\theta (\sigma_0 \wedge \sigma_0 \tau_0)\) can be written in the form \(B_2(\sigma_0, \sigma, \nabla_{\sigma_0} \theta, \theta)\). (Note that \(d = \sum_i e_i \wedge \nabla_{e_i}\) ) By Lemma 2.2, \(\nabla_{\sigma_0} \theta\) can be expressed in terms of \(\tau_{\sigma_0}\). Hence \(B_2(\sigma_0, \sigma, \nabla_{\sigma_0} \theta, \theta)\) can be rewritten in the form \(B_2(\sigma_0, \sigma, \nabla_{\sigma_0} \theta, \theta)\). Setting \(A = A_1 + A_2\) and \(B = B_1 + B_2\), we then arrive at (6.7). It is easy to verify that \(A\) and \(B\) have the claimed properties. Moreover, the quantities \(*\sigma, *\sigma_0, A\) and \(B\) etc. can all be given explicitly in terms of \(\tau_0\) and \(\sigma_0\). This is in part a consequence of (2.28) and (2.24).

By Lemma 6.1 the \(\sigma_0\)-gauged Laplacian flow can be written as follows
\[
\frac{\partial \sigma}{\partial t} = -\Delta_{\sigma_0} \theta - d(\Phi_{\sigma_0}(\theta)).
\] (6.15)

For convenience of presentation, we formulate a simple lemma, which is an easy consequence of Lemma 2.1 and the nature of the function \(A\).

**Lemma 6.2.** There hold
\[
|A(\sigma_0, \sigma, \theta, \gamma)| \leq \frac{C_0}{\sqrt{1 - |\sigma_0| |\theta|}} \|\sigma_0\|_{C^0} \quad \text{and hence} \quad \|A(\sigma_0, \sigma, \theta, \cdot)\|_{C^0} \leq C_0 \|\theta\|_{C^0}
\] (6.16)

for a universal positive constant \(C_0 \geq 1\), provided that \(\|\theta\|_{C^0} \leq \epsilon_0\), where \(\epsilon_0\) is from Lemma 2.1.

### 6.2. Short time solutions of the gauge fixed Laplacian flow.

Let a closed \(G_2\)-structure \(\sigma_0\) be given. The smoothness requirement for \(\sigma_0\) will be specified in each situation (or is clear from the context). In this subsection we prove an existence and uniqueness theorem for the \(\sigma_0\)-gauged Laplacian flow with \(C^{3+\mu}\) initial data. For simplicity of presentation, here we choose the background metric \(g_0\) of the last subsection to be the induced metric \(g_{\sigma_0}\). Thus, all norms in this subsection are measured w.r.t. \(g_{\sigma_0}\). The covariant derivative \(\nabla\) means \(\nabla_{\sigma_0}\), i.e., it is associated with \(g_{\sigma_0}\). Furthermore, \(e_i\) denotes a local orthonormal frame for \(g_{\sigma_0}\), and \(\tilde{e}_i\) its dual.

**Definition 6.2** For the convenience of presentation we introduce the following notation
\[
\nu_{\sigma, \sigma_0} = \Delta_{\sigma} \sigma + d(X_{\sigma_0}(\sigma - \sigma_0)) = -\Delta_{\sigma_0} \theta - d(\Phi_{\sigma_0}(\theta))
\] (6.17)
for a closed \(G_2\)-structure \(\sigma\), where \(\theta = \sigma - \sigma_0\) as before.

The following lemma provides an elementary estimate for this quantity.
**Lemma 6.3.** Assume \(\|\sigma - \sigma_0\|_{C^0} \leq \epsilon_0\), where \(\epsilon_0\) is from Lemma 2.1. There holds
\[
\|\nu_{\sigma, \sigma_0}\|_{C^0} \leq \eta_1 \|\sigma - \sigma_0\|_{C^2+\mu}, \|\nabla_{\tau_{\sigma_0}}\|_{C^0}, \|\tau_{\sigma_0}\|_{C^0},
\] (6.18)

where \(\eta_1\) is a universal continuous increasing (in each argument) positive function of its arguments with \(\eta_1(0, 0, \cdot) = 0\). We also have for \(0 < \mu < 1\)
\[
\|\nu_{\sigma, \sigma_0}\|_{\mu} \leq C \eta_2(\|\sigma - \sigma_0\|_{C^{3+\mu}}, \|\nabla_{\tau_{\sigma_0}}\|_{\mu}, \|\tau_{\sigma_0}\|_{C^0}),
\] (6.19)
where \(C\) is a positive constant depending only on \(\|\sigma_0\|_{C^{1+\mu}}\), and \(\eta_2\) is a universal positive function with the same properties as \(\eta_1\). Moreover, we have
\[
\|\nu_{\sigma, \sigma_0}\|_{C^{3+\mu}} \leq \tilde{C} \eta_3(\|\sigma - \sigma_0\|_{C^{3+\mu}}, \|\nabla_{\tau_{\sigma_0}}\|_{C^{3+\mu}}, \|\tau_{\sigma_0}\|_{C^{3+\mu}}),
\] (6.20)
where \(\tilde{C}\) is a positive constant depending only on \(\|\sigma_0\|_{C^{3+\mu}}\), and \(\eta_3\) is a universal positive function with the same properties as \(\eta_1\).

**Proof.** Obviously, there hold \(|d\tau_{\sigma_0}| \leq C_0|\tau_{\sigma_0}|\) and \(|\Delta_{\sigma_0} \theta| \leq C_0|\nabla^2 \theta|\) for a universal positive constant \(C_0\). On the other hand, we have for the functions \(A\) and \(B\) in the formula (6.7) for \(d(\Phi_{\sigma_0}(\theta))\)
\[
d(A(\sigma_0, \sigma, \theta, \nabla \theta)) = \sum_i e_i \wedge [(\nabla e_i)A(\sigma_0, \sigma, \theta, \nabla \theta) + (\nabla e_i)A(\sigma_0, \sigma, \theta, \nabla \theta)] + A(\sigma_0, \sigma, \nabla e_i \theta, \nabla \theta) + A(\sigma_0, \sigma, \theta, \nabla e_i \nabla \theta)
\] (6.21)

\[\begin{aligned}
&= \sum_i e_i \wedge [(\nabla e_i)A(\sigma_0, \sigma, \theta, \nabla \theta) + (\nabla e_i)A(\sigma_0, \sigma, \theta, \nabla \theta)] + A(\sigma_0, \sigma, \nabla e_i \theta, \nabla \theta) + A(\sigma_0, \sigma, \theta, \nabla e_i \nabla \theta)
\end{aligned}
\]
and
\[
d(B(\sigma_0, \sigma, \tau_{\sigma_0}, \theta)) = \sum_i e^i \wedge (\nabla e_i)B(\sigma_0, \sigma, \tau_{\sigma_0}, \theta) + (\nabla e_i)B(\sigma_0, \sigma, \tau_{\sigma_0}, \theta) + B(\sigma_0, \sigma, \tau_{\sigma_0}, \theta) + B(\sigma_0, \sigma, \tau_{\sigma_0}, \theta),
\]
(6.22)
where \((\nabla e_i)\) is a comparison with the \(k\)-th argument as the variable, while keeping the other arguments parallel. Note that by Lemma 2.2, \(\nabla \sigma_0\) can be expressed in terms of \(\tau_{\sigma_0}\). It follows that
\[
|d(\Phi_{\sigma_0}(\theta))| \leq C|\nabla \theta|(|\theta| \nabla \theta + |\tau_{\sigma_0}| |\theta|) + |\nabla \theta|^2 + |\theta| |\nabla \theta|
+ |\theta| |\nabla \tau_{\sigma_0}| + |\tau_{\sigma_0}| |\theta| + |\nabla \tau_{\sigma_0}|
\]
(6.23)
where the positive constant \(C\) depends only on \(|\sigma|_{C^0}\) and \(|g_{\sigma_0}^{-1}|_{C^0}\), which can be estimated by using Lemma 2.1 and the assumption \(|\sigma - \sigma_0|_{C^0} \leq \epsilon_0\). Obviously, the first claim of the lemma follows from (6.23). The second and third claims of the lemma follow from similar computations based on (6.24) and (6.25).

**Theorem 6.4.** (existence of the \(\sigma_0\)-gauged Laplacian flow) Assume \(\sigma_0 \in C^{2+\mu}\) for some \(0 < \mu < 1\). Let \(\delta_0 = \delta_0(\sigma_0) = \delta_0(2 + \mu, g_{\sigma_0})\) be from Lemma 6.7 below (for \(l = 2 + \mu\)), which depends only on \(g_{\sigma_0}\) (in terms of its Riemannian norm \(|g_{\sigma_0}|_{C^{1+\mu}}\)). Let \(\sigma_1\) be a closed \(C^{2+\mu}\) \(G_2\)-structure on \(M\) such that \(\nu_{\sigma_0, \sigma_1} \in C^{2+\mu}\) and
\[
|\sigma_1 - \sigma_0|_{C^0} \leq \frac{1}{4} \min\{\epsilon_0, \delta_0\}.
\]
(6.24)
For each positive constant \(K > 0\) there is a positive constant \(\rho(K, g_{\sigma_0}) \leq K\) depending only on \(K\) and \(g_{\sigma_0}\) (in terms of \(|g_{\sigma_0}|_{C^{1+\mu}}\)) with the following properties. Assume \(0 < T \leq 1\),
\[
|\nu_{\sigma_1, \sigma_0}|_{C^{2+\mu}} \leq \frac{1}{4} \min\{\epsilon_0, \delta_0\},
\]
(6.25)
\[
|\tau_{\sigma_0}|_{C^{3+\mu}} + |\sigma_1 - \sigma_0|_{C^{2+\mu}} \leq K,
\]
(6.26)
and
\[
|\nu_{\sigma_0, \sigma_1}|_{2} + t^{(1-\mu)/2}|\nu_{\sigma_1, \sigma_0}|_{C^{2+\mu}} \leq \rho(K, g_{\sigma_0}).
\]
(6.27)
Then there is a closed \(C^{2+\mu, (2+\mu)/2}\)-solution \(\sigma = \sigma(t)\) of the \(\sigma_0\)-gauged Laplacian flow on \([0, T]\) with \(\sigma(0) = \sigma_1\), such that
\[
|\sigma - \sigma_0|_{C^0} \leq \min\{\epsilon_0, \delta_0\}
\]
(6.28)
and
\[
|\sigma - \sigma_0|_{C^{2+\mu, (2+\mu)/2}} \leq 5K + \frac{1}{2} \min\{\epsilon_0, \delta_0\}.
\]
(6.29)

Let \(l > 2 + \mu\) be a non-integer. If \(\sigma_0 \in C^l\), then \(\sigma \in C^{l+1/2}\) for \(t > 0\). If in addition \(\sigma_1 \in C^l\), then we have \(\sigma \in C^{l+1/2}\) on \(M \times [0, T]\).

Finally, the solution depends smoothly on \(\sigma_1\) and \(\sigma_0\).

**Theorem 6.5.** \((C^{4+\mu, (4+\mu)/2} estimates)\) Assume \(\sigma_0 \in C^{4+\mu}\) and \(\sigma_1 \in C^{4+\mu}\), and everything as in Theorem 6.4, except that \(\delta_0 = \delta_0(4 + \mu, g_{\sigma_0})\) (from Lemma 6.7 with \(l = 4 + \mu\)), which depends only on \(g_{\sigma_0}\) (in terms of \(|g_{\sigma_0}|_{C^{3+\mu}}\)). In addition, assume
\[
|\tau_{\sigma_0}|_{C^{3+\mu}} + |\sigma_1 - \sigma_0|_{C^{3+\mu}} \leq K.
\]
(6.30)
Then all the conclusions of Theorem 6.4 hold true. Moreover, there holds
\[
|\sigma - \sigma_0|_{C^{4+\mu, (4+\mu)/2}} \leq C(K, g_{\sigma_0})(|\sigma_1 - \sigma_0|_{C^{4+\mu}} + |d\tau_{\sigma_0}|_{C^{2+\mu}}).
\]
(6.31)
for a positive constant \(C(K, g_{\sigma_0})\) depending only on \(K\) and \(g_{\sigma_0}\) (in terms of \(|g|_{C^{3+\mu}}\)).

**Theorem 6.6.** (global uniqueness) Let \(\sigma_0\) be a \(C^{2}\) \(G_2\)-structure on \(M\). Let \(\sigma = \sigma(t)\) and \(\bar{\sigma} = \bar{\sigma}(t)\) be two \(C^{2,1}\) solutions of the \(\sigma_0\)-gauged Laplacian flow on a common interval \([0, T]\), such that \(\sigma(0) = \bar{\sigma}(0)\). Then \(\sigma \equiv \bar{\sigma}\) on \([0, T]\). (See Appendix for the definition of \(C^{2,1}\).)
Remark For the purpose of obtaining existence and uniqueness of a short time solution of the Laplacian flow for a given initial $G_0$-structure $\sigma_0$, it suffices to consider the case $\sigma_0 = \sigma_1$, cf. the next section. In this case, the conditions in Theorem 5.4 and Theorem 6.5 are obviously simplified. Besides the independent interest of the gauge fixed Laplacian flow, we need to consider the general case of $\sigma_0$ for the purpose of obtaining regularity of solutions of the Laplacian flow and long time existence and convergence of the Laplacian flow starting near a torsion-free $G_2$-structure, cf. Sections 7, 8 and 9.

We need some preparations for the proofs of these theorems. Let $C^1_\circ(\Lambda^3 T^* M)$ denote the set of sections in $C^1_\circ(\Lambda^3 T^* M)$ with values in $\Lambda^3 T^* M$, and $C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M))$ denote the set of sections in $C^{1,1/2}_\circ(\Lambda^3 T^* M)$ with values in $\pi^*(\Lambda^3 T^* M)$. By Lemma 2.1, $C^1_\circ(\Lambda^3 T^* M)$ is a domain in $C^1_\circ(\Lambda^3 T^* M)$, and $C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M))$ is a domain in $C^{1,1/2}_\circ(\Lambda^3 T^* M)$. Consider $P(\sigma) = \Delta_{\sigma} \sigma - d(X_{\sigma_0}(\sigma - \sigma_0), \sigma)$, the $P$-operator of $\Delta_{\sigma} \sigma - d(X_{\sigma_0}(\sigma - \sigma_0), \sigma)$:

$$P : C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M)) \to C^{1,-2,(l-2)/2}_\circ(\pi^*(\Lambda^3 T^* M))$$

and the corresponding $P$-map

$$P : C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M)) \to C^{1,-2,(l-2)/2}_\circ(\pi^*(\Lambda^3 T^* M)) \times C^1_\circ(\Lambda^3 T^* M)$$

It is obviously a smooth map.

Lemma 6.7. Let $\sigma_0 \in C^1_\circ(\Lambda^3 T^* M)$ for a noninteger $l > 2$. Let $\delta_0 = \delta_0(l, g_{\sigma_0})$ be the positive constant from Theorem 5.1 with $g_* = g_{\sigma_0}$. (In this case, the dependence of $\delta_0$ on $\|g_0\|_{C^1}$, $\|g_{\sigma_0}\|_{C^1}$, and $g_*$ is reduced to the dependence on $l$ and $g_{\sigma_0}$, which is in terms of $\|g_{\sigma_0}\|_{C^{1,-1}}$.) Set $\delta_0 = \delta_0(l, g_{\sigma_0}) = C_0^{-1} \delta_0$, where $C_0$ is from Lemma 6.2. Let $\sigma \in C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M))$ such that $\|\sigma - \sigma_0\|_{C^0} \leq \min\{\epsilon_0, \delta_0\}$ with $\epsilon_0$ being from Lemma 2.1. By Lemma 2.1 we then have $\sigma \in C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M))$. Then the linearization of $P$ at $\sigma$:

$$D_\sigma P : C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M)) \to C^{1,-2,(l-2)/2}_\circ(\pi^*(\Lambda^3 T^* M)) \times C^1_\circ(\Lambda^3 T^* M)$$

is an isomorphism. Moreover, there hold

$$\|D_\sigma P\| \leq C$$

and

$$\|\|D_\sigma P\|^{-1}\| \leq C$$

for a positive constant $C = C(l, T, \|\tau_{\sigma_0}\|_{C^{1,-1}}, \|\sigma - \sigma_0\|_{C^{1,-1} \cdot (l-1/2)}, g_{\sigma_0})$, where the dependence on $g_{\sigma_0}$ is in terms of its Riemannian norm $\|g_{\sigma_0}\|_{C^{1,-1}}$.

Proof. There holds

$$D_\sigma P(\gamma) = (D_\sigma P, \gamma(\cdot, 0))$$

for all $\gamma$. By 6.6 we have

$$D_\sigma P(\gamma) = \frac{\partial \gamma}{\partial l} + D_\sigma \gamma + d(D_\sigma \Phi_{\sigma_0}(\gamma)).$$

It follows from 6.7 that

$$D_\sigma \Phi_{\sigma_0}(\gamma) = \Phi_0(\gamma) + \Phi_1(\nabla_{\sigma_0} \gamma),$$

where

$$\Phi_0 = D_\sigma A(\sigma_0, \cdot, \theta, \nabla \theta) + A(\sigma_0, \sigma, \cdot, \nabla \theta) + D_\sigma B(\sigma_0, \cdot, \tau_{\sigma_0}, \theta) + B(\sigma_0, \sigma, \tau_{\sigma_0}, \cdot),$$

and

$$\Phi_1 = A(\sigma_0, \sigma, \theta, \cdot).$$

and $\theta = \sigma - \sigma_0$ as before. By the assumption $\|\sigma - \sigma_0\|_{C^0} \leq \epsilon_0$ and Lemma 6.2 there holds $\|\Phi_1\|_{C^0} \leq C_0 \|\theta\|_{C^0}$. Applying Theorem 5.4 and employing the nature of the functions $A$ and $B$ (cf. (6.21) and (6.22)) we arrive at the desired conclusions.

Lemma 6.8. Let $\sigma_0 \in C^1_\circ(\Lambda^3 T^* M)$ and $\sigma \in C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M))$ for a noninteger $l > 2$. Assume $\|\sigma - \sigma_0\|_{C^0} \leq \epsilon_0$. Then the second derivative operator of $P$ at $\sigma$

$$D^2_\sigma P : C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M)) \times C^{1,1/2}_\circ(\pi^*(\Lambda^3 T^* M)) \to C^{1,-2,(l-2)/2}_\circ(\pi^*(\Lambda^3 T^* M)) \times C^1_\circ(\Lambda^3 T^* M)$$

satisfies the bound

$$\|D^2_\sigma P\| \leq C$$

for a positive constant $C = C(\|\tau_{\sigma_0}\|_{C^{1,-1}}, \|\sigma - \sigma_0\|_{C^{1,1/2}})$. 

Proof. We have
\[ D^2_{\sigma} P(\gamma, \gamma') = (D^2_{\sigma} P(\gamma, \gamma'), 0) \] (6.42)
and
\[ D^2_{\sigma} P(\gamma, \gamma') = d(D^2_{\sigma} \Phi_{\sigma_0}(\gamma, \gamma')). \] (6.43)

By (6.38) we have
\[ (D^2_{\sigma} \Phi_{\sigma_0})(\gamma, \gamma') = (D_{\sigma} \Phi_0)(\gamma', \gamma) + (D_{\sigma} \Phi_1)(\gamma', \gamma). \] (6.44)

By the formulas (6.39) for \( \Phi \) and (6.40) for \( \hat{\delta}_0 \) we have
\[ \| P \| \leq \| \sigma_0 - \sigma_0 \| + \| \nu \| \leq \min \left\{ \varepsilon_0, \delta_0 \right\} \] (6.45)
and
\[ \| \hat{\theta} \| \leq \| \sigma_0 - \sigma_0 \| + \| \nu \| \leq \min \left\{ \varepsilon_0, \delta_0 \right\} \] (6.46)

Proof of Theorem 6.4 Consider the above P-map \( P \) with \( l = 2 + \mu \). Let \( \hat{\delta}_0 = \hat{\delta}_0(2 + \mu, g_{\sigma_0}) \) be given by Lemma 6.7 for \( l = 2 + \mu \), as stated in the theorem. Set
\[ \hat{\sigma}_1 = \sigma_1 + t \nu \] (6.47)
and
\[ P(\hat{\sigma}_1) = (\hat{P}(\hat{\sigma}_1), \sigma_1) \] (6.48)

for a positive constant \( C \) depending only on \( K \) and \( g_{\sigma_0} \). (In this proof, the directly indicated dependence of various constants on \( g_{\sigma_0} \) are all in terms of \( |g_{\sigma_0}|_{C^{1+\mu}} \).)

By Lemma 6.8 and (6.49) we can apply the inverse function theorem for mappings of Banach spaces to deduce that \( P \) is a smooth diffeomorphism from an open neighborhood \( U_{\hat{\sigma}_1} \) of \( \hat{\sigma}_1 \) in \( C^{2+\mu} \) onto the open ball \( B_{\tau_1}(P(\hat{\sigma}_1)) \) in \( C^{\mu/2} \), for a positive number \( r \) having the same dependences as the above \( C \). Moreover, \( U_{\hat{\sigma}_1} \subset B_{\tau_2}(\hat{\sigma}_1) \) for a positive number \( r_2 \) with the same dependences as \( C \). In addition, we choose \( r_1 \) and \( r_2 \) such that \( r_2 \leq \frac{1}{2} \min \{ \epsilon_0, \delta_0 \} \). On the other hand, we have \( P(\hat{\sigma}_1) = (\hat{P}(\hat{\sigma}_1), \sigma_1) \) and
\[ \hat{P}(\hat{\sigma}_1) = \frac{\partial}{\partial \theta} \hat{\theta} + [\Delta_{\sigma_0} \hat{\theta} + d(\Phi_{\sigma_0}(\hat{\theta}))] \] (6.49)
Employing the formula (6.50) and calculating as in the proof of Lemma 6.9 we deduce
\[ \| \hat{P}(\hat{\sigma}_1) - (0, \sigma_1) \|_{C^{2+\mu}(2+\mu)/2} \leq C_1 \| t \nu \|_{C^{2+\mu}(2+\mu)/2}, \] (6.51)
where \( C_1 \) depends only on \( \| \sigma_0 \|_{C^{1+\mu}} \), \( \| \theta \|_{C^{2+\mu}} \), \( \| t \nu \|_{C^{1+\mu}(1+\mu)/2} \), and \( \| \tau \|_{C^{1+\mu}} = \tau_{\sigma_0} \). Since \( \nabla \sigma_0 \) can be expressed in terms of \( \tau_\sigma \) (by Lemma 2.2), we have \( \| \sigma_0 \|_{C^{1+\mu}} \leq C_2(\| \tau \| + \| \tau \|_{C^{\mu}}) \) for a universal positive constant \( C_2 \). Thus can be written
\[ \| t \nu \|_{C^{1+\mu}(1+\mu)/2} \leq \frac{1}{2} \min \{ T, T(1-\mu)/2, T(2-\mu)/2/2 \} \| \nu \|_{C^{2+\mu}} \leq T(1-\mu)/2 \| \nu \|_{C^{2+\mu}} \] (6.52)
and
\[ \| t \nu \|_{C^{2+\mu}(2+\mu)/2} \leq [t \nu]_{\mu} + \max \{ T, T(1-\mu)/2, (2-\mu)/2 \} \| \nu \|_{C^{2+\mu}} \leq [t \nu]_{\mu} + T(1-\mu)/2 \| \nu \|_{C^{2+\mu}}. \] (6.53)
We infer
\[ \| \hat{P}(\hat{\sigma}_1) - (0, \sigma_1) \|_{C^{2+\mu}(2+\mu)/2} \leq C_3([t \nu]_{\mu} + T(1-\mu)/2 \| \nu \|_{C^{2+\mu}}) \] (6.54)
for a positive constant $C_3$ depending only on $K$ and $g_{g_0}$. We set $\rho(K, g_{g_0}) = \min\{2^{-1}C_3^{-1}r_1, K\}$. Then the condition (5.21) implies that $(0, \sigma_1) \in B_{r_1}(\tilde{\rho}(\tilde{\gamma}_1))$. Consequently, we obtain a solution $\sigma = \mathcal{P}_{\tilde{\rho}(\tilde{\gamma}_1)}^{-1}(0, \sigma_1)$ of the $g_0$-gauged Laplacian flow on $[0, T)$ with $\sigma(0) = \sigma_1$. There holds
\[
\|\sigma - \sigma_0\|_{C^{2+\mu, (2+\mu)/2}} \leq \|\sigma - \tilde{\sigma}_1\|_{C^{2+\mu, (2+\mu)/2}} + \|\tilde{\sigma}_1 - \sigma_0\|_{C^{2+\mu, (2+\mu)/2}} \leq 5K + \frac{1}{2} \min\{\epsilon_0, \tilde{\delta}_0\}. \tag{6.55}
\]

On the other hand, we have
\[
\|\sigma - \sigma_0\|_{C^0} \leq \|\sigma - \tilde{\sigma}_1\|_{C^0} + \|\sigma_1 - \sigma_0\|_{C^0} + T\|\nu\|_{C^0} \leq r_2 + \min\{\epsilon_0, \tilde{\delta}_0\} \leq \min\{\epsilon_0, \tilde{\delta}_0\}. \tag{6.56}
\]

The $C^{1/2}$ regularity for $t > 0$ and the $C^{4+\mu, (4+\mu)/2}$ and $C^{1/2}$ regularities up to $t = 0$ (under the corresponding given assumptions) follow from the standard regularity arguments in coordinate charts, cf. [LSU] and [Y2].

The smooth dependence of $\sigma$ on $\sigma_0$ and $\sigma_1$ follows from the formula $\sigma = \mathcal{P}_{\tilde{\rho}(\tilde{\gamma}_1)}^{-1}(0, \sigma_1)$ and the inverse function theorem. 

**Proof of Theorem 6.7** Since $\hat{\delta}_0(4, g_{g_0}) \leq \tilde{\delta}_0(2, g_{g_0})$ (cf. Theorem 5.1), the conclusions of Part I hold true. To derive the estimate (6.31) we view the gauge fixed Laplacian flow (6.15) with the solution $\sigma$ as a linear equation in the form (5.22), i.e.
\[
\frac{\partial \theta}{\partial t} + \Delta \theta + d(\Phi_0(\theta) + \Phi_1(\nabla \theta)) = d\tau_{g_0}, \tag{6.57}
\]
with $\theta = \sigma - \sigma_0$, $\Delta = \Delta_{g_0}$, $\Phi_0(\theta) = B(\sigma_0, \sigma, \tau_{g_0}, \gamma)$ and $\Phi_1(\nabla \gamma) = A(\sigma_0, \sigma, \theta, \nabla \gamma)$. By (6.56) we deduce
\[
\|\Phi_1\|_{C^0} \leq C_0 \hat{\delta}_0 \leq \hat{\delta}_0(4 + \mu, g_{g_0}) \leq \hat{\delta}_0(3 + \mu, g_{g_0}), \tag{6.58}
\]
where $C_0$ is from Lemma 6.2. Hence we can apply Theorem 5.3 (or Theorem 5.1) to deduce
\[
\|\theta\|_{C^{3+\mu, (3+\mu)/2}} \leq C_1 \|\sigma_1 - \sigma_0\|_{C^{3+\mu}} + \|d\tau_{g_0}\|_{C^{1+\mu}}, \tag{6.59}
\]
where $C_1$ depends only on $\|\tau_{g_0}\|_{C^{2+\mu, (2+\mu)/2}}$ and $\|\sigma_1\|_{C^{2+\mu}}$. Combining this with the estimate (6.29) we deduce an estimate for $\|\theta\|_{C^{3+\mu, (3+\mu)/2}}$. Then we repeat the argument for $4 + \mu$ instead of $3 + \mu$ and arrive at the estimate (6.31).

**Proof of Theorem 6.7** Let $\sigma$ and $\tilde{\sigma}$ be two $C^{2,1}$ solutions of the $g_0$-gauged Laplacian flow on an interval $[0, T]$, such that $\hat{\sigma}(0) = \tilde{\sigma}_1$. Set $\theta = \sigma - \tilde{\sigma}_1, \tilde{\theta} = \tilde{\sigma} - \sigma_1$ and $\gamma = \tilde{\gamma} - \sigma$. By Lemma 6.1 and the Bochner-Weitzenböck formula (5.6) we have
\[
\frac{\partial \gamma}{\partial t} = -\nabla^* \nabla \gamma - R(\gamma) - d(\Phi_{g_0}(\tilde{\sigma} - \sigma_0) - \Phi_{g_0}(\sigma - \sigma_0)). \tag{6.60}
\]
Multiplying this equation by $\gamma$ and then integrating lead to
\[
\frac{d}{dt} \int_M |\gamma|^2 + \int_M |\nabla \gamma|^2 = -\int_M \nabla R(\gamma) \cdot \gamma - \int_M (\Phi_{g_0}(\tilde{\sigma} - \sigma_0) - \Phi_{g_0}(\sigma - \sigma_0)) \cdot \nabla \gamma. \tag{6.61}
\]
By (6.7) we deduce
\[
\Phi_{g_0}(\tilde{\sigma} - \sigma_0) - \Phi_{g_0}(\sigma - \sigma_0) = A(\sigma_0, \sigma, \theta, \nabla \gamma) + (A(\sigma_0, \tilde{\sigma}, \tilde{\theta}, \nabla \tilde{\theta}) - A(\sigma_0, \sigma, \theta, \nabla \theta)) + (B(\sigma_0, \tilde{\sigma}, \tilde{\tau_{g_0}}, \tilde{\theta}) - B(\sigma_0, \sigma, \tau_{g_0}, \theta)). \tag{6.62}
\]
It follows that
\[
|\Phi_{g_0}(\tilde{\sigma} - \sigma_0) - \Phi_{g_0}(\sigma - \sigma_0)| \leq C_1 |\gamma| + C_2 |\theta| \cdot |\nabla \gamma| \tag{6.63}
\]
on $M \times [0, T]$, with positive numbers $C_1$ and $C_2$. Combining (6.61) with (6.63) we then deduce
\[
\frac{d}{dt} \int_M |\gamma|^2 + \int_M |\nabla \gamma|^2 \leq \int_M (C_3 |\gamma|^2 + C_4 |\theta|^2 |\nabla \gamma|^2) \tag{6.64}
\]
for positive numbers $C_3$ and $C_4$. There holds for $0 < T_1 \leq T$
\[
\max_{[0, T_1]} |\theta| \leq C_5 T_1, \tag{6.65}
\]
where $C_5 = \max_{M \times [0, T]} |\partial \theta/\partial t|$. Now we assume $T_1 \leq C_4^{-1/2} C_5^{-1}$. Then (6.64) yields
\[
\frac{d}{dt} \int_M |\gamma|^2 \leq C_3 \int_M |\gamma|^2 \tag{6.66}
\]
on \([0, T_1]\), which implies \(\gamma = 0\) on \([0, T_1]\) because \(\gamma = 0\) at \(t = 0\). Then we repeat the above argument with the time 0 replaced by \(T_1\), and with the new definitions \(\theta = \sigma - \sigma(T_1)\) and \(\hat{\theta} = \hat{\sigma} - \sigma(T_1)\). After finitely many such steps we then conclude \(\gamma = 0\) on \([0, T]\). \(\square\)

7. Short time solutions of the Laplacian flow

In this section we apply the results from the last section to prove Theorem \[11\]. In Subsection 6.1 we prove the existence part, and in Subsection 6.2 we prove the uniqueness part.

7.1. Existence.

Let \(l > 4\) be a non-integer, and \(\sigma_0 \in C^l\) and \(\sigma_1 \in C^l\) be closed \(G_2\)-structures on \(M\). Let \(\sigma = \sigma(t)\) be a \(C^{l+1/2}\) solution of the \(\sigma_0\)-gauged Laplacian flow on an interval \([0, T]\) with the initial value \(\sigma_1\). We consider the ODE

\[
\frac{d}{dt}\phi(\cdot, t) = -X_{\sigma_0}(\sigma(t) - \sigma_0)(\phi(\cdot, t))
\]

with the initial condition

\[
\phi(\cdot, 0) = Id,
\]

where \(X_{\sigma_0}\) is given by \(\ref{6.3}\). By a difference quotient argument one easily shows that the solution \(\phi\) inherits the spacial regularity of \(X_{\sigma_0}(\sigma - \sigma_0)\), and improves a \(C^k\) regularity of it in the time direction to \(C^{k+1}\) regularity. Since \(X_{\sigma_0}(\sigma - \sigma_0) \in C^{l+1,(l-1)/2}\), we deduce that \(\phi \in C^{l+1,l/2}\).

**Lemma 7.1.** \(\hat{\sigma}(t) = \phi(\cdot, t)^*\sigma(t)\) is a \(C^{l-2,(l-1)/2}\) solution of the Laplacian flow on \([0, T]\) with \(\sigma(0) = \sigma_1\).

**Proof.** By the above regularity of \(\phi\) we obviously have \(\hat{\sigma} \in C^{l-2,(l-1)/2}\). (The derivative of \(\phi\) involved in \(\phi^*\sigma\) causes the drop of regularity.) Now we have with \(X_{\sigma_0} = X_{\sigma_0}(\sigma - \sigma_0)\)

\[
\frac{\partial \hat{\sigma}}{\partial t} = \phi(\cdot, t)^* (\mathcal{L}_{-X_{\sigma_0}} \sigma) + \phi(\cdot, t)^* (\Delta_\sigma \sigma + \mathcal{L}_{X_{\sigma_0}} \sigma)
\]

\[
= \Delta_{\phi(\cdot, t)^* \sigma} \phi(\cdot, t)^* \sigma
\]

\[
= \Delta \hat{\sigma}.
\]

(7.3)

It is obvious that \(\hat{\sigma}(0) = \sigma_1\). \(\square\)

7.2. Uniqueness.

Let \(\sigma\) be a function with closed \(G_2\)-structures as values on a time interval \([0, T]\) such that \(\sigma(0) = \sigma_1\). Analogous to the situation in [BX], we consider the following nonlinear evolution equation for diffeomorphisms

\[
\frac{\partial \phi}{\partial t} = Z_{\sigma_0, \sigma}(\phi) \circ \phi
\]

with the initial condition \(\phi(\cdot, 0) = Id\), where

\[
Z_{\sigma_0, \sigma}(\phi) = -X_{\sigma_0}((\phi(\cdot, t)^{-1})^* \sigma - \sigma_0).
\]

(7.4)

**Theorem 7.2.** Assume \(\sigma_0 \in C^l\) and \(\sigma \in C^{l+1/2}\) for a non-integer \(l > 2\). Then there is a unique \(C^{l+1,l+1/2}\) solution of the evolution equation \(\ref{7.4}\) on a time interval \([0, T]\) \((T_1 > 0)\) with \(\phi(\cdot, 0) = Id\).

**Proof.** We compute the linearization of \(Z_{\sigma_0, \sigma}\) at a given diffeomorphism \(\phi : M \to M\) in the direction of a vector field \(Y\) along \(\phi\). (Thus \(Y(p) \in T_{\phi(p)}M\) for each \(p \in M\).) Let \(\phi_s\) be a family of diffeomorphisms of \(M\) such that \(\phi_0 = \phi\) and

\[
\frac{d}{ds}\phi_s|_{s=0} = Y.
\]

(7.5)

(For example, \(\phi_s = \exp_{\phi}(sY)\) for small \(s\), where \(\exp\) is the exponential map of a Riemannian metric on \(M\).)

There hold

\[
\frac{d}{ds}\phi_s^{-1}|_{s=0} = -(\phi^{-1})_s Y
\]

(7.6)

and

\[
\frac{d}{ds}(\phi_s^{-1})^* \sigma|_{s=0} = -\mathcal{L}_Y((\phi^{-1})^* \sigma).
\]

(7.7)
Set $\tilde{\sigma} = (\phi^{-1})^\ast \sigma$, which is a closed $G_2$-structure. Then we have

$$D_\phi Z_{\sigma_0, \sigma}(Y) = -\frac{d}{ds} X_{\sigma_0}((\phi_s^{-1})^\ast \sigma - \sigma_0)|_{s=0} = X_{\sigma_0}(L_Y(\tilde{\sigma})) = X_{\sigma_0}(d(Y, \tilde{\sigma})) = X_{\sigma_0}(d(Y, \sigma_0)) + X_{\sigma_0}(d(Y, \ast \sigma_0)).$$

(7.9)

As is easy to verify, there holds $Y, \ast \sigma_0 = \ast(Y^\flat \ast \sigma_0)$ (cf. [B2]), where $\ast = \ast_{\sigma_0}$ and $Y^\flat$ is the 1-form dual to $X$ w.r.t. $g_{\sigma_0}$. By Lemma 4.1 we have

$$d * (Y^\flat \wedge \ast \sigma_0) = -\frac{3}{7} d_1^2 Y^\flat \sigma_0 - \frac{1}{2} * (d_1^2 Y^\flat \wedge \sigma_0) + d_2^2 Y^\flat + \zeta(Y^\flat)$$

(7.10)

with $\zeta = \zeta_{\sigma_0}$. By (6.3), Lemma 4.3, (7.10) and the Bochner-Weitzenböck formula we then infer

$$X_{\sigma_0}(d(Y, \sigma_0)) = ((-d_1^2 Y^\flat - d_2^2 Y^\flat) \# + X_{\sigma_0}(\zeta(Y^\flat))) = -((\Delta_{\sigma_0} Y^\flat) \# + \frac{1}{3} * d * \zeta_{\sigma_0} ((\ast(Y^\flat \wedge \ast \sigma_0)) + X_{\sigma_0}(\zeta(Y^\flat)))$$

$$= -\nabla^\ast \nabla Y - R(Y^\flat) \# + \frac{1}{3} * d * \zeta_{\sigma_0} ((\ast(Y^\flat \wedge \ast \sigma_0)) + X_{\sigma_0}(\zeta(Y^\flat))).$$

(7.11)

On the other hand, we have

$$X_{\sigma_0}(d(Y, \ast \sigma_0)) = A_2(\sigma_0, \ast \sigma_0, \nabla^2 Y) + A_1(\sigma_0, \nabla(\ast \sigma_0), \nabla Y) + A_0(\sigma_0, \nabla^2 (\ast \sigma_0), Y),$$

(7.12)

where $A_0$, $A_1$ and $A_2$ are universal pointwise functions, smooth in their first arguments, and linear in their second and third arguments. We arrive at

$$D_\phi Z_{\sigma_0, \sigma}(Y) = -\nabla^\ast \nabla Y + A_2(\sigma_0, \ast \sigma_0, \nabla^2 Y) + W_{\sigma_0, \sigma, \phi}(Y, \nabla Y),$$

(7.13)

where the first order linear differential operator $W_{\sigma_0, \sigma, \phi}(Y, \nabla Y)$ is the sum of $A_0$, $A_1$ and the lower order terms on the far right hand side of (7.11). Obviously, $D_\phi Z_{\sigma_0, \sigma}$ is strongly elliptic when $||\ast \sigma_0||_{C^m}$ is small enough, which is the case for small time because of the fact $n \sigma(0) = \sigma_0$. Now we can apply the general result on evolutions of mappings in [Y2] to deduce the desired existence and uniqueness of short time solutions. (The basic mechanism for the said general result is similar to Theorem 6.4 and its proof.)

Now we are ready to prove Theorem 7.11. (Note that only the existence part of Theorem 7.2 is needed.)

**Proof of Theorem 7.2** Let $\sigma_1$ be a $C^{4+\mu}$ closed $G_2$-structures on $M$. We choose a $C^\infty$ closed $G_2$-structure $\sigma_0$ sufficiently close to $\sigma_1$ in $C^{3+\mu}$, such that the conditions of Theorem 6.4 and Theorem 6.5 are satisfied for $K = 1$ and a suitable $T > 0$. Let $\sigma$ denote the unique $C^{4+\mu,(4+\mu)/2}$ solution of the $\sigma_0$-gauged Laplacian flow on $[0, T]$ with the initial value $\sigma_1$ as given by Theorem 6.4. Applying Lemma 7.1 we then obtain a closed $C^{2+\mu,(3+\mu)/2}$ solution $\tilde{\sigma}(t) = \phi(\cdot, t)^\ast \sigma(t)$ of the Laplacian flow with the initial value $\sigma_1$ on the time interval $[0, T]$ given by Theorem 6.4. The claimed $C^{1-2(t-1)/2}$ regularity follows from Theorem 6.4 and Lemma 7.1.

For a given $0 < \epsilon < T$. Let $\psi$ be the solution of (7.1) on $[\epsilon, T]$ with the initial value $Id$ at $t = \epsilon$. By Theorem 6.4, $\sigma$ is smooth for $t > 0$. Hence $\psi$ is smooth. Consequently, the pullback $\psi^\ast(\cdot, t)^\ast \sigma(t)$ is a smooth solution of the Laplacian flow on the time interval $[\epsilon, T]$. Obviously, it equals $\phi(\cdot, t)^\ast \sigma(t)$ for a family of diffeomorphisms $\phi(\cdot, t)$ of class $C^{4+\mu,(4+\mu)/2}$.

Renaming $\tilde{\sigma}$ we then obtain a desired solution of the Laplacian flow $\sigma = \sigma(t)$.

Next we show the uniqueness. Let $\gamma_1 = \gamma_1(t)$ and $\gamma_2 = \gamma_2(t)$ be two $C^{2+\mu,(2+\mu)/2}$ solutions of the Laplacian flow on a common interval $[0, T]$ for some $T > 0$, such that $\gamma_1(0) = \gamma_2(0)$. We set $\sigma_0 = \gamma_1(0) = \gamma_2(0)$. For $i = 1, 2$, let $\phi_i$ be the $C^{3+\mu,(3+\mu)/2}$ solution of the equation (7.4) on an interval $[0, T] \subset [0, T]$, with $\gamma_i$ playing the role of $\sigma$ and with the initial value $Id$, as provided by Theorem 7.2. We set for each $i$

$$\dot{\gamma}_i(t) = (\phi_i(\cdot, t)^{-1})^\ast \gamma_i(t),$$

(7.14)

which is of class $C^{2+\mu,(2+\mu)/2} \subset C^{2,1}$. Then we have for $i = 1, 2$

$$\frac{\partial \dot{\gamma}_i}{\partial t} = (\dot{\phi}_i(\cdot, t)^{-1})^\ast L(\dot{\phi}_i(\cdot, t)^{-1})^\ast \gamma_i = L_X_{\sigma_0}(\gamma_i - \sigma_0) \gamma_i + \Delta_{\gamma_i} \gamma_i$$

(7.15)
Thus, for each $i$, $\dot{\gamma}_i$ is a $C^{2,1}$ solution of the $\sigma_0$-gauged Laplacian flow $(6.3)$ on $[0, T_i]$. Obviously, we also have $\dot{\gamma}_i(0) = \sigma_0$ for each $i$. Hence Theorem 6.10 implies that $\dot{\gamma}_1$ and $\dot{\gamma}_2$ agree on $[0, T_0]$ for $T_0 = \min\{T_1, T_2\}$. Consequently, for each $i$, $\phi_{\gamma_i}$ satisfies on $[0, T_0]$ the same ODE
\[
\frac{\partial \phi}{\partial t} = -X_{\sigma_i}(\dot{\gamma} - \sigma_0) \circ \phi,
\]
where $\dot{\gamma}$ stands for $\dot{\gamma}_1 = \dot{\gamma}_2$. Since $\phi_{\gamma_1}$ and $\phi_{\gamma_2}$ have the same initial value, we deduce that $\phi_{\gamma_1} \equiv \phi_{\gamma_2}$ on $[0, T_0]$. By (7.13) we then infer that $\gamma_1 \equiv \gamma_2$ on $[0, T_0]$. Next we repeat the above argument with $T_0$ being the new time origin. This way we can extend the interval on which $\gamma_1$ and $\gamma_2$ agree. By a simple continuity argument we then conclude that $\gamma_1 \equiv \gamma_2$ on $[0, T]$. \[\square\]

8. Long time existence and convergence of the gauge fixed Laplacian flow

8.1. A Sobolev-type inequality.

Consider a compact manifold $N$ of dimension $n \geq 3$ equipped with a Riemannian metric $g$. The Sobolev constant $C_S(N, g)$ (for the exponent 2) is defined to be the smallest positive number for which the following Sobolev inequality holds
\[
\int_N |f|^{\frac{2n}{n-2}} dvol \leq C_S(N, g) \int_N |\nabla f|^2 dvol + V_g(N)^{-\frac{2}{n}} \int_N f^2 dvol \tag{8.1}
\]
for all $f \in C^1(N)$, where $V_g(N)$ denotes the volume of $(N, g)$. We have the following $L^1$ version of Moser type maximum principle.

**Theorem 8.1.** Let $T > 0$ and $f$ be a nonnegative Lipschitz continuous function on $M \times [0, T]$ satisfying
\[
\frac{\partial f}{\partial t} \leq -\Delta f + b f \tag{8.2}
\]
on $N \times [0, T]$ in the sense of distributions, where $\Delta$ denotes the Hodge Laplacian on functions and $b$ is a nonnegative constant. Then we have for each $p \in N$ and $0 < t \leq T$
\[
\max_{M \times [t, T]} |f| \leq t^{-\frac{n-2}{2}} C_n \left( \max\{b, \frac{n}{2}(1 + \frac{n}{2})^2 \right) \frac{1}{4} \left( \max\{C_S(N, g), TV_g(N)^{-\frac{2}{n}} \right) \frac{2}{7} \int_{M \times [0, T]} |f|, \tag{8.3}
\]
where $C_n$ is a positive constant depending only on $n$.

**Proof.** This is the global formulation of a corresponding local version in [Y5] (see also [Sa]). First we have
\[
\max_{M \times [t, T]} |f| \leq \left( 1 + \frac{2}{n} \right) \frac{1}{4} \left( \max\{C_S(N, g), TV_g(N)^{-\frac{2}{n}} \right) \frac{2}{7} \left( 2b + \frac{n}{2}(1 + \frac{n}{2})^2 \cdot \frac{1}{t} \right)^{\frac{2}{7}} \left( \int_0^T \int_N f^2(\cdot, s) dvol ds \right)^{\frac{1}{7}}, \tag{8.4}
\]
for all $0 < t \leq T$, where $c_n = \sum_0^\infty 2k(1 + \frac{n}{2})^{-k}$. This is the global formulation of a corresponding local version in [Y4]. We can adapt its proof in [Y4]. The cut-off function $\eta$ in that proof is not needed here, hence we can take $\eta \equiv 1$. The local Sobolev inequality used there is replaced by (8.1). Then the arguments there lead to (8.1) straightforwardly.

Applying (8.4) to $0 < t' < t \leq T$ with $t'$ playing the role of the time origin we infer
\[
\max_{M \times [t, T]} |f| \leq C T^{\frac{2}{7}} \left( 1 + \frac{1}{t - t'} \right)^{\frac{n-2}{2}} \left( \int_{M \times [t', T]} |f|^2 \right)^{\frac{1}{2}} \leq C T^{\frac{2}{7}} \left( 1 + \frac{1}{t - t'} \right)^{\frac{n-2}{2}} \left( \max_{M \times [t', T]} |f|^{1/2} \right)^{\frac{1}{2}} \left( \int_{M \times [0, T]} |f| \right)^{\frac{1}{2}}, \tag{8.5}
\]
where $C = (1 + \frac{2}{n})c_n^{n/2}$ and $T = \max\{C_S(N, g), TV_g(N)^{-\frac{2}{n}} \}$. We may assume that $\max_{M \times [t, T]} |f|$ is positive. (If it is zero, then the estimate (8.1) holds trivially.) Then we deduce for $i \geq 0$
\[
\frac{\max_{M \times [t, T]} |f|^{2^{-i}}}{\max_{M \times [t', T]} |f|^{2^{-i-1}}} \leq (CT^{\frac{2}{7}})^{2^{-i}} \left( 1 + \frac{1}{t - t'} \right)^{\frac{n-2}{2}} \left( \int_{M \times [0, T]} |f| \right)^{2^{-i-1}}, \tag{8.6}
\]
For a sequence of positive times $t = t_0 > t_1 > t_{k+1}, k \geq 0$ we infer from it
\[
\frac{\max_{M \times [t_i, T]} |f|}{\max_{M \times [t_i+1, T]} |f|^{2^{-i-1}}} \leq \Pi_{0 \leq i \leq k} \left( C \frac{\hat{T}^2}{\hat{T}^4} \right)^{2^{-i}} \left( 1 + \frac{1}{t_i - t_{i+1}} \right)^{\frac{2}{2+i}} \left( \int_{M \times [0, T]} |f| \right)^{2^{-i-1}}. \tag{8.7}
\]
Choosing $t_i = t(1 - \sum_{j \leq 2^{-i}})$ for $i \geq 1$ we have $1 + \frac{1}{t_i - t_{i+1}} = 1 + t^{-1}(2^{i+1}) \leq t^{-1}(T + 2^{i+1})$. Hence we deduce
\[
\frac{\max_{M \times [t_i, T]} |f|}{\max_{M \times [t_i+1, T]} |f|^{2^{-i-1}}} \leq \Pi_{0 \leq i \leq k} \left( C \frac{\hat{T}^2}{\hat{T}^4} \right)^{2^{-i}} \left( t^{-1}(T + 2^{i+1}) \right)^{\frac{2}{2+i}} \left( \int_{M \times [0, T]} |f| \right)^{2^{-i-1}}. \tag{8.8}
\]
Letting $k \to \infty$ we obtain
\[
\max_{M \times [t, T]} |f| \leq t^{-1} \frac{\hat{T}^2}{\hat{T}^4} \Pi_{0 \leq i \leq \infty} \left( T + 2^{i+1} \right)^{\frac{2}{2+i}} \left( \int_{M \times [0, T]} |f| \right), \tag{8.9}
\]
Replacing $T + 2^{i+1}$ by $(T + 1)2^{i+1}$ we then arrive at [8.1] with
\[
C_n = 4 \left( 1 + \frac{2}{n} \right) \sum_{i=0}^{2n} 2^{i(1 + \frac{2}{n})^{-1}} 2^{(n+2)} \sum_{i=0}^{2n} (i + 1)^{2^{-i-2}}. \tag{8.10}
\]
\[
\square
\]

8.2. Long time existence and convergence of the gauge fixed Laplacian flow: the statement and preliminaries.

Consider a $G_2$ structure $\sigma_0$. Let $\lambda_0 = \lambda_0(\sigma_0)$ denote the first eigenvalue of the Hodge Laplacian $\Delta_{\sigma_0}$ on exact 3-forms. It is obviously positive, because a harmonic form which is also exact must be trivial. There holds
\[
\int_M |d_{\sigma_0}^* \gamma|^2 \geq \lambda_0 \int_M |\gamma|^2 \tag{8.11}
\]
for all exact $C^1$ 3-forms $\gamma$, or more generally, $W^{1,2}$ 3-forms. Indeed, this is a consequence of the decomposition $\gamma = \sum a_i \gamma_i$, where the $\gamma_i$ are the exact forms among a complete set of $L^2$ orthonormal eigenforms of the Hodge Laplacian, see the proof of Theorem [5.4].

We have the following preliminary result.

**Lemma 8.2.** Let $0 < \mu < 1$, $K > 0$, and $\hat{\sigma}_0$ be a $C^{4+\mu}$ torison-free $G_2$-structure on $M$. Then there are positive constants $\lambda_0$ and $\rho$ depending only on $\hat{\sigma}_0$ and $K$, and $\hat{\sigma}_0, T_0$ and $c$ depending only on $\hat{\sigma}_0, \mu$ and $K$, with the following properties. Let $\sigma_0$ and $\sigma_1$ be $C^{4+\mu}$ $G_2$-structure on $M$ such that
\[
||\sigma_0 - \hat{\sigma}_0||_{C^0, \hat{\sigma}_0} \leq \epsilon_0, \tag{8.12}
\]
\[
||\sigma_0 - \hat{\sigma}_0||_{C^{4+\mu}, \hat{\sigma}_0} \leq K, \tag{8.13}
\]
\[
||\sigma_1 - \sigma_0||_{C^0, \sigma_0} \leq \frac{1}{4} \min \{\epsilon_0, \hat{\sigma}_0(\sigma_0)\}, \tag{8.14}
\]
\[
||\sigma_1 - \sigma_0||_{C^{4+\mu}, \sigma_0} \leq K, \tag{8.15}
\]
and
\[
||\sigma_1 - \sigma_0||_{C^{4+\mu}, \sigma_0} \leq \rho, \tag{8.16}
\]
where $\hat{\sigma}_0(\sigma_0)$ is from Theorem [6.2]. Then there hold $\lambda_0(\sigma_0) \geq \lambda_0$ and $\hat{\sigma}_0(\sigma_0) \geq \hat{\sigma}_0$. On the other hand, there is a unique $C^{4+\mu,(4+\mu)/2}$ solution $\sigma = \sigma(t)$ of the $\sigma_0$-fixed Laplacian flow on $M \times [0, T_0]$ with $\sigma(0) = \sigma_1$. There hold
\[
||\sigma - \sigma_0||_{C^0} \leq \min \{\epsilon_0, \hat{\sigma}_0(\sigma_0)\} \tag{8.17}
\]
and
\[
||\sigma - \sigma_0||_{C^{4+\mu,(4+\mu)/2}} \leq c. \tag{8.18}
\]
Proof. The bounds for $\lambda_0(\sigma_0)$ and $\dot{\sigma}_0(\sigma_0)$ follow from simple compactness arguments. The unique existence of the solution $\sigma = \sigma(t)$ on a uniform time interval $[0, T_0]$ and the estimates \((8.17)\) and \((8.18)\) follow from Theorem \(6.4\) and Theorem \(6.5\). Note that, by Lemma \(6.3\) \(\|\sigma_1, \sigma_0\|_{C^{2+\mu}}\) can be estimated in terms of $\hat{\sigma}_0, K$ and $\mu$, and $[\nu_{\sigma_1, \sigma_0}]_\mu$ can be estimated in terms of $\|\sigma_1 - \sigma_0\|_{C^{2+\mu}}$ multiplied by a positive constant depending only on $\sigma_0$ and $K$. Therefore the condition \((8.19)\) in Theorem \(6.4\) follows from \((8.16)\) and a suitable choice of $T_0$.

We sketch an alternative argument for obtaining the above uniform existence and estimates. Since $\sigma_0$ is torsion-free, there holds $\Delta_{\sigma_0} \sigma_0 = 0$. On the other hand, we obviously have $\Phi_{\sigma_0}(\sigma_0) = 0$. Hence $\sigma_0$ is a constant valued solution of the $\sigma_0$-gauged Laplacian flow on $[0, \infty)$. Now we choose e.g. $T_0 = 1$. (In this argument we can choose the value of $T_0$ first, and then determine other constants.) Applying the inverse function theorem at $\sigma_0$ (restricted to the time interval $[0, T_0]$) we then obtain the desired existence and estimates, provided that $\|\sigma_1 - \sigma_0\|_{C^{2+\mu}}$ is sufficiently small. (So we obtain this way a somewhat weaker result than the above one.) \(\square\)

Now we formulate the long time existence and convergence theorem for the gauge fixed Laplacian flow. Its proof will be presented in the next subsections.

**Theorem 8.3.** Let $0 < \mu < 1$, $K > 0$, and $\hat{\sigma}_0$ be a $C^{4+\mu}$ $G_2$-structure on $M$. Then there are positive constants $c$ and $\epsilon_0$ depending only on $\sigma_0, \mu$ and $K$ with the following properties. Let $\sigma_0$ and $\sigma_1$ be two cohomologous closed $C^{4+\mu}$ $G_2$-structures on $M$ satisfying \((8.12)\), \((8.13)\), \((8.14)\), \((8.15)\) and \((8.16)\). Assume that $\sigma_0$ is torsion-free. In addition, assume that

$$\int_M |\sigma_1 - \sigma_0|^2 \leq \epsilon_0,$$

where the metric $g_{\sigma_0}$ is used for the norm and the volume form. (The notation for the volume form is omitted. The reader is also advised to be aware that this $\epsilon_0$ is different from $\epsilon_0$ of Lemma \(2.1\).) Then the $\sigma_0$-gauged Laplacian flow

$$\frac{\partial \sigma}{\partial t} = \Delta_\sigma \sigma + d(X_{\sigma_0}(\sigma - \sigma_0))_\sigma$$

with initial value $\sigma_1$ has a unique $C^{4+\mu,(4+\mu)/2}$ solution $\sigma = \sigma(t)$ on $[0, \infty)$ which converges in $C^{4+\mu,(4+\mu)/2}$ to $\sigma_0$ at exponential rate as $t \to \infty$.

If $\sigma_0 \in C^1$ for a non-integer $l > 4$, then $\sigma(t) - \sigma_0$ converges in $C^{l+1,(l+1)/2}$ to $0$ at exponential rate.

### 8.3. $L^2$-decay.

Let $0 < \mu < 1$ and $K > 0$ be given. Consider $\hat{\sigma}_0, \sigma_0$ and $\sigma_1$ satisfying the conditions of Theorem \(8.3\). Let $\sigma = \sigma(t)$ be the solution of the $\sigma_0$-fixed Laplacian flow on $[0, T_0]$ with $\sigma(0) = \sigma_1$ as given by Lemma \(8.2\). We proceed to prove that $\sigma$ extends to a solution of the $\sigma_0$-gauged Laplacian flow on $[0, \infty)$ and converges to $\sigma_0$ as $t \to \infty$.

Henceforth we employ the metric $g_{\sigma_0}$ for all geometric measurements and operations.

**Lemma 8.4.** There holds

$$\|\sigma - \sigma_0\|_{C^0(M \times [\frac{T_0}{2}, T_0])} \leq C \int_M |\sigma_1 - \sigma_0|^2$$

for a positive constant $C = C(\sigma_0, K)$.

The proof of this lemma will be given below.

**Definition 7.1** For $0 < \epsilon \leq \min\{\epsilon_0, \dot{\sigma}_0\}$ let $I_{\sigma}$ denote the set of $T \geq T_0$ such that $\sigma$ extends to a $C^{4+\mu,(4+\mu)/2}$ solution of the $\sigma_0$-gauged Laplacian flow on $[0, T]$ with the following three properties

$$\|\sigma - \sigma_0\|_{C^0(M \times [0, T])} \leq \min\{\epsilon_0, \dot{\sigma}_0(\sigma_0)\},$$

$$\|\sigma - \sigma_0\|_{C^0(M \times [\frac{T_0}{2}, T])} \leq \epsilon,$$

and

$$\|\sigma - \sigma_0\|_{C^{4+\mu,(4+\mu)/2}(M \times [T - \frac{T_0}{2}, T])} \leq 2\epsilon$$

for all $T_0 \leq t \leq T$, where $c = c(\hat{\sigma}_0, \mu, K)$ is from Lemma \(8.2\).
Remark Alternatively, we can replace the condition (8.24) by the following one
\[
\|\sigma(t) - \sigma_0\|_{C^{1,\nu}} \leq 2c.
\] (8.25)
for all \(T_0 \leq t \leq T\). Then the proof below also goes through with some modifications.

Set \(\theta = \sigma - \sigma_0\) and \(\theta_0 = \sigma_1 - \sigma_0\). Note that (8.23) and Lemma 8.2 imply the following estimate
\[
\max_{M \times [0,T]} \{ |\nabla \theta|, |\nabla^2 \theta|, |\nabla^3 \theta|, |\nabla^4 \theta|, |\frac{\partial \theta}{\partial t}|, |\frac{\partial^2 \theta}{\partial t^2}|, |\nabla^2 \frac{\partial \theta}{\partial t}|, |\nabla^3 \frac{\partial \theta}{\partial t}| \} \leq 2c.
\] (8.26)

We derive various exponential decay estimates for \(\theta\), starting with the \(L^2\)-decay.

Since \(\sigma_0\) is torsion-free, we deduce from (6.15) and (6.7)
\[
\frac{\partial \theta}{\partial t} = -\Delta \theta - d(\Phi_{\sigma_0}(\theta)) = -dd^* \theta - d(\Phi_{\sigma_0}(\theta))
\] (8.27)
with
\[
\Phi_{\sigma_0}(\theta) = A(\sigma_0, \sigma, \theta, \nabla \theta).
\] (8.28)
(Note that e.g. \(\Delta = \Delta_{\sigma_0}\) and \(\nabla = \nabla_{\sigma_0}\).) Integrating (8.27) yields
\[
\theta = \sigma_1 - \sigma_0 - d \int_0^t (d^* \theta - \Phi_{\sigma_0}(\theta)).
\] (8.29)

Since \(\sigma_1\) and \(\sigma_0\) are cohomologous, it follows that \(\theta\) is exact.

We write \(\theta(t) = \theta(\cdot, t)\) and often abbreviate it to \(\theta\).

Lemma 8.5. There is a positive constant \(\epsilon_1 = \epsilon_1(\dot{\sigma}_0, K)\) with the following properties. Let \(T \in I_\epsilon\) with \(\epsilon \leq \epsilon_1\). Then there holds for each \(t \in [0, T]\)
\[
\int_M |\theta(t)|^2 \leq C e^{-\lambda_0 t} \int_M |\theta_0|^2.
\] (8.30)
for a positive constant \(C = C(\dot{\sigma}_0, K)\).

Proof. By (8.27), (8.28) and Lemma 8.2 we have
\[
\frac{d}{dt} \int_M |\theta|^2 = 2 \int_M \theta \cdot (-\Delta \theta - d(\Phi_{\sigma_0}(\theta))) = -2 \int_M |d^* \theta|^2 - \int_M dd^* \theta \cdot \Phi_{\sigma_0}(\theta)
\]\[
\leq -2 \int_M |d^* \theta|^2 + C_1 \max |\theta| \int_M |\nabla \theta|^2,
\] (8.31)
where \(C_1\) is a universal positive constant. Since \(dd^* = 0\), we can apply Bochner-Weitzenböck formula and the bound (8.20) (for controlling the curvature) to deduce
\[
\int_M |\nabla \theta|^2 \leq \int_M |d^* \theta|^2 + C_2 \int_M |\theta|^2
\] (8.32)
with \(C_2 = C_2(\dot{\sigma}_0, K)\). We set
\[
\epsilon_1 = \min\left\{ \frac{1}{C_1}, \frac{\lambda_0}{C_2 C_1} \right\}.
\] (8.33)

Then we deduce from (8.31) and (8.32), on account of the bound (8.23) and the assumption \(\epsilon \leq \epsilon_1\)
\[
\frac{d}{dt} \int_M |\theta|^2 \leq -(2 - C_1 \epsilon) \int_M |d^* \theta|^2 + C_1 C_2 \epsilon \int_M |\theta|^2
\]\[
\leq -(2 - C_1 \epsilon) \lambda_0 - C_1 C_2 \epsilon \int_M |\theta|^2
\] \[
\leq -\lambda_0 \int_M |\theta|^2,
\] (8.34)
as long as \(T_0/8 \leq t \leq T\). Consequently, we have for \(t \in [T_0/8, T]\)
\[
\int_M |\theta|^2 \leq e^{-\lambda_0(t-T_0/8)} \int_M |\theta(T_0/8)|^2.
\] (8.35)

To handle the time interval \([0, T_0/8]\) we argue as follows. By the computation in (8.31) and the bound (8.20) we have
\[
\frac{d}{dt} \int_M |\theta|^2 \leq -2 \int_M |d^* \theta|^2 + C_3 \int_M |\theta| \cdot |\nabla \theta|
\] (8.36)
for $C_3 = C_5(\hat{\sigma}_0, K)$. Employing this inequality, (8.32) and the Cauchy-Schwarz inequality we then deduce for all $0 \in [0, T]$
\[
\frac{d}{dt} \int_M |\theta|^2 \leq - \int_M |d^* \theta|^2 + C_4 \int_M |\theta|^2
\]
(8.37)
for $C_4 = C_4(\hat{\sigma}_0, K)$. It follows that
\[
\int_M |\theta(t)|^2 \leq C_5 \int_M |\theta_0|^2
\]
(8.38)
for $C_5 = C_5(\hat{\sigma}_0, K)$ and $0 \leq t \leq T_0$. Combining (8.35) and (8.38) we then arrive at (8.30).

**Remark** We’ll derive decay estimates for other quantities from the above $L^2$-decay of $\theta$. Alternatively, one can also adapt the above arguments to handle other quantities, as they all can be handled in terms of exact forms. However, that approach involves additional or stronger conditions for the initial date, which is not satisfactory.

### 8.4. $C^0$-decay and gradient $C^0$-decay.

Next we derive decay estimates for $||\theta(t)||_{C^0}$ and $||\nabla \theta(t)||_{C^0}$.

**Lemma 8.6.** Let $T \in I_\epsilon$ with $\epsilon \leq \epsilon_1$. Then there holds for each $t \in [T_0/8,T]$
\[
||\theta(t)||_{C^0} \leq Ce^{-\lambda_0 t} \int_M |\theta_0|^2.
\]
Moreover, there holds for each $t \in (0,T]$
\[
\int_{t^*}^t \int_M |\nabla \theta|^2 \leq Ce^{-\lambda_0 t} \int_M |\theta_0|^2,
\]
where $t^* = \max\{t - T_0, 0\}$.

**Proof.** Applying Bochner-Weitzenböck formula we deduce
\[
\frac{\partial}{\partial t} |\theta|^2 = -\Delta |\theta|^2 - |\nabla |\theta|^2| \leq -\Delta |\theta|^2 - |\nabla |\theta|^2| \leq -|\nabla |\theta|^2|.
\]
(8.41)
There holds
\[
d(A(\sigma_0, \sigma, \theta, \nabla \theta)) = \sum_i e^i \wedge \nabla e_i A(\sigma_0, \sigma, \theta, \nabla \theta)
\]
\[
= \sum_i e^i \wedge (\nabla e_i)_2 A(\sigma_0, \sigma, \theta, \nabla \theta) + \sum_i e^i \wedge A(\sigma_0, \sigma, \nabla e_i, \theta, \nabla \theta)
\]
\[
+ \sum_i e^i \wedge A(\sigma_0, \sigma, \theta, \nabla e_i, \nabla \theta),
\]
(8.42)
where $e_i$ stands for a local orthonormal frame and $e^i$ its dual, and $(\nabla e_i)_2$ means to take the covariant derivative of $A(\sigma_0, \sigma, \theta, \nabla \theta)$ with the second argument $\sigma$ as the variable, while keeping the other arguments parallel. Then we infer, on account of the bounds (8.22) and (8.26)
\[
\frac{\partial}{\partial t} |\theta|^2 \leq -|\nabla |\theta|^2| - |\theta||\nabla |\theta|^2| + |\theta||\nabla |\theta|^2| + |\theta||\nabla |\theta|^2|,
\]
(8.43)
where $C_6$ depends only on $\hat{\sigma}_0, \mu$ and $K$. There holds $C_6|\theta||\nabla |\theta|^2| \leq |\nabla |\theta|^2| + \frac{1}{4}C_6^2 |\theta|^2|\nabla |\theta|^2|$. Applying this and (8.26) we then infer
\[
\frac{\partial}{\partial t} |\theta|^2 \leq -|\nabla |\theta|^2| - |\theta||\nabla |\theta|^2| + C_7|\theta|^2
\]
(8.44)
for $t \in [T_0/8,T]$, with $C_7 = C_7(\hat{\sigma}_0, \mu, K)$. Applying Theorem 8.1 to $|\theta|^2$ over the interval $[t - T_0/8, t]$ (with $t - T_0/8$ as the new time origin) and appealing to the bound (8.26) we then obtain for $t \in [T_0/8,T]$
\[
|\theta(t)|^2 \leq C_8 \int_{t - T_0/8}^t \int_M |\theta|^2
data (8.45)
with $C_8 = C_8(\hat{\sigma}_0, K)$. Combining (8.45) and Lemma 8.7 we then arrive at (8.47).

Integrating (8.44) we infer (8.40).
Remark 1) The differential inequality \((8.41)\) is not strong enough to lead to
\[
\frac{\partial}{\partial t}|\theta| \leq -\Delta |\theta| + C|\theta|
\]
for a positive constant \(C\). This is because of the second order part contained in the term \(-d(A(\sigma_0, \sigma, \theta, \nabla \theta))\).

A differential inequality like \((8.46)\) would allow one to obtain the estimate \((8.47)\) in terms of the \(L^2\)-version \((8.1)\) of the maximum principle, which is weaker than the \(L^1\)-version \((8.4)\).

2) If we apply the \(L^1\)-version \((8.4)\) instead of the \(L^2\)-version \((8.4)\) to \((8.47)\), then we would obtain the following estimate
\[
\|\theta(t)\|_{C^0}^4 \leq C e^{-\lambda_0 t} \int_M |\theta_0|^2.
\]

This estimate is enough for deriving the convergence of the gauge fixed Laplacian flow and then the convergence of the Laplacian flow. However, it only leads to a Hölder continuity of the limit map of the Laplacian flow.

The above remarks also apply to the similar situations below. Now it is convenient to present the proof of Lemma 8.4.

Proof of Lemma 8.4. Here we deal with the solution \(\sigma = \sigma(t)\) on \([0, T_0]\) given by Lemma 8.2. First, arguing as in the proof of Lemma 8.5 using \((8.17)\) and \((8.18)\) instead of \((8.22)\) and \((8.26)\), we deduce \((8.44)\), with a different \(C_8\) depending only on \(\hat{\sigma}_0, \mu, K\). Then there holds
\[
\max_t |\nabla \theta|^2 \leq C e^{-\lambda_0 t} \int_M |\theta_0|^2
\]
for \(t \in [T_0/8, T]\) and \(C = C(\hat{\sigma}_0, \mu, K)\).

Next we derive a \(C^0\)-decay estimate for \(\nabla \theta\).

Lemma 8.7. Let \(T \in I_\epsilon\) with \(\epsilon \leq \epsilon_1\). Then there holds
\[
\max_t |\nabla \theta|^2 \leq C e^{-\lambda_0 t} \int_M |\theta_0|^2
\]
for \(t \in [T_0/8, T]\) and \(C = C(\hat{\sigma}_0, \mu, K)\).

Proof. We take the covariant derivative in the equation \((8.41)\) to obtain
\[
\frac{\partial}{\partial t} |\gamma|^2 = -\nabla^\ast \nabla \gamma - R_2 \gamma - \nabla R_1 \theta - \nabla d(A(\sigma_0, \sigma, \theta, \nabla \theta))
\]
for \(\gamma = \nabla \theta\), where \(R_1\) and \(R_2\) are some linear actions of the curvature operator. Employing \((8.42)\), we obtain a similar formula for \(\nabla d(A(\sigma_0, \sigma, \theta, \nabla \theta))\). On account of the bounds \((8.22)\) and \((8.26)\), we then deduce
\[
|\nabla d(A(\sigma_0, \sigma, \theta, \gamma))| \leq C_9 |\gamma| + |\nabla \gamma| + |\nabla^2 \gamma| + C_9 |\gamma| (|\gamma| + |\nabla \gamma|) \leq C_{10} (|\gamma| + |\theta|)
\]
for positive constants \(C_9\) and \(C_{10}\) depending only on \(\hat{\sigma}_0, \mu, K\). It follows that
\[
\frac{\partial}{\partial t} |\gamma|^2 \leq -\Delta |\gamma|^2 - 2|\nabla \gamma|^2 + C_{11} (|\gamma|^2 + |\theta|^2)
\]
with \(C_{11} = C_{11}(\hat{\sigma}_0, \mu, K)\). The extra term \(C_{11} |\theta|^2\) in this differential inequality can be handled by various means. One way is to combine \((8.44)\) and \((8.51)\) to deduce
\[
\frac{\partial}{\partial t} (|\theta|^2 + |\gamma|^2) \leq -\Delta (|\theta|^2 + |\gamma|^2) - (|\gamma|^2 + |\nabla^2 \theta|^2) + (C_8 + C_{11}) (|\theta|^2 + |\gamma|^2).\]

Applying Theorem 8.1, \((8.52)\) and the integral estimate \((8.49)\) as before we deduce \((8.43)\).

8.5. \(L^2\)-decay and \(C^0\)-decay for \(\frac{\partial \theta}{\partial t}\).

Let \(T \in I_\epsilon\) with \(\epsilon \leq \epsilon_1\). Integrating \((8.51)\) and \((8.52)\) and employing the previous \(L^2\)-decay estimates for \(\theta\) and \(\nabla \theta\) we deduce
\[
\int_{T_0/16}^t \int_M |\nabla^2 \theta|^2 \leq C_{12} e^{-\lambda_0 t} \int_M |\theta_0|^2
\]

\((8.53)\).
for \( t \leq [T_0/16, T] \) and \( C_{12} = C_{12}(\sigma_0, \mu, K) \). (Here we employ a simple cut-off function of \( t \) similar to the one used in the proof of Lemma 5.2.) Employing this estimate, the above estimates and the evolution equation (8.21) we then infer
\[
\int_{t \leq [T_0/16]} \int_M |\partial_\theta|^2 \leq C e^{-\lambda_0 t} \int_M |\theta_0|^2
\]
for \( t \in [T_0/16, T] \) and \( C = C(\sigma_0, \mu, K) \).

**Lemma 8.8.** Let \( T \in \mathcal{I}_\epsilon \) with \( \epsilon \leq \epsilon_1 \). Then there holds
\[
\max_t |\partial_\theta|^2 \leq C e^{-\lambda_0 t} \int_M |\theta_0|^2
\]
for \( T_0/8 \leq t \leq T \) and \( C = C(\sigma_0, \mu, K) \).

**Proof.** Set \( \vartheta \equiv \partial_\theta/\partial t \). Taking the time derivative in the evolution equation (8.21) we deduce
\[
\frac{\partial \vartheta}{\partial t} = -\nabla \vartheta - \mathcal{R} \vartheta - d(\frac{\partial}{\partial t} \Phi_0(\vartheta)).
\]
There holds
\[
\frac{\partial}{\partial t} \Phi_0(\vartheta) = D_2 A(\sigma_0, \sigma, \theta, \gamma)(\vartheta) + A(\sigma_0, \sigma, \vartheta, \gamma) + A(\sigma_0, \sigma, \theta, \nabla \vartheta).
\]
Following the pattern of computations in (8.22) we deduce
\[
|d(\frac{\partial}{\partial t} \Phi_0(\vartheta))| \leq C_{13} |\vartheta||\vartheta| + |\nabla \vartheta||\vartheta| + |\nabla \vartheta||\vartheta| + |\nabla \vartheta||\vartheta| + |\nabla \vartheta| + |\nabla \vartheta|
\]
\[
+ C_{13} (|\vartheta| + |\nabla \vartheta|)
\]
for \( C_{13} = C_{13}(\sigma_0, \mu, K) \). Employing (8.57), (8.55) and the bound (8.26) we then deduce
\[
\frac{\partial}{\partial t} |\vartheta|^2 \leq -\Delta |\vartheta|^2 - |\nabla \vartheta|^2 + C_{14} |\vartheta|(|\vartheta| + |\gamma| + |\theta|)
\]
for \( C_{14} = C_{14}(\sigma_0, \mu, K) \). Combining this with (8.52) we then infer
\[
\frac{\partial}{\partial t} (|\vartheta|^2 + |\gamma|^2 + |\theta|^2) \leq -\Delta (|\vartheta|^2 + |\gamma|^2 + |\theta|^2) - (|\nabla \vartheta|^2 + |\nabla \vartheta|^2 + |\nabla \vartheta|^2)
\]
\[
+ C_{15} (|\vartheta|^2 + |\gamma|^2 + |\theta|^2)
\]
for \( C_{15} = C_{15}(\sigma_0, \mu, K) \). Applying this differential inequality, Theorem 8.1 the integral estimate (8.54) and the above \( L^2 \)-decay estimates for \( \gamma \) and \( \gamma \) we arrive at (8.55).

### 8.6. \( C^{4+\mu,(4+\mu)/2} \)-decay.

**Lemma 8.9.** Let \( T \in \mathcal{I}_\epsilon \) with \( \epsilon \leq \epsilon_1 \). Assume \( T_0 \leq t \leq T \). Then there holds
\[
\|\theta\|_{C^{4+\mu,(4+\mu)/2}(M \times [t - \frac{T_0}{8}, t])} \leq C e^{-\lambda_0 t} \int_M |\theta_0|^2,
\]
with \( C = C(\sigma_0, \mu, K) \).

**Proof.** We view the evolution equation (8.21) with the given solution \( \theta \) as a linear equation in the form of (5.22), similar to (6.57). Thus we have
\[
\frac{\partial \theta}{\partial t} + \Delta \theta + d(\Phi_0(\theta) + \Phi_1(\nabla \theta)) = 0
\]
with \( \Phi_0 \equiv 0 \) and \( \Phi_1(\nabla \theta) = A(\sigma_0, \sigma, \theta, \nabla \theta) \). By Lemma 5.2 and the bound (8.22) there holds
\[
\|\Phi_1\|_{C^{0}(M \times [0, T_0])} \leq C_{0} \epsilon \leq C_{0} \epsilon_0 \leq \delta(\sigma_0).
\]
Now consider \( T_0 \leq t \leq T \). By the bounds (8.22) and (8.21) we have
\[
\|\Phi_1\|_{C^{3+\mu,(3+\mu)/2}(M \times [t - \frac{T_0}{16}, t])} \leq C_{16}
\]
for \( C_{16} = C_{16}(\sigma_0, K) \). Applying Theorem 5.3 with \( l = 4 + \mu, m = \mu, \epsilon = \frac{1}{8} T_0 \) and \( t = \frac{T_0}{8} \) as the new time origin we then arrive at
\[
\|\theta\|_{C^{4+\mu,(4+\mu)/2}(M \times [t - \frac{T_0}{8}, t])} \leq C_{17} \|\theta\|_{C^{\mu,(\mu)/2}(M \times [t - \frac{T_0}{8}, t])}
\]
(8.65)
for $C_{17} = C_{17}(\delta_0, K)$. Combining this with Lemma 8.6, 8.7 and 8.8 we then arrive at (8.61). □

Proof of Theorem 8.3

We define the number $\varepsilon_0$ in the theorem as follows

$$
\varepsilon_0 = (4C)^{-1} \min\{\varepsilon_1, c, K, \varepsilon_0, \delta_0, \rho\},
$$

where $C$ is the larger of the $C$ from Lemma 8.3 and the $C$ from Lemma 8.4 and $\rho = \rho(\delta_0, K)$ is from Lemma 8.1.

Claim 1 The set $I_{\varepsilon_1}$ is nonempty, indeed $T_0 \in I_{\varepsilon_1}$.

Indeed, the estimates (8.17) and (8.18) imply the conditions (8.22) and (8.23) for $T = T_0$, and Lemma 8.4, the assumption (8.19) and (8.66) imply the condition (8.13) for $\varepsilon = \varepsilon_1$ and $T = T_0$. It follows that $T_0 \in I_{\varepsilon_1}$.

Claim 2 The set $I_{\varepsilon_1}$ is closed.

This follows from elementary convergence and continuity arguments based on (8.14).

Claim 3 The set $I_{\varepsilon_1}$ is open in $[T_0, \infty)$.

To prove this claim, assume $T \in I_{\varepsilon_1}$. By Lemma 8.9, the assumption (8.19) and (8.66) we have for $T_0 \leq t \leq T$

$$
||\theta||_{C^{4+\mu, (4+\mu)/2}(M \times [t-\frac{T_0}{s}, t])} \leq \frac{1}{2} \min\{\varepsilon_1, c, K, \varepsilon_0, \delta_0, \rho\}.
$$

(8.67)

Applying Lemma 8.2 to the initial $G_2$-structure $\sigma(T - \frac{1}{2}T_0)$ with $T - \frac{1}{2}T_0$ as the time origin we then obtain a $C^{4+\mu, (4+\mu)/2}$-solution of the $\sigma_0$-gauged Laplacian flow on $[T - \frac{1}{2}T_0, T + \frac{1}{2}T_0]$. By its uniqueness property, it agrees with $\sigma(t)$ on $[T - \frac{1}{2}T_0, T]$. Hence it extends $\sigma(t)$ to a $C^{4+\mu, (4+\mu)/2}$ solution of the $\sigma_0$-gauged Laplacian flow on $[0, T + \frac{1}{2}T_0]$. By (8.67) and continuity we have

$$
||\theta||_{C^{4+\mu, (4+\mu)/2}(M \times [t-\frac{T_0}{s}, t])} \leq c
$$

for all $t \in [T_0, T']$, whenever $T' > T$ and $T' - T$ is sufficiently small. For such a $T'$ and a $t$, there are two possible cases to consider. One is that $t - \frac{T_0}{s} \geq T_0$, the other is that $t - \frac{T_0}{s} < T_0$. In the first case, we write the time interval $[t - \frac{T_0}{s}, t]$ as the union of $[t - \frac{T_0}{s}, t - \frac{T_0}{s}]$ and $[t - \frac{T_0}{s}, t]$. Then we can apply the estimates (8.68) to the both subintervals. By the triangular inequality we then deduce

$$
||\theta||_{C^{4+\mu, (4+\mu)/2}(M \times [t-\frac{T_0}{s}, t])} \leq 2c.
$$

(8.69)

In the latter case, we write $[t - \frac{T_0}{s}, t]$ as the union of $[t - \frac{T_0}{s}, T_0]$ and $[T_0, t]$. Then we can apply the estimate (8.68) to the second subinterval, while apply the estimate (8.18) in Lemma 8.2 to the first subinterval. By the triangular inequality we again arrive at (8.68). We conclude that $T'$ belongs to $I_{\varepsilon_1}$, whenever $T' > T$ and $T' - T$ is sufficiently small. It follows that $I_{\varepsilon_1}$ is open.

Combining the above three claims we infer that $I_{\varepsilon_1} = [T_0, \infty)$. Hence the solution $\sigma(t)$ has been extended to $[0, \infty)$. By Lemma 8.9 $\theta$ converges in $C^{4+\mu, (4+\mu)/2}$ to zero at exponential rate as $t \to \infty$.

Finally we assume that $\sigma_0 \in \mathcal{C}_l$ for a non-integer $l > 4$. Then $\sigma - \sigma_0 \in \mathcal{C}^{l+1, (l+1)/2}$ by the regularity property provided by Theorem 6.4. By the above $C^{4+\mu, (4+\mu)/2}$ convergence we have

$$
||\Phi_1||_{C^0(M \times [t-\frac{T_0}{s}, t])} \leq \delta_0(l + 1, \sigma_0),
$$

(8.70)

whenever $t$ is large enough. Hence we can argue as in the proof of Lemma 8.4 to obtain for large $t$ an estimate for $||\theta||_{C^{l+1, (l+1)/2}(M \times [t-1, t])}$ similar to (8.61). This is precisely the desired convergence of $\theta$ in $C^{l+1, (l+1)/2}$ to zero at exponential rate. (In particular, since $\sigma_0 \in C^{4+\mu, (4+m\mu)/2}$, $\sigma - \sigma_0$ converges in $C^{5+\mu, (5+\mu)/2}$ to 0 at exponential rate.) □

9. Long time existence and convergence of the Laplacian flow

9.1. Local structure of the moduli space of torsion-free $G_2$-structures.

To establish the long time existence and convergence of the Laplacian flow starting near a torsion-free $G_2$-structure, we’ll need some results on the local structure of the moduli space of torsion-free $G_2$-structures.
On the other hand, the said convergence of the Laplacian flow also reveals an interesting dynamic property of this moduli space. Consider a non-integer \( l > 1 \). (In the following presentation, \( l \) is allowed to be \( \infty \) with the convention \( \infty + 1 = \infty \), except for the norms and their associated objects.) Let \( T_l \) denote the space of torsion-free \( C^l \) \( G_2 \)-structures on \( M \), and \( \text{Diff}^{l+1}_0(M) \) be the group of \( C^{l+1} \) diffeomorphisms of \( M \) which are \( C^{l+1} \)-isotopic to the identity map. Obviously, this group acts on \( T_l \). The quotient \( T_l/\text{Diff}^{l+1}_0(M) \) is the moduli space of torsion-free \( C^l \) \( G_2 \)-structures. Let \( \pi_l : T_l \to T_l/\text{Diff}^{l+1}_0(M) \) be the projection. Occasionally, we abbreviate \( T_{\infty} \) \( \text{Diff}^{\infty}_0(M) \) and \( \pi_{\infty} \) to \( T, \text{Diff}(M) \) and \( \pi \) respectively.

Let \( \sigma_0 \) be a given torsion-free \( C^l \) \( G_2 \)-structure on \( M \). In this subsection, all the geometric operations and measurements are \( \text{w.r.t.} \ g_{\sigma_0} \). For \( 0 \leq \lambda \leq 7 \) let \( H^l \equiv H^l_{\sigma_0}(M) \subset C^l_0(\Lambda^3 T^* M) \) denote the space of harmonic \( j \)-forms on \( M \) \( \text{w.r.t.} \ \sigma_0 \), which represents the DeRham cohomology group \( H^j(M, \mathbb{R}) \).

**Definition 9.1** For \( l, r \geq 0 \) and \( \gamma \in C^l_0(\Lambda^3 T^* M) \), let \( B^l_r(\gamma) \) denote the open ball of center \( \gamma \) and radius \( r \) in \( C^l_0(\Lambda^3 T^* M) \). We set \( B^l_r(\gamma) = B^l_r(\gamma) \cap H^l_{\sigma_0} \) for \( \gamma \in H^l_{\sigma_0} \).

**Theorem 9.1.** Let \( 0 < \mu < 1 \), \( 2 + \mu \leq l \leq \infty \) (a non-integer), and let \( \sigma_0 \) be a given torsion-free \( C^l \) \( G_2 \)-structure on \( M \). Then there are a positive number \( r_0 \equiv r_0(\sigma_0, \mu) \leq \epsilon_0 \) (with \( \epsilon_0 \) from Lemma [L2]) depending only on \( \sigma_0 \) and \( \mu \), and a smooth embedding \( \Xi_{\sigma_0} : B^{2+\mu}_{r_0}(\sigma_0) \to C^l_0(\Lambda^3 T^* M) \) whose image consists of torsion-free \( G_2 \)-structures, such that \( \Xi_{\sigma_0}(\gamma) \) is cohomologous to \( \gamma \) for all \( \gamma \in B^{2+\mu}_{r_0}. \) (Since \( r_0 \leq \epsilon_0 \), \( B^{2+\mu}_{r_0} \) consists of \( G_2 \)-structures.) Moreover, \( \Xi_{\sigma_0}(B^{2+\mu}_{r_0}) \) provides a local slice of the space of torsion-free \( C^l \) \( G_2 \)-structures under the action of \( \text{Diff}^{l+1}_0(M) \). As a consequence, the collection of \( (B^{2+\mu}_{r_0}(\sigma_0), \Xi_{\sigma_0}) \) for all \( \sigma_0 \in T_l \) provides a natural smooth structure on \( T_l/\text{Diff}^{l+1}_0(M) \).

We also have for all \( h \in B^{2+\mu}_{r_0}(\sigma_0) \)

\[
\| \Xi_{\sigma_0}(h) - \sigma_0 \|_{C^l} \leq C \| h - \sigma_0 \|_{C^l} \tag{9.1}
\]

with a positive constant \( C \) depending only on \( \sigma_0 \) and \( l \).

This result is a refinement of Joyce’s result [J] on local moduli of torsion-free \( G_2 \)-structures, and can be viewed as the elliptic version of Theorem [L2]. For its proof we refer to [XY2].

We consider for each \( 0 \leq j \leq 7 \) the projection map \( H^j_l : C^l_0(\Lambda^j T^* M) \to H^j \) which sends each closed \( C^l \) \( j \)-form \( \gamma \) to the unique harmonic form (w.r.t. \( \sigma_0 \)) in the cohomology class of \( [\gamma]_l = \{ \gamma + \lambda \beta : \beta \) is a \( C^{l+1}(j-1) \) form on \( M \} \). The following result is a simple consequence of basic elliptic regularity.

**Lemma 9.2.** There holds \( H^j_l = H^j_{\sigma_0} \) on \( C^l_0(\Lambda^j T^* M) \) for \( l' \leq l \).

Set \( B^{2+\mu}_{r_0}(\sigma_0) = H^{2+\mu}_{2+\mu}(B^{2+\mu}_{r_0}(\sigma_0)) \) and \( \tilde{B}^{2+\mu}_r(\sigma_0) = H^{2+\mu}_{2+\mu}(B^{2+\mu}_{r_0}(\sigma_0)) = \tilde{B}^{2+\mu}_{r_0}(\sigma_0) \cap C^l \). We define the smooth projection maps

\[
\hat{\Xi}_{\sigma_0} : \Xi_{\sigma_0} \circ H^{2+\mu}_{2+\mu} : \tilde{B}^{2+\mu}_{r_0}(\sigma_0) \to T_{2+\mu} \tag{9.2}
\]

and

\[
\Pi_{\sigma_0} = \pi_{2+\mu} : \hat{\Xi}_{\sigma_0} : \tilde{B}^{2+\mu}_{r_0}(\sigma_0) \to T_{2+\mu}/\text{Diff}^{2+\mu}_{2+\mu}(M). \tag{9.3}
\]

Restricting them to \( \tilde{B}^{2+\mu}_{r_0}(\sigma_0) \) we then obtain the smooth projection maps

\[
\hat{\Xi}_{\sigma_0} : \tilde{B}^{2+\mu}_{r_0}(\sigma_0) \to T_l \tag{9.4}
\]

and

\[
\Pi_{\sigma_0} : \tilde{B}^{2+\mu}_{r_0}(\sigma_0) \to T_l/\text{Diff}^{l+1}_0(M). \tag{9.5}
\]

Note that the former equals \( \Xi_{\sigma_0} \circ H^j_l \), while the latter equals \( \pi_l \circ \hat{\Xi}_{\sigma_0} \).

**Lemma 9.3.** There holds for each \( 1 \leq j \leq 7 \) and all \( \gamma \in C^l_0(\Lambda^j T^* M) \)

\[
\max \{ \| H^l(\gamma) - \sigma_0 \|_{C^l}, \| H^l(\gamma) - \gamma \|_{C^l} \} \leq C \| \gamma - \sigma_0 \|_{C^l} \tag{9.6}
\]

for a positive constant \( C \) depending only on \( \sigma_0 \) and \( l \).

**Proof.** Let \( \beta \in C^{l+1}(\Lambda^2 T^* M) \) be the unique solution of the equation

\[
\Delta \beta = -d^* \gamma \tag{9.7}
\]

subject to the \( L^2 \)-orthogonality condition \( \beta \perp L^2 \gamma \). Set \( h = \gamma + d\beta \). Then there holds \( (d^* + d)h = 0 \), and hence \( h \in H^l \). It follows that \( h = H^l(\gamma) \). By basic elliptic estimates we have

\[
\| \beta \|_{C^{l+1}} \leq C_1 \| d^* \gamma \|_{C^{l+1}} \tag{9.8}
\]
for a positive constant $C_1$ and hence
\[ \|H_1(\gamma) - \gamma\|_{C^l} \leq C_2 \|d^*\gamma\|_{C^{l-1}} \]  
for a positive constant $C_2$, where $C_1$ and $C_2$ depend only on $\sigma_0$ and $l$. Next we observe $d^*\gamma = d^*(\gamma - \sigma_0)$ and hence $\|d^*\gamma\|_{C^{l-1}} \leq C_3 \|\gamma - \sigma_0\|_{C^l}$ for a universal positive constant $C_3$. The first estimate in (9.10) follows. The second estimate follows from the first by the triangular inequality.

\[ \Box \]

**Lemma 9.4.** There holds for all $\gamma \in \hat{\mathcal{B}}_{\gamma_0}^{2+\mu,l}(\sigma_0)$

\[ \max\{\|\hat{\Xi}_{\sigma_0}(\gamma) - \gamma\|_{C^l}, \|\hat{\Xi}_{\sigma_0}(\gamma) - \sigma_0\|_{C^l}\} \leq C\|\gamma - \sigma_0\|_{C^l} \]  
for a positive constant $C$ depending only on $\sigma_0$ and $l$.

**Proof.** The desired estimates follow from Theorem 9.1, Lemma 9.3 and the triangular inequality.

\[ \Box \]

**9.2. Convergence I.**

In this subsection, all the geometric operations and measurements are w.r.t. $\sigma_0$ as given below.

**Theorem 9.5.** Let $0 < \mu < 1$, $K > 0$ and $\sigma_0$ a given $C^{4+\mu}$ structure on $M$. Assume that $\sigma_0$ and $\sigma_1$ satisfy the conditions in Theorem 8.3. Then there is a unique $C^{2+\mu,(3+\mu)/2}$ solution $\sigma = \sigma(t)$ of the Laplacian flow on $[0, \infty)$ which takes the initial value $\sigma_1$. It converges in $C^{2+\mu,(2+\mu)}$ at exponential rate to a torsion-free $C^{2+\mu}$ $G_2$-structure $\sigma_\infty$ on $M$ which is $C^{3+\mu}$ isotopic to $\sigma_0$. Moreover, there holds

\[ \|\sigma_\infty - \sigma_0\|_{C^{2+\mu}} \leq C\|\sigma_1 - \sigma_0\|_{C^{3+\mu}} \]  
for a positive constant $C$ depending only on $\sigma_0$, $K$ and $\mu$. If $\sigma_0 \in C^l$ for an non-integer $l > 4$, then $\sigma_\infty \in C^{l-2}$ and is $C^{l-1}$ isotopic to $\sigma_0$, and $\sigma$ converges in $C^{l-2,(l-3)/2}$ to $\sigma_\infty$ at exponential rate.

**Proof.** By Theorem 8.3 we have a unique $C^{4+\mu,(4+\mu)/2}$ solution $\sigma = \sigma(t)$ of the $\sigma_0$-gauged Laplacian flow on $[0, \infty)$ which takes the initial value $\sigma_1$ and converges in $C^{4+\mu,(4+\mu)/2}$ at exponential rate to $\sigma_0$. As in Section 7, we consider the ODE (7.1) on $M \times [0, \infty)$ with the present $\sigma$ and the initial condition (7.2). Since $\theta = \sigma - \sigma_0$ converges to 0 in $C^{4+\mu,(4+\mu)/2}$ at exponential rate, $X_{\sigma_0}(\sigma - \sigma_0)$ converges to 0 in $C^{3+\mu,(3+\mu)/2}$ at exponential rate. Consequently, the $C^{3+\mu,(4+\mu)/2}$ solution $\phi$ of (7.1) exists for all time and converges in $C^{3+\mu,(4+\mu)/2}$ at exponential rate to a $C^{3+\mu}$ map $\phi_\infty$. Taking derivative in (7.1) we infer

\[ \nabla_{\dot{\phi}} d\phi = \nabla_{d\phi} X_{\sigma_0} \]  
with $X_{\sigma_0} = X_{\sigma_0}(\sigma - \sigma_0)$. Consequently, there exists for each $p \in M$ and $v \in T_pM$

\[ \frac{d}{dt}|d\phi(v)|^2 = 2d\phi(v) \cdot \nabla_{d\phi(v)} X_{\sigma_0} \geq -2|d\phi(v)|^2|\nabla X_{\sigma_0}(\phi(p))| \]  
Because $|d\phi(v)| = |v|$ at $t = 0$, integration then yields

\[ |d\phi(v)| \geq |v|e^{-\int_0^t \|\nabla X_{\sigma_0}\|_{C^0}} \]  
for $d\phi_\infty$ at time $t$. We conclude

\[ |d\phi_\infty(v)| \geq |v|e^{-\int_0^\infty \|\nabla X_{\sigma_0}\|_{C^0}} \]  
and hence $d\phi_\infty$ is an isomorphism everywhere. Consequently, $\phi_\infty$ is a local diffeomorphism. Since it is homotopic to the identity map, it is a diffeomorphism.

Now we set $\tilde{\sigma}(\cdot, t) = \phi(\cdot, t)^*\sigma(\cdot, t)$. By Lemma 7.1 $\tilde{\sigma}$ is a $C^{2+\mu,(3+\mu)/2}$ solution of the Laplacian flow on $[0, \infty)$ with the initial value $\sigma_1$. By the above reasoning and the convergence of $\sigma$ to $\sigma_0$, it converges in $C^{2+\mu,(3+\mu)/2}$ at exponential rate to $\tilde{\sigma}_\infty = \phi_\infty^*\sigma_0$. This $G_2$-structure is obviously torsion-free because $\sigma_0$ is so. This also follows from the Laplacian flow equation and the convergence of $\frac{\partial \sigma}{\partial t}$ to 0.

If $\sigma_0 \in C^l$ for $l > 4$, then the corresponding convergence result in Theorem 8.3 and the above reasoning imply that $\tilde{\sigma}$ converges in $C^{l-2,(l-1)/2}$ to $\tilde{\sigma}_\infty$ at exponential rate. Moreover, $\phi$ converges in $C^{l-1,l/2}$ to $\phi_\infty$. Hence $\tilde{\sigma}_\infty \in C^{l-2}$ and is $C^{l-1}$ isotopic to $\sigma_0$. 

\[ \Box \]
9.3. Convergence II.

Definition 9.2 Let \(0 < \mu < 1, 4 + \mu \leq l \leq \infty\) (a non-integer), and \(\gamma \in C^{4+\mu}_0(\Lambda^3 T^* M)\). For \(K > 0\) and \(r > 0\) we set

\[
B^r_{K,\gamma}(\gamma) = \{\gamma' \in B^r_K(\gamma) : \|\gamma' - \gamma\|_{C^{4+\mu}} \leq r, \|\gamma' - \gamma\|_{C^{2+\mu}} \leq K\}.
\]

It will be called a \(K\)-strong (or simply, strong) \(C^{2+\mu}\) ball (or neighborhood) in the \(C^l\) space.

Theorem 9.6. Let \(0 < \mu < 1, K > 0\) and \(\tilde{\sigma}_0\) a given \(C^{4+\mu}\) torsion-free \(G_2\)-structure on \(M\). There is a positive constant \(\tilde{r}_0 \leq \epsilon_0\) depending only on \(K, \mu\) and \(\tilde{\sigma}_0\) with the following properties. Let \(\sigma_1 \in B^\tilde{r}_0(\tilde{\sigma}_0)\) for a non-integer \(l \geq 4 + \mu\). Then there is a unique \(C^{l-2, (l-1)/2}\) solution of the Laplacian flow on \([0, \infty)\) with the initial value \(\sigma_1\). It converges in \(C^{l-2, (l-1)/2}\) at exponential rate to a torsion free \(C^{l-2}\) \(G_2\)-structure \(\sigma_\infty\), which is \(C^{l-1}\) isotopic to \(\hat{\tau}_{\tilde{\sigma}_0}(\sigma_1)\).

Proof. We apply the results in Subsection 9.1 with \(\tilde{\sigma}_0\) playing the role of \(\sigma_0\) there. Let \(\sigma_1 \in B^\tilde{r}_0(\tilde{\sigma}_0)\) for some \(\tilde{r}_0 > 0\). By Lemma 9.3 there hold

\[
\|H_1(\sigma_1) - \sigma_0\|_{C^{4+\mu}, \tilde{\sigma}_0} \leq C_1 K
\]

and

\[
\|H_1(\sigma_1) - \sigma_0\|_{C^{2+\mu}, \tilde{\sigma}_0} \leq C_1 \tilde{r}_0,
\]

where \(C_1\) depends only on \(\tilde{\sigma}_0\) and \(\mu\). Assume \(\tilde{r}_0 \leq C_1^{-1} \tilde{r}_0\). Then \(H_1(\sigma_1) \in B^{2+\mu/l}_0(\tilde{\sigma}_0)\). It follows that

\[
\sigma_1 - \sigma_0 \in C^{2+\mu, \tilde{\sigma}_0} \leq C_2 K, \quad \|\sigma_0 - \tilde{\sigma}_0\|_{C^{4+\mu}, \tilde{\sigma}_0} \leq C_2 K
\]

for a positive constant \(C_2\) depending only on \(\tilde{\sigma}_0\) and \(\mu\). We assume \(\tilde{r}_0 \leq C_2^{-1} \epsilon_0\) (with \(\epsilon_0\) from Lemma 2.1). Then the ratios between the norms measured in \(\tilde{\sigma}_0\) and the norms measured in terms of \(\sigma_0\) (in both directions) are bounded by positive constants depending only on \(\tilde{\sigma}_0\) and \(\mu\). Hence we have

\[
\|\sigma_1 - \sigma_0\|_{C^{2+\mu}, \tilde{\sigma}_0} \leq C_3 \tilde{r}_0
\]

and

\[
\|\sigma_1 - \sigma_0\|_{C^{4+\mu}, \tilde{\sigma}_0} \leq C_3 K
\]

for a positive constant \(C_3\) depending only on \(\tilde{\sigma}_0\) and \(\mu\).

Now we replace \(K\) in Theorem 9.5 by \(\max\{C_2, C_3\} K\) and obtain the corresponding \(\rho\) there. Then it is clear that we can define \(\tilde{r}_0\) according to the above two conditions and the conditions in Theorem 9.5 and Lemma 8.2. Then we can apply Theorem 9.1 to deduce the desired long time existence and convergence of the Laplacian flow with the initial value \(\sigma_1\). The claimed isotopy property of the limit also follows from the same theorem.

10. THE LIMIT MAP OF THE LAPLACIAN FLOW

Set \(F(\sigma_0, \sigma_1) = \sigma_\infty\), where \(\sigma_\infty\) is the limit given in Theorem 9.5. Note that by the uniqueness part of Theorem 9.5, this map is actually independent of \(\sigma_0\). We’ll keep \(\sigma_0\) as an argument for the following reasons. First, in the proof below, we’ll employ several quantities which depend on both \(\sigma_0\) and \(\sigma_1\). So it is natural to treat everything in the framework of two arguments \(\sigma_0\) and \(\sigma_1\). Second, since we construct our arguments without using the uniqueness part of Theorem 9.5, they have a broader scope of possible applications.

Theorem 10.1. Let \(\sigma_0, \sigma_0\) and \(\sigma_1\) be as in Theorem 9.5. Then the map \(\sigma_0, \sigma_1\) is a Lipschitz continuous function on \(\sigma_0\) and \(\sigma_1\) w.r.t. \(C^{4+\mu}\)-norm on \(\sigma_0\), \(C^{4+\mu}\)-norm on \(\sigma_1\), and \(C^{2+\mu}\)-norm on \(F(\sigma_0, \sigma_1)\). In general, \(\sigma_0, \sigma_1\) is Lipschitz continuous w.r.t. \(C^l\)-norm on \(\sigma_0\), \(C^l\)-norm on \(\sigma_1\), and \(C^{l-2}\)-norm on \(F(\sigma_0, \sigma_1)\), provided that \(\sigma_0 \in C^l\) and \(\sigma_1 \in C^l\) for \(l \geq 4 + \mu\).

Proof. For two initial \(G_2\)-structures \(\sigma_0\) and \(\sigma_1\), and two torsion-free reference \(G_2\)-structures \(\sigma_0\) and \(\sigma_0\) as in the situation of Theorem 9.5, we consider the corresponding solutions \(\sigma = \sigma(t)\) of the \(\sigma_0\)-gauged Laplacian flow, and \(\tilde{\sigma} = \tilde{\sigma}(t)\) of the \(\tilde{\sigma}_0\)-gauged Laplacian flow. Set \(\theta = \sigma - \sigma_0, \tilde{\theta} = \tilde{\sigma} - \tilde{\sigma}_0, \) and \(\gamma = \theta - \tilde{\theta}\). We first derive estimates for \(\gamma\). There holds

\[
\frac{\partial \gamma}{\partial t} = -\Delta \sigma_0 \gamma - d(\Phi_{\sigma_0}(\gamma)) + (\Delta \sigma_0 - \Delta \tilde{\sigma}_0)\tilde{\theta} - d(\Phi_{\tilde{\sigma}_0}(\tilde{\theta}) - \Phi_{\sigma_0}(\tilde{\theta})).
\]
We handle the two difference terms on the RHS of (10.1) in terms of integration. For example, there holds
\[ d(\Phi_{\sigma_0}(\bar{t}) - \Phi_{\sigma_0}(\bar{t})) = d \left( \int_0^1 \frac{d}{ds} \Phi_{\sigma_0 + s(\bar{t} - \sigma_0)} ds \right). \] (10.2)

Since \( \gamma \) is exact, we can apply the arguments in the proof of Theorem 8.3 to obtain decay estimates for \( \gamma \). Here we employ the exponential decay of \( \theta \) and \( \bar{t} \) to handle the integration terms resulting from the three difference terms on the RHS of (10.1). (Note that the non-homogeneous terms arising from the last two terms can be handled by elementary integration techniques). We deduce for \( 2 < l \leq 4 + \mu \)
\[ \|\gamma\|_{C^{l,1/2}(M \times [1-1,1])}^2 \leq C_1 e^{-(\bar{t} - \sigma_0)\|\bar{t} - \sigma_0\|_{L^2}^2 + \|\bar{t} - \sigma_1\|_{L^2}^2} \] (10.3)
with a positive constant \( C_1 \) depending only on \( \bar{t}_0, \mu \) and \( K \). All the constants below in this proof have this same dependence.

Next let \( \phi \) be the solution of the ODE (7.1) corresponding to the solution \( \sigma \) (with \( \phi(0) = Id \)), and let \( \bar{\phi} \) be the solution of the ODE (7.1) corresponding to the solution \( \bar{\sigma} \), i.e.
\[ \frac{d\bar{\phi}}{dt} = -X_{\sigma_0}(\bar{\phi})(\bar{\phi}), \] (10.4)
also with \( \bar{\phi}(0) = Id \). Then \( \phi^* \sigma \) is the solution of the Laplacian flow with the initial value \( \sigma_1 \), and \( \bar{\phi}^* \bar{\sigma} \) is the solution of the Laplacian flow with the initial value \( \bar{\sigma}_1 \) as studied in the proof of Theorem 9.5.

Now we embed \( M \) into a Euclidean space and set \( \psi = \phi - \bar{\phi} \). We deduce
\[ \frac{d\psi}{dt} = -(X_{\sigma_0}(\phi + \psi) - X_{\sigma_0}(\phi)) = -(X_{\sigma_0}(\phi + \psi) - X_{\sigma_0}(\bar{\phi})) - (X_{\sigma_0}(\bar{\phi})(\bar{\phi}) - X_{\sigma_0}(\bar{\phi})(\bar{\phi})) - X_{\sigma_0}(\bar{\gamma})(\bar{\phi}) \] (10.5)
and \( \psi(0) = 0 \). As above, we can handle the two difference terms in the bottom line of (10.5) by integration. For example, there holds
\[ (X_{\sigma_0}(\bar{\phi})(\bar{\phi}) - X_{\sigma_0}(\bar{\phi})(\bar{\phi})) = \int_0^1 \frac{d}{ds} X_{\sigma_0 + s(\bar{\sigma}_0 - \sigma_0)}(\bar{\phi})(\bar{\phi}) ds. \] (10.6)
(We can assume that \( \|\bar{\sigma}_0 - \sigma_0\|_{C^0} \leq \epsilon_0 \). Then \( \sigma_0 + s(\bar{\sigma}_0 - \sigma_0) \) are \( G_2 \)-structures and as smooth as \( \sigma_0 \) and \( \bar{\sigma}_0 \) for \( 0 \leq s \leq 1 \).) When treating the first one, we need to make sure to use quantities defined on \( M \). For each \( p \in M \) and \( t \geq 0 \) choose a shortest geodesic \( c(t), 0 \leq t \leq d(p, q) \). Now we can integrate along \( c(t) \) to get a desired formula for the first difference term. Employing these formulas and the exponential decay estimates for \( \theta, \bar{\theta} \) and \( \gamma \), and integrating (10.5), we deduce for all \( t > 0 \)
\[ \|\psi(\cdot, t)\|_{C^0} \leq C_2(\|\bar{\sigma}_0 - \sigma_0\|_{C^{1+\mu}} + \|\bar{\sigma}_1 - \sigma_1\|_{C^{1+\mu}}) \] (10.7)
with a positive constant \( C_2 \). Taking derivatives in (10.5) and arguing in similar fashions we then obtain for all \( t > 0 \)
\[ \|\psi(\cdot, t)\|_{C^{3+\mu}} \leq C_3(\|\bar{\sigma}_0 - \sigma_0\|_{C^{4+\mu}} + \|\bar{\sigma}_1 - \sigma_1\|_{C^{4+\mu}}) \] (10.8)
with a positive constant \( C_3 \).

Now we combine the above estimates to deduce for all \( t > 1 \)
\[ \|\phi^*\bar{\sigma}(\cdot, t) - \phi^*\sigma(\cdot, t)\|_{C^{2+\mu}} \leq \|\phi^*\gamma(\cdot, t)\|_{C^{2+\mu}} + \|\psi^*\bar{\sigma}(\cdot, t)\|_{C^{2+\mu}} \leq C_4(\|\bar{\sigma}_0 - \sigma_0\|_{C^{4+\mu}} + \|\bar{\sigma}_1 - \sigma_1\|_{C^{4+\mu}}) \] (10.9)
with a positive constant \( C_4 \). Taking the limit as \( t \to \infty \) we then deduce
\[ \|\mathcal{F}(\bar{\sigma}_0, \bar{\gamma}_1) - \mathcal{F}(\sigma_0, \sigma_1)\|_{C^{2+\mu}} \leq C_4(\|\bar{\sigma}_0 - \sigma_0\|_{C^{4+\mu}} + \|\bar{\sigma}_1 - \sigma_1\|_{C^{4+\mu}}). \] (10.10)
The general case of \( l \) is similar.

The definition domain of \( \mathcal{F} \) is a domain in a Banach space, as the following lemma displays.

Lemma 10.2. Let \( t > 0 \) be an non-integer. Set \( \mathcal{X}^l = \{ (\gamma_0, \gamma_1) : \gamma_0 \in C^l_0(\Lambda^3 T^* M), \gamma_1 \in C^l_0(\Lambda^3 T^* M), \gamma_1 - \gamma_0 \in dC^{l+1}(\Lambda^3 T^* M) \} \) and \( \mathcal{Y}^l = \mathcal{X}^l \cap (C^l_0(\Lambda^3 T^* M) \times C^l_0(\Lambda^3 T^* M)) \). Then \( \mathcal{X}^l \) is a closed subspace of \( C^l_0(\Lambda^3 T^* M) \times C^l_0(\Lambda^3 T^* M) \), and \( \mathcal{Y}^l \) is a domain of \( \mathcal{X}^l \).
Proof. First observe that $dC^{l+1}(\Lambda^3T^*M)$ is a closed subspace of $C^l_0(\Lambda^3T^*M)$. To show this, consider a sequence $\beta_k \in C^{l+1}(\Lambda^3T^*M)$ such that $d\beta_k \to f$ in $C^l_0(\Lambda^3T^*M)$. We solve the equation $\Delta \gamma_k = d\beta_k$ with $\gamma_k \perp \mathcal{H}^3$. Then we have $\Delta(\gamma_k - \gamma_k') = d\beta_k - d\beta_k'$. It follows that $\|\gamma_k - \gamma_{k'}\|_{C^{l+2}} \leq C\|d\beta_k - d\beta_{k'}\|_{C^l}$. Hence $\gamma_k \to \gamma$ in $C^{l+2}$ for some $\gamma$. But $\Delta \gamma_k = d\beta_k$ implies $d^* \gamma_k = d\beta_k$. Hence we infer $d^* \gamma = f$, which implies $f \in dC^{l+1}(\Lambda^3T^*M)$.

Obviously, the above closedness implies the desired closedness of $\mathcal{X}^l$. By Lemma 2.1, $\mathcal{X}^l$ is a domain of $\mathcal{X}^l$.

Lemma 10.3. Let $l > 2$ be a non-integer. Then $T^l$ is a smooth Banach submanifold of the Banach space $C^l_0(\Lambda^3T^*M)$.

We refer to [XY2] for the proof of this lemma.

Theorem 10.4. For given $\sigma_0, \mu$ and $K$ as in Theorem 9.3, let $U(\sigma_0, \mu, K)$ denote the neighborhood in $\mathcal{X}^{4+\mu}$ defined by the conditions in that theorem. Then the map $F : U(\sigma_0, \mu, K) \to T^{2+\mu}$ is smooth. Moreover, for each $l > 4 + \mu$, the restriction $F : U(\sigma_0, \mu, K) \cap C^l \to T^{l-2}$ is smooth.

Proof. This is a lengthy proof, which we break into three parts. In the first part, we decompose the difference form $\gamma$ in the proof of Theorem 9.5 and derive the associated estimates. In the second part, we decompose the difference map $\psi$ in that proof and derive the associated estimates. In the last part, we draw the final conclusions.

1) We employ the notations in the proof of Theorem 9.3 and set $p = (\sigma_0, \sigma_1), q = (\bar{\sigma}_0, \bar{\sigma}_1)$. First observe for the equation (10.11)

$$d(\Phi_{\sigma_0}(\bar{\theta}) - \Phi_{\sigma_0}(\theta)) = L_0\gamma + Q_0(\gamma, \gamma)$$

with

$$L_0\gamma = d(D_\sigma A(\sigma_0, \sigma, \theta, \nabla_\sigma \theta)(\gamma) + A(\sigma_0, \sigma, \gamma, \nabla_\sigma \theta) + A(\sigma_0, \sigma, \theta, \nabla_\sigma \theta))$$

and

$$Q_0(\gamma, \gamma) = d\int_0^1 tD_\sigma^2 A(\sigma_0, \sigma + \theta + t\gamma, \theta + t\gamma, \nabla_\sigma \theta + t\nabla_\sigma \gamma)(\gamma, \gamma)dsdt.$$  

Similarly, we can write the sum of the second and third terms on the RHS of (10.11) as the sum of a linearized term and a quadratic term:

$$(\Delta_{\sigma_0} - \Delta_{\sigma_0})\bar{\theta} - d(\Phi_{\sigma_0}(\bar{\theta}) - \Phi_{\sigma_0}(\theta)) = L_1(\bar{\sigma}_0 - \sigma_0) + Q_1(\bar{\sigma}_0 - \sigma_0, \bar{\sigma} - \sigma_0) + Q_2(\bar{\sigma}_0 - \sigma_0, \gamma),$$

where $L_1$ is the independent of the quantities with bar. It follows that

$$\frac{\partial \gamma_1}{\partial t} = -\Delta_{\sigma_0} \gamma + L_0\gamma + L_1(\bar{\sigma}_0 - \sigma_0) + Q_0(\gamma, \gamma) + Q_1(\bar{\sigma}_0 - \sigma_0, \bar{\sigma} - \sigma_0) + Q_2(\bar{\sigma}_0 - \sigma_0, \gamma).$$

Note that $L_0, L_1, Q_0, Q_1$ and $Q_2$ are time-dependent, and converge to zero at exponential rate in suitable norms as $t \to \infty$. Now we consider the equation

$$\frac{\partial \gamma_1}{\partial t} = -\Delta_{\sigma_0} \gamma_1 + L_0\gamma_1 + L_1(\bar{\sigma}_0 - \sigma_0)$$

with the initial condition $\gamma_1(0) = 0$.

2) We write $L_0$ in the following form

$$L_0\gamma' = d(\Phi_0(\gamma') - \Phi_1(\nabla_{\sigma_0} \gamma'))$$

with $\Phi_0(\gamma') = D_\sigma A(\sigma_0, \sigma, \theta, \nabla_\sigma \theta)(\gamma') + A(\sigma_0, \sigma, \gamma', \nabla_\sigma \theta)$ and $\Phi_1(\nabla_{\sigma_0} \gamma') = A(\sigma_0, \sigma, \theta, \nabla_\sigma \gamma')$. As in the proof of Lemma 9.7 we have $\|\Phi\|_{C^{0+\mu}} \leq C_0\|\theta\|_{C^{0+\mu}}$. By the estimate for $\theta$ we can then apply Theorem 5.3 (or Theorem 9.1) to obtain a unique $C^{4+\mu, (4+\mu)/2}$ solution $\gamma_1$ and a unique $C^{4+\mu, (4+\mu)/2}$ solution $\gamma_2$ on $[0, \infty)$. We also obtain the following estimates for all $t \geq 1$

$$\|\gamma_1\|_{C^{4+\mu}(M \times [t-1, t])} \leq C_1 e^{-\frac{1}{2} \lambda_0 t} (\|\bar{\sigma}_0 - \sigma_0\|_{C^{4+\mu}} + \|\bar{\sigma} - \sigma_0\|_{C^{4+\mu}})$$

and

$$\|\gamma_2\|_{C^{4+\mu}(M \times [t-1, t])} \leq C_1 e^{-\frac{1}{2} \lambda_0 t} (\|\bar{\sigma}_0 - \sigma_0\|_{C^{4+\mu}}^2 + \|\bar{\sigma} - \sigma_0\|_{C^{4+\mu}}^2)$$

with a positive constant $C_1$ depending only on $\sigma_0, \mu$ and $K$. Obviously, there holds $\gamma = \gamma_1 + \gamma_2$. 

\[\square\]
For $\sigma'_0 \in C^{3+\mu}(L^3T^*M)$ $\sigma'_1 \in C^{4+\mu}(L^3T^*M)$ such that $\sigma'_1 - \sigma'_0$ is exact, we consider the following general version of (10.10)

$$\frac{\partial \gamma'_1}{\partial t} = -\Delta_{\sigma_0} \gamma'_1 + L_0 \gamma'_1 + L_1 \sigma'_0$$

(10.21)

with the initial condition $\gamma'_1(0) = \sigma'_1 - \sigma'_0$. Set $p' = (\sigma'_0, \gamma'_1)$. Let $\Gamma_{1,p}(p')$ denote the unique $C^{4+\mu,(4+\mu)/2}$ solution on $M \times [0, \infty)$. We have the following generalization of (10.19)

$$\|\Gamma_{1,p}(p')\|_{C^{4+\mu}(M \times [0, t])} \leq C_1 e^{-\frac{1}{2} \mu t} (\|\sigma'_0\|_{C^{4+\mu}} + \|\sigma'_1\|_{C^{4+\mu}}).$$

(10.22)

We obviously have

$$\gamma_1 = \Gamma_{1,p}(q - p).$$

(10.23)

2) Next we consider the equation (10.5). By the estimate (10.7), we can achieve the following by assuming $||\dot{\sigma}_0 - \sigma_0||_{C^{1+\nu}} + ||\dot{\sigma}_1 - \sigma_1||_{C^{1+\nu}}$ to be small enough: for each $p \in M$ and $t \geq 0$, the distance between $\phi(p, t)$ and $\phi(p, t)$ is less than half of the injectivity radius of $M$ (w.r.t. $\sigma_0$). Then we can handle the difference terms in (10.5) by unique shortest geodesics. The resulting quantities then retain the previous regularity and estimates. This way, we decompose the far right hand side of (10.5) into a linearized part and a quadratic part and deduce

$$\frac{d\psi}{dt} = \hat{L}_0 \psi + \hat{L}_1 \gamma + \hat{L}_2 (\sigma_0 - \sigma_0) + \hat{Q}_0 (\psi, \psi) + \hat{Q}_1 (\gamma, \gamma) + \hat{Q}_2 (\sigma_0 - \sigma, \sigma_0 - \sigma_0) + \hat{Q}_3 (\psi, \gamma) + \hat{Q}_4 (\psi, \bar{\sigma}_0 - \sigma_0)$$

$$+ \hat{Q}_5 (\gamma, \bar{\sigma}_0 - \sigma_0),$$

(10.24)

where $\hat{L}_0, \hat{L}_1$ and $\hat{L}_2$ are independent of the quantities with bar. Note that the involved operators $\hat{L}_0, \hat{L}_1, \hat{L}_2, \hat{Q}_0$ etc. are all time-dependent and decay exponentially in suitable norms. We further write $L_2 \gamma = \hat{L}_2 \gamma_1 + \hat{L}_2 \gamma_2$. Then we have $\psi = \psi_1 + \psi_2$, where $\psi_1$ is the unique solution of the ODE

$$\frac{d\psi_1}{dt} = \hat{L}_0 \psi_1 + \hat{L}_1 \gamma_1 + \hat{L}_2 (\sigma_0 - \sigma_0)$$

(10.25)

with the initial condition $\psi_1(0) = 0$, and $\psi_2$ is the unique solution of the ODE

$$\frac{d\psi_2}{dt} = \hat{L}_0 \psi_2 + \hat{L}_1 \gamma_2 + \hat{Q}_0 (\psi, \psi) + \hat{Q}_1 (\gamma, \gamma) + \hat{Q}_2 (\sigma_0 - \sigma, \sigma_0 - \sigma_0) + \hat{Q}_3 (\psi, \gamma) + \hat{Q}_4 (\psi, \bar{\sigma}_0 - \sigma_0) + \hat{Q}_5 (\gamma, \bar{\sigma}_0 - \sigma_0),$$

(10.26)

with the initial condition $\psi_2(0) = 0$. Employing the decay estimates for all the involved quantities we obtain the limits $\psi_1^\infty$ and $\psi_2^\infty$ of $\psi_1$ and $\psi_2$ respectively as $t \to \infty$, which satisfy

$$||\psi_1^\infty||_{C^{3+\mu}} \leq C_2 ||\sigma_0 - \sigma_0||_{C^{1+\nu}} + ||\sigma_1 - \sigma_1||_{C^{1+\nu}}$$

(10.27)

and

$$||\psi_2^\infty||_{C^{3+\mu}} \leq C_2 ||\sigma_0 - \sigma_0||_{C^{1+\nu}} + ||\sigma_1 - \sigma_1||_{C^{1+\nu}}.$$  

(10.28)

There holds $\psi_\infty = \psi_1^\infty + \psi_2^\infty$. On the other hand, we have the following generalization of (10.25) (analogous to (10.21))

$$\frac{d}{dt} \Psi_{1,p}(p) = \hat{L}_0 \Psi_{1,p}(p) + \hat{L}_1 \Gamma_{1,p}(p) + \hat{L}_2 (\sigma_0')$$

(10.29)

with the initial condition $\Psi_{1,p}(p) = 0$, its limit $\Psi_{1,p}^\infty(p)$ as $t \to \infty$ and the estimate

$$||\Psi_{1,p}^\infty(p)||_{C^{3+\mu}} \leq C_2 ||\sigma_0 - \sigma_0||_{C^{1+\nu}} + ||\sigma_1 - \sigma_1||_{C^{1+\nu}}.$$  

(10.30)

Thus $\Psi_{1,p}^\infty$ is a bounded linear operator. There holds $\psi_1^\infty = \Psi_{1,p}^\infty(\sigma_0 - \sigma_0, \sigma_1 - \sigma_1)$.

3) Now we calculate

$$\mathcal{F}(q) - \mathcal{F}(p) = \mathcal{F}(\sigma_0, \sigma_1) - \mathcal{F}(\sigma_0, \sigma_1) = \bar{\phi}_{\infty}(\sigma_0) - \phi_{\infty}(\sigma_0) + \rho_{\infty}(\sigma_0)$$

$$= \phi_{\infty}(\sigma_0) + \bar{\psi}_{\infty}(\sigma_0) + \psi_{\infty}(\sigma_0) + \rho_{\infty}(\sigma_0).$$

(10.31)

By the estimates (10.28) and (10.27) we infer that $\mathcal{F}$ is differentiable at $p$. Moreover, we have

$$\mathcal{D}_p \mathcal{F} = \phi_{\infty}(\sigma_0) + \psi_{\infty}(\sigma_0) + \rho_{\infty}(\sigma_0).$$

(10.32)

where $\sigma_0(\gamma', \gamma'') = \gamma'$. Adapting the above arguments to handle the difference $\mathcal{D}_\mathcal{F} \mathcal{F} - \mathcal{D}_p \mathcal{F}$, we deduce that $\mathcal{D}_p \mathcal{F}$ is Lipschitz continuous. It follows that $\mathcal{F}$ is $C^1$ as a map into $C_0^{2+\mu}(L^3T^*M)$. By Lemma (10.3) it is also $C^1$ as a map into $T^{2+\mu}$.
The above scheme can easily be extended to higher order derivatives of $\mathcal{F}$, and we derive that $\mathcal{F}$ is $C^\infty$. Applying the $C^{l,l/2}$ estimates we then obtain the claimed smoothness of the $C^l$ restriction of $\mathcal{F}$. \hfill $\square$

Finally we set $\mathcal{F}(\sigma_1) = \sigma_\infty$, where $\sigma_\infty$ is the limit of the Laplacian flow given in Theorem 9.6. The following theorem contains Theorem 4.3 in Introduction as a special case.

**Theorem 10.5.** Let $0 < \mu < 1, K > 0, \delta_0$ a given $C^{4+\mu}$ $G_2$-structure on $M$, and $\hat{\tau}_0$ be given by Theorem 9.6. Let $4 + \mu \leq \ell \leq \infty$ be a non-integer. Then the limit map $\mathcal{F} : B^l_{K,\hat{\tau}_0}(\delta_0) \to T^{l-2}$ is smooth. Moreover, there holds

$$\pi_{l-2} \circ \mathcal{F} = \Pi. \quad (10.33)$$

(In the case $l = \infty$, we have the convention $\infty - 2 = \infty$.)

**Proof.** By the proof of Theorem 9.6 there holds $\mathcal{F}(\sigma_1) = \mathcal{F}(\hat{\Xi}_{\delta_0}(\sigma_1), \sigma_1)$. Hence the claimed smoothness follows from Theorem 10.2 and the smoothness of $\hat{\Xi}_{\delta_0}$. By Theorem 9.6, $\mathcal{F}(\sigma_1)$ is $C^{l-1}$ isotopic to $\hat{\Xi}_{\delta_0}(\sigma_1)$. It follows that $\pi_{l-2}(\mathcal{F}(\sigma_1)) = \pi_{l-2}(\hat{\Xi}_{\delta_0}(\sigma_1)) = \Pi(\sigma_1)$. \hfill $\square$

**Appendix: Space-time Function Spaces**

Let $M$ be a compact manifold of dimension $n \geq 1$. Let a background Riemannian metric $g_*$ on $M$ be given. We assume that it has the required smoothness in each individual situation below. The norms defined below depend on the choice of $g_*$, but we an easily relate the norms w.r.t. one background metric to those w.r.t. another background metric.

Each tensor bundle $E$ associated with the tangent bundle $TM$ is equipped with the natural metric induced from $g_*$ and the natural connection $\nabla$ induced from the Levi-Civita connection (still called the Levi-Civita connection). We’ll use these metric and connection in the definitions below.

In this Appendix, we define various Hölder spaces of $E$-valued functions (i.e. sections of $E$) used in this article. In particular, we define spacetime Hölder spaces which play a crucial role in our parabolic theory. We basically follow the definitions given in [Y3]. Note that it is only for convenience of presentation that we restrict to tensor bundles associated with the tangent bundle. Our theory extends straightforwardly to a general vector bundle over $M$, which is equipped with a metric and a metric-compatible connection.

From now on we fix a tensor bundle $E$.

**$C^k$-spaces.** Let $k \geq 0$ be an integer. We define the space $C^k(E)$ to be the space of continuous sections $\zeta$ of $E$ that have up to $k$-th order continuous covariant derivatives, and define the $C^k$ norm as follows

$$\| \zeta \|_{C^k(E)} = \sum_{i=0}^{k} \sup_{M} |\nabla^i \zeta|. \quad (10.34)$$

Equipped with this norm, the space $C^k(E)$ is a Banach space.

**Remark** We write this norm as $\| \zeta \|_{C^k(E), g_*}$, if we need to indicate the background metric $g_*$. We replace the subscript $g_*$ by $\sigma$ if $g_*$ is the induced metric of a $G_2$-structure $\sigma$, i.e. $g_* = g_\sigma$. Similar notations are also used for the other norms in this paper.

Next let $0 < \mu < 1$. We define the Hölder semi-norm $|\zeta|_\mu$ of a section $\zeta$ of $E$:

$$|\zeta|_\mu = \sup_{p,q \in M, 0 < d(p,q) \leq 1} \sup_{\gamma} \frac{|P_\gamma (\zeta(p)) - \zeta(\zeta(q))|}{d(p,q)^\mu}, \quad (10.35)$$

where $\gamma$ runs through all piecewise $C^1$-curves in $M$ going from $p$ to $q$ and having length not exceeding $2d(p,q)$, and $P_\gamma$ denotes the parallel transport along $\gamma$. (Alternatively, we can restrict to geodesics $\gamma$. Then we obtain an equivalent seminorm.) Note that the condition $|\zeta|_\mu < \infty$ can be interpreted as a fractional differentiability.

Let $l = k + \mu$ for an integer $k \geq 0$. The Hölder space $C^l(E)$ consists of sections $\zeta$ of $C^k(E)$ with $|\nabla^k \zeta|_\mu < \infty$. The norm $\| \zeta \|_{C^l}$ is defined as

$$\| \zeta \|_{C^l} = \| \zeta \|_{C^k} + |\nabla^k \zeta|_\mu. \quad (10.36)$$

Equipped with this norm, $C^l(E)$ is a Banach space.
c^{m,l}$-spaces. Let $I$ be a bounded closed interval of $\mathbb{R}$ with coordinate $t$. We abbreviate for the derivative $\frac{\partial}{\partial t}$ to $\partial_t$. Let $\pi$ be the projection of $M \times I$ onto $M$ and $\pi^* E$ be the pull-back of $E$ to $M \times I$.

For integers $m \geq 0, l \geq 0$, we define $C^{m,l}(\pi^* E)$ to be the space of sections $\zeta$ of $\pi^* E$ which have continuous partial derivatives of the form $\partial_t^i \nabla^j \zeta$ with $i + 2j \leq m$ and $j \leq l$. When dealing with parabolic equations of the type of the heat equation, it is natural to count one time derivative as two space derivatives. This is the underlying reason for the above factor 2 in front of $j$. This factor also appears below for the same reason.

The norm $\| \cdot \|_{C^{m,l}(\pi^* E)}$ is defined as follows

$$\| \zeta \|_{C^{m,l}(\pi^* E)} = \sup_{M \times I} \sum_{i + 2j \leq m, j \leq l} |\partial_t^i \nabla^j \zeta|.$$  (10.37)

**Remark** We often abbreviate the above notation to $\| \zeta \|_{C^{m,l}}$. We also write it as $\| \zeta \|_{C^{m,l}(M \times I)}$ if we need to emphasize the base domain. Similar abbreviations and notations are also used for other norms or spaces in this paper.

It is easy to show that equipped with the above norm, $C^{m,l}(\pi^* E)$ is a Banach space.

$c^{l,m/2}$-spaces. We now introduce “fractional” differentiability in both the time and space directions. For $0 < \mu < 1$, we define the $\mu$-Hölder semi-norm in the time direction

$$[\zeta]_{\mu, M \times I, I} = \sup_{p \in M, 0 \leq t_2 - t \leq 1} \frac{|\zeta(p, t_2) - \zeta(p, t_1)|}{|t_2 - t_1|^\mu},$$  (10.38)

and the $\mu$-Hölder semi-norm in the space direction

$$[\zeta]_{\mu, M \times I, M} = \sup_{t \in I} [\zeta(\cdot, t)]_\mu.$$  (10.39)

Now let $l$ and $m$ be nonnegative non-integers with $2l \geq m$. We define the space $c^{l,m/2}(\pi^* E)$ as the space of sections in $C^{[l], [m/2]}(\pi^* E)$ with finite $c^{l,m/2}$-norm, which is defined as follows

$$\| \zeta \|_{c^{l,m/2}(\pi^* E)} = \sum_{i + 2j \leq [l], j \leq [m/2]} \max_{M \times I} |\partial_t^i \nabla^j \zeta| + < \zeta >^l_{M \times I, M} + < \zeta >^{(m/2)}_{M \times I, M}.$$  (10.40)

with the $(l)$-Hölder semi-norm in the space direction

$$< \zeta >^l_{M \times I, M} = \sum_{i + 2j = [l], j \leq [m/2]} |\partial_t^i \nabla^j \zeta|_{[l], M \times I, M},$$

and the $(m/2)$-Hölder semi-norm in the time direction

$$< \zeta >^{(m/2)}_{M \times I, I} = \sum_{0 < m - i - 2j \leq [l]} |\partial_t^i \nabla^j \zeta|_{(m - i - 2j)/2, M \times I, I}.$$  (10.41)

It is easy to show that, equipped with the norm (10.40), $c^{l,m/2}$ is a Banach space.

Of particular importance is the case $l = m$, i.e. the spaces $C^{l/2}$. They are the natural spaces for formulating a priori estimates for solutions of parabolic equations, see Theorem 5.1. The formula for the $C^{l/2}$-norm takes a slightly simpler form:

$$\| \zeta \|_{C^{l/2}(\pi^* E)} = \sum_{i + 2j \leq l} \max_{M \times I} |\partial_t^i \nabla^j \zeta| + < \zeta >^l_{M \times I, M} + < \zeta >^{(l/2)}_{M \times I, I},$$  (10.42)

with the $(l)$-Hölder semi-norm in the space direction

$$< \zeta >^l_{M \times I, M} = \sum_{i + 2j = l} |\partial_t^i \nabla^j \zeta|_{l, M \times I, M}$$

and the $(l/2)$-Hölder semi-norm in the time direction

$$< \zeta >^{(l/2)}_{M \times I, I} = \sum_{0 < l - i - 2j < 2} |\partial_t^i \nabla^j \zeta|_{(l - i - 2j)/2, M \times I, I}.$$  (10.43)

For example, we have for $0 < \mu < 1$

$$\| \zeta \|_{C^{\mu,l/2}} = \sup_{M \times I} |\zeta| + [\zeta]_{\mu, M \times I, I} + [\zeta]_{\mu, M \times I, M},$$

$$\| \zeta \|_{C^{l+(1+\mu)/2}} = \sup_{M \times I} |\zeta| + |\nabla \zeta| + |\nabla \zeta|_{M \times I, I} + |\nabla \zeta|_{M \times I, M}.$$  (10.44)
and
\[ \|\zeta\|_{C^{2+(\mu, \frac{1}{2})/2}} = \max_{M \times I} (|\zeta| + |\partial_t \zeta| + |\nabla \zeta| + |\nabla^2 \zeta|) + ((|\partial_t \zeta|)_{\mu, M \times I, M} + |\nabla^2 \zeta|_{\mu, M \times I, M}) \]
\[ + (|\partial_t \zeta|)_{\mu, M \times I, I} + |\nabla \zeta|_{\mu, M \times I, I} + |\nabla^2 \zeta|_{\mu, M \times I, I} ). \]  

(10.46)

Finally we present another separate definition which is used in the formulation of some results in this paper.

**Definition 11.1** We define the inverse tensor \((g_2)^{-1}\) of a Riemannian metric \(g_2\) w.r.t. another Riemannian metric \(g_1\) as follows. There holds \(g_2(v_1, v_2) = g_1(Av_1, v_2)\) for a section \(A\) of \(T^*M \otimes TM = \text{Hom}(TM, TM)\).

Then we define
\[ \|g_2^{-1}\|_{C^1, g_1} = \|(g_2)^{-1}\|_{C^1, g_1}. \]

(10.47)

We write it as \(\|g_2^{-1}\|_{C^1}\), if the metric \(g_1\) is clear from the context. Note that the eigenvalues of \(g_2^{-1}\) w.r.t. \(g_1\) are the reciprocals of the eigenvalues of \(g_2\) w.r.t. \(g_1\).

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