EQUIDISTRIBUTION OF VALUES OF LINEAR FORMS ON QUADRATIC SURFACES.

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Abstract. In this paper we investigate the distribution of the set of values of a linear map at integer points on a quadratic surface. In particular, it is shown that subject to certain algebraic conditions, this set is equidistributed. This can be thought of as a quantitative version of the main result from [Sar11]. The methods used are based on those developed by A. Eskin, S. Mozes and G. Margulis in [EMM98]. Specifically, they rely on equidistribution properties of unipotent flows.

1. Introduction.

Consider the following situation. Let $X$ be a rational surface in $\mathbb{R}^d$, $R$ be a fixed region in $\mathbb{R}^s$ and $F : X \to \mathbb{R}^s$ be a polynomial map. An interesting problem is to investigate the size of the set

$$Z = \{ x \in X \cap \mathbb{Z}^d : F(x) \in R \},$$

consisting of integer points in $X$ such that the corresponding values of $F$, are in $R$. Suppose that the set of values of $F$ at the integer points of $X$, is dense in $\mathbb{R}^s$. In this case, the set $Z$ will be infinite. However, the set

$$Z_T = \{ x \in X \cap \mathbb{Z}^d : F(x) \in R, \|x\| \leq T \},$$

can be considered. This set will be finite, and its size will depend on $T$. Typically, the density assumption indicates that the set $Z$ might be equidistributed, within the set of all integer points in $X$. Namely, as $T$ increases, the size of the set $Z_T$, should be proportional to the appropriately defined volume, of the set

$$\{ x \in X : F(x) \in R, \|x\| \leq T \},$$

consisting of real points on $X$, with values in $R$ and bounded norm. Such a result, if it is obtained, can be seen as quantifying the denseness of the values of $F$ at integral points.

The situation described above is too general, but it serves as motivation for what is to come. So far, what is proven, is limited to special cases. For instance, when $M : \mathbb{R}^d \to \mathbb{R}^s$ is a linear map, classical methods can be used to establish necessary and sufficient conditions, which ensure the values of $M$ on $\mathbb{Z}^d$ are dense in $\mathbb{R}^s$. The equidistribution problem described above can also be considered in this case. It is straightforward to obtain an asymptotic estimate for the number of integer points with bounded norm whose values lie in some compact region of $\mathbb{R}^s$ (cf. [Cas72]).

When $Q : \mathbb{R}^d \to \mathbb{R}$ is a quadratic form the situation is that of the Oppenheim conjecture. In [Mar89], G. Margulis obtained necessary and sufficient conditions to ensure that the values of $Q$ on $\mathbb{Z}^d$ are dense in $\mathbb{R}$. Considerable work has gone into the equidistribution problem in this case, first by S.G. Dani and G. Margulis, who obtained an asymptotic lower bound for the number of integers with bounded height such that their images lie in a fixed interval (cf. [DM93]). Later, A. Eskin, G. Margulis and S. Mozes, gave the corresponding asymptotic upper bound for the same problem (cf. [EMM98]). The major ingredient, used in the proof of Oppenheim conjecture, is to relate the density of the values of a quadratic form at integers to the density of certain orbits inside a homogeneous space. This connection was first noted by M. S. Raghunathan in the late 70’s (appearing in print in [Dan81], for instance). It is, in this way, using tools from dynamical systems to study the orbit closures of subgroups corresponding to quadratic forms, that Margulis proved the Oppenheim conjecture. Similarly, the later refinement, due to Dani-Margulis, who considered the values of quadratic forms at primitive integral...
points in [DM90] and work on the equidistribution (quantitative) problem by Dani-Margulis and Eskin-Margulis-Mozes, were also obtained by studying the orbit closures of subgroups acting on homogeneous spaces.

Similar techniques were also used by A. Gorodnik in [Gor04], to study the set of values of a pair, consisting of a quadratic and linear form, at integer points and in [Sar11] to establish conditions, sufficient to ensure that the values of a linear map at integers lying on a quadratic surface are dense in the range of the map. The main result of this paper deals with the corresponding equidistribution problem and is stated in the following Theorem.

**Theorem 1.1.** Suppose $Q$ is a quadratic form on $\mathbb{R}^d$ such that $Q$ is non-degenerate, indefinite with rational coefficients. Let $M = (L_1, \ldots, L_s) : \mathbb{R}^d \to \mathbb{R}^s$ be a linear map such that:

1. The following relations hold, $d > 2s$ and $\text{rank} \left( Q|_{\ker(M)} \right) = d - s$.
2. The quadratic form $Q|_{\ker(M)}$ has signature $(r_1, r_2)$ where $r_1 \geq 3$ and $r_2 \geq 1$.
3. For all $\alpha \in \mathbb{R}^s \setminus \{0\}$, $\alpha_1 L_1 + \cdots + \alpha_s L_s$ is non rational.

Let $a \in \mathbb{Q}$ be such that the set $\{v \in \mathbb{Z}^d : Q(v) = a\}$ is non empty. Then there exists $C_0 > 0$ such that for every $\theta > 0$ and all compact $R \subset \mathbb{R}^s$ with piecewise smooth boundary, there exists a $T_0 > 0$ such that for all $T > T_0$,

$$(1 - \theta) C_0 \text{Vol}(R) T^{d - s - 2} \leq \left| \{ v \in \mathbb{Z}^d : Q(v) = a, M(v) \in R, \|v\| \leq T \} \right| \leq (1 + \theta) C_0 \text{Vol}(R) T^{d - s - 2},$$

where $\text{Vol}(R)$ is the $s$ dimensional Lebesgue measure of $R$.

**Remark 1.2.** The constant $C_0$ appearing in Theorem 1.1 is such that $C_0 \text{Vol}(R) T^{d - s - 2} \sim \text{Vol} \left( \{ v \in \mathbb{R}^d : Q(v) = a, M(v) \in R, \|v\| \leq T \} \right)$ where the volume on the right is the Haar measure on the surface defined by $Q(v) = a$.

**Remark 1.3.** Theorem 1.1 should hold with the condition that $\text{rank} \left( Q|_{\ker(M)} \right) = d - s$ replaced by the condition that $\text{rank} \left( Q|_{\ker(M)} \right) > 3$. Dealing with the more general situation requires taking into account the nontrivial unipotent part of $\text{Stab}_{SO(Q)}(M)$, as such lower bounds could probably be proved using methods of [DM93], but so far no way has been found to obtain the statement that would be needed in order to obtain an upper bound.

**Remark 1.4.** As in [DMM98] it would be possible to obtain a version of Theorem 1.1 where the condition that $\|v\| < T$ was replaced by $v \in TK_0$ where $K_0$ is an arbitrary deformation of the unit ball by a continuous and positive function. It should also be possible to obtain a version of Theorem 1.1 where the parameters $T_0$ and $C_0$ remain valid for any pair $(Q, M)$ coming from compact subsets of pairs satisfying the conditions of the Theorem.

**Remark 1.5.** The cases when the quadratic form $Q|_{\ker(M)}$ has signature $(2, 2)$ or $(2, 1)$ can be considered exceptional. There are asymptotically more integers than expected (by a factor of $\log T$) lying on certain surfaces defined by quadratic forms of signature $(2, 2)$ or $(2, 1)$. This leads to counterexamples of Theorem 1.1 in the cases when the quadratic form $Q|_{\ker(M)}$ has signature $(2, 2)$ or $(2, 1)$. Details of these examples are found in Section 6.

**Outline of the paper.** The proof of Theorem 1.1 rests on statements about the distribution of orbits in a certain homogeneous spaces. The philosophy is that equidistribution of the orbits corresponds to equidistribution of the points considered in Theorem 1.1. Consider the following Theorem of M. Ratner found in [Rat94].

**Ratner’s Equidistribution Theorem.** Let $G$ be a connected Lie group, $\Gamma$ a lattice in $G$ and $U = \{ u_t : t \in \mathbb{R} \}$ a one parameter unipotent subgroup of $G$. Then for all $x \in G/\Gamma$ the closure of the orbit $Ux$ has an invariant measure, $\mu_{Ux}$ supported on it and for all bounded continuous functions, $f$ on $G/\Gamma$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u_t x) = \int_{Ux} f \, d\mu_{Ux}.$$
Recall that in the proof of the quantitative Oppenheim conjecture (cf. [EMM98]) one needs to consider an unbounded function on the space of lattices. Similarly, in order to prove Theorem 1.1 one needs to consider an unbounded function $F$ on a certain homogeneous space. The basic idea is to try to apply Ratner’s Equidistribution Theorem to $F$ in order to show that the average of the values of $F$ evaluated along a certain orbit converges to the average of $F$ on the entire space. This is the fact that corresponds to the fact that integral points on the quadratic surface with values in $R$ are equidistributed. The main problem in doing this is that $F$ is unbounded and so one must obtain an ergodic theorem, taking a similar form to Ratner’s Equidistribution Theorem, but valid for unbounded functions. In order to do this one needs precise information about the behaviour of the orbits near the cusp. This information is obtained in Section 3 and comes in the form of non divergence estimates for certain dilated spherical averages. In order to obtain these estimates we use a certain function defined by Y. Benoist and J.F. Quint in [BQ12]. The required ergodic theorem is then proved in Section 4.

Finally in Section 5 the proof of Theorem 1.1 is completed using an approximation argument similar by Y. Benoist and J.F. Quint in [BQ12]. The required ergodic theorem is then proved in Section 4. Theorem 2.1 fails for certain pairs. In Section 6 these counterexamples are explicitly constructed.

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2. Set up.

2.1. Main results. For the rest of the paper the following convention is in place: $s, d$ and $p$ will be fixed natural numbers such that $2s < d$ and $0 < p < d$. Also, $r_1$ and $r_2$ will be varying, natural numbers such that $d - s = r_1 + r_2$. Let $L$ denote the space of linear forms on $R^d$ and let $C_{\text{Lin}}$ denote the subset of $L^2$ such that for all $M \in C_{\text{Lin}}$ condition 3 of Theorem 1.1 is satisfied. A quadratic form on $R^d$ is said to be defined over $Q$, if it has rational coefficients or is a scalar multiple of a form with rational coefficients. For $a$, a rational number let $Q(p, a)$ denote quadratic forms on $R^d$ defined over $Q$ with signature $(p, d - p)$ such that the set $\{v \in Z^d : Q(v) = a\}$ is non empty for all $Q \in Q(p, a)$. Define

$$C_{\text{Pairs}}(a, r_1, r_2) = \{(Q, M) : Q \in Q(p, a), M \in C_{\text{Lin}} \text{ and } Q|_{\ker(M)} \text{ has signature } (r_1, r_2)\}.$$ 

Note that for $r_1 \geq 3$ and $r_2 \geq 1$ the set $C_{\text{Pairs}}(a, r_1, r_2)$ consists of pairs satisfying the conditions of Theorem 1.1. Although the set $C_{\text{Pairs}}(a, r_1, r_2)$ and hence its subsets and sets derived from them depend on $a$, this dependence is not a crucial one, so from now on, most of the time this dependence will be omitted from the notation. For $M \in L^2$ and $R \subset R^d$ a connected region with smooth boundary let $V_M(R) = \{v \in R^d : M(v) \in R\}$. For $Q \in Q(p, d - p)$, $a \in Q$ and $K = R$ or $Z$ let $X_Q^+(K) = \{v \in K^d : Q(v) = a\}$. Denote the annular region inside $R^d$ by $A(T_1, T_2) = \{v \in R^d : T_1 \leq ||v|| \leq T_2\}$. Using this notation, we state the following (equivalent) version of Theorem 1.1 which will be proved in Section 5.

**Theorem 2.1.** Suppose that $r_1 \geq 3$, $r_2 \geq 1$ and $a \in Q$. Then for all $(Q, M) \in C_{\text{Pairs}}(a, r_1, r_2)$ there exists $C_0 > 0$ such that for every $\theta > 0$ and all compact $R \subset R^d$ with piecewise smooth boundary, there exists a $T_0 > 0$ such that for all $T > T_0$,

$$(1 - \theta) C_0 \text{Vol}(R) T^{d-s-2} \leq |X_Q^+(Z) \cap V_M(R) \cap A(0, T)| \leq (1 - \theta) C_0 \text{Vol}(R) T^{d-s-2}.$$

**Remark 2.2.** As remarked previously, the cases when $r_1 = 2$ and $r_2 = 2$ or $r_1 = 2$ and $r_2 = 1$ are interesting. In dimensions 3 and 4 there can be more integer points than expected lying on some surfaces defined by quadratic forms of signature $(2, 2)$ or $(2, 1)$, this means that the statement of Theorem 2.1 fails for certain pairs. In Section 6 these counterexamples are explicitly constructed. Moreover, it is shown that this set of pairs is big in the sense that it is of second category. We note that as in [EMM98] one could also show that this set has measure zero and one could prove the expected asymptotic formula as in Theorem 2.1 for almost all pairs.
Even though Theorem 2.1 fails when \( r_1 = 2 \) and \( r_2 = 2 \) or \( r_1 = 2 \) and \( r_2 = 1 \), we do have the following uniform upper bound, which will be proved in Section 4 and is analogous to Theorem 2.3 from [EMM98].

**Theorem 2.3.** Let \( R \subset \mathbb{R}^s \) be a compact region with piecewise smooth boundary and \( a \in \mathbb{Q} \).

(I) If \( r_1 \geq 3 \) and \( r_2 \geq 1 \), then for all \((Q,M) \in C_{\text{Pairs}}(a,r_1,r_2)\), there exists a constant \( C \) depending only on \((Q,M)\) and \( R \) such that for all \( T > 1 \),

\[
|X_{Q}(Z) \cap V_{M}(R) \cap A(0,T)| \leq CT^{d-s-2}.
\]

(II) If \( r_1 = 2 \) and \( r_2 = 1 \) or \( r_1 = r_2 = 2 \), then for all \((Q,M) \in C_{\text{Pairs}}(a,r_1,r_2)\) there exists a constant \( C \) depending only on \((Q,M)\) and \( R \) such that for all \( T > 2 \),

\[
|X_{Q}(Z) \cap V_{M}(R) \cap A(0,T)| \leq C(\log T)T^{d-s-2}.
\]

### 2.2. A canonical form.

For \( v_1, v_2 \in \mathbb{R}^d \) we will use the notation \( \langle v_1, v_2 \rangle \) to denote the standard inner product in \( \mathbb{R}^d \). For a set of vectors \( v_1, \ldots, v_n \in \mathbb{R}^d \) we will also use the notation \( \langle v_1, \ldots, v_n \rangle \) to denote the span of \( v_1, \ldots, v_n \) in \( \mathbb{R}^d \), although this could lead to some ambiguity, the meaning of the notation should be clear from the context.

For some computations it will be convenient to know that our system is conjugate to a canonical form. Let \( e_1, \ldots, e_d \) be the standard basis of \( \mathbb{R}^d \). Let \((Q_0, M_0)\) be the pair consisting of a quadratic form and a linear map defined by

\[
Q_0(v) = Q_{1,\ldots,s}(v) + 2v_{s+1}v_d + \sum_{i=s+2}^{s+r_1} v_i^2 - \sum_{i=s+r_1+1}^{d-1} v_i^2 \quad \text{and} \quad M_0(v) = (v_1, \ldots, v_s),
\]

where \( v_i = \langle v, e_i \rangle \) and \( Q_{1,\ldots,s}(v) \) is a non-degenerate quadratic form in variables \( v_1, \ldots, v_s \). By Lemma 2.2 of [Sar11] all pairs \((Q,M)\) such that \( \text{rank } (Q|_{\ker(M)}) = d-s \) and the signature of \( Q|_{\ker(M)} \) is \((r_1, r_2)\) are equivalent to the pair \((Q_0, M_0)\) in the sense that there exist \( g_d \in GL_d(\mathbb{R}) \) and \( g_s \in GL_s(\mathbb{R}) \) such that \((Q,M) = (Q_0, g_s M_0 g_s^{-d})\), where for \( g \in GL_d(\mathbb{R}) \) we write \( Q = Q_0^g \) if and only if \( Q_0(gv) = Q(v) \) for all \( v \in \mathbb{R}^d \). Moreover since \( R \subset \mathbb{R}^s \) is arbitrary, up to rescaling and possibly replacing \( R \) by \( g_s R \) we assume that \( g_d \in SL_d(\mathbb{R}) \) and that \( g_s \) is the identity. Let

\[
C_{SL}(a,r_1,r_2) = \{ g \in SL_d(\mathbb{R}) : (Q_0^g, M_0^g) \in C_{\text{Pairs}}(a,r_1,r_2) \}.
\]

For \( g \in C_{SL}(a,r_1,r_2) \), let \( G_a \) be the identity component of the group \( \{ x \in SL_d(\mathbb{R}) : Q_0^g(xv) = Q_0^g(v) \} \), \( \Gamma_g = G_g \cap SL_d(\mathbb{Z}) \), \( H_g = \{ x \in G_a : M_0^g(xv) = M_0^g(v) \} \) and \( K_g = H_g \cap g^{-1} Q_d(\mathbb{R}) g \). By examining the description of the subgroup \( H_g \), given in Section 2.3 of [Sar11], it is clear that \( K_g \) is a maximal compact subgroup of \( H_\mathbb{g} \). It is a standard fact that \( G_a \) is a connected semisimple Lie group and hence, has no nontrivial rational characters. Therefore, because \( Q_0^g \) is defined over \( \mathbb{Q} \), the Borel Harish-Chandra Theorem (cf. [PR94], Theorem 4.13) implies \( \Gamma_g \subset \text{a lattice in } G_a \). We will consider the dynamical system that arises from \( H_g \) acting on \( G_a/\Gamma_g \). For \( K = \mathbb{R} \) or \( \mathbb{Z} \), the shorthand \( X_{Q_0}^g(\mathbb{K}) = X_g(\mathbb{K}) \) will be used.

### 2.3. Equidistribution of measures.

Consider the function \( \alpha_s \), as defined in [EMM98]. It is an unbounded function on the space of unimodular lattices in \( \mathbb{R}^d \). It has the properties that it can be used to bound certain functions that we will consider and it is left \( K_g \)-invariant. Similar functions have been considered in [Sch09], where it is related to various quantities involving successive minima of a lattice. Let \( \Delta \) be a lattice in \( \mathbb{R}^d \). For any such \( \Delta \) we say that a subspace \( U \) of \( \mathbb{R}^d \) is \( \Delta \)-rational if \( \text{Vol}(U/U \cap \Delta) < \infty \). Let

\[
\Psi_i(\Delta) = \{ U : U \text{ is a } \Delta \text{-rational subspace of } \mathbb{R}^d \text{ with } \dim U = i \}.
\]

For \( U \in \Psi_i(\Delta) \) define \( d_U(U) = \text{Vol}(U/U \cap \Delta) \). Note that \( d_U(U) = \| u_1 \wedge \ldots \wedge u_i \| \) where \( u_1, \ldots, u_i \) is a basis for \( U \cap \Delta \) over \( \mathbb{Z} \) and the norm on \( \bigwedge^i(\mathbb{R}^d) \) is induced from the euclidean norm on \( \mathbb{R}^d \). Now
we recall the definition the function $\alpha$, as follows
\[
\alpha_i(\Delta) = \sup_{U \in \mathcal{U}_i(\Delta)} \frac{1}{d_\Delta(U)} \quad \text{and} \quad \alpha(\Delta) = \max_{0 \leq i \leq d} \alpha_i(\Delta).
\]
Here we use the convention that if $U$ is the trivial subspace then $d_\Delta(U) = 1$, hence $\alpha_0(\Delta) = 1$. Also note that if $\Delta$ is a unimodular lattice then $d_\Delta(\mathbb{R}^d) = 1$ and hence $\alpha_d(\Delta) = 1$.

In (2.2) and Theorem 2.3 we consider $\alpha$ as a function on $G_g/\Gamma_g$, this is done via the canonical embedding of $G_g/\Gamma_g$ into the space of unimodular lattices in $\mathbb{R}^d$, given by $x \Gamma_g \to x \mathbb{Z}^d$. Specifically, every $x \in G_g/\Gamma_g$ can be identified with its image under this embedding before applying $\alpha$ to it. For $f \in C_c(\mathbb{R}^d)$ and $g \in C_{SL}(r_1, r_2)$ we define the function $F_{f,g} : G_g/\Gamma_g \to \mathbb{R}$ by
\[
F_{f,g}(x) = \sum_{v \in X_g(\mathbb{Z})} f(xv).
\]
The function $\alpha$ has the property that there exists a constant $c(f)$ depending only on the support and maximum of $f$ such that for all $x$ in $G_g/\Gamma_g$,
\[
F_{f,g}(x) \leq c(f) \alpha(x).
\]
The last property is well known and follows from Minkowski’s Theorem on successive minima, see Lemma 2 of [Sch68] for example. Alternatively, see [HW08] for an up to date review of many related results.

We will be carrying out integration on various measure spaces defined by the groups introduced at the beginning of the section. With this in mind let us introduce the following notation for the corresponding measures. If $v$ denotes some variable, the notation $dv$ is used to denote integration with respect to Lebesgue measure and this variable. Let $\mu_g$ be the Haar measure on $G_g/\Gamma_g$, if $g \in C_{SL}(r_1, r_2)$ then since $\Gamma_g$ is a lattice in $G_g$ we can normalise so that $\mu_g(G_g/\Gamma_g) = 1$. In addition, $\nu_g$ will denote the measure on $K_g$ normalised so that $\nu_g(K_g) = 1$. Let $m^g_a$ denote the Haar measure on $X_g^a(\mathbb{R})$ defined by
\[
\int_{\mathbb{R}^d} f(v) \, dv = \int_{-\infty}^{\infty} \int_{X_g^a(\mathbb{R})} f(v) \, dm^g_a(v) \, da.
\]
The following Theorem provides us with our upper bounds and will be proved in Section 3.

**Theorem 2.4.** Let $g \in C_{SL}(r_1, r_2)$ be arbitrary and let $\Delta = g \mathbb{Z}^d$. Let $\{a_t : t \in \mathbb{R}\}$ denote a self adjoint one parameter subgroup of $SO(2,1)$ embedded into $H_1$ so that it fixes the subspace $\langle e_{d+2}, \ldots, e_{d-1} \rangle$ and only has eigenvalues $e^{-t}$, 1 and $e^t$.

(I) Suppose $r_1 \geq 3$ , $r_2 \geq 1$ and $0 < \delta < 2$, then
\[
\sup_{t > 0} \int_{K_1} \alpha(a_t k \Delta)^{\delta} \, d\nu_1(k) < \infty.
\]

(II) Suppose $r_1 = r_2 = 2$ or $r_1 = 2$, $r_2 = 1$, then
\[
\sup_{t > 1} \frac{1}{t} \int_{K_1} \alpha(a_t k \Delta) \, d\nu_1(k) < \infty.
\]

In Section 4 we will modify the results from Section 4 of [EMM98] and combine them with Theorem 2.4 to prove the following Theorem which will be a major ingredient of the proof of Theorem 2.4.

**Theorem 2.5.** Suppose $r_1 \geq 3$ and $r_2 \geq 1$. Let $A = \{a_t : t \in \mathbb{R}\}$ be a one parameter subgroup of $H_g$, not contained in any proper normal subgroup of $H_g$, such that there exists a continuous homomorphism $\rho : SL_2(\mathbb{R}) \to H_g$ with $\rho(D) = A$ and $\rho(SO(2)) \subset K_g$ where $D = \{\left(t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) : t > 0\}$. Let $\phi \in L^1(G_g/\Gamma_g)$ be a continuous function such that for some $0 < \delta < 2$ and some $C > 0$,
\[
|\phi(\Delta)| < C \alpha(\Delta)^{\delta}, \text{ for all } \Delta \in G_g/\Gamma_g.
\]
Then for all $\epsilon > 0$ and all $g \in C_{SL}(r_1, r_2)$ there exists $T_0 > 0$ such that for all $t > T_0$,
\[
\left| \int_{K_g} \phi(a_t(k)) d\nu_g(k) - \int_{G_u / F_g} \phi d\mu_g \right| \leq \epsilon.
\]

Remark 2.6. The condition that $A$ should not be contained in any proper normal subgroup of $H_g$ is only necessary in the case when $H_g \cong SO(2, 2)$, since in all other cases $H_g$ is simple.

3. The Upper Bounds.

In this section we prove Theorem 2.4. By definition $H_I \cong SO(r_1, r_2)$ and is embedded in $SL_d(\mathbb{R})$ so that it fixes $(e_1, \ldots, e_s)$. Let $\{a_t : t \in \mathbb{R}\}$ denote a self adjoint one parameter subgroup of $SO(2, 1)$ embedded into $H_I$ so that it fixes the subspace $(e_{s+2}, \ldots, e_{d-1})$. Moreover, suppose that the only eigenvalues of $a_t$ are $e^{-t}, 1$ and $e^t$. For $g \in C_{SL}(r_1, r_2)$, let $\Delta = g \mathbb{Z}^d$.

3.1. Proof of part I of Theorem 2.4. The aim is to construct a function $f : H_I \rightarrow \mathbb{R}$ which is contracted by the operator
\[
A_t f(h) = \int_{K_I} f(a_t h) d\nu_I(k).
\]
We say that $f$ is contracted by the operator $A_t$ if for any $c > 0$ there exists $t_0 > 0$ and $b > 0$ such that for all $h \in H_I$,
\[
A_{t_0} f(h) < cf(h) + b.
\]
This fact will be used in conjunction with Proposition 5.12 from [EMM98] which is stated below.

Proposition 3.1. Let $f : H_I \rightarrow \mathbb{R}$ be a strictly positive function such that:

1. For any $\epsilon > 0$ there exists a neighbourhood $V(\epsilon) \subset H_I$ of 1 such that
\[
(1 - \epsilon) f(h) \leq f(uh) \leq (1 + \epsilon) f(h)
\]
for all $h \in H_I$ and $u \in V(\epsilon)$.
2. The function $f$ is left $K_I$ invariant.
3. $f(1) < \infty$.
4. The function $f$ is contracted by the operator $A_t$.

Then $\sup_{t > 0} A_t f(1) < \infty$.

It is clear that in addition to satisfying properties 1–4, we have $\alpha(h\Delta)^{\delta} \leq f(h)$ for all $h \in H_I$, then the conclusion of Part I of Theorem 2.4 follows. We define the function in three stages. In the first stage we define a function on the exterior algebra of $\mathbb{R}^d$, then this function is used to define a function on the space of lattices in $\mathbb{R}^d$. Finally we use that function to define a function with the required properties.

3.1.1. A function on the exterior algebra of $\mathbb{R}^d$. Let $\bigwedge (\mathbb{R}^d) = \bigoplus_{i=1}^{d-1} \bigwedge^i (\mathbb{R}^d)$. We say that $v \in \bigwedge (\mathbb{R}^d)$ has degree $i$ if $v \in \bigwedge^i (\mathbb{R}^d)$. Let $\Omega_i = \{v_1 \wedge \cdots \wedge v_i : v_1, \ldots, v_i \in \mathbb{R}^d\}$ be the set of monomial elements of $\bigwedge (\mathbb{R}^d)$ with degree $i$. Define $\Omega = \bigcup_{i=1}^{d-1} \Omega_i$. Consider the representation $\rho : H_I \rightarrow GL(\bigwedge (\mathbb{R}^d))$. Since $H_I$ is semisimple this representation decomposes as a direct sum of irreducible subrepresentations. Associated to each of these subrepresentations is a unique highest weight. Let $\mathcal{P}$ denote the set of all these highest weights. For $\lambda \in \mathcal{P}$, denote by $U^\lambda$ the sum of all of the subrepresentations with highest weight $\lambda$ and let $\tau_\lambda : \bigwedge (\mathbb{R}^d) \rightarrow U^\lambda$ be the orthogonal projection.

Let $\epsilon > 0$. For $0 < i < d$ and $v \in \bigwedge^i (\mathbb{R}^d)$ the following function was defined by Benoist and Quint in [BQ12]. Let
\[
\varphi_\epsilon(v) = \begin{cases} 
\min_{\lambda \in \mathcal{P} \setminus \{0\}} \epsilon^{\tau_\lambda(v) \| \tau_\lambda(v) \|^2} & \text{if } \| \tau_0(v) \| \leq \epsilon^i \\
0 & \text{else,}
\end{cases}
\]
where for $0 < i < d$ we define $\gamma_i = (d - i)i$. In fact, the definition of $\varphi_\epsilon$ given here is a special case of the definition given in [BQ12]. In the definition of $\varphi_\epsilon$, given by Benoist and Quint, there is an extra
Lemma 3.6. Then for any δ, there exists a positive constant C such that for any 0 < ε < C⁻¹, u ∈ Ω₁, v ∈ Ω₂ and w ∈ Ω₃, with i₁ ≥ 0, i₂ > 0 and i₃ > 0 such that ϕₑ(u ∩ v) ≥ 1 and ϕₑ(u ∩ w) ≥ 1, one has:

1. If i₁ > 0 and i₁ + i₂ + i₃ < d, then
   \[ \min \{ \varphiₑ(u ∩ v), \varphiₑ(u ∩ w) \} \leq (Ce)^{1/2} \max \{ \varphiₑ(u), \varphiₑ(u ∩ v ∧ w) \}. \]

2. If i₁ = 0 and i₁ + i₂ + i₃ < d, then
   \[ \min \{ \varphiₑ(u ∩ v), \varphiₑ(u ∩ w) \} \leq (Ce)^{1/2} \varphiₑ(u ∩ v ∧ w). \]

3. If i₁ > 0, i₁ + i₂ + i₃ = d and \( \| u ∩ v ∧ w \| \geq 1 \), then
   \[ \min \{ \varphiₑ(u ∩ v), \varphiₑ(u ∩ w) \} \leq (Ce)^{1/2} \varphiₑ(u). \]

4. If i₁ = 0, i₁ + i₂ + i₃ = d and \( \| v ∧ w \| \geq 1 \), then
   \[ \min \{ \varphiₑ(v), \varphiₑ(w) \} \leq b₁. \]

We also need to obtain uniform bounds for the spherical averages of \( \varphiₑ \). In order to do this we use the following Lemma (Lemma 5.2) from [EMM08] will be used.

Lemma 3.5. Let V be a finite-dimensional real inner product space, A a self-adjoint linear transformation of V, K a closed connected subgroup of O(V), and S a closed subset of the unit sphere in V. Assume the only eigenvalues of A are -1, 0, and 1 and denote by W⁻, W⁰ and W⁺ the corresponding eigenspaces. Assume that Kv ∉ W⁰ for any v ∈ S and that there exists a self-adjoint subgroup H₁ of GL(V) with the following properties:

1. The Lie algebra of H₁ contains A.
2. H₁ is locally isomorphic to SO(3,1).
3. H₁ ∩ K is a maximal compact subgroup of H₁.

Then for any δ, 0 < δ < 2,

\[ \lim_{t \to \infty} \sup_{v \in S} \int_K \| \exp(tA)kv \|^{-δ} \, dν_{K}(k) = 0. \]

Using Lemma 3.5 we can obtain the following bound on the spherical averages.

Lemma 3.6. Suppose r₁ ≥ 3 and r₂ ≥ 1. Then for all ε > 0, 0 < δ < 2 and c > 0 there exists \( t₀ > 0 \) such that for all \( t > t₀ \) and all \( v \in F^c \setminus \{0\} \),

\[ \int_{K₁} \varphiₑ(aεkv)δ \, dν_{K₁}(k) < cε(γ)δ. \]
Proof. The subset $S = \{ v \in \wedge (\mathbb{R}^d) : \|v - \tau_0 (v)\| = 1 \}$ is a closed subset of the unit sphere in $\wedge (\mathbb{R}^d)$. We have $a_\ell = \exp (tA)$, for an appropriate choice of $A$ satisfying the conditions of Lemma 3.5.

We claim that for any $v \in S$, $Kv \not\subset W^0$. To see this, let

$$H_v = \{ h \in H_1 : hkv = kv \text{ for all } k \in K_1 \}.$$ 

Note that $K_1$ normalises $H_v$. Let $E_v$ be the subgroup generated by $K_1 \cup H_v$. By its definition $E_v$ also normalises $H_v$. Since $K_1$ is a maximal proper subgroup of $H_1$, in the case that $H_v \not\subset K_1$ we must have $E_v = H_1$. Therefore, $H_v$ is a normal subgroup of $H_1$. Since $r_1 \geq 3$ and $r_2 \geq 1$, $H_1$ is simple and hence $H_v = H_1$ or $H_v$ is trivial. Since $S \cap F = 0$, the first case is impossible. Therefore, for all $v \in S$, $H_v \subset K_1$. In particular this means that $\{ a_\ell : t \in \mathbb{R} \}$ is not contained in $H_v$. This implies the claim.

Then if $r_1 \geq 3$ and $r_2 \geq 1$ the conditions of Lemma 3.5 are satisfied. Hence, for any $\delta$ with $0 < \delta < 2,$

$$\lim_{t \to \infty} \sup_{v \in S} \int_{K_1} \|a_t kv\|^{-\delta} \, dv_I (k) = 0.$$ 

This implies that for all $c > 0$, there exists $t_0 > 0$, such that for all $t > t_0$ and all $v \in F^c \setminus \{0\},$

$$\int_{K_1} \|a_t kv\|^{-\delta} \, dv_I (k) < c \|v\|^{-\delta}.$$ 

Then the claim of the Lemma follows from Remark 3.2 \qed

3.1.2. A function on the space of lattices. For any lattice $\Lambda$, we say that $v \in \Omega$ is $\Lambda$-integral if one can write $v = v_1 \wedge \cdots \wedge v_i$ where $v_1, \ldots, v_i \in \Lambda$. Let $\Omega_i (\Lambda)$ and $\Omega (\Lambda)$ be the sets of $\Lambda$-integral elements of $\Omega_i$ and $\Omega$ respectively. Define $f_\epsilon : SL_d (\mathbb{R}) / SL_d (\mathbb{Z}) \to \mathbb{R}$ by

$$f_\epsilon (\Lambda) = \max_{v \in \Omega (\Lambda)} \varphi_\epsilon (v).$$ 

Note that, by Remark 3.2, for all $\epsilon > 0$ there exists some constant $c_\epsilon > 0$ such that for any unimodular lattice $\Lambda$, we have

$$\max_{v \in \Omega (\Lambda)} \|v\|^{-1} \leq \max_{0 < i < d} \left( \max_{v \in \Omega_i (\Lambda), \|\tau_0 (v)\| \leq \epsilon \gamma_i} \|v\|^{-1} + \max_{v \in \Omega_i (\Lambda), \|\tau_0 (v)\| > \epsilon \gamma_i} \|v\|^{-1} \right)$$

$$\leq c_\epsilon f_\epsilon (\Lambda) + \max_{0 < i < d} \epsilon^{-\gamma_i}. \quad (3.1)$$

Moreover, it follows from the definition of the $\alpha$ function that

$$\alpha (\Lambda) = \max \left\{ \max_{v \in \Omega (\Lambda)} \|v\|^{-1}, 1 \right\}. \quad (3.2)$$

The following Lemma is necessary to ensure that the function $f_\epsilon (h\Delta)$ is finite for all $h \in H_1$.

Lemma 3.7. For all $h \in H_1$, if $u \in \Omega (h\Delta), \text{ then } u \not\in F$.

Proof. Suppose for a contradiction that $u \in \Omega (h\Delta) \cap F$. Suppose that $u$ has degree $i$ for some $0 < i < d$ and let $u = u_1 \wedge \cdots \wedge u_i$ and $U = \{ u_1, \ldots, u_i \}$. Since $u \in \Omega (h\Delta)$, it follows that $U \cap h\Delta$ is a lattice in $U$. Moreover, because $u \in F$, $U \cap \Delta$ is also a lattice in $U$, or equivalently $g^{-1} U \cap \mathbb{Z}^d$ is a lattice in $g^{-1} U$. The subspace $g^{-1} U$ is $H_g$ invariant.

Conversely, it follows from Lemma 3.4 of [Sar11] that if $V$ is any $H_g$ invariant subspace, then either

1. $V \subseteq g^{-1} \langle e_1, \ldots, e_s \rangle$ or,
2. $V = g^{-1} \langle e_{s+1}, \ldots, e_d \rangle \oplus V'$ where $V' \subseteq g^{-1} \langle e_1, \ldots, e_s \rangle$.

Therefore, either $V$ or the orthogonal complement of $V$ is contained in $g^{-1} \langle e_1, \ldots, e_s \rangle$. By Corollary 3.2 of [Sar11], $g^{-1} \langle e_1, \ldots, e_s \rangle$ contains no subspaces defined over $\mathbb{Q}$. This implies that if $V$ is any $H_g$ invariant subspace then $V$ is not defined over $\mathbb{Q}$. In particular $V \cap \mathbb{Z}^d$ cannot be a lattice in $V$. This gives a contradiction. \qed
3.1.3. A function on \( H_f \). Define \( \tilde{f}_{\Delta, \epsilon} : H_f \rightarrow \mathbb{R} \) by

\[
\tilde{f}_{\Delta, \epsilon} (h) = f_{\epsilon} (h \Delta).
\]

In view of \( \boxed{3.1} \) and \( \boxed{3.2} \) the proof of part I of Theorem \( 2.3 \) will be complete provided that the conditions \( \boxed{11} - \boxed{14} \) from Proposition \( 3.1 \) are satisfied by the function \( \tilde{f}_{\Delta, \epsilon} \) for some \( \epsilon > 0 \). It is clear that \( \tilde{f}_{\Delta, \epsilon} \) is left \( K_f \) invariant. Also, since \( |\tau_1 (\rho (h^{-1}))|^{-1} \leq |\tau_\lambda (hv)|/|v| \leq |\tau_\lambda (\rho (h))| \) for all \( \lambda \in \mathcal{P}, v \in \Omega \) and \( h \in H_f \), \( \tilde{f}_{\Delta, \epsilon} \) also satisfies condition \( \boxed{1} \) of Proposition \( 3.1 \). From Remark \( 3.3 \) we get that \( \tilde{f}_{\Delta, \epsilon} (1) = \infty \) only if there exists \( v \in \Omega (\Delta) \cap \mathcal{F} \), but by Lemma \( 3.4 \) we know that no such \( v \) exists and so \( \tilde{f}_{\Delta, \epsilon} (1) < \infty \). It remains to show that \( \tilde{f}_{\Delta, \epsilon} \) is contracted by the operator \( A_{\epsilon} \). The proof is very similar to that of Proposition \( 5.3 \) in \( \text{[BQ12]} \).

**Lemma 3.8.** Suppose \( r_1 \geq 3 \) and \( r_2 \geq 1 \). There exists \( \epsilon > 0 \) such that for all \( 0 < \delta < 2 \), the function \( \tilde{f}_{\Delta, \epsilon} \) is contracted by the operator \( A_{\epsilon} \).

**Proof.** Fix \( c > 0 \). By Lemma \( 5.0 \) there exists \( t_0 > 0 \), so that for any \( v \in \mathcal{F} \setminus \{0\} \),

\[
(3.3) \quad \int_{K_f} \varphi_k (a_{t_0} kv)^{\delta} \, dv_f (k) < \frac{c}{d} \varphi_k (v)^{\delta}.
\]

Let \( m_0 = \|\rho (a_{t_0})\| = \|\rho (a_{t_0}^{-1})\| \). Then, for all \( v \in \wedge (\mathbb{R}^d) \),

\[
(3.4) \quad m_0^{-1} \leq \|a_{t_0} v\| \leq m_0.
\]

It follows from the definition of \( \varphi_k \) and \( 3.3 \) that

\[
(3.5) \quad m_0^{-1} \varphi_k (v) \leq \varphi_k (a_{t_0} v) \leq m_0 \varphi_k (v).
\]

Let

\[
\Psi (h \Delta) = \{ v \in \Omega (h \Delta) : f_{\epsilon} (h \Delta) \leq m_0^{\delta} \varphi_k (v) \}.
\]

Note that

\[
(3.6) \quad f_{\epsilon} (h \Delta) = \max_{\psi \in \Psi (h \Delta)} \varphi_k (\psi).
\]

Let \( C \) be the constant from Lemma \( 3.3 \). Assume that \( \epsilon \) is small enough so that

\[
(3.7) \quad m_0^4 C \epsilon < 1.
\]

There are now two cases.

**Case 1.** If \( f_{\epsilon} (h \Delta) \leq \max \{ b_1, m_0^2 \} \). In this case \( \boxed{3.5} \) and the fact that \( f_{\epsilon} \) is left \( K_f \) invariant, imply that \( f_{\epsilon} (a_{t_0} kh \Delta) \leq m_0 f_{\epsilon} (h \Delta) \). Hence

\[
(3.8) \quad \int_{K_f} f_{\epsilon} (a_{t_0} kh \Delta)^{\delta} \, dv_f (k) \leq \left( m_0 \max \{ b_1, m_0^2 \} \right)^{\delta}.
\]

**Case 2.** If \( f_{\epsilon} (h \Delta) > \max \{ b_1, m_0^2 \} \). This implies:

**Claim 3.9.** The set \( \Psi (h \Delta) \) contains only one element up to sign change in each degree.

We verify the claim as follows. Assume that for some \( 0 < i < d \), \( \Psi (h \Delta) \cap \Omega (h \Delta) \) contains two non-colinear elements, \( v_0 \) and \( w_0 \). Then because \( f_{\epsilon} (h \Delta) > m_0^2 \) and \( v_0 \) and \( w_0 \) are in \( \Psi (h \Delta) \), we have \( \varphi_k (v_0) \geq 1 \) and \( \varphi_k (w_0) \geq 1 \). We can write \( v_0 = u \wedge v \) and \( w_0 = u \wedge w \) where \( u \in \Omega_{i_1} (h \Delta) \), \( v \in \Omega_{i_2} (h \Delta) \) and \( w \in \Omega_{i_2} (h \Delta) \) with \( i_1 \geq 0 \) and \( i_2 > 0 \). There are four cases.

**Case 2.1.** If \( i_2 < i < d - i \). In this case

\[
(3.9) \quad f_{\epsilon} (h \Delta) \leq \left( m_0^4 C \epsilon \right)^{1/2} f_{\epsilon} (h \Delta),
\]

which contradicts \( \boxed{3.7} \).
Case 2.2. If $i_2 = i < d - i$. In this case $u = 1$. The same computation but using Lemma 3.14 part 2 still gives (3.4) which is still a contradiction.

Case 2.3. If $i_2 = d - i < i$. In this case $\|u \wedge v \wedge w\|$ is an integer. Therefore, the same computation but using Lemma 3.14 part 3 still gives (3.9).

Case 2.4. If $i_2 = i = d - i$. The same computation, using Lemma 3.14 part 4 gives

$$f_\epsilon(h\Delta) \leq b_1,$$

which is again a contradiction.

This completes the proof of the claim.

Suppose $v \in \Omega$ is arbitrary. If $v \notin \Psi(h\Delta)$, then $f_\epsilon(h\Delta) > m_0 \varphi_\epsilon(v)$, and by left $K_I$ invariance of $\varphi_\epsilon$, (3.5) and (3.6) for all $k \in K_I$ we have

$$\varphi_\epsilon(a_{k}v) \leq m_0 \varphi_\epsilon(v) \leq m_0^{-1} f_\epsilon(h\Delta) \leq m_0^{-1} \max_{\psi \in \Psi(h\Delta)} \varphi_\epsilon(\psi) \leq \max_{\psi \in \Psi(h\Delta)} \varphi_\epsilon(a_{k}k\psi).$$

If $v \in \Psi(h\Delta)$, then (3.10) holds for obvious reasons. Therefore (3.10) holds for all $v \in \Omega$. Thus using the definition of $f_\epsilon$ and (3.10) we get

$$\int_{K_I} f_\epsilon(a_{k}k\Delta)^{\delta} dv_I(k) = \int_{K_I} \max_{\psi \in \Psi(h\Delta)} \varphi_\epsilon(a_{k}k\psi)^{\delta} dv_I(k) \leq \sum_{\psi \in \Psi(h\Delta)} \int_{K_I} \varphi_\epsilon(a_{k}k\psi)^{\delta} dv_I(k).$$

Using Lemma 5.7 we see that for all $\psi \in \Psi(h\Delta)$, $\psi \notin F$ and hence $\psi - \tau_0(\psi) \notin F \setminus \{0\}$. Moreover, if $\varphi_\epsilon(a_{k}k\psi) \neq 0$, then $\varphi_\epsilon(a_{k}k\psi) = \varphi_\epsilon(a_{k}k(\psi - \tau_0(\psi)))$ and we can apply (3.3) to get

$$\int_{K_I} \varphi_\epsilon(a_{k}k\psi)^{\delta} dv_I(k) \leq \frac{c}{d} \varphi_\epsilon(\psi)^{\delta},$$

for each $\psi \in \Psi(h\Delta)$. If $\varphi_\epsilon(a_{k}k\psi) = 0$, then it is clear that (3.12) also holds. Using Claim 3.9 we obtain

$$\sum_{\psi \in \Psi(h\Delta)} \int_{K_I} \varphi_\epsilon(a_{k}k\psi)^{\delta} dv_I(k) \leq d \max_{\psi \in \Psi(h\Delta)} \int_{K_I} \varphi_\epsilon(a_{k}k\psi)^{\delta} dv_I(k),$$

the claim of the Lemma follows from (3.8), (3.9), (3.11) and (3.12).

3.2. Proof of part II of Theorem 2.4 This time the aim is to construct a function such that it satisfies the conditions of the following Lemma.

Lemma 3.10. Suppose $r_1 = 2$ and $r_2 = 1$ or $r_1 = r_2 = 2$. Let $f : H_I \to \mathbb{R}$ be a strictly positive continuous function such that:

1. For any $\epsilon > 0$, there exists a neighbourhood $V(\epsilon)$ of 1 in $H_I$ such that

$$1 - \epsilon f(h) \leq f(uh) \leq (1 + \epsilon) f(h)$$

for all $h \in H_I$ and $u \in V(\epsilon)$.

2. The function $f$ is left $K_I$ invariant.

3. $f(1) < \infty$.

4. There exists $t_0 > 0$ and $b > 0$, such that for all $h \in H_I$ and $0 \leq t \leq t_0$,

$$A_t f(h) \leq f(h) + b.$$

Then $\sup_{t > 1} \frac{1}{t} A_t f(1) < \infty$.

Proof. Since $SO(2,1)$ is locally isomorphic to $SL_2(\mathbb{R})$ and $SO(2,2)$ is locally isomorphic to $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, this follows directly from Lemma 5.13 from [EMM98].

The general strategy of this subsection is broadly the same as in the last one. First we define a certain function on the exterior algebra of $\mathbb{R}^d$ and then we use this function to define a function which has the properties demanded by Lemma 3.10.
3.2.1. Functions on the exterior algebra of $\mathbb{R}^d$. As before we work with a function on the exterior algebra of $\mathbb{R}^d$. This time the definition is simpler because in this case the vectors fixed by the action of $H_1$ cause no extra problems. For $\epsilon > 0$, $0 < i < d$ and $v \in \bigwedge^i(\mathbb{R}^d)$ we define

$$\bar{\varphi}_\epsilon (v) = \epsilon^{\gamma_i} \|v\|^{-1}.$$

If $v \in \bigwedge^0(\mathbb{R}^d)$ or $v \in \bigwedge^d(\mathbb{R}^d)$ then we set $\bar{\varphi}_\epsilon (v) = 1$. The following Lemma is the analogue of Lemma 3.4.

**Lemma 3.11.** Let $i_1 \geq 0$ and $i_2 > 0$ and $\Lambda$ be a unimodular lattice. Then for all $u \in \Omega_{i_1} (\Lambda), v \in \Omega_{i_2} (\Lambda)$ and $w \in \Omega_{i_2} (\Lambda)$, $\bar{\varphi}_\epsilon (u \wedge v) \bar{\varphi}_\epsilon (u \wedge w) \leq \epsilon^{2i_2} \bar{\varphi}_\epsilon (u) \bar{\varphi}_\epsilon (u \wedge v \wedge w)$.

**Proof.** This is a direct consequence of Lemma 5.6 from [EMM98] and the fact that $2\gamma_{i_1+i_2} - \gamma_{i_1} - \gamma_{i_1+i_2} = 2i_2$. Let $\pi^T$ denote the projection from $\bigwedge^k(\mathbb{R}^d)$ onto the contracting and expanding eigenspaces of $\Lambda$, respectively. Note that

$$\frac{d^2}{dt^2} F_v (t) = \int_{K_1} e^{-2t} \|\pi^+ (kv)\| \|v\|^{-1} \cdot \frac{d^2}{dt^2} F_v (t) \leq \frac{1}{\|v\|^2} \int_{K_1} e^{-2t} \|\pi^+ (kv)\| \|v\|^{-1}$$

or

$$\frac{d}{dt} F_v (t) = \int_{K_1} e^{-2t} \|\pi^+ (kv)\| \|v\|^{-1} \cdot \frac{d}{dt} F_v (t) \leq \frac{1}{\|v\|^2} \int_{K_1} e^{-2t} \|\pi^+ (kv)\| \|v\|^{-1}$$

and

$$(\frac{d}{dt} F_v (t)) \leq 1 \int_{K_1} e^{-2t} \|\pi^+ (kv)\| \|v\|^{-1} \cdot \frac{d}{dt} F_v (t) \leq \frac{1}{\|v\|^2} \int_{K_1} e^{-2t} \|\pi^+ (kv)\| \|v\|^{-1}.$$
that $\tilde{f}_{\Delta,\epsilon}$ is left $K_I$ invariant. Also, since $\|\rho(h^{-1})\|^{-1} \leq \|hv\| / \|v\| \leq \|\rho(h)\|$ for all $v \in \Omega$ and $h \in H_I$, $\tilde{f}_{\Delta,\epsilon}$ also satisfies condition (1) of Proposition 3.10. We also have that $\tilde{f}_{\Delta,\epsilon}(1) < \infty$. It remains to show that $\tilde{f}_{\Delta,\epsilon}$ satisfies condition (2) of Proposition 3.10.

Lemma 3.13. Suppose $r_1 = 2$ and $r_2 = 1$ or $r_1 = r_2 = 2$. Then there exists $\epsilon > 0$ and $t_0 > 0$, such that for all $0 \leq t < t_0$ and $h \in H_I$,

$$\int_{K_I} \tilde{f}_{\Delta,\epsilon}(ah) \, dv_I(k) \leq \tilde{f}_{\Delta,\epsilon}(h).$$

Proof. Let $m_0 = \|\rho(a_{t_0})\|$. Then, for all $v \in \bigwedge (R^d)$ and $0 \leq t < t_0$,

$$(3.15) \quad m_0^{-1} \leq \|av\| / \|v\| \leq m_0.$$ 

It follows from the definition of $\tilde{\varphi}_\epsilon$ and (3.15) that for all $0 \leq t < t_0$,

$$(3.16) \quad m_0^{-1} \tilde{\varphi}_\epsilon(v) \leq \tilde{\varphi}_\epsilon(ah) \leq m_0 \tilde{\varphi}_\epsilon(v).$$

Let $\Psi(h\Delta) = \left\{ v \in \Omega(h\Delta) : \max_{v \in \Omega(h\Delta)} \tilde{\varphi}_\epsilon(v) \leq m_0^2 \tilde{\varphi}_\epsilon(v) \right\}$. Now we show that: For $\epsilon$ small enough the set $\Psi(h\Delta)$ contains only one element up to sign change in each degree. To see this, assume that for some $0 < i < d$, $\Psi(h\Delta) \cap \Omega(h\Delta)$ contains two non-collinear elements, $v_0$ and $v_0$. We can write $v_0 = u \wedge v$ and $w_0 = u \wedge w$ where $u \in \Omega_i(h\Delta), v \in \Omega_i G(h\Delta)$ and $w \in \Omega_{i_2}(h\Delta)$ with $i_1 \geq 0$ and $i_2 > 0$. In this case

$$\tilde{f}_{\Delta,\epsilon}(h)^2 \leq d^2 m_0^4 \tilde{\varphi}_\epsilon(u \wedge v) \tilde{\varphi}_\epsilon(u \wedge w) \leq d^2 m_0^4 \tilde{\varphi}_\epsilon(h)^2,$$

by Lemma 3.11. Hence the claim is true since taking $\epsilon$ small enough gives a contradiction.

In view of this discussion we can suppose that $\Psi(h\Delta) = \{\psi_i\}_{i=1}^d$ where $\psi_i$ has degree $i$. Let $v \in \Omega_i(h\Delta)$ be arbitrary. If $v \notin \Psi(h\Delta)$, then $\max_{v \in \Omega_i(h\Delta)} \tilde{\varphi}_\epsilon(v) > m_0^2 \tilde{\varphi}_\epsilon(v)$, and by left $K_I$ invariance of $\tilde{\varphi}_\epsilon$ and (3.16) for all $k \in K_I$ we have

$$(3.17) \quad \tilde{\varphi}_\epsilon(a_{t_0}kv) \leq m_0 \tilde{\varphi}_\epsilon(v) \leq m_0^{-1} \max_{v \in \Omega_i(h\Delta)} \tilde{\varphi}_\epsilon(v) = m_0^{-1} \tilde{\varphi}_\epsilon(\psi_i) \leq \tilde{\varphi}_\epsilon(a_{t_0}\psi_i).$$

If $v \in \Psi(h\Delta)$, then (3.17) holds for obvious reasons. Therefore (3.17) holds for all $v \in \Omega$. Thus using the definition of $\tilde{f}_{\Delta,\epsilon}$ and (3.17) we get

$$(3.18) \quad \int_{K_I} \tilde{f}_{\Delta,\epsilon}(ah) \, dv_I(k) = \sum_{i=1}^d \int_{K_I} \max_{v \in \Omega_i(h\Delta)} \tilde{\varphi}_\epsilon(a_{t_0}kv) \, dv_I(k) \leq \sum_{i=1}^d \int_{K_I} \tilde{\varphi}_\epsilon(a_{t_0}\psi_i) \, dv_I(k).$$

By Lemma 3.12, there exists $t_0 > 0$, so that for any $v \in \bigwedge (R^d)$ and all $0 \leq t < t_0$,

$$(3.19) \quad \int_{K_I} \tilde{\varphi}_\epsilon(a_{t_0}\psi_i) \, dv_I(k) \leq \tilde{\varphi}_\epsilon(\psi_i),$$

for each $\psi_i \in \Psi(h\Delta)$. The claim of the Lemma follows from (3.18) and (3.19). \qed

4. Ergodic Theorems.

For subgroups $W_1$ and $W_2$ of $G_q$, let $X(W_1, W_2) = \{ g \in G_q : W_2g \subset gW_1 \}$. As in [EMM93], the ergodic theory is based on Theorem 3 from [DM93] reproduced below in a form relevant to the current situation.

Theorem 4.1. Suppose $r_1 \geq 2$ and $r_2 \geq 1$. Let $g \in C_{SL}(r_1, r_2)$ be arbitrary. Let $U = \{ u_t : t \in \mathbb{F} \}$ be a unipotent one parameter subgroup of $G_q$ and $\phi$ be a bounded continuous function on $G_q/\Gamma_g$. Let $D$ be a compact subset of $G_q/\Gamma_g$ and let $\epsilon > 0$ be given. Then there exist finitely many proper closed subgroups $H_1, \ldots, H_k$ of $G_q$, such that $H_i \cap \Gamma_g$ is a lattice in $H_i$ for all $i$, and compact subsets $C_1, \ldots, C_k$ of
Lemma 4.3. Suppose one parameter unipotent subgroup of \( H \) be a bounded continuous function on \( [KSS02] \). the identity is fixed as the base point for the flow and the condition that \( T \) exists a \( D \). Let \( \lambda \) be all \( T > T \). From Theorem 4.1, for all \( (4.4) \) \( \eta > 0 \) and \( \eta > 0 \) there exists a compact subset \( X \) \( \epsilon > 0 \). From Remark 4.2, \( H \) is a normal subgroup of \( H \) and hence \( H \) are defined over \( Q \). For a precise reference see Theorem 3.6.2 and Remark 3.4.2 of \( [PR94] \).

The next result is a reworking of Theorem 4.3 from \( [EMM98] \). The difference is that in Lemma 4.3 the identity is fixed as the base point for the flow and the condition that \( H \) be maximal is dropped.

**Lemma 4.3.** Suppose \( r_1 \geq 2 \) and \( r_2 \geq 1 \). Let \( g \in C_{SL}(r_1, r_2) \) be arbitrary. Let \( U = \{u_t : t \in \mathbb{R}\} \) be a one parameter unipotent subgroup of \( H_g \), not contained in any proper normal subgroup of \( H_g \). Let \( \phi \) be a bounded continuous function on \( G_g/\Gamma_g \). Then for all \( \epsilon > 0 \) and \( \eta > 0 \) there exists a \( T_0 > 0 \) such that for all \( T > T_0 \),

\[
\nu_g \left( \left\{ k \in K_g : \frac{1}{T} \int_0^T \phi(u_t k) dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right\} > \epsilon \right) \leq \eta.
\]

**Proof.** Let \( H_1, \ldots, H_k \) and \( C_1, \ldots, C_k \) be as in Theorem 4.1. Let \( \gamma \in \Gamma_g \), consider \( Y_i(\gamma) = K_g \cap X(H_i, U) \gamma \). Suppose that \( Y_i(\gamma) = K_g \), then \( U/k\gamma^{-1} \subset k\gamma^{-1}H_i \) for all \( k \in K_g \). In other words \( k^{-1}Uk \subset \gamma^{-1}H_i \gamma \) for all \( k \in K_g \).

The subgroup \( \langle k^{-1}Uk : k \in K_g \rangle \) is normalised by \( U \cup K_g \) and clearly \( \langle k^{-1}Uk : k \in K_g \rangle \subseteq \langle U \cup K_g \rangle \subseteq H_g \). If \( G \) is a simple Lie group with finite centre, with maximal compact subgroup \( K \), it follows from exercise A.3, chapter IV of \( [Hel01] \) that \( K \) is also a maximal proper subgroup of \( G \). This means that because \( H_g \) is semisimple with finite centre, any connected subgroup \( L \) of \( H_g \) containing \( K_g \) can be represented as \( L = H'K_g \) where \( H' \) is a connected normal subgroup of \( H_g \). Because \( U \) is not contained in any proper normal subgroup of \( H_g \), this implies that \( \langle U \cup K_g \rangle = H_g \). Therefore, \( \langle k^{-1}Uk : k \in K_g \rangle \) is a normal subgroup of \( H_g \) and because \( U \) is not contained in any proper normal subgroup of \( H_g \), we have \( \langle k^{-1}Uk : k \in K_g \rangle = H_g \). This and \( (4.4) \) imply that \( H_g \subset \gamma^{-1}H_i \gamma \). Note that \( \gamma \in SL_d(\mathbb{Z}) \) and by Remark 4.2, \( H_i \) is defined over \( Q \). Therefore, \( \gamma^{-1}H_i \gamma \) is defined over \( Q \), it follows from Theorem 7.7 of \( [PR94] \) that \( \gamma^{-1}H_i \gamma \cap SL_d(\mathbb{Q}) = \gamma^{-1}H_i \gamma \). Therefore Lemma 3.7 and Proposition 4.1 of \( [Sar11] \) imply that \( \gamma^{-1}H_i \gamma = G_g \) which is a contradiction and therefore \( Y_i(\gamma) \subseteq K_g \). This means for all \( 1 \leq i \leq k \), \( Y_i(\gamma) \) is a submanifold of strictly smaller dimension than \( K_g \) and hence

\[
\nu_g(\{ Y_i(\gamma) \}) = 0.
\]

Note that because \( C_i \subseteq X(H_i, U) \),

\[
K_g \cap \bigcup_{1 \leq i \leq k} C_i \Gamma_g \subseteq K_g \cap \bigcup_{1 \leq i \leq k} X(H_i, U) \Gamma_g = \bigcup_{1 \leq i \leq k} \bigcup_{\gamma \in \Gamma_g} Y_i(\gamma)
\]

and therefore \( (4.3) \) implies

\[
\nu_g \left( K_g \cap \bigcup_{1 \leq i \leq k} C_i \Gamma_g \right) = 0.
\]

Let \( D \) be a compact subset of \( G_g \) such that \( K_g \subset D \). Then from \( (4.4) \), it follows that, for all \( \eta > 0 \) there exists a compact subset \( F \) of \( D - \bigcup_{1 \leq i \leq k} C_i \Gamma_g \), such that

\[
\nu_g(F \cap K_g) \geq 1 - \eta.
\]

From Theorem 4.1 for all \( \epsilon > 0 \) there exists a \( T_0 > 0 \), such that for all \( x \in (F \cap K_g) / \Gamma_g \) and \( T > T_0 \),

\[
\left| \frac{1}{T} \int_0^T \phi(u_t x) dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| < \epsilon.
\]
Therefore if \( k \in K_g, T > T_0 \) and
\[
\left| \frac{1}{T} \int_0^T \phi(u_t k) dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| > \epsilon,
\]
then \( k \in K_g \setminus F \), but \( \nu_g(K_g \setminus F) \leq \eta \) by (4.5) and this implies (4.4).

**Lemma 4.4.** Suppose \( r_1 \geq 2 \) and \( r_2 \geq 1 \). Let \( g \in C_{SL}(r_1, r_2) \) be arbitrary. Let \( U = \{ u_t : t \in \mathbb{R} \} \) be a one parameter unipotent subgroup of \( H_g \), not contained in any proper normal subgroup of \( H_g \). Let \( \phi \) be a bounded continuous function on \( G_g/\Gamma_g \). Then for all \( \epsilon > 0 \) and \( \delta > 0 \) there exists a \( T_0 > 0 \) such that for all \( T > T_0 \),
\[
\left| \frac{1}{\delta T} \int_0^{(1+\delta)T} \int_{K_g} \phi(u_t k) \, d\nu_g(k) \, dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| < \epsilon.
\]

**Proof.** Let \( \phi \) be a bounded continuous function on \( G_g/\Gamma_g \). Lemma 4.3 implies for all \( \epsilon > 0 \), \( \eta > 0 \) and \( d > 0 \) there exists a \( T_0 > 0 \) such that for all \( T > T_0 \),
\[
(4.6) \quad \nu_g \left( \left\{ k \in K_g : \left| \frac{1}{\delta T} \int_0^{dT} \int_{K_g} \phi(u_t k) \, d\nu_g(k) \, dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| > \epsilon \right\} \right) \leq \eta.
\]
Using (4.6) with \( d = 1 \) and \( d = 1 + \delta \) we get that for all \( \epsilon > 0 \) and \( \eta > 0 \) there exists a subset \( C \subseteq K_g \) with \( \nu_g(C) \geq 1 - \eta \) such that for all \( k \in C \) the following holds
\[
\left| \int_0^T \phi(u_t k) \, dt - T \int_{G_g/\Gamma_g} \phi d\mu_g \right| < \epsilon T \quad \text{and} \quad \left| \int_0^{(1+\delta)T} \phi(u_t k) \, dt - (1 + \delta)T \int_{G_g/\Gamma_g} \phi d\mu_g \right| < (1 + \delta)T \epsilon.
\]
Hence for all \( k \in C \) we have
\[
\left| \int_T^{(1+\delta)T} \phi(u_t k) \, dt - \delta T \int_{G_g/\Gamma_g} \phi d\mu_g \right| = \left| \int_0^{(1+\delta)T} \phi(u_t k) \, dt - (1 + \delta)T \int_{G_g/\Gamma_g} \phi d\mu_g - \int_0^T \phi(u_t k) \, dt + T \int_{G_g/\Gamma_g} \phi d\mu_g \right| \leq \left| \int_0^T \phi(u_t k) \, dt - T \int_{G_g/\Gamma_g} \phi d\mu_g \right| + \left| \int_0^{(1+\delta)T} \phi(u_t k) \, dt - (1 + \delta)T \int_{G_g/\Gamma_g} \phi d\mu_g \right| \leq (2 + \delta)T \epsilon.
\]
This means that for all \( \delta > 0 \), \( \eta > 0 \) and \( \epsilon > 0 \),
\[
\nu_g \left( \left\{ k \in K_g : \left| \frac{1}{\delta T} \int_T^{(1+\delta)T} \phi(u_t k) \, dt - \int_{G_g/\Gamma_g} \phi d\mu_g \right| < \frac{(2 + \delta) \epsilon}{\delta} \right\} \right) \geq 1 - \eta.
\]
Since we can make \( \epsilon \) and \( \eta \) as small as we wish this implies the claim.

**Lemma 4.5.** Suppose \( r_1 \geq 2 \) and \( r_2 \geq 1 \). Let \( A = \{ a_t : t \in \mathbb{R} \} \) be a one parameter subgroup of \( H_g \), not contained in any proper normal subgroup of \( H_g \), such that there exists a continuous homomorphism \( \rho : SL_2(\mathbb{R}) \to H_g \) with \( \rho(D) = A \) and \( \rho(SO(2)) \subseteq K_g \) where \( D = \{ (t^I, 0 I, 0^T) : t > 0 \} \). Let \( \phi \) be a continuous function on \( G_g/\Gamma_g \) vanishing outside of a compact set. Then for all \( g \in C_{SL}(r_1, r_2) \) and \( \epsilon > 0 \) there exists \( T_0 > 0 \) such that for all \( t > T_0 \),
\[
\left| \int_{K_g} \phi(a_t k) \, d\nu_g(k) - \int_{G_g/\Gamma_g} \phi d\mu_g \right| \leq \epsilon.
\]
Proof. This is very similar to the proof of Theorem 4.4 from [EMM98] and some details will be omitted. Fix \( \epsilon > 0 \). Assume that \( \phi \) is uniformly continuous. Let \( u_t = \begin{pmatrix} \frac{2}{t} & 0 \\ 0 & 1 \end{pmatrix} \) and \( w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Then it is clear that 
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = b_t u_t k_t w,
\]
where \( b_t = (1 + t^{-2})^{-1/2} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \) and \( k_t = (1 + t^{-2})^{-1/2} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \).
By our assumptions on \( A \) there exists a continuous homomorphism \( \rho : SL_2(\mathbb{R}) \rightarrow H_g \) such that \( \rho(D) = A \) and \( \rho(SO(2)) \subset K_g \).
Let \( \rho(d_t) = d_t', \rho(b_t) = b_t', \rho(k_t) = k_t' \) and \( \rho(w) = w' \). Then for all \( t > 0 \) and \( g \in \mathcal{C}_{SL}(r_1, r_2) \),
\[
\int_{K_g} \phi(d_t' k) \, d\nu_g(k) = \int_{K_g} \phi(b_t' u_t' k') \, d\nu_g(k)
\]
(4.7)
\[
= \int_{K_g} \phi(b_t' u_t') \, d\nu_g(k),
\]
since \( k_t' \) and \( w' \in K_g \). It follows from (4.7) that for \( r, t > 0 \),
\[
\left| \int_{K_g} \phi(d_t' k) \, d\nu_g(k) - \int_{K_g} \phi(u_t' k) \, d\nu_g(k) \right|
\]
(4.8)
\[
\leq \left| \int_{K_g} (\phi(d_t' k) - \phi(d_t' k)) \, d\nu_g(k) \right| + \left| \int_{K_g} (\phi(d_t' k) - \phi(u_t' k)) \, d\nu_g(k) \right|
\]
\[
= \left| \int_{K_g} (\phi(d_t' k) - \phi(d_t' k)) \, d\nu_g(k) \right| + \left| \int_{K_g} (\phi(b_t' u_t' k) - \phi(u_t' k)) \, d\nu_g(k) \right|.
\]
By uniform continuity, the fact that \( \lim_{t \to \infty} b_t = I \) and (4.8) imply there exists \( T_1 > 0 \) and \( \delta > 0 \) such that for \( t > T_1 \) and \( |r - 1| < \delta \) we have
\[
\left| \int_{K_g} \phi(d_t' k) \, d\nu_g(k) - \int_{K_g} \phi(u_t' k) \, d\nu_g(k) \right| \leq \epsilon.
\]
Thus, if \( T > T_1 \) then
\[
\left| \int_{K_g} \phi(d_t' k) \, d\nu_g(k) - \frac{1}{\delta T} \int_T^{(1+\delta)T} \int_{K_g} \phi(u_t' k) \, d\nu_g(k) \, dt \right| \leq \epsilon.
\]
(4.9)
Combining (4.9) with Lemma 1.3 via the triangle inequality finishes the proof of the Lemma.

The section is completed by the proof of the main ergodic result whose proof follows that of Theorem 3.5 in [EMM98].

Proof of Theorem 2.2. Assume that \( \phi \) is non-negative. Let \( A(r) = \{ x \in G_g/\Gamma_g : \alpha(x) > r \} \). Choose a continuous non-negative function \( g_r \) on \( G_g/\Gamma_g \) such that \( g_r(x) = 1 \) if \( x \in A(r+1) \), \( g_r(x) = 0 \) if \( x \notin A(r) \) and \( 0 \leq g_r(x) \leq 1 \) if \( x \in A(r) \setminus A(r+1) \). Then
\[
\int_{K_g} \phi(a_t k) \, d\nu_g(k) = \int_{K_g} \phi(a_t k) g_r(a_t k) \, d\nu_g(k) + \int_{K_g} (\phi(a_t k) - \phi(a_t k) g_r(a_t k)) \, d\nu_g(k).
\]
(4.10)
Let \( \beta = 2 - \delta \) then for \( x \in G_g/\Gamma_g \),
\[
\phi(x) g_r(x) \leq C \alpha(x)^{2-\beta} g_r(x)
\]
\[
= C \alpha(x)^{2-\beta/2} g_r(x) \alpha(x)^{-\beta/2} \leq C r^{-\beta/2} \alpha(x)^{2-\beta/2}.
\]
The last inequality is true because \( g_r(x) = 0 \) if \( \alpha(x) \leq r \). Therefore
\[
\int_{K_g} \phi(a_t k) g_r(a_t k) \, d\nu_g(k) \leq C r^{-\beta/2} \int_{K_g} \alpha(a_t k)^{2-\beta/2} \, d\nu_g(k).
\]
(4.11)
Since \(g \in \mathcal{C}_{\text{SL}}(r_1, r_2), r_1 \geq 3 \text{ and } r_2 \geq 1\) Theorem 2.1 part I implies there exists \(B\) such that
\[
\int_{K_g} \alpha(a_t k)^{2-\beta/2} d\nu_g(k) = \int_{K_t^1} \alpha \circ g^{-1}(a_t kg)^{2-\beta/2} d\nu_t(k) \leq c(g) \int_{K_t^1} \alpha(a_t kg)^{2-\beta/2} d\nu_t(k) < B
\]
for all \(t \geq 0\). Then (4.11) implies that
\[
(4.12) \quad \int_{K_g} \phi(a_t k) g_r(a_t k) d\nu_g(k) \leq BCR^{-\beta/2}.
\]
For all \(\epsilon > 0\) there exists a compact subset, \(C\) of \(G_g/\Gamma_g\) such that \(\mu_g(C) \geq 1 - \epsilon\). The function \(\alpha\) is bounded on \(C\) and hence for all \(\epsilon > 0\),
\[
\lim_{r \to \infty} \mu_g(A(r)) = \lim_{r \to \infty} (\mu_g(\{x \in C : \alpha(x) > r\}) + \mu_g(\{x \in (G_g/\Gamma_g) \setminus C : \alpha(x) > r\})) \leq \epsilon.
\]
This means that
\[
(4.13) \quad \lim_{r \to \infty} \mu_g(A(r)) = 0.
\]
Note that
\[
(4.14) \quad \int_{G_g/\Gamma_g} \phi(x) g_r(x) d\mu_g(x) \leq \int_{A(r)} \phi(x) d\mu_g(x).
\]
Since \(\phi \in L^1(G_g/\Gamma_g)\), (4.12) and (4.14) imply that
\[
(4.15) \quad \lim_{r \to \infty} \int_{G_g/\Gamma_g} \phi(x) g_r(x) d\mu_g(x) = 0.
\]
Since the function \(\phi(x) - \phi(x) g_r(x)\) is continuous and has compact support, Lemma 4.5 implies for all \(\epsilon > 0\) and \(g \in \mathcal{C}_{\text{SL}}(r_1, r_2)\) there exists \(T_0 > 0\) such that for all \(t > T_0\),
\[
(4.16) \quad \left| \int_{K_g} (\phi(a_t k) - \phi(a_t k) g_r(a_t k)) d\nu_g(k) - \int_{G_g/\Gamma_g} (\phi(x) - \phi(x) g_r(x)) d\mu_g(x) \right| < \frac{\epsilon}{2}.
\]
It is straightforward to check that (4.11), (4.12), (4.14) and (4.16) imply the conclusion of the Theorem if \(r\) is sufficiently large. \(\Box\)

5. Proof of Theorem 2.1

The proof of Theorem 2.1 follows the same route as that of Sections 3.4-3.5 of [EMM98]. The main modification we make in order to handle the present situation is that we work inside the surface \(X_g(\mathbb{R})\) rather than in the whole of \(\mathbb{R}^d\). For \(t \in \mathbb{R}\) and \(v \in \mathbb{R}^d\) define a linear map \(a_t\) by

\[a_t v = (v_1, \ldots, v_s, e^{-t} v_{s+1}, v_{s+2}, \ldots, e^t v_d).\]

Note that the one parameter group \(\{\hat{a}_t : t \in \mathbb{R}\} = g^{-1} \{a_t : t \in \mathbb{R}\} g \subset H_g\) and that there exists a continuous homomorphism \(\rho : SL_2(\mathbb{R}) \to H_g\) with \(\rho(D) = \{\hat{a}_t : t \in \mathbb{R}\} \) and \(\rho(SO(2)) \subset K_g\) where \(D = \{(t, 0) : t > 0\}\). Moreover note that \(\{a_t : t \in \mathbb{R}\}\) is self adjoint, not contained in any normal subgroup of \(H_g\) and the only eigenvalues of \(a_t\) are \(e^{-t}, 1\) and \(e^t\). In other words, \(\{\hat{a}_t : t \in \mathbb{R}\}\) satisfies the conditions of Theorem 2.3 and Theorem 2.4. For any natural number \(n\), let \(S^{n-1}\) denote the unit sphere in a \(n\) dimensional Euclidean space and let \(\gamma_n = \text{Vol}(S^n)\) and \(c_{r_1, r_2} = \gamma_{r_1-1} \gamma_{r_2-1}\) then define

\[
(5.1) \quad C_1 = c_{r_1, r_2} g((2-r_1-r_2)/2) = c_{r_1, r_2} g((2-d+s)/2).
\]
5.1. Proof of Theorem 2.3 In Lemma 5.1 it is shown that it is possible to approximate certain integrals over $K_g$ by integrals over $R^{d-s-2}$. The integral over $R^{d-s-2}$ can be used like the characteristic function of $R \times A(T/2, T)$, in particular Theorem 2.3 is proved as an application of Lemma 5.1. It should be noted that Lemma 5.1 is analogous to Lemma 3.6 from [1] and its proof is similar.

Lemma 5.1. Let $f$ be a continuous function of compact support on $R^d_+ = \{ v \in R^d : \langle v, e_{s+1} \rangle > 0 \}$ and for $g \in C_{\text{SL}}(r_1, r_2)$ let

$$J_{f,g}(\ell_1, \ldots, \ell_s, r) = \frac{1}{\nu^{d-s-2}} \int_{R^{d-s-2}} f(\ell_1, \ldots, \ell_s, r, v_{s+2}, \ldots, v_{d-1}, v_d) dv_{s+2} \ldots dv_{d-1},$$

where $v_d = (a - Q_0^g(\ell_1, \ldots, \ell_s, 0, v_{s+2}, \ldots, v_{d-1}, 0))/2r$, so that $Q_0^g(\ell_1, \ldots, \ell_s, r, v_{s+2}, \ldots, v_{d-1}, v_d) = a$. Then for every $\epsilon > 0$ there exists $T_0 > 0$ such that for every $t$ with $e^t > T_0$ and every $v \in R^d_+$ with $\|v\| > T_0$,

$$\left| C_{\nu} e^{(d-s-2)t} \int_{K_g} f(\hat{a}_t k v) dv_g(k) - J_{f,g}(M_0^g(v), \|v\| e^{-t}) \right| < \epsilon.$$

Proof. By Lemma 2.2 of [1], for all $g \in C_{\text{SL}}(r_1, r_2)$ there exists a basis of $R^d$, denoted by $b_1, \ldots, b_d$ such that

$$Q_0^g(v) = Q_{1,\ldots,s}(v) + 2v_{s+1}v_d + \sum_{i=s+2}^{s+r_1} v_i^2 - \sum_{i=s+r_1+1}^{d-1} v_i^2,$$

and

$$\hat{a}_t(v) = (v_1, \ldots, v_s, e^{-t}v_{s+1}, v_{s+2}, \ldots, v_{d-1}, e^{t}v_d),$$

where $v_i = \langle v, b_i \rangle$ for $1 \leq i \leq d$ and $Q_{1,\ldots,s}(v)$ is a non degenerate quadratic form in variables $v_1, \ldots, v_s$. Let $E$ denote the support of $f$. Let $c_1 = \inf_{v \in E} \langle v, b_{s+1} \rangle$, $c_2 = \sup_{v \in E} \langle v, b_{s+1} \rangle$. From the definition of $\hat{a}_t$ it follows that $f(\hat{a}_t w) = 0$ unless

$$\langle w, b_{s+1} \rangle \leq \beta,$$

$$c_1 \leq \langle w, b_{s+1} \rangle e^{-t} \leq c_2,$$

$$\pi'(w) \in \pi'(E),$$

where $\beta$ depends only on $E$ and $\pi'$ denotes the projection onto the span of $b_1, \ldots, b_s, b_{s+2}, \ldots, b_{d-1}$. For $w$ satisfying (5.2) and (5.3) we have $\langle w, b_d \rangle = O(e^{-t})$. This, together with (5.4) and (5.3) imply that if $f(\hat{a}_t w) \neq 0$ and $t$ is large, then

$$\|w\| = \langle w, b_{s+1} \rangle + O(e^{-t}).$$

Note that by (5.6),

$$\langle \hat{a}_t w, b_{s+1} \rangle = \langle w, b_{s+1} \rangle e^{-t} = e^{-t} \|w\| + O(e^{-2t}),$$

and

$$\langle \hat{a}_t w, b_i \rangle = \langle w, b_i \rangle,$$

for $1 \leq i \leq s$, or $s+2 \leq i \leq d-1$.

Finally,

$$\langle \hat{a}_t w, b_d \rangle = (Q_0^g(w) - Q_0^g(\langle w, b_1 \rangle, \ldots, \langle w, b_s \rangle, 0, \langle w, b_{s+1} \rangle, \ldots, \langle w, b_{d-1} \rangle, 0))/2 \langle \hat{a}_t w, b_{s+1} \rangle$$

$$= (Q_0^g(w) - Q_0^g(\langle w, b_1 \rangle, \ldots, \langle w, b_s \rangle, 0, \langle w, b_{s+1} \rangle, \ldots, \langle w, b_{d-1} \rangle, 0))/2e^{-t} \|w\| + O(e^{-t}).$$

Hence, using (5.6), (5.7) and (5.8) together with the uniform continuity of $f$, applied with $w = kv$ for $v \in R^d_+$ and $k \in K_g$, we see that for all $\delta > 0$ there exists a $t_0 > 0$ so that if $t > t_0$ then

$$|f(\hat{a}_t kv) - f(v_1, \ldots, v_s, \|v\| e^{-t}, \langle kv, b_{s+1} \rangle, \ldots, \langle kv, b_{d-1} \rangle, v_d)| < \delta,$$

where $v_d$ is determined by

$$Q_0^g(v_1, \ldots, v_s, \|v\| e^{-t}, \langle kv, b_{s+1} \rangle, \ldots, \langle kv, b_{d-1} \rangle, v_d) = Q_0^g(v) = a.$$
Change basis by letting $f_{s+1} = (b_{s+1} + b_d)/\sqrt{2}$, $f_d = (b_{s+1} - b_d)/\sqrt{2}$ and $f_i = b_i$ for $1 \leq i \leq s$, or $s + 2 \leq i \leq d - 1$. In this basis $K_g \cong SO(r_1) \times SO(r_2)$ consists of orthogonal matrices preserving the subspaces $L_1 = (f_1, \ldots, f_s)$, $L_2 = (f_{s+1}, \ldots, f_{s+r_1})$ and $L_3 = (f_{s+r_1+1}, \ldots, f_d)$. For $i = 1, 2$ or 3, let $\pi_i$ denote the orthogonal projection onto $L_i$. Write $\rho_i = \|\pi_i(v)\|$; then the orbit $K_g v$ is product of a point and two spheres $\{v_1, \ldots, v_s\} \times \rho_2 S^{r_1-1} \times \rho_3 S^{r_2-1}$, where $S^{r_1-1}$ denotes the unit sphere in $L_2$ and $S^{r_2-1}$ the unit sphere in $L_3$.

Suppose $w \in K_g v$ is such that $f(\hat{a}_i w) \neq 0$. Then from (5.2) and (5.3) it follows that $\langle w, b_d \rangle = O(e^{-t})$. Now, set $w_i = \langle w, f_i \rangle$, then $w_{s+1} = 2^{-1/2} \langle w, b_{s+1} \rangle + O(e^{-t})$, $w_d = 2^{-1/2} \langle w, b_{s+1} \rangle + O(e^{-t})$ and for $1 \leq i \leq s$, or $s + 2 \leq i \leq d - 1$, $w_i = O(1)$. Hence for $i = 2$ or 3,

\begin{equation}
\rho_i = \|\pi_i(w)\| = 2^{-1/2} (\langle w, b_{s+1} \rangle + O(e^{-t})) = 2^{-1/2} \|w\| + O(e^{-t}),
\end{equation}

where the last estimate follows from (5.5).

By integrating (5.9) with respect to $K_g$ we see that for all $\epsilon > 0$ there exists a $t_0 > 0$ so that if $t > t_0$ then

\begin{equation}
\left| \int_{K_g} f(\hat{a}_i k v) d\nu_g(k) - \int_{K_g} f(v_1, \ldots, v_s, \|v\| e^{-t}, \langle kv, b_{s+1} \rangle, \ldots, \langle kv, b_{d-1} \rangle, v_d) d\nu_g(k) \right| < \epsilon.
\end{equation}

Equation (5.11) implies that if $f(\hat{a}_i kv) \neq 0$, then $kv$ is within a bounded distance from $\rho_2 f_{s+1} + \rho_3 f_d$. As $\|v\|$ increases so do the $\rho_i$ and the normalised Haar measure on $\rho_2 S^{r_1-1}$ near $\rho_2 f_{s+1}$ tends to $(1/\text{Vol} (\rho_2 S^{r_1-1})) d\nu_{s+2} \cdots d\nu_{s+r_1}$ and similarly the Haar measure on $\rho_3 S^{r_2-1}$ near $\rho_3 f_d$ tends to $(1/\text{Vol} (\rho_3 S^{r_2-1})) d\nu_{s+r_1+1} \cdots d\nu_{d-1}$. This means that as $\|v\|$ tends to infinity the second integral in (5.11) tends to

\begin{equation}
\frac{\rho_{r_1}^{-1-r_1} \rho_{r_2}^{-1-r_2}}{c_{r_1,r_2}} \int_{\mathbb{R}^{d-s-2}} f(v_1, \ldots, v_s, \|v\| e^{-t}, v_{s+1}, \ldots, v_d) d\nu_{s+2} \cdots d\nu_{d-1} = \frac{\|v\| e^{-t} d^{s-2}}{\rho_{r_1}^{-1-r_1} \rho_{r_2}^{-1-r_2} c_{r_1,r_2}} J_{f,g}(M_0^g(v), \|v\| e^{-t}).
\end{equation}

Because (5.10) implies that $\rho_{r_1}^{-1-r_1} \rho_{r_2}^{-1-r_2} = 2^{(s+2-d)/2} \|v\|^{d-s-2} + O(e^{-t})$ we can use (5.11) and (5.12) to get that for all $\epsilon > 0$ there exists a $t_0 > 0$ so that if $t > t_0$ and $\|v\| > t_0$ then

\begin{equation}
\left| \int_{K_g} f(\hat{a}_i k v) d\nu_g(k) - \frac{e^{(s+2-d)-1}}{C_1} J_{f,g}(M_0^g(v), \|v\| e^{-t}) \right| < \epsilon.
\end{equation}

By dividing through by the factor $e^{(s+2-d)-1}$ we obtain the desired conclusion. \hfill \Box

For $f_1$ and $f_2$ continuous functions of compact support on $\mathbb{R}^d_+ = \{v \in \mathbb{R}^d : \{v, e_{s+1} \} > 0\}$, define $J_{f_1,g} + J_{f_2,g} = J_{f_1+f_2,g}$ and $J_{f_1,g}J_{f_2,g} = J_{f_1 f_2,g}$. These operations make the collection of functions of the form $J_{f,g}$ into an algebra of real valued functions on the set $\mathbb{R}^s \times \{v \in \mathbb{R} : v > 0\}$. Denote by this algebra by $\mathcal{A}$. The following Lemma will be used in the proofs of Theorem 2.3 and Theorem 2.1.

**Lemma 5.2.** $\mathcal{A}$ is dense in $C_c(\mathbb{R}^s \times \{v \in \mathbb{R} : v > 0\})$.

**Proof.** Let $B$ be a compact subset of $\mathbb{R}^s \times \{v \in \mathbb{R} : v > 0\}$. Let $\mathcal{A}_B$ denote the subalgebra of $\mathcal{A}$ of functions with support $B$. It is straightforward to check that the algebra $\mathcal{A}_B$ separates points in $B$ and does not vanish at any point in $B$. Therefore, by the Stone-Weierstrass Theorem (cf. [Rud76], Theorem 7.32) $\mathcal{A}_B$ is dense in the space of continuous functions on $B$. Since $B$ is arbitrary this implies the claim. \hfill \Box

**Proof of Theorem 5.1.** Let $\epsilon > 0$ be arbitrary and $g \in C_{SL}(r_1, r_2)$. By Lemma 5.2 there exists a continuous non-negative function $f$ on $\mathbb{R}^d_+$ of compact support so that $J_{f,g} \geq 1 + \epsilon$ on $R \times [1, 2]$. Then
if $v \in \mathbb{R}^d$ satisfies $e^t \leq \|v\| \leq 2e^t$, $M_0^g(v) \in R$ and $Q_0^g(v) = a$ then $J_{f,g}(M_0^g(v), \|v\| e^{-t}) \geq 1 + \epsilon$. Then by Lemma 5.1 for sufficiently large $t$,

$$C_1 e^{(d-s-2)t} \int_{K_g} f(\hat{a}_k kv) d\nu_g(k) \geq 1$$

if $e^t \leq \|v\| \leq 2e^t$, $M_0^g(v) \in R$, and $Q_0^g(v) = a$. Then summing over $v \in X_g(\mathbb{Z})$, we get

$$|X_g(\mathbb{Z}) \cap V_M([a,b]) \cap A(e^t, 2e^t)| \leq \sum_{v \in X_g(\mathbb{Z})} C_1 e^{(d-s-2)t} \int_{K_g} f(\hat{a}_k kv) d\nu_g(k)$$

$$= C_1 e^{(d-s-2)t} \int_{K_g} F_{f,g}(\hat{a}_k) d\nu_g(k).$$

Note that

$$\int_{K_g} F_{f,g}(\hat{a}_k) d\nu_g(k) = \int_{K_1} F_{f,g}(g^{-1}a_k kg) d\nu_1(k).$$

By (2.22) we have the bound $F_{f,g}(x) \leq c(f) \alpha(x)$ for all $x \in G_g / T_g$ where $c(f)$ is a constant depending only on $f$. Since $g \in \mathcal{C}_{SL}(r_1, r_2)$, part I of Theorem 2.4 implies that if $r_1 \geq 3$ and $r_2 \geq 1$ then

$$\int_{K_1} F_{f,g}(g^{-1}a_k kg) d\nu_1(k) < c(f \circ g^{-1}) \int_{K_1} \alpha(a_k kg) d\nu_1(k) < \infty.$$

In the case when $r_1 = 2$ and $r_2 = 1$ or $r_1 = r_2 = 2$ part II of Theorem 2.4 implies that for all $g \in \mathcal{C}_{SL}(r_1, r_2)$ there exists a constant $C$ so that

$$\int_{K_1} F_{f,g}(g^{-1}a_k kg) d\nu_1(k) < c(f \circ g^{-1}) \int_{K_1} \alpha(a_k kg) d\nu_1(k) < Ct.$$

Hence, (5.13), (5.14) and (5.15) imply that as long as $r_1 \geq 3$ and $r_2 \geq 1$ there exists a constant $C_2$ such that

$$|X_g(\mathbb{Z}) \cap V_M(R) \cap A(e^t, 2e^t)| \leq C_2 e^{(d-s-2)t}.$$

Similarly, (5.13), (5.14) and (5.16) imply that if $r_1 = 2$ and $r_2 = 1$ or $r_1 = r_2 = 2$, then

$$|X_g(\mathbb{Z}) \cap V_M(R) \cap A(e^t, 2e^t)| \leq C_2 e^{(d-s-2)t}.$$

Since we can write $T = e^t$ and

$$A(0, T) = \lim_{n \to \infty} \left( A(0, T/2^n) \bigcup_{i=1}^n A(T/2^i, T/2^{i-1}) \right),$$

the Theorem follows by summing a geometric series. \hfill \Box

Theorem 2.3 has the following Corollary which is comparable with Proposition 3.7 from [EMM98] and will be used in the proof of Theorem 2.4.

**Corollary 5.3.** Let $f$ be a continuous function of compact support on $\mathbb{R}^d_x$. Then for every $\epsilon > 0$ and $g \in \mathcal{C}_{SL}(r_1, r_2)$ there exists $t_0 > 0$ so that for $t > t_0$,

$$e^{-(d-s-2)t} \sum_{v \in X_g(\mathbb{Z})} J_{f,g}(M_0^g(v), \|v\| e^{-t}) - C_1 \int_{K_g} F_{f,g}(\hat{a}_k) d\nu_g(k) < \epsilon.$$

**Proof.** Since $J_{f,g}$ has compact support, the number of non zero terms in the sum on the left hand side of (5.17) is bounded by $c e^{(d-s-2)t}$ because of Theorem 2.3. Hence summing the result of Lemma 5.1 over $v \in X_g(\mathbb{Z})$ proves (5.17). \hfill \Box
5.2. Volume estimates. For a compactly supported function $h$ on $\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\}$ we define

$$\Theta (h, T) = \int_{X_g (\mathbb{R})} h (M^0_\delta (v), v/T) \, dm_g (v).$$

For $\mathcal{X} \subseteq \mathbb{R}^d$ we will use the notation $\text{Vol}_{X_g} (\mathcal{X}) = \int_{X_g (\mathbb{R})} 1_{\mathcal{X} \cap X_g (\mathbb{R})} \, dm_g$ to mean the volume of $\mathcal{X}$ with respect to the volume measure on $X_g (\mathbb{R})$.

The following Lemma and its Corollary are analogous to Lemma 3.8 from [EMM98] and the proofs share some similarities, although it is here that the fact we are integrating over $X_g (\mathbb{R})$ rather than the whole of $\mathbb{R}^d$ becomes an important distinction. In Lemma 5.4 we compute $\lim_{T \to \infty} \frac{1}{T^{d-2}} \Theta (h, T)$, here it is crucial that $h$ is not defined on $\mathbb{R}^d \times \{0\}$; if it was, using the fact that $h$ can be bounded by an integrable function, one could directly pass the limit inside the integral and the limit would be 0. The basic strategy of Lemma 5.4 is that we evaluate the integral by an integrable function, one could directly pass the limit inside the integral and the limit would be 0.

**Lemma 5.4.** Suppose that $h$ is a continuous function of compact support in $\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\}$. Then

$$\lim_{T \to \infty} \frac{1}{T^{d-2}} \Theta (h, T) = C_1 \int_{K_g} \int_0^\infty \int_{\mathbb{R}^s} h (z, rke_0) r^{d-s-2} dz \frac{dr}{2r} dv_g (k),$$

where $e_0$ is a unit vector in $\mathbb{R}^d$ and $C_1$ is the constant defined by (5.1).

**Proof.** By Lemma 2.2 of [Sar11], for all $g \in C_{SL} (r_1, r_2)$ there exists a basis of $\mathbb{R}^d$, denoted by $f_1, \ldots, f_d$ such that

$$Q^r_0 (v) = \sum_{i=1}^{s_1} v_i^2 - \sum_{i=1}^s v_i^2 - \sum_{i=s_1+1}^{s_1+r_1} v_i^2 + \sum_{i=s_1+1}^d v_i^2$$

and $M^0_\delta (v) = J (v_1, \ldots, v_s)$, where $v_i = \langle v, f_i \rangle$ for $1 \leq i \leq d$, $J \in GL_s (\mathbb{R})$, $s_1$ is a non-negative integer such that $r_1 + s_1 = p$ and $s_2$ is a non-negative integer such that $r_2 + s_2 = d - p$. Let $L_1 = (v_1, \ldots, v_{s_1}, v_{s_1+1}, \ldots, v_{s_1+r_1})$, $L_2 = (v_{s_1+1}, \ldots, v_s, v_{s_1+r_1+1}, \ldots, v_d)$, $S^{p-1}$ be the unit sphere inside $L_1$ and $S^{d-p-1}$ be the unit sphere inside $L_2$. Let $\alpha \in S^{p-1}$ and $\beta \in S^{d-p-1}$. Using polar coordinates, we can parametrise $v \in X_g (\mathbb{R})$ so that

$$v_i = \begin{cases} \sqrt{a} \alpha_i \cosh t & \text{for } 1 \leq i \leq s_1 \\ \sqrt{a} \beta_i \sinh t & \text{for } s_1 + 1 \leq i \leq s \\ \sqrt{a} \alpha_i \theta_{s+s_1} \cosh t & \text{for } s_1 + 1 \leq i \leq s + r_1 \\ \sqrt{a} \beta_i \theta_{s+s_1} \sinh t & \text{for } s + r_1 + 1 \leq i \leq d. \end{cases}$$

In these coordinates we may write

$$dm_g (v) = \frac{a^{(d-2)/2}}{2} \cos \theta_{s+s_1} t \sinh^2 (\alpha, \beta) \, dtd\xi (\alpha, \beta) = P (e^t) \, dt \, d\xi (\alpha, \beta),$$

where $P (x) = \frac{a^{(d-2)/2}}{2a^{d-1}} x^{d-2} + O (x^{d-3})$ and $\xi$ is the Haar measure on $S^{p-1} \times S^{q-1}$. Making the change of variables, $r = \frac{\sqrt{a} t}{a}$, gives

$$\sqrt{a} \cosh t = Tr + a/4Tr \quad \text{and} \quad \sqrt{a} \sinh t = Tr - a/4Tr. \quad \text{(5.19)}$$

Let $L_1' = (v_{s_1+1}, \ldots, v_{s_1+r_1})$, $L_2' = (v_{s_1+r_1+1}, \ldots, v_d)$, $S^{r_1-1}$ be the unit sphere inside $L_1'$, $S^{r_2-1}$ be the unit sphere inside $L_2'$, $\alpha' \in S^{r_1-1}$ and $\beta' \in S^{r_2-1}$. We may write

$$d\xi (\alpha, \beta) = \delta (\alpha, \beta) d\alpha_1 \ldots d\alpha_s d\beta_1 \ldots d\beta_{s_1} d\xi' (\alpha', \beta')$$

where $\delta (\alpha, \beta)$ is the appropriate density function and $d\xi'$ is the Haar measure on $S^{r_1-1} \times S^{r_2-1}$. This gives

$$dm_g (v) = P \left( \frac{2Tr}{\sqrt{a}} \right) \delta (\alpha, \beta) \frac{dr}{r} d\alpha_1 \ldots d\alpha_s d\beta_1 \ldots d\beta_{s_1} d\xi' (\alpha', \beta'). \quad \text{(5.20)}$$
Let $z \in \mathbb{R}^s$. Make the further change of variables

$$\tag{5.21} (\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_{s-s}) = \frac{1}{Tr} J^{-1} z,$$

this means that

$$\tag{5.22} d\alpha_1 \ldots d\alpha_s \, d\beta_1 \ldots d\beta_{s-s} = \frac{1}{\det(J) (Tr)^s} dz.$$

Moreover, using (5.18), (5.19) and (5.21) gives

$$\tag{5.23} M^0(v) = z + O\left(1/T\right) \quad \text{and} \quad v/T = r (\alpha' + \beta') + O\left(1/T\right).$$

Since $h$ is continuous and compactly supported it may bounded by an integrable function and hence

$$\lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta(h, T) = \lim_{T \to \infty} \frac{1}{T^{d-s-2}} \int_{X_s(\mathbb{R})} h \left(M^0_0(v), v/T\right) dm_g(v)$$

$$= \int_{X_s(\mathbb{R})} \lim_{T \to \infty} \frac{1}{T^{d-s-2}} h \left(M^0_0(v), v/T\right) dm_g(v)$$

$$= \int_{S^{s-1} \times S^{s-1}} \int_0^\infty \int_{\mathbb{R}^s} h(z, r (\alpha' + \beta')) v^{d-s-2} \delta(\alpha', \beta') dz \frac{dr}{2r} d\xi'(\alpha', \beta'),$$

where in the last step follows from (5.21), the definition of $P(x)$, (5.22) and (5.23). Note that from the definition of $\delta$ it is clear that $\delta(\alpha', \beta') = 1$. Finally, let $e_0 = \frac{1}{\sqrt{2}} (f_1 + f_{p+1})$ and $r' = \sqrt{2r}$ to get that

$$\lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta(h, T) = C_1 \int_{K_s} \int_0^\infty \int_{\mathbb{R}^s} h(z, r' \underline{e}_0) v^{d-s-2} dz \frac{dr'}{2r'} d\nu_g(k).$$

\[\square\]

**Corollary 5.5.** For all $g \in C_{SL}(r_1, r_2)$ there exists a constant $C_3 > 0$ such that for all compact regions $R \subset \mathbb{R}^s$ with piecewise smooth boundary

$$\lim_{T \to \infty} \frac{1}{T^{d-s-2}} \text{Vol}_{X_s} \left(V_{M^0_0}(R) \cap A(T/2, T)\right) = C_3 \text{Vol}(R).$$

**Proof.** Let $\1$ denote the characteristic function of $R \times A(1/2, 1)$, then it is clear that

$$\lim_{T \to \infty} \frac{1}{T^{d-s-2}} \text{Vol}_{X_s} \left(V_{M^0_0}(R) \cap A(T/2, T)\right) = \lim_{T \to \infty} \frac{1}{T^{d-s-2}} \int_{X_s(\mathbb{R})} \1 \left(M_0(gv), v/T\right) dm_g(v)$$

$$= \lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta(\1, T).$$

Since $R$ has piecewise smooth boundary there exist regions $R_\delta^- \subset R \times A(1/2, 1) \subset R_\delta^+$ such that $\lim_{\delta \to 0} R_\delta^- = \lim_{\delta \to 0} R_\delta^+ = R$ and for all $\delta > 0$ we can choose continuous compactly supported functions $h_\delta^-$ and $h_\delta^+$ on $\mathbb{R}^s \times \mathbb{R}^d$ such that, where $0 \leq h_\delta^- \leq 1 \leq h_\delta^+ \leq 1$, $h_\delta^-(v) = 1(v)$ if $v \in R_\delta^-$ and $h_\delta^+(v) = 0$ if $v \notin R_\delta^+$. By Lemma 5.2

$$\lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta(h_\delta^-, T) \leq \liminf_{T \to \infty} \frac{1}{T^{d-s-2}} \int_{X_s(\mathbb{R})} \1 \left(M_0(gv), v/T\right) dm_g(v)$$

$$\leq \limsup_{T \to \infty} \frac{1}{T^{d-s-2}} \int_{X_s(\mathbb{R})} \1 \left(M_0(gv), v/T\right) dm_g(v) \leq \lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta(h_\delta^+, T).$$

It is clear that

$$\lim_{\delta \to 0} \lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta(h_\delta^-, T) = \lim_{\delta \to 0} \lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta(h_\delta^+, T) = \lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta(1, T),$$
hence we can apply Lemma 5.4 to get that
\[
\lim_{\tau \to \infty} \frac{1}{\tau^{d-s-2}} \Theta(1, T) = C_1 \int_{K_g} \int_{0}^{\infty} \int_{\mathbb{R}^s} \mathbb{1}_A(z, rk^{-1}e_0) \, r^{d-s-2} \, dz \, dr \, d\nu_g(k)
\]
\[
= C_1 \int_{K_g} \int_{0}^{\infty} \mathbb{1}_A(1/2, 1) \, (rk^{-1}e_0) \, r^{d-s-2} \, dr \, d\nu_g(k) = C_3 \text{Vol}(R).
\]
The last equality holds because
\[
\int_{K_g} \int_{0}^{\infty} \mathbb{1}_A(1/2, 1) \, (rk^{-1}e_0) \, r^{d-s-2} \, dr \, d\nu_g(k) < \infty
\]
as \(\mathbb{1}_A(1/2, 1)\) has compact support and \(K_g\) is compact. \(\Box\)

5.3. **Proof of Theorem 2.1.** By Theorem 4.9 of [PR94] there exist \(v_1, \ldots, v_j \in X_g(\mathbb{Z})\) such that \(X_g(\mathbb{Z}) = \bigcup_{i=1}^{j} \Gamma_g v_i\). Let \(P_i(g) = \{ x \in G_g : xv_i = v_i \}\) and \(\Lambda_i(g) = P_i(g) \cap \Gamma_g\). By Proposition 1.13 of [Hel00] there exist Haar measures \(\varrho_{\Lambda_i}, p_{\Lambda_i}\) on \(G_g/\Lambda_i(g), P_i(g)/\Lambda_i(g)\) and \(\Gamma_g/\Lambda_i(g)\) respectively such that, for \(f \in C_c(G_g/\Lambda_i(g))\), and hence for integrable functions on \(G_g/\Lambda_i(g)\),
\[
\int_{G_g/\Lambda_i(g)} f \varrho_{\Lambda_i} = \int_{X_g(\mathbb{R})} \int_{P_i(g)/\Lambda_i(g)} f(xp) \, dp_{\Lambda_i}(p) \, dm_g(x),
\]
and
\[
\int_{G_g/\Lambda_i(g)} f \varrho_{\Lambda_i} = \int_{G_g/\Gamma_g} \int_{\Gamma_g/\Lambda_i(g)} f(x\gamma) \, d\gamma_{\Lambda_i}(\gamma) \, d\mu_g(x).
\]
Note that \(\Gamma_g/\Lambda_i(g) = \Gamma_g v_i\) is discrete and its Haar measure \(d\gamma_{\Lambda_i}\) is just the counting measure and so
\[
\int_{\Gamma_g/\Lambda_i(g)} f(x\gamma) \, d\gamma_{\Lambda_i}(\gamma) = \sum_{v \in \Gamma_g v_i} f(xv).
\]
Therefore the normalisations already present on \(m_g\) and \(\mu_g\) induce a normalisation on \(p_{\Lambda_i}\). Moreover, it follows from the Borel Harish-Chandra Theorem (cf. [PR94], Theorem 4.13) that the measure of \(p_{\Lambda_i}(P_i(g)/\Lambda_i(g)) < \infty\), for each \(1 \leq i \leq j\). As in [EMM98] and [DM93] where the proofs rely on Siegel’s integral formula, here the proof relies on the following result.

**Lemma 5.6.** For all \(f \in C_c(X_g(\mathbb{R}))\) and \(g \in C_{SL}(r_1, r_2)\) there exists a constant
\[
C(g) = \sum_{i=1}^{j} p_{\Lambda_i}(P_i(g)/\Lambda_i(g)),
\]
such that
\[
C(g) \int_{X_g(\mathbb{R})} f \, dm_g = \int_{G_g/\Gamma_g} F_{f,g} \, d\mu_g.
\]
**Proof.** Note that for \(1 \leq i \leq j, G_g/P_i(g) \approx X_g(\mathbb{R})\). If \(f \in C_c(X_g(\mathbb{R}))\) then \(f\) is \(\Lambda_i(g)\) invariant and therefore can be considered as an integrable function on \(G_g/\Lambda_i(g)\) and so
\[
\int_{X_g(\mathbb{R})} \int_{P_i(g)/\Lambda_i(g)} f(xp) \, dp_{\Lambda_i}(p) \, dm_g(x) = \int_{P_i(g)/\Lambda_i(g)} \int_{X_g(\mathbb{R})} f \, dm_g.
\]
Now it follows from the definition of \(F_{f,g}\) (i.e. \(\Gamma_1), \Gamma_2), \Gamma_2), \Gamma_2, \Gamma_2, \Gamma_2\) and \(\Gamma_2\) that
\[
\int_{G_g/\Gamma_g} F_{f,g} \, d\mu_g = \sum_{i=1}^{j} \int_{G_g/\Gamma_g} \sum_{v \in \Gamma_g} f(xv) \, d\mu_g(x).
\]
which is the desired result. \(\Box\)
The final Lemma of this section is the counterpart of Lemma 3.9 from [EMM98] and again the proof there is mimicked.

**Lemma 5.7.** Let $f$ be a continuous function of compact support on $\mathbb{R}^d_+$. Then for all $g \in C_{SL} (r_1, r_2)$,

$$
\lim_{T \to \infty} \frac{1}{T^{d-s-2}} \int_{X_g (\mathbb{R})} J_{f,g} (M^0_{g} (v), \|v\| / T) \, dm_g (v) = C_1 C (g) \int_{G_g / T_g} F_{f,g} \, d\mu_g ,
$$

where $C_1$ is defined by (5.1) and $C (g)$ is defined in Lemma 5.6.

**Proof.** Let $v_i$ be the components of $v$ when written in the basis $b_1, \ldots, b_d$ from Lemma 5.1. Using the change of variables $(v_1, \ldots, v_d) \to (z_1, \ldots, z_s, r, v_{s+2}, \ldots, a)$ where $Q_0^0 (v_1, \ldots, v_d) = a$ we see that

$$
\int_{\mathbb{R}^d} f (v) \, dv = \int_{-\infty}^\infty \int_0^\infty \int_{\mathbb{R}^s} J_{f,g} (z, r) r^{d-s-2} \, dz \, dr\, da .
$$

Hence it follows from how $m_g$ is defined (i.e. (2.3)) that

$$
(5.29) \quad \int_{X_g (\mathbb{R})} f (v) \, dm_g (v) = \int_{0}^\infty \int_{\mathbb{R}^s} J_{f,g} (z, r) r^{d-s-2} \, dz \, dr .
$$

Lemma 5.4 and (5.29) imply that

$$
\lim_{T \to \infty} \frac{1}{T^{d-s-2}} \int_{X_g (\mathbb{R})} J_{f,g} (M^0_{g} (v), \|v\| / T) \, dm_g (v) = C_1 \int_{K_g} \left( \int_{X_g (\mathbb{R})} f (v) \, dm_g \right) \, d\nu_g (k) .
$$

Now the conclusion follows from Lemma 5.6. $\square$

The purpose of Lemma 5.7 is to relate the integral over $G_g / T_g$ to an integral over $X_g (\mathbb{R})$ in order that the integral over $X_g (\mathbb{R})$ can be approximated by an integral over $K_g$ via Theorem 2.5. Then the integral over $K_g$ can be approximated by the appropriate counting function via Corollary 5.3. We now proceed to put this into action in the proof of our main Theorem which is just a modification of the proof in [EMM98].

**Proof of Theorem 2.1.** By Lemma 5.1 the functional $\Psi$ on $C_c (\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\})$ given by

$$
\Psi (h) = \lim_{T \to \infty} \frac{1}{T^{d-s-2}} \Theta (h, T)
$$

is continuous. For all connected regions $R \subset \mathbb{R}^s$ with smooth boundary, if $\mathbb{1}$ denotes the characteristic function of $R \times A (1/2, 1)$, then for every $\epsilon > 0$ there exist continuous functions $h_+$ and $h_-$ on $\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\}$ such that for all $(r, v) \in \mathbb{R}^s \times \mathbb{R}^d \setminus \{0\}$,

$$
(5.30) \quad h_- (r, v) \leq \mathbb{1} (r, v) \leq h_+ (r, v)
$$

and

$$
(5.31) \quad | \Psi (h_+) - \Psi (h_-) | < \epsilon .
$$

Let $\mathcal{J}$ denote the space of linear combinations of functions on $\mathbb{R}^s \times \mathbb{R}^d$ of the form $J_{f,g} (r, \|v\|)$, where $f$ is continuous function of compact support on $\mathbb{R}^d_+$. Let $\mathcal{H}$ denote the collection of functions in $C_c (\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\})$ such that if $h \in \mathcal{H}$ then $h$ takes an argument of the form $(r, \|v\|)$. By Lemma 5.4, $\mathcal{J}$ is dense in $\mathcal{H}$ and since $h_+$ and $h_-$ belong to $\mathcal{H}$ we may suppose that $h_+$ and $h_-$ maybe written as a finite linear combination of functions from $\mathcal{J}$. The function $F_{f,g}$ defined by (2.1) obeys the bound (2.2) with $\delta = 1$, by (2.2). Moreover, Lemma 3.10 of [EMM98] implies that $F_{f,g} \in L_1 (G_g / T_g)$. Therefore, if $h' \in \{ h_+, h_- \}$, then for all $g \in C_{SL} (r_1, r_2)$, we can apply Theorem 2.5 with the function $F_{f,g}$, followed by Corollary 5.3 and Lemma 5.7 to get that there exists $t_0 > 0$, so that for all $\epsilon > 0$ and $t > t_0$,

$$
(5.32) \quad \left| \frac{C (g)}{e^{(d-s-2)t}} \sum_{v \in X_g (\mathbb{Z})} h' (M^0_{g} (v), \|v\| e^{-t}) - \Psi (h') \right| < \epsilon .
$$
From the definition of $\Psi (h)$ we see that for all $h \in C_c (\mathbb{R}^s \times \mathbb{R}^d \setminus \{0\})$ and $g \in C_{SL} (r_1, r_2)$ there exists $t_0 > 0$, so that for all $\epsilon > 0$ and $t > t_0$,

\begin{equation}
\left| \frac{1}{e^{(d-s-2)t}} \int_{X_g (\mathbb{R})} h (M^g_0 (v), ve^{-t}) \, dm_g (v) - \Psi (h) \right| < \epsilon.
\end{equation}

Clearly (5.30) implies

\begin{equation}
\frac{C (g)}{e^{(d-s-2)t}} \sum_{v \in X_g (\mathbb{Z})} h_- (M^g_0 (v), ve^{-t}) - \Psi (h_+) \leq \frac{C (g)}{e^{(d-s-2)t}} \sum_{v \in X_g (\mathbb{Z})} \{ (M^g_0 (v), ve^{-t}) - \Psi (h_+) 
\end{equation}

\begin{equation}
\leq \frac{C (g)}{e^{(d-s-2)t}} \sum_{v \in X_g (\mathbb{Z})} h_+ (M^g_0 (v), ve^{-t}) - \Psi (h_+).
\end{equation}

Apply (5.31) to the left hand side of (5.34) and then apply and (5.32) with suitable choices of $\epsilon$’s to get that for all $g \in C_{SL} (r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$,

\begin{equation}
\frac{C (g)}{e^{(d-s-2)t}} \sum_{v \in X_g (\mathbb{Z})} \{ (M^g_0 (v), ve^{-t}) - \Psi (h_+) \} \leq \frac{\theta}{2}.
\end{equation}

Similarly using (5.30), (5.31) and (5.33) we see that for all $g \in C_{SL} (r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$,

\begin{equation}
\frac{1}{e^{(d-s-2)t}} \int_{X_g (\mathbb{R})} \{ (M^g_0 (v), ve^{-t}) - \Psi (h_+) \} \, dm_g (v) \leq \frac{\theta}{2}.
\end{equation}

Hence using (5.35) and (5.36) we see that for all $g \in C_{SL} (r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$

\begin{equation}
\frac{C (g)}{\sum_{v \in X_g (\mathbb{Z})} \{ (M^g_0 (v), ve^{-t}) \} - \int_{X_g (\mathbb{R})} \{ (M^g_0 (v), ve^{-t}) \} \, dm_g (v) \leq \theta.
\end{equation}

This means that for all $g \in C_{SL} (r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$,

\begin{equation}
(1 - \theta) \int_{X_g (\mathbb{R})} \{ (M^g_0 (v), ve^{-t}) \} \, dm_g (v)
\end{equation}

\begin{equation}
\leq C (g) \sum_{v \in X_g (\mathbb{Z})} \{ (M^g_0 (v), ve^{-t}) \} \leq (1 + \theta) \int_{X_g (\mathbb{R})} \{ (M^g_0 (v), ve^{-t}) \} \, dm_g (v).
\end{equation}

Hence for all $(Q, M) \in C_{pairs} (r_1, r_2)$ there exists $t_0 > 0$, so that for all $\theta > 0$ and $t > t_0$,

\begin{equation}
(1 - \theta) \text{Vol}_{X_Q} (V_M (R) \cap A (T/2, T)) \leq C (M) \text{Vol}_Q (Z) \leq (1 + \theta) \text{Vol}_{X_Q} (V_M (R) \cap A (T/2, T)).
\end{equation}

The conclusion of the Theorem follows by applying Corollary 5.5 and summing a geometric series. \[\square\]

6. Counterexamples

In small dimensions there are slightly more integer points than expected on the quadratic surfaces defined by forms with signature $(1, 2)$ and $(2, 2)$. This fact was exploited in [EMM93] to show that the expected asymptotic formula for the situation they consider is not valid for these special cases. In a similar manner it is possible to construct examples that show that Theorem 1.1 is not valid in the cases that the signature of $H_g$ is $(1, 2)$ or $(2, 2)$. In this section, for the sake of brevity we restrict our
attention to the case when \( s = 1 \), but we note that similar arguments would hold in the case when \( s > 1 \). To start with make the following definitions

\[
Q_1(x) = -x_1x_2 + x_3^2 + x_4^2, \\
Q_2(x) = x_1x_2 + x_3^2 - x_4^2, \\
Q_3(x) = -x_1x_2 + x_3^2 + x_1^2 - \alpha x_5^2, \\
L_\alpha(x) = x_1 - \alpha x_2.
\]

We can now prove.

**Lemma 6.1.** Let \( \epsilon > 0 \), suppose \([a, b] = [1/2 - \epsilon, 1]\) or \([-1, -1/2 + \epsilon]\). Let \( a > 0 \), then for every \( T_0 > 0 \), the set of \( \beta \in \mathbb{R} \) for which there exists a \( T > T_0 \) such that

\[
|X_{Q_2}^a(\mathbb{Z}) \cap V_{L_\alpha}([a, b]) \cap A(0, T)| > T (\log T)^{1-\epsilon}
\]

is dense. Similarly if \( a = 0 \), then for every \( T_0 > 0 \), the set of \( \beta \in \mathbb{R} \) for which there exists \( T > T_0 \) such that

\[
|X_{Q_3}^a(\mathbb{Z}) \cap V_{L_\alpha}([a, b]) \cap A(0, T)| > T^2 (\log T)^{1-\epsilon}
\]

is dense.

**Proof.** Let \( S_i(\alpha, T, a) = \{x \in \mathbb{Z}^{d_i} : L_\alpha(x) = 0, Q_i(x) = a, \|x\| \leq T\} \) where \( d_i = 4 \) if \( i = 1 \) or \( 2 \) and \( d_i = 5 \) if \( i = 3 \). Lemma 3.14 of [EMM98] implies that

(6.1) \(|S_i(\alpha, T, a)| \sim T \log T \) for \( i = 1, 2 \) and \( \sqrt{\alpha} \in \mathbb{Q} \) and \( a > 0 \),

(6.2) \(|S_3(\alpha, T, 0)| \sim T^2 \log T \) for \( \sqrt{\alpha} \in \mathbb{Q} \).

Note that if \( i = 1, 2 \) and \( x \in S_i(\alpha, T, a) \setminus S_i(\alpha, T/2, a) \), then

(6.3) \[
\frac{T^2}{4} - (\alpha^2 + 1) x_2^2 \leq x_3^2 + x_4^2 \leq T^2 - (\alpha^2 + 1) x_2^2
\]

and

(6.4) \[
x_3^2 + x_4^2 = \alpha x_2^2 + a.
\]

Similarly if \( x \in S_3(\alpha, T, 0) \setminus S_3(\alpha, T/2, 0) \),

(6.5) \[
\frac{T^2}{4} - (\alpha^2 + 1) x_2^2 \leq x_3^2 + x_4^2 + x_5^2 \leq T^2 - (\alpha^2 + 1) x_2^2
\]

and

(6.6) \[
x_3^2 + x_4^2 + x_5^2 = \alpha (x_2^2 + x_5^2).
\]

Combining (6.3) and (6.4) gives

(6.7) \[
\frac{T^2 - 4a}{4(\alpha^2 + \alpha + 1)} \leq x_2^2 \leq \frac{T^2 - a}{\alpha^2 + \alpha + 1},
\]

Respectively, combining (6.5) and (6.6) gives

(6.8) \[
\frac{T^2 - (\alpha + 1) x_5^2}{4(\alpha^2 + \alpha + 1)} \leq x_2^2 \leq \frac{T^2 - (\alpha + 1) x_5^2}{\alpha^2 + \alpha + 1},
\]

which upon noting that \(-T \leq x_5 \leq T\) offers

(6.9) \[
\frac{T^2 - (\alpha + 1) T}{4(\alpha^2 + \alpha + 1)} \leq x_2^2 \leq \frac{T^2 + (\alpha + 1) T}{\alpha^2 + \alpha + 1}.
\]

Take

(6.10) \[
\beta_\pm = \alpha \pm \sqrt{\frac{\alpha^2 + \alpha + 1}{T^2}}.
\]
It is clear that \( L_{\beta \pm} (x) = L_\alpha (x) \pm \sqrt{a^2 + a^1} x_2 \) and hence if \( i = 1, 2 \) and \( x \in S_i (\alpha, T, a) \setminus S_i (\alpha, T/2, a) \), then (6.11) implies
\[
\frac{1}{4} - \frac{a}{T^2} \leq L_{\beta +} (x) \leq \frac{1}{4} - \frac{a}{T^2} \quad \text{and} \quad -\sqrt{1 - \frac{a}{T^2}} \leq L_{\beta -} (x) \leq -\sqrt{1 - \frac{a}{T^2}}.
\]

Similarly if \( x \in S_3 (\alpha, T, 0) \setminus S_3 (\alpha, T/2, 0) \), then (6.12) implies
\[
\frac{1}{4} - \frac{(a + 1)}{T} \leq L_{\beta +} (x) \leq \frac{1}{4} - \frac{(a + 1)}{T} \quad \text{and} \quad -\sqrt{1 - \frac{(a + 1)}{T}} \leq L_{\beta -} (x) \leq -\sqrt{1 - \frac{(a + 1)}{T}}.
\]

This means for all \( \epsilon > 0 \) there exists \( T_+ > 0 \) such that if \( T > T_+ \) then \( S_i (\alpha, T, a) \subset X_{Q_i}^0 (Z) \cap V_{L_\beta +} ([1/2 - \epsilon, 1]) \cap A(0, T) \) respectively there also exists \( T_- > 0 \) such that if \( T > T_- \) then \( S_i (\alpha, T, a) \subset X_{Q_i}^0 (Z) \cap V_{L_\beta -} ([1/2, 1 - \epsilon]) \cap A(0, T) \). By (6.1) and (6.2) for \( i = 1, 2 \) and large enough \( T \), \(|S_i (\alpha, T, a)| > T (log T)^{1-\epsilon}\) and \(|S_i (\alpha, T, a)| > CT^2 (log T)^{1-\epsilon}\). The set of \( \beta \) satisfying (6.10) for rational \( \alpha \) and large \( T \) is clearly dense and this proves the Lemma.

**Theorem 6.2.** Let \( j = 1, 2 \). For every \( \epsilon > 0 \) and every interval \([a, b]\) there exists a rational quadratic form \( Q \) and an irrational linear form \( L \) such that \( Stab_{SO(Q)} (L) \cong SO (j, 2) \) such that for an infinite sequence \( T_k \to \infty \),
\[
\left| X_{Q_i}^0 (Z) \cap V_L ([a, b]) \cap A(0, T_k) \right| > T_k^j (log T_k)^{1-\epsilon},
\]
where \( a_1 > 0 \) and \( a_2 = 0 \).

**Proof.** Since the interval \([a, b]\) must intersect either the positive or negative reals there is no loss of generality in assuming, after passing to a subset and rescaling that \([a, b] = [1/4, 5/4] \) or \([-5/4, -1/4]\). For a given \( S > 0 \) and \( i = 1, 2 \) let \( U_S \) be the set of \( \gamma \in \mathbb{R} \) for which there exist \( \beta \in \mathbb{R} \) and \( T > S \) with
\[
\left| X_{Q_i}^0 (Z) \cap V_{L_\beta} ([1/2, 1]) \cap A(0, T) \right| > CT \log T,
\]
and
\[
|\beta - \gamma| < T^{-2}.
\]
Then \( U_S \) is open and dense by Lemma 6.1. By the Baire category Theorem (cf. Rud87, Theorem 5.6) \( \bigcap_{k=1}^{\infty} U_{2k+1} \) is dense in \( \mathbb{R} \) and is in fact of second category and hence uncountable. Let \( \gamma \in \bigcap_{k=1}^{\infty} U_{2k+1} \setminus Q \), then there exist infinite sequences \( \beta_k \) and \( T_k \) such that (6.13) and (6.14) hold with \( \beta \) replaced by \( \beta_k \) and \( T \) by \( T_k \). Note that (6.14) implies that for \( \|x\| < T_k \),
\[
|L_{\beta_k} (x) - L_\gamma (x)| < \frac{1}{T_k} < \frac{1}{4},
\]
so that
\[
X_{Q_i}^0 (Z) \cap V_{L_{\beta_k}} ([1/2, 1]) \cap A(0, T_k) \subseteq X_{Q_i}^0 (Z) \cap V_{L_\gamma} ([1/4, 5/4]) \cap A(0, T_k)
\]
and hence \( \left| X_{Q_i}^0 (Z) \cap V_{L_\gamma} ([1/4, 5/4]) \cap A(0, T_k) \right| > CT_k \log T_k \) by (6.13). If \( i = 3 \) then we can carry out the same process but we replace \( U_S \) by the set \( W_S, \gamma \in \mathbb{R} \) for which there exist \( \beta \in \mathbb{R} \) and \( T > S \) with
\[
\left| X_{Q_3}^0 (Z) \cap V_{L_\beta} ([1/2, 1]) \cap A(0, T) \right| > CT^2 \log T,
\]
and
\[
|\beta - \gamma| < T^{-2}.
\]

\( \square \)
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