Entanglement is a physical property describing the inseparability of quantum systems composed by multiple elements. This core concept of contemporary physics dates back to 1935, when Einstein, Podolsky and Rosen proposed a gedanken experiment criticizing the nonlocality of quantum mechanics and pointing out at a possible incompleteness of the theory [1]. Nowadays entanglement is prepared regularly in a number of physical systems [2]. For instance, entangled quadratures of electromagnetic fields are generated by single- and two-mode squeezing in parametric amplifiers and oscillators [3, 4]; These squeezed states are the cornerstone of multipartite entangled quantum networks and have greatly promoted the development of quantum optics [5–8] and quantum information science [9–11]. They exhibit Gaussian statistics and their entanglement properties are completely characterized by their covariance matrix [12]. However, it has been shown that non-Gaussian entangled states – inseparable states with non-Gaussian statistics – are not only an indispensable element to realize universal quantum computing [13–17], but also demonstrate superior performance in many continuous variable quantum information protocols, such as quantum key distribution [18], quantum teleportation [19–21] and quantum metrology [22]. Non-Gaussian entangled states have been probabilistically created by photon addition and subtraction on Gaussian states [23–25], but a source of deterministic non-Gaussian entangled states is still missing.

The triple-photon state (TPS), a non-Gaussian entangled state produced by nondegenerate three-mode spontaneous parametric down-conversion, has been recently observed in a superconducting circuit [26], opening new possibilities for the development of quantum optics and the proposal of novel quantum information protocols [27, 28]. Other potential candidates of TPS sources also include, but are not limited to, nonlinear crystals [29], waveguides [30], optical fibers [31] and atomic ensembles [32]. There are two possible configurations in the generation of a triplet depending on the initial state of the system: some of the modes can be seeded with coherent states or it can be a fully spontaneous process. When one or few modes of the triplet are seeded, the generated state loses its non-Gaussian nature and its entanglement properties can be detected by Gaussian inseparability criteria [33–35]. On the other hand, we have shown recently that a spontaneously generated TPS is fully non-Gaussian – a Greenberger-Horne-Zeilinger state with super-Gaussian statistics –, and its entanglement features cannot be revealed by Gaussian criteria [27]. Towards a general certification of entanglement an infinite hierarchy of conditions based on positive partial transposition were proposed in [36–38]. By constructing Hermitian operators with special forms, equivalent or different forms of inseparability inequalities were derived [39–41]. Other types of criteria mainly include methods based on either entropy [42, 43] or fidelity of teleportation [44]. However, these criteria based on the two-mode correlations are useless for TPS with three-party high-order quantum correlations [26, 27]. Recently, sufficient conditions for full inseparability and genuine entanglement of TPS have been derived [45]. These conditions, however, are not necessary and do not fully reveal the entanglement nature of TPS. More importantly, a systematic framework to fully characterize the entanglement of TPS is still missing.

In this Letter, we propose a hierarchy of full separability criteria for tripartite continuous variable states. The inequalities in our criteria are based on the variances of a pair of linear combinations of high-order quadratures and built-up operators. For any biseparable state, the
sum of the variances of the corresponding hierarchical operators is bounded from below by the value of the mode-dependent photon number resulting from the sum uncertainty relations. Thus, these inequalities should be satisfied for fully separable tripartite states. We then study how strong these boundaries are for TPS. As with Gaussian states, the entanglement properties of arbitrary bipartitions of TPS can be jointly characterized by a series of high-order covariance matrices. Using this feature, we prove that the proposed inequalities are necessary and sufficient conditions for the separability of corresponding high-order covariance matrices, which establishes a systematic framework for the entanglement characterization of TPS. Based on our full separability conditions, we derive a hierarchy of stringent conditions which rule out mixtures of biseparable states resulting in genuine tripartite non-Gaussian entanglement criteria. Finally, we demonstrate the hierarchical entanglement structure of TPS by means of numerical simulations.

We start our analysis by considering the interaction Hamiltonian describing the nondegenerate three-mode spontaneous parametric down-conversion

$$\hat{H}_I = i\hbar \kappa (\hat{a}_1^\dagger \hat{a}_2 \hat{a}_3^{\dagger} \hat{a}_4 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4^{\dagger}),$$  

(1)

where $\kappa$ is the third-order nonlinear coupling constant. The annihilation operators $\hat{a}_1, \hat{a}_2, \hat{a}_3$ and $\hat{a}_4$ describe respectively the three down-converted modes and the pump mode. Under the dynamic evolution generated by the Hamiltonian (1) acting on input vacuum or thermal states, it has been demonstrated theoretically [27, 45] and experimentally [26] that third order is the lowest-order quantum correlation of TPS. Taking into account its symmetry and non-Gaussianity, TPS should present also 6th- and 9th-order quantum correlations, and even higher. This means that the well-developed entanglement criteria involving second-order correlations are no longer applicable to the TPS [35].

We introduce the high-order quadrature operators $\hat{q}_k^n = (\hat{a}_k^{\dagger n} + \hat{a}_k^n)/2$ and $\hat{p}_k^n = i(\hat{a}_k^{\dagger n} - \hat{a}_k^n)/2$ for the subsystem $k$, satisfying the commutation relation $[\hat{q}_k^n, \hat{p}_k^n] = i f_k^n$, where the superscript $n$ is a positive integer representing the hierarchy index and the full expressions of $f_k^n$ are given in the Supplementary Material [46]. Likewise, the built-up operators $\hat{q}_{lm}^n = (\hat{a}_l^{\dagger n} \hat{a}_m^{\dagger n} + \hat{a}_l^n \hat{a}_m^n)/2$ and $\hat{p}_{lm}^n = i(\hat{a}_l^{\dagger n} \hat{a}_m^n - \hat{a}_l^n \hat{a}_m^{\dagger n})/2$ are defined for the subsystems $l$ and $m$, which follow the commutation relation $[\hat{q}_{lm}^n, \hat{p}_{lm}^n] = i f_{lm}^n$ [46]. We also define the following linear combinations

$$\hat{u}_k^n = g_{k,n} \hat{q}_k^n - \frac{1}{g_{k,n}} \hat{q}_{lm}^n, \quad \hat{v}_k^n = g_{k,n} \hat{p}_k^n + \frac{1}{g_{k,n}} \hat{p}_{lm}^n,$$

(2)

for a given permutation $\{k, l, m\}$ of $\{1, 2, 3\}$, where $g_{k,n}$ is an arbitrary real number.

The standard approach to tripartite entanglement is to examine the separability of the three possible bipartitions of the system. Thus let us consider the tripartite density operator $\rho = \sum_i \eta_i \rho_{k,lm}^i$ with $\Sigma_i \eta_i = 1$. $\rho_{k,lm}^i$ indicates that $\rho$ is a mixture of states $i$ where the subsystems $l$ and $m$ may be entangled or not, but subsystem $k$ is not entangled with the rest. We denote this biseparable state as $k - lm$.

For the biseparable state $k - lm$, the total variance of the pair of operators in Eq. (2) is expressed as follows

$$\langle \Delta(\hat{u}_k^n)^2 \rangle + \langle \Delta(\hat{v}_k^n)^2 \rangle = \sum_i \eta_i [g_{k,n}^2 \langle \Delta \hat{q}_k^n \rangle^2 + g_{k,n}^2 \langle \Delta \hat{p}_k^n \rangle^2 + \frac{1}{g_{k,n}^2} \langle \Delta \hat{q}_{lm}^n \rangle^2 + \frac{1}{g_{k,n}^2} \langle \Delta \hat{p}_{lm}^n \rangle^2] - 2\langle \langle \hat{q}_k^n \hat{q}_{lm}^n \rangle - \langle \hat{q}_k^n \rangle \langle \hat{q}_{lm}^n \rangle \rangle + \sum_i \eta_i \langle \hat{u}_k^n \rangle^2 - \sum_i \eta_i \langle \hat{v}_k^n \rangle^2 + \sum_i \eta_i \langle \hat{v}_k^n \rangle^2 - \sum_i \eta_i \langle \hat{v}_k^n \rangle^2,$$

(3)

where $\langle \cdots \rangle_i$ represents the average over the state $\rho_{k,lm}^i$. Using the Cauchy-Schwarz inequality $\sum_i \eta_i \langle \hat{u}_k^n \rangle^2 \geq \sum_i \eta_i \langle \hat{q}_k^n \rangle_i^2$, we know that the last four terms in Eq. (3) are bounded below by zero. Thus, the total variance of the hierarchy of operators $\hat{u}_k^n$ and $\hat{v}_k^n$ is bounded below by $g_{k,n}^2 f_k^n + f_{lm}^n / g_{k,n}^2$, with $f_k^n = \langle f_k^n \rangle, f_{lm}^n = \langle f_{lm}^n \rangle$, and where we applied the sum uncertainty relation $\langle \Delta \hat{A} \rangle^2 + \langle \Delta \hat{B} \rangle^2 \geq \langle \langle [\hat{A}, \hat{B}] \rangle \rangle^2$. Note that we did not make any assumption about the quantum statistics of states in the derivation. Regarding full tripartite inseparability, we have the following theorem.

**Theorem 1.** – Violation of the three inequalities

$$F_1^n \equiv \langle \Delta(\hat{u}_k^n)^2 \rangle + \langle \Delta(\hat{v}_k^n)^2 \rangle - g_{k,n}^2 f_k^n - \frac{f_{lm}^n}{g_{k,n}^2} \geq 0, \quad (4a)$$

$$F_2^n \equiv \langle \Delta(\hat{u}_k^n)^2 \rangle + \langle \Delta(\hat{v}_k^n)^2 \rangle - g_{k,n}^2 f_k^n - \frac{f_{lm}^n}{g_{k,n}^2} \geq 0, \quad (4b)$$

$$F_3^n \equiv \langle \Delta(\hat{u}_k^n)^2 \rangle + \langle \Delta(\hat{v}_k^n)^2 \rangle - g_{k,n}^2 f_k^n - \frac{f_{lm}^n}{g_{k,n}^2} \geq 0, \quad (4c)$$

with any hierarchy index $n$ is sufficient to confirm fully inseparable tripartite entanglement.

**Proof.** – Inequality (4a) is a necessary condition for the separability of the bipartition $1 - 23$. Once inequality (4a) is violated with any hierarchy index $n$, we can conclude that the state cannot be described by 1–23. Similarly, the hierarchy of inequalities (4b) and (4c) are implied by biseparable states 2–13 and 3–12, respectively. Therefore, violating the three inequalities for any index $n$ negates all possible bipartitions, thus proving the full inseparability of the state.

The three hierarchical bounds given in Eq. (4) must be respected for the three biseparable states. This naturally raises a question: are these bounds strong enough to ensure that the tripartite state satisfying the three hierarchical inequalities is fully separable? So far, three types of continuous variable states have been identified regarding their statistical features: sub-Gaussian, Gaussian and super-Gaussian states [23–25, 27]. Here we restrict our study to spontaneous TPS, part of the family of super-Gaussian states. We find that the inequalities (4a)-(4c)
Hence we have the following statement of the uncertainty relation for these high-order covariance matrices \( V \) specifically, inequality (6) implies that every physical TPS must conform to this inequality. In that case, the criteria based on second-order moments are sufficient to detect entanglement [47, 48]. It is well known that the entanglement properties of Gaussian states can be fully characterized by its covariance matrix [12]. As an analogy, the TPS should have a hierarchy of three-dimensional covariance matrices due to the fact that there are only quantum correlations of order 3n among the three modes. However, these matrices may be non computable. Thus, we decompose them into three sets of two-dimensional high-order covariance matrices, which exactly correspond to the three bipartitions of the TPS. These decomposed covariance matrices have similar physical properties due to symmetry. In the following, we describe their basic characteristics.

We first collect the nonlinear quadrature operators in the vector \( \hat{\mathbf{R}}^n = \left( \hat{q}_k^n, \hat{p}_k^n, \hat{q}_{lm}^n, \hat{p}_{lm}^n \right) \) and write the commutation relations as

\[
\left[ \hat{R}_i^n, \hat{R}_j^n \right] = i\Omega_{ij}^n, \quad i, j = 1, \ldots, 4,
\]

where \( \Omega^n = \text{diag}(J_{k,n}^n, J_{lm,n}^n) \) with \( J_{k,n}^n = \text{diag}(\hat{f}_{k,n}^2 - \hat{f}_{k,n}^0) \) and \( J_{lm,n}^n = \text{diag}(\hat{f}_{lm,n}^2 - \hat{f}_{lm,n}^0) \). The high-order covariance matrices \( V^n \) is defined as \( V_{ij}^n = <\Delta \hat{R}_{ij}^n - <\hat{R}_{ij}^n>/2, 
\]

where \( \Delta \hat{R}_{ij}^n = \hat{R}_{ij}^n - <\hat{R}_{ij}^n> \) and \( \hat{R}_{ij}^n = \text{tr}[\hat{R}^n \rho] \), with \( \rho \) being the density operator of the system. Then we get \( V_{ij}^n + i\langle\Omega_{ij}^n\rangle/2 = (R_{ij}^n R_{ij}^n) \), where the commutation relation (5) and the TPS property (\( \hat{R}^n = 0 \)) are used [46]. Hence we have the following statement of the uncertainty principle for a TPS:

\[
V^n + i\langle\Omega^n\rangle \geq 0.
\]

Every physical TPS must conform to this inequality. In particular, inequality (6) implies that \( V^n > 0 \). Note that these high-order covariance matrices \( V^n \) with \( n \geq 1 \) only describe the correlation information corresponding to the bipartition \( k - lm \).

\( V^n \) is by definition a symmetric matrix, which can be divided into \( 2 \times 2 \) sub-blocks

\[
V^n = \begin{pmatrix}
A_k & C_{k-lm} \\
C_{k-lm}^T & B_{lm}
\end{pmatrix},
\]

where \( A_k \) and \( B_{lm} \) are local high-order covariance matrices related respectively to the subsystems \( k \) and \( lm \), and \( C_{k-lm} \) represents their correlation. Using Williamson’s theorem and a suitable singular value decomposition [49], we can always transform Eq. (7) into the following standard form [46]

\[
V_1^n = \begin{pmatrix}
n_1 & 0 & 0 & 0 \\
0 & n_2 & s_2 & 0 \\
0 & s_1 & m_1 & 0 \\
0 & 0 & m_2 & 0
\end{pmatrix},
\]

where the matrix elements satisfy the relations

\[
\frac{n_2}{n_1} = \frac{m_2}{m_1}, \quad 2(|s_1| - |s_2|) = \sqrt{n_1 m_1} - \sqrt{n_2 m_2},
\]

with \( n_1 = 2n_i - f_{k,n}^0 \) and \( m_i = 2m_i - f_{lm,n}^0 \) (\( i = 1, 2 \)).

The standard form (8) applies also to the bipartitions \( l - mk \) and \( m - kl \). Local symplectic transformations do not affect the separability of the corresponding bipartitions. This implies that tripartite states with the same three sets of hierarchical standard forms (8) have the same entanglement features. With these preliminaries, we now present the main theorem about the separability of the bipartition \( k - lm \).

**Theorem 2:** The necessary and sufficient condition for the separability of \( V_1^n \) is that the hierarchy of operators

\[
\hat{a}_k^n = g_{k,n} \hat{q}_{k,n}^n - s_1 \hat{q}_{lm,n}^n, \quad \hat{b}_k^n = g_{k,n} \hat{p}_{k,n}^n + s_2 \hat{p}_{lm,n}^n
\]

satisfies the inequality (4a), where \( g_{k,n} = \sqrt{m_1/n_1} \).

**Proof.**—Inequality (4a) is already a necessary condition for the separability of the bipartition \( k - lm \), so we only need to prove its sufficiency. Substituting \( \hat{a}_k^n \) and \( \hat{b}_k^n \) of Eq. (10) into inequality (4a) and using the standard form \( V_1^n \), we obtain the inequality \( g_{k,n}^2 (n_1 + n_2) + g_{k,n}^2 (m_1 + m_2) - 4|s_1| - 4|s_2| \geq 0 \). Combined with Eq. (9), one finds

\[
2|s_i| \leq \sqrt{n_i m_i}.
\]

Since the uncertainty principle is invariant under local standard transformations, the standard form \( V_1^n \) also satisfies Eq. (6), which can be further reduced to

\[
\text{det}(V_1^n + i\Omega^n)/2 = (f_{k,n}^2 (f_{lm,n}^2)^2 \text{det}(V_2^n + i\Omega^n)/2 \geq 0,
\]

where

\[
V_2^n = \begin{pmatrix}
n_1 & 0 & 0 & 0 \\
0 & n_2 & 0 & 0 \\
0 & 0 & m_1 & 0 \\
0 & 0 & 0 & m_2
\end{pmatrix}
\]

and \( \Omega = \text{diag}(J_{k,n}^1, J_{k,n}^1) \) [46]. This is equivalent to inserting a normalization coefficient into the commutation relations and it does not affect the separability of bipartition \( k - lm \). The standard form \( V_2^n \) together with the commutation relations \( V_2^n + i\Omega^n/2 \geq 0 \), suggest that the hierarchy of high-order covariance matrices \( V_2^n \) represents a set of independent bipartite Gaussian states in the linear quadratures. Inequality (11) ensures that every matrix \( V_2^n - 1/2 \) is semi-positive definite, implying that all Gaussian states represented by \( V_2^n \) are separable.
Theorem 2 also holds for the biseparable states $l-nk$ and $n-kl$. Thus, we have the following result: A super-Gaussian TPS represented by $3n$ high-order covariance matrices $V^n_k$ is fully separable iff three pairs of hierarchical operators violate inequalities $(4a)-(4c)$, respectively, with any hierarchy index $n$. This statement is one of the main results of the paper.

Full inseparability is however not the more general form of multipartite entanglement [50, 51]. It can only exclude any biseparable case rather than the general one in which the state can be described as a mixture

$$\rho = P_1 \sum_{i} \eta_i^{(1)} \rho_{k,lm} + P_2 \sum_{i} \eta_i^{(2)} \rho_{l,mk} + P_3 \sum_{j} \eta_j^{(3)} \rho_{m,kl},$$

where $\sum_i P_i = 1$. If a tripartite quantum state can not be described by this equation it is said to be genuinely entangled. Genuine entanglement and full inseparability are equivalent for pure states, and the former is stricter than the latter [48, 51]. Regarding the genuine tripartite entanglement, we have the following main result.

**Theorem 3:** A tripartite state is genuinely entangled if the inequality

$$W_n = F^n_1 + F^n_2 + F^n_3 + 4 \langle \hat{a}^n_1 \hat{a}^n_2 \hat{a}^n_3 \rangle - 4 \langle \hat{a}^n_1 \hat{a}^n_2 \hat{a}^n_3 \rangle + 2 \langle \hat{a}^n_1 \hat{a}^n_2 \rangle - 2 \langle \hat{a}^n_1 \hat{a}^n_2 \rangle \geq 0$$

(12)

is violated for any hierarchy index $n$.

This criterion of tripartite genuine entanglement is obtained using the full inseparability inequalities (4) in combination with the two-mode separability condition proposed by M. Hillery and M. S. Zubairy [40]. A detailed derivation is provided in the Supplementary Material [46].

Let us now discuss the main features of the proposed criteria. For any tripartite continuous variable state, criteria (4) and (12) are sufficient conditions to determine entanglement. For a hierarchy index $n=1$, the matrix elements of block $C_{k-lm}$ in the high-order covariance matrix (8) can be decomposed into the superposition of 3rd-order standardized moments—co-skewness—, i.e.,

$$\langle \hat{a}_k \hat{a}^\dagger_m \hat{a}_l \rangle = \langle \hat{a}_k \hat{a}^\dagger_m \hat{a}_l \rangle - \langle \hat{a}_k \hat{a}^\dagger_m \rangle \langle \hat{a}_l \rangle,$$

recently measured experimentally for TPS [26]. Inequality (11) indicates that even if there are non-Gaussian correlations among the three modes, they may not be entangled, which means that non-zero high-order standardized moments are a necessary but not sufficient condition for diagnosing non-Gaussian entanglement. Thus, based on the standard form (8), violations of criteria (4) and (12) imply fully inseparable and genuine tripartite non-Gaussian entanglement, respectively. In addition, the elements in the block $C_{k-lm}$ have state-independent properties [46], which significantly simplify the complexity of experimental measurements compared to other cases where determining non-Gaussian entanglement requires two different measurement protocols, non-Gaussianity and inseparability [23–25].

So far, there are other types of genuine tripartite entanglement criteria derived from the uncertainty principle [48, 51] and the Cauchy-Schwartz inequality [45]. No assumptions were made there about the statistical properties of the states in deriving these criteria, so in principle they apply to any tripartite continuous-variable state, Gaussian or non-Gaussian. In particular, A. Agustí et al. reported that violation of a proposed sufficient condition reveals non-Gaussian entanglement [45]. This criterion reveals 3rd-order—or $n=1$—TPS entanglement rather than a complete hierarchy of non-Gaussian entanglement. Our criteria is thus more general since it captures tripartite non-Gaussian entanglement in moments of order $3n$, detecting entanglement in a parameter range where 3rd-order criteria fail. Moreover, the high-order moments in our criteria can be measured by heterodyne detection, as opposed to the homodyne correlation experiments required in Ref. [45].

To conclude this Letter, we present the numerical verification of the proposed criteria. Using the interaction Hamiltonian (1), the master equations are solved numerically to deduce the final state of system at time $t$ considering that the initial state is vacuum for the triplets...
and a coherent mode $\alpha_p$ for the pump [27]. Figure 1(a) shows the evolution of $F^n_1$ versus the interaction strength $\xi = \kappa \alpha_t$, where $g_{1,n} = g_{2,n} = g_{3,n} = 1$. $F^n_1 = F^n_2 = F^n_3$ as expected from the symmetry of the TPS. $F^n_1<1,2,3 < 0$ in the parameter region demonstrating full inseparability related to the 3rd-, 6th- and 9th-order covariance matrices of the TPS. Figure 1(b) shows the evolution of $W_n$ versus the interaction strength. $W_n < 0$ in the parameter region demonstrating genuine entanglement. Starting from the vacuum, the genuine tripartite non-Gaussian entanglement is firstly loaded on the 3rd-order covariance matrices ($n = 1$) and then gradually transitioned to the higher-order covariance matrices ($n = 2, 3$) with the increase of the interaction strength. Therefore, TPS exhibit a hierarchical entanglement structure over a considerable range of parameters.

In summary, the continuous variable entanglement of triple-photon states produced by nondegenerate three-mode spontaneous parametric down-conversion can be fully characterized by three groups of hierarchies of high-order covariance matrices. We proposed three hierarchies of sufficient and necessary conditions for separability applicable to these high-order covariance matrices.

This provides a systematic framework for the study of triple-photon state entanglement. Furthermore, we derived a hierarchy of genuine tripartite non-Gaussian entanglement criterion. Through numerical simulations, we revealed that triple-photon states possess a hierarchy of fully inseparable and genuine tripartite non-Gaussian entanglement structures. This feature may have potential applications in quantum key distribution, quantum teleportation, or quantum computing, among others. We expect that the proposed criteria will help experimentalists to verify the entanglement features of non-Gaussian tripartite continuous variable states in future experiments.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
[3] A. Heidmann, R. J. Horowicz, S. Reynaud, E. Giacobino, C. Fabre, and G. Camy, Phys. Rev. Lett. 59, 255 (1987).
[4] D. Zhang, C. Li, Z. Zhang, Y. Zhang, Y. Zhang, and M. Xiao, Phys. Rev. A 96, 043847 (2017).
[5] H. P. Yuen, Phys. Rev. A 13, 2226 (1976).
[6] B. L. Schumaker and C. M. Caves, Phys. Rev. A 31, 3093 (1985).
[7] M. O. Scully and M. S. Zubairy, Quantum optics (American Association of Physics Teachers, 1999).
[8] G. S. Agarwal, Quantum optics (Cambridge University Press, 2012).
[9] L. Gerd et al., Quantum information with continuous variables of atoms and light (World Scientific, 2007).
[10] D. Petz, Quantum information theory and quantum statistics (Springer Science & Business Media, 2007).
[11] S. L. Braunstein and A. K. Pati, Quantum information with continuous variables (Springer Science & Business Media, 2012).
[12] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. 77, 513 (2005).
[13] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. 82, 1784 (1999).
[14] S. D. Bartlett, B. C. Sanders, S. L. Braunstein, and K. Nemoto, Phys. Rev. Lett. 88, 097904 (2002).
[15] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, Phys. Rev. Lett. 97, 110501 (2006).
[16] M. Ohliger, K. Kieling, and J. Eisert, Phys. Rev. A 82, 042336 (2010).
[17] M. Ohliger and J. Eisert, Phys. Rev. A 85, 062318 (2012).
[18] L. Hu, M. Al-amri, Z. Liao, and M. S. Zubairy, Phys. Rev. A 102, 012608 (2020).
[19] T. Opatrný, G. Kurizki, and D.-G. Welsch, Phys. Rev. A 61, 032302 (2000).
[20] S. Olivares, M. G. A. Paris, and R. Bonifacio, Phys. Rev. A 67, 032314 (2003).
[21] F. Dell’Ano, S. De Siena, L. Albano, and F. Illuminati, Phys. Rev. A 76, 022301 (2007).
[22] H. Strobel, W. Mussel, D. Linnemann, T. Zibold, D. B. Hume, L. Pezzè, A. Smerzi, and M. K. Oberthaler, Science 345, 424 (2014).
[23] H. Jeong, A. Zavatta, M. Kang, S.-W. Lee, L. S. Costanzo, S. Grandi, T. C. Ralph, and M. Bellini, Nat. Photonics 8, 564 (2014).
[24] O. Morin, K. Huang, J. Liu, H. Le Jeannic, C. Fabre, and J. Laurat, Nat. Photonics 8, 570 (2014).
[25] Y.-S. Ra, A. Dufour, M. Walschaers, C. Jacquard, T. Michel, C. Fabre, and N. Treps, Nat. Phys. 16, 144 (2020).
[26] C. W. S. Chang, C. Sabin, P. Forn-Diaz, F. Quijandría, A. M. Vadraj, I. Nsanzineza, G. Johansson, and C. M. Wilson, Phys. Rev. X 10, 011011 (2020).
[27] D. Zhang, Y. Cai, Z. Zheng, D. Barral, Y. Zhang, M. Xiao, and K. Bencheikh, Phys. Rev. A 103, 013704 (2021).
[28] Y. Zheng, O. Hahn, P. Stadler, P. Holmvall, F. Quijandría, A. Ferraro, and G. Ferrini, PRX Quantum 2, 010327 (2021).
[29] J. Douady and B. Boulanger, Opt. Lett. 29, 2794 (2004).
[30] M. G. Moebius, F. Herrera, S. Griesse-Nascimento, O. Reshef, C. C. Evans, G. G. Guerrero, A. Aspuru-Guzik, and E. Mazur, Opt. Express 24, 9932 (2016).
[31] A. Cavanna, J. Hammer, C. Okoth, E. Ortiz-Ricardo, H. Cruz-Ramírez, K. Garay-Palmett, A. B. U'Ren, M. H. Frosz, X. Jiang, N. Y. Joly, and M. V. Chekhova, Phys. Rev. A 101, 033840 (2020).

[32] K. Li, Y. Cai, J. Wu, Y. Liu, S. Xiong, Y. Li, and Y. Zhang, Adv. Quantum Technol. 3, 1900119 (2020).

[33] R. Simon, Phys. Rev. Lett. 84, 2726 (2000).

[34] P. van Loock and A. Furusawa, Phys. Rev. A 67, 052315 (2003).

[35] E. A. R. González, A. Borne, B. Boulanger, J. A. Levenson, and K. Bencheikh, Phys. Rev. Lett. 120, 043601 (2018).

[36] E. Shchukin and W. Vogel, Phys. Rev. Lett. 95, 230502 (2005).

[37] A. Miranowicz and M. Piani, Phys. Rev. Lett. 97, 058901 (2006).

[38] D. Zhang, D. Barral, Y. Cai, Y. Zhang, M. Xiao, and K. Bencheikh, Phys. Rev. Lett. 127, 150502 (2021).

[39] G. S. Agarwal and A. Biswas, New J. Phys. 7, 211 (2005).

[40] M. Hillery and M. S. Zubairy, Phys. Rev. Lett. 96, 050503 (2006).

[41] H. Nha and M. S. Zubairy, Phys. Rev. Lett. 101, 130402 (2008).

[42] S. P. Walborn, B. G. Taketani, A. Salles, F. Toscano, and R. L. de Matos Filho, Phys. Rev. Lett. 103, 160505 (2009).

[43] A. Saboia, F. Toscano, and S. P. Walborn, Phys. Rev. A 83, 032307 (2011).

[44] H. Nha, S.-Y. Lee, S.-W. Ji, and M. S. Kim, Phys. Rev. Lett. 108, 030503 (2012).

[45] A. Agustí, C. W. S. Chang, F. Quijandría, G. Johansson, C. M. Wilson, and C. Sabin, Phys. Rev. Lett. 125, 020502 (2020).

[46] See Supplemental Material at http://link.aps.org/ supplemental/XXXXXX for details on the commutation relations of operators, the basic properties of TPS, the derivation processes of standard form $V_n^\alpha$ and genuine tripartite entanglement criterion.

[47] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000).

[48] R. Y. Teh and M. D. Reid, Phys. Rev. A 90, 062337 (2014).

[49] R. A. Horn and C. R. Johnson, Matrix analysis (Cambridge university press, 2012).

[50] P. Hyllus and J. Eisert, New J. Phys. 8, 51 (2006).

[51] L. K. Shalm, D. R. Hamel, Z. Yan, C. Simon, and T. Jennewein, Nat. Phys. 9, 19 (2012).