A TRUDINGER–MOSER INEQUALITY WITH MEAN VALUE ZERO
ON A COMPACT RIEMANN SURFACE WITH BOUNDARY

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Abstract. In this paper, on a compact Riemann surface \((\Sigma, g)\) with smooth boundary \(\partial \Sigma\), we concern a Trudinger-Moser inequality with mean value zero. To be exact, let \(\lambda_1(\Sigma)\) denotes the first eigenvalue of the Laplace-Beltrami operator with respect to the zero mean value condition and \(\mathcal{S} = \{ u \in W^{1,2}(\Sigma, g) : \| \nabla_g u \|_2^2 \leq 1 \text{ and } \int_{\Sigma} u \, dv_g = 0 \}\), where \(W^{1,2}(\Sigma, g)\) is the usual Sobolev space, \(\| \cdot \|_2\) denotes the standard \(L^2\)-norm and \(\nabla_g\) represent the gradient. By the method of blow-up analysis, we obtain
\[
\sup_{u \in \mathcal{S}} \int_{\Sigma} e^{2\pi u^2 (1 + \alpha \| u \|^2_2)} \, dv_g < +\infty, \quad \forall \ 0 \leq \alpha < \lambda_1(\Sigma);
\]
when \(\alpha \geq \lambda_1(\Sigma)\), the supremum is infinite. Moreover, we prove the supremum is attained by a function \(u_\alpha \in C^\infty(\Sigma) \cap \mathcal{S}\) for sufficiently small \(\alpha > 0\). Based on the similar work in the Euclidean space, which was accomplished by Lu-Yang \([19]\), we strengthen the result of Yang \([29]\).

Mathematics subject classification (2020): 46E35, 58J05, 58J32.

Keywords and phrases: Trudinger-Moser inequality, Riemann surface, blow-up analysis, extremal function.

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