Low-energy modified gravity signatures on the large-scale structures

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Abstract

A large number of dark energy and modified gravity models lead to the same expansion history of the Universe, hence, making it difficult to distinguish them from observations. To make the calculations transparent, we consider $f(R)$ gravity with a pressureless matter without making any assumption about the form of $f(R)$. Using the late-time expansion history realizations constructed by Shafieloo et al. [1], we explicitly show that one of the Bardeen potential $\Psi$ is suppressed compared to $\Phi$ for any $f(R)$ model that lead to late-time acceleration. For an arbitrary $f(R)$ model, we explicitly show that $|\Psi+\Phi|$ and its time-derivative are about an order larger than the $\Lambda$CDM model at lower redshifts. We discuss the implications of the results for the cosmological observations.

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I. INTRODUCTION

Our current understanding of the cosmos is based on an enormous extrapolation of our limited knowledge of gravity since General Relativity (GR) has not been independently tested on galactic and cosmological scales [2–4]. On the largest scales, the biggest surprise from observational cosmology has been that the current Universe is accelerating [5, 6]. The observations of Type Ia supernovae suggests that the current Universe is undergoing a phase of accelerated expansion [7] which agrees with the observations of cosmic microwave background radiation [8–10].

Providing a fundamental understanding of the late-time accelerated expansion of the Universe is one of the most challenging problems in cosmology. GR alone can not explain the late-time acceleration of the Universe with ordinary matter or radiation. The presence of an exotic matter source energy referred to as dark energy can explain the late-time accelerated expansion [11–15]. The most straightforward candidate for the dark energy (DE) is the cosmological constant $\Lambda$ [16–19]. However, the estimated value of $\Lambda$ from observations shows that it is many orders smaller than the vacuum energy density predicted by particle physics [16, 17]. While the dark energy content of the Universe does not change with the evolution of the Universe in the case of Cosmological constant, in other dark energy models like Quintessence, K-essence, Phantom models, Chameleon scalar fields dark energy changes with time [12].

An alternative to dark energy is the modified gravity (MG) model, where the late-time acceleration is due to the large-scale modifications to GR. Several modified gravity models, like $f(R)$, Braneworld and Galileon models, have been proposed as the possible explanation for late time accelerated expansion of the universe [20–23]. Among the modified gravity models, $f(R)$ models (where $f$ is an arbitrary function of the Ricci scalar $R$) are popular owing to the simplicity of the dynamical equations and that they contain deviations from GR. More importantly, $f(R)$ models do not suffer from Ostr̕ogradsky instability [24].

Naturally, many phenomenological $f(R)$ models have been proposed in the literature that is consistent with local gravity tests and have stable late-time de Sitter point [25–29]. One of the drawbacks of these models is that they suffer from some fine-tuning problem as like the cosmological constant. In other words, one needs to tune the threshold value of the Ricci scalar $R_0$ to obtain the observed late-time acceleration.
As mentioned above, many different $f(R)$ models with fine-tuning can account for the late-time acceleration. Similarly, many different dark energy models, within GR, can also account for the late-time acceleration. This leads to the question: Are there signatures that distinguish dark energy and modified gravity models?

Such parameters have been constructed in the literature [30]. Since the degeneracy is apparent in the background equations, they showed that the same might not be the case for the first order perturbations. In particular, Song and Koyama provided consistency test, based on the first order scalar metric perturbations. They proposed that the modified gravity models can be mapped to modifications in Newton’s constant and obtained two parameters that can distinguish MG and dark energy models.

In this work, we focus on generic $f(R)$ model which leads to late-time expansion history that is consistent with observations. In particular, we use 6400 late-time expansion history realizations constructed by Shafieloo et al [1] from the latest Pantheon supernovae distance modulus compilation [7]. For each of these 6400 realizations, we obtain the evolution of $f(R)$ as a function of red-shift ($z$). We use the constructed $f(R)$ in the first order perturbation equations to obtain the observationally relevant quantities like $\Phi + \Psi$, $\Phi' + \Psi'$ and $\Psi/\Phi$. We explicitly show that these quantities evolve differently for $f(R)$ and dark-energy models. More specifically, we show that one of the Bardeen potential $\Psi$ is suppressed compared to $\Phi$ for any $f(R)$ model that lead to late-time acceleration. We show that $|\Psi + \Phi|$ and its time-derivative for $f(R)$ model are about an order larger than $\Lambda$CDM model at lower redshifts. To our knowledge, such an analysis has not been done earlier for an arbitrary $f(R)$. We then discuss the implication of our results relating to future observations.

In section (II), the two scenarios — GR with cosmological constant $\Lambda$ and $f(R)$ model where $f(R)$ is an arbitrary function of $R$ — are introduced. In section (III), we obtain the evolution of various background quantities using the model-independent data of Hubble parameter constructed by Shafieloo et al. in [1, 31]. In section (IV), we obtain the density perturbations and the scalar metrics perturbations in both the scenarios using the background evolution. In section (V), we discuss the difference in the growth of the first order quantities in these two scenarios and obtain the relevant variables for the cosmological observations. In section (VI), we conclude by briefly discussing the results.

In this work we use the natural units where $c = \hbar = 1$, $\kappa^2 = 8\pi G$, and the metric signature $(-, +, +, +)$. Greek alphabets denote the 4-dimensional space-time coordinates,
and Latin alphabets denote the 3-dimensional spatial coordinates. Overbarred quantities (like $\bar{p}(t)$, $\bar{f}(R)$, $\bar{F}(R)$) are evaluated for the FRW background. $H_0$ is the Hubble constant and is taken to be $70 \text{ Km/s Mpc}^{-1} = 1.467 \times 10^{-33} \text{eV}$ which falls between the value obtained from the PLANCK data [9, 10] and Supernovae Ia data [7].

II. FRAMEWORK AND THE TWO SCENARIOS

As mentioned in the introduction, we consider two scenarios — Dark Energy and $f(R)$ gravity — that explain the late-time acceleration of the Universe. In this section, we briefly discuss the two scenarios and use the expansion history realizations constructed by Shafieloo et al [1, 31] to obtain the evolution of $f(R)$ as a function of $z$.

1. **Scenario I:** In this scenario, we consider General Relativity with dark energy, where dark energy is represented by cosmological constant $\Lambda$. The field equations are given by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu},$$  \hspace{1cm} (1)

where $T_{\mu\nu}$ is the stress-tensor of the matter fields. We consider only the pressureless matter while keeping the successes of standard cosmology at early times.

2. **Scenario II:** In this scenario, we assume that the late-time acceleration is due to the large-scale modifications to GR which is given by $f(R)$ where $f$ is a continuous, and arbitrary function of $R$. The action and the corresponding field equations are given by:

$$S_{II} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + \int d^4x \sqrt{-g} \mathcal{L}_M$$  \hspace{1cm} (2)

$$FR_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \Box F = \kappa^2 T_{\mu\nu},$$  \hspace{1cm} (3)

where $F = \frac{\partial f}{\partial R}$. In the case of $f(R)$ gravity, unlike General Relativity, the trace of the field equation (3) is dynamical [24]:

$$R F(R) + 3 \Box F(R) - 2 f(R) = \kappa^2 T$$  \hspace{1cm} (4)

Thus, in $f(R)$ gravity, the scalar curvature $R$, which can be expressed in terms of the metric and its derivatives, plays a non-trivial role in the determination of the metric
itself. As a result, $f(R)$ gravity has 11 dynamical variables — 10 metric variables $(g_{\mu\nu})$ and $F(R)$. Note that the above equation points that $F(R)$ is a dynamical quantity as $F(R)$ is acted on by the differential operators.

Here again, we consider only the pressureless matter, while keeping the success of standard general relativity at early times. In other words, we assume that until around the redshift of 1.2 the Universe can be described by GR with cosmological constant with a dominant contribution from the pressureless matter.

To distinguish the above two scenarios in a model-independent manner, we do not assume any form of $f(R)$, i.e., $f(R)$ is an arbitrary function. Instead, we assume that both scenarios lead to the same background evolution of the Universe and have the same evolution of the Hubble parameter ($H(z)$). In other words, the input parameter for the two scenarios is $H(z)$ as a function of red-shift $z$. We use the model-independent data constructed by Shafieloo et al. in [1, 31]. In particular, we use 6400 late-time expansion history realizations constructed by Shafieloo et al [1] from the latest Pantheon supernovae distance modulus compilation [7].

It is important to note that the Pantheon dataset consists of 1048 supernovae in the redshift range [0.01, 2.26]. Pantheon dataset has 630, 832, and 1025 supernovae below $z = 0.3$, $z = 0.5$ and $z = 1$, respectively [7]. Due to the sparsity of the data beyond the redshift of 1.2, there is some uncertainty in the values of $H(z)$ beyond the redshift of 1.2 [1]. Hence, in our analysis, for the 6400 late-time expansion history realizations, we consider the evolution of the Hubble parameter in the range $0.01 < z < 1.2$.

In the next section, using the above realizations, we obtain $F$ as a function of $z$. In Sec. (IV), we use the evolution of $F(z)$ to obtain the first order scalar perturbations in both the scenarios. We use the evolution of $H(z)$ and $F(z)$ to obtain the first order scalar perturbations in both the scenarios.

III. BACKGROUND EVOLUTION IN THE TWO SCENARIOS

In this work, we consider spatially flat FRW line-element:

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,$$

where $a(t)$ is the scale factor, $\delta_{ij}$ is the Kronecker delta. As mentioned in Sec. (II), for both the scenarios, the matter content of the Universe is pressureless dust.
A. Scenario I

For the above background, the field equations (1) lead to:

\[ H^2(z) = \frac{\Lambda}{3} + \frac{\kappa^2}{3} \bar{\rho}(z) \]  

(6)

where \( \bar{\rho}(z) \) is the background matter density. We use the value of cosmological constant to be \( \Lambda = (4.24 \pm 0.11) \times 10^{-66} \text{eV}^2 \) from the PLANCK-2018 [10] and the value of \( H(z) \) from the late-time expansion history by Shafieloo et al [1]. Substituting these two quantities in the above Friedmann equation, we obtain the evolution of the background matter energy density \( \bar{\rho}(z) \).

It is important to note that for this scenario, exact analytical solution for the scale factor exists [11]:

\[ H(t) = \frac{2\alpha}{3} \coth(\alpha t); \quad \alpha = \sqrt{\frac{3\Lambda}{4}}; \quad \bar{\rho}(t) = \frac{\rho_0}{a^3(t)} = \rho_0(1 + z)^3; \quad \rho_0 = \frac{\Lambda}{2\kappa^2} \]  

(7)

![FIG. 1. Evolution of \( \kappa^2 \bar{\rho}(z) \) vs \( z \) for all the 6400 realizations constructed by Shafieloo et al [1] and exact analytical solution for the model.](image)

Figure (1) contains the evolution of \( \bar{\rho}(z) \) for all the 6400 realizations. It also contains the energy density evolution for the exact analytical solution (7) for \( H_0 = 70 \text{ km/s/MPc} \). It is
important to note that the curve corresponding to the exact solution depends on the value of $H_0$, and the ratio of energy density between matter and the cosmological constant. As mentioned above, for the rest of the analysis, we have used the above $H_0$ value.

**B. Scenario II**

For the FRW background, the field equations (3) lead to

$$H^2(z) \frac{d^2 F(z)}{dz^2} + \left( H(z) \frac{dH}{dz} + 2 \frac{H^2(z)}{(1+z)} \right) \frac{dF(z)}{dz} - 2 \frac{H(z)}{(1+z)} \frac{dH}{dz} F(z) + \frac{\kappa^2}{(1+z)^2} \bar{\rho}(z) = 0 \quad (8)$$

As mentioned above, due to the sparsity of the PATHEON data beyond the redshift of 1.2 [7], we assume that beyond the redshift of 1.2 the Universe can be described by GR with cosmological constant with dominant contribution from the pressureless matter $\bar{\rho}(z)$. In other words, the modifications to gravity begin to dominate the evolution of the Universe around $z \sim 1.2$. In order to obtain the evolution of $F(z)$ as a function of $z$ in the above equation, we use the following initial conditions:

$$F(z = 1.2) = 1, \quad \text{and} \quad \left. \frac{dF}{dz} \right|_{z=1.2} = 10^{-5} \quad (9)$$

We would like to mention the following points regarding the initial conditions: First, the analysis is independent of the choice of the initial value of $z$. Our choice of the initial value of $z$ is linked to the data set. Second, the condition $F(1.2) = 1$ implies that at $z = 1.2$, the gravity is described by General Relativity. The condition $dF(z = 1.2)/dz = 10^{-5}$ provides the initial choice of the rate of change of $F$. As we will show below the results do not depend on the choice of this initial condition.

For the above initial conditions, 6400 late-time expansion history realizations constructed by Shafieloo et al [1] gives the evolution of $H(z)$ and, from scenario I, we obtain the corresponding $\bar{\rho}(z)$. Using these the mid-point discretization scheme in Eq. (8), we obtain $F(z)$ as a function of $z$. Figure (2) contains the evolution of $F(z)$ for all the 6400 realizations.

This is the first key result regarding which we would like to stress the following points: First, it is remarkable that the range of $F(z)$ is narrow for all the 6400 realizations. This implies that the data pick a narrow range of evolution of $F(z)$. Second, the above analysis does not assume any form of $f(R)$. It is possible that many different $f(R)$ models, with fine-tuned parameters, may produce the same evolution. In Appendix (A), we have used
the popular $f(R)$ models [25–29] that lead to the late-time accelerated expansion and how they compare with the generic $F(z)$. Third, the evolution of $F(z)$ does not depend on the value of $dF/dz$ at $z = 1.2$. Appendix (A) contains the plots of evolution of $F(z)$ for different values of $dF/dz$ at $z = 1.2$. These plots clearly show $F(z)$ is independent of the initial condition on $dF/dz$.

IV. FIRST ORDER SCALAR PERTURBATIONS IN THE TWO SCENARIOS

We aim to distinguish between GR and modified gravity models using observations. To obtain the physical parameters that can be used to separate the two scenarios, we need to obtain perturbed quantities about the FRW background.

In the largest scales, it is a good approximation to assume that the perturbed part is small compared to the background. More specifically, the perturbed energy density is smaller than the (average) background density. The first order scalar perturbations about the FRW line-element in the Newtonian gauge is given by [32]:

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t) (1 - 2\Psi)\delta_{ij}dx^i dx^j,$$

where $\Phi \equiv \Phi(t, x^i)$ and $\Psi \equiv \Psi(t, x^i)$ are the scalar perturbations. As mentioned earlier, for both the scenarios, the matter content of the universe is represented by pressureless dust.
with energy momentum tensor:

\[ T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu\nu} \]  \hspace{1cm} (11)

where \( u_{\mu} \) is the four velocity, \( \rho(t, x^i) = \bar{\rho}(t) (1 + \delta(t, x^i)) \) is the energy density including the first order density perturbations, \( \delta(t, x^i) = \delta \rho(t, x^i)/\bar{\rho}(t) \) is the fractional amplitude of density perturbations, and \( p \) is the pressure of the fluid which is taken to be zero.

**A. Scenario I**

The first order perturbed Einstein’s equation in this Scenario leads to the following equations in the Fourier space [32]:

\[
\ddot{\delta}_{GR} + 2H\dot{\delta}_{GR} - \frac{\kappa^2}{2} \bar{\rho} \delta_{GR} = 0 \tag{12}
\]

\[
\frac{k^2}{a^2} \Phi_{GR} + \frac{\kappa^2}{2} \bar{\rho} \delta_{GR} = 0 \tag{13}
\]

\[
\Phi_{GR} - \Psi_{GR} = 0. \tag{14}
\]

Even though the result for this scenario is trivial, the procedure we follow is the same for both the scenarios: First, using the background density \( \bar{\rho}(z) \) in Eq. (12), we obtain the fractional amplitude of density perturbations \( \delta_{GR}(z) \). [We use the expressions in Appendix (A1), to convert the differential equations from \( t \) to \( z \).] Next, substituting \( \delta_{GR}(z) \) in Eq. (13), we obtain \( \Phi_{GR} \). In the case of GR, with single fluid, the two Bardeen potentials (\( \Phi_{GR}, \Psi_{GR} \)) are identical.

**B. Scenario II**

As mentioned earlier, the scalar curvature \( R \) satisfies the differential equation (4) and plays a non-trivial role in the determination of the metric itself. Hence, \( F(R) \) can be treated as a dynamical variable. For the perturbed FRW line-element (10), \( i \neq j \) component of the modified Einstein’s equations (8) leads to:

\[
\Phi_{MG} - \Psi_{MG} = -\frac{\delta F}{F} \quad \text{where} \quad \delta F = \left( \frac{\partial F}{\partial R} \right) \delta R \tag{15}
\]

The above expression shows that the two Bardeen potentials (\( \Phi_{MG}, \Psi_{MG} \)) are not identical and their difference depends on \( F(R) \). [We only list the key equations in this section and relegate the details to Appendix (B).]
The fractional amplitude of density perturbations $\delta(z)$ in $f(R)$ model is given by [33, 34]:

$$
\ddot{\delta}_{MG} + 2H\dot{\delta}_{MG} - \frac{\kappa_{\text{eff}}^2}{2} \dot{\rho} \delta_{MG} = 0 \tag{16}
$$

where

$$
\kappa_{\text{eff}}^2 = \frac{\kappa^2}{F} \left( 1 + 4 \frac{k^2}{a^2} \frac{\partial \ln F}{\partial R} \right) / \left( 1 + 3 \frac{k^2}{a^2} \frac{\partial \ln F}{\partial R} \right) \tag{17}
$$

Note that in the limit of $f(R) \to R$, $\kappa_{\text{eff}} \to \kappa$ and $\delta_{MG} \to \delta_{GR}$. Like in the earlier scenario, using the background density $\overline{\rho}(z)$, we obtain the fractional amplitude of density perturbations $\delta_{MG}(z)$. However, the Bardeen potentials satisfy the following coupled differential equations:

$$
\dot{\Psi}_{MG} + \left( H - \frac{F\dot{H}}{F} + \frac{F}{3F} \frac{k^2}{a^2} \right) \Phi_{MG} + \left( \frac{F\dot{H}}{F} + \frac{F}{3F} \frac{k^2}{a^2} \right) \Psi_{MG} + \frac{\kappa_{\text{eff}}^2 \overline{\rho}}{3F} \delta_{MG} = 0 \tag{18}
$$

$$
\dot{\Phi}_{MG} + \left( H - \frac{\dot{F}}{F} - \frac{F\dot{H}}{F} - \frac{F}{3F} \frac{k^2}{a^2} \right) \Psi_{MG} + \left( \frac{2\dot{F}}{F} + \frac{F\dot{H}}{F} - \frac{F}{3F} \frac{k^2}{a^2} \right) \Phi_{MG} - \frac{\kappa_{\text{eff}}^2 \overline{\rho}}{3F} \delta_{MG} = 0 \tag{19}
$$

Using $\delta_{MG}$ from (16), we numerically solve the above differential equations for 6400 realizations and is plotted in Figure (3).

FIG. 3. Evolution of $\Psi_{MG}/\Phi_{MG}$ and $\Psi_{GR}/\Phi_{GR}$ as a function of redshift $z$ for $k = H_0$.

This is the second key result regarding which we would like to stress the following points: First, for any arbitrary $f(R)$ that leads to late-time acceleration $\Phi_{MG} \neq \Psi_{MG}$ and the
evolution of $\Psi_{MG}/\Phi_{MG}$ deviates from the scenario I at the lower red shifts. Second, for almost all realizations, there is a turn around in the scalar perturbations around $z \sim 0.5$. In other words, for all realizations, around $z \sim 0.5$, $\Phi_{MG} < \Psi_{MG}$ goes to $\Phi_{MG} > \Psi_{MG}$. From Fig. (1), we can see that around $z \sim 0.5$, the background energy density of the pressureless matter $\rho(z)$ drops by half. To understand this further, we have plotted $\delta_{GR}(z)$ and $\delta_{MG}(z)$ as a function of $z$ in Fig. (4). It can be seen from the Figure that around $z \sim 0.5$, the deviations in the growth of structures for the Scenario 1 and 2 become prominent. In other words, the density perturbations in modified gravity grow faster compared to GR from around $z \sim 0.5$. Here again, we want to stress that the analysis is for any arbitrary $f(R)$. Third, for all the realizations, $\Phi_{MG}$ is very large compared to $\Psi_{MG}$ near $z \sim 0$. In other words, in the current epoch, only $\Phi_{MG}$ contributes to the density perturbations (or contribution of $\Psi_{MG}$ is negligible compared to $\Phi_{MG}$). We will discuss the implications of this in the next section. Lastly, while we have plotted the various parameters for $k = H_0$, the results are similar for other perturbation scales. In Appendix (C), we have plotted the scalar perturbations averaged over different all the realizations for both the scenarios. As can be seen, the profile for different perturbation modes are similar.

FIG. 4. Evolution of $\delta_{MG}$ and $\delta_{GR}$ as a function of $z$ for $k = H_0$, and all realizations.
V. CONFRONTING WITH OBSERVATIONS

In the previous section, we showed that even if both the scenarios lead to the same background evolution, the scalar perturbations in the both scenarios evolve differently. More importantly, for the $f(R)$ model, we showed that $\Psi_{\text{MG}}/\Phi_{\text{MG}}$ is negligible at the current epoch. Fig (5) contains the plot of $|\Psi_{\text{MG}}|$ and $|\Phi_{\text{MG}}|$ as a function of $z$. To further investigate, in Figure (6), we have plotted $|\Psi_{\text{MG}} + \Phi_{\text{MG}}|/|\Phi_{\text{MG}}|$ as a function of $z$.

For these two plots it is clear that, at the current epoch $|\Psi_{\text{MG}} + \Phi_{\text{MG}}| \simeq |\Phi_{\text{MG}}|$. However, for GR, $|\Psi_{\text{GR}} + \Phi_{\text{GR}}|/|\Phi_{\text{GR}}| = 2$ at all times.

The quantity $\Phi + \Psi$ determines the geodesic of a photon, which affects the weak gravitational lensing [32]. Figure (7) contains the evolution of $\Phi + \Psi$ for both the scenarios. This is the third key result regarding which we would like to stress the following: In the case of GR, the quantity $\Psi + \Phi$ is almost a constant. However, in the case of $f(R)$, $\Psi_{\text{MG}} + \Phi_{\text{MG}}$ is increased by an order at the current epoch, although, $\Psi_{\text{MG}}$ is negligible compared to $\Phi_{\text{MG}}$.

Since the Bardeen potential changes by an order in the case of $f(R)$, this change should potentially change the temperature fluctuations of the CMB photons. In other words, the rate of change of the $(\Phi + \Psi)$ w. r. t. $\eta$ contribute to the evolution of scalar perturbations in the temperature fluctuations in CMB in large scales — Integrated Sachs Wolfe effect [35].
Here $\eta$ is the conformal time which is related to the cosmic time via $\eta = \int dt/a$. Figure (V) shows the $\Phi' + \Psi'$ as a function of $z$. Here again we see that $|\Psi'_{MG} + \Phi'_{MG}|$ is about an order larger than $|\Psi'_{GR} + \Phi'_{GR}|$ at lower redshifts.

From the above plots, we see that it is possible to distinguish two scenarios using weak
FIG. 8. Evolution of $\Psi'_{MG} + \Phi'_{MG}$ and $\Psi'_{GR} + \Phi'_{GR}$ as a function of redshift $z$ for $k = H_0$.

gravitational lensing and Integrated Sachs Wolfe effect in a model-independent manner. In both the cases, $|\Psi_{MG} + \Phi_{MG}|$ and $|\Psi'_{MG} + \Phi'_{MG}|$ are about an order larger than $|\Psi_{GR} + \Phi_{GR}|$ and $|\Psi'_{GR} + \Phi'_{GR}|$ respectively towards the lower redshifts.

VI. CONCLUSIONS

In this work, we have investigated in detailed the two scenarios which can explain the late time acceleration of the universe namely General Relativity with cosmological constant and $f(R)$ gravity where $f$ is an arbitrary function of $R$. We have shown that in these two scenarios for which the background evolution is identical, the evolution of scalar perturbations is different. More specifically, we have shown that one of the Bardeen potential $\Psi_{MG}$ is suppressed compared to $\Phi_{MG}$. We explicitly showed that the difference in the evolution of the scalar perturbations could be used to distinguish these two scenarios using the cosmological observations such as weak lensing and Integrated Sachs Wolfe effect in a model-independent manner. To our knowledge, this is the first time such an analysis has been done for an arbitrary $f(R)$ model.

To study the evolution of various background and perturbed quantities, we have used the model independent data of the Hubble parameter constructed by Shafieloo et.al [1].
have analyzed in the redshift range $z = 0$ to 1.2. Due to the scarcity of PANTHEON data at redshift greater than 1.2, we have not included the high redshift data in our analysis. We assumed that the effect of modifications to gravity begins to contribute from $z = 1.2$ and have kept the success of standard general relativity at early times. The current analysis can be extended to higher redshifts once more data is available on the expansion history of the universe at higher redshifts.

In Appendix (A), we have compared generic $f(R)$ model with the popular $f(R)$ models in the literature. We have shown that the evolution of $F(z)$ constructed using the different realizations of the expansion history of the Universe can be described by various $f(R)$ models which have been proposed to explain the late time acceleration of the universe.

To keep the calculations transparent, we have assumed that pressureless matter contributes to the stress-tensor. Extending the analysis for multiple fields is possible. This is currently under investigation.

Our analysis shows that one of the Bardeen potentials is suppressed in the case of $f(R)$ theories. It is interesting to see whether this feature is common for all modified gravity theories. This is currently under investigation.

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Appendix A: Background Evolution in $f(R)$: Details

In this Appendix, we provide more details of the results from Section (III) for Scenario II.
1. Numerical Analysis in redshift space

The time derivatives in the evolution equations can be rewritten in terms of derivative with respect to redshift $z$, using the following relations

\[
\frac{d}{dt} = -H(1 + z) \frac{d}{dz}
\]

\[
\frac{d^2}{dt^2} = H(1 + z)^2 \frac{dH}{dz} \frac{d}{dz} + H^2(1 + z) \frac{d}{dz} + H^2(1 + z)^2 \frac{d^2}{dz^2}
\]

To numerically solve the equations, derivatives are rewritten using the central difference method.

\[
\frac{df(z)}{dz} = \frac{f_{i+1} - f_{i-1}}{2dz}
\]

\[
\frac{d^2f(z)}{dz^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{dz^2}
\]

2. Comparison of general $f(R)$ model with popular models

Many $f(R)$ models have been proposed that lead to the late time acceleration of the Universe [25–29]. In this appendix, we check how the evolution of the function $F$ that we obtained using the different realizations of the expansion history of the universe compared with three such $f(R)$ models. Table I gives the best fit for the constructed realizations of $F(z)$ in the range $0 < z < 1.2$ for these models.

| $F(R)$ | Best fit |
|--------|----------|
| $F(R) = 1 - 2\lambda n \frac{R}{R_0} \left[ 1 + \left( \frac{R}{R_0} \right)^2 \right]^{-(n+1)}$ | $F(R) = 1.032 - 2\lambda n \frac{R}{R_0} \left[ 1 + \left( \frac{R}{R_0} \right)^2 \right]^{-(n+1)}$ |
| Starobinsky [26] | $n = 0.8686$, $\lambda = 421.2$, $R_0 = H_0^2$ |
| $F(R) = 1 - \lambda n \left( \frac{R}{R_0} \right)^{n-1}$ | $F(R) = 1.032 - \lambda n \left( \frac{R}{R_0} \right)^{n-1}$ |
| Hu & Sawicki [25], Amendola et al. [27] | $n = -1.717$, $\lambda = -400.71$, $R_0 = H_0^2$ |

The best fit corresponding to these models are plotted in the Figure below along with the $F(R)$ for different realizations as a function of $R$. 

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FIG. 9. Best fit for the evolution of $F$ based on $f(R)$ models from the literature. Left panel: Starobinsky [26], Right panel: Hu, Sawicki [25], Amendola et al. [27].

Here we see that the evolution of $F(z)$ constructed using the different realizations of the expansion history of the universe can be described by various $f(R)$ models which have been proposed to explain the late time acceleration of the universe.

3. Evolution of $F(z)$ for different initial conditions

As mentioned in Sec. (III), we obtain the evolution of $F(z)$ for different values of the initial condition. It is important to note that the physical assumption that at redshift $z = 1.2$ the gravity is described by GR leads to the condition that $F(1.2) = 1$. Here, we show that the results obtained in Sec. (III) do not depend on this value. In the plots below we have plotted the evolution of $F$ for around 330 realizations for four other initial conditions:

$$\left. \frac{dF}{dz} \right|_{z=1.2} = 10^{-4}, 10^{-6}, 10^{-7}, 10^{-8} \quad (A5)$$

Thus, the choice of initial conditions does not have any bearing on the evolution of $F(z)$. 

FIG. 10. Evolution of $F(z)$ as a function of $z$ for the initial conditions $\frac{dF}{dz}\big|_{z=1.2} = 10^{-4}$ (Left panel), $10^{-6}$ (Right panel).

FIG. 11. Evolution of $F(z)$ as a function of $z$ for the initial conditions $\frac{dF}{dz}\big|_{z=1.2} = 10^{-7}$ (Left panel), $10^{-8}$ (Right panel).

Appendix B: Simplified first order evolution equations for $\Phi$ and $\Psi$ in Scenario II

In this appendix, we provide the complete set of first order scalar perturbation equations for an arbitrary $f(R)$. For the perturbed line-element (10), the modified Einstein’s equations (3) and the trace equation (4) lead to:

\[-\frac{\nabla^2 \Phi}{a^2} + 3H(\dot{H} + \dot{\Phi} + \ddot{\Phi}) + \frac{1}{2F} \left( 3H^2 + 3\dot{H} + \frac{\nabla^2 \Phi}{a^2} \right) \delta F - 3H\delta F + 3H\dot{F} + 3\dot{\Phi}(H\dot{\Phi} + \dot{\Psi}) + \kappa^2 \delta \rho \right] = 0, \tag{B1}
\]
\[3(\dot{H} + \dot{H} + \ddot{H}) + 6H(\dot{H} + \dot{\Phi} + \ddot{\Phi}) + 3\dot{H} + \frac{\nabla^2 \Phi}{a^2} - \frac{1}{2F} \left[ 3\delta F + 3H\delta F - 6H^2 \right. \]
\[\delta F - \frac{\nabla^2 \delta F}{a^2} - 3\dot{H} + 3\dot{F}(H\dot{\Phi} + \dot{\Psi}) - (3H\dot{\Phi} + 6\dot{\Phi})\dot{\Phi} + \kappa^2 \delta \rho \right] = 0 \tag{B2}
\]
\[
\begin{align*}
\delta F + 3H\delta F + \left( \frac{k^2}{a^2} - 4H^2 - 2\dot{H} \right) \delta F - 2\dot{F}\delta \Psi - \left( 8FH + 3\dot{F} \right) \dot{\Psi} \\
- \left( 2FH + \dot{F} \right) \dot{\Phi} - \left( 6H\dot{F} + 2\dot{F} + 4F\dot{H} + 8FH^2 - \frac{2Fk^2}{3a^2} \right) \dot{\Phi} - \frac{4Fk^2}{3a^2} \Psi - \frac{\kappa^2 \delta \rho}{3} = 0 \quad (B3)
\end{align*}
\]

\[
H\dot{\Phi} + \dot{\Psi} - \frac{1}{2F} (\delta F - H\delta F - \dot{F}\Phi) = 0 \quad (B4)
\]

\[
\Phi - \Psi + \frac{\delta F}{F} = 0 \quad (B5)
\]

\[
\delta F - F'\delta R = 0 \quad (B6)
\]

Substituting for \( \delta F \) using Eq. (B5), we get

\[
\begin{align*}
& \left( 3H + \frac{3\dot{F}}{F} \right) \dot{\Psi} + \left( 3H^2 + 3\dot{H} - 3H\frac{\dot{F}}{F} + \frac{k^2}{a^2} \right) \Psi \\
& + \left( 3H^2 - 3\dot{H} + 9H\frac{\dot{F}}{F} + \frac{k^2}{a^2} \right) \Phi + 3H\dot{\Phi} + \frac{\kappa^2 \delta \rho}{F} = 0, \quad (B7)
\end{align*}
\]

\[
\dot{\Phi} + \dot{\Psi} + \left( H - \frac{\dot{F}}{F} \right) \Psi + \left( H + 2\frac{\dot{F}}{F} \right) \dot{\Phi} = 0, \quad (B8)
\]

\[
\begin{align*}
\ddot{\Psi} + \ddot{\Phi} + 3 \left( H + \frac{\dot{F}}{F} \right) \dot{\Phi} + \left( 3H - \frac{\dot{F}}{F} \right) \dot{\Psi} + \left( 2H^2 - \frac{\dot{F}}{F} - \frac{H\dot{F}}{F} - \frac{k^2}{3a^2} \right) \Psi \\
+ \left( 2H^2 + \frac{3\dot{F}}{F} + \frac{3H\dot{F}}{F} + 4\dot{H} - \frac{k^2}{3a^2} \right) \Phi - \frac{\kappa^2 \delta \rho}{3F} = 0, \quad (B9)
\end{align*}
\]

\[
\begin{align*}
\ddot{\Phi} + \ddot{\Psi} + \left( 5H + \frac{3\dot{F}}{F} \right) \dot{\Phi} + \left( 4H^2 - \frac{3\dot{F}}{F} - \frac{3H\dot{F}}{F} + 2\dot{H} + \frac{k^2}{3a^2} \right) \Phi + \frac{\kappa^2 \delta \rho}{3F} = 0 \quad (B10)
\end{align*}
\]

Substituting \( \Phi \) in Eq. (B8) using Eq. (B7), we get

\[
\ddot{\Psi} + \left( H - \frac{F\dot{H}}{F} + \frac{F}{3F} \frac{k^2}{a^2} \right) \dot{\Phi} + \left( \frac{F\dot{F}}{F} + \frac{F}{3F} \frac{k^2}{a^2} \right) \Psi + \frac{\kappa^2 \delta \rho}{3F} = 0. \quad (B11)
\]

Similarly, substituting \( \Phi \) and \( \Psi \) in Eq. (B10) using Eq. (B9) and Eq. (B11), respectively, we get:

\[
\begin{align*}
\ddot{\Phi} + \left( H - \frac{\dot{F}}{F} - \frac{F\dot{H}}{F} - \frac{F}{3F} \frac{k^2}{a^2} \right) \dot{\Psi} + \left( \frac{\dot{F}}{F} + \frac{2\dot{F}}{F} - \frac{F}{3F} \frac{k^2}{a^2} \right) \dot{\Phi} - \frac{\kappa^2 \delta \rho}{3F} = 0. \quad (B12)
\end{align*}
\]

**Appendix C: Evolution of scalar perturbation for different values of \( k \)**

In Secs. (IV, V), we plotted the scalar perturbations for \( k = H_0 \). In this appendix, for completeness, we plot the evolution of \( \Psi_{GR} + \Phi_{GR} \) and \( \Psi_{MG} + \Phi_{MG} \) averaged over different
datasets for different $k$ values.

FIG. 12. Evolution of $\Psi(z) + \Phi(z)$ for both the scenarios for $k = H_0$ (Left panel) and $k = 2H_0$ (Right panel).

FIG. 13. Evolution of $\Psi(z) + \Phi(z)$ for both the scenarios for $k = 3H_0$ (Left panel) and $k = 4H_0$ (Right panel).

FIG. 14. Evolution of $\Psi(z) + \Phi(z)$ for both the scenarios for $k = 5H_0$ (Left panel) and $k = 6H_0$ (Right panel).
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