Non-Hermitian anisotropic $XY$ model with intrinsic rotation-time reversal symmetry

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We systematically study the non-Hermitian version of the one-dimensional anisotropic $XY$ model, which in its original form, is a unique exactly solvable quantum spin model for understanding the quantum phase transition. The distinguishing features of this model are that it has full real spectrum if all the eigenvectors are intrinsic rotation-time reversal ($RT$) symmetric rather than parity-time reversal ($PT$) symmetric, and that its Hermitian counterpart is shown approximately to be an experimentally accessible system, an isotropic $XY$ spin chain with nearest neighbor coupling. Based on the exact solution, exceptional points which separated the unbroken and broken symmetry regions are obtained and lie on a hyperbola in the thermodynamic limit. It provides a nice paradigm to elucidate the complex quantum mechanics theory for a quantum spin system.

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I. INTRODUCTION

In recent years, much effort has been devoted to establish a parity-time ($PT$) symmetric quantum theory as a complex extension of the conventional quantum mechanics1,2 since the seminal discovery by Bender1. A cornerstone of the theory is the fact that the non-Hermitian Hamiltonian with $PT$ symmetry can have an entirely real energy spectrum. It is also motivated by the interest in complex potentials in both theoretical and experimental aspects, since the imaginary potential could be realized by complex index in optics3-5. A natural question to ask is whether the non-Hermitian quantum theory prefers to the $PT$ symmetry. While there is as yet no answer to this question, we can gain some insight regarding the pseudo-Hermiticity of a non-Hermitian model without the $PT$ symmetry. On the other hand, the non-Hermitian quantum model in discrete system is a nice testing ground for the study of the non-Hermitian quantum mechanics because of its analytical and numerical tractability. As simplified models, tight-binding and quantum spin models can capture the essential features of many discrete systems. In general, the non-Hermiticity arises from $PT$ symmetric on-site imaginary potentials in tight-binding models6,9,10,14,25, and imaginary magnetic fields in quantum spin models26,27. In addition, the complex coupling constant between spins can also introduce the non-Hermiticity, which relates to the spin, the intrinsic degree of freedom. Then the spin rotation may take the role of the parity operation for constructing a pseudo-Hermitian spin system.

In this paper, we propose a pseudo-Hermitian model without $PT$ symmetry explicitly, but with intrinsic rotation-time ($RT$) reversal symmetry. It is a non-Hermitian version of the one-dimensional anisotropic $XY$ model. The original Hermitian $XY$ model was initially solved and has became the paradigm of the quantum spin system possessing a second-order quantum phase transition28. We will show that this model has full real spectrum if all the eigenvectors have $RT$ rather than parity-time reversal ($PT$) symmetry, and its Hermitian counterpart is an isotropic $XY$ spin chain. The result for such a concrete example may have profound theoretical and methodological implications.

This paper is organized as follows. In Section II we present the model Hamiltonian. Based on the solutions we investigate the phase diagram and analyze the symmetry of the ground state. In Section III we construct the Hermitian counterpart of the model and its approximate reduced form. Finally, we give a summary and discussion in Section IV.

II. HAMILTONIAN AND INTRINSIC $RT$ SYMMETRY

The model Hamiltonian of the non-Hermitian anisotropic one-dimensional spin-½ $XY$ model in a transverse magnetic field $\lambda$ for $N$ particles is given by

$$H = J \sum_{j=1}^{N} \left( \frac{1 + i \gamma}{2} \sigma_{j}^{x} \sigma_{j+1}^{x} - \frac{1 - i \gamma}{2} \sigma_{j}^{y} \sigma_{j+1}^{y} + \lambda \sigma_{j}^{z} \right)$$

where $\sigma_{j}^{\alpha}$ ($\alpha = x, y, z$) are the Pauli operators on site $j$, and satisfy the periodic boundary condition $\sigma_{j}^{\alpha} \equiv \sigma_{j+N}^{\alpha}$. For the sake of simplicity, we only concern the case of even $N$, the conclusion is available in the case of odd $N$. In comparison with the model proposed by Giorgi29, the present model is not $PT$ symmetric. The non-Hermiticity arises from the imaginary anisotropic parameters $\pm i \gamma$. The Hermitian version of the $XY$ model is completely solved by applying the Jordan-Wigner28, Fourier and Bogoliubov transformation28,29. The Jordan-Wigner transformation maps the Pauli operators into canonical fermions, while the Fourier transformation essentially decomposes the Hamiltonian into invariant subspaces due to the translational symmetry of the system. We will see that these transformations are applicable in solving the Hamiltonian [1] if we extend it to its complex versions. Before solving the Hamiltonian, it is profitable to investigate the symmetry of the system and its breaking in...
the eigenstates. By direct derivation, we have \([\mathcal{R}, H] \neq 0\) and \([\mathcal{T}, H] \neq 0\), but

\[
[\mathcal{R} \mathcal{T}, H] = 0.
\] (2)

The Hamiltonian is rotation-time (\(\mathcal{R} \mathcal{T}\)) reversal invariant, where the linear rotation operator \(\mathcal{R}\) has the function of rotating each spin by \(\pi/2\) about the z-axis

\[
\mathcal{R} \equiv \exp \left[ -i (\pi/4) \sum_{j=1}^{N} \sigma_{j}^{x} \right],
\] (3)

and the antilinear time reversal operator \(\mathcal{T}\) has the function \(\mathcal{T} i \mathcal{T} = -i\). Before we consider the general non-Hermitian \(XY\) model, we highlight key ideas on two limiting cases. When \(\gamma = 0\), the Hamiltonian (1) reduces to the ordinary \(XY\) model with external field in z direction

\[
H_{0} = \frac{J}{2} \sum_{j=1}^{N} (\sigma_{j}^{x} \sigma_{j+1}^{x} + \sigma_{j}^{y} \sigma_{j+1}^{y} + 2 \lambda \sigma_{j}^{z}),
\] (4)

which has full real spectrum and its eigenstates can always be written as the eigenstates of operator \(\mathcal{R} \mathcal{T}\). On the other hand, when \(\gamma \gg \lambda \) and \(1\), the Hamiltonian (1) reduces to

\[
H_{\infty} = \frac{i J \gamma}{2} \sum_{j=1}^{N} (\sigma_{j}^{x} \sigma_{j+1}^{x} - \sigma_{j}^{y} \sigma_{j+1}^{y}),
\] (5)

which has full imaginary spectrum. Then for an eigenstate \(|\psi_{n}\rangle\) of \(H_{\infty}\) with nonzero eigenvalue \(E_{n}\), i.e., \(H_{\infty} |\psi_{n}\rangle = E_{n} |\psi_{n}\rangle\), we have

\[
H_{\infty} \mathcal{R} \mathcal{T} |\psi_{n}\rangle = -E_{n} \mathcal{R} \mathcal{T} |\psi_{n}\rangle
\] (6)

due to the facts of Eq. (2) and \(E_{n}^{*} = -E_{n}\). The eigenstate \(|\psi_{n}\rangle\) obviously breaks the \(\mathcal{R} \mathcal{T}\) symmetry. These results strongly imply that the \(\mathcal{R} \mathcal{T}\) symmetry in the present model plays the same role as \(\mathcal{P} \mathcal{T}\) symmetry in the \(\mathcal{P} \mathcal{T}\) pseudo-Hermitian system. It motivates further study of such a model systematically.

A. Solutions

Now we consider the solution of the non-Hermitian \(XY\) Hamiltonian of Eq. (1). We note that the general solutions of the Hermitian anisotropic \(XY\) Hamiltonian are not restricted to the real anisotropic parameter. We will show that in the case of imaginary parameter, eigenstates and energies are still accessible. As the same procedures performed in solving the Hermitian Hamiltonian, we take the Jordan-Wigner transformation to replace the Pauli operators by the fermionic operators \(c_{j}\). We note that the parity of the number of fermions

\[
\Pi = \prod_{l=1}^{N} (\sigma_{l}^{z}) = (-1)^{N_{p}}
\] (8)

is a conservative quantity, i.e., \([H, \Pi] = 0\), where \(N_{p} = \sum_{j=1}^{N} c_{j}^{\dagger} c_{j}\). Then the Hamiltonian (1) can be rewritten

\[
H = \sum_{\eta=+,-} P_{\eta} H_{\eta} P_{\eta},
\] (9)

where

\[
P_{\eta} = \frac{1}{2} (1 + \eta \Pi)
\] (10)

is the projector on the subspaces with even (\(\eta = +\)) and odd (\(\eta = -\)) \(N_{p}\). The Hamiltonian in each invariant subspaces has the form

\[
H_{\eta} = J \sum_{j=1}^{N-1} \left( c_{j}^{\dagger} c_{j+1} + c_{j+1}^{\dagger} c_{j} + i \gamma c_{j}^{\dagger} c_{j+1}^{\dagger} + i \gamma c_{j+1} c_{j} \right)
\]

\[
- \eta \left( c_{N}^{\dagger} c_{1} + c_{1}^{\dagger} c_{N} + i \gamma c_{N}^{\dagger} c_{1}^{\dagger} + i \gamma c_{1} c_{N} \right)
\] (11)

\[
- 2 J \lambda \sum_{j=1}^{N} c_{j}^{\dagger} c_{j} + NJ \lambda.
\]

Taking the Fourier transformation

\[
c_{j} = \frac{1}{\sqrt{N}} \sum_{k_{\pm}} e^{ik_{\pm} j} c_{k_{\pm}}
\] (12)

for the Hamiltonians \(H_{\pm}\), we have

\[
H_{\eta} = -J \sum_{k_{n}} \left[ 2 (\lambda - \cos k_{n}) c_{k_{n}}^{\dagger} c_{k_{n}} + \gamma \sin k_{n} (c_{-k_{n}}^{\dagger} c_{k_{n}} + c_{k_{n}}^{\dagger} c_{-k_{n}}) - \lambda \right]
\] (13)

where \(k_{\pm} = 2 \pi (m + 1/2) / N, k_{-} = 2 \pi m / N, m = 0, 1, 2, ..., N - 1\).

So far the procedures are the same as those for solving the Hermitian version of \(H\). To diagonalize the non-Hermitian Hamiltonian, we introduce the Bogoliubov transformation in complex version:

\[
A_{k_{n}} = \cos \left( \frac{\theta}{2} \right) c_{k_{n}} - i \sin \left( \frac{\theta}{2} \right) c_{-k_{n}}
\]

\[
\overline{A}_{k_{n}} = \cos \left( \frac{\theta}{2} \right) c_{k_{n}}^{\dagger} + i \sin \left( \frac{\theta}{2} \right) c_{-k_{n}}
\] (14)
where
\[
\tan(\theta) = \frac{i\gamma \sin k_\eta}{\lambda - \cos k_\eta}. \tag{15}
\]

We would like to point out that this is the crucial step to solve the non-Hermitian Hamiltonian, which essentially establish the biorthogonal bases. Obviously, complex Bogoliubov modes \((\lambda_k, \lambda'_k)\) satisfy the canonical commutation relations
\[
\{A_{\lambda_k}, A_{\lambda'_k}\} = \delta_{\lambda_k, \lambda'_k}, \tag{16}
\]
\[
\{A_{\lambda_k}, A_{\lambda_k}\} = \{\lambda_k, \lambda'_k\} = 0;
\]
which result in the diagonal form of the Hamiltonian
\[
H_\eta = \sum_{\lambda_k} \epsilon(\lambda, k_\eta, \gamma) \left(\lambda_{\lambda_k} A_{\lambda_k} - \frac{1}{2}\right). \tag{17}
\]

Here the single-particle spectrum in each subspace is
\[
\epsilon(\lambda, k_\eta, \gamma) = -2J\sqrt{(\lambda - \cos k_\eta)^2 - \gamma^2 \sin^2 k_\eta}. \tag{18}
\]

Note that the Hamiltonian \(H_\eta\) is still non-Hermitian due to the fact that \(A_{\lambda_k} \neq A_{\lambda_k}^\dagger\). Accordingly, the eigenstates of \(H_\eta\) can be written as the form
\[
\prod_{\{k_\eta\}} \lambda_{\lambda_k} |G_\eta\rangle,
\]
which constructs the biorthogonal set associated with the eigenstates
\[
\langle G_\eta| \prod_{\{k_\eta\}} A_{\lambda_k} \tag{19}
\]
of the Hamiltonian \(H_\eta^\dagger\), where
\[
|G_\eta\rangle = \prod_{k>0} \left[\cos \left(\frac{\theta}{2}\right) + i \sin \left(\frac{\theta}{2}\right) c_{k} c_{-k}^\dagger \right] |\text{Vac}\rangle \tag{20}
\]
is the ground state of \(H_\eta\), and |Vac\rangle is the vacuum state of the fermion \(c_f\). In the following, we will investigate the phase diagram based on the properties of the solutions.

### B. Phase diagram

It is clear that when any one of the momentum \(k_\eta\) satisfies
\[
|\lambda - \cos k_\eta| < |\gamma \sin k_\eta|, \tag{22}
\]
the imaginary energy level appears in single-particle spectrum, which leads to the occurrence of complex energy level for the Hamiltonian \(H\), and the \(\mathcal{R}\mathcal{T}\) symmetry is broken in the corresponding eigenstates. This can be seen from the properties of the single-particle spectrum and the ground states \(|G_\eta\rangle\).

Firstly, we focus on the boundary between the broken and unbroken symmetry regions. There are totally \(2N\) equations in the form of \(|\lambda - \cos k_\eta| = |\gamma \sin k_\eta|\) for all the possible value of \(k_\eta\), only one of which determines the line along the boundary of the diagram within certain region in the parameters \(\lambda\) and \(\gamma\) plane. Then the phase boundary is dog-leg path for finite \(N\), and becomes a smooth loop for the infinite \(N\). In the thermodynamic limit, the momentum \(k_\eta\) becomes continuous, then the phase boundary as a curve can be given by the following set of parametric equations:
\[
\frac{\partial \epsilon(\lambda_c, k_\eta, \gamma_c)}{\partial \lambda_c} = 0, \tag{23}
\]
\[
\epsilon(\lambda_c, k_\eta, \gamma_c) = 0. \tag{24}
\]

Straightforward algebra gives the analytical boundary curve as
\[
\lambda_c^2 - \gamma_c^2 = 1, \tag{25}
\]
which is a hyperbola. In addition, the broken region does not include the line \(\gamma = 0\). In Fig. 1, we plot the phase diagrams for the systems with \(N = 4, 8,\) and \(30\), respectively. It is shown that as \(N\) increases, the boundary approaches to the hyperbola of Eq. (25).

Secondly, according to the non-Hermitian quantum theory, the occurrence of the exceptional point always accomplishes the \(\mathcal{R}\mathcal{T}\) symmetry breaking of an eigenstate.
For the concerned model, the symmetry of the groundstate \(|G_\eta\rangle\) can be indicator of the phase transition due to the fact that the groundstate energy becomes complex once the system is in broken region. In the following, we focus on the discussion about the symmetry of \(|G_\eta\rangle\) in the different regions.

Taking the combination of the Jordan-Wigner and Fourier transformations on the rotational operator in Eq. (3), we have

\[
\mathcal{R} = (-i)^{N/2} \prod_{\eta=\pm,k_\eta} \left[ 1 - \sqrt{2} e^{-i\pi/4} n_{k_\eta} \right],
\]

which are available in the both regions. However, the coefficients \(\cos (\theta/2)\) and \(\sin (\theta/2)\) experience a transition as following when the corresponding single-particle level changes from real to imaginary: We have \([\cos (\theta/2)]^* = \cos (\theta/2)\) and \([\sin (\theta/2)]^* = -\sin (\theta/2)\) for real levels and \([\cos (\theta/2)]^* = \sin (\theta/2)\) for the imaginary levels, respectively. This leads to the conclusion that the groundstate is not \(\mathcal{R}\mathcal{T}\) symmetric in the broken region, i.e.,

\[
\{ \begin{array}{ll}
\mathcal{R}\mathcal{T} |G_\eta\rangle & = |G_\eta\rangle; \quad \text{Unbroken} \\
\mathcal{R}\mathcal{T} |G_\eta\rangle & \neq |G_\eta\rangle; \quad \text{Broken}
\end{array} \}
\]

It shows that the \(\mathcal{R}\mathcal{T}\) symmetry in the present model plays the same role as \(\mathcal{P}\mathcal{T}\) symmetry in the \(\mathcal{P}\mathcal{T}\) pseudo-Hermitian system.

As a comparison, it is noted that the phase boundary of the Hermitian anisotropic \(XY\) model is an ellipse. It is worth pointing out that the phase transitions in the Hermitian and the non-Hermitian models are the different types of quantum phase transition. The typical quantum phase transition\[31\] describes an abrupt change in the ground state of a many-body system. For the Hermitian anisotropic \(XY\) model, the transition occurs when the order parameter, the energy gap between the ground and first excited states, goes to zero. For the present non-Hermitian model, zero gap also leads to the phase
boundary of the ground state, which tends to \( \lambda_c^2 - \gamma_c^2 = 1 \) in thermodynamic limit. It accords with the boundary of the unbroken \( RT \) symmetric region. It reveals two differences between the phase transitions in the non-Hermitian and the Hermitian \( XY \) model: (i) The energy gap can vanish for the former with the finite \( N \), while for the Hermitian \( XY \) model zero-gap is never achieved unless the thermodynamic limit is reached. (ii) There is no \( RT \) symmetry breaking in the quantum phase transition of the Hermitian \( XY \) model.

### III. HERMITIAN COUNTERPART

In the complex quantum theory, as an important perspective, the physical meaning of a non-Hermitian Hamiltonian has received a lot of attentions. When speaking of the physical significance of a non-Hermitian Hamiltonian, one of the ways is to seek its Hermitian counterparts, which possess the identical real spectrum. According to the complex quantum mechanics, a non-Hermitian Hamiltonian can be transformed into a Hermitian Hamiltonian by introducing a metric, a bounded positive-definite Hermitian operator. However, the obtained equivalent Hermitian Hamiltonian is usually quite complicated. On the one hand, it is tough to provide an explicit mapping of a pseudo-Hermitian Hamiltonian to its equivalent Hermitian Hamiltonian and on the other hand, the obtained Hermitian counterpart is unphysical, which usually contains the nonlocal interactions. In this section, we will provide a physical counterpart of the present model and analyze its behavior when approaches to the exceptional point.

A Hermitian Hamiltonian becomes a Hermitian counterpart of a non-Hermitian system if and only if they share the identical real spectrum. In this sense, the counterpart is not unique and seems to be easy constructed. However, it is expected that the obtained counterpart is physically relevant, or possessing the local interaction, which makes things difficult. To this end, our strategy is to consider the operator as the form

\[
h_\eta = \sum_{k_n} \epsilon (\lambda, k_n, \gamma) \left( d_{k_n}^\dagger d_{k_n} - \frac{1}{2} \right),
\]

where \( d_{k_n} \) is Fermion operator satisfying

\[
\{ d_{k_n}^\dagger, d_{k_n'} \} = \delta_{n, n'},
\]

\[
\{ d_{k_n}, d_{k_n'} \} = \{ d_{k_n}, d_{k_n}^\dagger \} = 0.
\]

Here \( h_+ \) and \( h_- \) represent the Hamiltonians in the invariant subspaces with even number of \( \sum_{k_+} d_{k_+}^\dagger d_{k_+} \) and odd number of \( \sum_{k_-} d_{k_-}^\dagger d_{k_-} \), respectively. Obviously, \( h_\eta \) is Hermitian when \( \epsilon (\lambda, k_n, \gamma) \) is real and has the identical spectrum with that of \( H_\eta \).

We employ the Fourier transformation

\[
d_{k_n} = \frac{1}{\sqrt{N}} \sum_j d_j e^{-ik_n j},
\]

and Jordan-Wigner transformation\(^{20}\)

\[
d_j = \frac{1}{2} \prod_{l<j} (\tau_i^+ \tau_j^+),
\]

to rewrite the Hamiltonian \( h_\eta \) by the Pauli spin operators \( \tau_i^\pm (\alpha = x, y, z) \). Here \( \{ \tau_i^\alpha \} \) and \( \{ \sigma_i^\alpha \} \) are different sets of spin operators, which have no direct relation, or more precisely, there is no explicit transformation to connect them. The essence is that we replace the pair of conjugate operators \((\overline{\alpha}_{k_n}, A_{k_n'})\) by \((d_{k_n}^\dagger, d_{k_n'})\), which ensures the identical spectrum of two Hamiltonians. However, they are distinct due to the fact that \((\overline{\alpha}_{k_n})^\dagger \neq A_{k_n}\).

Now we transform the Hamiltonians \( h_\eta \) into the following spin model

\[
h_\eta = -\frac{1}{2} \sum_{m>l} \kappa_{lm} \prod_{l<j<m} \tau_j^+ \tau_m^+ \tau_l^- + \text{H.c.} + \sum_l \kappa_l \tau_l^2,
\]

where

\[
\kappa_{lm} = \frac{J}{N} \sum_{k_n} \left( (\lambda - \cos k_n)^2 - \gamma^2 \sin^2 k_n \cos \left[ k_n (l - m) \right] \right)
\]

\[
= \kappa (l - m)
\]

is \( \eta \)-dependent coupling constant. Here \( h_\eta \) and \( h_- \) represent the Hamiltonians in the invariant subspaces with even and odd number of magnons. It looks complicated due to the long-range coupling and the extra phase \( \prod_{l<j<m} \tau_j^+ \). We will show that it can be reduced approximately to a simple model in the non-trivial parameter region.

First of all, we consider the strong field limit case, \( \lambda \gg 1 + \gamma^2 \). Taking the Taylor expansion, we get

\[
k_\eta (l - m) \approx \frac{J}{N} \sum_{k_n} \left( (\lambda - \cos k_n) \frac{(\lambda + \cos k_n) \gamma^2 \sin^2 k_n \cos \left[ n (l - m) \right]}{2\lambda^2} \right).
\]

Using the relation

\[
\sum_{k_n} e^{-ik_n (l - m)} = N \delta_{l,m}
\]

we have

\[
k_\eta (l - m) = J \sum_{n=0}^3 [\kappa (n) \delta_{l-m,n} + O (\nu^2)]
\]
where $\nu = \gamma^2/\lambda^2$ and

$$
\kappa (0) = \left(1 - \frac{\nu}{4}\right) \lambda J, \quad \kappa (1) = -\frac{J}{2} \left(1 + \frac{\nu}{8}\right),
\kappa (2) = \frac{\nu \lambda J}{8}, \quad \kappa (3) = \frac{\nu J}{16}.
$$

It is shown that $\kappa_\eta (l - m)$ is $N$ and $\eta$-independent, and decays as $|l - m|$ increases. Secondly, numerical simulation indicates that even the field $\lambda$ is not so strong, $\kappa (l - m)$ is still negligible for $|l - m| > 1$. Fig. 2 is the plot of $\kappa (l - m)$ in Eq. (35).

Then the Hamiltonian can be reduced approximately as

$$
\mathcal{H} = -\kappa (1) \sum_i (\tau_i^+ \tau_{i+1} + \text{H.c.}) + \kappa (0) \sum_i \tau_i^z,
$$

which is the ordinary $XY$ model. To demonstrate this equivalent, we make a comparison of $\mathcal{H}$ and the original Hamiltonian $H$ in Eq. (1). The energy levels are computed for small size systems. In Fig. 3 we see that they are in agreement well in the physical parameter region.

Secondly, it is worthy to point out that $\mathcal{H}$ does not contain the information of the exceptional point if we take the approximate form of $\kappa (0)$ and $\kappa (1)$ in Eq. (39). As well known, distinguished from phase transition in a Hermitian system, the non-analytic properties for the phase transition of a non-Hermitian are caused by the Hamiltonian becoming a Jordan block operator at the exceptional point. Nevertheless, the Jordan block cannot appear in a Hermitian Hamiltonian. Therefore it is interesting to see what happens to the equivalent Hamiltonian at the exceptional point. To investigate the critical behaviors of the equivalent Hamiltonian, the exact coupling constants $\kappa_\eta (l - m)$ should be considered. We focus on the feature of $\kappa_\eta (l - m)$ as $(\lambda, \gamma)$ turns to the critical point $(\lambda_c, \gamma_c)$ except the case of $\lambda_c = 1$ ($\gamma_c = 0$), which is not the exceptional point. To this end, we take the gradient of $\kappa_\eta (l - m)$ in the $\lambda - \gamma$ plane, i.e.,

$$
\nabla \kappa_\eta = \frac{\partial \kappa_\eta}{\partial \lambda} \hat{e}_1 + \frac{\partial \kappa_\eta}{\partial \gamma} \hat{e}_2
$$

where $\hat{e}_1$ and $\hat{e}_2$ denote the unit vectors, $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$. When $(\lambda, \gamma)$ turns to the critical point $(\lambda_c, \gamma_c)$, the dominant contribution to the summation is the term of $k_\eta = k_c$, where

$$
\cos k_c = \frac{1}{\lambda_c}.
$$

Then we have

$$
\lim_{(\lambda, \gamma) \to (\lambda_c, \gamma_c)} \nabla \kappa_\eta = \lim_{k_\eta \to k_c} \frac{J}{N} \cos |k_\eta (l - m)| \times \left(\frac{\lambda - \cos k_\eta}{\lambda - \cos k_c}\right)^2 \frac{\gamma^2 \sin^2 k_\eta \hat{e}_2}{\gamma^2 \sin^2 k_c} = \infty.
$$

It accords with the fact that the derivative of the energy diverges when the system tends to the exceptional point.

Then for a Hermitian matrix, the non-analytic behavior is not from the Jordan block but from the divergence of derivatives of matrix elements. The similar situation also occurs in another model\cite{27}, which may imply a general conclusion.

**IV. SUMMARY**

In summary, we have proposed a non-Hermitian model without $\mathcal{PT}$ symmetry explicitly, but with intrinsic $\mathcal{RT}$ symmetry, which is a non-Hermitian version of the anisotropic $XY$ model. Based on the exact solution, we have found the phase diagram for the finite $N$ system and the corresponding symmetry breaking in ground state. An analysis of the symmetry in the ground state shows that the $\mathcal{RT}$ symmetry is broken when the groundstate energy becomes complex, which is similar to that for a $\mathcal{PT}$ system. It indicates that the $\mathcal{PT}$ symmetric model is not the unique candidate of pseudo-Hermitian system. We also constructed a Hermitian counterpart which exhibiting the identical real spectrum to that of the original one within the unbroken region. It is an isotropic $XY$ spin chain with long-range coupling, which has been shown to have the following properties: The Hermitian counterpart can be reduced to a simple $XY$ model with nearest neighbor coupling when the system is not in the vicinity of the exceptional point. The derivatives of all the coupling strengths with respect to the system parameters diverge at the critical points. This provides an
interpretation for the critical behavior of level repulsion. The result for such a concrete example may have profound theoretical and methodological implications.

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