Stochastic Compositional Optimization with Compositional Constraints

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Abstract

Stochastic compositional optimization (SCO) has attracted considerable attention because of its broad applicability to important real-world problems. However, existing works on SCO assume that the projection within a solution update is simple, which fails to hold for problem instances where the constraints are in the form of expectations, such as empirical conditional value-at-risk constraints. We study a novel model that incorporates single-level expected value and two-level compositional constraints into the current SCO framework. Our model can be applied widely to data-driven optimization and risk management, including risk-averse optimization and high-moment portfolio selection, and can handle multiple constraints. We further propose a class of primal-dual algorithms that generates sequences converging to the optimal solution at the rate of $O(\frac{1}{\sqrt{N}})$ under both single-level expected value and two-level compositional constraints, where $N$ is the iteration counter, establishing the benchmarks in expected value constrained SCO.

Keywords: Stochastic gradient · Expectation constraint · Compositional optimization

1 Introduction

Over the past few years, the study of stochastic compositional optimization (SCO) problems has attracted increasing attention. The SCO problems are of the compositional form that

$$\min_{x \in \mathcal{X}} F(x) = \mathbb{E}_{\xi_1}[f_1(\mathbb{E}_{\xi_2}[f_2(x, \xi_2)], \xi_1)],$$

where $\mathcal{X} \subset \mathbb{R}^d_x$ is convex, and $f_2(x, \xi_2)$ and $f_1(y, \xi_1)$ are two functions depending on the random variables $\xi_2$ and $\xi_1$, respectively. The SCO problems have widespread applications in fields such as

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For the first instance, in reinforcement learning, the policy evaluation problem aims to find the value function by using empirically observed samples. Formulating the policy evaluation problem as a Bellman minimization problem, it can be further recast as a two-level SCO [32]. In another instance, the low-rank matrix estimation problem has been employed widely to solve many engineering applications, such as image analysis and recommendation systems. In the scenario where the data arrives sequentially, the online low-rank estimation problem can be reformulated as a two-level SCO [13]. Further, SCO has also been applied to handling Model-Agnostic Meta-Learning (MAML) [10, 11], which is a powerful approach to training new models from prior related tasks. It was shown by [7] that MAML falls into the two-level SCO category.

Due to SCO’s success in handling real-world applications, various algorithms have been developed to solve SCO [31, 32] by iteratively updating the solution using projected stochastic gradient descent (SGD) with an approximated gradient. Despite the rapid development of SCO in both theoretical and computational aspects, existing studies of constrained SCO rely on the assumption that the feasible region $X$ is simple so that the projections onto $X$ are efficient. Unfortunately, this assumption does not hold in many applications where the constraints have complicated forms, such as nonlinear and high-order polynomials, where the projection is computationally expensive. Further, the projection is even more challenging when the constraints are of expected value single-level or compositional functions whose explicit forms are not available. Meanwhile, such constraints are essential in practice when we want to incorporate risk-type constraints such as the conditional value-at-risk (CVaR) and high-moment constraints. However, existing approaches in SCO are not directly applicable to these cases as a result of these complicated constraints.

To overcome this challenge, we aim to solve SCO problems with single-level and two-level compositional expected value constraints in this paper. In particular, we consider the following expected value constrained SCO that

$$\min_{x \in X} F(x) = E_{\xi_1}[f_1(E_{\xi_2}[f_2(x, \xi_2)], \xi_1)],$$

subject to $G^{(j)}(x) \leq 0$, for $j = 1, 2, \cdots, m$. \hfill (1.2)

Here we assume that set $X$ is convex and projection onto it is simple. We assume $F(x)$ is convex, and each $G^{(j)}(x)$ is a convex single-level or two-level compositional function in the form of an expectation whose explicit form is unknown and needs to be estimated using observed samples.

Our model serves as a powerful tool for incorporating the data-driven constraints arising from large-scale datasets or online scenarios [34]. Specifically, consider a scenario where the decision-maker has access to a large-scale dataset consisting of $n$ data points $\{w_i\}_{i=1}^n$ and aims to satisfy the data-driven constraint $\frac{1}{n} \sum_{i=1}^n g(x, w_i) \leq 0$ in a decision-making process. When $n$ is large, it is computationally intractable to evaluate the empirical average $\frac{1}{n} \sum_{i=1}^n g(x, w_i)$ and make projections even if the each $g(x, w_i)$ is simple. Alternatively, we can reformulate it as a single-level expected value constraint $E_w [g(x, w)] \leq 0$ where $w$ is a random sample drawn uniformly from the dataset, which is captured by our model (1.2). Similarly, letting $\{w_{1,i}\}_{i=1}^{n_1}$ and $\{w_{2,j}\}_{j=1}^{n_2}$ be two datasets, and $g_1(z, w_{1,i})$ and $g_2(x, w_{2,j})$ be two functions associated with $w_{1,i}$ and $w_{2,j}$, respectively, we can reformulate the following empirical two-level constraint $\frac{1}{n_1} \sum_{i=1}^{n_1} g_1(\frac{1}{n_2} \sum_{j=1}^{n_2} g_2(x, w_{2,j}), w_{1,i}) \leq 0$ as a two-level compositional expected value constraint $E_{w_1}|g_1(E_{w_2}[g_2(x, w_2)], w_1)| \leq 0$, where $w_1$ and
weights, for any even number $p$ of the distributional properties of portfolios. High-moment optimization has aroused considerable interest because of its efficient characterization.

Letting $d$ be the decision variable and $w$ be its associated random loss with random variable $\zeta$, we denote by $\text{CVaR}_\alpha(x)$ the $\alpha$-quantile of CVaR. To control the risk of loss such that the $\alpha$-quantile conditional mean of $\ell(x, \zeta)$ is no more than $\gamma$, we can impose a constraint that

$$\{ x : \text{CVaR}_\alpha(x) \leq \gamma \} = \left\{ x : \inf_u \left\{ u + \frac{1}{1-\alpha} \mathbb{E}_\zeta \left[ (\ell(x, \zeta) - u)^+ \right] \right\} \leq \gamma \right\},$$

where $u$ is an auxiliary variable. When the distribution of $\zeta$ is unknown, the above constraint does not result in an explicit expression but must be written in the form of an expectation. Further, when the loss function $\ell(x, \xi)$ is convex in $x$, the set $\{ x : \text{CVaR}_\alpha(x) \leq \gamma \}$ is also convex in $x$.

(b) Risk-averse mean-deviation constraint. Risk-averse optimization has attracted significant attention from various communities because of its wide applications in finance and other industries. So far, risk-averse optimization studies have mainly focused on controlling the risk incurred by a single utility function. By contrast, we can incorporate the risk incurred by multiple utility functions into our decision-making process by employing its constrained-optimization counterpart. Letting $x \in \mathbb{R}^d$ be the decision variable and $\ell_j(x, \xi)$ for $j = 1, \cdots, K$ be $K$ heterogeneous utility functions, we consider the following set of first-order risk-averse mean-deviation constraints

$$\{ x : \mathbb{E}_\xi[\ell_j(x, \xi)] - \mathbb{E}_\zeta[\ell_j(x, \xi)] \geq \gamma_j, \text{ for } j = 1, \cdots, K \}. \tag{1.3}$$

The above set can be reformulated as a region consisting of two-level compositional expected value constraints:

$$\{ x : \mathbb{E}_\zeta[g_1^{(j)}(\mathbb{E}_\xi[g_2^{(j)}(x, \xi)], \xi)] \leq -\gamma_j, \text{ for } j = 1, 2, \cdots, K \}, \tag{1.4}$$

where $g_1^{(j)}((z, x, \xi)) = z + (\ell_j(x, \xi) + z)_+$ and $g_2^{(j)}(x, \xi) = (-\ell_j(x, \xi), x)$.

(c) Portfolio optimization with high-moment constraints. Since the early 60s, portfolio managers have been incorporating high-moment considerations into portfolio selection. Letting $w \in \mathbb{R}^d$ be a vector representing the random returns of $d$ assets and letting $x \in \mathbb{R}^d$ be the corresponding portfolio weights, for any even number $p$, the $p$-th order moment of the portfolio’s return is

$$M_p(x) = \mathbb{E}_w[|w^T x - \mathbb{E}_w[w^T x]|^p]. \tag{1.5}$$

High-moment optimization has aroused considerable interest because of its efficient characterization of the distributional properties of portfolios. For instance, the standardized 4-th order moment of a portfolio, called Kurtosis, is commonly adopted to capture the tail distribution of portfolios. Often, a portfolio with high Kurtosis preserves a long tail.
We consider the following portfolio optimization problem with $p$-th order moment constraints:

$$\max \mathbb{E}_w [w^\top x], \quad \text{s.t.} \quad M_p(x) \leq c_p, \quad \text{for} \quad p = 2, 4, 6, 8, \ldots.$$  

(1.6)

Each of the $p$-th order moment constraint can be recast as a two-level compositional expected value constraint:

$$\mathbb{E}_w [g_1(\mathbb{E}[g_2(x, w)], w)] \leq c_p,$$

where $g_1((z, x), w) = |w^\top x - z|^p$ and $g_2(x, w) = (w^\top x, x)$.  

(1.7)

1.2 Related Work

**Stochastic compositional optimization.** SCO was first studied by [9] who studied the asymptotic convergence behavior of a two-timescale algorithm that employs two sequences of step-sizes preserving different time scales, and established its almost sure convergence guarantee. Wang et al. [31] proposed an algorithm SCGD to tackle the two-level SCO and established the first non-asymptotic convergence result. The convergence rate guarantees were improved further by using acceleration techniques [32]. Subsequently, Ghadimi et al. [13] proposed a single-timescale algorithm that achieves an $O(\frac{1}{\sqrt{N}})$ convergence rate for nonconvex two-level SCO, and various methods were applied to tackle SCO for achieving fast convergence rates [7, 37]. The generalization to multi-level SCO was first examined by [35] and the $O(\frac{1}{\sqrt{N}})$ convergence rates for multi-level nonconvex problems were established by [2, 25]. Liu et al. [21] studied the compositional DC regime and extensively investigated a few applications in risk management. The convex regime was investigated by [39] where they established the $O(\frac{1}{\sqrt{N}})$ convergence rates for two-level and multi-level SCO by using Fenchel conjugates.

**Expected value constrained optimization.** Studies of constrained optimization are divided into two streams. On the one hand, constraints can be considered as penalty terms to the targeted optimization problem to avoid costly projections. Various penalization approaches have been studied extensively [4]. Often, these approaches require solving subproblems, which may be computationally expensive and weakens their applicability. On the other hand, tremendous amount of efforts have been made to handle constrained optimization problems directly. To tackle deterministic constraints, Nemirovski et al. [23] proposed a class of gradient-type methods to solve problems with functional constraints. The level-set method, also known as the bundle method, was studied by [18, 19]. In the scenario where the first-order information is absent, Lan et al. [16] developed a class of Frank-Wolfe based algorithms for solving deterministic optimization problems with complicated deterministic constraints. Recently, there has been a growing focus on solving problems with stochastic constraints. Lan and Zhou [17] first proposed a cooperative stochastic approximation (CSA) algorithm that updates the solution based on the estimated feasibility of stochastic constraints. Subsequently, Boob et al. [5] considered the smooth scenario and proposed a primal-dual based constraint extrapolation (ConEx) method that iteratively updates the primal and dual solutions by using accelerated SGD. ConEx generates a sequence that converges to the optimal solution at the rate of $O(\frac{1}{\sqrt{N}})$ and establishes the benchmark for expected value constrained optimization. Expected value constrained problem with minimax objectives was studied by [34], and Yu et al. [36] investigated the online scenario where an arbitrary objective function arrives.
at each online round. More detailed reviews in constrained optimization can be found in [5]. We remark that all above mentioned works focus on the single-level stochastic objective and constraints but cannot handle the compositional setting.

1.3 Contributions

Our work aims to incorporate various complicated expected value constraints into the current SCO framework to handle real-world applications. The contributions are three-fold.

- We identify a new model incorporating single-level and two-level compositional expected value constraints into the current SCO framework. Our model can handle complicated expectation constraints arising from real-world applications with online or large-scale datasets. We provide three applications of our model to risk management.
- We consider SCO with single-level expected value constraints, reformulate it as a saddle point optimization problem, and propose a primal-dual type algorithm that iteratively updates the primal and dual variables. We show that the solution path converges to the optimal solution at the optimal rate of $O(\frac{1}{\sqrt{N}})$.
- We further consider SCO with two-level compositional expected value constraints and propose a modified primal-dual algorithm to tackle it. We show that the solution path enjoys the same $O(\frac{1}{\sqrt{N}})$ rate of convergence to the optimal solution.

Notation. For any vectors $a, b \in \mathbb{R}^n$, we denote by $\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$ their inner product and denote by $\max\{a, b\} = (\max\{a_1, b_1\}, \ldots, \max\{a_n, b_n\})^\top \in \mathbb{R}^n$ their component-wise maximum. We denote by $\|a\| = \|a\|_2$ the Euclidean norm for any vector $a \in \mathbb{R}^n$ and denote by $\|A\| = \|A\|_2$ the Euclidean norm for any matrix $A \in \mathbb{R}^{n \times n'}$. For any non-smooth stochastic function $f(x, \omega)$, we denote by $f(x) = \mathbb{E}[f(x, \omega)]$, denote by $\partial f(x)$ the collection of its deterministic subgradients, and denote by $\partial f(x, \omega)$ its stochastic subgradient such that $\mathbb{E}_\omega[\partial f(x, \omega)] \in \partial f(x)$.

2 Stochastic Compositional Optimization with Single-level Expected Value Constraints

In this section, we consider the single-level expected value constrained stochastic compositional optimization (EC-SCO) problem

$$\min_{x \in \mathcal{X}} \quad F(x) = f_1 \circ f_2(x),$$

subject to $g^{(j)}(x) \leq 0, \text{ for } j = 1, 2, \cdots, m,$

where $F$ is convex, $\mathcal{X} \subset \mathbb{R}^{d_x}$ is a convex set, $f_1(y) = \mathbb{E}_{\xi_1}[f_1(y, \xi_1)] : \mathbb{R}^{d_\xi} \mapsto \mathbb{R}$ is convex and differentiable, $f_2(x) = \mathbb{E}_{\xi_2}[f_2(x, \xi_2)] : \mathbb{R}^{d_x} \mapsto \mathbb{R}^{d_\xi}$ is convex, and $g^{(j)}(x) = \mathbb{E}_{\zeta_j}[g^{(j)}(x, \zeta_j)] : \mathbb{R}^{d_x} \mapsto \mathbb{R}$ for $j = 1, 2, \cdots, m$ are convex functions. To be specific, we write $f_2(x) = (f_2^{(1)}(x), \cdots, f_2^{(d_\xi)}(x))^\top$ and assume each $f_2^{(j)}$ is a convex function. The functions $f_1, f_2,$ and $g$ are taken in the forms of expectation, whose explicit forms are unknown, and can only be estimated from observed samples.
To facilitate our discussion, we briefly review the Fenchel conjugate here. In particular, for any convex function $h(z) : \mathbb{R}^{d_z} \rightarrow \mathbb{R}$, its Fenchel conjugate is $h^*(z) = \max_{u \in \mathbb{R}^{d_z}} \langle u, z \rangle - h(u)$. With a slight abuse of notation, for any function $\overline{h}(z) : \mathbb{R}^{d_z} \mapsto \mathbb{R}^k$, where each component, $\overline{h}_j(z)$ for $j = 1, \ldots, k$, of the function is convex, we let $\overline{h}_j^*(z) = \max_{w \in \mathbb{R}^{d_z}} \langle w, z \rangle - \overline{h}_j(w)$, and let $\overline{h}^*(z) = (\overline{h}_1^*(z), \ldots, \overline{h}_k^*(z))^\top$. That is, we let the Fenchel conjugate of $\overline{h}(z)$ be $\overline{h}^*(z) : \mathbb{R}^{d_z} \rightarrow \mathbb{R}^k$, where the $j$-th component of $\overline{h}^*(z)$, $\overline{h}_j^*(z)$, is the Fenchel conjugate of $\overline{h}_j(z)$, the $j$-th component of $\overline{h}(z)$. For any convex smooth function $f$, we denote by $D_f(x, y) = f(y) - f(x) - \langle \nabla f(x), y - x \rangle$ its associated Bregman’s distance for ease of presentation. Given $x \in \mathbb{R}^{n_2}$, we denote by $g(x) = (g^{(1)}(x), \ldots, g^{(m)}(x))^\top \in \mathbb{R}^m$ the vector of constraint values and denote by $g(x, \zeta) = (g^{(1)}(x, \zeta_1), \ldots, g^{(m)}(x, \zeta_m))^\top \in \mathbb{R}^m$ the vector of the sample constraint values.

We first specify the stochastic sampling environment and assume access to a blackbox sampling oracle that returns stochastic zeroth- and first-order information for the objective functions $f_1, f_2$ and constraint function $g$ upon each query. To be specific, we assume that we have access to the following Sampling Oracle (SO) such that

- Given $x \in \mathcal{X}$, the SO returns a sample value $f_2(x, \xi_2) \in \mathbb{R}^{d_y}$ and a sample subgradient
  \[
  \partial f_2(x, \xi_2) \in \mathbb{R}^{d_z \times d_y}.
  \]

- Given $x \in \mathcal{X}$, the SO returns a sample value $g(x, \zeta) \in \mathbb{R}^m$ and a sample subgradient
  \[
  \partial g(x, \zeta) \in \mathbb{R}^{d_z \times m}.
  \]

- Given $y \in \mathbb{R}^{m_1}$, the SO returns a sample value $f_1(y, \xi_1) \in \mathbb{R}$ and a sample gradient
  \[
  \nabla f_1(y, \xi_1) \in \mathbb{R}^{d_y}.
  \]

Note that in the above SO, we do not impose any smoothness assumption for the inner level function $f_2$ and the constraint function $g$, and we let $\partial f_2(x, \xi_2)$ and $\partial g(x, \zeta)$ be their sample subgradients, respectively. We then introduce the Lagrangian dual function of problem (2.1) and briefly illustrate the challenges of solving the problem. Specifically, the Lagrangian dual function of problem (2.1) is

\[
\mathcal{L}(x, \lambda) = F(x) + \sum_{j=1}^{m} \lambda_j g^{(j)}(x), \quad \text{where } \lambda \geq 0.
\]

By the convexity of $F$ and $g_j$’s, the above Lagrangian $\mathcal{L}(x, y)$ is convex in $x$ and concave in $\lambda$. We assume that there exists a saddle point $(x^*, \lambda^*)$ such that

\[
(x^*, \lambda^*) = \arg\min_{x \in \mathcal{X}} \max_{\lambda \in \mathbb{R}_+^m} \mathcal{L}(x, \lambda),
\]

and we have that for any $(x, \lambda) \in \mathcal{X} \times \mathbb{R}_+^m$, it holds that

\[
\mathcal{L}(x^*, \lambda) \leq \mathcal{L}(x, \lambda) \leq \mathcal{L}(x, \lambda^*). \tag{2.2}
\]

Note that the existence of such a saddle point is guaranteed under the mild Slater condition [27].
With a slight abuse of notation, the first-order partial subgradients of \( \mathcal{L} \) satisfy
\[
\partial_x \mathcal{L}(x, \lambda) = \partial F(x) + \sum_{j=1}^{m} \lambda_j \partial g_j(x) \text{ and } \nabla_{\lambda} \mathcal{L}(x, \lambda) = g(x), \quad \forall x \in \mathcal{X}, \lambda \in \mathbb{R}^m.
\]

We then present the challenges of developing a primal-dual first-order algorithm \([20, 29, 33]\) to find a saddle point of the Lagrangian dual function.

- **Primal update:** At iteration \( t \), we first update the primal variable \( x_t \) by obtaining an estimator of \( \partial \mathcal{L}(x_t, \lambda_t) \). The key challenge to solving the EC-SCO problem lies in the unavailability of an unbiased estimator of the sub-gradient of \( F \). Specifically, by employing the chain rule, an unbiased estimator of \( \partial F(x) \) is
\[
\partial f_2(x, \xi_2) \nabla f_1(\mathbb{E}_{\xi_2}[f_2(x, \xi_2)], \xi_1).
\]

Unfortunately, the absence of knowledge about \( \mathbb{E}_{\xi_2}[f_2(x_t, \xi_2)] \) induces a bias when we use the plug-in sample estimator \( \nabla f_1(f_2(x_t, \xi_2), \xi_1) \) \([30]\). Without an unbiased estimator of \( \partial F(x) \), the primal update is challenging.

- **Dual update:** In general, primal-dual type approaches generate a sequence of dual variables \( \{\lambda_k\} \), which only converges under certain restrictive settings \([5, 22]\). By comparison with the classical single-level stochastic optimization problems, the compositional objective \( F(x) \) is much more complicated, and existing theories cannot be applied to prove the convergence of the dual variables.

To address the above-mentioned obstacles, in the next subsection, we develop an efficient algorithm to tackle problem \((2.1)\). Before presenting this algorithm, we impose some assumptions on the objective functions \( f_1 \) and \( f_2 \).

**Assumption 2.1.** Let \( C_{f_1}, C_{f_2}, \sigma_{f_1}, \sigma_{f_2}, L_{f_1} \), and \( D_X \) be positive constants. We assume that:

1. The set \( \mathcal{X} \) is convex and bounded such that \( \sup_{x \in \mathcal{X}} \|x - x^*\| \leq D_X \).
2. \( f_1, f_2 \) are convex, and have equivalent representations by their Fenchel conjugates \( f_1^*(\pi_1) \) and \( f_2^*(\pi_2) \) that for all \( x \in \mathcal{X}, y \in \mathbb{R}^{d_y} \),
   \[
f_1(y) = \max_{\pi_1 \in \Pi_1} \langle \pi_1, y \rangle - f_1^*(\pi_1), \text{ and } f_2(x) = \max_{\pi_2 \in \Pi_2} \langle \pi_2, x \rangle - f_2^*(\pi_2),
   \]
   where \( \Pi_1 \subset \mathbb{R}^{d_y} \) and \( \Pi_2 \subset \mathbb{R}^{d_x \times d_y} \) are the domains of \( f_1^* \) and \( f_2^* \), respectively.
3. \( f_1 \) is an \( L_{f_1} \)-smooth function with a Lipschitz continuous gradient such that
   \[
   \|\nabla f_1(y_1) - \nabla f_1(y_2)\| \leq L_{f_1} \|y_1 - y_2\|, \forall y_1, y_2 \in \mathbb{R}^{d_y}.
   \]
4. The sample value \( f_2(x, \xi_2) \) returned by the \( SO \) is unbiased and has a bounded second moment such that
   \[
   \mathbb{E}_{\xi_2}[f_2(x, \xi_2)] = f_2(x), \text{ and } \mathbb{E}_{\xi_2}[\|f_2(x, \xi_2) - f_2(x)\|^2] = \sigma_{f_2}^2, \forall x \in \mathcal{X}.
   \]
5. The sample value $f_1(y, \xi_1)$ returned by the $SO$ is unbiased and has a bounded second moment such that
\[
\mathbb{E}_{\xi_1}[f_1(y, \xi_1)] = f_1(y), \text{ and } \mathbb{E}_{\xi_1}[\|f_1(y, \xi_1) - f_1(y)\|^2] = \sigma_{f_1}^2, \forall y \in \mathbb{R}^{d_y}.
\]

6. The sample subgradient $\partial f_2(x, \xi_2)$ returned by the $SO$ is unbiased and has a bounded second moment such that
\[
\mathbb{E}_{\xi_2}[\partial f_2(x, \xi_2)] \in \partial f_2(x), \text{ and } \mathbb{E}_{\xi_2}[\|\partial f_2(x, \xi_2)\|^2] \leq C_{f_2}^2, \forall x \in \mathcal{X}.
\]

7. The sample gradient $\nabla f_1(y, \xi_1)$ returned by the $SO$ is unbiased and has a bounded second moment such that
\[
\mathbb{E}_{\xi_1}[\nabla f_1(y, \xi_1)] = \nabla f_1(y), \text{ and } \mathbb{E}_{\xi_1}[\|\nabla f_1(y, \xi_1)\|^2] \leq C_{f_1}^2, \forall y \in \mathbb{R}^{d_y}.
\]

As stated in (b) above, when $f_1$ and $f_2$ are convex, we have their equivalent representations through their Fenchel conjugates. Also, we recall that for any convex but possibly non-smooth function $f$, for a dual variable $\pi_y$ associated with a primal variable $y$ such that $\pi_y \in \arg\max_{\pi \in \Pi_y} \langle \pi, y \rangle - f^*(\pi)$, we have that $\pi_y$ is a subgradient of $f(y)$. In particular, we have
\[
\pi_y \in \arg\max_{\pi \in \Pi_y} \langle \pi, y \rangle - f^*(\pi) \iff \pi_y \in \partial f(y) \iff f(y) = \langle \pi_y, y \rangle - f^*(\pi_y). \quad (2.4)
\]

Further, we note that for any $L_f$-smooth convex function $f$, the Bregman distance function generated by its Fenchel conjugate function is $\frac{1}{L_f}$-strongly convex (see, for example, Theorem 5.26 in [3]) that
\[
\mathcal{D}_{f^*}(\bar{\pi}; \pi) = f^*(\bar{\pi}) - f^*(\pi) - \langle \nabla f^*(\pi), \bar{\pi} - \pi \rangle \geq \frac{1}{2L_f} \|\pi - \bar{\pi}\|^2, \quad \forall \pi, \bar{\pi} \in \text{dom}(f^*). \quad (2.5)
\]

Thus the $L_{f_1}$-smoothness assumption of $f_1$ implies that $\mathcal{D}_{f_1^*}$ is $\frac{1}{L_{f_1}}$-strongly convex.

Then, we make the following assumptions for the constraint functions $g^{(j)}$'s.

**Assumption 2.2.** Let $C_g$ and $\sigma_g$ be positive constants. The constraint functions $g^{(j)}$'s satisfy:

1. For $j = 1, \cdots, m$, each component $g^{(j)}$ is convex such that it admits an equivalent representation using its Fenchel conjugate that
\[
g^{(j)}(x) = \max_{\theta^{(j)} \in V^{(j)}} \langle x, \theta^{(j)} \rangle - [g^{(j)}(v)]^*, \quad \forall x \in \mathcal{X},
\]

where $V^{(j)} \subset \mathbb{R}^{n_2}$ is the domain of $[g^{(j)}]^*$.

2. The sample value $g(x, \zeta)$ returned by the $SO$ is unbiased and has a bounded second moment such that
\[
\mathbb{E}_\zeta[g(x, \zeta)] = g(x), \text{ and } \mathbb{E}_\zeta[\|g(x, \zeta) - g(x)\|^2] \leq \sigma_g^2, \quad \forall x \in \mathcal{X}.
\]

3. The sample sub-gradient $\partial g(x, \zeta)$ returned by the $SO$ is unbiased and has a bounded second moment such that
\[
\mathbb{E}_\zeta[\partial g(x, \zeta)] \in \partial g(x), \text{ and } \mathbb{E}_\zeta[\|\partial g(x, \zeta)\|^2] \leq C_g^2, \quad \forall x \in \mathcal{X}.
\]
In this paper, we focus on two scenarios: (i) the inner function $f_2$ is affine; (ii) the inner function $f_2$ is non-affine and the outer function is monotone non-decreasing, which is specified in the next assumption.

**Assumption 2.3.** For a non-affine function $f_2$, $\nabla f_1(x)$ is always component-wise nonnegative, i.e., $\nabla f_1(y) \geq 0$ for any $y \in \mathbb{R}^{d_y}$. That is, the outer function $f_1$ is monotone non-decreasing if $f_2$ is non-affine.

For ease of notation, in what follows, we omit the subscripts $\xi_1, \xi_2$, and $\zeta$ within $\mathbb{E}_{\xi_1}[\bullet], \mathbb{E}_{\xi_2}[\bullet]$, and $\mathbb{E}_\zeta[\bullet]$.

### 2.1 A Primal-Dual Algorithm

In this subsection, we propose our algorithm to tackle problem (2.1). Our algorithm runs iteratively. At each iteration, we alternately update the primal variable $x$ and dual variable $\lambda$. At the $t$-th iteration, we first query the $SO$ twice at the previous solution $x_t$ to obtain a sample of $f_2(x_t, \xi_{2,t}^1)$ and an independent sample of subgradient $\partial g(x_t, \xi_{2,t}^0)$. Then, given the previous estimator $y_t$ for $\mathbb{E}[f_2(x_t, \xi_2)]$ from the previous iteration, we update our estimator. In particular, we take an weighted average of $y_t$ and $f_2(x_t, \xi_{2,t}^1)$ such that for some $\tau_t \in (0, 1)$, we let

$$y_{t+1} = \frac{f_2(x_t, \xi_{2,t}^1) + \tau_t y_t}{1 + \tau_t}. \quad (2.6)$$

Next, given $y_{t+1}$, we query the $SO$ at $y_{t+1}$ to obtain a sample gradient $\nabla f_1(y_{t+1}, \xi_{1,t}^0)$. In addition, we query the $SO$ at $x_t$, where $x_t$ is the solution from the previous iteration, for a sample subgradient $\partial g(x_t, \zeta^0)$. Then, we update the primal solution $x_t$ by a projected stochastic gradient step such that

$$x_{t+1} = \arg\min_{x \in \mathcal{X}} \left\{ \langle \partial f_2(x_t, \xi_{2,t}^1) \nabla f_1(y_{t+1}, \xi_{1,t}^0) + \partial g(x, \zeta^0), x \rangle + \frac{\eta_t}{2} \|x - x_t\|^2 \right\}$$

$$= \text{Proj}_X \left[ x_t - \left\{ \langle \partial f_2(x_t, \xi_{2,t}^1) \nabla f_1(y_{t+1}, \xi_{1,t}^0) + \partial g(x_t, \zeta^0), \lambda_t \rangle / \eta_t \right\} \right], \quad (2.7)$$

where $\eta_t > 0$ is prespecified.

For the dual update part, we query an independent sample of $g(x_t, \zeta_t^1)$, and update the dual variable $\lambda_{t+1}$ by a projected stochastic gradient ascent step that

$$\lambda_{t+1} = \arg\max_{\lambda \in \mathbb{R}^m_+} \left\{ \langle g(x_t, \zeta_t^1), \lambda \rangle - \frac{\alpha_t}{2} \|\lambda_t - \lambda\|^2 \right\} = \left[ \lambda_t + g(x_t, \zeta_t^1) / \alpha_t \right]_+, \quad (2.8)$$

where $\alpha_t > 0$ is prespecified. We summarize the proposed algorithm in Algorithm 1.

### 2.2 Stochastic Sequential Dual Interpretation

We provide a stochastic sequential dual (SSD) interpretation [38, 39] of Algorithm 1, which is the key for our later analysis. The SSD interpretation relies on the Fenchel conjugate reformulations for
Algorithm 1 Expected Value Constrained Stochastic Compositional Gradient Descent (EC-SCGD)

**Input**: Step-sizes \( \{\alpha_t\}, \{\eta_t\}, \{\tau_t\} \), initial points \( x_0 \in \mathcal{X} \), \( y_0 = 0 \), \( \lambda_0 \in \mathbb{R}_+^m \), sampling oracle \( SO \) for \( t = 0, 1, 2, \ldots, N - 1 \) do

Query the \( SO \) at \( x_t \) twice for the sample values \( f_2(x_t, \xi_{2,t}^j) \) and sample subgradients \( \partial f_2(x_t, \xi_{2,t}^j) \), for \( i = 0, 1 \).

Update

\[
y_{t+1} = f_2(x_t, \xi_{2,t}^1) + \frac{\tau_t y_t}{1 + \tau_t}.
\]

Query \( SO \) once at \( y_{t+1} \) to obtain a sampled gradient \( \nabla f_1(y_{t+1}, \xi_0^t) \).

Query the \( SO \) once at \( x_t \) to obtain a sampled subgradient \( \partial g(x_t, \xi_0^t) \).

Update the main solution \( x_{t+1} \) by

\[
x_{t+1} = \text{Proj}_{\mathcal{X}} \left( x_t - \frac{\partial f_2(x_t, \xi_{2,t}^1) \nabla f_1(y_{t+1}, \xi_0^t) + \partial g(x_t, \xi_0^t) \lambda_t}{\eta_t} \right).
\]

Query the \( SO \) once at \( x_t \) to obtain \( g(x_t, \xi_1^t) \), update the dual variable \( \lambda_t \) by

\[
\lambda_{t+1} = \left[ \lambda_t + g(x_t, \xi_1^t)/\alpha_t \right]_+.
\]

end for

**Output**: \( \bar{x}_N = \frac{1}{N} \sum_{t=1}^N x_t \)

the convex objective and constraints. To build up the SSD framework, we first define the following composite Lagrangian functions for \( f_1, f_2 \) and constraint functions \( g^{(j)} \):

\[
\begin{align*}
\mathcal{L}_{f_2}(x, \pi_2) &= \pi_2^T x - f_2^*(\pi_2), \\
\mathcal{L}_F(x, \pi_1:2) &= \mathcal{L}_{f_1}(x, \pi_1, \pi_2) = \pi_1^T \mathcal{L}_{f_2}(x; \pi_2) - f_1^*(\pi_1), \\
\mathcal{L}_{g^{(j)}}(x, v^{(j)}) &= x^T v^{(j)} - [g^{(j)}]^*(v^{(j)}), \text{ for } j = 1, \cdots, m.
\end{align*}
\]

In what follows, we denote by \( v = [v^{(1)}, \cdots, v^{(m)}] \in \mathbb{R}^{d_x \times m} \) the collection of \( v^{(j)} \)'s, denote by \( \mathcal{L}_g(x, v) = [\mathcal{L}_{g^{(1)}}(x, v^{(1)}), \cdots, \mathcal{L}_{g^{(m)}}(x, v^{(m)})]^T \in \mathbb{R}^m \), and denote by \( V = [V^{(1)}, \cdots, V^{(m)}] \subset \mathbb{R}^{d_x \times m} \) the domain of \( g^* \) for ease of presentation. Let \( \lambda \in \mathbb{R}_+^m \) be the Lagrangian dual variable associated with the constraints \( g^{(j)} \)'s. We have the following overall composite Lagrangian function

\[
\mathcal{L}(x, \lambda, \pi_{1:2}, v) = \mathcal{L}_F(x, \pi_{1:2}) + \sum_{j=1}^m \lambda_j \mathcal{L}_{g^{(j)}}(x, v^{(j)}), \text{ where } \lambda \geq 0.
\]

Letting \( \Pi = \Pi_1 \times \Pi_2 \), problem (2.1) has an equivalent formulation as the following saddle point problem that

\[
\min_{x \in \mathcal{X}} \left( \max_{\pi_{1:2} \in \Pi} \mathcal{L}_F(x, \pi_{1:2}) + \max_{\lambda \in \mathbb{R}_+^m} \max_{v \in V} \lambda^T \mathcal{L}_g(x, v) \right).
\]

The existence of a saddle point to (2.1) and the first-order optimality condition of \((x^*, \lambda^*)\) ensure that there exist \( \pi_2^* \in \partial f_2(x^*), \pi_1^* = \nabla f_1(f_2(x^*)), \) and \( v^* \in \partial g(x^*) \) such that \((x^*, \lambda^*, \pi_{1:2}^*, v^*)\)
constitutes a saddle point to problem (2.10) (see, for example, Proposition 1 of [39]), i.e., for any \((x, \lambda, \pi_{1:2}, v) \in \mathcal{X} \times \mathbb{R}_+^m \times \Pi \times V\),
\[
\mathcal{L}(x, \lambda^*, \pi_{1:2}^*, v^*) \geq \mathcal{L}(x^*, \lambda^*, \pi_{1:2}^*, v^*) \geq \mathcal{L}(x^*, \lambda^*, \pi_{1:2}, v).
\] (2.11)

We then provide interpretations for both primal and dual update steps of Algorithm 1 under the stochastic sequential dual framework. Specifically, for the primal update, given a solution \(x_t\), we consider the inner layer \(f_2\) and let
\[
\pi_{2,t+1} \in \arg\max_{\pi_2 \in \Pi} \{\pi_2^T x_t - f_2^*(\pi_2)\}.
\] (2.12)

Here \(\pi_{2,t+1}\) is a subgradient of the deterministic function \(f_2\) at \(x_t\) that \(\pi_{2,t+1} \in \partial f_2(x_t)\). Correspondingly, we have that
\[
\mathcal{L}_{f_2}(x_t, \pi_{2,t+1}) = \pi_{2,t+1}^T x_t - f_2^*(\pi_{2,t+1}) = f_2(x_t).
\]

Under the stochastic setting, we let \(\pi_{2,t+1}^0 \in \partial f_2(x_t, \xi_{2,t}^0)\) be the stochastic dual variable such that
\[
\mathbb{E}[\pi_{2,t+1}^0 | x_t] \in \partial f_2(x_t).
\]

Then, recall (2.6) that \(y_{t+1} = (f_2(x_t, \xi_{2,t}^1) + \tau_t y_t)/(1 + \tau_t)\). Using \(y_{t+1}\) as an estimator for \(f_2(x_t)\), we let
\[
\pi_{1,t+1} = \arg\max_{\pi_1 \in \Pi} \{\pi_1^T y_{t+1} - f_1^*(\pi_1)\},
\] (2.13)

which equals the gradient of \(f_1\) at \(y_{t+1}\) that \(\pi_{1,t+1} = \nabla f_1(y_{t+1})\). We also define a stochastic dual variable \(\pi_{1,t+1}^0 = \nabla f_1(y_{t+1}, \xi_{1,t}^0)\) such that
\[
\mathbb{E}[\pi_{1,t+1}^0 | y_{t+1}] = \nabla f_1(y_{t+1}).
\]

Meanwhile, for the constraints \(g(x)\), we consider its Fenchel conjugate and let
\[
v_{t+1} \in \partial g(x_t), \quad v_{t+1}^i \in \partial g(x_t, \xi_{i}^0), \quad \text{for } i = 0, 1,
\]
\[
g(x_t) = \mathcal{L}_g(x_t, v_{t+1}), \quad \text{and } g(x_t, \xi_{i}^0) = \mathcal{L}_g(x_t, v_{t+1}^i).
\] (2.14)

Using the above dual interpretation, the primal update step (2.7) is equivalent to
\[
x_{t+1} = \arg\min_{x \in \mathcal{X}} \left\{\langle \pi_{2,t+1}^0 \pi_{1,t+1}^0 + v_{t+1}^0 \lambda_t, x \rangle + \frac{\eta_t}{2} \|x - x_t\|^2\right\}
\]
\[
= \text{Proj}_{\mathcal{X}} \left[ x_t - \frac{\pi_{2,t+1}^0 \pi_{1,t+1}^0 + v_{t+1}^0 \lambda_t}{\eta_t} \right].
\] (2.15)

For the dual update step (2.8), using the dual interpretation (2.14), it is equivalent to
\[
\lambda_{t+1} = \arg\max_{\lambda \in \mathbb{R}_+^m} \left\{\langle \mathcal{L}_g(x_t, v_{t+1}^1), \lambda \rangle - \frac{\alpha_t}{2} \|\lambda_t - \lambda\|^2\right\} = \left[\lambda_t + \frac{\mathcal{L}_g(x_t, v_{t+1})}{\alpha_t}\right].
\] (2.16)

The above dual interpretation helps us analyze the structures within the compositional objectives and expectation constraints. In particular, under the above SSD framework, we provide the convergence results of Algorithm 1 in the next subsection.
2.3 Convergence Analysis

After presenting Algorithm 1 and its dual interpretation, we now analyze its convergence properties. Letting \( \{x_t, \lambda_t\} \) be the sequence generated by Algorithm 1 and letting \( \{(\pi_{1:2}, v_t)\} \) be its corresponding dual sequence, we denote by \( \bar{x}_N = \frac{1}{N} \sum_{t=1}^{N} x_t \). To evaluate the performance of our algorithm, we consider the following objective optimality gap \( F(\bar{x}_N) - F(x^*) \) and feasibility residual \( \|g(\bar{x}_N)\| = \max\{g(\bar{x}_N), 0\} \). Clearly, we can see that for any \( x \in \mathcal{X} \), it is an optimal solution to problem (2.1) if and only if \( F(x) - F(x^*) = 0 \) and \( \|g(x)\| = 0 \).

In the remaining part of this section, we show that both the above metrics converge to zero and derive their respective convergence rates.

2.3.1 Convergence of gap function

Letting \( z_t = (x_t, \lambda_t, x_{1:2}, v_t) \) be the solution generated by Algorithm 1 and \( z = (x^*, \lambda, x_{1:2}, v) \in \mathcal{X} \times \mathbb{R}_+^m \times \Pi \times \mathcal{V} \) be a general feasible point, to facilitate our analysis, we consider the gap function

\[
Q(z_t, z) = \mathcal{L}(x_t, \lambda, x_{1:2}) - \mathcal{L}(x^*, \lambda_t, x_{1:2}, v_t).
\]

In what follows, we provide our first result to characterize the contraction of the above gap function.

**Theorem 2.1.** Suppose Assumptions 2.1, 2.2, and 2.3 hold. Suppose Algorithm 1 generates \( \{(x_t, \lambda_t, x_{1:2}, v_t)\}_{t=1}^{N} \) by setting \( \tau_t = t/2, \alpha_t = \alpha \), and \( \eta_t = \eta \) for \( t \leq N \). Let \( \lambda \in \mathbb{R}_+^m \) be a nonnegative bounded (random) variable such that \( \|\lambda\| \leq M_\lambda \) uniformly and let \( z = (x^*, \lambda, x_{1:2}, v) \) be a feasible point, then for any integer \( K \leq N \), we have

\[
\sum_{t=1}^{K} \mathbb{E}[Q(z_t, z)] + \frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda\|^2] \\
\leq \frac{95KC_f^2C_s^2}{2\eta} + \frac{\eta}{2} \|x_0 - x^*\|^2 + \frac{K}{\alpha} D_\mathcal{X}C_s^2 + \sqrt{K} \left( M_\lambda C_g + L_{f_1}C_s^2 + 3C_{f_1}\sigma_{f_2} \right) \\
+ \frac{\alpha}{2} \mathbb{E}[\|\lambda_0 - \lambda\|^2] + \sum_{t=1}^{K} 5C_g^2 \left( \frac{\mathbb{E}[\|\lambda_{t-1}\|^2]}{2\eta} + \frac{2M_{\lambda}^2}{\eta} + \frac{\mathbb{E}[\|\lambda - \lambda_t\|^2]}{2\eta} \right).
\] (2.18)

The above result serves as a crucial building block when analyzing the convergence behavior of Algorithm 1. A key observation is that the dual sequence \( \{\lambda_t\} \) appears on both sides of (2.18), so that it is not guaranteed to be bounded. In this case, we cannot readily establish the overall convergent property of the gap function \( Q(z_t, z) \). In the next two subsections, without imposing any additional assumptions, we show that the dual sequence \( \{\lambda_t\} \) is bounded by using (2.18), and further derive the convergence rates of both objective optimality gap and feasibility residual.

2.3.2 Boundedness of \( \{\lambda_t\} \)

To show the boundedness of \( \{\lambda_t\} \), we observe that (2.18) in Theorem 2.1 preserves a recursive structure. Specifically, in iteration \( K \), the dual solution \( \lambda_K \) can be bounded by the sum of dual solutions \( \{\lambda_{t}\}_{t=0}^{K-1} \) generated in previous iterations plus an extra constant. Here, we restate a technical result from Lemma 2.8 of [5] to study the boundedness of such sequences.

\[
12
\]
Lemma 1. Let \( \{a_t\} \) be a nonnegative sequence and \( m_1, m_2 \geq 0 \) be constants such that \( a_0 \leq m_1 \). Suppose the following relationship holds for all \( K \geq 1 \):
\[
 a_K \leq m_1 + m_2 \sum_{t=0}^{K-1} a_t.
\]
Then we have \( a_K \leq m_1(1 + m_2)^K \).

Using the above result, we then show the boundedness of \( \{\lambda_t\} \) in the following proposition.

Proposition 2. Suppose Assumptions 2.1, 2.2, and 2.3 hold. Let \( \{(x_t, \lambda_t, \pi_{1,2,t}, v_t)\}_{t=1}^N \) be the solution sequence generated by Algorithm 1 with \( \tau_t = \frac{t}{2} \), \( \alpha_t = \alpha = 2\sqrt{N} \), and \( \eta_t = \eta = \frac{15\sqrt{N}}{2} \) for all \( t \leq N \). Then for any \( N \geq 3C_\sigma^2/2 \) and any integer \( K \leq N \), we have
\[
\mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq 2Re^{2C_\sigma^2},
\]
where
\[
R = 6C_{f_1}^2 C_{f_2}^2 + \frac{D_X C_\sigma^2}{2} + 2C_\sigma^2 \|\lambda^*\|^2 + \frac{\sigma_g^2}{2} + L_{f_1} \sigma_{f_2} + 3C_{f_1} \sigma_{f_2} + \|\lambda^*\| \sigma_g + \frac{15}{4} \|x_0 - x^*\|^2 + \|\lambda_0 - \lambda^*\|^2.
\]

The above result implies that with properly chosen step-sizes, the dual sequence \( \|\lambda_t\|^2 \) generated by Algorithm 1 is indeed bounded in expectation, despite more randomness and bias incurred under the compositional setting.

2.3.3 Convergence of value and feasibility metrics

After proving the boundedness of \( \{\lambda_t\} \), we derive the convergence rates of the objective optimality gap and the feasibility residual in an ergodic sense in the following result.

Theorem 2.2. Suppose that Assumptions 2.1, 2.2, and 2.3 hold. Let \( \{(x_t, \lambda_t, \pi_{1,2,t}, v_t)\}_{t=1}^N \) be the solution sequence generated by Algorithm 1 with \( \lambda_0 = 0 \), \( \tau_t = \frac{t}{2} \), \( \alpha_t = \alpha = 2\sqrt{N} \), and \( \eta_t = \eta = \frac{15\sqrt{N}}{2} \) for all \( 0 \leq t \leq N \). Let \( R \) be the constant defined in (2.19). Letting \( \bar{x}_N = \sum_{t=1}^N x_t/N \), for any \( N \geq 3C_\sigma^2/2 \), we have
\[
\mathbb{E}[F(\bar{x}_N)] - F(x^*) \leq \frac{1}{\sqrt{N}} \left( 6C_{f_1}^2 C_{f_2}^2 + L_{f_1} \sigma_{f_2} + 3C_{f_1} \sigma_{f_2} + \frac{15}{4} \|x_0 - x^*\|^2 + \frac{(4\|\lambda^*\|^2 + 8Re^{2C_\sigma^2}C_\sigma^2)}{3} \right),
\]
and
\[
\mathbb{E}[\|g(\bar{x}_N)\|] \leq \frac{1}{\sqrt{N}} \left( 6C_{f_1}^2 C_{f_2}^2 + L_{f_1} \sigma_{f_2} + 3C_{f_1} \sigma_{f_2} + \frac{\eta}{2} \|x_0 - x^*\|^2 + \frac{D_X C_\sigma^2}{2} + \frac{\sigma_g^2}{2} \right)
\]
\[
+ \frac{(4\|\lambda^*\|^2 + 8Re^{2C_\sigma^2}C_\sigma^2)}{\sqrt{N}} + (2\|\lambda^*\|^2 + 4Re^{2C_\sigma^2}C_\sigma^2 + (2C_\sigma^2 + 1)(\|\lambda^*\| + 1)^2 + (\|\lambda^*\| + 1)\sigma_g).
\]
Proof: In our analysis, we set \( \lambda_0 = 0 \) for ease of presentation. Recall that \( \bar{x}_N = \frac{1}{N} \sum_{t=1}^{N} x_t \), for any fixed \((\lambda, \pi_{1:2}, v)\), by the convexity of \( \mathcal{L}(x, \lambda, \pi_{1:2}, v) \) with respect to \( x \), we have

\[
\frac{1}{N} \sum_{t=1}^{N} \mathcal{L}(x_t, \lambda, \pi_{1:2}, v) \geq \mathcal{L}(\bar{x}_N, \lambda, \pi_{1:2}, v). \tag{2.20}
\]

Meanwhile, we denote by

\[
\tilde{\pi}_2 \in \partial f_2(\bar{x}_N), \quad \tilde{\pi}_1 = \nabla f_1(f_2(\bar{x}_N)), \quad \text{and} \quad \tilde{v} \in \partial g(\bar{x}_N),
\]

the dual variables associated with \( \bar{x}_N \). By the definition of composite Lagrangian (2.9), we have

\[
F(\bar{x}_N) = \mathcal{L}_\mathcal{F}(\bar{x}_N, \tilde{\pi}_{1:2}) = \mathcal{L}(\bar{x}_N, 0, \tilde{\pi}_{1:2}, \tilde{v}). \tag{2.22}
\]

First, we derive the convergence rate of the objective optimality gap \( F(\bar{x}_N) - F(x^*) \). Let \( \bar{\lambda}_N = \frac{1}{N} \sum_{t=1}^{N} \lambda_t, \tilde{\pi}_{1:2,N} = \frac{1}{N} \sum_{t=1}^{N} \pi_{1:2,t}, \) and \( \bar{v}_N = \frac{1}{N} \sum_{t=1}^{N} v_t \). By setting \( \lambda = 0, \pi_{1:2} = \tilde{\pi}_{1:2}, \) and \( v = \tilde{v} \) within (2.21), and using the concavity of \( \mathcal{L}(x, \lambda, \pi_{1:2}, v) \) in \( \lambda, \pi_{1:2}, \) and \( v \), we have

\[
\frac{1}{N} \sum_{t=1}^{N} \left( \mathcal{L}(x_t, 0, \tilde{\pi}_{1:2}, \tilde{v}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2,t}, v_t) \right) \
\geq \mathcal{L}(\bar{x}_N, 0, \tilde{\pi}_{1:2}, \tilde{v}) - \mathcal{L}(x^*, \bar{\lambda}_N, \tilde{\pi}_{1:2,N}, \bar{v}_N) \
\geq \mathcal{L}(\bar{x}_N, 0, \tilde{\pi}_{1:2}, \tilde{v}) - \mathcal{L}(x^*, \lambda^*, \pi_{1:2}^*, v^*) = F(\bar{x}_N) - F(x^*), \tag{2.23}
\]

where the last inequality holds by the min-max relationship (2.11) that

\[
\mathcal{L}(x^*, \lambda^*, \pi_{1:2}^*, v^*) \geq \mathcal{L}(x^*, \bar{\lambda}_N, \tilde{\pi}_{1:2,N}, \bar{v}_N),
\]

and the last equality holds by (2.22). Therefore, by (2.23), setting the feasible point as \( (x^*, \lambda, \pi_{1:2}, v) = (x^*, 0, \tilde{\pi}_{1:2}, \tilde{v}) \), and applying Theorem 2.1 with the fact \( \|\lambda\| = 0 \) when \( \lambda = 0 \), we have

\[
\mathbb{E}[F(\bar{x}_N)] - F(x^*) \leq \frac{1}{N} \sum_{t=1}^{N} \mathbb{E}\left( \mathcal{L}(x_t, 0, \tilde{\pi}_{1:2}, \tilde{v}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2,t}, v_t) \right) \
\leq \frac{1}{\sqrt{N}} \left( 6C_{f_1}^2 C_{f_2}^2 + L_{f_1} \sigma_{f_2}^2 + 3C_{f_1} \sigma_{f_2} + \frac{15}{4} \|x_0 - x^*\|^2 \right) \
+ \frac{1}{3\sqrt{N}} \sum_{t=1}^{N} \left( \mathbb{E}[\|\lambda_{t-1}\|^2] C_g^2 \right) + \frac{1}{3\sqrt{N}} \sum_{t=1}^{N} \left( \mathbb{E}[\|\lambda_t\|^2] C_g^2 \right) \
\leq \frac{1}{\sqrt{N}} \left( 6C_{f_1}^2 C_{f_2}^2 + L_{f_1} \sigma_{f_2}^2 + 3C_{f_1} \sigma_{f_2} + \frac{15}{4} \|x_0 - x^*\|^2 \right) + \frac{(4\|\lambda^*\|^2 + 8R^2)C_g^2}{3\sqrt{N}},
\]

where the last inequality comes from the bound of \( \|\lambda_t\|^2 \) provided by Proposition 2 that

\[
\mathbb{E}[\|\lambda_t\|^2] \leq 2\|\lambda^*\|^2 + 2\mathbb{E}[\|\lambda_t - \lambda^*\|^2] \leq 2\|\lambda^*\|^2 + 4R^2, \text{ for all } t = 1, 2, \cdots, N. \tag{2.24}
\]

This establishes the convergence rate for the objective optimality gap \( F(\bar{x}_N) - F(x^*) \).
Second, we consider the feasibility residual \( \|g(\bar{x}_N)\|_2 \). Recall that \( \hat{\pi}_{1:2} \) and \( \hat{v} \) are the dual variables associated with \( \bar{x}_N \) such that \( \hat{\pi}_{1:2} \in \text{argmax}_{\pi_{1:2} \in \Pi} \mathcal{L}_F(\bar{x}_N, \pi_{1:2}) \) and \( \hat{v} \in \text{argmax}_{v \in \mathcal{V}} \mathcal{L}_g(\bar{x}_N, v) \), we have

\[
F(\bar{x}_N) = \mathcal{L}_F(\bar{x}_N, \hat{\pi}_{1:2}) \geq \mathcal{L}_F(\bar{x}_N, \pi_{1:2}^*) \text{ and } g(\bar{x}_N) = \mathcal{L}_g(\bar{x}_N, \hat{v}) \geq \mathcal{L}_g(\bar{x}_N, v^*).
\]

Since \( \lambda^* \geq 0 \), by the above inequalities and the min-max relationship (2.11), we can see

\[
\mathcal{L}(\bar{x}_N, \lambda^*, \hat{\pi}_{1:2}, \hat{v}) = \mathcal{L}_F(\bar{x}_N, \hat{\pi}) + \langle \lambda^*, \mathcal{L}_g(\bar{x}_N, \hat{v}) \rangle \\
\geq \mathcal{L}_F(\bar{x}_N, \pi_{1:2}^*) + \langle \lambda^*, \mathcal{L}_g(\bar{x}_N, v^*) \rangle = \mathcal{L}(\bar{x}_N, \lambda^*, \pi_{1:2}^*, v^*) \geq \mathcal{L}(x^*, \lambda^*, \pi_{1:2}^*, v^*),
\]

which further implies

\[
F(\bar{x}_N) + \langle \lambda^*, g(\bar{x}_N) \rangle - F(x^*) \geq 0.
\]

Meanwhile, due to the facts that \( \lambda^* \geq 0 \) and \( g(\bar{x}_N) \leq g(\bar{x}_N)_+ \), we have \( \langle \lambda^*, g(\bar{x}_N) \rangle \leq \langle \lambda^*, g(\bar{x}_N)_+ \rangle \), leading to

\[
F(\bar{x}_N) + \|\lambda^*\||g(\bar{x}_N)_+\| - F(x^*) \geq F(\bar{x}_N) + \langle \lambda^*, g(\bar{x}_N) \rangle - F(x^*) \geq 0. \tag{2.25}
\]

Let \( \tilde{\lambda} = (\|\lambda^*\|_2 + 1)g(\bar{x}_N)_+/\|g(\bar{x}_N)_+\| \). Consider another feasible point \( (x^*, \tilde{\lambda}, \hat{\pi}_{1:2}, \hat{v}) \), by (2.20) and the facts that \( g(\bar{x}_N)^Tg(\bar{x}_N)_+ = \|g(\bar{x}_N)_+\|^2 \) and \( \mathcal{L}(x^*, \lambda_t, \pi_{1:2}, v_t) \leq \mathcal{L}(x^*, \lambda^*, \pi_{1:2}^*, v^*) \), we have

\[
\langle g(\bar{x}_N), \tilde{\lambda} \rangle = (\|\lambda^*\|_2 + 1)\|g(\bar{x}_N)_+\|,
\]

which further yields that

\[
\frac{1}{N} \sum_{t=1}^{N} \left( \mathcal{L}(x_t, \tilde{\lambda}, \hat{\pi}_{1:2}, \hat{v}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2}, v_t) \right) \geq \mathcal{L}(\bar{x}_N, \tilde{\lambda}, \hat{\pi}_{1:2}, \hat{v}) - \mathcal{L}(x^*, \lambda^*, \pi_{1:2}^*, v^*) \geq F(\bar{x}_N) + \langle \tilde{\lambda}, g(\bar{x}_N) \rangle - F(x^*) = F(\bar{x}_N) + (\|\lambda^*\|_2 + 1)\|g(\bar{x}_N)_+\| - F(x^*).
\]

By rearranging the terms in the inequality above and applying (2.25), we obtain

\[
\|g(\bar{x}_N)_+\| \leq \frac{1}{N} \sum_{t=1}^{N} \left( \mathcal{L}(x_t, \tilde{\lambda}, \hat{\pi}_{1:2}, \hat{v}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2}, v_t) \right) \\
- \left( F(\bar{x}_N) + \|\lambda^*\||g(\bar{x}_N)_+\| - F(x^*) \right) \\
\leq \frac{1}{N} \sum_{t=1}^{N} \left( \mathcal{L}(x_t, \tilde{\lambda}, \hat{\pi}_{1:2}, \hat{v}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2}, v_t) \right).
\]

Next, we note that \( \|\tilde{\lambda}\| = \|\lambda^*\| + 1 \). By the inequality above, considering the feasible point \( (x^*, \tilde{\lambda}, \hat{\pi}_{1:2}, \hat{v}) \), and applying Theorem 2.1 with \( \alpha = 0 \), \( M_\tilde{\lambda} = \|\tilde{\lambda}\| = \|\lambda^*\| + 1 \), \( \alpha = 2\sqrt{N} \), and
Here we note that given any \( \lambda_0 \in \mathbb{R}_+^m \), our algorithm still enjoys the \( \mathcal{O}(\frac{1}{\sqrt{N}}) \) convergence rate for both objective optimality gap and feasibility residual. Notably, our approach is sufficiently general to handle multiple expected value constraints. It is also worth mentioning that the above result matches the optimal \( \mathcal{O}(\frac{1}{\sqrt{N}}) \) convergence rate for standard convex stochastic optimization, despite the complicated compositional objective and expected value constraints involved. This establishes the benchmark for stochastic compositional optimization under expected value constraints. Compared with [5] where only smooth constraints are studied, our primal-dual update framework can handle nonsmooth constraints, which find wide applications in real-world scenarios, such as Conditional Value-at-Risk (CVaR) in risk management [24]. Finally, we emphasize that the step-sizes \( \tau_t, \alpha_t, \) and \( \eta_t \) within our algorithm are problem-independent. That is, we can apply Algorithm 1 without knowing any of the problem-dependent parameters, such as the Lipschitz parameter \( L_{f_1} \).

### 3 Stochastic Compositional Optimization under Compositional Expected Value Constraints

In Section 2, we have presented a primal-dual algorithm for expected value constrained stochastic compositional optimization that achieves the optimal rate of convergence. It is natural to ask whether our analytical framework can also handle the compositional expected value constrained applications discussed in Section 1.1. We extend the framework by considering the following two-level compositional expected value constrained stochastic compositional optimization (CoC-SCO)
problem:

\[
\min_{x \in \mathcal{X}} F(x) = f_1 \circ f_2(x),
\]

subject to \( G(x) = g_1 \circ g_2(x) \leq 0, \)

where \( \mathcal{X} \subset \mathbb{R}^{d_x} \) is convex and closed, \( f_1(y) = \mathbb{E}_{\xi_1}[f_1(y, \xi_1)] : \mathbb{R}^{d_y} \mapsto \mathbb{R} \) is convex and differentiable, \( f_2(x) = \mathbb{E}_{\xi_2}[f_2(x, \xi_2)] : \mathbb{R}^{d_x} \mapsto \mathbb{R}^{d_y} \) is convex, \( F(x) : \mathbb{R}^{d_x} \mapsto \mathbb{R} \) is convex, \( G(x) : \mathbb{R}^{d_x} \mapsto \mathbb{R}^m \) is convex and differentiable, \( g_1(z) = \mathbb{E}_{\zeta_1}[g_1(z, \zeta_1)] : \mathbb{R}^{d_z} \mapsto \mathbb{R} \) is convex, \( g_2(x) = \mathbb{E}_{\zeta_2}[g_2(x, \zeta_2)] : \mathbb{R}^{d_z} \mapsto \mathbb{R}^{d_y} \) is convex. In more details, we write \( f_2(x) = (f_2^{(1)}(x), \ldots, f_2^{(d_y)}(x))^\top \in \mathbb{R}^{d_y}, \) \( G(x) = (G^{(1)}(x), \ldots, G^{(m)}(x))^\top \in \mathbb{R}^m, \) \( g_2(x) = (g_2^{(1)}(x), \ldots, g_2^{(d_y)}(x))^\top \in \mathbb{R}^{d_y}, \) and \( g_1(z) = (g_1^{(1)}(z), \ldots, g_1^{(m)}(z))^\top \in \mathbb{R}^m, \) and assume that \( f_2^{(j)}, G^{(j)}, g_2^{(j)}, \) and \( g_1^{(j)} \) are convex. As in the previous section, we consider that the functions \( f_1, f_2, g_1, \) and \( g_2 \) are in the form of expectations and can only be estimated from observed samples. Note that each \( G^{(j)} \) represents one compositional constraint and problem (3.1) allows us to handle \( m \) compositional constraints.

We first specify the sampling environment for CoC-SCO. In particular, we assume access to the following Compositional Sampling Oracle (SO) such that

- Given \( x \in \mathcal{X}, \) the SO returns a sample value \( f_2(x, \xi_2) \in \mathbb{R}^{d_y} \) and a sample subgradient \( \partial f_2(x, \xi_2) \in \mathbb{R}^{d_x \times d_y}. \)

- Given \( y \in \mathbb{R}^{d_y}, \) the SO returns a sample value \( f_1(y, \xi_1) \in \mathbb{R} \) and a sample gradient \( \nabla f_1(y, \xi_1) \in \mathbb{R}^{d_y}. \)

- Given \( x \in \mathcal{X}, \) the SO returns a sample value \( g_2(z, \xi_2) \in \mathbb{R}^{d_z} \) and a sample subgradient \( \partial g_2(x, \xi_2) \in \mathbb{R}^{d_x \times d_z}. \)

- Given \( z \in \mathbb{R}^{d_z}, \) the SO returns a sample value \( g_1(z, \xi_1) \in \mathbb{R}^m \) and a sample gradient \( \nabla g_1(z, \xi_1) \in \mathbb{R}^{d_z \times m}. \)

Here we do not impose any smoothness assumptions on the inner-level functions \( f_2 \) and \( g_2. \) For the compositional expected value constrained problem (3.1), with a slight abuse of notation, we write its Lagrangian dual function as

\[
\mathcal{L}(x, \lambda) = F(x) + \langle G(x), \lambda \rangle, \quad \text{where } \lambda \in \mathbb{R}^m_+.
\]

By the convexity of \( F \) and \( G^{(j)} \)'s, the above Lagrangian \( \mathcal{L}(x, y) \) is convex in \( x \) and concave in \( \lambda. \) We assume the Slater condition holds such that there exists a saddle point \((x^*, \lambda^*)\) satisfying

\[
\mathcal{L}(x^*, \lambda) \leq \mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*), \quad \forall (x, \lambda) \in \mathcal{X} \times \mathbb{R}^m_+.
\]  

Letting \( \partial F(x) \in \mathbb{R}^{d_x} \) and \( \partial G(x) \in \mathbb{R}^{d_x \times m} \) be the subgradients of \( F(x) \) and \( G(x), \) respectively, the first-order partial subgradients of \( \mathcal{L} \) satisfy

\[
\partial_x \mathcal{L}(x, \lambda) = \partial F(x) + \partial G(x) \lambda \quad \text{and} \quad \nabla_\lambda \mathcal{L}(x, \lambda) = G(x), \quad \forall x \in \mathcal{X}, \lambda \in \mathbb{R}^m_+.
\]
Compared with EC-SCO discussed in Section 2, the compositional expected value constrained problem is more challenging, because of the lack of unbiased estimators of \( \partial G(x) \) and \( G(x) \). Specifically, the unbiased estimators of \( \partial G(x) \) and \( G(x) \) are

\[
\partial g_2(x, \zeta_2) \nabla g_1(\mathbb{E}_{\zeta_2}[g_2(x, \zeta_2)], \zeta_1) \text{ and } g_1(\mathbb{E}_{\zeta_2}[g_2(x, \zeta_2)], \zeta_1),
\]

respectively. Lacking the knowledge of the expected value \( \mathbb{E}_{\zeta_2}[g_2(x, \zeta_2)] \), the plug-in estimators, \( \partial g_2(x, \zeta_2) \nabla g_1(g_2(x, \zeta_2), \zeta_1) \) and \( g_1(g_2(x, \zeta_2), \zeta_1) \), induce biases in both primal and dual update steps. Such biases would corrupt both our primal and dual updates. For the primal update step, in addition to the bias in estimating \( \partial F(x) \) discussed earlier, the biased estimator of \( \partial G(x) \) induces further biases to our estimation of \( \partial_x \mathcal{L}(x, \lambda) \). Furthermore, for the dual update step, the bias induced by estimating \( G(x) \) would compound with the randomness within updating \( \lambda_t \). Consequently, it is more challenging to analyze the boundedness of the dual sequence \( \{\lambda_t\} \) and the convergence behavior of the solution sequence \( \{x_t\} \) for problem (3.1).

To facilitate our discussion, in addition to Assumptions 2.1 and 2.3 for the objective functions \( f_1 \) and \( f_2 \), we impose the following convexity, smoothness, and boundedness assumptions for the constraint functions \( g_1 \) and \( g_2 \).

**Assumption 3.1.** Let \( C_{g_1}, C_{g_2}, \sigma_{g_1}, \sigma_{g_2} \) and \( L_{g_1} \) be positive scalars. The constraint functions \( g_1 \) and \( g_2 \) satisfy:

1. \( g_1, g_2 \) are convex such that they could be reformulated using the following Fenchel conjugates

   \[
g_1(z) = \max_{v_1 \in \mathcal{V}_1} v_1^\top z - g_1^*(v_1), \text{ and } g_2(x) = \max_{v_2 \in \mathcal{V}_2} v_2^\top x - g_2^*(v_2), \quad \forall \ x \in \mathcal{X}, \ z \in \mathbb{R}^{d_z},
\]

   where \( \mathcal{V}_1 \subset \mathbb{R}^{d_z \times m} \) and \( \mathcal{V}_2 \subset \mathbb{R}^{d_x \times d_z} \) are the domains of \( g_1^* \) and \( g_2^* \), respectively.

2. The sample value \( g_1(z, \zeta_1) \) returned by the \( \mathcal{SO}_{co} \) is unbiased and has a bounded second moment such that

   \[
   \mathbb{E}_{\zeta_1}[g_1(z, \zeta_1)] = g_1(z) \text{ and } \mathbb{E}_{\zeta_1}[\|g_1(z, \zeta_1) - g_1(z)\|^2] \leq \sigma_{g_1}^2, \quad \forall \ z \in \mathbb{R}^{d_z}.
   \]

3. The sample value \( g_2(x, \zeta_2) \) returned by the \( \mathcal{SO}_{co} \) is unbiased and has a bounded second moment such that

   \[
   \mathbb{E}_{\zeta_2}[g_2(x, \zeta_2)] = g_2(x) \text{ and } \mathbb{E}_{\zeta_2}[\|g_2(x, \zeta_2) - g_2(x)\|^2] \leq \sigma_{g_2}^2, \quad \forall \ x \in \mathcal{X}.
   \]

4. The sample subgradient \( \partial g_2(x, \zeta_2) \) returned by the \( \mathcal{SO}_{co} \) is unbiased and has a bounded second moment such that

   \[
   \mathbb{E}_{\zeta_2}[\partial g_2(x, \zeta_2)] \in \partial g_2(x) \text{ and } \mathbb{E}_{\zeta_2}[\|\partial g_2(x, \zeta_2)\|^2] \leq C_{g_2}, \quad \forall \ x \in \mathcal{X}.
   \]

5. The sample gradient \( \nabla g_1(y, \zeta_1) \) returned by the \( \mathcal{SO}_{co} \) is unbiased and has a bounded second moment such that

   \[
   \mathbb{E}_{\zeta_1}[\nabla g_1(z, \zeta_1)] = \nabla g_1(z) \text{ and } \mathbb{E}_{\zeta_1}[\|\nabla g_1(z, \zeta_1)\|^2] \leq C_{g_1}, \quad \forall \ z \in \mathbb{R}^{d_z}.
   \]
6. \( g_1 \) is an \( L_{g_1} \)-smooth function with a Lipschitz continuous gradient such that

\[
||\nabla g_1(z_1) - \nabla g_1(z_2)|| \leq L_{g_1} ||z_1 - z_2||, \quad \forall z_1, z_2 \in \mathbb{R}^{d_z}.
\]

7. For each constraint \( G^{(j)} \), its outer level function is monotone non-decreasing if its inner-level function is non-affine.

Note that (g) above is essentially a generalization of Assumption 2.3 to handle the mixture of affine and non-affine inner-level compositional constraints. Specifically, we allow some constraints \( G^{(j)} \) to have affine inner-level functions, while allowing others to have non-affine inner-level and monotone non-decreasing outer-level functions. Also, we only assume smoothness for the outer-level constraint \( g_1 \) while allowing the inner-level constraint \( g_2 \) to be non-smooth. We omit the subscripts \( \xi_1, \xi_2, \zeta_1, \) and \( \zeta_2 \) within \( E_{\xi_1[\cdot]}, E_{\xi_2[\cdot]}, E_{\zeta_1[\cdot]}, \) and \( E_{\zeta_2[\cdot]} \) for notational convenience.

The above assumptions are mild and our framework can handle the compositional expected value constrained risk management applications provided in Section 1.1 (b) and (c). Specifically, in the risk-averse mean-deviation application (b), when each utility function \( \ell_j(x, \xi) \) is concave and monotone non-decreasing in \( x \), and we observe that \( g_2^{(j)}(x, \xi) \) is convex in \( x \), and \( g_1^{(j)}((z, x), \xi) \) is convex and monotone nondecreasing in \( (z, x) \). Thus, this application can be handled after smoothing the \( \langle \cdot \rangle_+ \) operator within the outer-layer function. For the high-moment portfolio selection application (c), given any random return \( w \), \( g_2(x, w) \) is an affine function while \( g_1((z, x), w) \) is convex in \( (z, x) \). Consequently, by taking expectation over the random return \( w \), \( E_w[g_2(x, w)] \) is affine and \( E_w[g_1((z, x), w)] \) is convex in \( (z, x) \).

3.1 Algorithm

Now we propose our algorithm to solve problem (3.1). In iteration \( t \), we first construct an estimator \( y_{t+1} \) for \( E_{\xi_2}[f_2(x_t, \xi_2)] \), or the limit point of \( \{E_{\xi_2}[f_2(x_t, \xi_2)]\}_t \) to be precise, by letting \( y_{t+1} = \frac{f_2(x_t, \xi_{2,t}) + \tau_t y_t}{1 + \tau_t} \) for some \( \tau_t > 0 \). We query \( SO_{co} \) for a sample gradient \( \nabla f_1(y_{t+1}, \xi_{1,t}^0) \) and a sample subgradient \( \partial f_2(x_t, \xi_{2,t}^0) \). Letting \( z_t \) be an estimator of \( g_2(x_{t-1}) \), we query \( SO_{co} \) to obtain \( g_2(x_t, \xi_{2,t}^1) \), and update

\[
z_{t+1} = \frac{g_2(x_t, \xi_{2,t}^1) + \rho_t z_t}{1 + \rho_t},
\]

where \( \rho_t > 0 \) is prespecified. Next, we query \( SO_{co} \) at \( x_t \) to obtain a sample subgradient \( \partial g_2(x_t, \xi_{2,t}^0) \) and query \( SO_{co} \) at \( z_{t+1} \) to obtain another sample gradient \( \nabla g_1(z_{t+1}, \xi_{1,t}^0) \). We then employ \( \partial f_2(x_t, \xi_{2,t}^0)\nabla f_1(y_{t+1}, \xi_{1,t}^0) \) as an estimator of \( \partial F(x_t) \), use \( \partial g_2(x_t, \xi_{2,t}^0)\nabla g_1(z_{t+1}, \xi_{1,t}^0) \) as an estimator of \( \partial G(x_t) \), and update the primal solution \( x_t \) by a projected stochastic gradient step that

\[
x_{t+1} = \text{Proj}_{X} \left[ x_t - \frac{\partial f_2(x_t, \xi_{2,t}^0)\nabla f_1(y_{t+1}, \xi_{1,t}^0) + \partial g_2(x_t, \xi_{2,t}^0)\nabla g_1(z_{t+1}, \xi_{1,t}^0)\lambda_t}{\eta_t} \right],
\]

where \( \eta_t > 0 \) is prespecified. Finally, we consider the dual variable \( \lambda \in \mathbb{R}_+^{m} \), query \( SO_{co} \) to obtain \( g_1(z_{t+1}, \xi_{1,t}^1), \nabla g_1(z_{t+1}, \xi_{1,t}^1), \) and \( g_2(x_t, \xi_{2,t}^2) \), and set

\[
H_{t+1} = g_1(z_{t+1}, \xi_{1,t}^1) - \nabla g_1(z_{t+1}, \xi_{1,t}^1)z_{t+1} + \nabla g_1(z_{t+1}, \xi_{1,t}^1)g_2(x_t, \xi_{2,t}^2)
\]
and define the composite Lagrangians for constraint functions $g_1, g_2$ that

$$
\mathcal{L}_{g_2}(x, v_2) = v_2^\top x - g_2^*(v_2),
\mathcal{L}_{g_1}(x, v_1; 2) = v_1^\top \mathcal{L}_{g_1}(x, v_2) - g_1^*(v_1).
$$

After introducing Algorithm 2, we now present its interpretation under the SSD framework to facilitate our analysis. We first define the following composite Lagrangian functions for $f_1, f_2$:

$$
\mathcal{L}_{f_2}(x, \pi_2) = \pi_2^\top x - f_2^*(\pi_2),
\mathcal{L}_F(x, \pi_1; 2) = \pi_1^\top \mathcal{L}_{f_2}(x, \pi_2) - f_1^*(\pi_1),
$$

as an estimator of $G(x)$. We update $\lambda_{t+1}$ by a projected stochastic gradient step that

$$
\lambda_{t+1} = \arg\max_{\lambda \in \mathbb{R}_+^n} \left\{ \lambda^\top H_{t+1} - \frac{\alpha_t}{2} \| \lambda - \lambda_t \|^2 \right\} = \left[ \lambda_t + H_{t+1}/\alpha_t \right]_+.
$$

for some $\alpha_t > 0$. We summarize the details of the above process in Algorithm 2. We point out that given $x_t, \lambda_t, z_t$, and $y_t$, the subsequent updates $x_{t+1}, \lambda_{t+1}$ are independent, because we employ independent stochastic samples within our update scheme.

### 3.2 Stochastic Sequential Dual Interpretation

After introducing Algorithm 2, we now present its interpretation under the SSD framework to facilitate our analysis. We first define the following composite Lagrangian functions for $f_1, f_2$:

$$
\mathcal{L}_{f_2}(x, \pi_2) = \pi_2^\top x - f_2^*(\pi_2),
\mathcal{L}_F(x, \pi_1; 2) = \pi_1^\top \mathcal{L}_{f_2}(x, \pi_2) - f_1^*(\pi_1),
$$

and define the composite Lagrangians for constraint functions $g_1, g_2$ that

$$
\mathcal{L}_{g_2}(x, v_2) = v_2^\top x - g_2^*(v_2),
\mathcal{L}_{g_1}(x, v_1; 2) = v_1^\top \mathcal{L}_{g_1}(x, v_2) - g_1^*(v_1).
$$

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We then have the overall composite Lagrangian function

\[ \mathcal{L}(x, \lambda, \pi_{1:2}, v_{1:2}) = \mathcal{L}_F(x, \pi_{1:2}) + \lambda^\top \mathcal{L}_G(x, v_{1:2}). \]

Letting \( \Pi = \Pi_1 \times \Pi_2 \) and \( \mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \), problem (3.1) can be reformulated as a saddle point problem

\[ \min_{x \in \mathcal{X}} \left( \max_{\pi_{1:2} \in \Pi} \mathcal{L}_F(x, \pi_{1:2}) + \max_{\lambda \in \mathbb{R}_+^m, v_{1:2} \in \mathcal{V}} \lambda^\top \mathcal{L}_G(x, v_{1:2}) \right). \]  

(3.6)

There exist \( \pi_2^* \in \partial f_2(x^*), \pi_1^* = \nabla f_1(f_2(x^*)), v_2^* \in \partial g_2(x^*), \) and \( v_1^* = \nabla g_1(g_2(x^*)) \) such that \((x^*, \lambda^*, \pi_{1:2}^*, v_{1:2}^*)\) serves as a saddle point to (3.6). Specifically, for any feasible \((x, \lambda, \pi_{1:2}, v_{1:2}) \in \mathcal{X} \times \mathbb{R}_+^m \times \Pi \times \mathcal{V}, \)

\[ \mathcal{L}(x, \lambda^*, \pi_{1:2}^*, v_{1:2}^*) \geq \mathcal{L}(x^*, \lambda^*, \pi_{1:2}^*, v_{1:2}^*) \geq \mathcal{L}(x^*, \lambda, \pi_{1:2}, v_{1:2}). \]  

(3.7)

For any solution \( x_t \), we define \( \pi_{1:t}, \pi_{1:t+1}, \pi_{2:t}, \pi_{2:t+1} \) as

\[ \pi_{2:t+1} = \arg\max_{\pi_2 \in \Pi_2} \{ \pi_2^\top x_t - f_2^*(\pi_2) \}, \quad \pi_{2:t} = \partial f_2(x_t, \zeta_{2:t}), \quad \pi_{1:t+1} = \arg\max_{\pi_1 \in \Pi_1} \{ \pi_1^\top y_{t+1} - f_1^*(\pi_1) \}, \quad \pi_{1:t} = \nabla f_1(y_{t+1}, \xi_{1:t}). \]

Consequently, we have

\[ \mathcal{L}_{f_2}(x_t, \pi_{2:t+1}) = f_2(x_t), \quad \mathbb{E}[\pi_{2:t+1}^\top | x_t] = \partial f_2(x_t), \quad \text{and} \quad \mathbb{E}[\pi_{1:t+1}^\top | y_{t+1}] = \nabla f_1(y_{t+1}). \]

Then for the constraints \( g_1, g_2 \), we define

\[ v_{2:t+1}^0 \in \arg\max_{v_2 \in \mathcal{V}_2} \{ v_2^\top x_t - g_2^*(v_2) \}, \]  

(3.8)

and observe that \( v_{2:t+1}^0 \) is the subgradient of \( g_2 \) at \( x_t \). That is, \( v_{2:t+1}^0 \in \partial g_2(x_t) \). Correspondingly, we have

\[ \mathcal{L}_{g_2}(x_t, v_{2:t+1}) = v_{2:t+1}^\top x_t - g_2^*(v_{2:t+1}) = g_2(x_t). \]

Meanwhile, we let the stochastic dual variable be \( v_{2:t+1}^0 \in \partial g_2(x_t, \zeta_{2:t}^0) \), and observe that

\[ \mathbb{E}[v_{2:t+1}^0 | x_t] \in \partial g_2(x_t). \]

Then, recalling that \( z_{t+1} = \frac{g_2(x_t, \zeta_{1:t}^0) + \rho z_t}{1 + \rho} \), taking \( z_{t+1} \) as an estimator for \( g_2(x_t) \), we have that

\[ v_{1:t+1}^0 = \arg\max_{v_1 \in \mathcal{V}_1} \{ v_1^\top z_{t+1} - g_1^*(v_1) \} \]

is the gradient of \( g_1 \) at \( z_{t+1} \). In particular, we have

\[ v_{1:t+1}^0 = \nabla g_1(z_{t+1}). \]

We also define a stochastic dual variable \( v_{1:t+1}^0 = \nabla g_1(z_{t+1}, \zeta_{1:t}^0) \) such that

\[ \mathbb{E}[v_{1:t+1}^0 | z_{t+1}] = v_{1:t+1}. \]
After understanding the stochastic gradients under the stochastic sequential dual framework, the primal update step (3.3) can be viewed as

\[
x_{t+1} = \arg\min_{x \in \mathcal{X}} \left\{ \langle \pi_{0, t+1} \pi_{1, t+1} + v_{1, t+1}^{0} \lambda_t, x \rangle + \frac{\eta}{2} \|x - x_t\|^2 \right\}.
\]

(3.9)

For the dual update step (3.5), by the following lemma, we observe that \( H_{t+1} \) is an unbiased estimator of \( \mathcal{L}_G(x_t, v_{1:2, t+1}) \) with finite variance. This implies that (3.5) is equivalent to

\[
\lambda_{t+1} = \arg\max_{\lambda \in \mathbb{R}^p_t} \left\{ \lambda^\top H_{t+1} - \frac{\alpha_t}{2} \|\lambda\|^2 \right\},
\]

(3.10)

where \( \mathbb{E}[H_{t+1} | x_{t}, v_{1, t+1}, v_{2, t+1}] = \mathcal{L}_G(x_t, v_{1:2, t+1}) \).

**Lemma 3.** Suppose Assumptions 2.1, 2.3, and 3.1 hold. Let \( \{(x_t, \lambda_t, \pi_{1:2, t}, v_{1:2, t})\}_{t=1}^{N} \) be the solution sequence generated by Algorithm 2. We have

(a) \( H_t \) is an unbiased estimator of \( \mathcal{L}_G(x_{t-1}, v_{1:2, t}) \) such that

\[
\mathbb{E}[H_t | x_{t-1}, v_{1, t}, v_{2, t}] = \mathcal{L}_G(x_{t-1}, v_{1:2, t}).
\]

(3.11)

(b) The variance of \( H_t \) is upper bounded by a constant \( \sigma_H > 0 \) such that

\[
\text{Var}(H_t | x_{t-1}, v_{1, t}, v_{2, t}) \leq \sigma_H^2.
\]

(3.12)

**Proof:** (a) By recalling the definition of \( H_t \) (3.4) and noting that \( g_1(z_t, \zeta_{1, t-1}^{1}), \nabla g_1(z_t, \zeta_{1, t-1}^{1}) \), and \( g_2(x_{t-1}, \zeta_{2, t-1}^{2}) \) are independently sampled, we have

\[
\mathbb{E}[H_t | x_{t-1}, v_{1, t}, v_{2, t}]
= \mathbb{E}\left[ g_1(z_t, \zeta_{1, t-1}^{1}) - [v_{1, t}^\top z_t + [v_{1, t}^\top] g_2(x_{t-1}, \zeta_{2, t-1}^{2}) | x_{t-1}, v_{1, t}, v_{2, t} \right]
= g_1(z_t) + v_{1, t}^\top g_2(x_{t-1}) - v_{1, t}^\top z_t = v_{1, t}^\top \mathcal{L}_G(x_{t-1}, v_{2, t}) - g_1^*(v_{1, t})
= \mathcal{L}_G(x_{t-1}, v_{1:2, t}),
\]

where the third equality holds since \( v_{1, t} = \nabla g_1(z_t) \) and \( v_{2, t} \in \partial g_2(x_{t-1}) \) such that \( g_1(z_t) = v_{1, t}^\top z_t - g_1^*(v_{1, t}) \) and \( g_2(x_{t-1}) = \mathcal{L}_G(x_{t-1}, v_{2, t}) \). This proves part (a).

(b) Next, given \( x_{t-1}, v_{1, t}, v_{2, t} \), by the definition of \( H_t \) (3.4) and the fact that \( g_1(z_t, \zeta_{1, t-1}^{1}), v_{1, t} \), and \( g_2(x_{t-1}, \zeta_{2, t-1}^{2}) \) are independently sampled, we have that each component of \( H_t \) preserves a bounded second moment. Therefore, \( H_t \) has finite variance, completing the proof. \( \square \)

### 3.3 Convergence Analysis

We then analyze the convergence behavior of Algorithm 2. With a slight abuse notation, we denote by \((x_t, \lambda_t, \pi_{1:2, t}, v_{1:2, t})\) the solution generated by Algorithm 2 at the \( t \)-th iteration and denote by \((x^*, \lambda, \pi_{1:2}, v_{1:2}) \in \mathcal{X} \times \mathbb{R}_+^p \times \Pi \times \mathcal{V} \) a general feasible point. We start our analysis by decomposing...
the composite Lagrangian difference:
\[
\mathcal{L}(x_t, \lambda_t, \pi_t, v_t) - \mathcal{L}(x^*, \lambda_t, \pi_t, v_t)
= \mathcal{L}_F(x_t, \pi_{1:t}) - \mathcal{L}_F(x^*, \pi_{1:2:t}) + \lambda^T (\mathcal{L}_G(x_t, v_{1:t}) - \mathcal{L}_G(x^*, v_{1:2:t}))
+ \lambda_t^T (\mathcal{L}_G(x_t, v_{1:2:t}) - \mathcal{L}_G(x^*, v_{1:2:t})) + (\lambda - \lambda_t)^T \mathcal{L}(x_t, v_{1:2:t})
\]
(3.13)

By Lemma 3 that \( H_t \) is an unbiased estimator for \( \mathcal{L}_G(x_{t-1}, v_{1:2:t}) \) with finite variance, we first bound the term \((\lambda - \lambda_t)^T \mathcal{L}_G(x_t, v_{1:2:t}) \) in expectation in the next lemma.

**Lemma 4.** Suppose Assumptions 2.1, 2.3, and 3.1 hold. Let \( \{(x_t, \lambda_t, \pi_{1:2:t}, v_{1:2:t})\}_{t=1}^{\infty} \) be the solution sequence generated by Algorithm 2 with \( \tau_t = \rho_t = t/2, \alpha_t = \alpha > 0, \) and \( \eta_t = \eta > 0 \) for \( t = 1, 2, \cdots , N \). Let \( \lambda \in \mathbb{R}_+^n \) be a bounded random dual variable such that \( \|\lambda\| \leq M_\lambda \) uniformly, then there exists a constant \( \sigma_H > 0 \) such that for any \( K \leq N \),
\[
\mathbb{E}\left[\sum_{t=1}^{K} (\lambda - \lambda_t)^T \mathcal{L}_G(x_t, v_{1:2:t})\right] + \frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda\|^2]
\leq \sum_{t=1}^{K} \left( \frac{\sigma_H^2}{\alpha} - \frac{\alpha \mathbb{E}[\|\lambda_{t-1} - \lambda\|^2]}{4} + \frac{5C_{g1}^2 C_{g2}^2 \mathbb{E}[\|\lambda - \lambda_t\|^2]}{2\eta} + \frac{\eta \mathbb{E}[\|x_t - x_{t-1}\|^2]}{10} \right)
+ \sqrt{K}M_\lambda \sigma_H + \frac{\alpha}{2} \mathbb{E}[\|\lambda_0 - \lambda\|^2].
\]

Next, we show the contraction of the composite Lagrangian difference. Note that we first assume that the dual variable \( \lambda \) is bounded, and we justify this assumption in Proposition 5.

**Theorem 3.1.** Suppose Assumptions 2.1, 2.3, and 3.1 hold. Let \( \{(x_t, \lambda_t, \pi_{1:2:t}, v_{1:2:t})\}_{t=1}^{\infty} \) be the solution sequence generated by Algorithm 2 with \( \tau_t = \rho_t = t/2, \eta_t = \eta \) and \( \alpha_t = \alpha \) for \( 1 \leq t \leq N \). Letting \( \lambda \in \mathbb{R}_+^n \) be a bounded random variable such that \( \|\lambda\| \leq M_\lambda \), then for any integer \( K \leq N \) and any feasible point \( (x^*, \lambda, \pi_{1:2}, v_{1:2}) \), we have
\[
\sum_{t=1}^{K} \mathbb{E}\left[\mathcal{L}(x_t, \lambda, \pi_{1:2:t}, v_{1:2:t}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2:t}, v_{1:2:t})\right] + \frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda\|^2]
\leq \frac{40K C_{g1}^2 C_{g2}^2 M_\lambda^2}{\eta} + \frac{2\sqrt{K}L_{g1}\sigma_{g2} M_\lambda + 3\sqrt{K}C_{g1}\sigma_{g2} M_\lambda + \eta}{2}\|x_0 - x^*\|^2
+ \frac{95K C_{f1} C_{f2}^2}{2\eta} + \frac{K}{\alpha} (D_X^2 C_{g1}^2 C_{g2}^2 + \sigma_H^2) + \frac{\alpha}{2} \mathbb{E}[\|\lambda_0 - \lambda\|^2] + \sqrt{K}M_\lambda \sigma_H
+ 2\sqrt{K}L_{f1}\sigma_{f2}^2 + 3\sqrt{K} C_{f1} \sigma_{f2}^{\alpha} + \sum_{t=1}^{K} \frac{5C_{g1}^2 C_{g2}^2}{2\eta}\left( \mathbb{E}[\|\lambda - \lambda_t\|^2] + \mathbb{E}[\|\lambda_{t-1}\|^2] \right),
\]
where \( \sigma_H > 0 \) is provided in (3.12).
Proof: We provide bound on each term after decomposing (3.13). Consider any integer $K$ such that $K \leq N$. First, by Lemma 13 in Appendix Section B, we obtain

$$\sum_{t=1}^{K} \mathbb{E} \left[ \lambda^T \left( \mathcal{L}_G(x_t, v_{1:2}) - \mathcal{L}_G(x_t, v_{1:2,t}) \right) \right]$$

$$\leq \sum_{t=1}^{K} \frac{\eta}{10} \mathbb{E} \left[ \|x_t - x_{t-1}\|^2 \right] + \frac{40KC_g^2C_{g_2}^2M_\lambda^2}{\eta} + 2\sqrt{KLg_1}\sigma_{g_2}^2M_\lambda + 3\sqrt{KC_g}\sigma_{g_2}M_\lambda.$$  \hfill (3.14)

Second, by Lemma 14 in Appendix Section B, we have

$$\sum_{t=1}^{K} \mathbb{E} \left[ \mathcal{L}_F(x_t, \pi_{1:2,t}) - \mathcal{L}_F(x^*, \pi_{1:2,t}) + \lambda^T \left( \mathcal{L}_G(x_t, v_{1:2,t}) - \mathcal{L}_G(x^*, v_{1:2,t}) \right) \right]$$

$$\leq \frac{\eta}{2} \|x_0 - x^*\|^2 - \sum_{t=1}^{K} \frac{3\eta}{10} \mathbb{E} \left[ \|x_{t-1} - x_t\|^2 \right] + \sum_{t=1}^{K} \frac{5C_f^2C_{f_2}^2}{2\eta}$$

$$+ \sum_{t=1}^{K} \frac{\alpha}{4} \mathbb{E} \left[ \|\lambda_t - \lambda_{t-1}\|^2 \right] + \sum_{t=1}^{K} \frac{D^2\sigma_{g_1}^2C_{g_2}^2}{\alpha} + \sum_{t=1}^{K} \frac{5C_{g_1}^2C_{g_2}^2 \mathbb{E} \left[ \|\lambda_{t-1}\|^2 \right]}{2\eta}. \hfill (3.15)$$

Meanwhile, Lemma 9 in Appendix Section A.2 implies that

$$\sum_{t=1}^{K} \mathbb{E} \left[ \mathcal{L}_F(x_t, \pi_{1:2}) - \mathcal{L}_F(x_t, \pi_{1:2,t}) \right]$$

$$\leq \sum_{t=1}^{K} \frac{\eta}{10} \mathbb{E} \left[ \|x_t - x_{t-1}\|^2 \right] + \frac{40KC_f^2C_{f_2}^2}{\eta} + 2\sqrt{KLf_1}\sigma_{f_2}^2 + 3\sqrt{KC_f}\sigma_{f_2}. \hfill (3.16)$$

Finally, by Lemma 4 and the assumption that $\|\lambda\| \leq M_\lambda$, we have

$$\mathbb{E} \left[ \sum_{t=1}^{K} (\lambda - \lambda_t)^T \mathcal{L}_G(x_t, v_{1:2,t}) \right] + \frac{\alpha}{2} \mathbb{E} \left[ \|\lambda_K - \lambda\|^2 \right]$$

$$\leq \sum_{t=1}^{K} \left( \frac{\sigma_H^2}{\alpha} - \frac{\alpha \mathbb{E} \left[ \|\lambda_{t-1} - \lambda_t\|^2 \right]}{4} + \frac{5C_{g_1}^2C_{g_2}^2 \mathbb{E} \left[ \|\lambda - \lambda_t\|^2 \right]}{2\eta} + \frac{\eta \mathbb{E} \left[ \|x_t - x_{t-1}\|^2 \right]}{10} \right)$$

$$+ \frac{\alpha}{2} \mathbb{E} \left[ \|\lambda_0 - \lambda\|^2 \right] + \sqrt{KM_\lambda}\sigma_H. \hfill (3.17)$$

By substituting (3.14), (3.15), (3.16), and (3.17) into (3.13), and omitting some algebras, we obtain

$$\sum_{t=1}^{K} \mathbb{E} \left( \mathcal{L}(x_t, \lambda, \pi_{1:2}, v_{1:2}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2,t}, v_{1:2,t}) \right) + \frac{\alpha}{2} \mathbb{E} \left[ \|\lambda_K - \lambda\|^2 \right]$$

$$\leq \frac{40KC_{g_1}^2C_{g_2}^2M_\lambda^2}{\eta} + 2\sqrt{KLg_1}\sigma_{g_2}^2M_\lambda + 3\sqrt{KC_g}\sigma_{g_2}M_\lambda + \frac{\eta}{2} \|x_0 - x^*\|^2$$

$$+ \frac{95KC_f^2C_{f_2}^2}{2\eta} + \frac{K}{\alpha} \left( D^2\sigma_{g_1}^2C_{g_2}^2 + \sigma_H^2 \right) + \frac{\alpha}{2} \mathbb{E} \left[ \|\lambda_0 - \lambda\|^2 \right] + \sqrt{KM_\lambda}\sigma_H$$

$$+ 2\sqrt{KLf_1}\sigma_{f_2}^2 + 3\sqrt{KC_f}\sigma_{f_2} + \sum_{t=1}^{K} \frac{5C_{g_1}^2C_{g_2}^2}{2\eta} \left( \mathbb{E} \left[ \|\lambda - \lambda_t\|^2 \right] + \mathbb{E} \left[ \|\lambda_{t-1}\|^2 \right] \right).$$
This completes the proof.

We now show that the dual variables \{\lambda_t\} are bounded in the next proposition.

**Proposition 5.** Suppose Assumptions 2.1, 2.3, and 3.1 hold. Let \{\lambda_t\}^N_{t=1} be the sequence of dual variables generated by Algorithm 2 with \(\tau_t = \rho_t = t/2\), \(\alpha_t = 2\sqrt{N}\), and \(\eta_t = \frac{5\sqrt{N}}{2}\) for all \(t \leq N\). Then for any \(N \geq 2C_{g_1}^2C_{g_2}^2\) and any integer \(K \leq N\), we have

\[
\mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq 2Re^{6C_{g_1}^2C_{g_2}^2},
\]

where \(\sigma_H > 0\) is provided in (3.12),

\[
Q = 2L_{g_1}\sigma_{g_2}^2\|\lambda^*\| + 3C_{g_1}\sigma_{g_2}\|\lambda^*\| + \|\lambda^*\|\sigma_H + 2L_{f_1}\sigma_{f_2}^2 + 3C_{f_1}\sigma_{f_2},
\]

and \(R = 18C_{g_1}^2C_{g_2}^2\|\lambda^*\| + 19C_{f_1}^2C_{f_2}^2 + \frac{D_H^2C_{g_1}^2C_{g_2}^2 + \sigma_H^2}{2} \left[ \frac{5}{4}\|x_0 - x^*\|^2 + \|\lambda_0 - \lambda^*\|^2 + Q \right].
\]

In the above, we show that the dual sequence \{\lambda_t\} generated by Algorithm 2 is also bounded in expectation, serving as a building block for analyzing the convergence behavior of the solution path \(\{x_t\}\).

**Remark 6.** It is worth mentioning that by setting \(\eta_t = \frac{5C_{g_1}^2C_{g_2}^2\sqrt{N}}{2}\), the above analysis yields a similar bound \(\mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq 2R'e^6\), where \(R' > 0\) is a constant polynomially dependent on the primitives.

Letting \(\bar{x}_N = \frac{1}{N}\sum_{t=1}^{N} x_t\), by the boundedness property, we derive the convergence rates for the objective optimality gap \(F(\bar{x}_N) - F(x^*)\) and feasibility residual \(G(\bar{x}_N) = \max\{G(x), 0\}\) in the next theorem.

**Theorem 3.2.** Suppose Assumptions 2.1, 2.3, and 3.1 hold. Let \(\{(x_t, \lambda_t, \pi_{1:2:t}, v_{1:2:t})\}_{t=0}^{N}\) be the solution sequence generated by Algorithm 2 with \(\lambda_0 = 0\), \(\tau_t = \rho_t = t/2\), \(\alpha_t = 2\sqrt{N}\), and \(\eta_t = \frac{5\sqrt{N}}{2}\) for \(t = 1, \ldots, N\), and \(N \geq 2C_{g_1}^2C_{g_2}^2\). Letting \(\bar{x}_N = \frac{1}{N}\sum_{t=1}^{N} x_t\), we have

\[
\mathbb{E}[F(\bar{x}_N) - F(x^*)]
\leq \frac{1}{\sqrt{N}} \left( \frac{5}{4}\|x_0 - x^*\|^2 + 19C_{f_1}^2C_{f_2}^2 + \frac{D_H^2C_{g_1}^2C_{g_2}^2 + \sigma_H^2}{2} + 2L_{f_1}\sigma_{f_2}^2 + 3C_{f_1}\sigma_{f_2} \right)
+ \frac{4C_{g_1}^2C_{g_2}^2}{\sqrt{N}} \left( \|\lambda^*\|^2 + 2Re^{6C_{g_1}^2C_{g_2}^2} \right),
\]

and

\[
\mathbb{E}[\|G(\bar{x}_N)\|_2]
\leq \frac{1}{\sqrt{N}} \left( 16C_{g_1}^2C_{g_2}^2\|\bar{\lambda}\|^2 + 2L_{g_1}\sigma_{g_2}^2\|\bar{\lambda}\| + 3C_{g_1}\sigma_{g_2}\|\bar{\lambda}\| + \frac{5}{4}\|x_0 - x^*\|^2 + 19C_{f_1}^2C_{f_2}^2 \right)
+ \frac{1}{\sqrt{N}} \left( \frac{D_H^2C_{g_1}^2C_{g_2}^2 + \sigma_H^2}{2} + \|\bar{\lambda}\|^2 + \sigma_1\|\bar{\lambda}\| + 2L_{f_1}\sigma_{f_2}^2 + 3C_{f_1}\sigma_{f_2} \right)
+ \frac{3C_{g_1}^2C_{g_2}^2}{\sqrt{N}} \left( \|\bar{\lambda}\| + \|\lambda^*\|^2 + 2Re^{6C_{g_1}^2C_{g_2}^2} \right),
\]

25
where \( \tilde{\lambda} = (\|\lambda^*\| + 1) \frac{G(\bar{x}_N)}{\|G(\bar{x}_N)\|_2} \) is bounded such that \( \|\tilde{\lambda}\| = \|\lambda^*\| + 1 \) and \( \|\tilde{\lambda}\|^2 = (\|\lambda^*\| + 1)^2 \), and \( \sigma_H, R > 0 \) are constants defined in (3.12) and (3.18), respectively.

**Proof:** We set \( \lambda_0 = 0 \) throughout our analysis. We first consider the objective optimality gap \( F(\bar{x}_N) - F(x^*) \). We denote by

\[
\hat{\pi}_2 \in \partial f_2(\bar{x}_N), \hat{\pi}_1 = \nabla f_1(f_2(\bar{x}_N)), \hat{\nu}_2 \in \partial g_2(\bar{x}_N), \text{ and } \hat{\nu}_1 = \nabla g_1(g_2(\bar{x}_N)).
\]

Setting the feasible point as \( (x^*, 0, \hat{\pi}_{1:2}, \hat{\nu}_{1:2}) \) and following Theorem 2.2 with \( M_\lambda = 0 \) for \( \lambda = 0, \alpha_t = 2\sqrt{N}, \text{ and } \eta_t = \frac{5\sqrt{N}}{2} \), we obtain that

\[
\mathbb{E}[F(\bar{x}_N)] - F(x^*) \leq \frac{1}{N} \sum_{t=1}^{N} \mathbb{E} \left( \mathcal{L}(x_t, 0, \hat{\pi}_{1:2}, \hat{\nu}_{1:2}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2,t}, v_{1:2,t}) \right)
\leq \frac{1}{\sqrt{N}} \left( \frac{5}{4} \|x_0 - x^*\|^2 + 19C_f^2 C_f^2 + \frac{D_\lambda^2 C_{g_2}^2 C_{g_2}^2 + \sigma_H^2}{2} + 2L_{f_1} \sigma_{f_2}^2 + 3C_{f_1} \sigma_{f_2} \right)
\leq \frac{1}{\sqrt{N}} \left( \frac{5}{4} \|x_0 - x^*\|^2 + 19C_f^2 C_f^2 + \frac{D_\lambda^2 C_{g_2}^2 C_{g_2}^2 + \sigma_H^2}{2} + 2L_{f_1} \sigma_{f_2}^2 + 3C_{f_1} \sigma_{f_2} \right)
\leq \frac{1}{\sqrt{N}} \left( \mathbb{E}[\|\lambda_t\|^2] + \mathbb{E}[\|\lambda_{t-1}\|^2] \right).
\]

We then derive the convergence rate of the optimality gap by the bound of \( \|\lambda_t\|^2 \) in Proposition 5 that

\[
\mathbb{E}[\|\lambda_t\|^2] \leq 2\|\lambda^*\|^2 + 2\mathbb{E}[\|\lambda_t - \lambda^*\|^2] \leq 2\|\lambda^*\|^2 + 4R e^{6C_{g_1}^2 C_{g_2}^2}, \text{ for all } t = 1, 2, \cdots, N.
\]

Second, we consider the feasibility residual \( G(\bar{x}_N) \). By adopting the feasible point \( (x^*, \bar{\lambda}, \hat{\pi}_{1:2}, \hat{\nu}_{1:2}) \) where \( \bar{\lambda} = (\lambda^* + 1) \frac{G(\bar{x}_N)}{\|G(\bar{x}_N)\|_2} \) and following Theorem 2.2 with \( M_{\bar{\lambda}} = \|\bar{\lambda}\|, \alpha_t = 2\sqrt{N}, \text{ and } \eta_t = \frac{5\sqrt{N}}{2} \), we obtain

\[
\mathbb{E}[\|G(\bar{x}_N)\|_2] \leq \frac{1}{N} \sum_{t=1}^{N} \mathbb{E} \left( \mathcal{L}(x_t, \bar{\lambda}, \hat{\pi}_{1:2}, \hat{\nu}_{1:2}) - \mathcal{L}(x^*, \lambda_t, \pi_{1:2,t}, v_{1:2,t}) \right)
\leq \frac{1}{\sqrt{N}} \left( 16C_{g_1}^2 C_{g_2}^2 \|\bar{\lambda}\|^2 + 2L_{g_1} \sigma_{g_2}^2 \|\bar{\lambda}\| + 3\sqrt{K^2 C_{g_1}} \sigma_{g_2}^2 \|\bar{\lambda}\| + \frac{5}{4} \|x_0 - x^*\|^2 \right)
\leq \frac{1}{\sqrt{N}} \left( 16C_{g_1}^2 C_{g_2}^2 \|\bar{\lambda}\|^2 + 2L_{g_1} \sigma_{g_2}^2 \|\bar{\lambda}\| + 3\sqrt{K^2 C_{g_1}} \sigma_{g_2}^2 \|\bar{\lambda}\| + \frac{5}{4} \|x_0 - x^*\|^2 \right)
\leq \frac{1}{\sqrt{N}} \left( \mathbb{E}[\|\lambda_t - \lambda^*\|^2] + \mathbb{E}[\|\lambda_{t-1}\|^2] \right).
\]

We obtain the convergence rate of feasibility residual \( \|G(x^*)\|_2 \) by using the facts that

\[
\|\lambda_t\|^2 \leq 2\|\lambda^*\|^2 + 2\|\lambda_t - \lambda^*\|^2 \text{ and } \|\bar{\lambda} - \lambda_t\|^2 \leq 3\|\bar{\lambda}\|^2 + 3\|\lambda^*\|^2 + 3\|\lambda_t - \lambda^*\|^2,
\]

and applying \( \mathbb{E}[\|\lambda_t - \lambda^*\|^2] \leq 2R e^{6C_{g_1}^2 C_{g_2}^2} \) provided by Proposition 5. This completes the proof. \( \square \)

The above result implies that Algorithm 2 enjoys the optimal \( \mathcal{O}(\frac{1}{\sqrt{N}}) \) convergence rate for the more complicated compositional expected value constrained problems. Similar as Algorithm 1,
Algorithm 2 is parameter-free so that we can employ it without knowing any of the problem-dependent constants. Further, it is worth emphasizing that the primitives $C^2_{g_1}$ and $C^2_{g_2}$ grow in the same order as the number of constraints $m$. As pointed out by Remark 6, by refining the step-sizes as $\eta_t = \frac{5C^2_{g_1}C^2_{g_2}\sqrt{N}}{2}$ for $t \leq N$, we can bound the dual sequence by $E[\|\lambda_K - \lambda^*\|^2] \leq 2R'e^6$ for some $R'>0$ polynomially dependent on the primitives. Consequently, the hidden constant within $O(\frac{1}{\sqrt{N}})$ grows polynomially in the number of constraints $m$. This suggests the efficiency of our algorithm for handling real-world applications with large number of compositional constraints.

4 Conclusion

In this paper, we develop a novel model for SCO with single-level and two-level compositional expected value constraints. Our formulation is robust when handling multiple expected value constraints and finds wide applications in data-driven optimization and risk management. We propose a class of primal-dual algorithms and interpret our algorithms under the stochastic sequential dual framework. When both objective and constraints are convex, we show that our algorithms find the optimal solutions at the rate of $O(\frac{1}{\sqrt{N}})$ for both single-level and two-level compositional expected value constrained scenarios. Our results match the optimal convergence rate for standard convex stochastic optimization and establish benchmarks.

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Appendix

A Proof of Results in Section 2

Letting Assumptions 2.1, 2.2, and 2.3 hold and letting \((x^*, \lambda, \pi_{1:2}, v) \in X \times \mathbb{R}_+^m \times \Pi \times V\) be a general feasible point, we first introduce two technical lemmas to facilitate our analysis.

Lemma 7 (Lemma 3.8 of [15]). Assume function \(g\) is \(\mu\)-strongly convex with respect to some Bergman distance \(V\), i.e., \(g(y) - g(\bar{y}) - g'(\bar{y})^\top (y - \bar{y}) \geq \mu V(y, \bar{y})\). If \(\bar{y} \in \argmin_{y \in V} \{\pi^\top y + g(y) + \tau V(y, \bar{y})\}\), then

\[
(y - \bar{y})^\top \pi + g(\bar{y}) - g(y) \leq \tau V(y, \bar{y}) - (\tau + \mu) V(y, \bar{y}) - \tau V(\bar{y}, \bar{y}).
\]

A.1 Proof of Theorem 2.1

Proof: Recall that \(L(x, \pi_{1:2}, v, \lambda) = L_F(x, \pi_{1:2}) + \lambda^\top L_g(x, v)\) for any \((x, \lambda, \pi_{1:2}, v) \in X \times \mathbb{R}_+^m \times \Pi \times V\), we consider \(Q(z_t, z)\) and decompose it as

\[
Q(z_t, z) = L(x_t, \lambda_t, \pi_{1:2}, v_t) - L(x^*, \lambda_t, \pi_{1:2}, v_t)
= L_F(x_t, \pi_{1:2}) - L_F(x^*, \pi_{1:2}) + \lambda_t^\top L_g(x_t, v_t) - \lambda_t^\top L_g(x^*, v_t)
= L_F(x_t, \pi_{1:2}) - L_F(x^*, \pi_{1:2}) + \lambda_t^\top \left( L_g(x_t, v_t) - L_g(x^*, v_t) \right)
+ \lambda_t^\top \left( L_g(x_t, v_t) - L_g(x^*, v_t) \right) + (\lambda - \lambda_t)^\top L_g(x_t, v_t).
\]

(A.1)

We then provide bounds for the terms above in expectation. First, by Lemma 9 in Appendix Section A.2, we have

\[
\sum_{t=1}^K \mathbb{E} \left[ L_F(x_t, \pi_{1:2}) - L_F(x, \pi_{1:2}) \right] \leq \frac{40KC_1^2C_2^2}{\eta_0} + \sum_{t=1}^K \frac{\eta_{t-1}}{10} \mathbb{E} \left[ \|x_t - x_{t-1}\|^2 \right] + \sqrt{KL_1}\sigma_2^2 + 3\sqrt{KC_1}\sigma_2.
\]

(A.2)

Secondly, by Lemma 10 in Appendix Section A.3, we obtain that

\[
\sum_{t=1}^K \mathbb{E} \left[ L_F(x_t, \pi_{1:2}) - L_F(x^*, \pi_{1:2}) + \lambda_t^\top \left( L_g(x_t, v_t) - L_g(x^*, v_t) \right) \right] \leq \frac{\eta}{2} \|x_0 - x^*\|^2 + \sum_{t=1}^K \left( \frac{5C_1^2C_2^2}{2\eta} - \frac{3\eta}{10} \mathbb{E} \left[ \|x_{t-1} - x_t\|^2 \right] \right)
+ \sum_{t=1}^K \left( \frac{\alpha}{4} \mathbb{E} \|\lambda_t - \lambda_{t-1}\|^2 + \frac{D_2^2C_2^2}{\alpha} + \frac{5\mathbb{E} \|\lambda_{t-1}\|^2}{} \right).
\]

(A.3)
Thirdly, by Lemma 11 in Appendix Section A.4 and the condition that $\|\lambda\| \leq M_\lambda$, we obtain
\[
\mathbb{E}\left[ \sum_{t=1}^{K} \lambda^T \left( \mathcal{L}_g(x_t, v_t) - \mathcal{L}_g(x_t, v_{t-1}) \right) \right] \leq \sum_{t=1}^{K} \frac{10C_g^2M_\lambda^2}{\eta} + \frac{\eta \mathbb{E}[\|x_t - x_{t-1}\|^2]}{10}.
\] (A.4)

Finally, by Lemma 12 in Appendix Section A.5, we obtain that
\[
\mathbb{E}\left[ \sum_{t=1}^{K} (\lambda - \lambda_t)^T \mathcal{L}_g(x_t, v_t) \right] + \frac{\alpha}{2} \mathbb{E}[\|\lambda - \lambda\|^2] \\
\leq \sqrt{K}M_\lambda \sigma_g + \sum_{t=1}^{K} \left( \frac{5C_g^2\mathbb{E}[\|\lambda - \lambda_t\|^2]}{2\eta} + \frac{\eta}{10} \mathbb{E}[\|x_{t-1} - x_t\|^2] - \frac{\alpha}{4} \mathbb{E}[\|\lambda_t - \lambda_t\|^2] \right) + \frac{\alpha}{2} \mathbb{E}[\|\lambda_0 - \lambda\|^2] + \frac{K\sigma_g^2}{\alpha}
\] (A.5)

Summing (A.1) over $t = 1, 2, \cdots, K$, taking expectations, and combining (A.2), (A.3), (A.4), and (A.5), we conclude that
\[
\sum_{t=1}^{K} \mathbb{E}\left( \mathcal{L}(x_t, \lambda_t, \pi_{1:t}, v_t) - \mathcal{L}(x^*, \lambda_t, \pi_{1:t}, v_t) \right) + \frac{\alpha}{2} \mathbb{E}[\|\lambda - \lambda\|^2] \\
\leq \frac{95KC_1^2C_2^2}{2\eta} + \sqrt{KL_1\sigma_1^2} + 3\sqrt{KC_1\sigma_2} + \frac{\eta}{2} \mathbb{E}[\|x_0 - x^*\|^2] + \frac{K}{\alpha} D_1^2C_1^2 + \sqrt{K}M_\lambda \sigma_g \\
+ \frac{\alpha}{2} \mathbb{E}[\|\lambda_0 - \lambda\|^2] + \frac{K\sigma_g^2}{\alpha} + \sum_{t=1}^{K} \left( \frac{5C_g^2\mathbb{E}[\|\lambda_t\|^2]}{2\eta} + \frac{10C_g^2M_\lambda^2}{\eta} + \frac{5C_g^2\mathbb{E}[\|\lambda - \lambda_t\|^2]}{2\eta} \right),
\]
which completes the proof.

**Lemma 8** (Lemma 2 of [39]). Let $\mathcal{F}_t$ be a filtration and $\{\delta_t\}_{t=1}^{N}$ be a martingale noise sequence such that $\delta_j \in \mathcal{F}_{t+1}$ for $j = 1, 2, \cdots, t$, $\mathbb{E}[\delta_t|\mathcal{F}_t] = 0$, and $\mathbb{E}[\|\delta_t\|^2|\mathcal{F}_t] \leq \sigma^2$. For any random variable $\pi \in \Pi$ correlated with $\{\delta_t\}_{t=1}^{N}$, suppose it is bounded such that $\|\pi\| \leq M_\pi$ uniformly, then
\[
\mathbb{E}\left[ \sum_{t=1}^{N} \pi^T \delta_t \right] \leq \sqrt{N}M_\pi \sigma.
\]

**A.2 Lemma 9 and Its Proof**

**Lemma 9.** Suppose Assumptions 2.1, 2.2 and 2.3 hold. Let $\{(x_t, \lambda_t, \pi_{1:t}, v_t)\}_{t=1}^{N}$ be the solution sequence generated by Algorithm 1 with $\tau_t = t/2$ and $\eta_t = \eta$ for $t = 0, 1, \cdots, N$. Then for any integer $K \leq N$, we have
\[
\sum_{t=1}^{K} \mathbb{E}\left[ \mathcal{L}_F(x_t, \pi_{1:t}) - \mathcal{L}_F(x_t, \pi_{1:t}) \right] \\
\leq \sum_{t=1}^{K} \frac{\eta_t - 1}{10} \mathbb{E}[\|x_t - x_{t-1}\|^2] + \frac{40KC_1^2C_2^2}{\eta} + 2\sqrt{KL_1\sigma_1^2} + 3\sqrt{KC_1\sigma_2}.
\]
Proof: This result can be derived by combining Propositions 5 and 6 of [39]. Here we provide the proof sketch. By decomposing $\mathcal{L}_F(x_t, \pi_{1:2}) - \mathcal{L}_F(x_t, \pi_{1:2,t})$, we have

$$
\mathcal{L}_F(x_t, \pi_{1:2}) - \mathcal{L}_F(x_t, \pi_{1:2,t}) \\
= \mathcal{L}_{f_1}(x_t, \pi_1, \pi_2) - \mathcal{L}_{f_1}(x_t, \pi_1, \pi_{2,t}) + \mathcal{L}_{f_1}(x_t, \pi_1, \pi_{2,t}) - \mathcal{L}_{f_1}(x_t, \pi_{1,t}, \pi_{2,t}).
$$

We provide bound for each term after the above decomposition. First, we consider $\mathcal{L}_{f_1}(x_t, \pi_1, \pi_2) - \mathcal{L}_{f_1}(x_t, \pi_1, \pi_{2,t})$ and obtain

$$
\sum_{t=1}^{K} \mathbb{E}(\mathcal{L}_{f_1}(x_t, \pi_1, \pi_2) - \mathcal{L}_{f_1}(x_t, \pi_1, \pi_{2,t})) = \sum_{t=1}^{K} \mathbb{E}[\mathcal{L}_{f_2}(x_t, \pi_2) - \mathcal{L}_{f_2}(x_t, \pi_{2,t}), \pi_1]
$$

$$
= \sum_{t=1}^{K} \mathbb{E}\left[\langle (\pi_2 - \pi_{2,t})^\top x_t - f_2^*(\pi_2) + f_2^*(\pi_{2,t}), \pi_1 \rangle\right]
$$

$$
= \sum_{t=1}^{K} \mathbb{E}\left[\langle (\pi_2 - \pi_{2,t})^\top x_{t-1} - f_2^*(\pi_2) + f_2^*(\pi_{2,t}), \pi_1 \rangle\right]
$$

$$
+ \sum_{t=1}^{K} \mathbb{E}\left[\langle (\pi_2 - \pi_{2,t})^\top (x_t - x_{t-1}), \pi_1 \rangle\right],
$$

By recalling (2.12) that $\pi_{2,t} \in \text{argmax}_{\pi_2 \in \Pi_2} \{\langle \pi_2, x_{t-1} \rangle - f_2^*(\pi_2)\}$ and using Assumptions 2.3 that $\pi_1 \geq 0$ for non-affine $f_2$, we obtain for both affine ($\pi_2$ is a constant) and non-affine $f_2$,

$$
\mathbb{E}\left[\langle (\pi_2 - \pi_{2,t})^\top x_{t-1} - f_2^*(\pi_2) + f_2^*(\pi_{2,t}), \pi_1 \rangle\right] \leq 0,
$$

which further implies that

$$
\sum_{t=1}^{K} \mathbb{E}(\mathcal{L}_{f_1}(x_t, \pi_1, \pi_2) - \mathcal{L}_{f_1}(x_t, \pi_1, \pi_{2,t})) \leq \sum_{t=1}^{K} \mathbb{E}[\langle (\pi_2 - \pi_{2,t})\pi_1, x_t - x_{t-1} \rangle]
$$

$$
\leq \sum_{t=1}^{K} \left[ \frac{5\mathbb{E}[\|\pi_2 - \pi_{2,t}\pi_1\|^2]}{\eta} + \frac{\eta}{20} \mathbb{E}[\|x_t - x_{t-1}\|^2] \right]
$$

$$
\leq \sum_{t=1}^{K} \left[ \frac{20C^2}{\eta} \right] + \frac{\eta}{20} \mathbb{E}[\|x_t - x_{t-1}\|^2] \quad \text{(A.6)}
$$

The next part is similar to Proposition 6 of [39]. Consider the following decomposition:

$$
\mathbb{E}[\mathcal{L}_{f_1}(x_t, \pi_1, \pi_{2,t}) - \mathcal{L}_{f_1}(x_t, \pi_1, \pi_{2,t})] = \mathbb{E}[\{\pi_1 - \pi_{1,t}, f_2(x_t)\} - \{f_1^*(\pi_1) - f_1^*(\pi_{1,t})\}]
$$

$$
+ \mathbb{E}[\{\pi_1, f_2(x_{t-1})\} - \{f_1^*(\pi_1)\}] + \mathbb{E}[\{\pi_1, \xi_{2,t}\} - \{f_1^*(\pi_{1,t})\}]
$$

$$
+ \mathbb{E}[\{\pi_1, f_2(x_{t-1})\} - \{f_2(x_{t-1})\} + \mathbb{E}[\{\pi_1, f_2(x_{t-1}) - f_2(x_{t-1})\}] = 0 \quad \text{(A.7)}
$$

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Among these terms, the update rule of $\pi_{1,t}$, the strong convexity of $f_1^*$ with respect to $D^*_f$, and the choice of $\tau_t$ imply

$$
\mathbb{E}\left[\sum_{t=1}^{K} (\pi_{1, t} - \pi_{1, t-1}, f_2(x_{t-1}, \xi_{2,t}^{1})) - (f_1^*(\pi_1) - f_1^*(\pi_{1,t}))\right]
\leq \mathbb{E}\left[\sum_{t=1}^{K} \tau_t D^*_f(\pi_{1, t-1}) - (\tau_t + 1) D^*_f(\pi_{1, t-1}) - \tau_t D^*_f(\pi_{1, t}; \pi_{1, t-1})\right]
\leq -\mathbb{E}\left[\sum_{t=1}^{K} \tau_t D^*_f(\pi_{1, t}; \pi_{1, t-1})\right].
$$

The $1/L_f$-strong convexity of $D^*_f$, the choice of $\tau_t$ and the Cauchy-Schwartz inequality imply

$$
\mathbb{E}\sum_{t=1}^{K} \left[ (\pi_{1, t} - \pi_{1, t-1})(f_2(x_{t-1}, \xi_{2,t}^{1}) - f_2(x_{t-1})) - \tau_t D^*_f(\pi_{1, t}; \pi_{1, t-1})\right]
\leq \sum_{t=2}^{K} \frac{L_f \sigma_f^2}{\tau_t} + \sqrt{\mathbb{E}[||\pi_{1, t} - \pi_{1, 0}||^2]} \sqrt{\mathbb{E}[||f_2(x_0, \xi_{2,0}) - f_2(x_0)||^2]}
\leq 2L_f \sigma_f^2 \log K + 2C_f \sigma_f^2
\leq 2\sqrt{K}L_f \sigma_f^2 + 2\sqrt{KC_f \sigma_f^2}.
$$

Lemma 2 of [39] implies that

$$
\mathbb{E}\sum_{t=1}^{K} (\pi_{1, t} f_2(x_{t-1}) - f_2(x_{t-1}, \xi_{2,t}^{1})) \leq \sqrt{KC_f \sigma_f^2}.
$$

Similar to (A.6), we also have

$$
\mathbb{E}\sum_{t=1}^{K} (\pi_{1, t} - \pi_{1, t}, f_2(x_t) - f_2(x_{t-1})) \leq \sum_{t=1}^{K} \left[ \frac{20C_f^2 C_{f_2}^2}{\eta} + \frac{\eta}{20} \mathbb{E}[||x_t - x_{t-1}||^2]\right].
$$

Combining them with the decomposition in (A.7), we obtain

$$
\sum_{t=1}^{K} \mathbb{E}\left( \mathcal{L}_{f_1}(x_t, \pi_1, \pi_{2,t}) - \mathcal{L}_{f_1}(x_t, \pi_{1,t}, \pi_{2,t}) \right)
\leq \sum_{t=1}^{K} \left[ \frac{20C_f^2 C_{f_2}^2}{\eta} + \frac{\eta}{20} \mathbb{E}[||x_t - x_{t-1}||^2]\right] + 2\sqrt{K}L_f \sigma_f^2 + 3\sqrt{KC_f \sigma_f^2}.
$$

The desired result then follows from adding to preceding inequality to (A.6).
A.3 Lemma 10 and Its Proof

**Lemma 10.** Suppose Assumptions 2.1, 2.2, and 2.3 hold. Let \(\{(x_t, \lambda_t, \pi_{1:2}, v_t)\}_{t=1}^N\) be the solution sequence generated by Algorithm 1 with \(\tau_t = t/2\), \(\alpha_t = \alpha\), and \(\eta_t = \eta\) for all \(0 \leq t \leq N\). We have

\[
\sum_{t=1}^K \mathbb{E}\left[\mathcal{L}_F(x_t, \pi_{1:2}) - \mathcal{L}_F(x^*, \pi_{1:2}) + \lambda_t^\top \left(\mathcal{L}_g(x_t, v_t) - \mathcal{L}_g(x^*, v_t)\right)\right]
\]

\[
\leq \frac{\eta}{2} \mathbb{E}[\|x_{t-1} - x^*\|^2] - \frac{K}{10} \sum_{t=1}^K \mathbb{E}[\|x_{t-1} - x_t\|^2] + \sum_{t=1}^K \frac{5C_f^2 C_g^2}{2\eta} + \sum_{t=1}^K \frac{1}{\alpha} D_x^2 C_g^2 \sum_{t=1}^K \frac{5\mathbb{E}[\|\lambda_{t-1}\|^2]C_g^2}{2\eta}.
\]

**Proof.** Recall Algorithm 1, for \(t = 1, 2, \ldots, K\), we note that \(\pi_2, \pi_{1:t}, v_t \lambda_t \in \mathbb{R}^{dx}\) and obtain

\[
\mathcal{L}_F(x_t, \pi_{1:2}) - \mathcal{L}_F(x^*, \pi_{1:2}) + \lambda_t^\top \left(\mathcal{L}_g(x_t, v_t) - \mathcal{L}_g(x^*, v_t)\right)
\]

\[
= \left(\pi_2 \pi_{1:t}, x_t - x^*\right) + \left(v_t \lambda_t, x_t - x^*\right)
\]

\[
= \left(\pi_0^T \pi_{1:t}, x_t - x^*\right) + \left(v_0^T \lambda_{t-1}, x_t - x^*\right)
\]

\[
+ \left(\pi_{2:t} \pi_{1:t} - \pi_0^T \pi_{1:t}, x_t - x^*\right) + \left(v_t \lambda_t - v_0^T \lambda_{t-1}, x_t - x^*\right),
\]

and

\[
x_t = \arg\min_{x \in \mathcal{X}} \left\{\left(\pi_0^T \pi_{1:t} + v_0^T \lambda_{t-1}, x\right) + \frac{\eta_{t-1}}{2} \|x_{t-1} - x\|^2\right\}.
\]

By applying Lemma 7 to the above update rule, since \(x^* \in \mathcal{X}\), we have

\[
\left(\pi_0^T \pi_{1:t} + v_0^T \lambda_{t-1}, x_t - x^*\right) \leq \frac{\eta}{2} \|x_{t-1} - x^*\|^2 - \frac{\eta}{2} \|x_{t-1} - x_t\|^2 - \frac{\eta}{2} \|x_t - x^*\|^2,
\]

which further yields

\[
\mathcal{L}_F(x_t, \pi_{1:2}) - \mathcal{L}_F(x^*, \pi_{1:2}) + \lambda_t^\top \left(\mathcal{L}_g(x_t, v_t) - \mathcal{L}_g(x^*, v_t)\right)
\]

\[
\leq \frac{\eta}{2} \|x_{t-1} - x^*\|^2 - \frac{\eta}{2} \|x_{t-1} - x_t\|^2 - \frac{\eta}{2} \|x_t - x^*\|^2
\]

\[
+ \left(\pi_{2:t} \pi_{1:t} - \pi_0^T \pi_{1:t}, x_t - x^*\right) + \left(v_t \lambda_t - v_0^T \lambda_{t-1}, x_t - x^*\right).
\]

Then, we decompose \(\Delta_0^t\) by

\[
\Delta_0^t = \left(\pi_{2:t} \pi_{1:t} - \pi_0^T \pi_{1:t}\right)(x_{t-1} - x^*) + \left(\pi_{2:t} \pi_{1:t} - \pi_0^T \pi_{1:t}\right)(x_t - x_{t-1}).
\]

We note that given \(x_{t-1}, y_t\),

\[
\mathbb{E}\left[\pi_{2:t} \pi_{1:t} - \pi_0^T \pi_{1:t} \mid x_{t-1}, y_t\right] = 0,
\]

and

\[
\mathbb{E}[\|\pi_{2:t} \pi_{1:t} - \pi_0^T \pi_{1:t}\|^2 \mid x_{t-1}, y_t]\n\]

\[
\leq \mathbb{E}[\|\pi_{1:t} - \pi_0^T \pi_{1:t}\|^2 \mid x_{t-1}, y_t] \mathbb{E}[[\pi_{2:t} \pi_{1:t}\|^2 \mid x_{t-1}] + \mathbb{E}[[\|\pi_0^T \pi_{1:t}\|^2 \mid x_{t-1}, y_t]
\]

\[
\leq 2C_f^2 C_g^2.
\]

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By the independence between \((x_{t-1} - x^*)\) and the mean-zero term \((\pi_{2,t} \pi_{1,t} - \pi_{2,t}^0 \pi_{1,t}^0)\), we have 
\[
\mathbb{E}\left[ (\pi_{2,t} \pi_{1,t} - \pi_{2,t}^0 \pi_{1,t}^0)^\top (x_{t-1} - x^*) \right] = 0.
\]
Moreover, to handle the possible correlation between \((x_t - x_{t-1})\) and \((\pi_{2,t} \pi_{1,t} - \pi_{2,t}^0 \pi_{1,t}^0)\), we utilize the fact that \(ab \leq \frac{5a^2}{2} + \frac{b^2}{10}\) for all \(a, b \in \mathbb{R}\), and obtain
\[
\mathbb{E}\left[ \sum_{t=1}^{K} \Lambda_0^t \right] = \mathbb{E}\left[ \sum_{t=1}^{K} \langle \pi_{2,t} \pi_{1,t} - \pi_{2,t}^0 \pi_{1,t}^0, x_t - x_{t-1} \rangle \right]
\leq \sum_{t=1}^{K} \left( \frac{5C_f^2 C_g^2}{2\eta} + \frac{\eta\mathbb{E}[\|x_t - x_{t-1}\|^2]}{10} \right).
\] (A.9)

We then decompose \(\Lambda_0^t\) as
\[
\Lambda_0^t = \underbrace{\langle v_t (\lambda_t - \lambda_{t-1}), x_t - x^* \rangle}_{\Lambda_1^t} + \underbrace{\langle (v_t - v_t^0) \lambda_{t-1}, x_t - x^* \rangle}_{\Lambda_2^t}.
\]

By the fact that \(2ab \leq a^2 + b^2\) and Assumptions 2.1 and 2.2 that \(\|x_t - x^*\| \leq D_X\) and \(\|v_t\| \leq C_g\), for any \(\alpha > 0\), we have
\[
\mathbb{E}[\Lambda_1^t] \leq \frac{\alpha}{4} \mathbb{E}[\|\lambda_t - \lambda_{t-1}\|^2] + \frac{1}{\alpha} \mathbb{E}[\|v_t^\top (x_t - x^*)\|^2]
\leq \frac{\alpha}{4} \mathbb{E}[\|\lambda_t - \lambda_{t-1}\|^2] + \frac{1}{\alpha} D_X^2 C_g^2.
\] (A.10)

Meanwhile, since \(\mathbb{E}[v_t - v_t^0] = 0\) and \(\mathbb{E}[\|v_t - v_t^0\|^2] \leq C_g^2\), we obtain
\[
\mathbb{E}\left[ \sum_{t=1}^{K} \Lambda_2^t \right] = \mathbb{E}\left[ \sum_{t=1}^{K} \langle (v_t - v_t^0) \lambda_{t-1}, x_t - x^* \rangle \right] + \mathbb{E}\left[ \sum_{t=1}^{K} \langle (v_t - v_t^0) \lambda_{t-1}, x_t - x_{t-1} \rangle \right]
\leq \sum_{t=1}^{K} \mathbb{E}\left[ \frac{5\| (v_t - v_t^0) \lambda_{t-1} \|^2}{2\eta} \right] + \sum_{t=1}^{K} \frac{\eta\mathbb{E}[\|x_t - x_{t-1}\|^2]}{10}
\leq \sum_{t=1}^{K} \frac{5C_f^2 \mathbb{E}[\|\lambda_{t-1}\|^2]}{2\eta} + \sum_{t=1}^{K} \frac{\eta\mathbb{E}[\|x_t - x_{t-1}\|^2]}{10},
\] (A.11)

where the second equality holds by the independence between \((v_t - v_t^0) \lambda_{t-1}\) and \((x_{t-1} - x^*)\), and the second last inequality holds due to the fact that \(ab \leq \frac{5a^2}{2} + \frac{b^2}{10}\). Taking expectations on both sides of (A.8), summing over \(t = 1, \cdots, N\), and substituting (A.9), (A.10), and (A.11) into it, we
obtain
\[
\sum_{t=1}^{K} \mathbb{E}\left[ \mathcal{L}_F(x_t, \pi_{1:2,t}) - \mathcal{L}_F(x^*, \pi_{1:2,t}) + \lambda_t^T \left( \mathcal{L}_g(x_t, v_t) - \mathcal{L}_g(x^*, v_t) \right) \right] \\
\leq \sum_{t=1}^{K} \mathbb{E}\left[ \frac{\eta}{2} \| x_t - x^* \|^2 - \frac{\eta}{2} \| x_t - x_{t+1} \|^2 - \frac{\eta}{2} \| x_{t+1} - x^* \| + \Delta_t^\lambda + \Lambda_t^\lambda + \Lambda_2^t \right] \\
\leq \frac{\eta}{2} \mathbb{E}\left[ \| x_{t-1} - x^* \|^2 \right] - \sum_{t=1}^{K} \frac{3\eta}{10} \mathbb{E}\left[ \| x_{t-1} - x_t \|^2 \right] + \sum_{t=1}^{K} \frac{5C^2_{\lambda}C^2_{g}}{2\eta t} \\
+ \sum_{t=1}^{K} \frac{\alpha}{4} \mathbb{E}\left[ \| \lambda_t - \lambda_{t-1} \|^2 \right] + \sum_{t=1}^{K} \frac{D^2_{\lambda}C^2_{g}}{\alpha} + \sum_{t=1}^{K} \frac{5\mathbb{E}[\| \lambda_{t-1} \|^2]C^2_{g}}{2\eta},
\]
where the last inequality holds by the fact that \( \eta_t = \eta \) for \( t = 1, 2, \cdots, N \). This completes the proof. \( \square \)

### A.4 Lemma 11 and Its Proof

**Lemma 11.** Suppose Assumptions 2.1, 2.2 and 2.3 hold. Let \( \{(x_t, \lambda_t, \pi_{1:2,t}, v_t)\}_{t=1}^{K} \) be the solution sequence generated by Algorithm 1 with \( \tau_t = t/2 \), \( \alpha_t = \alpha \), and \( \eta_t = \eta \) for all \( 0 \leq t \leq N \). Let \( \lambda \in \mathbb{R}^m_+ \) be a nonnegative random variable whose norm is bounded such that \( \| \lambda \| \leq M_\lambda \) uniformly. Then for any \( K \leq N \), we have

\[
\mathbb{E}\left[ \sum_{t=1}^{K} \lambda^T (\mathcal{L}_g(x_t, v) - \mathcal{L}_g(x_t, v_t)) \right] \leq \frac{10M^2_\lambda C^2_g}{\eta} + \frac{\eta \mathbb{E}[\| x_t - x_{t-1} \|^2]}{10}.
\]

**Proof.** Recall that the update rule is that

\[
v_t = \operatorname{argmax}_{v \in \mathcal{V}} \left\{ \langle v, x_{t-1} \rangle - g^*(v) \right\}.
\]

By the fact that \( \lambda \geq 0 \), we have

\[
\lambda^T (\mathcal{L}_g(x_t, v) - \mathcal{L}_g(x_t, v_t)) \\
= \lambda^T \left( (v - v_t)^T x_{t-1} - (g^*(v) - g^*(v_t)) + (v - v_t)^T (x_t - x_{t-1}) \right) \\
\leq \lambda^T \left( (v - v_t)^T (x_t - x_{t-1}) \right).
\]

By taking expectations on both sides of the above inequality, we have

\[
\mathbb{E}\left[ \lambda^T (\mathcal{L}_g(x_t, v) - \mathcal{L}_g(x_t, v_t)) \right] \leq \mathbb{E}\left[ \frac{5\| v - v_t \|^2}{2\eta} + \frac{\eta \| x_t - x_{t-1} \|^2}{10} \right] \\
\leq \frac{5M^2_\lambda \mathbb{E}[\| v - v_t \|^2]}{2\eta t} + \frac{\eta \mathbb{E}[\| x_t - x_{t-1} \|^2]}{10} \leq \frac{10C^2_g M^2_\lambda}{\eta} + \frac{\eta \mathbb{E}[\| x_t - x_{t-1} \|^2]}{10},
\]

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where the first inequality uses the fact that \( ab \leq \frac{5a^2}{2} + \frac{b^2}{10} \), the second inequality holds by the condition that \( \|\lambda\| \leq M_\lambda \), and the third inequality holds by Assumption 2.2 that \( \|v - v_t\|^2 \leq 2\|v\|^2 + 2\|v_t\|^2 \leq 4C_g^2 \). Summing the above inequality over \( t = 1, \ldots, K \), we obtain

\[
\mathbb{E}\left[ \sum_{t=1}^{K} \lambda^\top (L_g(x_t, v_t) - L_g(x_t, v_t)) \right] \leq \frac{10C_g^2M_\lambda^2}{\eta} + \frac{\eta\mathbb{E}[\|x_t - x_{t-1}\|^2]}{10},
\]

which completes the proof.

\[\square\]

### A.5 Lemma 12 and Its Proof

**Lemma 12.** Suppose Assumptions 2.1, 2.2, and 2.3 hold. Let \( \{(x_t, \lambda_t, \pi_{1:2}, v_t)\}_{t=1}^{N} \) be the solution sequence generated by Algorithm 1 with \( \tau_t = t/2 \), \( \alpha_t = \alpha \), and \( \eta_t = \eta \) for all \( 0 \leq t \leq N \). Let \( \lambda \in \mathbb{R}^n \) be a bounded nonnegative random variable such that \( \|\lambda\| \leq M_\lambda \). For any integer \( K \leq N \), we have

\[
\mathbb{E}\left[ \sum_{t=1}^{K} (\lambda - \lambda_t)^\top L_g(x_t, v_t) \right] + \frac{\alpha K - 1}{2} \mathbb{E}[\|\lambda_K - \lambda\|^2] \\
\leq \sqrt{K}M_\lambda \sigma_g + \sum_{t=1}^{K} \left( \frac{5C_g^2\mathbb{E}[\|\lambda - \lambda_t\|^2]}{2\eta} + \frac{\eta}{10} \mathbb{E}[\|x_{t-1} - x_t\|^2] - \frac{\alpha}{4} \mathbb{E}[\|\lambda_{t-1} - \lambda_t\|^2] \right) \\
+ \frac{\alpha}{2} \mathbb{E}[\|\lambda_0 - \lambda\|^2] + \frac{K \sigma_g^2}{\alpha}.
\]

**Proof:** By decomposing \( (\lambda - \lambda_t)^\top L_g(x_t, v_t) \), we have,

\[
(\lambda - \lambda_t)^\top L_g(x_t, v_t) \\
= (\lambda - \lambda_t)^\top \left( L_g(x_t, v_t) - L_g(x_{t-1}, v_t) \right) + (\lambda - \lambda_t)^\top [L_g(x_{t-1}, v_t) - g(x_{t-1}, \zeta_{t-1}^1)] \\
+ (\lambda - \lambda_t)^\top g(x_{t-1}, \zeta_{t-1}^1) \\
= \underbrace{\langle v_t(\lambda - \lambda_t), x_{t-1} - x_t \rangle}_{T_{1,t}} + \underbrace{(\lambda - \lambda_t)^\top g(x_{t-1})}_{T_{2,t}} - \underbrace{(\lambda - \lambda_t)^\top g(x_{t-1}, \zeta_{t-1}^1)}_{T_{3,t}}.
\]

We provide bounds for \( T_{1,t}, T_{2,t}, \) and \( T_{3,t} \) in our analysis.

First, for \( T_{1,t} \), by using the fact that \( \langle a, b \rangle \leq \frac{5\|a\|^2}{2} + \frac{\|b\|^2}{10} \) and Assumption 2.2 that \( \|v_t\|^2 \leq C_g^2 \), we obtain

\[
\mathbb{E}[\langle v_t(\lambda - \lambda_t), x_{t-1} - x_t \rangle] \leq \frac{5\mathbb{E}[\|v_t(\lambda - \lambda_t)\|^2]}{2\eta} + \frac{\eta\mathbb{E}[\|x_{t-1} - x_t\|^2]}{10} \\
\leq \frac{5C_g^2\mathbb{E}[\|\lambda - \lambda_t\|^2]}{2\eta} + \frac{\eta\mathbb{E}[\|x_{t-1} - x_t\|^2]}{10}.
\]

Second, consider \( T_{2,t} \), by the independence between \( \lambda_{t-1} \) and the fact that \( \mathbb{E}[g(x_{t-1}) - g(x_{t-1}, \zeta_{t-1}^1)] = 0 \), we have

\[
\mathbb{E}[\lambda_{t-1}^\top (g(x_{t-1}) - g(x_{t-1}, \zeta_{t-1}^1))] = 0,
\]

and the fact that
which further implies that
\[
E[T_{2,t}] = E \left[ (\lambda - \lambda_{t-1})^T [g(x_{t-1}) - g(x_{t-1}, \zeta^1_{t-1})] \right] \\
+ E \left[ (\lambda_{t-1} - \lambda_t)^T [g(x_{t-1}) - g(x_{t-1}, \zeta^1_{t-1})] \right] \\
= E \left[ \lambda^T [g(x_{t-1}) - g(x_{t-1}, \zeta^1_{t-1})] \right] + E \left[ (\lambda_{t-1} - \lambda_t)^T [g(x_{t-1}) - g(x_{t-1}, \zeta^1_{t-1})] \right].
\] (A.14)

Recall that \( \lambda \) is random but bounded such that \( \| \lambda \| \leq M_\lambda \), by Lemma 8 and Assumption 2.2 that \( E[\|g(x_{t-1}) - g(x_{t-1}, \zeta^1_{t-1})\|^2] \leq \sigma_g^2 \), we obtain
\[
E \left[ \sum_{t=1}^{K} \lambda^T \left( g(x_{t-1}) - g(x_{t-1}, \zeta^1_{t-1}) \right) \right] \leq \sqrt{K} M_\lambda \sigma_g.
\] (A.15)

Meanwhile, by the fact that \( \langle a, b \rangle \leq \frac{\|a\|^2}{4} + \frac{\|b\|^2}{\alpha} \) for \( \alpha > 0 \), we bound the second term within the right side of (A.14) by
\[
E \left[ (\lambda_{t-1} - \lambda_t)^T [g(x_{t-1}) - g(x_{t-1}, \zeta^1_{t-1})] \right] \\
\leq \frac{\alpha E[\|\lambda_{t-1} - \lambda_t\|^2]}{4} + \frac{E[\|g(x_{t-1}) - g(x_{t-1}, \zeta^1_{t-1})\|^2]}{\alpha} \leq \frac{\alpha E[\|\lambda_{t-1} - \lambda_t\|^2]}{4} + \frac{\sigma_g^2}{\alpha}.
\] (A.16)

Summing (A.14) over \( t = 1, \ldots, K \) and applying (A.15) and (A.16), we obtain
\[
\sum_{t=1}^{K} E[T_{2,t}] \leq \sqrt{K} M_\lambda \sigma_g + \sum_{t=1}^{K} \left( \frac{\alpha E[\|\lambda_{t-1} - \lambda_t\|^2]}{4} + \frac{\sigma_g^2}{\alpha} \right).
\] (A.17)

Third, consider \( T_{3,t} \), by recalling (2.16) that
\[
\lambda_t = \arg\max_{\lambda \in \mathbb{R}^m} \left\{ \langle g(x_{t-1}, \zeta^1_{t-1}), \lambda \rangle - \frac{\alpha}{2} \|\lambda_t - \lambda\|^2 \right\},
\]
and by Lemma 7, we have
\[
T_{3,t} = (\lambda - \lambda_t)^T g(x_{t-1}, \zeta^1_{t-1}) = -(\lambda_t - \lambda)^T g(x_{t-1}, \zeta^1_{t-1}) \\
\leq \frac{\alpha}{2} \|\lambda_{t-1} - \lambda\|^2 - \frac{\alpha}{2} \|\lambda_{t-1} - \lambda_t\|^2 - \frac{\alpha}{2} \|\lambda_t - \lambda\|^2.
\] (A.18)

Finally, summing (A.12) over \( t = 1, \ldots, N \), taking expectations on both sides, and plugging in (A.13), (A.17), and (A.18), we conclude that
\[
E \left[ \sum_{t=1}^{K} (\lambda - \lambda_t)^T \mathcal{L}_g(x_t, v_t) \right] + \frac{\alpha}{2} E[\|\lambda_K - \lambda\|^2] \\
\leq \sqrt{K} M_\lambda \sigma_g + \sum_{t=1}^{K} \left( \frac{5C_g^2 E[\|\lambda - \lambda_t\|^2]}{2\eta} + \frac{\eta}{10} E[\|x_{t-1} - x_t\|^2] - \frac{\alpha}{4} E[\|\lambda_{t-1} - \lambda_t\|^2] \right) \\
+ \frac{\alpha}{2} E[\|\lambda_0 - \lambda\|^2] + \frac{K \sigma_g^2}{\alpha},
\]
which completes the proof. \( \square \)
B Proof of Results in Section 3

Lemma 13. Suppose Assumptions 2.1, 2.3, and 3.1 hold. Let \( \{(x_t, \lambda_t, \pi_{1:2}, t, v_{1:2})\} \) be the solution sequence generated by Algorithm 2 with \( \tau_t = \rho \), \( \eta_t = \eta \), and \( \alpha_t = \alpha \) for \( t = 1, 2, \ldots, N \). Letting \( \lambda \in \mathbb{R}^m_+ \) be a bounded random variable satisfying \( \|\lambda\| \leq \lambda \) uniformly, then for any \( K \leq N \), we have

\[
\sum_{t=1}^{K} \mathbb{E} \left[ \lambda^\top \left( \mathcal{L}_2(x_t, v_{1:2}) - \mathcal{L}_2(x_t, v_{1:2}) \right) \right] 
\leq \sum_{t=1}^{K} \frac{\eta}{10} \mathbb{E}[\|x_t - x_{t-1}\|^2] + \frac{60KC^2g_2C^2g_2M^2}{\eta} + 2\sqrt{K}Lg_1M\lambda^2g_2 + 4\sqrt{K}M\lambda Cg_1\sigma g_2.
\]

Proof: This result is similar to Lemma 9 with an additional nonnegative (random) variable \( \lambda \). We start with the following decomposition that

\[
\lambda^\top (\mathcal{L}_2(x_t, v_{1:2}) - \mathcal{L}_2(x_t, v_{1:2})) = \lambda^\top (\mathcal{L}_1(x_t, v_{1:2}) - \mathcal{L}_1(x_t, v_{1:2})) + \lambda^\top (\mathcal{L}_2(x_t, v_{1:2}) - \mathcal{L}_2(x_t, v_{1:2})).
\]

First, we recall (3.8) that \( v_{2:t} \in \arg\max_{v_2 \in V_2} \{v_2^\top x_{t-1} - g_2^*(v_2)\} \). By Assumption 3.1 (g) that for each compositional constraint \( G^{(j)} = g_1^{(j)} \circ g_2^{(j)}(x) \), the outer-level function \( g_1^{(j)} \) is monotone nondecreasing for non-affine inner-level function \( g_2^{(j)} \), we obtain

\[
\mathbb{E} \left[ \left( (v_2 - v_{2:t})^\top x_{t-1} - g_2^*(v_2) + g_2^*(v_{2:t}), v_1 \right) \right] \leq 0.
\]

Because \( \lambda \geq 0 \), by following (A.6) in Lemma 9, we have

\[
\sum_{t=1}^{K} \mathbb{E} \left[ \lambda^\top (\mathcal{L}_1(x_t, v_{1:2}) - \mathcal{L}_1(x_t, v_{1:2})) \right] \leq \sum_{t=1}^{K} \mathbb{E} \left[ ((v_2 - v_{2:t})v_1 \lambda, x_t - x_{t-1}) \right]
\leq \sum_{t=1}^{K} \left[ \frac{10\mathbb{E}[\|v_2 - v_{2:t}\|\lambda^2]}{\eta} + \frac{\eta}{20} \mathbb{E}[\|x_t - x_{t-1}\|^2] \right]
\leq \sum_{t=1}^{K} \left[ \frac{40C^2g_1C^2g_2M^2}{\eta} + \frac{\eta}{20} \mathbb{E}[\|x_t - x_{t-1}\|^2] \right],
\]

where the last inequality uses the fact that \( \|\lambda\| \leq \lambda \). Second, by Proposition 15 in [39], the \( L_{g_1} \) smoothness of \( g_1 \) implies \( \lambda^\top D g_1^*(v_1; \tilde{v}_1) \geq \frac{\lambda^\top (v_1 - \tilde{v}_1)}{2\|\lambda\|L_{g_1}} \geq \frac{\lambda^\top (v_1 - \tilde{v}_1)}{2M\lambda L_{g_1}} \), which further yields

\[
\sum_{t=1}^{K} \mathbb{E} \left[ \lambda^\top (v_1, t - v_{1:t-1})^\top (g_2(x_{t-1}, \zeta_{2,t-1}) - g_2(x_{t-1})) - \rho_t \lambda^\top D g_1^*(v_1, t - v_{1:t-1}) \right]
\leq \sum_{t=2}^{K} \frac{M\lambda L_{g_1}\sigma g_2}{\rho_t} + \sqrt{\mathbb{E}[\|\lambda\|^2]\|v_{1:1} - v_{1:0}\|^2} \sqrt{\mathbb{E}[\|g_2(x_0, \zeta_{2,0}) - g_2(x_0)\|^2]}
\leq 2L_{g_1}M\sigma g_2^2 \log K + 2M\lambda Cg_1\sigma g_2
\leq 2\sqrt{K}L_{g_1}M\lambda \sigma g_2^2 + 2\sqrt{K}M\lambda Cg_1\sigma g_2.
\]
By substituting the above bound into Proposition 6 of [39], we further obtain that
\[
\sum_{t=1}^{K} \mathbb{E} \left[ \lambda_t \left( L_{g_1}(x_t, v_1, v_2, t) - L_{g_1}(x_{t-1}, v_1, v_2, t) \right) \right] \\
\leq \sum_{t=1}^{K} \left[ \frac{20C^2_{g_1}C^2_{g_2}M^2_{\lambda}}{\eta} + \frac{\eta}{20} \mathbb{E} \left[ \|x_t - x_{t-1}\|^2 \right] \right] + 2\sqrt{K}L_{g_1}M\lambda_1\sigma_{g_2}^2 + 4\sqrt{K}M_\lambda C_{g_1}\sigma_{g_2}. 
\]

The desired result then follows by combining the preceding inequalities. □

**Lemma 14.** Suppose Assumptions 2.1, 2.3, and 3.1 hold, and Algorithm 2 generates \(\{(x_t, \lambda_t, \pi_t, v_t)\}_{t=1}^{N}\) by setting \(\rho_t = \tau_t = \frac{t}{T}, \eta_t = \eta, \) and \(\alpha_t = \alpha\) for \(t \leq N\). Then for any integer \(K \leq N\), we have
\[
\sum_{t=1}^{K} \mathbb{E} \left[ L_F(x_t, \pi_{1:2}, t) - L_F(x^*, \pi_{1:2}, t) + \lambda_t^\top \left( L_G(x_t, v_{1:2}, t) - L_G(x^*, v_{1:2}, t) \right) \right] \\
\leq \frac{\eta}{2} \|x_0 - x^*\|^2 - \sum_{t=1}^{K} \frac{3\eta - 1}{10} \mathbb{E} \left[ \|x_{t-1} - x_t\|^2 \right] + \sum_{t=1}^{K} \frac{5C^2_{f_1}C^2_{f_2}}{2\eta} + \sum_{t=1}^{K} \frac{C^2_{g_1}C^2_{g_2}}{\alpha} \mathbb{E} \left[ \|\lambda_t - \lambda_{t-1}\|^2 \right].
\]

**Proof:** Recall the update rule for \(x_t\) (3.9) that
\[
x_t = \arg\min_{x \in \mathcal{X}} \left\{ \langle \pi^0_{1:t}, \pi_{1:2}^0 + v^0_{1:t}, \lambda_{t-1}, x \rangle + \frac{\eta}{2} \|x_{t-1} - x\|^2 \right\}.
\]
By following analogous analysis in Lemma 10, and applying the three-point Lemma 7 to the above update rule, we have
\[
L_F(x_t, \pi_{1:2}, t) - L_F(x^*, \pi_{1:2}, t) + \lambda_t^\top \left( L_G(x_t, v_{1:2}, t) - L_G(x^*, v_{1:2}, t) \right) \\
= \langle x_t - x^* \rangle^\top \pi_{2:t} \pi_{1,t} + \langle x_t - x^* \rangle^\top v_{2:t} v_{1,t} \lambda_t \\
\leq \frac{\eta}{2} \|x_{t-1} - x^*\|^2 - \frac{\eta}{2} \|x_{t-1} - x_t\|^2 - \frac{\eta}{2} \|x_t - x^*\|^2 \\
+ \langle \pi_{2:t} \pi_{1:t} - \pi^0_{2:t} \pi^0_{1:t}, x_t - x^* \rangle + \tilde{A}_{1,t} + \tilde{A}_{2,t},
\]
where
\[
\tilde{A}_{1,t} = (x_t - x^*)^\top v_{2,t} v_{1,t} (\lambda_t - \lambda_{t-1}) \quad \text{and} \quad \tilde{A}_{2,t} = (x_t - x^*)^\top (v_{2,t} v_{1,t} - v^0_{2:t} v^0_{1:t}) \lambda_{t-1}.
\]
We note that a bound for \(\langle \pi_{2:t} \pi_{1:t} - \pi^0_{2:t} \pi^0_{1:t}, x_t - x^* \rangle\) has been provided in (A.9). Specifically, we have
\[
\mathbb{E} \left[ \sum_{t=1}^{K} \langle \pi_{2:t} \pi_{1:t} - \pi^0_{2:t} \pi^0_{1:t}, x_t - x^* \rangle \right] = \mathbb{E} \left[ \sum_{t=1}^{K} \langle \pi_{2:t} \pi_{1:t} - \pi^0_{2:t} \pi^0_{1:t}, x_t - x_{t-1} \rangle \right] \\
\leq \sum_{t=1}^{K} \left( \frac{5C^2_{f_1}C^2_{f_2}}{2\eta} + \frac{\eta \mathbb{E} \left[ \|x_t - x_{t-1}\|^2 \right]}{10} \right). 
\]
To bound $\widetilde{\Lambda}_{1,t}$, by following the analysis of (A.10), we have

$$
\mathbb{E}[\widetilde{\Lambda}_{1,t}] \leq \frac{\alpha}{4} \mathbb{E}[\|\lambda_t - \lambda_{t-1}\|^2] + \frac{1}{\alpha} \mathbb{E}[\|(x_t - x^*)^\top v_{2,t}v_{1,t}\|^2] \\
\leq \frac{\alpha}{4} \mathbb{E}[\|\lambda_t - \lambda_{t-1}\|^2] + \frac{D_X^2 C_{g1}^2 C_{g2}^2}{\alpha}.
$$

(B.3)

Meanwhile, we observe from Algorithm 2 that $\mathbb{E}[x_{t-1}^\top (v_{2,t}v_{1,t}^0 - v_{2,t}v_{1,t})\lambda_{t-1}] = 0$. By following the analysis of (A.11), we have

$$
\begin{align*}
\mathbb{E} \left[ \sum_{t=1}^{K} \widetilde{\Lambda}_{2,t} \right] &= \sum_{t=1}^{K} \mathbb{E}[(x_t - x_{t-1})^\top (v_{2,t}v_{1,t} - v_{0,t}v_{1,t})\lambda_{t-1}] \\
&\leq \sum_{t=1}^{K} \mathbb{E} \left[ \frac{5}{2\eta} \|(x_t^0 v_{0,t}^0 - v_{0,t}v_{1,t})\lambda_{t-1}\|^2 \right] + \sum_{t=1}^{K} \frac{\eta \mathbb{E}[\|x_t - x_{t-1}\|^2]}{10} \\
&\leq \sum_{t=1}^{K} \frac{5C_{g1}^2 C_{g2}^2 \mathbb{E}[\|\lambda_{t-1}\|^2]}{2\eta} + \sum_{t=1}^{K} \frac{\eta \mathbb{E}[\|x_t - x_{t-1}\|^2]}{10},
\end{align*}
$$

(B.4)

where the last inequality uses the fact that $\lambda_{t-1}$ and $v_{2,t}v_{1,t}^0 - v_{2,t}v_{1,t}$ are independent. Summing (B.1) over $t = 1, 2, \cdots, K$, and combining (B.2), (B.3), and (B.4), we obtain

$$
\begin{align*}
\sum_{t=1}^{K} \mathbb{E} &\left[ \mathcal{L}_F(x_t, \pi_{1,t}, \pi_{2,t}) - \mathcal{L}_F(x^*, \pi_{1,t}, \pi_{2,t}) + \lambda_t^\top \left( \mathcal{L}_G(x_t, v_t) - \mathcal{L}_G(x^*, v_t) \right) \right] \\
&\leq \frac{\eta}{2} \|x_0 - x^*\|^2 + \sum_{t=1}^{K} \left( \frac{\eta}{10} - \frac{\eta}{2} \right) \mathbb{E}[\|x_{t-1} - x_t\|^2] + \sum_{t=1}^{K} \frac{5C_{f1}^2 C_{f2}^2}{2\eta} \\
&\quad + \sum_{t=1}^{K} \frac{\alpha}{4} \mathbb{E}[\|\lambda_t - \lambda_{t-1}\|^2] + \sum_{t=1}^{K} \frac{D_X^2 C_{g1}^2 C_{g2}^2}{\alpha} + \sum_{t=1}^{K} \frac{5C_{g1}^2 C_{g2}^2 \mathbb{E}[\|\lambda_{t-1}\|^2]}{2\eta} \\
&= \frac{\eta}{2} \|x_0 - x^*\|^2 - \sum_{t=1}^{K} \frac{3\eta}{10} \mathbb{E}[\|x_{t-1} - x_t\|^2] + \sum_{t=1}^{K} \frac{5C_{f1}^2 C_{f2}^2}{2\eta} \\
&\quad + \sum_{t=1}^{K} \frac{\alpha}{4} \mathbb{E}[\|\lambda_t - \lambda_{t-1}\|^2] + \sum_{t=1}^{K} \frac{D_X^2 C_{g1}^2 C_{g2}^2}{\alpha} + \sum_{t=1}^{K} \frac{5C_{g1}^2 C_{g2}^2 \mathbb{E}[\|\lambda_{t-1}\|^2]}{2\eta}.
\end{align*}
$$

This completes the proof. □
B.1 Proof of Proposition 2

By applying Theorem 2.1 with \( \lambda = \lambda^* \), \( \pi_{1:2} = \pi_{1:2}^* \), and \( v = v^* \) defined in (2.11), we set \( M_{\lambda^*} = \|\lambda^*\| \) and obtain

\[
\sum_{t=1}^{K} \mathbb{E}\left( \mathcal{L}(x_t, \lambda^*, \pi_{1:2}^*, v^*) - \mathcal{L}(x_t, \lambda_t, \pi_{1:2,t}, v_t) \right) + \frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda^*\|^2] \\
\leq \frac{45K^2C_f^2C_f^2}{\eta} \sqrt{KL_{f_1}\sigma_{f_2}^2} + 3\sqrt{KC_f\sigma_{f_2}} + \frac{\eta}{2} \|x_0 - x^*\|^2 \\
+ \frac{KD_x^2C_g^2}{\alpha} + \frac{10KC_g^2\|\lambda^*\|^2}{\eta} + \sum_{t=1}^{K} \frac{5C_g^2\mathbb{E}[\|\lambda_{t-1} - \lambda^*\|^2]}{2\eta} + \sqrt{K}\|\lambda^*\|\sigma_g \\
+ \frac{\alpha}{2} \|\lambda_0 - \lambda^*\|^2 + \frac{K\sigma_g^2}{\alpha} + \sum_{t=1}^{K} \frac{5C_g^2\mathbb{E}[\|\lambda^* - \lambda_t\|^2]}{2\eta}.
\]

By using the fact that \( \|\lambda_t\|^2 \leq 2\|\lambda_t - \lambda^*\|^2 + 2\|\lambda^*\|^2 \), we further express the above inequality as

\[
\sum_{t=1}^{K} \mathbb{E}\left( \mathcal{L}(x_t, \lambda^*, \pi_{1:2}^*, v^*) - \mathcal{L}(x_t, \lambda_t, \pi_{1:2,t}, v_t) \right) + \frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda^*\|^2] \\
\leq \frac{45K^2C_f^2C_f^2}{\eta} \sqrt{KL_{f_1}\sigma_{f_2}^2} + 3\sqrt{KC_f\sigma_{f_2}} + \frac{\eta}{2} \|x_0 - x^*\|^2 \\
+ \frac{KD_x^2C_g^2}{\alpha} + \frac{15KC_g^2\|\lambda^*\|^2}{\eta} + \sum_{t=1}^{K} \frac{5C_g^2\mathbb{E}[\|\lambda_{t-1} - \lambda^*\|^2]}{\eta} + \sqrt{K}\|\lambda^*\|\sigma_g \\
+ \frac{\alpha}{2} \|\lambda_0 - \lambda^*\|^2 + \frac{K\sigma_g^2}{\alpha} + \sum_{t=1}^{K} \frac{5C_g^2\mathbb{E}[\|\lambda^* - \lambda_t\|^2]}{2\eta}.
\]

By using the min-max relationship (2.11) that

\[ \mathcal{L}(x_t, \lambda^*, \pi_{1:2}^*, v^*) - \mathcal{L}(x_t, \lambda_t, \pi_{1:2,t}, v_{1:t}) \geq 0, \]

dividing both sides of the above inequality by \( \frac{\alpha}{2} \), and setting \( \alpha = 2\sqrt{N} \) and \( \eta = \frac{15\sqrt{N}}{2} \), we obtain

\[ \mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq \frac{2}{\alpha} \sum_{t=1}^{K} \mathbb{E}\left( \mathcal{L}(x_t, \lambda^*, \pi_{1:2}^*, v^*) - \mathcal{L}(x_t, \lambda_t, \pi_{1:2,t}, v_t) \right) + \mathbb{E}[\|\lambda_K - \lambda^*\|^2] \\
\leq R_K + \frac{C_g^2\mathbb{E}[\|\lambda^* - \lambda_K\|^2]}{3N} + \sum_{t=0}^{K-1} \frac{C_g^2\mathbb{E}[\|\lambda^* - \lambda_t\|^2]}{N}, \]

where

\[ R_K = \frac{K}{N} \left( 6C_f^2C_f^2 + \frac{D_x^2C_f^2}{2} + 2C_g^2\|\lambda^*\|^2 + \frac{\sigma_g^2}{2} \right) + \frac{15}{4} \|x_0 - x^*\|^2 + \|\lambda_0 - \lambda^*\|^2 \\
+ \sqrt{\frac{K}{N}} \left( L_{f_1}\sigma_{f_2}^2 + 3C_f\sigma_{f_2} + \|\lambda^*\|\sigma_g \right). \]
Next, letting $R$ be the constant defined in (2.19), we note that $R_K \leq R$, and the above inequality further implies that
\[
\left(1 - \frac{C_g^2}{3N}\right)\mathbb{E}[||\lambda_K - \lambda^*||^2] \leq R + \frac{\sum_{t=0}^{K-1} C_g^2 \mathbb{E}[||\lambda^* - \lambda_t||^2]}{N}.
\]
Dividing both sides of the above inequality by $(1 - \frac{1}{3N})$, using the fact that $(1 - \frac{C_g^2}{3N}) \geq 1/2$ for $N \geq 3C_g^2/2$, and applying Lemma 1, we conclude that
\[
\mathbb{E}[||\lambda_K - \lambda^*||^2] \leq \left(1 - \frac{C_g^2}{3N}\right)^{-1} \left(R + \frac{\sum_{t=0}^{K-1} C_g^2 \mathbb{E}[||\lambda^* - \lambda_t||^2]}{N}\right)
\]
\[
\leq 2R + 2C_g^2 \sum_{t=0}^{K-1} \frac{\mathbb{E}[||\lambda^* - \lambda_t||^2]}{N} \leq 2R \left(1 + \frac{2}{N}\right)^K \leq 2R e^{2C_g^2},
\]
for all $1 \leq K \leq N$. This completes the proof. \qed

### B.2 Proof of Lemma 4

First, we decompose $(\lambda - \lambda_t)^\top L_G(x_t, v_{1:2,t})$ that
\[
(\lambda - \lambda_t)^\top L_G(x_t, v_{1:2,t}) = (\lambda - \lambda_t)^\top \left( L_G(x_t, v_{1:2,t}) - L_G(x_{t-1}, v_{1:2,t}) \right) + (\lambda - \lambda_t)^\top L_G(x_{t-1}, v_{1:2,t})
\]
\[
= \langle v_{2,t}v_{1,t}(\lambda - \lambda_t), x_t - x_{t-1} \rangle + (\lambda - \lambda_t)^\top L_G(x_{t-1}, v_{1:2,t}) - H_t + (\lambda - \lambda_t)^\top H_t. \tag{B.5}
\]

We now provide bounds for the three terms $T_{1,t}, T_{2,t},$ and $T_{3,t}$. First, for $T_{1,t}$, by the fact that $\langle a, b \rangle \leq \frac{5||a||^2}{2\eta} + \frac{7||b||^2}{10}$ for any vectors $a, b$, we have
\[
\sum_{t=1}^{K} \mathbb{E}[T_{1,t}] = \sum_{t=1}^{K} \mathbb{E}[\langle x_t - x_{t-1} \rangle^\top v_{2,t}v_{1,t}(\lambda - \lambda_t)]
\]
\[
\leq \sum_{t=1}^{K} \frac{5\mathbb{E}[||v_{2,t}v_{1,t}(\lambda - \lambda_t)||^2]}{2\eta} + \sum_{t=1}^{K} \frac{\eta\mathbb{E}[||x_t - x_{t-1}||^2]}{10} \tag{B.6}
\]
\[
\leq \sum_{t=1}^{K} \frac{5C_{g1}^2 C_{g2}^2 \mathbb{E}[||\lambda - \lambda_t||^2]}{2\eta} + \sum_{t=1}^{K} \frac{\eta\mathbb{E}[||x_t - x_{t-1}||^2]}{10},
\]
where the last inequality holds by Assumption 3.1 (d)-(e) that $||v_{1,t}||^2 \leq C_{g1}^2$ and $||v_{2,t}||^2 \leq C_{g2}^2$. Second, we consider $T_{2,t}$ and denote by $\tilde{\Delta}_{2,t} = L_G(x_{t-1}, v_{1:2,t}) - H_t$. Given $x_{t-1}$, we observe from Algorithm 2 that $H_t$ is conditionally independent of $\lambda_{t-1}$. Together with Lemma 3 (a) that $H_t$ is also an unbiased estimator for $L_G(x_{t-1}, v_{1:2,t})$, we have $\mathbb{E}[\lambda_{t-1}^\top \tilde{\Delta}_{2,t}] = 0$, which further implies that
\[
\mathbb{E}[T_{2,t}] = \mathbb{E}[\lambda^\top \tilde{\Delta}_{2,t}] - \mathbb{E}[\tilde{\Delta}_{2,t}^\top \lambda_{t-1}] + \mathbb{E}[(\lambda_{t-1} - \lambda_t)^\top \tilde{\Delta}_{2,t}]
\]
\[
= \mathbb{E}[\lambda^\top \tilde{\Delta}_{2,t}] + \mathbb{E}[(\lambda_{t-1} - \lambda_t)^\top \tilde{\Delta}_{2,t}].
\]

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By setting $M_{\lambda} = \|\lambda\|$ and recalling Lemma 3 (b) that $\operatorname{Var}(\Delta_{2,t}) \leq \sigma^2_H$, we further have

$$
\sum_{t=1}^{K} \mathbb{E}[T_{2,t}] = \sum_{t=1}^{K} \mathbb{E}[\lambda^\top \Delta_{2,t}] + \sum_{t=1}^{K} \mathbb{E}[(\lambda_{t-1} - \lambda_t)^\top \Delta_{2,t}]
\leq \sqrt{K} M_{\lambda} \sigma_H + \sum_{t=1}^{K} \mathbb{E}[(\lambda_{t-1} - \lambda_t)^\top \Delta_{2,t}]
\leq \sqrt{K} M_{\lambda} \sigma_H + \sum_{t=1}^{K} \left( \frac{\alpha \mathbb{E}[\|\lambda_{t-1} - \lambda_t\|^2]}{4} + \frac{\sigma^2_H}{\alpha} \right),
$$

where the first inequality holds by Lemma 8 in Appendix Section B.7, and the last inequality holds by the fact $\langle a, b \rangle \leq \frac{\|a\|^2}{4} + \|b\|^2$ for any vectors $a, b$.

Finally, for $T_{3,t}$, by applying the three-point Lemma 7 to the update rule that

$$
\lambda_t = \operatorname{argmax}_{\lambda \in \mathbb{R}^n} \left\{ \lambda^\top H_t - \frac{\alpha}{2} \|\lambda_{t-1} - \lambda\|^2 \right\},
$$

we have that for any $\lambda \in \mathbb{R}^n$,

$$
(\lambda - \lambda_t)^\top H_t = - (\lambda_t - \lambda)^\top H_t \leq \frac{\alpha}{2} \|\lambda_{t-1} - \lambda\|^2 - \frac{\alpha}{2} \|\lambda_{t-1} - \lambda_t\|^2 - \frac{\alpha}{2} \|\lambda_t - \lambda\|^2,
$$

which implies

$$
\mathbb{E}[T_{3,t}] \leq \sum_{t=1}^{K} \left( \frac{\alpha}{2} \|\lambda_{t-1} - \lambda\|^2 - \frac{\alpha}{2} \|\lambda_{t-1} - \lambda_t\|^2 - \frac{\alpha}{2} \|\lambda_t - \lambda\|^2 \right).
$$

Summing (B.5) over $t = 1, \cdots, K$, taking expectations, and plugging (B.6), (B.7), and (B.8) in, we obtain that for all $\lambda \in \mathbb{R}^n$,

$$
\mathbb{E}\left[ \sum_{t=1}^{K} (\lambda - \lambda_t)^\top L_G(x_t, v_{1,t}, v_{2,t}) \right]
\leq \frac{\alpha}{2} \mathbb{E}[\|\lambda_0 - \lambda\|^2] - \frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda\|^2] - \sum_{t=1}^{K} \frac{\alpha}{2} \mathbb{E}[\|\lambda_{t-1} - \lambda_t\|^2] + \sqrt{K} M_{\lambda} \sigma_H
+ \sum_{t=1}^{K} \left( \frac{\alpha \mathbb{E}[\|\lambda_{t-1} - \lambda_t\|^2]}{4} + \frac{\sigma^2_H}{\alpha} + \frac{5C_2^2 C_{2t}^2 \mathbb{E}[\|\lambda - \lambda_{t-1}\|^2]}{2\eta} + \frac{\eta \mathbb{E}[\|x_t - x_{t-1}\|^2]}{10} \right)
\leq \frac{\alpha}{2} \mathbb{E}[\|\lambda_0 - \lambda\|^2] - \frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda\|^2] + \sqrt{K} M_{\lambda} \sigma_H
+ \sum_{t=1}^{K} \left( \frac{\sigma^2_H}{\alpha} - \frac{\alpha \mathbb{E}[\|\lambda_{t-1} - \lambda_t\|^2]}{4} + \frac{5C_2^2 C_{2t}^2 \mathbb{E}[\|\lambda - \lambda_{t-1}\|^2]}{2\eta} + \frac{\eta \mathbb{E}[\|x_t - x_{t-1}\|^2]}{10} \right),
$$

which completes the proof. \qed
B.3 Proof of Proposition 5

For the feasible point \((x^*, \lambda, \pi_{1:2}, v_{1:2})\), we set \(\pi_{1:2} = \pi_{1:2}'\), \(v_{1:2} = v_{1:2}'\), and \(\lambda = \lambda^*\) as defined in (3.7). By applying Theorem 3.1 with \(M_{\lambda^*} = \|\lambda^*\|\), we obtain

\[
\sum_{t=1}^{K} \mathbb{E}(\mathcal{L}(x_t, \lambda^*, \pi_{1:2}', v_{1:2}') - \mathcal{L}(x^*, \lambda_t, \pi_{1:2,t}, v_{1:2,t})) + \frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq \frac{40KC_{g_1}^2C_{g_2}^2\|\lambda^*\|}{\eta} + 2\sqrt{KL_g\sigma_{g_2}^2\|\lambda^*\| + 3\sqrt{KC_{g_1}\sigma_{g_2}\|\lambda^*\| + \frac{\eta}{2}\|x_0 - x^*\|^2}
\]

\[
+ \frac{95KC_{g_1}^2C_{g_2}^2}{2\eta} + \frac{K}{\alpha}(D_X^2C_{g_1}^2C_{g_2}^2 + \sigma_{H}^2) + \frac{\alpha}{2}\|\lambda_0 - \lambda^*\|^2 + \sqrt{K}\|\lambda^*\|\sigma_H
\]

\[
+ 2\sqrt{KL_{f_1}\sigma_{f_2}^2} + 3\sqrt{KC_{f_1}\sigma_{f_2}^2} + \sum_{t=1}^{K} \frac{5C_{g_1}^2C_{g_2}^2}{2\eta}\left(\mathbb{E}[\|\lambda^* - \lambda_t\|^2] + \mathbb{E}[\|\lambda_{t-1}\|^2]\right).
\]

We note that \(\mathcal{L}(x_t, \lambda^*, \pi_{1:2}', v_{1:2}') - \mathcal{L}(x^*, \lambda_t, \pi_{1:2,t}, v_{1:2,t}) \geq 0\) by the definition of a saddle point (3.7). Thus, we have

\[
\frac{\alpha}{2} \mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq \frac{40KC_{g_1}^2C_{g_2}^2\|\lambda^*\|}{\eta} + \frac{\eta}{2}\|x_0 - x^*\|^2 + \frac{95KC_{g_1}^2C_{g_2}^2}{2\eta} + \frac{K}{\alpha}(D_X^2C_{g_1}^2C_{g_2}^2 + \sigma_{H}^2)
\]

\[
+ \frac{\alpha}{2}\|\lambda_0 - \lambda^*\|^2 + \sum_{t=1}^{K} \frac{5C_{g_1}^2C_{g_2}^2}{2\eta}\left(\mathbb{E}[\|\lambda^* - \lambda_t\|^2] + \mathbb{E}[\|\lambda_{t-1}\|^2]\right) + \sqrt{K}Q,
\]

where \(Q = 2L_g\sigma_{g_2}^2\|\lambda^*\| + 3C_{g_1}\sigma_{g_2}\|\lambda^*\| + \|\lambda^*\|\sigma_H + 2L_{f_1}\sigma_{f_2}^2 + 3C_{f_1}\sigma_{f_2}\) is defined in (3.18). By using the fact that \(\|\lambda_t\|^2 \leq 2\|\lambda_t - \lambda^*\|^2 + 2\|\lambda^*\|^2\), setting \(\alpha = 2\sqrt{N}\) and \(\eta = \frac{5\sqrt{N}}{2}\), and dividing both sides of the above inequality by \(\sqrt{N}\), we obtain

\[
\mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq \frac{16KC_{g_1}^2C_{g_2}^2\|\lambda^*\|}{N} + \frac{5}{4}\|x_0 - x^*\|^2 + \frac{19KC_{g_1}^2C_{g_2}^2}{N} + \frac{K}{2N}(D_X^2C_{g_1}^2C_{g_2}^2 + \sigma_{H}^2)
\]

\[
+ \|\lambda_0 - \lambda^*\|^2 + \frac{C_{g_1}^2C_{g_2}^2}{N}\sum_{t=1}^{K} \left(\mathbb{E}[\|\lambda^* - \lambda_t\|^2] + 2\mathbb{E}[\|\lambda_{t-1} - \lambda^*\|^2] + 2\|\lambda^*\|^2\right) + \frac{\sqrt{K}Q}{\sqrt{N}}
\]

\[
= R_K + \frac{C_{g_1}^2C_{g_2}^2}{N}\mathbb{E}[\|\lambda^* - \lambda_K\|^2] + \frac{3C_{g_1}^2C_{g_2}^2}{N}\sum_{t=1}^{K} \mathbb{E}[\|\lambda_{t-1} - \lambda^*\|^2],
\]

where

\[
R_K = \frac{K}{N}\left(18C_{g_1}^2C_{g_2}^2\|\lambda^*\| + 19C_{g_1}^2C_{g_2}^2 + \frac{D_X^2C_{g_1}^2C_{g_2}^2 + \sigma_{H}^2}{2}\right)
\]

\[
+ \frac{5}{4}\|x_0 - x^*\|^2 + \|\lambda_0 - \lambda^*\|^2 + \frac{\sqrt{K}Q}{\sqrt{N}}.
\]
By noting that \( R_K \leq R \) defined in (3.18), the above inequality further implies that

\[
(1 - \frac{C_1^2 C_2^2}{N}) \mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq R + \sum_{t=1}^{K} \frac{3C_1^2 C_2^2 \mathbb{E}[\|\lambda_{t-1} - \lambda^*\|^2]}{N}.
\]

Assuming \( N \geq 2C_1^2 C_2^2 \), for all \( K \leq N \), we have

\[
\mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq 2R + \sum_{t=0}^{K-1} \frac{6C_1^2 C_2^2}{N} \cdot \mathbb{E}[\|\lambda_t - \lambda^*\|^2].
\]

Finally, by applying Lemma 1, we recursively bound \( \mathbb{E}[\|\lambda_K - \lambda^*\|^2] \) by

\[
\mathbb{E}[\|\lambda_K - \lambda^*\|^2] \leq 2R e^{6C_1^2 C_2^2}
\]

for all \( K \leq N \), which completes the proof. \( \square \)