Integral Homology of $PGL_2$ over Elliptic Curves

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The Friedlander–Milnor Conjecture [1] asserts that if $G$ is a reductive algebraic group over an algebraically closed field $k$, then the comparison map

$$H^*_\text{et}(BG_k, \mathbb{Z}/p) \longrightarrow H^*(BG, \mathbb{Z}/p)$$

is an isomorphism for all primes $p$ not equal to the characteristic of $k$. Gabber’s rigidity theorem [2] implies that this map is indeed an isomorphism for the stable general linear group $GL$ (this is due to Suslin [6] for $k = \mathbb{C}$ and to Jardine [3] for arbitrary $k$). Similarly, a proof of an unstable version of rigidity would lead to a proof of the unstable Friedlander–Milnor Conjecture.

In this note we consider unstable rigidity for the group $PGL_2$ over an elliptic curve $E$. We assume that $E$ is defined by the equation $F(x, y) = 0$, where

$$F(x, y) = y^2 + a_1 xy + a_3 y - x^3 - a_2 x^2 - a_4 x - a_6,$$

and the $a_i$ lie in an infinite field $k$. Denote by $\overline{E}$ the projective curve $E \cup \{\infty\}$. Denote by $A$ the coordinate ring of the affine curve $E$. If $l \in k$ and $F(l, y) = 0$ has no rational solutions, denote by $k(\omega)$ the quadratic extension of $k$ inside the algebraic closure $\overline{k}$ for which $F(l, \omega) = 0$. Our main result is the following.

**THEOREM.** For all $i \geq 1$,

$$H_i(PGL_2(A), \mathbb{Z}) = \bigoplus_{p \in \overline{E}} H_i(PGL_2(k), \mathbb{Z}) \bigoplus_{p^2 \neq 0} H_i(k^\times, \mathbb{Z})$$

$$\bigoplus_{l \in k, F(l, y) = 0} H_i(k(\omega)^\times / k^\times, \mathbb{Z}).$$

Here, $p \sim -p$ means that we identify the factors corresponding to the points $p$ and $-p$.

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This is proved by examining the action of $PGL_2(A)$ on the Bruhat–Tits tree $X$ associated to a two-dimensional vector space over the function field of $E$ (see, e.g., Serre’s book [5]). The quotient graph $PGL_2(A)\backslash X$ is a tree (this is due to S. Takahashi [8]) and the various simplex stabilizers are easily described.

As a corollary, we have the following rigidity result.

**Corollary.** Suppose the field $k$ is algebraically closed and let $x, y$ be distinct points on $E$. Then the specialization homomorphisms $s_x, s_y : H_\bullet(PGL_2(A), \mathbb{Z}) \rightarrow H_\bullet(PGL_2(k), \mathbb{Z})$ coincide.

1. The Quotient and the Stabilizers

In [8], Takahashi described a fundamental domain for the $PGL_2(A)$-action on $X$; denote this subtree by $D$. There is a distinguished vertex $o \in D$. For each $l$ in $k \cup \{\infty\}$ there is a vertex $v(l)$ adjacent to $o$. The rest of $D$ may be described as follows. Let $D(l)$ denote the subtree of $D - \{o\}$ which contains $v(l)$. The tree $D$ is the union of $o$ and the various $D(l)$ (which are disjoint). The trees $D(l)$ are as follows.

1. Suppose $F(x, y) = 0$ has no rational solution with $x = l$. Then $D(l)$ consists only of $v(l)$ (see Figure 1).

2. Suppose $l = \infty$ or $F(x, y) = 0$ has a unique rational solution with $x = l$. Let $p$ be the point at infinity of $E$ or the rational point corresponding to the solution. Note that $p$ is a point of order 2. Then $D(l)$ consists of an infinite path $c(p, 1), c(p, 2), \ldots$ and an extra vertex $e(p)$ (see Figure 2).

3. Suppose $F(x, y) = 0$ has two different solutions such that $x = l$. Let $p, q$ be the corresponding points on $E$. Then $D(l)$ consists of two infinite paths $c(p, 1), c(p, 2), \ldots$ and $c(q, 1), c(q, 2), \ldots$ (see Figure 3).

The infinite path $c(p, 1), c(p, 2), \ldots$ is called a *cusp*. Note that there is a one-to-one correspondence between cusps and the rational points of $E$.

Since $X$ is contractible, we have a spectral sequence with $E^1$-term

$$E^1_{p,q} = \bigoplus_{\sigma^{(p)} \subset D} H_q(\Gamma_\sigma, \mathbb{Z}) \implies H_{p+q}(PGL_2(A), \mathbb{Z})$$
Figure 3. $F(l, y) = 0$ has two distinct solutions

where $\Gamma_\sigma$ is the stabilizer of the $p$–simplex $\sigma$ in $\text{PGL}_2(A)$.

The various stabilizers of the $\text{GL}_2(A)$ action were described in [8]. Denote these by $\tilde{\Gamma}_\sigma$.

**Proposition 1.1** (cf. [8], Theorem 5). *The stabilizers $\tilde{\Gamma}_\sigma$ are (up to isomorphism)*

\[
\begin{align*}
\tilde{\Gamma}_o & \cong k^x \\
\tilde{\Gamma}_{v(l)} & \cong \begin{cases} 
(k(\omega)^x & \text{in case (1)} \\
k^x \times k & \text{in case (2)} \\
k^x \times k^x & \text{in case (3)}
\end{cases} \\
\tilde{\Gamma}_{e(p)} & \cong \text{GL}_2(k) \\
\tilde{\Gamma}_{c(p, n)} & \cong (k^n \times_\theta k^x) \times k^x
\end{align*}
\]

*where $k^n \times_\theta k^x$ is the semidirect product of $k^n$ and $k^x$ and $\theta$ is the automorphism of $k^n$ given by coordinate-wise multiplication by elements of $k^x$. Furthermore, the stabilizer of an edge is the intersection of the stabilizers of its vertices.*

The groups $\tilde{\Gamma}_{c(p, n)}$ are of the form

\[
\left\{ \begin{pmatrix} p & q \\
0 & s \end{pmatrix} : p, s \in k^x, q \in k^n \right\}.
\]
Denote the diagonal subgroups of these $\tilde{\Gamma}$ by $\tilde{L}$. By Theorem 1.11 of [4], these stabilizers satisfy

$$H_\bullet(\tilde{\Gamma}, \mathbb{Z}) \cong H_\bullet(\tilde{L}, \mathbb{Z}),$$

the isomorphism being induced by the inclusion $\tilde{L} \to \tilde{\Gamma}$.

**Corollary 1.2.** The stabilizers $\Gamma_\sigma$ satisfy

$$H_\bullet(\Gamma_\sigma, \mathbb{Z}) \cong H_\bullet(\{1\}, \mathbb{Z})$$

$$H_\bullet(\Gamma_{v(l)}, \mathbb{Z}) \cong \begin{cases} H_\bullet(k(\omega)^\times / k^\times, \mathbb{Z}) & \text{in case (1)} \\ H_\bullet(k, \mathbb{Z}) & \text{in case (2)} \\ H_\bullet(k^\times, \mathbb{Z}) & \text{in case (3)} \end{cases}$$

$$H_\bullet(\Gamma_{e(p,n)}, \mathbb{Z}) \cong H_\bullet(k^\times, \mathbb{Z})$$

$$H_\bullet(\Gamma_{e(p)}, \mathbb{Z}) \cong H_\bullet(\text{PGL}_2(k), \mathbb{Z}).$$

**2. The Main Theorem**

Note that our spectral sequence consists of two columns and that each row $E^1_{i,0}$ is the chain complex $C_\bullet(\mathcal{D}, \mathcal{H}_q)$, where $\mathcal{H}_q$ is the coefficient system $\sigma \mapsto H_q(\Gamma_\sigma)$. Fix a positive integer $q \geq 1$. Note that $H_q(\Gamma_o, \mathbb{Z}) = 0$ and that $H_q(\Gamma_{e}, \mathbb{Z}) = 0$ for any edge incident with $o$. It follows that the chain complex $C_\bullet(\mathcal{D}, \mathcal{H}_q)$ is a direct sum of chain complexes

$$C_\bullet(\mathcal{D}, \mathcal{H}_q) = \bigoplus_{l \in k \cup \{\infty\}} C_\bullet(\mathcal{D}(l), \mathcal{H}_q).$$

**Proposition 2.1.** Suppose $F(x, y) = 0$ has no rational solutions with $x = l$. Then

$$H_i(\mathcal{D}(l), \mathcal{H}_q) = \begin{cases} H_q(k(\omega)^\times / k^\times, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

**Proof.** This is clear since $C_\bullet(\mathcal{D}(l), \mathcal{H}_q)$ consists only of the single group $H_q(k(\omega)^\times / k^\times, \mathbb{Z})$ sitting in degree zero. $\square$

**Proposition 2.2.** Suppose $l = \infty$ or $F(x, y) = 0$ has a unique rational solution with $x = l$. Then

$$H_i(\mathcal{D}(l), \mathcal{H}_q) = \begin{cases} H_q(\text{PGL}_2(k), \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

**Proof.** The stabilizer of $v(l)$ and of the edge joining $v(l)$ to $c(p,1)$ is isomorphic to $k$. The map $\Gamma_{v(l),c(p,1)} \to \Gamma_{v(l)}$ is an isomorphism on homology and the map $\Gamma_{v(l),c(p,1)} \to \Gamma_{e(p,1)}$ induces the zero map on homology. It follows that $H_\bullet(\mathcal{D}(l), \mathcal{H}_q) \cong H_\bullet(\mathcal{D}(l)', \mathcal{H}_q)$ where $\mathcal{D}(l)'$ is the tree obtained by deleting $v(l)$ and the edge joining it to $c(p,1)$. One checks easily that the map

$$C_1(\mathcal{D}(l), \mathcal{H}_q) \longrightarrow C_0(\mathcal{D}(l), \mathcal{H}_q)$$

is injective with cokernel $H_q(\text{PGL}_2(k), \mathbb{Z})$ (or equivalently, check that the relative homology groups $H_\bullet(\mathcal{D}(l), c(p); \mathcal{H}_q)$ vanish). $\square$
Proposition 2.3. Suppose $F(x, y) = 0$ has two distinct solutions with $x = l$. Then
$$H_i(D(l), \mathcal{H}_q) = \begin{cases} H_q(k^\times, \mathbb{Z}) & i = 0 \\ 0 & i > 0. \end{cases}$$

Proof. In this case, $D(l)$ is a tree and the stabilizer of each vertex and each edge is $k^\times$. The maps $\Gamma_e \to \Gamma_v$ induce isomorphisms on homology for each edge and vertex. It follows that $C_\bullet(D(l), \mathcal{H}_q)$ is a chain complex with constant coefficients. Since $D(l)$ is contractible, the result follows.

Theorem 2.4. For all $i \geq 1$,
$$H_i(PGL_2(A), \mathbb{Z}) \cong \bigoplus_{p \in \mathbb{E}} H_i(PGL_2(k), \mathbb{Z}) \oplus \bigoplus_{p \in \mathbb{E}, 2p \neq 0} H_i(k^\times, \mathbb{Z}) \oplus \bigoplus_{l \in k, F(l, y) = 0 \text{ has no solutions}} H_i(k(\omega)^\times / k^\times, \mathbb{Z}).$$

Proof. Note that since $D$ is contractible, $E_{0,0}^2 = \mathbb{Z}$ and $E_{1,0}^2 = 0$. The preceding propositions show that
$$H_0(D, \mathcal{H}_q) = \bigoplus_{l \in k \cup \{\infty\}} H_0(D(l), \mathcal{H}_q)$$
and $H_1(D, \mathcal{H}_q) = 0$. It remains to identify the various direct summands with points of $\mathbb{E}$.

Those $l$ for which $F(l, y) = 0$ has a unique solution (or $l = \infty$) correspond to points of order 2 in $\mathbb{E}$. Those $l$ for which $F(l, y) = 0$ has two distinct solutions correspond to pairs of points on $\overline{E}$. For such a pair $p, q$, the groups $H_i(k^\times, \mathbb{Z})$ arising from the cusps associated to $p, q$ are identified together since they are both adjacent to $v(l)$. The claimed direct sum decomposition follows.

3. Rigidity

Suppose now that the field $k$ is algebraically closed. Note that the projective curve $\overline{E}$ is isomorphic to the group $\text{Pic}^0 \overline{E}$ of degree zero line bundles on $\overline{E}$. Moreover, after a suitable linear change of coordinates, the points $p, q$ corresponding to the two solutions of $F(l, y) = 0$ may be assumed to satisfy $q = -p$.

Corollary 3.1. If $k$ is algebraically closed, then for all $i \geq 1$,
$$H_i(PGL_2(A), \mathbb{Z}) \cong \bigoplus_{L \in \text{Pic}^0 \overline{E}} H_i(PGL_2(k), \mathbb{Z}) \oplus \bigoplus_{L \in \text{Pic}^0 \overline{E}, 2L \neq 0} H_i(k^\times, \mathbb{Z}).$$

Proof. This is obvious once one notes that the factors $H_i(k(\omega)^\times / k^\times, \mathbb{Z})$ do not enter the picture when $k$ is algebraically closed.

As a consequence, we have the following rigidity result.
Corollary 3.2. Suppose \( k \) is algebraically closed and let \( x, y \) be distinct points on \( E \). Then the corresponding specialization homomorphisms

\[
s_x, s_y : H_i(PGL_2(A), \mathbb{Z}) \longrightarrow H_i(PGL_2(k), \mathbb{Z})
\]

coincide for all \( i \geq 0 \).

Proof. This follows from the direct sum decomposition of Corollary 3.1. Since the groups which appear as summands do not involve rational functions on \( \mathbb{E} \), the homomorphisms \( s_x \) and \( s_y \) must agree on each summand.

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References
[1] E. Friedlander, G. Mislin, Cohomology of classifying spaces of complex Lie groups and related discrete groups, Comment. Math. Helv. 49 (1984), 347–361.
[2] O. Gabber, K-theory of Henselian local rings and Henselian pairs, Contemp. Math. 126 (1992), 59–70.
[3] J. Jardine, Simplicial objects in a Grothendieck topos, Contemp. Math. 55 (1986), 193–239.
[4] Yu. Nesterenko, A. Suslin, Homology of the full linear group over a local ring, and Milnor’s K-theory, Math. USSR Izvestiya 34 (1990), 121–145.
[5] J.-P. Serre, Trees, Springer–Verlag, Berlin/ Heidelberg, New York, 1980.
[6] A. Suslin, On the K-theory of local fields, J. Pure Appl. Algebra 34 (1984), 301–318.
[7] A. Suslin, Algebraic K-theory of fields, Proc. ICM, Berkeley, 1986.
[8] S. Takahashi, The fundamental domain of the tree of \( GL(2) \) over the function field of an elliptic curve, Duke Math. J. 73 (1993), 85–97.

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