Quantum cosmology in the energy representation *

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Abstract

The Hawking minisuperspace model (closed FRW geometry with a homogeneous massive scalar field) provides a fairly non-trivial testing ground for fundamental problems in quantum cosmology. We provide evidence that the Wheeler-DeWitt equation admits a basis of solutions that is distinguished by analyticity properties in a large scale factor expansion. As a consequence, the space of solutions decomposes in a preferred way into two Hilbert spaces with positive and negative definite scalar product, respectively. These results may be viewed as a hint for a deeper significance of analyticity. If a similar structure exists in full (non-minisuperspace) models as well, severe implications on the foundations of quantum cosmology are to be expected.

Semiclassically, the elements of the preferred basis describe contracting and expanding universes with a prescribed value of the matter (scalar field)

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energy. Half of the basis elements have previously been constructed by Hawking and Page in a wormhole context, and they appear in a new light here. The technical tools to arrive at these conclusions are transformations of the wave function into several alternative representations that are based on the harmonic oscillator form of the matter energy operator, and that are called oscillator, energy and Fock representation. The framework defined by these may be of some help in analyzing the Wheeler-DeWitt equation for other purposes as well.
1 Introduction

The aim of this article is twofold: In the weaker sense it provides a formulation of minisuperspace quantum cosmology with a massive scalar field (the so-called Hawking model) in terms of representations that are based on eigenstates of the matter energy operator $E$. Since this object does not commute with the operator defining the Wheeler-DeWitt equation, its eigenstates do not satisfy the latter, but their harmonic oscillator form motivates a change of representation for the wave function and an according transformation of the form of the Wheeler-DeWitt equation. All issues concerning the excitation of the scalar field oscillator modes become particularly transparent in the "energy representation" and related versions thereof. Also, our formulation may facilitate attempts to solve the exact Wheeler-DeWitt equation for whatever purpose. In the post-inflationary regime the matter energy eigenstates can be extended to approximate solutions of the Wheeler-DeWitt equation at the level of the WKB-approximation.

In the stronger sense we present perspectives for the construction of exact wave functions that coincide with the approximate ones in the WKB-domain. Moreover, these states seem to have a preferred status, defined by analyticity properties in an expansion for large values of the scale factor. Although some of our conclusions are only conjectures, we seem to be able to define two exact Hilbert spaces of wave functions, playing a distinguished role. This is in some formal analogy to the one-particle Hilbert spaces of negative/positive frequencies in the flat Klein-Gordon equation. In this case the crucial property enabling us to decompose the space of solutions into two Hilbert spaces is Lorentz invariance. In contrast, the symmetries and covariance properties of the Wheeler-DeWitt equation do not suffice to provide such a decomposition. What we found is that the asymptotic analyticity structure of solutions may play a role analogous to Lorentz invariance. Accepting a preferred decomposition has of course implications for the conceptual basis of quantum cosmology. What we cannot answer at the moment is the question to what extent this structure will apply for the full (non-minisuperspace) theory.

The basic variables in the Hawking minisuperspace model are the scale factor $a$ of a closed Friedmann-Robertson-Walker universe and the value $\phi$ of a homogeneous, minimally coupled massive scalar field. The external parameters are the mass $m$ of the scalar field and a numerical constant $p$ representing the operator ordering ambiguity in the Wheeler-DeWitt equation. Apart from some general remarks, we will work entirely within this model.
In Section 2, we introduce as tools for the analysis a representation in terms of the harmonic oscillator eigenfunctions associated with the matter energy, and a suitable Fock space notation. We have designed this Section as self-contained as possible and provided various formulae that are helpful in dealing with different representations of the wave function. For later reference, we distinguish between position, oscillator, energy and Fock representation. In Section 3, we use the fact that matter energy is approximately conserved after inflation to write down energy eigenstates that approximately satisfy the Wheeler-DeWitt equation in the corresponding domain of minisuperspace. These states have been used by other authors as well, at the level of the WKB-approximation. The asymptotic structure for values of \( a \) larger than classically allowed is related to the classical domain by an appropriate WKB matching procedure. The same asymptotic structure appears when the semiclassical WKB-expansion method is applied straightforwardly.

In Section 4, we write down a fairly general expression for wave functions in the representation based on the original variables \( a \) and \( \phi \). We impose an analyticity requirement in terms of an expansion in inverse powers of \( a \) that seems to single out an exact version of the approximate states considered in Section 3. In Section 5, a similar procedure, when applied to the Wheeler-DeWitt equation in the oscillator and Fock representations, seems to generate identical results. Assuming these to hold, we end up with a basis of exact and distinguished wave functions \( \Xi^\pm_n(a, \phi) \), each describing (at the level of WKB-identification) an ensemble of collapsing/expanding universes with matter energy \( (n + \frac{1}{2})m \). Half of these wave functions have been constructed previously in a wormhole context by Hawking and Page \cite{3}, and the way they emerge in our framework sheds new light on them. Using the indefinite Klein-Gordon type scalar product \( Q \) on the space of solutions of the Wheeler-DeWitt equation in Section 6, we decompose this space into two Hilbert spaces, with positive/negative definite scalar product, respectively. If this decomposition is considered as a preferred one, relevant for interpretation, positive probabilities may be written down by means of conventional Hilbert space techniques. As an example, we expand the no-boundary wave function in terms of this basis in Section 7. Such an expansion may be called “energy representation” in the sense of representing a state entirely in terms of expansion coefficients with respect to a basis of solutions of the Wheeler-DeWitt equation that describe universes of definite energy. Thereby, also the role of the above-mentioned Hawking-Page solutions as providing only half of a basis is illustrated. Some concluding remarks, concerning the tunnelling wave function and the significance of the structure we found for the conceptual issues of
quantum cosmology are given in Section 8.

To conclude this introduction, we comment on the units used (see Ref. [2]). Let
\[ \sigma^2 = \frac{2G}{3\pi}. \]
In what follows a tilde shall denote ”true” physical quantities in units in which \( c = 1 \). The Planck mass and length are \( \tilde{m}_P = (\hbar/G)^{1/2} \) and \( \tilde{\ell}_P = \hbar/\tilde{m}_P \).

The FRW space-time metric is given by
\[
d s^2 = -\tilde{N}(t)^2 dt^2 + \tilde{a}(t)^2 d\sigma_3^2
\]
where \( d\sigma_3^2 \) is the metric on the round unit three-sphere. The scale factor and lapse are rescaled as \( \tilde{a} = \sigma a \) and \( \tilde{N} = \sigma N \). Let furthermore be \( \tilde{\phi} \) the (spatially homogeneous) scalar field (with dimension \( \hbar^{1/2}/\ell^{-1} \), \( \ell \) denoting length) and \( \tilde{m} \) its mass, and set \( \phi = \sigma \pi \sqrt{2} \tilde{\phi} \) and \( m = \sigma \tilde{m} \). (A general scalar field potential would be redefined as \( V(\phi) = 2\pi^2 \sigma^4 \tilde{V}(\tilde{\phi}) \). For the massive case we have \( \tilde{V}(\tilde{\phi}) = \tilde{m}^2 \tilde{\phi}^2/2\hbar^2 \) and \( V(\phi) = m^2 \phi^2/2\hbar^2 \)). According to this scheme we rescale the Planck mass and length as \( \tilde{m}_P = \sigma \tilde{m}_P = (2\hbar/3\pi)^{1/2} \approx 0.46 \) and \( \tilde{\ell}_P = (3\pi/2)^{1/2} \approx 2.17 \). The ratio between scalar field mass and Planck mass is thus \( \tilde{m}/\tilde{m}_P = m/m_P \approx 2.17 \). In order to account for the necessary amount of density fluctuations [4][5], we expect the mass parameter to be \( m \approx 10^{-6} \), hence much smaller than 1.

2  Representations for states

Let us as preliminaries write down the equations governing the classical dynamics of the Hawking model [3][7], i.e. the trajectories \((a(t), \phi(t))\) in the minisuperspace manifold \( \{(a, \phi)|a > 0\} \). If the space-time lapse \( N \) has been fixed (e.g. by assuming
a functional dependence $N \equiv N(a, \phi)$, the relation between the momenta and the
time-derivatives of the minisuperspace variables is given by

$$p_a = -\frac{a}{N} \frac{da}{dt}, \quad p_\phi = \frac{a^3}{N} \frac{d\phi}{dt}. \quad (2.1)$$

In the time gauge $N = 1$, the classical constraint equation reads

$$\dot{a}^2 + 1 = a^2 (\dot{\phi}^2 + m^2 \phi^2). \quad (2.2)$$

When evaluated at some initial time $t_0$, it may be interpreted as a restriction on the
set of initial conditions $(a(t_0), \phi(t_0), \dot{a}(t_0), \dot{\phi}(t_0))$. Once it is imposed for all times,
the time evolution equations reduce to

$$\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + m^2 \phi = 0. \quad (2.3)$$

The corresponding $\ddot{a}$-equation which arises from the Lagrangian formalism is automatically satisfied for all times on account of (2.2). In a general time gauge, the above equations apply after replacing $d/dt$ by $N^{-1} d/dt$.

We will not go into the details of the properties of classical trajectories. They
have been studied by a number of authors (see e.g. Refs. [6] [7]). Let us just note that
a typical (outgoing) classical trajectory leaves the inflationary domain $ma|\phi| \gg 1, |
\phi| \gg 1$ when $|\phi|$ settles to the order of unity. The subsequent evolution is of
the matter dominated type, with $\phi$ undergoing rapid oscillations. Eventually (when
the amplitude of $\phi$ falls into the domain of negative potential, $ma|\phi| < 1$), the scale
factor $a$ reaches its maximum value, and the universe recollapses again. During this
process the energy of the scalar field

$$E = \frac{a^3}{2} (\dot{\phi}^2 + m^2 \phi^2) \quad (2.4)$$

is approximately conserved. The amplitude of the oscillations of $\phi$ is given by

$$\phi_{\text{ampl}} \approx \frac{\sqrt{2E}}{ma^{3/2}}. \quad (2.5)$$

The maximum scale factor is $a_{\text{max}} \approx 2E$. (In order not to deal too much with
approximate quantities, one may simply define $E = a_{\text{max}}/2$). The value of $a$ at
which the trajectory enters (and leaves) the domain $|\phi| \lesssim 1$ (i.e. the value of the
scale factor at the end of inflation) is roughly given by \( a_{\text{min}} \approx (E/m^2)^{1/3} \). Hence, only if \( a_{\text{min}} \ll a_{\text{max}} \), i.e. \( E \gg m^{-1} \), we have a post-inflationary classical evolution at all.

Canonical quantization is achieved by rewriting the constraint in Hamiltonian form and performing the usual substitutions (see e.g. \( [8] \)). The result is the minisuperspace Wheeler-DeWitt equation

\[
\mathcal{H}\psi = 0,
\]

where the state is represented as a wave function \( \psi(a, \phi) \). Since in what follows different representations will be used, we distinguish between the notation of an ”operator” and its ”representation”, in particular if derivatives with respect to \( a \) are involved. Let \( \mathcal{D}_a \) be the operator that acts on a wave function \( \psi(a, \phi) \) as the partial derivative \( \partial_a \). Then the Wheeler-DeWitt operator is given by

\[
\mathcal{H} = \mathcal{D}_a \mathcal{D}_a + \frac{p}{a} \mathcal{D}_a + 2aE - a^2,
\]

where

\[
E = -\frac{1}{2a^3} \partial_{\phi\phi} + \frac{1}{2} a^3 m^2 \phi^2
\]

represents the total matter energy (cf. \( [2.4] \) for its classical analogue), and \( p \) is a parameter accounting for the operator ordering ambiguity. There are good arguments in favour of \( p = 1 \) (see e.g. Ref. \( [4] \)), but we will leave it unspecified. The last term \(-a^2\) in (2.7) represents the spatial curvature. In case of a spatially flat FRW model, one would omit it, and in case of an open FRW model one would change its sign. In analogy with ordinary quantum mechanics, the wave function \( \psi(a, \phi) \) can be said to be in the position representation (which is just the defining representation here).

To be explicit, the Wheeler-DeWitt equation in this representation reads

\[
(\mathcal{H}\psi)(a, \phi) \equiv \left( \partial_{aa} + \frac{p}{a} \partial_a - \frac{1}{a^2} \partial_{\phi\phi} + m^2 a^4 \phi^2 - a^2 \right) \psi(a, \phi) = 0,
\]

which is the standard form in which it is usually written down \( [2] \) and which determines its form in all other representations. (We will not make attempts to modify it, e.g. by introducing different operator orderings or taking square roots of operators as inspired by the hope of making expressions simple in some particular representation. In other words, we are not searching for an alternative wave equation, but just stick to (2.9) as the starting point, although written down in other representations).
Let us begin our analysis by exploiting the fact that for any value of $a$ the operator $E$ (as acting on functions of $\phi$) represents a quantum mechanical harmonic oscillator with frequency $m$ and mass $a^3$. Its eigenvalues are thus $E_n = (n + \frac{1}{2})m$ for non-negative integer $n$. Using the combination

$$\xi = m^{1/2}a^{3/2}\phi,$$  \hspace{1cm} (2.10)

we define an alternative representation of states by

$$\psi(a,\phi) \equiv m^{1/4}a^{3/4} \hat{\psi}(a,\xi).$$  \hspace{1cm} (2.11)

Since $\xi$ plays the role of an oscillator variable, one could call $\hat{\psi}(a,\xi)$ to be in the oscillator representation. The matter energy operator becomes essentially a unit harmonic oscillator,

$$E = \frac{m}{2} (-\partial_\xi + \xi^2).$$  \hspace{1cm} (2.12)

The operator $\mathcal{D}_a$ (which was $\partial_a$ in the representation $\psi(a,\phi)$) takes a different form now. Let $\mathcal{I}D_a$ be the operator that acts on a wave function in the representation $\hat{\psi}(a,\xi)$ as the partial derivative $\partial_a$. Then we have

$$\mathcal{D}_a = \mathcal{I}D_a + \frac{3}{2a} \mathcal{K}$$  \hspace{1cm} (2.13)

with

$$\mathcal{K} = \frac{1}{2} \{ \phi, \partial_\phi \} \equiv \phi \partial_\phi + \frac{1}{2} = \frac{1}{2} \{ \xi, \partial_\xi \} \equiv \xi \partial_\xi + \frac{1}{2}.$$  \hspace{1cm} (2.14)

The Wheeler-DeWitt operator in the oscillator representation $\hat{\psi}(a,\xi)$ is now still given by (2.7), but with $\mathcal{D}_a$ being represented as (2.13), and $\mathcal{I}D_a$ being represented as $\partial_a$. Thus, one may write

$$\mathcal{H} = \mathcal{I}D_a \mathcal{I}D_a + \frac{p}{a} \mathcal{I}D_a + \frac{3\mathcal{K}}{a} \mathcal{I}D_a + \frac{3(p - 1)}{2a^2} \mathcal{K} + \frac{9}{4a^2} \mathcal{K}^2 + 2aE - a^2$$  \hspace{1cm} (2.15)

which is valid in any representation (just as (2.7) is), as long as by $\mathcal{I}D_a$ the appropriate operator representation is understood.

The eigenfunctions of $E$ are just those of the unit harmonic oscillator with coordinate $\xi$. In terms of Hermite polynomials (which are generated by $e^{2\xi t - t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n H_n(t)$) they read

$$\Psi_n(\xi) = \frac{H_n(\xi)}{\sqrt{\pi} \sqrt{2^n n!}} e^{-\frac{1}{2} \xi^2}$$  \hspace{1cm} (2.16)
with \( n \) a non-negative integer. By expansion with respect to these, we define a further way of writing wave functions

\[
\hat{\psi}(a, \xi) = \sum_{n=0}^{\infty} f_n(a) \Psi_n(\xi),
\]

(2.17)

where the component functions \( f_n(a) \) may be regarded as providing the state in the energy representation (although we will encounter a further meaning of this word later on). This notation is justified by the fact that the operator \( E \) is now diagonal: it sends \( f_n(a) \) to \( E_n f_n(a) \). Its action may symbolically be written as \((E f)_n(a) = E_n f_n(a)\). Note that even \( n \) belongs to the even \((\psi(a, -\phi) = \psi(a, \phi))\) and odd \( n \) to the odd \((\psi(a, -\phi) = -\psi(a, \phi))\) sector of wave functions. Since the \( \Psi_n \) are an orthonormal basis, (2.17) may be inverted to give the oscillator excitations

\[
f_n(a) = \int_{-\infty}^{\infty} d\xi \, \Psi_n(\xi) \hat{\psi}(a, \xi)
\]

(2.18)
in terms of the wave function in the oscillator representation.

In performing (2.17) we have implicitly assumed that the wave function \( \psi \) is sufficiently well-behaved for large \( \xi \) (or \( \phi \)) so as to allow for such an expansion. A quite restrictive condition on general wave functions would be square integrability in the matter variable \( \xi \) (or \( \phi \), which is equivalent). Although this would offer a Hilbert space structure for any value of \( a \), the more interesting candidate wave functions of the universe are of distributional character with respect to this structure, and we just assume that the expansion (2.17) is possible. The formal squared norm of wave functions

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi \psi^*(a, \phi) \psi(a, \phi) = \int_{-\infty}^{\infty} d\xi \, \hat{\psi}^*(a, \xi) \hat{\psi}(a, \xi) = \sum_{n=0}^{\infty} f^*_n(a) f_n(a)
\]

(2.19)
may but need not be finite. The prefactor \( m^{1/4} a^{3/4} \) in (2.11) has been chosen so as to give these expressions a simple form. The formal hermiticity of operators like \( i\partial_{\phi}, i\partial_{\xi}, i\mathcal{K} \) and \( E \) with respect to the according scalar product is evident.

The Wheeler-DeWitt operator in this representation is given by (2.15) with \( \mathcal{D}_a = \partial_a, E \) acting diagonal with eigenvalues \( E_n \) and \( \mathcal{K} \) acting as (in a symbolic notation)

\[
(\mathcal{K} f)_n(a) = -\frac{1}{2} \sqrt{n(n-1)} f_{n-2}(a) + \frac{1}{2} \sqrt{(n+1)(n+2)} f_{n+2}(a),
\]

(2.20)
its square being given by

\[
(K^2 f)_n(a) = \frac{1}{4} \sqrt{n(n-1)(n-2)(n-3)} f_{n-4}(a) - \\
\frac{1}{4} \frac{1}{n(n-1) + (n+1)(n+2)} f_n(a) + \\
\frac{1}{4} \sqrt{(n+1)(n+2)(n+3)(n+4)} f_{n+4}(a).
\]

(2.21)

For completeness, we write down the Wheeler-DeWitt equation in this representation explicitly:

\[
(H f)_n(a) \equiv \left( \partial_{aa} + \frac{p}{a} \partial_a \right) f_n(a) + \frac{3}{a} \partial_a (K f)_n(a) + \frac{3(p-1)}{2a^2} (K f)_n(a) + \\
\frac{9}{4a^2} (K^2 f)_n(a) + (2aE_n - a^2) f_n(a) = 0.
\]

(2.22)

It involves differences with respect to \( n \) rather than derivatives. Note that the highest component function \( f_{n+4}(a) \) appears algebraically. Hence, the Wheeler-DeWitt equation in the energy representation is just a recursive expression for \( f_n(a) \) \((n \geq 4)\) in terms of \((f_0(a), \ldots, f_3(a))\).

The form (2.12) of \( E \) amounts to define formal annihilation and creation operators

\[
\mathcal{A} = a^{3/2} \sqrt{\frac{m}{2}} \phi + \frac{1}{a^{3/2} \sqrt{2m}} \partial_\phi = \frac{1}{\sqrt{2}} (\xi + \partial_\xi)
\]

(2.23)

\[
\mathcal{A}^\dagger = a^{3/2} \sqrt{\frac{m}{2}} \phi - \frac{1}{a^{3/2} \sqrt{2m}} \partial_\phi = \frac{1}{\sqrt{2}} (\xi - \partial_\xi)
\]

(2.24)

from which

\[
E = m \left( \mathcal{A}^\dagger \mathcal{A} + \frac{1}{2} \right) \equiv m \left( \mathcal{N} + \frac{1}{2} \right)
\]

(2.25)

and

\[
K = \frac{1}{2} \left( \mathcal{A}^2 - (\mathcal{A}^\dagger)^2 \right).
\]

(2.26)

The square of the latter turns out to be

\[
K^2 = \frac{1}{4} \left( \mathcal{A}^4 - \mathcal{N} (\mathcal{N} - 1) - (\mathcal{N} + 1)(\mathcal{N} + 2) + (\mathcal{A}^\dagger)^4 \right).
\]

(2.27)
The index $n$ of the components $f_n(a)$ represents the eigenvalues of the oscillator number operator $\mathcal{N} = \mathcal{A}^\dagger \mathcal{A}$. The dagger denotes hermitean conjugation in a formal sense, with respect to the scalar product associated with the squared norm expressions (2.19). By formally identifying $\Psi_n(\xi)$ with the abstract state $|n\rangle$ and using the commutator relation $[\mathcal{A}, \mathcal{A}^\dagger] = 1$, we find the usual structure defining a Fock space

\[\mathcal{A}|n\rangle = \sqrt{n} |n-1\rangle \quad \mathcal{A}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle.\]  

(2.28)

The $n$-the eigenstate is generated out of the ground state (note that $\mathcal{A}|0\rangle = 0$) by

\[|n\rangle = \frac{(\mathcal{A}^\dagger)^n}{\sqrt{n!}} |0\rangle.\]  

(2.29)

A given wave function may be written in the form

\[|\psi, a\rangle = \sum_{n=0}^{\infty} f_n(a) |n\rangle = \sum_{n=0}^{\infty} f_n(a) \frac{(\mathcal{A}^\dagger)^n}{\sqrt{n!}} |0\rangle \equiv \mathcal{F}(a, \mathcal{A}^\dagger)|0\rangle,\]  

(2.30)

thus defining a "Fock representation" for states. If $\langle \psi, a|$ is defined as $\sum_{n} \langle n| f_n^*(a)$, or equivalently as $\langle 0| \mathcal{F}^*(a, \mathcal{A})$, the formal squared norm (2.19) is given by $\langle \psi, a| \psi, a\rangle$. The analogue of (2.18) is

\[f_n(a) = \langle n| \psi, a\rangle,\]  

(2.31)

and the orthonormality of the oscillator basis carries over to $\langle r|s\rangle = \delta_{rs}$. If $|\psi, a\rangle$ is written as $\mathcal{F}(a, \mathcal{A}^\dagger)|0\rangle$, the operator $\mathcal{A}$ is formally represented as $\partial/\partial \mathcal{A}^\dagger$, whereas $\mathcal{A}^\dagger$ may be considered as a multiplication operator, and $\mathcal{D}_a$ is the partial derivative $\partial_a$. The relation between $\hat{\psi}(a, \xi)$ and $\mathcal{F}(a, \mathcal{A}^\dagger)$ has so far been given only through a number of intermediate steps, involving Hermite polynomials and an infinite sum.

It can be made more explicit by noting the identifications of functions of $\xi$ with abstract Fock space states

\[e^{ik\xi} \equiv \sqrt{2\pi} \sum_{n=0}^{\infty} i^n \Psi_n(k) \Psi_n(\xi) \longleftrightarrow \sqrt{\pi} \sqrt{2} e^{-\frac{1}{2}k^2} e^{i\sqrt{2}k\mathcal{A}^\dagger + \frac{1}{2}(\mathcal{A}^\dagger)^2} |0\rangle,\]  

(2.32)

for fixed $k$ and

\[\delta(\xi - \eta) \equiv \sum_{n=0}^{\infty} \Psi_n(\eta) \Psi_n(\xi) \longleftrightarrow \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}\eta^2} e^{\sqrt{2}\eta\mathcal{A}^\dagger - \frac{1}{2}(\mathcal{A}^\dagger)^2} |0\rangle\]  

(2.33)

for fixed $\eta$. A further interesting relation illustrating the appearence of Hermite polynomials is

\[\xi^n \longleftrightarrow \sqrt{\pi} \sqrt{2} (-i)^n 2^{-n/2} H_n(i\mathcal{A}^\dagger) e^{\frac{1}{2}(\mathcal{A}^\dagger)^2} |0\rangle.\]  

(2.34)
for non-negative integer \( n \). Using (2.33), one finds for a given wave function \( \hat{\psi}(a, \xi) \) that

\[
\mathcal{F}(a, A^\dagger) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\eta \hat{\psi}(a, \eta)e^{-\frac{1}{2}(\eta-\sqrt{2}A^\dagger)^2} e^{\frac{1}{2}(A^\dagger)^2}.
\] (2.35)

Inverting this relation is not that straightforward. Defining \( \tilde{\psi}(a, k) \) to be the inverse Fourier transform of \( \hat{\psi}(a, \xi) \), i.e.

\[
\hat{\psi}(a, \xi) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{\psi}(a, k)e^{ik\xi},
\] (2.36)

and making use of (2.32), equation (2.33) may be expressed alternatively as

\[
\mathcal{F}(a, A^\dagger) = \sqrt{\frac{\sqrt{2}}{\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{\psi}(a, k)e^{-\frac{1}{2}(k-i\sqrt{2}A^\dagger)^2} e^{-\frac{1}{2}(A^\dagger)^2}.
\] (2.37)

This may be inverted to give

\[
\tilde{\psi}(a, k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dA^\dagger}{\sqrt{2\pi}} \mathcal{F}(a, A^\dagger)e^{-\frac{1}{2}(A^\dagger)^2 + i\sqrt{2}k^2} e^{-\frac{1}{2}k^2}.
\] (2.38)

Combining this last equation with (2.36) and interchanging the \( A^\dagger \) and \( k \) integrations gives a direct formula for \( \tilde{\psi}(a, \xi) \) in terms of \( \mathcal{F}(a, A^\dagger) \), but with an integrand that exists only in a distributional sense.

The Wheeler-DeWitt operator in this representation is given by (2.15) with \( D_a = \partial_a \), \( E \) from (2.23) and \( K \) from (2.26). Note that, due to (2.27), the operator \( K^2 \) contains \( A^4 \). This makes the Wheeler-DeWitt equation in the Fock representation a fourth order differential equation for \( \mathcal{F}(a, A^\dagger) \).

We are thus able to represent states in various forms, namely as \( \psi(a, \phi) \), \( \hat{\psi}(a, \xi) \), \( f_n(a) \), \( |\psi, a\rangle \) and \( \mathcal{F}(a, A^\dagger) \). The latter three forms are of course closely related to each other, referring to the scalar field energy as a variable. The different versions of the Wheeler-DeWitt equation are equivalent, which makes the choice of representation a matter of convenience and taste.

3 Approximate solutions

The framework of representations as given in the last Section is “kinematic” in nature, i.e. it makes no reference to the Wheeler-DeWitt equation. It is clear that
an object like $|n\rangle$, for some fixed $n$, does not even approximately satisfy it. On the other hand, we know that classically the energy $E$ is approximately conserved in the post-inflationary regime. We thus expect that there are solutions of the Wheeler-DeWitt equation which behave to some accuracy like

$$ |\psi, a\rangle = h(a)|n\rangle \tag{3.1} $$

for values of $a$ which belong to the post-inflationary classical domain (i.e. between end of inflation at $a_{\text{min}} \approx (E_n/m^2)^{1/3}$ and maximum size at $a_{\text{max}} \approx 2E_n$).

Inserting (3.1) into the Wheeler-DeWitt equation in the Fock representation of (2.13), the reason why it cannot be an exact solution turns out to be that $K$ and $K^2$ as represented by (2.26) and (2.27) contain powers of $\mathcal{A}$ and $\mathcal{A}^\dagger$ that mix the elements $|r\rangle$ of the oscillator basis. These terms arise from the $a$-dependence of $E$ in (2.8). When $a$ is large, they can be neglected, and the harmonic oscillator dynamics follows the dynamics of the gravitational sector. This leads to the adiabatic approximation, a technique which is frequently applied (see e.g. Ref. [7]). In order to have a non-trivial classical domain at all ($a_{\text{min}} \ll a_{\text{max}}$) we must choose $n \gg m^{-2} \approx 10^{12}$, hence much larger than 1. Neglecting all non-trivial powers of $\mathcal{A}$ and $\mathcal{A}^\dagger$ in (2.15), we pick up the effective potential term $-\frac{9}{8a^2}(n^2 + n + 1)$ from $K^2$. Taking into account the operator ordering term as well, we end up with an approximate equation for the range $a \gg a_{\text{min}}$

$$ \left( \partial_{aa} + \frac{p}{a} \partial_a + U(a) \right) h(a) = 0 \tag{3.2} $$

with

$$ U(a) = 2aE_n - a^2 - \frac{9}{8a^2}(n^2 + n + 1). \tag{3.3} $$

From the point of view of the oscillator basis, this is just the projection of the Wheeler-DeWitt equation with (3.1) inserted onto the $n$-the basis element. Hence, (3.2) is identical to $\langle n|\mathcal{H} h(a)|n\rangle = 0$. It is the best we can do when all oscillators other than $|n\rangle$ are ignored.

By redefining $h(a) = a^{-p/2}g(a)$, one kills the first order derivative. In the range $a \gg a_{\text{min}}$, the $a^{-2}$ contribution to $U(a)$ (together with an additional term from the above redefinition) is small. Neglecting it, the new approximate equation reads

$$ \left( \partial_{aa} + 2aE_n - a^2 \right) g(a) = 0. \tag{3.4} $$

The solutions thereof may be expressed in terms of parabolic cylinder functions [10]. Since this is not very instructive, let us look at some possible ranges of $a$. For
\( a \ll a_{\text{max}} \), the potential in (3.4) is dominated by \( 2aE_n \), the according solutions being the two Airy functions \( \text{Ai}(-2E_n^{1/3}a) \) and \( \text{Bi}(-2E_n^{1/3}a) \). Since the arguments \(-2E_n^{1/3}a\) of these functions are much less than \(-1\), we are in the range in which they oscillate rapidly. Up to a multiplicative constant, the according asymptotic expressions for \( g(a) \) are

\[
a^{-1/4} \text{osc} \left( \frac{2}{3} (2E_n)^{1/2} a^{3/2} + \frac{\pi}{4} \right)
\]

with \( \text{osc} = \cos \) for \( \text{Bi} \) and \( \text{osc} = \sin \) for \( \text{Ai} \).

When \( a \) approaches \( a_{\text{max}} \), the \(-a^2\) contribution in the potential becomes important and will slow down the oscillations. Here we note that so far it was not necessary to restrict \( a \) to be less than \( a_{\text{max}} \). At \( a \approx a_{\text{max}} \approx 2E_n \), the potential vanishes. For \( a \gg a_{\text{max}} \), the overall behaviour is exponential, and we find two asymptotic solutions for \( g(a) \) of the type

\[
a^{-1/2} \exp \left( \pm \frac{1}{2} a^2 \mp E_n a \right).
\]

For given \( E_n \), this range is "classically forbidden", because all classical universes with matter energy \( E_n \) cannot extend to such sizes. Note that for \( n < \sim m^{-2} \), the asymptotic solutions (3.6) still exist, although they do not have an oscillatory domain of the type (3.5).

So far we have not stated anything about the allowed range of \( \phi \). The classical amplitude of oscillations is given by (2.5), and we expect the approximation for the oscillating regime (3.5) to hold if \( |\phi| \lesssim \phi_{\text{ampl}} \). In terms of \( \xi \), this means \( |\xi| \lesssim \sqrt{2n} \). This is in fact just the domain in which the oscillator basis \( \Psi_n(\xi) \) is considerably non-zero. (Note that the amplitude of the classical oscillations with energy \( mn \) corresponding to the Hamiltonian (2.12) is just \( \phi_{\text{ampl}} = \sqrt{2n} \)). In the exponential regime (3.6), an identical condition on \( \xi \) holds. Hence, for \( a \gg a_{\text{min}} \) the interesting values are concentrated in a stripe around the \( \phi = 0 \) axis that gets arbitrarily narrow as \( a \) increases.

In order to link the asymptotic form (3.6) with the oscillating behaviour (3.5), we may apply the standard WKB matching procedure between domains in which the potential has different sign [11]. Introducing the number

\[
\Theta_n = \frac{\pi}{2} E_n^2 + \frac{\pi}{4},
\]
we fix the normalization in the oscillatory region by defining two real approximate solutions of the type (3.5)
\[
\Omega_n^+, a = a^{-p/2-1/4} (2E_n)^{-1/4} \cos \left( \Theta_n - \frac{2}{3} (2E_n)^{1/2} a^{3/2} \right) |n\rangle \\
\Omega_n^-, a = a^{-p/2-1/4} (2E_n)^{-1/4} \sin \left( \Theta_n - \frac{2}{3} (2E_n)^{1/2} a^{3/2} \right) |n\rangle.
\]
(3.8) (3.9)

The result of applying the WKB matching procedure (which is too boring to be shown here) is that, in the exponential domain,
\[
\Omega_n^+, a = a^{-p/2-1/2} \left( \frac{2a}{E_n} \right)^{-E_n^2/2} \exp \left( \frac{1}{2} a^2 - E_n a + \frac{1}{4} E_n^2 \right) |n\rangle \\
\Omega_n^-, a = \frac{1}{2} a^{-p/2-1/2} \left( \frac{2a}{E_n} \right)^{E_n^2/2} \exp \left( - \frac{1}{2} a^2 + E_n a - \frac{1}{4} E_n^2 \right) |n\rangle.
\]
(3.10) (3.11)

For \( n \lesssim m^{-2} \) there is no oscillatory region, and we define the asymptotic normalization by these two expressions as well. The factor \( \frac{1}{2} \) in (3.11) may look a bit strange, but it is an immediate consequence of the matching procedure (it is actually part of the standard formulae) and the fact that we have arranged the normalization of the oscillating behaviour (3.8)–(3.9) in a symmetric way. Note that the Airy functions \( \text{Ai}(x) \) and \( \text{Bi}(x) \), when expanded for \( x \to -\infty \) and \( x \to \infty \) (i.e. in the oscillatory and in the exponential domain) display a precisely analogous factor \( \frac{1}{2} \) for \( \text{Ai}(x) \) (cf. also (3.5)).

Defining two other sets of approximate solutions \( \Xi_n^\pm \) by
\[
\Xi_n^\pm = e^{\pm i\Theta_n} (\Omega_n^+ \mp i\Omega_n^-),
\]
(3.12)

or, inversely,
\[
\Omega_n^+ = \frac{1}{2} \left( e^{-i\Theta_n} \Xi_n^+ + e^{i\Theta_n} \Xi_n^- \right) \\
\Omega_n^- = \frac{1}{2} \left( e^{-i\Theta_n} \Xi_n^+ - e^{i\Theta_n} \Xi_n^- \right),
\]
(3.13) (3.14)

we find, in the oscillatory regime,
\[
|\Xi_n^\pm\rangle = a^{-p/2-1/4} (2E_n)^{-1/4} \exp \left( \pm \frac{2i}{3} (2E_n)^{1/2} a^{3/2} \right) |n\rangle.
\]
(3.15)
The involved structure of prefactors was necessary in order to achieve this simple form. The significance of the normalization of this expression (in particular the factor \((2E_n)^{-1/4}\)) will become clear later.

Thus, we have a set of approximate solutions at hand that correspond to prescribed values of the matter energy. These functions have been used by Kiefer in his discussion of wave packets in the Hawking model \([7]\). Also, the exponential behaviour of \(\Omega^+_0(a, \phi)\) has been displayed by Page \([6]\) as the dominant part of the no-boundary wave function for small \(\phi\) and large \(a\). The \(\Omega^-_n\) also occur in the work of Hawking and Page \([3]\), to which we will refer later on.

The physical significance of any of the wave functions \(\Xi^\pm_n\) with \(n \gg m^{-2}\) at the WKB-level is to represent an ensemble of contracting/expanding classical universes with post-inflationary matter energy \(E_n\). The expressions \((3.15)\) may be viewed as semiclassical WKB-states around a family of expanding classical backgrounds with action \(S_0(a) = -\frac{2}{3}\sqrt{2Ea^{3/2}}\) (this is discussed in a bit more detail in Ref. \([12]\)). Note however that the classical matter energy \(E\) is approximately conserved only in the domain \(|\phi| \ll 1\), i.e. ”after” inflation. Hence, the individual \(|\Xi^\pm_n, a\rangle\) can be expected to contain essentially a single oscillator excitation (i.e. of \(|n\rangle\)) only if \(a \gg a_{\text{min}} \approx (E_n/m^2)^{1/3}\). For smaller values of \(a\) we expect \(|\Xi^\pm_n, a\rangle\) to be a non-trivial superposition of (virtually) all oscillators \(|r\rangle\) and thus all approximate expressions we gave for these wave functions to break down. Furthermore it is likely that these states are not well-behaved for small \(a\). This is because there are many classical trajectories to some given value of \(a_{\text{max}}\) which behave quite singular as \(a \to 0\) (namely of the ”collapse” type \(|\phi| \to \infty\)). Also, we cannot expect \(|\Xi^\pm_n, a\rangle\) to satisfy nice properties near the zero potential curve \(a^2m^2\phi^2 = 1\), along which usually ”nucleation” is assumed to occur. In the case \(n \gtrsim m^{-2}\) we can talk about pure tunnelling states that do not correspond to classical universes at all.

Although, so far, the wave functions have only been identified as approximate solutions of the Wheeler-DeWitt equation, it is reasonable to suppose the existence of exact solutions that behave qualitatively in the same way. In the range \(a \gg a_{\text{min}}\) this means that the excitations of oscillators other than \(|n\rangle\) may be non-zero but are small. Moreover, we expect to obtain a basis for the set of solutions: At some fixed value of \(a\), the degrees of freedom contained in \(\{\Omega^\pm_n\}\) correspond to two free functions. Hence, at the approximate level, any initial data \((\psi, \partial_a \psi)|_{a=a_{\text{ini}}}\) may be expanded in terms of the oscillators \(|n\rangle\) (i.e. in terms of the functions \(\Psi_n(m^{1/2}a^{3/2}\phi) \equiv \Psi_n(\xi)\)). This behaviour is expected to carry over to the exact case.
For large $a$, the oscillator mixing operators (powers of $A$ and $A^\dagger$) contained in $K$ and $K^2$ get suppressed. Hence we expect the approximation of the predominance of one oscillator $|n\rangle$ in a wave function to become arbitrarily accurate as $a \to \infty$. In this limit, we may treat all states belonging to non-negative integers $n$ at the same footing (including the pure tunnelling states $\Omega_n^\pm$ for small $n$). One must of course be aware that an exactification of our wave functions will leave a considerable amount of freedom. However, by combining the large-$a$ expansion with the assumption of certain analyticity properties, we will encounter an apparently distinguished way to single out a unique exact solution for any choice of the ± label and any $n$.

Let us at the end of this Section provide another argument leading to the same exponential behaviour of wave functions. One may apply the standard semiclassical WKB-expansion scheme \cite{13}, based on a Born-Oppenheimer approximation, by rewriting the Wheeler-DeWitt equation (2.9) in terms of units that make the Planck length explicit. At the end of the introduction we have displayed the relation between the ”true” quantities $\tilde{a}$, $\tilde{\phi}$, $\tilde{m}$ and the rescaled ones $a$, $\phi$ and $m$. Restoring the true units, one finds the kinetic part of the gravitational field multiplied by the Planck length squared and the curvature contribution multiplied by the inverse of the Planck length squared. The matter sector remains unaffected (up to a numerical factor of order unity). This structure may equivalently be written down by introducing a formal book-keeping parameter $\lambda$ (playing the role of the Planck length) that is treated as a small quantity (and reset equal to 1 in the end). The Wheeler-DeWitt equation (2.9) thus becomes

$$\left( \lambda^2 \left( \partial_{aa} + \frac{p}{a} \partial_a \right) - \frac{1}{a^2} \partial_{\phi\phi} + m^2 a^4 \phi^2 - \frac{a^2}{\lambda^2} \right) \psi(a,\phi) = 0.$$  \hspace{1cm} (3.16)

This may formally be achieved by replacing $a \to \lambda^{-1}a$, $\phi \to \lambda\phi$ and $m \to \lambda m$ in (2.9). According to the WKB-description we expand a wave function as

$$\psi(a,\phi) = \exp \left( i \left( \frac{S_0(a,\phi)}{\lambda^2} + S_1(a,\phi) + \lambda^2 S_2(a,\phi) + \lambda^4 S_3(a,\phi) + \ldots \right) \right).$$  \hspace{1cm} (3.17)
The equation at $O(\lambda^{-4})$ turns out to be just $\partial_\phi S_0(a, \phi) = 0$, telling us that $S_0 \equiv S_0(a)$. At $O(\lambda^{-2})$ we obtain the equation $(\partial_a S_0(a))^2 = -a^2$. Formally, this is the Hamilton-Jacobi equation for the pure gravitational field and thus fits into the general semiclassical scheme. On the other hand, it has no real solution, which just reflects the fact that an empty closed FRW universe does not exist. Proceeding straightforwardly, we write down the two imaginary solutions $S_0(a) = \mp ia^2/2$. The WKB-phase factor $e^{iS_0(a)} \equiv e^{\pm a^2/2}$ thus provides already the dominant large-$a$ behaviour of $\Omega^\pm_n(a, \phi)$. Rescaling the next order contribution as $e^{iS_1(a, \phi)} = a^{-1/2-n/2} \chi(a, \phi)$, we find at $O(\lambda^0)$ the Wheeler-DeWitt equation to state

\[(\pm \partial_a + E)\chi(a, \phi) = 0\]  

(3.18)

with $E$ from (2.8). This is the step where usually the effective Schrödinger equation (as a minisuperspace version of the Tomonaga-Schwinger equation) arises \[13\]. Obviously, we encounter a Euclidean type Schrödinger equation. In the adiabatic approximation, one neglects the $a$-dependence of $E$. By introducing the oscillator variable $\xi$ from (2.10), the energy operator becomes represented as (2.12). The adiabatic approximation amounts to keep the derivative $\partial_a$ unchanged in (3.18). Thus, factorizing $\chi \equiv m^{-1/4} a^{-3/4} \chi$ (cf. (2.11)) into a product $A(a) X(\xi)$ (in the usual context this would be a stationary state) yields immediately an eigenvalue equation, hence $X(\xi) = \Psi_n(\xi)$ for some non-negative integer $n$, as well as the ”time evolution” prefactor $A(a) = e^{\pm E_n a}$. Putting everything together, we have reproduced qualitatively the behaviour (3.10)–(3.11) of the wave functions $\Omega^\pm_n$. The additional factors of the type $a^{\mp E_n^2/2}$ are effects at $O(\lambda^2)$.

Thus, for large $a$, the approximate solutions $\Omega^\pm_n(a, \phi)$ may be considered as semiclassical states, built around a ”pure tunneling” background gravitational field (which nevertheless — in the way how it appears in a WKB-expansion — displays formal similarities to a true classical background). This way of looking at things may seem a bit strange, but since we are faced with a wave equation (as opposed to classical equations of motion) we cannot exclude that the tunneling region $a \gg a_{\text{max}}$ plays an important role in the structure of the space of solutions or in the conceptual foundations of a quantum theory of the universe. This may be in some correspondence with the idea of a Euclidean path-integral \[14\]|\[15\] or some other principle which provides some additional structure that is invisible for the semiclassical techniques. Anyway, it may be taken as a further motivation in favour of examining the limit $a \to \infty$ of the Wheeler-DeWitt equation.
4 Exact solutions in the position representation

Leaving the level of approximate wave functions, we will try now to define a large-$a$ expansion scheme that enables us to specify a set of exact solutions of the Wheeler-DeWitt equation. An appropriate ansatz is modelled according to the asymptotic behaviour of $\Omega^\pm_n$ as given by (3.6), multiplied by $|n\rangle$, i.e. by the function $\Psi_n(\xi)$. The dominant behaviour for large $a$ and fixed $\phi$ of the latter is given by the exponential $e^{-\xi^2/2} \equiv e^{-ma^3\phi^2/2}$ (see equations (2.10) and (2.16)). This is followed by the factor $e^{\pm a^2/2}$. Hence, in a large-$a$ expansion at fixed $\phi$, the leading order in the exponential is $a^3$, with a $\phi$-dependent coefficient. Separating the positive powers of $a$, as appearing in the exponential, from a pure $a^q$ term (which may represent $e^{q \ln a}$) and expanding the remainder in terms of $a^{-1}$, we write down as a general ansatz in the position representation

$$\psi(a, \phi) = a^q \exp \left( F_3(\phi) a^3 + F_2(\phi) a^2 + F_1(\phi) a \right) \sum_{r=0}^{\infty} \frac{G_r(\phi)}{a^r}. \quad (4.1)$$

This looks fairly general, the only severe restriction being that only one exponential of the above type is involved. In a paper on wormhole solutions, Hawking and Page [3] have effectively used a similar type of expansion, and some of the structure we will encounter appears there as well. By inserting the ansatz into the Wheeler-DeWitt equation (2.9) and separating powers of $a$, one obtains a sequence of ordinary differential equations for the functions $F_r(\phi)$ and $G_r(\phi)$, thereby expecting the freedom of choosing arbitrary integration constants.

It is important to note that in an expansion in negative powers of $a$, the $\phi$-dependent coefficient functions cannot in general be expected to display regular behaviour. This may be illustrated by expanding the function $(1 + a^4\phi^4)^{1/2}$ (which is $C^\infty$ on the domain $a > 0$, $\phi = \text{arbitrary}$) as $a^2\phi^2 + \frac{1}{2}a^{-2}\phi^{-2} + \ldots$. For fixed $\phi$, the domain of convergence of this series is $a > |\phi|^{-1}$, which breaks down at $\phi = 0$. The reason for limited convergence is that the function has singularities at $a = \pm \sqrt{\pm i\phi^{-1}}$, against which a condition like $C^\infty$ is insensitive.

However, we are free to impose analyticity properties on the coefficient functions in the ansatz for $\psi(a, \phi)$, as long as they are compatible with the Wheeler-DeWitt equation. By doing so, we touch upon a deeper structure which is not yet completely understood. We choose as a first condition the most natural requirement:

(i) The functions $F_r(\phi)$ and $G_r(\phi)$ are real analytic, i.e. they are real and analytic in a neighbourhood of the real axis in the complex $\phi$-plane. As a consequence, they
admit a Taylor expansion at \( \phi = 0 \).

Hence, by truncating the series in (4.1) at fixed \( \phi \) and sufficiently large \( a \) one should obtain a good numerical approximation for \( \psi \). There will be a function \( R(\phi) \) such that the series converges for all \( a > R(\phi) \). This defines a domain of convergence in minisuperspace. In order to exclude catastrophic behaviour of the leading \( a^3 \) term in the exponent, we require in addition

\[(ii) \quad F_3(\phi) \leq 0 \quad \text{for all } \phi.\]

The significance of this condition may be illustrated for the case of the approximate expressions for \( \Omega^\pm_n \). It allows for the dominant term \( e^{-\xi^2/2} \equiv e^{-ma^3\phi^2/2} \) which appears in \( \Psi_n(\xi) \), whereas it prevents a behaviour of the type \( e^{\xi^2/2} \equiv e^{ma^3\phi^2/2} \) which would otherwise be possible.

Remarkably, the ansatz (4.1) together with the two conditions (i) and (ii) seem to leave only a discrete freedom. We believe that the general solution is characterized by a non-negative integer \( n \) and a choice of sign, i.e. a two-valued label \( \pm \). The wave functions appearing in this way are recognized as exactifications of \( \Omega^\pm_n(a, \phi) \) (and henceforth called by the same name) that are distinguished by their analyticity properties.

Let us look at the sequence of equations generated by inserting (4.1) into (2.9) and dividing by (4.1). At the leading order \( O(a^4) \) we find

\[ F_3'(\phi)^2 = 9F_3(\phi)^2 + m^2 \phi^2. \]

Condition (ii) specifies a unique solution. Let us suppose that \( F_3(0) \neq 0 \). As a consequence, \( F_3'(\phi) \), which is given by a square root whose argument never vanishes, will always be non-zero. For \( \phi \to \pm \infty \) it will either tend to \( \infty \) or \( -\infty \). In both cases it is not possible for \( F_3(\phi) \) to be non-positive for all \( \phi \). Hence, we must have \( F_3(0) = 0 \). This fixes \( F_3'(0) = 0 \). Further differentiation of (4.2) generates the expansion

\[ F_3(\phi) = -m \left( \frac{1}{2} \phi^2 + \frac{9}{32} \phi^4 + \frac{27}{256} \phi^6 + \ldots \right), \]

condition (ii) fixing the sign. This also agrees to leading order with the behaviour of the exponential factor \( e^{-\xi^2/2} \equiv e^{-ma^3\phi^2/2} \) in \( \Psi_n(\xi) \). The behaviour of \( F_3(\phi) \) for large \( \phi \) is

\[ F_3(\phi) \approx -0.047 m e^{3\phi}, \]

where the constant has been determined by numerical methods.
At the next order $O(a^3)$ we find the equation

$$F_2'(\phi)F_3'(\phi) = 6F_2(\phi)F_3(\phi). \tag{4.5}$$

Since $F_3(\phi)$ is already uniquely determined, $F_2(\phi)$ is fixed up to a multiplicative constant. In particular, we find $F_2'(0) = 0$. The equation at $O(a^2)$, reads

$$2F_1'(\phi)F_3'(\phi) + F_2'(\phi)^2 = 6F_1(\phi)F_3(\phi) + 4F_2(\phi)^2 - 1. \tag{4.6}$$

Again, $F_1(\phi)$ is determined only up to an integration constant. Inserting $\phi = 0$, we get $4F_2(0)^2 = 1$. Thus, there are two possibilities $F_2(0) = \pm \frac{1}{2}$. For either sign, the solution of (4.5) is now uniquely determined to be

$$F_2(\phi) = \pm \left(\frac{1}{2} + \frac{3}{4} \phi^2 + \frac{45}{128} \phi^4 + \frac{9}{512} \phi^6 + \ldots\right). \tag{4.7}$$

The leading behaviour of $F_2(\phi)$ for small $\phi$ thus reproduces $e^{\pm a^2/2}$, which we already know from (3.6). For large $\phi$ we find

$$F_2(\phi) \approx \pm 0.231 e^{2\phi} \tag{4.8}$$

as the leading asymptotic behaviour.

At $O(a)$, the number $q$ as well as the operator ordering parameter $p$ and the first function of the series in (4.1) come into play. The equation reads

$$F_3''(\phi) + 2F_3'(\phi) \frac{G_0'(\phi)}{G_0(\phi)} + 2F_1'(\phi)F_2'(\phi) = 4F_1(\phi)F_2(\phi) + 3(2 + p + 2q)F_3(\phi). \tag{4.9}$$

Since, by condition (i), $G_0(\phi)$ is analytic, its Taylor expansion at $\phi = 0$ exists and starts with $k_1 \phi^n + k_2 \phi^{n+1} + \ldots$, where $n$ is some non-negative integer and $k_1 \neq 0$. In the limit $\phi \to 0$ we have $F_3'(\phi)G_0'(\phi)/G_0(\phi) \to -mn$, and (4.9) reduces in this limit to $F_1(0) = \mp(n + \frac{1}{2})m \equiv \mp E_n$. As a consequence, equation (4.10) admits a unique solution

$$F_1(\phi) = \mp E_n + \frac{3}{16m}(-1 \mp 4mE_n)\phi^2 + \frac{9}{256m}(3 \mp 2mE_n)\phi^4 + \frac{3}{4096m}(41 \mp 48mE_n)\phi^6 + \ldots \tag{4.10}$$

The leading term thus reproduces $e^{\mp E_n a}$ from (3.4). For large $\phi$ we find

$$F_1(\phi) \approx \mp 0.654 E_n e^{\phi} \tag{4.11}$$
as the dominant behaviour.

From now on we just describe the general structure of the subsequent steps. The limit $\phi \to 0$ of the $O(a^0)$ equation yields that $\phi^{-n}G_1(\phi)$ must be regular. Moreover, the leading order part of this equation fixes

$$q = \frac{3n}{2} + \frac{1}{4} - \frac{p}{2} + \frac{1}{2} \frac{E^2}{n}. \quad (4.12)$$

The first term $3n/2$ combines together with $\phi^n$ from $G_0(\phi)$ and a $m^{n/2}$ from the overall normalization into $\xi^n$ — cf. (2.10) —, which provides the leading order of the $n$-th Hermite polynomial $H_n(\xi)$. Taking into account the factor $a^{3/4}$ between the position and oscillator representation (see (2.11)), we exactly reproduce the contribution $a^{-p/2-1/2+\frac{E^2}{2}}$ from (3.10)–(3.11). This result is inserted into (4.9), by which $G_0(\phi)$ becomes unique up to the multiplicative constant $k_1$ which survives as an overall normalization freedom for $\psi(a,\phi)$, and we find

$$G_0(\phi) = k_1 \phi^n \left(1 + \frac{3}{16m^2} \left(3mE_n - m^2 + 1 \mp 2m^2E^2_n\right)\phi^2 + \ldots\right). \quad (4.13)$$

The overall pattern seems to persist at all orders. All equations are of the linear inhomogeneous type, leaving an integration constant which is determined at the next order by the analyticity requirement. We have checked this up to $G_7(\phi)$.

By re-writing the series in (4.1) as an exponential, one may re-arrange terms in a more explicit way. This is in fact what Hawking and Page [3] have done for the case $p = 1$. When translated to our formulation, their result seems to make explicit how the pattern determining the functions $G_r(\phi)$ persists to all orders. The uniqueness of the coefficient functions is not considered as an important issue in Ref. [3], but it is effectively achieved by throwing away $\ln(\phi)$-terms at each order. Since these authors intended to construct wormhole solutions, they considered only the exponentially decreasing behaviour $e^{-a^2/2}$. They arrive at a set of functions $\Psi_n(a,\phi)$, which we will denote as $\Psi_{HP}^n(a,\phi)$.

Hence, without having a rigorous proof, we conjecture that all $G_r(\phi)$ exist and are uniquely determined, once the $\pm$ label and $n$ have been chosen. The case of the exponentially decreasing sector (lower sign) for $p = 1$ seems to be covered by Ref. [3]. Moreover, recalling the physical discussion of the approximate wave functions in Section 3, we believe that the series (4.1) defines exact solutions $\Omega^\pm_n(a,\phi)$ of the Wheeler-DeWitt equation for all $(a,\phi)$, i.e. also outside the domain of convergence.
of the series (in case this domain does not agree with the whole of minisuperspace). We also note that, pulling an overall factor $\phi^n$ out of $\Omega_n^\pm(a, \phi)$, only even powers of $\phi$ remain, hence $\Omega_n^\pm(a, -\phi) = (-)^n \Omega_n^\pm(a, \phi)$. The functions $\Psi_n^{HP}(a, \phi)$ as displayed by Hawking and Page apparently coincide (up to normalization) with our $\Omega_n^-(a, \phi)$. (The leading order of their equation (72) is just the explicit formula for the $n$-th Hermite polynomial).

Due to the expansions of the coefficient functions around $\phi = 0$ we have an idea how $\Omega_n^\pm(a, \phi)$ behaves for small $\phi$ and large $a$. By looking at the expression (4.4) of $F_3(\phi)$ for large $\phi$, we expect that, for sufficiently large and fixed $a$, the dominant behaviour for large $\phi$ is $\exp(-0.047 ma^3 e^{3\phi})$. If this conclusion holds, the wave functions are actually more damped than one would expect from the factor $e^{-\xi^2/2} \equiv e^{-ma^3 \phi^2/2}$ of the oscillator basis (2.10) alone. One may in fact perform an independent analysis of solutions of the Wheeler-DeWitt equation in terms of large $\phi$, thus identifying the $\Omega_n^\pm(a, \phi)$ by means of the asymptotic expressions (4.4), (4.8) and (4.11). We will not go into these details but just complete our conjecture by noting that our wave functions are likely to be well-behaved as $|\phi| \to \infty$.

5 Exact solutions in the oscillator and Fock representations

So far we have considered an expansion in $a^{-1}$ at constant $\phi$. For actual computations the oscillator and Fock representations turn out to be more suitable. As we have seen, the series in (4.1) contains an overall factor $\phi^n$ which, together with $a^{3n/2}$ from (4.12) makes up an overall factor $\xi^n$. Since the remainder contains only even powers of $\phi$, the transformation to the variable $\xi$ — cf. (2.10) — amounts to substitute $\phi^2 \to m^{-1}a^{-3}\xi^2$, and even powers thereof. As a consequence, we never pick up half-integer powers of $a$ or negative powers of $\xi$. Moreover, keeping $\xi$ fixed means to follow a curve in minisuperspace whose $\phi$-coordinate value decreases towards the axis $\phi = 0$ as $a \to \infty$. This should not affect questions of existence and convergence very much (or even improve the situation). The finite sum $F_3(\phi)a^3 + F_2(\phi)a^2 + F_1(\phi)a$ reduces to the expression $\pm a^2/2 \pm E_n a$ plus a series of functions that contains only negative integer powers of $a$.

We can thus re-arrange the expression (4.1) in terms of $a$ and $\xi$. In order not to overcomplicate things, we assume the choice of the $\pm$ sector and of $n$ has already
been made, and explicitly insert the leading orders. Transforming $\psi(a, \phi)$ into the oscillator representation $\hat{\psi}(a, \xi)$ by (2.11), we end up with

$$\hat{\Omega}^\pm_n(a, \xi) = C^\pm_n a^{q^\pm_n} \exp \left( \pm \frac{1}{2} a^2 \mp E_n a \right) \sum_{r=0}^{\infty} \frac{F^\pm_{rn}(\xi)}{a^r},$$

where the functions $F^\pm_{rn}(\xi)$ are analytic and $C^\pm_n$ is an arbitrary overall normalization constant. The numbers $q_n$ may either be inferred from (4.12) by taking into account the correct transformation factors, yielding

$$q^\pm_n = -\frac{1}{2} - \frac{p}{2} \mp \frac{1}{2} E_n^2,$$

or left undetermined in order to be re-discovered in the oscillator or Fock formalism. In view of our conjecture, the $F^\pm_{rn}(\xi)$ should be uniquely determined. Note however that these functions arise from a rearrangement of orders in the series of (4.1), combined with a series stemming from the exponent. Computationally, they are related to $F^r(\phi)$ and $G^r(\phi)$ in a non-trivial way. Due to the discussion given in Section 3, we expect the eigenfunctions of $E$ to appear at leading order in $a^{-1}$. Hence, we set

$$F^\pm_{0n}(\xi) = \sqrt{n!} \Psi_n(\xi),$$

which will be justified simply by being consistent. The prefactor $\sqrt{n!}$ is for later convenience.

This formulation is still a bit awkward for general $n$. We just report briefly on the expansion for $n = 0$. Redefining the sum $\sum_r F^\pm_{rn}(\xi)/a^r$ as $\exp(\sum_r g^\pm_r(\xi)/a^r)$, one finds that all $g^\pm_r(\xi)$ are polynomials in $\xi$ containing only even powers and being of order $\frac{2}{3}(r + 3 - j)$ with $j = 0, 1$ or 2. The formal criterion necessary to single out this unique sequence of functions turns out to be the exclusion of homogeneous solutions for the $g^\pm_r(\xi)$ of the error function type. This just prevents a behaviour in the wave function that would contradict condition (ii) of Section 4.

Some simplification occurs by translating the above expression (5.1) into the Fock representation in which states are written as $F(a, \mathcal{A})|0\rangle$, and this is the setup we will consider now in more detail. The function (5.3) is just $(\mathcal{A})^n|0\rangle$. All other $F^\pm_{rn}(\xi)$ — which are of the form $G^\pm_{rn}(\mathcal{A})|0\rangle$ — are written as $(\mathcal{A})^n G^\pm_{rn}(\mathcal{A})|0\rangle$, thereby defining a set of functions $G^\pm_{rn}(\mathcal{A})$. The wave functions thus become

$$|\Omega^\pm_n, a\rangle = C^\pm_n a^{q^\pm_n} \exp \left( \pm \frac{1}{2} a^2 \mp E_n a \right) (\mathcal{A})^n \sum_{r=0}^{\infty} \frac{G^\pm_{rn}(\mathcal{A})}{a^r}|0\rangle,$$
with
\[ G_{0n}^\pm(\mathcal{A}^\dagger) \equiv 1. \] (5.5)

According to the structure exhibited so far, we expect only even powers of \( \mathcal{A}^\dagger \) to occur in \( G_{rn}^\pm(\mathcal{A}^\dagger) \). This is in accordance with \( \mathcal{K} \) and \( \mathcal{K}^2 \) from (2.26)–(2.27), as appearing in the Wheeler-DeWitt operator (2.15), being even in \( \mathcal{A} \) and \( \mathcal{A}^\dagger \).

Due to our construction, the combinations \((\mathcal{A}^\dagger)^n G_{rn}^\pm(\mathcal{A}^\dagger)\) can be expected to be analytic at \( \mathcal{A}^\dagger = 0 \). One may however ignore the reasoning of Section 4 and treat (5.4) and (5.5) as an ansatz by its own (leaving the numbers \( q_n^\pm \) and \( E_n \) unspecified as well). This is the strategy we will pursue in what follows. The procedure is again to separate orders of \( a \) and to determine the solutions by some additional requirement. One might impose analyticity of \((\mathcal{A}^\dagger)^n G_{rn}^\pm(\mathcal{A}^\dagger)\), but it turns out that a weaker condition does the job as well. We simply demand that

(iii) \( G_{rn}^\pm(\mathcal{A}^\dagger) \) admits a Laurent series at \( \mathcal{A}^\dagger = 0 \).

In other words, \( G_{rn}^\pm(\mathcal{A}^\dagger) \) may be expanded in integer (positive and negative) powers of \( \mathcal{A}^\dagger \). Logically, this replaces the condition (i) as used in Section 4 (whereas an analogue of condition (ii) is no longer necessary). The technical point of condition (iii) will be to exclude terms of the type \( \ln(\mathcal{A}^\dagger) \). The Wheeler-DeWitt equation in the Fock representation is of fourth order in derivatives with respect to \( \mathcal{A}^\dagger \). Although one might expect that this feature provides an additional complication, things actually become simpler. Writing a particular \(|\Omega_n^\pm, a\rangle\) as \( F(a, \mathcal{A}^\dagger)|0\rangle \), we apply the Wheeler-DeWitt operator and thereafter divide the result by \( F(a, \mathcal{A}^\dagger) \). This allows for a proper separation of orders of \( a \). Proceeding iteratively, one encounters only first order differential equations of a very simple type. Also, the solutions \( G_{rn}^\pm(\mathcal{A}^\dagger) \) turn out to be polynomials in positive and negative powers of \( (\mathcal{A}^\dagger)^2 \), hence are represented in terms of elementary functions. In other words, the Laurent series whose existence is required by condition (iii) are actually finite. This is a great simplification as compared to the procedure of Section 4, where the coefficient functions \( F_r(\phi) \) and \( G_r(\phi) \) emerged as infinite series.

The first non-trivial order \( O(a) \) is of purely algebraic type and yields \( E_n = (n + \frac{1}{2})m \), thus re-introducing the well-known eigenvalues of \( E \). At \( O(a^0) \) we encounter the differential equation

\[ 2m \mathcal{A}^\dagger \frac{d}{d\mathcal{A}^\dagger} G_{1n}^\pm(\mathcal{A}^\dagger) = \mp \frac{3n(n-1)}{2(\mathcal{A}^\dagger)^2} - E_n^2 \mp (1 + p + 2q_n^\pm) \pm \frac{3}{2}(\mathcal{A}^\dagger)^2, \] (5.6)
the general solution of which is

\[ G_{1n}^{\pm}(A^\dagger) = \pm \frac{3n(n-1)}{8m(A^\dagger)^2} + \kappa_1 - \frac{1}{2m} \left( E_n^2 \pm (1 + p + 2q_n^{\pm}) \right) \ln(A^\dagger) \pm \frac{3}{8m}(A^\dagger)^2, \quad (5.7) \]

where \( \kappa_1 \) is an arbitrary integration constant. Condition (iii) implies that the coefficient of the \( \ln(A^\dagger) \) term must vanish. This is an equation for \( q_n^{\pm} \), the solution immediately turning out to be (5.2). Inserting all results obtained so far into the equation at \( O(a^{-1}) \), we obtain a differential equation of the type

\[ A^\dagger g'(A^\dagger) = \rho(A^\dagger), \quad (5.8) \]

where \( g(A^\dagger) \) stands for \( G_{2n}^{\pm}(A^\dagger) \) and \( \rho(A^\dagger) \) for an expression that has already been determined (up to the constant \( \kappa_1 \)). Moreover, \( \rho(A^\dagger) \) contains only integer powers, ranging from \( (A^\dagger)^{-4} \) to \( (A^\dagger)^4 \). The solution is thus a function of equal type, including an additive integration constant \( \kappa_2 \), and a \( \ln(A^\dagger) \) term whose coefficient turns out to be

\[ \frac{1}{8m^3} \left( -9E_n^2 - 4m^2E_n^3 \mp 4m^2E_n \pm 8\kappa_1m^2 \right). \quad (5.9) \]

Again, this term has to vanish. Thus \( \kappa_1 \) is fixed, and the complete solution for \( G_{1n}^{\pm}(A^\dagger) \) is

\[ G_{1n}^{\pm}(A^\dagger) = \pm \frac{3n(n-1)}{8m(A^\dagger)^2} + \frac{E_n}{8m^2} \left( 4m^2 \pm 9 \pm 4m^2E_n^2 \right) \pm \frac{3}{8m}(A^\dagger)^2. \quad (5.10) \]

This pattern persists at all orders. At \( O(a^{-2}) \) the differential equation for \( G_{3n}^{\pm}(A^\dagger) \) is again of the type (5.8), with \( \rho(A^\dagger) \) being a finite sum of even integer orders of \( A^\dagger \), and involving the constant \( \kappa_2 \). The solution for \( G_{3n}^{\pm}(A^\dagger) \) thus consists of a finite sum of even integer orders of \( A^\dagger \), an additive integration constant \( \kappa_3 \) and a \( \ln(A^\dagger) \) term whose coefficient has to vanish, thus fixing \( \kappa_2 \), and so forth. We just display the highest and lowest order of the second coefficient function

\[ G_{2n}^{\pm}(A^\dagger) = \frac{9n(n-1)(n-2)(n-3)}{128m^2(A^\dagger)^4} + \ldots + \frac{9}{128m^2}(A^\dagger)^4 \quad (5.11) \]

and note that in general \( G_{rn}^{\pm}(A^\dagger) \) contains contributions from \( (A^\dagger)^{-2r} \) to \( (A^\dagger)^{2r} \). Moreover, the structure is such that the combination \( G_{rn}^{\pm}(A^\dagger) \equiv (A^\dagger)^n G_{rn}^{\pm}(A^\dagger) \) is a polynomial. This is reflected by the coefficients \( n(n-1) \) and \( n(n-1)(n-2)(n-3) \) in (5.10)–(5.11). Thus, despite the expansion in powers of \( a^{-1} \), the regular nature of
the functional dependence on $A^\dagger$ remains intact. As already mentioned above, such a feature is not at all generic for functions that are regular in two variables, but it serves here as part of the property singling out the wave functions $\Omega^\pm_\nu$.

Since the oscillatory domain is bounded in $a$ (and, moreover, exists only if $n \gg m^{-2}$), there is no analogous expansion there. However, due to the WKB matching procedure as applied in Section 3, we define another set of exact wave functions $\Xi^\pm_\nu$ by (3.12) (or, inversely by (3.13)–(3.14)), where $\Theta_\nu$ is still given by (3.7). In order to achieve the (approximate) oscillatory behaviour as in (3.15), we have to define the normalization constants in (5.1) and (5.4) so as to give $\Omega^\pm_\nu$ the asymptotic form (3.10)–(3.11), hence

$$ C^+_n = \frac{1}{\sqrt{n!}} \left( \frac{2}{E_n} \right)^{-E_n^2/2} e^{E_n^2/4}, \quad C^-_n = \frac{1}{2\sqrt{n!}} \left( \frac{2}{E_n} \right)^{E_n^2/2} e^{-E_n^2/4}. \quad (5.12) $$

This completely determines our wave functions $\Omega^\pm_\nu$ and $\Xi^\pm_\nu$. Due to their structure (the appearance of the oscillator basis $|n\rangle$ at leading order) it is clear that a rather large set of solutions of the Wheeler-DeWitt equation may be expanded into them. By assuming

$$ \psi = \sum_{n=0}^{\infty} (k^+_n \Omega^+_n + k^-_n \Omega^-_n) = \sum_{n=0}^{\infty} (c^+_n \Xi^+_n + c^-_n \Xi^-_n) \quad (5.13) $$

for some solution $\psi$ of the Wheeler-DeWitt equation, the $k^\pm_\nu$ (or $c^\pm_\nu$) may be computed by fixing some (large enough) $a$ and expanding the initial data $\psi$ and $\partial_a \psi$ in terms of the $\Omega^\pm_\nu$ at the fixed value of $a$ (which should be possible if the behaviour in $\phi$ is not too catastrophic). We will find a more convenient method later on, and the present argument just serves for the count of degrees of freedom contained in $\Omega^\pm_\nu$. Thus we treat $\{\Omega^\pm_\nu\}$ (or likewise $\{\Xi^\pm_\nu\}$) as a basis of the space $\mathcal{H}$ of wave functions, and a precise definition of which coefficients in (5.13) are allowed will be given later. The $\Omega^\pm_\nu$ are, by construction, real (complex conjugation $^\ast$ being defined by its action in the position representation), while the transformation of the basis (3.12) implies

$$ (\Xi^\pm_\nu)^* = \Xi^{\mp}_\nu. \quad (5.14) $$

If $n \gg m^{-2}$ and at the level of the WKB approximation at which (3.13) is valid, this property is in accordance with $\Xi^+_\nu$ and $\Xi^-_\nu$ representing an ensemble of collapsing and expanding universes, respectively.

If one accepts the wave functions $\Xi^\pm_\nu$ to play a distinguished role, one ends up with a distinguished decomposition of the space $\mathcal{H}$ of wave functions into the span
$\mathcal{H}^+$ of $\{\Xi^+_n\}$ and the span $\mathcal{H}^-$ of $\{\Xi^-_n\}$, hence $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Wave functions are thus uniquely decomposed as $\psi = \psi^+ + \psi^-$. It is important here to note that the differential structure of minisuperspace is not sufficient for identifying collapsing and expanding modes exactly. This is a major difference to the flat Klein Gordon equation (where a differential background structure, namely a timelike Killing vector field, enables one to define negative/positive frequency modes exactly in a Lorentz-invariant way) and it provides one of the most severe problems in constructing a consistent setup for quantum cosmology. However, if the analyticity structure of wave functions is accepted as a guiding principle in our model, we seem to have uniquely defined a decomposition of the space of wave functions which — in the WKB approximation — is identified with collapsing and expanding (incoming and outgoing) modes.

There is a formal relation between $\Omega^+_n$ and $\Omega^-_n$ that might provide a hint towards a deeper significance of the analytic structure we have exhibited so far. The Wheeler-DeWitt equation remains invariant under the substitution

$$a \rightarrow ia \quad m \rightarrow im,$$

leaving $\phi$ unchanged. Thus the quantities $ma^3$ and $m^{-1}a$ (and by definition $\xi$, which involves $m^{1/2}a^{3/2}$) remain unchanged as well, but we have $a^2 \rightarrow -a^2$, and $m^2 \rightarrow -m^2$. Under this substitution the wave functions $\Omega^+_n$ and $\Omega^-_n$, when written down in the large-$a$ expansion, are almost perfectly transformed into each other. This includes the structure of the exponential prefactors as well as the normalization (5.12) (cf. (3.10)–(3.11) and note that $E^2_n \rightarrow -E^2_n$ while $a/E_n \rightarrow a/E_n$), except for the prefactor $a^{-1/2-p/2}$ and the numerical factor $\frac{1}{2}$ in $\Omega^-_n$. (Also Hawking and Page have noted that this transformation carries their $\Psi^\text{HP}_0(a, \phi)$ — which is our $\Omega^-_0(a, \phi)$ — into the exponentially growing part of the no-boundary wave function as given by Page [6] — which is just our $\Omega^+_0(a, \phi)$). A similar relation exists between the Airy functions $\text{Ai}(x)$ and $\text{Bi}(x)$ in the expansion for large $x$, if $x \equiv a^{4/3}$ is set. One could thus try to define $\mathcal{H}^\pm$ as the "eigenspaces" under a suitable substitution operation. However, this forces us to treat $m$ as a variable rather than a fixed constant. We leave it open whether one would gain any new insight by doing so.

The crucial question in exploiting the emergence of the spaces $\mathcal{H}^\pm$ for further developments of the subject of quantum cosmology is certainly whether the structure showing up here carries over to the full (non-minisuperspace) Wheeler-DeWitt equation.
6 Scalar product and Hilbert spaces

There is a natural (Klein Gordon type) scalar product associated with the Wheeler-DeWitt equation \[1\] [12]. If \(\psi_1\) and \(\psi_2\) are two solutions of the latter that are well-behaved for large \(\phi\), the expression

\[
Q(\psi_1, \psi_2) = -\frac{i}{2} a^p \int_{-\infty}^{\infty} d\phi \left( \psi_1^*(a, \phi) \hat{\partial}_a \psi_2(a, \phi) \right)
\]  

(with \(f \hat{\partial}_a g \equiv f \partial_a g - (\partial_a f) g\)) is independent of \(a\) on account of (2.9). It defines an indefinite scalar product, the integrand being the \(a\)-component of a conserved current \([16]\). Due to its indefiniteness, it does not enable us to define a Hilbert space directly.

When wave functions are expressed in the energy representation as introduced in Section 2, we find

\[
Q(\psi_1, \psi_2) = -\frac{i}{2} a^p \sum_{n=0}^{\infty} \left( f_n^*(a) \hat{\partial}_a g_n(a) + \frac{3}{2a} \sqrt{n(n+1)} \left( f_{n-1}^*(a) g_{n+1}(a) - f_{n+1}^*(a) g_{n-1}(a) \right) \right),
\]  

(6.2)

where \(\hat{\psi}_1(a, \xi) = \sum_n f_n(a) \Psi_n(\xi)\) and \(\hat{\psi}_2(a, \xi) = \sum_n g_n(a) \Psi_n(\xi)\). In the Fock representation the scalar product reads

\[
Q(\psi_1, \psi_2) = -\frac{i}{2} a^p \langle \psi_1, a| \hat{\partial}_a + \frac{3}{a} \mathcal{K} |\psi_2, a\rangle,
\]  

(6.3)

where the derivative \(\hat{\partial}_a\), when acting to the left, does not include the prefactor \(a^p\).

Inserting the asymptotic expressions of our wave functions in the oscillatory domain, we find

\[
Q(\Omega^\pm_r, \Omega^\pm_s) = 0 \quad Q(\Omega^+_r, \Omega^-_s) = \delta_{rs}
\]  

(6.4)

and

\[
Q(\Xi^\pm_r, \Xi^\pm_s) = \pm \delta_{rs} \quad Q(\Xi^+_r, \Xi^-_s) = 0.
\]  

(6.5)

Using the asymptotic series for \(a \to \infty\), we find hints that these relations hold exactly for all \(r\) and \(s\). For all those combinations in which the exponential \(a\)-dependent prefactor decreases, they are evident (\(Q\) being evaluated at arbitrarily large \(a\)).
For all other cases, we have used the first few terms of the series to check them. In the following we will assume that they hold. They also explain the particular normalization we have chosen (see (3.13) and (5.12)).

Let us consider now the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ of the space of solutions of the Wheeler-DeWitt equation as induced by the wave functions $\Xi^\pm_n$. If any solution $\psi$ is expanded as in (5.13), the above normalization of the basis yields

$$c^\pm_n = \pm Q(\Xi^\pm_n, \psi).$$

(6.6)

Encoding the information contained in a wave function in terms of the numbers $c^\pm_n$, we can talk about an ”energy representation” in a sense quite more sophisticated than the notation of Section 2. Since the basis elements are solutions of the Wheeler-DeWitt equation, there is no dependence on additional variables, except for the ”true” physical labels $n$ and $\pm$.

Evidently, the scalar product $Q$ is positive/negative definite on $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively. Hence, admitting only wave functions $\psi$ whose coefficients satisfy

$$\sum_{n=0}^{\infty} (|c^+_n|^2 + |c^-_n|^2) < \infty$$

(6.7)

makes $(\mathcal{H}^+, Q)$ and $(\mathcal{H}^-, -Q)$ two Hilbert spaces that may be used as a starting point for descriptions how to compute probabilities for observations. Admitting more general wave functions, one should still be able to compute relative probabilities. Given a solution $\psi$ as in (5.13), the (relative) probabilities associated with the states $\Xi^\pm_n$ are $|Q(\Xi^\pm_n, \psi)|^2$. These numbers should be relevant when predictions for the energy contents are drawn. In the WKB-philosophy — which is relevant for quantitative predictions — this is quite clear for any basis being normalized as (6.6). (Any such basis gives rise to a decomposition into two Hilbert spaces). In Ref. [12] we have suggested a ”minimal” interpretational scheme for quantum cosmology based on $(\mathcal{H}, Q)$ as the only fundamental mathematically well-defined structure. However, here we suggest a different thing. If one accepts the basis wave functions $\Xi^\pm_n$ as distinguished objects (not just as wave functions which describe a semiclassical ensemble of universes with approximately conserved matter energy), one is faced with a distinguished Hilbert space structure, based on asymptotic analyticity properties. One might think of it to be introduced by some path-integral or to constitute a first principle by its own, and it constitutes an exact fundamental structure in addition to $(\mathcal{H}, Q)$. If an analogous feature may be found in the full
(non-minisuperspace) case as well, it should be of some relevance to the fundamental conceptual problems of quantum cosmology.

7 No-boundary wave function

In order to show an example, we estimate the coefficients of the expansion \( \psi_{NB} \) for the no-boundary wave function \([15]\) \( \psi_{NB} \) of the Hawking model \([2]\). It has been studied in great detail by Page \([6]\). The classical no-boundary trajectories start from a point \((a_0, \phi_0)\) on the zero potential line \( m^2 a^2 \phi^2 = 1 \) and are in the inflationary domain \( ma|\phi| \gg 1, |\phi| \gg 1 \) given by

\[
a = \frac{1}{2m\phi_0^{2/3}\phi^{1/3}} \exp \left( \frac{3}{2}(\phi_0^2 - \phi^2) \right) \approx \exp \left( \frac{3}{2}(\phi_0^2 - \phi^2) \right). \tag{7.1}
\]

This expression is valid as long as \( 1 \ll |\phi| \ll |\phi_0 - 1/(3\phi_0)| \). Since there is only one no-boundary trajectory hitting any point \((a, \phi)\) in the inflationary domain, we associate with this point the initial value \( \phi_0 \) of the latter, thus turning \( \phi_0 \) into a function of \( a \) and \( \phi \). The action of this congruence of trajectories is

\[
S = -\frac{1}{3m^2\phi^2} \left( m^2 a^2 \phi^2 - 1 \right)^{3/2} + \frac{\pi}{4} \approx -\frac{1}{3} m|\phi| a^3, \tag{7.2}
\]

the no-boundary wave function in the inflationary domain being (thereby generalizing Page’s expression to arbitrary \( p \))

\[
\psi_{NB}(a, \phi) \approx a^{-p/2-1} A(\phi_0) \cos(S(a, \phi)) \tag{7.3}
\]

with

\[
A(\phi_0) = \frac{1}{\sqrt{\pi m|\phi_0|}} \left( \sqrt{6} + 2 \left( \exp \left( \frac{1}{3m^2\phi_0^2} \right) - 1 \right) \right) \tag{7.4}
\]

and \( \phi_0 \) now interpreted as a function of \( a \) and \( \phi \). Its basic structure is that it is a product of the rapidly oscillating WKB-type function \( \cos(S) \) with a slowly varying prefactor. (The prefactor \( A(\phi_0) \) is in fact to some extent arbitrary. In the WKB-approximation the Wheeler-DeWitt implies that it is constant along the classical trajectories. The expression \( (7.4) \) corresponds to the solution arising from the no-boundary proposal for the Euclidean path-integral).
The no-boundary trajectory with initial value $\phi_0$ leaves the inflationary domain (i.e. attains $|\phi| \approx 1$) at $a_{\text{min}} \approx m^{-1}|\phi_0|^{-2/3}e^{3\phi_0^2/2}$, and subsequently undergoes the matter dominated era in which $\phi$ oscillates and the matter energy $E$ is approximately conserved. Since, on the other hand, $a_{\text{min}} \approx (n/m)^{1/3}$, we find $\phi_0^2 \approx 2/9 \ln(nm^2)$, which makes $\phi_0$ in (7.3) effectively a function of $n$. Since $|\phi_0| \gg 1$, we have $n \gg m^{-2}$, and thus a non-trivial classical domain. The prefactor (7.4) then becomes

$$A_n \approx \sqrt{\frac{3}{\pi m}} \left( \frac{1}{\sqrt{2 \ln(nm^2)}} \right) \left( \sqrt{6 + 2 \left( \exp\left( \frac{3}{2m^2 \ln(nm^2)} \right) - 1 \right)} \right)$$

(7.5)
as far as the contribution of trajectories representing universes with matter energy $E_n$ is concerned.

We will choose an indirect way to estimate the magnitude of the coefficients $c^+_n$ when $\psi_{\text{NB}}$ is expanded into $\Xi^\pm_n$ as in (5.13). Since we do not know precisely the behaviour of $\psi_{\text{NB}}(a, \phi)$ in just those regions in which we know $\Xi^\pm_n(a, \phi)$, the information necessary to perform the integration (6.6) is not easily accessible. We may instead first evaluate the relative probability distribution for trajectories labelled by $\phi_0$ in the position representation. Since $Q(\psi, \psi) = 0$ for real $\psi$, we consider the incoming/outgoing projections $\psi_{\text{NB}}^+ \approx \frac{1}{2} a^{-\nu/2-1} A e^{\pm iS}$. The expressions $Q(\psi_{\text{NB}}^+, \psi_{\text{NB}}^+)$, when computed according to (6.1), turn out to be integrals over the measure $\pm \frac{1}{4} m A^2 |\phi| d\phi$ (the additional $a$-dependence cancelling, as it should). Using $|\phi| d\phi \approx |\phi_0| d\phi_0$ (at constant $a$), this becomes $\pm P(\phi_0) d\phi_0$ ($\pm$ the relative probability for finding the universe represented by a trajectory in the interval between $\phi_0$ and $\phi_0 + d\phi_0$), where

$$P(\phi_0) = \frac{m}{4} |\phi_0| A^2(\phi_0).$$

(7.6)

This is the standard procedure of evaluating probabilities for WKB-type wave functions based on the conserved current. (An alternative interpretation [9] predicts probabilities that differ from these by the amount of proper time spent by trajectories in a domain of minisuperspace). By including an additional factor 2, we may restrict $\phi_0$ to be positive. Since in this case $\phi_0$ and $n$ are related uniquely by $\phi_0 \approx 1/2 \sqrt{2 \ln(nm^2)}$, we may compute $d\phi_0/dn \approx (9n\phi_0)^{-1}$. Due to the symmetry $\psi_{\text{NB}}(a, -\phi) = \psi_{\text{NB}}(a, \phi)$ we know that $c^+_n = 0$ for odd $n$. Supposing that $|c^+_n|$ for even $n$ may be approximated by continuous functions, we set $dn = 2$ and find that the relative probability for the universe to have energy quantum number $n$ is given by $P_n \approx 4 (9n\phi_0)^{-1} P(\phi_0)$. On the other hand, the expressions $Q(\psi^+_{\text{NB}}, \psi^+_{\text{NB}})$
are given by $\pm \sum_n |c_n|^\pm$, and the relative probability to find the universe containing matter energy $E_n$ in the contracting/expanding mode is $P_n^\pm = \frac{1}{2} P_n = |c_n^\pm|^2$ for even $n$ (and zero for odd $n$). We thus identify for even $n$ and $n \gg m^{-2}$

$$c_n^\pm = (c_n^-)^* \approx \frac{1}{3} \sqrt{\frac{2 P(\phi_0)}{n \phi_0}} K_n \approx \frac{1}{3} \sqrt{\frac{m}{2n}} A_n K_n \approx \frac{1}{\sqrt{6 \pi n}} \frac{1}{\sqrt{2 \ln(nm^2)}} \left( \sqrt{6} + 2 \left( \exp\left(\frac{3}{2m^2 \ln(nm^2)}\right) - 1 \right) \right) K_n \quad (7.7)$$

where $|K_n| = 1$. The first equality is exact, it stems from the fact that $\psi_{NB}$ is real. (Note that due to the reality of $\psi_{NB}$, both $\pm$ modes are of equal probability, although this number is infinite).

The density $P(\phi_0)$ as well as the probabilities $P_n$ display the well-known problems for the no-boundary wave function to predict sufficient inflation (i.e. a sufficiently large universe; see Refs. [17][18]). Due to the smallness of $m$, the exponentials prefer small $\phi_0$ and $n$, and it is only the flat behaviour of $P(\phi_0)$ for $\phi_0 \to \infty$ (or the dominant behaviour $P_n \sim n^{-1}$ as $n \to \infty$) that allow for large universes. However, trajectories with $m|\phi_0| \gtrsim 1$ correspond to classical universes above Planckian densities after nucleation, and it is not clear whether they should contribute (cf. Ref. [1]). These trajectories correspond to $n \gtrsim m^{-2} \exp\left(\frac{9}{2m^2}\right) \approx \exp(5 \times 10^{12})$ (and thus $a_{max}$ being given roughly by the same number, or, expressed in “true” units as displayed in the Introduction, $\tilde{a}_{max} \approx \exp(5 \times 10^{12}$) centimeters, or light years, or present Hubble scales, which makes no big difference due to the huge value of this number).

As a consequence of (7.7), we find in the energy representation $|\psi_{NB}, a\rangle = \sum_n f_n(a)|n\rangle$ that the oscillator components (for even $n \gg m^{-2}$ and in the range $a_{min} \approx (n/m)^{1/3} \ll a \ll a_{max} \approx 2mn$) are given by

$$f_n(a) \approx \frac{2}{3} a^{-n/2-1/4} \sqrt{\frac{m}{2n}} \frac{A_n}{\sqrt{2E_n}} \cos \left( \frac{2}{3} (2E_n)^{1/2} a^{3/2} + \delta_n \right). \quad (7.8)$$

Here, we have set $K_n = e^{i\delta_n}$. Due to the large value of $n$ one may of course set $E_n \approx mn$. The components for odd $n$ vanish. For $a \gtrsim a_{max}$ the $f_n(a)$ contain the exponential $e^{+a^2/2+E_n a}$ terms of (3.10)–(3.11). An analogous large-$a$ behaviour is expected to apply for the component functions with $n \gtrsim m^{-2}$ (which do not contribute to classical universes) as well.
The expression (7.8) may heuristically be checked by using the real part of (2.32) with \( k = \frac{1}{2} m^{1/2} a^{3/2} \) as an approximation for \( \cos(S) \). Invoking \( \Psi_n(x) \approx \Psi_n(0) \cos(x \sqrt{2n}) \) for small \( x \) and even \( n \), and \( \Psi_n(0) \approx (-)^{n/2}(2/n)^{1/4}\pi^{-1/2} \) for large even \( n \) (which follows from Stirling’s formula for \( n! \)), a behaviour roughly similar to (7.8) is recovered but without the phases \( \delta_n \) and an additional factor \( \frac{1}{2} \) in the Cosine. This reflects our lack of knowledge about the details of \( \psi_{NB}(a, \phi) \) in the domain where \( \Xi_n^\pm(a, \phi) \) is known, and \textit{vice versa}. It would be interesting to study this problem in more detail, and to find an estimate for the \( \delta_n \). In case these numbers vary rapidly with \( n \), the Cosine in (7.8) would introduce a chaotic type behaviour of the oscillator excitations.

In their work on wormholes Hawking and Page [3] have assumed that the no-boundary wave function (which increases as \( e^{a^2/2} \)) can be expanded in terms of their wave functions \( \Psi_{HP}^n(a, \phi) \), which are our \( \Omega_n^- \) and decrease as \( e^{-a^2/2} \). In our language, they have assumed \{\( \Omega_n^- \)\} to form a basis, in which case one would expect the expansion coefficients showing a tremendous increase with \( n \). In contrast, in our framework, these states are only half of a basis. Note that linear independence relies on the precise definition of a vector space. (For example, Sine and Cosine functions can or cannot be expanded into each other, depending on the interval on which they are considered). However, by just counting degrees of freedom, the existence of a framework in which \( \psi_{NB} \) may be expanded in terms of \{\( \Omega_n^- \)\} is very unlikely. (This would in fact imply that \( \Omega_n^+ \) may be expanded in terms of \( \Omega_n^- \), and the structure provided by the scalar product \( Q \) would break down). The explicit computation of the expansion coefficients \( k_n^\pm \) with respect to \( \Omega_n^\pm \) as defined by (5.13) seems to be difficult at the present status of our knowledge. For \( c_n^+ = (c_n^-)^* \), we find \( k_n^+ + ik_n^- = 2c_n^* e^{i\Theta_n} \) with \( \Theta_n \) from (3.7). Our estimate (7.7), together with \( K_n = e^{i\delta_n} \), leads to a real expression multiplied by \( e^{i(\Theta_n+\delta_n)} \). Since we do not know \( \delta_n \), we cannot compute the real and imaginary part of this phase factor directly. On the other hand, an estimate for \( \delta_n \) might be accessible by using further information about \( \psi_{NB} \). The mere fact that it contains an increasing \( e^{a^2/2} \) contribution implies that at least some \( \Theta_n + \delta_n \) are different from \( \frac{\pi}{2} \) (modulo 2\( \pi \)).

There is another interesting question related with the work of Hawking and Page. They assume that the regular superpositions of \{\( \Omega_n^- \)\} — due to their exponentially damped nature — provide quantum wormhole states. On the other hand, the second half of the basis \{\( \Omega_n^+ \)\} appears on quite an equal footing here: both types of states describe universes with matter energy \( E_n \). The only possible essential difference concerns observations when the universe is near its classical turning point
$a \approx a_{\text{max}}$. It is in particular the states $\Omega_n^-$ that do not seem to provide problems there. How does this fact relate to the interpretation of certain superpositions of these states as wormholes? Usually, the exponentially damped behaviour $e^{-a^2/2}$ is associated with wormhole states by definition. However, we do not have a satisfactory interpretation of the exponentially increasing behaviour $e^{a^2/2}$ (nor is it required by definition for any wave function, but just accepted as a grain of salt rather than a desired property when it emerges). This is a certain asymmetry (at least as long as no path-integral arguments are invoked) that might point towards a theoretical lack in our understanding of quantum cosmology. We are not able to give an answer to this question, but it seems worth pursuing it.

The predictive power contained in the coefficients (7.7) is — as far as practical quantitative features are concerned — equal to the results of the common WKB-philosophy. If, however, the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (with $\mathcal{H}$ being defined by the normalization condition (6.7)) is regarded as a distinguished one, we may embed $\psi_{\text{NB}}$ into a mathematically well-defined underlying structure. We have two fundamental Hilbert spaces ($\mathcal{H}^\pm, \pm Q$) and a wave function for which $Q(\Xi, \psi_{\text{NB}})$ is finite for any element $\Xi \in \mathcal{H}$, and it is just these numbers in which all physical information about $\psi_{\text{NB}}$ is encoded. The wave function $\psi_{\text{NB}}$ is not an element of $\mathcal{H}$.

Since both projections $\psi_{\text{NB}}^\pm$ onto $\mathcal{H}^\pm$ have infinite norm $Q(\psi_{\text{NB}}^\pm, \psi_{\text{NB}}^\pm)$, they are of distributional character (similar to momentum eigenstates in conventional quantum mechanics).

8 Concluding remarks

The behaviour of the real wave functions $\Omega_n^\pm(a, \phi)$ for large $a$ provides a relation between the expansion (5.13) of some $\psi$ into these (the coefficients $k_n^\pm$ being trivially connected to $c_n^\pm$ by (3.12)–(3.14)) and the question of boundedness of $\psi$. Due to the factors $e^{a^2/2}$, all $\Omega_n^\pm(a, \phi)$ are unbounded. Hence, a given wave function $\psi(a, \phi)$ seems to be bounded away from $a = 0$ (i.e. $|\psi(a, \phi)| < K < \infty$ in any domain $a > a_1$) if and only if $k_n^+ = 0$ for all $n$. (Note that this is in contradiction with the remark in Ref. [3] concerning the expectation that $\psi_{\text{NB}}$, which is not bounded, may be expanded in terms of $\Psi_n^{\text{HP}} \equiv \Omega_n^-$). It is not entirely clear to what extent the unboundedness of a wave function causes interpretational problems (e.g. near the classical turning point). Some authors consider boundedness as a condition necessary for interpretation [19, 7] and talk about a "final condition" for the wave
function. Such approaches could provide an additional justification for expecting the limit \( a \to \infty \) to play a conceptually fundamental role. At the technical level we do not know whether the degrees of freedom contained in the coefficients \( k_n \) may be arranged so as to cancel the expected singular behaviour of \( \Omega_n(a, \phi) \) for small \( a \) and make up a strictly bounded solution (although Hawking and Page, when constructing approximate wormhole states, provide a hint that this is possible). In the case of the tunnelling wave function emerging from the outgoing mode proposal \([20] [21]\), the boundedness of \( \psi_T \) is usually considered part of the definition. If such functions exist at all, this would immediately imply that \( \psi_T \) is a superposition of the \( \Omega_n \) alone. It might be worth thinking about whether a possible relation between \( \psi_T \) and the wormhole context in which Hawking and Page considered the wave functions \( \Psi_{HP} \equiv \Omega_n \) sheds some new light on the conceptual questions.

Let us close this article by adding some general speculations. If the structure encountered in the Hawking model carries over to some more sophisticated (preferably non-minisuperspace) model one would apply WKB-techniques in combination with decoherence arguments (traces in the Hilbert spaces \( \mathcal{H}^\pm \)) in order to identify states with physical observables and to recover the standard laws of physics. The \( Q \)-product should boil down to plus or minus the standard scalar product of quantum mechanics \([1] [12]\). The existence of two separate Hilbert spaces may be a hint that the \( \pm \) sectors decouple from any observational point of view: given a wave function, one is either in the + or in the − sector, no experience of a superposition is possible. One could call this a super-selection rule. Only within these sectors Hilbert space techniques apply, and the actual non-experience of various other superpositions is delegated to decoherence. The ultimate object to describe experience would be a reduced density matrix, as evaluated by standard Hilbert space methods, hence within a completely well-defined framework. It remains to be seen whether such an interpretation is still possible when observations near the turning point are concerned. There, one would heuristically expect to undergo a “transition” from a reduced density matrix belonging to the − sector to one belonging to the + sector.

Since the situation is a bit reminiscent of the one-particle Hilbert spaces of negative/positive frequency modes as emerging from the flat Klein Gordon equation, a further possible direction to pursue is to envisage a third-quantization \([22] [23]\) in terms of the preferred decomposition (cf. Ref. \([12]\)).
References

[1] B. DeWitt, "Quantum Theory of Gravity. I. The Canonical Theory", Phys. Rev. 160, 1113 (1967).

[2] S. W. Hawking, "The quantum state of the universe", Nucl. Phys. B 239, 257 (1984).

[3] S. W. Hawking and D. N. Page, "Spectrum of wormholes", Phys. Rev. D 42, 2655 (1990).

[4] J. J. Halliwell and S. W. Hawking, "Origin of structure in the universe", Phys. Rev. D 31, 1777 (1985).

[5] A. Linde, "Inflation and quantum cosmology", Physica Scripta T 36, 30 (1991).

[6] D. N. Page, "Hawking’s wave function of the universe", in: R. Penrose and C. J. Isham (eds.), Quantum concepts in space and time, Clarendon Press (Oxford, 1986), p. 274.

[7] C. Kiefer, "Wave packets in minisuperspace", Phys. Rev. D 38, 1761 (1988).

[8] J. J. Halliwell, "Introductory lectures on quantum cosmology", in: S. Coleman et. al. (eds.), Quantum cosmology and baby universes, World Scientific (Singapore, 1991), p. 159.

[9] S. W. Hawking and D. N. Page, "Operator ordering and the flatness of the universe", Nucl. Phys. B 264, 185 (1986).

[10] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, 1980, p. 1067.

[11] I. Bialynicki-Birula, M. Cieplak and J. Kaminski, Theory of Quanta, Oxford University Press, 1992, p. 286.

[12] F. Embacher, "On the interpretation of quantum cosmology", preprint UWThPh-1996-22, also gr-qc/9605019.

[13] C. Kiefer, "The semiclassical approximation to quantum gravity", in: J. Ehlers, H. Friedrich (eds.), Canonical Gravity: From Classical to Quantum, Springer (Berlin, 1994), p. 170.
[14] S. W. Hawking, ”The path-integral approach to quantum gravity”, in: S. W. Hawking and W. Israel (eds.), General relativity: An Einstein Centenary Survey, Cambridge University Press (Cambridge, 1979), p. 746.

[15] J. B. Hartle and S. W. Hawking, ”Wave function of the universe”, Phys. Rev. D 28, 2960 (1983).

[16] A. Vilenkin, ”Interpretation of the wave function of the universe”, Phys. Rev. D 39, 1116 (1989).

[17] L. P. Grishchuk and L. V. Rozhansky, ”On the beginning and the end of classical evolution in quantum cosmology”, Phys. Lett B 208, 369 (1988); ”Does the Hartle-Hawking wavefunction predict the universe we live in?”, Phys. Lett B 234, 9 (1990).

[18] A. Lukas, ”The no-Boundary wave function and the duration of the inflationary period”, Phys. Lett B 347, 13 (1995).

[19] H. D. Zeh, ”Time in quantum gravity”, Phys. Lett. A 126, 311 (1988).

[20] A. D. Linde, ”Quantum creation of the inflationary universe”, Lett. Nuovo Cimento 39, 401 (1984).

[21] ”Boundary conditions in quantum cosmology”, Phys. Rev. D 33, 3560 (1986).

[22] A. Hosoya and M. Morikawa, ”Quantum field theory of the universe”, Phys. Rev. D 39, 1123 (1989).

[23] M. McGuigan, ”Universe creation from the third-quantized vacuum”, Phys. Rev. D 39, 2229 (1989).