LARGE TIME BEHAVIOR IN THE LOGISTIC KELLER-SEGEL MODEL VIA MAXIMAL SOBOLEV REGULARITY

Xinru Cao*

Institute of Mathematical Sciences, Renmin University
Beijing 100872, China
and
Institut für Mathematik, Universität Paderborn
33098 Paderborn, Germany

(Communicated by Michael Winkler)

Abstract. The fully parabolic Keller-Segel system with logistic source
\[
\begin{align*}
\partial_t u &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, & (x,t) \in \Omega \times (0,T), \\
\tau \partial_t v &= \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\
\nabla u \cdot \nu &= \nabla v \cdot \nu = 0, & (x,t) \in \partial \Omega \times (0,T),
\end{align*}
\]
is considered in a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) under Neumann boundary conditions, where $\kappa \in \mathbb{R}$, $\mu > 0$, $\chi > 0$ and $\tau > 0$. It is shown that if the ratio $\frac{\chi}{\mu}$ is sufficiently small, then any global classical solution $(u,v)$ converges to the spatially homogenous steady state $(\kappa + \mu, \kappa + \mu)$ in the large time limit. Here we use an approach based on maximal Sobolev regularity and thus remove the restrictions $\tau = 1$ and the convexity of $\Omega$ required in [17].

1. Introduction. In this paper, we consider
\[
\begin{align*}
\partial_t u &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, & (x,t) \in \Omega \times (0,T), \\
\tau \partial_t v &= \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\
\nabla u \cdot \nu &= \nabla v \cdot \nu = 0, & (x,t) \in \partial \Omega \times (0,T),
\end{align*}
\]
where $\kappa \in \mathbb{R}$, $\mu > 0$, $\chi > 0$ and $\tau > 0$, $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary and $\nu$ denotes the outward normal vector on $\partial \Omega$. The initial distribution $(u_0,v_0)$ is a pair of nonnegative functions satisfying
\[
\begin{align*}
u_0 & \in C^0(\bar{\Omega}) \text{ with } u_0 \neq 0, \quad v_0 \in W^{1,\infty}(\Omega).
\end{align*}
\]
Partial differential system of \([\text{1}]\) is called Keller-Segel system, which describes cells’ migration and their movement tendency. Here the density of the cell population and chemical substance concentration are denoted by $u$ and $v$, respectively. Let $\kappa = \mu = 0$, \([\text{1}]\) becomes the classical Keller-Segel system. An interesting and challenging problem is to detect the generation of singularity of solutions, which has been proved for two- and higher-dimensional cases \([\text{2}]\) \([\text{9}]\). Conditions for global
existence and characteristics of the large time behavior of solutions can be found in [15, 3].

In contrast to the classical Keller-Segel system, a logistic source restraining the ultimate growth of cells’ population has been included in (1) if $\kappa > 0$, $\mu > 0$. One may expect that the interplay among diffusion, aggregation and logistic growth restriction can result in colorful dynamics [16, 6]. As far as we know, only few results concerning finite time blowup has been found except for that in [14], where $N \geq 5$ is required. It is also shown that the logistic source can prevent blow up whenever $N \leq 2$, or $\mu$ is sufficiently large [11, 8, 13].

Going beyond the boundedness results, the study of global dynamics is a natural continuation [17], we refer to [10, 12, 2] for Keller-Segel model including multiple species. We note that (1) can be seen as a subsystem in a multiple species model. In the case $\tau = 0$, the results from [10, 12, 2] can be summarized as: if the quotient $\frac{\chi}{\mu}$ is suitably small, (1) admits a global classical solution and it converges to $(\kappa \mu, \kappa \mu)$.

Considering the fully parabolic system, that is $\tau > 0$, [17] proves the same conclusion under the restrictions that $\tau = 1$ and $\Omega$ is convex, which are quite critical in the proof. Under these assumptions, the combination $y(x, t) = u + \frac{\chi}{\mu} |\nabla v|^2$ satisfies a scalar parabolic inequality

$$y_t \leq \Delta y - y + \frac{C}{\mu}$$

with some $C > 0$ for all $t > 0$ [13]. The comparison principle immediately yields that

$$\limsup_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \limsup_{t \to \infty} y(x, t) \leq \frac{C}{\mu}. \quad (4)$$

With this information, one can finally show convergence combined on the basis of estimates for the Neuman semigroup. However, if $\tau \neq 1$, the first step already fails; we can not find any combination like $y(x, t)$ satisfying a single parabolic inequality on its own. In a recent paper [1], the authors develop a functional approach to prove convergence for global bounded solutions if $\frac{\chi}{\mu}$ is small. This approach also works for $\tau \neq 1$.

It is our purpose in the present paper to investigate how the size of the quotient $\frac{\chi}{\mu}$ affects the global dynamics for any choice of $\tau > 0$. We find a replacement of (4):

$$\limsup_{t \to \infty} \|u(\cdot, t)\|_{L^p(\Omega)} \leq \frac{C}{\mu}$$

with sufficiently large $p$ and for some $C > 0$, which is sufficient for the conclusion in [17]. Our main result reads as follows:

**Theorem 1.1.** Suppose $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is bounded. Let $\chi > 0$, $\mu > 0$ and $\kappa \in \mathbb{R}$. Then there exists $\theta_0 > 0$ with the property that if

$$\frac{\chi}{\mu} < \theta_0,$$

then for all initial data $(u_0, v_0)$ fulfilling [2], the system (1) possesses a global classical solution

$$(u, v) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)))^2. \quad (7)$$

Moreover, $(u, v)$ satisfies

$$\|u(\cdot, t) - \frac{\kappa_+}{\mu} L^\infty(\Omega) \| \to 0, \quad \|v(\cdot, t) - \frac{\kappa_+}{\mu} L^\infty(\Omega) \| \to 0 \text{ as } t \to \infty. \quad (8)$$
2. Preliminaries. Before going into details, we first introduce the following local existence result for \( I \). The proof can be found in many previous work \([15]\).

**Lemma 2.1.** Suppose \( \Omega \subset \mathbb{R}^N \) with \( N \geq 1 \), is a bounded domain with smooth boundary, \( \mu > 0 \) and \( \chi > 0 \), and \( u_0 \in C^0(\Omega) \) and \( v_0 \in W^{1,q}(\Omega) \) (with some \( q > N \)) both are nonnegative. Then there exist \( T_{\text{max}} \in (0, \infty) \) and a pair nonnegative functions \( (u,v) \in (C^0(\Omega \times [0,T_{\text{max}}]) \cap C^{2,1}_r(\Omega \times (0,T_{\text{max}})))^2 \) classically solving \( \{1\} \) in \( \Omega \times (0,T_{\text{max}}) \). Moreover, if \( T_{\text{max}} < \infty \), then

\[
\limsup_{t \to T_{\text{max}}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{9}
\]

The main ingredient in this paper heavily relies on the maximal Sobolev regularity with time weighted function, as obtained in \([4, 18]\).

**Lemma 2.2.** Let \( \tau > 0 \), \( r \in (1, \infty) \), and \( T \in (0, \infty] \). We consider the following evolution equation

\[
\begin{aligned}
v_t &= \Delta v - v + u, \quad (x,t) \in \Omega \times (0,T), \\
\partial_v v &= 0, \quad (x,t) \in \partial \Omega \times (0,T), \\
v(x,0) &= v_0(x), \quad x \in \Omega.
\end{aligned} \tag{10}
\]

For each \( v_0 \in W^{1,r}(\Omega) \) and any \( u \in L^r((0,T);L^r(\Omega)) \), there exists a unique solution \( v \in W^{1,r}(0,T;L^r(\Omega)) \cap L^r(0,T;W^{2,r}(\Omega)) \).

Moreover, there exists \( C_r > 0 \), such that if \( t_0 \in (0,T) \), \( v(\cdot,t_0) \) satisfies \( v(\cdot,t_0) \in W^{1,r}(\Omega) \) with \( \frac{\partial v(\cdot,t_0)}{\partial \nu} = 0 \), we have

\[
\begin{aligned}
\int_0^T \int_\Omega e^\tau s |\Delta v(x,s)|^r dx ds &\leq C_r \int_0^T \int_\Omega e^\tau s u^r dx ds \\
&\quad + C_r \tau e^\tau t_0 \left( \|v(\cdot,t_0)\|_{L^r(\Omega)} + \|\Delta v(\cdot,t_0)\|_{L^r(\Omega)} \right), \tag{11}
\end{aligned}
\]

where \( C_r \) depends on \( r \).

3. Large time behavior of \( L^p \) norm. As already mentioned in the introduction, our first and the most important goal is to identify the large time behavior of \( \|u(\cdot,t)\|_{L^p(\Omega)} \). The proof is very similar to that of Lemma 3.1 in \([18]\). We have

**Lemma 3.1.** Let \( \theta := \frac{\tau}{p} \) and \( (u,v) \) be a solution of \( \{1\} \) on \((0,T_{\text{max}})\). For all \( p \in (1, \infty) \), we can find \( \theta_1 := \theta_1(p) > 0 \) and \( C := C(p) > 0 \) such that if \( \theta < \theta_1 \), then

\[
\|u(\cdot,t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0,T_{\text{max}}). \tag{12}
\]

Moreover, if \( T_{\text{max}} = \infty \), we have

\[
\limsup_{t \to \infty} \|u(\cdot,t)\|_{L^p(\Omega)} \leq \frac{C}{\mu}. \tag{13}
\]

**Proof.** First we see that for any \( a,b > 0 \), Young’s inequality provides \( k_p > 0 \) such that

\[
ab \leq \frac{1}{4} a^{\mu+1} + k_p b^{p+1}. \tag{14}
\]

Let \( C_{r+1} \) denote the constant from Lemma \( 2.2 \) for \( r \in (1, \infty) \). Now we can find \( \theta_1 > 0 \) small enough such that

\[
C_{p+1} k_p \theta^{p+1} \leq \frac{1}{2} \quad \text{for all } \theta < \theta_1. \tag{15}
\]
We multiply the first equation in (1) by $u^{p-1}$ to obtain that
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 = (p-1) \chi_{\Omega} \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \kappa \int_{\Omega} u^p - \mu \int_{\Omega} u^{p+1}
\]
for all $t \in (0, T_{\text{max}})$. Now Young’s inequality (14) yields that
\[
(\kappa + \frac{p+1}{\tau p}) \int_{\Omega} u^p \leq \frac{\mu}{4} \int_{\Omega} u^{p+1} + k_p \mu^{-p} (\kappa + \frac{p+1}{\tau p})^{p+1} |\Omega|,
\]
for all $t \in (0, \infty)$. Let $s_0 \in (0, \infty)$, using the Gronwall’s inequality to the above inequality we obtain that
\[
\int_{\Omega} u^p(\cdot, t) \leq e^{-\frac{\mu}{2} (t-s_0)} \int_{\Omega} u^p(\cdot, s_0) - \frac{\mu p}{2} \int_{s_0}^t e^{-\frac{\mu}{2} (t-s)} \int_{\Omega} u^{p+1} + k_p \mu^{-p} (\kappa + \frac{p+1}{\tau p})^{p+1} |\Omega| \int_{s_0}^t e^{-\frac{\mu}{2} (t-s)} ds
\]
for all $t \in (t_0, T_{\text{max}})$. An application of Lemma 2.2 implies
\[
\int_{\Omega} u^p(\cdot, t) \leq e^{-\frac{\mu}{2} (t-t_0)} \int_{\Omega} u^p(\cdot, t_0) + e^{-\frac{\mu}{2} (t-t_0)} C_{p+1} \|v(\cdot, t_0)\|_{W^{2,p+1}}^p + \frac{p \mu}{2} (\frac{1}{2} - c_1 \theta^{p+1}) \int_{t_0}^t e^{-\frac{\mu}{2} (t-s)} \int_{\Omega} u^{p+1} + c_1 \mu^{-p} (\kappa + \frac{p+1}{\tau p})^{p+1} |\Omega| \int_{t_0}^t e^{-\frac{\mu}{2} (t-s)} ds,
\]
for all $t \in (t_0, T_{\text{max}})$. In view of the condition (15), we see that the term $-p \mu (\frac{1}{2} - c_1 \theta^{p+1}) \int_{t_0}^t e^{-\frac{\mu}{2} (t-s)} \int_{\Omega} u^{p+1}$ is negative. Thus
\[
\int_{\Omega} u^p(\cdot, t) \leq e^{-\frac{\mu}{2} (t-t_0)} \int_{\Omega} u^p(\cdot, t_0) + e^{-\frac{\mu}{2} (t-t_0)} C_{p+1} \|v(\cdot, t_0)\|_{W^{2,p+1}}^p + \frac{c_1 \mu^{-p}}{\tau p} (\kappa + \frac{p+1}{\tau p})^{p+1} |\Omega| \int_{t_0}^t e^{-\frac{\mu}{2} (t-s)} ds
\]
for all \( t \in (t_0, T_{\text{max}}) \). This implies (12). Suppose that \( T_{\text{max}} = \infty \). Letting \( t \to \infty \), we obtain that
\[
\limsup_{t \to \infty} \int_\Omega u^p(\cdot, t) \leq \frac{C}{\mu^p}
\]
with some \( C > 0 \). Taking the \( p \)-th root on both sides, we finish the proof.

4. Large time behavior of \( L^\infty \) norm. Using the variation of constants formula to the second equation in (1) and the \( L^p-L^q \) estimate for the Neumann semigroup, we readily have the following:

**Lemma 4.1.** Suppose that \( T_{\text{max}} = \infty \), and
\[
\limsup_{t \to \infty} \| u(\cdot, t) \|_{L^p(\Omega)} \leq \frac{C}{\mu}
\]
for some \( C_1 > 0 \). Then for all
\[
\left\{ \begin{array}{lcl}
   r < \frac{Np}{(N-p)^+}, & p \leq N, \\
   r = \infty, & p > N,
\end{array} \right.
\]
there exists \( C_2 > 0 \) such that
\[
\limsup_{t \to \infty} \| \nabla v(\cdot, t) \|_{L^r(\Omega)} \leq \frac{C_2}{\mu}.
\]

**Lemma 4.2.** Let \( p > \frac{N}{2} \). Suppose that \( T_{\text{max}} = \infty \), and
\[
\limsup_{t \to \infty} \| u(\cdot, t) \|_{L^p(\Omega)} \leq \frac{C_1}{\mu}
\]
for some \( C_1 > 0 \). Then there is \( C_2 > 0 \) fulfilling
\[
\limsup_{t \to \infty} \| u(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{C_2}{\mu}.
\]

**Proof.** Assume that \( p \in (\frac{N}{2}, N) \) without loss of generality. First we find \( r < \frac{Np}{N-p} \), \( c_1 > 0 \) and \( t_0 > 0 \) such that
\[
\| u(\cdot, t) \|_{L^p(\Omega)} \leq \frac{c_1}{\mu},
\]
\[
\| \nabla v(\cdot, t) \|_{L^r(\Omega)} \leq \frac{c_1}{\mu}
\]
for all \( t > t_0 \). Let \( s_0 \in (t_0, \infty) \). Using the constants formula for the first equation in (1), we have
\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq \| e^{(t-s_0)\Delta} u(\cdot, s_0) \|_{L^\infty(\Omega)} + \chi \int_{s_0}^t \| e^{(t-s)\Delta} \nabla \cdot (u \nabla v)(\cdot, s) \|_{L^\infty(\Omega)} ds
\]
\[
+ \int_{s_0}^t \| e^{(t-s)\Delta} (ku - \mu u^2)(\cdot, s) \|_{L^\infty(\Omega)} ds
\]
for all \( t \in (s_0, s_0 + 2) \). We begin with
\[
\int_{s_0}^t \| e^{(t-s)\Delta} (ku - \mu u^2)(\cdot, s) \|_{L^\infty(\Omega)} ds \leq \int_{s_0}^t \sup_{s \geq 0} \| ku - \mu u^2 \|_{L^\infty(\Omega)} ds
\]
\[
\leq \int_{s_0}^t \frac{\kappa^2}{4\mu} ds \leq \frac{\kappa^2}{2\mu}
\]
for all \( t \in (s_0, s_0 + 2) \). By the \( L^p-L^q \) estimate for the Neumann heat semigroup, there exists a constant \( k_1 > 0 \) fulfilling
\[
\| e^{(t-s_0)\Delta} u(\cdot, s_0) \|_{L^\infty(\Omega)} \leq k_1 (t - s_0)^{-\frac{N}{2p}} \| u(\cdot, s_0) \|_{L^p(\Omega)} \tag{26}
\]
for all \( t \in (s_0, s_0 + 2) \). Let \( q \) satisfy \( \frac{1}{q} \in (\frac{1}{r'}, \frac{1}{r}) \), we can find \( r' > q \) such that \( \frac{1}{q} = \frac{1}{r'} + \frac{1}{r} \), and \( a = 1 - \frac{1}{r'} \in (0, 1) \). Using the Hölder inequality and the interpolation inequality, the second term can be estimated as
\[
\chi \int_{s_0}^t \| e^{(t-s)\Delta} \nabla v(\cdot, s) \|_{L^\infty(\Omega)} ds \\
\leq \chi \int_{s_0}^t (t - s)^{-\frac{1}{2} - \frac{N}{2p}} \| u \nabla v(\cdot, s) \|_{L^r(\Omega)} ds \\
\leq \chi \int_{s_0}^t (t - s)^{-\frac{1}{2} - \frac{N}{2p}} \| u(\cdot, s) \|_{L^{r'}(\Omega)} \| \nabla v(\cdot, s) \|_{L^{r}(\Omega)} ds \\
\leq \chi \int_{s_0}^t (t - s)^{-\frac{1}{2} - \frac{N}{2p}} \| u(\cdot, s) \|_{L^\infty(\Omega)} \| u(\cdot, s) \|_{L^{r}(\Omega)}^{-\frac{a}{1-a}} \| \nabla v(\cdot, s) \|_{L^{r}(\Omega)}^a ds \tag{27}
\]
for all \( t \in (s_0, s_0 + 2) \). Let \( M(t) = (t - s_0)^\frac{N}{2p} \| u(\cdot, t) \|_{L^\infty(\Omega)} \). Now we collect the above estimates (24-27) to see that
\[
M(t) \leq k_1 \left( \frac{c_1}{\mu} \right) + (t - s_0)^\frac{N}{2p} \kappa^2 \frac{\mu^2}{2p} \\
+ (t - s_0)^{-\frac{1}{2} - \frac{N}{2p} + \frac{N}{2p}(1-a)} c_1^{2-a} \chi \left( \sup_{t \in (s_0, s_0+2)} M(t) \right)^a \left( \frac{c_1}{\mu} \right)^{1-a}
\]
for all \( t \in (s_0, s_0 + 2) \). Let \( \overline{M} := \sup_{t \in (s_0, s_0+2)} M(t) \). Because \( \frac{1}{2} - \frac{N}{2q} + \frac{N}{2p}(1-a) > 0 \), we take the supremum on both sides of the above inequality to obtain that
\[
\overline{M} \leq k_1 \left( \frac{c_1}{\mu} \right) + 2^\frac{N}{2p} \kappa^2 \frac{\mu^2}{2p} + 2^{\frac{1}{2} - \frac{N}{2q} + \frac{N}{2p}(1-a)} c_1^{2-a} \theta \left( \frac{c_1}{\mu} \right)^{1-a} \overline{M}^a.
\]
Since \( a < 1 \), this implies that
\[
\overline{M} \leq \frac{C_2}{\mu}
\]
with some \( C_2 > 0 \). It also holds that
\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{C_2}{\mu}
\]
for all \( t \in (s_0 + 1, s_0 + 2) \). According to the choice of \( s_0 \), we conclude the assertion. \( \square \)

For convenience, we introduce the following notations
\[
U = u - \frac{\kappa_+}{\mu}, \quad V = v - \frac{\kappa_+}{\mu}, \tag{28}
\]
with
\[3374 \text{ XINRU CAO} \]
Proof. In view of the condition, Lemma 3.1 implies that $(U, V)$ satisfies
\[
\begin{align*}
U_t &= \Delta U - \chi \nabla \cdot (u \nabla V) - \max \{\kappa_+, \kappa_-\} U - \mu U^2, \\
\tau V_t &= \Delta V - V + U, \\
\nabla U \cdot \nu &= \nabla V \cdot \nu = 0, \\
U(x, 0) &= u_0(x) - \frac{\kappa_+}{\mu}, V(x, 0) = v_0(x) - \frac{\kappa_+}{\mu}, \quad x \in \Omega.
\end{align*}
\] (29)

Moreover, Lemmas 3.1 and 4.2 imply that there exists $c_0$ with $\kappa_+ > 0$ fulfilling the embedding
\[
D(A^\alpha) \hookrightarrow W^{2, \infty}(\Omega), \quad \text{if } 2\alpha - \frac{N}{p} > 2.
\] (30)

Moreover, $A$ generates a analytic semigroup $(e^{-tA})_{t \geq 0}$ and for all $\alpha > 0$ there is $c(p, \alpha) > 0$ such that
\[
\|A^\alpha e^{-tA} \varphi\|_{L^p(\Omega)} \leq c(p, \alpha)t^{-\alpha}\|\varphi\|_{L^p(\Omega)}
\] (31)
for all $t > 0$ and $\varphi \in L^p(\Omega)$. We now follow Lemmata 4.1, 4.2 and 5.1 in [17] to prove that:

Lemma 4.3. Suppose that $\kappa \neq 0$ and $p > \frac{N}{2}$. If it holds that $\frac{2}{p} < \theta_1(p)$. Then $T_{\text{max}} = \infty$. And there exists $C > 0$ such that we have
\[
\lim_{t \to \infty} \|\Delta v(\cdot, t)\|_{L^{\infty}(\Omega)} \leq \frac{C}{\mu}.
\] (32)

Proof. In view of the condition, Lemma 3.1 implies that $\|u(\cdot, t)\|_{L^p(\Omega)} (p > \frac{N}{2})$ is bounded. Thus we infer that $(u, v)$ is bounded and $T_{\text{max}} = \infty$ [11] Lemma 2.6]. Moreover, Lemmata 3.1 and 4.2 imply that there exists $c_1 > 0$ and $t_0 > 0$ fulfilling
\[
\|u(\cdot, t)\|_{L^{\infty}(\Omega)} \leq \frac{c_1}{\mu}, \\
\|\nabla v(\cdot, t)\|_{L^{\infty}(\Omega)} \leq \frac{c_1}{\mu}
\] (33) (34)
for all $t > t_0$. Now we fix $\eta \in (1, \frac{3}{2})$, then choose $\beta \in (\eta - 1, \frac{1}{2})$ and
\[
p > \frac{1}{2(\eta - 1)}.
\] (35)

Applying the constants variation formula to the first equation in (28) and employing the same argument used in [17] Lemma 4.2], we show that
\[
\lim_{t \to \infty} \|A^\alpha U(\cdot, t)\|_{L^p(\Omega)} \leq \frac{c_2}{\mu}
\] (36)
with some $c_2 > 0$. We again follow the idea of [17] Lemma 4.3] to find that
\[
\lim_{t \to \infty} \|A^\alpha V(\cdot, t)\|_{L^p(\Omega)} \leq \frac{c_3}{\mu}
\] (37)
with some $c_3 > 0$. By the embedding theorem (30), there is $c_4 > 0$ fulfilling
\[
\|\Delta v\|_{L^{\infty}(\Omega)} = \|\Delta V\|_{L^{\infty}(\Omega)} \leq \|V\|_{W^{2, \infty}(\Omega)} \leq c_4\|A^\alpha V\|_{L^p(\Omega)}.
\] (38)

The proof is complete.
5. **Refined estimate for** \( u \). In this section, we show that after suitably large time, \( u \) lies in a neighborhood of \( \frac{\kappa}{\mu} \) whose radius is measured by \( \theta \). We can prove it by using maximum principle and the pointwise bound of \( \Delta v \).

**Lemma 5.1.** Let \( \theta := \frac{\chi}{\mu} \). Let \((u,v)\) be a global classical solution of \((1)\) and \((U,V)\) be defined in \((29)\). If it holds that

\[
\limsup_{t \to \infty} \| \Delta v(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{C_1}{\mu},
\]

for some \( C_1 > 0 \). Then there exists \( C_2 > 0 \) such that

\[
\limsup_{t \to \infty} \| U(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{C_2 \theta}{\mu}. \tag{40}
\]

**Proof.** By the assumption, we can find \( t_0 > 0 \) and \( c_1 > 0 \) fulfilling

\[
\| \Delta v(\cdot, t) \|_{L^\infty(\Omega)} \leq c_1 \mu \quad \text{for all } t \geq t_0. \tag{41}
\]

We use \((41)\) and the first equation in \((1)\) to estimate that

\[
u_t = \Delta u - \chi \nabla u \cdot \nabla v - \chi u \Delta v + \kappa u - \mu u^2 \\
\leq \Delta u - \chi \nabla u \cdot \nabla v + \chi u \frac{c_1}{\mu} + \kappa u - \mu u^2 \\
\leq \Delta u - \chi \nabla u \cdot \nabla v + u (c_1 \theta + \kappa - \mu u) \tag{42}
\]

for all \( x \in \Omega \) and \( t > t_0 \), where we use \( \theta = \frac{\chi}{\mu} \). Let \( z := z(t) \) be the solution to

\[
\begin{cases}
  z'(t) = z(t) \left( c_1 \theta + \kappa - \mu z(t) \right), \\
  z(t_0) = \sup_{x \in \Omega} u(x,t_0).
\end{cases} \tag{43}
\]

It is easy to see that \( z(t_0) > 0 \) by the strong maximum principle. The comparison principle implies

\[
u(x,t) \leq z(t) \quad \text{for all } x \in \Omega, t > t_0. \tag{44}
\]

Thus we can derive that

\[
\limsup_{t \to \infty} \| u(\cdot, t) \|_{L^\infty(\Omega)} \leq \lim_{t \to \infty} z(t) = \frac{c_1 \theta}{\mu} + \frac{\kappa}{\mu},
\]

This leads to

\[
\limsup_{t \to \infty} \| U(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{c_1 \theta}{\mu}. \tag{45}
\]

Similarly (see also in \([17]\)), using the lower bound of \( \Delta v \) in the first equation in \((1)\) and constructing subsolution by the corresponding ODE show that

\[
\limsup_{t \to \infty} \| U_+(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{c_1 \theta}{\mu}, \tag{46}
\]

Combining \((45)\) and \((46)\), we establish \((40)\). \qed
6. **Decay of** \((U,V)\). In the last section, we prove that \(U\) is in a neighborhood of 0 after suitably large time. This enables us to show that \(U\) in fact decays in the large time limit if \(\theta\) is sufficiently small. At the same time, the decay of \(V\) is also obtained. Letting \(\lambda_1\) be the first non-zero eigenvalue of \(-\Delta\) associated with Neumann boundary conditions, we have the following:

**Lemma 6.1.** Suppose that \(\kappa \neq 0\). Let \(0 < \zeta < \min\{\frac{1}{\mu}, \lambda_1\}\), \(\theta := \frac{\chi}{\mu}\). If for some \(C_1 > 0\), it holds that

\[
\limsup_{t \to \infty} \|U(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{C_1 \theta}{\mu}.
\]  

(47)

Then there exist \(\theta_2 > 0\), \(C > 0\) such that for each solution of (1) with initial data fulfilling [2], we have \((U,V)\) defined as (28) satisfies

\[
\|U(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{-\zeta t},
\]

(48)

\[
\|V(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{-\zeta t}.
\]

(49)

for all \(t \geq 0\).

The proof follows from [17, Lemma 7.1]. We point out that the assumption \(\kappa \neq 0\) is critical to derive the exponential decay in the above lemma. In case \(\kappa = 0\), we have the following

**Proposition 1.** Suppose that \(\kappa = 0\). Let \(p > \frac{N}{2}\). If \(\frac{1}{\mu} < \theta_1(p)\), we have

\[
\|U(\cdot,t)\|_{L^p(\Omega)} \leq \frac{C}{t + 1},
\]

(50)

\[
\|V(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{C}{t + 1},
\]

(51)

for all \(t > 0\).

**Proof.** In view of the condition, we infer from Lemma 3.1 that \(\|u(\cdot,t)\|_{L^p(\Omega)}\) \((p > \frac{N}{2})\) is bounded, this implies the global boundedness of \((u,v)\) by standard iteration [1, Lemma 2.6]. For such global bounded solutions, [5, Theorem 1.1] implies the assertion. □

**Proof of Theorem 1.1.** Let \(p > \frac{N}{2}\), and \(\theta_1 := \theta_1(p)\) and \(\theta_2\) be defined as in Lemmata 3.1 and 6.1 respectively. Let \(\theta_0 = \min\{\theta_1, \theta_2\}\). The condition that \(\theta < \theta_0\) implies the boundedness and global existence of \((u,v)\). We obtain (8) directly by Lemmata 4.3, 5.1 and 6.1 and Proposition 1. □

**REFERENCES**

[1] X. Bai and M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics Indiana Univ. Math. J., 65 (2016), 553–583.

[2] T. Black, J. Lankeit and M. Mizukami, On the weakly competitive case in a two-species chemotaxis model, IMA Journal of Applied Mathematics, 81 (2016), 860–876, arXiv:1604.03529, 2016.

[3] X. Cao, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces Discrete Contin. Dyn. Syst., 35 (2015), 1891–1904.

[4] X. Cao, Boundedness in a three-dimensional chemotaxis-haptotaxis model, Z. Angew. Math. Phys., 67 (2016), Art. 11, 13 pp.

[5] X. Cao and M. Winkler, Sharp decay estimates in a bioconvection model with quadratic degradation in bounded domains, preprint, 2016.

[6] J. Lankeit, Chemotaxis can prevent thresholds on population density Discrete and Continuous Dynamical Systems-B, 20 (2015), 1499–1527.
3378 XINRU CAO

[7] N. Mizoguchi and M. Winkler, Blow-up in the two-dimensional parabolic Keller-Segel system. preprint, 2013.
[8] J. I. Tello and M. Winkler, A chemotaxis system with logistic source Comm. Partial Differential Equations, 32 (2007), 849–877.
[9] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, J. Math. Pures Appl., 100 (2013), 748–767.
[10] J. I. Tello and M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source Nonlinearity, 25 (2012), 1413–1425.
[11] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, Exponential attractor for a chemotaxis-growth system of equations Nonlinear Anal., 51 (2002), 119–144.
[12] C. Stinner, J. I. Tello and M. Winkler, Competitive exclusion in a two-species chemotaxis model J. Math. Biology, 68 (2014), 1607–1626.
[13] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source Comm. Partial Differential Equations, 35 (2010), 1516–1537.
[14] M. Winkler, Blow-up on a higher-dimensional chemotaxis system despite logistic growth restriction J. Math. Anal. Appl., 384 (2011), 261–272.
[15] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model J. Differential Equations, 248 (2010), 2889–2905.
[16] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities? J. Nonlinear Sci., 24 (2014), 809–855.
[17] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with logistic dampening J. Differential Equations, 257 (2014), 1056–1077.
[18] C. Yang, X. Cao, Z. Jiang and S. Zheng, Boundedness in a quasilinear fully parabolic Keller-Segel system of higher dimension with logistic source J. Math. Anal. Appl., 430 (2015), 585–591.

Received October 2016; revised January 2017.

E-mail address: caoxinru@gmail.com