TWISTED CONJUGATION ON CONNECTED SIMPLE LIE GROUPS AND TWINING CHARACTERS

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Abstract. This article discusses the twisted adjoint action \( \text{Ad}_{\kappa}^g : G \to G, x \mapsto g x \kappa (g^{-1}) \) given by a Dynkin diagram automorphism \( \kappa \in \text{Aut}(G) \), where \( G \) is compact, connected, simply connected and simple. The first aim is to recover the classification of \( \kappa \)-twisted conjugacy classes by elementary means, without invoking the non-connected group \( G \times \langle \kappa \rangle \). The second objective is to highlight several properties of the so-called twining characters \( \tilde{\chi}^{(\kappa)} : G \to \mathbb{C} \), as defined by Fuchs, Schellekens and Schweigert. These class functions generalize the usual characters, and define \( \kappa \)-twisted versions \( \tilde{R}^{(\kappa)}(G) \) and \( \tilde{R}_k^{(\kappa)}(G) (k \in \mathbb{Z}_{>0}) \) of the representation and fusion rings associated to \( G \). In particular, the latter are shown to be isomorphic to the representation and fusion rings of the orbit Lie group \( G(\kappa) \), a simply connected group obtained from \( \kappa \) and the root data of \( G \).

1. Introduction

Let \( G \) be a compact connected semi-simple Lie group, and \( \kappa \in \text{Aut}(G) \) an element induced by a Dynkin diagram automorphism of \( G \). The \( \kappa \)-twisted adjoint action of \( G \) on itself is defined as:

\[
\text{Ad}_{\kappa}^g : G \to G, x \mapsto g x \kappa (g^{-1}), \; g \in G.
\]

Early publications discussing this action are Gantmacher’s work on automorphisms of complex groups [12], and de Siebenthal’s study of compact non-connected Lie groups [28]. Since then, twisted conjugation appeared sporadically in the literature, mostly in the context of representations of non-connected groups and of twisted affine Lie algebras [8,17,27,30]. Many results in this direction can be found in Mohrdieck’s work on twisted conjugacy classes [23,24], in Wendt’s study of orbital integrals [31], as well as in Springer’s note [29].

Aside from representation theory, twisted conjugation has found several applications in mathematical physics and gauge theory in the last two decades. One particularly relevant paper is Fuchs, Schellekens and Schweigert’s [10] on twining characters, which introduced twisted analogs of the usual characters and recovered an important character formula of Jantzen, in order to solve the so-called “resolution of fixed points” problem in conformal field theory. Later, Mohrdieck and Wendt studied twisted conjugacy classes as D-branes for the Wess-Zumino-Witten model in [25] (see also [6]), as well as in the context of moduli spaces of principal bundles in [26]. More recently, Hong addressed these questions in [13,15], where he constructed an analog of the fusion ring for twining characters and developed a notion of twisted conformal blocks with Kumar.

Our interest in twisted conjugation stems from the study of certain symplectic invariants of twisted quasi-Hamiltonian spaces, as discussed in work of Boalch and Yamakawa [3], Knop [19] and Meinrenken [22]. These are finite-dimensional models for symplectic Banach manifolds...
endowed with Hamiltonian actions of twisted loop groups, and provide a differential-geometric viewpoint on the topics mentioned in the previous paragraph.

In light of these developments, the purpose of this paper is twofold: First, review twisted conjugation with direct proofs of the main structure results, and second, highlight certain properties of twining characters from this perspective. Some features of our approach are that:

(i) The action $\text{Ad}^\kappa$ is studied as an intrinsic action of $G$ on itself, without resorting to the theory of non-connected groups.
(ii) The classification of twisted conjugacy classes is recovered by elementary means, from a differential-geometric perspective.
(iii) Twining characters are viewed as functions on the group $G$, as opposed to formal characters of an affine Kac-Moody algebra.
(iv) The level $k$ fusion ring for twining characters is constructed without invoking twisted loop groups or affine Lie algebras.

Keeping the notation above, suppose henceforth that $G$ is simply connected and simple. Let $T \subseteq G$ be a maximal torus with fixed-point subgroup $T^\kappa \subseteq T$, and denote by $\mathfrak{R} = \mathfrak{R}(G,T)$ the root system of $G$. The upcoming sections are organized as follows. In section 2, we introduce the notations and associate two root systems to the data $(G,\kappa)$: the folded root system $\mathfrak{R}^f(\mathfrak{R})$ and the orbit root system $\mathfrak{R}(\kappa)$, both supported in the dual of $t^\kappa = \text{Lie}(T^\kappa)$. We then gather the properties of these root systems that form the basis of sections 3 and 4.

The structure of $\kappa$-twisted conjugacy classes in $G$ and their classification is undertaken in section 3. As in the classical theory, there are twisted versions of the Weyl group, of the conjugation map $G/T \times T \to G$ and of a fundamental alcove parametrizing the orbits of $\text{Ad}^\kappa$. The main results pertaining to these objects in [23,25,31] are re-derived after establishing certain key properties of the $\kappa$-twisted normalizer $N^\kappa_G(T^\kappa) = \{g \in G \mid \text{Ad}^\kappa_g(T^\kappa) = T^\kappa\}$. The next section deals with Fuchs, Schellekens and Schweigert’s twining characters $\tilde{\chi}(\kappa) : G \to \mathbb{C}$, which are the natural twisted generalization of the usual characters. Indeed, these class functions satisfy orthogonality relations, give a Fourier basis of the space of $\text{Ad}^\kappa(G)$-invariant $L^2$-functions, and generate $\tilde{R}^{(\kappa)}(G)$ and $\tilde{R}_k^{(\kappa)}(G)$, the twisted counterparts to the representation ring and the fusion ring at level $k \in \mathbb{Z}_{>0}$. Most notably, if $G_{(\kappa)}$ is the orbit Lie group (simply connected with root system $\mathfrak{R}(\kappa)$), we have that $\tilde{R}_k^{(\kappa)}(G) \simeq R_k(G_{(\kappa)})$. This is a recent result of Hong [13,14] that follows from Jantzen’s character formula [10,17,31], and that is recovered here with a different set of techniques.

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2. Diagram Automorphisms and Associated Root Systems

The aim of this section is to set-up the notations used in this work, and to present some reminders on the root systems that one can associate to an indecomposable root system $\mathfrak{R}$ and a Dynkin diagram automorphism $\kappa \in \text{Aut}(\Pi)$, where $\Pi \subseteq \mathfrak{R}$ is a choice of simple roots. For the sake of brevity, we do not give detailed proofs here, and refer the reader to [23, Ch.1].
2.1. Notation.

2.1.1. Groups and Lie algebras. Let $G$ be a compact, connected, simply connected and simple Lie group with Lie algebra $\mathfrak{g}$. We consider a fixed maximal torus $T \subseteq G$ throughout, with Lie algebra $\mathfrak{t}$ of dimension $r = \text{rk}(\mathfrak{g})$, and denote by $W = N_G(T)/T$ the Weyl group with respect to this torus. We assume throughout that we have an $\text{Ad}(G)$-invariant non-degenerate symmetric bilinear form $B$ on $\mathfrak{g}$, and we use the shorthand notation $\xi \cdot \zeta = B(\xi, \zeta)$ for $\xi, \zeta \in \mathfrak{g}$.

2.1.2. Roots and coroots. Below, $\mathfrak{R} \subseteq \mathfrak{t}^*$ is the root system $\mathfrak{R}(G, T)$, $\mathfrak{R}_+$ is a fixed choice of positive roots with cardinality $n_+ = |\mathfrak{R}_+|$, $\Pi = \{\alpha_i\}_{i=1}^r$ designates the simple roots, and $\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} \alpha$ is the half-sum of positive roots of $G$. We use real roots in this work: for $\xi \in \mathfrak{t}$ and $t = e^{\xi} \in T$, we write $t^\alpha = e^{2\pi i \langle \alpha, \xi \rangle} \in \mathbb{C}$ for the corresponding character. The (closed) fundamental Weyl chamber with respect to $\mathfrak{R}_+$ is $t_+ \subseteq \mathfrak{t}$, and the closed fundamental alcove is:

$$\mathfrak{A} = \{ \xi \in t_+ \mid \langle \xi, \theta \rangle \leq 1 \},$$

where $\theta \in \mathfrak{R}_+$ is the highest root of $G$. Letting $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathfrak{g}^*$ induced by $B \in S^2 \mathfrak{g}^*$, we assume that $B$ is normalized so that for any long root $\alpha \in \mathfrak{R}$, we have $\|\alpha\|^2 = \langle \alpha, \alpha \rangle = 2$. For $\alpha \in \mathfrak{R}$, the corresponding coroot is $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \langle \alpha, \cdot \rangle \in \mathfrak{t}$, and the coroots and simple coroots of $G$ are denoted by $\mathfrak{R}^\vee$ and $\Pi^\vee$.

2.1.3. Lattices. We denote by $\Lambda = \ker(\exp) \cap \mathfrak{t}$ the integral lattice of $G$, and by $\Lambda^* = \text{Hom}_\mathbb{Z}(\Lambda, \mathbb{Z})$ the character lattice of $T$. The root and coroot lattices are respectively denoted by $Q = \mathbb{Z}[\mathfrak{R}] \subseteq \mathfrak{t}^*$ and $Q^\vee = \mathbb{Z}[\mathfrak{R}^\vee] \subseteq \mathfrak{t}$, while the weight and coweight lattices are denoted by $P = \text{Hom}_\mathbb{Z}(Q^\vee, \mathbb{Z}) \subseteq \mathfrak{t}^*$ and $P^\vee = \text{Hom}_\mathbb{Z}(Q, \mathbb{Z}) \subseteq \mathfrak{t}$. For the other root systems introduced below, we will use the same letters and adequate sub/superscripts to denote the same objects. Since $G$ is simply connected, we sometimes use the identifications $\Lambda = Q^\vee$ and $\Lambda^* = P$ in the sequel, so that the maximal torus is $T = t/Q^\vee$, and the alcove is given by $\mathfrak{A} = t/W_{\text{aff}}$, where $W_{\text{aff}} = \Lambda \rtimes W$ is the affine Weyl group with $\Lambda$ acting on $t$ by translations.

2.2. Dynkin Diagram Automorphisms. Considering the automorphism group $\text{Aut}(G)$ of $G$, and its normal subgroup $\text{Inn}(G) \simeq G/\mathbb{Z}(G)$ of inner automorphisms, the group of outer automorphisms of $G$ is defined as the quotient $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$. The latter can be identified with the automorphism group $\text{Aut}(\Pi)$ of the Dynkin diagram of $\mathfrak{g}$, as we now briefly indicate. By a diagram automorphism $\kappa \in \text{Aut}(\Pi)$, we mean a bijection:

$$\kappa : \Pi \rightarrow \Pi, \quad \alpha_i \mapsto \alpha_{\kappa(i)},$$

that preserves the entries of the Cartan matrix associated to $\mathfrak{R}$:

$$\langle \alpha_{\kappa(j)}, \alpha_{\kappa(i)}^\vee \rangle = \langle \alpha_j, \alpha_i^\vee \rangle, \quad i, j = 1, \ldots, r.$$

Any such automorphism induces a unique element $\kappa \in \text{Aut}(\mathfrak{g}_C)$ acting on the Chevalley generators $e_{\pm \alpha} \in \mathfrak{g}_C$ associated to simple roots as:

$$(2.1) \quad \kappa(e_{\pm \alpha}) = e_{\pm \kappa(\alpha)}, \quad \alpha \in \Pi.$$  

It then follows that $\kappa$ preserves the real forms $\mathfrak{g} \subseteq \mathfrak{g}_C$ and $\mathfrak{t} \subseteq \mathfrak{t}_C$, and exponentiates to an automorphism of $G$ preserving $T$. Furthermore, the element $\kappa \in \text{Aut}(\mathfrak{g})$ preserves the fundamental chamber $t_+$, since it permutes the positive roots $\mathfrak{R}_+$. Using the identification $\text{Aut}(G) \equiv \text{Aut}(\mathfrak{g})$, equation (2.1) hence defines a group homomorphism $\text{Aut}(\Pi) \rightarrow \text{Aut}(G)$.
descending to an isomorphism with $\text{Out}(G)$. For a given $\kappa \in \text{Aut}(\Pi)$, we will also denote the corresponding elements in $\text{Out}(g)$ and $\text{Out}(G)$ by $\kappa$.

To study the $\kappa$-twisted conjugation $\text{Ad}^\kappa_g : G \to G$, $x \mapsto gx\kappa(g^{-1})$, it is sufficient to take $\kappa \in \text{Aut}(G)$ corresponding to a Dynkin diagram automorphism. To see this, notice that for $\kappa \in \text{Aut}(G)$ and $\text{Ad}_a \in \text{Inn}(G)$, the right action map $R_a : G \to G$, $x \mapsto xa^{-1}$ gives an equivariant diffeomorphism from $(\text{Ad}_a \circ \kappa)$-twisted conjugacy classes to $\kappa$-classes, since for any $g \in G$:

$$R_{a^{-1}} \circ \text{Ad}_g^{(\text{Ad}_a \circ \kappa)} = \text{Ad}_g^\kappa \circ R_{a^{-1}}.$$ 

We assume from now onwards that $\kappa \in \text{Out}(G)$ is a non-trivial automorphism, and therefore restrict $\mathcal{R}$ to the cases compiled in the table below.

| $\mathcal{R}$ | $\mathfrak{A}_n$, $n \geq 2$ | $\mathfrak{D}_{n+1}$, $n \geq 4$ | $\mathfrak{D}_4$ | $\mathfrak{E}_6$ |
|---------------|-----------------|-----------------|---------|---------|
| $\text{Out}(G)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $S_3$ | $\mathbb{Z}_2$ |

**Table 1.**

Let $\kappa \in \text{Out}(G)$ be a fixed diagram automorphism of $G$, and assume that the inner-product $B$ on $g$ is $\kappa$-invariant. The $\kappa$-fixed subgroups will be denoted by $G^\kappa \subseteq G$ and $T^\kappa \subseteq T$, the corresponding Lie algebras by $g^\kappa$ and $t^\kappa$ respectively, and the subgroup of elements of the Weyl group commuting with $\kappa$ by $W^\kappa \subseteq W$. The orthogonal projection $p : g \to g^\kappa$ is given by

$$p(\xi) = \frac{1}{|\kappa|} \sum_{j=1}^{\frac{|\kappa|}{\kappa}} \kappa^j(\xi), \quad \xi \in g,$$

and we use the same letter to denote its restriction $t \to t^\kappa$ as well as the projection $t^* \to (t^\kappa)^*$. Lastly, we will need certain subgroups of $T$ obtained from $\kappa \in \text{Out}(G)$. Considering the Ad- and $\kappa$-invariant inner product $B$ on $g$, we define $t_\kappa := (t^\kappa)^{-1}$ with respect to the restriction $B|_t$, as well as the subtorus $T_\kappa := \exp(t_\kappa) \subseteq T$. The orthogonal decomposition $t = t^\kappa \oplus t_\kappa$ yields:

$$T = T^\kappa \cdot T_\kappa.$$

The intersection $T^\kappa \cap T_\kappa \subseteq T$ is a finite subgroup, and will be of particular importance in the remaining of this paper.

2.3. **The Folded and the Orbit Root Systems.** We are in position to introduce the main constructions of this section:

**Definition 2.1.** To the data $(\mathcal{R}, \kappa)$, we associate the following two root systems:

- **The folded root system**, which is the image $p(\mathcal{R})$ under the orthogonal projection $p : t^* \to (t^\kappa)^*$:
  $$\mathcal{R}_F^{(\kappa)} = \{ p(\alpha) \mid \alpha \in \mathcal{R} \} \subseteq (t^\kappa)^*.$$

- **The orbit root system**, given by $\mathcal{O}_O^{(\kappa)} = (\mathcal{R}_F^{(\kappa)})^\vee$ for $\mathcal{R} = A_{2n-1}$, $D_{n+1}$ ($n \geq 3$), and $E_6$. For the case of $\mathcal{R} = A_{2n}$ with $n > 1$, we define $\mathcal{R}_O^{(\kappa)}$ to be the dual of the $B_n$ subsystem of $\mathcal{R}_F^{(\kappa)} = BC_n$. For $\mathcal{R} = A_2$, $\mathcal{R}_O^{(\kappa)}$ is the $A_1$ system with unique positive root $2(\alpha_1 + \alpha_2)$. 

Under our assumptions on $\mathcal{R}$ and $\kappa$, the definitions above give the table:

| $\mathcal{R}$ | $A_{2n-1}$ | $A_{2n}$, $n > 1$ | $A_2$ | $D_{n+1}$, $|\kappa| = 2$ | $D_4$, $|\kappa| = 3$ | $E_6$ |
|---------------|-------------|-------------------|-------|-------------------------|---------------------|-------|
| $\mathcal{R}_F^{(\kappa)}$ | $C_n$ | $BC_n$ | $A_1 \sqcup A_1$ | $B_n$ | $G_2$ | $F_4$ |
| $\mathcal{R}_O^{(\kappa)}$ | $B_n$ | $C_n$ | $A_1$ | $C_n$ | $G_2$ | $F_4$ |

Table 2.

Remark 2.2.

(1) For the proof that $\mathcal{R}_F^{(\kappa)} = p(\mathcal{R})$ is indeed a root system with Weyl group $W^{\kappa}$, see [8 Prop.13.2.2]. By definition, $W^{\kappa}$ is also the Weyl group of $\mathcal{R}_O^{(\kappa)}$.

(2) The case $\mathcal{R} = A_{2n}$ is the only one where $\mathcal{R}_F^{(\kappa)}$ is decomposable. When $n = 1$, $\mathcal{R}_F^{(\kappa)}$ is the disjoint union of two copies of $A_1$ (see next lemma). When $n > 1$, there are 3 root lengths in $\mathcal{R}_F^{(\kappa)} = BC_n$, with the long and intermediate roots constituting a $C_n$ subsystem, and the short and intermediate roots forming a $B_n$ subsystem.

(3) The terminology “orbit root system” for $\mathcal{R}_O^{(\kappa)}$ follows Fuchs-Schellekens-Schweigert [10] and Hong [13,14]. This is the root system denoted by $\mathcal{R}$ in [23] and by $\mathcal{R}^1$ in [31], which is obtained by rescaling the elements of $\mathcal{R}_F^{(\kappa)}$.

In order to lighten the notation, we drop the $(\kappa)$ superscripts for the remaining of this subsection, bearing in mind that we are discussing objects associated to a fixed automorphism $\kappa \in \text{Out}(G)$.

Lemma 2.3. Let $(\mathcal{R}, \kappa)$ be as in table 2 and denote by $\Pi^\kappa \subset \Pi$ the $\kappa$-invariant simple roots.

(i) For $\mathcal{R} \neq A_{2n}$, the simple roots of $\mathcal{R}_F$ are given by $\Pi_F = \{p(\alpha) \mid \alpha \in \Pi\}$, and those of $\mathcal{R}_O$ by:

$$\Pi_O = \{\alpha \mid \alpha \in \Pi^\kappa\} \cup \{\kappa p(\alpha) \mid \alpha \in \Pi \setminus \Pi^\kappa\}.$$ (ii) For $\mathcal{R} = A_{2n}$ with $n > 1$, the simple roots of the $B_n$ and $C_n$ subsystems of $\mathcal{R}_F$ are:

$$\Pi_{F,B_n} = \{p(\alpha_1), \ldots, p(\alpha_{n-1}), p(\alpha_n)\},$$

$$\Pi_{F,C_n} = \{p(\alpha_1), \ldots, p(\alpha_{n-1}), 2p(\alpha_n)\},$$

and the simple roots of $\mathcal{R}_O$ are given by:

$$\Pi_O = \{2p(\alpha_1), \ldots, 2p(\alpha_{n-1}), 4p(\alpha_n)\}.$$ (iii) For $\mathcal{R} = A_2$, we have $\mathcal{R}_F = \{\pm \frac{1}{2}(\alpha_1 + \alpha_2), \pm (\alpha_1 + \alpha_2)\}$ and $\mathcal{R}_O = \{\pm 2(\alpha_1 + \alpha_2)\}$.

Outline of proof. Let $\mathcal{R}_O^\kappa \subseteq \mathcal{R}$ denote the subset $\kappa$-invariant roots.

(i) When $\mathcal{R} = A_{2n-1}$, $D_{n+1}$ and $E_6$, a naive identification of the nodes in a $\kappa$-orbit of the diagram of $\mathcal{R}$ gives the diagram of $\mathcal{R}_F$, with simple roots given by $\Pi_F = \{p(\alpha) \mid \alpha \in \Pi\}$, and lengths of the other elements of $\mathcal{R}_F$ determined by whether or not $\kappa(\alpha) = \alpha$. The definition of the orbit root system says that:

$$\mathcal{R}_O = \left\{ \frac{2p(\alpha)}{(p(\alpha),p(\alpha))} \mid \alpha \in \mathcal{R} \right\} \subseteq (t^\kappa)^*,$$
which can be re-expressed as:
\[ \mathfrak{R}_O = \{ \alpha \mid \alpha \in \mathfrak{R}^c \} \cup \{ |\kappa|p(\alpha) \mid \alpha \in \mathfrak{R} \setminus \mathfrak{R}^c \}. \]

For the \( \kappa \)-fixed \( \alpha \in \mathfrak{R} \), we have \( p(\alpha) = \alpha \) with \( |\alpha|^2 = 2 \), while for the elements satisfying \( \kappa(\alpha) \neq \alpha \), we have that \( |p(\alpha)|^2 = \frac{1}{|\kappa|}|\alpha|^2 \). The set \( \Pi_O \) in the statement gives a set of simple roots for \( \mathfrak{R}_O \).

(ii) In the case of \( \mathfrak{R} = A_{2n} \) with \( n > 1 \), we have a partition \( \mathfrak{R} = \mathfrak{R}^c \sqcup \mathfrak{R}^{(d)} \sqcup \mathfrak{R}^{(q)} \), where:
\[ \mathfrak{R}^{(d)} = \{ \alpha \in \mathfrak{R} \mid \kappa(\alpha) \neq \alpha \text{ and } (\kappa(\alpha), \alpha) = 0 \}, \]
\[ \mathfrak{R}^{(q)} = \{ \alpha \in \mathfrak{R} \mid \kappa(\alpha) \neq \alpha \text{ and } (\kappa(\alpha), \alpha) = -1 \}. \]

Projecting onto \( (t^\ast)^\circ \), the elements of \( \mathfrak{R}^c = p(\mathfrak{R}^c) \) have length 2, those of \( p(\mathfrak{R}^{(d)}) \) have length 1, and those of \( p(\mathfrak{R}^{(q)}) \) have length \( \frac{1}{2} \). In \( \mathfrak{R}_F \), we see from explicit verifications that \( p(\mathfrak{R}^c \sqcup \mathfrak{R}^{(d)}) \) is a \( C_n \) subsystem with simple roots \( \Pi_{F,C_n} \), while \( p(\mathfrak{R}^{(q)} \sqcup \mathfrak{R}^{(d)}) \) constitutes a \( B_n \) subsystem with simple roots \( \Pi_{F,B_n} \). Here, the orbit root system can be described as:
\[ \mathfrak{R}_O = \{ 2\alpha \mid \alpha \in \mathfrak{R}^c \} \cup \{ 2p(\alpha) \mid \alpha \in \mathfrak{R}^{(d)} \}, \]
with \( 2\mathfrak{R}^c = 4p(\mathfrak{R}^{(q)}) \), and a set of simple roots is given by:
\[ \Pi_O = \left\{ \frac{2}{|\alpha|^2}\alpha \mid \alpha \in \Pi_{F,B_n} \right\}. \]

(iii) Follows from the definition.

We want to realize \( \mathfrak{R}_F \) and \( \mathfrak{R}_O \) as the root systems of Lie groups related to \( G \). The next proposition determines the (co)root and (co)weight lattices of these two root systems in terms of those of \( \mathfrak{R} \) (see also [23, Lemmas 1.1, 1.4]), which will yield the root data needed. Before doing so, recall that we introduced the subtorus \( T_\kappa \subseteq T \) at the end of 2.1 which gives rise to the finite subgroup \( T^c \cap T_\kappa \subseteq T^c \). We define the lattice:
\[ (2.2) \quad \Lambda_\kappa = \exp_{\kappa}^{-1}(T^c \cap T_\kappa) \subseteq t^c. \]

We now have the following facts:

**Proposition 2.4.** With the notations of subsection 2.1, we have that:

1. If \( \mathfrak{R} = A_{2n-1}, D_n \) and \( E_6 \), the lattices of \( \mathfrak{R}_F \) satisfy the inclusions:
\[ Q^\vee_Y = \Lambda^c \subseteq P^\vee_Y = (P^\vee)^c, \]
\[ Q_F = p(Q) \subseteq p(\Lambda^c) = P_F. \]

For the orbit root system \( \mathfrak{R}_O \), we have that \( \Lambda_\kappa = p(\Lambda) \), along with the inclusions:
\[ Q^\vee_O = p(\Lambda) \subseteq P^\vee_O = p(P^\vee), \]
\[ Q_O = Q^c \subseteq (\Lambda^c)^c = P_O. \]

2. If \( \mathfrak{R} = A_{2n} \), let \( P_{B_n}, Q^\vee_{B_n}, P^\vee_{B_n} \) and \( Q^\vee_{B_n} \) denote the lattices associated to the \( B_n \) subsystem of \( \mathfrak{R}_F = BC_n \). Then:
\[ Q^\vee_{B_n} \subseteq \Lambda^c \subseteq P^\vee_{B_n} = (P^\vee)^c, \]
\[ Q_{B_n} = p(Q) \subseteq p(\Lambda^c) \subseteq P_{B_n}. \]
where \([ \Lambda^\kappa : Q^\vee_{B_n} ] = [ P_{B_n} : p(\Lambda^* ) ] = 2 \). For the orbit root system, we have that \( \Lambda_{(\kappa)} = p(\Lambda) \), along with the inclusions:

\[
Q^\vee_0 = p(\Lambda) \subseteq p(P^\vee) \subseteq P^\vee_0,
\]

\[
Q_0 \subseteq Q^\kappa \subseteq (\Lambda^*)^\kappa = P_0,
\]

with \([ P^\vee_0 : p(P^\vee) ] = [ Q^\kappa : Q_0 ] = 2 \).

Outline of proof. Firstly, the simple roots in lemma 2.3 determine all the simple coroots and simple (co)weights involved, which allows to check the statements above explicitly. We illustrate for the case of \( \mathcal{R} \neq A_{2n} \). Let \( I = \{ 1, \cdots, r \} \) be the indexing set for the simple roots \( \Pi = \{ \alpha_i \}_{i \in I} \), and partition it as:

\[
I = I_0 \sqcup I_1 \sqcup \kappa(I_1) \sqcup \cdots \sqcup \kappa^{\lfloor \frac{n}{2} \rfloor}(I_1),
\]

with \( I_0 \subseteq I \) the \( \kappa \)-fixed indices. The \( \kappa \)-orbits in \( I \) are then partitioned as \( \bar{I} = \bar{I}_0 \sqcup \bar{I}_1 \) (with \( I_j = \bar{I}_j, \ j = 0, 1 \)). Next, let:

\[
\Pi^\vee = \{ \alpha^\vee_i \}_{i \in \bar{I}}, \quad \mathfrak{F} = \{ \varpi_i \}_{i \in \bar{I}}, \quad \mathfrak{F}^\vee = \{ \varpi^\vee_i \}_{i \in \bar{I}}
\]

denote the simple coroots, the fundamental weights and fundamental coweights of \( \mathcal{R} \) respectively. Using lemma 2.3 one checks that for the folded system \( \mathcal{R}_F \):

\[
\Pi^\vee_F = \{ \alpha^\vee_i \}_{i \in I_0} \sqcup \{ |\kappa|p(\alpha^\vee_i) \}_{i \in \bar{I}_1},
\]

\[
\mathfrak{F}_F = \{ p(\varpi_i) \}_{i \in \bar{I}},
\]

\[
\mathfrak{F}^\vee_F = \{ \varpi^\vee_i \}_{i \in I_0} \sqcup \{ |\kappa|p(\varpi^\vee_i) \}_{i \in \bar{I}_1},
\]

and then use these bases to show that:

\[
Q_F = Z[\Pi_F] = p(Q),
\]

\[
Q^\vee_F = Z[\Pi^\vee_F] = \Lambda \cap t^\kappa = \Lambda^\kappa,
\]

\[
P_F = Z[\mathfrak{F}_F] = p(\Lambda^*) = \text{Hom}(Q^\vee_F, \mathbb{Z}),
\]

\[
P^\vee_F = Z[\mathfrak{F}^\vee_F] = P^\vee \cap t^\kappa = \text{Hom}(Q^\vee_F, \mathbb{Z}).
\]

For the orbit root system \( \mathcal{R}_O \), one checks with \( \Pi_O = \{ \alpha_i \}_{i \in I_0} \sqcup \{ |\kappa|p(\alpha_i) \}_{i \in \bar{I}_1} \) that:

\[
\Pi^\vee_O = \{ p(\alpha^\vee_i) \}_{i \in \bar{I}},
\]

\[
\mathfrak{F}_O = \{ \varpi_i \}_{i \in I_0} \sqcup \{ |\kappa|p(\varpi_i) \}_{i \in \bar{I}_1},
\]

\[
\mathfrak{F}^\vee_O = \{ p(\varpi^\vee_i) \}_{i \in \bar{I}},
\]

and then that:

\[
Q_O = Z[\Pi_O] = Q \cap (t^\kappa)^* = Q^\kappa,
\]

\[
Q^\vee_O = Z[\Pi^\vee_O] = p(\Lambda) = \Lambda_{(\kappa)},
\]

\[
P_O = Z[\mathfrak{F}_O] = \Lambda^* \cap (t^\kappa)^* = (\Lambda^*)^\kappa = \text{Hom}(Q^\vee_O, \mathbb{Z}),
\]

\[
P^\vee_O = Z[\mathfrak{F}^\vee_O] = p(P^\vee) = \text{Hom}(Q^\vee_O, \mathbb{Z}).
\]

In the case of \( \mathcal{R} = A_{2n} \), the verifications are done using the simple roots \( \Pi_{F,B_n} \) of the \( B_n \) subsystem of \( \mathcal{R}_F = BC_n \) (dual to \( \mathcal{R}_O \)). From the expressions of the simple (co)roots and fundamental (co)weights, we also easily see that:

\[
[ \Lambda^\kappa : Q^\vee_{B_n} ] = [ P_{B_n} : p(\Lambda^* ) ] = [ P^\vee_O : p(P^\vee) ] = [ Q^\kappa : Q_0 ] = 2.
\]
Remark 2.5. The global idea underlying the computations above is as follows. Given $\Lambda = Q^y \subset t$ and $\Lambda^* = P \subset t^*$, we roughly have two non-equivalent operations on these groups that yield lattices in $t^*$ and $(t^*)^*$: we can either intersect with $t^*$ (resp. $(t^*)^*$) to get $\kappa$-invariants, or project onto $t^*$ (resp. $(t^*)^*$) using $p$, and one operation is dual to the other over $\mathbb{Z}$.

The maximal tori of the compact connected groups associated to $\mathfrak{r}_F$ and $\mathfrak{r}_O$ are given by:

$$T_F = t^*/\Lambda^* = T^*; \quad T_O = t^*/\Lambda(\kappa) = T^*/(T^* \cap T_\kappa).$$

On the one hand, the group $T_O$ is the maximal torus of a Lie group of type $\mathfrak{r}_O$ with fundamental group isomorphic to $\Lambda(\kappa)/Q_0^\vee = \{1\}$. This motivates our next definition:

Definition 2.6. With $G$ and $\kappa \in \text{Out}(G)$ as in the last paragraph, we define the orbit Lie group $G_{(\kappa)}$ to be the simply connected Lie group with maximal torus $T_{(\kappa)} = T^*/(T^* \cap T_\kappa)$ and root system $\mathfrak{r}_{(\kappa)} := \mathfrak{r}_O$.

On the other hand, the torus $T_F = T^*$ is the maximal torus of the fixed-point subgroup $G^\kappa \subseteq G$, which is always connected for $G$ simply connected [9, Cor.3.15]. We may state:

Proposition 2.7. With the notations of this subsection:

1. If $\mathfrak{r} = A_{2n}$, then the root system of $G^\kappa$ coincides with the $B_n$ subsystem of $\mathfrak{r}_F = BC_n$ and $\pi_1(G^\kappa) \simeq \mathbb{Z}_2$. In the remaining cases, the group $G^\kappa$ has root system $\mathfrak{r}_F$ and is simply connected.

2. The orbit Lie group $G_{(\kappa)}$ is related to $G$ by the isomorphism $L G_{(\kappa)} \simeq (L G)_{0}^\kappa$, where $L G$ denotes the Langlands dual of $G$.

Remark 2.8. The first statement above is proved in the case of $G$ not necessarily simply connected in [8, §§13.3,14.4] (see also [23, Prop.1.1]), and $\pi_1(G^\kappa)$ can be deduced from the proposition dealing with lattices. The second statement is discussed in [20] for complex groups (see also the more general [29, Cor.1], and [16, Rk.2.3]), and follows from the definition of Langlands duality along with the fact that the identity component $(L G)_{0}^\kappa$ is of the same type as $G^\kappa$.

We close this section with a lemma that will be used for several formulas, including the description of the fundamental alcove for the $\kappa$-twisted adjoint action on $G$ (see [13 §2] for a more extensive treatment).

Lemma 2.9.

(a) Let $\theta$ denote the highest root of $\mathfrak{r}$. The highest root $\theta_{(\kappa),1}$ and the highest short root $\theta_{(\kappa),s}$ of $\mathfrak{r}_{(\kappa)}$ are given by:

$$\theta_{(\kappa),1} = \begin{cases} |\kappa| \theta_s^\kappa, & \mathfrak{r} \neq A_{2n}; \\ 4\theta_s^\kappa = 2\theta, & \mathfrak{r} = A_{2n}, n > 1; \\ 2(\alpha_1 + \alpha_2), & \mathfrak{r} = A_2; \end{cases}$$

$$\theta_{(\kappa),s} = \theta = \begin{cases} \theta^\kappa, & \mathfrak{r} \neq A_{2n}; \\ 4\theta^\kappa, & \mathfrak{r} = A_{2n}, n > 1; \end{cases}$$

where $\theta^\kappa$ and $\theta_s^\kappa$ denote the highest root and the highest short root of $\mathfrak{r}_F$ (resp. of $\mathfrak{r}_{B_n}$) for $\mathfrak{r} \neq A_{2n}$ (resp. $\mathfrak{r} = A_{2n}$).
(b) The half-sums of positive roots of $\mathcal{R}$ and $\mathcal{R}_{(\kappa)}$ are equal:

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_{(\kappa)}+} \alpha.$$  

Proof. Part (a) can be checked using the tables on affine root systems in [4,7,18]. To see that $\rho$ is also the half-sum of positive roots of $\mathcal{R}_{(\kappa)}$, it suffices to write it as the sum of fundamental weights $\{\varpi_i\}_{i=1}^r \subset t^*$:

$$\rho = \sum_{i=1}^r \varpi_i = \sum_{i=\kappa(i)} \varpi_i + \sum_{i \neq \kappa(i)} |\kappa| p(\varpi_i).$$

The forms $\{\varpi_i | i = \kappa(i)\} \sqcup \{\kappa| p(\varpi_i) | i \neq \kappa(i)\}$ are the fundamental weights of $\mathcal{R}_{(\kappa)}$ used in the proof of proposition [2,4].

Remark 2.10. Note that since $\rho \in (t^e)^*$, there exists a regular element in $T^\kappa$ under the adjoint action $\text{Ad}$. This fact will be used several times in the next section.

3. Outer Weyl Group and Conjugacy Classes

In this section, we look at the generalizations to twisted conjugation of several results in Weyl’s classical theory of compact groups. In the first subsection, we study the substitute to the Weyl group $W(\kappa)$. The main result of the second subsection is proposition 3.10 and says that the $\kappa$-twisted conjugacy classes in $G$ are parametrized by the fundamental alcove of the orbit group $G_{(\kappa)}$.

3.1. The Structure of the Outer Weyl Group. At the end of subsection 2.2, we introduced the group $T_\kappa$ with Lie algebra $t_\kappa = (t^e)^{\perp} \subset t$, along with the finite subgroup $T^\kappa \cap T_\kappa$. By introducing the map $\phi_\kappa : T \to T$, $t \mapsto t\kappa(t^{-1})$, we can describe the subtori of $T$ associated to $\kappa$ as:

$$T^\kappa = \ker(\phi_\kappa), \quad T_\kappa = \text{im}(\phi_\kappa),$$

with a similar description for the Lie algebras $t^e$ and $t_\kappa$. The map $\phi_\kappa : T \to T$ will be convenient for the study of the outer Weyl group.

Definition 3.1. The $\kappa$-twisted normalizer of $T^\kappa$ in $G$ is defined as:

$$N^\kappa_G(T^\kappa) := \{ g \in G \mid \text{Ad}^\kappa_g(T^\kappa) = T^\kappa \},$$

and we define the outer Weyl group as the quotient $W(\kappa) := N^\kappa_G(T^\kappa)/T^\kappa$.

Recall that $W^\kappa \subset W$ is the subgroup of Weyl group elements commuting with $\kappa$. Since $N_{G^\kappa}(T^\kappa) \subset N^\kappa_G(T^\kappa)$, we have $W^\kappa \subset W(\kappa)$. The next lemma clarifies how $N^\kappa_G(T^\kappa)$ fits between $N_G(T)$ and $N_{G^\kappa}(T^\kappa)$:

Lemma 3.2. The $\kappa$-twisted normalizer $N^\kappa_G(T^\kappa)$ coincides with the following subgroup of $N_G(T)$:

$$N^\kappa_G(T^\kappa) = \{ g \in N_G(T) \mid \text{Ad}^\kappa_g(e) \in T^\kappa \}.$$  

Proof. First, we prove the inclusion $N^\kappa_G(T^\kappa) \subset N_G(T^\kappa)$. For $x \in T^\kappa$ and $g \in N^\kappa_G(T^\kappa)$, we see from the formula:

\begin{equation}
(3.1) \quad \text{Ad}^\kappa_g(x) = \text{Ad}_g(x) \cdot \text{Ad}_g^\kappa(e),
\end{equation}
that $\text{Ad}_g(x) \in T^\kappa$. Next, let $x \in T^\kappa \subseteq T$ be regular under $\text{Ad}$, so that its centralizer is $Z_G(x) = T$. Then $\text{Ad}_g(x) \in T^\kappa$ is again regular, with centralizer:

$$\text{Ad}_g(Z_G(x)) = Z_G(\text{Ad}_g(x)) = T,$$

which shows that $\text{Ad}_g(T) = T$, and therefore that $g \in N_G(T)$ for any $g \in N^\kappa_G(T^\kappa)$.

Secondly, we define:

$$H := \{ g \in N_G(T) \mid g\kappa(g^{-1}) \in T^\kappa \},$$

and we claim that $H = N^\kappa_G(T^\kappa)$. For any $x \in T^\kappa$ and $h \in H \subseteq N_G(T)$, we have that $\text{Ad}_h(x) \in T^\kappa$, since $\text{Ad}_h(x) \in T$ and:

$$\kappa(\text{Ad}_h(x)) = \text{Ad}_{\kappa(h)}(x) = \text{Ad}_{(h\kappa(h^{-1}))^{-1}} \text{Ad}_h(x) = \text{Ad}_h(x),$$

where we used $h\kappa(h^{-1}) \in T^\kappa$ on the first line. Consequently, $h \in N^\kappa_G(T^\kappa)$ by equation (3.1), and we obtain $H \subseteq N^\kappa_G(T^\kappa)$. The other inclusion follows from $N^\kappa_G(T) \subseteq N_G(T)$ and the fact that $\text{Ad}_g^\kappa(e) \in T^\kappa$ for any $g \in N^\kappa_G(T^\kappa)$, which concludes the proof. 

The second lemma that we prove will be used for the characterization of $W(\kappa)$:

**Lemma 3.3.** For any $w \in W^\kappa \subseteq W$, one may choose a representative $g \in N_G(T)$ such that $g\kappa(g^{-1}) \in T^\kappa$.

**Proof.** If $g_w \in N_G(T)$ is such that $g_w T = w \in W^\kappa$, then:

$$\kappa(w \cdot t) = w \cdot \kappa(t), \; t \in T \iff \text{Ad}_{\kappa(g_w)}(t) = \text{Ad}_{g_w}(t), \; \forall t \in T.$$ 

Taking $t \in T$ regular, we see that $g_w \kappa(g_w^{-1})$ must be an element of $T$. On the other hand, the fact that $T = T^\kappa \cdot T_\kappa$ implies that $g_w \kappa(g_w^{-1}) = t_0 s^{-1} \kappa(s)$ for some $t_0 \in T^\kappa$ and $s \in T$, so that $(sg_w)\kappa((sg_w)^{-1}) = t_0 \in T^\kappa$. The representative $g \in N_G(T)$ of $w \in W^\kappa$ such that $g\kappa(g^{-1}) \in T^\kappa$ is then given by $g = g_w s' \in N_G(T)$, where $s' = g_w^{-1} s g_w \in T$. 

In the upcoming characterization of $W(\kappa)$, we will see the appearance of the finite group $(T/T^\kappa)^\kappa$, which is in fact isomorphic to $T^\kappa \cap T_\kappa$:

**Lemma 3.4.** Let $\pi : T \to T/T^\kappa$ be the canonical projection. The map:

$$\psi : (T/T^\kappa)^\kappa \to T^\kappa \cap T_\kappa, \; \pi(t) \mapsto \phi_\kappa(t),$$

is an isomorphism.

**Proof.** Let $q : N^\kappa_G(T^\kappa) \to W(\kappa)$ be the canonical projection. It will be convenient to write $\psi$ as the composition $\psi_2 \circ \psi_1$ of two isomorphisms:

$$\psi_1 : (T/T^\kappa)^\kappa \to q(\phi_\kappa^{-1}(T^\kappa \cap T_\kappa)), \quad (3.2)$$

$$\psi_2 : q(\phi_\kappa^{-1}(T^\kappa \cap T_\kappa)) \to T^\kappa \cap T_\kappa, \quad (3.3)$$

with $q(\phi_\kappa^{-1}(T^\kappa \cap T_\kappa)) \subseteq W(\kappa)$. On the one hand, we have $\text{Ad}_x^\kappa = \phi_\kappa(t)x \in T$ for all $x \in T^\kappa$ and $t \in T$, which is used to check that:

$$N^\kappa_G(T^\kappa) \cap T = \phi_\kappa^{-1}(T^\kappa \cap T_\kappa).$$
The homomorphism $\pi_1: T \to T/T^\kappa$, $t \mapsto tT^\kappa$ is $\kappa$-equivariant, and an element $tT^\kappa \in T/T^\kappa$ is found to be $\kappa$-invariant iff $t \in \phi_\kappa^{-1}(T^\kappa \cap T_\kappa) = N_G^\kappa(T^\kappa) \cap T$. The isomorphism (3.2) is then given explicitly by:

$$\psi_1: (T/T^\kappa)^\kappa \longrightarrow q(\phi_\kappa^{-1}(T^\kappa \cap T_\kappa)), \pi(t) \mapsto q(t).$$

On the other hand, the representatives of the group $q(\phi_\kappa^{-1}(T^\kappa \cap T_\kappa))$ in $N_G^\kappa(T^\kappa) \cap T$ act on $T^\kappa$ by twisted conjugation, which coincides with multiplication by elements of $T^\kappa \cap T_\kappa$. The isomorphism 3.3 is given by:

$$\psi_2: q(\phi_\kappa^{-1}(T^\kappa \cap T_\kappa)) \longrightarrow T^\kappa \cap T_\kappa, tT^\kappa \mapsto \phi_\kappa(t).$$

The last lemma that we will need is the following:

**Lemma 3.5.** One has that:

$$T^\kappa \cap T_\kappa \cong \begin{cases} (\mathbb{Z}_2)^{\dim t_\kappa}, & |\kappa| = 2; \\ \mathbb{Z}_3, & |\kappa| = 3. \end{cases}$$

**Proof.** We consider the two cases separately:

- **For $|\kappa| = 2$:** Let $t = s\kappa(s^{-1}) \in T^\kappa \cap T_\kappa$ be an arbitrary element, with $s \in T$. For $|\kappa| = 2$, the automorphism $\kappa$ acts as inversion on $T^\kappa \cap T_\kappa$:

$$\kappa(t) = \kappa(s)s^{-1} = t^{-1} = t,$$

and it is easily checked that in fact:

$$T^\kappa \cap T_\kappa = \{ t \in T_\kappa \mid t = t^{-1} \}.$$ 

Since $T_\kappa \cong (S^1)^{\dim t_\kappa}$, we see that there are $\dim t_\kappa = (\dim t - \dim t^\kappa)$ generators of order 2 for $T^\kappa \cap T_\kappa$, and the claim follows.

- **For $|\kappa| = 3$:** In the case of $\mathfrak{R} = D_4$, that $T^\kappa \cap T_\kappa \cong \mathbb{Z}_3$ can be seen more easily with a concrete description. Letting $T = \{(z_1, z_2, z_3, z_4) \mid z_j \in S^1\}$, the automorphism $\kappa$ acts as $\kappa(z_1, z_2, z_3, z_4) = (z_2, z_3, z_1, z_4)$. Thus:

$$T^\kappa = \{(a, b, a, a) \mid a, b \in S^1\},$$

$$T_\kappa = \{(a, 1, b, a^{-1}b^{-1}) \mid a, b \in S^1\},$$

and there are only 3 possibilities for $t \in T^\kappa \cap T_\kappa$: $(e^{\pm \frac{2\pi i}{3}}, 1, e^{\pm \frac{2\pi i}{3}}, e^{\pm \frac{2\pi i}{3}})$ and the identity. 

The main result of this subsection is the following (cf. [23, §2.3], [31, Prop.2.4]):

**Theorem 3.6.** The outer Weyl group decomposes as the semi-direct product:

$$W^{(\kappa)} = (T^\kappa \cap T_\kappa) \rtimes W^\kappa.$$
Proof. Here, we use the notation of the proof of lemma 3.4. We show that we have the following split exact sequence:

\[ 1 \longrightarrow (T/T^\kappa)^\kappa \longrightarrow W(\kappa) \overset{\nu}{\longrightarrow} W^\kappa \longrightarrow 1, \]

where:

\[ \nu : W(\kappa) \longrightarrow W^\kappa, \ gT^\kappa \mapsto gT. \]

We prove each claim separately.

- **Exactness at \((T/T^\kappa)^\kappa\):** If \( tT^\kappa \in (T/T^\kappa)^\kappa \) then \( t \in \phi_{\kappa}^{-1}(T^\kappa \cap T_\kappa) \subset N_G^\kappa(T^\kappa) \) by equation (3.4), and we have a well defined map \( \iota : (T/T^\kappa)^\kappa \rightarrow W(\kappa), \ tT^\kappa \mapsto tT^\kappa \) that is clearly injective.

- **Exactness at \(W^\kappa\):** That \( \nu : W(\kappa) \longrightarrow W^\kappa \) is a well-defined homomorphism is easily seen from its definition above, and its surjectivity is a direct consequence of lemma 3.3.

- **Exactness at \(W(\kappa)\):** We show that \( \ker \nu \subset (T/T^\kappa)^\kappa \): If \( g \in N_G^\kappa(T^\kappa) \) is such that \( \nu(gT^\kappa) = eT \), then \( g \in N_G^\kappa(T^\kappa) \cap T = \phi_{\kappa}^{-1}(T^\kappa \cap T_\kappa) \) and \( q(g) = gT^\kappa \in (T/T^\kappa)^\kappa \) by (3.4). Since \( (T/T^\kappa)^\kappa \subset \ker \nu \) is obvious, we have \( \ker \nu = (T/T^\kappa)^\kappa \).

- **Splitting:** For any \( x \in T^\kappa \) and \( gT \in W^\kappa \), we have:

\[
\Ad_g(x) = (g\kappa(g^{-1}))^{-1} \Ad_g^\kappa(x) \in T,
\]

and proceeding as in the proof of lemma 3.3 we find that for any \( x \in T^\kappa \), there exists a \( t \in T \) such that \( \Ad_g^\kappa(x) \in T^\kappa \). We thus have an inclusion \( j : W^\kappa \rightarrow W(\kappa), \ gT \mapsto gT^\kappa \) such that \( \nu \circ j = \Id_{W^\kappa} \).

From the above, we obtain \( W(\kappa) \simeq (T/T^\kappa)^\kappa \times W^\kappa \), and the theorem follows from the isomorphism in lemma 3.4 \( \square \)

### 3.2. Twisted Conjugacy Classes and Fundamental Alcove

The orbit space \( G/\text{Ad}^\kappa(G) \) is controlled by the groups \( G(\kappa), \ W(\kappa), \ T^\kappa \) and \( T_\kappa \). The following lemma corresponds to [31 Lem.2.1] (c.f. [5 Lem.IV.4.4]), and we present it in terms of the orbit root system \( \mathfrak{R}(\kappa) \) instead of the folded root system.

**Lemma 3.7.** Define the conjugation map:

\[ c : T^\kappa \times (G/T^\kappa) \longrightarrow G, \ (x, gT^\kappa) \mapsto \Ad_g^\kappa(x). \]

The determinant of the differential \( dc \) at the point \( (t, gT^\kappa) \in T^\kappa \times (G/T^\kappa) \) is given by:

\[
\det (dc_{(t,gT^\kappa)}) = |T^\kappa \cap T_\kappa| \cdot |\tilde{\Delta}(t)|^2,
\]

where \( \tilde{\Delta} : T^\kappa \rightarrow \mathbb{C} \) is the function:

\[
\tilde{\Delta}(t) = \prod_{\tilde{\alpha} \in \mathfrak{R}(\kappa)_+} (1 - t^{-\tilde{\alpha}}).
\]

**Proof.** Recall that we have a \( \kappa \)- and \( \text{Ad} \)-invariant inner product \( B \) on \( \mathfrak{g} \) (2.1). Fixing \( (t, gT^\kappa) \in T^\kappa \times (G/T^\kappa), \) the connectedness of \( G \) implies that \( \text{Ad}_{t,g} \in \text{SO}(\mathfrak{g}), \) and in terms of the orthogonal decomposition

\[
(\mathfrak{g}/t^\kappa)_C = (t_\kappa)_C \oplus \bigoplus_{\alpha \in \mathfrak{R}} \mathfrak{g}_\alpha,
\]
where \( g_\alpha \) denotes the root space for \( \alpha \), a direct computation yields:
\[
\det \left( dc_{(t,g^T\kappa)} \right) = \det_{g/t^\kappa} (Ad_{t^{-1}} - \kappa) = \det (t_\kappa)_C (Ad_{t^{-1}} - \kappa) \prod_{\alpha \in \mathcal{R}} \det_{g_\alpha} (Ad_{t^{-1}} - \kappa).
\]

The determinant on \((t_\kappa)_C\) depends on \(|\kappa|\), and we see from an eigenspace decomposition that:
\[
\det (t_\kappa)_C (Ad_{t^{-1}} - \kappa) = |T^\kappa \cap T_\kappa| = \begin{cases} 
2^{\dim t_\kappa}, & \text{if } |\kappa| = 2; \\
3, & \text{if } |\kappa| = 3.
\end{cases}
\]

For the remaining factors, we first note that if \( e_\alpha \in g_\alpha \) is a root vector for \( \alpha \in \mathcal{R} \), then \( \kappa \) acts on it as follows (see [11, Eq. (12.48)] for an explanation):
\[
\kappa(e_\alpha) = \begin{cases} 
\epsilon(\kappa(\alpha)), & \text{if } \mathcal{R} = A_{2n-1}, D_n, E_6; \\
\epsilon(\kappa(\alpha))(-1)^{h_\alpha + 1}, & \text{if } \mathcal{R} = A_{2n}.
\end{cases}
\]

Secondly, we gather the terms in the product over \( \mathcal{R} \) by \( \kappa\)-orbits. Suppose for simplicity that \( \mathcal{R} \neq A_{2n} \) with \(|\kappa| = 2\), and that \( \alpha \in \mathcal{R}_+ \). We obtain:
\[
\det \left( (Ad_{t^{-1}} - \kappa)|_{g_\alpha \oplus g_{-\alpha}} \right) = |1 - t^{-\alpha}|^2, \text{ if } \kappa(\alpha) = \alpha,
\]
\[
\det \left( (Ad_{t^{-1}} - \kappa)|_{g_\alpha \oplus g_{-\alpha}} \right) = |1 - t^{-|\kappa|p(\alpha)}|^2, \text{ if } \kappa(\alpha) \neq \alpha.
\]

Using the description of \( R(\kappa) \) in the proof of lemma 2.3 we have:
\[
\det \left( dc_{(t,g^T\kappa)} \right) = |T^\kappa \cap T_\kappa| \times \prod_{\tilde{\alpha} \in R(\kappa)_+} |1 - t^{-\tilde{\alpha}}|^2 = |T^\kappa \cap T_\kappa| \cdot |\tilde{\Delta}(t)|^2.
\]

With the appropriate modifications, the same formula is obtained for the cases \( \mathcal{R} = A_{2n} \) and \( \mathcal{R} = D_4 \) with \(|\kappa| = 3\).

The second lemma needed follows from elementary considerations, and we omit its proof.

**Lemma 3.8.** For \( x \in G \), denote by:
\[
Z^\kappa_G(x) := \{ g \in G \mid Ad^\kappa_g(x) = x \}
\]
the stabilizer under \( \kappa\)-twisted conjugation, and let \( z^\kappa_\mathcal{C} = \text{Lie}(Z^\kappa_G(x)) \) be its Lie algebra. Then:

1. For \( x \in T^\kappa \), the group \( Z^\kappa_G(x) \) is preserved by \( \kappa \) and contains \( T^\kappa \) as a maximal torus.
2. Let \( \mathcal{C} = \text{Ad}^\kappa(G) \cdot x \). With respect to the \( \kappa\)- and \( \text{Ad}\)-invariant metric on \( G \), one has an orthogonal decomposition \( T_xG = z^\kappa_\mathcal{C} \oplus T_x\mathcal{C} \), with \( z^\kappa_\mathcal{C} = \ker(\text{Ad}_{x^{-1}} - \kappa) \), and such that in the left trivialization of \( T^\kappa_\mathcal{C} \):
\[
T_x\mathcal{C} = \left\{ ((\text{Ad}_{x^{-1}} - \kappa) \cdot \xi)_x^L \mid \xi \in \mathcal{C} \right\}.
\]

(For \( \xi \in \mathcal{C} \), we use \( \xi^L_{(t,g)} = \frac{d}{dt} (g \cdot \exp(t\xi))|_{t=0} \) for the left-invariant vector fields.)

The next proposition generalizes a classical result of the theory of compact connected Lie groups, and is in essence a rephrasing of [5] Prop.IV.4.3 and [29, Lem.2.1]. The analogous result for algebraic groups is proved in [23, §5.25] and [29, Lem.2].

**Proposition 3.9.** The \( \kappa\)-twisted conjugacy classes in \( G \) satisfy the following properties:
Proof.

(1) For any $g \in G$, there exist elements $x \in T^\kappa$ and $a \in G$ such that $g = \text{Ad}^\kappa_a(x)$. That is, any element of $G$ is $\text{Ad}^\kappa$-conjugate to an element of the torus $T^\kappa$.

(2) Two elements of $T^\kappa$ are twisted conjugate under an element of $G$ if and only if they are twisted conjugate under an element of $N^\kappa_G(T^\kappa)$.

(3) Any $\kappa$-twisted conjugacy class in $G$ intersects $T^\kappa$ in an orbit of $W^\kappa = N^\kappa_G(T^\kappa)/T^\kappa$. In particular, one has for any $x \in T^\kappa$ that:

$$\text{Ad}^\kappa(G) \cdot x \cap T^\kappa = W^\kappa \cdot x.$$  

This assertion is proved by establishing the surjectivity of the twisted conjugation map $c : T^\kappa \times (G/T^\kappa) \rightarrow G$. Let $\xi \in t^\kappa \cap t^\kappa_{\text{reg}}$ be a regular element with respect to usual conjugation, and set $x = e^\xi \in T^\kappa$. By lemma 3.7, we have $\det (d_c(x,gT^\kappa)) > 0$ for any $g \in G$, meaning that the element $\text{Ad}^\kappa_c(x) \in G$ is a regular value of the conjugation map $c$, which is orientation preserving at $(x,gT^\kappa) \in T^\kappa \times (G/T^\kappa)$. From the decomposition in theorem 3.6 of $W^\kappa$, we have:

$$\# (c^{-1}(\text{Ad}^\kappa_g(x))) = \# (W^\kappa \cdot x) \geq 1,$$

which shows that $\deg c > 0$, and that $c : T^\kappa \times (G/T^\kappa) \rightarrow G$ is a surjective map.

(2) Let $C = \text{Ad}^\kappa(G) \cdot x$ for $x \in T^\kappa$, and let $y = \text{Ad}^\kappa_{a\kappa}(x) \in C \cap T^\kappa$ for some $a \in G$. We will modify $a$ by an element $z \in Z^\kappa_G(x)$, in such a way that $az \in N^\kappa_G(T^\kappa)$ with $\text{Ad}^\kappa_{az}(x) = y$, which will prove the claim.

Consider a regular element $\xi \in t^\kappa$ with respect to the usual conjugation action. Viewing $\xi$ as an element of $T^\kappa = (G/T^\kappa)$, the differential $(\text{Ad}^\kappa_{a\kappa^{-1}})_y : T^\kappa \rightarrow T^\kappa$ sends it to $\text{Ad}^\kappa_{a\kappa^{-1}}(\xi) \in (T^\kappa G)^{\perp} = g^\kappa_G$ by lemma 3.8(2) (as $t^\kappa \subset g^\kappa_G$). Now by lemma 3.8(1), that $T^\kappa$ is a maximal torus in $Z^\kappa_G(x)$ implies that there exists an element $z \in Z^\kappa_G(x)$ such that $\text{Ad}^\kappa_{a\kappa^{-1}}(\text{Ad}^\kappa_{a\kappa^{-1}}(\xi)) \in t^\kappa$. Putting $b = az \in G$, we thus have:

$$\text{Ad}^\kappa_b(x) = \text{Ad}^\kappa_z(x) = y \in T^\kappa,$$

$$\text{Ad}^\kappa_{b\kappa^{-1}}(\xi) = \text{Ad}^\kappa_{b\kappa^{-1}}(\xi) \in t^\kappa.$$

Since $\xi, \text{Ad}^\kappa_{b\kappa^{-1}}(\xi) \in t^\kappa$ are conjugate, the exists some $\nu \in N_G(T)$ such that $\text{Ad}^\kappa_{b\nu^{-1}}(\xi) = \xi$, meaning that $b\nu \in T \subset N_G(T)$ by regularity of $\xi$. Thus, the first conclusion here is that:

$$b \in N_G(T).$$

On the other hand, $\text{Ad}^\kappa_{b\kappa^{-1}}(\xi) = \text{Ad}^\kappa_{b\kappa^{-1}}(\xi) \in t^\kappa$ implies that $\text{Ad}^\kappa_{b\kappa^{-1}}(\xi) = \xi$, and by $\text{Ad}$-regularity of $\xi \in t^\kappa$ that $\kappa(b) = bt$ for some $t \in T$. Proceeding then as in the proof of lemma 3.2 we find $\kappa(\text{Ad}^\kappa_b(x)) = \text{Ad}^\kappa_b(x) \in T$ and therefore $\text{Ad}^\kappa_b(x) \in T^\kappa$. Since $y = \text{Ad}^\kappa_b(x) \cdot b\kappa^{-1}(b^{-1}) \in T^\kappa$, the second conclusion is that:

$$b\kappa^{-1}(b^{-1}) \in T^\kappa.$$

By lemma 3.2 we established that $b \in N^\kappa_G(T^\kappa)$, which finishes the proof.

(3) By the first statement, any element $g \in G$ lies in a conjugacy class of the form $C_g = \text{Ad}^\kappa(G) \cdot x$ for some $x \in T^\kappa$. By the second statement, $y \in C_g \cap T^\kappa$ if and only if there is an element $w \in W^\kappa$ such that $y = w \cdot x$, hence the claim.

□
The last statement in this proposition gives an identification $G/\text{Ad}^\kappa(G) \simeq T^\kappa/W^\kappa$. To express this space of orbits as quotient of $t^\kappa$, it is sufficient to determine the pre-image under $\exp_{t^\kappa} : t^\kappa \rightarrow T^\kappa$ of an intersection $\text{Ad}^\kappa(G) \cdot e^\xi \cap T^\kappa = W^\kappa \cdot e^\xi$, where $\xi \in t^\kappa$. The map $\exp_{t^\kappa}$ is equivariant with respect to the action of $W^\kappa = N_{G^\kappa}(T^\kappa)/T^\kappa$ on $T^\kappa$ and $t^\kappa$, and using $W^\kappa \simeq (T^\kappa \cap T_\kappa) \rtimes W^\kappa$, we have:

$$\exp_{t^\kappa}^{-1}(W^\kappa \cdot e^\xi) = \exp_{t^\kappa}^{-1}(W^\kappa \cdot (T^\kappa \cap T_\kappa \cdot e^\xi)) = W^\kappa \cdot \exp_{t^\kappa}^{-1}((T^\kappa \cap T_\kappa) \cdot e^\xi).$$

Recalling that we introduced $\Lambda_{(\kappa)} = \exp_{t^\kappa}^{-1}(T^\kappa \cap T_\kappa)$ in section 2.3, this lattice has a natural action by translations on $t^\kappa$, so that $\exp_{t^\kappa}^{-1}((T^\kappa \cap T_\kappa) \cdot e^\xi) = \Lambda_{(\kappa)} \cdot \xi$. Defining the twisted affine Weyl group $W_{\text{aff}}^\kappa := \Lambda_{(\kappa)} \rtimes W^\kappa$, the space of $\kappa$-twisted conjugacy classes is then given by:

$$G/\text{Ad}^\kappa(G) \simeq T^\kappa/W^\kappa \simeq t^\kappa/W_{\text{aff}}^\kappa.$$

Using the notation of 2.9, we have an explicit description of $t^\kappa/W_{\text{aff}}^\kappa$.

**Proposition 3.10.** There is a unique fundamental domain containing the origin in $t^\kappa$ for the action of $W_{\text{aff}}^\kappa$, which is the alcove:

$$\mathfrak{A}^\kappa = \{ \xi \in t^\kappa \mid 0 \leq \langle \tilde{\alpha}, \xi \rangle, \forall \tilde{\alpha} \in \Pi_{(\kappa)}; \langle \theta_{(\kappa),1}, \xi \rangle \leq 1 \},$$

where $\theta_{(\kappa),1} \in \mathcal{R}_{(\kappa)+}$ is the highest root of $\mathcal{R}_{(\kappa)}$, and $\Pi_{(\kappa)} \subset (t^\kappa)^*$ are its simple roots.

**Example 3.11.** We illustrate this alcove in the case $G = SU(3)$ (see Fig. 3.1). Here $\mathcal{R} = A_2$, and $\kappa \neq 1$ is the automorphism permuting the simple roots $\alpha_1$ and $\alpha_2$, the dual subtorus $(t^\kappa)^* \subset t^\kappa$ is the line $\{ \lambda(\alpha_1 + \alpha_2) \}_{\lambda \in \mathbb{R}}$, and $\mathfrak{R}_{(\kappa)}$ has only one positive root $(\theta^2_{(\kappa)})_1 = 2(\alpha_1 + \alpha_2)$.

In the standard coordinates on $\mathbb{R}^2 \equiv t$, we have $\alpha_1^\gamma = \sqrt{2}/2(\sqrt{3}, -1)$ and $\alpha_2^\gamma = \sqrt{2}(0, 1)$, so that the fundamental alcove for twisted conjugation is:

$$\mathfrak{A}^\kappa = \{ \xi \in t^\kappa \mid 0 \leq 2(\alpha_1 + \alpha_2, \xi) \leq 1 \} = \{ \frac{1}{2}(\sqrt{3}, 1) \lambda \in \mathbb{R}^2 \mid 0 \leq \lambda \leq \sqrt{2} \}.$$

**Remark 3.12.** The group of affine reflections $W_{\text{aff}}^\kappa$ coincides with the affine Weyl group of the untwisted affine algebra corresponding to $\mathcal{R}_{(\kappa)}$.

For the sake of completeness, we conclude this section with Mohrdieck and Wendt’s description of the stabilizers $Z^\kappa_a = Z^\kappa_{\mathfrak{g}}(a)$ for $a \in T^\kappa$ and their Lie algebras $\mathfrak{z}^\kappa_a \subset \mathfrak{g}$ [25, Prop.4.1]:

**Proposition 3.13.** Let $\xi \in \mathfrak{R}^\kappa$, and let $\Pi_{(\kappa)} = \Pi_{(\kappa)} \cup \{-\theta_{(\kappa),1}\}$ label the vertices of the extended Dynkin diagram associated to $\mathcal{R}_{(\kappa)}$.

1. The Dynkin diagram of $(\mathfrak{z}^\kappa_a, t^\kappa)$ is obtained from that of $\Pi_{(\kappa)}$ by deleting the vertex $\alpha \in \Pi_{(\kappa)}$ if $\langle \alpha, \xi \rangle \neq 0$, and deleting the vertex $-\theta_{(\kappa),1}$ if $\langle \theta_{(\kappa),1}, \xi \rangle < 1$.

2. If $a \subseteq \mathfrak{g}$ denotes the subalgebra corresponding to the diagram in (1), then: $\mathfrak{z}^\kappa_a = t^\kappa + a$.

3. The stabilizer $Z^\kappa_{\mathfrak{g}} \subset G$ is the Lie group with the root system described in (1) and fundamental group $\pi_1(Z^\kappa_{\mathfrak{g}}) = \Lambda^\kappa/Q^\gamma_\xi$, where $Q^\gamma_\xi$ is the corresponding coroot lattice.
Remark 3.14. The stabilizers $Z^\kappa_G (\exp g (\xi)) \subseteq G$ and $Z_{G(\kappa)} (\exp g_{(\kappa)} (\xi)) \subseteq G_{(\kappa)}$ are not isomorphic for $\xi \in \mathfrak{A}^{(\kappa)}$. The simplest example is that of $\xi = 0$, for which: \[ Z^\kappa_G (e) = G \quad \text{and} \quad Z_{G(\kappa)} (e) = G_{(\kappa)}. \]

This also shows that the $\kappa$-twisted conjugacy class corresponding to $\xi \in \mathfrak{A}^{(\kappa)}$ is typically of higher dimension than the orbit $\text{Ad}(G_{(\kappa)}) \cdot \exp g (\kappa (\xi)) \subseteq G_{(\kappa)}$.

4. Twining Characters

For a compact connected Lie group $G$, let $\Lambda^*_+ = \Lambda^* \cap \mathfrak{t}^*_+$ be the integral dominant weights, and denote by $\mathcal{C}^\infty (G)^G$ the ring of smooth $\text{Ad}$-invariant functions on $G$. The representation ring $R(G)$ can be realized as the subring of $\mathcal{C}^\infty (G)^G$ generated by the irreducible characters $\{ \chi_\lambda \}_{\lambda \in \Lambda^*}$, and as a $\mathbb{Z}$-module, one has:

\[ R(G) \simeq \mathbb{Z}[\Lambda^*_+]. \]

When $G$ is also simply connected, we can also define the fusion ring at level $k \in \mathbb{N} \setminus \{0\}$ solely in terms of $\{ \chi_\lambda \}_{\lambda \in \Lambda^*_+}$, by taking:

\[ R_k (G) := R(G) / I_k (G) \simeq \mathbb{Z}[\Lambda^*_k], \]

where under the identification $\mathfrak{g} \equiv \mathfrak{g}^*$ given by the basic inner product, the level $k$ weights are given by $\Lambda^*_k = \Lambda^*_+ \cap k \mathfrak{A}$, and the fusion ideal is defined as:

\[ I_k (G) := \left\{ f \in R(G) \mid f \left( \exp \left( \frac{\lambda + \rho}{k + h^\vee} \right) \right) = 0 \text{ for all } \lambda \in \Lambda^*_k \right\}, \]

with $h^\vee = 1 + \rho \cdot \theta$ the dual coxeter number of $G$. We recall that for $G$ simply connected, $R_k (G)$ coincides with the ring of level $k$ projective representations of $LG$, the loop group associated to $G$ [21, Rk. A.6].
The main objects of this section are the $\kappa$-twisted analogues of the rings $R(G)$ and $R_k(G)$ for $G$ simply connected and simple, and we start by discussing the analogues of the irreducible characters $\{\chi_\lambda\}_{\lambda \in \Lambda^*_+}$.

4.1. **Construction and First Properties.** For a dominant weight $\lambda \in \Lambda^*_+$, let $(\rho_\lambda, V_\lambda)$ denote the corresponding irreducible highest weight unitary representation, with $\rho_\lambda : G \to U(V_\lambda)$ and normalized highest weight vector $v_\lambda \in V_\lambda$. If $(\rho, V)$ is a finite-dimensional unitary representation of $G$ such that its decomposition into irreducibles is of the form

$$V = \bigoplus_{j=1}^p V_\lambda^\otimes m_j,$$

with $\lambda_j \in (\Lambda^*_+)^\kappa = (\Lambda^*_+)^\kappa \cap (t^*)_+^\kappa$ for all $j$, we will call $(\rho, V)$ a $\kappa$-admissible representation.

For $\lambda \in (\Lambda^*_+)^\kappa$, Schur’s lemma gives the existence of a unique unitary automorphism $\tilde{\kappa}_\lambda \in \text{Aut}(V_\lambda)$ such that:

(i) $\rho_\lambda(\kappa(g)) = \tilde{\kappa}_\lambda \circ \rho_\lambda(g) \circ \tilde{\kappa}_\lambda^{-1}$, for all $g \in G$;

(ii) $\tilde{\kappa}_\lambda(v_\lambda) = v_\lambda$.

By extension, for any $\kappa$-admissible representation $(\rho, V)$ there is a unique unitary automorphism $\tilde{\kappa}_\nu \in \text{Aut}(V)$ preserving the highest weight vectors of all irreducible components of $V$ and satisfying equation (i) above. In light of these facts, and following the terminology of [10][13], we have:

**Definition 4.1.** Let $(\rho_\nu, V)$ be a $\kappa$-admissible representation of $G$. The $\kappa$-twining character of $(\rho_\nu, V)$ is the function $G \to \mathbb{C}$ given for any $g \in G$ by:

$$\tilde{\chi}_\nu^\kappa(g) := \text{tr}_V (\tilde{\kappa}_\nu \circ \rho_\nu(g)).$$

**Remark 4.2.** For $V$ and $W$ $\kappa$-admissible, it is easily verified that:

$$\tilde{\chi}_\nu^\kappa \otimes W = \tilde{\chi}_\nu^\kappa \chi_V^\kappa, \quad \tilde{\chi}_\nu^\kappa \otimes \chi_W^\kappa = \tilde{\chi}_\nu^\kappa \cdot \chi_V^\kappa \cdot \chi_W^\kappa.$$

We will use these identities implicitly throughout this section.

Let $L^2(G)$ be the space of $\mathbb{C}$-valued $L^2$-functions on $G$ with respect to the normalized Haar measure $dg$, and denote by $C^\infty(G)$ the smooth $\mathbb{C}$-valued functions. We use $L^2(G\kappa)^G$ and $C^\infty(G\kappa)^G$ to denote the subspaces of $\text{Ad}^\kappa$-invariant functions on $G$. As it would be expected, the irreducible twining characters $\{\tilde{\chi}_\nu^\kappa\}_{\lambda \in \Lambda^*_+}^\kappa$ generalize several properties of the characters $\{\chi_\lambda\}_{\lambda \in \Lambda^*_+}$. First, we remind the following facts on matrix coefficients [5], §§II.4, III.3:

**Lemma 4.3.** For any finite-dimensional unitary representation $(\rho_\nu, V)$ of $G$, denote the matrix coefficient corresponding to $\alpha \in V^*$ and $v \in V$ by:

$$m^V_{\alpha,v} : G \to \mathbb{C}, \quad g \mapsto \alpha(\rho_\nu(g) \cdot v),$$

and let $G$ act on $\text{End}(V)$ via the assignment:

$$g \cdot A := \rho_\nu(g) \circ A \circ \rho_\nu(g)^{-1}, \text{ for all } A \in \text{End}(V).$$

Denote by $M(G)$ the space of complex matrix coefficients on $G$.

1. The space $M(G)$ is dense in $C^0(G)$ and in $L^2(G)$. 

\[ \]
(2) For \((\rho_V, V)\) irreducible, one has the following identity, for all \(A \in \text{End}(V)\):
\[
\int_G dg \cdot (g \cdot A) = \frac{\text{tr}_V(A)}{\dim V} \text{Id}_V.
\]

(3) For irreducible representations \((\rho_V, V)\) and \((\rho_W, W)\), the matrix coefficients satisfy the following orthogonality relations:
\[
\int_G \langle a, \rho_V(g) \cdot v \rangle w \langle b, \rho_W(g) \cdot w \rangle w = \begin{cases} 
\frac{1}{\dim V} \langle a, b \rangle_V \langle v, w \rangle_V, & \text{if } V \simeq W; \\
0, & \text{if } V \not\simeq W,
\end{cases}
\]
for any \(a, v \in V\) and \(b, w \in W\), and \(\langle -, -, \rangle_V\) denoting the Hermitian inner product on \(V\).

The main result of this subsection is the following:

**Proposition 4.4.** With the notations of this section:

1. The averaging map \(\text{Av}^\kappa : L^2(G) \to L^2(G\kappa)^G\), \(f \mapsto \int_G dg \cdot (\text{Ad}^\kappa_g)^* f\) is an orthogonal projection.
2. The twining characters \(\{\tilde{\chi}_\lambda^\kappa\}_{\lambda \in (\Lambda_+^\kappa)^*}\) generate a dense subspace of \(L^2(G\kappa)^G\), and satisfy the orthogonality relations:
\[
\langle \tilde{\chi}_\lambda^\kappa, \tilde{\chi}_\mu^\kappa \rangle_{L^2} = \delta_{\lambda\mu}, \quad \lambda, \mu \in (\Lambda_+^\kappa)^*.
\]
3. For a second Dynkin diagram automorphism \(\tau \in \text{Out}(G)\), one has:
\[
(\tilde{\chi}_\lambda^\kappa * \tilde{\chi}_\mu^\tau)(x) = (\dim V \lambda)^{-1} \delta_{\lambda\mu} \tilde{\chi}_\lambda^\kappa(x) = (\dim V \lambda)^{-1} \delta_{\lambda\mu} \tilde{\chi}_\lambda^\tau(\kappa(x)),
\]
for any \(\lambda, \mu \in (\Lambda_+^\kappa)^* \cap (\Lambda_+^\tau)^*\), where convolution is given by:
\[
(\psi * \varphi)(x) = \int_G dg \cdot \psi(xg^{-1}) \varphi(g), \quad \psi, \varphi \in L^2(G).
\]
4. For \(\lambda \in (\Lambda_+^\kappa)^*\), the "spherical harmonic" \(\phi_\lambda : G \to \mathbb{C}, \quad g \mapsto \langle v_\lambda, \rho_\lambda(g) \cdot v_\lambda \rangle_{V_\lambda}\) satisfies the identity:
\[
\int_G dg \cdot \phi_\lambda(\text{Ad}^\kappa_g x) = (\dim V \lambda)^{-1} \tilde{\chi}^\kappa_\lambda(x).
\]

**Proof.**

1. The formula in the statement defines a bounded linear operator \(\text{Av}^\kappa : L^2(G) \to L^2(G)\) with image in \(L^2(G\kappa)^G\). That it is an orthogonal projection follows from the fact that it is self-adjoint and coincides with the identity on \(L^2(G\kappa)^G\).
2. That the \(\kappa\)-twining characters \(\{\tilde{\chi}_\lambda^\kappa\}_{\lambda \in (\Lambda_+^\kappa)^*}\) are \(\text{Ad}^\kappa\)-invariant follows from the cyclic property of the trace and the defining identity:
\[
\rho_\lambda(\kappa(g)) = \tilde{\kappa}_\lambda \circ \rho_\lambda(g) \circ \tilde{\kappa}_\lambda^{-1}.
\]

Next, let \(\alpha \in V_\lambda^*\), \(v \in V_\lambda\) and \(A = \tilde{\kappa}_\lambda \circ \rho_\lambda(g)\). Using lemma [4.3] we have for any \(x \in G\) that:
\[
(\text{Av}^\kappa m_{\alpha,v})_\lambda(x) = \alpha \left( \int_G (\rho_\lambda(\text{Ad}^\kappa_g x) \cdot v) \, dg \right) = \alpha \left( \int_G (g \cdot A)(\tilde{\kappa}_\lambda^{-1} \cdot v) \right)
\]
\[
= \alpha \left( \frac{\text{tr}_\lambda(A)}{\dim V_\lambda} \tilde{\kappa}_\lambda^{-1} \cdot v \right) = \frac{\alpha(\tilde{\kappa}_\lambda^{-1} \cdot v)}{\dim V_\lambda} \tilde{\chi}_\lambda^\kappa(x).
\]
This is used for the density of twining characters as follows. Given \( \varphi \in L^2(G\kappa)^G \), for any \( \varepsilon > 0 \) there is a function \( f \in M(G) \) such that \( \| \varphi - f \|_{L^2} < \varepsilon \), and such that for a finite subset \( L \subseteq \Lambda_+^* \):

\[
f(x) = \sum_{\lambda \in L} \sum_{i=1}^{k_\lambda} a_i m_{\alpha_i, v_i}(x),
\]

where \( \{k_\lambda\}_{\lambda \in L} \) are integers, and for all \( 1 \leq i \leq k_\lambda \): \( a_i \in \mathbb{C}, v_i \in V_\lambda, \alpha_i \in V_\lambda^* \). By equation (4.1), we have:

\[
A v^\kappa f(x) = \sum_{\lambda \in L \cap (\Lambda_+^*)^\kappa} c_\lambda \tilde{x}_\lambda^\kappa(x),
\]

for some constants \( c_\lambda \in \mathbb{C} \), and our initial function \( \varphi \in L^2(G\kappa)^G \) then satisfies the estimate:

\[
\| \varphi - A v^\kappa f \|_{L^2} = \| A v^\kappa (\varphi - f) \|_{L^2} \leq \| \varphi - f \|_{L^2} < \varepsilon,
\]

which establishes the density claim. For the orthogonality relations, let \( \{v_i\} \subset V_\lambda \) and \( \{w_i\} \subset V_\mu \) be orthonormal bases, so that:

\[
\langle \tilde{x}_\lambda^\kappa, \tilde{x}_\mu^\kappa \rangle_{L^2} = \sum_{i,j} \int_G \langle v_i, (\tilde{\kappa}_\lambda g) \cdot v_i \rangle_{V_\lambda} \langle w_j, (\tilde{\kappa}_\mu g) \cdot w_j \rangle_{V_\mu}.
\]

By lemma 4.3(3), the above vanishes if \( \mu \neq \lambda \), and otherwise:

\[
\langle \tilde{x}_\lambda^\kappa, \tilde{x}_\lambda^\kappa \rangle_{L^2} = \frac{1}{\dim V_\lambda} \sum_{i,j} \langle v_i, v_j \rangle_{V_\lambda} \langle (\tilde{\kappa}_\lambda g) \cdot v_i, (\tilde{\kappa}_\lambda g) \cdot v_j \rangle_{V_\lambda} = 1.
\]

(3) Let \( \lambda, \mu \in (\Lambda_+^*)^\kappa \cap (\Lambda_+^*)^\tau \) and let \( \{v_i\} \subset V_\lambda \) and \( \{w_i\} \subset V_\mu \) be orthonormal bases. If \( \mu \neq \lambda \), the integral:

\[
\langle \tilde{x}_\lambda^\kappa * \tilde{x}_\mu^\tau, \tilde{x}_\lambda^\kappa * \tilde{x}_\mu^\tau \rangle(x) = \sum_{i,j} \int_G \langle v_i, (\tilde{\kappa}_\lambda x g^{-1}) \cdot v_i \rangle_{V_\lambda} \langle w_j, (\tilde{\tau}_\mu x g^{-1}) \cdot w_j \rangle_{V_\mu} \, dg
\]

vanishes by 4.3(3). To compute \( \tilde{x}_\lambda^\kappa * \tilde{x}_\lambda^\kappa \), we use \( \tilde{x}_\lambda^\kappa(x) = \text{tr}_{V_\lambda}(x \tilde{\kappa}_\lambda) \) and notice that:

\[
\langle v_i, (x g^{-1} \tilde{\kappa}_\lambda) \cdot v_i \rangle_{V_\lambda} = \sum_{k=1}^{\dim V_\lambda} \langle v_i, x \cdot v_k \rangle_{V_\lambda} \langle \tilde{\kappa}_\lambda \cdot v_i, g \cdot v_k \rangle_{V_\lambda}.
\]

This identity along with lemma 4.3(3) yield:

\[
\langle \tilde{x}_\lambda^\kappa * \tilde{x}_\lambda^\kappa \rangle(x) = \sum_{i,j} \int_G \langle v_i, (x g^{-1} \tilde{\kappa}_\lambda) \cdot v_i \rangle_{V_\lambda} \langle v_j, (g \tilde{\tau}_\lambda) v_j \rangle_{V_\lambda} \, dg
\]

\[
= \sum_{i,j,k} \langle v_i, x \cdot v_k \rangle_{V_\lambda} \int_G \langle \tilde{\kappa}_\lambda \cdot v_i, g \cdot v_k \rangle_{V_\lambda} \langle v_j, (g \tilde{\tau}_\lambda) v_j \rangle_{V_\lambda} \, dg
\]

\[
= \frac{1}{\dim V_\lambda} \sum_{i,j,k} \langle v_i, x \cdot v_k \rangle_{V_\lambda} \langle v_j, \tilde{\kappa}_\lambda \cdot v_i \rangle_{V_\lambda} \langle v_k, \tilde{\tau}_\lambda \cdot v_j \rangle_{V_\lambda}
\]

\[
= \frac{1}{\dim V_\lambda} \sum_i \langle v_i, (x \tilde{\tau}_\lambda \tilde{\kappa}_\lambda) \cdot v_i \rangle_{V_\lambda} = \frac{1}{\dim V_\lambda} \text{tr}_{V_\lambda}(\rho_\lambda(x) \circ (\tilde{\tau}_\lambda \tilde{\kappa}_\lambda)).
\]
The operator \((\bar{\tau}_\lambda \bar{k}_\lambda) \in \text{Aut}(V_\lambda)\) preserves the highest weight vector of \(V_\lambda\) and satisfies:

\[
\rho_\lambda(\tau_\kappa(g)) = (\bar{\tau}_\lambda \bar{k}_\lambda) \circ \rho_\lambda(g) \circ (\bar{\tau}_\lambda \bar{k}_\lambda)^{-1}, \quad g \in G.
\]

By uniqueness, we must have \((\bar{\tau}_\lambda \bar{k}_\lambda) = (\tau_\kappa)\lambda\), and therefore:

\[
\bar{\chi}_\lambda^\kappa \ast \bar{\chi}_\mu^\tau = (\dim V_\lambda)^{-1} \delta_{\lambda \mu} \bar{\chi}_\lambda^\kappa.
\]

That one has \(\bar{\chi}_\lambda^{\tau \kappa}(g) = \hat{\chi}_\lambda^{\kappa \tau}(\kappa(g))\) for all \(g \in G\) and \(\lambda \in (\Lambda^*_+)^\kappa \cap (\Lambda^*_+)^\tau\) follows from:

\[
\text{tr}_{V_\lambda}((\bar{\tau}_\lambda \bar{k}_\lambda)\rho_\lambda(g)) = \text{tr}_{V_\lambda}(\bar{\tau}_\lambda \rho_\lambda(\kappa(g)) \bar{k}_\lambda) = \text{tr}_{V_\lambda}((\bar{k}_\lambda \bar{\tau}_\lambda)\rho_\lambda(\kappa(g))).
\]

(4) The identity of the statement is a special case of equation (4.1) with \(v = v_\lambda = \bar{k}_\lambda(v_\lambda)\) and \(\alpha = \langle \nu_\lambda, - \rangle_{V_\lambda}\).

\[\square\]

4.2. The Jantzen Character Formula. We now come to a Weyl-type character formula for the \(\bar{\chi}_\lambda^\kappa\), which we will need to integrate an integration formula. If \((T^\kappa)^{\kappa-\text{reg}}\) and \(G^{\kappa-\text{reg}}\) denote the submanifolds of \(\text{Ad}^\kappa\)-regular elements in \(T^\kappa\) and \(G\) (elements \(g\) with \(\dim Z_G^\kappa(g) = \dim T^\kappa\)), then proposition 3.9 and lemma 3.7 imply that the restriction of the conjugation map \(\cdot\) to \((T^\kappa)^{\kappa-\text{reg}} \times (G/T^\kappa)\) is a covering onto \(G^{\kappa-\text{reg}}\), with group of deck transformations \(W^\kappa\). Using Fubini’s theorem along with the fact that the connected components of \(G \setminus G^{\kappa-\text{reg}}\) and \(T^\kappa \setminus (T^\kappa)^{\kappa-\text{reg}}\) have measure zero, we obtain [31, Thm.2.5]:

**Lemma 4.5.** For a class function \(f \in L^1(G \kappa)^G\), one has that:

\[
\int_G f(x)dx = \frac{1}{|W^\kappa|} \int_{T^\kappa} f(t)|\Delta(t)|^2dt.
\]

We will also need some observations on the action of the induced automorphism \(\bar{k}_\lambda \in \text{Aut}(V_\lambda)\) on the highest weight irrep \(V_\lambda\), which are easily established using the weight space decomposition:

**Lemma 4.6.** For \(\lambda \in (\Lambda^*_+)^\kappa\), let \(P(V_\lambda)\) be the set of weights of the representation \(V_\lambda\), and let \(V^\kappa_\lambda = \ker(\bar{k}_\lambda - 1)\) and \(P(V_\lambda)^\kappa = \{\mu \in P(V_\lambda) \mid \kappa(\mu) = \mu\}\).

1. The set \(P(V_\lambda) \subset \Lambda^*\) is preserved by \(\kappa \in \text{Out}(G)\), and if \(u \in V_\lambda\) is a weight vector for \(\mu \in P(V_\lambda)\), then \(\bar{k}_\lambda(u) \in V_\lambda\) is a weight vector for \(\kappa(\mu) \in P(V_\lambda)\).

2. For \(t \in T^\kappa\), one has that:

\[
\bar{\chi}_\lambda^\kappa(t) = \text{tr}_{V^\kappa_\lambda}(\rho_\lambda(t)) = \sum_{\mu \in P(V_\lambda)^\kappa} m_\mu t^\mu,
\]

with \(m_\mu = \dim(V^\kappa_\lambda)\mu\), and that:

\[
\bar{\chi}_\lambda^\kappa(e) = \dim V^\kappa_\lambda.
\]

For a weight \(\lambda \in (\Lambda^*_+)^\kappa\), let \(V_\lambda\) denote the corresponding highest weight irrep of the orbit group \(G(\kappa)\), and let \(\sigma_\lambda : G(\kappa) \to \mathbb{C}\) be the associated irreducible character. Recalling that the half-sum of positive roots of \(\mathfrak{R}(\kappa)\) coincides with \(\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} \alpha\), we consider the functions:

\[
\bar{J}_\lambda(\xi) := \sum_{w \in W^\kappa} (-1)^{(w)} e^{2\pi i (w \cdot (\lambda + \rho), \xi)}, \quad \xi \in t^\kappa,
\]
We thus have \( \sigma_\lambda \left( \exp_{g(\kappa)}(\xi) \right) = \frac{\tilde{J}_\lambda(\xi)}{J_0(\xi)}, \)

where \( \exp_{g(\kappa)} : g(\kappa) \to G(\kappa) \) sends \( t^{\kappa} \) to \( T(\kappa) = T^\kappa/(T^\kappa \cap T_\kappa) \) (we make this distinction since \( \exp : g \to G \) is also used below, and sends \( t^{\kappa} \) to \( T^\kappa \)). The next result was first proved by Jantzen in [17], then recovered in the context of affine algebras in [10, Thm.1], and in the context of non-connected groups in [31, Thm.2.6].

**Proposition 4.7. Jantzen Character Formula** Let \( G \) be a compact, connected, simply connected and simple Lie group, let \( \kappa \in \text{Out}(G) \) be a Dynkin diagram automorphism and \( G(\kappa) \) the associated orbit Lie group. For an invariant dominant weight \( \lambda \in (\Lambda_+)^\kappa \), one has that:

\[
(\tilde{\chi}_\lambda^\kappa)_{|T^\kappa} = \pi^*(\sigma_\lambda|T_{(\kappa)}),
\]

where \( \pi : T^\kappa \to T_{(\kappa)} \) is the quotient map by the action of \( T^\kappa \cap T_\kappa \). In particular, for \( \xi \in t^{\kappa} \) regular:

\[
\tilde{\chi}_\lambda^\kappa \left( \exp(\xi) \right) = \sum_{w \in W^\kappa} (-1)^l(w) e^{2\pi i \langle \lambda(\kappa) - \rho, \xi \rangle} = \frac{\tilde{J}_\lambda(\xi)}{J_0(\xi)}.
\]

**Proof.** Since \( \tilde{\chi}_\lambda^\kappa \) is \( \text{Ad}^\kappa \)-invariant, \( (\tilde{\chi}_\lambda^\kappa)_{|T^\kappa} \) is invariant under the action of \( T^\kappa \cap T_\kappa \) by multiplication. Along with lemma 4.6(2), we thus have for any \( t \in T^\kappa \) that:

\[
(\tilde{\chi}_\lambda^\kappa)_{|T^\kappa}(t) = \sum_{\mu \in P(V_\lambda)^\kappa} m_{\mu} t^\mu = \sum_{\mu \in P(V_\lambda)^\kappa} m_{\mu} \pi(t)^\mu.
\]

On the one hand, this shows that \( P(V_\lambda)^\kappa \) is a subset of the weight lattice \( (\Lambda_+)^\kappa \) of \( G(\kappa) \), and on the other hand, it means that \( (\tilde{\chi}_\lambda^\kappa)_{|T^\kappa} = \pi^*(\sigma_{V|T_{(\kappa)}}) \), where \( \sigma_V : G(\kappa) \to \mathbb{C} \) the character of a \( G(\kappa) \)-module \( \hat{V} = \bigoplus_{\mu \in (\Lambda_+)^\kappa} V^\mu_{\kappa} \):

\[
\sigma_V = \sum_{\mu \in (\Lambda_+)^\kappa} q_\mu \sigma_\mu,
\]

with finitely many nonzero integers \( q_\mu \). Since the function \( \overline{\Delta}(t) = \prod_{\alpha \in \Phi(\kappa)} (1 - t^{-\alpha}) \) is the “Weyl denominator” on \( T_{(\kappa)} \), we have by lemma 4.5 and the orthogonality relations for twining characters that:

\[
\sum_{\mu \in (\Lambda_+)^\kappa} q_\mu^2 = \frac{1}{|W^\kappa|} \int_{T_{(\kappa)}} ds \cdot |\overline{\Delta}\sigma_V|^2 = \frac{1}{|W^\kappa|} \int_{T^\kappa} dt \cdot |\overline{\Delta}(\tilde{\chi}_\lambda^\kappa)_{|T^\kappa}|^2 = 1.
\]

We thus have \( \hat{V} = \hat{V}_\lambda \simeq V_\lambda^\kappa \), since the equation above shows that \( \hat{V} \) is irreducible, and since \( \lambda \in (\Lambda_+)^\kappa \) is the highest weight in \( P(V_\lambda)^\kappa \). This shows that \( (\tilde{\chi}_\lambda^\kappa)_{|T^\kappa} = \pi^*(\sigma_{\lambda|T_{(\kappa)}}) \), and for any
With $V_T G$ (in the left trivialization of $T G$), we define $\xi$, which is the second equation of the proposition.

Let $\kappa \in \text{Out}(G)$ be a Dynkin diagram automorphism. We have by the Chevalley restriction theorem:

The Jantzen character formula shows that $\kappa$ regular, we have by the character formula for $g$:

$$\kappa(\sum_{w \in W^k} (-1)^l(w) e^{2\pi i (w(\lambda+\rho),\xi)})$$

is an isomorphism. To obtain the open subset $U_\kappa$, we use the following version of the Chevalley restriction theorem:

$$\sigma_\chi \left( \exp_{\theta(k)}(\xi) \right) = \frac{\tilde{J}_\chi(\xi)}{\tilde{J}_0(\xi)} = \frac{\sum_{w \in W^k} (-1)^l(w) e^{2\pi i (w(\lambda+\rho),\xi)}}{\sum_{w \in W^k} (-1)^l(w) e^{2\pi i (w(\rho,\xi)}}$$

which is the second equation of the proposition.

The Jantzen character formula shows that $\pi^*: L^2(T_{(\kappa)})^{W^k} \to L^2(T_{(\kappa)})^{W_{(\kappa)}}$ is an isomorphism, and this is true for smooth class functions in particular. With a little more work, we also have the following version of the Chevalley restriction theorem:

**Proposition 4.8.** Let $G$ be a compact, connected, simply connected and simple Lie group, and $\kappa \in \text{Out}(G)$ a Dynkin diagram automorphism. The inclusion $\iota_{T^\kappa} : T^\kappa \hookrightarrow G$ induces the following isomorphism of algebras:

$$\iota^*_\kappa : \mathcal{C}_c^\infty(G\kappa)^G \longrightarrow \mathcal{C}_c^\infty(T^\kappa)^{W_{(\kappa)}} \simeq \mathcal{C}_c^\infty(T_{(\kappa)})^{W_{(\kappa)}}, \ f \mapsto f|_{T^\kappa}$$

where $\mathcal{C}_c^\infty(G\kappa)^G$ is the ring of smooth $\text{Ad}^{\kappa}$-invariant functions on $G$ with values in $\mathbb{C}$.

**Proof.** Using a $\text{Ad}^{\kappa}(G)$-invariant partition of unity on $G$, the problem is reduced to proving that for any $x \in G$, there exists a $\text{Ad}^{\kappa}(G)$-invariant open neighborhood $U_x$ such that the restriction map:

$$\mathcal{C}_c^\infty(U_x)^G \longrightarrow \mathcal{C}_c^\infty(U_x \cap T^\kappa)^{W_{(\kappa)}}, \ f \mapsto f|_{U_x \cap T^\kappa},$$

is an isomorphism. To obtain the open subset $U_x$, we construct a slice for the twisted adjoint action following the approach of [22, Prop.2.5], and to lighten the notation, we write $Z_x^\kappa$ for the stabilizer of $x$ with Lie algebra $\mathfrak{z}_x^\kappa$.

Let $F \subset \mathfrak{A}_{(\kappa)}$ be an open face of the fundamental alcove in $\mathfrak{t}^\kappa$, and let $\xi \in F$ with $x = e^\xi$. Let $V_\xi \subset \mathfrak{t}^\kappa$ be an open ball centred at $\xi$, and let $\mathfrak{A}_F^{\kappa}(\kappa)$ be the union of open faces in $\mathfrak{A}_{(\kappa)}$ containing $F$ in their closure. We define the following $\text{Ad}^{\kappa}(Z_x^\kappa)$-invariant open subset of $Z_x^\kappa$:

$$U_x := \text{Ad}^{\kappa}(Z_x^\kappa) \cdot \exp \left( V_\xi \cap \mathfrak{A}^{\kappa, F}_{(\kappa)} \right),$$

which for $V_\xi$ small enough gives a $Z_x^\kappa$-equivariant diffeomorphism:

$$x \exp_{\xi^F_x} : V_0 \longrightarrow U_x, \ \zeta \longmapsto x e^\zeta = e^{\kappa(\zeta)} x,$$

with $V_0 \subset \mathfrak{z}_x^\kappa$ a $Z_x^\kappa$-invariant open ball centred at the origin, and such that:

$$x \exp_{\xi^F_x} (\text{Ad}_{\kappa(g)} \zeta) = \text{Ad}_g \left( x \exp_{\xi^F_x}(\zeta) \right), \text{ for all } g \in Z_x^\kappa.$$

(In the left trivialization of $T G$, the twisted adjoint action of $g \in G$ on $\zeta \in \mathfrak{g}$ is given by $\text{Ad}_{\kappa(g)} \zeta$.) By construction, $U_x \subset Z_x^\kappa$ is a slice through $x$ for $\kappa$-twisted conjugation, and the map:

$$G \times Z_x^\kappa V_0 \longrightarrow G, \ [g, \zeta] \longmapsto \text{Ad}_g \left( x e^\zeta \right)$$
gives a Ad\(^{\kappa}(G)\)-equivariant diffeomorphism onto its image, which we denote henceforth by \(U_x\). Recalling that \(T^\kappa\) is a maximal torus of \(Z_x^\kappa\), we also obtain a \(W_x^{(\kappa)} = N_x^\kappa(T^\kappa)/T^\kappa\)-equivariant diffeomorphism:

\[
N^\kappa_G(T^\kappa) \times N^\kappa_x(T^\kappa) \rightarrow U_x \cap T^\kappa, \quad [g, \zeta] \mapsto \text{Ad}^\kappa_g \left( x\xi^\kappa \right).
\]

By the Chevalley restriction theorem for Lie algebras \([2 \text{ Thm.7.28}]\), we have an isomorphism 
\[
C^\infty_\mathbb{C}(\mathfrak{g}_\kappa) \cong C^\infty_\mathbb{C}(\mathfrak{t}^\kappa)^{W_x^{(\kappa)}},
\]
and therefore
\[
C^\infty_\mathbb{C}(V_0) \cong C^\infty_\mathbb{C}(V_0 \cap \mathfrak{t}^\kappa)^{W_x^{(\kappa)}},
\]
and the equivariant diffeomorphisms constructed above yield:
\[
C^\infty_\mathbb{C}(U_x) G \cong C^\infty_\mathbb{C}(U_x \cap T^\kappa)^{W(\kappa)}.
\]

4.3. Twisted Representation and Fusion Rings. We turn to the representation rings associated to the \(\kappa\)-admissible representations of \(G\) in this subsection, following the approach of \([21 \text{ App.4}]\) (proofs can be found in \([1 \text{ Pt.III}]\)).

**Definition 4.9.** Define the **twining representation ring** \(\hat{R}^{(\kappa)}(G)\) to be the subring of \(C^\infty_\mathbb{C}(G)\) generated by the twining characters of finite-dimensional \(\kappa\)-admissible representations of \(G\).

This is the ring of virtual \(\kappa\)-admissible representations of \(G\). As a \(\mathbb{Z}\)-module, it is spanned by the irreducible twining characters:

\[
\hat{R}^{(\kappa)}(G) \cong \mathbb{Z} \left[ (\Lambda^\kappa_+)^\kappa \right].
\]

The last two results of the previous subsection yield the following fact:

**Proposition 4.10.** Let \(G\) be a compact, connected, simply connected and simple Lie group, \(\kappa \in \text{Out}(G)\) a diagram automorphism and \(G^{(\kappa)}\) the corresponding orbit Lie group. One then has:

\[
\hat{R}^{(\kappa)}(G) \cong \hat{R}^{(\kappa)}(T^\kappa)^{W(\kappa)} \cong R(T^{(\kappa)})^{W^\kappa} \cong R(G^{(\kappa)}).
\]

As in the case of the usual representation ring \(R(G)\), the twining representation ring is equipped with an involution:

\[
(\cdot)^* : \hat{R}^{(\kappa)}(G) \rightarrow \hat{R}^{(\kappa)}(G),
\]

which acts as \(\hat{\chi}_V^{(\kappa)} \mapsto \hat{\chi}_V^{(\kappa)*}\) for any \(\kappa\)-admissible representation \(V\) of \(G\). There is also a trace map:

\[
\text{Tr}_0 : \hat{R}^{(\kappa)}(G) \rightarrow \mathbb{Z}, \quad f \mapsto \int_G dg \cdot \left( \overline{\hat{\chi}_0} f \right),
\]

that gives the coefficient of the basis element corresponding to the weight \(\lambda = 0\).

Now, consider the following rescaling of the inner product \(B\) on \(\mathfrak{g}\):

\[
\tilde{B} := \frac{2}{B(\theta^{(\kappa)}_l, \theta^{(\kappa)}_l)} B,
\]

where \(\theta^{(\kappa)}_l \in \mathfrak{R}^{(\kappa)}\) is the highest root of \(G^{(\kappa)}\). The restriction \(\tilde{B}^{\kappa}_{\psi_e} \in S^2(\mathfrak{t}^\kappa)^*\) coincides with the restriction of the **basic inner-product** on \(\mathfrak{g}^{\kappa}\) to \(\mathfrak{t}^\kappa\), which is the only inner-product on \(\mathfrak{t}^\kappa\).
such that $\tilde{B}^{-1}(\tilde{\alpha}, \tilde{\alpha}) = 2$ for any long root $\tilde{\alpha} \in \mathcal{R}_\kappa$, and restricting to a $\mathbb{Z}$-valued bilinear form on $\Lambda_\kappa$. In particular, the isomorphism:

$$\tilde{B}^\flat : \mathfrak{g} \to \mathfrak{g}^*, \xi \mapsto \tilde{B}(\xi, \cdot)$$

maps $\Lambda_\kappa$ onto $(\Lambda^*\kappa)^\kappa$, since $\tilde{B}_\kappa \in (\Lambda^*\kappa) \otimes (\Lambda^*\kappa)$. Fixing a positive integer $k \in \mathbb{Z}_{>0}$, we introduce the following objects:

- The isomorphism $\tilde{B}^\flat_k := k\tilde{B}^\flat$, with inverse:

$$\tilde{B}^\sharp_k : \mathfrak{g}^* \to \mathfrak{g}, \lambda \mapsto \frac{1}{k}(\tilde{B}^\flat)^{-1}(\lambda).$$

- The $k$-rescaled alcove in $(t^\kappa)^*$ (proposition 3.10):

$$\mathfrak{A}^\kappa_{(\kappa),k} := \tilde{B}^\sharp_k(\mathfrak{A}_{(\kappa)}).$$

- The set of $\kappa$-invariant level $k$ weights of $G$:

$$(\Lambda^*\kappa)^k := (\Lambda^*\kappa) \cap \mathfrak{A}^\kappa_{(\kappa),k}.$$

- The finite subgroup:

$$T^\kappa_{k+h^\kappa_{(\kappa)}} := \tilde{B}^\sharp_{k+h^\kappa_{(\kappa)}}\left( (\Lambda^*\kappa) / \Lambda_\kappa \right) \subseteq T_\kappa,$$

where $h^\kappa_{(\kappa)} = 1 + \langle \rho, \tilde{B}^\sharp(\theta_{(\kappa),1}) \rangle$ is the dual Coxeter number of $\mathfrak{R}_\kappa$, and where $T^\kappa_{k+h^\kappa_{(\kappa)}} / W^\kappa$ consists of the elements:

$$s_{\lambda} := \exp_{\mathfrak{g}_{(\kappa)}}\left( \tilde{B}^\sharp_{k+h^\kappa_{(\kappa)}}(\lambda + \rho) \right), \quad \lambda \in (\Lambda^*\kappa)^k.$$

- The shifted action $\bullet_k$ of $W^\kappa_{\text{aff}} = \Lambda_\kappa \times W^\kappa$ on $(t^\kappa)^*$, specified by the assignment:

$$w \bullet_k \lambda = \begin{cases} w \cdot (\lambda + \rho) - \rho, & w \in W^\kappa; \\ \lambda - \tilde{B}^\flat_{k+h^\kappa_{(\kappa)}}(w), & w \in \Lambda_\kappa, \end{cases}$$

for any $\lambda \in (t^\kappa)^*$.

Note that the shifted action of $W^\kappa_{\text{aff}}$ is generated by reflections in the affine hyperplanes:

$$\mathcal{H}^\kappa_{\alpha, m} = \left\{ \lambda \in (t^\kappa)^* \mid \langle \tilde{\alpha}, \tilde{B}^\sharp_{k+h^\kappa_{(\kappa)}}(\lambda + \rho) \rangle = m \right\}, \quad \tilde{\alpha} \in \mathcal{R}_\kappa, m \in \mathbb{Z},$$

and that the elements $s_{\lambda}$ are in bijection with those of

$$(\Lambda^*\kappa)^k_{k+h^\kappa_{(\kappa)}} = (\Lambda^*\kappa)^k + \rho.$$

An element $\lambda \in (\Lambda^*\kappa)^\kappa$ has a trivial stabilizer under $W^\kappa_{\text{aff}}$ if and only if $\lambda \in W^\kappa_{\text{aff}} \bullet_k (\Lambda^*\kappa)^k_k$, and is otherwise lying on some shifted affine wall $\mathcal{H}^\kappa_{\alpha, m}$. Lastly, using the Jantzen character formula, we have for any $\lambda \in \Lambda_\kappa \in (\Lambda^*\kappa)^\kappa$ and $w \in W^\kappa_{\text{aff}}$:

$$\tilde{\chi}^\kappa_{w \bullet_k \lambda} = (-1)^{l(w)} \tilde{\chi}^\kappa_{\lambda}. $$
**Definition 4.11.** With the notations above, define the level $k$ twining fusion ring as the quotient:

$$
\tilde{R}_k^{(\kappa)}(G) := \tilde{R}^{(\kappa)}(G)/I_k^{(\kappa)}(G),
$$

where $I_k^{(\kappa)}(G)$ is the level $k$ fusion ideal:

$$
I_k^{(\kappa)}(G) := \left\{ f \in \tilde{R}^{(\kappa)}(G) \mid f(t) = 0, \text{ for all } t \in \pi^{-1}(s_\lambda), \lambda \in (\Lambda^*)_k^{\kappa} \right\},
$$

with $\pi : T^\kappa \to T^{(\kappa)}$ the projection.

The $\mathbb{Z}$-linear maps $(\cdot)^* : \tilde{R}^{(\kappa)}(G) \to \tilde{R}^{(\kappa)}(G)$ and $\text{Tr}_0 : \tilde{R}^{(\kappa)}(G) \to \mathbb{Z}$ descend to the ring $\tilde{R}_k^{(\kappa)}(G)$, which has $\{\tilde{x}_\mu^\kappa\}_{\mu \in (\Lambda^*)_k^{\kappa}}$ as a basis over $\mathbb{Z}$, and:

$$
\tilde{R}_k^{(\kappa)}(G) \simeq \mathbb{Z}[(\Lambda^*)_k^{\kappa}].
$$

The isomorphism $\tilde{R}^{(\kappa)}(G) \simeq R(G^{(\kappa)})$ induces an isomorphism of $\tilde{R}^{(\kappa)}(G)$-modules $I_k^{(\kappa)}(G) \simeq I_k(G^{(\kappa)})$, with $I_k(G^{(\kappa)}) \subseteq R(G^{(\kappa)})$ the level $k$ fusion ideal of $G^{(\kappa)}$. We thus obtain a complete characterization of $\tilde{R}_k^{(\kappa)}(G)$ for $k \in \mathbb{Z}_{>0}$.

**Proposition 4.12.** Let $G$ be compact, connected, simply connected and simple, $\kappa \in \text{Out}(G)$ a diagram automorphism and $G^{(\kappa)}$ the corresponding orbit Lie group. The twining fusion ring $\tilde{R}_k^{(\kappa)}(G)$ at level $k \in \mathbb{Z}_{>0}$ satisfies the following properties:

1. It is isomorphic to the fusion ring of $G^{(\kappa)}$ at level $k$:

$$
\tilde{R}_k^{(\kappa)}(G) \simeq R_k(G^{(\kappa)}).
$$

2. Under the canonical projection $\Phi : \tilde{R}^{(\kappa)}(G) \to \tilde{R}_k^{(\kappa)}(G)$, the images of basis elements $\{\tilde{x}_\mu^\kappa\}_{\mu \in (\Lambda^*)_k^{\kappa}}$ of $\tilde{R}^{(\kappa)}(G)$ are given by:

$$
\Phi(\tilde{x}_\mu^\kappa) = \begin{cases} 
(-1)^{l(w)}\tilde{x}_{w\cdot \mu}^\kappa, & \text{if } w \cdot \mu \in (\Lambda^*)_k^{\kappa} \text{ for } w \in W^{(\kappa)} \setminus \{1\}; \\
0, & \text{if } (W^{(\kappa)})_0 \neq \{1\}.
\end{cases}
$$

The elements $\{\Phi(\tilde{x}_\mu^\kappa)\}_{\mu \in (\Lambda^*)_k^{\kappa}}$ constitute the spectrum of $\tilde{R}_k^{(\kappa)}(G)$.

3. For any $\lambda, \mu \in (\Lambda^*)_k^{\kappa}$, the fusion product on $\tilde{R}_k^{(\kappa)}(G)$ is given by:

$$
\tilde{x}_\lambda^\kappa \cdot \tilde{x}_\mu^\kappa = \sum_{\nu \in (\Lambda^*)_k^{\kappa}} N_{\lambda \mu \nu \kappa}^{(\kappa)} \tilde{x}_\nu^\kappa,
$$

where the fusion coefficients $N_{\lambda \mu \nu \kappa}^{(\kappa)} = \text{Tr}_0(\tilde{x}_\lambda^\kappa \cdot \tilde{x}_\mu^\kappa \cdot \tilde{x}_\nu^\kappa)$ are given by:

$$
N_{\lambda \mu \nu \kappa}^{(\kappa)} = \frac{1}{|T^{(\kappa)}_k + h^{(\kappa)}_\kappa|} \sum_{\alpha \in (\Lambda^*)_k^{\kappa}} |\tilde{J}_0(s_\alpha)|^2 \tilde{x}_\lambda^\kappa(s_\alpha) \tilde{x}_\mu^\kappa(s_\alpha) \tilde{x}_\nu^\kappa(s_\alpha).
$$

**Outline of proof.** Statement (1) follows from the discussion preceding the proposition. The same remarks are used to modify the proofs of propositions 8.3 and 9.4 of [1] to obtain statement (2), and statement (3) is obtained by modifying the proof of lemma 9.7 in [1] (see also [21, Prop. A.8]).
4.4. **Concluding comments.** As mentioned in the introduction, our main motivation behind the study of twisted conjugation is its link with twisted quasi-Hamiltonian manifolds \[3,19,22\]. Such a space is a triple \((M,\omega,\Phi)\), where \(M\) is a \(G\)-manifold, the moment map \(\Phi : M \to G\kappa\) is equivariant, and \(\omega \in \Omega^2(M)^G\) is an invariant 2-form satisfying conditions analogous to those of the symplectic structure of a Hamiltonian \(G\)-manifold. To avoid certain technical complications related to the connectedness of the fibres of \(\Phi : M \to G\kappa\), one takes the group \(G\) to be simply connected, which is why we focused on this case in the present paper.

In [32], we study the Duistermaat-Heckman measure associated to a twisted quasi-Hamiltonian manifold \((M,\omega,\Phi)\), a certain invariant measure on \(G\kappa\) encoding the volumes of the symplectic quotients of \(M\). Under certain regularity conditions, such a measure essentially corresponds to an element of the twining ring \(R^{(\kappa)}(G)\). Our interest in the fusion rings \(B^{(\kappa)}_k(G)\) is related to the geometric quantization [21] of the spaces \((M,\omega,\Phi)\), and will be the subject of future work.

**REFERENCES**

1. Arnaud Beauville, *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 75–96. MR 1360497
2. Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004, Corrected reprint of the 1992 original. MR 2273508
3. Phillip Boalch and Daisuke Yamakawa, *Twisted wild character varieties*, arXiv:1512.08091 (2015), Preprint.
4. Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1985 French original by Andrew Pressley. MR 1890629
5. Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1995, Translated from the German manuscript, Corrected reprint of the 1985 translation. MR 1410059
6. Alan L. Carey and Bai-Ling Wang, *Fusion of symmetric D-branes and Verlinde rings*, Comm. Math. Phys. 277 (2008), no. 3, 577–625. MR 2365446
7. R. W. Carter, *Lie algebras of finite and affine type*, Cambridge Studies in Advanced Mathematics, vol. 96, Cambridge University Press, Cambridge, 2005. MR 2188930
8. Roger W. Carter, *Simple groups of Lie type*, John Wiley & Sons, London-New York-Sydney, 1972, Pure and Applied Mathematics, Vol. 28. MR 0401935
9. J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000. MR 1738431
10. Jürgen Fuchs, Bert Schellekens, and Christoph Schweigert, *From Dynkin diagram symmetries to fixed point structures*, Comm. Math. Phys. 180 (1996), no. 1, 39–97. MR 1403859
11. Jürgen Fuchs and Christoph Schweigert, *Symmetries, Lie algebras and representations*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1997, A graduate course for physicists. MR 1473220
12. Felix Gantmacher, *Canonical representation of automorphisms of a complex semi-simple Lie group*, Rec. Math. (Moscou) 5(47) (1939), 101–146. MR 0000998
13. Jiuzu Hong, *Conformal blocks, Verlinde formula and diagram automorphisms*, arXiv:1610.02975 (2017), Preprint.
14. ______, *Fusion ring revisited*, arXiv:1708.07919 (2017), Preprint, to appear in Contemporary Math.
15. Jiuzu Hong and Shrawan Kumar, *Conformal blocks for Galois covers of algebraic curves*, arXiv:1807.00118 (2018), Preprint.
16. Jiuzu Hong and Linhui Shen, *Tensor invariants, saturation problems, and Dynkin automorphisms*, Adv. Math. 285 (2015), 629–657. MR 3406511
17. Jens Carsten Jantzen, *Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen*, Bonn. Math. Schr. (1973), no. 67, v–1-24. MR 0401935
18. Victor G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990. MR 1104219
19. Friedrich Knop, *Multiplicity free quasi-Hamiltonian manifolds*, arXiv:1612.03843 (2016), Preprint.
20. Shrawan Kumar, George Lusztig, and Dipendra Prasad, *Characters of simplylaced nonconnected groups versus characters of nonsimplylaced connected groups*, Representation theory, Contemp. Math., vol. 478, Amer. Math. Soc., Providence, RI, 2009, pp. 99–101. MR 2513268
21. E. Meinrenken, *Twisted K-homology and group-valued moment maps*, Int. Math. Res. Not. IMRN (2012), no. 20, 4563–4618. MR 2989614
22. ______, *Convexity for twisted conjugation*, Math. Res. Lett. 24 (2017), no. 6, 1797–1818. MR 3762696
23. Stephan Mohrdieck, *Conjugacy classes of non-connected semisimple algebraic groups*, Thesis, http://ediss.sub.uni-hamburg.de/volltexte/2000/172/pdf/diss.pdf (2000).
24. ______, *Conjugacy classes of non-connected semisimple algebraic groups*, Transform. Groups 8 (2003), no. 4, 377–395. MR 2015256
25. Stephan Mohrdieck and Robert Wendt, *Integral conjugacy classes of compact Lie groups*, Manuscripta Math. 113 (2004), no. 4, 531–547. MR 2129875
26. ______, *Conjugacy classes in affine Kac-Moody groups and principal G-bundles over elliptic curves*, J. Algebra 322 (2009), no. 6, 1859–1876. MR 2542823
27. Graeme Segal, *The representation ring of a compact Lie group*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 113–128. MR 0248277
28. Jean de Siebenthal, *Sur les groupes de lie compacts non connectes*, Commentarii mathematici Helvetici 31 (1956/57), 41–89.
29. T. A. Springer, *Twisted conjugacy in simply connected groups*, Transform. Groups 11 (2006), no. 3, 539–545. MR 2264465
30. Robert Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968. MR 0230728
31. Robert Wendt, *Weyl’s character formula for non-connected Lie groups and orbital theory for twisted affine Lie algebras*, J. Funct. Anal. 180 (2001), no. 1, 31–65. MR 1814422
32. Ahmed J. Zerouali, *Twisted Moduli Spaces and Duistermaat-Heckman Measures*, (2018), In preparation.