Tel Aviv University School of Mathematical Sciences.

**The Langlands-Shahidi Method for the metaplectic group and applications.**

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0 Introduction

Over a period of about thirty years Freydoon Shahidi has developed the theory of local coefficients and its applications. Nowadays this method is known as the Langlands-Shahidi method. The references \[16, 47, 48, 49, 50, 51, 52, 53, 55\] are among Shahidi’s works from the first half of this period. These works are used in this dissertation. The applications of this theory are numerous; see the surveys \[16, 54, 56\] and \[30\] for a partial list. Although this theory addresses quasi-split connected reductive algebraic groups, our aim in this dissertation is to extend this theory to \(S_{p_{2n}}(\mathbb{F})\), the metaplectic double cover of the symplectic group over a p-adic field \(\mathbb{F}\), which is not an algebraic group, and present some applications. We shall realize \(S_{p_{2n}}(\mathbb{F})\) as the set \(S_{p_{2n}}(\mathbb{F}) \times \{\pm 1\}\) equipped with the multiplication law
\[
(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2c(g_1, g_2)),
\]
where \(c(\cdot, \cdot)\) is Rao’s cocycle as presented in \[42\].

The properties of \(S_{p_{2n}}(\mathbb{F})\) enable the extension of the general representation theory of quasi-split connected reductive algebraic groups as presented in \[60\] and \[67\]. A great part of this extension is already available in the literature; see \[29, 11, 1, 2\] and \[72\] for example.

An analog to Bruhat decomposition holds in \(S_{p_{2n}}(\mathbb{F})\). If \(\mathbb{F}\) is a p-adic field, \(S_{p_{2n}}(\mathbb{F})\) is an \(l\)-group in the sense of Bernstein and Zelevinsky, \[8\]. Since Rao’s cocycle is continuous, it follows that there exists \(U\), an open compact subgroup of \(S_{p_{2n}}(\mathbb{F})\), such that \(c(U, U) = 1\). Thus, a system of neighborhoods of \((I_{2n}, 1)\) is given by open compact subgroups of the form \((V, 1)\), where \(V \subseteq U\) is an open compact subgroup of \(S_{p_{2n}}(\mathbb{F})\). Furthermore, in \(S_{p_{2n}}(\mathbb{F})\) the analogs of the Cartan and Iwasawa decompositions hold as well. If \(\mathbb{F}\) is not 2-adic then \(S_{p_{2n}}(\mathbb{F})\) splits over the standard maximal compact subgroup of \(S_{p_{2n}}(\mathbb{F})\). Over any local field (of characteristic different than 2) \(S_{p_{2n}}(\mathbb{F})\) splits over the unipotent subgroups of \(S_{p_{2n}}(\mathbb{F})\).

For a subset \(H\) of \(S_{p_{2n}}(\mathbb{F})\) we denote by \(\overline{H}\) its pre-image in \(S_{p_{2n}}(\mathbb{F})\). Let \(P = M \ltimes N\) be a parabolic subgroups of \(S_{p_{2n}}(\mathbb{F})\). \(\overline{P}\) has a “Levi” decomposition, \(\overline{P} = \overline{M} \ltimes \mu(N)\), where \(\mu\) is an embedding of \(N\) in \(S_{p_{2n}}(\mathbb{F})\) which commutes with the projection map.

During a course by David Soudry (2008-2009) dedicated to Waldspurger’s *La formule de Plancherel pour les groupes p-adiques (d’après Harish-Chandra)* which is a remake of Harish-Chandra’s theory as presented in \[60\], the author checked that the general theorems regarding Jacquet modules, \(L_2\)-representations, matrix coefficients, intertwining operators, Harish-Chandra’s \(c\)-functions etc. extend to the metaplectic group. Same holds for Harish-Chandra’s completeness theorem and the Knapp-Stein dimension theorem which follows from this theorem. In fact many of the geometric proofs that are given in \[8\] and \[9\] apply word for word to the metaplectic group.

We note that many of the properties mentioned in the last paragraph are common to general \(n\)-fold covering groups of classical groups. However, the following property is a special feature of \(S_{p_{2n}}(\mathbb{F})\). Let \(g, h \in S_{p_{2n}}(\mathbb{F})\). If \(g\) and \(h\) commute then the pre-images in \(S_{p_{2n}}(\mathbb{F})\) also commute. In particular, the inverse image of a commutative subgroup of \(S_{p_{2n}}(\mathbb{F})\) is commutative. This implies that the irreducible representations of \(T_{S_{p_{2n}}(\mathbb{F})}\), the inverse image of the maximal torus of \(S_{p_{2n}}(\mathbb{F})\), are one dimensional. As noted in \[17\] this is the reason that a Whittaker model for principal series representation of \(S_{p_{2n}}(\mathbb{F})\) is unique.
Furthermore, in Chapter 5 of this dissertation we prove the uniqueness of Whittaker model for $Sp_{2n}(F)$ in general. We emphasize that this uniqueness does not hold for general covering groups; see \[15\] and \[1\]. It is the uniqueness of Whittaker model that enables a straightforward generalization of the definition of the Langlands-Shahidi local coefficients to the metaplectic group (see \[5\] for an application of the theory of local coefficients in the context of non-unique Whittaker model).

0.1 Main results

Let $F$ be a local field of characteristic 0. Let $\psi$ be a non-trivial character of $F$. We regard $\psi$ also as a genuine non-degenerate character of the inverse image of the unipotent radical of a symplectic group.

**Theorem A.** Let $\pi$ be an irreducible admissible representation of $Sp_{2n}(F)$. Then, the dimension of the space of $\psi$-Whittaker functionals on $\pi$ is at most 1; see Chapter 5.

**Theorem B.** Let $\tau$ be an irreducible admissible generic representation of $GL_n(F)$. Let $P(F)$ be the Siegel parabolic subgroup of $Sp_{2n}(F)$ and let $w$ be a particular representative of the long Weyl element of $Sp_{2n}(F)$ modulo the long Weyl element of $P(F)$. Let $C^\psi_{Sp_{2n}(F)}(P(F), s, \tau, w)$ be the metaplectic analog to the Langlands-Shahidi local coefficient. Then, there exists an exponential function $c_F(s)$ such that

$$C^\psi_{Sp_{2n}(F)}(P(F), s, \tau, w) = c_F(s) \frac{\gamma(\tau, sym^2, 2s, \psi)}{\gamma(\tau, s + \frac{1}{2}, \psi)}.$$

Furthermore, $c_F(s) = 1$ provided that $F$ is a $p$-adic field of odd residual characteristic, $\tau$ is unramified and $\psi$ is normalized. See Sections 8.6 and 9.3.

Let $F$ be a $p$-adic field.

**Theorem C.** We keep the notation and assumptions of Theorem B. Let $\beta(s, \tau)$ the meromorphic function defined by the relation

$$A_{w^{-1}}A_w = \beta(s, \tau)Id.$$

Here $A_w$ is the intertwining operator defined on $\pi$, the representation of $Sp_{2n}(F)$ (parabolically) induced from $\tau$. Then, $\beta(s, \tau)$ has the same analytic properties as the Plancherel measure attached to $SO_{2n+1}(F)$ and $\tau$. In particular, if we assume in addition that $\tau$ is supercuspidal unitary then $\pi$ is irreducible if and only if $\pi'$ is irreducible. Here $\pi'$ is the representation of $SO_{2n+1}(F)$ (parabolically) induced from $\tau$. See Section 10.3.

**Theorem D.** Let $\pi$ be a principal series representation of $Sp_{2n}(F)$ induced from a unitary character. Then $\pi$ is irreducible. See Section 10.1.

**Theorem E.** For $1 \leq i \leq r$ let $\tau_i$ be an irreducible admissible supercuspidal unitary representation of $GL_{n_i}(F)$ and let $\sigma$ be an irreducible admissible supercuspidal generic genuine representation of $Sp_{2k}(F)$. Denote by $\pi$ the corresponding parabolic induction on $Sp_{2n}(F)$. Then, $\pi$ is reducible if and only if there exists $1 \leq i \leq r$ such that $\tau_i$ is self dual and

$$\gamma(\sigma \times \tau_i, 0, \psi)\gamma(\tau_i, sym^2, 0, \psi) \neq 0.$$

Here $\gamma(\sigma \times \tau_i, s, \psi)$ is the $\gamma$-factor attached to $\sigma$ and $\tau_i$ defined by analogy to the general definition of Shahidi; See section 10.2.
Theorem F. Let $\chi$ be a character of $F^*$. There exists a meromorphic function $\tilde{\gamma}(\chi, \psi, s)$ such that

$$\tilde{\gamma}(\chi, \psi^{-1}, s)\zeta(\phi, \chi, s) = \zeta(\phi, \chi^{-1}, 1 - s)$$

for every $\phi$, a Schwartz function on $F$. Here $\zeta$ is the Mellin Transform and

$$\tilde{\phi}(x) = \int_F \phi(y)\psi(xy)\gamma_{\psi}^{-1}(xy) dy,$$

where $\gamma_{\psi}$ is the normalized Weil factor attached to a character of second degree. Furthermore, $\tilde{\gamma}(\chi, \psi, s)$ has a relation to the $SL_2(F)$ local coefficient which is similar to the relation that the Tate $\gamma$-factor has with the $SL_2(F)$ local coefficient, i.e.,

$$\tilde{\gamma}^{-1}(\chi^{-1}, \psi^{-1}, 1 - s) = \epsilon'(\chi, s, \psi) \frac{\gamma(\chi^2, 2s, \psi)}{\gamma(\chi, s + \frac{1}{2}, \psi)},$$

where $\epsilon'(\chi, s, \psi)$ is an exponential factor which equals 1 if $\chi$ is unramified and $F$ is p-adic field of odd residual characteristic. See Section 8.5.

Theorem G. Let $F$ be a field of characteristic different then 2. The unique extension of Rao’s cocycle from $Sp_{2n}(\mathbb{F})$ to $GSp_{2n}(\mathbb{F})$ is given by

$$\tilde{c}(g, h) = v_{\lambda_h}(p(g))c(p(g)^{\lambda_h}, p(h)).$$

For details see Section 2.6.

0.2 An outline of the thesis

Chapters 1-4 are of a preliminary nature. In Chapter 1 we give the general notation to be used throughout this dissertation. Among these notations is $(\cdot, \cdot)_{\mathbb{F}}$, the quadratic Hilbert symbol.

In Chapter 2 we present the metaplectic group and prove some of its properties. In Section 2.1 we introduce some notations and facts related to symplectic groups. In particular we denote by $P_{T}^-(\mathbb{F})$ and $M_{T}^-(\mathbb{F})$ a standard parabolic subgroup and its Levi part.

In Section 2.2 we introduce Rao’s cocycle and prove some useful properties. The metaplectic group is also presented in this section along with its basic properties. In Section 2.3 we address the rank 1 case of Rao’s cocycle which is identical to Kubota’s cocycle; see [32]. For p-adic fields of odd residual characteristic we recall the explicit splitting of $SL_2(O_F)$. We use this splitting to describe some properties of the splitting of $Sp_{2n}(O_F)$; see Lemma 2.1. For all the p-adic fields we prove that the cocycle is trivial on small enough open compact subgroups of $SL_2(F)$. We describe explicitly the two isomorphisms between $SO_2(\mathbb{R})$ and $\mathbb{R}/4\pi \mathbb{Z}$; see Lemma 2.3. One of the explicit isomorphisms will be used in Section 8.2 where we compute the local coefficients of $SL_2(\mathbb{R})$. In Section 2.4 we deal with parabolic subgroups of $Sp_{2n}(\mathbb{F})$ which are defined to be an inverse image of parabolic subgroups of $Sp_{2n}(\mathbb{F})$. We prove that these groups possess an exact analog to Levi decomposition. For p-adic fields of odd residual characteristic we give in Lemma 2.7 an explicit isomorphism between $M_{T}^-(\mathbb{F})$ and $GL_{n_1}(\mathbb{F}) \times GL_{n_2}(\mathbb{F}) \times \cdots \times GL_{n_r}(\mathbb{F}) \times Sp_{2k}(\mathbb{F})$. As a by product of this isomorphism we obtain an explicit splitting of the Siegel parabolic subgroup. The main
ingredient of the proof of Lemma 2.4 is the existence of functions $\xi : F^* \to \{\pm 1\}$ with the property
\[ \xi(ab) = \xi(a)\xi(b)(a,b)_F. \]
See Lemma 2.6. In Section 2.5 we introduce the global metaplectic group. This group will appear again only in Chapter 9.

In Section 2.6 we describe explicitly the unique extension of Rao's cocycle to $GSp_{2n}(F)$. This extension is equivalent to a realization of the unique double cover of $GSp_{2n}(F)$ which extends the unique non-trivial double cover of $Sp_{2n}(F)$. For the rank 1 case, this theorem is proven in [32]. The main ingredient of this extension is a lift of an outer conjugation of $Sp_{2n}(F)$ by an element of $GSp_{2n}(F)$ to $\overline{Sp_{2n}(F)}$. A particular property of this lifting (see Corollary 2.1) will play a crucial role in Chapter 5 where we prove the uniqueness of Whittaker model.

In Chapter 3 we introduce $\gamma_\psi : F^* \to \{\pm 1, \pm i\}$, the normalized Weil factor of a character of second degree; see Theorem 2 of Section 14 of [69]. Since the complex case is trivial and since the real case is clear (see [40]) we mostly address the $p$-adic case. As may be expected, the main difficulty lies in 2-adic fields. We give some formulas (see Lemma 3.2) that we shall use in Section 8.1 where we compute the local coefficients for $SL_2(F)$ over $p$-adic fields. For the sake of completeness we explicitly compute all the Weil factors for $p$-adic fields of odd residual characteristic and for $Q_2$. It turns out that if $F$ is not 2-adic then $\gamma_\psi$ is not onto $\{\pm 1, \pm i\}$. The Weyl index defined on $Q_2$ is onto $\{\pm 1, \pm i\}$. Furthermore, if $F$ is a $p$-adic field of odd residual characteristic and if $-1 \in F^*2$ then $\gamma_\psi$ equals to one of the $\xi$ functions presented in Section 2.4.

Let $F$ be a $p$-adic field. In Chapter 4 we survey some facts from the existing and expected representation theory of $\overline{Sp_{2n}(F)}$ to be used in this dissertation. This theory is a straightforward generalization of the theory for algebraic groups. For $1 \leq i \leq r$ let $\tau_i$ be a smooth representation of $GL_{m_i}(F)$ and let $\pi$ be a genuine smooth representation of $Sp_{2n}(F)$. In Section 4.1 we construct a smooth genuine representation of $\overline{M}(F)$ from these representations. This is done, roughly speaking, by tensoring $\tau_1 \otimes \tau_2 \ldots \otimes \tau_r \otimes \pi$ with $\gamma_\psi$; see Lemma 4.1. Note that this is not quite the process used for general covering groups; see [1] for example. Next we define parabolic induction in an analogous way to the definition for algebraic groups. In Section 4.2 we give a rough condition for the irreducibility of unitary parabolic induction that follows from Bruhat theory. Namely, we explain which representations of $\overline{M}(F)$ are regular; see Theorem 4.2. In Section 4.3 we define the intertwining operator $A_w$ attached to a Weyl element $w$. This operator is the meromorphic continuation of a certain integral; see (1.10). Its definition and basic properties are similar to the analogous intertwining operator for algebraic groups. Chapter 4 culminates in Section 4.4 where we give the metaplectic analog to the Knapp-Stein dimension theorem; see [58] for the $p$-adic case. This Theorem describes the (dimension of) the commuting algebra of a parabolic induction via the properties of the meromorphic functions $\beta(\overline{s}, \tau_1, \ldots, \tau_r, \pi, w)$ defined by the relation
\[ A_w^{-1}A_w = \beta^{-1}(\overline{s}, \tau_1, \ldots, \tau_r, \pi, w)Id. \]
In more details, let $\pi$ be an irreducible admissible supercuspidal unitary genuine representation of $\overline{P}(F)$, let $W(\pi)$ be the subgroup of Weyl elements which preserve $\pi$, let $\Sigma_{P_{\pi}(F)}$ be the set of reflections corresponding to the roots of $TSp_{2n}(F)$ outside $M_{\pi}(F)$ and let $W'(\pi)$ be the subgroup of $W(\pi)$ generated by $w \in \Sigma_{P_{\pi}(F)} \cap W(\pi)$ which satisfies $\beta(\overline{s}, \pi, w) = 0$. 

8
denote by $I(\pi)$ the representation of $\overline{Sp_{2n}(F)}$ induced from $\pi$. The Knapp-Stein dimension theorem states that

$$\text{Dim}(\text{Hom}(I(\pi), I(\pi))) = [W(\pi) : W''(\pi)].$$

For connected reductive quasi split algebraic groups $\beta(\overline{\mathfrak{g}}, \pi, w)$ is closely related to the Plancherel measure; see Section 3 of the survey [54] for example. In Chapter 10 we shall compute this function in various cases.

Since $\overline{Sp_{2n}(F)}$ splits over unipotent subgroups of $Sp_{2n}(F)$ one can define a genuine $\psi$-Whittaker functionals attached to a smooth genuine representation $\pi$ of $\overline{Sp_{2n}(F)}$ in a closely related way to the definition in the linear case. Denote by $W_{\pi,\psi}$ the space of $\psi$-Whittaker functionals defined on $V_\pi$. In Theorem 5.1 of Chapter 5 we prove that if $\pi$ be an irreducible, admissible representation of $\overline{Sp_{2n}(F)}$. Then $\text{dim}(W_{\pi,\psi}) \leq 1$.

We emphasize that uniqueness of Whittaker model is not a general property of covering groups; see [15] for the failure of this uniqueness in the $GL_2(F)$ case. One of the key reasons that this uniqueness holds is that $T_{Sp_{2n}(F)}$ is commutative. We first explain why in the archimedean case the uniqueness proof is done exactly as in the linear case; see [22]. Then, in Section 5.1 we move to the non-archimedean case. Our method of proof is a method similar to the one in the linear case; see [57], [17] and [8]. In particular, we adopt the Gelfand-Kazhdan method to $\overline{Sp_{2n}(F)}$; see Theorem 5.4. We use there $\overline{h} \mapsto \tau_{\overline{h}}$, an involution on $\overline{Sp_{2n}(F)}$ which is a lift of $h \mapsto \tau h$, the involution used for the uniqueness proof in the symplectic case; see Lemma 5.1. It is somehow surprising to know that $\overline{\psi}$ extends $\tau$ in the simplest possible way, i.e., if $\overline{h} = (h, \epsilon)$ then $\overline{\tau} = (\tau, \epsilon)$. As mentioned before, the explicit computation and crucial properties of $\overline{\psi}$ follow from the results proven in Section 2.6. The main technical ingredient used for the uniqueness proof is Theorem 5.5. As indicated in the proof itself, provided that the relevant properties of $\overline{\psi}$ are proven, Theorem 5.5 is proved exactly as its linear analog which is Lemma 5.2.

Once the uniqueness of Whittaker model is established we define in Chapter 6 the metaplectic Langlands-Shahidi local coefficient

$$C^\psi_{Sp_{2n}(F)}(P_{\overline{T}(F)}, \overline{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma}, w)$$

in exactly the same way as in the linear case; see Theorem 3.1 of [48]. We note that the zeros of the local coefficient are among the poles $A_w$. Furthermore, since by definition

$$\beta(\overline{s}, \tau_1, \ldots, \tau_n, \overline{\sigma}, w) \equiv C^\psi_{Sp_{2n}(F)}(P_{\overline{T}(F)}, \overline{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma}, w) C^\psi_{Sp_{2n}(F)}(P_{\overline{T}(F)}, \overline{s}, (\otimes_{i=1}^r \tau_i) \otimes \overline{\sigma})^w, w^{-1}$$

the importance of the local coefficients for questions of irreducibility of parabolic induction is clear. Using the local coefficients we also define

$$\gamma(\overline{\sigma} \times \tau, s, \psi) = \frac{C^\psi_{Sp_{2n}(F)}(P_{\overline{m}\overline{k}(F)}, s, \tau \otimes \overline{\sigma}, j_{m,n}(\omega_{m-1})}{C^\psi_{Sp_{2n}(F)}(P_{\overline{m};0}(F), s, \tau, \omega_{m-1}^l)},$$

the $\gamma$-factor attached to $\overline{\sigma}$, an irreducible admissible genuine $\psi$-generic representation of $\overline{Sp_{2k}(F)}$, and $\tau$, an irreducible admissible generic representation of $GL_{m}(F)$; see [48]. Here $\omega_m$ and $j_{m,n}(\omega_{m-1}^l)$ are appropriate Weyl elements. This definition of the $\gamma$-factor is an
exact analog to the definition given in Section 6 of [53] for quasi-split connected reductive algebraic groups.

Most of Chapter 7 is devoted to the proof of the multiplicativity properties of $\gamma(\sigma \times \tau, s, \psi)$. See Theorems 7.1 and 7.2 of Section 7.1. It is the metaplectic analog to Part 3 of Theorem 3.15 of [53]. This multiplicativity is due to the multiplicativity of the local coefficients. The main ingredient of the proof of the multiplicativity is a certain decomposition of the intertwining operators; see Lemma 7.4. This decomposition resembles the decomposition of the intertwining operators in the linear case. The only (small) difference is that two Weyl elements may carry cocycle relations. Our choice of Weyl elements is such that these relations are non-trivial only in the field of real numbers and in 2-adic fields. In Lemma 7.4 of Section 7.2 we compute $\gamma(\sigma \times \tau, s, \psi)$ for principal series representations. Let $\eta_1, \eta_2, \ldots, \eta_k, \alpha_1, \alpha_2, \ldots, \alpha_m$ be $n$ characters of $\mathbb{F}^*$. If $\tau$ is induced from $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\sigma$ is induced from $\eta_1, \eta_2, \ldots, \eta_k$ (twisted by $\gamma^*$) then there exists $c \in \{\pm 1\}$ such that

$$
\gamma(\sigma \times \tau, s, \psi) = c \prod_{i=1}^k \prod_{j=1}^m \gamma(\alpha_j \times \eta_i^{-1}, s, \psi) \gamma(\eta_i \times \alpha_j, s, \psi).
$$

If $\mathbb{F}$ is a $p$-adic field of odd residual characteristic, and $\tau$ and $\sigma$ are unramified then $c = 1$. This computation is an immediate corollary of Theorems 7.1 and 7.2.

Assume that $\mathbb{F}$ is either $\mathbb{C}$, $\mathbb{R}$ or a $p$-adic field. In Sections 8.1, 8.2 and 8.3 of Chapter 8 we compute the local coefficients for principal series representations of $\overline{SL_2(\mathbb{F})}$. In this case $\chi$, the inducing representation is a character of $\mathbb{F}^*$ and there is only one non-trivial Weyl element. We prove that

$$
C_{\overline{SL_2(\mathbb{F})}}(\overline{B_{SL_2(\mathbb{F})}}, s, \chi, (0 \ 1 \ -1 \ 0)) = \epsilon'(\chi, s, \psi) \frac{\gamma(\chi^2, 2s, \psi)}{\gamma(\chi, s + \frac{1}{2}, \psi)},
$$

where $\epsilon'(\chi, s, \psi)$ is an exponential factor which equals 1 if $\chi$ is unramified, $\psi$ is normalized and $\mathbb{F}$ is $p$-adic field of odd residual characteristic. $\epsilon'(\chi, s, \psi)$ is computed explicitly for $p$-adic fields and for the field of real numbers; see Theorems 8.1, 8.2 and lemma 8.1. In this chapter only we write $C_{\psi_u}(\chi \otimes \gamma^{-1}_u, s)$ instead of $C_{\overline{SL_2(\mathbb{F})}}(\overline{B_{SL_2(\mathbb{F})}}, s, \chi, (0 \ 1 \ -1 \ 0))$, where $\psi_u(x) = \psi(ax)$. This notation emphasizes the dependence of the local coefficient on two additive characters, rather than on one in the algebraic case; one is the Whittaker character and the second is the character defining $\gamma^*$.

Section 8.1 is devoted to the $p$-adic case. In Section 8.1.1 we express the local coefficient as the following "Tate type" integral

$$
C_{\psi_u}(\chi \otimes \gamma^{-1}_u, s)^{-1} = \int_{\mathbb{F}^*} \gamma^{-1}_u(u) \chi(a) \psi_u(a) |u|^s d^* u.
$$

See Lemma 8.1 An immediate corollary of Lemma 8.1 is Lemma 8.3 which asserts that

$$
C_{\psi}(\chi \otimes \gamma^{-1}_\psi, s) \gamma(\psi(a) \chi(a) |a|^s = C_{\psi_u}(\chi \cdot (a, \cdot) \otimes \gamma^{-1}_\psi, s).
$$

In light of the computation of $C_{\psi}(\chi \otimes \gamma^{-1}_\psi, s)$ this means that the analytic properties of $C_{\psi_u}(\chi \otimes \gamma^{-1}_u, s)$ depend on $a$ if $\chi^2$ is unramified. This phenomenon has no analog in the linear case.
After proving some technical lemmas in Section 8.1.2 we compute the "Tate type" integral in Section 8.1.3. The difficult computations appear in the 2-adic case. The parameter that differentiates between 2-adic fields and p-adic fields of odd residual characteristic is $e(2, \mathbb{F})$ which is defined to be the ramification index of $\mathbb{F}$ over $\mathbb{Q}_2$ if $\mathbb{F}$ is a 2-adic field and 0 if $\mathbb{F}$ is of odd residual characteristic.

In Section 8.2 we compute the local coefficients for $SL_2(\mathbb{R})$. Since $SO_2(\mathbb{R})$ is commutative it follows from the Iwaswa decomposition that the $SO_2(\mathbb{R})$-types are one dimensional. This means that the intertwining operator maps a function of $\vartheta_n$ type to a multiple of a function of $\vartheta_n$ type. Thus, we may compute the local coefficients in a way similar to the computation of the local coefficients in the $SL_2(\mathbb{R})$ case. In fact, from a certain point we use Jacquet computations (see [24]) which ultimately use the results of Whittaker himself; see [70].

Section 8.3 which follows next is short and devoted to the complex case. Since $\gamma_\psi(\mathbb{C}^*) = 1$ and since $SL_2(\mathbb{C}) = SL_2(\mathbb{C}) \times \{\pm 1\}$ the local coefficients of this group are identical to the local coefficients of $SL_2(\mathbb{C})$. The $SL_2(\mathbb{C})$ computation is given in Theorem 3.13 of [51]. The surprising fact is that the $SL_2(\mathbb{C})$ computation agrees with the $SL_2(\mathbb{F})$ computations where $\mathbb{F}$ is either $\mathbb{R}$ or a p-adic field. This follows from the duplication formula of the classical $\Gamma$-function.

Section 8.4 is a detailed remark of our choice of parameterization, i.e., our choice to compute $C_\psi(\chi \otimes \gamma_\psi^{-1}, s)$ rather than $C_{\psi_0}(\chi \otimes \gamma_\psi^{-1}, s)$ for some other $a \in \mathbb{F}^*$ (for instance, in [7] $a = -1$ is used). We explain in what sense our parameterization is the correct one.

Let $\mathbb{F}$ be a p-adic field. In Section 8.5 we show that one can define a meromorphic function $\tilde{\gamma}(\chi, \psi, s)$ by a similar method to the one used by Tate (see [44]) replacing to role of the Fourier transform with $\phi \mapsto \tilde{\phi}$ defined on the space of Schwartz functions by

$$
\tilde{\phi}(x) = \int_{\mathbb{F}} \phi(y) \psi(xy) \gamma_\psi^{-1}(xy) dy.
$$

We Show that

$$
C_{SL_2(\mathbb{F})}^{SL_2(\mathbb{F})}(B_{SL_2(\mathbb{F})}, s, \chi, (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}), -1) = \tilde{\gamma}(\chi^{-1}, \psi^{-1}, 1 - s).
$$

This is an analog to the well known equality

$$
C_{SL_2(\mathbb{F})}^{SL_2(\mathbb{F})}(B_{SL_2(\mathbb{F})}, s, \chi, (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}), -1) = \gamma(\chi^{-1}, \psi^{-1}, 1 - s).
$$

We conclude this chapter in Section 8.6 where we prove that if $\tau$ is a principal series representation of $GL_m(\mathbb{F})$ then there exists an exponential function $c(s)$ such that

$$
C_{PSL_2(\mathbb{F})}^{PSL_2(\mathbb{F})}(B_{PSL_2(\mathbb{F})}, s, \tau, \pm_\omega^{-1}) = c(s) \gamma(\tau, \pm_\omega^{-1}, 2s, \psi) \gamma(\tau, s + \frac{1}{2}, \psi).
$$

If $\mathbb{F}$ is a p-adic field of odd residual characteristic, $\psi$ is normalized and $\tau$ is unramified then $c(s) = 1$. This computation uses the $SL_2(\mathbb{F})$ computations given in sections 8.1 8.2 and 8.3, and the multiplicativity of the local coefficients proven is Section 7.1, see Theorem 8.1.

In Chapter 9 we prove a global functional equation satisfied by $\gamma(\sigma \times \tau, s, \psi)$. See Theorem 9.1 in Section 9.2. This theorem is the metaplectic analog to the crude functional
equation proven by Shahidi in Theorem 4.1 of [48] (see also Part 4 of Theorem 3.15 of [53]).

The proof Theorem 9.1 requires some local computations of spherical Whittaker functions; see Section 9.1. These computations use the work of Bump, Friedberg and Hoffstein; see [7]. As a corollary of the Crude functional equation and the general results presented in [52] we prove in Section 9.3 that if $\tau$ is an irreducible admissible generic representation of $GL_m(F)$ then (0.1) holds.

In Chapter 10 we use the results of Chapters 7-9 combined with the Knapp-Stein dimension Theorem presented in Chapter 4 to prove certain irreducibility theorems for parabolic induction on the metaplectic group over p-adic fields. We assume that the inducing representations are unitary. In Theorem 10.1 of Section 10.1 we prove via the computation of the local coefficients the irreducibility of principal series representations of $Sp_{2n}(F)$ induced from unitary characters. The method used in Section 10.1 is generalized in Section 10.2 where we prove the following.

Let $n_1, n_2, \ldots, n_r, k$ be $r + 1$ non-negative integers whose sum is $n$. For $1 \leq i \leq r$ let $\tau_i$ be an irreducible admissible supercuspidal unitary representation of $GL_{n_i}(F)$ and let $\pi$ be an irreducible admissible supercuspidal genuine $\psi$-generic representation of $Sp_{2k}(F)$. Denote $\pi = (\otimes_{i=1}^{r}(\gamma^{-1} \psi \otimes \tau_i)) \otimes \sigma$ and $I(\pi) = Ind_{P_{n,k}(F)}^{Sp_{2n}(F)} \pi$. Then $I(\pi)$ is reducible if and only if there exists $1 \leq i \leq r$ such that $\tau_i$ is self dual and

$$\gamma(\pi \times \tau_i, 0, \psi) \gamma(\tau_i, sym^2, 0, \psi) \neq 0.$$ 

Denote now $\pi_i = (\gamma^{-1} \otimes \tau_i) \otimes \sigma$ and $I(\pi_i) = Ind_{P_{n_i,k}(F)}^{Sp_{2n_i+k}(F)} \pi_i$. An immediate corollary of the last theorem is that $I(\pi)$ is irreducible if and only if $I(\pi_i)$ is irreducible for each $1 \leq i \leq r$.

Let $SO_{2n+1}(F)$ be a split odd orthogonal group and let $P_{SO_{2n+1}}(F)$ be a parabolic subgroup of $SO_{2n+1}(F)$ whose Levi part is isomorphic to $GL_n(F)$. In Lemma 10.2 of Section 10.3 we prove that if $\tau$ is an irreducible admissible generic representation of $GL_n(F)$ then $\beta(s, \tau, \omega_n^{-1})$ has the same analytic properties as the Plancherel measure attached to $SO_{2n+1}(F)$, $P_{SO_{2n+1}}(F)$ and $\tau$. An immediate Corollary is Theorem 10.3 which states the following.

Let $\tau$ be an irreducible admissible self dual supercuspidal representation of $GL_n(F)$. Then, $Ind_{P_{n_0,0}(F)}^{Sp_{2n_0}(F)}(\gamma^{-1} \otimes \tau)$ is irreducible if and only if $Ind_{P_{SO_{2n+1}}(F)}^{SO_{2n+1}(F)} \tau$ is irreducible.

We then list a few corollaries that follow from Theorem 10.3 and from Shahidi’s work, [55].

Some of the results presented in this dissertation have already been published. The main results of Section 2.6 and Chapter 5 appeared in [64]. The main results of Sections 8.1 and 8.2 were published in [65]. Theorems 10.1 and 10.3 along with their proofs were outlined in the introduction of [65]. The main results of Chapters 7, 9 and 10 were presented in Seminar talks in Tel-Aviv University (September 2008), Ohio-State University (October 2008) and Purdue University (October 2008).
1 General notations

Assume that \( F \) either the field of real numbers or a finite extensions of \( \mathbb{Q}_p \). \( F \) is self dual: If \( \psi \) is a non-trivial (complex) character of \( F \), any non-trivial character of \( F \) has the form \( \psi_a(x) = \psi(ax) \) for a unique \( a \in F^* \). If \( F = \mathbb{R} \) we define \( \psi(x) = e^{ix} \).

Let \( F \) be a \( p \)-adic field. Let \( \mathcal{O}_F \) be the ring of integers of \( F \) and let \( \mathfrak{P}_F \) be its maximal ideal. Let \( q \) be the cardinality of the residue field \( \overline{F} = \mathcal{O}_F / \mathfrak{P}_F \). For \( b \in \mathcal{O}_F \) denote by \( \overline{b} \) its image in \( \overline{F} \). Fix \( \pi \), a generator of \( \mathfrak{P}_F \). Let \( \| \cdot \| \) be the absolute value on \( F \) normalized in the usual way: \( \| \pi \| = q^{-1} \). Normalize the Haar measure on \( F \) such that \( \mu(\mathcal{O}_F) = 1 \) and normalize the Haar measure on \( F^* \) such that \( d^*x = \frac{dx}{|x|} \). Define

\[
e(2, F) = \log_q[\mathcal{O}_F : 2\mathcal{O}_F] = -\log_q|2|.
\]

Note that if the residual characteristic of \( F \) is odd then \( e(2, F) = 0 \), while if the characteristic of \( F \) is 2, \( e(2, F) \) is the ramification index of \( F \) over \( \mathbb{Q}_2 \). We shall write \( e \) instead of \( e(2, F) \), suppressing its dependence on \( F \). We also define \( \omega = 2\pi^{-e} \in \mathcal{O}_F^\times \).

We shall often be interested in subgroups of \( \mathcal{O}_F^\times \) of the form \( 1 + \mathfrak{P}_F^n \), where \( n \) is a positive integer. By abuse of notation we write \( 1 + \mathfrak{P}_F^0 = \mathcal{O}_F^\times \). For \( \chi \), a character of \( F^* \), we denote by \( m(\chi) \) its conductor, i.e, the minimal integer such that \( \chi(1 + \mathfrak{P}_F^n) = 1 \). Note that if \( \chi \) is unramified then \( m(\chi) = 0 \). Let \( \psi \) be a non-trivial additive character of \( F \). The conductor of \( \psi \) is the minimal integer \( n \) such that \( \psi(\mathfrak{P}_F^n) = 1 \). \( \psi \) is said to be normalized if its conductor is 0. Assuming that \( \psi \) is normalized, the conductor of \( \psi_a \) is \( \log_q|a^{-1}| \). If the conductor of \( \psi \) is \( n \), we define \( \psi_0(x) = \psi(x \pi^n) \). \( \psi_0 \) is clearly normalized.

For any field (of characteristic different then 2) we define \((\cdot, \cdot)_F\) to be the quadratic Hilbert symbol of \( F \). The Hilbert symbol defines a non-degenerate bilinear form on \( F^* / F^* 2 \).

For future references we recall some of the properties of the Hilbert symbol:

\[
1. (a, -a)_F = 1 \quad 2. (aa', b)_F = (a, b)_F(a', b)_F \quad 3. (a, b)_F = (a, -ab)_F.
\]
2 The metaplectic group

Let $\mathbb{F}$ be a local field of characteristic 0. In this chapter we introduce $Sp_{2n}(\mathbb{F})$, the metaplectic group which is the unique non-trivial double cover of the symplectic group and describe its basic properties. Through this dissertation, we realize the metaplectic group via Rao’s cocycle, [42]. In Section 2.1 we list several notations and standard facts about the symplectic group and its subgroups. In Section 2.2 we introduce Rao’s cocycle and list several of its properties. In Section 2.3 we describe Kubota’s cocycle which is Rao’s cocycle for the $SL_2(\mathbb{F})$ case. Some of the properties of $Sp_{2n}(\mathbb{F})$ may be proven using the properties of $SL_2(\mathbb{F})$. Next, in Section 2.4 we describe the properties of parabolic subgroups of $Sp_{2n}(\mathbb{F})$, which are defined to be the inverse images in $Sp_{2n}(\mathbb{F})$ of parabolic subgroups of $Sp_{2n}(\mathbb{F})$. We show that these groups have an exact analog to the Levi decomposition. For $p$-adic fields of odd residual characteristic we give an explicit interesting isomorphism between a Levi subgroup and and $GL_{n_1}(\mathbb{F}) \times GL_{n_2}(\mathbb{F}) \cdots \times GL_{n_r}(\mathbb{F}) \times Sp_{2k}(\mathbb{F})$.

We do so by proving the existence of functions $\xi : \mathbb{F}^* \to \{\pm 1\}$ with the property $\xi(ab) = \xi(a)\xi(b)(a,b)_\mathbb{F}$. In Section 2.5 we define the adelic metaplectic group which is the unique non-trivial double cover of the adelic symplectic group. We conclude this chapter in Section 2.6 where we explicitly compute the unique extension of Rao’s cocycle to $GSp_{2n}(\mathbb{F})$. We use this extension to give a realization of the unique double cover of $GSp_{2n}(\mathbb{F})$ which extends the unique non-trivial double cover of $Sp_{2n}(\mathbb{F})$. In Chapter 5 we use this extension to prove certain properties of an involution on $Sp_{2n}(\mathbb{F})$ which plays a crucial role in the proof of the uniqueness of Whittaker model for irreducible admissible representations of $Sp_{2n}(\mathbb{F})$; see Lemma [5.1]

2.1 The symplectic group

Let $\mathbb{F}$ be a field of characteristic different then 2. Let $X = \mathbb{F}^{2n}$ be a vector space of even dimension over $\mathbb{F}$ equipped with $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$, a non degenerate symplectic form and let $Sp(X) = Sp_{2n}(\mathbb{F})$ be the subgroup of $GL(X)$ of isomorphisms of $X$ onto itself which preserve $\langle \cdot, \cdot \rangle$. Following Rao, [42], we shall write the action of $GL(X)$ on $X$ from the right. Let

$$E = \{e_1, e_2, \ldots, e_n, e_1^*, e_2^*, \ldots, e_n^*\}$$

be a symplectic basis of $X$; for $1 \leq i, j \leq n$ we have $\langle e_i, e_j \rangle = \langle e_i^*, e_j^* \rangle = 0$ and $\langle e_i, e_j^* \rangle = \delta_{i,j}$. In this base $Sp(X)$ is realized as the group

$$\{a \in GL_{2n}(\mathbb{F}) \mid aJ_{2n}a^t = J_{2n}\},$$

where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Through this dissertation we shall identify $Sp_{2n}(\mathbb{F})$ with this realization. For $0 \leq r \leq n$ define $i_{r,n}$ to be an embedding of $Sp_{2r}(\mathbb{F})$ in $Sp_{2n}(\mathbb{F})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} I_{n-r} & a \\ c & I_{n-r} \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
where \( a, b, c, d \in \text{Mat}_{r \times r}(\mathbb{F}) \), and define \( f_{r,n} \) to be an embedding of \( Sp_{2r}(\mathbb{F}) \) in \( Sp_{2n}(\mathbb{F}) \) by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ I_{n-r} & c \\ & & d \\ & & & I_{n-r} \end{pmatrix}.
\]

Let \( T_{GL_n}(\mathbb{F}) \) be the subgroup of diagonal elements of \( GL_n(\mathbb{F}) \), let \( Z_{GL_n}(\mathbb{F}) \) be the group of upper triangular unipotent matrices in \( GL_n(\mathbb{F}) \) and let \( B_{GL_n}(\mathbb{F}) = T_{GL_n}(\mathbb{F}) \times Z_{GL_n}(\mathbb{F}) \) be the standard Borel subgroup of \( GL_n(\mathbb{F}) \). Let \( T_{Sp_{2n}}(\mathbb{F}) \) be the subgroup of diagonal elements of \( Sp_{2n}(\mathbb{F}) \) and let \( Z_{Sp_{2n}}(\mathbb{F}) \) be the following maximal unipotent subgroup of \( Sp_{2n}(\mathbb{F}) \);

\[
\left\{ \begin{pmatrix} z & b \\ 0 & \tilde{z} \end{pmatrix} \mid z \in Z_{GL_n}(\mathbb{F}), b \in \text{Mat}_{n \times n}(\mathbb{F}), \ b' = z^{-1}b\tilde{z}^{-1} \right\},
\]

where for \( a \in GL_n \) we define \( \tilde{a} = {}^t a^{-1} \). The subgroup \( B_{Sp_{2n}}(\mathbb{F}) = T_{Sp_{2n}}(\mathbb{F}) \times Z_{Sp_{2n}}(\mathbb{F}) \) of \( Sp_{2n}(\mathbb{F}) \) is a Borel subgroup. We call it the standard Borel subgroup. A standard Levi subgroup (unipotent radical) is a Levi part (unipotent radical) of a standard parabolic subgroup. In particular a standard Levi subgroup contains \( T_{Sp_{2n}}(\mathbb{F}) \) and a standard unipotent radical is contained in \( Z_{Sp_{2n}}(\mathbb{F}) \).

Let \( n_1, n_2, \ldots, n_r, k \) be \( r+1 \) nonnegative integers whose sum is \( n \). Put \( \overrightarrow{t} = (n_1, n_2, \ldots, n_r; k) \). Let \( M_{\overrightarrow{t}} \) be the standard Levi subgroup of \( Sp_{2n}(\mathbb{F}) \) which consists of elements of the form

\[
[g_1, g_2, \ldots, g_r, h] = \text{diag}(g_1, g_2, \ldots, g_r, I_{k}, \tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_r, I_{k})i_{k,n}(h),
\]

where \( g_i \in GL_{n_i}(\mathbb{F}), h \in Sp_{2k}(\mathbb{F}) \). When convenient we shall identify \( GL_{n_i}(\mathbb{F}) \) with its natural embedding in \( M_{\overrightarrow{t}}(\mathbb{F}) \). Denote by \( P_{\overrightarrow{t}}(\mathbb{F}) \) the standard parabolic subgroup of \( Sp_{2n}(\mathbb{F}) \) that contains \( M_{\overrightarrow{t}}(\mathbb{F}) \) as its Levi part. Denote by \( N_{\overrightarrow{t}}(\mathbb{F}) \) the unipotent radical of \( P_{\overrightarrow{t}}(\mathbb{F}) \). We denote by \( P_{Sp_{2n}}(\mathbb{F}) \) or simply by \( P(\mathbb{F}) \), the Siegel parabolic subgroup of \( Sp_{2n}(\mathbb{F}) \):

\[
P_{(n,0)}(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ 0 & \tilde{a} \end{pmatrix} \mid a \in GL_n(\mathbb{F}), b \in \text{Mat}_{n \times n}(\mathbb{F}), b' = a^{-1}ba' \right\}.
\]

Note that \( M_{(n,0)}(\mathbb{F}) \cong GL_n(\mathbb{F}) \). A natural isomorphism is given by

\[
g \mapsto \hat{g} = \begin{pmatrix} g \\ \end{pmatrix}.
\]

Define

\[
V = \text{span}\{e_1, e_2, \ldots, e_n\}, \quad V^* = \text{span}\{e_1^*, e_2^*, \ldots, e_n^*\}.
\]

These are two transversal Lagrangian subspaces of \( X \). The Siegel parabolic subgroup is the subgroup of \( Sp(X) \) which consists of elements that preserve \( V^* \). Let \( S \) be a subset of \( \{1, 2, \ldots, n\} \). Define \( \tau_S, a_S \) to be the following elements of \( Sp(X) \):

\[
e_i \cdot \tau_S = \begin{cases} -e_i^* & i \in S \\ e_i & \text{otherwise} \end{cases}, \quad e_i^* \cdot \tau_S = \begin{cases} e_i & i \in S \\ e_i^* & \text{otherwise} \end{cases},
\]

\[
e_i \cdot a_S = \begin{cases} -e_i & i \in S \\ e_i & \text{otherwise} \end{cases}, \quad e_i^* \cdot a_S = \begin{cases} -e_i^* & i \in S \\ e_i^* & \text{otherwise} \end{cases}.
\]
The elements $\tau_{S_1}, a_{S_1}, \tau_{S_2}, a_{S_2}$ commute. Note that $a_S \in P(\mathbb{F})$, $a_S^2 = I_{2n}$, and that

$$\tau_{S_1} \tau_{S_2} = \tau_{S_1 \triangle S_2} a_{S_1 \cap S_2}, \tag{2.4}$$

where $S_1 \triangle S_2 = S_1 \cup S_2 \setminus S_1 \cap S_2$. In particular $\tau_S^2 = a_S$. For $S = \{1, 2, \ldots, n\}$ we define $\tau = \tau_S$, in this case $a_S = -I_{2n}$.

Denote by $W'_{Sp_{2n}}(\mathbb{F})$ the subgroup of $Sp_{2n}(\mathbb{Z})$ generated by the elements $\tau_S$, and $\hat{w}_\pi$, where $S \subseteq \{1, 2, \ldots, n\}$, and $w_\pi \in GL_n(\mathbb{F})$ is defined by $w_{\pi i,j} = \delta_{\pi(i),j}; \pi$ is a permutation in $S_n$. If $\mathbb{F}$ is a p-adic field then $W'_{Sp_{2n}}(\mathbb{F})$ is a subgroup of $Sp_{2n}(\mathbb{F})$. Note that $W'_{Sp_{2n}}(\mathbb{F})$ modulo its diagonal elements may be identified with the Weyl group of $Sp_{2n}(\mathbb{F})$ denoted by $W_{Sp_{2n}}(\mathbb{F})$. Define $W_{PT}(\mathbb{F})$ to be the subgroup of $W'_{Sp_{2n}}(\mathbb{F})$ which consists of elements $w$ such that

$$M_{PT}(\mathbb{F})^w = wM_{PT}(\mathbb{F})w^{-1}$$

is a standard Levi subgroup and

$$w(Z_{Sp_{2n}}(\mathbb{F}) \cap M_{PT}(\mathbb{F}))w^{-1} \subset Z_{Sp_{2n}}(\mathbb{F}).$$

This means that up to conjugation by diagonal elements inside the blocks of $M_{PT}(\mathbb{F})$, we have

$$w[g_{1, g_2, \ldots, g_r}, s]w^{-1} = [g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(r)}, s], \tag{2.5}$$

where $\pi$ is a permutation of $\{1, 2, \ldots, r\}$, and where for $g \in GL_n(\mathbb{F}), \epsilon = \pm 1$ we define

$$g^{(\epsilon)} = \begin{cases} g & \epsilon = 1 \\ \omega_n g \omega_n & \epsilon = -1 \end{cases},$$

where

$$\omega_n = \begin{pmatrix} 1 & 1 & \cdots \\ -1 & 1 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$ 

We may assume, and in fact do, that $W_{PT}(\mathbb{F})$ commutes with $i_{k,n}(Sp_{2k}(\mathbb{F}))$.

For $w \in W_{PT}(\mathbb{F})$, let $P_{PT}(\mathbb{F})^w$ be the standard parabolic subgroup whose Levi part is $M_{PT}(\mathbb{F})^w$, and let $N_{PT}(\mathbb{F})^w$ be its standard unipotent radical.

### 2.2 Rao’s cocycle

In [42], Rao constructs an explicit non-trivial 2-cocycle $c(\cdot, \cdot)$ on $Sp(X)$ which takes values in $\{\pm 1\}$. The set $Sp(X) = Sp(X) \times \{\pm 1\}$ is then given a group structure via the formula

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)). \tag{2.6}$$

It is called the metaplectic group. For any subset $A$ of $Sp_{2n}(\mathbb{F})$ we denote by $\overline{A}$ its inverse image in $Sp_{2n}(\mathbb{F})$. If $\mathbb{F}$ is either $\mathbb{R}$ or a p-adic field, this group is the unique non-trivial double cover of $Sp_{2n}(\mathbb{F})$. It is well known that other classical groups over local fields have more then one non-trivial double cover. For example, on $GSp_{2n}(\mathbb{F})$ one may define a double cover by the multiplication law

$$(g, \epsilon_1)(h, \epsilon_2) = (gh, \epsilon_1 \epsilon_2 (\lambda_g, \lambda_h)),$$

where $\lambda_g \in GSp_{2n}(\mathbb{F})$.
where λ₂ is the similitude factor of g. One may also define a double cover via the construction given in Section 2.6. The corresponding cocycles are clearly inequivalent. The first is trivial on $Sp_{2n}(\mathbb{F}) \times Sp_{2n}(\mathbb{F})$ while the second is an extension of Rao’s cocycle.

We now describe Rao’s cocycle. Detailed proofs can be found in [42]. Define

$$\Omega_j = \{ \sigma \in Sp(X) \mid \dim(V^* \cap V^*\sigma) = n - j \}. $$

Note that $P(\mathbb{F}) = \Omega_0$, $\tau_S \in \Omega_{|S|}$ and more generally, if $\alpha, \beta, \gamma, \delta \in Mat_{n \times n}(\mathbb{F})$ and $\sigma = \begin{pmatrix} a & b \\ \gamma & \delta \end{pmatrix} \in Sp(X)$ then $\sigma \in \Omega_{\text{rank}(\gamma)}$. The Bruhat decomposition states that each $\Omega_j$ is a single double coset in $P(\mathbb{F}) \backslash Sp(X) / P(\mathbb{F})$, that $\Omega_j^{-1} = \Omega_j$ and that $\bigcup_{j=0}^{n} \Omega_j = Sp(X)$. In particular, every element of $Sp(X)$ has the form $p\tau_S p'$, where $p, p' \in P(\mathbb{F})$, $S \subseteq \{1, 2, \ldots, n\}$.

Let $p_1, p_2 \in P(\mathbb{F})$. Rao defines

$$x(p_1\tau_S p_2) \equiv \det(p_1 p_2 |_{V^*})(\text{mod}(\mathbb{F}^*)^2), \quad (2.7)$$

and proves that it is a well defined map from $Sp(X)$ to $\mathbb{F}^*/(\mathbb{F}^*)^2$. Note that $x(a_S) \equiv (-1)^{|S|}$. More generally, if $p = \begin{pmatrix} a & b \\ 0 & \bar{a} \end{pmatrix} \in P(\mathbb{F})$ then $x(p) \equiv \det(a)$. We shall use the notation $\det(p) = \det(a)$. Also note that $x(\tau_S) \equiv 1$ and that for $g \in \Omega_j$, $p_1, p_2 \in P(\mathbb{F})$,

$$x(g^{-1}) \equiv x(g)(-1)^j, \quad x(p_1 p_2 g) \equiv x(p_1) x(g) x(p_2). \quad (2.8)$$

Theorem 5.3 in Rao’s paper states that a non-trivial 2-cocycle on $Sp(X)$ can be defined by

$$c(\sigma_1, \sigma_2) = (x(\sigma_1), x(\sigma_2)) \rho(-x(\sigma_1) x(\sigma_2), x(\sigma_1 \sigma_2)) \rho((-1)^d, d_F(\rho)) \rho(-1, -1)^{(l-1)} \rho^l h_F(\rho), \quad (2.9)$$

where $\rho$ is the Leray invariant $q(V^*, V^* \sigma_1, V^* \sigma_2^{-1})$, $d_F(\rho)$ and $h_F(\rho)$ are its discriminant, and Hasse invariant, and $2l = j_1 + j_2 - j - \dim(V)$, where $\sigma_1 \in \Omega_{j_1}$, $\sigma_2 \in \Omega_{j_2}$, $\sigma_1 \sigma_2 \in \Omega_j$. We use Rao’s normalization of the Hasse invariant. (Note that the cocycle formula just given differs slightly from the one that appears in Rao’s paper. There is a small mistake in Theorem 5.3 of [42]. A correction by Adams can be found in [33], Theorem 3.1).

An immediate consequence of Rao’s formula is that if $g$ and $h$ commute in $Sp_{2n}(\mathbb{F})$ then their pre-image in $Sp_{2n}(\mathbb{F})$ also commute (this may be deduced from more general ideas. See page 39 of [38]). In particular, a preimages in $Sp_{2n}(\mathbb{F})$ of a commutative subgroup of $Sp_{2n}(\mathbb{F})$ is also commutative. This does not hold for general covering groups; for example, using the cocycle constructed in Section 2.6, the reader may check that the inverse image of the diagonal subgroup of $GSp_{2n}(\mathbb{F})$ in $GSp_{2n}(\mathbb{F})$ is not commutative. See [1] for the same phenomenon in the $n$-fold cover of $GL_n(\mathbb{F})$.

If $\mathbb{F}$ is a local field then $Sp_{2n}(\mathbb{F})$ is a locally compact group. If $\mathbb{F}$ is a $p$-adic field, $Sp_{2n}(\mathbb{F})$ is an $l$-group in the sense of [8]; since $c(\cdot, \cdot)$ is continuous, it follows that there exists $U$, an open compact subgroup of $Sp_{2n}(\mathbb{F})$, such that $c(U, U) = 1$. Thus, a system of neighborhoods of $(I_{2n}, 1)$ is given by open compact subgroups of the form $(V, 1)$, where $V \subseteq U$ is an open compact subgroup of $Sp_{2n}(\mathbb{F})$. As an example we describe it explicitly in the $SL_2(\mathbb{F})$ case; see Lemma 2.2.
From (2.10) and from previous remarks we obtain the following properties of \( c(\cdot, \cdot) \); for \( \sigma, \sigma' \in \Omega_j \), \( p, p' \in P(\mathbb{F}) \) we have

\[
\begin{align*}
  c(\sigma, \sigma^{-1}) &= \left( x(\sigma), (-1)^{j} x(\sigma) \right) \mathbb{F} (-1, -1)^{\frac{j(j-1)}{2}} \mathbb{F} \quad \text{(2.10)} \\
  c(p, \sigma') &= c(\sigma, \sigma')(x(p), x(\sigma)) \mathbb{F} (x(p'), x(\sigma')) \mathbb{F} (x(p), x(\sigma')) \mathbb{F} \quad \text{(2.11)}
\end{align*}
\]

As a consequence of (2.11) we obtain

\[
c(p, \sigma) = c(\sigma, p) = \left( x(p), x(\sigma) \right) \mathbb{F} \quad \text{(2.12)}
\]

Another property of the cocycle noted in [42] is that

\[
c(\tau_{S_1}, \tau_{S_2}) = (-1, -1)^{\frac{j(j+1)}{2}} \mathbb{F} \quad \text{(2.13)}
\]

where \( j \) is the cardinality of \( S_1 \cap S_2 \). From (2.12), (2.11) and (2.13) we conclude that if \( S \) and \( S' \) are disjoint then for \( p, p' \in P(\mathbb{F}) \) we have

\[
c(p \tau_{S}, \tau_{S'} p') = (x(p), x(p')) \mathbb{F} \quad \text{(2.14)}
\]

It follows from (2.12) and (1.1) that

\[
(p, \epsilon_1)(\sigma, \epsilon)(p, \epsilon_1)^{-1} = (p \sigma^{-1}, \epsilon) \quad \text{(2.15)}
\]

for all \( \sigma \in Sp(X), p \in P(\mathbb{F}), \epsilon_1, \epsilon \in \{\pm 1\} \). Furthermore, assume that \( p \in P(\mathbb{F}), \sigma \in Sp(X) \) satisfy \( p \sigma^{-1} \in P(\mathbb{F}) \). Then

\[
(\sigma, \epsilon_1)(p, \epsilon) \sigma^{-1}, \epsilon \sigma^{-1} = (p, \epsilon) \sigma^{-1}, \epsilon \sigma^{-1} \quad \text{(2.16)}
\]

Indeed, due to (2.15) and the Bruhat decomposition we only need to show that if \( \tau_{S} p \tau_{S}^{-1} \in P(\mathbb{F}) \) then

\[
c(p, \tau_{S}) c(\tau_{S} p, \tau_{S}^{-1}) c(\tau_{S}, \tau_{S}^{-1}) = 1 \quad \text{(2.17)}
\]

Define \( j \) to be the cardinality of \( S \). From (2.10) it follows that \( c(\tau_{S}, \tau_{S}^{-1}) = (-1, -1)^{\frac{j(j-1)}{2}} \mathbb{F} \). From (2.12) it follows that \( c(p, \tau_{S}) = 1 \). It is left to show that if \( V^* \tau_{S}^{-1} p \tau_{S} = V^* \) then

\[
c(\tau_{S} p, \tau_{S}^{-1}) = (-1, -1)^{\frac{j(j+1)}{2}} \mathbb{F} \quad \text{(2.17)}
\]

Recall that the Leray invariant is stable under the action of \( Sp(X) \) on Lagrangian triplets; see Theorem 2.11 of [42]. Therefore,

\[
q(V^*, V^* \tau_{S} p, V^* \tau_{S}) = q(V^* \tau_{S}^{-1}, V^* \tau_{S} p \tau_{S}^{-1}, V^*) = q(V^* \tau_{S}^{-1}, V^*, V^*)
\]

is an inner product defined on the trivial space. (2.10) implies now (2.17).

We recall Corollary 5.6 in Rao’s paper. For \( S \subset \{1, 2, \ldots, n\} \) define

\[
X_S = \text{span}\{\epsilon_i, \epsilon_i^* \mid i \in S\}.
\]

We may now consider \( x_S \) and \( c_{X_S}(\cdot, \cdot) \) defined by analogy with \( x \) and \( c(\cdot, \cdot) \). Let \( S_1 \) and \( S_2 \) be a partition of \( \{1, 2, \ldots, n\} \). Suppose that \( \sigma_1, \sigma'_1 \in Sp(X_{S_1}) \) and that \( \sigma_2, \sigma'_2 \in Sp(X_{S_2}) \). Put \( \sigma = \text{diag}(\sigma_1, \sigma_2) \), \( \sigma' = \text{diag}(\sigma'_1, \sigma'_2) \). Rao proves that \( c(\sigma, \sigma') \) equals

\[
c_{S_1}(\sigma_1, \sigma'_1) c_{S_2}(\sigma_2, \sigma'_2) (x_{S_1}(\sigma_1), x_{S_2}(\sigma_2)) \mathbb{F} (x_{S_1}(\sigma'_1), x_{S_2}(\sigma'_2)) \mathbb{F} \quad \text{(2.18)}
\]
From (2.18) it follows that \((s, \epsilon) \mapsto (i_{r,n}(s), \epsilon)\) and \((s, \epsilon) \mapsto (j_{r,n}(s), \epsilon)\) are two embeddings of \(Sp_{2r}(F)\) in \(Sp_{2n}(F)\). We shall continue to denote these embeddings by \(i_{r,n}\) and \(j_{r,n}\) respectively. Note that the map \(g \mapsto (\hat{g}, 1)\) is not an embedding of \(GL_n(F)\) in \(Sp_{2n}(F)\), although, by (2.12), its restriction to \(Z_{GL_n(F)}\) is an embedding.

2.3 Kubota’s cocycle

For \(n=2\) Rao’s cocycle reduces to Kubota’s cocycle, [32]:

\[
c(g_1, g_2) = \left( x(g_1), x(g_2) \right)_F \left( -x(g_1)x(g_2), x(g_1g_2) \right)_F, \tag{2.19}
\]

where

\[
x \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \frac{c}{\sqrt{c}} & c \neq 0 \\ \frac{d}{\sqrt{c}} & c = 0 \end{cases}.
\]

For \(F\), a p-adic field of odd residual characteristic it is known (see page 58 of [36]) that \(SL_2(F)\) splits over \(SL_2(\mathbb{O}_F)\), the standard maximal compact subgroup of \(SL_2(F)\) and that \(\iota_2 : SL_2(\mathbb{O}_F) \to \{\pm 1\}\) defined by

\[
\iota_2 \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} (c, d) & 0 < c < 1 \\ 1 & \text{otherwise} \end{cases}
\]

is the unique map such that the map

\[
k \mapsto \kappa_2(k) = (k, \iota_2(k))
\]

is an embedding of \(SL_2(\mathbb{O}_F)\) in \(SL_2(F)\). More generally, it is known, see [36], that if \(F\) is a p-adic field of odd residual characteristic then \(Sp_{2n}(F)\) splits over \(Sp_{2n}(\mathbb{O}_F)\); there exists a map

\[
\iota_{2n} : Sp_{2n}(\mathbb{O}_F) \to \{\pm 1\}
\]

such that the map

\[
k \mapsto \kappa_{2n}(k) = (k, \iota_{2n}(k))
\]

is an embedding of \(Sp_{2n}(\mathbb{O}_F)\) in \(Sp_{2n}(F)\). Since \(\kappa_2\) is the unique splitting of \(SL_2(\mathbb{O}_F)\) in \(SL_2(F)\) and since \(Sp_{2n}(\mathbb{O}_F)\) is generated by various embedding of \(SL_2(\mathbb{O}_F)\) it follows that \(\iota_{2n}\) is also unique.

**Lemma 2.1.** The restrictions of \(\iota_{2n}\) to \(P(F) \cap Sp_{2n}(\mathbb{O}_F)\) and to \(W'_{Sp_{2n}(F)}\) are trivial.

**Proof.** Since for odd residue characteristic \((\mathbb{O}_F^*, \mathbb{O}_F^*)_F = 1\) (see Lemma 3.3 for this well known fact), we conclude, using (2.12), that \(\iota_{2n}\) restricted to \(P(F) \cap Sp_{2n}(\mathbb{O}_F)\) is a quadratic character and hence has the form \(p \mapsto \chi(\det p)\), where \(\chi\) is a quadratic character of \(\mathbb{O}_F^*\). By the inductivity property of Rao’s cocycle and by the formula of \(\iota_2\) we conclude that

\[
\iota_{2n}(i_{1,n}(SL_2(\mathbb{O}_F))) \cap P(F) = 1.
\]

Thus \(\chi = 1\). We now move to the second assertion. We note that the group generated by the elements of the form \(\tau_S\) is the group of elements of the form \(\tau_{S_1}a_{S_2}\), where \(S, S_1, S_2 \subseteq \mathbb{O}_F\).
\{1, 2, \ldots, n\}. The group \{\tilde{w}_\pi | \pi \in S_n\} is disjoint from that group and normalizes it. Hence we need only to show that for all \(S_1, S_2 \subseteq \{1, 2, \ldots, n\}\), and for all \(\pi \in S_n\)
\[\nu_{2n}(\tilde{w}_\pi a_{S_1} \tau_{S_2}) = 1.\]
The fact that \(\nu_{2n}(\tilde{w}_\pi a_{S_2}) = 1\) was proved already. We now show that \(\nu_{2n}(\tau_{S_1}) = 1\): The fact that for \(|S_1| = 1\): \(\nu_{2n}(\tau_{S_1}) = 1\) follows from the properties of \(\nu_2\) and the inductivity of Rao’s cocycle. We proceed by induction on the cardinality of \(S_1\); suppose that if \(|S_1| \leq l\) then \(\nu_{2n}(\tau_{S_1}) = 1\). Assume now that \(|S_1| = l + 1\). Write \(S_1 = S' \cup S''\), where \(S'\) and \(S''\) are two non-empty disjoint sets. By Lemma 2.2 we have
\[\nu_{2n}(\tau_{S_1}) = \nu_{2n}(\tau_{S'}) \nu_{2n}(\tau_{S''}) c(\tau_{S'}, \tau_{S''}) = 1.\]
Finally,
\[\nu_{2n}(\tilde{w}_\pi a_{S_2} \tau_{S_1}) = \nu_{2n}(\tilde{w}_\pi) \nu_{2n}(a_{S'}) \nu_{2n}(\tau_{S}) c(\tilde{w}_\pi, a_{S'}) c(\tilde{w}_\pi a_{S'}, \tau_S) = 1.\]
\[\square\]
For all the p-adic fields we have the following lemma; define

\[K_n = \{(1 + a \quad b \quad c \quad d) \in SL_2(\mathbb{F}) \mid a, b, c, d \in \mathbb{F}_p^n\}\]

**Lemma 2.2.** \(c(K_n, K_n) = 1\) for all \(n \geq 2e + 1\).

**Proof.** In Lemma 3.3 we prove that \(1 + \mathbb{F}_p^n \in \mathbb{F}_p^{*2}\) for all \(n \geq 2e + 1\). We shall use this fact here. Suppose that \(g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\), where \(a \in 1 + \mathbb{F}_p^n, n \geq 2e + 1\). Then, for all \(h \in SL_2(\mathbb{F})\), since \(a \in \mathbb{F}_p^{*2}\) we have
\[c(g, h) = c(h, g) = (a, x(h))_F = 1.\]
It is left to show that for \(g = \begin{pmatrix} a & \frac{ad - 1}{c} \\ c & d \end{pmatrix} \in K_n, h = \begin{pmatrix} x & \frac{aw - 1}{z} \\ z & w \end{pmatrix} \in K_n\), where \(c, z \neq 0\), we have \(c(g, h) = 1\). Note that
\[x(gh) = \begin{cases} cx + dz & \text{if } cx + dz \neq 0 \\ ax + (ad - 1)\frac{z}{c} & \text{if } cx + dz = 0 \end{cases}.\]
If \(cx + dz = 0\) then
\[c(g, h) = (c, z)_F(-cz, ax + (ad - 1)\frac{z}{c})_F = (c, z)_F(-cz, \frac{z}{c})_F = (-c, -z)_F.\]
The fact that \(c = -\frac{d}{z}\) and that \(\frac{z}{d} \in \mathbb{F}_p^{*2}\) implies now that \(c(g, h) = 1\). Suppose now that \(cx + dz \neq 0\). In this case
\[c(g, h) = (c, z)_F(-cz, cx + dz)_F = (-cz, x + \frac{dz}{c})_F = (-cz, 1 + \frac{dz}{cx})_F.\]
Since for all \(p, m \in \mathbb{F}_p, p \neq 1\), we have \((p, m)_F = (p, p(1 - m))_F\), we finally get
\[c(g, h) = (1 + \frac{dz}{cx}, \frac{d}{x} z^2)_F = 1.\]
\[\square\]

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Assume now that \( \mathbb{F} = \mathbb{R} \). The group \( SO_2(\mathbb{R}) \) is a maximal compact subgroup of \( SL_2(\mathbb{R}) \) and it is commutative. Every \( k \in SO_2(\mathbb{R}) \) can be written uniquely as

\[
k(t) = \begin{pmatrix}
\cos(t) & \sin(t) \\
-sin(t) & \cos(t)
\end{pmatrix}
\]

for some \( 0 \leq t < 2\pi \). We have:

\[
c(k(t), k(t)^{-1}) = \begin{cases} 
1 & t \neq \pi \\
-1 & t = \pi 
\end{cases}.
\]

If \( k(t) \neq \pm I_2 \) then

\[
c(k(t), -k(t)^{-1}) = \begin{cases} 
1 & \sin(t) > 0 \\
-1 & \sin(t) < 0 
\end{cases},
\]

and

\[
c(k(t), -I_2)^{-1} = \begin{cases} 
1 & \sin(t) > 0 \\
-1 & \sin(t) < 0 
\end{cases}.
\]

If none of \( (k(t_1), k(t_2), k(t_1)k(t_2) \) equals \( \pm I_2 \) then

\[
c((k(t_1), k(t_2)) = \begin{cases} 
-1 & \text{sign}(\sin(t_1)) = \text{sign}(\sin(t_2)) \neq \text{sign}(\sin(t_1 + t_2)) \\
1 & \text{otherwise}
\end{cases}.
\]

\( SL_2(\mathbb{R}) \) does not split over \( SO_2(\mathbb{R}) \). Indeed, Let \( H \) be any subgroup of \( SL_2(\mathbb{R}) \) which contains \( -I_2 \). Then, there is no \( \beta : H \to \{\pm 1\} \) such that

\[
\beta(ab) = c(a,b)\beta(a)\beta(b)
\]

since this function must satisfy

\[
1 = \beta((-I_2)^2) = c(-I_2, -I_2)\beta^2(-I_2)
\]

or, equivalently, \( \beta^2(-I_2) = -1 \). It is well known, see page 74 of [14] for example, that \( SO_2(\mathbb{R}) \simeq \mathbb{R}/4\pi\mathbb{Z} \). We determine all the isomorphisms between these two groups:

**Lemma 2.3.** Realize \( \mathbb{R}/4\pi\mathbb{Z} \) as the segment \( [0,4\pi) \). There are exactly two isomorphisms between \( \mathbb{R}/4\pi\mathbb{Z} \) and \( SO_2(\mathbb{R}) \). One is \( t \mapsto \phi(t) = (k(t), \theta(t)) \), where \( \theta : \mathbb{R}/4\pi\mathbb{Z} \to \{\pm 1\} \) is the unique function that satisfies

\[
\theta(t_1 + t_2) = \theta(t_1)\theta(t_2)c(k(t_1), k(t_2)). \tag{2.20}
\]

The second is \( t \mapsto \phi(-t) \). \( \theta \) is given by

\[
\theta(t) = \begin{cases} 
1 & 0 \leq t \leq \pi \text{ or } 3\pi < t < 4\pi \\
-1 & \pi < t \leq 3\pi
\end{cases}. \tag{2.21}
\]

**Proof.** Using the cocycle formulas given above, one may check directly that \( \theta \) as defined in \( (2.21) \) satisfies \( (2.20) \). The uniqueness of \( \theta \) follows from the fact that any function from \( \mathbb{R}/4\pi\mathbb{Z} \) to \( \{\pm 1\} \) that satisfies \( (2.20) \) must satisfy \( \theta(t) = c(k(t), k(t)) \). It is now clear that \( \phi \) is indeed an isomorphism. Let \( \phi' : \mathbb{R}/4\pi\mathbb{Z} \to SO_2(\mathbb{R}) \) be an isomorphism. Since \( \phi^{-1}\phi' \) is an automorphism of \( \mathbb{R}/4\pi\mathbb{Z} \) it follows that either \( \phi^{-1}\phi'(t) = t \) or \( \phi^{-1}\phi'(t) = -t \). This implies \( \phi'(t) = \phi(t) \) or \( \phi'(t) = \phi(-t) \). □
2.4 Some facts about parabolic subgroups of $\overline{Sp_{2n}(F)}$

Let $F$ be a local field. By a parabolic subgroup of $\overline{Sp_{2n}(F)}$ we mean an inverse image of a parabolic subgroup of $Sp_{2n}(F)$.

**Lemma 2.4.** Let $Q$ a parabolic subgroup of $Sp_{2n}(F)$. Write $Q = M \ltimes N$, a Levi decomposition. Then, there exists a unique function $\mu': N \to \{\pm 1\}$, such that $n \mapsto \mu(n) = (n, \mu'(n))$ is an embedding of $N$ in $\overline{Sp_{2n}(F)}$. Furthermore: $Q = M \ltimes \mu(N)$. By abuse of language we shall refer to the last equality as the Levi decomposition of $Q$.

**Proof.** Suppose first that $Q$ is standard. From the fact $Sp_{2n}(F)$ splits over $N$ via the trivial section it follows that $\mu'$ is a quadratic character of $N$. Since $N = N^2$ we conclude that $\mu'$ is trivial. Using (2.16) we get $Q = M \ltimes \mu(N)$. Assume now that $Q$ is a general parabolic subgroup. Then, $Q = (w, 1)Q'(w, 1)^{-1}$ for some $w \in Sp_{2n}(F)$, and a standard parabolic subgroup $Q'$. For $n \in Q$ define $n' = w^{-1}nw$. From the proof in the standard case it follows that

$$\mu'(n) = c(w, n')c(wn', w^{-1})c(w^{-1}, w) = c(nw, w^{-1})c(w^{-1}, w)$$

is the unique function mentioned in the lemma. The fact that $Q = M \ltimes \mu(N)$ follows also from the standard case.

**Lemma 2.5.** Let $F$ be a $p$-adic field. $\overline{Sp_{2k}(O_F)}$ is a maximal open compact subgroup of $\overline{Sp_{2n}(F)}$. For any parabolic subgroup $Q$ of $Sp_{2n}(F)$ we have

$$\overline{Sp_{2n}(F)} = (Q, 1)\overline{Sp_{2k}(O_F)} = \overline{Sp_{2k}(O_F)(Q, 1)}.$$ 

If $F$ is a $p$-adic field of odd residual characteristic then $\overline{Sp_{2n}(F)} = Q_{k, 2n}(Sp_{2n}(O_F))$.

By an abuse of natation we call the last decomposition an Iwasawa decomposition of $\overline{Sp_{2n}(F)}$.

**Proof.** This follows immediately from the analogous lemma in the algebraic case.

Let $F$ be a $p$-adic field of odd residual characteristic. Pick $\alpha \in \{\pm 1\}$ and $\chi$, a quadratic character of $O_F^*$. Define $\xi_{\alpha, \chi} : F^* \to \{\pm 1\}$ by

$$\xi_{\alpha, \chi}(e^{2\pi n}) = \chi(e)(\pi, \pi)^n_F$$
$$\xi_{\alpha, \chi}(e^{2\pi n+1}) = \alpha \chi(e)(\pi, \pi)^n_F(e, \pi)_F,$$

where $e \in O_F^*$ and $n \in \mathbb{Z}$.

**Lemma 2.6.** Let $F$ be a $p$-adic field of odd residual characteristic. There are exactly four maps $\xi : F^* \to \{\pm 1\}$ such that

$$\xi(ab) = \xi(a)\xi(b)(a, b)_F.$$  \hspace{1cm} (2.22)

Each of them has the form $\xi_{\alpha, \chi}$.
A closely related argument is presented in Lemma 4.1. Since (2.22) has this form. Indeed, define \( \alpha = \xi(\pi) \) and define \( \chi \) to be the restriction of \( \xi \) to \( \mathbb{O}_p^* \). Since \( (\mathbb{O}_p^*,\mathbb{O}_p^* F) = 1 \) it follows that \( \chi \) is a quadratic character. We shall show that \( \xi = \xi_{\alpha,\chi} \):

\[
\begin{align*}
\xi(\epsilon \pi^{2n}) &= \xi(\epsilon)\xi(\pi^{2n})(\epsilon, \pi^{2n})_F = \chi(\epsilon)\xi(\pi^{2n}), \\
\xi(\epsilon \pi^{2n+1}) &= \xi(\epsilon)\xi(\pi^{2n+1})(\epsilon, \pi)_F = \alpha \chi(\epsilon)\xi(\pi^{2n})(\epsilon, \pi)_F.
\end{align*}
\]

It remains to show that

\[
\xi(\pi^{2n}) = (\pi, \pi)^n_F
\]

for all \( n \in \mathbb{N} \). Since \( \xi(1) = 1 \) and since

\[
\xi(\pi^2) = \xi^2(\pi)(\pi, \pi)_F = \alpha^2(\pi, \pi)_F = (\pi, \pi)_F,
\]

it follows that for all \( n \in \mathbb{Z} \)

\[
\xi(\pi^{2n+2}) = \xi(\pi^{2n})\xi(\pi^2) = \xi(\pi^{2n})(\pi, \pi)_F.
\]

Similarly, \( \xi(\pi^{2n-2}) = \xi(\pi^{2n})(\pi, \pi)_F \) for all \( n \in \mathbb{Z} \). This completes the proof of (2.23).

\[\square\]

**Remark:** Since \( \mathbb{F}^* \cong \mathbb{O}_p^* \times \mathbb{Z} \) it follows that \( \mathbb{F}^*/\mathbb{F}^{*2} \cong \mathbb{O}_p^*/\mathbb{O}_p^{*2} \times \{\pm 1\} \). Thus, there is an isomorphism between the group of quadratic characters of \( \mathbb{F}^* \) and the group of quadratic characters of \( \mathbb{O}_p^* \times \{\pm 1\} \). The isomorphism is \( \eta \mapsto (\eta|_{\mathbb{O}_p^*}, \eta(\pi)) \). Since every quadratic character of \( \mathbb{F}^* \) has the form \( \eta(b) = \eta_a(b) = (a, b)_F \), we have proven that every \( \xi \) satisfying (2.22) has the form \( \xi = \xi_a \), where \( a \in \mathbb{F}^* \) and \( \xi_a \) is defined as follows

\[
\begin{align*}
\xi_a(\epsilon \pi^{2n}) &= (a, \epsilon)(\pi, \pi)^n, \\
\xi_a(\epsilon \pi^{2n+1}) &= \xi_a(\epsilon \pi^{2n})(\pi, a\epsilon).
\end{align*}
\]

**Lemma 2.7.** If \( \mathbb{F} \) is a \( p \)-adic field of odd residual characteristic then

\[
GL_{n_1}(\mathbb{F}) \times GL_{n_2}(\mathbb{F}) \ldots \times GL_{n_r}(\mathbb{F}) \times \overline{Sp_{2k}(\mathbb{F})} \cong M_{T_1}(\mathbb{F}).
\]

An isomorphism is given by

\[
(g_1, g_2, \ldots, g_r, \overline{h}) \mapsto (j_{n-k,n}(\overline{g}), 1)(i_{k,n}(h), \epsilon \xi_{\alpha,\chi}(\det g)),
\]

where for \( 1 \leq i \leq r \), \( g_i \in GL_{n_i}(\mathbb{F}) \), \( g = \text{diag}(g_1, g_2, \ldots, g_r) \in GL_{n-k}(\mathbb{F}) \), \( \overline{h} = (h, \epsilon) \in \overline{Sp_{2k}(\mathbb{F})} \). In particular, \( \overline{Sp_{2n}(\mathbb{F})} \) splits over the Siegel parabolic subgroup via the map

\[
p \mapsto (p, \xi_{\alpha,\chi}(\det p)).
\]

**Proof.** Once the existence of a map \( \xi : \mathbb{F}^* \to \{\pm 1\} \) satisfying (2.22) is established, this lemma reduces to a straightforward computation. One uses the fact, following from (2.12), that for \( p, p' \in P(\mathbb{F}) \) we have

\[
\xi_{\alpha,\chi}(\det(p))\xi_{\alpha,\chi}(\det(p'))c(p, p') = \xi_{\alpha,\chi}(\det(pp')).
\]

A closely related argument is presented in Lemma 4.1. \[\square\]

We shall not use this lemma.
2.5 The global metaplectic group

Let $\mathbb{F}$ be a number field, and let $\mathbb{A}$ be its Adele ring. For every place $\nu$ of $\mathbb{F}$ we denote by $\mathbb{F}_\nu$ its completion at $\nu$. We denote by $\hat{Sp}_{2n}(\mathbb{A})$ the restricted product $\prod'_\nu Sp_{2n}(\mathbb{F}_\nu)$ with respect to
\[
\left\{ \kappa_{2n}(Sp_{2n}(\mathbb{O}_{F_\nu})) \mid \nu \text{ is finite and odd} \right\}.
\]
$\hat{Sp}_{2n}(\mathbb{A})$ is clearly not a double cover of $Sp_{2n}(\mathbb{A})$. Put
\[
C' = \left\{ \prod _\nu (I, \epsilon_\nu) \mid \prod _\nu \epsilon_\nu = 1 \right\}.
\]
We define
\[
\hat{Sp}_{2n}(\mathbb{A}) = C' \setminus \hat{Sp}_{2n}(\mathbb{A})
\]
to be the metaplectic double cover of $Sp_{2n}(\mathbb{A})$. It is shown in page 728 of [28] that $k \mapsto C' \prod _\nu (k, 1)$ is an embedding of $Sp_{2n}(\mathbb{F})$ in $\hat{Sp}_{2n}(\mathbb{A})$.

2.6 Extension of Rao’s cocycle to $GSp(X)$.

Let $\mathbb{F}$ be a local field. Let $GSp(X)$ be the similitude group of $Sp(X)$. This is the subgroup of $GL_{2n}(\mathbb{F})$ which consists of elements which satisfy $< v_1 g, v_2 g > = \lambda_g < v_1, v_2 >$ for all $v_1, v_2 \in X$, where $\lambda_g \in \mathbb{F}^*$. $\mathbb{F}^*$ is embedded in $GSp(X)$ via
\[
\lambda \mapsto i(\lambda) = \begin{pmatrix} I_n & 0 \\ 0 & \lambda I_n \end{pmatrix}.
\]
Using this embedding we define an action of $\mathbb{F}^*$ on $Sp(X)$:
\[
(g, \lambda) \mapsto g^\lambda = i(\lambda^{-1})g i(\lambda).
\]
Let $\mathbb{F}^* \times Sp(X)$ be the semi-direct product corresponding to this action. For $g \in GSp(X)$ define
\[
p(g) = \begin{pmatrix} \alpha & \beta \\ \lambda^{-1} \gamma & \lambda^{-1} \delta \end{pmatrix} \in Sp(X),
\]
where $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GSp(X), \alpha, \beta, \gamma, \delta \in Mat_{n \times n}(\mathbb{F})$. Note that $g = i(\lambda_g) p(g)$. The map
\[
g \mapsto \iota(g) = (\lambda_g, p(g))
\]
is an isomorphism between $GSp(X)$ and $\mathbb{F}^* \times Sp(X)$.

We know that we can lift the outer conjugation $g \mapsto g^\lambda$ of $Sp(X)$ to $\overline{Sp(X)}$ (see page 36 of [38]), namely we can define a map $g^\lambda : Sp(X) \to \{\pm 1\}$ such that
\[
(g, \epsilon) \mapsto (g, \epsilon)^\lambda = (g^\lambda, \epsilon \nu_\lambda(g))
\]
is an automorphism of $\text{Sp}(X)$. In 2.6.1 we compute $v_\lambda$. We shall also show there that 

$$(\lambda, (g, \epsilon)) \mapsto (g, \epsilon)^\lambda$$

defines an action of $F^*$ on $\text{Sp}(X)$. Let us show how this computation enables us to extend $c(\cdot, \cdot)$ to a 2-cocycle $\tilde{c}(\cdot, \cdot)$ on $G\text{Sp}(X)$ which takes values in $\{\pm 1\}$ and hence write an explicit multiplication formula of $G\text{Sp}(X)$, the unique metaplectic double cover of $G\text{Sp}(X)$ which extends a non-trivial double cover of $\text{Sp}(X)$. We define the group $F^* \ltimes \text{Sp}(X)$ using the multiplication formula

$$(a, (g, \epsilon_1))(b, (h, \epsilon_2)) = (ab, (g, \epsilon_1)^b(h, \epsilon_2)).$$

We now define a bijection from the set $G\text{Sp}(X) \times \{\pm 1\}$ to the set $F^* \times \text{Sp}(X)$ by the formula

$$\tau(g, \epsilon) = (\lambda g, (p(g), \epsilon)),$$

whose inverse is given by

$$\tau^{-1}(\lambda, (h, \epsilon)) = (i(\lambda)h, \epsilon).$$

We use $\tau$ to define a group structure on $G\text{Sp}(X) \times \{\pm 1\}$. A straightforward computation shows that the multiplication in $G\text{Sp}(X)$ is given by

$$(g, \epsilon_1)(h, \epsilon_2) = (gh, v_{\lambda g}(p(g))c(p(g)^{\lambda h}, p(h)\epsilon_1\epsilon_2)).$$

Thus,

$$\tilde{c}(g, h) = v_{\lambda h}(p(g))c(p(g)^{\lambda h}, p(h))$$

serves as a non-trivial 2-cocycle on $G\text{Sp}(X)$ with values in $\{\pm 1\}$. We remark here that Kubota, [32] (see also [14]), used a similar construction to extend a non-trivial double cover of $\text{SL}_2(F)$ to a non-trivial double cover of $\text{GL}_2(F)$. For $n = 1$ our construction agrees with Kubota’s.

### 2.6.1 Computation of $v_\lambda(g)$

In [3] Barthel extended Rao’s unnormalized cocycle to $G\text{Sp}(X)$. One may compute $v_\lambda(g)$ using Barthel’s work and Rao’s normalizing factors. Instead, we compute $v_\lambda(g)$ using Rao’s (normalized) cocycle. Fix $\lambda \in F^*$. Since $(g, \epsilon) \mapsto (g, \epsilon)^\lambda$ is an automorphism, $v_\lambda$ satisfies:

$$v_\lambda(g)v_\lambda(h)v_\lambda(gh) = \frac{c(g^\lambda, h^\lambda)}{c(g, h)}.$$  (2.25)

We shall show that this property determines $v_\lambda$.

We first note the following:

$$\begin{pmatrix} a & b \\ 0 & \tilde{a} \end{pmatrix}^\lambda = \begin{pmatrix} a & \lambda b \\ 0 & \tilde{a} \end{pmatrix}, \quad x \begin{pmatrix} a & b \\ 0 & \tilde{a} \end{pmatrix}^\lambda = x^\lambda \begin{pmatrix} a & b \\ 0 & \tilde{a} \end{pmatrix}.$$  (2.26)

For $S \subseteq \{1, 2, \ldots, n\}$ define $a_S(\lambda) \in P(F)$ by

$$e_i \cdot a_S(\lambda) = \begin{cases} \lambda^{-1}e_i & i \in S \\ e_i & \text{otherwise} \end{cases}, \quad e_i^*a_S(\lambda) = \begin{cases} \lambda e_i^* & i \in S \\ e_i^* & \text{otherwise} \end{cases}.$$  (2.27)
Note that \( a_S = a_S(-1) \). One can verify that

\[
\tau_S^\lambda = a_S(\lambda) \tau_S = \tau_S a_S(\lambda^{-1}).
\] (2.28)

Since \( x(a_S(\lambda)) \equiv \lambda^{[S]} \), we conclude, using (2.26), (2.28), the Bruhat decomposition and the properties of \( x \) presented in Section 2.2 that \( \Omega_j^\lambda = \Omega_j \), and that for \( g \in \Omega_j \)

\[
x(g^\lambda) \equiv \lambda^j x(g).
\] (2.29)

**Lemma 2.8.** For \( p \in P(F) \), \( g \in \Omega_j \) we have

\[
v_\lambda(p) v_\lambda(g) v_\lambda(pg) = (x(p), \lambda^j)_F
\] (2.30)

and

\[
v_\lambda(g) v_\lambda(p) v_\lambda(gp) = (x(p), \lambda^j)_F
\] (2.31)

**Proof.** We prove (2.30) only. (2.31) follows in the same way. We use (2.25), (2.12), (2.29) and (1.1):

\[
v_\lambda(p) v_\lambda(g) v_\lambda(pg) = \frac{c(p^\lambda, g^\lambda)}{c(p, g)} = \frac{(x(p^\lambda), x(g^\lambda))_F}{(x(p), x(g))_F} = \frac{(x(p), x(g)\lambda^j)_F}{(x(p), x(g))_F} = (x(p), \lambda^j)_F
\]

\[\square\]

**Lemma 2.9.** There exists a unique \( t_\lambda \in F^*/F^* \) such that for all \( p \in P(F) \):

\[v_\lambda(p) = (x(p), t_\lambda)_F.\]

**Proof.** Substituting \( p' \in P(F) \) instead of \( g \) in (2.30) we see that \( v_\lambda|_{P(F)} \) is a quadratic character. Since \( N_{n,0}(F) \), the unipotent radical of \( P(F) \) is isomorphic to a vector space over \( F \), it follows that \( N_{n,0}(F)^2 = N_{n,0}(F) \). Thus, \( v_\lambda|_{N_{n,0}(F)} \) is trivial. We conclude that \( v_\lambda|_{P(F)} \) is a quadratic character of \( GL_n(F) \) extended to \( P(F) \). Every quadratic character of \( GL_n(F) \) is of the form \( g \mapsto \chi(|det(g)|) \), where \( \chi \) is a quadratic character of \( F^* \). Due to the non-degeneracy of the Hilbert symbol, every quadratic character of \( F^* \) has the form \( \chi(a) = (a, t_\lambda)_F \), where \( t_\lambda \in F^* \) uniquely determined by \( \chi \) up to multiplication by squares. \[\square\]

**Lemma 2.10.** For \( \sigma \in \Omega_j \) we have

\[v_\lambda(\sigma) = (x(\sigma), t_\lambda \lambda^j)_F v_\lambda(\tau_S),\]

where \( S \subseteq \{1, 2, \ldots, n\} \) is such that \( |S| = j \). In particular, if \( |S| = |S'| \) then \( v_\lambda(\tau_S) = v_\lambda(\tau_{S'}) \).

**Proof.** An element \( \sigma \in \Omega_j \) has the form \( \sigma = p \tau S p' \) where \( p, p' \in P(F), \ |S| = j \). Substituting \( g = \tau S p' \) in (2.30) yields

\[v_\lambda(p \tau S p') = v_\lambda(p) v_\lambda(\tau S p') (x(p), \lambda^j)_F.\]

Substituting \( g = \tau_S \) and \( p = p' \) in (2.31) yields

\[v_\lambda(\tau_S p') = v_\lambda(\tau_S) v_\lambda(p') (x(p'), \lambda^j)_F.\]
We conclude that \( \rho \) by (2.25) we have
\[
v_\lambda(p \tau S p') = (x(pp'), t_1 \lambda^j) v_\lambda(\tau S).
\]

Since \(|S| = |S'|\) implies \( p \tau S p^{-1} = \tau S' \), for some \( p \in P(\mathbb{F}) \), the last argument shows that \( v_\lambda(\tau S) = v_\lambda(\tau S') \).

It is clear now that once we compute \( t_\lambda \) and \( v_\lambda(\tau S) \) for all \( S \subseteq \{1, 2, \ldots, n\} \) we will find the explicit formula for \( v_\lambda \).

**Lemma 2.11.** \( t_\lambda = \lambda \) and \( v_\lambda(\tau S) = (\lambda, \lambda)_{\mathbb{F}}^{\frac{\left|S\right|\left(|S| - 1\right)}{2}} \).

**Proof.** Let \( k \) be a symmetric matrix in \( GL_n(\mathbb{F}) \). Put
\[
p_k = \begin{pmatrix} k & -I_n \\ 0 & k - 1 \end{pmatrix} \in P(\mathbb{F}), \quad n_k = \begin{pmatrix} I_n & k \\ 0 & I_n \end{pmatrix} \in N_{n,0}(\mathbb{F}),
\]
and note that \( x(n_k) \equiv 1 \), \( x(p_k) \equiv \det(k) \), and that
\[
\tau n_k \tau = n_{-k-1} \tau p_k.
\]

We are going to compute \( v_\lambda(\tau)v_\lambda(n_k \tau)v_\lambda(\tau n_k \tau) \) in two ways: First, by Lemma 2.10 and by (2.32) we have
\[
v_\lambda(\tau)v_\lambda(n_k \tau)v_\lambda(\tau n_k \tau) = v_\lambda(\tau)v_\lambda(\tau) v_\lambda(n_{-k-1} \tau p_k) = v_\lambda(n_{-k-1} \tau p_k).
\]

Since
\[
x(\tau n_k \tau) \equiv x(n_{-k-1} \tau p_k) \equiv \det(k),
\]
we obtain, using Lemma 2.10 again,
\[
v_\lambda(\tau)v_\lambda(n_k \tau)v_\lambda(\tau n_k \tau) = \left( \det(k), t_\lambda \lambda^n \right) v_\lambda(\tau).
\]

Second, by (2.25) we have
\[
v_\lambda(\tau)v_\lambda(n_k \tau)v_\lambda(\tau n_k \tau) = \frac{c(\tau, n_k \tau)}{c(\tau^\lambda, (n_k \tau)^\lambda)}.
\]

Recall that \( \tau = \tau_{\{1, 2, \ldots, n\}}. \) We shall compute the two terms on the right side of (2.35), starting with \( c(\tau, n_k \tau) \): Let \( \rho \) and \( l \) be the factors in (2.9), where \( \sigma_1 = \tau, \sigma_2 = n_k \tau \). Since,
\[
q(V^*, V^* \tau, V^*(n_k \tau)^{-1}) = q(V^*, V, V^*(-I_{2n} \tau n_{-k})) = q(V^*, V, V n_{-k}).
\]

We conclude that \( \rho = k, l = 0 \). Using (2.9), and (2.33) we observe that
\[
c(\tau, n_k \tau) = (-1, \det(k))_\varphi h_\varphi(k).
\]

We now turn to \( c(\tau^\lambda, (n_k \tau)^\lambda) \): Let \( \rho \) and \( l \) be the factors in (2.9), where \( \sigma_1 = \tau^\lambda, \sigma_2 = (n_k \tau)^\lambda \). Note that by (2.26) and (2.28), \( (n_k \tau)^\lambda = n_{kk} \lambda I_{2n} \tau \), hence
\[
q(V^*, V^* \tau^\lambda, V^*(n_k \tau)^{-1}) = q(V^*, V, V n_{-\lambda k}).
\]
Thus, \( \rho = mk, l = 0 \), and we get

\[
    c(\tau^\lambda, (n_k\tau)^\lambda) = \left( x(\lambda I_{2n}), x(\lambda I_{2n}) \right)_F (-1, x(\tau^\lambda(n_k\tau)^\lambda))_F h_F(\lambda k).
\]

We recall (2.32) and note now that

\[
    \tau^\lambda(n_k\tau)^\lambda = (\tau n_k\tau)^\lambda = (n_{-k^{-1}}\tau p_k)^\lambda = n_{-\lambda k^{-1}} \lambda I_{2n} \tau \begin{pmatrix} k & \lambda k \\ 0 & k^{-1} \end{pmatrix}.
\]

Hence, \( c(\tau^\lambda, (n_k\tau)^\lambda) = (\lambda^n, \lambda^n)_F (-1, \lambda^n \det(k))_F h_F(\lambda k) \), or, using (1.1):

\[
    c(\tau^\lambda, (n_k\tau)^\lambda) = (-1, \det(k))_F h_F(\lambda k).
\]

Using (2.33), (2.35), (2.36) and (2.37) we finally get

\[
    v_\lambda(\tau)(\det(k), t_\lambda \lambda^n)_F = \frac{h_F(\lambda k)}{h_F(k)}.
\]

By substituting \( k = I_n \) in (2.38), we get \( v_\lambda(\tau) = (\lambda, \lambda)_F \frac{n(n-1)}{2} \). We can now rewrite (2.38) as

\[
    (\det(k), t_\lambda \lambda^n)_F = \frac{h_F(\lambda k)}{h_F(k)} \left( \lambda, \lambda \right)_F \frac{n(n-1)}{2}.
\]

In order to find \( t_\lambda \) we note that for any \( y \in \mathbb{F}^* \) we can substitute \( k_y = \text{diag}(1, 1, \ldots, 1, y) \) in (2.39) and obtain

\[
    (y, t_\lambda \lambda^n)_F = \left( \lambda, \lambda \right)_F \frac{(n-1)(n-2)}{2} \lambda y^{n-1} \left( \lambda, \lambda \right)_F \frac{n(n-1)}{2}.
\]

For both even and odd \( n \) this is equivalent to \( (y, \lambda)_F = (y, t_\lambda)_F \). The validity of the last equality for all \( y \in \mathbb{F}^* \) implies that \( t_\lambda \equiv \lambda (\text{mod} (\mathbb{F}^*)^2) \).

We are left with the computation of \( v_\lambda(\tau_S) \) for \( S \subseteq \{1, 2, \ldots, n\} \). For such \( S \), define \( s_\tau \in Sp(X_S) \) by analogy with \( \tau \in Sp(X) \). We can embed \( Sp(X_S) \) in \( Sp(X) \) in a way that maps \( s_\tau \) to \( \tau_S \). We may now use (2.18) and repeat the computation of \( v_\lambda(\tau) \).

Joining Lemma 2.10 and Lemma 2.11 we write the explicit formula for \( v_\lambda \): For \( g \in \Omega_j \) we have

\[
    v_\lambda(g) = \left( x(g), \lambda^{j+1} \right)_F (\lambda, \lambda) \frac{j(j-1)}{2}.
\]

One can easily check now that \( v_\lambda(g)v_\eta(g^\lambda) = v_{\lambda\eta}(g) \), and conclude that the map \( (\lambda, (g, \epsilon)) \rightarrow (g, \epsilon)^\lambda \) defines an action of \( \mathbb{F}^* \) on \( Sp(X) \), namely that \( (g, \epsilon)^\lambda \equiv (g, \epsilon)^{\lambda\eta} \).

**Corollary 2.1.** Comparing (2.40) and (2.10), keeping (1.1) in mind, we note that

\[
    v_{-1}(g) = v_{-1}(g^{-1}) = c(g, g^{-1}).
\]

This fact will play an important role in the proof of the uniqueness of Whittaker models for \( Sp(X) \), see Chapter 5.
3 Weil factor attached to a character of a second degree

Let \( \mathbb{F} \) be a local field. More accurately, assume that \( \mathbb{F} \) is either \( \mathbb{R}, \mathbb{C} \) or a finite extension of \( \mathbb{Q}_p \). In this chapter we introduce \( \gamma_\psi \), the normalized Weil factor associated with a character of second degree of \( \mathbb{F} \) which takes values in \( \{ \pm i, \pm 1 \} \subset \mathbb{C}^* \). This factor is an ingredient in the definition of genuine parabolic induction on \( \text{Sp}_{2n} (\mathbb{F}) \). We prove certain properties of the Weil factor that will be crucial to the computation the local coefficients in Chapter 8. We also compute it for p-adic fields of odd residual characteristic and for \( \mathbb{Q}_2 \). The Weil factor is trivial in the field of complex numbers. In the real case it is computed in [40]. It is interesting to know that unless \( \mathbb{F} \) is a 2-adic field \( \gamma_\psi \) is not onto \( \{ \pm 1, \pm i \} \).

Let \( \psi \) be a non-trivial character of \( \mathbb{F} \). For \( a \in \mathbb{F}^* \) let \( \gamma_\psi (a) \) be the normalized Weil factor associated with the character of second degree of \( \mathbb{F} \) given by \( x \mapsto \psi_a (x^2) \) (see Theorem 2 of Section 14 of [69]). It is known that

\[
\gamma_\psi (ab) = \gamma_\psi (a) \gamma_\psi (b) (a, b)_\mathbb{F}.
\]

and that

\[
\gamma_\psi (b^2) = 1, \gamma_\psi (ab^2) = \gamma_\psi (a), \gamma_\psi (a) = 1.
\]

From Appendix A-1-1 of [40] it follows that for \( \mathbb{F} = \mathbb{R} \) we have

\[
\gamma_\psi (a) = \begin{cases} 1 & y > 0 \\ -\text{sign}(a)i & y < 0 \end{cases}.
\]

(recall that for \( \mathbb{F} = \mathbb{R} \) we use the notation \( \psi_b (x) = e^{ibx} \)). Unless otherwise stated, until the end of this chapter we assume that \( \mathbb{F} \) is a p-adic field. It is known that

\[
\gamma_\psi (a) = \| a \|^\frac{1}{2} \int_{\mathbb{F}} \psi_a (x^2) \frac{dx}{\int_{\mathbb{F}} \psi (x^2) dx},
\]

see page 383 of [7] for example. These integrals, as many of the integrals that will follow, should be understood as principal value integrals. Namely,

\[
\gamma_\psi (a) = \| a \|^\frac{1}{2} \lim_{n \to \infty} \int_{\mathbb{F}_p^n} \psi (ax^2) \frac{dx}{\int_{\mathbb{F}_p^n} \psi (x^2) dx}.
\]

For \( a \in \mathbb{F}^* \) define \( c_\psi (a) = \int_{\mathbb{F}} \psi (ax^2) dx \). With this notation:

\[
\gamma_\psi (a) = \| a \|^\frac{1}{2} c_\psi (a) c_\psi^{-1} (1).
\]

Recall the following definitions and notations from Chapter 11: \( e = e(\mathbb{F}) = [\mathbb{O}_\mathbb{F} : 2\mathbb{O}_\mathbb{F}], \omega = 2\pi^{-\epsilon} \in \mathbb{O}_\mathbb{F}^* \). If \( x \in \mathbb{O}_\mathbb{F} \) then \( \overline{x} \) is its image in the residue field \( \overline{\mathbb{F}} \).

**Lemma 3.1.** \( 1 + \mathbb{F}_p^{2e+1} \subseteq \mathbb{F}^{*2}, 1 + \mathbb{F}_p^{2e} \not\subseteq \mathbb{F}^{*2} \)

**Proof.** The lemma is known for \( \mathbb{F} \) of odd residual characteristic, see Theorem 3-1-4 of [71] for example. We now assume that \( \mathbb{F} \) is of even residual characteristic. The proof of the first assertion in this case resembles the proof for the case of odd residual characteristic. We must show the for every \( b \in 1 + \mathbb{F}_p^{2e+1} \) the polynomial \( f_b (x) = x^2 - b \in \mathbb{O}_\mathbb{F}[x] \) has a root in
This follows from Newton’s method: By Theorem 3-1-2 of [71], it is sufficient show that \( \left| \frac{f_p(1)}{f_p'(1)} \right| < 1 \), and it is clear. We now prove the second assertion. Pick \( b \in \mathbb{O}_F^* \) such that the polynomial
\[
p_b(x) = x^2 + \omega x - b \in \mathbb{F}[x]
\]
does not have a root in \( \mathbb{F} \). Such a polynomial exists: The map \( x \mapsto \phi(x) = x^2 + \omega x \) from \( \mathbb{F} \) to itself is not injective since \( \phi(0) = \phi(\omega) \) and since \( \omega \neq 0 \). Therefore, it is not surjective. Thus, there exists \( b \in \mathbb{O}_F^* \) such that \( x^2 + \omega x \neq b \) for all \( x \in \mathbb{F} \). Define now \( y = 1 + b\pi^{2e} \in 1 + \mathbb{F}^e \). We shall see that \( y \notin \mathbb{F}^e \). It is sufficient to show that there is no \( x_0 \in \mathbb{O}_F^* \) such that \( x_0^2 = y \). Suppose that such an \( x_0 \) exists. Since \( x_0^2 - 1 \equiv 0 \mod \mathbb{P}_F \) it follows that \( \omega x_0 \) is a root of
\[
f(x) = x^2 - 1 = (x-1)^2 \in \mathbb{F}[x].
\]
This polynomial has a unique root. Thus, we may assume \( x_0 = 1 + a\pi^k + c\pi^{k+1} \) for some \( k \geq 1, a \in \mathbb{O}_F^*, c \in \mathbb{O}_F \). If \( k < e \) then
\[
x_0^2 = 1 + a^2\pi^{2k} + e\pi^{2k+1}
\]
for some \( e' \in \mathbb{O}_F \). Hence \( x_0^2 \neq y \). If \( k > e \) then
\[
x_0^2 = 1 + \omega a\pi^{k+e} + e''\pi^{k+e+1}
\]
for some \( e'' \in \mathbb{O}_F \). Again \( x_0^2 \neq y \). So, \( k = e \) and
\[
x_0^2 = 1 + (a^2 + \omega a)\pi^{2e} + e''\pi^{2e+1}
\]
for some \( e''' \in \mathbb{O}_F \). Since we assumed that \( x_0^2 = y \) it follows that \( a^2 + \omega a = b \). This contradicts the fact that \( p_b(x) \) does not have root in \( \mathbb{F} \).

**Lemma 3.2.** Assume that \( \psi \) is normalized. For \( a \in \mathbb{O}_F^* \) we have
\[
\gamma_{\psi}(a) = c_{\psi}^{-1}(1) \left( 1 + \sum_{n=1}^{e} q^n \int_{\mathbb{O}_F^*} \psi(\pi^{-2n}x^2 a) \, dx \right) \quad (3.6)
\]
\[
\gamma_{\psi}(\pi a) = q^{-\frac{1}{2}} c_{\psi}^{-1}(1) \left( 1 + \sum_{n=1}^{e+1} q^n \int_{\mathbb{O}_F^*} \psi(\pi^{-1-2n}x^2 a) \, dx \right) \quad (3.7)
\]
\[
\gamma_{\psi}^{-1}(a) = c_{\psi}^{-1}(-1) \left( 1 + \sum_{n=1}^{e} q^n \int_{\mathbb{O}_F^*} \psi^{-1}(\pi^{-2n}x^2 a) \, dx \right) \quad (3.8)
\]
\[
\gamma_{\psi}^{-1}(\pi a) = q^{-\frac{1}{2}} c_{\psi}^{-1}(-1) \left( 1 + \sum_{n=1}^{e+1} q^n \int_{\mathbb{O}_F^*} \psi^{-1}(\pi^{-1-2n}x^2 a) \, dx \right) \quad (3.9)
\]

**Proof.** In order to prove (3.6) and (3.7), it is sufficient to show that
\[
c_{\psi}(a) = 1 + \sum_{n=1}^{e} q^n \int_{\mathbb{O}_F^*} \psi(\pi^{-2n}x^2 a) \, dx \quad (3.10)
\]
and that
\[
c_{\psi}(\pi a) = 1 + \sum_{n=1}^{e+1} q^n \int_{\mathbb{O}_F^*} \psi(\pi^{-1-2n}x^2 a) \, dx \quad (3.11)
\]

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We prove only the first assertion. The proof of the second is done in the same way. For \( a \in \mathcal{O}_F^* \) we have

\[
c_\psi(a) = \int_{\mathcal{O}_F^*} \psi(ax^2) \, dx + \int_{|x| > 1} \psi(ax^2) \, dx
\]

\[
= 1 + \sum_{n=1}^{\infty} \int_{|x| = q^n} \psi(ax^2) \, dx = 1 + \sum_{n=1}^{\infty} q^n \int_{\mathcal{O}_F^*} \psi(a\pi^{-2n}x^2) \, dx.
\]

It remains to show that for \( n > e \) we have \( \int_{\mathcal{O}_F^*} \psi(a\pi^{-2n}x^2) \, dx = 0 \). Indeed, by Lemma 3.1, for every \( u \in 1 + \mathbb{P}_F^{2e+1} \) we may change the integration variable \( x \mapsto x\sqrt{1 + u} \) and obtain

\[
\int_{\mathcal{O}_F^*} \psi(a\pi^{-2n}x^2) \, dx = \int_{\mathcal{O}_F^*} \psi(a\pi^{-2n}x^2)\psi(a\pi^{-2n}x^2u) \, dx.
\]

Hence

\[
\int_{\mathcal{O}_F^*} \psi(a\pi^{-2n}x^2) \, dx = \mu(\mathbb{P}_F^{2e+1})^{-1} \int_{\mathbb{P}_F^{2e+1}} \int_{\mathbb{P}_F^{2e+1}} \psi(a\pi^{-2n}x^2)\psi(a\pi^{-2n}x^2u) \, dx \, du
\]

\[
= q^{2e+1} \int_{\mathcal{O}_F^*} \psi(a\pi^{-2n}x^2) \left( \int_{\mathbb{P}_F^{2e+1}} \psi(a\pi^{-2n}x^2u) \, du \right) \, dx.
\]

For \( n > e \) the last inner integral vanishes. To prove (3.8) and (3.9) note that \( |\gamma_\psi(a)| = 1 \) implies \( \gamma_\psi(a)^{-1} = \gamma_\psi(a) \) and since \( c_\psi \) is defined via integrations over compact sets we have:

\[
\gamma_\psi^{-1}(a) = |a|^{\frac{1}{2}} c_\psi(-a) c_\psi^{-1}(-1).
\]

Since \( \gamma_\psi \) is determined by its values on \( \mathcal{O}_F^* \) and \( \pi\mathcal{O}_F^* \), the last lemma provides a description of \( \gamma_\psi \) for normalized \( \psi \). Furthermore, there is no loss of generality caused by the normalization assumption; recalling the definition of \( \gamma_\psi \) and (3.1) we observe that

\[
\gamma_\psi(a) = \frac{\gamma_\psi_0(a\pi^{-n})}{\gamma_\psi_0(\pi^{-n})} = \gamma_\psi_0(a, \pi^n)_F.
\]  

(3.12)

(recall that in Chapter I we have defined \( \psi_0(x) = \psi(x\pi^n) \), where \( n \) is the conductor of \( \psi \). \( \psi_0 \) is normalized)

**Lemma 3.3.** Assume that \( \psi \) is normalized.

1. \( \gamma_\psi(uk) = \gamma_\psi(k) \) for all \( k \in \mathcal{O}_F^* \), \( u \in 1 + \mathbb{P}_F^{2e} \).
2. \( \gamma_\psi(u) = 1 \) for all \( u \in 1 + \mathbb{P}_F^{2e} \).
3. \( (1 + \mathbb{P}_F^{2e}, \mathcal{O}_F^*F) = 1 \).
4. The map \( x \mapsto (x, \pi)_F \) defined on \( 1 + \mathbb{P}_F^{2e} \) is a non-trivial character.

**Remark:** A particular case of the first assertion of this lemma is the well known fact that that \( (\mathcal{O}_F^*, \mathcal{O}_F^*F)_F = 1 \) for all the p-adic fields of odd residual characteristic.
Proof. Due to (3.6), to prove the first assertion it is sufficient to show that
\[ \psi(\pi^{-2n}x^2k) = \psi(\pi^{-2n}x^2uk) \]
for all \( k, x \in \mathbb{O}_F^* \), \( u \in 1 + \mathbb{P}_F^2 \), \( n \leq e \) and this is clear. The second assertion is a particular case of the first. The third assertion follows from the first and from (3.1). We prove the forth assertion. By Lemma 3.1 we can pick \( u \in 1 + \mathbb{P}_F^2 \) which is not a square. By the non-degeneracy of the Hilbert symbol, there exists \( x \in \mathbb{F}^* \) such that \((x, u)_F = -1\). Write \( x = \pi^nk \) for some \( n \in \mathbb{Z} \), \( k \in \mathbb{O}_F^* \). Then, by the third assertion of this lemma we have \(-1 = (x, u)_F = (\pi, u)^n_F \). Thus, \( n \) is odd and the assertion follows.

The last two lemmas will be sufficient for our purposes, i.e., the computation of the local coefficients in Chapter 8. However, for the sake of completeness we prove the following:

**Lemma 3.4.** Suppose \( \mathbb{F} \) is of odd residual characteristic. If the conductor of \( \psi \) is even then for \( a \in \mathbb{O}_F^* \) we have:

\[
\begin{align*}
\gamma_{\psi}(a) &= 1 \\
\gamma_{\psi}(\pi a) &= \begin{cases} 
\gamma_{\psi}(\pi) & a \in \mathbb{F}^*^2 \\
-\gamma_{\psi}(\pi) & a \notin \mathbb{F}^*^2
\end{cases}.
\end{align*}
\]

If the conductor of \( \psi \) is odd then for \( a \in \mathbb{O}_F^* \) we have:

\[
\begin{align*}
\gamma_{\psi}(a) &= \begin{cases} 
1 & a \in \mathbb{F}^*^2 \\
-1 & a \notin \mathbb{F}^*^2
\end{cases} \\
\gamma_{\psi}(\pi a) &= \gamma_{\psi}(\pi).
\end{align*}
\]

Also

\[
\gamma_{\psi}(\pi) = q^{-\frac{1}{2}} \sum_{\pi \in \overline{\mathbb{F}}} \overline{\psi}(\pi^2) \times \begin{cases} 
-1 & -1 \notin \mathbb{F}^*^2 \text{ and the conductor of } \psi \text{ is odd} \\
1 & \text{otherwise}
\end{cases} \in \begin{cases} 
\{\pm 1\} & -1 \in \mathbb{F}^*^2 \\
\{\pm i\} & -1 \notin \mathbb{F}^*^2
\end{cases},
\]

where \( \overline{\psi} \) is the character of \( \overline{\mathbb{F}} \) defined by \( \overline{\psi}(\pi) = \psi_0(\pi^{-1}x) \).

Proof. The fact that \((\mathbb{O}_F^*, \mathbb{O}_F^*)_F = 1\) for all the \( p \)-adic fields of odd residual characteristic combined with the non-degeneracy of the Hilbert symbol implies that for \( a \in \mathbb{O}_F^* \) we have

\[
(a, \pi)_F = \begin{cases} 
1 & a \in \mathbb{F}^*^2 \\
-1 & a \notin \mathbb{F}^*^2
\end{cases}.
\]

It follows that

\[
\gamma_{\psi}(\pi)^2 = \gamma_{\psi}(\pi^2)(\pi, \pi)_F = (\pi, -1)_F = \begin{cases} 
1 & -1 \in \mathbb{F}^*^2 \\
-1 & -1 \notin \mathbb{F}^*^2
\end{cases}.
\]

This implies that

\[
\gamma_{\psi}(\pi) \in \begin{cases} 
\{\pm 1\} & -1 \in \mathbb{F}^*^2 \\
\{\pm i\} & -1 \notin \mathbb{F}^*^2
\end{cases}.
\]
By (3.18) and (3.12) it is sufficient to prove the rest of the assertions mentioned in this lemma for normalized characters. In this case (3.13) is simply the second part of Lemma 3.3. (3.14) follows now from (3.1), (3.13) and (3.18). It remains to prove that if \( \psi \) is normalized then

\[
\gamma_\psi(\pi) = q^{-\frac{1}{2}} \sum_{x \in F} \overline{\psi(x^2)}.
\]

Indeed, by (3.7) and by (3.10) we have

\[
\gamma_\psi(\pi) = q^{-\frac{1}{2}} \left( 1 + \int_{O_F^*} \psi(\pi^{-1}x^2) dx \right) = q^{-\frac{1}{2}} \left( 1 + q \sum_{x \in O_F^*/1+P_F} \int_{1+P_F} \psi(\pi^{-1}x^2y^2) dy \right)
\]

Note that

\[
\psi(\pi^{-1}x) = \psi(\pi^{-1}xy)
\]

for all \( x \in O_F, \ y \in 1 + P_F \). This shows that \( \overline{\psi} \) is well defined and that

\[
\gamma_\psi(\pi) = q^{-\frac{1}{2}} \left( 1 + \sum_{x \in O_F^*/1+P_F} \psi(\pi^{-1}x^2) \right) = q^{-\frac{1}{2}} \sum_{x \in F} \overline{\psi(x^2)}.
\]

Lemma 3.5. Let \( \psi \) be a non-trivial character of \( \mathbb{Q}_2 \). Denote its conductor by \( n \). Assume that \( a \in \mathbb{O}_{\mathbb{Q}_2}^* \). We have \( \psi(a2^{n-2}) \in \{ \pm i \} \). Also, if \( n \) is even then

\[
\gamma_\psi(a) = \begin{cases} 1 & a = 1 \mod 4 \\ \psi(-2^{n-2}) & a = -1 \mod 4 \end{cases}
\]

(3.19)

\[
\gamma_\psi(2a) = \gamma_\psi(2) \begin{cases} 1 & a = 1 \mod 8 \\ \psi(-2^{n-2}) & a = -1 \mod 8 \\ -1 & a = 5 \mod 8 \\ -\psi(-2^{n-2}) & a = -5 \mod 8 \end{cases},
\]

(3.20)

while if \( n \) is odd we have

\[
\gamma_\psi(a) = \begin{cases} 1 & a = 1 \mod 8 \\ \psi(-2^{n-2}) & a = -1 \mod 8 \\ -1 & a = 5 \mod 8 \\ -\psi(-2^{n-2}) & a = -5 \mod 8 \end{cases}
\]

(3.21)

\[
\gamma_\psi(2a) = \gamma_\psi(2) \begin{cases} 1 & a = 1 \mod 4 \\ \psi(-2^{n-2}) & a = -1 \mod 4 \end{cases}.
\]

(3.22)

Finally,

\[
\gamma_\psi(2) = \frac{\sqrt{2} \psi(2^{n-3})}{1 + \psi(2^{n-2})} \in \{ \pm 1 \}
\]

(3.23)
Proof. The computation of the Hilbert symbol for $\mathbb{Q}_2$ is well known (see Theorem 1 of Chapter 3 of [45] for example). For $a \in \mathcal{O}_Q^*$ we have

$$(2, a)_{\mathbb{Q}_2} = \begin{cases} 1 & a = \pm 1 \mod 8 \\ -1 & a = \pm 5 \mod 8 \end{cases} \tag{3.24}$$

Note that from (1.1) it follows that for any field of characteristic different then 2 we have

$$(2, 2)_F = (2, -1)_F = (2, -(1 - 2))_F = 1 \tag{3.25}$$

These facts, combined with (3.12) imply that it is sufficient to prove this lemma with the additional assumption that $\psi$ is normalized. From (3.10) it follows that if $a \in \mathcal{O}_Q^*$ then

$$c_\psi(a) = 1 + 2 \int_{\mathcal{O}_F^*} \psi(\frac{1}{4}x^2a) \, dx. \tag{3.26}$$

Next note that for any $x \in \mathcal{O}_Q^*$

$$\psi(\frac{1}{4}x^2a) = \psi(\frac{a}{4}). \tag{3.27}$$

Indeed, $|\mathcal{O}_Q^* : 1 + \mathbb{P}_Q^2| = |\mathbb{Q}_2^*| = 1$. Thus, $\mathcal{O}_Q^* = 1 + \mathbb{P}_Q^2$. Therefore, for any $x \in \mathcal{O}_Q^*$ there exists $t \in \mathbb{P}_Q^2$ such that $x = 1 + t$. This implies that

$$\psi(\frac{1}{4}x^2a) = \psi(\frac{a}{4}) \psi(\frac{at}{2} + \frac{t^2}{4}) = \psi(\frac{a}{4}).$$

(3.26) and (3.27) imply now that

$$\gamma_\psi(a) = \frac{1 + \psi(\frac{a}{4})}{1 + \psi(\frac{a}{4})}$$

By Lemma 3.3, $\gamma_\psi(\mathcal{O}_Q^*)$ is determined on $\mathcal{O}_Q^*/1 + \mathbb{P}_Q^2$. Since $\{\pm 1\}$ is a complete set of representatives of $\mathcal{O}_Q^*/1 + \mathbb{P}_Q^2$, (3.19) will follow once we show that $\psi(\frac{1}{4})$ is a primitive fourth root of 1. Since $\psi^2(\frac{1}{4}) = \psi(\frac{1}{2})$ it is sufficient to show that $\psi(\frac{1}{2}) = -1$. Since $\psi^2(\frac{1}{2}) = 1$ and since $\psi$ is normalized we only have to show that for $x \in \mathcal{O}_Q^*$, we have $\psi(\frac{a}{2}) = \psi(\frac{1}{2})$. Write $x = 1 + t$, where $t \in \mathbb{P}_Q^2$. We have: $\psi(\frac{a}{2}) = \psi(\frac{1}{2})(\frac{1}{2}) = \psi(\frac{1}{2})$. (3.20) follows now from (3.19) and from (3.24).

We now prove (3.23). From (3.25) it follows that in any field of characteristic different then 2, $\gamma_\psi^2(2) = 1$. From (3.17) and (3.26) it follows that

$$\gamma_\psi(2) = \frac{1 + \sum_{n=1}^{2} 2^n \int_{\mathcal{O}_F^*} \psi(2^{1-2n}x^2) \, dx}{\sqrt{2(1 + \psi(\frac{1}{4}))}}.$$

We have already seen that $\psi(\frac{1}{2}\mathcal{O}_Q^*) = -1$. This implies

$$\gamma_\psi(2) = \frac{4 \int_{\mathcal{O}_F^*} \psi(\frac{1}{8}x^2) \, dx}{\sqrt{2(1 + \psi(\frac{1}{4}))}}.$$
We now show that $\psi(\frac{1}{8}x^2) = \psi(\frac{1}{8})$ for all $x \in \mathbb{Q}_{Q_2}^*$ and conclude that

$$
\gamma_\psi(2) = \frac{\sqrt{2} \psi(\frac{1}{8})}{(1 + \psi(\frac{1}{4}))}.
$$

Indeed, for $x \in \mathbb{Q}_{Q_2}^*$ we write again $x = 1 + m$, where $m \in \mathbb{P}_{Q_2}$. With this notation:

$\psi(\frac{1}{8}x^2) = \psi(\frac{1}{8})\psi(\frac{2m}{8})\psi(\frac{m^2}{8})$. We finally note that if $\|m\| = \frac{1}{2}$ then $\psi(\frac{2m}{8}) = \psi(\frac{m^2}{8}) = -1$ and if $\|m\| < \frac{1}{2}$ then $\psi(\frac{2m}{8}) = \psi(\frac{m^2}{8}) = 1$. \hfill \Box

The last two lemmas give a complete description of $\gamma_\psi$ for p-adic field of odd residual characteristic and for $F = \mathbb{Q}_2$. From Lemma 3.4 it follows that for $F$, a p-adic field of odd residual characteristic we have

$$
|\gamma_\psi(F^*)| = \begin{cases} 2 & \text{if } -1 \in F^{*2} \\ 3 & \text{if } -1 \notin F^{*2}. \end{cases}
$$

(3.28)

Note that in the case where $-1 \in F^{*2}$ we must conclude that $\gamma_\psi$ is one of the four functions $\xi_{a,\chi}$ described in Lemma 2.6. Combining (3.28) with (3.3) we observe that if $F$ is either $\mathbb{R}$ or a p-adic field of odd residual characteristic then $\gamma_\psi$ is not onto $\{\pm 1, \pm i\}$. Lemma 3.5 shows that there exists a 2-adic field such that the cardinality of $\gamma_\psi(F^*)$ is 4.
4 Some facts from the representation theory of $\overline{Sp_{2n}(\mathbb{F})}$

Let $\mathbb{F}$ be a $p$-adic field. Throughout this section we list several general results concerning the representation theory of $\overline{Sp_{2n}(\mathbb{F})}$. Although $\overline{Sp_{2n}(\mathbb{F})}$ is not a linear algebraic group, its properties enable the extension of the theory presented in [60], [67], [8], and [9]. Among these properties are the obvious analogs of Bruhat, Iwasawa, Cartan and Levi decompositions and the fact that $\overline{Sp_{2n}(\mathbb{F})}$ is an $l$-group in the sense of [8]. These properties are common for $n$-fold covering groups.

We first note that since $\overline{Sp_{2n}(\mathbb{F})}$ is an $l$-group we have the same definitions of smooth and admissible representations as in the linear case; see page 18 of [1] or page 9 of [72] for example. We shall mainly be interested in smooth, admissible, genuine representations of $\overline{Sp_{2n}(\mathbb{F})}$. A representation $(V, \sigma)$ of $\overline{Sp_{2n}(\mathbb{F})}$ is called genuine if

$$\sigma(I_{2k}, -1) = -Id_V.$$  

It means that $\sigma$ does not factor through the projection map $Pr : \overline{Sp_{2n}(\mathbb{F})} \to Sp_{2n}(\mathbb{F})$. Same definition applies to representations of $\overline{MT}(\mathbb{F})$.

4.1 Genuine parabolic induction

For a representation $(\tau, V)$ of $GL_n(\mathbb{F})$ and a complex number $s$ we denote by $\tau(s)$ the representation of $GL_n(\mathbb{F})$ in $V$ defined by

$$g \mapsto \| \det(g) \|^s \tau(g).$$

Put $\overline{\tau} = (n_1, n_2, \ldots, n_r; k)$, where $k + \sum_{i=1}^r n_i = n$. Let $(\tau_1, V_{\tau_1}), (\tau_2, V_{\tau_2}), \ldots, (\tau_r, V_{\tau_r})$ be $r$ representations of $GL_{n_1}(\mathbb{F}), GL_{n_2}(\mathbb{F}), \ldots, GL_{n_r}(\mathbb{F})$ respectively. Let $(\overline{\sigma}, V_{\overline{\sigma}})$ be a genuine representation of $\overline{Sp_{2k}(\mathbb{F})}$. We shall now describe a representation of $\overline{MT}(\mathbb{F})$ constructed from these representations. We cannot repeat the algebraic construction since generally

$$\overline{MT}(\mathbb{F}) \not\simeq GL_{n_1}(\mathbb{F}) \times GL_{n_2}(\mathbb{F}) \times \ldots \times GL_{n_r}(\mathbb{F}) \times \overline{Sp_{2k}(\mathbb{F})}$$

(these groups are isomorphic in the case of $p$-adic fields of odd residual characteristic, see Lemma [2,7]). Instead we define

$$(\otimes_{i=1}^r (\gamma_i^{-1} \otimes \tau_i(s_i))) \otimes \overline{\sigma} : \overline{MT}(\mathbb{F}) \to GL((\otimes_{i=1}^r V_{\tau_i}) \otimes V_{\overline{\sigma}})$$

by

$$(\otimes_{i=1}^r (\gamma_i^{-1} \otimes \tau_i(s_i))) \otimes \overline{\sigma}(j_{n-k,n}(g), 1)(i_{k,n}(h), \epsilon) = \gamma_i^{-1}(\det(g)) \left( \otimes_{i=1}^r \tau_i(s_i)(g_i) \right) \otimes h, \epsilon, \quad (4.1)$$

where for $1 \leq i \leq r$, $g_i \in GL_{n_i}(\mathbb{F})$, $g = diag(g_1, g_2, \ldots, g_r) \in GL_{n-k}(\mathbb{F})$, $h \in Sp_{2k}(\mathbb{F})$ and $\epsilon \in \{\pm 1\}$. When convenient we shall use the notation

$$\pi(\overline{\sigma}) = \left( \otimes_{i=1}^r (\gamma_i^{-1} \otimes \tau_i(s_i)) \right) \otimes \overline{\sigma} \quad \pi = \pi(0).$$

Lemma 4.1. $\pi(\overline{\sigma})$ is a representation of $\overline{MT}(\mathbb{F})$. 

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Recalling that follows from the fact that (Z functor in the linear case; see Section 4.1 of [1] or Proposition 4.7 of [72]. This similarity in the role of Z due to (2.12), for p, p' ∈ P(F) we have

\[ \pi(\alpha')(\pi(\alpha)v) = \gamma^{-1}_{\psi}(\det(g))\gamma^{-1}_{\psi}(\det(g')) \left( \otimes_{i=1}^{r} \tau_{i}(g_{i}')\tau_{i}(g_{i})v_{i} \right) \otimes \overline{\sigma}(h', \epsilon')\overline{\sigma}(h, \epsilon)w. \]

Due to (2.12), for p, p' ∈ P(F) we have

\[ \gamma^{-1}_{\psi}(\det(p))\gamma^{-1}_{\psi}(\det(p'))c(p, p') = \gamma^{-1}_{\psi}(\det(pp')). \] (4.3)

Recalling that σ is genuine we now see that

\[ \pi(\alpha')(\pi(\alpha)v) = \epsilon \epsilon' \gamma^{-1}_{\psi}(\det(gg'))c(j_{n-k,n}(g_{i}'), j_{n-k,n}(g_{i}))c(h', h) \left( \otimes_{i=1}^{r} \tau_{i}(g_{i}', g_{i})v_{i} \right) \otimes \overline{\sigma}(h'h, 1)w. \] (4.4)

Next, we note that since (j_{n-k,n}(g_{i}'), 1) and (i_{k,n}(h), \epsilon) commute we have

\[ \alpha'\alpha = \left( j_{n-k,n}(g_{i}'), 1 \right) \left( i_{k,n}(h' h), \epsilon \epsilon' c(h', h)c(j_{n-k,n}(g_{i}'), j_{n-k,n}(g_{i})) \right). \]

(4.4) implies now that \( \pi(\alpha'\alpha)v = \pi(\alpha')(\pi(\alpha)v) \). \( \blacksquare \)

As in the linear case we note that if \( \tau_{1}, \tau_{2}, \ldots, \tau_{r} \) and \( \overline{\sigma} \) are smooth (admissible) representations then \( (\overline{\sigma})' \) is also smooth (admissible). Due to (2.14) it is possible to extend \( (\overline{\sigma})' \) to a representation of \( P_{\overline{\sigma}}(F) \) by letting \( (\overline{\sigma})'(N_{\overline{\sigma}}(F), 1) \) act trivially.

Assuming that \( \tau_{1}, \tau_{2}, \ldots, \tau_{r} \) and \( \overline{\sigma} \) are smooth we define smooth induction

\[ I(\pi(\overline{\sigma})) = I(\tau_{1}(s_{1}), \tau_{2}(s_{2}), \ldots, \tau_{r}(s_{r}), \overline{\sigma}) = Ind_{P_{\overline{\sigma}}(F)}^{Sp_{2n}(F)} \pi(\overline{\sigma}) \] (4.5)

and

\[ I(\pi) = I(\tau_{1}, \tau_{2}, \ldots, \tau_{r}, \overline{\sigma}) = Ind_{P_{\overline{\sigma}}(F)}^{Sp_{2n}(F)} \pi. \] (4.6)

All the induced representations in this dissertation are assumed to be normalized, i.e., if \( (\pi, V) \) is a smooth representation of \( H \), a closed subgroup of a locally compact group \( G \), then \( Ind_{H}^{G} \pi \) acts in the space of all right-smooth functions on \( G \) that take values in \( V \) and satisfy \( f(hg) = \sqrt{\det(h)}\pi(h)f(g) \), for all \( h \in H, g \in G \). Whenever we induce from a parabolic subgroup (a pre-image of a parabolic subgroup in a metaplectic group) we always mean that the inducing representation is trivial on its unipotent radical (on its embedding in the metaplectic group).

We claim that if the inducing representations are admissible, then \( (\overline{\sigma})' \) is also admissible. Indeed, the proof of Proposition 2.3 of [2] which is the p-adic analog of this claim applies to \( Sp_{2n}(F) \) as well since it mainly uses the properties on an l-group; see Proposition 4.7 of [72] or Proposition 1 of [1].

Next we note that similar to the algebraic case we can define the Jacquet functor, replacing the role of \( Z_{Sp_{2n}(F)}(1) \) with \( (Z_{Sp_{2n}(F)}, 1) \). The notion of a supercuspidal representation is defined via the vanishing of Jacquet modules along unipotent radicals of parabolic subgroups. The (metaplectic) Jacquet functor has similar properties to those of the Jacquet functor in the linear case; see Section 4.1 of [1] or Proposition 4.7 of [72]. This similarity follows from the fact that \( (Z_{Sp_{2n}(F)}, 1) \) is a limit of compact groups.
4.2 An application of Bruhat theory

Let \( \mathbf{t} = (n_1, n_2, \ldots, n_r; k) \) where \( n_1, n_2, \ldots, n_r, k \) are \( r + 1 \) non-negative integers whose sum is \( n \). Recall that \( W_{P_{\tau}}(\mathbb{F}) \) is the subgroup of Weyl group of \( SP_{2n}(\mathbb{F}) \) which consists of elements which maps \( M_{\mathbf{t}}(\mathbb{F}) \) to a standard Levi subgroup and commutes with \( i_{k,n}(SP_{2k}(\mathbb{F})) \); see Section 2.1.

**Lemma 4.2.** For \( 1 \leq i \leq r \) let \( g_i \) be an element of \( GL_{n_i}(\mathbb{F}) \) and let \( (h, \epsilon) \) be an element of \( SP_{2k}(\mathbb{F}) \). Denote again \( g = \text{diag}(g_1, g_2, \ldots, g_r) \in GL_{n-k}(\mathbb{F}) \). For \( w \in W_{P_{\tau}}(\mathbb{F}) \) we have

\[
(w, 1)(j_{n-k,n}(g), 1)(i_{k,n}(h), \epsilon)(w, 1)^{-1} = (w_{j_{n-k,n}(g)}w^{-1}, 1)(i_{k,n}(h), \epsilon)
\]

**Proof.** This follows from (2.16) and from the fact that \( w \) commutes with \( h \). \( \square \)

For \( 1 \leq i \leq r \) let \( \tau_i \) be an irreducible admissible supercuspidal representation of \( GL_{n_i}(\mathbb{F}) \). Let \( \overline{\sigma} \) be an irreducible admissible supercuspidal genuine representation of \( SP_{2k}(\mathbb{F}) \). Then,

\[
\pi = \left( \bigotimes_{i=1}^{r} (\gamma_i^{-1} \otimes \tau_i) \right) \otimes \overline{\sigma}
\]

is an irreducible admissible supercuspidal genuine representation of \( M_{\mathbf{t}}(\mathbb{F}) \). For \( w \in W_{P_{\tau}}(\mathbb{F}) \) denote by \( \pi^w \) the representation of \( M_{\mathbf{t}}(\mathbb{F})^w \) defined by

\[
(m, \epsilon) \mapsto \pi((w, 1)^{-1}(m, \epsilon)(w, 1)).
\]

Exactly as in the algebraic case, see Section 2 of [9], \( I(\pi) \) and \( I(\pi^w) \) have the same Jordan Holder series.

Recalling (2.5), we note that due to Lemma 4.2 and the fact that \( \gamma_\psi(a) = \gamma_\psi(a^{-1}) \) we have

\[
\left( \bigotimes_{i=1}^{r} \gamma_i^{-1} \otimes \tau_i(s_i) \right) \otimes \overline{\sigma}
\]

\[
(\psi((w, 1)^{-1}(j_{n-k,n}(g), 1)(i_{k,n}(h), \epsilon))(w, 1))
\]

\[
= \gamma_i^{-1}(\det(g))(\bigotimes_{i=1}^{r} \| \det(g_i) \|^{|\epsilon_i|s_i \tau_i(-1)}(g_i)\overline{\sigma}(h, \epsilon),
\]

where for \( 1 \leq i \leq r, g_i \in GL_{n_i}(\mathbb{F}), g = \text{diag}(g_1, g_2, \ldots, g_r) \in GL_{n-k}(\mathbb{F}), h \in SP_{2k}(\mathbb{F}), \epsilon \in \{\pm 1\} \) and where \( \tau_i(\epsilon_i)(g_i) = \begin{cases} 
\tau_i(g_i) & \epsilon_i = 1 \\
\tau_i(\omega_n \omega_n) & \epsilon_i = -1
\end{cases} \) where \( t_w^{-1}(\omega_n \omega_n) t_w \) is to be understood via the identification of \( GL_{n_i}(\mathbb{F}) \) with its image in the relevant block of \( M_{\mathbf{t}} \).

Hence, it makes sense to denote by \( \pi^w \) the representation

\[
\left( \bigotimes_{i=1}^{r} \gamma_i^{-1} \otimes (\tau_i^{(-1)}(\epsilon_i))_{(s_i,s_i)} \right) \otimes \overline{\sigma}.
\]

Note that \( \tau_i^{(-1)} \simeq \overline{\tau_i} \), where \( \overline{\tau_i} \) is the dual representation of \( \tau_i \); see Theorem 4.2.2 of [6] for example.

Define \( W(\pi) \) to be the following subgroup of \( W_{P_{\tau}}(\mathbb{F}) \):

\[
W(\pi) = \{ w \in W_{P_{\tau}}(\mathbb{F}) \mid \pi^w \simeq \pi \}.
\]

\( \pi \) is called regular if \( W(\pi) \) is trivial and singular otherwise. Bruhat theory, [4], implies the following well known result:
Theorem 4.1. If \( \pi \) is regular then

\[
\text{Hom}_{\text{Sp}_{2n}(\mathbb{F})}(I(\pi), I(\pi)) \simeq \mathbb{C}.
\] (4.9)

See the Corollary in page 177 of [21] for this theorem in the context of connected algebraic reductive p-adic groups, and see Proposition 6 in page 61 of [1] for this theorem in the context of an \( r \)-fold cover of \( GL_n(\mathbb{F}) \). This theorem follows immediately from the description of the Jordan-Holder series of a Jacquet module of a parabolic induction; see Proposition Theorem 5 in page 49 of [1]. This Jordan-Holder series is an exact analog to the Jordan-Holder series of a Jacquet module of a parabolic induction in the linear case. The proof of Proposition Theorem 5 in page 49 of [1] is done exactly as in the linear case. It uses Bruhat decomposition and a certain filtration (see Theorem 5.2 of [9]) which applies to both linear and metaplectic cases. An immediate corollary of Theorem 4.1 is the following.

Theorem 4.2. Let \( \vec{\nu} = (n_1, n_2, \ldots, n_r; k) \) where \( n_1, n_2, \ldots, n_r, k \) are \( r + 1 \) non-negative integers whose sum is \( n \). For \( 1 \leq i \leq r \) let \( \tau_i \) be an irreducible admissible supercuspidal representation of \( GL_{n_i}(\mathbb{F}) \). Let \( \sigma \) be an irreducible admissible supercuspidal genuine representation of \( \text{Sp}_{2k}(\mathbb{F}) \). Denote \( \pi = (\otimes_{i=1}^{r} \tau_i^{-1}) \otimes \sigma \). If \( \tau_i \neq \tau_j \) for all \( 1 \leq i < j \leq n \) and \( \tau_i \neq \tau_j \) for all \( 1 \leq i \leq j \leq n \) then \((4.9)\) holds. If we assume in addition that \( \tau_i \) is unitary for each \( 1 \leq i \leq r \) then \( I(\pi) \) is irreducible.

Proof. From (4.7) it follows that \( \pi \) is regular. Thus, \((4.9)\) follows immediately from Theorem 4.1. Note that since the center of \( \text{Sp}_{2k}(\mathbb{F}) \) is finite \( \sigma \) is unitary. Therefore, the assumption that \( \tau_i \) is unitary for each \( 1 \leq i \leq r \) implies that \( \pi \) is unitary. Hence \( I(\pi) \) is unitary. The irreducibility of \( I(\pi) \) follows now from \((4.9)\). \( \square \)

4.3 The intertwining operator

Let \( n_1, n_2, \ldots, n_r, k \) be \( r+1 \) nonnegative integers whose sum is \( n \). Put \( \vec{t} = (n_1, n_2, \ldots, n_r; k) \). For \( 1 \leq i \leq r \) let \( (\tau_i, V_{\tau_i}) \) be an irreducible admissible representation of \( GL_{n_i}(\mathbb{F}) \) and let \( (\sigma, V_{\sigma}) \) be an irreducible admissible genuine representation of \( \text{Sp}_{2k}(\mathbb{F}) \). Fix now \( w \in W_{\vec{t}}(\mathbb{F}) \). Define

\[
N_{\vec{t},w}(\mathbb{F}) = Z_{\text{Sp}_{2n}(\mathbb{F})} \cap (wN_{\vec{t}}(\mathbb{F})w^{-1}),
\]

where \( N_{\vec{t}}(\mathbb{F})^{-} \) is the unipotent radical opposite to \( N_{\vec{t}}(\mathbb{F}) \), explicitly in the \( \text{Sp}_{2n}(\mathbb{F}) \) case:

\[
N_{\vec{t}}(\mathbb{F})^{-} = J_{2n}N_{\vec{t}}(\mathbb{F})J_{2n}^{-1}.
\]

For \( g \in \text{Sp}_{2n}(\mathbb{F}) \) and \( f \in I(\pi(\sigma)) \) define

\[
A_wf(g) = \int_{N_{\vec{t},w}(\mathbb{F})} f((wt_w),1)^{-1}(n,1)g \, dn,
\] (4.10)

where \( t_w \) is a particular diagonal element in \( M_{\vec{t}}(\mathbb{F}) \cap W'_{\text{Sp}_{2k}(\mathbb{F})} \) whose entries are either 1 or -1. The exact definition of \( t_w \) will be given in Chapter 6; see (6.5). The appearance of \( t_w \) here is technical.

As in the algebraic case (see Section 2 of [38]) the last integral converges absolutely in some open set of \( \mathbb{C}^+ \) and has a meromorphic continuation to \( \mathbb{C}^+ \). In fact, it is a rational
function in $q^{s_1}, q^{s_2}, \ldots, q^{s_r}$. See Chapter 5 of [1] for the context of a p-adic covering group. We define its continuation to be the intertwining operator

$$A_w : I(\tau_1(s_1), \tau_2(s_2), \ldots, \tau_r(s_r), \sigma) \to I(\tau_1^{(e_1)}, \tau_2^{(e_2)}, \ldots, \tau_r^{(e_r)}),$$

Denote $\widetilde{\tau}^w = (e_1s_1, \ldots, e_rs_r)$ and $(\otimes_{i=1}^r \tau_i)^w = (\otimes_{i=1}^r \tau_i^{(e_i)})$.

4.4 The Knapp-Stein dimension theorem

We keep the notation and assumptions of the first paragraph of Section 4.2. From Theorem 4.1 it follows that outside a Zariski open set of values of $(q^{s_1}, \ldots, q^{s_r}) \in (\mathbb{C}^*)^r$,

$$\text{Hom}_{\text{Sp}_{2n}(F)}(I(\pi(\widetilde{\tau})), I(\pi(\widetilde{\tau}))) \simeq \mathbb{C}.$$ 

This implies that for every $w_0 \in W_{P^+}(F)$ there exists a meromorphic function $\beta(\widetilde{\tau}, \tau_1, \ldots, \tau_r, \sigma, w_0)$ such that

$$A_{w_0}^{-1} A_{w_0} = \beta^{-1}(\widetilde{\tau}, \tau_1, \ldots, \tau_r, \sigma, w_0) \text{Id}.$$ \hspace{1cm} (4.11)

Remark: In the case of connected reductive quasi-split algebraic group this function differs from the Plancherel measure by a well understood positive function; see Section 3 of [54] for example. For connected reductive quasi-split algebraic groups it is known that if (with addition to all the other assumptions made here) we assume that $\tau_1, \ldots, \tau_r, \sigma$ are unitary then $\beta(\widetilde{\tau}, \tau_1, \ldots, \tau_r, \sigma, w_0)$, as a function of $\widetilde{\tau}$ is analytic on the unitary axis. This is (part of) the content of Theorem 5.3.5.2 of [60] or equivalently Lemma V.2.1 of [67]. The proof of this property has a straightforward generalization to the metaplectic group.

Let $\Sigma_{P^+}(F)$ be the set of reflections corresponding to the roots of $T_{\text{Sp}_{2n}(F)}$ outside $M_{\Gamma}(F)$. $W_{P^+}(F)$ is generated by $\Sigma_{P^+}(F)$. Following [58] we denote by $W''(\pi)$ the subgroup of $W(\pi)$ generated by the elements $w \in \Sigma_{P^+}(F) \cap W(\pi)$ which satisfy

$$\beta(0, \tau_1, \ldots, \tau_r, \sigma, w) = 0.$$ 

The Knapp-Stein dimension theorem states the following:

**Theorem 4.3.** Let $\widetilde{\tau} = (n_1, n_2, \ldots, n_r; k)$ where $n_1, n_2, \ldots, n_r, k$ are $r + 1$ non-negative integers whose sum is $n$. For $1 \leq i \leq r$ let $\tau_i$ be an irreducible admissible supercuspidal unitary representation of $GL_{n_i}(F)$. Let $\sigma$ be an irreducible admissible supercuspidal genuine representation of $\text{Sp}_{2k}(F)$. We have:

$$\text{Dim}(\text{Hom}(I(\pi), I(\pi))) = [W(\pi) : W''(\pi)].$$ 

The Knapp-Stein dimension theorem was originally proved for real groups, see [31]. Harish-Chandra and Silberger proved it for algebraic p-adic groups; see [58], [59] and [60]. It is a consequence of Harish-Chandra’s completeness theorem; see Theorem 5.5.3.2 of [60]. Although Harish-Chandra’s completeness theorem was proved for algebraic p-adic groups its proof holds for the metaplectic case as well; see the remarks on page 99 of [11]. Thus, Theorem 4.3 is a straightforward generalization of the Knapp-Stein dimension theorem to the metaplectic group. The precise details of the proof will appear in a future publication of the author. We note here that the theorem presented in [58] is more general; it deals with square integrable representations. The version given here will be sufficient for our purposes.
5 Uniqueness of Whittaker model

Let \( \mathbb{F} \) be a local field and let \( \psi \) be a non-trivial character of \( \mathbb{F} \). We shall continue to denote by \( \psi \) the non-degenerate character of \( Z_{Sp_{2n}}(\mathbb{F}) \) given by

\[
\psi(z) = \psi\left(z_{(n,2n)} + \sum_{i=1}^{n-1} z_{(i,i+1)}\right).
\]

From [2,12] it follows that \( Z_{Sp_{2n}}(\mathbb{F}) \simeq Z_{Sp_{2n}}(\mathbb{F}) \times \{\pm 1\} \). We define a character of \( Z_{Sp_{2n}}(\mathbb{F}) \) by \((z,\epsilon) \mapsto \epsilon \psi(z)\), and continue to denote it by \( \psi \). We shall also denote by \( \psi \) the characters of \( Z_{GL_m}(\mathbb{F}) \) identified with \( i_{k,n}(j_{m,k}(Z_{GL_m}(\mathbb{F})),1) \) and of \( Z_{Sp_{2k}}(\mathbb{F}) \) identified with \( i_{k,n}(Z_{Sp_{2k}}(\mathbb{F})) \), obtained by restricting \( \psi \) as a character of \( Z_{Sp_{2n}}(\mathbb{F}) \).

Let \( (\pi, V_\pi) \) be a smooth representation of \( Sp_{2n}(\mathbb{F}) \) (of \( GL_n(\mathbb{F}) \)). By a \( \psi \)-Whittaker functional on \( \pi \) we mean a linear functional \( w \) on \( V_\pi \) satisfying

\[
w(\pi(z)v) = \psi(z)w(v)
\]

for all \( v \in V_\pi \), \( z \in Z_{Sp_{2n}}(\mathbb{F})(Z_{GL_n}(\mathbb{F})) \). Define \( W_{\pi,\psi} \) to be the space of Whittaker functionals on \( \pi \) with respect to \( \psi \). If \( \mathbb{F} \) is archimedean we add smoothness requirements to the definition of a Whittaker functional, see [22] or [17]. \( \pi \) is called \( \psi \)-generic or simply generic if it has a non-trivial Whittaker functional with respect to \( \psi \). If \( w \) is a non-trivial \( \psi \)-Whittaker functional on \((\pi, V)\) then one may consider \( W_w(\pi, \psi) \), the space of complex functions on \( Sp_{2n}(\mathbb{F}) \) (on \( GL_n(\mathbb{F}) \)) of the form

\[
g \mapsto w(\pi(g)v),
\]

where \( v \in V \). \( W_w(\pi, \psi) \) is a representation space of \( Sp_{2n}(\mathbb{F}) \) (of \( GL_n(\mathbb{F}) \)). The group acts on this space by right translations. It is clearly a subspace of \( \text{Ind}_{Z_{Sp_{2n}}(\mathbb{F})}^{Sp_{2n}(\mathbb{F})} \psi \) (of \( \text{Ind}_{Z_{GL_n}(\mathbb{F})}^{GL_n(\mathbb{F})} \psi \)).

From Frobenius reciprocity it follows that if \( \pi \) is irreducible and \( \dim(W_{\pi,\psi}) = 1 \) then \( W_w(\pi, \psi) \) is the unique subspace of \( \text{Ind}_{Z_{Sp_{2n}}(\mathbb{F})}^{Sp_{2n}(\mathbb{F})} \psi \) (of \( \text{Ind}_{Z_{GL_n}(\mathbb{F})}^{GL_n(\mathbb{F})} \psi \)) which is isomorphic to \( \pi \). In this case we drop the index \( w \) and we write \( W(\pi, \psi) \). One can identify \( \pi \) with \( W(\pi, \psi) \) which is called the Whittaker model of \( \pi \).

For quasi-split algebraic groups, uniqueness of Whittaker functional for irreducible admissible representations is well known, see [57], [17] and [8] for the p-adic case, and see [22] and [57] for the archimedean case. Uniqueness of Whittaker models for irreducible admissible representations of \( Sp_{2n}(\mathbb{F}) \) is a well known fact to the experts. Although this fact had been used many times, there is no written proof of this uniqueness in general (a uniqueness theorem for principal series representations in the case of p-adic fields can be found in [2]).

In this chapter we correct the situation and prove

**Theorem 5.1.** Let \( \pi \) be an irreducible admissible representation of \( Sp_{2n}(\mathbb{F}) \). Then,

\[
\dim(W_{\pi,\psi}) \leq 1.
\]

We note that uniqueness of Whittaker model does not hold in general for covering groups. See the introduction of [2] for a \( k \) fold cover of \( GL_n(\mathbb{F}) \), see [5] for an application of a theory of local coefficients in a case where Whittaker model is not unique and see [15] for a unique model for genuine representations of a double cover of \( GL_2(\mathbb{F}) \).
Uniqueness for $\text{SL}_2(\mathbb{R})$ was proven in [66]. To prove uniqueness for $\text{SP}_{2n}(\mathbb{R})$ for general $n$, it is sufficient to consider principal series representations. The proof in this case follows from Bruhat Theory. In fact, the proof goes almost word for word as the proof of Theorem 2.2 of [22] for minimal parabolic subgroups. The proof in this case is a heredity proof in the sense of [43]. The reason that this, basically algebraic, proof works for $\text{Sp}_{2n}(\mathbb{F})$ is not 2 then the inverse image in $\text{SP}_{2n}(\mathbb{F})$ of a maximal torus of $\text{Sp}_{2n}(\mathbb{F})$ is commutative. This implies that its irreducible representations are one dimensional. Uniqueness for $\text{SP}_{2n}(\mathbb{C})$ follows from the uniqueness for $\text{Sp}_{2n}(\mathbb{C})$ since $\text{Sp}_{2n}(\mathbb{C}) = \text{Sp}_{2n}(\mathbb{C}) \times \{\pm 1\}$. However, the p-adic case is not as easy. Although it turns out that one may use similar methods to those used in [17], [57], and [8] for quasi-split groups defined over $\mathbb{F}$, one has to use a certain involution on $\text{Sp}_{2n}(\mathbb{F})$ whose crucial properties follow from the results of Section 2.6.

5.1 Non-archimedean case

Until the rest of this Chapter, $\mathbb{F}$ will denote a p-adic field. We may assume that the representations of $\text{SP}_{2n}(\mathbb{F})$ in discussion are genuine, otherwise the proof is reduced at once to the well known $\text{Sp}_{2n}(\mathbb{F})$ case.

**Theorem 5.2.** If $(\pi, V_\pi)$ is an irreducible admissible representation of $\text{SP}_{2n}(\mathbb{F})$ then,

$$\dim(W_{\pi, \psi}) \cdot \dim(W_{\hat{\pi}, \psi^{-1}}) \leq 1.$$ 

The proof of Theorem 5.2 will show:

**Theorem 5.3.** Suppose $(\pi, V_\pi)$ is an irreducible admissible representation of $\text{SP}_{2n}(\mathbb{F})$. If from the existence of a non-trivial Whittaker functional on $\pi$ with respect to $\psi$ one can deduce the existence of a non-trivial Whittaker functional on $\hat{\pi}$ with respect to $\psi^{-1}$, then $\dim(W_{\pi, \psi}) \leq 1$.

**Corollary 5.1.** If $(\pi, V_\pi)$ is an irreducible admissible unitary representation of $\text{SP}_{2n}(\mathbb{F})$ then $\dim(W_{\pi, \psi}) \leq 1$.

**Proof.** We show that the conditions of Theorem 5.3 hold in this case. Indeed, if $(\pi, V_\pi)$ is an irreducible admissible unitary representation of $\text{SP}_{2n}(\mathbb{F})$ one can realize the dual representation in the space $\bar{V}_\pi$, which is identical to $V_\pi$ as a commutative group. The scalars act on $\bar{V}_\pi$ by $\lambda \cdot v = \bar{\lambda}v$. The action of $\hat{\pi}$ in this realization is given by $\hat{\pi}(g) = \pi(g)$. It is clear now that if $L$ is a non-trivial Whittaker functional on $\pi$ with respect to $\psi$ then $L$, acting on $\bar{V}_\pi$, is a non-trivial Whittaker functional on $\hat{\pi}$ with respect to $\psi^{-1}$. \qed

Since every supercuspidal representation $\pi$ is unitary, it follows from Corollary 5.1 that $\dim(W_{\pi, \psi}) \leq 1$. Furthermore, assume now that $\pi$ is an irreducible admissible representation of $\text{SP}_{2n}(\mathbb{F})$. Then $\pi$ is a subquotient of a representation induced from a supercuspidal representation of a parabolic subgroup: Let $H$ be a parabolic subgroup of $\text{SP}_{2n}(\mathbb{F})$. We may assume that $H = P_{\mathbb{F}}$ where $\mathbb{F} = (n_1, n_2, \ldots, n_r; k)$. Suppose that for $1 \leq i \leq r$, $\sigma_i$ is a supercuspidal representation of $GL_{n_i}(\mathbb{F})$, and that $\pi'$ is a supercuspidal representation of $\text{SP}_{2n}(\mathbb{F})$. Denote by $\psi_i$ and $\psi'$ the restriction of $\psi$ to $Z_{GL_{n_i}}(\mathbb{F})$ and $Z_{SP_{2n}}(\mathbb{F})$ respectively,
embedded in $\overline{M_H}$. Note that $\dim(W_{\sigma,\psi}) \leq 1$ and that $\dim(W_{\pi',\psi'}) \leq 1$. Let $\tau$ be the representation of $\overline{M_H}$ defined by
\[
(diag(g_1, g_2, \ldots, g_r, h, t^{-g_r^{-1}}, \ldots, t^{-g_1^{-1}}), \epsilon) \mapsto \otimes_{i=1}^{\frac{r+1}{2}} \sigma_i(g_i) \gamma(\det g_i) \otimes \pi(h, \epsilon),
\]
We extend $\tau$ from $M_H$ to $\overline{H}$, letting the unipotent radical act trivially. Define $\text{Ind}_{\overline{H}}^{\overline{G}2n(\mathbb{F})} \tau$ to be the corresponding induced representation. One may use the methods of Rodier, [13], extended by Banks, [2], to a non-algebraic setting and conclude that
\[
\dim(W_{\text{Ind}_{\overline{H}}^{\overline{G}2n(\mathbb{F})} \tau, \psi}) = \dim(W_{\pi',\psi}) \Pi_{i=1}^{\frac{r+1}{2}} \dim(W_{\sigma,\psi}).
\]
Now, if $V_2 \subseteq V_1 \subseteq \text{Ind}_{\overline{H}}^{\overline{G}2n(\mathbb{F})} \tau$ are two $\overline{G}2n(\mathbb{F})$ modules then clearly the dimension of the space of Whittaker functionals on $V_1$ and $V_2$ with respect to $\psi$ is not greater than $\dim(W_{\text{Ind}_{\overline{H}}^{\overline{G}2n(\mathbb{F})} \tau, \psi})$. It follows now that $\dim(W_{\pi',\psi}) \leq 1$. This is due to the exactness of twisted Jacquet functor (this is a basic property of twisted Jacquet functor. It is proven for the algebraic case in 5.12 of [8]. The proof given there continues to hold in the metaplectic case; see page 72 of [68] for example). Thus, we proved Theorem 5.1.

5.1.1 Proof of Theorem 5.2

Define the following map on $\overline{G}2n(\mathbb{F})$:
\[
g \mapsto \tau g = \sigma_0(t g) \sigma_0^{-1},
\]
where $\sigma_0 = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n \end{pmatrix}$, $\epsilon_n = \text{diag}(1, -1, 1, \ldots, (-1)^{n+1}) \in GL_n(\mathbb{F})$.

We note that $\sigma_0^{-1} = t^\sigma_0 = \sigma_0$, and that $\sigma_0 \in G\overline{G}2n(\mathbb{F})$, with similitude factor -1. Hence, $g \mapsto \tau g$ is an anti-automorphism of order 2 of $\overline{G}2n(\mathbb{F})$. We now extend $\tau$ to $\overline{G}2n(\mathbb{F})$. A similar lifting was used in [15] for $GL_2(\mathbb{F})$.

**Lemma 5.1.** The map $(g, \epsilon) \mapsto (\tau g, \epsilon) = \left( \tau g, \epsilon \right)$ is an anti-automorphism of order 2 of $\overline{G}2n(\mathbb{F})$. It preserves both $\overline{Z}_{\overline{G}2n(\mathbb{F})}$ and $\psi$ and satisfies $\tau(S\overline{G}2n(\mathbb{F}), \epsilon) = (S\overline{G}2n(\mathbb{F}), \epsilon)$.

**Proof.** We note that if $g \in \overline{G}2n(\mathbb{F})$ then $t g = J g^{-1} (-J)$. Hence,
\[
\tau g = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n \end{pmatrix} \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} g^{-1} \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n \end{pmatrix}^{-1}.
\]
Thus, the map
\[
(g, \epsilon) \mapsto \left( \tau g, \epsilon c(g, g^{-1}) v_{-1}(g^{-1}) c(p_c, \tilde{g}) c(p_c, \tilde{g}^{-1}) c(p_c, p_c^{-1}) \right),
\]
where $p_c = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n \end{pmatrix} \in P$, $\tilde{g} = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} g^{-1} \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$ is an anti-automorphism of $\overline{G}2n(\mathbb{F})$. We now show that
\[
c(g, g^{-1}) v_{-1}(g^{-1}) c(p_c, \tilde{g}) c(p_c, \tilde{g}^{-1}) c(p_c, p_c^{-1}) c(p_c, p_c^{-1}) = 1.
\]
Indeed, the fact that $c(p_c, \tilde{g}) c(p_c, p_c^{-1}) c(p_c, p_c^{-1}) = 1$ is a property of Rao’s cocycle and is noted in [5.13]. The fact that $c(g, g^{-1}) v_{-1}(g^{-1}) = 1$ is a consequence of the calculation of $v_{\lambda}(g)$ and is noted in [2.41]. The rest of the assertions mentioned in this Lemma are clear. □
Let $S(Sp_{2n}(\mathbb{F}))$ be the space of Schwartz functions on $Sp_{2n}(\mathbb{F})$. For $h \in Sp_{2n}(\mathbb{F})$, $\phi \in S(Sp_{2n}(\mathbb{F}))$ we define $\lambda(h)\phi$, $\rho(h)\phi$, and $\tau_\phi$ to be the following elements of $S(Sp_{2n}(\mathbb{F}))$:

$$
(\rho(h)\phi)(g) = \phi(gh), \quad (\lambda(h)\phi)(g) = \phi(h^{-1}g), \quad \tau_\phi(g) = \phi(\tau g).
$$

(5.2)

$S(Sp_{2n}(\mathbb{F}))$ is given an algebra structure, called the Hecke Algebra, by the convolution. Given $(\pi, V_\pi)$, a smooth representation of $Sp_{2n}(\mathbb{F})$, we define, as usual a representation of this algebra in the space $V_\pi$ by

$$
\pi(\phi)v = \int_{Sp_{2n}(\mathbb{F})} \phi(g)\pi(g)v \, dg.
$$

(5.3)

The following theorem, known as Gelfand-Kazhdan Theorem, in the context of $GL_n(\mathbb{F})$ (see [17] or [8]), will be used in the proof of Theorem 5.2.

**Theorem 5.4.** Suppose that $D$ is a functional on $S(Sp_{2n}(\mathbb{F}))$ satisfying

$$
D(\lambda(z_1)\rho(z_2)\phi) = \psi(z_2z_1^{-1})D(\phi)
$$

for all $\phi \in S(Sp_{2n}(\mathbb{F}))$, $z_1, z_2 \in Z_{Sp_{2n}(\mathbb{F})}$. Then $D$ is $\tau$ invariant, i.e., $D(\tau_\phi) = D(\phi)$ for all $\phi \in S(Sp_{2n}(\mathbb{F}))$.

We will give the proof of this theorem in [5.1.2]. Here we shall use this theorem to prove Theorem 5.2.

**Proof.** Since any irreducible admissible representation of $Sp_{2n}(\mathbb{F})$ may be realized as a dual representation, the proof of Theorem 5.2 amounts to showing that if $W_{\pi, \psi} \neq 0$ then $\dim(W_{\pi, \psi}^{-1}) \leq 1$. We shall use a similar argument to the one used in [61]. Theorem 2.1. Let $w$ be a non-trivial Whittaker functional on $(\pi, V_\pi)$ with respect to $\psi$. Suppose $\hat{w}_1$ and $\hat{w}_2$ are two non-trivial Whittaker functionals on $\hat{\pi}$ with respect to $\psi^{-1}$. The proof will be achieved once we show that $\hat{w}_1$ and $\hat{w}_2$ are proportional.

For $\phi \in S(Sp_{2n}(\mathbb{F}))$, let $\pi^*(\phi)w$ be a functional on $V_\pi$ defined by:

$$
(\pi^*(\phi)w)v = \int_{Sp_{2n}(\mathbb{F})} \phi(g)w(\pi(g^{-1})v) \, dg.
$$

(5.4)

Note that since $\phi \in S(Sp_{2n}(\mathbb{F}))$, $\pi^*(\phi)w$ is smooth even if $w$ is not. Thus, $\pi^*(\phi)w \in V_\pi$. By a change of variables we note that

$$
\hat{\pi}(h)(\pi^*(\phi)w) = \pi^*(\lambda(h)\phi)w.
$$

(5.5)

Define now $R_1$ and $R_2$ to be the following two functionals on $S(Sp_{2n}(\mathbb{F}))$:

$$
R_i(\phi) = \hat{w}_i(\pi^*(\phi)w), \quad i = 1, 2.
$$

(5.6)

Using (5.5), the facts that $w$, $\hat{w}_1$, $\hat{w}_2$ are Whittaker functionals, and by changing variables we observe that for all $z \in Z_{Sp_{2n}(\mathbb{F})}$

$$
R_i(\lambda(z)\phi) = \psi^{-1}(z)R_i(\phi), \quad R_i(\rho(z)\phi) = \psi(z)R_i(\phi).
$$

(5.7)
From Theorem 5.4 it follows now that $R_i(\phi) = R_i(\bar{\tau}\phi)$. Hence,
\[
\hat{w}_i(\bar{\pi}(h)\pi^*(\phi)w) = \hat{w}_i\left(\pi^*\left(\bar{\tau}(\lambda(h)\phi)\right)w\right).
\] (5.8)

Using a change of variables again we also note that
\[
\pi^*\left(\bar{\tau}(\lambda(h)\phi)\right)w = \pi^*\left(\bar{\tau}\phi\right)\left(\pi^*\left(\bar{\tau}h\right)w\right).
\] (5.9)

Joining (5.8) and (5.9) we obtain
\[
\hat{w}_i(\bar{\pi}(h)\pi^*(\phi)w) = \hat{w}_i\left(\pi^*\left(\bar{\tau}\phi\right)\left(\pi^*\left(\bar{\tau}h\right)w\right)\right).
\] (5.10)

In particular, if for some $\phi \in S(Sp_{2n}(F))$, $\pi^*(\phi)w$ is the zero functional, then for all $h \in Sp_{2n}(F)$:
\[
\hat{w}_i\left(\pi^*\left(\bar{\tau}\phi\right)\left(\pi^*\left(\bar{\tau}h\right)w\right)\right) = 0.
\]

In this case, for all $f \in S(Sp_{2n}(F))$:
\[
0 = \hat{w}_i\left(\int_{Sp_{2n}(F)} f(h)\hat{w}_i\left(\pi^*\left(\bar{\tau}\phi\right)\left(\pi^*\left(\bar{\tau}h\right)w\right)\right) dh\right).
\] (5.11)

By the definition of $\pi^*(\phi)w$ and by changing the order of integration, we have, for all $v \in V_\pi$:
\[
\int_{Sp_{2n}(F)} f(h)\hat{w}_i\left(\pi^*\left(\bar{\tau}\phi\right)\left(\pi^*\left(\bar{\tau}h\right)w\right)\right)(v)dh = \bar{\pi}(\tau\phi)(\pi^*(f)w)(v).
\] (5.12)

(5.11) and (5.12) yield that if $\pi^*(\phi)w = 0$ then for all $v \in V_\pi$, $f \in S(Sp_{2n}(F))$:
\[
\hat{w}_i\left(\pi\left(\bar{\tau}\phi\right)\left(\pi^*(f)w\right)\right)(v) = 0.
\] (5.13)

From the fact $\pi$ is irreducible one can conclude that
\[
\pi^*\left(S\left(Sp_{2n}(F)\right)\right)w = V_{\bar{\pi}}.
\]

Indeed, since $\pi^*\left(S\left(Sp_{2n}(F)\right)\right)w$ is an $Sp_{2n}(F)$ invariant subspace we only have to show that $\pi^*\left(S\left(Sp_{2n}(F)\right)\right)w \neq \{0\}$, and this is clear.

Hence, using a change of variables once more we see that for all $\xi \in V_{\bar{\pi}}$ we have
\[
0 = \hat{w}_i(\bar{\pi}(\bar{\tau}\phi)\xi) = \int_{Sp_{2n}(F)} \bar{\tau}\phi(g^{-1})\hat{w}_i(\bar{\pi}(g^{-1}\xi)) dg.
\]

For $g \in Sp_{2n}(F)$ define $\omega g = \bar{\tau}g^{-1}$, and for $\phi \in S(Sp_{2n}(F))$ define $\omega\phi(g) = \phi(\omega g)$. We have just seen that if $\pi^*(\phi)w = 0$ then
\[
(\pi^*(\omega\phi))\hat{w}_i = 0.
\] (5.14)

This fact and the fact that $\pi^*\left(S\left(Sp_{2n}(F)\right)\right)w = V_{\bar{\pi}}$ show that the following linear maps are well defined: For $i = 1, 2$ define $S_i : V_{\bar{\pi}} \rightarrow V_{\bar{\pi}}$ by $S_i(\pi^*(\phi)w) = ((\pi^*(\omega\phi))\hat{w}_i$. One can
easily check that \( S_1 \) and \( S_2 \) are two intertwining maps from \( \hat{\pi} \) to \( h \mapsto \hat{\pi}(\tau h^{-1}) \). The last representation is clearly isomorphic to \( h \mapsto \pi(\tau h^{-1}) \), which due to the irreducibility of \( \pi \) is irreducible. Schur’s Lemma guarantees now the existence of a complex number \( c \) such that \( S_2 = c S_1 \). So, for all \( \phi \in S(\widetilde{\text{Sp}}_{2n}(\mathbb{F})) \) and for all \( \xi \in V_\mathbb{F} \):

\[
\int_{\text{Sp}_{2n}(\mathbb{F})} \phi(g)(\hat{w}_2 - c\hat{w}_1)(\hat{\pi}(g^{-1})\xi) dg = 0.
\]

We can now conclude that \( \hat{w}_1 \) and \( \hat{w}_2 \) are proportional. \( \Box \)

**Remark:** Let \( \pi \) be a genuine generic admissible irreducible representation of \( \text{Sp}_{2n}(\mathbb{F}) \). Assume that \( \hat{\pi} \) is also generic (by Corollary 5.1 this assumption holds if \( \pi \) is a unitary, in particular if \( \pi \) is supercuspidal). Then, from the last step in the proof just given, it follows that

\[
\hat{\pi} \simeq h \mapsto \pi(\tau h^{-1}).
\]

In page 92 of \([38]\) a similar result is proven for \( \pi \), an irreducible admissible representation of \( \text{Sp}_{2n}(\mathbb{F}) \) provided that its character is locally integrable. Recently, Sun has proved this result for any genuine admissible irreducible representation of \( \text{Sp}_{2n}(\mathbb{F}) \), see \([63]\).

### 5.1.2 Proof of Theorem 5.4

Put \( G = \text{Sp}_{2n}(\mathbb{F}) \) and define \( H = Z_{\text{Sp}_{2n}(\mathbb{F})} \times Z_{\text{Sp}_{2n}(\mathbb{F})} \). Denote by \( \tilde{\psi} \) be the character of \( H \) defined by \( \tilde{\psi}(n_1, n_2) = \psi(n_1^{-1}n_2) \). \( H \) is acting on \( G \) by \( (n_1, n_2) \cdot g = n_1 g n_2^{-1} \). For \( g \in G \) we denote by \( H_g \) the stabilizer of \( g \) in \( H \). It is clearly a unimodular group. Let \( Y \) be an \( H \) orbit, that is, a subset of \( G \) of the form \( H \cdot g = Z_{\text{Sp}_{2n}(\mathbb{F})} g Z_{\text{Sp}_{2n}(\mathbb{F})} \), where \( g \) is a fixed element in \( G \). Let \( S(Y) \) be the space of Schwartz functions on \( Y \). \( H \) acts on \( S(Y) \) by

\[
(h \cdot \phi)(k) = \phi(h^{-1} \cdot k) \tilde{\psi}^{-1}(h).
\]

With this notation, the proof of Theorem 5.4 goes almost word for word as Soudry’s proof of Theorem 2.3 in [61]. The main ingredient of that proof was Theorem 6.10 of [8] which asserts that the following four conditions imply Theorem 5.4:

1. The set \( \{(g, h \cdot g) \mid g \in G, h \in H\} \) is a union of finitely many locally closed subsets of \( G \times G \).
2. For each \( h \in H \) there exists \( h_\tau \in H \) such that for all \( g \in G \): \( h \cdot \tau g = \tau(h_\tau \cdot g) \).
3. \( \tau \) is of order 2.
4. Let \( Y \) be an \( H \) orbit. Suppose that there exists a non zero functional on \( S(Y) \), satisfying \( D(h \cdot \phi) = D(\phi) \) for all \( \phi \in S(Y), h \in H \), then \( \tau Y = Y \) and for all \( \phi \in S(Y) \), \( D(\tau \phi) = D(\phi) \).

Of these four conditions, only the forth requires some work. In order to make Soudry’s proof work in our context we only have to change Theorem 2.2 of [61] to

**Theorem 5.5.** Fix \( g \in G \). If for all \( h \in H_g \) we have \( \tilde{\psi}^{-1}(h) = 1 \) then, there exists \( h^g \in H \) such that \( h^g \cdot g = \tau g \) and \( \tilde{\psi}^{-1}(h) = 1 \).

Before we prove this theorem we state and prove its analog for \( \text{Sp}_{2n}(\mathbb{F}) \). After the proof we give a short argument which completes the proof of Theorem 5.5.
Lemma 5.2. For a fixed $g \in Sp_{2n}(F)$ one of the following holds:
A. There exist $n_1, n_2 \in Z_{Sp_{2n}(F)}$ such that $n_1 g n_2 = g$ and $\psi(n_1 n_2) \neq 1$.
B. There exist $n_1, n_2 \in Z_{Sp_{2n}(F)}$ such that $n_1 g n_2 = g$ and $\psi(n_1 n_2) = 1$.

(in fact this lemma gives the uniqueness of Whittaker model in the linear case)

Proof. Due to the fact that $\tau$ preserves both $Z_{Sp_{2n}(F)}$ and $\psi$, it is enough to prove this lemma only for a complete set of representatives of $Z_{Sp_{2n}(F)} \setminus Sp_{2n}(F) / Z_{Sp_{2n}(F)}$. We recall the Bruhat decomposition:

$$Sp_{2n}(F) = \bigcup_{w \in W_{Sp_{2n}(F)}} Z_{Sp_{2n}(F)} T_{Sp_{2n}(F)} w Z_{Sp_{2n}(F)}.$$ 

We realize the set of Weyl elements as

$$\{ \overline{w} \sigma T_S \mid \sigma \in S_n, S \subseteq \{1, 2, \ldots, n\} \},$$

where for $\sigma \in S_n$ we define $w_\sigma \in GL_n(F)$ by $(w_\sigma)_{i,j} = \delta_{i,\sigma(j)}$, and $\overline{w}_\sigma = \begin{pmatrix} w_\sigma & 0 \\ 0 & w_\sigma \end{pmatrix} \in Sp_{2n}(F)$. Thus, as a complete set of representatives of $Z_{Sp_{2n}(F)} \setminus Sp_{2n}(F) / Z_{Sp_{2n}(F)}$ we may take

$$\{ d^{-1} \overline{w}_\sigma^{-1} \varphi_S \mid d \in T, \sigma \in S_n, S \subseteq \{1, 2, \ldots, n\} \},$$

where $\varphi_S = \tau_S a_S c = \begin{pmatrix} M_S & M_S c \\ -M_S c & M_S \end{pmatrix}$, where for $S \subseteq \{1, 2, \ldots, n\}$, $M_S \in Mat_{n \times n}(F)$ is defined by $(M_S)_{i,j} = \delta_{i,j} \delta_{i \in S}$.

Denote by $w_k$ the $k \times k$ invertible matrix defined by $(w_k)_{i,j} = \delta_{i+j,n+1}$. Suppose that $k_1, k_2, \ldots, k_p, k$ are non negative integers such that $k + \sum_{i=1}^p k_i = n$. Suppose also that $a_1, a_2, \ldots, a_p \in F^*$, and $\eta \in \{\pm 1\}$. For

$$\overline{w}_\sigma = diag(w_{k_1}, w_{k_1}, \ldots, w_{k_p}, I_k, w_{k_1}, w_{k_1}, \ldots, w_{k_p}, I_k),$$

$$d = diag(a_1 \epsilon_{k_1}, a_2 \epsilon_{k_2}, \ldots, a_p \epsilon_{k_p}, \eta I_k, a_1^{-1} \epsilon_{k_1}, a_2^{-1} \epsilon_{k_2}, \ldots, a_p^{-1} \epsilon_{k_p}, \eta I_k),$$

and $S = \{n-k+1, n-k, \ldots, n\}$ one checks that

$$\tau(d^{-1} \overline{w}_\sigma^{-1} \varphi_S) = d^{-1} \overline{w}_\sigma^{-1} \varphi_S.$$ 

Thus $d^{-1} \overline{w}_\sigma^{-1} \varphi_S$ is of type $B$.

We shall show that all other representatives are of type $A$: We will prove that in all the other cases one can find $n_1, n_2 \in Z_{Sp_{2n}(F)}$ such that

$$\overline{w}_\sigma d n_1 d^{-1} \overline{w}_\sigma^{-1} = \varphi_S n_2^{-1} \varphi_S^{-1},$$

and

$$\psi(n_1 n_2) \neq 1.$$
We shall use the following notations and facts: Denote by \( E_{p,q} \) the \( n \times n \) matrix defined by \( (E_{p,q})_{i,j} = \delta_{p,i} \delta_{q,j} \). For \( i, j \in \{1, 2, \ldots, n\} \), \( i \neq j \) we define the root subgroups of \( Sp_{2n}(\mathbb{F}) \):

\[
U_{i,j} = \{ u_{i,j}(x) = \begin{pmatrix} I_n + xE_{i,j} & 0 \\ 0 & I_n - xE_{j,i} \end{pmatrix} \mid x \in F \} \cong F, \quad (5.20)
\]

\[
V_{i,j} = \{ v_{i,j}(x) = \begin{pmatrix} I_n & xE_{i,j} + xE_{j,i} \\ 0 & I_n \end{pmatrix} \mid x \in F \} \cong F, \quad (5.21)
\]

\[
V_{i,i} = \{ v_{i,i}(x) = \begin{pmatrix} I_n & xE_{i,i} \\ 0 & I_n \end{pmatrix} \mid x \in F \} \cong F. \quad (5.22)
\]

If \( i < j \) we call \( U_{i,j} \) a positive root subgroup. If \( j = i + 1 \) we call \( U_{i,j} \) a simple root subgroup, and if \( j > i + 1 \) we call \( U_{i,j} \) a non-simple root subgroup. We call \( U_{i,j} = U_{j,i} \) the negative of \( U_{i,j} \). \( S_n \) acts on the set \( \{ U_{i,j} \mid i, j \in \{1, 2, \ldots, n\}, i \neq j \} \) by

\[
\varpi_{\sigma} u_{i,j}(x) \varpi_{\sigma}^{-1} = u_{\sigma(i), \sigma(j)}(x), \quad (5.23)
\]

and on the set \( \{ V_{i,j} \mid i, j \in \{1, 2, \ldots, n\} \} \) by

\[
\varpi_{\sigma} v_{i,j}(x) \varpi_{\sigma}^{-1} = v_{\sigma(i), \sigma(j)}(x). \quad (5.24)
\]

\( T \) acts on each root subgroup via rational characters:

\[
du_{i,j}(x)d^{-1} = u_{i,j}(xd_id_j^{-1}), \quad (5.25)
\]

and

\[
dv_{i,j}(x)d^{-1} = v_{i,j}(xd_d_j), \quad (5.26)
\]

where \( d = \text{diag}(d_1, d_2, \ldots, d_n, d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1}) \). We also note the following: If \( i \in S \) then

\[
\varphi_S v_{i,i}(x) \varphi_S^{-1} = v_{i,i}(x), \quad (5.27)
\]

if \( i \in S, j \notin S \) then

\[
\varphi_S v_{i,j}(x) \varphi_S^{-1} = u_{i,j}(x), \quad (5.28)
\]

if \( i \in S, j \in S, i \neq j \) then

\[
\varphi_S u_{i,j}(x) \varphi_S^{-1} = u_{i,j}(x) \quad (5.29)
\]

and if \( i \notin S, j \notin S, i \neq j \) then

\[
\varphi_S u_{i,j}(x) \varphi_S^{-1} = u_{j,i}(-x) = t_{u_{i,j}}(x)^{-1}. \quad (5.30)
\]

Consider the representative \( d^{-1} \varpi_{\sigma}^{-1} \varphi_S \). Assume first that \( S \) is empty. Suppose that there exists a simple root subgroup \( U_{k,k+1} \) that \( \sigma \) takes to the negative of a non-simple root subgroup. Then we choose \( n_1 = u_{k,k+1}(x), n_2 = t_{u_{\sigma(k), \sigma(k+1)}}(d_kd_{k+1}^{-1}) \). For such a choice, by \( (5.23), (5.25) \) and \( (5.30), (5.18) \) holds. Also, since \( \psi(n_1n_2) = \psi(n_1) = \psi(x) \), it is possible by choosing \( x \) properly, to satisfy \( (5.19) \). Suppose now that there exists a non-simple positive root subgroup \( U_{i,j} \) that \( \sigma \) takes to the negative of a simple root subgroup. Then, we choose \( n_1 = u_{i,j}(x), n_2 = t_{u_{\sigma(i), \sigma(j)}}(d_id_j^{-1}) \), and repeat the previous argument. Thus, \( d^{-1} \varpi_{\sigma}^{-1} \varphi_S \) is of type A unless \( \sigma \) has the following two properties: 1) If \( \sigma \) takes a simple root to a negative root, then it is taken to the negative of a simple root. 2) If \( \sigma \) takes a non-simple positive root to a negative root, then it is taken to the negative of a non-simple root.
An easy argument shows that if \( \sigma \) has these two properties, \( \overline{w}_\sigma \) must be as in (5.14) with \( k = 0 \). We assume now that \( \overline{w}_\sigma \) has this form. To finish the case \( S = \emptyset \) we show that unless \( d \) has the form (5.14), with \( k = 0 \) and \( k_1, k_2, \ldots, k_p \) corresponding to \( \overline{w}_\sigma \), then \( d^{-1}\overline{w}_\sigma^{-1}\varphi_\emptyset \) is of type A. Indeed, suppose that there exist \( d_k \) and \( d_{k+1} \) that belong to the same block in \( \overline{w}_\sigma \), such that \( d_k \neq -d_{k+1} \). Then, we choose \( n_1 = u_{k,k+1}(x), n_2 = u_{\sigma(k),\sigma(k+1)}(d_k d_{k+1}^{-1} x) \). For such a choice, as before, (5.18) holds, and \( \psi(n_1 n_2) = \psi(x(1 + d_k d_{k+1}^{-1})) \). Therefore, it is possible, by choosing \( x \) properly, to satisfy (5.19).

We may now assume \( |S| \geq 1 \). We show that if \( n \notin S \) then \( d^{-1}\overline{w}_\sigma^{-1}\varphi_S \) is of type A. Indeed, if \( \sigma(n) \in S \), in particular \( \sigma(n) \neq n \), then for all \( x \in \mathbb{F} \), if we choose \( n_1 = v_{n,n}(x), n_2 = v_{\sigma(n),\sigma(n)}(-x d_n^2) \), by (5.21), (5.25) and (5.27), (5.18) holds. Clearly, there exists \( x \in \mathbb{F} \) such that \( \psi(n_1 n_2) = \psi(n_1) = \psi(x) \neq 1 \). Suppose now that \( n \notin S \neq \emptyset \), and that \( \sigma(n) \notin S \). In this case we can find \( 1 \leq k \leq n - 1 \), such that

\[
\sigma(k) \in S, \text{ and } (k + 1) \notin S.
\]

We choose

\[
n_1 = u_{k,k+1}(x), \quad n_2 = v_{\sigma(k),\sigma(k+1)}(-x d_k d_{k+1}^{-1}).
\]

By (5.23), (5.25) and (5.28), (5.18) holds, and since \( \psi(n_1 n_2) = \psi(n_1) = \psi(x) \), we can satisfy (5.19) by properly choosing \( x \). We assume now \( n \in S \). We also assume \( \sigma(n) \in S \), otherwise we use the last argument. Fix \( n_1 = v_{n,n}(x), n_2 = v_{\sigma(n),\sigma(n)}(-x d_n^2) \). One can check that by (5.24), (5.26), and (5.27), (5.18) holds. Note that

\[
\psi(n_1 n_2) = \begin{cases} 
  x & \text{if } \sigma(n) \neq n \\
  x(d_n^2 - 1) & \text{if } \sigma(n) = n,
\end{cases}
\]

hence unless \( \sigma(n) = n, d_n = \pm 1, d^{-1}\overline{w}_\sigma^{-1}\varphi_S \) is of type A. We now assume \( \sigma(n) = n, d_n = \pm 1 \). If \( S = \{n\} \), we use a similar argument to the one we used for \( S = \emptyset \), analyzing the action of \( \sigma \) on \( \{1, 2, 3, \ldots, n - 1\} \) this time.

We are left with the case \( \sigma(n) = n, d_n = \pm 1, S \supseteq \{n\} \). If \( \sigma(n - 1) \notin S \) we repeat an argument we used already: We choose \( 1 \leq k \leq n - 2, n_1, n_2 \) as in (5.31) and (5.32). We now assume \( \sigma(n - 1) \in S \). We choose \( n_1 = u_{n-1,n}(x), n_2 = u_{\sigma(n-1),\sigma(n)}(-x d_{n-1} d_n^{-1}) \). Using (5.23), (5.26) and (5.29) we observe that (5.18) holds. Also,

\[
\psi(n_1 n_2) = \begin{cases} 
  x & \text{if } \sigma(n - 1) \neq n - 1 \\
  x(d_{n-1} d_n^{-1} - 1) & \text{if } \sigma(n - 1) = n - 1.
\end{cases}
\]

Thus, unless \( \sigma(n - 1) = n - 1 \) and \( d_{n-1} = d_n = \pm 1, d^{-1}\overline{w}_\sigma^{-1}\varphi_S \) is of type A. Therefore, we should only consider the case \( \sigma(n) = n, \sigma(n - 1) = n - 1, d_{n-1} = d_n = \pm 1, \{n - 1, n\} \subseteq S \). We continue in the same course: If \( S = \{n - 1, n\} \) we use similar argument we used for \( S = \emptyset \), analyzing the action of \( \sigma \) on \( \{1, 2, 3, \ldots, n - 2\} \). If \( S \supseteq \{n - 1, n\} \) we show that unless \( \sigma(n - 2) = n - 2 \in S \) and \( d_{n-2} = d_{n-1} = d_n = \pm 1 \) we are in type A etc'.

We now complete the proof of Theorem 5.5. We define types \( \overline{A} \) and \( \overline{B} \) for \( \overline{S}_{2n}(\mathbb{F}) \) by analogy with the definitions given in Lemma 5.2 and show that each element of

\[
\overline{Z}_{\overline{S}_{2n}(\mathbb{F})} \setminus \overline{S}_{2n}(\mathbb{F}) \setminus \overline{Z}_{\overline{S}_{2n}(\mathbb{F})}
\]

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is either of type \( \overline{A} \) or of type \( \overline{B} \). Given \( \overline{g} = (g, \epsilon) \in \overline{Sp_{2n}(F)} \), if \( g \) is of type A then there are \( n_1, n_2 \in Z_{Sp_{2n}(F)} \) such that \( n_1 g n_2 = g \) and \( \psi(n_1 n_2) \neq 1 \). Let \( \overline{n_i} = (n_i, 1) \). Clearly \( \overline{n_1 g n_2} = \overline{g} \) and \( \psi(\overline{n_1 n_2}) = \psi(n_1 n_2) \neq 1 \). If \( g \) is not of type A then by Lemma 5.2 it is of type B: There are \( n_1, n_2 \in Z_{Sp_{2n}(F)} \) such that \( n_1 g n_2 = \tau g \) and \( \psi(n_1 n_2) = 1 \). Define \( \overline{n_i} \) as before. Note that \( \psi(\overline{n_1 n_2}) = \psi(n_1 n_2) = 1 \). From Lemma 5.1 it follows that \( \overline{n_1 g n_2} = \overline{\tau g} \). This proves Lemma 5.2 for \( \overline{Sp_{2n}(F)} \), which is Theorem 5.5.
6 Definition of the local coefficient and of \( \gamma(\sigma \times \tau, s, \psi) \)

Unless otherwise is mentioned, through this chapter \( \mathbb{F} \) will denote a p-adic field. We shall define here the metaplectic analog of the local coefficients defined by Shahidi in Theorem 3.1 of [48] for connected reductive quasi split algebraic groups. Let \( n_1, n_2, \ldots, n_r, k \) be \( r+1 \) nonnegative integers whose sum is \( n \). Put \( \mathbf{t}^r = (n_1, n_2, \ldots, n_r, k) \). Let \( (\tau_1, V_{\tau_1}), (\tau_2, V_{\tau_2}), \ldots, (\tau_r, V_{\tau_r}) \) be \( r \) irreducible admissible generic representations of \( GL_{n_1}(\mathbb{F}), GL_{n_2}(\mathbb{F}), \ldots, GL_{n_r}(\mathbb{F}) \) respectively. It is clear that for \( s_i \in \mathbb{C}, \tau_i(s_i) \) is also generic. In fact, if \( \lambda \) is a \( \psi \)-Whittaker functional on \( (\tau_1, V_{\tau_1}) \) it is also a \( \psi \)-Whittaker functional on \( (\tau_1(s_i), V_{\tau_1}) \). Let \( (\sigma, \psi) \) be an irreducible admissible \( \psi \)-generic genuine representation of \( \tilde{Sp}_{2k}(\mathbb{F}) \). Let

\[
I(\pi^{(\sigma)}(\mathbf{t})) = I(\tau_1(s_1), \tau_2(s_2), \ldots, \tau_r(s_r), \sigma)
\]

be the parabolic induction defined in Section 4.1. Since the inducing representations are generic, then, by a theorem of Rodier, [43], extended to a non algebraic setting in [2], \( I(\pi^{(\sigma)}(\mathbf{t})) \) has a unique \( \psi \)-Whittaker functional. Define \( \lambda_{\tau_1}, \psi, \lambda_{\tau_2}, \psi, \ldots, \lambda_{\tau_r}, \psi \) to be non-trivial \( \psi \)-Whittaker functionals on \( V_{\tau_1}, V_{\tau_2}, \ldots, V_{\tau_r} \) respectively and fix \( \lambda_{\sigma}, \psi \), a non-trivial \( \psi \)-Whittaker functional on \( V_{\sigma} \). Define

\[
\epsilon(\mathbf{t}) = j_{n-k}(diag(\epsilon_{n_1}, \epsilon_{n_2}, \ldots, \epsilon_{n_r}, \epsilon_{n_1}, \epsilon_{n_2}, \ldots, \epsilon_{n_r}))
\]

where as in Section 5.1, \( \epsilon_n = diag(1, -1, 1, \ldots, (-1)^{n+1}) \in GL_n(\mathbb{F}) \). We fix \( J_{2n} \) as a representative of the long Weyl element of \( Sp_{2n}(\mathbb{F}) \) and

\[
\omega_n = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}
\]

as a representative of the long Weyl element of \( GL_n(\mathbb{F}) \). We now fix

\[
w_l(\mathbf{t}) = j_{n-k,n}(diag(\epsilon_{n_1} \omega_{n_1}, \epsilon_{n_2} \omega_{n_2}, \ldots, \epsilon_{n_r} \omega_{n_r}, \epsilon_{n_1} \omega_{n_1}, \epsilon_{n_2} \omega_{n_2}, \ldots, \epsilon_{n_r} \omega_{n_r})) k_{n}(-J_{2k})
\]

as a representative of the long Weyl element of \( P_{\mathbf{t}}(\mathbb{F}) \). We also define

\[
w_l(\mathbf{t}) = w_l(\mathbf{t}) J_{2n}.
\]

Note that \( w_l(\mathbf{t}) \) is a representative of minimal length of the longest Weyl element of \( Sp_{2n}(\mathbb{F}) \) modulo the Weyl group of \( M_{\mathbf{t}}(\mathbb{F}) \). It maps the positive roots outside \( M_{\mathbf{t}}(\mathbb{F}) \) to negative roots and maps the positive roots of \( M_{\mathbf{t}}(\mathbb{F}) \) to positive roots. The presence of \( \epsilon(\mathbf{t}) \) in the definition of \( w_l(\mathbf{t}) \) and \( w_l(\mathbf{t}) \) is to ensure that

\[
\psi\left( (w_l(\mathbf{t}), 1)^{-1} n(w_l(\mathbf{t}), 1) \right) = \psi(n) \quad (6.2)
\]

for all \( n \in M_{\mathbf{t}}(\mathbb{F}) w_l(\mathbf{t}) \cap Z_{Sp_{2n}(\mathbb{F})} \) (the reader may verify this fact by (2.16) and by a matrix multiplication). Consider the integral

\[
\lim_{r \to \infty} \int_{N_{\mathbf{t}}(\mathbb{F})} \left( \otimes_{i=1}^{r} \lambda_{\tau_i}, \psi^{-1} \right) \left( \otimes_{i=1}^{r} \lambda_{\sigma}, \psi \right) f(w_l(\mathbf{t}), 1)^{-1} (n, 1) \psi^{-1}(n) \ dn. \quad (6.3)
\]
By abuse of notations we shall write

\[ \int_{N_{\mathcal{T}}(\mathbb{F})} \left( \bigotimes_{i=1}^r \lambda_{i,\psi}^{-1} \otimes \lambda_{\sigma,\psi} \right) \left( f(w'_i(\tilde{t}_i), 1)^{-1}(n, 1) \right) \psi^{-1}(n) \, dn \]  

(6.4)

Since \( Z_{SP_{2n}}(\mathbb{F}) \) splits in \( SP_{2n}(\mathbb{F}) \) via the trivial section the integral converges exactly as in the algebraic case, see Proposition 3.1 of [48] (and see Chapter 4 of [1] for the context of a p-adic covering group). In fact, it converges absolutely in an open subset of \( p \)-adic covering group). Due to (2.15), it defines, again as in the algebraic case, a \( \psi \)-Whittaker functional on \( I(\tau_{1(s_1)}, \tau_{2(s_2)}, \ldots, \tau_{r(s_r)}, \tilde{\sigma}) \). We denote this functional by

\[ \lambda(\tilde{s}', (\bigotimes_{i=1}^r \tau_{i}) \otimes \tilde{\sigma}, \psi), \]

where \( \tilde{s}' = (s_1, s_2, \ldots, s_r) \). Also, since the integral defined in (6.3) is stable for a large enough \( r \) it defines a rational function in \( q^{s_1}, q^{s_2}, \ldots, q^{s_r} \).

**Remark:** One can check that \( w'_i(\tilde{t}_i) M_{\mathcal{T}}(\mathbb{F}) w_i(\tilde{t}_i) = M_{\mathcal{T}}(\mathbb{F}) \). Thus, in (6.3) and (6.4) we could have written \( N_{\mathcal{T}}(\mathbb{F}) \) instead of \( N_{\mathcal{T}}(\mathbb{F})^w \). However, in other groups this does not always happen and we find it appropriate to describe this construction so that it will fit the general case (see for example a \( GL_{m}(\mathbb{F}) \) maximal parabolic case in Section 7.1).

Fix now \( w \in W_{\mathcal{T}}(\mathbb{F}) \). Let \( t_w \) be the unique diagonal element in \( M_{\mathcal{T}}(\mathbb{F}) \cap W_{SP_{2n}}(\mathbb{F}) \) whose first entry in each block is 1 such that

\[ \psi((wt_w, 1)^{-1} n(wt_w, 1)) = \psi(n) \]

(6.5)

for each \( n \in M_{\mathcal{T}}(\mathbb{F})^w \cap Z_{SP_{2n}}(\mathbb{F}) \). (6.5) assures that

\[ \lambda(\tilde{s}'^w, (\bigotimes_{i=1}^r \tau_{i})^w \otimes \tilde{\sigma}, \psi) \circ A_w \]

is another \( \psi \)-Whittaker functional on \( I(\pi(\tilde{s}')) \), (here \( A_w \) is the intertwining operator defined in Section 4.3). It now follows from the uniqueness of Whittaker functional that there exists a complex number

\[ C_{\psi}^{SP_{2n}}(\mathbb{F}) (P_{\mathcal{T}}(\mathbb{F}), \tilde{s}', (\bigotimes_{i=1}^r \tau_{i}) \otimes \tilde{\sigma}, w) \]

defined by the property

\[ \lambda(\tilde{s}', (\bigotimes_{i=1}^r \tau_{i}) \otimes \tilde{\sigma}, \psi) = C_{\psi}^{SP_{2n}}(\mathbb{F}) (P_{\mathcal{T}}(\mathbb{F}), \tilde{s}', (\bigotimes_{i=1}^r \tau_{i}) \otimes \tilde{\sigma}, w) \lambda(\tilde{s}'^w, (\bigotimes_{i=1}^r \tau_{i})^w \otimes \tilde{\sigma}, \psi) \circ A_w. \]

(6.6)

It is called the **local coefficient** and it clearly depends only on \( \tilde{s}', w \) and the isomorphism classes of \( \tau_1, \tau_2, \ldots, \tau_r \) (not on a realization of the inducing representations nor on \( \lambda_{\tau_1, \psi}, \lambda_{\tau_2, \psi}, \ldots, \lambda_{\tau_r, \psi}, \psi^{-1}, \lambda_{\tilde{\sigma}, \psi} \)). Also it is clear by the above remarks that

\[ \tilde{s}' \mapsto C_{\psi}^{SP_{2n}}(\mathbb{F}) (P_{\mathcal{T}}(\mathbb{F}), \tilde{s}', (\bigotimes_{i=1}^r \tau_{i}) \otimes \tilde{\sigma}, w) \]

defines a rational function in function in \( q^{s_1}, q^{s_2}, \ldots, q^{s_r} \). Note that (6.6) implies that the zeros of the local coefficient are among the poles of the intertwining operator.

**Remark:** Assume that the residue characteristic of \( \mathbb{F} \) is odd. Then, by Lemma 2.1 \( \kappa_{2n}(w) = (wt_w, 1) \) for all \( w \in W_{\mathcal{T}}(\mathbb{F}) \) and \( \kappa_{2n}(w'_i(\tilde{t}_i)) = (w'_i(\tilde{t}_i), 1) \). Keeping the Adelic
context in mind, whenever one introduces local integrals that contain a pre-image of \( w \in W_{Sp_{2n}}(\mathbb{F}) \subset Sp_{2n}(\mathbb{F}) \) inside \( Sp_{2n}(\mathbb{F}) \) one should use elements of the form \( \kappa_{2n}(w) \).

Let \( \vec{t} = (n_1, n_2, \ldots, n_r; k) \) where \( n_1, n_2, \ldots, n_r, k \) are \( r + 1 \) non-negative integers whose sum is \( n \). For each \( 1 \leq i \leq r \) let \( \tau_i \) be an irreducible admissible generic representation of \( GL_{n_i}(\mathbb{F}) \) and let \( \sigma \) be an irreducible admissible generic genuine representation of \( Sp_{2k}(\mathbb{F}) \). From the definition of the local coefficients it follows that

\[
\beta(\vec{s}, \tau_1, \ldots, \tau_n, \sigma, w_0) = \beta(\vec{s}, \tau_1, \ldots, \tau_n, \sigma, w_0) - \beta(\vec{s}, \tau_1, \ldots, \tau_n, \sigma, w_0^{-1}),
\]

where \( \beta(\vec{s}, \tau_1, \ldots, \tau_n, \sigma, w_0^{-1}) \) is the function defined in (4.11). Recalling Theorem 4.3, the significance of the local coefficients for questions of irreducibility of a parabolic induction is clear. In Chapter 10 we shall compute \( \beta(\vec{s}) \) in various cases via the computations of the local coefficients.

Let \( \sigma \) be an irreducible admissible generic irreducible admissible generic genuine representation of \( Sp_{2k}(\mathbb{F}) \). Let \( \tau \) be an irreducible admissible \( \psi \)-generic representation of \( GL_m(\mathbb{F}) \). Put \( n = m + k \). We define:

\[
\gamma(\sigma \times \tau, s, \psi) = \frac{C_{Sp_{2k}(\mathbb{F})}(P_{m,k}(\mathbb{F}), s, \tau \otimes \sigma, j_{m,n}(\omega_{m}^{-1}))}{C_{Sp_{2n}(\mathbb{F})}(P_{m;0}(\mathbb{F}), s, \tau, \omega_{m}^{-1})},
\]

where \( \omega_{m} = \begin{pmatrix} \omega_m & 0 \\ -\omega_m & \omega_m \end{pmatrix} \). It is clearly a rational function in \( q^s \). Note that if \( k = 0 \) then \( \sigma \) is the non-trivial character of the group of 2 elements and \( \gamma(\sigma \times \tau, s, \psi) = 1 \).

This definition of the \( \gamma \)-factor is an exact analog to the definition given in Section 6 of [53] for quasi-split connected algebraic groups over a non-archimedean field. We note that similar definitions hold for the case \( \mathbb{F} = \mathbb{R} \). In this case the local coefficients are meromorphic functions.
7 Basic properties of $\gamma(\sigma \times \tau, s, \psi)$

In this chapter $F$ can be any of the local fields in discussion. Most of this chapter is devoted to the proof of a certain multiplicativity property of the $\gamma$-factor; see Theorems 7.1 and 7.2 of Section 7.1. This property in an analog of Part 3 of Theorem 3.15 of [53]. Aside from technicalities, the proof of the multiplicativity property follows from a decomposition of the intertwining operators; see Lemma 7.2. We remark that the proof of this property for $\gamma$-factors defined via the Rankin-Selberg integrals is harder; see Chapter 11 of [62] for example. In Section 7.2 as an immediate corollary of the multiplicativity of the $\gamma$-factor, we compute $\gamma(\sigma \times \tau, s, \psi)$ in the case where $\sigma$ and $\tau$ are principal series representations; see Corollary 7.4.

7.1 Multiplicativity of $\gamma(\sigma \times \tau, s, \psi)$

Let $\sigma$ be a genuine irreducible admissible $\psi$-generic representation of $\text{Sp}_{2k}(F)$. Let $\tau$ be an irreducible admissible $\psi$-generic representation of $GL_m(F)$. For two nonnegative integers $l, r$ such that $l + r = m$ denote by $P^0_{l,r}(F)$ the standard parabolic subgroup of $GL_m(F)$ whose Levy part is

$$M^0_{l,r}(F) = \begin{pmatrix} GL_l(F) & \cdot \\ \cdot & GL_r(F) \end{pmatrix}.$$ 

Denote its unipotent radical by $N^0_{l,r}(F)$. Let $\tau_l, \tau_r$ be two irreducible admissible $\psi$-generic representations of $GL_l(F)$ and $GL_r(F)$ respectively. In the p-adic case; see [50], Shahidi defines

$$\gamma(\tau_l \times \tau_r s, \psi) = \chi_{\tau_r}(-I_r)^{\frac{1}{2}} C^{GL_m(F)}_{\psi}(P^0_{l,r}(F), (\frac{s}{2}, -\frac{s}{2}, \tau_l \otimes \tau_r, \psi_{l,r}), \psi(\tau_r s, n, \psi_{l,r})),$$

(7.1)

where $\psi_{l,r} = \begin{pmatrix} \cdot & I_r \\ I_l & \cdot \end{pmatrix}$, $\chi_{\tau_r}$ is the central character of $\tau_r$ and $C^{GL_m(F)}_{\psi}(:, :, :, :)$, the $GL_m(F)$ local coefficient in the right-hand side defined via a similar construction to the one presented in Chapter 6. In the same paper the author proves that the $\gamma$-factor defined that way is the same arithmetical factor defined by Jacquet, Piatetskii-Shapiro and Shalika via the Rankin-Selberg method, see [26]. Due to the remark given in the introduction of [51], see page 974 after Theorem 1, we take (7.1) as a definition in archimedean fields as well. The archimedean $\gamma$-factor defined in this way agrees also with the definition given via the Rankin-Selberg method, see [25].

For future use we note the following:

$$\psi_{l,r}^{-1} = \psi_{l,r} = \psi_{l,r} M^0_{l,r}(F) \psi_{l,r} M^0_{r,l}(F),$$

(7.2)

and

$$\psi_{l,r} n \psi_{l,r} = \psi(n),$$

(7.3)

for all $n \in Z_{GL_m(F)} \cap M^0_{l,r}(F)$. $\psi_{l,r}$ is a representative of the long Weyl element of $GL_n(F)$ modulo the long Weyl element of $M^0_{l,r}(F)$.

**Theorem 7.1.** Assume that $\tau = \text{Ind}_{P^0_{l,r}(F)}^{GL_m(F)} \tau_l \otimes \tau_r$, where $\tau_l, \tau_r$ are two irreducible admissible generic representations of $GL_l(F)$ and $GL_r(F)$ respectively, where $l + r = m$, then

$$\gamma(\sigma \times \tau, s, \psi) = \gamma(\sigma \times \tau_l, s, \psi) \gamma(\sigma \times \tau_r, s, \psi).$$

(7.4)
Proof. With notations in Theorem 7.1 we have:

Lemma 7.1. With notations in Theorem 7.1 we have:

\[ C_{\psi}^{Sp_{2n}(\mathbb{F})}(P_{m,k}(\mathbb{F}), s, \tau \otimes \sigma, j_{m,n}(\omega^{-1}_m)) = C_{\psi}^{Sp_{2n}(\mathbb{F})}(P_{l,r;k}(\mathbb{F}), (s, s), \tau_l \otimes \tau_r \otimes \sigma, j_{m,n}(\omega^{-1}_m)) \]  

(7.6)

In particular, for \( m = n \), that is when \( k = 0 \) and \( \sigma \) is the non trivial representation of the group of two elements, we have

\[ C_{\psi}^{Sp_{2n}(\mathbb{F})}(P_{m,0}(\mathbb{F}), s, \tau, \omega^{-1}_m) = C_{\psi}^{Sp_{2n}(\mathbb{F})}(P_{l,r,0}(\mathbb{F}), (s, s), \tau_l \otimes \tau_r, \omega^{-1}_m) \]  

(7.7)

Proof. We find it convenient to assume that the inducing representations \( \tau_l, \tau_r \) and \( \sigma \) are given in their \( \psi \)-Whittaker model. In this realization \( f \mapsto f(I_l), f \mapsto f(I_r) \) and \( f \mapsto f(I_{2n}, 1) \) are \( \psi \)-Whittaker functionals on \( \tau_l, \tau_r \) and \( \sigma \) respectively. We realize the space on which \( \tau \) acts as a space of functions

\[ f : GL_m(\mathbb{F}) \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \rightarrow \mathbb{C} \]

which are smooth from the right in each variable and which satisfies

\[ f\left( \begin{pmatrix} a & * \\ b & \end{pmatrix} g \right) = |\det(a)|^{\frac{1}{2}} |\det(b)|^{\frac{1}{2}} \psi(n)a \psi(n') f(g, l_0 a, r_0 b), \]

for all \( g \in GL_m(\mathbb{F}), l_0, a \in GL_l(\mathbb{F}), r_0, b \in GL_r(\mathbb{F}), n \in Z_{GL_l}(\mathbb{F}), n' \in Z_{GL_r}(\mathbb{F}) \). In this realization \( \tau \) acts by right translations of the first argument (see pages 11 and 65 of [52] for similar realizations). According to the \( GL_m(\mathbb{F}) \) analog to the construction given in Chapter 6, i.e., Proposition 3.1 of [18] for \( GL_m(\mathbb{F}) \), and due to (7.2) and (7.3), a \( \psi \)-Whittaker functional on \( \tau \) is given by

\[ \lambda_{\tau, \psi}(f) = \int_{X_{\tau,j}(\mathbb{F})} f(w_{l,r} n, I_l, I_r) \psi^{-1}(n) \, dn. \]  

(7.8)

We realize the representation space of \( I(\tau(s), \sigma) \) as a space of functions

\[ f : Sp_{2n}(\mathbb{F}) \times Sp_{2k}(\mathbb{F}) \times GL_m(\mathbb{F}) \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \rightarrow \mathbb{C} \]

which are smooth from the right in each variable and which satisfy

\[ f((m,h)g, \begin{pmatrix} a & * \\ b & \end{pmatrix} g, n l_0, n' r_0) = f(s,y h, a, r_0 b) \]  

(7.9)

\[ \gamma^{-1}_\psi(d(m)) \det(m)^{\frac{2k+m-k}{2}+\frac{s}{2}} \det(b)^{\frac{s}{2}} \psi(n) \psi(n') \psi(n''), \]
for all \( s \in \overline{Sp_{2n}(\mathbb{F})}, \) \( h, y \in \overline{Sp_{2k}(\mathbb{F})}, \) \( \left( \begin{array}{c} a \\ b \end{array} \right) \in P_{l,r}^0(\mathbb{F}), m, g \in GL_m(\mathbb{F}), l_0 \in GL_l(\mathbb{F}), r_0 \in GL_r(\mathbb{F}), u \in (N_{m,k}(\mathbb{F}), 1), n \in \mathbb{Z}_{Sp_{2k}}(\mathbb{F}), n' \in Z_{GL_l}(\mathbb{F}), n'' \in Z_{GL_r}(\mathbb{F}). \) In this realization \( \overline{Sp_{2n}(\mathbb{F})} \) acts by right translations of the first argument. Due to (7.8), we have

\[
\lambda(s, \tau \otimes \sigma, \psi)(f) = \int_{N_{m,k}(\mathbb{F})} \lambda_{\tau, \psi}(f(w'(m;k)n, 1), (I_{2k}, 1), I_m, I_l) \psi^{-1}(n)dn = 
\int_{n \in N_{m,k}(\mathbb{F})} \int_{n' \in N_{l,r}^0(\mathbb{F})} f((j_{m,n}(-\epsilon_m \omega_m)n, 1), (I_{2k}, 1), \omega_{l,r}n', I_l, I_r) \psi^{-1}(n') \psi^{-1}(n) dn'dn.
\]

(7.10)

Note that \( \omega_m' = \left( \begin{array}{cc} I_m & -I_m \\ \omega_m & \omega_m \end{array} \right) \). Thus, \( x(-\epsilon_m \omega_m') = (-1)^m \). Due to (7.9) and (2.12), we observe that for \( n \in N_{m,k}(\mathbb{F}), n' \in M_{l,r}^0(\mathbb{F}) \)

\[
f((j_{m,n}(-\epsilon_m \omega_m)n, 1), (I_{2k}, 1), \omega_{l,r}n', I_l, I_r) = (-1, -1)^P \gamma^{-1}_\psi(-1^l) f((j_{m,n}(-\omega_{l,r}n' \epsilon_m \omega_m)n, 1), (I_{2k}, 1), I_m, I_l, I_r).
\]

We shall write

\[
-\omega_{l,r}n' \epsilon_m \omega_m = -\omega_{l,r} \epsilon_m \omega_m n' \gamma_m,
\]

where for \( g \in GL_m(\mathbb{F}) \) we define

\[
\tilde{g} = (\epsilon_m \omega_m)^{-1} g(\epsilon_m \omega_m).
\]

Since \( n \mapsto \tilde{n} \) maps \( N_{l,r}^0(\mathbb{F}) \) to \( N_{l,r}^0(\mathbb{F}) \), we get by (6.2) and (7.9):

\[
\lambda(s, \tau \otimes \sigma, \psi)(f) = (-1, -1)^P \gamma^{-1}_\psi(-1^l) \int_{N_{l,r}^0(\mathbb{F})} f((j_{m,n}(-\epsilon_m \omega_m)n, 1), (I_{2k}, 1), I_m, I_l, I_r) \psi^{-1}(n)dn.
\]

(7.11)

(Clearly the change of integration variable does not require a correction of the measure).

We turn now to \( I(\tau_\infty(\omega), \tau_\infty(\sigma)) \). We realize the space of this representation as a space of functions

\[
f: \overline{Sp_{2n}(\mathbb{F})} \times \overline{Sp_{2k}(\mathbb{F})} \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}
\]

which are smooth from the right in each variable and which satisfy

\[
f((j_{m,n} \left( \begin{array}{cc} a \\ b \end{array} \right), 1)i_{k,n}(h)us, ny, n'l_0, n''r_0) = f(s, yh, l_0a, r_0b)
\]

(7.12)

\[
\gamma^{-1}_\psi(\det \left( \begin{array}{c} a \\ b \end{array} \right)) \| \det \left( \begin{array}{c} a \\ b \end{array} \right) \|^{2k+m+1}\| \det(a)\|^{l} \| \det(b)\|^{r} \psi(n) \psi(n') \psi(n''),
\]

for all \( s \in \overline{Sp_{2n}(\mathbb{F})}, h, y \in \overline{Sp_{2k}(\mathbb{F})}, \left( \begin{array}{c} a \\ b \end{array} \right) \in M_{l,r}^0(\mathbb{F}), l_0 \in GL_l(\mathbb{F}), r_0 \in GL_r(\mathbb{F}), u \in (N_{l,r;k}(\mathbb{F}), 1), n \in \mathbb{Z}_{Sp_{2k}}(\mathbb{F}), n' \in Z_{GL_l}(\mathbb{F}), n'' \in Z_{GL_r}(\mathbb{F}). \) In this realization \( \overline{Sp_{2n}(\mathbb{F})} \) acts by right translations of the first argument. Recall that in (6.1) we have defined \( w_i'(\tilde{v}) \)
to be a particular representative of minimal length of the longest Weyl element of $\text{Sp}_{2n}(\mathbb{F})$ modulo the Weyl group of $M_\tau(\mathbb{F})$. Since

$$w'_l(l, k; r, k) = j_{m, n}(-\hat{g}_0 \omega_{l, r} \tilde{e}_m \omega'_m),$$

where

$$g_0 = \begin{pmatrix} (-l)^r & 0 \\ 0 & I_r \end{pmatrix},$$

We have:

$$\lambda((s, s), \tau \otimes \tau_r \otimes \sigma, \psi)(f) = \int_{N_{s, r, k}(\mathbb{F})} F(f_{m, n}(-\hat{g}_0 \omega_{l, r} \tilde{e}_m \omega'_m)n, 1), (I_{2k}, 1), I_l, I_r) \psi^{-1}(n) \, dn. \quad (7.13)$$

Note that

$$f \left( (j_{m, n}(-\hat{g}_0 \omega_{l, r} \tilde{e}_m \omega'_m)n, 1), (I_{2k}, 1), I_l, I_r \right) = (-1, -1)^{mlr} \gamma_{\psi} \chi_{\tau}^{*}(-I_{l}) f \left( (j_{m, n}(-\hat{g}_0 \omega_{l, r} \tilde{e}_m \omega'_m)n, 1), (I_{2k}, 1), I_l, I_r \right).$$

Thus,

$$\lambda((s, s), \tau \otimes \tau_r \otimes \sigma, \psi)(f) = (-1, -1)^{mlr} \gamma_{\psi} \chi_{\tau}^{*}(-I_{l}) \int_{N_{s, r, k}(\mathbb{F})} F(f_{m, n}(-\hat{g}_0 \omega_{l, r} \tilde{e}_m \omega'_m)n, 1), (I_{2k}, 1), I_l, I_r) \psi^{-1}(n) \, dn. \quad (7.14)$$

For $f \in I(\tau(s), \sigma)$ define

$$\tilde{f} : \text{Sp}_{2m}(\mathbb{F}) \times \text{Sp}_{2k}(\mathbb{F}) \times \text{GL}_l(\mathbb{F}) \times \text{GL}_r(\mathbb{F}) \to \mathbb{C}$$

by $\tilde{f}(s, y, r_0, l_0) = f(s, y, I_{m}, r_0, l_0)$. The map $f \mapsto \tilde{f}$ is an $\text{Sp}_{2m}(\mathbb{F})$ isomorphism from $I(\tau(s), \sigma)$ to $I(\tau(s), \tau_{-s}, \sigma)$. Comparing (7.11) and (7.14) we see that

$$\lambda(s, \tau \otimes \sigma, \psi)(f) = \chi_{\tau}^{*}(-I_{l}) \lambda((s, s), \tau \otimes \tau_r \otimes \sigma, \psi)(\tilde{f}). \quad (7.15)$$

We now introduce an intertwining operator

$$A_{j_{m, n}(\omega^{-1}_m)} : I(\tau(s), \sigma) \to I(\tau_{-s}, \sigma),$$

defined by

$$A_{j_{m, n}(\omega^{-1}_m)}(s, y, m, l_0, r_0) = \int_{j_{m, n}(N_{m, 0}(\mathbb{F})))} f((j_{m, n}(\tilde{e}_m \omega'_m)n, 1), s, y, m, l_0, r_0) \, dn.$$

Note that for $f \in I(\tau(s), \sigma)$ we have

$$A_{j_{m, n}(\omega^{-1}_m)} f : \text{Sp}_{2m}(\mathbb{F}) \times \text{Sp}_{2k}(\mathbb{F}) \times \text{GL}_l(\mathbb{F}) \times \text{GL}_r(\mathbb{F}) \to \mathbb{C}$$
Consider now we have, we have shown:

\[ A_{j,m,n}(\omega^{-1})(f)((j,m,n(\hat{m}),1)i_{k,n}(h)u_s,ny, \left( \begin{array}{c} a^* \\ b \end{array} \right) g,n'l_o,n''r_0) = \]

\[ \gamma^{-1}_\psi(\det(m))| \det(m)|^{\frac{2k+1}{2}}| \det(a)|^\frac{5}{2} \cdot \hat{\psi}(n)\psi(n')\psi(n'')\psi f(s,yh,g\hat{m},l_0a,r_0b), \]

(7.16)

for all \( s \in \mathcal{S}_n, h,y \in \mathcal{S}_2, (m,n) \in \mathcal{S}_1 \), \( m \in \mathcal{G}_m, l_0 \in \mathcal{G}_l \), \( r_0 \in \mathcal{G}_r \), \( u \in \left( N_{m,k}(\mathbb{F}),1 \right), n \in \mathcal{Z}_n \), \( n' \in \mathcal{Z}_n', n'' \in \mathcal{Z}_n'' \). Since \( \hat{g} = g \) and since \( \varphi_{l_r} = h_0\varphi_{r,l} \), where

\[ h_0 = \left( \begin{array}{c} (-I_r)^t \\ (-I_l)^r \end{array} \right), \]

we have,

\[ \lambda(-s, \hat{\tau} \otimes \overline{\sigma}, \psi)(f) \]

\[ = \int_{n \in \mathcal{N}_{m,k}(\mathbb{F})} f((j,m,n(\hat{m}),1),(I_{2k},1),h_0h_0\varphi_{l,r,n'},I_l,I_r)\psi^{-1}(n')\psi^{-1}(n)dn'dn \]

\[ = \chi^r_l(-I_l)\chi^l_r(-I_r)(-1,-1)^{mrl}_\psi \hat{\psi}^{-1}(-1)^{tr} \]

\[ \int_{n \in \mathcal{N}_{m,k}(\mathbb{F})} f((j,m,n(\hat{m}),1),(I_{2k},1),I_m,I_l,I_r)\psi^{-1}(n')\psi^{-1}(n)dn'dn \]

\[ = \chi^r_l(-I_l)\chi^l_r(-I_r)(-1,-1)^{mrl}_\psi \hat{\psi}^{-1}(-1)^{tr} \]

\[ \int_{n \in \mathcal{N}_{m,k}(\mathbb{F})} f((j,m,n(\hat{m}),1),(I_{2k},1),I_m,I_l,I_r)\psi^{-1}(n')\psi^{-1}(n)dn'dn. \]

We have shown:

\[ \lambda(-s, \hat{\tau} \otimes \overline{\sigma}, \psi)(f) = \chi^r_l(-I_l)\chi^l_r(-I_r)(-1,-1)^{mrl}_\psi \hat{\psi}^{-1}(-1)^{tr} \]

\[ \int_{n \in \mathcal{N}_{r,l,k}(\mathbb{F})} f((j,m,n(\hat{m}),1),(I_{2k},1),I_m,I_l,I_r)\psi^{-1}(n)dn. \]

(7.17)

Consider now

\[ \overline{A}_{j,m,n}(\omega^{-1}) : I(\tau_l(s),\tau_r(s),\overline{\sigma}) \rightarrow I(\overline{\tau}_l(-s),\overline{\tau}_r(-s),\overline{\sigma}), \]

defined by

\[ \overline{A}_{j,m,n}(\omega^{-1})(s,y,l_0,r_0) = \int_{j,m,n(\mathbb{N}_{m,n})} f((j,m,n(\hat{m}),1)s,yl_r)dn. \]

Note that for \( f \in I(\tau_l(s),\tau_r(s),\overline{\sigma}) \), we have

\[ \overline{A}_{j,m,n}(\omega^{-1})(f) : \mathcal{S}_n \times \mathcal{S}_2 \times \mathcal{G}_l \times \mathcal{G}_r \rightarrow \mathbb{C} \]

is smooth from the right in each variable and satisfies

\[ \overline{A}_{j,m,n}(\omega^{-1})(f)((j,m,n(\hat{b},a),1)i_{k,n}(h)us,ny,n'l_o,n''r_0) = f(s,yh,l_0\hat{a},r_0b) \]

\[ \gamma^{-1}_\psi(\det(\left( \begin{array}{c} b \\ a \end{array} \right))| \det(\left( \begin{array}{c} b \\ a \end{array} \right))|^{\frac{2k+m+1}{2}}| \det(a)|^\frac{5}{2} \cdot \hat{\psi}(n)\psi(n')\psi(n'') \]

(7.18)
for all \( s \in Sp_{2n}(F) \), \( h, y \in Sp_{2k}(F) \), \( \left( \begin{array}{c} b \\ o \end{array} \right) \in M_{r,l}^0 \), \( l_0 \in GL_l(F) \), \( r_0 \in GL_r(F) \), \( u \in (N_{r,l,k}(F), 1) \), \( n \in Z_{Sp_{2k}(F)} \), \( n' \in Z_{GL_l(F)} \), \( n'' \in Z_{GL_r(F)} \). Similar to (7.14) we have:

\[
\lambda((-s,-s), \tau_r \otimes \hat{\tau}_l, \psi)(f) = (-1, -1)^{mlr} \gamma_{2}(1)^{r}(1)^{-1}(1) \chi_{\tau_{r}(-r_{l})}
\]

\[
\int_{I_{r,l,k}(F)} f \left( (j_{m,n}(-g_{0})\tilde{r}_{r,l}, \tilde{c}_{n,m} n, 1), (I_{2k}, 1), I_{r}, \tilde{\psi}(1) \right) \psi^{-1}(n)
\]

(7.19)

For \( f \in I(\tilde{\tau}_{-(s)} \otimes \tilde{\sigma}) \) define

\[
\tilde{f} : Sp_{2m}(F) \times Sp_{2k}(F) \times GL_l(F) \times GL_r(F) \to C
\]

by

\[
\tilde{f}(s, y, r_0, l_0) = f(s, y, I_m, r_0, l_0).
\]

The map \( f \mapsto \tilde{f} \) is an \( Sp_{2n}(F) \) isomorphism from \( I(\tilde{\tau}_{-(s)} \otimes \tilde{\sigma}) \) to \( I(\tilde{\tau}_{-(s)} \otimes \hat{\tau}_l \otimes \psi) \). By (7.17) and (7.20) we have,

\[
\lambda((-s, \tilde{\tau} \otimes \tilde{\sigma}, \psi)(f) = \chi_{\tilde{\tau}}^r(-I_{l}) \lambda((-s, -s), \tilde{\tau}_r \otimes \hat{\tau}_l \otimes \tilde{\sigma}, \psi)(\tilde{f}).
\]

(7.20)

We use (7.15), (7.20) and the fact that for all \( f \in I(\tilde{\tau}_{s} \otimes \tilde{\sigma}) \), we have

\[
\tilde{A}_{jm,n(\omega_m^{-1})}(\tilde{f}) = A_{j_{m,n}(\omega_m^{-1})}(f),
\]

to complete the lemma:

\[
\frac{C_{Sp_{2n}(F)}^{\psi}(P_{m,k,0}(F), s, \tau \otimes \tilde{\sigma}, j_{m,n}(\omega_m^{-1}))}{\lambda((-s, \tilde{\tau} \otimes \tilde{\sigma}, \psi)(\tilde{f}))} = \frac{\lambda(s, \tau \otimes \tilde{\sigma}, \psi)(f)}{\lambda((s, s), \tilde{\tau}_l \otimes \tilde{\tau}_r \otimes \tilde{\sigma}, \psi)((\tilde{A}_{j_{m,n}(\omega_m^{-1})}(f)))}
\]

\[
C_{Sp_{2n}(F)}^{\psi}(P_{l,r,k}(F), (s, s), \tilde{\tau}_l \otimes \tilde{\tau}_r \otimes \tilde{\sigma}, j_{m,n}(\omega_m^{-1})) = \phi_{\psi}^{-1}(r, l, \tau_r) c_1(s) c_2(s) c_3(s),
\]

where

\[
c_1(s) = C_{Sp_{2n}(F)}^{\psi}(P_{l,r,k}(F), (s, s), \tilde{\tau}_l \otimes \tilde{\tau}_r \otimes \tilde{\sigma}, j_{m,n}(w_1^{-1})),
\]

\[
c_2(s) = C_{Sp_{2n}(F)}^{\psi}(P_{l,r,k}(F), (s, -s), \tilde{\tau}_l \otimes \tilde{\tau}_r \otimes \tilde{\sigma}, j_{m,n}(w_2^{-1})),
\]

\[
c_3(s) = C_{Sp_{2n}(F)}^{\psi}(P_{l,r,k}(F), (-s, s), \tilde{\tau}_r \otimes \tilde{\tau}_l \otimes \tilde{\sigma}, j_{m,n}(w_3^{-1})).
\]
and where

$$w_1 = \begin{pmatrix} I_l & -\omega_r \\ -\omega_r & I_l \end{pmatrix}, \quad w_2 = \overline{w}_{l,r}, \quad w_3 = \begin{pmatrix} I_r & -\omega_l \\ \omega_l & I_r \end{pmatrix},$$

and

$$\phi_\psi(r, l, \tau_r) = (-1, -1)^{\frac{\ell^2}{2}} \chi_{\tau_r}(-I_r)^{\ell} \tau_r^{-1} ((-1)^{r_1}).$$

In particular:

$$C_{\psi, j}^{\mathcal{S}_{2m}(E)}(P_{l,r,0}(E), (s, s), \tau_l \otimes \tau_r, \omega_{m^{-1}}) = \phi_\psi^{-1}(r, l, \tau_r)c_1(s)c_2(s)c_3(s),$$

where

$$c_1'(s) = C_{\psi, j}^{\mathcal{S}_{2m}(E)}(P_{l,r,0}(E), (s, s), \tau_l \otimes \tau_r, w_{l^{-1}}),$$

$$c_2'(s) = C_{\psi, j}^{\mathcal{S}_{2m}(E)}(P_{l,r,0}(E), (s, -s), \tau_l \otimes \tau_r, w_{l^{-1}}),$$

$$c_3'(s) = C_{\psi, j}^{\mathcal{S}_{2m}(E)}(P_{l,r,0}(E), (-s, s), \tau_l \otimes \tau_r, w_{l^{-1}}).$$

**Proof.** We keep the realizations used in Lemma 7.1 and most of its notations. Consider the following three intertwining operators:

$$A_{j, m, n}^{(w_1^{-1})} : I(\tau_l(s), \tau_r(-s), \overline{\sigma}) \rightarrow I(\tau_l(s), \tau_r(-s), \overline{\sigma})$$

$$A_{j, m, n}^{(w_2^{-1})} : I(\tau_l(s), \tau_r(-s), \overline{\sigma}) \rightarrow I(\hat{\tau}_r(-s), \tau_l(s), \overline{\sigma})$$

$$A_{j, m, n}^{(w_3^{-1})} : I(\hat{\tau}_r(-s), \tau_l(s), \overline{\sigma}) \rightarrow I(\hat{\tau}_r(-s), \hat{\tau}_l(-s), \overline{\sigma})$$

Suppose that we show that

$$A_{j, m, n}^{(w_3^{-1})} \circ A_{j, m, n}^{(w_2^{-1})} \circ A_{j, m, n}^{(w_1^{-1})} = \phi_\psi(r, l, \tau_r)A_{j, m, n}^{(w_m^{-1})},$$

This will finish the lemma at once since

$$C_{\psi, j}^{\mathcal{S}_{2m}(E)}(P_{l,r,0}(E), (s, s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, j, m, n(\omega_m')) = \frac{\lambda((s, s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi)(f)}{\lambda((-s, -s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi)(A_{j, m, n}(w_m^{-1})(f))} = \frac{\lambda((s, s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi)(f)}{\lambda((-s, -s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi)(A_{j, m, n}(w_m^{-1})(f))} \frac{\lambda((-s, -s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi)(A_{j, m, n}(w_m^{-1})(f))}{\lambda((-s, -s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi)(A_{j, m, n}(w_2^{-1})(f))} \frac{\lambda((-s, -s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi)(A_{j, m, n}(w_2^{-1})(f))}{\lambda((-s, -s), \tau_l \otimes \tau_r \otimes \overline{\sigma}, \psi)(A_{j, m, n}(w_1^{-1})(f))}.$$
Thus, we prove (7.22). It is sufficient to prove it for $Re(s) > 0$ where $A_{j,m,n}^{(\omega_{m^{-1}})}$ is given by an absolutely convergent integral. Our argument will use Fubini’s theorem, whose use will be justified by (7.23).

$$A_{j,m,n(w_3^{-1})} \circ A_{j,m,n(w_2^{-1})} \circ A_{j,m,n(w_1^{-1})}(f)(s, y, l, r) =$$

$$\int_{N_{w_3}(\mathbb{F})} A_{j,m,n(w_3^{-1})} \circ A_{j,m,n(w_2^{-1})} \circ A_{j,m,n(w_1^{-1})}(f)(\left(j_{m,n}(t_{w_1}w_3)n_3, 1\right)s, y, l, r)dn_3 =$$

$$\int_{N_{w_3}(\mathbb{F})} \int_{N_{w_2}(\mathbb{F})} A_{j,m,n(w_3^{-1})} \circ A_{j,m,n(w_2^{-1})}(f)(\left(j_{m,n}(t_{w_2}w_3)n_2, 1\right) \left(j_{m,n}(t_{w_3}w_3)n_3, 1\right)s, y, l, r)dn_2dn_3 =$$

$$\int_{N_{w_3}(\mathbb{F})} \int_{N_{w_2}(\mathbb{F})} \int_{N_{w_1}(\mathbb{F})} f((j_{m,n}(t_{w_1}w_1)n_1, 1)(j_{m,n}(t_{w_2}w_2)n_2, 1)(j_{m,n}(t_{w_3}w_3)n_3, 1)s, y, l, r)dn_1dn_2dn_3,$$

where, as a straightforward computation will show, $N_{w_1}(\mathbb{F})$ is the group of elements of the form

$$\begin{pmatrix} I_l & 0 & z \\ I_r & z & 0_k \\ I_k & I_l & 0_k \\ I_t & I_r & I_k \end{pmatrix}, \quad z \in Mat_{sym}^{I_{xI}}(\mathbb{F}),$$

$N_{w_2}(\mathbb{F})$ is the group of elements of the form

$$\begin{pmatrix} I_l & z \\ I_r & I_l \\ I_k & I_t \\ -z^t & I_r & I_k \end{pmatrix}, \quad z \in Mat_{I_{xI}}^{sym}(\mathbb{F}),$$

$N_{w_3}(\mathbb{F})$ is the group of elements of the form

$$\begin{pmatrix} I_r & 0_r & z \\ I_l & 0_r & z \\ I_t & I_r & 0_k \\ I_l & I_t & I_k \end{pmatrix}, \quad z \in Mat_{I_{xI}}^{sym}(\mathbb{F}),$$

and where $t_{w_1} = \left( \begin{array}{c} I_l \\ \epsilon_r \end{array} \right)$, $t_{w_2} = I_{m^{-1}}$, $t_{w_3} = \left( \begin{array}{c} I_r \\ \epsilon_l \end{array} \right)$. We consider the first argument of $f$ in (7.23): By (2.12) and (2.16) we have:

$$(j_{m,n}(t_{w_1}w_1)n_1, 1)(j_{m,n}(t_{w_2}w_2)n_2, 1)(j_{m,n}(t_{w_3}w_3)n_3, 1)$$

$$(j_{m,n}(t_{w_1}w_1), 1)(n_1, 1)(j_{m,n}(t_{w_2}w_2), 1)(n_2, 1)(j_{m,n}(t_{w_3}w_3), 1)(n_3, 1)$$

$$(j_{m,n}(t_{w_1}w_1), 1)(j_{m,n}(t_{w_2}w_2), 1)(j_{m,n}(t_{w_3}w_3), 1)(n'_1n'_2n_3, 1)$$

$$(I_{2n}, \epsilon)(j_{m,n}(t_{w_1}w_1t_{w_2}w_2t_{w_3}w_3), 1)(n'_1n'_2n_3, 1)$$

$$(I_{2n}, \epsilon)(j_{m,n}(I_l)(\epsilon_{m^{-1}})^t)(\epsilon_{m\omega_{m^{-1}}})(n'_1n'_2n_3, 1).$$
where
\[ n'_1 = j_{m,n}(w_3 t w_3 w_2 t w_2)\left( (n_1) j_{m,n}(w_2 t w_2 w_3 t w_3)\right)^{-1}, \quad n'_2 = j_{m,n}(w_3 t w_3)\left( (n_2) j_{m,n}(w_3 t w_3)\right)^{-1}, \]
and where
\[ \epsilon = c(t_{w_1} w_1, t_{w_2} w_2) c(t_{w_1} w_1 t_{w_2} w_2, t_{w_3} w_3). \]

We compute:
\[
\begin{align*}
\left( j_{m,n}(w_3 t w_3)\right)^{-1} &= \begin{pmatrix}
I_l & z & 0_l \\
I_r & I_k & 0_k \\
im(I_l) & I_r & I_k
\end{pmatrix}, \\
\left( j_{m,n}(w_3 t w_3)\right)^{-1} &= \begin{pmatrix}
I_l & z & 0_l \\
im(I_l) & I_k & 0_k \\
I_r & I_k & I_l
\end{pmatrix},
\end{align*}
\]
\[
\begin{align*}
\left( j_{m,n}(w_3 t w_3)\right)^{-1} &= \begin{pmatrix}
I_l & z & 0_l \\
I_r & I_k & 0_k \\
-I^t & I_r & I_k
\end{pmatrix}, \\
\left( j_{m,n}(w_3 t w_3)\right)^{-1} &= \begin{pmatrix}
I_l & z^t & 0_l \\
I_r & I_k & 0_k \\
-I^t & I_r & I_k
\end{pmatrix},
\end{align*}
\]
where \( z^t = z \omega_t t w_3 \). Hence we can change the three integrals in (7.23) to a single integration on \( j_{m,n}(N_{m;0}(\mathbb{F})) \) without changing the measure and obtain
\[
\begin{align*}
A_{j_{m,n}(w_3^{-1})} A_{j_{m,n}(w_2^{-1})} A_{j_{m,n}(w_1^{-1})}(f)(s, y, l, r) \\
= \epsilon \int_{j_{m,n}(N_{m;0})} f \left( j_{m,n}(\begin{pmatrix}
I_l & z & 0_l \\
im(I_l) & I_k & 0_k \\
I_r & I_k & I_l
\end{pmatrix}) (\epsilon m^{-1}(\omega') \gamma(n, 1)) s, y, l, r \right) \, dn \\
= \epsilon \chi_r(-I)^t (\gamma^{-1}(\gamma^t)) A_{j_{m,n}(\omega^{-1})}(f)(s, y, l, r).
\end{align*}
\]

It is left to show that \( \epsilon = (-1, -1)^2(\mathbb{F})^2 \). Indeed, we have
\[
\begin{align*}
t_{w_1} w_1 &= \begin{pmatrix}
I_l & 0_l \\
e^r \omega_r & I_l
\end{pmatrix} \begin{pmatrix}
I_l & -I_r \\
I_l & I_l
\end{pmatrix}, \\
t_{w_2} w_2 &= \begin{pmatrix}
I_l & 0_l \\
e^r \omega_r & I_l
\end{pmatrix} \begin{pmatrix}
I_l & I_r \\
I_l & I_l
\end{pmatrix},
\end{align*}
\]
Thus, by (2.12) we have \( c(t_{w_1} w_1, t_{w_2} w_2) = (-1, -1)^2(\mathbb{F})^2 \). Since
\[
\begin{align*}
t_{w_1} w_1 t_{w_2} w_2 &= \begin{pmatrix}
I_l & 0_l \\
e^r \omega_r & I_l
\end{pmatrix} \begin{pmatrix}
I_l & I_r \\
I_l & I_l
\end{pmatrix} \begin{pmatrix}
I_l & -I_r \\
I_l & I_l
\end{pmatrix}, \\
t_{w_3} w_3 &= \begin{pmatrix}
I_l & 0_l \\
e^r \omega_r & I_l
\end{pmatrix} \begin{pmatrix}
I_l & -I_l \\
I_l & I_l
\end{pmatrix} \begin{pmatrix}
I_l & \epsilon \omega_l \\
I_l & \epsilon \omega_r
\end{pmatrix},
\end{align*}
\]
we conclude, using (2.14), that \( c(t_{w_1} w_1 t_{w_2} w_2, t_{w_3} w_3) = (-1, -1)^2(\mathbb{F})^2 + \frac{p^2 (1-1)}{2} \).
Lemma 7.3. Keeping the notations of the previous lemmas we have:

\[
C_{\psi,\tau}(P_{r,k}(F), (s, s), \tau_1 \otimes \tau_r \otimes \sigma, j_{m,n}(w^{-1})) = C_{\psi,\tau}(P_{r,k}(F), s, \tau_r \otimes \sigma, j_{r,k}(\omega_r^{-1})).
\]

(7.26)

\[
C_{\psi,\tau}(P_{r,k}(F), (s, -s), \tau_1 \otimes \tau_r \otimes \sigma, j_{m,n}(w_2^{-1})) = C_{\psi,\tau}(P_{r,k}(F), (s, -s), \tau_r \otimes \tau_r, \omega_r^{-1}).
\]

(7.27)

\[
C_{\psi,\tau}(P_{r,k}(F), (-ss, s), \tau_1 \otimes \tau_1 \otimes \sigma, j_{m,n}(w_3^{-1})) = C_{\psi,\tau}(P_{r,k}(F), s, \tau_1 \otimes \sigma, j_{l,l+k}(\omega_r^{-1})).
\]

(7.28)

In particular:

\[
C_{\psi,\tau}(P_{r,0}(F), (s, s), \tau_1 \otimes \tau_r, w_1^{-1}) = C_{\psi,\tau}(P_{r,0}(F), s, \tau_r, \omega_r^{-1}).
\]

\[
C_{\psi,\tau}(P_{r,0}(F), (s, -s), \tau_1 \otimes \tau_r, w_2^{-1}) = C_{\psi,\tau}(P_{r,0}(F), (s, -s), \tau_r \otimes \tau_r, \omega_r^{-1}).
\]

\[
C_{\psi,\tau}(P_{r,0}(F), (-ss, s), \tau_s \otimes \tau_1 \otimes \sigma, w_3^{-1}) = C_{\psi,\tau}(P_{r,0}(F), s, \tau_1, \omega_r^{-1}).
\]

Proof. We prove (7.26) and (7.27) only. (7.28) is proven exactly as (7.26). We start with (7.26): As in Lemma 7.1 we realize \( I(\tau_r(s), \sigma) \) as a space of functions

\[
f : S_{2(r+k)}(F) \times S_{2k}(F) \times GL_r(F) \to \mathbb{C}
\]

which are smooth from the right in each variable and which satisfies

\[
f(j_{r+k}(b)) = \gamma^{-1}_b |(\det(b))| (\det(b)) |^{k+r+\frac{1}{2}} \psi(n) \psi(n') f(s, yh, r_0 b),
\]

(7.29)

for all \( s \in S_{2(r+k)}(F), h, y \in S_{2k}(F), b, r_0 \in GL_r(F), u \in (N_{l,k}(F), 1), n \in Z_{Sp_{2k}}(F) \), \( n' \in Z_{GL_m}(F) \). For \( f \in I(\tau_r(s), \sigma) \), \( g \in S_{2n}(F) \) we define

\[
f_g : S_{2(r+k)}(F) \times S_{2k}(F) \times GL_r(F) \to \mathbb{C}
\]

by

\[
f_g(s, y, r) = f(i_{r+k,n}(g)s, y, I_1, r).
\]

Recalling (7.12) we note that \( f_g \in I(\tau_r(s), \sigma) \). We want to write the exact relation between \( \lambda((s, s), \tau_1 \otimes \tau_r \otimes \sigma, \psi) \) and \( \lambda((s, \tau_r \otimes \sigma, \psi) \). To do so we consider the left argument of \( f \) in (7.13): We decompose \( n \in N_{l,r+k}(F) \) as \( n = n'n'' \), where

\[
n' \in i_{r+k,n}(N_{r,k}(F)), n'' \in U_0(F) = \left\{ u = \begin{pmatrix} I_l & \ast & \ast & \ast & \ast \\ I_r & 0 \times k & 0 \times k & 0 \times k & 0 \times k \\ I_k & 0 \times r & 0 \times r & 0 \times r & 0 \times r \\ I_l & \ast & \ast & \ast & \ast \\ I_k & \ast & \ast & \ast & \ast \end{pmatrix} \mid u \in S_{2k}(F) \right\}.
\]

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We have

\[
(w'_1(l, r, k)n, 1) = \left( \begin{array}{ccc}
I_l & \epsilon_r \omega_r & \epsilon_l \\
I_k & I_l & -\epsilon_r \omega_r \\
I_k & I_l & I_k
\end{array} \right) \left( \begin{array}{c}
I_r \\
I_k \\
I_r
\end{array} \right) (n'n', 1)
\]

Thus, for an appropriate \( \epsilon \) independent of \( n' \) and \( n'' \), we have:

\[
(w'_1(l, r, k)n, 1) = \left( \begin{array}{c}
i_{r+k,n}(j_{r,r+k}(-\tilde{\epsilon}_r \omega'_r))n'
\end{array} \right) \left( \begin{array}{ccc}
I_r & \epsilon_l \\
I_k & I_r \\
I_k & -\epsilon_l
\end{array} \right) \left( \begin{array}{c}
n'' \\
\epsilon
\end{array} \right).
\]

We denote the right element of the last line by \( g(n'') \), and we see that

\[
\lambda((s, s), \tau_l \otimes \tau_r \otimes \sigma, \psi)(f) = \int_{U_0(F)} \int_{N_{r.k}(F)} f_{g(n''')} (j_{r,r+k}(-\tilde{\epsilon}_r \omega'_r))n' (I_{2k}, 1, I_r) \psi^{-1}(n') \psi^{-1}(n'') dn' dn''
\]

\[
= \int_{U_0(F)} \lambda(s, \tau_r \otimes \sigma, \psi) (f_{g(n'')}) \psi^{-1}(n'') dn''.
\]

For \( f \in I(\tau_l(s), \tau_r(-s), \sigma), s \in \mathcal{P}_{2n}(F) \) we define \( f_g \) as we did for \( I(\tau_l(s), \tau_r(s), \sigma) \). In this case \( f_g \in I(\tilde{\tau}_r(-s), \sigma) \). Exactly as (7.30) we have

\[
\lambda((s, -s), \tau_l \otimes \tilde{\tau}_r \otimes \sigma, \psi)(f) = \int_{U_0(F)} \lambda(-s, \tilde{\tau}_r \otimes \sigma, \psi) (f_{g(n''')} \psi^{-1}(n'')) dn''.
\]

Let

\[
A_{j_{r,r+k}(-\omega'_r)^{-1}} : I(\tau_r(s), \sigma) \rightarrow I(\tilde{\tau}_r(-s), \sigma)
\]

be as in lemma (7.2) and let

\[
A_{j_{r,r+k}(\omega'_r)^{-1}} : I(\tau_r(s), \sigma) \rightarrow I(\tilde{\tau}_r(-s), \sigma)
\]
be the intertwining operator defined by
\[ A_{j,r+k}(\omega^{-1}_r)(f) = \int_{J_{r+k}(\mathbb{F})} f(j_{r+k}(\epsilon_r \omega_r^{-1} n), (I_2 k, 1), I_r) \psi^{-1}(n) \, dn \]

Using the fact that \( A_{j,r+k}(\epsilon_r \omega_r^{-1})^{-1}(f_g) = (A_{j,m,n}(w_1^{-1})(f))_g \) we prove \((7.26)\):

\[
\begin{align*}
C_{\psi}^{Sp_{2n}(\mathbb{F})} \left( P_{l;_r,k}(\mathbb{F}), (s, s), \tau_r \otimes \tau_r \otimes \sigma, j_{m,n}(w_1^{-1}) \right) \\
= \lambda((s, s), \tau_r \otimes \tau_r \otimes \sigma) (A_{j,m,n}(w_1^{-1})(f)) \\
= \int_{U_0(\mathbb{F})} \lambda(s, \tau_r \otimes \sigma) (f_g(n)) \psi^{-1}(n) \, dn \\
= \int_{U_0(\mathbb{F})} \lambda(-s, \tau_r \otimes \sigma) (A_{j,r+k}(\epsilon_r \omega_r^{-1})^{-1}(f_g(n))) \psi^{-1}(n) \, dn \\
= \int_{U_0(\mathbb{F})} C_{\psi}^{Sp_{2(r+k)}(\mathbb{F})} \left( P_{l;_r,k}(\mathbb{F}), (s, s), \tau_r \otimes \sigma, j_{r+k}(\omega_r^{-1}) \right) \lambda(-s, \tau_r \otimes \sigma) (A_{j,r+k}(\epsilon_r \omega_r^{-1})^{-1}(f_g(n))) \psi^{-1}(n) \, dn \\
= C_{\psi}^{Sp_{2(r+k)}(\mathbb{F})} \left( P_{l;_r,k}(\mathbb{F}), (s, \tau_r \otimes \sigma, j_{r+k}(\omega_r^{-1}) \right) 
\end{align*}
\]

To prove \((7.27)\) one uses similar arguments. The key point is that for \( f \in I(\tau_1(\sigma), \tilde{\tau}_r(-s), \sigma) \), \( g \in Sp_{2n}(\mathbb{F}) \), the function
\[
f_g : GL_m(\mathbb{F}) \times GL_l(\mathbb{F}) \times GL_r(\mathbb{F}) \to \mathbb{C}
\]
defined by
\[
f_g(a, l, r) = \| \det a \|^{-2(k+m+1)/2} \gamma_{\psi}(a) f(\tilde{m,n}(a), 1) g, (I_2 k, 1), l, r, \]
lies in \( I(\tau_1(\sigma), \tilde{\tau}_r(-s)) \).
These three lemmas provide the proof of Theorem 7.1. We outline them: First one proves an analog to Lemma 7.1 and shows that

\[ C_{\psi}^{Sp_{2n}(F)}(P_{m,k}(F), s, \tau \otimes \overline{\tau}, j_{m,n}(\omega_{m}^{-1})) \]

\[ = C_{\psi}^{Sp_{2n}(F)}(P_{m,0}(F), s, \tau, \omega_{m}^{-1}) \]

\[ C_{\psi}^{Sp_{2n}(F)}(P_{r,k}(F), (s, s), \tau \otimes \tau_r \otimes \overline{\tau}, j_{m,n}(\omega_{m}^{-1})) \]

\[ = C_{\psi}^{Sp_{2n}(F)}(P_{r,0}(F), (s, s), \tau_1 \otimes \tau_r, \omega_{m}^{-1}) \]

\[ C_{\psi}^{Sp_{2n}(F)}(P_{r,k}(F), (s, s), \tau_1 \otimes \tau_r \otimes \overline{\tau}, j_{m,n}(\omega_{m}^{-1})) \]

\[ = C_{\psi}^{Sp_{2n}(F)}(P_{r,0}(F), (s, s), \tau_1 \otimes \tau_r, \omega_{m}^{-1}) \]

\[ \gamma(\overline{\psi} \times \tau, s, \psi) \]

\[ = C_{\psi}^{Sp_{2n}(F)}(P_{l,0(k)}(F), (s, s), \tau_1 \otimes \tau_r \otimes \overline{\tau}, j_{m,n}(\omega_{m}^{-1})) \]

\[ = C_{\psi}^{Sp_{2n}(F)}(P_{l,0(k)}(F), (s, s), \tau_1 \otimes \tau_r, \omega_{m}^{-1}) \]

\[ C_{\psi}^{Sp_{2n}(F)}(P_{l,1(k)}(F), (s, s), \tau_1 \otimes \tau_r \otimes \overline{\tau}, j_{m,n}(\omega_{m}^{-1})) \]

\[ = C_{\psi}^{Sp_{2n}(F)}(P_{l,0}(F), (s, s), \tau_1, \omega_{m}^{-1}) \]

\[ = \gamma(\overline{\psi} \times \tau, s, \psi) \gamma(\overline{\psi} \times \tau_r, s, \psi). \]

The proof of Theorem 7.5 is achieved through similar steps to those used in the proof of Theorem 7.1. We outline them: First one proves an analog to Lemma 7.1 and shows that

\[ C_{\psi}^{Sp_{2n}(F)}(P_{m,k}(F), s, \tau \otimes \overline{\tau}, j_{m,n}(\omega_{m}^{-1})) = C_{\psi}^{Sp_{2n}(F)}(P_{m,0}(F), s, \tau, \omega_{m}^{-1}) \]

\[ (7.32) \]

Then one gives an analog to Lemma 7.2. Using a decomposition of \( A_{j_{m,n}(\omega_{m}^{-1})} \), one shows that

\[ C_{\psi}^{Sp_{2n}(F)}(P_{m,l,r}(F), (s, 0), \tau \otimes \tau_l \otimes \overline{\tau}_r, j_{m+l,n}(\omega_{m}^{-1})) = (-1, -1)^{m} k_1(s) k_2(s) k_3(s), \]  \[ (7.33) \]

where

\[ k_1(s) = C_{\psi}^{Sp_{2n}(F)}(P_{m,l,r}(F), (s, 0), \tau \otimes \tau_l \otimes \overline{\tau}_r, j_{m+l,n}(w_4^{-1})) \],

\[ k_2(s) = C_{\psi}^{Sp_{2n}(F)}(P_{l,m,r}(F), (0, s), \tau_l \otimes \tau \otimes \overline{\tau}_r, j_{m+l,n}(w_5^{-1})) \],

\[ k_3(s) = C_{\psi}^{Sp_{2n}(F)}(P_{m,l,r}(F), (0, -s), \tau_l \otimes \tau \otimes \overline{\tau}_r, j_{m+l,n}(w_6^{-1})) \]

and where

\[ w_4 = \overline{w}_{m,l}, \quad w_5 = \begin{pmatrix} I_l & \omega_m \\ -\omega_m & I_l \end{pmatrix}, \quad w_6 = \overline{w}_{l,m}. \]

The third step is, an analog to Lemma 7.3.

\[ C_{\psi}^{Sp_{2n}(F)}(P_{m,l,r}(F), (s, 0), \tau \otimes \tau_l \otimes \overline{\tau}_r, j_{m+l,n}(w_4^{-1})) = C_{\psi}^{GL_{m+l}(F)}(P_{m,l}(F), (\frac{s}{2} - \frac{s}{2}, \tau \otimes \tau_l, \overline{\omega}_{m,l}), \]

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Combining (7.32), (7.33), (7.31) we have:

\[
\gamma(\sigma \times \tau, s, \psi) = (-1)^{m_1} (\gamma(\sigma \times \tau, s, \psi)
\]

C_{\psi-1}^{GL_{m+1}(F)} \left( P_{m,l}^0(F) \left( \frac{s - \sigma}{2}, \frac{r - \sigma}{2} \right), \tau \otimes \tau, \omega_{l,m} \right) C_{\psi-1}^{GL_{m+1}(F)} \left( P_{m,l}^0(F) \left( \frac{s - \sigma}{2}, \frac{r - \sigma}{2} \right), \tau \otimes \tau, \omega_{l,m} \right).

(7.34)

With (7.31) we finish.

7.2 Computation of $\gamma(\sigma \times \tau, s, \psi)$ for principal series representations

Assume that $F$ is either $\mathbb{R}$, $\mathbb{C}$ or a p-adic field. Let $\eta_1, \eta_2, \ldots, \eta_k$ be $k$ characters of $F^*$ and let $\gamma^{-1} \otimes \chi = (\gamma^{-1} \circ \text{det}) \otimes \chi$ be the character of $T_{Sp_{2k}}(F)$ defined by

\[
(\text{diag}(t_1, t_2, \ldots, t_k, t_1^{-1}, t_2^{-1}, \ldots, t_k^{-1}), \epsilon) \mapsto \epsilon \gamma^{-1}(t_1, t_2, \ldots, t_k) \prod_{i=1}^k \eta_i(t_i).
\]

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be $m$ characters of $F^*$ and let $\mu$ be the character of $T_{GL_m}(F)$ defined by

\[
\text{Diag}(t_1, t_2, \ldots, t_m) \mapsto \prod_{i=1}^m \alpha_i(t_i).
\]

Define $\sigma$ and $\tau$ to be the corresponding principal series representations:

\[
\sigma = I(\chi) = \text{Ind}_{T_{Sp_{2k}}(F)}^{GL_{2k}(F)} \gamma^{-1} \otimes \chi, \quad \tau = I(\mu) = \text{Ind}_{BGL_m(F)}^{GL_m(F)} \mu.
\]

Lemma 7.4. There exists $c \in \{ \pm 1 \}$ such that

\[
\gamma(\sigma \times \tau, s, \psi) = c \prod_{i=1}^k \prod_{j=1}^m \gamma(\alpha_j \otimes \eta_i^{-1}, s, \psi) \gamma(\eta_i \otimes \alpha_j, s, \psi).
\]

Proof. Note that $\tau \simeq \text{Ind}_{F_{m-1}^0(F)}^{GL_{m-1}(F)} \alpha_1 \otimes \tau'$, where $\tau' = \text{Ind}_{BGL_{m-1}(F)}^{GL_{m-1}(F)} \otimes_{j=1}^{m-1} \alpha_j$. Theorem 7.1 implies that

\[
\gamma(\sigma \times \tau, s, \psi) = \gamma(\sigma \otimes \alpha_1, s, \psi) \gamma(\sigma \otimes \tau', s, \psi).
\]

Repeating this argument $m - 1$ more times we observe that

\[
\gamma(\sigma \times \tau, s, \psi) = \prod_{j=1}^{m} \gamma(\sigma \times \alpha_j, s, \psi).
\]

(7.36)

Next we note that $\sigma' = \text{Ind}_{T_{Sp_{2k-1}}(F)}^{GL_{2k-1}(F)} (\gamma^{-1} \otimes \eta_1) \otimes \sigma'$, where $\sigma' = \text{Ind}_{B_{Sp_{2k-1}}(F)}^{GL_{2k-1}(F)} (\gamma^{-1} \otimes \otimes_{j=1}^{k-1} \eta_j)$. By using Theorem 7.2 we observe that for all $1 \leq j \leq m$. There exists $c' \in \{ \pm 1 \}$ such that

\[
\gamma(\alpha_j, s, \psi) = c' \gamma(\sigma' \otimes \alpha_j, s, \psi) \gamma(\sigma' \otimes \alpha_j, s, \psi) \gamma(\eta_1 \otimes \alpha_j, s, \psi).
\]

(7.37)

By Repeating this argument $k - 1$ more times for each $1 \leq j \leq m$ and by using (7.36) we finish.
8 Computation of the local coefficients for principal series representations of $SL_2(\mathbb{F})$

Assume that $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. Let $\psi$ be a non-trivial character of $\mathbb{F}$ and let $\chi$ be a character of $\mathbb{F}^\times$. For $s \in \mathbb{C}$ let

$$I(\chi \otimes \gamma^{-1}_{\psi}, s) = \text{Ind}_{B_{SL_2(\mathbb{F})}^{-1}}^{SL_2(\mathbb{F})} \gamma^{-1}_{\psi} \otimes \chi(s)$$

be the corresponding principal series representation of $SL_2(\mathbb{F})$. Its space consists of smooth complex functions on $SL_2(\mathbb{F})$ which satisfy:

$$f \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \epsilon \right) g = \epsilon \|a\|^s \chi(a) \gamma^{-1}_{\psi}(a) f(g),$$

for all $a \in \mathbb{F}^\times$, $b \in \mathbb{F}$, $g \in SL_2(\mathbb{F})$. $SL_2(\mathbb{F})$ acts on this space by right translations.

$\lambda(s, \chi, \psi_a)$, the $\psi_a$ Whittaker functional on $I(\chi \otimes \gamma^{-1}_{\psi}, s)$, defined in Chapter 6, is the analytic continuation of

$$\int_{\mathbb{F}} f \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1 \right) \psi^{-1}_{\psi}(x) dx. \quad (8.1)$$

The intertwining operator corresponding to the unique non-trivial Weyl element of $SL_2(\mathbb{F})$,

$$A(s) : I(\chi \otimes \gamma^{-1}_{\psi}, s) \rightarrow I(\chi^{-1} \otimes \gamma^{-1}_{\psi}, -s)$$

is defined by the meromorphic continuation of

$$(A(s)(f))(g) = \int_{\mathbb{F}} f \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, 1 \right) g \right) dx. \quad (8.2)$$

We shall prove in this chapter, see Theorems 8.1, 8.2 and Lemma 8.14 that there exists an exponential function, $\overline{\epsilon}(\chi, s, \psi)$ such that

$$C_{\psi}^{SL_2(\mathbb{F})} \left( B_{SL_2(\mathbb{F})}, s, \chi, \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right) = \overline{\epsilon}(\chi, s, \psi) \frac{L(\chi, s + \frac{1}{2})}{L(\chi^{-1}, -s + \frac{1}{2})} \frac{L(\chi^{-2}, -2s + 1)}{L(\chi^{2}, 2s)}. \quad (8.3)$$

Furthermore, if $\mathbb{F}$ is a p-adic field of odd residual characteristic, $\psi$ is normalized and $\chi$ is unramified then $\overline{\epsilon}(\chi, s, \psi) = 1$. Recall that

$$\gamma(\chi, s, \psi) = \epsilon(\chi, s, \psi) \frac{L(\chi^{-1}, 1 - s)}{L(\chi, s)},$$

where $\epsilon(\chi, s, \psi)$ is an exponential factor. Thus, (8.3) can be written as

$$C_{\psi}^{SL_2(\mathbb{F})} \left( B_{SL_2(\mathbb{F})}, s, \chi, \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \right) = \epsilon'(\chi, s, \psi) \frac{\gamma(\chi^{2}, 2s, \psi)}{\gamma(\chi, s + \frac{1}{2}, \psi)}, \quad (8.4)$$

where $\epsilon'(\chi, s, \psi)$ is an exponential factor which equals 1 if $\chi$ is unramified and $\mathbb{F}$ is p-adic field of odd residual characteristic.
In this chapter only, we shall write

\[ C_{\psi_a}(\chi \otimes \gamma_{-1}, s) \]

Instead of

\[ C_{\psi_a}^{SL_2(\mathbb{F})}(B_{SL_2(\mathbb{F})}, s, \chi, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \].

This notation emphasizes the dependence of the local coefficient on two additive characters, rather than on one in the algebraic case.

This chapter is organized as follows. In Section 8.1 we present the p-adic computation. Our computations include the often overlooked case of 2-adic fields. In Section 8.2 the computation for the real case is given. In both two sections the exponential factor is computed explicitly. The complex case is addressed in Section 8.3. Since \( SL_2(\mathbb{C}) = SL_2(\mathbb{C}) \times \{ \pm 1 \} \) and \( \gamma_{-1}(\mathbb{C}^*) = 1 \) the local coefficients for this group are identical to the local coefficients of \( SL_2(\mathbb{C}) \). The \( SL_2(\mathbb{C}) \) computation is given in Theorem 3.13 of [51]. It turns out that the \( SL_2(\mathbb{C}) \) computation of the local coefficients agrees with (8.4). Next we give two detailed remarks; one on the chosen parameterizations and the other on the existence of a non-archimedean metaplectic \( \tilde{\gamma} \)-factor defined by a similar way to the definition of the Tate \( \gamma \)-factor (see [44]) whose relation to the computed local coefficients is similar to the relation that the Tate \( \gamma \)-factor has with the local coefficients of \( SL_2(\mathbb{F}) \). These are Sections 8.4 and 8.5. We conclude this chapter in Section 8.6 where we show that if \( \tau \) is a principal series representation then

\[ C_{\psi}(\chi \otimes \gamma_{-1}^{-1}, s) = k_{\chi} q^{d_{\chi} s} \frac{L(\chi, s + \frac{1}{2})}{L(\chi^{-1}, -s + \frac{1}{2})} \frac{L(\chi^2, -2s + 1)}{L(\chi^2, 2s)}. \] \hspace{1cm} (8.5)

Furthermore, if \( \chi^2 \) is unramified we have:

\[ k_{\chi} = \gamma_{-1}^{-1}(\pi^n) c_{\psi_0}(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) \frac{G(\chi, \psi)}{G(\chi, \psi)} q^{-m(\chi) \frac{n}{2}}, \quad d_{\chi} = m(\chi) - 2e + n, \]
where \( n \) is the conductor of \( \psi \).

Recall that for \( \mathbb{F} \), a \( p \)-adic field,

\[
L_{\mathbb{F}}(\chi, s) = \begin{cases} 
\frac{1}{1 - \chi(\pi) q^{-s}} & \chi \text{ is unramified} \\
1 & \text{otherwise}
\end{cases}
\]

For unramified characters and \( \mathbb{F} \) of odd residual characteristic (8.5) can be proved by using the existence of a spherical function, see [7].

We proceed as follows: In Subsection 8.1.1 we reduce the computation of the local coefficient to a computation of a certain "Tate type" integral; see Lemma 8.1. In Subsection 8.1.2 we collect some facts that will be used in Subsection 8.1.3 where we compute this integral.

8.1.1 The local coefficient expressed as an integral

**Lemma 8.1.** For \( \text{Re}(s) >> 0 \):

\[
C_{\psi_a}(\chi \otimes \gamma_\psi^{-1}, s)^{-1} = \int_{\mathbb{F}^*} \gamma_\psi^{-1}(u) \chi(u) \psi_a(u) d^*u.
\]

This integral should be understood as a principal value integral, i.e.,

\[
\int_{\mathbb{F}^* \setminus N \mathbb{F}} \gamma_\psi^{-1}(u) \chi(u) \psi_a(u) d^*u,
\]

for \( N \) sufficiently large.

**Proof.** We recall that the integral in the right side of (8.1) converges in principal value for all \( s \), and furthermore, for all \( f \in I(\chi \otimes \gamma_\psi^{-1}, s) \) there exists \( N_f \) such that for all \( N > N_f \):

\[
\lambda(s, \chi, \psi_a)(f) = \int_{\mathbb{F}^* \setminus N \mathbb{F}} f \left( \left( \begin{array}{cc} 0 & 1 \\ \pm & \chi \end{array} \right) \right) \psi_a^{-1}(x) dx.
\]

Assume that \( \text{Re}(s) \) is so large such that the integral in the right hand side of (8.2) converges absolutely for all \( f \in I(\chi \otimes \gamma_\psi^{-1}, s) \). Define \( f \in I(\chi \otimes \gamma_\psi^{-1}, s) \) to be the following function:

\[
f(g) = \begin{cases} 
0 & g \in \mathcal{B} \\
\epsilon \chi_{s+1}(b) \gamma_\psi^{-1}(b) \phi(x) & g = \left( \begin{array}{cc} b & c \\ 0 & b^{-1} \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \epsilon
\end{cases}
\]

where \( \phi \in S(\mathbb{F}) \) is the characteristic function of \( \mathcal{O}_F \). It is sufficient to show that

\[
\lambda(-s, \chi^{-1}, \psi_a)(A(s)f) = \lambda(s, \chi, \psi_a)(f) \int_{\mathbb{F}^*} \gamma_\psi^{-1}(u) \chi(u) \psi_a(u) d^*u. \tag{8.6}
\]

Indeed,

\[
\lambda(-s, \chi^{-1}, \psi_a)(A(s)f) = \int_{\mathbb{F}} \int_{\mathbb{F}^*} f \left( \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \right) \psi_a^{-1}(x) du dx.
\]
By matrix multiplication and by (2.19) we have:

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & -u^{-1} \\
0 & -u
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & x-u^{-1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1
\end{pmatrix}.
\]

Hence, for \( N \) sufficiently large:

\[
\lambda(-s, \chi^{-1}, \psi_a)(A(s)f)
= \int_{\mathbb{F}^N} \int_{\mathbb{F}^*} \chi(s)(-u^{-1})\|u\|^{-1}\gamma_\psi(-u)^{-1}f\left(\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
1 & x-u^{-1} \\
0 & 1
\end{pmatrix}, 1\right) \psi_a^{-1}(x) du dx
= \int_{\mathbb{F}^N} \int_{\mathbb{F}^*} \chi(s)(-u)\|u\|^{-1}\gamma_\psi(-u)^{-1}f\left(\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
1 & x-u \\
0 & 1
\end{pmatrix}, 1\right) \psi_a^{-1}(x) du dx.
\]

Since the map \( y \mapsto f\left(\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
1 & y \\
0 & 1
\end{pmatrix}, 1\right) \) is supported on \( O_F \) and since we may assume that \( N > 0 \) we have:

\[
\lambda(-s, \chi^{-1}, \psi_a)(A(s)f)
= \int_{|x|<q^N} \int_{0<|u|<q^N} \chi(s)(-u)\gamma_\psi(-u)^{-1}f\left(\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
1 & x-u \\
0 & 1
\end{pmatrix}, 1\right) \psi_a^{-1}(x) d^* u dx.
\]

By changing the order of integration and by changing \( x \mapsto x + u \) we obtain

\[
\lambda(-s, \chi^{-1}, \psi_a)(A(s)f)
= \int_{|x|<q^N} f\left(\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix}, 1\right) \psi_a^{-1}(x) dx \int_{0<|u|<q^N} \gamma_\psi^{-1}(-u)\chi(s)(-u)\psi_a^{-1}(u) d^* u.
\]

(8.6) is proven once we change \( u \mapsto -u \).

The next lemma reduces the proof of Theorem 8.1 to case where \( \psi \) is normalized.

**Lemma 8.2.** Define \( n \) to be the conductor of \( \psi \). We have:

\[
C_\psi(\chi \otimes \gamma_\psi^{-1}, s) = \gamma_\psi^{-1}(\pi^n)\chi(\pi^{-n})q^n C_\psi(\chi \otimes \gamma_\psi^{-1}, s).
\]

**Proof.** Recalling the definition of \( \gamma_\psi \) we observe that

\[
\gamma_\psi(a) = \frac{\gamma_{\psi_0}(a\pi^{-n})}{\gamma_{\psi_0}(\pi^{-n})}.
\]

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Thus, by Lemma 8.1 we have

\[ C_\psi(\chi \otimes \gamma_\psi^{-1}, s)^{-1} = \int_{\mathbb{F}^*} \gamma_\psi^{-1}(u)\chi(s)(u)\psi(u) d^* u \]

\[ = \gamma_{\psi_0}(\pi^n) \int_{\mathbb{F}^*} \gamma_{\psi_0}^{-1}(u\pi^{-n})\chi(s)(u)\psi^{-1}(u) d^* u \]

\[ = \gamma_{\psi_0}(\pi^n) \int_{\mathbb{F}^*} \gamma_{\psi_0}^{-1}(u)\chi(s)(u\pi^n)\psi^{-1}(u) d^* u \]

\[ = \gamma_{\psi_0}(\pi^n)\chi(s)(\pi^n) \int_{\mathbb{F}^*} \gamma_{\psi_0}^{-1}(u)\chi(s)(u)\psi^{-1}(u) d^* u \]

\[ = \gamma_{\psi_0}(\pi^n)q^{-ns}\chi(\pi^n)C_{\psi_0}(\chi \otimes \gamma_\psi^{-1}, s)^{-1}. \]

\[ \square \]

We note that in Theorem 8.1, we compute

\[ C_\psi(\chi \otimes \gamma_\psi^{-1}, s) \]

rather than

\[ C_{\psi_0}(\chi \otimes \gamma_\psi^{-1}, s), \]

that is, we use the same additive character in the definition of \( I(\chi \otimes \gamma_\psi^{-1}, s) \) and in the Whittaker functional. The advantage of this choice is explained in Section 8.4. We conclude this section with a lemma that describes the relation between these two local coefficients.

**Lemma 8.3.**

\[ C_\psi(\chi \otimes \gamma_\psi^{-1}, s)\gamma_\psi(a)\chi(s)(a) = C_{\psi_0}(\chi \cdot (a, \cdot) \otimes \gamma_\psi^{-1}, s). \]

**Proof.** By Lemma 8.1

\[ C_{\psi_0}(\chi \otimes \gamma_\psi^{-1}, s)^{-1} = \int_{\mathbb{F}^*} \gamma_\psi^{-1}(u)\chi(s)(u)\psi(au) d^* u \]

\[ = \int_{\mathbb{F}^*} \gamma_\psi^{-1}(a^{-1}u)\chi(s)(a^{-1}u)\psi(u) d^* u \]

\[ = \gamma_\psi^{-1}(a)\chi(s)(a^{-1}) \int_{\mathbb{F}^*} \gamma_\psi^{-1}(u)\chi(s)(u)(a,u)\psi(u) d^* u \]

\[ = \gamma_\psi^{-1}(a)\chi(s)(a^{-1})C_\psi(\chi \cdot (a, \cdot) \otimes \gamma_\psi^{-1}, s)^{-1}. \]

\[ \square \]

### 8.1.2 Some lemmas.

**Lemma 8.4.** Assume that \( \chi^2 \) is unramified and that \( m(\chi) \leq 2e \). Then \( m(\chi) \) is even.

**Proof.** The lemma is trivial for \( \text{F} \) of odd residual characteristic. Assume that \( \text{F} \) is of even residual characteristic. Since \( \mathbb{O}_F^* / (1 + \mathbb{P}_F) \simeq \mathbb{F}^* \) is a cyclic group of odd order it follows that if \( \chi(1 + \mathbb{P}_F) = 1 \) then \( \chi \) is unramified. Thus, \( m(\chi) \neq 1 \) and it is left to show that for any \( 2 \leq k \leq e - 1 \) if \( \chi(1 + \mathbb{P}_F^{2k+1}) = 1 \) then \( \chi(1 + \mathbb{P}_F^{2k}) = 1 \). For \( l \in \mathbb{N} \), define \( G_l = \mathbb{O}_F^* / 1 + \mathbb{P}_F^l \).
The number of quadratic characters of $G_l$ is $b_l = |G_l : G_l^2|$. We want to prove that for any $2 \leq k \leq e-1$, $b_{2k} = b_{2k+1}$. Note that for any $l \geq 1$, $G_l \cong \mathbb{F}^* \times H_l$, where $H_l = 1 + \mathbb{P}_F/1 + \mathbb{P}^d_F$ is a commutative group of order $2^{2l}$ where $F = [\mathbb{F} : \mathbb{Z}]$ is the residue class degree of $\mathbb{F}$ over $\mathbb{Q}_2$. Thus, there exists $c_l \in \mathbb{N}$ such that $H_l = \prod_{j=1}^{c_l} \mathbb{Z}/2^{e(j)}\mathbb{Z}$ and $G_l/G_l^2 = H_l/H_l^2 = (\mathbb{Z}/2\mathbb{Z})^{c_l}$. Hence, the proof is done once we show that for any $2 \leq k \leq e-1$, $c_{2k} = c_{2k+1}$. We shall prove that for any $2 \leq k \leq e-1$, $c_{2k} = c_{2k+1} = f_k$. Note that

$$| \{ x \in H_l \mid x^2 = 1 \} | = 2^{c_l}.$$  

Also note that $H_l$ may be realized as

$$\{ 1 + \sum_{j=1}^{l-1} a_j \pi^j \mid a_j \in A \},$$

where $A$ is a set of representatives of $O_F/\mathbb{P}_F$ which contains $0$. Hence, it is sufficient to show that for $x = 1 + \sum_{j=1}^{2k-1} a_j \pi^j$, where $a_j \in A$ we have $x^2 \in 1 + \mathbb{P}_F^{2k}$ if and only if $a_j = 0$ for all $1 \leq j \leq k-1$, and that for $x = 1 + \sum_{j=1}^{2k-1} a_j \pi^j$, where $a_j \in A$ we have $x^2 \in 1 + \mathbb{P}_F^{2k+1}$ if and only if $a_j = 0$ for all $1 \leq j \leq k$. We prove only the first claim, the second is proven in the same way. Suppose $x \neq 1$ and that $x = 1 + \sum_{j=1}^{2k-1} a_j \pi^j$, where $a_j \in A$. Note that

$$(1 + \sum_{j=1}^{2k-1} a_j \pi^j)^2 = 1 + \sum_{i=1}^{2k-1} a_i^2 \pi^{-2i} + \sum_{i=1}^{2k-1} \omega a_i \pi^{-e+i} + \sum_{1 \leq i < j \leq 2k-1} \omega a_i a_j \pi^{-i+j+e},$$

(recall that $\omega = 2\pi^{-e} \in \mathbb{O}_F^*$). Since $k < e$ we have $\|x^2 - 1\| = q^{-2r}$, where $r$ is the minimal index such that $a_r \neq 0$. The assertion is now proven.

The following sets will appear in the computations of $\int_{\mathbb{F}^*} \gamma^{-1}_\psi(u)\chi(s)(u)\psi(u)\text{d}^*u$. For $n \in \mathbb{N}$ define

$$H(n, \mathbb{F}) = \{ x \in \mathbb{O}_F^* \mid \| 1 - x^2 \| \leq q^{-n} \},$$  \hspace{1cm} (8.7)

$$D(n, \mathbb{F}) = \{ x \in \mathbb{O}_F^* \mid \| 1 - x^2 \| = q^{1-n} \}. $$ \hspace{1cm} (8.8)

**Lemma 8.5.** Suppose $\mathbb{F}$ is of odd residual characteristic. Then

$$D(1, \mathbb{F}) = \{ x \in \mathbb{O}_F^* \mid \mathbb{F} \notin \{ 0, \pm 1 \} \},$$  \hspace{1cm} (8.9)

$$H(1, \mathbb{F}) = \{ x \in \mathbb{O}_F^* \mid \mathbb{F} \in \{ \pm 1 \} \}. $$ \hspace{1cm} (8.10)

Suppose $\mathbb{F}$ is of even residual characteristic. Then

$$D(1, \mathbb{F}) = \{ x \in \mathbb{O}_F^* \mid \mathbb{F} \notin \{ 0, 1 \} \},$$  \hspace{1cm} (8.11)

$$H(1, \mathbb{F}) = \{ x \in \mathbb{O}_F^* \mid \mathbb{F} = 1 \}. $$ \hspace{1cm} (8.12)

for $1 \leq k \leq e$:

$$D(2k, \mathbb{F}) = \emptyset, \quad H(2k, \mathbb{F}) = H(2k-1, \mathbb{F}) = 1 + \mathbb{P}_F^k,$$  \hspace{1cm} (8.13)

for all $1 \leq k < e$:

$$D(2k+1, \mathbb{F}) = 1 + \mathbb{P}_F^k \setminus 1 + \mathbb{P}_F^{k+1}, $$ \hspace{1cm} (8.14)

and

$$D(2e+1, \mathbb{F}) = \{ 1 + b\pi^e \mid b \in \mathbb{O}_F^*, \overline{b} \neq \overline{w} \}, $$ \hspace{1cm} (8.15)

$$H(2e+1, \mathbb{F}) = \{ 1 + b\pi^e \mid b \in \mathbb{O}_F, \overline{b} \in \{ 0, \overline{w} \} \}. $$ \hspace{1cm} (8.16)
Proof. We start with $F$ of odd residual characteristic. Suppose $x \in \mathbb{O}_F^*$. Then $|1 - x^2| = 1$ is equivalent to $1 - x^2 \neq 0$. This proves (8.9). (8.10) follows immediately since $H(1, F) = \mathbb{O}_F^* \setminus D(1, F)$. Suppose now that $F$ is of even residual characteristic. (8.11) and (8.12) are proven in the same way as (8.9) and (8.10). Note that in the case of even residual characteristic $t = -t$ for all $t \in F$. Assume now $x = 1 + b\pi^m$, $m \geq 1$, $b \in \mathbb{O}_F^*$. We have:

$$|x^2 - 1| = \|b w \pi^{m+e} + b^2 \pi^{2m}\| = \begin{cases} q^{-2m} & 1 \leq m < e \\ q^{-2c} |b + w| & m = e \\ q^{m+e} & m > e \end{cases}$$

the rest of the assertions mentioned in this lemma follow at once. □

Lemma 8.6. Assume $\chi$ is ramified and that $\psi$ is normalized. Fix $x \in \mathbb{O}_F^*$, and $n$, a non-negative integer.

$$\int_{\mathbb{O}_F^*} \psi(\pi^{-n} u) \, du = \begin{cases} 1 - q^{-1} & n = 0 \\ -q^{-1} & n = 1 \\ 0 & n > 1 \end{cases} . \quad (8.17)$$

If $n > 0$ then

$$\int_{\mathbb{O}_F^*} \psi(\pi^{-m(\chi)} u (1 - \pi^n x^2)) \chi(u) \, du = \chi^{-1}(1 - \pi^n x^2) G(\chi, \psi), \quad (8.18)$$

and

$$\int_{\mathbb{O}_F^*} \psi(\pi^{-n} u (1 - x^2)) \, du = \begin{cases} 1 - q^{-1} & x \in H(n, F) \\ -q^{-1} & x \in D(n, F) \\ 0 & \text{otherwise} \end{cases} . \quad (8.19)$$

If $n \neq m(\chi)$ then:

$$\int_{\mathbb{O}_F^*} \psi(\pi^{-n} u) \chi(u) \, du = 0. \quad (8.20)$$

$$\int_{\mathbb{O}_F^*} \psi(\pi^{-n} u (1 - x^2)) \chi(u) \, du = \begin{cases} \chi^{-1}(\pi^n-m(\chi)(1-x^2)) G(\chi, \psi) & n \geq m(\chi) \text{ and} \\ 0 & x \in D(n-m(\chi)+1, F) , \text{ otherwise} \end{cases} \quad (8.21)$$

Suppose $2n \geq m(\chi)$. Then

$$\int_{\mathbb{O}_F^*} \chi(1 + \pi^n u) \, du = \begin{cases} 1 - q^{-1} & n \geq m(\chi) \\ -q^{-1} & n = m(\chi) - 1 \\ 0 & n \leq m(\chi) - 2 \end{cases} . \quad (8.22)$$

Suppose in addition that $\chi'$ is another non-trivial character and that $m(\chi) - n \neq m(\chi')$. Then

$$\int_{\mathbb{O}_F^*} \chi'(x) \chi(1 + \pi^n u) \, du = 0. \quad (8.23)$$
Proof. To prove (8.17) note that
\[
\int_{\mathcal{O}_F^*} \psi(\pi^{-n}u)\,du = \int_{\mathcal{O}_F^*} \psi(\pi^{-n}u)\,du - \int_{\mathfrak{P}_F} \psi(\pi^{-n}u)\,du,
\]
and recall that for any compact group $G$ and a character $\beta$ of $G$ we have
\[
\int_G \beta(g)\,dg = \begin{cases} Vol(G) & \beta \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}.
\]
The proof of (8.18) follows from the definition of $G(\chi, \psi)$ by changing of $u \mapsto u(1-\pi^n x^2)^{-1}$. To prove (8.19) note that the conductor of the additive character $u \mapsto \psi(\pi^{-n}u(1-x^2))$ is $n - \log_q |1-x^2|$ and repeat the proof of (8.17). We now prove (8.20): If $n < m(\chi)$ then
\[
\int_{\mathcal{O}_F^*} \psi(\pi^{-n}u)\chi(u)\,du = \sum_{t \in \mathcal{O}_F^*/1+\mathfrak{P}_F^{m(\chi)}} \psi(\pi^{-nt})\chi(t) \int_{1+\mathfrak{P}_F^{m(\chi)}} \chi(u)\,du = 0,
\]
and if $n > m(\chi)$ then
\[
\int_{\mathcal{O}_F^*} \psi(\pi^{-n}u)\chi(u)\,du = \sum_{t \in \mathcal{O}_F^*/1+\mathfrak{P}_F^{m(\chi)}} \chi(t) \int_{1+\mathfrak{P}_F^{m(\chi)}} \psi(\pi^{-nt})\,du = \sum_{t \in \mathcal{O}_F^*/1+\mathfrak{P}_F^{m(\chi)}} \chi(t) \psi(\pi^{-nt}) \int_{\mathfrak{P}_F^{m(\chi)}} \psi(\pi^{-nk})\,dk = 0.
\]
To prove (8.21) write
\[
\int_{\mathcal{O}_F^*} \psi(\pi^{-n}(1-x^2))\chi(u)\,du = \int_{\mathcal{O}_F^*} \psi(\pi^{-m(\chi)}(1-x^2)\pi^{m(\chi)-n}u)\chi(u)\,du.
\]
By (8.20), the integral vanishes unless $(1-x^2)\pi^{m(\chi)-n} \in \mathcal{O}_F^*$. This implies the second case of (8.21). Changing the integration variable $u \mapsto u((1-x^2)\pi^{m(\chi)-n})^{-1}$ proves the first case. We move to (8.22): Since $2n \geq m(\chi)$ we have
\[
(1+\pi^n(x+y)) = (1+\pi^n x)(1+\pi^n y) \pmod{1+\mathfrak{P}_F^{m(\chi)}},
\]
for all $x, y \in \mathcal{O}_F$. This implies that $u \mapsto \chi(1+\pi^n u)$ is an additive character of $\mathcal{O}_F$. It is trivial if $n \geq m(\chi)$ otherwise its conductor is $m(\chi) - n$. The rest of the proof of (8.22) is the same as the proof of (8.17). The proof of (8.23) is now a repetition of the proof of (8.20). \qed

8.1.3 Computation of $\int_{\mathfrak{F}_*} \gamma^{-1}_\psi (u) \chi_{(s)}(u) \psi(u)\,d^*u$

In this subsection we assume that $\psi$ is normalized. By Lemmas 8.1 and 8.2 the proof of Theorem 8.1 amounts to computing
\[
\int_{\mathfrak{F}_*} \gamma^{-1}_\psi (u) \chi_{(s)}(u) \psi(u)\,d^*u.
\]
We write
\[
\int_{\mathbb{F}} \gamma^{-1}_\psi(u) \chi(s)(u) \psi(u) d^s u = A(\mathbb{F}, \psi, \chi, s) + B(\mathbb{F}, \psi, \chi, s),
\]
where
\[
A(\mathbb{F}, \psi, \chi, s) = \int_{\mathcal{O}_\mathbb{F}_s} \gamma^{-1}_\psi(u) \chi(s)(u) d^s u = \sum_{n=0}^{\infty} (q^{-s} \chi(\pi))^n \int_{\mathcal{O}_\mathbb{F}_s} \gamma^{-1}_\psi(\pi^n u) \chi(u) du
\]
and
\[
B(\mathbb{F}, \psi, \chi, s) = \int_{\|u\| > 1} \gamma^{-1}_\psi(u) \chi(s)(u) \psi(u) d^s u = \sum_{n=1}^{\infty} (\chi^{-1}(\pi q^n)) J_n(\mathbb{F}, \psi, \chi),
\]
where
\[
J_n(\mathbb{F}, \psi, \chi) = \int_{\mathcal{O}_\mathbb{F}_s} \gamma^{-1}_\psi(\pi^{-n} u) \psi(\pi^{-n} u) \chi(u) du.
\]
By (8.39), if \( k \in \mathbb{N}_{odd} \)
\[
J_k(\mathbb{F}, \psi, \chi) = q^{-\frac{k}{2}} c^{-1}_\psi(-1) \left( \int_{\mathcal{O}_\mathbb{F}_s} \psi(\pi^{-k} u) \chi(u) du + \sum_{n=1}^{e+1} q^n \int_{\mathcal{O}_\mathbb{F}_s} \int_{\mathcal{O}_\mathbb{F}_s} \psi(u(\pi^{-k} - \pi^{-2n} x^2)) \chi(u) du dx \right),
\]
while by (8.38), if \( k \in \mathbb{N}_{even} \)
\[
J_k(\mathbb{F}, \psi, \chi) = c^{-1}_\psi(-1) \left( \int_{\mathcal{O}_\mathbb{F}_s} \psi(\pi^{-k} u) \chi(u) du + \sum_{n=1}^{e} q^n \int_{\mathcal{O}_\mathbb{F}_s} \int_{\mathcal{O}_\mathbb{F}_s} \psi(u(\pi^{-k} - \pi^{-2n} x^2)) \chi(u) du dx \right).
\]
Due to Lemma 8.1 and 8.24, in order to compute \( C_\psi(\chi \otimes \gamma^{-1}_\psi, s) \), it is sufficient to compute \( A(\mathbb{F}, \psi, \chi, s) \) and \( B(\mathbb{F}, \psi, \chi, s) \).

**Lemma 8.7.** \( A(\mathbb{F}, \psi, \chi, s) = 0 \) unless \( \chi^2 \) is unramified, in which case
\[
A(\mathbb{F}, \psi, \chi, s) = c^{-1}_\psi(-1) \chi(-1)(1-q^{-1}) q^{\frac{m(\chi)}{2}} (1-\chi^2(\pi) q^{-2s})^{-1} G(\chi, \psi) \begin{cases} \chi(\pi) q^{-s} & m(\chi) = 2e + 1 \\ 1 & m(\chi) \leq 2e \end{cases}.
\]

**Proof.** First we show that
\[
\int_{\mathcal{O}_\mathbb{F}_s} \gamma^{-1}_\psi(u) \chi(u) du
\]
\[
= c^{-1}_\psi(-1) \left( \int_{\mathcal{O}_\mathbb{F}_s} \chi(u) du + \chi(-1) \int_{\mathcal{O}_\mathbb{F}_s} \chi(x^{-2}) dx \sum_{n=1}^{e} q^n \int_{\mathcal{O}_\mathbb{F}_s} \psi(\pi^{-2n} u) \chi(u) du \right),
\]
and that
\[
\int_{\mathcal{O}_\mathbb{F}_s} \gamma^{-1}_\psi(\pi u) \chi(u) du
\]
\[
= q^{-\frac{1}{2}} c^{-1}_\psi(-1) \left( \int_{\mathcal{O}_\mathbb{F}_s} \chi(u) du + \chi(-1) \int_{\mathcal{O}_\mathbb{F}_s} \chi(x^{-2}) dx \sum_{n=1}^{e+1} q^n \int_{\mathcal{O}_\mathbb{F}_s} \psi(\pi^{-1-2n} u) \chi(u) du \right).
\]
Indeed, by Lemma 3.2
\[
\int_{O_F^\times} \gamma_\psi^{-1}(u)\chi(u)\, du
\]
\[
= c_\psi^{-1}(-1) \int_{O_F^\times} \left(1 + \sum_{n=1}^{e} q^n \int_{O_F^\times} \psi^{-1}(\pi^{-2n}x^2 u)\, dx\right) \chi(u)\, du
\]
\[
= c_\psi^{-1}(-1) \left(\int_{O_F^\times} \chi(u)\, du + \sum_{n=1}^{e} q^n \int_{O_F^\times} \psi(-\pi^{-2n}x^2 u)\chi(u)\, du\right)
\]
\[
= c_\psi^{-1}(-1) \left(\int_{O_F^\times} \chi(u)\, du + (\pi^{-1}) \int_{O_F^\times} \chi(\pi^{-2})\, dx \sum_{n=1}^{e} q^n \int_{O_F^\times} \psi(\pi^{-2n} u)\chi(u)\, du\right).
\]
(8.31) follows in the same way. If \(\chi^2\) is ramified then (8.25), (8.30) and (8.31) implies that \(A(F, \psi, \chi, s) = 0\). From now on we assume \(\chi^2\) is unramified. The fact, proven in Lemma 3.1, that \(1 + \mathbb{P}_F^{2e+1} \subset F^2\) implies that \(m(\chi) \leq 2e + 1\). Consider first the case \(m(\chi) < 2e + 1\). By Lemma 3.3
\[
\int_{O_F^\times} \gamma_\psi^{-1}(\pi u)\chi(u)\, du = \gamma_\psi^{-1}(\pi) \int_{O_F^\times} \gamma_\psi^{-1}(u)\chi_\pi(u)\, du
\]
\[
= \gamma_\psi^{-1}(\pi) \sum_{t \in O_F^\times / 1+\mathbb{P}_F^{2e}} \chi_\pi(t) \int_{1+\mathbb{P}_F^{2e}} \gamma_\psi^{-1}(ut)\chi_\pi(u)\, du
\]
\[
= \gamma_\psi^{-1}(\pi) \sum_{t \in O_F^\times / 1+\mathbb{P}_F^{2e}} \gamma_\psi^{-1}(t)\chi_\pi(t) \int_{1+\mathbb{P}_F^{2e}} (u, \pi)\, du = 0.
\]
By (8.30) and (8.17):
\[
\int_{O_F^\times} \gamma_\psi^{-1}(u)\, du = c_\psi^{-1}(-1)(1 - q^{-1}).
\]
Recalling (8.25), this lemma follows for unramified \(\chi\). Assume now that \(1 \leq m(\chi) \leq 2e\). By Lemma 8.4 \(m(\chi)\) is even. From (8.30), (8.18) and (8.20) it follows that
\[
\int_{O_F^\times} \gamma_\psi^{-1}(u)\chi(u)\, du = \chi(-1)c_\psi^{-1}(-1)(1 - q^{-1})q^{\frac{m(\chi)}{2}}G(\chi, \psi).
\]
By (8.25) we are done in this case also. Finally, assume \(m(\chi) = 2e + 1\). In this case, by Lemma 3.3
\[
\int_{O_F^\times} \gamma_\psi^{-1}(u)\chi(u)\, du = \sum_{t \in O_F^\times / 1+\mathbb{P}_F^{2e}} \gamma_\psi^{-1}(t)\chi(t) \int_{1+\mathbb{P}_F^{2e}} \chi(u)\, du = 0.
\]
Also, from (8.31), (8.18) and (8.20) it follows that
\[
\int_{O_F^\times} \gamma_\psi^{-1}(\pi u)\chi(u)\, du = c_\psi^{-1}(-1)\chi(-1)(1 - q^{-1})q^{\frac{m(\chi)}{2}}G(\chi, \psi).
\]
With (8.25) we now finish. \(\square\)

Lemma 8.8. \(B(F, \psi, \chi, s) = \sum_{n=1}^{\max\{2e+1, m(\chi)\}} (\chi^{-1}(\pi)q^s)^n J_n(F, \psi, \chi)\)
Proof. Put $k = \max(2e + 1, m(\chi))$. By \eqref{8.26}, one should prove that $J_n(F, \psi, \chi) = 0$ for all $n > k$. Indeed,

\[
\int_{\mathcal{O}_F^e} \gamma_\psi^{-1}(u) \psi(\pi^{-n}u)(u, \pi^n)_F \chi(u) \, du = \sum_{t \in \mathcal{O}_F^e/(1 + \mathcal{O}_F^e)} \chi(t) \gamma_\psi^{-1}(t)(t, \pi^n)_F \int_{1 + \mathcal{O}_F^e} \psi(\pi^{-n}tu) \, du = 0.
\]

\[\square\]

Lemma 8.9. If $\chi$ is unramified then

\[
B(F, \psi, \chi, s) = c_\psi^{-1}(-1) \left( \chi^{-2e-1}(\pi)q^{(2e+1)s-\frac{3}{2}} + (1 - q^{-1}) \sum_{k=1}^{e} (\chi^{-1}(\pi)q^s)_{2k} \right).
\]

Proof. First note that by Lemma \ref{8.8} we have

\[
B(F, \psi, \chi, s) = \sum_{n=1}^{2e+1} (\chi^{-1}(\pi)q^s)^n J_n(F, \psi, \chi).
\]

Assume first that $F$ of is of odd residual characteristic. In this case we only want to show that $J_1(F, \psi, \chi) = q^{-\frac{e}{2}}$. By \eqref{8.28} we have

\[
J_1(F, \psi, \chi) = q^{-\frac{e}{2}} \left( \int_{\mathcal{O}_F^e} \psi(u\pi^{-1}) \, du + q \int_{\mathcal{O}_F^e} \int_{\mathcal{O}_F^e} \psi^{-1}(u\pi^{-1}(u^2 - 1)) \, du \, dx \right).
\]

Using \eqref{8.17} and \eqref{8.19} we now get

\[
J_1(F, \psi, \chi) = -q^{\frac{e}{2}} + q^2 \left( \mu(H(1, F)) (1 - q^{-1}) - q^{-1}(\mu(D(1, F))) \right).
\]

Since from Lemma \ref{8.9} it follows that $\mu(D(1, F)) = 1 - 3q^{-1}$ and $\mu(H(1, F)) = 2q^{-1}$, we have completed the proof of this lemma for $F$ of odd residual characteristic. We now assume that the residue characteristic of $F$ is even. Note that by Lemma \ref{8.3} if $m(\chi) < 2e + 1$ and $0 \leq k \leq e - 1$, then:

\[
J_{2k+1}(F, \psi, \chi) = \gamma_\psi^{-1}(\pi) \int_{\mathcal{O}_F^e} \gamma_\psi^{-1}(u) \chi(u) \psi(u\pi^{-2k-1}) = \gamma_\psi^{-1}(\pi) \sum_{t \in \mathcal{O}_F^e/(1 + \mathcal{O}_F^e)} \psi(t\pi^{-2k-1}) \gamma_\psi^{-1}(t) \chi(t) \int_{1 + \mathcal{O}_F^e} (u, \pi)_F \, du = 0.
\]  

(8.32)

Therefore,

\[
B(F, \psi, \chi, s) = \sum_{k=1}^{e} (\chi^{-1}(\pi)q^s)^{2k} \int_{\mathcal{O}_F^e} \gamma_\psi^{-1}(u) \psi(u\pi^{-2k}) \, du
\]

\[
+ \chi^{-2e-1}(\pi)q^{(2e+1)s} \int_{\mathcal{O}_F^e} \gamma_\psi^{-1}(\pi u) \psi(u\pi^{-2e-1}) \, du.
\]

(8.33)

Next, we use \eqref{8.28} and \eqref{8.17} and note that for $1 \leq k \leq e$:

\[
J_{2k}(F, \psi, \chi) = c_\psi^{-1}(-1) \sum_{n=1}^{e} q^n \int_{\mathcal{O}_F^e} \left( \int_{\mathcal{O}_F^e} \psi((\pi^{-2k} - x^2\pi^{2n})u) \, du \right) \, dx.
\]

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Note that if $k \neq n$ then $|\pi^{1-k} + x^2 \varpi^{1-n}| = q^{\max(2k,2n)} \geq q^2$. Therefore if $k \neq n$ the conductor of the character $u \mapsto \psi((\pi^{1-k} - x^2 \varpi^{1-n})u)$ is at least 2. Hence, by (8.19) and by the same arguments used for (8.17) we get:

$$
\int_{\mathbb{O}_F^*} \gamma_{\psi}^{-1}(u)\psi(u\pi^{-2k})\,du = c_{\psi}^{-1}(-1)q^k \int_{\mathbb{O}_F^*} \int_{\mathbb{O}_F^*} \psi((1-x^2)\pi^{-2k}u)\,du
$$

(8.34)

$$
= c_{\psi}^{-1}(-1)q^k \left(-q^{-1}\mu(D(\mathbb{F},2k)) + (1-q^{-1})\mu(H(\mathbb{F},2k))\right).
$$

Similarly,

$$
\int_{\mathbb{O}_F^*} \gamma_{\psi}^{-1}(\pi u)\psi(u\pi^{-2e-1})\,du = c_{\psi}^{-1}(-1)q^{e+1} \left(-q^{-1}\mu(D(\mathbb{F},2e+1)) + (1-q^{-1})\mu(H(\mathbb{F},2e+1))\right).
$$

(8.35)

From Lemma 8.5 it follows that for $1 \leq k \leq e$

$$
\mu(D(2k,\mathbb{F})) = 0, \quad \mu(H(2k,\mathbb{F})) = q^{-k},
$$

(8.36)

and that

$$
\mu(D(2e+1,\mathbb{F})) = \frac{q - 2}{q^{e+1}}, \quad \mu(H(2e+1,\mathbb{F})) = \frac{2}{q^{e+1}}.
$$

(8.37)

Combining this with (8.34), (8.35) and (8.33) the lemma for the even residual characteristic case follows.

The last lemma combined with the computation given in Lemma 8.7 for unramified characters gives explicit formulas for $A(\mathbb{F},\psi,\chi,s)$ and $B(\mathbb{F},\psi,\chi,s)$ for these characters. A straightforward computation will now give the proof of Theorem 8.1 for unramified characters.

**Lemma 8.10.** Suppose that $m(\chi) > 0$ and that $\chi^2$ is unramified. Then:

$$
B(\mathbb{F},\psi,\chi,s)
$$

$$
= c_{\psi}^{-1}(-1)G(\chi,\psi)\chi(-1)q^{\frac{m(\chi)}{2}} \left((-1)q^{-\frac{m(\chi)}{2}} \sum_{k=1}^{\frac{m(\chi)}{2}} (\chi(\pi)^{-1} q^s)^{2k} - \chi(\pi)^{m(\chi)-2e-2q^{2e+2-m(\chi)} s}\right),
$$

where the sum in the right-hand side is to be understood as 0 if $m(\chi) \geq 2e$.

**Proof.** We first assume that $\mathbb{F}$ is of odd residual characteristic. Since $1 + \mathbb{F} \subseteq \mathfrak{O}_F^\ast$, $m(\chi) = 1$ ($\mathfrak{O}_F^\ast/1 + \mathbb{F} \simeq \mathbb{F}^\ast$, therefore $u \mapsto (u,\pi)$ is the only non-trivial quadratic character of $\mathfrak{O}_F^\ast$). Due to Lemma 8.8 it is sufficient to show that

$$
J_1(\mathbb{F},\psi,\chi) = -q^{-\frac{1}{2}} c_{\psi}^{-1}(-1)G(\chi,\psi)\chi(-1).
$$

By (8.28) and by (8.21) we get

$$
J_1(\mathbb{F},\psi,\chi) = q^{-\frac{1}{2}} c_{\psi}^{-1}(-1) \left(G(\chi,\psi) + q \int_{\mathfrak{O}^\ast} \int_{\mathfrak{O}^\ast} \psi(u\pi^{-1}(1-x^2))\chi(u)\,du\,dx\right)
$$

$$
= q^{-\frac{1}{2}} c_{\psi}^{-1}(-1)G(\chi,\psi) \left(1 + q \int_{D(1,\mathbb{F})} \chi^{-1}(1-x^2)\,dx\right).
$$

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Fix $x \in D(1, \mathbb{F})$. For any $u \in 1 + \mathbb{F}_p$ we have $1 - x^2u^2 = (1 - x^2)(1 + \frac{x^2}{1 - x^2} \pi)$ for some $b \in \mathbb{O}_x$. This implies that $1 - x^2u^2 = 1 - x^2 (\mod 1 + \mathbb{F}_p)$. Therefore, $\chi^{-1}(1 - x^2) = \chi^{-1}(1 - x^2u^2)$.

Thus,

$$J_1(\mathbb{F}, \psi, \chi) = q^{-\frac{1}{2}} c_\psi^{-1}(-1) G(\chi, \psi) \left( 1 + \sum_{t \in \mathbb{O}_p^*, t \neq \pm 1} \chi^{-1}(1 - t^2) \right)$$

$$= q^{-\frac{1}{2}} c_\psi^{-1}(-1) G(\chi, \psi) \sum_{t \in \mathbb{F}} \chi'(1 - t^2),$$

Where $\chi'$ is the only non-trivial quadratic character of $\mathbb{F}$. Theorem 1 of Chapter 8 of [23] combined with Exercise 3 of the same chapter implies that $\sum_{t \in \mathbb{F}} \chi'(1 - t^2) = -\chi(-1)$.

From now on we assume that $\mathbb{F}$ is of even residual characteristic. We start with the case $m(\chi) \leq 2e$. By Lemma 8.8 and by (8.32) it is sufficient to prove that $J_{2e+1}(\mathbb{F}, \psi, \chi) = 0$ and that for all $1 \leq k \leq e$ we have:

$$J_{2k}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) q^\frac{m(\chi)}{2} G(\chi, \psi) \chi(-1) \begin{cases} q - 1 & k \leq e \\ -1 & k = e + 1 \\ 0 & k \geq e + 2 \end{cases} \tag{8.38}$$

By (8.28) and by (8.20) we have

$$J_{2e+1}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) \sum_{n=1}^{e+1} q^{n-\frac{1}{2}} \left( \int_{\mathbb{O}_p^*} \int_{\mathbb{O}_p^*} \psi((\pi^{-2e-1} - \pi^{-1-2n} x^2)u) \chi(u) du dx \right).$$

Since for $n < e + 1, x \in \mathbb{O}_p$ the conductor of the character $u \mapsto \psi((\pi^{-2e-1} - \pi^{-1-2n} x^2)u)$ is $2e + 1 \neq m(\chi)$ it follows from (8.20) that

$$J_{2e+1}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) q^{e+\frac{1}{2}} \left( \int_{\mathbb{O}_p^*} \int_{\mathbb{O}_p^*} \psi(\pi^{-2e-1}u(1 - x^2)) \chi(u) du dx \right).$$

Using (8.21) we obtain

$$J_{2e+1}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) q^{e+\frac{1}{2}} G(\chi, \psi) \int_{D(2e+2 - m(\chi))} \chi^{-1}(\pi^{m(\chi)-2e-1}(1 - x^2)) dx.$$

Since by Lemma 8.4 $m(\chi)$ is even, it follows from Lemma 8.5 that $D(2e + 2 - m(\chi)) = \emptyset$. We conclude that $J_{2e+1}(\mathbb{F}, \psi, \chi) = 0$ (note that we used only the facts that $0 < m(\chi) < 2e+1$ and that $m(\chi) \in \mathbb{N}_{even}$). Suppose now that $1 \leq k \leq e$ and that $2k \neq m(\chi)$. By (8.29) we have:

$$J_{2k}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) \sum_{n=1}^{e} q^{n} \left( \int_{\mathbb{O}_p^*} \int_{\mathbb{O}_p^*} \psi((\pi^{-2k} - \pi^{-2n} x^2)u) \chi(u) du dx \right).$$

For $n < k$ the conductor of $u \mapsto \psi((\pi^{-2k} - \pi^{-2n} x^2)u)$ is $2k \neq m(\chi)$. Hence, by (8.21) the first $k - 1$ terms in the last equation vanish and we have

$$J_{2k}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) \sum_{n=k}^{e} q^{n} \left( \int_{\mathbb{O}_p^*} \int_{\mathbb{O}_p^*} \psi((\pi^{-2k} - \pi^{-2n} x^2)u) \chi(u) du dx \right). \tag{8.39}$$

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Since for $n > k$ the conductor of $u \mapsto \psi((\pi^{-2k} - \pi^{-2n}x^2)u)$ is $2n$, it follows from (8.20) and (8.21) that for $e \geq k > \frac{m(\chi)}{2}$

$$J_{2k}(F, \psi, \chi) = c_{\psi}^{-1}(-1)q^kG(\chi, \psi) \int_{D(2k-m(\chi)+1)} \chi^{-1}(\pi^{m(\chi)-2k}(1-x^2)) \, dx.$$  

By Lemma 8.5 in order to prove (8.38) for $e \geq k > \frac{m(\chi)}{2}$, it is sufficient to show that

$$
\int_{1+P_{\chi}^{m(\chi)-2k}} \chi^{-1}(\pi^{m(\chi)-2k}(1-x^2)) \, dx = q^{\frac{m(\chi)}{2}} \int_{D(\chi)^{2k}} \chi^{-1}((-u^2(1+u^2w\pi e^{\frac{m(\chi)}{2}}-k)) \, du.
$$

By changing $x = 1 + u\pi^{k-m(\chi)}$ we have

$$
\int_{D(2k-m(\chi)+1)} \chi^{-1}(\pi^{m(\chi)-2k}(1-x^2)) \, dx = q^{\frac{m(\chi)}{2}} \chi^{-1}(1+u\pi e^{\frac{m(\chi)}{2}}-k) \, du.
$$

(8.22) implies now (8.40). We now prove (8.38) for $1 \leq k < \frac{m(\chi)}{2}$. Equation (8.39) and arguments we have already used imply in this case:

$$
J_{2k}(F, \psi, \chi) = c_{\psi}^{-1}(-1)q^{\frac{m(\chi)}{2}} \left( \int_{D(\chi)^{2k}} \psi(\pi^{-m(\chi)}u(\pi^{m(\chi)-2k}-x^2)) \chi(u) \, du \right)
$$

$$
= c_{\psi}^{-1}(-1)q^{\frac{m(\chi)}{2}} G(\chi, \psi) \int_{D(\chi)^{2k}} \chi(\pi^{m(\chi)-2k}-x^2) \, dx
$$

$$
= c_{\psi}^{-1}(-1)q^{\frac{m(\chi)}{2}} G(\chi, \psi) \chi(-1) \int_{D(\chi)^{2k}} \chi(x^2-\pi^{m(\chi)-2k}) \, dx.
$$

It is left to show that

$$
\int_{D(\chi)^{2k}} \chi(x^2-\pi^{m(\chi)-2k}) \, dx = \begin{cases} 1 - q^{-1} \frac{m(\chi)}{2} + k \leq e \\ -q^{-1} \frac{m(\chi)}{2} + k = e + 1 \\ 0 \frac{m(\chi)}{2} + k \geq e + 2 
\end{cases}.
$$

This is done by changing $x = u + \pi^{\frac{m(\chi)}{2}}$ and by using (8.22). Finally, we prove (8.38) for the case $2k = m(\chi)$: By (8.20), (8.21) and (8.18) we have

$$
J_{m(\chi)}(F, \psi, \chi) = c_{\psi}^{-1}(-1) \left( G(\chi, \psi) + \sum_{n=1}^{e} q^n \int_{D(\chi)^{2k}} \psi(u\pi^{-m(\chi)}(1-\pi^{m(\chi)-2n}x^2)) \chi(u) \, du \right)
$$

$$
= c_{\psi}^{-1}(-1) G(\chi, \psi) \left( 1 + \sum_{n=1}^{m(\chi)-1} q^n \int_{D(\chi)^{2k}} \chi(1-\pi^{m(\chi)-2n}x^2) \, dx + q^{\frac{m(\chi)}{2}} \int_{D(1,\chi)} \chi(1-x^2) \, dx \right).$$
(8.38) will follow in this case once we prove:

\[
\int_{O_2^*} \chi(1 - \pi^{m(\chi)-2n}x^2) \, dx = \begin{cases} 
1 - q^{-1} & n \leq e - \frac{m(\chi)}{2} \\
- q^{-1} & n = e - \frac{m(\chi)}{2} + 1 \\
0 & n > e - \frac{m(\chi)}{2} + 1
\end{cases} \tag{8.41}
\]

and

\[
\int_{O_2^* \setminus 1+P_2} \chi(1 - x^2) \, dx = \begin{cases} 
\chi(-1)(1 - 2q^{-1}) & m(\chi) \leq e \\
- q^{-1}(1 + \chi(-1)) & m(\chi) = e + 1 \\
0 & m(\chi) \geq e + 2
\end{cases} \tag{8.42}
\]

As for the first assertion: Since \(\chi^2\) is unramified:

\[
\int_{O_2^*} \chi(1 - \pi^{m(\chi)-2n}x^2) \, dx = \int_{O_2^*} \chi(x^{-2} - \pi^{m(\chi)-2n}) \, dx.
\]

We change \(x \mapsto x^{-1}\) and then \(k = x - \pi^{m(\chi)/2-n}\):

\[
\int_{O_2^*} \chi(1 - \pi^{m(\chi)-2n}x^2) \, dx = \int_{O_2^*} \chi(k^2(1 + \pi^{m(\chi)/2-n}e\omega k^{-1})) \, dk.
\]

We use once more the fact the \(\chi^2\) is unramified and we change \(k = \omega u^{-1}\):

\[
\int_{O_2^*} \chi(1 - \pi^{m(\chi)-2n}x^2) \, dx = \int_{O_2^*} \chi(1 + \pi^{m(\chi)/2-n}e\omega u) \, du.
\]

(8.22) now implies (8.41). As for (8.42), we change \(k = 1 - x\) and then \(u = -k^{-1}\) and get:

\[
\int_{O_2^* \setminus 1+P_2} \chi(1 - x^2) \, dx = \chi(-1) \int_{0^* \setminus 1+P_2} \chi(1 + \omega u) \, du.
\]

It is clear that if \(m(\chi) \leq e\) then

\[
\int_{O_2^* \setminus 1+P_2} \chi(1 + \omega u) \, du = \mu(O_2^* \setminus 1+P_2) = 1 - \frac{2}{q}.
\]

If \(2e \geq m(\chi) > e\) we write

\[
\int_{O_2^* \setminus 1+P_2} \chi(1 - x^2) \, dx = \chi(-1) \left( \int_{O_2^*} \chi(1 + \omega u) \, du - \int_{1+P_2} \chi(1 + \omega u) \, du \right).
\]

By (8.22) we have

\[
\int_{O_2^*} \chi(1 + \omega u) \, du = \begin{cases} 
-q^{-1} & m(\chi) = e + 1 \\
0 & m(\chi) \geq e + 2
\end{cases}
\]

We now compute \(\int_{1+P_2} \chi(1 + \omega u) \, du\): We change \(u = 1 + \omega u\pi k\). Since

\[(1 + \omega u) = (1 + \pi e)(1 + \pi e+1 k) \pmod{P_2^{2e+1}},\]
we have 

\[ \int_{1+\mathbb{P}_F} \chi(1 + \omega \pi e u) \, du = \chi(3) \int_{\mathcal{O}_F} \chi(1 + \pi^{e+1} k) \, dk. \]

As in the proof of (8.22), since \( m(\chi) \leq 2e \), we conclude that \( k \mapsto \chi(1 + \pi^{e+1} k) \) is a character of \( \mathcal{O}_F \). It is trivial if \( m(\chi) = e + 1 \) and non-trivial otherwise. Also, since \(-3 \in 1 + \mathbb{P}_F^{2e}\), we have \( \chi(3) = \chi(-1) \). Thus,

\[ \int_{1+\mathbb{P}_F} \chi(1 + \omega \pi e u) \, du = \begin{cases} \chi(-1)q^{-1} & m(\chi) = e + 1 \\ 0 & m(\chi) \geq e + 2 \end{cases}. \]

We move to the case \( m(\chi) = 2e + 1 \). By Lemma (8.38) it is sufficient to show that

\[ J_n(\mathbb{F}, \psi, \chi) = 0 \]

for all \( 2 \leq n \leq 2e + 1 \) and that

\[ J_1(\mathbb{F}, \psi, \chi) = -q^{-\frac{1}{e}}c_{\psi}^{-1}(-1)G(\chi, \psi)\chi(-1). \]

From the fact that \( m(\chi) = 2e + 1 \) it follows that for all \( 1 \leq k \leq e \):

\[ J_{2k}(\mathbb{F}, \psi, \chi) = \int_{\mathcal{O}_F^\times} \gamma_{\psi}^{-1}(u)\chi(u)\psi(\pi^{-2k}u) \, du = 0. \]

Next we show that \( J_{2e+1}(\mathbb{F}, \psi, \chi) = 0 \). By (8.28), (8.20), (8.21) and (8.18):

\[ \int_{D(1, \mathbb{F})} \chi(1 - x^2) \, dx = 0, \]

As in the proof of (8.44) (using \( u = x - \pi^{e+1-n} \) rather then \( k = x - \pi^{m(\chi)-n} \)) for all \( 1 \leq n \leq e \):

\[ \int_{\mathcal{O}_F^\times} \chi(1 - \pi^{2(e+1-n)}x^2) \, dx = \int_{\mathcal{O}_F^\times} \chi(1 + \pi^{2(e+1-n)}u) \, du. \]

(8.22) implies now that

\[ \int_{\mathcal{O}_F^\times} \chi(1 - \pi^{2(e+1-n)}x^2) \, dx = \begin{cases} -q^{-1} & n = 1 \\ 0 & n > 1 \end{cases}. \]

It is clear now that if we prove that

\[ \int_{D(1, \mathbb{F})} \chi(1 - x^2) \, dx = 0, \]

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we will conclude that $J_{2e+1}(\mathbb{F}, \psi, \chi) = 0$. As in the proof of \[S.12\]

$$\int_{D(1, \mathbb{F})} \chi(1 - x^2) \, dx = \chi(-1) \left( \int_{\mathbb{O}_\mathbb{F}^*} \chi(1 + w \pi^e u) \, du - \int_{1 + \mathbb{F}} \chi(1 + w \pi^e u) \, du \right),$$

and

$$\int_{1 + \mathbb{F}} \chi(1 + w \pi^e u) \, du = 0.$$

We show that

$$\int_{\mathbb{O}_\mathbb{F}^*} \chi(1 + w \pi^e u) \, du = 0,$$

although the map $u \mapsto \chi(1 + w \pi^e u)$ is not a character of $\mathbb{O}_\mathbb{F}$:

$$\int_{\mathbb{O}_\mathbb{F}^*} \chi(1 + w \pi^e u) \, du = \sum_{t \in \mathbb{O}_\mathbb{F}^*/1 + \mathbb{F}} \int_{1 + \mathbb{F}} \chi(1 + w \pi^e tu) \, du.$$

For any $t \in \mathbb{O}_\mathbb{F}^*$, $k \in \mathbb{O}_\mathbb{F}$:

$$1 + w \pi^e t(1 + \pi k) = (1 + w \pi^e t)(1 + w \pi^{e+1} tk) \ (mod \ 1 + \mathbb{F}^{2e+1}).$$

Hence, by changing $u = 1 + \pi k$ we get

$$\int_{\mathbb{O}_\mathbb{F}^*} \chi(1 + w \pi^e u) \, du = q^{-1} \sum_{t \in \mathbb{O}_\mathbb{F}^*/1 + \mathbb{F}} \chi(1 + w \pi^e t) \int_{\mathbb{O}_\mathbb{F}} \chi(1 + w \pi^{e+1} tk) \, dk.$$

Since for all $t \in \mathbb{O}_\mathbb{F}^*$, $k \mapsto \chi(1 + w \pi^{e+1} tk)$ is a non-trivial character of $\mathbb{O}_\mathbb{F}$, all the integrations in the right wing vanish.

Finally, we compute $J_{2k+1}(\mathbb{F}, \psi, \chi)$ for $0 \leq k \leq e - 1$: By \[S.28\] and \[S.20\] we observe:

$$J_{2k+1}(\mathbb{F}, \psi, \chi) = q^{e+\mathbb{F}^{-1}} c_{\psi}^{-1}(-1) \int_{\mathbb{O}_\mathbb{F}^*} \psi(\pi^{2e+1} u(\pi^{2(e-k)} - x^2)) \chi(u) \, du \, dx \ (8.46)$$

$$= q^{e+\mathbb{F}^{-1}} c_{\psi}^{-1}(-1) G(\chi, \psi) \int_{\mathbb{O}_\mathbb{F}^*} \chi(\pi^{2(e-k)} - x^2) \, dx.$$

We finish by showing:

$$\int_{\mathbb{O}_\mathbb{F}^*} \chi(\pi^{2(e-k)} - x^2) \, dx = \begin{cases} -\chi(-1)q^{-1} & k = 0 \\ \chi(-1) & k > 0 \end{cases}.$$

We note that

$$\int_{\mathbb{O}_\mathbb{F}^*} \chi(\pi^{2(e-k)} - x^2) \, dx = \chi(-1) \int_{\mathbb{O}_\mathbb{F}^*} \chi(x^2 - \pi^{2(e-k)}) \, dx.$$

We now change $u = x - \pi^{e-k}$ and use \[S.22\].

The last lemma combined with the computation given in Lemma\[S.7\] for these characters gives explicit formulas for $A(\mathbb{F}, \psi, \chi, s)$ and $B(\mathbb{F}, \psi, \chi, s)$ for a character $\chi$ which satisfies $m(\chi) > m(\chi^2) = 0$. By a simple computation one can now finish the proof of Theorem \[S.1\] for a ramified $\chi$ provided that $\chi^2$ is unramified.

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Lemma 8.11. If $\chi^2$ is ramified then $B(\mathbb{F}, \psi, \chi, s)$ is a non-zero monomial in $q^s$.

Proof. The fact that $B(\mathbb{F}, \psi, \chi, s)$ is non-zero follows from the fact, proven in Lemma 8.7 that $A(\mathbb{F}, \psi, \chi, s) = 0$ and from the fact, following from Lemma 8.1 that

$$
\int_{\mathbb{F}^*} \gamma_\psi^{-1}(u) \chi(u) \psi(u) d^u
$$

can not be identically 0 as a function of $s$. Assume first that $m(\chi) > 2e + 1$. By Lemma 8.8 it is enough to show that $J_n(\mathbb{F}, \psi, \chi) = 0$ for all $1 \leq n < m(\chi)$: Put $k = \max(n, 2e + 1)$. We have

$$
J_n(\mathbb{F}, \psi, \chi) = \sum_{t \in \mathbb{O}_F^*/1 + \mathbb{P}_F^k} \gamma_\psi^{-1}(\pi^{-n} t) \chi(t) \psi(\pi^{-n} t) \int_{1 + \mathbb{P}_F^k} \chi(u) du.
$$

Since $k < m(\chi)$ that last integral vanishes. This shows that

$$
B(\mathbb{F}, \psi, \chi, s) = \chi^{-m(\chi)}(\pi) q^{m(\chi)/2} J_{m(\chi)}(\mathbb{F}, \psi, \chi).
$$

In order to complete the proof of this lemma for $\mathbb{F}$ of odd residual characteristic it is left to consider the case $m(\chi) = 1$. In fact, there is nothing to prove here since by Lemma 8.8 we get:

$$
B(\mathbb{F}, \psi, \chi, s) = \chi^{-1}(\pi) q^s J_1(\mathbb{F}, \psi, \chi).
$$

From now on we assume that $\mathbb{F}$ is of even residual and that $m(\chi) \leq 2e + 1$. Assume first that $m(\chi) < 2e + 1$ and that $m(\chi)$ is odd. By Lemma 8.8 once we prove $J_n(\mathbb{F}, \psi, \chi) = 0$ for all $n < 2e + 1$ we will conclude:

$$
B(\mathbb{F}, \psi, \chi, s) = \chi(\pi)^{-2e-1} q^{(2e+1)/2} J_{2e+1}(\mathbb{F}, \psi, \chi).
$$

By (8.32), $J_n(\mathbb{F}, \psi, \chi) = 0$ for all $0 < n < 2e + 1$, $n \in \mathbb{N}_{odd}$. Let $1 \leq k \leq e$. Since $m(\chi)$ is odd, by (8.20) and (8.20) we get:

$$
J_{2k}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) \sum_{n=1}^e q^n \int_{\mathbb{O}_F^k} \int_{\mathbb{O}_F^k} \psi((\pi^{-2k} - \pi^{-2n} x^2) u) \chi(u) du dx.
$$

For $n \neq k$, $x \in \mathbb{O}_F^k$, the conductor of the character $u \mapsto \psi((\pi^{-2k} - \pi^{-2n} x^2) u)$ is $\max(2k, 2n)$. Again, since $m(\chi)$ is odd, (8.20) implies

$$
J_{2k}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) q^k \int_{\mathbb{O}_F^k} \int_{\mathbb{O}_F^k} \psi(u \pi^{-2k}(1 - x^2)) \chi(u) du dx.
$$

Using (8.21) we conclude that if $2k < m(\chi)$ then $J_{2k}(\mathbb{F}, \psi, \chi) = 0$, and if $2k > m(\chi)$ then

$$
J_{2k}(\mathbb{F}, \psi, \chi) = c_\psi^{-1}(-1) q^k G(\chi, \psi) \int_{D(2k-m(\chi)+1, \mathbb{F})} \chi^{-1}(\pi^m(\chi) - 2k(1 - x^2)) dx.
$$

Lemma 8.5 implies that $D(2k-m(\chi)+1, \mathbb{F}) = \emptyset$. Therefore, we have shown: $J_{2k}(\mathbb{F}, \psi, \chi) = 0$ for all $1 \leq k \leq e$.

We now assume that $2 \leq m(\chi) \leq 2e$ and that $m(\chi)$ is even. Again, $J_n(\mathbb{F}, \psi, \chi) = 0$ for all $n < 2e + 1$, $n \in \mathbb{N}_{odd}$. In this case, the proof of the fact that $J_{2e+1}(\mathbb{F}, \psi, \chi) = 0$ is a repetition of the proof that $J_{2e+1}(\mathbb{F}, \psi, \chi) = 0$ given in Lemma 8.10 for the case $m(\chi) \leq 2e$. 

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By Lemma 8.23 it is enough now to prove that if $1 \leq k \leq e$, and if $2k \neq 2e + 2m(\chi^2) - m(\chi)$ then $J_{2k}(\mathbb{F}, \psi, \chi) = 0$ to conclude that

$$B(\mathbb{F}, \psi, \chi, s) = \chi(\pi)^{-\left(2e + 2m(\chi^2) - m(\chi)\right)} q^{\left(2e + 2m(\chi^2) - m(\chi)\right) s} f_{\left(2e + 2m(\chi^2) - m(\chi)\right)}(\mathbb{F}, \psi, \chi).$$

Assume that $2k < m(\chi)$. As in the proof of (8.38) for the case $2k < m(\chi)$ we have:

$$J_{2k}(\mathbb{F}, \psi, \chi) = \chi(-1) c_{\psi}^{-1}(-1) q^{\frac{m(\chi)}{2}} G(\chi, \psi) \int_{\mathcal{O}_{\mathbb{F}}} \chi^{-1}(x^2 - \pi^{m(\chi) - 2k}) dx.$$ 

By changing $x = u + \pi \frac{m(\chi)}{2} - k$ and then $wu^{-1} \mapsto u$ we obtain

$$J_{2k}(\mathbb{F}, \psi, \chi) = \chi(w^{-2}) c_{\psi}^{-1}(-1) q^{\frac{m(\chi)}{2}} G(\chi, \psi) \int_{\mathcal{O}_{\mathbb{F}}} \chi^{2}(u) \chi^{-1}(1 + \pi e + \frac{m(\chi)}{2} - k) du.$$ 

It follows from (8.23) that if $2k \neq 2e + 2m(\chi^2) - m(\chi)$ then $J_{2k}(\mathbb{F}, \psi, \chi) = 0$. Assume now $2k > m(\chi)$. As in the proof of (8.38) for the case $2k > m(\chi)$ we have:

$$J_{2k}(\mathbb{F}, \psi, \chi) = c_{\psi}^{-1}(-1) q^{k} G(\chi, \psi) \int_{\mathcal{O}_{\mathbb{F}} \setminus \mathbb{F}_{p}} \chi^{-1}(\pi^{m(\chi) - 2k}(1 - x^2)) \ dx.$$ 

Changing $x = 1 + u \pi \frac{m(\chi)}{2}$ and then $wu^{-1} \mapsto u$ we reduce this computation to the previous case. As in the proof of (8.38) for the case $2k = m(\chi)$ we have

$$J_{m(\chi)}(\mathbb{F}, \psi, \chi) = c_{\psi}^{-1}(-1) G(\chi, \psi) \left(1 + \sum_{n=1}^{\frac{m(\chi)}{2} - 1} q^{n} \int_{\mathcal{O}_{\mathbb{F}}} \chi^{-1}(1 - \pi^{m(\chi) - 2n} x^2) \ dx + q^{\frac{m(\chi)}{2}} \int_{D(1, \mathbb{F})} \chi^{-1}(1 - x^2) \ dx \right).$$ 

It is sufficient now to show that if $m(\chi) \neq 2e + 2m(\chi^2) - m(\chi)$ then

$$\sum_{n=1}^{\frac{m(\chi)}{2} - 1} q^{n} \int_{\mathcal{O}_{\mathbb{F}}} \chi^{-1}(1 - \pi^{m(\chi) - 2n} x^2) \ dx = -1, \quad (8.47)$$

and

$$\int_{\mathcal{O}_{\mathbb{F}} \setminus \mathbb{F}_{p}} \chi^{-1}(1 - x^2) \ dx = 0. \quad (8.48)$$

We prove (8.47). Let $1 \leq n \leq \frac{m(\chi)}{2} - 1$. We change $x^{-1} = x'$, $u' = x' - \pi \frac{m(\chi)}{2} \omega_{n}$. Note that $x' = u'(1 + u'^{-1} \pi \frac{m(\chi)}{2} \omega_{n}^{-1})$ and that $x'^{2} - \pi^{m(\chi) - 2n} = u'^{2} (1 + u'^{-1} \omega_{n} e + \frac{m(\chi)}{2} \omega_{n}^{-1}).$ Thus,

$$\chi^{-1}(1 - \pi^{m(\chi) - 2n} x^2) = \chi^{-1}(x'^{2}) \chi^{-1}(x'^{2} - \pi^{m(\chi) - 2n}) = \chi^{2}(u') \chi^{-1}(x'^{2} - \pi^{m(\chi) - 2n}) = \chi^{2}(u') \chi^{2}(1 + u'^{-1} \pi \frac{m(\chi)}{2} \omega_{n}^{-1}) \chi^{-1}(u'^{2}) \chi^{-1}(1 + u'^{-1} \omega_{n} e + \frac{m(\chi)}{2} \omega_{n}^{-1}).$$

Now setting $u' = u^{-1}$ implies:

$$\int_{\mathcal{O}_{\mathbb{F}}} \chi^{-1}(1 - \pi^{m(\chi) - 2n} x^2) \ dx = \int_{\mathcal{O}_{\mathbb{F}}} \chi^{2}(1 + \pi \frac{m(\chi)}{2} \omega_{n}^{-1} u) \chi^{-1}(1 + \pi e + \frac{m(\chi)}{2} \omega_{n}^{-1} \omega_{n} u) \ du = 86$$
The fact that \( u \mapsto \phi_1(u) = \chi^{-1}(1 + \pi^{e + \frac{m(\chi)}{2}} - nw u) \) is an additive character of \( \mathcal{O}_F \) follows from the same argument used in the proof of (8.22). However, by the same argument, \( u \mapsto \phi_2(u) = \chi^2(1 + \pi^{-\frac{m(\chi)}{2}} - n u) \) is an additive character of \( \mathcal{O}_F \) if and only if \( m(\chi) - 2n \geq m(\chi^2) \).

We first show that if \( m(\chi) > 2n > m(\chi) - m(\chi^2) \) then

\[
\int_{\mathcal{O}_F} \phi_1(u)\phi_2(u)\,du = 0. 
\]

(8.49)

Define \( k = 2n - m(\chi) + m(\chi^2) \). We have:

\[
\int_{\mathcal{O}_F} \phi_1(u)\phi_2(u)\,du = \sum_{t \in \mathcal{O}_F/1 + \mathbb{P}_F} \int_{1 + \mathbb{P}_F} \chi(1 + \pi^{m(\chi) - n}tu)\chi^{-1}(1 + \pi^{m(\chi) - n}w tu)\,du.
\]

In the right hand side we change \( u = 1 + x\pi^k \), \( x \in \mathcal{O}_F \). We have

\[
1 + ut\pi^{\frac{m(\chi)}{2} - n} = (1 + t\pi^{\frac{m(\chi)}{2} - n})(1 + tx\pi^{\frac{m(\chi)}{2} - n + k}) \quad (mod \, 1 + \mathbb{P}_F^{m(\chi)}),
\]

and

\[
1 + u\omega t\pi^{e + \frac{m(\chi)}{2} - n} = (1 + \omega t\pi^{e + \frac{m(\chi)}{2} - n})(1 + \omega tx\pi^{e + \frac{m(\chi)}{2} - n + k}) \quad (mod \, 1 + \mathbb{P}_F^{m(\chi)}).
\]

Thus, we get

\[
\int_{\mathcal{O}_F} \phi_1(u)\phi_2(u)\,du = q^{-k} \sum_{t \in \mathcal{O}_F/1 + \mathbb{P}_F} \beta_1^t(1)\beta_2^t(1) \int_{\mathcal{O}_F} \beta_1^t(x)\beta_2^t(x)\,dx,
\]

(8.50)

where \( \beta_1^t(x) = \chi^{-1}(1 + t\omega x\pi^{e + \frac{m(\chi)}{2} - n + k}) \) and \( \beta_2^t(x) = \chi^2(1 + tx\pi^{e + \frac{m(\chi)}{2} - n + k}) \). Since

\[
m(\chi) + 2k - 2n = 2n - m(\chi) + 2m(\chi^2) > m(\chi^2)
\]

and

\[
2e + m(\chi) + 2k - 2n > 2e \geq m(\chi),
\]

we conclude, using the proof of (8.22), that for \( t \in \mathcal{O}_F^* \), \( \beta_1^t \) and \( \beta_2^t \) are two characters \( \mathcal{O}_F \). \( \beta_2^t \) is a non-trivial character and its conductor its \( \frac{m(\chi)}{2} - n \). \( \beta_1^t \) is trivial if \( \frac{3m(\chi)}{2} - n - e - m(\chi^2) \leq 0 \), otherwise its conductor is \( \frac{3m(\chi)}{2} - n - e - m(\chi^2) \). Since we assume \( m(\chi) - m(\chi^2) \neq e \) we observe that \( \beta_1^t \) and \( \beta_2^t \) have different conductors. In particular, \( \beta_1^t/\beta_2^t \) is a non-trivial character of \( \mathcal{O}_F \). (8.50) now implies (8.49).

We now assume that \( 2n \leq m(\chi) - m(\chi^2) \). \( \phi_1 \) and \( \phi_2 \) are now both characters of \( \mathcal{O}_F \). \( \phi_1 \) is trivial if \( n \leq e - \frac{m(\chi)}{2} \). Otherwise its conductor is \( n + \frac{m(\chi)}{2} - e \). \( \phi_2 \) is trivial if \( n \leq \frac{m(\chi)}{2} - m(\chi^2) \). Otherwise its conductor is \( n - \frac{m(\chi)}{2} + m(\chi^2) \). Recall that we assumed \( m(\chi^2) \neq m(\chi) - e \). Hence, if either \( \phi_1 \) or \( \phi_2 \) are non-trivial then the conductor of \( \phi_1\phi_2 \) is \( n + \max(m(\chi^2) - \frac{m(\chi)}{2}, \frac{m(\chi)}{2} - e) \). From (8.47) it follows now that if \( m(\chi^2) > m(\chi) - e \) then

\[
\int_{\mathcal{O}_F} \chi^2(1 + \pi^{\frac{m(\chi)}{2} - n}u)\chi^{-1}(1 + \pi^{e + \frac{m(\chi)}{2} - n}u)\,du = \begin{cases} 
1 - q^{-1} & 1 \leq n \leq \frac{m(\chi)}{2} - m(\chi^2) \\
-q^{-1} & n = \frac{m(\chi)}{2} - m(\chi^2) + 1 \\
0 & n > \frac{m(\chi)}{2} - m(\chi^2) + 1 \end{cases}
\]
and if \( m(\chi^2) < m(\chi) - e \) then

\[
\int_{\mathbb{O}_F^*} \chi^2(u) \chi^{-1}(1 + m(\chi) - e) \, du = \begin{cases} 0 & n > e \frac{m(\chi)}{2} + 1 \\
 & \\
1 - q^{-1} & 1 \leq n \leq e - \frac{m(\chi)}{2} \\
- q^{-1} & n = e - \frac{m(\chi)}{2} + 1 \\
0 & 
\end{cases}
\]

Both cases imply \( (8.44) \). We now prove \( (8.45) \). As in the proof of \( (8.42) \), we change \( u' = 1 - x \) and then \( u = u'^{-1} \). Since \( 1 - x^2 = -u'^2(1 - \omega \pi e u'^{-1}) \), we get

\[
\int_{\mathbb{O}_F^* \setminus 1 + \mathbb{P}_F} \chi^{-1}(1 - x^2) \, dx = \chi(-1) \left( \int_{\mathbb{O}_F^*} \chi^2(u) \chi^{-1}(1 - u \omega \pi e) \, du - \int_{1 + \mathbb{P}_F} \chi^2(u) \chi^{-1}(1 - u \omega \pi e) \, du \right).
\]

Since \( m(\chi) - e \neq m(\chi^2) \), \( (8.23) \) implies

\[
\int_{\mathbb{O}_F^*} \chi^2(x) \chi^{-1}(1 - \omega x \pi e) \, dx = 0.
\]

If \( m(\chi^2) > m(\chi) - e \) we have

\[
\int_{1 + \mathbb{P}_F} \chi^2(x) \chi^{-1}(1 - x \omega \pi e) \, dx = \sum_{t \in 1 + \mathbb{P}_F^{m(\chi) - e}} \chi^2(t) \chi^{-1}(1 - t \omega \pi e) \int_{1 + \mathbb{P}_F^{m(\chi) - e}} \chi^2(x) \, dx = 0.
\]

If \( m(\chi^2) < m(\chi) - e \) we have

\[
\int_{1 + \mathbb{P}_F} \chi^2(x) \chi^{-1}(1 - \omega x \pi e) \, dx = \sum_{t \in 1 + \mathbb{P}_F^{m(\chi^2)}} \chi^2(t) \chi^{-1}(1 - t \omega \pi e) \int_{1 + \mathbb{P}_F^{m(\chi^2)}} \chi^{-1}(1 - \omega t x \pi e) \, dx.
\]

The proof of \( (8.45) \) will be finished once we show that for all \( t \in \mathbb{O}_F^* \):

\[
\int_{1 + \mathbb{P}_F^{m(\chi^2)}} \chi^{-1}(1 - \omega t x \pi e) = 0.
\]

We change \( x = 1 + u \pi m(\chi^2) \). Since

\[
1 - \omega t x \pi e = (1 - \omega t \pi^e)(1 - \omega t u \pi^{e+m(\chi^2)})(\mod 1 + \mathbb{P}_F^{2e}),
\]

we get

\[
\int_{1 + \mathbb{P}_F^{m(\chi^2)}} \chi^{-1}(1 - \omega t x \pi e) = q^{-m(\chi^2)} \chi^{-1}(1 + t \pi^e) \int_{\mathbb{O}_F^*} \chi^{-1}(1 - \omega t u \pi^{e+m(\chi^2)}) \, du.
\]

The integral on the right hand side vanishes since \( u \mapsto \chi^{-1}(1 - \omega t u \pi^{e+m(\chi^2)}) \) is a non-trivial character of \( \mathbb{O}_F^* \).

It is left to prove the lemma for the case \( m(\chi) = 2e + 1 \). By Lemma \( 8.8 \) and by \( (8.43) \), we only have to show that for all \( 0 \leq k \leq e \), unless \( k + 1 = m(\chi^2) \), \( J_{2k+1}(\mathbb{F}, \psi, \chi) = 0 \), to conclude that

\[
B(\mathbb{F}, \psi, \chi, s) = \chi(\pi)^{-\frac{2m(\chi^2) - 1}{s}} q^{\frac{2m(\chi^2) - 1}{s}} J_{2m(\chi^2) - 1}(\mathbb{F}, \psi, \chi).
\]
Assume first that $0 \leq k < e$. The same argument we used for (8.46) shows that

$$J_{2k+1}(F, \psi, \chi) = q^{e+1} c_\psi^{-1}(-1) G(\chi, \psi) \int_{\bar{F}_\psi} \chi^{-1}(\pi^{2(e-k)} - x^2) dx.$$  

We change $x = \pi^{e-k} - u'$. We have $\pi^{2(e-k)} - x^2 = -u'^2(1 - u'^{-1}\pi^{2e-k}\omega)$. Thus,

$$\int_{\bar{F}_\psi} \chi^{-1}(\pi^{2(e-k)} - x^2) dx = \chi^{-1}(-1) \int_{\bar{F}_\psi} \chi^{-1}(u'^2) \chi^{-1}(1 - u'^{-1}\pi^{2e-k}\omega) du'.$$

Next we change $u = -u'^{-1}\omega$ and obtain:

$$\int_{\bar{F}_\psi} \chi^{-1}(\pi^{2(e-k)} - x^2) dx = \chi(-\omega^{-2}) \int_{\bar{F}_\psi} \chi^2(u) \chi^{-1}(1 + u^{2e-k}) du.$$

(8.23) implies now that if $k + 1 \neq m(\chi^2)$ then

$$\int_{\bar{F}_\psi} \chi^{-1}(\pi^{2(e-k)} - x^2) dx = 0.$$

Since $m(\chi) = 2e + 1$ implies $1 < m(\chi^2) < e + 1$ it is left to show that if $m(\chi^2) < e + 1$ then $J_{2e+1} = 0$. As in the proof of (8.44)

$$J_{2e+1}(F, \psi, \chi) = q^{-\frac{e}{2}} c_\psi^{-1}(-1) G(\chi, \psi) \left( 1 + \sum_{n=1}^{e} \frac{q^n}{e^n} \int_{\bar{F}_\psi} \chi^{-1}(1 - \pi^{2(e+1-n)} x^2) dx \right) + q^{e+1} \int_{D(1,F)} \chi^{-1}(1 - x^2) dx.$$

As we have seen before:

$$\int_{D(1,F)} \chi^{-1}(1 - x^2) dx$$

$$= \chi(-\omega^{-2}) \left( \int_{\bar{F}_\psi} \chi^2(u) \chi^{-1}(1 + w\pi^eu) du - \int_{1+\bar{F}_\psi} \chi^2(u) \chi^{-1}(1 + w\pi^eu) du \right).$$

We show that the last two integrals vanish. Similar steps to those used in the proof of (8.45) shows that

$$\int_{\bar{F}_\psi} \chi^2(u) \chi^{-1}(1 + w\pi^eu) du$$

$$= \sum_{t \in \bar{F}_{/1+\bar{F}}} \chi^2(t) \int_{1+\bar{F}_{/wπt}} \chi^{-1}(1 + w\pi^eu) du$$

$$= \sum_{k \in \bar{F} / 1+\bar{F}_{/wπ}} \chi^2(t) \chi^{-1}(1 + w\pi^eu t) \int_{\bar{F}_\psi} \chi^{-1}(1 + w\pi^eu t+k) dk.$$

Since $\frac{m(\chi)}{2} \leq e + m(\chi^2) < m(\chi)$, (8.22) implies that all the integrals in the last sum vanish. The fact that

$$\int_{1+\bar{F}_\psi} \chi^2(u) \chi^{-1}(1 + w\pi^eu) du = 0$$
is proven by a similar argument. Last, we have that to show that 
\[ 1 \leq m(\chi^2) \leq e \] implies that
\[ \sum_{n=1}^{e} q^n \int_{0}^{\frac{1}{e}} \chi^{-1}(1 - \pi^2(e+1-n)x^2) \, dx = -1. \]
This is done by a similar argument to the one we use in the proof of (8.47). □

In Lemma 3.7 we proved that if \( \chi^2 \) is ramified then \( A(\mathbb{F}, \psi, \chi, s) = 0 \). Thus, the last lemma completes the proof of Theorem 8.1 for these characters.

8.2 Real case

8.2.1 Notations and main result

Any non-trivial character of \( \mathbb{R} \) has the form \( \psi_b(x) = e^{ibx} \) for some \( b \) in \( \mathbb{R}^* \). Any character of \( \mathbb{R}^* \) has the form \( \chi = \chi_{x,n}(a) = (\text{sign}(a))^n |a|^x \) for some \( x \in \mathbb{C}, n \in \{0,1\} \). We may assume that \( x = 0 \), i.e., that \( \chi \) is either the trivial character of the sign character. Thus, \( \chi(s) \) will denote the character \( a \mapsto (\text{sign}(a))^n |a|^s \) where \( n \in \{0,1\} \).

Lemma 8.12.
\[
C_{\psi_b}(\chi \otimes \gamma_{\psi_a}^{-1}, s) = e^{-\frac{i\pi \chi(-1)\text{sign}(a)}{4}} \frac{\Gamma(\frac{1-s}{2} + \frac{\text{sign}(ab)\chi(-1)}{4})\Gamma(\frac{1+s}{2} - \frac{\text{sign}(ab)\chi(-1)}{4})}{\Gamma(s)}. 
\]

We shall prove Lemma 8.12 in the next subsection. An immediate corollary of this lemma and of the classical duplication formula
\[
\Gamma(-\frac{s}{2} + \frac{1}{4})\Gamma(-\frac{s}{2} + \frac{3}{4}) = 2^{s+1/2} \sqrt{\pi} \Gamma(-s + \frac{1}{2}) 
\]
is the following:

Theorem 8.2.
\[
C_{\psi_a}(\chi \otimes \gamma_{\psi_a}^{-1}, s) = e^{-\frac{i\pi \chi(-1)\text{sign}(a)}{4}} \frac{\pi^s}{2^{s-\frac{1}{2}}} \frac{L(\chi, s + \frac{1}{2}) L(\chi^2, -2s + 1)}{L(\chi, -s + \frac{1}{2}) L(\chi^2, 2s)}. 
\]

Recall that the local \( L \)-function for \( \mathbb{R} \) is defined by
\[
L_{\mathbb{R}}(\chi_{0,n}, s) = \begin{cases} 
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) & n = 0 \\
\pi^{-\frac{n+1}{2}} \Gamma\left(\frac{s+n+1}{2}\right) & n = 1 
\end{cases}. 
\]

8.2.2 Proof of Lemma 8.12

We shall see that the computation of the local coefficient for \( SL_2(\mathbb{R}) \) is done by the same methods as the computation for \( SL_2(\mathbb{R}) \). Namely, we shall use the Iwasawa decomposition, \( SL_2(\mathbb{R}) = B(\mathbb{R})SO_2(\mathbb{R}) \), and the fact that, as an inverse image of a commutative group, \( SO_2(\mathbb{R}) \) is commutative; see Section 2.3.
All the characters of $SO_2(\mathbb{R})$ are given by

$$\vartheta_n(k(t), \epsilon) = e^{in\phi^{-1}(k(t), \epsilon)},$$

where $2n \in \mathbb{Z}$ and $\phi : \mathbb{R}/4\pi \mathbb{Z} \to SO_2(\mathbb{R})$ is the isomorphism from Lemma 2.3. Denote by $I(\chi \otimes \gamma_{\psi_a}^{-1}, s)_n$ the subspace of $I(\chi \otimes \gamma_{\psi_a}^{-1}, s)$ on which $SO_2(\mathbb{R})$ acts by $\vartheta_n$. From the Iwasawa decomposition it follows that

$$\dim I(\chi \otimes \gamma_{\psi_a}^{-1}, s)_n \leq 1.$$  

Furthermore, since $B(\mathbb{R}) \cap SO_2(\mathbb{R})$ is a cyclic group of order 4 which is generated by $(-I_2, 1)$, it follows that $I(\chi \otimes \gamma_{\psi_a}^{-1}, s)_n \neq \{0\}$ if and only if $e^{in\phi^{-1}(-I_2, 1)} = \chi(-1)\gamma_{\psi_a}^{-1}(-1)$. Recall that

$$\gamma_{\psi_a}(y) = \begin{cases} 1 & y > 0 \\ -\text{sign}(a)i & y < 0 \end{cases}.$$  

Therefore, the last condition is equivalent to $e^{in\pi} = i\chi(-1)\text{sign}(a)$. Thus, we proved:

**Lemma 8.13.** $\dim I(\chi \otimes \gamma_{\psi_a}^{-1}, s)_n = 1$ if and only if $n \in \frac{\chi(-1)\text{sign}(a)}{2} + 2\mathbb{Z}$.

Let $n$ be as in Lemma 8.13. Define $f_{s,\chi,a,n}$ to be the unique element of $I(\chi(s) \otimes \gamma_{\psi_a}^{-1})_n$ which satisfies $f_{s,\chi,a,n}(I_2, 1) = 1$. Since $A(s)I(\chi(s) \otimes \gamma_{\psi_a}^{-1})_n \subset I(\eta(\chi, -s) \otimes \gamma_{\psi_a}^{-1})_n$, it follows that

$$\frac{A_s(f_{s,\chi,a,n})}{A_s(f_{s,\chi,a,n})(I_2, 1)} = f_{-s,\chi,a,n}.$$  

Therefore:

$$C_{\psi}(\chi \otimes \gamma_{\psi_a}^{-1}, s) = \frac{\lambda(\psi, \chi(s) \otimes \gamma_{\psi_a}^{-1})(f_{s,\chi,a,n})}{\lambda(\psi, \eta(\chi, -s) \otimes \gamma_{\psi_a}^{-1})(f_{-s,\chi,a,n})} (A_s(f_{s,\chi,a,n})(I_2, 1))^{-1}. \quad (8.51)$$

We have:

$$A_s(f_{s,\chi,a,n})(I_2, 1) = \int_{\mathbb{R}} f_{s,\chi,a,n} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \, dx = \int_{\mathbb{R}} f_{s,\chi,a,n}(u_x, k_x) \, dx,$$

where

$$u_x = \left( \begin{array}{c} \frac{1}{\sqrt{1+x^2}} \\ \frac{-x}{\sqrt{1+x^2}} \end{array} \right), \quad k_x = \left( \begin{array}{c} \frac{-x}{\sqrt{1+x^2}} \\ \frac{1}{\sqrt{1+x^2}} \end{array} \right).$$

Note that $c(u_x, k_x) = 1$, and that since $\frac{1}{\sqrt{1+x^2}} > 0$, $\phi^{-1}(k_x, 1) = t$, where $0 < t < \pi$ is the unique number that satisfies $e^{it} = \frac{i-x}{\sqrt{1+x^2}}$. Therefore,

$$A_s(f_{s,\chi,a,n})(I_2, 1) = \int_{\mathbb{R}} \left( 1 + x^2 \right)^{-\frac{n}{2}} \left( \frac{i-x}{\sqrt{1+x^2}} \right)^n \, dx = e^{\frac{\pi n}{2}} \int_{\mathbb{R}} \left( 1 + ix \right)^{-\frac{n+1}{2}} \left( 1 - ix \right)^{-\frac{n-1}{2}} \, dx.$$  

By Lemma 53 of [66] we conclude:

$$A_s(f_{s,\chi,a,n})(I_2, 1) = e^{\frac{\pi n}{2}} \frac{\pi 2^{1-s}}{\Gamma \left( \frac{s+1+n}{2} \right) \Gamma \left( \frac{s+1-n}{2} \right)} \Gamma(s). \quad (8.52)$$
For a similar computation, see [13]. In the rest of the proof we assume that $b > 0$. We compute $C_{\psi_b}(\chi \otimes \gamma_{\psi_a}^{-1}, s)$:

$$\lambda(s, \chi, \psi_b)(f_{s, \chi, a, n}) = \int_{\mathbb{R}} f_{s, \chi, a, n} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \psi_b^{-1}(x) dx$$

$$= \int_{\mathbb{R}} \left( 1 + x^2 \right)^{-\frac{s+n}{2}} \left( \frac{i-x}{1+x^2} \right)^n e^{-ibx} dx$$

$$= e^{i\pi n} \int_{\mathbb{R}} \|x + i\|^{-n-(s+1)}(x + i)^{-n} e^{-ibx} dx$$

$$= e^{i\pi n} b^s \int_{\mathbb{R}} \|x + bi\|^{-n-(s+1)}(x + bi)^{-n} e^{-ix} dx.$$

The rest of the computation goes word for word as the computations in pages 283-285 of [24]:

$$\lambda(s, \chi, \psi_b)(f_{s, \chi, a, n}) = 2^{1+s} e^{i\pi n} \Gamma^{-1}(n + \frac{s+1+n}{2}) W_{\frac{s}{2}, -\frac{s}{2}}(2b),$$

where

$$W_{k,r}(u) = e^{-\frac{1}{2}u} u^k \frac{1}{\Gamma(\frac{1}{2} + r - k)} \int_{0}^{\infty} v^{r-\frac{1}{2}-k} (1 + \frac{v}{u})^{r-\frac{1}{2}-k} e^{-v} dv$$

is the Whittaker function defined in Chapter 16 of [70]. This function satisfies

$$W_{k,r} = W_{-k,-r}.$$

It follows that

$$\frac{\lambda(s, \chi, \psi_b)(f_{s, \chi, a, n})}{\lambda(-s, \chi, \psi_b)(f_{-s, \chi, a, n})} = 2^{-s} \Gamma(\frac{s+1+n}{2})^{-1} \Gamma(\frac{s+1-n}{2})^{-1}.$$ (8.53)

Plugging (8.53) and (8.52) into (8.51) and using Lemma 8.13 we conclude that

$$C_{\psi_b}(\chi \otimes \gamma_{\psi_a}^{-1}, s) = \frac{e^{i\pi n}}{2\pi} \frac{\Gamma(-s+1+n)}{\Gamma(\frac{s+1+n}{2})} \frac{\Gamma(s+1-n)}{\Gamma(\frac{s+1-n}{2})},$$ (8.54)

where $n \in \frac{\lambda(-1) \text{sign}(a)}{2} + 2\mathbb{Z}$. We now compute $C_{\psi_{-b}}(\chi \otimes \gamma_{\psi_a}^{-1}, s)$ and finish the prove of Lemma 8.12.

$$\lambda(s, \chi, \psi_{-b})(f_{s, \chi, a, n}) = e^{i\pi n} b^s \int_{\mathbb{R}} \|x + bi\|^{-n-(s+1)}(x + bi)^{-n} e^{-ix} dx$$

$$= b^s \int_{\mathbb{R}} \|x + bi\|^{-n-(s+1)}(x - bi)^{-n} e^{-ix} dx.$$

We note that $(x - bi)^{-n} = \|x + bi\|^{-2n}(x + bi)^n$. Thus, repeating the previous computation we get:

$$\frac{\lambda(s, \chi, \psi_{-b})(f_{s, \chi, a, n})}{\lambda(-s, \chi, \psi_{-b})(f_{-s, \chi, a, n})} = 2^{-s} \Gamma(\frac{s+1-n}{2})^{-1} \Gamma(\frac{s+1-n}{2})^{-1},$$

which implies

$$C_{\psi_{-b}}(\chi \otimes \gamma_{\psi_a}^{-1}, s) = \frac{e^{-i\pi n}}{2\pi} \frac{\Gamma(-s+1+n)}{\Gamma(\frac{s+1+n}{2})} \frac{\Gamma(s+1-n)}{\Gamma(\frac{s+1-n}{2})},$$ (8.55)
where \( n \in \frac{\chi(-1)\text{sign}(a)}{2} + 2\mathbb{Z} \).

**Remark:** When one computes
\[
C_{\psi}^{SL_2(\mathbb{R})}(B_{SL_2(\mathbb{R})}, s, \chi, (0 \ 1 - 1 0))
\]
one obtains the same expressions on the right-hand sides of (8.54) and (8.55), only that one should assign 
\( n \in 2\mathbb{Z} \) if \( \chi = \chi_{0,0} \) and \( n \in 2\mathbb{Z} + 1 \) if \( \chi = \chi_{0,1} \).

### 8.3 Complex case

Since \( SL_2(\mathbb{C}) = SL_2(\mathbb{C}) \times \{\pm 1\} \) and since \( \gamma_\psi(\mathbb{C}^*) = 1 \) it follows that
\[
C_{\psi}^{SL_2(\mathbb{C})}(B_{SL_2(\mathbb{C})}, s, \chi, (0 \ 0 1 - 1)) = C_{\psi}^{SL_2(\mathbb{C})}(B_{SL_2(\mathbb{C})}, s, \chi, (0 \ 1 0 1)).
\]

Theorem 3.13 of [51] states that
\[
C_{\psi}^{SL_2(\mathbb{C})}(B_{SL_2(\mathbb{C})}, s, \chi, (0 \ 1 - 1 0)) = c'_s L(\chi^{-1}, 1 - s) L(\chi, s),
\]
where \( c'_s \) is an exponential factor. Recall that any character of \( \mathbb{C}^* \) has the form
\[
\chi(re^{i\theta}) = \chi_n(s_0)(re^{i\theta}) = r^{s_0}e^{in\theta},
\]
for some \( s_0 \in \mathbb{C}, n \in \mathbb{Z} \). We may assume that \( s_0 = 0 \) or equivalently that \( \chi = \chi_{n,0} \). The corresponding local L-function is defined by
\[
L_\mathbb{C}(\chi_{n,0}, s) = (2\pi)^{-s} |n|^{1/2} \Gamma(s + \frac{|n|}{2}).
\]

**Lemma 8.14.** There exists an exponential factor \( c(s) \) such that
\[
C_{\psi}^{SL_2(\mathbb{C})}(B_{SL_2(\mathbb{C})}, s, \chi, (0 \ 1 - 1 0)) = c(s) \frac{\gamma(\chi^2, 2s, \psi)}{\gamma(\chi, s + \frac{1}{2}, \psi)}.
\]

**Proof.** Due to (8.56) and (8.57) we only have to show that
\[
\frac{\Gamma(1 + \frac{|n|}{2} - s)}{\Gamma(\frac{|n|}{2} + s)} = 2 \frac{\Gamma(1 + n - 2s)\Gamma(\frac{1}{2} + \frac{|n|}{2} + s)}{\Gamma(n + 2s)\Gamma(\frac{1}{2} + \frac{|n|}{2} - s)}
\]

This fact follows from the classical duplication formula
\[
\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).
\]
8.4 A remark about the chosen parameterization

From Theorem 8.1, Lemma 8.3 and Lemma 8.12 it follows that if $\chi^2$ is unramified

$$C_\psi(\chi \otimes \gamma_\psi^{-1}, s)$$

and

$$C_{\psi^2}(\chi \otimes \gamma_\psi^{-1}, s)$$

might not have the same of zeros and poles. This phenomenon has no analog in connected reductive algebraic groups over a local field. Many authors consider $\psi^{-1}$-Whittaker functionals rather than $\psi$-Whittaker functionals, see [7] for example. It follows from Theorem 8.1, Lemma 8.3 and Lemma 8.12 that $C_{\psi^{-1}}(\chi \otimes \gamma_\psi^{-1}, s)$ will contain twists of $\chi$ by the quadratic character $x \mapsto (x, -1)_F$ whose properties depend on $F$. The situation is completely clear in the real case. The case of $p$-adic fields of odd residual characteristic is also easy since a character of $\mathbb{F}_p^*$, $x \mapsto (x, -1)_F$ is trivial if $-1 \in \mathbb{F}_p^2$, otherwise it is non-trivial but unramified. Thus, Theorem 8.1 and Lemma 8.3 give a formula for $C_{\psi^{-1}}(\chi \otimes \gamma_\psi^{-1}, s)$.

However, in the case of $p$-adic fields of even residual characteristic, the properties of $x \mapsto (x, -1)_F$ depend so heavily on the field, that it is not possible to give a general formula for $C_{\psi^{-1}}(\chi \otimes \gamma_\psi^{-1}, s)$. If $\chi^2$ is ramified it follows from Theorem 8.1 and from Lemma 8.3 that $C_{\psi^{-1}}(\chi \otimes \gamma_\psi^{-1}, s)$ is a monomial in $q^{-s}$. But, if $\chi^2$ is unramified, things are much more complicated. For example, assume that $\psi$ is normalized and that $\chi$ is the trivial character of $\mathbb{F}_p^*$. By the same techniques we used in Section 8.1 the following can be proved: If $e$ is odd then

$$C_{\psi^{-1}}(\chi \otimes \gamma_\psi^{-1}, s) = c_\psi(-1)q^{(1-e)s}L(\chi^{-1}, -2s + 1)\frac{L(\chi, 2s)}{L(\chi, 2s)}.$$

If $e$ is even, we have:

$$C_{\psi^{-1}}(\chi \otimes \gamma_\psi^{-1}, s) = \frac{q - 1}{q} \left(1 - \frac{q^{-2s}}{q}\right)^{-1} + \sum_{1 \leq k \leq \frac{q}{2}} q^{2ks} + P_{F}(s), \quad (8.59)$$

where $P_{F}(s)$ is defined to be

$$\sum_{\frac{q}{2} < k \leq e} q^{2ks} \left(\left(c_1(F, 2k)(-q^{-1}) + c_2(F, 2k)(1 - q^{-1})\right)q^k\right) +$$

$$q^{s(2e+1)} \left(\left(c_1(F, 2e + 1)(-q^{-1}) + c_2(F, 2e + 1)(1 - q^{-1})\right)q^{e+\frac{1}{2}}\right),$$

where

$$c_1(F, n) = \mu(\{x \in O_F^n | \|x^2 + 1\| = q^{1-n}\}), \quad (8.60)$$

$$c_2(F, n) = \mu(\{x \in O_F^n | \|x^2 + 1\| \leq q^{-n}\}). \quad (8.61)$$

For the 6 ramified quadratic extensions of $\mathbb{Q}_2$, (8.59) gives the following: If $F$ is either $\mathbb{Q}_2(\sqrt{2})$, $\mathbb{Q}_2(\sqrt{-2})$, $\mathbb{Q}_2(\sqrt{6})$, $\mathbb{Q}_2(\sqrt{-6})$ we have:

$$C_{\psi^{-1}}(\chi \otimes \gamma_\psi^{-1}, s) = c_\psi(-1)q^{-s}L(\chi^{-1}, -2s + 1)\frac{L(\chi, 2s)}{L(\chi, 2s)},$$

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where $m \geq m(\chi)$. We have seen that for $k \geq \max(m(\chi), 2e + 1)$

$$C_\psi(\chi \otimes \gamma_\psi^{-1}, s)^{-1} = \int_{0<|u|\leq q^k} \gamma_\psi^{-1}(u) \chi(s)(u) \psi(u) d^*u$$  \hfill (8.64)
whenever the integral on the right-hand side converges, that is, for \( Re(s) > 0 \).

The similarity between the integrals in the right-hand sides of (8.63) and (8.64) and the fact that
\[
C_{SL^2(F)}(B_{SL^2(F)}, s, \chi, (0, 1, 0))^{-1} = \gamma(\chi^{-1}, \psi^{-1}, 1 - s)
\] (8.65)
raise the question whether one can define a meromorphic function \( \tilde{\gamma}(\chi, \psi, s) \) by a similar method to the one used by Tate, replacing to role of the Fourier transform with \( \phi \mapsto \tilde{\phi} \) defined on \( S(F) \) by
\[
\tilde{\phi}(x) = \int_F \phi(y)\psi(xy)\gamma^{-1}(xy) dy
\]
(we define \( \gamma^{-1}(0) = 0 \)). We shall show that such a definition is possible and that there exists a metaplectic analog to (8.65), namely that
\[
C_{SL^2(F)}(B_{SL^2(F)}, s, \chi, (0, 1, 0))^{-1} = \tilde{\gamma}(\chi^{-1}, \psi^{-1}, 1 - s).
\] (8.66)

Lemma 8.15. For any \( \phi \in S(F) \)
\[
\int_{F^*} \tilde{\phi}(x)\chi(x) d^* x
\] converges absolutely for \( 0 < Re(s) < 1 \). This integral extends to a rational function in \( q^{-s} \).

Proof. For \( A \subseteq F \), denote by \( 1_A \) the characteristic function of \( A \). Regarding the convergence of
\[
\int_{F^*} \tilde{\phi}(x)\chi(x) d^* x,
\]
we may assume that \( \phi = 1_{\mathbf{F}^*} \) or that \( \phi = 1_{a + \mathbf{F}^*} \), where \( \|a\| \geq q^{2e+1-n} \) since these functions span \( S(F) \).

Fix \( n \in \mathbb{Z} \), \( a \in F^* \) such that \( \|a\| \geq q^{2e+1-n} \).

\[
\tilde{1}_{a + \mathbf{F}^*}(y) = \int_{a + \mathbf{F}^*} \psi(xy)\gamma^{-1}(xy) dx = \psi(ay) \int_{\mathbf{F}^*} \psi(xy)\gamma^{-1}((a + x)y) dx
\]
\[
= |y|^{-1}\psi(ay) \int_{|x| \leq |a|q^{-n}} \psi(x)\gamma^{-1}(ay(1 + xy^{-1}a^{-1})) dx.
\]

Note that for \( x, y \) and \( a \) in the last integral we have
\[
|xy^{-1}a^{-1}| \leq \|q^{-n}a^{-1}\| \leq q^{-2e-1}.
\]

Therefore,
\[
\tilde{1}_{a + \mathbf{F}^*}(y) = |y|^{-1}\psi(ay)\gamma^{-1}(ay) \int_{|x| \leq |a|q^{-n}} \psi(x) dx.
\]

This implies that for \( y \in F^* \)
\[
\tilde{1}_{a + \mathbf{F}^*}(y) = \begin{cases} 0 & |y| > q^n \\psi(ay)\gamma^{-1}(ay)q^{-n} & |y| \leq q^n \end{cases}.
\] (8.67)
Thus,
\[
\int_{\mathbb{F}^*} |1_{a+c} \chi(x)(x)||d^*x = q^{-n} \int_{0 < |x| \leq q^n} |x|^{Re(s)} d^*x = q^{-n} \sum_{k=-n}^{\infty} q^{-k} Re(s).
\]
This shows that
\[
\int_{\mathbb{F}^*} \widetilde{1_{a+c}}(x)(x)|d^*x
\]
converges absolutely for $Re(s) > 0$. Furthermore, for $Re(s) > 0$,
\[
\int_{\mathbb{F}^*} \widetilde{1_{a+c}}(x)(x)|d^*x = q^{-n} \int_{0 < |x| \leq q^n} \psi(ax) \gamma^{-1}_\psi(ax)(x) d^*x
\]
\[
= \psi(x) \gamma^{-1}_\psi(x) \chi(x)(x) d^*x.
\]
It was shown in Section 8.1.3 that the last integral is a rational function in $q^{-s}$.

We now move to $\phi = 1_{\mathbb{P}_\phi}$. By an argument that we used already, for $y \in \mathbb{F}^*$ we have
\[
\widetilde{1_{\mathbb{P}_\phi}}(y) = \|y\|^{-1} = \int_{|x| \leq |y| q^{-n}} \psi(x) \gamma^{-1}_\psi(x) dx.
\]
Put $\|y\| = q^m$. Assume first that $m \leq n$:
\[
\widetilde{1_{\mathbb{P}_\phi}}(y) = q^{-m} \int_{|x| \leq q^m q^{-n}} \gamma^{-1}_\psi(x) dx = q^{-m} \sum_{k=-n}^{\infty} q^{-k} \int_{\mathbb{O}_\phi} \gamma^{-1}_\psi(uq^k) du.
\]
It was shown in lemma 8.7 that
\[
\int_{\mathbb{O}_\phi} \gamma^{-1}_\psi(uq^k) du = \begin{cases} 0 & k \in \mathbb{N}_{\text{odd}} \\ c^{-1}_\psi(-1)(1-q^{-1}) & k \in \mathbb{N}_{\text{even}} \end{cases}
\]
This implies
\[
\widetilde{1_{\mathbb{P}_\phi}}(y) = q^{-m} c^{-1}_\psi(-1)(1-q^{-1}) \sum_{k \in \mathbb{N}_{\text{even}}, k \geq n-m} q^{-k} = \frac{1}{c\psi(-1)(1+q^{-1})q^m} \begin{cases} 1 & n-m \in \mathbb{N}_{\text{even}} \\ q^{-1} & n-m \in \mathbb{N}_{\text{odd}} \end{cases}.
\]
If $m > n$ then
\[
\widetilde{1_{\mathbb{P}_\phi}}(y) = q^{-m} \int_{|x| \leq q^m q^{-n}} \psi(x) \gamma^{-1}_\psi(x) dx.
\]
Put
\[
c_m = \int_{|x| \leq q^m q^{-n}} \psi(x) \gamma^{-1}_\psi(x) dx.
\]
with this notation, if $\|y\| = q^m > q^n$ then
\[
\widetilde{1_{\mathbb{P}_\phi}}(y) = c_m \|y\|^{-1}.
\]
It was shown in Lemma 8.8 that $c_m$ stabilizes, more accurately, that there exists $c \in \mathbb{C}$ such that $c_m = c$ for all $m > n + 2e + 1$.  

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Since \( \widetilde{1_{P_p}}(x) \) is bounded, it now follows that there exists two positive constants \( c_1 \) and \( c_2 \) such that

\[
\int_{\mathbb{R}^*} |\widetilde{1_{P_p}}(x)\chi(s)(x)| d^s x \leq c_1 \int_{0<|x|\leq q^n} \|x\|Re(s) d^s x + c_2 \int_{|x|>q^n} \|x\|Re(s)-1 d^s x = c_1 \sum_{k=-n}^{\infty} q^{-nRe(s)} + c_2 \sum_{k=n}^{\infty} q^n(Re(s)-1).
\]

This shows that

\[
\int_{\mathbb{R}^*} \widetilde{1_{P_p}}(x)\chi(s)(x) d^s x
\]

converges absolutely for \( 1 > Re(s) > 0 \). From (8.68) and (8.69) it follows that there exist complex constants such that for \( 1 > Re(s) > 0 \) we have

\[
\int_{\mathbb{R}^*} \widetilde{1_{P_p}}(x)\chi(s)(x) d^s x = k_1 \sum_{k \leq m, k \in Z_{even}} \int_{|x|=q^k} \chi(s)(x) d^s x + k_2 \sum_{k \leq m, k \in Z_{odd}} \int_{|x|=q^k} \chi(s)(x) d^s x + \sum_{n+2e+1}^{\infty} c_k \int_{|x|=q^k} \chi(s-1)(x) d^s x + c \sum_{k>n+2e+1} \int_{|x|=q^k} \chi(s-1)(x) d^s x.
\]

It is clear that the right hand side of the last equation is rational in \( q^{-s} \).

From the lemma just proven it follows that it makes sense to consider \( \zeta(\tilde{\phi}, \chi, s) \), although \( \tilde{\phi} \) does not necessarily lie in \( S(\mathbb{F}) \).

**Theorem 8.3.** There exists a meromorphic function \( \tilde{\gamma}(\chi, \psi, s) \) such that

\[
\tilde{\gamma}(\chi, \psi^{-1}, s)\zeta(\phi, \chi, s) = \zeta(\tilde{\phi}, \chi^{-1}, 1-s)
\]

(8.70)

For every \( \phi \in S(\mathbb{F}) \). In fact, it is a rational function in \( q^s \). Furthermore, (8.66) holds.

**Proof.** It is sufficient to prove (8.70) for \( 0 < Re(s) < 1 \). By lemma 8.15 in this domain both \( \zeta(\phi, \chi, s) \) and \( \zeta(\tilde{\phi}, \chi^{-1}, 1-s) \) are given by absolutely convergent integrals. The proof goes word for word as the well known proof of (8.62); see Theorem 7.2 of [41] for example, replacing the role of the Fourier transform with \( \phi \mapsto \tilde{\phi} \). We now prove (8.66). Due to Lemmas 8.1 and 8.8 it is sufficient to prove that

\[
\tilde{\gamma}(\chi^{-1}, \psi^{-1}, 1-s) = \int_{0<|x|\leq q^{max(m(\chi), 2e+1)}} \gamma_{\psi^{-1}}(u)\chi(s)(u)\psi(u) d^s u.
\]

(8.71)

To prove (8.71) we use \( \phi = 1_{1+P_p} \), where \( k \geq max(m(\chi), 2e+1) \). An easy computation shows that

\[
\zeta(1_{1+P_p}, \chi^{-1}, 1-s) = q^{-k}.
\]

Thus, from (8.70) it follows that

\[
\tilde{\gamma}(\chi^{-1}, \psi^{-1}, 1-s) = q^k \zeta(1_{1+P_p}, \chi, s).
\]

(8.71) follows now from (8.67).
Many properties of the Tate $\gamma$-factor follow from the properties of the Fourier transform. What properties of $\overline{\gamma}$ follow from the properties of the transform $\phi \mapsto \overline{\phi}$ is left for a future research.

### 8.6 Computation of $C_{\psi}^{Sp_{2m}(F)}(P_{m,0}(F), s, \tau, \omega_m^{-1})$ for principal series representations

In this subsection we assume that $F$ is either $\mathbb{R}$, $\mathbb{C}$ or a $p$-adic field. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be $m$ characters of $F^*$ and let $\mu$ be the character of $T_{GL_m}(F)$ defined by

$$Diag(t_1, t_2, \ldots, t_m) \mapsto \prod_{i=1}^m \alpha_i(t_i).$$

We also regard $\mu$ as a character of $B_{GL_m}(F)$. Define $\tau$ to be the corresponding principal series representation:

$$\tau = I(\mu) = Ind_{B_{GL_m}(F)}^{GL_m(F)} \mu.$$

**Lemma 8.16.** There exists $d \in \{\pm 1\}$ such that

$$C_{\psi}^{Sp_{2m}(F)}(P_{m,0}(F), s, \tau, \omega_m^{-1}) = d \prod_{i=1}^m C_{\psi}^{SL_2(F)}(B_{SL_2(F)}, s, \alpha_i, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \prod_{i=1}^{m-1} C_{\psi}^{GL_{m+1-i}(F)}(P_{1,m-i}(F), (s, -s), \alpha_i \otimes \overline{\tau}_i, \omega^{-1}_{1,m-i})$$

where for $1 \leq i \leq m-2$, $\tau_i = Ind_{B_{GL_{m-i}(F)}}^{GL_{m-i}(F)} \otimes_{j=i+1}^m \alpha_j$ and $\tau_{m-1} = \alpha_m$. Furthermore, $d = 1$ if $F$ is a $p$-adic field of odd residual characteristic and $\tau$ is unramified.

**Proof.** We prove this lemma by induction. For $m = 1$ there is nothing to prove. Suppose now that the theorem is true for $m - 1$. With our enumeration this means that there exists $d' \in \{\pm 1\}$ such that

$$C_{\psi}^{Sp_{2(m-1)}(F)}(P_{m-1,0}(F), s, \tau_1, \omega_{m-1}) = d' \prod_{i=2}^m C_{\psi}^{SL_2(F)}(B_{SL_2(F)}, s, \alpha_i, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \prod_{i=2}^{m-1} C_{\psi}^{GL_{m-i}(F)}(P_{1,m-i}(F), (s, -s), \alpha_i \otimes \overline{\tau}_i, \omega^{-1}_{1,m-i})$$

and that $d' = 1$ if $F$ is a $p$-adic field of odd residual characteristic and $\tau$ is unramified. The proof is done now once we observe that since $\tau \simeq Ind_{P_{1,m-1}}^{GL_m(F)} \alpha_1 \otimes \tau_1$ it follows from Lemmas 7.1, 7.2 and 7.3 that

$$C_{\psi}^{Sp_{2m}(F)}(P_{m,0}(F), s, \tau, \omega_m^{-1}) = d'' C_{\psi}^{SL_2(F)}(B_{SL_2(F)}, s, \alpha_1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$$

$$C_{\psi}^{Sp_{2(m-1)}(F)}(P_{m-1,0}(F), s, \tau_1, \omega_{m-1}) C_{\psi}^{GL_{m}(F)}(P_{1,m-1}(F), (s, -s), \alpha_1 \otimes \overline{\tau}_1, \omega^{-1}_{1,m-1}).$$

for some $d'' \in \{\pm 1\}$. If $F$ is a $p$-adic field of odd residual characteristic and $\tau$ is unramified then $d'' = 1$. \qed
Theorem 8.4. There exists an exponential function $c(s)$ such that

$$C_{Sp_m(F)}^\psi(P_{m,0(F)}, s, \tau, \omega_{m-1}^{-1}) = c(s) \frac{\gamma(\tau, \text{sym}^2, 2s, \psi)}{\gamma(\tau, s + \frac{1}{2}, \psi)} \quad (8.73)$$

If $F$ is a $p$-adic field of odd residual characteristic, $\psi$ is normalized and $\tau$ is unramified then $c(s) = 1$.

In Section 9.3 we shall show that (8.73) holds for every irreducible admissible generic representation $\tau$ of $GL_m(F)$; see Theorem 9.3.

Proof. We keep the notations of Lemma 8.16. From (7.1) and from the known properties of $\gamma(\tau, s, \psi)$ (see [50] or [26]) it follows that for every $1 \leq i \leq m - 1$ there exists $d_i \in \{\pm 1\}$ such that

$$C_{GL_{m+1-i}(F)}^\psi(P_{1,m-i}(F), (s, -s), \alpha_i \otimes \tau_i, \omega_{m-1}^{-1}) = d_i \prod_{j=i+1}^m \gamma(\alpha_i \alpha_j, 2s, \psi). \quad (8.74)$$

and that $d_i = 1$ provided that $F$ is a $p$-adic field and $\tau$ is unramified. From (8.4) it follows that

$$\prod_{i=1}^m C_{SL_2(F)}^\psi(B_{SL_2(F)}, s, \alpha_i, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = c'(s) \prod_{i=1}^m \frac{\gamma(\alpha_i^2, 2s, \psi)}{\gamma(\alpha_i, s + \frac{1}{2}, \psi)}, \quad (8.75)$$

where $c'(s)$ is an exponential function that equals 1 if $F$ is a $p$-adic field of odd residual characteristic, $\psi$ is normalized and $\tau$ is unramified. Plugging (8.74) and (8.75) into (8.73) we get

$$C_{Sp_{2m}(F)}^\psi(P_{m,0(F)}, s, \tau, \omega_{m-1}^{-1}) = c(s) \prod_{i=1}^m \left( \frac{\gamma(\alpha_i^2, 2s, \psi)}{\gamma(\alpha_i, s + \frac{1}{2}, \psi)} \prod_{j=i+1}^m \gamma(\alpha_i \alpha_j, 2s, \psi) \right).$$

By definition, (8.73) now follows. \qed
9 An analysis of Whittaker coefficients of an Eisenstein series

In this chapter we prove a global functional equation satisfied by \( \gamma(\sigma \times \tau, s, \psi) \); see Theorem 9.1 of Section 9.2. It is a metaplectic analog to Theorem 4.1 of [44] (see also Part 4 of Theorem 3.15 of [53]). The argument presented here depends largely on the theory of Eisenstein series developed by Langlands, [35], for reductive groups. Moeglin and Waldspurger extended this theory to covering groups; see [37]. This chapter is organized as follows. In Section 9.1 we introduce some unramified computations which follow from [7] and [10]. These computations will be used in Section 9.2 where we prove a global functional equation by analyzing the \( \psi \)-Whittaker coefficient of a certain Eisenstein series. As a consequence of a particular case of the global functional equation we show in Theorem 9.3 of Section 9.3 that if \( F \) is a p-adic field and \( \tau \) is an irreducible admissible generic representation of \( GL_m(F) \) then there exists an exponential function \( c(s) \) such that

\[
\mathcal{C}^{SP_{2m}(F)}_{\psi}(\mathbf{P}_{m,0}(F), s, \tau, \omega^{-1}_m) = c(s) \frac{\gamma(\tau, sym^2, 2s, \psi)}{\gamma(\tau, s + \frac{1}{2}, \psi)}.
\]

9.1 Unramified computations

We keep all the notations we used in Lemma 7.4 but we add the following restrictions; we assume that \( F \) is a p-adic field of odd residual characteristic, that \( \psi \) is normalized and that \( \chi \) and \( \mu \) are unramified. We define the local unramified \( L \)-function of \( \sigma \times \tau \) with respect to \( \psi \):

\[
L_{\psi}(\sigma \times \tau, s) = \prod_{1 \leq i \leq k} \prod_{1 < j \leq m} L(\eta_i \alpha_j, s)L(\eta^{-1}_i \alpha_j, s).
\]  

(9.1)

The subscript \( \psi \) is in place due to the dependence on \( \gamma^{-1}_\psi \) in the definition of \( \sigma \).

Similar to the algebraic case, \( I(\chi(s)) \) has a one dimensional \( \kappa_{2k}(Sp_{2k}(O_F)) \) invariant subspace. Let \( f^0_{\chi(s)} \) be the normalized spherical vector of \( I(\chi(s)) \), i.e., the unique \( \kappa_{2k}(SP_{2k}(O_F)) \) invariant vector with the property \( f^0_{\chi(s)}(I_{2k}, 1) = 1 \). For \( f \in I(\chi(s)) \) the corresponding Whittaker function is defined by

\[
W_f(g) = \frac{1}{C_{\chi(s)}} \int_{Z_{Sp_{2k}(F)}} f((J_{2k} u, 1)g) \psi^{-1}(u) du,
\]

where

\[
C_{\chi(s)} = \prod_{i=1}^{k} (1 + \eta_i(\pi)q^{-s+\frac{1}{2}}) \prod_{1 < i < j \leq k} \left((1 - q^{-1} \eta_i(\pi) \eta_j(\pi)^{-1})(1 - q^{-1} \eta_i(\pi) \eta_j(\pi) q^{-2s})\right).
\]

With the normalization above, Theorem 1.2 of [7] states that \( W_{f^0_{\chi(s)}} = W^0_{\chi(s)} \), where \( W^0_{\chi(s)} \) is the normalized spherical function in \( W(I(\chi(s)), \psi) \). To be exact we note that in [7], the \( \psi^{-1} \)-Whittaker functional is computed. This difference manifests itself only in the \( SL_2(F) \) computation presented in page 387 of [7]. Consequently the left product defining \( C_{\chi(s)} \) presented in [7] differs slightly from the one given here. It is the same difference discussed in Lemma 8.3 and Section 8.4.
Let $f^0_\mu$ be the normalized spherical vector of $I(\mu)$. Define
\[ W_f(g) = \frac{1}{D_\mu} \int_{Z_{GL_m}(\mathbb{F})} f(\omega_m u g) \psi^{-1}(u) \, du, \]
where
\[ D_\mu = \prod_{1 \leq i < j \leq m} (1 - q^{-1} \alpha_i \alpha_j^{-1}). \]

Denote by $W^0_\mu$ the normalized spherical function of $W(I(\mu), \psi)$. Theorem 5.4 of \cite{10} states that $W^0_\mu = W^0_{\mu'}$. Let $\lambda_{r, \psi}$ and $\lambda_{\sigma, \psi}$ be Whittaker functionals on $I(\mu)$ and $I(\chi)$ respectively. Note that
\[ \lambda_{r, \psi}(\tau(g) f) = W_f(g), \quad \lambda_{\sigma, \psi}(\sigma(s) f) = W_f(s). \tag{9.2} \]

Similar to Chapter \[7.1\] we realize
\[ I_1 = Ind_{P_{m,k}(\mathbb{F})}^{Sp_{2n}(\mathbb{F})} (\gamma^{-1}_\psi \otimes \tau(s)) \otimes \sigma \]
as a space of complex functions on $Sp_{2n}(\mathbb{F}) \times GL_m(\mathbb{F}) \times Sp_{2k}(\mathbb{F})$ which are smooth from the right in each argument and which satisfy
\[ f((j_{m,n}(g), 1) i_{k,n}(y) h, b g_0, (b', \epsilon) y_0) = e^{\gamma_\psi^{-1}(\det(g) \det(b'))} |\det(g)|^{s+n+k+1} \delta_{GL_m(\mathbb{F})}(b) \delta_{Sp_{2n}(\mathbb{F})}(b') \mu(b) \chi(b) f(h, g_0 y, y_0 y). \]

For all $g, g_0 \in GL_m(\mathbb{F})$, $y, y_0 \in Sp_{2k}(\mathbb{F})$, $n \in (N_{m,k}, 1)$, $h \in Sp_{2n}(\mathbb{F})$, $b \in B_{GL_m(\mathbb{F})}$, $(b', \epsilon) \in B_{Sp_{2k}(\mathbb{F})}$. We realize
\[ I'_1 = Ind_{P_{m,k}(\mathbb{F})}^{Sp_{2n}(\mathbb{F})} (\gamma^{-1}_\psi \otimes W_s(\tau, \psi)) \otimes W(\sigma, \psi) \]
as we did in Lemma \[7.1\]. An isomorphism $T_1 : I_1 \to I'_1$ is given by
\[ (T_1 f)(h, g, y) = \frac{1}{C_\chi D_\mu} \int_{n_1 \in Z_{GL_m}(\mathbb{F})} \int_{n_2 \in Z_{Sp_{2k}(\mathbb{F})}} f(s, \omega_n m_1 g, (J_k n_2, 1) y) \psi^{-1}(n_1) \psi^{-1}(n_2). \]

Let $f_{I_1}^0 \in I_1$ be the unique function such that
\[ f_{I_1}^0((I_{2n}, 1), I_m, (I_{2k}, 1)) = 1 \]
and such that for all $o \in \kappa_{2n}(Sp_{2n}(\mathbb{F}))$, $g \in GL_m(\mathbb{F})$, $y \in Sp_{2k}(\mathbb{F})$ we have
\[ f_{I_1}^0(o, g, y) = f_\mu^0(g) \cdot f_{\chi}^0(y). \]

Let $f_{I'_1}^0 \in I'_1$ be the unique function such that
\[ f_{I'_1}^0((I_{2n}, 1), I_m, (I_{2k}, 1)) = 1 \]
and such that for all $o \in \kappa_{2n}(Sp_{2n}(\mathbb{F}))$, $g \in GL_m(\mathbb{F})$, $y \in Sp_{2k}(\mathbb{F})$ we have
\[ f_{I'_1}^0(o, g, y) = W_\mu^0(g) \cdot W_{\chi}^0(y). \]

According \[9.2\]: $T_1(f_{I'_1}^0) = f_{I_1}^0$. We denote by $\lambda'(s, \tau \otimes \sigma, \psi)$ the Whittaker functional on $I'_1$ constructed in the usual way.

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Lemma 9.1.

\[ \lambda'(s, \tau \otimes \overline{\sigma}, \psi) f^0_{I_1} = \frac{L(\tau, s + \frac{1}{2})}{L(\tau, \text{sym}^2, 2s + 1) L_{\psi}(\overline{\sigma} \otimes \tau, s + 1)}. \]  

(9.3)

Proof. For \( f \in I_1 \) we have:

\[ \lambda'(s, \tau \otimes \overline{\sigma}, \psi)(T_1(f)) \]

\[ = \frac{1}{C_x D_\mu} \int_{N_{Sp_{2n}(F)}} f((J_{2u}u, 1), (I_{2k}, 1)) \psi^{-1}(u) du. \]

In particular

\[ \lambda'(s, \tau \otimes \overline{\sigma}, \psi) f^0_{I_1} = \frac{1}{C_x D_\mu} \int_{N_{Sp_{2n}(F)}} f^0_{I_1}((J_{2u}u, 1), (I_{2k}, 1)) \psi^{-1}(u) du. \]

Let \( \mu(s) \otimes \chi \) be the character of \( T_{Sp_{2n}(F)} \) defined by

\[ (j_{m,n}(t)i_{k,n}(t')) \mapsto |\det(t)|^s \mu(t) \chi(t'), \]

where \( t \in T_{GL_{mn}(F)}, t' \in T_{Sp_{2k}(F)} \). We realize \( I(\mu(s) \otimes \chi) = Ind_{B_{Sp_{2n}(F)}}^{T_{Sp_{2n}(F)}} \gamma_{\psi}^{-1} \otimes \mu(s) \otimes \chi \), in the obvious way. For the Whittaker functional defined on this representation space,

\[ \lambda(s, \chi \otimes \mu)(f) = \int_{N_{Sp_{2n}(F)}} f(J_{2u}u, 1) \psi^{-1}(u) du, \]

we have

\[ \lambda(s, \mu \otimes \chi)(f^0_{I_1(\mu_s \otimes \chi)}) = C_{\mu(s) \otimes \chi}. \]  

(9.5)

The isomorphism \( T_2 : I_1 \rightarrow I(\mu(s) \otimes \chi) \) defined by

\[ (T_2 f)(h) = f(h, I_m, (I_{2k}, 1)), \]

whose inverse is given by

\[ (T_2^{-1} f)(h, g, y) = \gamma_{\psi}(\det(g)) |\det(g)|^{n+k+1} f((j_{m,n}(g)i_{k,n}(y)h), \]

has the property:

\[ T_2(f^0_{I_1}) = f^0_{I_1(\mu_s \otimes \chi)}. \]  

(9.6)

Using (9.4), (9.5) and (9.6) we observe that

\[ C_{\mu(s) \otimes \chi} = \lambda(s, \chi \otimes \mu)(f^0_{I_1(\chi_s \otimes \mu)}) = \int_{N_{Sp_{2n}(F)}} (T_2(f^0_{I_1}))(J_{2u}u, 1) \psi^{-1}(u) du = \]

\[ \int_{N_{Sp_{2n}(F)}} f^0_{I_1}((J_{2u}u, 1), (I_{2k}, 1)) \psi^{-1}(u) du = C_x D_\mu \lambda'(s, \tau \otimes \overline{\sigma}, \psi) f^0_{I_1}. \]

(9.7)

Since

\[ \frac{C_{\mu(s) \otimes \chi}}{C_x D_\mu} = \frac{L(\tau, s + \frac{1}{2})}{L(\tau, \text{sym}^2, 2s + 1) L_{\psi}(\overline{\sigma} \otimes \tau, s + 1)}, \]

the lemma is proved.
Remark: In the case $k = 0$ [9.3] reduces to
\[
\lambda'(s, \tau \otimes \sigma, \psi) f^0_{I_1} = \frac{L(\tau, s + \frac{1}{2})}{L(\tau, s^2, 2s + 1)}.
\]
This case appears in the introduction of [7].

Let $A_{j,m,n}(\omega_{m}^{-1})$, $A'_{j,m,n}(\omega_{m}^{-1})$, $A_{j,m,n}(\omega_{m}^{-1})$ be the intertwining operators defined on $I(\mu_{(s)} \otimes \chi)$, $I_1$, $I_1'$ respectively.

**Lemma 9.2.**

\[
\lambda'(-s, \tilde{\tau} \otimes \tilde{\sigma}, \tilde{\psi})(A'_{j,m,n}(\omega_{m}^{-1})(f^0_{I_1})) = \frac{L(\tilde{\tau}, s + \frac{1}{2})L(\tilde{\sigma} \otimes s, s)L(\tau, s^2, 2s)}{L(\tilde{\tau}, s, 2s) = 0 \ (9.3)} \text{ reduces to}
\]

**Proof.** Application of Lemma 3.4 of [7] to the relevant Weyl element proves that
\[
A_{j,m,n}(\omega_{m}^{-1})(f^0_{I_1}) = K_{\mu_{(s)} \otimes \chi} f^0_{I_1} f^0_{I_1},
\]
where
\[
K_{\mu_{(s)} \otimes \chi} = \frac{L(\tilde{\sigma} \otimes s)}{L(\tilde{\sigma} \otimes s, s) L(\tau, s^2, 2s)}.
\]

We define
\[
\tilde{T}_1 : \text{Ind}_{\text{P}_{m,1}^n(F)}^{\text{Sp}_{2n}(F)} \gamma^{-1} \otimes \tau_{(-s)} \otimes \tilde{\sigma} \rightarrow \text{Ind}_{\text{P}_{m,1}^n(F)}^{\text{Sp}_{2n}(F)} \gamma^{-1} \otimes W(-s)(\tau, \tilde{\psi}) \otimes W(\tilde{\sigma}, \tilde{\psi})
\]
and
\[
\tilde{T}_2 : \text{Ind}_{\text{P}_{m,1}^n(F)}^{\text{Sp}_{2n}(F)} \gamma^{-1} \otimes \tilde{\sigma} \rightarrow \text{Ind}_{\text{P}_{m,1}^n(F)}^{\text{Sp}_{2n}(F)} \gamma^{-1} \mu_{(-s)} \otimes \chi
\]
by analogy with $T_1$ and $T_2$. Note that $\tilde{T}_1$ commutes with $A_{j,m,n}$ and that $\tilde{T}_2$ commutes with $A_{j,m,n}(\omega_{m}^{-1})$. Therefore,
\[
\tilde{T}_1(A_{j,m,n}(\omega_{m}^{-1})(I_1)) \subseteq A'_{j,m,n}(\omega_{m}^{-1})(I_1')
\]
and
\[
\tilde{T}_2 : (A_{j,m,n}(\omega_{m}^{-1})(I_1)) \subseteq A_{j,m,n}(\omega_{m}^{-1})(I_{1'}) \otimes \chi.
\]

We denote by $\tilde{f}^0_{I_1}$ and $\tilde{f}^0_{I_1'}$ the spherical functions of $A_{j,m,n}(\omega_{m}^{-1})(I_1)$ and $A'_{j,m,n}(\omega_{m}^{-1})(I_1')$ respectively. Since $T_1, T_2, \tilde{T}_1, \tilde{T}_2$ map a normalized spherical function to a normalized spherical function and since a straightforward computation shows that
\[
\tilde{T}_1 \tilde{T}_2^{-1} A_{j,m,n}(\omega_{m}^{-1}) = A'_{j,m,n}(\omega_{m}^{-1}) \tilde{T}_1 \tilde{T}_2^{-1},
\]
we have
\[
A'_{j,m,n}(\omega_{m}^{-1})(f^0_{I_1'}) = \tilde{T}_1 \tilde{T}_2^{-1} A_{j,m,n}(\omega_{m}^{-1}) \tilde{T}_2 \tilde{T}_1^{-1} (f^0_{I_1'}) = K_{\mu_{(s)} \otimes \chi} f^0_{I_1'}.
\]
From this and from [9.1] we conclude that
\[
\lambda'(-s, \tilde{\tau} \otimes \tilde{\sigma}, \tilde{\psi})(A'_{j,m,n}(\omega_{m}^{-1})(f^0_{I_1})) = K_{\mu_{(s)} \otimes \chi} \lambda'(-s, \tilde{\tau} \otimes \tilde{\sigma}, \tilde{\psi})(f^0_{I_1'}) = K_{\mu_{(s)} \otimes \chi} C_\mu \otimes \chi C_\chi D_\mu.
\]
This finishes the proof of this lemma. \qed
Remark: Combining (9.3) and (9.8) we get

\[ C_{\psi}^{\mathcal{P}^{2m}(\mathbb{F})}(\mathcal{P}_{m;0}(\mathbb{F}), s, \psi, j_m, n(\omega_{m}^{-1}))) = \frac{\lambda'(s, \tau \otimes \overline{\tau}, \psi)(f_{1}^0)}{\lambda'(-s, \tau \otimes \overline{\tau}, \psi)(A'_{j_m,n(\omega_{m}^{-1})}(f_{1}^0))} \]  
(9.9)

In particular,

\[ C_{\psi}^{\mathcal{P}^{2m}(\mathbb{F})}(\mathcal{P}_{m;0}(\mathbb{F}), s, \psi, j_m, n(\omega_{m}^{-1})) = \frac{L(\tau, s + \frac{1}{2})L(\overline{\tau}, sym^2, -2s + 1)L_{\psi}(\overline{\tau} \otimes \overline{\tau}, 1 - s)}{L(\overline{\tau}, -s + \frac{1}{2})L(\tau, sym^2, 2s)L_{\psi}(\overline{\tau} \otimes \tau, s)}. \]  
(9.10)

Dividing (9.9) by (9.10) we get

\[ \gamma(\sigma \times \tau, s, \psi) = \frac{L_{\psi}(\sigma \times \sigma, 1 - s)}{L_{\psi}(\sigma \times \tau, s)}. \]  
(9.11)

Thus, the computations presented in this subsection provide an independent proof of Corollary [7,1] in the case where \( \mathbb{F} \) is a p-adic field of odd residual characteristic and \( \overline{\sigma} \) and \( \tau \) are unramified.

9.2 Crude functional equation.

Throughout this section, \( \mathbb{F} \) will denote a number field. For every place \( \nu \) of \( \mathbb{F} \), denote by \( \mathbb{F}_\nu \) the completion of \( \mathbb{F} \) at \( \nu \). Let \( \mathbb{A} \) be the adele ring of \( \mathbb{F} \). We fix a non-trivial character \( \psi \) of \( \mathbb{F} \setminus \mathbb{A} \). We write \( \psi(x) = \prod_{\nu} \psi_{\nu}(x_{\nu}) \), where for almost all finite \( \nu \), \( \psi_{\nu} \) is normalized. As in the local case, \( \psi \) will also denote a character of \( Z_{GL_{m}}(\mathbb{A}), Z_{Sp_{2m}(\mathbb{A})} \) and of their subgroups.

Let \( \tau \) and \( \overline{\sigma} \) be a pair of irreducible automorphic cuspidal representations of \( GL_{m}(\mathbb{A}) \) and \( Sp_{2m}(\mathbb{A}) \) respectively. Let \( \tau \) and \( \overline{\sigma} \) act in the spaces \( V_{\tau} \) and \( V_{\overline{\sigma}} \) respectively. We assume that \( \overline{\sigma} \) is genuine and globally \( \psi \)-generic, i.e., that

\[ \int_{n \in Z_{Sp_{2n}(\mathbb{F})} \setminus Z_{Sp_{2n}(\mathbb{A})}} \phi_{\overline{\sigma}}(n, 1)\psi^{-1}(n) \, dn \neq 0 \]  
(9.12)

for some \( \phi_{\overline{\sigma}} \in V_{\overline{\sigma}} \). Fix isomorphisms \( T_1 : \otimes_{\nu}^\prime \tau_{\nu} \rightarrow \tau \) and \( T_2 : \otimes_{\nu}^\prime \overline{\sigma}_{\nu} \rightarrow \overline{\sigma} \). Here, for each place \( \nu \) of \( \mathbb{F} \), \( \tau_{\nu} \) and \( \overline{\sigma}_{\nu} \) are the local components. Outside a finite set of places \( S \), containing the even places and those at infinity, \( \tau_{\nu} \) and \( \overline{\sigma}_{\nu} \) come together with a chosen spherical vectors \( \alpha_{\tau_{\nu}}^0 \) and \( \beta_{\overline{\sigma}_{\nu}}^0 \) respectively. We may assume, and in fact do, that \( \psi_{\nu} \) is normalized for all \( \nu \notin S \).

Let \( T = T_1 \otimes T_2 \). We identify \( \otimes_{\nu}^\prime \tau_{\nu} \otimes \otimes_{\nu}^\prime \overline{\sigma}_{\nu} \) with \( \otimes_{\nu}^\prime (\tau_{\nu} \otimes \overline{\sigma}_{\nu}) \) in the obvious way. We also identify the image of \( T \) with the space of cusp forms on \( GL_{m}(\mathbb{A}) \times Sp_{2k}(\mathbb{A}) \) generated by the functions \((g, h) \mapsto \phi_{\tau}(g)\phi_{\overline{\sigma}}(h)) \), here \( g \in GL_{m}(\mathbb{A}), h \in Sp_{2k}(\mathbb{A}), \phi_{\tau} \in V_{\tau} \) and \( \phi_{\overline{\sigma}} \in V_{\overline{\sigma}} \). \( T \) then is an isomorphism \( T : \otimes_{\nu}^\prime (\tau_{\nu} \otimes \overline{\sigma}_{\nu}) \rightarrow \tau \otimes \overline{\sigma} \). Denote for \( \phi \in V_{\tau \otimes \overline{\sigma}} \).

\[ W_{\phi}(g, h) = \int_{n_1 \in Z_{GL_{m}(\mathbb{F})} \setminus Z_{GL_{m}(\mathbb{A})}} \int_{n_2 \in Z_{Sp_{2n}(\mathbb{F})} \setminus Z_{Sp_{2n}(\mathbb{A})}} \phi(n_1 g, (n_2, 1) h) \psi^{-1}(n_1) \psi^{-1}(n_2) \, dn_2 \, dn_1. \]
By our assumption (9.12), there exists \( \phi \in V_{\tau \otimes \overline{\sigma}} \) such that \( W_\phi \neq 0 \) is not the zero function. Note that the linear functional

\[
\lambda_{\tau \otimes \overline{\sigma}, \psi}(\phi) = W_\phi(I_m, (I_{2n}, 1))
\]

is a non-trivial (global) \( \psi \)-Whittaker functional on \( V_{\tau \otimes \overline{\sigma}} \), i.e.,

\[
\lambda_{\tau \otimes \overline{\sigma}, \psi}(\tau \otimes \overline{\sigma}(n_1, n_2)\phi) = \psi(n_1)\psi(n_2)\lambda_{\tau \otimes \overline{\sigma}, \psi}(\phi),
\]

for all \((n_1, n_2) \in GL_m(\mathbb{A}) \times \overline{Sp}_{2k}(\mathbb{A})\). The last fact and the local uniqueness of Whittaker functional imply that

**Lemma 9.3.** There exists a unique, up to scalar, global \( \psi \)-Whittaker functional on \( \tau \otimes \overline{\sigma} \):

\[
\phi \mapsto \lambda_{\tau \otimes \overline{\sigma}, \psi}(\phi) = \int_{Z_{GLm}(\mathbb{F})Z_{GLm}(\mathbb{A})} \int_{Z_{Sp_{2k}(\mathbb{F})}Z_{Sp_{2k}(\mathbb{A})}} \phi(n_1, (n_2, 1))\psi^{-1}(n_1)\psi^{-1}(n_2)dn_2dn_1.
\]

For each \( \nu \) let us fix a non-trivial \( \psi_\nu \) Whittaker functional \( \lambda_{\tau_\nu \otimes \overline{\sigma}_\nu, \psi_\nu} \) on \( V_{\tau_\nu \otimes \overline{\sigma}_\nu} \) at each place \( \nu \), such that if \( \tau_\nu \otimes \overline{\sigma}_\nu \) is unramified then

\[
\lambda_{\tau_\nu \otimes \overline{\sigma}_\nu, \psi_\nu}(\alpha^0_{\tau_\nu} \otimes \beta^0_{\overline{\sigma}_\nu}) = 1.
\]

Then, by normalizing \( \lambda_{\tau_\nu \otimes \overline{\sigma}_\nu, \psi_\nu} \) at one ramified place, we have

\[
\lambda_{\tau \otimes \overline{\sigma}, \psi}(\phi) = \prod_{\nu} \lambda_{\tau_\nu \otimes \overline{\sigma}_\nu, \psi_\nu}(v_{\tau_\nu} \otimes v_{\overline{\sigma}_\nu}),
\]

where \( \phi = T(\otimes \nu(v_{\tau_\nu} \otimes v_{\overline{\sigma}_\nu})) \), i.e., \( \phi \) corresponds to a pure tensor.

We shall realize each local representation

\[
I_\nu(\tau_\nu(s), \overline{\sigma}_\nu) = \text{Ind}_{\overline{Sp}_{2n}(\mathbb{F}_\nu)}^{Sp_{2n}(\mathbb{F}_\nu)}(\gamma_\psi^{-1} \otimes \tau_\nu(s)) \otimes \overline{\sigma}_\nu
\]

as the space of smooth from the right functions

\[
f : Sp_{2n}(\mathbb{F}_\nu) \to V_{\tau_\nu} \otimes V_{\overline{\sigma}_\nu}
\]

satisfying

\[
f((j_{m,n}(\overline{g}), 1)i_{k,n}(y)nh) = \gamma_\psi^{-1}(g)\det(g)^{\frac{n+1}{2}}\tau_\nu(g) \otimes \overline{\sigma}_\nu(y)f(h)
\]

for all \( g \in GL_m(\mathbb{F}_\nu), y \in Sp_{2k}(\mathbb{F}_\nu), n \in (N_{m,k}(\mathbb{F}_\nu), 1), h \in Sp_{2n}(\mathbb{F}_\nu)\). For each place where \( \tau_\nu \) and \( \overline{\sigma}_\nu \) are unramified we define \( f^{0,s}_{\nu} \in I_\nu(\tau_\nu(s), \overline{\sigma}_\nu) \) to be the normalized spherical function, namely, \( f^{0,s}_{\nu}(I_{2n}, 1) = \alpha^0_{\nu} \otimes \beta^0_{\overline{\sigma}_\nu} \). We shall realize the global representation

\[
I(\tau(s), \overline{\sigma}) = \text{Ind}_{\overline{Sp}_{2n}(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\gamma_\psi^{-1} \otimes \tau(s)) \otimes \overline{\sigma}
\]

as a space of functions

\[
f : Sp_{2n}(\mathbb{A}) \times GL_m(\mathbb{A}) \times Sp_{2k}(\mathbb{A}) \to V_{\tau \otimes \overline{\sigma}}
\]

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smooth from the right in the first variable such that
\[ f((j_{m,n}(g), 1)i_{k,n}(y)nh, g_0, y_0) = \gamma^{-1}_\psi(g) \left| \det(g) \right|^{s+n+\frac{k+1}{2}} \tau(g) \otimes \overline{\sigma}(y)f(h, g_0 g, y_0 y), \]
for all \( g, g_0 \in GL_m(\mathbb{A}), y, y_0 \in \overline{Sp_{2n}(\mathbb{A})}, n \in (N_{m,k}(\mathbb{A}), 1), h \in \overline{Sp_{2n}(\mathbb{A})}, \) and such that for all \( h \in \overline{Sp_{2n}(\mathbb{A})} \) the map \( (g, y) \mapsto f(h, g, y) \) lies in \( V_{\tau \otimes \sigma}. \)

\( I(\tau(s), \sigma) \) is spanned by functions of the form \( f(g) = T((\otimes_{\nu} f_\nu(g_\nu)), \) where \( f_\nu \in I_\nu(\tau_\nu(s), \sigma_\nu) \) and for almost all \( \nu: f_\nu = f_\nu^0 \) (for a fixed \( g, f(g) \) is a cuspidal automorphic form corresponding to a pure tensor).

We note that for \( (p, 1) \in (P_{m,k}(\mathbb{F}), 1) \) we have \( f((p, 1)g) = f(g). \) This follows from the fact that \( \prod_{\nu} \gamma^{-1}_\psi(a) = 1 \) for all \( a \in \mathbb{F}^*. \) Hence, it makes sense to consider Eisenstein series: For a holomorphic smooth section \( f_s \in I(\tau(s), \sigma) \) define
\[ E(f_s, g) = \sum_{\gamma \in P_{m,k}(\mathbb{F}) \setminus Sp_{2n}(\mathbb{F})} f_s((\gamma, 1)g, I_m, (I_{2k}, 1)) \]
It is known that the series in the right-hand side converges absolutely for \( Re(s) >> 0, \) see Section II.1.5 of [37] and that it has a meromorphic continuation to the whole complex plane, see Section IV.1.8 of [37]. We continue to denote this continuation by \( E(f_s, g). \)

We introduce the \( \psi \)-Whittaker coefficient
\[ E_\psi(f_s, g) = \int_{u \in Z_{Sp_{2n}(\mathbb{F}) \setminus Sp_{2n}(\mathbb{A})}} E(f_s, (u, 1)g) \psi^{-1}(u) \, du. \]
Note that no question of convergence arises here since \( Z_{Sp_{2n}(\mathbb{F}) \setminus Sp_{2n}(\mathbb{A})} \) is compact. It is also clear that \( E_\psi(f_s, g) \) is meromorphic in the whole complex plane.

**Lemma 9.4.** \( \prod_{\nu \in S} L_{\psi_\nu}(\sigma_\nu \otimes \tau_\nu, s) \) converges absolutely for \( Re(s) >> 0. \) This product has a meromorphic continuation on \( \mathbb{C}. \) We shall denote this continuation by \( L_\psi^S(\sigma \otimes \tau, s). \) We have:
\[ E_\psi(f_s, (I_{2n}, 1)) = \frac{L_s^S(\tau, s + \frac{1}{2})}{L_s^S(\tau, sym^2, 2s + 1)L_\psi^S(\sigma \otimes \tau, s + 1)} \prod_{\nu \in S} \lambda(s, \tau_\nu \otimes \sigma_\nu, \psi)(f_\nu). \quad (9.13) \]

Recall that \( \prod_{\nu \in S} L(\tau_\nu, s) \) and \( \prod_{\nu \in S} L(\tau_\nu, sym^2, s) \) converge absolutely for \( Re(s) >> 0 \) and that these products have a meromorphic continuation on \( \mathbb{C}; \) see [34]. These continuations are denoted by \( L^S(\tau, s) \) and \( L^S(\tau, sym^2, s) \) respectively.

**Proof.** Recall that in Section 24.1 we have denoted by \( W_{Sp_{2n}} \) the Weyl group of \( Sp_{2n}(\mathbb{F}). \) We now and denote by \( W_{M_{m,k}} \) the Weyl group of \( M_{m,k}. \) We fix \( \Omega, \) a complete set of representatives of \( W_{M_{m,k}} \setminus W_{Sp_{2n}}. \) Recall the Bruhat decomposition
\[ Sp_{2n}(\mathbb{F}) = \bigcup_{w \in \Omega} P_{m,k}(\mathbb{F})wB_{Sp_{2n}}(\mathbb{F}). \]
Clearly for \( w \in \Omega: \)
\[ P_{m,k}(\mathbb{F})wB_{Sp_{2n}}(\mathbb{F}) = P_{m,k}(\mathbb{F})wZ_{Sp_{2n}}(\mathbb{F}). \]
Also, for \( w \in \Omega, p_1, p_2 \in P_{m,k}(\mathbb{F}), u_1, u_2 \in Z_{Sp_{2n}}(\mathbb{F}) \) we have: If \( p_1wu_1 = p_2wu_2 \) then
\[
 u_2u_1^{-1} \in Z_w(\mathbb{F}) = Z_{Sp_{2n}}(\mathbb{F}) \cap w^{-1}Z_{Sp_{2n}}(\mathbb{F})w.
\]
Thus, every element \( \gamma \) of \( Sp_{2n}(\mathbb{F}) \) can be expressed as \( g = pwu \), where \( p \in P_{m,k}(\mathbb{F}) \) and \( w \in \Omega \) are determined uniquely and \( u \in Z_{Sp_{2n}}(\mathbb{F}) \) is determined uniquely modulo \( Z_w(\mathbb{F}) \) from the left. Note that if \( W_{M_{m,k}}w_1 = W_{M_{m,k}}w_2 \) it does not follow that \( Z_w(\mathbb{F}) = Z_w(\mathbb{F}) \). This is why we started from fixing \( \Omega \). Thus, for \( f_s \in I(\tau_s, \sigma) \) and \( Re(s) > 0 \) we have:
\[
 E_\psi(f_s, (I_{2n}, 1)) = \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} f_s((\gamma u, 1), (I_{2k}, 1)) \psi^{-1}(u) du
\]
\[
 = \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) \sum_{w \in \Omega} \sum_{n \in Z_w(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{F})} f_s((wnu, 1), (I_{2k}, 1)) du
\]
\[
 = \sum_{w \in \Omega} \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) \sum_{n \in Z_w(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{F})} f_s((wnu, 1), (I_{2k}, 1)) du
\]
\[
 = \sum_{w \in \Omega} \int_{u \in Z_w(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s(wu, (I_{2k}, 1)) du.
\]
We now choose \( w_0 = w'(m; k) \) (see (6.1)) as the representative of \( J_{2n} \) in \( \Omega \). We note that \( Z_{w_0} = Z_{Sp_{2n}} \cap M_{n,k} \). By the same argument used in page 182 of [10] one finds that
\[
 \int_{u \in Z_w(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s(wu, (I_{2k}, 1)) = 0,
\]
for all \( w \in \Omega, w \neq w_0 \). Thus, from (9.14) we have:
\[
 E_\psi(f_s, (I_{2n}, 1)) = \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \setminus M_{n,k}(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s(w_0u, (I_{2k}, 1)) du
\]
\[
 = \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \setminus M_{n,k}(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s(w_0u, (I_{2k}, 1)) du
\]
\[
 = \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \setminus M_{n,k}(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s(w_0u, (I_{2k}, 1)) du
\]
\[
 = \int_{n \in N_{m,k}(\mathbb{A})} \psi^{-1}(n) \int_{u \in Z_{Sp_{2n}}(\mathbb{F}) \setminus M_{n,k}(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(u) f_s(w_0u, (I_{2k}, 1)) du dn
\]
\[
 = \int_{n \in N_{m,k}(\mathbb{A})} \psi^{-1}(n) \int_{n_1 \in Z_{GL_n}(\mathbb{F}) \setminus Z_{GL_n}(\mathbb{A})} \int_{n_2 \in Z_{Sp_{2n}}(\mathbb{F}) \setminus Z_{Sp_{2n}}(\mathbb{A})} \psi^{-1}(n_2) \psi^{-1}(n_1) f_s(w_0n, n_1, (n_2, 1)) du dn.
\]
Recall that \( S \) is a finite set of places of \( \mathbb{F} \), such that for all \( \nu \notin S \), \( \nu \) is finite and odd, \( \tau_\nu \otimes \sigma_\nu \) is unramified and \( \psi_\nu \) is normalized. Assume now that \( f_s \) corresponds to the following pure tensor of holomorphic smooth sections, \( f_s(g) = T(\otimes_\nu f_{s,\nu}(g_\nu)) \), where \( f_{s,\nu} \in V_{\nu}(\tau_\nu, \sigma) \) and for \( \nu \notin S: f_{s,\nu} = f_0^{0,s} \). By Lemma 9.3, we have
\[
 E_\psi(f_s, (I_{2n}, 1)) = \prod_\nu \int_{n \in N_{m,k}(\mathbb{F}_\nu)} \lambda_{\tau_\nu \otimes \sigma_\nu, \psi_\nu} \left( \rho(w_0n) f_{s,\nu} \right) \psi^{-1}(n) dn = \prod_\nu \lambda(s, \tau_\nu \otimes \sigma_\nu, \psi)(f_{s,\nu})
\]
The global functional equation for the Eisenstein series states that

\[ E_\psi(f_s, (I_{2n}, 1)) = \frac{L^S(\tau, s + \frac{1}{2})}{L^S(\tau, \text{sym}^2, 2s + 1) \prod_{\nu \notin S} L^S(\sigma^\nu \otimes \tau^\nu, s + 1)} \prod_{\nu \in S} \lambda(s, \tau^\nu \otimes \sigma^\nu, \psi)(f_{s, \nu}). \]  

(9.18)

We claim that we may choose \( f_s \) as above such that for all \( \nu \in S \)

\[ \lambda(s, \tau^\nu \otimes \sigma^\nu, \psi)(f_{s, \nu}) = 1 \]  

(9.19)

for all \( s \in \mathbb{C} \). Indeed, we choose \( f_{s, \nu} \) which is supported on the open Bruhat cell

\[ \mathcal{P}_{m,k}(\mathbb{F}_\nu)(w'_1(m; k)Z_{m,k}(\mathbb{F}_\nu), 1) \]

which satisfies

\[ f_{s, \nu}(w'_1(m; k)z, 1)(g, y) = \phi(z)W_{\tau^\nu}(g)W_{\psi}(y). \]

Here \( z \in Z_{m,k}(\mathbb{F}_\nu), g \in GL_m(\mathbb{F}_\nu), y \in S_{2k}(\mathbb{F}_\nu) \), \( \phi \) is a properly chosen smooth compactly supported function on \( Z_{m,k}(\mathbb{F}_\nu) \) and \( W_{\tau^\nu}(I_m) = W_{\psi}(I_{k, 1}) = 1 \). We have

\[ \lambda(s, \tau^\nu \otimes \sigma^\nu, \psi)(f_{\nu}) = \int_{Z_{m,k}(\mathbb{F}_\nu)} \phi(z)\psi^{-1}(z_{n, n}) \, dn. \]

We now choose \( \phi \) such that (9.19) holds. For such a choice we have

\[ E_\psi(f_s, (I_{2n}, 1)) = \frac{L^S(\tau, s + \frac{1}{2})}{L^S(\tau, \text{sym}^2, 2s + 1) \prod_{\nu \notin S} L^S(\sigma^\nu \otimes \tau^\nu, s + 1)} \prod_{\nu \in S} \lambda(s, \tau^\nu \otimes \sigma^\nu, \psi)(f_{s, \nu}). \]

The absolute convergence of \( \prod_{\nu \notin S} L^S(\sigma^\nu \otimes \tau^\nu, s) \) for \( \text{Re}(s) > 0 \) is clear now. Furthermore, the fact that this product has a meromorphic continuation to \( \mathbb{C} \) follows from the meromorphic continuations of \( E_\psi(f_s, (I_{2n}, 1)), L^S(\tau, s) \) and \( L^S(\tau, \text{sym}^2, s) \). Finally, the validity of (9.13) for all \( s \) follows from (9.18).

\[ \square \]

**Theorem 9.1.**

\[ \prod_{\nu \in S} \gamma(\sigma^\nu \times \tau^\nu, s, \psi^\nu) = \frac{L^S(\sigma \times \tau, s)}{L^S(\sigma \times \tau, 1 - s)}. \]  

(9.20)

**Proof.** The global functional equation for the Eisenstein series states that

\[ E(f_s, g) = E(A(f_s, g)), \]

where \( A \) is the global intertwining operator; see Section IV.1.10 of [37]. We compute the \( \psi \)-Whittaker coefficient of both sides of the last equation. By (9.8) we have

\[ \frac{L^S(\tau, s + \frac{1}{2})}{L^S(\tau, \text{sym}^2, 2s + 1)} \prod_{\nu \in S} \lambda(s, \tau^\nu \otimes \sigma^\nu, \psi)(f_{s, \nu}) \]

\[ = \frac{L^S(\tau, -s + \frac{1}{2}) L^S(\sigma \otimes \tau, s)}{L^S(\tau, \text{sym}^2, 2s)} L^S(\tau, \text{sym}^2, 2s + 1) \prod_{\nu \in S} \lambda(-s, \tau^\nu \otimes \sigma^\nu, \psi)(A_{J_{m,n}}(f_{l, 1}^n), (f_{l, 1}^n)). \]
Or equivalently, by the definition of the local coefficients
\[
\prod_{\nu \in S} C_{\psi_{\nu}}^{Sp_{2m}(F)}(P_{m;0}(F_{\nu}), s, \tau_{\nu} \otimes \sigma_{\nu}, j_{m,n}(\omega_{m}^{-1})) = \frac{L^{S}(\hat{\tau}, -s + \frac{1}{2}) L^{S}(\tau, \text{sym}^{2}, 2s) L^{S}_{\psi}(\sigma \otimes \tau, s)}{L^{S}(\tau, s + \frac{1}{2}) L^{S}(\hat{\tau}, \text{sym}^{2}, -2s + 1) L^{S}_{\psi}(\sigma \otimes \hat{\tau}, 1 - s)}.
\]

(9.21)

In particular, for \(k = 0\)
\[
\prod_{\nu \in S} C_{\psi_{\nu}}^{Sp_{2m}(F)}(P_{m;0}(F_{\nu}), s, \tau_{\nu}, j_{m,n}(\omega_{m}^{-1})) = \frac{L^{S}(\hat{\tau}, -s + \frac{1}{2}) L^{S}(\tau, \text{sym}^{2}, 2s)}{L^{S}(\tau, s + \frac{1}{2}) L^{S}(\hat{\tau}, \text{sym}^{2}, -2s + 1)}. \quad (9.22)
\]

Dividing (9.21) and (9.22) we get (9.20).

\[
\prod_{\nu \in S} C_{\psi_{\nu}}^{Sp_{2m}(F)}(P_{m;0}(F_{\nu}), s, \tau, \omega_{m}^{-1}) = \prod_{\nu \in S} C_{\psi_{\nu}}^{Sp_{2m}(F)}(P_{m;0}(F_{\nu}), s, \tau, \omega_{m}^{-1}) = \frac{L^{S}(\hat{\tau}, -s + \frac{1}{2}) L^{S}(\tau, \text{sym}^{2}, 2s)}{L^{S}(\tau, s + \frac{1}{2}) L^{S}(\hat{\tau}, \text{sym}^{2}, -2s + 1)}. \quad (9.20)
\]

9.3 Computation of \(C_{\psi}^{Sp_{2m}(F)}(P_{m;0}(F), s, \tau, \omega_{m}^{-1})\) for generic representations

**Theorem 9.2.** Let \(F\) be a \(p\)-adic field and let \(\tau\) be an irreducible admissible supercuspidal representation of \(GL_{m}(F)\). There exists an exponential function \(c_{F}(s)\) such that
\[
C_{\psi}^{Sp_{2m}(F)}(P_{m;0}(F), s, \tau, \omega_{m}^{-1}) = c_{F}(s) \frac{\gamma(\tau, \text{sym}^{2}, 2s, \psi)}{\gamma(\tau, s + \frac{1}{2}, \psi)}.
\]

**Proof.** Since \(\tau\) is supercuspidal it is also generic. Proposition 5.1 of [53] implies now that there exists a number field \(K\), a non-degenerate character \(\tilde{\psi}\) of \(Z_{GL_{n}(K)} \setminus Z_{GL_{n}(A)}\) and an irreducible cuspidal representation \(\pi \simeq \otimes_{\nu} \pi_{\nu}\) of \(GL_{n}(A)\) such that
1. \(K_{\nu_{0}} = F\) for some place \(\nu_{0}\) of \(K\).
2. \(\tilde{\psi}_{\nu_{0}} = \psi\).
3. \(\pi_{\nu_{0}} = \tau\).
4. For any finite place \(\nu \neq \nu_{0}\) of \(K\), \(\pi_{\nu}\) is unramified.

Define \(S\) to be the finite set of places of \(K\) which consists of \(\nu_{0}\), of the infinite and even places and of the finite places where \(\tilde{\psi}_{\nu}\) is not normalized. From the fourth part of Theorem 3.5 of [52] it follows that
\[
\prod_{\nu \in S} \gamma_{F_{\nu}}(\pi_{\nu}, s, \psi_{\nu}) = \frac{L^{S}(\pi, s)}{L^{S}(\pi, 1 - s)}
\]
and that
\[
\prod_{\nu \in S} \gamma_{F_{\nu}}(\pi_{\nu}, s, \text{sym}^{2}, \psi_{\nu}) = \frac{L^{S}(\pi, \text{sym}^{2}, s)}{L^{S}(\pi, \text{sym}^{2}, 1 - s)}.
\]

Therefore, (9.22) can be written as
\[
\prod_{\nu \in S} C_{\psi_{\nu}}^{Sp_{2m}(F_{\nu})}(P_{m;0}(F_{\nu}), s, \pi_{\nu}, j_{m,n}(\omega_{m}^{-1})) = \prod_{\nu \in S} \frac{\gamma(\pi_{\nu}, \text{sym}^{2}, 2s, \psi)}{\gamma(\pi_{\nu}, s + \frac{1}{2}, \psi)}.
\]

This implies that this theorem will be proven once we show that for all \(\nu \in S, \nu \neq \nu_{0}\), there exists an exponential function \(c_{\nu}(s)\) such that
\[
C_{\psi_{\nu}}^{Sp_{2m}(F_{\nu})}(P_{m;0}(F_{\nu}), s, \pi_{\nu}, j_{m,n}(\omega_{m}^{-1})) = c_{\nu}(s) \frac{\gamma_{F_{\nu}}(\pi_{\nu}, \text{sym}^{2}, 2s, \psi)}{\gamma_{F_{\nu}}(\pi_{\nu}, s + \frac{1}{2}, \psi)}.
\]
Since for all $\nu \in S$, $\nu \neq \nu_0$, $\pi_\nu$ is the generic constituent of a principal series representation series, this follows from Theorem 8.4.

**Theorem 9.3.** Let $F$ be a $p$-adic field and let $\tau$ be an irreducible admissible generic representation of $GL_m(F)$. There exists an exponential function $c_F(s)$ such that

$$C_{Sp^{2m}(F)}(P_{m;0}(F), s, \tau, \omega_m^{-1}) = c_F(s) \frac{\gamma(\tau, \text{sym}^2, 2s, \psi)}{\gamma(\tau, s + \frac{1}{2}, \psi)}.$$  \hfill (9.23)

**Proof.** By Chapter II of [S], $\tau$ may be realized as a sub-representation of $\tau' = \text{Ind}_{Q(F)}^{GL_m(F)}(\otimes_{i=1}^r \tau_i)$, where $Q(F)$ is a standard parabolic subgroup of $GL_m(F)$ whose Levi part, $M(F)$, is isomorphic to $GL_{n_1}(F) \times GL_{n_2}(F) \times \cdots \times GL_{n_r}(F)$ and where for all $1 \leq i \leq r$, $\tau_i$ is an irreducible admissible supercuspidal representation of $GL_{n_i}(F)$. Since for all $1 \leq i \leq r$, $\tau_i$ has a unique Whittaker model it follows from the heredity property of the Whittaker model that $\tau'$ has a unique Whittaker model; see [43]. This implies that $\tau$ is the generic constituent of $\tau'$. Hence,

$$C_{Sp^{2m}(F)}(P_{m;0}(F), s, \tau, \omega_m^{-1}) = C_{Sp^{2m}(F)}(P_{m;0}(F), s, \tau', \omega_m^{-1}).$$

Thus, it is sufficient to prove (9.23) replacing $\tau$ with $\tau'$. By similar arguments to those used in Lemma 8.16 and Theorem 8.4 of Section 8.6 one shows that there exists $d \in \{\pm 1\}$ such that

$$C_{Sp^{2m}(F)}(P_{m;0}(F), s, \tau', \omega_m^{-1}) = d \prod_{i=1}^r \left( C_{Sp^{n_i}(F)}(P_{n_i;0}(F), s, \tau_i, \omega_{n_i}^{-1}) \prod_{j=i+1}^r \gamma(\tau_i \times \tau_j, 2s, \psi) \right).$$

Since for $1 \leq i \leq r$, $\tau_i$ are irreducible admissible supercuspidal representations it follows from Theorem 9.2 that there exists an exponential factor, $c_F(s)$, such that

$$C_{Sp^{2m}(F)}(P_{m;0}(F), s, \tau', \omega_m^{-1}) = c_F(s) \prod_{i=1}^r \left( \frac{\gamma(\tau_i, \text{sym}^2, 2s, \psi)}{\gamma(\tau_i, s + \frac{1}{2}, \psi)} \prod_{j=i+1}^r \gamma(\tau_i \times \tau_j, 2s, \psi) \right).$$

Using the known multiplicativity of the symmetric square $\gamma$-factor (see Part 3 of Theorem 3.5 of [52]) we finish. \hfill □
10 Irreducibility theorems

Let \( \mathbb{F} \) is a p-adic field. In this chapter we prove several irreducibility theorems for parabolic induction on the metaplectic group. Through this chapter we shall assume that the inducing representations are smooth irreducible admissible cuspidal unitary genuine generic.

In Section 10.1 we prove the irreducibility of principal series representations (induced from a unitary characters); see Theorem 10.1. A generalization of the argument given in this theorem is used in Section 10.2 where we prove a general criterion for irreducibility of parabolic induction; see Theorem 10.2. As a corollary we reduce the question of irreducibility of parabolic induction on \( \text{Sp}_{2n}(\mathbb{F}) \) to irreducibility of representations of metaplectic groups of smaller dimension induced from representations of maximal parabolic groups; see Corollary 10.2. In Section 10.3 we reduce the question of irreducibility of representations of \( \text{Sp}_{2n}(\mathbb{F}) \) induced from the Siegel parabolic subgroup to the question of irreducibility of a parabolic induction on \( \text{SO}_{2n+1}(\mathbb{F}) \); see Theorem 10.3. We then use [55] to give a few corollaries.

Our method here is an application of Theorem 4.3 to the computations and results presented in Chapters 7.1, 8 and 9. The link between Theorem 4.3 and the local coefficients is explained in Chapter 6; see (6.7). Throughout this chapter we shall use various definitions and notation given in previous chapters. Among them are \( \pi_w \) and the notion of a regular and of a singular representation (see Section 4.2), \( W_{\mathcal{P}}(\mathcal{T}^{-\tau}(\mathbb{F})) \) (see Section 2.1), \( W(\pi) \) (see (4.8)) and \( \Sigma_{\mathcal{P}^{-\tau}(\mathbb{F})} \) (see Section 4.4).

10.1 Irreducibility of principal series representations of \( \text{Sp}_{2n}(\mathbb{F}) \) induced from unitary characters.

Lemma 10.1. Let \( \beta_1 \) and \( \beta_2 \) be two characters of \( \mathbb{F}^* \). Denote \( \beta = \beta_1 \beta_2^{-1} \). If \( \mathbb{F} \) is a p-adic field then

\[
C_{\psi}^{GL_2(\mathbb{F})}(B_{GL_2(\mathbb{F})}, (s_1, s_2), \beta_1 \otimes \beta_2, (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})) = \frac{\beta(\pi^n(\beta)-n)q^{n-m(\beta)(n_1-s_2)}}{G(\beta, \psi^{-1})} \frac{L(\beta^{-1}, 1 - (s_1 - s_2))}{L(\beta, (s_1 - s_2))},
\]

where \( n \) is the conductor of \( \psi \). If \( \mathbb{F} = \mathbb{R} \) then

\[
C_{\psi}^{GL_2(\mathbb{R})}(B_{GL_2(\mathbb{R})}, (s_1, s_2), \beta_1 \otimes \beta_2, (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})) = \beta_2(-1)e^{-is\beta(-1)}(2\pi)^{-(s_1-s_2)}L(\beta^{-1}, 1 - (s_1 - s_2))L(\beta, (s_1 - s_2))
\]

This lemma can be proved by similar computations to those presented in Sections 8.1 and 8.2. The p-adic computations here are much easier than those presented in this dissertation. The lemma also follows from more general known results; see Lemma 2.1 of [49] for the p-adic case and see Theorem 3.1 of [51] for the real case.

Theorem 10.1. Let \( \alpha_1, \ldots, \alpha_n \) be \( n \) unitary characters of \( \mathbb{F}^* \). Let \( \alpha \) be the character of \( \mathcal{T}_{Sp_{2n}(\mathbb{F})} \) defined by

\[
(\text{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}), e) \mapsto e\gamma^{-1}_\psi(\prod_{i=1}^n a_i) \prod_{i=1}^n \alpha_i(a_i).
\]

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Then $I(\alpha)$ is irreducible.

Proof. Since $\alpha$ is unitary, $I(\alpha)$ is also unitary. Therefore, the irreducibility of $I(\alpha)$ will follow once we show that

$$\text{Hom}_{\text{Sp}_{2n}(F)}(I(\alpha), I(\alpha)) \simeq \mathbb{C}.$$ 

For $1 \leq i < j \leq n$ define

$$w(i, j) = \left( \begin{array}{ccc} I_{i-1} & \cdots & 1 \\ \cdots & & \cdots \\ 1 & \cdots & I_{n-j+1} \end{array} \right)$$

and

$$w'(i, j) = \tau(i, j)w(i, j).$$

A routine exercise shows that

$$\Sigma_{B_{\text{Sp}_{2n}(F)}} = \{w(i, j) \mid 1 \leq i < j \leq n\} \cup \{\tau_r \mid 1 \leq r \leq n\} \cup \{w'(i, j) \mid 1 \leq i < j \leq n\}. \tag{10.3}$$

Note that,

$$w(i, j)\text{diag}(a_1, \ldots, a_n)w^{-1}(i, j) = \text{diag}(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_{j-1}, a_i, a + j + 1, \ldots, a_n),$$

$$w(\tau)\text{diag}(a_1, \ldots, a_n)w^{-1}(\tau) = \text{diag}(a_1, \ldots, a_{r-1}, a_r^{-1}, a_{r+1}, \ldots, a_n)$$

and that

$$w'(i, j)\text{diag}(a_1, \ldots, a_n)w'^{-1}(i, j) = \text{diag}(a_1, \ldots, a_{i-1}, a_j^{-1}, a_{i+1}, \ldots, a_{j-1}, a_i^{-1}, a + j + 1, \ldots, a_n).$$

Therefore $w(i, j) \in W(\alpha) \iff \alpha_i = \alpha_j, \tau(\tau) \in W(\alpha) \iff \alpha_r$ is quadratic and $w'(i, j) \in W(\alpha) \iff \alpha_i = \alpha_j^{-1}$. Furthermore, $W(\alpha)$ is generated by $\Sigma_{B_{\text{Sp}_{2n}(F)}} \cap W(\alpha)$. Thus, using Theorem 4.3 and (6.7), the proof of this theorem amounts to showing that

$$C_{P_{2n}(F)}(P_{2n}(F), s, (\otimes_{\ell=1}^n \alpha_\ell) \otimes, w)C_{P_{2n}(F)}(P_{2n}(F), s, (\otimes_{\ell=1}^n \alpha_\ell)^{w-1}) = 0 \tag{10.4}$$

for all $w \in \Sigma_{B_{\text{Sp}_{2n}(F)}} \cap W(\alpha)$. We prove it for each of the three types in the right-hand side of (10.3).

Suppose that $w(i, j) \in W(\alpha)$. We write:

$$w(i, j) = w(i, i+1)w(i+1, i+2) \cdots w(j-1, j)w(j-2, j-1) \cdots w(i+1, i). \tag{10.5}$$

We claim that the expression in the right-hand side of (10.5) is reduced. Indeed,

$$\Sigma_{B_{\text{Sp}_{2n}(F)}} = \{w(i, i+1) \mid 1 \leq i < n\} \cup \{\tau(1)\} \subset \Sigma_{B_{\text{Sp}_{2n}(F)}}.$$
is the subset of reflections corresponding to simple roots and the length of \( w_{(i,j)} \) is \( 2(j-i) - 1 \) (any claim about the length of a given Weyl element \( w \) may be verified by counting the number of positive root subgroups mapped by \( w \) to negative root subgroups). Thus, we may use the same argument as in Lemma 7.2 and conclude that there exists \( c \in \{ \pm 1 \} \) such that

\[
C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^n, (\otimes_{i=1}^n \alpha_i), w_{(i,j)}) = \left( \prod_{k=1}^{j-2} f_k(s)f_k^*(s) \right) C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{\vec{w}}, (\otimes_{i=1}^n \alpha_i)^{\vec{w}}, w_{(j-1,j)}),
\]

where

\[
f_k(s) = C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{w(k)}, (\otimes_{i=1}^n \alpha_i)^{w(k)}, w_{(k,k+1)}),
\]

\[
f_k^*(s) = C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{w'(k)}, (\otimes_{i=1}^n \alpha_i)^{w'(k)}, w_{(k,k+1)}),
\]

where

\[
w(k) = w(i,i+1)w_{(i+1,i+2)} \cdots w_{(k-1,k)},
\]

\[
\vec{w} = w(i,i+1)w_{(i+1,i+2)} \cdots w_{(j-1,j)}w_{(j-2,j-1)},
\]

\[
w'(k) = w(i,i+1)w_{(i+1,i+2)} \cdots w_{(j-1,j)}w_{(j-2,j-1)}w_{(j-1,j)}w_{(j-2,j-1)} \cdots w_{(k-1,k+2)}.
\]

Since all the local coefficients in the right-hand side of (10.6) correspond to simple reflections we may use the same argument as in Lemma 7.3 and conclude that

\[
C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{w(k)}, (\otimes_{i=1}^n \alpha_i)^{w(k)}, w_{(k,k+1)}) = C_{\text{GL}2(F)}(B_{\text{GL}2(F)}, (s_i, s_{i+k}), \alpha_i \otimes \alpha_{i+k}, \omega_2, (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})),
\]

\[
C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{\vec{w}}, (\otimes_{i=1}^n \alpha_i)^{\vec{w}}, w_{(j-1,j)}) = C_{\text{GL}2(F)}(B_{\text{GL}2(F)}, (s_i, s_j), \alpha_i \otimes \alpha_j, \omega_2, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})),
\]

\[
C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{w'(k)}, (\otimes_{i=1}^n \alpha_i)^{w'(k)}, w_{(k,k+1)}) = C_{\text{GL}2(F)}(B_{\text{GL}2(F)}, (s_{i+k}, s_j), \alpha_{i+k} \otimes \alpha_j, \omega_2, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})).
\]

Since \( \alpha_1, \ldots, \alpha_k \) are unitary, (10.7) and (10.1) implies that for \( i \leq k \leq j - 2, \)

\[
C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{w(k)}, (\otimes_{i=1}^n \alpha_i)^{w(k)}, w_{(k,k+1)})
\]

and

\[
C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{w'(k)}, (\otimes_{i=1}^n \alpha_i)^{w'(k)}, w_{(k,k+1)})
\]

are holomorphic at \( \vec{s} = 0 \). Also, since \( w_{(i,j)} \in W(\alpha) \) implies that \( \alpha_i = \alpha_j, \)

(10.7) and (10.1) imply that

\[
C_{\text{Sp}2n(F)}(B_{\text{Sp}2n(F)}, \vec{s}^{\vec{w}}, (\otimes_{i=1}^n \alpha_i)^{\vec{w}}, w_{(j-1,j)})
\]

vanishes for \( \vec{s} = 0 \). Recalling (10.6) we now conclude that if \( w = w_{(i,j)} \in W(\alpha) \) then (10.4) holds.

Suppose now that \( \tau_{(r)} \in W(\alpha) \). We write

\[
\tau_{(r)} = w_{(r,r+1)}w_{(r+1,r+2)} \cdots w_{(n-1,n)}w_{(n-2,n-1)} \cdots w_{(r+1,r)}.
\]
The reader may check that the expression in the right-hand side of (10.8) is reduced. We now use the same arguments we used for $w = w_{(i,j)}$: We decompose

$$C_{\psi}^{Sp_{2n}(F)}(B_{Sp_{2n}(F)}, s, (\otimes_{i=1}^{n} \alpha_{i}), \tau_{(r)})$$

into $1 + 2(n - i)$ local coefficients. $2(n - i)$ of them are of the form (10.1). These factors are holomorphic at $s = 0$. The additional local coefficient, the one corresponding to $\tau_{(n)}$ is

$$C_{\psi}^{SL_{2}(F)}(B_{SL_{2}(F)}, s, r, (\begin{smallmatrix}0 & 1 \\ -1 & 0 \end{smallmatrix})).$$

Theorem 8.1 implies that there exists $c \in \mathbb{C}^\ast$ such that

$$C_{\psi}^{SL_{2}(F)}(B_{SL_{2}(F)}, s, \chi, (\begin{smallmatrix}0 & 1 \\ -1 & 0 \end{smallmatrix})) C_{\psi}^{SL_{2}(F)}(B_{SL_{2}(F)}, -s, \chi^{-1}, (\begin{smallmatrix}0 & 1 \\ -1 & 0 \end{smallmatrix})) = c \frac{L_{F}(\chi^{2}, -2s + 1)}{L_{F}(\chi^{2}, 2s + 1)} \frac{L_{F}(\chi^{2}, 2s)}{L_{F}(\chi^{2}, -2s)}.$$

Since $\tau_{(r)} \in W(\alpha)$ implies that $\alpha_{r}$ is quadratic, we now conclude that (10.4) holds for $w = \tau_{(r)}$.

Finally, assume that $w'_{(i,j)} \in W(\alpha)$. We write it as a reduced product of simple reflections:

$$w'_{(i,j)} = w_{(j,j+1)} w_{(i+1,i+2)} \cdots w_{(n-1,n)} \tau_{(n)} w_{(n-2,n-1)} \cdots w_{(j+1,i)}$$

$$w_{(i,i+1)} w_{(i+1,i+2)} \cdots w_{(j-1,j)} w_{(j-2,j-1)} \cdots w_{(i+1,i)}$$

$$w_{(j,j+1)} w_{(j+1,j+2)} \cdots w_{(n-1,n)} \tau_{(n)} w_{(n-2,n-1)} \cdots w_{(j+1,i)}.$$

We then decompose

$$C_{\psi}^{Sp_{2n}(F)}(B_{Sp_{2n}(F)}, s, (\otimes_{i=1}^{n} \alpha_{i}), w'_{(i,j)})$$

into $1 + 2(n - i)$ local coefficients coming either from $GL_{2}(F)$ or from $SL_{2}(F)$. All these local coefficients are holomorphic at $s = 0$. The factor corresponding to $w_{(i,j-1)}$ equals

$$C_{\psi}^{GL_{2}(F)}(B_{GL_{2}(F)}, s, -s, \chi, \alpha_{i} \otimes \alpha_{j}^{-1}, (\begin{smallmatrix}0 & 1 \\ -1 & 0 \end{smallmatrix})).$$

Since $w'_{(i,j)} \in W(\alpha)$ implies that $\alpha_{i} = \alpha_{j}^{-1}$ we conclude, using (10.1), that (10.4) holds for $w = w'_{(i,j)}$, provided that $w'_{(i,j)} \in W(\alpha)$.

\[ \square \]

**Remarks:**

1. Assume that $F$ is a p-adic field of odd residual characteristic. For the irreducibility of principal series representations of $SL_{2}(F)$, induced from unitary characters see [39]. For the irreducibility of principal series representations induced from unitary characters to the $C^{1}$ cover of $Sp_{4}(F)$ see [72]. During the final preparations of this manuscript the author of these lines encountered a preprint which proves Theorem 10.1 using the theta correspondence; see [19].

2. One can show that Theorem 4.3 applies also to the field of real numbers in the case of a parabolic induction from unitary characters of $B_{Sp_{2n}(R)}$. Thus, repeating the same argument used in this section, replacing Theorem 8.1 with Theorem 8.2 and (10.1) with (10.2), one concludes that Theorem 10.1 applies for the real case as well.

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10.2 Irreducibility criteria for parabolic induction

Theorem 10.2. Let $\mathbf{t} = (n_1, n_2, \ldots, n_r; k)$ where $n_1, n_2, \ldots, n_r, k$ are $r + 1$ non-negative integers whose sum is $n$. For $1 \leq i \leq r$ let $\tau_i$ be an irreducible admissible supercuspidal unitary representation of $GL_n(F)$ and let $\pi$ be an irreducible admissible supercuspidal $\psi$-generic genuine representation of $Sp_{2n}(F)$. Denote $\pi = (\bigotimes_{i=1}^r (\gamma_i^{-1} \otimes \tau_i)) \otimes \pi$. $I(\pi)$ is reducible if and only if there exists $1 \leq i \leq r$ such that $\tau_i$ is self dual and

$$C_{\psi}^{Sp_{2(k+n_i)}(F)}(P_{n_i; k}(F), 0, \tau_i \otimes \pi, j_{n_i, k+n_i}(\omega_{n_i}^{-1})) \neq 0 \quad (10.9)$$

Proof. Since $j_{n_i, n}(\omega_{n_i}^{-1})$ is of order two as a Weyl element it follows that if $\tau_i$ is self dual then

$$C_{\psi}^{Sp_{2(k+n_i)}(F)}(P_{n_i; k}(F), 0, \tau_i \otimes \pi, j_{n_i, k+n_i}(\omega_{n_i}^{-1})) = 0$$

if and only if

$$C_{\psi}^{Sp_{2(k+n_i)}(F)}(P_{n_i; k}(F), s, \tau_i \otimes \pi, j_{n_i, k+n_i}(\omega_{n_i}^{-1})) C_{\psi}^{Sp_{2n}(F)}(P_{n_i; k}(F), -s, \tau_i \otimes \pi, j_{n_i, k+n_i}(\omega_{n_i}^{-1}))$$

vanishes at $s = 0$. Thus, since $I(\pi)$ is unitary, we only have to show that

$$\dim(Hom_{Sp_{2n}(F)}(I(\pi), I(\pi))) > 1 \quad (10.11)$$

if and only if there exits $1 \leq i \leq r$ such that $\tau_i$ is self dual and (10.10) does not vanish at $s = 0$.

Suppose first that there exits $1 \leq i \leq r$ such that $\tau_i$ is self dual and (10.10) does not vanish at $s = 0$. Since for any $w \in W_{P_{\mathfrak{r}}}(F)$, $I(\pi)$ and $I(\pi^w)$ have the same Jordan Holder series we may assume that $i = r$. It follows from (4.7) that $w_0 = j_{n_r, n}(\omega_{n_r}^{-1}) \in \sigma_{P_{\mathfrak{r}}}(F) \cap W(\pi)$. Since $w_0$ is a simple reflection we may use a similar argument to the one used in Lemma [4.3] and conclude that

$$C_{\psi}^{Sp_{2n}(F)}(P_{\mathfrak{r}}(F), \mathbf{s}, (\bigotimes_{i=1}^r \tau_i) \otimes \pi, w_0) = C_{\psi}^{Sp_{2(k+n_i)}(F)}(P_{n_r; k}(F), s_r, \tau_i \otimes \pi, w_0),$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_r)$. Thus, our assumption implies that

$$C_{\psi}^{Sp_{2n}(F)}(P_{\mathfrak{r}}(F), \mathbf{s}, (\bigotimes_{i=1}^r \tau_i) \otimes \pi, w_0) C_{\psi}^{Sp_{2n}(F)}(P_{\mathfrak{r}}(F), \mathbf{s}, (\bigotimes_{i=1}^r \tau_i) \otimes \pi, w_0)$$

(10.12)

does not vanish at $s = 0$. Theorem [4.3] and (6.7) imply now that (10.11) holds.

We now assume that for any $1 \leq i \leq r$, if $\tau_i$ is self dual then (10.10) vanishes at $s = 0$. Again, by theorem [4.3] and (6.7) we only have to show that (10.12) vanishes at $\mathbf{s} = 0$ for any $w_0 \in \Sigma_{P_{\mathfrak{r}}(F)} \cap W(\pi)$. Similar to the proof of Theorem [10.11] there are three possible types of $w_0 \in \Sigma_{P_{\mathfrak{r}}(F)} \cap W(\pi)$:

Type 1. $\tau_i \simeq \tau_j$ for some $1 \leq i < j \leq r$ then the if and only if the Weyl element that inter change the $GL_{n_i}(F)$ and the $GL_{n_j}(F)$ blocks lies in $\Sigma_{P_{\mathfrak{r}}(F)} \cap W(\pi)$.

Type 2. $\tau_i \simeq \hat{\tau}_j$ for some $1 \leq i < j \leq r$ if and only if the Weyl element that inter change the $GL_{n_i}(F)$ with the "dual" $GL_{n_j}(F)$ blocks lies in $\Sigma_{P_{\mathfrak{r}}(F)} \cap W(\pi)$.

Type 3. $\tau_i$ is self dual for some $1 \leq i \leq r$ if and only if the Weyl element that inter change the $GL_{n_i}(F)$ with its "dual" block lies in $\Sigma_{P_{\mathfrak{r}}(F)} \cap W(\pi)$.

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In fact, by switching from $\pi$ to $\pi^w$ for some $w \in W_{\mathcal{P}}(\mathbb{F})$, we may assume that there are no elements in $\Sigma_{\mathcal{P}}(\mathbb{F}) \cap W(\pi)$ of type 2. Indeed, let $I \subseteq \{1, 2, \ldots, r\}$ such that

$$\{1, 2, \ldots, r\} = \bigcup_{i \in I} A_i,$$

where $A_i$ are the equivalence classes

$$A_i = \{1 \leq j \leq r \mid \tau_i \simeq \tau_j \text{ or } \tau_i \simeq \tilde{\tau}_j\}.$$

By choosing $w \in W_{\mathcal{P}}(\mathbb{F})$ properly we may assume that $\tau_i \simeq \tau_j$ for all $j \in A_i$. Thus, we only prove that (10.12) vanishes at $\mathbf{w}^t = 0$ for any $w_0 \in \Sigma_{\mathcal{P}}(\mathbb{F}) \cap W(\pi)$ of type 1 or 3.

Assume that $\tau_i \simeq \tau_j$. Let $w_0 \in \Sigma_{\mathcal{P}}(\mathbb{F}) \cap W(\pi)$ be the corresponding Weyl element. We decompose (10.12) into a product of local coefficients corresponding to simple reflections which may be shown to be equal to local coefficients of the form

$$C^{GL_{np+nq}(\mathbb{F})}_\psi (P^0_{np,nq}(\mathbb{F}), (s_p, s_q), \tau_p \otimes \tau_q, \psi_{q,p}) C^{GL_{nq+n_p}(\mathbb{F})}_\psi (P^0_{nq,n_p}(\mathbb{F}), (s_q, s_p), \tau_q \otimes \tau_p, \psi_{p,q}).$$

(10.13)

All these factors are analytic at $(0, 0)$; see Theorem 5.3.5.2 of [60]. One of these factors corresponds to $(p, q) = (i, j)$. Since by assumption $\tau_i \simeq \tau_j$, the well-known reducibility theorems for parabolically induced representation of $GL_n(\mathbb{F})$ (see the first remark on page 1119 of [18], for example) implies that the factor that corresponds to $p = i, q = j$ vanishes at $(0, 0)$. This shows that (10.12) vanishes at $\mathbf{w}^t = 0$ for any $w_0 \in \Sigma_{\mathcal{P}}(\mathbb{F}) \cap W(\pi)$ of type 1.

Assume that $\tau_i$ is self dual. Let $w_0 \in \Sigma_{\mathcal{P}}(\mathbb{F}) \cap W(\pi)$ be the corresponding Weyl element. We decompose (10.12) into a product which consist of elements of the form (10.13) and of factor of the form (10.10). All the factors of the form (10.13) are analytic and $(0, 0)$. Since $\tau_i$ is self dual, by our assumption the other factor vanishes at $s = 0$. This shows that (10.12) vanishes at $\mathbf{w}^t = 0$ for any $w_0 \in \Sigma_{\mathcal{P}}(\mathbb{F}) \cap W(\pi)$ of type 3.

**Corollary 10.1.** We keep the notations and assumptions of Theorem 10.2. $I(\pi)$ is reducible if and only if there exists $1 \leq i \leq r$ such that $\tau_i$ is self dual and

$$\gamma(\mathbf{w} \times \tau_i, 0, \psi) = \gamma(\tau_i, \text{sym}^2, 0, \psi) \neq 0$$

(10.14)

**Proof.** Let $\tau$ be an irreducible admissible generic representation of $GL_n(\mathbb{F})$. From the definition of $\gamma(\mathbf{w} \times \tau, s, \psi)$, (6.8), and from Theorem 9.3 it follows that

$$C^{SP_{2n}(\mathbb{F})}_\psi (P_{m,k}(\mathbb{F}), s, \tau \otimes \mathbf{w}, j_{m,n}(\omega_{m}^{-1})) = c(s) \gamma(\mathbf{w} \times \tau, s, \psi) \frac{\gamma(\tau, \text{sym}^2, 2s, \psi)}{\gamma(\tau, s + \frac{1}{2}, \psi)}$$

for some exponential factor $c(s)$. By (6.1.4) in page 108 of [50] we have

$$\gamma(\mathbf{w}, 1 - s, \psi) \gamma(\tau, s, \psi) = \tau(-I_m) \in \{\pm 1\}.\tag{10.15}$$

Therefore, if we assume in addition that $\tau$ is self dual we know that $\gamma(\tau, \frac{1}{2}, \psi) \in \{\pm 1\}$. This implies that (10.9) may be replaced with (10.14).

The following two corollaries follow immediately from Theorem 10.2.
Corollary 10.2. With the notations and assumptions of Theorem 10.1, \( I(\pi) \) is irreducible if and only if \( I(\tau_i, \sigma) \) is irreducible for every \( 1 \leq i \leq r \).

Corollary 10.3. Let \( \vec{r} = (n_1, n_2, \ldots, n_r; 0) \) where \( n_1, n_2, \ldots, n_r \) are \( r \) non-negative integers whose sum is \( n \). For \( 1 \leq i \leq r \) let \( \tau_i \) be an irreducible admissible supercuspidal unitary representation of \( GL_{n_i}(\mathbb{F}) \). Denote \( \pi = \otimes_{i=1}^r (\tau^{-1}_i \otimes \tau_i) \). \( I(\pi) \) is reducible if and only if there exists \( 1 \leq i \leq r \) such that \( \tau_i \) is self dual and \( \gamma(\tau_i, sym^2, 0) \neq 0 \).

10.3 A comparison with \( SO_{2n+1}(\mathbb{F}) \)

Let \( SO_{2n+1}(\mathbb{F}) \) be the special orthogonal group:

\[
SO_{2n+1}(\mathbb{F}) = \{ g \in GL_{2n+1}(\mathbb{F}) \mid gJ_n'g^* = J_n', \ det(g) = 1 \},
\]

where \( J_n' = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \). Denote by \( B_{SO_{2n+1}}(\mathbb{F}) \), \( N_{SO_{2n+1}}(\mathbb{F}) \) the standard Borel subgroup and its unipotent radical respectively (see page 2 of [62] for example). Let \( \psi \) be a non-trivial character of \( \mathbb{F} \). We continue to denote by \( \psi \) the character of \( N_{SO_{2n+1}}(\mathbb{F}) \) defined by

\[
\psi(u) = \psi\left( \sum_{k=1}^{n} u_{k,k+1} \right).
\]

We also view \( \psi \) a character of any subgroup of \( N_{SO_{2n+1}}(\mathbb{F}) \). Let \( P_{SO_{2n+1}}(\mathbb{F}) \) be the standard parabolic subgroup of \( SO_{2n+1}(\mathbb{F}) \) whose Levi part and unipotent radical are

\[
M_{SO_{2n+1}}(\mathbb{F}) = \left\{ \begin{pmatrix} g & \ast \\ 1 & g^* \end{pmatrix} \mid g \in GL_n(\mathbb{F}) \right\} \cong GL_n(\mathbb{F}),
\]

\[
U_{SO_{2n+1}}(\mathbb{F}) = \left\{ \begin{pmatrix} I_n & x & z \\ 1 & x' & \ast \\ \ast & \ast & I_n \end{pmatrix} \in SO_{2n+1}(\mathbb{F}) \right\},
\]

where \( g^* = J_n'g^{-1}, x' = -xJ_n' \). Define

\[
\omega_n' = \begin{pmatrix} \omega_n \\ (-1)^n \omega_n \end{pmatrix} \in SO_{2n+1},
\]

and let \( \tau \) be a generic representation of \( GL_n(\mathbb{F}) \) identified with \( M_{SO_{2n+1}}(\mathbb{F}) \). The local coefficient

\[
C^\psi_{SO_{2n+1}}(P_{SO_{2n+1}}(\mathbb{F}), s, \tau, \omega_n'^{-1})
\]

is defined in the same way as in Chapter 6 via Shahidi’s general construction; see Theorem 3.1 of [45]. From the second part of Theorem 3.5 of [52] it follows there exists \( c \in \mathbb{C}^* \) such that

\[
C^\psi_{SO_{2n+1}}(P_{SO_{2n+1}}(\mathbb{F}), s, \tau, \omega_n'^{-1}) = c_\gamma(\tau, sym^2, 2s, \psi).
\]

Furthermore, if \( \tau \) is unramified then \( c = 1 \). In Theorem 9.3 we have proven that

\[
C^{Sp_{2m}(\mathbb{F})}(\overline{P}_{m;0}(\mathbb{F}), s, \tau, \omega_m'^{-1}) = c_\gamma(s) \frac{\gamma(\tau, sym^2, 2s, \psi)}{\gamma(\tau, s + \frac{1}{2}, \psi)}.
\]
where \(c_T(s)\) is an exponential factor which equals 1 if \(\mathbb{F}\) is a p-adic field of odd residual characteristic, \(\psi\) is normalized and \(\tau\) is unramified. Recalling (10.15) we have proved the following.

**Lemma 10.2.** Let \(\tau\) be an irreducible admissible generic representation of \(GL_n(\mathbb{F})\). There exists an exponential function \(c(s)\) such that

\[
C_{\psi}^{SO_{2n+1}(\mathbb{F})}(P_{SO_{2n+1}(\mathbb{F})}, s, \tau, \omega_n^{-1}) = c(s)C_{\psi}^{Sp_{2n}(\mathbb{F})}(P_{n,0}(\mathbb{F}), s, \tau, \omega_n^{-1})
\]

(10.16)

where \(c(s) = 1\) provided that \(\mathbb{F}\) is a p-adic field of odd residual characteristic, \(\psi\) is normalized and \(\tau\) is unramified.

**Theorem 10.3.** Let \(\tau\) be an irreducible admissible self dual supercuspidal representation of \(GL_n(\mathbb{F})\). Then,

\[
I(\tau) = Ind_{P_{n,0}(\mathbb{F})}^{Sp_{2n}(\mathbb{F})}((\gamma^-_n \circ \text{det}) \otimes \tau)
\]

is irreducible if and only if

\[
I'(\tau) = Ind_{P_{SO_{2n+1}(\mathbb{F})}}^{SO_{2n+1}(\mathbb{F})}\tau
\]

is irreducible.

**Proof.** In both cases we are dealing with a representation induced from a singular representation of a maximal parabolic subgroup. Therefore, applying Theorem 4.3 and (6.7) to these representations, the theorem follows from Lemma 10.2. \(\square\)

**Remarks:**

1. One can replace the assumption that \(\tau\) is self dual and replace it with the assumption that \(\tau\) is unitary, since by Theorem 4.9 the commuting algebras of these representations are one dimensional if \(\tau\) is not self dual.

2. Theorem 10.3 may be proved without a direct use of Lemma 10.2. One just has to recall Corollary 10.3 and the well known fact that \(I'(\tau)\) is irreducible if and only if \(\gamma(\tau, \text{sym}^2, 0) \neq 0\); see \([55]\). However, the last fact follows also from the Knapp-Stein dimension theory and from the theory of local coefficients. In fact, Lemma 10.2 gives more information than Theorem 10.3. This Lemma implies that \(\beta(s, \tau, \omega_n^{-1})\) has the same analytic properties as the Plancherel measure attached to \(SO_{2n+1}(\mathbb{F}), P_{SO_{2n+1}(\mathbb{F})}\) and \(\tau\).

3. In general, we expect the same connection between the the parabolic inductions

\[
Ind_{Q'(\mathbb{F})}^{Sp_{2n}(\mathbb{F})}((\tau \otimes \gamma^-_n \circ \text{det}) \otimes \sigma) \quad \text{and} \quad Ind_{Q'(\mathbb{F})}^{SO_{2n+1}(\mathbb{F})}((\tau \otimes \theta_{\psi}(\sigma))
\]

where \(\mathbb{F}\) is a p-adic field, \(Q(\mathbb{F})\) is the standard parabolic subgroup of \(Sp_{2n}(\mathbb{F})\) which has \(GL_m(\mathbb{F}) \times Sp_{2k}(\mathbb{F})\) as its Levi part, \(Q'(\mathbb{F})\) is the standard parabolic subgroup of \(SO_{2n+1}(\mathbb{F})\) which has \(GL_m(\mathbb{F}) \times SO_{2k+1}(\mathbb{F})\) as its Levi part \((r + m = n)\), \(\tau\) is an irreducible supercuspidal generic representation of \(GL_m(\mathbb{F})\) and \(\sigma\) is an irreducible genuine supercuspidal generic representation of \(Sp_{2k}(\mathbb{F})\). Here \(\theta_{\psi}(\sigma)\) is the generic representation of \(SO_{2k+1}(\mathbb{F})\) obtained from \(\sigma\) by the local theta correspondence. See \([27]\) and \([12]\) for more details on the theta correspondence between generic representations of \(Sp_{2k}(\mathbb{F})\) and \(SO_{2k+1}(\mathbb{F})\). In a recent work of Hanzer and Muic, a progress in this direction was made; see \([20]\).
Corollary 10.4. Let $\tau$ be an irreducible admissible self dual supercuspidal representation of $GL_m(\mathbb{F})$. Let $\sigma$ be a generic genuine irreducible admissible supercuspidal representation of $Sp_{2k}(\mathbb{F})$. If $I(\tau)$ is irreducible then $I(\tau, \sigma)$ is irreducible if and only if $\gamma(\sigma \times \tau, 0, \psi) = 0$.

Proof. Recalling Theorem 10.2, we only have to show that $\gamma(\sigma \times \tau, 0, \psi) = 0$ if and only if
\[
C_{Sp_{2m}}(\mathbb{F})(P_{m,0}(\mathbb{F}), s, \tau, \omega_{m}^{j-1}) \\psi
\]
is analytic and non-zero in $s = 0$. The analyticity of this local coefficient at $s = 0$ follows since by (10.16),
\[
C_{Sp_{2m}}(\mathbb{F})(P_{m,0}(\mathbb{F}), s, \tau, \omega_{m}^{j-1}) C_{SO_{2m+1}}(\mathbb{F})(P_{SO_{2m+1}}(\mathbb{F}), -s, \tau, \omega_{m}^{j-1})
\]
has the same analytic properties as
\[
C_{SO_{2m+1}}(\mathbb{F})(P_{SO_{2m+1}}(\mathbb{F}), s, \tau, \omega_{m}^{j-1}) C_{SO_{2m+1}}(\mathbb{F})(P_{SO_{2m+1}}(\mathbb{F}), -s, \tau, \omega_{m}^{j-1})
\]
which is known to be analytic in $s = 0$; see Theorem 5.3.5.2 of [60] (note that that last assertion does not rely on the fact that $\tau$ is self dual). The fact that
\[
C_{Sp_{2m}}(\mathbb{F})(P_{m,0}(\mathbb{F}), s, \tau, \omega_{m}^{j-1}) \neq 0
\]
follows from Theorem 4.3 and the assumption that $I(\tau)$ is irreducible. \qed

The corollaries below follow from [55]:

Corollary 10.5. Let $\tau$ be as in Theorem 10.3. Assume that $n \geq 2$. Then $I(\tau)$ is irreducible if and only if
\[
I''(\tau) = Ind_{P_{n,0}(\mathbb{F})}^{Sp_{2n}(\mathbb{F})} \tau
\]
is reducible.

Proof. Theorem 1.2 of [55] states that $I''(\tau)$ is irreducible if and only if $I'(\tau)$ is reducible. \qed

Corollary 10.6. Let $\tau$ be as in Theorem 10.3. If $n$ is odd then $I(\tau)$ is irreducible.

Proof. Corollary 9.2 of [55] states that under the conditions in discussion $I''(\tau)$ is reducible. \qed
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