Dynamical Emergence of a Potts-Nematic Superfluid in a Hexagonal $sp^2$ Optical Lattice

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Nematicity is a long-range orientational order associated with rotation-symmetry breaking in the presence of translational invariance, borne out of the description of classical liquid crystals (I). This order also emerges in interacting electrons and has been found to largely intertwine with multi-orbital correlation in high-temperature superconductivity, where Ising nematicity arises from a four-fold rotation symmetry $C_4$ broken down to $C_2$ (2, 3). Here we report an observation of a bosonic superfluid with a three-state ($\mathbb{Z}_3$) quantum nematic order, dubbed “Potts-nematicity”, in a system of ultracold atoms loaded in an excited band of a hexagonal optical lattice described by an $sp^2$-orbital hybridized model (4, 5). This Potts-nematic superfluid spontaneously breaks a
three-fold rotation symmetry of the lattice, qualitatively distinct from the Ising nematicity. Our field theory analysis shows that the Potts-nematic order is stabilized by intricate renormalization effects enabled by strong inter-orbital mixing present in the hexagonal lattice. This discovery paves a way to investigate quantum vestigial orders (6) in multi-orbital atomic superfluids.

In electronic materials, the existence of nematic order has been established in high temperature superconductors such as cuprates (7) and iron-based superconductors (2, 3, 6, 8). The quantum liquid crystal phase is of great importance to the fundamental understanding of high temperature superconductivity. The investigation of intertwined vestigial orders in multi-orbital superconductivity that incorporates nematicity has been attracting much attention (6) in recent years. In these superconducting materials, an Ising nematic order is most predominantly observed, where the nematic orientation has only two choices. In such systems, what drives the nematic order has ambiguity for it is difficult to separate the electron correlation effects from material structural transitions (3).

The system of ultracold neutral atoms confined in optical lattices has a large degree of controllability. The backaction from atoms to the confining laser potential is typically negligible, making the structural transition avoidable. As an effort to build an optical lattice emulator for multi-orbital physics, excited band condensation of cold atoms has been achieved in one (9, 10) and two-dimensional lattices (11–14). A crucial difference of such condensates from the ground-state condensate is the physics is generically described by a multi-component order parameter that respects crystalline symmetries (4, 5), distinctive from single-component (15) or multi-component spinor condensates (16). At the level of effective field theory description, this atomic system shares similarity as multi-orbital iron-based superconductors and enjoys more controllability. Interaction driven orbital orders such as chiral symmetry breaking (5, 11, 12, 14), and dynamical phase-sliding (10) have been reported in such bosonic superfluids of cold atoms.
But many-body correlation effects beyond mean field theory have not been observed so far in such experimental systems.

Here we report observation of Potts-nematic superfluid in a system of ultracold atoms loaded into the second band of a hexagonal optical lattice. The emergence of this novel phase is not captured by a simple mean field theory. We first prepare an atomic Bose-Einstein condensate (BEC) in the ground band which respects all symmetries of the lattice, and then project the condensate onto the band-maxima of the second band using a lattice quench (see Fig. 1). The phase coherence in the state will immediately disappear and then reemerge within a few milliseconds. During this process of phase-coherence reformation, the quantum state spontaneously chooses one orientation, giving rise to three-state Potts nematicity, which is qualitatively distinct from the commonly observed Ising nematic order in multi-orbital superconductors. In the dynamical evolution, the lifetime of the Potts-nematic superfluid is around 20 ms. The emergence and disappearance of the Potts-nematic order in dynamics are found to coincide with the high-band condensation. Our theory analysis shows that the Potts-nematic superfluid is captured by an orbital-\(sp^2\) (with \(s, p_x,\) and \(p_y\) hybridized) lattice model (see Fig. 1b) (17–19), yet with strong many-body renormalization effects caused by inter-orbital mixing between \(p_x\) and \(p_y\). This effect is absent in the square lattice (20) but unavoidable in the hexagonal lattice, which makes the \(p_x-p_y\) orbital Josephson coupling generically renormalize from the positive to the negative side in our field theory analysis. This work opens up a wide window to explore rich correlated vestigial orders in orbital-mixed atomic superfluids (17–19, 21–24). The Potts-nematic atomic BEC may find potential applications in making non-interferometric gyroscope using its response to external rotation symmetry breaking.

Our experiment is based on a \(^{87}\text{Rb}\) BEC with \(10^5\) atoms in a quasi 2D hexagonal optical lattice, composed of two classes of tube-shaped lattice sites, denoted as \(A\) and \(B\) (see Fig. 1). The lattice potential is formed by three intersecting far-red-detuned laser beams in the x-y plane.
with an enclosing angle of 120°. Each beam is formed by combining two linearly polarized light with polarization directions oriented in the x-y plane (denoted as in-light) and along the z-axis (denoted as out-light), respectively. The in- and out-light form an inversion symmetric honeycomb lattice, and a simple triangular lattice, respectively, whose lattice depths are separately tunable. The out-to-in light intensity ratio is denoted as $\tan^2 \alpha$. The well-depths at $A$ and $B$ sites are made different by aligning two lattices in a way that $A$ ($B$) sites of the honeycomb lattice are enhanced (weakened) by the potential minima (maxima) of the triangular lattice or the other way around, which is controllable by choosing relative phases between the in- and out-light, denoted as $\theta_{1,2,3}$ (Methods). We first adiabatically loaded BEC into the ground band optical lattice. The phase differences are initially set to be $\theta_{1,2,3} = (2\pi/3, 4\pi/3, 0)$, for which $B$ sites are deeper than the $A$ sites. The ground state BEC forms at the $\Gamma$ point, which respects all lattice symmetries. In real space atoms mainly reside in the $s$-orbitals of $B$ sites. We then switch the phase differences rapidly (within 0.1 ms) to the reverse case with $\theta_{1,2,3} = (4\pi/3, 2\pi/3, 0)$, making $A$ sites much lower than $B$. In this way the BEC state is directly projected onto the excited band. By selecting an appropriate $\alpha = 14°$, a second-band population-ratio of 50% is achieved, as measured by band mapping techniques (Fig. 1). In this work, we choose laser intensity such that $s$-orbital of $B$ sites are near resonance with $p_{x,y}$-orbitals of $A$ sites in the final lattice, and consequently the second, third, and forth bands are close-by in energy (Methods).

The quantum tunnelings at the final stage is then described by an $sp^2$-orbital-hybridized model,

$$H_0 = t_{sp} \sum_{r \in B} \sum_{a=1,2,3} \left[ \hat{s}_r^+(\vec{p}_r + \vec{d}_a \cdot \vec{e}_a) + H.c. \right] - \mu_s \sum_{r \in B} \hat{s}_r^+ \hat{s}_r - \mu_p \sum_{r' \in A} \vec{p}_{r'}^+ \cdot \vec{p}_{r'},$$

(1)

Here, $\hat{s}$ and $\hat{p}$ represent quantum mechanical annihilation operators for $s$- and $p$-orbitals, and the shorthand notation $\vec{p} = (\hat{p}_x, \hat{p}_y)$. The unit vectors $\vec{e}_1 = (-1, 0)$, $\vec{e}_2 = (1/2, -\sqrt{3}/2)$, and $\vec{e}_3 = (1/2, \sqrt{3}/2)$ and corresponding $\vec{d}_a = (2\lambda/3\sqrt{3})\vec{e}_a$ mark the relative position between
the two sub-lattices (Fig. 1), with \( \lambda \) the laser wavelength. The many-body quantum effects are modeled by the \( s \)-orbital interaction, \( H_{\text{int},s} = U_s/2 \sum_{r \in A} \hat{s}_{r}^\dagger \hat{s}_{r} \), and the \( p \)-orbital interaction, \( H_{\text{int},p} = \sum_{r \in A} \left\{ J \left[ \hat{p}_{r,x} \hat{p}_{r,y} \hat{p}_{x,r} \hat{p}_{y,r} + H.c. \right] + \frac{1}{2} \sum_{\alpha,\beta \in \{x,y\}} U_{p,\alpha\beta} \hat{p}_{\alpha,r}^\dagger \hat{p}_{\beta,r} \hat{p}_{\beta,r} \hat{p}_{\alpha,r} \right\} . \) (2)

In the language of group theory, \( s \)-orbital transforms according to a one-dimensional representation of the lattice symmetry group \( C_{3v} \) (\( A_1 \)), and \( p \)-orbitals correspond to the two-dimensional representation (E). The \( p \)-orbital couplings are constrained by \( U_{p,xx} = U_{p,yy} \equiv U_p \), \( U_{p,xy} = U_{p,yx} \equiv U_p \perp \), \( J = (U_p \parallel - U_p \perp)/2 \), according to symmetry analysis. By loading the BEC into the excited band in our hexagonal lattice, a quantum many-body system with \( sp^2 \)-orbital hybridization is achieved, which is a versatile platform to host rich physics such as large-gap topological phases (25, 26), exotic orbital frustration (22, 23), and novel carbon structure (27) analogies. 

Right after the lattice switch we have BEC reside symmetrically on the \( \Gamma \) point of the second band. We then hold the system for 5 ms, and take the measurements of momentum distribution of the BEC through time-of-flight (TOF). We repeat the same experiment for 600 times, and then perform statistics on the independently obtained TOF images. The results are shown in Fig. 2. From the averaged signal shown in Fig. 2a, it is evident that the condensate forms at the \( M \) points in the Brillouin zone. The kinetic energy decrease in the lattice is expected to be absorbed by the continuous degrees of freedom along the tube, where interaction effects play an important role. To diagnose the Potts-nematic order, we divide the momentum space into three regions marked as \( \square \), \( \bigcirc \), and \( \triangle \), related to each other by a \( C_3 \) rotation. The total population in these three different regions are denoted as \( n_{\square} \), \( n_{\bigcirc} \), and \( n_{\triangle} \), correspondingly. We define a complex valued Potts nematic contrast (PNC) as

\[
PNC = \frac{n_{\square} + e^{i2\pi/3}n_{\bigcirc} + e^{i4\pi/3}n_{\triangle}}{n_{\square} + n_{\bigcirc} + n_{\triangle}},
\] (3)
which vanishes only when the $C_3$ symmetry is unbroken. When the symmetry is completely broken, PNC takes discrete values from $(1, e^{2i\pi/3}, e^{4i\pi/3})$. The occurrence of PNC collected from consecutive experimental runs (Fig. 2b) explicitly shows that the superfluid state randomly acquires one of the three orientations. We then divide the experimental TOF images into three classes according to their PNC values, and then take the average within each class. The post-classification averaged results are shown in Fig. 2a. From these results, the Bragg-peaks of the momentum distribution form a reciprocal lattice of the hexagonal lattice, which means the lattice translation symmetry remains unbroken. We thus conclude the observed excited-band superfluid has Potts-nematic order.

Since the observed Potts-nematicity occurs in the excited band, it has finite lifetime and eventually decays in the dynamical evolution. In Fig. 3, we show the rise and disappearance of the Potts-nematic order in the quantum dynamics. The observation implies three different stages of dynamical evolution. At the first stage right after the excited band BEC is loaded, the effective mass is negative at the $\Gamma$ point causing strong dynamical instability (15), which immediately (within 1 ms) destroys the phase coherence in the lattice directions. At a second stage atoms start to re-condense in the excited band around several milliseconds after getting excited, and the reformed condensate remains stable up to about 20 ms. The existence of the Potts-nematic order is found to coincide with the excited-band condensation. The intermediate-time nematic order defines two consecutive dynamical phase transitions, whose dynamical criticality is worth further investigation.

To gain insight into the mechanism supporting the Potts-nematic order in the $sp^2$-orbital hybridized superfluid, we provide a mean field theory analysis assuming a plane-wave condensate. Taking a trial condensate wavefunction with $\langle s_r \rangle = \phi_s e^{i k_r r}$, $\langle p_{x,y,r} \rangle = \phi_{x,y} e^{i k_r r}$, with $\phi_s, \phi_{x,y}$ the variational parameters. For each lattice momentum $k$ we minimize the energy by varying $\phi_s, \phi_{x,y}$, and the resultant energy is denoted as $E(k)$ and shown in Fig. 4. With the orbital Josephson
coupling $J > 0$ (Eq. 2), both the kinetic tunnelings and interaction energy favor a condensate at $K$ points which breaks the time-reversal symmetry but respects the rotation symmetry. The corresponding condensate has a $p_x + ip_y$ character as in the square lattice (5, 11). With the Josephson coupling $J < 0$, minimizing the kinetic and the interaction energies meet frustration, as interaction energy then favors $p$-orbital polarization. Once the Josephson coupling is beyond a certain threshold $J < J_c \sim (-t_{sp}) < 0$, the competition between kinetic and interaction energies leads to a condensate at $M$ points, breaking the lattice rotation symmetry. It is worth noting here that at the field theory tree level (28) the Josephson coupling is always positive for repulsive atoms. The observation of the Potts-nematic order in the experiment is thus beyond the simple mean field theory and requires considering renormalization effects (Supplementary Information). We analyze the renormalization under one-loop approximation (Methods), and find that the coupling $J$ generically renormalizes to the negative side in our system due to the strong orbital mixing between $p_x$ and $p_y$, unavoidable in the hexagonal lattice (Fig. 4). The renormalization effects then stabilize the Potts-nematic order. This is in sharp contrast to the chiral $p$-orbital condensate in the square lattice (11, 20), where the physics is captured within a simple mean field theory in absence of $p_x$-$p_y$ orbital mixing.

### Methods

**Lattice construction protocol.** Our experimental setup is a hexagonal lattice composed of two sets of triangular sub-lattices with different well depths, whose lattice sites are denoted as $A$ and $B$, respectively. The lattice potential is formed by three intersecting far-red-detuned laser beams in the x-y plane with an enclosing angle of 120°, as shown in Fig. [1]. Each laser beam is elliptically polarized and is formed by combining two linearly polarized beam whose polarization directions are respectively oriented within x-y plane (denoted as in-light) and along the z-axis (denoted as out-light). Choosing the same intensity for the three laser beams, the in-
light and out-light components generate an inversion symmetric honeycomb lattice and a simple triangular lattice, respectively. Quantitatively, the total optical potential takes a form,

$$V(r) = -V_{\text{out}} \sum_{\langle i,j \rangle} \cos [(k_i - k_j) \cdot r - (\theta_i - \theta_j)] + \frac{1}{2}V_{\text{in}} \sum_{\langle i,j \rangle} \cos [(k_i - k_j) \cdot r] . \quad (4)$$

Here the indices are $i(j) \in \{1, 2, 3\}$, $k_1 = (\sqrt{3} \pi, -\pi)/\lambda$, $k_2 = (-\sqrt{3} \pi, -\pi)/\lambda$, and $k_3 = (0, 2\pi)/\lambda$, and the summation is limited to $\langle 1, 2 \rangle$, $\langle 2, 3 \rangle$, $\langle 3, 1 \rangle$. The laser intensities of the in- and out-light are separately controllable, and the resultant potential strengths are denoted by $V_{\text{in}}$ and $V_{\text{out}}$, whose ratio $\tan^2 \alpha = V_{\text{out}}/V_{\text{in}}$ has been introduced in the main text to describe the relative intensity. The relative phase between in- and out-light $\theta_i$ determines the spatial alignment of the two sets of optical potentials. The well depths of $A$ and $B$ sites are made different by carefully aligning two lattices in a way that $A$ sites of the honeycomb lattice are enhanced by the potential minima of the triangular lattice while $B$ are weakened by the potential barriers, or vice versa. With $\theta_{1,2,3} = (2\pi/3, 4\pi/3, 0)$ the $A$ sites are shallower than $B$ sites and it is reversed with $\theta_{1,2,3} = (4\pi/3, 2\pi/3, 0)$. The fast swap between two configurations can be achieved within 0.1ms. In the experiment, we choose $V_{\text{in}} + V_{\text{out}}$ to be thirty times of photon-recoil-energy, for which $s$-orbitals on $B$-sites are near resonance with $p$-orbitals on $A$ sites in the final lattice configuration. The relative phases and laser intensities are carefully stabilized to avoid lattice potential deformation in the experiment (Supplementary Information).

**Loading and detection procedure.** A BEC of $^{87}$Rb is prepared in a hybrid trap with the harmonic trapping frequencies $(\omega_x, \omega_y, \omega_z) = 2\pi \times (28\text{Hz}, 55\text{Hz}, 60\text{Hz})$. Then the optical lattice is adiabatically ramped up within 80ms. In this stage, $B$ sites are chosen as the deeper ones. After holding for 1ms, we quickly swap the depths of $A$ and $B$, such that $A$ sites suddenly becomes deeper than $B$. By selecting appropriate lattice depths, we are able to load 50% of the atoms to the second band, as shown in Fig.1e. The atomic population in different bands is measured using the standard band mapping technique.
After the swap process, we hold the atomic system up to tens of milliseconds and measure the momentum distribution through time-of-flight (TOF without band mapping). The momentum distribution of the atoms is shown in Fig. 2a. In the first 1.5 ms, the coherence of atoms gradually disappears, and re-emerges within a few milliseconds. More technical details of the experimental platform have been provided in our earlier work (29).

**Mean field theory.** Although the mean field theory is imprecise, it still helps us gain insight about the underlying mechanism at phenomenological level. The energy dispersion of the second band is derived from the $sp^2$ model (Eq. [1]) to be

$$
\epsilon(k) = -\frac{(\mu_s + \mu_p)}{2} - \sqrt{\frac{(\mu_s - \mu_p)^2}{4} + t_{sp}^2 \left[ 3 - \cos(k \cdot a_1) - \cos(k \cdot a_2) - \cos(k \cdot a_3) \right]},
$$

where $a_{1,2,3} = d_{1,2,3} - d_{2,3,1}$, which has band minima at $K$ points of the hexagonal lattice. To incorporate the interaction energy, we take a trial condensate wavefunction with $\langle \hat{s}_r \rangle = \phi_s e^{ikr}$, $\langle \hat{p}_{x,y} \rangle = \phi_{x,y} e^{ikr}$. Minimizing the kinetic energy would lead to a condensate at the $K$ points, and the resultant phase difference between $p_x$ and $p_y$ components is $\pm \pi/2$. The interaction energy (per unit cell) is given by

$$
E_{\text{int}} = U_s |\phi_s|^4/2 + U_p ||(|\phi_x|^4 + |\phi_y|^4)/2 + (U_p - 2J)|\phi_x|^2|\phi_y|^2 + 2J \text{Re}[(\phi_x^* \phi_y)^2]
$$

(6)

When the orbital Josephson coupling $J$ is positive, minimizing $E_{\text{int}}$ leads to $\phi_x = e^{\pm \pi/2} \phi_y$. In this case, minimizing the total energy always produces a chiral condensate at $K$ points (Supplementary Information). When the coupling $J$ is negative, minimizing the interaction leads to a phase difference of 0, or $\pi$ between the two $p$-orbital components, which is inconsistent with chiral condensation at $K$ points. The interaction then makes time-reversal symmetric condensates energetically more favorable. When the interaction energy dominates over the kinetic energy, the time-reversal invariant condensates at $M$ points of the hexagonal lattice becomes
the stable ground state within the \( sp^2 \) model. This requires \( J < J_c < 0 \), and the critical value \( J_c \) is at the order of tunneling \( t_{sp} \) to compensate the kinetic energy cost.

**Field theoretical renormalization effects.** We now explain why the orbital Josephson coupling is negative despite its tree-level estimate (28) is positive for repulsive atoms. We construct the field theoretical action under the standard path integral formalism for the multi-orbital superfluid as

\[
S[\Phi] = \int d\tau dz \sum_{rr'} \Phi^\dagger_r(z, \tau) \left[ \frac{h^2}{2M} \frac{\partial^2}{\partial z^2} + \mathcal{H}_{rr'} \right] \Phi_{r'}(z, \tau) \\
+ \int d\tau dz \frac{1}{2} U_s \sum_{r \in B} \phi^\dagger_{s,r} \phi^\dagger_{s,r} \phi_{s,r} \phi_{s,r} \\
+ \int d\tau dz \sum_{r \in A} \left( J \left[ \phi^\dagger_{x,r} \phi^\dagger_{y,r} \phi_{y,r} \phi_{y,r} + H.c. \right] + \frac{1}{2} \sum_{\alpha, \beta \in \{x,y\}} U_{p,\alpha \beta} \phi^\dagger_{\alpha,r} \phi^\dagger_{\beta,r} \phi_{\beta,r} \phi_{\alpha,r} \right).
\]

Here \( \Phi_r(z, \tau) \) is a compact notation for \([\phi_{x,r}(z, \tau), \phi_{y,r}(z, \tau), \phi_{s,r}(z, \tau)]^T\), which are fluctuating fields associated with annihilation operators \([\hat{p}_{x,r}, \hat{p}_{y,r}, \hat{s}_r]\) in the path integral formalism, and \( \mathcal{H} \) the tunneling matrix given by Eq. (1). We have incorporated the continuous degrees of freedom along the tube (\( z \)-direction) in this field theory. Considering the \( C_{3v} \) symmetry, we have \( U_{p,xx} = U_{p,yy} \equiv U_{p||}, U_{p,xy} = U_{p,yy} \equiv U_{p\perp}, J = (U_{p||} - U_{p\perp})/2 \). Introducing the Fourier components of the fields as \( \phi_{a \in \{x,y,s\},r}(z, \tau) = \int \frac{d^3k d\omega}{(2\pi)^2} \phi_a(k, \omega) e^{i(k_x r_x + k_y r_y + k_z z - \omega \tau)} \), the non-interacting Green functions defined by \( G_{ab}(k, \omega) = \langle \phi^\dagger_a(k, \omega) \phi_b(k, \omega) \rangle \) are given by the Fourier transform of \( \left[ \partial_\tau - \frac{h^2}{2M} \frac{\partial^2}{\partial z^2} + \mathcal{H} \right]^{-1} \), with \( \omega \) the Matsubara frequency. Introducing a running energy scale \( \Lambda \) which is continuously decreased from an initial \( \Lambda_0 \), the couplings in a renormalized mean field theory can be derived by continuously integrating out the high energy modes with momentum \( |k_z| \in [\Lambda - \Delta \Lambda, \Lambda] \) (30). At field theory one-loop level, the
The renormalization of the coupling constants is obtained as,

\[
\frac{\Delta U_s}{\Delta \Lambda} = -[U_s(\Lambda)]^2 I_{ssss}(\Lambda) \tag{8}
\]

\[
\frac{\Delta [U_p + 2J]}{\Delta \Lambda} = -[U_p(\Lambda) + 2J(\Lambda)]^2 [I_{xxxx}(\Lambda) + I_{xyxy}(\Lambda)] \tag{9}
\]

\[
\frac{\Delta U_{p \perp}}{\Delta \Lambda} = -[U_{p \perp}(\Lambda)]^2 [I_{xxxx}(\Lambda) - I_{xyxy}(\Lambda)], \tag{10}
\]

where we have introduced

\[
I_{abcd}(\Lambda) = \frac{1}{\pi} \int \frac{dk_x dk_y d\omega}{(2\pi)^3} G_{ab}(k_x, k_y, \Lambda, \omega) G_{cd}(-k_x, -k_y, -\Lambda, -\omega).
\]

We find that all the three involved integrals \(I_{ssss}, I_{xxxx},\) and \(I_{xyxy}\) are positive (Supplementary Information). It is worth emphasizing here that the term \(I_{xyxy}\) comes from the orbital mixing between \(p_x\) and \(p_y\) mediated by the \(s\) orbital, which vanishes at the limit of \(t_{sp} \to 0\). This orbital mixing makes the \(p\)-orbital condensate in the hexagonal lattice drastically distinctive from that in the square lattice. Keeping the leading terms proportional to \(1/\Lambda^2\) in \(I_{abcd}\) we obtain an invariant in the renormalization,

\[
C_{\text{Renorm}} = \frac{I_{xxxx} + I_{xyxy}}{U_{p \parallel}(\Lambda) - 2J(\Lambda)} - \frac{I_{xxxx} - I_{xyxy}}{U_{p \parallel}(\Lambda) + 2J(\Lambda)}. \tag{11}
\]

With a bare positive coupling \(J(\Lambda_0) > 0\), we have \(C_{\text{Renorm}} > 0\), which implies the running couplings would generically renormalize to a point of

\[
U_{p \parallel} = U_{p \perp} = \frac{2I_{xyxy}}{C_{\text{Renorm}}}, J = 0,
\]

and then flow to the negative side of \(J\). This feature is generic provided that \(I_{xyxy}\) (or equivalently the off-diagonal Green’s function \(G_{xyy}\)) is finite, or in physical words the orbital mixing between \(p_x\) and \(p_y\) is finite. The characteristic renormalization flow is shown in Fig. 4. The renormalization theory explains why the orbital Josephson coupling \(J\) is negative in the renormalized mean field theory, as required by the Potts-nematicity.
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Figure 1: Experimental preparation of the excited band condensate in the hexagonal optical lattice. a, illustrates the arrangement of the laser beams forming the hexagonal lattice. There are three laser beams in the x-y plane with laser-wavelength $\lambda = 1064$ nm forming a bipartite optical hexagonal lattice (see Methods). The three angles $\theta_1$, $\theta_2$, $\theta_3$ represent the relative phases of the elliptical polarization of the light. b, the geometry of the hexagonal lattice. The lattice is formed taking $\{\theta_1, \theta_2, \theta_3\} = \{4\pi/3, 2\pi/3, 0\}$ or $\{2\pi/3, 4\pi/3, 0\}$. The lattice is composed of two sets of triangular sub-lattices with different well depths, whose lattice sites are denoted as $A$ and $B$, respectively. c, The first three Brillouin regions of this bipartite optical hexagonal lattice. d, The time sequence implemented in the experiment to load atoms from the lowest to the second band. e, The measured time-evolution of the atomic population in the ground and first-excited bands normalized by their sum. Here we average over five experimental runs for each data point.
Figure 2: **Potts-nematic superfluid in the hexagonal optical lattice.** 

- **a**, the averaged momentum distribution. We introduce a Potts nematic contrast, $\text{PNC} = \frac{n_\Box + e^{i2\pi/3} n_\bigcirc + e^{i4\pi/3} n_\bigtriangleup}{n_\Box + n_\bigcirc + n_\bigtriangleup}$, where $n_\Box, n_\bigcirc, n_\bigtriangleup$ correspond to momentum distributions in three separate regions as marked in **a**, related to each other by a three-fold lattice rotational symmetry. The middle panel of **a** shows the momentum distribution averaged over 600 experimental images. In the other three panels, we first classify the experimental images into three classes according to the polar angle of the nematic contrast $\text{arg(PNC)} \in (-\pi/3, \pi/3)$, $(\pi/3, \pi)$, or $(\pi, 4\pi/3)$, and then take the average within each class.

- **b**, the statistical occurrence of the nematic contrast. The nematic contrast extracted from the experimental data shows the spontaneous breaking of the three fold lattice rotation symmetry, i.e., the emergence of the Potts nematic order in this atomic superfluid system.
Figure 3: **Time evolution of the Potts-nematic order and dynamical phase transitions.** **a,** Dynamical evolution of momentum distribution. In **a,** we average over the experimental results having a Potts nematic contrast (PNC) with \( \arg(\text{PNC}) \in (-\pi/3, \pi/3) \). **b,** Evolution of the PNC and the condensate fraction. The timepoint we quench the lattice (see Fig. [1]) is set to be 0 in this plot. The condensate in the second band does not immediately form after the quench but instead appears about several milliseconds later. The emergence of Potts-nematic order coincides with the second band condensation. The rise and disappearance of the Potts nematic order define two dynamical phase transitions in the quantum dynamics. Here we average over ten experimental images at each time point.
Figure 4: Theoretical quantum phase transitions with varying the orbital Josephson coupling. The orbital Josephson coupling $J$ is introduced in Eq. (2). a shows the Gross-Pitaevskii energy $\mathcal{E}(k)$ for a plane-wave condensate at a lattice momentum $k$. Here we choose $t_{sp}$ as an energy unit. The chemical potentials are set at $\mu_s/t_{sp} = 0.1$, $\mu_p = 0$, the interaction strengths are $U_s/t_{sp} = U_{p,\parallel} = 0.5$, $J/t_{sp} = 0.4$ and $-0.4$ in a and b, respectively, and $U_{p,\perp}$ is fixed respecting the lattice rotation symmetry. The energy $\mathcal{E}(k)$ has minima at K points in a, and at M points in b. The ground state condensates are chiral and Potts-nematic correspondingly. c, The sketch of the renormalization of the $p$-orbital couplings to low energy. The feature of $J$ renormalizing to the negative side is generic for the hexagonal lattice whose inter-orbital coupling between $p_x$ and $p_y$ is unavoidable (Supplementary Information). This mixing shows up in the off-diagonal term of Green’s function causing intricate one-loop renormalization in field theory analysis, without which the observed Potts nematicity cannot be stabilized. In c, the couplings are in arbitrary units.
S-1 Experimental details

S-1.1 Creation of the controllable hexagonal lattice

The lattice potential is formed by three intersecting red-detuned laser beams in the x-y plane with an enclosing angle of $120^\circ$. Each laser beam is formed by combining two linearly polarized light with polarization directions oriented in the lattice plane (denoted as in-light) and along the z-axis (denoted as out-light), respectively. The electric field experienced by the atoms is given by

$$ E(r, t) = E_{\text{out}} \sum_{j=1,2,3} e_z \cos (k_j \cdot r - \theta_{j,\text{out}} - \omega t) + E_{\text{in}} \sum_{j=1,2,3} (\hat{k}_j \times e_z) \cos (k_j \cdot r - \theta_{j,\text{in}} - \omega t), $$

(S1)

where we have $k_1 = (\sqrt{3} \pi, -\pi)/\lambda$, $k_2 = (-\sqrt{3} \pi, -\pi)/\lambda$, $k_3 = (0, 2\pi)/\lambda$, $\hat{k}_j = k_j/|k_j|$, and $E_{j,\text{in(out)}}$ and $\theta_{j,\text{in(out)}}$ are the electric field amplitude and the phase of the in-light (out-light) of each beam. The corresponding laser intensity is then

$$ I(r) = |E(r)|^2, $$

given as

$$ I(r) = I_0 + I_{\text{out}} \sum_{\langle i,j \rangle} \cos [(k_i - k_j) \cdot r - (\theta_{i,\text{out}} - \theta_{j,\text{out}})] - \frac{1}{2} I_{\text{in}} \sum_{\langle i,j \rangle} \cos [(k_i - k_j) \cdot r - (\theta_{i,\text{in}} - \theta_{j,\text{in}})], $$

(S2)

where $|E(r)|^2$ denotes the time average, $I_0 = 3 (E_{\text{out}}^2 + E_{\text{in}}^2) / 2$, $I_{\text{out(in)}} = E_{\text{out(in)}}^2$ and the summation is limited to $\langle 1, 2 \rangle$, $\langle 2, 3 \rangle$ and $\langle 3, 1 \rangle$. With large red-detuning in our experiment, the resultant optical potential on atoms takes the form

$$ V(r) = -V_{\text{out}} \sum_{\langle i,j \rangle} \cos [(k_i - k_j) \cdot r - (\theta_{i,\text{out}} - \theta_{j,\text{out}})] + \frac{1}{2} V_{\text{in}} \sum_{\langle i,j \rangle} \cos [(k_i - k_j) \cdot r - (\theta_{i,\text{in}} - \theta_{j,\text{in}})]. $$

(S3)
By choosing a convenient set of coordinates, the optical potential further simplifies to

$$
V(r) = -V_{\text{out}} \sum_{\langle i,j \rangle} \cos \left[ (k_i - k_j) \cdot r + (\theta_{j,\text{out}} - \theta_{j,\text{in}}) - (\theta_{i,\text{out}} - \theta_{i,\text{in}}) \right] \\
+ \frac{1}{2} V_{\text{in}} \sum_{\langle i,j \rangle} \cos \left[ (k_i - k_j) \cdot r \right].
$$

(S4)

The $V_{\text{out}}$ and $V_{\text{in}}$ terms correspond to a simple triangular lattice and an inversion symmetric honeycomb lattice, respectively. It is worth remarking here that the alignment of these two only depend on the relative phases between the two polarization directions within each laser beam, which is stabilized using a feedback control in our experiment (Section S-1.2). With this optical potential, the relative position of the two lattices is controllable by tuning the phases $\theta_j \equiv \theta_{j,\text{out}} - \theta_{j,\text{in}}$. Given the cyclic constraint $(\theta_1 - \theta_2) + (\theta_2 - \theta_3) + (\theta_3 - \theta_1) = 0$, we have two independent degrees of freedom from which the two-dimensional relative position between the triangular and the honeycomb lattices is tunable to arbitrary degree. In the experiment we set $\theta_3 = 0$ for simplicity, yet without compromise of the lattice controllability. In order to make $\mathcal{A}$ sites deeper than $\mathcal{B}$, we choose $\{\theta_1, \theta_2, \theta_3\} = \{4\pi/3, 2\pi/3, 0\}$, for which the potential minima (maxima) of the triangular lattice locates at the $\mathcal{A}$ ($\mathcal{B}$) sites of the honeycomb lattice. This is reversed with $\{\theta_1, \theta_2, \theta_3\} = \{2\pi/3, 4\pi/3, 0\}$.

**S-1.2 Feedback stabilization of relative phases**

In the experiment, we use the system shown in Fig. S1 for phase stabilization. For each laser beam, we first split an inclined linearly polarized beam into two components, whose polarization directions are respectively along the x-y plane (in-light) and the z-direction (out-light). The out-light goes along an extra optical path which is controlled by a piezoelectric (PZT) mounted mirror and stabilized with a proportion-integral (PI) controller system, and then combines with the in-light. In this way, an elliptically polarized laser beam with a controllable relative phase is obtained. To reinforce the phase stability, we split out a small fraction of the elliptically polarized
polarized light before it enters the vacuum chamber, from which the relative phase is measured. The phase error is collected in real time for a feedback control on the PZT that controls the extra optical path added to the out-light. This forms a feedback loop for relative phase stabilization. With this feedback control, the relative phases \( \theta_{1,2,3} \) are highly controllable and fast switch of the phases is achieved in a reliable way. To verify this feedback control, we also directly measure the polarization of the dominant fraction of light before it enters the chamber, which confirms that the phase fluctuation is suppressed down to a level below \( \pi/200 \). With this experimental setup the phase-switching can be reached within 100 \( \mu s \).

To confirm the residual phase fluctuation is tolerable, we also carry out the experiment setting \( \theta_1 \) away from \( 2\pi/3 \) by amount of \( \pi/30 \), where we find the three nematic states still emerge. This means the phase stabilization achieved in the experiment is sufficient for the study of Potts nematic order.

**S-1.3 Population of the first excited band**

To excite atoms from the ground to the excited band, we adapt the swapping technique developed for the bipartite square lattice \((11)\) to our hexagonal lattice. The Bose-Einstein condensate is firstly loaded into the ground band of the lattice with \( B \) sites much deeper than \( A \), such that atoms mainly reside on the \( B \) sites. Then \( A \) and \( B \) sites are rapidly swapped to a final configuration and \( A \) sites become the deeper ones. In order to maximize the efficiency of this high-band loading protocol, we carefully choose a combination of lattice depth \( V_0 = (V_{\text{in}} + V_{\text{out}}) \) and out-to-in intensity ratio \( \tan^2 \alpha \) such that \( s \)-orbital of \( B \) sites are near resonance with \( p_x,p_y \)-orbitals of \( A \) sites in the final lattice. Quantitatively, we optimize the following wavefunction overlap,

\[
\eta(V_0, \alpha) \propto \int_{\text{unit cell}} dx dy \psi_{1,\Gamma}(x, y; V_0, \alpha) \cdot \psi_{2,\Gamma}(x, y; V_0, \alpha),
\]

with \( \psi_{1,\Gamma} \) (\( \psi_{2,\Gamma} \)) the \( \Gamma \) point Bloch function associated with the ground (first excited) band. The dependence of \( \eta \) on \( V_0 \) and \( \alpha \) is shown in Fig. S1b which provides important guidance to
optimize the excited band loading efficiency in our experiment.

Figure S1:  

a, Schematic of the phase control system. An inclined linear polarized beam is incident into a polarization-beam-splitter (PBS) where it splits into an in-light beam (red, building an inversion symmetric honeycomb lattice) and an out-light beam (blue, building a simple triangular lattice). The out-light beam goes through an additional optical path, which is controlled by a piezoelectric (PZT) mounted mirror. After the in- and out-light beams recombine we have an elliptically polarized light. A small fraction of the elliptically polarized light is split out by using a beam splitter (BS) for phase-stabilization purpose. With a half-wave plate and a polarizer, the relative phase $\theta_{1,2,3}$ is reflected by the intensity collected by the photodetector. We utilize a proportion-integral (PI) controller and a Notch-filter to build a feedback control that stabilizes $\theta_{1,2,3}$ to the desired value. 

b, The wavefunction overlap $\eta$ (Eq. (S5)) with different lattice depth and light intensity ratio (parameterized by the angle $\alpha$). At small $\alpha$, the overlap $\eta$ increases for larger $\alpha$. When $\alpha$ reaches a certain critical value $\alpha_c$, the second and third bands touch at the $\Gamma$ point, and $\eta$ then suddenly drops down to zero due to level crossing.

S-1.4 Adjustment and calibration of lattice depth

In order to achieve the three-fold rotation symmetry of the lattice (Fig. 1), we need to enforce the balance of laser-intensities in the three laser beams. In the experiment, we block one of the three laser beams and adjust the other two, which then form a one-dimensional optical lattice. Its lattice depth is precisely determined by measuring the Kapiztka-Dirac effect of the confined cold atoms. In this way, we are able to calibrate optical imperfection in the experiment, and maintain a balance in the laser intensity among the three directions.
Figure S2: **Absence of chiral condensation in the experiment.** *a*, Illustration of the Brillouin zones. Here we introduce a chiral contrast $\chi = \frac{n_\Box - n_\bigcirc}{n_\Box + n_\bigcirc}$, where $n_\Box$ and $n_\bigcirc$ include the number of atoms near the two rotation symmetric $K$ points as shown in *a*. *b*, Statistical occurrence of the chiral contrast in the 600 experimental images whose PNC has been shown in Fig. 2. It is evident that the BEC has the Potts nematic order rather than a chiral order.
To confirm our controllability of the laser-intensity symmetry is sufficient, we also deliberately make the laser intensities a bit asymmetric with a relative difference up to 5%. In such experiments, the three nematic states are still observed. This implies that the residual imperfection potentially existent in the experiment that may affect the laser intensity symmetry is negligible for the study of Potts nematic phase.

S-2 Theoretical analysis

S-2.1 Ruling out the simple mean field theory description

In this supplementary section, we rigorously rule out the possibility of describing the experimental observation using simple mean field theory. From the experimental observation, it is evident that atoms condense at a single lattice momentum in the excited band. In the simple mean field theory treatment, the condensate energy is obtained by replacing the annihilation/creation operators in the Hamiltonian by their expectation values,
\[
\langle \hat{s}_r \rangle = \phi_s e^{i \mathbf{k}_0 \cdot \mathbf{r}},
\]
\[
\langle \hat{\rho}_{x,y,r} \rangle = \phi_{x,y} e^{i \mathbf{k}_0 \cdot \mathbf{r}},
\]
where \( \phi_s, \phi_{x,y}, \) and \( \mathbf{k}_0 \) can be taken as variational parameters to minimize the mean field energy. We can re-parametrize the minimization problem by taking
\[
\begin{align*}
\phi_s &= \sqrt{n + \Delta n_{sp}/2} e^{i (\theta + \theta_{sp}/2)} \\
\phi_x &= \sqrt{n - \Delta n_{sp}/4 + \Delta n_{xy}/2} e^{i (\theta - \theta_{sp}/2 + \theta_{xy}/2)} \\
\phi_y &= \sqrt{n - \Delta n_{sp}/4 - \Delta n_{xy}/2} e^{i (\theta - \theta_{sp}/2 - \theta_{xy}/2)}
\end{align*}
\]

Here \( n \) corresponds to total atom number in one unit cell, \( \Delta n_{sp} \) is the atom number difference between \( s- \) and \( p- \) orbitals, \( \Delta n_{xy} \) is the difference between the two \( p- \) orbitals, \( \theta_{sp} \), and \( \theta_{xy} \) parameterize the relative phase among the three orbitals. The mean field energy \( E_{MF} \) contains interaction \( E_{int} \) and kinetic \( E_K \) terms, i.e., \( E_{MF} = E_{int} + E_K \). We consider a minimization procedure with \( n \) and \( \Delta n_{sp} \) first being fixed. From Eq. (5,6), we find that for any choice of \( n \) and \( \Delta n_{sp} \), both of the kinetic energy \( E_K \) and the interaction energy \( E_{int} \) are minimized by taking \( k_0 \) to be a rotation symmetric \( K \) point, \( \Delta n_{xy} = 0 \), and \( \theta_{xy} = \pi/2 \) for any choice of \( n \) and \( \Delta n_{sp} \), if the orbital Josephson coupling \( J > 0 \). This means the ground state has to be a \( K \) point.
condensate, which contradicts with experimental observation (see Fig. 2 and Fig. S2). And in the simple mean field theory treatment, $J$ is always positive for $^{87}$Rb atoms with repulsive interaction. But the experimental observation corresponds to $M$ point condensation. Therefore we rule out the possibility of using a simple mean field theory to describe our experiment.

S-2.2 Field theoretical renormalization

In this supplementary section, we provide details of the field theory renormalization analysis. With the action given in Methods, the partition function of the system reads

$$Z = \int \mathcal{D}\Phi^* D\Phi \exp(-S[\Phi^*, \Phi]).$$

The non-interacting Green function is given by $G(k, \omega) = [-i\omega + \mathcal{H}(k)]^{-1}$, with

$$\mathcal{H} = \begin{bmatrix}
\frac{\hbar^2 k_3^2}{2M} - \mu_p & 0 & \frac{\sqrt{3}}{2} t_{sp} (e^{ik_3} - e^{-ik_2}) \\
0 & \frac{\hbar^2 k_2^2}{2M} - \mu_p & \frac{1}{2} t_{sp} (e^{ik_3} + e^{-ik_2} - 2) \\
\frac{\sqrt{3}}{2} t_{sp} (e^{-ik_3} - e^{ik_2}) & \frac{1}{2} t_{sp} (e^{-ik_3} + e^{ik_2} - 2) & \frac{\hbar^2 k_2^2}{2M} - \mu_s
\end{bmatrix},$$

with $k_3 = k \cdot a_3$, and $k_2 = k \cdot a_2$. Here $M$ is the atomic mass, and $t_{sp}$ is the tunneling between $s$-orbitals on $B$ sites and the nearby $p$-orbitals on $A$ sites.

To proceed we introduce two functions for compactness,

$$h(k_x, k_y) = \sqrt{3} - \cos k_1 - \cos k_2 - \cos k_3,$$

$$\Delta(k_x, k_y) = \sqrt{[\mu_s - \mu_p]^2/4 + [h(k_x, k_y) t_{sp}]^2}.$$

The Green function is diagonalized to be $G(k, \omega) = \mathcal{U} \mathcal{D}(k, \omega) \mathcal{U}^\dagger \mathcal{U}^\dagger$, with

$$\mathcal{U} = \begin{bmatrix}
\frac{(e^{-ik_3} + e^{ik_2} - 2)}{2 h(k_x, k_y)} & \frac{\sqrt{3}(e^{ik_3} - e^{-ik_2})}{2} & 0 \\
\frac{\sqrt{3}(e^{-ik_3} - e^{ik_2})}{2} & \frac{h(k_x, k_y)}{e^{ik_3} + e^{-ik_2} - 2} & 0 \\
0 & 0 & 1
\end{bmatrix},$$

$$\mathcal{U}' = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\vartheta/2) & -\sin(\vartheta/2) \\
0 & \sin(\vartheta/2) & \cos(\vartheta/2)
\end{bmatrix},$$

$$\mathcal{D}(k, \omega) = \text{diag}[f_0(k, \omega), f_+(k, \omega), f_-(k, \omega)].$$

26
Here $\vartheta$ is defined by $\cos(\vartheta) = (\mu_\pm - \mu_\mp)/2\Delta(k_x,k_y)$, and $\sin(\vartheta) = \frac{h(k_x,k_y)p}{\Delta(k_x,k_y)}$, and $f_\gamma = 0,+, - = [-i\omega + \epsilon_\gamma(k)]$, with $\epsilon_0 = \frac{h^2 k^2}{2M} - \mu_p$, and $\epsilon_\pm = \frac{h^2 k^2}{2M} - \frac{\mu_+ + \mu_-}{2} \pm \Delta(k_x,k_y)$. The Green functions $G_{\alpha\beta}(k, \omega) \equiv \langle \phi^\dagger_\alpha(k, \omega) \phi_\beta(k, \omega) \rangle$, are obtained as

\[
\begin{align*}
G_{xx}(k, \omega) &= \sin^2(\varphi)f_0 + \cos^2(\varphi) \left[ \cos^2(\vartheta/2)f_+ + \sin^2(\vartheta/2)f_- \right], \\
G_{yy}(k, \omega) &= \cos^2(\varphi)f_0 + \sin^2(\varphi) \left[ \cos^2(\vartheta/2)f_+ + \sin^2(\vartheta/2)f_- \right], \\
G_{xy}(k, \omega) &= \frac{\sqrt{3} (e^{-ik_1} + e^{ik_2} - 2)(e^{ik_1} - e^{-ik_2})}{4|h(k_x,k_y)|^2} \left[ \cos^2(\vartheta/2)f_+ + \sin^2(\vartheta/2)f_- - f_0 \right], \\
G_{ss}(k, \omega) &= \cos^2(\vartheta/2)f_+ + \sin^2(\vartheta/2)f_+,
\end{align*}
\]

with $\varphi$ defined by $\cos^2(\varphi) = \frac{3|e^{ik_1} - e^{-ik_2}|^2}{4|h(k_x,k_y)|^2}$. The orbital mixing between $p_x$ and $p_y$, $G_{xy}$, is mediated by the $s$-orbital, which vanishes at the limit of $t_{sp} \to 0$.

As we integrate out the high energy modes with momentum $|k_z| \in [\Lambda - \Delta \Lambda, \Lambda]$, the renormalization of couplings among the low-energy modes is determined according to $e^{-S_{\text{eff}}[\Phi^_, \Phi^\text{\textbullet}]} = \int D\Phi^_z D\Phi^\text{\textbullet} e^{-S[\Phi^_, \Phi^\text{\textbullet}]}$, with $\Phi^_z$ ($\Phi^\text{\textbullet}$) referring to high (low) energy modes. Keeping one-loop Feynman diagrams, the renormalization is obtained in terms of kernel integrals

\[
I_{abcd}(\Lambda) = \frac{1}{\pi} \int \frac{dk_x dk_y d\omega}{(2\pi)^3} G_{ab}(k_x, k_y, \Lambda, \omega)G_{cd}(-k_x, -k_y, -\Lambda, -\omega),
\]

as shown in Methods. Here we discuss some several key properties of these integrals. Due to time-reversal symmetry, we have $G(k, \omega) = G^*(\mathbf{-k}, -\omega)$. It follows that these terms

\[
I_{abab}(\Lambda) > 0.
\]  

(S10)

Considering $p$-orbitals form a two-dimensional representation ($E$) of the $C_{3v}$ rotation group, we have

\[
I_{xxxx} = I_{yyyy}, \quad I_{xyyx} = I_{xxxx} - I_{xxyy} - I_{xyyx}.
\]  

(S11)

We have used these symmetries in the derivation which help simplify the form of the renormalization equations. The integral over the frequency $\omega$ can be carried out analytically using
\[
\int \frac{d\omega}{2\pi} f_\gamma(k, \omega) f_{\gamma'}(-k, -\omega) = \frac{\Theta(\epsilon_{\gamma'}(-k)) - \Theta(\epsilon_\gamma(k))}{\epsilon_\gamma(k) + \epsilon_{\gamma'}(-k)}
\]
with \(\Theta\) the heaviside step function. Then we know that the leading order term in the kernel integrals \(I_{abcd}\) scale as \(1/\Lambda^2\) for high-energy modes. Introducing a running scale \(l\) by \(\Lambda = \Lambda_0 l^{-1}\), the leading renormalization of the low-energy couplings is described by a flow equation

\[
\frac{dU_s}{dl} = -\Lambda_0 [U_s(\Lambda)]^2 [I_{ssss}(\Lambda_0)]
\]
\[
\frac{d[U_p|| + 2J]}{dl} = -\Lambda_0 [U_{p||}(\Lambda) + 2J(\Lambda)]^2 [I_{xxxx}(\Lambda_0) + I_{xyxy}(\Lambda_0)]
\]
\[
\frac{dU_{p\perp}}{dl} = -\Lambda_0 [U_{p\perp}(\Lambda)]^2 [I_{xxxx}(\Lambda_0) - I_{xyxy}(\Lambda_0)].
\]

Then we get an invariant in the renormalization,

\[
C_{\text{Renorm}} = \frac{I_{xxxx} + I_{xyxy}}{U_{p||}(\Lambda) - 2J(\Lambda)} - \frac{I_{xxxx} - I_{xyxy}}{U_{p||}(\Lambda) + 2J(\Lambda)}.
\]

With a bare positive coupling \(J(\Lambda_0) > 0\), we have \(C_{\text{Renorm}} > 0\), and the running couplings would always renormalize to a point of \(U_{p||} = U_{p\perp} = \frac{2I_{xyxy}}{C_{\text{Renorm}}}, J = 0\), and then flow to the negative side of \(J\). We thus establish the tendency of the Josephson coupling renormalizing to a negative value with the field theory analysis. This renormalization effect is generic in presence of \(p_x-p_y\) orbital mixing and is absent otherwise.