Vortices and hierarchy of states in double-layer fractional Hall effect

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21 July 1994

Abstract
Vortex solutions in $U(1) \times U(1)$ Chern-Simons theory coupled to a pair of hard-core bosons representing two layers of electrons are analysed. It is shown that there is such a range of parameters ($\alpha \beta < \gamma^2$) in which finite charge $(1, 0)$ or $(0, 1)$ vortices can not exist. Instead the minimal flux quanta are $(1, 1)$ hybrids. Their statistical dominance is shown to have an influence on Haldane-Halderin hierarchy of states. The form of Coulomb potential and its influence on the range of physically acceptable states is discussed.

1 Introduction
The two-dimensional electron system can have a rich structure. One of the most significant discoveries is the fractional Hall effect in strong magnetic fields [1]. The standard theoretical approach is through variational analysis with trial functions [2]. There are also field-theoretical studies of the subject based on generalisations of Ginzburg-Landau model with a use of effective Chern-Simons theory [3]. More microscopical treatment of the phenomenon was introduced by Ezawa and Iwazaki [4]. Recently there was discovered the even-denominator filling factor in double-layer electron system [5]. The system was proposed to be described by $U(1) \times U(1)$ Chern-Simons theory coupled to a pair of hard-core bosons representing two layers of electrons [6]. Nonperturbative excitations are topological vortices with some winding numbers on both of the layers $(p, q)$. If $(1, 0)$ and $(0, 1)$ vortices condense the hierarchy of states was shown [6] to begin by

$$\nu^{(1)} = \frac{\nu^{(0)}}{1 + \frac{\nu^{(0)}}{4q}},$$

(1)

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with characteristic $4q$ instead of $2q$ as in a single-layer system ($q$ is an integer). By careful analysis we show that there is such a range of statistical parameters in which minimal vortex excitations with finite charge are $(1,1)$ vortices living on both of the layers. In this respect this work is a generalisation of our previous paper [7]. The physical consequence of this fact is that the denominator is once again $2q$.

Another question is the form of Coulomb potential. To have finite Coulomb energy of the condensates one needs regularising background. If the background charge density is just opposite to the condensate density on a given layer the whole spectrum of states $(k,l,m)$ is allowed. If instead the background is uniform then only states with $k = l$ survive switching on the Coulomb interaction.

2 The model, condensates and vortex solutions

The model we consider in this paper was introduced by Ezawa and Iwazaki [6]. It consists of two species of polarised electrons represented by hard-core bosons coupled to $U(1) \times U(1)$ Chern-Simons field the effect of which is via singular gauge transformation equivalent to imposing the Fermi-Dirac statistics.

$$L = i\psi^* D_0 \psi - \frac{1}{2M} D_k \psi^* D_k \psi + i\phi^* D_0 \phi - \frac{1}{2M} D_k \phi^* D_k \phi - V(\psi, \phi)$$

$$- \frac{1}{4X} \varepsilon^{\mu\nu\lambda} a_\mu^{(1)} \partial_\nu a_\lambda^{(1)} - \frac{1}{4Y} \varepsilon^{\mu\nu\lambda} a_\mu^{(2)} \partial_\nu a_\lambda^{(2)}$$

$$- \frac{1}{4Z} \varepsilon^{\mu\nu\lambda} a_\mu^{(1)} \partial_\nu a_\lambda^{(2)} - \frac{1}{4Z} \varepsilon^{\mu\nu\lambda} a_\mu^{(2)} \partial_\nu a_\lambda^{(1)} + V(\psi^*, \phi^*, \phi) + \Delta V^C . \quad (2)$$

The coefficients $X, Y, Z$ are defined in an entangled way by another set of parameters

$$X = \frac{\alpha\beta - \gamma^2}{\beta}, \quad Y = \frac{\alpha\beta - \gamma^2}{\alpha}, \quad Z = \frac{\gamma^2 - \alpha\beta}{\gamma} . \quad (3)$$

These parameters take values $\alpha = \pi k, \beta = \pi l, \gamma = \pi m$ with $k, l, m$ odd integers which provide electrons $\psi, \phi$ with desired statistics. The potential term has the form characteristic for bosonic end perturbation theory with respect to the parameters $\alpha, \beta, \gamma$, see [4] and e.g. [8]

$$V = \frac{1}{2M} [\alpha | \psi |^4 + 2\gamma | \psi |^2 \phi |^2 + \beta | \phi |^4] , \quad (4)$$

This potential is a field-theoretical remnant of the two-particle delta-type potential [3]. Another potential term is a nonlocal Coulomb interaction

$$\Delta V^C = \Delta V^C_{11} + \Delta V^C_{12} + \Delta V^C_{22} . \quad (5)$$

The particular terms are given by

$$\Delta V^C_{11} = \frac{e^2}{2\varepsilon} \int d^2 x \ d^2 y \ [\psi^* \psi(x) - \rho_1^0] \frac{1}{|x - y|} [\psi^* \psi(y) - \rho_1^0] ,$$
\[ \Delta V_{12}^C = \frac{e^2}{\varepsilon} \int d^2 x \ d^2 y \left[ \psi^\ast \psi(x) - \rho_1^{(1)} \right] \frac{1}{\sqrt{(x-y)^2 + d^2}} \left[ \phi^\ast \phi(y) - \rho_2^{(2)} \right], \]
\[ \Delta V_{22}^C = \frac{e^2}{2\varepsilon} \int d^2 x \ d^2 y \left[ \phi^\ast \phi(x) - \rho_2^{(2)} \right] \frac{1}{|x-y|} \left[ \phi^\ast \phi(y) - \rho_2^{(2)} \right], \]

where \( \varepsilon \) is a dielectric constant and \( d \) is the interlayer distance. We have taken into account uniform background the charge density of which is opposite to the medium charge density of electrons. The Lagrangian in this form does make sense only if \( \alpha \beta - \gamma^2 \neq 0 \) - the case we refer to as nondegenerate, but the model itself can be made more general. Couplings between electrons and gauge fields are established by

\[
D_{\mu} \psi = \partial_{\mu} \psi - ia_{\mu}^{(1)} \psi + ieA_{\mu}^{\text{ext}} \psi, \quad D_{\mu} \phi = \partial_{\mu} \phi - ia_{\mu}^{(2)} \phi + ieA_{\mu}^{\text{ext}} \phi. \tag{7}
\]

In what follows we will consider only external magnetic field and put \( A_0^{\text{ext}} = 0 \). Variation with respect to Lagrange multipliers \( a_0^{(I)} \), \( I = 1, 2 \) leads to constraints

\[
B_1 = 2\alpha \rho_1 + 2\gamma \rho_2, \quad B_2 = 2\gamma \rho_1 + 2\beta \rho_2, \tag{8}
\]

where the electronic densities are \( \rho_1 = \psi^\ast \psi \), \( \rho_2 = \phi^\ast \phi \). The Hamiltonian of the model is

\[
H = \frac{1}{2M} D_k \psi^* D_k \psi + \frac{1}{2M} D_k \phi^* D_k \phi + V + \Delta V^C \tag{9}
\]

For the following we will switch off the Coulomb interaction and later on regard it as a correction. In a uniform external magnetic field Bogomol’nyi decomposition \[9\] leads to

\[
H = \int d^2 x \left[ \frac{1}{2M} |(D_1 + iD_2)\psi|^2 + \frac{1}{2M} |(D_1 + iD_2)\phi|^2 + \frac{eB^{\text{ext}}}{2M} (\rho_1 + \rho_2) \right. \\
+ V - \frac{1}{2M} (\rho_1 B_1 + \rho_2 B_2) - \nabla \times \vec{J}_1 - \nabla \times \vec{J}_2 \right]. \tag{10}
\]

If we neglect boundary terms and take into account Gauss’ laws \[8\] the second line will vanish identically. The term \( \rho_1 + \rho_2 \) in the first line can be integrated out if we impose an extra constraint of definite number of particles \( N \). Thus we are left with the Hamiltonian

\[
H = \frac{1}{2} \omega_c N + \int d^2 x \left[ \frac{1}{2M} |D_+ \psi|^2 + \frac{1}{2M} |D_+ \phi|^2 \right], \tag{11}
\]

where \( \omega_c = \frac{eB^{\text{ext}}}{M} \) is a cyclotron frequency. Minimal energy static solutions satisfy self-dual equations

\[
D_+ \psi \equiv \partial_+ \psi - ia_+^{(1)} \psi + ieA_+^{\text{ext}} \psi = 0, \quad D_+ \phi = 0. \tag{12}
\]
in addition to Gauss’ laws \( \mathbb{F} \). Solutions to these equations are also solutions of full Euler-Lagrange equations of the model only if Lagrange multipliers are equal to

\[
\alpha_0^{(1)} = -\alpha \rho_1 - \gamma \rho_2, \quad \alpha_0^{(2)} = -\gamma \rho_1 - \beta \rho_2. \tag{13}
\]

For a solution with constant densities \( \rho_1, \rho_2 \) the phases of scalar fields can be put constant and Eqs.\( \mathbb{F} \) imply that

\[
B_1 = eB^{\text{ext}}, \quad B_2 = eB^{\text{ext}}. \tag{14}
\]

These and the constraints \( \mathbb{F} \) lead to the following values of uniform condensates in the nondegenerate case

\[
\rho_0^1 = \frac{1}{2} eB^{\text{ext}} \frac{\beta - \gamma}{\alpha \beta - \gamma^2}, \quad \rho_0^2 = \frac{1}{2} eB^{\text{ext}} \frac{\alpha - \gamma}{\alpha \beta - \gamma^2}. \tag{15}
\]

For positive value of \( B^{\text{ext}} \), which we take for definiteness, the nonzero condensates exist if and only if \( \alpha - \gamma, \beta - \gamma \) and \( \alpha \beta - \gamma^2 \) are simultaneously positive or negative. The total electronic filling factor in this case is

\[
\nu^{(0)} = \frac{2 \pi (\rho_1 + \rho_2)}{eB^{\text{ext}}} = \frac{\pi (\alpha + \beta - 2 \gamma)}{\alpha \beta - \gamma^2} = \frac{k + l - 2 m}{kl - m^2}, \tag{16}
\]

Thus for e.g. \((k, l, m) = (3, 3, 1)\) we have the experimentally observed filling factor \( \frac{1}{3} \).

In the degenerate case there is a continuous family of constant solutions characterised by a constraint

\[
\rho \equiv \rho_1 + \rho_2 = \frac{eB^{\text{ext}}}{2 \gamma}, \tag{17}
\]

only if in addition \( \alpha = \beta = \gamma \). The filling factor is equal to \( \nu^{(0)} = \frac{1}{m} \) with \( m \) being an odd integer. A free parameter in the solution signals existence of a zero mode.

Thanks to the form of Coulomb potential all the uniform solutions \( \mathbb{F} \) have zero electrostatic energy. On the other hand contrary to the assumption in \( \mathbb{F} \) one could think it to be more natural that the positive charge density of the background is constant everywhere and in particular the same for the two layers \( \rho^0 = \frac{1}{2} (\rho^0_1 + \rho^0_2) \). In such a case the Coulomb potentials should be taken as

\[
\begin{align*}
\Delta \hat{V}_{11}^C &= \frac{e^2}{2 \varepsilon} \int d^2 x \, d^2 y \left[ \psi^* \psi(x) - \rho^0 \right] \frac{1}{|x - y|} \left[ \psi^* \psi(y) - \rho^0 \right], \\
\Delta \hat{V}_{12}^C &= \frac{e^2}{\varepsilon} \int d^2 x \, d^2 y \left[ \psi^* \psi(x) - \rho^0 \right] \frac{1}{\sqrt{(x - y)^2 + d^2}} \left[ \phi^* \phi(y) - \rho^0 \right], \\
\Delta \hat{V}_{22}^C &= \frac{e^2}{2 \varepsilon} \int d^2 x \, d^2 y \left[ \phi^* \phi(x) - \rho^0 \right] \frac{1}{|x - y|} \left[ \phi^* \phi(y) - \rho^0 \right].
\end{align*} \tag{18}
\]
With this background any constant solution such that $\rho_1 \neq \rho_2$ would have infinite energy. Thus the only physical states would be those with $\alpha = \beta$ or in other words one could observe only states of the type $(k, k, m)$, which characterise by filling factors $\nu^{(0)} = \frac{1}{s}$, where $s$ is any integer (even or odd). In the degenerate case there would be no longer any free parameter but rather the separate charge densities would be locked at $\rho_1 = \rho_2 = \frac{eB_{ext}}{4\gamma}$. Thus the zero mode can be expected to become massive thanks to Coulomb energy as it was observed experimentally [5].

Now let us take into account a possibility of nonuniform vortex solutions. Let us write the scalar fields in the form $\psi = \sqrt{\rho_1} \exp i\omega_1$, $\phi = \sqrt{\rho_2} \exp i\omega_2$. The self-dual equations (12) combined together with Gauss’ laws (8) lead to the following modified Toda equations

$$\frac{1}{2} \nabla^2 \ln \rho_1 = 2\alpha \rho_1 + 2\gamma \rho_2 - eB_{ext} + \varepsilon_{mn} \partial_m \partial_n \omega_1 ,$$

$$\frac{1}{2} \nabla^2 \ln \rho_2 = 2\gamma \rho_1 + 2\beta \rho_2 - eB_{ext} + \varepsilon_{mn} \partial_m \partial_n \omega_2 , \quad (19)$$

The phases $\omega_I$ are single-valued except of finite sets of singular points. Outside of these points one can neglect the last terms in above equations. It is easy to see that if we impose a condition $\rho_2 = \frac{\alpha - \gamma}{\beta - \gamma} \rho_1$ inspired by the uniform solutions (15) we will obtain an equation

$$\frac{1}{2} \nabla^2 \ln \rho_1 = \frac{2(\alpha \beta - \gamma^2)}{\beta - \gamma} \rho_1 - eB_{ext} , \quad (20)$$

which is known to possess multivortex solutions. Thus in any case there are static multivortex solutions with vortices with winding numbers $(1,1)$ located at arbitrary points of the plane. We will call them hybrid anyons. Now the question is whether we can also expect solutions with separate vortices of the types $(1,0)$ and $(0,1)$. Clearly if we put artificially $\gamma = 0$ then eqs.(13) decouple from each other and become similar to (20), so in this limit there exist such isolated vortices. The opposite limit of $\alpha = \beta = 0$ and $\gamma \neq 0$ characterises by existence of only hybrid $(1,1)$ vortices as physical excitations as was shown in [7]. In a more general case we have to analyse carefully asymptotics of such potentially interesting solutions. We substitute to Eqs.(19) following forms $\rho_I = \rho_I^0 (1 + f_I)$ and linearise them with respect to $f_I$’s

$$(\alpha \beta - \gamma^2) \nabla^2 f_1 = 2\alpha (\beta - \gamma) f_1 + 2\gamma (\alpha - \gamma) f_2 ,$$

$$(\alpha \beta - \gamma^2) \nabla^2 f_2 = 2\gamma (\beta - \gamma) f_1 + 2\beta (\alpha - \gamma) f_2 , \quad (21)$$

Of course there is a solution with $f_1 = f_2$ which corresponds to the exact solution with only hybrid vortices. To look for a more general solution let us take $f_2 = f_1 + u$:

$$\nabla^2 u = \frac{2(\alpha - \gamma)(\beta - \gamma)}{\alpha \beta - \gamma^2} u ,$$

5
\( \nabla^2 f_1 - 2f_1 = \frac{2\gamma (\alpha - \gamma)}{\alpha \beta - \gamma^2} u \),

(22)

The first equation is an eigenvalue problem which is solved by an asymptotics of Bessel or modified Bessel function dependent on whether the eigenvalue on R.H.S. is negative or positive respectively. Its sign is the same as the sign of its denominator \( \alpha \beta - \gamma^2 \). If this determinant is negative then both \( f_1 \) and \( f_2 \) have an admixture of slowly vanishing solution

\[
\begin{align*}
  f_1 &= f - \frac{\alpha - \gamma}{\alpha + \beta - 2\gamma} u \\
  f_2 &= f + \frac{\beta - \gamma}{\alpha + \beta - 2\gamma} u ,
\end{align*}
\]

(23)

where \( f \) is a general exponentially vanishing solution of uniform equation \( \nabla^2 f - 2f = 0 \). The charge density fluctuations due to \( u \) are

\[
\delta \rho_1 = -\frac{1}{2} e B_{ext} \frac{(\alpha - \gamma)(\beta - \gamma)}{(\alpha \beta - \gamma^2)(\alpha + \beta - 2\gamma)} u , \quad \delta \rho_2 = -\delta \rho_1 .
\]

(24)

A total electric charge of such a vortex does not have any contribution from these fluctuations since they exactly cancel each other. Nevertheless separate charges on each of the layers are divergent and not well localised. This signals that if we take into account Coulomb interaction such solutions even if substantially distorted can have much higher energy then \( (1,1) \) vortices.

For a positive determinant \( u \) vanishes exponentially and there are no problems with charge normalisation. Thus we can see that the line in the parameter space \( \alpha \beta = \gamma^2 \) is a border between an area where as physical excitations there are only hybrid \( (1,1) \) vortices \( (\alpha \beta < \gamma^2) \) and that in which there is a possibility of finite charge and energy separate \( (1,0) \) and \( (0,1) \) vortices \( (\alpha \beta > \gamma^2) \). In more physical terms if the interaction \( \gamma \) between the two layers is large enough as compared to \( \alpha \) and \( \beta \) vortex can not form only on one of the layers. Only the double-layer \( (1,1) \) composites are allowed.

Similar analysis can be performed at the bifurcation point \( \alpha = \beta = \gamma \). Eqs. (22) are replaced by

\[
\begin{align*}
  \nabla^2 u &= 0 \\
  \nabla^2 f_1 - f_1 &= \frac{1}{2} u .
\end{align*}
\]

(25)

A general solution is given by \( f_{1,2} = f + \frac{1}{2} u_0 \). \( u_0 \) is a constant which is a parameter connected with the zero mode - it only changes asymptotic values keeping \( \rho_1 + \rho_2 \) fixed. This symmetry is broken and the parameter is locked at \( u_0 = 0 \) if we take the Coulomb interaction of the form \( [13] \). \( f \) is an exponentially vanishing asymptotics of modified Bessel function. Thus if \( \alpha = \beta = \gamma \) and both condensates are nonvanishing the smallest quasiparticle excitations are hybrid \( (1,1) \) vortices. This perturbative result is in agreement with conclusion in \( [10] \ Eq.(2.19) \) if we take into account that in their paper boundary conditions are
imposed in such a way that one of the condensates vanishes. Our boundary conditions can be obtained by global SU(2) transformation.

Here we have just analysed properties of eventual single isolated rotationally symmetric (1, 0) or (0, 1) vortex solution. Now we will try to directly compare properties of (1, 1) vortex with a pair of (1, 0) and (0, 1) vortices.

3 Splitting of hybrid vortex into separate (1, 0) and (0, 1) vortices

Keeping in mind the whole area $\alpha \beta < \gamma^2$ we will restrict in this section for a sake of simplicity to the case $\alpha = \beta = 0$ and $\gamma > 0$. Let the unperturbed hybrid vortex be

$$\psi = \phi = F(r) e^{i\theta} .$$

Now we take perturbations of the phases of the scalar field to be $\alpha_1$, $\alpha_2$ and those of the moduli: $F_{h_1}$, $F_{h_2}$.

$$\psi + \delta\psi = F(r)[1 + h_1(r, \theta)] e^{i\theta + i\alpha_1(r, \theta)} ,$$
$$\phi + \delta\phi = F(r)[1 + h_2(r, \theta)] e^{i\theta + i\alpha_2(r, \theta)} .$$

Linearisation of the self-dual equations (12) with respect to perturbations of the scalar fields and those of gauge potentials $c^{(l)}_k$, $I = 1, 2$, yields

$$c_{\theta}^{(l)} = \partial_r h_I - \frac{1}{r} \partial_\theta \alpha_I ,$$
$$c_{r}^{(l)} = -\frac{1}{r} \partial_\theta h_I - \partial_r \alpha_I$$

for $I = 1, 2$. Once the perturbations of the Higgs field are known, $c_{k}^{(l)}$ can be calculated from the above equations. To have a unique solution we have to fix the gauge

$$\partial_k c_k^{(l)} = \partial_r c_r^{(l)} + \frac{1}{r} c_r^{(l)} + \frac{1}{r} \partial_\theta c_\theta^{(l)} = 0 , \; I = 1, 2 .$$

Upon substitution of (28) to this gauge condition we will obtain

$$\nabla^2 \alpha_1 = 0 , \; \nabla^2 \alpha_2 = 0 .$$

Similar substitution of (28) to Gauss’ laws (8) will lead to

$$\nabla^2 h_1 = 4\gamma F^2(r) h_2 , \; \nabla^2 h_2 = 4\gamma F^2(r) h_1 .$$

We would like to describe decay of the hybrid vortex in the direction of $x$-axis on two separate vortices. That is the reason for symmetry requirements on $h_I$

$$h_1(x, y) = h_2(-x, y) , \; h_1(x, y) = h_1(x, -y) .$$
They restrict Fourier transforms with respect to $\theta$ to following forms
\[ h_1(r, \theta) = H_0(r) + H_1(r) \cos \theta + \delta h_1 , \]
\[ h_2(r, \theta) = H_0(r) - H_1(r) \cos \theta + \delta h_2 , \tag{33} \]
where $\delta h_I$ are tails of Fourier seria beginning at terms $\cos 2\theta$ and $\sin 2\theta$. Substitution of these seria to Eqs.(31) leads to equations
\[ \frac{d^2}{dr^2} H_0 + \frac{1}{r} \frac{d}{dr} H_0 = 4\gamma H_0 , \]
\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) H_1 = -4\gamma H_1. \tag{34} \]
Thus $H_0$ is an exponentially vanishing function while an asymptotics of $H_1$ is equal to zero or proportional to the asymptotics of Bessel function and slowly vanishing. But why should $H_1$ be nonzero?

Its asymptotics near $r = 0$ can be read from Eq.(34), namely $H_1(r) \sim \frac{\xi}{r}$. With such an asymptotics the scalar fields for small $r = 0$ and small $\xi$ tend to
\[ \psi \sim re^{i\theta} - \xi , \quad \phi \sim re^{i\theta} + \xi , \tag{35} \]
and zeros are shifted to $\xi$ and $-\xi$ respectively. Thus if $(1,0)$ and $(0,1)$ vortices are already not exactly on top of each other then $\xi \neq 0$ and also $H_{(1)}$ has to be nonzero.

Putting all the above together let us calculate changes of electron numbers
\[ \delta Q_I \equiv \int d^2 x \left[ F^2(r)(1 + h_I)^2 - F^2(r) \right] . \tag{36} \]
With the forms (33) the above integral can be finally rewritten as
\[ \delta Q_I = \int d^2 x F^2(r)[2H_0 + H_0^2] + \int d^2 x F^2(r)[H_1^2 + (\delta h_1)^2] . \tag{37} \]
The first integral is finite. To show that the second one is infinite it is enough to calculate its first term
\[ \int d^2 x F^2(r)H_1^2(r) \sim \text{const } R , \tag{38} \]
where $R$ is a radial cut-off. Thus we can see that $\delta Q_I$’s are badly divergent. This divergence is not an artifact of linearisations we have done since it depends only on the asymptotics of $H_{(1)}$ for large $r$ where the approximations are valid.

In other words the lowest energy hybrid $(1,1)$ vortex can not decay into separate $(1,0)$ and $(0,1)$ vortices because such a decay would violate conservation of the numbers of particles. The decay could be possible only from an excited state of hybrid vortex. A pair of separated $(1,0)$ and $(0,1)$ vortices must be higher energy configuration than a single hybrid vortex. Once again we are lead to conclusion that hybrids will dominate statistics of the system.
4 Condensation of vortices and hierarchy of states

Let us consider the nondegenerate case $\alpha\beta - \gamma^2 \neq 0$. We perform a singular gauge transformation

$$\omega_I \rightarrow \omega_I + \alpha_I, \quad a^{(I)}_{\mu} \rightarrow a^{(I)}_{\mu} + \partial_\mu \alpha_I,$$

where positions of vortices are encoded in singular shifts of phases

$$\alpha_I = \sum_{s_I} \Theta(z - z_{s_I}),$$

on the fields in the Lagrangian

$$L = i\psi^* D_0 \psi - \frac{1}{2M} D_k \psi^* D_k \psi + i\phi^* D_0 \phi - \frac{1}{2M} D_k \phi^* D_k \phi - V(\psi, \phi) - \frac{1}{4X} \varepsilon^{\mu\nu\lambda} a^{(1)}_{\mu} \partial_\nu a^{(1)}_{\lambda} - \frac{1}{4Y} \varepsilon^{\mu\nu\lambda} a^{(2)}_{\mu} \partial_\nu a^{(2)}_{\lambda} - \frac{1}{4Z} \varepsilon^{\mu\nu\lambda} a^{(1)}_{\mu} \partial_\nu a^{(2)}_{\lambda} - \frac{1}{4Z} \varepsilon^{\mu\nu\lambda} a^{(2)}_{\mu} \partial_\nu a^{(1)}_{\lambda} - V,$$

Under such a transformation which is intended as an introduction of new vortex degrees of freedom the Lagrangian changes by

$$\Delta L_{\text{vortex}} = -\pi \left( \frac{a^{(1)}_{\mu} K_1^{\mu}}{X} + \frac{a^{(2)}_{\mu} K_2^{\mu}}{Y} + \frac{a^{(2)}_{\mu} K_1^{\mu} + a^{(1)}_{\mu} K_2^{\mu}}{Z} \right)$$

$$- \frac{1}{2\pi} \left( \frac{\partial_\mu a_1 K_1^{\mu}}{X} + \frac{\partial_\mu a_1 K_1^{\mu}}{Y} + \frac{\partial_\mu a_1 K_2^{\mu} + \partial_\mu a_2 K_1^{\mu}}{Z} \right),$$

where vortex currents are

$$K_1^{\mu} = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu \partial_\lambda \alpha_I = \sum_{s_I} \delta^{(2)}(z - z_{s_I}),$$

with the parametrisation of vortex positions by $x_{s_i}^\mu = (t, x_k^i)$. With this form of the vortex current one can easily rewrite the change in the Lagrangian (42) as

$$\Delta L_{\text{vortex}} = -\pi \left( \frac{a^{(1)}_{\mu} K_1^{\mu}}{X} + \frac{a^{(2)}_{\mu} K_2^{\mu}}{Y} + \frac{a^{(2)}_{\mu} K_1^{\mu} + a^{(1)}_{\mu} K_2^{\mu}}{Z} \right)$$

$$- \pi \left[ \frac{1}{X} \sum_{p_1 > s_1} \hat{\Theta}(z_{p_1} - z_{s_1}) + \frac{1}{Y} \sum_{p_2 > s_2} \hat{\Theta}(z_{p_2} - z_{s_2}) + \frac{1}{Z} \sum_{s_1, s_2} \hat{\Theta}(z_{s_1} - z_{s_2}) \right].$$

The charges of vortices with respect to the C-S fields can be read from the constraint equations

$$\Delta \rho_1 = -\frac{1}{2X} \Delta B_1 - \frac{1}{2Z} \Delta B_2,$$

$$\Delta \rho_2 = -\frac{1}{2Z} \Delta B_1 - \frac{1}{2Y} \Delta B_2,$$
where by $\triangle$ we denote deviations with respect to uniform background. Thus upon integrations and use of quantisation conditions we find that charges of vortices with winding numbers $(p, q)$, if they exist, are given by

$$
q_1[p, q] = \pi \frac{\beta p - \gamma q}{\alpha \beta - \gamma^2}, \\
q_2[p, q] = \pi \frac{\alpha q - \gamma p}{\alpha \beta - \gamma^2},
$$

(46)

while their electric charges are

$$
eQ[p, q] = -e(q_1 + q_2) = -e\pi \frac{(\beta - \gamma)p + (\alpha - \gamma)q}{\alpha \beta - \gamma^2}.
$$

(47)

Let us concentrate on the case $\alpha \beta - \gamma^2 < 0$ in which dominating vortices are those with winding numbers $(1, 1)$ - the hybrid anyons. The vortex currents can be identified

$$
K_{\mu} = K_{\mu}^0 = K^\mu,
$$

and the charges $q_1 = \pi \frac{\beta - \gamma}{\alpha \beta - \gamma^2}$ and $q_2 = \pi \frac{\alpha - \gamma}{\alpha \beta - \gamma^2}$ are positively definite. The vortex part of the Lagrangean can be rewritten as

$$
\triangle L_{\text{vortex}} = -q_1 a_\mu^{(1)} K^\mu - q_2 a_\mu^{(2)} K^\mu + \frac{\hat{\kappa}}{\pi} \sum_{p < q} \dot{\Theta}(z_p - z_q),
$$

(48)

where the statistical parameter $\hat{\kappa}$ is defined modulo integer multiplicity of $2\pi$

$$
\hat{\kappa} = -\pi^2 \left( \frac{\alpha + \beta - 2\gamma}{\alpha \beta - \gamma^2} \right) - 2\pi q.
$$

(49)

Upon variation of the effective Lagrangian with respect to $a^{(1)}_0$ we get a new set of constraint equations

$$
\rho_1 = -\frac{1}{2X} B_1 - \frac{1}{2Z} B_2 + q_1 \rho_{\text{vortex}}, \\
\rho_2 = -\frac{1}{2Z} B_1 - \frac{1}{2Y} B_2 + q_2 \rho_{\text{vortex}}.
$$

(50)

With the identification $K^0 = \rho_{\text{vortex}}$ these equations are to be understood in terms of mean values. For large enough separations of vortices the Lagrangian (48) can be supplemented by kinetic term similarly as in [6]

$$
\triangle H_{\text{vortex}} = \frac{1}{2M_v} \sum_p [p^k_p - q_1 a_k^{(1)} - q_2 a_k^{(2)} + c_k]^2.
$$

(51)

The auxillary vector field is given by $c_k(z) = \frac{\hat{\kappa}}{2} \sum_p \partial_k \Theta(z - z_p)$ or by a constraint

$$
\varepsilon_{mn} \partial_m c_n = 2\hat{\kappa} \rho_{\text{vortex}}.
$$

(52)

It is worthwhile to mention that the form of the Hamiltonian (51) is justified only if separations of vortices are large as compared to sizes of their cores. In the
derivarion vortices were represented by infinitely thin delta-like distributions of magnetic flux and charge. Short range interactions of hybrid (1, 1) anyons were analysed in [7] in the case of only mutual interactions. The conclusions can be easily generalised to this case. The main result is that statistical interactions mediated in (51) by the fields $c_k$ are effectively switched off at short distances. Instead there are short range charge-flux interactions as if vortices were finite-width charged selenoids. They can lead to magnetic trapping and quantum mechanically to scattering resonances characterised by a set of Landau levels.

The Hamiltonian (51) leads to the following field-theoretical description

$$H_{vortex} = \frac{1}{2M_v} | (\partial_k - iq_1a_k^{(1)} - iq_2a_k^{(2)} + ic_k)u |^2 ,$$

The self-dual equation for vortex field $D_+u = 0$ in a uniform state implies that $q_1B_1 + q_2B_2 + \varepsilon_{mnk}c_mc_n = 0$. This and the relations $B_1 = B_2 = eB_{ext}$ lead to the filling factor for vortices

$$\nu_{vortex} \equiv \frac{2\pi\rho_{vortex}}{QB_{ext}} = \frac{\pi}{\kappa} .$$

From equations (50) we get the electronic filling factor

$$\nu^{(1)} = \frac{1}{\xi + \frac{q}{2q}} , \quad \xi = \frac{1}{\nu^{(0)}} .$$

This formula is the same as in the single layer quantised Hall effect [2]. As such it has to be compared with that for the double-layer effect when there are single vortices of the types (1, 0) and (0, 1). Such a formula was derived in the article by Ezawa and Iwazaki [6].

$$\nu^{(1)} = \frac{1}{\xi + \frac{q}{2q}} = \frac{1}{\xi + \frac{p + r}{4q}} ,$$

where $p, r$ are integers and can be replaced by $q = p + r$. The characteristic difference between formulas (55) and (56) is the multiplier 2 instead of 4 when there is dominance by hybrid vortices (1, 1).

5 Conclusions

There are two characteristic areas in the parameter space. If $\alpha\beta > \gamma^2$ vortices with winding numbers (1, 0) and (0, 1) dominate while for $\alpha\beta < \gamma^2$ quite as well as for the bifurcation point $\alpha = \beta = \gamma$ the most important type are (1, 1) hybrids. This dominance by one type or the other can be observed thanks to characteristic changes in Haldane-Halderin hierarchy of states.

The hybrid (1, 1) vortices can be thought of as composites of separate (1, 0) and (0, 1) vortices. There is always present charge-flux interaction between them.
being a remnant of long range statistical Aharonov-Bohm-type interaction. This interaction acts also between whole hybrid vortices and can lead to magnetic trapping and at the quantum level to scattering resonances characterised by Landau levels. If $\alpha \beta < \gamma^2$ then $(1,0)$ and $(0,1)$ vortices are additionally attracted by potential force to form a hybrid vortex.

The distribution of positively charged background may be important for the range of physical states. If the background is the same for the two layers the Coulomb interaction will allow only states of the type $(k, k, m)$ with two charge densities equal one another.

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