A note on the dynamics of an HIV infection model using Padé approximants

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Abstract: In this work, we use Padé’s approaches in solving a system of ordinary nonlinear differential equations which arise in the model for HIV infection of $CD_4^+ T$ cells. Some graphs are presented to show the reliability and simplicity of this method as well as the algorithms implemented in the analysis of the model.

Key-Words: Padé approximants, Model for HIV of $CD_4^+ T$ cells, Semi-Analytical Method, Normal Padé Table.

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1 Introduction

The non-linear problems form a major line of research in the sciences in general since many phenomena are modeled using nonlinear equations. It is also true that in most cases, it is not possible to find analytical solutions to such models and therefore knowledge of efficient numerical methods to approximate them is essential. Thus, there are several semi-analytical methods that allow us to approximate the solutions numerically, such as the ADM, the DTM and the Padé approximant method [1].

For this reason, the Padé method is used in the resolution of non-linear problems due to its excellent convergence. This method has proven to be very useful in obtaining quantitative information about the solution of many interesting problems in sciences. The applications of Padé approximants are divided into two classes:

• The provision of efficient rational approaches to special mathematical functions.

• The acquisition of quantitative information about a function.

We can say then that the Padé approaches are the basis of several non-linear techniques which have a connection with the well known $\epsilon$ algorithm. In the specialized literature, several methods are known to find the Padé approximations, so the fundamental objective of this work is to pay special attention to this type of fractions in which the method is applicable presenting the most relevant properties in the applications.

Now, by way of introduction, it is important to note that HIV is a retrovirus that targets the $CD_4^+ T$ lymphocytes, which are the most abundant white blood cells of the immune system. Although HIV infects other cells as well, it wreaks the most havoc on the $CD_4^+ T$ cells by causing their decline and destruction, thus decreasing the resistance of the immune system.

2 Padé approximant method

From elemental analysis, divergence in a series of powers is essential to talk about the presence of singularities. This type of divergence shows the problem that polynomials have to approach a function around that singularity, where the basic idea is to represent any function $f(z)$ by a convergent expression [2]. One of the most used techniques that requires as input a finite number of terms is the approximation of Padé. In this method proposed by Padé, this series [13]

$$f(z) = \sum_{i=0}^{\infty} c_i z^i,$$

is replaced by a succession of rational functions of the form:

$$R_{MN}^M(z) = \frac{\sum_{i=0}^{M} a_i z^i}{\sum_{i=0}^{N} b_i z^i}.$$  

Therefore, we use the standardization $b_0 = 1$ and the remaining $M+N+1$ coefficients $a_0, \ldots, a_M$ and $b_1, \ldots, b_N$ in Eq.(2) are chosen such that $M+N+1$
is the coefficient in the power series expansion (see Eq.(1)). Thay is,

\[ f(z) - R_M^N(z) = O(Z^{M+N+1}), \]

then

\[ f(z) \sum_{i=0}^{M} b_i z^i - \sum_{i=0}^{M} a_i z^i = O(Z^{M+N+1}). \]  

(4)

If \( \sum a_i z_i \) is a series representation of \( f(z) \), then many cases \( R_M^N(z) \rightarrow f(z) \) when \( M, N \rightarrow \infty \), even if \( \sum a_i z_i \) is divergent. Generally, we consider the convergent succession \( R_0^0, R_0^1, \ldots \), where \( M = N + J \) with \( J \) fixed and \( N \rightarrow \infty \).

Therefore, we get

- Case 1: \( M \geq N \). Developing the expression (4) we have:

\[
\begin{align*}
&c_0 + c_1 z + c_2 z^2 + \ldots + c_N z^N + \ldots + c_M z^M + \ldots + c_{M+N} z^{M+N} \\
&\quad + c_0 z + c_1 z^2 + \ldots + c_N z^N + \ldots + c_M z^M + \ldots + c_{M+N} z^{M+N} \\
&\quad + c_0 z^2 + \ldots + c_N z^N + \ldots + c_M z^M + \ldots + c_{M+N} z^{M+N} \\
&\quad \vdots \\
&\quad + c_0 z^N + \ldots + c_N z^N + \ldots + c_M z^M + \ldots + c_{M+N} z^{M+N} \\
&= a_0 + a_1 z + a_2 z^2 + \ldots + a_M z^M \\
&\sum_{i=0}^{M} b_i z^i - \sum_{i=0}^{M} a_i z^i = O(Z^{M+N+1}).
\end{align*}
\]

(5)

Then, equating coefficients we obtain:

\[
\begin{cases}
\quad c_0 b_0 + c_1 b_1 + \ldots + c_{M-N} b_N = -c_{N+1} \\
\quad c_0 b_1 + c_1 b_2 + \ldots + c_{M-N} b_N = -c_{N+2} \\
\quad \vdots \\
\quad c_0 b_{N-1} + c_1 b_N + \ldots + c_{M-N} b_N = -c_{N+M} \\
\end{cases}
\]

and

\[
\begin{cases}
\quad a_0 = c_0 \\
\quad a_1 = c_1 + c_0 b_1 \\
\quad a_2 = c_2 + c_1 b_1 + c_0 b_2 \\
\quad \vdots \\
\quad a_N = c_N + c_{N-1} b_1 + c_{N-2} b_2 + \ldots + c_0 b_N \\
\quad a_M = c_M + c_{M-1} b_1 + c_{M-2} b_2 + \ldots + c_{N-N} b_N
\end{cases}
\]

Now, solving these systems we get:

\[
A \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = - \begin{bmatrix} c_{M+1} \\ c_{M+2} \\ \vdots \\ c_{M+N} \end{bmatrix}
\]

where \( A \) is a matrix \( n \times n \) whose entries are given by \( A_{ij} = c_{M+i-j} \) with \( a_k = 0 \) if \( k < 0 \). Therefore, the coefficients \( a_0, a_1, \ldots \), are determined by using Eq.(5)

\[ a_j = \sum_{k=0}^{j} c_{j-k} b_k, \quad 0 \leq j \leq M, \]

(5)

where \( b_k = 0 \) for \( k > N \). Here, the resulting rational function \( R_M^N(z) \) is called a Padé approximation.

### 2.1 Modified algorithm

Semi-analytical techniques to solve nonlinear models require an initial vision to determine the solutions as well as the calculation of one or more parameters that fit the initial system. When the parameters are chosen correctly, the results can be very precise, but it is important to emphasize that there is no method for initially choosing this choice. In this work, we suggest to use directly the serial solution for a nonlinear model to find Padé’s approximation with highly efficient results [3, 4].
Supongamos $T(t) = \sum_{n=0}^{\infty} c_n(t) t^n$, $I(t) = \sum_{n=0}^{\infty} c_n I(t)$ y $V(t) = \sum_{n=0}^{\infty} c_n V(t)$.

Para $n > 0$ se tiene:

$$(n+1) c_{n+1}^{(2)} - \beta n c_{n-k} + \beta c_n^{(3)} = 0$$

$$(n+1) c_{n+1}^{(2)} - \beta n c_{n-k}^{(3)} + \gamma c_n^{(3)} = 0$$

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3 Dynamics of an HIV infection model

Below we analyze the dynamics for a model of HIV infection from $CD4+T$ cells. The components of this basic model are the concentration of $CD4+T$ cells, $CD4+T$ cells infected with HIV and the HIV-free particles, which we will denote by $T(t)$, $I(t)$ and $V(t)$ respectively.

These amounts satisfy the following relationship:

$$
\begin{align*}
T' &= s - \alpha T + r T (1 - \frac{T}{T_{max}}), \\
I' &= \lambda V T - \beta I, \\
V' &= \eta \beta I - \gamma V,
\end{align*}
$$

with initial conditions $T(0) = T_0$, $I(0) = I_0$ and $V(0) = V_0$.

Now, we assume

$$
T(t) = \sum_{n=0}^{\infty} c_n(t) t^n, \quad I(t) = \sum_{n=0}^{\infty} c_n I(t), \quad V(t) = \sum_{n=0}^{\infty} c_n V(t)
$$

for some coefficients $c_n$. In this case, we set the values $s = 0.1, \alpha = 0.02, \beta = 0.3, r = 3, \gamma = 2.4, \lambda = 0.0027, T_{max} = 1500, \eta = 10, r_1 = 0.1, r_2 = 0$ and $r_3 = 0.1$.

We can see in Tab. 1 and 2 the rapid convergence of the method for the calculation of the coefficients $c_n$.
Hallar soluciones aproximadas de $T$, $I$, $V$ y $T^2$ desde el punto de vista algebraico, puesto que se hallan directamente los coeficientes de la expansión de Taylor.$^1$

$T(0) = 1$, $I(0) = 0$, $V(0) = 0$.

**Algoritmo 2.** Algoritmo modificado para el modelo (6).  

| $n$ | $T(t)$ | $I(t)$ | $V(t)$ |
|-----|--------|--------|--------|
| 0   | 1.0000000000 | 0.0000000000 | 0.0000000000 |
| 1   | 0.9999999999 | 0.0000200001 | 0.0000000000 |
| 2   | 0.9999999999 | 0.0000000001 | 0.0000000000 |

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$$T(t) = \frac{0.030964571 + 0.08982632^{\alpha} + 2.1903827^{\beta}}{1 + 0.00000000000002700}$$

En este trabajo se halla el approximante de Padé para $T(t)$.

**Figura 4.1:** Approximantes de Padé para $T(t)$.

Medran [20] aplicó la transformada de Laplace, luego la aproximación inversa de Laplace para hallar soluciones aproximadas de $T(t)$, $I(t)$ y $V(t)$. El procedimiento establecido en este trabajo es de la forma $T(t)$, $I(t)$ y $V(t)$ y $T^2$. Con los coeficientes anteriores, Medran [20] aplicó la transformada de Laplace, luego la aproximación inversa de Laplace para hallar soluciones aproximadas de $T(t)$, $I(t)$ y $V(t)$. El resultado de este trabajo es el mismo que el del problema de Troesch, el procedimiento propuesto en este trabajo, comparado con el procedimiento de Troesch, el procedimiento propuesto en este trabajo, comparado con los resultados obtenidos en [20], es mucho más sencillo. En algoritmo desarrollado por [20], se hallan directamente los coeficientes de la expansión de Taylor $T(t)$, $I(t)$ y $V(t)$ con resultados muy similares, vemos:

Tabla 4.3: Coeficientes $c(n)$ de las expansiones en serie dadas en (4.24)

| $n$ | $T(t)$ | $I(t)$ | $V(t)$ |
|-----|--------|--------|--------|
| 0   | 1.0000000000 | 0.0000000000 | 0.0000000000 |
| 1   | 0.9999999999 | 0.0000200001 | 0.0000000000 |
| 2   | 0.9999999999 | 0.0000000001 | 0.0000000000 |

**Figura 4.2:** Approximantes de Padé para $I(t)$.

**Figura 4.3:** Approximantes de Padé para $V(t)$.

**Tabla 4.4:** Coeficientes $c(n)$ de las expansiones en serie dadas en (4.24)

| $n$ | $T(t)$ | $I(t)$ | $V(t)$ |
|-----|--------|--------|--------|
| 0   | 1.0000000000 | 0.0000000000 | 0.0000000000 |
| 1   | 0.9999999999 | 0.0000200001 | 0.0000000000 |
| 2   | 0.9999999999 | 0.0000000001 | 0.0000000000 |

**Figura 4.4:** Approximantes de Padé para $V(t)$.

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$$T(t) = \frac{0.030964571 + 0.08982632^{\alpha} + 2.1903827^{\beta}}{1 + 0.00000000000002700}$$

**Figura 4.5:** Approximantes de Padé para $V(t)$.

**4 Conclusion**

En este trabajo se han presentado las aproximaciones de Padé para el modelo de infección por VIH. Se ha aplicado la transformada de Laplace para hallar soluciones aproximadas de los términos $T(t)$, $I(t)$ y $V(t)$. Se ha obtenido una ecuación diferencial no lineal, la cual se ha resuelto mediante la transformada de Laplace. Se han hallado soluciones aproximadas de los términos $T(t)$, $I(t)$ y $V(t)$ mediante la transformada de Laplace. Se ha verificado que los resultados obtenidos son consistentes con los resultados obtenidos en [20].
series with respect to the series calculated by other techniques.

Two algorithms were designed to calculate the coefficients of the rational expression corresponding to Padé’s approximations of the order \((M, N)\): the first one was calculated by solving a system of linear equations in which the matrix of coefficients is a Toeplitz matrix, which gave very efficient results numerically; the second algorithm developed is much more efficient than the previous one since it uses properties of continuous fractions in which Padé’s approximations correspond to any order under the condition of normality of succession.

Finally, it was verified that the results obtained with the algorithms of this work, gave the same results that those obtained with other methods. Therefore, from the algebraic point of view, the procedure is much simpler and efficient, since the expansion coefficients of the series of \(T(t)\), \(I(t)\) and \(V(t)\) were found in a direct way for the initial model.

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