Existence and stability of the doubly nonlinear anisotropic parabolic equation

Huashui Zhan

Zhaosheng Feng

The University of Texas Rio Grande Valley

Follow this and additional works at: https://scholarworks.utrgv.edu/mss_fac

Part of the Mathematics Commons

Recommended Citation

Zhan H, Feng Z. Existence and stability of the doubly nonlinear anisotropic parabolic equation. J Math Anal Appl. 2021;497(1):124850. doi:10.1016/j.jmaa.2020.124850

This Article is brought to you for free and open access by the College of Sciences at ScholarWorks @ UTRGV. It has been accepted for inclusion in Mathematical and Statistical Sciences Faculty Publications and Presentations by an authorized administrator of ScholarWorks @ UTRGV. For more information, please contact justin.white@utrgv.edu, william.flores01@utrgv.edu.
Abstract

In this paper, we are concerned with a doubly nonlinear anisotropic parabolic equation, in which the diffusion coefficient and the variable exponent depend on the time variable \( t \). Under certain conditions, the existence of weak solution is proved by applying the parabolically regularized method. Based on a partial boundary value condition, the stability of weak solution is also investigated.

Keywords: Anisotropic parabolic equation, Partial boundary value condition, Weak solution, Characteristic function method

1. Introduction

Since the significant disruption that is being caused by the coronavirus pandemic, we are aware that all communities must resolutely work together to battle the pandemic amid globalization. A growing number of mathematical models have been developed by health care systems, academic institutions and others to help forecast coronavirus spread, deaths, and medical supply needs, including ventilators, hospital beds and intensive care units, timing of patient surges and more. Mathematically, a model of infectious disease can be regarded as a special reaction-diffusion process. Motivated by this fact, in this study we consider a kind of reaction-diffusion equation, namely, a doubly nonlinear parabolic anisotropic equation:

\[ \text{(1)} \]
\[
    u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( a_i(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t) - 2} \frac{\partial B(u)}{\partial x_i} \right) + \sum_{i=1}^{N} g_i(x,t) \frac{\partial B(u)}{\partial x_i},
    \]

\[(x,t) \in Q_T,
\]

where \( Q_T = \Omega \times (0,T), \Omega \subset \mathbb{R}^N \) is a bounded domain with a \( C^2 \) boundary \( \partial \Omega \), \( 0 \leq a_i(x,t) \in C^1(\overline{Q}_T), \)

\[1 < p_i(x,t) \in C^1(\overline{Q}_T), \] and has been extensively studied in the past decades \[13\].

Compared with the isotropic-type equations, equation (1) is much closer to a diffusion process such as the epidemic of coronavirus disease. If

\[a_i(x,t) > 0, \quad (x,t) \in \Omega \times [0,T] \text{ and } a_i(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0,T], \quad i = 1,2,\cdots,N;\]

we conjecture that it inevitably leads to

\[u(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0,T),\]

which was partially proved in \[22\].

The biological explanation of condition (2) lies in the fact that if \( u(x) \) represents the velocity of spreading progress of an infectious disease such as coronavirus disease, condition (2) implies that the virus (or disease) can not transmit across \( \partial \Omega \), when the region remains under lockdown.

A special case of equation (1) is the so-called evolutionary \( p(x) \)-Laplacian equation, which takes the form:

\[u_t = \text{div}(|\nabla u|^{p(x,t)-2} \nabla u), \quad (x,t) \in Q_T,
\]

and has been extensively studied in the past decades \[1,2,5,6,9,14,18,23\] etc. Equation (1) can also be regarded as a generalized version of the polytropic infiltration equation:

\[u_t = \text{div}(|\nabla u^m|^{p-2} \nabla u^m), \quad (x,t) \in Q_T,
\]

where \( m > 0 \) and \( p > 1 \). For more details and recent results on equation (3), we refer the reader to \[4,11,13,19,20,29\] and the references therein.

Recently, a number of issues considered the anisotropic equation \[7,18\]:

\[u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( |u_{x_i}|^{p_i - 2} u_{x_i} \right) + f(x,t,u,\nabla u), \quad (x,t) \in Q_T,
\]
with the initial-boundary value conditions

$$u(x, t) = u_0(x), \ x \in \Omega,$$

$$u(x, t) = 0, \ (x, t) \in \partial \Omega \times [0, T).$$

A more general anisotropic equation \([10], [17], [21], [25], [26], [27]\):

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( a_i(x)|u_{x_i}|^{p_i(x)-2}u_{x_i} \right) + f(x, t, u, \nabla u), \ (x, t) \in Q_T,$$

was studied on the stability and the well-posedness. Here what interests us most is that if we take the diffusion coefficient \(a_i(x)\) and consider a partial boundary value condition

$$u(x, t) = 0, \ (x, t) \in \Sigma_p \subseteq \partial \Omega \times [0, T),$$

can the stability of equation (1) be achieved too? From \([25], [26], [27]\), we know that when \(\Sigma_p = \Sigma_1 \times (0, T)\), where \(\Sigma_1\) is a submanifold of \(\partial \Omega\) (or \(\Sigma_1 = \emptyset\)), the stability of weak solution of equation (3) can be true.

For equation (1), the diffusion coefficient \(a_i(x, t)\), the variable exponent \(p_i(x, t)\) and the convection coefficient \(g'(x, t)\) are all dependent on the time variable \(t\). Distinguished from \([25], [26], [27]\), we will show that \(\Sigma_p\) is a submanifold of \(\partial \Omega \times (0, T)\) and is generally not a cylinder as \(\Sigma_1 \times (0, T)\).

Assume that \(B(u)\) is a strictly increasing function. For examples, \(B(u)\) can be chosen as \(u^m, e^u - 1, \ln(1+u)\) and

$$B(u) = \begin{cases} u^{m_1}, & \text{if } 0 \leq u < 1, \\ u^{m_2}, & \text{if } u \geq 1 \end{cases}$$

with \(m_1 \neq m_2\). For convenience, we denote

$$p_- = \min_{(x, t) \in Q_T} \{p_1(x, t), p_2(x, t), \ldots, p_{N-1}(x, t), p_N(x, t)\}, \ p_- > 1,$$

$$p_+ = \max_{(x, t) \in Q_T} \{p_1(x, t), p_2(x, t), \ldots, p_{N-1}(x, t), p_N(x, t)\}.$$

Definition 1

We say that \(u(x, t)\) is a weak solution of equation (1), if
\[ u \in L^\infty(Q_T), \quad \frac{\partial}{\partial t} \int_0^u \sqrt{b(s)} \, ds \in L^2(Q_T), \quad a_i(x,t) \left[ \frac{\partial B(u)}{\partial x_i} \right]^{p_i(x,t)} \in L^1(Q_T), \quad i = 1, 2, \ldots, N, \tag{7} \]

and for any function \( \varphi \in C(0,T;W_0^{1,p}(\Omega)) \) there holds

\[
\iint_{Q_T} \left[ \frac{\partial u}{\partial t} \varphi(x,t) + \sum_{i=1}^N a_i(x,t) \left[ \frac{\partial B(u)}{\partial x_i} \right]^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \right] dx \, dt = \sum_{i=1}^N \iint_{Q_T} g_i(x,t) \frac{\partial B(u)}{\partial x_i} \varphi(x,t) dx \, dt. \tag{8} \]

The initial value condition (4) is satisfied in the sense of

\[
\lim_{t \to 0} \int_\Omega \left| u(x,t) - u_0(x) \right| \sqrt{\int_0^t \sqrt{b(s)} \, ds} \, dx = 0, \tag{9} \]

and the partial boundary value condition (6) is true in the sense of trace.

Let us summarize our main results as follows. For convenience, we use \( c \) to represent a constant that may change from line to line throughout the whole paper,

Theorem 2

Suppose that \( p_- \geq 2 \) and

\[
\frac{\partial p_i(x,t)}{\partial t} \leq 0, \quad i = 1, 2, \ldots, N. \tag{10} \]

Suppose that \( a_i(x,t) \) satisfies condition (2) and one of the following conditions:

\[
(i). \quad \left| \frac{\partial a_i(x,t)}{\partial t} \right| \leq c a_i(x,t), \quad i = 1, 2, \ldots, N. \tag{11} \]

\[
(ii). \quad \frac{\partial a_i(x,t)}{\partial t} \leq 0, \quad i = 1, 2, \ldots, N. \tag{12} \]
Suppose that $u_0(x) \geq 0$ satisfies

$$u_0 \in L^\infty(\Omega), \ a_i(x,0)u_0(x) \in W^{1,p_i(x,0)}(\Omega), \ i = 1,2,\cdots,N. \tag{13}$$

Then there is a nonnegative solution of equation (1) under condition (4).

Theorem 3

Suppose that for $i = 1,2,\cdots,N$, $a_i(x,t) \equiv a(x,t)$ satisfies condition (2) and for the large $n$ there holds

$$\int_0^T \int_{\Omega_1}^n \left( \int_{\Omega_1}^n \left| \frac{\partial a(x,t)}{\partial x_i} \right|_{p_i(x,t)} dx \right) \frac{1}{p_i(x,t)} \ dt \leq c, \tag{14}$$

$$\int_0^T \int_{\Omega} g_i(x,t)q_i(x,t) a(x,t) \left( \frac{1}{p_i(x,t)} \right) dx \leq c, \tag{15}$$

where $q_i(x,t) = \frac{p_i(x,t)}{p_i(x,t)-1}, \ p_{it}^+ = \max_{x \in \Omega} p_i(x,t), \ q_{it}^+ = \max_{x \in \Omega} q_i(x,t)$ and

$$\Omega_1 = \left\{ x \in \partial \Omega : a(x,t) > \frac{1}{n} \right\}, \ t \in [0,T).$$

Suppose that $u(x,t)$ and $v(x,t)$ are two weak solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$ respectively, and with a partial homogeneous boundary value condition

$$u(x,t) = v(x,t) = 0, \ (x,t) \in \Sigma_p =$$

$$\left\{ (x,t) \in \partial \Omega \times (0,T) : \sum_{i=1}^N g_i(x,t) \frac{\partial a(x,t)}{\partial x_i} \neq 0 \right\}. \tag{16}$$

Then we have

$$\int_\Omega |u(x,t) - v(x,t)|dx \leq c \int_\Omega |u_0(x) - v_0(x)|dx, \ a.e. \ t \in [0,T). \tag{17}$$
It is remarkable that Theorem 3 can be generalized to the case of \( a_i(x,t) \neq a_j(x,t) \) as \( i \neq j \). The proof can be processed in an analogous manner.

The rest of the paper is organized as follows. Proofs of Theorem 2, Theorem 3 are presented in Sections 2 and 3, respectively. The characteristic function method is introduced in Section 4. We show that this method can also be applied to study the stability for other degenerate parabolic equations. A brief conclusion is given in Section 5.

2. Proof of Theorem 2

For simplicity, we assume that \( B(u) \) is a \( C^1 \) strictly monotone increasing function. We prove Theorem 2 by starting to consider a parabolically regularized system:

\[
\begin{align*}
\dot{u}_t &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( (a_i(x,t) + \varepsilon) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \right) + \\
&\quad \sum_{i=1}^{N} g^i(x,t) \frac{\partial B(u)}{\partial x_i}, (x,t) \in Q_T,
\end{align*}
\]

(18)

\[
\begin{align*}
u(x,0) &= u_0(x) + \varepsilon, x \in \Omega, \\
u(x,t) &= \varepsilon, (x,t) \in \partial \Omega \times (0,T).
\end{align*}
\]

(19) (20)

Proof of Theorem 2

Since \( u_0(x) \geq 0 \) satisfies (13), similar to the evolutionary \( p \)-Laplacian equation [24], by using the monotone convergence method, we can prove that there exists a constant \( M \) such that the solution \( u_\varepsilon \in L^1(0,T; W^{1,p(x)}(\Omega)) \) of the initial-boundary value problem (18)-(20) satisfies

\[
\| u_\varepsilon \|_{L^\infty(Q_T)} \leq M.
\]

(21)

For more results on the existence of weak solutions to the initial-boundary value problem (18)-(20), we refer to [5], [6].

Denote

\[
\int_{0}^{r} B(s) ds = \mathbb{B}(r).
\]

(22)
existence and stability of the doubly nonlinear anisotropic parabolic equation

\[ \int_{\Omega} B(u(x,t)) \, dx + \sum_{i=1}^{N} \int_{Q_t} (a_i(x,t) + \varepsilon) \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| p_i(x,t) \, dx \, dt \]

\[ = \int_{\Omega} B(u_0(x)) \, dx + B(\varepsilon) \int_{\Omega} \left| u(x,t) - u_0(x) \right| \, dx \]

\[ + \sum_{i=1}^{N} \int_{Q_t} g^i(x,t) \frac{\partial B(u_\varepsilon)}{\partial x_i} \left[ B(u_\varepsilon) - B(\varepsilon) \right] \, dx \, dt, \]

where \( Q_t = \Omega \times (0, t) \) for any \( t \in [0, T] \).

Since

\[ \int_{Q_t} \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| \left[ B(u_\varepsilon) - B(\varepsilon) \right] \, dx \, dt \]

\[ = -\frac{1}{2} \int_{Q_t} \left[ \frac{\partial g^i(x,t)}{\partial x_i} \right] \left[ B(u_\varepsilon) - B(\varepsilon) \right]^2 \, dx \, dt, \]

from (22) we have

\[ \sum_{i=1}^{N} \int_{Q_t} a_i(x,t) \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| p_i(x,t) \, dx \, dt \]

\[ \leq c \int_{Q_t} \left( a_i(x,t) + \varepsilon \right) \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| p_i(x,t) \, dx \, dt \]

\[ \leq c. \]

Multiplying both sides of (18) by \([B(u_\varepsilon) - B(\varepsilon)]_t\) and integrating it over \( Q_t \) gives

\[ \int_{Q_t} \left[ B(u_\varepsilon) - B(\varepsilon) \right]_t u_\varepsilon \, dx \, dt \]

\[ = -\sum_{i=1}^{N} \int_{Q_t} \left( a_i(x,t) + \varepsilon \right) \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| p_i(x,t) - 2 \frac{\partial}{\partial x_i} B(u_\varepsilon) \frac{\partial}{\partial x_i} B(u_\varepsilon)_t \, dx \, dt \]

\[ + \sum_{i=1}^{N} \int_{Q_t} g^i(x,t) \frac{\partial B(u_\varepsilon)}{\partial x_i} \left[ B(u_\varepsilon) - B(\varepsilon) \right]_t \, dx \, dt. \]
Note that

\[
\frac{\partial B(u_\varepsilon)}{\partial x_i} \left| p(x,t)^{-2} \frac{\partial B(u_\varepsilon)}{\partial x_i} \frac{\partial}{\partial x_i} B(u_\varepsilon)_t \right|
\]

\[
= \frac{1}{2} \frac{\partial}{\partial t} \int_0^s s \frac{p_i(x,t)^{-2}}{2} \, ds - \frac{1}{2} \int_0^s \frac{\partial}{\partial t} s \frac{p_i(x,t)^{-2}}{2} \, ds
\]

\[
= \frac{1}{2} \frac{\partial}{\partial t} \int_0^s s \frac{p_i(x,t)^{-2}}{2} \, ds - \frac{1}{4} \int_0^s s \frac{p_i(x,t)^{-2}}{2} \ln s \frac{\partial p_i}{\partial t} \, ds
\]

\[
= \frac{1}{2} \frac{\partial}{\partial t} \int_0^s s \frac{p_i(x,t)^{-2}}{2} \, ds - \frac{1}{p_i(x,t)} \frac{\partial}{\partial x_i} B(u_\varepsilon) \left| p_i(x,t) \right| \ln \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| \frac{\partial p_i}{\partial t} \, ds
\]

\[
+ \frac{2}{p_i(x,t)} \frac{\partial p_i}{\partial t} \int_0^s s \frac{p_i(x,t)^{-2}}{2} \, ds.
\]

Then, we have

\[
(26)
\]
\[-\iint_{Q_t} \left( a_i(x,t) + \varepsilon \right) \frac{\partial B(u_\varepsilon)}{\partial x_i} \left| \frac{p_i(x,t)}{p_i(x,t)} - 2 \frac{\partial B(u_\varepsilon)}{\partial x_i} \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| dxdt \]

\[= -\frac{1}{2} \iint_{Q_t} \frac{\partial}{\partial t} \left( a_i(x,t) + \varepsilon \right) \left\{ \int_0^{p_i(x,t)} \frac{s^{p_i(x,t)-2}}{2} ds \right\} dxdt \]

\[+ \frac{1}{2} \iint_{Q_t} \int_0^{p_i(x,t)} \frac{s^{p_i(x,t)-2}}{2} ds \frac{\partial a_i(x,t)}{\partial t} dxdt \]

\[+ \frac{1}{4} \iint_{Q_t} [a_i(x,t) + \varepsilon] \left\{ \int_0^{p_i(x,t)} \frac{p_i(x,t)}{2} \ln s \frac{\partial p_i}{\partial t} ds dxdt \right\} \]

\[= \frac{1}{2} \int_{\Omega} \frac{2}{p_i(x,t)} \left[ \left( a_i(x,t) + \varepsilon \right) \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| \frac{p_i(x,t)}{p_i(x,t)} - \left( a_i(x,0) + \varepsilon \right) \left| \frac{\partial B(u_0)}{\partial x_i} \right| \frac{p_i(x,0)}{p_i(x,0)} \right] dx \]

\[+ \frac{1}{2} \iint_{Q_t} \left\{ \left| \frac{\partial}{\partial x_i} B(u_\varepsilon) \right| \frac{p_i(x,t)}{p_i(x,t)} \right\} \frac{\partial a_i(x,t)}{\partial t} dxdt \]

\[+ \iint_{Q_t} \frac{\partial p_i}{\partial t} [a_i(x,t) + \varepsilon] \left( \frac{1}{p_i(x,t)} \right) \frac{\partial B(u_\varepsilon)}{\partial x_i} \left| \frac{p_i(x,t)}{p_i(x,t)} \right| \frac{\partial B(u_\varepsilon)}{\partial x_i} \ln s \frac{\partial p_i}{\partial t} ds dxdt \]

\[+ \iint_{Q_t} \frac{\partial p_i}{\partial t} [a_i(x,t) + \varepsilon] \frac{2}{p_i(x,t)} \left( \frac{s^{p_i(x,t)-2}}{2} ds \frac{\partial a_i(x,t)}{\partial t} \right) dxdt \]

\[\leq c. \]

To derive (26) from (10)-(12), we have used the following facts. From condition (i), i.e. \[\left| \frac{\partial a_i(x,t)}{\partial t} \right| \leq a_i(x,t), \]

we get

\[\iint_{Q_t} \left\{ \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| \frac{p_i(x,t)}{p_i(x,t)} \right\} \frac{\partial a_i(x,t)}{\partial t} dxdt \leq c \iint_{Q_t} [a_i(x,t)] \left| \frac{\partial B(u_\varepsilon)}{\partial x_i} \right| \frac{p_i(x,t)}{p_i(x,t)} dxdt \leq c. \]
From condition (ii), i.e. $\frac{\partial u(x,t)}{\partial t} \leq 0$, we have

$$\iint_{Q_t} \frac{1}{s} \frac{\partial (u^2)}{\partial t} dxdt \leq 0.$$ 

Let

$$Q_1 = \left\{ (x,t) \in Q_t : \left| \frac{\partial B(u)}{\partial x_i} \right| < 1 \right\}, \quad Q_2 = Q_t \setminus Q_1.$$ 

In view of $\frac{\partial u}{\partial t} \leq 0$, we deduce

$$\iint_{Q_1} \frac{\partial p_i}{\partial t} [a_i(x,t) + \varepsilon] \frac{1}{p_i(x,t)} \left| \frac{\partial B(u)}{\partial x_i} \right| \ln \left| \frac{\partial B(u)}{\partial x_i} \right| dsdxdt \leq c + \iint_{Q_2} \frac{\partial p_i}{\partial t} [a_i(x,t) + \varepsilon] \frac{1}{p_i(x,t)} \left| \frac{\partial B(u)}{\partial x_i} \right| \ln \left| \frac{\partial B(u)}{\partial x_i} \right| dsdxdt \leq c$$

and

$$-\iint_{Q_1} \frac{\partial p_i}{\partial t} [a_i(x,t) + \varepsilon] \frac{2}{p_i(x,t)} \ln \left| \frac{\partial B(u)}{\partial x_i} \right| dsdxdt \leq c \int_{Q_t} \left| \frac{\partial B(u)}{\partial x_i} \right| dxdt \leq c.$$
It follows from $p_i(x,t) \geq p_- \geq 2$ that
\[
\iint_{Q_t} g^i(x,t) \frac{\partial B(u_\epsilon)}{\partial x_i} [B(u_\epsilon) - B(\epsilon)]_t \, dx \, dt \\
\leq \iint_{Q_t} \left[ c(\delta) \left| g^i(x,t) \frac{\partial B(u_\epsilon)}{\partial x_i} \right|^2 + \delta \left| B(u_\epsilon) \right|_t^2 \right] \, dx \, dt \\
\leq c(\delta) \left( \iint_{Q_t} \left| g^i(x,t) a_i(x,t) - \frac{2}{p_i(x,t)} \frac{p_i(x,t)}{p_i(x,t) - 2} \right| \, dx \, dt \right)^{\frac{1}{p_i}} \\
\iint_{Q_t} a_i(x,t) \left| \frac{\partial B(u_\epsilon)}{\partial x_i} \right| \, dx \, dt \right) \left( \frac{1}{p_i} \right)^{\frac{1}{p_i}} \\
+ \iint_{Q_t} \delta \left| B'(u_\epsilon) u_{et} \right|^2 \, dx \, dt \\
\leq c + \frac{1}{2} \iint_{Q_t} b(u_\epsilon) \left| u_{et} \right|^2 \, dx \, dt,
\]
where the small constant $\delta$ satisfies $\delta b(M) \leq \frac{1}{2}$.

From (24)-(27), we can derive
\[
\iint_{Q_t} (B(u_\epsilon))_t u_{et} \, dx \, dt = \iint_{Q_t} b(u_\epsilon) \left| u_{et} \right|^2 \, dx \, dt \leq c 
\]
and
\[
\frac{\partial}{\partial t} \int_0^u \sqrt{b(s)} \, ds \rightarrow \frac{\partial}{\partial t} \int_0^u \sqrt{b(s)} \, ds, \text{ in } L^2(Q_T).
\]

From inequalities (21), (23) and (28), it implies
\[
u_\epsilon \rightharpoonup u, \text{ weakly star in } L^\infty(Q_T),
\]
and there exists an $N$-dimensional vector $\zeta = (\zeta_1, \cdots, \zeta_N)$ satisfying

$$|\zeta| \in L^1 \left(0, T; L^{\frac{p(x)}{p(x)-1}}(\Omega)\right)$$

such that

$$a_i(x, t) \left| \frac{\partial B(u_\epsilon)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(u_\epsilon)}{\partial x_i} \to \zeta_i \text{ in } L^1(Q_T), \ i = 1, 2, \cdots, N.$$

In order to prove $u$ to be the solution of equation (1), we shall prove that

$$\sum_{i=1}^{N} \iint_{Q_T} a_i(x, t) \left| \frac{\partial B(u_\epsilon)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(u_\epsilon)}{\partial x_i} \cdot \varphi_{x_i} dx dt = \iint_{Q_T} \zeta \cdot \nabla \varphi dx dt$$

(30)

for any $\varphi \in C^1_0(Q_T)$.

Note that

$$\iint_{Q_T} \left[ u_\epsilon \varphi + \sum_{i=1}^{N} (a_i(x, t) + \epsilon) \left| \frac{\partial B(u_\epsilon)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(u_\epsilon)}{\partial x_i} \nabla \varphi_{x_i} \right] dx dt$$

$$+ \sum_{i=1}^{N} \iint_{Q_T} \left[ \frac{\partial \varphi(x, t)}{\partial x_i} g^i(x, t) B(u_\epsilon) + \frac{\partial g^i(x, t)}{\partial x_i} B(u_\epsilon) \varphi(x, t) \right] dx dt = 0.$$  (31)

Due to $a_i(x, t)|_{\partial \Omega \times [0, T]} = 0$ and $a_i(x, t) > 0$ for $(x, t) \in \Omega \times [0, T]$ , in view of $\varphi(x, t) \in C^1_0(Q_T)$ , we obtain $\max_{\text{supp } \varphi} \frac{|\varphi(x, t)|}{a_i(x, t)} \geq c > 0$ , and
\[
\varepsilon \left| \int_{Q_T} \left( \frac{\partial B(u_{\varepsilon})}{\partial x_i} \right)^{p_i(x,t)-2} \frac{\partial B(u_{\varepsilon})}{\partial x_i} \cdot \frac{\partial \varphi}{\partial x_i} \right| dx dt \right|
\]

\[
\leq \varepsilon c \sup_{\text{supp} \varphi} \frac{|\varphi_{x_i}|}{a_i(x,t)} \int_{Q_T} b_i(x,t) \left( \left| \frac{\partial B(u_{\varepsilon})}{\partial x_i} \right|^{p_i(x,t)} + 1 \right) dx dt
\]

\[
\rightarrow 0, \text{ as } \varepsilon \rightarrow 0.
\]

This further leads to

\[
\int_{Q_T} \zeta \cdot \nabla \varphi dx dt = \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N} \int_{Q_T} a_i(x,t) \left| \frac{\partial B(u_{\varepsilon})}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u_{\varepsilon})}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt
\]

\[
= \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N} \int_{Q_T} \left( a_i(x,t) + \varepsilon \right) \left| \frac{\partial B(u_{\varepsilon})}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u_{\varepsilon})}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt
\]

\[
- \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^{N} \int_{Q_T} \frac{\partial B(u_{\varepsilon})}{\partial x_i} \left| \frac{\partial B(u_{\varepsilon})}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u_{\varepsilon})}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt
\]

Since \( u_{\varepsilon} \rightarrow u \), we see \( B(u_{\varepsilon}) \rightarrow B(u) \) and

\[
\int_{Q_T} (u \varphi_t + \zeta \cdot \nabla \varphi) dx dt
\]

\[
+ \sum_{i=1}^{N} \int_{Q_T} \left[ \frac{\partial \varphi_t}{\partial x_i} g^i(x,t) B(u) + \frac{\partial g^i(x,t)}{\partial x_i} B(u) \varphi(x,t) \right] dx dt = 0. \tag{32}
\]

Let \( 0 \leq \psi \in C_0^\infty(Q_T) \) and \( \psi = 1 \) on \( \text{supp} \varphi \). In view of \( v \in L^\infty(Q_T) \) and \( b_i(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)} \in L^1(Q_T) \) for \( i = 1, 2, \cdots, N \), we have

\[
\left\| \nabla \varphi(x,t) \right\|_{L^2(Q_T)} \leq \left\| \nabla \varphi_0(x) \right\|_{L^2(Q_T)} + \left\| \int_0^t \int_{Q_T} \left( a_i(x,t) + \varepsilon \right) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt \right\|_{L^2(Q_T)}
\]

\[
+ \left\| \int_0^t \int_{Q_T} \frac{\partial B(u)}{\partial x_i} \left( a_i(x,t) + \varepsilon \right) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx dt \right\|_{L^2(Q_T)} \rightarrow 0, \quad \varepsilon \rightarrow 0
\]
\[
\int\int_{Q_T} \psi a_i(x, t) \left( \left| \frac{\partial B(u^e)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(u^e)}{\partial x_i} - \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(v)}{\partial x_i} \right) \cdot \left( \frac{\partial B(u^e)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) \, dx \, dt \geq 0.
\]

Choosing \( \varphi = \psi B(u^e) \) in (31) yields

\[
\int\int_{Q_T} \left[ \frac{\partial u^e}{\partial t} \psi B(u^e) + \sum_{i=1}^{N} (a_i(x, t) + \varepsilon) \left| \frac{\partial B(u^e)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(u^e)}{\partial x_i} \right] \psi B(u^e) \, dx \, dt = 0.
\]

It follows from (33)-(34) that

\[
\int\int_{Q_T} \psi_t B(u^e) \, dx \, dt - \sum_{i=1}^{N} \int\int_{Q_T} (a_i(x, t) + \varepsilon) \left| \frac{\partial B(u^e)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(u^e)}{\partial x_i} \, \psi x_i B(u^e) \, dx \, dt
\]

\[
- \sum_{i=1}^{N} \int\int_{Q_T} (a_i(x, t) + \varepsilon) \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(v)}{\partial x_i} \left( \frac{\partial B(u^e)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) \psi \, dx \, dt
\]

\[
- \sum_{i=1}^{N} \int\int_{Q_T} g^i(x, t) B(u^e) \left( \frac{\partial B(u^e)}{\partial x_i} \psi + B(u^e) \psi x_i \right) \, dx \, dt
\]

\[
- \sum_{i=1}^{N} \int\int_{Q_T} \frac{\partial g^i(x, t)}{\partial x_i} B(u^e) \psi \, dx \, dt \geq 0.
\]

Letting \( \varepsilon \to 0 \), we have

\[
(36)
\]
Taking $\varphi = \psi B(u)$ in (32), we get

\[
\int_Q \psi B(u) \, dx \, dt = \sum_{i=1}^N \int_Q \psi_i \frac{\partial B(u)}{\partial x_i} \, dx \, dt - \sum_{i=1}^N \int_Q B(u) \zeta_i \psi x_i \, dx \, dt
\]

- \sum_{i=1}^N \int_Q a_i(x, t) \zeta x_i \frac{\partial B(v)}{\partial x_i} \psi dx \, dt

- \sum_{i=1}^N \int_Q g^i(x, t) B(u) \left( \frac{\partial B(u)}{\partial x_i} \psi + B(u) \psi x_i \right) \, dx \, dt

- \sum_{i=1}^N \int_Q \frac{\partial g^i(x, t)}{\partial x_i} B(u) \psi dx \, dt \geq 0. \tag{37}

By combining (36) and (37), we have

\[
\sum_{i=1}^N \int_Q \psi \left( \zeta_i - a_i(x, t) \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x, t) - 2} \frac{\partial B(v)}{\partial x_i} \right) \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) \, dx \, dt \geq 0. \tag{38}
\]

In particular, taking $v = B^{-1}(B(u) - \lambda \varphi)$ and $\lambda > 0$, we find
Existence and stability of the doubly nonlinear anisotropic parabolic equation

\[ \lambda \sum_{i=1}^{N} \int_{Q_T} \psi \left( \zeta_i - a_i(x,t) \left| \frac{\partial}{\partial x_i} (B(u) - \lambda \varphi) \right|^{p_i(x,t)-2} \frac{\partial}{\partial x_i} (B(u) - \lambda \varphi) \right) \frac{\partial \varphi}{\partial x_i} \, dx \, dt \geq 0 \]

and so

\[ \lambda \sum_{i=1}^{N} \int_{Q_T} \psi \left( \zeta_i - a_i(x,t) \left| \frac{\partial}{\partial x_i} (B(u) - \lambda \varphi) \right|^{p_i(x,t)-2} \frac{\partial}{\partial x_i} (B(u) - \lambda \varphi) \right) \frac{\partial \varphi}{\partial x_i} \, dx \, dt \geq 0. \] (39)

When \( \lambda \) goes to zero, we have

\[ \sum_{i=1}^{N} \int_{Q_T} \psi \left( \zeta_i - a_i(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \right) \frac{\partial \varphi}{\partial x_i} \, dx \, dt \geq 0. \] (39)

Similarly, when \( \lambda < 0 \), we get

\[ \sum_{i=1}^{N} \int_{Q_T} \psi \left( \zeta_i - a_i(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \right) \frac{\partial \varphi}{\partial x_i} \, dx \, dt \leq 0. \]

Accordingly, we obtain

\[ \sum_{i=1}^{N} \int_{Q_T} \psi \left( \zeta_i - a_i(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \right) \frac{\partial \varphi}{\partial x_i} \, dx \, dt = 0. \]

Since \( \psi = 1 \) on \( \text{supp} \varphi \), we arrive at (30).

The initial value condition (4) in the sense of (9) can be derived from (29). We omit the details here. Consequently, \( u(x) \) satisfies equation (1) in the sense of Definition 1. \( \square \)

3. Proof of Theorem 3

To discuss the stability of solutions of equation (1), we need to introduce the following technical lemma.

Let \( p(x) \in C^1(\Omega) \), and denote \( p^+ = \max_{x \in \Omega} p(x) \) and \( p^- = \max_{x \in \Omega} p(x) \).

Lemma 4

\[ [12], [16] \]
(I) The space \( L^{p(x)}(\Omega), \| \cdot \|_{L^{p(x)}(\Omega)} \), \( W^{1,p(x)}(\Omega), \| \cdot \|_{W^{1,p(x)}(\Omega)} \) and \( W^{1,p(x)}_0(\Omega) \) are reflexive Banach spaces.

(II) Let \( p_1(x) \) and \( p_2(x) \) be real functions with \( \frac{1}{p_1(x)} + \frac{1}{p_2(x)} = 1 \) and \( p_1(x) > 1 \). Then the conjugate space of \( L^{p_1(x)}(\Omega) \) is \( L^{p_2(x)}(\Omega) \). And for any \( u \in L^{p_1(x)}(\Omega) \) and \( v \in L^{p_2(x)}(\Omega) \), we have

\[
\left| \int_{\Omega} uv \, dx \right| \leq 2 \| u \|_{L^{p_1(x)}(\Omega)} \| v \|_{L^{p_2(x)}(\Omega)}.
\]

(III) If \( \| u \|_{L^{p(x)}(\Omega)} = 1 \), then \( \int_{\Omega} \| u \|^{p(x)} \, dx = 1 \):

- if \( \| u \|_{L^{p(x)}(\Omega)} > 1 \), then \( \| u \|_{L^{p(x)}(\Omega)} \leq \int_{\Omega} |u|^{p(x)} \, dx \leq \| u \|_{L^{p(x)}(\Omega)} \); and
- if \( \| u \|_{L^{p(x)}(\Omega)} < 1 \), then \( \| u \|_{L^{p(x)}(\Omega)} \leq \int_{\Omega} |u|^{p(x)} \, dx \leq \| u \|_{L^{p(x)}(\Omega)} \).

For any large integer \( n \), we define an odd function \( S_n(s) \) by

\[
S_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2 s^2 e^{1-n^2 s^2}, & 0 \leq s \leq \frac{1}{n}, \end{cases}
\]

and let

\[
H_n(s) = \int_0^s S_n(s) \, ds.
\]

Then

\[
\lim_{n \to 0} S_n(s) = \text{sgn}(s) \quad \text{and} \quad \lim_{n \to 0} sS'_n(s) = 0, \quad s \in (-\infty, +\infty).
\]

Meanwhile, since \( a_t(x, t) \equiv a(x, t) \geq 0 \), we define

\[
\varphi_n(x, t) = \begin{cases} 1, & \text{if } x \in \Omega_{\frac{2}{n}t}, \\ n \left( a(x, t) - \frac{1}{n} \right), & \text{if } x \in \Omega_{\frac{1}{n}t} \setminus \Omega_{\frac{2}{n}t}, \\ 0, & \text{if } x \in \Omega \setminus \Omega_{\frac{1}{n}t}, \end{cases}
\]

where \( \Omega_M = \{ x \in \Omega : a(x, t) > \lambda \} \) for any \( \lambda > 0 \).

Proof of Theorem 3
Supposed that $u(x,t)$ and $v(x,t)$ are two weak solutions of equation (1). After a process of limit, we can choose $\varphi_n S_n(B(u) - B(v))$ as a test function. In view of $a_i(x,t) \equiv a(x,t)$, we have

$$
\int_0^t \int_\Omega \varphi_n(x,t) S_n(B(u) - B(v)) \frac{\partial(u - v)}{\partial t} dx dt
$$

$$
+ \sum_{i=1}^N \int_0^t \int_\Omega a(x,t) \left( \frac{\partial B(u)}{\partial x_i} \right)^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} - \left( \frac{\partial B(v)}{\partial x_i} \right)^{p_i(x,t)-2} \frac{\partial B(v)}{\partial x_i} \right)
\cdot \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) S'_n(B(u) - B(v)) \varphi_n(x,t) dx dt
$$

$$
+ \sum_{i=1}^N \int_0^t \int_\Omega a(x,t) \left( \frac{\partial B(u)}{\partial x_i} \right)^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} - \left( \frac{\partial B(v)}{\partial x_i} \right)^{p_i(x,t)-2} \frac{\partial B(v)}{\partial x_i} \right)
\cdot S_n(B(u) - B(v)) \frac{\partial \varphi_n(x,t)}{\partial x_i} dx dt
$$

$$
= - \sum_{i=1}^N \int_0^t \int_\Omega g^i(x,t)(B(u) - B(v)) \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) S'_n(B(u) - B(v)) \varphi_n(x,t) dx dt
$$

$$
- \sum_{i=1}^N \int_0^t \int_\Omega g^i(x,t)(B(u) - B(v)) S_n(B(u) - B(v)) \frac{\partial \varphi_n(x,t)}{\partial x_i} dx dt
$$

$$
+ \sum_{i=1}^N \int_0^t \int_\Omega \frac{\partial g^i(x,t)}{\partial x_i}(B(u) - B(v)) S_n(B(u) - B(v)) \varphi_n(x,t) dx dt.
$$

Note that the second term in the left hand side of (40) satisfies

$$
\sum_{i=1}^N \int_0^t \int_\Omega a(x,t) \left( \frac{\partial B(u)}{\partial x_i} \right)^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} - \left( \frac{\partial B(v)}{\partial x_i} \right)^{p_i(x,t)-2} \frac{\partial B(v)}{\partial x_i} \right)
\cdot \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) S'_n(B(u) - B(v)) \varphi_n(x,t) dx dt \geq 0.
$$

To evaluate the third term on the left hand side of (40), we use
\[
\frac{\partial \varphi_n(x, t)}{\partial x_i} = \begin{cases} 
0, & \text{if } x \in \Omega_{1,t}^n, \\
\frac{n \partial a(x,t)}{\partial x_i}, & \text{if } x \in \Omega_{1,t}^n \setminus \Omega_{1/2,t}^n, \\
0, & \text{if } x \in \Omega \setminus \Omega_{1/2,t}^n.
\end{cases}
\]

In view of condition (14), by the straightforward calculations we can deduce that

\[
\sum_{i=1}^{N} \int_0^t \int_{\Omega} a(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} - \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(v)}{\partial x_i} \right) \frac{\partial \varphi_n(x,t)}{\partial x_i} S_n(B(u) - B(v)) \, dx \, dt 
\]

\[
\leq C \sum_{i=1}^{N} \int_0^t \left[ \left( \int_{\Omega_{1/2,t}^n \setminus \Omega_{1/2,t}^n} a(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)} \, dx \right)^{\frac{1}{q_i^+}} + \left( \int_{\Omega_{1/2,t}^n \setminus \Omega_{1/2,t}^n} a(x,t) \left| \frac{\partial a(x,t)}{\partial x_i} \right|^{p_i(x,t)} \, dx \right)^{\frac{1}{q_i^+}} \right] \, dt 
\]

\[
\leq C \sum_{i=1}^{N} \int_0^t \left[ \left( \int_{\Omega_{1/2,t}^n \setminus \Omega_{1/2,t}^n} a(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)} \, dx \right)^{\frac{1}{q_i^+}} + \left( \int_{\Omega_{1/2,t}^n \setminus \Omega_{1/2,t}^n} a(x,t) \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)} \, dx \right)^{\frac{1}{q_i^+}} \right] \, dt 
\]
\[ \int_0^t n \left( \int_{\Omega_{\frac{1}{n}}^{\frac{1}{n}}} \frac{1}{\partial x_i} \varphi_{p_i} dx \right) \frac{1}{\varphi_{p_i}} dt \]

\[ \leq c \sum_{i=1}^N \int_0^t \left[ \left( \int_{\Omega_{\frac{1}{n}}^{\frac{1}{n}}} a(x, t) \left( \frac{1}{\partial x_i} \varphi_{p_i} \right) \right) \frac{1}{\varphi_{p_i}} dt \right] \]

\[ \rightarrow 0, \text{ as } n \to 0. \]

It follows from Hölder's inequality and (15) that

\[ -\sum_{i=1}^N \int_0^t \int_{\Omega} g^i(x, t)(B(u) - B(v)) \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) S_n'(B(u) - B(v)) \varphi_n(x, t) dx dt \]

\[ = -\sum_{i=1}^N \int_0^t \int_{\Omega} g^i(x, t)(B(u) - B(v)) S_n'(B(u) - B(v)) \]

\[ \cdot a(x, t) \frac{1}{\varphi_{p_i}(x, t)} a(x, t) \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) \varphi_n(x, t) dx dt \]

\[ \leq \sum_{i=1}^N \left( \int_0^t \right) \]

\[ \left( \int_{\Omega} \left[ g^i(x, t)(B(u) - B(v)) S_n'(B(u) - B(v)) \right] \varphi_{p_i}(x, t) \frac{1}{\varphi_{p_i}} dx dt \right) \frac{1}{\varphi_{p_i}} \]

\[ \cdot \left( \int_0^t \int_{\Omega} a(x, t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right| \frac{1}{\varphi_{p_i}} + \left| \frac{\partial B(v)}{\partial x_i} \right| \frac{1}{\varphi_{p_i}} \right) dx dt \right) \frac{1}{\varphi_{p_i}} \]

\[ \rightarrow 0, \text{ as } n \to 0, \]

where \( p_i = p_i^+ \) or \( p_i^- \) depends on whether
\[
\left( \int_0^t \int_\Omega a(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)} + \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)} \right) \, dx \, dt \right) \leq 1
\]

or

\[
\left( \int_0^t \int_\Omega a(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)} + \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)} \right) \, dx \, dt \right) > 1.
\]

Recall the partial boundary value condition (16), i.e.

\[
u(x,t) = \nu(x,t) = 0, \ (x,t) \in \Sigma_p = \\
\left\{(x,t) \in \partial \Omega \times (0,T) : \sum_{i=1}^N g^i(x,t) \frac{\partial a(x,t)}{\partial x_i} \neq 0 \right\}.
\]

Then we have

\[
\lim_{n \to \infty} \left| - \sum_{i=1}^N \int_0^t \int_\Omega g^i(x,t)(B(u) - B(v))S_n(B(u) - B(v)) \frac{\partial \varphi_n(x,t)}{\partial x_i} \, dx \, dt \right|
\]

\[
\leq \lim_{n \to \infty} \int_0^t \int_{\Omega \setminus \Omega_\frac{T}{2}} \left| (B(u) - B(v))S_n(B(u) - B(v)) \right| \left| \sum_{i=1}^N g^i(x,t) \frac{\partial a(x,t)}{\partial x_i} \right| \, dx \, dt
\]

\[
\leq c \int_0^t \int_{\Sigma_{1 \frac{T}{2}}} \left| B(u) - B(v) \right| \, d\Sigma \, dt
\]

\[
= 0
\]

and

\[
(45)
\]
Since $B(r) \geq 0$ is monotone, it follows that

$$\lim_{n \to \infty} \int_0^t \int_\Omega \varphi_n(x,t) S_n (B(u) - B(v)) \frac{\partial (u - v)}{\partial t} dx dt$$

$$= \int_0^t \int_\Omega \text{sgn}(B(u) - B(v)) \frac{\partial (u - v)}{\partial t} dx dt$$

$$= \int_0^t \int_\Omega \text{sgn}(u - v) \frac{\partial (u - v)}{\partial t} dx dt$$

$$= \int_\Omega |u(x,t) - v(x,t)| dx - \int_\Omega |u_0(x) - v_0(x)| dx.$$

By (41)-(46), letting $n \to \infty$ in (40) yields

$$\int_\Omega |u(x,t) - v(x,t)| dx \leq \int_\Omega |u_0(x) - v_0(x)| dx + c \int_0^t \int_\Omega |u - v| dx dt, \ t \in [0,T).$$

Using Gronwall’s inequality, we obtain

$$\int_\Omega |u(x,t) - v(x,t)| dx \leq \int_\Omega |u_0(x) - v_0(x)| dx, \ t \in [0,T).$$

□

4. Weak characteristic method

We can generalize the method described in the preceding section to prove the stability of weak solutions.

Let $\chi(x,t)$ be a nonnegative $C^1(\overline{Q_T})$ function as
\( \chi(x, t) > 0 \), if \((x, t) \in Q_T = \Omega \times (0, T)\),

and

\( \chi(x, t) = 0 \), if \((x, t) \in \Gamma_T = \partial \Omega \times [0, T)\).

If we denote

\[ \chi_t = \chi(x, t), \ x \in \Omega, \]

for \( t \in [0, T) \), then \( \chi_t \) is the weak characteristic function of \( \Omega \) as defined in [26]. Likewise, we can simply call \( \chi(x, t) \) a weak characteristic function of \( Q_T \).

We define

\[ \varphi_n(x, t) = \begin{cases} 
1, & \text{if } x \in D_{\frac{1}{n}t}, \\
\frac{1}{n}(\chi(x, t) - \frac{1}{n}), & \text{if } x \in D_{\frac{1}{n}t} \setminus D_{\frac{2}{n}t}, \\
0, & \text{if } x \in \Omega \setminus D_{\frac{1}{n}t}.
\end{cases} \]

Then

\[
\frac{\partial \varphi_n(x, t)}{\partial x_i} = \begin{cases} 
0, & \text{if } x \in D_{\frac{2}{n}t}, \\
\frac{1}{n} \frac{\partial \chi(x, t)}{\partial x_i}, & \text{if } x \in D_{\frac{1}{n}t} \setminus D_{\frac{2}{n}t}, \\
0, & \text{if } x \in \Omega \setminus D_{\frac{1}{n}t},
\end{cases}
\]

where \( D_{\lambda t} = \{ x \in \Omega : \chi(x, t) > \lambda \} \) for \( \lambda > 0 \).

Theorem 5

Suppose that there is a weak characteristic function \( \chi(x, t) \) of \( Q_T \) satisfying.

\[
\int_0^T \left( \int_{D_{\frac{1}{n}t} \setminus D_{\frac{2}{n}t}} a_i(x, t) \left| \frac{\partial \chi(x, t)}{\partial x_i} \right|^{p_i(x, t)} \frac{1}{v_i^t} \right) dx \ dt \leq c, \ i = 1, 2, \cdots, N, \tag{47}
\]

\[
\int_0^T \left( \int_{D_{\frac{1}{n}t} \setminus D_{\frac{2}{n}t}} \frac{1}{v_i^t} \right) dx \ dt \leq c, \ i = 1, 2, \cdots, N, \tag{48}
\]
where \( q_i(x,t) \cdot p^*_i \) and \( q^*_i \) are the same as given in Theorem 3. Suppose that \( u(x,t) \) and \( v(x,t) \) are two weak solutions of equation (1) with the initial values \( u_0(x) \) and \( v_0(x) \) respectively, with a partial homogeneous boundary value condition

\[
\begin{align*}
  u(x,t) = v(x,t) = 0, \quad (x,t) & \in \left\{ x \in \partial \Omega \times (0, T) : \sum_{i=1}^{N} g^i(x,t) \frac{\partial \chi(x,t)}{\partial x_i} \neq 0 \right\}. 
\end{align*}
\]

(49)

Then we have

\[
\int_{\Omega} |u(x,t) - v(x,t)| \, dx \leq c \int_{\Omega} |u_0(x) - v_0(x)| \, dx, \quad \text{a.e. } t \in [0, T].
\]

(50)

Proof of Theorem 5

Choose \( \varphi \), \( S_n(B(u) - B(v)) \) as a test function. Then we have

(51)
\[
\int_0^t \int_\Omega \varphi_n(x,t) S_n(B(u) - B(v)) \frac{\partial (u - v)}{\partial t} \, dx \, dt
\]

\[
+ \sum_{i=1}^N \int_0^t \int_\Omega a_i(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} - \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(v)}{\partial x_i} \right) \cdot \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) S'_n(B(u) - B(v)) \varphi_n(x,t) \, dx \, dt
\]

\[
+ \sum_{i=1}^N \int_0^t \int_\Omega a_i(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} - \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(v)}{\partial x_i} \right) \cdot S_n(B(u) - B(v)) \frac{\partial \varphi_n(x,t)}{\partial x_i} \, dx \, dt
\]

\[
= - \sum_{i=1}^N \int_0^t \int_\Omega g^i(x,t)(B(u) - B(v)) \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) S'_n(B(u) - B(v)) \varphi_n(x,t) \, dx \, dt
\]

\[
- \sum_{i=1}^N \int_0^t \int_\Omega g^i(x,t)(B(u) - B(v)) S_n(B(u) - B(v)) \frac{\partial \varphi_n(x,t)}{\partial x_i} \, dx \, dt
\]

\[
+ \sum_{i=1}^N \int_0^t \int_\Omega \frac{\partial g^i(x,t)}{\partial x_i}(B(u) - B(v)) S_n(B(u) - B(v)) \varphi_n(x,t) \, dx \, dt.
\]

As discussing in the proof of Theorem 3, we have

\[
\sum_{i=1}^N \int_0^t \int_\Omega a_i(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} - \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(v)}{\partial x_i} \right) \cdot \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) S'_n(B(u) - B(v)) \varphi_n(x,t) \, dx \, dt \geq 0.
\]

In view of condition (47), we can deduce

\[
(52)
\]

\[
(53)
\]
\[
\sum_{i=1}^{N} \int_{0}^{t} a_i(x, t) \left( \frac{\partial B(u)}{\partial x_i} \right)^{p_i(x, t)-2} \frac{\partial B(u)}{\partial x_i} - \left( \frac{\partial B(v)}{\partial x_i} \right)^{p_i(x, t)-2} \frac{\partial B(u)}{\partial x_i} \right)
\]

\[
\frac{\partial \varphi_n(x, t)}{\partial x_i} S_n(B(u) - B(v)) dx dt \bigg|_{\Omega} = \sum_{i=1}^{N} \int_{0}^{t} \int_{D_{\frac{1}{n}} \setminus D_{\frac{2}{n}}} a_i(x, t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(u)}{\partial x_i} - \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x, t)-2} \frac{\partial B(u)}{\partial x_i} \right)
\]

\[
\leq \sum_{i=1}^{N} \int_{0}^{t} n \int_{D_{\frac{1}{n}} \setminus D_{\frac{2}{n}}} a_i(x, t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x, t)-1} + \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x, t)-1} \right)
\]

\[
\frac{\partial \chi(x, t)}{\partial x_i} S_n(B(u) - B(v)) \bigg|_{\Omega} dx dt \bigg|_{\Omega} \leq c \sum_{i=1}^{N} \int_{0}^{t} \left[ \left( \int_{D_{\frac{1}{n}} \setminus D_{\frac{2}{n}}} a(x, t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x, t)} dx \right)^{\frac{1}{q_i^t}} \right.
\]

\[
+ \left( \int_{D_{\frac{1}{n}} \setminus D_{\frac{2}{n}}} a(x, t) \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x, t)} dx \right)^{\frac{1}{q_i^t}} \bigg] dt \to 0, \text{ as } n \to 0.
\]

Similar to the derivation of (43), using Hölder's inequality and (48), we obtain

(54)
\[- \sum_{i=1}^{N} \int_{0}^{t} \int_{\Omega} g^i(x,t)(B(u) - B(v)) \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) S'_n(B(u) - B(v)) \varphi_n(x,t) \, dx \, dt \]

\[- \sum_{i=1}^{N} \int_{0}^{t} \int_{\Omega} g^i(x,t)(B(u) - B(v)) S'_n(B(u) - B(v)) \cdot a_i(x,t) - \frac{1}{p_i(x,t)} a_i(x,t) \left( \frac{\partial B(u)}{\partial x_i} - \frac{\partial B(v)}{\partial x_i} \right) \varphi_n(x,t) \, dx \, dt \]

\[\leq \sum_{i=1}^{N} \left( \int_{0}^{t} \int_{\Omega} \left[ g^i(x,t)(B(u) - B(v)) S'_n(B(u) - B(v)) a_i(x,t) - \frac{1}{p_i(x,t)} \right] q_i(x,t) \, dx \, dt \right)^{\frac{1}{q_i}} \]

\[\cdot \left( \int_{0}^{t} \int_{\Omega} a_i(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)} + \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)} \right) \, dx \, dt \right)^{\frac{1}{p_i}} \].

Note that the right hand side of (54) goes to 0 as \( n \to 0 \). Here, \( p_i = p_i^+ \) or \( p_i^- \) depends on whether

\[\left( \int_{0}^{t} \int_{\Omega} a_i(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)} + \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)} \right) \, dx \, dt \right) \leq 1 \]

or

\[\left( \int_{0}^{t} \int_{\Omega} a_i(x,t) \left( \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)} + \left| \frac{\partial B(v)}{\partial x_i} \right|^{p_i(x,t)} \right) \, dx \, dt \right) > 1. \]

In view of (49), we get

(55)
Existence and stability of the doubly nonlinear anisotropic parabolic equation

\[
\lim_{n \to \infty} - \sum_{i=1}^{N} \int_{0}^{t} \int_{\Omega} g^i(x,t)(B(u) - B(v))S_n(B(u) - B(v)) \frac{\partial \varphi_n(x,t)}{\partial x_i} \, dx \, dt \\
\leq \lim_{n \to \infty} \int_{0}^{t} \int_{\frac{1}{n} \cdot D_{\frac{1}{2}}} \left| (B(u) - B(v))S_n(B(u) - B(v)) \right| \\
= c \int_{0}^{t} \int_{\Sigma_{1t}} |B(u) - B(v)| \, d\Sigma \, dt \\
= 0.
\]

Similarly, we can deduce that both (45) and (46) hold too. From (51)-(55), we arrive at the desired result (50). \(\square\)

We can see that by choosing different appropriate characteristic function of \(Q_T\), we can obtain the corresponding stability results under various conditions. For example,

i) If we take \(\chi(x,t) = \chi_{[\tau,s]}(t) \prod_{j=1}^{N} a_j(x,t)\), where \(\chi_{[\tau,s]}(t)\) is the characteristic function of \([s,t] \subset (0,T)\), then

\[
\frac{\partial \chi(x,t)}{\partial x_i} = \chi_{[\tau,s]}(t) \prod_{j=1}^{N} a_j(x,t) \sum_{k=1}^{N} \frac{a_{kx_i}}{a_k(x,t)},
\]

\[
\left| \frac{\partial \chi(x,t)}{\partial x_i} \right|_{p_i(x,t)} = \chi_{[\tau,s]}(t) \prod_{j=1}^{N} a_j(x,t) \sum_{k=1}^{N} \frac{a_{kx_i}}{a_k(x,t)}_{p_i(x,t)},
\]

where \(a_{kx_i} = \frac{\partial a_k(x,t)}{\partial x_i}, k = 1, 2, \ldots, N\).

By virtue of Theorem 5, we obtain

Corollary 6

Suppose that

\[
\int_{0}^{T} \left( \int_{\frac{1}{n} \cdot D_{\frac{1}{2}}} \left( \int_{\frac{1}{n} \cdot D_{\frac{2}{n}}} a_i(x,t) \left| \sum_{k=1}^{N} \frac{a_{kx_i}}{a_k(x,t)} \right|_{p_i(x,t)} \, dx \right) \right) \, dt \leq c, i = 1, 2, \ldots, N.
\]
Suppose that \( u(x, t) \) and \( v(x, t) \) are two weak solutions of equation (1) with the initial values \( u_0(x) \) and \( v_0(x) \) respectively, with a partial homogeneous boundary value condition

\[
u(x, t) = v(x, t) = 0, \quad (x, t) \in \{ (x, t) \in \partial \Omega \times (0, T) : \prod_{j=1}^{N} a_j(x, t) \sum_{i,k=1}^{N} g^i a_{k,i} (x, t) / a_k(x, t) \neq 0 \}.
\]

Then we have the stability of weak solution in the sense of (17).

ii) If we take \( \chi(x, t) = \chi_{[\tau, \tilde{t}]}(t) d^{\alpha - 1}(x) \), where \( d(x) = \text{dist}(x, \partial \Omega) \) is the distance function from the boundary \( \partial \Omega \) and \( \alpha \geq 1 \) is a constant, then

\[
\frac{\partial \chi(x, t)}{\partial x_i} = \alpha \chi_{[\tau, \tilde{t}]}(t) d^{\alpha - 1}(x), \quad \frac{\partial \chi(x, t)}{\partial x_i} \bigg|_{p_i(x,t)} = \alpha \chi_{[\tau, \tilde{t}]}(t) d^{\alpha - 1}(x) \bigg|_{p_i(x,t)}.
\]

According to \textbf{Theorem 5}, we can also obtain

\textbf{Corollary 7}

Suppose that

\[
\int_0^T \int_D 1 - \frac{(\alpha - 1) p_i}{n_i} \sum_{i=1}^{N} a_i(x, t) d^\alpha(x) \frac{1}{p_i} \Bigg|_{D} dt \leq c, \quad i = 1, 2, \ldots, N.
\]

Suppose that \( u(x, t) \) and \( v(x, t) \) are two weak solutions of equation (1) with the initial values \( u_0(x) \) and \( v_0(x) \) respectively, with a partial homogeneous boundary value condition

\[
u(x, t) = v(x, t) = 0, \quad (x, t) \in \{ x \in \partial \Omega \times (0, T) : \sum_{i} g^i(x, t) n_i \neq 0 \},
\]

where \( n = \{n_i\} \) is the outer normal vector of \( \Omega \). Then we have the stability of weak solution in the sense of (17).

\section{Conclusion}

In this study, we applied an analytical method to study the stability of weak solution for a doubly nonlinear anisotropic parabolic equation, where the diffusion coefficient and the variable exponent depend on the time variable \( t \). Under certain parametric choices, it includes the heat equation, reaction-diffusion
equations, non-Newtonian fluid equation and electrorheological fluid equation and the epidemic model of diseases as particular cases.

When \( a_i(x,t) \) is a strictly monotone increasing function, it excludes the strongly degenerate hyperbolic-parabolic equation, for which only under the entropy conditions, the uniqueness of weak solution can be guaranteed \([2],[13],[28]\). However, only under the condition \( B'(u) = b(u) \geq 0 \) or \( a_i(x,t) \) is degenerate in the interior of \( \Omega \), how to prove the uniqueness of weak solution to equation (1) is still an interesting and challenging problem. In addition, if there is an external forcing term \( f(u) \geq 0 \) in equation (1), i.e.

\[
 u_t = \sum_{i=1}^{N} \left( a_i(x,t) \left| \frac{\partial B(u)}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial B(u)}{\partial x_i} \right) + \sum_{i=1}^{N} g^i(x,t) \frac{\partial B(u)}{\partial x_i} + f(u), \quad (x,t) \in Q_T,
\]

we conjecture that weak solutions may blow-up in finite time. How to show such a blow-up behavior and the long time behavior of solutions to equation (56) seems more interesting and helpful from the physical and biological point of view. We will continue to work on this problem in a subsequent work.

Notes

Submitted by G. Chen

References

1. Acerbi E., Mingione G. Regularity results for stationary electrorheological fluids. Arch. Ration. Mech. Anal. 2002;164:213–259. [Google Scholar]

2. Amaziane B., Pankratov L., Piatnitski A. Nonlinear flow through double porosity media in variable exponent Sobolev spaces. Nonlinear Anal., Real World Appl. 2009;10:2521–2530. [Google Scholar]

3. Andreianov B., Bendahmane M., Karlsen K.H., Ouaro S. Well-posedness results for triply nonlinear degenerate parabolic equations. J. Differ. Equ. 2009;247:277–302. [Google Scholar]

4. Andreucci D., Cirmi G.R., Leonardi S., Tedeev A.F. Large time behavior of solutions to the Neumann problem for a quasilinear second order degenerate parabolic equation in domains with noncompact boundary. J. Differ. Equ. 2001;174:253–288. [Google Scholar]

5. Antontsev S., Shmarev S. Anisotropic parabolic equations with variable nonlinearity. Publ. Mat. 2009;53:355–399. [Google Scholar]

6. Antontsev S., Shmarev S. Parabolic equations with double variable nonlinearities. Math. Comput. Simul. 2011;81:2018–2032. [Google Scholar]

7. Bahrouri A., Radulescu V.D., Repovs D.D. A weighted anisotropic variant of the Caffarelli-Kohn-Nirenberg inequality and applications. Nonlinearity. 2018;31(4):1516–1534. [Google Scholar]

8. Barbu L., Enache C. Maximum principles, Liouville-type theorems and symmetry results for a general class of quasilinear anisotropic equations. Adv. Nonlinear Anal. 2016;5(4):395–405. [Google Scholar]

9. Bendahmane M., Wittbold P., Zimmermann A. Renormalized solutions for a nonlinear parabolic equation with variable exponents and \( L^1 \)-data. J. Differ. Equ. 2010;249:1483–1515. [Google Scholar]

10. Buhrii O., Buhrii N. Nonlocal in time problem for anisotropic parabolic equations with variable exponents of nonlinearities. J. Math. Anal. Appl. 2019;473:695–711. [Google Scholar]
11. Droniou J., Eymard R., Talbot K.S. Convergence in \( C([0, T]; L^2(\Omega)) \) of weak solutions to perturbed doubly degenerate parabolic equations. J. Differ. Equ. 2016;260:7821–7860. [Google Scholar]

12. Fan X.L., Zhao D. On the spaces \( L^{p(x)}(\Omega) \) and \( W^{m,p(x)} \) J. Math. Anal. Appl. 2001;263:424–446. [Google Scholar]

13. Gianni R., Tedeev A.F., Vesprì V. Asymptotic expansion of solutions to the Cauchy problem for doubly degenerate parabolic equations with measurable coefficients. Nonlinear Anal. 2016;138:111–126. [Google Scholar]

14. Khanghahi R.M., Razani A. Solutions for a singular elliptic problem involving the \( p(x) \)-Laplacian. Filomat. 2018;32:4841–4850. [Google Scholar]

15. Kobayasi K., Ohwa H. Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle. J. Differ. Equ. 2012;252:137–167. [Google Scholar]

16. Kováčik O., Rákosník J. On spaces \( L^{p(x)}(\mathbb{R}^n) \) and \( W^{k,p(x)}(\mathbb{R}^n) \). Czechoslov. Math. J. 1991;41:592–618. [Google Scholar]

17. Liu B.C., Xin Q., Dong M. Blow-up analyses in parabolic equations with anisotropic nonstandard damping source. J. Math. Anal. Appl. 2018;458:242–264. [Google Scholar]

18. Mashiyev R.A., Buhrii O.M. Existence of solutions of the parabolic variational inequality with variable exponent of nonlinearity. J. Math. Anal. Appl. 2011;377:450–463. [Google Scholar]

19. Shang H., Cheng J. Cauchy problem for doubly degenerate parabolic equation with gradient source. Nonlinear Anal. 2015;113:323–338. [Google Scholar]

20. Tedeev A.F. The interface blow-up phenomenon and local estimates for doubly degenerate parabolic equations. Appl. Anal. 2007;86(6):755–782. [Google Scholar]

21. Tersenov A.S., Tersenov A. Existence of Lipschitz continuous solutions to the Cauchy-Dirichlet problem for anisotropic parabolic equations. J. Funct. Anal. 2017;272:3965–3986. [Google Scholar]

22. Vazquez J.L. Oxford University Press; London: 2006. Smoothing and Decay Estimates for Nonlinear Diffusion Equation. [Google Scholar]

23. Wang Y. Intrinsic Harnack inequalities for parabolic equations with variable exponents. Nonlinear Anal. 2013;83:12–30. [Google Scholar]

24. Wu Z., Zhao J., Yun J., Li F. World Scientific Publishing; Singapore: 2001. Nonlinear Diffusion Equations. New York. [Google Scholar]

25. Zhan H. The stability of the solutions of an anisotropic diffusion equation. Lett. Math. Phys. 2019;109:1145–1166. [Google Scholar]

26. Zhan H., Feng Z. The well-posedness problem of an anisotropic parabolic equation. J. Differ. Equ. 2020;268:389–413. [Google Scholar]

27. Zhan H., Feng Z. Solutions of evolutionary equation based on the anisotropic variable exponent Sobolev space. Z. Angew. Math. Phys. 2019;70(110):1–25. [Google Scholar]

28. Zhan H., Feng Z. Partial boundary value condition for a nonlinear degenerate parabolic equation. J. Differ. Equ. 2019;267:2874–2890. [Google Scholar]

29. Zou W., Li L. Existence and uniqueness of solutions for a class of doubly degenerate parabolic equations. J. Math. Anal. Appl. 2017;446:1833–1862. [Google Scholar]