Abstract: The method of semigroups is a unifying, widely applicable, general technique to formulate and analyze fundamental aspects of fractional powers of operators $L$ and their regularity properties in related functional spaces. The approach was introduced by the author and José L. Torrea in 2009 (arXiv:0910.2569v1). The aim of this chapter is to show how the method works in the particular case of the fractional Laplacian $L^s = (-\Delta)^s$, $0 < s < 1$. The starting point is the semigroup formula for the fractional Laplacian. From here, the classical heat kernel permits us to obtain the pointwise formula for $(-\Delta)^s u(x)$. One of the key advantages is that our technique relies on the use of heat kernels, which allows for applications in settings where the Fourier transform is not the most suitable tool. In addition, it provides explicit constants that are key to prove, under minimal conditions on $u$, the validity of the pointwise limits

$$\lim_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x) \quad \text{and} \quad \lim_{s \to 0^+} (-\Delta)^s u(x) = u(x).$$

The formula for the solution to the Poisson problem $(-\Delta)^s u = f$ is found through the semigroup approach as the inverse of the fractional Laplacian $u(x) = (-\Delta)^{-s} f(x)$ (fundamental solution). We then present the Caffarelli–Silvestre extension problem, whose explicit solution is given by the semigroup formulas that were first discovered by the author and Torrea. With the extension technique, an interior Harnack inequality and derivative estimates for fractional harmonic functions can be obtained. The classical Hölder and Schauder estimates

$$(-\Delta)^{\pm s} : C^\alpha \to C^{\alpha \mp 2s}$$

are proved with the method of semigroups in a rather quick, elegant way. The crucial point for this will be the characterization of Hölder and Zygmund spaces with heat semigroups.

Keywords: method of semigroups; fractional Laplacian; extension problem; regularity estimates

Classification: 35R11; 26A33; 58J35
1 Introduction

Fractional powers, both positive and negative, as well as complex, of linear operators appear in many areas of mathematics. In particular, the fractional powers of the Laplacian are nowadays classical objects. Fractional operators appear in potential theory [10, 18, 63, 87], probability [3, 8, 12, 15, 16, 17, 19, 37, 72], fractional calculus and hypersingular integrals [49, 77, 78, 79, 80], harmonic analysis [5, 16, 49, 81, 88, 89], functional analysis [6, 53, 60, 62, 67, 99], and pseudo-differential operators [43, 52, 54, 55, 56, 57].

In recent years, the fractional Laplacian or, more generally, nonlocal equations of fractional order, gained a lot of attention from the partial differential equations research community. It can be said that the main driving force for this has been the fundamental work of Luis A. Caffarelli and his collaborators, see [23, 24, 25, 26, 27, 30, 84, 85], just to mention a few.

To introduce the notion of fractional Laplacian, let $u$ be a function in the Schwartz class $S = S(\mathbb{R}^n)$, $n \geq 1$. The Fourier transform of $u$, denoted by $\hat{u}$, is also in $S$. For the Laplacian $-\Delta$ on $\mathbb{R}^n$ we have

$$\hat{(-\Delta)u}(\xi) = |\xi|^{2s}\hat{u}(\xi) \quad \text{for every } \xi \in \mathbb{R}^n.$$  

The fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, is then defined in a natural way as

$$(-\Delta)^s u(\xi) = |\xi|^{2s}\hat{u}(\xi).$$  

1.1 A few applications

Let us begin by briefly describing problems in probability, financial mathematics, elasticity and biology where fractional powers of differential operators appear.

1. Lévy processes. Let $(X_t : t \geq 0)$ be a symmetric $2s$-stable ($0 < 2s \leq 2$) $\mathbb{R}^n$-valued Lévy process starting at 0. By the Lévy-Khintchine formula [3, 12] the characteristic function of $X_t$ is $E(e^{i\xi \cdot X_t}) = e^{-t\kappa^2|\xi|^{2s}}$, $\xi \in \mathbb{R}^n$, $t \geq 0$, for some positive constant $\kappa$ that for simplicity we take equal to 1. For $u \in S$ set $T_t u(x) = E(u(X_t + x))$, $x \in \mathbb{R}^n$, $t \geq 0$. Then, by Fubini’s Theorem, $T_t u(\xi) = e^{-t|\xi|^{2s}}\hat{u}(\xi)$. Therefore, the function $v(x, t) = T_t u(x)$ solves the fractional diffusion equation

$$\begin{cases}
\partial_t v = -(-\Delta)^s v & \text{in } \mathbb{R}^n \times (0, \infty) \\
v(x, 0) = u(x) & \text{on } \mathbb{R}^n.
\end{cases}$$

There is a Markov process corresponding to the fractional powers of the Dirichlet Laplacian $-\Delta_D$ in a smooth bounded domain $\Omega$. The process can be
obtained as follows: we first kill a Wiener process $W$ at $\tau_\Omega$, the first exit time of $W$ from $\Omega$, and then we subordinate the killed Wiener process using an $s$-stable subordinator $T_t$. This subordinated process has generator $(-\Delta_D)^s$, see [87].

II. Financial mathematics. For a symmetric $2s$-stable Lévy process $X_t$ with $X_0 = x$ consider the optimal stopping time $\tau$ to maximize the function

$$u(x) = \sup_\tau \mathbb{E}[\varphi(X_\tau) : \tau < \infty]$$

where $\varphi \in C_0(\mathbb{R}^n)$. Then $u$ is a solution to the free boundary problem

$$
\begin{cases}
  u(x) \geq \varphi(x) & \text{in } \mathbb{R}^n \\
  (-\Delta)^s u(x) \geq 0 & \text{in } \mathbb{R}^n \\
  (-\Delta)^s u(x) = 0 & \text{in } \{u(x) > \varphi(x)\}. 
\end{cases}
$$

This obstacle problem arises as a pricing model for American options [36, 84, 85].

III. Elasticity, biology. An equivalent formulation of the problem Antonio Signorini posed in [83] consists in finding the configuration of an elastic membrane in equilibrium that stays above some given thin obstacle. In mathematical terms, given $\varphi \in C_0(\mathbb{R}^n)$, the solution to the Signorini problem is the function $U = U(x, y)$, $x \in \mathbb{R}^n$, $y \geq 0$, that satisfies

$$
\begin{cases}
  \partial_{yy} U + \Delta_x U = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
  U(x, 0) \geq \varphi(x) & \text{on } \mathbb{R}^n \\
  \partial_y U(x, 0) \leq 0 & \text{on } \mathbb{R}^n \\
  \partial_y U(x, 0) = 0 & \text{in } \{U(x, 0) > \varphi(x)\},
\end{cases}
$$

see for example [23, 42]. A simple observation gives an equivalent description of the problem as an obstacle problem for the fractional Laplacian. The solution to $\partial_{yy} U + \Delta_x U = 0$ with boundary data $u(x) := U(x, 0)$ is given by convolution with the Poisson kernel in the upper half space:

$$U(x, y) = e^{-y(-\Delta_x)^{1/2}} u(x).$$

Taking the derivative of $U$ with respect to $y$ and evaluating it at $y = 0$ gives

$$
\partial_y U(x, y)|_{y=0} = (-\Delta_x)^{1/2} u(x).
$$

Hence, the Signorini problem can be rewritten as

$$
\begin{cases}
  \partial_{yy} U + \Delta_x U = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
  U(x, 0) \geq \varphi(x) & \text{on } \mathbb{R}^n \\
  (-\Delta_x)^{1/2} U(x, 0) \geq 0 & \text{on } \mathbb{R}^n \\
  (-\Delta_x)^{1/2} U(x, 0) = 0 & \text{in } \{U(x, 0) > \varphi(x)\}.
\end{cases}
$$
In other words, the Signorini problem is equivalent to the obstacle problem (2) for \( s = 1/2 \) through the relation \( u(x) = U(x,0) \) given by (3).

Consider next a Signorini problem where the Laplacian \(-\Delta_x\) is replaced by another partial differential operator \( L \) in a domain \( \Omega \subseteq \mathbb{R}^n \). For example, \( L \) can be the Dirichlet Laplacian \(-\Delta_D\) (meaning the elastic membrane is kept at zero level on \( \partial \Omega \)) or the heat operator \( \partial_t - \Delta \) (this becomes a model for semipermeable walls, like a cell membrane on \( y = 0 \), see [42]). The associated Poisson semigroup

\[
U(x,y) = e^{-yL^{1/2}} u(x)
\]

is the solution to

\[
\begin{cases}
\partial_{yy} U - LU = 0 & \text{in } \Omega \times (0,\infty) \\
U \big|_{y=0} = u & \text{on } \Omega
\end{cases}
\]

and satisfies

\[
\partial_y U \big|_{y=0} = -L^{1/2}u.
\]

Then the Signorini problem for \( L \) in place of \(-\Delta_x\) can be formulated for \( u = U \big|_{y=0} \) in an equivalent way as an obstacle problem for \( L^{1/2} \):

\[
\begin{cases}
u \geq \varphi & \text{in } \Omega \\
L^{1/2}u \geq 0 & \text{in } \Omega \\
L^{1/2}u = 0 & \text{in } \{ u > \varphi \},
\end{cases}
\]

see [4, 20, 90, 92].

Our list of problems above does not pretend to be exhaustive at all. Just to mention some more, there are applications in fluid mechanics [30, 35], fractional kinetics and anomalous diffusion [71, 86, 101], strange kinetics [82], fractional quantum mechanics [64, 65], Lévy processes in quantum mechanics [75], plasmas [2], electrical propagation in cardiac tissue [20], and biological invasions [9].

### 1.2 The method of semigroups

Consider the situation where we have derived a model (usually a nonlinear PDE problem) that involves a fractional power of some partial differential operator \( L \). As we saw before, \( L \) can be a Laplacian or a heat operator, or even an operator on a manifold [7, 39] or a lattice in the case of discrete models [34]. Then we are faced at least with the following basic questions.

**(I) Definition and pointwise formula for fractional operators.** For a general operator \( L \), classical functional analysis gives several ways to define \( L^s \)
according to its analytical properties. Nevertheless, a pure abstract formula is not useful to treat concrete PDE problems and a more or less explicit pointwise expression for $L^s u(x)$ is needed in many cases. The starting point for the method of semigroups is the formula

$$L^s u = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tL}u - u) \frac{dt}{t^{1+s}} \quad 0 < s < 1$$

where $\Gamma(-s) = \frac{\Gamma(s)}{(-s)}$ is the Gamma function evaluated at $-s$. Here $v = e^{-tL}u$ is the heat diffusion semigroup generated by $L$ acting on $u$, namely, $v$ the solution to the heat equation for $L$ with initial temperature $u$:

$$\begin{align*}
\partial_t v &= -Lv \quad \text{for } t > 0 \\
|v|_{t=0} &= u.
\end{align*}$$

The semigroup formula for $L^s$ is classical, see [6, 60, 62, 99]. The definition is motivated by the numerical identity

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}} \quad \text{for any } \lambda \geq 0 \quad (4)$$

that can be easily checked with a simple change of variables. In a similar way, starting from the numerical identity

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} \frac{dt}{t^{1-s}} \quad \text{for any } \lambda > 0, \ s > 0 \quad (5)$$

we can write down the solution to $L^s u = f$ as

$$u = L^{-s} f = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tL} f \frac{dt}{t^{1-s}}.$$

Again, this semigroup formula for $L^{-s}$ is classical, see [6, 60, 62, 99]. It turns out these are quite concrete and useful ways of defining and understanding fractional operators. Indeed, when a heat kernel is available for the semigroup $e^{-tL}$, then pointwise formulas for both positive and negative powers of $L$ can be obtained, see [13, 14, 29, 32, 33, 38, 39, 45, 70, 76, 90, 91, 92, 93, 94]. For degenerate cases like the usual derivative or discrete derivatives see [11, 11].

In this chapter we will explain how these formulas work only for the case of $L = -\Delta$, as developed in [90, 91]. Sections 2 and 3 are devoted to show how the
semigroup definitions of \((-\Delta)^s\) and \((-\Delta)^{-s}\) follow from the above-mentioned numerical formulas and how, with the help of the classical heat semigroup kernel, one can obtain the well-known nonlocal pointwise formula

\[
(-\Delta)^s u(x) = c_{n,s} \text{ P. V.} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, dz
\]

and similarly for \((-\Delta)^{-s} f(x)\). Obviously, these formulas are very well-known [63, 89] and can be deduced through several other techniques. Nevertheless, we present the semigroup ideas in this simple case so the reader can use them in other applications.

(II) The nonlocal nature. The fractional Laplacian is a nonlocal operator. Indeed, the value of \((-\Delta)^s u(x)\) for a given \(x \in \mathbb{R}^n\) depends on the values of \(u\) at infinity. Also, in general, if \(u\) has compact support then \((-\Delta)^s u\) has noncompact support. This basic property may create some issues. For example, the classical local PDE methods from the calculus of variations based on integration by parts and localization using test functions cannot be directly applied to the study of nonlinear problems for \((-\Delta)^s\). Even the notion of viscosity solution needs to be redefined to take into account the values of solutions at infinity [26]. L. A. Caffarelli and L. Silvestre showed in [24] that any fractional power of the Laplacian can be characterized as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension PDE problem. From a probabilistic point of view, the extension problem corresponds to the property that all symmetric stable processes can be obtained as traces of degenerate Bessel diffusion processes, see [72]. Consider the function

\[
U(x,y) = U(x,y) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}
\]

that solves the degenerate elliptic boundary value problem

\[
\begin{aligned}
\Delta_x U + \frac{a}{y} U_y + U_{yy} &= 0 & \quad & x \in \mathbb{R}^n, \ y > 0 \\
U(x,0) &= u(x) & \quad & x \in \mathbb{R}^n
\end{aligned}
\]

where \(a = 1 - 2s\). Then, for any \(x \in \mathbb{R}^n\),

\[
\lim_{y \to 0^+} y^{1-2s} U_y(x,y) = c_s (-\Delta)^s u(x)
\]

see [24]. The constant \(c_s > 0\) was computed explicitly for the first time in [90, 91]. We can interpret this result as saying that the new variable \(y\) added to extend \(u\) to the upper half space through \(U\) encodes the values of \(u\) at infinity needed to compute \((-\Delta)^s u\). The extension problem localizes the fractional Laplacian: it is enough to know \(U\) in some upper half ball around \((x,0)\) to already get \((-\Delta)^s u(x)\). The nonlinear problems for the nonlocal fractional Laplacian can then be localized by adding a new variable. Now one can exploit the classical
PDE tools and ideas that are available for these equations \[44\]. The work of Caffarelli and Silvestre \[24\] presented applications to Harnack inequalities and monotonicity formulas for \((-\Delta)^s\) by applying such local PDE techniques in the extension problem. Since then, \[24\] has created an explosion of results on problems with fractional Laplacians, see \[23, 30\] for a couple of important examples.

In general, fractional power operators \(L^s\) are nonlocal operators. It would be very useful in applications to have an analogous to the Caffarelli–Silvestre characterization for \(L^s\) as a Dirichlet-to-Neumann map via an extension problem. This open problem was solved in \[90, 91\]. The author and Torrea discovered an extension problem for fractional operators on Hilbert spaces. Later on, J. E. Galé, P. J. Miana and the author found an extension problem characterization for fractional powers of operators in Banach spaces and, more generally, generators of integrated semigroups, see \[47\]. In addition, \[47\] included the case of complex fractional power operators. The semigroup point of view turned out to be fundamental. As a matter of fact, when \(L = -\Delta\) in \[47, 90, 91\] then one recovers the extension PDE of \[24\]. Some of the main novelties of \[47, 90, 91\] were the analysis of the extension PDE by means of Bessel functions and the explicit semigroup formulas for the solution

\[
U(y) = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-tL} u \frac{dt}{t^{1+s}}
\]

These were new even for the case of the fractional Laplacian. As it could be expected, these general extension problems found many applications such as free boundary problems \[2, 4\], fractional derivatives \[11\], master equations \[14, 38, 92\], fractional elliptic PDEs \[29, 95, 100\], fractional Laplacians on manifolds \[7, 28, 31, 39, 46\] and in infinite dimensions \[74\], symmetrization \[40, 45\], nonlocal Monge–Ampère equations \[70\], numerical analysis \[73\], biology \[93\] and inverse problems \[48, 51\].

On the other hand, an extension problem for higher powers of fractional operators in Hilbert spaces using heat semigroups was proved in \[76\], see also \[98\] for the particular case of the fractional Laplacian on \(\mathbb{R}^n\). The fractional powers of the Laplacian can also be characterized by means of a wave extension problem, see \[61\]. In such scenario the wave and Schrödinger groups (instead of the heat semigroup) play a key, fundamental role.

We will not go into more details about all these general cases here, but we will only show how the semigroup ideas, techniques and formulas of \[47, 90, 91\]
work for the extension problem in the fractional Laplacian case, see Section 4. Applications to Harnack inequalities and derivative estimates for $s$-harmonic functions are given in Section 5 by following [23, 24].

(III) Regularity theory for fractional operators. Clearly the Fourier transform definition of the fractional Laplacian does not seem to be the most useful formulation to prove regularity estimates in Hölder and Zygmund spaces. One strategy that has been followed for this problem is to make heavy use of the pointwise formulas for $(-\Delta)^s$ and $(-\Delta)^{-s}$, see, for example, [84, 85]. This is only natural as pointwise formulas clearly allow us to handle differences of the form $|(-\Delta)^s u(x_1) - (-\Delta)^s u(x_2)|$.

We present here a semigroup method towards proving regularity estimates, where only the semigroup formulas for the fractional operators are needed. We will first show that Hölder and Zygmund spaces are characterized by means of the growth of time derivatives of the heat semigroup $\partial_t^k e^{t\Delta}$, see Section 6. The proof of such characterization is obviously nontrivial. It will be shown in Section 7 how the semigroup descriptions of Hölder–Zygmund spaces and fractional Laplacians allow for a quick, elegant proof of Hölder and Schauder estimates. We believe this is the first time these results have been presented and proved in such a systematic, complete way for the case of the fractional Laplacian.

If we now think about fractional powers of other differential operators $L$, we may ask for the “right” Schauder estimates for $L^s$. More precisely, what is the proper/adapted Hölder space to look for regularity properties of $L^s$? The semigroup approach then comes at hand: one can define regularity spaces associated to $L$ in terms of the growth of heat semigroups $\partial_t^k e^{-tL}$ in complete analogy to the case of the classical Hölder–Zygmund spaces. As we mentioned, passing from a semigroup formulation to a pointwise description of such spaces is a nontrivial task that must be carefully handled in each particular situation. Despite this, the great advantage is that the regularity properties of fractional powers $L^s$ on these spaces will follow at once using the semigroup representations. See, for example, [29] for the fractional Laplacian, [69, 93] for Schrödinger operators $L = -\Delta + V$, [50, 68] for the Ornstein–Uhlenbeck operator $L = -\Delta + \nabla \cdot x$, [14, 38, 92] for fractional powers of parabolic operators, [76] for the fractional Laplacian on the torus, and [13] for Bessel operators and radial solutions to the fractional Laplacian.

As we said before, the fractional Laplacian is a classical object in mathematics, and many of the results we will present here can be proved in several different ways and with other techniques. An exhaustive list of classical and modern references dealing with them is out of the scope of this chapter and the
reader is invited to explore the references mentioned at the beginning of this section as well as those contained in other chapters of this volume.

2 Fractional Laplacian: semigroups, pointwise formulas and limits

Recall the Fourier transform definition of the fractional Laplacian given in (1). It is obvious that \((-\Delta)^0 u = u, (-\Delta)^1 u = -\Delta u\) and, for any \(s_1, s_2\) we have \((-\Delta)^{s_1} \circ (-\Delta)^{s_2} u = (-\Delta)^{s_1+s_2} u\). Even though \(|\xi|^{2s}\hat{u}(\xi)\) is a well defined function of \(\xi \in \mathbb{R}^n\), we still have \((-\Delta)^s u / \in \mathcal{S}\) because \(|\xi|^{2s}\) creates a singularity at \(\xi = 0\). On the other hand, (1) implies that for any multi-index \(\gamma \in \mathbb{N}_0^n\),

\[
D^\gamma_x (-\Delta)^s = (-\Delta)^s D^\gamma_x. \tag{6}
\]

In particular, if \(u \in \mathcal{S}\) then \((-\Delta)^s u \in C^\infty(\mathbb{R}^n)\).

To compute \((-\Delta)^s u(x)\) for each point \(x \in \mathbb{R}^n\) one could try to take the inverse Fourier transform in (1). In fact, since \(|\xi|^{2s}\hat{u}(\xi) \in L^1(\mathbb{R}^n)\), one can make sense to \((-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s}\hat{u}(\xi))(x)\). But here we are going to avoid this and, instead, apply the method of semigroups. If we choose \(\lambda = |\xi|^2\), for \(\xi \in \mathbb{R}^n\), in the numerical formula (4), multiply it by \(\hat{u}(\xi)\) and recall (1), then

\[
(-\Delta)^s u(\xi) = |\xi|^{2s}\hat{u}(\xi) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t|\xi|^2}\hat{u}(\xi) - \hat{u}(\xi)) \frac{dt}{t^{1+s}}.
\]

Thus, by inverting the Fourier transform, we obtain the semigroup formula for the fractional Laplacian (see [6] [60] [62] [99], also [17] [90] [91])

\[
(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+s}}. \tag{7}
\]

The family of operators \(\{e^{t\Delta}\}_{t \geq 0}\) is the classical heat diffusion semigroup generated by \(\Delta\). Consider the solution \(v = v(x, t)\), for \(x \in \mathbb{R}^n\) and \(t \geq 0\), of the heat equation on the whole space \(\mathbb{R}^n\) with initial temperature \(u\):

\[
\begin{aligned}
\partial_t v &= \Delta v & \text{for } x \in \mathbb{R}^n, \ t > 0 \\
v(x, 0) &= u(x) & \text{for } x \in \mathbb{R}^n.
\end{aligned}
\]
If we apply the Fourier transform in the variable $x$ for each fixed $t$ then

$$
\hat{v}(\xi, t) = e^{-t|\xi|^2} \hat{u}(\xi) = e^{t\Delta} \hat{u}(\xi) \quad (8)
$$

so that $u \mapsto e^{t\Delta}u$ is the solution operator. It is well known that

$$
v(x, t) \equiv e^{t\Delta}u(x) = G_t * u(x) = \int_{\mathbb{R}^n} G_t(x - z) u(z) \, dz
$$

where $G_t(x)$ is the Gauss–Weierstrass heat kernel:

$$
G_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}. \quad (9)
$$

Observe that $G_t$ defines an approximation of the identity. Moreover,

$$
e^{t\Delta}1(x) = \int_{\mathbb{R}^n} G_t(x) \, dx \equiv 1 \quad \text{for any } x \in \mathbb{R}^n, \ t > 0. \quad (10)
$$

**Remark 1** (Maximum principle). The semigroup formula (7) and the positivity of the heat kernel (9) easily imply the maximum principle for the fractional Laplacian. Indeed, if $u \geq 0$ and $u(x_0) = 0$ at some point $x_0 \in \mathbb{R}^n$ then

$$
(-\Delta)^s u(x_0) = \frac{1}{\Gamma(-s)} \int_0^\infty e^{t\Delta} u(x_0) \frac{dt}{t^{1+s}} \leq 0.
$$

Moreover, $(-\Delta)^s u(x_0) = 0$ if and only if $e^{t\Delta} u(x_0) = 0$, that is, only when $u \equiv 0$. For another proof using pointwise formulas, see [84, 85]. For maximum principles for fractional powers of elliptic operators using semigroups, see [95].

The semigroup formula (7) and the heat kernel (9) permit us to compute the pointwise formula for the fractional Laplacian. The technique avoids the inverse Fourier transform in (1) and gives the constants explicitly.

**Theorem 1** (Pointwise formulas). Let $u \in \mathcal{S}$, $x \in \mathbb{R}^n$ and $0 < s < 1$.

(i) If $0 < s < 1/2$ then

$$
(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+2s}} \, dz
$$

and the integral is absolutely convergent.

(ii) If $1/2 \leq s < 1$, for any $\delta > 0$,

$$
(-\Delta)^s u(x) = c_{n,s} \lim_{\varepsilon \to 0^+} \int_{|x-z| > \varepsilon} \frac{u(x) - u(z)}{|x - z|^{n+2s}} \, dz
$$
\[ u(x) - u(z) - \nabla u(x) \cdot (x - z) \chi_{|x-z|<\delta}(z) \]

where the second integral is absolutely convergent.

The constant \( c_{n,s} > 0 \) in the formulas above is explicitly given by

\[
c_{n,s} = 4^s \frac{\Gamma(n/2 + s)}{|\Gamma(-s)|\pi^{n/2}} = \frac{s(1-s)4^s\Gamma(n/2 + s)}{|\Gamma(2-s)|\pi^{n/2}}.
\]

In particular, \( c_{n,s} \sim s(1-s) \) as \( s \to 0^+ \) and \( s \to 1^- \).

**Sketch of proof.** The detailed proof using the heat kernel can be found in [90, 91]. We write down the formula for \( e^{t\Delta}u(x) \) in (7) and use (10) to get

\[
(-\Delta)^s u(x) = \frac{1}{|\Gamma(-s)|} \int_0^\infty \int_{\mathbb{R}^n} G_t(x-z)(u(x) - u(z)) \, dz \, dt \, \frac{dt}{t^{1+s}}.
\]

By (9) and the change of variables \( r = |x-z|^2/(4t) \),

\[
\frac{1}{|\Gamma(-s)|} \int_0^\infty G_t(x-z) \, dt \, \frac{dt}{t^{1+s}} = c_{n,s} \cdot \frac{1}{|x-z|^{n+2s}}.
\]

When \( 0 < s < 1/2 \) the double integral in (12) is absolutely convergent and Fubini’s theorem gives (i). If \( 1/2 \leq s < 1 \) then one needs to use the fact that, for any \( i = 1, \ldots, n \) and \( 0 < \varepsilon < \delta \),

\[
\int_{|z|<\delta} z_i G_t(z) \, dz = \int_{\varepsilon<|z|<\delta} z_i G_t(z) \, dz = 0
\]

for all \( t > 0 \), for the gradient term to appear in (ii).

The pointwise formulas in Theorem 4 are valid for \( u \in \mathbb{S} \). However, the integrals are well defined for less regular functions. As a matter of fact, we can relax the requirement on \( u \) at infinity by asking that

\[
\|u\|_{L^s} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, dx < \infty.
\]

When \( 0 < s < 1/2 \), we only need \( u \) to be \( C^{2s+\varepsilon} \) at \( x \), for \( 2s + \varepsilon \leq 1 \), for the singular part of the integral (that is, when \( z \) is near \( x \)) to be finite:

\[
\int_{|x-z|<1} \frac{|u(x) - u(z)|}{|x-z|^{n+2s}} \, dz \leq [u]_{C^{2s+\varepsilon}(x)} \int_{|x-z|<1} \frac{|x-z|^{2s+\varepsilon}}{|x-z|^{n+2s}} \, dz < \infty.
\]
Similarly, when $1/2 \leq s < 1$, the requirement $u \in C^{1,2s+\varepsilon-1}$ at $x$, for $2s+\varepsilon-1 \leq 1$ would suffice as well.

To define the fractional Laplacian for less regular functions, we need to understand what is $(-\Delta)^s$ in the sense of distributions. The fractional Laplacian is a symmetric operator on $L^2(\mathbb{R}^n)$: when $u, f \in \mathcal{S}$,

$$
\int_{\mathbb{R}^n} (-\Delta)^s u(x) f(x) \, dx = \int_{\mathbb{R}^n} |\xi|^{2s} \hat{u}(\xi) \hat{f}(\xi) \, d\xi = \int_{\mathbb{R}^n} u(x)(-\Delta)^s f(x) \, dx.
$$

One may then think in the following way. For $u \in \mathcal{S}'$ (a tempered distribution) and $f \in \mathcal{S}$ one could define the distribution $(-\Delta)^s u$ as $\langle (-\Delta)^s u, f \rangle = \langle u, (-\Delta)^s f \rangle$. The problem here is that $(-\Delta)^s f \notin \mathcal{S}$, so this identity makes no sense for $u \in \mathcal{S}'$. First we need to characterize the set $(-\Delta)^s(\mathcal{S})$, see [34, 84, 85].

**Lemma 1.** Let $f \in \mathcal{S}$. Then $(-\Delta)^s f$ belongs to the class $\mathcal{S}_s$ defined by

$$
\mathcal{S}_s = \{ \psi \in C^\infty(\mathbb{R}^n) : (1 + |x|^{n+2s}) D^\gamma \psi(x) \in L^\infty(\mathbb{R}^n), \text{ for every } \gamma \in \mathbb{N}_0^n \}.
$$

The class $\mathcal{S}_s$ of Lemma 1 is endowed with the topology induced by the countable family of seminorms

$$
\rho_\gamma(\psi) = \sup_{x \in \mathbb{R}^n} |(1 + |x|^{n+2s}) D^\gamma \psi(x)|, \quad \gamma \in \mathbb{N}_0^n.
$$

Denote by $\mathcal{S}'_s$ the dual space of $\mathcal{S}_s$. Observe that $\mathcal{S} \subset \mathcal{S}_s$, so that $\mathcal{S}'_s \subset \mathcal{S}'$. The suitable space for the distributional definition of the fractional Laplacian is $\mathcal{S}'_s$.

**Definition 1.** Let $u \in \mathcal{S}'_s$. We define $(-\Delta)^s u \in \mathcal{S}'$ as

$$
((-\Delta)^s u)(f) = u((-\Delta)^s f) \quad \text{for every } f \in \mathcal{S}.
$$

In terms of pairings, we write $\langle (-\Delta)^s u, f \rangle_{\mathcal{S}'_s,\mathcal{S}_s} = \langle u, (-\Delta)^s f \rangle_{\mathcal{S}'_s,\mathcal{S}_s}$.

If $u \in \mathcal{S}$ then this distributional definition coincides with the one given in terms of the Fourier transform. Also, $(-\Delta)^s$ maps $\mathcal{S}'_s$ into $\mathcal{S}'$ continuously. Recall the definition of the space $L_s$ given in [14]. We have $L_s = L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'_s$. The proof of the following result is based on an approximation argument and the details can be found in [34, 85].

**Theorem 2** (Pointwise formula for less regular functions). Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $u \in L_s$, $0 < s < 1$. If $u \in C^{2s+\varepsilon}(\Omega)$ (or $C^{1,2s+\varepsilon-1}(\Omega)$ if $s \geq 1/2$) for some $\varepsilon > 0$ then $(-\Delta)^s u$ is a continuous function in $\Omega$ and $(-\Delta)^s u(x)$ is given by the pointwise formulas of Theorem 1 for every $x \in \Omega$. 

Remark 2 (Semigroup formula for less regular functions). It is an exercise to verify that the semigroup formula (7), that was initially derived for functions $u \in S$, also holds for the class of less regular functions $u$ considered in Theorem 2, for all $x \in \Omega$. Indeed, it is easy to work out the computations in the proof of Theorem 1 in a reverse order, namely, by starting with the pointwise integro-differential formula for $(-\Delta)^s u(x)$ and using the heat kernel identity (13) to end up with (7).

We turn our attention to the pointwise limits for $s \to 1^-$ and $s \to 0^+$ in the case of less regular functions. The explicit value of the constant $c_{n,s}$ in (11), that we found through the method of semigroups, plays a crucial role. Observe as well that we require minimal regularity on $u$ for $(-\Delta)^s u(x)$ to be defined through the integral formula and for the pointwise limits to have sense.

**Theorem 3.** If $u \in C^2(B_2(x)) \cap L^\infty(\mathbb{R}^n)$ at some $x \in \mathbb{R}^n$ then

$$
\lim_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x).
$$

**Sketch of proof.** The full details of the proof can be found in [90, 91]. We can write $(-\Delta)^s u(x)$ as

$$
c_{n,s} \int_{|x-z|>\delta} \frac{u(x) - u(z)}{|x-z|^{n+2s}} \, dz + c_{n,s} \int_{|x-z|<\delta} \frac{u(x) - u(z) - \nabla u(x) \cdot (x-z)}{|x-z|^{n+2s}} \, dz.
$$

Since $u$ is bounded the first term above converges to zero as $s \to 1^-$. The second term can be written as

$$
I - II := c_{n,s} \int_0^\delta r^{-1-2s} \int_{|z'|=1} \left[ \frac{r^2}{2} \langle D^2 u(x) z', z' \rangle - R_1 u(x, rz') \right] dS_{z'} \, dr
$$

$$
- c_{n,s} \int_0^\delta r^{-1-2s} \int_{|z'|=1} \frac{r^2}{2} \langle D^2 u(x) z', z' \rangle dS_{z'} \, dr
$$

where $R_1 u(x, rz') = u(x - rz') - u(x) + \nabla u(x) \cdot (rz')$. By the regularity of $u$, $|I| \leq C\varepsilon$ and using that

$$
\int_{|z'|=1} \langle D^2 u(x) z', z' \rangle \, dz' = \frac{(n/2 + 1)\pi^{n/2}}{\Gamma(n/2 + 2)} \Delta u(x)
$$

the conclusion follows.
Recall the definition of the space \( L_s \) from \([14]\), for \( 0 \leq s \leq 1 \).

**Theorem 4.** If \( u \in C^\alpha(B_2(x)) \cap L_0 \) for some \( x \in \mathbb{R}^n \) and \( 0 < \alpha < 1 \) then
\[
\lim_{s \to 0^+} (-\Delta)^s u(x) = u(x).
\]

**Sketch of proof.** The complete details of the proof are found in \([90]\). We have
\[
(-\Delta)^s u(x) = c_{n,s} \int_{|x-z| < R} \frac{u(x) - u(z)}{|x-z|^{n+2s}} dz + c_{n,s} \int_{|x-z| > R} \frac{u(x) - u(z)}{|x-z|^{n+2s}} dz
\]
for \( R = 1 + |x| \). Since \( u \) is regular, the first term above converges to zero as \( s \to 0^+ \). The second term is split into two: one with \( u(x) \) (that will converge to \( u(x) \) as \( s \to 0^+ \)) and the other one with \( u(y) \) (that will converge to zero as \( s \to 0^+ \)).

\[\square\]

### 3 Inverse fractional Laplacian: semigroups, pointwise formula and the Poisson problem

If we apply the Fourier transform to solve the Poisson equation
\[
(-\Delta)^s u = f \quad \text{in} \, \mathbb{R}^n
\]
we find that \( |\xi|^{2s} \hat{u}(\xi) = \hat{f}(\xi) \). The inverse of the fractional Laplacian, or **negative power of the Laplacian** \((-\Delta)^{-s}, s > 0\), is defined for \( f \in \mathcal{S} \) as
\[
(-\Delta)^{-s} f(\xi) = |\xi|^{-2s} \hat{f}(\xi) \quad \text{for} \, \xi \neq 0.
\]
In principle, we need the restriction \( 0 < s < n/2 \) because when \( s \geq n/2 \) the multiplier \( |\xi|^{-2s} \) does not define a tempered distribution, see \([84, 85, 89]\). This operator is also known as the **fractional integral operator** in the harmonic analysis literature \([41, 89]\).

To find a pointwise expression for \((-\Delta)^{-s} f(x)\) at a point \( x \in \mathbb{R}^n \) one could try to compute the inverse Fourier transform in \((15)\). This is a delicate task as the Fourier multiplier \( |\xi|^{-2s} \) is not in \( L^2(\mathbb{R}^n) \), see \([89]\). Instead, we apply the method of semigroups. We start by choosing \( \lambda = |\xi|^2 \), \( \xi \neq 0 \), in the numerical formula \([5]\) to find that, for a.e. \( \xi \in \mathbb{R}^n \),
\[
(-\Delta)^{-s} f(\xi) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t|\xi|^2} \hat{f}(\xi) \frac{dt}{t^{1-s}}.
\]
Therefore, by inverting the Fourier transform, we obtain the semigroup formula for the inverse fractional Laplacian (see [6, 60, 62, 89, 99], also [90])

\[ (-\Delta)^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta} f(x) \frac{dt}{t^{1-s}}. \tag{16} \]

The proof of the following result using heat kernels when \( 0 < s < n/2 \) is contained in [90]. The proof of the case \( s = n/2 \) is an original contribution of the author for this volume. Indeed, it has not been published elsewhere yet.

**Theorem 5** (Fundamental solution). Let \( f \in \mathcal{S} \), \( x \in \mathbb{R}^n \) and \( 0 < s \leq n/2 \). In the case when \( s = n/2 \) assume in addition that \( \int_{\mathbb{R}^n} f = 0. \) Then

\[ (-\Delta)^{-s} f(x) = \int_{\mathbb{R}^n} K_{-s}(x-z)f(z) \, dz. \tag{17} \]

Here

\[
K_{-s}(x) = \begin{cases} 
1 & \text{if } 0 < s < n/2 \\
1 & \text{if } s = n/2 \\
\Gamma(n/2)(4\pi)^{n/2}(-2 \log |x| - \gamma) & \text{if } s = n/2 
\end{cases}
\]

where

\[ \gamma = -\int_0^\infty e^{-r} \log r \, dr \approx 0.577215 \]

is the Euler–Mascheroni constant and

\[ c_{n,-s} = \frac{\Gamma(n/2-s)}{4^s \Gamma(s) \pi^{n/2}} \text{ for } 0 < s < n/2. \]

The reader should compare the explicit constant \( c_{n,-s} \) in Theorem 5 (that we found through the method of semigroups, see [90]) with the constant \( c_{n,s} \) for the fractional Laplacian given in (11).

**Proof of Theorem 5 in the case \( 2s = n \).** As we mentioned above, this proof is an original contribution of the author for this volume. From (16) and (9), the change of variables \( r = 1/(4t) \) and the fact that \( f \) has zero mean,

\[
(-\Delta)^{-s} f(x) = \frac{1}{\Gamma(n/2)(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} e^{-r|z|^2} f(x-z) \, dz \, dr
\]

\[ = \frac{1}{\Gamma(n/2)(4\pi)^{n/2}} \int_0^\infty \int_{\mathbb{R}^n} (e^{-r|z|^2} - \chi(0,1)(r)) f(x-z) \, dz \, dr. \tag{18} \]
The second double integral in (18) is absolutely convergent. Indeed,

\[ \int_{0}^{\infty} \left| e^{-r}|z|^2 - \chi_{(0,1)}(r) \right| \frac{dr}{r} = \int_{0}^{\infty} \left( 1 - e^{-r} \right) \frac{dr}{r} + \int_{\infty}^{\infty} e^{-r} \frac{dr}{r} \]

\[ = \int_{0}^{\infty} (1 - e^{-r}) \frac{d}{dr}(\log r) \, dr + \int_{\infty}^{\infty} e^{-r} \frac{d}{dr}(\log r) \, dr \]

\[ = (1 - e^{-|z|^2}) \log(|z|^2) - \int_{0}^{\infty} e^{-r} \log r \, dr - e^{-|z|^2} \log(|z|^2) + \int_{\infty}^{\infty} e^{-r} \log r \, dr \]

\[ \leq C + C |\log |z|| \in L_{1}^{1}(\mathbb{R}^{n}). \]

Thus we can apply Fubini’s theorem in (18). By following the computation we just did above we get the formula for the kernel:

\[ K_{-n/2}(z) = \frac{1}{\Gamma(n/2)(4\pi)^{n/2}} \left[ \int_{0}^{1} (e^{-r}|z|^2 - 1) \frac{dr}{r} + \int_{1}^{\infty} e^{-r}|z|^2 \frac{dr}{r} \right] \]

\[ = \frac{1}{\Gamma(n/2)(4\pi)^{n/2}} \left[ - \log(|z|^2) + \int_{0}^{\infty} e^{-r} \log r \, dr \right]. \]

\[ \square \]

**Theorem 6** (Distributional solvability). Let \( f \in L^{\infty}(\mathbb{R}^{n}) \) with compact support and \( 0 < s < \min\{1, n/2\} \). Define

\[ u(x) = (-\Delta)^{-s} f(x) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} e^{t \Delta} f(x) \frac{dt}{t^{1-s}}. \]

Then \( u \) is given by the pointwise formula (17), \( u \in L^{\infty}(\mathbb{R}^{n}) \) with

\[ \|u\|_{L^{\infty}(\mathbb{R}^{n})} \leq C_{n,s} \|f\|_{L^{\infty}(\mathbb{R}^{n})} \]

for some positive constant \( C_{n,s} \), and

\[ |u(x)| \to 0 \quad \text{as} \quad |x| \to \infty. \]

In addition, \((-\Delta)^{s} u = f\) in the sense of distributions.

The proof of Theorem 6 follows the exact same lines as the proof of the one dimensional case presented in [34, Theorem 9.9].
4 Extension problem: semigroup approach, weak formulation

As we mentioned in the introduction, the extension problem for the fractional Laplacian is a characterization of \((-\Delta)^s\) as the Dirichlet-to-Neumann map for a local degenerate elliptic PDE. This localization technique was introduced and exploited in the PDE context by Caffarelli and Silvestre \[24\]. The method of semigroups for this problem that the author and Torrea developed in \[90, 91\] (see also \[47\]) provided new insights, explicit formulas for the solution, the useful Bessel functions analysis and, ultimately, a unified approach. To present the extension problem, for \(0 < s < 1\), we let

\[
a = 1 - 2s \in (-1, 1).
\]

**Theorem 7** (Extension problem for positive powers). Let \(u \in S\). The unique solution \(U = U(x,y) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}\) to

\[
\begin{align*}
\Delta U + \frac{a}{y} U_y + U_{yy} &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\
U(x,0) &= u(x) \quad \text{on } \mathbb{R}^n
\end{align*}
\]

that weakly vanishes as \(y \to \infty\) is given by the following formulas

\[
U(x,y) = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{t\Delta} u(x) \frac{dt}{t^{1+s}}
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} e^{-\frac{y^2}{4t}} \Delta u(x) \frac{dt}{t^{1-s}}
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{t\Delta} ((-\Delta)^s u)(x) \frac{dt}{t^{1-s}}
\]

\[
= \frac{\Gamma(n/2 + s)}{\Gamma(s) \pi^{n/2}} \int_{\mathbb{R}^n} \frac{y^{2s}}{y^2 + |x - z|^2 (n+2s)/2} u(z) dz.
\]

Moreover \(U \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))\) satisfies

\[
- \lim_{y \to 0^+} y^a U_y(x,y) = \frac{\Gamma(1-s)}{4^s - 1/2 \Gamma(s)} (-\Delta)^s u(x)
\]

and

\[
- \lim_{y \to 0^+} \frac{U(x,y) - U(x,0)}{y^{2s}} = \frac{\Gamma(1-s)}{4^s \Gamma(1+s)} (-\Delta)^s u(x).
\]
The first three formulas for $U$ in (20) are due to [90, 91], while the last one was found in [24]. The explicit constants appearing in the limits (21) and (22) were first discovered in [90, 91]. Observe that when $s = 1/2$ the first two formulas in (20) reduce to the so-called Bochner subordination formula see [15, 16], also [58, 88]. Nevertheless, the third formula in (20) is original from [90, 91] even for the case $s = 1/2$.

The idea for solving explicitly the extension problem by using Bessel functions was introduced in [90, 91]. For each $y > 0$ we apply the Fourier transform in the variable $x$ to (19) to get an ODE of the form

$$\begin{cases} f''_\xi(y) + \frac{a}{y} f'_\xi(y) = \lambda f_\xi(y) & \text{for } y > 0 \\ f_\xi(0) = \hat{u}(\xi) \end{cases}$$

where $\lambda = |\xi|^2$ and $f_\xi(y) = \hat{U}(\xi, y)$. This is a Bessel differential equation. The condition that $U$ weakly vanishes as $y \to \infty$ translates in the fact that $f_\xi(y) \to 0$ as $y \to \infty$. Therefore the unique solution is (see [90, 91])

$$\hat{U}(\xi, y) = \frac{21-s}{\Gamma(s)} (y|\xi|)^s \mathcal{K}_s(y|\xi|) \hat{u}(\xi)$$

(23)

where $\mathcal{K}_s$ is the modified Bessel function of the second kind or Macdonald’s function (see [66]). Notice that $\mathcal{K}_{1/2}(z) = (\frac{\pi}{2z})^{1/2} e^{-z}$ so, when $s = 1/2$, $\hat{U}(\xi, y) = e^{-y|\xi|} \hat{u}(\xi)$, which is the classical Poisson semigroup for the harmonic extension of $u$ to the upper half space [3]. By inverting the Fourier transform in (23) we obtain the functional calculus identity

$$U(x, y) = \frac{21-s}{\Gamma(s)} (y(-\Delta)^{1/2})^s \mathcal{K}_s(y(-\Delta)^{1/2}) u(x).$$

Let us recall the following integral formula for the Bessel function (see [66]):

$$\mathcal{K}_s(z) = \frac{1}{2} \left( \frac{z}{2} \right)^s \int_0^\infty e^{-t} e^{-z^2/(4t)} \frac{dt}{t^{1+s}}.$$

We choose $z = y|\xi|$ and apply the change of variables $y^2/(4t) \to t$ to get

$$\hat{U}(\xi, y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-t|\xi|^2} (|\xi|^{2s} \hat{u}(\xi)) \frac{dt}{t^{1-s}}.$$

Because of (8), this is in fact the second to last formula for $U$ in (20).

This Bessel function analysis was applied in the extension problem for other fractional operators in [11, 14, 29, 39, 73, 90].
The second identity in (20) follows from the first one by the change of variables $y^2/(4t) \rightarrow t$. The third one is obtained by computing the Fourier transform of $U$ in $x$ in the first formula

$$\hat{U}(\xi, y) = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-t|\xi|^2} \hat{u}(\xi) \frac{dt}{t^{1+s}}$$

and performing the change of variables $y^2/(4t|\xi|^2) \rightarrow t$. The last convolution formula in (20), found for the first time in [24], follows immediately from the first one, see [90, 91] for the details.

Now that several identities for $U$ were given, all the properties established in Theorem 7 are easy to verify. For example, using that $\Delta e^{t\Delta} u = \partial_t e^{t\Delta} u$ plus an integration by parts,

$$\Delta U(x, y) = -\frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty \partial_t \left( e^{-y^2/(4t)} \right) \frac{e^{t\Delta} u(x) dt}{t^{1+s}}$$

$$= -\frac{a}{y} U_y(x, y) - U_{yy}(x, y).$$

The second identity in (20) immediately gives that $\lim_{y \rightarrow 0^+} U(x, y) = u(x)$.

Estimates for $U$ in terms of $u$ are easy to obtain by using the semigroup formulas and the fact that, for any $y > 0$,

$$\frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4t)} \frac{dt}{t^{1+s}} = 1. \quad (24)$$

For example, it is readily seen that $\|U(\cdot, y)\|_{L^p(\mathbb{R}^n)} \leq \|u\|_{L^p(\mathbb{R}^n)}$, for all $y \geq 0$, for any $1 \leq p \leq \infty$.

By using (24) and the semigroup formula for the fractional Laplacian (7),

$$y^a U_y(x, y) = \frac{y^{1-2s}}{4^s \Gamma(s)} \int_0^\infty \partial_y \left( y^{2s} e^{-y^2/(4t)} \right) \left( e^{t\Delta} u(x) - u(x) \right) \frac{dt}{t^{1+s}}$$

$$\rightarrow \frac{2s}{4^s \Gamma(s)} \int_0^\infty \left( e^{t\Delta} u(x) - u(x) \right) \frac{dt}{t^{1+s}} = -\frac{\Gamma(1-s)}{4^s-1/2 \Gamma(s)} (-\Delta)^s u(x)$$

as $y \rightarrow 0^+$. Similarly, one can check that (22) holds.

At this point the reader can observe that the semigroup approach gives not only clear proofs, but can also avoid the use of the Fourier transform and the special symmetries of the Laplacian. Indeed, it relies only on heat semigroups.
and kernels. In addition, as we have already mentioned, the methods have a wide applicability in a variety of different contexts.

The extension problem can be written in an equivalent way as an extension problem for the negative powers of the fractional Laplacian \((-\Delta)^{-s}\). This is an immediate consequence of the third formula for \(U\) in (20) and the results of Theorem 7, see [90, 91]. For such explicit statement for negative powers \(L^{-s}\) in other contexts like manifolds and discrete settings, see [34, 39, 45, 70].

**Theorem 8** (Extension problem for negative powers). Let \(f \in S\). The unique smooth solution \(U = U(x, y) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}\) to the Neumann extension problem

\[
\begin{aligned}
\Delta U + \frac{4}{y^2} U_y + U_{yy} &= 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
-\gamma^s U_y(x, y) \big|_{y=0} &= f(x) & \text{on } \mathbb{R}^n
\end{aligned}
\]

that weakly vanishes as \(y \to \infty\) is given by the formula

\[
U(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{t\Delta} f(x) \frac{dt}{t^{1-s}}.
\]

Moreover,

\[
\lim_{y \to 0^+} U(x, y) = \frac{4^{s-1/2} \Gamma(s)}{\Gamma(1-s)} (-\Delta)^{-s} f(x).
\]

We consider next weak solutions to the extension problem. It is easy to check that if \(u \in S\) then

\[
[u]^2_{H^s(\mathbb{R}^n)} \equiv ||(-\Delta)^{s/2} u||^2_{L^2(\mathbb{R}^n)} = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(z))^2}{|x-z|^{n+2s}} \, dx \, dz.
\]

The fractional Sobolev space \(H^s(\mathbb{R}^n), 0 < s < 1\), is defined as the completion of \(C^\infty_c(\mathbb{R}^n)\) under the norm

\[
\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + [u]^2_{H^s(\mathbb{R}^n)}.
\]

Then \(H^s(\mathbb{R}^n)\) is a Hilbert space with inner product

\[
\langle u, v \rangle_{H^s(\mathbb{R}^n)} = \langle u, v \rangle_{L^2(\mathbb{R}^n)} + \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(z))(v(x) - v(z))}{|x-z|^{n+2s}} \, dx \, dz.
\]

The dual of \(H^s(\mathbb{R}^n)\) is denoted by \(H^{-s}(\mathbb{R}^n)\). The definition of the fractional Laplacian can be extended to functions in \(H^s(\mathbb{R}^n)\). For any \(u \in H^s(\mathbb{R}^n)\) we define \((-\Delta)^s u\) as the element on \(H^{-s}(\mathbb{R}^n)\) that acts on \(v \in H^s(\mathbb{R}^n)\) via

\[
\langle (-\Delta)^s u, v \rangle_{H^{-s}, H^s} = \langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} v \rangle_{L^2(\mathbb{R}^n)}
\]
The extension problem can be posed in this $L^2$ setting. Notice that
\[
\Delta U + \frac{4}{y} U_y + U_{yy} = y^{-a} \text{div}_x,y(y^a \nabla_{x,y} U).
\]

The weighted Sobolev space $H^s_a \equiv H^1(\mathbb{R}^n \times (0, \infty), y^a dx dy)$ where $a = 1 - 2s$
is defined as the completion of $C_c^\infty(\mathbb{R}^n \times [0, \infty))$ under the norm
\[
\|U\|_{H^s_a}^2 = \|U\|_{L^2(\mathbb{R}^n \times (0, \infty), y^a dx dy)}^2 + \|\nabla_{x,y} U\|_{L^2(\mathbb{R}^n \times (0, \infty), y^a dx dy)}^2.
\]

Since $a \in (-1, 1)$, the weight $\omega(x, y) = y^a$ belongs to the Muckenhoupt class $A_2(\mathbb{R}^n \times (0, \infty))$, see [41] for details about these weights and [44, 97] for weighted Sobolev spaces. Then $H^s_a$ is a Hilbert space with inner product
\[
\langle U, V \rangle_{H^s_a} = \int_{0}^{\infty} \int_{\mathbb{R}^n} y^a U V \, dx \, dy + \int_{0}^{\infty} \int_{\mathbb{R}^n} y^a \nabla_{x,y} U \cdot \nabla_{x,y} V \, dx \, dy.
\]

Given $u \in L^2(\mathbb{R}^n)$ we say that $U \in H^1_a$ is a weak solution to the extension problem [19] if for every $V \in C_c^\infty(\mathbb{R}^n \times (0, \infty))$
\[
\int_{0}^{\infty} \int_{\mathbb{R}^n} y^a \nabla_{x,y} U \cdot \nabla_{x,y} V \, dx \, dy = 0
\]
and $\lim_{y \to 0^+} U(x, y) = u(x)$ in $L^2(\mathbb{R}^n)$. The proof of the following extension theorem in weak form is just a simple verification.

**Theorem 9.** Let $u \in H^s(\mathbb{R}^n)$. The unique weak solution $U \in H^1_a$ to the extension problem [19] is given by [20]. Moreover $U(\cdot, y) \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap C([0, \infty); L^2(\mathbb{R}^n))$ satisfies [21] and [22] in the sense of $H^{-s}(\mathbb{R}^n)$. In addition, for any $V \in C_c^\infty(\mathbb{R}^n \times [0, \infty))$,
\[
\int_{0}^{\infty} \int_{\mathbb{R}^n} y^a \nabla_{x,y} U \cdot \nabla_{x,y} V \, dx \, dy = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} \langle (-\Delta)^s u, V(\cdot, 0) \rangle_{H^{-s}, H^s}
\]
and also
\[
I_s(U) \equiv \int_{0}^{\infty} \int_{\mathbb{R}^n} y^a |\nabla_{x,y} U|^2 \, dx \, dy = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx.
\]
Moreover, $U$ is the unique minimizer of the energy functional $I_s(V)$ among all functions $V \in H^1_a$ such that $\lim_{y \to 0^+} V(x, y) = u(x)$ in $L^2(\mathbb{R}^n)$. 

5 Applications of the extension problem: Harnack inequality and derivative estimates

This section is devoted to show how the extension problem can be used to prove an interior Harnack inequality and derivative estimates for fractional harmonic functions. These original ideas are due to Caffarelli–Silvestre [24] and Caffarelli–Salsa–Silvestre [23]. Obviously, such results for the fractional Laplacian are classical [63], see also [8, 18, 26, 59, 80, 87] for other formulations, techniques and nonlocal operators. Harnack inequalities using the extension technique for fractional powers of operators in divergence form were systematically developed in [95]. For applications of the extension method to other fractional operators see [14, 29, 39, 46, 70, 76, 90, 91, 92, 93, 94]. Harnack inequalities for degenerate elliptic equations like the extension equation (19) were first proved by Fabes, Kenig and Serapioni in [44].

The following reflection lemma will be needed in the proof of Harnack inequality, see [24], also [95]. In what follows $\Omega$ denotes a domain in $\mathbb{R}^n$ that can be unbounded.

**Lemma 2.** Fix $Y > 0$. Suppose that a function $U = U(x, y) : \Omega \times (0, Y) \rightarrow \mathbb{R}$ satisfies

$$\text{div}(y^a \nabla_{x,y} U) = 0 \quad \text{in} \quad \Omega \times (0, Y)$$

in the weak sense, with

$$- \lim_{y \to 0^+} y^a U_y(x, y) = 0 \quad \text{on} \quad \Omega.$$

Namely, suppose that $U$ and $\nabla_{x,y} U$ belong to $L^2(\Omega \times (0, Y), y^a dx dy)$ and that for every test function $V \in C_c^\infty(\Omega \times [0, Y))$ we have

$$\int_0^\infty \int_\Omega y^a \nabla_{x,y} U \cdot \nabla_{x,y} V \, dx \, dy = 0$$

and

$$- \lim_{y \to 0^+} \int_\Omega y^a U_y(x, y) V(x, y) \, dx = 0.$$

Then the even reflection of $U$ in the variable $y$ defined as $\tilde{U}(x, y) = U(x, |y|)$, for $y \in (-Y, Y)$, is a weak solution to

$$\text{div}(|y|^a \nabla_{x,y} \tilde{U}) = 0 \quad \text{in} \quad \Omega \times (-Y, Y).$$
Theorem 10 (Interior Harnack inequality). Let $\Omega'$ be a domain such that $\Omega' \subset\subset \Omega$. There exists a constant $c = c(\Omega, \Omega', s) > 0$ such that for any solution $u \in H^s(\mathbb{R}^n)$ to

$$\begin{cases}
(-\Delta)^s u = 0 & \text{in } \Omega \\
u \geq 0 & \text{in } \mathbb{R}^n
\end{cases}$$

we have

$$\sup_{\Omega'} u \leq c \inf_{\Omega'} u.$$ 

Moreover, solutions $u \in H^s(\mathbb{R}^n)$ to $(-\Delta)^s u = 0$ in $\Omega$ are locally bounded and locally $\alpha$-Hölder continuous in $\Omega$, for some exponent $0 < \alpha < 1$ that depends only on $n$ and $s$. More precisely, for any compact set $K \subset \Omega$ there exists $C = C(c, K, \Omega) > 0$ such that

$$\|u\|_{C^{0,\alpha}(K)} \leq C\|u\|_{L^2(\mathbb{R}^n)}.$$ 

If, in addition, $u \in L^\infty(\mathbb{R}^n)$ then

$$[u]_{C^{0}(K)} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}.$$ 

Sketch of proof. Let $U$ be the extension of $u$ given by Theorem 9. If $u \geq 0$ in $\mathbb{R}^n$ then, by any of the formulas in (20), $U \geq 0$. Moreover, $U$ verifies the hypotheses of Lemma 2. Hence the reflection $\tilde{U}$ is a nonnegative weak solution to

$$\text{div}(|y|^a \nabla x,y \tilde{U}) = 0 \quad \text{in } \Omega \times (-2,2).$$

The interior Harnack inequality for divergence form degenerate elliptic equations with $A_2$ weights applies to $\tilde{U}$ (see [44]), and hence, to $u$. The estimates follow from (20) and (24).

The extension equation in (19) is translation invariant in the variable $x$. Thus, an argument based on Caffarelli’s incremental quotient lemma [22, Lemma 5.6] can be used to prove interior derivative estimates for fractional harmonic functions. Details of this argument are found in [23]. For similar techniques applied to parabolic problems, see [92].

Corollary 1 (Interior derivative estimates). Let $u \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution to

$$(-\Delta)^s u = 0 \quad \text{in } B_2.$$ 

Then $u$ is smooth in the interior of $B_2$ and, for any multi-index $\gamma \in \mathbb{N}_0^n$ there is a constant $C = C(|\gamma|, n, s) > 0$ such that

$$\sup_{x \in B_1} |D^\gamma u(x)| \leq C\|u\|_{L^\infty(\mathbb{R}^n)}.$$
6 Semigroup characterization of Hölder and Zygmund spaces

As we explained in the introduction, the Hölder and Schauder estimates for the fractional Laplacian can be proved in a rather quick and elegant way by means of a characterization of Hölder and Zygmund spaces in terms of heat semigroups. In this way one can avoid the use of pointwise formulas or the Schauder estimates for the Laplacian as done by Silvestre in [84, 85]. Moreover, the semigroup method allows us to reach the endpoint case of $\alpha + 2s$ being an integer (where the appropriate regularity spaces turn out to be different than the rather "natural" endpoint Lipschitz or $C^k$ spaces), and case of $L^\infty$ right hand side. See [29, 84, 85, 89, 102] for considerations about endpoint spaces.

Let $\alpha > 0$ and take any $k \geq \lfloor \alpha/2 \rfloor + 1$. Define

$$\Lambda^\alpha = \left\{ u \in L^\infty(\mathbb{R}^n) : [u]_{\Lambda^\alpha} = \sup_{x \in \mathbb{R}^n, t > 0} |e^{-\alpha/2} \partial^k e^t \Delta u(x)| < \infty \right\}$$

under the norm $\|u\|_{\Lambda^\alpha} = \|u\|_{L^\infty(\mathbb{R}^n)} + [u]_{\Lambda^\alpha}$. It can be seen that this definition is independent of $k$ and that the norms for different $k$ are all equivalent.

The Zygmund space $\Lambda_* = \Lambda^1_*$ is the set of functions $u \in L^\infty(\mathbb{R}^n)$ such that

$$[u]_{\Lambda_*} = \sup_{x, h \in \mathbb{R}^n} \frac{|u(x+h) + u(x-h) - 2u(x)|}{|h|} < \infty$$

under the norm $\|u\|_{\Lambda_*} = \|u\|_{L^\infty(\mathbb{R}^n)} + [u]_{\Lambda_*}$, see [102]. Note that $C^{0,1}(\mathbb{R}^n) \subsetneq \Lambda_*$ continuously. For any integer $k \geq 2$ we denote

$$\Lambda^k_* = \left\{ u \in C^{k-1}(\mathbb{R}^n) : D^\gamma u \in \Lambda_* \text{ for all } |\gamma| = k-1 \right\}$$

with norm $\|u\|_{\Lambda^k_*} = \|u\|_{C^{k-1}(\mathbb{R}^n)} + \max_{|\gamma| = k-1} |D^\gamma u|_{\Lambda_*}$. Then $C^{k-1,1}(\mathbb{R}^n) \subsetneq \Lambda^k_*$. The spaces $\Lambda^\alpha$, given in terms of the rate of growth of the heat semigroup, coincide with the classical Hölder and Zygmund spaces. The following result for $0 < \alpha < 2$ can be found in [96, Theorem 4] and [21]. For any $\alpha > 0$ and when $\mathbb{R}^n$ is replaced by the torus $\mathbb{T}^n$, see for example [76]. In [89, 96] a similar characterization is proved by using Poisson semigroups, and [92] contains the case of parabolic Hölder–Zygmund spaces. See [68] for the regularity spaces associated with the Ornstein–Uhlenbeck operator, as [38] for those related to the fractional powers of the parabolic harmonic oscillator.

Theorem 11. If $\alpha > 0$ then

$$\Lambda^\alpha = \begin{cases} 
C^{[\alpha],\alpha-\lfloor \alpha \rfloor}(\mathbb{R}^n) & \text{if } \alpha \text{ is not an integer} \\
\Lambda^k_* & \text{if } \alpha = k \text{ is an integer}
\end{cases}$$
7 Hölder and Schauder estimates with the method of semigroups

This Section is devoted to show how the fractional Laplacian interacts with Hölder and Zygmund spaces. For this we apply the method of semigroups in combination with Theorem 11. We sketch part of the (rather simple) proofs here. In particular, the technique avoids the use of pointwise formulas, or the boundedness of Riesz transforms on Hölder spaces, or the Hölder and Schauder estimates for the Laplacian as in [84, 85], see also [93]. Similar proofs but in different contexts can be found in [29, 38, 50, 69, 76, 92].

The first result establishes that the fractional Laplacian \((-\Delta)^s\) behaves as an operator of order \(2s\) in the scale of Hölder spaces.

**Theorem 12 (Hölder estimates).** Let \(u \in C^{k,\alpha}(\mathbb{R}^n)\), for \(k \geq 0\) and \(0 < \alpha \leq 1\).

(i) If \(0 < 2s < \alpha\) then \((-\Delta)^su \in C^{k,\alpha-2s}(\mathbb{R}^n)\) and

\[
\|(-\Delta)^su\|_{C^{k,\alpha-2s}(\mathbb{R}^n)} \leq C\|u\|_{C^{k,\alpha}(\mathbb{R}^n)}.
\]

(ii) If \(0 < \alpha < 2s\) and \(k \geq 1\) then \((-\Delta)^su \in C^{k-1,\alpha-2s+1}(\mathbb{R}^n)\) and

\[
\|(-\Delta)^su\|_{C^{k-1,\alpha-2s+1}(\mathbb{R}^n)} \leq C\|u\|_{C^{k,\alpha}(\mathbb{R}^n)}.
\]

The constants \(C > 0\) above depend only on \(n, s, k\) and \(\alpha\).

The idea for (i) is as follows. In view of (6), it is enough to prove (i) for \(k = 0\). By Theorem 11, we only have to show that \((-\Delta)^su \in \Lambda^{\alpha-2s}\) for \(u \in \Lambda^\alpha\) and \(0 < 2s < \alpha \leq 1\). For this, by using (7), one can write

\[
\Gamma(-s)t\partial_t e^{t\Delta}(-\Delta)^su(x) = \int_0^\infty t\partial_t e^{t\Delta}(e^{r\Delta}u(x) - u(x)) \frac{dr}{r^{1+s}}.
\]

On one hand, since \(\{e^{t\Delta}\}_{t \geq 0}\) is a semigroup and \(u \in \Lambda^\alpha\),

\[
\left| \int_0^t t\partial_t e^{t\Delta}(e^{r\Delta}u(x) - u(x)) \frac{dr}{r^{1+s}} \right| = \left| \int_0^t t\partial_t e^{t\Delta} \left[ \int_0^r \partial_\rho e^{\rho\Delta}u(x) d\rho \right] \frac{dr}{r^{1+s}} \right|
\]

\[
= t \left| \int_0^t \int_0^r \partial_w^2 e^{\rho\Delta}u(x) \big|_{w=t+\rho} d\rho \frac{dr}{r^{1+s}} \right|
\]
On the other hand,

\[
\left| \int_{t}^{\infty} t \partial_t e^{t\Delta} \left( e^{\tau \Delta} u(x) - u(x) \right) \frac{dr}{r^{1+s}} \right| \\
\leq t \int_{t}^{\infty} \left| \partial_x e^{w \Delta} u(x) \right| \frac{dr}{r^{1+s}} + \left| t \partial_t e^{t\Delta} u(x) \right| \int_{t}^{\infty} \frac{dr}{r^{1+s}}
\leq C[u]_{\Lambda^\alpha} t^{(\alpha-2s)/2}.
\]

If \( k \geq 1 \) then Theorem 11 shows that (ii) is a consequence of the fact that \((-\Delta)^s : \Lambda^{k+\alpha} \to \Lambda^{k+\alpha-2s}\). The latter can be accomplished with parallel arguments to those used to prove (i).

The semigroup formula from Theorem 6 and the characterization in Theorem 11 permit us to prove the Schauder estimates \((-\Delta)^s : C^{\alpha} \to C^{\alpha+2s}\) in a rather simple way. This same result, only for the case when \( \alpha + 2s \) is not an integer, was obtained using pointwise formulas and the Schauder estimates for the Laplacian in [84, 85]. In our case the semigroup method permits us to include the case when \( \alpha + 2s \in \mathbb{N} \). See [29, 38, 50, 69, 76, 92] for similar proofs in different contexts.

**Theorem 13** (Schauder–Zygmund estimates). Let \( f \in C^{0,\alpha}(\mathbb{R}^n) \) with compact support, for some \( 0 < \alpha \leq 1 \), and define \( u \) as in Theorem 6.

(i) If \( \alpha + 2s < 1 \) then \( u \in C^{0,\alpha+2s}(\mathbb{R}^n) \) and

\[
\| u \|_{C^{0,\alpha+2s}(\mathbb{R}^n)} \leq C \left( \| u \|_{L^\infty(\mathbb{R}^n)} + \| f \|_{C^{0,\alpha}(\mathbb{R}^n)} \right).
\]

(ii) If \( 1 < \alpha + 2s < 2 \) then \( u \in C^{1,\alpha+2s-1}(\mathbb{R}^n) \) and

\[
\| u \|_{C^{1,\alpha+2s-1}(\mathbb{R}^n)} \leq C \left( \| u \|_{L^\infty(\mathbb{R}^n)} + \| f \|_{C^{0,\alpha}(\mathbb{R}^n)} \right).
\]

(iii) If \( 2 < \alpha + 2s < 3 \) then \( u \in C^{2,\alpha+2s-2}(\mathbb{R}^n) \) and

\[
\| u \|_{C^{2,\alpha+2s-2}(\mathbb{R}^n)} \leq C \left( \| u \|_{L^\infty(\mathbb{R}^n)} + \| f \|_{C^{0,\alpha}(\mathbb{R}^n)} \right).
\]

(iv) If \( \alpha + 2s = k \), \( k = 1, 2 \), then \( u \in \Lambda^k \) and

\[
\| u \|_{\Lambda^k} \leq C \left( \| u \|_{L^\infty(\mathbb{R}^n)} + \| f \|_{C^{0,\alpha}(\mathbb{R}^n)} \right).
\]

The constants \( C > 0 \) above depend only on \( n \), \( s \) and \( \alpha \).

As a direct consequence of the solvability result in Theorem 6, \( u \) is the unique bounded classical solution to \((-\Delta)^s u = f\) that vanishes at infinity.
From Theorem \[11\] the statement of Theorem \[13\] reduces to show that \((-\Delta)^{-s}: \Lambda^\alpha \to \Lambda^{\alpha+2s}\), and this is very easy to prove. Indeed, for any \(k \geq \lfloor (\alpha+2s)/2 \rfloor +1\),

\[
|t^k \partial_t^k e^{t\Delta} [(-\Delta)^{-s} f](x)| = C_s t^k \left| \int_0^\infty \partial^k_w e^{w\Delta} f(x) \Big|_{w=t+r} \frac{dr}{r^{1-s}} \right| \leq C [f]_{\Lambda^\alpha} t^{(\alpha+2s)/2}.
\]

With the semigroup method we can prove the Schauder estimates in Hölder–Zygmund spaces for the case when the right hand side is just bounded, see the details in \[20\].

**Theorem 14** (Schauder–Hölder–Zygmund estimates). Let \(f \in L^\infty(\mathbb{R}^n)\) with compact support and define \(u\) as in Theorem \[6\]

(i) If \(2s < 1\) then \(u \in C^{0,2s}(\mathbb{R}^n)\) and

\[
\|u\|_{C^{0,2s}(\mathbb{R}^n)} \leq C \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\right).
\]

(ii) If \(2s = 1\) then \(u \in \Lambda_s\) and

\[
\|u\|_{\Lambda_s} \leq C \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\right).
\]

(iii) If \(2s > 1\) then \(u \in C^{1,2s-1}(\mathbb{R}^n)\) and

\[
\|u\|_{C^{1,2s-1}(\mathbb{R}^n)} \leq C \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\right).
\]

The constants \(C > 0\) above depend only on \(n\) and \(s\).

As a direct consequence of the solvability result in Theorem \[6\], \(u\) is the unique bounded solution to \((-\Delta)^s u(x) = f(x)\), for a.e. \(x \in \mathbb{R}^n\), that vanishes at infinity.

For the next result, we do not assume that the right hand side has compact support. The idea for the proof, see \[81, 82\], also \[92\] for the parabolic case, is to choose \(\eta \in C^\infty_c(B_2)\) such that \(0 \leq \eta \leq 1\), \(\eta = 1\) in \(B_1\), \(|\nabla \eta| \leq C\) in \(\mathbb{R}^n\), and write \(f = \eta f + (1-\eta)f = f_1 + f_2\) and \(u = u_1 + u_2\), where \(u_1\) is the solution to \((-\Delta)^s u_1 = f_1\) in \(\mathbb{R}^n\). Then, since \(f_1\) has compact support, \(u_1\) is given as in Theorem \[6\] and, therefore, Theorems \[13\] and \[14\] apply to it. On the other hand, \((-\Delta)^s u_2 = 0\) in \(B_1\), so by Corollary \[1\] we can bound, for any \(k\) and \(\alpha\),

\[
\|u_2\|_{C^{k,\alpha}(B_{1/2})} \leq C \|u - u_1\|_{L^\infty(\mathbb{R}^n)} \leq C \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\right).
\]

where \(C > 0\) depends only on \(n, k\) and \(\alpha\). For a proof of part (a) of the following result in the case when \(\alpha + 2s\) is not an integer and by using pointwise formulas see \[81, 82\].
**Theorem 15** (Schauder–Zygmund estimates). Let $u \in L^\infty(\mathbb{R}^n)$.

(a) Assume that $(-\Delta)^su = f \in C^{0,\alpha}(\mathbb{R}^n)$ for some $0 < \alpha \leq 1$. Then $u$ satisfies the estimates $(i)-(iv)$ from Theorem 13.

(b) Assume that $(-\Delta)^su = f \in L^\infty(\mathbb{R}^n)$. Then $u$ satisfies the estimates $(i)-(iii)$ from Theorem 14.
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