Low-frequency electromagnetic field in a Wigner crystal

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Long-wave low-frequency oscillations are described in a Wigner crystal by generalization of the reverse continuum model for the case of electronic lattice. The internal self-consistent long-wave electromagnetic field is used to describe the collective motions in the system. The eigenvectors and eigenvalues of the obtained system of equations are derived. The velocities of longitudinal and transversal sound waves are found.

Keywords:
Wigner crystal; self-consistent electromagnetic field; jellium model; long wave electromagnetic field; velocity of sound.

1 Introduction

The ordered two- and three-dimensional structures of charged particle systems are investigated in numerous recent studies [1–3], in particular, Wigner crystallization in a quasi-3D electronic system [4] and the holes in semiconductors [5] are also actively studied. In the paper [6] the authors dealt with a two-dimensional Wigner crystal classically as with elastic medium. In this case the elastic restoring force in the equation of motion is

$$\int D_{\alpha\beta}(r - r')u_{2\beta}(r', t) d^2r'$$

and the external driving force on the total charge density is given by

$$-n_e e E,$$

where $D_{\alpha\beta}(r)$ is the real-space dynamic matrix tensor, $u_{2}(r, t)$ is the two-dimensional displacement of the Wigner crystal, $n_e$ is its density, $E$ is the external electric field.

In this paper we extend the idea presented in [6] to a three-dimensional boundless Wigner crystal [7, 8]. We consider a low-frequency long-wave electromagnetic field in the Wigner crystal. A usual definition of an acoustic wave is the following: an acoustic wave is a joint collective motion of the valence electrons and ions of lattice in a self-consistent electromagnetic field [9, p. 345]. However, we have electron lattice and assume that ions are free as in the jellium model. We consider a mean electromagnetic field in an acoustic wave and ignore a dissipation.

2 Low-frequency waves in a Wigner crystal

In the case of long acoustic waves, it is possible to rewrite the elastic restoring force in the form

$$C_{\alpha\beta\chi\delta} \frac{\partial^2 u_{\delta}}{\partial x_{\beta} \partial x_{\chi}}$$

[10, p. 152]. Here $C_{\alpha\beta\chi\delta}$ is the tensor of the elastic modules of an electron subsystem. To simplify consideration, we assume that the Wigner crystal is isotropic one. Then using Lame
parameters it is possible to write down $C_{\alpha\beta\chi\delta} = \lambda\delta_{\alpha\beta}\delta_{\chi\delta} + \mu (\delta_{\alpha\delta}\delta_{\beta\chi} + \delta_{\alpha\chi}\delta_{\beta\delta})$ \[11\]. The similar consideration was given in \[12\] for metals. It is known that in the case of an ideal crystal without defects of type of vacancies or interstitials, that is exactly the case we investigate, velocity of environment points coincides with the derivative of their displacement with respect to time $\mathbf{v}_n = \partial \mathbf{u}/\partial t$ \[13\]. The electrons oscillate around their equilibrium positions in the lattice. Short-acting forces, correlating separate oscillations, act between them. In the study of long waves it is possible to unite equation of motion of elastic subsystem, where short-acting interelectronic potential $V(u_{\alpha\alpha})$ appears \[10\], that gives the elastic modules, with the long-wave self-consistent electric field. In the absence of external magnetic field the relativistic term of the Lorentz force can be omitted. Therefore, an equation of motion for electronic component can be written in the form

$$
\rho_e \frac{d^2 u_\alpha}{dt^2} = (\lambda + \mu) \frac{\partial^2 u_\alpha}{\partial x_\alpha \partial x_\chi} + \mu \frac{\partial^2 u_\alpha}{\partial x_\chi \partial x_\chi} - e_n E_\alpha. \tag{1}
$$

A self-consistent electric field satisfies the Maxwell equations with a hydrodynamic approximation for the current components $j_i = e_a n_{a\alpha} \mathbf{v}_a$, where $e_a$ is the corresponding charge \[14\]. For simplicity all ions are assumed to be identical and have valence $Z$. We don’t consider the piezoelectric or the magnetic matters. We ignore thermal motion of the ions, and using the jellium model we have the the equation of motion for an ionic component

$$
\rho_i \frac{dv_{i\alpha}}{dt} = Z e_n E_\alpha. \tag{2}
$$

It means that the Wigner crystal is an elastic electron environment that contains “raisins”-ions. Equations \(1\), \(2\) and the Maxwell equations for a self-consistent electromagnetic field will allow us to unite the consideration of solid and collective effects.

We consider the adiabatic sound waves of small amplitude starting from the obtained system of equations. For this purpose we linearize equations of the system near the equilibrium state, where all variables, namely, field strengths and component velocities, are equal to zero. Then the first Maxwell equation \[14\] takes the form

$$
\hat{\mathbf{E}} = c \nabla \times \mathbf{B} - 4\pi (Z e_n \delta_0 \mathbf{v}_i - e n_{e0} \mathbf{v}_e), \tag{3}
$$

where $n_{a\delta}$ is the equilibrium value of density of the proper particles. The Faraday equation is linear and, therefore, remains the same

$$
\hat{\mathbf{B}} = -c \nabla \times \mathbf{E}. \tag{4}
$$

In this system it is convenient to pass to the Fourier-components by the following rule

$$
A(\mathbf{x}, t) = \int d^3k A(\mathbf{k}, t) e^{i \mathbf{k} \cdot \mathbf{x}}/(2\pi)^3. \tag{5}
$$

Then we obtain the system of five linear homogeneous equations

$$
\hat{\mathbf{E}} = ic[\mathbf{k}, \mathbf{B}] - 4\pi (Z e_n \delta_0 \mathbf{v}_i - e n_{e0} \mathbf{v}_e), \quad \hat{\mathbf{B}} = -ic[\mathbf{k}, \mathbf{E}], \quad \dot{\mathbf{u}} = \mathbf{v}_e,
\rho_\delta \dot{\mathbf{v}}_i = Z e_n \delta_0 \mathbf{E}, \quad \rho_{e0} \dot{\mathbf{v}}_{e\alpha} = - (\lambda + \mu) k_\alpha (\mathbf{k} \mathbf{u}) - \mu u_{\alpha} k^2 - e n_{e0} E_\alpha. \tag{6}
$$

It is observed that the system for potential and vortical oscillations is divided into two subsystems. We project all variables on the wave vector $\mathbf{k}$ and introduce the notations for projections using the rule $\mathbf{E}k/k = E^\parallel$. Then the system takes the form

$$
\hat{E}^\parallel = -4\pi (Z e_n \delta_0 v^\parallel_i - e n_{e0} v^\parallel_e), \quad \dot{B}^\parallel = 0,
\dot{u}^\parallel = v^\parallel, \quad \rho_\delta \dot{v}^\parallel_i = Z e_n \delta_0 E^\parallel, \quad \rho_{e0} \dot{v}^\parallel_e = - (\lambda + 2\mu) u^\parallel k^2 - e n_{e0} E^\parallel. \tag{7}
$$
It is obvious that the magnetic field does not influence on the potential motion. It is convenient to introduce the plasma frequency of corresponding particles as \( \Omega_a = \sqrt{4\pi (e_an_{a0})^2/\rho_{a0}} \). We will choose new variables so that the resulting differential equation has no dimensions using the rules

\[
\left( E^\parallel, u^\parallel, v_e^\parallel, v_i^\parallel \right) \leftrightarrow \left( \frac{eE^\parallel}{ms^2k}, \frac{u^\parallel k}{s}, \frac{v_e^\parallel M}{smZ} \right), \quad t \leftrightarrow tks,
\]

where \( s^2 = \frac{(\lambda+2\mu)}{\rho_{a0}} \). In the dimensionless variables the latter system takes the form

\[
\dot{E}^\parallel = \Omega_e^2 v_e^\parallel - \Omega_i^2 v_i^\parallel, \quad \dot{u}^\parallel = v_e^\parallel, \quad \dot{v}_e^\parallel = \Omega_i^2 v_i^\parallel, \quad \dot{v}_i^\parallel = -u^\parallel - E^\parallel.
\]

In general, the system (8) gives a biquadratic characteristic equation, but we have to take into account the smallness of frequency of sound oscillations. As we study long acoustic waves (with a condition \( \Omega_e \gg ks \)), the solution with frequency \( \omega \approx \Omega_e \) (here we have taken into account, that \( \Omega_i \ll \Omega_e \)), that corresponds to high-frequency plasma waves, is not interesting. The eigenvectors show that electric field, displacement and both velocities perform coupled oscillations. The eigenvalues \( \Lambda = i\omega \), corresponding to low-frequency branch of oscillations, are given by

\[
\Lambda = \pm \frac{i}{2} \sqrt{\Omega_e^2 + \Omega_i^2 + 1 - \sqrt{(\Omega_e^2 + \Omega_i^2 + 1)^2 - 4\Omega_i^2}},
\]

which in limit \( \Omega_e \gg ks \) gives the solution

\[
\Lambda \approx \pm i\Omega_i/\Omega_e.
\]

Now we return to the dimensional variables and rewrite the frequency of sound with a dispersion as

\[
\omega^2 = \frac{(\lambda + 2\mu)}{\rho_{a0}} k^2.
\]

The velocity of longitudinal sound

\[
u_s^\parallel = \sqrt{\frac{(\lambda + 2\mu)}{\rho_{a0}}}
\]

is determined by mass of ions, in analogy to the ion sound in a two-temperature plasma [14].

Further, we consider the transversal oscillations. For this purpose we project equations (6) on \( (\delta_{\alpha\beta} k^2 - k_{\alpha}k_{\beta}) \), introducing notations by the rule \( (\delta_{\alpha\beta} k^2 - k_{\alpha}k_{\beta})E_\beta = E_\parallel \). It is convenient to pass from induction of the magnetic field to the new variable \( \mathbf{B}^\parallel \rightarrow [\mathbf{\frac{s}{k}}, \mathbf{B}^\perp] = \mathbf{Z} [15] \). In analogy with the longitudinal subsystem we pass to dimensionless variables using the rules

\[
(E^\perp, Z, u^\perp, v_e^\perp, v_i^\perp) \leftrightarrow \left( \frac{eE^\perp}{ms^2k}, \frac{eZ}{ms^2k}, \frac{u^\perp k}{s}, \frac{v_e^\perp M}{smZ} \right), \quad t \leftrightarrow tks, \quad c \leftrightarrow c/s,
\]

where \( s^2 = \frac{\mu}{\rho_{a0}} \). In the dimensionless variables we have the system

\[
\dot{E}^\perp = icZ + \Omega_e^2 v_e^\perp - \Omega_i^2 v_i^\perp, \quad \dot{Z} = iE^\perp,
\]

\[
\dot{u}^\perp = v_e^\perp, \quad \dot{v}_e^\perp = E^\perp, \quad \dot{v}_i^\perp = -u^\perp - E^\perp.
\]

(13)
The eigenvectors show that all physical variables perform coupled oscillations. The eigenvalues that correspond to low-frequency branch of oscillations are

$$\Lambda = \pm \frac{i}{2} \sqrt{\Omega_e^2 + \Omega_i^2 + 1 + c^2 - \sqrt{(\Omega_e^2 + \Omega_i^2 + 1 + c^2)^2 - 4 (\Omega_i^2 + c^2)}}.$$  \hspace{1cm} (14)

In the limit $\Omega_e \gg kc$ for the condition $\Omega_i \gg kc$ this gives the transversal sound

$$\Lambda \approx \pm i \Omega_i / \Omega_e.$$  \hspace{1cm} (15)

However, in the area $\Omega_i \ll kc \ll \Omega_e$ there is the quadratic dispersion $\Lambda \approx \pm ic / \Omega_e$.

Returning to the dimensional variables we obtain low-frequency mode

$$\omega^2 = \frac{\mu}{\rho_0} k^2 + c^2 \frac{\mu}{\rho_0 \Omega_e^2} k^4.$$  \hspace{1cm} (16)

Equation (16) demonstrates non-linearity for middle wavelengths (see Appendix). Expressions (14) and (16) approach to the sound, in supposition $\Omega_i \gg kc$, with dispersion

$$\omega^2 = \frac{\mu}{\rho_0} k^2.$$  \hspace{1cm} (17)

Therefore, the velocity of transversal sound is

$$u_s^\perp = \sqrt{\frac{\mu}{\rho_0}}.$$  \hspace{1cm} (18)

It is easy to see that the velocities satisfy the well-known requirement $u^\parallel / u_s^\perp = \sqrt{2 + \lambda / \mu} > \sqrt{2}$ \cite[13, p. 125]{13}. \hspace{1cm}

Now, starting from systems (8) and (13), it is easy to show that expressing velocities of subsystems of charges from equations of motion through the fields and substituting them in the corresponding wave equation obtained from the Maxwell equations, the sound oscillations reduce to oscillations of the electric field in the crystal. The quanta of the obtained sound oscillations are the quanta of the electromagnetic field in the crystal. That gives correct statistics for phonons, as bosons with zero chemical potential. Therefore, it is possible in standard way to introduce operators of annihilation and creation of phonons as quanta of the electromagnetic field after the decomposition of the vector potential of electromagnetic field on plane waves $\hat{A}_n (x) = c \sum_{k,\alpha} \left( \frac{2 \pi \hbar}{\omega_V} \right)^{1/2} (c_{k\alpha} + c_{-k,\alpha}^+) e_{k\alpha} e^{ikx}$ \cite[16]{16}, where the index $\alpha$ denotes longitudinal and two transversal polarizations.

### 3 Conclusion

Using the jellium (continuum) model of a solid by introducing an elasticity of electronic subsystem for a Wigner crystal, we have found the longitudinal and two transversal low-frequency oscillation branches. We have shown that these sound oscillations in a Wigner crystal can be considered as the coupled waves of the electromagnetic field and charges which it is possible to study as waves of the electromagnetic field in an environment. It gives phonons with necessary statistics after the standard quantization of the field.
Appendix

For numerical estimation we take the average inter-particle spacing of order $a \sim 100a_B$ [17], where $a_B \approx 5.29 \cdot 10^{-9}$ cm is the Bohr radius. Then an electronic plasma frequency is $\Omega_e \sim 1.5 \cdot 10^{14}$ s$^{-1}$ and let an ionic one be $\Omega_i \sim \Omega_e/100$. We estimate shear modulus using Coulomb interaction as $\mu \sim e^2/a^4 \sim 3 \cdot 10^6$ g/(cm$ \cdot $s$^2$) and from [18] obtain $u_s^\perp \sim 2 \cdot 10^5$ cm/s. In this supposition we get the numerical estimation for low-frequency mode (16)

$$\omega = 2\sqrt{10^{10}k^2 + 4 \cdot 10^6k^4}$$

and plot this dispersion dependence.

![Dispersion curve](image)

Figure 1: Low-frequency transversal mode.

Figure 1 shows the transition from linear sound dispersion for small wavevectors to nonlinear dispersion for middle ones.

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References

[1] A. Radzvilavicius, E. Anisimovas, J. Phys.: Condens. Matter. 23, No. 7, 075302 (2011).

[2] S. Pankov, V. Dobrosavljevic, Physica B. 403, 1440 (2008).

[3] D.A. Baiko, D.G. Yakovlev, H.E. De Witt, W.L. Slattery, Phys. Rev. E 61, 1912 (2000).

[4] B.A. Piot, Z. Jiang, C.R. Dean, L.W. Engel, G. Gervais, L.N. Pfeiffer, K.W. West, Nature Physics 4, 936 (2008).

[5] M. Bonitz, V.S. Filinov, V.E. Fortov, P.R. Levashov, H. Fehske, Phys. Rev. Lett. 95, 235006 (2005).

[6] Zhu Xuejun, P.B. Littlewood, A.J. Millis, Phys. Rev. B. 50, 4600 (1994).
[7] E.P. Wigner, Phys. Rev. 46, 1002 (1934).
[8] D. Pines, Elementary Excitations in Solids, Mir, Moscow, 1965 (in Russian).
[9] Physical Acoustics, W. Mason (Eds.). Vol. 4, Part A, Mir, Moscow, 1969 (in Russian).
[10] O. Madelung, Theory of Solid, Nauka, Moscow, 1980 (in Russian).
[11] K. Feng, Z.-C. Shi, Mathematical Theory of Elastic Structures, Springer, New York, 1981.
[12] A.A. Stupka, Metallofizika i Noveishie Tekhnologii 34, 605 (2012) (in Russian).
[13] L.D. Landau, E.M. Lifshitz, Elasticity Theory, Nauka, Moscow, 1987 (in Russian).
[14] Electrodynamics of Plasma, A.I. Akhiezer (Eds.), Nauka, Moscow, 1974 (in Russian).
[15] A. Sokolovsky, A. Stupka, Journal of Physical Studies. 10, No. 1, 12 (2006) (in Ukrainian).
[16] A.I. Sokolovsky, A.A. Stupka, Z.Yu. Chelbaevsky, Ukrainian Journal of Physics 55, 20 (2010).
[17] N. Drummond, Z. Radnai, J. Trail, M. Towler, R. Needs, Phys. Rev. B. 69, 085116 (2004).