Simple closed geodesics on regular tetrahedra in spaces of constant curvature

Darya Sukhorebska*

Abstract. In this survey results on the behavior of simple closed geodesics on regular tetrahedra in three-dimensional spaces of constant curvature are presented.

Keywords: closed geodesics, regular tetrahedron, hyperbolic space, spherical space
MSC: 53C22, 52B10

Contents
1 Introduction
2 Historical notes and main results
3 Closed geodesics on a regular tetrahedron in $E^3$
4 Simple closed geodesics on regular tetrahedra in $S^3$
4.1 Main definitions and examples
4.2 Properties of a simple closed geodesic on a regular tetrahedron in $S^3$
4.3 An estimation for the angle $\alpha$ for which there is no simple closed geodesic of type $(p,q)$
4.4 An estimation for the angle $\alpha$ for which there is a simple closed geodesic of type $(p,q)$
4.5 Necessary and sufficient condition for the existence of a simple closed geodesic
5 Simple closed geodesics on regular tetrahedra in $H^3$
5.1 Necessary conditions for a closed geodesic to be simple
5.2 Uniqueness of a simple closed geodesic of type $(p,q)$
5.3 Existence of a simple closed geodesic of type $(p,q)$ on a regular tetrahedron
5.4 Existence of a simple closed geodesic of type $(p,q)$ on a generic tetrahedron
5.5 The number of simple closed geodesics

1 Introduction
A closed geodesic is called simple if it is not self-intersecting and does not go along itself. At the end of the XIX century, working on three body problem, Poincaré [1] stated a problem about the existence of geodesic lines on smooth convex two-dimensional surfaces. Since then methods to find closed geodesics on regular surfaces of positive or negative curvature were created. In 1927

*The author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure, and supported by IMU Breakout Graduate Fellowship.
Birkhoff [3] proved that there exists at least one simple closed geodesic on an n-dimensional Riemannian manifold homeomorphic to a sphere. In contrast to this, there are non-smooth convex closed surfaces in Euclidean space that are free from simple closed geodesics. From the generalization of the Gauss-Bonnet theorem for polyhedra follows a necessary condition for the existence of a simple closed geodesic on convex polyhedra in $E^3$. This condition doesn’t hold for most convex polyhedra, but it holds for regular polyhedra, in particular regular tetrahedra.

In this survey we present results on the behavior of simple closed geodesics on regular tetrahedra in three dimensional spaces of constant curvature. D. Fuchs and E. Fuchs supplemented and systematized the results on closed geodesics on regular polyhedra in $E^3$ (see [37] and [38]). Protasov [40] obtained the condition for the existence of simple closed geodesics on an arbitrary tetrahedron in Euclidean space.

Borisenko and Sukhorebska studied simple closed geodesics on regular tetrahedra in three dimensional hyperbolic and spherical spaces (see [46], [48] and [49]). In Euclidean space the faces of a tetrahedron have zero Gaussian curvature, and the curvature of a tetrahedron is concentrated only on its vertices. In hyperbolic or spherical space the Gaussian curvature of faces is $k = -1$ or $1$, and the curvature of a tetrahedron is determined not only by its vertices, but also by its faces. In hyperbolic space the planar angle $\alpha$ of a face of a regular tetrahedron satisfies $0 < \alpha < \pi/3$. In spherical space the planar angle $\alpha$ satisfies $\pi/3 < \alpha \leq 2\pi/3$. In both cases the intrinsic geometry of a tetrahedron depends on planar angle. The behavior of closed geodesics on a regular tetrahedron in a three-dimensional space of constant curvature $k$ differs depending on the sign of $k$.

The author expresses her heartfelt thanks to Prof. Alexander A. Borisenko for setting the problem and for the valuable discussion.

2 Historical notes and main results

In [1] Henri Poincare studied the properties of the solutions of the three-body problem, in particular periodical and asymptotic solutions. He found that the key difficulty of this problem can be formulated as an independent problem of describing geodesics lines on a convex surfaces. In [2] Poincare shows existence of a simple (without points of self-intersection) closed geodesic on a convex smooth surface $S$ that is an embedding of two-dimensional sphere into Euclidean space $E^3$ with induced metric. He considered the shortest simple closed curve dividing $S$ into two pieces of equal total Gaussian curvature. Moreover Poincare stated a conjecture on the existence of at least three simple closed geodesic on a smooth closed convex two-dimensional surface in $E^3$. Later in 1927, Birkhoff proved that there exists at least one simple closed geodesic on a n-dimensional Riemannian manifold homeomorphic to a sphere $\mathbb{R}^n$.

In 1929, Lusternik and Schnirelmann [4], [5] published the proof of Poincare’s conjecture. However, their proof contained some gaps. These were filled in by W. Ballmann in 1978 [6] and independently by I. Taimanov in 1992 [7]. In 1951-1952, Lusternik and Fet [8], [9] proved the existence of a closed geodesic on an n-dimensional Riemannian manifold homeomorphic to a sphere $\mathbb{R}^n$.

Using ideas of Birkhoff it was proved that every Riemannian metric on a two dimensional sphere carries infinitely many geometrically distinct closed geodesics, cf. Franks [10] and Bangert [11]. The methods of the proof are restricted to surfaces. The condition of existence an infinitely many closed geodesics on a compact simply-connected manifold of arbitrary dimension is more complicated. In 1969 D. Gromoll and W. Meyer [12] shows that there always exist infinitely many distinct periodic geodesics on an arbitrary compact manifold $M$, provided some weak topological condition holds: if the sequence of Betti numbers of the free loop space $LM$ of $M$ is unbounded. Ziller [13] proved that this condition on free loop space holds for symmetric spaces of rank $> 1$. Rademacher [14] showed that for a $C^4$-regular metric on a compact
Riemannian manifold with finite fundamental group there are infinitely many geometrically distinct closed geodesics.

In 1898 Hadamard [15] showed that on a closed surface of negative curvature any closed curve, that is not homotopic to zero, could be deformed into the convex curve of minimal length within its free homotopy group. This minimal curve is unique and it is a closed geodesic. Then it is interesting to estimate the number of closed geodesics, depending on the length of these geodesics, on a compact manifold of negative curvature. Huber [16], [17] proved that on a complete closed two-dimensional manifold of constant curvature $-1$ the number of closed geodesics of length at most $L$ has the order of growth $e^L/L$ as $L \to \infty$. For compact $n$-dimensional manifolds of negative curvature this result was generalized by Sinai [18], Margulis [19], Gromov [20] and others.

In Rivin’s work [22], and later in Mirzakhani’s work [23], it’s proved that on a complete hyperbolic (constant negative curvature) Riemannian surface of genus $g$ and with $n$ cusps the number of simple closed geodesics of length at most $L$ is asymptotic to (positive) constant times $L^{g-6+2n}$ as $L \to \infty$. You can also see [24], [25] for details.

Theorems about geodesic lines on a convex two-dimensional surfaces play an important role in geometry “in the large” of convex surfaces in spaces of constant curvature. Important results on this subject were obtained by Cohn-Vossen [26], Alexandrov [27], Pogorelov [29]. In one of the earliest work Pogorelov proved that on a closed convex surface of the Gaussian curvature $\leq k$, $k > 0$, each geodesic of length $< \pi/\sqrt{k}$ realized the shortest path between its endpoint [30]. Toponogov [31] proved that on $C^2$-regular closed surface of curvature $\geq k > 0$ the length of a simple closed geodesic is at most $2\pi/\sqrt{k}$. Vaigant and Matukevich [32] proved that on this surface a geodesic of length $\geq 3\pi/\sqrt{k}$ has point of self-intersection.

Geodesics have also been studied on non-smooth surfaces, including convex polyhedra in $E^3$. Since geodesic is the locally shortest curve then it can not pass through any point for which the full angle is less then $2\pi$ (see [27]). Gruber [33] showed, that in the sense of Baire categories [34] most convex surfaces (no regularity required) do not contain closed geodesic. Pogorelov [35] generalized Lusternik and Schnirelmann’s result showing that on any closed convex surfaces there is at least three closed quasi-geodesics. Whereas a geodesic has exactly $\pi$ surface angle to either side at each point, a quasi-geodesic has at most $\pi$ surface angle to either side at each point. Unlike to geodesics, quasi-geodesics can pass through the vertices with the full angle $< 2\pi$ on surface [28].

On a convex polyhedron a geodesic has following properties: 1) it consists of line segments on faces of the polyhedron; 2) it forms equal angles with edges of adjacent faces; 3) a geodesic cannot pass through a vertex of a convex polyhedron [27]. Galperin [39] presented a necessary condition for existence a simple closed geodesic on a convex polyhedron in $E^3$. It is based on a generalization of Gauss-Bonnet theorem for polyhedra. The curvature of a convex polyhedron in $E^3$ is concentrated on its vertices. Let $\theta_1, \ldots, \theta_n$ be the full angles around the vertices $A_1, \ldots, A_n$ of a convex polyhedron. The curvature of the vertex $A_i$ is $\omega_i = 2\pi - \theta_i$, $i = 1, \ldots, n$. If there is a simple closed geodesic on a convex polyhedron, it is necessary that there is a subset $I \subset \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} \omega_i = 2\pi.$$ 

This condition doesn’t hold for most polyhedra, but it holds for regular polyhedra. D. Fuchs and E. Fuchs supplemented and systematized the results on closed geodesics on regular polyhedra in three-dimensional Euclidean space (see [37] and [38]). K. Lawson and others [39] obtain a complete classification of simple closed geodesics on the eight convex polyhedra (deltahedra) whose faces are all equilateral triangles.

Protasov [40] obtained a condition for the existence of simple closed geodesics on arbitrary tetrahedron in Euclidean space and evaluated from above the number of these geodesics in terms
of the difference from $\pi$ the sum of the angles at a vertex of the tetrahedron. In particular, it is proved that a simplex has infinitely many different simple closed geodesics if and only if all the faces are equal triangles. A. Akopyan and A. Petrunin \[41\] showed that if closed convex surface $M$ in $\mathbb{E}^3$ contains arbitrarily long simple closed geodesic, then $M$ is a tetrahedron whose faces are equal triangles.

**Definition 1.** A simple closed geodesic on a tetrahedron has type $(p,q)$ if it has $p$ vertices on each of two opposite edges of the tetrahedron, $q$ vertices on each of other two opposite edges, and $(p+q)$ vertices on each of the remaining two opposite edges.

On a regular tetrahedron in Euclidean space, for each ordered pair of coprime integers $(p,q)$ there exists a whole class of simple closed geodesics of type $(p,q)$, up to the isometry of the tetrahedron. On the development of the tetrahedron geodesics in each class are parallel to one another. Furthermore, in each class there is a simple close geodesic passing through the midpoints of two pairs of opposite edges of the tetrahedron \[40\].

O’Rourke and Vilcu \[42\] considered simple closed quasi-geodesics on tetrahedra in $\mathbb{E}^3$. In work \[43\] Davis and others consider geodesics which begin and end at vertices (and do not touch other vertices) on a regular tetrahedron and cube. It was proved that a geodesic as above never begins and ends at the same vertex and computed the probabilities with which a geodesic starting from a given vertex ends at every other vertex. Fuchs \[44\] obtain similar results for regular octahedron and icosahedron (in particular, such a geodesic never ends at the point where it begins).

Denote by $M_k^n$ a simply-connected complete Riemannian $n$-dimensional manifold of constant curvature $k \in \{-1, 0, 1\}$. A polyhedron in $M_k^n$ is a surface obtained by gluing finitely many geodesic polygons from $M_k^2$. In particular, a regular tetrahedron in $M_k^3$ is a closed convex polyhedron all of whose faces are regular geodesic triangles from $M_k^2$ and all vertices are regular trihedral angles. From Alexandrov’s gluing theorem \[28\] it follows that the polyhedron in $M_k^3$ with the induced metric is a compact Alexandrov surface $A(k)$ with the curvature bounded below by $k$. Note, that in $\mathbb{E}^3(M_0^3)$ the curvature of a tetrahedron is concentrated only on its vertices. In hyperbolic or spherical space, the Gaussian curvature of faces is $k = -1$ or $1$ respectively, and the curvature of a tetrahedron is determined not only by its vertices, but also by its faces.

In \[45\] Rouyer and Vilcu studied the existence or non-existence of simple closed geodesics on most (in the sense of Baire category \[34\]) Alexandrov surfaces. In particular it was proved that most surfaces in $A(-1)$ have infinitely many, pairwise disjoint, simple closed geodesics, and most surfaces in $A(1)$ have no simple closed geodesics.

As we say before, on a regular tetrahedron in Euclidean space $\mathbb{E}^3$, for each ordered pair of coprime integers $(p,q)$ there exists infinitely many simple closed geodesics of type $(p,q)$, that are parallel each other on the development of the tetrahedron. It’s follows from the fact, that the development of a tetrahedron along the geodesic is contained in the standard triangular tiling of the plane. Moreover, the vertices of the tiling can be labeled in such a way that for any development the labeling of vertices of the tetrahedron matches the labeling of vertices of the tiling. This is something that holds only for regular tetrahedron and only in $\mathbb{E}^3$ \[37\]. In general there is no tiling of a plane by regular triangles.

In spherical space $S^3$ the planar angle $\alpha$ of the face satisfies $\pi/3 < \alpha \leq 2\pi/3$. The intrinsic geometry of a tetrahedron depends on $\alpha$. If the planar angle $\alpha = 2\pi/3$, then the tetrahedron is a unit two-dimensional sphere. Hence there are infinitely many simple closed geodesics on it and they are great circles of the sphere. In the following we consider $\alpha$ such that $\pi/3 < \alpha < 2\pi/3$. In \[48\] Borisenko and Sukhorebska proved that on a regular tetrahedron in spherical space there exists the finite number of simple closed geodesics. The length of all these geodesics is less than $2\pi$. 

4
For any coprime integers \((p, q)\) it was found the numbers \(\alpha_1\) and \(\alpha_2\), depending on \(p, q\) and satisfying the inequalities \(\pi/3 < \alpha_1 < \alpha_2 < 2\pi/3\), such that

1) if \(\pi/3 < \alpha < \alpha_1\), then on a regular tetrahedron in spherical space with the planar angle \(\alpha\) there exists unique simple closed geodesic of type \((p, q)\), up to the rigid motion of this tetrahedron, and it passes through the midpoints of two pairs of opposite edges of the tetrahedron;

2) if \(\alpha_2 < \alpha < 2\pi/3\), then on a regular tetrahedron with the planar angle \(\alpha\) there is not simple closed geodesic of type \((p, q)\).

In [49] Borisenko proved necessary and sufficient condition of existence a simple closed geodesic on a regular tetrahedron in \(S^3\). We will consider it in details in Section 4.

Unlike in \(S^3\), on a regular tetrahedron in hyperbolic space \(\mathbb{H}^3\) there are infinitely many simple closed geodesics. Recall that the planar angle \(\alpha\) of a regular tetrahedron in \(\mathbb{H}^3\) satisfies \(0 < \alpha < \pi/3\). In [40] Borisenko and Sukhorebska proved that on a regular tetrahedron in hyperbolic space for any coprime integers \((p, q)\), \(0 \leq p < q\), there exists unique simple closed geodesic \(\gamma\) of type \((p, q)\), up to the rigid motion of the tetrahedron. \(\gamma\) passes through the midpoints of two pairs of opposite edges of the tetrahedron. These geodesics of type \((p, q)\) exhaust all simple closed geodesics on a regular tetrahedron in hyperbolic space. As a part of the proof it was found a constant \(d(\alpha) > 0\) for \(\alpha \in (0, \pi/3)\) such that the distances from the vertices of the regular tetrahedron to a simple closed geodesic is greater then \(d(\alpha)\). Note, that this property holds only for simple closed geodesics on regular tetrahedra in \(\mathbb{H}^3\). In \(\mathbb{E}^3\) or \(S^3\) for any \(\varepsilon > 0\) there is a simple closed geodesic \(\gamma\) on a regular tetrahedron such that the distance from a vertex of the tetrahedron to \(\gamma\) is \(< \varepsilon\).

Furthermore, in [16] it was proved, that the number of simple closed geodesics of length bounded by \(L\) is asymptotic to \(c(\alpha)L^2\), when \(L \to \infty\). If \(\alpha \to 0\), then \(c(\alpha) \to c_0 > 0\). On the other hand, when the planar angle \(\alpha\) of a regular tetrahedron in hyperbolic space is zero, the vertices of the tetrahedron become cusps. Then the tetrahedron becomes a noncompact surface homeomorphic to a sphere with four cusps, with a complete regular Riemannian metric of constant negative curvature. The genus of this surface is zero. In work of Rivin [22] it was shown that the number of simple closed geodesics on this surface has order of growth \(L^2\).

In [49] Borisenko proved, that if planar angles of any tetrahedron in hyperbolic space are at most \(\pi/4\), then for any pair of coprime integers \((p, q)\) there exists a simple closed geodesic of type \((p, q)\). This situation is differ from Euclidean space, where there are no simple closed geodesics on a generic tetrahedron [36].

### 3 Closed geodesics on a regular tetrahedron in \(\mathbb{E}^3\)

Consider a regular tetrahedron \(A_1A_2A_3A_4\) in Euclidean space with the edge of length 1.

Fix the point of a geodesic on a tetrahedron’s edge and roll the tetrahedron along the plane in such way that the geodesic always touches the plane. The traces of the faces form the development of the tetrahedron on a plane and the geodesic is a line segment inside the development.

A development of a regular tetrahedron in \(\mathbb{E}^3\) is a part of the standard triangulation of Euclidean plane. Denote the vertices of the triangulation in accordance with the vertices of the tetrahedron (see Figure 1). We introduce a rectangular Cartesian coordinate system with the origin at \(A_1\) and the \(x\)-axis along the edge \(A_1A_2\) containing \(X\). Then the vertices \(A_1\) and \(A_2\) has the coordinates \((l, k\sqrt{3})\), and the coordinates of \(A_3\) and \(A_4\) are \((l + 1/2, k\sqrt{3} + 1/2)\), where \(k, l\) are integers.

Choose two identically oriented edges \(A_1A_2\) of the triangulation, which don’t belong to the same line. Take two points \(X(\mu, 0)\) and \(X'(\mu + q + 2p, q\sqrt{3})\) on them, where \(0 < \mu < 1\) such that the segment \(XX'\) doesn’t contain any vertex of the triangulation. The segment \(XX'\) corresponds to the simple closed geodesic \(\gamma\) of type \((p, q)\) on a regular tetrahedron in Euclidean space.
space. If \((p, q)\) are coprime integers then \(\gamma\) does not repeat itself. On a tetrahedron \(\gamma\) has \(p\) vertices on each of two opposite edges of the tetrahedron, \(q\) vertices on each of other two opposite edges, and \((p + q)\) vertices on each of the remaining two opposite edges, so \(\gamma\) has type \((p, q)\).

The length of \(\gamma\) is equal
\[
L = 2\sqrt{p^2 + pq + q^2}.
\]  (3.1)

Note, that the segments of a geodesic lying on the same face of the tetrahedron are parallel to each other. It follows that any closed geodesic on a regular tetrahedron in Euclidean space does not have points of self-intersection.

Figure 1

If \(q = 0\) and \(p = 1\), then geodesic consists of four segments that consecutively intersect four edges of the tetrahedron, and doesn’t go through the one pair of opposite edges.

**Theorem 1.**

1) On a regular tetrahedron in Euclidean space, for each ordered pair of coprime integers \((p, q)\) there exists a whole class of simple closed geodesics of type \((p, q)\), up to the isometry of the tetrahedron. On the development of the tetrahedron geodesics in each class are parallel one another.

2) In each class there is a simple close geodesic passing through the midpoints of two pairs of opposite edges of the tetrahedron.

**Proof.** For each pair of coprime integers \((p, q)\) construct the segment connecting points \(X(\mu_0, 0)\) and \(X'(\mu_0 + q + 2p, q\sqrt{3})\). Chose \(\mu_0 \in (0, 1)\) such that \(XX'\) doesn’t contain any vertex of the triangulation. Then \(XX'\) corresponds to the simple closed geodesic \(\gamma\) of type \((p, q)\) on a regular tetrahedron in Euclidean space.

Consider the segments parallel to \(XX'\). They are characterized by the equation
\[
y = \frac{q\sqrt{3}}{q + 2p}(x - \mu).
\]

We can change \(\mu\) until the line touches a vertex of the tiling. Then for each pair \((p, q)\) there are \(\mu_1, \mu_2 \in (0, 1)\) such that \(\mu_1 \leq \mu_0 \leq \mu_2\) and for all \(\mu \in (\mu_1, \mu_2)\) the segment joining \(X(\mu_0, 0)\) and \(X'(\mu + q + 2p, q\sqrt{3})\) corresponds to the simple closed geodesic of type \((p, q)\) on a regular tetrahedron. Therefore the part 1) of the theorem is proved.

To prove 2) consider the lines
\[
\gamma_i : y = \frac{q\sqrt{3}}{q + 2p}(x - \mu_i), i = 1, 2
\]  (3.2)
pass through the vertices of the tiling. It means that there exist the integer numbers $c_1$ and $c_2$ such that the points $P_1 \left( \frac{c_1}{2}(q + 2p)/2q + \frac{\mu_1}{2}, \frac{c_1}{2}\sqrt{3}/2 \right)$ and $P_2 \left( \frac{c_2}{2}(q + 2p)/2q + \frac{\mu_2}{2}, \frac{c_2}{2}\sqrt{3}/2 \right)$ are the vertices of the tiling and $\gamma_1$ passes through $P_1$ and $\gamma_2$ passes through $P_2$.

Consider the closed geodesic $\gamma_0$ parallel to $\gamma$ such that the equation of $\gamma_0$ is

$$y = \frac{q\sqrt{3}}{q + 2p} \left( x - \frac{\mu_1 + \mu_2}{2} \right).$$

It passes through the point

$$P_0 \left( \frac{c_1 + c_2}{2} \frac{q + 2p}{2q} + \frac{\mu_1 + \mu_2}{2}, \frac{c_1 + c_2}{2} \frac{\sqrt{3}}{2} \right).$$

Consider three cases: 1) both of the points $P_1$ and $P_2$ belong to the line $A_1A_2$; 2) both of the points $P_1$, $P_2$ belong to the line $A_3A_4$; 3) the point $P_1$ belongs to the line $A_1A_2$ and the point $P_2$ belongs to the line $A_3A_4$. In each of this cases it is easy to show that $P_0$ is a midpoint of some edge of the tiling.

Then let us proof that if geodesic passes through the midpoint of the one edge, then it passes through the midpoints of two pairs of the opposite edges. Assume that a closed geodesic $\gamma_0$ passes through the midpoint of the edge $A_1A_2$. Then the equation of $\gamma_0$ is

$$y = \frac{q\sqrt{3}}{q + 2p} \left( x - \frac{1}{2} \right). \tag{3.3}$$

The vertices $A_3$ and $A_4$ belong to the line $y = (2k + 1)\sqrt{3}/2$, and their first coordinate is $x_v = l + 1/2$ $(k, l \in \mathbb{Z})$. Substituting the coordinates of the points $A_3$ and $A_4$ to equation (3.3), we get

$$q(2l - 2k - 1) = 2p(2k + 1). \tag{3.4}$$

If $q$ is even then there exist $k$ and $l$ satisfying equation (3.4). It follows that $\gamma_0$ passes through the vertex of the tiling. It contradicts the properties of $\gamma_0$, therefore $q$ is an odd integer.

The points $X_1 (1/2, 0)$ and $X'_1 \left( q + 2p + 1/2, q\sqrt{3} \right)$ satisfy equation (3.3). These points are the midpoints of the edge $A_1A_2$ on the tetrahedron. Suppose that the point $X_2$ is the midpoint of $X_1X'_1$. Then the coordinates of $X_2$ are $\left(q/2 + p + 1/2, q\sqrt{3}/2 \right)$. Substituting $q = 2k + 1$, we obtain $X_2 (k + p + 1, (k + 1/2)\sqrt{3})$. Since the second coordinate of $X_2$ is $(k + 1/2)\sqrt{3}$, where $k$ is an integer, then the point $X_2$ belongs to the line, that contains the vertices $A_3$ and $A_4$. Since the first coordinate of $X_2$ is an integer, it follows that $X_2$ is the midpoint of the edge $A_3A_4$.

Let $Y_1 \left( q/4 + p/2 + 1/2, q\sqrt{3}/4 \right)$ be the midpoint of $X_1X_2$. Substituting $q = 2k + 1$, we obtain $Y_1 \left( (k + p + 1)/2 + 1/4, (k/2 + 1/4)\sqrt{3} \right)$. From the second coordinate we have that $Y_1$ belongs to the line passing in the middle of the horizontal lines $y = k\sqrt{3}/2$ and $y = (k + 1)\sqrt{3}/2$. Looking at the first coordinate of $Y_1$, which has $1/4$ added, we can see that $Y_1$ is the center of $A_1A_3$, or $A_3A_2$, or $A_2A_4$, or $A_4A_1$.

In the similar way consider the midpoint $Y_2 \left( 3q/4 + 3p/2 + 1/2, 3q\sqrt{3}/4 \right)$ of $X_2X'_1$. Then $Y_2$ is the midpoint of the edge that is opposite to the edge with $Y_1$.

**Corollary 3.1.** The development of the tetrahedron obtained by unrolling along a closed geodesic consists of four equal polygons. Two adjacent polygons can be transformed into each other by rotating them through an angle $\pi$ around the midpoint of their common edge.

**Proof.** For any closed geodesic $\gamma$ we get the equivalent closed geodesic $\gamma_0$ that passes through the midpoints of two pairs of the opposite edges on the tetrahedron. Let the points $X_1$, $X_2$ and $Y_1$, $Y_2$ on $\gamma_0$ be the midpoints of the edges $A_1A_2$, $A_4A_3$ and $A_2A_3$, $A_1A_4$ respectively.
Consider the rotation of the regular tetrahedron through $\pi$ around the line passing through the points $X_1$ and $X_2$. This rotation is the isometry of the regular tetrahedron. The points $Y_1$ and $Y_2$ are swapped. Furthermore the segment of $\gamma_0$ that starts at $X_1$ on the face $A_1A_2A_4$ is mapped to the segment of $\gamma_0$ that starts from the point $X_1$ on $A_1A_2A_3$. It follows that the segments $X_1Y_1$ and $X_1Y_2$ are swapped. For the same reason after the rotation the segments $X_2Y_1$ and $X_2Y_2$ of $\gamma_0$ are also swapped.

From this rotation we get that the development of the tetrahedron along the segment $Y_1X_1Y_2$ of the geodesic is central-symmetric with respect to the point $X_1$. And the development along $Y_1X_2Y_2$ is central-symmetric with respect to $X_2$.

Now consider the rotation of the regular tetrahedron through $\pi$ around the line passing through the points $Y_1$ and $Y_2$. For the same argument as above we obtain that the development of the tetrahedron along the segment $X_1Y_1X_2$ of geodesic is central-symmetric with respect to $Y_1$, and the development along $X_2Y_2X_1$ is central-symmetric with respect to $Y_2$ (see Figure 2).

**Lemma 3.1.** Let $\gamma$ be a simple closed geodesic of type $(p,q)$ on a regular tetrahedron in Euclidean space such that $\gamma$ intersects the midpoints of two pairs of opposite edges. Then the distance $h$ from the tetrahedron’s vertices to $\gamma$ satisfies the inequality

$$h \geq \frac{\sqrt{3}}{4\sqrt{p^2 + pq + q^2}}.$$  

**Proof.** Suppose $\gamma$ intersects the edge $A_1A_2$ at the midpoint $X$. Then geodesic $\gamma$ is unrolled into the segment $XX'$ lying at the line

$$y = \frac{q\sqrt{3}}{q + 2p} \left( x - \frac{1}{2} \right).$$

The segment $XX'$ intersects the edges $A_1A_2$ at the points

$$(x_b, y_b) = \left( \frac{2(q + 2p)k + q}{2q}, k\sqrt{3} \right),$$

where $k \leq q$. Since $XX'$ does not pass through vertices of the tiling, $x_b$ can not be an integer. Hence on the edge $A_1A_2$ the distance from the vertices to the points of $\gamma$ is not less than $1/2q$.

Analogically on the edge $A_3A_2$ the distance from the vertices of the tetrahedron to the points of $\gamma$ is not less than $1/2p$.

Choose the points $B_1$ at the edge $A_2A_1$ and $B_2$ at the edge $A_2A_3$ such that the length $A_2B_1$ is $1/2q$ and the length $A_2B_2$ equals $1/2p$. Let $A_2H$ be a height of the triangle $B_1A_2B_2$. Then

$$|A_2H| = \frac{\sqrt{3}}{4\sqrt{p^2 + pq + q^2}}.$$

The distance $h$ from the vertex $A_2$ to $\gamma$ is not less than $|A_2H|$. 

\[\square\]
The pair of coprime integers \((p,q)\) determines the combinatorial structure of a simple closed geodesic and hence the order of intersections with the edges of the tetrahedron.

In [40] the generalization of a simple closed geodesic on a polyhedron was proposed. A polyline on a tetrahedron is a curve consisting of line segments which connect points consecutively on the edges of this tetrahedron. An abstract geodesic on a tetrahedron is a closed polyline with the following properties:
1) it does not have points of self-intersection and adjacent segments of it lie on different faces;
2) it crosses more than three edges and does not pass through the vertices of the tetrahedron.

For any two tetrahedra we can fix a one-to-one correspondence between their vertices, and label the corresponding vertices of the tetrahedra identically. Then two closed geodesics on these tetrahedra are called equivalent if they intersect identically labelled edges in the same order.

**Proposition 1.** [40] For every abstract geodesic \(\tilde{\gamma}\) on a tetrahedron in Euclidean space there exists an equivalent simple closed geodesic \(\gamma\) on a regular tetrahedron in Euclidean space.

A vertex of a geodesic \(\gamma\) is called a link node if it and two neighbouring vertices of \(\gamma\) lie on the edges of the same vertex \(A_i\) of the tetrahedron, and these three vertices are vertices of the geodesic that are closest to \(A_i\).

**Proposition 2.** [40] Let \(\gamma_1^1\) and \(\gamma_1^2\) be the segments of a simple closed geodesic \(\gamma\), starting at a link node on a regular tetrahedron, let \(\gamma_2^1\) and \(\gamma_2^2\) be the next segments and so on. Then for each \(i = 2,\ldots,2p+2q-1\) the segments \(\gamma_i^1\) and \(\gamma_i^2\) lie on the same face of the tetrahedron, and there are no other geodesic points between them. The segments \(\gamma_1^{2p+2q}\) and \(\gamma_2^{2p+2q}\) meet at the second link node of the geodesic.

4 Simple closed geodesics on regular tetrahedra in \(S^3\)

4.1 Main definitions and examples.

A spherical triangle is a convex polygon on a unit sphere bounded by three the shortest lines. A regular tetrahedron \(A_1A_2A_3A_4\) in three-dimensional spherical space \(S^3\) is a closed convex polyhedron such that all its faces are regular spherical triangles and all its vertices are regular trihedral angles. A planar angle \(\alpha\) of a regular tetrahedron in \(S^3\) satisfies the conditions \(\pi/3 < \alpha \leq 2\pi/3\). Note, than there exist a unique (up to the rigid motion) tetrahedron in spherical space with the given planar angle. The length of the edges is equal to

\[
a = \arccos \left( \frac{\cos \alpha}{1 - \cos \alpha} \right),
\]

(4.1)

\[
\lim_{\alpha \to \pi/3} a = 0; \quad \lim_{\alpha \to \pi/2} a = \pi/2; \quad \lim_{\alpha \to 2\pi/3} a = \pi - \cos^{-1}1/3.
\]

(4.2)

If \(\alpha = 2\pi/3\), then a tetrahedron is a unit two-dimensional sphere. There are infinitely many simple closed geodesics on it. In the following we assume that \(\alpha\) satisfies \(\pi/3 < \alpha < 2\pi/3\).

Spherical space \(S^3\) of curvature 1 is realized as a unite tree-dimensional sphere in four-dimensional Euclidean space. Hence the regular tetrahedron \(A_1A_2A_3A_4\) is located in an open hemisphere. Consider Euclidean space tangent to this hemisphere at the center of circumscribed sphere of the tetrahedron. A central projection of the hemisphere to this tangent space maps the regular tetrahedron from \(S^3\) onto the regular tetrahedron in Euclidean tangent space. A simple closed geodesic \(\gamma\) on \(A_1A_2A_3A_4\) is mapped into abstract geodesic on a regular tetrahedron in \(E^3\). Proposition [40] states that there exists a simple closed geodesic on a regular tetrahedron in Euclidean space equivalent to this generalized geodesic. It follows, that a simple closed geodesic
on a regular tetrahedron in $S^3$ is also characterized uniquely by a pair of coprime integers $(p, q)$ and has the same combinatorial structure as a closed geodesic on a regular tetrahedron in $E^3$.

**Lemma 4.1.** \[48\]

1) On a regular tetrahedron with the planar angle $\alpha \in (\pi/3, 2\pi/3)$ in spherical space there exist three different simple closed geodesics of type $(0, 1)$. They coincide under isometries of the tetrahedron.

2) Geodesics of type $(0, 1)$ exhaust all simple closed geodesics on a regular tetrahedron with the planar angle $\alpha \in [\pi/2, 2\pi/3]$ in spherical space.

3) On a regular tetrahedron with the planar angle $\alpha \in (\pi/3, \pi/2)$ in spherical space there exist three different simple closed geodesics of type $(1, 1)$.

**Proof.**

1) Consider a regular tetrahedron $A_1A_2A_3A_4$ in $S^3$ with planar angle $\alpha \in (\pi/3, 2\pi/3)$. Let $X_1$ and $X_2$ be the midpoints of $A_1A_4$ and $A_3A_2$, and $Y_1$, $Y_2$ be the midpoints of $A_1A_2$ and $A_1A_3$ respectively. Join these points consecutively with the segments through the faces. Since the points $X_1$, $Y_1$, $X_2$ and $Y_2$ are midpoints, then the triangles $X_1A_1Y_1$, $Y_1A_2X_2$, $X_2A_3Y_2$ and $Y_2A_1X_1$ are equal. It follows that the closed polyline $X_1Y_1X_2Y_2$ is a simple closed geodesic of type $(0, 1)$ on a regular tetrahedron in spherical space (see Figure 3). Choosing the midpoints of other pairs of opposite edges, we can construct other two geodesics of type $(0, 1)$ on the tetrahedron.

![Figure 3](image)

2) Consider a regular tetrahedron with planar angle $\alpha \geq \pi/2$. Since a geodesic is a line segment inside the development of the tetrahedron, then it cannot intersect three edges of the tetrahedron, coming out from the same vertex, in succession.

If a simple closed geodesic on the tetrahedron is of type $(p, q)$, where $p = q = 1$ or $1 < p < q$, then this geodesic intersect three edges, with the common vertex, in succession (see \[49\]). Only a simple closed geodesic of type $(0, 1)$ intersects two tetrahedron’s edges, that have a common vertex, and doesn’t intersects the third edge. It follows that on a regular tetrahedron in spherical space with planar angle $\alpha \in [\pi/2, 2\pi/3]$ there exist only three simple closed geodesic of type $(0, 1)$ and no other geodesics.

3) Consider a regular tetrahedron $A_1A_2A_3A_4$ in $S^3$ with planar angle $\alpha \in (\pi/3, \pi/2)$. As above, the points $X_1$, $X_2$, $Y_1$, $Y_2$ are the midpoints of $A_1A_4$, $A_3A_2$, $A_4A_2$ and $A_1A_3$ respectively.

Unfold two adjacent faces $A_1A_4A_3$ and $A_4A_2A_1$ into the plain and draw a geodesic line segment $X_1Y_1$. Since $\alpha < \pi/2$, then the segment $X_1Y_1$ is contained inside the development and intersects the edge $A_1A_2$ at right angle. Then unfold another two adjacent faces $A_1A_3A_4$ and $A_1A_2A_3$ and construct the segment $Y_1X_2$. In the same way join the points $X_2$ and $Y_2$ within the faces $A_2A_3A_4$ and $A_3A_4A_1$, and join $Y_2$ and $X_1$ within $A_1A_2A_4$ and $A_1A_4A_2$ (see Figure 4).

Since the points $X_1$, $Y_1$, $X_2$ and $Y_2$ are the midpoints of their edges, then the triangles $X_1A_4Y_1$, $Y_1A_2X_2$, $X_2A_3Y_2$ and $Y_2A_1X_1$ are equal. Hence, the segments $X_1Y_1$, $Y_1X_2$, $X_2Y_2$, $Y_2X_1$ form a simple closed geodesic of type $(1, 1)$ on the tetrahedron.

10
Two other simple closed geodesics of type (1, 1) on a tetrahedron can be constructed similarly by connecting the midpoints of other pairs of opposite edges of the tetrahedron.

In the following we assume that \( \alpha \) satisfying \( \pi/3 < \alpha < \pi/2 \).

### 4.2 Properties of a simple closed geodesic on a regular tetrahedron in \( S^3 \).

**Lemma 4.2.** The length of a simple closed geodesic on a regular tetrahedron in spherical space is less than 2\( \pi \).

In this Lemma was proved using Proposition 2 about the construction of a simple closed geodesic on a regular tetrahedron. However, Lemma 4.2 can be considered as the particular case of the result proved by A. Borisenko [47] about the generalization of V. Toponogov theorem [31] to the case of two-dimensional Alexandrov space.

**Lemma 4.3.** [48] On a regular tetrahedron in spherical space a simple closed geodesic intersects midpoints of two pairs of opposite edges.

**Proof.** Let \( \gamma \) be a simple closed geodesic on a regular tetrahedron \( A_1A_2A_3A_4 \) in \( S^3 \). As we show above there exists a simple closed geodesic \( \tilde{\gamma} \) on a regular tetrahedron in Euclidean space such that \( \tilde{\gamma} \) is equivalent to \( \gamma \). From Theorem 4.1 we assume \( \tilde{\gamma} \) intersects the midpoints \( \tilde{X}_1 \) and \( \tilde{X}_2 \) of the edges \( A_1A_2 \) and \( A_3A_4 \) on the tetrahedron in \( E^3 \). Denote by \( X_1 \) and \( X_2 \) the vertices of \( \gamma \) at the edges \( A_1A_2 \) and \( A_3A_4 \) on the tetrahedron in \( S^3 \) such that \( X_1 \) and \( X_2 \) are equivalent to the points \( \tilde{X}_1 \) and \( \tilde{X}_2 \).

Consider the development of the tetrahedron along \( \gamma \) starting from the point \( X_1 \) on a two-dimensional unite sphere. The geodesic \( \gamma \) is unrolled into the line segment \( X_1X_1' \) of length less than 2\( \pi \) inside the development. Denote by \( T_1 \) and \( T_2 \) the parts of the development along \( X_1X_2 \) and \( X_2X_1' \) respectively.

Let \( M_1 \) and \( M_2 \) be midpoints of the edges \( A_1A_2 \) and \( A_3A_4 \) respectively on the tetrahedron in \( S^3 \). Rotation by the angle \( \pi \) over the line \( M_1M_2 \) is an isometry of the tetrahedron. Then the development of the tetrahedron is centrally symmetric with the center \( M_2 \).

On the other hand, symmetry over \( M_2 \) swaps the parts \( T_1 \) and \( T_2 \). The point \( X_1' \) at the edge \( A_1A_2 \) of \( T_2 \) is mapped into the point \( \tilde{X}_1' \) at the edge \( A_2A_1 \) containing \( X_1 \) on \( T_1 \), and the lengths of \( A_2X_1 \) and \( \tilde{X}_1'A_4 \) are equal.

The image of the point \( X_1 \) on \( T_1 \) is a point \( \tilde{X}_1 \) at the edge \( A_1A_2 \) on \( T_2 \). Since \( M_2 \) is a midpoint of \( A_3A_4 \), then the symmetry maps the point \( X_2 \) at \( A_3A_4 \) onto the point \( \tilde{X}_2 \) at the same edge \( A_3A_4 \) such that the lengths of \( A_4X_2 \) and \( \tilde{X}_2A_3 \) are equal. Thus, the segment \( X_1X_1' \) is mapped into the segment \( \tilde{X}_1'\tilde{X}_1 \) inside the development.
Suppose the segments $\hat{X}_1\hat{X}_2$ and $X_1X_2$ intersect at the point $Z_1$ inside $T_1$. Then the segments $\hat{X}_2\hat{X}_1$ and $X_2X_1'$ intersect at the point $Z_2$ inside $T_2$, and the point $Z_2$ is centrally symmetric to $Z_1$ with respect to $M_2$ (see Figure 5). Inside the polygon on the sphere we obtain two circular arcs $X_1X_1'$ and $\hat{X}_1\hat{X}_1$ intersecting in two points. Therefore $Z_1$ and $Z_2$ are antipodal points on the sphere and the length of the geodesic segment $Z_1X_2Z_2$ equals $\pi$.

Now consider the development of the tetrahedron along $\gamma$ starting from the point $X_2$. This development also consists of spherical polygons $T_2$ and $T_1$, but in this case they are glued by the edge $A_1A_2$ and centrally symmetric with respect to $M_1$.

Similarly apply the symmetry over $M_1$. The segments $X_2X_1X_2'$ and $\hat{X}_2\hat{X}_1\hat{X}_2'$ are swapped inside the development. Since the symmetries over $M_1$ and over $M_2$ correspond to the same isometry of the tetrahedron, then the arcs $X_2X_1X_2'$ and $\hat{X}_2\hat{X}_1\hat{X}_2'$ also intersect at the points $Z_1$ and $Z_2$. It follows that the length of geodesic segment $Z_1X_1Z_2$ is also equal to $\pi$. Hence the length of the geodesic $\gamma$ on a regular tetrahedron in spherical space is $2\pi$, that contradicts to Lemma 4.2. We get that the segments $\hat{X}_1\hat{X}_2$ and $X_1X_2$ on $T_1$ either don’t intersect or coincide.

If the $X_1X_2$ and $\hat{X}_1\hat{X}_2$ don’t intersect, then they form the quadrilateral $X_1X_2\hat{X}_2\hat{X}_1$ inside $T_1$. Since $\gamma$ is closed geodesic, then $\angle A_1X_1X_2 + \angle A_2\hat{X}_1\hat{X}_2 = \pi$. Furthermore, $\angle X_1X_2A_3 + \angle \hat{X}_1\hat{X}_2A_4 = \pi$. We obtain the convex quadrilateral on a sphere with the sum of inner angles $2\pi$. It follows that the integral of the Gaussian curvature over the interior of $X_1X_2\hat{X}_2\hat{X}_1$ on a sphere is equal zero. Hence, the segments $X_1X_2$ and $\hat{X}_1\hat{X}_2$ coincide under the symmetry of the development. Then the points $X_1$ and $X_2$ of geodesic $\gamma$ are the midpoints of the edges $A_1A_2$ and $A_3A_4$ respectively.

Similarly it can be proved that $\gamma$ intersect the midpoints of the second pair of the opposite edges of the tetrahedron.

Corollary 4.1. If two simple closed geodesic on a regular tetrahedron in spherical space intersect the edges of the tetrahedron in the same order, then they coincide.

4.3 An estimation for the angle $\alpha$ for which there is no simple closed geodesic of type $(p,q)$.

Theorem 2. On a regular tetrahedron with the planar angle $\alpha$ in spherical space such that

$$\alpha > 2 \arcsin \sqrt{\frac{p^2 + pq + q^2}{4(p^2 + pq + q^2) - \pi^2}},$$

where $(p,q)$ is a pair of coprime integers, there is no simple closed geodesic of type $(p,q)$.

Proof. Let $A_1A_2A_3A_4$ be a regular tetrahedron in $S^3$ with planar angle $\alpha \in (\pi/3, \pi/2)$ and let $\gamma$ be a simple closed geodesic of type $(p,q)$ on it.
Each face of the tetrahedron is a regular spherical triangle. Consider a two-dimensional unit sphere containing the face $A_1A_2A_3$. Construct Euclidean plane $\Pi$ passing through the points $A_1$, $A_2$ and $A_3$. The intersection of the sphere with $\Pi$ is a small circle. Draw rays starting at the sphere’s center $O$ to the points at the spherical triangle $A_1A_2A_3$. The edges of $\triangle A_1A_2A_3$ are the chords joining the vertices of the spherical triangle. From (4.1) it follows that the length $\tilde{a}$ of an edge of $\tilde{\triangle} A_1A_2A_3$ equals

$$\tilde{a} = \frac{\sqrt{4 \sin^2(\alpha/2) - 1}}{\sin(\alpha/2)}.$$ 

(4.4)

The segments of the geodesic $\tilde{\gamma}$ lying inside $A_1A_2A_3$ are mapped into the straight line segments inside $\tilde{\triangle} A_1A_2A_3$ (see Figure 6).

In the similar way the other tetrahedron faces $A_2A_3A_4$, $A_2A_4A_1$ and $A_1A_4A_3$ are mapped into the plane triangles $\tilde{\triangle} A_2A_3A_4$, $\tilde{\triangle} A_2A_4A_1$ and $\tilde{\triangle} A_1A_4A_3$ respectively. Since the spherical tetrahedron is regular, the constructed plane triangles are equal. We can glue them together identifying the edges with the same labels. Hence we obtain the regular tetrahedron in Euclidean space. Since the segments of $\gamma$ are mapped into the straight line segments within the plane triangles, then they form an abstract geodesic $\tilde{\gamma}$ on the regular tetrahedron in $\mathbb{E}^3$, and $\tilde{\gamma}$ is equivalent to $\gamma$.

Let us show that the length of $\gamma$ is greater than the length of $\tilde{\gamma}$. Consider an arc $MN$ of the geodesic $\gamma$ within the face $A_1A_2A_3$. The rays $OM$ and $ON$ intersect the plane $\Pi$ at the points $M$ and $N$ respectively. The line segment $\tilde{M}$ and $\tilde{N}$ lying into $\triangle A_1A_2A_3$ is the image of the arc $MN$ under the geodesic map (see Figure 6). Suppose that the length of the arc $MN$ is equal to $2\varphi$, then the length of the segment $\tilde{M}\tilde{N}$ equals $2\sin \varphi$. Thus the length of $\gamma$ on a regular tetrahedron in spherical space is greater than the length of its image $\tilde{\gamma}$ on a regular tetrahedron in Euclidean space.

From Proposition [4] we know that on a regular tetrahedron in Euclidean space there exists a simple closed geodesic $\tilde{\gamma}$ equivalent to $\tilde{\gamma}$. On the development of the tetrahedron the geodesic $\tilde{\gamma}$ is a straight line segment, and the generalized geodesic $\tilde{\gamma}$ is a polyline, then the length of $\tilde{\gamma}$ is less than the length of $\tilde{\gamma}$.

This implies that on a regular tetrahedron $A_1A_2A_3A_4$ in $\mathbb{S}^3$ with planar angle $\alpha$ the length $L_{p,q}$ of a simple closed geodesic $\gamma$ of type $(p,q)$ is greater than the length of a simple closed geodesic $\tilde{\gamma}$ of type $(p,q)$ on a regular tetrahedron with the edge length $\tilde{a}$ in $\mathbb{E}^3$. From the
equations (3.1) and (4.4) we get, that

\[ L_{p,q} > 2\sqrt{p^2 + pq + q^2} \frac{\sqrt{4\sin^2(\alpha/2) - 1}}{\sin(\alpha/2)}. \]

If \( \alpha \) such that the following inequality holds

\[ 2\sqrt{p^2 + pq + q^2} \frac{\sqrt{4\sin^2(\alpha/2) - 1}}{\sin(\alpha/2)} > 2\pi, \tag{4.5} \]

then the necessary condition for the existence of a simple closed geodesic of type \((p, q)\) on a regular tetrahedron with face’s angle \(\alpha\) in spherical space is failed. Therefore, if

\[ \alpha > 2 \arcsin \sqrt{\frac{p^2 + pq + q^2}{4(p^2 + pq + q^2) - \pi^2}}, \]

then there is no simple closed geodesics of type \((p, q)\) on the tetrahedron with planar angle \(\alpha\) in spherical space.

**Corollary 4.2.** On a regular tetrahedron in spherical space there exist a finite number of simple closed geodesics.

**Proof.** If the integers \((p, q)\) go to infinity, then

\[ \lim_{p,q \to \infty} 2\arcsin \sqrt{\frac{p^2 + pq + q^2}{4(p^2 + pq + q^2) - \pi^2}} = 2\arcsin \frac{1}{2} = \frac{\pi}{3}. \]

From the inequality (4.3) we get, that for the large numbers \((p, q)\) a simple closed geodesic of type \((p, q)\) could exist on a regular tetrahedron with the planar angle \(\alpha\) closed to \(\pi/3\) in spherical space.

The pairs \(p = 0, q = 1\) and \(p = 1, q = 1\) don’t satisfy the condition (4.3). Geodesics of this types are described in Lemma 4.1.

### 4.4 An estimation for the angle \(\alpha\) for which there is a simple closed geodesic of type \((p, q)\).

In previous sections we assumed that the Gaussian curvature of faces of a regular tetrahedron in spherical space is equal 1. In this case the length \(a\) of the edges of the regular tetrahedron was the function of \(\alpha\) given by (4.1). In current section we will assume that the faces of the tetrahedron are spherical triangles with the angle \(\alpha\) on a sphere of radius \(R = 1/a\). Then the length of the tetrahedron edges equals 1, and the faces curvature is \(a^2\).

Since \(\alpha > \pi/3\), then we can write \(\alpha = \pi/3 + \varepsilon\), where \(\varepsilon > 0\). Taking into account Lemma 4.1 we also expect \(\varepsilon < \pi/6\).

**Theorem 3.** Let \((p, q)\) be a pair of coprime integers, \(0 \leq p < q\), and let \(\varepsilon\) satisfy

\[ \varepsilon < \min \left\{ \frac{\sqrt{3}}{4c_0\sqrt{p^2 + q^2 + pq}} \left( \sum_{i=0}^{\lfloor p+q \rfloor} \sum_{j=0}^{\lfloor p+q \rfloor} c(i) + \sum_{j=0}^{\lfloor p+q \rfloor} c(j) \right)^2 : \frac{1}{8\cos \frac{\pi}{12}(p + q)^2} \right\}, \tag{4.6} \]

where

\[ c_0 = \frac{3 - \frac{(p+q+2)}{\pi \cos \frac{\pi}{12}(p+q)^2} - 16 \sum_{i=0}^{\lfloor p+q \rfloor} \tan^2 \left( \frac{\pi i}{2(p+q)} \right)}{1 - \frac{(p+q+2)}{2\pi \cos \frac{\pi}{12}(p+q)^2} - 8 \sum_{i=0}^{\lfloor p+q \rfloor} \tan^2 \left( \frac{\pi i}{2(p+q)} \right)}, \]
\[ c_i(i) = \cos \frac{\pi}{12} (p + q)^2 (4 + \pi^2 (2i + 1)^2) \],
\[ (p + q - i - 1)^2, \]
\[ c_{\alpha}(j) = 4 \left( 8\pi (p + q)^2 \cos \frac{\pi}{12} \tan^2 \frac{\pi j}{2(p + q)} + 1 \right). \]

Then on a regular tetrahedron in spherical space with the planar angle \( \alpha = \pi/3 + \varepsilon \) there exists a unique, up to the rigid motion of the tetrahedron, simple closed geodesic of type \( (p,q) \).

First let us prove some auxiliary lemmas.

**Lemma 4.4.** The edge length of a regular tetrahedron in spherical space of curvature 1 satisfies the inequality
\[ a < \pi \sqrt{2 \cos(\pi/12)} \sqrt{\varepsilon}, \tag{4.7} \]
where \( \alpha = \pi/3 + \varepsilon \) is the planar angle of the face of the tetrahedron.

**Proof.** From (4.1) we have
\[ \sin a = \frac{\sqrt{4 \sin^2(\alpha/2) - 1}}{2 \sin^2(\alpha/2)}. \]
Substituting \( \alpha = \pi/3 + \varepsilon \), we get
\[ \sin a = \frac{\sqrt{\sin(\varepsilon/2) \cos(\pi/6 - \varepsilon/2)}}{\sin^2(\pi/6 + \varepsilon/2)}. \]
Since \( \varepsilon < \pi/6 \), then
\[ \cos(\pi/6 - \varepsilon/2) < \cos(\pi/12), \quad \sin(\pi/6 + \varepsilon/2) > \sin(\pi/6) \quad \text{and} \quad \sin(\varepsilon/2) < \varepsilon/2. \]
Using this estimations we obtain
\[ \sin a < 2 \sqrt{2 \cos(\pi/12)} \sqrt{\varepsilon}. \]
The inequality \( a < \pi/2 \) implies that \( \sin a > (2/\pi)a \). Then
\[ a < \pi \sqrt{2 \cos(\pi/12)} \sqrt{\varepsilon}. \]

Consider a parametrization of a two-dimensional sphere \( S^2 \) of radius \( R \) in \( \mathbb{E}^3 \):
\[
\begin{cases}
x = R \sin \varphi \cos \theta \\
y = R \sin \varphi \sin \theta \\
z = -R \cos \varphi
\end{cases}
\tag{4.8}
\]
where \( \varphi \in [0, \pi], \theta \in [0, 2\pi] \). Let the point \( P \) have the coordinates \( \varphi = r/R, \theta = 0 \), where \( r/R < \pi/2 \), and the point \( X_1 \) correspond to \( \varphi = 0 \). Apply a central projection of the hemisphere \( \varphi \in [0, \pi/2], \theta \in [0, 2\pi] \) onto the tangent plane at \( X_1 \) (see Figure 7).

**Lemma 4.5.** Under the central projection of the hemisphere of radius \( R = 1/a \) onto the tangent plane at \( X_1 \), the angle \( \alpha = \pi/3 + \varepsilon \) with the vertex \( P (R \sin(r/R), 0, -R \cos(r/R)) \) on hemisphere is mapped to the angle \( \hat{\alpha}_r \) on a plane, which satisfies the inequality
\[ \left| \hat{\alpha}_r - \pi/3 \right| < \pi \tan^2(r/R) + \varepsilon. \tag{4.9} \]
Proof. Construct the planes $\Pi_1$ and $\Pi_2$ through the center of a hemisphere and the point $P \left( R \sin \left( \frac{r}{R} \right), 0, -R \cos \left( \frac{r}{R} \right) \right)$:

$\Pi_1 : a_1 \cos (r/R) \cdot x + \sqrt{1 - a_1^2} \cdot y + a_1 \sin (r/R) \cdot z = 0$;

$\Pi_2 : a_2 \cos (r/R) \cdot x + \sqrt{1 - a_2^2} \cdot y + a_2 \sin (r/R) \cdot z = 0$,

where $|a_1|, |a_2| \leq 1.$ (4.10)

If the angle between this two planes $\Pi_1$ and $\Pi_2$ equals $\alpha$, then

$$\cos \alpha = a_1 a_2 + \sqrt{(1 - a_1^2)(1 - a_2^2)}.$$ (4.11)

The tangent plane to $S^2$ at $X_1$ is given by $z = -R$. The planes $\Pi_1$ and $\Pi_2$ intersect the tangent plane along the lines, that form the angle $\hat{\alpha}$ (see Figure 7), and

$$\cos \hat{\alpha}_r = \frac{a_1 a_2 \cos^2 (r/R) + \sqrt{(1 - a_1^2)(1 - a_2^2)}}{\sqrt{1 - a_1^2 \sin^2 (r/R)} \sqrt{1 - a_2^2 \sin^2 (r/R)}}.$$ (4.12)

From the equations (4.11) and (4.12) we get

$$| \cos \hat{\alpha}_r - \cos \alpha | < \frac{|a_1 a_2 \sin^2 (r/R)|}{\sqrt{1 - a_1^2 \sin^2 (r/R)} \sqrt{1 - a_2^2 \sin^2 (r/R)}}.$$ (4.13)

Inequalities (4.10) and (4.13) implies that

$$| \cos \hat{\alpha}_r - \cos \alpha | < \tan^2 (r/R).$$ (4.14)

It is true that

$$| \cos \hat{\alpha}_r - \cos \alpha | = 2 \sin \frac{\hat{\alpha}_r - \alpha}{2} \sin \frac{\hat{\alpha}_r + \alpha}{2}.$$ 

Then $\alpha > \pi/3$ and $\hat{\alpha}_r < \pi$ together with the inequities

$$\sin \frac{\hat{\alpha}_r + \alpha}{2} > \sin \frac{\pi}{6} \text{ and } \sin \frac{\hat{\alpha}_r - \alpha}{2} > \frac{2}{\pi} |\hat{\alpha}_r - \alpha|.$$
implies that
\[
\frac{2}{\pi} \left| \frac{\hat{\alpha}_r}{2} - \alpha \right| < | \cos \hat{\alpha}_r - \cos \alpha |.
\]

From (4.15), (4.14) and \( \alpha = \pi/3 + \varepsilon \) we obtain
\[
\left| \hat{\alpha}_r - \pi/3 \right| < \pi \tan^2 (r/R) + \varepsilon.
\]

On a sphere (4.8) let us consider the arc of length one starting at the point \( P \) with the coordinates \( \varphi = r/R, \theta = 0 \), where \( r/R < \pi/2 \). Apply the central projection of this arc to the plane \( z = -R \), which is tangent to the sphere at the point \( X_1(\varphi = 0) \) (see Figure 8).

**Lemma 4.6.** Under the central projection of the hemisphere of radius \( R = 1/a \) onto the tangent plane at \( X_1 \), the arc of the length one starting from the point \( P (R \sin(r/R), 0, -R \cos(r/R)) \) is mapped to the segment of length \( \hat{l}_r \) satisfying the inequality
\[
\hat{l}_r - 1 < \frac{\cos(\pi/12) \cdot (4 + \pi^2 (2r + 1)^2)}{(1 - (2\pi)a(r + 1))^2} \cdot \varepsilon.
\]  

**Proof.** The point \( P (R \sin(r/R), 0, -R \cos(r/R)) \) on the sphere \( S^2 \) is mapped to \( \hat{P} (R \tan(r/R), 0, -R) \) on the tangent plane \( z = -R \).

Take the point \( Q(Ra_1, Ra_2, Ra_3) \) on a sphere such that the spherical distance \( PQ \) equals 1. Then \( \angle POQ = 1/R \), where \( O \) is a center of the sphere \( S^2 \) (see Figure 8). We obtain the following conditions for the constants \( a_1, a_2, a_3 \):
\[
a_1 \sin(r/R) - a_3 \cos(r/R) = \cos(1/R); \tag{4.16}
\]
\[
a_1^2 + a_2^2 + a_3^2 = 1. \tag{4.17}
\]

The central projection into the plane \( z = -R \) maps the point \( Q \) to the point \( \hat{Q} \left( -\frac{a_1}{a_3} R, -\frac{a_2}{a_3} R, -R \right) \). The length of \( \hat{P}\hat{Q} \) equals
\[
|\hat{P}\hat{Q}| = R \sqrt{(a_1/a_3 - \tan(r/R))^2 + a_2^2/a_3^2} \; \tag{4.18}
\]
Using the Lagrange multipliers method to find the local extremum of the length $\hat{P}\hat{Q}$, we get, that the minimum of $|\hat{P}\hat{Q}|$ reaches when $Q$ has the coordinates

$$(R \sin ((r - 1)/R), 0, R \cos ((r - 1)/R)).$$

Then

$$|\hat{P}\hat{Q}|_{\min} = R |\tan(r/R) - \tan((r - 1)/R)| = \frac{R \sin(1/R)}{\cos(r/R) \cos((r - 1)/R)}.$$ 

Note that $|\hat{P}\hat{Q}|_{\min} > 1$.

The maximum of $|\hat{P}\hat{Q}|$ reaches at the point $Q (R \sin ((r + 1)/R), 0, R \cos ((r + 1)/R))$. This maximum value equals

$$|\hat{P}\hat{Q}|_{\max} = R |\tan(r/R) - \tan((r + 1)/R)| = \frac{R \sin(1/R)}{\cos(r/R) \cos((r + 1)/R)}.$$ 

Since $R = 1/a$, then the length $\hat{t}_r$ of the projection of $PQ$ satisfies

$$\hat{t}_r < \frac{\sin a}{a \cos(ar) \cos(a(r + 1))}.$$ 

From $\sin a < a$, we obtain

$$\hat{t}_r - 1 < \frac{2 - \cos a - \cos(a(2r + 1))}{2 \cos(ar) \cos(a(r + 1))}. (4.19)$$

The equation (4.7) implies that

$$1 - \cos a = \frac{\sin^2 a}{1 + \cos a} \leq 8 \cos(\pi/12) \varepsilon. (4.20)$$

Similarly from the inequality (4.7) we have

$$1 - \cos (a(2r + 1)) \leq 2 \pi^2 \cos(\pi/12)(2r + 1)^2 \varepsilon; (4.21)$$

Estimate the denominator of the (4.19) using the inequality $\cos x > 1 - (2/\pi)x$ where $x < \pi/2$. Using (4.20) and (4.21), we get

$$\hat{t}_r - 1 < \frac{4 \cos(\pi/12) + \pi^2 \cos(\pi/12)(2r + 1)^2}{(1 - (2/\pi)a(r + 1))^2} \cdot \varepsilon.$$

\[\square\]

Proof. of Theorem 3. Fix a pair of coprime integers $(p, q)$ such that $0 < p < q$. Consider a simple closed geodesic $\gamma$ of type $(p, q)$ on a regular tetrahedron $\tilde{A}_1A_2\tilde{A}_3A_4$ with the edge of the length 1 in $\mathbb{R}^3$. Assume that $\gamma$ passes through the midpoints $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{Y}_1$, $\tilde{Y}_2$ of the edges $\tilde{A}_1A_2$ and $\tilde{A}_3A_4$ and $A_1A_3$, $A_4A_2$ respectively.

Consider the development $\tilde{T}_{pq}$ of the tetrahedron along $\gamma$ starting from the point $\tilde{X}_1$. The geodesic unfolds to the segment $\tilde{X}_1\tilde{Y}_1\tilde{X}_2\tilde{Y}_2\tilde{X}_1'$ inside the development $\tilde{T}_{pq}$. From Corollary 3.1 we know, that the parts of the development along geodesic segments $\tilde{X}_1\tilde{Y}_1$, $\tilde{Y}_1\tilde{X}_2$, $\tilde{X}_2\tilde{Y}_2$ and $\tilde{Y}_2\tilde{X}_1'$ are equal, and any two adjacent polygons can be transformed into each other by a rotation through an angle $\pi$ around the midpoint of their common edge.

Now consider a two-dimensional sphere $S^2$ of radius $R = 1/a$, where $a$ depends on $\alpha$ according to (4.1). On this sphere we take the several copies of regular spherical triangles with
the angle $\alpha \in (\pi/3, \pi/2)$ at vertices. Fold this triangles up in the same order as the faces of the Euclidean tetrahedron were unfolded along $\tilde{\gamma}$ into the plane. In other words, we construct a polygon $T_{pq}$ on a sphere $S^2$ formed by the same sequence of regular triangles as the polygon $\tilde{T}_{pq}$ in $\mathbb{E}^3$. Denote the vertices of $T_{pq}$ in accordance with to the vertices of $\tilde{T}_{pq}$. By the construction the spherical polygon $T_{pq}$ has the same properties of the central symmetry as the Euclidean $\tilde{T}_{pq}$. Since the groups of isometries of regular tetrahedra in $S^3$ and in $\mathbb{E}^3$ are equal, then $T_{pq}$ corresponds to the development of a regular tetrahedron with the planar angle $\alpha$ in spherical space.

Denote by $X_1, X'_1$ and $X_2, Y_1, Y_2$ the midpoints of the edges $A_1A_2, A_3A_4, A_1A_3, A_4A_2$ on $T_{pq}$ respectively. These midpoints correspond to the points $\bar{X}_1, X'_1$ and $\bar{X}_2, \bar{Y}_1, \bar{Y}_2$ on the Euclidean development $\tilde{T}_{pq}$. Construct the great circle arcs $X_1Y_1, Y_1X_2, X_2Y_2$ and $Y_2X'_1$. The central symmetry of $T_{pq}$ implies that these arcs form the one great arc $X_1X'_1$ on $S^2$. If $\alpha$ such that $X_1X'_1$ lies inside $T_{pq}$, then $X_1X'_1$ correspond to the simple closed geodesic of type $(p, q)$ on a regular tetrahedron with the planar angle $\alpha$ in $S^3$.

In what follows we consider the part of the polygon $T_{pq}$ only along $X_1Y_1$ but we also denote it as $T_{pq}$ for convenience. This part consists of $p + q$ regular spherical triangles with the edges of length 1. The polygon $T_{pq}$ is contained inside the open hemisphere if

$$a(p + q) < \pi/2, \quad (4.22)$$

Since $\alpha = \pi/3 + \varepsilon$, then the condition $(4.7)$ implies that $(4.22)$ holds if

$$\varepsilon < \frac{1}{8 \cos(\pi/12)(p + q)^2}. \quad (4.23)$$

In this case the length of the arc $X_1Y_1$ is less than $\pi/2a$, so $X_1Y_1$ satisfies the necessary condition from Lemma 4.2.

Apply a central projection of the $T_{pq}$ into the tangent plane $T_{X_1}S^2$ at the point $X_1$ to the sphere $S^2$. The image of the spherical polygon $T_{pq}$ on $T_{X_1}S^2$ is a polygon $\tilde{T}_{pq}$.

Denote by $\tilde{A}_i$ the vertex of $\tilde{T}_{pq}$, which is an image of the vertex $A_i$ on $T_{pq}$. The arc $X_1Y_1$ maps into the line segment $\tilde{X}_1\tilde{Y}_1$ on $T_{X_1}S^2$, that joins the midpoints of the edges $\tilde{A}_1\tilde{A}_2$ and $\tilde{A}_1\tilde{A}_3$. If for some $\alpha$ the segment $\tilde{X}_1\tilde{Y}_1$ lies inside the polygon $\tilde{T}_{pq}$, then the arc $X_1Y_1$ is also containing inside $T_{pq}$ on the sphere.

The vector $\tilde{X}_1\tilde{Y}_1$ equals

$$\tilde{X}_1\tilde{Y}_1 = \tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_s + \tilde{a}_{s+1}, \quad (4.24)$$

where $\tilde{a}_i$ are the sequential vectors of the $\tilde{T}_{pq}$ boundary, $\tilde{a}_0 = \tilde{X}_1\tilde{A}_2$, $\tilde{a}_{s+1} = \tilde{A}_1\tilde{Y}_1$, and $s = \left[\frac{p+q}{2}\right] + 1$ (if we take the boundary of $\tilde{T}_{pq}$ from the other side of $\tilde{X}_1\tilde{Y}_1$, then $s = \left[\frac{p+q}{2}\right]$) (see Figure 9).

On the other hand at Euclidean plane $T_{X_1}S^2$ there exists a development $\bar{T}_{pq}$ of a regular Euclidean tetrahedron $\bar{A}_1\bar{A}_2\bar{A}_3\bar{A}_4$ with the edge of length 1 along a simple closed geodesic $\bar{\gamma}$. The development $\bar{T}_{pq}$ is equivalent to $T_{pq}$, and then it’s equivalent to $\tilde{T}_{pq}$. The segment $\tilde{X}_1\tilde{Y}_1$ lies inside $\bar{T}_{pq}$ and corresponds to the segment of $\bar{\gamma}$ (see Figure 9).

Let the development $\bar{T}_{pq}$ be placed such that the point $\tilde{X}_1$ coincides with $\bar{X}_1$ of $\bar{T}_{pq}$, and the vector $\tilde{X}_1\tilde{A}_2$ has the same direction with $\bar{X}_1\bar{A}_2$. Similarly to the above we have

$$\tilde{X}_1\tilde{Y}_1 = \tilde{a}_0 + \tilde{a}_1 + \cdots + \tilde{a}_s + \tilde{a}_{s+1}, \quad (4.25)$$

where $\tilde{a}_i$ are the sequential vectors of the $\tilde{T}_{pq}$ boundary, $s = \left[\frac{p+q}{2}\right] + 1$ and $\tilde{a}_0 = \tilde{X}_1\tilde{A}_2$, $\tilde{a}_{s+1} = \tilde{A}_1\tilde{Y}_1$ (see Figure 9).
Suppose the minimal distance from the vertices of $\tilde{T}_{pq}$ to the segment $\tilde{X}_1\tilde{Y}_1$ is reached at the vertex $\tilde{A}_k$ and equals $\tilde{h}$ from the formula (3.5). Let us estimate the distance $\hat{h}$ between the segment $\hat{X}_1\hat{Y}_1$ and the corresponding vertex $\hat{A}_k$ on $\hat{T}_{pq}$. A geodesic on a regular tetrahedron in $\mathbb{R}^3$ intersects at most three edges starting from the same tetrahedron’s vertex. It follows, that the interior angles of the polygon $\tilde{T}_{pq}$ are not greater than $4\pi/3$. Hence the angles of the corresponding vertices on $\hat{T}_{pq}$ are not greater than $4\hat{\alpha}_i$. Applying (4.9) for $1 \leq i \leq s$ we get that the angle between $\hat{a}_i$ and $\tilde{a}_i$ satisfies the inequality

$$\angle(\hat{a}_i, \tilde{a}_i) < \sum_{j=0}^{i} 4 \left( \pi \tan^2 \frac{j}{R} + \varepsilon \right).$$

(4.26)

Since $R = 1/a$, then using (4.7) we obtain

$$\tan \frac{j}{R} < \tan \left( j\pi \sqrt{2 \cos \frac{\pi}{12}} \right).$$

(4.27)

The inequality (4.22) holds if the following condition fulfills

$$\tan \left( j\pi \sqrt{2 \cos \frac{\pi}{12}} \right) < \tan \frac{\pi j}{2(p+q)}.\tag{4.28}$$

If $\tan x < \tan x_0$, then $\tan x < \frac{\tan x_0}{x_0} x_0$. From (4.28) it follows

$$\tan \left( j\pi \sqrt{2 \cos \frac{\pi}{12}} \right) < 2(p+q) \tan \frac{\pi j}{2(p+q)} \sqrt{2 \cos \frac{\pi}{12}} \sqrt{\varepsilon}.\tag{4.29}$$

Therefore from (4.27) and (4.29) we get

$$\tan \frac{j}{R} < 2(p+q) \tan \frac{\pi j}{2(p+q)} \sqrt{2 \cos \frac{\pi}{12}} \sqrt{\varepsilon}.\tag{4.30}$$

Using (4.26) and (4.30) we obtain the final estimation for the angle between the vectors $\hat{a}_i$ and $\tilde{a}_i$:

$$\angle(\hat{a}_i, \tilde{a}_i) < \sum_{j=0}^{i} 4 \left( 8\pi (p+q)^2 \cos \frac{\pi}{12} \tan^2 \frac{\pi j}{2(p+q)} + 1 \right) \varepsilon.\tag{4.31}$$

Now estimate the length of the vector $\hat{a}_i - \tilde{a}_i$. The following inequality holds

$$|\hat{a}_i - \tilde{a}_i| \leq \left| \frac{\hat{a}_i}{|\hat{a}_i|} - \tilde{a}_i \right| + |\hat{a}_i - \frac{\hat{a}_i}{|\hat{a}_i|}|.\tag{4.32}$$
Since $\hat{a}_i$ is a unite vector, then
\[
\left| \frac{\hat{a}_i}{|a_i|} - \hat{a}_i \right| \leq \angle(\hat{a}_i, \hat{a}_i) \quad \text{and} \quad \left| \hat{a}_i - \frac{\hat{a}_i}{|a_i|} \right| \leq \hat{h}_i - 1. \tag{4.33}
\]

From the inequality (4.15) we get
\[
\left| \hat{a}_i - \frac{\hat{a}_i}{|a_i|} \right| < \frac{\cos \frac{\pi}{12} (4 + \pi^2 (2i + 1)^2)}{(1 - \frac{2}{\pi} a(i + 1))^2} \cdot \varepsilon. \tag{4.34}
\]

Estimate the denominator in (4.34) using (4.22). Then
\[
\left| \hat{a}_i - \frac{\hat{a}_i}{|a_i|} \right| < \frac{\cos \frac{\pi}{12} (p + q) (4 + \pi^2 (2i + 1)^2)}{(p + q - i - 1)^2} \cdot \varepsilon. \tag{4.35}
\]

From (4.32), (4.31) and (4.35) we obtain
\[
|\hat{a}_i - \hat{a}_i| \leq \left( c_l(i) + \sum_{j=0}^{i} c_a(j) \right) \varepsilon, \tag{4.36}
\]

where
\[
c_l(i) = \frac{\cos \frac{\pi}{12} (p + q)^2 (4 + \pi^2 (2i + 1)^2)}{(p + q - i - 1)^2}, \tag{4.37}
\]
\[
c_a(j) = 4 \left( 8\pi(p + q)^2 \cos \frac{\pi}{12} \tan^2 \frac{\pi j}{2(p + q)} + 1 \right). \tag{4.38}
\]

We estimate the length of $\hat{Y}_1 \hat{Y}_1$ using (4.36)
\[
|\hat{Y}_1 \hat{Y}_1| < \sum_{i=0}^{s+1} |\hat{a}_i - \hat{a}_i| < \sum_{i=0}^{s+1} \left( c_l(i) + \sum_{j=0}^{i} c_a(j) \right) \varepsilon. \tag{4.39}
\]

From (4.31) it follows that the angle $\angle \hat{Y}_1 \hat{X}_1 \hat{Y}_1$ satisfies
\[
\angle \hat{Y}_1 \hat{X}_1 \hat{Y}_1 < \sum_{i=0}^{s+1} c_a(i) \varepsilon. \tag{4.40}
\]

The distance between the vertices $\hat{A}_k$ and $\tilde{A}_k$ equals
\[
|\hat{A}_k \tilde{A}_k| < \sum_{i=0}^{k} \left( c_l(i) + \sum_{j=0}^{i} c_a(j) \right) \varepsilon. \tag{4.41}
\]

We drop a perpendicular $\hat{A}_k \hat{H}$ from the vertex $\hat{A}_k$ into the segment $\hat{X}_1 \hat{Y}_1$. The length of $\hat{A}_k \hat{H}$ equals $\hat{h}$. Then we drop the perpendicular $\hat{A}_k \tilde{H}$ into the segment $\hat{X}_1 \tilde{Y}_1$ and the length of $\tilde{A}_k \tilde{H}$ equals $\tilde{h}$ (see Figure 10).

Let the point $F$ on $\hat{X}_1 \tilde{Y}_1$ be such that the segment $\tilde{A}_k F$ is perpendicular to $\hat{X}_1 \tilde{Y}_1$. Then the length of $\tilde{A}_k F$ is at least $\tilde{h}$. Let $G$ be the point of intersection of $\tilde{X}_1 \tilde{Y}_1$ and the extension of $\hat{A}_k \hat{H}$ and $\tilde{A}_k \tilde{H}$. Let $FK$ be perpendicular to $\hat{H}G$ (see Figure 10). Then the length of $FK$ is not greater than the length of $\hat{A}_k \tilde{A}_k$, and $\angle KFG = \angle \hat{Y}_1 \hat{X}_1 \hat{Y}_1$. From the triangle $GFK$ we obtain
\[
|FG| = \frac{|FK|}{\cos \angle \hat{Y}_1 \hat{X}_1 \hat{Y}_1}. \tag{4.42}
\]
Applying the inequality \( \cos x > 1 - \frac{2}{\pi} x \) for \( x < \frac{\pi}{2} \), to (4.42), we obtain

\[
|FG| < \frac{|\hat{A}_k \tilde{A}_k|}{1 - 2 \frac{\pi}{\pi} \angle\hat{Y}_1 \hat{X}_1 \hat{Y}_1}.
\]

(4.43)

The inequalities (4.40), (4.41) and (4.43) imply

\[
|FG| < \frac{\sum_{k=0}^k \left( c_l(i) + \sum_{j=0}^i c_\alpha(j) \right) \varepsilon}{1 - \sum_{s=0}^s \left( 64\pi(p + q)^2 \cos \frac{\pi}{12} \tan^2 \left( \frac{\pi i}{2(p+q)} \right) + \frac{8}{\pi} \right) \varepsilon}.
\]

(4.44)

Applying (4.23) to the denominator in (4.44) we obtain

\[
|FG| < \frac{\sum_{k=0}^k \left( c_l(i) + \sum_{j=0}^i c_\alpha(j) \right) \varepsilon}{1 - \sum_{s=0}^s \left( (p+q+2) \cos \frac{\pi}{12} \tan^2 \left( \frac{\pi i}{2(p+q)} \right) - 8 \sum_{s=0}^{s+1} \tan^2 \left( \frac{\pi i}{2(p+q)} \right) \right) \varepsilon}.
\]

(4.45)

Therefore we have

\[
\tilde{h} \leq \tilde{A}_k F \leq \hat{h} + |\hat{H}G| + |\hat{A}_k \tilde{A}_k| + |FG|;
\]

(4.46)

Note, that \(|\hat{H}G| < |\hat{Y}_1 \hat{Y}_1|\). Lemma 3.1 implies that

\[
\tilde{h} > \frac{\sqrt{3}}{4\sqrt{p^2 + q^2 + pq}}.
\]

From (4.46) it follows, that

\[
\hat{h} > \frac{\sqrt{3}}{4\sqrt{p^2 + q^2 + pq}} - \left| \hat{Y}_1 \hat{Y}_1 \right| - |\hat{A}_k \tilde{A}_k| - |FG|.
\]

(4.47)

Applying the estimations (4.39), (4.41), (4.45) and the identity \( s = \left[ \frac{p+q}{2} \right] + 1 \), we obtain

\[
\hat{h} > \frac{\sqrt{3}}{4\sqrt{p^2 + q^2 + pq}} - c_0 \sum_{i=0}^{\left[ \frac{p+q}{2} \right] + 2} \left( c_l(i) + \sum_{j=0}^i c_\alpha(j) \right) \varepsilon,
\]

(4.48)

where \( c_l(i) \) is from (4.37), and \( c_\alpha(j) \) is from (4.38) and

\[
c_0 = \frac{3 - \frac{(p+q+2)}{2\pi \cos \frac{\pi}{12} (p+q)^2}}{1 - \frac{(p+q+2)}{2\pi \cos \frac{\pi}{12} (p+q)^2}} - 16 \sum_{i=0}^{\left[ \frac{p+q}{2} \right] + 2} \tan^2 \left( \frac{\pi i}{2(p+q)} \right) + 8 \sum_{i=0}^{\left[ \frac{p+q}{2} \right] + 2} \tan^2 \left( \frac{\pi i}{2(p+q)} \right).
\]
The inequality (4.48) implies that if \( \varepsilon \) satisfies the condition

\[
\varepsilon < \frac{\sqrt{3}}{4c_0 \sqrt{p^2 + q^2 + pq} \sum_{i=0}^{t+2} \left( c_l(i) + \sum_{j=0}^{t} c_a(j) \right)},
\]

(4.49)

then the distance from the vertices of the polygon \( \hat{T}_{pq} \) to \( \hat{X}_1 \hat{Y}_1 \) is nonzero.

Since we use the estimation (4.23), we get, that if

\[
\varepsilon < \min \left\{ \frac{\sqrt{3}}{4c_0 \sqrt{p^2 + q^2 + pq} \sum_{i=0}^{t+2} \left( c_l(i) + \sum_{j=0}^{t} c_a(j) \right)}, \frac{1}{8 \cos \frac{\pi}{12} (p + q)^2} \right\},
\]

(4.50)

then the segment \( \hat{X}_1 \hat{Y}_1 \) lies inside the polygon \( \hat{T}_{pq} \). This implies that the arc \( X_1Y_1 \) on a sphere lies inside the polygon \( T_{pq} \). The arc \( X_1Y_1 \) corresponds to a simple closed geodesic \( \gamma \) of type \( (p, q) \) on a regular tetrahedron with the planar angle \( \alpha = \pi/3 + \varepsilon \) in spherical space. From Corollary 4.1 we get, that this geodesic is unique, up to the rigid motion of the tetrahedron.

Note, that the geodesic \( \gamma \) is invariant under the rotation of the tetrahedron of the angle \( \pi \) over the line passing through the midpoints of the opposite edges of the tetrahedron. The rotation of the tetrahedron through the angle \( 2\pi/3 \) or \( 4\pi/3 \) over the altitude dropped from the vertex to the center of its opposite face changes \( \gamma \) into another simple closed geodesic of type \( (p, q) \).

Rotating over the lines connecting other vertices of the tetrahedron with the center of the opposite faces doesn’t give us any new geodesic. So if \( \varepsilon \) satisfies the condition (4.50), then on a regular tetrahedron with the planar angle \( \alpha = \pi/3 + \varepsilon \) in a spherical space there exist three different simple closed geodesics of type \( (p, q) \), disregarding isometries of the tetrahedron.

4.5 Necessary and sufficient condition for the existence of a simple closed geodesic.

Let \( T(\alpha) \) be a regular tetrahedron with planar angles \( \alpha \) in spherical space \( S^3 \) of curvature 1. Consider a development \( R_{pq}(\alpha) \) of \( T(\alpha) \) in \( S^3 \) along a simple closed geodesic \( \gamma_{pq} \) of type \( (p, q) \), for \( \alpha \in (\pi/3, \pi/3 + \varepsilon) \), where \( \varepsilon \) is from Theorem 3. It follows from Lemma 3.1 that the development \( R_{pq}(\alpha) \) has four points of symmetry \( X_1(\alpha), X_2(\alpha), Y_1(\alpha), Y_2(\alpha) \) and \( X'_1(\alpha) \) that correspond to the midpoints of two pairs of opposite edges of the tetrahedron. The geodesic \( \gamma_{pq} \) passes through these midpoints.

Now for fixed \( (p, q) \) consider a one-parameter family of closed polygons \( R_{pq}(\alpha) \), where \( \alpha \in (\pi/3, 2\pi/3) \). Then \( R_{pq}(\alpha) \) may have overlaps on the sphere. However \( R_{pq}(\alpha) \) is considered as an abstract polygon homeomorphic to a disc, with intrinsic metric, since each interior point of this polygon has a neighbourhood isometric to the interior of a disc on the unit sphere \( S^2 \). This polygon is locally isometrically immersed in the sphere \( S^2 \) (see Figure 11). The development \( R_{pq}(\alpha) \) also has a symmetry property for any \( \alpha \in (\pi/3, 2\pi/3) \) with corresponding points \( X_1(\alpha), X_2(\alpha), Y_1(\alpha), Y_2(\alpha) \) and \( X'_1(\alpha) \) on them.

Then consider rectifiable curves \( \sigma_{pq}(\alpha) \) on \( R_{pq}(\alpha) \) that connect the points \( X_1(\alpha), X'_1(\alpha) \) and pass through \( X_2(\alpha), Y_1(\alpha), Y_2(\alpha) \). If \( X_1(\alpha)X'_1(\alpha) \) lies inside the development \( R_{pq}(\alpha) \), then \( \sigma_{pq}(\alpha) \) corresponds to the simple closed geodesic on regular tetrahedron \( T(\alpha) \). From Theorem 3 follows that this is true if \( \alpha \) is close to \( \pi/3 \). Then from Lemma 4.2 we get that length of \( \sigma_{pq}(\alpha) \) is less then \( 2\pi \). In [19] Borisenko proved that this condition is also sufficient for existence a simple closed geodesic on a regular tetrahedron in \( S^3 \).

The infimum \( L_{pq}(\alpha) \) of the lengths of the curves \( \sigma_{pq}(\alpha) \) is referred to as the length of the abstract shortest curve in the development.
Theorem 4. On a regular tetrahedron in spherical space of curvature one there exist a simple closed geodesic of type \((p,q)\) if and only if the length of the abstract shortest curve on the development is less than \(2\pi\).

Proof. 1. Necessity. If there exist a simple closed geodesic of type \((p,q)\) on a tetrahedron \(T(\alpha)\), then by unfolding along this geodesic we obtain \(R_{p,q}(\alpha)\). The geodesic unfolds into an arc of great circle, which lies inside \(R_{p,q}(\alpha)\), connects the points \(X_1(\alpha)\) and \(X'_1(\alpha)\) and passes through the points of symmetry of \(R_{p,q}(\alpha)\). Lemma 4.2 implies that \(L_{p,q}(\alpha)\) equals the length of this geodesic and \(L_{p,q}(\alpha)\) is less than \(2\pi\) (see Figure 12).

2. Sufficiency. Let us proof the monotonicity of \(L_{p,q}(\alpha)\). Let the infimum \(L_{p,q}(\alpha)\) is attained on a curve \(\sigma_{p,q}(\alpha)\) on \(R_{p,q}(\alpha)\). Consider the geodesic mapping of the sphere \(S^3\) onto the Euclidean tangent space \(T_O S^3\), where \(O\) is a center of the inscribed sphere in the tetrahedron \(T(\alpha)\). Then \(T(\alpha)\) is mapped onto the regular tetrahedron \(\hat{T}(\alpha)\) in \(E^3\) and the curve \(\sigma_{p,q}(\alpha)\) is mapped onto \(\hat{\sigma}_{p,q}(\alpha)\).

Let \(\hat{T}(\alpha(\lambda)) = \lambda \hat{T}(\alpha)\) be the tetrahedron homothetic to \(T(\alpha)\) with center \(O\) and ratio \(\lambda < 1\), so that \(\alpha(\lambda) < \alpha\). This homothety takes \(\hat{\sigma}_{p,q}(\alpha)\) to a curve \(\hat{\sigma}_{p,q}(\alpha(\lambda))\).

Consider the inverse geodesic mapping of \(T_O S^3\) onto \(S^3\). It takes \(\hat{T}(\alpha(\lambda))\) to a regular tetrahedron \(T(\alpha(\lambda))\) where \(\alpha(\lambda) < \alpha\). The curve \(\hat{\sigma}_{p,q}(\alpha(\lambda))\) is mapped to \(\sigma_{p,q}(\alpha(\lambda))\) that belongs to our class of curves. Let us show that the length of the curve \(\sigma_{p,q}(\alpha(\lambda))\) is less than \(L_{p,q}(\alpha)\) for \(\lambda < 1\).

The curve \(\hat{\sigma}_{p,q}(\alpha)\) consists of the finite number of segments with endpoints on edges of the regular tetrahedron. Consider one of these segments \(\hat{z}(\alpha)\) on the face \(A_1 A_2 A_3\) of \(\hat{T}(\alpha)\). The
family of segments $\lambda \hat{z}(\alpha)$ on $\lambda T(\alpha)$ is homothetic to $\hat{z}(\alpha)$ with respect to the center $O$. The great circle arc $z(\lambda) = z(\alpha(\lambda))$ is the inverse geodesic images of $\lambda \hat{z}(\alpha)$. We show that the length of $z(\lambda)$ is monotonically increasing function of $\lambda$.

![Figure 13](image)

Denote by $A_x$ and $A_y$ the endpoints of $\hat{z}(\alpha)$ on $A_1A_2$ and $A_1A_3$ respectively. Then

\[ |A_xA_y|^2 = |A_1A_x|^2 + |A_1A_y|^2 - |A_1A_x||A_1A_y|. \]

The radius of the inscribed sphere of the tetrahedron $T(\alpha)$ with edge length $a$ equals $r = a/(2\sqrt{6})$. The distance from the center of $T(\alpha)$ to the points $A_x$ and $A_y$ can be found from the triangles $\triangle A_1\hat{O}A_x$, where $\hat{O}$ is the center of the face $A_1A_2A_3$ (see Figure 13):

\[ |\hat{O}A_x|^2 = |A_1A_x|^2 + \frac{a^2}{3} - a|A_1A_x|. \]

From the triangle $\triangle O\hat{O}A_x$ we get

\[ |\hat{O}A_x|^2 = \frac{3}{8} a^2 + |A_1A_x|^2 - a|A_1A_x|. \]

From the triangle $\triangle O\hat{O}A_y$ we have

\[ |\hat{O}A_y|^2 = \frac{3}{8} a^2 + |A_1A_y|^2 - a|A_1A_y|. \]

From the triangles $\triangle OA_xA$ and $\triangle OA_yA$, where $S$ is the center of the sphere $S^3$, we obtain

\[ |SA_x|^2 = 1 + |\hat{O}A_x|^2; \quad |SA_y|^2 = 1 + |\hat{O}A_y|^2. \]

From $\triangle A_xSA_y$ we obtain

\[ \cos z = \frac{(1 + |\hat{O}A_x|^2) + (1 + |\hat{O}A_y|^2) - |A_xA_y|^2}{2\sqrt{1 + |\hat{O}A_x|^2} \sqrt{1 + |\hat{O}A_y|^2}}, \]

where $z$ is the angle at the vertex $S$.

Similarly, for the homothetic tetrahedra $\lambda T(\alpha)$, we have

\[ \cos z(\lambda) = \frac{(1 + \lambda^2|\hat{O}A_x|^2) + (1 + \lambda^2|\hat{O}A_y|^2) - \lambda^2|A_xA_y|^2}{2\sqrt{1 + \lambda^2|\hat{O}A_x|^2} \sqrt{1 + \lambda^2|\hat{O}A_y|^2}}. \]

The derivative of $z(\lambda)$ at $\lambda = 1$ is positive. This implies that the length of $\sigma_{p,q}(\alpha(\lambda))$ is less than the length of $\sigma_{p,q}(\alpha)$ for $\lambda < 1$. Hence $L_{p,q}(\alpha(\lambda)) < L_{p,q}(\alpha)$ for $\lambda < 1$ and $\alpha(\lambda) < \alpha$. 

25
For $\pi/3 < \alpha < \pi/3 + \varepsilon$, where $\varepsilon$ is from Theorem 3, there is a simple closed geodesic of
\[ \alpha \]
type $(p,q)$ on a regular tetrahedron in $S^3$. This geodesic unfolds into a curve $\sigma_{p,q}(\alpha)$ of length
\[ L_{p,q}(\alpha) < 2\pi \]
inside the development $R_{p,q}(\alpha)$.

Now increase the angle $\alpha$ starting from $\alpha_1$. As long as $\sigma_{p,q}(\alpha)$ lies inside the development $R_{p,q}(\alpha)$, it corresponds to a simple closed geodesic on a regular tetrahedron $T(\alpha)$. Let $\beta$ be the first value of $\alpha$ for which $\sigma_{p,q}(\alpha)$ attains the boundary of $R_{p,q}(\alpha)$. This value exists by Theorem 2 which implies that there is $\alpha_2 \in (\pi/3, \pi/2)$ such that there is no simple closed
geodesics on $T(\alpha)$ for $\alpha > \alpha_2$.

The point of intersection of $\sigma_{p,q}(\beta)$ with the boundary of the development $R_{p,q}(\beta)$ is a vertex of the tetrahedron. Since $R_{p,q}(\beta)$ consists of congruent polygons, then the segment $\sigma_{p,q}(\beta)$ ‘-touches’ the boundary of $R_{p,q}(\beta)$ at four vertices. The property of symmetry of $R_{p,q}(\beta)$ implies that these ‘touchings’ alternate so there are two of them from each side of $\sigma_{p,q}(\beta)$ (see Figure 14).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{Figure 14}
\end{figure}

The segment $\sigma_{p,q}(\alpha)$ cannot ‘touch’ the boundary of the development $R_{p,q}(\beta)$ at five points. Otherwise the curve $\sigma_{p,q}(\alpha)$ passes twice through some vertex of $T(\beta)$. For any line segment, the full angle on one side is $\pi$. The full angle at any vertex is less than $2\pi$. Thus the segments $l_1$ and $l_2$ of the curve $\sigma_{p,q}(\alpha)$ intersect at a nonzero angle at that vertex. The geodesics $\sigma_{p,q}(\alpha)$ with $\alpha < \beta$ and $\alpha$ close to $\beta$ also intersect themselves, which contradicts the fact that these geodesics are simple.

The case when two points of intersection (for example, the vertices $A_2$ and $A_3$) merge is also impossible. These two vertices are not connected by an edge, because if we take $\alpha < \beta$ and let $\lim \alpha = \beta$, then we see that the length of the edge connecting these two vertices of intersection tends to 0. As $\alpha \to \beta$, the full angles at $A_2$ and $A_3$ tend to angles $\geq \pi$, for otherwise the geodesics $\sigma_{p,q}(\alpha)$ would cross the boundary of the development for some $\alpha < \beta$. Without loss of generality, we can assume that $\beta \leq \beta_0 = 2 \arcsin \sqrt{\pi/18} < 2\pi$, since there are only three simple closed geodesics for $\beta \geq \pi/2$ (see Lemma 4.1). This bound follows from the case $p = 2, q = 1$ of inequality (4.3) from Theorem 2. For the full angles at the vertices $A_2$ to tend to limits $\geq \pi$, it is necessary that at least three triangles meet at $A_2$ and that for $\alpha$ close to $\beta$ two edges meeting at $A_2$ belong to triangles in the development traversed by the line segment $\sigma_{p,q}(\alpha)$. The same we observed for $A_3$. Then four different edges of triangles would meet at the merged vertex. Thus, four edges come out of a vertex of the tetrahedron, which is a contradiction.

As a result, for $\alpha = \beta$ the segment $\sigma_{p,q}(\alpha)$ ‘touch’ the boundary of $R_{p,q}(\beta)$ at four points, which correspond to the vertices of the tetrahedron. The curve $\sigma_{p,q}(\alpha)$ divides the tetrahedron into two regions homeomorphic to a circle. Each interior point has a neighborhood isometric to a disc on the sphere $S^2$ of curvature 1, and the boundary is a digon. The edges of this digon have the same length, the full angles at both vertices are $3\beta - \pi$, and the geodesic curvature of the digon is 0. Therefore, the perimeter of the digon is $2\pi$. Hence the length of $\sigma_{p,q}(\alpha)$ is $2\pi$, which implies that $L_{p,q}(\alpha) = 2\pi$.

If a simple closed geodesic exists for a fixed $\alpha$, then $L_{p,q}(\alpha)$ is equal to the length of this
geodesic, and therefore it is $< 2\pi$ for $\alpha < \beta$. If $\alpha > \beta$, then, due to the monotonicity of $L_{p,q}(\alpha)$, the length of $L_{p,q}(\alpha)$ is greater than $2\pi$, and there are no simple closed geodesics of type $(p,q)$ on the tetrahedron $T(\alpha)$.

**Corollary 4.3.** [19] *If the edge $a$ of a regular tetrahedron in spherical space satisfies the inequality*

$$a < 2 \arcsin \frac{\pi}{\sqrt{p^2 + pq + q^2 + \sqrt{(p^2 + pq + q^2)^2 + 2\pi^2}}} \quad (4.51)$$

*then this tetrahedron has a simple closed geodesic of type $(p,q)$.*

**Proof.** Let $O$ be the centre of the inscribed and circumscribed spheres of a regular tetrahedron $T(\alpha)$ in spherical space $S^3$.

Consider a geodesic mapping of the open hemisphere of $S^3$ containing $T(\alpha)$ onto the tangent space $T_O S^3$. The tetrahedron $T(\alpha)$ is mapped to a regular tetrahedron $\hat{T}(\alpha)$ with center at $O$ in Euclidean space $T_O S^3$. The midpoints of the edges are mapped to the midpoints. Let $\hat{a}$ be the edge length of $\hat{T}(\alpha)$.

Let $\hat{\gamma}_{p,q}(\alpha)$ be a simple closed geodesic of type $(p,q)$ that passes through the midpoints of two pairs of opposite edges of $\hat{T}(\alpha)$. Then the length of $\hat{\gamma}_{p,q}(\alpha)$ is equal

$$\hat{L}_{p,q}(\alpha) = 2\hat{a}\sqrt{p^2 + pq + q^2} \quad (4.52)$$

Take $\alpha$ such that $\hat{L}_{p,q}(\alpha) < 2\pi$. The inverse image $\gamma_{p,q}(\alpha)$ of the geodesic $\hat{\gamma}_{p,q}(\alpha)$ on $T(\alpha)$ has length less than $\hat{L}_{p,q}(\alpha)$, and therefore less than $2\pi$. The curve $\gamma_{p,q}(\alpha)$ belongs to the class of admissible curves $\sigma_{p,q}(\alpha)$ in the definition of $L_{p,q}(\alpha)$. Therefore, $L_{p,q}(\alpha) < 2\pi$, and Theorem 4 implies that there exists a simple closed geodesic of type $(p,q)$ on $T(\alpha)$. It remains to use the inequality

$$2\hat{a}\sqrt{p^2 + pq + q^2} < 2\pi$$

to obtain a bound on $\alpha$, or, equivalently, on $a$. Formula 4.1 implies that

$$2\sin(a/2)\cos(a/2) = 1.$$

We apply a geodesic mapping of the sphere $S^3$ from its centre $S$ onto the tangent space $T_O S^3$. Consider the triangle $\triangle SOB$, where $B$ is the midpoint of $A_1A_2$. Let $\hat{B}$ be the image of $B$ under the geodesic mapping (Figure 15); then

$$|OB| = \tan |OB|.$$

The edge $A_1A_2$ of the spherical triangle maps to the edge $\hat{A}_1\hat{A}_2$ of the regular tetrahedron in Euclidean space, and $\hat{A}_1\hat{A}_2$ is perpendicular to $\hat{O}\hat{B}$. From the triangle $\triangle S\hat{A}_1\hat{B}$ we obtain

$$\frac{\hat{a}}{2} = |\hat{A}_1\hat{B}| = |S\hat{B}| \tan \frac{a}{2} = \frac{\tan(a/2)}{\cos |OB|} \quad (4.53)$$

From the triangle $\triangle PA_1A_2$ on a face of the tetrahedron in spherical space, where $P$ is the centre of the inscribed and circumscribed circles of the face, we obtain

$$\cos a = \cos^2 R_{bas} - \frac{1}{2} \sin^2 R_{bas},$$

where $R_{bas} = |PA_1| = |PA_2|$. Hence

$$\cos R_{bas} = \sqrt{\frac{1 + 2 \cos a}{3}}. \quad (4.54)$$

27
From $\triangle A_4PA_1$ (Figure 16) we obtain
\begin{equation}
\cos a = \cos(R + r) \cos R_{bas},
\end{equation}
where $R$ is the radius of the circumscribed sphere of the tetrahedron $A_1A_2A_3A_4$, $r$ is the radius of the inscribed ball, and $|A_4P| = R + r$. Then (4.55) implies that
\begin{equation}
\cos R > \frac{\cos a}{\cos R_{bas}}.
\end{equation}
From $\triangle O\!A_1B$ we obtain
\begin{equation}
\cos R = \cos |OB| \cos(a/2).
\end{equation}
Expressions (4.56) and (4.57) implies that
\begin{equation}
\frac{1}{\cos |OB|} = \frac{\cos(a/2)}{\cos R} < \frac{\cos(a/2) \cos R_{bas}}{\cos a}.
\end{equation}
From (4.53), (4.54) and (4.58) we get
\begin{equation}
\frac{\hat{a}}{2} < \frac{\sin(a/2)}{\cos a} \sqrt{1 + \frac{2 \cos a}{3}} \leq \frac{\sin(a/2)}{\cos a}.
\end{equation}
Therefore, from (4.52) and (4.59) we obtain the following estimation for the length of a simple closed geodesic $\hat{\gamma}_{p,q}(\alpha)$ of type $(p, q)$ on $\hat{T}(\alpha)$:
\begin{equation}
\hat{L}_{p,q}(\alpha) \leq 4\frac{\sin(a/2)}{\cos a} \sqrt{p^2 + pq + q^2}.
\end{equation}
Remind, that from Theorem 4 it follows that if $\hat{L}_{p,q}(\alpha) < 2\pi$, then there exist a simple closed geodesic of type $(p, q)$ on $T(\alpha)$ in $S^3$. Resolving the quadratic inequality
\begin{equation}
4\frac{\sin(a/2)}{\cos a} \sqrt{p^2 + pq + q^2} < 2\pi
\end{equation}
with respect to $\sin(a/2)$, we obtain the required inequality.

5 Simple closed geodesics on regular tetrahedra in $\mathbb{H}^3$

5.1 Necessary conditions for a closed geodesic to be simple.

We assume that the Gaussian curvature of hyperbolic space (Lobachevsky space) $\mathbb{H}^3$ equals $-1$. A regular tetrahedron in $\mathbb{H}^3$ is a closed convex polyhedron all of whose faces are regular geodesic.
triangles and all vertices are regular trihedral angles. The planar angle $\alpha$ of the face satisfies the inequality $0 < \alpha < \pi/3$ and the length $a$ of edges is equal to

$$a = \text{arcosh} \left( \frac{\cos \alpha}{1 - \cos \alpha} \right).$$

(5.1)

Consider the Cayley-Klein model of hyperbolic space. In this model points are represented by the points in the interior of the unit ball. Geodesics in this model are the chords of the ball. Assume that the center of the circumscribed sphere of a regular tetrahedron coincides with the center of the model. Then the regular tetrahedron in hyperbolic space is represented by a regular tetrahedron in Euclidean space.

**Lemma 5.1.** If a geodesic on a regular tetrahedron in hyperbolic space intersects three edges meeting at a common vertex consecutively, and intersects one of these edges twice, then this geodesic has a point of self-intersection.

**Proof.** Let $A_1A_2A_3A_4$ be a regular tetrahedron in $\mathbb{H}^3$. Suppose the geodesic $\gamma$ intersects $A_4A_1$, $A_4A_2$ and $A_4A_3$ consecutively at the points $X_1$, $X_2$, $X_3$ respectively and then intersects the edge $A_4A_1$ again at the point $Y_1$.

Suppose that the length of $A_4X_1$ is less than the length of $A_4Y_1$.

Unfold the faces $A_1A_2A_4$, $A_4A_2A_3$ and $A_4A_3A_1$ to the hyperbolic plane. Consider the Cayley-Klein model of the hyperbolic plane and place the vertex $A_4$ at the center of the model. Then the part $X_1X_2X_3Y_1$ of the geodesic is the straight line segment on the development. We obtain a triangle $X_1A_4Y_1$ on the development.

Figure 16

Figure 17

29
Let $\rho(X)$ be the distance function between the vertex $A_4$ and a point $X$ on $\gamma$. It is known that if $\gamma$ be a geodesic in a complete simply connected Riemannian manifold $M$ of nonpositive curvature, then the function $\rho(X)$ of a distance from the fixed point $A$ on $M$ to the points $X$ on $\gamma$ is a convex function. The minimum of $\rho(X)$ is achieved at the point $H_0$ such that $A_4H_0$ is orthogonal to $\gamma$ and $\angle H_0A_4Y_1 > 3\alpha/2$.

Let $Z_1$ be the point on the segment $H_0Y_1$ such that $\angle H_0A_4Z_1 = 3\alpha/2$. On the opposite side of $H_0$ we choose the point $Z_2$ such that $\angle H_0A_4Z_2 = 3\alpha/2$. The point $Z_2$ also lies on the face at the vertex $A_4$ of tetrahedron.

Since $\angle H_0A_4Z_1 = \angle H_0A_4Z_2 = 3\alpha/2$, it follows that the points $Z_1$ and $Z_2$ correspond to the same point $Z$ on the generatrix $A_4Z$ opposite to $A_4H_0$ on the tetrahedron. This point is the self-intersection point of the geodesic $\gamma$ (Figure 17).

**Lemma 5.2.** [14] Let $d$ be the minimum distance from the vertices of a regular tetrahedron in hyperbolic space to a simple closed geodesic on the tetrahedron. Then

$$d > \frac{1}{2} \ln \left( \frac{\sqrt{2\pi^3} + (\pi - 3\alpha)^3}{\sqrt{2\pi^3} - (\pi - 3\alpha)^3} \right),$$

(5.2)

where $\alpha$ is the planar angle of a face of the tetrahedron.

**Proof.** Let $\gamma$ be a simple closed geodesic on a regular tetrahedron $A_1A_2A_4A_3$ in hyperbolic space $\mathbb{H}^3$. Assume that minimum distance $d$ from the vertices of the tetrahedron to $\gamma$ is achieved at the vertex $A_4$ on the face $A_2A_4A_3$. Draw a generatrix $A_4H$ orthogonal to $\gamma$ at the point $H_0$. Denote by $\beta$ the angle $\angle A_4HA_2$. Without loss of generality we assume that $0 \leq \beta \leq \alpha/2$.

We draw a generatrix $A_4K$ such that the planar angle between $A_4K$ and $A_4H$ equals $3\alpha/2$. Then $A_4K$ lies in the face $A_1A_4A_3$ and $\angle A_1A_4K = \alpha/2 - \beta$. Note that if $\beta = \alpha/2$, then $A_4K$ coincides with $A_4A_1$. If $\beta = 0$, then $A_4K$ coincides with the altitude in a face of the tetrahedron and has the smallest length $h$ (Figure 18).

We cut the trihedral angle at $A_4$ along the generatrix $A_4K$ and develop it to the hyperbolic plane in the Cayley-Klein model. We put the vertex $A_4$ at the centre of the boundary circle. The trihedral angle unfolds into a convex polygon $K_1A_4K_2A_3A_2A_1$. The angle $K_1A_4K_2$ equals $3\alpha$. The segment $A_4H$ corresponds to the bisector of the angle $K_1A_4K_2$. The geodesic $\gamma$ is a straight line orthogonal to $A_4H$ at $H_0$.

On the lines $A_4K_1$ and $A_4K_2$ choose the points $P_1$ and $P_2$ respectively such that $|A_4P_1| = |A_4P_2| = h$. The line segment $P_1P_2$ is orthogonal to $A_4H$ at the point $H_p$, and

$$\tanh |A_4H_p| = \cos(3\alpha/2) \tanh h.$$
If \( d \leq |A_4H_p| \), then \( \gamma \) lies above the segment \( P_1P_2 \), and therefore \( \gamma \) intersects the lines \( A_4K_1 \) and \( A_4K_2 \) at the points \( Z_1 \) and \( Z_2 \) respectively. When we fold the development back to the tetrahedron, the segments \( A_4K_1 \) and \( A_4K_2 \) are mapped to the segment \( A_4K \) on the tetrahedron, and \( Z_1' \) and \( Z_2' \) are mapped to the same point \( Z \) on \( A_4K \). This point \( Z \) is point of self intersection of the geodesic \( \gamma \).

Therefore, in order that \( \gamma \) have no points of self-intersection, it is necessary that \( d > |A_4H_p| \). This implies

\[
\tanh d > \cos(3\alpha/2) \tanh h. \tag{5.3}
\]

The altitude \( h \) of the face of the tetrahedron satisfies

\[
\tanh h = \tanh a \cos \alpha/2 = \cos \alpha/2 \frac{\sqrt{2 \cos \alpha - 1}}{\cos \alpha}. \tag{5.4}
\]

Combining (5.4) and (5.3), we obtain

\[
\tanh d > \cos \alpha/2 \cos(3\alpha/2) \frac{\sqrt{2 \cos \alpha - 1}}{\cos \alpha}, \tag{5.5}
\]

Now we estimate the expression on the right-hand side of (5.5) from below. Consider the function \( \sqrt{2 \cos \alpha - 1} \):

\[
2 \cos \alpha - 1 = 4 \sin (\pi/6 - \alpha/2) \sin (\pi/6 + \alpha/2). 
\]

Since the function \( \sin(\pi/6 + \alpha/2) \) increases on the interval \((0, \pi/3)\), then

\[
\sin(\pi/6 + \alpha/2) > 1/2 \text{ when } \alpha \in (0, \pi/3). 
\]

The function \( \sin(\pi/6 - \alpha/2) \) increases on the interval \((0, \pi/3)\). It is known that \( \sin y > (2/\pi)y \) when \( 0 < y < \pi/2 \). These imply

\[
\sin(\pi/6 - \alpha/2) > \frac{1}{\pi} (\pi/3 - \alpha). 
\]

We obtain

\[
\sqrt{2 \cos \alpha - 1} > \sqrt{\frac{2}{3\pi}} (\pi - 3\alpha). \tag{5.6}
\]

The function \( \cos(3\alpha/2) \) is decreasing for \( 0 < \alpha < \pi/3 \). It is true that \( \cos y > 1 - (2/\pi)y \), when \( 0 < y < \pi/2 \). Therefore,

\[
\cos(3\alpha/2) > \frac{1}{\pi} (\pi - 3\alpha). \tag{5.7}
\]

We have \( \cos \alpha/2 > \sqrt{3}/2 \) when \( 0 < \alpha < \pi/3 \).

These inequalities, together with (5.6) and (5.7), give the following bound

\[
\tanh d > \frac{1}{\sqrt{2\pi}} (\pi - 3\alpha)^{3/2}. \tag{5.8}
\]

The inequality (5.8) implies inequality (5.2), as required.

5.2 Uniqueness of a simple closed geodesic of type \((p, q)\).

For a regular tetrahedron in hyperbolic space the following analogue of Lemma 4.3 holds.

Lemma 5.3. A simple closed geodesic on a regular tetrahedron in hyperbolic space passes through the midpoints of two pairs of the opposite edges on the tetrahedron.
Proof. Let \( \gamma \) be a simple closed geodesic on a regular tetrahedron \( T \) in hyperbolic space \( \mathbb{H}^3 \). Consider the Cayley-Klein model of \( \mathbb{H}^3 \) and place the tetrahedron so that the center of circumscribed sphere of the tetrahedron coincides with the center of the model. Then \( T \) is represented by a regular tetrahedron \( \tilde{T} \) in Euclidean space \( \mathbb{E}^3 \).

A simple closed geodesic \( \gamma \) on \( T \) is represented by an abstract geodesic on \( \tilde{T} \). From Proposition \[\text{[1]}\] we get that this generalized geodesic is equivalent to a simple closed geodesic \( \tilde{\gamma} \) on \( \tilde{T} \) in \( \mathbb{E}^3 \). From Theorem \[\text{[1]}\] we assume that \( \tilde{\gamma} \) passes through the midpoints of two pairs of the opposite edges on this tetrahedron.

Label the vertices of tetrahedron \( T \) and corresponding vertices of \( \tilde{T} \) with \( A_1, A_2, A_3 \) and \( A_4 \). Suppose that \( \tilde{\gamma} \) passes through the midpoints \( X_1, X_2 \) of the edges \( A_1A_2 \) and \( A_3A_4 \). Consider the development of \( \tilde{T} \) along \( \tilde{\gamma} \) starting from \( X_1 \). From Corollary \[\text{[3,1]}\] it follows that this development is central-symmetric with respect to the point \( X_2 \).

Let \( X_1, X_2 \) be the corresponding points on \( \gamma \) on the edges \( A_1A_2 \) and \( A_3A_4 \) of \( T \). Consider the development of \( T \) onto hyperbolic plane along \( \gamma \) starting from the point \( X_1 \). Then \( \gamma \) is a line segment \( X_1X'_1 \) on the development.

Denote by \( M_1 \) and \( M_2 \) the midpoints of the edges \( A_1A_2 \) and \( A_3A_4 \) respectively. Since the rotation of the tetrahedron through \( \pi \) around \( M_1M_2 \) in hyperbolic space is the isometry of the tetrahedron then, the development of \( T \) along \( X_1X_2X'_1 \) on hyperbolic plane is central symmetric with the center at \( M_2 \).

Denote by \( T_1 \) and \( T_2 \) the parts of the development along segments \( X_1X_2 \) and \( X_2X'_1 \) respectively. The central symmetry of the development around the point \( M_2 \). Then the point \( X'_1 \) belongs to the edge \( A_1A_2 \) of the \( T_1 \), and the lengths of \( A_2X_1 \) and \( X'_1A_1 \) are equal.

The edge \( A_1A_2 \) containing \( X'_1 \) is mapped onto \( A_2A_1 \) with the point \( X_1 \). Then the point \( X'_1 \) is mapped into itself with the opposite orientation. The point \( X_2 \) on \( A_3A_4 \) is mapped to the point \( X'_2 \) on \( A_3A_4 \) such that the lengths of \( A_4X_2 \) and \( X'_2A_3 \) are equal. Moreover, \( \angle X_1X_2A_4 = \angle X'_1X'_2A_4 \). Since the geodesic is closed, then \( \angle A_1X_1X_2 = \angle A_1X'_1X'_2 \) (Figure \[\text{[19]}\]).

We obtain the quadrilateral \( X_1X_2X'_2X'_1 \) inside \( T_1 \) the sum of whose interior angles is \( 2\pi \). Then the integral of the Gaussian curvature over the interior of \( X_1X_2X'_2X'_1 \) in hyperbolic plane is zero. This implies that the rotation takes the part \( X'_2X'_1 \) of the geodesic to the part \( X_1X_2 \). Hence the points \( X_1 \) and \( X_2 \) are the midpoints of the corresponding edges (Figure \[\text{[19]}\]).

In the same way it can be proved that \( \gamma \) passes through the midpoints of other two opposite edges on the regular tetrahedron in \( \mathbb{H}^3 \).

\textbf{Corollary 5.1.} If two closed geodesics on the regular tetrahedron in hyperbolic space intersect the edges of the tetrahedron in the same order, then they coincide.
5.3 Existence of a simple closed geodesic of type \((p, q)\) on a regular tetrahedron.

**Theorem 5.** \([46]\) On a regular tetrahedron in hyperbolic space for each ordered pair of coprime integers \((p, q)\), there exists unique, up to the rigid motion of the tetrahedron, simple closed geodesic of type \((p, q)\). The geodesics of type \((p, q)\) exhaust all simple closed geodesics on a regular tetrahedron in hyperbolic space.

**Proof.** Let \(\tilde{\gamma}\) be a simple closed geodesic on a regular tetrahedron \(A_1A_2A_3A_4\) in Euclidean space. Assume, that \(\tilde{\gamma}\) passes through the midpoints \(X_1, X_2, Y_1\) and \(Y_2\) of the edges \(A_1A_2, A_3A_4, A_1A_3\) and \(A_2A_4\) respectively.

Consider the development \(\tilde{T}\) of the tetrahedron along \(\tilde{\gamma}\) starting from the point \(\tilde{X}_1\) to the point \(\tilde{X}_1'\). The polygon \(\tilde{T}\) consist of four equal polygons and any two adjacent polygons can be transformed into each other by a rotation through an angle \(\pi\) around the midpoint of their common edge - the points \(\tilde{X}_2, \tilde{Y}_1\) and \(\tilde{Y}_2\). The interior angles of \(\tilde{T}\) are equal to \(\pi/3\), \(2\pi/3\), \(\pi\), or \(4\pi/3\). Moreover, the angle of \(4\pi/3\) is obtained if \(\tilde{\gamma}\) intersects three edges having a common vertex consecutively.

Now we take a regular triangles on hyperbolic plane with angle \(\alpha\) at the vertices. Put these triangles in the same order in which the faces of the tetrahedron were unfolded in Euclidean space along \(\tilde{\gamma}\).

In other words, we construct a polygon \(T\) on hyperbolic plane that is formed by the same sequence of regular triangles as the polygon \(\tilde{T}\) on Euclidean plane. Label the vertices of \(T\) according to the vertices of \(\tilde{T}\). Then the polygon \(T\) corresponds to a development of a regular tetrahedron with the planar angle \(\alpha\) in hyperbolic space (see Figure 20).

Moreover, \(T\) has the same property of central symmetry with respect to the midpoint of the same edge as the polygon \(\tilde{T}\). Denote by \(X_1, X_2, Y_1, Y_2\) and \(X'_1\) the midpoints of the edges \(A_1A_2, A_3A_4, A_1A_3\) and \(A_2A_4\) of \(T\) respectively. We draw the geodesic line segment \(X_1X'_1\).

By construction, the interior angles at the vertices of \(T\) are equal to \(\alpha\), \(2\alpha\), \(3\alpha\) or \(4\alpha\), according to the development on Euclidean plane.

First assume that \(\alpha \in (0, \pi/4]\). Then the polygon \(T\) is convex and the segment \(X_1X'_1\) lies inside \(T\). Furthermore, \(X_1X'_1\) passes through the points \(X_2, Y_1, Y_2\) that are the centers of symmetry of \(T\). Therefore, \(X_1X'_1\) is a simple closed geodesic \(\gamma\) on the regular tetrahedron with the planar angle \(\alpha \in (0, \pi/4]\) in hyperbolic space.

Now we increase the angle \(\alpha\) starting from \(\alpha = \pi/4\). Then the polygon \(T\) is not convex because it contains the interior angles \(4\alpha > \pi\).

Let \(\alpha_0\) be the supremum of \(\alpha\) for which the segment \(X_1X_2\) lies inside \(T\). Suppose \(\alpha_0 < \pi/3\).

For all \(\alpha < \alpha_0\) the segment \(X_1X'_1\) lies entirely inside \(T\) and it is a simple closed geodesic \(\gamma\) on the regular tetrahedron in \(\mathbb{H}^3\). The distance \(d\) from the vertices of the tetrahedron to \(\gamma\) satisfies \([5.2]\). Therefore there exists \(\alpha_1 = \alpha_0 + \varepsilon\) such that the segment \(X_1X_2\) lies entirely inside \(T\). This contradicts the maximality of \(\alpha_0\). Thus \(\alpha_0 = \pi/3\).
It follows that for any \( \alpha \in (0, \pi/3) \) there is a simple closed geodesic of type \((p, q)\) on a regular tetrahedron with a planar angle \( \alpha \) in hyperbolic space.

From Corollary 5.1 it follows the uniqueness of a simple closed geodesic of type \((p, q)\) on a regular tetrahedron in \( \mathbb{H}^3 \). This geodesic has \( p \) points on each of two opposite edges of the tetrahedron, \( q \) points on each of another two opposite edges, and \((p + q)\) points on each edge of the third pair of opposite edges. For any coprime integers \((p, q)\), \( 0 \leq p < q \), there exist three simple closed geodesic of type \((p, q)\) on a regular tetrahedron in \( \mathbb{H}^3 \). They coincide by the rotation of the tetrahedron by the angle \( 2\pi/3 \) or \( 4\pi/3 \) about the altitude constructed from a vertex to the opposite face.

Since any simple closed geodesic on a regular tetrahedron in \( \mathbb{H}^3 \) is equivalent to a simple closed geodesic on a regular tetrahedron in \( \mathbb{E}^3 \), there are not other simple closed geodesics on a regular tetrahedron in \( \mathbb{H}^3 \).

\[\square\]

5.4 Existence of a simple closed geodesic of type \((p, q)\) on a generic tetrahedron.

In Euclidean space \( \mathbb{E}^3 \), there is no simple closed geodesic on a generic tetrahedron. Protasov [40] gave an upper bound for the number of simple closed geodesics depending on the largest deviation from \( \pi \) of the sum of planar angles at the vertices of the tetrahedron. The situation in hyperbolic space is quite different provided that the planar angles of the tetrahedron are sufficiently small. Borisenko proved the following result.

Theorem 6. [49] If the planar angles of a tetrahedron in hyperbolic space are at most \( \pi/4 \), then for any pair of coprime natural numbers \((p, q)\) there exist three simple closed geodesics of type \((p, q)\), disregarding isometries of the tetrahedron.

Proof. Let \( \tilde{\gamma} \) be a simple closed geodesic on a regular tetrahedron \( A_1A_2A_3A_4 \) in Euclidean space. Consider the development \( \tilde{T} \) of the tetrahedron along \( \tilde{\gamma} \) starting from the point \( \tilde{X}_1 \) on \( A_1A_2 \) to the point \( \tilde{X}_1' \).

Consider a generic tetrahedron in hyperbolic space. For convenience we can also label the tetrahedron’s vertices with \( A_1, A_2, A_3 \) and \( A_4 \). Develop this tetrahedron onto the hyperbolic plane in the same order as the development \( \tilde{T} \) is unfolded, starting from the edge \( A_1A_2 \).

As it was shown in the proof of Theorem 5 at most four facets can meet at one vertex of the development. Hence if \( \alpha \leq \pi/4 \), then the development is a convex polygon.

However, there are at most two facets meeting at each of the vertices \( A_1, A_2, A_1', A_2' \), where \( A_1A_2 \) is starting edge and \( A_1'A_2' \) is finishing. Therefore the angles at these vertices are at most \( \pi/2 \).

Consider the quadrilateral \( A_1A_2A_2'A_1' \). Take points \( X(s) \) on \( A_1A_2 \) and \( X'(s) \) on \( A_1'A_2' \) such that \( X(0) = A_1, X'(0) = A_1' \), and the lengths of \( A_1X(s) \) and \( A_1'X'(s) \) are both equal to \( s \) (Figure 21).

For \( s = 0 \), the sum of the angles \( \angle A_1 \) and \( \angle A_1' \) measured from inside the polygon is less than \( \pi \). For \( s = |A_1A_2| \), the sum of \( \angle A_2 \) and \( \angle A_2' \) measured from outside the polygon is greater than \( \pi \). Therefore, there is \( s_0 \) such that the sum of \( \angle X(s_0) \) and \( \angle X'(s_0) \) equals \( \pi \). The line segment \( X(s_0)X'(s_0) \) on the development corresponds to a simple closed geodesic of type \((p, q)\) on the tetrahedron in \( \mathbb{H}^3 \).

Since for any ordered pair of coprime integers \((p, q)\) there exist three simple closed geodesics of type \((p, q)\) on a regular tetrahedron in \( \mathbb{E}^3 \), disregarding isometries of the tetrahedron, then similarly we can construct three simple closed geodesic of type \((p, q)\) on a tetrahedron in \( \mathbb{H}^3 \) with planar angle at most \( \pi/4 \).
5.5 The number of simple closed geodesics.

Let $N(L, \alpha)$ be a number of simple closed geodesics of length not greater than $L$ on a regular tetrahedron with planar angle $\alpha$ in hyperbolic space. In [46] it was shown that

$$N(L, \alpha) = c(\alpha)L^2 + O(L \ln L),$$

where $O(L \ln L) \leq CL \ln L$ when $L \to +\infty$, and

$$c(\alpha) = \frac{27}{16} \left( \frac{1 - \sqrt{3}}{1 - \sqrt{3} \left(1 - \frac{2\alpha}{\pi}\right)} \left(1 - \frac{2\alpha}{\pi}\right) + \ln \frac{1 + \sqrt{3} \left(1 - \frac{3\alpha}{\pi}\right)}{1 - \sqrt{3} \left(1 - \frac{3\alpha}{\pi}\right)} \right)^2,$$

$$\lim_{\alpha \to \frac{\pi}{3}} c(\alpha) = +\infty; \quad \lim_{\alpha \to 0} c(\alpha) = \frac{27}{16} \left( \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \right)^2. \quad (5.9)$$

This result was proved using Proposition 2 about the structure of a simple closed geodesic on a regular tetrahedron.

In current paper we improve the constant $c(\alpha)$ using the estimations obtained in [46].

**Lemma 5.4.** If the length of a simple closed geodesic of type $(p, q)$ on a regular tetrahedron in hyperbolic space is not greater than $L$, then

$$L \geq 2(p + q) \ln \left( 2\sqrt{3} \left(1 - \frac{3\alpha}{\pi}\right) + 1 \right),$$

where $\alpha$ is the plane angle of a face of the tetrahedron.

**Proof.** Let $\gamma$ be a simple closed geodesic of type $(p, q)$, $0 \leq p < q$, on a regular tetrahedron $A_1A_2A_3A_4$ in hyperbolic space.

Assume that $\gamma$ has $q$ points on the edges $A_1A_2$ and $A_3A_4$, $p$ points on $A_1A_4$ and $A_2A_3$, and $p + q$ points on $A_2A_4$ and $A_1A_3$. Denote by $B_1, \ldots, B_{p+q}$ points of $\gamma$ on $A_1A_3$ and by $B'_1, \ldots, B'_{p+q}$ points of $\gamma$ on $A_2A_4$.

Consider the development of the faces $A_3A_1A_4$ and $A_1A_4A_2$ onto the plane. The geodesic segment starting at the point $B_i$, where $i = 1, \ldots, p$, goes through the edge $A_1A_4$ to the point $B'_{q+i}$. Similarly on the development of the faces $A_1A_2A_3$ and $A_2A_3A_4$ there are $p$ segments of $\gamma$ connecting $B_i' \text{ and } B_{q+i}$, $i = 1, \ldots, p$ and passing through the edge $A_2A_3$.

On the faces $A_4A_1A_2$ and $A_1A_2A_3$ the geodesic segments $B_iB'_{q-(i-1)}$, $i = 1, \ldots, q$, pass through the edge $A_1A_2$. Similarly on the development of the faces $A_2A_4A_3$ and $A_4A_3A_1$ there are $q$ geodesic segments $B_{p+i}B'_{(p+q)-(i-1)}$, $i = 1, \ldots, q$ (see Figure 22).
Therefore the geodesic $\gamma$ consists of $2(p + q)$ segments, that connect opposite edges of the tetrahedron. Let us evaluate from below the length of these segments. Consider the quadrilateral obtaining by unfolding of the faces $A_2A_1A_4$ and $A_1A_4A_3$. The minimum distance between points on the edges $A_2A_4$ and $A_1A_3$ is achieved at $H_1H_2$ perpendicular to these edges. Since the planar angle of the tetrahedron $\alpha < \pi/3$, then $H_1H_2$ lies inside the quadrilateral $A_3A_1A_4A_2$ and passes through the midpoint $M$ of the edge $A_1A_4$ (see Figure 23). From the

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure23}
\caption{Figure 23}
\end{figure}

triangle $A_4MH_1$ we have

$$\sinh |MH_1| = \sinh(a/2) \sin \alpha$$

Using (5.1) we get

$$\sinh |MH_1| = \cos(\alpha/2)\sqrt{2}\cos \alpha - 1.$$  

Using

$$2 \cos \alpha - 1 = \frac{\cos(3\alpha/2)}{\cos(\alpha/2)}$$

we get

$$\sinh |MH_1| = \sqrt{\cos(\alpha/2)\cos(3\alpha/2)}.$$

The inequality (5.7) together with $\cos \alpha/2 > \sqrt{3}/2$ implies

$$\sinh |MH_1| \geq \sqrt{\frac{\sqrt{3}}{2} \left( 1 - \frac{3\alpha}{\pi} \right)}, \quad (5.10)$$

36
Consider the function $\text{arsinh}(x)$:

$$2\text{arsinh}(x) = 2 \ln \left( x + \sqrt{x^2 + 1} \right) = \ln(2x^2 + 1 + 2x\sqrt{x^2 + 1}) > \ln(4x^2 + 1).$$

This implies

$$|H_1H_2| \geq \ln \left( 2\sqrt{3} (1 - 3\alpha/\pi) + 1 \right).$$

We obtain that the length $L$ of a simple closed geodesic $\gamma$ of type $(p, q)$ satisfies

$$L \geq 2(p + q) \ln \left( 2\sqrt{3} (1 - 3\alpha/\pi) + 1 \right).$$

Euler’s function $\phi(n)$ is equal to the number of positive integers not greater than $n$ and prime to $n \in \mathbb{N}$. From [50, Th. 330] we know

$$\sum_{n=1}^{x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \ln x), \quad (5.11)$$

where $O(x \ln x) <Cx \ln x$, when $x \rightarrow +\infty$.

Denote by $\psi(x)$ the number of pairs of coprime integers $(p, q)$ such that $p < q$ and $p+q \leq x$, $x \in \mathbb{R}$. Suppose $\hat{\psi}(y)$ is equal to the number of pairs of coprime integers $(p, q)$ such that $p < q$ and $p + q = y$, $y \in \mathbb{N}$. From the definitions we get

$$\psi(x) = \sum_{y=1}^{x} \hat{\psi}(y), \quad (5.12)$$

If $(p, q) = 1$ and $p + q = y$, then $(p, y) = 1$ and $(q, y) = 1$. Consider Euler’s function $\phi(y)$. We obtain that the set of integers not greater than and prime to $y$ are separated into the pairs of coprime integers $(p, q)$ such that $p < q$ and $\phi(y)$ is even and $\hat{\psi}(y) = \phi(y)/2$. From (5.12) we have

$$\psi(x) = \frac{1}{2} \sum_{y=1}^{x} \phi(y).$$

The formula (5.11) implies

$$\psi(x) = \frac{3}{2\pi^2} x^2 + O(x \ln x), \quad (5.13)$$

where $O(x \ln x) <Cx \ln x$ when $x \rightarrow +\infty$.

Using this asymptotic it can be proved following result.

**Theorem 7.** Let $N(L, \alpha)$ be the number of simple closed geodesics of length not greater than $L$ on a regular tetrahedron with plane angles of the faces equal to $\alpha$ in hyperbolic space. Then

$$N(L, \alpha) = c(\alpha)L^2 + O(L \ln L), \quad (5.14)$$

where

$$c(\alpha) = \frac{9}{8\pi^2 \ln \left( 2\sqrt{3} (1 - 3\alpha/\pi) + 1 \right)},$$

$$\lim_{\alpha \rightarrow \frac{\pi}{3}} c(\alpha) = +\infty; \quad \lim_{\alpha \rightarrow 0} c(\alpha) = \frac{9}{8\pi^2 \ln \left( 2\sqrt{3} + 1 \right)}.$$

and $O(L \ln L) \leq CL \ln L$ when $L \rightarrow +\infty$. \hfill \Box
Proof. To each ordered pair of coprime integers \((p, q)\), \(p < q\) there correspond three different geodesics on the regular tetrahedron. We have

\[
N(L, \alpha) = 3\psi \left( \frac{L}{2 \ln \left( 2\sqrt{3} \left( 1 - \frac{3\alpha}{\pi} \right) + 1 \right)} \right)
\]

Using (5.13), we get

\[
N(L, \alpha) = \frac{9}{8\pi^2 \ln \left( 2\sqrt{3} \left( 1 - \frac{3\alpha}{\pi} \right) + 1 \right)} L^2 + O(L \ln L),
\]

when \(L \to +\infty\). \(\square\)

In work [22] of Rivin it was shown that for any hyperbolic structure on a sphere with \(n\) boundary components, the number of simple closed geodesics of length bounded by \(L\) on it grows like \(L^{2n-6}\) as \(L \to \infty\).

From Lemma 5.2 we know that the distances from the vertices of the tetrahedron to a simple closed geodesic is greater then \(d(\alpha)\), where \(d_0(\alpha)\) is from (5.2). This estimation holds also for a generic tetrahedron in hyperbolic space.

We can consider tetrahedron as a non-compact surface with regular Riemannian metric of constant negative curvature with \(4\) boundary components. From Lemma 5.1 it follows that there is no simple closed geodesic that is boundary parallel. From (5.14) we get, that the number \(N(L, \alpha)\) of simple closed geodesics of length \(\leq L\) on a regular tetrahedron is asymptotic to \(L^2\) when \(L \to +\infty\).

If the planar angle \(\alpha\) of the tetrahedron goes to zero, then the vertices of the tetrahedron tend to infinity. The limiting surface is homeomorphic to a sphere with \(4\) cusps. It is shown in [21] that any cusp on hyperbolic surface has a neighborhood bounded by horocycle curve of length 2. There is no simple closed geodesic intersecting this neighborhood. Then the asymptotic of the number of simple closed geodesics on a sphere with \(n\) cusps is equal to the asymptotic of the number of simple closed geodesics on a sphere with \(n\) boundary components.

References

[1] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Tome I, Paris, Gauthier-Viltars, 1892.

[2] H. Poincaré, *Sur les lignes geodesiques des surfaces convexes*, Trans. Amer. Math. Soc. 6 (1905), 237-274.

[3] G. D. Birkhoff, *Dynamical systems*, Am. Math. Soc. Colloq. Publ., 9, Providence R.I., 1927.

[4] L. A. Lyusternik and L. G. Shnirelman, *Sur le probleme de trois geodesique fermees sur les surfaces de genre 0*, C. R. Acad. Sci. Paris, 189 (1929), 269-271.

[5] L. A. Lyusternik, L. G. Shnirelman, *Topological methods in variational problems and their application to the differential geometry of surfaces*, Uspekhi Mat. Nauk, 2:1(17) (1947), 166-217.

[6] W. Ballmann, *Der Satz von Lusternik und Schnirelmann*, Beiträge zur Differentialgeometrie, Heft 1, 1-25, Bonner Math. Schriften 102, Universität Bonn, 1978.
[7] I. A. Taimanov, *Closed extremals on two-dimensional manifolds*, Russian Math. Surveys, **47**:2 (1992), 163-211

[8] L. A. Lyusternik, A. I. Fet, *Variational problems on closed manifolds*, Dokl. Akad. Nauk. SSSR, **81** (1951), 17-18.

[9] A. I. Fet, *Variational Problems on Closed Manifolds*, Mat. Sb. (N.S.), **30**(72):2 (1952), 271-316 English translation in Amer. Math. Society, Translation No. 90 (1953).

[10] J. Franks, *Geodesics on $S^2$ and periodic points of annulus homeomorphisms*, Invent. Math., **108** (1992), 403-418.

[11] V. Bangert, *On the existence of closed geodesics on two-spheres*, International Journal of Mathematics, **4**:1 (1993), 1-10. https://doi.org/10.1142/S0129167X93000029

[12] D. Gromoll, W. Meyer, *Periodic geodesics on compact Riemannian manifolds*, J. Differential Geom. **3** (1969), 493-510.

[13] W. Ziller, *The free loop space of globally symmetric spaces*, Invent. Math., **41**:1 (1977), 1-22.

[14] H. B. Rademacher, *On the average indices of closed geodesics*, J. Diff. Geom. **29**:1 (1989), 65-83.

[15] J. Hadamard, *Les surfaces à courbures opposées et leurs lignes géodésiques*, J. Math. Pures et Appl. **4**:5 (1898), 27-74.

[16] H. Huber, *Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen*, Mathematische Annalen, **138**:1 (1959), 1-26.

[17] H. Huber, *Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen II*, Mathematische Annalen, **143**, 1961, 463-464.

[18] Ya. G. Sinai, *Asymptotic behavior of closed geodesics on compact manifolds with negative curvature*, Izv. Akad. Nauk SSSR Ser. Mat., **30**:6 (1966), 1275-1296 (in Russian)

[19] G. A. Margulis, *Applications of ergodic theory to the investigation of manifolds of negative curvature*, Funktsional. Anal. i Prilozhen., **3**:4, 1969, 89-90. English translation: Funct Anal Its Appl., **3**(1969), 335-336. https://doi.org/10.1007/BF01076325

[20] M. Gromov, *Three remarks on geodesic dynamics and fundamental group*, preprint SUNY (1976), reprinted in L’Enseignement Mathematique, **46** (2000), 391-402.

[21] G. McShane, I. Rivin. *Simple curves on hyperbolic tori*, C. R. Acad. Sci. Paris Ser. I. Math., **320**:12 (1995).

[22] I. Rivin, *Simple curves on surfaces*, Geometriae Dedicata, **87**:1-3 (2001), 345-360.

[23] M. Mirzakhani, *Growth of the number of simple closed geodesics on hyperbolic surfaces*, Annals of Mathematics, **168** (2008), 97-125.

[24] I. A. Rivin, *Simpler Proof of Mirzakhani’s Simple Curve Asymptotics*, Geom Dedicata, **114** (2005), 229-235. https://doi.org/10.1007/s10711-005-7153-1

[25] V. Erlandsson, J. Souto, *Mirzakhani’s Curve Counting and Geodesic Currents*, Progress in Mathematics, **345** (2022). https://doi.org/10.1007/978-3-031-08705-9
[26] S. Cohn-Vossen, *Some problems of differential geometry in the large*, 1959 (in Russian).

[27] A. D. Alexandrov, *Convex Polyhedra*, Springer, 2005.

[28] A. D. Alexandrov, *Selected Works Part II: Intrinsic Geometry of Convex Surfaces*, 1978

[29] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, Providence, R.I: AMS, 1973.

[30] A. V. Pogorelov, *One theorem about geodesic lines on a closed convex surface*, Rec. Math. [Mat. Sbornik], 18(60):1 (1946), 181-183.

[31] V. A. Toponogov, *Estimation of the length of a convex curve on a two-dimensional surface*, Sibirsk. Mat. Zh., 4:5 (1963), 1189-1183 (in Russian).

[32] V. A. Vaigant, O. Yu. Matukevich, *Estimation of the length of a simple geodesic on a convex surface*, Siberian Math. J, 42:5 (2001), 833-845.

[33] P. Gruber, *A typical convex surface contains no closed geodesic*, J. Reine Angew. Math. 416 (1991), 195-205.

[34] J. Itoh, J. Rouyer, C. Vilcu, *Moderate smoothness of most Alexandrov surfaces*, Int. J. Math. (2015), http://dx.doi.org/10.1142/S0129167X15400042, in press, [arXiv:1308.3862 [math.MG]].

[35] A.V. Pogorelov, *Quasi-geodesic lines on a convex surface*, Mat. Sb., 25(62):275-306, 1949. English transl., Amer. Math. Soc. Transl. 74 (1952).

[36] G. Galperin, *Convex Polyhedra without Simple Closed Geodesics*, Regul. Chaotic Dyn., 8:1 (2003), 45-58.

[37] D. B. Fuchs, E. Fuchs, *Closed geodesics on regular polyhedra*, Mosc. Math. J., 7:2 (2007), 265-279.

[38] D. B. Fuchs, *Geodesics on a regular dodecahedron*, Preprints of Max Planck Institute for Mathematics, Bonn, 91 (2009), 1-14.

[39] K. Lawson, J. Parish , C. Traub, A. Weyhaupt, *Coloring graphs to classify simple closed geodesics on convex deltahedra*, International Journal of Pure and Aplied Mathematics, 89 (2013), 1-19.

[40] V. Yu. Protasov, *Closed geodesics on the surface of a simplex*, Sbornik: Mathematics, 198:2 (2007), 243-260.

[41] A.Akopyan, A., Petrunin, A. *Long Geodesics on Convex Surfaces*, Math Intelligencer, 40 (2018), 26-31. https://doi.org/10.1007/s00283-018-9795-5

[42] J. O'Rourke, C. Vilcu, *Simple Closed Quasigeodesics on tetrahedra*, Information, 13(2022), 238. https://doi.org/10.3390/info13050238

[43] D. Davis, V. Dods, C. Traub, J. Yang, *Geodesics on the regular tetrahedron and the cube*, Discrete Mathematics, 340:1 (2017), 3183-3196. https://doi.org/10.1016/j.disc.2016.07.004

[44] D. Fuchs, *Geodesics on Regular Polyhedra with Endpoints at the Vertices*, Arnold Math J. 2 (2016), 201-211. https://doi.org/10.1007/s40598-016-0040-z

[45] J. Rouyer, C. Vilcu, *Simple closed geodesics on most Alexandrov surfaces*, Advances in Mathematics, 278 (2015), 103-120. https://doi.org/10.1016/j.aim.2015.04.003
[46] A A Borisenko, D D Sukhorebska, Simple closed geodesics on regular tetrahedra in Lobachevsky space, Sb. Math., 211:5 (2020), 617-642, DOI:10.1070/SM9212.

[47] A A Borisenko, The estimation of the length of a convex curve in two-dimensional Alexandrov space, Journal of Mathematical Physics, Analysis, Geometry, 16:3 (2020), p. 221-227.

[48] A.A. Borisenko, D. D. Sukhorebska, Simple closed geodesics on regular tetrahedra in spherical space, Sb. Math, 212:8 (2021), 1040-1067, DOI:10.1070/SM9433

[49] A. A. Borisenko, A necessary and sufficient condition for the existence of simple closed geodesics on regular tetrahedra in spherical space, Sb. Math., 213:2 (2022), 161-172 DOI:10.4213/sm9576

[50] G. H. Hardy, E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1975.

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, Kharkiv, 61103, Ukraine
Mathematisches Institut, WWU Münster, Einsteinstrasse 62, D-48149, Münster
E-mail address: suhdaria0109@gmail.com, s.darya@uni-muenster.de