Online Learning of Competitive Equilibria in Exchange Economies

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Abstract

The sharing of scarce resources among multiple rational agents is one of the classical problems in economics. In exchange economies, which are used to model such situations, agents begin with an initial endowment of resources and exchange them in a way that is mutually beneficial until they reach a competitive equilibrium (CE). CE allocations are Pareto efficient and fair. Consequently, they are used widely in designing mechanisms for fair division. However, computing CEs requires the knowledge of agent preferences which are unknown in several applications of interest. In this work, we explore a new online learning mechanism, which, on each round, allocates resources to the agents and collects stochastic feedback on their experience in using that allocation. Its goal is to learn the agent utilities via this feedback and imitate the allocations at a CE in the long run. We quantify CE behavior via two losses and propose a randomized algorithm which achieves $\tilde{O}(\sqrt{T})$ loss after $T$ rounds under both criteria. Empirically, we demonstrate the effectiveness of this mechanism through numerical simulations.

1 Introduction

An exchange economy (EE) is a classical microeconomic construct used to model situations where multiple rational agents share a finite set of scarce resources. Such scenarios arise frequently in applications in operations management, urban planning, crowdsourcing, wireless networks, and compute clusters [14, 19, 23, 27, 29, 46]. In an EE, $n$ agents begin with an initial endowment of $m$ resource types; agents then exchange these resources amongst themselves based on a price system. This exchange process allows two agents to trade different resource types if they find it mutually beneficial to do so. Under certain conditions, continually trading in this manner results in a competitive equilibrium, where the allocations have desirable Pareto-efficiency and fairness properties. EEs have attracted much research attention, historically since they are a tractable model to study human behavior and price determination in real-world economies, and more recently for designing multi-resource fair-division mechanisms [7, 8, 10, 15, 17, 48].

One of the most common use cases for fair division occurs in the context of shared computational resources. For instance, in a compute cluster shared by an organization or university, we wish to allocate resources such as CPUs, memory, and GPUs to different users (agents) who share this cluster in a way that is Pareto-efficient (so that the resources are put into good use) and fair.
(for long-term user satisfaction). Fair-division mechanisms are a staple in many popular multi-
tenant cluster management frameworks, such as Mesos [28], Quincy [30], Kubernetes [11], and
Yarn [51]. Due to this strong practical motivation, a recent line of work has studied such fair-
division mechanisms for resource sharing in a compute cluster [13, 24, 25, 42], with some of them
based on exchange economies and their variants [26, 35, 49, 55].

However, prior work in exchange economies and fair division typically assumes knowledge of
the agent preferences, in the form of a utility function which maps an allocation of the $m$
resource
types to the value the agent derives from the allocation. For instance, in the above example, an
application developer would need to quantify how well her application performs for each allocation
of CPU/memory/GPU she receives. At best, doing so requires the laborious and erroneous task
of profiling their application [18, 40], and at worst, it can be impossible [44, 52]. However, having
received an allocation, application developers find it easier to report feedback on the utility they
achieved. Moreover, in many real-world systems, this can often be automated [28].

1.1 Our contributions

In this work, we study a multi-round mechanisms for computing CE in an exchange economy so as
to generate fair and efficient allocations when the utilities are unknown a priori. At the beginning
of each round, we allocate some resources to each agent; at the end of the round, agents report
feedback on the allocation they received. The mechanism then uses this information to determine
allocations in future rounds. Our contributions are as follows:

1. We formalize this online learning task with two loss functions: the first directly builds on the
definition of a CE, while the latter is motivated by fairness and Pareto-efficiency considerations
that arise in fair division.

2. We develop a randomized online mechanism which learns utilities over rounds of allocations
while simultaneously striving to achieve Pareto-efficient and fair allocations. We show that this
mechanism achieves sublinear loss for both loss functions with both in-expectation and high-
probability upper bounds (Theorems 4.1 and 4.2), under a general family of utility functions.
To the best of our knowledge, this is the first work that studies CE without knowledge of user
utilities; as such new analysis techniques are necessary.

3. We corroborate these theoretical insights with empirical simulations.

1.2 Related work

Our work builds on a rich line of line of literature at the intersection of microeconomics and
machine learning. The richness is not surprising—many real world systems are economic and
multi-agent in nature, and decisions taken by or for one agent should be weighed against the
considerations of others, especially when these agents have competing goals. The majority of this
work is in auction-like settings. As in our work, several authors study online learning formulations
to handle situations where agent preferences are not known a priori, but can be learned from
repeated interactions [4, 6, 9, 21, 31, 32]. Our setting departs from this existing work as we wish to
learn agent preferences in an exchange economy, with a focus on designing fair division mechanisms.

Since the seminal work of Varian [49], the fair division of multiple resource types has received
significant attention in the game theory, economics, and computer systems literatures. One of the
most common perspectives on this problem is as an exchange economy (or as a Fisher market, which is a special case of an exchange economy) [7, 8, 10, 15, 26, 39, 48, 49]. In addition to this theoretical literature, fair-allocation mechanisms have been deployed in practical data-center resource-allocation tasks [55]. Additionally, there have been applications of other market-based resource allocation schemes for data centers and power grids [11, 28, 35, 51, 54].

Another line of work has studied fair division when the resources in question are perfect complements; some examples include dominant resource fairness and its variants [20, 24, 25, 26, 37, 42]. The assumption of perfect complement resource types leads to computationally simple mechanisms. However, in many practical applications, there is ample substitutability between resources, and hence the above mechanisms can be inappropriate. For example, in compute clusters, CPUs and GPUs are often interchangeable for many jobs, albeit with different performance characteristics.

In all of the above cases, an important requirement for the mechanisms is that agent utilities be known ahead of time. Some work has attempted to skirt this requirement by making explicit assumptions on the utility, but it is not clear that these assumptions hold in practice [36, 55]. A recent paper by Kandasamy et al. [33] is perhaps unique as providing a general method for learning agent utilities for fair division using feedback. However, they only study a single-resource setting and do not explore situations where one user can exchange a resource of one type for a different type of resource from another user, so that both are better off after the exchange. Indeed, learning in the multi-resource settings is significantly more challenging than the single-resource setting, requiring different formalisms, algorithms, and analysis techniques.

2 Exchange Economies, Competitive Equilibria, and Fair Division

We first briefly review some background material that will be necessary for our problem formulation. We begin with exchange economies and their competitive equilibria.

2.1 Exchange economies

In an exchange economy, we have $n$ agents and $m$ divisible resource types. Each agent $i \in [n]$ has an endowment, $e_i = (e_{i1}, \ldots, e_{im})$, where $e_{ij}$ denotes the amount of resource $j$ agent $i$ brings to the economy for trade. Without loss of generality we will assume $\sum_{i \in [n]} e_{i1} = 1$ so that the space of resources may be denoted by $[0, 1]^m$.

We will denote an allocation of these resources to the $n$ agents by $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in [0, 1]^m$ and $x_{ij}$ denotes the amount of resource $j$ that was allocated to agent $i$. The set of all feasible allocations is $\mathcal{X} = \{x : \sum_{i=1}^m x_{ij} \leq 1, x_{ij} \geq 0, \forall i \in [n], j \in [m]\}$. An agent’s utility function is $u_i : [0, 1]^m \rightarrow [0, 1]$, where $u_i(x_i)$ quantifies her valuation for an allocation $x_i$ she receives. $u_i$ is non-decreasing, i.e., $u_i(x_i) \leq u_i(x'_i)$ for all $x_i \leq x'_i$ (elementwise); this simply states that the agent always prefers more resources.

In an exchange economy, agents exchange resources based on a price system. We will denote a price vector by $p$, where $p \in \mathbb{R}_+^m$ and $1^T p = 1$ (the normalization accounts for the fact that only relative prices matter). Here $p_j$ denotes the price for resource $j$. Given a price vector $p$, an agent $i$ has a budget $p^T e_i$, which is the monetary value of her endowment according to the prices in $p$. As this is an economy, a rational agent will then seek to maximize her utility under her budget.
Precisely, she may purchase some resources \(x_i \in d_i(p)\), where

\[
d_i(p) = \arg \max_{x_i \in [0,1]^m} u_i(x_i) \quad \text{subject to} \quad p^\top x_i \leq p^\top e_i
\]

While generally \(d_i(p)\) is a set, for simplicity we will assume it is a singleton and treat \(d_i\) as a function which outputs an allocation for agent \(i\). This is justified under very general conditions [39, 50]. We will refer to \(d_i(p)\) chosen in the above manner as the agent \(i\)’s demand for prices \(p\).

**Competitive equilibria—definition, existence and uniqueness:** A seemingly natural way to allocate resources to agents is to set prices \(p\) for each resource, and have the agents maximize their utility under this price system. That is, we allocate \(x(p) = (x_1, \ldots, x_n)\). Unfortunately, such an allocation may be infeasible, and even if it were, it may not be Pareto-efficient (to be described shortly). However, under certain conditions, we may be able to find a **competitive equilibrium** (CE), where the prices have both properties. We formally define a CE below.

**Definition 2.1** (Competitive (Walrasian) Equilibrium). A CE is an allocation and price-vector pair \((x^*, p^*)\) such that (i) the allocation is feasible and (ii) all agents maximize their utilities under the budget induced by prices \(p^*\). Precisely,

\[
\sum_{i \in [n]} x^*_{i,j} \leq \sum_{i \in [n]} e_{ij} = 1, \quad \forall \ j \in [m], \quad x^*_i = d_i(p^*), \quad \forall \ i \in [n].
\]

Some definitions of a CE require that the first condition above be an exact equality (e.g., [39]); when the utilities are strictly increasing (which will be the case in the sequel), both definitions coincide [50].

CEs are not always guaranteed to exist, and when they do, they may not be unique. One condition that guarantees a unique CE is that the resources are gross substitutes; that is, if we increase the price of good \(j\) while keeping all other prices constant, then the demand \(\{d_i(p)\}_j\) for all goods \(\ell \neq j\) will have increased. One class of utilities that guarantee this condition are the **constant elasticity of substitution** (CES) utilities [39, 50]. We describe it below as it will be useful going forward.

**Example 2.2** (CES utilities). A CES utility takes the form \(u_i(x) = (\sum_{j=1}^m \theta_{ij} x_i^\rho)^{1/\rho}\) where \(\rho\) is the elasticity of substitution, and \(\theta_i = (\theta_{i1}, \ldots, \theta_{im})\) is an agent-specific parameter. When \(\rho = 1\), this corresponds to linear utilities where goods are perfect substitutes. As \(\rho \to \infty\), the utilities approach perfect complements. CES utilities are ubiquitous in the microeconomics literature; due to this flexibility in interpolating between perfect substitutability and complementarity, they are able to approximate real world utility functions [56].

If the utilities of all agents satisfy gross substitutability, then a CE exists uniquely. Besides this property, several other works have studied conditions for uniqueness and existence of CEs, which we will not delve into. An interested reader may refer to [39, 50]. Next, we will describe how exchange economies are used in designing fair-division mechanisms.

### 2.2 Fair division

We will first describe the fair division problem independently. We wish to allocate \(m\) divisible resource types among \(n\) agents. Each agent \(i \in [n]\) has an endowment (a.k.a entitlement or fair
share) $e_i = (e_{i1}, \ldots, e_{im})$. For example, in a shared compute cluster, $e_i$ may denote agent $i$’s contribution to this cluster. Moreover, as described above, each agent $i$ has a utility function $u_i : [0,1]^m \rightarrow \mathbb{R}$.

In a mechanism for fair division, each agent truthfully submits her utility $u_i$ to the mechanism. The mechanism then returns an allocation $x \in X$, so as to satisfy two requirements: sharing incentive (SI) and Pareto-efficiency (PE). We say that an allocation $x = (x_1, \ldots, x_m)$ satisfies SI if the utility an agent receives is at least as much as her utility when using her endowment, i.e. $u_i(x_i) \geq u_i(e_i)$. This simply states that she is never worse off than if she had kept her endowment to herself, so she has the incentive to participate in the fair division mechanism.

Next, we say that a feasible allocation $x$ is PE if the utility of one agent can be increased only by decreasing the utility of another. To define this rigorously, we will say that an allocation $x$ dominates another allocation $x'$, if $u_i(x_i) \geq u_j(x'_i)$ for all $i \in [n]$ and there exists some $i \in [n]$ such that $u_i(x_i) > u_j(x'_i)$. An allocation is Pareto-efficient if it is not dominated by any other point. We will denote the set of Pareto-efficient allocations by $\mathcal{PE}$. One advantage of the PE requirement, when compared to other formalisms which maximize social or egalitarian welfare, is that it does not compare the utility of one agent against that of another. The utilities are useful solely to specify an agent’s preferences over different allocations.

**EEs in fair division:** The above problem description for fair division naturally renders itself to a solution based on EEs. In particular, by treating the above environment as an exchange economy, we may compute its equilibrium to determine the allocations for each agent. The SI property follows from the fact that each agent is maximizing her utility under her budget, and an agent’s endowment (trivially) falls under her budget. The PE property follows from the first theorem of welfare economics [39, 50]. Several prior works have used this connection to design fair-division mechanisms for many practical applications [15, 49, 55].

**Computing a CE:** In order to realize a CE allocation in a fair-division mechanism, we need to be able to compute a CE given agent utilities. The most general way to do this is tatonnement [50]. Another promising approach is called proportional response dynamics [55, 56] which converges faster when $e_i = \alpha_i \mathbf{1}_m$ for all $i \in [n]$ (with $\sum_i \alpha_i = 1$) and under certain classes of utility functions, including, but not limited to, CES utilities.

### 3 Online Learning Problem Formulation

In this section, we formalize online learning an equilibrium in an exchange economy, when the agent utilities are unknown a priori. We consider a multi-round setting. At the beginning of each round $t$, a centralized mechanism selects $(x_t, p_t)$, where $x_t = (x_{t1}, \ldots, x_{tn}) \in X$ are the allocations for each agent for the current round, and $p_t$ are the prices. The agents, having experienced their allocation, report stochastic feedback $\{y_{t,i}\}_{i \in [n]}$, where $y_{t,i}$ is $\sigma$ sub-Gaussian and $\mathbb{E}[y_{t,i}|x_{t,i}] = u_i(x_{t,i})$. The mechanism then uses this information to compute allocations and prices for future rounds.

As described in Section 1, this set up is motivated by use cases in data centre resource allocation, where jobs (agents) cannot state their utility up front, but given an allocation, can report feedback on their performance in an automated way.

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1Unlike some previous works on fair division [24, 33, 42], we do not study strategic considerations, where agents may attempt to manipulate outcomes in their favor by falsely submitting their utilities.
Going forward, we will slightly abuse notation when referring to the allocations. When \( i \in [n] \) indexes an agent, \( x_i = (x_{i1}, \ldots, x_{im}) \in [0, 1]^m \) denotes the allocation to agent \( i \). When \( t \) indexes a round, \( x_t = (x_{t1}, \ldots, x_{tn}) \in X \) will refer to an allocation to all agents, where \( x_{ti} = (x_{ti1}, \ldots, x_{ti,m}) \in [0, 1]^m \) denotes \( i \)'s allocation in that round. The intended meaning should be clear from context.

### 3.1 Losses

We study two losses for this setting. We will first describe them both and then illustrate the differences. For \( a \in \mathbb{R} \), denote \( a^+ = \max(0, a) \).

The first of them is based directly on the definition of an equilibrium (Definition 2.1). We define the CE loss \( \ell^{CE} \) of an allocation–price pair \((x, p)\) as the sum, over all agents, of how much they are worse off than some Pareto-efficient allocation. One intuitive interpretation of the CE loss is that it can be bounded above by the FD loss as such an allocation is Pareto-efficient; moreover, by simply allocating each agent their endowment we have zero \( \ell^{CE} \). In \( \ell^{FD} \), we require both to be simultaneously small which necessitates a clever allocation that accounts for agents' endowments and utilities. One intuitive interpretation of the PE loss is that it can be bounded above by the SI loss and the utility achieved by allocation \( x \) as the minimum sum, over all agents, of how much they are worse off than their endowment utilities. We define the PE loss \( \ell^{PE} \) for an allocation \( x \) as the sum, over all agents, of the difference between the maximum attainable utility under price \( p \) and the utility achieved by allocation \( x \). The \( T \)-round loss \( L_T^{CE} \) for an online mechanism is the sum of \( \ell^{CE}(x_t, p_t) \) losses over \( T \) rounds. We have:

\[
\ell^{CE}(x, p) \overset{\text{def}}{=} \sum_{i=1}^{n} \left( \max_{x_i' \in \mathcal{X}, p_i' \leq p_i} u_i(x_i') - u_i(x) \right)^+, \quad L_T^{CE} \overset{\text{def}}{=} \sum_{t=1}^{T} \ell^{CE}(x_t, p_t).
\]

It is straightforward to see that for a CE pair \((x^*, p^*)\), we have \( \ell^{CE}(x^*, p^*) = 0 \).

Our next loss function is motivated by the sharing incentive and Pareto-efficiency desiderata for fair division. We define the SI loss \( \ell^{SI} \) for an allocation \( x \) as the sum, over all agents, of how much they are worse off than their endowment utilities. We define the PE loss \( \ell^{PE} \) for an allocation \( x \) as the minimum sum, over all agents, of how much they are worse off than some Pareto-efficient utilities. Next, we define the FD loss \( \ell^{FD} \) as the maximum of \( \ell^{SI} \) and \( \ell^{PE} \). Finally, we define the \( T \)-round loss \( L_T^{FD} \) for an online mechanism as the sum of \( \ell^{FD}(x_t) \) losses over \( T \) rounds. We have:

\[
\ell^{SI}(x) \overset{\text{def}}{=} \sum_{i=1}^{n} (u_i(e_i) - u_i(x_i))^+, \quad \ell^{PE} \overset{\text{def}}{=} \inf_{x' \in \mathcal{PE}} \sum_{i=1}^{n} (u_i(x_i') - u_i(x_i))^+, \quad \ell^{FD}(x) \overset{\text{def}}{=} \max(\ell^{PE}(x), \ell^{SI}(x)), \quad L_T^{FD} \overset{\text{def}}{=} \sum_{t=1}^{T} \ell^{FD}(x_t).
\]

Observe that individually achieving either small \( \ell^{SI} \) or small \( \ell^{PE} \) is trivial: if an agent’s utility is strictly increasing, then by allocating all the resources to this agent we have zero \( \ell^{PE} \) as such an allocation is Pareto-efficient; moreover, by simply allocating each agent their endowment we have zero \( \ell^{SI} \). In \( \ell^{FD} \), we require both to be simultaneously small which necessitates a clever allocation that accounts for agents' endowments and utilities. One intuitive interpretation of the PE loss is that it can be bounded above by the \( L_1 \) distance to the Pareto-front in utility space; i.e. denoting the set of Pareto-efficient utilities by \( \mathcal{U}_{PE} = \{ \{u_i(x_i)\}_{i \in [n]} \in \mathbb{R}^n \} \subset \mathbb{R}^n \), and letting \( u(x) = (u_1(x_1), \ldots, u_n(x_n)) \in \mathbb{R}^n \), we can write, \( \ell^{PE}(x) \leq \min_{u \in \mathcal{U}_{PE}} \|u - u(x)\|_1 \).

The FD loss is more interpretable as it is stated in terms of the SI and PE requirements for fair division. On the other hand, the CE loss is less intuitive. Moreover, in EEs, while prices help us determine the allocations, they do not have value on their own. Given this, the CE loss has the somewhat undesirable property that it depends on the prices \( p_t \). That said, since the CE loss is based directly on the definition of a CE, it captures other properties of a CE that are not considered in \( \ell^{FD} \); we provide one such example in Appendix D. It is also worth mentioning that either loss cannot be straightforwardly bounded in terms of the other.
3.2 Assumptions

To make the learning problem tractable, we will make additional assumptions on the problem. First we will assume that utilities belong to the following parametric class of functions $\mathcal{P}$. To define $\mathcal{P}$, let $\phi : [0, 1]^m \rightarrow [0, 1]^m$ be a featurization, where $\phi(x_i) = (\phi_1(x_{i1}), \phi_2(x_{i2}), \ldots, \phi_m(x_{im}))$; here, $\phi_j$ is an increasing function which maps the allocation $x_{ij}$ of resource $j$ to agent $i$ to a feature value.

Next, let $\mu : \mathbb{R}_+ \rightarrow [0, 1]$ be an increasing function. Finally, let $\Theta \subset [\theta_{\text{min}}, \infty)^m$, where $\theta_{\text{min}} > 0$, be a space of positive coefficients bounded away from 0. We now define $\mathcal{P}$ as follows:

$$\mathcal{P} = \left\{ \{v_i\}^n_{i=1};~ v_i(x_i) = \mu \left( \sum_{\Theta} \phi(x_i) \right) \text{ for some } \theta_i \in \Theta, \forall i \in [n] \right\}$$

We assume that the features $\phi$ and the function $\mu$ are known. An agent’s utility takes the form $u_i(x_i) = \mu(\theta_i^+ \phi(x_i))$ where the true parameters $\theta_i \in \Theta$ are unknown and need to be learned by the mechanism. We will additionally assume the following regularity conditions: $\mu$ is continuously differentiable, is Lipschitz-continuous with constant $L_{\mu}$ and that $C_{\mu} = \inf_{\theta \in \Theta, x \in X} \mu \left( \theta^\top \phi(x) \right) > 0$.

We consider the above class of functions for the following reasons. First, it subsumes several families of utilities studied previously in the literature, including the CES utilities from Example 2.2 [7, 8, 10, 15, 48], and other application-specific utilities [55]. Second, we are able to efficiently compute a CE on this class [56]. Third, it also allows us to leverage techniques for estimating generalized linear models in our online learning mechanism [12, 22]. Our results also apply when $\mu, \phi, \Theta$ can be defined separately for each agent, but we assume they are the same to simplify the exposition.

Finally, we will assume that for each sets of utilities in $\mathcal{P}$, a CE exists uniquely. This assumption is true on the CES utilities and, as mentioned previously, under very general conditions. Assuming a unique equilibrium simplifies our analysis as it avoids having to deal with degenerate cases.

4 Algorithm and Theoretical Results

Now we present a randomized online learning algorithm for learning the agents’ utilities and generating fair and efficient allocations. The algorithm, outlined in Algorithm 1, takes input parameters $\{\delta_i\}_{i=1}^\infty$ whose values we will specify shortly. It begins with an initialization phase for $m^2$ sub-phases (line 3). On each sub-phase, each user will receive each resource once entirely, but only that resource; i.e. for $m$ rounds in each sub-phase, user $i$ will receive allocations $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ and in the remaining rounds she will receive $0_m$. Each sub-phase will take $\max(n, m)$ rounds and hence the initialization phase will take $\max(m^3, nm^2)$ rounds in total. This initialization phase is necessary to ensure that some matrices we will define subsequently are well conditioned.

After the initialization phase, the algorithm operates on each of the remaining rounds as follows. For each user, it first computes quantities $\hat{Q}_{t,i} \in \mathbb{R}^{m \times m}$ and $\hat{c}_{t,i} \in \mathbb{R}^m$ as defined in lines 16, and 17. As we will explain shortly, $\hat{c}_{t,i}$ can be viewed as an estimate of $\theta_i^+$ based on the data from the first $t - 1$ rounds. The algorithm then samples $\theta_i' \in \mathbb{R}^m$ from a normal distribution with mean $\hat{c}_{t,i}$ and co-variance $\alpha_i^2 Q_{t,i}$, where, $\alpha_i$ is defined as:

$$\alpha_i^2 = \frac{\kappa^2 \sigma^2}{C^2_{\mu}} m \log(t) \log \left( \frac{m}{\delta_t} \right), \quad \kappa = 3 + 2 \log \left( 1 + 2 \phi(1) \| \hat{c}_{t,i} \|^2 \right).$$
Algorithm 1 A randomized algorithm for learning in EEs

1: **Input:** Input space \( \mathcal{X} = [0, 1]^{n \times m} \), parameters \( \{ \delta_t \}_{t \geq 1} \).
2: \( t \leftarrow 0 \)
3: for \( \ell = 1, \ldots, m^2 \) do  
4: \hspace{1em} for \( k = 1, \ldots, \max(m, n) \) do  
5: \hspace{2em} \( t \leftarrow t + 1 \)
6: \hspace{2em} \( x_t \leftarrow (0_m, \ldots, 0_m) \)
7: \hspace{2em} for \( h = 1, \ldots, \min(m, n) \) do  
8: \hspace{3em} if \( m < n \) then
9: \hspace{4em} \( x_{t,h+k-1,j} \leftarrow 1 \) for all \( j \in [m] \).
10: \hspace{3em} else
11: \hspace{4em} \( x_{t,i,h+k-1} \leftarrow 1 \) for all \( i \in [n] \).
12: \hspace{2em} Allocate \( x_t \) and observe rewards \( \{ y_{t,i} \}_{i \in [n]} \).
13: \hspace{1em} while True do  
14: \hspace{2em} \( t \leftarrow t + 1 \)
15: \hspace{2em} for \( i = 1, 2, \ldots, n \) do
16: \hspace{3em} Compute \( Q_{t,i} \) \( \overset{\text{def}}{=} \sum_{s=1}^{t-1} \phi(x_{s,i})\phi(x_{s,i})^\top \)
17: \hspace{3em} Compute \( \theta_{t,i} \) \( \overset{\text{def}}{=} \arg \min_{\theta \in \Theta} \| \sum_{s=1}^{t-1} \phi(x_{s,i}) (\mu (\theta^\top \phi(x_{s,i})) - y_{s,i}) \|_{Q_{t,i}} \)
18: \hspace{3em} Sample \( \theta'_{t,i} \sim N(\theta_{t,i}, \alpha^2 Q_{t,i}^{-1}) \).  
19: \hspace{3em} \( \theta_{t,i} \leftarrow \arg \min_{\theta' \in \Theta} \| \theta'_{t,i} - \theta' \| \).  
20: \hspace{3em} Choose allocations and prices \( x_t, p_t = \text{CE}(\{ u_{t,i} \}_{i=1}^n) \), where \( u_{t,i}(\cdot) = \mu(\theta_{t,i}^\top \phi(\cdot)) \)
21: \hspace{2em} Observe rewards \( \{ y_{t,i} \}_{i \in [n]} \).

The sampling distribution, which is centered at our estimate \( \hat{\theta}_{t,i} \), is designed to balance the exploration-exploitation trade-off on this problem. Next, it projects the sampled \( \theta'_{t,i} \) onto \( \Theta \) to obtain \( \theta_{t,i} \).

Then, in line (20), the algorithm obtains an allocation and price pair \( x_t, p_t \) by computing the competitive equilibrium on the \( \theta_{t,i} \) values obtained above, i.e. by pretending that \( u_{t,i}(\cdot) = \mu(\theta_{t,i}^\top \phi(\cdot)) \) is the utility for user \( i \). Having allocated \( x_t \), the mechanism collects the rewards \( \{ y_{t,i} \}_{i \in [n]} \) from each user and then repeats the same for the remaining rounds. This completes the description of the algorithm.

**Computation of \( \hat{\theta}_{it} \):** It is worth explaining steps 16–17 used to obtain the estimate \( \hat{\theta}_{it} \) for user \( i \)'s parameter \( \theta_i^\ast \). Recall that for each agent \( i \), the mechanism receives stochastic rewards \( y_{t,i} \) where \( y_{t,i} \) is a \( \sigma \) sub-Gaussian random variable with \( \mathbb{E}[y_{t,i}] = u_i(x_{t,i}) \) in round \( t \). Therefore, given the allocation-reward pairs \( \{(x_{s,i}, y_{s,i})\}_{s=1}^{t-1} \), the maximum quasi-likelihood estimator \( \hat{\theta}_{it}^{\text{MLE}} \) for \( \theta_i \) is defined as the maximizer of the quasi-likelihood \( \mathcal{L}(\theta) = \sum_{s=1}^{t-1} \log p_\theta(y_{s,i}|x_{s,i}) \), where \( p_\theta(y_{t,i}|x_{t,i}) \) is as defined below. Here, \( \mu(\nu) = \frac{\partial^2 \psi(\nu)}{\partial \nu^2} \) and \( c(\cdot) \) is a normalising term. We have:

\[
p_\theta(y_{t,i}|x_{t,i}) = \exp \left( y_{t,i}^\top \phi(x_{t,i}) - b(\theta^\top \phi(x_{t,i})) + c(y_{t,i}) \right).
\]

(6)
Upon differentiating, we have that \( \hat{\theta}_{t,i}^{\text{MLE}} \) is the unique solution of the estimating equation:

\[
\sum_{s=1}^{t-1} \phi(x_{si}) \left( \mu \left( \hat{\theta}_{t,i}^{\text{MLE}} \right) \phi(x_{s,i}) \right) \left( \mu \left( \hat{\theta}_{t,i}^{\text{MLE}} \right) \phi(x_{s,i}) \right) - y_{si} = 0.
\]

In other words, \( \theta_{t,i}^{\text{MLE}} \) would be the maximum likelihood estimate for \( \theta^*_i \) if the rewards \( y_{t,i} \) followed an exponential family likelihood as shown in (6). Our assumptions are more general; we only assume the rewards are sub-Gaussian centred at \( \mu \left( \theta^*_i \phi(x_{t,i}) \right) \). However, this estimate is known to be consistent under very general conditions, including when the rewards are sub-Gaussian [12, 22]. Since \( \hat{\theta}_{t,i}^{\text{MLE}} \) might be outside of the set of feasible parameters \( \Theta \), this motivates us to perform the projection in the \( Q_{t,i}^{-1} \) norm to obtain \( \bar{\theta}_{t,i} \) as defined in line 17. Here, \( Q_{t,i} \), defined in line (16), is the design matrix obtained from the data in the first \( t - 1 \) steps.

4.1 Upper bounds on the loss

The following two theorems are the main results of this paper, bounding the loss terms \( L_{\text{FD}}, L_{\text{CE}} \) for Algorithm 1. In the first theorem, we are given a target failure probability of at most \( \delta \). By choosing \( \delta_t \) appropriately, we obtain an infinite horizon algorithm for which both loss terms are \( O(\sqrt{T}) \) with probability at least \( 1 - \delta \). In the second theorem, we are given a time horizon \( T \) and we obtain an algorithm whose expected losses are \( \tilde{O}(\sqrt{T}) \).

**Theorem 4.1.** Assume the conditions in Section 3.2. Let \( \delta > 0 \) be given. Choose \( \delta_t = \frac{2\delta}{n\pi^2 t^2} \). Then, the following upper bounds on \( L_{\text{FD}}, L_{\text{CE}} \) hold for Algorithm 1 with probability at least \( 1 - \delta \).

\[
L_{\text{FD}}(T), L_{\text{CE}}(T) \in O \left( nm\sqrt{T} \left( \log(nT/\delta) + \log(T) \right) \right),
\]

**Theorem 4.2.** Assume the conditions in Section 3.2. Let \( T > \max(m^3, nm^2) \) be given. Choose \( \delta_t = \frac{1}{T} \). Then, the follow upper bounds on \( L_{\text{FD}}, L_{\text{CE}} \) hold for Algorithm 1.

\[
E[L_{\text{FD}}(T)], E[L_{\text{CE}}(T)] \in O \left( nm\sqrt{T} (\log(T)) \right),
\]

Above, probabilities and expectations are with respect to both the randomness in the observations and the sampling procedure. Both theorems show that we can learn with respect to both losses at \( O(nm\sqrt{T}) \) rate. This result is consistent with previous upper and lower bounds for \( m \) dimensional linear and generalised linear bandits where the minimax rate is known to be \( \tilde{O}(m\sqrt{T}) \) [16, 22, 38, 43, 45]. The additional \( n \) term is required since we need to learn for \( n \) agents separately.

4.2 Discussion

**Algorithm design:** While our formulation of finding a CE is distinctly different from a vanilla optimization task, there are several similarities. For instance, in a CE, each agent is maximizing their utility under a budget constraint. Therefore, a seemingly natural idea is to adopt an upper confidence bound (UCB) based procedure, which is the most common approach for stochastic optimization under bandit feedback [5]. However, adopting a UCB-style method for our problem
proved to be unfruitful. To illustrate this, consider using a UCB of the form $\mu(\hat{\theta}_i^T \phi(\cdot)) + U_i(\cdot)$, where $U_i$ quantifies the uncertainty in the current estimate. Unfortunately, a CE is not guaranteed to exist for utilities of the above form, which means that finding a suitable allocation can be difficult. An alternative idea is to consider UCBs of the form $\mu(\hat{\theta}_i^T \phi(\cdot))$ where $\hat{\theta}_i$ is an upper confidence bound on $\theta_i^*$ (recall that both $\theta_i^*$ and $\phi$ are non-negative). While CEs are guaranteed to exist for such UCBs, $\hat{\theta}_i$ is not guaranteed to uniformly converge to $\theta_i^*$, resulting in linear loss.

Instead, our algorithm takes inspiration from classical Thompson sampling (TS) procedure for multi-armed bandits in the Bayesian paradigm [47]. The sampling step in line 18 is akin to sampling from the posterior beliefs in TS. It should be emphasized however, that the sampling distributions on each round cannot be interpreted as the posterior of some prior belief on $\theta_i^*$. In fact, they were designed so as to put most of their mass inside a frequentist confidence set for $\theta_i^*$.

**Proof sketch:** Our proof uses several novel ideas but also relies on some high level intuitions from prior work which provide frequentist guarantees for Thompson sampling [2, 34, 41]. Our proof for bounding $L_{CE}^T$ first defines high probability events $A_{t,i}, B_{t,i}$ for each agent $i$ and round $t$. $A_{t,i}$ captures the event that the estimated $\hat{\theta}_i$ is close to $\theta_i^*$ in $Q_{t,i}$ norm. We upper bound $P(A_{t,i}^c)$ using the properties of the maximum quasi-likelihood estimator on GLMs [12, 22] and a martingale argument. $B_{t,i}$ captures the event that the sampled $\theta_i$ is close to $\hat{\theta}_i$ in $Q_{t,i}$ norm. Given these events, we then bound the instantaneous losses $\ell_{CE}(x_t, p_t)$ by a super martingale with bounded differences. The final bound is then obtained by an application of the Azuma inequality. Another key ingredient in this proof is to show that the sampling step also explores sufficiently—the $B_{t,i}$ event only captures exploitation; since the sampling distribution is a multi-variate Gaussian, this can be conveniently argued using an upper bound on the standard normal tail probability. While bounding $L_{FD}^T$ uses several results and techniques as above, it cannot be directly related to $L_{CE}^T$. However, we can use similar techniques and intuitions as $L_{CE}^T$.

5 Simulations

In this section, we present our simulation results. We evaluate only Algorithm 1 as we are not aware of other natural baselines for this problem. We focus our evaluation on the CE loss; computing the FD loss is computationally expensive as it requires taking an infimum over the Pareto-front. Our evaluation considers two types of utilities.

1. CES utilities: These are as described in Example 2.2.

2. Amdahl’s utilities: The Amdahl’s utility function, described in Zahedi et al. [55], is used to model the performance of jobs distributed across heterogeneous machines in a data center. This utility is motivated by Amdahl’s Law [3], which models a job’s speed up in terms of the fraction of work that can be parallelized. To define this utility, let $0 < f_{ij} < 1$ denote the parallel fraction of user $i$’s job on machine type $j$. Then, an agent’s Amdahl utility is,

$$ u_i(x) = \sum_{j=1}^{m} \theta_{ij} \phi_{ij}(x_{ij}), \quad \text{where, } \phi_{ij}(x_{ij}) = \frac{x_{ij}}{f_i + (1 - f_i)x_{ij}}. $$

(7)

Here, $\phi_{ij}(x_{ij})$ is the relative speedup produced by the allocation $x_{ij}$. 

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Figure 1: The CE loss $L_{CE}^T$ vs the number of rounds $T$. The top row shows results for the simulations with $m = 3$ resource types and $n = 5$ agents with CES utilities. In the three figures, we present results for $\rho = 0.5$, $\rho = 0.75$, and $\rho = 1$ respectively (see Example 2.2). In the bottom row, we simulate $m = 2$ resource types and $n = 8$ agents with Amdahl’s utilities. The three figures correspond to $f = 0.2$, $f = 0.3$, and $f = 0.5$ (see (7)). All figures show results which are averaged over 10 runs, and the shaded region shows the standard error at each time $T$.

Note that both the CES and Amdahl utilities belong to our class $\mathcal{P}$ given in (4).

Our first set of experiments consider an environment with $m = 3$ resource types and $n = 5$ agents, all of whom have CES utilities. We conduct three experiments with different values for the elasticity of substitution $\rho$. Our second set of experiments consider an environment with $m = 2$ resource types and $n = 8$ agents, all of whom have Amdahl’s utilities. We conduct three experiments with different values for the parallel fraction $f_{ij}$. All experiments are run for $T = 2000$ rounds, where we set $\delta = \frac{1}{T}$. The results are given in Figure 1. They show that the CE loss grows sublinearly with $T$ which indicates that the algorithm is able to learn utilities and compute CE equilibria.

**Experimental details:** To compute the CE in line 20 of Algorithm 1, we use the proportional response dynamics procedure from [56] with 10 iterations. To compute $L_{CE}^T$, we need to maximize each agent’s utility subject to a budget. We approximate this by empirically as follows: first, we randomly sample feasible allocations $y$ from a simplex, accept those that cost no more price than the agent’s endowment, and then take the maximum. We sampled up to 50 accepted samples in each round. All experiments are run on a AWS EC2 p3.2xlarge machine.

6 Conclusion

We introduced and studied the problem of online learning a competitive equilibrium in an exchange economy, without a priori knowledge of agents’ utilities. We quantify the learning performance via two losses, the first motivated from the definition of competitive equilibrium, and the second by
fairness and Pareto-efficiency considerations in fair division. We develop a randomized algorithm which achieves \( \tilde{O}(nm\sqrt{T}) \) loss after \( T \) rounds under both loss criteria, and corroborate these theoretical results with simulations. While our work takes the first step towards sequentially learning a market equilibrium in general exchange economies, and interesting avenue for future work would be to study learning approaches in broader classes of agent utilities and market dynamics.

Our work addresses the technical challenge of efficient and fair allocations when agents’ utilities are unknown, with a specific fairness notion on sharing incentives. However, we emphasize that there are other notions of fairness in the fair division literature, with varying connections to prior sociopolitical framing. On a broader societal level, this work, as with many other fair allocation algorithms, if being applied to scenarios where the choice of fairness criteria is not appropriate can lead to potential negative impact. Thus, we emphasize that whether our model is applicable to certain applications should be carefully evaluated by domain experts, along with awareness of the tradeoffs involved.

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A Some Technical Lemmas

We first provide some useful technical lemmas.

**Lemma A.1.** Suppose that \( Z \) is a \( \chi^2_m \) random variable, i.e. \( Z = \sum_{k=1}^m Z_k^2 \), where for all \( k \), \( Z_k \) are i.i.d. random variables drawn from \( \mathcal{N}(0, 1) \). Then,

\[
P(Z > m + \alpha) = \begin{cases} 
  e^{-\frac{\alpha^2}{2m}} & \alpha > m \\
  e^{-\frac{\alpha^2}{8m}} & \alpha \leq m.
\end{cases}
\]

**Proof.** Suppose that \( X \) is sub-exponential random variable with parameters \((\nu, b)\) and expectation \( \mu \). Applying well known tail bounds for sub-exponential random variables (e.g. [53]) yields:

\[
P(X > \mu + \alpha) = \begin{cases} 
  e^{-\frac{\alpha^2}{2\nu^2}} & 0 \leq \alpha \leq \frac{\nu^2}{b} \text{, and} \\
  e^{-\frac{\alpha^2}{b}} & \alpha > \frac{\nu^2}{b}.
\end{cases}
\]

The lemma follows from the fact that a \( \chi^2_m \) random variable is sub-exponential with parameters \((\nu, b) = (2, 4)\).

**Lemma A.2.** (Lower bound for normal distributions) Let \( Z \) be a random variable \( Z \sim \mathcal{N}(0, 1) \), then

\[
P(Z > t) \geq \frac{1}{t + \sqrt{t^2 + 4}} \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}}.
\]

**Proof.** First, from Abramowitz et al. [1] (7.1.13) we have,

\[
e^{x^2} \int_x^\infty e^{-t^2} dt \geq \frac{1}{x + \sqrt{x^2 + 2}}.
\]

Set \( t = \sqrt{2}x \), then the above equation yields:

\[
P(Z > t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{x^2}{2}} dx \geq \frac{1}{t + \sqrt{t^2 + 4}} \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}},
\]

which completes the proof.

**Lemma A.3.** (Azuma-Hoeffding inequality[53]) Let \((Z_s)_{s \geq 0}\) be a super martingale w.r.t. a filtration \((\mathcal{F}_t)_{t \geq 0}\). Let \((B_t)_{t \geq 0}\) be predictable processes w.r.t. \((\mathcal{F}_t)_{t \geq 0}\), such that \(|Z_s - Z_{s-1}| \leq B_s\) for all \( s \geq 1 \) almost surely. Then for any \( \delta > 0 \),

\[
P\left(Z_T - Z_0 \leq \sqrt{2 \log \left( \frac{1}{\delta} \right) \sum_{t=1}^T B_t^2} \right) \geq 1 - \delta.
\]

**Lemma A.4.** \( \forall x \in [0, c], c > 0, \) we have \( x \leq \frac{c}{\log(1+c)} \log(1 + x) \).

**Proof.** The result follows immediately from the fact that the function \( f(x) = \frac{x}{\log(1+x)} \) is non-decreasing on \((0, \infty)\).
Lemma A.5. (Lemma 1, Filippi et al. [22]) Let \( \mathcal{F}_k, k \geq 0 \) be a filtration, \( (m_k; k \geq 0) \) be an \( \mathbb{R}^d \)-valued stochastic process adapted to \( \mathcal{F}_k \). Assume that \( \eta_k \) is conditionally sub-Gaussian in the sense that there exists some \( R > 0 \) such that for any \( \gamma \geq 0, k \geq 1, \mathbb{E}[\exp(\gamma \eta_k) | \mathcal{F}_{k-1}] \leq \exp \left( \frac{\gamma^2 R^2}{2} \right) \) almost surely. Then, consider the martingale \( \xi_t = \sum_{k=1}^{t} m_{k-1} \eta_k \) and the process \( M_t = \sum_{k=1}^{t} m_{k-1} m_k^\top \). Assume that with probability one, the smallest eigenvalue of \( M_d \) is lower bounded by some positive constant \( \lambda_0 \), and that \( \|m_k\| \leq c_m \) almost surely for any \( k \geq 0 \). Then, the following holds true: for any \( 0 < \delta < \min(1, d/e) \) and \( t > \max(d, 2) \), with probability at least \( 1 - \delta \),

\[
\|\xi_t\|_{M_d^{-1}} \leq \kappa R \sqrt{2d \log(t) \log(d/\delta)},
\]

where \( \kappa = \sqrt{3 + 2 \log(1 + \frac{c_m^2}{\lambda_0})} \).

B Bounding \( L^{CE} \)

First, consider any round \( t \). We will let \( \mathcal{F}_t = \sigma \left( \{ (x_{is}, y_{is})_{i=1,s=1}^{n,t-1} \} \right) \) denote the \( \sigma \)-algebra generated by the observations in the first \( t-1 \) rounds. Clearly, \( \{ \mathcal{F}_t \}_{t \geq 0} \) is a filtration. We will denote \( \mathbb{E}_t[\cdot | \mathcal{F}_t] = \mathbb{E}[\cdot] \) to be the expectation when conditioning on the past observations up to round \( t-1 \).

Similarly, define \( P_t(\cdot) \overset{\text{def}}{=} P(\cdot | \mathcal{F}_t) = \mathbb{E}[\cdot | \mathcal{F}_t] \).

Recall that \( \{ \delta_t \}_{t \geq 0} \) are inputs to the algorithm. Similarly, let \( \{ \delta_{2t} \}_{t \geq 0} \) be a sequence. We will specify values for both sequences later in this proof. Given these, further define the following quantities on round \( t \):

\[
\beta_{1t} \overset{\text{def}}{=} \frac{2}{C_\mu} \kappa \sigma \sqrt{2m \log(t)} \sqrt{\log \left( \frac{m}{\delta_t} \right)}
\]

\[
\beta_{2t} \overset{\text{def}}{=} \sqrt{\alpha_t (m + \gamma_{2t})} \quad \text{where,} \quad \gamma_{2t} \overset{\text{def}}{=} \max \left( 8 \log \left( \frac{1}{\delta_{2t}} \right), \sqrt{8m \log \left( \frac{1}{\delta_{2t}} \right)} \right)
\]

\[
\beta_{3t} \overset{\text{def}}{=} L_\mu (\beta_{1t} + \beta_{2t}).
\]

Here, recall that \( L_\mu \) is the Lipschitz constant of \( \mu(\cdot) \), \( C_\mu \) is such that \( C_\mu \overset{\text{def}}{=} \inf_{\theta \in \Theta, x \in \mathcal{X}} \mu (\theta^\top \phi(x)) \), and \( \alpha_t \) is a sequence that is defined and used in Algorithm 1.

Next, we consider the following two events:

\[
A_{it} \overset{\text{def}}{=} \{ \|\theta_i^* - \tilde{\theta}_t\|_{Q_{it}} \leq \beta_{1t} \}, \quad B_{it} \overset{\text{def}}{=} \{ \|\tilde{\theta}_t - \theta_i\|_{Q_{it}} \leq \beta_{2t} \},
\]

where \( Q_{it} \overset{\text{def}}{=} \sum_{s=1}^{t-1} \phi(x_{is}) \phi(x_{is})^\top \) is a design matrix that corresponding to the first \( t-1 \) steps.

Lastly, define

\[
\rho_{it}(x) \overset{\text{def}}{=} \|\phi(x)\|_{Q_{it}^{-1}} = \sqrt{\phi^\top(x) Q_{it}^{-1} \phi(x)},
\]

and

\[
S_{it} \overset{\text{def}}{=} \{ x \in \mathcal{X} : u_i(x_{it}^*) - u_i(x) \geq \beta_{3t} \rho_{it}(x) \},
\]

where \( x_{it}^* = \arg \max_{y \in \mathcal{X}, p_i \geq p_i} u_i(y) \). Here, we used \( \mathcal{X} \) to denote the set of feasible allocations for one agent: \( \{ x \in \mathbb{R}^m : 0 \leq x \leq 1 \} \).
Intuitively, $\bar{x}_{it}$ is the best true optimal affordable allocation for agent $i$ in round $t$ under the price function $p_t$. Since the set $\{y \in \mathcal{X}, p_t^\top y \leq p_t^\top e_i\}$ is a compact set, the maximum is well defined.

Now we begin our analysis with the following lemmas.

**Lemma B.1.** For any round $t > t_0$, $P_t(A_{it}) \geq 1 - \delta_{it}$.

**Proof.** Define function $g_{it}(\theta) = \sum_{s=1}^{t-1} \mu \left( \theta^\top \phi(x_{is}) \right) \phi(x_{is})$. Then by the fundamental theorem of calculus, we have

$$g_{it}(\theta) - g_{it}(\bar{\theta}_{it}) = G_{it}(\theta^*_i - \bar{\theta}_{it}),$$

where $G_{it} = \int_0^1 \nabla g_{it}(s\theta^*_i + (1-s)\bar{\theta}_{it}) \, ds$, and

$$\nabla g_{it}(\theta) = \sum_{s=1}^{t-1} \phi(x_{is}) \phi(x_{is})^\top \mu'(\theta^\top \phi(x_{is})).$$

By the definition of $C_\mu$ and $Q_{it}$, we have that $G_{it} \succeq C_\mu Q_{it} \succeq m^2 I$, where the last inequality follows due to the initialisation scheme. Therefore, $G_{it}$ is invertible and moreover,

$$G_{it}^{-1} \preceq \frac{1}{C_\mu} Q_{it}^{-1}. \quad (8)$$

We can write,

$$\theta^*_i - \bar{\theta}_{it} = G_{it}^{-1} \left( g_{it}(\theta) - g_{it}(\bar{\theta}_{it}) \right). \quad (9)$$

Therefore, we have,

$$\begin{align*}
(\theta^*_i - \bar{\theta}_{it})^\top Q_{it} (\theta^*_i - \bar{\theta}_{it}) & = (g_{it}(\theta) - g_{it}(\bar{\theta}_{it}))^\top G_{it}^{-1} Q_{it} G_{it}^{-1} \left( g_{it}(\theta) - g_{it}(\bar{\theta}_{it}) \right) \\
& \leq \frac{1}{C^2_\mu} \left( g_{it}(\theta) - g_{it}(\bar{\theta}_{it}) \right) Q_{it}^{-1} \left( g_{it}(\theta) - g_{it}(\bar{\theta}_{it}) \right) \\
& = \frac{1}{C^2_\mu} \left\| \left( g_{it}(\theta) - g_{it}(\bar{\theta}_{it}) \right) \right\|_{Q_{it}^{-1}},
\end{align*}$$

where the first equality follows from Eq (8), and the inequality follows from Eq (9).

Therefore,\n
$$\|\theta^*_i - \bar{\theta}_{it}\|_{Q_{it}} \leq \frac{1}{C_\mu} \left\| \left( g_{it}(\theta) - g_{it}(\bar{\theta}_{it}) \right) \right\|_{Q_{it}^{-1}} \leq \frac{2}{C_\mu} \left\| \left( g_{it}(\theta) - g_{it}(\bar{\theta}_{it}^{MLE}) \right) \right\|_{Q_{it}^{-1}} \leq \frac{2}{C_\mu} \left\| \sum_{s=1}^{t-1} \phi(x_{is})^\top (Y_{is} - \mu \left( \phi(x_{is})^\top \theta^*_i \right) \right\|_{Q_{it}^{-1}},$$

where the second inequality is from the triangle inequality, and the last equality is from the definition of $\theta_{it}^{MLE}$ and $g_{it}$.

Let $A_{it}$ denote the event that

$$\left\| \sum_{s=1}^{t-1} \phi(x_{is})^\top (Y_{is} - \mu \left( \phi(x_{is})^\top \theta^*_i \right) \right\|_{Q_{it}^{-1}} \leq \kappa \sigma \sqrt{2m \log(t)} \sqrt{\log\left( \frac{d}{\delta_t} \right)},$$

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then we have $A_{it}$ holds with probability at least $\delta_t$ by Lemma A.5.

\[ \Box \]

**Lemma B.2.** For any round $t > t_0$, $P_t(B_{it}) \geq 1 - \delta_{2t}$.

**Proof.** First,

\[
P(B_{it}^c) = P(||\theta_{it} - \bar{\theta}_{it}||_Q < \beta_{2t})
\]

\[
= P(||\theta_{it} - \bar{\theta}_{it}||_{\alpha_t^{-2}Q_{it}} > \alpha_t^{-\frac{1}{2}} \beta_{2t})
\]

\[
= P(\sqrt{Z} > \sqrt{M_t \gamma_{2t}}),
\]

where $Z = (\theta_{it} - \bar{\theta}_{it})^\top \alpha_t^{-2}Q_{it} (\theta_{it} - \bar{\theta}_{it})$.

Note that $Z$ is a $\chi^2_m$ random variable. This follows from the fact that $\theta_{it} \sim N(\bar{\theta}_{it}, \alpha_t^{-2}Q_{it}^{-1})$, therefore we have

\[
\alpha_t^{-1}Q_{it}^{1/2} (\theta_{it} - \bar{\theta}_{it}) \sim N(0, I_m).
\]

Denote $y = \alpha_t^{-1}Q_{it}^{1/2} (\theta_{it} - \bar{\theta}_{it})$, then $Z = y^\top y$ thus a $\chi^2_m$ random variable.

Therefore, by Lemma A.1, and the definition that $\gamma_{2t} = \max\left(8 \log\left(\frac{1}{\delta_{2t}}\right), \sqrt{\log\left(\frac{1}{\delta_{2t}}\right)}\right)$, we have

\[
P(B_{it}^c) = P(Z > n + \gamma_{2t}) \leq \delta_{2t},
\]

which completes the proof.

\[ \Box \]

**Lemma B.3.** Let $x$ be arbitrary such that $x \in X$. Then,

\[
P_t(u_{it}(x) > u_i(x)|A_{it}) \geq q_0,
\]

with $q_0 = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2 + \sqrt{6}}} \approx 0.075$.

**Proof.** First, notice that

\[
u_{it}(x) > u_i(x) \iff \mu\left(\theta_{it}^\top \phi(x)\right) > \mu\left(\theta_i^*\right)\top \phi(x)
\]

\[
\iff \theta_{it}^\top \phi(x) > (\theta_i^*)\top \phi(x)
\]

\[
\iff \left(\frac{\theta_{it} - \bar{\theta}_{it}}{\alpha_t^2Q_{it}^{-1}}\phi(x)\right) > \left(\frac{\theta_i^* - \bar{\theta}_{it}}{\alpha_t^2Q_{it}^{-1}}\phi(x)\right).
\]

Since $\theta_{it} \sim N(\bar{\theta}_{it}, \alpha_t^{-2}Q_{it}^{-1})$, we have

\[
(\theta_{it} - \bar{\theta}_{it})\top \phi(x) \sim N(0, \alpha_t^2 \phi(x)\top Q_{it}^{-1}\phi(x)), \quad \iff \quad (\theta_{it} - \bar{\theta}_{it})\top \phi(x) \sim N(0, \alpha_t^2 \rho_{it}(x)).
\]

\[
\iff \left(\frac{\theta_{it} - \bar{\theta}_{it}}{\alpha_t^2Q_{it}^{-1}}\phi(x)\right) \sim N(0, 1).
\]

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From the above we have that
\[ P_t(u_{it}(x) > u_i(x) | A_{it}) = P_t \left( Z > \frac{(\theta_{it} - \bar{\theta}_{it})^\top \phi(x)}{\alpha_t \rho_{it}(x)} | A_{it} \right), \]
where \( Z \sim N(0, 1) \) is sampled independently of the observations, since the randomness in Algorithm 1 can be assumed to be independent of the randomness in the observations. Therefore, under the event \( A_{it} \),
\[
\left| \frac{(\theta_{it} - \bar{\theta}_{it})^\top \phi(x)}{\alpha_t \rho_{it}(x)} \right| = \left| \frac{(\theta_{it} - \bar{\theta}_{it})^\top Q_{it}^{\frac{1}{2}} Q_{it}^{-\frac{1}{2}} \phi(x)}{\alpha_t \rho_{it}(x)} \right| \\
\leq \frac{\|\theta_{it} - \bar{\theta}_{it}\|_Q \|\phi(x)\|_{Q_{it}^{-1}}}{\alpha_t \rho_{it}(x)} \\
\leq \frac{\beta_{it}}{\alpha_t} = \frac{\sqrt{8}}{\alpha_0}.
\]
Here, the first inequality follows from the definition of the matrix norm and the definition of \( A_{it} \), and the second inequality follows from the definition of \( \beta_{it} \).

Therefore, by Lemma A.2, we have
\[
P_t(u_{it}(x) > u_i(x) | A_{it}) \\
= P_{Z \sim N(0, 1)}(Z > \frac{\sqrt{8}}{\alpha_0}) \\
\geq \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\frac{2}{\sqrt{8/\alpha_0^2}} + \sqrt{4 + \sqrt{8/\alpha_0^2}}}} e^{-\frac{4}{\alpha_0^2}}.
\]
Setting \( \alpha_0^2 = 4 \), we have
\[
P_t(u_{it}(x) > u_i(x) | A_{it}) \geq \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2 + \sqrt{6}}} e^{-1} \approx 0.075,
\]
which completes the proof.

**Lemma B.4.** Let \( \theta_1, \theta_2 \in \Theta \in \mathbb{R}^m \). Let \( Q \succeq 0, Q \in \mathbb{R}^{m \times m} \) be a positive semi-definite matrix, and \( \rho_Q(x) = \sqrt{\phi(x)^\top Q^{-1} \phi(x)} \). Then,
\[
\left| \mu(\theta_1^\top \phi(x)) - \mu(\theta_2^\top \phi(x)) \right| \leq L_{\mu} \|\theta_1 - \theta_2\|_Q \cdot \rho_Q(x).
\]

**Proof.** This follows from the Lipschitz properties of \( \mu \) and the following simple calculations:
\[
\left| \mu(\theta_1^\top \phi(x)) - \mu(\theta_2^\top \phi(x)) \right| \\
= L_{\mu}|(\theta_1 - \theta_2)^\top \phi(x)| \\
\leq L_{\mu}\|\theta_1 - \theta_2\|_Q \|\phi(x)\|_{Q^{-1}} \\
= L_{\mu}\|\theta_1 - \theta_2\| \cdot \rho_Q(x).
\]

\[\blacksquare\]
Lemma B.5. For any round \( t > t_0 \), \( P_i(x_{it} \notin S_{it}) \geq q_0(1 - \delta_t) - \delta_{2t} \).

Proof. First, when event \( B_{it} \) holds, by lemma B.4, we have that for all \( x \),

\[
|u_{it}(x) - \bar{u}_{it}(x)| \leq L \mu \beta_{2t} \rho_{it}(x).
\]

Note that by definition, \( u_{it}(x) = \mu \left((\theta_{it})^\top \phi(x)\right) \) and \( \bar{u}_{it}(x) = \mu \left((\bar{\theta}_{it})^\top \phi(x)\right) \). Therefore,

\[
\bar{u}_{it}(x) - u_{it}(x) > -L\mu \beta_{2t} \rho_{it}(x).
\] (10)

On the other side, under event \( A_{it} \), by lemma B.4, we have that for all \( x \),

\[
|u_{i}(x) - \bar{u}_{it}(x)| \leq L \mu \beta_{1t} \rho_{it}(x).
\] (11)

Moreover, recall that by definition for any \( x \in S_{it} \),

\[
u_{i}(x_{it}^*) - u_{i}(x) \geq \beta_{3t} \rho_{it}(x).
\] (12)

Therefore, consider any \( x \in S_{it} \), and under the condition that \( A_{it} \cap B_{it} \cap \{u_{it}(x_{it}^*) > u_{i}(x_{it}^*)\} \), we have

\[
u_{it}(x_{it}^*) - u_{it}(x) > u_{i}(x_{it}^*) - u_{i}(x)
= (u_{i}(x_{it}^*) - u_{i}(x)) + (u_{i}(x) - \bar{u}_{it}(x)) + (\bar{u}_{it}(x) - u_{it}(x))
> 0,
\] (13)

where the last inequality follows from combining equations Eq (10), Eq (11), Eq (12) and the definition of \( \beta_{3t} \). Hence, Eq (13) implies that, under the same condition, \( x_{it} \notin S_{it} \) since by construction, \( x_{it} \) maximizes \( u_{it} \) under the budget, thus

\[
u_{it}(x_{it}) \geq u_{it}(x_{it}^*),
\]

This further implies that,

\[
P_i(x_{it} \notin S_{it}) \geq P_i \left( u_{it}(x_{it}^*) > u_{it}(x), \forall x \in S_{it} \right)
\geq P_i \left( u_{it}(x_{it}^*) > u_{it}(x), \forall x \in S_{it} | A_{it} \cap B_{it} \{u_{it}(x_{it}^*) > u_{i}(x)\} \right)
\times P(A_{it} \cap B_{it} \{u_{i}(x_{it}^*) > u_{i}(x)\})
= P(A_{it} \cap B_{it} \{u_{i}(x_{it}^*) > u_{i}(x)\})
\geq P(A_{it} \cap \{u_{i}(x_{it}^*) > u_{i}(x)\}) - P(B_{it}^c)
= P(\{u_{i}(x_{it}^*) > u_{i}(x)\} | A_{it}) P(A_{it}) - P(B_{it}^c)
\geq q_0(1 - \delta_t) + \delta_{2t}.
\]

Here, the second and third inequality both from the law of total probability and rearranging terms, and the last inequality follows from Lemma B.1, Lemma B.2 and Lemma B.3, which completes the proof. \( \blacksquare \)

Lemma B.6. For \( t \geq \max(t_0, t_0') \),

\[
\mathbb{E}_i[\ell_{it}] \leq \frac{5}{q_0} \beta_{3t} \mathbb{E}_i[\rho_{it}(x_{it})] + \delta_t + \delta_{2t}.
\]

Here, \( \ell_{it} = (u_{i}(x_{it}^*) - u_{i}(x_{it}))^+ \), and \( t_0' \) is chosen such that, \( \forall t > t_0' \), \( \delta_t < \frac{1}{4} \), \( \delta_{2t} < \frac{q_0}{4} \).
Proof. First, define
\[ x_{it}' = \arg \min_{x: p_{it}' < p_i, x \notin S_{it}} \rho_{it}(x). \]
This implies that,
\[ \mathbb{E}_t[\rho_{it}(x_{it})] \geq \mathbb{E}_t[\rho_{it}(x_{it})|x_{it} \notin S_{it}]P(x \notin S_{it}) \geq \rho_{it}(x_{it}')P(x \notin S_{it}). \]
Therefore, by Lemma B.5, we have
\[ \rho_{it}(x_{it}') \leq \frac{\mathbb{E}_t[\rho_{it}(x_{it})]}{q_0(1 - \delta_t) - \delta_{2t}}. \]
Select \( t_0' \) such that, \( \forall t \geq t_0' \), \( \delta_t = \frac{1}{4} \), and \( \delta_{2t} \leq \frac{2q_0}{T} \), then we have:
\[ \rho_{it}(x_{it}') \leq \frac{2}{q_0} \mathbb{E}_t[\rho_{it}(x_{it})]. \]
Also, under \( A_{it} \cap B_{it} \),
\[ \|\theta^* - \theta_{it}\|_Q < \|\theta^* - \bar{\theta}_{it}\|_Q + \|\bar{\theta}_{it} - \theta_{it}\|_Q \leq \beta_{1t} + \beta_{2t}, \]
where the first inequality follows from triangle inequality, and the second one follows from the definitions of \( A_{it} \) and \( B_{it} \). Hence,
\[ |\mu_{it}(x) - \mu_i(x) \leq L\mu(\beta_{1t} + \beta_{2t})\rho_{it}(x) = \beta_{3t}\rho_{it}(x). \]
Therefore, we have
\[ \ell_{it} = u_i(x_{it}^*) - u_i(x_{it}) = u_i(x_{it}^*) - u_i(x_{it}') + u_i(x_{it}') - u_i(x_{it}) \leq 2\beta_{3t}\rho_{it}(x_{it}') + \beta_{3t}\rho_{it}(x_{it}) \leq \frac{4}{q_0} \beta_{3t}\mathbb{E}[\rho_{it}(x_{it})] + \beta_{3t}\rho_{it}(x_{it}). \]
which further yields
\[ \mathbb{E}_t[\ell_{it}] \leq \mathbb{E}_t[\ell_{it}|A_{it} \cap B_{it}] + \mathbb{E}_t[\ell_{it}|A_{it}^c \cup B_{it}^c]P(A_{it}^c \cup B_{it}^c) \leq \frac{4}{q_0} \beta_{3t}\mathbb{E}[\rho_{it}(x_{it})] + \beta_{3t}\mathbb{E}[\rho_{it}(x_{it})] + \delta_t + \delta_{2t} \]
\[ \leq \frac{5}{q_0} \beta_{3t}\mathbb{E}[\rho_{it}(x_{it})] + \delta_t + \delta_{2t}, \]
which completes the proof.

Lemma B.7. Let \( \delta' > 0 \). Define \( L_{iT} = \sum_{t=1}^T \ell_{it} \). Then, with probability at least \( 1 - \delta' \),
\[ L_{iT} \leq \sum_{t=1}^T (\delta_t + \delta_{2t}) + \tilde{O}\left( m^2 \sqrt{T} \left( \log(T) + \log\left( \frac{1}{\delta'} \right) \right) \right). \]
\textbf{Proof.} First, define for \( s > 1, \)
\[
u_{is} = \ell_{is} - \frac{5\beta_{3s}}{q_0} \rho_{is}(x_{is}) - (\delta_s + \delta_{2s}),
\]
and \( v_{it} = \sum_{s=1}^{t} u_{is}, \) with \( v_{i0} = 0 \) and \( u_{i0} = 0. \) We show that \( \{v_{it}\}, t \geq 0 \) is a super-martingale with respect to the filtration \((F_t)_{t \geq 0}.\)

First, define for \( q_0 \)
\[
\mathbb{E}_t[u_{it}] = \mathbb{E}_t[\ell_{it}] - \frac{5\beta_{3t}}{q_0} \mathbb{E}_t[\rho_{it}(x_{it})] - (\delta_t + \delta_{2t}) \leq 0.
\]
Moreover,
\[
|v_{it} - v_{i,t-1}| \leq |\ell_{it}| + \frac{5\beta_{3t}}{q_0} |\rho_{it}(x_{it})| + (\delta_t + \delta_{2t})
\leq 1 + \frac{5\beta_{3t}}{q_0} \|\phi(1)\|_2 + 1
\leq \frac{7\beta_{3t}}{q_0} \|\phi(1)\|_2 \triangleq D_t.
\]

Therefore, by Lemma A.3, with probability at least \( 1 - \delta', \) we have that
\[
v_{iT} - v_{i,0} \leq \sqrt{s \log \left( \frac{1}{\delta'} \sum_{t=1}^{T} D_t^2 \right)} \leq \frac{7\beta_{3t}}{q_0} \|\phi(1)\|_2 \sqrt{\frac{2 \log(1/\delta')}{{m^2}}} T.
\]

Therefore, we have
\[
L_{iT} = \sum_{s=1}^{T} \ell_{is} \leq \sum_{t=1}^{T} (\delta_t + \delta_{2t}) + \frac{5\beta_{3t}}{q_0} \sum_{t=1}^{T} \rho_{it}(x_{it}) + \frac{7\beta_{3t}}{q_0} \|\phi(1)\|_2 \sqrt{\frac{2 \log(1/\delta')}{{m^2}}} T. \tag{15}
\]

Now it remains to bound \( \sum_{t=1}^{T} \rho_{it}(x_{it}). \) Since \( Q_{it} \geq m^2 I, \) by the definition of \( \rho_{it}(x_{it}), \) we have
\[
\rho_{it}x_{it} \leq \|\phi(1)\|_2 \frac{1}{m}.
\]
Hence, by Lemma A.4 and rearranging terms, we have
\[
\sum_{t=t_0}^{T} \rho_{it}^2(x_{it}) \leq \frac{\|\phi(1)\|_2^2}{m^2} \log \left( \frac{1}{1 + \|\phi(1)\|_2^2} \right) \sum_{t=t_0}^{T} \log \left( 1 + \phi^T(x_{it})Q_{it}^{-1}\phi(x_{it}) \right). \tag{16}
\]

Also notice that,
\[
\sum_{t=t_0}^{T} \log \left( 1 + \phi^T(x_{it})Q_{it}^{-1}\phi(x_{it}) \right) = \log \Pi_{t=t_0}^{T} \left( 1 + \|\phi(x_{it})\|_{Q_{it}^{-1}}^2 \right) = \log \frac{\det(Q_{iT})}{\prod_{t=0}^{T} (Q_{it})}.
\]

Note that the trace of \( Q_{i,t+1} \) is upper-bounded by \( t \cdot \|\phi(1)\|_2, \) then given that the trace of the positive definite matrix \( Q_{iT} \) is equal to the sum of its eigenvalues, we have that \( \det(Q_{iT}) \leq \left( t \cdot \|\phi(1)\|_2 \right)^m. \) Moreover, \( \det(Q_{i,t_0}) \geq (m^2)^m, \) therefore,
\[
\sum_{t=t_0}^{T} \log \left( 1 + \phi^T(x_{it})Q_{it}^{-1}\phi(x_{it}) \right) \leq m \log \left( \frac{\|\phi(1)\|_2^2 T}{m^2} \right).
\]
Combining with Eq (16), and applying Cauchy-Schwartz inequality, we have
\[
\sum_{t=t_0}^T \rho_t(x_{it}) \leq \sqrt{T \sum_{t=t_0}^T \rho_t^2(x_{it})} \leq \sqrt{T \sum_{t=t_0}^T \frac{\|\phi(1)\|_2^2}{m^2} \log \left( \frac{1}{1 + \|\phi(1)\|_2^2} \right) \log \left( \frac{\|\phi(1)\|_2^2 T}{m^2} \right)}.
\tag{17}
\]

Putting together Eq (15) and Eq (17), with the fact that \(\|\phi(1)\|_2 = O(\sqrt{m})\), e.g. \(\|\phi(1)\|_2 \leq c_\phi \sqrt{m}\), we have that with probability at least \(1 - \delta\),
\[
L_{iT} \leq T \sum_{t=1}^T (\delta_1 + \delta_2) + \beta_{3T} \sqrt{T} \frac{\sqrt{T}}{q_0} \left( 5c_\phi m \sqrt{\log \left( \frac{1}{1 + \|\phi(1)\|_2^2} \right) \log \left( \frac{\|\phi(1)\|_2^2 T}{m^2} \right)} + 7c_\phi^2 \sqrt{m} \sqrt{2 \log \left( \frac{1}{\delta} \right)} \right) \\
= T \sum_{t=1}^T (\delta_1 + \delta_2) + \tilde{O} \left( m \sqrt{T} \left( \log(T) + \log \left( \frac{1}{\delta} \right) \right) \right),
\]
where the last step comes from the fact that \(\beta_{3T} = \tilde{O}(m)\). This completes the proof. \(\blacksquare\)

### B.1 Proof of Theorem 4.1 for \(L_{CE}\)

**Corollary B.8.** With probability at least \(1 - \delta\),
\[
L_{CE}^T \leq n T \sum_{t=1}^T (\delta_1 + \delta_2) + \tilde{O} \left( nm \sqrt{T} \left( \log(T) + \log \left( \frac{1}{\delta} \right) \right) \right)
\]

**Proof.** This a direct result from Lemma B.7 and the definition of \(L_{CE}^T\): With probability at least \(1 - \delta\),
\[
L_{CE}^T = \sum_{i=1}^n \sum_{t=1}^T \left( \max_{y : \vec{p}(y) \leq \vec{p}(e_i)} u_i(y) - u_i(x_{it}) \right) ^+ \\
\leq nm \sum_{t=1}^T (\delta_1 + \delta_2) + \tilde{O} \left( nm \sqrt{T} \left( \log(T) + \log \left( \frac{1}{\delta} \right) \right) \right).
\]

**Proof.** Choose \(\delta_1 = \delta_2 = \frac{2\delta}{n \pi \sigma^2}\). Then,
\[
\sum_{t=1}^T (\delta_1 + \delta_2) \leq \frac{2}{3} \delta.
\]

Also choose \(\delta' = \frac{\delta}{3}\), then by Corollary B.8, with probability at least \(1 - \delta\),
\[
L_{CE}^T = O \left( nm \sqrt{T} \left( \log \left( \frac{\delta}{3} \right) + \log(T) \right) \right).
\]

\(\blacksquare\)
B.2 Proof of Theorem 4.2 for $L^{CE}$

Proof. Choose $\delta_1 = \delta_2 = \delta' = \frac{1}{T}$. Denote the event where $L^{CE}_T = O \left(nm^2 \sqrt{T} \left( \log \left( \frac{\delta}{3} \right) + \log(T) \right) \right)$ holds as $\mathcal{E}$.

Then, by Lemma B.7,

$$
\mathbb{E}[L^{CE}_T] = \mathbb{E}[L^{CE}_T|\mathcal{E}] + \mathbb{E}[L^{CE}_T|\mathcal{E}^c] P(\mathcal{E}^c) \leq 2 + O \left(nm^2 \sqrt{T} \left( \log \left( \frac{\delta}{3} \right) + \log(T) \right) \right),
$$

which completes the proof. ■

C Bounding $L^{FD}$

Recall that the definition of $\ell^{FD}$ is directly based on the requirements of Pareto efficiency and fair share: $\ell^{FD}(x) \overset{\text{def}}{=} \max(\ell^{PE}(x), \ell^{SI}(x))$, where $\ell^{PE}(x) = \min_{x' \in \mathcal{P}} \sum_{i=1}^n (u_i(x') - u_i(x))^+$; and $\ell^{SI}(x) = \sum_{i=1}^n (u_i(x) - u_i(x^*))^+$.

To bound $L^{FD}$, we first provide a useful lemma which shows that $\ell^{SI}$ is a weaker notion than $\ell^{CE}$.

Lemma C.1. For any allocation $x$ and price $p$, $\ell^{SI}(x) \leq \ell^{CE}(x, p)$.

Proof. This simply uses the fact that an agent’s endowment is always affordable under any price vector $p$. Therefore,

$$
\ell^{SI}(x) = \sum_{i=1}^n (u_i(x) - u_i(x^*))^+
\leq \sum_{i=1}^n \left( \max_{y \cdot p^T e_i \geq p^T e_i} u_i(y) - u_i(x) \right)^+, \forall p
= \ell^{CE}(x, p),
$$

Having lemma C.1 at hand, the key remaining task is to bound $\ell^{PE}$. We will show that this can be achieved by an analogous analysis as in Section B.1, but with some key differences.

First, we define $\hat{S}_{it}$ (in comparison to $S_{it}$ used in Section B.1):

$$
\hat{S}_{it} \overset{\text{def}}{=} \{ x \in \mathcal{X} : u_i(x^*_t) - u_i(x) \geq \beta_{it} \rho_{it}(x) \},
$$

where $x^* \in \mathbb{R}^{n \times m}$ is the unique equilibrium allocation. Note that $\hat{S}_{it}$ shares a similar spirit as $S_{it}$, which is used in Section B.1, but with a different referencing point $x^*$.

We show a key lemma which provides a lower bound on $P(x \notin \hat{S}_{it})$.

Lemma C.2. For any round $t > t_0$, $P_t(x_{it} \notin \hat{S}_{it}) \geq q_0(1 - \delta_t) - \delta_{2t}$.

Proof. First, when event $B_{it}$ holds, by lemma B.4, we have that for all $x$,

$$
|u_{it}(x) - \bar{u}_{it}(x)| \leq L \beta_{2t} \rho_{it}(x).
$$
Note that by definition, \( u_{it}(x) = \mu \left( (\theta_{it})^\top \phi(x) \right) \), and \( \bar{u}_{it}(x) = \mu \left( (\bar{\theta}_{it})^\top \phi(x) \right) \). Therefore, 

\[
\bar{u}_{it}(x) - u_{it}(x) > -L_\mu \beta_{2t} \rho_{it}(x). \tag{18}
\]

On the other side, under event \( A_{it} \), by lemma B.4, we have that for all \( x \),

\[
|u_{i}(x) - \bar{u}_{it}(x)| \leq L_\mu \beta_{2t} \rho_{it}(x). \tag{19}
\]

Moreover, recall that by definition for any \( x \in \hat{S}_{it} \),

\[
u_i(x^*_i) - \nu_i(x) \geq \beta_{3t} \rho_{it}(x). \tag{20}
\]

Therefore, consider any \( x \in \hat{S}_{it} \), and under the condition that \( A_{it} \cap B_{it} \cap \{u_{it}(x^*_i) > u_{i}(x^*_i)\} \), we have

\[
\nu_{it}(x^*_i) - \nu_{it}(x) > \nu_{i}(x^*_i) - \nu_{i}(x)
= (\nu_{i}(x^*_i) - \nu_{i}(x)) + (\nu_{i}(x) - \bar{\nu}_{it}(x)) + (\bar{\nu}_{it}(x) - \nu_{it}(x))
> 0,
\]

where the last inequality follows from combining equations Eq (18), Eq (19), Eq (20) and the definition of \( \beta_{3t} \). Moreover, recall that \( x_{it} \) maximizes \( u_{it} \) under the budget, thus

\[
u_{it}(x_{it}) \geq \nu_{it}(x^*_i),
\]

Therefore, Eq (21) implies that, \( x_{it} \notin \hat{S}_{it} \). This further implies,

\[
P_t(x_{it} \notin \hat{S}_{it}) \geq P_t \left( \nu_{it}(x^*_i) > \nu_{it}(x), \forall x \in \hat{S}_{it} \right)
\geq P_t \left( \nu_{it}(x^*_i) > \nu_{it}(x), \forall x \in \hat{S}_{it} | A_{it} \cap B_{it} \{u_{it}(x^*_i) > u_{i}(x)\} \right)
\cdot P(A_{it} \cap B_{it} \{u_{it}(x^*_i) > u_{i}(x)\})
= P(A_{it} \cap B_{it} \{u_{it}(x^*_i) > u_{i}(x)\})
\geq P(A_{it} \{u_{i}(x^*_i) > u_{i}(x)\}) - P(B_{it}^c)
= P(\|u_{it}(x^*_i) > u_{i}(x)\| A_{it}) P(A_{it}) - P(B_{it}^c)
\geq q_0 (1 - \delta_t) + \delta_{2t}.
\]

Here, the second and third inequality both from the law of total probability and rearranging terms, and the last inequality follows from Lemma B.1, Lemma B.2 and Lemma B.3, which completes the proof. \( \blacksquare \)

**Lemma C.3.** At any round \( t > t_0 \), define \( x'^{\mu}_{it} \equiv \arg\min_{x \in \hat{S}_{it}, p_{it} \geq p_{i}} \rho_{it}(x_{it}) \), then we have

\[
l^{\text{PE}}(x_{it}) \leq \sum_{i \in [n]} b^{\text{PE}}_{it},
\]

with probability at least \( 1 - \delta_t - \delta_{2t} \), and \( b^{\text{PE}}_{it} = 2\beta_{3t} \rho_{it}(x'^{\mu}_{it}) + \beta_{3t} \rho_{it}(x_{it}) \).
Proof. Begin with the definition of $\ell^{PE}$, we have

$$\ell^{PE}(x_t) = \min_{x \in PE} \sum_{i \in [n]} (u_i(x_i) - u_i(x_{it}))$$

$$\leq \sum_{i \in [n]} (u_i(x_i^*) - u_i(x_{it}))$$

$$= \sum_{i \in [n]} (u_i(x_i^*) - u_i(x_{it}^*)) + \sum_{i \in [n]} (u_i(x_{it}^*) - u_i(x_{it}))$$

$$\leq \beta_{3t} \sum_{i \in [n]} \rho_{it}(x_{it}^*) + \sum_{i \in [n]} (u_i(x_{it}^*) - u_i(x_{it})),$$

where the last inequality follows from this definition. Moreover, we have

$$u_i(x_{it}^*) \leq u_{it}(x_{it}^*) + \beta_{3t} \rho_{it}(x_{it}^*),$$

and

$$u_i(x_{it}) \geq u_{it}(x_{it}) - \beta_{3t} \rho_{it}(x_{it}).$$

under the event $A_{it} \cap B_{it}$, by Eq (18) and Eq (19). Putting these together yields

$$\ell^{PE}(x_t) \leq 2 \beta_{3t} \sum_{i \in [n]} \rho_{it}(x_{it}^*) + \beta_{3t} \sum_{i \in [n]} \rho_{it}(x_{it}) \defeq \sum_{i \in [n]} b_{it}^{PE},$$

which completes the proof.  

Now we show that the above result leads to the lemma below, which shows a analogous guarantee as we obtained in lemma B.6.

**Lemma C.4.** For $t \geq \max(t_0, t'_0)$,

$$\mathbb{E}_t[\sum_{i \in [n]} \ell^{FD}_{it}] \leq \frac{5}{q_0} \beta_{3t} \mathbb{E}_t[\sum_{i \in [n]} \rho_{it}(x_{it})] + n(\delta_t + \delta_{2t}).$$

Here, $t'_0$ is chosen such that, $\forall t > t'_0$, $\delta_t < \frac{1}{4}$, $\delta_{2t} < \frac{q_0}{4}$.

**Proof.** First, note that, by the definition of $x_{it}^*$,

$$\mathbb{E}_t[\rho_{it}(x_{it})] \geq \mathbb{E}_t[\rho_{it}(x_{it}) | x_{it} \notin \mathcal{S}_{it}] P(x \notin \mathcal{S}_{it}) \geq \rho_{it}(x_{it}^*) P(x \notin \mathcal{S}_{it}).$$

Moreover, combining the above with Lemma C.2, we have

$$\rho_{it}(x_{it}^*) \leq \frac{\mathbb{E}_t[\rho_{it}(x_{it})]}{q_0(1 - \delta_t) - \delta_{2t}}.$$

Select $t'_0$ such that, $\forall t > t'_0$, $\delta_t = \frac{1}{4}$, and $\delta_{2t} = \frac{q_0}{4}$, then we have:

$$\rho_{it}(x_{it}^*) \leq \frac{2}{q_0} \mathbb{E}_t[\rho_{it}(x_{it})].$$
Also, under $A_{it} \cap B_{it}$,

$$\|\theta^*_i - \theta_{it}\|_{Q_{it}} \leq \|\theta^*_i - \theta_{it}\|_{Q_{it}} + \|\theta_{it} - \theta_{it}\|_{Q_{it}} \leq \beta_{1t} + \beta_{2t},$$

where the first inequality follows from triangle inequality, and the second one follows from the definitions of $A_{it}$ and $B_{it}$. Hence,

$$|\mu_{it}(x) - \mu_{i}(x)| \leq L_{\mu}(\beta_{1t} + \beta_{2t}) \rho_{it}(x) = \beta_{3t} \rho_{it}(x).$$

Therefore, we have

$$\sum_{i \in [n]} b_{it}^{PE} \leq 2 \beta_{3t} \sum_{i \in [n]} \rho_{it}(x'_{it}) + \beta_{3t} \sum_{i \in [n]} \rho_{it}(x_{it})$$

$$\leq \frac{4}{q_0} \beta_{3t} \mathbb{E} \left[ \sum_{i \in [n]} \rho_{it}(x_{it}) \right] + \beta_{3t} \sum_{i \in [n]} \rho_{it}(x_{it}).$$

Moreover, by Lemma C.1 and eq (14) which holds under the same condition of $A_{it} \cap B_{it}$, we have

$$\sum_{i \in [n]} \ell_{it}^{FD} \leq \sum_{i \in [n]} \max \{ \ell_{it}^{PE}, \ell_{it}^{SI} \}$$

$$\leq 2 \beta_{3t} \sum_{i \in [n]} \rho_{it}(x''_{it}) + \beta_{3t} \sum_{i \in [n]} \rho_{it}(x_{it})$$

$$\leq \frac{4}{q_0} \beta_{3t} \mathbb{E} \left[ \sum_{i \in [n]} \rho_{it}(x_{it}) \right] + \beta_{3t} \sum_{i \in [n]} \rho_{it}(x_{it}).$$

Therefore, we have

$$\mathbb{E}_t \left[ \sum_{i \in [n]} \ell_{it}^{FD} \right] \leq \mathbb{E}_t \left[ \sum_{i \in [n]} b_{it}^{PE} \right] \leq \frac{4}{q_0} \beta_{3t} \mathbb{E} \left[ \sum_{i \in [n]} \rho_{it}(x_{it}) \right] + \beta_{3t} \mathbb{E}_t \left[ \sum_{i \in [n]} \rho_{it}(x_{it}) \right] + n(\delta_t + \delta_{2t})$$

$$\leq \frac{5}{q_0} \beta_{3t} \mathbb{E}_t \left[ \sum_{i \in [n]} \rho_{it}(x_{it}) \right] + n(\delta_t + \delta_{2t}),$$

which completes the proof. 

With the above lemmas at hand, we are now ready to provide a proof of Theorem 4.1 for $L_{T}^{FD}$.

### C.1 Proof of Theorem 4.1 for $L_{T}^{FD}$

**Proof.** Lemma C.4 shows a analog guarantee as we obtained in lemma B.6 for the $L_{T}^{FD}$ loss function. Therefore, following the same steps in lemma B.7, we have that with probability at least $1 - \delta'$
where $\delta'$ will be specified momentarily,

$$L^{FD}_T = \sum_{t=0}^{T} \sum_{i=1}^{n} \ell^{FD}_{it}$$

$$\leq n \sum_{t=1}^{T} (\delta_t + \delta_{2t}) + \frac{n\beta_3}{q_0} \sqrt{T} \left( 5c_{i,\phi} \sqrt{m} \left( \frac{1}{\log(1 + \frac{\|\phi(1)\|^2T}{m^2})} \log \left( \frac{\|\phi(1)\|^2T}{m^2} \right) \right) \right)$$

$$+ 7c_\phi^2 \sqrt{m} \sqrt{2 \log(\frac{1}{\delta'})}$$

$$= n \sum_{t=1}^{T} (\delta_t + \delta_{2t}) + O \left( nm\sqrt{T} \left( \log(T) + \log(\frac{1}{\delta'}) \right) \right). \tag{22}$$

Choose $\delta_t = \delta_{2t} = \frac{2\delta}{n\pi^2T^2}$. Then,

$$\sum_{t=1}^{T} (\delta_t + \delta_{2t}) \leq \frac{2}{3} \delta.$$ 

Also choose $\delta' = \delta / 3$, then by Eq (22), with probability at least $1 - \delta$,

$$L^{FD}_T = O \left( nm\sqrt{T} \left( \log\left( \frac{\delta}{3} \right) + \log(T) \right) \right).$$

\[\square\]

### C.2 Proof of Theorem 4.2 for $L^{FD}$

**Proof.** The theorem results follow from eq (22). Choose $\delta_{1t} = \delta_{2t} = \delta' = \frac{1}{T}$ and denote the event where $L^{FD}_T = O \left( nm^2\sqrt{T} \left( \log \left( \frac{\delta}{3} \right) + \log(T) \right) \right)$ holds as $E$. Then,

$$E[L^{FD}_T] = E[L^{PE}_T | E] + E[L^{FD}_T | E^c]P(E^c) \leq O \left( nm^2\sqrt{T} \left( \log \left( \frac{\delta}{3} \right) + \log(T) \right) \right),$$

which completes the proof. \[\square\]

### D On the Loss Functions

We provide an example to demonstrate that the FD loss (3), while more interpretable than the CE loss (2), may not capture all properties of an equilibrium.

For this consider the following example with $n = 3$ agents and $m = 2$ resources where the endowments of agent 1, agent 2, and agent 3 are $e_1 = (0.45, 0.05)$, $e_2 = (0.45, 0.05)$, and $e_3 = (0.1, 0.9)$ respectively. Their utilities are:

$$u_1(x_1) = 0.1x_{11} + x_{12}, \quad u_2(x_2) = 0.2x_{21} + x_{22}, \quad u_3(x_3) = x_{31} + 0.1x_{32}.$$
The utilities of the three users if they were to simply use their endowment is, \( u_1(e_1) = 0.1 \times 0.45 + 0.05 = 0.095 \), \( u_2(e_2) = 0.14 \), and \( u_3(e_3) = 0.19 \). We find that while agents 1 and 2 benefit more from the second resource, they have more of the first resource in their endowments and vice versa for agent 3. By exchanging resources, we can obtain a more efficient allocation.

The unique equilibrium prices for the two goods are \( p^\star = (1/2,1/2) \) and the allocations are \( x_1^\star = (0,0.5) \) for agent 1, \( x_2^\star = (0,0.5) \) for agent 2, and \( x_3^\star = (1.0,0.0) \) for agent 3. The utilities of the agents under the equilibrium allocations are \( u_1(x_1) = 0.5 \), \( u_2(x_2) = 0.5 \), and \( u_3(x_3) = 1.0 \). Here, we find that by the definition of CE, \( \ell^{\text{PE}}(x^\star,p^\star) = 0 \). It can also be verified that \( \ell^{\text{FD}}(x^\star,p^\star) = 0 \).

In contrast, consider the following allocation for the 3 users: \( x_1 = (0.35,0.49) \) for agent 1, \( x_2 = (0.35,0.49) \) for agent 2, and \( x_3 = (0.3,0.02) \) for agent 3. Here, the utilities are \( u_1(x_1) = 0.1 \times 0.35 + 0.49 = 0.525 \), \( u_2(x_2) = 0.56 \), and \( u_3(x_3) = 0.3002 \). This allocation is both PE (as the utility of one user can only be increased by taking resources from someone else), and SI (as all three users are better off than having their endowments). Therefore, \( \ell^{\text{FD}}((x_1,x_2,x_3)) = 0 \). However, user 3 might complain that their contribution of resource 2 (which was useful for users 1 and 2) has not been properly accounted for in the allocation. Specifically, there do not exist a set of prices \( p \) for which \( \ell^{\text{PE}}(x,p) = 0 \). This example illustrates the role of prices in this economy: it allows us to value the resources relative to each other based on the demand.