HECKE-CLIFFORD ALGEBRAS AND SPIN HECKE ALGEBRAS II: 
THE RATIONAL DOUBLE AFFINE TYPE

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Abstract. The notion of rational spin double affine Hecke algebras (sDaHa) and rational double affine Hecke-Clifford algebras (DaHCa) associated to classical Weyl groups are introduced. The basic properties of these algebras such as the PBW basis and Dunkl operator representations are established. An algebra isomorphism relating the rational DaHCa to the rational sDaHa is obtained. We further develop a link between the usual rational Cherednik algebra and the rational sDaHa by introducing a notion of rational covering double affine Hecke algebras.

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1. Introduction

1.1. The rational Cherednik algebra (see Etingof-Ginzburg [EG] and also Drinfeld [Dr] for a more general deformation construction) is a degenerate version of the double affine Hecke algebra [Ch], and it admits a polynomial representation via the Dunkl operators [Dun]. A similar degeneration in the case of affine Hecke algebras was introduced and studied earlier in [Dr] in the type A case and by Lusztig in general [Lu1, Lu2]. The rational Cherednik algebra has a rich representation theory and it affords various interesting connections to integrable systems, noncommutative geometry, etc. We refer to Etingof [Et] and Rouquier [Rou] for reviews and extensive references. The rational Cherednik algebra with one particular parameter being zero,
denoted by $\mathcal{H}_W$ in this Introduction, is known to have a large center [EG] (cf. Gordon [Gor]).

In [W1], the second author introduced (degenerate) spin Hecke algebras of affine and double affine type as well as double affine Hecke-Clifford algebra, associated to I. Schur’s spin symmetric group [Sch]. These algebras were shown to be closely related to the affine Hecke-Clifford algebra of Nazarov [Naz]. The spin affine Hecke algebras and affine Hecke-Clifford algebras associated to all classical Weyl groups have been recently constructed by the authors [KW].

1.2. In this paper we shall construct three classes of closely related (super) algebras associated to each classical finite Weyl group $W$: the rational double affine Hecke-Clifford algebra (DaHCa) $\tilde{\mathcal{H}}_W$, the rational spin double affine Hecke algebra (sDaHa) $\tilde{\mathcal{H}}^{-}_W$, and the rational covering double affine Hecke algebra (cDaHa) $\tilde{\mathcal{H}}^\sim_W$. We show that the algebras $\tilde{\mathcal{H}}^c_W$ and $\tilde{\mathcal{H}}^{-}_W$ are Morita super-equivalent (in the terminology of [W2]) and that $\tilde{\mathcal{H}}^\sim_W$ has both the rational Cherednik algebra $\mathcal{H}_W$ and the sDaHa $\tilde{\mathcal{H}}^{-}_W$ as its natural quotients. Some basic properties including the PBW basis theorem and Dunkl operator realizations of these algebras are further established.

We expect that these algebras afford very interesting representation theory and connections with noncommutative geometry.

1.3. In [Mo, KW] a double cover $\tilde{W}$ of the finite Weyl group $W$ associated to a distinguished 2-cocycle

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 1$$

was considered. Denote $\mathbb{Z}_2 = \{1, z\}$. From now on, let $W$ be one of the classical Weyl groups. In this paper, we define the algebras $\tilde{\mathcal{H}}^c_W$, $\tilde{\mathcal{H}}^{-}_W$ and $\tilde{\mathcal{H}}^\sim_W$ for every $W$ of type $A_{n-1}, D_n, B_n$ in terms of explicit generators and relations, where the number of parameters in each of these algebras is the number of conjugacy classes of reflections in $W$. The compatibility among the defining relations for $\tilde{\mathcal{H}}^c_W$ (which would imply the PBW basis property and similar compatibility for $\tilde{\mathcal{H}}^{-}_W$ and $\tilde{\mathcal{H}}^\sim_W$ when combined with other results) requires lengthy but elementary case-by-case verifications. In a suitable sense, the defining relations are naturally and uniquely dictated by the compatibility of these relations.

As is well known, the rational Cherednik algebra $\mathcal{H}_W$ has a triangular decomposition with the group algebra $\mathbb{C}W$ as its middle term. We show that the algebras $\tilde{\mathcal{H}}^c_W$, $\tilde{\mathcal{H}}^{-}_W$ and $\tilde{\mathcal{H}}^\sim_W$ also afford triangular decompositions which contain $\mathbb{C}_n \times \mathbb{C}W$, $\mathbb{C}W^-$ and $\tilde{\mathbb{C}}W$ respectively as the middle terms, where $\mathbb{C}_n$ denotes the Clifford algebra of the reflection representation of $W$ with a natural $W$-action. For instance, the rational DaHCa $\tilde{\mathcal{H}}^c_W$ and sDaHa
have the following triangular decompositions:
\[ \mathcal{H}_c \cong \mathbb{C}[x_1, \ldots, x_n] \otimes (\mathbb{C}_n \times \mathbb{C}W) \otimes \mathbb{C}[y_1, \ldots, y_n] \]
\[ \mathcal{H}^- \cong \mathbb{C}[\xi_1, \ldots, \xi_n] \otimes \mathbb{C}W^- \otimes \mathbb{C}[y_1, \ldots, y_n] \]

where \( \mathbb{C}[\xi_1, \ldots, \xi_n] \) is a noncommutative algebra with \( \xi_i \xi_j = -\xi_j \xi_i \) for \( i \neq j \). The relations between \( \mathbb{C}W^- \) and \( \mathbb{C}[\xi_1, \ldots, \xi_n] \) involve subtle signs similar to those appearing in the spin affine Hecke algebras defined in [KW, W1].

We further show that the algebras \( \mathcal{H}_c \) and \( \mathcal{H}^- \) have large centers which contain \( \mathbb{C}[y_1, \ldots, y_n] \) and \( \mathbb{C}[x_1^2, \ldots, x_n^2] \) respectively as subalgebras. In particular, the algebras \( \mathcal{H}_c \) and \( \mathcal{H}^- \) are module-finite over their centers.

The group algebra \( \mathbb{C}W \) and the spin Weyl group algebra \( \mathbb{C}W^- \) appear as natural quotients of \( \mathbb{C}\tilde{W} \) by the ideals \( \langle z \mp 1 \rangle \) respectively. We show that these quotient maps, denoted by \( \Upsilon_{\pm} \), extend to the setup of double affine Hecke algebras. All these statements can be summarized in the following commutative diagram with the vertical arrows being natural inclusions:

\[ \mathbb{C}W \xrightarrow{\Upsilon_+} \mathbb{C}\tilde{W} \xrightarrow{\Upsilon_-} \mathbb{C}W^- \downarrow \downarrow \downarrow \]
\[ \mathcal{H}_c \xrightarrow{\Upsilon_+} \mathcal{H}^- \xrightarrow{\Upsilon_-} \mathcal{H}^- \]

In [KW], we established a superalgebra isomorphism \( \Phi : \mathbb{C}_n \times \mathbb{C}W \cong \mathbb{C}_n \otimes \mathbb{C}W^- \) (which actually holds also for exceptional Weyl groups), generalizing the type \( A \) result of Sergeev and Yamaguchi. In this paper, we shall establish a Morita super-equivalence between \( \mathcal{H}_c \) and \( \mathcal{H}^- \), or more explicitly, a superalgebra isomorphism between \( \mathcal{H}_c \) and the tensor algebra \( \mathbb{C}_n \otimes \mathcal{H}^- \) which extends the isomorphism \( \Phi \) (see [W1] for the type \( A \) case). This can be summarized conveniently in the following commutative diagram with the vertical arrows being natural inclusions:

\[ \mathbb{C}_n \times \mathbb{C}W \xrightarrow{\cong}_\Phi \mathbb{C}_n \otimes \mathbb{C}W^- \downarrow \downarrow \downarrow \]
\[ \mathcal{H}_c \xrightarrow{\cong}_\Phi \mathcal{H}^- \]

1.4. As our constructions in a way rely on a choice of orthonormal basis of \( \mathfrak{g} \), they do not seem to be easily extendable to the exceptional Weyl groups. Also, in contrast to the setup of rational Cherednik algebras in [EG], our Hecke algebras do not seem to afford an extra parameter in a natural way to trivialize their center.

There has been another attempt (cf. Chmutova [Chm]) to generalize the rational Cherednik algebras and more generally symplectic reflection algebras by adding a twist with a 2-cocycle of a finite group. But the
approach therein does not produce intrinsically interesting new algebras with nontrivial 2-cocycles of the Weyl groups as in our approach.

1.5. The paper is organized as follows. In Section 2, we recall some facts about the distinguished double covers of the Weyl groups. For more detailed treatment, consult [KW]. We introduce in Section 3 the rational DaHCa $\tilde{\mathfrak{g}}_W$ and establish its PBW basis property. In Section 4 the Dunkl operator representations of $\tilde{\mathfrak{g}}_W$ are obtained. Section 5 and 6 are the counterparts for the sDaHa $\tilde{\mathfrak{g}}_W^-$ of Section 3 and 4 respectively. In addition, the superalgebra isomorphism $\Phi$ relating the sDaHa and DaHCa is established in Section 5. Finally, in Section 7 the rational cDaHa $\tilde{\mathfrak{g}}_W^\sim$ is introduced and it provides a link between the sDaHa $\tilde{\mathfrak{g}}_W^+$ and the usual DaHa $\tilde{\mathfrak{g}}_W$. Finally, in the Appendix (Section 8), we present the proofs of several lemmas in Section 3 and 4.

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2. The spin Weyl groups

In this section, we recall from [KW] some preliminary setups which lead to Theorem 2.1 below, but here we will restrict ourselves to classical Weyl groups only. This is all we need in the subsequent sections.

2.1. A double covering of Weyl groups. Let $W$ be an (irreducible) finite Weyl group of classical type (i.e. of type $A, B, D$) with the following presentation:

$$\langle s_1, \ldots, s_n \mid (s_is_j)^{m_{ij}} = 1, \ m_{ii} = 1, \ m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2}, \text{for } i \neq j \rangle. \quad (2.1)$$

The integers $m_{ij}$ are specified by the Coxeter-Dynkin diagrams whose vertices correspond to the generators of $W$ below. By convention, we only mark the edge connecting $i, j$ with $m_{ij} \geq 4$. We have $m_{ij} = 3$ for $i \neq j$ connected by an unmarked edge, and $m_{ij} = 2$ if $i, j$ are not connected by an edge.
We shall be concerned about a distinguished double covering \( \tilde{W} \) of \( W \):

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 1.
\]

We denote by \( \mathbb{Z}_2 = \{1, z\} \), and by \( \tilde{t}_i \) a fixed preimage of the generators \( s_i \) of \( W \) for each \( i \). The group \( \tilde{W} \) is generated by \( z, \tilde{t}_1, \ldots, \tilde{t}_n \) with relations (besides the obvious relation that \( z \) is central of order 2) listed in the following table, which corresponds to setting the \( \alpha_i \) for all \( i \) in Karpilovsky [Kar, Table 7.1] to be \( z \).

| \( W \) | Defining Relations for \( W \) |
|------|------------------|
| \( A_n \) | \( t_i^2 = 1, 1 \leq i \leq n, \)
|       | \( \tilde{t}_i \tilde{t}_{i+1} \tilde{t}_i = \tilde{t}_{i+1} \tilde{t}_i \tilde{t}_{i+1}, 1 \leq i \leq n-1 \)
|       | \( \tilde{t}_i \tilde{t}_j = z \tilde{t}_j \tilde{t}_i \) if \( m_{ij} = 2 \)
| \( B_n \) | \( t_i^2 = 1, 1 \leq i \leq n, \)
| \( (n \geq 2) \) | \( t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, 1 \leq i \leq n-2 \)
|       | \( \tilde{t}_i \tilde{t}_j = z \tilde{t}_j \tilde{t}_i, 1 \leq i < j \leq n-1, m_{ij} = 2 \)
|       | \( \tilde{t}_i \tilde{t}_n = \tilde{t}_n \tilde{t}_i, 1 \leq i \leq n-2 \)
|       | \( (\tilde{t}_{n-1} \tilde{t}_n)^2 = z (\tilde{t}_n \tilde{t}_{n-1})^2 \)
| \( D_n \) | \( t_i^2 = 1, 1 \leq i \leq n \)
| \( (n \geq 4) \) | \( \tilde{t}_i \tilde{t}_j \tilde{t}_i = \tilde{t}_j \tilde{t}_i \tilde{t}_j \) if \( m_{ij} = 3 \)
|       | \( \tilde{t}_i \tilde{t}_j = z \tilde{t}_j \tilde{t}_i, 1 \leq i < j \leq n, m_{ij} = 2 \)

**TABLE 1:** The defining relations of \( \tilde{W} \)

The quotient algebra \( \mathbb{C}W^- := \mathbb{C}\tilde{W}/\langle z+1 \rangle \) of \( \mathbb{C}\tilde{W} \) by the ideal generated by \( z+1 \) will be called the spin Weyl group algebra associated to \( W \). Denote by \( t_i \in \mathbb{C}W^- \) the image of \( \tilde{t}_i \). The spin Weyl group algebra \( \mathbb{C}W^- \) has a natural superalgebra (i.e. \( \mathbb{Z}_2 \)-graded algebra) structure by letting each \( t_i \) be odd. The algebra \( \mathbb{C}W^- \) is generated by \( t_1, \ldots, t_n \) with the labeling as in the Coxeter-Dynkin diagrams and the explicit relations summarized in the following table.

| Type of \( W \) | Defining Relations for \( \mathbb{C}W^- \) |
|------|------------------|
| \( A_n \) | \( t_i^2 = 1, t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \)
|       | \( (t_i t_j)^2 = -1 \) if \( |i - j| > 1 \)
| \( B_n \) | \( t_1, \ldots, t_{n-1} \) satisfy the relations for \( \mathbb{C}W^-_{A_{n-1}} \)
|       | \( t_n^2 = 1, (t_i t_n)^2 = -1 \) if \( i \neq n-1, n \)
|       | \( (t_{n-1} t_n)^4 = -1 \)
| \( D_n \) | \( t_1, \ldots, t_{n-1} \) satisfy the relations for \( \mathbb{C}W^-_{A_{n-1}} \)
|       | \( t_n^2 = 1, (t_i t_n)^2 = -1 \) if \( i \neq n-2, n \)
|       | \( t_{n-2} t_n t_{n-2} = t_n t_{n-2} t_n \)

**TABLE 2:** The defining relations of \( \mathbb{C}W^- \)
By definition, the quotient by the ideal $\langle z - 1 \rangle$ of the group algebra $\mathbb{C}\tilde{W}$ is isomorphic to $\mathbb{C}W$.

2.2. A superalgebra isomorphism. Denote by $\mathfrak{h} = \mathbb{C}^n$ the natural representation (respectively the reflection representation) of the Weyl group $W$ of type $A_{n-1}$ (respectively of type $B_n$, and $D_n$). Note that $\mathfrak{h}$ carries a $W$-invariant nondegenerate bilinear form $(-,-)$, which gives rise to an identification $\mathfrak{h}^* \cong \mathfrak{h}$ and also a bilinear form on $\mathfrak{h}^*$ which will be again denoted by $(-,-)$.

Denote by $\mathcal{C}_n$ the Clifford algebra associated to $(\mathfrak{h}, (-,-))$. We shall denote by $\{c_i\}$ the generators in $\mathcal{C}_n$ corresponding to a standard orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$ and denote by $\{\beta_i\}$ the elements of $\mathcal{C}_n$ corresponding to the simple roots $\{\alpha_i\}$ normalized with $\beta_i^2 = 1$.

More explicitly, $\mathcal{C}_n$ is generated by $c_1, \ldots, c_n$ subject to the relations

$$c_i^2 = 1, \quad c_ic_j = -c_jc_i \quad (i \neq j). \quad (2.2)$$

For type $A_{n-1}$, we have $\beta_i = \frac{1}{\sqrt{2}}(c_i - c_{i+1}), 1 \leq i \leq n - 1$. For type $B_n$, we have an additional $\beta_n = c_n$, and for type $D_n$, $\beta_n = \frac{1}{\sqrt{2}}(c_{n-1} + c_n)$.

The action of $W$ on $\mathfrak{h}$ and $\mathfrak{h}^*$ preserves the bilinear form $(-,-)$ and thus it acts as automorphisms of the algebra $\mathcal{C}_n$. This gives rise to a semi-direct product $\mathcal{C}_n \rtimes CW$. Moreover, the algebra $\mathcal{C}_n \rtimes CW$ naturally inherits the superalgebra structure by letting elements in $W$ be even and each $c_i$ be odd.

Given two superalgebras $A$ and $B$, we view the tensor product of superalgebras $A \otimes B$ as a superalgebra with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}(aa' \otimes bb') \quad (a, a' \in A, b, b' \in B)$$

where $|b|$ denotes the $\mathbb{Z}_2$-degree of $b$, etc. Also, we shall use short-hand notation $ab$ for $(a \otimes b) \in A \otimes B$, $a = a \otimes 1$, and $b = 1 \otimes b$.

Theorem 2.1. [KW] We have an isomorphism of superalgebras:

$$\Phi : \mathcal{C}_n \rtimes CW \stackrel{\sim}{\longrightarrow} \mathcal{C}_n \otimes CW^-$$

which extends the identity map on $\mathcal{C}_n$ and sends $s_i \mapsto -\sqrt{-1}\beta_i t_i$. The inverse map $\Psi$ is the extension of the identity map on $\mathcal{C}_n$ which sends $t_i \mapsto \sqrt{-1}\beta_i s_i$. (In the terminology of [W2], the superalgebras $\mathcal{C}_n \rtimes CW$ and $CW^-$ are Morita super-equivalent.)

Remark 2.2. Theorem 2.1 was formulated and proved in [KW] for every finite Weyl group including the exceptional types, and the type $A$ case was due to Sergeev and Yamaguchi. See [KW] for more detail.
3. Rational Double Affine Hecke-Clifford Algebras (DaHCA)

In this section, we introduce the rational double affine Hecke-Clifford algebras associated to the Weyl group $W$ of type $A$, $D$, and $B$, and then establish the PBW property. The type $A$ case was treated in [W1].

3.1. The definition of the algebras $\tilde{\mathcal{H}}_W$.

3.1.1. The algebra $\tilde{\mathcal{H}}_W$ of type $A_{n-1}$. We will identify $\mathbb{C}[\mathfrak{h}^*] \cong \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[y_1, \ldots, y_n]$, where the $x_i$’s and $y_i$’s correspond to the standard orthonormal basis $\{e_i\}$ for $\mathfrak{h}^*$ and its dual basis $\{e_i^*\}$ for $\mathfrak{h}$.

The following algebra $\tilde{\mathcal{H}}_{A_{n-1}}$ was introduced in [W1] under the notation $A_u$. We recall it for convenience and usage in the subsequent subsections. For $x,y$ in an algebra $A$, we denote as usual that

$$[x,y] = xy - yx \in A.$$ 

**Definition 3.1.** Let $u \in \mathbb{C}$ and $W = W_{A_{n-1}} \equiv S_n$. The rational double affine Hecke-Clifford algebra (DaHCA) of type $A_{n-1}$, denoted by $\tilde{\mathcal{H}}_W$ or $\tilde{\mathcal{H}}_{A_{n-1}}$, is the algebra generated by $x_i, y_i, c_i$, $1 \leq i \leq n$ and $W$, subject to the relation (2.2) among $c_i$’s and the following relations (where we identify $\mathfrak{h}^* = \mathbb{C}x_1 + \cdots + \mathbb{C}x_n$ and $\mathfrak{h} = \mathbb{C}y_1 + \cdots + \mathbb{C}y_n$):

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad y_i c_j = c_j y_i \quad (\forall i \neq j)$$  \hspace{1cm} (3.1)

$$x_i c_i = -c_i x_i, \quad x_i c_j = c_j x_i \quad (i \neq j)$$  \hspace{1cm} (3.2)

$$wxw^{-1} = w(x) \quad (\forall x \in \mathfrak{h}^* \quad \forall w \in W)$$  \hspace{1cm} (3.3)

$$wyw^{-1} = w(y) \quad (\forall y \in \mathfrak{h} \quad \forall w \in W)$$  \hspace{1cm} (3.4)

$$wcw^{-1} = w(c) \quad (\forall c \in \mathbb{C} \quad \forall w \in W)$$  \hspace{1cm} (3.5)

$$[y_j, x_i] = u(1 + c_j c_i)s_{ji} \quad (i \neq j)$$  \hspace{1cm} (3.6)

$$[y_i, x_j] = -u \sum_{k \neq i}(1 + c_k c_i)s_{ki}.$$  \hspace{1cm} (3.7)

Alternatively, we may view $u$ as a formal variable and $\tilde{\mathcal{H}}_W$ as a $\mathbb{C}(u)$-algebra. Similar remarks apply to all DaHCA, sDaHa, and cDaHa introduced in this paper.

3.1.2. The algebra $\tilde{\mathcal{H}}_W$ of type $D_n$. Let $W = W_{D_n}$. Regarding elements in $W$ as even signed permutations of $1, 2, \ldots, n$ as usual, we identify the generators $s_i \in W$, $1 \leq i \leq n-1$, with transposition $(i, i + 1)$, and $s_n \in W$ with the transposition of $(n-1, n)$ coupled with the sign changes at $n-1, n$. For $1 \leq i \neq j \leq n$, we denote by $s_{ij} \equiv (i,j) \in W$ the transposition of $i$ and $j$, and $\pi_{ij} \equiv (i,j) \in W$ the transposition of $i$ and $j$ coupled with the sign changes at $i,j$. By convention, we have

$$\pi_{n-1,n} \equiv (n-1,n) = s_n, \quad \pi_{ij} \equiv (i,j) = s_{jn}s_{i,n-1}s_n s_{i,n-1}s_{jn}.$$
Definition 3.2. Let \( u \in \mathbb{C} \) and \( W = W_{D_n} \). The rational double affine Hecke-Clifford algebra of type \( D_n \), denoted by \( \mathcal{S}_W^\dagger \), is the algebra generated by \( x_i, y_i, c_i, 1 \leq i \leq n \) and \( W \), subject to the relation (2.2) among \( c_i \)'s, (3.3–3.5) with the current \( W \), and (3.8–3.9) below:

\[
[y_j, x_i] = u((1 + c_j c_i)s_{ij} - (1 - c_j c_i)\overline{s}_{ij}) \quad (i \neq j) \quad (3.8)
\]

\[
[y_i, x_i] = -u \sum_{k \neq i} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\overline{s}_{ki}). \quad (3.9)
\]

3.1.3. The algebra \( \mathcal{S}_W^\dagger \) of type \( B_n \). Let \( W = W_{B_n} \). We identify \( W \) as usual with the signed permutations on \( 1, \ldots, n \). Regarding \( W_{D_n} \) as a subgroup of \( W \), we have \( s_{ij}, \overline{s}_{ij} \in W \) for \( 1 \leq i \neq j \leq n \). Further denote \( \tau_i \equiv (i) \in W \) the sign change at \( i \) for \( 1 \leq i \leq n \). By definition, we have

\[
\tau_n \equiv (n) = s_n, \quad \tau_i \equiv (i) = s_i s_n s_is_m.
\]

Definition 3.3. Let \( u, v \in \mathbb{C} \), and \( W = W_{B_n} \). The rational double affine Hecke-Clifford algebra of type \( B_n \), denoted by \( \mathcal{S}_W^\dagger \) or \( \mathcal{S}_{B_n}^\dagger \), is the algebra generated by \( x_i, y_i, c_i, 1 \leq i \leq n \) and \( W \), subject to the relations (2.2) for \( c_i \)'s, (3.3–3.5) with the current \( W \), and (3.10–3.11) below:

\[
[y_j, x_i] = u((1 + c_j c_i)s_{ij} - (1 - c_j c_i)\overline{s}_{ij}) \quad (i \neq j) \quad (3.10)
\]

\[
[y_i, x_i] = -u \sum_{k \neq i} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\overline{s}_{ki}) - \sqrt{2}v\tau_i. \quad (3.11)
\]

When it is necessary to indicate the dependence of the algebra \( \mathcal{S}_W^\dagger \) on \( u \) and \( v \), we will write \( \mathcal{S}_W^\dagger(u,v) \) for \( \mathcal{S}_W^\dagger \).

Remark 3.4. The factor \( \sqrt{2} \) in (3.11) is inserted to make the definition of \( \mathcal{S}_{B_n}^\dagger \) compatible with the notion of sDaHa \( \mathcal{S}_{B_n}^\dagger \) below under a Morita super-equivalence \( \Phi \) (cf. Theorem 5.5).

3.2. The PBW basis for \( \mathcal{S}_W^\dagger \). For any classical Weyl group \( W \), the algebra \( \mathcal{S}_W^\dagger \) is a superalgebra by letting elements of \( W \) and \( x_i, y_i \) for all \( i \) be even, and each \( c_i \) be odd.

Theorem 3.5. Let \( W \) be \( W_{A_n-1}, W_{D_n} \), or \( W_{B_n} \). The multiplication of the subalgebras \( \mathbb{C}[\mathfrak{h}^*], \mathbb{C}[\mathfrak{h}], \mathcal{C}, \) and \( \mathbb{C}W \) induces a vector space isomorphism

\[
\mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{C} \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}] \longrightarrow \mathcal{S}_W^\dagger.
\]

Equivalently, the elements \( \{x^\alpha e^\varepsilon w y^\gamma | \alpha, \gamma \in \mathbb{Z}^n_+, \varepsilon \in \mathbb{Z}_2^n, w \in W \} \) form a linear basis for \( \mathcal{S}_W^\dagger \) (the PBW basis).

Proof. Recall that \( W \) acts diagonally on \( V = \mathfrak{h}^* \oplus \mathfrak{h} \). The strategy of proving the theorem is similar as [EGI] Proof of Th. 1.3 with one crucial modification as first observed in \([W1]\).

Clearly \( K := \mathcal{C} \otimes \mathbb{C}W \) is a semisimple algebra. Observe that \( E := V \otimes \mathbb{C}K \) is a natural \( K \)-bimodule (even though \( V \) is not) with the right \( K \)-module
structure on $E$ given by right multiplication and the left $K$-module structure on $E$ by letting

\[
\begin{align*}
    w.(v \otimes a) &= \ v^w \otimes wa \\
    c_i.(x_j \otimes a) &= (-1)^{\delta_{ij}} x_j \otimes (c_i a) \\
    c_i.(y_j \otimes a) &= y_j \otimes (c_i a).
\end{align*}
\]

where $v \in V$, $w \in W$, $a \in K$.

The rest of the proof can proceed in the same way as in [EG, Proof of Th. 1.3]. It boils down to the verifications of Lemmas 3.7, 3.8 and 3.9 below, on the conjugation invariance (by $c_i$ and $W$) of the defining relations (3.6–3.7), (3.8–3.9), or (3.10–3.11) for type $A$, $D$ or $B$ respectively, and on the Jacobi identities among the generators $x_i$’s and $y_i$’s.

**Remark 3.6.** Note that $\mathfrak{c}_n \rtimes \mathbb{C}W$ is actually a subalgebra of $\mathcal{H}_W$ and the tensor product in the above theorem indicates that $\mathcal{H}_W$ has the structure of an algebra with triangular decomposition:

\[\mathcal{H}_W \cong \mathbb{C}[h^*] \otimes (\mathfrak{c}_n \rtimes \mathbb{C}W) \otimes \mathbb{C}[h].\]

The verifications of Lemmas 3.7, 3.8 and 3.9 below are postponed to the Appendix.

**Lemma 3.7.** Let $W = W_{A_{n-1}}, W_{D_n}$ or $W_{B_n}$. Then the relations (3.6–3.7), (3.8–3.9), or (3.10–3.11) are invariant under the conjugation by $c_i$, $1 \leq i \leq n$.

**Lemma 3.8.** The relations (3.6–3.7), (3.8–3.9), or (3.10–3.11) respectively are invariant under the conjugation by elements in $W_{A_{n-1}}, W_{D_n}$ or $W_{B_n}$ respectively.

**Lemma 3.9.** Let $W = W_{A_{n-1}}, W_{D_n}$ or $W_{B_n}$. Then the Jacobi identity holds for any triple among $x_i, y_i$, $1 \leq i \leq n$ in $\mathcal{H}_{W}$.

4. **The Dunkl Operators for DaHCA**

4.1. **The Dunkl representations.** The algebra $\mathcal{H}_W$ is a superalgebra by letting elements of $W$ and $x_i, y_i$ for all $i$ be even, and each $c_i$ be odd. Recall that $\mathcal{H}_W$ admits the triangular decomposition:

\[\mathcal{H}_W \cong \mathbb{C}[h^*] \otimes K \otimes \mathbb{C}[h]\]

where we have denoted

\[K = \mathfrak{c}_n \rtimes \mathbb{C}W.\]

In contrast to the usual DaHa $\mathcal{H}_W$, the DaHCA $\mathcal{H}_W$ has no automorphism which switches the subalgebras $\mathbb{C}[h]$ and $\mathbb{C}[h^*]$. Denote by $\mathcal{H}_x$ and $\mathcal{H}_y$ the subalgebras of $\mathcal{H}_W$ generated by $K$ and $x_1, \ldots, x_n$, and generated by $K$ and $y_1, \ldots, y_n$ respectively.
A $K$-module $M$ can be extended to either $\mathcal{H}_x$-module or $\mathcal{H}_y$-module by demanding the action of $x_i$’s and $y_i$’s to be trivial respectively. We define

$$M_x := \text{Ind}_{\mathcal{H}_x} \mathcal{H}_W M, \quad M_y := \text{Ind}_{\mathcal{H}_y} \mathcal{H}_W M.$$ 

Below we will always use the following identification of vector spaces:

$$M_x = \mathbb{C}[y_1, \ldots, y_n] \otimes M, \quad M_y = \mathbb{C}[x_1, \ldots, x_n] \otimes M.$$ 

Then the action of $\mathcal{H}_W$ on $M_x$ (resp. $M_y$) is transferred to $\mathbb{C}[y_1, \ldots, y_n] \otimes M$ (resp. $\mathbb{C}[x_1, \ldots, x_n] \otimes M$) as follows. On $\mathbb{C}[y_1, \ldots, y_n] \otimes M$, $K$ acts diagonally.

More explicitly, $K$ acts on $\mathbb{C}[x_1, \ldots, x_n] \otimes M$ by

$$w.(x_j \otimes m) = x_j^w \otimes wm$$

$$c_i.(x_j \otimes m) = (-1)^{\delta_{ij}} x_j \otimes c_i m$$

where $c_i \in \mathbb{C}_n$, $w \in W$. Moreover, $y_i$ acts by left multiplication in the first tensor factor, and the action of $x_i$ is given by the so-called Dunkl operators (which are generalizations of $[\text{Dun}]$). Similarly, on $\mathbb{C}[x_1, \ldots, x_n] \otimes M$, $x_i$ acts by left multiplication, and $y_i$ acts by another version of Dunkl operators.

In the remainder of this section we shall describe these Dunkl operators explicitly.

**Remark 4.1.** A canonical choice for a $K$-module is $\mathbb{C}_n$, whose $K$-module structure is defined by letting $\mathbb{C}_n$ act by left multiplication and $W$ act as usual (cf. Section 2.2).

**4.2. The Dunkl Operators for $\tilde{\mathcal{H}}_{A_{n-1}}$.** We first prepare a few lemmas. We shall denote the action of $\sigma \in W$ on $\mathbb{C}[h]$ and $\mathbb{C}[h^*]$ by $f \mapsto f^\sigma$.

**Lemma 4.2.** Let $W = W_{A_{n-1}}$. Then the following holds in $\tilde{\mathcal{H}}_{W}$ for $l \in \mathbb{Z}_+$ and $i \neq j$:

$$[y_i, x_j^l] = u \left( \frac{x_j^l - x_i^l}{x_j - x_i} + \frac{x_j^l - (-x_i)^l}{x_j + x_i} c_i c_j \right) s_{ij},$$

$$[y_i, x_i^l] = -u \sum_{k \neq i} \left( \frac{x_i^l - x_k^l}{x_i - x_k} + \frac{x_i^l - (-x_k)^l}{x_i + x_k} c_k c_i \right) s_{ki}.$$ 

It is understood here and in similar ratios of operators below that $\frac{h}{g} = \frac{1}{g} \cdot h$.

**Proof.** This lemma is a type $A$ counterpart of Lemma 4.8 for type $B$ below. A proof can be simply obtained by modifying the proof of Lemma 4.8 with the removal of those terms involving $\tau_i, \tau_{ki}, \tau_i$ therein. We skip the details. 

□
Lemma 4.3. Let \( W = W_{A_{n-1}} \), and \( f \in \mathbb{C}[x_1, \ldots, x_n] \). Then the following identity holds in \( \tilde{\mathcal{H}}_W \):
\[
[y_i, f] = -u \sum_{k \neq i} \left( \frac{f - f_{sk_i}}{x_i - x_k} + \frac{f_{ckc_i} - c_kc_if_{sk_i}}{x_i + x_k} \right) s_{ki}.
\]

Proof. It suffices to check the formula for every monomial \( f \) of the form \( x_1^{l_1} \cdots x_n^{l_n} \), which follows by Lemma 4.2 and an induction on \( a \) based on the identity
\[
[y_i, x_1^{l_1} \cdots x_a^{l_a} x_{a+1}] = [y_i, x_1^{l_1} \cdots x_a^{l_a}] x_{a+1} + x_1^{l_1} \cdots x_a^{l_a} [y_i, x_{a+1}].
\]

Now we are ready to compute the Dunkl operator for \( y_i \)'s.

Theorem 4.4. Let \( W = W_{A_{n-1}} \) and \( M \) be a \( K \)-module. The action of \( y_i \) on the \( \tilde{\mathcal{H}}_W \)-module \( \mathbb{C}[x_1, \ldots, x_n] \otimes M \) is realized as a Dunkl operator as follows. For any polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n] \) and \( m \in M \), we have
\[
y_i \circ (f \otimes m) = -u \sum_{k \neq i} \left( \frac{f - f_{sk_i}}{x_i - x_k} + \frac{f_{ckc_i} - c_kc_if_{sk_i}}{x_i + x_k} \right) \otimes s_{ki}m.
\]

Proof. We calculate that
\[
y_i \circ (f \otimes m) = [y_i, f] \otimes m + f \otimes y_im = [y_i, f] \otimes m.
\]
Now the result follows from Lemma 4.3.

Lemma 4.5. Let \( W = W_{A_{n-1}} \). Then the following holds in \( \tilde{\mathcal{H}}_W \) for \( l \in \mathbb{Z}_+ \) and \( i \neq j \):
\[
[y_i^l, x_i] = \frac{y_j^l - y_i^l}{y_j - y_i} (1 + c_jc_i) s_{ij},
\]
\[
[y_i^l, x_i] = -u \sum_{k \neq i} \frac{y_j^l - y_k^l}{y_i - y_k} (1 + c_kc_i) s_{ki}.
\]

Proof. This lemma is a type \( A \) counterpart of Lemma 4.11 for type \( B \) below, with the removal of those terms involving \( \tilde{s}_{ij}, \tilde{s}_{ki}, \tau_i \) therein. We leave a detailed proof to the reader.

Lemma 4.6. Let \( W = W_{A_{n-1}} \), and \( f \in \mathbb{C}[y_1, \ldots, y_n] \). Then the following identity holds in \( \tilde{\mathcal{H}}_W \):
\[
[f, x_i] = -u \sum_{k \neq i} \frac{f - f_{sk_i}}{y_i - y_k} (1 + c_kc_i) s_{ki}.
\]

Proof. It suffices to check the formula for every monomial \( f \), which can be done as for the formula in Lemma 4.3 using now Lemma 4.5 replacing Lemma 4.2 therein.

Now we are ready to compute the Dunkl operator for \( x_i \)'s.
Theorem 4.7. Let $W = W_{A_{n-1}}$ and $M$ be a $K$-module. The action of $x_i$ on $\mathbb{C}[y_1, \ldots, y_n] \otimes M$ is realized as a Dunkl operator as follows. For any polynomial $f \in \mathbb{C}[y_1, \ldots, y_n]$ and $m \in M$, we have

$$x_i \circ (f \otimes m) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes (1 + c_k c_i) s_{ki} m.$$ 

Proof. We observe that

$$x_i \circ (f \otimes m) = [x_i, f] \otimes m + f \otimes x_i m = [x_i, f] \otimes m.$$ 

Now the result follows by Lemma 4.6. □

4.3. The Dunkl Operators for $\tilde{\mathcal{H}}_{B_n}$. We first prepare a few lemmas. The proofs of Lemmas 4.8, 4.9, and 4.11 are postponed to the Appendix.

Lemma 4.8. Let $W = W_{B_n}$. Then the following holds in $\tilde{\mathcal{H}}_W$ for $l \in \mathbb{Z}_+$ and $i \neq j$:

$$[y_i, x_j^l] = u \left( \frac{x_j^l - x_i^l}{x_j - x_i} + \frac{x_j^l - (-x_i)^l}{x_j + x_i} c_i c_j \right) s_{ij}$$

$$- u \left( \frac{x_j^l - (-x_i)^l}{x_j + x_i} - \frac{x_j^l - x_i^l}{x_j - x_i} c_i c_j \right) \tilde{s}_{ij},$$

$$[y_i, x_i^l] = - u \sum_{k \neq i} \left( \frac{x_i^l - x_k^l}{x_i - x_k} + \frac{x_i^l - (-x_k)^l}{x_i + x_k} c_k c_i \right) s_{ki}$$

$$- u \sum_{k \neq i} \left( \frac{x_i^l - (-x_k)^l}{x_i + x_k} - \frac{x_i^l - x_k^l}{x_i - x_k} c_k c_i \right) \tilde{s}_{ki} - \sqrt{2} u \frac{x_i^l - (-x_i)^l}{2x_i^l} \tau_i.$$ 

Lemma 4.9. Let $W = W_{B_n}$, and $f \in \mathbb{C}[x_1, \ldots, x_n]$. Then the following holds in $\tilde{\mathcal{H}}_W$:

$$[y_i, f] = - u \sum_{k \neq i} \left( \frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f - f^{\tilde{s}_{ki}}}{x_i + x_k} c_k c_i \right) s_{ki}$$

$$- u \sum_{k \neq i} \left( \frac{f - f^{\tilde{s}_{ki}}}{x_i + x_k} - \frac{f - f^{s_{ki}}}{x_i - x_k} c_k c_i \right) \tilde{s}_{ki} - \sqrt{2} u \frac{f - f^{\tau_i}}{2x_i} \tau_i.$$ 

Now we are ready to compute the Dunkl operator for $y_i$’s.

Theorem 4.10. Let $W = W_{B_n}$ and $M$ be a $K$-module. The action of $y_i$ on $\mathbb{C}[x_1, \ldots, x_n] \otimes M$ is realized as follows. For any polynomial $f \in$
\( \mathbb{C}[x_1, \ldots, x_n] \) and \( m \in M \), we have
\[
y_i \circ (f \otimes m) = -u \sum_{k \neq i} \left( \frac{f - f^{s_{ki}}}{x_i - x_k} + \frac{f - f^{s_{ki}}}{x_i + x_k} c_k c_i \right) \otimes s_{ki} m
\]
\[
- u \sum_{k \neq i} \left( \frac{f - f^{s_{ki}}}{x_i - x_k} - \frac{f - f^{s_{ki}}}{x_i + x_k} c_k c_i \right) \otimes \overline{s}_{ki} m
\]
\[
- \sqrt{2} v \frac{f - f^{\tau_i}}{2x_i} \otimes \tau_i m.
\]

Proof. We observe that
\[
y_i \circ (f \otimes m) = [y_i, f] \otimes m + f \otimes y_i m = [y_i, f] \otimes m.
\]
Now the result follows from Lemma 4.11. \( \square \)

Lemma 4.11. Let \( W = W_{B_n} \). Then the following holds in \( \tilde{\mathcal{H}}_W \) for \( l \in \mathbb{Z}_+ \) and \( i \neq j \):
\[
[y_j^l, x_i] = u \left( y_j^l - y_i^l \right) (1 + c_j c_i) s_{ij} - \frac{y_j^l - (-y_i)^l}{y_j + y_i} (1 - c_j c_i) \overline{s}_{ij}.
\]
\[
[y_i^l, x_i] = -u \sum_{k \neq i} \frac{y_i^l - y_k^l}{y_i - y_k} (1 + c_k c_i) s_{ki}
\]
\[
- u \sum_{k \neq i} \frac{y_i^l - (-y_k)^l}{y_i + y_k} (1 - c_k c_i) \overline{s}_{ki} - \sqrt{2} v \frac{y_i^l - (-y_i)^l}{2y_i} \tau_i.
\]

Similarly as before, we can derive the next lemma from Lemma 4.11.

Lemma 4.12. Let \( W = W_{B_n} \), and \( f \in \mathbb{C}[y_1, \ldots, y_n] \). Then the following identity holds in \( \tilde{\mathcal{H}}_W \):
\[
[f, x_i] = -u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} (1 + c_k c_i) s_{ki}
\]
\[
- u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i + y_k} (1 - c_k c_i) \overline{s}_{ki} - \sqrt{2} v \frac{f - f^{\tau_i}}{2y_i} \tau_i.
\]

Now we are ready to compute the Dunkl operator for \( x_i \)'s.

Theorem 4.13. Let \( W = W_{B_n} \). The action of \( x_i \) on \( \mathbb{C}[y_1, \ldots, y_n] \otimes M \) is realized as follows. For any polynomial \( f \in \mathbb{C}[y_1, \ldots, y_n] \) and \( m \in M \), we have
\[
x_i \circ (f \otimes m) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes (1 + c_k c_i) s_{ki} m
\]
\[
+ u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i + y_k} \otimes (1 - c_k c_i) \overline{s}_{ki} m + \sqrt{2} v \frac{f - f^{\tau_i}}{2y_i} \otimes \tau_i m.
\]
Proof. We observe that
\[ x_i \circ (f \otimes m) = [x_i, f] \otimes m + f \otimes x_i m = [x_i, f] \otimes m. \]
Now the result follows from Lemma 4.12. \( \square \)

4.4. The Dunkl Operators for \( \tilde{\mathfrak{g}}_{D_n}^f \). Below, the actions of \( x_i \)'s and \( y_i \)'s are realized as Dunkl operators. Due to the similarity of the bracket relations \([-,-]\) in \( \tilde{\mathfrak{g}}_{D_n}^f \) and \( \tilde{\mathfrak{g}}_{B_n}^f \) (e.g. compare the type \( D \) relation (3.9) with the type \( B \) relation (3.11)), the formulas below for type \( D_n \) are obtained from their type \( B_n \) counterparts in the previous subsection by dropping the terms involving the parameter \( v \). The proofs are the same as for the type \( B \), and thus will be skipped.

Lemma 4.14. Let \( W = W_{D_n} \), and \( f \in \mathbb{C}[x_1, \ldots, x_n] \). Then the following holds in \( \tilde{\mathfrak{g}}_{W}^f \):
\[
[y_i, f] = -u \sum_{k \neq i} \left( f - f^s_{ki} \frac{x_i - x_k}{x_i + x_k} + f - f^\sigma_{ki} \frac{c_k c_i}{x_i + x_k} \right) s_{ki}
- u \sum_{k \neq i} \left( f - f^\sigma_{ki} \frac{x_i + x_k}{x_i - x_k} - f - f^s_{ki} \frac{c_k c_i}{x_i - x_k} \right) \sigma_{ki}.
\]

Lemma 4.15. Let \( W = W_{D_n} \), and \( f \in \mathbb{C}[y_1, \ldots, y_n] \). Then the following identity holds in \( \tilde{\mathfrak{g}}_{W}^f \):
\[
[f, x_i] = -u \sum_{k \neq i} \frac{f - f^s_{ki}}{y_i - y_k} (1 + c_k c_i) s_{ki} - u \sum_{k \neq i} \frac{f - f^\sigma_{ki}}{y_i + y_k} (1 - c_k c_i) \sigma_{ki}.
\]

Theorem 4.16. Let \( W = W_{D_n} \), and let \( M \) be a \( K \)-module. The action of \( y_i \) on \( \mathbb{C}[x_1, \ldots, x_n] \otimes M \) is realized as a Dunkl operator as follows. For any polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n] \) and \( m \in M \), we have
\[
y_i \circ (f \otimes m) = -u \sum_{k \neq i} \left( f - f^s_{ki} \frac{x_i - x_k}{x_i + x_k} + f - f^\sigma_{ki} \frac{c_k c_i}{x_i + x_k} \right) \otimes s_{ki} m
- u \sum_{k \neq i} \left( f - f^\sigma_{ki} \frac{x_i + x_k}{x_i - x_k} - f - f^s_{ki} \frac{c_k c_i}{x_i - x_k} \right) \otimes \sigma_{ki} m.
\]

Theorem 4.17. Let \( W = W_{D_n} \), and let \( M \) be a \( K \)-module. The action of \( x_i \) on \( \mathbb{C}[y_1, \ldots, y_n] \otimes M \) is realized as follows. For any \( f \in \mathbb{C}[y_1, \ldots, y_n] \) and \( m \in M \), we have
\[
x_i \circ (f \otimes m) = u \sum_{k \neq i} \frac{f - f^s_{ki}}{y_i - y_k} \otimes (1 + c_k c_i) s_{ki} m
+ u \sum_{k \neq i} \frac{f - f^\sigma_{ki}}{y_i + y_k} \otimes (1 - c_k c_i) \sigma_{ki} m.
\]
4.5. **The even center for $\tilde{\mathfrak{S}}_W$.** Recall that the even center $\mathcal{Z}(A)$ of a superalgebra $A$ consists of the even central elements of $A$. It turns out the algebra $\tilde{\mathfrak{S}}_W$ has a large center.

**Proposition 4.18.** Let $W$ be $W_{A_{n-1}}, W_{D_n}$ or $W_{B_n}$. The even center $\mathcal{Z}(\tilde{\mathfrak{S}}_W^c)$ contains $\mathbb{C}[y_1, \ldots, y_n]^W$ and $\mathbb{C}[x_1^2, \ldots, x_n^2]^W$ as subalgebras. In particular, $\tilde{\mathfrak{S}}_W^c$ is module-finite over its even center.

**Proof.** Let $f \in \mathbb{C}[y_1, \ldots, y_n]^W$. Then by the definition of $\tilde{\mathfrak{S}}_W^c$, $f$ commutes with $\mathfrak{c}_n$, $W$, and $y_i$ for all $1 \leq i \leq n$. Since $f = f^w$ for all $w \in W$, it follows by Lemmas 4.6, 4.12 or 4.15 (for type $A$, $D$ or $B$ respectively) that $[f, x_i] = 0$ for each $i$. Hence $f$ commutes with $\mathfrak{c}_n$, $W$, and $\mathbb{C}[x_1, \ldots, x_n]$. Therefore $f$ is in the even center $\mathcal{Z}(\tilde{\mathfrak{S}}_W^c)$.

Suppose now that $f \in \mathbb{C}[x_1^2, \ldots, x_n^2]^W$, then by the definition of $\tilde{\mathfrak{S}}_W^c$, $f$ commutes with $\mathfrak{c}_n$, $W$, and $x_i$ for all $1 \leq i \leq n$. By Lemma 4.3, 4.9 or 4.14 (for type $A$, $D$ or $B$ respectively), we have $[y_i, f] = 0$ for each $i$. Therefore $f$ is in the even center.

The module-finiteness over the even center now follows from the PBW property of $\tilde{\mathfrak{S}}_W^c$ (see Theorem 3.5). \qed

5. **Rational spin double affine Hecke algebras (sDaHa)**

In this section, we introduce the rational spin double affine Hecke algebras associated to the Weyl group $W$ of type $A_{n-1}$, $D_n$ and $B_n$, and then establish their PBW property.

5.1. **Elements in $\mathbb{C}W^-$ of order 2.** Recall that the spin group algebra $\mathbb{C}W^-$ has a presentation with generator $t_i$ given in Section 2. Introduce the following notation

\[
\begin{align*}
t_{i+j} &= \begin{cases} t_it_{i+1}\cdots t_j, & \text{if } i \leq j \\ 1, & \text{otherwise} \end{cases} \\
t_{ij} &= \begin{cases} t_{i-1}\cdots t_{j-1}, & \text{if } i \geq j \\ 1, & \text{otherwise} \end{cases}
\end{align*}
\]

Define the following odd elements in $\mathbb{C}W^-$ of order 2, which are analogs of reflections in $W$, for $1 \leq i < j \leq n$:

\[
\begin{align*}
t_{ij} &\equiv [i, j] = (-1)^{j-i}t_{j-1}\cdots t_{i+1}t_i\cdots t_{j-1} \\
t_{ji} &\equiv [j, i] = -[i, j] \\
t_{ij} &\equiv \overline{t_{ij}} = \begin{cases} (-1)^{j-i-1}t_{j-1}\cdots t_{i-1}t_{i+1}t_i\cdots t_{j-1}, & \text{for type } D_n \\ (-1)^{j-i}t_{j-1}\cdots t_{i-1}t_{i+1}t_i\cdots t_{j-1}, & \text{for type } B_n \end{cases} \\
t_{ji} &\equiv \overline{t_{ji}} = \overline{[i, j]} \\
t_i &\equiv \overline{[i]} = (-1)^{n-i}t_i\cdots t_{n-1}t_n\cdots t_i (1 \leq i \leq n).
\end{align*}
\]
Note the natural inclusions of algebras $\mathbb{C}W_{A_{n-1}}^- \leq \mathbb{C}W_{D_n}^- \leq \mathbb{C}W_{B_n}^-$. In particular, $t_1, \ldots, t_{n-1}$ and $t_n t_{n-1} t_n$ generate a subalgebra of $\mathbb{C}W_{B_n}^-$ which is isomorphic to $\mathbb{C}W_{D_n}^-$ (where $-t_n t_{n-1} t_n$ corresponds to the $n$-th generator for $\mathbb{C}W_{D_n}^-$. Hence, the notations $\langle i, j \rangle, \langle i, j \rangle$ here are consistent with such a subalgebra structure. Although we will not use it in this paper, we can show for $i < j$ that $\langle i, j \rangle = [j, n][i, n-1]t_n[i, n-1][j, n]$.

5.2. The algebra $\tilde{\mathcal{H}}_W$ of type $A_{n-1}$. The following algebra $\tilde{\mathcal{H}}_{A_{n-1}}$ was introduced in [11] under the notation of $\mathcal{B}_u$. We recall the definition here for the convenience of the subsequent subsections.

**Definition 5.1.** Let $u \in \mathbb{C}$, and let $W = W_{A_{n-1}}$. The rational spin double affine Hecke algebra of type $A_{n-1}$, denoted by $\tilde{\mathcal{H}}_W$ or $\tilde{\mathcal{H}}_{A_{n-1}}$, is the algebra generated by $\xi_i, y_i$ for $1 \leq i \leq n$ and $\mathbb{C}W^-$, subject to the following relations:

\[
y_i y_j = y_j y_i, \quad \xi_i \xi_j = -\xi_j \xi_i \quad (i \neq j)
\]

\[
t_i y_i = y_{i+1} t_i, \quad t_i \xi_i = -\xi_{i+1} t_i
\]

\[
t_i y_j = y_j t_i, \quad t_i \xi_j = -\xi_j t_i, \quad (j \neq i, i+1)
\]

\[
[y_j, \xi_i] = -u[i, j] \quad (i \neq j)
\]

\[
[y_i, \xi_i] = u \sum_{k \neq i} [i, k].
\]

5.3. The algebra $\tilde{\mathcal{H}}_W$ of type $D_n$.

**Definition 5.2.** Let $u \in \mathbb{C}$, and let $W = W_{D_n}$. The rational spin double affine Hecke algebra of type $D_n$, denoted by $\tilde{\mathcal{H}}_W$ or $\tilde{\mathcal{H}}_{D_n}$, is the algebra generated by $\xi_i, y_i$ for $1 \leq i \leq n$ and $\mathbb{C}W^-$, subject to the relations (5.1–5.3) and the following additional relations:

\[
t_n y_n = -y_{n-1} t_n, \quad t_n \xi_n = -\xi_{n-1} t_n
\]

\[
t_n y_j = y_j t_n, \quad t_n \xi_j = -\xi_j t_n, \quad (j \neq n-1, n)
\]

\[
[y_j, \xi_i] = -u[i, j] + u[i, j] \quad (i \neq j)
\]

\[
[y_i, \xi_i] = u \sum_{k \neq i} ([i, k] + [i, k]).
\]

5.4. The algebra $\tilde{\mathcal{H}}_W$ of type $B_n$.

**Definition 5.3.** Let $u, v \in \mathbb{C}$, and $W = W_{B_n}$. The rational spin double affine Hecke algebra of type $B_n$, denoted by $\tilde{\mathcal{H}}_W$ or $\tilde{\mathcal{H}}_{B_n}$, is the algebra generated by $\xi_i, y_i$ for $1 \leq i \leq n$ and $\mathbb{C}W^-$, subject to the relations (5.1–5.3) and the following additional relations:

\[
t_n y_n = -y_n t_n, \quad t_n \xi_n = -\xi t_n
\]

\[
t_n y_j = y_j t_n, \quad t_n \xi_j = -\xi_j t_n, \quad (j \neq n)
\]
\[
[y_j, \xi_i] = -u[i, j] + u[i, j] \quad (i \neq j)
\]
\[
[y_i, \xi_i] = u \sum_{k \neq i} \left([i, k] + [i, k]\right) + v[i].
\]

We write \(\tilde{\mathcal{H}_W}(u, v)\) for \(\tilde{\mathcal{H}_W}\) to indicate the dependence on \(u, v\) if necessary.

5.5. Isomorphism of superalgebras. The algebra \(\tilde{\mathcal{H}_W}\) contains several distinguished subalgebras: the skew-polynomial algebra \(C[\xi_1, \ldots, \xi_n]\), the spin Weyl group algebra \(\mathbb{C}W^{-}\), and the polynomial algebra \(\mathbb{C}[y_1, \ldots, y_n]\). The algebra \(\tilde{\mathcal{H}_W}\) has a superalgebra structure with \(y_i\) even and \(\xi_i, t_i\) odd for all \(i\).

Lemma 5.4. Let \(W\) be one of the Weyl groups \(W_{A_{n-1}}, W_{D_n}\) or \(W_{B_n}\). The map \(\Phi: \mathbb{C}_n \rtimes \mathbb{C}W \to \mathbb{C}_n \otimes \mathbb{C}W^{-}\) (which is an isomorphism by Theorem 2.1) sends

\[
(c_k - c_i)s_{ik} \mapsto -\sqrt{-2}[k, i] \quad (5.6)
\]
\[
(c_k + c_i)s_{ik} \mapsto -\sqrt{-2}[k, i] \quad (5.7)
\]
\[
c_i \tau_i \mapsto -\sqrt{-1}[i] \quad (5.8)
\]

for \(i \neq k\), whenever it is applicable.

Proof. We may assume that \(i > k\) without loss of generality.

We prove (5.6) by induction on \(i\). First, (5.6) for \(i = k + 1\) holds by Theorem 2.1. Assuming that (5.6) holds for \(i\), i.e. \(\Phi((c_k - c_i)s_{ik}) = -\sqrt{-2}[k, i]\), we have by Theorem 2.1 and the definition of \([k, i]\) that

\[
\Phi((c_k - c_{i+1})s_{i+1,k}) = \Phi(s_i(c_k - c_i)s_{ik}s_i)
\]
\[
= (-\sqrt{-1}t_i(-\sqrt{-2}[k, i])(-\sqrt{-1}t_i)
\]
\[
= \sqrt{-2}t_i[k, i]t_i = -\sqrt{-2}[k, i + 1].
\]

We now prove (5.8) by a similar downward induction on \(i\), whose initial case \(i = n\) is taken care of by Theorem 2.1. Assume that (5.8) holds for \(i + 1 \leq n\), i.e. \(\Phi(c_{i+1}\tau_{i+1}) = -\sqrt{-1}[i + 1]\). Then, by Theorem 2.1 and the definition of \([k, i]\), we have

\[
\Phi(c_i \tau_i) = \Phi(s_i c_{i+1}\tau_{i+1}s_i)
\]
\[
= (-\sqrt{-1}t_i(-\sqrt{-1}[i + 1])(-\sqrt{-1}t_i)
\]
\[
= \sqrt{-1}t_i[i + 1]t_i = -\sqrt{-1}[i].
\]

Now, we prove (5.7) by downward induction first on \(k\) and then on \(i\), for \(W = W_{D_n}\). The initial case \(i = n, k = n - 1\) holds by Theorem 2.1. Then, it follows by the induction assumption that \(\Phi((c_{k+1} + c_n)s_{n,k+1}) =\)
which uses (5.6) and the definition of \([k, n]\) that
\[
\Phi((c_k + c_n)_{\pi_{nk}}) = \Phi(s_k(c_{k+1} + c_n)_{\pi_{n,k+1}s_k})
\]
\[
= (-\sqrt{2}i_{k-1}) \cdot (-\sqrt{2}[k + 1, n]) \cdot (-\sqrt{2}i_{k-1})
\]
\[
= \sqrt{2}t_k[k + 1, n] t_k = -\sqrt{2}[k, n].
\]
This in turn becomes the initial step when \(i = n\) for proving (5.7) by downward induction on \(i\) (with fixed \(k < n\)). By induction assumption (5.7) holds for \(i > k + 1\). Then
\[
\Phi((c_k + c_i-1)_{\pi_{i-1,k}}) = \Phi(s_i-1(c_k + c_i)_{\pi_{k}s_i-1})
\]
\[
= (-\sqrt{2}i_{k-1})(-\sqrt{2}[k, i])(-\sqrt{2}i_{k-1})
\]
\[
= -\sqrt{2}t_{i-1}[k, i] t_{i-1} = -\sqrt{2}[k, i - 1].
\]
This completes the proof of (5.7) for type \(D\).

The formula (5.7) for \(W = W_{B_n}\) is similarly proved by double downward inductions on \(k\) and then on \(i\). The only difference from the type \(D\) case is that for type \(B\) we have to check the initial case when \(k = n - 1\) and \(i = n\), which uses (5.6) and (5.8):
\[
\Phi((c_{n-1} + c_n)_{\pi_{n-1,n}}) = \Phi(\tau_n(c_{k+1} - c_n)s_{n-1,n}\tau_n)
\]
\[
= (-\sqrt{2}\tau_n t_n) \cdot (-\sqrt{2} t_{n-1}) \cdot (-\sqrt{2}\tau_n t_n)
\]
\[
= \sqrt{2}t_n t_{n-1} = -\sqrt{2}[n - 1, n].
\]
Thus the lemma is proved. \(\square\)

Recall the isomorphism of superalgebras \(\Phi : \mathcal{C}_n \times \mathbb{C}W \to \mathcal{C}_n \otimes \mathbb{C}W^-\) and its inverse \(\Psi\) given in Theorem 2.1.

**Theorem 5.5.** Let \(W\) be one of the Weyl groups \(W_{A_n-1}, W_{D_n}\) or \(W_{B_n}\). Then,

1. there exists an isomorphism of superalgebras
   \[
   \Phi : \tilde{\mathcal{H}}_W \longrightarrow \mathcal{C}_n \otimes \tilde{\mathcal{H}}_W
   \]
   which extends \(\Phi : \mathcal{C}_n \times \mathbb{C}W \to \mathcal{C}_n \otimes \mathbb{C}W^-\) and sends
   \[
   y_i \leftrightarrow y_i, \ x_i \leftrightarrow \sqrt{2}e_i, \ s_i \leftrightarrow -\sqrt{2}i_{\alpha}, \ c_i \leftrightarrow c_i, \ \forall i;
   \]
2. the inverse
   \[
   \Psi : \mathcal{C}_n \otimes \tilde{\mathcal{H}}_W \longrightarrow \tilde{\mathcal{H}}_W
   \]
   extends \(\Psi : \mathcal{C}_n \otimes \mathbb{C}W^- \to \mathcal{C}_n \times \mathbb{C}W\) and sends
   \[
   y_i \leftrightarrow y_i, \ \xi_i \leftrightarrow \frac{1}{\sqrt{2}}e_i, \ t_i \leftrightarrow \sqrt{2}i_{\alpha}, \ c_i \leftrightarrow c_i, \ \forall i.
   \]

In the terminology of \([W2]\), \(\tilde{\mathcal{H}}_W\) and \(\tilde{\mathcal{H}}_W\) are Morita super-equivalent by Theorem 5.5.
Proof. Recall that $\Phi$ extends the isomorphism $C_n \rtimes CW \xrightarrow{\sim} C_n \otimes CW$. Among all the relations (3.1-3.11) for $\mathcal{H}_W^c$, it is easy to check that (3.1-3.5) are preserved by $\Phi$. So it remains to check that $\Phi$ preserves the relations (3.6-3.7), (3.8-3.9), and (3.10-3.11) for $W = W_{A_{n-1}}, W_{D_n},$ and $W_{B_n}$ respectively.

We shall verify in detail that $\Phi$ preserves (3.10-3.11) with $W = W_{B_n}$. Indeed, by Lemma 5.4, we have for $i \neq j$ that

$$\Phi(\text{l.h.s. of (3.10)}) = \sqrt{-2}[y_j, c_i \xi_i]$$

$$= \sqrt{-2}c_i(-u[i, j] + u[i, j])$$

$$= \Phi(u((1 + c_j c_i) s_{ji} - (1 - c_j c_i) \overline{s}_{ij}))$$

$$= \Phi(\text{r.h.s. of (3.10)}).$$

Also, by Lemma 5.4, we have

$$\Phi(\text{l.h.s. of (3.11)}) = \sqrt{-2}[y_i, c_i \xi_i]$$

$$= \sqrt{-2}uc_i \sum_{k \neq i} (i, k) + \sqrt{-2}v c_i [i]$$

$$= \Phi\left(-u \sum_{k \neq i} ((1 + c_k c_i) s_{ki} + (1 - c_k c_i) \overline{s}_{ki} - \sqrt{2} v_{si})\right)$$

$$= \Phi(\text{r.h.s. of (3.11)}).$$

By dropping the terms involving $v$ in the above equations, we verify that the relations (3.8-3.9) with $W = W_{D_n}$ are preserved by $\Phi$. By further dropping the terms involving $[ij], \overline{s}_{ij}$ etc., we can also verify (3.6-3.7) with $W = W_{A_{n-1}}$.

So, the homomorphism $\Phi$ is well defined. Similarly, one shows that $\Psi$ is a well-defined algebra homomorphism. For example, the relation $t_n \xi_n = -\xi_{n-1} t_n$ in $\mathcal{H}_W^c$ for $W = W_{D_n}$ is preserved by $\Psi$, since

$$\Psi(t_n \xi_n) = \sqrt{-1} \sqrt{2}(c_{n-1} + c_n) s_n = \frac{1}{\sqrt{-2}}c_n x_n$$

$$= \frac{1}{2}(c_{n-1} + c_n) c_{n-1} x_{n-1} s_n$$

$$= \frac{1}{2} c_{n-1} x_{n-1} (-c_{n-1} - c_n) s_n = -\Psi(\xi_{n-1} t_n).$$

On the other hand, the relation $t_n \xi_n = -\xi_n t_n$ in $\mathcal{H}_W^c$ for $W = W_{B_n}$ is preserved by $\Psi$, since

$$\Psi(t_n \xi_n) = \sqrt{-1} c_n s_n \frac{1}{\sqrt{-2}} c_n x_n = \frac{1}{\sqrt{2}} x_n s_n = -\frac{1}{\sqrt{2}} c_n x_n c_n s_n = -\Psi(\xi_n t_n).$$

Since $\Phi$ and $\Psi$ are inverses on generators, they are (inverse) algebra isomorphisms. $\square$
5.6. **The PBW property for** \( \check{\mathcal{H}}_W \). We have the following PBW type property for the algebra \( \check{\mathcal{H}}_W \).

**Theorem 5.6.** Let \( W \) be one of the Weyl groups \( W_{A_{n-1}} \), \( W_{D_n} \) or \( W_{B_n} \). The multiplication of the subalgebras induces an isomorphism of vector spaces

\[
\mathbb{C}[\xi_1, \ldots, \xi_n] \otimes \mathbb{C}W^- \otimes \mathbb{C}[y_1, \ldots, y_n] \longrightarrow \check{\mathcal{H}}_W.
\]

Equivalently, the set \( \{ \xi^\sigma \gamma \} \) forms a basis for \( \check{\mathcal{H}}_W \), where \( \sigma \) runs over a basis for \( \mathbb{C}W^- \), and \( \alpha, \gamma \in \mathbb{Z}^n_+ \).

**Proof.** It follows from the defining relations for \( \check{\mathcal{H}}_W \) that \( \check{\mathcal{H}}_W \) is spanned by the elements \( \xi^\alpha \sigma y^\gamma \) where \( \sigma \) runs over a basis for \( \mathbb{C}W^- \), and \( \alpha, \gamma \in \mathbb{Z}^n_+ \). By the isomorphism \( \Psi : \mathbb{C}_n \otimes \check{\mathcal{H}}_W \longrightarrow \check{\mathcal{H}}_W \) in Theorem 5.5, we see that the image \( \Psi(\xi^\alpha \gamma) \) are linearly independent in \( \check{\mathcal{H}}_W \) by the PBW property for \( \check{\mathcal{H}}_W \), see Theorem 3.5). So the elements \( \xi^\alpha \gamma \) are linearly independent in \( \check{\mathcal{H}}_W \).

Therefore, the set \( \{ \xi^\alpha \gamma \} \) forms a basis for \( \check{\mathcal{H}}_W \). \( \Box \)

6. **The Dunkl Operators for sDaHa**

Denote by \( h_\xi \) the subalgebra of \( \check{\mathcal{H}}_W \) generated by \( \xi_i \)'s \( (1 \leq i \leq n) \) and \( \mathbb{C}W^- \). For a \( \mathbb{C}W^- \)-module \( V \), it can be extended to a \( h_\xi \)-modules by letting the actions of \( \xi_i \)'s on \( V \) be trivial. We define

\[
V_\xi := \text{Ind}_{h_\xi}^{\check{\mathcal{H}}_W} V \cong \mathbb{C}[y_1, \ldots, y_n] \otimes V.
\]

We will always identify \( V_\xi = \mathbb{C}[y_1, \ldots, y_n] \otimes V \). On \( \mathbb{C}[y_1, \ldots, y_n] \otimes V \), the element \( t_i \in \mathbb{C}W^- \) acts as \( s_i \otimes t_i \), \( y_i \) acts by left multiplication, and \( \xi_i \) acts as Dunkl operators, which we will describe in this section.

Under Lemma 5.4 and the superalgebra isomorphism \( \Phi : \check{\mathcal{H}}_W \rightarrow \mathbb{C}_n \otimes \check{\mathcal{H}}_W \) in Theorem 5.5, the results in this section are fairly straightforward counterparts of those in Section 4, and we omit the proofs.

6.1. **The Dunkl Operator for** \( \check{\mathcal{H}}_{A_{n-1}} \). The following lemma is the counterpart of Lemma 4.6

**Lemma 6.1.** Let \( W = W_{A_{n-1}} \), and \( f \in \mathbb{C}[y_1, \ldots, y_n] \). Then the following identity holds in \( \check{\mathcal{H}}_W \):

\[
[f, \xi_i] = -u \sum_{k \neq i} \frac{f - f^s_{ki}}{y_i - y_k} [k, i].
\]

The following is the counterpart of Theorem 4.7.
Proposition 6.2. Let \( W = W_{A_{n-1}} \) and \( V \) be a \( \mathbb{C}W^- \)-module. The action of \( \xi_i \) on \( \mathbb{C}[y_1, \ldots, y_n] \otimes V \) is realized as a Dunkl operator as follows. For any polynomial \( f \in \mathbb{C}[y_1, \ldots, y_n] \) and \( v \in V \), we have
\[
\xi_i \circ (f \otimes v) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes [k, i]v.
\]

6.2. The Dunkl Operator for \( \tilde{\mathcal{H}}_{\tilde{B}_n} \). The following lemma is the counterpart of Lemma 4.11.

Lemma 6.3. Let \( W = W_{B_n} \) and \( l \in \mathbb{Z}_+ \). Then we have
\[
[y_j, \xi_i] = u \frac{y_j - y_i}{y_j - y_i} [j, i] + u \frac{y_j - (-y_i)^j}{y_j + y_i} [j, i],
\]
\[
[y_i, \xi_i] = -u \sum_{k \neq i} \frac{y_i - y_k}{y_i - y_k} [k, i] + u \sum_{k \neq i} \frac{y_i - (-y_k)^l}{y_i + y_k} [k, i] + v \frac{y_i - (-y_i)^j}{2y_i} [i].
\]

The following lemma is the counterpart of Lemma 4.12.

Lemma 6.4. Let \( W = W_{B_n} \). The following identity holds in \( \tilde{\mathcal{H}}_{\tilde{W}} \):
\[
[f, \xi] = -u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} [k, i] + u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i + y_k} [k, i] + v \frac{f - f^{r_i}}{2y_i} [i].
\]

The following is the counterpart of Theorem 4.13.

Proposition 6.5. Let \( W = W_{B_n} \), \( V \) be a \( \mathbb{C}W^- \)-module. The action of \( \xi_i \) on \( \mathbb{C}[y_1, \ldots, y_n] \otimes V \) is realized as a Dunkl operator as follows. For any polynomial \( f \in \mathbb{C}[y_1, \ldots, y_n] \) and \( v \in V \), we have
\[
\xi_i \circ (f \otimes v) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes [k, i]v
\]
\[
- u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i + y_k} \otimes [k, i]v - v \frac{f - f^{r_i}}{2y_i} \otimes [i]v.
\]

6.3. The Dunkl Operator for \( \tilde{\mathcal{H}}_{\tilde{D}_n} \).

Proposition 6.6. Let \( W = W_{D_n} \), \( V \) be a \( \mathbb{C}W^- \)-module. The action of \( \xi_i \) on \( \mathbb{C}[y_1, \ldots, y_n] \otimes V \) is realized as a Dunkl operator as follows. For any polynomial \( f \in \mathbb{C}[y_1, \ldots, y_n] \) and \( v \in V \), we have
\[
\xi_i \circ (f \otimes v) = u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i - y_k} \otimes [k, i]v - u \sum_{k \neq i} \frac{f - f^{s_{ki}}}{y_i + y_k} \otimes [k, i]v.
\]
6.4. The even center for $\tilde{\mathcal{C}}^e_W$.

**Proposition 6.7.** Let $W$ be one of the Weyl groups $W_{A_{n-1}}$, $W_{D_n}$ or $W_{B_n}$. The even center for $\tilde{\mathcal{C}}^e_W$ contains $\mathbb{C}[y_1, \ldots, y_n]^W$ and $\mathbb{C}[\xi_1^2, \ldots, \xi_n^2]^W$. In particular, $\tilde{\mathcal{C}}^e_W$ is module-finite over its even center.

**Proof.** By the isomorphism $\Phi : \tilde{\mathcal{C}}^e_W \rightarrow \mathcal{C}_n \otimes \tilde{\mathcal{C}}^e_W$ (see Theorems 3.5 and Proposition 4.18) we have

\[
\mathbb{C}[y_1, \ldots, y_n]^W \subseteq \Phi(Z(\tilde{\mathcal{C}}^e_W)) = Z(\mathcal{C}_n \otimes \tilde{\mathcal{C}}^e_W),
\]

\[
\mathbb{C}[\xi_1^2, \ldots, \xi_n^2]^W \subseteq \Phi(Z(\tilde{\mathcal{C}}^e_W)) = Z(\mathcal{C}_n \otimes \tilde{\mathcal{C}}^e_W).
\]

The first statement follows by noting that $\mathbb{C}[y_1, \ldots, y_n]^W$ and $\mathbb{C}[\xi_1^2, \ldots, \xi_n^2]^W$ actually lie in $\tilde{\mathcal{C}}^e_W$. The second statement now follows from the PBW property of $\tilde{\mathcal{C}}^e_W$ (Theorem 5.6). \qed

7. Rational covering double affine Hecke algebras (cDaHa)

In this section, the rational covering double affine Hecke algebras (cDaHa) $\tilde{\mathcal{C}}^e_W$ associated to classical Weyl groups $W$ are introduced. It has as its natural quotients the usual rational DaHa $\tilde{\mathcal{C}}^e_W$ (which will be recalled below) and the rational sDaHa $\tilde{\mathcal{C}}^e_W$ introduced in Section 5.

7.1. "Reflections" in $\tilde{W}$. Recall the distinguished double cover $\tilde{W}$ of a Weyl group $W$ with generators $\tilde{t}_i$'s from Section 2.1.

Introduce the notation

\[
\tilde{t}_{i|j} = \begin{cases} 
\tilde{t}_i \tilde{t}_{i+1} \cdots \tilde{t}_j, & \text{if } i < j \\
1, & \text{otherwise},
\end{cases}
\]

\[
\tilde{t}_{ij} = \begin{cases} 
\tilde{t}_i \tilde{t}_{i-1} \cdots \tilde{t}_j, & \text{if } i > j \\
1, & \text{otherwise}.
\end{cases}
\]

Define the following elements in $\tilde{W}$, which are distinguished preimages of reflections in $W$ under the canonical map $\tilde{W} \rightarrow W$, for $1 \leq i < j \leq n$:

\[
\{i, j\} = z^{j-i-1} \tilde{t}_{i|j}, \quad \{j, i\} = z^{i-j} \tilde{t}_{ij},
\]

\[
\{i, j\} = \begin{cases} 
\{j, i\} & \text{if } i < j, \\
z^{j-i-1} \tilde{t}_{j|n-1} \tilde{t}_{i|n-2} \tilde{t}_{n-2} \tilde{t}_{n-1} - \tilde{t}_{n-1}, & \text{for type } D_n \\
z^{j-i-1} \tilde{t}_{j|n-1} \tilde{t}_{i|n-2} \tilde{t}_{n-2} \tilde{t}_{n-1} \tilde{t}_{i|n-1}, & \text{for type } B_n
\end{cases}
\]

\[
\{i\} = \begin{cases} 
\{i, j\} & \text{if } i < j, \\
z^{n-i} \tilde{t}_i \cdots \tilde{t}_{n-1} \tilde{t}_{n-1} \cdots \tilde{t}_i & \text{if } 1 \leq i \leq n.
\end{cases}
\]

We have $\{i, j\} \in \tilde{W}_{A_{n-1}}$, $\{i, j\} \in \tilde{W}_{D_n}$ for $1 \leq i < j \leq n$, and $\{i\} \in \tilde{W}_{B_n}$ for $1 \leq i \leq n$, while noting that we have a sequence of subgroups $\tilde{W}_{A_{n-1}} \leq \tilde{W}_{D_n} \leq \tilde{W}_{B_n}$. The next lemma is straightforward from the definitions, and it helps to explain our choices of notations (recall $s_{ij} = (i, j)$ and $\overline{s}_{ij} = (i, j)$).
Lemma 7.1. Let $W$ be $\tilde{W}_{A_{n-1}}, \tilde{W}_{D_n}$, or $\tilde{W}_{B_n}$. The canonical quotient map $\Upsilon_+ : \mathbb{C} \tilde{W} \to \mathbb{C} W$ sends (for $i \neq j$)
\[
\{i,j\} \mapsto (i,j), \quad \{i,j\} \mapsto (i,j), \quad \{i\} \mapsto \tau_i,
\]
and the canonical quotient map $\Upsilon_- : \mathbb{C} \tilde{W} \to \mathbb{C} W^-$ sends (for $i \neq j$)
\[
\{i,j\} \mapsto [i,j], \quad \{i,j\} \mapsto [i,j], \quad \{i\} \mapsto [i]
\]
whenever it makes sense for the given $W$.

7.2. The rational Cherednik algebras. Recall that $\mathfrak{h} = \mathbb{C}^n$, and we have identified $\mathbb{C}[\mathfrak{h}] = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[y_1, \ldots, y_n]$. Below we shall recall the definition [EG] of rational double affine Hecke algebras (also called rational Cherednik algebras) associated to the classical Weyl groups in a more concrete form.

Let $t, u \in \mathbb{C}$. Let $W$ be one of the Weyl groups $W_{A_{n-1}}, W_{D_n}$, or $W_{B_n}$ respectively. The rational Cherednik algebra $\mathcal{H}_W$ is the algebra generated by $x_i, y_i$ $(1 \leq i \leq n)$ and $W$, subject to the common relations (7.1–7.2), and the additional relations (7.3–7.4) for type $A$, (7.5–7.6) for type $D$, (7.7–7.8) for type $B$, respectively:

\[
\begin{align*}
\sigma x &= x^\sigma \sigma, \quad \sigma y = y^{\sigma^*} \sigma \quad (\sigma \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h}) \quad (7.1) \\
[y_j, x_i] &= us_{ij} \quad (i \neq j) \quad (7.2) \\
[y_i, x_i] &= t \cdot 1 - u \sum_{k \neq i} s_{ki} \quad (7.3) \\
[y_j, x_i] &= u(s_{ij} - \mathfrak{s}_{ij}) \quad (i \neq j) \quad (7.4) \\
[y_i, x_i] &= t \cdot 1 - u \sum_{k \neq i} (s_{ki} + \mathfrak{s}_{ki}) \quad (7.5) \\
[y_j, x_i] &= u(s_{ij} - \mathfrak{s}_{ij}) \quad (i \neq j) \quad (7.6) \\
[y_i, x_i] &= t \cdot 1 - u \sum_{k \neq i} (s_{ki} + \mathfrak{s}_{ki}) - v \tau_i. \quad (7.7) \\
[y_i, x_i] &= t \cdot 1 - u \sum_{k \neq i} (s_{ki} + \mathfrak{s}_{ki}) - v \tau_i. \quad (7.8)
\end{align*}
\]

The algebra $\mathcal{H}_W$ has the following well-known PBW property: the multiplication of the subalgebras induces a vector space isomorphism
\[
\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C} W \otimes \mathbb{C}[\mathfrak{h}] \stackrel{\cong}{\rightarrow} \mathcal{H}_W.
\]
Equivalently, $\{x_\alpha y_\gamma w^z | \alpha, \gamma \in \mathbb{Z}^n, w \in W \}$ form a PBW basis for $\mathcal{H}_W$.

7.3. The rational covering double affine Hecke algebra $\tilde{\mathcal{H}}_{W}$. Recall that the group $\tilde{W}$ from Section 2 has the defining relations given in Table 1, Section 2 and $\tilde{W}$ contains a central element $z$ of order 2.

Definition 7.2. Let $W = W_{A_{n-1}}$, and let $t, u \in \mathbb{C}$. The rational covering double affine Hecke algebra of type $A_{n-1}$, denoted by $\tilde{\mathcal{H}}_{W}$ or $\tilde{\mathcal{H}}_{A_{n-1}}$, is the
algebra generated by $\tilde{x}_i, \tilde{y}_i$ ($1 \leq i \leq n$) and $z, \tilde{t}_1, \ldots, \tilde{t}_{n-1}$ subject to the relations for $\tilde{W}$, and the following relations: $z$ is central and

$$\tilde{x}_i \tilde{x}_j = z \tilde{x}_j \tilde{x}_i, \quad \tilde{y}_i \tilde{y}_j = \tilde{y}_j \tilde{y}_i \quad (i \neq j)$$ \hfill (7.9)

$$\tilde{t}_i \tilde{x}_j = z \tilde{x}_j \tilde{t}_i, \quad \tilde{t}_i \tilde{y}_j = \tilde{y}_j \tilde{t}_i \quad (j \neq i, i+1)$$ \hfill (7.10)

$$\tilde{t}_i \tilde{x}_{i+1} = z \tilde{x}_i \tilde{t}_i, \quad \tilde{t}_i \tilde{y}_{i+1} = \tilde{y}_i \tilde{t}_i$$ \hfill (7.11)

$$[\tilde{y}_j, \tilde{x}_i] = uz \{i, j\} \quad (j \neq i)$$

$$[\tilde{y}_i, \tilde{x}_i] = -uz \sum_{k \neq i} \{i, k\}.$$

**Definition 7.3.** Let $W = W_{D_n}$, and let $u \in \mathbb{C}$. The rational covering double affine Hecke algebra of type $D_n$, denoted by $\tilde{\mathcal{H}}_W^u$ or $\tilde{\mathcal{H}}_{D_n}$, is the algebra generated by $\tilde{x}_i, \tilde{y}_i$ ($1 \leq i \leq n$) and $z, \tilde{t}_1, \ldots, \tilde{t}_n$, subject to the relations for $\tilde{W}$, relations (7.9, 7.11), and the following additional relations: $z$ is central and

$$\tilde{t}_n \tilde{x}_j = z \tilde{x}_j \tilde{t}_n, \quad \tilde{t}_n \tilde{y}_j = \tilde{y}_j \tilde{t}_n \quad (i \neq n - 1, n)$$

$$\tilde{t}_n \tilde{x}_n = -\tilde{x}_n \tilde{t}_n, \quad \tilde{t}_n \tilde{y}_n = -\tilde{y}_n \tilde{t}_n$$

$$[\tilde{y}_j, \tilde{x}_i] = uz \left( \{i, j\} - \{i, j\} \right) \quad (j \neq i)$$

$$[\tilde{y}_i, \tilde{x}_i] = -uz \sum_{k \neq i} \left( \{i, k\} + \{i, k\} \right).$$

**Definition 7.4.** Let $W = W_{B_n}$, and let $u, v \in \mathbb{C}$. The rational covering double affine Hecke algebra of type $B_n$, denoted by $\tilde{\mathcal{H}}_W^u$ or $\tilde{\mathcal{H}}_{B_n}$, is the algebra generated by $\tilde{x}_i, \tilde{y}_i$ ($1 \leq i \leq n$) and $z, \tilde{t}_1, \ldots, \tilde{t}_n$, subject to the relations for $\tilde{W}$, relations (7.9, 7.11), and the following additional relations: $z$ is central and

$$\tilde{t}_n \tilde{x}_i = z \tilde{x}_i \tilde{t}_n, \quad \tilde{t}_n \tilde{y}_i = \tilde{y}_i \tilde{t}_n \quad (i \neq n)$$

$$\tilde{t}_n \tilde{x}_n = -\tilde{x}_n \tilde{t}_n, \quad \tilde{t}_n \tilde{y}_n = -\tilde{y}_n \tilde{t}_n$$

$$[\tilde{y}_j, \tilde{x}_i] = uz \left( \{i, j\} - \{i, j\} \right) \quad (j \neq i)$$

$$[\tilde{y}_i, \tilde{x}_i] = -uz \sum_{k \neq i} \left( \{i, k\} + \{i, k\} \right) - vz\{i\}.$$

7.4. **PBW basis for $\tilde{\mathcal{H}}_W^u$.** The following result provides a link between the rational Cherednik algebra $\tilde{\mathcal{H}}_{W}^{t=0}$ with the specialization $t = 0$ and the rational sDaHa via the notion of rational cDaHa.

**Proposition 7.5.** Let $W = W_{A_{n-1}}, W_{D_n}$, or $W_{B_n}$. Then the quotient of the rational cDaHa $\tilde{\mathcal{H}}_W^u$ by the ideal $(z - 1)$ (respectively, by the ideal $(z + 1)$) is isomorphic to the rational Cherednik algebra $\tilde{\mathcal{H}}_{W}^{t=0}$ (respectively, the rational sDaHa $\tilde{\mathcal{H}}_W$).
Proof. We will merely construct the isomorphisms of superalgebras explicitly, while noting that the verification follows directly from the definitions of the various algebras involved.

The canonical isomorphism map \(\Upsilon_+ : \mathbb{C}\tilde{W}/(z-1) \to \mathbb{C}W\) (cf. Lemma 7.1) can be extended to the isomorphism of superalgebras

\[
\Upsilon_+ : \tilde{\mathfrak{H}}\sim W/\langle z-1 \rangle \longrightarrow \mathfrak{H}_t = 0 W,
\]

\(\tilde{t}_i \mapsto s_i, \tilde{x}_i \mapsto x_i, \tilde{y}_i \mapsto y_i\).

Also, the canonical isomorphism map \(\Upsilon_- : \mathbb{C}\tilde{W}/(z+1) \to \mathbb{C}W^-\) (cf. Lemma 7.1) can be extended to the isomorphism of superalgebras \(\Upsilon_+ : \tilde{\mathfrak{H}}\sim W/(z+1) \to \tilde{\mathfrak{H}}W\) by sending \(\tilde{t}_i \mapsto t_i, \tilde{x}_i \mapsto \xi_i, \tilde{y}_i \mapsto y_i\).

The next theorem follows from Proposition 7.5, the PBW basis Theorem 5.6 for \(\tilde{\mathfrak{H}}W\), and the PBW property for \(\mathfrak{H}W\) (cf. [EG]), by the same type of argument for [W2 Proposition 3.10] or [KW Theorem 5.5].

**Theorem 7.6.** Let \(W = W_{A_{n-1}}, W_{D_n},\) or \(W_{B_n}\). Then the elements \(\tilde{x}^\alpha \tilde{w} \tilde{y}^\gamma\), where \(\alpha, \gamma \in \mathbb{Z}^n_+\) and \(\tilde{w} \in \tilde{W}\), form a basis for \(\tilde{\mathfrak{H}}W\).

### 8. Appendix: Proofs of Several Lemmas

8.1. **Proof of Lemma 3.7.** We will show that the relations (3.10) and (3.11) are invariant under the conjugation by elements \(c_l, 1 \leq l \leq n\). The verifications for the invariants of other relations under the conjugation by \(c_l\) are similar and will be omitted.

Consider the relation (3.10) first. Clearly, (3.10) is invariant under the conjugation by \(c_l, l \neq i, j\). Moreover, we calculate that

\[
c_i (\text{r.h.s. of (3.10)})c_i = u((c_ic_j - 1)s_{ji} - (-c_ic_j - 1)s_{ij}) = -[y_j, x_i] = c_i (\text{l.h.s. of (3.10)})c_i,
\]

\[
c_j (\text{r.h.s. of (3.10)})c_j = u((c_jc_i + 1)s_{ij} - (-c_jc_i + 1)s_{ij}) = [y_j, x_i] = c_j (\text{l.h.s. of (3.10)})c_j.
\]

Thus, (3.10) is conjugation-invariant by all \(c_l\).
Next, we will show that the relation (3.11) is invariant under the conjugation by each $c_l$. Indeed, we have

$$c_i(\text{r.h.s. of (3.11)})c_i = -\sqrt{2}vc_i\tau_i c_i - u\sum_{k \neq i} c_i((1 + c_k c_i) s_{ki} + (1 - c_k c_i) \overline{s}_{ki}) c_i$$

$$= \sqrt{2}v\tau_i - u\sum_{k \neq i} ((c_k c_i - 1)s_{ki} + (-c_i c_k - 1)\overline{s}_{ki})$$

$$= \sqrt{2}v\tau_i + u\sum_{k \neq i} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\overline{s}_{ki})$$

$$= -[y_i, x_i] = c_i(\text{l.h.s. of (3.11)})c_i.$$ 

For $j \neq i$, we have

$$c_j(\text{r.h.s. of (3.11)})c_j = -\sqrt{2}v\tau_i - uc_j((1 + c_j c_i) s_{ji} + (1 - c_j c_i) \overline{s}_{ji}) c_j$$

$$- u\sum_{k \neq i,j} c_j((1 + c_k c_i) s_{ki} + (1 - c_k c_i) \overline{s}_{ki})c_j$$

$$= -\sqrt{2}v\tau_i - uc_j(c_j c_i + 1)s_{ji} + (-c_j c_i + 1)\overline{s}_{ji})c_j$$

$$- u\sum_{k \neq i,j} ((1 + c_k c_i) s_{ki} + (1 - c_k c_i) \overline{s}_{ki})c_j$$

$$= c_j(\text{l.h.s. of (3.11)})c_j.$$

Therefore, the lemma is proved.

8.2. Proof of Lemma 3.8 We will show below that the relations (3.10–3.11) are invariant under the conjugation by elements in $W_{B_n}$. The proof can be readily modified to yield the Weyl group invariance of the relations (3.6–3.7) and (3.8–3.9) in type $A$ and $D$ cases respectively, and we leave the details to the interested reader.

(i) We check the invariance of (3.10) under $W_{B_n}$.

Consider first the conjugation invariance by the transposition $s_{lk}$. If $\{l,k\} \cap \{i,j\} = \emptyset$, then we have

$$s_{lk}(\text{r.h.s. of (3.10)})s_{lk} = u((1 + c_j c_i) s_{ji} - (1 - c_j c_i) \overline{s}_{ij})$$

$$= [y_j, x_i] = s_{lk}(\text{l.h.s. of (3.10)})s_{lk}.$$ 

If $\{l,k\} \cap \{i,j\} = \{i\}$, then we may assume $l = j$ and we have

$$s_{jk}(\text{r.h.s. of (3.10)})s_{jk} = u((1 + c_k c_i) s_{ki} - (1 - c_k c_i) \overline{s}_{ik})$$

$$= [y_k, x_i] = s_{jk}(\text{l.h.s. of (3.10)})s_{jk}.$$ 

We leave an entirely analogous computation when $\{l,k\} \cap \{i,j\} = \{j\}$ to the reader.

Now, if $\{l,k\} = \{i,j\}$, then

$$s_{ji}(\text{r.h.s. of (3.10)})s_{ji} = u((1 + c_i c_j) s_{ij} - (1 - c_i c_j) \overline{s}_{ij})$$

$$= [y_i, x_j] = s_{ji}(\text{l.h.s. of (3.10)})s_{ji}.$$
So (3.10) is invariant under the conjugation by each transposition $s_{ik}$.

It remains to show that (3.10) is invariant under the conjugation by the simple reflection $s_n = \tau_n$. Observe that (3.10) is clearly invariant under the conjugation by $s_n$ for $n \neq j, i$. Moreover, if $j = n$ then we have

$$s_n(\text{r.h.s. of (3.10)})s_n = u((1 - c_j c_i)\overline{\tau}_{ji} - (1 + c_j c_i)s_{ij}) = -[y_j,x_i] = s_n(\text{l.h.s. of (3.10)})s_n.$$ 

If $i = n$, then we have

$$s_n(\text{r.h.s. of (3.10)})s_n = u((1 - c_j c_i)\overline{\tau}_{ji} - (1 + c_j c_i)s_{ij}) = -[y_j,x_i] = s_n(\text{l.h.s. of (3.10)})s_n.$$ 

This completes (i).

(ii) We check the invariance of (3.11) under $W_{B_n}$.

Consider first the conjugation invariance by a transposition $s_{jl}$. If $\{j, l\} \cap \{i\} = \emptyset$, then we have

$$s_{jl}(\text{r.h.s. of (3.11)})s_{jl} = -us_{jl}((1 + c_j c_i)s_{ji} + (1 - c_j c_i)\overline{\tau}_{ji})s_{jl}$$
$$-us_{jl}((1 + c_l c_i)s_{li} + (1 - c_l c_i)\overline{\tau}_{li})s_{jl}$$
$$-u\sum_{k \neq i, j, l} s_{jl}((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\overline{\tau}_{ki})s_{jl} - \sqrt{2}v_{\tau_l}$$
$$= [y_l, x_i] = s_{jl}(\text{l.h.s. of (3.11)})s_{jl}.$$ 

If $\{j, l\} \cap \{i\} = \{i\}$, we may assume that $j = i$, and then we have

$$s_{il}(\text{r.h.s. of (3.11)})s_{il} = -us_{il}((1 + c_l c_i)s_{li} + (1 - c_l c_i)\overline{\tau}_{li})s_{il}$$
$$-u\sum_{k \neq i, l} s_{il}((1 + c_k c_i)s_{kl} + (1 - c_k c_i)\overline{\tau}_{kl})s_{il} - \sqrt{2}v_{\tau_l}$$
$$= [y_l, x_i] = s_{il}(\text{l.h.s. of (3.11)})s_{il}.$$ 

It remains to show that (3.11) is invariant under the conjugation by the simple reflection $s_n \equiv \tau_n \in W_{B_n}$. If $i \neq n$, we have

$$s_n(\text{r.h.s. of (3.11)})s_n = -\sqrt{2}v_{\tau_i} - us_n((1 + c_n c_i)s_{ni} + (1 - c_n c_i)\overline{\tau}_{ni})s_n$$
$$-u\sum_{k \neq i, n} s_n((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\overline{\tau}_{ki})s_n$$
$$= -\sqrt{2}v_{\tau_i} - u((1 - c_n c_i)\overline{\tau}_{ni} + (1 + c_n c_i)s_{ni})$$
$$-u\sum_{k \neq i, n} ((1 + c_k c_i)s_{ki} + (1 - c_k c_i)\overline{\tau}_{ki})$$
$$= [y_i, x_i] = s_n(\text{l.h.s. of (3.11)})s_n.$$
If $i = n$, then
\[ s_n(\text{r.h.s. of (3.11)})s_n \]
\[ = -\sqrt{2} v_{\tau n} - u \sum_{k \neq n} ((1 - c_k c_n) s_{kn} + (1 + c_k c_n) s_{kn}) \]
\[ = [y_n, x_n] = s_n(\text{l.h.s. of (3.11)})s_n. \]
This completes the proof of (ii). Hence the lemma is proved.

8.3. Proof of Lemma 3.9. We will establish the Jacobi identity for $W = W_{B_n}$. The proof can be easily modified for the cases of type $A$ and $D$, and we leave the details to the reader.

The Jacobi identity trivially holds among triple $x_i$’s or triple $y_i$’s.

Now, we consider the triple with two $y$’s and one $x$. The case with two identical $y_i$ is trivial. So we first consider $x_i$, $y_j$, and $y_l$ where $i, j, l$ are all distinct. The Jacobi identity holds in this case since
\[ [x_i, [y_j, y_l]] + [y_l, [x_i, y_j]] + [y_j, [y_l, x_i]] \]
\[ = 0 + [y_l, u((1 + c_j c_i) s_{ji} - (1 - c_j c_i) s_{ij})] \]
\[ = 0 + [y_l, u((1 + c_j c_i) s_{ji} - (1 - c_j c_i) s_{ij})] = 0. \]
Now for $i \neq j$, we have
\[ [x_i, [y_l, y_j]] + [y_j, [x_i, y_l]] + [y_l, [y_j, x_i]] \]
\[ = 0 + [y_l, u \sum_{k \neq i} ((1 + c_k c_i) s_{ki} + (1 - c_k c_i) s_{ki}) + \sqrt{2} v_{\tau i}] \]
\[ + [y_l, u((1 + c_j c_i) s_{ji} - (1 - c_j c_i) s_{ij})] \]
\[ = [y_l, u \sum_{k \neq i, j} ((1 + c_k c_i) s_{ki} + (1 - c_k c_i) s_{ki})] \]
\[ + [y_l, u((1 + c_j c_i) s_{ji} + (1 - c_j c_i) s_{ji})] \]
\[ = 0 + u((1 + c_j c_l) y_j s_{ji} + (1 - c_j c_l) y_l s_{ji}) \]
\[ - u((1 + c_j c_l) s_{ji} y_j + (1 - c_j c_l) s_{ji} y_l) \]
\[ + u((1 + c_j c_l) y_l s_{ji} - (1 - c_j c_l) s_{ji} y_l) \]
\[ - u((1 + c_j c_l) s_{ji} y_l - (1 - c_j c_l) s_{ji} y_l) = 0. \]

Now we consider the Jacobi identity with one $y$ and two $x$’s. The case with all distinct indices can be easily verified as above. Moreover, for $i \neq j$, we have
\[ [x_i, [y_i, x_j]] + [x_j, [x_i, y_l]] + [y_l, [x_j, x_i]] \]
\[ = [x_i, u((1 + c_i c_j) s_{ij} - (1 - c_i c_j) s_{ij})] \]
+ \left[ x_j, u \sum_{k \neq i} \left( (1 + c_k c_j) s_{ki} + (1 - c_k c_j) s_{ki} \right) + \sqrt{2 v r_i} \right] + 0

= [x_i, u ((1 + c_i c_j) s_{ij} - (1 - c_i c_j) s_{ij})]
+ [x_j, u ((1 + c_j c_i) s_{ij} + (1 - c_j c_i) s_{ij})]
= u ((1 - c_i c_j) x_i s_{ij} - (1 + c_i c_j) x_i s_{ij})
- u ((1 + c_i c_j) s_{ij} x_i - (1 - c_i c_j) s_{ij} x_i)
+ u ((1 - c_j c_i) x_j s_{ij} + (1 + c_j c_i) x_j s_{ij})
- u ((1 + c_j c_i) s_{ij} x_j + (1 - c_j c_i) s_{ij} x_j) = 0.

This completes the verification of the Jacobi identity for any triples.

8.4. Proof of Lemma 4.8. We will proceed by induction. For \( l = 1 \), then the equations hold by (3.10) and (3.11). Now assume that the statement is true for \( l \). Then

\[ [y_i, x_j^{l+1}] = [y_i, x_j^l] x_j + x_j^l [y_i, x_j] \]

= \( u \left( \frac{x_j^l - x_i^l}{x_j - x_i} + \frac{x_j^l - (-x_i)^l c_i c_j}{x_j + x_i} \right) s_{ij} x_j \)
- \( u \left( \frac{x_j^l - (-x_i)^l}{x_j + x_i} - \frac{x_j^l - x_i^l c_i c_j}{x_j - x_i} \right) s_{ij} x_j \)
+ \( x_j^l u ((1 + c_i c_j) s_{ij} - (1 - c_i c_j) s_{ij}) \)
= \( u \left( \frac{x_j^{l+1} - x_i^{l+1}}{x_j - x_i} + \frac{x_j^l - (-x_i)^l c_i c_j}{x_j + x_i} \right) s_{ij} \)
- \( u \left( \frac{x_j^{l+1} - (-x_i)^l+1}{x_j + x_i} - \frac{x_j^l - x_i^{l+1} c_i c_j}{x_j - x_i} \right) s_{ij} \)

\[ [y_i, x_i^{l+1}] = [y_i, x_i^l] x_i + x_i^l [y_i, x_i] \]

= \(-u \sum_{k \neq i} \left( \frac{x_i^l - x_k^l}{x_i - x_k} + \frac{x_i^l - (-x_k)^l c_k c_i}{x_i + x_k} \right) s_{ki} x_i \)
- \( u \sum_{k \neq i} \left( \frac{x_i^l - (-x_k)^l}{x_i + x_k} - \frac{x_i^l - x_k^l c_k c_i}{x_i - x_k} \right) s_{ki} x_i \)
- \( \sqrt{2 v} x_i^l \tau_i - \tau_i x_i^l \)
- \( u x_i^l \sum_{k \neq i} ((1 + c_k c_i) s_{ki} + (1 - c_k c_i) s_{ki}) - \sqrt{2 v} u x_i^l \tau_i \)
= \(-u \sum_{k \neq i} \left( \frac{x_i^{l+1} - x_k^{l+1}}{x_i - x_k} + \frac{x_i^{l+1} - (-x_k)^{l+1} c_k c_i}{x_i + x_k} \right) s_{ki} \).
\[-u \sum_{k \neq i} \left( \frac{x_i^l + (-x_k)^l}{x_i + x_k} - \frac{x_i^{l+1} - x_k^{l+1}}{x_i - x_k} c_k c_i \right) s_{ki} \]
\[-\sqrt{2} x_i^{l+1} \tau_i - \tau_i x_i^{l+1} \]

This completes the proof of the identities.

8.5. **Proof of Lemma 4.9.** It suffices to check the formula for every monomial $f$. First, we consider the monomial $g = \prod_{j \neq i} x_j^a_j$. By induction and Lemma 4.8, we can show that the formula holds for the monomial of the form $g = \prod_{j \neq i} x_j^a_j$ (the detail of the induction step does not differ much from the following calculation). Now consider the monomial $f = x_i^l g$. By Lemma 4.8 we have

\[
[y_i, f] = [y_i, x_i^l] g + x_i^l[y_i, g] \\
= -u \sum_{k \neq i} \left( \frac{x_i^l - x_k^l}{x_i + x_k} + \frac{x_i^l - (-x_k)^l}{x_i + x_k} c_k c_i \right) s_{ki} g \\
- u \sum_{k \neq i} \left( x_i^l - (-x_k)^l - x_i^l - x_k^l c_k c_i \right) s_{ki} g \\
- \sqrt{2} x_i^l \frac{x_i^l - (-x_i^l)^l}{2x_i} \tau_i g \\
- u \sum_{k \neq i} x_i^l \left( g - g_{s_{ki}} \frac{g - g_{s_{ki}}}{x_i + x_k} + \frac{g - g_{s_{ki}}}{x_i + x_k} c_k c_i \right) s_{ki} \\
- u \sum_{k \neq i} x_i^l \left( g - g_{s_{ki}} \frac{g - g_{s_{ki}}}{x_i + x_k} - \frac{g - g_{s_{ki}}}{x_i + x_k} c_k c_i \right) \tilde{s}_{ki} \\
= - u \sum_{k \neq i} \left( f - f_{s_{ki}} \frac{f - f_{s_{ki}}}{x_i + x_k} + \frac{f - f_{s_{ki}}}{x_i + x_k} c_k c_i \right) s_{ki} \\
- u \sum_{k \neq i} \left( f - f_{s_{ki}} \frac{f - f_{s_{ki}}}{x_i + x_k} - \frac{f - f_{s_{ki}}}{x_i + x_k} c_k c_i \right) \tilde{s}_{ki} - \sqrt{2} \tau_i^l \tau_i \]

So the lemma is proved.

8.6. **Proof of Lemma 4.11.** We will proceed by induction. For $l = 1$, the equations hold by (3.10) and (3.11). Now assume that the statement is true
for \( l \). Then

\[
[y_j^{l+1}, x_i] = y_j [y_j^l, x_i] + [y_j, x_i] y_j^l
\]

\[
= uy_j \left( \frac{y_j^l - y_j^l}{y_j - y_i} (1 + c_j c_i) s_{ij} - \frac{y_j^l - (-y_j^l)^l}{y_j + y_i} (1 - c_j c_i) \bar{s}_{ij} \right) + u \left( (1 + c_j c_i) s_{ji} - (1 - c_j c_i) \bar{s}_{ij} \right) y_j^l
\]

\[
= u \left( \frac{y_j^{l+1} - y_j^{l+1}}{y_j - y_i} (1 + c_j c_i) s_{ij} - \frac{y_j^{l+1} - (-y_j^{l+1})}{y_j + y_i} (1 - c_j c_i) \bar{s}_{ij} \right).
\]

On the other hand, we have

\[
[y_i^{l+1}, x_i] = y_i [y_i^l, x_i] + [y_i, x_i] y_i^l
\]

\[
= -u \sum_{k \neq i} y_i \frac{y_i^l - y_k}{y_i - y_k} (1 + c_k c_i) s_{ki} - u \sum_{k \neq i} y_i \frac{y_i^l - (-y_k)^l}{y_i + y_k} (1 - c_k c_i) \bar{s}_{ki} - \sqrt{2} v y_i \frac{y_i^l - (-y_i^l)^l}{2 y_i} \tau_i
\]

\[
= -u \sum_{k \neq i} \frac{y_i^{l+1} - y_k^{l+1}}{y_i - y_k} (1 + c_k c_i) s_{ki} - \sqrt{2} v y_i \frac{y_i^{l+1} - (-y_i^{l+1})}{2 y_i} \tau_i.
\]

This completes the proof of the lemma.

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