ANALYSIS OF NONLINEAR FRACTIONAL DIFFUSION EQUATIONS WITH A RIEMANN-LIOUVILLE DERIVATIVE

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Abstract. In this paper, we consider a nonlinear fractional diffusion equations with a Riemann-Liouville derivative. First, we establish the global existence and uniqueness of mild solutions under some assumptions on the input data. Some regularity results for the mild solution and its derivatives of fractional orders are also derived. Our key idea is to combine the theories of Mittag-Leffler functions, Banach fixed point theorem and some Sobolev embeddings.

1. Introduction. Fractional derivatives and fractional calculus was studied by many mathematicians and physicists because of applications in potential theory, physics, electrochemistry, biophysics, viscoelasticity, biomedicine, control theory and signal processing, see e.g. [26, 21] and the references therein. They play important role in different fields of science (anomalous diffusion, non-exponential relaxation, etc.), and fractional differential equations have been derived by considering different physical systems (continuous time random walk processes, diffusion in comb-like structures, viscoelastic models, etc.).

The time fractional diffusion equation was first introduced by Nigmatullin [24], which models a diffusion in media with fractal geometry. In [30], Metzler-Klafter showed that non-Markovian diffusion processes with a memory can be modeled by a time fractional diffusion equation. These time fractional equations have become a

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primary component in the fields of partial differential equations and has attracted much attention; see for example [27, 18, 3, 13, 21, 15, 19, 30].

1.1. Model settings. Let $\Omega$ be a $C^2$ bounded open set of $\mathbb{R}^N$ with sufficient smooth boundary $\Gamma$, and $T > 0$. In this paper, we consider the following time fractional diffusion equation

$$
\begin{cases}
RL \partial_t^\beta u + Au = F(t, u(t, x)), & \text{in } (0, T) \times \Omega, \\
u(t, x) = 0, & \text{on } (0, T) \times \Gamma, \\
t^{1-\beta}u(t, x)|_{t=0} = \varphi(x), & \text{in } \Omega,
\end{cases}
$$

(1)

where $RL \partial_t^\beta$ represents the Riemann-Liouville derivative of fractional order $\beta$ given by

$$ RL \partial_t^\beta u(t, x) \defeq \frac{d}{dt} \left( RL I_{1-\beta} t u(t, x) \right) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{\psi(s)}{(t-s)^\beta} ds, \quad 0 < \beta < 1, $$

and

$$ RL \partial_t^\alpha u(t, x) = \frac{d}{dt} u(t, x), \quad \beta = 1. $$

Note from the above definition, $RL \partial_t^\beta$ is non-locally defined for $0 < \beta < 1$. In this work, $A$ is an uniformly elliptic operator on $\Omega$ with smooth coefficients defined by

$$ Ah(x) = -\sum_{n=1}^N \frac{\partial}{\partial x_n} \left( \sum_{j=1}^N a_{ij}(x) \frac{\partial}{\partial x_j} h(x) \right) + a_\sharp(x) h(x), \quad x \in \Omega, $$

where $a_{ij} \in C^1(\Omega)$, $a_\sharp \in C(\overline{\Omega}; [0, +\infty))$ and $a_{ij} = a_{ji}, 1 \leq i, j \leq N$. Suppose that, there is a positive constant $\Lambda$ such that for $x \in \Omega$, $z = (z_1, z_2, ..., z_k) \in \mathbb{R}^N$, we have

$$ \sum_{1 \leq i,j \leq N} z_i a_{ij}(x) z_j \geq \Lambda |z|^2. $$

Here, a special example of $A$ is the negative Laplace operator $-\Delta$.

The case $\beta = 1$ corresponds to the standard parabolic equation, which was shown to enable investigations of random walk processes in particular and sub-diffusive phenomena in general. The case $0 < \beta < 1$ corresponds to the time-fractional diffusion equation, which can be instantly interpreted from the basis of continuous time random walk (CTRW) [15]. Assume that the length of a given jump and the waiting time elapsing between two consecutively successive jumps (of a wandering particle) are drawn from a probability density function (pdf) $\psi(x, t)$. In the context of the characteristic waiting time $\tau$ diverges but the jump length variance $\Sigma^2$ is finite, and a long-tailed waiting time probability density function

$$ \int_{\mathbb{R}} \psi(x, t) dx \sim (\tau/t)^{1+\beta}, \quad 0 < \beta < 1, $$

was introduced; see Subsection 3.4 in Metzler-Klafter [22]. In the Fourier-Laplace space, the pdf $W$ of being in $x$ at time $t$ is given by

$$ W(k, \zeta) = \frac{W_0(k)/\zeta}{1 + K_{\beta} k^{1-\beta}}, \quad K_{\beta} = \frac{\Sigma^2}{2\tau^2}, $$
(\(W_0\) denotes the propagator at the time \(t = 0\)) which consequently produces a linear diffusion equation for the propagator \(W\) containing the time fractional derivative in the Riemann–Liouville or Caputo sense

\[
RL \partial_t^\beta W(x, t) - K_\beta \Delta W(x, t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} W_0(x), \quad t > 0,
\]

where \(K_\beta\) is called the generalized diffusion constant.

In the model, \(F\) is a given function standing for the density of sources, which can be explained by borrowing ideas from CTRW; see [20, 1]. The initial state with the inverse power-law, the condition \(t^{1-\beta} u(t, x)|_{t=0} = \varphi(x)\) is natural and suitable (see Remark 3 for more explanations). This condition or the fractional integral condition \(I_t^{1-\beta} u(0, x) = \varphi(x)\) are suitable and can be transformed into each others; see [30].

Our model in this paper comes from the fractional ODE introduced in the recent paper [5]. The authors applied the Banach contraction principle and Schaefer’s fixed point theorem to establish the existence and uniqueness of solutions for the initial value problem

\[
RL \partial_t^\beta u = F(t, u(t)), \quad t > 0, \quad t^{1-\beta} u(t)|_{t=0} = \varphi,
\]

where the state spaces or the solution spaces that one considers are Euclidean spaces \(\mathbb{R}^N\). For the corresponding PDE model of (1), Sandev-Metzler-Tomovski [28] considered a linear problem, i.e. \(f\) does not depend on \(u\), on a one-dimensional bounded domain, where the authors used the Fourier–Laplace transform method to find the fundamental solution. The existence of solutions was proved using spectral expansions. Studying fractional PDE models requires us to investigate functional spaces on which \(A\) operates, and these are usually infinite dimensional spaces. To the best of the authors’ knowledge, this is one of the first works that analyze the nonlinear problem (1).

1.2. Contributions. Our current study focuses on the theory of regularity structures for Problem (1). Regularity theory enables us to improve the smoothness and stability of solutions in various solution spaces and this leads to efficient ways for numerical simulations. If the Riemann-Liouville derivative \(RL \partial_t^\beta\) is replaced by the Caputo derivative in Problem (1) with the initial datum \(\varphi\) then we have

\[
\begin{aligned}
C \partial_t^\beta u + Au &= F(t, u(t)), \quad \text{in } (0, T) \times \Omega, \\
u(t, x) &= 0, \quad \text{on } (0, T) \times \Gamma, \\
u(x, 0) &= \varphi(x), \quad \text{in } \Omega,
\end{aligned}
\]

where \(C \partial_t^\beta\) in the first equation of (2) represents the Caputo derivative of fractional order \(\beta\). Problem (2) was considered by many authors [13, 27, 19, 30]. If \(F = 0\) (the homogeneous source case) the homogeneous solution of (2) belongs to the space \(L^\infty(\Omega)\) when the input data \(\varphi \in L^2(\Omega)\). If \(f \neq 0\), solutions of the respective problem are given by

\[
u_C(t, x) = E_{\beta,1}(-t^\beta A)\varphi(x) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)F(s, u(s, x))ds,
\]

where the operators \(E_{\beta,1}(-t^\beta A)\) and \(E_{\beta,\beta}(-t^\beta A)\) are bounded on \(L^2(\Omega)\). In fact, under a suitably global Lipschitz assumption on \(F\) with the Lipschitz coefficient
\[ K, \text{ one can establish an existence of solutions in } L^\infty(0, T; L^2(\Omega)) \text{ without any restrictions on } T \text{ and } K. \] However, properties of solutions of Problem (1) are very different. Here, its solutions (1) are given by

\[ u_{RL}(t, x) = \Gamma(\beta)t^{\beta-1}E_{\beta, \beta}(-t^\beta A)\varphi(x) + \int_0^t (t-s)^{\beta-1}E_{\beta, \beta}(-(t-s)^\beta A)F(s, u(s, x))ds, \]

which is singular at \( t = 0 \). Therefore, the function \( u_{RL} \) may not belong to \( L^\infty(0, T; L^2(\Omega)) \); see Remark 2 for furthermore discussions.

In this paper, we will find solutions in the subspace \( C^{\alpha, \mu}_{w, n}(\Omega; \mathbb{H}^\sigma(\Omega)) \). To obtain a solution in this subspace without any restrictions on \( T \) and \( K \), we need the asymptotic behavior of a special function, called the Kummer function or the hypergeometric function [11]. If the initial state \( \varphi \) is taken from the Hilbert scales space \( \mathbb{H}^\gamma(\Omega) \) of the spatial variable \( x \) we obtain the existence in \( C^{\alpha, \mu}_{w, n}(\Omega; \mathbb{H}^\sigma(\Omega)) \), but if \( \varphi \) is considered in a Sobolev space \( W^q(\Omega) \) we obtain the existence in \( L^p(0, T; L^q(\Omega)) \). Our main tool for the existence of the mild solution is to apply the Banach fixed point theorem and to do this we need to construct embeddings between the Hilbert and Sobolev-Slobodecki spaces. Researchers are interested on integer order PDEs in Sobolev-Slobodecki spaces; see for example, [6, 7, 8, 12]. However, the results on regularity for fractional PDEs on Sobolev-Slobodecki spaces are very limited. Studying the regularity estimate for Sobolev-Slobodecki spaces for fractional PDEs is a new and challenging direction. In recent work [19], B. Li and X. Xie considered some regularity results of solutions to time fractional diffusion equations. They introduce Sobolev-Slobodecki spaces but however the error analysis for these space was not addressed. Our current work is one of the first investigations on regularity estimates for Sobolev-Slobodecki spaces for fractional PDEs.

1.3. Organization. The outline of this paper is as follows. In section 2 we introduce some terminology used throughout this work. In section 3, we state the existence and uniqueness of the mild solution of Problem (1). In section 4, we give regularity result for mild solutions and its derivative. Theorem 4.1 gives the regularity result of the solution \( u \) if the initial state \( \varphi \in W^q(\Omega) \). Theorem 4.2 gives the regularity result of the Riemann-Liouville fractional derivative of the mild solution \( (RL_t^\beta u) \).

2. Preliminaries.

2.1. Hilbert and Sobolev settings. Throughout this paper, we consider the operator \( A \) acting from \( D(A) = W^{1, 2}_0(\Omega) \cap W^{2, 2}(\Omega) \) to \( L^2(\Omega) \). Note that \( A \) is a symmetric uniformly elliptic operator, hence it possesses a non-negative, non-decreasing and discrete spectrum \( 0 < \rho_1 \leq \rho_2 \leq \ldots \leq \rho_n \searrow \infty \). The corresponding eigenvectors of \( A \) are denoted by \( b_n \in D(A) \), which satisfy \( Ab_n(x) = \rho_n b_n(x) \) for \( x \in \Omega \). Let us define the following space

\[ \mathbb{H}^\gamma(\Omega) := \left\{ v = \sum_{n=1}^\infty \hat{v}_n b_n \in L^2(\Omega) : \|v\|_{\mathbb{H}^\gamma} := \left( \sum_{n=1}^\infty \|\hat{v}_n\|^2 \rho_n^\gamma \right)^{\frac{1}{2}} < \infty \right\}, \]

where \( \hat{v}_n := \int_\Omega v(x)b_n(x)dx \) denotes the \( n \)-th Fourier coefficient of \( v \).

In this paper, let \( W^{\gamma,q}(\Omega), \gamma \in \mathbb{R}_+ \) be the Sobolev-Slobodecki space (see Subsection 3.1.3 in [9], Subsection 2.1. in [10]). Let \( W^{1, 2}_0(\Omega) \) be the Lions–Magenes
space which was introduced in Subsection 2.1 in [4]. Then, from [9, 10, 4] the space \( H^\gamma(\Omega) \) is defined by

\[
H^\gamma(\Omega) = \begin{cases} 
W_0^{\gamma,2}(\Omega), & \text{if } \gamma \in (0, \frac{1}{2}) \cup \left( \frac{1}{2}, 1 \right), \\
W_0^{\frac{1}{2},2}(\Omega), & \text{if } \gamma = \frac{1}{2}, \\
W_0^{1,2}(\Omega) \cap W^{\gamma,2}(\Omega), & \text{if } (1, 2].
\end{cases}
\]

Let us also denote by \( \gamma \) where

\[
\delta
\]

is equal to the Kronecker delta \( \delta \) where we recall

\[
Ab
\]

one-dimensional equations

\[
\hat{A}
\]

Laplace operator is defined by

\[
W
\]

Let

\[
\mathrm{Mild}
\]

concept. 2.2. \textbf{Mild concept.} In order to find a representation formula for solutions, we write \( u(t, x) = \sum_{n=1}^{\infty} \hat{u}_n(t) b_n(x) \), then substitute this series into the first equation of Problem (1). Multiplying two sides of the result by \( b \) and taking the integral over \( \Omega \), we obtain

\[
\int_{\Omega} RL \partial_t^\beta \sum_{n=1}^{\infty} \hat{u}_n(t) b_n(x) b_m(x) dx + \int_{\Omega} \sum_{n=1}^{\infty} \rho_n \hat{u}_n(t) b_n(x) b_m(x) dx = \int_{\Omega} F(u(t, x)) b_n(x) b_m(x) dx,
\]

where we recall \( Ab_n = \rho_n b_n \). By noting that the \( m \)-th Fourier coefficient of \( b_n \) is equal to the Kronecker delta \( \delta_{mn} \), and we then have the following system of one-dimensional equations

\[
RL \partial_t^\beta \hat{u}_n(t) = -\rho_n \hat{u}_n(t) + \hat{F}(t, u)_n,
\]

where \( \hat{F}(t, u)_n \) denotes the \( n \)-th Fourier coefficient of \( F(t, u(t, x)) \). It is suitable to consider solutions and the nonlinear term on the Lebesgue space \( L^p(0, T; H^\mu(\Omega)) \) for some \( p, \mu \). Hence, the right hand side of (4) belongs to \( L^p(0, T; \mathbb{R}) \). Consequently, the Riemann-Liouville derivative on the left hand side of (4) is integrable. Therefore, acting on the Riemann-Liouville integral of fractional order \( \beta \), and then applying Lemma 3.1 in [5] we derive

\[
\hat{u}_n(t) = \Gamma(\beta) t^{\beta-1} E_{\beta, \beta}(-t^{\beta} \rho_n) \varphi_n + \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(-(t-s)^{\beta} \rho_n) F(s, u) ds,
\]

where the notation \( E_{\beta, \beta} \) stands for the Mittag-Leffler function defined by \( E_{a,b}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(ak+b)} \), for \( a > 0, b \in \mathbb{R}, z \in \mathbb{C} \). Thus we deduce that

\[
u(t, x) = \Gamma(\beta) t^{\beta-1} E_{\beta, \beta}(-t^{\beta}A) \varphi(x) + \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(-(t-s)^{\beta}A) F(s, u(s, x)) ds,
\]

(5)
where, for a function \( h : \Omega \to \mathbb{R} \), the operator \( E_{\beta,\beta'}(-t^\beta A) \), \( \beta', \beta \in \mathbb{R} \), is formulated by
\[
E_{\beta,\beta'}(-t^\beta A)h := \sum_{n=1}^{\infty} E_{\beta,\beta'}(-t^\beta \rho_n)\hat{h}_n b_n.
\]

**Definition 2.1.** A function \( u \) in \( L^p(0,T;L^q(\Omega)) \) for some \( p,q \geq 1 \) is said to be a mild solution of Problem (1) if it satisfies equation (5).

**Remark 1.** In fact, some definitions of mild solutions of initial value problems also require mild solutions to satisfy the condition \( t^{1-\beta}u(t,x)|_{t=0} = \varphi \). However, this point can be checked easily, so we do not attach this condition in our definition 2.1. This paper is to focus on establishing the smoothnesses of solutions and its derivatives.

**Remark 2.** In [13, 27], if we replace \( RL_t^{\beta} \) by the Caputo fractional derivative \( C_t^{\beta} \), then the mild solutions of Problem (1) has a slightly different form
\[
u(t,x) = E_{\beta,1}(-t^\beta A)\varphi(x) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)F(s,u(s,x))ds.
\] (7)

Since \( E_{\beta,1}(-t^\beta A) \) is a bounded operator on \( L^2(\Omega) \), constructing the existence of solutions has some advantages than the form (5). In fact, since \( t^{\beta-1} \) is singular at \( t = 0 \), the term \( \Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(-t^\beta A)\varphi \) may not belong to \( L^\infty(0,T;L^2(\Omega)) \) for \( \varphi \in L^2(\Omega) \). This can be seen in the following example.

Take \( \varphi(x) = b_1(x) \) and then \( \varphi \in L^2(\Omega) \) and \( \|\varphi\| = 1 \). Thus, by noting that \( E_{\beta,\beta}(0) = 1/\Gamma(\beta) \) we then have
\[
\lim_{t \searrow 0^+} \|\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(-t^\beta A)\varphi\|_{L^2(\Omega)} = \lim_{t \searrow 0^+} \Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(-t^\beta \rho_1) = \infty.
\]
Hence, the function \( (t,x) \mapsto \Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(-t^\beta A)\varphi(x) \) does not belong to \( L^\infty(0,T;L^2(\Omega)) \).

**Remark 3.** It is important to explain the initial condition given in Problem (1), namely,
\[
t^{1-\beta}u(t,x)|_{t=0} = \varphi(x), \quad x \in \Omega.
\] (8)
Assume that the above condition is replaced by
\[
u(0,x) = \varphi(x). \quad x \in \Omega
\] (9)
Then, by assuming that the solutions are smooth enough, one can not obtain the original equation \( RL_t^{\beta}u + Au = F(t,u(t,x)) \) from (7) and (9) unless \( \varphi = 0 \). This comes from noting that
\[
RL_t^{\beta}c = \frac{c}{\Gamma(1-\beta)}t^{-\beta}, \quad t > 0,
\]
where \( c \) is a constant. Note the condition (8) or \( I_t^{1-\beta}u(0, x) = \varphi(x) \) are suitable and can be transformed into each others; for discussions for ODEs see [30].
3. Well-posedness of the problem. In this section, we will study the existence and uniqueness of mild solutions to problem (1). First, throughout this paper, we assume the global Lipschitz continuity and the time Hölder continuity on the nonlinear term. More precisely, we suppose that \( F : \mathbb{H}^\sigma(\Omega) \to \mathbb{H}^\sigma(\Omega) \), \( F(0) = 0 \), and

\[
\| F(t, v_1) - F(t, v_2) \|_{\mathbb{H}^\sigma(\Omega)} \leq K \| v_1 - v_2 \|_{\mathbb{H}^\sigma(\Omega)},
\]

where \( K : [0, T] \to \mathbb{R}_+ \) and \( \sigma_1, \sigma_2 \) are real numbers. In the special case \( \sigma = 0 \) and \( \sigma - \eta = 0 \), the Lipschitz continuity can be rewritten as

\[
\| F(t, v_1) - F(t, v_2) \|_{L^2(\Omega)} \leq K \| v_1 - v_2 \|_{L^2(\Omega)}.
\]

Our results in this section present the well-posedness of the problem. One can think that one should be able to find solutions in the space \( C((0, T]; \mathbb{H}^\sigma(\Omega)) \). However, in order to establish an existence without any restrictions on the final time \( T \) and the Lipschitz coefficient \( K \), we will consider the solution space \( C_{w,1}^{\alpha,\mu}(0, T]; \mathbb{H}^\sigma(\Omega)) \) instead of \( C((0, T]; \mathbb{H}^\sigma(\Omega)) \). Here, \( C_{w,1}^{\alpha,\mu}(0, T]; \mathbb{H}^\sigma(\Omega)) \) denotes the weighted space of all functions \( v \in C((0, T]; \mathbb{H}^\sigma(\Omega)) \) such that

\[
\| v \|_{C_{w,1}^{\alpha,\mu}(0, T]; \mathbb{H}^\sigma(\Omega))} := \sup_{t \in (0, T]} t^{\alpha} e^{-\mu t} \| v(t, \cdot) \|_{\mathbb{H}^\sigma(\Omega)} < \infty.
\]

In the technical approach, we should recall the Kummer function or the hypergeometric function which is defined by

\[
\mathcal{F}_1(\omega_1, \omega_2, z) := \frac{\Gamma(\omega_2)}{\Gamma(\omega_2 - \omega_1)} \int_0^1 (1 - s)^{\omega_2 - \omega_1 - 1} s^{\omega_1 - 1} e^{zs} ds,
\]

\( \omega_2 > \omega_1 > 0, z \in \mathbb{C} \). Whentz = 0, we derive the so called Euler’s integral of the first kind \( \int_0^1 (1 - s)^{\alpha_1 - 1} s^{\alpha_2 - 1} ds \). This integral can be expressed in terms of Gamma functions as

\[
\int_0^1 (1 - s)^{\alpha_1 - 1} s^{\alpha_2 - 1} ds = \frac{\Gamma(\alpha_1) \Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}.
\]

Thanks to chapter 13 in [2], the asymptotic behavior of \( \mathcal{F}_1 \) is given by

\[
\mathcal{F}_1(\omega_1, \omega_2, z) := \frac{\Gamma(\omega_2)}{\Gamma(\omega_1)} z^{-(\omega_2 - \omega_1)} e^{z} \left( 1 + O \left( \frac{1}{|z|} \right) \right).
\]

Therefore, we can obtain the following lemma which will be useful in our main results (this lemma can be found also in [11], Lemma 8, page 9).

**Lemma 3.1.** Let \( \omega_1 > -1, \omega_2 > -1 \) such that \( \omega_1 + \omega_2 \geq -1, d > 0 \) and \( t \in [0, T] \). For \( \mu > 0 \), the following limit holds

\[
\lim_{\mu \to \infty} \left( \sup_{t \in [0, T]} t^d \int_0^1 s^{\omega_1}(1 - s)^{\omega_2} e^{\mu t(1 - s)} ds \right) = 0.
\]

Now, we are in the position to introduce the main contributions of this work. Our main results address the existence and regularity of the mild solution.

**Theorem 3.2.** Assume that \( 1/2 < \beta < 1, \sigma \geq 0, \alpha \in (0, 1), \) and \( \gamma_0 \geq \sigma \). If \( \phi \in \mathbb{H}^{\gamma_0}(\Omega) \) and \( F \) is defined by (10) with \( 0 \leq \sigma - \eta < 2 \), then Problem (1) has a unique solution \( u \) in \( C_{w,1}^{\alpha,\mu_0}(0, T]; \mathbb{H}^\sigma(\Omega)) \) with some \( \mu_0 > 0 \). Moreover, there exist positive constants \( C_1, C_2 \) independently of \( t, x \) such that

\[
\| u(t, \cdot) \|_{\mathbb{H}^\sigma(\Omega)} \leq C_1 t^{-\alpha} e^{\mu_0 t},
\]

where
and for $1/2 < \beta < 1$, $1 - \beta < \alpha < 1/2$ that

$$
\|u(t, \cdot)\|_{E^\gamma(\Omega)} \leq C_2 t^{-\alpha} e^{\mu t} \|\varphi\|_{E^\gamma(\Omega)}.
$$

(14)

**Proof.** Let us define the mapping $\mathfrak{M} : C_{w^\infty}^{\alpha, \mu}((0, T]; \mathbb{H}^\sigma(\Omega)) \to C_{w^\infty}^{\alpha, \mu}((0, T]; \mathbb{H}^\sigma(\Omega))$, $\mu > 0$, by the following equality

$$
\mathfrak{M}v(t, x) := \Gamma(\beta) t^{\beta-1} E_{\beta, \beta}(-t^\beta A) \varphi(x)
+ \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(-t^\beta A) F(s, v(s, x)) ds.
$$

(15)

In what follows, we shall prove the existence of a unique solution of Problem (1). This is based on the Banach principal argument. Indeed, for $v_1, v_2 \in C_{w^\infty}^{\alpha, \mu}((0, T]; \mathbb{H}^\sigma(\Omega))$, we have

$$
\begin{align*}
\|\mathfrak{M}v_1 - \mathfrak{M}v_2\|_{C_{w^\infty}^{\alpha, \mu}((0, T]; \mathbb{H}^\sigma(\Omega))} &= \sup_{t \in (0, T]} t^\alpha e^{-\mu t} \left\| \int_0^t (t-s)^{\beta-1} E_{\beta, \beta}(-t^\beta A) \left[ F(s, v_1(s, \cdot)) - F(s, v_2(s, \cdot)) \right] ds \right\|_{\mathbb{H}^\sigma(\Omega)} \\
&\leq \sup_{t \in (0, T]} t^\alpha e^{-\mu t} \int_0^t (t-s)^{\beta-1} \left\| E_{\beta, \beta}(-t^\beta A) \left[ F(s, v_1(s, \cdot)) - F(s, v_2(s, \cdot)) \right] \right\|_{\mathbb{H}^\sigma(\Omega)} ds \\
&\leq C \sup_{t \in (0, T]} t^\alpha e^{-\mu t} \int_0^t (t-s)^{\beta-1} (t-s)^{\frac{\beta(\sigma - \eta)}{\beta}} \left\| F(s, v_1(s, \cdot)) - F(s, v_2(s, \cdot)) \right\|_{\mathbb{H}^\sigma(\Omega)} ds \\
&\leq CK \| v_1 - v_2 \|_{C_{w^\infty}^{\alpha, \mu}((0, T]; \mathbb{H}^\sigma(\Omega))} \| v_1 - v_2 \|_{C_{w^\infty}^{\alpha, \mu}((0, T]; \mathbb{H}^\sigma(\Omega))} \\
&\quad \cdot \int_0^t (t-s)^{\beta(1-\frac{\sigma - \eta}{\beta}) - 1}s^{-\alpha} e^{-\mu(t-s)} ds,
\end{align*}
$$

(16)

where we have used Lemma 5.2 in the second estimate, and $1 - \frac{\sigma - \eta}{\beta} > 0$ as $0 \leq \sigma - \eta < 2$. Hence, we derive the estimate

$$
\|\mathfrak{M}v_1 - \mathfrak{M}v_2\|_{C_{w^\infty}^{\alpha, \mu}((0, T]; \mathbb{H}^\sigma(\Omega))} \leq L_\mu \| v_1 - v_2 \|_{C_{w^\infty}^{\alpha, \mu}((0, T]; \mathbb{H}^\sigma(\Omega))},
$$

where a simple integration by substitution yields that

$$
\begin{align*}
\lim_{\mu \to \infty} L_\mu := K \lim_{\mu \to \infty} \left( \sup_{t \in (0, T]} t^\alpha \int_0^t (t-s)^{\beta(1-\frac{\sigma - \eta}{\beta})} s^{-\alpha} e^{-\mu(t-s)} ds \right) \\
&= K \lim_{\mu \to \infty} \left( \sup_{t \in (0, T]} t^\alpha \int_0^1 (1-s)^{\beta(1-\frac{\sigma - \eta}{\beta})} s^{-\alpha} e^{-\mu(1-s)} ds \right) \\
&= K \lim_{\mu \to \infty} \left( \sup_{t \in (0, T]} t^\beta(1-\frac{\sigma - \eta}{\beta}) \int_0^1 (1-s)^{\beta(1-\frac{\sigma - \eta}{\beta})} s^{-\alpha} e^{-\mu(1-s)} ds \right) \\
&= 0,
\end{align*}
$$

where Lemma 3.1 was used. Hence, there exists $\mu_0 > 0$ such that $\mathfrak{M}$ is a contraction mapping on $C_{w^\infty}^{\alpha, \mu_0}((0, T]; \mathbb{H}^\sigma(\Omega))$. Thus, it possesses a unique fixed point, which is also a unique mild solution of Problem (1) in the space $C_{w^\infty}^{\alpha, \mu_0}((0, T]; \mathbb{H}^\sigma(\Omega))$.

We now prove the inequalities (13), (14). Since $u \in C_{w^\infty}^{\alpha, \mu_0}((0, T]; \mathbb{H}^\sigma(\Omega))$, the first one is clear. We will prove the second one which corresponds to the conditions $1/2 < \beta < 1$, $1 - \beta < \alpha < 1/2$. In this case, we have $1 - \alpha - \beta < 0$. Hence, one can find a number $\gamma_0$ such that

$$
\frac{2(1-\alpha-\beta)}{\beta} + \sigma \leq \gamma_0 \leq \sigma \leq \gamma_0 + 2.
$$

(17)
We see \( \sigma - \gamma_0' \in [0, 2] \). Besides, we recall that \( \gamma_0 \geq \sigma \), so we have \( \gamma_0' \leq \gamma_0 \) which guarantees the embedding \( H^\sigma(\Omega) \hookrightarrow H^{\gamma_0}(\Omega) \), see [27]. Namely, \( \|\varphi\|_{H^{\gamma_0}(\Omega)} \leq M\|\varphi\|_{H^\sigma(\Omega)} \) with some constant \( M \) independent of the function \( \varphi \) and the space variable \( x \). Therefore, by applying Lemma 3.1 for \( \delta = \sigma - \gamma_0' \in [0, 2] \), we obtain

\[
\|\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(-t^\beta A)\varphi\|_{H^\sigma(\Omega)} \leq \Gamma(\beta)t^{\beta-1}t^{-\beta(\sigma-\gamma_0')/2}\|\varphi\|_{H^\gamma(\Omega)} \\
\leq C\Gamma(\beta)t^{\beta}\left(1-\frac{\sigma-\gamma_0'}{2}\right)^{-1}\|\varphi\|_{H^{\gamma_0}(\Omega)}.
\]

This and the technique in (16) yields

\[
\|u(t, \cdot)\|_{H^\sigma(\Omega)} \\
\leq \|\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(-t^\beta A)\varphi\|_{H^\sigma(\Omega)} + \int_0^t (t-s)^{\beta-1}\|E_{\beta,\beta}(-(t-s)^\beta A)F(s, u(s, \cdot))\|_{H^\sigma(\Omega)}\,ds \\
\leq C\Gamma(\beta)t^{\beta}\left(1-\frac{\sigma-\gamma_0'}{2}\right)^{-1}\|\varphi\|_{H^{\gamma_0}(\Omega)} + C\int_0^t (t-s)^{\beta}\left(1-\frac{2\alpha}{\beta}\right)^{-1}\|u(s, \cdot)\|_{H^\sigma(\Omega)}\,ds.
\]

It follows from (17) that

\[
\beta \left(1 - \frac{\sigma - \gamma_0'}{2}\right) - 1 + \alpha \geq \beta \left(1 + \frac{1 - \alpha - \beta}{\beta}\right) - 1 + \alpha = 0.
\]

Thus we have the following estimate

\[
t^\alpha e^{-\mu t}t^{\beta}\left(1-\frac{\sigma-\gamma_0'}{2}\right)^{-1}\|\varphi\|_{H^{\gamma_0}(\Omega)} \leq M_1\|\varphi\|_{H^{\gamma_0}(\Omega)}
\]

with some constant \( M_1 \). We recall that \( \beta(1 - (\sigma - \eta)/2) - 1 > -1 \). On the other hand, the number \( -2\alpha > -1 \) as \( \alpha < 1/2 \) which helps to claim that the Euler’s integral of the first kind \( \int_0^1 (1-s)^{\beta(1-(\sigma-\eta)/2)-1}s^{-2\alpha}\,ds \) finitely exists. By applying the Hölder inequality, and then using \( e^{-2\mu(t-s)} < 1 \), we can find some positive constant \( M_2 \) such that

\[
t^\alpha e^{-\mu t}\int_0^t (t-s)^{\beta(1-\frac{2\alpha}{\beta})^{-1}}\|u(s, \cdot)\|_{H^\sigma(\Omega)}\,ds \\
\leq \left(\int_0^t (t-s)^{\beta(1-\frac{2\alpha}{\beta})^{-1}}s^{-2\alpha}e^{-2\mu(t-s)}\,ds\right)^{\frac{1}{2}} \\
\leq \left(\int_0^t (t-s)^{\beta(1-\frac{2\alpha}{\beta})^{-1}}s^{-2\alpha}\|u(s, \cdot)\|_{H^\sigma(\Omega)}^2\,ds\right)^{\frac{1}{2}} \\
\leq \left(\int_0^t (t-s)^{\beta(1-\frac{2\alpha}{\beta})^{-1}}s^{-2\alpha}\|u(s, \cdot)\|_{H^\sigma(\Omega)}^2\,ds\right)^{\frac{1}{2}} \\
\leq M_2\left(\int_0^t (t-s)^{\beta(1-\frac{2\alpha}{\beta})^{-1}}s^{-2\alpha}\|u(s, \cdot)\|_{H^\sigma(\Omega)}^2\,ds\right)^{\frac{1}{2}}.
\]
Taking the estimate (19), (21), and (22) together gives that
\[
\mathcal{U}_{\alpha, \mu}(t) \leq M^2 C^2 (\Gamma(\beta))^2 \| \varphi \|^2_{H_0^\sigma(\Omega)} + M^2 C^2 K^2 \int_0^t (t-s)^{\beta(1-\sigma-\eta)} U_{\alpha, \mu}(s) ds,
\]
where
\[
\mathcal{U}_{\alpha, \mu}(t) := \left( t^\alpha e^{-\mu t} \| u(t, \cdot) \|_{H^\sigma(\Omega)} \right)^2.
\]
Employing the fractional Grönwall inequality given in Lemma 5.1, we directly obtain the inequality (14).

4. Regularity of the solution. In this section, we derive the regularity result of the mild solution and its fractional derivative. The following theorem gives a regularity result in the \( L^q \) setting.

Theorem 4.1. Given \( \frac{1}{2} < \beta < 1 \). Let the numbers \( N \geq 2, q \geq 1 \) satisfy \( \frac{N}{2} - \frac{N}{q} < \frac{1}{2} \).

Assume that \( \max \left( \frac{N}{2} - \frac{N}{q}; 0 \right) < \sigma < \frac{1}{2}, 1 - \beta < \alpha < \frac{1}{2}, \) and \( \gamma_0 \geq \sigma \). If \( \varphi \in W^{\theta, r}(\Omega) \) with \( \theta \geq \sigma, r \geq \frac{2N}{N + 2(\theta - \sigma)} \), and \( F \) is defined by (10) with \( 0 \leq \sigma - \eta < 2 \), then Problem (1) has a unique solution
\[
u \in C_{\omega, \mu}^{\alpha, \mu}(0, T; H^\sigma(\Omega)) \cap L^p(0, T; L^q(\Omega)),
\]
for some \( \mu > 0 \) and \( 1 \leq p < \frac{1}{\alpha} \). Furthermore, for \( t > 0 \)
\[
\| u(t, \cdot) \|_{L^q(\Omega)} \lesssim t^{-\alpha} \| \varphi \|_{W^{\theta, r}(\Omega)}.
\]
The hidden constant, in the latter inequality, is independent of the problem data.

Proof. First, we will verify that \( \varphi \) fulfills the respective assumption given in Theorem 3.2. Actually, it is necessary to show that \( \varphi \in H^\sigma(\Omega) \). From \( 0 < \sigma < 1/2 \), we have \( H^\sigma(\Omega) = W^{\sigma, 2}(\Omega) \), see e.g. \([9, 10, 25]\). Moreover, we observe that
\[
\theta - \sigma \geq \frac{N}{r} - \frac{N}{2} \quad \text{as} \quad r \geq \frac{2N}{N + 2(\theta - \sigma)},
\]
which ensures the embedding \( W^{\theta, r}(\Omega) \hookrightarrow W^{\sigma, 2}(\Omega) \). We then deduce that
\[
W^{\theta, r}(\Omega) \hookrightarrow W^{\sigma, 2}(\Omega) = H^\sigma(\Omega).
\]

Thus \( \varphi \) satisfies the condition in Theorem 3.2. Therefore, the existence of the solution \( \nu \in C_{\omega, \mu}^{\alpha, \mu}(0, T; H^\sigma(\Omega)) \) can be obtained by applying Theorem 3.2. The inequality (14) also holds. We now prove \( \nu \in L^p(0, T; L^q(\Omega)) \). Since \( \sigma \geq 0 \), the embedding \( H^\sigma(\Omega) \hookrightarrow W^{\sigma, 2}(\Omega) \) is ensured. The assumption \( \sigma > \frac{N}{2} - \frac{N}{q} \) guarantees the embedding \( W^{\sigma, 2}(\Omega) \hookrightarrow L^q(\Omega) \). In summery, we obtain the following Sobolev embedding
\[
H^\sigma(\Omega) \hookrightarrow W^{\sigma, 2}(\Omega) \hookrightarrow L^q(\Omega).
\]
Therefore, \( \|h\|_{L^2(\Omega)} \leq M_3 \|h\|_{H^s(\Omega)} \) for all \( h \in H^s(\Omega) \), where the constant \( M_3 \) depends only on \( \sigma, q \) and \( \Omega \). As a consequence of (27) and (28), we have the following\[ \|u(t, \cdot)\|_{L^4(\Omega)} \lesssim \|\int_0^t (t-s)^{\beta-1} \|E_{\beta,\beta}(-(t-s)^{\beta}A)F(s, u(s, \cdot))\|_{L^q(\Omega)} ds \| \]
\[ \lesssim \|\int_0^t (t-s)^{\beta-1} \|E_{\beta,\beta}(-(t-s)^{\beta}A)F(s, u(s, \cdot))\|_{L^q(\Omega)} ds \| \]
\[ \lesssim \|\varphi\|_{L^r(\Omega)} + \int_0^t (t-s)^{\beta(1-\frac{\sigma}{2})-1}s^{\alpha\eta}e^{\mu_0 s} ds \lesssim \|\varphi\|_{W^{\theta,r}(\Omega)}^{p} \left( t^{\beta-1} + \int_0^t (t-s)^{\beta(1-\frac{\sigma}{2})-1}s^{\alpha\eta}e^{\mu_0 s} ds \right) := \mathcal{J}_p \|\varphi\|_{W^{\theta,r}(\Omega)}^{p}. \] The integral \( \mathcal{J}_p \) can be estimated using (11) of the Euler’s integral of the first kind. Indeed, one can check the following computations
\[ \int_0^T \mathcal{J}_p dt = \int_0^T \left( t^{\beta-1} + \int_0^t (t-s)^{\beta(1-\frac{\sigma}{2})-1}s^{\alpha\eta}e^{\mu_0 s} ds \right)^{p} dt \leq \int_0^T \left( t^{\beta-1} + e^{\mu_0 T} \int_0^t (t-s)^{\beta(1-\frac{\sigma}{2})-1}s^{\alpha\eta} ds \right)^{p} dt \]
\[ = \int_0^T \left( t^{\beta-1} + e^{\mu_0 T} t^{\beta(1-\frac{\sigma}{2})-\alpha} \frac{\Gamma(\beta(1-(\sigma-\eta)))\Gamma(1-\alpha)}{\Gamma(\beta(1-(\sigma-\eta)) + 1-\alpha)} \right)^{p} dt \]
\[ \leq M_4 \int_0^T \left( t^{\beta-1} + t^{\beta(1-\frac{\sigma}{2})-\alpha} \right)^{p} dt, \] with some constant \( M_4 \) independent of \( t \). Here, we recall from \( 1 - \beta < \alpha < 1/2 \) and \( 0 \leq \sigma - \eta < 1 \) so that \( \min\{\beta-1; \beta(1-\frac{\sigma}{2})-\alpha\} \geq -\alpha \), and hence one can seek a constant \( M_5 > 0 \) satisfying \( t^{\beta-1} + t^{\beta(1-\frac{\sigma}{2})-\alpha} \leq M_5 t^{-\alpha} \). Then, we derive the estimate
\[ \|u\|_{L^p(0,T;L^q(\Omega))} \lesssim \|\varphi\|_{W^{\theta,r}(\Omega)}^{p} \left( \int_0^T t^{-\alpha p} dt, \right) \] which associated with assumption \( 1 \leq p < 1/\alpha \) gives that \( u \in L^p(0,T;L^q(\Omega)) \). The desired inequality is consequently obtained from the latter estimate.}

**Theorem 4.2.** Given \( \frac{1}{2} < \beta < 1 \). Let the numbers \( N \geq 2 \), \( q \geq 1 \) satisfy \( \frac{N}{2} - \frac{N}{q} < \frac{1}{2} \). Assume that \( \max\left( \frac{N}{2} - \frac{N}{q}; 0 \right) < \sigma < \frac{1}{2} \), \( 1 - \beta < \alpha < \frac{1}{2} \). Let \( \varphi \in W^{\theta,r}(\Omega) \) with
\[ \theta \geq \sigma, \quad r \geq \frac{2N}{N+2(\theta-\sigma)}, \]
\( F \) be defined by (10) with \( 0 \leq \sigma - \eta < 2 \), and \( u \in C^{\alpha,\eta}_{w} ((0,T]; H^s(\Omega)) \cap L^p(0,T;L^q(\Omega)) \) be the solution of Problem (1). Then, for \( \sigma - \eta < \xi \leq \sigma \) and each \( t > 0 \),
\[ RL\partial_t^\beta u(t, \cdot) \in L^{\frac{2N}{N-2(\sigma-\eta)}}(\Omega), \]
which is formulated by

\[ RL \partial_t^\beta u(t, x) = -\Gamma(\beta)t^{\beta-1}AE_{\beta,\beta}(-t^\beta A)\varphi(x) \]

\[ - \int_0^t (t-s)^{\beta-1}AE_{\beta,\beta}(-(t-s)^\beta A)F(s, u(s, x))ds. \]  

(29)

Moreover, we have that

\[ \| RL \partial_t^\beta u(t, .) \|_{L^{\frac{2n}{n-2(\sigma-\xi)}(\Omega)}} \lesssim (t^{\frac{\beta}{2}} - 1 + t^{\frac{\beta}{2} - \frac{(\sigma-\xi)}{2}}) \| \varphi \|_{W^{\sigma,2}(\Omega)}. \]  

(30)

The hidden constant, in the latter inequality, is independent of \( x, t. \)

Proof. In order to prove the existence and establish regularity of the derivative \( \partial_t u, \)
using inequality (26) plays a major role. We recall that the eigenvectors \( \{b_n\}_{n \geq 1} \)
forms an orthonormal basis of \( L^2(\Omega). \) Let us denote by \( V_k, V_{k,k'}, k' > k, \) the linear spans of \( \{b_1, b_2, ..., b_k\}, \{b_{k+1}, b_{k+2}, ..., b_{k'}\} \) respectively. The notations \( E^{(k)}_{\beta,\beta}(-t^\beta A), \)
\( E^{(k,k')}_{\beta,\beta}(-t^\beta A) \) stand for the restrictions of the operator \( E_{\beta,\beta}(-t^\beta A) \) on the spaces \( V_k, V_{k,k'}, \) namely,

\[ E^{(k)}_{\beta,\beta}(-t^\beta A) = E_{\beta,\beta}(-t^\beta A)|_{V_k}, \quad E^{(k,k')}_{\beta,\beta}(-t^\beta A) = E_{\beta,\beta}(-t^\beta A)|_{V_{k,k'}.} \]

On the \( n \)-th dimension, according to the definition of the Riemann-Liouville and some simple computations, one can check that

\[ RL \partial_t^\beta (\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(-t^\beta \rho_n) = -\Gamma(\beta)\rho_n t^{\beta-1}E_{\beta,\beta}(-t^\beta \rho_n), \]

see e.g. the formula (4.10.16) in section 4.10.3 [14], and thus

\[ RL \partial_t^\beta \left( \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta \rho_n)F(s, u(s, x))ds \right) \]

\[ = -\rho_n \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta \rho_n)F(s, u(s, x))ds. \]

Henceforth, this implies

\[ RL \partial_t^\beta \sum_{n=1}^k \tilde{b}_n(t)b_n(x) = \sum_{n=1}^k [\partial_t \tilde{b}_n(t)]b_n(x) = \mathcal{G}^{(k)}_{\beta,\varphi}(t, x) + \mathcal{H}^{(k)}_{\beta,\varphi}(t, x), \]

where

\[ \mathcal{G}^{(k)}_{\beta,\varphi}(t, x) := -\Gamma(\beta)t^{\beta-1}AE^{(k)}_{\beta,\beta}(-t^\beta A)\varphi(x), \]

\[ \mathcal{H}^{(k)}_{\beta,\varphi}(t, x) := -\int_0^t (t-s)^{\beta-1}AE^{(k)}_{\beta,\beta}(-(t-s)^\beta A)F(s, u(s, x))ds. \]

For each \( t > 0, \) we shall prove that the sequences \( \{\mathcal{G}^{(k)}_{\beta,\varphi}(t, \cdot)\}, \{\mathcal{H}^{(k)}_{\beta,\varphi}(t, \cdot)\} \) are convergent in \( L^q(\Omega) \) using the argument of Cauchy sequences. Let us consider the first sequence. It follows from \( \sigma - \xi \geq 0 \) that the embedding \( H^{\sigma-\xi}(\Omega) \hookrightarrow W^{\sigma,2}(\Omega) \) holds. Moreover, the embedding \( W^{\sigma,2}(\Omega) \hookrightarrow L^{2N/(N-2(\sigma-\xi))}(\Omega) \) is clear. In summary, we obtain

\[ H^{\sigma-\xi}(\Omega) \hookrightarrow W^{\sigma,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2(\sigma-\xi)}}(\Omega). \]  

(31)

Henceforth, we obtain the following inequality

\[ \| \mathcal{G}^{(k')}_{\beta,\varphi}(t, \cdot) - \mathcal{G}^{(k)}_{\beta,\varphi}(t, \cdot) \|_{L^{\frac{2N}{n-2(\sigma-\xi)}}(\Omega)} \lesssim \| \mathcal{G}^{(k')}_{\beta,\varphi}(t, \cdot) - \mathcal{G}^{(k)}_{\beta,\varphi}(t, \cdot) \|_{H^{\sigma-\xi}(\Omega)}. \]
On the other hand, the quantity $E^{(k',k)}_{\beta,\beta'}(-t^\beta A)\varphi$ can be estimated in the same way as (18). In addition, we also note that $\|Ah\|_{H^{\sigma}(\Omega)} = \|Ah\|_{H^{\sigma}(\Omega)}$. Therefore, combining the above arguments help us to ensure the following estimates

\[
\left\| \mathcal{G}^{(k')}_{\beta,\varphi}(t,\cdot) - \mathcal{G}^{(k')}_{\beta,\varphi}(t,\cdot) \right\|_{H^{\sigma-\xi}(\Omega)} \lesssim \int_0^t (t-s)^{\sigma-1} \left\| F(s, u(s, \cdot)) \right\|_{H^{\sigma-\xi}(\Omega)} ds
\]

\[
\lesssim C K \int_0^t (t-s)^{\sigma-1} \left\| u(s, \cdot) \right\|_{H^{\sigma}(\Omega)} ds
\]

where we note that $\|\varphi\|_{H^{\sigma}(\Omega)} \lesssim \|\varphi\|_{W^{s,\sigma}(\Omega)}$ by (27).

For the second sequence, we note that its $L^{2N/(N-2(\sigma-\eta))(\Omega)}$-norm can be estimated by its $L^{2N/(N-\xi)-(\Omega)}$ norm. To estimate this sequence on $H^{\sigma-\xi}(\Omega)$, we also use $\|Ah\|_{H^{\tau}(\Omega)} = \|Ah\|_{H^{\tau}(\Omega)}$ and employ the techniques in (16). Indeed, using Lemma 5.2 and the Lipschitz continuity (10), one can deduce the following chain

\[
\left\| \mathcal{H}^{(k')}_{\beta,F}(t,\cdot) - \mathcal{H}^{(k')}_{\beta,F}(t,\cdot) \right\|_{L^{\frac{2N}{N-2(\sigma-\eta)}}(\Omega)}
\]

\[
\lesssim C K \int_0^t (t-s)^{\frac{\sigma-1}{2}} \left\| F(s, u(s, \cdot)) \right\|_{H^{\sigma}(\Omega)} ds
\]

where the boundedness (27) was used. Here, the powers of the functions inside the latter integral are strictly greater than $-1$ since $\xi - (\sigma - \eta) > 0$. Now, it follows from $\varphi \in W^{\sigma,\tau}(\Omega)$ that $\lim_{k,k' \to \infty} \|\varphi\|_{W^{s,\sigma}(\Omega)} = 0$, and hence by taking the estimates (32) and (33) together we consequently have the limits

\[
0 \leq \lim_{k,k' \to \infty} \left( \left\| \mathcal{G}^{(k')}_{\beta,\varphi}(t,\cdot) - \mathcal{G}^{(k')}_{\beta,\varphi}(t,\cdot) \right\|_{L^{\frac{2N}{N-2(\sigma-\eta)}}(\Omega)} + \left\| \mathcal{H}^{(k')}_{\beta,F}(t,\cdot) - \mathcal{H}^{(k')}_{\beta,F}(t,\cdot) \right\|_{L^{\frac{2N}{N-2(\sigma-\eta)}}(\Omega)} \right)
\]

\[
\lesssim \lim_{k,k' \to \infty} \left( t^{\frac{\sigma-1}{2}} + \int_0^t (t-s)^{\frac{\sigma-1}{2}} ds \right) \left\| \varphi \right\|_{W^{s,\sigma}(\Omega)} = 0.
\]

Accordingly, we get the following equality

\[
\left\{ \begin{array}{l}
\lim_{k,k' \to \infty} \left\| \mathcal{G}^{(k')}_{\beta,\varphi}(t,\cdot) - \mathcal{G}^{(k')}_{\beta,\varphi}(t,\cdot) \right\|_{L^{\frac{2N}{N-2(\sigma-\eta)}}(\Omega)} = 0,
\lim_{k,k' \to \infty} \left\| \mathcal{H}^{(k')}_{\beta,F}(t,\cdot) - \mathcal{H}^{(k')}_{\beta,F}(t,\cdot) \right\|_{L^{\frac{2N}{N-2(\sigma-\eta)}}(\Omega)} = 0.
\end{array} \right.
\]
Therefore, we conclude that the infinite series
\[ \{G_{\beta,\varphi}(t,\cdot)\}, \{H_{\beta,F}(t,\cdot)\} \]
are Cauchy sequences in the Banach space
\[ L^{2N/(N-2(\sigma-\xi))}(\Omega) \]

Therefore, we conclude that the infinite series
\[
G_{\beta,\varphi}(t,\cdot) = -\Gamma(\beta)t^{\beta-1}\sum_{n=1}^{\infty} \rho_n E_{\beta,\beta}(-t^\beta \rho_n)\hat{\varphi}_n b_n = -\Gamma(\beta)t^{\beta-1}A E_{\beta,\beta}(-t^\beta A)\varphi,
\]
\[
H_{\beta,F}(t,\cdot) = -\sum_{n=1}^{\infty} \int_0^t (t-s)^{\beta-1} \rho_n E_{\beta,\beta}(-(t-s)^\beta \rho_n)F(s,u)_n ds b_n = -\int_0^t (t-s)^{\beta-1}A E_{\beta,\beta}(-(t-s)^\beta A)F(s,u(\cdot),t)ds,
\]
are convergent in \( L^{2N/(N-2(\sigma-\xi))}(\Omega) \). As a consequence, the derivative of first order \( \partial_t u \) exists in this space, and furthermore that
\[
RL\partial_t^{\beta} u(t,x) = G_{\beta,\varphi}(t,x) + H_{\beta,F}(t,x).
\]

Consequently, (29) is obtained.

Finally, we shall prove the inequality (30), which is based on (32) and (33). Indeed, by letting \( k = 0 \) and taking the limit as \( k' \to \infty \), we can deduce from these estimates and (11) that
\[
\left\| RL \partial_t^{\beta} u(t,\cdot) \right\|_{L^{N-2(\sigma-\xi)}(\Omega)} \leq \left\| G_{\beta,\varphi}(t,\cdot) \right\|_{L^{N-2(\sigma-\xi)}(\Omega)} + \left\| H_{\beta,F}(t,\cdot) \right\|_{L^{N-2(\sigma-\xi)}(\Omega)} \lesssim \left( \frac{\beta \Gamma(\beta)}{\Gamma(\beta + 1)} \right) \left( \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} \right) \|\varphi\|_{W^{\sigma,r}(\Omega)}.
\]

The proof is so completed. \( \square \)

5. Conclusion. This paper has treated a nonlinear fractional diffusion equations with a Riemann-Liouville derivative. Our key idea is to combine the theories of Mittag-Leffler functions, the Banach fixed point theorem, the fractional Grönwall inequality, and appropriate Sobolev embeddings.

Under the assumptions \( 1/2 < \beta < 1, \sigma \geq 0, \varphi \in H^0(\Omega) \) for \( \gamma_0 \geq \sigma \), and \( F \) is globally Lipschitz continuous as in (10), the main result is to establish the global existence and uniqueness of mild solutions in \( C_{\text{wrt}}(\alpha,\mu)(0,T;H^\sigma(\Omega)) \) with \( \alpha \in (0,1) \) and some \( \mu_0 > 0 \). Moreover, we showed it to be stable in the inverse power-law
\[
\left\| u(t,\cdot) \right\|_{H^\sigma(\Omega)} \leq C_2 t^{-\alpha} e^{\mu_0 t} \left\| \varphi \right\|_{H^\gamma_0(\Omega)}, \quad t > 0,
\]
for \( 1 - \beta < \alpha < 1/2 \).

The second result is to construct regularity of the solution in \( L^p(0,T;L^q(\Omega)) \). Under the assumptions \( N \geq 2, 1 \leq p < \frac{1}{\sigma}, q \geq 1, \frac{N}{2} - \frac{N}{q} < \frac{1}{2}, \max\left( \frac{N}{2} - \frac{N}{q} ; 0 \right) < \frac{1}{2} \),
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σ < \frac{1}{2}, \varphi \in W^{\theta,r}(\Omega) \text{ with } \theta \geq \sigma, r \geq \frac{2N}{N + 2(\theta - \sigma)}, \text{ and } F \text{ is globally Lipschitz continuous as in (10), we have showed that}

u \in C^{\alpha,\mu}_{wei}((0,T]; H^\sigma(\Omega)) \cap L^p(0,T; L^q(\Omega))

for 1 - \beta < \alpha < \frac{1}{2}. Moreover, for \sigma - \eta < \xi \leq \sigma \text{ and each } t > 0, RL\partial_t^\beta u(t,\cdot) \in L^{\frac{2N}{N - 2(\theta - \sigma)}}(\Omega), \text{ which satisfies stability in the below inverse power-law}

\|RL\partial_t^\beta u(t,\cdot)\|_{L^{\frac{2N}{N - 2(\theta - \sigma)}}(\Omega)} \lesssim (t^\beta \xi^{\frac{\xi - (\sigma - \eta)}{2} - \alpha})\|\varphi\|_{W^{\theta,r}(\Omega)}, t > 0.

Appendix. We recall a version of the fractional Grönnwall inequality. Its proof can be found in [29]. The following lemma recalls Corollary 2 in the paper [29].

Lemma 5.1 (Fractional Grönnwall inequality). Assume \beta > 0, the function \varphi is nonnegative, locally integrable, and

\varphi(t) \leq B + D \int_0^t (t - \tau)^{\beta - 1}\varphi(s)ds,

on (0,T), where C, D are positive constants. Then,

\varphi(t) \leq B E_{\beta,1}(D\Gamma(\beta)t^\beta), \quad \text{on } (0,T).

In the next lemma, we present an important estimate for the solution operator \(E_{\beta,\beta}(-t^\beta A)\) which is given by (6). The proof of this lemma is based on the inequality 2.7 in [18].

Lemma 5.2. Let 0 < \beta < 1, \delta \in [0,2], t > 0 and h \in H^\gamma(\Omega). Then, we have that

\|E_{\beta,\beta}(-t^\beta A)h\|_{H^{\gamma+\delta}(\Omega)} \leq Ct^{-\frac{\delta}{2}}\|h\|_{H^\gamma(\Omega)}.

Proof. By applying the inequality 2.7 in [18], we can bound \(E_{\beta,\beta}(-t^\beta \rho_n)\) by \((1 + t^\beta \rho_n)\). Therefore, it follows from \((1 + t^\beta \rho_n) \geq (1 + t^\beta \rho_n)^{\delta/2} \geq (t^\beta \rho_n)^{\delta/2}\) that

\|E_{\beta,\beta}(-t^\beta A)h\|_{H^{\gamma+\delta}(\Omega)} = \left(\sum_{n=1}^{\infty} (E_{\beta,\beta}(-t^\beta \rho_n)\hat{h}_n)^2 \rho_n^{\gamma+\delta}\right)^{\frac{1}{2}} \leq \left(C^2 \sum_{n=1}^{\infty} ((t^\beta \rho_n)^{-\delta/2}\hat{h}_n)^2 \rho_n^{\gamma+\delta}\right)^{\frac{1}{2}}.

This completes the proof since the right side of the above estimate equals:

\[ Ct^{-\frac{\delta}{2}}\|h\|_{H^\gamma(\Omega)}. \]

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