ON THE ENTROPY OF FLOWS WITH REPARAMETRIZED
GLUING ORBIT PROPERTY

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Abstract. We show that a flow or a semiflow with a weaker reparametrized
form of gluing orbit property is either minimal or of positive topological en-
tropy.

1. Introduction

The gluing orbit property introduced in [7], [5] and [3] is a much weaker variation
of the well-studied specification property. It is satisfied by a larger class of systems
but is still productive, as indicated by a series of recent works (see also [1], [2], [4],
[8] and [9]). A system with gluing orbit property may have zero topological entropy,
which is different from those with specification property. However, it seems that
such a system should be quite simple. In [6], we show that it must be minimal. In
[6] we have made an effort to show that our results holds in both discrete-time and
continuous-time cases. In communication with Paulo Varandas, we realize that such
results should be more practical for flows if a reparametrization is allowed. Then
we have to overcome a bunch of technical difficulties. Finally, we are convinced
that the result extends to a even more general case.

Theorem 1.1. Let \((X, f)\) be a flow or semiflow with weak reparametrized gluing
orbit property. Then either it is minimal, or it has positive topological entropy.

2. Preliminaries

Let \((X, d)\) be a compact metric space. Denote by \(f^t\) a flow or a semiflow on \(X\).

Definition 2.1. For \(L \geq 1\), we call a strictly increasing continuous function \(\gamma : [0, \infty) \to [0, \infty)\) an \(L\)-reparametrization if \(\gamma(0) = 0\) and
\[
L^{-1} \leq \frac{\gamma(t_1) - \gamma(t_2)}{t_1 - t_2} \leq L \text{ for any } t_1, t_2 \in [0, \infty), t_1 \neq t_2.
\]

Definition 2.2. We call the finite sequence of ordered pairs
\[
\mathcal{C} = \{(x_j, m_j) \in X \times [0, \infty) : j = 1, \cdots, k\}
\]
an orbit sequence of rank \(k\). A gap for an orbit sequence of rank \(k\) is a \((k-1)\)-tuple
\[
\mathcal{G} = \{t_j \in [0, \infty) : j = 1, \cdots, k-1\}.
\]

2010 Mathematics Subject Classification. Primary: 37B05, 37B40, 37C50. Secondary: 37B20.

Key words and phrases. flow, gluing orbit property, minimality, topological entropy,
reparametrization.
Let $\gamma$ be a reparametrization. For $\varepsilon > 0$, we say that $(\mathcal{C}, \mathcal{G}, \gamma)$ is be $\varepsilon$-shadowed by $z \in X$ if for every $j = 1, \ldots, k$,

$$(f^{\gamma(s_j+t)}(z), f^t(x_j)) < \varepsilon \text{ for every } t \in [0, m_j],$$

where

$$s_1 = 0 \text{ and } s_j = \sum_{i=1}^{j-1}(m_i + t_i) \text{ for } j = 2, \ldots, k.$$

**Definition 2.3.** We say that $(X, f)$ have reparametrized gluing orbit property, if for every $\varepsilon > 0$ there is $M(\varepsilon) > 0$ such that for any orbit sequence $\mathcal{C}$, there are a gap $\mathcal{G}$ with $\max \mathcal{G} \leq M(\varepsilon)$ and a $(1 + \varepsilon)$-reparametrization $\gamma$ such that $(\mathcal{C}, \mathcal{G}, \gamma)$ can be $\varepsilon$-shadowed.

**Definition 2.3** is the equivalent to the definition of reparametrized gluing orbit property introduced in [1]. It is natural to expect that most results that hold for gluing orbit property also hold for reparametrized gluing, where the reparametrization tends to identity as $\varepsilon$ goes to 0. However, we perceive that our result only requires a weaker condition, where the reparametrization can be more flexible when $\varepsilon$ gets smaller.

**Definition 2.4.** We say that $(X, f)$ has weak reparametrized gluing orbit property if for every $\varepsilon > 0$ there is $M = M(\varepsilon) > 0$ such that for any orbit sequence $\mathcal{C}$, there are a gap $\mathcal{G}$ with $\max \mathcal{G} \leq M$ and an $M$-reparametrization $\gamma$ such that $(\mathcal{C}, \mathcal{G}, \gamma)$ can be $\varepsilon$-shadowed.

It is clear that reparametrized gluing implies weak reparametrized gluing.

### 3. Proof of the Theorem

Throughout this section, we assume that $(X, f)$ has weak reparametrized gluing property and it is not minimal. We shall show that the topological entropy $h(f) > 0$.

**Lemma 3.1.** There are $z \in X$, $\varepsilon > 0$ and $\tau > 0$ such that

$$d(f^t(z), z) \geq \varepsilon \text{ for any } t \geq \tau.$$

**Proof.** As $f$ is not minimal, there is a point whose orbit is not dense. We can find $x, y \in X$ and $\delta > 0$ such that

$$d(f^t(x), y) \geq \delta \text{ for any } t \geq 0.$$

Let $0 < \varepsilon < \frac{1}{3}\delta$ and $M := M(\varepsilon)$. For each $n \in \mathbb{Z}^+$, consider

$$\mathcal{C}_n := \{(y, 0), (x, n)\}.$$

There are $\tau_n \in [0, M]$ and an $M$-reparametrization $\gamma_n$ such that $(\mathcal{C}_n, \{\tau_n\}, \gamma_n)$ is $\varepsilon$-shadowed by $z_n$. This implies that for any $t \geq M^2$ and $n \geq Mt$,

$$d(f^t(z_n), y) = d(f^{\gamma_n(t_n)}(z_n), y) \geq d(f^{\tau_n}(x), y) - d(f^{\gamma_n(t_n)}(z_n), f^{\tau_n}(x)) > 2\varepsilon,$$

where

$$t_n := \gamma_n^{-1}(t) \in [M^{-1}t, Mt] \subset [\tau_n, \tau_n + n].$$

Let $z$ be a subsequential limit of $\{z_n\}$. Then

$$d(f^t(z), y) \geq \liminf_{n \to \infty} d(f^t(z_n), y) \geq 2\varepsilon \text{ for any } t \geq M^2.$$

Note that

$$d(z, y) \leq \liminf_{n \to \infty} d(z_n, y) \leq \varepsilon.$$
So for \( \tau := M^2 \),
\[
d(f^t(z), z) \geq d(f^t(z), y) - d(z, y) \geq \epsilon \text{ for any } t \geq \tau.
\]

\[\square\]

**Lemma 3.2.** There are \( x, y \in X, \epsilon > 0 \) and \( T > 0 \) such that
\[
d(f^t(x), x) \geq \epsilon \text{ for any } t \geq T,
\]
\[
d(f^t(y), x) \geq \epsilon \text{ for any } t \geq T,
\]
\[
d(f^t(y), y) \geq \epsilon \text{ for any } t \geq T, \text{ and}
\]
\[
d(f^t(x), y) \geq \epsilon \text{ for any } t \geq 0.
\]

**Proof.** By Lemma 3.1, there is \( x \in X, \epsilon_0 > 0 \) and \( \tau > 0 \) such that
\[
d(f^t(x), x) \geq \epsilon_0 \text{ for every } t \geq \tau.
\]

Let \( \epsilon_1 := \frac{1}{4} \epsilon_0 \) and \( M := M(\epsilon_1) > 1 \). For each \( n \), there are \( \tau_n \in [0, M] \) and an \( M \)-reparametrization \( \gamma_n \) such that
\[
\{(x, M\tau), (x, n)\}, \{\tau_n\}, \gamma_n
\]
is \( \epsilon_1 \)-shadowed by \( y_n \). Let
\[
T := M(M\tau + M) > M\tau > \tau.
\]

For any \( t \geq T \) and \( n \geq Mt \),
\[
d(f^t(y_n), x) = d(f^{\gamma_n(t_n)}(y_n), x) \geq d(f^{t_n}(x), x) - d(f^{\gamma_n(t_n)}(y_n), f^{t_n}(x)) > 2\epsilon_1,
\]
where
\[
t_n := \gamma_n^{-1}(t) \in [M^{-1}t, Mt] \subset [M\tau + \tau_n, M\tau + \tau_n + n] \text{ and } t_n > \tau.
\]

As \( \tau \leq \gamma_n(M\tau + \tau_n) \leq T \) for each \( n \), there is a subsequence such that
\[
\gamma_{n_k}(M\tau + \tau_{n_k}) \to \tau_0 \geq \tau \text{ as } n_k \to \infty.
\]

Let \( y \) be a subsequential limit of \( \{y_{n_k}\} \). Note that \( d(x, y) < \epsilon_1 \). So we have for any \( t \geq T \):
\[
d(f^t(y), x) \geq \liminf_{n \to \infty} d(f^t(y_n), x) \geq 2\epsilon_1,
\]
\[
d(f^t(y), y) \geq d(f^t(y), x) - d(x, y) \geq \epsilon_1, \text{ and}
\]
\[
d(f^t(x), y) \geq d(f^t(x), x) - d(x, y) \geq 2\epsilon_1. \tag{1}
\]

For any \( t \geq 0 \),
\[
d(f^{T_0}(y), x) \leq \limsup_{n_k \to \infty} d(f^{\gamma_{n_k}(M\tau + \tau_{n_k})}(y_{n_k}), x) \leq \epsilon_1 < \epsilon_0 \leq d(f^{T_0+t}(x), x).
\]

This guarantees that \( f^t(x) \neq y \) for any \( t \geq 0 \). Let
\[
\epsilon := \min\{d(f^t(x), y) : 0 \leq t \leq T\}.
\]

Then \( \epsilon \in (0, \epsilon_1) \). Together with (1) we have
\[
d(f^t(x), y) \geq \epsilon \text{ for every } t \geq 0.
\]

\[\square\]

**Proposition 3.3.** \((X, f)\) has positive topological entropy.
Proof. Let \( x, y \in X, \varepsilon > 0 \) and \( T > 0 \) be as in Lemma 3.2. Let \( 0 < \varepsilon_0 < \frac{1}{3} \varepsilon \) and \( M := M(\varepsilon_0) > 1 \). Denote
\[
T_2 := M^2(M + T) + T \quad \text{and} \quad T_1 := T + M^2(T_2 + M) > T_2.
\]
Let
\[
Q_1 := \{(y, T_1)\} \quad \text{and} \quad Q_2 := \{(x, T_2), (x, T_2)\}.
\]
Let \( n \in \mathbb{Z}^+ \). For each \( \xi = \{\omega_k(\xi)\}_{k=1}^n \in \{1, 2\}^n \), consider
\[
\mathcal{G}_\xi := \{Q_{\omega_k(\xi)} : k = 1, \cdots, n\} = \{(x_j(\xi), m_j(\xi)) : j = 1, \cdots, n(\xi)\},
\]
where
\[
n(\xi) = \sum_{k=1}^n \omega_k(\xi).
\]
There are
\[
\mathcal{G}_\xi = \{t_j(\xi) : j = 1, \cdots, n(\xi) - 1\}
\]
with \( \max \mathcal{G}_\xi \leq M \) and an \( M \)-reparametrization \( \gamma_\xi \) such that \((\mathcal{G}, \mathcal{G})\) is \( \varepsilon_0 \)-shadowed by \( z_\xi \in X \). For each \( \xi \), denote
\[
s_1(\xi) := 0 \quad \text{and} \quad s_j(\xi) := \sum_{i=1}^{j-1} (m_i(\xi) + t_i(\xi)) \quad \text{for} \quad j = 2, \cdots, n(\xi).
\]
Then
\[
s_{n(\xi)}(\xi) < nT_3 \quad \text{for every} \quad \xi \in \{1, 2\}^n,
\]
where
\[
T_3 := \max\{T_1 + M, 2T_2 + 2M\}.
\]
We claim that if \( \xi \neq \xi' \) then there is
\[
s \leq \max\{\gamma_\xi(s_{n(\xi)}(\xi)), \gamma_{\xi'}(s_{n(\xi')} (\xi'))\} < nMT_3
\]
such that
\[
d(f^* (z_\xi), f^* (z_{\xi'})) > \varepsilon_0.
\]
Assume that \( x_j(\xi) = x_j(\xi') \) for \( j = 1, \cdots, l - 1, x_l(\xi) = y \) and \( x_l(\xi') = x \).
For \( j < k \), denote
\[
r_j := \begin{cases} \gamma_{\xi}^{-1}(\gamma_{\xi'}(s_j(\xi'))) - s_j(\xi), & \text{if } \gamma_\xi(s_j(\xi)) \leq \gamma_{\xi'}(s_j(\xi')); \\ \gamma_{\xi'}^{-1}(\gamma_\xi(s_j(\xi))) - s_j(\xi'), & \text{if } \gamma_\xi(s_j(\xi)) > \gamma_{\xi'}(s_j(\xi')). \end{cases}
\]
Our discussion can be split into the following cases.

(1) When \( l = 1 \).
Then
\[
d(z_\xi, z_{\xi'}) \geq d(x, y) - d(z_\xi, x) - d(z_{\xi'}, y) > \varepsilon_2.
\]
We can take \( s = 0 \).

(2) When \( l \geq 2 \) and there is \( k \leq l \) with \( r_k \geq T \).
We may assume that \( k \) is the smallest index with \( r_k \geq T \). As \( r_1 = 0 \), we have \( k \geq 2 \) and hence \( r_{k-1} < T \). We assume that \( \gamma_\xi(s_k(\xi)) \leq \gamma_{\xi'}(s_k(\xi')) \). Argument for the subcase \( \gamma_\xi(s_k(\xi)) > \gamma_{\xi'}(s_k(\xi')) \) is analogous.
(2.1) When $\gamma_l(s_k(\xi)) \leq \gamma_{l'}(s_{k-1}(\xi') + m_{k-1}(\xi'))$.

Note that

$$\gamma_l(s_k(\xi)) \geq \gamma_l(s_{k-1}(\xi) + m_{k-1}(\xi))$$

$$\geq \gamma_l(s_{k-1}(\xi) + T_2)$$

$$\geq \gamma_l(s_{k-1}(\xi) + r_{k-1} + T_2 - T)$$

$$\geq \gamma_l(s_{k-1}(\xi) + r_{k-1} + M^{-1}(T_2 - T)$$

$$> \gamma_{l'}(s_{k-1}(\xi')) + MT$$

$$\geq \gamma_{l'}(s_{k-1}(\xi') + T).$$

There is $r \in (T, m_{k-1}(\xi')]$ such that

$$\gamma_l(s_k(\xi)) = \gamma_{l'}(s_{k-1}(\xi') + r).$$

Then

$$d(f^{\gamma_l(s_k(\xi))}(z_\xi), f^{\gamma_{l'}(s_{k-1}(\xi'))}(z_{\xi'}))$$

$$\geq d(f^r(x_{k-1}(\xi')), x_k(\xi)) - d(f^{\gamma_l(s_k(\xi))}(z_\xi), x_k(\xi))$$

$$- d(f^{\gamma_{l'}(s_{k-1}(\xi'))+r}(z_{\xi'}), f^r(x_{k-1}(\xi')))$$

$$> \varepsilon_0.$$  

We can take $s = \gamma_l(s_k(\xi))$.

(2.2) When $\gamma_l(s_k(\xi)) > \gamma_{l'}(s_{k-1}(\xi') + m_{k-1}(\xi'))$.

We have

$$r_k = \gamma^{-1}_l(\gamma_{l'}(s_{k-1}(\xi')) - s_k(\xi))$$

$$\leq M((\gamma_{l'}(s_{k-1}(\xi') + m_{k-1}(\xi')) - \gamma_l(s_k(\xi)))$$

$$\leq M((\gamma_{l'}(s_{k-1}(\xi') + m_{k-1}(\xi')) + M) - \gamma_{l'}(s_{k-1}(\xi') + m_{k-1}(\xi')))$$

$$\leq M^3 < T_2.$$  

Then $r_k \in (T, T_2)$ implies that

$$d(f^{\gamma_{l'}(s_{k-1}(\xi'))}(z_\xi), f^{\gamma_{l'}(s_{k-1}(\xi'))}(z_{\xi'}))$$

$$\geq d(f^{r_k}(x_k(\xi)), x_k(\xi')) - d(f^{\gamma_{l'}(s_{k-1}(\xi'))+r_k}(z_\xi), f^{r_k}(x_k(\xi)))$$

$$- d(f^{\gamma_{l'}(s_{k-1}(\xi'))}(z_{\xi'}), x_k(\xi'))$$

$$> \varepsilon_0.$$  

We can take $s = \gamma_{l'}(s_k(\xi))$.

(3) When $l \geq 2$ and $r_l < T$.

(3.1) When $\gamma_{l'}(s_{l-1}(\xi')) \leq \gamma_{l'}(s_{l}(\xi))$.

Note that $r_l < T < T_2$. We have

$$d(f^{\gamma_{l'}(s_{l}(\xi))}(z_\xi), f^{\gamma_{l'}(s_{l}(\xi))}(z_{\xi'}))$$

$$\geq d(y, f^{r_l}(x)) - d(f^{\gamma_{l'}(s_{l}(\xi))}(z_\xi), y) - d(f^{\gamma_{l'}(s_{l}(\xi'))+r_l}(z_{\xi'}), f^{r_l}(x))$$

$$> \varepsilon_0.$$  

We can take $s = \gamma_{l'}(s_{l}(\xi))$. 
(3.2) When $\gamma_{\xi}(s_{l}(\xi)) < \gamma_{\xi'}(s_{l}(\xi'))$.

Note that by the definitions of $C_{\xi}$ and $\mathcal{C}_{\xi'}$, we must have

$$x_{i+1}(\xi') = x, m_{i}(\xi') = T_{2} \text{ and } n(\xi') \geq l + 1.$$  

Then in this case we have

$$\gamma_{\xi}(s_{l+1}(\xi')) = \gamma_{\xi}(s_{l}(\xi') + m_{l}(\xi'))$$

$$\geq \gamma_{\xi}(s_{l}(\xi')) + M^{-1}T_{2}$$

$$> \gamma_{\xi}(s_{l}(\xi)) + MT$$

$$\geq \gamma_{\xi}(s_{l}(\xi) + T),$$

$$\gamma_{\xi}(s_{l+1}(\xi')) = \gamma_{\xi}(s_{l}(\xi') + m_{l}(\xi') + t_{l}(\xi'))$$

$$\leq \gamma_{\xi}(s_{l}(\xi')) + M(T_{2} + M)$$

$$= \gamma_{\xi}(s_{l}(\xi) + r_{l}) + M(T_{2} + M)$$

$$\leq \gamma_{\xi}(s_{l}(\xi) + T + M^{2}(T_{2} + M)).$$

$$= \gamma_{\xi}(s_{l}(\xi) + T).$$

So there is $r \in (T, T_{1}]$ such that

$$\gamma_{\xi}(s_{l+1}(\xi')) = \gamma_{\xi}(s_{l}(\xi) + r).$$

This yields that

$$d(f^{r}(\gamma_{\xi}(s_{l+1}(\xi')))(z_{\xi}), f^{r}(\gamma_{\xi}(s_{l}(\xi')))(z_{\xi}))$$

$$\geq d(f^{r}(y), x) - d(f^{r}(\gamma_{\xi}(s_{l}(\xi) + r)(z_{\xi}), f^{r}(y)) - d(f^{r}(\gamma_{\xi}(s_{l+1}(\xi')))(z_{\xi}), x)$$

$$> \varepsilon_{0}.$$  

We can take $s = \gamma_{\xi}(s_{l+1}(\xi')).$  

Above argument shows that

$$E := \{z_{\xi} : \xi \in \{1, 2\}^{n}\}$$

is an $(nMT_{3}, \varepsilon_{0})$-separated subset of $X$ that contains $2^{n}$ points. Hence

$$h(f) \geq \limsup_{n \to \infty} \frac{\ln s(nMT_{3}, \varepsilon_{0})}{nMT_{3}} \geq \limsup_{n \to \infty} \frac{n \ln 2}{nMT_{3}} = \frac{\ln 2}{MT_{3}} > 0.$$  

\[\square\]

Acknowledgments

We would like to thank Paolo Varandas for suggestions and comments. The author is supported by NSFC No. 11571387.

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