§1. Introduction.

Let $N$ be a connected nilpotent Lie group with associated Lie Algebra $\mathfrak{n}$ of left-invariant vector fields. We shall assume that $\mathfrak{n}$ is stratified in the sense of Folland \cite{F}. That is, there is a direct sum decomposition of finite-dimensional vector spaces $V_i$, with

$$\mathfrak{n} = \bigoplus_{j=1}^{m+1} V_j, [V_j, V_i] \subseteq V_{i+j}, i + j \leq m + 1,$$

and $[V_j, V_i] = 0$ if $i + j > m + 1$. Furthermore, we assume that $V_1$ generates the Lie algebra $\mathfrak{n}$. Let $\dim V_1 = p$, and let $B = \{X_1, \ldots, X_p\}$ be a basis for $V_1$. We study the PDE (and associated operator $L$),

$$Lu = \sum_{i=1}^{p} X_i^2 u = 0.$$  

In particular, we wish to study the analytic hypo-ellipticity of the operator $L$. The $C^\infty$ hypo-ellipticity of $L$ follows by a celebrated theorem of Hörmander \cite{Ho}.

An important counter example of Baouendi-Goulaouic \cite{BG} demonstrates that $L$ need not be analytic hypoelliptic. However, there are situations where (1.1) can be analytic hypoelliptic, like on the Heisenberg group and for the related operators of the type of Baouendi-Grushin. This was demonstrated by Treves \cite{Tr1} using a microlocal approach and a parametrix construction, and by Tartakoff \cite{T} using energy estimates. See \cite{Tr2} for a precise statement and a historical overview. After the \cite{BG} example, several examples have been found by Oleinik \cite{O}, Hanges-Himonas \cite{HH1-2}, Helffer \cite{H}, Pham-The-Lai and Robert \cite{LR} and by Christ \cite{C1-C2}. More recently there have been articles by Hoshiro \cite{Hos} and the Costins’ \cite{CC}. We will put these results in perspective in the sequel.
The point of view we adopt is that from [Tr2]. We will thus be working on the co-tangent bundle of \( N, T^*N \), and also we will have reason later to work on the complexified co-tangent bundle, \( \mathbb{C}T^*N \). On \( T^*N \), we will focus on the symbols of the vector fields \( V_1 \). The characteristic set \( \Sigma \) of (1.1) then has a stratification given by the vanishing of the symbols of the vector fields in \( V_1 \) and by the vanishing of the symbols of the vector fields we systematically get by taking brackets of the vector fields in \( V_1 \), [Tr2]. We assume that one such stratum is not symplectic. That is to be more precise: The equation defining the non-symplectic stratum is given by the vanishing of a co-vector.

Thus we are assuming that one can find a curve \( \gamma(t) \) on \( N \), and more precisely in the base projection of \( \Sigma \) on \( N \), such that the tangent line bundle of \( \gamma(t) \) is orthogonal for the fundamental symplectic form to \( TS \) restricted to \( \gamma, TS|_\gamma \), where \( S \) is a Poisson stratum of \( \Sigma \) that is not symplectic. Of course it may happen, as in the example [BG], that \( \Sigma \) itself is non-symplectic.

Our approach is via the theory of representations due to Kirillov. We may naturally attach elements \( \lambda \in n^* \) (the dual Lie algebra) to the vanishing co-vectors that define the non-symplectic strata. One then induces representations for these elements \( \lambda \). These representations give rise to Schrödinger equations on some \( \mathbb{R}^n \), \( n \geq 1 \). The principal difficulty arises when \( n \geq 2 \). The Schrödinger equations then have potentials which in general go off to negative infinity and are thus not confining. Bounds for such potentials are difficult to come by, though we ask for very weak bounds. Nevertheless, we are able to obtain bounds in an important case, thereby obtaining solutions to (1.1) which are smooth but not analytic. Previous works on this topic when examined under the view point in this article shows that the majority of the examples considered lead to representations that act on functions on \( L^2(\mathbb{R}) \), that is the so-called Gelfand-Kirillov dimension is 1 ([H] for example is not restricted to the Kirillov dimension being 1). Therefore instead of a true PDE Schrödinger, the analysis is that of an ODE Schrödinger. However, our article shows that to understand completely the conjecture (1.1) in [Tr2], the analysis of Schrödinger equations in \( \mathbb{R}^n \) with degenerate potentials is unavoidable. Our treatment of the Schrödinger equations uses machinery from potential theory, in particular Harnack’s inequality. Thus we rely extensively on positivity. This is a technical drawback and prevents us from capturing the oscillatory solutions which will necessarily arise in the general case. However, we state a theorem of Treves that indicates that the bounds we desire are valid in full generality.

Since, we are assuming that the curves \( \gamma(t) \) lie in the base, we are also ruling out the interesting example of Métérior [M]. The idea of using representations in the study of (1.1) is not new and goes back to Rothschild-Stein [RS], and subsequently was developed by Rockland [Ro] and Helffer-Nourrigat [HN] all in the
$C^\infty$-hypoellipticity context. The novel point here is to associate representation theory with the notion of strata [Tr2].

Lastly, we do not have general theorems when the vector fields are not free. The notion of lifting vector fields introduced in [RS], has to be handled with care as we will show by examples. In any case the bounds for the Schrödinger equation allow us to easily construct new classes of sums of squares which are composed of degenerate vector fields and which have smooth, non-analytic solutions.

The paper is organized as follows. In Section 2, we show how the induced representations are linked to the co-vectors that define the strata of [Tr2]. These give rise to Schrödinger operators. It is then shown how solutions to these Schrödinger equations if they satisfy a certain growth bound, yield non-analytic solutions of (1.1). In Section 3, under additional conditions we obtain polynomial growth bounds on the solutions to the Schrödinger equations of Section 2. The last section, section 4, contains several examples, which clarify and focus on what remains to be done.

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§2. Induced Representations and Strata.

We will freely use the theory of induced representations as developed by Kirillov [K]. An excellent account is given in the book by Corwin and Greenleaf [CG]. We assume that

\begin{equation}
\mathfrak{n} = \bigoplus_{j=1}^{m+1} V_j,
\end{equation}

and that the co-vector, the vanishing of which defines the non-symplectic stratum occurs in $V_{s+1}^*$, the dual vector space to $V_{s+1}$ in (2.1). This co-vector can always be picked as a basis element for $V_{s+1}^*$. This particular co-vector will be denoted by $\lambda_1$,

\[ \lambda_1 = (0,0,\ldots,0,\lambda_{s+1,\alpha},0,\ldots,0), \lambda_{s+1,\alpha} \neq 0. \]

More precisely, $\lambda_1 \in \mathfrak{n}^*$, with $\lambda_1|_{V_i} = 0$, $i \neq s+1$, and we select a basis $\{X_{s+1,j}\}$, $j = 1,\ldots, \dim V_{s+1}$ for $V_{s+1}$, such that in this basis $\lambda_1(X_{s+1,j}) = 0$, $j \neq \alpha$, and $\lambda_1(X_{s+1,\alpha}) = \lambda_{s+1,\alpha}$. 

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Next we pick an element in $V_{m+1}^*$. This element is typically all or some of the non-vanishing co-vectors in the set of equations that describe the characteristic set $\Sigma$ for the operator $L$ defined in (1.1) (see remark (2.6) and section 4). We call this element $\lambda_2$, that is

$$\lambda_2 = (0, 0, \ldots, 0, \lambda_{m+1, \beta}), \lambda_{m+1, \beta} \neq 0.$$  

Again more precisely, $\lambda_2 \in n^*$, such that, $\lambda_2|_{V_i} = 0$ if $i \neq m + 1$, and we select a basis for $V_{m+1}$ which we denote by, $\{X_{m+1,j}\}, j = 1, \ldots, \dim V_{m+1}$ such that in this basis $\lambda_2(X_{m+1,j}) = 0$ for $j \neq \beta$, and $\lambda_2(X_{m+1,\beta}) = \lambda_{m+1, \beta}$.

To keep the notation simple we have assumed that $\lambda_2$ has a single non-zero component. This assumption is unnecessary as can be seen by examining the ensuing proof. See also remark (2.6). Now define,

$$\lambda = \lambda_1 + \lambda_2 \quad (2.2)$$

One of the reasons for the existence of non-analytic solutions to (1.1) seems to be $s < m$. This fails for the Heisenberg group, and so we have analytic solutions. We do not aim for maximum generality, and thus to keep the combinatorial aspects simple we will make an assumption. The first assumption has been discussed a few sentences above.

Assumptions:

$$s < m \quad (2.3)$$

$$\langle \lambda_i, [V_k, V_j] \rangle = 0, i = 1, 2 \quad (2.4)$$

where $\langle, \rangle$ is the duality bracket between tangent vectors and co-vectors.

All the examples in [Tr2] of non-analytic hypoellipticity that ensue from a nilpotent group situation, are seen to satisfy both assumptions. In fact in all the known examples ensuing from groups [BG], [C1,2], [HH1, 2], [H], [LR], [O], [Hos], [CC] a very strong form of (2.4) is assumed, $[V_k, V_j] = 0$ for $k, j > 1$, except Example (3.5) in [Tr2] due to N. Hanges. This example satisfies (2.4) but not the strong condition $[V_k, V_j] = 0$ for $k, j > 1$.

It may occur to the reader to lower the value of $m$, by modding out the ideal $g$,

$$g = \bigoplus_{j=s+3}^{m+1} V_j,$$

from the Lie algebra in (2.1). One can do this at a price. Modding out $g$ leads us to a new Lie Algebra $p = n/g$, and its associated sums of squares operator $\tilde{L}$. 

It is possible to construct explicit examples of non-analytic solutions to $\tilde{L}$ that have a smaller Gevrey order than non-analytic solutions that can be constructed for the original operator $L$ associated to the Lie algebra $n$. Thus some care has to be exercised when using inductive arguments based on modding out of ideals, if the aim is also simultaneously to obtain solutions which have optimal Gevrey properties.

We now wish to study the representation $\pi$, and in particular the derived representation, $d\pi_\lambda$, and $d\pi_\lambda(L)$. We have,

**Theorem 2.1.** Under the assumption (2.3), (2.4) and for $\lambda$ as in (2.2),

$$d\pi_\lambda(L) = \sum_{j=1}^{n} \left( \frac{\partial}{\partial t_j} + i (\lambda_{s+1, \alpha} \tilde{p}_j(t) + \lambda_{m+1, \beta} \tilde{q}_j(t)) \right)^2 - \sum_{k=1}^{r} (\lambda_{m+1, \beta} q_k(t) + \lambda_{s+1, \alpha} p_k(t))^2.$$  

Each $(\tilde{q}_j(t))_{j=1}^{n}$, $(q_k(t))_{k=1}^{r}$ are homogeneous polynomials of degree $m$, with real coefficients. $(\tilde{p}_j(t))_{j=1}^{n}$ and $(p_k(t))_{k=1}^{r}$ are homogeneous polynomials of degree $s$, with real coefficients.

**Remarks.** We will give an explicit algorithm for the Gelfand-Kirillov dimension $n$, and also for the number $r$. We will then show that under additional structural hypotheses on the Lie Algebra $n$, the magnetic potentials $\tilde{p}_i(t)$ and $\tilde{q}_i(t)$ vanish. This structural hypotheses is automatically fulfilled when $n = 1$, that is the ODE situation. Thus no magnetic potentials arise in any of the references listed.

**Proof.** We begin by giving the algorithm for $n$ and $r$.

**Case 1: $s \not= 1, m \not= 1$.** We point out that $s = 0$ could possibly occur in this case. Now in this case $\lambda_1|_{V_2} = 0$. Let $B$ be the basis for $V_1$ selected to define the operator $L$.

Define

$$S_1 = \{ X_k \in B : \exists Y_k \in n, \exists \langle \lambda_1, [X_k, Y_k] \rangle \neq 0 \}.$$ 

Note since $\lambda_1|_{V_2} = 0$, we conclude $Y_k \notin V_1$. Once the elements of $S_1$ have been picked and set aside we start afresh and define

$$S_2 = \{ X_k \in B : \exists \tilde{Y}_k \in n, \exists \langle \lambda_2, [X_k, \tilde{Y}_k] \rangle \neq 0 \}.$$ 

Let
(2.6) \[ S = S_1 \cup S_2, \#S = n. \]

Note \( S \neq \phi \), since \( n \) is stratified.

Define

(2.7) \[ r = \dim V_1 - n = p - n. \]

When \( s = 1 \), \( S_1 \) is defined differently, but the definition for \( S_2 \) remains the same, and so does the definition of \( S, n \) and \( r \).

**Case 2:** \( s = 1 \). We define the pair set

\[ P_0 = \left\{ \{X_k, X_j\} | X_k, X_j \in \mathcal{B}, \exists \langle \lambda_1, [X_k, X_j] \rangle \neq 0 \right\}. \]

Define \( X_{k_1} \), to be any vector field, that appears in the most number of pairs (elements of \( P_0 \)) in \( P_0 \). There could be several candidates, so simply pick any one.

Then \( X_{k_1} \in S_1 \). From \( P_0 \), eliminate all pairs containing \( X_{k_1} \). We get a new pair set \( P_1 \), with

\[ P_1 \subseteq P_0. \]

We repeat the process, and from the pairs in \( P_1 \), let \( X_{k_2} \) be that vector field that appears in the most pairs of \( P_1 \). Then \( X_{k_2} \in S_1 \). Delete all pairs in \( P_1 \) which contain \( X_{k_2} \), we arrive at \( P_2, P_2 \subseteq P_1 \). Containing this way, we arrive at \( P_j \), where each pair contains vector fields that appear only once. Pick from each pair any one vector field and add it to \( S_1 \). The process now stops and we have \( S_1, S_2 \) is defined as in Case 1, and \( S = S_1 \cup S_2 \).

**Case 3:** \( m = 1 \). Now necessarily \( s = 0 \), and \( S_1 = \phi \). We repeat the construction of Case 2, with \( \lambda_1 \) replaced by \( \lambda_2 \) in every instance, to construct \( S_2 \)

Let, \( W \) be the vector subspace of \( V_1 \) defined by, taking \( \mathcal{B} \setminus S \) as a basis. Form the vector subspace \( \mathfrak{h} \) of \( n \),

(2.8) \[ \mathfrak{h} = W \oplus \bigoplus_{j=2}^{m+1} V_j. \]

**Claim:** \( \mathfrak{h} \) is a maximal sub-ordinate sub-algebra for \( \lambda \) given by (2.2).

It is clear from the construction that \( [\mathfrak{n}, \mathfrak{h}] \subseteq \mathfrak{h} \), and thus \( \mathfrak{h} \) is a sub-algebra. In fact \( [\mathfrak{n}, \mathfrak{h}] \subseteq \mathfrak{h} \) so \( \mathfrak{h} \) is an ideal. We now verify,

(2.9) \[ \langle \lambda, [\mathfrak{h}, \mathfrak{h}] \rangle = 0 \]
(2.4) will be used now to simplify the argument. Let $h_i \in \mathfrak{h}, i = 1, 2$. Then,

$$h_i = w_i + x_i, \ w_i \in W, x_i \in \bigoplus_{j \geq 2} V_j.$$ 

Thus,

$$[h_1, h_2] = [w_1, x_2] + [w_2, x_1] + [w_1, w_2] + [x_1, x_2].$$

Now (2.4) asserts that

$$\langle \lambda, [x_1, x_2] \rangle = 0.$$ 

We have,

$$\langle \lambda, [h_1, h_2] \rangle = \langle \lambda, [w_1, w_2] \rangle + \langle \lambda, [w_2, x_1] \rangle + \langle \lambda, [w_1, x_2] \rangle.$$ 

**Case 1:** $s \neq 1, m \neq 1$. Since $[w_1, w_2] \in \mathfrak{v}_2$, the first term on the right vanishes. By the construction of $S$ and $W$ the second and third terms on the right above also vanish. We conclude (2.9).

**Case 2:** $s = 1$. Now $[w_2, x_1], [w_1, x_2] \in \bigoplus_{j \geq 3} V_j$.

By the construction of $S_2$,

$$\langle \lambda, [w_1, x_2] \rangle = \langle \lambda, [w_2, x_1] \rangle = 0.$$ 

Again by the construction of $S_2$, in the case $s = 1$, and by the construction of $W$, we see easily

$$\langle \lambda, [w_1, w_2] \rangle = 0.$$ 

**Case 3:** $m = 1$. This case is treated exactly as in Case 2.

$\mathfrak{h}$ is maximal, since inclusion of any $X_k \in S$ in $\mathfrak{h}$, clearly violates $\langle \lambda, [\mathfrak{h}, \mathfrak{h}] \rangle = 0$, by the construction of $S$. Let $H = \exp \mathfrak{h}$. Since $\mathfrak{h}$ is an ideal, $H$ is a normal subgroup of $N$. Our representation will act on $C^\infty(N/H)$. It follows from (2.6) and (2.8) that,

$$\dim(N/H) = \# S = n.$$ 

We will identify the homogeneous space (group) $N/H$, with $\mathbb{R}^n$. We consider the one-dimensional representation on the subgroup $H, P_\lambda$,

$$P_\lambda(h) = e^{i \langle \lambda, h \rangle},$$ 

where $\exp h \in H$. We now induce a representation on $N$, using $P_\lambda$, to get $\pi_\lambda$,

$$\pi_\lambda = \operatorname{ind}_{H \uparrow N} P_\lambda.$$
The number $n$ introduced above is called the Gelfand-Kirillov dimension of $\pi_\lambda$. Let $f \in \mathcal{S}(\mathbb{R}^n)$ in the Schwartz class. Let $t \in \mathbb{R}^n$. Assume exponential coordinates on $N$ have been picked so that the first $n$ components correspond exactly to the $n$-elements of $\mathcal{S}$. Thus for $f \in \mathcal{S}(N/H)$, we may write,

$$f(t) = f(t, 0), t \in \mathbb{R}^n, (0) \in \mathbb{R}^d,$$

where $d = \dim N - n$. From now on $a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+r}, n + r = p$ will denote the exponential coordinates corresponding to the basis vectors $\mathcal{B}$ of $V_1$. When we write $a_j, j \geq p + 1$, we mean simply exponential coordinates of the basis vectors for $V_j, j \geq 2$. The last coordinates $a_j, j \geq p + 1$ have an insignificant role to play. Also all quadratic terms $a_i a_j$ no matter what $i, j$ and cubic terms etc. will have no role to play, so we do not take much care in explicitly writing them down.

Let $R_a$ be the right regular representation. We have, for $a, b \in N$, and $h \in C^\infty(N)$

$$R_a h(b) = h(b \cdot a)$$

Since the group law is always abelian in the first $p = \dim V_1$ variables, we have,

$$\quad (t, 0) \cdot a = (t_1 + a_1, \ldots, t_n + a_n, a_{n+1}, \ldots, a_{n+r}, \cdot, \cdot, \cdot)$$

where the remaining components $\cdot$ are to be computed using the Campbell-Hausdorff formula, and $r$ is given by (2.7). We re-write the right side of (2.11) as

$$\quad (0, \ldots, 0, a_{n+1}, \ldots, a_{n+r}, \cdot, \cdot, \cdot) \cdot (t_1 + a_1, \ldots, t_n + a_n, 0)$$

where the component $(0)$ in the first term in (2.11) is a vector with $n$ components. Again the components $\cdot$ are given by the Campbell-Hausdorff formula. What matters now are the $s + 1, \alpha$ component and $m + 1, \beta$ component in the first vector in (2.12), since we have to use (2.10). We have, in (2.12),

$$\frac{\text{component}}{\text{of degree } s} = \sum_{j=1}^{n} \tilde{p}_j(t)a_j + \sum_{j=1}^{r} p_j(t)a_{j+n}$$

$$\quad + \sum_{j \geq p+1} b_j(t)a_j + \sum_{j_1, j_2 \ldots, j_k} p_{j_1, j_2 \ldots, j_k}(t)a_{j_1}a_{j_2} \cdots a_{j_k}.$$  

The natural gradation on $n$ shows that $\tilde{p}_j(t)$ and $p_j(t)$ are homogeneous polynomials of degree $s$. Similarly we have in (2.12),

$$\frac{\text{component}}{\text{of degree } m} = \sum_{j=1}^{n} \tilde{q}_j(t)a_j + \sum_{j=1}^{r} q_j(t)a_{j+n}$$

$$\quad + \sum_{j \geq p+1} b'_j(t)a_j + \sum_{j_1, j_2 \ldots, j_k} q_{j_1, j_2 \ldots, j_k}(t)a_{j_1}a_{j_2} \cdots a_{j_k}.$$
Again the polynomials $\tilde{q}_j(t)$ and $q_j(t)$ are homogeneous of degree $m$, with real coefficients. Thus from (2.10), (2.12) - (2.14), we have for $i = \sqrt{-1}$,

$$\pi_\lambda(a) f(t) = \exp \left( i\lambda_{s+1,\alpha} \left( \sum_{k=1}^{n} \tilde{p}_k(t) a_k + \sum_{j=1}^{r} p_j(t) a_{j+n} \right) \right.$$

$$+ i\lambda_{m+1,\beta} \left( \sum_{k=1}^{n} \tilde{q}_k(t) a_k + \sum_{j=1}^{r} q_j(t) a_{j+n} \right) \left. \right)$$

(2.15)

$$\left. + h.o.t \right) f(t_1 + a_1, \ldots, t_n + a_n)$$

where h.o.t stands for the higher order terms in (2.13) and (2.14), which eventually will not count towards the derived representation.

In fact in (2.15) we only display and will only need the linear terms in $a_j$, for $1 \leq j \leq p = n + r$. Our goal is to compute for $t \in \mathbb{R}^n$,

$$\frac{\partial \pi_\lambda(a)}{\partial a_j} f(t) \bigg|_{a=0}, j = 1, 2, \ldots, p.$$ 

(2.16)

This will allow us to compute, $d\pi_\lambda(X_k) f(t)$ for every $X_k \in \mathcal{B}$, the basis for $V_1$.

From (2.15) and (2.16) we see, for $j = 1, 2, \ldots, n$,

$$\frac{\partial \pi_\lambda(a)}{\partial a_j} f(t) \bigg|_{a=0} = \frac{\partial f}{\partial t_j} + i \left( \lambda_{s+1,\alpha} \tilde{p}_j(t) + \lambda_{m+1,\beta} \tilde{q}_j(t) \right).$$

(2.17)

For $j = n + 1, \ldots, n + r$, from (2.15) we have,

$$\frac{\partial \pi_\lambda(a)}{\partial a_j} f(t) \bigg|_{a=0} = i \left( \lambda_{s+1,\alpha} p_j(t) + \lambda_{m+1,\beta} q_j(t) \right).$$

(2.18)

Thus, from (2.17), (2.18),

$$d\pi_\lambda \left( \sum_{k=1}^{p} X_k^2 \right) f(t) = \sum_{j=1}^{n} \left( \frac{\partial}{\partial t_j} + i \left( \lambda_{s+1,\alpha} \tilde{p}_j(t) + \lambda_{m+1,\beta} \tilde{q}_j(t) \right) \right)^2 f(t)$$

$$- \sum_{k=1}^{r} \left( \lambda_{s+1,\alpha} p_k(t) + \lambda_{m+1,\beta} q_k(t) \right)^2 f(t)$$

This establishes the theorem.
Proposition 2.2. If the vector fields in $S$ commute amongst themselves, or if $\#S = 1$, then the magnetic potentials, $\tilde{\mathbf{p}}_j(t), \tilde{\mathbf{q}}_j(t)$ vanish for $1 \leq j \leq n$.

Proof. We again need to use the Campbell-Hausdorff formula. We have in (2.11), that the first $n$ components commute among themselves. Thus the components represented by $\bullet$ in (2.11) will have the property that those terms in $\bullet$, which are linear in $a_j$, will only involve $a_j, n + 1 \leq j \leq n + r$ out of the set $\{a_1, \ldots, a_p\}, p = n + r$. That is the variables $a_1, \ldots, a_n$ will not appear in the linear terms in $\bullet$. In fact the linear terms arise on consideration of $[X, Y], X = (x, 0), Y = (a)$. The Campbell-Hausdorff formula consists of commutators of $[X, Y]$. Since $[X, Z] = 0, Z = (a_1, \ldots, a_n, 0)$, we see that the mixed terms in the $j$-component of $[X, Y], j \geq p + 1$, will only be of the type $a_k t_j, n + 1 \leq k \leq p, 1 \leq j \leq n$. As seen earlier it is only the terms that are linear in $a_k, 1 \leq k \leq p$, that matter in the derived representation. Thus the $s + 1, \alpha$ component in (2.11) is of the form,

$$
(2.19) \quad a_{s+1,\alpha} + \sum_{j=1}^{p} p_j(t)a_{j+n} + \sum_{j \geq p+1} b_j(t)a_j + \text{h.o.t.}
$$

$h.o.t$ involves quadratic and higher terms in $a_j$. We re-write (2.19) in the form (2.12). The $s + 1, \alpha$ term in the first term in (2.12) must certainly contain (2.19) by the Campbell-Hausdorff formula. To see the additional linear terms involving $a_j, 1 \leq j \leq p$, in the $s + 1, \alpha$ component, we proceed by induction based on the grading of $n$. We claim that the additional linear terms in $a_j$ also do not involve any $a_j, 1 \leq j \leq n$. This is certainly obvious for the terms of the lowest grade, in (2.12), the starting point of the induction. For the higher gradations its a consequence of (2.19), the Campbell-Hausdorff formula and the inductive hypothesis that any linear term in $a_j, 1 \leq j \leq p$, in a fixed vector space $V_k$ (gradation) involves only $a_j, n + 1 \leq j \leq p$, and that there are no zero order terms in $a_j$, that is pure polynomials in $t$. Thus, the only additional linear terms involving $a_j, 1 \leq j \leq p$, in the $s + 1, \alpha$ component will again be,

$$
\sum_{j=1}^{p} p_j(t)a_{j+n}.
$$

Thus in (2.13) $\tilde{\mathbf{p}}_j(t) = 0$. Similarly in (2.14) $\tilde{\mathbf{q}}_j(t) = 0, 1 \leq j \leq n$. Our proposition is proved.

We now show how to construct a non-analytic solution to $L$. 

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Theorem 2.3. Assume for $\lambda_{s+1,\alpha} \in \mathbb{C}$ and $\lambda_{m+1,\beta} \in \mathbb{R}$, $\lambda_{m+1,\beta} \neq 0$, there exists a non-trivial, smooth solution $f$ in all $\mathbb{R}^n$ to,

$$Af = \left[ \sum_{j=1}^{n} \left( \frac{\partial}{\partial t_j} + i (\lambda_{s+1,\alpha} \tilde{p}_j(t) + \lambda_{m+1,\beta} \tilde{q}_j(t)) \right)^2 - \sum_{k=1}^{r} (\lambda_{s+1,\alpha} p_k(t) + \lambda_{m+1,\beta} q_k(t))^2 \right] f = 0.$$

Furthermore, assume that, there exist $C_1, C_2 > 0$, such that,

$$(2.20) \quad |f(t)| \leq C_1 e^{C_2|t|^{s+1}}.$$ 

Then there exists a smooth solution $u$ to (1.1) in some neighborhood of the origin which satisfies the following properties.

(1) If $f(0) = 0$, the solution $u$ to (1.1), vanishes to infinite order in the coordinate $a_{m+1,\beta}$.

(2) If $f(0) \neq 0$, then there is a solution $u$ to (1.1) whose Gevrey order is exactly $(m+1)/(s+1)$, in the coordinate $a_{m+1,\beta}$ at the origin.

Remarks. We now see clearly that the non-analyticity is an interplay of the difference between the non-symplectic stratum and the center, i.e. $s < m$. On the Heisenberg group, $m = s = 1$, that is there are no non-symplectic strata. In [BG], $m = 1, s = 0$ and so on.

It is crucial to point out that we must complexify $\lambda_{s+1,\alpha}$ or else we may not have any solutions to the Schrödinger equation with the prescribed bound if we restrict to real values of $\lambda_{s+1,\alpha}$. However, $\lambda_{m+1,\beta}$ is picked real. Complex values of $\lambda_{s+1,\alpha}$ of course do not contribute to the Plancherel formula for $N$, and it is only the real values of $\lambda \in n^*$ that plays a role in the $C^\infty$ theory. The observation that one needs to complexify parameters of the representation to understand analytic hypoellipticity was also made by B. Helffer [H].

Proof of Theorem 2.3. Let $\rho > 0$. We recall that there is a natural dilation $\delta(\rho)$ on the graded Lie Algebra $n$, [F] and [RS]. Since $n^*$ is dual to $n$, it inherits the same gradation as $n$, and thus

$$(2.21) \quad \delta(\rho) \lambda = (0, \ldots, 0, \rho^{s+1} \lambda_{s+1,\alpha}, 0, \ldots, 0, \rho^{m+1} \lambda_{m+1,\beta}).$$

Next, we have for $X_j \in B, a \in N, t \in \mathbb{R}^n$,

$$(2.22) \quad X_j \pi_\lambda(a) f(t) = \pi_\lambda(a) d\pi_\lambda(X_j) f(t).$$
The left side of (2.22) is
\[
\frac{\partial}{\partial \tau} \pi_\lambda (a e^{\tau} X_j) f \bigg|_{\tau=0} = \pi_\lambda (a) \frac{d}{d\tau} \pi_\lambda (e^{\tau} X_j) f \bigg|_{\tau=0} = \pi_\lambda (a) d\pi_\lambda (X_j) f.
\]
Thus (2.22) is established. Define for \(a \in N\),
\[
(2.23) \quad u(a) = \int_0^\infty \left[ \pi_\delta(\rho) \lambda (a) f_\rho(t) \right] \left|_{t=0} \right. e^{-M \rho^{s+1}} d\rho
\]
where \(f_\rho(t) = f(\rho t)\), and \(M > 0\) is a suitable large number. From (2.15), the definition of \(\delta(\rho)\) and the hypothesis (2.20), we see that if \(a \in U, U\) a suitably picked neighborhood of the origin in \(N\), and \(M\) is picked large enough, the integral in (2.23) will converge absolutely. Now by (2.22),
\[
\sum_{j=1}^p X_j^2 u = \int_0^\infty \left[ \pi_\delta(\rho) \lambda (a) d\pi_\delta(\rho) \lambda \left( \sum_{j=1}^p X_j^2 \right) f_\rho(t) \right] \left|_{t=0} \right. e^{-M \rho^{s+1}} d\rho.
\]
But,
\[
(2.24) \quad d\pi_\delta(\rho) \lambda \left( \sum_{j=1}^p X_j^2 \right) f_\rho \equiv 0.
\]
This is because,
\[
d\pi_\delta(\rho) \lambda \left( \sum_{j=1}^p X_j^2 \right) f_\rho = \sum_{j=1}^n \left( \frac{\partial}{\partial t_j} + i \left( \rho^{s+1} \lambda_{s+1,\alpha} \tilde{p}_j(t) + \rho^{m+1} \lambda_{m+1,\beta} \tilde{q}_j(t) \right) \right)^2 f_\rho
\]
\[
- \sum_{j=1}^r \left( \rho^{s+1} \lambda_{s+1,\alpha} p_j(t) + \rho^{m+1} \lambda_{m+1,\beta} q_j(t) \right)^2 f_\rho.
\]
Since \(\tilde{p}_j, p_j\) are homogeneous of degree \(s\), and \(\tilde{q}_j\) and \(q_j\) homogeneous of degree \(m\), by changing variables \(w = \rho t\), we may write the expression above as, \(\rho^2 A f(w)\). By hypothesis this vanishes. Thus, \(u(a)\) is a solution. We now denote the variable \(a_{m+1,\beta}\) on \(N\), by simply \(a_{m+1}\).

We calculate,
\[
\frac{\partial^\sigma u}{\partial a^\sigma_{m+1}} (0, \ldots, 0, a_{m+1}) \bigg|_{a=0}.
\]
Let $\tilde{a} = (0, 0, \ldots, 0, a_{m+1, \beta})$. Then we claim,

\begin{equation}
\pi_{\delta(\rho)}(\tilde{a}) f_{\rho}(t) = e^{i\lambda_{m+1, \beta} \rho^{m+1} a_{m+1}} f_{\rho}(t).
\end{equation}

Since $\tilde{a} \in V_{m+1}$, it commutes with every element in $N$. Thus, if $t = (t_1, \ldots, t_n)$,

\[(t, 0) \cdot \tilde{a} = (t, 0, \ldots, 0, a_{m+1, \beta}) = (0, 0, \ldots, 0, a_{m+1, \beta}) \cdot (t, 0).
\]

Using (2.10), (2.25) follows.

Thus, combining (2.23) and (2.25),

\begin{equation}
u(0, a_{m+1}) = \int_{0}^{\infty} e^{i\lambda_{m+1, \beta} \rho^{m+1} a_{m+1}} f(0) e^{-M\rho^{s+1}} d\rho.
\end{equation}

Note now that it is crucial $\lambda_{m+1, \beta} \in \mathbb{R}$ for the integral to converge. If $f(0) = 0$, then (2.26) vanishes with all derivatives. If $f(0) \neq 0$, then

\[\frac{\partial^{\sigma} u}{\partial a_{m+1}^{a_{m+1}}} (0, a_{m+1}) \bigg|_{a=0} = f(0) \lambda_{m+1, \beta}^{\sigma} \int_{0}^{\infty} \rho^{(m+1)\sigma} e^{-M\rho^{s+1}} d\rho.
\]

Changing variables we see easily that for $C > 0$,

\[\left| \frac{\partial^{\sigma} u}{\partial a_{m+1}^{a_{m+1}}} (0, a_{m+1}) \right| \sim C^{\sigma} (\sigma!)^{(m+1)/(s+1)}.
\]

We also used the fact that $\lambda_{m+1, \beta} \neq 0$ in the last step. \[\blacksquare\]

The next two propositions seek to establish a link between non-symplectic strata in the characteristic set $\Sigma$ of $L$ and the appearance of potential wells in the Schrödinger equations attached to the derived representations $d\pi_{\chi}(L)$. We have only been able to establish this link when the characteristic set itself is non-symplectic as in the situation of [BG] and that too in a stronger situation. The difficulty in establishing this link when the non-symplectic stratum is immersed as in [HH2] is in the combinatorial aspects of the Campbell-Hausdorff formula.

**Proposition 2.4.** Assume that the vector space $V_1$ contains an element from the center of $\mathfrak{n}$. Then the characteristic set $\Sigma$ of $L$ is non-symplectic. The converse statement does not hold.

**Proof.** Since symplectic properties of $\Sigma$ remain invariant under linear combinations, we may suppose that in our basis $\mathcal{B}$ for $V_1$, the element $X_{\rho}$ is in the center of $\mathfrak{n}$. 


Denote the symbols of the vector fields $X_i$ in the basis $\mathcal{B}$ by $\sigma(X_i)$. By hypothesis 
$[X_i, X_p] = 0$ for all $X_i \in \mathcal{B}$. Thus, $\{\sigma(X_i), \sigma(X_p)\} = 0$, for every $X_i \in \mathcal{B}$. Here $\{,\}$ denotes the Poisson bracket. Since $\Sigma$ is defined by the set of equations $\sigma(X_i) = 0$, $X_i \in \mathcal{B}$, we conclude right away that the presence of the element $X_p$ from the center of $\mathfrak{n}$ forces $\Sigma$ to be non-symplectic.

Now on the other hand consider 
\[
X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}, \quad X_3 = \frac{\partial}{\partial x_4} + x_1 \frac{\partial}{\partial x_5}.
\]

The characteristic set of the operator $L = \sum_{i=1}^{3} X_i^2$ is non-symplectic. However in the Lie algebra generated by the $X_i$, the vector space $V_i$ does not contain an element from the center. Note by considering a solution of the form $u(x_1, x_2, x_3)$ the situation reduces to the example in [BG] and thus the operator $L$ is not analytic hypoelliptic.

---

**Theorem 2.5.** Assume that the basis $\mathcal{B}$ contains an element from the center of $\mathfrak{n}$. Furthermore set $\lambda_{s+1, \alpha} = \mu$ and $\lambda_{m+1, \beta} = 1$. Then we have 
\[
d\pi_{\lambda}(L) = \sum_{j=1}^{n} \left( \frac{\partial}{\partial t_j} + i\tilde{q}_j(t) \right)^2 - \sum_{k=1}^{r-1} q_k^2(t) - \mu^2.
\]

**Proof.** Let us assume that the basis element $X_p \in \mathcal{B}$ is an element of the center of $\mathfrak{n}$. We select $\lambda_1$, such that, $\lambda_1(X_p) = \lambda_{s+1, \alpha} = \lambda_{1, p} = \mu$ and $\lambda_1(X_i) = 0$, $i \neq p$ where $X_i \in \mathcal{B}$. Further let $\lambda_1|_{V_i} = 0$, $i \neq 1$. Thus we are in the situation $s = 0$. Since $s = 0$, by construction $S_1 = \phi$. Now $X_p \notin S_2$, because $\langle \lambda_2, [Z, X_p] \rangle = 0$ for any $Z \in \mathfrak{n}$. Thus $X_p \in \mathcal{B}\backslash S$ and thus $X_p \in W$. We conclude that $\lambda_1|_W \neq 0$. This means that $\langle \lambda_1, h \rangle \neq 0$. Thus $\langle \lambda_1, h \rangle$ contributes to the exponential in (2.10) and thus to (2.15) which is the induced representation $\pi_{\lambda}$. From (2.12) we conclude that (2.13) takes the form

(2.27) \quad (\text{component})_{1, p} = a_p.

Next we focus on (2.14). Since $X_p$ lies in the center once again we see right away from the Campbell-Hausdorff formula that the second sum on the right in (2.14) will contain no linear terms involving $a_p$. That is (2.14) in the current situation takes the form

(2.28) \quad (\text{component})_{m+1, \beta} = \sum_{j=1}^{n} \tilde{q}_j(t)a_j + \sum_{j=1}^{r-1} q_j(t)a_j + \sum_{j=p+1} b'_j(t)a_j + \sum q_{j_1, j_2, \ldots, j_k}(t)a_{j_1}a_{j_2} \ldots a_{j_k}.

Recall that $p = r + n$, and the second sum on the right above stops at $r - 1$ because $X_p$ commutes with every element of $n$. Using now (2.27) and (2.28) instead of (2.17) and (2.18) and setting, $\lambda_{m+1, \beta} = 1$, $\lambda_{1, p} = \mu$ we easily get our conclusion.

\section*{Remark 2.6.} Though the results of this section were derived under the assumption

$$\lambda_2 = (0, 0 \ldots, 0, \lambda_{m+1, \beta})$$

an examination of the proofs shows that one could take for $\lambda_2 \in V_{m+1}^{\ast}$,

$$\lambda_2 = (0, \ldots, 0, \lambda_{m+1, \beta_1}, \lambda_{m+1, \beta_2}, \ldots, \lambda_{m+1, \beta_k}, 0)$$

provided, (2.4) holds. By allowing this flexibility it is technically easier to verify (2.20). This remark will become clear when we consider examples in Section 4.

\section*{§3. Bounds for the Schrödinger Equation.}

We now try to prove the bound given in (2.20) for the spectral problem in Theorem (2.3). We begin with some remarks.

When $n = 1$, the ODE case, we observed by Proposition (2.2), that the spectral problem is, for $a_j, b_j \in \mathbb{R}$,

$$\frac{d^2 f}{dt^2} - \sum_{j=1}^{r} (\lambda a_j t^s + \mu b_j t^m)^2 f = 0.$$ 

with $s < m$. We pick $\mu = -1$, and we see easily that we get the problem for $\lambda \in \mathbb{C}, a_j, b_j \in \mathbb{R}$,

$$-\frac{d^2 f}{dt^2} + \sum_{j=1}^{r} (b_j t^m - \lambda a_j t^s)^2 f = 0.$$ 

This problem has been the principal problem treated in the literature [BG], [C1, 2], [HH1, 2], [Hos], [LR], [H], [CC], [O]. Thus under (2.3) and (2.4) and when $n = 1$, the problem of analytic hypoellipticity on a nilpotent group is essentially settled, though all authors treated $r = 1$. The main point is when $n = 1$, all potentials of the Schrödinger equation are confining. For $n \geq 2$, we get for the first time non-confining potentials and the problem is far more difficult even though we are asking for weak bounds (2.20). We have been only able to treat special cases of the Schrödinger equation. At the end of this section we present a theorem of F. Treves that hints that (2.20) is true in general. Unfortunately we have been unable to capitalize on Treves’s theorem.

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We assume that (2.3), (2.4) holds and the vector fields in $S$ commute. Then by Section 2, we arrive at the Schrödinger equation, (after setting $\lambda_{m+1,\beta} = -1, \lambda_{s+1,\alpha} = \sigma$)

$$-\Delta f + \sum_{j=1}^{r} (\sigma p_j(t) - q_j(t))^2 f(t) = 0.$$ 

Let us assume that for all pairs $q_j(t), p_j(t)$ if $q_j(t) \neq 0$, then $p_j(t) \equiv 0$, and if $p_j(t) \neq 0$, then $q_j \equiv 0$. Then the equation above becomes,

$$-\Delta f + \left( \sum_{j=1}^{r'} q_j^2(t) + \sigma^2 \sum_{j=1}^{r''} p_j^2(t) \right) f = 0.$$

Notice for $\sigma \in \mathbb{R}$, we need not have a solution with the prescribed growth (2.20) and we are forced to pick $\sigma \in \mathbb{C}$. In fact we take $\sigma = i\sqrt{\lambda}, \lambda > 0$ and we get the problem, $t \in \mathbb{R}^n$

$$-\Delta f + (q(t) - \lambda p(t)) f = 0, \lambda > 0,$$

$q \geq 0$, homogeneous polynomial of degree $2m, p(t) \geq 0$, homogeneous polynomial of degree $2s, m > s$, and both polynomials having real coefficients. Unlike the ODE case $n = 1$, we cannot anymore expect $f(t)$ to decay.

**Example (3.1)** Let $h(t)$ be a homogeneous harmonic polynomial in $\mathbb{R}^n$ of degree $k$. Let,

$$f(t) = e^{-h^2/2},$$

then $f(t)$ is a solution to,

$$-\Delta f + (q(t) - p(t)) f(t) = 0,$$

$q(t) = h^2|\nabla h|^2, p(t) = |\nabla h|^2$, thus $\deg p = 2(k - 1), \deg q = 2(2k - 1)$. In $\mathbb{R}^2$, if we select $h(t) = t_1^2 - t_2^2$, we notice that along the ray $t_1 = t_2, f(t) \equiv 1$. Thus decay in the eigenfunction problem above is not possible and (2.20) does not demand it either.

We have,

**Theorem 3.2.** Let us consider in all $\mathbb{R}^n$,

$$-\Delta f + (q(t) - \lambda p(t)) f(t) = 0,$$

$q(t) \geq 0$ is a homogeneous polynomial of degree $2m, p(t) \geq 0$ is a non-trivial, homogeneous polynomial of degree $2s, m > s$. We have the following. If
(a) $q(t)$ does not vanish on the unit sphere $S^{n-1}$, 
or
(b) $q(t)$ vanishes on $S^{n-1}$ and the maximal order of vanishing of $q(t)$ restricted to $S^{n-1}$ is $j$, and

\begin{equation}
2s - 2m + j \frac{1}{j} + s < 0.
\end{equation}

Then there exists $\lambda > 0$, and a solution $f(t)$ to the Schrödinger equation above, such that,

1. $f(t) > 0$, $f(0) = 1$,
2. $0 < f(t) < C(1 + |t|)^{(m+1)(n-1)}$, $t \in \mathbb{R}^n$.

The case $j = 0$ is included in (3.1), and under assumption (a) we can even conclude that $f(t)$ decays.

The proof relies very strongly on Harnack’s inequality. In fact we need a scale invariant version.

**Lemma 3.3.** Let $g \geq 0$. Let $g$ satisfy in $\mathbb{R}^n$,

\[-\Delta g + V(t)g = 0.\]

Let $B_R(t_0)$ be a ball centered at $t_0$ of radius $R$.

Assume,

\begin{equation}
R \sup_{B_{2R}(t_0)} |V(t)|^{1/2} \leq C_1.
\end{equation}

Then,

\[\sup_{B_R(t_0)} g \leq C(C_1, n) \inf_{B_R(t_0)} g.\]

Proof. This is a special case of Theorem 8.20 in Gilbarg-Trudinger [GT]. Set $w(t) = g(Rt + t_0)$.

Then,

\[-\Delta w + R^2 V(Rt + t_0)w(t) = 0\]

$t \in B_2(0)$, where $B_2(0)$ is the ball of radius 2 around the origin. Using (3.2) and the result in [GT], our conclusion follows.

\[\blacksquare\]
We will display the proof of theorem (3.2) for $n = 2$, and indicate the changes necessary for the higher dimensional version. We do so to keep the ideas transparent, the idea being the same in all dimensions. Perhaps this will help to eventually get rid of assumption (3.1) which is our goal. The proof in higher dimension involves notational complications and use of Whitney’s lemma.

We begin with an approximate problem. Let $\phi_N$ be a smooth, radial cut-off function for $N$ large, given by,

$$
\phi_N(t) = \begin{cases} 
1, & |t| < N \\
0, & |t| > 2N.
\end{cases}
$$

Fix, $\lambda \in \mathbb{R}$. We will also need polar coordinates $(r, \theta)$. Then $q(t) = r^{2m}q_1(\theta), p(t) = r^{2s}p_1(\theta)$.

Define

$$
(3.3) \quad T = \{t : q(t) - \lambda p(t) < 1, |t| \geq 100\}.
$$

The set $T = \bigcup_{\sigma=1}^{\ell} T_{\sigma}$, where each $T_{\sigma}$ is a tapering tube, that is, there is a central axis, and transverse to this axis the diameter of $T_{\sigma}$ falls off. Assume that we have rotated axes, so that $q_1(0) = 0$ and let this correspond to $T_1$. Let $k$ be the vanishing order of $q_1$ at $\theta = 0$. Then the set $T_1$ is bounded by the curves for large $r$, given by,

$$
r^k |\theta|^k = c(\lambda r^{2s-2m+k} + r^{k-2m}).
$$

Taking, $k - th$ roots, we have the bounding curves can be taken to be, for large $r$,

$$
(3.4) \quad t_2 = \pm c(1 + |\lambda|)^{1/k} r^{(2s-2m+k)/k}, t = (t_1, t_2).
$$

By (3.1), $(2s - 2m + k)/k < 0$.

We now consider an approximate problem,

$$
(3.5) \quad -\Delta f_N + (q(t) - \lambda_N p(t)\phi_N(t)) f_N(t) = 0.
$$

The new potential $q(t) - \lambda_N p(t)\phi_N(t)$ is bounded below so we hope to find a non-negative solution to (3.5) by a variational approach.

**Lemma 3.4.** Let the weaker condition $2s - 2m + j < 0$ hold. Then there exists $\lambda_N > 0, f_N > 0$, with

$$
(3.6) \quad \int_{\mathbb{R}^n} |\nabla f_N|^2 + \int_{\mathbb{R}^n} |f_N|^2 < +\infty
$$
and such that \( f_N \), satisfies (3.5).

**Proof.** We will see that \( f_N \) satisfies a stronger condition than (3.6). Fix a value \( \lambda \), and consider the sets

\[
S_1 = \{ t : q(t) - \lambda p(t) \phi_N(t) \geq 1 \} \\
S_2 = \{ t : q(t) - \lambda p(t) \phi_N(t) < 1 \}.
\]

We shall drop the subscript \( N \) in \( f_N \) from now on.

Now,

\[
- \Delta f + (q - \lambda p \phi_N) f = - \Delta f + (q - \lambda p \phi_N)(\chi_{S_1} + \chi_{S_2}) f \\
= - \Delta f + [(q - \lambda p \phi_N) \chi_{S_1} + \chi_{S_2}] f + [(q - \lambda p \phi_N) \chi_{S_2} - \chi_{S_2}] f
\]

(3.7) \[ = - \Delta f + V_1(t)f - V_2(t)f, \]

where,

\[
V_1(t) = (q - \lambda p \phi_N) \chi_{S_1} + \chi_{S_2} \geq 1, \\
V_2(t) = \chi_{S_2} - (q - \lambda p \phi N) \chi_{S_2} \geq 0.
\]

Moreover,

(3.9) \[ \| V_2 \|_{L^\infty} \leq c(\lambda, N) \]

Define the Hilbert space \( \mathcal{H} \), to be the completion of the vector space \( W \),

\[
W = \{ \phi \in C_0^\infty(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla \phi|^2 + \int_{\mathbb{R}^n} |\phi|^2 V_1 < +\infty \},
\]

with norm

\[
\| \phi \|_{\mathcal{H}}^2 = \int_{\mathbb{R}^n} |\nabla \phi|^2 + |\phi|^2 V_1.
\]

Let \( \mathcal{H}^* \) be the dual to \( \mathcal{H} \). Then, \( A = (-\Delta + V_1)^{-1} \) is a bounded operator from \( \mathcal{H}^* \rightarrow \mathcal{H} \), and in fact \( \| A \| \leq 1 \). We show that \( A(V_2 f) = Kf \) is a compact operator on \( \mathcal{H} \). To demonstrate this, note by (3.9), and since \( V_1 \equiv 1 \) on the support of \( V_2 \),

\[
K_1 : \mathcal{H} \rightarrow \mathcal{H}^*, K_1 f = V_2 f,
\]

is a bounded operator. Now \( S_2 \subseteq T \), for \( r \geq 100 \). Thus \( S_2 \) is bounded by the curves (3.4) which are tapering in the \( t_2 \) direction. This tapering will produce the desired
compactness. In the general \( \mathbb{R}^n \) case it is seen that the components of the set \( T \), tapers at least in one distinguished direction and that is all that is required in the ensuing proof. We now show \( K_1 \) is a compact operator. For \( \varepsilon > 0 \), we prove, there exists \( R_\varepsilon \), such that for \( C > 0, C = C(\lambda, N) \),

\[
\int_{|t| > R_\varepsilon} |f|^2 V_2 < C \varepsilon \|f\|_H^2
\]  

(3.10)

Once (3.10) is established, then an application of Rellich’s lemma on the compact part \( \{|t| < R_\varepsilon\} \) and a diagonal process establishes the compactness. Since the support of \( V_2 \) is contained in \( T \), we have from (3.9),

\[
\int_{|t| > R_\varepsilon} |f|^2 V_2 \leq \sum_{\sigma=1}^{\ell} \int_{\{|t| > R_\varepsilon\} \cap T_\sigma} |f|^2.
\]  

(3.11)

So it’s enough to establish (3.10) for \( \sigma = 1 \). Let \( \beta = (2m - 2s - k)/k > 0 \). Pick \( \alpha > 0, \alpha < 1/2 \) and \( 2\alpha < \beta \). Let \( \psi \in C_0^\infty(\mathbb{R}) \),

\[
\psi(\tau) = \begin{cases} 1, & |\tau| \leq 2 \\ 0, & |\tau| \geq 4. \end{cases}
\]

Note first that, because \( \alpha < \beta \),

\[
B_1 \cap \{|t| > C_1(\lambda)\} \subseteq \{(t_1, t_2) : |t_2||t_1|^{\alpha} \leq 1\}.
\]  

(3.12)

This follows because the set on the left in (3.12) by virtue of (3.4) is bounded by the curves, \( t_2 = \pm c(1 + |\lambda|)^{1/k}|t_1|^{-\beta} \). For \( |t| \geq c_1(\lambda) \), we have for \( \alpha < \beta, c(1 + |\lambda|)^{1/k}|t_1|^{-\beta} \leq |t_1|^{-\alpha} \). (3.12) follows. Going back to the right side of (3.11), we have, by (3.12),

\[
\int_{\{|t| > R_\varepsilon\} \cap T_1} |f|^2 \leq \int_{\{|t| > R_\varepsilon\} \cap T_1} |f\psi(|t_2||t_1|^{\alpha})|^2 dt_1 dt_2.
\]

For fixed \( t_1 \), we apply Poincare’s inequality to the \( t_2 \) variable to the integral on the right above. We see that,

\[
\int_{\{|t| > R_\varepsilon\} \cap T_1} |f\psi|^2 dt_2 \leq cR_\varepsilon^{-2\alpha} \int_{\{|t| > R_\varepsilon\}} |\nabla t_2 f|^2 \psi^2 dt_2
\]

\[
+ c \int_{\{|t| > R_\varepsilon\} \cap \{|t_2||t_1|^{\alpha} < 2\}} |\psi'|^2 |f|^2 |t_1|^{2\alpha} dt_2.
\]  

(3.13)
Now, for \( \alpha' \), such that \( \alpha < \alpha' < 1/2 \) we have for \( R_\varepsilon \) large,

\[
|t_1|^{2\alpha} < \varepsilon |t_1|^{2\alpha'}.
\]

We now show that if \( 2 < |t_2||t_1|^\alpha < 4 \),

\[
|t_1|^{2\alpha'} \leq q(t) - \lambda p(t).
\]

**Case 1:** If \( s = 0 \), \( p(t) = c \). Now when \( 2 < |t_2||t_1|^\alpha < 4 \), \( q(t) \sim |t_2|^k|t_1|^{2m-k} \sim |t_1|^{2m-(1+\alpha)k} \). Thus we need to have, \( 2m - (1 + \alpha)k > 2\alpha' \). Since \( k \neq 2m \), by choosing \( \alpha, \alpha' \) sufficiently small and positive we easily verify (3.15).

**Case 2:** If \( s \neq 0 \), then we have for \( \alpha' \) small, \( 2\alpha' < 2s \), and thus on \( 2 < |t_2||t_1|^\alpha < 4 \), \( |t_1| \geq R_0(\lambda) \),

\[
|t_1|^{2\alpha'} + \lambda p(t) \leq c|t_1|^{2s} < |t_1|^{2m-k}|t_2|^k \sim q(t).
\]

The last inequality holds whenever \( |t_2||t_1|^\beta \geq c \). Since \( |t_1| \geq 10 \), and \( |t_2||t_1|^\alpha \geq c, |t_2||t_1|^\beta \geq c \).

Thus (3.15) holds in all cases. Combining (3.14) and (3.15), for \( |t| \geq R_\varepsilon \),

\[
|t_1|^{2\alpha} \leq \varepsilon (q(t) - \lambda p(t)\phi_N(t)).
\]

Thus for large \( R_\varepsilon \), in (3.13),

\[
\int_{\{|t|>R_\varepsilon\} \cap T_1} |f\psi|^2 dt_2 \leq c\varepsilon \left( \int_{\{|t_1|^\alpha|t_2|\leq 4\} } (|\nabla f|^2 + |f|^2 V_1) dt_2 \right).
\]

Integrating both sides above in the \( t_1 \) variable, we easily get (3.10). Thus the self-adjoint, compact operator, \((-\Delta + V_1)^{-1}(V_2 f) = Kf\), has a largest positive eigenvalue \( \mu_1^{-1} \), and eigenfunction \( f \in \mathcal{H} \), such that

\[
Kf = \mu_1^{-1} f.
\]

We re-write this equation as,

\[
-\Delta + V_1 f - \mu_1 V_2 f = 0.
\]

We now choose \( \lambda \), so that \( \mu_1 = 1 \), and then with \( \mu_1 = 1 \), from (3.7) we see that \( f \) will satisfy our Schrödinger equation. The value of \( \lambda \), which gives \( \mu_1 = 1 \), is the desired \( \lambda_N \). Since \( f \in \mathcal{H} \), and \( V_1 \geq 1 \), we also see,

\[
\int_{\mathbb{R}^2} (|\nabla f|^2 + |f|^2) \leq \|f\|^2_{\mathcal{H}} < +\infty.
\]
To see we can select $\lambda_N$, so that $\mu_1 = 1$, we write (3.17) in a variational framework.

\begin{equation}
\mu_1 = \inf_{\phi \in C_0^\infty(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla \phi|^2 + V_1 \phi^2}{\int_{\mathbb{R}^2} \phi^2 V_2}.
\end{equation}

We claim:

(a) If $\lambda \to +\infty$, $\mu_1 \to 0$.

(b) If $\lambda \to -\infty$, $\mu_1 \to +\infty$.

Thus from (a), (b), there exists a $\lambda_N$, for which $\mu_1 = 1$. This will be necessarily positive. This is because, for the eigenfunction $f$ and the $\lambda_N$, we must have,

$$-\Delta f + q(t)f - \lambda_N p(t)\phi_N f = 0.$$  

Thus,

$$\lambda_N = \frac{\int_{\mathbb{R}^2} |\nabla f|^2 + q f^2}{\int_{\mathbb{R}^2} p(t)\phi_N f^2}.$$  

Thus $\lambda_N > 0$, since $q(t), p(t),$ and $\phi_N(t) \geq 0$, with $p(t) \neq 0$. It also follows from (3.18) that $f > 0$. We now check our claim.

**Case (a):** Since,

$$V_2(t) = \chi_{S_2} - (q - \lambda p)\phi_N \chi_{S_2},$$

as $\lambda \to +\infty$, the support of $V_2$ will contain a fixed ball $B$ of radius $r_0$, such that $V_2 \geq M$ on this ball. The ball $B$ can be fixed since, for $\lambda_1 < \lambda_2$,

$$\{t : q(t) - \lambda_1 p(t)\phi_N(t) < 1\} \subseteq \{t : q(t) - \lambda_2 p(t)\phi_N(t) < 1\}.$$  

On this ball $V_2 \to +\infty$ as $\lambda \to \infty$. Pick $\phi \in C_0^\infty(B)$. Then substituting this test function into (3.18), we notice on $B$, $V_1 \equiv 1$, for large $\lambda$, from the definition of $V_1$, while the denominator goes to positive infinity, thus $\mu_1 \to 0$.

**Case (b):** Let $\lambda \to -\infty$. Now notice for $\lambda \leq 0$, $q - \lambda p = q + |\lambda|p \geq 0$, and thus $\|V_2\|_\infty \leq 2$. Secondly by examining the proof that led to (3.15), for $\lambda \leq 0$, we see easily we have for $\epsilon > 0$, there exist $R_0$, independent of $\lambda$, such that, for $\lambda \leq 0$ and $2 < |t_2||t_1| \alpha < 4$ and $|t| > R_0$,

$$|t_1|^{2\alpha} \leq \epsilon(q(t) + |\lambda|p(t)\phi_N(t)) = \epsilon V_1.$$  

Thus for any $\phi \in C_0^\infty$, we see if $\lambda \leq 0$,

\begin{equation}
\int_{\{|t| > R_0\}} |\phi|^2 V_2 \leq C\epsilon\|\phi\|^2_\mathcal{H}
\end{equation}
with $C$ independent of $\lambda$ and even $N$. Now for $|t| < R_0$, as $\lambda \to -\infty$,

\begin{equation}
|S_2 \cap \{|t| < R_0\}| \to 0.
\end{equation}

Thus,

\begin{align*}
\int_{\mathbb{R}^2} |\phi|^2 V_2 &= \int_{T \cap \{|t| > R_0\}} + \int_{T \cap \{|t| \leq R_0\}} |\phi|^2 V_2 \\
&\leq c \varepsilon \|\phi\|_{L^2}^2 + \left( \int_{T \cap \{|t| \leq R_0\}} V_2^2 \right)^{1/2} \left( \int_{\{|t| \leq R_0\}} |\phi|^4 \right)^{1/2}.
\end{align*}

From (3.20),

\begin{align*}
&\leq c \varepsilon \|\phi\|_{L^2}^2 + c \varepsilon \left[ \int_{\{|t| \leq R_0\}} (|\nabla \phi|^2 + \phi^2) \right] \\
&\leq c \varepsilon \|\phi\|_{L^2}^2.
\end{align*}

Thus, for any $\phi \in C_0^\infty(\mathbb{R}^2)$, as $\lambda \to -\infty$, with a uniform $c$ we have,

\[ \frac{c}{\varepsilon} \leq \frac{\|\phi\|_{L^2}^2}{\int_{\mathbb{R}^2} |\phi|^2 V_2} . \]

Claim (b) follows and our lemma is proved.

\[ \blacksquare \]

**Lemma 3.5.** Under (3.1), there exists a finite, positive constant $C > 0$, independent of $N$, such that the eigenvalues $\lambda_N$ of Lemma (3.4) satisfy,

\begin{equation}
C^{-1} < \lambda_N < C
\end{equation}

**Proof.** The upper bound in (3.21) is easy to come by. Re-writing (3.18) with $\mu_1 = 1$, as we have done earlier, we see,

\begin{equation}
\lambda_N = \inf_{\phi \in C_0^\infty(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} (|\nabla \phi|^2 + q\phi^2)}{\int_{\mathbb{R}^2} p\phi_N \phi^2} .
\end{equation}

Now choose,

\[ \phi(t) = \begin{cases} 
1 & |t| \leq 1 \\
0, & |t| \geq 2
\end{cases} . \]
and insert in (3.22). One sees that $\phi_N \equiv 1$, on the support of $\phi$, and hence we get easily $\lambda_N \leq C$, with $C$ independent of $N$.

We now establish the lower bound for $\lambda_N$. We know from Lemma (3.4), that for the eigenfunction $f_N \equiv f$, we have,

$$
\lambda_N = \frac{\int_{\mathbb{R}^2} |\nabla f|^2 + qf^2}{\int_{\mathbb{R}^2} p(t)\phi_N f^2}
$$

(3.23)

We will show, for $C > 0$, independent of $N, f$

$$
\int_{\mathbb{R}^2} p(t)\phi_N f^2 \leq c \int_{\mathbb{R}^2} qf^2
$$

(3.24)

Combining (3.23) and (3.24) the lower bound for $\lambda_N$ follows. First we note, by Lemma (3.3) and by the fact just established that $\lambda_N \leq C$,

$$
\int_{|t| \leq 100} p\phi_N f^2 \leq c \sup_{|t| \leq 100} f^2 \leq c \inf_{|t| \leq 100} f^2 \leq c \int_{|t| \leq 100} f^2 q.
$$

(3.25)

Thus we show (3.24) for $|t| \geq 100$. We revert to polar coordinates again. Fix $r$, and a circle $|t| = r$. We show for $r \geq 100$.

$$
\int_0^{2\pi} p\phi_N f^2 d\theta \leq c \int_0^{2\pi} qf^2 d\theta,
$$

(3.26)

with $c$ independent of $r, N, f$. Multiplying (3.26) by $r$, and integrating over $r \geq 100$, we get,

$$
\int_{|t| \geq 100} p\phi_N f^2 dt \leq c \int_{|t| \geq 100} qf^2 dt.
$$

(3.27)

Combining (3.27) and (3.25), (3.24) follows. We now use the notation $r\theta = (r \cos \theta, r \sin \theta)$. Using the homogeneity of $p, q$, we write (3.26) as,

$$
\phi_N(r)r^{2s} \int_0^{2\pi} f^2(r\theta)p_1(\theta) d\theta \leq cr^{2m} \int_0^{2\pi} f^2(r\theta)q_1(\theta) d\theta.
$$

(3.28)

For a fixed $r \geq 100$,

$$
\Gamma_r = \{ \theta : ar^{2s} \geq r^{2m}q_1(\theta) \},
$$

(3.29)
where \( a = \| p_1 \|_{L^\infty(S^1)} \). Now \( r\Gamma_r \) (the dilate of \( \Gamma_r \) by a factor of \( r \)) consists of a union of \( \ell \) arcs, each arc transversal to one of the tapering tubes \( \tau_\sigma \), defined earlier of which there are \( \ell \) in number. Let us set \( r\Gamma_r \cap T_1 = G_{r,1} \). Now note on the complement of \( \Gamma_r \) in \( S^1, \Gamma^c_r \) we have,

\[
\phi_N(r^2s) \int_{\Gamma^c_r} f^2(r\theta)p_1(\theta)d\theta \leq r^{2m} \int_{\Gamma^c_r} f^2(r\theta)q_1(\theta)d\theta.
\]

So to prove (3.28), it is enough to show,

(3.30) \[ \phi_N(r^2s) \int_{\Gamma_{r,1}} f^2(r\theta)p_1(\theta)d\theta \leq cr^{2m} \int_{\Gamma_{r,1}} f^2(r\theta)q_1(\theta)d\theta, \]

where \( \Gamma_{r,1} = \{ \theta : r\theta \in G_{r,1} \} \).

Now let \( \tilde{\Gamma}_{r,1} \), be an arc, such that \( |\tilde{\Gamma}_{r,1}| = 5|\Gamma_{r,1}| \), and \( \tilde{\Gamma}_{r,1} \) disjoint from \( \Gamma_{r,1} \) but having common end-point with \( \Gamma_{r,1} \). Thus \( \tilde{\Gamma}_{r,1} \) is adjacent to \( \Gamma_{r,1} \) with five times the length of \( \Gamma_{r,1} \). Now the arc \( r\Gamma_{r,1} \) lies between the curves, \( |t_2| = c|t_1|^{(2s-2m+k)/k} \), \( c = c(a) \) (see also (3.4)) and follows from the definition (3.29). Thus, \( (\bullet) \) (denotes arc-length)

(3.31) \[ r|\Gamma_{r,1}| \approx r^{(2s-2m+k)/k} \approx r|\tilde{\Gamma}_{r,1}|. \]

Moreover using (3.29) again, we see on \( \tilde{\Gamma}_{r,1} \),

(3.32) \[ ar^{2s} \leq r^{2m}q_1(\theta) \leq car^{2s}, \]

\( c \) independent of \( r, \theta \). In fact for \( (t_1, t_2) \in \tilde{\Gamma}_{r,1} \),

\[ |t_1|^{(2s-2m+k)/k} \leq |t_2| \leq 5|t_1|^{(2s-2m+k)/k}. \]

Now fix a ball \( B, B \supseteq (r\Gamma_{r,1}) \cup (r\tilde{\Gamma}_{r,1}) \) and \( \text{diam } B = |r\Gamma_{r,1} \cup (r\tilde{\Gamma}_{r,1})| \). Since we have (3.32), on \( B, |V| = |(q - \lambda N p)| \leq cr^{2s} \). By (3.31), it follows that if \( d = \text{diameter of } B \),

\[ d|V|^{1/2} \leq cr^{((2s-2m+k)/k+s)} \leq c, \]

since (3.1) holds and \( r \geq 100 \). Thus Lemma (3.3) applies. We have, for \( r \geq 100 \),

(3.33) \[ \sup_{\Gamma_{r,1}} f^2(r\theta) \leq c \inf_{\Gamma_{r,1}} f^2(r\theta). \]

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Going back to (3.30), using (3.33)

\[
\phi_N(r) r^{2s} \int_{\Gamma_{r,1}} f^2(r\theta)p_1(\theta)d\theta \leq r^{2s} \inf_{\widetilde{\Gamma}_{r,1}} f^2(r\theta) \int_{\Gamma_{r,1}} p_1(\theta)d\theta \\
\leq c r^{2s}\|\Gamma_{r,1}\| \inf_{\widetilde{\Gamma}_{r,1}} f^2(r\theta).
\]

(3.34)

Thus the right side of (3.34) is bounded by

\[
\leq c r^{2s} \inf_{\widetilde{\Gamma}_{r,1}} f^2(r\theta)|\widetilde{\Gamma}_{r,1}|.
\]

(3.35)

But on \(\widetilde{\Gamma}_{r,1}\) from (3.32) \(r^{2s} \leq r^{2m}q_1(\theta)\). Hence (3.35) is bounded by,

\[
c \int_{\widetilde{\Gamma}_{r,1}} f^2(r\theta)q_1(\theta)r^{2m}d\theta.
\]

This finishes the proof of (3.30).

\[\blacksquare\]

**Lemma 3.6.** The following form of inequality (3.26) also holds for \(r \geq r_0, r_0\) independent of \(N\), and for some \(\varepsilon > 0\),

\[
r^{2s+\varepsilon} \int_0^{2\pi} f(r\theta)d\theta \leq c \int_0^{2\pi} q_1(\theta)r^{2m}f(r\theta)d\theta.
\]

Proof. The proof is simply re-writing the proof of the previous lemma. In fact in the proof of the previous lemma we used a weaker form of (3.1),

\[
\frac{2s - 2m + j}{j} + s \leq 0.
\]

When \(j = 0\), since \(q_1(\theta) > c > 0\), the corollary is obvious. Thus assume \(j > 0\).

Define, for \(\varepsilon > 0\) (that will be picked later),

\[
\Gamma_r = \{\theta : ar^{2s+\varepsilon} \geq r^{2m}q_1(\theta)\}.
\]

The tubes, for example the tube \(T_1\) is bounded by the curves \(|t_2| = |t_1|^{(2s+\varepsilon-2m+k)/k}\).

Thus as before,

\[
r|\Gamma_{r,1}| \approx r^{(2s+\varepsilon-2m+k)/k} \approx r|\widetilde{\Gamma}_{r,1}|.
\]

Again on \(\widetilde{\Gamma}_{r,1}\),

\[
ar^{2s+\varepsilon} \leq r^{2m}q_1(\theta) \leq c a r^{2s+\varepsilon},
\]

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Now fix a ball $B$ with diameter $|\Gamma_{r,1} \cup (\tilde{\Gamma}_{r,1})|$, and $B \supseteq r\Gamma_{r,1} \cup (r\tilde{\Gamma}_{r,1})$. On $B$, the potential $V(t) = q(t) - \lambda \phi_N(t)p(t)$, satisfies, $|V|^{1/2} \leq cr^{s+\varepsilon/2}$. If $d = \text{diam } B$, then on $B, d|V|^{1/2} \leq cr^n, \eta = ((2s - 2m + k)/k) + s + \varepsilon/2 + \varepsilon/k$. Choose $\varepsilon > 0$ so that $\eta \leq 0$, which we can because of (3.1). Thus we may apply Harnack’s inequality, Lemma (3.3) again to conclude for $f$, an identical inequality as in (3.33). Thus,

$$r^{2s+\varepsilon} \int_{\Gamma_{r,1}} f(r\theta) d\theta \leq r^{2s+\varepsilon} \inf_{\tilde{\Gamma}_{r,1}} f(r\theta)|\tilde{\Gamma}_{r,1}|$$

$$\leq \int_{\tilde{\Gamma}_{r,1}} f(r\theta) q_1(\theta) r^{2m} d\theta,$$

because on $\tilde{\Gamma}_{r,1}, r^{2m} q_1(\theta) \sim r^{2s+\varepsilon}$.

In $\mathbb{R}^n$, one proves this lemma by constructing the arcs $\Gamma_{r,1}$ in the direction transversal to the direction of tapering. One direction always exists in which the diameter tapers and we can apply the argument in that direction.

Our $f_N \equiv f$, satisfies $f_N > 0,$ and solves, (3.5)

$$-\Delta f_N + (q_1(\theta)r^{2m} - \lambda_N p_1(\theta)r^{2s}\phi_N(r))f_N = 0.$$

We may normalize $f_N$, such that $f_N(0) = 1$ and we assume this from now on. We integrate both sides of the equation above in $\theta$. We set,

(3.36) $F(r) = \frac{1}{2\pi} \int_0^{2\pi} f_N(r\theta)d\theta$

By our normalization

(3.37) $F(0) = 1,$

and we also see that,

(3.38) $F'(0) = 0.$

Further $F$ satisfies the ODE,

$$-(F'' + \frac{1}{r}F') + \left[ \frac{r^{2m} \int_0^{2\pi} q_1(\theta)f_N(r\theta)d\theta}{\int_0^{2\pi} f_N(r\theta)d\theta} \right]$$

$$- \lambda_N \phi_N(r) \frac{r^{2s} \int_0^{2\pi} p_1(\theta)f_N(r\theta)d\theta}{\int_0^{2\pi} f_N(r\theta)d\theta} ] F(r) = 0.$$
We set,

\[
a(r) = \frac{\int_0^{2\pi} q_1(\theta)f_N(r\theta)d\theta}{\int_0^{2\pi} f_N(r\theta)d\theta},
\]

(3.39)

\[
b(r) = \frac{\int_0^{2\pi} p_1(\theta)f_N(r\theta)d\theta}{\int_0^{2\pi} f_N(r\theta)d\theta}.
\]

Our ODE becomes

\[
-(F'' + \frac{1}{r}F') + (a(r)r^{2m} - \lambda_N\phi_N(r)b(r)r^{2s})F = 0.
\]

(3.40)

Now, by the Schwartz inequality and (3.36)

\[
|F(r)|^2 \leq c \int_0^{2\pi} |f_N(r\theta)|^2 d\theta
\]

\[
|F'(r)|^2 \leq c \int_0^{2\pi} |\partial f_N/\partial r|^2 d\theta.
\]

Thus from (3.6) we have,

\[
\int_0^{\infty} (|F'(r)|^2 + |F|^2) r dr < +\infty.
\]

(3.41)

**Lemma 3.7.** Let \( F(r) \) on \([0, \infty) \) be defined by (3.36). Then \( F(r) \) satisfies (3.37), (3.38), (3.39), (3.40) and (3.41). Moreover, for some \( C > 0 \),

\[
0 < F(r) \leq C,
\]

\( C \) independent of \( N \).

**Proof.** Since \( f_N > 0 \), from (3.36) it follows that \( F(r) > 0 \). That \( F(r) \) satisfies (3.36), - (3.41) has been proved already, all that remains is the upper bound for \( F(r) \). First for \( c > 0 \), independent of \( N \) we have,

\[
0 \leq b(r) \leq c, 0 \leq a(r) \leq c,
\]

(3.42)

\[
c_1r^{2s+\varepsilon-2m} \leq a(r), r \geq R_0 > 0
\]

where \( c_1 > 0, R_0 \) are independent of \( N \). The first two inequalities for \( a(r), b(r) \) follows from the definitions (3.39), and the last inequality in (3.42) follows from Lemma (3.6), once we re-write \( a(r) \) as,

\[
a(r) = r^{2s-2m+\varepsilon} \frac{\int_0^{2\pi} r^{2m} q_1(\theta)f(r\theta)d\theta}{\int_0^{2\pi} r^{2s+\varepsilon} f(r\theta)d\theta}.
\]
Thus from (3.42), for \( r \geq R_0 \),
\[
(3.43) \quad r^{2m} \alpha(r) \geq c_1 r^{2s+\varepsilon}.
\]
Now using (3.42), (3.43), our potential satisfies for \( r \geq r_1 \), \((r_1\text{ independent of } N)\),
\[
(3.44) \quad c_1 r^{2s+\varepsilon} \leq a(r)r^{2m} - \lambda_N \phi_N(r)b(r)r^{2s} \leq c_2 r^{2m}.
\]
We now show our solution \( F(r) \) cannot have a local maximum for \( r > c_1 \), with \( c_1 \) independent of \( N \). Assume that a local maximum is attained at \( r = r_0 > 0 \).

Then, \( F'(r_0) = 0, F''(r_0) \leq 0 \). Thus, from the ODE for \( F \), since \( F > 0 \),
\[
a(r)r^{2m} - \lambda_N \phi_N(r)b(r)r^{2s} \leq 0,
\]
and using (3.42) and (3.43), we have,
\[
c_1 r^{2s+\varepsilon} \leq \lambda_N r^{2s} \leq C r^{2s}.
\]
Thus \( r_0 \leq c_0, c_0 \text{ independent of } N \). Fix \( \varepsilon < 1/2 \). We claim that \( F(r) \leq \varepsilon \) for \( r > r_1 > c_1 \). If there exists \( r_2 \) such that \( F(r) > \varepsilon \) for \( r > r_2 \) we contradict (3.41). Thus there exists \( r_2 \) such that \( F(r) \leq \varepsilon \) for \( r > r_2 \), or there exists intervals \([r_k, r_{k+1}]\), \( r_k > c_1 \) such that \( F(r_k) = F(r_{k+1}) = \varepsilon \) and \( F \) has a local maximum in the interior of \([r_k, r_{k+1}]\) by Rolle’s theorem. If the second case occurs, we contradict the fact that \( F(r) \) has no local maxima for \( r > c_1 \). Our claim is proved. Since \( F(0) = 1 \), and \( F(r) \leq \varepsilon < 1/2 \) for \( r > r_2 \), either \( F(r) \leq 1 \) for all \( r \) or \( F(r) \) achieves its maximum at \( r_0 > 0 \), which is also a local maximum. But then this \( r_0 \leq c_0, c_0 \text{ independent of } N \). This proves our lemma. Notice we have also proved that \( F(r) \to 0 \) as \( r \to \infty \). This fact can also be deduced by refined estimates in Davies[D].

\[\square\]

**Lemma 3.8.** There exists a constant \( C > 0 \), independent of \( N \), such that, for \( t \in \mathbb{R}^2 \),
\[
|f_N(t)| \leq C(1 + |t|)^{(m+1)}.
\]

**Proof.** We use polar coordinates once again. Define, for fixed \( r, r \geq 100 \),
\[
E_\theta = \{ \theta : f_N(r\theta) > (1 + r)^{(m+1)} \}.
\]
The open set \( E_\theta \) can be written as a disjoint union of open arcs on \( S^1 \). Thus, \( E_\theta = \cup I_\sigma \), where \( I_\sigma \) are disjoint open arcs on \( S^1 \). Now, for \( r \geq 100 \), using Lemma (3.7),
\[
|E_\theta|^m \leq \int_{E_\theta} f_N(r\theta) d\theta \leq 2\pi \int_0^\infty f_N(r\theta) d\theta \leq c
\]
Thus,
\[ | \cup I_\sigma | \leq cr^{-(m+1)}, \]
and in particular,
\[ |I_\sigma| \leq cr^{-(m+1)}. \]
Thus on the circle \(|t| = r\), the arc \(rI_\sigma\), has measure \(r^{-m}\). Now at the end-points of arc \(rI_\sigma\) on the circle \(|t| = r\), \(f_N(r\theta) \leq r^{m+1}\). Since \(f_N\) satisfies (3.5), we may apply Harnack’s inequality, Lemma (3.3) to \(f_N\) on \(rI_\sigma\). The potential in (3.5) on \(rI_\sigma\) is bounded by \(r^{2m}\), and \(rI_\sigma\) has length at most \(r^{-m}\), so (3.2) applies. Thus,
\[ \sup_{\theta \in I_\sigma} f(r\theta) \leq Cr^{m+1}. \]
Thus we have our conclusion for \(r \geq 100\). For \(r \leq 100\) since Harnack’s inequality applies to (3.5), and since \(f_N(0) = 1\), and \(\lambda_N\) bounded, the conclusion of our lemma is again immediate.

For \(\mathbb{R}^n\), we define, for fixed \(r\), \(|t| = r\) as before
\[ E_\omega = \{ \omega \in S^{n-1} : f_N(r\omega) > (1 + r)^{(m+1)(n-1)} \}. \]
The proof then follows the \(\mathbb{R}^2\) case, but now using the properties of Whitney cubes which replace the arcs \(I_\sigma\). That is we can cover \(S^{n-1}\) by coordinate patches, such that \(S^{n-1}\) is approximately Euclidean on each patch. We then decompose \(E_\omega\) into \(n-1\) dimensional cubes on \(S^{n-1}\) using the Whitney lemma [St].

**Proof of Theorem 3.2.** The uniform bounds of Lemma (3.8) and the bounds on \(\lambda_N\) from Lemma (3.5) allow us to take a limit of a suitable subsequence of \(N\), with \(N \to \infty\). Since \(f_N(0) = 1\), for all \(N\), we get by standard elliptic estimates a \(0 < \lambda < \infty\), and a function \(f(t), f(0) = 1, f > 0\), which satisfies the Schrödinger equation of our theorem.

The next proposition handles the case of equality in (3.1) when \(s = 0\).

**Proposition 3.9.** Assume that \(s = 0\), \(n = 2\) and we have equality in (3.1). Then for every \(\lambda > 0\) we can construct an oscillatory and bounded solution \(f(t)\) with \(f(0) = 1\), or we have a solution \(f(t)\) that satisfies \(f(0) = 1\) and for some \(C > 0\)
\[ 0 < f(t) < e^{C|t|}. \]
Proof. Since \( s = 0 \) and we have equality in (3.1), necessarily \( j = 2m \). Thus \( q(t) \) vanishes identically or after a rotation of coordinates \( q(t) = t_1^{2m} \). If \( q(t) \) vanishes identically we have the spectral problem
\[
\Delta f + \lambda f = 0
\]
We select \( f(t) = \cos \lambda t_1^{1/2} \) as a solution which has the claimed properties. In case \( q(t) = t_1^{2m} \) we may separate variables. Let \( \psi(z) \) denote the first eigenfunction with eigenvalue \( \lambda_0 \) of the problem
\[
-\psi''(z) + z^{2m} \psi(z) = \lambda_0 \psi, \quad -\infty < z < \infty.
\]
We are seeking a solution to the spectral problem
\[
-\Delta f + (t_1^{2m} - \lambda)f = 0
\]
If \( \lambda > \lambda_0 \), we take \( f(t) = \psi(t_1) \cos \mu t_2, \mu = |\lambda - \lambda_0|^{1/2} \) as a solution. If \( \lambda \leq \lambda_0 \) we take \( f(t) = \psi(t_1)e^{\mu t_2} \) as a solution. This proves the proposition.

We may combine the previous proposition and theorem (3.2) to give a complete solution to the spectral problem when \( s = 0 \) and \( n = 2 \).

Proposition 3.10. Assume that \( q(t), t \in \mathbb{R}^2 \) is a non-negative homogeneous polynomial of degree \( 2m \) with real coefficients. Then for some \( \lambda > 0 \) there exists in all \( \mathbb{R}^2 \) a solution \( f(t) \) to the spectral problem
\[
-\Delta f + (q(t) - \lambda)f = 0,
\]
where \( f(0) = 1 \) and \( f(t) \) satisfies with some \( C > 0 \) the growth condition
\[
|f(t)| \leq e^{C|t|}
\]
which is the growth condition which satisfies the requirement (2.20) for \( s = 0 \).

We are now in a position to combine all of our results and state a concrete theorem about analytic hypoellipticity

Theorem 3.11. Assume that the basis \( \mathcal{B} \) for \( V_1 \) contains an element from the center of \( \mathfrak{n} \). Assume that \( \# S = 2 \) and the elements of \( S \) commute amongst themselves. Then the sums of squares operator (1.1) has a solution \( u \) that is Gevrey of order exactly \( m + 1 \) at the origin.

Proof. Theorem (2.5) and proposition (2.2) reduces the spectral problem to that considered in proposition (3.10). Proposition (3.10) guarantees for us a solution
with the requisite growth properties that allows us to apply Theorem (2.3). The-orem (2.3) allows us to conclude the existence of the solution $u$ with the given Gevrey property.

We believe Theorem (3.2) holds in full generality without the assumption (3.1) and with the weaker conclusion (2.20) which is enough. The following unpublished result of F. Treves seems to support this guess.

**Theorem 3.12 (F. Treves).** Let $p(t)$, be any homogeneous polynomial on $\mathbb{R}^n$ of degree $2s$, with real coefficients. Then there exist a solution $f(t)$ to,

$$
\Delta f + p(t)f(t) = 0,
$$

in all $\mathbb{R}^n$, such that, for some $C_1, C_2 > 0$,

$$
|f(t)| \leq C_1 \exp(C_2|t|^{s+1}).
$$

§4. Examples and further remarks.

**Example (4.1)** We consider

$$
X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}, X_3 = \frac{\partial}{\partial x_3}, X_4 = x_1^7x_2^2 \frac{\partial}{\partial x_4}, X_5 = x_1^2x_2^7 \frac{\partial}{\partial x_5}.
$$

Let $L = \sum_{i=1}^5 X_i^2$. We define,

$$
(4.1) \quad u(x_1, x_2, x_3, x_4, x_5) = \int_0^\infty f(\rho x_1, \rho x_2)e^{\lambda^{1/2}\rho^3x_3 + i\rho^{10}(x_4 + x_5)e^{-M\rho^3}dp.
$$

$u$ will satisfy $Lu = 0$, provided $f$ satisfies in $\mathbb{R}^2$,

$$
(4.2) \quad -\Delta f + \left[(x_1^{14}x_2^4 + x_1^4x_2^{14}) - \lambda(x_1^2 + x_2^2)^2\right] f = 0
$$

In (4.2), $m = 9, s = 2$ and $j = 4$. Thus (3.1) is satisfied. Thus Theorem (3.2) applies and we have a solution $f(x_1, x_2), f(0) = 1$, with $x = (x_1, x_2),

$$
(4.3) \quad 0 < f(x) \leq C(1 + |x|)^{10}
$$

Thus the integral in (4.10) is convergent.
Further,
\[
\frac{\partial^\sigma u(0)}{\partial x_4^\sigma} = i^\sigma \int_0^\infty \rho^{10\sigma} e^{-M\rho^3} \, dp.
\]
Thus the solution is exactly in Gevrey 10/3 = \((m + 1)/(s + 1)\) at the origin.

Note that \(\xi_3 = 0\), defines the non-symplectic stratum, corresponding to \(\frac{\partial}{\partial x_3}\). Now if one were to consider the Lie Algebra \(\mathfrak{n}\), with the same bracket relations as the \(X_i\) above, one finds,
\[
\mathfrak{n} = \bigoplus_{i=1}^{10} V_i.
\]
The co-vector corresponding to \(\frac{\partial}{\partial x_3}\) when restricted to \(V_3\) does not vanish, but vanishes when restricted to \(V_i, i \neq 3\) i.e. \(s = 2\). Next \(\dim V_{10} = 2\), corresponding to \(\frac{\partial}{\partial x_4}\) and \(\frac{\partial}{\partial x_5}\), and \(m = 9\) clearly. Note too the strong form of assumption (2.4) holds, \([V_i, V_j] = 0, i, j > 1\). Next Remark (2.4) is also clear. We have the right to induce representations using
\[
\lambda_2 = (0, 0, \ldots, 0, \lambda_{10,1}, \lambda_{10,2})
\]
where \((0, \ldots, 0, \lambda_{10,1}, 0)\) is a co-vector to \(\frac{\partial}{\partial x_4}\) and \((0, \ldots, 0, 0, \lambda_{10,2})\) is a co-vector to \(\frac{\partial}{\partial x_5}\). Given this flexibility, one may desire to set \(\lambda_{10,1} \neq 0\) but \(\lambda_{10,2} = 0\). This is equivalent to searching for a solution \(u\) of the form \(u(x_1, x_2, x_3, x_4)\) and we are led to find an \(f(x_1, x_2)\) a solution to,
\[
(4.4) \quad -\Delta f + \left[ x_1^{14} x_2^4 - \lambda(x_1^2 + x_2^2)^2 \right] f = 0.
\]
This operator is less positive then (4.2) and Theorem (3.2) does not apply to (4.4). Thus to apply Theorem (3.2) we must keep both \(\lambda_{10,1}, \lambda_{10,2}\) non-zero and set them equal to 1. Thus in the parameter \(\lambda_{m+1,\beta}\) there is great flexibility, and given that we have not proved Theorem (3.2) in full strength it is advisable to induce in all the parameters \(\lambda_{m+1,\beta}\) so that Theorem (3.2) applies. Further \(\mathcal{S} = \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}, \#\mathcal{S} = 2\), and the elements of \(\mathcal{S}\) commute. Notice also \(\sum_{i=1}^{3} X_i^2\) has a characteristic set that is symplectic.

**Lifting:** We have seen in Example (4.1) that the results of Section 3 apply to even degenerate vector fields. However, we have been unable to adapt the lifting mechanism in [RS] to Section 2. The difficulty is to verify formula (2.23) for the degenerate case, though (2.23) and (4.1) are very close. Furthermore, lifting may introduce non-symplectic strata where non-existed.

**Example (4.2)**
\[
X_1 = \frac{\partial}{\partial x_1}, X_2 = x_1^2 \frac{\partial}{\partial x_2}.
\]
The characteristic set of $X_1, X_2$ is symplectic. The corresponding lifted vector fields are,

$$Y_1 = \frac{\partial}{\partial x_1}, Y_2 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4} + x_2^2 \frac{\partial}{\partial x_2}.$$ 

The vanishing of $\xi_4 = 0$, the co-vector corresponding to $\frac{\partial}{\partial x_4}$, defines a non-symplectic stratum, and $L = Y_1^2 + Y_2^2$ is in fact non-analytic hypoelliptic. In fact the Gelfand-Kirillov dimension is 1, and this operator gives an ODE as in [CC]. In fact by considering a solution $u$ of the form $u = u(x_1, x_2, x_4)$ to $L$ we easily reduce matters to the ODE considered in [CC]. However, it is also clear that if there are non-symplectic strata, then after lifting there will be non-symplectic strata, and thus the co-vectors in the degenerate fields determine the parameters in $n^*$ for the induced representation in an invariant way.

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