On the ubiquity of the Lévy integral, its relationship with the generalised 
Euler-Jacobi series, and their asymptotics beyond all orders.

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We present here an overview of the history, applications and important properties of a function 
which we refer to as the Lévy integral. For certain values of its characteristic parameter the Lévy 
integral defines the symmetric Lévy stable probability density function. As we discuss however the 
Lévy integral has applications to a number of other fields besides probability, including random 
matrix theory, number theory and asymptotics beyond all orders. We exhibit a direct relation-
ship between the Lévy integral and a number theoretic series which we refer to as the generalised 
Euler-Jacobi series. The complete asymptotic expansions for all natural values of its parameter are 
presented, and in particular it is pointed out that the intricate exponentially small series become 
dominant for certain parameter values.

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Dedicated to the memory of Elliott Montroll

I. INTRODUCTION

The objective of this article is to pay homage to an integral which has occupied the minds of a host of great 
mathematicians including Cauchy, Lévy, Hardy, Littlewood, Ramanujan, and Pólya and which continues to arise in 
modern contexts such asymptotics beyond all orders and random matrix theory. This function, \( F_\alpha(z) \), which we refer 
to as the Lévy integral is defined via

\[
F_\alpha(z) = \int_0^\infty \exp(-t^\alpha) \cos(zt) \, dt \quad \alpha > 0.
\]

For \( \alpha < 1 \) the integral is defined only for real \( z \).

The number of seemingly disparate topics in which \( F_\alpha(z) \) appears in a fundamental way is remarkable. The reader 
familiar with the theory of probability will immediately recognise the function \((1/\pi)F_\alpha(z)\) as defining the symmetric 
Lévy stable probability density function when \( \alpha \leq 2 \). For \( \alpha = 1, 2 \) the function \((1/\pi)F_\alpha(z)\) is identified respectively 
with the Cauchy and Gaussian probability densities. However, \( F_\alpha(z) \) also has an intimate connection to Waring’s 
problem in number theory, is of great interest in asymptotics beyond all orders, and appears in certain problems 
arising in random matrix theory and wave phenomena.

The analytic properties of \( F_\alpha(z) \) are of long standing interest. Bernstein in 1919 investigated the zeros of \( F_\alpha(z) \), 
showing that it possesses an infinite number of real zeros when \( \alpha = 4, 6, 8, \ldots \). He appears to have been the first to 
realise that \( F_\alpha(z) \) might become negative for some values of \( z \) when \( \alpha > 2 \). In such cases \((1/\pi)F_\alpha(z)\) does not define 
a probability density function. In 1923 Pólya extended the work of Bernstein by systematically investigating 
the zeros of \( F_\alpha(z) \) and showed that when \( \alpha = 4, 6, 8, \ldots \), \( F_\alpha(z) \) has an infinite number of real zeros but no complex 
zeros, whereas if \( \alpha > 1 \) is not even integer then \( F_\alpha(z) \) has an infinite number of complex zeros and a finite number, 
not less than \( 2[\alpha/2] \), of real zeros. Lévy in 1923 proved the non-negativity of \( F_\alpha(z) \) for all real \( z \) when \( 0 < \alpha \leq 2 \), 
and was responsible for the development of the theory of stable distributions, and so we refer to the function 
\( F_\alpha(z) \) as the Lévy integral for all positive \( \alpha \) regardless of whether it defines a probability density function or not. It 
has been observed that for all positive rational \( \alpha \) the function \( F_\alpha(z) \) can be expressed as a sum of generalised 
hypergeometric functions, and in fact \( F_\alpha(z) \) is itself a specific type of Fox H-function. There are however only a 
handful of specific values of \( \alpha \) for which an identification with more familiar special functions is known.

The present authors have recently constructed the complete asymptotic expansions of \( F_\alpha(z) \) for all natural \( \alpha \), 
and in particular have demonstrated that there exist very intricate series of exponentially small terms lying beyond all

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orders of the asymptotic power series. Interestingly, when \( \alpha \) is an even integer the asymptotic power series vanishes and these exponentially small series are liberated from their subdominance. This provides a powerful case study for the newly emerging field of asymptotics beyond all orders\[1\], the discussion of which we shall return to after discussing how \( F_\alpha(z) \) arises in the fields of probability, random matrix theory, diffraction integrals, and number theory.

II. LÉVY STABLE PROBABILITY DENSITY FUNCTIONS

The theory of stable distributions is a branch of probability theory of fundamental importance. At a time when it was fashionable to find the most general conditions possible in order for the central limit theorem to be satisfied, Paul Lévy was interested in finding elegant examples of when it is not. Indeed Lévy’s stable probability density functions have divergent moments for \( \alpha \neq 2 \), and the definition of stability is manifestly one of self similarity rather than universality. Lévy defines a probability density function \( p(x) \) to be stable if it satisfies the following property \[2\] \[3\] \[4\]. Let \( X_1 \) and \( X_2 \) be independent random variables with the same probability density function \( p(x) \), then

\[
c_1X_1 + c_2X_2 = cX
\]

for all positive \( c_1 \) and \( c_2 \), where \( X \) has the same probability density function as \( X_1 \) and \( X_2 \). The constant \( c \) is determined by the constants \( c_1 \) and \( c_2 \). The symmetric solutions of this condition have characteristic functions of the form \( \exp(-|q|^\alpha) \) \[5\], which is the Fourier transform of \( (1/\pi)F_\alpha(z) \).

The function \( F_\alpha(z) \) was studied by Cauchy in the early 1850s \[6\] in relation to the theory of errors, and it is evident that he was aware that these functions satisfy Lévy stability. Cauchy appears unaware however that \( F_\alpha(z) \) may become negative for some values of \( z \) when \( \alpha > 2 \) and so fail the non negativity requirement of a probability density function.

The divergence of the moments of \( F_\alpha(z) \) for \( \alpha \) not an even natural can immediately be seen from the non-analyticity of \( \exp(-|q|^\alpha) \), since if the characteristic function of a probability density function is analytic then it possesses a power series whose coefficients are proportional to the moments of that probability density function. We can summarise\[3\] the behaviour of the moments of \( (1/\pi)F_\alpha(z) \) as a function of \( \alpha \) as

\[
<z^{2m}> = \begin{cases} 
\frac{(-1)^{m+j/2(2m)!}}{j}, & \alpha = \frac{2m}{j} \in \{2, 4, 6, \ldots\} \\
+\infty, & \alpha < 2m \text{ and } \alpha \in \cup_{k=0}^{\infty}(4k, 4k + 2) \\
-\infty, & \alpha < 2m \text{ and } \alpha \in \cup_{k=0}^{\infty}(4k + 2, 4k + 4) \\
0, & \text{otherwise.}
\end{cases}
\]

The stable probability density functions have been experiencing a tsunami of popularity in a wide variety of applications in the past decade, from physics to biology to economics \[14\]. The reason for their utility stemming precisely from the divergence of the moments for \( \alpha \neq 2 \). This implies that for \( \alpha < 2 \) no characteristic scale exists and random walks constructed using \( (1/\pi)F_\alpha(z) \) as the step length probability density function display statistically self similar trajectories, and as stressed by Mandelbrot the fractal dimension of these trajectories is \( \alpha \) \[13\] \[16\] \[17\]. Thus \( F_\alpha(z) \) plays a fundamental role in the increasingly popular pursuit of modeling phenomena which display structure on many scales.

III. RANDOM MATRIX THEORY AND THE CUSP DIFFRACTION CATASTROPHE

It has been noted \[1\] that \( F_4(y) \) is proportional to the special case of the Pearcey integral \( P(x, y) \) in which its first argument is set to zero. Specifically,

\[
P(0, y) = 2e^{i\pi/8}F_4(y).
\]

The Pearcey integral is the canonical form of the oscillatory integral describing the cusp diffraction catastrophe \[18\] \[19\] \[20\] and is defined by

\[
P(X, Y) = \int_{-\infty}^{\infty} \exp[i(u^4 + Xu^2 + Yu)] \, du,
\]

which can be expressed in an alternative more general form via analytic continuation

\[
P(x, y) = 2e^{i\pi/8}\int_{0}^{\infty} \exp(-t^4 - xt^2) \cos(yt) \, dt.
\]
This integral was investigated by Pearcey in his investigation of the electromagnetic field near a cusp \[24\] and is utilised in many short wavelength problems such as wave propagation and optical diffraction \[18\] \[19\] \[24\]. The function \(P(x, y)\) was known before Pearcey however, and was discussed by Lord Rayleigh in 1879 in connection with the diffraction of light \[22\]. Brillouin, as early as 1916, obtained an asymptotic expansion of \(P(x, y)\) using steepest descents \[23\]. As discussed in \[18\], \(P(0, y)\) displays dominant exponentially small oscillatory behaviour. As we shall discuss later, this behaviour is shared by \(F_0(z)\) for all even natural \(\alpha\) larger than 2.

Brézin and Hikami \[24\] have recently demonstrated that \(F_0(z)\) appears in the solution of a certain problem in random matrix theory. They consider an \(N \times N\) Hamiltonian matrix, \(H = H_0 + V\), which is the sum of a given nonrandom Hermitian matrix, \(H_0\), and a random Gaussian Hermitian matrix \(V\). The addition of \(H_0\) breaks the unitary invariance of the probability measure that defines the Gaussian unitary ensemble,

\[
P(H) \propto e^{-(N/2)\text{Tr}V^2} \propto e^{-(N/2)\text{Tr}(H^2 - 2H_0 H)}. \tag{7}
\]

They investigate the eigenvalue density of \(H\) when the eigenvalues of the nonrandom matrix, \(H_0\), are chosen to be \(\pm a\), each value being \(N/2\) times degenerate. In this case when \(a > 1\) a gap appears in the eigenvalue spectrum of \(H\) which closes as \(a \to 1\).

An important quantity in random matrix theory is the \(n\)-point correlation function, \(R_n(\lambda_1, \lambda_2, ..., \lambda_n)\), given by

\[
R_n(\lambda_1, \lambda_2, ..., \lambda_n) = \left\langle \prod_{j=1}^{n} \frac{1}{N} \text{Tr} \delta(\lambda_j - H) \right\rangle, \tag{8}
\]

where \(\lambda_j\) is an eigenvalue of \(H\).

The presence of nonzero \(H_0\) implies that the usual orthogonal polynomial methods are no longer available for the determination of the \(n\)-point correlation function. Brézin and Hikami show however that the function \(R_n(\lambda_1, \lambda_2, ..., \lambda_n)\) is again expressible as the determinant of an \(n \times n\) matrix whose elements are given by a kernel \(K_N(\lambda_1, \lambda_2)\), just as in standard random matrix theory. The following contour integral expression for the kernel is arrived at

\[
K_N(\lambda_1, \lambda_2) = (-1)^{(N/2)+1} \int_{-\infty}^{\infty} \frac{dt}{2\pi} \int_{|u|>a} \frac{du}{2\pi i} \left( \frac{t^2 + a^2}{u^2 - a^2} \right)^{N/2} \times \frac{1}{u - it} e^{-(N/2)u^2 - (N/2)t^2 - N\pi i\lambda_1 + Nu\lambda_2}. \tag{9}
\]

For the critical value of \(a = 1\) Brézin and Hikami then find that in the scaling limit in which \(N\) is large and \(N^{3/4}/\lambda\) is finite, the kernel becomes

\[
\hat{K}(x, y) = \frac{\hat{\phi}'(x)\hat{\psi}'(y) - \hat{\phi}''(x)\hat{\psi}(y) - \hat{\phi}(x)\hat{\psi}''(y)}{x - y}, \tag{10}
\]

where

\[
\hat{\phi}(x) = \frac{\sqrt{\pi}}{2} F_4(\sqrt{2}x), \tag{11}
\]

and

\[
\hat{\psi}(x) = -\frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1} (2n)!}{n!(4n+1)!}. \tag{12}
\]

They then go on to show that this behaviour is universal in the sense that \(\hat{K}(x, y)\) is independent of the eigenvalue distribution of the non random Hermitian matrix \(H_0\) provided that a gap closes at the origin in the eigenvalue density of the total matrix \(H\).

Brézin and Hikami then discuss a situation in which one considers a random Hermitian matrix coupled to a non random complex matrix, \(H_0\), rather than an Hermitian matrix. In this case the probability measure of the total matrix becomes complex and so loses its probabilistic interpretation. By appropriately selecting the eigenvalues of this non random matrix Brézin and Hikami discuss a multcritical generalisation of the closing gap phenomena discussed above, the kernel of which is now expressible in an analogous manner in terms of \(F_{2k}(z)\), with each natural \(k\) larger than 2 corresponding to a particular choice of the eigenvalues of \(H_0\). As we mentioned in the introduction, and will discuss in more detail presently, it is precisely this subclass of the functions \(F_\alpha(z)\) that display dominant transcendental asymptotics.
IV. THE LÉVY INTEGRAL, THE GENERALISED EULER-JACOBI SERIES AND WARING’S PROBLEM.

Waring’s conjecture \cite{25} asserts that every natural number can be expressed as the sum of \( s \) integers all to the \( k \)th power. Let the function \( g(k) \) denote the least value of \( s \) such that the proposition is true for all natural numbers, and \( G(k) \) the least value of \( s \) such that the proposition is true for all but a finite number of natural numbers. Waring’s problem naturally decomposes into two parts, that of proving the existence of \( g(k) \) for all \( k \), and that of the determination of the explicit structure of \( g(k) \) and \( G(k) \). Hilbert\cite{26} proved the existence of \( g(k) \) in 1909, and although much progress took place last century \cite{25} the investigation of \( G(k) \) continues to this day.

A fundamental arithmetic function that arises in the investigation of Waring’s problem \cite{25,27} is \( r_{k,s}(n) \) which for \( k \) even is the number of representations of \( n \in \mathbb{N} \) in the form

\[
n = a_1^k + a_2^k + \ldots + a_s^k, \quad a_i \in \mathbb{Z},
\]

with representations which differ only in the order of the \( a_i \) reckoned as distinct. The generating function of the arithmetic function \( r_{k,s}(n) \) which was investigated by Hardy and Littlewood \cite{27} in relation to Waring’s problem, takes the elegant form

\[
\sum_{n=0}^{\infty} r_{k,s}(n)e^{-an} = [2S_k(a) - 1]^s
\]

where the function \( S_k(a) \) is special case of a function which we refer to as the generalised Euler-Jacobi series, defined by

\[
S_{k,q}(a) = \sum_{n=0}^{\infty} e^{-an^{p/q}}, \quad p, q \in \mathbb{N}, \quad \Re(a) > 0.
\]

The series \( S_{k,q}(a) \) was studied in detail in \cite{7}. Application of Euler’s summation formula \cite{6,7,28} results in the following inversion, which is the generalisation of the familiar transformation identity for the Jacobi theta function \( \vartheta_3(a) = 2S_2(a) - 1 \).

\[
S_{k,q}(a) = \frac{\Gamma((q/p + 1)/a^{q/p})}{a^{q/p}} + \frac{1}{2} + \frac{2}{a^{q/p}} \sum_{n=1}^{\infty} F_{p/q} \left( \frac{2n\pi}{a^{q/p}} \right).
\]

Hence we see that there exists a startlingly direct relationship between the number theoretic Euler-Jacobi series \( S_{p/q}(a) \), and the Lévy integral \( F_{p/q}(z) \). For this reason, Hardy and Littlewood devoted an entire section of \cite{27} to the investigation of \( F_{\alpha}(z) \), where it is noted that \( F_{\alpha}(z) \) can be written as a sum of hypergeometric functions and that it displays dominant exponential asymptotics, two observations which we show are indeed true for all even natural \( \alpha \). Without the modern theory of the generalised hypergeometric functions at their disposal, as appears in \cite{29} for instance, an investigation along the hypergeometric route however is dismissed by Hardy and Littlewood as containing too many formal complications. In \cite{7} the hypergeometric representations are given for \( F_{p/q}(z) \) and hence \( S_{p/q}(a) \), and the complete asymptotic expansions are investigated for various cases of \( \alpha \). Ramanujan was interested in the series \( S_{p/q}(a) \) and derived a small \( a \) asymptotic expansion for it as discussed by Berndt \cite{30}. This expansion consists only of the component power series however which vanishes identically when \( \alpha \) is an even natural, all exponentially small terms having been neglected. Berndt uses the Mellin transform technique rather than utilising the inversion formula \cite{16} and hypergeometric representation of \( F_{\alpha}(z) \) to recover Ramanujan’s result.

V. THE EXPONENTIAL ASYMPTOTICS OF THE LÉVY INTEGRAL.

The question of the asymptotics of \( F_{\alpha}(z) \) appears to have been first investigated thoroughly by Burwell \cite{31}, who utilised the technique of steepest descents to obtain the first terms in the dominant asymptotic expansion but was not concerned with obtaining the exponentially small subdominant terms. By expressing \( F_{\alpha}(z) \) as a sum of generalised hypergeometric functions the complete large \( z \) asymptotic expansions of \( F_{\alpha}(z) \) were calculated in \cite{7} for \( \alpha = 3, 4, 5, 6, 7 \), which then yielded the small \( a \) asymptotic expansions of \( S_{p/q}(a) \). A more general problem has recently been solved
by the present authors. The \( d \) dimensional isotropic Lévy stable probability density function possesses the following integral representation, where \( J_{(d/2)-1}(qr) \) is a Bessel function,

\[
p^d_\alpha(r) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d^d q \, e^{-ir \cdot q} \exp(-|q|^\alpha)
\]

(17)

\[
p^d_\alpha(r) = \frac{r}{(2\pi r^{(d/2)})} \int_0^\infty dq \, J_{(d/2)-1}(qr)q^{(d/2)}e^{-q^\alpha}.
\]

(18)

When \( d = 1 \) this clearly reduces to \((1/\pi)F_\alpha(z)\). It is shown in [10] that \( p^d_\alpha(r) \) can be written in terms of generalised hypergeometric functions for all positive rational \( \alpha \). These hypergeometric representations are then used to construct the complete asymptotic expansions for \( F_{p/q}(z) \) for all natural \( \alpha \). Utilising the useful notation, \( f(\pm z)^\pm \equiv f(z) \pm f(-z) \) for some function \( f \), the Lévy integral for rational \( \alpha \) can be identified with a sum of generalised hypergeometric functions as follows

\[
F_{p/q}(z) = \frac{\pi q^{\frac{\alpha}{2} + \frac{d}{2}}}{p^2} (2\pi)^{\frac{(p-q-2)}{2}} \sum_{l=0}^{p-1} (-1)^l \left( \frac{q^\alpha z^p}{p^p} \right)^{\frac{2l}{\alpha}}
\]

\[
\times \frac{\Gamma(l+\frac{2l+1}{p})}{\Gamma(l+\frac{2l+1}{p}+1)} \left. F_{l-1} \left( \frac{1}{p}, \frac{1}{p}, \ldots, \frac{1}{p} \right) \left( \frac{\alpha}{z}, \ldots, \frac{\alpha}{z} \right) \right|_{\alpha \equiv \pm j \frac{q^\alpha z^p}{p^p}}.
\]

(19)

where \( p, q \) are natural numbers. The complete asymptotic expansions of \( F_\alpha(z) \) and \( S_\alpha(\alpha) \) for all natural \( \alpha \) are a straightforward corollary of the asymptotics for \( p^d_\alpha(r) \) given in [10], and we present the expansion for \( F_\alpha(z) \) below. In this way, this article can be considered as an update of the work presented in [10], in that we now display the structure of the asymptotics for all \( \alpha \in \mathbb{Z}_{\geq 3} \), rather than discussing specific cases. We remark that the complete asymptotic expansion of \( p^d_\alpha(r) \) presented in [10] is only extremely weakly dependent on the dimension \( d \), none of the overall structure changing as one moves from \( d = 1 \) to \( d > 1 \). Thus all of the interesting asymptotic behaviour possessed by \( p^d_\alpha(r) \) is retained by \( F_\alpha(z) \) and in this sense the asymptotic behaviour of \( p^d_\alpha(r) \) for natural \( \alpha \) is universal with regard to the spatial dimension \( d \).

\[
F_\alpha(z) \sim \frac{\alpha}{z} \sum_{m=1}^{\infty} (-)^m + \frac{\Gamma((m+1)\alpha)}{\Gamma(m) \sin \left( \frac{m\pi\alpha}{2} \right)}
\]

\[
+ \sqrt{2} \frac{2}{(2^{\alpha - 1} \alpha^{(\alpha - 2)} \Gamma(\alpha - 1))^{1/2(\alpha - 1)}} \sum_{m=0}^{[\alpha/2]-[\alpha/4]-1} 2^m N_m \left( \frac{\alpha}{z} \right)^{m\alpha/(\alpha - 1)}
\]

\[
\times \sum_{k=0}^{[\alpha/2]-[\alpha/4]-1} \exp \left[ \left( \alpha - 1 \right) \sin \left( \frac{4(k + 1)\pi}{2(\alpha - 1)} \right) \left( \frac{z}{\alpha} \right)^{\alpha/(\alpha - 1)} \right]
\]

\[
\times \cos \left( \alpha - 1 \right) \cos \left( \frac{(4k + 1)\pi}{2(\alpha - 1)} \right) \left( \frac{z}{\alpha} \right)^{\alpha/(\alpha - 1)} - \frac{(4k + \alpha)\pi}{2(\alpha - 1)} + \frac{(4k + 2 - \alpha)\pi}{4(\alpha - 1)}
\]

\[
- \sqrt{2} \frac{\pi}{(2^{\alpha - 1} \alpha^{(\alpha - 2)} \Gamma(\alpha - 1))^{1/2(\alpha - 1)}} \sum_{m=0}^{\infty} 2^m N_m \left( \frac{\alpha}{z} \right)^{m\alpha/(\alpha - 1)} (-)^m
\]

\[
\times \exp \left[ \left( \alpha - 1 \right) \left( \frac{z}{\alpha} \right)^{\alpha/(\alpha - 1)} \right],
\]

(20)

\[
- \frac{\pi}{\alpha} < \arg(z) < \frac{\pi}{\alpha}, \quad |z| \to \infty, \quad \alpha \in \mathbb{Z}_{\geq 3}.
\]

Here \([x]\) refers to the integer part of the real number \( x \), and \( \delta_{(\alpha-4[\alpha/4])2} \) is the Kronecker delta. The \( N_k \) are defined by the following recursion relation
\[ N_k = \sum_{s=1}^{2(\alpha-1)} \sum_{r=0}^{2(\alpha-1)-s} \left( \frac{\alpha}{2(\alpha-1)} (r + s - k) - (\alpha - 1) + \frac{\alpha}{4(\alpha - 1)} \right)_{\alpha-1} \times \left( \frac{\alpha}{2(\alpha-1)} (r + s - k) - (\alpha - 1) - \frac{(\alpha - 2)}{4(\alpha - 1)} \right)_{\alpha} \times (-1)^{s+r} \frac{(2(\alpha-1))^{2s-2}}{r!} \frac{(2(\alpha-1) - s - r)!}{\alpha^{2\alpha-1} k^{k-s}} N_{k-s}, \]

\[ N_s = 0 \quad s < 0, \quad N_0 = 1. \]  

where \((x)_{\alpha}\) denotes the Pochhammer symbol.

We note that for the case of \(\alpha = 4\), (20) is in agreement with the corresponding result for the Pearcey integral given in [18]. We remark also that in the process of extending Pólya’s work on the zeros of \(F_{\alpha}(z)\), Senouf [32] has obtained the \(k = 0\) portion of the complete asymptotic expansion (20) for even \(\alpha\).

The small \(\alpha\) asymptotics of the series \(S_{\alpha}(a)\) is now given simply by utilising its relationship with the Lévy integral [10]. Since the cases \(\alpha = 1, 2\) are simply the Cauchy and Gaussian probability density functions, up to a normalisation factor of \(\pi\), whose asymptotic behaviour is trivial to obtain, (20) gives us a complete understanding of the asymptotic behaviour of \(F_{\alpha}(z)\) for all natural \(\alpha\). It is clear from the form of the power series component of the full asymptotic expansion, given by the first series of (20), that when \(\alpha\) is even every term vanishes identically and so the exponentially small terms become dominant. We note that when \(\alpha\) is an even integer the hypergeometric representation of \(F_{\alpha}(z)\) simplifies in such a way that only hypergeometric functions with no numerator parameters appear [10], and it is known, see e.g. [29], that such hypergeometric functions always display purely transcendental asymptotics. Thus the function \(F_{\alpha}(z)\) displays qualitatively different behaviour depending on whether the parameter \(\alpha\) is even or odd. Also, the presence of \([\alpha/4]\) in the summation limits and the Kronecker delta show that the structure of the asymptotics of \(F_{\alpha}(z)\) depends sensitively not only on the parity of \(\alpha\), but on the arithmetic residue of \(\alpha\) modulo 4. The mechanism by which this dependence arises in the construction of (20) is discussed in [10]. An interesting property of (20) is that the number of exponentially decaying oscillatory series increases as \(\alpha\) increases, so that the asymptotics becomes ever more intricate as \(\alpha\) becomes large. This phenomena was observed in [3, 4, 5, 6, 7, 8] for the specific cases \(\alpha = 3, 4, 5, 6, 7,\) and predicted to be true for general \(\alpha\). Our result (21) now confirms this.

**VI. CONCLUDING REMARKS**

The theory of stable distributions is a fundamental branch of probability, and consequently the importance of the Lévy integral for \(\alpha\) less than two is widely recognised. What is equally as intriguing as its probabilistic applications is the fact that the Lévy integral also appears to be of importance when \(\alpha\) is larger than 2 and it no longer defines a probability density function. Its delicate asymptotics when \(\alpha\) is an even natural number is qualitatively different to anything exhibited when \(\alpha < 2\). From examples such as its appearance in random matrix theory, and its relationship with the Pearcey integral, one might well expect that the future growth of interest in the Lévy integral might lie not in probability, in which it is well established, but in problems in which its exponential asymptotics are utilised. In particular it appears that the subclass of \(F_{\alpha}(z)\) where \(\alpha\) is an even natural number, which exhibits dominant exponential asymptotics, will be of interest.

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