A stronger no-cloning theorem

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Abstract
It is well known that (non-orthogonal) pure states cannot be cloned so one may ask: how much or what kind of additional (quantum) information is needed to supplement one copy of a quantum state in order to be able to produce two copies of that state by a physical operation? For classical information, no supplementary information is required. However for pure quantum (non-orthogonal) states, we show that the supplementary information must always be as large as it can possibly be i.e. the clone must be able to be generated from the additional information alone, independently of the first (given) copy.

1 Introduction
It is well known that (non-orthogonal) pure states cannot be cloned [1] i.e. if \( \{ |\psi_i \rangle \} \) is a set of pure states containing at least one non-orthogonal pair then no physical operation can achieve the transformation \( |\psi_i \rangle \rightarrow |\psi_i \rangle |\psi_i \rangle \). Evidently two copies of a quantum state contain strictly more “information” about the state than is available from just one copy so in view of the impossibility of cloning one may ask: what additional information is needed to supplement one copy \( |\psi_i \rangle \) in order to be able to produce two copies \( |\psi_i \rangle |\psi_i \rangle \)? For classical information, no supplementary information at all is needed and one might guess that as the set \( \{ |\psi_i \rangle \} \) becomes “more classical” the necessary supplementary information should decrease in some suitable way. However we prove below that this is not the case: we show (for mutually non-orthogonal states) that the supplementary information must always be as large as it can possibly be i.e. the second copy \( |\psi_i \rangle \) can always be generated from the supplementary information alone, independently of the first (given) copy. Thus in effect, cloning of \( |\psi_i \rangle \) is possible only if the second copy is fully provided as an additional input.

Using our techniques we will also give simple proof of the Pati-Braunstein no-deleting principle [2] (for the mutually non-orthogonal states) and discuss its relation to cloning.

2 No cloning
We now give a precise formulation of our main result. By a physical operation we will mean a trace preserving completely positive map. Note that this definition excludes the collapse of wavefunction in a quantum measurement, as a valid physical process. (This will be relevant to our later discussion of the no-deleting principle.)

**Theorem 1** Let \( \{ |\psi_i \rangle \} \) be any finite set of pure states containing no orthogonal pairs of states. Let \( \{ \rho_i \} \) be any other set of (generally mixed) states indexed by the same labels. Then there is a physical operation

\[
|\psi_i \rangle \otimes \rho_i \rightarrow |\psi_i \rangle |\psi_i \rangle
\]
if and only if there is a physical operation

$$\rho_i \rightarrow |\psi_i\rangle$$

i.e. the full information of the clone must already be provided in the ancilla state $\rho_i$ alone.

**Remark**[3]. If the set $\{|\psi_i\rangle\}$ contains some orthogonal pairs then the unassisted clonability of orthogonal states spoils the simplicity of the statement of theorem 1. As an example consider

\[
|\psi_1\rangle = |0\rangle \quad |\alpha_1\rangle = |a\rangle
\]
\[
|\psi_2\rangle = |1\rangle \quad |\alpha_2\rangle = |a\rangle
\]
\[
|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |\alpha_3\rangle = |b\rangle
\]

where $|a\rangle$ and $|b\rangle$ are orthogonal. Then clearly $|\psi_i\rangle \alpha_i \rightarrow |\psi_i\rangle |\psi_i\rangle$ is possible (as $\{|\psi_i\rangle |\alpha_i\rangle\}$ is an orthonormal set) but $|\alpha_i\rangle \rightarrow |\psi_i\rangle$ is not possible (as $|\alpha_1\rangle = |\alpha_2\rangle$ but $|\psi_1\rangle \neq |\psi_2\rangle$).

Indeed the $|\alpha_i\rangle$ states here provide reliable distinguishability of $i$ values in the case that this is not already provided by the $|\psi_i\rangle$’s themselves. QED.

To prove the theorem we will use the following lemma which is proved as lemma 1 of [3].

**Lemma 1** Let $\{|\alpha_i\rangle\}$ and $\{|\beta_i\rangle\}$ be two sets of pure states (indexed by the same labels). Then the two sets have equal matrices of inner products (i.e. $\langle \alpha_i | \alpha_j \rangle = \langle \beta_i | \beta_j \rangle$ for all $i$ and $j$) if and only if the two sets are unitarily equivalent (i.e. there exists a unitary operation $U$ on the direct sum of the state spaces of the two sets with $U |\alpha_i\rangle = |\beta_i\rangle$ for all $i$).

**Proof of theorem**[3] Suppose that there is a physical operation $\rho_i \rightarrow |\psi_i\rangle$. Then clearly $|\psi_i\rangle \otimes \rho_i \rightarrow |\psi_i\rangle |\psi_i\rangle$ is allowed.

Conversely suppose that there is a physical operation

$$|\psi_i\rangle \otimes \rho_i \rightarrow |\psi_i\rangle |\psi_i\rangle.$$  \hspace{1cm} (1)

Consider first the case that $\rho_i$ are pure states, $|\alpha_i\rangle$ say. The physical operation eq. (1) may be expressed as a unitary operation if we include an environment space, initially in a fixed state denoted $|A\rangle$. For clarity we include an extra register, initially in a fixed state $|0\rangle$, that is to receive the clone of $|\psi_i\rangle$. Then eq. (1) may be written as a unitary transformation

$$|\psi_i\rangle |0\rangle |\alpha_i\rangle |A\rangle \rightarrow |\psi_i\rangle |\psi_i\rangle |C_i\rangle$$

where $|C_i\rangle$ (generally depending on $i$) is the output state of the two registers that initially contained $|\alpha_i\rangle |A\rangle$. Hence by unitarity, the two sets $\{|\psi_i\rangle |\alpha_i\rangle\}$ and $\{|\psi_i\rangle |\psi_i\rangle |C_i\rangle\}$ have equal matrices of inner products and then, so do the sets $\{|\alpha_i\rangle\}$ and $\{|\psi_i\rangle |C_i\rangle\}$ (by a simple cancellation of $\langle \psi_i | \psi_j \rangle$ from the two initial matrices). Thus by lemma 1 these two sets are unitarily equivalent so $|\psi_i\rangle$ can be generated from $|\alpha_i\rangle$ alone (by applying the unitary equivalence and discarding the $|C_i\rangle$ register).

If $\rho_i$ are mixed we express them as probabilistic mixtures of pure states

$$\rho_i = \sum_k p_k^{(i)} |\alpha_k^{(i)}\rangle \langle \alpha_k^{(i)}|.$$
Then a physical operation achieves

$$|\psi_i\rangle \otimes \rho_i \rightarrow |\psi_i\rangle |\psi_i\rangle$$

for all $i$ iff it achieves

$$|\psi_i\rangle \otimes |\alpha_k^{(i)}\rangle \rightarrow |\psi_i\rangle |\psi_i\rangle$$

for all $i$ and $k$. (2)

By the pure state analysis above, a physical operation effecting eq. (2) exists iff there is a physical operation effecting

$$|\alpha_k^{(i)}\rangle \rightarrow |\psi_i\rangle$$

for all $i$ and $k$.

and then we get $\rho_i \rightarrow |\psi_i\rangle$ too. QED.

In the particular case of cloning assisted by classical information (i.e. the states $\rho_i$ are required to be mutually commuting) we deduce that this supplementary data must contain the full identity of the states as classical information (rather than just quantum information). Indeed if the $\rho_i$ are classical then they can be copied any number of times so if we can make one clone of $|\psi_i\rangle$ from $\rho_i$, we can make arbitrarily many clones and hence determine the the identity of $|\psi_i\rangle$ i.e. the classical information of the label $i$ must be contained in the supplementary classical information.

### 3 No deleting

Our techniques may also be used to give a simple proof of the Pati-Braunstein no-deleting principle [4] for sets $\{|\psi_i\rangle\}$ that contain no orthogonal pairs. The issue here is the following. Suppose we have two copies $|\psi_i\rangle |\psi_i\rangle$ of a state and we wish to delete one copy by a physical operation:

$$|\psi_i\rangle |\psi_i\rangle \rightarrow |\psi_i\rangle |0\rangle$$

(where $|0\rangle$ is any fixed state of the second register). As before, any such physical operation may be expressed as a unitary operation if we include an environment space, initially in a fixed state $|A\rangle$ say. Then eq. (3) is equivalent to the unitary transformation

$$|\psi_i\rangle |\psi_i\rangle |A\rangle \rightarrow |\psi_i\rangle |0\rangle |A_i\rangle$$

(4)

where the final state $|A_i\rangle$ of the environment may depend on $|\psi_i\rangle$ in general. One way of achieving this is to simply swap (a constant) part of the environment into the second register but then the second copy of $|\psi_i\rangle$ remains in existence (albeit in the environment now). The no-deleting principle states that the second copy of $|\psi_i\rangle$ can never be “deleted” in the sense that $|\psi_i\rangle$ can always be resurrected from $|A_i\rangle$. Note however, that if wavefunction collapse is also allowed as a valid physical process then deletion is possible. (We perform a complete measurement on $|\psi_i\rangle$ and rotate the seen post-measurement state to $|0\rangle$ by a unitary transformation that depends on the measurement outcome.)

To see the no-deleting principle with our methods, note that the unitarity of eq. (4) implies that the sets $\{|\psi_i\rangle |\psi_i\rangle |A\rangle\}$ and $\{|\psi_i\rangle |0\rangle |A_i\rangle\}$ have equal matrices of inner products and then as before, so do the sets $\{|\psi_i\rangle\}$ and $\{|A_i\rangle\}$. Thus lemma 1 states that these sets are unitarily equivalent, which is just the no-deleting principle.
4 Further remarks

Deleting and cloning have a common feature: in cloning we saw that the existence of the first copy $|\psi_i\rangle$ provided no assistance in constructing the second copy from the supplementary information. Similarly for deletion, the existence of the first copy provides no assistance in deleting the second copy – in effect the only way to delete the second copy is to transform it out into the environment (i.e. $|0\rangle |A_i\rangle$ in eq. (4) is a unitary transform of $|\psi_i\rangle |A\rangle$ alone) and this again makes no use of the first copy. Considering no-cloning and no-deleting together (and excluding wavefunction collapse as a valid physical process) we see that quantum information (of non-orthogonal states) has a quality of “permanence”: creation of copies can only be achieved by importing the information from some other part of the world where it had already existed; destruction (deletion of a copy) can only be achieved by exporting the information out to some other part of the world where it must continue to exist. This property is different from the preservation of information by any reversible dynamics. For example the classical reversible C-NOT operation can imprint copies of a bit $b$ into a standard state via $|b\rangle |0\rangle \rightarrow |b\rangle |b\rangle$ and also delete copies via $|b\rangle |b\rangle \rightarrow |b\rangle |0\rangle$ but in both cases the first copy is used in an essential way in the process and the information content of one copy is the same as that of two copies. In contrast, in the quantum (non-orthogonal) case, copying and deleting can only occur independently of the first copy and then reversibility of dynamics implies that the information of the second copy must have already separately existed in the environment (for cloning) or continue to exist separately in the environment (for deletion). But in any reasonable intuitive sense, $|\psi_i\rangle |\psi_i\rangle$ does not have double the information content of $|\psi_i\rangle$ and one might interpret this as an overlap of information content of the two copies. Then curiously, this common part cannot be taken from a single copy and merely extended, to give the second copy.

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References

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[2] Thanks to H. R. Thomann and A. Winter for pointing out an error in an earlier version of theorem 1.

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[4] Pati, A. and Braunstein, S. Nature 404 164-165.