Dynamics of layered reinforced concrete beam on visco-elastic foundation with different resistances of concrete and reinforcement to tension and compression

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Abstract. Originally, fundamentals of the theory of limit equilibrium and dynamic deformation of building metal and reinforced concrete structures were created by A. A. Gvozdev [1] and developed by his followers [4, 5, 6, 7, 11, 12]. Forming the basis for the calculation, the model of an ideal rigid-plastic material has enabled to determine in many cases the ultimate load bearing capacity and upper (kinematically possible) or lower (statically valid) values for a wide class of different structures with quite simple methods. At the same time, applied to concrete structures the most important property of concrete to significantly differently resist tension and compression was not taken into account [10]. This circumstance was considered in [3] for reinforced concrete beams under conditions of quasistatic loading. The deformation is often accompanied by resistance of the environment in construction practice [8, 9]. In [2], the dynamics of multi-layered concrete beams on visco-elastic foundation under the loadings of explosive type is considered. In this work we consider the case which is often encountered in practical applications when the loadings weakly change in time.

1. Introduction
In the scientific literature the calculation of reinforced concrete structures is often limited to the case of the simplest forms of the rod cross-section and simplest conditions of loading and fixing. Modern technological capabilities to create flexible sets of hybrid laminated reinforced structures, where various grades of concrete, reinforcing elements and reinforcement structures can be implemented in layers of cross-section, are not taken into account. In this paper, the problem of limit equilibrium and dynamic deformation of arbitrary reinforced rods based on concrete is examined by the ideas of the model of perfectly plastic deals.

2. Methods
In this paper we used a classical model of ideal rigid-plastic body for all materials (concrete and reinforcement) in accordance with the diagram shown in Figure 1.
We assume that the structure of the reinforcement in layers 1 and 3 of the beam cross section varies according to the intensity and properties of the phase materials. For all considered sections we assume the classical kinematical hypotheses of Kirchhoff-Lyav to be valid, according to which the deformation will have the expression

\[ \varepsilon(x, z) = \varepsilon_0(x) + z\kappa(x), \quad \varepsilon_0(x) = \frac{du_0}{dx}, \quad \kappa(x) = -\frac{d^2w}{dx^2}, \]

where the \( u_0(x) \) is movement along the reporting axis \( x \) of the beam, \( w(x) \) is a deflection.

3. Results
Consider an ideal-plastic section of the beam, for which

\[ \varepsilon_0(x) < 0, \quad \varepsilon_0 + h_3\kappa(x) > 0. \]

Both inequalities will be satisfied if

\[ \varepsilon_0(x) < 0, \quad \kappa(x) > 0. \]

The state of stress will correspond to figure 2.
Then, for the considered part of the beam the longitudinal force \( \overline{N} \) is defined by the equality

\[
\overline{N} = -2\sigma_{01}\int_0^{h_1} b_1(z)dz - 2\sigma_{02}\int_{h_1}^{h_2} b_2(z)dz + 2\sigma_{02}\int_{h_2}^{h_3} b_2(z)dz + 2\sigma_{03}\int_{h_2}^{h_3} \frac{b_3(z)}{h_2}dz + 2\sigma_{03}\int_{h_2}^{h_3} b_3(z)dz, \\
+ 2\sigma_{03}\int_{h_2}^{h_3} b_3(z)dz = -2b_1(\sigma_{02} + \sigma_{01}^+) \xi_1 + 2b_2(\sigma_{02} h_2 + \sigma_{02}^+ h_1) - 2\sigma_{01}\int_0^{h_1} b_1(z)dz + 2\sigma_{03}\int_{h_2}^{h_3} b_3(z)dz.
\]

The expression for the bending moment \( \overline{M} \) will be

\[
\overline{M} = -2\sigma_{01}\int_0^{h_1} b_1(z)dz - 2\sigma_{02}\int_{h_1}^{h_2} b_2(z)dz + 2\sigma_{02}\int_{h_1}^{h_2} b_2(z)dz + 2\sigma_{03}\int_{h_2}^{h_3} b_2(z)dz + 2\sigma_{03}\int_{h_2}^{h_3} \frac{b_3(z)}{h_2}dz + 2\sigma_{03}\int_{h_2}^{h_3} b_3(z)dz, \\
+ 2\sigma_{03}\int_{h_2}^{h_3} b_3(z)dz = -2b_1(\sigma_{02} + \sigma_{01}^+) \xi_1 + 2b_2(\sigma_{02} h_2 + \sigma_{02}^+ h_1) - 2\sigma_{01}\int_0^{h_1} b_1(z)dz + 2\sigma_{03}\int_{h_2}^{h_3} b_3(z)dz.
\]
Figure 2. The beam cross section. Numbers 1, 2, 3 refer to the layers of the beam.

In the case of transverse bending ($N = 0$), we have

$$
\tau_1 = \frac{b_2^0 (\sigma_{02} H_2 + \sigma_{02} H_1) - \sigma_{01} \int_0^{H_1} b_1(z) d\tau + \sigma_{03} \int_0^{H_3} b_3(z) d\tau}{b_2^0 (\sigma_{02} + \sigma_{02})},
$$

and for $\tau_1$ the ratio $H_1 \leq \tau_1 \leq H_2$ must be satisfied.

Substituting $\tau_1$ in (1), we obtain the expression for the limit bending moment

$$
M^+ = \left(\frac{b_2^0 (\sigma_{02} H_2 + \sigma_{02} H_1) - \sigma_{01} \int_0^{H_1} b_1(\tau) d\tau + \sigma_{03} \int_0^{H_3} b_3(\tau) d\tau}{b_2^0 (\sigma_{02} + \sigma_{02})}\right)^2 + b_2^0 (\sigma_{02} H_1^2 + \sigma_{02} H_2^2) - 2\sigma_{01} \int_0^{H_1} b_1(\tau) \tau d\tau + 2\sigma_{03} \int_0^{H_3} b_3(\tau) \tau d\tau.
$$

For an ideal-plastic section of the beam with the deformed state

$$
\tau_0(\tau) > 0, \quad \tau_0 + H_3 \quad \kappa(\tau) < 0, \quad \tau_0(\tau) > 0, \quad \kappa(\tau) < 0.
$$

Similarly, it is possible to determine the limiting bending moment $M^-_0$

$$
M^-_0 = \left(\frac{b_2^0 (\sigma_{02} H_2 + \sigma_{02} H_1) - \sigma_{01} \int_0^{H_1} b_1(\tau) d\tau - \sigma_{03} \int_0^{H_3} b_3(\tau) d\tau}{b_2^0 (\sigma_{02} + \sigma_{02})}\right)^2 - b_2^0 (\sigma_{02} H_1^2 + \sigma_{02} H_2^2) + 2\sigma_{01} \int_0^{H_1} b_1(\tau) \tau d\tau - 2\sigma_{03} \int_0^{H_3} b_3(\tau) \tau d\tau.
$$

Thus, in the general case the beam is divided into sections corresponding to the deformed states $\tau_0 < 0, \kappa < 0$ or $\tau_0 > 0, \kappa > 0$ and $\tau_0 = \kappa = 0$. The last correspond to the rigid undeformed state. The number of rigid sections and the plastic hinges separating them will depend on the conditions of fastening of the beams, the type of distributed and concentrated loads and the law of resistance of the supporting environment.

Consider a reinforced concrete beam of length $l$, lying on visco-elastic foundation and cantilever-fixed on the left edge of $\tau = 0$. 

The beam is loaded by a distributed transverse load \( q(x,t) \), its own weight \( q_s = mg \), concentrated load \( Q(t) \) and moment \( M_2(t) \) at the end of \( \pi = \bar{t} \). Here, \( m \) is the mass of length unit of the beam. For a given beam, the mass will be determined by

\[
m = \int_0^{h_1} (\rho_a \mu_1 + \rho_c (1 - \mu_1)) \bar{b}_1(z) \, dz + 2b_0 \rho_c (h_2 - \bar{h}_1) + \int_{h_2}^{h_3} (\rho_a \mu_3 + \rho_c (1 - \mu_3)) \bar{b}_3(z) \, dz
\]

where \( \rho_a, \rho_c \) are the density of reinforcement materials and binding concrete, \( \mu \) are the coefficients of reinforcement.

Represent the current external loads in the form of

\[
q(x,t) = q_1(x) \psi_1(t), \quad Q(t) = Q_1 \psi_2(t), \quad M(t) = M_1 \psi_3(t).
\]

For beams of constant cross section, loaded with a uniformly distributed load (\( \varphi(x) = 1 \)), if \( M_2(\bar{t}) \geq -M_0^+ \), the movement will be provided, as a rigid rod, with the formation of plastic hinge in the left edge \( x = 0 \). Then for the deflection \( \bar{w} \) we will have the expression

\[
\bar{w}(x,\bar{t}) = \bar{w}_0(\bar{t}) \frac{x}{\bar{t}}, \quad 0 \leq x \leq \bar{t}
\]

and taking into account that according to the conditions of dynamic equilibrium the main point regarding the support \( \pi = 0 \) of all active forces, including inertia forces must be equal to zero, we obtain the equation

\[
M_0^+ = -M_2(\bar{t}) + \bar{Q}(\bar{t}) + \int_0^{\bar{t}} \left[ \bar{q}(\bar{x},\bar{t}) + q_s - m \ddot{\bar{w}} - \bar{r}\dot{\bar{w}} \right] \, d\bar{x},
\]

where \( \bar{r} = \frac{\partial \bar{M}}{\partial \bar{w}} = \bar{M}_0(\bar{t}), \quad \bar{q}_s = \bar{k}_1 \bar{w} + \bar{k}_2 \dot{\bar{w}}, \quad \bar{k}_1, \bar{k}_2 \) are the coefficients of the elastic and viscous resistance of the base, and then we assume that \( \bar{k}_1 > 0, \bar{k}_2 > 0, \bar{b}_0 = \bar{b}_3(h_3) \) are the length of the base of the beam cross section. A point denotes a partial derivative in time \( \bar{t} \).
Figure 4. The law of change $\bar{q}(\tau, \bar{r})$

From (2), we get the equation to determine the deflection

$$\ddot{w}_0(\bar{r}) + \frac{b_0 k_2}{m} \dot{w}_0(\bar{r}) + \frac{b_0 k_1}{m} w_0(\bar{r}) = \gamma(\bar{r}),$$

where

$$\gamma(\bar{r}) = \frac{3}{2m} \bar{q}_1 \psi_1(\bar{r}) + \frac{3\bar{q}}{2} + \frac{3}{m l} (-\bar{M}_0^+ + \bar{Q}_1 \psi_2(\bar{r}) - \bar{M}_{12} \psi_3(\bar{r})).$$

Discarding in (3) the inertial member $\ddot{w}_0(\bar{r})$, we obtain the equation of quasi-static change of the beam overlimited deflection

$$\frac{b_0 k_2}{m} \ddot{w}_0(\bar{r}) + \frac{b_0 k_1}{m} \dot{w}_0(\bar{r}) = \gamma(\bar{r}),$$

where $\gamma(\bar{r})$ is defined by equation (4).

Taking the value $\ddot{w}_0(\bar{r}) = 0$, we obtain the equation for the first limiting amplitudes of the considered beam

$$\frac{3}{2m} \bar{q}_1 \psi_1(\bar{r}) + \frac{3\bar{q}}{2} + \frac{3}{m l} (-\bar{M}_0^+ + \bar{Q}_1 \psi_2(\bar{r}) - \bar{M}_{12} \psi_3(\bar{r})) = 0.$$

From relation (6) we determine the point in time $\bar{\tau}_0$ when the loads reach the limit value and the plastic hinge occurs at the pinch point of the beam.

If at $\bar{r} \geq \bar{\tau}_0$ the loads monotonically quasistatically temporarily increase, then integrating the equation (5) under the initial conditions

$$\ddot{w}_0(\bar{\tau}_0) = 0,$$

for the deflection we get the expression

$$\ddot{w}_0(\bar{r}) = \frac{m}{b_0 k_2} e^{-\frac{\bar{r}}{k_2}} \int_{\bar{\tau}_0}^{\bar{r}} \gamma(\bar{\tau}) e^{\frac{\bar{r}}{k_2}} d\bar{\tau}.$$

If we accept that the law of pressure variation on the beam $\bar{q}(\bar{x}, \bar{r}) = \bar{q}_1 \psi_1(\bar{r})$ changes according to the law as shown in figure 4
\[ \psi_1(t) = \begin{cases} \frac{t}{\tau_1}, & \text{if } \tau_0 \leq t \leq \tau_1, \\ e^{-\left(\frac{t}{\tau_1} - 1\right)}, & \text{if } \tau_1 \leq t \leq \infty, \end{cases} \] (8)

and the acting external moment and concentrated force at \( x = \tau \) are absent

\[ M_2(t) = 0, \quad Q(t) = 0. \]

Then before reaching the first load limit \( q_0 \) at the time \( 0 \leq t < \tau_0 \) the beam will be rigid, and for the time moment of formation of the plastic hinge and the value of the first limit load it is true that

\[ \tau_0 = \frac{2m}{3q} \left( \frac{3M_0}{m^2} - \frac{3g}{2} \right) \tau_1, \quad q_0 = \frac{2m}{3} \left( \frac{3M_0}{m^2} - \frac{3g}{2} \right). \] (9)

From relations (7) and assumptions \( 0 < \tau_0 < \tau_1 \) the following inequalities have the form of

\[ 0 < \frac{3M_0}{m^2} - \frac{3g}{2} < 1. \]

In case of load increase according to the law (8) after reaching the first limit load, for the time interval \( \tau_0 \leq t \leq \tau_1 \) the relations (4), (5) take the form of

\[ f_1 \dot{w}_0(t) + f_2 w_0(t) = f_4 t + f_3 \] (10)

where

\[ f_1 = \frac{b_0 t_2}{m}, \quad f_2 = \frac{b_0 t_1}{m}, \quad f_3 = \frac{3g}{2} - \frac{3M_0}{m^2}, \quad f_4 = \frac{3q_1}{2m \tau_1}. \]

Integrating equation (10) under the initial conditions (7) we obtain an expression for deflection for the time interval \( \tau_0 \leq t \leq \tau_1 \)

\[ w(t) = g_1 e^{\frac{3g}{2} t} + g_2 t + g_4, \] (11)

where

\[ g_1 = \left( -\frac{f_3 \tau_0 + f_4}{f_2} + \frac{f_3 f_1}{f_2} \right) e^{\frac{3g}{2} \tau_0}, \quad g_2 = \frac{f_2}{f_1}, \quad g_3 = \frac{f_3}{f_2}, \quad g_4 = \frac{f_4}{f_2} - \frac{f_3 f_1}{f_2}. \]

The deflection rate for the indicated interval will be equal to

\[ \dot{w}(t) = g_1 g_2 e^{\frac{3g}{2} t} + g_3. \]

We denote the deflection by \( w_{0r} \) at time \( t = \tau_1 \)

\[ w(\tau_1) = g_1 e^{\frac{3g}{2} \tau_1} + g_2 \tau_1 + g_4 = w_{0r}. \] (12)

For time interval \( \tau_1 \leq t \leq \infty \) the relations (4), (5) take the form

\[ f_1 \dot{w}(t) + f_2 w(t) = j_4 e^{-t} + j_3, \] (13)

where
\[ j_3 = \frac{3\pi}{2} - \frac{3}{m^2\tau_0^2}, \quad j_4 = \frac{3\pi_1}{2m\tau}. \]

The solution to (13) with conditions (12) has the form
\[
\bar{m}(\bar{T}) = v_1 e^{v_2 \bar{T}} + v_3 e^{-\bar{T}} + v_4,
\] (14)

where
\[
v_1 = \left(\frac{\bar{m}_0 - j_3}{f_2} + \frac{j_4}{f_1 - f_2} e^{-\tau_1}\right) \frac{1}{\bar{T}}, \quad v_2 = -\frac{f_2}{f_1}, \quad v_3 = -\frac{j_4}{f_1 - f_2}, \quad v_4 = \frac{j_3}{f_2}.
\]

If \(0 \leq \bar{T} < \tau_0\), the beam remains rigid, then the equation to determine the moment has the form
\[
\frac{\partial^2 M}{\partial \bar{T}^2} = \bar{q}(\bar{x}, \bar{T}) + \bar{q}_s,
\] (15)

integrating which with the initial conditions
\[
\bar{M}(\bar{T}, \bar{T}) = 0, \quad \frac{\partial \bar{M}}{\partial \bar{x}}(\bar{T}, \bar{T}) = 0,
\] (16)

we obtain an expression for the moment that is valid when \(0 \leq \bar{T} < \tau_0\)
\[
\bar{M}(\bar{T}) = \frac{\bar{q}_1 \bar{T} + \bar{m} g \tau_1}{2\tau_1} + \frac{\bar{q}_1 \bar{T} + \bar{m} g \tau_1}{\tau_1} - \frac{3(\bar{q}_1 \bar{T} + \bar{m} g \tau_1) \bar{T}^2}{2\tau_1}.
\]

The equation of beam deflection in the case of plastic hinge in the place of pinch of a beam has the form
\[
\frac{\partial^2 \bar{M}}{\partial \bar{T}^2} = \bar{q}(\bar{x}, \bar{T}) + \bar{q}_s - \frac{\bar{k}_1 \bar{b}_0 \bar{x} \bar{m}_0(\bar{T})}{\bar{T}} - \frac{\bar{k}_2 \bar{b}_0 \bar{x} \bar{m}_0(\bar{T})}{\bar{T}},
\] (17)

adding to which the expression for deflection (10), we obtain
\[
\frac{\partial^2 \bar{M}}{\partial \bar{T}^2} = \bar{x} \left( g_5 e^{g_2 \bar{T}} + g_6 \bar{T} + g_7 \right) + g_8 \bar{T} + g_9,
\] (18)

where
\[
g_5 = -\frac{k_1 b_0}{l}, \quad g_6 = -\frac{k_2 b_0}{l}, \quad g_7 = -\frac{k_1 b_0}{l}, \quad g_8 = \frac{\bar{q}_1}{\tau_1}, \quad g_9 = \bar{m} g.
\]

Integrating (18) under the initial conditions (16), we obtain the expression for the moment that is valid for \(\tau_0 \leq \bar{T} < \tau_1\)
\[
\bar{M} = \bar{x} \left( \frac{\rho_6}{6} e^{g_2 \bar{T}} + \frac{g_4}{6} \bar{T} + \frac{g_6}{6} \right) \bar{T} + e^{g_2 \bar{T}} \left( \frac{g_4}{2} + \frac{g_6}{2} \right) \bar{T} + e^{g_2 \bar{T}} \left( \frac{g_4}{2} + \frac{g_6}{2} \bar{T} + g_6 \bar{T} \right) + \frac{e^{g_2 \bar{T}}}{2} \left( \frac{g_4}{2} + \frac{g_8}{2} \right) \bar{T} + \frac{e^{g_2 \bar{T}}}{2} \left( \frac{g_4}{2} + \frac{g_6}{2} \right) \bar{T}.
\]

Substituting in (17) the expression for the deflection (13), we obtain
\[
\frac{\partial^2 \bar{M}}{\partial \bar{T}^2} = \bar{x} \left( v_5 e^{v_2 \bar{T}} + v_6 e^{-\bar{T}} + v_7 \right) + v_8 e^{-\bar{T}} + v_9,
\] (19)
where
\[ v_5 = -\frac{k_1 b_0}{l} v_1 - \frac{k_2 b_0}{l} v_1 v_2, \quad v_6 = -\frac{k_1 b_0}{l} v_3 + \frac{k_2 b_0}{l} v_3, \quad v_7 = -\frac{k_1 b_0}{l} v_4, \quad v_8 = q_1 e^{\tau}, \quad v_9 = mg. \]

Integrating (19) under the initial conditions (16), we obtain the expression for the moment that is valid for \( \tau_1 \leq \tau < \infty \)
\[
\mathcal{M} = \frac{l^3}{6} \left( \frac{v_6}{6} e^{\tau} + \frac{v_6}{6} e^{-\tau} + \frac{v_9}{6} \right) + \frac{l^2}{2} \left( \frac{v_8}{2} e^{\tau} + \frac{v_9}{2} \right) - \frac{l}{2} \left( \frac{v_6}{2} e^{\tau} + \frac{v_6}{2} e^{-\tau} + \frac{v_9}{2} \right) e^{-\tau} + \frac{v_8}{2} e^{\tau} + \frac{v_8}{2} e^{-\tau}.
\]

4. Conclusions

Thus, the obtained expressions are determined for the deflections of a beam and the expression of the moments. This solution is true for the entire length of the beam \( 0 \leq x \leq l \) if the bending moment \( \mathcal{M}(x, t) \) is within \( -\mathcal{M}_0 \leq \mathcal{M}(x, t) \leq \mathcal{M}_0 \).

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