Vibration of non-uniform rod using Differential Transform Method

S Shali1,4, S R Nagaraja2 and P Jafarali3
1Department of Civil Engineering, Amrita School of Engineering, Bengaluru, Amrita Vishwa Vidyapeetham, Amrita University, India
2Department of Mechanical Engineering, Amrita School of Engineering, Bengaluru, Amrita Vishwa Vidyapeetham, Amrita University, India
3Associate Research Scientist, Kuwait Institute for Scientific Research, Kuwait
Email: shali_135@yahoo.co.in

Abstract. The present paper analyses the vibration characteristics of non-uniform rods using Differential Transform Method. The method is very effective in solving ordinary and partial differential equations. The governing equations are transformed into a set of polynomials and the solution of these algebraic equations gives the desired frequency. Method is implemented for fix-free and fix-fix end conditions of a non-uniform rod. It is seen that the lower modes are sensitive to taper ratio in case of fix-free rod and fix-fix case frequency is same as that of uniform rod for different taper ratios. The method has proved to be accurate, simple and effective for eigenvalue analysis.

1. Introduction

The vibration of non-uniform rods and beams are of great practical interest and has been studied extensively. These studies are important in the field of architecture, civil engineering and aeronautics. Exact solutions exist for specific conditions. In the absence of exact solutions, the problem can be solved using approximate or numerical methods. Abrate [1] computed the fundamental frequency of a fix-free and free-free non-uniform rod by using Rayleigh-Ritz approach. Transfer matrix approach is used for the computation of natural frequency of non-uniform rods by Bapat [2]. Analytical solution for the vibration of rods using appropriate transformations was presented by Kumar [3]. Bulent Yardimoglu and Levent Aydin [9] analysed tapering rods of cross section variations as power of the sinusoidal functions wherein the differential equation is reduced to associated Legendre equation and solved.

The objective of the present paper is to present the application of DTM to solve for the free vibration of non-uniform rods. Polynomial variation of area is considered with fix-free and fix-fix end condition.

2. Differential Transform Method

Differential Transform Method (DTM) is a transformation technique based on Taylor series expansion, used to solve the ordinary and partial differential equation approximately. The method was first used in 1986 by Zhou to solve initial value problem in electric circuit analysis. The method reduces the
governing differential equation and the boundary conditions to a set of algebraic equations according to certain transformation rules. Hence DTM is treated as an iterative procedure to get higher order series.

The differential transform of a function \( f(x) \) which is analytic in domain \( D \) is represented as;

\[
\overline{F}[k] = \frac{1}{k!} \left( \frac{d^k f}{dx^k} \right)_{x=x_0}
\]

(1)

Where \( x=x_0 \) represents any point within the domain \( D \). The inverse differential transform can written in the form,

\[
f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{d^k f}{dx^k} \right)_{x=x_0} (x-x_0)^k
\]

(2)

Function \( f(x) \) is represented by a finite series in practical cases, hence,

\[
f(x) = \sum_{k=0}^{N} \overline{F}[k] (x-x_0)^k
\]

(3)

Which implies that \( \sum_{k=N+1}^{\infty} \overline{F}[k] (x-x_0)^k \) is taken negligibly small and \( N \) is decided on the convergence of the eigenvalues.

Fundamental transformation rules are listed in table 1.

**Table 1** Fundamental transformations of DTM.

| Original function | Transformed function |
|-------------------|----------------------|
| \( f(x) = g(x) \pm h(x) \) | \( F[k] = G[k] \pm H[k] \) |
| \( f(x) = c.g(x) \) | \( F[k] = c.G[k] \) |
| \( f(x) = g(x) \cdot h(x) \) | \( F[k] = \sum_{l=0}^{k} G[l] H[k-l] \) |
| \( f(x) = \frac{d^n g(x)}{dx^n} \) | \( F[k] = (k+1)(k+2)\ldots(k+n)G[k+n] \) |
| \( f(x) = x^n \) | \( F[k] = \delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} \) |

3. Free vibration of non-uniform rod

The governing differential equation for the longitudinal motion of a rod with varying cross section is,

\[
\frac{\partial}{\partial x} \left[ EA(x) \frac{\partial u}{\partial x} \right] = \rho A(x) \frac{\partial^2 u}{\partial t^2}
\]

(4)

Where \( u(x, t) \) is deformation, \( A(x) \) is area of cross section at a position \( x \), \( \rho \) is the mass density and \( E \) is Young’s modulus of the material. Assuming a solution of the form, \( u(x, t) = U(x) e^{i\omega t} \), where \( \omega \) is the natural frequency of the rod, equation (4) can be written as;
\[
\frac{d}{dx} \left[ EA(x) \frac{dU}{dx} \right] = -\omega^2 U(x) \rho A(x) \tag{5}
\]

Introducing the non dimensional parameters;
\[\xi = \frac{x}{L}; \quad U(\xi) = \frac{U(x)}{L}; \quad A(\xi) = \frac{A(x)}{A_0}; \quad \Omega^2 = \omega^2 L^2 \frac{\rho}{E} \tag{6}\]

Where \(\Omega^2\) is the non dimensional natural frequency of the rod, the differential equation (5) can be rewritten as;
\[
\frac{d}{d\xi} \left[ A(\xi) \frac{dU(\xi)}{d\xi} \right] + \Omega^2 A(\xi) U(\xi) = 0 \tag{7}
\]

For a rod the boundary conditions stated as

Fix-Free case; \( U(0,t) = 0 \) and \( \frac{dU}{dx} \Big|_{x=L} = 0 \) \tag{8}

Fix-Fix case; \( U(0,t) = 0 \) and \( U(L,t) = 0 \) \tag{9}

In the present study the variation of area of the rod is taken in the form of \( A(x) = A_0 \left(1 + a x / L \right)^2 \).

The non dimensional boundary conditions becomes,

Fix-Free: \( U(\xi) = 0 \); for \( \xi = 0 \) and \( \frac{dU(\xi)}{d\xi} = 0 \) for \( \xi = 1 \) \tag{10}

Fix-Fix: \( U(\xi) = 0 \); for \( \xi = 0 \) and \( U(\xi) = 0 \) for \( \xi = 1 \) \tag{11}

4. Free vibration of non-uniform rod using DTM

Differential Transformation Method is applied to the governing equations and the boundary conditions which will result in a set of algebraic equations. The solution of these algebraic equations gives the required non-dimensional frequency of the rod.

4.1 DTM formulation

The free vibration analysis was done for a rod with area of cross section variation given as \( A(x) = A_0 \left(1 + a x / L \right)^2 \). In terms of non-dimensional quantity area can be expressed as \( A(\xi) = A_0 \left(1 + a \xi \right)^2 \).

Applying the transformation rules, the equation (7) can be written as;
\[
\sum_{r=0}^{k} A(k-r)(r+1)(r+2)U(r+2) + \sum_{r=0}^{k} A(k-r+1)(k-r+1)(r+1)U(r+1) + \sum_{r=0}^{k} \Omega^2 A(k-r)U(r) = 0 \tag{12}
\]

Where \( A(k) = \delta(k) + 2a\delta(k-1) + a^2 \delta(k-2) \) \tag{13}

Substituting the transformation of area term the expression (12) takes the form;
The equation (14) can be solved for the frequency \( \Omega^2 \) with the transformed end condition for fix free rod as;

\[
\sum_{k=0}^{n} k U[k] = 0
\]

(15)

4.2 Solution Procedure

The algebraic equations derived by the differential transform method can be rearranged to form an eigen value problem and the eigen value is the non-dimensional frequency to be calculated. From the boundary condition at \( x=0 \) (\( \xi=0 \)), we have the transformed end condition as \( U[0]=0 \). To start with take \( U[1]=s \), which is an arbitrary constant. Using \( U[0] \) and \( U[1] \), the subsequent values \( U[2], U[3], \ldots \) can be found in terms of \( s \) by taking \( k=0,1,2,\ldots n \). The resulting algebraic expression can be solved by using the end condition at \( x=L \), (\( \xi=1 \)). Hence we get a polynomial equation in terms of \( \Omega^2 \) and \( s \) which can be written as;

\[
A(\Omega) s = 0
\]

(16)

For non trivial solutions, equate \( A(\Omega)=0 \) which gives the required frequency. Hence \( \Omega = \Omega^N_n \) which corresponds to the frequency of \( n^{th} \) mode can be obtained. The value of \( N \) is decided based on the desirable accuracy that is required. \( |\Omega^N_n - \Omega^N_{n-1}| \leq \epsilon \) Where \( \epsilon \) is a small value taken as 0.001 in the present analysis.

5. Results

The non uniform rod was analysed for the natural frequencies using the approach of DTM. Rod with two support conditions was analysed. The derived equations are solved in Mathematica.

5.1 Non dimensional Frequency

For the end condition of fix-free, with a value of taper ratio \( a=0.6 \), we have:

\[
U[0] = 0
\]
\[
U[1] = s
\]
\[
U[2] = -0.6s
\]
\[
U[3] = (0.35999 - 0.1667\Omega^2)s
\]
\[
U[4] = (-0.125999 + 0.09999\Omega^2)s
\]
\[\ldots U[n] \]

\( U[k] \) values are substituted in equation (14) and solved for \( \Omega^2 \) using the end condition for \( \xi=1 \). For \( N=26 \), we get the non dimensional frequency as
As the values of \(N\) increases, higher modes can be determined. The calculated frequency of fix-free rod for different values of \(a\) are presented in Table 2.

**Table 2** Non-dimensional frequency of rod with fix-free support condition

| \(a\) | \(\Omega_1\) | \(\Omega_2\) | \(\Omega_3\) |
|------|------------|------------|------------|
| 0    | 0.57079    | 4.71238    | 7.85397    |
| 0.1  | 1.51069    | 4.69303    | 7.79953    |
| 0.2  | 1.45689    | 4.676762   | 7.83271    |
| 0.5  | 1.32429    | 4.65219    | 7.81354    |
| 0.6  | 1.287342   | 4.63159    | 7.80597    |
| 0.8  | 1.22154    | 4.61679    | 7.79689    |

The non-uniform rod analysed for fix-fix end condition for different taper ratios are tabulated in Table (3). The values computed by DTM is compared with the exact values corresponding to \(a=0\) which represent a rod of uniform cross section which is in very good agreement. The frequency values for non-uniform rod with fix-fix end condition is computed to be the same as uniform rod with different taper ratios. The same conclusions were derived in previous works also [1], [2]. For the case of fix-free end condition, the frequency values are found to be decreasing as the taper ratio increases. Also it is noticed that the lower mode frequency are affected more than the higher modes with the taper ratio.

**Table 3** Non-dimensional Frequency of rod with fix-fix support condition

| \(a\) | \(\Omega_1\) | \(\Omega_2\) | \(\Omega_3\) |
|------|------------|------------|------------|
| 0    | 3.14159    | 6.2832     | 9.42477    |
| 0.1  | 3.14159    | 6.2833     | 9.4248     |
| 0.2  | 3.14159    | 6.2832     | 9.4248     |
| 0.5  | 3.14162    | 6.2822     | 9.4255     |
| 0.6  | 3.14159    | 6.2832     | 9.4248     |
| 0.8  | 3.1416     | 6.2831     | 9.4247     |
5.2 Convergence of frequency values
The convergence plot for the frequency corresponding to taper ratio 0.6 for the first three modes is shown in figure (1).

![Convergence plot of non-dimensional frequency of rod for a=0.6.](image)

**Figure 1** The Convergence plot of non-dimensional frequency of rod for a=0.6.

6. Conclusions

DTM was used to compute the non-dimensional frequency of non-uniform rod with different values of taper ratio. The formulation was done with fix-free and fix-fix end condition. The results obtained clearly shows that for the fix-fix condition the frequency values are same as that of an uniform rod. Also it is the lower modes that are sensitive more to the taper ratio when compared to the higher modes.

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