Using multiobjective optimization to map the entropy region of four random variables

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Abstract

Mapping the structure of the entropy region in higher dimensions is an important open problem, as even partial knowledge about this region has far reaching consequences in other areas in mathematics like information theory, cryptography, probability theory and combinatorics. Presently the only known method of exploring the entropy region is the one of Zhang and Yeung from 1998. Using some non-trivial properties of the entropy function, their method is transformed to solving high dimensional linear multiobjective optimization problems.

Benson’s outer approximation algorithm is a fundamental tool for solving such optimization problems. An improved version of Benson’s algorithm is presented which requires solving one scalar linear program in each iteration rather than two or three as in previous versions. During the algorithm design special care was taken for numerical stability. The implemented algorithm was used to verify previous statements about the entropy region, as well as to explore it further. Experimental results demonstrate the viability of the improved Benson’s algorithm for determining the extremal set of medium-sized numerically ill-posed optimization problems. With larger problem sizes, two limitations of Benson’s algorithm were observed: the inefficiency of the scalar LP solver, and the unexpectedly large number of intermediate vertices.

Keywords: multiobjective programming, effective solutions, entropy region, Benson’s algorithm.

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1 Introduction

Exploring the 15 dimensional entropy region formed by the entropies of the non-empty subsets of four random variables is an intriguing research problem. The entropy function maps the nonempty subsets of a finite set of random variables into the Shannon entropies of the marginal distributions. The range of the entropy function is the entropy region; it is a subset of a high-dimensional Euclidean space indexed by the non-empty subsets of the random variables. Inequalities that hold for the points of the region are called information theoretic. The entropy region is bounded by hyperplanes corresponding to the well-known Shannon information inequalities.

Presently the only available method which goes beyond the standard Shannon inequalities is the one of Zhang and Yeung from 1998. The method starts with a description of “copy steps,” which determine a (usually very) high dimensional linearly constrained region. The projection of this polytope onto the 15 dimensional space of the original entropies contains the entropy region, and, quite frequently, its facets yield new (linear) entropy inequalities. Using some non-trivial properties of the entropy region, this problem is transformed into the problem of finding all extremal vertices of a 10-dimensional projection of a high dimensional polytope – which problem lies in the realm of linear multiobjective optimization.
Benson’s outer approximation algorithm is a fundamental tool for solving multiobjective linear optimization problems. Compared to the original version and its refinements, we introduce a modification which leads to a significant improvement. The improvement is based on the observation that the scalar LP instance, which is used to decide whether an objective point is on the boundary of the projection or not, can also provide a separating facet when the point is outside the facet. In all earlier published versions of the algorithm, a separate LP instance was used to find such a facet. This improvement is of independent interest as it applies to all versions of Benson’s algorithm.

The implemented algorithm was used successfully to check earlier results on the entropy region. It also generated hundreds of new entropy inequalities, and was essential in formulating a general conjecture about the limits of the Zhang–Yeung method. The experiments indicated the shortcomings of the implemented variant of the algorithm, and raised an interesting theoretical question about the structure of high dimensional polytopes.

1.1 The entropy region

The Shannon-inequalities bound the \( 2^N - 1 \)-dimensional entropy region; this bounding polytope is known as the Shannon bound. If \( N = 2 \), then the entropy region and the Shannon bound coincide; if \( N = 3 \), then the Shannon bound is the closure of the entropy region, and there are missing points on the boundary. (In fact, the boundary looks like a fractal, its exact structure is unknown.) In the case of \( N \geq 4 \), the Shannon bound strictly exceeds the closure of the entropy region [1].

To map the structure of the entropy region for \( N \geq 4 \) is an intriguing open problem. Even partial knowledge about this region has important consequences in several mathematical and engineering disciplines. N. Pippenger argued in [2] that linear information inequalities encode the fundamental laws of Information Theory, which determine the limits of information transmission and data compression. In communication networks the capacity region of any multi-source network coding can be expressed in terms of the entropy region, see the thorough review on network coding in [3]. Information inequalities have a direct impact on the converse setting with multiple receivers [1]. In cryptography, such inequalities are used to establish bounds on the complexity of secret sharing schemes [4]. In probability theory, the implication problem of conditional independence among subvectors of a random vector can be rephrased as the investigation of the lower dimensional faces of the entropy region [5, 6]. Guessing number of games on directed graphs are related to network coding, where new bounds on the entropy region provide sharper bounds [7]. Information theoretic inequalities surface in additive combinatorics [8], and are intimately related to Kolmogorov complexity [9], determinant inequalities and group-theoretic inequalities [10].

The very first information theoretic inequality which showed that the entropy region is strictly contained in the Shannon bound was found by Zhang and Yeung in 1998 [11]. Since then, many other inequalities have been found based on their idea [12, 13]. Until now this is the only technique at our disposal: all other proposed techniques were shown to be equivalent to the Zhang-Yeung method [14].

1.2 Mapping the entropy region is an optimization problem

Section 2 outlines why the technique of Zhang and Yeung sketched above is equivalent to solving a multiobjective linear optimization problem, with the main focus on describing the general form of these problems. Using some non-trivial properties of the entropy region, in the case of \( N = 4 \) random variables the objective space of the optimization can be reduced to be 10 dimensional. The exact details of how to generate the optimization problem are given in the Appendix. Solving these linear optimization problems are especially challenging as

\[ a) \text{ the size of the problem grows exponentially, and becomes prohibitively large very soon; } \]

\[ 1 \text{ A. H. Hamel, A. Löhne and B. Rudloff in [19] have observed the same improvement independently. } \]
b) while the linear constraints form a sparse matrix, there are many non-trivial linear combi-
nations among them; consequently

c) the whole system is numerically ill-posed.

When \( N \geq 5 \), the corresponding optimization problems have less structure, are two order of
magnitude larger, and even in the simplest case no existing optimization technique seems to be
able to handle them.

1.3 Benson’s algorithm revisited

Benson’s outer approximation algorithm [15] is a fundamental tool for solving linear multiobjective
optimization problems. It works in the low dimensional objective space, which in the \( N = 4 \) case
has 10 dimensions, rather in the much larger (several hundred dimensional) problem space. It
was a natural choice to use Benson’s algorithm for solving the optimization problem described
in Section 2 and in Section 3 we describe an improved version of Benson’s algorithm. Material
of Section 2 is of independent interest as the improvement applies to other variants of Benson’s
algorithm as well. The original version [15] and other published variants [16, 17, 18] use two scalar
LP instances in each iteration step, while this version requires solving a single LP instance of
the same size in each iteration. In typical applications of Benson’s algorithm the computation
time outside the LP solver is negligible, thus the total running time of the algorithm is reduced
considerably. The same improvement can be applied to the “dual” optimization problem as defined
in [18], thus both the primal and the dual problem require solving the same number of scalar LP
instances. As a consequence of this optimization, the discrepancy between the running times of
the primal and dual variants, as observed in [17], vanishes. The same improvement of Benson’s
algorithm has been observed independently, and put into a wider context by Hamel, Löhne, and
Rudloff in [19].

After sketching the general idea behind the improved version, we prove its correctness in Section
3.2. The termination of the algorithm is immediate from the facts that the extremal polytope has
finitely many facets and finitely many vertices, and that each iteration generates either a new
vertex or a new facet of the extremal polytope.

The modified algorithm is detailed in Section 3.4. It uses the double description method [20]
for vertex enumeration. Section 3.5 discusses the modifications of the simplex-based LP solver we
employed which improves efficiency and numerical stability.

1.4 The results

Section 4 describes the experimental results. The algorithm was run successfully for all 133 copy
strings described in [12]. For each of those strings, all extremal solutions of the corresponding
multiobjective linear optimization problem was generated. The 10 dimensional extremal solutions
corresponded to the “strongest” entropy inequalities this copy string could yield.

Copy strings leading to larger optimization problems were also considered. As the result, the
total number of known computer-generated information-theoretic entropy inequalities grew from
214 in [12] to more than 470.

We also ran the algorithm successfully on a couple of significantly larger problems, where the
symmetry of the problem allowed to reduce the dimension of the objective space (the dimension
of the extremal polytope) from 10 to three. Results achieved here were essential in formulating a
general conjecture about the limits of the Zhang–Yeung method [21].

Section 5 concludes the paper where we also discuss the shortcomings of the described variant of
Benson’s algorithm. The double description method, which is used to enumerate the intermediate
vertices and facets, seems to be the bottleneck when the extremal polytope has several thousand
vertices. It is an interesting open question how many extra vertices the intermediate polytopes
might have compared to the final number of vertices and facets. We settle this question (at least
up to constant multipliers) when the dimension of the objective space is three, but in higher
dimensions, there is a huge discrepancy between the lower and upper bounds.
2 How mapping the entropy region leads to multiobjective optimization

The entropy region is a subset of the non-negative orthant of the $2^N - 1$-dimensional Euclidean space, and its closure (in the usual Euclidean topology) is denoted by $\bar{\Gamma}^*_N$. It is known that $\bar{\Gamma}^*_N$ is a convex, closed, pointed cone, and the entropy region misses only boundary points, see [22].

The region $\bar{\Gamma}^*_N$ is bounded by linear facets corresponding to the so-called Shannon entropy inequalities. If $N \leq 3$, $\bar{\Gamma}^*_N$ is exactly the polytope determined by these Shannon inequalities, while for $N \geq 4$, it is a proper subset. Hyperplanes that cut into the Shannon polytope and contain all entropic points on one side are called non-Shannon (entropy) inequalities. The first such inequality was found by Zhang and Yeung in 1998 [11]. Since then, many other inequalities have been found. For a thorough discussion and new results, see [12, 13, 14]. The Zhang–Yeung method can be outlined as follows [12, 11, 14]. The process starts with four random variables $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$. They are split into two groups. An independent copy of the first group is created over the second group, and these new auxiliary random variables are added to the pool of existing (random) variables. Due to independence, several linear equations hold for their entropies. This copy step is then repeated as described by the copy string. The process yields an extension of the original set of random variables and a list of linear dependencies among their entropies. Then all Shannon inequalities for the entropies of this extended set are collected, and all linear dependencies are added. As the Shannon inequalities are linear, this results in a large set of (homogeneous) linear inequalities among the entropies of the subsets of the original and the auxiliary random variables. Finally, it is checked whether this set of homogeneous linear inequalities has any (new) consequences on the fifteen entropies of the original four variables, $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$.

Taking the dual view, consequences of a set of homogeneous linear inequalities are their non-negative linear combinations. From all the combinations, we are only interested in those where all coefficients are zero, except for the 15 non-empty subsets of the original four random variables. The collection of these latter combinations is a convex, closed 15-dimensional pointed polyhedral cone yielding all required consequences. The result we would like to get is the extremal rays of this cone, that is, the best possible entropy inequalities, which are not superseded by any linear combination of others. This view leads naturally to multiobjective optimization as follows. In our case the problem space consists of the (non-negative) combining coefficients (denoted by $x$) of the Shannon inequalities, that is, the non-negative orthant of $\mathbb{R}^n$ where $n$ is the number of inequalities. The constraints are the (linear) conditions on $x \in \mathbb{R}^n$ expressing the fact that in the combined inequality, all entropies containing any auxiliary random variable should vanish. That is,

\[ A'x = 0, \quad \text{where } x \geq 0, \ x \in \mathbb{R}^n \]

for an $m$ by $n$ matrix $A'$ compiled from the Shannon inequalities and linear dependencies. The objective space is $\mathbb{R}^{15}$, which corresponds to the 15 entropies of the original four random variables. The objective of the optimization is the vector of the coefficients of these entropies in the resulting combined inequality, thus it can be written as $P'x$, where $P'$ is a $15 \times n$ matrix. The region whose extremal rays provide the solution for our problem is the pointed polyhedral cone

\[ C' = \{ P'x : A'x = 0, \ x \geq 0 \}, \]

Using information-theoretic considerations as discussed in [9, 23, 24, 25], it is more convenient to look at $C'$ in a different coordinate system. Let us consider the cone

\[ \mathcal{C} = UC' = \{ UP'x : A'x = 0, \ x \geq 0 \}, \]

where $U$ is a $15 \times 15$ unimodular matrix (for details, please see the Appendix). There are many advantages of considering $\mathcal{C}$ instead of $C'$. First, we can set one $U$-coordinate – the so-called Ingleton coordinate – to 1, cutting the pointed cone $\mathcal{C}$ to a polytope $\mathcal{P}$. Rather than searching for the extreme rays in $\mathcal{C}$, now we can search for the vertices of $\mathcal{P}$. 

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Second, the other 14 $U$-coordinates are all non-negative entropy expressions. Consequently, the coordinates of $y \in \mathcal{P}$ in these directions are necessarily non-negative, that is, $\mathcal{P}$ lies in the non-negative orthant of $\mathbb{R}^{14}$. Furthermore, if $y \in \mathcal{P}$ and $y' \geq y$ coordinatewise, then $y' \in \mathcal{P}$. Thus the vertices of $\mathcal{P}$ are exactly the extremal points of $\mathcal{P}$, where $y \in \mathcal{P}$ is extremal if no $y' \leq y$, different from $y$, is in $\mathcal{P}$.

Third, the polytope $\mathcal{P}$ is known to be the direct product of a 10-dimensional polytope $\mathcal{Q}$ and the non-negative orthant of $\mathbb{R}^4$, see \cite{23}. Therefore, the extremal vertices of $\mathcal{P}$ are the extremal vertices of $\mathcal{Q}$ with four zero coordinates added. One can get the points of $\mathcal{Q}$ directly by merging these five additional constraints on $x$ (that the Ingleton coordinate in $UP'x$ should be 1, and the four additional coordinates in $UP'x$ should be zero) to the original constraints $A'x = 0$.

The problem of finding the minimal set of entropy inequalities which generate (via non-negative linear combinations) all other entropy inequalities resulting from a given copy string has thus been transformed into the following multiobjective linear optimization problem.

**Optimization Problem** Given the $m \times n$ matrix $A$, the $p \times n$ matrix $P$ with $p = 10$ generated from the copy string, find the extremal vertices of the $p$-dimensional polytope

$$\mathcal{Q} = \{Px : Ax = b, \, x \geq 0\},$$

knowing that $\mathcal{Q}$ is in the non-negative orthant of $\mathbb{R}^p$, and the column vector $b$ contains 1 in the Ingleton row, and zero elsewhere.

Indeed, given the linear constraints $Ax = b$, $x \geq 0$ in the $n$-dimensional problem space $\mathbb{R}^n$, one has to simultaneously minimize the $p$ linear objectives given by the matrix $Px$. The solution of the optimization problem is the complete list of the extremal points of $\mathcal{Q}$, that is, those $p$-tuples from $\mathcal{Q}$ where no coordinate can be decreased without increasing some other coordinate and still remain in $\mathcal{Q}$. This list gives the coefficients of the minimal set of entropy inequalities generating all consequences of the copy string.

### 3 Improved variant of Benson’s outer algorithm

Benson’s algorithm ~\cite{15} solves the Optimization Problem defined in Section 2 working in the $p$-dimensional objective space, the range of $Px$, rather than in its much larger domain $\mathbb{R}^n$. The outline of this algorithm is as follows. It starts from a polytope $S_0$ containing $\mathcal{Q}$. In each iteration the algorithm maintains a convex bounding polytope $S_n$ by listing all of its vertices and all of its facets. If all vertices of $S_n$ are on the boundary of $\mathcal{Q}$, then we are done, as the extremal vertices of $\mathcal{Q}$ are among the vertices of $S_n$. Otherwise, we select a vertex of $S_n$ that is not in $\mathcal{Q}$, connect it to an internal point of $\mathcal{Q}$, and find the intersection of this line with the boundary of $\mathcal{Q}$. Let the intersection point be $\hat{y}_n$. Then, we find a facet of $\mathcal{Q}$ which is adjacent to $\hat{y}_n$. We add this facet to $S_n$ to get $S_{n+1}$, determine the new vertices of $S_{n+1}$, and iterate the method. The algorithm always terminates after finitely many iterations.

Several improvements have been suggested to the original algorithm: see, among others, \cite{16,17,18}. The first paper discusses how $S_0$ is chosen, which may have a heavy impact on the performance of the algorithm. Other papers suggest improvements related to the steps where we need to find the new vertices of $S_{n+1}$ and decide whether a new vertex is on the boundary of $\mathcal{Q}$.

The improvement of Benson’s algorithm this paper describes is achieved by merging the steps of finding a boundary point and finding the adjacent facet of $\mathcal{Q}$. We note that the same improvement was found independently in \cite{19}.

#### 3.1 Extremal points

The transpose of matrix $M$ is denoted by $M^T$. Vectors are usually denoted by small letters, and are considered single column matrices. For two vectors $x$ and $y$ of the same dimension, $xy$ denotes their inner product, which is the same as the matrix product $x^Ty$. 
Recall that $A$ is an $m \times n$ matrix mapping $\mathbb{R}^n$, the problem space, to $\mathbb{R}^m$, and $P$ is a $p \times n$ matrix mapping the problem space into $\mathbb{R}^p$, the objective space. (In our case $n$ is the number of Shannon inequalities generated from the copy string, and $p$ is 10, see Section 2.) Let $\mathcal{A}$ be the polytope

$$\mathcal{A} = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \}$$

for the fixed vector $b$. Then

$$\mathcal{Q} = \{ Px : x \in \mathcal{A} \}.$$

For better clarity, $y$ will denote points of the objective space $\mathbb{R}^p$, while points in the problem space will be denoted by $x$. The point $y \in \mathcal{Q}$ is extremal if no other point in $\mathcal{Q}$ “supersedes” it, that is, from $y' \leq y$, $y' \in \mathcal{Q}$ it follows that $y' = y$. The point $y$ is weakly extremal if there is no $y' < y$ in $\mathcal{Q}$ (i.e. when all coordinates of $y'$ are smaller than the corresponding coordinate in $y$).

We want to generate the set of extremal vertices of $\mathcal{Q}$, but instead of $\mathcal{Q}$, we consider another polytope, $\mathcal{Q}^+$, which has the same set of extremal points, and is easier to handle [15]. This polytope is defined as

$$\mathcal{Q}^+ = \{ y \in \mathbb{R}^p : y \geq y' \text{ for some } y' \in \mathcal{Q} \}. \quad (2)$$

In fact, $\mathcal{Q}^+$ is the Minkowski sum of $\mathcal{Q}$ and the non-negative orthant of $\mathbb{R}^p$, thus it is a convex closed polytope. The following facts can be found, e.g., in [17] Proposition 4.3.

**Proposition 1.** a) The extremal points of $\mathcal{Q}^+$ and $\mathcal{Q}$ are the same.

b) The weakly extremal points of $\mathcal{Q}^+$ are exactly its boundary points. \qed

### 3.2 Finding a boundary point and a supporting hyperplane at the same time

The crucial step in Benson’s algorithm is to find a boundary point of $\mathcal{Q}^+$, and then to find the supporting hyperplane at that point. In the original version, it is achieved by solving two appropriately chosen scalar LP problems [15] [17] [18]. In our version, we need to solve only one scalar LP instance to do both jobs.

To describe the procedure, let $q \in \mathbb{R}^p$ be an internal point of $\mathcal{Q}^+$, and let $d \in \mathbb{R}^p$ be some direction in the $p$-dimensional space. Consider the ray starting at $q$ with the direction $-d$, that is, points of the form $q - \lambda d \in \mathbb{R}^p$ where $\lambda \geq 0$ is a non-negative real number. If not the whole ray is in $\mathcal{Q}^+$, then there is a $\lambda > 0$ so that $q - \lambda d \in \mathcal{Q}^+$ if and only if $\lambda \leq \lambda$. By the next proposition, this threshold $\lambda$ can be found by solving a scalar LP problem.

**Proposition 2.** Suppose $q \in \mathcal{Q}^+$ is an internal point, and not the whole ray $\{ q - \lambda d : \lambda \geq 0 \}$ is in $\mathcal{Q}^+$. Let $\lambda$ be the solution of the LP problem

$$\max_{\lambda,x} \{ \lambda : q - \lambda d \geq Px, Ax = b, x \geq 0 \}. \quad \text{P}(q,d)$$

Then $\hat{y} = q - \hat{\lambda}d$ is a boundary point of $\mathcal{Q}^+$. Let moreover $(\hat{u}, \hat{v})$ be a place where the dual problem

$$\min_{u,v} \{ bu + qv : A^T u + P^T v \geq 0, dv = 1, v \geq 0 \}. \quad \text{D}(q,d)$$

takes the same extremal value. Then $\{ y \in \mathbb{R}^p : y = \hat{y} \}$ is a supporting hyperplane to $\mathcal{Q}^+$ at $\hat{y}$.

**Proof.** The first part of the proposition is clear: $q - \lambda d$ is in $\mathcal{Q}^+$ if it is $\geq Px$ for some $x \geq 0$ with $Ax = b$. The LP problem $P(q,d)$ simply searches for the largest $\lambda$ with this property. From this, it also follows that $\hat{y}$ is a boundary point of $\mathcal{Q}^+$: there are points arbitrarily close to $\hat{y}$ which are not in $\mathcal{Q}^+$.

The dual problem $D(q,d)$ has the same optimal value as the primal one, thus

$$\min_{u,v} \{ bu + qv : A^T u + P^T v \geq 0, dv = 1, v \geq 0 \} = \hat{\lambda}. \quad (3)$$

Fixing the $\hat{v}$ part of the solution, the minimum is still $\hat{\lambda}$ as $u$ runs over its domain $\mathbb{R}^m$, and therefore

$$\max_u \{ -bu : A^T(-u) \leq P^T \hat{v} \} = q\hat{v} - \hat{\lambda}.$$
Consequently its dual also has the same optimal value:
\[
\min_{x} \{ x(P^T \hat{v}) : Ax = b, \ x \geq 0 \} = q\hat{v} - \hat{\lambda}.
\]
Now \( \hat{v} \geq 0 \) by (3), and an arbitrary point \( y \) of \( Q^+ \) can be written as \( z + Px \) where \( z \geq 0, \ z \in \mathbb{R}^p \) and \( x \geq 0, \ Ax = b, \ x \in \mathbb{R}^n \), and so
\[
(z + Px)^T \hat{v} = z\hat{v} + x(P^T \hat{v}) \geq 0 + q\hat{v} - \hat{\lambda}.
\]
On the other hand, \( d\hat{v} = 1 \) by (3) again, thus
\[
q\hat{v} - \hat{\lambda} = q\hat{v} - \hat{\lambda}(d\hat{v}) = (q - \hat{\lambda}d)\hat{v} = \hat{y}\hat{v}.
\]
This means that for any point \( y \) of \( Q^+ \), we have \( y\hat{v} \geq \hat{y}\hat{v} \), furthermore \( \hat{y} \) is in \( Q^+ \). Thus \( \{ y \in \mathbb{R}^p : \ y\hat{v} = \hat{y}\hat{v} \} \) is a supporting hyperplane to \( Q^+ \), as was claimed.

We remark that if \( \{ y \in \mathbb{R}^p : \ yw = \hat{y}w \} \) is a supporting hyperplane to \( Q^+ \) at \( \hat{y} \in Q^+ \), then, necessarily, \( w \geq 0 \). Indeed, if \( w_i \) were negative, then letting \( e_i \) be the unit vector with 1 at the \( i \)-th coordinate, \( \hat{y} + e_i \in Q^+ \), and \( (\hat{y} + e_i)w = \hat{y}w + w_i < \hat{y}w \), which is a contradiction.

### 3.3 Initial bounding polytope

Following the ideas of Burton and Ozlen in [16], we extend the objective space by positive ideal elements. As they showed, this extension does not restrict the applicability of the algorithm, but simplifies the intermediate polytopes.

First of all, we know that points of \( Q \) have non-negative coordinates. Thus \( Q^+ \) is included in the non-negative orthant of \( \mathbb{R}^p \), that is, it is part of the “ideal” \( p \)-dimensional simplex with the the origin and the \( p \) “positive endpoints” of the coordinate axes of \( \mathbb{R}^p \) as its vertices.

During the algorithm we will be dealing with two kinds of “objective” points: ordinary (finite) points \( y = \langle y_1, \ldots, y_p \rangle \) lying in the non-negative orthant (that is, \( y_i \geq 0 \)), and ideal points at the positive endpoints of the rays \( m = \langle m_1, \ldots, m_p \rangle \) in the non-negative orthant (that is, \( m_i \geq 0 \) for all \( i \)). Using homogeneous coordinates as suggested in [16] we can handle both types uniformly: ordinary points have coordinates \( (y, 1) = \langle y_1, \ldots, y_p, 1 \rangle \), while ideal points can be conveniently written as \( (m, 0) \), indicating that rays are invariant under multiplication by a (positive) scalar. A hyperplane is a \( p + 1 \)-tuple \( \langle w, d \rangle = \langle w_1, \ldots, w_p, d \rangle \). A point \( (z, j) \) is on a hyperplane if \( (z, j)^T \cdot \langle w, d \rangle = zw + jd = 0 \), and it is on its non-negative side if \( (z, j)^T \cdot \langle w, d \rangle \geq 0 \). All ideal points are on the non-negative side of the hyperplane \( \langle w, d \rangle \) if and only if \( w_i \geq 0 \) for each \( i \). The line connecting the points \( (y, 1) \) and \( (m, 0) \) intersects the hyperplane \( \langle w, d \rangle \) in the (ordinary) point \( (y + \lambda m, 1) \), where the scalar \( \lambda \) is determined by the condition
\[
(y + \lambda m, 1)^T \cdot \langle w, d \rangle = y^T w + \lambda m^T w + d = 0,
\]
where \( m^T w \neq 0 \), as otherwise \( (m, 0) \) is on the hyperplane.

Benson’s algorithm starts with an internal point \( q \in Q^+ \), and an initial bounding polytope \( S_0 \supseteq Q^+ \). In the course of the algorithm, \( q \) will be connected to the vertices of the polytope \( S_n \), and these lines will ideally intersect the boundary of \( Q^+ \) in the relative interior of some \( p - 1 \)-dimensional facet. If this happens, then the supporting hyperplane to \( Q^+ \) at the intersection point is uniquely determined, and is, in fact, a facet of \( Q^+ \). This preferable event occurs if the internal point \( q \) is in general position, that is, not in any hyperplane determined by any \( p \) points among the union of the vertices of \( Q^+ \) and \( S_0, \ldots, S_n \). As points of the objective space not in general position have measure zero, choosing \( q \) randomly from a set with positive measure will ensure that the above event will happen with probability 1. We will discuss the choice of \( q \) in detail later.

The choice of the initial surrounding polytope \( S_0 \) is quite natural. It is the \( p \)-dimensional ideal simplex with vertices \( (0, \ldots, 0, 1) \) (the origin), and the ideal points \( (e_i, 0) \), where the coordinates of the ray \( e_i \) are zero except for the \( i \)-th coordinate. \( p \) facets of this simplex are the coordinate
hyperplanes with the equation \((e_i, 0)\). The \(p + 1\)-st facet is the ideal plane containing all ideal points. As \(\mathcal{Q}^+\) is not empty, this facet is, in fact, part of \(\mathcal{Q}^+\), thus the ideal points \((e_i, 0)\) are boundary points of \(\mathcal{Q}^+\). Moreover, it follows from the remark after the proof of Proposition 1 that ideal points are on the non-negative side of any supporting hyperplane of \(\mathcal{Q}^+\).

### 3.4 Details of Benson’s algorithm

Benson’s algorithm constructs a sequence \(S_n\) of bounding polytopes. We discussed in Section 3.3 how to select the initial polytope \(S_0\). We also have an internal point \(q \in \mathcal{Q}^+\) in general position; we will return later to how it can be generated. With each polytope \(S_n\) we also maintain two sets: a set of hyperplanes such that the intersections of the non-negative sides of these hyperplanes is exactly \(S_n\), and the list of vertices of \(S_n\), indicating whether each vertex is known to be a boundary point of \(\mathcal{Q}^+\) or not.

Suppose we have obtained \(S_n\), \(n \geq 0\), and we proceed to generate \(S_{n+1}\).

1. Let us look at the vertices of \(S_n\), and select one which is not marked as a boundary point of \(\mathcal{Q}^+\). If no such vertex can be found, then the algorithm terminates: according to Proposition 1 these vertices are the extremal vertices of \(\mathcal{Q}^+\), thus the extremal vertices of \(\mathcal{Q}\).

2. Let \(y\) be the vertex selected. Then, the boundary point of \(\mathcal{Q}^+\) needs to be found on the line segment \(y - q\) by solving the scalar LP problem \(P(q, q - y)\). The solution is denoted by \(\hat{\lambda}\).

   According to Proposition 2 if \(\hat{\lambda} = 1\), then \(q - (q - y) = y\) is on the boundary of \(\mathcal{Q}^+\). If so, it is marked as such, and return to step 1.

3. Otherwise, \(0 < \hat{\lambda} < 1\). Let us compute \(\hat{y} = q - \hat{\lambda}(y - q)\), which point is on the boundary of \(\mathcal{Q}^+\). By the second part of Proposition 2 if \((\hat{u}, \hat{v})\) is the solution to the dual problem \(D(q, q - y)\), then the hyperplane \(h = (\hat{v}, -\hat{y}^T \hat{v})\) written in homogeneous coordinates is a supporting hyperplane to \(\mathcal{Q}^+\) at \(\hat{y}\), and \(\mathcal{Q}^+\) is on its non-negative side. Also, as \(q\) is in general position, \(h\) is a facet of \(\mathcal{Q}^+\). We can therefore add \(h\) to the hyperplanes of \(S_n\) to create the polytope \(S_{n+1}\).

4. Next, the vertices of \(S_{n+1}\) need to be computed using the double description method, see [20] as follows. Vertices of \(S_n\) on the non-negative side of \(h\) remain vertices of \(S_{n+1}\). All other vertices of \(S_{n+1}\) are the intersection points of the relative interiors of some edges (two dimensional face) of \(S_n\) and \(h\).

Please note that in Step 1 all considered edges of \(S_n\) intersect \(h\) at different vertices of \(S_{n+1}\), so there is no need to check for equality between them. This leaves us to determine whether a pair of vertices \((v_1, v_2)\) of \(S_n\) is an edge of \(S_n\). To this end we use the following observation from [16].

**Proposition 3.** Vertices \(v_1\) and \(v_2\) of \(S_n\) are connected by an edge if and only if every other vertex is missed by some facet of \(S_n\) containing both \(v_1\) and \(v_2\).

Proposition 3 suggests the following fast combinatorial test: if one considers all facets of \(S_n\) adjacent to both \(v_1\) and \(v_2\), and takes the intersection of the adjacency lists of these faces, then if this intersection contains \(v_1\) and \(v_2\) only, they form an edge of \(S_n\), otherwise, they do not. To perform this test we also need to maintain the adjacency lists of vertices and facets:

5. Adjust the adjacency lists of vertices and facets of \(S_{n+1}\).

While the algorithm works with a mixture of ordinary and ideal points, fortunately, the initial ideal points are on the boundary of the polytope \(\mathcal{Q}^+\), and because of this, the algorithm introduces no other ideal points. Ideal points may occur in steps 4 and 5 only, when calculating the intersection of the edge \(v_1 - v_2\) of \(S_n\) with \(h\); here \(v_1\) or \(v_2\), but not both may be one of the ideal points.

Also, some of the ideal vertices might be adjacent to the new facet \(h\), thus they may appear on the adjacency list of \(h\) (and vice versa).
The easiest way of finding a random internal point \( q \in Q^+ \) is to get any feasible \( x \in A \) (that is, \( x \geq 0, Ax = b \)), and then increasing all coordinates of \( Px \) by some positive random amount. To find such a feasible \( x \) requires solving an LP problem, but we can improve the algorithm further if we postpone finding \( q \) until it is actually needed, which is when we determine the first new facet of \( S_1 \) in Step 2. As the only vertex of \( S_0 \) that is not a boundary point of \( Q^+ \) is the origin, we can direct a ray from the origin in a random direction \( d > 0 \), find the intersection point of the ray and \( Q^+ \) by solving the LP problem \( P(0, -d) \), and then set \( q \) on that ray at some random distance behind the intersection point. This way, we not only get the internal point \( q \), but also the supporting hyperplane at the intersection of the segment \( 0 - q \) and \( Q^+ \).

If \( Q^+ \) is not empty, then any ray \( 0 + \lambda d \) with \( d > 0 \) cuts into it. Therefore, for example, we can choose \( d \) so that every coordinate is a uniform random value between 1 and 2. If the LP problem \( P(0, -d) \) has no solution, then \( Q^+ \) is empty; otherwise, let \( \lambda > 0 \) be the solution, \( r \) be a uniform random value between 1 and 2, and we can set \( q = (r + \lambda)d \), having \( \tilde{y} = \lambda d \) as the point at the boundary of \( Q^+ \).

### 3.5 The scalar LP problems

The scalar LP problems which are to be solved repeatedly during the algorithm have very similar structures. Figure 1 shows their general form. We indicated that there is only one row in the matrix \( A \) where \( b \) is not zero. The internal point \( q \) is fixed throughout the algorithm; only the direction \( d \) changes in each iteration. The uniformity of the LP problems can be used to speed up the initializations required by the LP solver.

### 3.6 Tweaks and modifications

Due to the nature of the original combinatorial problem, the optimization problem is ill-posed, and special care needs to be taken to maintain numerical stability. We dismissed using integer arithmetic as being prohibitively expensive both in the LP solver and during computing the vertices of the approximating polytopes. Instead, the LP solver and vertex computation were carefully modified to achieve better numerical stability.

The applied scalar LP solver is a standard but finely tuned simplex method. During the execution of the simplex method, one walks through the vertices of a large dimensional polytope while maintaining different descriptions of the polytope. Each vertex during the walk is determined by a set of the original facets (equations) whose intersection the vertex is. In simplex terminology, this set of facets is known as a base. Knowing the base is sufficient to regenerate the internal state at any step directly from the original facet equations. Therefore, after a predetermined number of steps we do two things: we save the actual base, and recalculate the description of the polytope. When we discover any problem caused by accumulating numerical errors, we return to the last
saved base, and continue the method from that point on. We decided to save the base after each 60 steps, which seemed to be a good choice for the size of the problems to be solved.

As described in Section 3.4, Benson’s algorithm advances by computing the vertices of the new approximating polytope $S_{n+1}$ from the vertices (and facets) of $S_n$, and from a new facet of $Q^+$. The input for this computation comes from the LP solver which gives the equation of the facet. In our case, all facets had rational coefficients with small denominators, and so they could be represented either exactly or with extremely small error. When computing the new vertices of $S_{n+1}$, we lose some of this exactness as the new vertices are computed from the intersections of an old edge and a new facet, and the angle between the edge and the facet can be very small. When this happens repeatedly, the coordinates of a vertex can be very far from their exact values. Therefore, rather than carrying these errors forward, we decided to compute the coordinates of each new vertex directly from the facets they are incident to.

Last, but not least, the modified Benson’s algorithm requires a solution $\hat{v}$ of the dual LP problem $D(q,d)$ only when the optimum $\hat{\lambda}$ is not one. This means that we can abort the LP solver as soon as it finds a feasible solution with $\lambda = 1$, further improving the speed of the algorithm.

4 Experimental results

The modified Benson’s algorithm as described in the previous section has been successfully used to investigate the entropy region $\bar{\Gamma}_N$ by generating hundreds of new entropy inequalities. We show some of the results for three different data sets. In the first two of the sets, the objective space had 10 dimensions, while in the third one, this dimension was reduced to 3 using the internal symmetry of the original problem.

The algorithm was coded in C with an embedded scalar LP solver, a school-book simplex method with the tweaks discussed in Section 3.6. The compiled code was run on a dedicated personal computer with a 1GHz AMD Athlon 64 X2 Dual Core processor and 4 gigabytes of RAM. In fact, the program used only a single core (thus two instances could be run simultaneously on the dual-core machine without affecting the running time), and the combined memory usage was never above 2 gigabytes.

4.1 Results for $p = 10$

The paper of Dougherty et al [12] lists all extremal non-Shannon entropy inequalities which are consequences of at most three copy steps using no more than four auxiliary variables. They used 133 different copy strings to determine 214 new entropy inequalities. The modified Benson algorithm, as outlined in Section 3, was used to generate all extremal vertices of the polyhedrons determined by these copy strings. The results confirmed their report, and did not find any new entropy inequalities which were not consequences of the ones on their list.

Table 1 gives some representative data. Size is the size of the matrix $A$ (columns and rows), followed by the number of vertices and facets. Time is the running time of the algorithm in hours, minutes and seconds. The running time is included here to indicate how it changes with the size of the problem, and does not indicate any comparison with other implementations of Benson’s algorithm. The typical number of the extremal vertices is around a couple of hundreds, with the only exception shown in the last row of Table 1, where the number of extremal vertices is 2506. Also, it might be worth mentioning that in this case, one of the intermediate polytopes had more than 22,000 vertices, which is an about 10-fold increase.

As the size of the matrix $A$ does not vary too much from case to case, we expected the running time of the algorithm to depend only on the number of iterations, that is, on the sum of the number of vertices and facets. The plot on Figure 2 confirms this expectation, indicating a linear dependence. The variance is due to the fact that occasionally, the LP solver failed and had to resume the work from an earlier stage, as it was discussed in Section 3.6.

There appears to be no easy way to predict the number of iterations, and thus the expected running time. Similar copy strings require a widely varying number of iterations, and have a
Table 1: Representative results for the Dougherty et al list \((p = 10)\)

| Copy string | Size     | Vertices | Facets | Time     |
|-------------|----------|----------|--------|----------|
| rs=cd:ab;tu=cr:ab;v=(cs):abtu | 4055×370 | 19       | 58     | 1:10:10  |
| rs=ad:bc;tu=ar:bc;v=r:abst   | 4009×370 | 40       | 103    | 3:24:37  |
| rs=cd:ab;tv=(cr):ab;uv=cs:abt | 3891×358 | 30       | 102    | 3:34:31  |
| rs=cd:ab;tu=cr:ab;v=t:adr    | 3963×362 | 167      | 235    | 9:20:19  |
| rs=cd:ab;tv=dr:ab;v=b:adsu   | 4007×370 | 318      | 356    | 13:20:08 |
| rs=cd:ab;tv=dr:ab;u=a:bcstu  | 4007×370 | 318      | 356    | 14:34:42 |
| rs=cd:ab;tv=cs:ab;v=a:bcrt   | 4007×370 | 297      | 648    | 22:02:39 |
| rs=cd:ab;tv=a:bcst;uv=bt:acr | 3913×362 | 779      | 1269   | 37:15:33 |
| rs=cd:ab;tv=cs:ab;v=a:bcstu  | 3987×362 | 4510     | 7966   | 427:43:30|
| rs=cd:ab;tv=cs:ab;v=a:bcrtu  | 3893×362 | 10387    | 13397  | 716:36:32|

Table 2: Some extended copy strings with \(p = 10\)

| Copy string | Size     | Vertices | Facets | Time     |
|-------------|----------|----------|--------|----------|
| r=c:ab;s=r:ac;t=r:ad   | 561×80   | 5        | 20     | 0:01     |
| rs=cd:ab;tu=r:ad;u=s:ad  | 1509×172 | 40       | 132    | 6:19     |
| rs=cd:ab;tv=a:bcst;u=(cs):abt | 1569×178 | 47       | 76     | 6:51     |
| rs=cd:ab;tv=a:bcst;u=b:adst | 1512×178 | 177      | 261    | 17:40    |
| rs=ab;st=cd:ab;u=t:acr   | 1346×161 | 209      | 436    | 29:18    |
| rs=cd:ab;tu=a:bcst;v=a:bcrt | 1512×177 | 355      | 591    | 38:59    |
| rs=cd:ab;tv=a:bcst;u=a:bcrst | 1511×178 | 363      | 599    | 1:04:32  |
| rs=cd:ab;tv=a:bcst;u=c:abct | 1369×166 | 355      | 591    | 1:07:01  |
| rs=cd:ab;tv=a:bcst;u=(at):bcst | 1555×177 | 484      | 676    | 1:39:30  |
| rs=cd:ab;tv=a:bcst;u=a:bdrt | 1509×177 | 880      | 1238   | 4:30:26  |
| rs=cd:ab;tv=a:bcst;u=a:bdrt | 1513×177 | 2506     | 2708   | 5:11:25  |

varying number of extremal vertices. Let us also mention that in each of the 133 cases, the polytope \(Q^+\) factors to an 8-dimensional polytope and the non-negative quadrant of \(R^2\). The fact that the objective space is practically 8 dimensional rather than 10 might have an impact on the observed speed of the algorithm.

The implemented algorithm was used to check consequences of copy strings beyond the ones considered in \([12]\). This resulted in increasing the total number of known computer-generated entropy inequalities from 214 in \([12]\) to more than 470. Some representatives are shown in Table 2. As the copy string becomes more involved, the generated problem becomes larger, and we have also observed a significant increase in the number of vertices of the intermediate polytopes. This unexpected increase makes the double description method highly inefficient, and it actually becomes the bottleneck in the algorithm. In most of the cases enlisted in Table 2, we had to set the bound on the number of facets and vertices to \(2^{16}\), as smaller values were exhausted very fast. This in turn meant that the size of the incidence matrix of vertices and facets was \(2^{16} \times 2^{16}\), and that executing the combinatorial test of Proposition 3 for each pair \(v_1\) and \(v_2\) of several thousands of vertices took a considerable amount of time, and became comparable to the time taken by the LP solver. In about 20% of the extended cases investigated, even this limit was too small, and the algorithm aborted.
Figure 2: Running time vs. problem size

4.2 Results for $p = 3$

Table 3 shows statistics for a set of even larger problems. In this case, the optimization problem was determined by the total number of random variables employed during the copy steps. Relying on the high symmetry in the chosen problems and with some careful preprocessing, we were able to reduce the problem size significantly: 20 random variables had more than $10^6$ entropies, and

| Random variables | Size          | Vertices | Facets | Running time |
|------------------|---------------|----------|--------|--------------|
| 10               | 692×99        | 11       | 13     | 0:01         |
| 12               | 1298×150      | 26       | 24     | 0:55         |
| 14               | 2175×233      | 53       | 43     | 9:38         |
| 16               | 3373×338      | 100      | 78     | 2:18:45      |
| 18               | 4942×474      | 171      | 129    | 6:55:40      |
| 20               | 5772×635      | 278      | 208    | 23:20:17     |

Table 3: More random variables with $p = 3$

more than $4.8 \cdot 10^7$ Shannon inequalities. The symmetry in the problems also made it possible to reduce the dimensions of the objective space from 10 to 3.

Problems listed in this section had their running time determined almost exclusively by the LP solver. Solving the numerous numerically ill-posed scalar LP problems of such sizes appears to be at the limit of the simplex-based LP solver implemented in our software. It would be an interesting research problem to investigate which LP solvers can handle problems of this size efficiently and with sufficient accuracy.

5 Conclusion

Exploring the 15-dimensional entropy region formed by the entropies of the non-empty subsets of four random variables is an intriguing research problem. Presently the only available method which goes beyond the standard Shannon inequalities is the one of Zhang and Yeung from 1998 [11]. Using non-trivial properties of the entropy region, the Zhang–Yeung method was transformed into
the problem of finding all extremal vertices of a 10-dimensional projection of a high dimensional polytope. A modified variant of Benson’s algorithm was used to solve this linear multiobjective optimization problem. As opposed to the original algorithm [15] and its refinements [16, 17, 18], where each iteration requires solving two scalar LP problems, this variant uses only one scalar LP instance in each iteration. This improvement, applicable to other variants of Benson’s algorithm as well, might be of independent interest, and was observed independently in [19].

The implemented algorithm was used successfully to check the results on the 133 copy strings of [12], and confirm that no entropy inequality has been missed. Copy strings leading to larger optimization problems were also investigated. This resulted in more than doubling the number of known computer-generated entropy inequalities.

The algorithm was also run on some extremely symmetrical, but significantly larger data sets, where the objective space could be reduced to three dimensions. The results achieved were essential in forming a general conjecture about the limits of the Zhang–Yeung method [21].

A compelling theoretical problem arose as the experimental results were evaluated. It has been observed that some of the intermediate polytopes $S_n$ (bounded by a certain subset of the facets of $Q^+$) had significantly more vertices than $Q^+$ itself. It is worth noting that this phenomenon cannot occur when the objective space has at most three dimensions. By the well-known Euler formula, the number of vertices of any convex 3-dimensional polytope is bounded by $2f - 4$, where $f$ is the number of its faces. Thus any intermediate polytope can have at most $2M$ vertices, where $M$ is a bound on the total number of steps the algorithm might take. In the 3-dimensional case, setting this limit in the software to a couple of thousand appears to be a safe choice. However, the situation changes when the dimension $p$ of the objective space increases. Let us suppose that $M$ is an upper bound on the number of final vertices and facets. Then an obvious upper bound on the number of vertices of any intermediate polytope is $Mp$, as any vertex is determined by the $p$ facets it is incident to. There are $p$-dimensional convex polytopes with $M$ facets and $O(M^{[p/2]})$ vertices, but it is not known what happens if we bound the number of facets plus the number of vertices of the final polytope, and would like to determine how many extra vertices an intermediate polytope might have. The following example shows that in three dimensions, this increase can be linear, which is, apart from the constant, is the worst amount one can expect. In the example, the final polyhedron consists of two regular $2n$-polygon-based pyramids joined together at their bases. This polyhedron has $2n + 2$ vertices and $4n$ faces. Leaving out every other facet of the top pyramid increases the number of vertices to $3n + 2$. It would be interesting to see if similar examples can be construed in higher dimensions.

The shortcomings of using the double description method have been observed earlier [26], and alternative algorithms have been suggested to generate the extremal rays of pointed cones. In our case, however, these algorithms could not be used directly, as Benson’s algorithm generates facets and extremal vertices simultaneously, and so we do not know all the facets in advance. The procedure outlined in Section 3.2 can be considered as an “oracle call,” which, given any point in the objective space, returns whether the point is outside the polytope $Q^+$, and if yes, also provides a facet of $Q^+$ that separates the polytope from this point. The challenge then becomes to devise an algorithm which calls the oracle, and finds all vertices of $Q^+$ in time and space linear in the final number of vertices plus the number of facets.

Investigating the entropy region of five random variables instead of four appears to be significantly harder. The very first obstacle is the lack of our understanding even of the Shannon region of this 31-dimensional space. There does not seem to be any suitable coordinate transformation similar to the unimodular matrix $U$ which would simplify the structure of some cross-sections of this region as $U$ did in the four-variable case. This simplification is what made the application of Benson’s algorithm possible, as we had an immediate surrounding polytope (the non-negative orthant of $\mathbb{R}^{10}$), and as all vertices of the projected polytope were extremal vertices, they comprised the solution of a global multiobjective optimization problem.
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Appendix A

A.1 Shannon inequalities

Let $\langle x_i : i \in I \rangle$ be a collection of random variables. For $A \subseteq I$, we let $x_A = \langle x_i : i \in A \rangle$. The Shannon inequalities say that the entropy function is a monotone submodular function on the subsets of $I$, that is,

$$H(x_A) \leq H(x_B) \quad \text{when} \ A \subseteq B \subseteq I,$$

and

$$H(x_{A \cup B}) + H(x_{A \cap B}) \leq H(x_A) + H(x_B),$$

for all subsets $A, B$ of $I$. Moreover, there is a minimal subset of these inequalities from which all others follow: consider the inequalities from [4] where $B = I$, and $A$ is missing only one element of $I$; and the inequalities from [5] where both $A$ and $B$ has exactly one element not in $A \cap B$.

A.2 Independent copy of random variables

If we split a set of random variables into two disjoint groups $\langle x_i : i \in I \rangle$ and $\langle y_j : j \in J \rangle$, and create $\langle x'_i : i \in I \rangle$ as an independent copy of $\langle x_i \rangle$ over $\langle y_j \rangle$, then the entropy of certain subsets of these variables can be computed from other subsets as follows. Let $A, B \subseteq I$ and $C \subseteq J$. Then,

$$H(x'_A x_B y_C) = H(x'_B x_A y_C),$$

which is due to the complete symmetry between $x'_i$ and $x_i$. The fact that $x'_I$ and $x_I$ are independent over $y_J$ translates into the following entropy equality:

$$H(x'_A x_B y_J) = H(x'_A y_J) + H(x_B y_J) - H(y_J)$$

for all subsets $A, B \subseteq I$.

A.3 Copy strings

As explained in Section 2, we start from four random variables, split them into two parts, create an independent copy of the first part over the second, add the newly created random variables to the group, and then repeat this process. To save on the number of variables created, in each step certain newly generated variables are discarded, or two or more new variables are merged into a single one. This process is described by a copy string, which has the following form:

$$rs=cd:ab; t=(cr):ab; u=t:acs$$

This string describes three iterations separated by semicolons. The initial four variables are $a, b, c$ and $d$, and the newly created variables are $r, s, t$ and $u$. In a copy step, variables after the colon are in the “over” set. We keep the copied image of those variables only which are after the equality sign, and the copies are marked by the letters before the equality sign. Thus in the first iteration, we make a copy of $cd$ over $ab$, and the copy of $c$ and $d$ will be named $r$ and $s$, respectively. When variables are enclosed in parentheses, their copies are merged into a single variable. For example, in the second iteration we keep the copies of $c$ and $r$ only, discard the copies of $d$ and $s$, and merge these two copies into $t$. 
A.4 A unimodular matrix

As explained in Section 2, it is advantageous to look at the 15 entropies of the four random variables in another coordinate system. The new coordinates can be computed using the unimodular matrix shown in Table 4. Columns represent the entropies of the subsets of the four random variables $a$, $b$, $c$ and $d$, as indicated in the top row. The value of the “Ingleton row” should be set to 1, and rows marked by the letter “z” vanish for all extremal vertices, thus they should be set to 0.

| Ingleton | $a$ | $b$ | $c$ | $d$ | $ab$ | $ac$ | $ad$ | $bc$ | $bd$ | $cd$ | $abcd$ |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-------|
|           | -1  | -1  | 0   | 0   | 1   | 1   | 1   | 1   | -1  | -1  | 0     |
|           | 0   | -1  | 0   | 0   | 1   | 0   | 1   | 0   | 0   | -1  | 0     |
|           | 0   | 0   | -1  | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0     |
|           | 0   | -1  | 0   | 0   | 1   | 0   | 1   | 0   | 0   | -1  | 0     |
|           | -1  | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 0     |
|           | 0   | -1  | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 0     |
|           | -1  | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 0     |
|           | 0   | -1  | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 0     |
|           | -1  | 0   | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 0     |
|           | 0   | -1  | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 0     |
|           | 0   | 0   | 1   | 1   | 0   | 0   | 0   | 0   | -1  | 0   | 0     |
|           | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | -1  | 0   | 0     |
|           | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | -1  | 0   | 0     |
|           | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | -1  | 0   | 0     |

Table 4: The unimodular matrix

Appendix B

This section lists entropy inequalities which were found during the experiments described in Section 4 and have all coefficients less than 100. Each entry in the list contains nine integers representing the coefficients $c_0$, $c_1$, ..., $c_8$ for the non-Shannon information inequality of the form

$$c_0(I(c, d) - I(a, b) + I(a, b | c) + I(a, b | d)) +$$

$$+ c_1I(a, b | c) + c_2I(a, b | d) +$$

$$+ c_3I(a, c | b) + c_4I(b, c | a) + c_5I(a, d | b) + c_6I(b, d | a) +$$

$$+ c_7I(c, d | a) + c_8I(c, d | b) \geq 0.$$ 

Here $I(A, B) = H(A) + H(B) - H(AB)$ is the mutual information, $I(A, B | C) = H(AC) + H(BC) - H(ABC) - H(C)$ is the conditional mutual information. The expression after $c_0$ is the Ingleton value. Following the list of coefficients is the copy string without the equality signs. If the inequality, or one of its permuted variants, appears in the list of [12], then their number of the inequality is given after the / symbol.

1) $2103200000 rs cd:ab; t b:acr/5$
2) $2103132000 rs cd:ab; t c:abrs; u a:bdst/30$
3) $2103100300 rs bd:ac; tu cd:abs; v b:acrst$ 
4) $2202100300 rs bd:ac; t c:adr/4$
5) $2202103000 rs cd:ab; t b:acs/9$
6) $2203100000 rs ac:bd; t b:adr/8$
7) $2212110000 r c:abs; r rad/1$
8) $2215041000 rs cd:ab; t c:abrs; u a:dst/27$
9) $2302100000 rs ac:bd; t d:abr/7$
10) $2312031000 rs cd:ab; t r rad/2$
11) $3105500000 rs cd:ab; t a:bcru ; u t:acr/94$
12) $3115410000 rs cd:ab; t abcru ; a:bdst/58$
13) $3115401000 rs cd:ab; t c:bcru ; u t:bras/96$
14) $3204202000 rs cd:ab; t b:adr; u t:br/162$
15) $3206200000 rs ac:bd; t a:dr; u b:adrt/60$
16) $3214210000 rs cd:ab; t c:as; u t:adr/41$
17) $3305111300 rs cd:ab; t c:abrs; u t:adr/102$
18) $3306111000 rs ac:bd; t b:ars; u b:adr/142$
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