A simple third order compact finite element method for 1D BVP

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Abstract

A simple third order compact finite element method is proposed for one-dimensional Sturm-Liouville boundary value problems. The key idea is based on the interpolation error estimate, which can be related to the source term. Thus, a simple posterior error analysis or a modified basis functions based on original piecewise linear basis function will lead to a third order accurate solution in the $L^2$ norm, and second order in the $H^1$ or the energy norm. Numerical examples have confirmed our analysis.

keywords: compact scheme, finite element method, high order FEM, posterior analysis.

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1 Introduction

In this paper, we discuss a new third order finite element method for the following Sturm-Liouville boundary value problem

$$-\nabla \cdot \left( \beta(x) \nabla u(x) \right) + q(x) u(x) = f(x), \quad x_l < x < x_r,$$

(1)

with linear boundary conditions at $x = a$ and $x = b$, such as Dirichlet, Neumann, and Robin condition. To guarantee the well-posedness of the problem, we assume that

$$\beta(x) \in L^\infty(x_l, x_r), \quad \beta(x) \geq \beta_0 > 0, \quad q(x) \in L^\infty(x_l, x_r), \quad q(x) \geq 0, \quad f(x) \in L^2(x_l, x_r),$$

(2)

where $\beta_0$ is a constant. With one of Dirichlet boundary condition, that is, the solution is specified at $x = x_r$ or $x = x_r$, then the problem has a unique solution $u(x) \in H^2(x_l, x_r)$ from the Lax-Milgram lemma, see for example, [1,4].

It is often difficult if not possible to obtain the analytic solution to the problem. In many situations, it is easier to get a numerical solution using modern computers. Basic finite difference (FD) or finite element (FE) methods have been well developed and understood. In a FD or FE method, a mesh is constructed with a mesh size parameter $h$ so that an approximation to the true solution can be approximated. Commonly used ones are second order accurate meaning that the errors of the computed solution in approximating the true is of $O(h^2)$ in some norms, see for example, [2,3].

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Naturally, we hope to have better than second order methods, like third order \(O(h^3)\) or higher without too much additional cost and effort. There are many high order compact finite difference methods in the literature but almost none high order compact finite element methods. In this paper, we propose a third order accurate finite element method based on a novel idea which is dependent on existing \(P_1\) (using piecewise linear basis functions) finite element method. The new idea is based on the interpolation theory to add a local compact support to the basis function so that the high order term in the error is of \(O(h^3)\).

2 The classical FEM using \(P_1\) finite element

In the classical finite element method using a piecewise linear function to approximate the solution to (1), a mesh is generated

\[ x_l = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_{n-1} < x_n = x_r, \]  

where \(n\) is a parameter and the mesh parameter is defined as \(h = \max_{1 \leq i \leq n} |x_{i+1} - x_i|\). Note that \(h \sim O(1/n)\).

In the finite element method, the weak form is below

\[
a(u, v) = L(v), \quad \forall v(x) \in H^1_0(x_l, x_r),
\]

\[
a(u, v) = \int_{x_l}^{x_r} \left( \beta(x)u'(x)v'(x) + q(x)u(x)v(x) \right) dx + R_1(u, v),
\]

\[
L(v) = \int_{x_l}^{x_r} fvdx + R_2(v),
\]

where \(R_1\) and \(R_2\) are contributions from boundary conditions, \(a(u, v)\) is a bilinear form, and \(L(v)\) is a linear form. For the homogenous Dirichlet boundary conditions \(u(x_l) = u(x_r) = 0\), \(R_1 = R_2 = 0\).

Once we have a mesh, we construct a set of basis functions based on the mesh, such as the piecewise linear functions \((i = 1, 2, \cdots, n - 1)\)

\[
\phi_i(x) = \begin{cases} 
\frac{x - x_{i-1}}{h} & \text{if } x_{i-1} \leq x < x_i, \\
\frac{x_{i+1} - x}{h} & \text{if } x_i \leq x < x_{i+1}, \\
0 & \text{otherwise}, 
\end{cases}
\]

often called the hat functions, see the right diagram for a hat function.

In the literature, we denote all the piecewise linear functions over the mesh as a finite element space \(V_h\),

\[
V_h = \left\{ v_h(x), \quad v_h(x) = \sum_{j=1}^M c_j \phi_j(x), \quad c_j \in \mathcal{R}, \quad j = 1, 2, \cdots, M \right\},
\]

where \(M\) is \(n - 1\), or \(n\), or \(n + 1\), depending on the boundary conditions. The finite element
solution then is a special combination of the basis functions
\[ u_h(x) = \sum_{j=1}^{M} c_j^* \phi_j(x), \] (6)
that minimizes the energy error in the space \( V_h \). The coefficients \( c_j^* \)'s are determined from the linear system of equations \( AU = F \) with
\[ A = \{a_{ij}\}, \quad a_{ij} = a(\phi_i(x), \phi_j(x)), \quad F = \{F_i\}, \quad F_i = L(\phi_i(x)), \] (7)
where \( U = [c_1, c_2, \ldots, c_M]^T \).

It is well know that the finite element solution converges the true solution as \( h \to 0 \) according to the following error estimates
\[ \|u - u_h\|_{L^2} \leq Ch^2, \quad \|u - u_h\|_{H^1} \leq Ch, \quad \|u - u_h\|_e \leq Ch, \] (8)
where \( C \) is a generic error constant. Thus the finite element method is second order accurate in the \( L^2 \) norm and first order accurate in the \( H^1 \) norm.

3 A new third order method based on a posterior error analysis

Now we consider the special case that \( \beta(x) = \beta \) a constant and \( q(x) = 0 \). In this case, the ODE is simply \( u''(x) = -f(x)/\beta \). It is well-known that the finite element solution is the exact at nodal points for the special case. Thus the interpolation function is the same as the finite element solution,
\[ u_I^h(x) = \sum_{j=1}^{M} u(x_j)\phi_j(x) = \sum_{j=1}^{M} c_j^* \phi_j(x) = u_h(x), \] (9)
From the classical interpolation theory, see for example, [], we know that on an element \( e_k = [x_k, x_{k+1}] \), the following is true
\[ u(x) = u_I^h(x) + \frac{1}{2}(x - x_k)(x - x_{k+1})u''(x) + O(h^3) \]
\[ = u_h(x) - \frac{1}{2}(x - x_k)(x - x_{k+1})\frac{f(x)}{\beta} + O(h^3). \] (10)
Thus we obtain a third method using a posterior error estimate with a simple correction term that can be easily calculated.

**Theorem 1** Let \( u(x) \in H^2(x_l, x_r), f(x) \in L^2(x_l, x_r), \) and \( u_h(x) \) be the finite element solution obtained using the \( P_1 \) finite element space, then
\[ u_h^{new}(x) = u_h(x) - \frac{1}{2\beta}(x - x_k)(x - x_{k+1})f(x), \quad x_k < x < x_{k+1}, \] (11)
is a third order approximation to the true solution \( u(x) \) and we have the following error estimates
\[ \|u - u_h^{new}\|_{L^2} \leq Ch^3, \quad \|u - u_h^{new}\|_{H^1} \leq Ch^2, \quad \|u - u_h^{new}\|_e \leq Ch^2. \] (12)
Note also that \( u_h^{new}(x_k) = u(x_k) \) for all \( k \)'s. That is, the solution has one more order accuracy.
4 A new third order compact FE method for variable coefficients

The above third order method based on a posterior error analysis is valid only if \( u_h(x_k) = u(x_k) \) for all \( k \)'s, which is not be true in general if \( \beta(x) \) is not a constant. Note that, the third second order method by adding auxiliary points is not compact. In this section, we propose a new compact finite element method based on the interpolation function to construct a bubble function with compact support.

The main idea is to use the interpolation theory and

\[
    u''(x) = -\frac{\beta'(x)}{\beta(x)} u'(x) + \frac{q(x)}{\beta(x)} u(x) - \frac{f(x)}{\beta(x)},
\]

(13)
to get a third order approximation to \( u(x) \) and then use new estimate to construct the basis functions and the FEM solution. We know that

\[
    u(x) = u'_h(x) + \frac{1}{2} u''(x)(x - x_k)(x - x_{k+1}) + O(h^3)
\]

(14)

Note that the added terms has compact support.

The new modified finite element solution is defined as

\[
    u^n_h(x) = u_h(x) - \frac{1}{2} \beta'(x) (x - x_k)(x - x_{k+1}) u'_h(x) + \frac{1}{2} q(x) \beta'(x) (x - x_k)(x - x_{k+1}) u'_h(x)
\]

(15)

\[
    + \frac{1}{2} f(x) \beta'(x) (x - x_k)(x - x_{k+1}) u_h(x) - \frac{1}{2} f(x) \beta'(x) (x - x_k)(x - x_{k+1}),
\]

where \( u_h(x) \) is defined as before \( u_h(x) = \sum_{j=1}^{M} c_j \phi_j(x) \). We use the weak form again to determine the linear coefficient \( c^*_j \).

**Theorem 2** Let \( u(x) \in H^3(x_l, x_r) \), \( f(x) \in H^1(x_l, x_r) \), \( \beta(x) \in C^1(x_l, x_r) \cap H^2(x_l, x_r) \), \( q(x) \in C(x_l, x_r) \cap H^1(x_l, x_r) \), and \( u^n_h(x) \) be the finite element solution obtained using the above formulation, then \( u^n_h(x) \) is a third order approximation to the true solution \( u(x) \) with the following error estimates

\[
    \|u - u^n_h\|_{L^2} \leq C h^3, \quad \|u - u^n_h\|_{H^1} \leq C h^2, \quad \|u - u^n_h\|_e \leq C h^2.
\]

(15)

**Proof:** First, we define the new basis function in the interval \((x_k, x_{k+1})\) as

\[
    \phi^n_i(x) = \phi_i(x) - \frac{1}{2} \beta'(x) (x - x_k)(x - x_{k+1}) \phi'_i(x)
\]

(16)

\[
    + \frac{1}{2} q(x) \beta'(x) (x - x_k)(x - x_{k+1}) \phi_i(x) - \frac{1}{2} f(x) \beta'(x) (x - x_k)(x - x_{k+1}),
\]
where $\phi_i(x)$ is an original hat function. Note that the new basis function depends on the $\beta(x)$, $q(x)$ and $f(x)$. We enlarge the piecewise linear space over the mesh to the new space below

$$V_h^{\text{new}} = \left\{ v_h^{\text{new}}(x), \quad v_h^{\text{new}}(x) = \sum_{j=1}^{M} c_j \phi_j^{\text{new}}(x), \quad c_j \in \mathcal{R}, \quad j = 1, 2, \cdots, M \right\}. \quad (17)$$

Thus the degree of the freedom of $V_h^{\text{new}}$ is the same as that of $V_h$. If we take a special element in $V_h^{\text{new}}$ in $(x_k, x_{k+1})$ as

$$v_h^{\text{new}}(x) = u_I^h(x) - \frac{1}{2} \frac{\beta'(x)}{\beta(x)} (x - x_k)(x - x_{k+1}) \frac{d}{dx} u_I^h(x)$$

$$+ \frac{1}{2} \frac{q(x)}{\beta(x)} (x - x_k)(x - x_{k+1}) u_I^h(x) - \frac{1}{2} \frac{f(x)}{\beta(x)} (x - x_k)(x - x_{k+1}),$$

where $u_I^h(x)$ is the linear interpolation function. Thus we have

$$\|u - v_h^{\text{new}}\|_{L^2} \leq Ch^3, \quad \|u - v_h^{\text{new}}\|_{H^1} \leq Ch^2, \quad \|u - v_h^{\text{new}}\|_{e} \leq Ch^2.$$

Since the finite element solution is the energy norm and the equivalence of the energy norm and the $H^1$ norm, the theorem follows directly.

**Remark 1** The new method and the convergence theorem are also valid when $\beta(x)$ is a constant. So the discussion in Section 3 is included. Nevertheless, the algorithms are different, one is a posterior error estimate while the other is a new finite element method.

5 Numerical experiments

In this section, we show two numerical experiments for the one-dimensional Sturm-Liouville problem. We present the $L^2$ and $H^1$ errors. The order of convergence is estimated using the following formula

$$\text{Order} = \left| \frac{\log(\|E_{N_1}\|_{\infty}/\|E_{N_2}\|_{\infty})}{\log(N_2/N_1)} \right|,$$

with two different $N$'s.

**Example 5.1**

In this example, we show a example for 1D Poisson problem, with the following analytic solution

$$u(x) = \sin(kx). \quad (18)$$

In Table 1 and 2, we show grid refinement results for the problem with $k = 5\pi$ and $k = 50\pi$, respectively. In these tables, the first column is the mesh size, the second column is the infinity error of the computed solution and the computed convergence order on the right, and the third is the error and convergence order of the derivative of the solution, respectively. In all of the cases, we see a clean third order convergence in the solution and second order convergence in the derivative.
Table 1: A grid refinement analysis of the third order compact finite element method for Example 5.1 with $k = 5\pi$.

| $N$ | $\|E\|_{L^2}$ | Order | $\|E\|_{H^1}$ | Order |
|-----|----------------|-------|----------------|-------|
| 8   | 9.4674E-01     |       | 1.5922E+01     |       |
| 16  | 7.6248E-02     | 3.63  | 2.8671E+00     | 2.47  |
| 32  | 1.1364E-02     | 2.75  | 8.0230E-01     | 1.84  |
| 64  | 1.4521E-03     | 2.97  | 2.0772E-01     | 1.95  |
| 128 | 1.8229E-04     | 2.99  | 5.2407E-02     | 1.99  |
| 256 | 2.2810E-05     | 3.00  | 1.3132E-02     | 2.00  |
| 512 | 2.8532E-06     | 3.00  | 3.2849E-03     | 2.00  |
| 1024| 3.7524E-07     | 2.93  | 8.2137E-04     | 2.00  |

Table 2: A grid refinement analysis of the third order compact finite element method for Example 5.1 with $k = 50\pi$.

| $N$  | $\|E\|_{L^2}$       | Order | $\|E\|_{H^1}$ | Order |
|------|----------------------|-------|--------------|-------|
| 64   | 7.7242E+00           |       | 3.8035E+02   |       |
| 128  | 1.2320E-01           | 5.97  | 4.4494E+01   | 3.10  |
| 256  | 2.1751E-02           | 2.50  | 1.2243E+01   | 1.86  |
| 512  | 2.8265E-03           | 2.94  | 3.2238E+00   | 1.93  |
| 1024 | 3.5579E-04           | 2.99  | 8.1745E-01   | 1.98  |
| 2048 | 4.4562E-05           | 3.00  | 2.0510E-01   | 1.99  |
| 4096 | 5.8690E-06           | 2.92  | 5.1322E-02   | 2.00  |
Example 5.2

In this example, we show an example for 1D elliptic problem, with the following analytic solution

\[ u(x) = \sin(k_1 x) \cos(k_2 x). \]  

(19)

The coefficient \( \beta(x) \) and \( q(x) \) are chosen as

\[ \beta(x) = e^x, \quad q(x) = x^2. \]  

(20)

In Table 3 and 6, we show grid refinement results for the problem with different \( k_1 \) and \( k_2 \). In all of the cases, we see a clean third order convergence in the solution and second order convergence in the derivative.

Table 3: A grid refinement analysis of the third order compact finite element method for Example 5.2 with \( k_1 = 5\pi \) and \( k_2 = 0 \).

| \( N \) | \( \|E\|_{L^2} \) | Order | \( \|E\|_{H^1} \) | Order |
|--------|----------------|--------|----------------|--------|
| 8      | 1.5393E+00     |        | 1.7273E+01     |        |
| 16     | 9.5685E-02     | 0.01   | 3.0023E+00     | 2.52   |
| 32     | 1.2644E-02     | 0.92   | 8.0654E-01     | 1.90   |
| 64     | 1.5437E-03     | 0.03   | 2.0990E-01     | 1.95   |
| 128    | 1.8826E-04     | 0.04   | 5.2686E-02     | 1.99   |
| 256    | 2.3235E-05     | 0.02   | 1.3196E-01     | 2.00   |
| 512    | 2.8846E-06     | 0.01   | 3.3004E-03     | 2.00   |
| 1024   | 3.6254E-07     | 0.00   | 8.2518E-04     | 2.00   |

Table 4: A grid refinement analysis of the third order compact finite element method for Example 5.2 with \( k_1 = 50\pi \) and \( k_2 = 0 \).

| \( N \) | \( \|E\|_{L^2} \) | Order | \( \|E\|_{H^1} \) | Order |
|--------|----------------|--------|----------------|--------|
| 64     | 7.9628E+00     |        | 3.8417E+02     |        |
| 128    | 1.2781E-01     | 5.96   | 4.4780E+01     | 3.10   |
| 256    | 2.2066E-02     | 2.53   | 1.2249E+01     | 1.87   |
| 512    | 2.8471E-03     | 2.95   | 3.2242E+00     | 1.93   |
| 1024   | 3.5711E-04     | 3.00   | 8.1755E-01     | 1.98   |
| 2048   | 4.4621E-05     | 3.00   | 2.0512E-01     | 1.99   |
| 4096   | 5.6445E-06     | 2.98   | 5.1325E-02     | 2.00   |

6 Conclusions and acknowledgements

A new third order compact finite element method is proposed in this paper. For constant coefficient, the method is a simple posterior error estimate technique and the third order
Table 5: A grid refinement analysis of the third order compact finite element method for Example 5.2 with $k_1 = 5\pi$ and $k_2 = 5\pi$.

| $N$ | $\|E\|_{L^2}$   | Order | $\|E\|_{H^1}$   | Order |
|-----|-----------------|-------|-----------------|-------|
| 8   | 9.5172E+00      |       | 1.2581E+02      |       |
| 16  | 6.2881E-01      | 3.92  | 1.6649E+01      | 2.92  |
| 32  | 4.3173E-02      | 3.86  | 2.9322E+00      | 2.51  |
| 64  | 6.0059E-03      | 2.85  | 8.0410E-01      | 1.87  |
| 128 | 7.4785E-04      | 3.01  | 2.0788E-01      | 1.95  |
| 256 | 9.2580E-05      | 3.01  | 5.2486E-02      | 1.99  |
| 512 | 1.1502E-05      | 3.01  | 1.3149E-02      | 2.00  |
| 1024| 1.4323E-06      | 3.01  | 3.2889E-03      | 2.00  |

Table 6: A grid refinement analysis of the third order compact finite element method for Example 5.2 with $k_1 = 50\pi$ and $k_2 = 50\pi$.

| $N$   | $\|E\|_{L^2}$   | Order | $\|E\|_{H^1}$   | Order |
|-------|-----------------|-------|-----------------|-------|
| 128   | 3.9235E+00      |       | 3.8229E+02      |       |
| 256   | 6.2757E-02      | 5.97  | 4.4637E+01      | 3.10  |
| 512   | 1.0954E-02      | 2.52  | 1.2245E+01      | 1.87  |
| 1024  | 1.4185E-03      | 2.95  | 3.2238E+00      | 1.93  |
| 2048  | 1.7822E-04      | 2.99  | 8.1748E-01      | 1.98  |
| 4096  | 2.2338E-05      | 3.00  | 2.0511E-01      | 1.99  |
| 8192  | 3.0299E-06      | 2.90  | 5.1324E-02      | 2.06  |
convergence has been proved. For variable coefficients, the new third order compact finite element method has been proposed and confirmed numerically. Rigorous proof of the optimal convergence is also presented. The degree of freedom and the structure of the resulting linear system of equations of the new third order compact finite element method are the same as the traditional finite element method using piecewise linear functions over the mesh.

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