ABOUT ONE MODULUS INEQUALITY OF THE VÄISÄLÄ TYPE

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May 10, 2014

Abstract

The present paper is devoted to the study of space mappings, which are more general than quasiregular. The analogue of the known Väisälä inequality for the special classes of curves and mappings was proved for the open, discrete, differentiable a.e. mappings having \( N, N^{-1} \) and \( ACP^{-1} \) properties.

2010 Mathematics Subject Classification: Primary 30C65; Secondary 30C62

1 Introduction

Here are some definitions. Everywhere below, \( D \) is a domain in \( \mathbb{R}^n, n \geq 2 \), \( m \) is the Lebesgue measure in \( \mathbb{R}^n \), \( m_1 \) is the linear Lebesgue measure on \( \mathbb{R} \). A mapping \( f : D \to \mathbb{R}^n \) is said to be a discrete if the pre-image \( f^{-1}(y) \) of any point \( y \in \mathbb{R}^n \) consists of isolated points, and an open if the image of any open set \( U \subset D \) is open in \( \mathbb{R}^n \). The notation \( f : D \to \mathbb{R}^n \) assumes that \( f \) is continuous.

We write \( f \in W^{1,n}_{\text{loc}}(D) \), iff all of the coordinate functions \( f_j, f = (f_1, \ldots, f_n) \), have the partitional derivatives which are locally integrable in the degree \( n \) in \( D \). Recall that a mapping \( f : D \to \mathbb{R}^n \) is said to be a mapping with bounded distortion, if the following conditions hold:
1) \( f \in W^{1,n}_{\text{loc}}(D) \), 2) a Jacobian \( J(x, f) := \det f'(x) \) of the mapping \( f \) at the point \( x \in D \) preserves the sign almost everywhere in \( D \), 3) \( \|f'(x)\|^n \leq K \cdot |J(x, f)| \) at a.e. \( x \in D \) and some constant \( K < \infty \), where \( \|f'(x)\| := \sup_{h \in \mathbb{R}^n : |h| = 1} |f'(x)h| \), see §3. I in [Re], or definition 2.1 of the 2. I in [Ri].

A curve \( \gamma \) in \( \mathbb{R}^n \) is a continuous mapping \( \gamma : \Delta \to \mathbb{R}^n \) where \( \Delta \) is an interval in \( \mathbb{R} \). Its locus \( \gamma(\Delta) \) is denoted by \( |\gamma| \). Given a family of curves \( \Gamma \) in \( \mathbb{R}^n \), a Borel function \( \rho : \mathbb{R}^n \to [0, \infty] \) is called admissible for \( \Gamma \), abbr. \( \rho \in \text{adm} \Gamma \), if curvilinear integral of the first type \( \int_{\gamma} \rho(x)|dx| \) satisfies the condition
\[
\int_{\gamma} \rho(x)|dx| \geq 1
\]
for each (locally rectifiable) $\gamma \in \Gamma$. Given $p \geq 1$, the $p$–modulus $M_p(\Gamma)$ of $\Gamma$ is defined as

$$M_p(\Gamma) = \inf_{\rho \in \text{adm} \Gamma} \int_{\mathbb{R}^n} \rho^p(x) dm(x)$$

interpreted as $+\infty$ if $\text{adm} \Gamma = \emptyset$. The properties of it are analogous to the properties of the measure of Lebesgue $m$ in $\mathbb{R}^n$. Namely, $M_p(\emptyset) = 0$, $M_p(\Gamma_1) \leq M_p(\Gamma_2)$ whenever $\Gamma_1 \subset \Gamma_2$ and $M_p \left( \bigcup_{i=1}^{\infty} \Gamma_i \right) \leq \sum_{i=1}^{\infty} M_p(\Gamma_i)$, see Theorem 6.2 in [Vα2].

Set $l(f'(x)) := \inf_{h \in \mathbb{R}^n; |h| = 1} |f'(x)h|$. Recall that inner dilatation of the order $p$ of the mapping $f$ at a point $x$ is defined as

$$K_{I,p}(x, f) = \begin{cases} \frac{|J(x, f)|}{l(f'(x))}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases}$$

Let $K_{I,p}(f) = \text{ess sup}_{x \in D} K_{I,p}(x, f)$. Note that $K_{I,n}(f) < K^{n-1} < \infty$ whenever $f$ is a mapping with bounded distortion, see (2.7) and (2.8) of 2.1 in [Ri].

Suppose that $\alpha$ and $\beta$ are curves in $\mathbb{R}^n$. Then a notation $\alpha \subset \beta$ denotes that $\alpha$ is a subpath of $\beta$. In what follows, $I$ denotes an open, a closed or a semi–open interval on the real axes. The following definition can be found in the section 5 of Ch. II in [Re].

Let $I = [a, b]$. Given a rectifiable path $\gamma : I \to \mathbb{R}^n$ we define a length function $l_\gamma(t)$ by the rule $l_\gamma(t) = S(\gamma, [a, t])$, where $S(\gamma, [a, t])$ is a length of the path $\gamma_{|a, t]}$. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a rectifiable curve in $\mathbb{R}^n$, $n \geq 2$, and $l(\alpha)$ be its length. A normal representation $\alpha^0$ of $\alpha$ is defined as a curve $\alpha^0 : [0, l(\alpha)] \to \mathbb{R}^n$ which is can be got from $\alpha$ by change of parameter such that $\alpha(t) = \alpha^0(\gamma, [a, t])$.

Let $f : D \to \mathbb{R}^n$ be such that $f^{-1}(y)$ does not contain a nondegenerate curve for any $y \in \mathbb{R}^n$, $\beta : I_0 \to \mathbb{R}^n$ be a closed rectifiable curve and $\alpha : I \to D$ such that $f \circ \alpha \subset \beta$. If the length function $l_\beta : I_0 \to [0, l(\beta)]$ is a constant on $J \subset I$, then $\beta$ is a constant on $J$ and consequently a curve $\alpha$ to be a constant on $J$. Thus, there exists a unique function $\alpha^* : l_\beta(I) \to D$ such that $\alpha = \alpha^* \circ (l_\beta|_I)$. We say that $\alpha^*$ to be a $f$–representation of $\alpha$ by the respect to $\beta$ if $\beta = f \circ \alpha$.

If $\alpha : [a, b] \to D$ is a closed curve, we say that $f$ winds $\alpha$ $m$ times around itself if $f \circ \alpha = \beta$ is rectifiable and if the following condition is satisfied: Let $\beta^0 : [0, c] \to \mathbb{R}^n$ be the normal representation of $\beta$, let $\alpha^* : [0, c] \to D$ be $f$–representation of $\alpha$ with respect to $\beta$, and let $h = c/m$. Then $\beta^0(t + jh) = \beta^0(t)$ and $\alpha^*(t + jh) \neq \alpha^*(t)$ whenever $0 \leq t < t + jh < c$ and $j \in \{1, \ldots, m - 1\}$.

In 1972 in the work of J. Väisälä was proved the following, see e.g. Theorem 3.9 in [Vα3].

**Theorem 1.1.** Let $f : D \to \mathbb{R}^n$ be a non–constant mapping with bounded distortion. Suppose that $\Gamma$ is a curve family in $D$, that $m$ is a positive integer, and that $f$ winds every
path of \( \Gamma \) \( m \) times around itself. Then

\[
M_n(f(\Gamma)) \leq \frac{K_{I,n}(f)}{m} \cdot M_n(\Gamma). \tag{1.1}
\]

The goal of the present paper is to prove the analogue of the for more general classes of mappings. Recall that a mapping \( f : D \to \mathbb{R}^n \) is said to have the \( N \)-property (of Luzin) if \( m(f(S)) = 0 \) whenever \( m(S) = 0 \) for all such sets \( S \subset \mathbb{R}^n \). Similarly, \( f \) has the \( N^{-1} \)-property if \( m(f^{-1}(S)) = 0 \) whenever \( m(S) = 0 \).

We say that a property \( P \) holds for \( p \)-almost every (a.e.) curves \( \gamma \) in a family \( \Gamma \) if the subfamily of all curves in \( \Gamma \) for which \( P \) fails has \( p \)-modulus zero.

A curve \( \gamma \) in \( D \) is called here a lifting of a curve \( \tilde{\gamma} \) in \( \mathbb{R}^n \) under \( f : D \to \mathbb{R}^n \) if \( \tilde{\gamma} = f \circ \gamma \). Suppose that \( f^{-1}(y) \) does not contain a non–degenerate curve for any \( y \in \mathbb{R}^n \). We say that a mapping \( f \) is absolute continuous on curves in the inverse direction, abbr. \( ACP_{p}^{-1} \), if for \( p \)-a.e. closed curves \( \gamma \) a lifting \( \tilde{\gamma} \) of \( \gamma \) is rectifiable and the corresponding \( f \)–representation \( \gamma^* \) of \( \gamma \) is absolutely continuous.

The result of the paper is the following statement.

**Theorem 1.2.** Let \( p \geq 1 \), let a mapping \( f : D \to \mathbb{R}^n \) be a differentiable a.e., discrete mapping, having \( N, N^{-1} \) and \( ACP_{p}^{-1} \) properties. Suppose that \( \Gamma \) is a curve family in \( D \), that \( m \) is a positive integer, and that \( f \) winds every path of \( \Gamma \) \( m \) times around itself. Then

\[
M_p(f(\Gamma)) \leq \frac{1}{m} \cdot \int_D K_{I,p}(x, f) \cdot \rho^p(x) \, dm(x) \tag{1.2}
\]

for every \( \rho \in \text{adm} \Gamma \).

Note that the Theorem follows from Theorem at \( p = n \) as corollary. In fact, every non–constant mapping with bounded distortion is discrete and has \( N \)–property, see Theorems 6.2 and 6.3 of Ch. II in [Re] (see also Theorem 4.1 and Proposition 4.14 Ch. I in [Ri]); is differentiable a.e., see Lemma 3 in [Va1]; has \( N^{-1} \)–property, see Theorem 8.2 in [BI]; and has \( ACP_{n}^{-1} \)–property, see Lemma 6 in [Pol].

### 2 Proof of the main result

A mapping \( \varphi : X \to Y \) between metric spaces \( X \) and \( Y \) is said to be a **Lipschitzian** provided

\[
\text{dist} (\varphi(x_1), \varphi(x_2)) \leq M \cdot \text{dist}(x_1, x_2)
\]

for some \( M < \infty \) and for all \( x_1 \) and \( x_2 \in X \). The mapping \( \varphi \) is called **bi–lipschitz** if, in addition,

\[
M^* \text{dist} (x_1, x_2) \leq \text{dist} (\varphi(x_1), \varphi(x_2))
\]

for some \( M^* > 0 \) and for all \( x_1 \) and \( x_2 \in X \). Later on \( X \) and \( Y \) are subsets of \( \mathbb{R}^n \) with the Euclidean distance.
The following proposition can be found in [MRSY₁], see Lemma 3.20, see also Lemma 8.3 Ch. VIII in [MRSY₂].

**Proposition 2.1.** Let \( f : D \to \mathbb{R}^n \) be a differentiable a.e. in \( D \), and have \( N^- \) and \( N^{-1} \)-properties. Then there is a countable collection of compact sets \( C^*_k \subset D \) such that \( m(B_0) = 0 \) where \( B_0 = D \setminus \bigcup_{k=1}^{\infty} C^*_k \) and \( f|_{C^*_k} \) is one-to-one and bi-lipschitz for every \( k = 1, 2, \ldots \). Moreover, \( f \) is differentiable at \( C^*_k \) and \( J(x, f) \neq 0 \).

Given a set \( E \) in \( \mathbb{R}^n \) and a closed curve \( \gamma : \Delta \to \mathbb{R}^n \), we identify \( \gamma \cap E \) with \( \gamma(\Delta) \cap E \). If \( \gamma \) is rectifiable, then we set

\[
l(\gamma \cap E) = m_1(E_\gamma),
\]

where \( E_\gamma = l_\gamma(\gamma^{-1}(E)) \); here \( l_\gamma : \Delta \to \Delta_\gamma \) as in the previous section. Note that \( E_\gamma = \gamma^{0-1}(E) \), where \( \gamma^0 : \Delta_\gamma \to \mathbb{R}^n \) is the normal representation of \( \gamma \) and

\[
l(\gamma \cap E) = \int_\Delta \chi_E(\gamma(t)) |dx| := \int_{\Delta_\gamma} \chi_{E_\gamma}(s)ds.
\]

The statement mentioned bellow can be found in Chapter IX of [MRSY₂], see Theorem 9.1.

**Proposition 2.2.** Let \( E \) be a set in a domain \( D \subset \mathbb{R}^n \), \( n \geq 2 \). Then \( E \) is measurable if and only if \( \gamma \cap E \) is measurable for \( p \)-a.e. closed curve \( \gamma \) in \( D \). Moreover, \( m(E) = 0 \) if and only if

\[
l(\gamma \cap E) = 0
\]

for \( p \)-a.e. closed curve \( \gamma \) in \( D \).

**Proof of the Theorem** Let \( B_0 \) and \( C^*_k \) be as in Proposition 2.1. Setting by induction \( B_1 = C^*_1, B_2 = C^*_2 \setminus B_1 \ldots, \)

\[
B_k = C^*_k \setminus \bigcup_{l=1}^{k-1} B_l,
\]

we obtain the countable covering of \( D \) consisting of mutually disjoint Borel sets \( B_k, k = 1, 2, \ldots, \) with \( m(B_0) = 0 \). By the construction and \( N^- \)-property, \( m(f(B_0)) = 0 \). Thus, by Proposition 2.2 \( \gamma^0(s) \notin f(B_0) \) for \( p \)-a.e. curves \( \tilde{\gamma} \) in \( f(D) \), where \( \gamma^0_\gamma \) is a normal representation of \( \gamma \). Moreover, by \( ACP_p^{p-1} \)-property, the \( f \)-representation \( \gamma^* \) of a curve \( \gamma \) is rectifiable and absolutely continuous for \( p \)-a.e. closed curves \( \tilde{\gamma} \) in \( f(D) \) such that \( f \circ \gamma = \tilde{\gamma} \).

Let \( \rho \in \text{adm} \Gamma \) and

\[
\tilde{\rho}(y) = \frac{1}{m} \cdot \chi_{f(D\setminus B_0)}(y) \sup_C \sum_{x \in C} \rho^*(x), \tag{2.1}
\]

where

\[
\rho^*(x) = \begin{cases} 
\rho(x)/l(f'(x)), & x \in D \setminus B_0, \\
0, & x \in B_0
\end{cases}
\]
and $C$ runs over all subsets of $f^{-1}(y)$ in $D \setminus B_0$ such that $\text{card } C \leq m$. Note that

$$\tilde{\rho}(y) = \frac{1}{m} \cdot \sup_{i=1}^{s} \rho_{k_i}(y), \quad (2.2)$$

where $\sup$ in $(2.2)$ is taken over all $\{k_1, \ldots, k_s\}$ such that $k_i \in \mathbb{N}$, $k_i \neq k_j$ if $i \neq j$, all $s \leq m$, and

$$\rho_{k}(y) = \begin{cases} \rho^* \left( f_k^{-1}(y) \right), & y \in f(B_k), \\ 0, & y \notin f(B_k) \end{cases}.$$  

Here $f_k = f|_{B_k}$, $k = 1, 2, \ldots$ is injective and $f(B_k)$ is Borel. Thus, the function $\tilde{\rho}(y)$ is Borel, see e.g. 2.3.2 in [Fe].

Given a rectifiable path $\beta$ we denote through $c$ the length of $\beta$, $c := l(\beta)$, and through $\beta^0$ it’s normal representation. Using the definitions of the paths $\Gamma$ and $f(\Gamma)$, for every curve $\beta \in f(\Gamma)$ we obtain

$$\int_{\beta} \tilde{\rho}(y) dy = \int_{0}^{c} \tilde{\rho}(\beta^0(t)) dt =$$

$$= \frac{m}{m} \int_{0}^{\frac{c}{m}} \tilde{\rho}(\beta^0(t)) dt = \sum_{j=1}^{m} \int_{\frac{j-1}{m}}^{\frac{j}{m}} \tilde{\rho}(\beta^0(t)) dt = = \sum_{j=1}^{m} \int_{0}^{\frac{h}{m}} \tilde{\rho}(\beta^0(t)) dt, \quad h = c/m. \quad (2.3)$$

If $0 < t < h$, then $\alpha^*(t), \alpha^*(t+h), \ldots, \alpha^*(t+(m-1)h)$ are distinct points in $f^{-1}(\beta^0(t))$. Hence

$$\tilde{\rho}(\beta^0(t)) \geq \frac{1}{m} \sum_{j=0}^{m-1} \rho^* (\alpha^*(t+jh)) \quad (2.4)$$

for $t \in (0, h)$. By Proposition 2.2 we can consider that $\beta^0(t) \notin f(B_0)$ a.e. $t \in [0, c]$. Since $\beta$ is rectifiable, $\beta(t)$ is a differentiable a.e. Besides that, a curve $\alpha^*$ is rectifiable and absolutely continuous for $p$-a.e. $\beta$, moreover, $\alpha^*(t) \notin B_0$ for a.e. $t \in [0, c]$. Thus, the derivatives $f'(\alpha^*(t))$ and $\alpha'^* (t)$ exist for a.e. $t \in [0, c]$. Taking into account the formula of the derivative of the superposition of functions, and that the modulus of the derivative of the curve by the natural parameter equals to 1, we have that

$$1 = \left| (f \circ \alpha^*)' (t) \right| = \left| f'(\alpha^*(t)) \alpha'^*(t) \right| =$$

$$= \left| f'(\alpha^*(t)) \cdot \left| \frac{\alpha'^*(t)}{|\alpha'^*(t)|} \right| \cdot |\alpha'^*(t)| \geq l (f'(\alpha^*(t))) \cdot |\alpha'^*(t)| \quad (2.5)$$

for $p$-a.e. curves $\beta \in f(\Gamma)$, $\beta = f \circ \alpha$. It follows from (2.5) that a.e.

$$\rho^* (\alpha^*(t)) = \frac{\rho(\alpha^*(t))}{l (f'(\alpha^*(t)))} \geq \rho(\alpha^*(t)) \cdot |\alpha'^*(t)|. \quad (2.6)$$
By absolutely continuity of $\alpha^*$, Theorem 4.1 in [V2], (2.3), (2.4) and (2.6) we obtain that
\[
\int_{\beta} \tilde{\rho}(y) |dy| \geq \sum_{j=0}^{m-1} \int_0^h \rho(\alpha^*(t + jh)) \cdot |\alpha^*(t + jh)| dt = \int_\alpha \rho(x) |dx| \geq 1. \tag{2.7}
\]

Thus, $\tilde{\rho} \in \text{adm} f(\Gamma) \setminus \Gamma_0$, where $M_p(\Gamma_0) = 0$. Consequently
\[
M_p(f(\Gamma)) \leq \int_{f(D)} \tilde{\rho}^p(y) \, dm(y). \tag{2.8}
\]

By 2.3.5 for $m = n$ in [Fe], we obtain that
\[
\int_{B_k} K_{I,p}(x, f) \cdot \rho^p(x) \, dm(x) = \int_{f(D)} \rho_p^p(y) \, dm(y). \tag{2.9}
\]

By Hölder inequality for series,
\[
\left(\frac{1}{m} \cdot \sum_{i=1}^s \rho_{k_i}(y)\right)^p \leq \frac{1}{m} \cdot \sum_{i=1}^s \rho_{k_i}^p(y) \tag{2.10}
\]
for each $1 \leq s \leq m$ and every $\{k_1, \ldots, k_s\}, k_i \in \mathbb{N}, k_i \neq k_j, \text{if } i \neq j$.

Finally, by the Lebesgue positive convergence theorem, see Theorem 12.3 in §12.1 in [Sa], we conclude from (2.8)–(2.10) that
\[
\frac{1}{m} \cdot \int_D K_{I,p}(x, f) \cdot \rho^p(x) \, dm(x) = \frac{1}{m} \cdot \int_{f(D)} \sum_{k=1}^{\infty} \rho_{k}^p(y) \, dm(y) \geq 0
\]
\[
\geq \frac{1}{m} \cdot \int_{f(D)} \sup_{k_1, \ldots, k_s \in \mathbb{N}, k_i \neq k_j \text{if } i \neq j} \sum_{i=1}^s \rho_{k_i}^p(y) \, dm(y) \geq \int_{f(D)} \tilde{\rho}^p(y) \, dm(y) \geq M_p(f(\Gamma)).
\]

The proof is complete. \[ \square \]

**Remark 2.1.** The above investigations are closely related with the so-called mappings with finite length distortion, introduced by O. Martio together with V. Ryazanov, U. Srebro and E. Yakubov, see [MRSY1]–[MRSY2], see also works [BGMV] and [KO].

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