A VARIANCE-REDUCED STOCHASTIC ACCELERATED PRIMAL DUAL ALGORITHM

A PREPRINT

Bugra Can
Department of Management Sciences and Information Systems
Rutgers Business School
Piscataway, NJ, 08854.
bc600@scarletmail.rutgers.edu

Mert Gürbüzbalaban
Department of Management Sciences and Information Systems
Rutgers Business School
Piscataway, NJ, 08854
mert.gurbuzbalaban@rutgers.edu

Necdet Serhat Aybat
Industrial and Manufacturing Engineering Department
Penn State University
University Park, PA, 16802-4400,
nsa10@psu.edu

February 22, 2022

ABSTRACT

In this work, we consider strongly convex strongly concave (SCSC) saddle point (SP) problems
\[
\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^d} f(x,y)
\]
where \( f \) is \( L \)-smooth, \( f(\cdot, y) \) is \( \mu \)-strongly convex for every \( y \), and \( f(x, \cdot) \) is \( \mu \)-strongly concave for every \( x \). Such problems arise frequently in machine learning in the context of robust empirical risk minimization (ERM), e.g., distributionally robust ERM, where partial gradients are estimated using mini-batches of data points. Assuming we have access to an unbiased stochastic first-order oracle, we consider the stochastic accelerated primal dual (SAPD) algorithm recently introduced in Zhang et al. [2021] for SCSC SP problems as a robust method against the gradient noise. In particular, SAPD recovers the well-known stochastic gradient descent ascent (SGDA) as a special case when the momentum parameter is set to zero and can achieve an accelerated rate when the momentum parameter is properly tuned, i.e., improving the \( \kappa \triangleq L/\mu \) dependence from \( \kappa^2 \) for SGDA to \( \kappa \). We propose efficient variance-reduction strategies for SAPD based on Richardson-Romberg extrapolation and show that our method improves upon SAPD both in practice and in theory.

Keywords Saddle point algorithms · Variance reduction methods

1 Introduction

Saddle point (SP) problems arise frequently in many key settings in machine learning. Examples include but are not limited to robust training of machine learning models [Gürbüzbalaban et al., 2020], Duchi and Namkoong [2021], training Generative Adversarial Networks (GANs) [Arjovsky et al., 2017], design of fair classifiers [Nouiehed et al., 2019], and empirical risk minimization (ERM) problems such as regression and classification [Palaniappan and Bach, 2016].
Furthermore, ERM problems with constraints on the model parameters result in constrained stochastic optimization problems which can also be cast into saddle-point problems using Lagrangian duality.

Motivated by such applications, we consider the strongly convex/strongly concave (SCSC) saddle-point (SP) problem

$$\min_{x} \max_{y} f(x, y),$$  \hspace{1cm} (1)

where \(f : \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R}\) is smooth and strongly convex in \(x\) and strongly concave in \(y\). As we discuss in the numerical experiments section, our algorithms and results are also directly applicable to the convex/concave setting by adding an appropriate quadratic regularizer.

In a different setting, SCSC problems of the form given in (1) arise as subproblems when solving weakly convex-weakly concave SP problems using inexact proximal point method, e.g., see [1].

We consider (1) assuming we only have access to unbiased stochastic estimates of the true gradients \(\nabla_x f(x, y)\) and \(\nabla_y f(x, y)\); as often in machine learning applications gradients are not exactly computed but are estimated from randomly sampled subsets (mini-batches) of data. This is also the case in privacy-preserving empirical risk minimization where noise is added to the gradients to preserve privacy of the user data [Chaudhuri et al. 2011]. For solving these SP problems, stochastic first-order (SFO) methods that rely on stochastic gradient information have been very popular in practice due to their favorable scalability properties; but, they come with a number of challenges. In particular, even though the problem (1) is well-studied when the gradients are deterministic, many aspects of the stochastic setting with inexact gradient information are relatively understudied. In fact, accelerated SFO methods achieving acceleration beyond bilinear problems are quiet recent in the literature, e.g., [Zhao 2021, Fallah et al. 2020, Zhang et al. 2021, Dubey et al. 2016, Wang et al. 2019, Thekumparampil et al. 2019, Chen et al. 2017, Hsieh et al. 2019] and the references therein.

To that end, the recently proposed stochastic accelerated primal-dual (SAPD) method [Zhang et al. 2021], based on momentum-averaging, can achieve the best iteration complexity bound among single-loop methods for SCSC problems of the form: \(\min_{x} \max_{y} g(x) + \langle f(x, y), y \rangle - h(y)\), where \(f\) is a smooth SCSC function admitting an SFO oracle as we assume for the SP problem in (1), and \(g, h\) are closed, strongly convex functions with efficient proximal maps; furthermore, SAPD achieves the optimal complexity for bilinear SP problems with \(f(x, y) = \langle Ax, y \rangle\) for some \(A \in \mathbb{R}^{d_y \times d_x}\). In particular, SAPD, which is an extension of the APD algorithm from deterministic to stochastic first-order oracle setting, recovers the well-known stochastic gradient descent ascent (SGDA) as a special case when the momentum parameter is set to zero and can achieve an accelerated rate when the momentum parameter is properly tuned, i.e., improving the \(\kappa \triangleq L/\mu\) dependence from \(\kappa^2\) for SGDA to \(\kappa\).

Due to persistent nature of the noise on the gradients, the performance of the stochastic SP methods differs from their deterministic counterparts and depends heavily on the statistical properties of the limiting point/distribution generated as a solution to the problem in (1). In particular, addition of a momentum-averaging step has a typical affect to increase the stationary variance of the iterates in the context of stochastic momentum methods such as SAPD [Zhang et al. 2021] and in this context developing efficient variance-reduction strategies becomes a key for achieving better practical performance.

**Contributions.** In this paper, we consider the SAPD algorithm with constant (primal and dual) stepsize for solving SCSC SP problems of the form (1). Employing constant stepsize has at least two major benefits: (i) only one value needs to be tuned as opposed to \(a, b\) and \(c\) parameters needed to be tuned for decaying step size sequence of the form \(a/(b + ck)\) for \(k \geq 0\); (ii) the “bias term”, which characterizes how fast the effects of initial conditions on the iterates are forgotten, decays exponentially (see Proposition 2).

Our contributions are as follows:

- In Proposition 2 we consider a parametrization of the primal stepsize \(\tau\) and dual stepsize \(\sigma\) in terms of the momentum parameter \(\theta\) and obtain convergence guarantees for SAPD for that particular choice of parameters. For this specific parametrization of \(\tau, \sigma\) in \(\theta\), working directly with the expected distance square (EDS) metric, rather than the more stronger expected gap metric as in Zhang et al. [2021], we were able to provide a sharper result for the EDS metric in terms of constants, compared to those available for SAPD in Zhang et al. [2021]. Under this parametrization, the corresponding linear convergence rate for the bias term is equal to the momentum parameter \(\theta\) as also shown in Zhang et al. [2021]. For ill-conditioned problems we expect a slower convergence for the bias as selecting small \(\tau, \sigma > 0\) requires \(\theta \in (0, 1)\) chosen close to 1.

- Under some assumptions on the gradient noise structure, we show that SAPD iterates admit an invariant distribution (Theorem 3). Since SAPD iterates result in a Markov chain that is non-reversible, standard tools for reversible Markov Chains (such as those arising in the study of stochastic gradient descent methods [Dieuleveut et al. 2017]) are not directly applicable. We achieve this result by showing that so-called “minorization and drift conditions” hold for the Markov chain corresponding to the SAPD iterates.
In Lemma 4, we characterize the second and fourth moments of the stationary distribution in terms of its dependency to the momentum parameter $\theta$. Building on this lemma, in Theorem 5, we show that the expected iterates contain a bias that is proportional to $O(1 - \theta)$ as $\theta \rightarrow 1$. Motivated by this result, based on Richardson-Romberg extrapolation type arguments, we propose a variance-reduction scheme: running SAPD twice independently with different parameters $\theta_1$ and $\theta_2$, one can remove the bias term proportional to $O(1 - \theta)$ by considering a linear combination of the two iterate sequences corresponding to two appropriately chosen parameters $\theta_1$ and $\theta_2$. We call this method variance-reduced SAPD (VR-SAPD). Although it is known that Richardson extrapolation can be very efficient for optimization algorithms such as stochastic gradient descent [Dieuleveut et al. 2017] and Frank-Wolfe methods [Bach 2021], to our knowledge, our paper provides the first application of this technique to the accelerated stochastic primal-dual algorithms for saddle-point problems. There are also variance-reduction techniques for finite-sum problems where the objective is in the form of an average of finitely many component functions, e.g., [Palaniappan and Bach 2016, Chavdarova et al. 2019, Alacaoglu and Malitsky 2021]; however, these techniques are not available to our setting as our objective is more general and we do not assume that it has a finite-sum structure.

In Theorem 6, we characterize the gap between the expected ergodic average of SAPD iterates and the expectation of SAPD iterates in the limit as the number of iterations $k$ grows. We show that this gap is of the order $O(1/k)$ and is vanishing asymptotically; therefore, it implies that the same variance-reduction mechanism can be applied to the ergodic averages of the SAPD iterates as well.

We showcase the efficiency of our proposed algorithm VR-SAPD on distributionally robust logistic regression (DRLR) problems on three datasets where we compare VR-SAPD with SAPD [Zhang et al. 2021], S-OGDA [Fallah et al. 2020], and Stochastic Mirror Prox (SMP) [Juditsky et al. 2011] algorithms. For the DRLR problem, we adopted $f_2$-divergence to define the uncertainty set around the uniform distribution; therefore, the problem domain is the Euclidean space, and in this case SMP reduces to the stochastic gradient descent ascent (SGDA) method which is commonly used in machine learning practice, e.g., for training GANs [Kontonis et al. 2020]. Our results show that VR-SAPD reduces the variance of the SAPD iterates considerably and results in a significant performance improvement.

Notations. Let $\mathbb{R}_+$ be the set of positive real numbers. For any vector $v \in \mathbb{R}^d$, $\|x\|$ denotes the Euclidean norm, and for any matrix $M \in \mathbb{R}^{d \times d}$, $\|M\|_2$ denotes the spectral norm, i.e., the largest singular value of $M$. Let $E$ and $F$ be two real vector spaces. We use $E \otimes F$ to denote the tensor product of $E$ and $F$. The tensor product of two vectors $x \in E$ and $y \in F$ is denoted as $E \otimes F \ni x \otimes y = xy^\top$. Similarly, $E^{\otimes k}$ is the $k$-th tensor power of $E$ and $x^{\otimes k} \in E^{\otimes k}$ is the $k$-th tensor product of $x \in E$. We use $N$ for the set of non-negative integers, and $N_+ \triangleq N \setminus \{0\}$. Let $n \in N_+$, then $C^n(\mathbb{R}^d, \mathbb{R}^m)$ is the set of $n$-times continuously differentiable functions from $\mathbb{R}^d \rightarrow \mathbb{R}^m$. Let $f \in C^n(\mathbb{R}^d, \mathbb{R})$, $D^n f$ is the $n$-th differential of $f$. If $f \in C^n(\mathbb{R}^d, \mathbb{R})$, then $\nabla^{(n)} f(z)$ denotes the tensor of order $n$ at the point $z \in \mathbb{R}^d$ arising in the Taylor expansion of $f$. For any vector $z \in \mathbb{R}^d$ and any matrix $M \in \mathbb{R}^{d \times d}$, we define $\nabla^{(3)} f(z) M = \sum_{\ell=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} M_{j\ell} \frac{\partial^3 f}{\partial z_j \partial z_\ell \partial z_k}(z)$. For any matrices $M, N \in \mathbb{R}^{d \times d}$, $M \otimes N$ is defined as the endomorphism of $\mathbb{R}^{d \times d}$ such that $M \otimes N: P \rightarrow MPN^\top$. Lastly, $(x_+, y_+)$ is the unique solution of the SP problem in (1) and $\nabla^{(2)} f_*$ and $\nabla^{(3)} f_*$ are used for $\nabla^{(2)} f(x_+, y_+)$ and $\nabla^{(3)} f(x_+, y_+)$, respectively, for simplicity.

2 Preliminaries and Background

SAPD Algorithm. Recently, Zhang et al. [Zhang et al. 2021] have proposed the (SAPD) algorithm to solve SCSC SP problems of the form (1). SAPD updates are given as follows:

$$\begin{align*}
\hat{y}_k &= (1 + \theta)\nabla_y f(x_k, y_k) - \theta \nabla_y f(x_{k-1}, y_{k-1}), \\
y_{k+1} &= y_k + \sigma \hat{y}_k, \\
x_{k+1} &= x_k - \tau \nabla_x f(x_k, y_{k+1}).
\end{align*}$$  (2a, 2b, 2c)

Basically, SAPD admits three positive parameters $(\tau, \sigma, \theta)$ where $\tau$ is the primal stepsize, $\sigma$ is the dual stepsize, and $\theta$ is the momentum parameter. In our work, we are particularly interested in designing a method for the SP problem in (1) that would improve the variance of the SAPD iterate sequence $\{(x_k, y_k)\}_{k \geq 0}$. We call our method Variance Reduced SAPD (VR-SAPD). For analysis purposes, first state our assumptions on $f$ and $\nabla f$.

Assumption 1. We assume $f \in C^4$, i.e., four times continuously differentiable with uniformly bounded 3-rd and 4-th order derivatives. For any $\hat{y} \in \mathbb{R}^{d_y}$, $f(\cdot, \hat{y})$ is $L_{yx}$-smooth and $\mu_y$-strongly convex for some $L_{yx}, \mu_y > 0$. Furthermore, for any $\hat{x} \in \mathbb{R}^{d_x}$, $f(\hat{x}, \cdot)$ is $\mu_y$-strongly concave, and there exist constants $L_{yx}, L_{yy} > 0$ such that

$$\|\nabla_y f(x, y) - \nabla_y f_*(\hat{x}, \hat{y})\| \leq L_{yx}\|x - \hat{x}\| + L_{yy}\|y - \hat{y}\|,$$
for all $x, \tilde{x} \in \mathbb{R}^{d_x}$ and $y, \tilde{y} \in \mathbb{R}^{d_y}$.

We also make the following standard assumption on $\hat{\nabla} f$, which says that we have access to unbiased stochastic estimates of the gradient.

**Assumption 2.** Let $\left\{x_k, y_k\right\}$ be the SAPD iterate sequence. For all $k \geq 0$, we have access to $\hat{\nabla}_y f(x_k, y_k, \zeta_k^y)$ and $\hat{\nabla}_x f(x_k, y_{k+1}, \zeta_k^x)$ which are unbiased estimates of $\nabla_y f(x_k, y_k)$ and $\nabla_x f(x_k, y_{k+1})$, respectively\footnote{$\zeta_k^y$ and $\zeta_k^x$ are determining the noise structure, and for simplicity of the notation we suppress them and use $\hat{\nabla}_y f(x_k, y_k)$ and $\hat{\nabla}_x f(x_k, y_{k+1})$ instead.}, i.e.,

$$
\mathbb{E}[\hat{\nabla}_y f(x_k, y_k, \zeta_k^y)|x_k, y_k] = \nabla_y f(x_k, y_k),
$$

and 

$$
\mathbb{E}[\hat{\nabla}_x f(x_k, y_{k+1}, \zeta_k^x)|x_k, y_{k+1}] = \nabla_x f(x_k, y_{k+1}),
$$

such that the $\{\zeta_k^y\}$ and $\{\zeta_k^x\}$ are independent sequences and also independent from each other. Furthermore, for $p \in \{2, 3, 4\}$, there exists $\delta_p(\cdot) > 0$ such that

$$(3a)$$

$$
\mathbb{E}[\|\hat{\nabla}_y f(x_k, y_k, \zeta_k^y) - \nabla_y f(x_k, y_k)\|^p] \leq \delta_p(\epsilon),
$$

and 

$$(3b)$$

$$
\mathbb{E}[\|\hat{\nabla}_x f(x_k, y_{k+1}, \zeta_k^x) - \nabla_x f(x_k, y_{k+1})\|^p] \leq \delta_p(\epsilon).
$$

We also assume that both noise, i.e., $\hat{\nabla}_x f - \nabla_x f$ and $\hat{\nabla}_y f - \nabla_y f$, are stationary, and independent from the past.

We note that $\tilde{f}$ being twice continuously differentiable and existence of second moments for the gradient noise would be sufficient to obtain convergence guarantees in $L^2$ for the accumulation of SAPD sequence $\{x_k, y_k\}$ in the vicinity of the unique saddle point—see Theorem 1 below; however, in this paper, we require further smoothness up to fourth order to be able to obtain stronger convergence guarantees for the fourth moments which is then used in our variance reduction mechanism (see Lemma 4 and Theorem 5).

The next result, which is a direct consequence of Theorems 2.6 and 2.9 in Zhang et al. [2021], summarizes the known iteration complexity results for SAPD with constant primal and dual stepsizes.

**Theorem 1 (Zhang et al. [2021]).** Consider the problem in [1]. Under Assumptions 1 and 2 for $p = 2$, given any $\epsilon > 0$, there exists $\bar{\theta}_* \in (0, \bar{\theta}_1)$ such that the SAPD iterate sequence $\{x_k, y_k\}$ generated using $\tau = \frac{1 - \theta}{\bar{\mu}_x}$ and $\sigma = \frac{1 - \theta}{\bar{\mu}_y}$ for any $\theta \in [\bar{\theta}_*, \bar{\theta}_1]$ satisfies

$$
\mathbb{E}\left[\max\{\sup_{x, y} f(x_N, y) - f(x, y_N)\}, \|z_N - z^*\|^2\right] \leq \epsilon,
$$

for all $N \geq N_\epsilon \in \mathbb{N}_+$ such that

$$
N_\epsilon = \mathcal{O}\left(\left(\frac{L_{xx}}{\bar{\mu}_x} + \frac{L_{yx}}{\bar{\mu}_x \bar{\mu}_y} + \frac{L_{yy}}{\bar{\mu}_y}\right) \log \left(\frac{1}{\epsilon}\right) + \left(\frac{1}{\bar{\mu}_x} + \frac{1}{\bar{\mu}_y}\right) \frac{\delta_2(\epsilon)}{\epsilon}\right).
$$

Here $\mathcal{O}(\cdot)$ notation hides $\log(1/\epsilon)$ factor for the variance term, which can be removed adopting a restart approach proposed in Fallah et al. [2020]. Let

$$
\kappa_x \triangleq L_{xx}/\mu_x, \quad \kappa_y \triangleq L_{yy}/\mu_y, \quad \kappa_{yx} \triangleq L_{yx}/\sqrt{\mu_x \mu_y},
$$

and define $\kappa \triangleq \max\{\kappa_x, \kappa_y, \kappa_{yx}\}$. In the above result, the first part $\mathcal{O}(\kappa \log 1/\epsilon)$ is related to the bias term; it is essential to emphasize that the bias term for SGDA using constant step size is $\mathcal{O}(\kappa^2 \log 1/\epsilon)$. Thus, SAPD accelerates the convergence of the bias term compared to the popular alternative SGDA.

**Dynamical system representation of SAPD.** Through defining the following concatenations,

$$
z_k \triangleq \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad \xi_k \triangleq \begin{bmatrix} z_k \\ \zeta_k \end{bmatrix}, \quad \hat{\Phi}_k \triangleq \begin{bmatrix} \hat{\nabla}_x f(x_k, y_{k+1}) \\ \hat{\nabla}_y f(x_k, y_{k+1}) \end{bmatrix},
$$

the SAPD recursion in (2) can be rewritten as

$$
\xi_{k+1} = M \xi_k + \mathcal{N} \hat{\Phi}_k,
$$

with the associated matrices

$$
M = \begin{bmatrix} I_d & 0_d \\ 0_d & 0_d \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} -\tau I_{d_x} & 0_{d_x \times d_y} & 0_{d_x \times d_y} \\ 0_{d_y \times d_x} & \sigma(1 + \theta) I_{d_y} & -\theta \sigma I_{d_y} \\ 0_{d_y \times d_x} & 0_{d_y \times d_y} & 0_{d_y \times d_y} \end{bmatrix},
$$

with $d \triangleq d_x + d_y$. In some places, we will also use the notation $\xi_k^{(\tau, \sigma, \theta)}$ interchangeably with $\xi_k$ to emphasize the iterates’ dependency on the parameters of the algorithm.
SAPD iterates as a Markov Chain. We notice that under Assumption 2, the iterates \( \{\xi_k^{(\tau,\sigma,\theta)}\}_{k \in \mathbb{N}} \) define a time-homogeneous Markov chain. Let \( \mathcal{R}^{(\tau,\sigma,\theta)} \) be the Markov kernel of iterates \( \{\xi_k^{(\tau,\sigma,\theta)}\}_{k \in \mathbb{N}} \) on \((\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d}))\), where \( \mathcal{B}(\mathbb{R}^{2d}) \) is the sigma-algebra generated by the Borel sets in \( \mathbb{R}^{2d} \), i.e., for all \( A \in \mathcal{B}(\mathbb{R}^{2d}) \) and \( k \geq 0 \),
\[
\mathcal{R}^{(\tau,\sigma,\theta)}(\xi_k, A) = \mathbb{P}\{\xi_{k+1} \in A | \xi_k\}, \quad \forall \xi_k \in \mathbb{R}^{2d},
\]
almost surely, the map \( \xi \mapsto \mathcal{R}^{(\tau,\sigma,\theta)}(\xi, A) \) is Borel measurable, and \( \mathcal{R}^{(\tau,\sigma,\theta)}(\xi_0, \cdot) \) is a probability measure on \((\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d}))\) for any \( \xi_0 \in \mathbb{R}^{2d} \). Then, for all \( k \geq 1 \), the Markov kernel associated with \( \mathcal{R}^{(\tau,\sigma,\theta)} \) of \( \xi_k \) is recursively defined for any \( \xi_0 \in \mathbb{R}^{2d} \) and \( A \in \mathcal{B}(\mathbb{R}^{2d}) \) as
\[
\mathcal{R}^{(\tau,\sigma,\theta)}(\xi_0, A) = \int_{\mathbb{R}^{2d}} \mathcal{R}^{(\tau,\sigma,\theta)}(\xi_0, d\xi) \mathcal{R}^{(\tau,\sigma,\theta)}(\xi, A),
\]
where \( \mathcal{R}^{(\tau,\sigma,\theta)}_1 = \mathcal{R}^{(\tau,\sigma,\theta)} \). We also define \( \lambda \mathcal{R}^{(\tau,\sigma,\theta)}_k \) for any probability measure \( \lambda \) on \((\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d}))\) by
\[
\lambda \mathcal{R}^{(\tau,\sigma,\theta)}_k(A) \triangleq \int_{\mathbb{R}^{2d}} \lambda(d\xi) \mathcal{R}^{(\tau,\sigma,\theta)}(\xi, A), \quad \forall A \in \mathcal{B}(\mathbb{R}^{2d}).
\]
The above definition implies that for any probability measure \( \lambda \) on \( \mathbb{R}^{2d} \) and \( k \in \mathbb{N}_+ \), \( \lambda \mathcal{R}^{(\tau,\sigma,\theta)}_k \) is the distribution of \( \xi_k^{(\tau,\sigma,\theta)} \) initialized from \( \xi_0 \sim \lambda \), i.e., \( \xi_0 \) drawn from \( \lambda \). For any function \( \Psi : \mathbb{R}^{2d} \to \mathbb{R} \) and \( k \in \mathbb{N}_+ \), the measurable function \( \mathcal{R}^{(\tau,\sigma,\theta)}_k \Psi : \mathbb{R}^{2d} \to \mathbb{R} \) is defined as
\[
\mathcal{R}^{(\tau,\sigma,\theta)}_k \Psi(\xi_0) \triangleq \int_{\mathbb{R}^{2d}} \Psi(\xi) \mathcal{R}^{(\tau,\sigma,\theta)}_k(\xi_0, d\xi), \quad \forall \xi_0 \in \mathbb{R}^{2d}.
\]
For any measure \( \lambda \) on \((\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d})) \) and any measurable function \( h : \mathbb{R}^{2d} \to \mathbb{R} \), we denote \( \int_{\mathbb{R}^{2d}} h(\xi)d\lambda(\xi) \) by \( \lambda(h) \) (whenever exists) for simplicity. Notice that this notation implies the equality \( \lambda(\mathcal{R}^{(\tau,\sigma,\theta)}_k h) = (\lambda \mathcal{R}^{(\tau,\sigma,\theta)})(h) \) for any measure \( \lambda \) on \( \mathcal{B}(\mathbb{R}^{2d}) \) and any measurable function \( h : \mathbb{R}^{2d} \to \mathbb{R} \) and \( k \in \mathbb{N}_+ \). A distribution \( \pi^{(\tau,\sigma,\theta)} \) is called an invariant measure for \( \mathcal{R}^{(\tau,\sigma,\theta)} \) if \( \pi^{(\tau,\sigma,\theta)} \mathcal{R}^{(\tau,\sigma,\theta)} = \pi^{(\tau,\sigma,\theta)} \) and we say the Markov chain \( \{\xi_k^{(\tau,\sigma,\theta)}\}_{k \in \mathbb{N}} \) is stationary if it admits an invariant measure \( \pi^{(\tau,\sigma,\theta)}_k \), i.e., \( \xi_0 \sim \pi^{(\tau,\sigma,\theta)}_k \) is distributed according to \( \pi^{(\tau,\sigma,\theta)}_k \). It also means that the distribution of \( \xi_k^{(\tau,\sigma,\theta)} \) is also \( \pi^{(\tau,\sigma,\theta)}_k \) for all \( k \in \mathbb{N} \). Depending on the context, we use the notation \( \pi_k^{(\tau,\sigma,\theta)} \) interchangeably with \( \pi^{(\tau,\sigma,\theta)}_k \) for simplicity. Similarly, when all the parameters of the SAPD algorithm are parametrized as function of \( \theta \), then we adopt the notation \( \pi^{(\tau,\sigma,\theta)}_k \triangleq \pi^{(\tau_0,\sigma_0,\theta)}_k \) and similarly, \( \xi^{(\tau,\sigma,\theta)}_k \triangleq \xi^{(\tau_0,\sigma_0,\theta)}_k \) and \( \mathcal{R}^{(\tau,\sigma,\theta)} \triangleq \mathcal{R}^{(\tau_0,\sigma_0,\theta)} \).

3 Variance Reduced SAPD

Since \( f \) is SCSC, the SP problem in (1) admits a unique saddle point \( z_* = (x_*, y_*) \). In the following result, we obtain convergence guarantees for the SAPD iterates \( (x_k, y_k) \) to the saddle-point in \( L^2 \). Recently, convergence results for very general choice of SAPD parameters are given in [Zhang et al. 2021]. Our result considers a specific choice of parameters, for which we can prove the existence of an invariant distribution for the iterates. For this specific choice of \( \tau, \sigma \) in \( \theta \), we were able to provide a sharper result for the expected distance square metric in terms of constants, compared to those available for SAPD in [Zhang et al. 2021].

Proposition 2. Let the parameters \((\tau, \sigma, \theta)\) of SAPD algorithm (2) be chosen as,
\[
\tau_0 \triangleq \frac{1 - \theta}{\mu_x \theta}, \quad \sigma_0 \triangleq \frac{1 - \theta}{\mu_y \theta}, \quad \theta \in [\hat{\theta}, 1),
\]
where \( \hat{\theta} \triangleq \max\{\hat{\theta}_1, \hat{\theta}_2\} \in (0, 1) \) and
\[
\hat{\theta}_1 \triangleq \frac{1}{1 + (\kappa_x + 4\kappa_y^2)^{-1}} < 1, \quad \hat{\theta}_2 \triangleq \frac{2}{\sqrt{(1 + \frac{1}{8\kappa_y^2})^2 + \frac{1}{\kappa_y^2}} + \frac{1}{\kappa_y^2}} < 1,
\]
where \( \kappa_x = L_{xx}/\mu_x, \kappa_y = L_{yy}/\mu_y \) and \( \kappa_{yx} = L_{yx}/\sqrt{\mu_x \mu_y} \), the iterates \( \{z_k\}_{k \in \mathbb{N}_+} \) satisfies,
\[
\mathbb{E}\left[\mu_x \|x_k - x_*\|^2 + \frac{\mu_y (1 + \theta)}{2} \|y_k - y_*\|^2\right] \leq \theta^k \Delta_0 + (1 - \theta^k) \bar{\lambda}_0 \delta_0^2,
\]
where \( z_k \triangleq [x_k^T, y_k^T] \), \( \Delta_0 \triangleq \mu_x \|x_0 - x_*\|^2 + \mu_y \|y_0 - y_*\|^2 \),
\[
\bar{\lambda}_0 \triangleq \frac{1}{\theta} \left( \frac{2}{\mu_x} + \frac{4}{\mu_y} \right)(1 + \theta^2 + \theta^2) \right).
\]
We are going to show that the invariant measure exists for the given choice of parameters in Proposition 2. One complication is that the Markov Chain corresponding to SAPD iterates becomes non-reversible due to the momentum term; this can also be seen from the fact that the matrices $M$ and $N$ in the representation (5) are non-symmetric. Consequently, standard tools for showing the existence of a stationary measure for reversible Markov chains are not applicable to our setting. However, the results in Hairer and Mattingly [2011] do not require reversibility and are applicable to our setting. More specifically, Hairer and Mattingly [2011] shows the existence of a stationary distribution for a Markov Chain provided that the so-called “minorization and drift” conditions hold with certain parameters (see the appendix for the details). In the following, we introduce an assumption on the gradient noise which says that the gradient noise admits a continuous density. This assumption allows us to show that the “minorization condition” holds.

**Assumption 3.** The gradient noise admits a continuous density, i.e., there is a density $p(\xi, z)dz$ such that $p(\xi, z)dz \equiv P(z_{k+1} \in dz | z_k = \xi)$ is continuous in $(\xi, z)$.

The minorization condition from [Hairer and Mattingly, 2011] for the transition kernel $R$ of SAPD iterations shown in (2) requires the existence of a Lyapunov function $V : \mathbb{R}^d \rightarrow [0, \infty)$ and constants $K \geq 0$ and $\zeta \in (0, 1)$ such that

$$V(x_k) \leq \zeta V(x) + K,$$

for every $x \in \mathbb{R}^d$. For this purpose, we devised the following Lyapunov function

$$V(x_k) = \frac{\theta}{1-\theta} \left( \frac{\mu_x}{4} \left( \| x_k - x^* \|^2 + \| x_{k-1} - x^* \|^2 \right) + \frac{\mu_y}{8} (1 + \theta) (\| y_k - y^* \|^2 + \| y_{k-1} - y^* \|^2) \right),$$

for which we can show that (8) holds for some $\zeta \in (0, 1)$ and $K \geq 0$ (see the appendix for more details). Then, building on the results of [Hairer and Mattingly, 2011], we establish in the following result that the distributions of SAPD iterates converge to the invariant distribution, and also give a rate result for this convergence provided that the variance of the gradient noise is not too large. The proof is deferred to the appendix.

**Theorem 3.** Consider the SAPD algorithm with parameters given by (6). Suppose that Assumptions 1, 2 and 3 hold, and the variance bound $\delta^2_{(2)}$ of the noise is small such that

$$(1 + \theta) \left( \frac{4}{\mu_x} + \frac{8}{\mu_y} (1 + \theta)^2 + 2^2 \right) \delta^2_{(2)} \leq R,$$

where $R$ is a constant specified in the Appendix. Then for any initialization $\xi_0 \in \mathbb{R}^d$, SAPD iterates admit a unique invariant measure $\pi_\theta(\cdot)$, i.e., $\lambda_{\xi_0} R_k(\theta)$, the distribution of $\xi_k(\theta)$, converges to $\pi_\theta(\cdot)$, where $\lambda_{\xi_0}$ denotes the Dirac distribution at $\xi_0$. Moreover, there exists $C > 0$ such that

$$\| R_k(\cdot) \psi - \pi_\theta(\cdot) \psi \| \leq C \left( \frac{2\theta}{1+\theta} \right)^k \| \psi - \pi_\theta(\cdot) \psi \|,$$

for any initialization $\xi_0$ and every measurable function $\psi$ such that $\| \psi \| < \infty$, where $\| \psi \| \triangleq \sup_{\xi} |\psi(\xi)|$ is the weighted supremum norm.

Since $\theta \in (\hat{\theta}, 1)$, by taking limit superior of both sides in (7) as $k \rightarrow \infty$, it can be seen that we have

$$\limsup_{k \rightarrow \infty} \mathbb{E}[\| z_k - z^* \|^2] = O(1 - \theta) \delta^2_{(2)}.$$

Since the distribution of the SAPD iterates $z_k$ converges to the stationary distribution based on Theorem 3, we can replace the limit superior with a limit to obtain

$$\mathbb{E}[\| z_\infty - z^* \|^2] = \lim_{k \rightarrow \infty} \mathbb{E}[\| z_k - z^* \|^2] = O(1 - \theta) \delta^2_{(2)},$$

with the convention that $z_\infty$ is distributed according to the stationary distribution. This is summarized in Lemma 4 below, where we also characterize the fourth moments of the stationary distribution as a function of $\theta$ as $\theta \rightarrow 1$. In particular, obtaining the fourth moments require a careful non-trivial Lyapunov analysis of the SAPD algorithm that involves higher-order interactions between the gradient noise and the iterates; the proof is deferred to the appendix.

**Lemma 4.** Under the premise of Theorem 3, $\mathbb{E}[\| z_\infty - z^* \|^2] = O(1 - \theta)$ and $\mathbb{E}[\| z_\infty - z^* \|^4] = O((1 - \theta)^2)$, where $z_\infty$ is a random variable distributed according to the stationary distribution of the SAPD iterates.

Building on this lemma, in the next result, we show that the expected iterates contain a bias that is $O(1 - \theta)$ as $\theta \rightarrow 1$. The bias stems mainly from the fact that we use constant primal and dual stepizes in SAPD as opposed to decaying stepizes. Our proof is based on relating the bias to the equilibrium covariance of the iterates and characterizing the equilibrium covariance as the solution to a set of coupled algebraic Lyapunov equations (whose solutions can then be approximated as $\theta \rightarrow 1$).
Theorem 5. Under the premise of Theorem \[ \text{5} \] the limit of expected SAPD iterate sequence, i.e.,
\[ \bar{\xi}^{(\theta)}(t) \triangleq \lim_{k \to \infty} E[\xi_k^{(\theta)}], \]
exists at stationarity such that
\[ \bar{\xi}^{(\theta)}(t) - \xi_s = (1 - \theta) \left[ \frac{\nabla^{(2)} f_s}{\nabla^{(3)} f_s} \right]^{-1} \left( \frac{\nabla^{(3)} f_s M}{\nabla^{(3)} f_s} \right) + \mathcal{O} \left( (1 - \theta)^{3/2} \right), \] (11)
as \( \theta \to 1 \) where \( \xi_s = (z_s, z_s) \). \( M \) is a fixed matrix with an explicit formula provided in the appendix, \( \nabla^{(2)} f_s \) is the Hessian matrix at the saddle point \( z_s \), and \( \nabla^{(3)} f \) is the tensor of order 3 that contains the third-order partial derivatives of \( f \) at \( z_s \).

Theorem \[ \text{5} \] shows that the expected iterates admit a bias that is \( \mathcal{O}(1 - \theta) \). Following a similar approach to Richardson-Romberg extrapolation techniques adopted for stochastic gradient descent methods for unconstrained optimization in Dieuleveut et al \[ \text{2017} \], our result in \[ (11) \] suggests that if we generate two SAPD sequence \( \{ \xi_k^{(\theta_1)} \} \) and \( \{ \xi_k^{(\theta_2)} \} \), then through appropriately forming a weighted sequence using the two, one can eliminate a significant portion of the bias. Indeed, if we choose the parameters as \( \theta_2 = 2\theta_1 - 1 \), then the from Theorem \[ \text{5} \] we obtain
\[ 2\bar{\xi}^{(\theta_1)} - \bar{\xi}^{(\theta_2)} = \xi_s + \mathcal{O} \left( (1 - \theta)^{3/2} \right), \]
i.e., the bias improves from \( \mathcal{O}(1 - \theta) \) to \( \mathcal{O} \left( (1 - \theta)^{3/2} \right) \). That being said, the variance of \( \{ 2\bar{\xi}^{(\theta_1)} - \bar{\xi}^{(\theta_2)} \}_{k} \) sequence could be larger than that of \( \{ \xi_k^{(\theta_1)} \} \). For controlling the variance, a standard idea in stochastic approximation methods is to consider the average of the iterates, i.e., consider \( \{ \bar{\xi}_k^{(\theta)} \}_{k \in \mathbb{N}^+} \) such that for \( k \geq 1 \),
\[ \bar{\xi}_k^{(\theta)} \triangleq \frac{1}{k} \sum_{i=0}^{k-1} \xi_i^{(\theta)} , \] (12)
Often, based on central limit theorem-type arguments, \( \bar{\xi}_k^{(\theta)} \) improves the variance of \( \xi_k^{(\theta)} \) due to its averaged nature.

Next, in the following result, we study the gap between the mean of the averaged iterate \( \bar{\xi}_k^{(\theta)} \) and \( \bar{\xi}^{(\theta)} \) defined in Theorem \[ \text{5} \]. The result shows that this gap decays polynomially fast in \( k \) and is of the order \( \mathcal{O}(1/k) \).

Theorem 6. Under the premise of Theorem \[ \text{5} \] let \( \xi_0 \) be the initialization of the SAPD iterates \( \{ \xi_k^{(\theta)} \}_{k \in \mathbb{N}} \). For any \( \xi_0 \) and \( k \geq 1 \),
\[ E[\bar{\xi}^{(\theta)}_k] = \bar{\xi}^{(\theta)} + \frac{1 + \theta}{1 - \theta} \left( 1 - \left( \frac{2\theta}{1 + \theta} \right)^k \right) \frac{C}{k} \| \xi_0 - \bar{\xi}^{(\theta)} \| , \] (13)
where \( C > 0 \) is the constant defined in Theorem \[ \text{5} \] and \( \bar{\xi}^{(\theta)}_k \) is the averaged iterate defined in \[ (12) \].

From the equations \[ (11) \] and \[ (13) \], we obtain that
\[ E[\bar{\xi}^{(\theta)}_k] - \xi_s = \frac{1 + \theta}{1 - \theta} \frac{C}{k} \| \xi_0 - \bar{\xi}^{(\theta)} \| + \eta \left[ \frac{\nabla^{(3)} f_s M}{\nabla^{(3)} f_s} \right]^{-2} \right] \left( \frac{\nabla^{(3)} f_s M}{\nabla^{(3)} f_s} \right) + \eta \theta , \] (14)
where \( \| \eta \| < C \|(1 - \theta)^{3/2} + \eta \| \) for some constants \( C_3, C_4 > 0 \). In particular, given that the error between \( E[\bar{\xi}^{(\theta)}_k] \) and \( \bar{\xi}^{(\theta)} \) is vanishing, we could apply the same variance reduction ideas to the averaged sequence \( \{ \bar{\xi}^{(\theta)}_k \} \) to reduce the variance. In the following section, we provide numerical examples to illustrate the benefits of our variance reduced SAPD method.

4 Numerical Experiments

In this section, we compare the performance of VR-SAPD with SAPD Zhang et al. \[ \text{2021} \], S-OGDA Fallah et al. \[ \text{2020} \], and SMP Juditsky et al. \[ \text{2011} \] on the \( \ell_2 \)-regularized distributionally robust learning problem
\[ \min \max_{x \in \mathcal{B}} f_0(x, y) \triangleq \frac{1}{2} \| x \|^2 + \sum_{i=1}^{n} y_i \phi_i(x) , \] (DRO)
where $S \triangleq \{ x \in \mathbb{R}^d : \| x \|^2 \leq D_x \}$ for some diameter $D_x > 0$ and $\phi_i : \mathbb{R}^d \to \mathbb{R}$ is a convex loss function corresponding to the $i$-th data point for $i = 1, \ldots, n$. While for the classic empirical risk minimization problem the weights are selected as $y_i = 1/n$ for every $i = 1, \ldots, n$, the robust formulation in (DRO) allows for $\{ y_i \}_{i=1}^n$ to come from an uncertainty set around the uniform weights $\mathcal{P}_r \triangleq \{ y \in \mathbb{R}_+^n : 1^\top y = 1, \| y - 1/n \|^2 \leq \frac{\epsilon}{D_y} \}$. whose radius is controlled by the parameter $r$, where $1$ is the vector of all ones --see the experimental section in Zhang et al. [2021] for more details.

The DRO formulation is affine in the variable $y$; hence, not strongly concave in $y$. However, in a similar spirit to Nesterov’s smoothing technique [Nesterov] [2013], we can introduce a dual regularizer to have a SCSC approximation to (DRO). This corresponds to approximating the original problem in the primal form that is non-smooth with a smooth primal optimization problem. Indeed, we will implement VR-SAPD algorithm on

$$
\min_{x \in S} \max_{y \in \mathcal{P}_r} f(x, y) \triangleq \frac{\mu}{2} \| x \|^2 + \sum_{i=1}^n y_i \phi_i(x) - \frac{\mu y}{2} \| y \|^2, \tag{Reg-DRO}
$$

where $\mu_y$ is chosen appropriately, i.e., $\mu_y = \Theta(\epsilon)$. In particular, if we choose $\mu_y = \frac{\epsilon}{2 D_y}$ where $D_y \triangleq \sup_{y \in \mathcal{P}_r} \| y \|^2 = 1$ for (Reg-DRO) and find a solution $(\bar{x}, \bar{y})$ satisfying $\mathbb{E}[\sup_{(x,y) \in S \times \mathcal{P}_r} \{ f(\bar{x}, \bar{y}) - f(x, y) \}] < \frac{\epsilon}{4}$, then it can be shown that $(\bar{x}, \bar{y})$ also satisfies $\mathbb{E}[\sup_{(x,y) \in S \times \mathcal{P}_r} \{ f_0(\bar{x}, \bar{y}) - f_0(x, y) \}] < \epsilon$. In the following, we consider the binary logistic loss function $\phi_i(x) = \log(1 + \exp(-b_i a_i^\top x))$ calculated at the $i$-th data point from $\{a_i, b_i\}_{i \in \{1, \ldots, n\}}$, and we set $r = 2\sqrt{n}$. Notice that the objective of (Reg-DRO) admits Lipschitz constants $L_{xy} = L_{yx} = \| A \|_2$, $L_{xx} = \max_{i=1,\ldots,n} \left\{ \frac{1}{4} \| a_i \|_2^2 \right\}$, and $L_{yy} = 0$ where $A \in \mathbb{R}^{n \times d}$ is the data matrix with rows $\{a_i\}_{i=1}^n$ and columns $\{A_j\}_{j=1}^d$. To stay feasible at each iteration, after performing $y_{k+1}$ and $x_{k+1}$ steps as in [2], we computed the Euclidean projection onto the constraint sets $S$ and $\mathcal{P}_r$. In our theoretical results, for the simplicity of the presentation, we considered SAPD without projection steps. However, our convergence results to the stationary distribution extend in a straightforward manner when such projection steps are included.

To illustrate our results from Section 3 we consider two sequences $\xi_k^{(\theta_1)}$ and $\xi_k^{(\theta_2)}$, both initialized with the same point $\xi_0$, where such that $\theta_2 = 2\theta_1 - 1$; and then consider the performance of the interpolated sequence $2\xi_k^{(\theta_1)} - \xi_k^{(\theta_2)}$.

![Figure 1: Comparison of S-OGDA, SMP, SAPD, and VR-SAPD on the Arcene data set in terms of the relative expected distance squared $\mathbb{E}[\| z_k - z_0 \|^2 / \| z_0 - z_* \|^2]$ as a performance metric.](image)

We consider the algorithms VR-SAPD, SAPD, S-OGDA, and SMP on three datasets:

---

[2] This is because the projection is a non-expansive operation [Bertsekas] [2015] which does not increase the distance to the solution and therefore the minorization and drift conditions will still hold for our distance-based Lyapunov function [2] after minor modifications to the proof of Theorem 4.
Figure 2: Comparison of S-OGDA, SMP, SAPD, and VR-SAPD on the Dry Bean data set in terms of the relative expected distance squared $E\|z_k - z_*\|^2/\|z_0 - z_*\|^2$ as a performance metric.

Figure 3: Comparison of S-OGDA, SMP, SAPD, and VR-SAPD on the MNIST data set in terms of the relative expected distance squared $E\|z_k - z_*\|^2/\|z_0 - z_*\|^2$ as a performance metric.

1. Dry Bean dataset [Koklu and Ozkan 2020] with $(n, d) = (9528, 16)$;
2. Arcene dataset [Guyon et al. 2004] with $(n, d) = (97, 10000)$;
3. MNIST dataset [LeCun et al. 1998] for hand-written digit classification with $(n, d) = (1000, 400)$.

Throughout the experiments, we set the regularization parameter $\mu_x$ through cross-validation and defined the parameters $(\tau_\theta, \sigma_\theta)$ of both VR-SAPD and SAPD as given in Proposition 2, tuning the parameter as $\theta = 0.95$. The gradients are estimated from mini-batches of data sampled at random with replacement where we have a batch size of 10. At Dry
Beans, we set $\mu_x = 0.01$, $\mu_y = 10$. We normalize each feature column $A_j \in \mathbb{R}^n$ using $A_j = \frac{A_j - \min(A_j)}{\max(A_j) - \min(A_j)}$, where max and min are taken over the elements of $A_j$. For the Arcene data set, we set $\mu_x = 0.02$ and $\mu_y = 10$ and normalize the data as $A \leftarrow A / \min(\sqrt{d}, \sqrt{n})$. Lastly, we set $\mu_x = 0.1$, $\mu_y = 10$ and mini-batch size 10 for MNIST. We set $b_i = 1$ if the i-th data point is number four and set $b_i = -1$, otherwise.

We generated two sequences $\{z_k^{(\theta_1)}\}_{k \in \mathbb{N}}$ and $\{z_k^{(\theta_2)}\}_{k \in \mathbb{N}}$ with $\theta_1 = \theta$ and $\theta_2 = 2\theta - 1$. For VR-SAPD, we considered the extrapolated sequence $z_k = 2z_k^{(\theta_1)} - z_k^{(\theta_2)}$. The step-size of S-OGDA and SMP are taken as $\frac{1}{L}$ and $\frac{1}{\sqrt{L}}$, respectively, where $L = \max\{L_{xx} + \mu_x, \mu_y, L_{xy}, L_{yx}\}$. We generated 50 random sample paths to compute the mean and the variance of $\{\|z_k - z_\ast\|^2\}_{k = 1}^K$ generated by the algorithms, each running for $K = 1000$ iterations, where we closely approximate $z_\ast$ by running the deterministic APD algorithm [Hamedani and Aybat 2021] until the change the loss function is below $\%0.01$. Our results for each dataset are summarized in Figures 1, 2 and 3 where we compared the methods in terms of the expected relative change in the distance squared, i.e., $\mathbb{E}[\|z_k - z_\ast\|^2/\|z_0 - z_\ast\|^2]$, as a performance metric —all the algorithms are initialized from $x_0 = 2 \mathbf{1}_{d_x}$ and $y_0 = 1/d_y \mathbf{1}_{d_y}$. For each method, we also display the standard deviation as the shaded region around the average of 50 sample paths. Our results show that our variance reduction techniques result in a smaller asymptotic variance for the SAPD iterates as intended. We also see that VR-SAPD iterates admit a smaller variance in general compared to SAPD and improves upon the other methods SMP and S-OGDA. These results showcase the benefits of our variance reduction techniques.

5 Conclusion

Under some assumptions, we showed that with constant stepsize, the distribution of the SAPD iterates admit a stationary distribution and characterized its dependency to the SAPD parameters. Based on these results, we introduced a variance-reduced stochastic accelerated primal-dual method (VR-SAPD), which is based on Richardson-Romberg extrapolation. We also illustrated the efficiency and benefits of our accelerated variance-reduced method on robust training of logistic regression models.

6 Acknowledgements

This work was funded by the grants ONR N00014-21-1-2244, ONR N00014-21-1-2271, NSF CCF-1814888 and NSF DMS-2053485. We thank Xuan Zhang for providing us with the MATLAB implementations of SAPD, S-OGDA and SMP for our numerical experiments.

References

Xuan Zhang, Necdet Serhat Aybat, and Mert Gürbüzbalaban. Robust Accelerated Primal-Dual Methods for Computing Saddle Points. arXiv e-prints, art. arXiv:2111.12743, November 2021.

Mert Gürbüzbalaban, Andrzej Ruszczyński, and Landi Zhu. A stochastic subgradient method for distributionally robust non-convex learning. arXiv preprint arXiv:2006.04873, 2020.

John C Duchi and Hongseok Namkoong. Learning models with uniform performance via distributionally robust optimization. The Annals of Statistics, 49(3):1378–1406, 2021.

Martin Arjovsky, Soumith Chintala, and Léon Bottou. Wasserstein generative adversarial networks. In Doina Precup and Yee Whye Teh, editors, Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pages 214–223. PMLR, 06–11 Aug 2017. URL https://proceedings.mlr.press/v70/arjovsky17a.html

Maher Nouiehed, Maziar Sanjabi, Tianjian Huang, Jason D Lee, and Meisam Razaviyayn. Solving a class of non-convex min-max games using iterative first order methods. Advances in Neural Information Processing Systems, 32:14934–14942, 2019.

Balamurugan Palaniappan and Francis Bach. Stochastic variance reduction methods for saddle-point problems. In Advances in Neural Information Processing Systems, pages 1416–1424, 2016.

Mingrui Liu, Hassan Rafique, Qihang Lin, and Tianbao Yang. First-order convergence theory for weakly-convex-weakly-concave min-max problems. Journal of Machine Learning Research, 22(169):1–34, 2021.

Kamalika Chaudhuri, Claire Monteleoni, and Anand D Sarwate. Differentially private empirical risk minimization. Journal of Machine Learning Research, 12(3), 2011.
Renbo Zhao. Accelerated stochastic algorithms for convex-concave saddle-point problems, 2021.

Alireza Fallah, Asuman Ozdaglar, and Sarath Pattathil. An optimal multistage stochastic gradient method for minimax problems. In 2020 59th IEEE Conference on Decision and Control (CDC), pages 3573–3579, 2020. doi:10.1109/CDC42340.2020.9304033

Avinava Dubey, Sashank J. Reddi, Barnabas Poczos, Alexander J Smola, Eric P Xing, and Sinead A Williamson. Variance reduction in stochastic gradient langevin dynamics. Advances in neural information processing systems, 29: 1154–1162, 2016.

Zhe Wang, Kaiyi Ji, Yi Zhou, Yingbin Liang, and Vahid Tarokh. Spiderboost and momentum: Faster variance reduction algorithms. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d’Alché-Buc, E. Fox, and R. Garnett, editors, Advances in Neural Information Processing Systems, volume 32. Curran Associates, Inc., 2019. URL https://proceedings.neurips.cc/paper/2019/file/512c5cad6c37edeb98ae91c8a76c3a291-Paper.pdf

Kiran Koshy Thekumpampill, Prateek Jain, Praneeth Netrapalli, and Sewoong Oh. Efficient algorithms for smooth minimax optimization. arXiv preprint arXiv:1907.01543, 2019.

Yunmei Chen, Guanghui Lan, and Yuyuan Ouyang. Accelerated schemes for a class of variational inequalities. Mathematical Programming, 165(1):113–149, 2017.

Yu-Guan Hsieh, Franck Iutzeler, Jérôme Malick, and Panayotis Mertikopoulos. On the convergence of single-call stochastic extra-gradient methods. arXiv preprint arXiv:1908.08465, 2019.

Erfan Yazdandoost Hamedani and Necdet Serhat Aybat. A primal-dual algorithm with line search for general convex-concave saddle point problems. SIAM Journal on Optimization, 31(2):1299–1329, 2021.

Aymeric Dieuleveut, Alain Durmus, and Francis Bach. Bridging the Gap between Constant Step Size Stochastic Gradient Descent and Markov Chains. arXiv e-prints, art. arXiv:1707.06386, July 2017.

Francis Bach. On the effectiveness of richardson extrapolation in data science. SIAM Journal on Mathematics of Data Science, 3(4):1251–1277, 2021. doi:10.1137/21M1397349. URL https://doi.org/10.1137/21M1397349

Tatjana Chavdarova, Gauthier Gidel, François Fleuret, and Simon Lacoste-Julien. Reducing noise in gan training with variance reduced extragradient. Advances in Neural Information Processing Systems, 32:393–403, 2019.

Ahmet Alacaoglu and Yura Malitsky. Stochastic variance reduction for variational inequality methods. arXiv preprint arXiv:2102.08352, 2021.

Anatoli Juditsky, Arkadii S. Nemirovskii, and Claire Tauvel. Solving variational inequalities with stochastic mirror-prox algorithm, 2011.

Vasilis Kontonis, Sihan Liu, and Christos Tzamos. Convergence and sample complexity of sgd in gans. arXiv preprint arXiv:2012.00732, 2020.

Martin Hairer and Jonathan C Mattingly. Yet another look at harris’ ergodic theorem for markov chains. In Seminar on Stochastic Analysis, Random Fields and Applications VI, pages 109–117. Springer, 2011.

Y. Nesterov. Introductory Lectures on Convex Optimization: A Basic Course. Springer Science & Business Media, 2013.

Dimitri Bertsekas. Convex optimization algorithms. Athena Scientific, 2015.

Murat Koklu and Ilker Ali Ozkan. Multiclass classification of dry beans using computer vision and machine learning techniques. Computers and Electronics in Agriculture, 174:105507, 07 2020. doi:10.1016/j.compag.2020.105507

Isabelle Guyon, Steve Gunn, Asa Ben-Hur, and Gideon Dror. Result analysis of the nips 2003 feature selection challenge. Advances in neural information processing systems, 17, 2004.

Yann LeCun, Léon Bottou, Yoshua Bengio, and Patrick Haffner. Gradient-based learning applied to document recognition. Proceedings of the IEEE, 86(11):2278–2324, 1998.

I Borno and Z Gajic. Parallel algorithm for solving coupled algebraic lyapunov equations of discrete-time jump linear systems. Computers & Mathematics with Applications, 30(7):1–4, 1995.

W Govaerts and JD Pryce. A singular value inequality for block matrices. Linear algebra and its applications, 125: 141–148, 1989.

Kemin Zhou, JC Doyle, and Keither Glover. Robust and optimal control. Control Engineering Practice, 4(8):1189–1190, 1996.
7 Appendix

7.1 Proof of Proposition 2

In order to prove Proposition 2, we first provide a technical lemma about a specific SAPD parameter selection, and next we give an adaptation of a convergence result from Zhang et al. [2021].

It has been shown in Remark 2.4 of [Zhang et al., 2021] that the SAPD parameters $\tau, \sigma > 0$ and $\theta \in (0, 1)$ are admissible if they satisfy the inequalities given in (15) for some $\alpha > 0$. In the next result, we show that our choice of parameters satisfy these inequalities for some $\alpha > 0$.

Lemma 7. The SAPD parameters $\tau, \sigma > 0$ and $\theta \in (0, 1)$ chosen as in (6) satisfy

$$\min \{ \tau \mu_x, \sigma \mu_y \} \geq \frac{1 - \theta}{\theta}, \quad \begin{bmatrix} \frac{1}{\tau} - L_{xx} & 0 & -L_{yx} \\ 0 & \frac{1}{\sigma} - \alpha & -L_{yy} \\ -L_{yx} & -L_{yy} & \frac{2\alpha^2}{\sigma} \end{bmatrix} \geq 0, \quad (15)$$

for $\alpha = \frac{1 - \theta}{2\theta} \in (0, \frac{1}{\sigma})$.

Proof. We show that when $\theta \in (0, 1)$ sufficiently close to 1, these inequalities hold for the specific parameter choice given in (6), i.e., $\tau = \frac{1 - \theta}{\mu_x \theta}$ and $\sigma = \frac{1 - \theta}{\mu_y \theta}$ for any $\theta \in [\hat{\theta}, 1)$, satisfies (15) when $\alpha = \frac{1 - \theta}{2\theta}$. With this choice of parameters, it is clear that $\tau$ and $\sigma$ satisfy the first inequality on the left-hand side of (15). It remains to prove that the matrices in (15) are also satisfied. If we write

$$\begin{bmatrix} \frac{1}{\tau} - L_{xx} & 0 & -L_{yx} \\ 0 & \frac{1}{\sigma} - \alpha & -L_{yy} \\ -L_{yx} & -L_{yy} & \frac{2\alpha^2}{\sigma} \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau} - L_{xx} & 0 & -L_{yx} \\ 0 & 0 & -L_{yy} \\ -L_{yx} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_{yy} \end{bmatrix}, \quad (16)$$

then it suffices to show that the eigenvalues of the matrices $G_1$ and $G_2$ are non-negative. The characteristic function of $G_1$ can be computed as

$$\det(G_1 - \lambda I_3) = -\lambda \left[ \lambda^2 - \lambda \left( \frac{1}{\tau} - L_{xx} + \frac{\alpha \theta}{2\theta} \right) + \frac{\alpha \theta}{2\theta} \left( \frac{1}{\tau} - L_{xx} \right) - L_{yx}^2 \right],$$

which has 3 roots, one being at $\lambda = 0$. Notice that $\theta \geq \hat{\theta}_1 \geq \frac{1}{1 + \frac{\mu_y}{\mu_x}}$ implies $0 \leq \frac{\mu_y}{1 - \theta} - L_{xx} = \frac{1}{\tau} - L_{xx}$. Moreover, since $\alpha = \frac{1 - \theta}{2\theta} = \frac{\mu_x}{2\mu_y}$, we have

$$\frac{\alpha \theta}{2\theta} \left( \frac{1}{\tau} - L_{xx} \right) - L_{yx}^2 = \frac{\mu_y}{4(1 - \theta)} \left[ \theta (\mu_x + L_{xx} + \frac{4L_{yx}^2}{\mu_y}) - (L_{xx} + \frac{4L_{yx}^2}{\mu_y}) \right],$$

which is non-negative for $\theta \geq \hat{\theta}_1$. This implies that the sum and the products of the roots of $\det(G_1 - \lambda I_3)$ are positive for $\theta \geq \hat{\theta}_1$, i.e we have $G_1 \succeq 0$ if $\theta \geq \hat{\theta}_1$. Similarly, we write the characteristic function of $G_2$,

$$\det(G_2 - \lambda I_3) = -\lambda \left[ \lambda^2 - \lambda \left( \frac{\alpha}{2\theta} + \frac{1}{\sigma} - \alpha \right) + \frac{\alpha}{2\theta} \left( \frac{1}{\sigma} - \alpha \right) - L_{yy}^2 \right],$$

which has 3 roots, one being at $\lambda = 0$. By a straightforward computation considering the trace of $G_2$, it follows that the sum of the roots of $\det(G_2 - \lambda I_3)$ is positive; therefore, the eigenvalues of $G_2$ are non-negative if the following inequality holds

$$0 \leq \frac{\alpha}{2\theta} \left( \frac{1}{\sigma} - \alpha \right) - L_{yy}^2 \leq \frac{\mu_y^2}{8(1 - \theta)} \left[ \theta^2 + \theta \left( 1 + \frac{8L_{yx}^2}{\mu_y^2} \right) - \frac{8L_{yx}^2}{\mu_y^2} \right].$$

The polynomial on the right-side is non-negative for $\theta \geq \hat{\theta}_2$ in which case we have $G_2 \succeq 0$. This proves that the conditions (15) hold for $\tau = \frac{1 - \theta}{\mu_x \theta}, \sigma = \frac{1 - \theta}{\mu_y \theta}$ and $\alpha = \frac{1 - \theta}{2\theta}$ as long as $\theta \geq \max(\hat{\theta}_1, \hat{\theta}_2)$. This completes the proof. □

Although some conclusions of [Theorem 2.1, Zhang et al., 2021] requires a compactness assumption on the domain of the objective, it is worth emphasizing that the particular result [ineq. (2.3), Zhang et al., 2021] is still valid even when the domain is unbounded. We will use this result to prove Proposition 2. For the sake of completeness, we state a simplified version of [Theorem 2.1, Zhang et al., 2021] below which is applicable to (1) where the domain is unbounded.
**Theorem 8** (Adaptation of [Theorem 2.1, Zhang et al. [2021]].) Suppose Assumption [1] and Assumption [2] hold. Let \( \{x_k, y_k\}_{k \geq 0} \) be the SAPD iterate sequence generated according to (2), using \( \tau, \sigma > 0 \) and \( \theta \in (0, 1) \) that satisfy (15) for some \( \alpha \in (0, \frac{1}{2}) \). Then the following inequality holds for any \( x_0 \in \mathbb{R}^{d_x} \) and \( y_0 \in \mathbb{R}^{d_y} \),

\[
E \left[ \frac{1}{2\tau} \|x_N - x_*\|^2 + \frac{1 - \alpha\sigma}{2\sigma} \|y_N - y_*\|^2 \right] \leq \frac{\theta N}{2} \left( \frac{1}{\tau} \|x_0 - x_*\|^2 + \frac{1}{\sigma} \|y_0 - y_*\|^2 \right) + \frac{\theta(1 - \theta)}{1 - \theta} \hat{\Xi}_{\tau, \sigma, \theta} \delta_{(2)},
\]

where the constant \( \hat{\Xi}_{\tau, \sigma, \theta} \triangleq \tau + 2\sigma ((1 + \theta)^2 + \theta^2) \).

**Proof.** The result immediately follows from [Theorem 2.1, Zhang et al. [2021]] and [Remark 2.4, Zhang et al. [2021]]. More precisely, in the proof of [Theorem 2.1, Zhang et al. [2021]], since \( (x^*, y^*) \) is a saddle point, setting \( x = x^* \) and \( y = y^* \) in [(A.17), (A.28) and (A.29), Zhang et al. [2021]] and adding \( \rho^{-N+1} \Delta_N (x^*, y^*) \) to both sides implies that

\[
\frac{1}{2\theta^2} \|x_N - x_*\|^2 + \frac{1 - \alpha\sigma}{2\sigma} \|y_N - y_*\|^2 \leq \theta N - 1 \left( \frac{1}{\tau} \|x_0 - x_*\|^2 + \frac{1}{\sigma} \|y_0 - y_*\|^2 + \sum_{k=0}^{N-1} \theta^{-k} F_k \right),
\]

where

\[
F_k \triangleq -\left\langle \nabla_x f(x_k, y_{k+1}) - \nabla_x f(x_k, y_k), x_k - x_* - \tau \nabla_x f(x_k, y_{k+1}) \right\rangle + \left\langle \tilde{q}_k - q_k, y_{k+1} - y_* \right\rangle,
\]

with

\[
q_k \triangleq (1 + \theta) \nabla_y f(x_k, y_k) - \theta \nabla_y f(x_{k-1}, y_{k-1}).
\]

Using the fact that \( y_{k+1} = y_k + \sigma \tilde{q}_k \) and by adding the term \( \tau \nabla_x f(x_k, y_{k+1}) \) to the dot product in (19) and subtracting the same term, we obtain

\[
E[F_k] = -E \left[ \left\langle \nabla_x f(x_k, y_{k+1}) - \nabla_x f(x_k, y_k), x_k - x_* - \tau \nabla_x f(x_k, y_{k+1}) \right\rangle \right]
+ \tau E \left[ \|\nabla_x f(x_k, y_{k+1}) - \nabla_x f(x_k, y_k)\|^2 \right] + E \left[ \left( \tilde{q}_k - q_k, y_k - y_* \right) \right] + \sigma E \left[ \|\tilde{q}_k - q_k\|^2 \right].
\]

By Assumption [2] the estimates \( \tilde{q}_k \) and \( \tilde{q}_k \) are unbiased estimates for \( \nabla_x f(x_k, y_{k+1}) \) and \( q_k \) and are independent from the iterates \( x_k, y_k \); therefore, the expectation of the inner products above are zero. This leads to,

\[
E[F_k] = \tau E \left[ \|\nabla_x f(x_k, y_{k+1}) - \nabla_x f(x_k, y_k)\|^2 \right] + \sigma E \left[ \|\tilde{q}_k - q_k\|^2 \right].
\]

Furthermore, we can bound the last term using the inequality \((a + b)^2 \leq 2(a^2 + b^2)\) as follows:

\[
E[\|\tilde{q}_k - q_k\|^2] = E[\| (1 + \theta) (\nabla_y f(x_k, y_k) - \nabla_y f(x_{k-1}, y_{k-1})) - \theta (\nabla_y f(x_k, y_k) - \nabla_y f(x_{k-1}, y_{k-1})) \|^2]
\leq 2 \left( (1 + \theta^2) E[\| \tilde{q}_k - q_k \|^2] + \theta^2 \| \nabla_y f(x_{k-1}, y_{k-1}) - \nabla_y f(x_{k-1}, y_{k-1}) \|^2 \right)
= 2(1 + \theta^2 + \theta^2) \delta_{(2)}^2.
\]

Thus,

\[
E[F_k] \leq (\tau + 2\sigma ((1 + \theta)^2 + \theta^2)) \delta_{(2)}^2, \quad \forall k \geq 0.
\]

Combining this bound with (18) gives the desired result.

The proof of Proposition [3] immediately follows from Lemma [7] and Theorem [8].

### 7.2 Proof of Theorem [3]

The proof is based on the Harris’ ergodic theorem for Markov chains. Particularly, Hairer and Mattingly have shown in [Hairer and Mattingly [2011]] that if the following Condition [1] and Condition [2] hold for a transition kernel \( P_{\tau, \sigma, \theta} \) of a Markov chain \( \{\xi_k^{(\tau, \sigma, \theta)}\} \in \mathbb{R}^{d_x} \), then the Markov chain admits a unique invariant measure (Theorem 2.1 of [Hairer and Mattingly [2011]]). Throughout the proof, we drop \( \theta \) dependency on the notation and set \( \xi_k^{(\tau, \sigma, \theta)} \rightarrow \xi_k \) to denote the Markov chain \( \{\xi_k^{(\tau, \sigma, \theta)}\} \) generated by SAPD, for simplicity.

\footnote{This conditions are also known as the drift and minorization conditions.}
**Condition 1.** There exists a function $\mathcal{V} : \mathbb{R}^{2d} \rightarrow [0, \infty]$ and constants $K \geq 0$ and $\zeta \in (0, 1)$ such that

$$\mathcal{R}^{(\tau, \sigma, \theta)}(\xi) \leq \xi \mathcal{V}(\xi) + K, \quad \forall \xi \in \mathbb{R}^{2d}.$$  

**Condition 2.** There exists a constant $\varphi \in (0, 1)$ and a probability measure $\nu$ on $\mathcal{B}(\mathbb{R}^{2d})$ such that

$$\inf_{\xi \in \mathbb{R}^{2d}} \mathcal{R}^{(\tau, \sigma, \theta)}(\xi, A) \geq \varphi \nu(A), \quad \forall A \in \mathcal{B}(\mathbb{R}^{2d}),$$

for some $R > 2K/(1 - \zeta)$, where $C_R \triangleq \{ \xi \in \mathbb{R}^{2d} : \mathcal{V}(\xi) \leq R \}$. $K$ and $\zeta$ are the constants from Condition 1.

Our proof relies on showing that the Markov kernel associated with the SAPD iterations satisfies Condition 1 and Condition 2 with appropriate constants $R$, $K$ and $\zeta$, which will imply that the Markov chain admits a unique invariant measure based on Hairer and Mattingly (2011). Firstly, we show the Condition 1 is satisfied for $\mathcal{R}^{(\tau, \sigma, \theta)}$ using Theorem 8.

We first observe that the inequality (17) is valid for any initialization $(x_0, y_0)$. Secondly, this inequality holds if the expectation is conditioned on the previous iterate since the noise on the gradient is independent from the iterates. Consequently, we obtain the following inequalities:

$$E \left[ \frac{1}{2\tau} \|x_2 - x_*\|^2 + \frac{1}{2\tau} \|y_2 - y_*\|^2 \mid (x_1, y_1) \right] \leq \frac{\theta}{2} \left[ \frac{1}{2\tau} \|x_1 - x_*\|^2 + \frac{1}{\sigma} \|y_1 - y_*\|^2 \right] + \theta \tilde{\Xi}_{\tau, \sigma, \theta} \delta(2),$$

(23a)

$$E \left[ \frac{1}{2\tau} \|x_1 - x_*\|^2 + \frac{1}{2\tau} \|y_1 - y_*\|^2 \mid (x_0, y_0) \right] \leq \frac{\theta}{2} \left[ \frac{1}{2\tau} \|x_0 - x_*\|^2 + \frac{1}{\sigma} \|y_0 - y_*\|^2 \right] + \theta \tilde{\Xi}_{\tau, \sigma, \theta} \delta(2),$$

(23b)

In Lemma 9, we provide the conditions for which $\mathcal{R}^{(\tau, \sigma, \theta)}$ satisfies Condition 1.

**Lemma 9.** Suppose Assumptions 7 and 2 hold and the SAPD parameters $\tau, \sigma > 0$ and $\theta \in (0, 1)$ satisfy the matrix inequality in (15) for some $\alpha \in (0, \frac{1}{\sigma})$ such that $1 - \alpha \sigma > \theta$. Let

$$\mathcal{V}_\alpha^{(\tau, \sigma, \theta)}(\xi_k) \triangleq \frac{1}{4\tau} \left( \|x_k - x_*\|^2 + \|x_{k-1} - x_*\|^2 + \frac{1}{1 - \alpha \sigma} (\|y_k - y_*\|^2 + \|y_{k-1} - y_*\|^2) \right).$$

Then the Markov kernel $\mathcal{R}^{(\tau, \sigma, \theta)}$ of SAPD satisfies the following inequality for any $\xi \in \mathbb{R}^{2d}$:

$$\mathcal{R}^{(\tau, \sigma, \theta)}(\mathcal{V}_\alpha^{(\tau, \sigma, \theta)})(\xi) \leq \xi \mathcal{V}_\alpha^{(\tau, \sigma, \theta)}(\xi) + K,$$

(24)

with the constants $\zeta$ and $K$ chosen as

$$\zeta = \frac{\theta}{1 - \alpha \sigma}, \quad K = \theta \tilde{\Xi}_{\tau, \sigma, \theta} \delta(2).$$

(25)

**Proof.** The desired result immediately follows from the inequalities in (23). Indeed, we have

$$E[\mathcal{V}_\alpha^{(\tau, \sigma, \theta)}(\xi_2) \mid \xi_1] = E \left[ \frac{1}{4\tau} \left( \|x_2 - x_*\|^2 + \|x_1 - x_*\|^2 \right) \right] \leq \frac{\theta}{4} \left[ \frac{1}{\tau} \|x_1 - x_*\|^2 + \frac{1}{\tau} \|y_1 - y_*\|^2 \right] + \theta \tilde{\Xi}_{\tau, \sigma, \theta} \delta(2).$$

Thus, since $1 - \alpha \sigma \in (0, 1)$, we obtain

$$E[\mathcal{V}_\alpha^{(\tau, \sigma, \theta)}(\xi_2) \mid \xi_1] \leq \frac{\theta}{1 - \alpha \sigma} \mathcal{V}_\alpha^{(\tau, \sigma, \theta)}(\xi_1) + \theta \tilde{\Xi}_{\tau, \sigma, \theta} \delta(2).$$

Because the noise is stationary, the Markov chain is time-homogeneous; hence, the operator $\mathcal{R}^{(\tau, \sigma, \theta)}$ is the same throughout the iterations which gives the desired result.

Based on Lemma 9, in order to show Condition 2, it is sufficient to show that given $\varphi \in (0, 1)$, there exists a measure $\tilde{\nu}$ defined on the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^{2d})$ satisfying the inequality

$$\inf_{\xi \in \mathbb{R}^{2d}} \{ p(\xi, B) : \mathcal{V}_\alpha^{(\tau, \sigma, \theta)}(\xi) \leq R \} \geq \varphi \tilde{\nu}(B), \quad \forall B \in \mathcal{B}(\mathbb{R}^{2d}),$$

(26)

for some

$$R > 2\theta \delta(2) \tilde{\Xi}_{\tau, \sigma, \theta} \left( 1 - \frac{\theta}{1 - \alpha \sigma} \right).$$

14
This is because if such a measure \( \nu \) defined on \( B(\mathbb{R}^d) \) exists, then we can always define a measure \( \nu \) on \( B(\mathbb{R}^{2d}) \) satisfying Condition 1. Based on the product measure. More specifically, any Borel set in \( B(\mathbb{R}^{2d}) \) is of the form \( B \times C \) where \( B, C \in B(\mathbb{R}^d) \) are Borel sets in \( \mathbb{R}^d \). Introducing the set

\[
D_R \triangleq \left\{ z \in \mathbb{R}^d : \exists z_0 \in \mathbb{R}^d \quad \text{s.t.} \quad \mathcal{V}_\alpha^{(\tau,\sigma,\theta)}(\xi) \leq R \quad \text{for} \quad \xi = \left[ z^T, z_0^T \right]^T \right\}.
\]

It can be seen that if (26) holds for \( \tilde{\nu} \), then we can define the measure

\[
\nu(B \times C) \triangleq \begin{cases} \tilde{\nu}(B) & \text{if} \; C \subseteq D_R \\ 0 & \text{otherwise}, \end{cases}
\]

which will satisfy the minorization condition, i.e., Condition 2 on the Markov kernel of \( \{\xi_k\} \).

In the next lemma, we show that if the distribution function is continuous then for any given constant \( \varphi \in (0, 1) \), we can find a probability measure \( \tilde{\nu} \) and a set of the form \( \{ \xi \in \mathbb{R}^{2d} : \mathcal{V}_\alpha^{(\tau,\sigma,\theta)}(\xi) \leq R \} \) for some \( R > 0 \) on which the inequality \( p(\xi, A) \geq \varphi \tilde{\nu}(A) \) is satisfied for any \( A \in B(\mathbb{R}^d) \).

**Lemma 10.** Under Assumption 3 for any \( \varphi \in (0, 1) \) fixed, there exists a positive constant \( R > 0 \) and a probability measure \( \tilde{\nu} \) on \( B(\mathbb{R}^d) \) such that for every \( z \in \mathbb{R}^d \),

\[
\inf_{\xi \in \mathbb{R}^{2d}} \left\{ p(\xi, z) : \mathcal{V}_\alpha^{(\tau,\sigma,\theta)}(\xi) \leq R \right\} \geq \varphi \tilde{\nu}(z).
\]

**Proof.** We consider the probability measure

\[
\tilde{\nu}(z) \triangleq p(\xi_*, z) \frac{1_{||z - z_*|| \leq M}}{\int_{z: ||z - z|| \leq M} p(\xi_*, z)dz},
\]

where \( 1_{||z - z_*|| \leq M} = 1 \) if \( ||z - z_*|| \leq M \) and 0 otherwise. Suppose \( M > 0 \) is chosen such that the denominator is not zero. Notice that for sufficiently large \( M \), the integral at the denominator can be arbitrarily close to 1. We will show that for any \( \varphi \in (0, 1) \), there exists positive constants \( M \) and \( R \) such that (27) holds for every \( z \) for this choice of \( \tilde{\nu} \). In particular, given \( \varphi \in (0, 1) \), we simply take

\[
M \triangleq \inf \left\{ m > 0 : \int_{z: ||z - z|| \leq m} p(\xi_*, z)dz \geq \sqrt{\varphi} \right\},
\]

and

\[
R \triangleq \sup \left\{ r > 0 : \inf_{\xi \in \mathbb{R}^{2d} : \mathcal{V}_\alpha^{(\tau,\sigma,\theta)}(\xi) \leq r} p(\xi, z) \geq \sqrt{\varphi} p(\xi_*, z), \quad \forall z : ||z - z|| \leq M \right\},
\]

then, by definition of \( M \) and \( R \), it can be seen that for every \( z \) satisfying \( ||z - z|| \leq M \),

\[
\inf_{\xi \in \mathbb{R}^{2d} : \mathcal{V}_\alpha^{(\tau,\sigma,\theta)}(\xi) \leq R} p(\xi, z) \geq \varphi \tilde{\nu}(z);
\]

otherwise, for any \( z \) such that \( ||z - z|| > M \), we have \( \tilde{\nu}(z) = 0 \), and (28) holds trivially. This implies (27) and we conclude. \( \square \)

The Proposition 11 utilizes the results obtained at Lemma 9 and Lemma 10 in order to prove the existence of the invariant measure.

**Proposition 11.** Under the premise of Lemma 9 suppose \( \tau, \sigma > 0 \) and \( \theta \in (0,1) \) satisfy the matrix inequality in (15) for some \( \alpha \in (0, \frac{1}{\sigma}) \) such that \( 1 - \alpha \sigma > \theta \). Let \( \varphi \in (0, 1) \) and \( M > 0 \) be such that \( \int_{z : ||z - z|| \leq M} p(\xi_*, z)dz \geq \sqrt{\varphi} \) and \( R > 0 \) be such that

\[
\inf_{\xi \in \mathbb{R}^{2d}, z \in \mathbb{R}^d} \left\{ \frac{p(\xi, z)}{p(\xi_*, z)} : \mathcal{V}_\alpha^{(\tau,\sigma,\theta)}(\xi) \leq R, \quad ||z - z|| \leq M \right\} > \sqrt{\varphi}.
\]

If the SAPD parameters \( (\tau, \sigma, \theta) \) and the noise variance \( \delta_{(2)}^2 \) satisfy

\[
2\delta_{(2)}^2 \tilde{\Xi}^{(\tau,\sigma,\theta)} \left( \frac{1 - \theta}{1 - \alpha \sigma} \right)^{-1} \leq R,
\]

then
where $\tilde{\Xi}_{\tau,\sigma,\theta}$ is defined in Lemma 9, then the SAPD iterates starting from initialization $\xi_0 \in \mathbb{R}^{2d}$ admit a unique invariant measure $\pi_{*}(\tau,\sigma,\theta)$, i.e., the distribution of $\xi_k$, $\lambda_{\xi_0} R_k^{(\theta)}$, converges to $\pi_{*}$ where $\lambda_{\xi_0}$ denotes the Dirac distribution at $\xi_0$. Moreover, there exists $C > 0$ such that

$$
\|R_k^{(\tau,\sigma,\theta)} \psi - \pi_{*}(\tau,\sigma,\theta)(\psi)\| \leq C \left(\frac{\theta}{1 - \alpha \sigma}\right)^k \|\psi - \pi_{*}(\tau,\sigma,\theta)(\psi)\|,
$$

(31)

for every measurable function $\psi$ such that $\|\psi\| < \infty$, where $\|\psi\| = \sup_{\xi} \frac{|\psi(\xi)|}{1 + \|\psi(\tau,\sigma,\theta)(\xi)\|}$ is the weighted supremum norm.

Proof. Since $1 - \alpha \sigma > \theta$, from Lemma 9 we get Condition 1, i.e.,

$$
R(\tau,\sigma,\theta) Y(\tau,\sigma,\theta)(\xi) \leq \frac{\theta}{1 - \alpha \sigma} Y(\tau,\sigma,\theta)(\xi) + \theta \tilde{\Xi}_{\tau,\sigma,\theta}^2(\xi).
$$

Therefore, when (30) holds, Lemma 10 implies Condition 2, i.e.,

$$
\inf_{\xi \in \mathbb{R}^{2d}} \{p(\xi, z) : Y(\tau,\sigma,\theta)(\xi) \leq R\} \geq \varphi(\tilde{\nu}(z)).
$$

Therefore, we can conclude from [Theorem 1.2., Hairer and Mattingly (2011)] that there exists an invariant measure $\pi_{*}(\tau,\sigma,\theta)$ to which $R_k^{(\tau,\sigma,\theta)}$ converges according to (31). □

Remark 1. In Proposition 17, the constant $C$ is not explicitly available. However, explicit constants can be provided if the convergence is analyzed in a different metric. In particular, if the parameters satisfy the inequality $1 > \frac{\theta}{1 - \alpha \sigma} + \frac{2\theta \tilde{\Xi}_{\tau,\sigma,\theta}^2}{R} \triangleq \gamma_{R,\delta,\theta}$, then for any $\varphi_0 \in (0, \varphi)$,

$$
\|R_k^{(\tau,\sigma,\theta)} \psi - \pi_{*}(\tau,\sigma,\theta)(\psi)\| \leq \hat{p} \|\psi - \pi_{*}(\tau,\sigma,\theta)(\psi)\|,
$$

holds with $\hat{p} \triangleq 1 - \min \left\{\varphi - \varphi_0, \frac{R \beta}{2 + R \beta} (1 - \gamma_{R,\delta,\theta})\right\}$ and $\beta = \frac{2 \varphi_0}{\theta \tilde{\Xi}_{\tau,\sigma,\theta}^2(\xi)}$ for every measurable function $\psi$ such that $\|\psi\| < \infty$, where $\|\psi\| \triangleq \inf_{\xi \in \mathbb{R}} \|\psi + c\|_\beta$ and $\|\psi\|_\beta \triangleq \sup_{\xi \in \mathbb{R}^{2d}} |\psi(\xi)| / |1 + \|\psi(\tau,\sigma,\theta)(\xi)\||$ is the $\beta$-weighted supremum norm.

Lastly, notice the choice of parameters given in (6) together with $\alpha = \frac{1 - \theta}{2 \sigma}$ satisfy the condition $1 - \alpha \sigma = 1 - \frac{1 - \theta}{2} \geq \theta$. Therefore, we conclude that the assumption of Lemma 9 holds with $\Xi_0 \triangleq \Xi_{\tau,\sigma,\theta}$; hence, Theorem 5 directly follows from the Proposition 11.

7.3 Proof of Lemma 4

Throughout the proof, $E[\|z_k - z_*\|^p]$ and $E[\|z - z_*\|^p]$ denote the $p$-th moments of $z_k - z_*$ when $z_k \sim \lambda R_k$, where $z_0 \sim \lambda$ and $z_{-1} \triangleq [z_{-1}^T, y_{-1}^T]^T \sim \lambda$ for $\lambda$ being the initial distribution, and the $p$-th moment of $z - z_*$ when $z \sim \pi_*$ the invariant measure, respectively.

We first recall the inequality in (18) from the proof of Theorem 8, i.e.,

$$
\frac{1}{2\theta} \|x_N - x_*\|^2 + \frac{1 - \alpha \sigma}{2\theta \sigma} \|y_N - y_*\|^2 \leq \Theta^{N-1} \left(\frac{1}{2\tau} \|x_0 - x_*\|^2 + \frac{1}{2\sigma} \|y_0 - y_*\|^2 + \sum_{k=0}^{N-1} \theta^{-k} F_k\right),
$$

(32)

where $F_k$ is defined in (19).

This inequality holds for all $N \geq 1$ whenever $\tau, \sigma > 0$ and $\theta \in (0, 1)$ satisfy (15) for some $\alpha \in [0, \frac{1}{2}]$. Since Proposition 2 shows that choosing $(\tau, \sigma, \theta)$ as in (6) and $\alpha = \frac{1 - \theta}{2 \sigma} < \frac{1}{2}$ satisfies (15), we can conclude that for all $N \geq 1$, (32) is valid for $(\tau, \sigma, \theta)$ as in (6) and $\alpha = \frac{1 - \theta}{2 \sigma}$, which yields the following inequality for $N = 1$,

$$
\frac{\mu_x}{2} \|x_1 - x_*\|^2 + \frac{\mu_y}{4} (1 + \theta) \|y_1 - y_*\|^2 \leq \frac{\mu_x}{2} \|x_0 - x_*\|^2 + \frac{\mu_y}{4} \|y_0 - y_*\|^2 + (1 - \theta) F_0.
$$

(33)

Notice that the inequality $\theta < \frac{2\theta}{1 + \theta}$ holds for all $\theta < 1$; therefore, (33) implies that

$$
\frac{\mu_x}{2} \|x_1 - x_*\|^2 + \frac{\mu_y}{4} (1 + \theta) \|y_1 - y_*\|^2 \leq \left(\frac{2\theta}{1 + \theta}\right) \left(\frac{\mu_x}{2} \|x_0 - x_*\|^2 + \frac{\mu_y}{4} (1 + \theta) \|y_0 - y_*\|^2\right) + (1 - \theta) F_0.
$$

(34)
The selection of the parameters $\theta \in (0,1)$ and $\tau, \sigma > 0$ in (1) imply that $\tau = O(1 - \theta)$ and $\sigma = O(1 - \theta)$. Furthermore, Assumption 2 says that the gradient estimates are conditionally independent from the sequence $\{x_k, y_k\}$ generated by SAPD; therefore, if we take the expectation of $F_0$, it follows from (22) that
\[
\mathbb{E}[F_0] \leq \left( \tau + 2\sigma (1 + \theta)^2 + \theta^2 \right) \delta_0^2 = O\left( (1 - \theta) \delta_0^2 \right),
\] (35)
with the convention that $z_0 \sim \pi_\ast$ and $z_{-1} \sim \pi_\ast$ are random variables drawn from invariant measure $\pi_\ast$. Since $\pi_\ast$ is the invariant measure, we have $z_1 \sim \pi_\ast$ as well and therefore $\mathbb{E}[\|x_1 - x_*\|^2] = \mathbb{E}[\|x_0 - x_*\|^2]$ and $\mathbb{E}[\|y_1 - y_*\|^2] = \mathbb{E}[\|y_0 - y_*\|^2]$; moreover, Theorem 3 implies that these quantities are finite. Let $\mu \triangleq \min\{\mu_x, \mu_y\}$. Hence, taking expectation of both sides of (34) with respect to the invariant measure $\pi_\ast$ and using (35) yields the desired result
\[
\mathbb{E}[\|z - z_*\|^2] \leq \frac{4}{\mu} \mathbb{E}[F_0] = O\left( (1 - \theta) \delta_0^2 \right). \tag{36}
\]
Next, we will show $\mathbb{E}[\|z - z_*\|^4] = O\left( (1 - \theta)^2 \right)$. For this purpose, we are going to bound $|F_0|$. Let $\tilde{y}_k \triangleq y_{k+1}$, and consider $F_k$ and $q_k$ defined in (19) and (20). Notice that the Cauchy-Schwarz and triangular inequalities yield for any $k \geq 0$,
\[
|F_k| \leq \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\| \|x_k - x_* - \tau \nabla_x f(x_k, \tilde{y}_k)\| + \tau \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\|\|\nabla_y f(x_k, \tilde{y}_k)\| + \tau \|\nabla_y f(x_k, \tilde{y}_k) - \nabla_y f(x_k, y_k)\|\|\nabla_x f(x_k, \tilde{y}_k)\| + \tau \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\|\|\nabla_y f(x_k, \tilde{y}_k)\|
\]
\[
\leq \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\| \|x_k - x_*\| + \tau \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\|\|\nabla_y f(x_k, \tilde{y}_k)\| + \tau \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\|\|\nabla_y f(x_k, \tilde{y}_k)\|
\]
\[
+ \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\| \|\nabla_y f(x_k, \tilde{y}_k)\| + \tau \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\|\|\nabla_y f(x_k, \tilde{y}_k)\|.
\] (37)
Next, we derive some bounds that will be used later to upper bound $\mathbb{E}[|F_0|^2]$. First, using $\nabla f(x_*, y_*) = 0$ and the definition of $q_k$ from (20), we get
\[
\|q_k\| = \|(1 + \theta) (\nabla_y f(x_k, y_k) - \nabla_y f(x_*, y_*)) - \theta (\nabla_y f(x_k, y_k) - \nabla_y f(x_*, y_*))\| = O\left( \max_{i \in \{k,k-1\}} \|z_i - z_*\| \right),
\] (38)
where we used Assumption 1 for the last equality. Furthermore, recalling the definition of $\tilde{q}_k$ from (2), we also have
\[
\|\tilde{q}_k - q_k\| = O\left( \max_{i \in \{k,k-1\}} \{\|\nabla_y f(x_i, y_i) - \nabla_y f(x_i, y_k)\|\} \right),
\] (39)
together with the fact that $y_{k+1} = y_k + \sigma \tilde{q}_k$ implies that
\[
\|\tilde{y}_k - y_*\| \leq \|y_k - y_* + \sigma q_k\| + \sigma \|\tilde{q}_k - q_k\|
\]
\[
= O\left( \max_{i \in \{k,k-1\}} \{\|z_i - z_*\|\} \right) + \sigma O\left( \max_{i \in \{k,k-1\}} \{\|\nabla_y f(x_i, y_i) - \nabla_y f(x_i, y_k)\|\} \right),
\]
where we used (38). Consequently, we also get
\[
\|\nabla_x f(x_k, \tilde{y}_k)\| = \|\nabla_x f(x_k, \tilde{y}_k) - \nabla_x f(x_k, y_k)\|
\]
\[
= O\left( \max_{i \in \{k,k-1\}} \{\|z_i - z_*\|\} \right) + \sigma O\left( \max_{i \in \{k,k-1\}} \{\|\nabla_y f(x_i, y_i) - \nabla_y f(x_i, y_k)\|\} \right),
\] (40)
where we again used Assumption 1 in the last equality.

Suppose the distribution of $\xi_0 = (z_0, z_{-1})$ is the stationary distribution of the Markov chain, where $z_0 = (x_0, y_0)$ and $z_{-1} = (x_{-1}, y_{-1})$. Setting $k = 0$ in (37) and using the inequality $\sum_{i=1}^N \gamma_i^2 \leq N \sum_{i=1}^N \gamma_i^2$ for any $\gamma_i \in \mathbb{R}^N$, we can bound $\mathbb{E}[|F_0|^2]$ based on the tower law of expectation and the fact that gradient noise is conditionally independent given the gradient arguments as follows:
\[
\mathbb{E}[|F_0|^2] \leq 6 \delta_0^2 \mathbb{E}[\|x_0 - x_*\|^2] + 6\tau^2 \delta_0^2 \mathbb{E}[\|\nabla_x f(x_0, \tilde{y}_0)\|^2] + 6\tau^2 \delta_0^4
\]
\[
+ 6(1 + \theta)^2 \delta_0^2 \mathbb{E}[\|y_0 - y_*\|^2] + 6\tau^2 (1 + \theta)^2 \delta_0^2 \mathbb{E}[\|q_0\|^2] + 6\sigma^2 \delta_0^4.
\]
Thus, combining the terms, we get
\[
\mathbb{E}[|F_0|^2] \leq 6\left[ \mathbb{E}[\|z - z_*\|^2] + \tau^2 \mathbb{E}[\|\nabla_x f(x_0, \tilde{y}_0)\|^2] + \sigma^2 \mathbb{E}[\|q_0\|^2] + (\tau^2 + \sigma^2) \delta_0^4 \right].
\]
Using (35), (38) and (40) together with $\tau = O(1 - \theta)$ and $\sigma = O(1 - \theta)$, we get
\[
\mathbb{E}[|F_0|^2] = O\left( \mathbb{E}[\|z - z_*\|^2] \right) + O\left( (1 - \theta)^2 \right) + O\left( (1 - \theta)^2 \mathbb{E}[\|z - z_*\|^2] \right) = O(1 - \theta),
\]
after neglecting the terms with second or higher orders of \((1 - \theta)\) as \(\theta \to 1\).

On the other hand, for \(\tau = \frac{1 - \theta}{\mu_x}, \delta = \frac{1 - \theta}{\mu_y}, \alpha = \frac{1 - \theta}{2\delta}, \) and \(\bar{\mu} = \min\{\mu_x, \mu_y\},\) \[\tag{32}\] implies that

\[
\frac{\bar{\mu}}{2} \|z_N - z_*\|^2 \leq \theta^N \left( \mu_x \|x_0 - x_*\|^2 + \mu_y \|y_0 - y_*\|^2 \right) + 2\theta^{N-1}(1 - \theta) \sum_{k=0}^{N-1} \theta^{-k} F_k,
\]

for all \(N > 0\). Therefore taking the square of both sides of the inequality above yields

\[
\frac{\bar{\mu}^2}{4} \|z_N - z_*\|^4 \leq 2\theta^{2N} \left( \mu_x \|x_0 - x_*\|^2 + \mu_y \|y_0 - y_*\|^2 \right)^2 + 4\theta^{2N-2}(1 - \theta)^2 \left( \sum_{k=0}^{N-1} \theta^{-k} F_k \right)^2.
\]

We can use Cauchy-Schwarz inequality to get \((\sum_{k=0}^{N-1} \theta^{-k} F_k)^2 \leq \left( \sum_{k=0}^{N-1} \theta^{-k} \right) \left( \sum_{k=0}^{N-1} \theta^{-k} F_k^2 \right)\) implying

\[
\frac{\bar{\mu}^2}{4} \mathbb{E}[\|z_N - z_*\|^4] \leq 2\theta^{2N} \left( \mu_x \|x_0 - x_*\|^2 + \mu_y \|y_0 - y_*\|^2 \right)^2 + 4\theta^{2N}(1 - \theta)^2 \mathbb{E}[\sum_{k=0}^{N-1} \theta^{-k} F_k^2].
\]

This inequality holds for every initialization \((x_0, y_0)\). Notice that the inequality \((37)\) holds in a.s. sense for any \(k \geq 0\); hence, \((38)\) implies that \(\mathbb{E}[||F_k||^2] = O(\mathbb{E}[\max_{k \leq k_N} \|z_k - z_*\|^2])\). Therefore, if we assume the initial distribution of the Markov chain \(\{x_k, y_k\}_{k \in \mathbb{N}}\) is Dirac measure at \((x_0, y_0)\), then taking the expectation of both sides gives the following inequality for each \(N \geq 0,\)

\[
\frac{\bar{\mu}^2}{4} \mathbb{E}[\|z_N - z_*\|^4] = 2\theta^{2N} \left( \mu_x \|x_0 - x_*\|^2 + \mu_y \|y_0 - y_*\|^2 \right)^2 + 4\theta^{2N}(1 - \theta)^2 \mathbb{E}[\max_{k \leq k_N} \|z_k - z_*\|^2].
\]

On the other hand, we know that \(\{\mathbb{E}[\|z_k - z_*\|^2]\}_{k \in \mathbb{N}}\) is a uniformly bounded sequence from Theorem 3, therefore, Theorem 3 suggests \(\mathbb{E}[\|z_N - z_*\|^4] \to \mathbb{E}[\|z_\infty - z_*\|^4]\) as \(N \to \infty\) with \(z_\infty\) being distributed according to the stationary distribution. This also implies \(\mathbb{E}[\|z_\infty - z_*\|^4]\) is bounded. In the rest of the proof, to show \(\mathbb{E}[\|z_\infty - z_*\|^4] = O((1 - \theta)^2)\), we follow a similar approach we used above to derive a bound on the 2-norm of \(z - z_*\) when \(z \sim \pi_*\) follows the stationary distribution, i.e., \(\mathbb{E}[\|z_\infty - z_*\|^2]\).

Firstly, we define the function \(V(z) \triangleq \frac{\mu_x}{2} \|x - x_*\|^2 + \frac{\mu_y(1 + \theta)}{4} \|y - y_*\|^2\) for \(z = (x, y) \in \mathbb{R}^d\), which satisfies the following inequality due to \((33)\):

\[
V^2(z_1) \leq \left( \frac{2\theta}{1 + \theta} \right)^2 V^2(z_0) + \frac{4\theta(1 - \theta)}{1 + \theta} V(z_0) F_0 + (1 - \theta)^2 F_0^2.
\]

Let \(z_0\) be drawn from the stationary distribution \(\pi_*\), then we have \(\mathbb{E}[V^2(z_1)] = \mathbb{E}[V^2(z_0)]\) since \(\pi_*\) is the invariant measure for \(\{z_k\}_{k \in \mathbb{N}}\). Notice that \(\mathbb{E}[V(z)] = O(\mathbb{E}[\|z - z_*\|^2])\) which is bounded by the arguments above; hence, we obtain,

\[
\mathbb{E}[V^2(z_0)] \left( \frac{1 + 3\theta}{1 + \theta} \right) \leq 4\theta \mathbb{E}[V(z) F_0] + (1 - \theta)^2 \mathbb{E}[F_0^2].
\]

As we obtained \((21)\), we can also get

\[
\mathbb{E}[F_0 | z_0] = -\mathbb{E}[\hat{\nabla}_x f(x_0, y_1) - \nabla_x f(x_0, y_0), x_0 - x_* - \tau \hat{\nabla}_x f(x_0, y_1) | z_0] + \mathbb{E}[\hat{q}_0 - q_0, y_0 - y_0 | z_0]
\]

\[
= \tau \mathbb{E}[\|\hat{\nabla}_x f(z_0, y_1) - \nabla_x f(z_0, y_1)\|^2 | z_0] + \sigma \mathbb{E}[\|\hat{q}_0 - q_0\|^2 | z_0] = \mathcal{O} \left( (1 - \theta) \delta^2_{(2)} \right),
\]

which follows from Assumption 2 and \((39)\). Moreover, we also have \(\mathbb{E}[V(z)] = O(\mathbb{E}[\|z - z_*\|^2]) = O(1 - \theta)\). Thus,

\[
\mathbb{E}[V(z_0) F_0] = \mathbb{E}[V(z_0) \mathbb{E}[F_0 | z_0]] = \mathcal{O} \left( (1 - \theta) \delta^2_{(2)} \right) \mathbb{E}[V(z_0)] = \mathcal{O} \left( (1 - \theta)^2 \delta^2_{(2)} \right).
\]

Since we have already shown \(\mathbb{E}[\|F_0\|^2] = O(1 - \theta)\); the result directly follows from \((42)\), i.e., when \(z \sim \pi_*\), it holds that

\[
\mathbb{E}[\|z - z_*\|^4] = \mathcal{O}(\mathbb{E}[V^2(z)]) = \mathcal{O}\left((1 - \theta)^2\right).
\]

(43)
7.4 Proof of Theorem 5

By Theorem 3, the distribution of the iterates \( \xi_k^{(t)} \) converges to the invariant measure \( \pi^{(t)} \). In particular, if we take \( \psi(\xi) = \xi \) in Theorem 3, then it follows that
\[
\bar{\xi}^{(t)} = \lim_{k \to \infty} R_k^{(t)} \psi = \lim_{k \to \infty} E[\xi_k^{(t)}] = E[\xi] ,
\]
where \( \xi \) is a random variable distributed according to the invariant measure \( \pi^{(t)} \). Therefore, for computing \( \bar{\xi}^{(t)} \), without loss of generality, we can assume that the initialization \( \xi_1 \) is drawn from \( \pi^{(t)} \); in which case \( \xi_1^{(t)} \) will also be distributed according to \( \pi^{(t)} \) for every \( k \geq 1 \).

**Lemma 12.** Consider the setting of Theorem 5. Suppose the initialization \( \xi_1 \) is distributed according to invariant measure \( \pi^{(t)} \). Then, we have
\[
E[z_1 - z_*] = -\frac{1}{2} \nabla^2 f_*^{-1} \nabla^3 f_* E[(z_1 - z_*)^2] + O \left( (1 - \theta)^{3/2} \right) .
\]
(44)

**Proof.** Let \( \nabla f(z) = \nabla f(x, y) \) for simplicity. Recall that \( \nabla f(z_*) = 0 \) and consider the second-order Taylor expansion of \( \nabla f \) with integral remainder of \( \nabla f \) around \( z_* \),
\[
\nabla f(z) = \nabla f(z_*) + \nabla^2 f_* (z - z_*) + \frac{1}{2} \nabla^3 f_* (z - z_*)^2 + r_1(z).
\]
(45)

Since \( f \) has uniformly bounded 4-th order derivative, \( r_1 \) satisfies the condition
\[
\sup_{z \in \mathbb{R}^d} \left\{ \frac{\|r_1(z)\|}{\|z - z_*\|^3} \right\} < \infty.
\]
(46)

Therefore, we can write the following approximations for each \( x_k, y_k \) and \( \tilde{y}_k \equiv y_{k+1} \) defined in (2):
\[
\nabla f(x_k, \tilde{y}_k) = \nabla^2 f_* \left[ x_k - x_* \right] \tilde{y}_k - y_* + \frac{1}{2} \nabla^3 f_* \left[ x_k - x_* \right] \tilde{y}_k - y_*^2 + r_1(x_k, \tilde{y}_k),
\]
(47a)
\[
\nabla f(z_k) = \nabla^2 f_* (z_k - z_*) + \frac{1}{2} \nabla^3 f_* (z_k - z_*)^2 + r_1(z_k),
\]
(47b)
\[
\nabla f(z_{k-1}) = \nabla^2 f_* (z_{k-1} - z_*) + \frac{1}{2} \nabla^3 f_* (z_{k-1} - z_*)^2 + r_1(z_{k-1}).
\]
(47c)

Taking expectation of both sides of the equation (5) yields
\[
E[\xi_2] = M E[\xi_1] + N E[\hat{\Phi}_1],
\]
which implies that
\[
E[z_2 - z_*] = E[z_1 - z_*] + \frac{-\nabla^2 f(x_1, \tilde{y}_1)}{\sigma (1 + \theta) \nabla f(z_1) - \sigma \theta \nabla f(z_0)} .
\]
(48)

Since the measure \( \pi^{(t)} \) is invariant for \( \mathcal{R} \), clearly we have \( E[z_2] = E[z_1] \); thus, (48) yields the following system of equations:
\[
E[z_2 - z_*] = E[z_1 - z_*],
\]
(49a)
\[
E[\nabla_x f(x_1, \tilde{y}_1)] = 0,
\]
(49b)
\[
(1 + \theta) E[\nabla_y f(z_1)] = \theta E[\nabla_y f(z_0)].
\]
(49c)

Define the variables,
\[
\begin{align*}
& w_1^0 \triangleq \nabla_x f(x_1, \tilde{y}_1) - \nabla_x f(x_1, \tilde{y}_1), \quad w_1^1 \triangleq \nabla_y f(z_1) - \nabla_y f(z_1) , \quad w_1^2 \triangleq \nabla_y f(z_0) - \nabla_y f(z_0), \\
& w_1^3 \triangleq \nabla_y f(z_0). 
\end{align*}
\]
(50)

and introduce new notations,
\[
H_1 \triangleq \nabla_x^2 f(z_1), \quad H_2 \triangleq \nabla_{xy}^2 f(z_1), \quad H_3 \triangleq \nabla_{yy}^2 f(z_1).
\]

Recall that \( \tilde{y}_1 = y_1 + \sigma \tilde{q}_1 \), and \( \tilde{q}_1 = (1 + \theta) \nabla_y f(z_1) - \theta \nabla_y f(z_0) \); therefore, (49c) yields,
\[
E[\tilde{y}_1] = E[y_1 + \sigma \tilde{q}_1] = E[y_1] + \sigma E[(1 + \theta) \nabla_y f(z_1)] - \sigma \theta E[\nabla_y f(z_0)] = E[y_1].
\]
(51)
If we set \( S \triangleq \mathbb{E}\left[ x_1 - x_s \right]^{\otimes 2} - \mathbb{E}\left[ z_1 - z_s \right]^{\otimes 2} \), then using the Taylor expansions in (47) and (51), the expectation of the elements of \( \hat{\phi}_1 = [\tilde{\nabla}_x f(x_1, \tilde{y}_1)^\top, \tilde{\nabla}_y f(z_1)^\top, \tilde{\nabla}_y f(z_0)^\top]^{\top} \) can be written as follows:

\[
\mathbb{E}[\tilde{\nabla}_x f(x_1, \tilde{y}_1)] = [H_1, H_2]\mathbb{E}[z_1 - z_s] + \frac{1}{2} \Pi_1 \nabla(3) f_s \mathbb{E}[(z_1 - z_s)^{\otimes 2}] + \frac{1}{2} \Pi_1 \nabla(3) f_s S + \Pi_1 \mathbb{E}[r_1(x_1, \tilde{y}_1)],
\]

\[
\mathbb{E}[\tilde{\nabla}_y f(z_1)] = [H_2^{\top}, H_3] \mathbb{E}[z_1 - z_s] + \frac{1}{2} \Pi_2 \nabla(3) f_s \mathbb{E}[(z_1 - z_s)^{\otimes 2}] + \Pi_2 \mathbb{E}[r_1(z_1)],
\]

\[
\mathbb{E}[\tilde{\nabla}_y f(z_0)] = [H_2^{\top}, H_3] \mathbb{E}[z_0 - z_s] + \frac{1}{2} \Pi_2 \nabla(3) f_s \mathbb{E}[(z_0 - z_s)^{\otimes 2}] + \Pi_2 \mathbb{E}[r_1(z_0)],
\]

where \( \Pi_1 \) and \( \Pi_2 \) are projection matrices defined as \( \Pi_1 z_k = x_k \) and \( \Pi_2 z_k = y_k \) for any \( z_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix} \in \mathbb{R}^d \) and we used the fact that \( \mathbb{E}[w_i^0] = \mathbb{E}[w_i^1] = \mathbb{E}[w_i^2] = 0 \).

Since \( \pi_s^{(\theta)} \) is invariant measure, \( \mathbb{E}[(\xi_2 - \xi_s)^{\otimes 2}] = \mathbb{E}[(\xi_1 - \xi_s)^{\otimes 2}] \) which also implies \( \mathbb{E}[(z_1 - z_s)^{\otimes 2}] = \mathbb{E}[(z_0 - z_s)^{\otimes 2}] \), where the expectations are taken with respect to \( \pi_s^{(\theta)} \). We also recall that \( \mathbb{E}[z_1 - z_s] = \mathbb{E}[z_0 - z_s] \). Therefore, the Taylor expansion of the partial derivatives (52) together with the conditions (49) give the following equalities:

\[
0 = [H_1, H_2] \mathbb{E}[z_1 - z_s] + \frac{1}{2} \Pi_1 \nabla(3) f_s \mathbb{E}[(z_1 - z_s)^{\otimes 2}] + \frac{1}{2} \Pi_1 \nabla(3) f_s S + \Pi_1 \mathbb{E}[r_1(x_1, \tilde{y}_1)],
\]

\[
0 = (1 + \theta) \mathbb{E}[\tilde{\nabla}_y f(z_1)] - \theta \mathbb{E}[\tilde{\nabla}_y f(z_0)]
\]

\[
= [H_2^{\top}, H_3] \mathbb{E}[z_1 - z_s] + \frac{1}{2} \Pi_2 \nabla(3) f_s \mathbb{E}[(z_1 - z_s)^{\otimes 2}] + \Pi_2 ((1 + \theta) \mathbb{E}[r_1(z_1)] - \theta \mathbb{E}[r_1(z_0)]).
\]

Introducing

\[
e_1 \triangleq [r_1(x_1, \tilde{y}_1)^\top \Pi_1^{\top} + \frac{1}{2} ((\Pi_1 \nabla(3) f_s)^\top ((1 + \theta) r_1(z_1) - \theta r_1(z_0))^\top \Pi_2^{\top}] \in \mathbb{R}^d,
\]

these inequalities yield

\[
0 = \nabla(2) f_s \mathbb{E}[z_1 - z_s] + \frac{1}{2} \nabla(3) f_s \mathbb{E}[(z_1 - z_s)^{\otimes 2}] + \mathbb{E}[e_1].
\]

This gives the characterization of the gap \( \mathbb{E}[z_1] - z_s \) with respect to variance \( \mathbb{E}[(z_1 - z_s)^{\otimes 2}] \). For completing the proof, (54) shows that it suffices to prove that the error term \( \mathbb{E}[e_1] = \mathcal{O} \left( 1 - \theta \right)^{3/2} \). In the rest of the proof, we will study the components of the vector \( \mathbb{E}[e_1] \) to show this. First, note that Hölder’s inequality and (43) imply that

\[
\mathbb{E}[(z_1 - z_s)^{3}] = \mathcal{O} \left( 1 - \theta \right)^{3/2}.
\]

The Jensen’s inequality, (46) and (55) imply

\[
\mathbb{E}[r_1(z_i)] \leq \mathbb{E}[\|r_1(z_i)\|] = \mathcal{O} \left( \mathbb{E}[\|z_1 - z_s\|^{3}] \right) = \mathcal{O} \left( 1 - \theta \right)^{3/2} \quad \text{for each} \quad i \in \{0, 1\}.
\]

Note that the remainder term \( \mathbb{E}[r_1(x_1, \tilde{y}_1)] \) can be bounded as

\[
\mathbb{E}[r_1(x_1, \tilde{y}_1)] \leq \mathbb{E}[\|r_1(x_1, \tilde{y}_1)\|] = \mathcal{O} \left( \mathbb{E}[\|x_1 - \tilde{y}_1\|^{3}] \right) = \mathcal{O} \left( \|z_1 - z_s\|^{3} + \sigma^{3} \right) \mathcal{O} \left( \|z_1 - z_s\|^{3} \right),
\]

where we used (45). In order to bound \( \mathbb{E}[r_1(x_1, \tilde{y}_1)] \), we now consider \( \|\tilde{q}_1\|^{2} \).

\[
\|\tilde{q}_1\|^{2} = \left( 1 + \theta \right) \|\tilde{\nabla}_y f(z_1) - \nabla_y f(z_1) \| + \left( 1 + \theta \right) \|\tilde{\nabla}_y f(z_1) - \nabla_y f(z_1) \| - \theta \|\tilde{\nabla}_y f(z_0) - \nabla_y f(z_0) \| + \theta \|\tilde{\nabla}_y f(z_0) - \nabla_y f(z_0) \|
\leq 4 \left( 1 + \theta \right)^{2} \|\tilde{\nabla}_y f(z_1) - \nabla_y f(z_1) \|^{2} + 4 \theta^{2} \|\tilde{\nabla}_y f(z_0) - \nabla_y f(z_0) \|^{2}
+ 4 \left( 1 + \theta \right)^{2} \|\tilde{\nabla}_y f(z_1) - \nabla_y f(z_1) \|^{2} + 4 \theta^{2} \|\tilde{\nabla}_y f(z_0) - \nabla_y f(z_0) \|^{2}.
\]

Recall that the function \( g^{1/2} \) is convex and monotonically increasing for \( t > 0 \); hence, it satisfies the property \( (a + b + c + d)^{3/2} \leq \frac{1}{4} (a^{3/2} + b^{3/2} + c^{3/2} + d^{3/2}) \) for any \( a, b, c, d > 0 \). Therefore, if both sides of (58) are raised to the power of 3/2, then we obtain

\[
\mathbb{E}[\|\tilde{q}_1\|^{3}] = \left( 1 + \theta \right)^{3} \mathcal{O} \left( \delta_{\theta}^{3} + \mathbb{E}[\|z_1 - z_s\|^{3}] \right).
\]
which follows from Assumption 1 and Assumption 2 — note Assumption 1 implies $\|\nabla_y f(z_i) - \nabla_y f(z_*)\|^2 \leq 2\max(L_{yz}^x, L_{yz}^y)\|z_i - z_*\|^2$ for each $i \in \{0, 1\}$. Thus, (57) and (59) together imply that
$$\|E[r_1(x_1, \tilde{y}_1)]\| = \mathcal{O}\left(E[\|z_1 - z_*\|^3] + \sigma^3(1 + \theta)^3\delta^3(3)\right).$$

Recall that $\sigma = \frac{1 - \theta}{\mu_y\theta}$; thus, using (55), we get
$$\|E[r_1(x_1, \tilde{y}_1)]\| = \mathcal{O}\left((1 - \theta)^{3/2} + (1 - \theta)^3\delta^3(3)\right) = \mathcal{O}\left((1 - \theta)^{3/2}\right).$$

Lastly, let us consider $\|E[\Pi_1 \nabla^{(3)} f_s]\|$. Recall that $\Pi_1$ is the projection of the vector $\nabla^{(3)} f_s$ onto its first $d_x$-coordinates; hence we can write $\|\Pi_1 \nabla^{(3)} f_s\|^2 = \sum_{i=1}^{d_x} \sum_{j=1}^{d_x} \sum_{k=1}^{d_x} \partial^3 f(x_1, \ldots, x_k)\|S_{ij}\|^2 = \mathcal{O}(\|S\|_F^2)$ where $\|\cdot\|_F$ is the Frobenius norm and $S_{ij}$ is the $(i, j)$-th element of the matrix $S$. The property $\|S\|_F^2 \leq \text{rank}(S)\|S\|_2^2$ implies $\|E[\Pi_1 \nabla^{(3)} f_s]\| = \mathcal{O}(\|S\|_2)$. Therefore the eigenvalues of $S$ are relevant for our analysis. Writing $S$ explicitly as follows,
$$S = \begin{bmatrix} 0_{d_x} & E[(\tilde{y}_1 - y_1)(x_1 - x_*)_1] & \cdots & E[(\tilde{y}_1 - y_1)(x_1 - x_*)_d] \\ E[(\tilde{y}_1 - y_1)(x_1 - x_*)_1] & 0_{d_x} & \cdots & E[(\tilde{y}_1 - y_1)(x_1 - x_*)_d] \\ \vdots & \vdots & \ddots & \vdots \\ E[(\tilde{y}_1 - y_1)(x_1 - x_*)_1] & E[(\tilde{y}_1 - y_1)(x_1 - x_*)_2] & \cdots & 0_{d_x} \end{bmatrix} + \sigma^2 \begin{bmatrix} 0 & 0 \\ 0 & E[\tilde{q}\tilde{q}^\top] \end{bmatrix},$$

and using Jensen’s inequality, we note $\|E[(x_1 - x_*)_1]\|_2 \leq \|E[(x_1 - x_*)_1]\|_2 \leq \|E[(x_1 - x_*)_1]\|_2$ since the spectral norm $\|\cdot\|_2$ is sub-multiplicative. Next, we compute this expectation using conditioning on $z_1$ and (58), i.e.,
$$\mathbb{E}\left[\left\|x_1 - x_*\right\|\left\|\tilde{q}_1\right\|_2 \right| z_1 = \|z_1 - z_*\|](1 + \theta)(\|z_1 - z_*\| + \|z_0 - z_*\| + \epsilon(2))). \quad (62)$$

Since $z_0 \sim \pi_x^{(\theta)}$ and $\pi_x^{(\theta)}$ is the stationary distribution; thus, we also have $z_1 \sim \pi_x^{(\theta)}$, which implies $\mathbb{E}[\|z_0 - z_*\|^2] = \mathbb{E}[\|z_1 - z_*\|^2] = \mathcal{O}(1 - \theta)$ based on (56). Therefore, using Hölder’s inequality, we get
$$\mathbb{E}\left[\left\|z_1 - z_*\right\|\left\|z_0 - z_*\right\| \right] \leq \left(\mathbb{E}[\|z_1 - z_*\|^2]\right)^{\frac{1}{2}} \left(\mathbb{E}[\|z_0 - z_*\|^2]\right)^{\frac{1}{2}} = \mathcal{O}(1 - \theta). \quad (63)$$

Finally, using Hölder’s inequality one more time, we obtain $\mathbb{E}[\|z_1 - z_*\|] = \mathcal{O}(\sqrt{1 - \theta})$. This bound together with (62) implies that $\mathbb{E}\left[\left\|x_1 - x_*\right\|\left\|\tilde{q}_1\right\|_2 \right] = \mathcal{O}(\sqrt{1 - \theta})$. Consequently, we have also
$$\|E[(x_1 - x_*)_1]\|_2 = \mathcal{O}(\sqrt{1 - \theta}). \quad (64)$$

Let $v_1, v_2 \in \mathbb{R}^d$ be some arbitrary vectors such that $\|v_1\| = \|v_2\| = 1$, then using (61), the quadratic form $v^\top S v$ of the vector $v = [v_1^\top, v_2^\top]^\top$ can be written as follows:
$$v^\top S v = [2\sigma v_1^\top E[(x_1 - x_*)_1]\{y_1 - y_*)_1^\top v_2 + 2\sigma v_2^\top E[\tilde{q}_1 (y_1 - y_*)^\top + (y_1 - y_*)\tilde{q}_1^\top] v_2 + 2\sigma v_2^\top E[\tilde{q}_1 \tilde{q}_1^\top] v_2 | \leq 2\sigma\|E[(x_1 - x_*)_1]\|_2 + 2\sigma\|E[(y_1 - y_*)\tilde{q}_1]\|_2 + 2\sigma^2\|E[\tilde{q}_1 \tilde{q}_1^\top]\|_2, \quad (65)$$

where we used (i) $v_1^\top E[(x_1 - x_*)_1]\{y_1 - y_*)_1^\top v_2 \leq \|E[(x_1 - x_*)_1]\|_2$ which follows from the variational definition of the spectral norm, (ii) the definition of the spectral norm, i.e.,
$$v_2^\top E[\tilde{q}_1 (y_1 - y_*)^\top + (y_1 - y_*)\tilde{q}_1^\top] v_2 \leq \|E[\tilde{q}_1 (y_1 - y_*)^\top + (y_1 - y_*)\tilde{q}_1^\top]\|_2 \leq \|E[(y_1 - y_*)\tilde{q}_1]\|_2,$$

(iii) similarly, $v_2^\top E[\tilde{q}_1 \tilde{q}_1^\top] v_2 \leq \|E[\tilde{q}_1 \tilde{q}_1^\top]\|_2 \leq \|E[\tilde{q}_1 \tilde{q}_1^\top]\|_2 \leq \|E[\tilde{q}_1 \tilde{q}_1^\top]\|_2$. Using the identical arguments for deriving (64), we also get $\|E[(y_1 - y_*)\tilde{q}_1]\|_2 = \mathcal{O}(\sqrt{1 - \theta})$. Finally, from (58), we get $\|E[\tilde{q}_1]\|_2 = (1 + \theta)^2\mathcal{O}\left(\delta^2(2) + \mathcal{O}(\|z_1 - z_*\|^2)\right)$. Therefore, since $\sigma = \mathcal{O}(1 - \theta)$, all these bounds can be combined to result in $\|S\|_2 = \mathcal{O}\left((1 - \theta)^{3/2}\right)$. Combining this with (60) and (56), we conclude from the definition (53) of $e_1$ that
$$\|E[e_1]\| = \|E[e_1]\| = \mathcal{O}\left((1 - \theta)^{3/2}\right).$$

Plugging this into (54), we conclude.
Equipped with Lemma 12, now we are ready to complete the proof of Theorem 5. We recall that without loss of generality, we assume $\xi_1$ is distributed according to the invariant measure. In this case, the iterates $\xi_k = [z_k^T, z_{k-1}^T]^T$ are also distributed according to the invariant measure and we have $\mathbb{E}[\xi_k - \xi_*] = \mathbb{E}[\xi_1 - \xi_*]$ for every $k$. From the equations (49), we conclude that $\mathbb{E}[z_k - z_*] = \mathbb{E}[z_{k-1} - z_*]$ for each $k \in \mathbb{N}$. Consequently, we have

$$\mathbb{E}[\xi_1 - \xi_*] = \left[ \begin{array}{c} \mathbb{E}[z_1 - z_*] \\ \mathbb{E}[z_0 - z_*] \end{array} \right] = \left[ \begin{array}{c} \mathbb{E}[z_1 - z_*] \\ \mathbb{E}[z_1 - z_*] \end{array} \right].$$

(66)

Therefore it suffices to characterize $\mathbb{E}[(z_1 - z_*)]$ which itself depends on $\mathbb{E}[(z_1 - z_*)^{\otimes 2}]$ according to Lemma 12. For this purpose, we will next study $\mathbb{E}[(z_1 - z_*)^{\otimes 2}]$. We first write the second-order Taylor expansion of $\nabla f$ around $z_*$ with a remainder term,

$$\nabla f(x, y) = \nabla f_* + \nabla^{(2)} f_* \begin{bmatrix} x - x_* \\ y - y_* \end{bmatrix} + r_2(x, y), \quad \forall \ z = (x, y) \in \mathbb{R}^d.$$

(67)

Since $f$ has uniformly bounded 3-rd order partial derivatives, $r_2$ satisfies the condition $\sup_{z \in \mathbb{R}^d} \|r_2(z)\| < \infty$. Since $\nabla f_* = 0$, for $x_k, y_k$, and $\tilde{y}_k \triangleq y_{k+1}$ defined by SAPD, we have

$$\nabla f(x_k, \tilde{y}_k) = \nabla^{(2)} f_* \begin{bmatrix} x_k - x_* \\ \tilde{y}_k - y_* \end{bmatrix} + r_2(x_k, \tilde{y}_k)$$

$$\nabla f(x_k, y_k) = \nabla^{(2)} f_* (z_k - z_*) + r_2(z_k)$$

$$\nabla f(x_{k-1}, y_{k-1}) = \nabla^{(2)} f_* (z_{k-1} - z_*) + r_2(z_{k-1}).$$

Using the fact that $\tilde{y}_k - y_* = y_k - y_* + \sigma \tilde{q}_k$ for any $k \geq 0$, we get

$$\nabla_x f(x_1, \tilde{y}_1) = [H_1, H_2](z_1 - z_*) + \sigma H_2 \tilde{q}_1 + \Pi_1 r_2(x_1, \tilde{y}_1) + w_1^0$$

(68)

$$\nabla_y f(x_1, y_1) = [H_2^T, H_3](z_1 - z_*) + \Pi_2 r_2(z_1) + w_1^1$$

(69)

$$\nabla_y f(x_0, y_0) = [H_2^T, H_3](y_0 - z_*) + \Pi_2 r_2(z_0) + w_1^2,$$

(70)

where $H_1 = \nabla^{(2)} f_* x$, $H_2 = \nabla^{(2)} f_* y$, $H_3 = \nabla^{(2)} f_*$, the projection matrices $\Pi_i$'s are as defined in the proof of Lemma 12 and the variables $w_1^0, w_1^1$ and $w_1^2$ are as given in (50). Let

$$M_0 \triangleq \begin{bmatrix} \nabla^{(2)} f_* & 0_d \times d_x & 0_d \times d_y \\ 0_d \times d_x & H_2^T & H_3 \end{bmatrix}, \quad e_2 \triangleq \begin{bmatrix} \sigma \bar{H}_2 \bar{q}_1 + \Pi_1 r_2(x_1, \tilde{y}_1) \\ \Pi_2 r_2(z_1) \\ \Pi_2 r_2(z_0) \end{bmatrix}, \quad w_1 \triangleq \begin{bmatrix} w_1^0 \\ w_1^1 \\ w_1^2 \end{bmatrix}, \quad \xi_* \triangleq \begin{bmatrix} x^*_1 \\ y^*_1 \\ x^*_a \\ y^*_a \end{bmatrix},$$

(71)

then using the dynamical system notation in (5), we get

$$\xi_2 - \xi_* = (M + NM_0)\underbrace{(\xi_1 - \xi_*) + Ne_2 + Nw_1}_{L},$$

where

$$M = \begin{bmatrix} I_d & 0_d \\ I_d & 0_d \end{bmatrix}, \quad N = \begin{bmatrix} -\tau I_{d_x} & 0_d \times d_y \\ 0_d \times d_x & \sigma(1 + \theta) I_{d_y} \\ 0_d \times d_x & 0_d \times d_y \end{bmatrix},$$

(72)

and

$$L \triangleq M + NM_0 = \begin{bmatrix} I_d + A \nabla^{(2)} f_* & B \nabla^{(2)} f_* \\ I_d & 0_d \end{bmatrix},$$

with the matrices

$$A \triangleq \begin{bmatrix} -\tau I_{d_x} & 0_d \times d_y \\ 0_d \times d_x & \sigma(1 + \theta) I_{d_y} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0_d \times d_x & 0_d \times d_y \\ 0_d \times d_x & -\theta \sigma I_{d_y} \end{bmatrix}.$$

Using the fact that the noise $w_1$ satisfies $\mathbb{E}[w_1 \mid \xi_1, \tilde{y}_1] = 0$, we obtain from (74),

$$\mathbb{E}[(\xi_2 - \xi_*)^{\otimes 2}] = LE[(\xi_1 - \xi_*)^{\otimes 2}] L^T + NE[w_1^{\otimes 2}] N^T + \Delta,$$

(73)
where $\Delta$ has the following form:

$$\Delta \triangleq \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2 & \Delta_3 \end{bmatrix} = LE[(\xi_1 - \xi_\star)e_2^T]N^T + NE[e_2(\xi_1 - \xi_\star)^TL + NE[e_2e_2^T]N^T,$$

(74)

for $\Delta_i \in \mathbb{R}^{d \times d}$ for each $i \in \{1, 2, 3\}$. Define $\Sigma \triangleq \mathbb{E}[(\xi_1 - \xi_\star)^{\otimes 2}]$ and $W \triangleq \mathbb{E}[w_1^{\otimes 2}]$; furthermore, let $W_{ij} \triangleq \mathbb{E}[(w_i^1)(w_j^1)^T]$ for $i, j \in \{0, 1, 2\}$. $\Sigma_1 \triangleq \mathbb{E}[(z_1 - z_\star)(z_1 - z_\star)^T], \Sigma_2 \triangleq \mathbb{E}[(z_1 - z_\star)(z_0 - z_\star)^T]$, and $\Sigma_3 \triangleq \mathbb{E}[(z_0 - z_\star)^{\otimes 2}] = \Sigma_1$. Firstly notice that

$$NWN^T = \begin{bmatrix} \bar{W} & 0_d \\ 0_d & 0_d \end{bmatrix}$$

for $\bar{W}$ such that

$$\bar{W} = TWT^\top, \quad T \triangleq \begin{bmatrix} -\tau I_{d_x} & 0_{d_x \times d_y} & 0_{d_x \times d_y} \\ 0_{d_y \times d_x} & \sigma(1 + \theta)I_{d_y} & -\theta \sigma I_{d_y} \end{bmatrix}.$$ (75)

Therefore, the equation (73) yields the following system of equations,

$$\Sigma_1 = (I + A\nabla^2 f_s)\Sigma_1 (I + A\nabla^2 f_s)^\top + (I + A\nabla^2 f_s)\Sigma_2 \nabla^2 f_s B^T + B\nabla^2 f_s \Sigma_2^2 (I + A\nabla^2 f_s)^\top,$$

$$+ B\nabla^2 f_s \Sigma_1 \nabla^2 f_s B^T + \Delta_1 + \bar{W},$$

(76a)

$$\Sigma_2 = \Sigma_1 (I + A\nabla^2 f_s)^\top + \Sigma_2 \nabla^2 f_s B^T + \Delta_2,$$

(76b)

$$\Sigma_3 = \Sigma_1 + 0_d + \Delta_3;$$

(76d)

thus, $\Delta_3 = 0_d$. If we introduce the matrix $C \triangleq A + B = \begin{bmatrix} -\tau I_{d_x} & 0_{d_x \times d_y} \\ 0_{d_y \times d_x} & \sigma I_{d_y} \end{bmatrix}$, and the notation $H \triangleq \nabla^2 f_s$ for simplicity of the presentation, then using (76c) within (76a) we get the first equation below and substituting (76b) into (76c) we get the second equation below:

$$\Sigma_1 - BH\Sigma_1 HB^T = \Sigma_2 - BH\Sigma_2 HB^T + \Sigma_2 HC^T + CH\Sigma_2 HB^T + \Delta_1 - \Delta_2(I + AH)^\top + \bar{W};$$

(77a)

$$\Sigma_2 - BH\Sigma_2 HB^T = \Sigma_1 - BH\Sigma_1 HB^T + CH\Sigma_1 + BH\Sigma_1 HC^T + BH\Delta_2 + \Delta_2.$$ (77b)

Next, we estimate $\Sigma_1$ by approximately solving the coupled matrix equations (77) known as coupled Lyapunov equations (see e.g. [Borno and Gajic (1995)]). For this purpose, first we are going to study the terms $BH\Sigma_i HB^T$ and $BH\Sigma_i HC^T$ and show that their spectral norms are $\mathcal{O}((1 - \theta)^3)$ for $i = 1, 2$. From the definitions of $B$ and $C$, the following equalities hold for any matrix $X \in \mathbb{R}^{d \times d}$:

$$BHXB^T = (\theta \sigma)^2 \begin{bmatrix} 0_{d_x \times d_y} & 0_{d_x \times d_y} \\ 0_{d_y \times d_x} & 0_{d_y \times d_x} \end{bmatrix} X_4,$$

$$BHXC^T = \theta \sigma \begin{bmatrix} 0_{d_x \times d_y} & 0_{d_x \times d_y} \\ \tau X_3 & -\sigma X_4 \end{bmatrix},$$

(78)

where $X_3 \in \mathbb{R}^{d_y \times d_x}$ and $X_4 \in \mathbb{R}^{d_x \times d_y}$ are defined as the blocks of the matrix $HXH$ satisfying $\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = HXH$. Since in Theorem [5] we set $\tau = \frac{1 - \theta}{\mu \sigma}$ and $\sigma = \frac{1 - \theta}{\mu \sigma}$, (78) implies that $\|BHXB^T\|_2 = \mathcal{O}((1 - \theta)^2 \|X_4\|_2) = \mathcal{O}((1 - \theta)^2 \|X\|_2)$ and $\|BHXC^T\|_2 = \mathcal{O}((1 - \theta)^2 \max_{i \in \{3, 4\}} \|X_i\|_2) = \mathcal{O}((1 - \theta)^2 \|X\|_2)$ for each $i \in \{1, 2\}$, where we bound the spectral norm of a matrix from the spectral norm of a submatrix (see [Govaerts and Przybylo (1989)] for similar inequalities). Hence, we get $\|BH\Sigma_i HB^T\|_2 = \mathcal{O}((1 - \theta)^2 \|\Sigma_i\|_2)$ and $\|BH\Sigma_i HC^T\|_2 = \mathcal{O}((1 - \theta)^2 \|\Sigma_i\|_2)$ for each $i \in \{1, 2\}$. Note Jensen’s inequality implies

$$\|\Sigma_1\|_2 \leq \mathbb{E}[\|z_1 - z_\star\|z_1 - z_\star\|^T]\|Z_1 - Z_\star\|_2 = \mathbb{E}[\|Z_1 - Z_\star\|_2^2];$$

(79)

similarly, we also have

$$\|\Sigma_2\|_2 \leq \mathbb{E}[\|z_1 - z_\star\|(z_0 - z_\star)^T]\|Z_1 - Z_\star\|_2 = \mathbb{E}[\|z_1 - z_\star\|_2 \|Z_0 - Z_\star\|_2] \leq \mathbb{E}[\|z_1 - z_\star\|_2^2]^{1/2} \mathbb{E}[\|Z_0 - Z_\star\|_2^2]^{1/2}$$

(80)

by Hölder and Cauchy-Schwarz inequalities. Finally, since $z_1$ and $z_0$ are distributed according to the stationary distribution, Lemma [4] implies that $\mathbb{E}[\|z_1 - z_\star\|^2] = \mathcal{O}(1 - \theta)$; therefore, (79) and (80) imply the desired results for each $i = 1, 2$,

$$\|BH\Sigma_i HB^T\|_2 = \mathcal{O}((1 - \theta)^3), \quad \|BH\Sigma_i HC^T\|_2 = \mathcal{O}((1 - \theta)^3).$$ (81)
Consequently, we can write the system in (77) as follows:

\[
\begin{align*}
\Sigma_1 &= \Sigma_2 + \Sigma_2 HC^T + \Delta_1 - \Delta_2 (I + AH)^T + \hat{W} + \mathcal{O} \left( (1 - \theta)^3 \right) \\
\Sigma_2 &= \Sigma_1 + C \hat{H} \Sigma_1 + BH \Delta_2^T + \Delta_1 + \mathcal{O} \left( (1 - \theta)^3 \right) \cdot
\end{align*}
\]

(82a) (82b)

If we eliminate \( \Sigma_2 \) within the system in (82), we get the following equation in terms of \( \Sigma_1 \):

\[
\Sigma_1 = \Sigma_1 + C \hat{H} \Sigma_1 + \Sigma_1 HC^T + C \hat{H} \Sigma_1 HC^T + BH \Delta_2^T + \Delta_2 + \mathcal{O} \left( (1 - \theta)^3 \right)
\]

The arguments we used for deriving (81) also imply that \( \|C \hat{H} \Sigma_1 HC^T\|_2 = \mathcal{O} \left( (1 - \theta)^2 \|\Sigma_1\|_2 \right) \); thus, we get \( \|C \hat{H} \Sigma_1 HC^T\|_2 = \mathcal{O} \left( (1 - \theta)^3 \right) \). Furthermore, (76c) implies that \( \|\Delta_2\|_2 = \mathcal{O} (\|\Sigma_1\|_2 + \|\Sigma_2\|_2) \); hence, we also get \( \|BH \Delta_2 HC^T\|_2 = \mathcal{O} \left( (1 - \theta)^3 \right) \). Therefore,

\[
C \hat{H} \Sigma_1 + \Sigma_1 HC^T = -\hat{W} - \left( \frac{(2H \Delta_2 B^T + BH \Delta_2^T + \Delta_1)}{E} \right) + \mathcal{O} \left( (1 - \theta)^3 \right).
\]

(83)

The matrix on the left-hand-side of the equation depends on \( \theta \). Particularly, if we define the matrix

\[
C_0 \triangleq \begin{bmatrix}
-\frac{1}{\mu_x} I_{d_x} & 0_{d_x \times d_y} \\
0_{d_y \times d_x} & \frac{1}{\mu_y} I_{d_y}
\end{bmatrix},
\]

(84)

then we have \( C = \frac{1 - \theta}{\theta} C_0 \) and (83) can be written as

\[
((C_0 H) \otimes I_d + I_d \otimes (C_0 H)) \Sigma_1 = Q, \quad Q \triangleq -\left( \frac{\theta}{1 - \theta} \hat{W} + \frac{\theta}{1 - \theta} E \right) + \mathcal{O} \left( (1 - \theta)^2 \right),
\]

(85)

where \( \otimes \) denotes the tensor product for matrices, i.e., for any matrices \( M, N \in \mathbb{R}^{d \times d} \), \( M \otimes N : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d} \) such that \( M \otimes N : P \rightarrow MPN^T \). The equation in (85) is known as the Lyapunov equation, well-studied in the control literature. \( \Sigma_1 \) satisfying (85) is unique, if the real part of the eigenvalues of the matrix \( C_0 H \) are all strictly less than 0, in which case the solution can be expressed as

\[
\Sigma_1 \triangleq -\int_0^\infty e^{(C_0 H) t} Q e^{(C_0 H)^T} \, dt,
\]

(86)

(see e.g. Zhou et al. [1996]). In this case, the operator \( ((C_0 H) \otimes I_d + I_d \otimes (C_0 H)) \) is invertible with an inverse \( \Gamma \triangleq ((C_0 H) \otimes I_d + I_d \otimes (C_0 H))^{-1} \), i.e., we have \( \Sigma_1 = \Gamma Q \) where \( \Gamma \) is the linear (integral) operator defined by (86).

In the following lemma, we show that the real part of the eigenvalues of the operator \( (C_0 H) \otimes I_d + I_d \otimes (C_0 H) \) are indeed strictly less than zero. This result implies that the inverse \( \Gamma \) exists.

**Lemma 13.** The matrix \( C_0 H \) is stable, i.e., the real parts of the eigenvalues of the matrix \( C_0 H \) are all negative, where \( H = \nabla^{(2)} f_x \) is the Hessian of \( f \) at the saddle point \( z \), and \( C_0 \) is the matrix defined in (84).

**Proof.** We introduce the matrix \( P = \begin{bmatrix}
\frac{1}{\mu_x} I_{d_x} & 0_{d_x \times d_y} \\
0_{d_y \times d_x} & \frac{1}{\mu_y} I_{d_y}
\end{bmatrix} \) where \( i \) is the imaginary unit satisfying \( i^2 = -1 \). Then, \( C_0 H \) is similar to the matrix \( \tilde{H} \) defined as

\[
\tilde{H} \triangleq P^{-1} (C_0 H) P = \begin{bmatrix}
\frac{1}{\mu_x} H_1 & i \frac{1}{\sqrt{\mu_x \mu_y}} H_2^T \\
\frac{1}{\sqrt{\mu_x \mu_y}} H_2 & \frac{1}{\mu_y} H_3
\end{bmatrix}, \quad \text{where} \quad H = \begin{bmatrix} H_1 \\ H_2^T \\ H_3 \end{bmatrix}
\]

such that \( H_1 = \nabla^{(2)} f_x, H_2 = \nabla^{(2)} y f_x, H_3 = \nabla^{(2)} y f_x \). Hence, \( C_0 H \) is diagonalizable if and only if \( \tilde{H} \) is diagonalizable. Moreover, \( C_0 H \) and \( \tilde{H} \) share the same eigenvalues as they are similar. The matrix \( \tilde{H} \in \mathbb{C}^{d \times d} \) can be decomposed into \( \tilde{H}_r \in \mathbb{R}^{d \times d} \) and \( \tilde{H}_c \in \mathbb{R}^{d \times d} \), i.e., real and imaginary parts, as follows:

\[
\tilde{H} = \begin{bmatrix}
\frac{1}{\mu_x} H_1 & 0_{d_x \times d_y} & \frac{i}{\sqrt{\mu_x \mu_y}} H_2^T \\
0_{d_y \times d_x} & \frac{1}{\mu_y} H_3 & 0_{d_y} \\
\frac{1}{\sqrt{\mu_x \mu_y}} H_2 & \frac{1}{\mu_y} H_3 & 0_{d_y}
\end{bmatrix}.
\]

Let \( \lambda \) be the eigenvalue of \( \tilde{H} \) with the corresponding right eigenvector \( v \), i.e., \( \tilde{H} v = \lambda v \); then its complex conjugate, \( \bar{\lambda} \), i.e., \( \bar{\lambda} \bar{\lambda} = |\lambda|^2 \), is also an eigenvalue of the Hermitian transpose (conjugate transpose) of \( \tilde{H} \) with the left eigenvector \( \bar{v}^* \).
We observe from (94) that this requires obtaining a bound on the error term 
\( \Delta \). Then, we can decompose 
\( \bar{\Delta} \)

Each matrix \( \Delta \) in the rest of the proof, we analyze the order of 
so that

Based on Lemma 13, the integral in the representation (86) is finite, and the inverse \( \Gamma = ((C_0 H) \otimes I_d + I_d \otimes (C_0 H))^{-1} \) exists where we can write

Recall that \( \bar{W} = T W T^\top \), where \( T \) is defined in (75) and \( W = \mathbb{E}[w_1^{\otimes 2}] \). Clearly, \( \bar{W} \) depends on \( (1 - \theta) \). In order to see this, first we decompose \( T \) using

so that

Then, we can decompose \( \bar{W} \) using \( \bar{W} = T W T^\top \) as follows:

where \( W_1 \triangleq T_1 W T_1^\top \), \( W_2 \triangleq T_1 W T_2^\top + T_2 W T_1^\top \) and \( W_3 \triangleq T_2 W T_2^\top \), and the matrices \( W_1 \), \( W_2 \) and \( W_3 \), by definition, do not depend on \( \theta \).

Hence, the equation (90) becomes:

Since \( \frac{1 - \theta}{\theta} = 1 - \theta + O((1 - \theta)^2) \) as \( \theta \to 1 \), we can write

In the rest of the proof, we analyze the order of \( \| E \|_2 \) with respect to \( (1 - \theta) \). Recall the definition of \( E \) from (83), i.e.,

where \( \Delta_1 \) and \( \Delta_2 \) are the submatrices of the matrix \( \Delta \) defined in (74), i.e.,

Each matrix \( \Delta_i \) is a submatrix of \( \Delta_1 \); therefore, their spectral norm is bounded by \( \| \Delta_i \|_2 \), i.e., \( \| \Delta_i \|_2 \leq \| \Delta \|_2 \) for each \( i \in \{ 1, 2 \} \) —see for example Govaert and Pryce [1989]. Therefore, it is sufficient to bound \( \| \Delta_2 \|_2 \) rather than each \( \Delta_i \). We observe from (83) that this requires obtaining a bound on the error term \( e_2 \), defined in (77), which depends on \( q_1 \),
\(r_2(z_0), r_2(z_1), \) and \(r_2(x_1, \bar{y}_1)\). In particular, since \(r_2(.)\) is the remainder of the first-order Taylor expansion of \(\nabla f(z_i)\) around \(z_*\), we have

\[
\|r_2(z_i)\| = O \left( \|z_i - z_*\|^2 \right),
\]

for each \(i \in \{0, 1\}\). Similarly, we have \(\|r_2(x_1, \bar{y}_1)\| = O \left( \|x_1 - x_*\|^2 + \|\bar{y}_1 - y_*\|^2 \right)\). Since \(\bar{y}_1 = y_1 + \sigma \bar{q}_1\), we also get

\[
\|r_2(x_1, \bar{y}_1)\|^2 = O \left( (\|z_1 - z_*\|^2 + 2^{\|\bar{q}_1\|^2}) \right) = O \left( \|z_1 - z_*\|^4 + 2^{\|\bar{q}_1\|^4} \right),
\]

where we used the Cauchy-Schwarz inequality and the definition of \(\bar{y}_1\). Consequently, we can bound \(\|e_2\|^2\) using (95):

\[
\|e_2\|^2 = O \left( \|r_2(x_1, \bar{y}_1)\|^2 + \|r_2(z_1, \bar{y}_1)\|^2 \right) = O \left( \|z_1 - z_*\|^4 + 2^{\|\bar{q}_1\|^4} \right) + \|z_0 - z_*\|^4 + \|z_1 - z_*\|^4.
\]

Notice that the inequality (58) directly implies bounds on \(E[\|\bar{q}_1\|^2]\) and \(E[\|\bar{q}_1\|^2]\) as given below:

\[
E[\|\bar{q}_1\|^2] = O \left( (1 + \theta)^2 (\delta^2(z_2) + E[\|z_1 - z_*\|^2]) \right), \quad E[\|\bar{q}_1\|^4] = O \left( (1 + \theta)^4 (\delta^4(z_4) + E[\|z_1 - z_*\|^4]) \right),
\]

(see the derivation of (59) for further details). Therefore, by taking the expectation of both sides of equation (96), and using the fact that \(\sigma = O(1 - \theta)\), the following relation

\[
E[\|e_2\|^2] = O \left( (1 + \theta)^2 (\delta^2(z_2) + E[\|z_1 - z_*\|^2] + (1 - \theta)^4 (\delta^4(z_4) + E[\|z_1 - z_*\|^4]) + E[\|z_1 - z_*\|^4] \right),
\]

(97)

as \(\theta \to 1\) where we used (36) and (43) in the last equality. We are now ready to bound \(\|\Delta\|_2\) which will in turn give us the order of \(E[\|E\|_2]\) with respect to \((1 - \theta)\). Using the sub-multiplicative property of the spectral norm after computing \(\|\Delta\|_2\) from (94), and then using Jensen’s inequality to bound both \(\|E[\|\xi_1 - \xi_*\|e_2^T\|_2]\|_2\) and \(\|E[\|e_2e_2^T\|_2]\|_2\), we obtain

\[
\|\Delta\|_2 = O \left( (1 + \theta)^5 \right) \left( (1 - \theta)^5 / (1 - \theta) \right) = O((1 - \theta)^{5/2}).
\]

Inserting this bound into (93) and using the fact \(\|\Delta_i\| \leq \|\Delta\|\) for \(i = 1, 2\); we obtain \(E = O((1 - \theta)^{5/2})\). Then, we conclude from (92) that

\[
\Sigma_1 = - (1 - \theta)^3 \Gamma(W_1 + W_2 + W_3) + O \left( (1 - \theta)^{3/2} \right),
\]

(99)

as \(\theta \to 1\) where \(W_i = O(1)\) for \(i = 1, 2, 3\). Recalling that \(\Sigma_1 = E[(z_1 - z_*)e_2^T]\) by definition, inserting (99) into Lemma 12 and using (66), we conclude that (11) holds for the matrix \(M = \frac{1}{2} \Gamma(W_1 + W_2 + W_3)\) where \(M = O(1)\) as \(\theta \to 1\). This completes the proof.

7.5 Proof of Theorem 6

Let \(\phi : \mathbb{R}^d \to \mathbb{R}\) be a measurable function with \(\|\phi\| < \infty\), where \(\|\cdot\|\) denotes the weighted supremum norm defined in Theorem 3. We define the function \(\psi(\theta) : \mathbb{R}^d \to \mathbb{R}\) such that for any initialization \(z_0 \in \mathbb{R}^d\) of the Markov chain, we set

\[
\psi(\theta)(z_0) = \sum_{k=0}^{\infty} (R_k^\theta \phi)(z_0) = \sum_{k=0}^{\infty} \left( E[\phi(z_k^\theta)] - E[\phi(z_\infty^\theta)] \right).
\]

(100)

Notice that the Theorem 3 and the triangle inequality imply that there exists \(C > 0\) such that for all \(k \geq 0\), we have

\[
\| \sum_{i=0}^{k} (R_i^\theta \phi - \pi_*^\theta(\phi)) \| \leq \sum_{i=0}^{k} \|R_i^\theta \phi - \pi_*^\theta(\phi) \| \leq C \sum_{i=0}^{k} \frac{2\theta}{1 + \theta} \| \phi - \pi_*^\theta(\phi) \|
\]

(101)
for any measurable function \( \phi \) such that \( \| \phi \| < \infty \). Recalling that \( \frac{2\theta}{1+\theta} < 1 \), this shows that the function \( \xi \to \sum_{i=0}^{k} (R_i^\theta (\phi - \pi_k^\theta (\phi))) \) is uniformly bounded over \( k \in \mathbb{N} \). Hence, the function \( \psi(\theta) \) is well-defined as a converging series with a finite weighted supremum norm.

We will adapt the notation \( R = R(\theta) \) and \( \tilde{\phi} = \phi - \pi_k^\theta (\phi) \) in the rest of the proof for simplicity. We next show that \( \psi = \psi(\theta) \) is the unique solution to the system:

\[
(Id - R) \psi = \tilde{\phi}, \quad \pi_k^\theta (\psi) = 0,
\]

where \( Id \) denotes the identity operator, i.e., \( Id \psi = \psi \) for any \( \psi \). Noticing that for any given \( \xi_0 \) and \( \phi \), \( \pi_k^\theta (\phi) \) is a constant and \( R \) is a probability measure, we get \( R \pi_k^\theta (\phi) = \pi_k^\theta (\phi) \); thus, it is immediate from the definition of \( \psi(\theta) \) in \( 100 \) that \( (Id - R) \psi(\theta) = \tilde{\phi} \) holds. Let us next show that \( \pi_k^\theta (\psi(\theta)) = 0 \). Since we have \( (Id - R) \psi(\theta) = \tilde{\phi} \), we also get \( (R_{k-1} - R_k) \psi(\theta) = R_{k-1} \tilde{\phi} \). Hence we can write, \( (Id - R_k) \psi(\theta) = (Id + R_1 + \ldots + R_{k-1}) \tilde{\phi} \); therefore, in the limit \( k \to \infty \), we obtain

\[
\lim_{k \to \infty} (\psi(\theta) - R_k \psi(\theta)) = \lim_{k \to \infty} \sum_{i=0}^{k-1} R_i \tilde{\phi} = \psi(\theta),
\]

which yields that \( \lim_{k \to \infty} R_k \psi(\theta) = 0 \). On the other hand, since \( \psi(\theta) \) has finite weighted supremum norm, it follows from Theorem 3 that \( \pi_k^\theta (\psi(\theta)) = \lim_{k \to \infty} R_k \psi(\theta) = 0 \).

To see uniqueness, suppose there exists another function \( \tilde{\psi}(\theta) \) such that \( (Id - R) \tilde{\psi}(\theta) = \tilde{\phi} \), then we can write \( (\psi(\theta) - \tilde{\psi}(\theta)) = R(\psi(\theta) - \tilde{\psi}(\theta)) \). Thus for all \( k \in \mathbb{N} \) we obtain \( (\psi(\theta) - \tilde{\psi}(\theta)) = R_k (\psi(\theta) - \tilde{\psi}(\theta)) \). On the other hand, Theorem 3 implies \( (\psi(\theta) - \tilde{\psi}(\theta)) = \pi_k^\theta (\psi(\theta) - \tilde{\psi}(\theta)) = 0 \) which shows \( \psi(\theta) = \tilde{\psi}(\theta) \).

After presenting the properties of \( \psi(\theta) \), we are now ready to prove Theorem 4. Let \( \phi(\xi) : \mathbb{R}^d \to \mathbb{R} \) be a function with finite weighted supremum norm, and define \( Z_k \equiv \frac{1}{k} \sum_{i=0}^{k-1} \phi(\xi_i(\theta)) \). Recall that we have already shown \( (Id - R_k) \psi(\theta) = \sum_{i=0}^{k-1} R_i (\phi - \pi_k^\theta (\phi)) \); hence, \( Z_k \) can be written as

\[
E[Z_k] = \pi_k^\theta (\phi) + \frac{1}{k} \sum_{i=0}^{k-1} E[\phi(\xi_i(\theta)) - \pi_k^\theta (\phi)]
\]

\[
= \pi_k^\theta (\phi) + \frac{1}{k} \sum_{i=0}^{k-1} (R_i \phi(\xi_0) - \pi_k^\theta (\phi)) = \pi_k^\theta (\phi) + \frac{\psi(\theta)(\xi_0) - R_k \psi(\theta)(\xi_0)}{k}.
\]

Recall that \( \xi(\theta) = E[\xi(\theta)] \); therefore, choosing the function \( \phi \) as \( \phi(\xi) = \xi - \bar{\xi} \), we have \( \pi_k^\theta (\phi) = 0 \). We have shown above that \( (Id - R_k) \psi(\theta) = (Id + R_1 + \ldots + R_{k-1})(\phi - \pi_k^\theta (\phi)) = \sum_{i=0}^{k-1} (R_i \phi - \pi_k^\theta (\phi)) \); therefore, \( 101 \) and \( \pi_k^\theta (\phi) = 0 \) together imply that

\[
\frac{\psi(\theta)(\xi_0) - R_k \psi(\theta)(\xi_0)}{k} \leq C \sum_{i=0}^{k-1} \left( \frac{2\theta}{1+\theta} \right)^i \| \phi \| = \frac{1 + \theta}{1 - \theta} \left( 1 - \left( \frac{2\theta}{1+\theta} \right)^k \right) \frac{C}{k} \| \phi \|.
\]

Since \( E[Z_k] = E[\xi(\theta)] - \bar{\xi} \); consequently, we immediately get the desired result in \( 13 \).