Small model property reflects in games and automata

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Abstract. Small model property is an important property that implies decidability. We show that the small model size is directly related to some important resources in games and automata for checking provability.

1 Introduction

Dependent types is one of the popular logic-based approaches developed in the field of functional programming. With the help of such types it is possible to more precisely capture the behaviour of programs. Intuitionistic first order logic is the primary form of dependent types and the Curry-Howard isomorphism strictly relates functional program synthesis and construction of proofs in intuitionistic first order logic.

One of the well established ways towards understanding different aspects of logic, proofs and proof search is through correspondence with different representations, e.g. ones that are more abstract like games and tableaux or ones that are more detailed like linear logic. One of the most fruitful ideas fulfilling the pattern is the game based approach, in the spirit of Ehrenfeucht-Fra"ıssé games [4,3]. Another game-based technique was introduced for intuitionistic first order logic [6]. The duality between proof-search and countermodel search [1] has been interpreted there in terms of games and was used to make one unified game that yields either a proof or a Kripke countermodel.

We extend the game based approach [6] to classes that have the finite model property which, for algorithmically well-behaved classes, implies decidability [2, p. 240]. A stronger property, the small model property, that also gives an upper bound on the complexity of the satisfiability problem, is also studied. As it turns out these two properties are equivalent for many interesting classes.
We show in the current work a correspondence between the limit of the model size given by the small model property and some resources in automata and games used for the description of logic. Section 2 contains preliminaries and definitions. Section 3 discusses the automata and Theorem 1 bounds the size of the set of eigenvariables with a number dependent on the number of subformulas in the formula, the limit on the model size and the number of variables in the initial formula. Section 4 covers games and Theorem 2 shows that a strategy can be constructed that uses a number of maximal variables at most equal to the limit on the model size; the maximal variable would be understood as the one having a maximal, by inclusion, set of known facts.

The paper is structured as follows. Section 2 contains preliminaries and definitions necessary to understand the following sections and discusses basic facts about the small model property. Section 3 defines a quasiorder on variables capturing the notion of variable with more facts. Using this order we show that the size of the small countermodel defined in the small model property is also a limit on the number of maximal variables in Afrodite strategy. Section 4 shows a limit on the size of the set of eigenvariables of an Arcadian automaton, which depends on the size of the small model, number of subformulas in the original formula and the number of its variables.

2 Preliminaries

We work in intuitionistic first-order logic with no function symbols or constants. The logic is the same as in previous works on games [6] and automata [7]. There is a set of predicates $\mathcal{P}$ and every predicate $P \in \mathcal{P}$ has a defined arity. First order variables are noted as $X, Y, \ldots$ (with possible annotations) and form an infinite set $\mathcal{X}_1$. The formulas are understood as abstract syntax trees and the possible formulas are generated with the grammar

$$\tau, \sigma ::= P(X, \ldots, X) \mid \tau \land \sigma \mid \tau \lor \sigma \mid \tau \rightarrow \sigma \mid \forall X.\tau \mid \exists X.\tau \mid \bot.$$ 

We define the set $\text{FV}(\tau)$ of free variables of a formula $\tau$ as

- $\text{FV}(P(X_1, \ldots, X_n)) = \{X_1, \ldots, X_n\}$,
- $\text{FV}(\tau_1 \ast \tau_2) = \text{FV}(\tau_1) \cup \text{FV}(\tau_2)$ where $\ast \in \{\land, \lor, \rightarrow\}$,
\[
FV(\forall X.\tau) = FV(\tau) \setminus \{X\} \quad \text{where} \quad \forall \in \{\exists, \forall\},
\]
\[
FV(\bot) = \emptyset.
\]
We assume that there is an infinite set \(X_p\) of proof term variables usually noted as \(x, y, \ldots\) that can be used to form the following terms.

\[
M, N, P ::= x \mid \langle M, N \rangle \mid \pi_1 M \mid \pi_2 M \mid \lambda x : \varphi . M \mid MN \mid \lambda XM
\]
\[
MX \mid \text{in}_{1,\varphi \lor \psi} M \mid \text{in}_{2,\varphi \lor \psi} M \mid \text{case } M \text{ of } [x : \varphi ] N, [y : \psi ] P \mid \text{pack } M, Y \text{ to } \exists X. \varphi \mid \text{let } x : \varphi \text{ be } M : \exists X. \varphi \text{ in } N \mid \bot \bot \varphi M
\]

The free variables in terms are defined by structural recursion on the terms, i.e.
\[
FV(\lambda x : \varphi . M) = FV(\varphi) \cup FV(M).
\]
The inference rules for the logic are shown in Figure 1.

2.1 Models
We follow the definition of Kripke model from the work of Sørensen and Urzyczyn [5]: A Kripke model is a triple \(\langle C, \leq, \{A_c : c \in C\}\rangle\) where \(C \neq \emptyset\) is a set of states, \(\leq\) is a partial order on \(C\) and \(A_c = \langle A_c, P_1^{A_c}, \ldots, P_n^{A_c}\rangle\) are structures such that if \(c \leq c'\) then \(A_c \subseteq A_{c'}\) and for all \(i\) the relation \(P_i^{A_c} \subseteq P_i^{A_{c'}}\) holds. A valuation \(\rho\) maps variables to elements of \(A_c\). The satisfaction relation \(c, \rho \models \varphi\) is defined in the usual way:

\begin{align*}
&c, \rho \models P(t_1, \ldots, t_n) \quad \text{iff } A_c, \rho \models P(t_1, \ldots, t_n) \quad \text{classically}, \\
&c, \rho \models \tau \lor \sigma \quad \text{iff } A_c, \rho \models \tau \text{ or } A_c, \rho \models \sigma, \\
&c, \rho \models \tau \land \sigma \quad \text{iff } A_c, \rho \models \tau \text{ and } A_c, \rho \models \sigma, \\
&c, \rho \models \tau \rightarrow \sigma \quad \text{iff for all } c' \geq c \text{ if } c', \rho \models \tau \text{ then } c', \rho \models \sigma, \\
&c, \rho \models \forall a \varphi \quad \text{iff for all } c' \geq c \text{ if } a \in A_{c'} \text{ then } c', \rho[a/a] \models \tau, \\
&c, \rho \models \exists a \varphi \quad \text{iff for some } a \in A_c \text{, } c, \rho[a/a] \models \tau.
\end{align*}

Proposition 1 (completeness, Theorem 8.6.7 of [5]). The Kripke models as defined above are complete for the intuitionistic predicate logic, i.e. \(\Gamma \vdash \tau\) iff \(\Gamma \models \tau\).

2.2 The finite model property and the small model property
We focus on classes of formulas that have finite model property. Our definitions closely follow that of Börger et al. [2]:

[1]

[2]

[3]
Fig. 1. The rules of the intuitionistic first-order logic. 

Definition 1 (finite model property). A class of formulas $X$ has the finite model property when, for all formulas $\tau \in X$, if $\tau$ is satisfiable, there exists a finite model $M$ such that $M \models \tau$. 

Since all classical theories can be easily expressed as intuitionistic theories by explicitly including the law of excluded middle, so there are many interesting classes that have finite model property.

Although the finite model property in the book by Börger et al. [2] is strongly attached to decidability of a particular fragment of logic, this is not a property that implies this computational feature.
The following property is what actually takes place in the fragment considered in the book.

**Definition 2 (small model property).** A class $X$ has the small model property when there exists a computable function $s_X$ such that for all formulas $\tau \in X$, if $\tau$ is satisfiable, there exists a finite model $M$ of size $s_X(\tau)$ such that $M \models \tau$.

This definition was used in the book [2] in the context of classical logic. It can also be used for intuitionistic first order logic, but the finiteness concerns a different but relevant notion of size. We say that a model $M = \langle C, \leq, \{A_c : c \in C\} \rangle$ is finite when $C$ is finite and $A_c$ is finite for all $c \in C$. The number $u = |C| + |\bigcup_{c \in C} A_c|$ is the size of the model $M$.

**Lemma 1.** For all formulas $\tau$ from a class $X$ that has the finite model property either $\vdash \neg \tau$ or there exists a finite model $M$ and a state $s$ such that $s, M \models \tau$.

**Proof.** If $\neg \tau$ or $\tau$ is true the proof follows by the completeness theorem and by definition. Otherwise there exists a model $M$ of class $X$ such that $s, M \not\models \tau$, and a state $s' > s$ such that $s', M \models \tau$ (if $M$, $s$ and $s'$ do not exist either $\tau$ or $\neg \tau$ would be valid in the model). Note that it does not necessarily mean that a proof for $\neg \tau$ exists.

But, since Kripke models are monotonous, $(s', M) \in X$ and we have a model of $\tau$ in the class $X$: the part of the original model starting in $s'$.

**Definition 3 (effective class).** We say that a model $M$ is a model of a class of formulas $X$ when for each $\varphi \in X$ it holds that $M \models \varphi$.

A class of formulas $X$ is effective iff, it is decidable that given a final model $M$ whether $M$ is a model of the class $X$.

For example every class that has a finite number of axioms or axiom schemes, as well as prefix classes from the book of Börger et al [2] is effective.

**Proposition 2.** An effective class $X$ has finite model property iff it has small model property.

**Proof.** The implication from right to left is trivial. For the other one: we show how to compute $s_X(\tau)$. Given $\tau$ we run two processess in...
parallel: one generates finite models and checks whether they are
models of $X$ and then if $\tau$ is satisfied in them, and another one gen-
erates proof and checks if one of them proves $\neg \tau$. If the first process
succeeds, we return the size of the model found, and if the second
succeeds we return 1. Lemma 1 shows that one of the processess suc-
cceeds. This function is correct as it returns the size of a finite model
if it exists, and otherwise the formula is not satisfiable, so the return
value does not matter.

In the proof above we can also make the function return the smallest
finite model by enumerating the models ordered by size, but it is not
needed in this paper.

3 Small model size and small Afrodite
strategies

In this section we show that from a finite countermodel of a given
size we can construct a small Afrodite strategy. First we introduce
games, strategies and introduce tools to replace some variables in
a strategy. Then we use these tools to show a limit on the set of
eigenvariables in the game depending on the size of the small model,
proving the Theorem 1.

3.1 Better variables

Notation. We define substitutions applied to a formula: $\varphi[x/y]$ is the
formula $\varphi$ with all free occurrences of $x$ replaced by $y$ in a capture-
avoiding fashion. The disjuncts of a formula $\alpha \lor \beta$ are $\alpha$ and $\beta$, and
for a formula that is not a disjunction the whole formula is called a
disjunct. We understand the formula $\alpha \lor \beta \lor \gamma$ to mean $\alpha \lor (\beta \lor \gamma)$,
but understanding it as a disjunction of three disjuncts would also
be possible with minor technical changes.

Games. We show how the small model property can be expressed
in terms associated with the notion of intuitionistic games for first-
order logic as defined in Section 5 of the work by Sørensen and
Urzczyn [6]. The game describes a search for a proof and has two
players: Eros, tryig to prove the judgement and Afrodite, showing
that it can’t be proven. We write $\Gamma \vdash \tau \rightsquigarrow_{\text{move}} \Gamma' \vdash \tau'$ to state that the positions $\Gamma \vdash \tau$ and $\Gamma' \vdash \tau'$ are connected with a turn. A game is a sequence of positions connected by turns, i.e. a sequence $P_1, \ldots, P_n, \ldots$ such that $P_i \rightsquigarrow_{\text{move}} P_{i+1}$ for each $i \in \mathbb{N}$. Possible moves are shown in Fig. 2. We omit the subscript “move” when it is not needed or clear from the context. A game starts in a position $\Gamma \vdash \tau$ and begins with Eros’ move, followed by Afrodite’s move which determines the next turn. If Eros reaches a final position he wins, otherwise the game is infinite and Afrodite wins. We call $\Gamma \vdash \tau$ the precedent and $\Gamma' \vdash \tau'$ the antecedent of the move. Some turns have players associated with them: if Afrodite makes a choice in a move we call the precedent an Afrodite’s position, and if Eros makes a choice we call it an Eros position. A disjunct of $\tau$ is an aim. In order to avoid confusion with classical provability we write $\Gamma \vdash_{\text{IFOL}} \tau$ to denote that $\tau$ is provable from $\Gamma$ in first-order intuitionistic logic. If the exact proof $p$ is important we use the notation $\Gamma \vdash_{\text{IFOL},p} \tau$.

Strategies. A strategy is a tree of nodes labeled by positions linked by edges labeled by turns, which we call moves. In each position either one or none of the players makes a choice. In a position with no choice the next position is determined by game rules (see Figure 2) and the corresponding turn must appear in the strategy. For Afrodite strategy the tree consists of non-final positions and all paths are infinite as well as the tree has at least one move in each Afrodite’s position and all the possible moves (up to renaming of fresh and bound variables) in Eros positions. A final position is a position in which $\tau \in \Gamma$ or $\bot \in \Gamma$. For Eros strategy all paths end at a final position and the tree has at least one move in each Eros’ position and all the possible moves (up to renaming of fresh and bound variables) in Afrodite positions. It should be obvious that if Eros cannot make a move that introduces something new to the game, he is forced to replay one of the previous moves and Afrodite wins.

Ordering of variables. Intuitively speaking we would like to capture the fact that one variable is “better” than the other if all the information that was known about the “worse” variable is kept and possibly extended with new facts. More formally we say $x_1 \preceq_{\Gamma} x_2$ when for every formula $\tau$
if $\Gamma \vdash_{\text{IFOL}} \tau$ then $\Gamma \vdash_{\text{IFOL}} (\tau[x_1/x_2])$.

The relations $\prec_\Gamma$ and $\sim_\Gamma$ are defined in the following way:

\[ x_1 \sim_\Gamma x_2 \quad \text{when} \quad x_1 \preceq_\Gamma x_2 \land x_2 \preceq_\Gamma x_1, \]

\[ x_1 \prec_\Gamma x_2 \quad \text{when} \quad x_1 \preceq_\Gamma x_2 \land x_2 \not\sim_\Gamma x_1. \]

In cases when $\Gamma$ is clear we omit it for brevity.

**Proposition 3.** For any $\Gamma$, the relation $\preceq_\Gamma$ is a quasiorder, but not a partial order.

**Proof.** The relation $\preceq_\Gamma$ is trivially reflexive and transitivity follows immediately from definition with help of an observation that $\tau[x_1/x_2][x_2/x_3] = \tau[x_1/x_3]$, so it is a quasiorder.

If we choose two distinct fresh variables $x_\alpha$ and $x_\beta$, i.e. not in $\text{FV}(\Gamma)$, we have $x_\alpha \preceq x_\beta$ and $x_\beta \preceq x_\alpha$, but $x_\alpha \neq x_\beta$, so $\preceq$ is not a partial order. \qed

This leads to the conclusion that the only important variables are those that are **maximal in the $\preceq$ relation**, as we can replace all the other variables with their maximal counterparts.

**Proposition 4.** Let $\tau$ be a formula such that $\text{FV}(\tau) = x_1, \ldots, x_n$. If $\Gamma \vdash_{\text{IFOL}} \tau$, then $\Gamma \vdash_{\text{IFOL}} \tau[x_1/x_1', \ldots, x_n/x_n']$, where, for all $i$, $x_i \preceq x_i'$.

**Proof.** We apply the definition of $\preceq$ for each $x_i$ in turn. \qed

**Proposition 5.** If Eros or Afrodite has a strategy in position $\mathcal{P} = \Gamma \vdash \tau$ and if at some position $\mathcal{P}_{i_0}$ and all subsequent positions in that strategy we have $x'_1 \preceq x_1, \ldots, x'_n \preceq x_n$ then we can replace all occurrences of variables $x'_i$ with $x_i$ at $\mathcal{P}_{i_0}$ and the same player still has strategy in position $\mathcal{P}_{i_0}$.

**Proof.** The replacement is done while looking at the whole game tree and with maximum knowledge (i.e. trueness of predicates through the whole game tree) about variables, which is not a problem since our aim is to construct the strategy for Afrodite, which implies knowledge of all the possible turns.
Suppose Afrodite has a strategy in $\mathcal{P}$. Let us focus on a path in the tree of the strategy

$$\Gamma_1 \vdash \tau_1 \leadsto \Gamma_2 \vdash \tau_2 \leadsto \ldots \leadsto \Gamma_n \vdash \tau_n \leadsto \ldots$$

Each of the moves may add something to $\Gamma$, but nothing is removed and we can separate the newly added facts:

$$\Gamma_2 = \Gamma_1, \psi_1 \quad \ldots \quad \Gamma_{n+1} = \Gamma_n, \psi_n$$

Another view of the new facts would be to separately keep track of those referring to the variables $x_i$:

$$\Gamma_n = \Gamma_1, \Delta_n, \hat{\Delta}_n$$

where $\Delta_n$ has all the facts that reference the variables $x_i$ and $\hat{\Delta}_n$ the others. To make the notation concise we write $\Gamma, \Delta_1, \Delta_2 \vdash \tau$ as a shorthand for $\Gamma, (\Delta_1 \cup \Delta_2) \vdash \tau$.

Instead of taking the original path we can take the following one

$$\Gamma_1 \vdash \tau_1 \leadsto \Gamma_2' \vdash \tau_2' \leadsto \ldots \leadsto \Gamma_n' \vdash \tau_n' \leadsto \ldots$$

where $\Gamma_n' = \Gamma_{n-1}, \psi_{n-1}', \psi_n' = \psi[x'_i/x_i]$ and $\tau_n' = \tau_n[x'_i/x_i]$. Or, viewed in the terms of $\Delta$s,

$$\Gamma_n' = \Gamma_1, \Delta_n', \hat{\Delta}_n$$

where $\Delta_n' = \Delta_n[x'_i/x_i]$.

To make this construction sound we need to show that $\Gamma_1, \Delta_n', \Delta_n \vdash x'_i \preceq x_i$ and that the move $(\Gamma_1, \Delta_n', \hat{\Delta}_n \vdash \tau_n') \leadsto (\Gamma_1, \Delta_n', \hat{\Delta}_n' \vdash \tau_n')$ is possible. The first part follows directly from Proposition 4. For the second part we show how to adapt the original move $\Gamma_n \vdash \tau_n$. The possible moves are listed in Fig. 2. Only two of them have direct interaction with non-fresh variables: in (a4) and (b5) Eros is free to choose any variable and the replacement variables $x_1, \ldots, x_n$ are already available, so would not lead him to winning the game, otherwise he could have played this move in the original strategy.

The other case is when Eros has a strategy in $\mathcal{P}$. The substitution is almost the same as in the previous case except some nonfinal positions might become final, as the set of facts known about $x'_i$ is bigger or equal to those that were known about $x_i$, as $x'_i \preceq x_i$. Final positions remain final by Proposition [4] Nonfinal positions might become final, but it only makes Eros win faster. \hfill \Box
3.2 Construction of the strategy

Small strategies of Afrodite. With the aim of relating the size of the Afrodite strategy and the size of the small model we define a notion of a small strategy. Proposition 5 suggest the following definition. Since we know that using only the maximal variables is sufficient in the game, we define small strategy of Afrodite for a formula τ from class X as a strategy that has at most \( s_X(\tau) \) classes of abstraction of maximal variables. Given a small countermodel \( M \) of a formula we aim to construct a small winning Afrodite strategy \( S \), i.e. one that gives at least one possible response for each possible Eros’ move. For a given turn \( t = \Gamma \vdash \tau \) we need to choose a response to Eros moves. We associate a state \( s \in M \) with each turn \( t \). Our strategy has the following invariant that holds at each turn:

\[
\exists_{\rho:FV(\Gamma) \to A} (\rho, s \models \Gamma) \land \forall_{\rho:FV(\Gamma) \to A} (\rho, s \models \Gamma \rightarrow \rho, s \not\models \tau),
\]

and the sets of maximal variables corresponds to states of the small countermodel. The part of the invariant quantified with \( \exists \) is called the existential part and the part quantified with \( \forall \) is called the universal part.

Fig. 2 lists possible moves and the choices players make. We define a strategy for Afrodite and she makes a choice in cases marked with * in the figure. Afrodite should choose in the indicated moves:

(a1) We choose \( \Gamma, \gamma \vdash \tau \) when \( \rho, s \models \Gamma, \gamma \).
(a2) We choose \( \beta \) when \( \rho, s \models \Gamma, \beta \).
(b2) We choose \( \beta \) when \( \rho, s \models \beta \).

In case of (b1) and (b4) the current model state \( s \) might needs to be advanced to some subsequent state to keep the invariant.

We still need to show that each move preserves the invariant.

Proposition 6. At each position \( P : \Gamma \vdash \tau \) the invariant (1) holds.

Proof. We assume the notation of Figure 2 and show that each move preserves the invariant (1).

(a1) We have two possibilities:
Moves manipulating assumptions:

*a1*) If \( \alpha \) is an assumption \( \beta \rightarrow \gamma \) then Afrodite chooses between positions \( \Gamma, \gamma \vdash \tau \) and \( \Gamma \vdash \beta \).
*a2*) If \( \alpha \) is an assumption \( \beta \lor \gamma \) then Afrodite chooses between positions \( \Gamma, \beta \vdash \tau \) and \( \Gamma, \gamma \vdash \tau \).

a3) If \( \alpha \) is an assumption \( \beta \land \gamma \) then the next position is \( \Gamma, \beta, \gamma \vdash \tau \).

a4) If \( \alpha \) is an assumption \( \forall x \varphi \) then Eros chooses a variable \( y \) and the next position is \( \Gamma, \varphi[y/x] \vdash \tau \).

Moves manipulating the proof goal:

a5) If \( \alpha \) is an assumption \( \exists x \varphi \) then the next position is \( \Gamma, \varphi[y/x] \vdash \tau \) where \( y \) is a fresh variable.

b1) If \( \alpha \) is an aim of the form \( \beta \rightarrow \gamma \) the next position is \( \Gamma, \beta \vdash \gamma \).

*b2*) If \( \alpha \) is an aim of the form \( \beta \lor \gamma \) then Afrodite chooses between positions \( \Gamma \vdash \beta \) and \( \Gamma \vdash \gamma \).

b3) If the aim \( \alpha \) is an atom or a disjunction the next position is \( \Gamma \vdash \alpha \).

b4) If \( \alpha \) is an aim of the form \( \forall x \varphi \) the next position is \( \Gamma \vdash \varphi[y/x] \) where \( y \) is fresh.

b5) If \( \alpha \) is an aim of the form \( \exists x \varphi \) then Eros chooses a variable \( y \) and the next position is \( \Gamma \vdash \varphi[y/x] \).

Fig. 2. Table of moves in position \( \Gamma \vdash \tau \) for the intuitionistic game [6, fig. 11, p. 32]. In each move Eros chooses a formula \( \alpha \) - either an assumption or an aim, and the move is selected from this table according to the \( \alpha \) chosen.

- In case \( \rho, s \models \Gamma, \gamma \): we choose \( \Gamma, \gamma \vdash \tau \). The existential part of the invariant follows directly from the invariant of the previous step. For the universal part suppose the opposite, i.e \( \rho, s \not\models \tau \), so for given \( \rho \) we either have contradiction with \( \rho, s \not\models \tau \) from invariant of the previous step or \( \rho, s \not\models \gamma \), but then we would not choose this move for the strategy.
- Otherwise \( \rho, s \not\models \Gamma, \gamma \) and we choose \( \Gamma \vdash \beta \). The existential part of the invariant remains true as \( \Gamma \) does not change. For universal part suppose \( \rho, s \models \beta \), but then \( \rho, s \models \gamma \rightarrow \beta \), which is in contradiction with the invariant from the previous step.

(a2) Once again we have two possibilities:
- In case \( \rho, s \models \Gamma, \beta \): we choose \( \Gamma, \beta \vdash \tau \). The existential part of the invariant follows directly from the invariant of the previous step. The universal part is the same as in the corresponding point of the move (a1).
- Otherwise \( \rho, s \not\models \Gamma, \gamma \) and the proof is the same as in the corresponding point of (a1).
(a3) The existential part is true because $\rho, s \models \beta, \gamma$ follows from $\rho, s \models \beta \land \gamma$. The universal part is proven by simply applying the definition of $\models$.

(a4) We can choose any value for $\rho(y)$. Existential part: from the invariant in the previous move we have $\rho, s \models \forall x \varphi$, so we apply definition of $\models$ to get $\rho, s \models \varphi[y/x]$. Universal part: suppose that $\rho, s \models \Gamma, \varphi[y/x]$ and $\rho, s \models \tau$. But this means $\rho, s \models \Gamma$ which implies have a contradiction with $\rho, s \models \neg \tau$ from the previous move.

(a5) Since $\rho, s \models \exists x \varphi$ we know that there exists $\hat{x}$ such that $\rho[\hat{x}/x], s \models \varphi$. In the existential part we just need to take $\rho(y) = \rho(\hat{x})$. Universal part: identical with the universal part of (a4).

(b1) Using the assumption we have a state $s' \geq s$ such that $\rho, s' \models \Gamma, \beta$ but $\rho, s' \models \neg \gamma$. We advance $s$ to $s'$. The existential part is trivially true. For the universal part: suppose $\rho, s' \models \Gamma, \beta$ and $\rho, s' \models \gamma$. This is in contradiction with $\rho, s \models \beta \rightarrow \gamma$.

(b2) We have the following cases:

- $\rho, s \models \neg \beta$: the set of assumptions does not change so the existential part is proven by applying the existential part from the previous move. For the universal part, $\rho, s \models \beta$ is exactly the assumption of the case under investigation.

- otherwise $\rho, s \models \beta$. We choose the position $\Gamma \models \gamma$; the existential part is the same as in the previous step. For the universal part suppose $\rho, s \models \gamma$: then $\rho, s \models \beta \land \gamma$ contradicts the invariant $\rho, s \models \neg \beta \land \gamma$ from the previous move.

(b3) The existential part is the same as in the previous move. The universal part is the same as in the second bullet of (b2).

(b4) We can choose any value for $\rho'(y)$. The existential part is true since $\Gamma$ is the same as previously and the valuation of $y$ does not affect it. The universal part is true since $\rho', s \models \varphi[y/x]$. Then by definition $\rho, s \models \varphi$, which is in contradiction with the invariant from the previous move.

(b5) The existential part is the same as in the previous move. For the universal part suppose $\rho, s \models \varphi[y/x]$. Then by definition of $\models$ we have $\rho, s \models \exists x \varphi$, which is a contradiction with the invariant of the previous step.

□
The constructed strategy is small: elements of $A_s$ correspond to $\simeq$-classes and the valuation $\rho$ proves that all the variables fit in $s_X(\varphi)$ classes as the size of the model is $s_X(\varphi)$. This proves the following:

**Theorem 1.** For all classes $X$ that have the small model property and all formulas $\tau \in X$, if a strategy of Afrodite exists for $\tau$ then a small strategy of Afrodite for $\tau$ also exists.

### 4 Small model size and the Arcadian automata

Here we show a limit on resources of Arcadian automata [7] checking derivability of a formula $\varphi$ from an effectively axiomatized class $X$ that has the finite model property. Theorem 2 shows a limit on the size of the set of eigenvariables of the automaton in terms of the numbers of variables and subformulas in $\varphi$ and the size of the small model. We reason only about automata that are translated from a formula as defined in Section 4 of [7]. In Section 4.1 we introduce Arcadian automata, show how to replace variables in their runs in Section 4.2, and limit the size of the set of eigenvariables in 4.3.

#### 4.1 Arcadian automata

**Notation** We already know that $\simeq$-maximal variables play a crucial role. Given a set of facts $\Gamma$ we denote by $\bar{\Gamma}$ the set obtained from $\Gamma$ by selecting only those facts $\gamma$ that do have only maximal variables in $\text{FV}(\Gamma)$. An **Arcadian automaton** is a tuple $\langle A, Q, q^0, I, i, \text{fv} \rangle$, where $A$ is a finite tree, $Q$ and $I$ are sets of states and instructions with $i$ mapping states to instructions, $\text{fv} : A \to P(A)$ describes the binding of variables and $q^0$ and $\varphi^0$ are the initial state and node. The function $\text{fv}$ satisfies the condition that for all $v$ either $v$ is a leaf or $\text{fv}(v) = \bigcup_{w \in B(v)} \text{fv}(w)$ where $B(v) = \{w \mid v \text{ succ } w\}$ and $\succ$ is the usual predicate of being a successor. An **instantaneous description** is $\langle q, \kappa, V, w, w', S \rangle$ where $q \in Q$ and $\kappa \in A$ are the current state and node, $V$ is a set of eigenvariables, $w$ and $w'$ are interpretations of bindings and $S$ is the store. For more details see [7].

#### 4.2 Better variables in Arcadian automata

**Equivalent positions** We say that the position $\Gamma \vdash \tau$ and $\Gamma' \vdash \tau'$ are equivalent when $\bar{\Gamma} = \bar{\Gamma'}$ and $\tau = \tau'$. 


Proposition 7. Suppose $\Gamma, \hat{\Gamma} \vdash_{\text{IFOL}} M : \tau$ where for some $\alpha$ and $\alpha' \geq \alpha$ such that $\alpha \in \text{FV}(\Gamma)$ and $\alpha \not\in \text{FV}(\hat{\Gamma})$. If $\Gamma = x_1 : \tau_1, \ldots, \Gamma \vdash_{\text{IFOL}} x_n : \tau_n$ then $\hat{\Gamma}, \Gamma, \Gamma' \vdash_{\text{IFOL}} M[x_1/x'_1] \ldots [x_n/x'_n] : \tau[\alpha/\alpha']$ where $\Gamma' = x'_1 : \tau_1[\alpha/\alpha'], \ldots, x'_n : \tau_n[\alpha/\alpha']$ and $x'_1, \ldots, x'_n$ are fresh variables, i.e. $x'_i \notin \text{FV}(M)$.

Proof. Proof is by induction over the length of the proof of $\tau$. We look at the last rule in the proof. In most of the cases the conclusion follows by simple application of the inductive hypothesis, but there are three rules that change the environment, namely ($\lor E$), ($\rightarrow I$) and ($\exists E$) and the proof is more subtle for them. Let us focus on the ($\rightarrow I$) rule. If $(x_i : \tau_i) \in \hat{\Gamma}$ we do not need to change anything, in the other case we know that $(x_i : \tau_i) \in \Gamma$ and we apply the induction hypothesis and use the assumption $(x'_i : \tau_i[\alpha/\alpha']) \in \Gamma'$ for the $\lambda$-abstraction. We can now remove the variable $x_i : \tau_i$ as it is not referenced in $M[x_1/x'_1] \ldots [x_n/x'_n] : \tau[\alpha/\alpha']$.

The induction base is the $\text{var}$ rule, since the proof must begin with this rule, and the correctness of replacing $\alpha$ with $\alpha'$ follows immediately from the definition of $\Gamma'$.

\[\square\]

Proposition 8. If $\Gamma \vdash_{\text{IFOL}} M : \tau$, $\alpha$ and $\alpha'$ are variables in $M$ such that $\alpha \preceq \alpha'$ then $\Gamma \vdash_{\text{IFOL}} M[\alpha/\alpha'] : \tau[\alpha/\alpha']$ and $\alpha \not\in \text{FV}(\tau)$.

Proposition 9. If $\Gamma \vdash_{\text{IFOL}, p} M : \tau$ then there exists $M'$ and $p'$ such that $\Gamma \vdash_{\text{IFOL}, p'} M' : \tau$ and in each step $\Gamma' \vdash \tau'$ of $p'$ only maximal variables are mentioned in $\tau'$.

Proof. Apply Proposition 8 sequentially to each nonmaximal variable.

\[\square\]

4.3 Loquacious runs

Note that our logic has the subformula property ([7]). This means that there is only a limited number of possible targets $\tau$. Let us review fragments of an automaton run $p_0 \rightarrow_\alpha \Gamma \vdash \tau \rightarrow_\beta \Gamma' \vdash \tau \rightarrow_\gamma p_1$. If $\Gamma \vdash \tau$ and $\Gamma' \vdash \tau$ are equivalent, then a part of the run $\rightarrow_\beta$ is removable, i.e. there exists a run of the same automaton $p_p \rightarrow_\alpha \Gamma \vdash \tau \rightarrow_\gamma \tilde{p}_1$ where $\tilde{\gamma}$ and $\tilde{p}_1$ are obtained from $\gamma$ and $p_1$ by replacing some variables by their maximal counterparts. Otherwise that fragment is not removable, as new fact about the maximal variables are discovered, but the number of such non-removable runs is limited:
there are at most \( s_X(\varphi) \) maximal variables. Suppose \( v(\varphi) \) is the number of variables in \( \varphi \) and \( f(\varphi) \) is the number of subformulas in \( \varphi \). The maximum size of an environment \( \Gamma \) is \( \mu = f(\varphi) \cdot s_X(\varphi)^{v(\varphi)} \). Each non-trivial step has to either add something to the environment or change the target \( \tau \) as otherwise the previous state is repeated. We have at most \( \mu \) possible targets, so after at most \( \mu \) steps the target repeats and in the worst case each step introduces a new variable, so the maximum size of \( V \) in the automaton is \( \mu^2 \).

This proves the following:

\textbf{Theorem 2.} Let \( \tau \) be a formula from an effective class \( X \) that has the small model property. For a given accepting run of an Arcadian automaton for that formula there exists an accepting run of the same automaton with the same result that has the property \( |V| \leq \mu^2 \), where \( V \) is the working domain of the automaton and \( \mu \) is the maximum size of the environment defined in the previous paragraph.

\section{Conclusion}

The small model size is directly related to important resources in games and automata for checking provability. In terms of games, the elements of models directly correspond to abstraction classes of maximal elements of a quasiorder on eigenvariables that captures the relation of having more information available about a variable.

For automata the number of such maximal elements can be directly related to the size of set of eigenvariables \( V \); the dependency is exponential, caused by the necessity of representing the eigenvariables that correspond to non-maximal elements of the quasiorder.

These observations lead to an idea for implementing proof theory bases proves in a manner that would not be substantially less powerful than those based on model theory. More specifically we suggest that \( V \) should not be represented syntactically but rather as an abstraction class of the quasiorder.

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