GRAVITATING GLOBAL MONOPOLES IN EXTRA DIMENSIONS
AND THE BRANE WORLD CONCEPT

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Multidimensional configurations with Minkowski external space-time and a spherically symmetric
global monopole in extra dimensions are discussed in the context of the brane world concept. The
monopole is formed with a hedgehog-like set of scalar fields $\phi^i$ with a symmetry-breaking potential
$V$ depending on the magnitude $\phi^2 = \phi^i \phi^i$. All possible kinds of globally regular configurations are
singled out without specifying the shape of $V(\phi)$. These variants are governed by the maximum
value $\phi_m$ of the scalar field, characterizing the energy scale of symmetry breaking. If $\phi_m < \phi_{cr}$
(where $\phi_{cr}$ is a critical value of $\phi$ related to the multidimensional Planck scale), the monopole
reaches infinite radii while in the “strong field regime”, when $\phi_m \geq \phi_{cr}$, the monopole may end
with a cylinder of finite radius or possess two regular centers. The warp factors of monopoles with
both infinite and finite radii may either exponentially grow or tend to finite constant values far
from the center. All such configurations are shown to be able to trap test scalar matter, in striking
contrast to RS2 type five-dimensional models. The monopole structures obtained analytically are
also found numerically for the Mexican hat potential with an additional parameter acting as a
cosmological constant.

1. Introduction

According to a presently popular idea, our observable Universe can be located on a four-dimensional
surface, called the brane, embedded in a higher-dimensional manifold, called the bulk. This “brane
world” concept, suggested in the 80s [1], is broadly discussed nowadays, mainly in connection with
the recent developments in supersymmetric string/M-theories [2]. A reason why we do not see any
extra dimensions is that the observed matter is confined to the brane, and only gravity propagates
in the whole bulk. There are numerous applications of the brane world concept to particle physics,
astrophysics and cosmology, such as the hierarchy problem and the description of dark matter and
dark energy [3].

Most of the studies are restricted to infinitely thin branes with delta-like localization of matter. A
well-known example is Randall and Sundrum’s second model (RS2) [4] in which a single Minkowski
brane is embedded in a 5D anti-de Sitter (AdS) bulk.

Thin branes can, however, be only treated as a rough approximation since any fundamental
underlying theory, be it quantum gravity, string or M-theory, must contain a fundamental length
beyond which a classical space-time description is impossible. It is therefore necessary to justify
the infinitely thin brane approximation as a well-defined limit of a smooth structure, a thick brane,
able to be coupled to coupled gravitational and matter field equations. Such a configuration
is then required to be globally regular, stable and properly concentrated around a 3D surface
which is meant to describe the observed spatial dimensions. Topological defects, emerging in phase
transitions with spontaneous symmetry breaking (SSB), are probably the best candidates for this role.

It should be mentioned that the evolution of the Universe, according to modern views, contained
a sequence of phase transitions with SSB. A decisive step toward cosmological applications of the
SSB concept was made in 1972 by Kirzhnits [5]. He assumed that, as in the case of solid substances,
a symmetry of a field system, existing at sufficiently high temperatures, could be spontaneously
broken as the temperature falls down. A necessary consequence of such phase transitions is the
appearance of topological defects. The first quantitative analysis of the cosmological consequences
of SSB was given by Zel’dovich, Kobzarev and Okun’ [6]. Later on, the SSB phenomenon and
various topological defects were widely used in inflationary Universe models and in attempts to
explain the origin of the large-scale structure of the Universe, see, e.g., [7, 8].

The properties of global topological defects are generally described with the aid of a multiplet of
scalar fields playing the role of an order parameter. If a defect is to be interpreted as a brane world,
its structure is determined by the self-gravity of the scalar field system and may be described by a
set of Einstein and scalar equations.

In this paper we analyze the gravitational properties of candidate (thick) brane worlds with 4D
Minkowski metric as global topological defects in extra dimensions. Our general formulation covers
such particular cases as a brane (domain wall) in 5D space-time (one extra dimension), a global
cosmic string with winding number \( n = 1 \) (two extra dimensions), and global monopoles (three
or more extra dimensions). We restrict ourselves to Minkowski branes since most of the existing
problems are clearly seen even in these comparatively simple systems; on the other hand, in the
majority of physical situations, the inner curvature of the brane itself is much smaller than the
curvature related to brane formation, therefore the main qualitative features of Minkowski branes
should survive in curved branes.

Brane worlds as thick domain walls in a 5D bulk have been discussed in many papers (see,
e.g., [9] and references therein). Such systems were analyzed in a general form in Refs. [10, 11],
without specifying the symmetry-breaking potential; it was shown, in particular, that all regular
configurations should have an AdS asymptotic. So all possible thick branes are merely regularized
versions of the RS2 model, with all concomitant difficulties in material field confinement. Thus,
it has been demonstrated [11] that a test scalar field has a divergent stress-energy tensor (SET)
infinite far from the brane, at the AdS horizon. A reason for that is the repulsive gravity of RS2
and similar models: gravity repels matter from the brane and pushes it towards the AdS horizon.
To overcome this difficulty, it is natural to try a greater number of extra dimensions. This was one
of the reasons for us to consider higher-dimensional bulks.

We study the simplest possible realization of this idea, assuming a static, spherically symmetric
configuration of the extra dimensions and a thick Minkowski brane as a concentration of the scalar
field SET near the center. The possible trapping properties of gravity for test matter are then
determined by the behavior of the so-called warp factor (the metric coefficient acting as a gravita-
tional potential) far from the center, and we indeed find classes of regular solutions where gravity
is attracting.

Some of our results repeat those obtained in Refs. [12, 13] which have discussed global and gauge
(‘t Hooft-Polyakov type) monopoles in extra dimensions; see a more detailed comparison in Sec. 7.
The paper is organized as follows. In Sec. 2 we formulate the problem, introduce space-times with global topological defects in the extra dimensions, write down the equations and boundary conditions and demonstrate a connection between the possibility of SSB and the properties of the potential at a regular center.

In Sec. 3 we briefly discuss the trapping problem for RS2 type domain-wall models and show that they always possess repulsive gravity and are unable to trap matter in the form of a test scalar field.

Sec. 4 is devoted to a search for regular global monopole solutions in higher dimensions by analyzing their asymptotic properties far from the center. All regular configurations are classified by the behavior of the spherical radius $r$ and by the properties of the potential. This leads to separation of “weak gravity” and “strong gravity” regimes, related to maximum values of the scalar field magnitude.

In the weak gravity regime, the spherical radius $r$ tends to infinity along with the distance from the center. Such moderately curved configurations exist without any restrictions of fine-tuning type.

If the scalar field magnitude exceeds some critical value, the radius $r$ either tends to a finite value far from the center or returns to zero at a finite distance from the center, thus forming one more center which should also be regular. Some cases require fine tuning of the parameters of the potential, so one may believe that static configurations can only exist if the scalar and gravitational forces are somewhat mutually balanced.

In Sec. 5 we show that, in contrast to domain walls, global monopoles in different regimes do provide scalar field trapping on the brane.

Sec. 6 is a brief description of numerical experiments with the Mexican hat potential admitting shifts up and down, equivalent to introducing a bulk cosmological constant. Their results confirm and illustrate the conclusions of Sec. 4.

Sec. 7 summarizes the results.

2. Problem setting

2.1. Geometry

We consider a $(D = d_0 + d_1 + 1)$-dimensional space-time with the structure $\mathbb{M}^{d_0} \times \mathbb{R}_u \times \mathbb{S}^{d_1}$ and the metric

$$ds^2 = e^{2\gamma(u)} \eta_{\mu\nu} dx^\mu dx^\nu - \left( e^{2\alpha(u)} du^2 + e^{2\beta(u)} d\Omega^2 \right).$$

Here $\eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - (d\vec{x})^2$ is the Minkowski metric in the subspace $\mathbb{M}^{d_0}$, $\eta_{\mu\nu} = \text{diag}(1, -1, ..., -1)$; $d\Omega$ is a linear element on a $d_1$-dimensional unit sphere $\mathbb{S}^{d_1}$; $\alpha$, $\beta$ and $\gamma$ are functions of the radial coordinate $u$ with the definition domain $\mathbb{R}_u \subseteq \mathbb{R}$, to be specified later. The Riemann tensor has a diagonal form, and its nonzero components are

$$R^\rho_{\mu\sigma} = -e^{-2\alpha} \gamma^2 \delta^\rho_{\mu\sigma},$$

$$R^{ab}_{\ cd} = \left( e^{-2\beta} - e^{-2\alpha} \beta^2 \right) \delta^{ab}_{\ cd},$$

$$R^\mu_{\ ab\nu} = -\delta^\mu_b e^{-\gamma - \alpha} \left( e^{\gamma - \alpha} \gamma' \right),$$

$$R^{aa}_{\ ab} = -\delta^a_b e^{-\beta - \alpha} \left( e^{\beta - \alpha} \beta' \right),$$

$$R^{\mu\nu}_{\ a\ b} = -\delta^\mu_b \delta^\nu_a e^{-2\alpha} \gamma' \beta'. \quad (2)$$

Here

$$\delta^\mu_{\ \rho\sigma} = \delta^\mu_\rho \delta_\sigma - \delta^\mu_\sigma \delta_\rho, \quad (3)$$
and similarly for $\delta^{ab}_{cd}$. Greek indices $\mu, \nu, \ldots$ correspond to $d_0$-dimensional space-time, and Latin indices $a, b, \ldots$ to $d_1$ angular coordinates on $S^{d_1}$. We mostly bear in mind the usual dimension $d_0 = 4$ but keep $d_0$ arbitrary for generality.

A necessary condition of regularity is finiteness of the Kretschmann scalar $K = R^{AB}_C R^{CD}_A$. (Capital indices $A, B, \ldots$ correspond to all $D$ coordinates.) In our case, $K$ is a sum of squares of all nonzero $R^{AB}_{CD}$. Hence, in regular configurations all components of the Riemann tensor [2] are finite.

For the Ricci tensor we have

$$R^\mu_\nu = -\delta^\mu_\nu e^{-2\alpha} \left[ \gamma'' + \gamma'(-\alpha' + d_0 \gamma' + d_1 \beta') \right],$$

$$R^a_u = -e^{-2\alpha} \left[ d_0 (\gamma'' + \gamma'^2 + \alpha' \gamma') + d_1 (\beta'' + \beta'^2 - \alpha' \beta') \right],$$

$$R^m_m = e^{-2\beta} (d_1 - 1) \delta^m_m - \delta^m_m e^{-2\alpha} \left[ \beta'' + \beta'(-\alpha' + d_0 \gamma' + d_1 \beta') \right].$$

(4)

2.2. Topological defects

A global defect with a nonzero topological charge can be constructed as a multiplet of $d_1 + 1$ real scalar fields $\phi^k$, in the same way as, e.g., in [14]. It comprises a hedgehog configuration in $\mathbb{R}^n \times S^{d_1}$:

$$\phi^k = \phi(u) n^k(x^a),$$

$n^k$ is a unit vector in the $d_1 + 1$-dimensional Euclidean target space of the scalar fields: $n^k n^k = 1$.

The total Lagrangian of the system is taken in the form

$$L = \frac{R}{2 \kappa^2} + \frac{1}{2} g^{AB} \partial_A \phi^k \partial_B \phi^k - V(\phi),$$

(5)

where $R$ is the $D$-dimensional scalar curvature, $\kappa^2$ is the $D$-dimensional gravitational constant, and $V$ is a symmetry-breaking potential depending on $\phi^2(u) = \phi^a \phi_a$.

In case $d_1 = 0$ there is only one extra dimension. The topological defect is a flat domain wall. Combined with $d_0 = 4$, it is widely considered with reference to our Universe. Regular thick Minkowski branes supported by scalar fields with arbitrary potentials were analyzed in [10, 11], see also Sec. 3 of the present paper.

The case $d_1 = 1$ is a global cosmic string with the winding number $n = 1$. If $d_0 = 2$, it is a cosmic string in four dimensions, whose gravitational properties are reviewed in [15]. The case $d_0 = 4$ corresponds to a string in extra dimensions.

$d_1 = 2$, $d_0 = 1$ is the case of a global monopole in our 4-dimensional space-time. We have analyzed it in detail in [16]. The case $d_1 > 2$, $d_0 = 1$ is its multidimensional generalization to static, spherically symmetric space-times with $d_1$-dimensional rather than 2-dimensional coordinate spheres [14]. It was shown that such a heavy multidimensional global monopole leads to a multidimensional cosmology where the symmetry-breaking potential at late times can mimic both dark matter and dark energy.

In case $d_0 = 4$, $d_1 > 2$ we have a multidimensional global monopole entirely in the extra space-like dimensions. Different models of this kind were studied in Refs. [12, 13, 18, 17]. In particular, such a monopole in extra dimensions was used in an attempt to explain the origin of inflation [17].

2.3. Field equations

We are using the Einstein equations in the form

$$R^B_A = -\kappa^2 \tilde{T}^B_A, \quad \tilde{T}^B_A = T^B_A - \frac{\delta^B_A}{D-2} T^C_C,$$
where \( T^B_A \) is the stress-energy tensor of the scalar field multiplet. For our hedgehog configuration,
\[
\begin{align*}
\tilde{T}^\nu_\mu &= -2V \delta^\nu_\mu/(D-2), \\
\tilde{T}^u_\nu &= -2V/(D-2) - e^{-2\alpha} \phi^2, \\
\tilde{T}^b_a &= -2V \delta^b_a/(D-2) - e^{-2\beta} \delta^b_a \phi^2.
\end{align*}
\]

So far we did not specify the radial coordinate \( u \). For our purposes, the most helpful is the \textbf{Gaussian} gauge such that the real distance \( l \) along the radial direction is taken as a coordinate:
\[
u \equiv l, \quad \alpha \equiv 0, \quad (6)
\]
and the metric is
\[
ds^2 = e^{2\gamma(l)} \eta_{\mu\nu} dx^\mu dx^\nu - (dl^2 + e^{2\beta(l)} d\Omega^2).
\]

Then two independent components of the Einstein equations take the following form (the prime now denotes \( d/dl \)):
\[
\begin{align*}
\gamma'' + &d_0 \gamma' + d_1 \beta' \gamma' = -\frac{2\mathcal{R}^2}{D-2} V, \\
\beta'' + &d_0 \beta' \gamma' + d_1 \beta'^2 = (d_1 - 1 - \mathcal{R}^2 \phi^2) e^{-2\beta} - \frac{2\mathcal{R}^2}{D-2} V.
\end{align*}
\]

The Einstein equation \( G^l_i = -\mathcal{R}^2 T^l_i \) (where \( G^B_A \) is the Einstein tensor) is free of second-order derivatives:
\[
(d_0 \gamma' + d_1 \beta')^2 - d_0 \gamma'^2 - d_1 \beta'^2 = \mathcal{R}^2 (\phi'^2 - 2V) + d_1 e^{-2\beta}(d_1 - 1 - \mathcal{R}^2 \phi^2).
\]

The scalar field equations
\[
\nabla^A \nabla_A \phi^k + \partial V/\partial \phi^k = 0
\]
combine to yield an equation for \( \phi(l) \):
\[
\phi'' + (d_0 \gamma' + d_1 \beta') \phi' - d_1 e^{-2\beta} \phi = \frac{dV}{d\phi}.
\]

Due to the Bianchi identities, it is a consequence of the Einstein equations (8)–(10). On the other hand, (11) is a first integral of Eqs. (8), (9) and (11).

In our analytical study, we do not specify any particular form of \( V(\phi) \). We, however, suppose that \( V \) has a maximum at \( \phi = 0 \) and a minimum at some \( \phi = \eta > 0 \), so that \( V'(0) = V'(\eta) = 0 \). For convenience, we do not single out a cosmological constant which may be identified with a constant component of the potential \( V \) or, in many cases, with its minimum value.

The parameter \( \eta \) (as the scalar field itself) has the dimension \( [l^{-(D-2)/2}] \) and thus specifies a certain length scale \( \eta^{-2/(D-2)} \) and energy scale \( \eta^{2/(D-2)} \) (we are using the natural units, such that \( c = \hbar = 1 \)). In the conventional case \( D = 4 \), \( \eta \) has the dimension of energy and characterizes the SSB energy scale.
2.4. Regularity conditions. A regular center

For the geometry to be regular, we should require finite values of all Riemann tensor components \[ R_{abcd} \] in the Gaussian gauge \[ \text{Eq. (6)} \] the regularity conditions are as simple as that

\[ \beta', \beta'', \gamma', \gamma'' \text{ should be finite.} \] (12)

For \( d_1 > 0 \), in addition to (12), a special regularity condition is needed at a center, which is a singular point of the spherical coordinates in \( \mathbb{R}_u \times S^{d_1} \). A center is a point where the radius \( r \equiv e^\beta \) turns to zero. The regularity conditions there, also following from finiteness of the Riemann tensor components \[ R_{abcd} \], are the same as in a usual static, spherically symmetric space-time: in terms of an arbitrary \( u \) coordinate, they read

\[ \gamma = \gamma_c + O(r^2), \quad e^{\beta - \alpha} |\beta'| = 1 + O(r^2) \quad \text{as} \quad r \to 0, \] (13)

where \( \gamma_c \) is a constant which can be set to zero by a proper choice of scales of the coordinates \( x^\mu \). The second condition in (13) follows, for \( d_1 > 1 \), from finiteness of the Riemann tensor components \( R_{abcd} \), see (2). Its geometric meaning is local Euclidity at \( r = 0 \) that implies \( dr^2 = dl^2 \), i.e., a correct circumference to radius ratio for small circles. In the special case \( d_1 = 1 \), so that the factor space \( \mathbb{R}_u \times S^{d_1} \) is two-dimensional, we evidently have \( R_{abcd} = 0 \), but the second condition in (13) should still be imposed to avoid a conical singularity.

It is natural to put \( l = 0 \) at a regular center, so that \( l \) is the distance from the center.

Regularity of the Ricci tensor components \( R^B_A = R^{AC}B_{AC} \) implies regularity of the stress-energy tensor \( T^{AB}_A \), whence it follows that, at any regular point and with any radial coordinate,

\[ |V| < \infty, \quad e^{-\beta} |\phi| < \infty, \quad e^{-\alpha} |\phi'| < \infty. \] (14)

2.5. Boundary conditions

**Domain walls.** In case \( d_1 = 0 \), the metric (1) or (7) describes a plane-symmetric 5D space-time, the coordinate \( l \) ranges over the whole real axis, and the broken symmetry is \( \mathbb{Z}_2 \), mirror symmetry with respect to the plane \( l = 0 \). The topological defect is a domain wall separating two vacua, corresponding to two values of a single real scalar field \( \phi \), say, \( \phi = \pm \eta \). Accordingly, we assume that \( \phi(l) \) is an odd function whereas \( \gamma(l) \) and \( V(\phi) \) are even functions, and the conditions at \( l = 0 \) are

\[ \gamma(0) = \gamma'(0) = \phi(0) = 0. \] (15)

We thus have three initial conditions for the third-order set of equations (8), (10) [Eq. (11) is their consequence] since in this case the unknown function \( \beta \) is absent.

**Global strings and monopoles.** For \( d_1 > 0 \), the regular center requirement leads to the following boundary conditions for Eqs. (8)–(10) at \( l = 0 \):

\[ \phi(0) = \gamma(0) = \gamma'(0) = r(0) = 0, \quad r'(0) = 1. \] (16)

We have five initial conditions for a fifth-order set of equations. However, \( l = 0 \), being a singular point of the spherical coordinate system (not to be confused with a space-time curvature singularity), is also a singular point of our set of equations. As a result, the requirements of the theorem on the solution existence and uniqueness for our set of ordinary differential equations are violated. It turns out that the derivative \( \phi'(0) \) remains undetermined by (16). If we set \( \phi'(0) = 0 \), we obtain a trivial (symmetric) solution with \( \phi \equiv 0 \) and a configuration without a topological
defect. In case $V(0) = 0$ we arrive at the flat $D$-dimensional metric: we have then $\gamma \equiv 0$ and $r \equiv l$ in (7). If, however, $V(0) \neq 0$, the corresponding exact solutions to the Einstein equations for $d_0 > 1, d_1 > 1$ are yet to be found. A direct inspection shows that it cannot be de Sitter or AdS space: the constant curvature metrics are not solutions to the vacuum Einstein equations with a cosmological constant.

Nontrivial solutions take place if $\phi'(0) \neq 0$ and can correspond to SSB. One can notice that the very possibility of SSB appears as a result of violation of the solution uniqueness at $r = 0$ provided that a maximum of the potential $V(\phi)$ at $\phi = 0$ corresponds to the center. The lacking boundary condition which may lead to a unique solution can now follow from the requirement of regularity at the other extreme of the range of $l$ whose nature is in turn determined by the shape of the potential.

In what follows, assuming a regular center, we will try to find all possible conditions at the other extreme of the range $\mathbb{R}_l$ of the Gaussian radial coordinate, providing the existence of globally regular models with the metric (7). In other words, we seek solutions with such asymptotics that the quantities (2) are finite. All other regularity conditions, such as (14), then follow.

In doing so, we will not restrict in advance the possible shape of the potential $V(\phi)$. The cases under consideration will be classified by the final values of $r = e^\beta$ (infinite, finite or zero) and $V$ (positive, negative or zero). The scalar field $\phi$ is assumed to be finite everywhere.

Without loss of generality, we assume $\phi'(0) > 0$ near $l = 0$, i.e., that $\phi$ increases, at least initially, as we recede from the center.

3. Domain walls and the problem of matter confinement

Below we will mostly consider configurations with $d_1 \geq 2$ which correspond to a global monopole in the spherically symmetric space $\mathbb{R}_u \times S^{d_1}$. Before that, let us briefly discuss the problem of matter confinement on the brane and its difficulty in the 5-dimensional case.

The metric coefficient $e^{2\gamma}$ in (1), sometimes called the warp factor, actually plays the role of a gravitational potential that determines an attractive or repulsive nature of gravity with respect to the brane. If it forms a potential well with a bottom on (or very near) the brane, there is a hope that matter, at least its low-energy modes, will be trapped. It has been shown, in particular, that spin-1/2 fields are localized due to an increasing warp factor in (1+4)- and (1+5)-dimensional models [19, 20]. It was also repeatedly claimed that, in (1+4) dimensions, a brane with an exponentially decreasing warp factor (as, e.g., in the RS2 model) can trap spin 0 and 2 fields. Our calculation for a scalar field shows that it is not the case.

A gravitational trapping mechanism suggested in Refs. [21] was characterized there as a universal one, suitable for all fields. It is based on non-exponential warp factors, which increase from the brane and approach finite values at infinity. This mechanism was exemplified in [22] with a special choice of two so-called “smooth source functions” in the SET, describing a continuous distribution of some phenomenological matter and vanishing outside the brane.

Our analysis uses more natural assumptions: a scalar field system admitting SSB, without any special choice of the symmetry breaking potential, under the requirement of global regularity.

Let us briefly show, following Refs. [10, 11] (but in other coordinates) that this approach in (4+1) dimensions always leads to a decaying warp factor whatever be the choice of $V(\phi)$ and that such a system cannot trap a test scalar field. So consider a domain wall in 5 dimensions, so that $l \in \mathbb{R}$, in our equations we put $d_1 = 0$, the unknown $\beta(l)$ is absent, while Eqs. (8) and (11) for $\gamma$ and the single scalar field $\phi$ read

$$\gamma'' + d_0 \gamma^2 = -\frac{2\kappa^2}{d_0 - 1}V,$$

(17)
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\[ \phi'' + d_0 \gamma' \phi' - \frac{dV}{d\phi} = 0. \]  
(18)

Their first integral reduces to

\[ \gamma^2 = \frac{\dot{x}^2}{d_0(d_0 - 1)} (2V - \phi'^2), \]  
(19)

The initial conditions at \( l = 0 \) corresponding to \( \mathbb{Z}_2 \) symmetry (broken for the scalar field but preserved for the geometry) have the form

Excluding \( V \) from (17) and (19) and integrating subject to (15), we obtain

\[ (d_0 - 1) \gamma'(l) = -\kappa^2 \int_0^l \phi'^2 dl, \]  
(20)

and we conclude that \( \gamma'(l) \) is negative at all \( l > 0 \) and describes gravitational repulsion from the brane; moreover, \( e^{-\gamma} \) monotonically grows with growing \( l \). The only possible regular solution corresponds to \( |\gamma'(\infty)| < \infty \). Since in this case \( \gamma''(\infty) = 0 \), it follows from Eq. (17) that \( V(\infty) < 0 \), corresponding to a negative cosmological constant \( \Lambda = \kappa^2 V(\infty) \). So the only possible regular asymptotic is AdS, with

\[ e^\gamma \approx a e^{-hl}, \quad a, h = \text{const}, \quad h = -\sqrt{-\Lambda/6}. \]  
(21)

The constant \( a \) depends on the particular shape of \( V(\phi) \). At \( l = \infty \), there is an AdS horizon (\( e^\gamma = 0 \)), which, like a black hole horizon, attracts matter and prevents its trapping by the brane.

Let us show this for \( d_0 = 4 \) and a test scalar field \( \chi \) with the Lagrangian

\[ L_{\chi} = \frac{1}{2} \partial_A \chi^* \partial^A \chi - \frac{1}{2} m_0^2 \chi^* \chi - \frac{1}{2} \lambda \phi^2 \chi^* \chi, \]  
(22)

where \( \chi^* \) is the complex conjugate field, and the last term describes a possible interaction between \( \chi \) and the wall scalar field \( \phi \); \( \lambda \) is a coupling constant. The field \( \chi(x^A) \) satisfies the linear homogeneous (modified Fock-Klein-Gordon) equation

\[ \frac{1}{\sqrt{g}} \partial_A \left( \sqrt{g} g^{AB} \partial_B \chi \right) + (\lambda \phi^2 + m_0^2) \chi = 0. \]  
(23)

Its coefficients depend on \( l \) only, and \( \chi(x^A) \) may be sought for in the form

\[ \chi(x^A) = X(l) \exp(-ip_\mu x^\mu), \quad \mu = 0, 1, 2, 3. \]  
(24)

where \( p_\mu = (E, \vec{p}) \) is a constant 4-momentum. The function \( X(l) \) determines the \( \chi \) field distribution across the brane and satisfies the equation

\[ X'' + 4 \gamma' X' + \left[ e^{-2\gamma}(E^2 - \vec{p}^2) - \lambda \phi^2 - m_0^2 \right] X = 0. \]  
(25)

The \( \chi \) field is able to describe particles localized on the brane only if its SET, \( T_{\mu}^\nu[\chi] \), is finite in the whole 5-space and decays sufficiently rapidly at large \( l \). As an evident necessary condition of localization, the \( \chi \) field energy per unit 3-volume of the brane should be finite, i.e.,

\[ E_{\text{tot}}[\chi] = \int_{-\infty}^{\infty} T_{l} \sqrt{g} dl = \int_{0}^{\infty} e^{4\gamma} \left[ e^{-2\gamma}(E^2 + \vec{p}^2)X^2 + (m_0^2 + \lambda \phi^2)X^2 + X'^2 \right] dl < \infty. \]  
(26)

The inequality implies a finite norm of the \( \chi \) field defined as

\[ \|\chi\|^2 = \int_{-\infty}^{\infty} \sqrt{g} \chi^* \chi dl = \int_{-\infty}^{\infty} e^{4\gamma} X^2 dl. \]  
(27)
At large $l$, since $e^{-2\gamma} \to \infty$, the terms with $\lambda$ and $m_0$ in Eq. (25) may be neglected, and the equation determining the behavior of $\chi$ at large $l$ may be written as

$$X'' - 4hX' + P^2 e^{2hl} X = 0, \quad P^2 = \frac{E^2 - \vec{p}^2}{a^2 h^2}. \quad (28)$$

It is solved in terms of Bessel functions, and the solution has the asymptotic form

$$X = Ce^{3hl/2} \sin(P e^{hl} + \varphi_0), \quad l \to \infty, \quad (29)$$

where $C$ and $\varphi_0$ are integration constants. We see that the magnitude (29) not only does not vanish as $l \to \infty$ but oscillates with an increasing amplitude. As a result, the SET components $T^{\nu}_{\mu}[\chi]$ are infinite at $l = \infty$. Moreover, the integral (26) behaves as $\int e^{hl} dl$ and diverges. Though, the normalization integral (27) converges since the integrand behaves as $e^{-hl}$. The latter result is sometimes treated as a sufficient condition for localization, but, in our view, it is not true since the very existence of the brane configuration is put to doubt if the test field SET is somewhere infinite.

Thus a test scalar field with any mass tends to infinity as $l \to \infty$ and develops an infinite SET; even its interaction with the $\phi$ field that supports the brane does not improve the situation. We conclude that a single extra dimension is insufficient for providing gravitational attraction of matter to a regular isolated brane.

4. A search for regular asymptotics

Let us now consider the field equations (8)–(11) for global monopoles, assuming $d_1 \geq 2$. The string case $d_1 = 1$ is left aside since it has some peculiarities which need a special study.

A: Solutions with the asymptotic $r \to \infty$

Let us denote

$$\nabla = \frac{2\kappa^2 V}{D - 2}, \quad \nabla_\infty = \nabla \bigg|_{r \to \infty}. \quad (30)$$

Evidently, $l \to \infty$ as $r \to \infty$ since otherwise we would have $\beta' \to \infty$, violating the regularity conditions. The derivatives $\beta'$ and $\gamma'$ should tend to certain constant values, to be denoted $\beta'_\infty$ and $\gamma'_\infty$, respectively. Both $\beta''$ and $\gamma''$ vanish as $l \to \infty$. Moreover, in Eq. (9), the second term of the right-hand side also vanishes. Therefore, in the leading order of magnitude, Eqs. (8) and (9) take the form

$$\gamma'_\infty(d_0\gamma'_\infty + d_1\beta'_\infty) = -\nabla_\infty, \quad \beta'_\infty(d_0\gamma'_\infty + d_1\beta'_\infty) = -\nabla_\infty. \quad (31)$$

Consider separately the cases $\nabla_\infty \neq 0$ and $\nabla_\infty = 0$.

A1: $\nabla_\infty \neq 0$

Eqs. (31) immediately give

$$\beta'_\infty = \gamma'_\infty = \sqrt{-\nabla_\infty/(D - 1)}, \quad \nabla_\infty < 0. \quad (32)$$
An evident necessary condition of the existence of regular configurations is $\nabla_\infty \leq 0$. We thus obtain $e^\gamma \sim e^{\phi l} \cdot e^{\beta_\infty l}$, and the metric takes the asymptotic form

$$ds^2 \approx C_1 e^{2\beta_\infty l} \eta_{\mu\nu} dx^\mu dx^\nu - dl^2 - C_2 e^{2\beta_\infty l} d\Omega^2,$$

with some positive constants $C_1$ and $C_2$. Eq. (10) holds automatically provided $\phi'(\infty) = 0$, as should be the case if we assume a finite asymptotic value of $\phi$. Finally, in Eq. (11) all terms but $dV/d\phi$ manifestly vanish as $l \to \infty$, hence $dV/d\phi$ vanishes as well, which should be the case if the field $\phi$ reaching an extremum of the potential $V$.

The condition of finiteness of $\phi$ as $l \to \infty$ separates a family of regular solutions among the continuum of integral curves leaving the regular center with different slopes $\phi'(0)$. As is confirmed by numerical experiments, if the potential has only one extremum (minimum) $V_\infty < 0$, then there can be only one regular solution with $r \to \infty, l \to \infty$. However, there can be numerous regular solutions if the potential has several extremum points $V_\infty < 0$.

In particular, if the initial maximum of the potential is located below the zero level, $V(0) \leq 0$, then there can be a continuum of regular integral curves starting from the regular center and returning to $\phi = 0$ at $l \to \infty$. As can be verified numerically (see Sec. 4), there is a bunch of such curves parameterized by $\phi'(0) \in (0, \phi'_s)$, where $\phi'(0) = \phi'_s$ corresponds to a limiting regular curve (separatrix), also starting at $\phi(0) = 0$ but ending at the minimum $V(\eta)$.

The metric (33) solves the Einstein equations with the stress-energy tensor $T_{\mu\nu}^A = \delta_{AB} V_\infty$, having the structure of a (negative) cosmological term. Moreover, according to (2), the Riemann tensor has the structure of a constant curvature space at large $l$. In other words, such solutions have an anti-de Sitter (AdS) asymptotic far from the center. However, the metric (33) is not a solution to our equations in the whole space even in case $\phi = \text{const.}$ As was already mentioned, for $d_0 > 1$ and $d_1 > 1$, the constant curvature metrics (dS$_D$ and AdS$_D$) are not solutions to the vacuum Einstein equations with a cosmological constant.

**A2: $\nabla_\infty = 0$**

Eqs. (34) are solved either by $\beta_\infty = \gamma'_\infty = 0$ or by $d_0 \gamma'_\infty + d_1 \beta'_\infty = 0$. When, however, we substitute the second condition to Eq. (10), taking into account that $\phi' \to 0$ at large $l$, we get $d_0 \gamma'_\infty^2 + d_1 \beta'_\infty^2 = 0$ and return to $\beta'_\infty = \gamma'_\infty = 0$. Thus both $\beta'$ and $\gamma'$ vanish at infinity, and we can try to seek them as expansions in inverse powers of $l$:

$$\beta' = \frac{\beta_1}{l} + \frac{\beta_2}{l^2} + \ldots, \quad \gamma' = \frac{\gamma_1}{l} + \frac{\gamma_2}{l^2} + \ldots,$$

(34)

Then $O(l^{-2})$ is the senior order in the Einstein equations, and, to avoid a contradiction, $r^{-2} = e^{-2\beta}$ should be of the order $O(l^{-2})$ or smaller. Moreover, since we assume that $\phi$ tends to a finite value $\phi_\infty > 0$, we have $\phi' = o(1/l)$, and the scalar field equation (11) shows that $dV/d\phi = O(l^{-2})$ or smaller, i.e., $\phi_\infty$ should be an extremum of $V(\phi)$. If $\phi(l)$ grows monotonically to $\phi_\infty > 0$, then $\phi_\infty$ is a minimum of $V$ since according to (11) $dV/d\phi < 0$ as $\phi \to \phi_\infty$. However, if $V(0) = 0$, one cannot exclude that $\phi$ returns to zero as $l \to \infty$, see item (c) below.

In case $\phi \to \phi_\infty > 0$, since $V_\infty = dV/d\phi(\phi_\infty) = 0$, $V(\phi)$ is decomposed as

$$V(\phi) = \frac{1}{2} V_{\phi\phi}(\phi_\infty) \cdot (\phi - \phi_\infty)^2 + \ldots,$$

(35)

(where $V_{\phi\phi} = d^2V/d\phi^2$), so that $V = o(l^{-2})$. As a result, Eqs. (8)–(10) with (34) lead to

$$\gamma_1(-1 + d_0 \gamma_1 + d_1 \beta_1) = 0,$$

(36)
$$\beta_1(-1 + d_0 \gamma_1 + d_1 \beta_1) = \frac{l^2}{r^2}(d_1 - 1 - \phi_\infty^2),$$

$$\frac{(d_0 \gamma_1 + d_1 \beta_1)^2 - d_0 \gamma_1^2 - d_1 \beta_1^2}{d_1} = \frac{r^2}{2}(d_1 - 1 - \phi_\infty^2).$$

Now, it can be easily verified that one should necessarily put $\beta_1 = 1$. Indeed, for any $\beta_1 \neq 0$ we have $r = e^\beta \sim l^\beta$. So $\beta_1 < 1$ is excluded since it leads to $r \ll l$, contrary to the above requirement. But if we suppose $\beta_1 > 1$, then $l^2/r^2 \to 0$ as $l \to \infty$, and Eq. (35) leads either (if $\gamma_1 = 0$) to $\beta_1^2 = 0$ or (if $\gamma_1 \neq 0$ and then $d_0 \gamma_1 + d_1 \beta_1 = 0$) to $d_0 \gamma_1^2 + d_1 \beta_1^2 = 1$. Both opportunities contradict the assumption $\beta_1 > 1$.

Thus $\beta_1 = 1$, hence $r \approx kl$, $k = \text{const} > 0$, at large $l$.

Eq. (36) now leaves two opportunities, $\gamma_1 = 0$ and $\gamma_1 = -(d_1 - 1)/d_0$, and we consider them separately in items (a) and (b). Item (c) describes a case when the expansions (34) do not work. (a) $\gamma_1 = 0$. Then Eq. (37) yields

$$k^2 = 1 - \frac{\phi_\infty^2}{d_1 - 1};$$

and Eq. (10) in the same order of magnitude is satisfied automatically. The metric takes the following asymptotic form:

$$ds^2 = e^{2\gamma_\infty} \eta_{\mu\nu}dx^\mu dx^\nu - dl^2 - k^2 l^2 d\Omega^2,$$

where $\gamma_\infty$ is a constant (we cannot turn it to zero by rescaling the coordinates $x^\mu$ since such an operation has already been done for making $\gamma = 0$ at the center).

Thus the whole metric has a flat asymptotic, up to a solid angle deficit in the spherical part due to $k^2 \neq 1$. Such a deficit is a common feature of topological defects in cases when they have (almost) flat asymptotics. Its appearance due to cosmic strings and global monopoles in space-times without extra dimensions is discussed in detail in [8]. For a global monopole in extra dimensions in the particular case $d_0 = 4$, $d_1 = 2$ it was treated by Benson and Cho [18]. We would like to stress that the situation of a quasi-flat asymptotic with a solid angle deficit is not general. It takes place only for potentials with zero value at the minimum, $V(\phi_\infty) = 0$, and even in this case not always — see item B. Namely, this geometry requires

$$|\phi_\infty| < \phi_\text{cr} := \sqrt{d_0 - 1/\kappa},$$

i.e., $\phi_\infty$ should be smaller than the critical value $\phi_\text{cr}$ related to the $D$-dimensional Planck length. As $\phi_\infty$ approaches $\phi_\text{cr}$, $k \to 0$, the deficit absorbs the whole solid angle, and the above geometry disappears.

The scalar equation (11) shows how $\phi$ approaches $\phi_\infty$: in the senior order, it reads

$$- \frac{d_1 \phi_\infty}{k^2 l^2} = V_{\phi\phi}(\phi_\infty) \cdot (\phi - \phi_\infty).$$

Assuming $V_{\phi\phi}(\phi_\infty) 
eq 0$, we obtain $\phi - \phi_\infty \sim 1/l^2$.

(b) $\gamma_1 = -(d_1 - 1)/d_0$. Now Eq. (37) leads to $\phi_\infty^2 = d_1 - 1$, i.e., $\phi_\infty = \phi_\text{cr}$, while a substitution to (10) gives $(d_1 - 1)(d_0 + d_1 - 1) = 0$, contrary to our assumption $d_1 - 1 > 0$. So this opportunity does not lead to a regular asymptotic.

(c) If $V(0) = 0$, then a regular integral curve, starting at $l = 0$ and $\phi = 0$, can finish again with $\phi \to 0$ as $l \to \infty$. Assuming large $l$ and $r$, the scalar field equation (11) for $|\phi| \ll 1$ reduces to

$$\phi'' + (d_0 \gamma' + d_1 \beta')\phi' - V_2 \phi = 0,$$

(43)
where $V_2 = V_{\phi \phi}(0)$. Since, by assumption, $\phi = 0$ is a maximum of $V(\phi)$, we assume $V_2 < 0$.

If we further assume that the function $s(l) = e^{d_0 \gamma + d_1 \beta}$ satisfies the condition $s''/s \to 0$ as $l \to \infty$ (which is the case, e.g., for any power-behaved function), the solution to Eq. (13) is an oscillating function at large $l$:

$$\phi \approx \phi_0 e^{-(d_0 \gamma + d_1 \beta)/2} \cos \left[ \sqrt{|V_2|/(l - l_0)} \right], \quad l \to \infty,$$

(44)

where $\phi_0$ and $l_0$ are arbitrary constants. Substituting this to Eq. (8) and averaging $(\cos)^2 \to 1/2$, we get

$$e^{d_0 \gamma} \approx \frac{d_0 \gamma^2 |V_2| \phi_0^2}{2(D - 2)} \int l \ dl / r^{d_1}, \quad l \to \infty.$$

(45)

It is easy to verify that for $d_1 > 2$, when the integral in (45) converges, the asymptotic form of the solution for $r = e^\beta$ and $\gamma$ is $r \approx l$ and

$$\gamma = \gamma_\infty - \gamma_1/l^{d_1-2}, \quad \gamma_1, \gamma_\infty = \text{const},$$

i.e., we have a flat asymptotic.

In the special case $d_1 = 2$, the integral diverges logarithmically, and the solution may be approximated as (again) $r \approx l$ and $e^\gamma \approx \text{const} \cdot \ln l$. This “logarithmic” asymptotic resembles the behavior of cylindrically symmetric solutions in standard general relativity.

**B: Solutions with the asymptotic $r \to r_* > 0$**

A regular solution evidently cannot terminate at finite $r$ and $l < \infty$. Therefore we seek regular asymptotic $l \to \infty$, where $r$ and $\phi$ tend to finite limits, $r_*$ and $\phi_*$, hence the quantities $\beta', \beta'', \phi', \phi''$ vanish.

Moreover, in a regular solution, $\gamma'$ should tend to a finite limit as $l \to \infty$, hence $\gamma'' \to 0$. As a result, Eqs. (13) and (14) at large $l$ lead to

$$d_0 \gamma^2 = -\nabla_* = \frac{1}{r_*^2} (\varphi_*^2 \phi_*^2 - d_1 + 1),$$

(46)

where $\nabla_* = \nabla(\phi_*)$. We see that $\nabla_* \leq 0$ and, in addition, the scalar field should be critical or larger, $\phi_* \geq \phi_{cr}$. According to (46), at large $l$

$$\pm \gamma' \approx h := \sqrt{-\nabla_*/d_0} \geq 0,$$

(47)

and Eq. (10), as in the previous cases, simply verifies that the solution is correct in the leading order of magnitude. The scalar field equation gives a finite asymptotic value of $V_\phi \equiv dV/d\phi$:

$$V_\phi(\phi_*) = -d_1 \phi_* r_*^{-2}.$$

(48)

This value is negative if $\phi_* > 0$.

We obtain different asymptotic regimes for negative, positive and zero values of $\gamma'$.

**B1: $e^\gamma \sim e^{-hl}$, $h > 0$.** The metric has the asymptotic form

$$ds^2 = C^2 e^{-2hl} \eta_{\mu
u} dx^\mu dx^\nu - dl^2 - r_*^2 d\Omega^2.$$
The extra-dimensional part of the metric describes an infinitely long cylindrical tube, but the vanishing function \( g_{tt} = e^{2\gamma} \) resembles a horizon. The substitution \( e^{-hl} = \rho \) (converting \( l = \infty \) to a finite coordinate value, \( \rho = 0 \)) brings the metric \( [49] \) to the form

\[
d s^2 = C^2 \rho^2 \eta_{\mu\nu} dx^\mu dx^\nu - \frac{d\rho^2}{h^2 \rho^2} - \gamma^2 d\Omega^2. \tag{50}
\]

Thus \( \rho = 0 \) is a second-order Killing horizon in the 2-dimensional subspace parameterized by \( t \) and \( \rho \), it is of the same nature as, e.g., the extreme Reissner-Nordström black hole horizon, or the anti-de Sitter horizon in the second Randall-Sundrum brane world model. A peculiarity of the present horizon is that the spatial part of the metric, which at large \( l \) takes the form \( \rho^2 (d\vec{x})^2 \), is degenerate at \( \rho = 0 \). The volume of the \( d_0 \)-dimensional spacetime vanishes as \( l \to \infty \). And it will remain degenerate even if we pass to Kruskal-like coordinates in the \( (t, \rho) \) subspace. But the \( D \)-dimensional curvature is finite there, indicating that a transition to negative values of \( \rho \) (where the old coordinate \( l \) no longer works) is meaningful. \(^1\)

One more observation can be made. According to \( [16] \), the potential \( V \) is necessarily negative at large \( l \). On the other hand, Eq. \( [8] \) may be rewritten in an integral form:

\[
e^{d\alpha + d\beta} \gamma' = -\int_0^l e^{d\alpha + d\beta} V dl. \tag{51}
\]

The lower limit of the integral corresponds to a regular center, where the left-hand side of \( [51] \) vanishes. As \( l \to \infty \), it vanishes as well due to \( \gamma \to -\infty \). Thus the integral in the right-hand side, taken from zero to infinity, is zero. It means that the potential \( V(\phi) \) has an alternate sign and is positive in a certain part of the range \( (0, \phi_*) \).

Thus purely scalar solutions of monopole type may contain second-order horizons. The degenerate nature of the spatial metric at the horizon does not lead to a curvature singularity, and the solutions may be continued in a Kruskal-like manner. Nevertheless, we do not consider these solutions as describing viable monopole configurations because the zero volume of the corresponding spatial section will make the density of any additional (test) matter become infinite. It is then impossible to neglect its back reaction, which will evidently destroy such a configuration.

**B2:** \( e^\gamma \sim e^{hl} \), \( h > 0 \). The metric has the asymptotic form

\[
ds^2 = C^2 e^{2hl} \eta_{\mu\nu} dx^\mu dx^\nu - dt^2 - r^2 d\Omega^2, \quad C = \text{const} > 0. \tag{52}
\]

So, in the spherically symmetric extra-dimensional part of the metric, we have an infinitely long \( d_1 \)-dimensional cylindrical “tube” with an infinitely growing gravitational potential \( g_{tt} = e^{2\gamma} \).

At this cylindrical asymptotic, according to \( [17] \) and \( [18] \), the potential \( V \) tends to a negative value and has a negative slope. Moreover, in Eq. \( [51] \), the integral is negative and diverges at large \( l \) due to growing \( e^\gamma \).

\(^1\)One may wonder why we here do not obtain simple (first-order) horizons, like those in the Schwarzschild and de Sitter metrics, while such horizons generically appeared in the special case \( d_0 = 1 \), which corresponds to spherically symmetric global monopoles in general relativity, considered in detail in Refs. \([16\,14] \).

The reason is that, in case \( d_0 = 1 \), \( \delta^{\mu\nu} \rho_\sigma \) \( [9] \) is zero, and the corresponding component of the Riemann tensor is also zero regardless of the values of \( \gamma' \). In terms of the Gaussian coordinate \( l \), a simple horizon occurs at some finite \( l = l_h \) near which \( g_{tt} = e^{2\gamma} \sim (l - l_h)^2 \), so that \( \gamma' \to \infty \). When \( d_0 = 1 \) this does not lead to a singularity since then only the combinations \( \gamma'' + \gamma'^2 \) and \( \beta' \gamma' \) are required to be (and really are) finite. In case \( d_0 > 1 \), instead of a horizon, we would have a curvature singularity at finite \( l \), a situation excluded in the present study.

We thus have a general result for the metric \( [1] \): for \( d_0 > 1 \), horizons can only be of order 2 and higher.
Regular solutions with $\gamma'(\infty) > 0$ naturally arise if the potential $V(\phi)$ is negative everywhere. One can notice, however, that when $V(0)$ is above zero, by (51), near the center ($l = 0$) the function $\gamma(l)$ decreases due to $V > 0$ while at large $l$ it grows. So it has a minimum at some $l > 0$.

**B3:** $\nabla_*= 0$. This case contains one more asymptotic where the extra space ends with a regular tube.

Indeed, we can once again use the expansions (54), but now with $\phi_*$ instead of $\phi_\infty$ and $\beta_1 = 0$ in accord with $r \to r_*$. Eq. (59) [order $O(1)$] shows that $\lambda^2\phi_*^2 = d_1 - 1$, i.e., $\phi_* = \phi_{cr}$. Eq. (11) [order $O(1)$] gives a finite value of the derivative $dV/d\phi(\phi_*) = -d_1\phi_*/r_*^2$. Further, Eq. (58) [$O(l^{-2})$] yields $\gamma_1(d_0\gamma _1 - 1)/l^2 = -V$, showing that $V = O(l^{-2})$ (or is even smaller). Since $V = dV/d\phi(\phi_*) \cdot (\phi - \phi_*) + o(\phi - \phi_*$), we have to conclude that $\phi - \phi_* = O(l^{-2})$ or smaller.

Now, assuming $V(\phi) = V_2/l^2 + \ldots$, we can find $V_2$ directly as the senior term in $dV/d\phi(\phi_*) \cdot (\phi - \phi_*)$ and, independently, from Eq. (11) [$O(l^{-2})$], getting the following two expressions:

$$V_2 = -d_1\frac{\phi_*\phi_2}{r_*^2} \quad \text{and} \quad V_2 = -(D-2)\frac{\phi_*\phi_2}{r_*^2},$$

whence it follows $d_1 = D-2$, or $d_0 = 1$. Such a “critical” asymptotic ($\phi \to \phi_{cr}$, $g_{tt} \to 0$, $r \to \text{const}$) was indeed found for $d_0 = 1$ in our papers [16, 14] describing $(d_1 + 2)$-dimensional spherically symmetric global monopoles, but, as we see, it does not exist in the case under consideration, $d_0 > 1$.

The only remaining opportunity is that $\phi - \phi_* = o(l^{-2})$ and $\gamma \to \gamma_* = \text{const}$, i.e., a solution tending at large $l$ to the following simple “flux-tube” solution, valid for any $d_0$ and $d_1$:

$$r = \text{const}, \quad \gamma = \text{const}, \quad \phi = \phi_{cr}, \quad V = 0, \quad dV/d\phi = -d_1\phi_{cr}/r^2. \quad (53)$$

Such a solution can exist if the potential $V(\phi)$ has the properties $V(\phi_{cr}) = 0$ and $dV/d\phi(\phi_{cr}) < 0$, and the last equality in (53) then relates the constant radius $r$ to $dV/d\phi(\phi_{cr})$.

**C: Solutions with the asymptotic $r \to 0$**

The limit $r \to 0$ means a center, and for it to be regular, the conditions (12) should hold, hence, for our system, the initial conditions (11), where $l = 0$ should be replaced with, say, $l = l_0 > 0$.

Recall now that the conditions (11) determine the solution to the field equations for a given potential $V(\phi)$ up to the value of $\phi'$. In particular, if there is one more center at $l = l_0$, then, starting from it and choosing $\phi'(l_0) = -\phi'(0)$, we shall have the same solution in terms of $l_0 - l$ instead of $l$. We thus obtain a solution with two regular centers which is symmetric with respect to the middle point $l = l_0/2$, to be called the equator. To be smooth there, it should satisfy the conditions

$$\beta' = \gamma' = \phi' = 0 \quad \text{at} \quad l = l_0/2, \quad (54)$$

which implicitly restrict the shape of the potential. Given a potential $V(\phi)$, the conditions (54) create, in general, three relations among $l_0$, $\phi'(0)$ and the free parameters of $V(\phi)$ (if any). Excluding $l_0$ and $\phi'(0)$, we must obtain a single “fine tuning” condition for the parameters of the potential.

A necessary condition for the existence of such a solution is that $V(\phi)$ should have a variable sign. This follows from Eq. (11) with integration over the segment $(0, l_0/2)$: the integral vanishes since $\gamma' = 0$ at both ends.

Moreover, as follows from Eqs. (9) and (10) with (54),

$$r_e^{-2}(d_1 - 1 - \lambda^2\phi_e^2) = \frac{D-2}{d_1}V_e = \beta''_e + \nabla_e, \quad (55)$$
Gravitating global monopoles in extra dimensions and the brane world concept

leading to $d_1\beta'' = (d_0 - 1)\nabla e$ (where the index “e” refers to values at the equator). If $r = e^\beta$ is assumed to grow monotonically from zero to its maximum value at the equator, we have $\beta'' e < 0$, hence $\nabla e < 0$, and (55) implies that $\phi_e > \phi_{cr}$, i.e., the scalar field at the equator should exceed its critical value.

The existence of asymmetric solutions with two regular centers, corresponding to $\phi'(l_0) \neq -\phi'(0)$, is also not excluded. In this case, there would be, in general, no equator since $\beta$ and $\phi$ would have maxima at different $l$; moreover, there would be, in general, $\gamma(l_0) \neq \gamma(0) = 0$, and $\gamma(l)$ could even have no extremum. However, since $\gamma' = 0$ at both centers, the integral in (51) taken from 0 to $l_0$ should vanish, hence, again, $V$ would have an alternating sign.

The whole configuration with two regular centers has the topology $M^{d_0} \times S^{d_1+1}$, with closed extra dimensions in the spirit of Kaluza-Klein models. The main difference from them is that now all variables essentially depend on the extra coordinate $l$.

The main properties of all regular asymptotics found, which lead to a classification of possible global monopole configurations in extra dimensions, are summarized in Table 2. The word “attraction” corresponds to an increasing warp factor far from the brane.

| Notation | $r$ | $V(\phi)$ | $\phi$ | $\gamma$ | Asymptotic type |
|----------|-----|------------|--------|----------|----------------|
| A1       | $\infty$ | $V(\eta) < 0$ | $\eta < \phi_{cr}$ | $\infty$ | AdS, attraction |
| A2(a)    | $\infty$ | 0 | $\eta < \phi_{cr}$ | const | flat, solid angle deficit |
| A2(c), $d_1 > 2$ | $\infty$ | 0 | 0 | const | flat |
| A2(c), $d_1 = 2$ | $\infty$ | 0 | 0 | $\infty$ | “logarithmic”, attraction |
| B1       | $r_*$ | $V_* < 0$ | $\phi_* > \phi_{cr}$ | $-\infty$ | double horizon, repulsion |
| B2       | $r_*$ | $V_* < 0$ | $\phi_* > \phi_{cr}$ | $\infty$ | attracting tube |
| B3       | $r_*$ | 0 | $\phi_* = \phi_{cr}$ | const | trivial tube |
| C        | 0 | $V(0)$ | 0 | const | second center |

5. Scalar field trapping by global monopoles

Consider a test scalar field with the Lagrangian (22) in the background of global monopole configurations described in Sec. 4. After variable separation (24), the field equation for a $\vec{p}$-mode of the scalar field $\chi$ reads

$$X'' + (d_0 \gamma' + d_1 \beta')X' + (e^{-2\gamma}p^2 - \mu^2)X = 0,$$

where $p^2 = p_\mu p^\mu = E^2 - \vec{p}^2$ is the $d_0$-momentum squared and $\mu^2 = m_0^2 + \lambda \phi^2$ is the effective mass squared. The trapping criterion consists, as before, in the requirements that the $\chi$ field SET should vanish far from the brane and the total $\chi$ field energy per unit volume of the brane should be finite, i.e.,

$$E_{tot}[\chi] = \int \sqrt{g}d^{d_1+1}x \left[ e^{-2\gamma}(E^2 + \vec{p}^2)X^2 + \mu^2 X^2 + X'^2 \right] dl < \infty.$$
The first requirement means that each term in the square brackets in (57) must vanish at large $l$.

Let us now check whether these requirements can be met at different kinds of asymptotics listed in Table 1.

**A1:** attracting AdS asymptotic, $\beta \sim \gamma \sim k l$, $k > 0$. At large $l$, Eq. (56) reduces to the equation with constant coefficients $X'' + (D-1)X - \mu^2 X = 0$, and its solution vanishing as $l \to \infty$ is

$$X \sim \exp(-al), \quad a = \frac{1}{2} \left( (D-1)k + \sqrt{(D-1)^2 k^2 + 4\mu^2} \right).$$

It is straightforward to make sure that the trapping requirements are satisfied for all momenta $\vec{p}$ and all $\mu^2 \geq 0$.

**A2(a):** a quasi-flat asymptotic with a solid angle deficit. At large $l$, Eq. (26) reduces to $X'' + d_1 X/l + P^2 X = 0$, where $P^2 = p^2 \exp(-2\gamma) - \mu^2$ and $\gamma_\infty$ is the limiting value of $\gamma$ at $l = \infty$. In terms of $Y = l^{d_1/2} X$ this equation is (at large $l$) rewritten as $Y'' + P^2 Y = 0$, while the trapping condition (57) implies $\int l^{d_1} X^2(l) \, dl < \infty$. Therefore only an exponentially falling $Y(l)$ is suitable. In other words, the trapping condition is $P^2 < 0$, or

$$p^2 < m^2_{cr} := \mu^2 \exp(2\gamma_\infty),$$

where now $\mu^2 = m^2_0 + \lambda^2 \eta^2$. One should note that $p^2 = E^2 - \vec{p}^2$ is nothing else but the observable mass of a free $\chi$-particle if the observer watches its motion in the Minkowski section $l = 0$ of our manifold, i.e., on the brane. So the condition (59) means that the brane traps all scalar particles of masses smaller than the critical value $m_{cr}$ depending on the model parameters.

**A2(c), $d_1 > 2$:** this case differs from the previous one only by the asymptotic value of $\phi$ which is now zero, so that $\mu = m_0$.

**A2(c), $d_1 = 2$:** a “logarithmic” asymptotic, $\exp(\gamma) \sim \ln l$. Since $\exp(-2\gamma) \sim 1/(\ln l)^2 \to 0$, the term with $p^2$ drops out from Eq. (30), which then leads to the decreasing solution $X \sim l^{-1} \exp(-u)$, and a $\chi$-particle is trapped provided $\mu = m_0 > 0$.

**B1:** a horizon. As was remarked previously, we do not regard this configuration viable and omit it in our discussion.

**B2:** an attracting tube, $r \to r_*$ and $\gamma \approx h l$, $h > 0$ as $l \to \infty$. Eq. (50) takes the form $X'' + d_0 h X' - \mu^2 X = 0$ and has the decreasing solutions

$$X \sim \exp(-al), \quad a = \frac{1}{2} \left( d_0 h + \sqrt{d_0^2 h^2 + 4\mu^2} \right).$$

As in item A1, it is easy to verify that the trapping conditions hold provided $\mu^2 > 0$.

**B3:** a trivial tube, both $\beta$ and $\gamma$ tend to constants as $l \to \infty$. In Eq. (30) the term with $X'$ drops out at large $l$, and an exponentially decreasing solution exists under the condition (59) where $\mu^2 = m^2_0 + \lambda^2 \phi^2_{cr}$.

**C:** these configurations have no large $l$ asymptotic and are not interpreted in terms of branes.

A conclusion is that scalar particles of any mass and momentum are trapped by global monopoles with A1 and B2 asymptotics with exponentially growing warp factors and A2(c) with a logarithmic asymptotic; they are trapped under the restrictions (59) on the particle’s observable mass by monopoles with A2 and B3 asymptotics whose warp factors tend to constant limits far from the brane.
6. Numerical results: Mexican hat potential

In this section we present the results of our numerical calculations which confirm the classification of regular solutions given above. In doing so, we have been using the “Mexican hat” potential in the form (Fig. 1)

\[ V = \frac{\lambda \eta^4}{4} \left[ \varepsilon + \left( 1 - \frac{\phi^2}{\eta^2} \right)^2 \right]. \] (61)

Figure 1: Mexican hat potential

It has two extremum points in the range \( \phi \geq 0 \): a maximum at \( \phi = 0 \) and a minimum at \( \phi = \eta \). The SSB energy scale is characterized by \( \eta^{1/(D-2)} \), while \( \sqrt{\lambda} \eta \) determines, as usual, a length scale. The non-conventional parameter \( \varepsilon \), introduced in (61), moves the potential up and down, which is equivalent to adding a cosmological constant to the usual Mexican hat potential.

Given the potential (61), the nature of the solutions essentially depends on its two dimensionless parameters: \( \varepsilon \), fixing the extremal values of the potential with respect to zero, and \( \kappa^2 \eta^2 \) characterizing the gravitational field strength. As we remember from Sec. 4, the asymptotic \( r \to \infty \) only exists when \( \phi_{\infty} < \phi_{cr} \), which is the same as \( \kappa^2 \eta^2 < d_1 - 1 \).

If \( \varepsilon > 0 \), the potential (61) is always positive, and, in accordance with Sec. A1, regular solutions are absent.

In the conventional case \( \varepsilon = 0 \), in the range \( 0 < \kappa^2 \eta^2 < d_1 - 1 \), there are asymptotically flat regular solutions with a solid angle deficit (class A2).

The most complex case \( 0 > \varepsilon > -1 \) contains a variety of possibilities. Regular solutions with the asymptotic \( r \to \infty \) as \( l \to \infty \) having \( \gamma'_{\infty} > 0 \) (case A1) exist in some range \( 0 < \eta < \eta_s \), where the separating value \( \eta_s \) depends on \( d_0 \), \( d_1 \) and \( \varepsilon \). As an example, such a regular solution with \( \kappa^2 \eta^2 = 5 \), \( \varepsilon = -0.75 \), \( d_0 = 4 \), \( d_1 = 3 \) is presented in Fig. 2.

Depending on the parameters of the potential, there are regular solutions with the asymptotic \( r \to r_s < \infty \) and \( \gamma'_{\infty} > 0 \) (case B2) in some range \( \eta_{s1} < \eta < \eta_{s2} \), see Fig. 3.

Here \( \varepsilon = -0.9 \), \( d_0 = 4 \), \( d_1 = 3 \). The curves are given for \( \kappa^2 \eta^2 = 10, 12, 15, 20, 30, 40 \) and 45 (upside down). The red dotted curves (\( \kappa^2 \eta^2 = 10 \) and \( \kappa^2 \eta^2 = 45 \)) correspond to singular configurations. So for \( \varepsilon = -0.9 \), \( d_0 = 4 \), \( d_1 = 3 \) the lower bound of this parameter leading to regular models is somewhere between 10 and 12 while the upper bound is between 30 and 45.

An example of a regular solution with the asymptotic \( r \to r_s < \infty \) and \( \gamma'_{\infty} < 0 \) (class B1), corresponding to a second-order Killing horizon, is shown in Fig. 4. The value \( \kappa \eta^2 = 17.37 \) is fine-tuned to the parameters \( \varepsilon = -0.75 \), \( d_0 = 4 \), \( d_1 = 2 \) of this particular solution.

Other examples of fine-tuned regular solutions, namely, type C with two regular centers (\( r \to 0 \), \( \phi \to 0 \), \( \gamma' \to 0 \) at \( l \to l_0 \)), are presented in Fig. 5. For all three curves \( d_0 = 4 \), \( d_1 = 2 \). The red,
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Figure 2: A regular solution with an AdS asymptotic (type A1) for the potential (61) with \( \kappa \eta^2 = 5 \), \( \varepsilon = -0.75 \), \( d_0 = 4 \), \( d_1 = 3 \)

Figure 3: Regular solutions with the asymptotic B2 (attracting tube), such that \( r \to r_* < \infty \) and \( \gamma'_\infty > 0 \)

green and blue curves correspond to \( \varepsilon = -0.15 \), \(-0.5\), and \(-0.9626\), respectively. The fine-tuned values of \( \kappa^2 \eta^2 \) are approximately 2.637, 6.17 and 100.

In case \( \varepsilon \leq -1 \), the maximum \( V(0) \leq 0 \) is at or below the zero level, and there is a possibility for the integral curves to start and finish at the same value \( \phi(0) = \phi(\infty) = 0 \). We then observe a whole family of such regular curves in the range \( 0 < \phi'(0) < \phi'_c \), see Fig. 6. For the particular example presented (\( \varepsilon = -1.5 \), \( \kappa \eta^2 = 1 \), \( d_0 = 4 \), \( d_1 = 3 \)), the values of \( \phi'(0) \) for the dotted curves ending with \( \phi = 0 \) are 0.2, 0.3 and 0.4 (bottom up). The limiting red curve with \( \phi'(0) = \phi'_c = 0.4401425 \) (separatrix) is a regular solution ending at the minimum of the potential: \( \phi \to \eta \) as \( l \to \infty \).

The Mexican hat potential (61), with its only two extrema at \( \phi = 0 \) and \( \phi = \eta \), cannot demonstrate the whole variety of solutions which appear with more sophisticated potentials having additional maxima and/or minima. Thus, for instance, class A solutions may have a large \( r \) asymptotic at any such extremum.

7. Concluding remarks

We have obtained as many as seven classes of regular solutions to the field equations describing a Minkowski thick brane with a global monopole in extra dimensions, see Table 1.

Some of these classes, namely, A1 with an AdS asymptotic and B2 ending with an attracting tube, possess an exponentially growing warp factor \( e^{2\gamma} \) at large \( l \) and are shown to trap linear test scalar fields modes of any mass and momentum.
Others — A2(a) and A2(c) for $d_1 > 2$, ending with a flat metric at large $l$ — have a warp factor tending to a constant whose value is determined by the shape of the potential $V(\phi)$. They are also shown to trap a test scalar field but the latter's observable mass is restricted above by a value depending on the particular model of the global monopole.

Lastly, for $d_1 = 2$, i.e., a three-dimensional global monopole in the extra dimensions, class A2(c) solutions possess a logarithmically growing warp factor. All test scalar field modes are trapped by this configuration, but the slow growth of $\gamma(l)$ probably means that the test field is strongly smeared over the extra dimensions.

All such configurations, in sharp contrast to RS2-like domain walls in 5 dimensions, are able to trap scalar matter. It is certainly necessary to check whether or not nonzero-spin fields are trapped as well and Newton's law of gravity holds on the brane in conformity with the experiment. We hope to consider these subjects in our future publications.

In addition to the trapping problem, a shortcoming of RS2 type Minkowski branes is that they are necessarily fine-tuned. Many of the global monopole solutions, at least those existing in the weak gravity regime (class A), are free of this shortcoming and are thus better for thick brane model building.

Some results and conclusions of this paper have been previously found in Refs. [12, 13]. The main difference of our approach from theirs is their boundary condition which is, in our notation, $\phi = \eta$. This excludes the cases when the solution ends at a maximum or slope of the potential, such as, e.g., symmetric solutions with two regular centers. Another difference is that they consider solutions with an exponentially decreasing warp factor as those leading to matter confinement on the brane. In our view, such solutions with second-order horizons do not represent viable models of a brane world. We conclude that the present paper gives the most complete classification of all
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Figure 6: Regular solutions starting and terminating at $\phi(0) = \phi(\infty) = 0$. The limiting red curve with $\phi'(0) = \phi'_c = 0.4401425$ (separatrix) terminates at $\phi(\infty) = \eta$.

regular solutions for global monopoles in extra dimensions, which, even without gauge fields, seem to be promising as brane world models.

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