We present a method for the fast computation of the eigenpairs of a bijective positive symmetric linear operator $L$. The method is based on a combination of operator adapted wavelets (gamblets) with hierarchical subspace correction. First, gamblets provide a raw but fast approximation of the eigensubspaces of $L$ by block-diagonalizing $L$ into sparse and well-conditioned blocks. Next, the hierarchical subspace correction method, computes the eigenpairs associated with the Galerkin restriction of $L$ to a coarse (low dimensional) gamblet subspace, and then, corrects those eigenpairs by solving a hierarchy of linear problems in the finer gamblet subspaces (from coarse to fine, using multigrid iteration). The proposed algorithm is robust for the presence of multiple (a continuum of) scales and is shown to be of near-linear complexity when $L$ is an (arbitrary local, e.g. differential) operator mapping $H_0(\Omega)$ to $H^{-s}(\Omega)$ (e.g. an elliptic PDE with rough coefficients).

Keywords. Multiscale eigenvalue problem, Gamblet decomposition, multigrid iteration, subspace correction, numerical homogenization.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

Solving eigenvalue problems is one of the fundamental and difficult tasks in modern science and engineering. Although high-dimensional eigenvalue problems appear in physics, chemical sciences, data science, image processing and machine learning, the class of eigenpair
solving eigenvalues is not as diverse as that of linear solvers (which comprises many efficient algorithms such as geometric and algebraic multigrid, domain decomposition, etc.) . Eigenvalue problems may also involve operators with multiple scales and the interplay between those coupled scales and the nonlinearity from the eigenvalue problem poses significant challenges in materials science [1, 4, 21, 12].

Recently, two-level [20, 12] and multilevel [9, 10, 11, 18, 19] correction methods have been proposed to reduce the complexity of solving eigenpairs associated with low eigenvalues by first solving a coarse mesh/scale approximation, which can then be corrected by solving linear systems (corresponding to linearized eigenvalue problems) on a hierarchy of finer meshes/scales. Although the multilevel correction approach has produced multigrid methods for linear and nonlinear eigenvalue problems [6, 9, 10, 11, 18, 19], the regularity estimates which guarantee linear complexity do not hold for PDEs with rough coefficients and a naive application of the correction approach [6, 9, 11, 18, 19] to multiscale eigenvalue problem may converge very slowly. For two-level methods [20] this lack of robustness can be alleviated by numerical homogenization [12] (i.e. by using LOD basis functions as a coarse space). For multilevel methods, gamblets [13, 14, 16, 17, 15] (operator-adapted wavelets satisfying three desirable properties: scale orthogonality, well-conditioned multi-resolution decomposition, and localization) provide a natural multiresolution basis ensuring robustness to rough coefficients.

The aim of this paper is therefore to produce a fast multilevel numerical method for multiscale eigenvalue problems (e.g. for PDEs that may have rough and highly oscillatory coefficients) by integrating the multilevel correction approach with gamblet multi-resolution decomposition. In this merger, the gamblet decomposition provides a hierarchy of subspaces acting as hierarchies of coarse spaces for the multilevel correction method. The overall computation cost is that of solving a sequence of linear problems over this hierarchy (using a gamblet based multigrid approach [22]). We also note that recently, Hou et. al. [7] proposed to compute the leftmost eigenpairs of a sparse symmetric positive matrix by combining the implicitly restarted Lanczos method with the multiresolution matrix decomposition obtained from the gamblet transform and local eigenfunctions as measurement functions.

This paper is structured as follows. Section 2 summarizes the gamblet decomposition, its properties, and the gamblet based multigrid method [22] (please refer to [13, 14, 16, 17, 15] for the detailed construction). Section 3 presents the gamblet based multilevel method for multiscale eigenvalue problems. Section 4 provides the numerical analysis of the method. Some concluding remarks are presented in the final section.

2 Gamblet Decomposition and Gamblet based Multigrid Method

2.1 The setting

Let $(V, \| \cdot \|), (V^*, \| \cdot \|_*)$ and $(V_0, \| \cdot \|_0)$ be Hilbert spaces such that $V \subset V_0 \subset V^*$ and such that the natural embedding $i : V_0 \rightarrow V^*$ is compact and dense. Let $(V^*, \| \cdot \|_*)$ be the dual of $(V, \| \cdot \|)$ using the dual pairing obtained from the Gelfand triple. Writing $[\cdot, \cdot]$ for the duality pairing between $V^*$ and $V$ (derived from the Riesz duality between $V_0$ and itself),
there exists a symmetric positive linear bijection $\mathcal{L}$ mapping $V$ to $V^*$ such that
\[ \|u\|^2 = [\mathcal{L}u, u] \text{ for } u \in V. \quad (2.1) \]
The corresponding inner product on $V$ is defined by
\[ \langle u, v \rangle := [\mathcal{L}u, v] \text{ for } u, v \in V, \quad (2.2) \]
and $\| \cdot \|_*$ is the corresponding dual-norm on $V^*$, i.e.
\[ \|\phi\|^2 = \sup_{v \in V, v \neq 0} \frac{[\phi, v]}{\|v\|} \text{ for } \phi \in V^*. \quad (2.3) \]
Given $g \in V^*$, we will consider the solution $u$ of the variational problem
\[ \langle u, v \rangle = [g, v], \quad \text{for } v \in V. \quad (2.4) \]

**Example 2.1.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ (of arbitrary dimension $d \in \mathbb{N}^*$) with uniformly Lipschitz boundary. Given $s \in \mathbb{N}$, let
\[ \mathcal{L} : \mathcal{H}_0^s(\Omega) \to \mathcal{H}^{-s}(\Omega) \quad (2.5) \]
be a continuous linear bijection between $\mathcal{H}_0^s(\Omega)$ and $\mathcal{H}^{-s}(\Omega)$. Assume $\mathcal{L}$ to be symmetric, positive and local, i.e. $[\mathcal{L}u, v] = [u, \mathcal{L}v]$ and $|[\mathcal{L}u, u]| \geq 0$ for $u, v \in \mathcal{H}_0^s(\Omega)$ and $[\mathcal{L}u, v] = 0$ if $u, v$ have disjoint supports in $\Omega$. In this example $V, V^*$ and $V_0$ are $\mathcal{H}_0^s(\Omega)$, $\mathcal{H}^{-s}(\Omega)$ and $L^2(\Omega)$ endowed with the norms $\|u\|^2 = \int_{\Omega} \mathcal{L}u$, $\|\phi\|^2_\ast = \int_{\Omega} \phi \mathcal{L}^{-1} \phi$ and $\|u\|_0 = \|u\|_{L^2(\Omega)}$.

**Example 2.2.** Consider Example 2.1 with $s = 1$, $\mathcal{L} = -\text{div} (a(x) \nabla \cdot )$ and $a(x)$ is a symmetric, uniformly elliptic $d \times d$ matrix with entries in $L^\infty(\Omega)$ such that for all $x \in \Omega$ and $\ell \in \mathbb{R}^d$,
\[ \lambda_{\min}(a)|\ell|^2 \leq \ell^T a(x) \ell \leq \lambda_{\max}(a)|\ell|^2. \quad (2.6) \]

Note that
\[ \|v\|^2 = \int_{\Omega} (\nabla v)^T a \nabla v \quad \text{for } v \in \mathcal{H}_0^1(\Omega), \quad (2.7) \]
and the solution of (2.1) is the solution of the PDE
\[ \begin{cases} -\text{div} (a(x) \nabla u(x)) = g(x) & x \in \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases} \quad (2.8) \]

**2.2 Gamblets**

**Measurement functions.** Let $\mathcal{I}^{(1)}, \ldots, \mathcal{I}^{(q)}$ be a hierarchy of labels and let $\phi_i^{(k)}$ be a hierarchy of nested elements of $V^*$ such that
\[ \phi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k+1)}} \pi_{i,j}^{(k,k+1)} \phi_j^{(k+1)} \quad \text{for } k \in \{1, \ldots, q-1\} \text{ and } i \in \mathcal{I}^{(k)}, \quad (2.9) \]
for some rank $|\mathcal{I}^{(k)}|$, $\mathcal{I}^{(k)} \times \mathcal{I}^{(k+1)}$ matrices $\pi^{(k,k+1)}$ and such that the $(\phi_i^{(q)})_{i \in \mathcal{I}^{(q)}}$ are linearly independent and $\pi^{(k,k+1)} \pi^{(k+1,k)} = I_{\mathcal{I}^{(k)}}$ for $k \in \{1, \ldots, q-1\}$ (writing $I_{\mathcal{J}}$ for the $\mathcal{J} \times \mathcal{J}$ identity matrix and $\pi^{(k+1,k)}$ for $(\pi^{(k,k+1)})^T$).
**Operator adapted pre-wavelets.** For \( k \in \{1, \ldots, q\} \), let \( \Theta^{(k)} \) be the symmetric positive definite matrix with entries \( \Theta^{(k)}_{i,j} := [\phi_i^{(k)}, \phi_j^{(k)}] \) and (writing \( \Theta^{(k),-1} \) for the inverse of \( \Theta^{(k)} \)) let

\[
\psi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k)}} \Theta^{(k),-1}_{i,j} \phi_j^{(k)} \quad \text{for } i \in \mathcal{I}^{(k)}.
\]  

(2.10)

The elements \( \psi_i^{(k)} \) form a bio-orthogonal system with respect to the elements \( \phi_i^{(k)} \), i.e.

\[
[\phi_i^{(k)}, \psi_j^{(k)}] = \delta_{i,j}
\]

and

\[
u^{(k)} := \sum_{i \in \mathcal{I}^{(k)}} [\phi_i^{(k)}, \nu] \psi_i^{(k)},
\]

(2.11)

is the \( \langle \cdot, \cdot \rangle \) orthogonal projection of \( \nu \in V \) onto

\[
\mathcal{M}^{(k)} := \text{span}\{\psi_i^{(k)} \mid i \in \mathcal{I}^{(k)}\}.
\]

(2.12)

Furthermore \( A^{(k)} := \Theta^{(k),-1} \) can be identified as the stiffness matrix of the \( \psi_i^{(k)} \), i.e.

\[
A^{(k)}_{i,j} = \langle \psi_i^{(k)}, \psi_j^{(k)} \rangle \quad \text{for } i, j \in \mathcal{I}^{(k)}.
\]

(2.13)

The \( \psi_i^{(k)} \) are nested pre-wavelets in the sense that \( \mathcal{W}^{(k)} \subset \mathcal{W}^{(k+1)} \) and

\[
\psi_i^{(k)} = \sum_{j \in \mathcal{I}^{(k+1)}} R^{(k,k+1)}_{i,j} \psi_j^{(k+1)},
\]

(2.14)

where \( R^{(k,k+1)} = A^{(k)} \pi^{(k,k+1)} \Theta^{(k+1)} \) acts as a restriction/interpolation matrix.

**Operator adapted wavelets.** Let \( \mathcal{J}^{(k)} \) be hierarchy of labels such that (writing \( |\mathcal{J}^{(k)}| \) for the cardinal of \( \mathcal{J}^{(k)} \)) \( |\mathcal{J}^{(k)}| = |\mathcal{I}^{(k)}| - |\mathcal{I}^{(k-1)}| \). For \( k \in \{2, \ldots, q\} \), let \( W^{(k)} \) be a \( \mathcal{J}^{(k)} \times \mathcal{I}^{(k)} \) matrix such that (writing \( W^{(k),T} \) for the transpose of \( W^{(k)} \))

\[
\text{Ker}(\pi^{(k-1),k}) = \text{Im}(W^{(k),T}) \text{ and } W^{(k)}W^{(k),T} = I_{\mathcal{J}^{(k)}}.
\]

(2.15)

Define

\[
\chi_i^{(k)} := \sum_{j \in \mathcal{I}^{(k)}} W^{(k)}_{i,j} \psi_j^{(k)} \quad k \in \{2, \ldots, q\} \text{ and } i \in \mathcal{J}^{(k)}.
\]

(2.16)

Then \( \nu^{(k)} - \nu^{(k-1)} \) is the \( \langle \cdot, \cdot \rangle \) orthogonal projection of \( \nu \in V \) onto

\[
\mathcal{M}^{(k)} := \text{span}\{\chi_i^{(k)} \mid i \in \mathcal{J}^{(k)}\}.
\]

(2.17)

Furthermore \( \mathcal{W}^{(k)} \) is the \( \langle \cdot, \cdot \rangle \)-orthogonal complement of \( \mathcal{W}^{(k-1)} \) in \( \mathcal{W}^{(k)} \), i.e. \( \mathcal{W}^{(k)} = \mathcal{W}^{(k-1)} \oplus \mathcal{W}^{(k)} \),

\[
\mathcal{W}^{(q)} = \mathcal{W}^{(1)} \oplus \mathcal{W}^{(2)} \oplus \cdots \oplus \mathcal{W}^{(q)},
\]

(2.18)

and writing \( \mathcal{W}^{(q+1)} \) for the \( \langle \cdot, \cdot \rangle \)-orthogonal complement of \( \mathcal{W}^{(q)} \) in \( V \), \( \nu = u^{(1)} + (u^{(2)} - u^{(1)}) + \cdots + (u^{(q+1)} - u^{(q)}) \) is the multiresolution decomposition of \( \nu \) over \( V = \mathcal{W}^{(1)} \oplus \mathcal{W}^{(2)} \oplus \cdots \oplus \mathcal{W}^{(q+1)} \). For \( k \in \{2, \ldots, q\} \), \( B^{(k)} = W^{(k)}A^{(k)}W^{(k),T} \) is the stiffness matrix of the \( W^{(k)} \), i.e.

\[
B^{(k)}_{i,j} = \langle \chi_i^{(k)}, \chi_j^{(k)} \rangle \quad \text{for } i, j \in \mathcal{J}^{(k)}.
\]

(2.19)
Quantitative estimates. Under general stability conditions on the $\phi^{(i)}_i$ these operator adapted wavelets satisfy the quantitative estimates of Property 2.1. We will first state those estimates and provide an example of their validity in the general setting of Example 2.1.

To simplify notations, from now on, we will write $\mathcal{J}^{(1)} := I^{(1)}$, $\chi^{(1)} := \psi^{(1)}_i$, $B^{(1)} := A^{(1)}$ and $\Omega^{(1)} := \Omega^{(1)}$.

**Property 2.1.** The following properties are satisfied for some constant $C > 0$ and $H \in (0, 1)$:

1. Approximation:
   \[
   \|u - u^{(k)}\| \leq CH^k\|u - u^{(k)}\| \quad \text{for } u \in V,
   \]
   and
   \[
   \|u - u^{(k)}\| \leq CH^k\|Lu\|_0 \quad \text{for } u \in L^{-1}V_0.
   \]

2. Uniform bounded condition number: Writing $\text{Cond}(B)$ for the condition number of a matrix $B$ we have for $k \in \{1, \ldots, q\}$
   \[
   C^{-1}H^{-2(k-1)}I_{\mathcal{J}^{(k)}} \leq B^{(k)} \leq CH^{-2k}I_{\mathcal{J}^{(k)}} \quad \text{and } \text{Cond}(B^{(k)}) \leq CH^{-2}.
   \]

3. Near Linear Complexity: The wavelets $\psi^{(k)}_i$, $\chi^{(k)}_i$ and stiffness matrices $A^{(k)}$, $B^{(k)}$ can be computed to precision $\varepsilon$ (in $\|\cdot\|$-energy norm for elements of $V$ and in Frobenius norm for matrices) in $O(N \log N)$ complexity.

**Example 2.3.** Consider Example 2.1. Let $I^{(q)}$ be the finite set of $q$-tuples of the form $i = (i_1, \ldots, i_q)$. For $1 \leq k < r$ and a $r$-tuple of the form $i = (i_1, \ldots, i_r)$, write $i^{(k)} := (i_1, \ldots, i_k)$. For $1 \leq k \leq q$ and $i = (i_1, \ldots, i_q) \in I^{(q)}$, write $I^{(k)} := \{i^{(k)} : i \in I^{(q)}\}$. Let $\delta, h, \varepsilon \in (0, 1)$. Let $(\tau^{(k)}_i)_i \in I^{(k)}$ be uniformly Lipschitz convex sets forming a nested partition of $\Omega$, i.e. such that $\Omega = \bigcup_{i \in I^{(k)}} \tau^{(k)}_i$, $k \in \{1, \ldots, q\}$ is a disjoint union except for the boundaries, and $\tau^{(k)}_i = \bigcup_{j \in I^{(k+1)}} \tau^{(k+1)}_j$, $k \in \{1, \ldots, q-1\}$. Assume that each $\tau^{(k)}_i$ contains a ball of center $x^{(k)}_i$ and radius $\delta h^k$, and is contained in the ball of center $x^{(k)}_i$ and radius $\delta^{-1}h^k$. Writing $|\tau^{(k)}_i|$ for the volume of $\tau^{(k)}_i$, take
   \[
   \phi^{(k)}_i := 1_{x^{(k)}_i}\left|\tau^{(k)}_i\right|^{-\frac{1}{2}}.
   \]

The nesting relation (2.9) is then satisfied with $\pi^{(k+1)}_{i,j} := |\tau^{(k+1)}_j|\left|\tau^{(k)}_i\right|^{-\frac{1}{2}}$ for $j^{(k)} = i$ and $\pi^{(k+1)}_{i,j} = 0$ otherwise. For $i = (i_1, \ldots, i_{k+1}) \in I^{(k+1)}$ write $i^{(k)} := (i_1, \ldots, i_k) \in I^{(k)}$ and note that $\pi^{(k+1)}$ is cellular in the sense that $\pi^{(k+1)}_{i,j} = 0$ for $j^{(k)} \neq i$. Choose $(\mathcal{J}^{(k)})_{2 \leq k \leq q}$ to be a finite set of $k$-tuples of the form $j = (j_1, \ldots, j_k)$ such that $j^{(k-1)} := (j_1, \ldots, j_{k-1}) \in I^{(k-1)}$ and $|\mathcal{J}^{(k)}| = |\mathcal{J}^{(k-1)}| - |\mathcal{J}^{(k-1)}|$. Choose $W^{(k)}$ as in (2.15) and cellular in the sense that $W^{(k)}_{i,j} = 0$ for $i^{(k-1)} \neq j^{(k-1)}$ (see (2.18) for examples). Then corresponds to a multi-resolution decomposition of $H^1_0(\Omega)$ that is adapted to the operator $L$. 

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is then summarized in Algorithm 1 and we have the decomposition label the elements \( \tilde{O} \) mesh/grid in the setting of Example 2.2) used to discretize the operator spanning by the finite-elements \( \tilde{A} \) gamblets 

**Fast gamblet transform** The acceleration of Algorithm 1 to \( O(N \log^{3d} \frac{N}{\varepsilon}) \) complexity based on the truncation and localization of the computation of the interpolation matrices \( R^{(k,k+1)} \) enabled by the exponential decay of the gamblets and the uniform bound on \( \text{Cond}(B^{(k)}) \). In the setting of Examples 2.1 and 2.3 this acceleration is equivalent to localizing the computation of gamblets \( \psi^{(k)} \) to a sub-domain centered on \( \tau^{(k)} \) and of diameter \( O(H^k \ln \frac{1}{\varepsilon}) \). We refer to \[13, 14, 15\] for a detailed description of this acceleration.

**Algorithm 1** The Gamblet Transform.

1: \( \psi^{(q)} = \tilde{\psi}_i \)
2: \( A_{i,j}^{(q)} = \langle \psi^{(q)}, \tilde{\psi}^{(q)} \rangle \)
3: for \( k = q \) to 2 do
4: \( B^{(k)} = W^{(k)} A^{(k)} W^{(k),T} \)
5: \( \chi^{(k)} = \sum_{j \in I^{(k)}} W_{i,j} \psi^{(k)} \)
6: \( R^{(k-1,k)} = \pi^{(k-1,k)} (I^{(k)} - A^{(k)} W^{(k),T} B^{(k),-1} W^{(k)}) \)
7: \( A^{(k-1)} = R^{(k-1,k)} A^{(k)} R^{(k,k-1)} \)
8: \( \psi^{(k-1)} = \sum_{j \in I^{(k)}} R^{(k-1,k)} \tilde{\psi}^{(k)} \)
9: end for

**Discrete case** From now-on we will consider the situation where \( V \) is finite-dimensional and \( \mathcal{W}(q) = V \). In the setting of Example 2.1 we will identify \( V \) with the linear space spanned by the finite-elements \( \psi_i \) (e.g. piecewise linear or bi-linear tent functions on a fine mesh/grid in the setting of Example 2.2) used to discretize the operator \( L \), use \( I^{(q)} \) to label the elements \( \tilde{\psi}_i \) and set \( \psi^{(q)} = \tilde{\psi}^{(q)} \) for \( i \in I^{(q)} \). The gamblet-transform \[13, 14, 15\] is then summarized in Algorithm 1 and we have the decomposition

\[
V = \mathcal{W}^{(1)} + \mathcal{W}^{(2)} + \cdots + \mathcal{W}^{(q)}.
\]
Algorithm 2 Gamblet based Multigrid ($k$-th Level Iteration)

For $k = 1$, $\text{MG}(1, z_0, g^{(1)})$ is the solution obtained from a direct method. Namely

$$A^{(1)} \text{MG}(1, z_0, g^{(1)}) = g^{(1)}. \quad (2.28)$$

For $k > 1$, $\text{MG}(k, z_0, g^{(k)})$ is obtained recursively in three steps,

1. Presmoothing: For $1 \leq \ell \leq m_1$, let

   $$z_\ell = z_{\ell-1} + \frac{1}{\Lambda(k)}(g^{(k)} - A^{(k)} z_{\ell-1}). \quad (2.29)$$

2. Error Correction: Let $g^{(k-1)} := R^{(k-1,k)}(g^{(k)} - A^{(k)} z_0)$ and $q_0^{(k-1)} = 0$. For $1 \leq i \leq p$, let

   $$q_i^{(k-1)} = \text{MG}(k-1, q_i^{(k-1)}, g^{(k-1)}). \quad (2.30)$$

   Then $z_{m_1+1} := z_{m_1} + R^{(k,k-1)} q_p^{(k-1)}$.

3. Postsmoothing: For $m_1 + 2 \leq \ell \leq m_1 + m_2 + 1$, let

   $$z_\ell = z_{\ell-1} + \frac{1}{\Lambda(k)}(g^{(k)} - A^{(k)} z_{\ell-1}). \quad (2.31)$$

Then the output of the $k$-th level iteration is

$$\text{MG}(k, z_0, g^{(k)}) := z_{m_1+m_2+1}. \quad (2.32)$$

### 2.3 Gamblet based Multigrid Method

From now on we will consider the multiresolution decomposition (2.27) and the stiffness matrices $A^{(k)}$ and interpolation matrices $R^{(k-1,k)}$ computed in Algorithm 1 (or more precisely their numerical approximations, using the fast gamblet transform [13, 14, 17, 15], to a degree that is sufficient to obtain grid-size accuracy in the resolution of the discretization of (2.4). We will write $R^{(k,k-1)} := (R^{(k-1,k)})^T$ for the restriction matrix associated with the interpolation matrix $R^{(k-1,k)}$.

For $g^{(k)} \in \mathbb{R}^{T(k)}$ consider the linear system

$$A^{(k)} z = g^{(k)}. \quad (2.33)$$

Algorithm 2 provides an approximation $\text{MG}(k, z_0, g^{(k)})$ of the solution $z$ of (2.33) based on an initial guess $z_0$ and a number of iterations $k$. In that algorithm, $m_1$ and $m_2$ are nonnegative integers (and $p = 1$ or 2. $p = 1$ corresponds to a $V$-cycle method and $p = 2$ corresponds to a $W$-cycle method), $\Lambda^{(k)}$ is an upper bound for the spectral radius of $A^{(k)}$.

Under Condition 2.1 we take $\Lambda^{(k)} = CH^{-2k}$ where $C$ and $H$ are the constants appearing in the bound $A^{(k)} \leq CH^{-2k} I_T^{(k)}$.

Items 1 and 2 of Property 2.1 (i.e. the bounds on approximation errors and condition
numbers) imply the following results \[22\].

**Theorem 2.6** (Convergence of the \(k\)-th Level Iteration). Let \(m_1, m_2\) and \(k\) be the parameters and level number of Algorithm 2. For any \(0 < \theta < 1\), there exists \(m_1\) and \(m_2\) independent from \(k\) such that

\[
\|z - MG(k, z_0, g)\| \leq \theta \|z - z_0\|, \quad (2.34)
\]

**Proposition 2.7.** The work involved in the \(k\)-th Level Iteration defined by Algorithm 2 is \(O\left(N_k \ln N_k\right)\), where \(N_k := \dim(\mathfrak{U}^{(k)})\).

### 3 Gamblet Subspace Correction Method for Eigenvalue Problem

We will now describe the gamblet based multilevel correction method. Consider the setting of Section 2.1 and write \(\langle \cdot, \cdot \rangle_0\) for the scalar product associated with the norm \(\| \cdot \|_0\) placed on \(V_0\). Since \([\cdot, \cdot]\) is the dual product between \(V^*\) and \(V\) induced by the Gelfand triple \(V \subset V_0 \subset V^*\) we will also write \([u, v] := \langle u, v \rangle_0\) for \(u, v \in V_0\).

Consider the eigenvalue problem: Find \((\lambda, v) \in \mathbb{R} \times V\) such that

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\ell \leq \cdots, \quad \lim_{\ell \to \infty} \lambda_\ell = \infty,
\]

with associated eigenfunctions

\(v_1, v_2, \cdots, v_\ell, \cdots\),

where \([v_i, v_j] = \delta_{ij}\) (\(\delta_{ij}\) denotes the Kronecker function). In the sequence \(\{\lambda_j\}\), the \(\lambda_j\) are repeated according to their geometric multiplicity. For our analysis, we state the following definition for the smallest eigenvalue (cf. \[3, 5\])

\[
\lambda_1 = \min_{\theta \neq w \in V} \frac{\langle w, w \rangle}{[w, w]}.
\]

**Definition 3.2**. Define the subspace approximation problem for eigenvalue problem \(3.1\) on \(\mathfrak{U}^{(k)}\) as follows: Find \((\bar{\lambda}^{(k)}, \bar{v}^{(k)}) \in \mathbb{R} \times \mathfrak{U}^{(k)}\) such that \([\bar{v}^{(k)}, \bar{v}^{(k)}] = 1\) and

\[
\langle \bar{v}^{(k)}, w \rangle = \bar{\lambda}^{(k)}[\bar{v}^{(k)}, w], \quad \forall w \in \mathfrak{U}^{(k)}.
\]

Write \(N_k := \dim(\mathfrak{U}^{(k)})\). From \[2, 3, 5\], the discrete eigenvalue problem \(3.3\) has eigenvalues:

\[
0 < \lambda_1^{(k)} \leq \lambda_2^{(k)} \leq \cdots \leq \lambda_j^{(k)} \leq \cdots \leq \lambda_{N_k}^{(k)},
\]

and corresponding eigenfunctions

\(\bar{v}_1^{(k)}, \bar{v}_2^{(k)}, \cdots, \bar{v}_j^{(k)}, \cdots, \bar{v}_{N_k}^{(k)}\)
where $[\tilde{v}_i^{(k)}, \tilde{v}_j^{(k)}] = \delta_{i,j}, 1 \leq i, j \leq N_k$.

Define

$$\eta(\mathcal{V}^{(k)}) = \sup_{f \in \mathcal{V}_0 \|f\|_0 = 1} \inf_{w \in \mathcal{V}^{(k)}} \|\mathcal{L}^{-1}f - w\|. \quad (3.4)$$

Property 2.1 implies $\eta(\mathcal{V}^{(k)}) \leq CH^k$.

Let $M(\lambda_i)$ denote the eigenspace corresponding to the eigenvalue $\lambda_i$, namely,

$$M(\lambda_i) := \{v \in V \mid \langle v, w \rangle = \lambda_i[v, w], \forall w \in V\}, \quad (3.5)$$

and define

$$\delta_k(\lambda_i) = \sup_{v \in M(\lambda_i), \|v\|_0 = 1} \inf_{w \in \mathcal{V}^{(k)}} \|v - w\|. \quad (3.6)$$

**Proposition 3.1.** Property 2.1 implies

$$\eta(\mathcal{V}^{(k)}) \leq CH^k, \quad \delta_k(\lambda_i) \leq C\lambda_i H^k \text{ and } \delta_k(\lambda_i) \leq \lambda_i \eta(\mathcal{V}^{(k)}), \quad (3.7)$$

for $k \in \{1, \ldots, q\}$ where $C$ and $H$ are the constants appearing in Property 2.1.

For simplicity we will from now on restrict the presentation to the identification of simple eigenpairs $(\lambda,v)$ (the numerical method and results can naturally be extended to multiple eigenpairs). Let $E : V \to M(\lambda_i)$ be the spectral projection operator \cite{2} defined by

$$E = \frac{1}{2\pi i} \int_{\Gamma} (z - \mathcal{L})^{-1}dz, \quad (3.8)$$

where $\Gamma$ is a Jordan curve in $\mathbb{C}$ enclosing the desired eigenvalue $\lambda_i$ and no other eigenvalues.

The following lemma gives the error estimates for the Gamblet subspace approximation, which is a direct application of the subspace approximation theory for eigenvalue problems, see \cite[Lemma 3.6, Theorem 4.4]{2} and \cite{5}.

**Lemma 3.1.** The eigenpair approximations $(\bar{\lambda}_i^{(k)}, \bar{v}_i^{(k)})$ $(i = 1, 2, \ldots, N_k)$ satisfy

$$\|E\bar{v}_i^{(k)} - \bar{v}_i^{(k)}\| \leq (1 + C_i \eta(\mathcal{W}^{(k)})) \delta_k(\lambda_i), \quad (3.9)$$

$$\|E\bar{v}_i^{(k)} - \bar{v}_i^{(k)}\|_0 \leq C_i \eta(\mathcal{W}^{(k)}) \|E\bar{v}_i^{(k)} - \bar{v}_i^{(k)}\|, \quad (3.10)$$

$$|\lambda_i - \bar{\lambda}_i^{(k)}| \leq C_i \|E\bar{v}_i^{(k)} - \bar{v}_i^{(k)}\|^2. \quad (3.11)$$

for some constants $C_i$ depending only on $\lambda_i$.

**Remark 3.1.** Proposition 3.1 and Lemma 3.1 imply

$$|\lambda_i - \bar{\lambda}_i^{(k)}| + \|E\bar{v}_i^{(k)} - \bar{v}_i^{(k)}\|_0 \leq (1 + \lambda_i) C_i \eta(\mathcal{W}^{(k)}) \|E\bar{v}_i^{(k)} - \bar{v}_i^{(k)}\|. \quad (3.12)$$
3.1 One Correction Step

To describe the multilevel correction method we first present a “one step correction”. Given an eigenpair approximation \((\lambda^{(k,\ell)}, v^{(k,\ell)}) \in \mathbb{R} \times \mathcal{V}^{(k)}\) Algorithm 3 produces an improved eigenpair approximation \((\lambda^{(k,\ell+1)}, v^{(k,\ell+1)}) \in \mathbb{R} \times \mathcal{V}^{(k)}\). In this algorithm, the superscript \((k, \ell)\) denotes the \(\ell\)-th correction step in the \(k\)-th level Gamblet space.

**Remark 3.2.** The linear system (3.22) is solved by using Algorithm 2 whose convergence is obtained in [22] based on Property 2.1.

**Theorem 3.1.** Assume that the eigenpair approximation \((\lambda^{(k,\ell)}, v^{(k,\ell)})\) satisfies \(\|v^{(k,\ell)}\| = 1\) and

\[
|\tilde{\lambda}^{(k)} - \lambda^{(k,\ell)}| + \|\tilde{v}^{(k)} - v^{(k,\ell)}\|_0 \leq C\eta(\mathcal{V}^{(1)}) \|\tilde{v}^{(k)} - v^{(k,\ell)}\|
\]

for some constant \(C\) depending on the desired eigenpair \((\lambda, v)\) but independent of the level \(k\) of the Gamblet space, and the multigrid iteration for the linear equation (3.22) has the following uniform contraction rate

\[
\|\tilde{\gamma}^{(k,\ell+1)} - \tilde{\gamma}^{(k,\ell)}\| \leq \theta\|\tilde{v}^{(k,\ell)} - \tilde{\gamma}^{(k,\ell)}\|
\]

with \(\theta < 1\) independent from \(k\) and \(\ell\). Then the eigenpair approximation \((\lambda^{(k,\ell+1)}, v^{(k,\ell+1)}) \in \mathbb{R} \times \mathcal{V}^{(k)}\) produced by Algorithm 3 satisfies

\[
|\tilde{\lambda}^{(k)} - \lambda^{(k,\ell+1)}| + \|\tilde{v}^{(k)} - v^{(k,\ell+1)}\|_0 \leq C\eta(\mathcal{V}^{(1)}) \|\tilde{v}^{(k)} - v^{(k,\ell+1)}\|,
\]

where the constant \(C\) is the same as in (3.13) and \(\gamma\) is defined as follows

\[
\gamma = \left(\begin{array}{c}
1 + C\eta(\mathcal{V}^{(1)})
\end{array}\right) \left(\begin{array}{c}
\theta + (1 + \theta) \tilde{\lambda}^{(k)} + \frac{1}{\lambda_1} C\eta(\mathcal{V}^{(1)})
\end{array}\right).
\]

**Proof.** From (3.2), (3.3) and (3.22), we have for \(w \in \mathcal{V}^{(k)}\)

\[
\langle \tilde{v}^{(k)} - \tilde{\gamma}^{(k,\ell+1)}, w \rangle = \left\langle \tilde{\lambda}^{(k)} w - \lambda^{(k,\ell)} v^{(k,\ell)}, w \right\rangle
\]

\[
\leq \lambda^{(k)} \|v^{(k,\ell)}\| \|w\|_0 + |\tilde{\lambda}^{(k)} - \lambda^{(k,\ell)}| \|v^{(k,\ell)}\|_0 \|w\|_0
\]

\[
\leq (\lambda^{(k)} + \|v^{(k,\ell)}\|_0) (|\tilde{\lambda}^{(k)} - \lambda^{(k,\ell)}| + \|\tilde{v}^{(k)} - v^{(k,\ell)}\|_0) \|w\|_0
\]

\[
\leq \frac{\lambda^{(k)} + 1}{\lambda_1} \left(\begin{array}{c}
|\tilde{\lambda}^{(k)} - \lambda^{(k,\ell)}| + \|\tilde{v}^{(k)} - v^{(k,\ell)}\|_0\|w\|
\end{array}\right)
\]

Taking \(w = \tilde{v}^{(k)} - \tilde{\gamma}^{(k,\ell+1)}\) we deduce from (3.13) that

\[
\|\tilde{v}^{(k)} - \tilde{\gamma}^{(k,\ell+1)}\| \leq \frac{\lambda^{(k)} + 1}{\lambda_1} C\eta(\mathcal{V}^{(1)}) \|\tilde{v}^{(k)} - v^{(k,\ell)}\|.
\]

Using (3.14) and (3.18) we deduce that

\[
\|\tilde{v}^{(k)} - \tilde{v}^{(k,\ell+1)}\| \leq \|\tilde{v}^{(k)} - \tilde{v}^{(k,\ell+1)}\| + \|\tilde{v}^{(k,\ell+1)} - \tilde{v}^{(k,\ell+1)}\|
\]

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\[
\begin{align*}
&\leq \|v^{(k)} - \bar{v}^{(k,\ell+1)}\| + \theta \|\bar{v}^{(k,\ell+1)} - v^{(k,\ell)}\| \\
&\leq \|v^{(k)} - \bar{v}^{(k,\ell+1)}\| + \theta \|\bar{v}^{(k,\ell+1)} - \bar{v}^{(k)}\| + \theta \|\bar{v}^{(k)} - v^{(k,\ell)}\| \\
&\leq (1 + \theta)\|\bar{v}^{(k)} - \bar{v}^{(k,\ell+1)}\| + \theta \|\bar{v}^{(k)} - v^{(k,\ell)}\| \\
&\leq \left(\theta + (1 + \theta) \frac{\tilde{\lambda}^{(k)}}{\lambda_1} + C\eta(\mathfrak{U}(1))\right)\|\bar{v}^{(k)} - v^{(k,\ell)}\|.
\end{align*}
\] (3.19)

The eigenvalue problem (3.23) can be seen as a low dimensional subspace approximation of the eigenvalue problem (3.3). Using Lemma 3.1, Remark 3.1 and the subspace approximation theory for eigenvalue problems we obtain that
\[
\|\bar{v}^{(k)} - v^{(k,\ell+1)}\| \leq \left(1 + C\eta(\mathfrak{U}(1,k))\right) \inf_{w^{(1,k)} \in \mathfrak{U}(1,k)} \|\bar{v}^{(k)} - w^{(1,k)}\| \\
\leq (1 + C\eta(\mathfrak{U}(1)))\|\bar{v}^{(k)} - \bar{v}^{(k,\ell+1)}\| \\
\leq \gamma \|\bar{v}^{(k)} - v^{(k,\ell)}\|,
\] (3.20)

and
\[
\|\tilde{\lambda}^{(k)} - \lambda^{(k,\ell+1)}\| + \|v^{(k)} - v^{(k,\ell+1)}\|_0 \leq C\eta(\mathfrak{U}(1,k))\|\bar{v}^{(k)} - v^{(k,\ell+1)}\| \\
\leq C\eta(\mathfrak{U}(1))\|\bar{v}^{(k)} - v^{(k,\ell+1)}\|.
\] (3.21)

Then we obtained the desired results (3.15) and (3.16) and the proof is complete. \(\square\)

Remark 3.2. For example, from Remark 3.1, we can set the constant \(C = (1 + \lambda_1)C_1\) in (3.13) for the \(i\)-th eigenpair. From the definition (3.17), Theorem 2.6 and Lemma 3.1, it is easy to know that \(\gamma\) can be smaller than 1 when \(\eta(\mathfrak{U}(1))\) is small enough. Furthermore, we can increase the multigrid smoothing steps \(m_1\) and \(m_2\) to reduce \(\theta\) and then \(\gamma\).

3.2 Multilevel Method for Eigenvalue Problem

In this subsection, we introduce the multilevel method based on the subspace correction method defined in Algorithm 4 and the properties of Gamblet spaces. This type of multilevel method can obtain the same order of accuracy as the direct solve of the eigenvalue problem on the finest level Gamblet space. The multilevel method is defined by Algorithm 4.

Theorem 3.2. Assume the conditions of Theorem 3.1 hold. After implementing Algorithm 4, the resulting eigenpair approximation \((\lambda^{(q)}, v^{(q)})\) has the following error estimates
\[
\|v^{(q)} - v^{(q)}\| \leq 2 \sum_{k=1}^{q-1} \gamma^{(q-k)\varpi} \delta_k(\lambda),
\] (3.24)
\[
|\tilde{\lambda}^{(q)} - \lambda^{(q)}| + \|v^{(q)} - v^{(q)}\|_0 \leq C\eta(\mathfrak{U}(1)) \sum_{k=1}^{q-1} \gamma^{(q-k)\varpi} \delta_k(\lambda),
\] (3.25)

where \(\varpi\) is the number of subspace correction steps in Algorithm 4 and \(C\) is the same constant as in (3.13) of Theorems 3.1.
Algorithm 3 One Correction Step

1. Let \( \tilde{v}^{(k, \ell+1)} \in \mathfrak{V}(k) \) be the solution of the linear system

\[
\langle \tilde{v}^{(k, \ell+1)}, w \rangle = \lambda^{(k, \ell)} [v^{(k, \ell)}, w], \quad \forall w \in \mathfrak{V}(k).
\]

Approximate \( \tilde{v}^{(k, \ell+1)} \) by \( \hat{v}^{(k, \ell+1)} \) obtained from \( m \) multigrid iterations in \( \mathfrak{V}(k) \) with initial guess \( v^{(k, \ell)} \).

2. With the coarsest Gamblet space \( \mathfrak{V}(1) \), define

\[
\mathfrak{V}(1, k) = \mathfrak{V}(1) + \text{span}\{ \hat{v}^{(k, \ell+1)} \}
\]

and solve the eigenvalue problem: Find \((\lambda^{(k, \ell+1)}, v^{(k, \ell+1)}) \in \mathbb{R} \times \mathfrak{V}(1, k)\) such that

\[
\langle v^{(k, \ell+1)}, w \rangle = \lambda^{(k, \ell+1)} [v^{(k, \ell+1)}, w], \quad \forall w \in \mathfrak{V}(1, k).
\]

Let \( \text{EigenMG} \) be the function summarizing the action of the steps described above, i.e.

\[
(\lambda^{(k, \ell+1)}, v^{(k, \ell+1)}) = \text{EigenMG} (\mathfrak{V}(1), \lambda^{(k, \ell)}, v^{(k, \ell)}, \mathfrak{V}(k)).
\]

**Proof.** Define \( e_k := \bar{v}^{(k)} - v^{(k)} \). From step 1 in Algorithm 4 it is obvious \( e_1 = 0 \). Then the assumption (3.13) in Theorem 3.1 is satisfied for \( k = 1 \). From Theorem 3.1 and recursive argument, the convergence rate (3.15) is valid for all \( k = 1, \ldots , q \) and \( \ell = 0, \ldots , \varpi - 1 \).

For \( k = 2, \ldots , q \), from Lemma 3.1 and Theorem 3.1 we have

\[
\| e_k \| \leq \gamma^{\varpi} \| \bar{v}^{(k)} - v^{(k-1)} \| \\
\leq \gamma^{\varpi} (\| \bar{v}^{(k)} - v^{(k-1)} \| + \| v^{(k-1)} - \bar{v}^{(k-1)} \|) \\
\leq \gamma^{\varpi} (\| v^{(k)} - v \| + \| v^{(k-1)} - \bar{v}^{(k-1)} \| + \| \bar{v}^{(k-1)} - v^{(k-1)} \|) \\
\leq \gamma^{\varpi} (2\delta_{k-1}(\lambda) + \| e_{k-1} \|).
\]

(3.26)

By iterating inequality (3.26), the following inequalities hold

\[
\| e_q \| \leq 2(\gamma^{\varpi} \delta_{q-1}(\lambda) + \cdots + \gamma^{(q-1)\varpi} \delta_1(\lambda)) \leq 2 \sum_{k=1}^{q-1} \gamma^{(q-k)\varpi} \delta_k(\lambda).
\]

(3.27)

For such choice of \( \varpi \), we arrive at the desired result (3.24) and (3.25) can be obtained by Lemma 3.1 and 3.24).

**Corollary 3.3.** Let \( \gamma \) be as in (3.17). Given the uniform contraction rate \( 0 < \theta < 1 \) (obtained from Theorem 2.6) and given the bound \( \eta(\mathfrak{V}(1)) \leq CH \) (obtained from Property 2.7, which is implied by Theorem 2.4) select \( 0 < H < 1 \) small enough so that \( 0 < \gamma < 1 \) and then choose the integer \( \varpi > 1 \) to satisfy

\[
\frac{\gamma^{\varpi}}{H} < 1.
\]

(3.28)
Algorithm 4 Multilevel Scheme

1. Define the following eigenvalue problem in $\mathcal{V}(1)$: Find $(\lambda^{(1)}, v^{(1)}) \in \mathbb{R} \times \mathcal{V}(1)$ such that $[v^{(1)}, v^{(1)}] = 1$ and 
   $$\langle v^{(1)}, v^{(1)} \rangle = \lambda^{(1)} [v^{(1)}, v^{(1)}], \quad \forall v^{(1)} \in \mathcal{V}(1).$$
   Solve this eigenvalue problem to get the initial eigenpair approximation $(\lambda^{(1)}, v^{(1)}) \in \mathbb{R} \times \mathcal{V}(1)$.

2. For $k = 2, \ldots, q$, do the following iterations
   - Set $\lambda^{(k,0)} = \lambda^{(k-1)}$ and $v^{(k,0)} = v^{(k-1)}$.
   - Perform the following subspace correction steps for $\ell = 0, \ldots, \varpi - 1$:
     $$\begin{align*}
     (\lambda^{(k,\ell+1)}, v^{(k,\ell+1)}) &= \text{EigenMG}(\mathcal{V}(1), \lambda^{(k,\ell)}, v^{(k,\ell)}, \mathcal{V}(k)).
     \end{align*}$$
   - Set $\lambda^{(k)} = \lambda^{(k,\varpi)}$ and $v^{(k)} = v^{(k,\varpi)}$.

   End Do

Finally, we obtain an eigenpair approximation $(\lambda^{(q)}, v^{(q)}) \in \mathbb{R} \times \mathcal{V}(q)$ in the finest Gamblet space.

Then the resulting eigenpair approximation $(\lambda^{(q)}, v^{(q)})$ obtained by Algorithm 4 has following error estimates

$$\| E v^{(q)} - v^{(q)} \| \leq C H^q, \quad |\lambda - \lambda^{(q)}| \leq C H^{2q}, \quad \| E v^{(q)} - v^{(q)} \|_0 \leq C H^{q+1}, \quad (3.29)$$

where the constant $C$ depends on $C_i$ ($i$ is the index of the eigenpair) and $1/(1 - \frac{\gamma}{\varpi H})$.

Proof. From Lemma 3.1, Theorem 3.2, (3.28) and optimal error property of orthogonal projection, we have following estimates

$$\begin{align*}
\| E v^{(q)} - v^{(q)} \| &\leq \| E \bar{v}^{(q)} - v^{(q)} \| \leq \| E \bar{v}^{(q)} - \bar{v}^{(q)} \| + \| \bar{v}^{(q)} - v^{(q)} \| \\
&\leq C \delta_q(\lambda) + C \sum_{k=1}^{q-1} \gamma^{(q-k)\varpi} \delta_k(\lambda) \leq C H^q + C \sum_{k=1}^{q-1} \gamma^{(q-k)\varpi} H^k \\
&\leq C H^q \sum_{k=0}^{q-1} \left( \frac{\gamma}{H} \right)^k \leq C H^q \frac{1 - \left( \frac{\gamma}{H} \right)^q}{1 - \frac{\gamma}{H}} \leq C H^q. \quad (3.30)
\end{align*}$$

Then from (3.10), (3.11), (3.25) and (3.30), the desired result (3.29) can be obtained and the proof is complete.

Remark 3.3. From Algorithm 3, we can find that the main computational work is to solve the linear equation (3.22) by the multigrid method defined in Algorithm 2. Based on this understanding and Proposition 2.7, the computational work can be bounded by $O(N \text{polylog}(N))$ (cf. [13, 14, 16, 22]).

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4 Numerical Results

In this section, numerical examples are presented to illustrate the efficiency of the multi-level method proposed in this paper. In both examples, Gamblet based multilevel method achieves high accuracy with a few iterations, while geometric multigrid still has $O(1)$ errors.

4.1 SPE10

In the first example, we solve the eigenvalue problem \((2.8)\), where $\Omega = [-1, 1] \times [-1, 1]$ and the coefficient matrix $a(x)$ is taken from the data of the SPE10 benchmark (http://www.spe.org/web/csp/) which is shown in Figure 1. Here, we calculate the first 6 eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_6$. In order to use the multilevel method defined by Algorithm 4, we choose $H = 2/4$, $q = 6$. The multigrid method iteration step defined in Algorithm 2 with $m_1 = m_2 = 2$ and $p = 1$ is adopted to solve the linear equation \((3.22)\).

Figure 1: coefficient $a(x)$ from SPE10 benchmark, in log$_{10}$ scale

The numerical results for eigenvalue approximations are shown in Table 1 where the high accuracy is obtained by the multilevel method. For comparison, we also state the corresponding numerical results in Table 2 by the standard geometric multigrid which only has $O(1)$ accuracy.

Table 1: Relative errors for gamblet based multigrid

| level | $\lambda_1^{(k)} - \lambda_1$ | $\lambda_2^{(k)} - \lambda_2$ | $\lambda_3^{(k)} - \lambda_3$ | $\lambda_4^{(k)} - \lambda_4$ | $\lambda_5^{(k)} - \lambda_5$ | $\lambda_6^{(k)} - \lambda_6$ |
|-------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| k=2   | 1.3299e-2                     | 3.0490e-2                     | 1.3287e-1                     | 1.9856e-1                     | 1.7584e-1                     | 3.2068e-1                     |
| k=3   | 2.9030e-3                     | 4.3332e-3                     | 1.4580e-2                     | 2.6290e-2                     | 1.4929e-2                     | 4.5882e-2                     |
| k=4   | 1.2450e-3                     | 1.0300e-3                     | 1.7021e-3                     | 4.2904e-3                     | 2.1925e-3                     | 5.5244e-3                     |
| k=5   | 3.8248e-4                     | 2.4465e-4                     | 2.1443e-4                     | 4.5721e-4                     | 3.1176e-4                     | 8.8591e-4                     |
| k=6   | 4.8006e-10                    | 4.7992e-10                    | 2.5963e-7                     | 1.8019e-5                     | 1.1164e-6                     | 9.8976e-5                     |
Table 2: Relative errors for geometric multigrid

| level | $\lambda_1^{(k)} - \lambda_1$ | $\lambda_2^{(k)} - \lambda_2$ | $\lambda_3^{(k)} - \lambda_3$ | $\lambda_4^{(k)} - \lambda_4$ | $\lambda_5^{(k)} - \lambda_5$ | $\lambda_6^{(k)} - \lambda_6$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| k=2   | 5.5962e-1       | 7.6254e-1       | 1.4811e0        | 1.5747e0        | 2.2069e0        | 2.7635e0        |
| k=3   | 3.7138e-1       | 5.8835e-1       | 1.1421e0        | 1.1555e0        | 1.4946e0        | 1.9345e0        |
| k=4   | 3.6455e-1       | 5.3818e-1       | 8.6044e-1       | 1.0038e0        | 1.4564e0        | 1.5450e0        |
| k=5   | 3.6204e-1       | 5.0609e-1       | 7.0450e-1       | 9.8467e-1       | 1.4357e0        | 1.4290e0        |
| k=6   | 3.5602e-1       | 5.0429e-1       | 6.9040e-1       | 9.7660e-1       | 1.4217e0        | 1.4122e0        |

4.2 Random Checkerboard

In the second example, we also solve the eigenvalue problem (2.8), where $\Omega = [0, 1] \times [0, 1]$ and the matrix $a(x)$ is a realization of random coefficients taking values 10 or $1/10$ with probability 1/2 at small scale $\varepsilon = 1/64$, see Figure 2. Here, we also calculate the first 6 eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_6$. The parameters for Algorithm 4 are $H = 1/2$, $q = 6$, and for Algorithm 2 are $m_1 = m_2 = 2$ and $p = 1$.

Figure 2: Random Checkerboard coefficient, in log_{10} scale

The corresponding numerical results for eigenvalue approximations are shown in Table 3 where the high accuracy is also obtained by the multilevel method. For comparison, Table 4 gives the corresponding numerical results by the standard geometric multigrid which also only has $O(1)$ accuracy.

Table 3: Relative errors using Gamblet based multigrid

| level | $\lambda_1^{(k)} - \lambda_1$ | $\lambda_2^{(k)} - \lambda_2$ | $\lambda_3^{(k)} - \lambda_3$ | $\lambda_4^{(k)} - \lambda_4$ | $\lambda_5^{(k)} - \lambda_5$ | $\lambda_6^{(k)} - \lambda_6$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| k=2   | 7.1092e-3       | 4.9591e-3       | 4.5251e-3       | 1.0614e-2       | 1.2118e-2       | 1.4976e-2       |
| k=3   | 6.4277e-3       | 2.3379e-3       | 2.0046e-3       | 2.4096e-2       | 2.7884e-2       | 2.9295e-2       |
| k=4   | 4.6439e-3       | 1.6492e-3       | 1.2202e-3       | 1.0138e-3       | 1.1059e-3       | 9.3171e-3       |
| k=5   | 2.7618e-3       | 9.1602e-4       | 6.3657e-3       | 4.8176e-4       | 4.7756e-4       | 3.9240e-4       |
| k=6   | 8.0415e-11      | 1.057e-10       | 5.9263e-11      | 3.9927e-10      | 5.545e-10       | 1.3302e-8       |
Table 4: Relative errors using geometric multigrid

| k  | 2.6698e0 | 2.3886e1 | 2.2749e0 | 2.6224e0 | 2.9668e0 | 2.5762e0 |
|----|----------|----------|----------|----------|----------|----------|
| k=2| 2.5627e0 | 2.3037e1 | 2.1802e0 | 2.5191e0 | 2.8135e0 | 2.4229e0 |
| k=3| 2.0945e0 | 1.8806e1 | 1.8073e0 | 2.0624e0 | 2.2931e0 | 1.9413e0 |
| k=4| 3.8291e-1| 4.0754e-1| 4.2985e-1| 4.9242e-1| 5.0825e-1| 4.4099e-1|
| k=5| 3.4186e-3| 3.9342e-3| 2.5185e-3| 1.0877e-2| 9.7022e-3| 6.5403e-3|

5 Concluding Remarks

In this paper, we propose a Gamblet based multilevel method to solve multiscale eigenvalue problems. The idea here is to use the Gamblet based subspace correction method to transform the solution of the eigenvalue problem to a series of solutions of the underlying boundary value problems, which can be solved by some Gamblet based multigrid iteration steps, and solutions of eigenvalue problems defined on a very low dimensional Gamblet space.

Although the local linear elliptic operators in Example 2.1 have been used as prototypical examples, the proposed theory and algorithms have been presented in the abstract setting of linear operators on Hilbert spaces to emphasize the generality of the proposed method (which could also be applied to Graph Laplacians with well behaved gamlets).

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