Global Convergence of Policy Gradient Primal–Dual Methods for Risk-Constrained LQRs

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Abstract—While the techniques in optimal control theory are often model-based, the policy optimization (PO) approach directly optimizes the performance metric of interest. Even though it has been an essential approach for reinforcement learning problems, there is little theoretical understanding of its performance. In this article, we focus on the risk-constrained linear quadratic regulator problem via the PO approach, which requires addressing a challenging nonconvex constrained optimization problem. To solve it, we first build on our earlier result that an optimal policy has a time-invariant affine structure to show that the associated Lagrangian function is coercive, locally gradient dominated, and has a local Lipschitz continuous gradient, based on which we establish strong duality. Then, we design policy gradient primal–dual methods with global convergence guarantees in both model-based and sample-based settings. Finally, we use samples of system trajectories in simulations to validate our methods.

Index Terms—Gradient descent, policy optimization (PO), reinforcement learning, risk-constrained linear quadratic regulator (RC-LQR), stochastic control.

I. INTRODUCTION

The techniques in conventional optimal control theory often require an explicit dynamical model. Such a model-based idea is relatively easy to provide theoretical guarantees but is usually sensitive to modeling inaccuracy. The policy optimization (PO) method, as an end-to-end approach, directly searches for an optimal control policy to minimize a performance metric of interest and has advantages in scenarios where the dynamical model is complex and difficult to identify. In fact, it has been proved to be an essential approach for applications of reinforcement learning [1], [2], [3], [4], [5], e.g., robotic in-hand manipulation [4], [5].

However, there are only a few theoretical guarantees on PO methods as they often involve challenging nonconvex optimization problems. To study their convergence and sample complexities, there has recently been a resurgence interest in PO methods for classical control problems [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]. For example, the seminal work [6] studies the well-known linear quadratic regulator (LQR) problem via PO methods. Though an optimal policy can be simply parameterized by a gain matrix, the quadratic cost is nonconvex in the gain matrix space. A major contribution of [6] shows that the cost function is globally gradient dominated (aka Polyak–Lojasiewicz condition [17], [18]) with respect to (w.r.t.) the policy gain, which is indispensable to prove the global convergence of their PO methods.

Since the LQR problem only focuses on the quadratic regulation performance, the closed-loop system may be largely jeopardized by low-probability yet significant events, which is not allowed for safety-critical applications. To remedy it, risk-aware controllers have become natural choices [19], [20], [21], [22], [23], [24], [25], [26]. In [25], a finite-horizon LQR problem with a variance-like constraint was first proposed, which is then extended to the infinite-horizon version in our previous work [26]. While both are solved via the model-based dynamic programming (DP), this article studies the risk-constrained LQR (RC-LQR) problem of [26] under the PO framework in both model-based and sample-based settings. In fact, various constrained LQ problems have also been studied via PO methods, e.g., the linear exponential quadratic Gaussian (LEQG) [13] and distributed LQG [14].

In sharp contrast to those PO works [6], [7], [8], [9], [10], [11], [12], the nonconvex variance-like constraint results in a fundamentally different optimization landscape. In particular, we lack the global gradient dominance property. Thus, a natural question is whether there still exists a good PO method that yields a globally optimal policy for the infinite-horizon RC-LQR problem. We provide a positive answer in this article. As the finite-horizon version [25], an optimal policy has also been shown in [26] to have an affine structure in the form of $u^*(x) = -K^*x + b^*$ with a gain matrix $K^*$ and a vector $b^*$. We take this as a starting point, and propose here a novel primal–dual method
where the primal and dual iterations alternatively.compute an optimal policy-multiplier pair.

Even though the primal–dual method is conceptually simple, it is challenging to establish theoretical guarantees because of the following.

1) The optimization landscape of the Lagrangian function is yet unclear (in fact, we only obtain that the Lagrangian under a fixed multiplier is locally gradient dominated, meaning that there may exist multiple optimal policies for the Lagrangian).

2) The strong duality does not trivially hold in a nonconvex constrained optimization problem (note that the strong duality is the key to primal–dual methods and is usually established for convex problems [27]).

3) Exact gradients of both the Lagrangian and the dual function are unavailable.

Our main contribution here lies in satisfactorily addressing the above issues, and further showing that the Lagrangian is also coercive with a locally Lipschitz gradient, which along with local gradient dominance establishes the global convergence of our primal–dual method.

Clearly, the RC-LQR can be regarded as a special case of the long-studied constrained Markov decision problems (CMDPs) [28]. Strong duality for CMDPs has been proved, but only if the state–action space is finite [28] or the cost is uniformly bounded [29], neither of which holds in the RC-LQR of this article. To the best of our knowledge, we are the first to formally prove the strong duality for such a class of continuous CMDPs with quadratic costs.

Even though a similar policy gradient primal–dual framework has been adopted to solve continuous CMDPs in [29], [30], [31], [32], and [33], none of them can achieve global convergence. For example, the primal–dual methods in [29] and [30] have only been shown to converge to a neighborhood of the global optimum and can even lead to constraint violations. Even though it has been resolved in [31], [32], and [33], their optimization landscape lends them to resort to function approximations for optimal policies and thus can only achieve local convergence.

In comparison, an optimal policy of our RC-LQR problem has an exact affine structure in the state feedback. While for finite CMDPs, the primal–dual methods are relatively easy and can ensure the convergence to a globally optimal policy [34], [35], [36]. It is worth mentioning that there are also other PO-based works [37], [38], [39], [40] that do not follow a primal–dual framework, e.g., they leverage the interior-point method [39] and trust region method [40] to directly solve the constrained problem. Again, they still lack provable global convergence.

The rest of this article is organized as follows. In Section II, we formulate the infinite-horizon RC-LQR problem. In Section III, we approach it by proposing policy gradient primal–dual methods and recognizing the local gradient dominance property, based on which we prove the strong duality. In Sections IV and V, we propose primal–dual methods with convergence guarantees in model-based and sample-based settings, respectively. In Section VI, we conduct simulations to validate our theoretical results. Section VII concludes this article. Finally, five appendixes are presented.

II. PROBLEM FORMULATION

Consider a discrete-time linear time-invariant stochastic system

\[ x_{t+1} = Ax_t + Bu_t + w_t \]  

where \( x_t \in \mathbb{R}^n \) and \( u_t \in \mathbb{R}^m \) are the state and control vectors, respectively, and \( \{w_t\} \) is an independently and identically distributed noise sequence.

The infinite-horizon LQR problem aims to find a sequence of control policies \( \{\pi_t\} \) to minimize a time-average cost, i.e.,

\[
\min_{\pi_t} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R u_t) \right]
\]

subject to (1) and \( u_t = \pi_t(h_t, x_t) \) (2)

where \( h_t = \{x_0, u_0, \ldots, x_{t-1}, u_{t-1}\} \) is the system history trajectory. The expectation is taken over the statistics of the noise sequence \( \{w_t\} \). Throughout this article, we make the following standard assumption [41].

Assumption 1: \( Q \) is positive semidefinite and \( R \) is positive definite. The pair \((A, B)\) is controllable and \( (A, Q^{1/2}) \) is observable.

Under Assumption 1, solving (2) yields a unique optimal policy \( \pi_t^* (x_t, h_t) = -K x_t \) if the sequence \( \{w_t\} \) has zero mean.

Clearly, the LQR is risk-neutral as it only minimizes the quadratic cost, and the state may be substantially influenced by extreme noises, especially if \( w_t \) has a heavy-tailed distribution. To address it, the finite-horizon RC-LQR problem has been proposed in [25], which has been extended to the infinite-horizon case in our recent work [26] with the following form:

\[
\min_{\pi_t} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (x_t^\top Q x_t + u_t^\top R u_t) \right]
\]

subject to (1), \( u_t = \pi_t(h_t, x_t) \) and

\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (x_t^\top Q x_t - \mathbb{E}[x_t^\top Q x_t | h_t])^2 \right] \leq \rho \] (3)

where \( \rho > 0 \) is a user-defined constant to reflect our risk tolerance and \( w_t \) has a finite fourth-order moment for tractability.

Different from [25] and [26], however, in this article we re-solve the RC-LQR problem (3) via PO methods. Following the notations of [25], let the mean and covariance of stationary noise \( \{w_t\} \) be given by \( \bar{w} = \mathbb{E}[w_t] \) and \( W = \mathbb{E}[(w_t - \bar{w})(w_t - \bar{w})^\top] \geq 0 \), respectively. Define

\[ M_3 = \mathbb{E}[(w_t - \bar{w})(w_t - \bar{w})^\top Q(w_t - \bar{w})] \] (4)

\[ M_4 = \mathbb{E}[(w_t - \bar{w})^2 Q(w_t - \bar{w}) - \text{tr}(W Q)]^2. \] (5)

By [26, Th. 1], an optimal policy of (3) has a time-invariant affine structure, i.e., \( \pi_t^* (x_t, h_t) = -K^* x_t + l^* \), which also stabilizes the system (1) in the mean square sense. Thus, there is no loss of optimality to solve (3) by focusing on the parameterized policies in the form of \( u(x) = -K x + l \), leading to the following
optimization problem:

\[
\begin{align*}
\text{minimize } K,l & : J(K, l) := \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (x_t^T Q x_t + u_t^T R u_t) \right] \\
\text{subject to } & (1), u_t = -K x_t + l, \text{ and} \\
J_e(K, l) := \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (4 x_t^T Q W Q x_t + 4 x_t^T Q M_3) \right] \leq \bar{\rho}
\end{align*}
\]

with \( \bar{\rho} = \rho - m_4 + 4 \text{tr} \{(W Q)^2 \} \).

The PO method for the risk-neutral LQR in (2) is shown to be globally convergent by random search [11], [12] and policy gradient methods [6], [10]. However, the nonconvex constraint in (6) renders our problem much more involved, and we resort to the duality theory to establish global convergence.

III. PRIMAL–DUAL METHODS FOR THE RC-LQR

In this section, we solve the RC-LQR problem (6) via the primal–dual method. We first show that its Lagrangian function is coercive and locally gradient dominated. Then, we establish strong duality.

A. Overview of Our Policy Gradient Primal–Dual Method

Let \( X = [K, l] \) be the decision vector of (6) and define the set of stabilizing policy by

\[ S = \{ [K, l] | \rho(A - BK) < 1, K \in \mathbb{R}^{m \times n}, l \in \mathbb{R}^m \}. \]

Let \( \mu \geq 0 \) denote a Lagrange multiplier of (6), \( Q_\mu = Q + 4 \mu Q W Q \) and \( S = 2 \mu Q M_3 \). Then, the Lagrangian is given as

\[ L(X, \mu) = J(X) + \mu (J_e(X) - \bar{\rho}) \]

\[ = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} c_\mu(x_t, u_t) \right] \quad (7) \]

where

\[ c_\mu(x_t, u_t) = x_t^T Q_\mu x_t + 2 x_t^T S + u_t^T R u_t - \mu \bar{\rho} \quad (8) \]

is a reshaped cost with a nonnegative weight \( \mu \) to balance the quadratic cost and the risk. Define the dual function as

\[ D(\mu) = \min_{X \in S} L(X, \mu). \quad (9) \]

In the following, we refer to (6) as the primal problem and

\[ \max_{\mu \geq 0} D(\mu) \quad (10) \]

as its dual problem. Our primal–dual method is alternatively updated as

\[ X^k \in \arg\min_{X \in S} L(X, \mu^k) \quad (11a) \]

\[ \mu^{k+1} = [\mu^k + \zeta^k \cdot d^k]_+ \quad (11b) \]

where the stepsize \( \zeta^k > 0 \), \( d^k \) is a subgradient of \( D(\mu) \) at \( \mu^k \) and \( [x]_+ = \max\{0, x\} \) for any \( x \in \mathbb{R} \).

To achieve its global convergence, the strong duality property between the primal problem and the dual problem is essential. Since (6) is nonconvex, it does not trivially hold. Even though the Lagrangian in (7) is the LQR cost with a linear term, its nonconvex optimization landscape is yet unclear. Thus, computing the primal update in (11a) is itself challenging. In the rest of this section, we show the following.

1) \( L(X, \mu) \) is coercive over \( S \) and locally gradient dominated in Section III-B, which is key to establish that a critical point of (11a) is globally optimal.

2) \( L(X, \mu) \) and its gradient are locally Lipschitz in Section III-C, which implies a linear convergence rate of gradient methods for solving (11a).

3) The strong duality property indeed holds in Section III-D. Combining these results prove the global convergence of (11). Note that all the proofs on the properties of the Lagrangian are provided in Appendixes A and B.

B. Coercivity and Local Gradient Dominance of the Lagrangian

We first derive closed-form expressions for the Lagrangian and its gradient. For any \( X \in S \), the state of the system (1) has a stationary distribution, the mean \( \bar{x}_X \) and covariance \( \Sigma_K \) of which satisfy

\[ \bar{x}_X = (A - BK) \bar{x}_X + Bl + \bar{w} \quad (12) \]

\[ \Sigma_K = W + (A - BK) \Sigma_K (A - BK)^\top \quad (13) \]

Then, we define the value function under \( X \) associated with the reshaped cost \( c_\mu(x_t, u_t) \) as

\[ V_X(x) = \mathbb{E} \left[ \sum_{t=0}^\infty (c_\mu(x_t, u_t) - L(X, \mu)) \bigg | x_0 = x \right] \]

where \( \mathbb{E}[\cdot] \) takes expectation under a fixed policy \( X \in S \). Moreover, let \( P_K \geq 0 \) satisfy the following Lyapunov equation:

\[ P_K = Q_\mu + K^\top R K + (A - BK)^\top P_K (A - BK) \quad (14) \]

and define

\[ E_K = (R + B^\top P_K B)K - B^\top P_K A \]

\[ R_K = R + B^\top P_K B, \quad V = (I - (A - BK))^{-1}. \]

We show that \( V_X(x) \) is quadratic and provide a closed-form of \( L(X, \mu) \).

Lemma 1: For any \( X \in S \), it follows that

(i) \[ V_X(x) = x^\top P_K x + g_X x + z_X \]

(ii) \[ L(X, \mu) = \text{tr} \{ P_K \left( W + (Bl + \bar{w})(Bl + \bar{w})^\top \right) \} \]

\[ + g_X^\top (Bl + \bar{w}) + l^\top R l - \mu \bar{\rho} \]

where \( g_X = 2(-l^\top E_K + S^\top + \bar{w}^\top P_K (A - BK) V) \) and \( z_X \) is a constant irrespective of \( x \).

Moreover, the gradient of \( L(X, \mu) \) w.r.t. \( X \) is explicitly given in the following lemma.

Lemma 2: For any \( X \in S \), the gradient of \( L(X, \mu) \) in \( X \) is

\[ \nabla_X L(X, \mu) = 2[E_K, G_X] \Phi_X \quad (15) \]
where $G_X = R_K l + B^T P_K \bar{w} + \frac{1}{2} B^T g_X$ and $\Phi_X$ is an ergodic matrix

$$\Phi_X = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \begin{bmatrix} x_t \\ -1 \end{bmatrix} \begin{bmatrix} x_t \\ -1 \end{bmatrix}^T \right]$$

$$= \begin{bmatrix} \Sigma_K + \bar{x} X \bar{x}^T \\ -\bar{x} X \end{bmatrix} > 0.$$  \hfill (16)

Since $\Phi_X > 0$, letting $\nabla_X L(X, \mu) = 0$ yields a unique critical point

$$X^*(\mu) = [K_\mu, l_\mu]$$  \hfill (17)

with $K_\mu = R_{K_\mu}^{-1} B^T P_{K_\mu} A$ and $l_\mu = -R_{K_\mu}^{-1} B^T V^T (P_{K_\mu} \bar{w} + S)$. Now, we are ready to show two important properties of the Lagrangian.

**Lemma 3 (Coercivity):** Under a fixed $\mu > 0$, $L(X, \mu)$ is coercive in $X$ in the sense that $\lim_{X \to \partial S} L(X, \mu) = +\infty$, where $\partial S$ denotes the boundary of $S$, and has a compact $\alpha$-sublevel set

$$S_\alpha = \{X | L(X, \mu) \leq \alpha\}.$$  \hfill (18)

**Definition 1:** For a differentiable function $f(x) : \mathbb{R}^n \to \mathbb{R}$ with a finite global minimum $f^*$, it is gradient dominated over a set $X \subseteq \text{dom}(f)$ if

$$f(x) - f^* \leq \lambda_X \|\nabla f(x)\|^2 \quad \forall x \in X$$  \hfill (19)

If $X = \text{dom}(f)$, it reduces to the Polya–Lojasiewicz condition [17, 18], which is key to the global convergence of [6]. In this article, we can only show that (19) holds for some proper subset of $\text{dom}(f)$, and for distinction refer to them as the global and local gradient dominance, respectively.

For a given policy $X \in S$, define a truncated value function

$$V_X^T(x) = \mathbb{E} \left[ \sum_{t=0}^{T-1} (c_\mu(x_t, u_t) - L(X, \mu)) | x_0 = x \right]$$

and an advantage function $A_X^T(x, u) = c_\mu(x, u) - L(X, \mu) + \mathbb{E}[V_X^T(x_{t+1}) | x_t = x, u_t = u] - V_X^T(x)$. Then, the Lagrangian difference between the two stabilizing policies can be described by the advantage function.

**Lemma 4:** Let $\{x_t^i\}$ and $\{u_t^i\}$ be sequences generated by the stabilizing policy $X^i \in S$. For any $X \in S$, it follows that

(i) $L(X^i, \mu) - L(X, \mu) = -\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} A_X^T(x_t^i, u_t^i) \right]$

(ii) $\lim_{T \to \infty} A_X^T(x_t^i - K' x^i + l')$}

$$= ((K' - K)x - (l' - l) + R^{-1}_K (E_K x - G_X))^T \times R_K ((K' - K)x - (l' - l) + R^{-1}_K (E_K x - G_X)) - (E_K x - G_X)^T R^{-1}_K (E_K x - G_X)$$  \forall x \in \mathbb{R}^n.$$

Lemma 4 is consistent with [6, Lemma 10], though we focus on the ergodic cost here.

**Lemma 5:** For any $X \in S$, it holds that

$$L(X, \mu) - L^*(\mu) \leq \frac{\|\Phi^*\|}{4\sigma(\mathcal{R}) \sigma(\Phi_X)^2} \cdot \{\nabla_X L \triangledown_X L\}$$  \hfill (20)

where $\nabla_X L = \nabla_X L(X, \mu)$ is given in (15), $\Phi^*$ is the ergodic matrix (16) under $X^*(\mu)$ of (17), $\sigma(\cdot)$ returns the minimum eigenvalue of a positive definite matrix, and $L^*(\mu) = \min_{X \in S} L(X, \mu)$.

Since $\lim_{T \to \infty} \sigma(\Phi_X) = 0$ [cf., (12) and (16)], the coefficient on the right-hand side of (20) is unbounded, in contrast to the case of the LQR [6], where it is a finite constant, i.e., their quadratic cost is globally gradient dominated. The good news here is that it is also finite over the $\alpha$-sublevel set $S_\alpha$ in (18), as established below.

**Lemma 6 (Local gradient dominance):** For any $\mu > 0$, $L(X, \mu)$ is gradient dominated over its $\alpha$-sublevel set, i.e.,

$$L(X, \mu) - L^*(\mu) \leq \lambda_\alpha \cdot \{\nabla_X L \triangledown_X L\} \quad \forall X \in S_\alpha$$

where $\lambda_\alpha = \|\Phi^*\|/(4\sigma(\mathcal{R}) \cdot \sigma_\alpha)$ is a constant over $S_\alpha$ in (18) and $\sigma_\alpha = \min_{X \in S_\alpha} \sigma(\Phi_X) > 0$.

**Proof:** Since $\Phi_X > 0$ is continuous in $X$, then $\sigma(\Phi_X)$ can be lower bounded by a positive constant over the compact set $S_\alpha$. The result then follows.

Since Lemma 6 holds for any $\alpha > 0$, joint use of coercivity is sufficient for finding a global minimizer of (11a).

**Theorem 1:** For any $\mu > 0$, the critical point $X^*(\mu)$ in (17) is the unique global minimizer of $L(X, \mu)$.

**Proof:** It is straightforward from (17) and Lemma 6.

### C. Locally Lipschitz Gradient of the Lagrangian

For a fixed $\mu$, we show in this subsection that both $L(X, \mu)$ and its gradient are locally Lipschitz continuous.

**Lemma 7:** For any pair of stabilizing policies $X$ and $X'$, the gap of their Lagrangians is given as

$$L(X', \mu) - L(X, \mu)$$

$$= \text{tr}\{(K' - K)^T R_K (K' - K)(\Sigma_{K'} + \bar{x}' \bar{x}'^T)$$

$$+ 2(K' - K)^T E_K (\Sigma_{K'} + \bar{x}' \bar{x}'^T) - 2G_X^T (K' - K) \bar{x}'$$

$$- 2(l' - l)^T R_K (K' - K) \bar{x}' + (l' - l)^T R_K (l' - l)$$

$$- 2(l' - l)^T E_K \bar{x}' + 2(l' - l)^T G_X\}$$

where $\bar{x}'$ denotes $\bar{x}_{X'}$ in (12) for notational simplicity.

**Lemma 8 (Locally Lipschitz Lagrangian and gradient):** For any $X \in S$, there exist positive scalars $\{\xi_X, \beta_X, \gamma_X\}$ such that for any $X' \in S$ and $\|X' - X\| \leq \gamma_X$, it holds

$$|L(X', \mu) - L(X, \mu)| \leq \xi_X \|X' - X\|,$$

and

$$\|\nabla_X L(X', \mu) - \nabla_X L(X, \mu)\| \leq \beta_X \|X' - X\|.$$
the standard LQR cost. Then, Sections III-B and III-C recover the results in [6, Sec. 3].

D. Strong Duality

In this section, we show that the strong duality between the primal problem (6) and dual problem (10) holds.

\textbf{Lemma 9:} Both the policy \(X^*(\mu)\) in (17) and the constraint function \(J_c(X^*(\mu))\) are continuous over \(\mu \in [0, \infty)\).

\textbf{Proof:} For \(\mu \geq 0\) and \(X \in \mathcal{S}\), it follows from (14) that the Lyapunov equation yields a unique \(P_K > 0\), which jointly with (17) implies that \(X^*(\mu)\) is continuous in \(\mu \geq 0\). The continuity of \(J_c(X^*(\mu))\) can be established by using the arguments in [10, Lemma 3.6].

Note that the continuity in Lemma 9 is a strong result and usually lacks in the primal–dual framework. Particularly, it holds only if \(X^*(\mu)\) in (11) is unique, which is not the case for a general nonconvex optimization problem. We now formally prove the strong duality result under Slater’s condition, which essentially follows from [25, Th. 3] and [27, Ch. 6].

\textbf{Assumption 2 (Slater’s condition):} There exists a policy \(\bar{X} \in \mathcal{S}\) such that \(J_c(\bar{X}) < \bar{\rho}\).

\textbf{Theorem 2 (Strong duality):} Under Assumption 2, there is no duality gap between the primal problem (6) and the dual problem (10).

\textbf{Proof:} Define

\[
\mu^* = \inf\{\mu \geq 0 | J_c(X^*(\mu)) \leq \bar{\rho}\} \tag{21}
\]

where \(X^*(\mu) \in \text{argmin}_{X \in \mathcal{S}} \mathcal{L}(X, \mu)\). By [27, Prop. 6.1.5], it is sufficient to show that a) \(\mu^*\) is finite, and b) the policy-multiplier pair \((X^*, \mu^*)\) with \(X^* = X^*(\mu^*)\) satisfies the following optimality conditions:

\[
\mathcal{L}(X^*, \mu^*) = \min_{X \in \mathcal{S}} \mathcal{L}(X, \mu^*)
\]

\[
J_c(X^*) \leq \bar{\rho}
\]

\[
\mu^*(J_c(X^*) - \bar{\rho}) = 0. \tag{22}
\]

(a) By Assumption 2, there exists a constant \(a > 0\) such that \(J_c(\bar{X}) + a \leq \bar{\rho}\). We prove by contradiction and assume that for all \(\mu \geq 0\), \(J_c(X^*(\mu)) > \bar{\rho}\). Then,

\[
J_c(\bar{X}) \geq D(\mu) - \mu(J_c(\bar{X}) - \bar{\rho})
\]

\[
= J(X^*(\mu)) + \mu(J_c(X^*(\mu)) - J_c(\bar{X}))
\]

\[
\geq J(X^*(\mu)) + \mu(J_c(X^*(\mu)) - \bar{\rho} + a)
\]

\[
> J(X^*(\mu)) + \mu a.
\]

Letting \(\mu \to \infty\) implies that \(J(\bar{X}) \to \infty\), which contradicts Slater’s condition that \(\bar{X} \in \mathcal{S}\). Thus, \(\mu^*\) in (21) is finite.

(b) Clearly, we only need to verify that \(\mu^*(J_c(X^*) - \bar{\rho}) = 0\). If \(\mu^* = 0\), then it trivially holds. If \(\mu^* > 0\), it follows from (21) that \(J_c(X^*(0)) > \bar{\rho}\). Since \(\mu^*\) is finite, there must exist a \(\mu^* > 0\) such that \(J_c(X^*(\mu^*)) \leq \bar{\rho}\). The continuity of \(J_c(X^*(\mu))\) in Lemma 9 implies that \(J_c(X^*) = \bar{\rho}\).

IV. POLICY GRADIENT PRIMAL–DUAL ALGORITHM FOR THE MODEL-BASED SETTING

In the model-based setting, we assume that all the parameters in (1) are known and propose three gradient-based methods with linear convergence to solve (11a). Then, we develop a primal–dual method in the form of (11) with global convergence to solve (6).

A. Policy Gradient Methods for Solving (11a)

To solve (11a), we consider three widely used policy gradient methods [6], [7], [10]. Let \(X'\) be the one-step updated policy and \(\eta\) be the stepsize. The update rules are given by

\textbf{Policy Gradient (PG):} \(X' = X - \eta \nabla_X \mathcal{L}\)

\textbf{Natural PG (NPG):} \(X' = X - \eta R_K^{-1} \nabla_X \mathcal{L} \cdot \Phi_X^{-1}\)

\textbf{Gauss–Newton (GN):} \(X' = X - \eta R_K^{-1} \nabla_X \mathcal{L} \cdot \Phi_X^{-1}\)

where \(\nabla_X \mathcal{L}\) and \(\Phi_X\) can be computed via (15) and (16), respectively. The NPG update is related to the gradient over a Riemannian manifold, while the GN update is one type of quasi-Newton update.

For simplicity, we follow [6], [7], [10], [11], [12], [13], [14], and [15] to assume the access of an initial stabilizing policy \(X^{(0)} \in \mathcal{S}\). Note that this can be relaxed via the PO methods (see, e.g., [8], [9]).

The key to the linear convergence of (23) is to find an appropriate stepsize such that (23) yields a stabilizing \(X'\) and decreases the Lagrangian per iteration, which is formally stated below. Note that the proof is given in Appendix C.

\textbf{Theorem 3:} Define the compact sublevel set

\[
\mathcal{S}_0 = \{X | \mathcal{L}(X, \mu) \leq \mathcal{L}(X^{(0)}, \mu)\}
\]

and \(\sigma_0 = \min_{X \in \mathcal{S}_0} \mathcal{L}(\Phi_X)\). If \(X \in \mathcal{S}_0\) and \(\eta\) in (23) is appropriately selected, then there exists a finite \(\beta \in (0, 1)\) such that

\[
\mathcal{L}(X', \mu) - \mathcal{L}^*(\mu) \leq (1 - \beta) \mathcal{L}(X, \mu) - \mathcal{L}^*(\mu)\]

Moreover, a) \(0 < \eta \leq 1/2\) and \(\beta = 2\eta_0 \sigma_0 / \|\Phi^*\|\) for the GN update; b) \(0 < \eta \leq 1/(2R_K)\) and \(\beta = 2\eta_0 \sigma_0 (R) / \|\Phi^*\|\) for the NPG update; and c) \(\eta\) is a polynomial in problem parameters

\[\beta = 2\eta_0^2 \sigma_0 (R) / \|\Phi^*\|\] for the PG update.

We provide a comparison of the three methods of (23) in Table I. Since the NPG and GN updates use more information, e.g., \(\Phi_X\) and \(R_K\), they tend to use less conservative stepsizes and achieve better convergence rates. Even though the PG update is given in the simplest form in (23), the updates of \(K\) and \(l\) in GN

| Table I | COMPARISON OF THREE GRADIENT METHODS (I—BEST, III—WORST) |
|---------|---------------------------------------------------------|
| PG      | III | III | III |
| NPG     | II  | II  | II  |
| GN      | I   | I   | I   |

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of stepsizes \( \{\zeta^k\} \).
1: for \( k = 1, 2, \ldots \) do
2: Solve \( X^k = \arg\min_{X \in S} \mathcal{L}(X, \mu^k) \) via (23).
3: Compute a subgradient \( d^k \) by (24) and Lemma 10.
4: Update the multiplier by \( \mu^{k+1} = [\mu^k + \zeta^k \cdot d^k]_+ \).
5: end for

and NPG can be decoupled, e.g., the NPG is rewritten as

\[ K' = K - 2\eta G_K, \quad \lambda' = \lambda - 2\eta G_X \]

which can reduce the computational complexity per update. Nonetheless, their computational complexities are essentially the same as \( O(n^3) \). Interestingly, the GN update with stepsize \( \eta = 1/2 \) is equivalent to the policy iteration and achieves a superlinear convergence rate [10], which is also confirmed via a simulation in Section VI.

B. Model-Based Primal–Dual Algorithm

By duality theory [42, 43], a subgradient in (11b) is

\[ d^k = J_c(X^k) - \bar{\rho} \tag{24} \]

where \( X^k \) is given in (11a) and \( J_c(X^k) \) is computed by the following lemma.

Lemma 10: For a stabilizing policy \( X \in S \), we have

\[ J_c(X) = \text{tr}\{P_c(W + (BI + \bar{w})(BI + \bar{w})^\top)\} + g^\top(\bar{B}I + \bar{w}) \]

where \( P_c > 0 \) is a unique solution of the Lyapunov equation

\[ P_c = 4QWQ + (A - BK) \top P_c(A - BK) \]

and \( g^\top = 2((\bar{B}I + \bar{w})^\top P_c(A - BK) + 2M_c^\top)QV \).

Proof: The proof is similar to that of Lemma 1.

Our model-based primal–dual method is summarized in Algorithm 1. In general, the primal iteration will not converge to a feasible solution unless the subdifferential of the dual function is a singleton [27, 44]. Fortunately, Theorem 1 implies that \( X^k \) is the unique minimizer of \( \mathcal{L}(X, \mu^k) \). Since \( X^k \) is always able to stabilize the system, the subgradient (actual gradient) \( d^k \) and \( \mu^k \) are uniformly bounded. Jointly with the concavity of \( D(\mu) \), it follows from [26, Th. 3] that Algorithm 1 converges globally.

Theorem 4: Let \( \bar{\mu}^k = \frac{1}{k} \sum_{i=1}^k \mu^i \) and \( \zeta^k = O(k^{-1/2}) \). Under Assumption 2, Algorithm 1 yields

\[ D^* - D(\bar{\mu}^k) \leq O(k^{-1/2}) \]

where the maximum of the dual function \( D^* = \max_{\mu \geq 0} D(\mu) \) is finite.

Proof: It is similar to that of [26, Th. 3] and omitted for saving space.

By Lemma 10, a simple bisection method could be adopted to solve \( \mu^* \) in (21) for the model-based setting. However, it is unclear how to adopt it for the sample-based setting as the constraint function can only be randomly evaluated as well.

Algorithm 1: Model-Based Primal–Dual Algorithm for the RC-LQR.

**Input:** A randomly initialized multiplier \( \mu^1 \geq 0 \), and a set of stepsizes \( \{\zeta^k\} \).
1: for \( k = 1, 2, \ldots \) do
2: Solve \( X^k = \arg\min_{X \in S} \mathcal{L}(X, \mu^k) \) via (23).
3: Compute a subgradient \( d^k \) by (24) and Lemma 10.
4: Update the multiplier by \( \mu^{k+1} = [\mu^k + \zeta^k \cdot d^k]_+ \).
5: end for

V. POLICY GRADIENT PRIMAL–DUAL ALGORITHM FOR THE SAMPLE-BASED SETTING

If \( (A, B) \) in (1) is unknown, both \( \nabla_X \mathcal{L}(X, \mu) \) in (15) and \( d^k \) in (24) cannot be computed directly. In the sample-based setting, we estimate them via system trajectories and develop a sampled-based primal–dual algorithm with global convergence.

Specifically, assume that there is an oracle to return noisy values of \( \mathcal{L}(X, \mu) \) and \( J_c(X) \) viz.,

\[ \hat{\mathcal{L}}(X, \mu) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} c_t(x_t, u_t), \]

\[ \hat{J}_c(X) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} (4x_t^\top QWQx_t + 4x_t^\top QM_3) \tag{25} \]

where \( \{x_t\} \) and \( \{u_t\} \) denote the states and control inputs, respectively, of a sampled trajectory under the policy \( X \in S \). In practice, \( T \) is often selected to be finite as the resulted approximation error of (25) decreases exponentially to zero w.r.t. \( T \) [11].

A. Random Search for Solving (11a)

We adopt the random search of Algorithm 2 to estimate \( \nabla_X \mathcal{L} \) via the oracle (25). The smoothing radius \( r \) in step 4 is used to control its estimation error. Motivated by [11], we shall show that with a large probability, Algorithm 2 converges and \( \{X^{(i)}\} \) remains in the following compact sublevel set:

\[ S_{10} = \{X \mid \mathcal{L}(X, \mu) - \mathcal{L}(X^{(0)}, \mu) \leq 10\Delta_0\} \tag{26} \]

where \( \Delta_0 = \mathcal{L}(X(0), \mu) - \mathcal{L}(\mu) \).

Let

\[ \beta_0 = \sup_{X \in S_{10}} \beta_X, \quad \xi_0 = \sup_{X \in S_{10}} \xi_X, \quad \gamma_0 = \inf_{X \in S_{10}} \gamma_X \]

and for notational simplicity, let \( \lambda_0 = \lambda_{S_{10}} \) [cf., (19)] and \( \theta_0 = \min\{1/(2\beta_0), \xi_0/\gamma_0\} \). Moreover, we make the following assumption in the rest of this section.

Assumption 3: The noise sequence \( \{w_t\} \) is uniformly bounded, i.e., \( \|w_t\| \leq \nu \), where \( \nu \) is a positive constant.

Then, we define

\[ G_\infty = \sup_{X \in S_{10}} \|\nabla_X \mathcal{L}\|_2 \]

Algorithm 2: Random Search Algorithm for (11a).

**Input:** An initial policy \( X^{(0)} \in S \), the number of iterations \( N \), a smoothing radius \( r \), the stepsize \( \eta \), a multiplier \( \mu \).
1: for \( i = 0, 1, \ldots, N - 1 \) do
2: Sample \( U^{(i)} \in \mathbb{R}^{m \times n} \) uniformly from a unit sphere \( S \) and let \( \hat{X} = X^{(i)} + rU^{(i)} \).
3: Obtain a noisy Lagrangian \( \hat{\mathcal{L}}(\hat{X}, \hat{\mu}) \) from the oracle.
4: Compute a gradient estimate

\[ \hat{\nabla}_X \hat{\mathcal{L}} = \hat{\mathcal{L}}(\hat{X}, \hat{\mu}) \sum_{i=1}^n U^{(i)} \]

5: Update the policy by \( X^{(i+1)} = X^{(i)} - \eta \hat{\nabla}_X \hat{\mathcal{L}} \).
6: end for

**Output:** A policy \( X^{(N)} \).
Algorithm 3: Sample-Based Primal–Dual Algorithm for the RC-LQR.

Input: A multiplier $\mu^0 \geq 0$, and a set of stepsizes $\{\zeta^k\}$.

1. for $k = 1, 2, \ldots$ do
2. Solve (11a) by Algorithm 2 and obtain $\tilde{X}^k$.
3. Obtain a noisy $\tilde{J}_e(\tilde{X}^k)$ from the oracle.
4. Compute a subgradient estimate $\tilde{d}^k = \tilde{J}_e(\tilde{X}^k) - \tilde{\rho}$.
5. Update the multiplier by $\mu^{k+1} = [\mu^k + \zeta^k \cdot \tilde{d}^k]_+.
6. end for

$$G_2 = \sup_{X \in S_{10}} E_C [\|\nabla_X J - E_C [\nabla_X J|X]\|_2]$$ (27)

both of which are finite under Assumption 3.

Theorem 5: Suppose that the stepsize $\eta$ and the smoothing radius $r$ are chosen such that

$$\eta \leq \min \left\{ \frac{\epsilon}{2400 \alpha \beta G_2}, \frac{1}{2} \gamma_0 G_\infty \right\} \text{ and }$$

$$r \leq \min \left\{ \frac{\theta_0}{800 \alpha \beta \sqrt{15} \gamma_0 G_\infty}, \frac{1}{30}, \frac{1}{30} \gamma_0 \right\} .$$

For any error tolerance $\epsilon$ such that $\epsilon \log(120 \Delta_0 / \epsilon) < 10 \Delta_0 / 3$ and $\gamma_0 \geq 4 \lambda_0 \log(120 \Delta_0 / \epsilon)/\eta$, Algorithm 2 yields that

$$J(X^{(N)}, \mu) - J^*(\mu) \leq \epsilon$$ (28)

with a probability greater than $3/4$.

The proof is given in Appendix D. In view of [45], the convergence probability in Theorem 5 can be improved to $1 - \delta$ for any $0 < \delta < 1$ by focusing on $S_0 = \{ X | J(X, \mu) - J(X^{(0)}, \mu) \leq 10 \delta^3 \Delta_0 \}$.

B. Sample-Based Primal–Dual Algorithm

In this section, we let $\tilde{X}^k = X^{(N)}$ and assume that (28) holds for the sake of simplifying our presentation (see also, e.g., [9]). The oracle (25) is adopted to compute a subgradient estimate

$$\tilde{d}^k = \tilde{J}_e(\tilde{X}^k) - \tilde{\rho}$$

with the estimation error resulting from the oracle computation (25) and the gap between $\tilde{X}^k$ and $X^k$.

Now, we present our sample-based primal–dual method in Algorithm 3. Due to the use of biased subgradient estimate, we can obtain the global convergence to a value close to $D^*$.

Theorem 6: Let $\bar{\mu}^k = \frac{1}{k} \sum_{i=1}^k \mu^i$ and $\zeta^k = \text{poly}^{-1}(\epsilon, \nu) \cdot (k^{-1/2})$ in Algorithm 3, where poly$(\epsilon, \nu)$ is a polynomial of degree 4 and given in Appendix E. Under Assumptions 2 and 3, it holds that

$$D^* - D(\bar{\mu}^k) \leq \text{poly}(\epsilon, \nu)(k^{-1/2} + 4/3).$$

VI. SIMULATION

In this section, we use simulation to illustrate the effectiveness of our RC-LQR, and the convergence of the policy gradient primal–dual methods in both model-based and sample-based settings.
Fig. 2. Relative Lagrangian error of model-based gradient methods in (23) with $\mu = 2$. (a) Constant stepsize. (b) Backtracking line search (except GN).

First, we validate the convergence results in Theorem 3 on the three gradient methods in (23). We adopt the relative Lagrangian error $\frac{L(X^i, \mu) - L^*(\mu)}{L^*(\mu)}$ to examine the convergence behaviors of (23). Fig. 2 validates their linear convergence rates of Theorem 3 and Table I. As expected, it is also observed that the use of a backtracking line search increases the convergence rate. Then, we validate the sublinear convergence result in Theorem 4, where the GN is applied to minimize the Lagrangian in Algorithm 1. Let the initial multiplier be $\mu_1 = 0$ and the diminishing stepsize be $\zeta_k = 1/(15\sqrt{k})$. Fig. 3 displays how the relative optimality gap $|J(X^k) - J(X^*)|/J(X^*)$ and the constraint violation $\max\{J_c(X^k) - \bar{\rho}, 0\}/\bar{\rho}$ decrease to zero. Clearly, both converge fast under our model-based policy gradient primal–dual method. Note that both the objective function $J(X)$ and the constraint function $J_c(X)$ are quadratic, and converge with a similar behavior.

C. Sample-Based Setting

In the sample-based setting, we use trajectory samples of the system (1) to compute (25) and conduct 20 independent trials. First, we examine the convergence performance of Algorithm 2 and set the smoothing radius to $r = 0.2$, the sample horizon of the oracle $T = 100$, and the constant stepsize $\eta = 1 \times 10^{-5}$. Moreover, we display the relative Lagrangian error for $\mu = 2$ in Fig. 4, where the bold centerline denotes the trial mean and the shaded region indicates the variance size. As expected by Theorem 5, Algorithm 2 converges to a small relative error of 3% with a small variance.

Then, we verify the convergence result of our sample-based primal–dual method in Theorem 6 by performing Algorithm 3. Let the initial multiplier be $\mu_1 = 0$ and the diminishing stepsize be $\zeta_k = 1/(15\sqrt{k})$. Fig. 5 illustrates that both the relative optimality gap and the constraint violation are eventually close to zero.

VII. CONCLUSION

In this article, we have proposed a policy gradient primal–dual framework with global convergence guarantees to solve the RC-LQR problem with a variance-like constraint. Specifically,
we have shown here strong duality, to establish the global convergence, which in fact can be extended to the case of multiple constraints. Such a framework can also be utilized to study linear quadratic tracking.

APPENDIX A
PROOFS OF SOME RESULTS IN SECTION III

A Proof of Lemma 1
Let \( u_t = -K x_t + l \). Then, \( V_X(x) \) satisfies
\[
V_X(x) = \mathbb{E} \left[ \sum_{t=0}^{\infty} (x_t^T (Q_\mu + K^T R) x_t + (2S^T - 2I^T R) x_t + l^T R l - \mu \bar{\rho}) - \mathcal{L}(X, \mu) \bigg| x_0 = x \right].
\]

By using backward DP [41, Ch. 3], it can be shown that \( V_X(x) \) has a quadratic form [26], i.e., \( V_X(x) = x^T P_K x + g_K x + z_X \), where \( P_K, g_K, z_X \) are to be determined.

By the Bellman equation [47, (3.14)], it holds for any \( X \in \mathcal{S} \) that \( V_X(x_t) = \mathbb{E}_{w_t} [c_{\mu}(x_t, u_t) + V_X(x_{t+1}) | x_t, u_t = -K x_t + l] \). That is,
\[
\begin{align*}
x_t^T P_K x_t + g_K x_t + z_X &= x_t^T (Q_\mu + K^T R) x_t + (2S^T - 2I^T R) x_t + l^T R l \\
+ \mathbb{E} \left[ (A - BK) x_t + Bl + w_t \right] P_K \left[ (A - BK) x_t + Bl + w_t \right] \\
- \mathcal{L}(X, \mu) - \mu \bar{\rho} + \mathbb{E} [g_X ((A - BK) x_t + Bl + w_t)] + z_X \\
&= x_t^T (Q_\mu + K^T R) x_t + (2S^T - 2I^T R) x_t + l^T R l \\
+ 2[ -l^T E_K + S^T + \bar{w}^T P_K (A - BK) + g_X (A - BK)] x_t \\
+ \mathbb{E} [P_K (W + (B l + \bar{w}) (B l + \bar{w})^T)] \\
+ g_X (B l + \bar{w}) + l^T R l - \mathcal{L}(X, \mu) - \mu \bar{\rho} + z_X.
\end{align*}
\]
The proof follows as the equality holds for \( x_t \in \mathbb{R}^n \).

B Proof of Lemma 2

By Lemma 1 and observing that \( \mathcal{L}(X, \mu) \) is quadratic in \( l \), we can compute \( \nabla_l \mathcal{L}(X, \mu) \) in terms of \( G_X \) and \( E_K \) as
\[
\nabla_l \mathcal{L}(X, \mu) = 2G_X - 2E_K \bar{x}_X.
\]

We aim to show that \( \nabla_l \mathcal{L}(X, \mu) = 2E_K \Sigma_K - \nabla_l \mathcal{L}(X, \mu) \bar{x}_X \). First, we express \( \mathcal{L}(X, \mu) \) using the stationary distribution \( \tau \) of the state as
\[
\begin{align*}
\mathcal{L}(X, \mu) &= \mathbb{E}_{x \sim \tau} \left[ x^T Q_\mu x + 2x^T S + u_t^T R u_t - \mu \bar{\rho} \right] \\
&= \mathbb{E} \left[ Q_K (\Sigma_K + \bar{x}_X \bar{x}_X^T) \right] + 2(S^T - l^T R) \bar{x}_X + l^T R l - \mu \bar{\rho}.
\end{align*}
\]
Then, its gradient in \( l \) is given as
\[
\nabla_l \mathcal{L} = \nabla_l \mathbb{E} \left[ Q_K (\Sigma_K + \bar{x}_X \bar{x}_X^T) \right] + 2(S^T - l^T R) \bar{x}_X + 2R l \\
+ \nabla_l \left\{ (2S^T - 2I^T R) \bar{x}_X \right\} + 2R l \\
= 2B^T V^T (Q_\mu + K^T R) \bar{x}_X \\
+ 2B^T V^T (S - K^T R l) - 2R K (\bar{x}_X - l).
\]

Similar to [6, Lemma 1], one can show that
\[
\begin{align*}
\nabla_K \mathcal{L} &= \nabla_K \mathbb{E} \left[ (Q_\mu + K^T R K) (\Sigma_K + \bar{x}_X \bar{x}_X^T) \right] \\
+ \nabla_K \left\{ (2S^T - 2I^T R) \bar{x}_X \right\} \\
&= 2E_K \Sigma_K + 2R (K \bar{x}_X - l) \bar{x}_X \\
- 2B^T V^T (Q_\mu + K^T R K) \bar{x}_X - K^T R l + S) \bar{x}_X \\
&= 2E_K \Sigma_K - \nabla_l \mathcal{L}(X, \mu) \bar{x}_X.
\end{align*}
\]
Combining \( \nabla_x \mathcal{L}(X, \mu) = \nabla_K \mathcal{L} \nabla_l \mathcal{L} \), and the definition of \( \Phi_X \) in (16), we have \( \nabla_x \mathcal{L}(X, \mu) = [E_K \ G_X] \Phi_X \).

Since \( \Sigma_K > W > 0 \), and
\[
\Phi_X = \begin{bmatrix} I & -\bar{x}_X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Sigma_K & 0 \\ 0 & -\bar{x}_X^T \end{bmatrix}
\]
we have that \( \Phi_X \) is strictly positive definite.

C Proof of Lemma 3

By (12), it follows that
\[
\bar{x}_X = (I - A + BK)^{-1} (Bl + w).
\]
This implies that \( \|\bar{x}_X\| \to \infty \) if either \( \rho(A - BK) \to 1 \) or \( ||w|| \to \infty \), i.e., \( \mathcal{L}(X, \mu) \) is coercive over \( \mathcal{S} \).

Since the critical point in (17) is unique, we conclude that \( \mathcal{S}_\alpha \) in (18) is compact.

D Proof of Lemma 4

i) From the definition of \( \mathcal{L}(X, \mu) \) in (7), it follows that
\[
\begin{align*}
\mathcal{L}(X, \mu) &= \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (c_t(x_t, u_t) - \mathcal{L}(X, \mu)) \right] \\
&= \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} (c_t(x_t, u_t) - \mathcal{L}(X, \mu) + V^T_X (x_t) - V^T_X (x_t')) \right] \\
&= \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} A^T_X (x_t, u_t') \right]
\end{align*}
\]
where the expectation is w.r.t. the noise sequence. The second equality follows from the boundedness of \( V^T_X (x_0) \). The third equality follows by telescoping the sum appropriately. The last equality holds by the definition of the advantage function.

ii) Under \( u = -K' x + l' \), we have
\[
\begin{align*}
\lim_{T \to \infty} A^T_X (x, u) &= c_t (x, u) - \mathcal{L}(X, \mu) + \mathbb{E}_w[V_X (A x + B u + w)] - V_X (x) \\
&= x^T Q_\mu x + 2S^T x + (-K' x + l')^T R (-K' x + l') \\
- \mathcal{L}(K, l, \mu) + \mathbb{E}_w[V_X ((A - BK') x + Bl' + w)] - V_X (x).
\end{align*}
\]
The proof is completed by reorganizing the above terms.

**E Proof of Lemma 5**

It follows from Lemma 4 that

$$\lim_{t \to \infty} A^T_X(x, u) = \left( (K' - K)x - (l' - l) + R_{K'}^{-1}(E_K x - G_X) \right)^\top$$

$$\times R_K((K' - K)x - (l' - l) + R_{K'}^{-1}(E_K x - G_X))$$

$$- (E_K x - G_X)^\top R_{K'}^{-1}(E_K x - G_X)$$

$$\geq - (E_K x - G_X)^\top R_{K'}^{-1}(E_K x - G_X).$$

Let \( \{x^*_t\} \) and \( \{u^*_t\} \) be sequences generated by following the policy \( X^*(\mu) \) in (17). Then, it follows that

$$\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu) = - \lim_{t \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} A^T_X(x^*_t, u^*_t) \right]$$

$$\leq \lim_{t \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \text{tr}(\{E_K x^*_t - G_X\}^\top R_{K'}^{-1}(E_K x^*_t - G_X)) \right]$$

$$= \lim_{t \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \text{tr}(\{x^*_t\} \{x^*_t\}^\top [E_K G_X]^\top R_{K'}^{-1}[E_K G_X]) \right]$$

$$= \text{tr}(\{\Phi^*_t\} [E_K G_X]^\top R_{K'}^{-1}[E_K G_X])$$

$$\leq \left\| \Phi^*_t \right\| \text{tr}(\{E_K G_X\}^\top R_{K'}^{-1}[E_K G_X])$$

$$\leq \left\| \Phi^*_t \right\| R_{K'}^{-1} \text{tr}(\{E_K G_X\}^\top [E_K G_X]^\top)$$

$$= (\left\| \Phi^*_t \right\| / \mathcal{g}(R)) \text{tr}(\{E_K G_X\}^\top [E_K G_X]^\top). \quad (30)$$

**Lemma 2** implies that \( 4 \text{tr}(\{\Phi X \}^\top [E_K G_X]^\top) \leq \text{tr}(\nabla \mathcal{L}^\top \nabla \mathcal{L}) \). Together with (30), the Lagrangian difference is upper bounded by

$$\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu)$$

$$= \left\| \frac{\Phi^*_t}{\mathcal{g}(R)} \right\| \text{tr}(\{E_K G_X\}^\top [E_K G_X]^\top)$$

$$\leq \frac{\left\| \Phi^*_t \right\|}{4 \mathcal{g}(R) \mathcal{g}(\Phi)} \text{tr}(\nabla \mathcal{L}^\top \nabla \mathcal{L}).$$

**F Proof of Lemma 7**

Let \( \tau \) be the stationary distribution of the state under \( X' \). By Lemma 4, it follows that

$$\mathcal{L}(X', \mu) - \mathcal{L}(X, \mu) = \lim_{t \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} A^T_X(\tau^t, u^t) \right]$$

$$= \mathbb{E}_{x \sim \tau} \left[ ((K' - K)x - (l' - l) + R_{K'}^{-1}(E_K x - G_X))^\top \right.$$

$$\times R_K((K' - K)x - (l' - l) + R_{K'}^{-1}(E_K x - G_X))$$

$$- (E_K x - G_X)^\top R_{K'}^{-1}(E_K x - G_X) \right].$$

The rest of the proof follows by (12) and (13).

**APPENDIX B**

**Proof of Lemma 8**

We first build a connection between \( \mathcal{L}(X, \mu) \) and the standard LQR cost.

**Lemma 11**: Define \( C(K, \mu) = \text{tr}(P_K W) \). Then,

$$C(K, \mu) = \text{tr}\{ (Q_{\mu}^* + K^T R_K) \Sigma_K \} \leq \mathcal{L}(X, \mu) + S^T Q_{\mu}^{-1} S.$$

**Proof**: Comparing the definition of \( \mathcal{L}(X, \mu) \) with \( C(K, \mu) \), it follows that

$$\mathcal{L}(X, \mu) = \text{tr}\{(Q_{\mu} + K^T R_K) \Sigma_K \}$$

$$+ (2S^T - 2l^T R_K \bar{x}_X + l^T l + l^T)$$

$$= \text{tr}\{(Q_{\mu} + K^T R_K) \Sigma_K \} + (Q_{\mu} \bar{x}_X + S)^T Q_{\mu}^{-1}(Q_{\mu} \bar{x}_X + S)$$

$$+ (K \bar{x}_X - l)^T R(K \bar{x}_X - l) - S^T Q_{\mu}^{-1} S$$

$$\geq C(K, \mu) - S^T Q_{\mu}^{-1} S.$$

Then, the results in [6] are utilized in our analysis. Define

$$c_1 = \mathcal{g}(Q_{\mu}) \mathcal{g}(W) / (4C(K, \mu) \|B\| (\|A - BK\| + 1))$$

and \( c_2 = (1 - \rho(A - BK)) / \|B\| \).

**Lemma 12** ([6]): Let \( X \in \mathbb{S} \). Then, we have that

i) if \( \|\Sigma_K\| \leq C(K, \mu) / \mathcal{g}(Q_{\mu}), \|P_K\| \leq (C(K, \mu) / \mathcal{g}(W)) \),

ii) \( C(K' - K, \mu) \leq c_1 \|K\| \|K\| \),

iii) \( \|P_{K'} - P_K\| \leq \|K\| \|R\| (C(K, \mu) / \mathcal{g}(Q_{\mu}) \mathcal{g}(W))^2 \times (\|K\| \|B\| (\|A - BK\| + 1) + 1) \|K - K'\| \).

**Lemma 13**: Let \( V' = (I - A + BK')^{-1} \) and \( K' - K \leq c_2 \). Then, \( \|V' - V\| \leq 2\|B\| \|K' - K\| / (1 - \rho(A - BK)) \).

**Proof**: The proof follows from the matrix inverse perturbation theorem [48].

**Lemma 14**: For \( X \in \mathbb{S} \), we have the following relationship

$$\text{tr}(E_K^T E_K) \leq \text{tr}(\{E_K G_X^\top\}^\top [E_K G_X^\top])$$

$$\leq \| R_K \| (\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu)) / \| \sigma^*_o \| \). \quad \text{where} \ \phi^*_o \ \text{is defined in \text{Lemma 6}}.$$

**Proof**: Let \( X' = X - R_{K'}^{-1}[E_K G_X] \), we have

$$\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu) \geq \mathcal{L}(X, \mu) - \mathcal{L}(X', \mu)$$

$$= \text{tr}(\{E_K G_X^\top\}^\top R_{K'}^{-1}[E_K G_X^\top])$$

$$\geq \mathcal{g}(\Phi^*_o) \text{tr}(\{E_K G_X^\top\}^\top [E_K G_X^\top]) / \| R_K \|$$

$$\geq \| \sigma^*_o \| \text{tr}(\{E_K G_X^\top\}^\top [E_K G_X^\top]) / \| R_K \|. $$

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The last inequality follows since the GN method yields a decrease, i.e., $X' \in S_0$.

**Lemma 15:** For all $K'$ such that $||K' - K|| \leq \min(c_1, c_2)$, we have $\|g_{K', l} - g_{X'}\| \leq p_1(K, l)||K' - K||$, where $p_1(K, l)$ is polynomial in $C(K, \mu)$, $\|A\|$, $\|B\|$, $\gamma(W)$, $\gamma(Q_\mu)$, $\gamma(R)$, $\|l\|$, and $\|\tilde{\omega}\|$.

**Proof:** By definition, we have

$$g_{K', l} - g_{X'} = 2[-l^T E_{K'} + S^T + \tilde{w}^T P_{K'}(A - BK')V' - 2[-l^T E_{K'} + S^T + \tilde{w}^T P_{K'}(A - BK')V$$

$$= 2[-l^T (E_{K'} - E_{K})V'$$

$$+ \tilde{w}^T (P_{K'}(A - BK') - P_{K}(A - BK'))V'$$

$$+ 2[-l^T E_{K} + S^T + \tilde{w}^T P_{K}(A - BK'))(V' - V).$$

(31)

By Lemma 12 and the fact that $\|V'\| \leq \|V' - V\| + \|V\| \leq (1 + 2\|B\||K' - K|)/(1 - \rho(A - BK))$, it follows that $2l^T (E_{K'} - E_{K})V' \leq p(K)(||K' - K||$, where $p(K)$ is polynomial in $C(K, \mu)$, $\|A\|$, $\|B\|$, $\gamma(W)$, $\gamma(Q_\mu)$, and $\gamma(R)$. To bound the second term of (31), we note that

$$\|P_{K'}(A - BK') - P_{K}(A - BK')\|$$

$$\leq \|P_{K'} - P_{K}\| ||A - BK'|| + \|P_{K}\| ||B|| ||K' - K||.$$

Again by Lemma 12, it can be bounded by the product of $||K' - K||$ and polynomials of related parameters. The third term in (31) can be analyzed analogously. Combining the above completes the proof.

Next, we derive the Lipschitz constants of $\mathcal{L}(K, l, \mu)$ w.r.t. $K$ and $l$, respectively.

**Lemma 16:** Suppose that $||K' - K|| \leq \min(c_1, c_2)$. Then, $\mathcal{L}(K, l', \mu) - \mathcal{L}(K, l, \mu) \leq p_2(K, l)||K' - K||$, where $p_2(K, l)$ is a polynomial in $C(K, \mu)$, $\|A\|$, $\|B\|$, $\gamma(W)$, $\gamma(Q_\mu)$, $\gamma(R)$, $\|l\|$, $\|\tilde{\omega}\|$, and $\|W\|$.

**Proof:** Note that

$$\mathcal{L}(K, l) = (g_{K, l} - G_X)^T (Bl + \tilde{w})$$

$$+ \text{tr}(P_{K'} - P_{K})(W + (Bl + \tilde{w})(Bl + \tilde{w}))$$

$$\leq n\|P_{K'} - P_{K}\|(\|W\| + \|Bl + \tilde{w}\|^2)$$

$$+ ||g_{K'} - G_X|| ||Bl + \tilde{w}||.$$

By Lemmas 12 and Lemma 15, the proof follows.

**Lemma 17:** If $||l' - l|| \leq \delta$ for some $\delta > 0$, then $||\mathcal{L}(K, l', \mu) - \mathcal{L}(K, l, \mu)|| \leq p_3(K, l)||l' - l||$, where $p_3(K, l)$ is a polynomial in $C(K, \mu)$, $\|A\|$, $\|B\|$, $\gamma(W)$, $\gamma(Q_\mu)$, $\gamma(R)$, $\|l\|$, $\|\tilde{\omega}\|$, $\delta$, and $\|W\|$.

**Proof:** By direct calculation, we have that

$$\mathcal{L}(K, l, \mu) - \mathcal{L}(K, l, \mu)$$

$$= (l' - l)^T (R + B^TP_K B)l' + l^T (R + B^TP_K B)(l' - l)$$

$$+ 2\tilde{w}^T P_K B(l' - l) + g_{l'}^T B(l' - l) + g_{l}^T (Bl + \tilde{w})$$

and $\|g_{l'} - g_{l}\| \leq 2||l' - l|| ||E_K|| ||V||$. Then, the proof is completed by combining the above lemmas.

Finally, we prove that the Lagrangian is locally Lipschitz.

**Lemma 18:** There exist positive scalars $(\alpha_1, b_1)$ that depend on the policy $X = [K l]$ such that $|\mathcal{L}(X', \mu) - \mathcal{L}(X, \mu)| \leq \alpha_1||X' - X||$ for all policies $X'$ satisfying $||X' - X|| \leq b_1$.

**Proof:** By Lemmas 16 and 17, for $||X' - X|| \leq \min(c_1, c_2)$ and $||l' - l|| \leq \delta$, it follows that

$$\mathcal{L}(X', \mu) - \mathcal{L}(X, \mu)$$

$$= L(K, l', \mu) - L(K, l, \mu) + L(K, l', \mu) - L(K, l, \mu)$$

$$\leq p_2(K, l')||K' - K||_F + p_3(K, l)||l' - l||_F$$

$$\leq \sqrt{2}\max\{p_2(K, l'), p_3(K, l)\}||X' - X||_F$$

where $||\cdot||_F$ denotes the Frobenius norm.

Letting $a_1 = \sqrt{2}\max\{p_2(K, l'), p_3(K, l)\}$, and $b_1 = \min(c_1, c_2\), the proof follows.

In the following, we establish the Lipschitz property for the gradient $\nabla_X \mathcal{L}(X, \mu)$. Similarly, we first derive a bound.

**Lemma 19:** Suppose that $||K' - K|| \leq \min(c_1, c_2)$. Then, it follows that $|\Phi_{K, l} - \Phi_{X'}| \leq p_4(K, l)||K' - K||$, where $p_4(K, l)$ is a polynomial in $C(K, \mu)$, $\|A\|$, $\|B\|$, $\gamma(W)$, $\gamma(Q_\mu)$, $\gamma(R)$, $\|l\|$, and $\|\tilde{\omega}\|$.

**Proof:** Note that

$$|\Phi_{K, l} - \Phi_{X}| \leq \text{tr}(\Phi_{K, l} - \Phi_{X})$$

$$\leq n||\Sigma_{K'} - \Sigma_{K}|| + ||\tilde{x}_{K'} - \tilde{x}_K||.$$

For the first term, it has been shown in [6, Lemma 16] that

$$||\Sigma_{K'} - \Sigma_{K}|| \leq 4(C(K, \mu)\gamma(Q_\mu))^2 \|B\|(||A - BK|| + 1) ||K' - K||.$$

Since $||\tilde{x}_{K'} - \tilde{x}_K|| = ||\tilde{x}_{K'} - \tilde{x}_K|| + \tilde{x}_K(\tilde{x}_{K'} - \tilde{x}_K)$, we have

$$||\tilde{x}_{K'} - \tilde{x}_K|| = (||V' - V|| ||Bl + \tilde{w}||) \leq 2\|B\||||Bl + \tilde{w}||||K' - K||$$

and that $||\tilde{x}_{K'}|| \leq ||\tilde{x}_K|| + ||\tilde{x}_{K'} - \tilde{x}_K|| \leq (2 ||B|| ||K' - K|| + 1)||Bl + \tilde{w}||/(1 - \rho(A - BK))$. Then, the proof follows by reorganizing the above terms.

Next, we establish the Lipschitz constants for $\nabla_K \mathcal{L} \text{ and } \nabla_l \mathcal{L}$, respectively.

**Lemma 20:** Suppose that $||K' - K|| \leq \min(c_1, c_2)$. Then it follows that $||\nabla_K \mathcal{L} - \nabla_K \mathcal{L}|| \leq p_5(K, l)||K' - K||$, where $p_5(K, l)$ is a polynomial in $C(K, \mu)$, $\|A\|$, $\|B\|$, $\gamma(W)$, $\gamma(Q_\mu)$, $\gamma(R)$, $||l||$, and $||\tilde{\omega}||$.

**Proof:** We have that

$$||\nabla_K \mathcal{L} - \nabla_K \mathcal{L}|| = 2||E_{K'} - E_K\||\Phi_{K, l} - \Phi_{X}||$$

$$\leq 2\|(E_{K'} - E_K\|| ||G_{K'} - G_{X}\|| ||\Phi_{K, l}||$$

$$+ 2(||E_{K'}\|| ||G_{X}\|)(||\Phi_{K, l}||)$$

Note that $||\Phi_{X}||$ can be bounded as

$$||\Phi_{X}|| \leq 1 + \text{tr}(\Sigma_K + \tilde{x}_K\tilde{x}_K) \leq 1 + n||\Sigma_K|| + ||\tilde{x}_K||^2$$

$$\leq 1 + nC(K, \mu)\gamma(Q_\mu) + \left(||Bl + \tilde{w}||^2 \right)/\left(1 - \rho(A - BK)\right)^2.$$
Also, we have
\[ \|G_{K,l} - G_X\| \leq \|B\|\|P_{K'} - P_K\|Bl + \bar{w}\|
\[ + \|g_{K,l} - G_X\|/2 \leq p_0(K, l)\|K' - K\| \]
with \(p_0(K, l)\) being polynomial in related parameters. By Lemma 14, we obtain that
\[ \|[E_K G_X]\| \leq \sqrt{\|R_K\|}(\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu))/\sigma_0. \]

Combining the above inequalities, the proof is completed. \(\blacksquare\)

**Lemma 21:** For \(\|l' - l\| \leq \delta\), we have \(\|\nabla_l^\tau L - \nabla_l L\| \leq p_1(K, l)\|l' - l\|\), where \(p_1(K, l)\) is a polynomial in \(C(K, \mu), \|A\|, \|B\|, \|g(W), g(Q_\mu), g(R)\|, \|l\|, \|\bar{w}\|,\) and \(\delta\).

**Proof:** Note that
\[ \|\nabla_l^\tau L - \nabla_l L\| = 2\|[E_K G_{K,l}]\|\Phi_{K,l} - [E_K G_X]\|\Phi_X\|
\[ \leq 2\|[G_{K,l} - G_X]\|\|\Phi_{K,l} - \Phi_X\| + 2\|[E_K G_X]\|\|\Phi_{K,l} - \Phi_X\|. \]

Then, combining \(\|G_{K,l} - G_X\| \leq \|B\|\|g_{K,l} - G_X\|/2 \leq \|B\|\|\Phi_{K,l} - \Phi_X\|/\|l' - l\|\), and
\[ \|\Phi_{K,l} - \Phi_X\| \leq \|\Phi_{K,l} - \Phi_X\| \leq \|\Phi_{K,l} - \Phi_X\| \]
\[ = \|\bar{x}_l - \bar{x}_l\| \]
\[ = \|\bar{x}_l - \bar{x}_l\| \]
\[ \leq (2\|Bl + \bar{w}\| + \|B\|\delta)\|\Phi\|\|V\|\|l' - l\|
\]
completes the proof. \(\blacksquare\)

**Lemma 22:** There exist positive scalars \((a_2, b_2)\) that depend on the current policy \(X = [K]\), such that for all policies \(X'\) satisfying \(\|X' - X\| \leq a_2\), we have \(\|\nabla_{X'} L - \nabla_{X} L\| \leq a_2 \|X' - X\|\).

**Proof:** The proof is similar to that of Lemma 18. \(\blacksquare\)

So far, we have shown that the Lagrangian and its gradient are locally Lipschitz in Lemmas 18 and 22, respectively. Letting \(\gamma_X = \min\{b_1, b_2\}, \xi_X = a_1, \) and \(\beta_X = a_2\) completes the proof of Lemma 8.

**APENDIX C**

**PROOF OF THEOREM 3**

**A. Proof of the GN Update**

We prove that i) under the given stepsize, \(X' \in \mathcal{S}\), and ii) \(X'\) stays in the compact sublevel set \(S_X = \{X' | \mathcal{L}(X', \mu) \leq \mathcal{L}(X, \mu)\}\).

Suppose that i) holds (to be proved subsequently). Hence, \(\mathcal{L}(X', \mu)\) is well defined. By Lemma 7, one can show that
\[ \mathcal{L}(X', \mu) - \mathcal{L}(X, \mu) \]
\[ = (4\eta^2 - 4\eta)\text{tr}\{[E_K G_X]\|R_{K'}^{-1}[E_K G_X]\|\Phi\}
\[ \leq -2\eta\text{tr}\{[E_K G_X]\|R_{K'}^{-1}[E_K G_X]\|\Phi\}
\[ \leq -2\eta\Phi\cdot \text{tr}\{[E_K G_X]\|R_{K'}^{-1}[E_K G_X]\}
\[ \leq -2\eta\Phi||\|\Phi\|\|\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu)\|
\]
where the last inequality follows from (30).

Clearly, it leads to that \(\mathcal{L}(X', \mu) \leq \mathcal{L}(X, \mu)\). Thus, \(X'\) is contained in \(S_0 \subseteq \mathcal{S}\), and thus,
\[ \mathcal{L}(X', \mu) - \mathcal{L}(X, \mu) \leq -2(2\eta\sigma_0||\Phi^*||)\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu)\).

We have so far shown that for any \(0 < \eta \leq 1/2\), if the resulting policy \(X'\) is stabilizing, then \(X' \in S_X\).

To complete the proof, we prove i) by contradiction. Suppose that there exists a stepsize \(0 < q' \leq 1/2\) for which the resulting policy is not stabilizing. Consider the ray \(\{X - \eta\nabla_X L \cdot \Phi_{X'}^{-1}\eta > 0\} \). Let
\[ \eta = \sup\{\eta > 0 | \mathcal{L}(X - \eta\nabla_X L \cdot \Phi_{X'}^{-1}\eta > 0\} \].

Then, it follows from coercivity that there must exist a stabilizing \(X'_\eta = X - \eta\nabla_X L \cdot \Phi_{X'}^{-1}\eta \) in the ray with \(\eta \in (\eta, \eta')\) such that \(\mathcal{L}(X'_\eta) > \mathcal{L}(X, \mu)\). This leads to a contradiction since we can only have \(X'_\eta \in S_X\) by the previous analysis.

**B. Proof of the NPG Update**

The stability issue of \(X'\) is addressed similarly as in the proof of the GN update. By Lemma 7, we obtain that
\[ \mathcal{L}(X', \mu) - \mathcal{L}(X, \mu) = -4\eta\text{tr}\{[E_K G_X]^\tau[E_K G_X]\Phi\}
\[ + 4\eta^2\text{tr}\{[E_K G_X]^\tau[R_K E_K G_X]\Phi\}
\[ \leq (4\eta^2\|R_K\| - 4\eta)\text{tr}\{[E_K G_X]^\tau[E_K G_X]\Phi\}
\[ \leq -2\eta\text{tr}\{[E_K G_X]^\tau[E_K G_X]\Phi\}
\[ \leq -2\eta\Phi\cdot \text{tr}\{[E_K G_X]^\tau[E_K G_X]\}
\[ \leq -2\eta\Phi\cdot \text{tr}\{[E_K G_X]^\tau[E_K G_X]\}
\]
where the last inequality follows from (30).

Clearly, the Lagrangian decreases as long as \(0 < \eta \leq 1/(2\|R_K\|)\). Hence, we obtain that \(\sigma/(\Phi^*) \geq \sigma_0\).

Since the iteration \(K' = K - 2\eta E_K\) yields \(\|R_{K'}\| \leq \|R_K\|\) [6], it suffices to set the stepsize as \(0 < \eta \leq 1/(2\|R_K\|)\).

The proof is thus completed.

**C. Proof of the PG Update**

First, we determine a stepsize \(\eta\) such that the Lagrangian decreases after one-step gradient descent.

Define the following quantities
\[ c_3 = \frac{\sigma(W)\sigma(\Phi_X)\|\phi\|^2(Q_\mu)}{48n\|\nabla_K L\|^2\|B\|\|\|A - BK\| + 1\|C^2(K, \mu)\}
\]
\[ c_4 = \frac{(1 - \rho(A - BK))^2}{24\|B\|^2\|Bl + \bar{w}\|^2 + \|B\|^2\|\phi\|^2(3 - \rho(A - BK))\|\nabla_K L\|^2\}
\]
\[ c_5 = \frac{1}{12(2\|Bl + \bar{w}\|^2 + \|B\|^2\|\phi\|^2\|\nabla_{l}\|^2)}
\]
\[ c_6 = 3\left(1 + n\frac{C(K, \mu)}{\sigma(Q_\mu)} + \left(\frac{\|Bl + \bar{w}\|^2}{1 - \rho(A - BK)}\right)^2\|R_K\|\right)^{-1}
\]

**Lemma 23:** Suppose that
\[ \eta \leq \min\left\{\frac{c_1}{\|\nabla_K L\|^2}, \frac{c_2}{\|\nabla_{l}\|^2}, c_3, c_4, c_5, c_6, \delta\right\}. \]
Then, the PG update $X' = X - \eta \nabla X \mathcal{L} = X - 2\eta [E_K \ G_X] \Phi_X$ yields that
\[
\mathcal{L}(X', \mu) - \mathcal{L}^*(\mu) \leq \left(1 - 2\eta \frac{\sigma(\Phi_X)^2 \sigma(R)}{\|\Phi^*\|} \right) (\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu)).
\]

**Proof:** The stability issue of $X'$ is addressed similarly as in the proof of the GN update. By Lemma 7, one can show
\[
\mathcal{L}(X', \mu) - \mathcal{L}(X, \mu) = -4\eta \eta \operatorname{tr}(\Phi' \Phi_X [E_K \ G_X]^\top [E_K \ G_X])
\]
\[
+ 4\eta^2 \operatorname{tr}(\Phi' \Phi_X [E_K \ G_X]^\top [E_K \ G_X])
\]
\[
\leq -4\eta \operatorname{tr}(\Phi_X [E_K \ G_X]^\top [E_K \ G_X])
\]
\[
+ 4\eta \|\Phi' - \Phi_X\| \operatorname{tr}(\Phi_X [E_K \ G_X]^\top [E_K \ G_X])
\]
\[
+ 4\eta^2 \|\Phi'\| L_K \operatorname{tr}(\Phi_X [E_K \ G_X]^\top [E_K \ G_X])
\]
\[
\leq -4\eta (1 - \frac{\|\Phi' - \Phi_X\|}{\sigma(\Phi_X)^2}) \|R_K\| \operatorname{tr}(\nabla X^\top \nabla X). \]

By Lemma 5, we obtain that
\[
(\mathcal{L}(X', \mu) - \mathcal{L}(X, \mu))/(\mathcal{L}(X, \mu) - \mathcal{L}^*(\mu))
\]
\[
\leq -4\eta \frac{\sigma(R) \sigma(\Phi_X)^2}{\|\Phi^*\|} \left(1 - \frac{\|\Phi' - \Phi_X\|}{\sigma(\Phi_X)^2}\right). \]

Thus, it suffices to determine $\eta$ to yield a decreasing cost. To this end, we first bound the norm $\|\Phi' - \Phi_X\|$.

Note that
\[
\|\Phi' - \Phi_X\| \leq \|\Phi' - K,\mu\| + \|K,\mu - \Phi_X\|. \quad (33)
\]

By Lemma 19, if $\|K' - K\| = \eta \|\nabla K \mathcal{L}\| \leq \min(c_1, c_2)$, then the first term of (33) can be bounded by
\[
\|\Phi' - K,\mu\| \leq 4n \left(\frac{C(K,\mu)}{\sigma(Q_{\mu})}\right)^2 \frac{2\|B\| (\|A - BK\| + 1)}{\sigma(W)} \|K' - K\|
\]
\[
+ \frac{4\|B\| \|B' + \tilde{w}\|^2}{(1 - \rho(A - BK))^2} \|K' - K\| (1 + \|B\| \|K' - K\|)
\]
\[
\leq 4n \left(\frac{C(K,\mu)}{\sigma(Q_{\mu})}\right)^2 \frac{2\|B\| (\|A - BK\| + 1)}{\sigma(W)} \|\nabla K \mathcal{L}\| \times \eta
\]
\[
+ \frac{\|B\| \|B' + \tilde{w}\|^2}{(1 - \rho(A - BK))^2} (6 - 2\rho(A - BK)) \|\nabla K \mathcal{L}\| \times \eta. \]

If $\|l' - l\| = \eta \|\nabla l \mathcal{L}\| \leq \delta$, then the second term of (33) is bounded by
\[
\|\Phi_{K,l} - \Phi_X\| \leq \frac{2\|B\| \|B' + \tilde{w}\| + \|B\| \|\delta\| \|B\| \|\nabla l \mathcal{L}\|}{(1 - \rho(A - BK))^2} \times \eta.
\]

Under the given stepsize $\eta$, it can be shown that
\[
\|\Phi' - \Phi_X\| \leq (1/12 + 1/12 + 1/12) \|\Phi_X\| = \|\Phi_X\|/4.
\]

Then, we bound $\|\Phi'\|$ by
\[
\|\Phi'\| \leq \|\Phi' - \Phi_X\| + \|\Phi_X\|
\]
\[
\leq \frac{1}{4} \sigma(\Phi_X) + 1 + n \frac{C(K,\mu)}{\sigma(Q_{\mu})} + \left(\frac{\|B\| + \tilde{w}\|}{1 - \rho(A - BK)}\right)^2
\]
\[
\leq \frac{1}{4} \|\Phi'\| + 1 + n \frac{C(K,\mu)}{\sigma(Q_{\mu})} + \left(\frac{\|B\| + \tilde{w}\|}{1 - \rho(A - BK)}\right)^2
\]
which implies that
\[
\|\Phi'\| \leq \frac{4}{3} \left(1 + n \frac{C(K,\mu)}{\sigma(Q_{\mu})} + \left(\frac{\|B\| + \tilde{w}\|}{1 - \rho(A - BK)}\right)^2 \right).
\]

Hence, it follows that
\[
1 - \frac{\|\Phi' - \Phi_X\| / \sigma(\Phi_X) - \eta \|\Phi'\| \|R_K\|}{\|\Phi_X\|} \geq 1 - 1/4 - 1/4 = 1/2
\]
which completes the proof.

To find a constant stepsize, it suffices to quantify the lower bound of the terms in (32). By Lemma 23, we focus on the sublevel set $S_0 = \{X | \mathcal{L}(X, \mu) \leq \mathcal{L}(X(0), \mu)\}$. The following inequalities hold:
\[
\sigma(\Phi_X) \geq \sigma_0, \quad 1 - \rho(A - BK) \geq \sigma(W) / (2n C(K,\mu))
\]
\[
C(K,\mu) \leq \mathcal{L}(X(0), \mu) = S^\top Q_{\mu}^{-1} S \leq \mathcal{L}(X(0), \mu) + S^\top Q_{\mu}^{-1} S.
\]

Since $S_0$ is compact, $\|l\|$ is bounded by a constant related to $S_0$. The remaining terms are the gradient norm $\nabla l \mathcal{L}$ and $\nabla K \mathcal{L}$, which are bounded by $\nabla X \mathcal{L}$.

By the definition of $\nabla X \mathcal{L}$, it follows that
\[
\|\nabla X \mathcal{L}\|^2 \leq \operatorname{tr}(\Phi_X [E_K \ G_X]^\top [E_K \ G_X])
\]
\[
\leq \|\Phi_X\|^2 \operatorname{tr}(\Phi_X [E_K \ G_X]^\top [E_K \ G_X]).
\]

By Lemma 14, $\|\nabla X \mathcal{L}\|$ has an upper bound over $S_0$. Thus, the stepsize $\eta$ has a lower bound polynomial in the problem parameters.

**APPENDIX D**

**PROOF OF THEOREM 5**

Our proof is based on [11, Th. 1]. To guarantee the convergence of random search, it requires a) gradient dominance, b) locally Lipschitz continuity, and c) boundedness of gradient norms $G_{\infty}$ and $G_2$. Thus, we only need to establish c) by using Assumption 3. Since $G_2 \leq G_{\infty}^2$, it suffices to bound $G_{\infty}$.

We first show that $\mathcal{L}(X, \mu)$ is bounded over $S_{10}$. By the linear dynamics (1), the state $x_t$ can be written as
\[
x_t = (A - BK)^t x_0 + \sum_{k=0}^{t-1} (A - BK)^{t-k} (B l + \tilde{w}_t).
\]
By Assumption 3, it holds that
\[
\hat{L}(X, \mu) \leq \max_{\sigma \in \Sigma} \limsup_{t \to \infty} x_t^T (Q_\mu + K^T R K)x_t \\
+ 2(S - 2K^T R l)^T x_t + l^T R l.
\] (34)

Noting that \(\lim_{t \to \infty} (A - BK)^t x_0 = 0\), we have that
\[
\lim_{t \to \infty} \|x_t\| \leq \|\sum_{k=0}^{\infty} (A - BK)^k (B l + w_t)\| \\
\leq \|B l + v\|/(1 - \rho(A - BK)) \\
\leq 2nC(K, \mu)\|B l + v\|/(\sigma(W)\sigma(Q_\mu))
\] (35)

where the last inequality follows from Lemma 12.

Inserting (35) into (34) and noting \(\|Q_\mu + K^T R K\| \leq \|P_K\|\), then \(\hat{L}(X, \mu)\) can be bounded by
\[
\hat{L}(X, \mu) \leq 4n^2 C^3(K, \mu)\|B l + v\|^2/(\sigma(W)\sigma^2(Q_\mu)) + l^T R l \\
+ 4nC(K, \mu)\|B l + v\|\|S\| + \|K\| R \|l\|)/(\sigma(W)\sigma(Q_\mu))
\]
where \(C(K, \mu), \|K\|,\) and \(\|l\|\) are uniformly bounded over \(S_{\mu}\).

Thus, \(\hat{L}(X, \mu)\) is bounded. Similarly, it follows from the Lipschitz property that \(\|\hat{L}(X,rU,\mu) - \hat{L}(X+rU,\mu)\| \leq F\) with some constant \(F > 0\).

For a given radius \(r < \gamma_0\) and a unit perturbation \(U \in S\), the gradient estimate is bounded as
\[
\|\nabla_x \hat{L}\| = \frac{n}{r^2} \|\hat{L}(X + rU, \mu)\| = \frac{n}{r^2} (\|\hat{L}(X + rU, \mu)\| + F) \\
\leq \frac{n}{r^2} (\|\hat{L}(X, \mu)\| + r\xi_0 + F) \\
\leq \frac{n}{r^2} (\|10\hat{L}(X^{(0)}, \mu)\| + \gamma_0\xi_0 + F).
\]
Then, the proof follows directly from [11, Th. 1].

**APPENDIX E**

**PROOF FOR THEOREM 6**

Let \(S_\epsilon = \{X|\hat{L}(X, \mu) - \hat{L}(\mu) \leq \epsilon\}\). We first derive a uniform upper bound for \(\|d^k\|\) w.r.t. \(\epsilon, \nu\) via
\[
\|d^k\| \leq \sup_{(\nu, X) \in S_\epsilon} \|\hat{J}_\epsilon(X) - \hat{p}\|
\] (36)
which follows from (28). Then, we obtain the following result.

**Lemma 24**: \(\|d^k\| \leq \psi_0(\epsilon, \nu, \nu)\), where \(\psi_0(\epsilon, \nu, \nu)\) is a polynomial in \(\epsilon\) and \(\nu\) of degree 4.

**Proof**: For any \(X \in S_\epsilon\), \(J_\epsilon(X)\) is bounded by
\[
\hat{J}_\epsilon(X) \leq \max_{\{\nu_0\}} \limsup_{t \to \infty} \{(4x_0^T QW Q x_t + 4x_0^T Q M_3) \\
- \max_{\{\nu_0\}} \limsup_{t \to \infty} \{(4\|x_t\|^2\|Q W W\| + 4\|x_t\|\|Q M_3\|) \\
- 4n^2 C^2(K, \mu)\|B\|\|l\| + \nu\|Q W W\|/(\sigma(W)\sigma^2(Q_\mu)) \\
+ 8nC(K, \mu)\|B\|\|l\| + \nu\|Q M_3\|/(\sigma(W)\sigma(Q_\mu))\}
\] (37)

where the last inequality follows from (35). Moreover, it follows from Lemma 11 that \(C(K, \mu) \leq L(X, \mu) + S^T Q_\mu^{-1} S\). Thus, it suffices to prove an upper bound for \(\|l\|\) in (37) over \(S_\epsilon\).

By the definition of \(\hat{L}(X, \mu)\), it holds that
\[
\hat{L}(X, \mu) = E_{X \sim \tau} \left[ \begin{array}{c} [x] \top [Q + K^T R K R l - S] \\
[l^T R - S] \\
[l^T R - \mu \hat{p}] \end{array} \right] \Phi_X \]
\[
\geq \sigma_0 \|\tau (Q + K^T R K R l - \mu \hat{p})\| \\
\geq \sigma_0 (\mu(\|l\| - \mu \hat{p}))
\]
where the first inequality follows from \(S_\epsilon \subseteq S_0\) and the definition \(\sigma_0 = \min_{X \in S_0} \sigma(\Phi_X)\). Hence, \(\|l\|\) is bounded by \(\|l\| \leq (\hat{L}(X, \mu)/\sigma_0 + \mu \hat{p})/\sigma(R) \leq (\hat{L}(X, \mu) + \epsilon + \epsilon_0 \mu \nu)/\sigma(R)\).

Inserting the bound of \(C(K, \mu)\) and \(\|l\|\) into (37) and noting \(\epsilon \leq D(\|q\|) \leq L(\|q\|) = \sigma(\|q\|)\) yields
\[
\hat{J}_\epsilon(X) \leq \frac{4n^2 \|Q W W\|}{\sigma^2(W)\sigma^2(Q_\mu)} \|D^\epsilon + \epsilon + \epsilon_0 \mu \nu\| \|v(\sigma(R)\sigma_0)^{1/2} \|
\]
\[
\times (\|B\|\|D^\epsilon + \epsilon + \epsilon_0 \mu \nu\| \|v(\sigma(R)\sigma_0)^{1/2} \|)^2 \\
+ \frac{8n\|Q M_3\|}{\sigma(W)\sigma(Q_\mu)} \|D^\epsilon + \epsilon + \epsilon_0 \mu \nu\| \|v(\sigma(R)\sigma_0)^{1/2} \|)
\]

We note that the bound is also polynomial in \(\mu\). As in [43, Sec. 4.2], we can focus on a bounded set of \(\mu\) since \(\mu^*\) in (21) is finite, which can be achieved by projection. Thus, without loss of generality, we assume that \(\|\mu^k\| \leq \epsilon\). Then, \(J_\epsilon(X)\) is uniformly bounded by a polynomial of \(\epsilon\) and \(\nu\), and the proof is completed.

Now, we prove Theorem 6 using standard subgradient arguments (see [43]).

By the definition of the projection and subgradient, it holds
\[
\|\mu^{i+1} - \mu^*\| \leq \|\mu^i - \mu^* + \zeta^i \hat{d}\| \\
= \|\mu^i - \mu^*\| + 2\zeta^i \hat{d}^i(\mu^i - \mu^*) + 2\zeta^i (\hat{d}^i - d^i) (\mu^i - \mu^*) \\
+ (\zeta^i)^2 \|\hat{d}\|^2 \\
\leq \|\mu^i - \mu^*\| + 2\zeta^i (D(\mu^i) - D^*) + 8\zeta^i \cdot p_s e + (\zeta^i)^2 p_s^2
\]
where the inequality follows from Lemma 24 and the boundedness of \(\|\mu^k\|\).

Then, rearranging it yields that
\[
D^* - D(\mu^i) \leq \|\mu^i - \mu^*\| - \|\mu^{i+1} - \mu^*\| \\
\leq 2\zeta^i \|\mu^{i+1} - \mu^*\| + 4p_s e + \zeta^i p_s^2/2
\]

Summing up and noting \(\zeta^i \geq \zeta^{i+1}\), it follows that
\[
\sum_{i=1}^k (D^* - D(\mu^i)) \leq -\frac{1}{2\zeta^k} \|\mu^{k+1} - \mu^*\|^2
\]
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where the last inequality follows by letting $\zeta^k = \frac{1}{p_\epsilon} \sqrt{\frac{2}{k}}$.

$D^* - D(\mu)$

$\leq \frac{2}{k} \epsilon^2 + \sum_{i=1}^{k} \zeta^i + 4p_\epsilon \frac{e}{\sqrt{k}}$

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