Absence of self-averaging and of homogeneity in the large-scale galaxy distribution

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Abstract - The properties of the galaxy distribution at large scales are usually studied using statistics which are assumed to be self-averaging inside a given sample. We present a new analysis able to quantitatively map galaxy large-scale structures while testing for the stability of average statistical quantities in different sample regions. We find that the newest samples of the Sloan Digital Sky Survey provide unambiguous evidence that galaxy structures correspond to large-amplitude density fluctuations at all scales limited only by sample sizes. The two-point correlations properties are self-averaging up to approximately 30 Mpc/h and are characterized by a fractal dimension $D = 2.1 \pm 0.1$. Then at all larger scales probed density fluctuations are too large in amplitude and too extended in space to be self-averaging inside the considered volumes. These inhomogeneities are compatible with a continuation of fractal correlations but incompatible with: i) a homogeneity scale smaller than 100 Mpc/h, ii) predictions of standard theoretical models, iii) mock galaxy catalogs generated from cosmological $N$-body simulations.

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Introduction. - Understanding the large-scale structure of the Universe as mapped by galaxy distribution represents one of the cornerstones of modern cosmology. It provides the basic test for theories of structure formation in the Universe. A primary question in the statistical analysis of three-dimensional galaxy catalogs (where, in addition to the angular coordinates, the redshift is measured and through Hubble’s law [1] the distance of each object) concerns the determination of a scale where the distribution becomes homogeneous. Such a scale $\lambda_0$ can be defined to be the one beyond which counts of galaxies in three-dimensional spherical volumes of radius $r$ grow as $r^3$ [2].

A decade ago, by measuring the conditional density, i.e., the local galaxy density seen by a galaxy in a spherical volume of radius $r$ around itself [2,3], some of us found that galaxy correlations are power law with an exponent $\gamma \approx 1$ up to the sample sizes, i.e., $\sim 30$ Mpc/h$^1$, corresponding to a fractal dimension $D = 3 - \gamma \approx 2$ [4–6]. These results were in contrast with the analysis of the same samples by, e.g., [7–9], who found $\lambda_0 \approx 10$ Mpc/h and $\gamma = 1.8$. The reason for these differences lies in the *a priori* assumption of homogeneity, inside a given sample, of the standard statistical analysis [2–4].

At larger scales, with weaker statistical significance, there was an evidence compatible with the fact that power law correlations in the conditional density extend up to $r \sim 100$ Mpc/h or more [4,10]. These results generated a debate in the field [11–13] because even though galaxy structures were found in many different catalogs to extend to scales of the order of hundreds of megaparsecs, the characteristic length scale $\lambda_0$ statistically describing their correlations was determined to be a few megaparsecs [7–9]. While for some this was a paradox [3–5],

\[H_0 = 100h \text{ km/s/Mpc}, \quad 0.4 \leq h \leq 0.7, \text{ for the Hubble’s constant.}\]
for others [7–9,11,12] the explanation was that large-scale structures have small amplitude relative to the average density. However, this interpretation is problematic as in the range of scales where the conditional density shows power law correlations the sample density is not well defined while density fluctuations have large amplitude [2–4]. The determination of the crossover scale \( \lambda_0 \), where the conditional density from a power law turns to a constant, allowing a meaningful determination of the average density, has been thus an important task of galaxy correlations studies in the last decade [14].

Two new galaxy catalogs, the Sloan Digital Sky Survey (SDSS −15) and the Two-degree Field Galaxy Redshift Survey (2dF) [16], have recently provided great advances in the mapping of the local universe both for the number of objects measured in continuously growing volumes and for the determination of several parameters for each of them. Several studies [17–20] of different samples of these surveys confirmed the small-scale correlations measured by [4,6]. In addition it has been claimed that a slow crossover toward homogeneity occurs [17] with the average conditional density in spheres at \( \sim 20 \text{ Mpc}/\text{h} \) having twice the amplitude of the asymptotic density reached at \( r > 70 \text{ Mpc}/\text{h} \) [18]. It was however noticed that galaxy structures could bias the determination of correlations in these samples introducing uncontrolled systematic effects [19–21]. Recently in the 2dF it has been found that [22,23] galaxy distribution characterized by large-amplitude fluctuations with a large spatial extension, whose size is only limited by the sample’s boundaries. In addition at scales \( r < 40 \text{ Mpc}/\text{h} \), it has been observed a well-defined and statistically stable power law behavior of the average number of galaxies in spheres in agreement with previous determinations.

A complementary method to characterize structures is provided by galaxy counts as a function of the radial distance from us or of the apparent luminosity [24]. These show large fluctuations around the average behavior both in redshift [25] and angular surveys [26,27]. There have been controversies as to whether these are due to real clustering or to incompleteness of the catalogs [11,24]. Recent results support the conclusion that the local galaxy distribution is characterized by large-scale structures with significant correlations on scales \( r > 50 \text{ Mpc}/\text{h} \) [28,29].

In this paper we use a new method which is able to establish the link between the small-scales \( r < 30 \text{ Mpc}/\text{h} \) correlations and the large-scales \( r > 30 \text{ Mpc}/\text{h} \) fluctuations in galaxy counts and which clarifies how the latter influence the determination of the former. Using it, we can test whether sample means, variances and correlations are well defined, i.e., whether they are statistically stable in different sub-volumes of the given sample. By applying this method to the SDSS data [30] we detect large density fluctuations of spatial extension limited by the samples’ sizes. We show that these introduce systematic biases in the determination of correlations.

The data. – We consider the SDSS-DR6 main-galaxy (MG) sample [30] containing redshifts for about 800000 galaxies which covers an area of 7425 square degrees on the celestial sphere. To query the DR6 database we constrain the flags indicating the type of object so that we select only the galaxies from the MG sample. We then consider galaxies in the redshift range \( 10^{-4} < z < 0.3 \). The redshift confidence parameter is constrained to be \( z_{\text{conf}} > 0.35 \) with flags indicating no significant redshift determination errors. In addition we apply the filtering condition \( m_r < 17.77 \), using Petrosian apparent magnitudes in the \( r \) filter which are corrected for galactic absorption, and thus taking into account the target magnitude limit for the MG sample in the DR6 [31]. In this way we have selected 479417 objects. We considered also more stringent limits in apparent magnitude, to test whether a possible incompleteness of the survey at bright and/or faint apparent magnitudes could generate a fake signal. To this aim, we used we have \( 14.5 < m_r < 17.5 \) and we selected 370893 objects, i.e., about 25% less than with less conservative constraints.

We have considered a rectangular angular fields, with uniform coverage, in the SDSS internal angular coordinates (\( \eta, \lambda \)) limited by \(-6\degree < \eta < 36\degree \) and \(-48\degree < \lambda < 32.5\degree \).

To construct volume-limited (VL) samples we have applied a standard procedure [32]. We compute metric distances \( R(z) \) using the standard cosmological parameters \( \Omega_M = 0.3 \) and \( \Omega_\Lambda = 0.7 \). Secondly the galaxy absolute magnitude is determined to be \( M_r = m_r - 5 \log_{10} [R(z) \cdot (1 + z)] - K_r(z) - 25 \), where \( K_r(z) \) is the K-correction. We determine the \( K_r(z) \) term from [33]. Finally, we have considered two different VL samples (named VL1 and VL2) defined by two chosen limits in absolute magnitude and metric distance: for VL1 \( R \in [100, 300] \text{ Mpc}/\text{h}, M \in [-22, -20] \) and for VL2 \( R \in [200, 600] \text{ Mpc}/\text{h}, M \in [-23, -21.5] \). The number of galaxies is about \( 4 \cdot 10^4 \) in VL1 and \( 3 \cdot 10^4 \) in VL2. Different cuts in absolute magnitude do not introduce substantial differences in the results presented in this paper. When more conservative limits in apparent magnitude are applied, the main results presented below are affected only in the fact that statistics is less robust.

The Millennium project [34] has performed several cosmological simulations of standard theoretical models. Amount of dark matter and cosmological parameters are given in agreement with standard models. The dark-matter simulations have about \( 10^{10} \) particles. From these galaxies are identified according to semi-analytics models of galaxy formation [35]. We have cut a sample with exactly the same geometry as the SDSS VL1 sample and a sample close to the geometrical parameters of the SDSS VL2 applying the same absolute magnitude limits in \( r \)-filter as for the SDSS data. In the SDSS we use a redshift space analysis while in mock catalogs a real space one. The difference between the real and redshift space analysis is relevant for very small scales, i.e., \( r < 5 \text{ Mpc}/\text{h} \) [14,22,23].
Statistical methods. – Statistical properties are determined by making averages over the whole sample volume [2]. In so doing, one implicitly assumes that a certain quantity measured in different regions of the sample is statistically stable, i.e., that fluctuations in different sub-regions are described by the same probability density function (PDF). However measurements in different sub-regions may show systematic differences, which depend, for instance, on the spatial position of the specific sub-regions. In this case the considered statistic is not statistically stationary in space, the PDF systematically differs in different sub-regions and its whole-sample average value is not a meaningful descriptor [2].

Such systematic differences may be related to two different possibilities: i) that the underlying distribution is not translationally and/or rotationally invariant, ii) that the volumes considered are not large enough for fluctuations to be self-averaging [36]. On general grounds, we expect the galaxy distribution to satisfy the condition of statistical stationarity in space to avoid special points or directions [1,2]. Hence the question we face in a finite-volume analysis concerns whether it is large enough to obtain statistically stable results. Note that stationary stochastic distributions satisfy the condition of spatial statistical isotropy and homogeneity also when they have zero average density in the infinite-volume limit [2]. This condition is called the Conditional Cosmological Principle [2] to differentiate it from the stronger Cosmological Principle [2] which requires exact homogeneity and deterministic rotational and translation invariance [1,2].

For the case of galaxy surveys there is an intrinsic preferred direction which is set by the radial position from the observer, i.e., the Earth. It is thus necessary to show that statistical quantities do not depend on the radial distance from us. To this aim, in a given sample, a simple approach is to determine the number \( N(r;R) \) of galaxies in spheres of radius \( r \), centered on a galaxy whose distance from the origin is \( R \): we call it the scale length (SL) analysis. This is found to be very efficient in mapping large-scale structures which appear as large fluctuations of \( N(r;R) \). For instance by studying it in various angular slices of the SDSS samples we identify a giant filament covering, in the largest contiguous angular area of the survey, more than 400 Mpc/h at \( R \approx 500 \) Mpc/h. In different sky directions the SL analysis reveals a variety of structures, showing that large density fluctuations are quite typical.

Averaged over the whole sample the quantity \( N(r;R) \) gives an estimate of the average conditional number of galaxies in spheres of radius \( r \). An estimator making the weakest a priori assumptions about the properties of the distribution outside the sample volume is \([2-4]\)

\[
N(r) = \frac{1}{M(r)} \sum_{i=1}^{M(r)} N_i(r), \tag{1}
\]

where \( N_i(r) \) is the number of galaxies seen by the \( i \)-th center-point and the number of centers \( M(r) \) varies with \( r \) because only those galaxies for which the sphere is fully included in the sample volume are considered as centers [2]. Even in this case, there is an intrinsic selection effect related to the geometry of the samples, which are portions of spheres: when \( r \) is large only a part of the sample is explored by the volume average. Hence for large sphere radii \( M(r) \) decreases and the location of the galaxies contributing to the average in eq. (1) is mostly at radial distance \( \sim [R_{\text{min}} + r, R_{\text{max}} - r] \) from the radial boundaries of the sample at \( [R_{\text{min}}, R_{\text{max}}] \).

When eq. (1) scales as \( N(r) \sim r^D \) and \( D = 3 \) the distribution is homogeneous, while for \( D < 3 \) it is fractal [2,24]. Furthermore fluctuations \( \delta^2(r) = \overline{[N(r) - N(r)]^2}/\overline{N(r)} \) are small for a homogeneous distribution with any kind of small-amplitude correlations \( \delta^2(r) \ll 1 \) and large for a fractal one \( \delta^2(r) \sim 1 \) [2,24]. To study fluctuations we determine the PDF of \( N_i(r) \), which is expected to converge to a Gaussian when \( r \gg \lambda_0 \) [2].

Results. – In the VL1 sample, the SL analysis (fig. 1) detects large density fluctuations without a clear radial-distance dependent trend. Correspondingly the PDF has a regular shape characterized by a peak with a long \( N \) tail and it is sufficiently statistically stable in different non-overlapping sub-samples of equal volume. This occurs except for the largest sphere radii, i.e., for \( r > 30 \) Mpc/h, for which the number of independent centers becomes too small.
the two-point correlation function \[ \xi(r) \], can be written as \[ \xi(r) + 1 = \frac{N(r, \Delta r)}{V(r, \Delta r)} \frac{V(r)}{N} \].

The first ratio in the r.h.s. of eq. (2) is the average conditional density, i.e., the number of galaxies in shells of thickness \( \Delta r \) averaged over the whole sample, divided by the volume \( V(r, \Delta r) \) of the shell. The second ratio in the r.h.s. of eq. (2) is the average density estimated in a sample containing \( N \) galaxies and with volume \( V \).

When measuring this function we implicitly assume, in a given sample, that i) fluctuations are self-averaging in different sub-volumes [2], ii) the linear dimension of the sample volume is \( V^{1/3} \gg \lambda_0 [2,3] \), i.e., the distribution has reached homogeneity inside the sample volume. When the latter condition is not verified the \( \xi(r) \) analysis is biased by systematic finite-size effects even if fluctuations are self-averaging [2,3]. To show how non–self-averaging fluctuations inside a given sample bias the \( \xi(r) \) analysis, we consider the estimator

\[
\xi(r; R, \Delta R) + 1 = \frac{N(r, \Delta r)}{V(r, \Delta r)} \frac{V(r^*)}{N(r^*; R, \Delta R)},
\]

where the second ratio on the r.h.s. is the density of points in spheres of radius \( r^* \) averaged over the centers lying in a shell of thickness \( \Delta R \) around the radial distance \( R \). If the distribution is homogeneous, i.e., \( r^* > \lambda_0 \) and statistically stationary, eq. (3) should be statistically independent of the range of radial distances \( (R, \Delta R) \) considered. For instance we consider, in the VL2 sample, \( \Delta R = 40 \text{ Mpc}/h \) and \( R = 240 \text{ Mpc}/h \) or \( R = 520 \text{ Mpc}/h \), with \( r^* > 50 \text{ Mpc}/h \). We thus find large variations in the amplitude of \( \xi(r) \) (fig. 3). This is simply an artifact generated by the large density fluctuations on scales of the order of the sample sizes. The results that the estimator eq. (2) (or others based on pair-counting [2,39]) has nearly the same amplitude in different samples, e.g., [7–9,32,37], despite the large fluctuations of \( N(r; R) \), are simply explained by the fact that \( \xi(r) \) is a ratio between the local conditional density and the sample average density: both vary in the same way when the radial distance is changed and thus the amplitude is nearly constant.

On the other hand, eq. (1), \textit{averaged over the volumes where the PDF has a statistically stable shape}, shows in both considered samples a power law behavior for \( r < 30 \text{ Mpc}/h \) corresponding to a fractal dimension \( D = 2.1 \pm 0.1 \) in agreement with [4,6,17,19,20] (fig. 3). Due to the non–self-averaging nature of fluctuations at larger scales, i.e., due to limited volumes, we are not able to determine correlations for \( r > 30 \text{ Mpc}/h \).

\section*{Discussion}
According to standard models of cosmological structure formation, gravitational clustering gives rise to non-linear perturbations from homogeneous initial conditions in the early universe [1]. If the initial amplitude of fluctuations is normalized to the anisotropies of the Cosmic Microwave Background Radiation (CMBR) [40], then the homogeneity scale is about \( \lambda_0^{n_{\text{LSS}}} = 10 \text{ Mpc}/h \) [21], i.e., twice the value at which \( \xi(r) = 1 \) [1,2].

Indeed in mock galaxy catalogs generated from \( N \)-body simulations of standard cosmological models [34,35], \( N(r; R) \) does not show, for \( r > \lambda_0^{n_{\text{LSS}}} \), large fluctuations or systematic trends as a function of \( R \) (fig. 4). Because
Different selection in luminosity [34]. The case of VL1 contains fainter galaxies than VL2. The amplitude of the mock VL1 and VL2 samples has been rescaled by the same factor for seek of clarity. Bottom panel: standard two-point correlation function in the VL2 sample estimated by eq. (3). The correlation function in the VL2 sample estimated by eq. (3): the flattail of \( R \) is not amplified by gravitational clustering. Therefore for \( r > \lambda_0^m \), the shape of the theoretical \( \xi^m(r) \) must be the same as the initial one [1]. This is characterized by a length scale \( r_c \), where \( \xi^m(r_c) = 0 \), which is fixed by the physics of the early universe and estimated from CMBR anisotropies to be \( r_c \approx 100 \text{ Mpc/h} \) [39,40]. For \( r > r_c \), \( \xi^m(r) \) becomes negative, corresponding to super-homogeneous correlations characterized by the most rapid possible decay of fluctuations [41,42]. This theoretical framework applies to the whole mass distribution, where dark matter is supposed to provide us the main contribution. Galaxies would form on the largest peaks of the density field. Standard models of galaxy formation describe this physical phenomenon as a selection mechanism [43]. This leaves the scale \( r_c \) unperturbed and slightly changes \( \lambda_0^m \) [39,44]. We find \( \lambda_0^m > 100 \text{ Mpc/h} \). This raises a fundamental inconsistency for the relation between galaxy structures and CMBR anisotropies as no physical mechanism is known, which by sampling a super-homogeneous density field transforms it into a strongly inhomogeneous one [2,39,44].

Conclusion. – In summary, by applying the SL analysis to the newest SDSS galaxy samples, we measure large density fluctuations of spatial extension limited by sample sizes. At scales \( r < 30 \text{ Mpc/h} \) we detect statistically stable fractal correlations with \( D = 2.1 \pm 0.1 \). On larger scales, \( r > 30 \text{ Mpc/h} \), we find that the galaxy distribution is strongly inhomogeneous and fluctuations are not self-averaging in the samples considered. This situation is compatible with fractal power law correlations extending to such length scales but incompatible with homogeneity at \( \lambda_0 \leq 100 \text{ Mpc/h} \). Indeed, in a portion of a fractal, large structures are expected to be present at any scale, fluctuations being self-averaging only if the sample volume is large enough [2]. These results have important consequences on the theoretical interpretation of the
large-scale universe, where models, normalized to CMBR anisotropies, predict there is not enough time to form structures with relative density fluctuations larger than unity on scales larger than \( \lambda_0^m \approx 10 \text{ Mpc}/h \) [1,34]. This length scale is more than ten times smaller than our lower limit to \( \lambda_0 \). Indeed the latter is of the order of the scale \( r_w \) where theoretical models predict matter distribution to have negative correlations, a situation which is in contrast with the results from the data analyzed here. Thus the large-scale inhomogeneities detected in the SDSS samples are incompatible with the predictions of standard theoretical models relating the early-universe physics, with CMBR normalization, to structures in the present universe. Moreover we found that for \( r < \lambda_0^m \), mock galaxy catalogs have different correlations from real galaxy data, i.e., \( D = 1.1 \pm 0.1 \) instead of \( D = 2.1 \pm 0.1 \). Thus structures generated by N-body simulations are intrinsically different from observed ones.

Recent weak lensing observations suggest that dark and visible matters trace the same structures [45]. If the total matter density field were inhomogeneous on 100 Mpc/h scales, this would imply either a new kind of evolution scenario within the open Friedmann model [46] or new kinds of spatial averaging of Einstein’s equations [47,48].

The application of the SL analysis to forthcoming galaxy samples should allow determining the nature of galaxy structures on scales \( > 100 \text{ Mpc}/h \).

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