Sparse bounds on variational norms along monomial curves

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Abstract
Consider a monomial curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ and a family of truncated Hilbert transforms along $\gamma$, $\mathcal{H}_\gamma$. This paper addresses the possibility of the pointwise sparse domination of the $r$-variation of $\mathcal{H}_\gamma$ - namely, whether the following is true:

$$V^r \circ \mathcal{H}_\gamma f(x) \lesssim S_f(x)$$

where $f$ is a nonnegative measurable function, $r > 2$ and $S_f(x) = \sum_{Q \in \mathcal{Q}} \langle f \chi_Q \rangle_{Q,p}$ for some $p$ and some sparse collection $\mathcal{Q}$ depending on $f,p$.

Contents
1 Introduction 1
2 General discussion and acknowledgement 3
3 Proof of Theorem 1 3
4 Control of variational norm 8

1 Introduction
Consider the following monomial curve $\gamma_P(t) = (t, t^2, \ldots, t^d)$. (1.1)

Let $\mathcal{I}$ be a countable set and consider the family $\mathcal{A} = \{A_t\}_{t \in \mathcal{I}}$. Let $r > 0$. Then the $r$-variation of $\mathcal{A}$ is,

$$V^r \mathcal{A} = V^r \{A_t\}_{t \in \mathcal{I}} = \sup_{t_1 < \cdots < t_N} \left( \sum_{j=1}^{N-1} |A_{t_{j+1}} - A_{t_j}|^r \right)^{1/r},$$

where the supremum runs over all finite increasing subsequences of indices in $\mathcal{I}$. This paper focuses on such $r$-variation of the family of truncated Hilbert transforms along a monomial curve $\gamma$.

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function. The Hilbert transform of $f$ along a curve $\gamma$ is defined as follows,

$$H_\gamma f(x) = pv \int_{-\infty}^{\infty} f(x - \gamma(t)) \frac{dt}{t}.$$ 

The corresponding family of truncated Hilbert transforms along $\gamma$ is then, $\mathcal{H}_\gamma = \{H_\gamma^s\}_{s>0}$ where $H_\gamma^s f(x) = \int_{|t| > s} f(x - \gamma(t)) \frac{dt}{t}$. Define the $r$-variation of the truncated Hilbert transforms of $f$ along $\gamma$, using a set $\mathcal{I}$ (of positive reals), as follows,

$$V^r \{H_\gamma^s f(x)\}_{s \in \mathcal{I}} = \sup_{s_1 < \cdots < s_N} \left( \sum_{j=1}^{N-1} |H_\gamma^s f(x) - H_\gamma^{s_j} f(x)|^r \right)^{1/r},$$

(1.2)
where the supremum again runs over all finite subsequences \( s_{i_1} < \cdots < s_{i_N} \).

For instance, if \( I = \{ 2^j \}_{j \in \mathbb{Z}} \), the sum in 1.2 is simply, \( \left( \sum_{j=1}^{N-1} \left| \sum_{t \in 2^j} f(x - \gamma(t)) \frac{dt}{T} \right|^{1/r} \right)^r \).

Define \( \mathcal{T} f(x) := V^r \{ H_z^\gamma f(x) \}_{z \in \mathbb{Z}} \) (1.3) with \( r > 2 \) and \( \gamma \) satisfying 1.1. This paper is devoted to a result on sparse domination in bilinear form of such \( \mathcal{T} \). In order to formulate the result, some concepts need to be defined.

### 1.1 Definitions

#### 1.1.1 The cubes

In this exposition, "a cube" in always means a cube in \( \mathbb{R}^d \) whose sides are parallel to the coordinate axes. Two families of cubes that reflect well the the geometry of the curve \( \gamma \) will be utilized, one is the collection of \( \gamma \)-cubes, and the other - their cousins - the collection of dyadic \( \gamma \)-cubes. Their definitions, stated below, are largely borrowed from [1].

**Definition 1.** [1] A \( \gamma \)-cube \( Q \subset \mathbb{R}^d \) is a cube whose side-lengths \( \bar{l}(Q) = (l_1, \ldots, l_d) = (l^\alpha_1, \ldots, l^\alpha_d) \) for some \( l = l_Q > 0 \).

**Definition 2.** [1] A dyadic \( \gamma \)-cube \( Q \subset \mathbb{R}^d \) is a cube whose side-lengths \( \bar{l}(Q) = (2^{|k\alpha_1|}, \ldots, 2^{|k\alpha_d|}) \) for some \( k \in \mathbb{Z} \). If \( Q \) is one such dyadic \( \gamma \)-cube, define \( l_Q := 2^k \) for the largest possible \( k \).

The collection of all \( \gamma \)-cubes is denoted by \( Q^\gamma \) and the collection of all dyadic \( \gamma \)-cubes \( D^\gamma \). If \( Q_0 \) be a cube, then \( Q^\gamma(Q_0) \) denotes the collection of all \( \gamma \)-cubes \( Q \subset Q_0 \) and \( D^\gamma(Q_0) \) the collection of all dyadic \( \gamma \)-cubes \( Q \subset Q_0 \).

The \( \gamma \)-cubes and dyadic \( \gamma \)-cubes are essentially equivalent in the following sense. If \( Q \) is a dyadic \( \gamma \)-cube then there exists a \( \gamma \)-cube \( \bar{Q} \) such that \( Q \subset \bar{Q} \) and \( l_Q \leq 2l_{\bar{Q}} \), and vice versa.

The usefulness of working with dyadic \( \gamma \)-cubes is that, there exist \( 3^d \) universal shifted dyadic \( \gamma \)-grids:

\[
\left\{ 2^{|k\alpha_1|} \left[ m_1 + \frac{j_1}{3}, m_1 + 1 + \frac{j_1}{3} \right] \times \cdots \times 2^{|k\alpha_d|} \left[ m_d + \frac{j_d}{3}, m_d + 1 + \frac{j_d}{3} \right] \right\}
\]

where \( k \in \mathbb{Z}, \bar{m} \in \mathbb{Z}^d, \bar{j} \in \{0, 1, 2\}^d \). Now let \( D^\gamma \) denote a generic grid of the above type - and temporarily allow a slight abuse of notation - then \( D^\gamma \) satisfies the following properties [1]:

1) \( D^\gamma = \bigcup_k D^\gamma_k \) and each "generation" \( D^\gamma_k \) partitions \( \mathbb{R}^d \).
2) If \( Q_1, Q_2 \in D^\gamma \) and \( l_{Q_1} \leq l_{Q_2} \), then either \( Q_1 \subset Q_2 \) or \( Q_1 \cap Q_2 = \emptyset \).
3) If \( Q \in D^\gamma \) and \( Q^{(1)} \in D^\gamma \) is the smallest dyadic \( \gamma \)-cube such that \( Q \subseteq Q^{(1)} \), then \( l_{Q^{(1)}} \leq C(d, \gamma)l_Q \).
4) For every \( x \in \mathbb{R}^d \), there exists a chain \( \{ Q_i \} \subset D^\gamma \) containing \( x \) such that \( \lim_{i \to \infty} l_{Q_i} = 0 \).

It's these properties that enables one to have a Calderon-Zygmund decomposition using these dyadic \( \gamma \)-cubes.

**Dilation with cubes.** If \( Q \subset \mathbb{R}^d \) is a cube with side-lengths \( \bar{l}(Q) = (l_1, \ldots, l_d) \), then \( \lambda Q \) is another cube with the same center as \( Q \) and side-lengths \( \bar{l}(\lambda Q) = (\lambda l_1, \ldots, \lambda l_d) \). One should be careful that if \( Q \) is a \( \gamma \)-cube or a dyadic \( \gamma \)-cube then \( \lambda Q \) might not be a \( \gamma \)-cube nor a dyadic \( \gamma \)-cube.

#### 1.1.2 Monotonic functions

Order vectors in \( \mathbb{R}^d \) in the following manner, \( \bar{x} \succeq \bar{y} \) if \( x_i \geq y_i \). This is not a complete ordering. Consider functions \( f : \mathbb{R}^d \to \mathbb{R} \) such that exactly one of the following holds,

\[
\begin{align*}
f(\bar{x}) &\geq f(\bar{y}) \quad \text{whenever } \bar{x} \succeq \bar{y}, \\
f(\bar{x}) &\leq f(\bar{y}) \quad \text{whenever } \bar{x} \succeq \bar{y}.
\end{align*}
\]
These inequalities are thought to hold almost everywhere. Call such functions monotonic functions. These functions are not integrable but locally integrable. If $f$ is a monotonic function and its weak derivative, $Df$ exists, then exclusively, either

$$(Df(x) - Df(y)) \cdot (x - y) \geq 0$$

happens, or

$$(Df(x) - Df(y)) \cdot (x - y) \leq 0.$$ 

happens, for a.e $x \geq y$. There is a sub-class of these vector fields that satisfy exclusively either of these two inequalities for a.e $x, y$, called "monotonic vector fields" on $\mathbb{R}^d$. See [6] for more on these vector fields.

### 1.1.3 Other definitions

Given a full-body $B \subset \mathbb{R}^d$, then $\|B\|$ denotes its full measure, and given $y \in \mathbb{R}^d$, then $|y|$ denotes its vector length. Finally, denote $\langle f \rangle_{Q,p}^p = |Q|^{-1} \int_Q f(x)^p \, dx$ and $\langle f \rangle_Q = \langle f \rangle_{Q,1}.$

A family $S$ of cubes in $\mathbb{R}^d$ is said to be $\eta$-sparse, $0 < \eta < 1$, if for every $Q \in S$, there exists $E_Q \subset Q$ such that $|E_Q| \geq \eta |Q|$ and the sets $\{E_Q\}_{Q \in S}$ are pairwise disjoint. See [11] for instance for this well-known definition.

### 1.2 Main theorem

**Theorem 1.** Let $d \geq 2$. Let $T$ be defined as in 1.3 with $\gamma = \gamma_p$. Let $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be a monotonic, measurable function. Fix $Q_0 \in \mathcal{D}^\gamma$. Let $1 < p < \infty$. Then there exist $\kappa = \kappa(d, \gamma)$ and a sparse collection $S$ of $\gamma$-cubes, such that for a.e $x$,

$$|T(f \chi_{\kappa Q_0})(x)| \leq \sum_{Q \in S} \langle f \rangle_{Q, \kappa Q}^\kappa Q_{\chi Q}(x).$$

**Remark:** The implicit constants will be made clear in the body of the proof; they are dependent on natural parameters of the dimensions, the $L^p$ exponents, the exponents of the curves and various operator norms associated with $T$.

### 2 General discussion and acknowledgement

The approach of this paper roughly follows the general principle laid out in [11], where in order for one to obtain a pointwise sparse bound of a sub-linear operator $T$, one needs to obtain a control of its operator norm as well as the operator norm of its associated, so-called tail maximal operator. A more relaxed approach of this principle has just been uploaded [12]. However, this apparatus fails to yield a result in the "variation-sparse-low-dimensional" situation. As a matter of fact, any positive pointwise sparse result for variational norm along a monomial curve should lead to a contradiction.

Related works in this "variation-sparse" direction of consist of the papers [4], [7] and [9]. This list is certainly not complete.

### 2.1 Acknowledgement

In the previous version, there is a mistake in controlling the tail maximal operator, which leads to an incorrect pointwise result, which would hold true for a smaller sub-class of functions, regrettably. The author thanks Professor Lacey for pointing that out.

### 3 Proof of Theorem 1

#### 3.1 The tail maximal operator

Let $T$ be an operator of functions on $\mathbb{R}^d$. The tail maximal operator $\mathcal{M}_T$ associated with $T$ is defined by,

$$\mathcal{M}_T f(x) = \sup_{Q_{\kappa x}} \sup_{\xi \in Q} |T f \chi_{\mathbb{R}^d \setminus \kappa Q}(\xi)|$$
where the supremum is taken over all the cubes containing \( x \). A localized version, subject to a cube \( Q_0 \), of such operator, is,

\[
\mathcal{M}_{T,Q_0} f(x) = \sup_{Q \subset Q_0} \text{ess sup} \{ |T f \chi_{\kappa Q_0 \setminus \kappa Q}(\xi)| \}.
\]

The constant \( \kappa \) will be chosen beforehand. These definitions are originated in [11]. One needs two divergent points for the definitions of the tail maximal operators used here.

**Definition 3.** Let \( C \) be the smallest constant of all \( C(d, \gamma) \) satisfying the property (3) in **Definition 2**. Let \( \kappa := C + 1 \). The operators \( \mathcal{M}_T, \mathcal{M}_{T,Q_0} \) are called the \( \gamma \)-tail maximal operators if the cubes \( Q, Q_0 \) used are all dyadic \( \gamma \)-cubes with this defined \( \kappa \).

Regardless of the nature of the cubes used, it's clear that \( \mathcal{M}_{T,Q_0} f(x) \leq \mathcal{M}_T f(x) \).

### 3.2 A preliminary result

**Lemma 4.** Let \( 1 < p < \infty \). For every dyadic \( \gamma \)-cube \( Q_0 \) and a.e \( x \in Q_0 \),

\[
|T(f \chi_{\kappa Q_0})(x)| \leq \|T : L^p \to L^p\| f(x) + \mathcal{M}_{T,Q_0} f(x).
\]

**Proof:** The following route is standard; see [11]. Suppose \( x \in \text{Int}(Q_0) \) is a point of approximate continuity of \( T(f \chi_{\kappa Q_0}) \) and a Lebesgue point of \( f \). That means, if \( B(x, s) \) denotes a ball of radius \( s \) centered at \( x \), then for every \( \epsilon > 0 \), the sets

\[
E_s^x(x) = \{ y \in B(x, s) : |T(f \chi_{\kappa Q_0})(y) - T(f \chi_{\kappa Q_0})(x)| \leq \epsilon \}
\]

satisfy \( \lim_{s \to 0} \frac{|E_s^x(x)|}{|B(x, s)|} = 1 \). Let \( Q(x, s) \) be the smallest dyadic \( \gamma \)-cube centered at \( x \) containing \( B(x, s) \) and choose \( s \) so small that \( Q(x, s) \subset Q_0 \). Then, from the definition of \( \mathcal{M}_{T,Q_0} \), for a.e \( y \in E_s^x(x) \),

\[
|T(f \chi_{\kappa Q_0})(x)| \leq |T(f \chi_{\kappa Q(x, s)})(y)| + \mathcal{M}_{T,Q_0} f(x) + \epsilon,
\]

which implies,

\[
|T(f \chi_{\kappa Q_0})(x)| \leq \text{ess inf}_{y \in E_s^x(x)} |T(f \chi_{\kappa Q(x, s)})(y)| + \mathcal{M}_{T,Q_0} f(x) + \epsilon
\]

\[
\leq \|T : L^p \to L^p\| \left( \int_{Q(x, s)} f^p \right)^{\frac{1}{p}} + \mathcal{M}_{T,Q_0} f(x) + \epsilon.
\]

Now let \( s \to 0 \) and \( \epsilon \to 0 \) to obtain 3.1. \( \square \)

### 3.3 Control of the tail maximal operator

One needs the following concepts of of maximal truncation of the Hilbert transform

\[
H_\gamma f(x) = \sup_{0<s<\rho} \left| \int_{s \leq |t| \leq \rho} f(x - \gamma(t)) \frac{dt}{t} \right|
\]

and of a single scale average operator along \( \gamma \),

\[
A_\lambda f(x) = \int_{\lambda/2 < |t| \leq \lambda} f(x - \gamma(t)) \frac{dt}{t}.
\]

When working with dyadic scales, one can also write, \( A_\lambda f(x) = \int_{2^{j} < |t| \leq 2^{j+1}} f(x - \gamma(t)) \frac{dt}{t} \).

Let \( f \) be as in the hypotheses of **Theorem 1**. Fix \( Q_0 \in \mathcal{D}^\gamma \). Fix \( Q \subset Q_0 \cap \mathcal{D}^\gamma \) and consider \( x, \xi \in Q \). Assume for a moment that \( \mathcal{I} = \{2^j\}_{j \in \mathbb{Z}} \). For the sake of convenience, the notation for the \( \gamma \)-tail maximal operator remains unchanged.

Let \( jQ \) be the unique integer such that \( lQ = 2^{jQ} \). Let \( jM \) be the unique integer such that the smallest dyadic \( \gamma \)-cube containing \( \kappa Q_0 \) has the length \( 2^{jM} \); such a cube is contained in \( \kappa^2 Q_0 \).
Given a consecutive finite sequence of integers \( j_1 < \cdots < j_N \) that contains \( j_Q, j_M \), one has, for \( z \in Q \),

\[
\sum_{j=j_1}^{j_N} \left| \int_{2^{j-1}<|t|<2^j+1} f_{\chi_{x}\sim Q}(z - \gamma(t)) \frac{dt}{t} \right|^r \leq \sum_{j=j_Q}^{j_M-1} \left| \int_{2^{j-1}<|t|<2^j+1} f(z - \gamma(t)) \frac{dt}{t} \right|^r ,
\]

while a simple \( l^1 \) domination then leads to,

\[
\left( \sum_{j=j_1}^{j_N} \left| \int_{2^{j-1}<|t|<2^j+1} f(z - \gamma(t)) \frac{dt}{t} \right| \right)^{1/r} \leq \sum_{j=j_Q}^{j_M-1} \left| \int_{2^{j-1}<|t|<2^j+1} f(z - \gamma(t)) \frac{dt}{t} \right| . \tag{3.2}
\]

These imply, for \( x, \xi \in Q \),

\[
\left( \sum_{j=j_1}^{j_N} \left| \int_{2^{j-1}<|t|<2^j+1} f_{\chi_{x}\sim Q}(\xi - \gamma(t)) \frac{dt}{t} \right| \right)^{1/r} \leq \sum_{j=j_Q}^{j_M-1} \left| \int_{2^{j-1}<|t|<2^j+1} f(\xi - \gamma(t)) \frac{dt}{t} \right| \leq \sum_{j=j_Q}^{j_M-1} \left| \int_{2^{j-1}<|t|<2^j+1} f(x - \gamma(t)) \frac{dt}{t} \right| + \sum_{j=j_Q}^{j_M-1} \left| \int_{2^{j-1}<|t|<2^j+1} f(x - \gamma(t)) \frac{dt}{t} \right| . \tag{3.3}
\]

Let \( y = x - \xi \). Then \( y = (y_1, \cdots, y_d) \) with \( |y| \leq l_Q^2 \) - denote this relation as \( |y| \leq l_Q \), then the first term on the RHS of 3.3 is, \( \sum_{j=j_Q}^{j_M-1} (|\tau_y A^j f(x)| - |A^j f(x)|) \), and,

\[
\sum_{j=j_Q}^{j_M-1} (|\tau_y A^j f(x)| - |A^j f(x)|) \leq \sum_{j=j_Q}^{j_M-1} \sup_{|y| \leq l_Q} |\tau_y A^j f(x) - A^j f(x)|. \tag{3.4}
\]

Now one has,

\[
\left| \int_{a<|t|<b} f(x - \gamma(t)) \frac{dt}{t} \right| = \left| \int_{a}^{b} f(x - \gamma(t)) \frac{dt}{t} - \int_{a}^{b} f(x - \gamma(-t)) \frac{dt}{t} \right| .
\]

With \( f \) being monotonic and \( \gamma = \gamma_p \), the term inside the bracket on the RHS of the equality above is either all nonnegative or all nonpositive for all intervals \([a, b] \). That means, for the last term in 3.3,

\[
\sum_{j=j_Q}^{j_M-1} \left| \int_{2^{j-1}<|t|<2^j+1} f(x - \gamma(t)) \frac{dt}{t} \right| \leq \sum_{j=j_Q}^{j_M-1} \left| \int_{2^{j-1}<|t|<2^j+1} f_{\chi_{x}\sim Q_b}(x - \gamma(t)) \frac{dt}{t} \right| \leq H_a^*(f_{\chi_{x}\sim Q_b})(x) . \tag{3.5}
\]

From 3.4,3.5 and the definition of \( \mathcal{M}_{\tau, Q_0} \), one has for \( x \in Q \),

\[
\mathcal{M}_{\tau, Q_0} f(x) \leq \sup_{j_1 \cdots < j_N} \sum_{j=j_Q}^{j_M-1} \sup_{|y| \leq l_Q} |\tau_y A^j f(x) - A^j f(x)| + H_a^*(f_{\chi_{x}\sim Q_b})(x) , \tag{3.6}
\]

where the supremum runs over all finite increasing consecutive subsequences \( j_1 < \cdots < j_N \) that contains \( j_Q, j_M \).

### 3.3.1 The trapezoid

For a dimension \( d \), let \( \Omega(d) \) be the trapezoid with vertices

\[
(0,0), (1,1), \left( \frac{2}{d+1}, \frac{2(d-1)}{d(d+1)} \right), \left( \frac{d^2 - d + 2}{d(d+1)}, \frac{d - 1}{d+1} \right) .
\]

Geometrically, it’s a trapezoid that lies on the lower side of the line \( y = x \) on the plane. See [2] for the origin of this trapezoid.
3.3.2 Control of the tail maximal operator, continued

The following previously known results are needed.

**Theorem A.** [3] Let $1 < p < \infty$. If $f$ is compactly supported, then $\|H^p f\|_p \leq \|f\|_p$.

To bound the first term in 3.6, one notes that, from van der Corput’s lemma and Plancherel’s theorem, if $|y_i| \leq 1$,

$$\|A^\gamma - \tau_y A^\gamma\| : L^2 \to L^2 \| \leq |y|^\eta$$

for some $\eta_0 = \eta_0(\gamma) > 0$. One also has from [2] the following result.

**Theorem B.** [2] If $(1/p, 1/q) \in \Omega(d)$, then $A^\gamma$ is of the restricted weak type $(p, q)$.

In particular, **Theorem B** says that $A^\gamma$ is of the restricted weak type $(p, p)$. Interpolation between these two estimates, which is allowed by a result in [14], gives, for $1 < p < \infty$,

$$\|A^\gamma - \tau_y A^\gamma : L^p \to L^p \| \leq |y|^\eta$$

for some $\eta > 0$. By a simple change of variables, this last estimate then implies,

$$\|A^\gamma - \tau_y A^\gamma : L^p \to L^p \| \leq \left( \frac{1}{\lambda^3} \cdot \frac{1}{2^\alpha} \right)^\eta \quad (3.7)$$

whenever $|y_i| < \lambda^\alpha$. See [1] for a similar approach. Now 3.7 implies the following norm bound for the first term in 3.6,

$$\left\| \sum_{j=1}^{N-1} \sup_{y: |y| \leq 1} (\tau_y A^\gamma_j - A_j : L^p \to L^p \| \right\| \leq \sum_{i=0}^{\alpha} \left( \frac{1}{2^i} \cdot \frac{1}{2^\alpha} \right)^\eta \| < C(\alpha, \eta). \quad (3.8)$$

Then **Theorem A, 3.6, 3.8** altogether imply that for a fixed dyadic $\gamma$-cube $Q_0$, then for $1 < p < \infty$,

$$\|M_{\gamma, Q_0} f\| \leq \|f \chi_{\gamma Q_0}\|_p + \|f \chi_{\gamma^2 Q_0}\|_p. \quad (3.9)$$

3.3.3 Passing to dyadic powers

For the case of a general countable set $\mathcal{I}$ of positive reals, suppose $t_{i_1} < \cdots < t_{i_N}$ in $\mathcal{I}$ is an increasing subsequence. If $2^J \leq t_{i_1} \leq \cdots < t_{i_N} \leq 2^{J+1}$ for some $J$, then argue as in 3.2, one has,

$$\left( \sum_{j=1}^{N-1} \int_{t_{j_1} < |t| \leq t_{j_1+1}} g(z - \gamma(t)) \frac{dt}{t} \right)^{1/r} \leq \sum_{j=1}^{N-1} \int_{t_{j_1} < |t| \leq t_{j_1+1}} g(z - \gamma(t)) \frac{dt}{t} \leq \int_{2^J < |t| \leq 2^{J+1}} g(z - \gamma(t)) \frac{dt}{t}. \quad (3.10)$$

for any monotonic $g$. If $[t_{i_1}, t_{i_N}]$ includes multiple dyadic powers, then one splits the subsequence further into parts that are strictly included in some dyadic intervals and applies the domination in 3.10 again to these parts. More precisely, if $2^{J_1}, \cdots, 2^{J_K}$ are the dyadic powers interlacing with $t_{i_1}, \cdots, t_{i_N}$ such that $2^{J_1} \leq t_{i_1} < t_{i_N} \leq 2^{J_K}$, then

$$\left( \sum_{j=1}^{N-1} \int_{t_{j_1} < |t| \leq t_{j_1+1}} g(z - \gamma(t)) \frac{dt}{t} \right)^{1/r} \leq \sum_{j=1}^{N-1} \int_{t_{j_1} < |t| \leq t_{j_1+1}} g(z - \gamma(t)) \frac{dt}{t} \leq \int_{2^{J_1} < |t| \leq 2^{J_K}} g(z - \gamma(t)) \frac{dt}{t}. \quad (3.11)$$

Hence 3.11, 3.10 imply that 3.9 still holds for the general case of countable set $\mathcal{I}$.
3.4 Proof of Theorem 1, continued

It is true that

\[ \mathcal{T}(f_1 + f_2) \lesssim \mathcal{T}f_1 + \mathcal{T}f_2. \]  

(3.12)

Indeed, consider the inequality \((a + b)^s \lesssim a^s + b^s\) for \(a, b \geq 0\) and \(s > 0\), which is equivalent to \((1 + t)^s \lesssim 1 + t^s\), where \(t > 1\) and \(s > 0\), which in turn holds with the constant \(2^s\), by a simple calculus. Take an increasing subsequence \(t_1 < \cdots < t_N\), then,

\[
\left( \sum_{i=1}^{N-1} \left| \int_{t_i < |t| \leq t_{i+1}} f_1(x - \gamma(t)) + f_2(x - \gamma(t)) \frac{dt}{t} \right|^r \right)^{1/r} 
\lesssim \left( \sum_{i=1}^{N-1} 2^r \left| \int_{t_i < |t| \leq t_{i+1}} f_1(x - \gamma(t)) \frac{dt}{t} \right|^r \right)^{1/r} 
\lesssim 2^{1 + 1/r} \left( \sum_{i=1}^{N-1} \left| \int_{t_i < |t| \leq t_{i+1}} f_1(x - \gamma(t)) \frac{dt}{t} \right|^r \right)^{1/r} + \left( \sum_{i=1}^{N-1} \left| \int_{t_i < |t| \leq t_{i+1}} f_2(x - \gamma(t)) \frac{dt}{t} \right|^r \right)^{1/r}. 
\]

Assume now that for \(1 < p < \infty\),

\[ \| \mathcal{T} : L^p \rightarrow L^p \| < \infty. \]  

(3.13)

This will be proved later in Section 4. For the following Lemma 5, let \(D^\gamma\) denote one of the finitely many dyadic \(\gamma\)-grids. The approach is fairly routine.

**Lemma 5.** There exists a 1/2-sparse family \(F \subseteq D^\gamma(Q_0)\) such that for a.e \(x \in Q_0\)

\[ |\mathcal{T}(f_{x,Q_0})(x)| \leq \sum_{Q \in F} \langle f_{x,Q_0} \rangle_{\kappa^2 Q_0, p} \chi_{Q}(x). \]  

(3.14)

**Proof.** Firstly, one wants to show that there exist pairwise disjoint \(P_i \in D^\gamma(Q_0)\) such that,

\[ |\mathcal{T}(f_{x,Q_0})| \chi_{Q_0} \leq \langle f_{x,Q_0} \rangle_{\kappa^2 Q_0, p} + \sum_i |\mathcal{T}(f_{x,P_i})| \chi_{P_i}. \]  

(3.15)

for a.e \(x \in Q_0\) and \(\sum_i |P_i| \leq (1/2)|Q_0|\). To achieve that, let

\[ E = \{ x \in Q_0 : f(x) > C \langle f \rangle_{\kappa^2 Q_0, p} \} \cup \{ x \in Q_0 : M_{\mathcal{T},Q_0} f(x) > C \langle f \rangle_{\kappa^2 Q_0, p} \}, \]

with \(C\) be sufficiently large so that \(|E| \leq \frac{|Q_0|}{2^{2+\s}}|Q_0|\). This is possible due to 3.9. Then one applies the Calderon-Zygmund decomposition to the function \(\chi_F\) using the dyadic \(\gamma\)-cubes on \(Q_0\) at the height of \(\frac{1}{2^{2+\s}}\) to produce pairwise disjoint dyadic \(\gamma\)-cubes \(P_i \in D^\gamma(Q_0)\) such that

\[ |E \setminus \cup_i P_i| = 0 \text{ and } P_i \cap E^c + \mathcal{O} \text{ and } \sum_i |P_i| \leq (1/2)|Q_0|. \]

That means,

\[ \forall \ a.e \ x \in Q_0, f_{x,Q_0}\setminus P_i \leq C \langle f \rangle_{\kappa^2 Q_0, p} \text{ and } \text{ess sup}_{\xi \in P_i} |\mathcal{T}(f_{x,Q_0})|_{\kappa P_i} |(\xi)| \leq C \langle f \rangle_{\kappa^2 Q_0, p}. \]  

(3.16)

By 3.12, one has for a.e \(x \in Q_0\)

\[ |\mathcal{T}(f_{x,Q_0})|_{\kappa Q_0} \leq |\mathcal{T}(f_{x,Q_0})|_{\kappa Q_0} \setminus P_i + \sum_i |\mathcal{T}(f_{x,Q_0})|_{\kappa P_i} + \sum_i |\mathcal{T}(f_{x,P_i})| \chi_{P_i}, \]  

(3.17)

of which the first two terms of the RHS are dominated by,

\[ |\mathcal{T}(f_{x,Q_0})|_{\kappa Q_0} \setminus P_i + \sum_i |\mathcal{T}(f_{x,Q_0})|_{\kappa P_i} \chi_{P_i} \leq \langle f \rangle_{\kappa^2 Q_0, p}, \]

through 3.1, 3.13, 3.16, which then together with 3.17 leads to 3.15.
The next steps to get to 3.14 are the iterations of the type of estimate in 3.15 to obtain the sparse family \( F = \{ P_i \}_{i \in \mathbb{Z}} \), where \( P^0 = Q_0 \), \( \{ P_1 \} = \{ P \} \) and \( \{ P_i \} \) are the dyadic \( \gamma \)-cubes obtained at the \( j \)th step of the process.

Take a partition of \( \mathbb{R}^d \) using cubes \( R_i \in \{ \mathcal{D}^j \}_{j \in \{ 0, 1, 2 \}^d} \) such that \( Q_0 \subset \kappa R_i \) for each \( R_i \). Then for each \( R_i \), generate a sparse family \( \mathcal{F}_i \subset \mathcal{D}^\gamma (R_i) \) similarly as in 3.14. Then note that if \( Q \) is a dyadic \( \gamma \)-cube with \( l_Q = 2^k \), then \( 3Q \subset \hat{Q} \) where \( \hat{Q} \) is a \( \gamma \)-cube concentric with \( Q \) whose side-lengths \( \hat{l}(\hat{Q}) = (2^{(k+2)\alpha_1}, \ldots, 2^{(k+2)\alpha_d}) \). Let \( S \) be the collection of such \( \hat{Q} \) for each \( Q \in \cup_i \mathcal{F}_i \). Then 1.4 holds for this family \( S \) that is \( 1/(2 \cdot 4^{d-1}) \)-sparse.

4 Control of variational norm

The following discussion uses the ideas and notations introduced in [5], [10] and [13].

Let \( 1 < p < \infty \). It was observed in [10] that in order to obtain \( \| T f \|_p = \| V^r \{ H^\gamma f \} \|_p \lesssim \| f \|_p \), it’s sufficient obtain a similar control of its short 2-variation operator,

\[
\| S_2 \{ H^\gamma f \} \|_{L^p(\mathbb{R}^d)} \lesssim 1.
\] (4.1)

Let \( s \in [1, 2] \) and \( j \in \mathbb{Z} \). Define \( \nu_{0,s}, \nu_{j,s} \) respectively by

\[
\langle \nu_{0,s}, f \rangle = \int_{s \leq |u| \leq 2s} f(\gamma(u)) \frac{du}{u}.
\]

and

\[
\langle \nu_{j,s}, f \rangle = \langle \nu_{0,s}, f(2^j \cdot) \rangle = \int_{s \leq |u| \leq 2s} f(\gamma(2^j u)) \frac{du}{u}.
\]

One observes similarly as in [10] that,

\[
S_2 \{ H^\gamma f \} = \left( \sum_{j \in \mathbb{Z}} |V_{2^j}(T f)(x)|^2 \right)^{1/2} \leq \left( \sum_{j \in \mathbb{Z}} \| \{ \nu_{j,s} \ast f(x) \}_{s \in [1, 2]} \|_{L^1(\mathbb{R}^d)}^2 \right)^{1/2},
\] (4.2)

From the results in [10], this then entails that a good control of \( S_2 \{ \nu_{j,s} \ast f(x) \}_{s \in [1, 2]} \) is sufficient. Let \( D = \text{diag}(\alpha_1, \ldots, \alpha_d) \) and \( tDx = (t^{\alpha_1} x_1, \ldots, t^{\alpha_d} x_d) \). Denote \( G_{j,k} f(x) = \nu_{j,s} \ast f(x) \). Let \( \psi \) be a nonnegative smooth function such that \( \int_{\mathbb{R}^d} \psi (t^{-2^k} \cdot) = 1 \) for all \( t > 0 \). Define the following rescaled truncated Littlewood-Paley operators,

\[
\Pi_{j,k} f(x) = \psi_{j,k}(\xi) \hat{f}(\xi) = \psi(\xi |D^{-2^{-k}}| \xi) \hat{f}(\xi).
\]

Then \( E_{j,k} f(x) = \sum_{k \in \mathbb{Z}} \nu_{j,s} \ast \Pi_{j,k} f(x) = \sum_{k \in \mathbb{Z}} f \ast (\nu_{j,s} \ast \psi_{j,k})(x) = \sum_{k \in \mathbb{Z}} G_{j,k} f(x) \). Denote also that \( \tilde{E}_{j,k} f(x) = \frac{1}{t} (\nu_{j,s} \ast \Pi_{j,k} f)(x) \). As discussed - and in view of 4.2 and \( t^1 \) domination - it’s sufficient to obtain the following type of estimate:

\[
\| S_2 \{ E_{j,s} f(x) \}_{s \in [1, 2]} \|_{L^p(\mathbb{R}^d)} \lesssim 1
\] (4.3)

for \( 1 < p < \infty \) and uniformly for \( j \in \mathbb{Z} \). On the other hand, following from [13], one has,

\[
S_2 \{ E_{j,s} f(x) \}_{s \in [1, 2]} \leq \sum_{k \in \mathbb{Z}} S_2 \{ G_{j,k} f(x) \}_{s \in [1, 2]} \leq \sum_{k \in \mathbb{Z}} \left| \frac{d}{ds} \right|_{L^1(\mathbb{R}^d)}(\nu_{j,s} \ast \Pi_{j,k} f(x)) \frac{ds}{s}
\] (4.4)

where \( G_{j,k} f(x)^2 = \int_1^2 |G_{j,k} f(x)|^2 \frac{ds}{s} \) and \( \tilde{G}_{j,k} f(x)^2 = \int_1^2 |\tilde{E}_{j,k} f(x)|^2 \frac{ds}{s} \). This entails,

\[
\left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \left( \int_1^2 \left( |\nu_{j,s} \ast \Pi_{j,k} f(x)|^2 \frac{ds}{s} \right) \right) \frac{ds}{s} \right) \right)^{1/2} \leq \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \left( \int_1^2 \left( \left| \nu_{j,s} \ast \Pi_{j,k} f(x) \right|^2 \frac{ds}{s} \right) \right)^{1/2} \right)^{1/2}
\]
Then based on the Littlewood-Paley theory and this last display, it’s also sufficient to obtain the estimates of the following type, uniformly for \( s \in [1, 2] \):

\[
\| (\sum_{j \in \mathbb{Z}} |\nu_{j,s} \ast \Pi_{j,k} f|^2)^{1/2} \|_p \lesssim \| f \|_p
\]  

(4.5)

and

\[
\left\| \left( \sum_{j \in \mathbb{Z}} \frac{d}{ds} (\nu_{j,s} \ast \Pi_{j,k} f)^2 \right)^{1/2} \right\|_p \lesssim \| f \|_p.
\]

(4.6)

Following are two approaches in proving 4.1 - the first shows 4.3 and the second shows 4.5, 4.6.

### 4.1 Approach 1

The first approach is analogous to a similar work in [10]. Hence the notations used follow closely to those appeared in [10]. The reader is recommended to refer to the said work for some shortened steps below.

One first wants to smoothen up the cut-off at \( s \) and \( 2 \) for the measures \( \nu_{s} \)'s. Let \( \mu \) to be such that \( \langle \mu, f \rangle = \int_{|u| \leq 2} f(\gamma(u)) \frac{du}{u} \), and consider the smoothened version of \( \nu_{0,s} \) as a smoothened, \( s^{D} \)-dilated, truncated version of \( \mu \), as follows:

\[
\nu_{0,s} = \nu_{0,s}^1 + \nu_{0,s}^2 = \sum_{m \in \mathbb{Z}} \nu_{0,s,m}^1 + \sum_{m \in \mathbb{Z}} \nu_{0,s,n}^2,
\]

where

\[
\langle \nu_{0,s,m}^1, f \rangle = \int \frac{\gamma(|u|)}{u} \phi(2^m |u-s|) \frac{du}{u},
\]

\[
\langle \nu_{0,s,n}^2, f \rangle = \int \frac{\gamma(|u|)}{u} \phi(2^n |2-u|) \frac{du}{u},
\]

for some \( \theta, \tilde{\theta} \), both smooth functions, and some smooth \( \phi \) supported in \((1/2, 2)\); all nonnegative.

One still has, \( \nu_{s,m}^1, \nu_{s,n}^2 \) as \( 2^{D} \)-dilated versions of \( \nu_{0,s,m}^1, \nu_{0,s,n}^2 \), respectively. Finally, split \( E_{j,k,s} \) into \( \tilde{E}_{j,k,s,m} \) and \( \tilde{E}_{j,k,s,n} \), and \( E_{j,k,s} \) into \( \tilde{E}_{j,k,s,m} \) and \( \tilde{E}_{j,k,s,n} \). Similarly as in [10], observe that,

\[
s \frac{d}{ds} \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle) = u \frac{d}{du} \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle)
\]

which gives, for \( 1 \leq s \leq u \leq 2 \),

\[
\frac{d}{ds} \left( \nu_{j,s,m}^1 \right) (2^j s) D \xi = \frac{d}{ds} \int \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle) \theta(u) \phi(2^m (u-s)) \frac{du}{u}
\]

\[
= \frac{d}{ds} \left\{ \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle) \theta(u) \phi(2^m (u-s)) \right\} \frac{du}{u}
\]

\[
+ \int \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle) \frac{d}{ds} \left\{ \phi(2^m (u-s)) \right\} \frac{du}{u}
\]

\[
= \frac{d}{du} \left( \frac{1}{s} \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle) \right) \frac{d}{du} \left\{ \phi(2^m (u-s)) \right\} \frac{du}{u}
\]

\[
+ \int \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle) \frac{d}{ds} \left\{ \phi(2^m (u-s)) \right\} \frac{du}{u}
\]

which, from integration by parts, in turn gives,

\[
\frac{d}{ds} \left( \nu_{j,s,m}^1 \right) (2^j s) D \xi = - \int \frac{1}{s} \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle) \frac{d}{du} \left\{ \phi(2^m (u-s)) \right\} \frac{du}{u}
\]

\[
+ \int \exp(i \langle \xi (2^j s)^{\alpha_1} + \cdots + \xi_d (2^j s)^{\alpha_d} \rangle) \frac{d}{ds} \left\{ \phi(2^m (u-s)) \right\} \frac{du}{u}.
\]
One has a similar conclusion for the branch $-2 \leq u \leq s \leq -1$. A direct application of Van der Corput’s lemma yields, for $1 \leq s \leq 2$, 

$$\left| \frac{d}{ds}(\nu_{j,s,m}^{*})(2^j s^D \xi) \right| \leq \gamma_d \min\{1, 2^m |2^j D \xi|^{-1/d}\} \tag{4.7}$$

Similarly, 

$$\left| \frac{d}{ds}(\nu_{j,s}^{*})(2^j s^D \xi) \right| \leq \min\{1, 2^n |2^j D \xi|^{-1/d}\} \tag{4.8}$$

Furthermore, the cancellation nature of $\nu_{j,s}$’s implies that, 

$$\left| (\nu_{j,s})^{*}(2^j s^D \xi) \right| \leq \min\{|2^j D \xi|^{-1/d}, |2^j D \xi|^{1/d}\} \tag{4.9}$$

which allows one to obtain for all $1 \leq s \leq 2$, 

$$\| (\sum_{j \in \mathbb{Z}} |\nu_{j,s}^{*} \ast \Pi_j \xi f|^{2})^{1/2} \|_2 \leq 2^{-c|k|} \| f \|_2 \tag{4.10}$$

for some $c = c(\gamma, d) > 0$. See also the discussion below in Subsection 4.3. Then 4.7, 4.8, 4.9, 4.10, the second inequality in 4.4 and Plancherel’s theorem imply,

$$\| S_2 \{E^*_{j,k,s,m} f\}_{s \in [1, 2]} \|_2 \leq \gamma_d 2^{-c |m|} 2^{-c |j|} \| f \|_2$$

$$\| S_2 \{E^2_{j,k,s,m} f\}_{s \in [1, 2]} \|_2 \leq \gamma_d 2^{-c |m|} 2^{-c |j|} \| f \|_2 \tag{4.11}$$

for some $c = c(\gamma, d) > 0$.

Denote $K^*_{j,k,s,m} = \frac{d}{d}(\nu_{j,s}^{*} \ast \tilde{\nu}_{j,k})$ and $K^2_{j,k,s,m} = \frac{d^2}{d^2}(\nu_{j,s}^{*} \ast \tilde{\nu}_{j,k})$. Then $K^*_{j,k,s,m}, K^2_{j,k,s,m}$ are the kernels of $E^*_{j,k,s,m}, E^{2}_{j,k,s,m}$ respectively. Let $p$ be a metric that is homogeneous with respect to $(t^D)$ and $C_0 > 1$ sufficiently large (see [8]). Then it follows from the work in [10] (in particular, the proof of Theorem 1.5) as well as in [13] that,

$$\int_{1}^{2} \int_{\rho(x) \geq C \rho(y)} \left| K^*_{j,k,s,m}(x-y) - \tilde{K}^*_{j,k,s,m}(x) \right| dx \frac{ds}{s} \leq |k|, \tag{4.12}$$

$$\int_{1}^{2} \int_{\rho(x) \geq C \rho(y)} \left| K^2_{j,k,s,m}(x-y) - \tilde{K}^2_{j,k,s,m}(x) \right| dx \frac{ds}{s} \leq |k|.$$

See also the proof of 4.6 in the Approach 2 below. Still according to [10], 4.12 then in turn implies, for $1 < p < \infty$

$$\| S_2 \{E^*_{j,k,s,m} f\}_{s \in [1, 2]} \|_p \leq \gamma_d \| f \|_p$$

$$\| S_2 \{E^2_{j,k,s,m} f\}_{s \in [1, 2]} \|_p \leq \gamma_d \| f \|_p \tag{4.13}$$

Interpolation between 4.11, 4.13 gives, for $1 < p < \infty$

$$\| S_2 \{E^*_{j,k,s,m} f\}_{s \in [1, 2]} \|_{\gamma_d,p} \leq 2^{-c |m|} 2^{-c |j|} \| f \|_{\gamma_d,p}$$

$$\| S_2 \{E^2_{j,k,s,m} f\}_{s \in [1, 2]} \|_{\gamma_d,p} \leq 2^{-c |m|} 2^{-c |j|} \| f \|_{\gamma_d,p}$$

for some $c = c(d, p) > 0$. Summing these estimates over $k \in \mathbb{Z}; m, n \in \mathbb{Z}_+$, one then obtains the following form of 4.3:

$$\| S_2 \{E_{j,s} f\}_{s \in [1, 2]} \|_{\gamma_d,p} \leq 2^{-c |j|} \| f \|_{\gamma_d,p}.$$

### 4.2 Approach 2

The second approach shows 4.5, 4.6. Fix $s \in [1, 2]$ and $k \in \mathbb{Z}$. 

10
For 4.6, one can use the discussion in the proof of Lemma 6.1 in [10] to view that the kernels \( \tilde{K} = \{ \tilde{K}_{j,k,s} = \frac{d}{ds}(\nu_{j,s} \ast \psi_j,k) \}_{j \in \mathbb{Z}} \) of the convolution operators \( \{ f \ast \left( \frac{d}{ds}(\nu_{j,s} \ast \psi_j,k) \right) \}_j \) as having values in the \( l^2 \) space. Then an \( L^p \) bound of \( \left( \sum_{j \in \mathbb{Z}} |\nu_{j,s} \ast \Pi_{j,k} f|^2 \right)^{1/2} \) can be obtained from an upper bound of \( M \) of the following modulus of continuity of \( \tilde{K}_{j,k,s} \):

\[
\sup_{y \in \mathbb{R}^n, \rho(x) \geq C \rho(y)} \| \tilde{K}(x-y) - \tilde{K}(x) \|_{l^2} dx \leq M
\]

where \( \rho, C \) have the same meaning as in 4.12. It follows from [13] that one can take \( M \leq |\nu_{0,1}| \leq 1. \) This implies, for all \( 1 < p < \infty, \)

\[
\left\| \left( \sum_{j \in \mathbb{Z}} \frac{d}{ds}(\nu_{j,s} \ast \Pi_{j,k} f) \right)^2 \right\|_p^{1/2} \leq \| f \|_p
\]  

(4.14)

which is 4.6.

For 4.5, one can follow the above route in a similar manner, by getting a bound for the modulus of continuity of the right kernel. Then one get a similar estimate as in 4.14 for \( \left( \sum_{j \in \mathbb{Z}} |\nu_{j,s} \ast \Pi_{j,k} f|^2 \right)^{1/2}, \)

for \( 1 < p < \infty. \) Interpolate this estimate with the one from 4.10, one will obtain the following form of 4.5:

\[
\left\| \left( \sum_{j \in \mathbb{Z}} |\nu_{j,s} \ast \Pi_{j,k} f|^2 \right)^{1/2} \right\|_p \leq 2^{-c|k|} \| f \|_p
\]

for some \( c = c(\gamma,d,p) > 0. \)

4.3 Discussion of 4.9, 4.10

It was showed in the proof of Corollary 5.1 in [5] that the cancellation nature of a family of measure \( \{ \sigma_j \}_{j \in \mathbb{Z}} \) (\( \sigma_j(0) = 0 \)) that allows

\[
|\sigma_j(\xi)| \leq \min\{|2^j D \xi|^{-1/d}, |2^j D \xi|^{1/d}\}.
\]

There, \( \langle \sigma_j, f \rangle = \int_{2^{j-1} \leq |t| \leq 2^{j+1}} f(\Gamma(t)) d\lambda, \) and the considered \( \gamma \) belongs to this \( \Gamma \) class of curves. Every step of the derivation of the above fact holds for \( \nu_{j,s} \) in place of \( \sigma_j \) for all \( 1 \leq s \leq 2. \) This gives 4.9. One can then construct a partition of unity out of \( \psi_{j,k} \) as follows. Let \( \{ \omega_k \}_{k \in \mathbb{Z}} \) be smooth partition of unity on \( \mathbb{R} \), adapted to the intervals \( [2^{-k}, 2^{-k+1}] \). In particular, one wants \( 0 \leq \omega_k \leq 1, \sum_k \omega_k^2(t) = 1 \) if \( t > 0 \) and \( \text{supp}(\omega_k) \subset [2^{-k-1}, 2^{-k+1}] \). Finally, for each \( j \in \mathbb{Z}, \) let \( \{ \omega_k^j \}_{k \in \mathbb{Z}} \) be the \( 2^{jD} \)-dilated version of \( \{ \omega_k \}_{k \in \mathbb{Z}} \). Hence one can think of \( \{ \omega_k^j \}^2 \) as an appropriately smooth cut-off of \( \psi_{j,k}. \) Then by Plancherel’s theorem,

\[
\sum_{j \in \mathbb{Z}} \| \nu_{j,1} \ast \Pi_{j,k} f \|^2 \leq \sum_{j \in \mathbb{Z}} \int_{\Delta_k^*} |\tilde{\nu}_{j,1}^2(2^j D \xi)|\langle \tilde{\xi} \rangle^2(\xi) d\xi
\]

where \( \Delta_k = \{ 2^{-k-1} \leq |2^j D \xi| \leq 2^{-k+1} \}. \) If \( k < 0 \) then \( |2^j D \xi| \geq 2^{-k+1}. \) Hence by 4.9,

\[
\sum_{j \in \mathbb{Z}} \| \nu_{j,1} \ast \Pi_{j,k} f \|^2 \leq \sum_{j \in \mathbb{Z}} \int_{\Delta_k^*} \left| \tilde{\nu}_{j,1}^2(2^j D \xi) \right|^2(\xi) d\xi \leq 2^{k/3} \| f \|^2_2.
\]

Argue similarly for the cases, \( k > 1 \) and \( k = 0, 1, \) one obtains, for all \( k \in \mathbb{Z}, \)

\[
\sum_{j \in \mathbb{Z}} \| \nu_{j,1} \ast \Pi_{j,k} f \|^2 \leq 2^{-|k|/3} \| f \|^2_2.
\]

The same is true for all other \( \nu_{j,s}. \) All this then implies 4.10.

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