Cosmology with Planck T-E correlation coefficient

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Tensions in cosmological parameters measurement motivate a revisit of the effects of instrumental systematics. In this article, we focus on the Pearson’s correlation coefficient of the cosmic microwave background temperature and polarization E modes $R_{\ell}^{TE}$ which has the property of not being biased by multiplicative instrumental systematics. We build a $R_{\ell}^{TE}$-based likelihood for the Planck data, and present the first constraints on ΛCDM parameters from the correlation coefficient. Our results are compatible with parameters derived from a power spectra based likelihood. In particular the value of the Hubble parameter $H_0$ characterizing the expansion of the Universe today, $67.5 \pm 1.3$ km/s/Mpc, is consistent with the ones inferred from standard CMB analysis. We also discuss the consistency of the Planck correlation coefficient with the one computed from the most recent ACTPol power spectra.

I. INTRODUCTION

Increasingly precise constraints on ΛCDM parameters from different probes have revealed tensions between measurements from early and late time Universe, including the well-known $H_0$ tension. The value of $H_0$ inferred from Cosmic Microwave Background (CMB) temperature, polarization and lensing anisotropies by the Planck Collaboration is $67.36 \pm 0.54$ km/s/Mpc [1]. High resolution CMB observations from ground based experiments such as ACTPol and SPT3G have also independently measured $H_0$ and found $67.9 \pm 1.5$ km/s/Mpc [2] and $68.8 \pm 1.5$ km/s/Mpc [3] respectively. All of these measurements are inconsistent with the latest measurement from the Cepheids-calibrated cosmic distance ladder $H_0 = 73.2 \pm 1.3$ km/s/Mpc [4].

This discrepancy could indicate the need for new physics beyond ΛCDM. Many models have already been proposed to solve the Hubble tension. Modifications to early time physics, such as new physics that change the physical size of the sound horizon at recombination, are investigated and the different ways of solving the current tension have been reviewed in [5, 6]. Upcoming cosmic variance limited measurements of CMB polarization for a wide multipole range from Simons Observatory [7] and CMB-S4 [8] will put severe constraints on these models.

Another hypothesis which could explain these tensions is the presence of remaining systematic effects not accounted for in the data analysis. The presence of bias in the cosmic distance ladder data analysis is actively studied, in particular biases coming from supernovae light curves standardization [9, 10]. An alternative calibration of cosmic distances, using the Tip of the Red Giant Branch (TRGB) is also explored, giving another measurement of the Hubble parameter [11, 12].

In this paper we focus on possible instrumental systematic effects that can affect the measurement of CMB anisotropies. A way of testing this possibility is to use observables that are less sensitive to the exact instrument model. One of these observables is the Pearson’s correlation coefficient of T and E modes $R_{\ell}^{TE} = C_{\ell}^{TE} / \sqrt{C_{\ell}^{TT} C_{\ell}^{EE}}$. It was mentioned in [13], in which the authors give geometrical interpretations of the correlation coefficient as a cosine or as a decorrelation angle. Correlation coefficients have also been used to look at the correlations between galaxy surveys and CMB-derived lensing power spectrum in an unbiased way [14]. $R_{\ell}^{TE}$ statistical properties have been discussed in details in [15]. This observable has been shown to be robust against multiplicative instrumental systematics, such as beam error, calibration, polarization efficiency uncertainties or unmodelled transfer functions.

This work presents the first estimation of the cosmological parameters from $R_{\ell}^{TE}$. The nearly Gaussian nature of the correlation coefficient, at scales where we can ensure a high EE signal-to-noise ratio, makes possible the use of a Gaussian likelihood function to constrain cosmological parameters and to compare with constraints from the combination of $C_{\ell}^{TT}$, $C_{\ell}^{EE}$ and $C_{\ell}^{TE}$ power spectra.

The paper is organized as follows. In section II we illustrate the effect of different systematic effects on cosmological parameters constrained from the CMB power spectra. In section III we discuss the construction of a $R_{\ell}^{TE}$-based likelihood. In section IV we show the results of a $R_{\ell}^{TE}$-based analysis for the Planck PR4 dataset. We conclude in section V.

II. EFFECT OF SYSTEMATICS IN POWER SPECTRA BASED LIKELIHOOD

In this section, we introduce the dataset and likelihood used in the rest of the paper, we then study the impact of systematics on cosmological parameter estimation using a set of biased simulations.
A. Dataset and $C_{\ell}$-based likelihood

We use the latest Planck data release (PR4). The maps were produced using the NPIPE processing pipeline that jointly analyzes data from Planck High Frequency Instrument (HFI) and Planck Low Frequency Instrument (LFI) [16].

Cosmological parameters are estimated using the Planck HiLLiPoP (High-L Likelihood Polarized for Planck) likelihood. The code is publicly available on github\(^1\). This is a multifrequency likelihood for the Planck cosmology channels : 100, 143 and 217 GHz. The dataset consists of two split map for each frequency. From these maps, we obtain 15 cross power spectra from which we compute the six cross-frequency spectra used in the likelihood. The foreground model includes galactic dust, CIB, tSZ & kSZ contributions, tSZ-CIB correlation and Poisson-like distributed point source foregrounds. A more detailed description is given in [17]. The likelihood assumes that the $TT$, $TE$ and $EE$ CMB power spectra follow a Gaussian distribution, which is a good assumption at high multipoles ($\ell > 30$). The likelihood has been tested extensively against the Plik likelihood [13].

In section II B, we study the impact of systematics on simulated Planck data. Cosmological constraints from the Planck PR4 dataset are presented in section IV.

B. Bias on the cosmological parameters

The correlation coefficient is designed to be insensitive to any multiplicative bias. To illustrate the effect of this kind of bias on cosmological parameters estimated from the combination of the $TT$, $TE$ and $EE$ power spectra, we generate a set of biased simulations and fit for the cosmological parameters. The simulations are generated at the power spectra level from the best fit HiLLiPoP cosmology and foreground model, we then use the Planck PR4 covariance matrix to add scatter to the spectra. Fiducial cosmology has been set to $h_{90} = 1.04065$, $\Omega_b h^2 = 0.02231$, $\Omega_c h^2 = 0.1193$, $\ln(10^{10} A_s) = 3.045$, $n_s = 0.9619$ and $\tau = 0.0566$.

We include systematics into our simulated dataset such that the observed temperature $\tilde{a}_{\ell m}^T$ and polarization E-modes $\tilde{a}_{\ell m}^E$ are given by $\tilde{a}_{\ell m}^T = a_{\ell m}^T + \frac{\lambda}{\ell} \tilde{C}_\ell^T$, and $\tilde{a}_{\ell m}^E = a_{\ell m}^P + \frac{\lambda}{\ell} \tilde{C}_\ell^E$.

The measured power spectra are

$$
\tilde{C}_\ell^T = (\epsilon_{\ell}^T)^2 \tilde{C}_\ell^T, \\
\tilde{C}_\ell^E = (\epsilon_{\ell}^P)^2 \tilde{C}_\ell^E, \\
\tilde{C}_\ell^{EE} = (\epsilon_{\ell}^P)^2 C_\ell^{EE}
$$

(1)

Systematic effects can be constant over multipoles (e.g. polarization efficiency) or scale-dependant (e.g. transfer functions).

We obtain the posterior distributions of cosmological parameters using the Markov Chain Monte Carlo (MCMC) algorithm implemented in cobaya [18] with the $C_{\ell}$-based HiLLiPoP likelihood. Figure 1 displays the distributions of the six standard ΛCDM parameters for a set of different transfer functions. We consider three different kinds of systematics :

a. $\epsilon_{\ell}^T = \text{cte}, \epsilon_{\ell}^T = 1$ (Fig. 1a)

b. $\epsilon_{\ell}^T = \epsilon_{\ell}^P(\ell)$, $\epsilon_{\ell}^T = 1$ (Fig. 1b)

c. $\epsilon_{\ell}^P = 1$, $\epsilon_{\ell}^T = \epsilon_{\ell}^T(\ell)$ (Fig. 1c)

For illustration purpose we choose a transfer function such as

$$
\epsilon(\ell) = \begin{cases} 
\epsilon_{\ell \text{min}} & \text{if } \ell < \ell_{\text{min}} \\
\epsilon_{\ell \text{min}} + \Delta \epsilon \cdot \sin^2 \left( \frac{\pi \ell}{\ell_{\text{max}} - \ell_{\text{min}}} \right) & \text{if } \ell_{\text{min}} \leq \ell \leq \ell_{\text{max}} \\
\epsilon_{\ell \text{max}} & \text{if } \ell > \ell_{\text{max}} 
\end{cases}
$$

(2)

where $\Delta \epsilon = \epsilon_{\ell \text{max}} - \epsilon_{\ell \text{min}}$ and $\Delta \ell = \ell_{\text{max}} - \ell_{\text{min}}$ with $\ell_{\text{min}} = 100$. The transfer function will smoothly increase from $\epsilon_{\ell \text{min}} = 0.95$ to $\epsilon_{\ell \text{max}} = 1$ between $\ell_{\text{min}}$ and $\ell_{\text{max}}$.

Adding an inconsistency between temperature and polarization leads to significant shifts for all the cosmological parameters for the three different models used in this analysis. The effect is clearly noticeable for baryon density and particularly for the polarization efficiency case (1a). Using the biased simulations, we obtain constraints on $\Omega_b h^2$ that are more than 3σ discrepant with the unbiased $C_\ell$s-derived constraint for $\epsilon_{\ell} \leq 0.97$. $H_0$ is particularly affected by a temperature transfer function : the $H_0$ constraint is shifted towards lower values (1c). The $R_{TT}^{\text{EL}}$ correlation coefficient could therefore be an interesting consistency test, it could help avoiding large bias in our cosmological parameter measurements arising from multiplicative systematic effects.

III. A LIKELIHOOD FOR THE PEARSON’S CORRELATION COEFFICIENT OF T AND E MODES

In this section we recall some of the statistical properties of $R_{\ell}^{TE}$ and we propose and validate a multifrequency likelihood for estimating cosmological parameters based on the correlation coefficients.
FIG. 1: 1D posterior distributions of the six ΛCDM parameters using a simulated dataset (TT, TE, EE) biased with a constant polarization efficiency (1a), with a polarization transfer function (1b) and with a temperature transfer function (1c). The transfer function model is described in Eq. (2). We vary the $\ell_{\text{max}}$ parameter from 200 to 1000. The dashed black lines correspond to the fiducial model. Gray bands correspond to $\pm 1\sigma$ $R_{i}^{\text{TE}}$ constraints.
FIG. 2: ΛCDM parameters constraints for a set of $N_{\text{sim}} = 100$ simulations (gray). The average posterior distribution is shown in red. The distance from the average to the input value (in sigma units) is displayed in the top right corner of each panel. Mean posterior distributions are consistent with input values of the simulations.

A. Statistic of the multifrequency correlation coefficient

An estimator for the correlation coefficient can be obtained using the estimated $TT$, $EE$ and $TE$ power spectra: $\hat{R}^{\text{TE}}_b = \hat{C}^{\text{TE}}_b / \sqrt{\hat{C}^{\text{TT}}_b \hat{C}^{\text{EE}}_b}$. By definition, the measured $\tilde{R}^{\text{TE}}_\ell$ correlation coefficient is unaffected by any of the systematic effects we discuss in section II. (i.e. $\tilde{R}^{\text{TE}}_\ell = R^{\text{TE}}_\ell$).

This estimator is affected by a subdominant bias that can be corrected at first order as: $\hat{R}^{\text{TE},c}_b = \hat{R}^{\text{TE}}_b (1 - \alpha_b)$.

An analytical expression for $\alpha_b$ is given in appendix A.

The expression for the covariance matrix of the $R^{\text{TE}}_b$ estimator was also derived in [15], the generalization of the expression to a multifrequency case is given by

$$
\Gamma(\hat{R}^{\text{TE}}_b, \hat{R}^{\text{TE}}_{b'}) = \Gamma(C^{\text{TE}}_b, C^{\text{TE}}_{b'}) + \frac{1}{4} \left[ \Gamma(C^{\text{TT}}_b, C^{\text{TT}}_{b'}) + \Gamma(C^{\text{EE}}_b, C^{\text{EE}}_{b'}) \right]
$$

$$
- \frac{1}{2} \left[ \Gamma(C^{\text{TE}}_b, C^{\text{TT}}_{b'}) + \Gamma(C^{\text{TT}}_b, C^{\text{TE}}_{b'}) \right]
$$

$$
+ \Gamma(C^{\text{TE}}_b, C^{\text{EE}}_{b'}) + \Gamma(C^{\text{EE}}_b, C^{\text{TE}}_{b'})
$$

$$
+ \frac{1}{4} \left[ \Gamma(C^{\text{TT}}_b, C^{\text{EE}}_{b'}) + \Gamma(C^{\text{EE}}_b, C^{\text{TT}}_{b'}) \right]$$

where $\Gamma(X, Y) = \text{cov}(X^{\nu_1 \times \nu_2}, Y^{\nu_3 \times \nu_4})/(X^{\nu_1 \times \nu_2} \cdot Y^{\nu_3 \times \nu_4})$.

Derivation of Eq. (3) is developed in appendix A.

B. $R^{\text{TE}}_\ell$-based likelihood construction

We adapt the HiLLiPoP $C_\ell$-based likelihood introduced in section II in order to fit cosmological parameters from the correlations coefficient. We use the multi-frequency covariance matrix given in Eq. (3) and assume that the correlation coefficients follow a multivariate Gaussian distribution.

The likelihood is defined as follows:

$$
\ln \mathcal{L} \sim -\frac{1}{2} (\Delta R^{\text{vec}})^T \Xi^{-1} (\Delta R^{\text{vec}})
$$

where $\Xi$ is the $R^{\text{TE}}_\ell$ multi-frequency covariance matrix and $\Delta R^{\text{vec}} = R^{\text{vec, data}} - R^{\text{vec, th}}$ is the residual correlation coefficient vector including the 6 cross-frequency spectra. The $R^{\text{TE}}_\ell$ covariance matrix is computed from the Planck PR4 power spectra covariance matrices using Eq. (3).

The data vector is computed from the $C_\ell$s data vector using the unbiased estimator $\hat{R}^{\text{TE},c}_b$. The model for the correlation coefficient is constructed using theoretical and foregrounds $TT$, $EE$, and $TE$ power spectra. For a cross
FIG. 3: 2D posterior distributions for $H_0$, $\Omega_b h^2$ and $\Omega_c h^2$ constrained using $C_\ell$-based likelihood (blue) or the $R_{\ell}^{TE}$-based likelihood (red). The expected 1σ scatter between parameters estimated from the two different likelihoods is displayed with gray dotted lines.
TABLE I: Multipole ranges used for each power spectrum and for the $R_{\ell}^{TE}$ correlation coefficient.

| Frequencies (GHz) | $\ell_{\min}$ | $\ell_{\max}$ | $\ell_{\min}$ | $\ell_{\max}$ | $\ell_{\min}$ | $\ell_{\max}$ | $\ell_{\min}$ | $\ell_{\max}$ |
|------------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 100×100          | 30           | 1200         | 30           | 1000         | 30           | 1200         | 50           | 1000         |
| 100×143          | 30           | 1500         | 30           | 1250         | 30           | 1500         | 50           | 1250         |
| 100×217          | 100          | 1500         | 400          | 1250         | 300          | 1500         | 400          | 1250         |
| 143×143          | 30           | 2000         | 30           | 1750         | 30           | 1750         | 50           | 1500         |
| 143×217          | 100          | 2500         | 400          | 1750         | 300          | 1750         | 400          | 1500         |
| 217×217          | 100          | 2500         | 400          | 2000         | 300          | 2000         | 400          | 1500         |

TABLE II: Cosmological parameter constraints (mean values and 1σ errors) derived from the $C_{\ell}$-likelihood and the $R_{\ell}^{TE}$-likelihood.

| Parameter | $C_{\ell}^{TT}$, $C_{\ell}^{TE}$, $C_{\ell}^{EE}$ | $R_{\ell}^{TE}$ |
|-----------|---------------------------------|----------------|
| $H_0$ [km/s/Mpc] | 67.3 ± 0.5                   | 67.5 ± 1.3 |
| $\Omega_h^2$   | 0.02233 ± 0.00014             | 0.02235 ± 0.00037 |
| $\Omega_s^2$   | 0.1194 ± 0.0012               | 0.1192 ± 0.0028 |
| ln($10^{10}A_s$) | 3.040 ± 0.015                | 3.098 ± 0.152 |

The posterior distributions for $H_0$, and the density parameters $\Omega_b h^2$ and $\Omega_s h^2$ are shown in Fig. 3. We display the expected 1σ fluctuations between $C_{\ell}$-derived parameters and $R_{\ell}^{TE}$-derived ones in gray. This was derived from the set of simulations introduced in section II. Cosmological parameters estimated from the correlation coefficient $R_{\ell}^{TE}$ are consistent with the one estimated from power spectra. We do not detect the effect of any multiplicative systematic in the data.

The posterior distributions for $H_0$, and the density parameters $\Omega_b h^2$ and $\Omega_s h^2$ are shown in Fig. 3. We display the expected 1σ fluctuations between $C_{\ell}$-derived parameters and $R_{\ell}^{TE}$-derived ones in gray. This was derived from the set of simulations introduced in section II. Cosmological parameters estimated from the correlation coefficient $R_{\ell}^{TE}$ are consistent with the one estimated from power spectra. We do not detect the effect of any multiplicative systematic in the data.

Table II presents mean values and 1σ errors for the cosmological parameters discussed in this paper. As expected, using $R_{\ell}^{TE}$ worsens the constraints on cosmological parameters with respect to the $C_{\ell}$-based likelihood. The $R_{\ell}^{TE}$-derived errors on $H_0$ and $\Omega_b h^2$ are 2.6 times wider than the $C_{\ell}$-derived ones and the error on $\Omega_s h^2$ is 2.3 times larger. The correlation coefficient is poorly sensitive to the parameters describing the shape of the initial matter power spectrum, $A_s$ and $n_s$. Interestingly, while the amplitude parameter $A_s$ appears to cancel in the ratio of power spectra, it can be measured through the effect of lensing on the power spectra [19].

The posterior distribution of log($10^{10}A_s$) is displayed in Fig. 4. The $R_{\ell}^{TE}$-derived error on ln($10^{10}A_s$) is 10 times larger than the $C_{\ell}$-derived one.

We note that our value of the Hubble parameter $H_0 = 67.5 ± 1.3$ km/s/Mpc, derived from the correlation coefficient is consistent with other CMB based measurements of $H_0$. The Hubble parameter we obtain using $R_{\ell}^{TE}$ is still 3.1σ discrepant with the latest measurement from Cepheids-calibrated cosmic distance ladder [4].

B. CMB-only $R_{\ell}^{TE}$

We produce CMB-only power spectra, marginalizing over foreground parameters. We follow the method used in [20]. The multi-frequency vector is modelled as

$$C_{b}^{\text{model}} = A \cdot C_{b}^{\text{CMB}} + C_{b}^{\theta_{fg}}$$  \hspace{1cm} (6)
FIG. 5: (Top) : CMB-only $R_{\ell}^{\text{TE}}$ correlation coefficient for Planck PR4 data (red) and ACT data (green) after foreground marginalization. The best-fit correlation coefficient (gray dashed line) is computed with CAMB from the HiLLiPoP $C_\ell$-likelihood best fit cosmology — (Bottom) : Residual plot with respect to the binned correlation coefficient best-fit.

with $A$ a matrix that projects the CMB bandpowers to their corresponding elements in each cross-frequency spectrum, and $C_b^{fg}(\theta_{fg})$ the foregrounds power spectra corresponding to a set of foreground parameters $\theta_{fg}$.

Instead of sampling simultaneously CMB bandpowers and nuisance parameters, we split the joint probability distribution $p(C_{\text{CMB}}^{\text{CM}}|\theta_{fg}, \text{data})$ into two conditional probability densities $p(C_{\text{b}}^{\text{CM}}|\theta_{fg}, \text{data})$ and $p(\theta_{fg}|C_{\text{CMB}}^{\text{CM}}, \text{data})$. The main advantage here is that the CMB bandpowers sampling is simple because $C_{\text{CMB}}^{\text{CM}}$ follows a multivariate normal distribution with mean $\hat{C}_{\text{b}}$ and covariance $Q$ given by

$$\hat{C}_{\text{b}} = [A^T\Sigma^{-1}A]^{-1}[A^T\Sigma^{-1}(C_{\text{b}}^{\text{data}} - C_{\text{b}}^{fg}(\theta_{fg}))] \quad (7)$$

$$Q^{-1} = A^T\Sigma^{-1}A \quad (8)$$

with $\Sigma$ the $C_\ell$s multifrequency covariance matrix. We alternate CMB bandpowers sampling (using the known probability distribution described in Eqs. [7,8]) and Metropolis-Hastings sampling for the nuisance parameters using a Gaussian likelihood. We then compute the correlation coefficient and its covariance matrix using the CMB-only power spectra and the $Q$ covariance matrix.

In Fig. 5, we compare the CMB-only $R_{\ell}^{\text{TE}}$ with the best-fit model obtained using the $C_\ell$-based likelihood. The correlation coefficient is compatible with the model with $\chi^2/\text{dof} = 52.2/52$ (PTE = 0.47).

Higher resolution experiments such as Atacama Cosmology Telescope (ACT) bring more information about the correlation coefficient at small scales. We display
on Fig 5 the ACT CMB-only $R_{TE}^T$, computed from the power spectra provided by the ACT collaboration [21] and publicly available.² Before computing the correlation coefficient, we remove points for which the EE SNR is lower than 3. We find that the ACT correlation coefficient is in good agreement with the Planck PR4 best-fit, with $\chi^2$/dof = 39.0/36 (PTE = 0.33).

V. CONCLUSION

Given the current context of tensions between early and late Universe measurements, it is important to assess the robustness of the cosmological parameter constraints. In this work, we have studied the impact of remaining multiplicative systematic effects that would have been neglected in the data treatment, resulting in a bias at the power spectra level. We have shown that this kind of bias can significantly shift the cosmological parameter constraints determined from the combination of $C_{\ell}^{TT}$, $C_{\ell}^{TE}$ and $C_{\ell}^{EE}$.

To prevent the cosmological parameter measurements from being biased, we have proposed to use a likelihood of parameter constraints determined from the combination this kind of bias can significantly shift the cosmological parameter measurements. We have obtained the first constraints on cosmology only observable that is insensitive to multiplicative biases.

Given the current context of tensions between early and late Universe measurements, it is important to assess the robustness of the cosmological parameter constraints. We have shown that it gives cosmological parameters that are consistent with other CMB-based measurements. Using $R_{TE}^T$ increases the error on the cosmological parameters with respect to the errors derived from the combination of $TT$, $TE$, and $EE$ power spectra. However, we have measured the Hubble parameter $H_0 = 67.5 \pm 1.3$ km/s/Mpc with an associated error which is similar to the error obtained by ground-based CMB experiments such as ACTPol [2]. The correlation coefficient provides a good consistency check on Planck data. We have not observed the effect of any multiplicative bias in Planck PR4 data and we have obtained a value for the Hubble parameter which is still discrepant with late Universe measurements.

Next generation of ground based CMB experiments such as Simons Observatory [7] or CMB-S4 [8] will produce data with a high EE signal-to-noise ratio for a wide range of multipoles. This will allow the computation of $R_{TE}^T$ up to smaller scales, increasing its constraining power and relevance. Ground-based experiments are also typically more affected by transfer functions due to atmospheric and ground pick up filtering. We expect that observables such as $R_{TE}^T$ will provide a strong consistency check on these data.

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Appendix A: Statistical properties of $R_{TE}^T$ estimator

In the high signal-to-noise regime, the correlation coefficient estimator can be developed as:

$$\hat{R}_b^{TE} = R_b^{TE} \frac{1 + \frac{\Delta C_b^{TT}}{C_b^{TT}}}{\sqrt{(1 + \frac{\Delta C_b^{TT}}{C_b^{TT}})(1 + \frac{\Delta C_b^{EE}}{C_b^{EE}})}}$$

$$= R_b^{TE} \left(1 + \frac{\Delta C_b^{TT}}{C_b^{TT}}\right) \cdot \left(1 - \frac{1}{2} \frac{\Delta C_b^{TT}}{C_b^{TT}} + \frac{3}{8} \left(\frac{\Delta C_b^{TT}}{C_b^{TT}}\right)^2\right) \cdot \left(1 - \frac{1}{2} \frac{\Delta C_b^{EE}}{C_b^{EE}} + \frac{3}{8} \left(\frac{\Delta C_b^{EE}}{C_b^{EE}}\right)^2\right)$$ (A1)

where $R_b^{TE}$, $C_b^{TT}$, $C_b^{EE}$ and $C_b^{TE}$ are the spectra we want to estimate.

Such an estimator is not an unbiased estimator (i.e. $\langle \hat{R}_b^{TE} \rangle \neq R_b^{TE}$), but is affected by a bias term such that:

$$\langle \hat{R}_b^{TE} \rangle = R_b^{TE}(1 + \alpha_b)$$

The bias term $\alpha_b$ can be easily computed taking the mean value of Eq. (A1):

$$\alpha_b = \frac{3}{8} \left(\frac{\text{cov}(C_b^{TT}, C_b^{TT})}{C_b^{TT}^2} + \frac{\text{cov}(C_b^{EE}, C_b^{EE})}{C_b^{EE}^2}\right) - \frac{1}{2} \left(\frac{\text{cov}(C_b^{TT}, C_b^{TT})}{C_b^{TT}C_b^{TE}} + \frac{\text{cov}(C_b^{EE}, C_b^{EE})}{C_b^{EE}C_b^{TE}}\right) + \frac{1}{4} \frac{\text{cov}(C_b^{TT}, C_b^{EE})}{C_b^{TT}C_b^{EE}}.$$ (A2)

From Eq. (A1) we can compute the $R_{TE}^T$ covariance matrices defined by

$$\text{cov}(\Delta R_b^{TE, \nu_1 \times \nu_2}, \Delta R_b^{TE, \nu_3 \times \nu_4}) = \langle \Delta R_b^{TE, \nu_1 \times \nu_2} \Delta R_b^{TE, \nu_3 \times \nu_4} \rangle;$$ (A3)

with $\nu_1 \times \nu_2$ and $\nu_3 \times \nu_4$ two cross frequencies and $\Delta R_b^{TE} = \hat{R}_b^{TE} - R_b^{TE}$ the deviation to the mean value. $\Delta R_b^{TE}$ can be expressed analytically (at second order) using Eq. (A1)
Appendix B: HiLLiPoP power spectra model

In section II we present the HiLLiPoP $C_p$-based likelihood. In section III, we construct a $R_{T,E}$-based likelihood. In this appendix, we explain how the model in Eq. (5) is computed.

We work with three frequencies: 100, 143 and 217 GHz (indexed by $\nu_i, \nu_j$ for $(i, j) \in \{0, 1, 2\}^2$) and two split maps (indexed by $R, S$ with $(R, S) \in \{(A, B)\}^2$). We also define $X$ and $Y$ such as $(X, Y) \in \{T, E\}^2$. For two frequencies $(\nu_i, \nu_j)$ with $j \geq i$ and two maps $(R, S)$, we model the power spectra as

$$C_{\ell}^{XY;\nu_i^R \times \nu_j^S} = A_{pl} c_i^{R} c_j^{S} \left( C_{\ell}^{XY,\text{CMB}} + C_{\ell}^{XY,\text{fg},\nu_i^R \times \nu_j^S} \right), \quad (B1)$$

where $A_{pl}$ is a global amplitude calibration parameter and $c_i^{R}, c_j^{S}$ are calibration parameters at map level. This set of parameters is sampled in the likelihood as nuisance parameters.

We take the weighted average of cross-map power spectra to compute the cross-frequency power spectra, ignoring the auto power spectra.

$$C_{\ell}^{XY;\nu_i^R \times \nu_j^S} = A_{pl} \sum_{(R,S) \in \{(A,B)\}^2} w_{\ell}^{XY;\nu_i^R \times \nu_j^S} c_i^{R} c_j^{S} \left( C_{\ell}^{XY,\text{CMB}} + C_{\ell}^{XY,\text{fg},\nu_i^R \times \nu_j^S} \right) \left( 1 - \delta_{\nu_i \nu_j} (1 - \delta_{RA \delta SB}) \right), \quad (B2)$$

where $w_{\ell}^{XY;\nu_i^R \times \nu_j^S}$ are the weights associated to the $\nu_i^R \times \nu_j^S$ XY power spectrum and $\delta$ is the Kronecker delta. We can express Eq. (B2) as Eq. (5) using the following definition

$$A_{XY;\nu_i^R \times \nu_j^S} = \sum_{(R,S) \in \{(A,B)\}^2} w_{\ell}^{XY;\nu_i^R \times \nu_j^S} c_i^{R} c_j^{S} \left( 1 - \delta_{\nu_i \nu_j} (1 - \delta_{RA \delta SB}) \right). \quad (B3)$$

We obtain the equation describing the model used to compute the power spectra in the likelihood

$$C_{\ell}^{XY;\nu_i^R \times \nu_j^S} = A_{pl} A_{XY;\nu_i^R \times \nu_j^S} \left( C_{\ell}^{XY,\text{CMB}} + C_{\ell}^{XY,\text{fg},\nu_i^R \times \nu_j^S} \right). \quad (B4)$$
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