Quot schemes for flags and Gromov invariants for flag varieties

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1 Introduction

To define Gromov-Witten invariants arise in mirror symmetry, there are two general rigorous methods so far [14][11]. In particular Kontsevich introduced the notion of stable maps for a compactification of moduli spaces. For Grassmannians, however, there is a natural compactification of the space $\text{Mor}_d(\mathbb{P}^1, \text{Gr}(n, r))$ of all holomorphic maps from $\mathbb{P}^1$ to Grassmannians with a given degree $d$ where $\text{Gr}(n, r)$ is the Grassmannian of all rank $r$ quotient vector spaces of $\mathbb{C}^n$. We may see $\text{Mor}_d(\mathbb{P}^1, \text{Gr}(n, r))$ as the set of all rank $r$, degree $d$, quotient bundles of $\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^1}$. It is not a compact space. Hence we come to a Grothendieck’s Quot scheme $\text{Quot}_d(\text{Gr}(n, r))$, the set of all rank $r$, degree $d$, quotient sheaves of $\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^1}$. It is proven to be a smooth projective variety by S. A. Strømme [16]. Bertram and Franco-Reina used Grothendieck’s Quot schemes for Gromov-Witten invariants and quantum cohomology of Grassmannians respectively [3][6]. In this paper using Quot schemes and a localization theorem we study Gromov-Witten invariants for partial flag varieties. The strategy is the following. We extend A. Bertram’s result of Gromov-Witten invariants for special Schubert varieties of Grassmannians to the case of partial flag varieties. To do so a Grothendieck’s Quot scheme is generalized for flags and proven to be an irreducible, rational, smooth, projective variety following Strømme [16]. The Hilbert schemes for flags have already studied [13]. On a partial flag manifold there is an action by a special linear group. It induces an action on the Quot scheme for flags. There is another action on it by the multiplicative group $\mathbb{C}^\times$. It is induced from the action $\mathbb{C}^\times$ on $\mathbb{P}^1$. The analogous action by $\mathbb{C}^\times$ does not
exist on the Kontsevich moduli space. These two actions are commutative.
These together give isolated fixed points. Using a localization by action
an explicit formula of Gromov invariant for special Schubert classes with
a certain condition is given. Note that Kontsevich also uses torus actions
on his moduli space of stable maps \[11\]. In his case he has to deal with
summation over trees since the fixed subsets are rather complicated. For
projective spaces the formula derived in the sequel is shown to agree to the
residue formula \[10\]. The author does not know how to directly relate with
the result of Givental and Astashkevich-Sadov’s computation \[8\] \[1\]. Now we
state our main results.

For given integers \(s_0 = 0 < s_1 = n - r_1 < \cdots < s_l = n - r_l < n = s_{l+1}\),
a flag variety \(F_l := F(s_1, s_2, \ldots, s_l; n)\) is, by definition, the set of all flags
of complex subspaces \(V_1 \subseteq V_2 \subseteq \cdots \subseteq V = \mathbb{C}^n, \dim V_i = s_i\). There are
universal vector bundles \(S_i\) and universal quotient bundles \(Q_i\) over \(F_l\) with
fibers \(\mathbb{C}^{s_i}\) and \(\mathbb{C}^n / \mathbb{C}^{s_i}\) respectively. We are interested in a moduli space, the
set \(\text{Mor}_d(\mathbb{P}^1, F_l)\) of all morphisms \(\varphi\) from \(\mathbb{P}^1\) to \(F_l\) with \(<\mathbb{P}^1, c_1(\varphi^*Q_k) >= d_k, d = (d_1, d_2, \ldots, d_l)\). Since \(F_l\) is the fine moduli space such that the
associated flag functor is equivalent to \(\text{Mor}(\cdot, F_l)\) where the image of the
functor at a scheme \(S\) is the set of all flag quotient bundles of \(V \otimes \mathcal{O}_S\)
with ranks \(r_i\). From the point of view as above, \(\text{Mor}_d(\mathbb{P}^\infty, F_l)\) is realized
as the set of all flag quotient bundles \((F_1, \ldots, F_l)\) of \(V \otimes \mathcal{O}_{\mathbb{P}^\infty}\) with rank \(r_i\)
and degrees \(d_i\) and hence it can be compactified by collecting flag
quotient sheaves. More precisely,

**Theorem 1** There is a smooth compactification \(f\text{Quot}_d(F_l)\) of \(\text{Mor}_d(\mathbb{P}^1, F_l)\).
The underlying set of \(f\text{Quot}_d(F_l)\) is the set of all flag quotient sheaves
\((F_1, \ldots, F_l)\) of \(V \otimes \mathcal{O}_{\mathbb{P}^1}\) over \(\mathbb{P}^1\) where rank of \(F_i\) is \(r_i\) and its degree is \(d_i\). Over the irreducible, rational, projective variety \(f\text{Quot}_d(F_l) \times \mathbb{P}^\infty\) there
are tautological bundles \(E_i\) and sheaves \(Q_i\), \(i = 1, \ldots, l\). They form exact
sequences

\[0 \rightarrow E_i \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^\infty} \times \mathcal{O}_{\mathbb{P}^\infty} \rightarrow Q_i \rightarrow 0.\]

The induced sheaf morphisms \(E_i \rightarrow Q_i\) are identically zero.

This fine moduli space \(f\text{Quot}_d(F_l)\) will give the Gromov-Witten invariants defined in \[12\] \[11\].

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Theorem 2 Let $p_1, \ldots, p_N$ be fixed $N$ distinct points in $\mathbb{P}^k$. For $i=1, \ldots, N$, let $\alpha_i$ be integers in $\{s_1, \ldots, s_{l+1}\}$ and let $\beta_i$ be positive integers less than $s_{\alpha_i} - s_{\alpha_i - 1}$. Then the number of morphism $\varphi$ from $\mathbb{P}^k$ to $F_l$ such that each $\varphi(p_i)$ is in each the Poincaré dual Schubert subvariety to the classes $c_{\beta_i}(S_{\alpha_i})$ and $\langle \mathbb{P}^k, \varphi^*Q\gamma \rangle = \gamma$, $k = 1, \ldots, l$, is well-defined and it is

$$\int f_{\text{Quot}_d} \wedge_i c_{\beta_i}(E).$$

The integration is not depend on the choices of the point $p_1, \ldots, p_N$ in $\mathbb{P}^k$.

By the torus action on $f_{\text{Quot}_d}(F_l)$ induced from the standard $\mathbb{C}^\times$-action on $\mathbb{P}^k$ and the standard $(\mathbb{C}^\times)^k$ on $F_l$ one can apply Bott’s residue formula to the above integration to get

Theorem 3 The integration in the theorem 2 is

$$\sum_{\text{all integers as in (1)}} \prod_i (\sigma_{\alpha_i}) \prod(\text{characters as in (2)}) \prod(\text{characters as in (3)}).$$

The notations will be explained as follows.

Consider a sequence of data by nonnegative integers $d_{i,j}$ and $a_{i,j}$:

$$(d_{1,1}, a_{1,1}, \ldots; d_{1,s_1}, a_{1,s_1}) \cdots (d_{l,1}, a_{l,1}, \ldots; d_{l,s_l}, a_{l,s_l}) \quad (1)$$

such that $d_{i,j} - a_{i,j} \geq d_{i+1,j} - a_{i+1,j} \geq 0$, $a_{i,j} \geq a_{i+1,j}$ and $\sum_{j=1}^{s_i} d_{i,j} = d_i$.

Set $b_{i,j} := d_{i,j} - a_{i,j}$. Then we consider

$$(p - a_{i,j})h + \lambda_{j'} - \lambda_j, \text{ for } 0 \leq p \leq a_{i,j'} - 1, \ 1 \leq j, j' \leq s_i,$$

$$(b_{i,j} - p)h + \lambda_{j'} - \lambda_j, \text{ for } 0 \leq p \leq b_{i,j'} - 1, \ 1 \leq j, j' \leq s_i, \quad (2)$$

$$(p - a_{i,j})h + \lambda_m - \lambda_j, \text{ for } 0 \leq p \leq d_{i,j}, \ 1 \leq j \leq s_i, \ s_i + 1 \leq m \leq n,$$

and

$$(p - a_{i,j})h + \lambda_{j'} - \lambda_j, \text{ for } 0 \leq p \leq a_{i+1,j'} - 1, \ 1 \leq j, j' \leq s_{l+1},$$

$$(b_{i,j} - p)h + \lambda_{j'} - \lambda_j, \text{ for } 0 \leq p \leq b_{i+1,j'} - 1, \ 1 \leq j, j' \leq s_i, \quad (3)$$

$$(p - a_{i,j})h + \lambda_m - \lambda_j, \text{ for } 0 \leq p \leq d_{i,j}, \ 1 \leq j \leq s_i, \ s_{i+1} + 1 \leq m \leq n.$$
Finally let $\sigma_k^i$ be the $k$-th elementary symmetry function of $a_{i,j}h + \lambda_j$, $j = 1, \ldots, s_i$.

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2 flag-Quot schemes

All schemes will be assumed to be algebraic schemes over an algebraically closed field $k$ of characteristic 0 and all sheaves will be quasi-coherent. The space of all rank $r$, degree $d$, quotients of trivial sheaf $V \otimes \mathcal{O}_{\mathbb{P}^1}$—or equivalently, all subsheaves of rank $s = n - r$, degree $-d$—is a smooth, rational, irreducible, projective variety $Qout(V \otimes \mathcal{O}_{\mathbb{P}^1}, \text{Hilbert polynomial } rx + r + d)$ \cite{9}\cite{16}. Let us denote by $\text{Quot}_d(s, V)$ the Quot scheme. It could be considered as a compactification of the space $\text{Mor}_d(\mathbb{P}^1, \text{Gr}(n, r))$ of all degree $d$ holomorphic maps from the projective line $\mathbb{P}^1$ to the Grassmannian $\text{Gr}(n, r)$ which is the set of all rank $r$ quotient spaces of $V$ \cite{6}. It is equipped with a universal locally free sheaf $E$ and a universal quotient sheaf $Q$ over $\mathbb{P}^1 \times \text{Quot}$ with an exact sequence $0 \rightarrow E \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^1 \times \text{Quot}} \rightarrow Q \rightarrow 0$. They are flat over $\text{Quot}$. For the special Schubert varieties, Gromov invariants can be defined via Quot schemes as enumerative invariants \cite{3}\cite{6}. To extend their results to flag varieties (not necessary complete flag varieties), we will construct Quot schemes for flags, a compactification of $\text{Mor}_d(\mathbb{P}^k, \mathbb{F} \ltimes)$. 

Let $s = (s_1, \ldots, s_l)$, $s_{i+1} > s_i > 0$, and $d = (d_1, \ldots, d_l)$ be multi-indices of nonnegative integers. For this moduli problem, first we introduce a moduli functor $\mathcal{F}_d^s$. A contravariant functor $\mathcal{F}_d^s$ from the category of schemes to the category of sets is defined to be: for a scheme $S$, $\mathcal{F}_d^s(S) := \text{the set of all flag subsheaves } (E_1, E_2, \ldots, E_l, E_n = V \otimes \mathcal{O}_{\mathbb{P}^1 \times S}) \text{ over } \mathbb{P}^1 \times S \text{ where sheaf } E_i \text{ is subsheaf of } E_j \text{ if } i < j, \text{ sheaves are flat over } S, \text{ and the rank of } E_i \text{ is } s_i \text{ and its degree is } -d \text{ over } \mathbb{P}^1$. The functor $\mathcal{F}_d^s$ can be defined by quotients in a
more transparent way, for different data $0 \to E \to V \otimes O_{P^g \times S}$ could give the same data $V \otimes O_{P^g \times S} \to F \to t [3].$

Now we will show the functor is representable. First some notations; for a sheaf $F$ over $P^g \times S$, denote by $F(\pi) F \otimes \pi^* O(\pi)$, where $\pi$ is the projection from $P^g \times S$ to $P^g$. For the second projection from $P^g \times S$ to $S$ we will use $\pi_S$.

Taking advantage of the existence of Quot schemes, the obvious candidate for the scheme representing the functor $F^g_d$ is the appropriate subsheaf of $Quot_{d_1}(s_1, n) \times \ldots \times Quot_{d_l}(s_l, n)$.

Over the product of Quot schemes there are universal subsheaves $E_i$ and quotient sheaves $Q_i$ induced each from $Quot_{d_1}(s_1, n)$. Define $f Quot_d(F_l)$ as the degeneracy loci of

$$
\bigoplus_{i=1}^{l-1} \left( (\pi_{\Pi Quot_{d_1}(s_1, n)})^* E_i(m) \to V/((\pi_{\Pi Quot_{d_2}(s_1, n)})^* (E_{i+1}(m))) \right)
$$

for any $m \geq \max_i \{d_i\} - 1$. Note that $(\pi_{\Pi Quot_{d_1}(s_1, n)})^* E_i(m)$ and $V/((\pi_{\Pi Quot_{d_2}(s_1, n)})^* (E_{i+1}(m)))$ are locally free so that there is no problem giving a scheme structure on $f Quot_d(F_l)$.

For shorthand, we will write $Quot_{d_1}(s_1, n)$ by $Quot_i$ and $f Quot_d(F_l)$ by $f Quot$, then now we are ready for

Proposition 1 The functor $F^g_d$ is representable by the (unique) projective scheme, $f Quot_d(F_l)$.

Proof The statement means that for any scheme $S$, $F^g_d(S) = Mor(S, f Quot)$ in a functorial way. Let $(E_1, \ldots, E_l) \in F^g_d(S)$. Then we have the morphism $g : S \to Quot_1 \times \ldots \times Quot_l$ from the fine moduli property of the Quot schemes. We see that $g^* (\pi_{\Pi Quot_{d_1}})^* E_i(m) \cong (\pi_S)^* E_i(m)$ naturally $[13]$. The fact that $E_i \subset E_{i+1}$ implies that $(\pi_S)^* E_i(m) \subset (\pi_S)^* E_{i+1}(m)$. Hence $g^* (\pi_{\Pi Quot_{d_1}})^* E_i(m) \subset g^* (\pi_{\Pi Quot_{d_1}})^* E_{i+1}(m)$ and $g$ factor through $f Quot$. □

Over $f Quot$ there are exact sequences of universal sheaves

$$
0 \to E_i \to V_{P^1 \times f Quot} \to Q_i \to 0
$$

and surjections $Q_i \to Q_{i+1}$. Each $E_i$ are locally free since it is a subsheaf of a locally free sheaf over $P^1 \times S$ and it is flat over $f Quot$. For given $f \in Mor(S, f Quot)$, the corresponding quotient sheaves over $P^1 \times S$ is just the pull back $(id \times f)^*(Q_i)$. 

5
2.1 Irreducibility and Smoothness

To show the Quot scheme for flags is irreducible and smooth, one can simply adapt Strømme’s proof \[16\].

**Theorem 4**  
\(f\text{Quot}\) is an irreducible, rational, nonsingular, projective variety.

**Proof**  
We will work by the language of quotients rather than sub-sheaves. For \(m = 0, -1\) and \(i = 1, \ldots, l\), let \(Q^i_m\) be \((\pi f\text{Quot})^* Q_i(m)\), a locally free sheaf over \(f\text{Quot}\) of rank \((m + 1)r_i + d_i\), and let \(X^i_m \to f\text{Quot}\) be the associated principal \(GL((m+1)r_i+d_i)\)-bundle. One has a smooth morphism \(\rho:\)

\[
P^i_{f\text{Quot}, i = 1} (X^i_{-1} \times f\text{Quot} X^i_0) =: Y
\]

We will show that \(Y\) is an irreducible and smooth variety after finding an isomorphism to a smooth irreducible affine quasi-variety. Since \(\rho\) is smooth, we conclude \(f\text{Quot}\) is a smooth, irreducible, projective variety. Let \(N^i_m := V^{(m+1)r_i + d_i}\) for \(m = 0, -1\) and \(i = 1, \ldots, l\). Here \(V^r\) is, by definition, the \(r\)-dimensional vector space over the ground field \(k\).

Let \(W := \text{Hom}(V, V^{r+1}) \times \text{Hom}_{P^1}(\pi^* V^{d_i}(-1), \pi^* V^{r+1})\)

\[
\times \text{Hom}(V^{d_i}, V^{d_j}) \times \text{Hom}(V^{r_i}, V^{r_j+2}) \times \text{Hom}_{P^1}(\pi^* V^{d_j}(-1), \pi^* V^{r_j+2})
\]

\[
\times \cdots \times \text{Hom}(V^{d_{i-1}}, V^{d_i}) \times \text{Hom}(V^{r_i}, V^{r_j+2}) \times \text{Hom}_{P^1}(\pi^* V^{d_i}(-1), \pi^* V^{r_j+2}).
\]

Let \(\bar{X} :=\) associated affine space. On \(P^1 \times \bar{X}\), there is a tautological diagram:

\[
\begin{array}{ccc}
\pi^* V^{d_i}(-1) & \to_{v_1} & \pi^* V^{r_i+1} \\
& & \uparrow \mu_1 \\
\vdots & & \\
\pi^* V^{d_j}(-1) & \to_{v_2} & \pi^* V^{r_j+2} \\
& & \uparrow \mu_2 \\
\pi^* V^{d_i}(-1) & \to_{v_1} & \pi^* V^{r_i+1} \\
& & \uparrow \mu_1 \\
& & \pi^* V \\
\end{array}
\]

Let \(Z \subset \bar{X}\) be the nonsingular irreducible quasi-variety defined by the conditions

i) \(v_i\) is injective on each fiber over \(X\),
ii) the induced map \( \varphi_i : \pi_i^* V \to coker (v_i) \) is surjective, and
iii) \( coker(v_i) \) is flat over \( Z \) with rank \( r_i \) and degree \( d_i \) on the fibers of \( \pi_Z \).

The above conditions are open conditions in algebraic geometry. Let 
\[ W' := Hom(V^{d_1}, V^{r_2 + d_2}) \times \cdots \times Hom(V^{d_l}, V^{r_l + d_l}) \].
Let \( X \) be a closed subvariety of \( Z \) of codimension \( \sum_{i=1}^{l} d_i(r_i+1 + d_i+1) \) defined by the inverse image of a morphism

\[ W \to W' \] measuring commutativity of the diagram

\[ \begin{array}{ccc}
\pi_i^* V^d(-1) & \to_d & \pi_i^* V^{r+d} \\
\uparrow a & & \uparrow b \\
\pi_i^* V^d(-1) & \to_c & \pi_i^* V^{r+d}
\end{array} \]

by \( d \circ a - b \circ c \).

\( X \) is a product of hypersurfaces defined by irreducible quadratic polynomials. It is smooth away where the right vertical arrow is zero map. And so \( X \) is smooth and irreducible. There is a natural morphism \( g : X \to fQuot \) by the construction of \( X \).

By the construction, \( g^* Q_m^i = (N_m^i) \otimes O_X = ((\mathbb{P} + \infty) \nabla_j + [j]) O_X \). Therefore we have a morphism \( s \) in the diagram

\[ \begin{array}{ccc}
X & \to_s & \prod_{fQuot,i} fQuot_i \times fQuot_i X_0) =: Y \\
g \searrow & & \swarrow \rho \\
& fQuot &
\end{array} \]

We will show \( X \) and \( Y \) are isomorphic finding the inverse of \( s \) which complete the proof. We are given isomorphisms on \( Y \)

\[ \lambda_m : \rho^* Q_m^i \to (N_m^i) \).

By a proposition (1,1) in \([\text{ref}]\), we get a diagram on \( \mathbb{P}^1 \times Y \) except \( \uparrow \)

\[ \begin{array}{ccc}
0 & \to & \pi_Y^* N_{i-1}^i(-1) \\
\uparrow & & \uparrow \\
0 & \to & \pi_Y^* N_{i-1}^i \to (1 \times \rho)^* Q_i \to 0 \\
& & \uparrow \\
& & \pi_Y^* N_{i-1}^i \to (1 \times \rho)^* Q_{i-1} \to 0.
\end{array} \]

Note here that \( \pi_Y^* (1 \times \rho)^* Q_{i-1} = \pi_Y^* \rho^* Q_{i-1}^i \equiv \pi_Y^* (N_{i-1}^i)_Y. \)

Since a morphism from free sheaf is determined and can be defined by a morphism between the space of global sections, there exists a unique lifting
as indicated by the vertical arrow ↑. By the defining property of $X$, there is an induced morphism which is the inverse of $s$.

Since $X$ is an irreducible smooth affine quasi-variety and $g$ is smooth of relative dimension $\sum_{i=1}^{l}(d_i^2 + (r_i + d_i)^2)$, $R$ is smooth and irreducible. It’s dimensions is $d_1(n - r_2) + d_2(r_1 - r_3) + \cdots + d_{l-1}(r_{l-2} - r_l) + d_l r_{l-1}$

$$+ nr_1 + r_1 r_2 + r_2 r_3 + \cdots + r_{l-1} r_1 - r_2^2 - r_3^2 - \cdots - r_l^2.$$ The rationality will be from Bialynski-Birula’s theorem after considering an action [16]. The action will be studied in the following section 3.

\[\Box\]

2.2 Gromov-Witten invariants and flag-Quot schemes

From now on we will work over the complex number field $\mathbb{C}$ to consider complex manifolds. We shall recall the definition of Gromov-Witten invariants for homogeneous projective variety $X$. The variety is always smooth. Denote by $\overline{M}_N(X,d)$ the moduli stack of stable maps of degree $d$ and genus 0. The stack is represented by a smooth compact oriented orbifold. The same notations will be taken for the stack and the coarse moduli orbifold. It has morphisms, contraction $\pi_X$ and evaluations $ev_i$ at the $i$-th marked point:

$$\overline{M}_N(X,d) \rightarrow_{ev_i} X$$

$$\overline{M}_N \rightarrow M_N$$

where $\overline{M}_N$ is the coarse moduli space of stable $n$ marked points of genus zero. $\overline{M}_N$ is a smooth compact oriented manifold.

The (tree level) Gromov-Witten classes $I_{N,d}^X : H^*(X,\mathbb{Q})^\otimes N \rightarrow H^*(\overline{M}_N,\mathbb{Q})$ are defined as follows:

$$I_{N,d}^X(a_1 \otimes \cdots \otimes a_N) := (\pi_X)_!(ev_1^*(a_1) \otimes \cdots \otimes ev_N^*(a_N)).$$

In the sequel we are interested in $I_{N,d}^X(a_1 \otimes \cdots \otimes a_N)[p] \in \mathbb{Q}$ where $[p]$ is the homology class defined by a point $p$ in $\overline{M}_N$. If we choose any $N$ ordered distinct points $p_i$ in $\mathbb{P}^{\overline{d}r}$, we can naturally embed $Mor_d(\mathbb{P}^{\overline{d}r},X)$ into $(\pi_X)^{-1}(p)$ for any generic point $p$. The boundary $(\pi_X)^{-1}(p) \setminus Mor_d(\mathbb{P}^{\overline{d}r},X)$ does not matter much, namely

**Proposition 2** Let $Y_i$ be Schubert subvarieties of $X$. Then $\bigcap_{i=1}^{N} ev_i^{-1}(g_i Y_i) = \bigcap_{i=1}^{N} ev_i^{-1}(g_i Y_i) \cap Mor_d(\mathbb{P}^1, X)$ for generic $g_i$.

**Proof** The proof follows from the following general setting. \(\square\)
Proposition 3  \( M \) be open subvariety of a variety \( \bar{M} \). Suppose a connected algebraic group \( G \) acts transitively on another variety \( X \). Given a morphism \( f: \bar{M} \to X \) subvarieties \( Y_i \) of pure dimension, for generic \( g_i \in G \),

\[
\bigcap_i f^{-1}(g_i Y_i) = \bigcap_i (f^{-1}(g_i Y_i)) \cap M
\]

provided dimensional condition \( \sum_i \text{codim} Y_i = \dim \bar{M} \).

Proof  Apply Kleiman’s theorem in Fulton’s Book [7]. For generic \( g_i \), \( (\bar{M} \setminus M) \cap \bigcap_i f^{-1}(g_i Y_i) = \emptyset \) and for generic \( g_i \), \( (\bigcap_i f^{-1}(g_i Y_i)) \cap M \) is proper. Hence for generic \( g_i \), \( (\bigcap_i f^{-1}(g_i Y_i)) \cap M \) is proper and \( (\bar{M} \setminus M) \cap \bigcap_i f^{-1}(g_i Y_i) = \emptyset \). \[\square\]

Since \( \text{Mor}_d(\mathbb{P}^c, X) \) is a nonsingular quasi-projective variety, we can consider its Chow group \( A_*(\text{Mor}_d(\mathbb{P}^c, X)) \) with products. Let us use the same notation \( \text{ev}_i \) for the restriction of the evaluation map to \( \text{Mor}_d(\mathbb{P}^c, X) \). For \([\text{ev}^{-1}_1(Y_1)] \cdot ... \cdot [\text{ev}^{-1}_1(Y_1)] \in A_0(\text{Mor}_d(\mathbb{P}^c, X)) \), in \( \mathbb{Z} \) is

\[
\int_{\text{Mor}_d(\mathbb{P}^c, X)} [\text{ev}^{-1}_1(Y_1)] \cdot ... \cdot [\text{ev}^{-1}_1(Y_1)]
\]

after summing up the coefficients of cycles of points in \( \text{Mor}_d(\mathbb{P}^c, X) \), which is well-defined in these intersections. It is equal to \( I_X^{N,d}(a_1 \otimes ... \otimes a_N) \) for the Poincare dual classes \( a_i \) of \( Y_i \) because \( (\pi^X)^{-1}(p) \) is a projective variety and has a resolution of singularities to avoid the intersection theory of algebraic (smooth) stacks.

We would like to do a similar thing in Quot schemes following Bertram [3].

Proposition 4 (c.f. Bertram) Suppose \( Y \subset X \) is an irreducible subvariety of codimension \( c \) and suppose \( Z \subset \text{Mor}_d(\mathbb{P}^1, X) \) is an irreducible subvariety. Then for any \( p \in \mathbb{P}^1 \) and a generic translate \( g \), the intersection \( Z \cap \text{ev}^{-1}_p(gY) \) is either empty or has codimension \( c \) in \( Z \) where \( \text{ev}_p \) denoted the evaluation map at \( p \).

Proof  Apply Kleiman’s theorem in Fulton’s Book [7]. \[\square\]

Corollary 1 Let \( c_i = \text{codim}_X Y_i \) in the setting of the above definition. If \( \sum_{i=1}^{N} c_i > \dim(\text{Mor}_d(C, X)) \), then, for generic elements \( g_1, ..., g_N \), \( \bigcap_{i=1}^{N} \text{ev}^{-1}_p(g_i Y_i) = \emptyset \). If \( \sum_{i=1}^{N} c_i = \dim(\text{Mor}_d(C, X)) \), then, for generic elements \( g_1, ..., g_N \), \( \bigcap_{i=1}^{N} \text{ev}^{-1}_p(g_i Y_i) \) is isolated or empty.
The points $p_i$ in the above could not be distinct.

Let $(\mathcal{E}_i)_p$ be the restriction of the sheaf $\mathcal{E}_i$ at $p$ in $\mathbb{P}^1$. Consider a commutative diagram

$$
\begin{array}{ccc}
\text{Mor}(\mathbb{P}^1, Fl) & \rightarrow & \text{Mor}(\mathbb{P}^1, Gr(n,r_i)) \\
\downarrow ev_p & & \downarrow ev_p \\
Fl & \rightarrow & Gr(n,r_i)
\end{array}
$$

where $ev_p$ is the evaluation map at $p \in \mathbb{P}^1$. Let $W$ be the subspace of $V^*$ used for defining $Z$, i.e., the special Schubert varieties associated to $W$. Let $V_d(p, Z)$ be the degenerate locus of the sheaf homomorphism $W \otimes \mathcal{O}_{Quot} \rightarrow (\mathcal{E}_i)_p^*$. In this setting we have

**Proposition 5** $V_d(p, Z)$ represents the $(s_i + 1 - \dim(W))$-th Chern class of $(\mathcal{E}_i)_p^*$ over $fQuot$.

**Proof** When the flag variety $Fl$ is a Grassmannian, it is proven by A. Bertram [3]. For the general case, just consider the morphism $fQuot \rightarrow Quot_i$ from the embedding $fQuot \rightarrow \prod Quot_i$ followed by the projection $\prod Quot_i \rightarrow Quot_i$. It is smooth since both schemes are smooth and the induced homomorphism between tangent spaces is surjective after looking at $\mathbb{P}^1$ in the proof of the theorem. This implies the degeneracy locus has the expected dimension and $[V_d(p, Z)]$ in the Chow ring of the smooth projective variety $fQuot$ is the $(\text{codim}V_d(p, Z))$-th Chern class of $(\mathcal{E}_i)_p^*$ which complete the proof. \hfill \Box

Using a Plücker embedding and a stratification $Quot_d(\mathbb{P}^1, \mathbb{P}^n) = \coprod_{0 \leq m \leq d} C_m \times Mor_{d-m}(\mathbb{P}^1, \mathbb{P}^n)$ (by locally closed schemes) where $C_m$ is the $m$-th symmetric product of $\mathbb{P}^1$, one can extend Bertram’s result for flag varieties.

**Proposition 6** Let $Z_i$ be a special Schubert variety with $c_i$ codimension $\leq s_{k_i+1} - s_{k_i} - 1$ representing a Chern class of $S_{k_i}^*$. Suppose $\sum_{i=1}^N c_i \geq \dim(fQuot)$, then $\bigcap_{i=1}^N ev_{d,p_i}^{-1}(g_iZ_i) = \bigcap_{i=1}^N V_d(p_i, g_iZ_i)$ for distinct points $p_i \in \mathbb{P}^1$.

**Proof** We will use induction on the total degree $|d| = d_1 + \cdots + d_l$. When $|d| = 0$, it can be done by Kleiman’s theorem. Let $\tilde{C}_m = \prod_{i=1}^l C_{m_i}$, where $m = (m_1, \ldots, m_l)$ is a multi-index. Using the Plücker morphisms on
Quot schemes, let us consider the morphism

\[ J : \text{fQuot} \hookrightarrow \prod_{i=1}^{l} \text{Quot}_{d_i}(s_i, n) \rightarrow \prod_{i=1}^{l} \text{Quot}_{d_i}(\mathbb{P}^1, \mathbb{P}^{M_i}) \]

\[ = \text{Mor}_{d}(\mathbb{P}^1, \prod_{i=1}^{l} \mathbb{P}^{M_i}) \cup \bigcup_{|m|=1}^{d} \tilde{C}_m \times \text{Mor}_{d-m}(\mathbb{P}^1, \prod_{i=1}^{l} \mathbb{P}^{M_i}). \]

We would like to show that

\[ \bigcap_{i=1}^{N} V_d(p_i, g_i Z_i) \cap J^{-1} \left( \bigcup_{|m|=1}^{d} \tilde{C}_m \times \text{Mor}_{d-m}(\mathbb{P}^1, \prod_{i=1}^{l} \mathbb{P}^{M_i}) \right) = \emptyset. \]

Then we are done. To do so one has to show, for each \( m, |m| > 0, \)

\[ \emptyset = \bigcap_{i=1}^{N} V_d(p_i, g_i Z_i) \cap J^{-1} \left( \tilde{C}_m \times \text{Mor}_{d-m}(\mathbb{P}^1, \prod_{i=1}^{l} \mathbb{P}^{M_i}) \right). \quad (5) \]

For any subset \( P \) of \( \{p_1, ..., p_N\} \) let

\[ A_P = \{ \text{quotient sheaves } Q = (Q_1, ..., Q_l) \in \text{LHS of } (\mathfrak{L}) \mid \dim_{k(p_i)}(Q_{k_i})_{p_i} \otimes k(p_i) < r_{k_i} \text{ iff } p_i \in P \} \]

\[ A_P \subset \bigcap_{p_i \notin P} J^{-1}(\tilde{C}_m \times \text{ev}_{d-m,p_i}^{-1}(g_i Z_i)). \]

But for generic \( g_i, \bigcap_{p_i \notin P} J^{-1}(\tilde{C}_m \times \text{ev}_{d-m,p_i}^{-1}(g_i Z_i)) = \emptyset \) since \( \bigcap_{p_i \notin P} \text{ev}_{d-m,p_i}^{-1}(g_i Z_i) = \emptyset \) by dimension counting in \( \text{fQuot}(d - m; s_1, ..., s_l; n): \)

\[ \sum_{p_i \notin P} \text{codim}(\text{ev}_{d-m,p_i}^{-1}(g_i Z_i)) \geq \dim(\text{fQuot}_d(\mathfrak{L})) \]

\[ - \sum_{p_i \in P} (s_{k_i+1} - s_{k_i-1} - 1) \]

\[ \dim(\text{fQuot}_d(\mathfrak{L})) \]

\[ - \sum_{p_i \in P} (s_{k_i+1} - s_{k_i-1}). \]

\[ \geq \dim(\text{fQuot}_{d-m}(\mathfrak{L})). \]

Since \( p_i \) are distinct, we conclude the last inequality above. \( \square \)

The proof of the theorem follows from what are done.
3 A Formula by Localization

3.1 Equivariant action on $E \rightarrow fQuot \times \mathbb{P}^1$

By the standard action of $SL(n) \times PGL(2)$ on $V \times \mathbb{P}^k$, the group acts on the space of stalks of $V \otimes \mathcal{O}_{\mathbb{P}^k}\mathcal{U} \times \mathbb{P}^k$ and hence on the subsheaves $E$ and $fQuot$. The action on the sheaves is equivariant. In particular the maximal complex torus action of $T \times \mathbb{C}^\times$ will formulate integrations of wedges products of Chern classes of $(E_1)^\sqrt{\cdot}$ as certain finite sums of characters using the localization theorem [4].

For simplicity of notations let us do it for the Quot schemes Quot. Consider the action by $T \times \mathbb{C}^\times$ on $Quot \times \mathbb{P}^1$, then the action has a lift on the total space of the vector bundle $E$. Let $E_p$ be the restriction of the sheaf $E$ at $p$ in $\mathbb{P}^1$. The action has the lifting to vector bundles $E_r$ and $E_\infty$. It means $E_{r(\infty)}$ is an equivariant vector bundle and its equivariant Chern classes can be considered. For other points, say $p$, transitive $PSL(2)$-action on $\mathbb{P}^1$ will show $E_r, E_\infty$, and $E^\sqrt{\cdot}$ are isotropic:

$$
\begin{array}{cccc}
E & \downarrow & E & \downarrow & E & \downarrow & E & \downarrow \\
\downarrow & Quot \times \mathbb{P}^1 & \rightarrow_g & Quot \times \mathbb{P}^1 & Quot & \rightarrow_g & Quot
\end{array}
$$

where $g \cdot 0 = p$. In particular the Chern classes of $E_p$ are independent to $p$ since the map induced by $g$ is homotopic to identity. Let $\frac{1}{2}h$ (resp. $\lambda_i$) is (are) the $\mathbb{C}^\times$ (resp. $T$) characteristic classes. Then,

$$
\int_{Quot} \phi(c_{i_1}(E_{p_1}), \ldots, c_{i_m}(E_{p_m})) = \int_{Quot} \phi(c_{i_1}(E_0), \ldots, c_{i_m}(E_0)) = [\text{push forward of } \phi \text{ of equivariant classes of } (c_1, \ldots, c_r) \text{ at } E_0]_{h=\lambda_i=0} = [\text{localization into components } P \text{ of the fixed subset, i.e., } \sum_P \text{push forward } i_P^P \phi_{E(v_P)}]_{h=\lambda_i=0},
$$

(6)

where $i_P$ is the inclusion $P \subset Quot$ and $E(v_P)$ is the equivariant Euler class of the normal bundle of $P$ in $Quot$. The last expression is independent to $h$ and $\lambda_i$, without letting them zeros, if the quasi-homogeneous degree of $\phi$ given by degrees of the Chern classes agrees the dimension of $Quot$. The complete analog hold for $fQuot$. 

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It is easy to see that the fixed subset consists of finite points. Therefore $E(v_P)$ in (3) is the equivariant Euler class of the normal space over the point $P \in Quot$. It is the product of the complex characters of the representation of $T \times \mathbb{C}^x$ in the irreducible complex one dimensional subspaces of the tangent space of Quot at $p$. In the following subsection, we devote ourselves to spell out the all fixed points and all characteristics of the representation to finish the proof of the theorem 3.

### 3.2 Computation

Let us use the standard maximal torus $T \times \mathbb{C}^x$ in the picture of flag manifolds $Fl$ and the projective line $\mathbb{P}^1$. Fix a sequence of $(e_{k_1}, e_{k_2}, \ldots, e_{k_l})$ where $\{e_{i}\}_{i=1}^n$ is the standard basis of $V$. Then, for data (3) in introduction, one may associate a flag of subsheaves

$$\mathcal{O}(-d_{i,j}) \to \mathcal{O}(-d_{i+1,j})$$

by the global section

$$x^{(a_{i,j}-a_{i+1,j})}y^{(b_{i,j}-b_{i+1,j})},$$

It is a fixed point by the action. For such a (4) and a sequence, we can associate any fixed point in $fQuot$. We have found all fixed points.

Note that the tangent space at $x$ of a scheme $X$ is the first order infinitesimal deformation $\text{Mor}_x(D, X)$, the set of all morphisms sending the closed point of Spectrum of the ring $D$ of dual numbers to $x$. Therefore, at a subsheaf $\mathcal{S}$ over $\mathbb{P}$ of $V \otimes \mathcal{O}_{\mathbb{P}^\infty}$, the tangent space of Quot schemes is the set of flat families of quotient sheaves over the Spec $D$ whose fiber over the closed point of Spec $D$ is $\mathcal{S}$. It is $\text{Hom}(\mathcal{S}, V \otimes \mathcal{O}_{\mathbb{P}^\infty}/\mathcal{S})$.

For the flag-Quot scheme consider the following equivariant short exact sequence at a fixed point $\mathcal{S}$ of $fQuot$

$$0 \to T_3fQuot \left( d_1, \ldots, d_i; s_1, \ldots, s_l; n \right)$$

$$\to T_3 \left\{ \text{Quot}_{d_1}(s_1, n) \times \cdots \times \text{Quot}_{d_i}(s_l, n) \right\} \to \prod_{i=1}^l H\text{om}(\mathcal{S}_i, Q_{i+1}) \to 0.$$

At the fixed point associated to (4) we find all characters of irreducible subspace of $T_3Quot$ by the torus action. They are, for all $1 \leq i \leq l$,

$$(p - a_{i,j})h + \lambda_{j'} - \lambda_j, \text{ for } 0 \leq p \leq a_{i,j'} - 1, 1 \leq j, j' \leq s_i,$$

$$(b_{i,j} - p)h + \lambda_{j'} - \lambda_j, \text{ for } 0 \leq p \leq b_{i,j'} - 1, 1 \leq j, j' \leq s_i,$$

$$(p - a_{i,j})h + \lambda_m - \lambda_j, \text{ for } 0 \leq p \leq d_{i,j}, 1 \leq j \leq s_i, s_i + 1 \leq m \leq n,$$
\[
\bigoplus_{i=1}^l \text{Hom}(S_i, Q_i) = \bigoplus_{i=1}^l \bigoplus_{j=1,j'=1}^{s_i,s_i} \text{Hom}(x^{a_{i,j}} y^{b_{i,j}} O_j(-d_{i,j}), O_j'/x^{a_{i,j'}} y^{b_{i,j'}} O_j)
\]

\[
\bigoplus_{j=1}^{s_i,n} \bigoplus_{j=1,m=s_i+1}^{s_i,n} \text{Hom}(x^{a_{i,j}} y^{b_{i,j}} O_j(-d_{i,j}), O_m)].
\]

Characters from \(\prod_{i=1}^{l-1} \text{Hom}(S_i, Q_{i+1})\) are, for \(1 \leq i \leq l-1\),

\[(p-a_{i,j})h + \lambda_{j'} - \lambda_j, \text{ for } 0 \leq p \leq a_{i+1,j'} - 1, 1 \leq j \leq s_i, 1 \leq j' \leq s_{i+1},\]

\[(b_{i,j} - p)h + \lambda_{j'} - \lambda_j, \text{ for } 0 \leq p \leq b_{i+1,j'} - 1, 1 \leq j \leq s_i, 1 \leq j' \leq s_{i+1},\]

\[(p-a_{i,j})h + \lambda_m - \lambda_j, \text{ for } 0 \leq p \leq d_{i,j}, 1 \leq j \leq s_i, s_{i+1} + 1 \leq m \leq n.\]

The fiber space of \(S_i\) at the point 0 has characters

\[a_{i,j}h + \lambda_j\]

for \(0 \leq j \leq s_i\). Therefore the \(k\)-th Chern character is the \(k\)-th symmetric function in those characters. Let us denote it by \(\sigma_k^i\).

The proof of theorem 3 follows from the proposition 5.

### 3.3 Projective spaces

In this section we will relate the our result to the residue formula of intersection pairing in [10] for projective spaces. The author does not know for the other cases.

Let \(x\) be the Chern class of \(\mathcal{O}_{\mathbb{P}^n}(-1)\). Then the Gromov-Witten invariant \(I_{N,d}^{\mathbb{P}^n}(x^{\otimes (n+1)d+n})\) is

\[
\sum_{0 \leq i \leq n} \sum_{0 \leq k \leq d} \prod_{0 \leq p \leq d} \prod_{p \neq k} \prod_{0 \leq q \leq d} \prod_{0 \leq j \neq i \leq n} \frac{(\lambda_i + kh)^{(n+1)d+n}}{((q-k)h + \lambda_j - \lambda_i)}
\]  

(7)

**Proposition 7** \(\sum_{N=0}^{\infty} \frac{1}{N!} q^d I_{N,d}^{\mathbb{P}^n}(x^{\otimes N})\) is a global residue

\[
\frac{1}{2\pi} \oint \frac{f(x)dx}{x^{n+1} - q}
\]

where \(q\) is a formal variable.
Proof  The identity
\[
\frac{1}{2\pi} \oint \frac{x^{(n+1)d+n} dx}{x^{(n+1)(d+1)}} = \left[ \frac{1}{2\pi} \oint \frac{x^{(n+1)d+n}}{\prod_{0 \leq i \leq n} \left( x - \lambda_i - kh \right)} \right]_{\lambda_i, h = 0} \tag{7}
\]
implies the proof. □

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