Weighted Essentially Non-Oscillatory stochastic Galerkin approximation for hyperbolic conservation laws

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Abstract

In this paper we extensively study the stochastic Galerkin scheme for uncertain systems of conservation laws, which appears to produce oscillations already for a simple example of the linear advection equation with Riemann initial data. Therefore, we introduce a modified scheme that we call the weighted essentially non-oscillatory (WENO) stochastic Galerkin scheme, which is constructed to prevent the propagation of Gibbs phenomenon into the stochastic domain by applying a slope limiter in the stochasticity. In order to achieve a high order method, we use a spatial WENO reconstruction and also compare the results to a scheme that uses WENO reconstruction in both the physical and the stochastic domain. We evaluate these methods by presenting various numerical test cases where we observe the reduction of the total variation compared to classical stochastic Galerkin.

Keywords: Uncertainty Quantification, Polynomial chaos, Stochastic Galerkin, Multielement, Gibbs Oscillations, Slope Limiter, Hyperbolicity, WENO reconstruction

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1. Introduction

Many physical problem settings can be described by hyperbolic systems of conservation laws, however, crucial data or parameters might not be available exactly due to measurement errors and thus have non-deterministic effects on the approximation of the deterministic systems. The modeling of the propagation of this uncertainty into the solution is the topic of Uncertainty Quantification (UQ). In this context, UQ methods \cite{2,16,17,29,36} are gaining more and more popularity, whereas we distinguish between two approaches, namely the so-called non-intrusive and intrusive schemes.

The most widely known non-intrusive UQ method is (Multi-Level) Monte Carlo \cite{12,14,19} which is based on statistical sampling methods and can be easily implemented and adopted to any type of uncertain conservation law, but comes with potentially high cost due to repeated application of e.g. finite volume methods (FVM). These so-called MC-FVM schemes have been studied in \cite{23,26} showing a slow convergence rate that is improved by Multi-Level MC-FVM algorithms for conservation laws \cite{24,25}. Another non-intrusive UQ scheme is stochastic collocation \cite{47}, further developments of this method are described in \cite{20,27,33}.

Intrusive UQ methods aim to increase the overall efficiency but require a modification of the underlying solver for the deterministic problem (as FVM). The most popular methods are the Intrusive Polynomial Moment
(IPM) method \cite{30} and the stochastic Galerkin (sG) scheme. They both rely on the generalized Polynomial Chaos (gPC) expansion \cite{1,6,11,15} which is theoretically based on the Polynomial Chaos expansion from \cite{44}. IPM expands the stochastic solution in the so-called entropic variables, which results in a hyperbolic gPC system that yields a good approximation quality, however, it is necessary to know a strictly convex entropy solution beforehand. Stochastic Galerkin expands the solution in the conserved variables such that the gPC system results in weak formulation of the equations with respect to the stochastic variable.

For many non-hyperbolic equations, the underlying random field is sufficiently smooth in the stochasticity such that the sG method is superior to Monte Carlo type methods since the gPC approach exhibits spectral convergence \cite{4,11,48}. The biggest challenge of UQ methods for hyperbolic conservation laws lies in the fact that discontinuities in the physical space propagate into the solution manifold \cite{5}. The naive usage of sG for nonlinear hyperbolic problems even typically fails \cite{2,30} since the polynomial expansion of discontinuous data yields huge oscillations that results in the loss of hyperbolicity. In order to resolve this problem, we apply the hyperbolicity limiter from \cite{22,32} to the classical sG approach. In addition to that, the authors of \cite{42} introduced the so-called Multi-Element approach, where the random space is divided into disjoint elements in order to define local gPC approximations. Further developments of this method can be found in \cite{39,40,43}.

Within this article, we show a simple example of the linear transport equation with uncertain wave speed, supplemented with Riemann initial data, that still produces oscillations in the gPC expansion. For this reason, we propose a robust numerical method that is able to deal with this kind of Gibbs oscillations. We combine the hyperbolicity limiter and Multi-Element ansatz with a weighted essentially non-oscillatory (WENO) reconstruction \cite{9,15,28,31,38} in the physical space to deduce a high-order method and apply a slope limiter in the stochastic variable. Furthermore, we consider a full two-dimensional WENO reconstruction in both the physical and stochastic domain, motivated by the stochastic finite volume method from \cite{37}. We compare the performance of both methods by considering the total variation for various numerical test cases.

The paper is structured as follows. In Section \ref{sec:2} we describe our problem setting, that is the system of uncertain conservation laws which we discretize in the stochastic domain by the stochastic Galerkin scheme. We then demonstrate the propagation of Gibbs phenomenon by an introductory example which yields to the definition of the stochastic slope limiter. Section \ref{sec:3} formulates the weighted essentially non-oscillatory stochastic Galerkin scheme such as a full 2D WENO reconstruction of the conservation law. Finally, we show some numerical results in Section \ref{sec:4} demonstrating the reduction of the total variation for our methods compared to classical stochastic Galerkin.

2. Modeling Uncertainties

We consider stochastic conservation laws of the form

\[
\frac{\partial}{\partial t} u(t,x,\xi) + \frac{\partial}{\partial x} f(u(t,x,\xi),\xi) = 0, \quad \text{for } x \in X, t > 0, \xi \in \Omega, \tag{2.1}
\]

with physical domain $X \subset \mathbb{R}$, stochastic domain $\Omega \subset \mathbb{R}$ and initial conditions given by

\[
u(0,x,\xi) = u(0)(x,\xi), \quad \text{for } x \in X, \xi \in \Omega.
\]

Depending on $X$, additional boundary conditions have to prescribed. The solution $u \in \mathbb{R}^d$ is depending on a one-dimensional random variable $\xi$ with probability space $(\Omega,F,P)$ and probability density function $f_\xi(\xi) : \Omega \to \mathbb{R}_+$. We abuse notation and denote the random space of this uncertainty by $\Omega := \xi(\Omega)$ and write $\xi$ also for the realizations of the random variable.
2.1. Stochastic Galerkin

We seek for an approximate solution by a finite-term generalized Polynomial Chaos (gPC) expansion (see e.g. [13])

\[ u(t, x, \xi) \approx \sum_{k=0}^{K_\Omega} u_k(t, x) \phi_k(\xi), \quad (2.2) \]

where the polynomials \( \phi_k \) of degree \( k \) are supposed to satisfy the orthogonality relation

\[ \int_\Omega \phi_k(\xi) \phi_l(\xi) f_\Xi(\xi) \, d\xi = \delta_{kl}, \quad \forall \, k, l \in \{0, \ldots, K_\Omega\}. \quad (2.3) \]

Inserting (2.2) into (2.1) and applying a Galerkin projection in the stochastic space leads to the so called stochastic Galerkin system

\[ \frac{\partial}{\partial t} u_l + \frac{\partial}{\partial x} \int_\Omega f \left( \sum_{k=0}^{K_\Omega} u_k \phi_k \right) \phi_l f_\Xi \, d\xi = 0, \quad l = 0, \ldots, K_\Omega. \]

2.2. Multielement Ansatz

For discontinuous solutions, the gPC approach may converge slowly or even fail to converge, cf. [30, 40]. As presented in [21, 42], we therefore apply the Multielement approach, where \( \Omega \) is divided into disjoint elements with local gPC approximations of (2.1).

We assume that \( \Omega = (\xi_L, \xi_R) \) and define a decomposition of \( \Omega \) into \( N_\Omega \) Multielements \( D_j = (d_j, d_{j+1}) \) of width \( \Delta \xi = \frac{\xi_L - \xi_R}{N_\Omega} \). Moreover, we introduce an indicator variable \( \chi_j : \Omega \rightarrow \{0, 1\} \) on every random element

\[ \chi_j(\xi) := \begin{cases} 1 & \text{if } \xi \in D_j, \\ 0 & \text{else}, \end{cases} \quad (2.4) \]

for \( j = 1, \ldots, N_\Omega \) and \( \omega \in \Omega \). If we let \( \{ \phi_{k,j}(\xi) \}_{k=0}^{\infty} \) be orthonormal polynomials with respect to a conditional probability density function on the Multielement \( D_j \), as in [21], the global approximation (2.2) can be written as

\[ u(t, x, \xi) = \sum_{j=1}^{N_\Omega} u_j(t, x, \xi) \chi_j(\xi) \approx \sum_{j=1}^{N_\Omega} \sum_{k=0}^{K_\Omega} u_{k,j}(t, x) \phi_{k,j}(\xi) \chi_j(\xi). \quad (2.5) \]

As \( N_\Omega, K_\Omega \rightarrow \infty \), the local approximation converges to the global solution in \( L_2(\Omega) \), cf. [3].

The calculation of the expected value and the variance can be found in [21].

Remark 2.1. The stochastic Galerkin scheme can be applied to every Multielement separately due to the disjoint decomposition of the random space.

2.3. Propagation of the Gibbs phenomenon

As introductory example, we consider the one-dimensional hyperbolic problem

\[ u_t + a(\xi) u_x = 0 \quad (2.6) \]

with \( d = 1, \, x \in (0, \infty) \) and uncertain wave speed

\[ a(\xi) = 1.5 + 0.5\xi, \quad (2.7) \]
where \( \xi \sim U(-1, 1) \), i.e., \( \xi \) is uniformly distributed in \([-1, 1]\). Hence, the density function is given by \( f_{\xi} = 0.5 \) and the basis functions \( \phi_k \) by the Legendre polynomials orthonormalized wrt. (2.3). We further use non-smooth initial data

\[
u(0, x, \xi) = \begin{cases} 1 & \text{for } 0 \leq x \leq 0.5, \\ 0 & \text{for } 0.5 < x, \end{cases}
\]  

and constant boundary data

\[ u(t, 0, \xi) = 1. \]  

We approximate the solution by the Polynomial Chaos (gPC) expansion (2.2)

\[ u(t, x, \xi) \approx \sum_{k=0}^{K_\Omega} u_k(t, x) \phi_k(\xi). \]  

Inserting (2.10) in (2.6) and applying a Galerkin projection in the stochastic space leads to

\[
\frac{\partial}{\partial t} u_l(t, x) + \sum_{k=0}^{K_\Omega} a_{l,k} \frac{\partial}{\partial x} u_k(t, x) = 0, \quad \text{for } l = 0, \ldots, K_\Omega,
\]

with

\[
a_{l,k} = \int_{-1}^{1} a(\xi) \phi_l(\xi) \phi_k(\xi) f_{\xi}(\xi) \, d\xi.
\]

Collecting all \( u_l \) into a vector \( U = (u_0, \ldots, u_{K_\Omega})^T \) and the \( a_{l,k} \) into a matrix \( A \) yields the system

\[
\frac{\partial}{\partial t} U(t, x) + A \frac{\partial}{\partial x} U(t, x) = 0.
\]  

Note that this system is hyperbolic because \( A = A^T \).

The Galerkin projection of the initial conditions (2.8) gives

\[
u_l(0, x) = \int_{-1}^{1} u^{(0)}(x, \xi) \phi_l(\xi) f_{\xi}(\xi) \, d\xi = \int_{-1}^{1} \phi_l(\xi) f_{\xi}(\xi) \, d\xi, \quad \text{for } l = 0, \ldots, K_\Omega
\]

and therewith

\[
U(0, x) = \begin{pmatrix} u^{(0)}(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

For the boundary conditions (2.9), the Galerkin projection leads to

\[
u_l(t, 0, \xi) = \int_{-1}^{1} u(t, 0, \xi) \phi_l(\xi) f_{\xi}(\xi) \, d\xi = \int_{-1}^{1} \phi_l(\xi) f_{\xi}(\xi) \, d\xi, \quad \text{for } l = 0, \ldots, K_\Omega
\]

hence

\[
U(t, 0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
Since (2.11) is a linear hyperbolic system, it can be decoupled into $K$ scalar transport equations and the analytic solution can be explicitly given. For this purpose, we introduce a new variable $W := T^{-1}U$, where $T$ is the matrix of eigenvectors of $A$ (columnwise). With $\lambda_j$ being the corresponding (real) eigenvalues, the components of $W$ satisfy
\[
\frac{\partial}{\partial t}W_j + \lambda_j \frac{\partial}{\partial x}W_j = 0, \quad \text{for } j = 0, \ldots, K
\]
and the solution of the Galerkin system is given by
\[
W_j(t, x) = \begin{cases} (T^{-1}U(x - \lambda_j t, 0))_j & \text{for } x - \lambda_j t \geq 0, \\ (T^{-1}U(0, t - x \lambda_j))_j & \text{for } x - \lambda_j t \leq 0. \end{cases}
\]
For the following considerations, we use $K = 2$. From (2.3) we deduce the Legendre polynomials
\[
\phi_0(\xi) = 1, \quad \phi_1(\xi) = \sqrt{3} \xi, \quad \phi_2(\xi) = \frac{\sqrt{5}}{2}(3\xi^2 - 1),
\]
and further
\[
A = \begin{pmatrix} \frac{3}{2} & \frac{\sqrt{5}}{2} & 0 \\ \frac{\sqrt{5}}{2} & \frac{3}{2} & \frac{\sqrt{15}}{2} \\ 0 & \frac{\sqrt{15}}{2} & \frac{3}{2} \end{pmatrix}.
\]
The eigenvalues of $A$ are $\frac{3}{2} - \frac{\sqrt{15}}{10}$, $\frac{3}{2}$, and $\frac{3}{2} + \frac{\sqrt{15}}{10}$. Interestingly, the corresponding $\xi$s (via (2.7)) are the zeros of the third order Legendre polynomial and our finite-term expansion (2.10) is exact for those $\xi$s. Accordingly, the gPC Galerkin approximation given by (2.10) here equals the quadratic interpolating polynomial in the stochastic space with sampling points $\xi \in \{-\sqrt{3}, 0, \sqrt{3}\}$. Now, the problem is that the discontinuity with respect to $x$ in the initial conditions (2.8) carries over to the stochastic domain. This leads to overshoots (Gibbs phenomena) in the gPC Galerkin approximation as illustrated in Figure 1 for $t = 0.5$ and a choice of $\xi$s inside and outside of the convex hull of the sampling points.

![Figure 1](image-url)

Figure 1: Comparison of the gPC Galerkin approximation (dashed) and the exact solution (solid) for the given $\xi$s at $t = 0.5$.

Note that this effect does not result from the fact that we made a global ansatz in the stochastic space but from the lack of regularity (see below). Moreover, if we want to refine the grid in the stochastic space close to the discontinuity, we are faced with the problem that the position of the jump in the stochastic space depends on $x$ and $t$ as illustrated in Figure 2. A short computation shows that the discontinuity is located at
\[
\xi_{\text{jump}} = \frac{2x - 1}{t} - 3,
\]
which yields that for $t = 0.5$, the position of the discontinuity in the stochastic space takes all values in $[-1, 1]$ for $x \in [1, 1.5]$.

Now, as noted above, we demonstrate that a grid refinement in the sense of a Multielement approach does only marginally improve the so-found approximation. For this, we briefly consider the case where the
Figure 2: Plots of the exact solution depending on $\xi$ and the given $x$ coordinates at $t = 0.5$.

The stochastic domain is equally partitioned into three intervals and we apply the gPC Galerkin approach in each of them. With respect to the stochastic domain, this results in a discontinuous Galerkin approximation of the entire solution and the approximation in each interval can be independently computed analogously to the procedure described above. Now, we look at the so-found approximations for some $\xi_j \in \left[\frac{-1}{3}, \frac{1}{3}\right]$ and compare them with the global gPC Galerkin approximations from above at $3\xi_j \in [-1, 1]$. Certainly, we are looking at approximations to different solutions this way, but we are interested in another property of the solution manifold: As one can easily see from the plots in Figure 3, the width of the overshoots in the compared approximations reduced but the height remained, which is a well-known issue for Gibbs phenomena. Thus, solely the refinement of the grid in the stochastic space cannot reduce the overshoots in the solution manifold.

Figure 3: Comparison of the overshoots in the global gPC Galerkin approximation (dashed) and the approximations by the Multielement approach (solid) for the given $\xi_j$s at $t = 0.5$.

2.4. Stochastic Slope Limiter

We now consider a slope limiter within the stochastic space to reduce the oscillations described in the previous subsection. We present the minimod limiter \[7, 8\], given by the following troubled cell indicator for the $j$th Multielement $D_j$

$$TC_j(u) = \begin{cases} 1 & u_{1,j} \neq m(u_{0,j}, u_{0,j+1} - u_{0,j}, u_{0,j} - u_{0,j-1}), \\ 0 & \text{else}, \end{cases}$$

where $m(\cdot, \cdot, \cdot)$ is the minmod function and $u_{k,j}$, $k = 0, 1$, are the $k$th coefficients of the local gPC representations of the solution $u$ in the $j$th Multielement as in (2.5). At the boundary, we copy the entries of the
first (last) Multielement. Finally, we replace each coefficient of $\mathbf{u}|_{D_j}$ for $j = 1, \ldots, N_\Omega$ by

$$
\mathbf{u}|_{D_j} = \begin{pmatrix}
\hat{u}_{0,j} \\
\hat{u}_{1,j} \\
\vdots \\
\hat{u}_{K\Omega,j}
\end{pmatrix} = \begin{pmatrix}
(u_{0,j})^T \\
m(u_{1,j}, u_{0,j+1} - u_{0,j}, u_{0,j} - u_{0,j-1})^T \\
(0, \ldots, 0)^T \\
(0, \ldots, 0)^T
\end{pmatrix}
$$

if $\mathcal{T}_{C_j}(\mathbf{u}) = 1,$

else.

(2.13)

Hence, we assume that the oscillations in the solution vector are mainly generated in the part with linear uncertainty [35].

**Remark 2.2.** The troubled cell indicator (2.12) is defined for an expansion of $\mathbf{u}$ in monomials. If the SG basis polynomials $\phi_0$ and $\phi_1$ are given by other functions we need to adapt this definition using a basis transformation.

If $K_\Omega \geq 2,$ we only apply the slope limiter if $|u_{1,j}| \geq M|D_j|^2$ in addition to $\mathcal{T}_{C_j}(\mathbf{u}) = 1$ to achieve the TVBM property. The constant $M$ is chosen according to [31] as

$$M = \sup \left\{ |\partial_\xi^2 \mathbf{u}^{(0)}(x, \hat{\xi})| \mid \hat{\xi} \in \Omega, x \in X, \partial_\xi \mathbf{u}^{(0)}(x, \hat{\xi}) = 0 \right\}.$$

The slope limited gPC Galerkin approximation can now be applied to the linear advection example from Section 2.3. Figure 4 compares the gPC approximation in 3 Multielements with its slope limited version which completely reduces the overshoots. Thus, the application of the minmod slope limiter in the stochastic space (2.13) has been sufficient to prevent Gibbs phenomenon.

![Figure 4: Comparison of the overshoots in the gPC Galerkin approximation (solid) with 3 Multielements and the approximations by its slope limited approach (dashed) for the given $\xi$s at $t = 0.5$.](image)

In Figure 5 we kept refining the number of Multielements to 10, where Gibbs phenomenon still causes overshoots of the same height as before. The slope limiter is able to eliminate these oscillations.

### 3. Weighted Essentially Non-Oscillatory stochastic Galerkin scheme (WENOsG)

The previous example demonstrated the significance of limiting techniques not only on the spatial but also on the stochastic grid. In this section we formulate a stochastic Galerkin scheme that includes the stochastic slope limiter while preserving high order in space and time. To this end, we subdivide the spatial domain $X = [x_L, x_R] \subset \mathbb{R}$ into $N_x$ cells $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ of width $\Delta x = \frac{x_R - x_L}{N_x}$.
If the random variable is uniformly distributed, we can use a Gauß-Lobatto rule on $\Omega$ will be used in the numerical flux function. Later on, we also require quadrature in the stochastic space.

For this sake, we consider a semi-discretized finite volume scheme for the solution $u$ given by

$$
 \partial_t \int_{C_i} u \, dx + f(u(t, x_{i+\frac{1}{2}}), \xi) - f(u(t, x_{i-\frac{1}{2}}), \xi) = 0, \quad i = 1, \ldots, N_x.
$$

In the following, we denote the $x$ cell mean in $C_i$ by

$$
 u(t, C_i, \xi) = \frac{1}{f_i \, dx} \int_{C_i} u \, dx.
$$

Since $u$ is discontinuous at $x_{i+\frac{1}{2}}$, we replace the evaluation of $f$ at these points with a numerical flux function $\hat{f}$ that approximately solves the Riemann problem at the cell interface. In our numerical studies, we use the global Lax-Friedrichs flux

$$
 \hat{f}(u^-, u^+) = \frac{1}{2} \left( f(u^-) + f(u^+) - c(u^+ - u^-) \right).
$$

The numerical viscosity constant $c$ is taken as the global estimate of the absolute value of the largest eigenvalue of the Jacobian $\frac{\partial f(u)}{\partial u}$. The values

$$
 u^-(x_{i+\frac{1}{2}}, \xi) := \lim_{x \uparrow x_{i+\frac{1}{2}}} u(x, \xi), \quad u^+(x_{i+\frac{1}{2}}, \xi) := \lim_{x \downarrow x_{i+\frac{1}{2}}} u(x, \xi)
$$

denote the left and right limits of the piecewise polynomial solution at the interface $x_{i+\frac{1}{2}}$.

As a quadrature rule in the physical domain, we apply a Gauß-Lobatto rule on $C_i$ with $Q_X + 1$ points and weights $(\tilde{x}_q, \tilde{w}_q)$, for $q = 0, \ldots, Q_X$ and where $Q_X$ is chosen such that the quadrature integrates the WENO polynomial (will be defined in (3.5)) exactly. Gauß-Lobatto includes the endpoints, i.e. cell interfaces which will be used in the numerical flux function. Later on, we also require quadrature in the stochastic space.

If the random variable is uniformly distributed, we can use a Gauß-Lobatto on $D_j$ with order $K_\Omega$, i.e., $Q_\Omega + 1$ points and weights $(\xi_\rho, \hat{\omega}_\rho)$, $\rho = 0, \ldots, Q_\Omega$, where $Q_\Omega = \left\lfloor \frac{K_\Omega + 1}{2} \right\rfloor$. For other distributions we use the corresponding Gauß quadrature based on the orthogonal basis polynomials and weighted by the conditional probability density function. We scale the quadrature weights such that

$$
 \sum_{q=0}^{Q_X} \sum_{\rho=0}^{Q_\Omega} \tilde{w}_q \hat{\omega}_\rho = 1, \quad \int_{D_j} f g(x) \, dx \approx \sum_{\rho=0}^{Q_\Omega} g(\xi_\rho) \hat{\omega}_\rho.
$$

Figure 5: Comparison of the overshoots in the gPC Galerkin approximation (solid) with 10 Multielements and the approximations by its slope limited approach (dashed) for the given $\xi$ at $t = 0.5$. We deploy a $R$-th order SSP Runge Kutta scheme in time and a finite volume method combined with the Weighted Essentially Non-Oscillatory (WENO) reconstruction in space (cf. [15, 31, 38]) to obtain a high-order scheme for which we apply the stochastic slope limiter (2.13). Note that in our numerical experiments, we use the so called CWENOZ method from [9], which will be explained in (3.5). For systems of conservation laws we additionally use the hyperbolicity-preserving limiting process introduced in [21, 32] to preserve hyperbolicity of the solution vector.
The time discretization of the semi-discrete system (3.1) is performed using a $R$-th order SSP Runge-Kutta method, see [33, 34]. In each Runge Kutta stage we apply the stochastic slope limiter (2.13) to the gPC polynomial. Afterwards we perform a polynomial reconstruction in order to derive the values of the numerical flux at the left and right limits of cell interfaces (3.3).

We reconstruct the solution vector $u(t, x, \xi)$ as a polynomial of order (at most) $R$ for each quadrature node of $\xi$ and in all spatial cells $C_i$. This can be done by the CWENOZ scheme explained in [9], which combines the CWENO method from [18], ensuring uniform accuracies within the cells, and WENOZ [10] for an optimal choice of the nonlinear WENO weights. For all $i = 1, \ldots, N_x$, $\rho = 1, \ldots, Q_\Omega$, given the point values $u(t, x_i, \hat{\xi}_\rho)$, we represent the solution in each cell $C_i$ and for each $\hat{\xi}_\rho$ as a polynomial of degree $K_X := 2r - 2$ such that $K_X + 1 \leq R$, more precisely

$$u(t, x, \hat{\xi}_\rho)|_{C_i} = \sum_{k=0}^{K_X} p_{C_i}^{(k)} \frac{(x - x_{i+1})^k}{k!},$$  

(3.5)

where $p_{C_i}^{(k)}$ are the coefficients obtained by the CWENOZ algorithm from [9]. This polynomial will then be evaluated at the quadrature nodes $x = \hat{x}_0$ and $x = \hat{x}_{Q_X}$ in $C_i$ to obtain the limits (3.3).

**Remark 3.1.** We can remap a solution vector $u(t_n, C_i, \hat{\xi}_\rho)$ in each Multielement $D_j$ to its gPC moments using

$$u_{k,j}(t,C_i) = \int_{D_j} u(t, C_i, \xi) \phi_{k,j}(\xi) f d\xi = \sum_{\rho=0}^{Q_\Omega} u(t, C_i, \hat{\xi}_\rho) \phi_{k,j}(\hat{\xi}_\rho) \hat{\omega}_\rho,$$

(3.6)

for every $k = 0, \ldots, K_\Omega$ and $j = 1, \ldots, N_\Omega$.

### 3.1. Algorithm

We summarize our results in the following algorithm of the Weighted Essentially Non-Oscillatory stochastic Galerkin scheme.

**Algorithm 1** Weighted Essentially Non-Oscillatory stochastic Galerkin scheme (WENOsG)

At the initial state, compute the cell means at the $\xi$ quadrature points $u(0, C_i, \hat{\xi}_\rho)$ and derive the stochastic Galerkin moments $u_{k,j}(0, C_i)$ in every Multielement $D_j$ via (3.6). For the remaining time steps $t_n$ perform:

for $s = 1, \ldots, S$  

# explicit Runge Kutta stages

1. Update the stochastic modes $u_{k,j}^s(t_n^s, C_i)$ according to one Runge Kutta stage.

2. Apply the slope limiter (2.13) on $u^s(t_n^s, C_i, \hat{\xi}_\rho)|_{D_j} = \sum_{k=0}^{K_\Omega} u_{k,j}^s(t_n^s, C_i) \phi_{k,j}(\hat{\xi}_\rho)$.

3. Perform a WENO reconstruction on $u^s(t_n^s, C_i, \hat{\xi}_\rho)|_{D_j}$ and evaluate the polynomials (3.5) at the spatial quadrature nodes $x = \hat{x}_0$ and $x = \hat{x}_{Q_X}$ in $C_i$.

4. Compute the differential operator of the Runge Kutta stage using Lax Friedrichs (3.2) in (3.1).

end for

After all Runge Kutta stages, compute the value of $u(t_{n+1}, C_i, \hat{\xi}_\rho)$, apply steps 2 - 4 and remap the result to its sG moments via (3.6) to deduce the final solution vector at $t_{n+1}$.

**Remark 3.2.** If we consider hyperbolic systems of equations, we have to apply a hyperbolicity-preserving limiter in addition to the slope limiter in step 2. For more details see [32].
3.2. Full 2D WENO reconstruction of uncertain hyperbolic conservation laws

In our numerical results, we compare Algorithm 1 to a method using WENO reconstruction in both the physical and stochastic space. This is motivated by the Stochastic Finite Volume Method from [37], but now using a 2D WENO scheme instead of applying 1D reconstructions to the physical and stochastic space consecutively. Therefore, we consider a full discretized finite volume scheme for $u$

$$\frac{\partial}{\partial t} \int_{D_j} \int_{C_i} u dx f_\Xi d\xi + \int_{D_j} f(u(t, x_{i+\frac{1}{2}}, \xi)) - f(u(t, x_{i-\frac{1}{2}}, \xi)) f_\Xi d\xi = 0, \quad (3.7)$$

where $i = 1, \ldots, N_x$ and $j = 1, \ldots, N_\Omega$. Using the quadrature (3.4), we can formulate the scheme for the $x-\xi$ cell mean

$$u(t, C_i, D_j) = \frac{1}{f_{C_i} dx f_{D_j} f_\Xi d\xi} \int_{D_j} \int_{C_i} u dx f_\Xi d\xi$$

in each cell $C_i \times D_j$ as

$$\frac{\partial}{\partial t} u(t, C_i, D_j) + \frac{1}{\Delta x} \sum_{\rho=0}^{Q_0} \left( f(u(t, x_{i+\frac{1}{2}}, \hat{\xi}_\rho)) - f(u(t, x_{i-\frac{1}{2}}, \hat{\xi}_\rho)) \right) \hat{\omega}_\rho = 0, \quad (3.8)$$

which is approximated by a $R$-th order SSP Runge-Kutta method. The numerical solution is then represented by a polynomial WENO reconstruction in $x$ and $\xi$. Given the $x-\xi$ cell means, we obtain by the CWENOZ algorithm

$$u(t, x, \xi) \bigg|_{C_i \times D_j} = \sum_{\kappa=0}^{K_X} \sum_{k=0}^{K_\Omega} p_{C_i \times D_j}^{(\kappa, k)} \varphi_\kappa(x) \phi_k(\xi) \quad (3.9)$$

with basis polynomials $(\varphi_\kappa)_{\kappa=0:K_X}$ on the physical cell $C_i$. We use Legendre polynomials in our numerical calculations in Section 4. Finally, we replace the evaluation of the flux at the interfaces in (3.8) by the numerical Lax-Friedrichs flux (3.2), deriving the left and right limits with help of the reconstructed polynomial (3.9).

The scheme is summarized in the following algorithm. Similar to Remark 3.2 we have to apply the hyperbolic slope limiter from [32] if we consider systems of conservation laws ($d > 1$).

**Algorithm 2** Full 2D WENO reconstruction of uncertain hyperbolic conservation laws (2D WENO)

At the initial state, compute the $x-\xi$ cell means $u(0, C_i, D_j)$. For the remaining time steps $t_n$ perform:

1. Update the cell means $u^s(t_n^s, C_i, D_j)$ according to one Runge Kutta stage.
2. Perform a WENO reconstruction on $u^s(t_n^s, x, \xi) \big|_{C_i \times D_j}$ and evaluate the polynomials (3.9) for the derivation of the Lax-Friedrichs flux (3.2).
3. Compute the differential operator of the Runge Kutta stage in (3.8).

After all Runge Kutta stages, compute the value of the cell mean $u(t_{n+1}, C_i, D_j)$.

4. Numerical Results

4.1. Linear Advection

We apply the Weighted Essentially Non-Oscillatory stochastic Galerkin scheme from Algorithm 1 to the linear advection problem

$$u_t + a(\xi)u_x = 0$$
with uncertain wave speed $a(\xi) = 1.5 + 0.5\xi$, $\xi \sim \mathcal{U}(-1, 1)$ as in Section 2.3 and where $x \in X = [0.4, 2]$.

At first, we again consider the initial conditions (2.8) and compare the WENOsg scheme to a stochastic Galerkin scheme with WENO reconstruction in $x$ (cf. (3.5)) but without the stochastic slope limiter. We refer to this method as the standard stochastic Galerkin (sG) scheme in the following. The results are shown for 3 Multielements in Figure 6 and for 10 Multielements in Figure 7. We observe the Gibbs phenomenon at the boundaries of the stochastic domain, whereas a refinement to 10 Multielements reduced the width but not the height of the overshoots. This validates our theoretical results from Section 2.3. The oscillations are completely eliminated through the application of the stochastic slope limiter.

We compare the different approaches by calculating the $L_1$ error to the analytical solution and the total variation of the numerical solution over the spatial domain $X$ and one of the Multielements $D_j$ (due to the uniform decomposition of the random space). The total variations with respect to $x$ and $\xi$ are given by

\[
TV_x(u) = \int_{D_j} \sum_{k=1}^{Q_X \cdot N_x} \left| u(t, x_k, \xi) - u(t, x_{k-1}, \xi) \right| f_{\xi} d\xi, \tag{4.1}
\]
\[
TV_\xi(u) = \int_X \sum_{\rho=1}^{Q_x} \left| u(t, x, \xi_\rho) - u(t, x, \xi_{\rho-1}) \right| dx, \tag{4.2}
\]

where $x_k$ with $k = 0, \ldots, Q_X \cdot N_x$ enumerates all quadrature points in $x$ over each of the cells $C_i$ with $i = 1, \ldots, N_x$.

The results are given in Table 1. We choose a fine discretization of 2000 space cells in the physical domain and consider the convergence within the stochastic space. We additionally compare the WENOsg scheme to the 2D WENO method described in Section 3.2 and Algorithm 2. Since the stochastic Galerkin solution represents the best approximating polynomial due to the underlying Galerkin projection, we do not expect

![Figure 6](image6.png)

Figure 6: Comparison of the WENOsg approximation (dashed) with 3 Multielements for the linear advection problem and the sG approach (solid) for the given $\xi$s at $t = 0.5$, 2000 space cells, $K_\Omega = 2$, $K_X = 2$. Example (2.8).

![Figure 7](image7.png)

Figure 7: Comparison of the WENOsg approximation (dashed) with 10 Multielements for the linear advection problem and the sG approach (solid) for the given $\xi$s at $t = 0.5$, 2000 space cells, $K_\Omega = 2$, $K_X = 2$. Example (2.8).
to improve the $L_2$ or $L_1$ error. However, the oscillations and therefore the total variations (4.1) and (4.2) should be minimized by the application of our slope limiter.

Indeed, we observe that the $L_1$ errors of the new methods are slightly higher as for standard stochastic Galerkin while the total variation and hence the oscillation are reduced. The total variation with respect to the uncertainty $TV_\xi$ is even smaller in WENO$G$ and 2D WENO as for the analytical solution. This situation arises due to the lack of information about $\xi$ that is transported within the numerical schemes (cf. Figure 8). Especially the WENO$G$ method was able to improve the total variation wrt. $x$ tremendously, more precisely, it only increased the total variation of the analytic solution for 3 Multielements by 0.4% while standard Galerkin yield an increase of almost 24%. In this test case, the 2D WENO reconstruction performed not as good as the WENO$G$ scheme and sometimes even has a larger total variation in $x$ than standard stochastic Galerkin. In the upcoming examples, the two schemes will show comparable results. Note that we have used $Q_\Omega = 1000$ quadrature nodes in $\xi$ and $Q_x = 4$ quadrature nodes in $x$ for the calculations of the total variation.

| $L_1$ error | $N_\Omega = 3$ | $N_\Omega = 10$ | $N_\Omega = 30$ |
|-------------|----------------|----------------|----------------|
| analytic    | --             | --             | --             |
| sG          | 0.0265         | 0.0076         | 0.0031         |
| WENO$G$     | 0.0329         | 0.0101         | 0.0040         |
| 2D WENO     | 0.0392         | 0.0140         | 0.0090         |

| $TV_\xi$ | $N_\Omega = 3$ | $N_\Omega = 10$ | $N_\Omega = 30$ |
|-----------|----------------|----------------|----------------|
| analytic  | --             | --             | --             |
| sG        | 0.1664         | 0.0503         | 0.0167         |
| WENO$G$   | 0.2422         | 0.0688         | 0.0200         |
| 2D WENO   | 0.0572         | 0.0188         | 0.0086         |

| $TV_x$ | $N_\Omega = 3$ | $N_\Omega = 10$ | $N_\Omega = 30$ |
|--------|----------------|----------------|----------------|
| analytic | 1.0          | 1.0          | 1.0          |
| sG     | 1.3118        | 1.2791        | 1.0795        |
| WENO$G$ | 1.0037        | 1.0044        | 1.0080        |
| 2D WENO | 1.2316        | 1.7983        | 1.3712        |

| percentage above $TV_x$ – analytic | $N_\Omega = 3$ | $N_\Omega = 10$ | $N_\Omega = 30$ |
|-----------------------------------|----------------|----------------|----------------|
| analytic                          | --             | --             | --             |
| sG                                | 23.8%          | 21.8%          | 7.4%           |
| WENO$G$                           | 0.4%           | 0.4%           | 0.8%           |
| 2D WENO                           | 18.9%          | 44.4%          | 27.1%          |

Table 1: $L_1$ error and total variation for linear advection with and without stochastic slope limiter (SL) for 2000 space cells, $K_\Omega = 2$ and $K_X = 2$. Example (2.8).
Next, we analyze if the troubled cell indicator identifies cells for smooth test cases. We consider the following initial state
\[ u(0, x, \xi) = \sin(2\pi(x + 0.1\xi)), \quad x \in [0, 1] \]
with periodic boundary conditions. Thus, the linear advection describes a sinus wave traveling with uncertain speed \( a(\xi) \). In each of the settings given in Table 1, the slope limiter was not required to be applied to any space cell.

### 4.2. Burgers’ Equation

In this numerical example we consider the Burgers’ equation
\[ \partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0. \tag{4.3} \]
We compare the WENO\textsubscript{s}G scheme to standard stochastic Galerkin and the 2D WENO method from Algorithm\textsubscript{2} as we did for the linear advection problem in the previous subsection. For the nonlinear Burgers’ equation, we consider the following initial state
\[ u(0, x, \xi) = \sin(2\pi(x + 0.1\xi)), \tag{4.4} \]
where \( x \in X = [0, 1] \) and the uncertainty is uniformly distributed, i.e. \( \xi \sim \mathcal{U}(-1, 1) \). Then we compute the solution on a \( N_x = 2000 \) space grid until \( t = 0.4 \) and derive the \( L_1 \) error to a reference solution obtained by Monte Carlo sampling at the quadrature nodes in \( \xi \) and the total variations \( (4.1) \) and \( (4.2) \).

The results can be found in Table 2, where we now observe similar values for WENO\textsubscript{s}G and 2D WENO. The \( L_1 \) errors are again slightly higher for these two modified methods compared to plain stochastic Galerkin. They all converge while increasing the number of Multielements. However, the total variation with respect to \( x \) is almost the same as in the reference solution, while standard stochastic Galerkin yield an increase of around 14%. Thus, the overshoots could be completely eliminated which can be verified in Figure 9 showing the solution for 10 Multielements in the \( x - \xi \) plane. Here, the sG approach in the left picture has huge oscillations that vanish in the WENO\textsubscript{s}G approximation presented in the right picture. The total variation wrt. \( \xi \) shows a similar behavior as for the linear advection example in Table 1 and Figure 8, hence, it is reduced by the modified schemes due to the lack of information that is transported along the uncertainty. We have used \( Q_\Omega = 1000 \) quadrature nodes in \( \xi \) and \( Q_X = 4 \) quadrature nodes in \( x \) for the calculations of the total variations.

### 4.3. Compressible Euler Equations

The one-dimensional compressible Euler equations for the flow of an ideal gas are given by
\[
\begin{align*}
\partial_t \rho + \partial_x m &= 0, \\
\partial_t m + \partial_x \left( \frac{m^2}{\rho} + p \right) &= 0, \\
\partial_t E + \partial_x \left( (E + p) \frac{m}{\rho} \right) &= 0,
\end{align*}
\tag{4.5}
\]
where \( \rho \) describes the density, \( m \) the momentum and \( E \) the energy of the gas. The three equations model the conservation of mass, momentum and energy. The pressure \( p \) reads
\[
p = (\gamma - 1) \left( E - \frac{1}{2} \frac{m^2}{\rho} \right)
\]
with the adiabatic constant \( \gamma > 1 \).
L1 error \[ N_\Omega = 3 \quad N_\Omega = 10 \quad N_\Omega = 30 \]

|       | \[ N_\Omega = 3 \] | \[ N_\Omega = 10 \] | \[ N_\Omega = 30 \] |
|-------|--------------------|--------------------|--------------------|
| reference | -                  | -                  | -                  |
| sG     | 0.0227             | 0.0064             | 0.0018             |
| WENOsG | 0.0283             | 0.0085             | 0.0028             |
| 2D WENO| 0.0284             | 0.0143             | 0.0053             |

Total variation \[ TV_\xi \]

|       | \[ N_\Omega = 3 \] | \[ N_\Omega = 10 \] | \[ N_\Omega = 30 \] |
|-------|--------------------|--------------------|--------------------|
| reference | -                  | -                  | -                  |
| sG     | 0.2257             | 0.0677             | 0.0226             |
| WENOsG | 0.3187             | 0.0949             | 0.0306             |
| 2D WENO| 0.1132             | 0.0339             | 0.0112             |

Percentage above \[ TV_\xi \] reference

|       | \[ N_\Omega = 3 \] | \[ N_\Omega = 10 \] | \[ N_\Omega = 30 \] |
|-------|--------------------|--------------------|--------------------|
| sG     | 14.3\%             | 14.2\%             | 14.2\%             |
| WENOsG | 0\%                | 0\%                | 0\%                |
| 2D WENO| 1.6\%              | 0.1\%              | 0.1\%              |

Table 2: \( L_1 \) error and total variation for Burgers’ equation with and without stochastic slope limiter (SL) for 2000 space cells, \( K_\Omega = 2 \) and \( K_X = 2 \). Example 4.4.

We consider the Euler Equations with an uncertain shock test case as in [30], which is given by the following initial conditions:

\[
\rho(0, x, \xi) = \begin{cases} 
1, & x < 0.5 + 0.05\xi, \\
0.125, & x \geq 0.5 + 0.05\xi, 
\end{cases} \\
m(0, x, \xi) = 0, \\
E(0, x, \xi) = \begin{cases} 
0.25, & x < 0.5 + 0.05\xi, \\
2.5, & x \geq 0.5 + 0.05\xi, 
\end{cases} \quad (4.6)
\]

where \( \xi \sim U(-1, 1) \) and \( x \in X = [0, 1] \). Moreover we choose \( \gamma = 1.4 \) and compute the solution on a \( N_x = 2000 \) grid until \( t = 0.1 \). For this test case, we again compare the plain stochastic Galerkin approach to WENOsG and 2D WENO described in Algorithm 1 and Algorithm 2 respectively. Note that we have to apply the hyperbolicity-preserving limiter from [32] in order to ensure the hyperbolicity of the underlying system. The \( L_1 \) error and total variations (4.1) and (4.2) for the density \( \rho \) are shown in Table 3.

As before, we observe the smallest \( L_1 \) error for standard stochastic Galerkin and that each method is converging if we increase the number of Multielements. Only this time, the 2D WENO method has a better error than WENOsG. The results for the total variation in \( \xi \) are similar to the previous scalar examples. In this test case, the modified methods only marginally improved the total variation with respect to \( x \). Table 3 shows almost the same percentage for each of the schemes and even an increase for 2D WENO and 10 Multielements. This is additionally demonstrated in Figure 10 where the plain stochastic Galerkin approach and the 2D WENO method are illustrated for 10 Multielements in the \( x - \xi \) plane. They only show minor oscillations compared to the previous examples, however, the overshoots at the boundaries of the Multielement in sG could be eliminated using the full WENO reconstruction. We have used \( Q_\Omega = 1000 \) quadrature nodes in \( \xi \) and \( Q_X = 4 \) quadrature nodes in \( x \) for the calculations of the total variation.
5. Conclusions and Outlook

In this article, we demonstrated the propagation of Gibbs phenomenon into the stochastic domain of the stochastic Galerkin system based on an uncertain linear advection example. This lead to the formulation of our modified stochastic Galerkin scheme, including the stochastic slope limiter, which is supposed to reduce overshoots in the solution manifold. The scheme is combined with a Multielement ansatz, a WENO finite volume method and Runge Kutta time stepping, giving the so called WENO stochastic Galerkin scheme altogether a stable high order approximation of the solution of the conservation law. Combined with the hyperbolicity-preserving limiter from [32], the method is able to be used on any hyperbolic system of equations. We additionally considered a similar numerical scheme using full 2D WENO reconstruction in both the physical and stochastic domain, which is motivated by [37].

We applied the two methods to the scalar linear advection and nonlinear Burgers’ equation as well as to the system of Euler Equations and compared the results to the standard stochastic Galerkin scheme using a WENO reconstruction in the physical space. An analysis of the total variations within the physical and stochastic domain verified the reduction of Gibbs oscillations for example up to 23% for the linear advection
Table 3: $L_1$ error and total variation of density $\rho$ for Euler equations with and without stochastic slope limiter (SL) for 2000 space cells, $K_{\Omega} = 2$ and $K_X = 2$. Example [4.6].

|        | $N_\Omega = 3$ | $N_\Omega = 10$ | $N_\Omega = 30$ |
|--------|----------------|-----------------|-----------------|
| $L_1$ error |                 |                 |                 |
| reference | –              | –               | –               |
| sG       | 0.0040         | 0.00003         | 0.123e-04       |
| WENO sG  | 0.0092         | 0.0025e-03      | 0.806e-04       |
| 2D WENO  | 0.0062         | 0.0025e-03      | 0.817e-04       |

|        | $N_\Omega = 3$ | $N_\Omega = 10$ | $N_\Omega = 30$ |
|--------|----------------|-----------------|-----------------|
| $TV_\xi$ |                 |                 |                 |
| reference | 0.0291         | 0.0068          | 0.0028          |
| sG       | 0.0322         | 0.0092          | 0.0030          |
| WENO sG  | 0.0033         | 0.0008          | 0.0002          |
| 2D WENO  | 0.0179         | 0.0018          | 0.0002          |

|        | $N_\Omega = 3$ | $N_\Omega = 10$ | $N_\Omega = 30$ |
|--------|----------------|-----------------|-----------------|
| $TV_x$ |                 |                 |                 |
| reference | 0.8781         | 0.8781          | 0.8781          |
| sG       | 1.0622         | 0.9407          | 0.9124          |
| WENO sG  | 1.0501         | 0.9225          | 0.8912          |
| 2D WENO  | 1.0134         | 1.0218          | 0.9068          |

|        | $N_\Omega = 3$ | $N_\Omega = 10$ | $N_\Omega = 30$ |
|--------|----------------|-----------------|-----------------|
| percentage above $TV_x$ – reference |                 |                 |                 |
| reference | –              | –               | –               |
| sG       | 17.3%          | 6.6%            | 3.5%            |
| WENO sG  | 16.4%          | 4.5%            | 1.3%            |
| 2D WENO  | 13.3%          | 13.9%           | 2.4%            |

problem compared to plain stochastic Galerkin, coming with the price of an slightly higher $L_1$ error. This underlines the necessity of our stochastic slope limiter for discontinuous solutions in the $x - \xi$ plane. The WENO sG and 2D WENO methods behave differently on our numerical test cases which indicates variable choices on the investigated problem setting.

Future work should include multi dimensions in space as well as in the uncertainty. Due to the curse of dimensionality in the stochastic Galerkin system, this scheme is mainly applied to low dimensional random variables, where it is able to outperform non-intrusive methods such as Multi-Level Monte Carlo [24] and stochastic collocation [47]. In this context, it is important to find a range of applicability for the general stochastic Galerkin method, see for example [21].

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