PAPER

Topological dynamics and current-induced motion in a skyrmion lattice

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Abstract
We study the Thiele equation for current-induced motion in a skyrmion lattice through two soluble models of the pinning potential. Comprised by a Magnus term, a dissipative term and a pinning force, Thiele’s equation resembles Newton’s law but in virtue of the topological character to the first, it differs significantly from Newtonian mechanics and because the Magnus force is dominant, unlike its mechanical counterpart—the Coriolis force—skyrmion trajectories do not necessarily have mechanical counterparts. This is important if we are to understand skyrmion dynamics and tap into its potential for data-storage technology. We identify a pinning threshold velocity for the one-dimensional pinning potential and for a two-dimensional attractive potential we find a pinning point and the skyrmion trajectories toward that point are spirals whose frequency (compare Kepler’s second law) and amplitude-decay depend only on the Gilbert constant and potential at the pinning point. Other scenarios, e.g. other choices of initial spin velocity, a repulsive potential, etc are also investigated.

1. Introduction
The experimental discovery in 2009 of a hexagonal skyrmion lattice in MnSi under an external vertical magnetic field generated a convergence of efforts to understand better the interplay between the ferromagnetic exchange and Dzyaloshinskii–Moriya couplings in conjunction with crystalline field interactions in B20 compounds, i.e. magnetic materials lacking inversion symmetry (or chiral magnets) [1, 2]. A skyrmion is a planar topological spin texture whose spins are distributed in a circularly symmetric and continuous manner with the spin at the center pointing downward while all spins at the edge are pointing upward. Of particular interest for its application to information-storage technology, specifically the racetrack memory [3], is the effect of current on the magnetic texture since a relatively small current density is able to induce skyrmion motion, thus fueling hopes that ultra-low current densities might be feasible in the manipulation of magnetic structures [4, 5]. But this hope is not without fears since the mechanisms responsible for pinning and current-induced skyrmion motion are presently not well understood [6]. Moreover, experimentally, only very slow translation motion of skyrmions has been observed.

The standard tool of magnetization dynamics is the Landau–Lifshitz–Gilbert (LLG) equation

\[
\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{M} = -\gamma \mathbf{M} \times \mathbf{B}^{\text{eff}} + \frac{\alpha}{M} \mathbf{M} \times \left( \frac{\partial}{\partial t} + \frac{\beta}{\alpha} (\mathbf{v} \cdot \nabla) \right) \mathbf{M}.
\]

where \( \mathbf{B}^{\text{eff}} = \frac{1}{\hbar} \frac{\partial F}{\partial \mathbf{M}} \) is the effective field; \( \gamma = \frac{g \beta}{\hbar} > 0 \) the gyromagnetic ratio, \( \mathbf{v} \) the spin velocity parallel to the spin current, \( \mathbf{F} \) the free energy density, \( \alpha \) the Gilbert damping constant, and \( \beta \) the coupling between the spin-polarized current and local magnetization due to nonadiabatic effects [7, 8]. An immediate consequence of equation (1) is that the magnitude of the magnetization \( M^2 \) is conserved in time. The LLG equation has found application in a variety of systems such as ferromagnets, vortex filaments, and moving space curves and structures such as spin waves and solitons, to name a few [9]. As a time-dependent nonlinear equation, the LLG
We begin with a one-dimensional sinusoidal form for the equation onto the relevant translational modes and in this way obtained an equation that could be regarded as a dynamical force equation but derived from a torque equation. To obtain it, we first assume a steady-state rigid texture $M = M(r - \mathbf{v}_d t)$, $\mathbf{v}_d$ standing for the skyrmion drift velocity. Then we cross multiply equation (1) by $M$, followed by a scalar multiplication of the result by $\frac{1}{M} \partial_r M$ and finally we integrate over the skyrmion area:

$$
\mathbf{G} \times (\mathbf{v}_d - \mathbf{v}_g) + \mathbf{D}(\beta \mathbf{v}_d - \alpha \mathbf{v}_g) = \nabla V,
$$

where $G_k = \frac{1}{4\pi} \delta_{ik} \int d^2r \frac{1}{M} \mathbf{M} \cdot \partial_r \mathbf{M} \times \partial_r \mathbf{M}$, is the dimensionless gyro-coupling vector and $\mathbf{D}_g = \frac{1}{2\pi} \int_{S_{UC}} d^2r \frac{1}{M} \partial_r \mathbf{M} \cdot \partial_r \mathbf{M}$ the dissipative dyadic. The gyro-term can be traced back to the Berry phase and pushes a moving a skyrmion perpendicularly to its direction of motion and is also referred to as the Magnus force [5, 11, 12]. The Magnus force is the counterpart of the Coriolis force in dynamics. The latter is a small correction to the dynamical equations [13], whereas the former, as we will see, dominates the dynamics of our skyrmion system. The Magnus force makes a spinning ball swerve one way as it passes the air; the Coriolis force is a fictitious force due to motion in moving noninertial frame. If we view equation (1) as an equation in the reference frame of the current, it seems more fitting to compare the first term with the Coriolis force. The dissipation term, which sums up the skyrmion’s tendency toward a region of lower energy, originates from Gilbert damping. At the right-hand side we inserted a term due to a potential $V$, which models the pinning potential. Internal details of the skyrmion are ignored. Since the skyrmion is assumed to be perfectly rigid, it is not possible to deduce the pinning forces due to cancellation of forces for such a structure. Pinning is important not only in magnetics but also in superconductivity [14], soliton theory [15] and meteorology. For a skyrmion of winding number $Q = -1$ [1], $G \equiv \mathbf{g} \mathbf{n} = -4\pi \mathbf{n}$, $\mathbf{n}$ being the normal to the thin film and $\mathbf{D}_g = 5.577\pi \delta_{ij}$, $I = j = 1, 2$.

It is obvious that the Thiele equations are already a vast simplification over the original LLG equation. Nevertheless it still is nonlinear, albeit one involving only first-order derivatives. Unlike Newton’s equations of motion, Thiele’s equations are not time-reversal invariant. Moreover, the quantities $g$ and $\mathbf{D}_g$ are of topological origin in contrast with the dynamical parameters entering into Newton’s equations. It is important then to gain familiarity with the Thiele equations if we are to understand current-induced motion in chiral magnets [16]. What is notable about this system is the topological character of the Magnus and dissipative parameters, a significant departure from mechanical systems. In the early 90s ideas about a topological quantum mechanics were in vogue [17]; we might now speak of a topological dynamics for the present system. In this paper, we present two models for which exact solutions of the Thiele equations can be derived. We find that these results are in excellent agreement with numerical results. Our findings allow us to identify key features of the dynamics. Insights from Newtonian mechanics do not necessarily translate into analogous situations for the Thiele case (for instance Kepler’s second law does not hold in one model; in the other model Coriolis deflection occurs without forward motion).

The current-induced dynamics of skyrmions is a subject of much relevance at this time. Some recent works bearing on the topic of particle-like motion of skyrmions moving over pinning sites may overlap with our work and we briefly enumerate some of them. The pinning mechanism was studied by Liu and Li [18] while pinning and creep in chiral magnets was elucidated by Lin et al [19]. The relation between skyrmion and superconducting vortices was explored by Reichhardt et al [20] while Lin et al [21] treated the effects of a current pulse in the creation of a skyrmion. Reichhardt et al explored various aspects of skyrmion dynamics: effects of random quenched disorder [22], ac driven skyrmions over asymmetric quasi-one-dimensional substrates [23] and two-dimensional periodic substrates [24]. The effects of a small hole in a magnetic layer were explained by Muller and Rosch [25]. The creation of skyrmions with polarized current was investigated by Lin et al [26].

## 2. One-dimensional potential

We begin with a one-dimensional sinusoidal form for $V; V = -V_0 \cos(2\pi x/\lambda)$ and assume $\lambda$ much larger than skyrmion size [27]. Let us also assume constant spin current $\mathbf{v}_s$ in the $x$-direction so $\mathbf{v}_y = 0$. With $v_{sx}$ known, the drift velocities $v_x^d, v_y^d$ can be solved from

$$
v_x^d = \frac{g^2 + \alpha \beta D^2}{g^2 + \alpha^2 D^2} v_{sx} + \frac{\alpha D}{g^2 + \alpha^2 D^2} \frac{2\pi}{\lambda} \sin \frac{2\pi x}{\lambda},
$$

$$
v_y^d = \frac{g(\beta - \alpha) D}{g^2 + \alpha^2 D^2} v_{sx} + \frac{g}{g^2 + \alpha^2 D^2} \frac{2\pi}{\lambda} \sin \frac{2\pi x}{\lambda}.
$$

(3a)
It is interesting to see upward motion on the right-hand side, which represents a dissipative term, being proportional to $\alpha$ in the right-hand side of equation (3). Close to the pinning point, this latter is much larger than the Coriolis effect, which requires longitudinal motion in order to observe transverse motion. Making use of the formula in contrast with the Coriolis effect, which requires longitudinal motion in order to observe transverse motion. Both the exact and numerical solutions agree as before. The de

| $X$ | $Y$ |
|-----|-----|
| 0.1 | 3   |
| 0.2 | 2.5 |
| 0.3 | 2.2 |
| 0.4 | 1.9 |
| 0.5 | 1.6 |
| 0.6 | 1.3 |

| $X$ | $T$ |
|-----|-----|
| 0.5 | 2   |
| 1   | 1.5 |
| 1.5 | 1   |
| 2   | 0.5 |

**Figure 1.** (a) Skyrmion trajectory for $v_0 = 0.6$, $V_0 = 10$, $\alpha = 0.1$, $\beta = \alpha/2$ with the origin as starting point: $X = x/\lambda$, $Y = y/\lambda$. Numerical integration gives the same graph. (b) For the exact solution of figure 1(a) we display the position, $X(t)$, $Y(t)$, and velocity $\dot{X}$ as functions of time. For clarity we plot $2\dot{X}(t)$ instead of $X$. See text for discussion.

Since the solutions are translationally invariant, we take, for simplicity, the initial position to be the origin. Making use of the formula $\sqrt{A^2 - 1} \int \frac{dx}{A + \sin x} = 2 \tan^{-1} \frac{1 + A \tan \frac{x}{\sqrt{A^2 - 1}}}{\sqrt{A^2 - 1}}$ we find

$$\tan \frac{\pi x}{\lambda} = \frac{r}{\sqrt{r^2 - 1}} \frac{\sin \frac{\pi}{\lambda} \sqrt{r^2 - 1}}{t - \sin \frac{\pi}{\lambda} \sqrt{r^2 - 1}} \frac{\cos \frac{\pi}{\lambda} \sqrt{r^2 - 1}}{t - \cos \frac{\pi}{\lambda} \sqrt{r^2 - 1}}$$

in which $M = \frac{g(\beta - \alpha)D}{\beta + \alpha D}v_0$, $X = \frac{g(\beta - \alpha)D}{\beta + \alpha D}v_0$, $V = \frac{2\pi}{\lambda} V_0 \frac{\alpha D}{\beta + \alpha D}$, $U = \frac{2\pi}{\lambda} V_0 \frac{g}{\beta + \alpha D}$ have dimensions of velocity whereas the ratio $r = \frac{\sqrt{r^2 - 1}}{t - \sin \frac{\pi}{\lambda} \sqrt{r^2 - 1}}$ is dimensionless. Equation (4) hold when $r > 1$. When $r < 1$, we must make the replacements $\sin \rightarrow \sinh$, $\cos \rightarrow \cosh$ and $\tan \rightarrow \tanh$. Since $-1 < \arctanh \xi < 1$, $x$ has a limit point when $r < 1$. This limiting point does not appear when $r > 1$.

There is another way to look at the case $r = 1$. The drift velocity is positive for all $x$ provided

$$\frac{g + \alpha D}{\beta + \alpha D} V_0 > \frac{2\pi}{\lambda} V_0 \frac{\alpha D}{\beta + \alpha D} \sin \frac{2\pi}{\lambda} \frac{g + \alpha D}{\beta + \alpha D} \geq 0$$

so a threshold spin velocity $v_{\text{pin threshold}}$ is required:

$$v_{\text{pin threshold}} \geq \frac{2\pi}{\lambda} V_0 \frac{\alpha D}{\beta + \alpha D}.$$ The case $r = 1$ corresponds to equality. For this case the second equation of equation (4) still holds but the first is replaced by $\frac{\pi x}{\lambda} = \cot^{-1} \left( \frac{g + \alpha D}{\beta + \alpha D} \right)$; one recognizes that pinning still occurs in this instance.

Figure 1(a) displays the trajectory and figure 1(b) the position-time graphs for spin velocity $v_0 = 0.6$ in the $x$-direction (this corresponds to $r < 1$) and $V_0 = 10$, $\alpha = 0.1$, $\beta = \alpha/2$. We use these latter parameters for figures 1–3. The $r = 1$ case for the parameters given corresponds to $v_0 = 0.690$ 37. The starting point is always the origin. Equation (4) and numerical integration yield the same graphs. The first term of equation (2), which is the Magnus term, shows that the motion along the $y$-direction is due to the gyro-term and is large as comparison of the $X$ and $Y$ displacements on figure 1(b) indicates.

Figure 1(b) shows that the motion along the $x$-direction approaches a fixed or pinning point as the velocity $\dot{X}$ approaches zero asymptotically; whereas there continues to be a drift upward. We can think of the first term on the right-hand side of equation (3a) as the force component of the Magnus force opposed by the second term on the right-hand side, which represents a dissipative term, being proportional to $\alpha D$. At the pinning point, these forces balance each other exactly. For equation (3b) the first term is the dissipative force, being proportional $g(\beta - \alpha)D$, whereas the second term is the force component of the Magnus force in the $y$-direction. Close to the pinning point, this latter is much larger than the first.

Figure 2(a), for spin velocity $v_0 = 0.7$ to the right, corresponds to $r > 1$. There is no pinning in this case. Both the exact and numerical solutions agree as before. The deflection upward by the Magnus force is large as figure 2(b) shows. It is due to the term linear in time in the equation for $Y(t)$ in equation (4); the first term, $UX/V$, is responsible for the small periodic downward dips in the right-hand plot for $Y(t)$. Each of the almost horizontal steps in figure 2(a) corresponds to a crossing into the potential barrier (but there is no tunneling here).

Figures 2(c) and (d) show two instances quite removed from the $r = 1$ case. In the first where $v_0 = 0.01$, the downward line is just the second half of equation (4), viz. $y = \frac{\pi}{\lambda} \frac{\alpha D}{\beta + \alpha D}(x - \frac{\pi}{\lambda} t)$. For the upward drift the velocity is $-\frac{\pi}{\lambda} \frac{\beta - \alpha}{\beta + \alpha D} v_0 > 0$, involves the quantities $g$ and $D$. It is interesting to see upward motion without horizontal motion, in contrast with the Coriolis effect, which requires longitudinal motion in order to observe transverse motion.
displacement. Compared with figure 1(a) the trajectory is here shows sharp change of direction. The flat red line on the x-axis in figure 2(c) shows the trajectory when the Magnus force is set to zero. Clearly the vertical motion is entirely due to the Magnus force. In figure 2(d) with \( v_s = 1.5 \), the downward dips correspond to crossings into the pinning potential. The dips and upward movements are commensurate with each other unlike the case with figure 2(a). For large velocities this trend is maintained.

3. Two-dimensional attractive potential

Consider next the attractive two-dimensional potential

\[
V(x, y) = -U_0 \exp \left(-\frac{|x|}{\lambda} - \frac{|y|}{\lambda}\right).
\]

This potential has cusps along the x- and y-axes. As above we assume a spin current only in the x-direction. The Thiele equations are

\[
-gv_{dx} + \alpha Dv_{dy} = -gv_{sx} + \frac{U_0}{\lambda} e^{\frac{|y|}{\lambda}} \operatorname{sgn}(y),
\]

\[
\alpha Dv_{dx} + gv_{dy} = D\beta v_{sx} + \frac{U_0}{\lambda} e^{\frac{|x|}{\lambda}} \operatorname{sgn}(x).
\]

Let \( \mathcal{N} = \frac{\epsilon^2 + \alpha^2 D^2}{\epsilon^2 + \alpha^2 D^2} v_{sx} \), \( \mathcal{M} = \frac{\epsilon (\beta - \alpha) D}{\epsilon^2 + \alpha^2 D^2} v_{xy} \) and set \( \alpha = \frac{g}{\epsilon^2 + \alpha^2 D^2}, \beta = \frac{\alpha D}{\epsilon^2 + \alpha^2 D^2} \). Let \( X = x/\lambda, Y = y/\lambda \), measure time with the same magnitude as \( \lambda \), and define \( U = U_0/\lambda \). We find

\[
\dot{X} = \mathcal{N} + (\epsilon \operatorname{sgn}(X) - \alpha \operatorname{sgn}(Y))U e^{-i|X| - i|Y|}, \quad \dot{Y} = \mathcal{M} + (\alpha \operatorname{sgn}(X) + \beta \operatorname{sgn}(Y))U e^{-i|X| - i|Y|}.
\]
where $Z$.

With small we can also expect straight trajectories. In regions where this is large, we might expect curved trajectories.

Far from the origin the trajectory is straight. To see this, multiply the line defects. For comparison we have shown in effect of the cusps is evident and occurs only at the coordinate axes. These cusps might be suitable in modeling gradient factor in equation 

Integrating we have

$$Y - Y_0 = \frac{\alpha \text{sgn}(X) + \beta \text{sgn}(Y)}{\beta \text{sgn}(X) - \alpha \text{sgn}(Y)} (X - X_0) + \left( \frac{\alpha \text{sgn}(X) + \beta \text{sgn}(Y)}{\beta \text{sgn}(X) - \alpha \text{sgn}(Y)} \right) t. \quad (8)$$

This holds for a given quadrant where signs of $X$ and $Y$ stay constant ‘during’ integration.

Again from the first of equation (7) we have $\dot{X} \text{sgn}(X) = \lambda X \text{sgn}(X) + (\beta - \alpha \text{sgn}(X) \text{sgn}(Y)) \beta e^{-|X|-|Y|}$ and from the second, $\dot{Y} \text{sgn}(Y) = M \text{sgn}(Y) + (\alpha \text{sgn}(X) \text{sgn}(Y) + \beta) \beta e^{-|X|-|Y|}$; adding and, staying in a fixed quadrant, we obtain $|X| + |Y| = \lambda X \text{sgn}(X) + M \text{sgn}(Y) + 2\beta e^{-|X|-|Y|}$. In all cases below $N > 0 > M$.

With $Z = |X| + |Y|$ we have finally

$$e^Z = e^{Z_0 + (\lambda X \text{sgn}(X) + M \text{sgn}(Y)) t} + \frac{\ell U}{N \text{sgn}(X) + M \text{sgn}(Y)} (e^{\ell U (\lambda X \text{sgn}(X) + M \text{sgn}(Y)) t} - 1), \quad (9)$$

where $Z_0 = |X_0| + |Y_0|$. Equations (8) and (9) are the parametric equations of the trajectory for a given quadrant. They are valid for any quadrant and for any choice of initial velocity in the $x$-direction and initial position. For small times the trajectory is clearly straight. When $\lambda X \text{sgn}(X) + M \text{sgn}(Y)$, which is proportional to $\alpha$, is small we can also expect straight trajectories. In regions where this is large, we might expect curved trajectories.

Far from the origin the trajectory is straight. To see this, multiply the first equation of equation (7) by $Y$ and the second by $X$ and subtract. Far away from the origin we obtain $\frac{d}{dt} \tan^{-1} \frac{Y}{X} \rightarrow \frac{MX - NY}{X^2 + Y^2}$, that is, if $\phi$ is the polar angle, then $\frac{d\phi}{dt} \rightarrow 0$ as $|X|$, $|Y| \rightarrow \infty$.

Figure 3(a) gives the trajectory for $U = 40$ and $v_i = 0.0002$ (to the right) for the starting point $S: (-6, 0)$. The plots of equations (8) and (9) compares well with the Mathematica graph. Although the spin velocity $v_\text{r}$ is to the right, the direction of motion in the beginning (i.e., third) and following (second) quadrants are dictated by the gradient factor in equation (8), $\frac{\text{sgn}(X) + \text{sgn}(Y)}{\text{sgn}(X) - \text{sgn}(Y)} \frac{\text{sgn}(X) + \text{sgn}(Y)}{\alpha \text{sgn}(X) - \text{sgn}(Y)}$. Note that this is a ratio of quantities (of topological origin for the case of $g$). After the motion has entered into the first quadrant, time has now become large ($\sim 10^2$, see figure 3(b)) and the trajectory veers off only to assume a straight path outward to infinity. The effect of the cusps is evident and occurs only at the coordinate axes. These cusps might be suitable in modeling line defects. For comparison we have shown in figure 3(a) the trajectory when the Magnus force is set to zero.
(dotted--dashed curve): the motion is entirely caused by current flow, dissipation and the attractive force. The role of the Magnus force is clearly dominant and defines the trajectory around the origin. In other words it is responsible for circling the potential center.

Figure 3(c) shows that trajectory from the starting point \( S = (−6, −1) \) for \( U = 20 \) and \( v_s = 2 \). As in figure 3(a), the exact and numerical results agree well with each other. The cusps are again evident. What appears striking here is the turn-around trajectory about the center of the potential at the origin. In fact a close-up of the trajectory around the origin, as shown in the inset of figure 3(d), indicates that the motion is very much like the first two parts of figure 3(a): they are straight-line segments whose gradients are given by the ratio \( \frac{\text{sgn}(X) + \alpha \text{Dsgn}(Y)}{\alpha \text{Dsgn}(X) - \text{sgn}(Y)} \), in which \( g \) is of topological origin. Because the turn-around occurs much closer to the source of the potential than in figure 3(a) we see a faster reversal of the motion. At the point \( T \) in figure 3(c), the drift is purely horizontal, i.e. the \( y \)-velocity vanishes. From the second equation of equation (7) we infer that this is where the dissipative (first term) force component is balanced by the Magnus (second) term.

Figure 4 shows other scenarios which indicate that whenever particles approach the origin they are bound to undergo the phenomenon already seen in figure 3: straight-line trajectories whose gradients are given by the ratio of quantities \( g \) and \( D_g \), being of topological origin. The dashed curve in figure 4(a) represents motion with Magnus force set to zero: at \( S \) it is almost perpendicular to the actual trajectory, clearly displaying the distinctive characteristic of the Magnus force. The red dotted--dashed trajectory with starting point at \( S'(1, −1, 8) \) is given to show that the position of the starting point (whether to the left or right of the origin) does not hold special significance in the dynamics. A trajectory that goes around the origin is seen in this case but if \( S' \) were further away from the origin, the particle would only go to the right without circling the origin. In many cases we chose a starting point to the left of the origin simply to take advantage of the fact that this would force the particle to come in close proximity of the potential center.

There is no special role played by the direction of the initial spin velocity. When we have vertical spin velocity \( v_y \) instead of horizontal velocity the same formulas (8) and (9) apply except for the replacement \( N \rightarrow −M, M \rightarrow N \). Figure 4(c) shows the trajectory corresponding to figure 4(b) but in which \( v_y = 0.01 \) while horizontal spin velocity is zero. We see that the trajectory is just a rotation of the trajectory of figure 4(b) by 90°.

4. Two-dimensional repulsive potential

The formulas (8) and (9) apply to the repulsive case as well. We only have to reverse the sign of \( U_0 \). As examples we take figures 4(b) and (c), change the sign of \( U_0 \) and display the results in figure 5. These graphs show the effect of the repulsive force since in the region where the potential is effective, i.e. near the origin, there is a tendency to avoid the origin as figure 5 shows whereas we see a certain tendency toward the origin in figure 4. The dashed trajectories in figure 5 correspond to motion when the Magnus force is set to zero. In both cases we see pinning, i.e. motion comes to a stop at a certain point (the origin in figure 5(a) and the \( y \)-axis in figure 5(b)). We do not see this pinning phenomenon in the attractive case. However, it should be noted that a situation of zero Magnus force does not occur for physical skyrmions.

5. Pinning

The LLG equation for the attractive potential (5) does not have a pinning point even though the origin is clearly a local minimum. This is because of the cusps. To see pinning we take two identical attractive potentials \( V(x, y) \),
one centered at the origin as in equation (5), and another at (3.5, 0). We choose $\mathcal{U} = 2$, $v_{\alpha \epsilon} = 0.03$ and consider the region $3.5 > x > 0$, $y < 0$, where the total potential is perfectly smooth. The Thiele equations take the form

\[
X^\prime = N + (\alpha + \beta) \mathcal{U} e^{-x+y} + (\alpha - \beta) \mathcal{U} e^{-3.5+x+y}, \quad (10a)
\]
\[
Y^\prime = M + (\alpha - \beta) \mathcal{U} e^{-x+y} - (\alpha + \beta) \mathcal{U} e^{-3.5+x+y}. \quad (10b)
\]

At a pinning point $(X_0, Y_0)$ both velocity components vanish: $X_0' = Y_0' = 0$. We can solve algebraically these equations with left-hand sides of equation (10) set to zero and obtain the pinning point:

\[
X_0 = 1.825 \quad \text{and} \quad Y_0 = -0.6133.
\]

Figure 5. The same as figures (b) and (c) but for the repulsive case, i.e. for $\mathcal{U} = -30$. Dashed trajectories correspond to $g = 0$.

These first-order equations can be easily solved exactly. The solutions have the time dependence factor $e^{-\omega t}$, where $\omega \equiv -\beta (W + V) \pm 2i \sqrt{\mathcal{V} W}$, since $\beta \ll 1$. This describes a spiral with frequency $2 \sqrt{\mathcal{V} W}$ and amplitude decays in time through the factor $e^{-\beta \sqrt{W + V} t}$. The first result is a clear departure from Kepler’s $(T^2 \propto r^3)$ second law. From the definition of $\beta$ we infer that the decay only depends on the Gilbert constant $\alpha$, but not on $\beta$. Moreover the frequency depends only on the strength of $\mathcal{V}$ at the pinning point. The result is shown in figure 6 for $\mathcal{U} = 2$, $v_{\alpha \epsilon} = 0.03$. The left-hand graph is obtained from equation (10). The right-hand graph with starting point at $(0.1, 0)$ is obtained by numerical integration. Equations (10) are applicable only in the smooth region $3.5 > x > 0$, $y < 0$. 

Figure 6. With two attractive potentials centered at the origin (0,0) and (3.5,0), it is possible to pin a skyrmion. We choose $\mathcal{U} = 2$, $v_{\alpha \epsilon} = 0.03$. On the right we have the result of equation (10). For comparison, on the left we use numerical integration from the starting point $S = (0.1, 0)$. Note the effect of the cusps at the axes.
Unfortunately application of the above procedure to a similar pair of repulsive potentials or a pair of potentials of opposite strengths did not yield a pinning point. In these cases a point of zero velocity might exist but it was not a stable point. Moreover since we have limited ourselves to soluble models we have not looked into multiple attractive potentials or similar systems.

6. Conclusions

In summary we studied the Thiele equation for current-induced motion in a skyrmion lattice through two soluble models of the pinning potential. Thiele’s equation is composed of a Magnus force, responsible for transverse motion relative to the current velocity, a dissipation force along the current velocity and the pinning force. The first has topological origin whereas the third is imposed externally. In the first, one-dimensional model the Magnus force was found to dominate the dynamics and even transverse motion without corresponding skyrmion motion in the spin direction was possible. We saw a threshold velocity below which motion in the current direction is not allowed and which can be interpreted in terms of balance between the Magnus and the pinning forces. In the second two-dimensional case we saw the occurrence of straight trajectories in which the interplay of the Magnus and dissipative forces and, hence, the topological character of \( g \) are evident. Because of the peculiarities of the model, pinning onto a point was not possible; however with two attractive potentials separated from each other, a pinning point could be found. The trajectory close to the pinning point is a spiral whose frequency and amplitude decay depend only on the Gilbert constant and the strength of the potential at the pinning point. Kepler’s second law did not hold in this system. We did not inquire into the effect of mass which can be a natural point for departure in the future [28]. We also examined cases in which (a) the initial velocity was vertical instead of horizontal as above, (b) or to the right of the potential center instead of to the left and (c) where the potential was repulsive.

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