The Renormalization Group method for simple operator problems in quantum mechanics

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Abstract

A simple backreaction problem in quantum mechanics, the full quantum anharmonic oscillator, and quantum parametric resonance are studied using Renormalization Group techniques for global asymptotic analysis. In this short note this technique is adapted for the first time to operator problems.

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I. INTRODUCTION AND MOTIVATIONS

In many different contexts perturbation theory fails miserably because of the growth of higher order terms, contrary to the basic perturbative assumption. This secularity is present in both classical and quantum theories, and pervades the motivation for the search for analytical methods to improve on perturbative expansions.

There is a set of problems of particular interest to us where improvement on perturbative expansions has shown itself to be of immediate necessity, and of direct physical meaning, namely, (p)reheating in inflationary cosmology [1–4]. We shall not deal directly with this problem, though, and we will content ourselves with an analysis of two simple quantum mechanical problems. The choice of problems and treatment thereof, however, will be inspired by the mentioned physical setting.

The method we use is novel, and subsumes many other previously known ones. It is the Renormalization Group (RG) method for global asymptotic analysis, as advocated by Goldenfeld and collaborators [5,6], which we extend to operator problems in this short note. The key idea of the RG method for global asymptotic analysis is the introduction of a time parameter, additional to the initial value point, in such a way that the perturbation expansion is valid in the vicinity of the introduced time parameter. The coupling constants/initial conditions (depending on your viewpoint and background one or another of these descriptions will be more suitable) are turned into running constants, that is to say that these constants are suitably modified by the change of the introduced time parameter. On the other hand, the solution itself cannot depend on the additional, new time parameter, so derivation with respect to the latter of the perturbative solution will impose evolution equations for the running constants. These equations are then solved for the running constants, and on substitution in the perturbative expansion, together with the choice that the time parameter is constantly updated to be time itself, we obtain an improved solution.

This method has the clear advantage over multiple scales perturbation analysis [7,8] that no a priori determination of the scales that appear in the problem is necessary, and a naive perturbation expansion is enough as a starting point. Many examples and illustrations of this advantage of the RG method can be found in the works of Goldenfeld and collaborators.

The multiple scales method itself has been applied to operator problems [9,10], but not the RG method, and this paper is the first example, to the best of our knowledge, of the application of the RG method to operator problems. We shall choose the quantum anharmonic oscillator and the phenomenon of quantum parametric resonance as our case studies, because of the important rôle they have traditionally had as theoretical laboratories for new perturbative methods, and because of their paradigmatic character in the context of cosmological (p)reheating. Our results are comparable to all other methods, and since they are obtained with a modest effort, we think that the RG method is highly competitive in the operator context as well. There are a number of areas where its usefulness might be proved, such as quantum optics, but that we leave for further work.

II. SETTING OF THE PROBLEM

The problem we shall first address is the back-reaction problem of the quantum anharmonic oscillator; that is to say, the problem posed by the backreaction of quantum
fluctuations on the expected value of the position. Consider thus a lagrangian of the form
\[ L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 - \frac{1}{4} \lambda m q^4. \] (2.1)

The equation of motion in Heisenberg’s picture reads
\[ \frac{d^2 q}{dt^2} + \omega^2 q + \lambda q^3 = 0, \] (2.2)

which is, of course, an operator equation, the quantum Duffing equation. Let us now decompose the q operator into a c-number times the unit operator and a fluctuation operator, in such a way that the expectation value of q is equal to the mentioned c-number:
\[ q = \varphi + \xi, \] (2.3)

where \( \varphi = \langle q \rangle, \langle \xi \rangle = 0 \), the unit operator is omitted, and \( \langle \cdot \rangle \) stands for the expectation value of an operator over a given state, which, as we are in Heisenberg’s picture, is a time independent state, and carries the initial value data for the operator evolution equation. Both the c-number \( \varphi \) and the operator \( \xi \) are time dependent, of course.

Taking the expectation value of the operator equation (2.2) we obtain
\[ \frac{d^2 \varphi}{dt^2} + \omega^2 \varphi + \lambda \left( \varphi^3 + 3 \varphi \langle \xi^2 \rangle + \langle \xi^3 \rangle \right) = 0, \] (2.4)

and using this result, together with (2.2), we can write
\[ \frac{d^2 \xi}{dt^2} + \left( \omega^2 + 3 \lambda \varphi^2 \right) \xi = 3 \lambda \varphi \left( \langle \xi^2 \rangle - \xi^2 \right) + \lambda \left( \langle \xi^3 \rangle - \xi^3 \right). \] (2.5)

It is clear that the pair of equations (2.4) and (2.5) is equivalent to the operator equation (2.2) together with an initial condition provided by the state over which the expectation values are computed.

So far we have followed the computations in Ehrenfest’s theorem [11], and the result is, as is well known in elementary quantum mechanics, that the evolution of the expectation value of the q operator is not given by the classical evolution equations: there are quantum corrections because the potential is not quadratic.

The customary way of handling this kind of problem is to revert to the interaction picture and use time independent perturbation theory. We shall depart from this well-trodden route and study the large time asymptotics of the expectation value from a truncation of the exact equations (2.4) and (2.5). The truncation, which we will call the one-loop approximation because of the analogy to the one-loop truncation in quantum field theory, leaves us the following coupled equations, one c-number and the other operator valued:
\[ \frac{d^2 \varphi}{dt^2} + \left( \omega^2 + 3 \lambda \langle \xi^2 \rangle \right) \varphi = -\lambda \varphi^3, \]
\[ \frac{d^2 \xi}{dt^2} + \left( \omega^2 + 3 \lambda \varphi^2 \right) \xi = 0. \] (2.6)
Notice that for these truncated equations the equation for fluctuations is linear, with a time-dependent frequency related to $\varphi^2$. We could now try a naive perturbation expansion for these equations, of the form

$$\varphi = \varphi_0 + \lambda \varphi_1 + O(\lambda^2),$$
$$\xi = \xi_0 + \lambda \xi_1 + O(\lambda^2),$$

which, when inserted in eqns (2.6), yields

$$\frac{d^2\varphi_0}{dt^2} + \omega^2 \varphi_0 = 0,$$
$$\frac{d^2\varphi_1}{dt^2} + \omega^2 \varphi_1 = -3\langle \xi_0^2 \rangle \varphi_0 - \varphi_0^3,$$
$$\frac{d^2\xi_0}{dt^2} + \omega^2 \xi_0 = 0,$$
$$\frac{d^2\xi_1}{dt^2} + \omega^2 \xi_1 = -3\varphi_0^2 \xi_0,$$

(2.7)

etc. It now becomes apparent that even within this simple truncation the perturbative expansion cannot work, since secular terms will appear and the perturbative hypothesis, namely, that the terms in higher order in $\lambda$ are smaller than the previous ones, will stop holding after a time of the order $1/\lambda$. This is the issue we shall now address by means of the RG method.

**III. APPLICATION OF THE RG METHOD**

The zeroth order equations in the series (2.7) are solved by

$$\varphi_0 = R \cos(\omega t + \theta),$$
$$\xi_0 = l \left( \alpha^\dagger e^{i\omega t} + \alpha e^{-i\omega t} \right),$$

(3.1)

where $\alpha$ and $\alpha^\dagger$ are formally adjoint to each other. The commutation relations among these two operators are fixed by imposing canonical commutation relations for $q$ and its conjugate operator $p$. We obtain

$$[\alpha, \alpha^\dagger] = \frac{\hbar}{2m\omega l^2},$$

(3.2)

and, therefore, choosing $l = \sqrt{\hbar/(2m\omega)}$, $\alpha$ and $\alpha^\dagger$ can be understood as annihilation and creation operators for a harmonic oscillator, implying that we have a way of defining a number operator $N = \alpha^\dagger \alpha$ that will provide us with information about the magnitude of the fluctuations.

So far everything has been straightforward. Problems first arise in computing the first order corrections, since secular terms will be generated. Consider a time $\tau$ for which the first order corrections vanish. The solutions to the first order equations will be
\[ \varphi_1 = \frac{3iR}{16\omega}(t-\tau) \times \left( e^{i\omega t} \left( R^2 e^{i\theta} + 4l^2 (\langle \alpha^\dagger \rangle^2) e^{-i\theta} + 8l^2 \left( \langle N \rangle + \frac{1}{2} \right) e^{i\theta} \right) - e^{-i\omega t} \left( R^2 e^{-i\theta} + 4l^2 (\alpha^2) e^{i\theta} + 8l^2 \left( \langle N \rangle + \frac{1}{2} \right) e^{-i\theta} \right) \right) + \text{r.t.,} \]  

(3.3)

and

\[ \xi_1 = \frac{3iR^2}{8\omega}(t-\tau) \left( e^{i\omega t} \left( 2\alpha^\dagger + \alpha e^{2i\theta} \right) - e^{-i\omega t} \left( 2\alpha + \alpha^\dagger e^{-2i\theta} \right) \right) + \text{r.t.,} \]  

(3.4)

where r.t. stands for regular terms, that is to say those which are bounded when \( t-\tau \) tends to infinity.

It is clear that for small \( t-\tau \) the perturbative expansion has no a priori reason not to hold. Now, the crucial point is to realize that the choice of \( \tau \) corresponds to a choice of initial time for perturbations, which, in other words, means that a particular value for the magnitudes \( R \) and \( \theta \) and a particular choice for the (constant) operators \( \alpha \) and \( \alpha^\dagger \) has been chosen. If we were to allow those to change with \( \tau \), we would be able to readjust the perturbative series as we went away from a given \( \tau \) for a fixed \( t \) to a different \( \tau' \), around which the perturbative expansion for \( t-\tau' \) would now proceed. The change in \( R, \theta, \alpha \) and \( \alpha^\dagger \) would compensate the change in \( \tau \).

On the other hand, the solution to the differential equations which we intend to solve by a perturbative expansion cannot depend on \( \tau \). Once initial values \( R \) and \( \theta \) and initial operators \( \alpha \) and \( \alpha^\dagger \) are given, the evolution of \( \varphi \) and \( \xi \) is fixed. Therefore, we obtain the first order RG equations by allowing the stated dependence of the (number and operator) parameters in \( \tau \), deriving the expressions \( \varphi_0 + \lambda \varphi_1 \) and \( \xi_0 + \lambda \xi_1 \) with respect to \( \tau \), and setting those derivatives equal to zero. To keep consistency with the perturbative expansion, the terms of the form \( \lambda \partial R/\partial \tau \) and similar are discarded, since they are of order \( \lambda^2 \), which is being disregarded.

Reordering of the first order RG equations and separation of the coefficients of \( e^{i\omega t} \) and \( e^{-i\omega t} \), together with some manipulation of the operator equations, lead us to the following closed set of five coupled differential equations for five ordinary functions of \( \tau \):

\[
\begin{align*}
\frac{1}{R} \frac{dR}{d\tau} &= \frac{3i\lambda l^2}{4\omega} \left( \langle (\alpha^\dagger)^2 \rangle e^{-2i\theta} - \langle \alpha^2 \rangle e^{2i\theta} \right) \\
\frac{d\theta}{d\tau} &= \frac{3\lambda}{8\omega} \left( R^2 + 4l^2 + 2l^2 \left( \langle (\alpha^\dagger)^2 \rangle e^{-2i\theta} + \langle \alpha^2 \rangle e^{2i\theta} \right) + 8l^2 \langle N \rangle \right) \\
\frac{d\langle \alpha^2 \rangle}{d\tau} &= -\frac{3i\lambda R^2}{8\omega} \left( 4\langle \alpha^2 \rangle + 2e^{-2i\theta} \langle N \rangle + e^{-2i\theta} \right) \\
\frac{d\langle (\alpha^\dagger)^2 \rangle}{d\tau} &= \frac{3i\lambda R^2}{8\omega} \left( 4\langle (\alpha^\dagger)^2 \rangle + 2e^{2i\theta} \langle N \rangle + e^{2i\theta} \right) \\
\frac{d\langle N \rangle}{d\tau} &= -\frac{3i\lambda R^2}{8\omega} \left( \langle (\alpha^\dagger)^2 \rangle e^{-2i\theta} - \langle \alpha^2 \rangle e^{2i\theta} \right)
\end{align*}
\]

(3.5)

These equations, after a little reorganising in order to write them as a set of real equations with no free parameters, can be solved numerically without much further ado, leading for predictions for the long time behaviour of \( R(\tau) \) and \( \theta(\tau) \).
Consider now that we constantly update $\tau$ to be $t$. The secular terms will disappear in $\varphi_1$, and their effect will show up through $R$ and $\theta$, now understood as functions of $t$:

$$\varphi = R(t) \cos(\omega t + \theta(t)) + \epsilon(\text{r.t.}),$$

where r.t. stands for regular terms, i.e., well controlled small perturbations to the first term.

Moreover, it can be seen directly, from the analysis of equations (3.5), that

$$R \frac{dR}{d\tau} + 2i^2 \frac{d\langle N \rangle}{d\tau} = 0,$$

which is the equation that implements the conservation of energy to this order.

The numerical solution for the initial conditions $\varphi(0) = \sqrt{8\omega/(3\lambda)}$, $\langle N \rangle = 0$, $\theta(0) = 0$, $\langle \alpha^2 \rangle = 0$ and $\langle (\alpha^\dagger)^2 \rangle = 0$ is as shown in figure 1, where $\varphi(t)$ is depicted. The units used are arbitrary.

IV. DIRECT APPLICATION OF THE RG METHOD TO THE QUANTUM DUFFING EQUATION

In the previous section we have demonstrated the usefulness of the RG method in eliminating the secular terms for the truncated equations obtained at one loop. We shall now consider the whole quantum Duffing equation, eqn. (2.2), and apply the same techniques for the complete problem.

Let us perform a simple perturbation expansion in the $\lambda$ coupling constant, $q = q_0 + \lambda q_1 + O(\lambda^2)$. The solution to order 0 is simply

$$q_0 = \sqrt{\frac{\hbar}{2m\omega}} (\beta e^{-i\omega t} + \beta^\dagger e^{i\omega t}).$$

We have written $\beta$ and $\beta^\dagger$ in order to make the distinction with the operators $\alpha^\dagger$ and $\alpha$, which were creation and annihilation operators for fluctuations in the previous section, whereas $\beta^\dagger$ and $\beta$ are creation and annihilation operators for the full oscillator problem. The first order equation presents resonance and, therefore, secular terms. Writing down just the singular (i.e., secular) part of $q_1$, which we call $q_{1,s}$,

$$q_{1,s} = \left( \frac{\hbar}{2m\omega} \right)^{3/2} \frac{i(t - \tau)}{2\omega} \left( (\beta^\dagger)^2 + \beta^\dagger \beta \beta^\dagger + (\beta^\dagger)^2 \beta \right) e^{i\omega t} - \left( \beta^2 \beta^\dagger + \beta^\dagger \beta + \beta \beta^\dagger \right) e^{-i\omega t},$$

(4.2)

Let us now allow a dependence in $\tau$ of the pair of operators $\beta$ and $\beta^\dagger$. Imposing the RG condition $dq/d\tau = 0$, we obtain

$$\frac{d\beta}{d\tau} + \frac{i\lambda \hbar}{4m\omega^2} (\beta^2 \beta^\dagger + \beta \beta^\dagger \beta + \beta^\dagger \beta^2) = O(\lambda^2),$$

(4.3)

and the hermitian conjugate thereof. We notice that $\mathcal{N} = \beta^\dagger \beta$ and $[\beta, \beta^\dagger]$ are constants under the flow of $\tau$, which allows us to solve these equations in the form
\[
\beta(\tau) = \beta(0)e^{-3i\lambda N\tau/(4m\omega^2)},
\]
\[
\beta^\dagger(\tau) = e^{3i\lambda N\tau/(4m\omega^2)}\beta(0),
\] (4.4)

which, on being substituted in the perturbative expansion of \(q\), together with the change \(\tau \to t\), gives us
\[
q(t) = \sqrt{\frac{\hbar}{2m\omega}} \left( e^{-i\omega t\beta(0)e^{-3i\lambda N\tau/(4m\omega^2)}} + e^{i\omega t\beta^\dagger(0)} \right).
\] (4.5)

We have thus obtained an asymptotic expression for this operator. On computing \(\langle n-1|q(t)|n\rangle\), we see that the energy difference between levels comes out as \(E_n - E_{n-1} = \hbar\omega(1 + (3\lambda\hbar n)/(4m\omega^3) + O(\lambda^2))\), consistent with all previous computations of this quantity.

It has to be observed that our result is identical to the one obtained by Bender and Bettencourt [9], as is only to be expected, given the equivalence of the multiple scales method and the RG methods for a wide class of differential equations, to which the (classical) Duffing equation belongs. On the other hand, note the simplicity of our approach, where no a priori scale has to be assumed.

In order to stress this latter point, let us consider the second order computation for this problem. The source term for \(q_2\) is given by \(- (q_0^2 q_1 + q_0 q_1 q_0 + q_1 q_0^2)\), where we have to consider the full \(q_1\) and not just the singular part. In this source term there will be terms that will give rise to secularities of the form \(e^{\pm 3i\omega t}(t - \tau)\) and \(e^{\pm i\omega t}(t - \tau)^2\). These we shall be able to ignore, because the renormalization to first order takes care of them. As a matter of fact, this is precisely what the statement of perturbative renormalizability amounts to in our case: that no divergences of a different form arise in the process of renormalization, that is to say, that all divergent (secular) terms can be taken care of by renormalization of the terms \(\beta e^{-i\omega t}\) and \(\beta^\dagger e^{i\omega t}\).

Another (simple) technicality in the problem at hand is that, since we have checked that \([\beta, \beta^\dagger]\) is constant to order \(\lambda^2\), we can use the commutator in the \(\lambda\) and \(\lambda^2\) terms, thus making the computation somewhat easier.

This results in the following expression for the secular relevant part of \(q_2\), \(q_{2, sr}\):
\[
q_{2, sr} = \frac{-3i\lambda^5(t - \tau)}{16\omega^3} \left( (5\lambda N^2 - 1)\beta^\dagger e^{i\omega t} - \beta(5\lambda N^2 - 1)e^{-i\omega t} \right),
\] (4.6)

whence the improved RG equation reads
\[
\frac{d\beta}{d\tau} + \frac{3i\lambda\hbar}{4m\omega^2}\beta N - \frac{3i\lambda^2h^2}{64m^2\omega^5}\beta (5\lambda N^2 - 1) = O(\lambda^3),
\] (4.7)

thus giving us
\[
\beta(\tau) = \beta(0) \exp \left( \frac{-3i\lambda\hbar N\tau}{(4m\omega^2)} + \frac{3i\lambda^2h^2(5\lambda N^2 - 1)\tau}{64m^2\omega^5} \right),
\] (4.8)

and, as a consequence,
\[
E_n - E_{n-1} = \hbar\omega \left( 1 + (3\lambda\hbar n)/(4m\omega^3) - 3\lambda^2h^2(5n^2 - 1)/(64m^2\omega^6) + O(\lambda^3) \right).
\] (4.9)
V. QUANTUM PARAMETRIC RESONANCE

As a last example of the usefulness of the RG method for quantum mechanical problems, we shall now illustrate its application to the phenomenon of quantum parametric resonance. Consider then the following Hamiltonian:

\[
H = \frac{1}{2m} P^2 + \frac{1}{2} m \omega_0^2 (A + 2q \cos(\omega_0 t)) X^2,
\]  

(5.1)

where \( A \), \( q \) and \( \omega_0 \) are constants. The evolution of any given state is computed by acting on it with the evolution operator \( U(t, t_0) \), which satisfies

\[
i\hbar \frac{\partial U(t, t_0)}{\partial t} = H(t) U(t, t_0),
\]

(5.2)

with \( U(t_0, t_0) = I \), the identity operator.

Let us divide the Hamiltonian into an unperturbed and a perturbation part:

\[
H = H_0 + H_1 = \left( \frac{1}{2m} P^2 + \frac{1}{8} m \omega_0^2 X^2 \right) + \left( \frac{1}{2} m \omega_0^2 ((A - \frac{1}{4}) + 2q \cos(\omega_0 t)) X^2 \right).
\]

(5.3)

The reason for this decomposition lies in our previous knowledge that resonance will definitely set in if \( A \) is equal to \( 1/4 \), but this is not essential for the final results.

The evolution operator can be written as

\[
U(t, t_0) = e^{-i(H_0/t \hbar)} U_I(t, t_0),
\]

(5.4)

in such a way that the interaction picture evolution operator obeys the following equation:

\[
i\hbar \frac{\partial U_I(t, t_0)}{\partial t} = H_I(t) U_I(t, t_0),
\]

(5.5)

and in our case

\[
H_I = \frac{1}{2} m \omega_0^2 ((A - \frac{1}{4}) + 2q \cos(\omega_0 t)) X_I^2
\]

\[
= \frac{\hbar \omega_0}{2} ((A - \frac{1}{4}) + 2q \cos(\omega_0 t)) \left( e^{-i\omega_0(t-t_0)/2} a + e^{i\omega_0(t-t_0)/2} a^\dagger \right)^2.
\]

(5.6)

The constant operators \( a \) and \( a^\dagger \) are the annihilation and creation operators at time \( t_0 \).

We now perform the usual perturbative expansion for the interaction picture evolution operator, restricting ourselves to the Born formula, \( U_I = 1 - \frac{i}{\hbar} \int_{t_0}^{t} ds H_I(s) \). However, this leads to secular terms, and in order to eliminate them, we shall rather use this approximation close to the time \( t = \tau \), by using the initial condition \( U_I(\tau, t_0) = \alpha(\tau) \), such that

\[
U_I(t, t_0) = \alpha(\tau) - \frac{i}{\hbar} \int_{\tau}^{t} ds H_I(s) \alpha(\tau) + \text{higher order terms}.
\]

(5.7)

Retaining only the secular terms, we obtain

\[
U_I(t, t_0) = \alpha(\tau) - \frac{i\omega_0}{2} (t - \tau) \left( (A - \frac{1}{4}) (aa^\dagger + a^\dagger a) + q(e^{i\omega_0 t_0} a^2 + e^{-i\omega_0 t_0} (a^\dagger)^2) \right) \alpha(\tau) + \text{r. t.}
\]

(5.8)
For the sake of simplicity, let us set \( t_0 = 0 \), without any loss of generality. We know that \( U_t \) cannot depend on the choice of \( \tau \), and we are thus led to the RG equation to first order

\[
\frac{\partial \alpha}{\partial \tau} + \frac{i \omega_0}{2} \left( (A - \frac{1}{4})(a a^\dagger + a^\dagger a) + q(a^2 + (a^\dagger)^2) \right) \alpha(\tau) = 0 ,
\]

which, on being solved, provides us with an improved expression for the interaction picture evolution operator, \( U_I(t, 0) = \exp(-itH_{\text{eff}}/\hbar) \), where \( H_{\text{eff}} \) is the large time asymptotic effective Hamiltonian read off directly from the RG equation:

\[
H_{\text{eff}} = \frac{\hbar \omega_0}{2} \left( (A - \frac{1}{4})(a a^\dagger + a^\dagger a) + q(a^2 + (a^\dagger)^2) \right)
= 2 \left( \frac{1}{2m} (A - \frac{1}{4} - q) P^2 + \frac{1}{8} m \omega_0^2 (A - \frac{1}{4} + q) X^2 \right).
\]

This is the first important result of our computation: we have resummed the effect of the variable frequency into an effective large time constant Hamiltonian. If it happens that, for small positive \( q, \frac{1}{4} - q < A < \frac{1}{4} + q \), this effective Hamiltonian corresponds to an inverted oscillator, thus marking the principal instability band (to the order we have computed).

It now behooves us to compute the creation of particles due to this instability. In order to do this, we shall first write down the integral kernel that corresponds to \( U_I \) in the position representation, \( K(x, t; x', t') \), using standard results for quadratic Hamiltonians \([12]\).

Let \( \gamma = \omega_0 \sqrt{q^2 - (A - 1/4)^2} \) and \( \varphi = \sqrt{(q - 1/4 + A)/(q + 1/4 - A)} \). The integral kernel \( K(x, t; x', 0) := \langle x | U_I(t, 0) | x' \rangle \) is computed to be (asymptotically):

\[
K(x, t; x', 0) = \left( \frac{i m \omega_0 \varphi}{4 \pi \hbar \sinh(\gamma t)} \right)^{1/2} \exp \left( \frac{-i m \omega_0 \varphi}{4 \hbar \sinh(\gamma t)} \left( (x^2 + x'^2) \cosh(\gamma t) - 2 xx' \right) \right). \tag{5.11}
\]

It is now feasible to compute the asymptotic value of \( \langle n | U(t, 0) | 0 \rangle \) through simple tabulated integrals, and we can calculate the transition probability from the ground state to even states:

\[
P_{0 \to 2l}(t) = \frac{(2l)!}{((l!)^2 2^{2l})} \frac{2 \varphi}{\sqrt{4 \varphi^2 + (1 + \varphi^2)^2 \sinh^2(\gamma t)}} \left( \frac{(1 + \varphi^2)^2 \sinh^2(\gamma t)}{4 \varphi^2 + (1 + \varphi^2)^2 \sinh^2(\gamma t)} \right)^l. \tag{5.12}
\]

It is easy to check that unitarity is preserved. An analogous computation leads to the rate of particle production,

\[
\mathcal{N}(t) = \frac{(1 + \varphi^2)^2}{4 \varphi^2} \sinh^2(\gamma t) . \tag{5.13}
\]

These results can be compared with the computations of Shtanov et al. \([2]\), and coincide completely for the specific case at hand. Shtanov et al. arrive to this result through Bogolyubov transformations (to identify the function giving particle creation) and Krylov-Bogolyubov averaging (to perform the asymptotic analysis). This coincidence comes as no surprise given the first order equivalence of Krylov-Bogolyubov averaging to two-timing (a particular instance of the multiple-scales method) for a wide class of differential equations \([8]\), and the (again first-order) equivalence of the multiple-scales and RG methods for many instances of equations.
VI. CONCLUSIONS

We have performed several operator computations in quantum mechanics using the RG method for global asymptotic analysis. This method has the serious advantage that unitarity is built in, and that computations are simpler and more direct than in other techniques for asymptotic analysis. These ideas should be useful in a wide realm of applications. In the special case of quantum parametric resonance we derive explicitly an (asymptotic) effective hamiltonian, which is an inverted harmonic oscillator whenever the system is in the instability region: the instability associated with the parametric resonance is turned into the unboundedness of the interaction hamiltonian, thus demonstrating the basic equivalence (asymptotically) of such different systems. Even so, unitarity is preserved throughout our computation, and the asymptotic results we obtain are well-behaved with respect to this fundamental property of quantum mechanical evolution. We have performed an analysis of the first instability band only for the quantum Mathieu equation. However, it is possible within this method to examine the whole forbidden/allowed band structure of this model, following in the quantum context the study carried out for the classical case by Goldenfeld and collaborators. As a matter of fact, for quadratic hamiltonians the whole instability analysis can be reduced to classical mechanics, i.e., to classical Mathieu (or similar) equations. What we emphasise as novel in our results is the interpretation of instabilities as being due to effective large time inverted oscillator hamiltonians. Furthermore, analogous analyses can be carried for non-quadratic hamiltonians, even time dependent, where the quantum mechanical character of the problem would show itself to its fullest.
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FIGURES

FIG. 1. Backreaction: $\varphi(t)$. 

