Riemann-Liouville and higher dimensional Harday operators for non-negative decreasing function in $L^{p(x)}$ spaces

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Abstract. In this paper one-weight inequalities with general weights for Riemann-Liouville transform and $n-$ dimensional fractional integral operator in variable exponent Lebesgue spaces defined on $\mathbb{R}^n$ are investigated. In particular, we derive necessary and sufficient conditions governing one-weight inequalities for these operators on the cone of non-negative decreasing functions in $L^{p(x)}$ spaces.

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1. Introduction

We derive necessary and sufficient conditions governing the one-weight inequality for the Riemann-Liouville operator

$$R_\alpha f(x) = \frac{1}{x^\alpha} \int_0^x \frac{f(t)}{(x-t)^1-\alpha}dt \quad 0 < \alpha < 1,$$

and $n-$dimensional fractional integral operator

$$I_\alpha g(x) = \frac{1}{|x|^\alpha} \int_{|y|<|x|} \frac{g(t)}{|x-t|^{n-\alpha}}dt \quad 0 < \alpha < n,$$

on the cone of non-negative decreasing function in $L^{p(x)}$ spaces.

In the last two decades a considerable interest of researchers was attracted to the investigation of the mapping properties of integral operators in so called Nakano spaces $L^{p(\cdot)}$ (see e.g., the monographs [3], [7] and references therein). Mathematical problems related to these spaces arise in applications to mechanics of the continuum medium. For example, M. Ruzička [19] studied the problems in the so called rheological and electrorheological fluids, which lead to spaces with variable exponent.

Weighted estimates for the Hardy transform

$$(Hf)(x) = \int_0^x f(t)dt, \quad x > 0,$$
in $L^p$ spaces were derived in the papers [8] for power-type weights and in [11], [12], [15], [6], [17] for general weights. The Hardy inequality for non-negative decreasing functions was studied in [3], [4].

Weighted problems for the Riemann-Liouville transform in $L^p$ spaces were explored in the papers [10], [11], [2], [14] (see also the monograph [18]).

Historically, one and two weight Hardy inequalities on the cone of non-negative decreasing functions defined on $\mathbb{R}_+$ in the classical Lebesgue spaces were characterized by M. A. Arino and B. Muckenhoupt [1] and E. Sawyer [22] respectively.

It should be emphasized that the operator $I_\alpha f(x)$ is the weighted truncated potential. The trace inequality for this operator in the classical Lebesgue spaces was established by E. Sawyer [21] (see also the monograph [13], Ch.6 for related topics).

In general, the modular inequality

\[
\int_0^1 \left| \int_0^x f(t) dt \right|^{q(x)} v(x) dx \leq c \int_0^1 \left| f(t) \right|^{p(t)} w(t) dt \tag{*}
\]

for the Hardy operator is not valid (see [23], Corollary 2.3, for details). Namely the following fact holds: if there exists a positive constant $c$ such that inequality (*) is true for all $f \geq 0$, where $q$, $p$, $w$ and $v$ are non-negative measurable functions, then there exists $b \in [0, 1]$ such that $w(t) > 0$ for almost every $t < b$; $v(x) = 0$ for almost every $x > b$, and $p(t)$ and $q(x)$ take the same constant values almost everywhere for $t \in (0; b)$ and $x \in (0; b) \cap \{v \neq 0\}$.

To get the main result we use the following pointwise inequities

\[
c_1(Tf)(x) \leq (R_\alpha f)(x) \leq c_2(Tf)(x),
\]
\[
c_3(Hg)(x) \leq (I_\alpha g)(x) \leq c_4(Hg)(x),
\]

for non-negative decreasing functions, where $c_1$, $c_2$, $c_3$ and $c_4$ are constants are independents of $f$, $g$ and $x$, and

\[
Tf(x) = \frac{1}{x} \int_0^x f(t) dt, \quad Hg(x) = \frac{1}{|x|^n} \int_{|y|<|x|} g(y) dy.
\]

In the sequel by the symbol $Tf \approx Tg$ we means that there are positive constants $c_1$ and $c_2$ such that $c_1 Tf(x) \leq Tg(x) \leq c_2 Tf(x)$. Constants in inequalities will be mainly denoted by $c$ or $C$; the symbol $\mathbb{R}_+$ means the interval $(0, +\infty)$.

2. PRELIMINARIES

We say that a radial function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is decreasing if there is a decreasing function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $g(|x|) = f(x), \ x \in \mathbb{R}^n$. We will denote $g$
again by \( f \). Let \( p : \mathbb{R}^n \to \mathbb{R}^n \) be a measurable function, satisfying the conditions 
\[ p^- = \text{ess inf}_{x \in \mathbb{R}^n} p(x) > 0, \quad p^+ = \text{ess sup}_{x \in \mathbb{R}^n} p(x) < \infty. \]

Given \( p : \mathbb{R}^n \to \mathbb{R}_+ \) such that \( 0 < p^- \leq p^+ < \infty \), and a non-negative measurable function (weight) \( u \) in \( \mathbb{R}^n \), let us define the following local oscillation of \( p \):
\[ \varphi_{p(\cdot),u(\delta)} = \text{ess sup}_{x \in B(0,\delta) \cap \text{supp } u} p(x) - \text{ess inf}_{x \in B(0,\delta) \cap \text{supp } u} p(x), \]
where \( B(0,\delta) \) is the ball with center 0 and radius \( \delta \).

We observe that \( \varphi_{p(\cdot),u(\delta)} \) is non-decreasing and positive function such that
\[ \lim_{\delta \to \infty} \varphi_{p(\cdot),u(\delta)} = p^+ u - p^- u, \quad (1) \]
where \( p^+ \) and \( p^- \) denote the essential infimum and supremum of \( p \) on the support of \( u \), respectively.

By the similar manner it is defined (see [3]) the function \( \psi_{p(\cdot),u(\eta)} \) for an exponent \( p : \mathbb{R}_+ \to \mathbb{R}_+ \) and weight \( v \) on \( \mathbb{R}_+ \):
\[ \varphi_{p(\cdot),v(\varepsilon)} = \text{ess sup}_{x \in B(0,\varepsilon) \cap \text{supp } v} p(x) - \text{ess inf}_{x \in (0,\varepsilon) \cap \text{supp } v} p(x), \]

Let \( D(\mathbb{R}_+) \) be the class of non-negative decreasing functions on \( \mathbb{R}_+ \) and let \( DR(\mathbb{R}^n) \) be the class of all non–negative radially decreasing functions on \( \mathbb{R}^n \). Suppose that \( u \) is measurable a.e. positive function (weight) on \( \mathbb{R}^n \). We denote by \( L^{p(x)}(u, \mathbb{R}^n) \), the class of all non–negative functions on \( \mathbb{R}^n \) for which
\[ S_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} u(x) d\mu(x) < \infty. \]

For essential properties of \( L^{p(x)} \) spaces we refer to the papers [16], [20] and the monographs [7], [5].

Under the symbol \( L^{p(x)}(u, \mathbb{R}_+) \) we mean the class of non-negative decreasing functions on \( \mathbb{R}_+ \) from \( L^{p(x)}(u, \mathbb{R}^n) \cap DR(\mathbb{R}^n) \).

Now we list the well-known results regarding one-weight inequality for the operator \( T \). For the following statement we refer to [1].

**Theorem A.** Let \( r \) be constant such that \( 0 < r < \infty \). Then the inequity
\[ \int_0^\infty v(x)(Tf(x))^r dx \leq c \int_0^\infty v(x)(f(x))^r dx, \quad f \in L^r(v, \mathbb{R}_+), \quad f \downarrow \quad (2) \]
for a weight \( v \) holds, if and only if there exists a positive constant \( C \) such that for all \( s > 0 \)
\[
\int_s^\infty \left(\frac{s}{x}\right)^r v(x) dx \leq C \int_0^s v(x) dx.
\tag{3}
\]

Condition (3) is called \( B_r \) condition and was introduced in [1].

**Theorem B**. Let \( v \) be a weight on \((0, \infty)\) and \( p : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that \( 0 < p^- \leq p^+ < \infty \), and assume that \( \psi_{p^+}(0^+) = 0 \). The following facts are equivalent:

(a) There exists a positive constant \( c \) such that for any \( f \in D(\mathbb{R}^+) \),
\[
\int_0^\infty (Tf(x))^{p(x)} v(x) dx \leq C \int_0^\infty (f(x))^{p(x)} v(x) dx.
\tag{4}
\]

(b) For any \( r, s > 0 \),
\[
\int_r^\infty \left(\frac{r}{sx}\right)^{p(x)} v(x) dx \leq C \int_0^s \frac{v(x)}{s^{p(x)}} dx.
\tag{5}
\]

(c) \( p|_{\text{supp } v} \equiv p_0 \) a.e and \( v \in B_{p_0} \).

**Proposition 2.1.** For the operators \( T, H, R_\alpha \) and \( I_\alpha \), the following relations hold:

(a) \( R_\alpha f \approx Tf, \quad 0 < \alpha < 1, \quad f \in D(\mathbb{R}^+) \);

(b) \( I_\alpha g \approx Hg, \quad 0 < \alpha < n, \quad g \in DR(\mathbb{R}^n) \).

**Proof.** (a) Upper estimate. Represent \( R_\alpha f \) as follows:
\[
R_\alpha f(x) = \frac{1}{x^\alpha} \int_0^{x/2} \frac{f(t)}{(x - t)^{1-\alpha}} dt + \frac{1}{x^\alpha} \int_{x/2}^{x} \frac{f(t)}{(x - t)^{1-\alpha}} dt = S_1(x) + S_2(x).
\]

Observe that if \( t < x/2 \), then \( x/2 < x - t \). Hence
\[
S_1(x) \leq c \frac{1}{x} \int_0^{x/2} f(t) dt \leq cTf(x),
\]

where the positive constant \( c \) does not depend on \( f \) and \( x \). Using the fact that \( f \) is decreasing we find that
\[
S_2(x) \leq cf(x/2) \leq cTf(x).
\]
Lower estimate follows immediately by using the fact that \( f \) is non-negative and the obvious estimate \( x - t \leq x \) and \( 0 < t < x \).

(b) Upper estimate. Let us represent the operator \( I_\alpha \) as follows:

\[
I_\alpha g(x) = \frac{1}{|x|^\alpha} \int_{|y|<|x|/2} \frac{g(y)}{|x-y|^{n-\alpha}} dy + \frac{1}{|x|^\alpha} \int_{|x|/2<|y|<|x|} \frac{g(y)}{|x-y|^{n-\alpha}} dy
\]

=: \( S_1'(x) + S_2'(x) \).

Since \( |x|/2 \leq |x-y| \) for \( |y| < |x|/2 \) we have that

\[
S_1'(x) \leq c \frac{|x|^{\alpha-n}}{|x|^{n-\alpha}} \int_{|y|<|x|/2} g(y) dy \leq c H g(x).
\]

Taking into account the fact that \( f \) is radially decreasing on \( \mathbb{R}^n \) we find that there is a decreasing function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that

\[
S_2'(x) \leq f(|x|/2) \cdot \frac{1}{|x|^\alpha} \int_{|x|/2<|y|<|x|} |x-y|^{\alpha-n} dy
\]

Let \( F_x = \{ y : |x|/2 < |y| < |x| \} \). Then we have

\[
\int_{F_x} |x-y|^{\alpha-n} dy = \int_0^\infty \left| \{ y \in F_x : |x-y|^{\alpha-n} > t \} \right| dt
\]

\[
\leq \int_0^{|x|^{\alpha-n}} \left| \{ y \in F_x : |x-y|^{\alpha-n} > t \} \right| dt + \int_{|x|^{\alpha-n}}^\infty \left| \{ y \in F_x : |x-y|^{\alpha-n} > t \} \right| dt
\]

=: \( I_1 + I_2 \).

It is easy to see that

\[
I_1 \leq \int_0^{|x|^{\alpha-n}} |B(0,|x|)| dt = c |x|^\alpha,
\]

while using the fact that \( \frac{n}{n-\alpha} > 1 \) we find that

\[
I_2 \leq \int_{|x|^{\alpha-n}}^\infty \left| \{ y \in F_x : |x-y| \leq t^{\alpha-n} \} \right| dt \leq c \int_{|x|^{\alpha-n}}^\infty t^{\frac{n}{n-\alpha}} dt = c_{\alpha,n} |x|^\alpha.
\]

Finally we conclude that

\[
S_2'(x) \leq c f(|x|/2) \leq c H f(x).
\]
Lower estimate follows immediately by using the fact that $f$ is non-negative and the obvious estimate $|x - y| \leq |x|$, where $0 < |y| < |x|$.

We will also need the following statement:

**Lemma 2.2.** Let $r$ be a constant such that $0 < r < \infty$. Then the inequality

$$
\int_{\mathbb{R}^n} (Hf(x))^r u(x) dx \leq C \int_{\mathbb{R}^n} (f(x))^r u(x) dx,
$$

$f \in L^r_{\text{dec}}(u, \mathbb{R}^n)$  \hspace{1cm} (6)

holds, if and only if there exists a positive constant $C$ such that for all $s > 0$,

$$
\int_{|x| > s} |x|^{r(1-n)} u(x) dx \leq C \int_{|x| < s} |x|^{r(1-n)} u(x) dx.
$$

(7)

**Proof.** We shall see that inequality (6) is equivalent to the inequality

$$
\int_0^\infty \tilde{u}(t) (T \bar{f}(t))^r dt \leq C \int_0^\infty \tilde{u}(t) (\bar{f}(t))^r dt,
$$

where $\tilde{u}(t) = t^{(n-1)(1-r)} \bar{u}(t)$, $\bar{f}(t) = t^{n-1} f(t)$ and $\bar{u}(t) = \int_{S_0} u(t\bar{x}) d\sigma(\bar{x})$.

Indeed, using polar the coordinates in $\mathbb{R}^n$ we have

$$
\int_{\mathbb{R}^n} (Hf(x))^r u(x) dx = \int_{\mathbb{R}^n} u(x) \left( \frac{1}{|x|^n} \int_{|y| < |x|} f(y) dy \right)^r dx
$$

$$
= \int_0^\infty t^{n-1} \left( \frac{1}{|t|^n} \int_{|y| < |t|} f(y) dy \right)^r \left( \int_{S_0} u(t\bar{x}) d\sigma(\bar{x}) \right) dt
$$

$$
= C \int_0^\infty t^{n-1} t^{-nr} t^r \left( \frac{1}{t} \int_0^t \tau^{n-1} f(\tau) d\tau \right)^r \tilde{u}(t) dt
$$

$$
= C \int_0^t \tilde{u}(t) (\bar{f}(t))^r \bar{u}(t) \left( \int_0^t \bar{f}(\tau) d\tau \right)^r dt
$$

$$
\leq C \int_0^t \tilde{u}(t) (\bar{f}(t))^r dt
$$

$$
= C \int_{\mathbb{R}^n} (f(x))^r u(x) dx.
$$
To formulate the main results we need to prove

**Proposition 3.1.** Let $u$ be a weight on $\mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\varphi_{p(\cdot), u(0^+)} = 0$. The following statements are equivalent:

(a) There exists a positive constant $C$ such that for any $f \in DR(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} (Hf(x))^{p(x)} u(x) dx \leq C \int_{\mathbb{R}^n} (f(x))^{p(x)} u(x) dx.
$$

(b) For any $r, s > 0$,

$$
\int_{|x| > r} \left( \frac{r}{s|x|^{ln}} \right)^{p_0} u(x) dx \leq C \int_{B(0,r)} \frac{|x|^{(1-n)p_0} u(x)}{s^{p_0}} dx.
$$

(c) $p_{|\text{supp } u|} \equiv p_0 \quad \text{a.e and } u \in B_{p_0}$.

**Proof.** We use the arguments of [3]. To show that (a) implies (b) it is enough to test the modular inequality (8) for the function $f_{r,s}(x) = \frac{1}{s} \chi_{B(0,r)}(x) |x|^{1-n}$, $s, r > 0$. Indeed, it can be checked that

$$
Hf_{r,s}(x) = \begin{cases} \frac{1}{|x|^{ln}} \int_{|y| \leq |x|} |y|^{1-n} dy, & \text{if } |x| \leq r; \\ \frac{1}{|x|^{ln}} \int_{|y| \leq r} |y|^{1-n} dy, & \text{if } |x| > r. \end{cases}
$$

Further, we find that

$$
\int_{|x| > r} u(x) (Hf_{r,s})^{p(x)} dx \leq \int_{\mathbb{R}^n} u(x) (Hf_{r,s})^{p(x)} dx \leq C \int_{\mathbb{R}^n} u(x) \left( \frac{1}{s} \chi_{B(0,r)}(x) |x|^{1-n} \right)^{p(x)} dx.
$$

Therefore

$$
\int_{|x| > r} u(x) \left( \frac{r}{s|x|^{ln}} \right)^{p(x)} dx \leq C \int_{B(0,r)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}} dx.
$$

To obtain (c) from (b) we are going to prove that condition (b) implies that $\varphi_{p(\cdot), u(\delta)}$ is a constant function, namely $\varphi_{p(\cdot), u(\delta)} = p_u^+ - p_u^-$ for all $\delta > 0$. This fact and the hypothesis on $\varphi_{p(\cdot), u(\delta)}$ implies that $\varphi_{p(\cdot), u(\delta)} \equiv 0$, and hence due to (1),

$$
p_{|\text{supp } u|} \equiv p_u^+ - p_u^- \equiv p_0 \quad \text{a.e.}.
$$
Finally (9) means that \( u \in B_{p_0} \). Let us suppose that \( \varphi_{p(\cdot),u} \) is not constant. Then one of the following conditions hold:

(i) there exists \( \delta > 0 \) such that

\[
\alpha = \text{ess sup}_{x \in B(0,\delta) \cap \text{supp } u} p(x) < p_u^+ < \infty,
\]

and hence, there exists \( \epsilon > 0 \) such that

\[
\{|x| > \delta : p(x) \geq \alpha + \epsilon\} \cap \text{supp } u > 0,
\]

or

(ii) there exists \( \delta > 0 \) such that

\[
\beta = \text{ess inf}_{x \in B(0,\delta) \cap \text{supp } u} p(x) > p_u^- > 0,
\]

and then, for some \( \epsilon > 0 \),

\[
\{|x| > \delta : p(x) \leq \beta - \epsilon\} \cap \text{supp } u > 0.
\]

In the case (i) we observe that condition (b) for \( r = \delta \), implies that

\[
\int_{|x| > \delta} \left( \frac{\delta}{s} \right)^{p(x)} \frac{u(x)}{|x|^{np(x)}} \, dx \
\leq C \int_{B(0,\delta)} |x|^{(1-n)p(x)} u(x) \, dx.
\]

Then using (10) we obtain, for \( s < \min(1, \delta) \),

\[
\left( \frac{\delta}{s} \right)^{\alpha + \epsilon} \int_{\{|x| > \delta : p(x) \geq \alpha + \epsilon\}} \frac{u(x)}{|x|^{np(x)}} \, dx \leq C \int_{B(0,\delta)} u(x) |x|^{(1-n)p(x)} \, dx,
\]

which is clearly a contradiction if we let \( s \downarrow 0 \). Similarly in the case (ii) let us consider the same condition (b) for \( r = \delta \), and fix now \( s > 1 \). Taking into account (11) we find that:

\[
\frac{1}{s^{\beta - \epsilon}} \int_{\{|x| > \delta : p(x) \leq \beta - \epsilon\}} \left( \frac{\delta}{|x|^n} \right)^{p(x)} u(x) \, dx \leq C \int_{B(0,\delta)} |x|^{(1-n)p(x)} u(x) \, dx,
\]

which is a contradiction if we let \( s \uparrow \infty \).

Finally, the fact that condition (c) implies (a) follows from [1, Theorem 1.7] \( \square \)

**Theorem 3.2.** Let \( u \) be a weight on \((0, \infty)\) and \( p : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( 0 < p^- \leq p^+ < \infty \). Assume that \( \psi_{p(\cdot),u(0^+)} = 0 \). The following facts are equivalent:

(i) There exists a positive constant \( C \) such that for any \( f \in D(\mathbb{R}_+) \),

\[
\int_{\mathbb{R}_+} \left( R_\alpha f(x) \right)^{p(x)} u(x) \, dx \leq C \int_{\mathbb{R}_+} \left( f(x) \right)^{p(x)} u(x) \, dx.
\]

(ii) condition (5) holds;
(iii) condition (c) of Theorem B is satisfied.

Proof. Proof follows by using Theorems B and Proposition 2.1(a). □

**Theorem 3.3.** Let \( u \) be a weight on \( \mathbb{R}^n \) and \( p : \mathbb{R}^n \to \mathbb{R}_+ \) such that \( 0 < p^- \leq p^+ < \infty \), and assume that \( \varphi_{p(\cdot),u(0^+)} = 0 \). The following facts are equivalent:

(i) There exists a positive constant \( C \) such that for any \( f \in DR(\mathbb{R}^n) \),
\[
\int_{\mathbb{R}^n} (I_\alpha f(x))^{p(x)} u(x) dx \leq C \int_{\mathbb{R}^n} (f(x))^{p(x)} u(x) dx.
\]

(ii) condition (9) holds;

(iii) condition (c) of Proposition 3.1 holds.

Proof. Proof follows by using Propositions 3.1 and Proposition 2.1 (b). □

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