Abstract. Given a pair of distinct non-CM normalized eigenforms having integer Fourier coefficients $a_1(n)$ and $a_2(n)$, we count positive integers $n$ with $(a_1(n), a_2(n)) = 1$ and make a conjecture about the density of the set of primes $p$ for which $(a_1(p), a_2(p)) = 1$. We also study the average order of the number of prime divisors of $(a_1(p), a_2(p))$.

1. Introduction

1.1. Motivation and the first result. Given two integer-valued sequences $a_1(n)$ and $a_2(n)$, an interesting question is, how the sequence of the gcd’s $(a_1(n), a_2(n))$ behaves; and in particular, how often $a_1(n)$ and $a_2(n)$ are coprime. For example, if $a_1(n) = n$ and $a_2(n) = \phi(n)$, the Euler’s $\phi$-function, then the density of such integers is zero. This follows from the beautiful result of Erdös [Erd49] given below:

$$|\{n \leq x : (n, \phi(n)) = 1\}| = (1 + o(1)) \frac{e^{-\gamma}x}{L_3(x)},$$

(1.1)

where $\gamma$ is the Euler constant and $L_3(x) = \log \log \log x$. In analogy with this result, V. Kumar Murty [Mur07] has shown that if $f$ is a normalized eigenform with integer Fourier coefficients $a_f(n)$, then

$$|\{n \leq x : (n, a_f(n)) = 1\}| = O \left( \frac{x}{L_3(x)} \right),$$

(1.2)

where the implied constant depends only on the two forms $f_1$ and $f_2$ as above. Then we have,

(a) $|\{n \leq x : (n, (a_1(n), a_2(n))) = 1\}| \ll \frac{x}{L_3(x)},$

(1.3)

(b) For any positive integer $d > 1$, we have,

$$|\{n \leq x : (d, (a_1(n), a_2(n))) = 1\}| \ll \frac{xL_3(x)}{L_2(x)},$$

(1.4)

where $L_2(x) = \log \log x$.

Theorem 1.1. Let $f_1$ and $f_2$ as above. Then we have,

Here the implied constant depends only on the two forms $f_1$ and $f_2$ in part (a), and only on $f_1, f_2$ and $d$ in part (b).

2020 Mathematics Subject Classification. Primary: 11F30; Secondary: 11F80, 11N37, 11N64, 11N99.

Key words and phrases. Fourier coefficients of cusp forms, coprimality of integer sequences, Chebotarev Density Theorem.
From part (b) of Thm. 1.1 the following is immediate (by choosing \( d = 2 \), for example).

**Corollary 1.2.** Let \( f_1 \) and \( f_2 \) be as in Thm. 1.1. Then we have

\[
\left| \{ n \leq x : (a_1(n), a_2(n)) = 1 \} \right| \ll \frac{xL_3(x)}{L_2(x)},
\]

where the implied constant depends only on the two forms \( f_1 \) and \( f_2 \). In particular, the density of the set of integers \( n \) such that \( a_1(n) \) and \( a_2(n) \) are coprime is zero.

**Remark 1.3.** A definition prevalent in the literature (see, e.g., [Iwa97, Chap. 6]) is that a form \( f \) is a Hecke eigenform or a Hecke form if it satisfies the condition that it is a common eigenfunction of the Hecke operators \( T_n \) with \( (n, N_f) = 1 \), where \( N_f \) is the level of the form \( f \). This weaker condition does not, in general, imply the stronger condition that \( f \) is a common eigenfunction of all the Hecke operators though these two are equivalent in the case the form is primitive (i.e., a newform). In the above theorem, the condition that the two forms are common eigenfunctions of the Hecke operators \( T_n \) for all positive integers \( n > 1 \) is essential since we need multiplicativity of the coefficients of the forms in the proof (see §6). Since multiplicativity of the coefficients is not required in the proof of the other results stated below, the results remain true even if one takes the forms to be Hecke eigenforms in the sense described above.

**Remark 1.4.** A question that arises naturally is whether we can find asymptotic formulae for the sums in (1.3), (1.4), and (1.5), or for the simpler sum in (1.2). Obtaining asymptotics or even good lower bounds for these sums seems to be a very difficult task and any progress on this question, even under reasonable conjectures, will be very interesting. In this connection, one should recall that for the sum in (1.2), the analogous question for primes is a well-known unsolved problem. Indeed, for a non-CM eigenform \( f \in S_k(N) \), it is expected that primes \( p \) for which \( (p, a_f(p)) > 1 \) is extremely rare and a probabilistic model suggests that the number of such primes up to \( x \) should be of the order \( O(\log \log x) \) (see [Gou97, §3]).

### 1.2. Restriction to primes: a probabilistic heuristic

Let \( f_1 \) and \( f_2 \) be as before. Another interesting question is how frequently \( a_1(p) \) and \( a_2(p) \) are coprime as \( p \) varies over the primes. In other words, we are interested in the order of growth of the function

\[
C(x; f_1, f_2) = |\{p \leq x : (a_1(p), a_2(p)) = 1\}|.
\]

As there are eigenforms, e.g., the Ramanujan Delta function, whose all but finitely many Fourier coefficients at primes are even, we need to assume further that there are infinitely many primes \( p \) such that one of \( a_1(p) \) and \( a_2(p) \) is odd. Under this condition, we describe a probabilistic heuristic to guess the answer. One of our crucial intermediate results is the asymptotic formula (4.4) which implies that for any fixed integer \( m > 1 \),

\[
\pi_{f_1,f_2}(x, m) \sim \delta(m)\pi(x), \quad \text{as } x \to \infty.
\]

Here \( \pi_{f_1,f_2}(x, m) \) denotes the number of primes \( p \) up to \( x \) such that \( m \) divides both \( a_1(p) \) and \( a_2(p) \); and \( \delta = \delta_{f_1,f_2} \) is an arithmetical function determined by the two forms \( f_1 \) and \( f_2 \) which we have defined and studied in some depth in §3. In view of the asymptotic relation (1.6), \( \delta(m) \) can be interpreted as the “probability” (in the sense of density) that a “random” prime \( p \) has the property that \( m \) divides both \( a_1(p) \) and \( a_2(p) \). Therefore, assuming that the conditions of divisibility by different primes do not influence each other (i.e., the “events” are independent), it seem reasonable to conjecture that the function \( \delta \) should be multiplicative. Indeed, we have shown that \( \delta \) is multiplicative on the set of integers that are supported on primes that are sufficiently large (see Prop. 3.6) for forms of general level and we have been able to establish multiplicativity over the entire set of natural numbers when the forms have level one (see Prop. 3.7). Moreover, the above assumption of independence suggests that the
“probability” that for a “random” prime \( p \), \( a_1(p) \) and \( a_2(p) \) are coprime should be given by the infinite product \( \alpha = \alpha_{f_1, f_2} \) defined by

\[
\alpha := \prod_{\ell \text{ prime}} (1 - \delta(\ell)).
\]

Since we know that the sum \( \sum_{\ell} \delta(\ell) \) converges (see Prop. 3.5) and that \( 0 < \delta(\ell) < 1 \) for all primes \( \ell \) (see §3), it is clear that the above infinite product converges to some real number in the interval \( (0, 1) \). The above discussion leads us to the following.

**Conjecture:** Under the assumptions stated above, we have the asymptotic relation

\[
C(x; f_1, f_2) = | \{ p \leq x : (a_1(p), a_2(p)) = 1 \} | \sim \alpha \pi(x) \quad \text{as} \quad x \to \infty,
\]

where \( \pi(x) \) denotes, as usual, the number of primes up to \( x \).

Proving this conjecture, even under GRH (the Generalized Riemann Hypothesis) for all Artin \( L \)-functions, seems to be out of the reach of the current knowledge. The difficulty lies in the fact that the exponent of \( x \) in the error term in (4.4) grows too rapidly in terms of \( m \) and after expressing the coprimality condition using the Möbius function, the error term becomes unmanageable. See also Remark 4.3. However, it is still possible to give an upper bound of the expected order of magnitude under GRH as stated below.

**Theorem 1.5.** Under the above assumptions on two forms and under GRH, one has the upper bound

\[
C(x; f_1, f_2) \leq (\alpha' + o(1)) \pi(x),
\]

where

\[
\alpha' = \sum_{n=1}^{\infty} \mu(n) \delta(n).
\]

Note that \( \alpha' = \alpha \) if \( \delta \) is multiplicative on the full set of positive integers, and hence, in particular, when the two forms are of level one.

**Remark 1.6.** We remark that the full strength of GRH is not essential to prove the above theorem. Indeed, an analysis of the proof shows that a quasi-GRH, which refers to the assertion that no Artin \( L \)-function has a zero in the region \( \Re(s) > 1 - \delta \) for some fixed \( \delta \) with \( 0 < \delta \leq 1/2 \), is sufficient for our purpose. However, one should note that the exponent of \( x \) in the error term in (2.3) will now depend on \( \delta \) (see Remark 2.2), and therefore the error terms in part (b) of both Prop. 4.1 and Prop. 4.2 will change accordingly.

### 1.3. Numerical verification of the conjecture

We have tested the conjecture numerically using SAGE [SAGE] for every pair of eigenforms \( f_i \) and \( f_j \) \((1 \leq i, j \leq 3, i \neq j)\), where

\[
\begin{align*}
f_1(z) &= q - 4q^2 - 15q^3 - 16q^4 - 19q^5 + 60q^6 + \cdots \in S_6(11), \\
f_2(z) &= q - 5q^2 - 7q^3 + 17q^4 - 7q^5 + 35q^6 + \cdots \in S_4(13), \\
f_3(z) &= q + 10q^2 - 73q^3 - 28q^4 - 295q^5 - 730q^6 + \cdots \in S_8(13).
\end{align*}
\]

Let us denote the \( n \)th Fourier coefficient of \( f_i \) by \( a_i(n) \), for \( 1 \leq i \leq 3 \). For a large integer \( x \), the ratio \( R(x; f_i, f_j) := |\{ p \leq x : (a_i(p), a_j(p)) = 1 \}|/\pi(x) \) is an approximation of the density of the set of primes for which \( a_i(p) \) and \( a_j(p) \) are coprime and the data, for different pairs \((f_i, f_j)\), are presented in the second column of the table below, where we have taken \( x = 100000 \). Note that for a large prime \( \ell \), \( \delta(\ell) \) is close to zero and so only the first few terms in the infinite product \( \alpha = \prod_{\ell} (1 - \delta(\ell)) \) should make a significant contribution as the other factors are very close to 1. Thus, we can approximate \( \alpha' \) by taking the product of primes \( \ell \) up to \( L \), for a suitably large integer \( L \). Due to constraints on our computational resources we took \( L = 100 \). Furthermore, since \( \delta \) is hard to compute on the exceptional primes (see §3) and finding the exceptional primes is a difficult task in itself, we have also approximated \( \delta(\ell) \), for primes \( \ell \) up to \( L \), by the ratio \( \delta(y, \ell) := \pi_{f_i, f_j}(y, \ell)/\pi(y) \) for a large integer \( y \). Thus \( \alpha_{L, y}(f_i, f_j) := \prod_{\ell \leq L} (1 - \delta_y(\ell)) \) should
be a crude approximation of the constant $\alpha_{f_i, f_j}$ and the third column gives this approximated value for different pairs $(f_i, f_j)$, where we have taken $y = 50000$. The closeness of the data in columns 2 and 3 of the table inspires some confidence in the truth of this conjecture but further numerical investigations will be welcome.

| Pair of forms $(f_i, f_j)$ | $R(x; f_i, f_j)$ | $\alpha_{L,y}(f_i, f_j)$ |
|---------------------------|------------------|---------------------------|
| $(f_1, f_2)$              | 0.40763          | 0.40757                   |
| $(f_1, f_3)$              | 0.42212          | 0.42414                   |
| $(f_2, f_3)$              | 0.13178          | 0.13265                   |

1.4. Number of prime divisors of $(a_1(p), a_2(p))$. We have studied some related questions about the sequence $\{(a_1(p), a_2(p)) : p \text{ prime}\}$ which are interesting in their own right. The first one concerns the average order of $\omega((a_1(p), a_2(p)))$, where $\omega(n)$ denotes the number of distinct prime divisors of $n$.

If $\alpha(n)$ is an arithmetic function, then it is natural to study the growth of the function $\omega(\alpha(n))$. For instance, it was shown by Murty and Murty [MM84(a), Thm. 6.2] that both $\omega(\phi(n))$ and $\omega(\sigma(n))$, where $\sigma(n)$ is the sum of divisors of $n$, have normal order $\frac{1}{2} (\log \log n)^2$, and if $a_f(n)$’s are the Fourier coefficients of a normalized eigenform $f$, then under GRH, Murty and Murty [MM84(a)] proved that $\omega(a_f(p))$ has normal order $\log \log p$, where $p$ runs over the set of primes. In a subsequent paper [MM84(b)], they proved an analogue of the Erdős-Kac Theorem for $\omega(a_f(p))$ as $p$ runs over primes. Inspired by these results we have studied the sequence $\omega((a_1(p), a_2(p)))$, as $p$ varies over the set of primes. Before stating our main result we first make a few comments to put matters into perspective.

We shall use the notation $\sum' \omega$ to denote sums over primes $p$ with $a_1(p) a_2(p) \neq 0$. First of all, we recall a classical bound which follows easily from PNT but can be proved independently (see [Ram15, §5]):

$$\omega(n) \ll \frac{\log n}{\log \log n},$$

where the implied constant is absolute; and using this we easily obtain the unconditional bound

$$\sum'_{p \leq x} \omega((a_1(p), a_2(p))) \ll \frac{x}{\log \log x},$$

where the implied constant depends only on the two forms. By the work of Murty and Murty (see [MM84(a), Thm. 3.1]), we have, under GRH,

$$\sum'_{p \leq x} \omega((a_1(p), a_2(p))) \ll \sum'_{p \leq x} \omega(a_1(p)) \ll \frac{x \log \log x}{\log x}.$$

We now state our main result.

**Theorem 1.7.** With the same assumptions and notation as above and under GRH, we have the bounds

$$\sum'_{p \leq x} \omega((a_1(p), a_2(p))), \quad \sum'_{p \leq x} \omega^2((a_1(p), a_2(p))) \ll \frac{x}{\log x},$$

where the implied constant depends only on the two forms.

**Corollary 1.8.** For any function $h : \mathbb{N} \to [0, \infty)$ that increases to infinity, however slowly, the subset of primes $\{p : (a_1(p), a_2(p)) > h(p)\}$ has density zero.
Remark 1.9. The upper bound in the above theorem is of the right order of magnitude as the lower bound is also \( \frac{x}{\log x} \). This is clear from (4.6). Thus, we have, in fact,

\[
\sum_{p \leq x}' \omega((a_1(p), a_2(p))) \asymp \frac{x}{\log x}; \quad \sum_{p \leq x}' \omega^2((a_1(p), a_2(p))) \asymp \frac{x}{\log x}.
\]

Obtaining a precise asymptotic formula for these sums appears to be quite difficult, but if we replace the function \( \omega \) with a function that counts only small enough prime divisors of \((a_1(p), a_2(p))\), then we can extract a main term. To be precise, let us define for a positive real number \( u \),

\[
\omega_u(n) := \sum_{p|n, p \leq u} 1.
\]

Then we have:

**Theorem 1.10.** Under GRH, and under the same assumptions and with the same notation as above, there exist explicit constants \( c_1, c_2 > 0 \) such that for any \( \varepsilon > 0 \) we have the following.

(a) For any \( u, x^\varepsilon \leq u < x^{1/12-\varepsilon} \),

\[
\sum_{p \leq x}' \omega_u((a_1(p), a_2(p))) = c_1 \frac{x}{\log x} + O(x^{1-\varepsilon}).
\]

(b) For any \( u, x^\varepsilon \leq u < x^{1/24-\varepsilon} \),

\[
\sum_{p \leq x}' \omega^2_u((a_1(p), a_2(p))) = c_2 \frac{x}{\log x} + O(x^{1-\varepsilon}).
\]

Furthermore, the constants \( c_1 \) and \( c_2 \) are given by

\[
c_1 = \sum_\ell \delta(\ell),
\]

and

\[
c_2 = \sum_{\ell_1 < \ell_2 \atop \ell_1 \neq \ell_2} \delta(\ell_1 \ell_2) + \sum_\ell \delta(\ell),
\]

where \( \delta \) is defined by (3.5) and \( \ell, \ell_1, \ell_2 \) run over the set of all primes. The implied constants in both (a) and (b) depend only on the two forms.

Remark 1.11. As in the case of Theorem 1.5, an analysis of the proof shows that a quasi-GRH statement is sufficient for proving Theorem 1.7 and Theorem 1.10.

1.5. Structure of the paper and the basic ideas of the proofs. The main tools used as black boxes in the proofs are Deligne’s theorem on Galois representations attached to eigenforms and the Chebotarev Density Theorem, especially the effective version due to Lagarias and Odlyzko. These results are recalled in §2. In §3, we use the above machinery to determine the size of the image of the mod-\( m \) Galois representation attached to the pair of eigenforms and to obtain the asymptotic size of the function \( \delta(m) \). Here the results of Ribet and Loeffler on the image of the Galois representations associated to a collection of modular forms play a major role. The main result in the next section is an asymptotic formula (see Prop. 4.1) for the number of primes \( p \) up to \( x \) such that both \( a_1(p) \) and \( a_2(p) \) are divisible by a fixed positive integer \( m \). The idea here is to translate the divisibility condition into a statement about the traces of \( \bar{\rho}_{f_i,m}(\text{Frob}_p) \), where \( \bar{\rho}_{f_i,m} \) denotes mod-\( m \) Galois representation associated to the form \( f_i \) by Deligne’s theorem. Thereafter, in §5, we again obtain some intermediate technical results of analytic nature which are required in the latter sections. Finally, in §6, §7 and §8, we finish the proofs by applying techniques from Analytic Number Theory; in particular, those developed and
employed by by P. Erdős, V. Kumar Murty and M. Ram Murty (see, e.g., [Erd35], [Erd49], [MM84(a)], [MM84(b)], [Mur07]).

1.6. Notation and conventions. By density of a subset $S$ of the set of primes, we mean the natural density; i.e., $\lim_{x \to \infty} \frac{|S \cap [1,x]|}{\pi(x)}$ ($|A|$ denotes the size of a subset $A$ of $\mathbb{N}$), if the limit exists. Here $\pi(x)$ denotes the number of primes less than or equal to $x$ for any real number $x \geq 2$. The letter $\varepsilon$ will denote a positive real number which can be taken to be as small as we want and in different occurrences it may assume different values. The notation $f(y) = O(g(y))$ or $f(y) \ll g(y)$, where $g$ is a positive function, mean that there is a constant $c > 0$ such that $|f(y)| \leq cg(y)$ for any $y$ in the concerned domain. The dependence of this implied constant on some parameter(s) may sometimes be displayed by a suffix (or suffixes) and may sometimes be suppressed but it will be clear from the context. For example, the implied constant will often depend on the pair of forms under consideration. For two positive functions $f$ and $g$, the notation $f(x) \asymp g(x)$ means both the bounds $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold. The notation $f(y) = o(g(y))$, where $g$ is a positive function, means that $f(y)/g(y) \to 0$ as $y \to \infty$ and $f(y) \sim g(y)$ means $f(y) - g(y) = o(g(y))$. The letters $p, q, \ell, \ell_1, \ell_2$ etc. will denote prime numbers throughout. We write $L_i(x) = \log x$ and for $i \geq 2$, define $L_i(x)$ inductively by $L_i(x) = \log L_{i-1}(x)$. PNT and CDT denote the Prime Number Theorem and the Chebotarev Density Theorem, respectively.

2. Background materials

2.1. Chebotarev Density Theorem. Let $K$ be a finite Galois extension of $\mathbb{Q}$ with the Galois group $G$ and degree $n_K$. For an unramified prime $p$, we denote by $\text{Frob}_p$, a Frobenius element of $K$ at $p$ in $G$. For a subset $C$ of $G$, stable under conjugation, we define

$$\pi_C(x) := \{p \leq x : p \text{ unramified in } K \text{ and } \text{Frob}_p \in C\}.$$ Let $d_K$ denote the absolute value of the discriminant of $K/\mathbb{Q}$. An effective version of the Chebotarev Density Theorem (denoted by CDT henceforth) was established by Lagarias and Odlyzko [LO77] and it asserts that there is a constant $c_1 > 0$ such that for every $x \geq 2$ with $\log x \geq c_1 n_K (\log d_K)^2$, we have

$$\left| \pi_C(x) - \frac{|C|}{|G|} \pi(x) \right| \leq \frac{|C|}{|G|} \pi(x^\beta) + O\left( \|C\| x \exp \left( -c' \sqrt{\frac{\log x}{n_K}} \right) \right),$$

(2.1)

where the constant $c' > 0$ is effectively computable, $\|C\|$ denotes the number of conjugacy classes contained in $C$ and $\pi(x) = \int_2^x \frac{dt}{\log t}$, the logarithmic integral function. Here $\beta$ is the possibly existing real “exceptional zero”, also called “the Landau-Siegel zero” of the Dedekind zeta function $\zeta_K(s)$ in the strip

$$1 - \frac{1}{4 \log d_K} \leq \Re(s) < 1;$$

and if $\beta$ does not exist, then the corresponding term is omitted from (2.1). By the works of Heilbronn [Hei72] and Stark [Sta74], we know that (see [Sta74, Eq. (27)])

$$\beta \leq 1 - \frac{c_0}{d_K^{1/n_K}},$$

(2.2)

where $c_0 > 0$ is an effective constant. We also need a conditional version of CDT, that is, under the assumption of GRH. This was first obtained by Lagarias and Odlyzko (op. cit.). We quote the following from [Ser81, Thm. 4].

**Proposition 2.1.** Suppose the Dedekind zeta function $\zeta_K(s)$ satisfies the Riemann Hypothesis. Then for every $x \geq 2$,

$$\pi_C(x) = \frac{|C|}{|G|} \pi(x) + O\left( \frac{|C|}{|G|} x^{1/2} (\log d_K + n_K \log x) \right).$$

(2.3)
Remark 2.2. If we assume a milder version of the hypothesis; namely that $\zeta_k(s)$ does not vanish if $\Re(s) > 1 - \delta$ for some $\delta \in (0, 1/2)$ then we obtain a similar asymptotic formula, the only difference being that the exponent of $x$ becomes $1 - \delta$.

2.2. mod-$m$ Galois representations. In this section, we recall some of the fundamental results on Galois representations associated with modular forms. Let $G_Q = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the Galois group of an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. The following result is due to Deligne.

Theorem 2.3. [Del71] Let $k \geq 2, N \geq 1$ and let $\ell$ be a prime. Then to any normalized eigenform $f \in S_k(N)$ with integer Fourier coefficients $a_f(n)$ one can attach a continuous two-dimensional Galois representation of the rationals

$$\rho_{f,\ell} : G_Q \to \text{GL}_2(\mathbb{Z}_\ell)$$

such that $\rho_{f,\ell}$ is odd and irreducible. Also, for all primes $p \nmid N\ell$ the representation $\rho_{f,\ell}$ is unramified at $p$ and

$$\text{tr}(\rho_{f,\ell}(\text{Frob}_p)) = a_f(p), \quad \det(\rho_{f,\ell}(\text{Frob}_p)) = p^{k-1}.$$

By reduction and semi-simplification, we obtain a mod-$\ell$ Galois representation, namely

$$\overline{\rho}_{f,\ell} : G_Q \to \text{GL}_2(\overline{\mathbb{F}}_\ell),$$

where $\overline{\mathbb{F}}_\ell := \mathbb{Z}/\ell\mathbb{Z}$.

Let $m$ be a positive integer with prime factorization $m = \prod_{j=1}^r \ell_j^{e_j}$. Using the $\ell_j$-adic representations associated to $f$, we construct an $m$-adic representation

$$\rho_{f,m} : G_Q \to \text{GL}_2(\prod_{1 \leq j \leq r} \mathbb{Z}_{\ell_j}).$$

For each $1 \leq j \leq r$, we have the natural projection $\mathbb{Z}_{\ell_j} \to \mathbb{Z}/\ell_j \mathbb{Z}$, and hence we get the reduction $\overline{\rho}_{f,m}$ of $m$-adic representation given by

$$\overline{\rho}_{f,m} : G_Q \to \text{GL}_2(\prod_{1 \leq j \leq r} \mathbb{Z}/\ell_j \mathbb{Z}) \cong \text{GL}_2(\mathbb{Z}/m\mathbb{Z}).$$

Furthermore, if $p \nmid mN$ is a prime, then $\overline{\rho}_{f,m}$ is unramified at $p$ and

$$\text{tr} (\overline{\rho}_{f,m}(\text{Frob}_p)) \equiv a_f(p) \pmod{m}, \quad \det (\overline{\rho}_{f,m}(\text{Frob}_p)) \equiv p^{k-1} \pmod{m}.$$

3. Algebraic preliminaries

Let $f_1 \in S_{k_1}(N_1)$ and $f_2 \in S_{k_2}(N_2)$ be as in the introduction and suppose they have integer Fourier coefficients $a_1(n)$ and $a_2(n)$, respectively. Suppose $\overline{\rho}_{f_i,m} : G_Q \to \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$, for $i = 1, 2$, denotes the mod-$m$ Galois representation associated to $f_i$. Then we consider the product representation $\overline{\rho}_m$ of $\overline{\rho}_{f_1,m}$ and $\overline{\rho}_{f_2,m}$, defined by

$$\overline{\rho}_m : G_Q \to \text{GL}_2(\mathbb{Z}/m\mathbb{Z}) \times \text{GL}_2(\mathbb{Z}/m\mathbb{Z}),$$

$$\sigma \mapsto (\overline{\rho}_{f_1,m}(\sigma), \overline{\rho}_{f_2,m}(\sigma)).$$

Let $\mathcal{A}_m$ denote the image of $G_Q$ under $\overline{\rho}_m$ and let

$$H_m := \ker(\overline{\rho}_m) = \{ \sigma \in G_Q : \overline{\rho}_{f_1,m}(\sigma) = \overline{\rho}_{f_2,m}(\sigma) = \text{Id} \},$$

where $\text{Id}$ denotes the identity element of the group $\text{GL}_2(\mathbb{Z}/m\mathbb{Z})$. Therefore,

$$\frac{G_Q}{H_m} \cong \mathcal{A}_m.$$ (3.1)

Since $\overline{\rho}_{f_i,m}$ is a continuous homomorphism for each $i = 1, 2$, $\overline{\rho}_m$ is continuous. Hence $H_m$ is an open and closed normal subgroup of $G_Q$ (the target group of $\overline{\rho}_{f_i,m}$ being equipped with the discrete topology).
By the fundamental theorem of Galois theory, the fixed field of $H_m$, say $L_m$, is a finite Galois extension of $\mathbb{Q}$ and

$$
\frac{G_\mathbb{Q}}{H_m} \cong \text{Gal}(L_m/\mathbb{Q}).
$$

Combining (3.1) and (3.2), we have

$$
\text{Gal}(L_m/\mathbb{Q}) \cong \mathcal{A}_m.
$$

(3.3)

Let $\mathcal{C}_m$ be a subset of $\mathcal{A}_m$ defined by

$$
\mathcal{C}_m = \{(A, B) \in \mathcal{A}_m : \text{tr}(A) = \text{tr}(B) = 0\}.
$$

(3.4)

Let us now define the following function on the set of positive integers which plays an important role in this work.

**Definition 3.1.** For an integer $m > 1$, define

$$
\delta(m) := \frac{|\mathcal{C}_m|}{|\mathcal{A}_m|}.
$$

(3.5)

and $\delta(1) := 1$.

**Remark 3.2.** Since the trace of the image of complex conjugation is always zero, $\mathcal{C}_m \neq \emptyset$, and hence $\delta(m) > 0$ for every integer $m$. Furthermore, because the identity element lies in $\mathcal{A}_\ell$ but not in $\mathcal{C}_\ell$ for any odd prime $\ell$, we always have $\delta(\ell) < 1$. The prime 2 is special and there are normalized eigenforms all whose coefficients at primes are even; e.g., the Ramanujan Delta function.

3.1. Sizes of $\mathcal{A}_\ell$ and $\mathcal{C}_\ell$ for large primes $\ell$. It is clear that for any prime $\ell$ and any integer $n \geq 1$, $\mathcal{A}_{\ell^n}$ is contained in the set

$$
\{(A, B) \in \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \times \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(A) = v^{k_1-1}, \det(B) = v^{k_2-1}, v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times\}
$$

and it follows from the work of Loeffler [Loe17, Thm. 3.2.2] (see also [Rib75, Thm. (6.1)] for an earlier result for forms of level 1), that outside a finite set of exceptional primes determined by the two forms, $\mathcal{A}_{\ell^n}$ is exactly the above set; i.e., the image of the product mod-$\ell^n$ representation of two eigenforms (that are not twists of each other) is as large as possible if $\ell$ does not belong to a finite set. In other words, there is a positive integer $M = M(f_1, f_2)$ such that for every prime $\ell > M$ and every integer $n \geq 1$, we have,

$$
\mathcal{A}_{\ell^n} = \{(A, B) \in \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \times \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) : \det(A) = v^{k_1-1}, \det(B) = v^{k_2-1}, v \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times\}.
$$

(3.6)

Clearly, we also have, for $\ell > M$,

$$
\mathcal{C}_{\ell^n} = \{(A, B) \in \mathcal{A}_{\ell^n} : \text{tr}(A) = \text{tr}(B) = 0\}.
$$

(3.7)

In the next two lemmas, we compute the cardinalities of $\mathcal{A}_\ell$ and $\mathcal{C}_\ell$ for $\ell > M$. We first set

$$
d = (\ell - 1, k_1 - 1, k_2 - 1).
$$

**Lemma 3.3.** For any prime $\ell > M$,

$$
|\mathcal{A}_\ell| = \frac{1}{d} (\ell - 1)^3 (\ell^2 + \ell)^2.
$$

Proof. First we consider the group $\Lambda$, defined by

$$
\Lambda = \{(v^{k_1-1}, v^{k_2-1}) : v \in \mathbb{F}_\ell^\times\}.
$$

(3.8)
Therefore
\[|\mathcal{C}_\ell| = \sum_{(t_1,t_2)\in \Lambda} |\{(A,B) \in \text{GL}_2(\mathbb{F}_\ell) \times \text{GL}_2(\mathbb{F}_\ell) : (\det(A), \det(B)) = (t_1, t_2)\}| \]
\[= \sum_{(t_1,t_2)\in \Lambda} |\{A \in \text{GL}_2(\mathbb{F}_\ell) : \det(A) = t_1\}| \cdot |\{B \in \text{GL}_2(\mathbb{F}_\ell) : \det(B) = t_2\}|. \]

Since
\[\text{GL}_2(\mathbb{F}_\ell) = \bigcup_{t \in \mathbb{F}_\ell^\times} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \text{SL}_2(\mathbb{F}_\ell),\]
it follows that for any fixed \(t \in \mathbb{F}_\ell^\times\), the cardinality of the set \(\{A \in \text{GL}_2(\mathbb{F}_\ell) : \det(A) = t\}\) does not depend on \(t\) and is equal to \(|\text{SL}_2(\mathbb{F}_\ell)| = (\ell - 1)(\ell^2 + \ell)\). Thus
\[|\mathcal{C}_\ell| = (\ell - 1)^2(\ell^2 + \ell)|\Lambda|. \]

To compute the cardinality of \(\Lambda\), consider the surjective group homomorphism
\[\phi : \mathbb{F}_\ell^\times \to \Lambda\text{ defined by }\phi(v) = (v^{k_1-1}, v^{k_2-1}).\]

By using the fact that \(d = (\ell - 1, k_1 - 1, k_2 - 1)\) one can easily see that \(\text{Ker}(\phi) = \{v \in \mathbb{F}_\ell^\times : v^d = 1\}\), a cyclic subgroup of \(\mathbb{F}_\ell^\times\) of order \(d\). Therefore
\[|\Lambda| = \frac{1}{d}(\ell - 1) \quad (3.9)\]
and this completes the proof.

\begin{lemma}
For any prime \(\ell > M\),
\[|\mathcal{C}_\ell| = \frac{1}{d}d^2(\ell - 1)(\ell^2 + 1). \]
\end{lemma}

\begin{proof}
From the definition of \(\mathcal{C}_\ell\), we can write
\[|\mathcal{C}_\ell| = \sum_{(t_1,t_2)\in \Lambda} |\{A \in \text{GL}_2(\mathbb{F}_\ell) : \det(A) = t_1, \text{tr}(A) = 0\}| \cdot |\{B \in \text{GL}_2(\mathbb{F}_\ell) : \det(B) = t_2, \text{tr}(B) = 0\}|, \quad (3.10)\]

where \(\Lambda\) is defined by (3.8). For \(t \in \mathbb{F}_\ell^\times\), we can easily obtain the following equality by an elementary counting argument:
\[|\{A \in \text{GL}_2(\mathbb{F}_\ell) : \det(A) = t, \text{tr}(A) = 0\}| = \begin{cases} \ell^2 + \ell, & \text{if } -t \text{ is quadratic residue,} \\ \ell^2 - \ell, & \text{if } \ell \text{ is non-residue.} \end{cases} \quad (3.11)\]

Let \((t_1,t_2)\in \Lambda\). Since \(k_1\) and \(k_2\) are even, we see that \(-t_1\) is a quadratic residue (respectively, non-residue) if and only if \(-t_2\) is a quadratic residue (respectively, non-residue). We split the sum on the right hand side of (3.10) into two parts depending on \(-t_1\) is a quadratic residue or not and obtain
\[|\mathcal{C}_\ell| = (\ell^2 + \ell)^2 \sum_{(t_1,t_2)\in \Lambda} 1 + (\ell^2 - \ell)^2 \sum_{(t_1,t_2)\in \Lambda} 1. \quad (3.12)\]

Since the group homomorphism
\[\Lambda \to \{\pm 1\} \text{ defined by } (t_1,t_2) \mapsto \frac{t_1}{\ell}, \]

\begin{lemma}
For any prime \(\ell > M\),
\[|\mathcal{C}_\ell| = \frac{1}{d}d^2(\ell - 1)(\ell^2 + 1). \]
\end{lemma}

\begin{proof}
From the definition of \(\mathcal{C}_\ell\), we can write
\[|\mathcal{C}_\ell| = \sum_{(t_1,t_2)\in \Lambda} |\{A \in \text{GL}_2(\mathbb{F}_\ell) : \det(A) = t_1, \text{tr}(A) = 0\}| \cdot |\{B \in \text{GL}_2(\mathbb{F}_\ell) : \det(B) = t_2, \text{tr}(B) = 0\}|, \quad (3.10)\]

where \(\Lambda\) is defined by (3.8). For \(t \in \mathbb{F}_\ell^\times\), we can easily obtain the following equality by an elementary counting argument:
\[|\{A \in \text{GL}_2(\mathbb{F}_\ell) : \det(A) = t, \text{tr}(A) = 0\}| = \begin{cases} \ell^2 + \ell, & \text{if } -t \text{ is quadratic residue,} \\ \ell^2 - \ell, & \text{if } \ell \text{ is non-residue.} \end{cases} \quad (3.11)\]

Let \((t_1,t_2)\in \Lambda\). Since \(k_1\) and \(k_2\) are even, we see that \(-t_1\) is a quadratic residue (respectively, non-residue) if and only if \(-t_2\) is a quadratic residue (respectively, non-residue). We split the sum on the right hand side of (3.10) into two parts depending on \(-t_1\) is a quadratic residue or not and obtain
\[|\mathcal{C}_\ell| = (\ell^2 + \ell)^2 \sum_{(t_1,t_2)\in \Lambda} 1 + (\ell^2 - \ell)^2 \sum_{(t_1,t_2)\in \Lambda} 1. \quad (3.12)\]

Since the group homomorphism
\[\Lambda \to \{\pm 1\} \text{ defined by } (t_1,t_2) \mapsto \frac{t_1}{\ell}, \]
is surjective, where \( (\cdot) \) denotes the Legendre symbol, the subgroup consisting of quadratic residue elements of \( \Lambda \) is of index two. Therefore,
\[
\sum_{(t_1, t_2) \in \Lambda \atop t_1 \text{ residue}} 1 = \sum_{(t_1, t_2) \in \Lambda \atop t_1 \text{ non-residue}} 1 = \left| \frac{\Lambda}{2} \right| = \frac{1}{2d} (\ell - 1),
\]
where we have used (3.9) in the last equality. Substituting this in (3.12) gives the desired result. \( \square \)

We record two simple consequences of the two foregoing lemmas in the following proposition.

**Proposition 3.5.** For every prime \( \ell > M \),
\[
\delta(\ell) \leq \frac{3}{\ell^2}, \tag{3.13}
\]
and as \( \ell \) varies over primes,
\[
\delta(\ell) \sim \frac{1}{\ell^2} \quad \text{as} \quad \ell \to \infty.
\]

### 3.2. Multiplicativity of \( \delta(m) \).

From the explicit descriptions of \( \mathcal{A}_\ell \) and \( \mathcal{C}_\ell \) for large primes \( \ell \) given in (3.6) and (3.7), it is clear that (see, e.g., [MM84(a), Lemma 5.4]) if \( f_1 \) and \( f_2 \) are two eigenforms as before then the following result holds:

**Proposition 3.6.** For all primes \( \ell_1, \ell_2 > M \) with \( \ell_1 \neq \ell_2 \) and any positive integers \( n_1 \) and \( n_2 \), we have
\[
\delta(\ell_1^{n_1} \ell_2^{n_2}) = \delta(\ell_1^{n_1}) \delta(\ell_2^{n_2}).
\]

However, in the following we prove that if the forms are of level 1, then the function \( \delta \) is multiplicative on the entire set of positive integers. This result is of independent interest and it may be useful in other investigations.

**Proposition 3.7.** Let \( f_1 \) and \( f_2 \) be two normalized eigenforms of level 1 and both have rational integral coefficients. Then \( m \mapsto \delta(m) \) is multiplicative; i.e., if \( m = m_1 m_2 \) and \( (m_1, m_2) = 1 \), then
\[
\delta(m) = \delta(m_1) \delta(m_2).
\]

**Proof.** From the definition of the function \( \delta \), it is sufficient to show that as a function of \( m \), \( |\mathcal{A}_m| \) and \( |\mathcal{C}_m| \) are multiplicative. We first show that the function \( m \mapsto |\mathcal{A}_m| \) is multiplicative. By (3.3) and the fundamental theorem of Galois theory, we know that
\[
|\mathcal{A}_m| = [L_m : \mathbb{Q}]. \tag{3.14}
\]
For \( i = 1, 2 \), \( \bar{\rho}_{m_i} \) is ramified only at the primes dividing \( m_i \) (since the level is 1), and therefore, it follows that \( L_{m_i} \) is ramified only at the primes dividing \( m_i \). Therefore,
\[
L_{m_1} \cap L_{m_2} = \mathbb{Q}. \tag{3.15}
\]
Now, from the definition of mod-\( m \) representations, we have \( H_{m_1 m_2} = H_{m_1} \cap H_{m_2} \) and it easily follows that \( L_m \) is the compositum of \( L_{m_1} \) and \( L_{m_2} \). Therefore (3.14) yields
\[
|\mathcal{A}_m| = [L_{m_1} L_{m_2} : \mathbb{Q}] = \frac{[L_{m_1} : \mathbb{Q}][L_{m_2} : \mathbb{Q}]}{[L_{m_1} \cap L_{m_2} : \mathbb{Q}]} \tag{3.16}
\]
and using (3.15), we obtain
\[
|\mathcal{A}_m| = |\mathcal{A}_{m_1}| |\mathcal{A}_{m_2}|. \tag{3.17}
\]
Next, note that the natural reduction map \( G(\mathbb{Z}/m\mathbb{Z}) \to G(\mathbb{Z}/m_1\mathbb{Z}) \times G(\mathbb{Z}/m_2\mathbb{Z}) \) is an isomorphism, where \( G(R) \) denotes \( GL_2(R) \times GL_2(R) \) for any commutative ring \( R \). Let \( \psi \) be the restriction of the above map to \( \mathcal{A}_m \subset G(\mathbb{Z}/m\mathbb{Z}) \). We see that the image of the map \( \psi \) lies in \( \mathcal{A}_{m_1} \times \mathcal{A}_{m_2} \) and the map
\[
\psi : \mathcal{A}_m \to \mathcal{A}_{m_1} \times \mathcal{A}_{m_2}
\]
is an injection and hence, by (3.17), it is an isomorphism. If we further restrict \( \psi \) to \( \mathcal{C}_m \), then it is easy to see that it gives a bijection of sets
\[
\mathcal{C}_m \rightarrow \mathcal{C}_{m_1} \times \mathcal{C}_{m_2}.
\]
(3.18)
This completes the proof of the proposition. \( \square \)

**Remark 3.8.** In the case of higher level \( N > 1 \), there can be primes dividing the respective levels where the mod-\( m \) representations are ramified and hence the conclusion that \( L_{m_1} \cap L_{m_2} = \mathbb{Q} \) will be false in general. We can say, however, that the map \( \psi \) is an injective homomorphism when restricted to \( \mathcal{A}_m \) and is an injective map of sets when restricted to \( \mathcal{C}_m \) and thus we can conclude that if \( (m_1, m_2) = 1 \),
\[
|\mathcal{A}_{m_1m_2}| \leq |\mathcal{A}_{m_1}| |\mathcal{A}_{m_2}| \quad \text{and} \quad |\mathcal{C}_{m_1m_2}| \leq |\mathcal{C}_{m_1}| |\mathcal{C}_{m_2}|.
\]  
(3.19)
Also, note that for \( i = 1, 2 \), if \( \psi_i : G(\mathbb{Z}/m_i \mathbb{Z}) \rightarrow G(\mathbb{Z}/m_i \mathbb{Z}) \) is the natural projection map then \( \psi_i(\mathcal{A}_{m_1m_2}) = \mathcal{A}_m \) and \( \psi_i(\mathcal{C}_{m_1m_2}) = \mathcal{C}_m \). Therefore,
\[
|\mathcal{A}_{m_1m_2}| \geq |\mathcal{A}_m| \quad \text{and} \quad |\mathcal{C}_{m_1m_2}| \geq |\mathcal{C}_m|.
\]  
(3.20)

### 3.3. Sizes of \( \mathcal{A}_m \) and \( \mathcal{C}_m \) for a general integer \( m \)

By calculations as in §3.1, one can show that
\[
|\mathcal{A}_m| \ll \ell^m \quad \text{and} \quad |\mathcal{C}_m| \ll \ell^m
\]  
(3.21)
for every prime \( \ell \) and every integer \( n \geq 1 \); and for \( \ell \) large enough, we have
\[
|\mathcal{A}_m| \asymp \ell^m \quad \text{and} \quad |\mathcal{C}_m| \asymp \ell^m.
\]

Now let \( m \) be any positive integer. By considering its prime factorization and applying the first bounds in Remark 3.8, we obtain from (3.21) the following bounds:
\[
|\mathcal{A}_m| \ll m^7 \quad \text{and} \quad |\mathcal{C}_m| \ll m^5.
\]  
(3.22)

### 4. Asymptotic formula for \( \pi_{f_1,f_2}(x, m) \) and \( \pi_{f_1,f_2}^*(x, m) \)

Let \( f_1 \) and \( f_2 \) be two non-CM eigenforms as in the previous section. For a positive integer \( m \) and a real number \( x \geq 2 \), define
\[
\pi_{f_1,f_2}(x, m) := \sum_{p \leq x : (p,m) = 1 \atop m|(a_1(p),a_2(p))} 1.
\]  
(4.1)
To obtain an asymptotic formula for \( \pi_{f_1,f_2}(x, m) \), our aim is to apply CDT for the finite Galois extension \( L_m/\mathbb{Q} \), \( L_m \) being the fixed field of the kernel of the mod-\( m \) representation \( \bar{\rho}_m = (\bar{\rho}_{f_1,m}, \bar{\rho}_{f_2,m}) \). Clearly, the representation \( \bar{\rho}_m \) is unramified at a prime \( p \) such that \( (p,m) = 1 \), where \( N = \text{lcm}(N_1, N_2) \), and hence \( p \) is unramified in \( L_m \). Moreover, for such a prime \( p \)
\[
\text{tr} (\bar{\rho}_m(\text{Frob}_p)) \equiv (a_1(p) \mod m, a_2(p) \mod m).
\]
Thus, we can write
\[
\pi_{f_1,f_2}(x, m) = |\{ p \leq x : p \text{ unramified in } L_m, \bar{\rho}_m(\text{Frob}_p) \in \mathcal{C}_m \}| + O(1),
\]  
(4.2)
where \( \mathcal{C}_m \) is defined by (3.4) and term \( O(1) \) is to account for the presence of possible prime divisors of \( N \) at which \( \bar{\rho}_m \) is unramified. Note that \( \bar{\rho}_m \) is ramified at primes \( p|m \) because a non-trivial power of the mod \( p \) cyclotomic character is a component of its determinant, which is ramified at \( p \). Now we state the main result of this section.

**Proposition 4.1.** Let \( f_1 \in S_{k_1}(N_1) \) and \( f_2 \in S_{k_2}(N_2) \) be as before. Let \( N = \text{lcm}(N_1, N_2) \) and \( m \geq 1 \) be an integer. Then we have:
we conclude the following.

\[ \pi_{f_1, f_2}(x, m) = \delta(m) \text{Li}(x) + O \left( \delta(m) \text{Li}(x^{\beta}) \right) + O \left( m^5 x \exp \left( -c' \frac{\log x}{m^7} \right) \right), \tag{4.3} \]

where \( c' > 0 \) is an effectively computable constant.

(b) Under GRH, for any \( x \geq 2 \), we have

\[ \pi_{f_1, f_2}(x, m) = \delta(m)\pi(x) + O \left( m^5 x^{1/2} \log(mN) \right). \tag{4.4} \]

Here the \( O \)-constants are absolute in both (a) and (b).

**Proof.** Note that the map \( \tilde{\rho}_m \) descends to an isomorphism \( \tilde{\rho}_m : G \xrightarrow{\sim} \mathcal{A}_m \), where \( G \) denotes \( \text{Gal}(L_m/\mathbb{Q}) \). We take \( C \) to be the subset

\[ C := \{ \sigma \in G : \tilde{\rho}_m(\sigma) \in \mathcal{C}_m \}, \]

of \( G \) which is clearly conjugacy-invariant. Now we apply CDT (see (2.1) and (2.3)) to obtain the two statements above. Note that the number of conjugacy classes in \( C \) is at most \( |C| = |\mathcal{C}_m| \). Here we have used the fact that \( n_{L_m} \ll m^7 \). This follows from combining (3.14) and (3.22). We have also used a consequence of an inequality of Hensel (see [Ser81, Prop. 5, p. 129]) that says,

\[ \log d_{L_m} \leq |\mathcal{A}_m| \log(mN|\mathcal{A}_m|). \tag{4.5} \]

Next we define

\[ \pi_{f_1, f_2}^*(x, m) = \{ p \leq x : a_1(p) a_2(p) \neq 0, a_1(p) \equiv a_2(p) \equiv 0 \pmod{m} \}. \]

It is well known (see [Ser81, p. 175]) that

\[ |\{ p \leq x : a_i(p) = 0 \}| = \begin{cases} O \left( \frac{x}{(\log x)^{3/2-\varepsilon}} \right), & \text{for any } \varepsilon > 0, \\ O(x^{3/4}), & \text{under GRH}; \end{cases} \tag{4.6} \]

and hence using Prop. 4.1 we conclude the following.

**Proposition 4.2.** Let the assumptions be as in Prop. 4.1. Then

(a) for \( \log x \gg m^{21}(\log(mN))^2 \)

\[ \pi_{f_1, f_2}^*(x, m) = \delta(m) \text{Li}(x) + O \left( \delta(m) \text{Li}(x^{\beta}) \right) + O \left( \frac{x}{(\log x)^{3/2-\varepsilon}} \right), \]

for any small \( \varepsilon > 0 \).

(b) Under GRH, we have

\[ \pi_{f_1, f_2}^*(x, m) = \delta(m)\pi(x) + O \left( m^{5} x^{1/2} \log(mN) \right) + O(x^{3/4}). \]

Here the \( O \)-constants are absolute in both (a) and (b).

**Remark 4.3.** Note that the error terms that appear in the conditional versions of the above propositions are quite large in terms of \( m \). This makes handling the sum of \( \omega((a_1(p), a_2(p))) \) over primes \( p \) difficult since the technique we use in proving Thm. 1.7 requires summing \( \pi_{f_1, f_2}^*(x, \ell) \) over primes \( \ell \). This is the reason we are unable to obtain an asymptotic formula in Thm. 1.7. Herein lies also the difficulty in proving Conjecture (1.8).
5. Preparation for the proof of Theorem 1.1

The bound in (4.5), together with (2.2), implies that the Landau-Siegel zero $\beta$, if it exists, satisfies

$$\beta \leq 1 - \frac{c_0}{Nm^8},$$

where $c_0$ is as in (2.2). Therefore, in such a case we can choose a constant $c > 0$ such that

$$\beta \leq 1 - \frac{1}{m^c}$$

uniformly over all $m \geq 2$ and we can assume without loss of generality that $c \geq 21$. For example, we can take

$$c = \max \left\{ 21, 9 + \frac{|\log(N/c_0)|}{\log 2} \right\}. \tag{5.1}$$

For this and the next section only, we set

$$y = y(x) = L_2(x)^{\eta}, \text{ where } 0 < \eta < \min \left\{ \frac{1}{21}, \frac{1}{c - 2} \right\}. \tag{5.2}$$

Here $c$ is as above and if $\beta$ does not appear in Prop. 4.1, then we set $c = 21$.

Below we will frequently use standard estimates such as

$$\sum_{p \leq x} \frac{1}{p} \ll L_2(x), \text{ or } \sum_{p \leq x} \frac{\log p}{p} \ll \log x.$$

**Lemma 5.1.** With the assumptions and notation as above, we have

(a) for any $\ell \leq y$

$$\sum_{p \leq x, \ell | (a_1(p), a_2(p))} \frac{1}{p} = \delta(\ell) L_2(x) + O \left( \frac{\ell^{c-2}}{L_2(x)} \right) + O(L_3(x)). \tag{5.3}$$

(b) Under GRH

$$\sum_{p \leq x, \ell | (a_1(p), a_2(p))} \frac{1}{p} = \delta(\ell) L_2(x) + O (\ell^5 \log \ell).$$

**Proof.** By partial summation, we write

$$\sum_{p \leq x, \ell | (a_1(p), a_2(p))} \frac{1}{p} = \frac{1}{x} \pi_{f_1,f_2}(x, \ell) + \int_2^x \frac{1}{t^2} \sum_{\ell | (a_1(p), a_2(p))} \pi_{f_1,f_2}(t, \ell) dt + O(1),$$

where the error term is present because of the primes dividing $N\ell$. Since

$$\frac{1}{x} \pi_{f_1,f_2}(x, \ell) \leq \frac{1}{x} \pi(x) = O(1),$$

we have,

$$\sum_{p \leq x, \ell | (a_1(p), a_2(p))} \frac{1}{p} = \int_2^x \frac{1}{t^2} \sum_{\ell | (a_1(p), a_2(p))} \pi_{f_1,f_2}(t, \ell) dt + O(1). \tag{5.4}$$

Now subdividing the interval $[2, x]$, we write

$$\sum_{p \leq x, \ell | (a_1(p), a_2(p))} \frac{1}{p} = \int_2^T \frac{1}{t^2} \pi_{f_1,f_2}(t, \ell) dt + \int_T^x \frac{1}{t^2} \pi_{f_1,f_2}(t, \ell) dt + O(1),$$
where $T$ is to be chosen later. The first integral on the right hand side is

$$
\ll \int_2^T \frac{1}{t^2} \pi(t) \, dt \ll L_2(T).
$$

To estimate the second integral, we assume that $T$ is large enough so that (4.3) can be applied for $h = \ell$; i.e., we need

$$
\ell^{21} (\log(\ell N))^2 \ll \log T.
$$

Since $y$ satisfies (5.2), we have,

$$
\ell^{21} (\log(\ell N))^2 \leq y^{21} (\log(yN))^2 \ll L_2(x)^{21\eta} L_3(x)^2 \ll L_2(x)^{1-\varepsilon}
$$

for some suitable $\varepsilon > 0$; and hence, we can take $T = \log x$. Therefore, the second integral is

$$
\int_{\log x}^x \frac{1}{t^2} \left( \delta(\ell) \text{Li}(t) + O \left( \delta(\ell) \text{Li}(t^\beta) \right) + O \left( \ell^5 t \exp \left( -c' \sqrt{\frac{\log t}{\ell^7}} \right) \right) \right) \, dt.
$$

Now we see that

$$
\int_{\log x}^x \delta(\ell) \frac{\text{Li}(t)}{t^2} \, dt = \int_{\log x}^x \delta(\ell) \left( \frac{1}{t \log t} + O \left( \frac{1}{t (\log t)^2} \right) \right) \, dt
$$

$$
= \delta(\ell) L_2(x) + O(L_3(x)).
$$

Next we consider the integral involving $\beta$. From (5.1) we know that $\beta \leq 1 - \frac{\ell}{\ell^c}$. Therefore,

$$
\int_{\log x}^x \frac{\text{Li}(t^\beta)}{t^2} \, dt \ll \int_{\log x}^x \frac{1}{t^{2-\beta} \log t} \, dt \ll \int_{\log x}^x \frac{1}{t \log t} \exp \left( -\frac{\log t}{\ell^c} \right) \, dt
$$

$$
\ll \ell^c (L_2(x))^{-1} \exp \left( -\frac{L_2(x)}{\ell^c} \right) \ll \ell^c (L_2(x))^{-1}.
$$

Finally,

$$
\int_{\log x}^x \ell^c \frac{1}{t} \exp \left( -c' \sqrt{\frac{\log t}{\ell^7}} \right) \, dt = \ell^c \int_{L_2(x)}^{\log x} \exp \left( -c' \sqrt{\frac{u}{\ell^7}} \right) \, du
$$

$$
\ll \ell^{c+7/2} \sqrt{L_2(x)} \exp \left( -c' \sqrt{\frac{L_2(x)}{\ell^7}} \right)
$$

$$
\ll \ell^{19} (L_2(x))^{-1}
$$

and this completes the proof of part (a).

The proof of part (b) is very similar after applying (4.4) in (5.4).

Given a positive integer $n$ and a prime $\ell$, we define

$$
v(\ell, n) = |\{p^\alpha : p^\alpha \parallel n \text{ and } \ell \parallel (a_1(p^\alpha), a_2(p^\alpha))\}|.
$$

(5.5)

In the next two lemmas, we obtain asymptotic formulae for the partial sums of $v(\ell, n)$ and $v^2(\ell, n)$ which will play an important role in proving Thm. 1.1.

**Lemma 5.2.** With the same assumption and notation as above, we have,

(a) for any $\ell \leq y$,

$$
\sum_{n \leq x} v(\ell, n) = \delta(\ell) x L_2(x) + O(x L_3(x));
$$

(b) for any $\ell \leq y$,

$$
\sum_{n \leq x} v^2(\ell, n) = \delta(\ell) x^2 L_2(x) + O(x^2 L_3(x)).
$$
(b) and under GRH, we have, for any prime $\ell$,
\[
\sum_{n \leq x} v(\ell, n) = \delta(\ell)x L_2(x) + O\left(\ell^5 x \log \ell\right).
\]

**Proof.** We write
\[
\sum_{n \leq x} v(\ell, n) = \sum_{\ell | (\alpha(p), \alpha(q))} \sum_{n \leq x} 1 = \sum_{\ell | (\alpha(p), \alpha(q))} \sum_{n \leq x} 1.
\]

We split the sum into two parts, the one with $\alpha = 1$ and the other with $\alpha \geq 2$. Since the contribution from all the terms with $\alpha \geq 2$ is $O(x)$, so we can write
\[
\sum_{n \leq x} v(\ell, n) = \sum_{\ell | (\alpha(p), \alpha(q))} \sum_{n \leq x} 1 + O(x). \tag{5.6}
\]

Simplifying further, we use the easily proved asymptotic formula
\[
\sum_{n \leq x} 1 = \frac{x}{p} + O\left(\frac{x}{p^2}\right) + O(1)
\]
to obtain
\[
\sum_{\ell | (\alpha(p), \alpha(q))} \sum_{n \leq x} 1 = \sum_{\ell | (\alpha(p), \alpha(q))} \left\{ \frac{x}{p} + O\left(\frac{x}{p^2}\right) + O(1) \right\} = x \left( \sum_{\ell | (\alpha(p), \alpha(q))} \frac{1}{p} \right) + O(x).
\]

Now use Lemma 5.1 for the sum appeared in the right hand side of this expression to obtain
\[
\sum_{\ell | (\alpha(p), \alpha(q))} \sum_{n \leq x} 1 = \begin{cases} \delta(\ell)x L_2(x) + O\left(\frac{\ell^5 x - 2}{L_2(x)}\right) + O(xL_3(x)), & \ell \leq y, \\
\delta(\ell)x L_2(x) + O\left(\ell^5 x \log \ell\right), & \text{under GRH.}\end{cases} \tag{5.7}
\]

Finally, substituting (5.7) in (5.6) and recalling the choice of $y$ in (5.2), we finish the proof. \hfill \square

**Lemma 5.3.** We have,
(a) for any $\ell \leq y$,
\[
\sum_{n \leq x} v^2(\ell, n) = \delta(\ell)^2 x (L_2(x))^2 + O\left(\delta(\ell)x L_2(x)L_3(x)\right);
\]
(b) and under GRH, we have, for any prime $\ell$,
\[
\sum_{n \leq x} v^2(\ell, n) = \delta(\ell)^2 x (L_2(x))^2 + O\left(\ell^{10} x (\log \ell)^2\right) + O\left(\ell^3 x L_2(x) \log \ell\right).
\]

**Proof.** We have,
\[
\sum_{n \leq x} v^2(\ell, n) = \sum_{n \leq x} \sum_{\ell | (\alpha(p), \alpha(q))} 1.
\]

We split the above sum into three parts: the first one with $\alpha = \beta = 1$, the second one with exactly one of $\alpha$ and $\beta = 1$; and the third one with $\min\{\alpha, \beta\} > 1$. In view of Lemma 5.1, the second sum will contribute $O(xL_2(x))$ whereas the contribution from the last sum is $O(x)$. Therefore, introducing the notation
\[
D(\ell) := \{(p, q) : \ell | (\alpha(p), \alpha(q)), \ell | (\alpha(q), \alpha(p))\},
\]
we may write
\[
\sum_{n \leq x} v^2(\ell, n) = \sum_{n \leq x} \sum_{(p,q) \in D(\ell)} 1 + O(xL_2(x))
\]
\[
= \sum_{n \leq x} \sum_{p \leq x, p||n} 1 + \sum_{n \leq x} \sum_{(p,q) \in D(\ell)} 1 + O(xL_2(x)).
\]
(5.8)

The first sum on the right of the above equation is already examined in (5.7). So we now simplify the latter sum.

\[
\sum_{n \leq x} \sum_{(p,q) \in D(\ell)} \frac{1}{pq} = 2 \sum_{p \leq \sqrt{x}} \frac{1}{p} \sum_{q \leq \sqrt{x}} \frac{1}{q} - \left( \sum_{p \leq \sqrt{x}} \frac{1}{p} \right)^2.
\]
(5.10)

and using part (a) of Lemma 5.1 for each individual sum, we obtain

\[
\sum_{(p,q) \in D(\ell)} \frac{1}{pq} = \delta(\ell)^2L_2(x)^2 + O(\delta(\ell)L_2(x)L_3(x)) + O(\ell^{2c-2}).
\]
(5.11)

Noting that \(\ell \leq y\) and combining (5.2), (5.8), (5.9) and (5.11) completes the proof of part (a).

To prove part (b), we use Lemma 5.1 (b) in (5.10) and proceed as before.

\[\Box\]

6. Proof of Theorem 1.1

Recall that the parameter \(y\) is defined by (5.2).

To prove part (a), we first write

\[
\sum_{n \leq x} \frac{1}{p} = \sum_{n \leq x} \frac{1}{p} = \sum_{n \leq x} \frac{1}{p} + \sum_{\ell | n \Rightarrow \ell \leq y} 1.
\]
(6.1)

We denote the first and the second sum on the right hand side of (6.1) by \(S_1\) and \(S_2\), respectively. Now to estimate \(S_1\), we need the following standard and easily proved lemma (the sieve of Eratosthenes).

**Lemma 6.1.** For any real numbers \(x \geq 3\) and \(y \geq 2\), we have

\[
\sum_{1 \leq n \leq x \atop p|n \Rightarrow p>y} 1 = x \prod_{p \leq y} \left(1 - \frac{1}{p}\right) + O(2^y).
\]
(6.2)
Using the above lemma, we can now write

$$S_1 \ll \sum_{
 \leq x \atop p|n} 1 \ll x \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \ll \frac{x}{L_3(x)}.$$  \hspace{1cm} (6.3)

For estimating $S_2$, we first note that the Fourier coefficients are multiplicative since $f_1$ and $f_2$ are both eigenforms. It follows that if $(n, (a_1(n), a_2(n))) = 1$, then $v(\ell, n) = 0$ for all primes $\ell|n$, where $v(\ell, n)$ is defined by (5.5). This assertion will be false if we do not have multiplicativity over the entire set of positive integers and this is why we need to restrict to forms that are eigenfunctions of all the Hecke operators. Thus

$$S_2 = \sum_{(n, (a_1(n), a_2(n))) = 1} 1 \leq \sum_{\ell \leq y} \sum_{n \leq x, \ell|n} 1 \leq \sum_{\ell \leq y} \sum_{n \leq x, v(\ell, n) = 0} 1.$$  \hspace{1cm} (6.4)

The idea for estimating the inner sum $\sum_{n \leq x, v(\ell, n) = 0} 1$ for a given prime $\ell \leq y$ is encapsulated in the following simple yet crucial lemma. The idea of this lemma is not new. See, e.g., [Mur07].

**Lemma 6.2.** Suppose $(a_n)$ is a sequence of real numbers such that

$$\sum_{n \leq x} a_n = c(x)x + E_1(x) \quad \text{and} \quad \sum_{n \leq x} a_n^2 = c(x)^2x + E_2(x),$$

for large enough $x$, where we assume that $c(x)$ is a function that never vanishes. Then we have,

$$\sum_{n \leq x, a_n = 0} 1 \leq c(x)^2 \left(E_2(x) - 2c(x)E_1(x)\right).$$

**Proof.** This is clear once we observe that

$$\sum_{n \leq x} (a_n - c(x))^2 \geq \sum_{n \leq x, a_n = 0} c(x)^2,$$

by non-negativity. \hfill \Box

As mentioned in Remark 3.2, we recall that for any prime $\ell$, the image of complex conjugation lies in $G_{\ell}$ because the trace of this image is always zero. This shows that $G_{\ell} \neq \phi$ and hence $\delta(\ell)$ is never zero. Now applying the above lemma with $a_n = v(\ell, n)$ in conjunction with Lemma 5.2 and Lemma 5.3 (with $c(x) = \delta(\ell)L_2(x)$), we obtain

$$\sum_{n \leq x, v(\ell, n) = 0} 1 \ll \frac{xL_3(x)}{\delta(\ell)L_2(x)}.$$  \hspace{1cm} (6.5)

Substituting the above bound in (6.4), recalling that $\delta(\ell) \sim \ell^{-2}$ and the size of the parameter $y$ given in (5.2), we obtain

$$S_2 \ll \frac{x}{L_2(x)^{6/7}}.$$  \hspace{1cm} (6.6)

Finally, substituting estimates (6.3) and (6.6) in (6.1) completes the proof.
To prove part (b), we first note that if \((d, (a_1(n), a_2(n))) = 1\), then \(v(\ell, n) = 0\) for all primes \(\ell \mid d\). Thus
\[
\sum_{n \leq x} 1 \leq \sum_{\substack{n \leq x \\ell \mid d \\nu(\ell, n) = 0}} 1 = \sum_{\ell \mid d} \sum_{n \leq x} 1.
\]
Now (6.5) yields the result.

7. Proof of Theorem 1.5

We shall denote \((a_1(p), a_2(p))\) by \(a_p\). Our goal is to estimate the number of primes \(p\) up to \(x\) for which \(a_p = 1\). Motivated by the theory of sieves, we first make a simple yet crucial observation:
\[
C(x; f_1, f_2) = \left| \{ p \leq x : a_p = 1 \} \right| \leq \left| \{ p \leq x : (a_p, P(y)) = 1 \} \right|,
\]
where \(P(y) = \prod_{\ell < y} \ell\), \(\ell\) running over primes, for some parameter \(y\) to be chosen later, subject to the conditions \(y < x\) and that \(y\) goes to infinity along with \(x\). Our goal is to estimate the sum on the right accurately. This is usually done using sieves. However, since the density function \(\delta\) is not known to multiplicative on the entire set of integers in the general case, we cannot apply standard results from Sieve Theory directly. Instead, we start from the scratch by expressing the coprimality condition by the Möbius function. Thus we write,
\[
\sum_{p \leq x} 1 = \sum_{\substack{p \leq x \\left( a_p, P(y) \right) = 1}} \mu(d) = \sum_{d \mid P(y)} \mu(d) \sum_{\substack{p \leq x \\left( a_p \equiv 0 \mod d \right)}} 1.
\]
Now, under the assumption of GRH, we have, by (4.4),
\[
\sum_{p \leq x \left( a_p \equiv 0 \mod d \right)} 1 = \delta(d)\pi(x) + O \left( d^5 x^{1/2} \log(dN) \right).
\]
Therefore,
\[
\sum_{p \leq x \left( a_p, P(y) \right) = 1} 1 = \pi(x) \sum_{d \mid P(y)} \mu(d)\delta(d) + O \left( x^{1/2} \sum_{d \mid P(y)} \mu^2(d)d^5 \log(dNx) \right).
\]
We first treat the sum in the error term. First of all, for \(d \mid P(y)\),
\[
\log d \leq \log P(y) = \sum_{\ell < y} \log \ell \ll y,
\]
by PNT. Therefore,
\[
\sum_{d \mid P(y)} \mu^2(d)d^5 \log(dN) \ll (y \log x) \sum_{d \mid P(y)} \mu^2(d)d^5.
\]
Now,
\[
\sum_{d \mid P(y)} \mu^2(d)d^5 = \prod_{\ell < y} (1 + \ell)^5 \ll \prod_{\ell < y} \ell^5 \ll \exp((5 + \varepsilon)y),
\]
for any fixed $\epsilon > 0$, again by PNT. Therefore,
\[
x^{1/2} \sum_{d \mid P(y)} \mu^2(d)d^\epsilon \log(dN x) = O \left( x^{1/2} \exp((5 + \epsilon) y)(y + \log x) \right).
\] (7.2)

We now treat the sum in the main term. Note that
\[
\sum_{d \mid P(y)} \mu(d)d^\epsilon = \alpha' - \sum_{d \not\mid P(y)} \mu(d)d^\epsilon.
\] (7.3)

Now to handle this new sum, we need to overcome the problem of the lack of multiplicativity coming from the small primes. We first recall the definition of $M = M(f_1, f_2)$ in the beginning of §3.1 and the bound $\delta(\ell) \leq 3/\ell^2$ for $\ell > M$ (see (3.13)). We observe that every $d$ in the above sum can be factored uniquely as $d = d_1d_2$, where
\[
d_1 = \prod_{p \mid d \leq M} p
\]
and $d_2 = d/d_1$. We also make another observation that for any two positive integers $a$ and $b$, $\delta(ab) \leq \delta(a)$. This does not follow directly from the definition of $\delta$ but the observation is clear once we interpret $\delta$ as a density using Prop. 4.1; namely,
\[
\delta(a) = \lim_{x \to \infty} \frac{\pi_{f_1,f_2}(x,a)}{\pi(x)},
\]
since, trivially, $\pi_{f_1,f_2}(x,ab) \leq \pi_{f_1,f_2}(x,a)$. Using the above observations and recalling the standard notations $P^+(n)$ and $P^-(n)$ for the largest and the smallest prime factor of a positive integer $n$, respectively, we write
\[
\left| \sum_{d \not\mid P(y)} \mu(d)d^\epsilon \right| \leq \sum_{d \not\mid P(y)} \mu^2(d)d^\epsilon
\]
\[
= \sum_{P^+(d) > y} \mu^2(d)d^\epsilon
\]
\[
\leq \sum_{P^+(d_1) \leq M} \sum_{P^-(d_2) > M} \sum_{P^+(d_2) > y} \mu^2(d_1d_2)d^\epsilon
\]
\[
\leq 2M \sum_{c > y} \frac{3\omega(c)}{c^2}
\]
\[
\ll y^{-1}(\log y)^2,
\]
by a well-known classical estimate and partial summation. By this bound and (7.1), (7.2), and (7.3), we finally obtain
\[
\sum_{p \leq x \atop (a_p, P(y)) = 1} \pi(x) = \alpha' \pi(x) + O(y^{-1}(\log y)^2 \pi(x)) + O \left( x^{1/2} \exp((5 + \epsilon) y)(y + \log x) \right).
\]

Now, if we choose $y = \frac{1}{12} \log x$ and $0 < \epsilon < 1/100$, we see that both the error terms are $o(\pi(x))$. 
8. Proofs of Theorem 1.7 and Theorem 1.10

For this section, we set $z = x^{1/12-\eta}$ for some fixed real number $\eta \in (0,1/100)$. We first prove two lemmas.

**Lemma 8.1.** Under GRH, we have
\[ \sum_{\ell \leq z} |\pi_{f_1,f_2}^* (x, \ell) - \delta(\ell)\pi(x)| = o(\pi(x)). \]

**Proof.** From part (b) of Prop. 4.2 we obtain
\[ \sum_{\ell \leq z} |\pi_{f_1,f_2}^* (x, \ell) - \delta(\ell)\pi(x)| = O\left( x^{3/4} \sum_{\ell \leq z} \ell^{5} \log(\ell Nx) \right) + O\left( x^{3/4} \sum_{\ell \leq z} 1 \right). \]

Because of our choice of $z$, both the error terms on the right hand side are $o(\pi(x))$ and this completes the proof. \qed

**Lemma 8.2.** Under GRH, we have
\[ \sum_{\ell} \pi_{f_1,f_2}^* (x, \ell) \ll \frac{x}{\log x}. \]

**Proof.** First, we write
\[ \sum_{\ell} \pi_{f_1,f_2}^* (x, \ell) = \sum_{\ell \leq z} \pi_{f_1,f_2}^* (x, \ell) + \sum_{\ell > z} \pi_{f_1,f_2}^* (x, \ell). \]

Since
\[ \sum_{\ell \leq z} \pi_{f_1,f_2}^* (x, \ell) \leq \sum_{\ell \leq z} |\pi_{f_1,f_2}^* (x, \ell) - \delta(\ell)\pi(x)| + \sum_{\ell \leq z} \delta(\ell)\pi(x), \]

applying Lemma 8.1, Prop. 3.5, and PNT, we obtain the bound
\[ \sum_{\ell \leq z} \pi_{f_1,f_2}^* (x, \ell) \ll \frac{x}{\log x}. \]

Thus in order to complete the proof it suffices to show that \( \sum_{\ell > z} \pi_{f_1,f_2}^* (x, \ell) \ll \frac{x}{\log x} \). Now
\[ \sum_{\ell > z} \pi_{f_1,f_2}^* (x, \ell) = \sum_{p \leq x} \sum_{\ell \geq z \atop \ell \mid (a_1(p),a_2(p))} 1, \]

and using the fact that \((a_1(p),a_2(p)) \ll x^{(k-1)/2}\) for \(p \leq x\) we have
\[ \sum_{\ell \geq z \atop \ell \mid (a_1(p),a_2(p))} 1 \ll \frac{\log x}{\log z} = O(1), \quad (8.1) \]

which yields the desired result. \qed

8.1. **Proof of Theorem 1.7.** We observe that under GRH, the following bound holds:
\[ \sum_{p \leq x} \omega((a_1(p),a_2(p))) \ll \frac{x}{\log x}. \quad (8.2) \]

Indeed,
\[ \sum_{p \leq x} \omega((a_1(p),a_2(p))) = \sum_{p \leq x} \sum_{\ell \mid (a_1(p),a_2(p))} 1 = \sum_{\ell} \sum_{p \leq x \atop \ell \mid (a_1(p),a_2(p))} 1 = \sum_{\ell} \pi_{f_1,f_2}^* (x, \ell). \]
Now the estimate (8.2) is clear after invoking Lemma 8.2.
By the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\) for any two real numbers \(a\) and \(b\), we can write
\[
\sum_{p \leq x}^\prime \omega^2 ((a_1(p), a_2(p))) \ll \sum_{p \leq x}^\prime \left( \omega ((a_1(p), a_2(p))) - \omega_{\sqrt{z}}((a_1(p), a_2(p))) \right)^2 + \sum_{p \leq x}^\prime \omega^2((a_1(p), a_2(p))).
\]

Now, \(\omega((a_1(p), a_2(p))) - \omega_{\sqrt{z}}((a_1(p), a_2(p)))\) is the number of distinct prime divisors of \((a_1(p), a_2(p))\) lying between \(\sqrt{z}\) and \(2x^{(k-1)/2}\) and hence from (8.1)
\[
\sum_{p \leq x}^\prime \left( \omega ((a_1(p), a_2(p))) - \omega_{\sqrt{z}}((a_1(p), a_2(p))) \right)^2 \ll \frac{x}{\log x}.
\]

Thus to complete the proof it remains to show that
\[
\sum_{p \leq x}^\prime \omega^2_{\sqrt{z}}((a_1(p), a_2(p))) \ll \frac{x}{\log x}
\]
and this follows from part (b) of Thm. 1.10 which is proved in the next section.

8.2. Proof of Theorem 1.10. To prove part (a), we write
\[
\sum_{p \leq x}^\prime \omega_u((a_1(p), a_2(p))) = \sum_{\ell \leq u} \pi_{f_1,f_2}^\ast(x, \ell)
\]
\[
= \pi(x) \left( \sum_{\ell} \delta(\ell) - \sum_{\ell > u} \delta(\ell) \right) + \sum_{\ell \leq u} \left( \pi_{f_1,f_2}^\ast(x, \ell) - \delta(\ell) \pi(x) \right) .
\]
(8.3)

By Prop. 3.5, we know that the series \(\sum_{\ell} \delta(\ell)\) is convergent and we denote the sum by \(c_1\). Obviously, \(c_1 > 0\). Also, by Prop. 3.5 and partial summation, we have the bound
\[
\sum_{\ell > u} \delta(\ell) \ll \frac{1}{u}.
\]

Thus applying Prop. 4.2 in (8.3) and using PNT, we obtain
\[
\sum_{p \leq x}^\prime \omega_u((a_1(p), a_2(p))) = c_1 \pi(x) + O \left( \frac{x}{u \log x} \right) + O \left( \frac{x^{1/2} (\log x)}{\log u} \frac{u^6}{\log u} \right) + O \left( x^{3/4} \frac{u}{\log u} \right).
\]

This completes the proof because of our choice of \(u\). The proof of part (b) is omitted as one just needs to follow the same idea that has been used for proving part (a).

Acknowledgements. The authors thank E. Ghate, V. M. Patankar, C. S. Rajan and J. Sengupta for helpful discussions. The authors thank the anonymous referee for a careful reading of the manuscript and they are grateful for several suggestions from the referee that led to a substantial improvement in the quality of this article. This project was initiated when the first named author visited the Tata Institute of Fundamental Research, Mumbai in January, 2019 where the second and the third authors were postdoctoral fellows at that time. The authors thank the institute for providing excellent working condition. The open-source mathematics software SAGE (www.sagemath.org) has been used for numerical computations in this work.

The research of the second author was supported by the grant no. 692854 provided by the European Research Council (ERC) while the third author was supported by Israeli Science Foundation grant 1400/19.
REFERENCES

[Del71] P. Deligne, *Formes modulaires et représentations l-adiques*, Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363, Exp. No. 355, 139–172. Lecture Notes in Math., 175, Springer, Berlin, 1971.

[DS05] F. Diamond and J. Shurman, A first course in modular forms, Graduate Texts in Mathematics, 228, *Springer-Verlag*, New York, 2005.

[Erd35] P. Erdős, *On the normal number of prime factors of \( p - 1 \) and some related problems concerning Euler’s \( \phi \)-function*, Quart. J. Math. Oxford 6 (1935), 205–213.

[Erd49] P. Erdős, *Some asymptotic formulas in number theory*, J. Indian Math. Soc. (N.S.) 12 (1948), 75–78.

[Gou97] F. Q. Gouvêa, *Non-ordinary primes: a story*, Experiment. Math. 6 (1997), no. 3, 195–205.

[Hei72] H. Heilbronn, *On real simple zeros of Dedekind \( \zeta \)-functions*, Proceeding of the Number Theory Conference (Univ. Colorado, Boulder, Colorado, 1972), 108–110.

[Iwa97] H. Iwaniec, Topics in classical automorphic forms, Graduate Studies in Mathematics, 17, *American Mathematical Society*, Providence, RI, 1997.

[LO77] J. Lagarias and A. Odlyzko, *Effective versions of the Chebotarev density theorem*, in: Algebraic Number Fields, pp. 409–464, ed. A. Fröhlich, Academic Press, New York, 1977.

[Loe17] D. Loeffler, *Images of adelic Galois representations for modular forms*, Glasg. Math. J. 59 (2017), no. 1, 11–25.

[MM84(a)] M. R. Murty and V. K. Murty, *Prime divisors of Fourier coefficients of modular forms*, Duke Math. J. 51 (1984), no. 1, 57–76.

[MM84(b)] M. R. Murty and V. K. Murty, *An analogue of the Erdős-Kac theorem for Fourier coefficients of modular forms*, Indian J. Pure Appl. Math. 15 (1984), no. 10, 1090–1101.

[Mur07] V. K. Murty, *A variant of Lehmer’s conjecture*, J. Number Theory 123 (2007), no. 1, 80–91.

[Ram15] S. Ramanujan, *Highly composite numbers* [Proc. London Math. Soc. (2) 14 (1915), 347–409]. In Collected Papers of Srinivasa Ramanujan, AMS Chelsea (Providence, RI, 2000), 78–128.

[Rib75] K. Ribet, *On \( l \)-adic representations attached to modular forms*, Invent. Math. 28 (1975), 245–275.

[SAGE] The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 9.2)*, 2020. http://www.sagemath.org.

[Ser81] J.-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. 54 (1981), 323–401.

[Sta74] H. M. Stark, *Some effective cases of the Brauer–Siegel theorem*, Invent. Math. 23 (1974), 135–152.