Pointwise estimates for solutions of singular quasi-linear parabolic equations

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Abstract

For a class of singular divergence type quasi-linear parabolic equations with a Radon measure on the right hand side we derive pointwise estimates for solutions via the nonlinear Wolff potentials.

1 Introduction and main results

In this note we give a parabolic version of a by now classical result by Kilpeläinen-Malý [7], who proved pointwise estimates for solutions to quasi-linear \( p \)-Laplace type elliptic equations with measure in the right hand side. The estimates are expressed in terms of the nonlinear Wolff potential of the right hand side. These estimates were subsequently extended to fully nonlinear equations by Labutin [8] and fully nonlinear and subelliptic quasi-linear equations by Trudinger and Wang [16]. The pointwise estimates proved to be extremely useful in various regularity and solvability problems for quasilinear and fully nonlinear equations [7, 8, 13, 14, 16]. An immediate consequence of these estimates is the sufficient condition of local boundedness of weak solutions which, as examples show, is optimal.

For the heat equations the corresponding result was recently given in [5]. The degenerate case \( p > 2 \) was studied recently in [10]. Here we provide the pointwise estimates for the singular supercritical case \( \frac{n}{n+1} < p < 2 \).

Let \( \Omega \) be a domain in \( \mathbb{R}^n \), \( T > 0 \). Let \( \mu \) be a Radon measure on \( \Omega \). Let \( \frac{n}{n+1} < p < 2 \). We are concerned with pointwise estimates for a class of non-homogeneous divergence type quasi-linear parabolic equations of the type

\[
\frac{\partial u}{\partial t} - \text{div} A(x,t,u,\nabla u) = \mu \quad \text{in} \quad \Omega_T = \Omega \times (0,T), \quad \Omega \subset \mathbb{R}^n.
\]

We assume that the following structure conditions are satisfied:

\[
\begin{align*}
A(x,t,u,\zeta) \zeta & \geq c_1 |\zeta|^p, & \zeta \in \mathbb{R}^n, \\
|A(x,t,u,\zeta)| & \leq c_2 |\zeta|^{p-1},
\end{align*}
\]

with some positive constants \( c_1, c_2 \). The model example involves the parabolic \( p \)-Laplace equation

\[
u_t - \Delta_p u = \mu, \quad (x,t) \in \Omega_T.
\]

Before formulating the main results, let us remind the reader of the definition of a weak solution to equation (1.1).

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We say that $u$ is a weak solution to (1.1) if $u \in V(\Omega_T) := C([0, T]; L^2_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega))$ and for any compact subset $K$ of $\Omega$ and any interval $[t_1, t_2] \subset (0, T)$ the integral identity

\begin{equation}
\int_K u_\varphi dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \{-u_\varphi + A(x, t, u, \nabla u)\nabla \varphi\} dx \, d\tau = \int_{t_1}^{t_2} \int_K \varphi \mu(dx) \, dr.
\end{equation}

for any $\varphi \in W^{1,2}_{\text{loc}}(0, T; C(K)) \cap L^p_{\text{loc}}(0, T; \overset{\text{top}}{W}^{1,p}(K))$. Note that $\varphi$ is required to be continuous with respect to the spatial variable so that the right hand side of (1.5) is well defined.

The crucial role in our results is played by the truncated version of the Wolff potential defined by

\begin{equation}
W^\mu_p(x, R) = \int_0^R \left( \frac{\mu(B_r(x))}{r^{n-p}} \right)^{\frac{1}{p-\beta}} \, dr.
\end{equation}

In the sequel, $\gamma$ stands for a constant which depends only on $n, p, c_1, c_2$ which may vary from line to line.

**Theorem 1.1.** Let $u$ be a weak solution to equation (1.1). Let $\beta = p + n(p - 2) > 0$. There exists $\gamma > 0$ depending on $n, c_1, c_2$ and $p$, such that for almost all $(x_0, t_0) \in \Omega_T$ there exists $R_0 > 0$ satisfying the condition $R_0 < \min\{1, \frac{3}{2}(T - t_0)^{\frac{2}{3}}\}$ such that for all $R \leq R_0$ the following estimates hold

\begin{enumerate}[(i)]
\item $u(x_0, t_0) \leq \gamma \left\{ R^2 + \left( \frac{1}{R^{n+\beta}} \int_{B_R \times (0, T)} u_+ \, dx \, dt \right)^{\frac{1}{p-\beta}} + W^\mu_p(x_0, 2R) \right\}$;
\item $u(x_0, t_0) \leq \gamma \left\{ R^2 + \left( \frac{1}{R^{n}} \text{ess sup}_{0 < \tau < T} \int_{B_R} u_+ \, dx \right) + W^\mu_p(x_0, 2R) \right\}$.
\end{enumerate}

Estimate (i) is not homogeneous in $u$ which is usual for such type of equations [2, 4]. The proof of Theorem 1.1 is based on a suitable modifications of De Giorgi’s iteration technique [1] following the adaptation of Kilpeläinen-Malý technique [7] to parabolic equations with ideas from [11, 15].

The following test for local boundedness of solutions to (1.1) is an immediate consequence of Theorem 1.1.

**Corollary 1.2.** Let $u$ be a weak solution to equation (1.1). Let $\frac{2n}{n+1} < p < 2$. Assume that there exists $R > 0$ such that

\[ \sup_{x \in \Omega} W^\mu_p(x, R) < \infty. \]

Then $u \in L^\infty_{\text{loc}}(\Omega_T)$.

**Remark 1.3.**
1. The range of $p$ in the above results is optimal for the validity of the Harnack inequality (cf. [2]), however for weak solutions of the considered class it is plausible to conjecture that the results are valid for $\frac{2n}{n+2} < p < 2$, although it may require some additional global information as in [2, Chapter V].

2. Stationary solutions of (1.1) solve the corresponding elliptic equation of $p$-Laplace type with measure on the right hand side, and the Kilpeläinen-Malý upper bound is valid for them [7, 12]. The difference with our result in Theorem 1.1(ii) is in the additional term $R^2$ on the right hand side of the estimate. It is not clear yet whether this is a result of the employed technique or it lies in the essence of the problem, as even for the homogeneous structure ($\mu = 0$) a similar term is present in the estimates (cf. [7, Chapter V]).

The rest of the paper is organized as follows. In Section 2 we give auxiliary energy type estimates. Section 3 contains the proof of Theorem 1.1. In Section 4 we provide an application giving the global supremum estimate for the solution to a simple initial boundary problem.
2 Integral estimates of solutions

We start with some auxiliary integral estimates for the solutions of (1.1) which are formulated in the next lemma.

Define

\[ G(u) = \begin{cases} 
    u & \text{for } u > 1, \\
    u^{2-2\lambda} & \text{for } 0 < u \leq 1.
\end{cases} \tag{2.1} \]

Set

\[ Q^\delta_p(y,s) = B_p(y) \times (s - \delta^{2-p}p^s), \quad s + \delta^{2-p}p^s) \subset \Omega_T, \quad p \leq R. \]

Lemma 2.1. Let the conditions of Theorem [7] be fulfilled. Let \( u \) be a solution to (1.1). Let \( \xi \in C_0^\infty(Q^\delta_p(y,s)) \) be such that \( \xi(x,t) = 1 \) for \( (x,t) \in Q^\delta_p(y,s) \) and \( |\nabla \xi| \leq \frac{\gamma}{\delta}, \quad |\xi| \leq c\delta^{p-2}p^{-p} \) for some \( c > 0 \). Then there exists a constant \( \gamma > 0 \) depending only on \( n, p, c_1, c_2 \) such that for any \( l, \delta > 0 \), any cylinder \( Q^\delta_p(y,s) \) there holds

\[ \delta^2 \int_{L(t)} G \left( \frac{u(x,t) - l}{\delta} \right) \xi^k(x,t) dx + \int_{L} \left( 1 + \frac{u - l}{\delta} \right)^{-1+\lambda} \left( \frac{u - l}{\delta} \right)^{-2\lambda} |\nabla u|^p \xi(x,\tau) dx d\tau \]
\[ \leq \gamma \frac{\delta^p}{\rho^p} \int_{L} \left( \frac{u - l}{\delta} \right)^{k+1} dx d\tau + \gamma \frac{\delta^p}{\rho^p} \int_{L} \left( 1 + \frac{u - l}{\delta} \right)^{1-\lambda} \left( \frac{u - l}{\delta} \right)^{2\lambda} \xi^{k+1} dx d\tau \]
\[ + \gamma \frac{\rho^p}{\delta^p} L^2(B_p(y)), \tag{2.2} \]

where \( L = Q^\delta_p(y,s) \cap \{u > l\}, \quad L(t) = L \cap \{\tau = t\} \) and \( \lambda \in (0,1/2), \quad k > p \).

Proof. Further on, we assume that \( u_t \in L^2_{\text{loc}}(\Omega_T) \), since otherwise we can pass to Steklov averages. First, note that

\[ \int_{l}^{w} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds \leq \gamma \delta, \tag{2.3} \]

and

\[ \int_{l}^{w} dw \int_{l}^{w} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds = \int_{l}^{w} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} (u - s) ds \]
\[ \leq \frac{1}{2} (u - l) \int_{l}^{w} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \frac{s - l}{\delta} \right)^{-2\lambda} ds = \frac{\delta^2}{2} \left( \frac{u - l}{\delta} \right)^{2\lambda} \int_{0}^{\frac{w}{2l}} (1+z)^{-1+\lambda} \lambda_z^{-2\lambda} dz \]
\[ \leq \gamma \delta^2 G \left( \frac{u - l}{\delta} \right). \tag{2.4} \]

Let \( \eta \) be the standard mollifier in \( \mathbb{R}^n \) and as usual \( \eta_\sigma(x) = \eta(\frac{x}{\sigma}) \). Set \( u_\sigma = \eta_\sigma * u \). Let \( \varepsilon > 0 \).

Test (1.3) by \( \varphi \) defined by

\[ \varphi(x,t) = \left[ \int_{l}^{u_\sigma(x,t)} \left( 1 + \frac{s - l}{\delta} \right)^{-1+\lambda} \left( \varepsilon + \frac{s - l}{\delta} \right)^{-2\lambda} ds \right] \xi(x,t)^k, \tag{2.5} \]

and \( t_1 = s - \delta^{2-p}p^s, \quad t_2 = t \).

Using first (2.3) on the right hand side, then passing to the limit \( \sigma \to 0 \) on the left, after applying the Schwarz inequality we obtain for any \( t > 0 \)
By (2.4) we obtain the required (2.2).

Proof. Set \( \rho \). The next lemma is a direct consequence of Lemma 2.1. Let the conditions of Lemma 2.1 be fulfilled. Let \( \lambda < 0 \) and \( \xi \). We further assume that \( \int_{\Omega} \nabla\psi \cdot \xi \, dx \mathbf{1}_S \), and \( \psi \) is a characteristic function of the set \( \mathbf{1}_S \). Let \( \xi \in C_0^\infty(B_j) \), \( \mathbf{1}_{B_{j+1}} \leq \xi_j \leq \mathbf{1}_{B_j} \) with \( |\nabla\xi_j| \leq 2\rho_j^{-1} \), where here and below \( \mathbf{1}_S \) stands for the characteristic function of the set \( S \).

The sequences of positive numbers \( (l_j)_{j \in \mathbb{N}} \) and \( (\delta_j)_{j \in \mathbb{N}} \) are defined inductively as follows.

Set \( l_{-1} = 0 \), \( l_0 = \rho_j^2 \), \( \delta_{-1} = l_0 - l_{-1} = \rho_j^2 \). Let \( B \geq 1 \) be a number satisfying

\[
B^{2-p}(2R_0)^{\beta} \leq \min\{t_0, T - t_0\}
\]

which will be fixed later.

Let \( j \geq 1 \). Suppose we have already chosen the values \( l_i \), \( i = 1, 2, \ldots, j \) and \( \delta_i = l_{i+1} - l_i \) in such a way that \( \delta_i \leq B\rho_j^{-n} \), \( i = 0, 1, 2, \ldots, j - 1 \).

Let \( l \in (l_j + \rho_j^2, l_j + B\rho_j^{-n}) \), \( \delta_j(l) = l - l_j \).

Take the interval

\[
I_j = \left( t_0 - B^{2-p}\rho_j^\beta, t_0 + B^{2-p}\rho_j^\beta \right).
\]
Let us divide $I_j$ into the equal parts by the points $\tau_{j,m}^*$, $m = 1, 2, \ldots, M^*(j)$, in such a way that

\begin{equation}
\frac{1}{4} \delta_{j-1}^2 p_j \leq \tau_{j,m+1}^* - \tau_{j,m}^* \leq \frac{1}{2} \delta_{j-1}^2 p_j, \quad \text{and } M^*(j) \leq \gamma B^{2-p} p_j^{n(p-2)} \delta_{j-1}^{-2}.
\end{equation}

Let $\bar{\theta} \in C^\infty(\mathbb{R})$ be such that $0 \leq \bar{\theta} \leq 1$, $\bar{\theta}(s) = 0$ if $|s| \geq 1$, $\bar{\theta}(s) = 1$ if $|s| \leq 2^{1-p}$ and $|\bar{\theta}'(s)| \leq \gamma(p)$. Set

$$
\theta_{j,m}^*(t, l) = \bar{\theta} \left( \delta_j(l)^{p-2} p_j^{-p}(t - \tau_{j,m}^*) \right)
$$

Note that

$$
A_{j,m}^*(l) = \sum_{m=1}^{M^*(j)} \theta_{j,m}^*(t, l) \geq 1.
$$

For a fixed $j \geq 1$ and every $m = 1, 2, \ldots, M^*(j)$ we define

\begin{equation}
A_{j,m}(l) = \frac{\delta_j(l)^{p-2}}{p_j^{n(p-2)}} \int_{L_j} \frac{u - l_j}{\delta_j(l)} \xi_l(x)^k \rho_j \left[ \theta_{j,m}^*(t, l) \right]^{k-p} dx dt
\end{equation}

where

$$
L_j = \{ (x, t) \in \Omega_T : u(x, t) > l_j \}, \quad L_j(t) = \{ x \in \Omega : u(x, t) > l_j \},
$$

$G$ is defined in (2.11) and $k$ will be fixed later.

It follows from (3.3) that

\begin{equation}
A_{j,m}^*(l_j + B \rho_j^{-n}) \leq 3B^{-1} \sup_{0 < t < T} \int_{B_{R_0}} |u| dx \leq 3B^{-1} c_{R_0},
\end{equation}

where

$$
c_{R_0} = \sup_{0 < t < T} \int_{B_{R_0}} |u| dx.
$$

Fix a number $\kappa \in (0, 1)$ depending on $n, p, c_1, c_2$, which will be specified later. Choose $B$ such that

\begin{equation}
B^{-1} c_{R_0} \leq \frac{\kappa}{6}.
\end{equation}

This implies that $A_{j}^*(l_j + B \rho_j^{-n}) \leq \frac{\kappa}{2}$. Note that there exists a small enough $R_0$ such that (3.1) and (3.5) are consistent (with some $B \geq 1$).

If

\begin{equation}
A_j^*(l_j + \frac{\delta_j - 1}{2}) \leq \kappa,
\end{equation}

we set $\delta_j = \frac{\delta_j + 1}{2}$ and $l_{j+1} = l_j + \frac{\delta_j - 1}{2}$.

Note that $A_j^*(l)$ is continuous as a function of $l$. So if

\begin{equation}
A_j^*(l_j + \frac{\delta_j - 1}{2}) > \kappa,
\end{equation}

there exists $l \in (l_j + \frac{\delta_j - 1}{2}, l_j + B \rho_j^{-n})$ such that $A_j^*(l) = \kappa$. In this case we set $l_{j+1} = l$ and $\delta_j = l_{j+1} - l_j$ in both cases.

Note that our choices guarantee that

\begin{equation}
A_j(l_{j+1}) \leq \kappa.
\end{equation}
Next we construct the sequence \((\theta_{j,N}(t))_{N\in\mathbb{N}}\) of the cut-off functions.

Since \(\frac{1}{2}\delta_{j-1} \leq \delta_j \leq B\rho_j^{-n}\), we can choose a subset \(\{m(1), \ldots, m(M(j))\}\) of the set \(\{1, \ldots, M^*(j)\}\) such that

\[
(3.9) \quad \sum_{N=1}^{M(j)} \left[ \theta_{j,m(N)}^*(t,l_{j+1}) \right]^k \geq 1 \quad \text{for all } t \in (t_0 - B^2 - p\rho_j^\beta, t_0 + B^2 - p\rho_j^\beta),
\]

\[
(3.10) \quad \max_{1 \leq N \leq M(j)} A_{j,m(N)}^*(l_{j+1}) \leq \varepsilon \quad \text{if } l_{j+1} = l_j + \frac{\delta_{j-1}}{2},
\]

\[
(3.11) \quad \max_{1 \leq N \leq M(j)} A_{j,m(N)}^*(l_{j+1}) = \varepsilon \quad \text{if } l_{j+1} > l_j + \frac{\delta_{j-1}}{2}.
\]

Set \(\theta_{j,N}(t) = \theta_{j,m(N)}^*(t,l_{j+1})\) and \(A_{j,N}(l_{j+1}) = A_{j,m(N)}^*(l_{j+1})\) for \(1 \leq N \leq M(j)\). Then (3.9)–(3.11) imply that

\[
(3.12) \quad A_j(l_{j+1}) = \max_{1 \leq N \leq M(j)} A_{j,N}(l_{j+1}) \leq \varepsilon \quad \text{if } l_{j+1} = l_j + \frac{\delta_{j-1}}{2},
\]

\[
(3.13) \quad A_j(l_{j+1}) = \max_{1 \leq N \leq M(j)} A_{j,N}(l_{j+1}) = \varepsilon \quad \text{if } l_{j+1} > l_j + \frac{\delta_{j-1}}{2},
\]

\[
(3.14) \quad \sum_{N=1}^{M(j)} \theta_{j,m(N)}^*(t,l_{j+1})^k \geq 1 \quad \text{for all } t \in (t_0 - B^2 - p\rho_j^\beta, t_0 + B^2 - p\rho_j^\beta).
\]

For a fixed \(N\) such that \(1 \leq N \leq M(j)\) using (3.14) define the finite sequence \(N(i), i = 1, 2, \ldots, i(N,j), 1 \leq N(i) \leq M(j-1)\) so that \(k - p > 1\) and there exists \(\gamma > 0\) such that

\[
(3.15) \quad \sum_{i=1}^{i(N,j)} \theta_{j-1,N(i)}(t)^k \geq \theta_{j,N}(t), \quad i(N,j) \leq \gamma \left( \frac{\delta_j}{\delta_{j-1}} \right)^{2-p}.
\]

### 3.2 Main lemma

The following lemma is a key in the Kilpeläinen-Malý technique [7].

**Lemma 3.1.** Let the conditions of Theorem 1.1 be fulfilled. There exists \(\gamma > 0\) depending on the data, such that for all \(j \geq 1\) we have

\[
(3.16) \quad \delta_j \leq \frac{1}{2} \delta_{j-1} + \gamma \rho_j^\beta + \gamma \left( \frac{1}{\rho_j^{p-n}} \mu(B_j) \right)^{\frac{1}{p-n}}.
\]

**Proof.** Fix \(j \geq 1\). Without loss assume that

\[
(3.17) \quad \delta_j > \frac{1}{2} \delta_{j-1}, \quad \delta_j > \rho_j^\beta,
\]

since otherwise (3.10) is evident. The second inequality in (3.17) guarantees that \(A_j(l_{j+1}) = \varepsilon\).

Next we claim that under conditions (3.17) there is a \(\gamma > 0\) such that

\[
(3.18) \quad \delta_j^p \rho_j^{-(p+n)} \int_{L_j} \xi \theta_{j,N} dx \, d\tau \leq \gamma \varepsilon.
\]

Indeed, for \((x,t) \in L_j\) one has

\[
(3.19) \quad \frac{u(x,t) - l_{j-1}}{\delta_{j-1}} = 1 + \frac{u(x,t) - l_j}{\delta_{j-1}} \geq 1.
\]
where

\[ H \leq \delta_j^{-p-2} \rho_j^{-p} \left( \int_{L_j} \left( \frac{u - l_j}{\delta_j} \right) \xi_j^{k-p} \theta_j^{k-p} \right) \]

and

\[ L_j = L_j^{p-1} \]

which proves the claim.

Recall that

\[ A_j = \frac{\delta_j^{-p-2}}{\rho_j^{p+q}} \left( \int_{L_j} \left( \frac{u - l_j}{\delta_j} \right) \xi_j^{k-p} \theta_j^{k-p} \right) \]

(3.20)

\[ + \text{ess sup} \int \frac{1}{\rho_j p} \left( \int_{u > l_j} G \left( \frac{u - l_j}{\delta_j} \right) \xi_j^{k-p} \theta_j^{k-p} \right) dx dt. \]

Let us estimate the terms in the right hand side of (3.20). For this we decompose \( L_j \) as \( L_j = L_j' \cup L_j'' \),

\[ L_j' = \left\{ (x, t) \in (x, t) \in L_j : \frac{u(x, t) - l_j}{\delta_j} < \varepsilon \right\}, \quad L_j'' = L_j \setminus L_j', \]

where \( \varepsilon \in (0, 1) \) depending on \( n, p, c_1, c_2 \) is small enough to be determined later. By (3.18) we have

\[ \frac{\delta_j^{-p-2}}{\rho_j^{p+q}} \int_{L_j'} \left( \frac{u - l_j}{\delta_j} \right) \xi_j^{k-p} \theta_j^{k-p} dxd\tau \]

\[ \leq \frac{\delta_j^{-p-2}}{\rho_j^{p+q}} \int \xi_j^{k-p} \theta_j^{k-p} dxd\tau \]

(3.22)

\[ \leq \varepsilon \frac{\delta_j^{-p-2}}{\rho_j^{p+q}} \sum_{i=1}^{i(N, j)} \left( \int_{L_{j-1}} \left( \frac{u - l_{j-1}}{\delta_{j-1}} \right) \xi_j^{k-p} \theta_j^{k-p} \right) \]

\[ \leq \varepsilon \frac{\delta_j^{-p-2}}{\rho_j^{p+q}} \left( \int_{L_{j-1}} \left( \frac{u - l_{j-1}}{\delta_{j-1}} \right) \xi_j^{k-p} \theta_j^{k-p} \right) \]

\[ \leq \varepsilon \left( \frac{\delta_{j-1}}{\delta_j} \right)^{2-p} \left( \int_{L_{j-1}} \left( \frac{u - l_{j-1}}{\delta_{j-1}} \right) \xi_j^{k-p} \theta_j^{k-p} \right) \]

\[ \leq \varepsilon \left( \frac{\delta_{j-1}}{\delta_j} \right)^{2-p} \left( \int_{L_{j-1}} \left( \frac{u - l_{j-1}}{\delta_{j-1}} \right) \xi_j^{k-p} \theta_j^{k-p} \right) \]

(3.23)

\[ \psi_j(x, t) = \frac{1}{\delta_j} \left( \int_{l_j}^{u(x, t)} \left( 1 + \frac{s - l_j}{\delta_j} \right)^{-\frac{p}{p-1}} \left( \frac{s - l_j}{\delta_j} \right)^{-\frac{p}{p-1}} ds \right) \]

and

\[ \rho(\lambda) = \frac{p}{p-1-\lambda}. \]

The following inequalities are easy to verify

\[ c \psi_j(x, t) \rho(\lambda) \leq \frac{u(x, t) - l_j}{\delta_j} \text{ for } (x, t) \in L_j, \text{ and} \]

\[ \frac{u(x, t) - l_j}{\delta_j} \leq c(\varepsilon) \psi_j(x, t) \rho(\lambda), \quad (x, t) \in L_j''. \]

Hence

\[ \frac{\delta_j^{-p-2}}{\rho_j^{p+q}} \int_{L_j} \left( \frac{u - l_j}{\delta_j} \right) \xi_j^{k-p} \theta_j^{k-p} dxd\tau \leq \gamma(\varepsilon) \frac{\delta_j^{-p-2}}{\rho_j^{p+q}} \int_{L_j''} \psi_j^\rho(\lambda) \xi_j^{k-p} \theta_j^{k-p} dxd\tau. \]
The integral in the second terms of the right hand side of (3.26) is estimated by using the Gagliardo–Nirenberg inequality in the form [9, Chapter II, Theorem 2.2] as follows

\[ \gamma \left( \frac{\delta^{p-2}}{\rho_j^{p+n}} \right) \int_{L_j''} \psi_j^{p+k_p} \xi_j^{k-p} \theta_{j,N} dx dt \]

\[ \leq \gamma \left( \sup_t \frac{1}{\rho_j} \int_{L_j(t)} \psi_j^{p+\alpha} \xi_j^{\theta_{j,N}} dx \right)^\frac{\nu_p}{\alpha} \left( \frac{1}{\rho_j^\alpha} \int \left| \nabla \left( \psi_j \xi_j^{\frac{\theta_{j,N}}{\theta_{j,N}+\theta_{j,N}^{(k_p+\alpha)}}} \theta_{j,N}^{(k_p+n)} \right) \right|^p dx dt \right). \]

(3.27)

Let us estimate separately the first factor in the right hand side of (3.27),

\[ \sup_t \frac{1}{\rho_j} \int_{L_j(t)} \psi_j^{p+\alpha} \xi_j^{\theta_{j,N}} dx \]

by (3.17)

\[ \leq c^{-\frac{1}{\nu_p}} \int_{L_j(t)} u - l_j^{\alpha} \xi_j^{\theta_{j,N}} dx \]

\[ \leq c^{-\frac{1}{\nu_p}} \int_{L_j(t)} G \left( \frac{u - l_j^{\alpha}}{\delta_j} \right) \xi_j^{\theta_{j,N}} dx \]

by (3.19) and (3.20)

\[ \leq c^{-\frac{1}{\nu_p}} \int_{L_j(t)} G \left( \frac{u - l_j^{\alpha}}{\delta_j} \right) \xi_j^{\theta_{j,N}} dx \]

by (3.21)

\[ \leq 2^n c^{-1} \left( \frac{\delta_j}{\delta_j} \right)^{p-1} A_j \leq \gamma \alpha. \]

(3.28)

Combining (3.26), (3.27) and (3.28) we obtain

\[ \delta_j^{p-2} \rho_j^{-n} \int_{L_j''} \left( \frac{u - l_j^{\alpha}}{\delta_j} \right) \xi_j^{k-p} \theta_{j,N} dx dt \]

\[ \leq \gamma (\varepsilon) \left( \frac{\delta_j}{\rho_j} \right)^{p-2} \rho_j^{-n} \int \left| \nabla \left( \psi_j \xi_j^{\frac{\theta_{j,N}}{\theta_{j,N}+\theta_{j,N}^{(k_p+n)}}} \theta_{j,N}^{(k_p+n)} \right) \right|^p dx dt. \]

(3.29)

For the last term in the above inequality we estimate by (3.18) and (3.21)

\[ \delta_j^{p-2} \rho_j^{-n} \int_{L_j} \psi_j^p |\nabla \xi_j|^p \theta_{j,N} dx d\tau \leq \delta_j^{p-2} \rho_j^{-n-p} \int_{L_j} \psi_j^p \xi_j^{\theta_{j,N}} \sum_{i=1}^{(k_p+n)} \theta_{j-1} \theta_{j-1,N(i)} dx d\tau \]

\[ \leq \gamma \delta_j^{p-2} \rho_j^{-n-p} \max_{1 \leq i \leq M(j-1)} \int_{L_j} \left( \frac{u - l_j^{\alpha}}{\delta_j} \right)^{p-\lambda} \xi_j^{k-p} \theta_{j-1} \theta_{j-1,N(i)} dx d\tau \]

by (3.11)

\[ \leq \gamma \delta_j^{p-2} \rho_j^{-n-p} \max_{1 \leq i \leq M(j-1)} \int_{L_j} \left( \frac{u - l_j^{\alpha}}{\delta_j} \right)^{p-\lambda} \xi_j^{k-p} \theta_{j-1} \theta_{j-1,N(i)} dx d\tau \]

\[ \leq \gamma A_j \leq \gamma \alpha. \]

By Lemma 2.2

\[ \frac{1}{\rho_j^\alpha} \int_{L_j} G \left( \frac{u - l_j^{\alpha}}{\delta_j} \right) \xi_j^{k-p} \theta_{j,N} dx d\tau \]

\[ \leq \gamma \delta_j^{p-2} \rho_j^{-n} \int_{L_j} \left( 1 + \frac{u - l_j^{\alpha}}{\delta_j} \right)^{1-2\lambda(p-1)} \left( \frac{u - l_j^{\alpha}}{\delta_j} \right)^{2\lambda(p-1)} \xi_j^{k-p} \theta_{j,N} dx d\tau \]

\[ + \gamma \frac{\rho_j^{-n}}{\delta_j^{p-1}} \mu(B_{\rho_j}(y)). \]

(3.31)
Using the decomposition \((3.21)\) and the first inequality in \((3.17)\) we have
\[
\delta_j^{p-2} \rho_j^{-(n+p)} \int_{L_j} \left( \frac{u - l_j}{\delta_j} \right)^{1-2\lambda(p-1)} \left( \frac{u - l_j}{\delta_j} \right)^{2\lambda(p-1)} \xi_j^{k-p} \theta_j^{k-p} dx d\tau
\leq \gamma_1\varepsilon^{2\lambda(p-1)} \delta_j^{p-2} \rho_j^{-(n+p)} \int_{L_j} \xi_j^{\theta_j, N} dx d\tau
+ \gamma(\varepsilon) \delta_j^{p-2} \rho_j^{-(n+p)} \max_{1 \leq i \leq M(j-1)} \int_{L_{j-1}} \left( \frac{u - l_{j-1}}{\delta_{j-1}} \right)^{2\lambda(p-1)} \xi_{j-1}^{k-p} \theta_{j-1, N} dx d\tau
\leq \gamma_1\varepsilon^{2\lambda(p-1)} \kappa + \gamma(\varepsilon) \kappa.
\]
(3.32)

Thus we obtain the following estimate for the first term of \(A_j(l_{j+1})\):
\[
\delta_j^{p-2} \rho_j^{-(n+p)} \int_{L_j} \left( \frac{u - l_j}{\delta_j} \right) \xi_j(x)^{k-p}[\theta_j, N(t)]^{k-p} dx d\tau
\leq \gamma_1\varepsilon^{2\lambda(p-1)} \kappa + \gamma(\varepsilon) \kappa \left( \kappa + \delta_j^{1-p} \rho_j^{p-n} \mu(B_j) \right).
\]
(3.33)

Let us estimate the second term in the right hand side of \((3.32)\). By \((3.31)\) we have
\[
\sup_t \rho_j^{n} \int_{L_j(t)} G \left( \frac{u - l_j}{\delta_j} \right) \xi_j^{\theta_j, N} dx
\leq \delta_j^{p-2} \rho_j^{-(n+p)} \int_{L_j} \left( 1 + \frac{u - l_j}{\delta_j} \right)^{1-2\lambda(p-1)} \left( \frac{u - l_j}{\delta_j} \right)^{2\lambda(p-1)} \xi_j^{k-p} \theta_j^{k-p} dx d\tau + \gamma \delta_j^{1-p} \rho_j^{p-n} \mu(B_j)
\]
(by using the decomposition \((3.21)\) and \((3.31)\))
\[
(3.34) \leq \gamma_1\varepsilon^{2\lambda(p-1)} \kappa + \gamma(\varepsilon) \kappa \left( \kappa + \delta_j^{1-p} \rho_j^{p-n} \mu(B_j) \right) + \gamma \delta_j^{1-p} \rho_j^{p-n} \mu(B_j).
\]

Combining \((3.32)\) and \((3.34)\) and choosing \(\varepsilon\) appropriately we can find \(\gamma_1\) and \(\gamma\) such that
\[
\kappa \leq \gamma_1 \kappa \left( \kappa + \delta_j^{1-p} \rho_j^{p-n} \mu(B_j) \right) + \gamma \delta_j^{1-p} \rho_j^{p-n} \mu(B_j).
\]
\[
(3.35) \leq \gamma_1 \kappa \left( \kappa + \delta_j^{1-p} \rho_j^{p-n} \mu(B_j) \right) + \gamma \delta_j^{1-p} \rho_j^{p-n} \mu(B_j).
\]

Now choosing \(\kappa < 1\) such that
\[
\kappa = \frac{1}{2\gamma_1}
\]
we have
\[
(3.36) \quad \delta_j \leq \gamma \left( \rho_j^{p-n} \mu(B_j) \right)^{\frac{1}{2\gamma_1}}
\]
which completes the proof of the lemma. \(\Box\)

In order to complete the proof of Theorem \(3.1\) we sum up \((3.10)\) with respect to \(j\) from 1 to \(J - 1\)
\[
l_j \leq \gamma \delta_0 + \gamma \sum_{j=1}^{\infty} \delta_j^2 + \gamma \sum_{j=1}^{\infty} \left( \rho_j^{p-n} \mu(B_j) \right)^{\frac{1}{2\gamma_1}}
\]
\[
(3.38) \leq \gamma (\delta_0 + R^2 + W_p^\mu(x_0, R)).
\]

It remains to estimate \(\delta_0\). There are two cases to consider. Either \(\delta_0 = \frac{1}{2} \rho_0^2 = \frac{1}{2} R^2\), or \(l_1\) and \(\delta_0\) are defined by \(A_0^* (l_1) = \kappa\) with \(\kappa\) fixed in the proof of Lemma \(3.1\) by \((3.36)\). It follows that there exists \(m\) such that \(A_0^* (l_1) = \kappa\).

Using the decomposition \((3.21)\) with \(\varepsilon\) chosen via \(\kappa\), and Lemma \(2.2\) together with \((3.32)\) one can see that
\[
\sup_t R^{-n} \int_{\{u > l_0\}} G \left( \frac{u - l_0}{\delta_0} \right) \xi_0(x) \theta_{0, m}(t) dx \leq \kappa / 2 + \gamma \delta_0^{p-2} \int_{\{u > l_0\}} \left( \frac{u - l_0}{\delta_0} \right) dx d\tau
+ \gamma \delta_0^{1-p} R^{n-p} \mu(B_R).
\]
Then by (3.20) we conclude that either

\begin{equation}
\delta_0^{p-2} \frac{R^{n+p}}{R^n} \int_{u \geq l_0} \left( \frac{u - l_0}{\delta_0} \right) \xi_0(x) \rho \theta_{0,m}(t) \xi^{k-p} dx dt \geq \gamma,
\end{equation}

or else

\begin{equation}
\delta_0^{p-1} \leq \gamma R^{-n} \mu(B_R).
\end{equation}

In case of (3.39) we obtain

\begin{equation}
\delta_0^{3-p} \leq \gamma \int u_+ \xi_0 \theta_{0,m} dx dt.
\end{equation}

Combining this with the first case and choosing $R_0$ such that $B^{2-p} R_0^3 \leq \min\{t_0, T - t_0\}$ we have

\begin{equation}
\delta_0 \leq \gamma \left\{ \left( \frac{1}{R^{p+n}} \int_{B_R \times (t_0 - B^{2-p} R^3, t_0 + B^{2-p} R^3)} u_+ dx dt \right)^{\frac{1}{p-1}} + R^2 \left( \frac{1}{R^{n-p}} \mu(B_R) \right)^{\frac{1}{p-1}} \right\}.
\end{equation}

Hence the sequence $(l_j)_{j \in \mathbb{N}}$ is convergent, and $\delta_j \rightarrow 0$ ($j \rightarrow \infty$), and we can pass to the limit $J \rightarrow \infty$ in (3.38). Let $l = \lim_{j \rightarrow \infty} l_j$. From (3.38) we conclude that

\begin{equation}
\frac{1}{R^{p+n}} \int_{B_{R_j} \times (t_0 - B^{2-p} R_j^3, t_0 + B^{2-p} R_j^3)} (u - l)_+ \leq \gamma \kappa \delta_j^{3-p} \rightarrow 0 \quad (j \rightarrow \infty).
\end{equation}

Choosing $(x_0, t_0)$ as a Lebesgue point of the function $(u - l)_+$ we conclude that $u(x_0, t_0) \leq l$ and hence $u(x_0, t_0)$ is estimated from above by

\begin{equation}
u(x_0, t_0) \leq \gamma \left\{ R^2 + \left( \frac{1}{R^{p+n}} \int_{B_R \times (t_0 - B^{2-p} R^3, t_0 + B^{2-p} R^3)} u_+ dx dt \right)^{\frac{1}{p-1}} + W^\mu_p(x_0, 2R) \right\}.
\end{equation}

Applicability of the Lebesgue differentiation theorem follows from [6]. Chap. II, Sec. 3. This completes the proof of the first assertion of Theorem [11]. Estimate (ii) is immediate consequence of (3.44). □

4 Example of application

In this section we give an application of our main result, Theorem [11], to the weak solution of the following model initial boundary value problem.

\begin{equation}
\begin{aligned}
&u_t - \Delta u = \mu, \quad (x, t) \in Q = B_R \times (0, T), \\
u(x, t) = 0, \quad (x, t) \in \partial Q = \partial B_R \times (0, T), \\
u(x, 0) = 0, \quad x \in B_R,
\end{aligned}
\end{equation}

where $\mu$ is a (positive) Radon measure on $B_R$.

Before formulating the result we need to clarify what we understand by the weak solution to the initial boundary value problem (4.1). We assume that

\begin{equation}
\begin{aligned}
u \in C([0, T]; L^2(B_R)) \cap L^p((0, T); W^{1,p}(B_R)) \quad \text{and} \quad u(t, \cdot) \rightarrow 0 \quad \text{in} \quad L^2(B_R) \quad \text{as} \quad t \rightarrow 0.
\end{aligned}
\end{equation}

**Proposition 4.1.** Let $u$ be the weak solution to problem (4.1). If $T \geq 4^{\frac{1}{2-p}} R^{\frac{n+1}{p-1}} \mu(B_R)$ then

\begin{equation}
\begin{aligned}
\text{ess sup}_{B_R \times (T/4, 3T/4)} u(x, t) \leq \gamma \left\{ \left( \frac{T}{R^3} \right)^{\frac{1}{p-1}} + R^2 + \sup_{B_R} W^\mu_p(x, 2R) \right\}.
\end{aligned}
\end{equation}

Otherwise,

\begin{equation}
\begin{aligned}
\text{ess sup}_{B_R \times (T/4, 3T/4)} u(x, t) \leq \gamma \left\{ \left( \frac{R^p}{T} \right)^{\frac{1}{p-1}} \left( \sup_{B_R} W^\mu_p(x, 2R) \right)^{\frac{p-1}{p}} + R^2 + \sup_{B_R} W^\mu_p(x, 2R) \right\}.
\end{aligned}
\end{equation}
Proof. We start with a proof of the following inequality

\[
\sup_{0 < t < T} \int_{B_R \times \{t\}} |u| \, dx \leq T \mu(B_R).
\]

As in the proof of Lemma 2.1 we need the test function for (4.1) to be continuous with respect to the spatial variable to make it \(\mu\)-measurable. It is clear that \(u\) can be approximated by the functions \(u_n\), which are step functions with respect to \(t\) on \((0, T)\) with values in \(C_0^\infty(B_R)\). Without loss \(u_n \to u\) pointwise almost everywhere and in \(L^p((0, T); W^{1,p}(B_R))\). Now, by \(v_h\) we will denote the symmetric Steklov average \(v_h(x, t) = \frac{1}{2h} \int_{t-h}^{t+h} v(x, \tau) \, d\tau\).

Taking \(t_1 > 0\) and \(t_2 = t \leq T\) in the integral identity (1.5), testing it with \(\varphi = \frac{(u_h)_h}{||(u_h)_h|| + \varepsilon}_h\), \(\varepsilon > 0\), and noting that \(|\varphi| \leq 1\) we obtain

\[
\int_{t_1}^{t_2} \int_{B_R} (u_h)_h \frac{(u_h)_h}{||(u_h)_h|| + \varepsilon} \, dx \, d\tau + \int_{t_1}^{t_2} \int_{B_R} (\nabla u |\nabla u|^{p-2})_h \frac{\varepsilon \nabla (u_h)_h}{||(u_h)_h|| + \varepsilon} \, dx \, d\tau \leq (t - t_1) \mu(B_R).
\]

In the above inequality we first pass to the limit \(n \to \infty\). Next passing to the limit \(h \to 0\) we obtain

\[
\int_{B_R} (|u(x, t)| - |u(x, t_1)|) \, dx - \varepsilon \int_{B_R} \ln \frac{|u(x, t)| + \varepsilon}{|u(x, t_1)| + \varepsilon} \, dx + \varepsilon \int_{t_1}^{t_2} \int_{B_R} \frac{|\nabla u|^p}{||u||^2} \, dx \, d\tau \leq (t - t_1) \mu(B_R).
\]

Passing to the limit \(\varepsilon \to 0\) in the above inequality and then \(t_1 \to 0\) we prove (4.4).

Next, in the proof of Theorem 1.1 we choose \(B = T \mu(B_R)\).

Let \(\sigma \in (0, 1]\). We divide the cylinder \(Q\) into the union of the cylinders

\[
Q_{\sigma R}(\bar{x}, \bar{t}) = B_{\sigma R}(\bar{x}) \times (\bar{t} - B^{2-p}R^3, \bar{t} + B^{2-p}R^3).
\]

We are going to apply estimate (ii) of Theorem 1.1 for the points \((\bar{x}, \bar{t})\) in the form

\[
u(\bar{x}, \bar{t}) \leq \gamma \left\{ R^2 + \left( \frac{1}{R^n} \text{ess sup}_{0 < t < T} \int_{B_R} u_{\gamma} \, dx \right) \right\}.
\]

For this \(\sigma\) has to satisfy the condition

\[
B^{2-p}R^3 \leq \frac{T}{4}.
\]

So we choose \(\sigma = \min \left\{ 1, \left(\frac{1}{4}\right) \frac{R}{T} - \frac{1}{T} \mu(B_R) \right\} \).

In case \(T \geq \frac{1}{4} \frac{R}{T} - \frac{1}{T} \mu(B_R) \) we obtain (4.2) from (4.5) and (4.4). In the opposite case we have

\[
u(\bar{x}, \bar{t}) \leq \gamma (\sigma R)^{-n} \int_{B_{\sigma R}(\bar{x})} |u| \, dx + \gamma W_p^\mu(\bar{x}, 2R) + \gamma R^2
\]

\[
\leq \gamma (\sigma R)^{-n} T \mu(B_R) + \gamma W_p^\mu(\bar{x}, 2R) + \gamma R^2
\]

\[
\leq \left( \frac{1}{4} \right)^{-\frac{n}{p}} T \cdot \frac{\gamma (\sigma R)^{-n} \mu(B_R)}{1 - \frac{n}{p}} + \gamma W_p^\mu(\bar{x}, 2R) + \gamma R^2.
\]

Taking supremum over the cylinder \(B_R \times (1/4T, 3/4T)\) we obtain (4.3).

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