The van der Waals-like Phase Transition in the FRW Universe

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In this paper, we treat the Friedmann-Robertson-Walker (FRW) universe as a thermodynamic system and study its thermodynamic properties especially the van der Waals (vdW)-like (or P-V) phase transition. The thermodynamic pressure \( P \) is, almost for the first time, defined by the work density \( W \) introduced by Hayward. We further take the Einstein gravity and the 4D Einstein-Gauss-Bonnet (4D EGB) gravity as two examples to write down the equations of state \( P = P(V, T) \). We find that in Einstein gravity with a perfect fluid, there is no vdW-like phase transition for the FRW universe, while in the 4D EGB gravity with a perfect fluid the FRW universe can have vdW-like phase transition, which to our best knowledge was never found previously in the literature of the FRW thermodynamics and is the first example beyond the asymptotically AdS black holes. We also find that the corresponding critical exponents are the same as those in the vdW system, so they satisfy the scaling laws.

I. INTRODUCTION

The Friedmann-Robertson-Walker (FRW) universe is a thermodynamic system similar to a dynamical black hole. In theory, a dynamical black hole has many well-known properties, e.g. the existence of an apparent horizon, the Hawking temperature and Hawking radiation \([1]\), the Bekenstein-Hawking entropy, the quasi-local energy such as the Misner-Sharp energy, the unified first law \([2]\) of thermodynamics, etc. These mentioned properties are all shared by the FRW universe \([3–11]\).

On the other hand, some other properties of black holes have not yet been known to exist for the FRW universe. Take the asymptotically AdS black hole as an example, in both Einstein gravity and modified gravity, it usually has a thermodynamic equation of state \( P = P(V, T) \), where \( V \) is the thermodynamic volume, \( T \) is the Hawking temperature, and \( P \) is the thermodynamic pressure that is usually defined by \( P := -\Lambda/(8\pi) \) \([12–16]\). In some situations such an equation is characterized the van der Waals (vdW)-like of phase transition (or P-V phase transition \([17–22]\)) with a critical point in the P-V phase diagram. However, for the FRW universe, so far there has been rarely any investigations on the equation of state, vdW-like phase transitions and criticality. In our recent paper \([23]\), using the above-mentioned definition of the thermodynamic pressure \( P \), we derived the equation of state for the FRW universe with a perfect fluid in Einstein gravity, and no vdW-like phase transition was found.

Out of curiosity, in this paper we would like to find a reasonable way to construct a vdW-like equation of state that can describe phase transitions of the FRW universe. The construction of such an equation of state depends on: (i) the definition of the thermodynamic quantities \((P, V, T)\); (ii) the choice of the gravitational theory; and (iii) the properties of the matter field or source. In this paper, we treat the matter field as a perfect fluid and follow the definitions of the thermodynamic volume \( V \) and the Hawking temperature \( T \) in \([7, 8]\). We will mainly focus on changing the definition of \( P \) and using modified gravity. The thermodynamic pressure \( P \) is the conjugate variable of the thermodynamic volume, and from the first law of thermodynamics for the FRW universe, one will find that it is exactly the work density \( W \) of the matter field

\[
P \equiv W = -\frac{1}{2} h_{ab} T^{ab}, \tag{1.1}
\]

where \( h_{ab} \) and \( T^{ab} \) are the 0,1-components of the metric and the stress-tensor \([8]\) with \( a, b = 0, 1, (x^0 = t, x^1 = r) \); and our discussion will be mainly based on Einstein gravity and 4D Einstein-Gauss-Bonnet (4D EGB) gravity.

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1 Note that investigations have shown that the existence of the apparent horizon is the main cause of having a self-consistent thermodynamics \([2, 3, 9, 11]\). Although the FRW thermodynamics was historically inspired by that of dynamical black holes, the self-consistency of the former in principle does not rely on the latter.
This paper is organized as follows. In Sec.II, we will introduce the new definition of the thermodynamic pressure $P$ and demonstrate briefly that it does not give any vdW-like properties in the framework of Einstein gravity with a perfect fluid. In Sec.III, we study the thermodynamics of the FRW universe in the 4D EGB gravity with a perfect fluid, we will obtain its equation of state and investigate its vdW-like phase transition and criticality. Sec.IV is for conclusions and discussions.

II. THE THERMODYNAMIC PRESSURE AND THE EQUATION OF STATE OF THE FRW UNIVERSE IN EINSTEIN GRAVITY

In this section, in the framework of Einstein gravity, we will introduce the FRW universe with a perfect fluid and the definition of its thermodynamic pressure. Then we will obtain the first law of thermodynamics and the equation of state, which shows no vdW-like phase transition.

In this paper, we investigate the thermodynamics of a homogeneous and isotropic universe, so we use the FRW metric as an ansatz in the following. In the co-moving coordinate system $\{t, r, \theta, \varphi\}$, the line element of the FRW universe can be written as

$$\text{d}s^2 = -\text{d}t^2 + a^2(t) \left[ \frac{\text{d}r^2}{1 - kr^2} + r^2(\text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2) \right], \quad (2.1)$$

where $a(t)$ is the time-dependent scale factor. The matter field is treated as the perfect fluid with stress-tensor

$$T_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu + p_m g_{\mu\nu}, \quad (2.2)$$

where $\rho_m$ is the energy density, $p_m$ is the pressure and $u_\mu$ is the 4-velocity of the perfect fluid. Starting from Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \quad (2.3)$$

the Friedmann’s equations can be derived

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3} \rho_m, \quad \dot{H} - \frac{k}{a^2} = -4\pi(\rho_m + p_m). \quad (2.4)$$

For later convenience, we rewrite the FRW metric in the following form

$$\text{d}s^2 = h_{ab}\text{d}x^a\text{d}x^b + R^2(\text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2), \quad (2.5)$$

where $a, b = 0, 1$ with $x^0 = t, x^1 = r$ and $R \equiv a(t)r$ is the areal radius of the FRW universe. In this new form, one can easily obtain the apparent horizon of the FRW universe by solving $h_{ab}\partial_a R \partial_b R = 0$ [9], which gives

$$R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \quad (2.6)$$

whose time derivative is

$$\dot{R}_A = -HR_A^3 \left( \dot{H} - \frac{k}{a^2} \right). \quad (2.7)$$

From (2.4), (2.6) and (2.7), one can express the energy density and the pressure in terms of the apparent horizon radius and its time derivative

$$\rho_m = \frac{3}{8\pi R_A^2}, \quad (2.8)$$

$$p_m = \frac{\dot{R}_A}{4\pi HR_A^3} - \frac{3}{8\pi R_A^2}. \quad (2.9)$$

Then the work density of the matter field defined by Hayward [2] can be written as

$$W := -\frac{1}{2}h_{ab}T^{ab} = \frac{1}{2}(\rho_m - p_m) = \frac{3}{8\pi R_A^2} - \frac{\dot{R}_A}{8\pi HR_A^3}, \quad (2.10)$$
which will be used in this paper to define the thermodynamic pressure.

The surface gravity at the apparent horizon can be calculated from [9]

\[ \kappa \equiv \frac{1}{2\sqrt{-h}} \nabla_a (\sqrt{-h} h^{ab} \nabla_b R) \bigg|_{R=R_A}, \]

(2.11)

which gives

\[ \kappa = -\frac{1}{R_A} \left( 1 - \frac{\dot{R}_A}{2H R_A} \right). \]

(2.12)

We treat \( \dot{R}_A \) as a small quantity, so that the surface gravity \( \kappa \) is negative, i.e. the apparent horizon of the FRW universe is an inner trapping horizon [2]. Then the Hawking temperature associated with the apparent horizon should be

\[ T \equiv \frac{|\kappa|}{2\pi} = \frac{1}{2\pi R_A} \left( 1 - \frac{\dot{R}_A}{2H R_A} \right). \]

(2.13)

Furthermore, in Einstein gravity we have the Bekenstein-Hawking entropy

\[ S \equiv \frac{A}{4} = \pi R_A^2, \]

(2.14)

and the Misner-Sharp energy [24, 25]

\[ M \equiv \frac{R_A}{2} \]

(2.15)

of the FRW universe, both of which are very similar to those of a Schwarzschild black hole, and we define the thermodynamic volume to be

\[ V \equiv \frac{4}{3} \pi R_A^3. \]

(2.16)

With the above quantities defined, the following relation can be easily checked:

\[ dM = -TdS + WdV. \]

(2.17)

Compared with the standard form of the first law of thermodynamics

\[ dU = TdS - PdV, \]

(2.18)

one can see that the work density \( W \) and the Misner-Sharp energy \( M \) should be interpreted as the thermodynamic pressure \( P \) and the minus of the internal energy \( -U \), i.e.

\[ P := W, \]

\[ U := -M. \]

(2.19, 2.20)

From (2.9) and (2.13) by eliminating \( \dot{R}_A \) we get

\[ p_m = -\frac{T}{R_A} + \frac{1}{8\pi R_A^2}. \]

(2.21)

From (2.8) and (2.21), we further obtain the equation of state for the FRW universe in Einstein gravity

\[ P = \frac{T}{2R_A} + \frac{1}{8\pi R_A^2}, \]

(2.22)

or

\[ \left( P - \frac{1}{8\pi R_A^2} \right) R_A = \frac{T}{2}. \]

(2.23)
which is to some extent similar to the van der Waals equation. It is then natural to ask whether this system has a vDW-like phase transition, whose necessary condition is that the equation

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0,$$

or equivalently

$$\left(\frac{\partial P}{\partial R_A}\right)_T = \left(\frac{\partial^2 P}{\partial R_A^2}\right)_T = 0,$$

has a critical-point solution $T = T_c$, $P = P_c$, $R_A = R_A$. By substituting (2.22) into (2.25), one can easily check that no such solution exists, and thus there is no vDW-like phase transition in the FRW universe of Einstein gravity with a perfect fluid.

III. THE VDW-LIKE PHASE TRANSITION OF THE FRW UNIVERSE IN THE 4D EINSTEIN-GAUSS-BONNET GRAVITY

We would like to study whether vDW-like phase transition may exist for the FRW universe, if we use modified gravity instead of Einstein gravity. In particular, in this section we will investigate this question in the 4D EGB gravity.

A. A Brief Introduction of the 4D Einstein-Gauss-Bonnet Gravity

The action of the traditional Einstein-Gauss-Bonnet gravity is written as

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g} (R + \alpha \mathcal{G}) + S_m,$$

where $\mathcal{G}$ is the Gauss-Bonnet term

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}.$$

High order curvature terms such as the Gauss-Bonnet term also appear in the effective low-energy action of heterotic string theory, where it can be viewed as the corrections of the large $N$ expansions of the boundary CFTs in the strong coupling limit according to AdS/CFT correspondence. Note that the Gauss-Bonnet term usually does not have dynamical effects and does not contribute to the field equations in the 4D spacetime, because it is a topological term (or Euler characteristic class) that just modifies the boundary dynamics, which is associated with holographic counterterms.

In recent years, a 4D Einstein-Gauss-Bonnet gravity, which is often called the novel 4D EGB gravity, has drawn much attention. The action in the novel 4D EGB gravity can be written as

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left(R + \frac{\alpha}{D-4} \mathcal{G}\right) + S_m,$$

where the coupling constant $\alpha$ in (3.1) is replaced by $\alpha/(D - 4)$, and it is actually the dimensional regularization method especially the so-called minisuperspace regularization. The 4D case is defined as the $D \to 4$ limit, and in this way the Gauss-Bonnet term can impose dynamical effects on the equation of motion. In the novel 4D EGB gravity, the Gauss-Bonnet term has nontrivial contributions and shows many interesting properties, for example, the novel 4D EGB gravity can help to explain inflation, the current cosmic acceleration and even alleviate the Hubble tension. The novel 4D EGB gravity admits static spherically symmetric spacetimes such as AdS black holes, which has sparked extensive investigations.

However, the novel 4D EGB gravity has been criticized to be an ill-defined theory at the action level, since it can result in divergence in the equations of motion and break diffeomorphism in a general 4D spacetime. One can see this problem from the field equation derived from the action (3.3) 

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{\alpha}{D-4}H_{\mu\nu} = 8\pi T_{\mu\nu},$$

where $H_{\mu\nu} = \partial_{\mu} \mathcal{G}/\partial R_A - \partial_{\nu} \mathcal{G}/\partial R_A$. 

While the novel 4D EGB gravity has its own merits, it is not entirely free of criticism. Nevertheless, it has shown great potential in the study of gravitational theories and their applications in cosmology and string theory.
where
\[ H_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R_{GB} - 2RR_{\mu\nu} + 4R_{\mu\gamma}R^\gamma_{\nu} + 4R^\delta_{\mu\gamma\delta\lambda}R^\gamma_{\nu\delta\lambda} - 2R_{\mu\gamma\delta\lambda}R^\gamma_{\nu\delta\lambda}. \] (3.5)

For a general spacetime, the Gauss-Bonnet part in the above field equation (3.4) can be written as [49, 53, 54]
\[ \frac{-\alpha}{D-4}H_{\mu\nu} = \frac{\alpha}{D-4}((D-4)A_{\mu\nu} + W_{\mu\nu}) = \alpha A_{\mu\nu} + \frac{\alpha W_{\mu\nu}}{D-4}, \] (3.6)
where
\[ A_{\mu\nu} = \frac{D-3}{(D-2)^2} \left[ \frac{2D}{D-1}RR_{\mu\nu} - \frac{D-2}{D-3}C_{\mu\nu\lambda\rho}\rho^\lambda - 4R_{\mu\rho}R_{\nu\rho} + 2g_{\mu\nu}R_{\rho\lambda}R^{\rho\lambda} - \frac{D+2}{2(D-1)}g_{\mu\nu}R^2 \right], \] (3.7)
\[ W_{\mu\nu} = 2C_{\mu\alpha\beta\gamma}C_{\nu}^{\alpha\beta\gamma} - \frac{1}{4}g_{\mu\nu}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}, \] (3.8)
in which \( C \) is the Weyl tensor. The second term on the right hand side of (3.6) obviously can be problematic in the \( D \to 4 \) limit. This problem can be avoided, if we restrict our discussion within conformally flat spacetimes (e.g. the FRW universe [52–58]), since their \( W_{\mu\nu} \) vanishes in all dimensions. However, the problem returns, if we add further perturbation. Take the Minkowski background as the simplest example, with the perturbation \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), the second-order terms in the vacuum field equation obey [59]

\[ \text{[GR terms of } O(h^2)] + \frac{\alpha}{D-4} \left[ -2\nabla_\gamma \nabla_\alpha h_{\beta\nu} \nabla^\gamma \nabla^\alpha h_{\mu\beta} + 2\nabla_\gamma \nabla_\beta h_{\nu\alpha} \nabla^\gamma \nabla^\beta h_{\mu\alpha} + 2\nabla^\gamma \nabla^\beta h_{\nu\alpha} \nabla_\mu \nabla_\alpha h_{\beta\gamma} \\
+ 2\nabla^\gamma \nabla^\beta h_{\mu\beta} \nabla_\nu \nabla_\gamma h_{\beta\alpha} - 2\nabla^\gamma \nabla^\beta h_{\nu\alpha} \nabla_\mu \nabla_\gamma h_{\beta\alpha} - 2\nabla^\gamma \nabla^\beta h_{\nu\alpha} \nabla_\mu \nabla_\beta h_{\alpha\gamma} - 2\nabla^\gamma \nabla^\beta h_{\nu\alpha} \nabla_\mu \nabla_\beta h_{\alpha\gamma} \\
+ 2\nabla^\mu \nabla^\mu h_{\alpha\beta} \nabla_\nu \nabla_\gamma h_{\gamma\alpha\beta} + \eta_{\mu\nu}(2\nabla_\delta \nabla_\beta h_{\nu\alpha} \nabla^\delta \nabla^\gamma h^{\alpha\beta} - 2\nabla^\delta \nabla^\gamma h^{\alpha\beta} - 2\nabla^\delta \nabla^\gamma h^{\alpha\beta}) \right] = 0, \] (3.9)

which, with a \( D - 4 \) on the denominator, demonstrates that the theory is ill-defined in the \( D \to 4 \) limit. To tackle this kind of problem, some regularized 4D EGB theories [46, 47, 60, 61] have been proposed, e.g. a generalized form of the regularized action has been given in [60] as:
\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ R - 2\Lambda - \beta e^{2\phi}[R + 6(\nabla \phi)^2] - 2\Lambda e^{4\phi} \\
- \alpha[\phi x - 4G^\mu\nu \nabla_\mu \phi \nabla_\nu \phi - 4\Box \phi(\nabla \phi)^2 - 2(\nabla \phi)^4] \right\} + S_m, \] (3.10)
which is well-defined directly in the 4D spacetime. A special form of the above action that will be used is [61]
\[ S = \int d^4x \sqrt{-g} \left\{ R - 2\Lambda + \alpha[4G^\mu\nu \nabla_\mu \phi \nabla_\nu \phi - \phi \Box - 4\Box \phi(\nabla \phi)^2 + 2(\nabla \phi)^4] \right\} + S_m. \] (3.11)

B. The Equation of State for the FRW Universe

The stress-tensor \( T_{\mu\nu} \) is still taken as the perfect fluid (2.2), using which one can obtain the modified Friedmann’s equations [9] from the field equations (3.4)\footnote{It should be noted that from the regularized 4D EGB actions e.g. (3.10), (3.11), one can obtain the same Friedmann’s equations, e.g. for the \( k = 0 \) case, both the regularized and novel 4D EGB theories give (see [60, 61] for details)
\[ (1+\alpha H^2)H^2 = \frac{8\pi}{3}\rho_m, \quad (1+2\alpha H^2)\dot{H} = -4\pi(\rho_m + p_m), \] (3.12)
so the evolution and thermodynamic properties of the FRW universe will not change in the regularized theory. As a side remark, it has been pointed out in [60] that these equations can be also obtained from holographic cosmology [62, 63], generalized uncertainty principle [64], or quantum corrected entropy-area relation of the FRW universe [65].}
\[ \left[ 1 + \alpha \left( H^2 + \frac{k}{a^2} \right) \right] \left( H^2 + \frac{k}{a^2} \right) = \frac{8\pi}{3}\rho_m, \] (3.13)
\[ \left[ 1 + 2\alpha \left( H^2 + \frac{k}{a^2} \right) \right] \dot{H} - \frac{k}{a^2} = -4\pi(\rho_m + p_m). \] (3.14)
By using the expressions of the apparent horizon (2.6) and (2.7), we obtain from (3.14):

\[ \rho_m = \left(1 + \frac{\alpha}{R_A^2}\right) \frac{3}{8\pi R_A^3}, \]  
(3.15)

\[ p_m = \left(1 + \frac{2\alpha}{R_A^2}\right) \frac{1}{4\pi H R_A^3} - \left(1 + \frac{\alpha}{R_A^2}\right) \frac{3}{8\pi R_A^3}. \]  
(3.16)

We again define the work density of the matter field in the same way as (2.10), the temperature of the FRW universe the same as the Hawking temperature (2.13), and the thermodynamic volume as (2.16). From the literature [41], the entropy for an AdS black hole in 4D EGB gravity is given by:

\[ S = \pi R_A^2 + 4 \pi \alpha \ln \left(\frac{R_A}{\sqrt{|\alpha|}}\right) = \frac{A}{4} + 2 \pi \alpha \ln \left(\frac{A}{A_0}\right), \]  
(3.17)

where \( A_0 \equiv 4\pi|\alpha| \). The formula of the entropy solely depends on the type of the gravity model, so the expression (3.17) holds as long as we stay in 4D EGB gravity, no matter whether it is for a black hole or a FRW universe. That is to say, (3.17) can also be used as the entropy of the FRW universe whose horizon area is \( A \) [65]. Furthermore we adopt the generalized Misner-Sharp energy associated with the FRW apparent horizon in the 4D EGB gravity from [24]

\[ M \equiv \frac{R_A}{2} + \frac{\alpha}{2R_A}, \]  
(3.18)

which also has an additional \( \alpha \)-dependent term.

With the above quantities defined, we have

\[ dM = -TdS + WdV, \]  
(3.19)

which formally resembles (2.17) in GR. Now in order to get the standard form of the first law of thermodynamics, one should again define

\[ P := W; \]  
\[ U := -M. \]  
(3.20)

Then, in the same way as we derived (2.22), here we obtain the FRW equation of state in the 4D EGB gravity:

\[ P = \frac{T}{2R_A} + \frac{1}{8\pi R_A^2} + \frac{\alpha T}{R_A^3} - \frac{\alpha}{8\pi R_A^3}, \]  
(3.22)

or

\[ \left(P - \frac{1}{8\pi R_A^2} + \frac{\alpha}{8\pi R_A^3}\right) R_A = \left(\frac{1}{2} + \frac{\alpha}{R_A^3}\right) T, \]  
(3.23)

which is also somewhat similar to the van der Waals equation 4.

C. The vdW-like Phase Transition of the FRW Universe

For the equation of state (3.22), the critical condition (2.24) can be written as

\[ 2\pi T_c R_c^3 + R_c^2 + 12\pi \alpha R_c T_c - 2\alpha = 0, \]  
(3.24)

\[ 4\pi T_c R_c^3 + 3R_c^2 + 48\pi \alpha R_c T_c - 10\alpha = 0. \]  
(3.25)

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3 Note that, the entropy \( S \) includes two terms, the usual Bekenstein-Hawking term \( A/4 \) and the additional logarithmic term \( 2\pi \alpha \ln(A/A_0) \), which also appears in the corrections of thermal or quantum fluctuation to the black hole entropy [65–71], loop quantum gravity [72–75] and gravity with conformal anomaly [76].

4 If \( \alpha < 0 \), the thermodynamic pressure is independent of the temperature when \( R_A = \sqrt{-2\alpha} \), which coincides with the expression of the smallest horizon radius of the 4D EGB AdS black holes [40]. However, it is not yet clear to us how to interpret this particular value of \( R_A \) in the context of the FRW universe.
If $\alpha > 0$, the critical radius and temperature can not be both positive, so there is not any physical solution in this case. If $\alpha < 0$, there is a critical point

$$R_c = \sqrt{(6 - 4\sqrt{3})\alpha}, \quad T_c = \frac{\sqrt{6 + 4\sqrt{3}}}{12\pi\sqrt{-\alpha}}, \quad P_c = -\frac{15 + 8\sqrt{3}}{288\pi\alpha}.$$  \hspace{1cm} (3.26)

A dimensionless constant can be acquired from the above three values:

$$\rho = \frac{2P_cR_c}{T_c} = 6 + \frac{\sqrt{3}}{12}.$$  \hspace{1cm} (3.27)

Near the critical point, there are four critical exponents ($\hat{\alpha}, \beta, \gamma, \delta$) defined in the following way [42]:

$$C_V = T \left( \frac{\partial S}{\partial T} \right)_V \propto |t|^{-\hat{\alpha}},$$  \hspace{1cm} (3.28)

$$\eta = \frac{V_l - V_s}{V_c} \propto |t|^\beta,$$  \hspace{1cm} (3.29)

$$\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T \propto |t|^{-\gamma},$$  \hspace{1cm} (3.30)

$$p \propto v^\delta,$$  \hspace{1cm} (3.31)

where

$$t = \frac{T - T_c}{T_c}, \quad p = \frac{P - P_c}{P_c}, \quad v = \frac{V - V_c}{V_c}. \hspace{1cm} (3.32)$$

In most cases, the four critical exponents satisfy the following four scaling laws

$$\hat{\alpha} + 2\beta + \gamma = 2, \quad \hat{\alpha} + \beta(1 + \delta) = 2, \quad \gamma(1 + \delta) = (2 - \hat{\alpha})(\delta - 1), \quad \gamma = \beta(\delta - 1), \hspace{1cm} (3.33)$$

in which there are actually two independent relations. In the following, we will calculate the four critical exponents and check whether they satisfy the scaling laws.

The entropy of the FRW universe in the 4D EGB gravity is also a function of the thermodynamic volume $V$, so $C_V$ is zero, which means the first critical exponent $\hat{\alpha}$ is zero. To get the other three critical exponents conveniently, one can expand the thermodynamic pressure or the equation of state (3.22) around the critical point

$$p = a_{10}t + a_{11}tv + a_{03}v^3 + O(tv^2, v^4),$$  \hspace{1cm} (3.34)

where the coefficients are

$$a_{10} = \left( \frac{\partial p}{\partial t} \right)_c = \frac{T_c}{P_c} \left( \frac{\partial P}{\partial t} \right)_c \equiv \frac{T_c(R_c^2 + 2\alpha)}{2P_cR_c^3} < 0,$$  \hspace{1cm} (3.35)

$$a_{11} = \left( \frac{\partial^2 p}{\partial t \partial v} \right)_c = \frac{R_cT_c}{3P_c} \left( \frac{\partial^2 P}{\partial T \partial R_A} \right)_c = -\frac{T_c(R_c^2 + 6\alpha)}{6P_cR_c^3} > 0,$$  \hspace{1cm} (3.36)

$$a_{03} = \frac{1}{3!} \left( \frac{\partial^3 p}{\partial v^3} \right)_c = \frac{R_c^3}{162P_c} \left( \frac{\partial^3 P}{\partial R_A^3} \right)_c = \frac{R_c^2 + 6\alpha}{648\pi P_cR_c^3} < 0.$$  \hspace{1cm} (3.37)

The Gibbs free energy is defined as usual

$$G := U + PV - TS,$$  \hspace{1cm} (3.38)

so we have

$$dG = -SdT + VdP,$$  \hspace{1cm} (3.39)

and thus the Maxwell’s equal area law still holds. The values of $P$ at the two endpoints of the coexistence line are the same

$$p^* = a_{10}t + a_{11}tv_s + a_{03}v_s^3 = a_{10}t + a_{11}tv_l + a_{03}v_l^3,$$  \hspace{1cm} (3.40)
or
\[ a_{11}(v_l - v_s)t + a_{03}(v_l^3 - v_s^3) = 0, \quad (3.41) \]
where the labels ‘s’ and ‘l’ stand for ‘small’ and ‘large’ respectively. Another relation is
\[ \int v dp = \int_{v_s}^{v_l} v \left( \frac{dp}{dv} \right)_t \, dv = 0, \quad (3.42) \]
so we have
\[ 2a_{11}(v_l^2 - v_s^2)t + 3a_{03}(v_l^3 - v_s^3) = 0. \quad (3.43) \]
From the above two relations (3.41) and (3.43), one can get a nontrivial solution
\[ v_l = \sqrt{- \frac{a_{11}}{a_{03}} t}, \quad v_s = -\sqrt{- \frac{a_{11}}{a_{03}} t}, \quad (3.44) \]
and
\[ \eta = v_l - v_s = 2 \sqrt{- \frac{a_{11}}{a_{03}} t} \propto |t|^{1/2}, \quad (3.45) \]
which shows that the second critical exponent $\beta$ is 1/2. Interestingly, because $a_{11} > 0, a_{03} < 0$, we have $t > 0$, which means that the coexistence phases in the $P-V$ diagram appear above the critical temperature $T > T_c$. This behavior is different from that of a standard van der Waals system and that of an AdS black hole, where coexistence phases appear below the critical temperature $T < T_c$.

The third critical exponent is from the isothermal compressibility near the critical point
\[ \kappa_T = -\frac{1}{V_c} \left. \left( \frac{\partial V}{\partial P} \right)_T \right|_c \propto - \left( \frac{\partial p}{\partial v} \right)_t^{-1} = - \frac{1}{a_{11}} t \propto t^{-1}, \quad (3.46) \]
which shows $\gamma = 1$.

The shape of the isothermal line of the critical temperature $t = 0$ is
\[ p \propto v^3, \quad (3.47) \]
which provides the fourth critical exponent $\delta = 3$.

In summary, the four critical exponents are:
\[ \tilde{\alpha} = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3, \quad (3.48) \]
which are the same as those in the van der Waals system and satisfy the scaling laws (3.33).

### IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we have studied the thermodynamic properties, especially the van der Waals-like phase transition, of the FRW universe with a perfect fluid. The thermodynamic pressure $P$ of the FRW universe is defined as the work density $W$, which is a natural definition directly read out from the first law of thermodynamics. We have derived the equations of state for the FRW universe with a perfect fluid both in Einstein gravity and in the novel 4D EGB gravity. Impressively, in the latter case, the equation of state exhibits van der Waals-like phase transitions. To our best knowledge of the literature, this is the first time that such phase transitions are found in a case that is not asymptotically AdS black holes. The phase transitions occur above the critical temperature, which is different from the conventional van der Waals system and is different from most of the black hole systems. In the end, we have calculated the four critical exponents, which are the same as those in the usual van der Waals system and thus satisfy the scaling laws.

We would like to discuss a few more open questions related to our work. A natural question is whether van der Waals-like phase transitions can be found for FRW universe in other modified theories of gravity and/or filled with other fields (e.g. a scalar field). An example directly related to our work is the $(\partial \phi)^4$ model [77], which has been found to be equivalent to the 4D EGB gravity from the perspective of scattering amplitudes (at least in the leading order). 5 Besides the FRW universe, we have also investigated the thermodynamic properties of another dynamical

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5 See Appendix for the discussion of the $(\partial \phi)^4$ theory.
spacetime, the McVittie spacetime, and found that it has a Hawking-Page-like phase transition instead of a van der Waals-like phase transition in Einstein gravity [78]. We will continue the investigation of phase transitions of dynamical spacetimes (including the FRW universe, the McVittie spacetime, etc) in the future.

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Appendix A: the \((\partial \phi)^4\) Theory

It has been shown in [77] that the amplitudes of the novel 4D EGB gravity can be obtained from the minimally coupled \((\partial \phi)^4\) theory

\[ S = M_p^2 \int d^4x \sqrt{-g} \left[ \frac{1}{2} R - \frac{1}{2} (\partial \phi)^2 + \hat{\alpha} (\partial \phi)^4 \right], \quad (A1) \]

so it is natural to ask whether they have the same thermodynamic behaviors.

Note that the much more complicated action (3.11) of the regularized 4D EGB theory also has a \((\partial \phi)^4\) term, and it leads to the same Friedmann’s equations as the novel 4D EGB gravity. Thus naively, (A1) should be thermodynamically the same, if it were somehow equivalent to (3.11). However, this is not the case. In [77], it has been shown that both (3.11) and (A1) can be derived from

\[ S \propto \int d^4x \sqrt{-g} e^{N\phi} \left\{ R + \alpha \mathcal{G} + N(N-1)(\partial \phi)^2 - \alpha N(N-1)[4G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \
+ 2(N-2)\Box \phi (\partial \phi)^2 + (N-1)(N-2)(\partial \phi)^4] \right\}, \quad (A2) \]

but their derivations are based on two completely different limits that involves infinities.\(^7\) Therefore, we have no reason to anticipate the same Friedmann’s equations from (A1), and hence in principle its phase transitions (if any) should also be different.

In the following, let us do some preliminary analysis of the thermodynamic behavior of the \((\partial \phi)^4\) theory. From (A1), one can get

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Phi_{\mu\nu}, \quad (A3) \]

\[ \Box \phi - 4\hat{\alpha} [\Box \phi (\partial \phi)^2 + \partial_\mu \phi \partial^\mu (\partial \phi)^2] = 0, \quad (A4) \]

where

\[ \Phi_{\mu\nu} \equiv \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial \phi)^2 - \hat{\alpha} [4\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} (\partial \phi)^2] (\partial \phi)^2, \quad (A5) \]

which, up to a constant factor, can be treated as the energy-momentum tensor:

\[ T_{\mu\nu} \equiv \frac{1}{8\pi} \Phi_{\mu\nu}. \quad (A6) \]

It can be rewritten in the form of a perfect fluid

\[ T_{\mu\nu} = (\rho_\phi + p_\phi) U_\mu U_\nu + p_\phi g_{\mu\nu}, \quad (A7) \]

\(^6\) We use \(\hat{\alpha}\) instead of \(\tilde{\alpha}\) to avoid confusion with the critical exponent \(\tilde{\alpha}\) in (3.28).

\(^7\) Roughly speaking (3.11) is obtained from the \(N \to 0, \alpha \to \infty\) limit (keeping \(\alpha N\) finite) of (A2), and (A1) is from the \(N \to 0, \alpha \to 0, \phi \to \infty\) limit (keeping \(\alpha/N\) and \(\phi\sqrt{N}\) finite). See [77] for details.
where
\[ \rho_\phi \equiv -\frac{1}{16\pi}(\partial \phi)^2 + \frac{3}{8\pi} \hat{\alpha}(\partial \phi)^4, \]
\[ p_\phi \equiv -\frac{1}{16\pi}(\partial \phi)^2 + \frac{\hat{\alpha}}{8\pi}(\partial \phi)^4, \]
\[ U_\mu \equiv \frac{\partial \phi}{\sqrt{-g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi}}. \]

By substituting the FRW metric (2.1) into the field equation (A3) and assuming \( \phi = \phi(t) \), one obtains the Friedmann’s equations
\[ H^2 + \frac{k}{a^2} = \frac{8\pi}{3} \rho_\phi, \]
\[ \dot{H} - \frac{k}{a^2} = -4\pi(p_\phi + p_\phi), \]
so the work density (thermodynamic pressure) of the scalar field is
\[ W_\phi := -\frac{1}{2}(T_\phi^0 + T_\phi^1) = \frac{1}{2} \rho_\phi - \rho_\phi = \frac{3}{8\pi}(H^2 + \frac{k}{a^2}) + \frac{1}{8\pi}(H - \frac{k}{a^2}) = \frac{T}{2R_A} + \frac{1}{8\pi R_A^2}, \]
where (2.6), (2.7), (2.13) have been used for the last step. (A13) can be seen as the equation of state for the scalar field, which is the same as (2.22), so it should have no van der Waals-like phase transition.

One can further supplement the action (A1) with perfect fluid
\[ S = M_P^2 \int \left[ \frac{1}{2}R - \frac{1}{2}(\partial \phi)^2 + \hat{\alpha}(\partial \phi)^4 \right] \sqrt{-g} d^4x + S_m, \]
where \( S_m \) stands for the action of the supplement perfect fluid. From the above action, one can get the field equation
\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} + 8\pi \tilde{T}_{\mu\nu}, \]
where
\[ \tilde{T}_{\mu\nu} = (\rho_m + p_m)u_{\mu}u_{\nu} + p_m g_{\mu\nu}, \]
with \( u_\mu = -\delta_{0\mu} \). In this case, the scalar equation is the same as (A4), while the Friedmann’s equations have become
\[ H^2 + \frac{k}{a^2} = \frac{8\pi}{3}(\rho_m + \rho_\phi), \]
\[ \dot{H} - \frac{k}{a^2} = -4\pi(\rho_m + \rho_\phi + p_m + p_\phi). \]

Obviously, the total work density of the two components is formally the same as (A13)
\[ W_t = \frac{1}{2}(\rho_t - p_t) = \frac{1}{2} (\rho_m + \rho_\phi - p_m - p_\phi) = \frac{T}{2R_A} + \frac{1}{8\pi R_A^2}, \]
which has no phase transition. One can also study separately the work density of the perfect fluid:
\[ W_m = \frac{1}{2}(\rho_m - p_m) = \frac{T}{2R_A} + \frac{1}{8\pi R_A^2} - \frac{\hat{\alpha}}{8\pi}(\partial \phi)^4. \]

The last term is not written as a function of \( R_A \) or \( T \), and we should replace the \( \hat{\phi} \) there with the solution to the scalar equation (A4).

The scalar equation (A4) can be written as
\[ (1 - 12\hat{\alpha}X)\ddot{X} + 6H(1 - 4\hat{\alpha}X)X = 0, \]
where $X \equiv (\partial \tilde{\phi})^2$. After integration, one can get

$$(1 - 4\alpha X)^2 X a^6 = C,$$  \hspace{1cm} (A22)$$

where $C$ is an integral constant. The general solution to the above equation is complicated and beyond the scope of this appendix, which we will leave for future work. In the following we study the special case of $C = 0$, which gives

$$X = \frac{1}{4\tilde{\alpha}} \quad \text{or} \quad X = 0.$$  \hspace{1cm} (A23)$$

Substituting them into (A20) gives

$$W_m = \frac{T}{2R_A} + \frac{1}{8\pi R_A^2} - \frac{1}{128\tilde{\alpha}},$$  \hspace{1cm} (A24)$$

or

$$W_m = \frac{T}{2R_A} + \frac{1}{8\pi R_A^2},$$  \hspace{1cm} (A25)$$

which do not have an inflection point in the $W - R_A$ diagram (i.e. the set of equations $\left(\frac{\partial W}{\partial R_A}\right)_T = 0$, $\left(\frac{\partial^2 W}{\partial R_A^2}\right)_T = 0$ has no solution), and thus the van der Waals-like phase transition does not exist. It should be noted that, in this case we have only studied one component of a double-component thermodynamic system, where the existence of an inflection point is merely a necessary but not sufficient condition for the existence of the van der Waals-like phase transition. In other words, even if the reflection point exists, there may still be no phase transition. Note that the minimally coupled $(\partial \tilde{\phi})^4$ theory is just a special form of the K-essence [79, 80], which is indeed very different from Gauss-Bonnet gravity.

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