Noncommutative line bundle and Morita equivalence

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Abstract

Global properties of abelian noncommutative gauge theories based on $\star$-products which are deformation quantizations of arbitrary Poisson structures are studied. The consistency condition for finite noncommutative gauge transformations and its explicit solution in the abelian case are given. It is shown that the local existence of invertible covariantizing maps (which are closely related to the Seiberg-Witten map) leads naturally to the notion of a noncommutative line bundle with noncommutative transition functions. We introduce the space of sections of such a line bundle and explicitly show that it is a projective module. The local covariantizing maps define a new star product $\star'$ which is shown to be Morita equivalent to $\star$.

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1 Introduction

Noncommutativity plays a prominent role in physics ever since the birth of quantum mechanics. When trying to replace the notions and concepts of commutative geometry in the more general noncommutative framework the basic strategy is to not consider the manifold itself but rather the algebra of functions on it. In the noncommutative realm the algebra of functions is replaced by an arbitrary associative algebra \( \mathfrak{A} \). In the spirit of the Gel’fand-Naimark theorem we shall keep referring to the elements of \( \mathfrak{A} \) as “functions on the noncommutative space.” Similarly, the celebrated Serre-Swan theorem allows us to replace the notion of vector bundle by the one of projective module of sections. Noncommutative Yang-Mills theories are naturally formulated in terms of projective modules [1]. In string theory \( D \)-branes are a realization of vector bundles on quite general sub-manifolds of space-time. In the presence of background fields these become noncommutative, see [2, 3] and references therein.

In this paper we deal with the situation when our noncommutative space \( \mathfrak{A} \) is a deformation quantization of some Poisson manifold \( M \). Since every deformation quantization originates from one of Kontsevich type, we can restrict ourselves to this case. Based on our previous studies of (abelian) noncommutative gauge theories [4, 5, 6, 7] we give the explicit form of finite noncommutative gauge transformations and develop the concept of a noncommutative line bundle, in the sense of deformation quantization, that is closer to the spirit of algebraic topology then a formulation in terms of projective modules.

The type of noncommutative gauge theory that we take as a starting point in this article can be based on a few basic ideas which we shall briefly review below: the concept of covariant functions, the requirement of locality (in the gauge potential) and accompanying gauge equivalence and consistency conditions.

It is natural to introduce local gauge transformations of a field \( \Psi \) on a noncommutative Space \( \mathfrak{A} \) in analogy to the commutative case by

\[
\hat{\delta}\Psi = i\Lambda \star \Psi, \quad \Psi, \Lambda \in \mathfrak{A}.
\]  

(1)

Multiplying a field from the right by a function yields a new field that transforms again according to (1). This is not the case, however, if we multiply a

\[1]\text{Here and in the following we use capital letters to denote noncommutative quantities.}\]
field from the left by a function $f$, simply because the local gauge parameter $\Lambda$ will not in general commute with $f$:

$$\hat{\delta}(f \star \Psi) = if \star \Lambda \star \Psi \neq i\Lambda \star (f \star \Psi); \quad (\hat{\delta}f = 0).$$

Consider the case of a coordinate function $x^i$ (i.e. a generator of $\mathfrak{A}$): In complete analogy to ordinary gauge theory (where the gauge parameter does not commute with derivatives) one needs to introduce noncommutative gauge potentials $A^i$ and covariant coordinates $X^i = x^i + A^i$ with transformation properties

$$\hat{\delta}X^i = i[\Lambda \star X^i] \Leftrightarrow \hat{\delta}A^i = i[\Lambda \star x^i] + i[\Lambda \star A^i]. \quad (2)$$

Covariant functions in general are introduced via an invertible gauge-dependent map $D : \mathfrak{A} \to \mathfrak{A}$ that transforms under gauge transformations such that $\hat{\delta}(Df) = i[\Lambda \star Df]$. The gauge-dependent map $A = D - \text{id}$ plays the role of a generalized noncommutative gauge potential. The more familiar noncommutative gauge potentials $A^i$ are obtained by evaluating $A$ on a coordinate function $x^i$. The covariant coordinates $X^i$ generate a new algebra $\mathfrak{A}'$ with product $\star'$ that can be computed on any pair of functions via $D(f \star' g) = Df \star Dg$. In this way we have a (physically desirable) back-reaction of the gauge fields on the noncommutative space.

If a classical (commutative) limit of the theory exists, as is for instance the case, when the noncommutative structure is given by a star product, one may ask about the relation between the noncommutative and corresponding classical gauge fields. It turns out that one can find maps $A[a], \Lambda_\alpha[a]$ that express the noncommutative gauge field and gauge parameter in terms of their classical counterparts $a, \lambda$ such that a gauge equivalence condition

$$A[a + \delta_\lambda a] = A[a] + \hat{\delta}_{\Lambda_\alpha[a]} A[a] + o(\lambda^2) \quad (3)$$

holds: Classical gauge transformations $\delta_\lambda$ induce noncommutative ones. These and similar mappings, require that all noncommutative quantities are local functions of the classical gauge potential and its derivatives. Extending the gauge equivalence condition to fields $\Psi$ that transform in the fundamental

$$\Psi[\psi + \delta_\lambda \psi, a + \delta_\lambda a] = \Psi[\psi, a] + i\Lambda_\alpha[a] \star \Psi[\psi, a] + o(\lambda^2) \quad (4)$$

on can derive a consistency condition that involves only the gauge parameters:

$$[\Lambda_\alpha[a] \star \Lambda_\beta[a]] + i\delta_\alpha \Lambda_\beta[a] - i\delta_\beta \Lambda_\alpha[a] = \Lambda_{[\alpha, \beta]}[a]. \quad (5)$$
This condition is of central importance in the present work, since it defines a noncommutative group law as we shall see. (We have written the nonabelian version of the consistency condition – in the abelian case $[\alpha, \beta] = 0$.)

We are particularly interested in noncommutative structures that are given by a star product, i.e. an associative algebra $\mathfrak{A} = (C^\infty(M)[[\hbar]], \star)$ that is the deformation quantization of a Poisson structure over some manifold $M$. For an arbitrary Poisson structure $\theta$ a star product $\star$ exists and can be expressed in terms of the Kontsevich formality map [8]. In [4, 6, 7] we have used this map to construct explicit solutions $D[a]$ and $\Lambda_\lambda[a]$ to the gauge equivalence

$$D[a + \delta_\lambda a](f) = D[a](f) + i[\Lambda_\lambda[a] \star D[a](f)]$$

and consistency condition

$$[\Lambda_\alpha[a] \star \Lambda_\beta[a] + i\delta_\alpha\Lambda_\beta[a] - i\delta_\beta\Lambda_\alpha[a] = 0.$$  

(For the following the existence of $\star$, $D[a]$ and $\Lambda_\lambda[a]$ that satisfy (3) and (4) is more important than the explicit form of these objects.) The map $D[a]$ is formally invertible and defines an equivalent star product $\star'$ via $D[a](f \star' g) = D[a]f \star D[a]g$ on the patch where the local gauge potential $a$ is defined. The new star product $\star'$ itself can be defined globally since it only depends on $a$ via its gauge-invariant field strength $f = da$: The star product $\star'$ is the deformation quantization by Kontsevich’s formula of the Poisson structure $\theta'$ which, for some choice of local coordinates and using matrix notation has the explicit form $\theta' = \sum_{n=0}^\infty (-1)^n \theta(f\theta)^n$. The star products $\star$ and $\star'$ are “patch-wise” equivalent. We will show in this article that the corresponding algebras $\mathfrak{A}$, $\mathfrak{A}'$ are in fact Morita equivalent [3].

The Kontsevich formality maps can be computed explicitly on open subsets of $\mathbb{R}^n$ with diagrammatic techniques that resemble those of Feynman diagrams; they can then be consistently globalized using maps that in fact closely resemble the $D[a]$. In the following we shall assume that this has already be done, i.e., we have a globally defined star product $\star$. In our previous work we have claimed that our formulas for $D[a]$, $\Lambda_\lambda[a]$ and also the noncommutative gauge potential can be globalized by noncommutative gauge transformations between patches; this will be made precise in this article.

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2 In general, the local gauge potential $a$ can be a formal power series in the deformation parameter $\hbar$. To conform with the mathematics literature it can be taken to start with a term of order $\hbar$; the expression for $\theta'$ is then also a formal power series.

3 See [8] for the relevance of Morita equivalence in string theory.
2 Finite gauge transformations

Let us consider finite classical gauge transformations
\[ \psi \mapsto \psi_g = g\psi, \quad a \mapsto a_g = a + igdg^{-1}. \]  
(In the nonabelian case the latter would read \( a \mapsto a_g = gag^{-1} + igdg^{-1} \).) Corresponding finite versions of the gauge equivalence conditions for the covariant functions and fields are

\[ \mathcal{D}_{[a]}(f) * G_g[a] = G_g[a] * \mathcal{D}_{[a]}(f), \]  
\[ \Psi[\psi, a_g] = G_g[a] * \Psi[\psi, a], \]

where \( \mathcal{D}_{[a]} \) is an invertible map. Any covariant function, i.e. a \( f_{[a]} \) which under a gauge transformation \( a \mapsto a_g \) transforms as

\[ f_{[a_g]} = G_g[a] * f_{[a]} * (G_g[a])^{-1}, \]

is necessarily of the form \( \mathcal{D}_{[a]}(f') \) with an invariant \( f' \). In fact, using equation (9) in the form

\[ \text{Ad} \circ \mathcal{D}_{[a]} = \mathcal{D}_{[a]} \circ \mathcal{D}_{[a]}^{-1} \]

it is easy to see that \( f' \equiv \mathcal{D}_{[a]}^{-1}(f_{[a]}) \) is invariant.

Evaluating two consecutive gauge transformations \( g_1, g_2 \) on \( \Psi = \Psi[\psi, a] \), \( \Psi \xrightarrow{g_1} G_{g_1}[a] * \Psi \xrightarrow{g_2} G_{g_2}[a_{g_2}] * G_{g_2}[a] * \Psi = G_{g_1 \cdot g_2}[a] * \Psi \), we obtain the gauge consistency condition

\[ G_{g_1}[a_{g_2}] * G_{g_2}[a] = G_{g_1 \cdot g_2}[a]. \]

One should note the appearance of the shifted gauge potential \( a_{g_2} \) in this formula – at first sight this seems to preclude the use of (13) to define transition functions of a noncommutative bundle, but it turns out that this is exactly what is needed. The finite noncommutative gauge transformation corresponding to \( g = e^{i\lambda} \) can be computed by evaluating \( e^{\delta \lambda} \) on a field \( \Psi \) using the gauge equivalence condition

\[ \delta \lambda \Psi[\psi, a] = i\Lambda \lambda[a] * \Psi[\psi, a] \]

repeatedly. One has to be careful though, because \( \delta \lambda \) not only affects \( \Psi \) but also \( a \) in \( \Lambda \lambda[a] \). This difficulty can be bypassed without having to resort to path-ordered exponentials with the following trick:

\[ (-\delta \lambda + i\Lambda \lambda[a] \Psi) = 0 \Rightarrow e^{\delta \lambda} \Psi[\psi, a] = (e^{\delta \lambda} e^{-\delta \lambda + i\Lambda \lambda[a] \Psi}) \Psi[\psi, a]. \]
Using the Baker-Campbell-Hausdorff formula the exponentials can be combined and one finds that there are no free $\delta$'s acting on the $\Psi[\psi, a]$. Thus $(G_{(e^{i\lambda})}[a])^* = (e^{\delta \lambda} e^{-\delta \lambda + i\Lambda \lambda}[a]^*)$. Using that the gauge transformation $\delta \lambda$ in this formula is actually just a shorthand for the functional expression

$$\delta \lambda = \int \delta \lambda(a(x)) \frac{\delta}{\delta a(x)} dx$$

and noting that for suitably chosen star product we can always insert a trailing $\star$ in this integral, we find

$$G_{(e^{i\lambda})}[a] = (e_{\star} \delta \lambda \star e_{\star}^{-\delta \lambda + i\Lambda \lambda}[a]) =: e_{\star} \Lambda'[\lambda][a],$$

where we have introduced the star exponential $e_{\star}$ by $(e_{\star} f \star g)$.

$G_{(e^{i\lambda})}[a]$ is clearly $\star$-invertible; its inverse is given by $G_{(e^{-i\lambda})}[a + d\lambda] = e_{\star}^{-\Lambda'[\lambda][a]}$.

The $G_g[a]$ play the role of “noncommutative group elements”$^5$ As in the case of $\star$ and $D[a]$ the existence of $G_g[a]$ is more important for the following then the explicit formula for it.

### 3 Noncommutative line bundle

#### 3.1 Classical line bundle

Let us recall that a classical (complex) line bundle is uniquely determined by a covering $\{U_k\}$ of a Manifold $M$ and a collection of transition functions $g_{jk} \in C^\infty(U_j \cap U_k)$ satisfying relations

$$g_{ij}g_{jk} = g_{ik},$$

$$g_{jk}g_{kj} = 1,$$

on all intersections $U_i \cap U_j \cap U_k$ and $U_j \cap U_k$ respectively. (Here and in the following there is no sum over repeated indices.)

A collection of functions $\psi_k \in C^\infty_c(U_k)$ satisfying

$$\psi_j = g_{jk} \psi_k$$

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$^4$For constant $\theta$ this is clear by partial integration for the general case, see $^{[10]}$.

$^5$It would be nice to get a better understanding of the relation to $^{[11]}$, where the gauge group of noncommutative gauge theory was studied in the operator formalism.
on the overlaps $U_j \cap U_k$ define a section $\psi = (\psi_k)$. A set of local 1-forms $a_k$, satisfying

$$a_j = a_k + d\lambda_{jk}, \quad d\lambda_{jk} \equiv ig_{jk}dg_{kj}$$

(21)
defines a connection on the line bundle. From (18) it follows that

$$\lambda_{ij} + \lambda_{jk} + \lambda_{ki} = 2\pi n, \quad n \in \mathbb{Z}.$$  

(22)

A connection always exists by the following construction: Consider a partition of unity $\{h_k^2\}$,

$$\sum_k h_k^2 = 1,$$

(23)

subordinate to the covering $\{U_k\}$ and define local 1-forms

$$a'_j = \sum_k ih_k^2g_{jk}dg_{kj}.$$  

(24)

These satisfy (21) by virtue of (18). The cohomological class $[da'] = [da]$ in $H^2(M)$ is the Chern class of the line bundle and is characterized by the number $n$ in (22).

Two line bundles $\{\tilde{g}_{jk}\}, \{g_{jk}\}$ are equivalent if there exists a collection of non-vanishing functions $f_k \in C^\infty(U_k)$ such that

$$\tilde{g}_{jk} = \zeta_jg_{jk}\zeta_k^{-1}.$$  

(25)

The corresponding expressions for the sections and gauge potentials are

$$\tilde{\psi}_k = \zeta_k\psi_k, \quad \tilde{a}_k = a_k + i\zeta_k d\zeta_k^{-1}.$$  

(26)

3.2 Noncommutative transition functions

The finite noncommutative gauge transformations introduced in section 2 depend on a local gauge potential defined in an open subset of $\mathbb{R}^n$. Assuming that we have the structure of a classical line bundle on a given Poisson manifold $M$ we can use the consistency condition to construct a “noncommutative” line bundle – the quantization (in the sense of deformation quantization) of the given line bundle. We choose a covering $\{U_k\}$ of $M$ such that the patches and all their non-empty intersections are diffeomorphic to $\mathbb{R}^n$. The $C^\infty$-functions on all these open subsets of $M$ become formal power
series in a deformation parameter. Obviously it makes sense to consider the restriction of the star product to any patch or any intersection of patches.

Choosing \(g_1 = g_{ij}, \; g_2 = g_{jk}, \; g_1 \cdot g_2 = g_{ik}\) and \(a = a_k\) in the consistency relation (13) gives the following relation in the intersection \(U_i \cap U_j \cap U_k\):

\[
G_{g_{ij}}[(a_k)_{g_{jk}}] \star G_{g_{jk}}[a_k] = G_{g_{ij}}[a_j] \star G_{g_{jk}}[a_k] = G_{g_{ik}}[a_k]
\] (27)

where we have used \((a_k)_{g_{jk}} = a_k + d\lambda_{jk} = a_j\). For the special case \(g_1 = g_{kj}, \; g_2 = g_{jk}\) with \(g_{kj}g_{jk} = 1\) we find an expression for the inverse of \(G_{g_{jk}}[a_k]:\)

\[
G_{g_{kj}}[(a_k)_{g_{jk}}] \star G_{g_{jk}}[a_k] = G_{g_{kj}}[a_j] \star G_{g_{jk}}[a_k] = 1
\] (28)

Similarly in the gauge equivalence relation (9) put \(g = g_{jk}\) and \(a = a_k\), then

\[
\mathcal{D}_{(a_k)_{g_{jk}}} \star G_{g_{jk}}[a_k] = \mathcal{D}_{[a_j]} \star G_{g_{jk}}[a_k] = G_{g_{jk}}[a_k] \star \mathcal{D}_{[a_k]}. \tag{29}
\]

It is consistent to use an abbreviated notation

\[
G_{jk} \equiv G_{g_{jk}}[a_k], \quad \mathcal{D}_k \equiv \mathcal{D}_{[a_k]}, \tag{30}
\]

The fundamental relations are then

\[
G_{ij} \star G_{jk} = G_{ik}, \quad G_{kj} \star G_{jk} = 1 \tag{31}
\]

and

\[
\mathcal{D}_j \star G_{jk} = G_{jk} \star \mathcal{D}_k. \tag{32}
\]

(There is no summation over \(j\) or \(k\) in these formulas.)

In view of (31), the \(G_{jk}\) play the role of noncommutative transition functions. The collection of \(G_{jk} \in C^\infty(U_j \cap U_k)[[\hbar]]\) is a good candidate for a noncommutative line bundle in the sense of deformation quantization. As we will see later the \(D_k\) play the role of a local connection.

We shall now study the freedom in the construction of the \(G_{jk}\). The \(G_{jk}\) depend explicitly on a classical connection \(a_k\). For a given classical line bundle, i.e. fixed \(g_{jk}\), the choice of different \(a_k\) only changes the star product on to one in the same equivalence class. The reason is that the new \(a_k\) differ from the old ones by a global one-form \(b\). The equivalence is given by \(\mathcal{D}[b]\). For an equivalent classical line bundle with transition functions \(\tilde{g}_{jk} = \zeta_j g_{jk} \zeta_k^{-1}\) and local connection forms \(\tilde{a}_k = a_k + d\zeta_k\) we find new transition functions for the noncommutative line bundle of the form

\[
\tilde{G}_{jk} = G_{\zeta_j}[a_j] \star G_{jk} \star (G_{\zeta_k}[a_k])^{-1}. \tag{33}
\]
Here we have twice used the consistency relation (13). The additional freedom from deformation quantization is that we should consider equivalence classes of star products on each patch. This is related to the further requirement that the equivalence classes of noncommutative line bundles should be independent of the choice of coordinates in the individual patches \[7\]. Such coordinate changes in patches $U_j$ and $U_k$ induce equivalence maps $\Sigma_j, \Sigma_k$ such that $\text{Ad}_\ast G_{jk}$ becomes $\Sigma_j \circ \text{Ad}_\ast G_{jk} \circ (\Sigma_k')^{-1}$. The classical freedom discussed above is included as a special case.

### 3.3 Quantized line bundle

We have explicitly constructed a candidate for a noncommutative line bundle starting from a classical line bundle and a solution $G_{g[a]}$ to the consistency relation. Now we would now like to collect the essential properties of a noncommutative line bundle in the framework of deformation quantization:

Let \( \{g_{jk}\} \) be a classical line bundle on a Poisson manifold \((M, \theta)\) and let \( \star \) be a star product (deformation quantization of \( \theta \)). We define a noncommutative line bundle \((G, \star)\) to be a set of noncommutative transition functions $G_{jk}$ of the following form:

The $G_{jk} \equiv G_{g_{jk}[a_k]} \in C_\infty^\infty(U_j \cap U_k)[[\hbar]]$ are local functions of the classical transition functions $g_{jk}$ and of a given connection $a_k$ and their derivatives satisfying

$$
G_{ij} \star G_{jk} = G_{ik}, \quad G_{kk} = 1, \quad (34)
$$

with the commutative limit \((\theta = 0)\): $G_{jk} \rightarrow g_{jk}$.

We call two line bundles \((G, \star), (G', \star')\) equivalent if they are based on equivalent classical line bundles, $g_{jk} \sim g'_{jk}$, and there are equivalence maps $\Sigma_k$ on every patch $U_k$ relating $\star$ and $\star'$.

(Here $\star$ and $\star'$ can possibly be quantizations of two different Poisson structures $\theta$ and $\theta'$.)

In place of connection forms we introduce invertible maps $D_k$ that define covariant functions $D_k(f)$ on every patch:

The $D_k \equiv D_{[a_k]} : C_\infty^\infty(U_k)[[\hbar]] \rightarrow C_\infty^\infty(U_k)[[\hbar]]$ are local functions of the given connection $a_k$ and its derivatives, with the properties that

$$
D_j(f) \star G_{jk} = G_{jk} \star D_k(f), \quad (35)
$$
(or, equivalently, $\text{Ad}_* G_{jk} = D_j \circ D_k^{-1}$) and that

$$D_k(f \star' g) = D_k f \star D_k g$$

(36)

defines a new star product $\star'$. Let us note that if this is the case, $\star'$ is independent of the patch $U_k$, i.e., it depends on the $a_k$ only via the global 2-form $f = da$.

Choosing appropriate representatives in the equivalence classes one can introduce the tensor product of two noncommutative line bundles. Namely

$$G''_{jk} = D_k (G'_{jk}) \star G_{jk}$$

(37)
satisfies again the consistency condition with star product $\star$. Here $G'_{jk} = G'_{g'_{jk}} [a'_k]$ satisfies the consistency condition with the star product $\star'$ (36).

The line bundle based on $G''$ is equivalent to the one based on $G_{g'g}[a' + a]$.

4 Sections

A section $\Psi = (\Psi_k)$ is a collection of functions $\Psi_k \in C^\infty(U_k)[[\hbar]]$ satisfying consistency relations

$$\Psi_j = G_{jk} \star \Psi_k$$

(38)
on all intersections $U_j \cap U_k$. With this definition the space of sections $\mathcal{E}$ is a right $\mathfrak{A}$ module. We shall use the notation $\mathcal{E}_\mathfrak{A}$ for it. The right action of the function $f \in \mathfrak{A}$ is the regular one

$$\Psi.f = (\Psi_k \star f).$$

(39)

Using the maps $D_k$ it is easy to turn $\mathcal{E}$ into a left $\mathfrak{A}' = (C^\infty_c(M)[[\hbar]], \star')$ module $\mathfrak{A}' \mathcal{E}$. The left action of $\mathfrak{A}'$ is given by

$$f.\Psi = (D_k(f) \star \Psi_k).$$

(40)

It is easy to check using (35) that the left action is compatible with (38), in fact this was the reason why we introduced covariant functions in the first place. From the property (36) of the maps $D_k$ we find

$$f.(g.\Psi) = (f \star' g).\Psi.$$ 

(41)

Together we have a bimodule structure $\mathfrak{A}' \mathcal{E}_\mathfrak{A}$ on the space of sections.
5 Connection

Let us note that using the Hochschild complex we can introduce a natural differential calculus on the algebra \( A \). The \( p \)-cochains, elements of \( C^p = \text{Hom}_C(\mathfrak{A}^p, \mathfrak{A}) \) play the role of \( p \)-forms and the derivation \( d : C^p \to C^{p+1} \) is given on \( C \in C^p \) as

\[
(dC)(f_1, f_2, \ldots, f_{p+1}) = f_1 \ast C(f_2, \ldots, f_{p+1}) - C(f_1 \ast f_2, \ldots, f_{p+1}) \\
+ C(f_1, f_2 \ast f_3, \ldots, f_{p+1}) - \ldots + (-1)^p C(f_1, f_2, \ldots, f_p \ast f_{p+1}) \\
+ (-1)^{p+1} C(f_1, f_2, \ldots, f_p) \ast f_{p+1}.
\]

(42)

We also have the cup product \( C_1 \cup C_2 \) of two cochains \( C_1 \in C^p \) and \( C_2 \in C^q \);

\[
(C_1 \cup C_2)(f_1, \ldots, f_{p+q}) = C_1(f_1, \ldots, f_p) \ast C_2(f_{p+1}, \ldots, f_q).
\]

(43)

The cup product extends to a map from \( (\mathcal{E} \otimes_{\mathfrak{A}} C^p) \otimes_{\mathfrak{A}} C^q \) to \( \mathcal{E} \otimes_{\mathfrak{A}} C^{p+q} \).

A connection \( \nabla : \mathcal{E} \otimes_{\mathfrak{A}} C^p \to \mathcal{E} \otimes_{\mathfrak{A}} C^{p+1} \) can now be defined by a formula similar to (42) using the natural extension of the left and right module structure of \( \mathcal{E} \) to \( \mathcal{E} \otimes_{\mathfrak{A}} C^p \). Namely, for a \( \Phi \in \mathcal{E} \otimes_{\mathfrak{A}} C^p \) we have

\[
(\nabla \Phi)(f_1, f_2, \ldots, f_{p+1}) = f_1 \Phi(f_2, \ldots, f_{p+1}) - \Phi(f_1 \ast f_2, \ldots, f_{p+1}) \\
+ \Phi(f_1, f_2 \ast f_3, \ldots, f_{p+1}) - \ldots + (-1)^p \Phi(f_1, f_2, \ldots, f_p \ast f_{p+1}) \\
+ (-1)^{p+1} \Phi(f_1, f_2, \ldots, f_p) \ast f_{p+1}.
\]

(44)

It is easy to check that \( \nabla \) satisfies the graded Leibniz rule with respect to the cup product and thus defines a bona fide connection on the module \( \mathcal{E}_{\mathfrak{A}} \).

On the sections the connection \( \nabla \) introduced here is simply the difference between the two actions of \( C^\infty_\mathfrak{A}(M)[[\hbar]] \) on \( \mathcal{E} \):

\[
(\nabla \Psi)(f) = f.\Psi - \Psi.f = (\mathcal{D}_k(f) \ast \Psi_k - \Psi_k \ast f).
\]

(45)

In terms of \( \mathcal{A}_k \equiv \mathcal{D}_k - \text{id} \) this reads \( \nabla \Psi = (d\Psi_k + \mathcal{A}_k \ast \Psi_k) \).

A simple computation using the definition (44) reveals that the corresponding curvature \( K_\nabla \equiv \nabla^2 : \mathcal{E} \otimes_{\mathfrak{A}} C^p \to \mathcal{E} \otimes_{\mathfrak{A}} C^{p+2} \) measures the difference between the two star products \( \ast' \) and \( \ast \). On a section \( \Psi \) it is given by

\[
(K_\nabla \Psi)(f, g) = (\mathcal{D}_k(f \ast' g - f \ast g) \ast \Psi_k)
\]

(46)
6 Projective module

Now we show that in the case of a compact Poisson manifold \( M \) the space of sections \( \mathcal{E} \) as a right \( \mathfrak{A} \)-module is projective of finite type. We need to introduce a \(*\)-covariant partition of unity: We start with a partition of unity \( \sum_i \rho_i = 1, \rho_i \in \mathcal{C}^\infty(M) \), subordinate to the finite covering \( \{ U_i \}_{i=1}^n \) of \( M \). We define functions \( h_i \in \mathcal{C}^\infty(\mathcal{E}(M)[[\hbar]] \) with support in \( U_i \) as \( h_i = \sqrt{\rho_i} \). In the sense of formal power series the \( h_i \) are \(*\)-invertible on \( U_i \). On any patch \( U_k \) we have the following property

\[
\sum_i \mathcal{D}_k(h_i) \ast \mathcal{D}_k(h_i) = 1. \tag{47}
\]

There is actually no need to know the maps \( \mathcal{D}_k \) on all of \( \mathfrak{A} \). It is enough to know the family of functions \( H_{kj} = \mathcal{D}_k(h_j) \) that feature in the \(*\)-covariant partition of unity. These functions can also be introduced abstractly by the following two requirements for all \( k \)

\[
\sum_j H_{kj} \ast H_{kj} = 1, \quad H_{ik} \ast G_{ij} = G_{ij} \ast H_{jk}. \tag{48}
\]

Maps \( \tilde{\mathcal{D}}_k \) with property (35) can always be found by a formula analogously to (24):

\[
\tilde{\mathcal{D}}_k(f) = \sum_j H_{kj} \ast H_{kj} \ast G_{kj} \ast f \ast G_{jk}. \tag{49}
\]

Equation (48) together with (34) can be taken as a starting point for an abstract definition of a noncommutative line bundle even for an associative algebra which not necessarily given by a star product.

Next we define a right \( \mathfrak{A} \)-module morphism \( \epsilon : \mathcal{E} \to \mathfrak{A}^n \). It sends a section \( \Psi = (\Psi_j) \) to the \( n \)-tuple \( (t_j) \) with

\[
t_j = \mathcal{D}_j(h_j) \ast \Psi_j \in \mathfrak{A}. \tag{50}
\]

There is also a surjection \( \pi : \mathfrak{A}^n \to \mathcal{E} \) which sends the \( n \)-tuple \( (t_k) \) to the section \( \Psi = (\Psi_k) \) with

\[
\Psi_k = \sum_j G_{kj} \ast \mathcal{D}_j(h_j) \ast t_j. \tag{51}
\]
This is obviously also a right $\mathcal{A}$-module morphism and it is also easy to check that $\Psi_k$ thus defined satisfies (38). The composition $\pi \circ \epsilon$ is the identity on $\mathcal{E}$ as we can check using (47):

$$
\sum_j G_{kj} \ast \mathcal{D}_j(h_j) \ast \mathcal{D}_j(h_j) \ast \Psi_j = \sum_j \mathcal{D}_k(h_j) \ast \mathcal{D}_k(h_j) \ast G_{kj} \ast \Psi_j
$$

$$
= \sum_j \mathcal{D}_k(h_j) \ast \mathcal{D}_k(h_j) \ast \Psi_k = \Psi_k.
$$

The opposite composition $\epsilon \circ \pi : \mathcal{A}^n \to \mathcal{A}^n$ is the projection defined by the projector $P$

$$
P_{ij} = \mathcal{D}_i(h_i) \ast G_{ij} \ast \mathcal{D}_j(h_j),
$$

with

$$
P_{ij} = \sum_k P_{ik} \ast P_{kj}.
$$

This makes $\mathcal{E}_{\mathcal{A}}$ to a projective right $\mathcal{A}$-module of finite type. The sections $\Psi \in \mathcal{E}$ can now be identified through the embedding (50) with elements $t = (t_j)$ of the free module $\mathcal{A}^n$ satisfying

$$
\sum_j P_{tj} \ast t_j = t_i.
$$

Within this identification the right $\mathcal{A}$ and the left $\mathcal{A}'$ actions on $\epsilon(\mathcal{E}) \subset \mathcal{A}^n$ look like

$$
t.f = (t_j \ast f)
$$

and

$$
f.t = ((\mathcal{D}_j(h_j) \ast \mathcal{D}_j(f) \ast \mathcal{D}_j(h_j)^{-1}) \ast t_j).
$$

Let us note that $\mathcal{E}$ naturally has also the structure of a left projective $\mathcal{A}'$-module. Namely all the construction above can be easily modified as follows. We start with an equivalent line bundle $(G'_{jk} = \mathcal{D}_j^{-1}(G_{jk})^*,*)$ and take for sections of this line bundle collections of functions

$$
\Psi'_k = (\mathcal{D}_k^{-1}(\Psi_k)).
$$

Thus

$$
\Psi'_k = \Psi'_j \ast G'_{jk}.
$$

This gives an equivalent description of $\mathcal{A}'\mathcal{E}_{\mathcal{A}}$ as a bimodule. Using now a $\ast'$-covariant partition of unity $h'_k$ we have the projector

$$
P'_{jk} = \mathcal{D}_j^{-1}(h'_j) \ast G'_{jk} \ast \mathcal{D}_k^{-1}(h'_k),
$$

and the identification $\mathcal{A}'\mathcal{E} \cong \mathcal{A}'^n P'$ of left $\mathcal{A}'$ modules.
7 Morita equivalence

According to the definition of the two star products \( \star \) and \( \star' \), these are equivalent in each individual patch \( U_k \). Obviously they are equivalent on the whole of \( M \) in the special case if the classical line bundle \( \{g_{ij}\} \) that we take as a starting point is trivial. Then there is a globally defined connection one-form \( a \) and consequently a globally defined covariantizing map \( D \). We recall that in the case of a line bundle which is based on the particular choice of transition functions \( G_{ij} \) described in section 2, \( \star' \) is the deformation quantization of the Poisson tensor \( \theta' = \theta - \theta f \theta + \theta f \theta f \theta + \ldots \).

Here we want to show that in the case of a general line bundle \( \{g_{ij}\} \) the two star products \( \star \) and \( \star' \) define Morita equivalent algebras. For this let us introduce a new space of sections \( \overline{E} \). Sections in \( \overline{E} \) are collections of functions \( \overline{\Psi} = (\overline{\Psi}_k) \) satisfying the opposite consistency relations

\[
\overline{\Psi}_j = \overline{\Psi}_k \star G_{kj} \quad (60)
\]

on all intersections \( U_j \cap U_k \). There are left module morphisms \( \overline{\tau} : \overline{E} \to \mathfrak{A}^n \) and \( \overline{\pi} : \mathfrak{A}^n \to \overline{E} \) given by

\[
(\overline{\psi}_j) \mapsto (\overline{t}_j) = (\overline{\psi}_j \star D_j(h_j)), \quad (61)
\]

\[
(\overline{t}_k) \mapsto (\overline{\psi}_k) = (\sum_j \overline{t}_j \star D_j(h_j) \star G_{jk}). \quad (62)
\]

The sections in \( \overline{E} \) can be identified with elements \( \overline{t} = (\overline{t}_j) \) of the free module \( \mathfrak{A}^n \) satisfying \( \sum_i \overline{t}_i \star P_{ij} = \overline{t}_j \). Clearly with this definition the roles of the algebras \( \mathfrak{A} \) and \( \mathfrak{A}' \) interchange and the space of sections \( \overline{E} \) becomes a \( \mathfrak{A} \mathfrak{A}' \) bimodule. One way to show the Morita equivalence of \( \mathfrak{A} \) and \( \mathfrak{A}' \) is to prove that

\[
\mathfrak{A} \mathfrak{E} \otimes_{\mathfrak{A}} \mathfrak{E} \mathfrak{A}' \cong \mathfrak{A} \mathfrak{A}' \mathfrak{E} \quad (63)
\]

and

\[
\mathfrak{A} \mathfrak{E} \otimes_{\mathfrak{A}} \mathfrak{E} \mathfrak{A} \cong \mathfrak{A} \mathfrak{E} \mathfrak{A} \quad (64)
\]

as bimodules. This will be done in the rest of this section.

We start with the following observation: For any \( M \in M_n(\mathfrak{A}) \),

\[
\sum_{ij} P_{ri} \star M_{ij} \star P_{js} = D_r(h_r) \star D_r(f_M) \star D_r(h_r)^{-1} \star P_{rs}, \quad (65)
\]
with the globally defined function

$$f_M = \sum_{ij} D_i^{-1}(D_i(h_i) * M_{ij} * D_j(h_j) * G_{ji}) \in C^\infty_c(M)[[h]].$$  \hspace{1cm} (66)$$

The proof uses $G_{js} = G_{ji} * G_{ir} * G_{rs}$ in $U_j \cap U_i \cap U_r \cap U_s$ and $G_{ri} * (\ldots) * G_{ir} = D_r(D_i^{-1}(\ldots))$. Similarly

$$\sum_{ij} P_{ri} * M_{ij} * P_{js} = P_{rs} * D_s(h_s)^{-1} * D_s(f_M) * D_s(h_s).$$  \hspace{1cm} (67)$$

Note that $P_{rs}$ is a section in $\mathcal{E}$ for every fixed $s$ under the embedding $\tilde{\epsilon}$. In this language, (67) is just the left $\mathfrak{A}'$-action (56) of $f_M$ on this section. Using an analogous formula for the right $\mathfrak{A}'$-action on $\tilde{\mathcal{E}}$ we see that (67) is the right $\mathfrak{A}'$-action of $f_M$ on $P_{rs}$ viewed as a section in $\tilde{\mathcal{E}}$ for fixed $r$ under the embedding $\tilde{\epsilon}$. With slight abuse of notation we can summarize (65) and (67) in matrix form as

$$f_M \cdot P = P * M * P = P \cdot f_M.$$  \hspace{1cm} (68)$$

Again under the embeddings $\epsilon$ and $\tilde{\epsilon}$, the tensor product of two arbitrary sections in $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively, takes the form

$$\sum_{ijs} P_{ri} * M_{ij} * P_{js} \otimes A P_{sm}.$$  \hspace{1cm} (70)$$

This implies that sections in $\mathcal{E} \otimes A \tilde{\mathcal{E}}$ have the form

$$\sum_{ij} P_{ri} * M_{ij} * P_{js} \otimes A P_{sm}.$$  \hspace{1cm} (70)$$

with $M \in M_n(\mathfrak{A})$ and that $\Psi_{ij} = (P_{ri} \otimes \mathfrak{A} P_{js})$ is a generating family for $\mathcal{E} \otimes A \tilde{\mathcal{E}}$. Using (65) and (67) in the form (68), we find the following chain of equalities (in matrix notation)

$$f_M \cdot P \otimes A P = P * M * P \otimes A P = P \otimes A P \cdot f_M$$  \hspace{1cm} (71)$$

which proofs (63) and, in view of the remarks at the end of the previous section, also (64). To summarize, we have shown that the algebras $(\mathfrak{A}, *)$ and $(\mathfrak{A}', *)'$ are Morita equivalent by an explicit construction of equivalence bimodules $\mathcal{E}$ and $\tilde{\mathcal{E}}$. 

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8 Discussion

Here we would like to make some comments, without going into details, relating our discussion of Morita equivalence to earlier works [13], [12] on the deformation of projective modules and the Morita equivalence of \(*\)-products which are deformation quantizations of some given Poisson structure \(\theta\). For this purpose we have to assume that the connection \(a\) enters all expressions \((G_g[a], D_{[a]}, \theta', \ldots)\) in the main body of our paper in the form \(h a\). Furthermore we want to focus on the case where all constructions of our paper are done with the explicit choice of noncommutative transition functions \(G_{jk}\) and of covariantizing maps \(D_k\) as described in 2. Different choices would lead to projective modules which are equivalent in the sense of [13]. The choice of a different covariant partition of unity would have the same effect.

In the case of a nontrivial classical line bundle, the Poisson structures \(\theta\) and \(\theta' = \sum_{n=0}^{\infty} (-h)^n \theta(f \theta)^n\) live in two different equivalence classes in the sense of Kontsevich [8]. Let us recall that equivalence classes of star products are in one to one correspondence with equivalence classes of Poisson structures. In the case of a trivial line bundle \(\theta\) and \(\theta'\) are equivalent.

We can now modify the action [12] of the Picard group \(\text{Pic}(C^\infty(M)) \cong \text{Pic}(M) \cong H^2(M, \mathbb{Z})\) on the equivalence classes of the star products quantizing \(\theta\). An element \(f \in H^2(M, \mathbb{Z})\) simply sends the equivalence class \([\star]\) of star products corresponding to the equivalence class of Poisson structures \([\theta]\) to the equivalence class \([\star']\) corresponding to the equivalence class of Poisson structures \([\theta']\). With the modified action of \(\text{Pic}(M)\) two star-products quantizing the same Poisson structure are also Morita equivalent if and only if they are related by the action of an element of \(\text{Pic}(C^\infty(M))\). This can be shown either directly or follows from the comparison of the action of \(\text{Pic}(C^\infty(M))\) introduced here with the one of the paper [12]. Since our projector \(P\), see equation (52), is a deformation of the classical full projector \(P_0 = P(h = 0)\) it is also a full one [13]. We can use it in the construction of the \(\text{Pic}(C^\infty(M))\) action of [12]. When we do this the two actions when applied to \(\star\) give equivalent star products on \(M\). The corresponding equivalence map \(\tilde{s}\) sends the function \(f\) to the function \(\tilde{f} = \sum_i D_i^{-1}(D_i(h_i) \star f \star D_i(h_i))\). The function \(\tilde{f}\) is simply the function \(f_M\) of the previous section for the diagonal matrix \(M = \text{diag}(f, \ldots, f)\).

\(^6\)For simplicity we have written the formula for \(\theta'\) for some choice of local coordinates, matrix multiplication is implied and \(f\) is the coefficient matrix of the curvature form \(d a\).
Let us finish with a note concerning an alternative proof of Morita equivalence of $A$ and $A'$. From the discussion of the previous section it also follows that $A' \cong P \ast M(A) \ast P$ with the full projector $P$. This is just another way to express the Morita equivalence of $A$ and $A'$. 

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