A GEOMETRIC CLASSIFICATION OF IMMERSIONS OF 3-MANIFOLDS INTO 5-SPACE

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Abstract. In this paper we define two regular homotopy invariants $c$ and $i$ for immersions of oriented 3-manifolds into $\mathbb{R}^5$ in a geometric manner. The pair $(c(f), i(f))$ completely describes the regular homotopy class of the immersion $f$. The invariant $i$ corresponds to the 3-dimensional obstruction that arises from Hirsch-Smale theory and extends the one defined in [10] for immersions with trivial normal bundle.

1. Introduction

Hirsch-Smale theory [5] reduces the problem of regular homotopy classification of immersions to homotopy theory. However, this homotopy theoretic problem is usually hard to deal with. In the case of immersions of oriented 3-manifolds into $\mathbb{R}^5$ this homotopy theoretic problem was solved by Wu [14] using algebraic topological methods (also see [1]).

However, it remains a problem to determine the regular homotopy class of a given immersion from its geometry. By geometry we mean the structure of the "singularities" of the map. For example, double points are such "singularities", and indeed, Smale [11] showed that for $n > 1$ the regular homotopy class of an immersion $S^n \hookrightarrow \mathbb{R}^{2n}$ is completely determined by the number of its double points (modulo 2 if $n$ is odd). A similar classification was carried out by Ekholm [2] for immersions of $S^k$ into $\mathbb{R}^{2k-1}$ for $k \geq 4$.

The whole picture changes when we consider immersions of $S^3$ into $\mathbb{R}^5$. Hughes and Melvin [6] showed that there are infinitely many embeddings $S^3 \hookrightarrow \mathbb{R}^5$ that are pairwise not regularly homotopic to each other. Therefore one can not determine the regular homotopy class from the "singularities" since an embedding has no such. Ekholm and Szücs [3] came over this problem using "singular Seifert surfaces" bounded by the immersions. For an immersion $f: M^3 \hookrightarrow \mathbb{R}^5$ a singular Seifert surface is a generic map $F: W^4 \to \mathbb{R}^5$ of a compact orientable manifold $W^4$ with boundary $M^3$ such that $\partial F = f$. In [3] it is shown that for $M^3 = S^3$ the Smale invariant of $f$ can be computed from the singularities of $F$. Later Saeki, Szücs and Takase [10] generalized these results for immersions $f: M^3 \hookrightarrow \mathbb{R}^5$ with trivial normal bundle (for oriented $M^3$). The invariant introduced in [10] corresponds to the 3-dimensional obstruction to a regular homotopy between two such immersions. Our present paper generalizes the results of [10] to arbitrary immersions $f: M^3 \hookrightarrow \mathbb{R}^5$.

We will consider the set $\text{Imm}(M^3, \mathbb{R}^5)_{\chi}$ of immersions with fixed normal Euler class $c(\nu_f) = \chi \in H^2(M^3; \mathbb{Z})$ and construct a $\mathbb{Z}_{2\delta(\chi)}$-valued regular homotopy
invariant $i$ for this set of immersions, where $d(\chi)$ denotes the divisibility of $\chi$. The construction of the invariant $i$ will also make use of a singular Seifert surface $F$. In [10] $F$ had to be an immersion near the boundary, but we (have to and) will allow arbitrary generic maps. If $\chi = 0$ and $F$ is an immersion near the boundary then the construction of the invariant $i$ agrees with the one introduced in [10]. We will also show that whenever $f, g: M^3 \to \mathbb{R}^5$ are regularly homotopic on a neighborhood of the 2-skeleton of $M^3$ then $i(f) = i(g)$ iff $f$ and $g$ are regularly homotopic. This shows that $i$ corresponds to the 3-dimensional obstruction to a regular homotopy between $f$ and $g$. (Note that there is an invariant which determines the regular homotopy class of the restriction of an immersion to a neighborhood of the 2-skeleton of $M^3$. This invariant was called the Wu invariant in [10], see below.)

Regular homotopy classes of immersions of oriented 3-manifolds into $\mathbb{R}^5$ endowed with the connected sum operation form a semigroup whose structure we will also determine. Finally, an exact sequence will be defined that relates $\text{Imm}[M^3, \mathbb{R}^5]$ to $\text{Imm}[M^3, \mathbb{R}^6]$ and $[M^3, S^2]$.

2. Preliminaries

First let us recall the result of Wu [14] that classifies immersions of an oriented 3-manifold $M^3$ into $\mathbb{R}^5$ up to regular homotopy.

**Theorem 2.1.** The normal Euler class $\chi$ of an immersion $f: M^3 \to \mathbb{R}^5$ is of the form $2c$ for some $c \in H^2(M^3; \mathbb{Z})$ and for any $c \in H^2(M^3; \mathbb{Z})$ there is an immersion $f$ such that $\chi = 2c$. Furthermore,

$$\text{Imm}[M^3, \mathbb{R}^5]_{\chi} \approx \coprod_{c \in H^2(M^3; \mathbb{Z}), \ 2c = \chi} H^3(M^3; \mathbb{Z})/(2\chi \cup H^1(M^3; \mathbb{Z})),$$

where $\text{Imm}[M^3, \mathbb{R}^5]_{\chi}$ is the set of regular homotopy classes of immersions with normal Euler class $\chi \in H^2(M^3; \mathbb{Z})$ and $\cup$ represents the cup product, moreover the symbol $\approx$ denotes a bijection.

**Remark 2.2.** For $\chi \in H^2(M^3; \mathbb{Z})$ let $d(\chi) \in \mathbb{Z}$ denote the divisibility of $\chi$, so that $\chi$ equals $d(\chi)$ times a primitive class in $H^2(M^3; \mathbb{Z})$ modulo torsion, and $d(\chi) = 0$ if $\chi$ is of finite order. Then Poincaré duality implies that

$$H^3(M^3; \mathbb{Z})/(2\chi \cup H^1(M^3; \mathbb{Z})) \approx \mathbb{Z}_{2d(\chi)}.$$

If $f$ is an immersion of $M^3$ into $\mathbb{R}^5$ with normal Euler class $\chi$ then let us introduce the notation $d(f)$ for $d(\chi)$.

**Notation 2.3.** For $\chi \in H^2(M^3; \mathbb{Z})$ let $\Gamma_2(\chi)$ denote the set $\{c \in H^2(M^3; \mathbb{Z}) : 2c = \chi\}$. Throughout this paper we will use the notation $M^3_0$ for the punctured 3-manifold $M^3 \setminus D^3$, where $D^3 \subset M^3$ is a closed 3-disc. Then the 2-skeleton $\text{sk}_2(M^3)$ is a deformation retract of $M^3_0$.

Theorem [14] can also be applied to the open manifold $M^3_0$. Since $H^3(M^3_0; \mathbb{Z}) = 0$ we obtain a bijection

$$\bar{c}: \text{Imm}[M^3_0, \mathbb{R}^5]_{\chi} \to \Gamma_2(\chi).$$

Thus for an immersion $f: M^3 \to \mathbb{R}^5$ the invariant $c(f) = \bar{c}(f|M^3_0) \in \Gamma_2(\chi)$ describes the regular homotopy class of $f|M^3_0$. Following [10] we will call $c(f)$ the Wu invariant of the immersion $f$. 


To get a complete description of \( \text{Imm}[M^3, \mathbb{R}^5]_\chi \) we will construct a \( \mathbb{Z}_{2d(\chi)} \)-valued invariant \( i \) such that the map
\[
(c, i) : \text{Imm}[M^3, \mathbb{R}^5]_\chi \to \Gamma_2(\chi) \times \mathbb{Z}_{2d(\chi)}
\]
will be a bijection. The invariant \( i \) is constructed in a geometric manner and is an extension of the invariant defined in [10] for \( \chi = 0 \).

Next let us recall Theorem 1.1(a) in [3]. Let \( f : S^3 \hookrightarrow \mathbb{R}^5 \) be an immersion and \( V^4 \) an arbitrary compact oriented 4-manifold with \( \partial V^4 = S^3 \). The map \( f \) extends to a generic map \( F : V^4 \to \mathbb{R}^5 \) which has no singular points near the boundary \( \partial V^4 \) since the normal bundle \( \nu_f \) of \( f \) is trivial. This map \( F \) has isolated cusps, each one having a sign. Let us denote by \( \#\Sigma^{1,1}(F) \) their algebraic number and let \( \Omega(f) \) be the Smale invariant of \( f \). The following formula was proved in [3].

**Theorem 2.4.**
\[
\Omega(f) = \frac{1}{2}(3\sigma(V^4) + \#\Sigma^{1,1}(F)).
\]

The proof of this theorem relies on the following proposition ([3], Lemma 3).

**Lemma 2.5.** Let \( X^4 \) be a closed oriented 4-manifold and \( g : X^4 \to \mathbb{R}^5 \) a generic map. Then \( 3\sigma(X^4) + \#\Sigma^{1,1}(g) = 0 \).

For the sake of completeness we will also recall from [10] the definition of the invariant \( i \) for immersions with trivial normal bundle. First we need a preliminary definition.

**Definition 2.6.** Let \( M^3 \) be a closed oriented 3-manifold. We denote by \( \alpha(M^3) \) the dimension of the \( \mathbb{Z}_2 \) vector space \( \tau H_1(M^3; \mathbb{Z}) \oplus \mathbb{Z}_2 \), where \( \tau H_1(M^3; \mathbb{Z}) \) is the torsion subgroup of \( H_1(M^3; \mathbb{Z}) \).

**Definition 2.7.** Let \( f : M^3 \hookrightarrow \mathbb{R}^5 \) be an immersion with trivial normal bundle. Let \( W^4 \) be any compact oriented 4-manifold with \( \partial W^4 = M^3 \) and \( F : W^4 \to \mathbb{R}^5 \) a generic map nonsingular near the boundary such that \( F|\partial W^4 = f \). (We can choose such a generic map \( F \) since \( f \) is an immersion with trivial normal bundle.) Denote the algebraic number of cusps of \( F \) by \( \#\Sigma^{1,1}(F) \). Then let
\[
i(f) = \frac{3}{2}(\sigma(W^4) - \alpha(M^3)) + \frac{1}{2}\#\Sigma^{1,1}(F).
\]

It is proved in [10] that \( i(f) \) is always an integer and a regular homotopy invariant.

In the following sections we will extend the above regular homotopy invariant \( i \) to arbitrary immersions. If \( f : M^3 \hookrightarrow \mathbb{R}^5 \) has non-trivial normal bundle then we have to give up the assumption that the singular Seifert-surface \( F \) is an immersion near the boundary. Thus we will use an arbitrary generic map \( F : W^4 \to \mathbb{R}^5 \) such that \( \partial F = f \). The singular set \( \Sigma^1(F) \) of such an \( F \) is a 2-dimensional submanifold of \( W^4 \) with boundary \( C(F) = \partial \Sigma^1(F) \subset M^3 \). If we orient \( \ker(dF)|C(F) \) so that it points into \( W^4 \) and project it into \( TM^3 \) then we obtain a normal field \( \nu(F) \) along \( C(F) \). We will define the rotation of \( \nu(F) \) around \( C(F) \) modulo \( 4d(f) \) and denote this by \( R(F) \). The double of the extended invariant will be defined to be
\[
I(f) = 3(\sigma(W^4) - \alpha(M^3)) + \#\Sigma^{1,1}(F) + R(F) \in \mathbb{Z}_{4d(f)}.
\]

We will show that \( I(f) \) is always even, thus it defines an element \( i(f) \in \mathbb{Z}_{2d(f)} \) using the natural embedding \( \mathbb{Z}_{2d(f)} \hookrightarrow \mathbb{Z}_{4d(f)} \).
Throughout this paper $M^3$ will denote a fixed closed connected and oriented 3-manifold.

**Notation 3.1.** A pair $(C, \nu)$ will always stand for an oriented 1-dimensional submanifold $C$ of $M^3$ and a nowhere vanishing normal field $\nu$ along $C$.

**Definition 3.2.** Let $\chi \in H^2(M^3; \mathbb{Z})$ and let $C_0$ and $C_1$ be 1-dimensional oriented submanifolds of $M^3$ with normal fields $\nu_0$ and $\nu_1$ such that $PD[C_0] = PD[C_1] = \chi$. (Here $PD$ denotes Poincaré duality.) Then we can define the *rotation difference* $\text{rd}((C_0, \nu_0), (C_1, \nu_1)) \in \mathbb{Z}_{2d(\chi)}$ of $(C_0, \nu_0)$ and $(C_1, \nu_1)$ as follows. Since $[C_0] = [C_1]$ and $H_1(M^3; \mathbb{Z}) \cong H^2(M^3; \mathbb{Z}) \cong [M^3, \mathbb{C}P^\infty]$, there exists an oriented cobordism $K^2 \subset M^3 \times I$ between $C_0 \subset M^3 \times \{0\}$ and $C_1 \subset M^3 \times \{1\}$. Let $\nu$ be a generic normal field along $K^2$ that extends $\nu_0$ and $\nu_1$. Then a sign can be given to each zero of $\nu$ since $M^3$ is oriented. Now we define $\text{rd}((C_0, \nu_0), (C_1, \nu_1))$ to be the algebraic number of zeroes of $\nu$ modulo $2d(\chi)$. Equivalently, $\text{rd}((C_0, \nu_0), (C_1, \nu_1))$ is the self intersection of $K$ in $M^3 \times I$ modulo $2d(\chi)$ if perturbed in the direction of $\nu$.

**Remark 3.3.** The rotation difference is the obstruction to the existence of a framed cobordism between the framed submanifolds $(C_0, \nu_0)$ and $(C_1, \nu_1)$ of $M^3$. Using the Pontrjagin construction this corresponds to the obstruction to a homotopy between two maps of $M^3$ to $S^2$. This situation was first examined in [9]. It is easy to see that $\text{rd}((C_0, \nu_0), (C_1, \nu_1)) = 0$ iff $(C_0, \nu_0)$ and $(C_1, \nu_1)$ are framed cobordant. Thus we obtain a bijection

$$[M^3, S^2] \cong \bigoplus_{\chi \in H^2(M^3; \mathbb{Z})} H^3(M^3; \mathbb{Z})/2\chi \cup H^1(M^3; \mathbb{Z}).$$

**Proposition 3.4.** In Definition 3.3 above the rotation difference is well defined, i.e., it does not depend on the choice of $K$ and $\nu$.

**Proof.** Let $K$, $\nu$ and $K'$, $\nu'$ be as in Definition 3.3. We glue together $M^3 \times I$ and $-M^3 \times I$ along their boundaries so that we obtain the double $D(M^3 \times I) = M^3 \times S^1$. Place $K$ into the half of $M^3 \times S^1$ corresponding to $M^3 \times I$ and $K'$ into the other half. Then we obtain a closed oriented surface $F = K \cup -K'$ in $M^3 \times S^1$ and a normal field $\mu = \nu \cup \nu'$ along $F$. Since $H^*(S^1; \mathbb{Z})$ is a torsion free $\mathbb{Z}$-module we can apply Künneth’s theorem and we get that

$$H^2(M^3 \times S^1; \mathbb{Z}) \cong H^1(M^3; \mathbb{Z}) \otimes H^1(S^1; \mathbb{Z}) \oplus H^2(M^3; \mathbb{Z}) \otimes H^0(S^1; \mathbb{Z}).$$

Thus the Poincaré dual of $F$ can be written in the form

$$PD[F] = x \times \alpha + y \times 1 \in H^2(M^3 \times S^1; \mathbb{Z}),$$

where $x \in H^1(M^3; \mathbb{Z})$ and $y \in H^2(M^3; \mathbb{Z})$, moreover $\alpha$ denotes the generator of $H^1(S^1; \mathbb{Z})$ and $1$ the generator of $H^0(S^1; \mathbb{Z})$ given by the orientation of $S^1$. Note that for $I \in S^1$ the dual class of $M^3 \times \{1\} \subset M^3 \times S^1$ is

$$PD[M^3 \times \{1\}] = PD[M^3] \times PD[\{1\}] = 1 \times \alpha \in H^1(M^3 \times S^1; \mathbb{Z}).$$

Moreover,

$$PD[F \cap (M^3 \times \{1\})] = PD[C_1 \times \{1\}] = PD[C_1] \times PD[\{1\}] = \chi \times \alpha.$$
On the other hand
\[ PD[F \cap (M^3 \times \{1\})] = PD[F] \cup PD[M^3 \times \{1\}] = (x \times \alpha + y \times 1) \cup (1 \times \alpha) = x \times \alpha^2 + y \times \alpha. \]
Since \( \alpha^2 = 0 \) we get that \( y \times \alpha = \chi \times \alpha \). Using K"unneth’s theorem again we obtain the equality \( y = \chi \). Thus we get that
\[ PD[F] \cup PD[F] = (x \times \alpha + \chi \times 1)^2 = (2x \cup \chi) \times \alpha \]
since \( \alpha^2 = \chi^2 = 0 \) and \( x \cup \chi = \chi \cup x \) because the degree of \( \chi \) is 2. So the self intersection of \( F \) in \( M^3 \times S^1 \) equals \((2\chi \cup x) \times \alpha, [M^3 \times S^1]\) = \((2\chi \cup x, [M^3]) \in 2d(\chi)Z\). If we perturb \( F \) in the direction of \( \mu \) we get that the self intersection of \( K \) with respect to \( \nu \) equals the self intersection of \( K' \) with respect to \( \nu' \) modulo 2d(\chi).

**Proposition 3.5.** If \([C_0] = [C_1] = [C_2] \in H_1(M^3, Z)\) then
\[ rd((C_0, \nu_0), (C_1, \nu_1)) + rd((C_1, \nu_1), (C_2, \nu_2)) = rd((C_0, \nu_0), (C_2, \nu_2)). \]

**Definition 3.6.** For each \( a \in H_1(M^3; Z) \) fix a pair \((C_a, \nu_a)\) such that \([C_a] = a\). Then for \([C] = a\) let \( r(C, \nu) = rd((C, \nu), (C_a, \nu_a))\).

**Corollary 3.7.** If \([C_0] = [C_1]\) then \( r(C_0, \nu_0) - r(C_1, \nu_1) = rd((C_0, \nu_0), (C_1, \nu_1))\).

**Definition 3.8.** We can define the mod 2 rotation difference \( rd_2((C_0, \nu_0), (C_1, \nu_1))\) for unoriented \( C_0 \) and \( C_1 \) just as in Definition 3.4 but allowing the cobordism \( K \) to be non-orientable and counting the self intersection of \( K \) in \( M \times I \) only modulo 2. The proof that this is well defined is analogous to the oriented case. It is clear that the epimorphism \( Z_{2d(\chi)} \to Z_2 \) takes \( rd \) to \( rd_2 \). The mod 2 rotation \( r_2 \) is defined just like \( r \).

Unfortunately we will have to lift the invariants \( rd \) and \( r \) to \( Z_{4d(\chi)} \). To be able to do this we need more structure on \( M^3 \) then just a framed submanifold. We will use this additional structure to restrict the homology class of the cobordism \( K \) so that the surface \( F \) in the proof of Proposition 3.4 will represent an even homology class and thus \( x \) will always be even (since \( \chi \) is even). So the self intersection of \( F \) will be divisible by 4d(\chi) instead of just 2d(\chi).

**Notation 3.9.** Fix a cohomology class \( \chi \in H^2(M^3; Z) \). Let \( e^3_M \) denote the 3-dimensional trivial bundle over \( M^3 \) and let \( t, v \in \Gamma(e^3_M) \) be two generic non-zero sections of \( e^3_M \). Furthermore, suppose that the 2-dimensional oriented subbundle \( t^+ \subset e^3_M \) has Euler class \( \chi \). If we project \( v \) into \( t^+ \) we obtain a section \( w \in \Gamma(t^+) \) that vanishes along a curve \( C \subset M^3 \) and we orient \( C \) so that \( PD[C] = c(t^+) \). In particular, \( t \) and \( v \) are linearly dependent exactly at the points of \( C \). Finally let \( \nu \) be a non-zero normal field along \( C \). In the future we will denote such a structure on \( M^3 \) by a quadruple \((C, \nu, t, v)\) and the set of these structures by \( N(M^3, \chi) \).

**Remark 3.10.** Since \( PD[C]|_2 = w_2(e^3_M) = 0 \in H^2(M^3; Z_2) \), the cohomology class \( \chi = PD[C] \) is of the form \( 2c \) for some \( c \in H^2(M^3; Z) \). This can be seen from the long exact sequence associated to the coefficient sequence \( Z \to Z \to Z_2 \). Thus \( N(M^3, \chi) = \emptyset \) if \( \chi \) is not of the form \( 2c \).

**Definition 3.11.** Suppose that \( a_0 = (C_0, \nu_0, t_0, v_0) \) and \( a_1 = (C_1, \nu_1, t_1, v_1) \) are elements of \( N(M^3, \chi) \), where \( \chi = 2c \). Then we will define their rotation difference \( Rd(a_0, a_1) \in Z_{4d(\chi)} \) as follows. We will consider \( a_i \) to be in \( N(M^3 \times \{i\}, \chi) \) for \( i = 0, 1 \). Let \( t_i, v \in \Gamma(e^3_{M \times I}) \) be generic non-zero sections extending \( t_i \) and \( v_i \) for
$i = 0, 1$. Denote by $K$ the 2-dimensional submanifold of $M^3 \times I$ where $t$ and $v$ are linearly dependent. Let $w$ denote the projection of $v$ into the 2-dimensional oriented subbundle $t^\perp < \varepsilon^3_{M \times I}$. Then $w$ is zero exactly at the points of $K$, thus it defines an orientation of $K$. With this orientation $K$ is an oriented cobordism between $C_0$ and $C_1$. Let $\nu$ denote a normal field of $K$ that extends both $\nu_0$ and $\nu_1$. Now we define $\text{Rd}(a_0, a_1)$ to be the algebraic number of zeroes of $\nu$ modulo $4d(\chi)$. Equivalently, $\text{Rd}(a_0, a_1)$ is the self intersection of $K$ in $M^3 \times I$ modulo $4d(\chi)$ if perturbed in the direction of $\nu$.

**Proposition 3.12.** In Definition 3.11 the rotation difference is well defined. I.e., it does not depend on the extensions $t, v$ and $\nu$.

**Proof.** Let $t, v, \nu$ and $t', v', \nu'$ be as in Definition 3.11. The sections $t, v$ are linearly dependent over $K$ and $t', v'$ are dependent over $K'$. Just as in the proof of Proposition 3.12 we will place $K, \nu$ and $K', \nu'$ in the two halves of the double $D(M^3 \times I) = M^3 \times S^1$ and place $t, v$ and $t', v'$ in the two halves of the trivial bundle $\varepsilon^3_{M \times S^1}$. Let $F$ denote the oriented surface $K \cup -K'$ and by $\mu$ the normal field along $F$ obtained from $\nu$ and $\nu'$. Moreover, let $T = t \cup t'$ and $V = v \cup v'$. Then $T, V \in \Gamma(\varepsilon_{M \times S^1}^3)$ are linearly dependent exactly over $F$, thus $PD[F]|_2 = w_2(\varepsilon_{M \times S^1}^3) = 0 \in H^2(M^3 \times S^1; \mathbb{Z}_2)$. Using the coefficient sequence $\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_2$ we get that $PD[F]$ is of the form $2b$ for some $b \in H^2(M^3 \times S^1; \mathbb{Z})$. Since $PD[F]$ is of the form $x \times a + \chi \times 1$ where $\chi = 2c$ we get that there exists an element $z \in H^1(M^3; \mathbb{Z})$ such that $x = 2z$. Thus $PD[F]|_2 = (4z \times \chi) \times \alpha$ which implies that the self intersection of $F$ is divisible by $4d(\chi)$. If we perturb $F$ in the direction of $\mu$ we get that the self intersection of $K$ with respect to $\nu$ equals the self intersection of $K'$ with respect to $\nu'$ modulo $4d(\chi)$. □

**Remark 3.13.** The surface $K$ represents the dual of the Stiefel-Whitney class of the bundle $\varepsilon^3_{M \times I}$ relative to the sections $t_i, v_i$ given over $M^3 \times \{0, 1\}$. I.e.,

$$PD[K]|_2 = w_2(\varepsilon^3_{M \times I}; t_i, v_i) \in H^2(M^3 \times I, ((M^3 \setminus C_0) \times \{0\}) \cup ((M^3 \setminus C_1) \times \{1\}); \mathbb{Z}_2)$$

since $v$ and $t$ are linearly independent over $((M^3 \setminus C_0) \times \{0\}) \cup ((M^3 \setminus C_1) \times \{1\})$. Using Lefschetz duality we get that the relative homology class $[K]|_2 \in H^2(M^3 \times I, C_0 \times \{0\} \cup C_1 \times \{1\}; \mathbb{Z}_2)$ is independent of the choice of $t$ and $v$. If we choose a simplicial subdivision of $M^3$ so that $\text{sk}_1(M^3) \cap C_i = \emptyset$ for $i = 0, 1$ then $w_2$ is the obstruction to extending the map $(t_i, v_i): \text{sk}_1(M^3 \times \{0, 1\}) \rightarrow V_2(\mathbb{R}^3)$ to $\text{sk}_2(M^3 \times I)$. So the homology class $[K]|_2$ and thus $\text{Rd}(a_0, a_1)$ depends only on the homotopy class of the map

$$(t_i, v_i)|_{\text{sk}_1(M^3)}: \text{sk}_1(M^3) \rightarrow V_2(\mathbb{R}^3)$$

for $i = 0, 1$. For the sake of completeness we note that if the extension $t$ is given then

$$PD[K] = e(t^\perp; w_i) \in H^2(M^3 \times I, (M^3 \setminus C_0) \times \{0\} \cup (M^3 \setminus C_1) \times \{1\}; \mathbb{Z}).$$

So we have obtained the following proposition.

**Proposition 3.14.** Suppose that $(C_0, \nu_0)$ and $(C_1, \nu_1)$ are framed submanifolds of $M^3$ and let $a_0, b_0, a_1, b_1 \in N(M^3, \chi)$ be of the form $a_i = (C_i, \nu_i, t_i^a, v_i^a)$ and $b_i = (C_i, \nu_i, t_i^b, v_i^b)$ for $i = 0, 1$. Moreover, suppose that $\text{sk}_1(M^3) \cap C_i = \emptyset$ and
Proof. Let $t^a$ and $v^a$ be generic extensions of $t_i^a$, respectively $v_i^a$ over $M^3 \times I$ and denote by $K^a$ the submanifold of $M^3 \times I$ where $t^a$ and $v^a$ are linearly dependent. We obtain the sections $t^b$ and $v^b$ of $\varepsilon^{3}_{M \times I}$ and the submanifold $K^b \subset M^3 \times I$ in a similar way. Then, according to Remark 3.13, we get that

$$PD[K^a]|_{2} = w_{2}(\varepsilon^{3}_{M \times I}; t_i^a, v_i^a) = w_{2}(\varepsilon^{3}_{M \times I}; t_i^a, v_i^a) = PD[K^b]|_{2},$$

since $w_2$ is the obstruction to extending a map into $V_2(\mathbb{R}^3)$ from $sk_1(M^3 \times I)$ to $sk_2(M^3 \times I)$ and $(t_i^a, v_i^a)|_{sk_1(M^3 \times \{i\})}$ is homotopic to $(t_i^b, v_i^b)|_{sk_1(M^3 \times \{i\})}$. Thus if $F$ denotes the submanifold of $M^3 \times S^1 = D(M^3 \times I)$ obtained by piecing together $K^a$ and $K^b$ we get that $PD[F]|_{2} = w_{2}(\varepsilon^{3}_{M \times S_1}) = 0$, so we can proceed as in the proof of Proposition 3.12. \hfill \square

**Proposition 3.15.** If $a_i = (C_i, \nu_i, t_i, v_i) \in N(M^3, \chi)$ for $i = 0, 1$ then

$$rd((C_0, \nu_0), (C_1, \nu_1)) = Rd(a_0, a_1) \mod 2d(\chi).$$

**Proposition 3.16.** If $a_0, a_1, a_2 \in N(M^3, \chi)$ then

$$Rd(a_0, a_1) + Rd(a_1, a_2) = Rd(a_0, a_2).$$

**Definition 3.17.** For each $\chi \in H^2(M^3; \mathbb{Z})$ of the form $\chi = 2c$ fix an element $a_\chi \in N(M^3, \chi)$. Then for each $a \in N(M^3, \chi)$ define the notation $R(a) \in \mathbb{Z}_{4d(\chi)}$ to be $Rd(a, a_\chi)$.

**Corollary 3.18.** If $a_0, a_1 \in N(M^3, \chi)$ then $Rd(a_0, a_1) = R(a_0) - R(a_1)$.

4. The Orientation of $\Sigma^1$

Now let us recall a special case of Lemma 6.1 of [3]. Let $F: W^4 \to \mathbb{R}^5$ be a generic map of a compact orientable manifold. Then the singularity set $\Sigma(F)$ of $F$ is a 2-dimensional submanifold of $W^4$ which is not necessarily orientable.

**Lemma 4.1.** The line bundles $det(T\Sigma(F))$ and $ker(dF)$ over $\Sigma(F)$ are isomorphic.

**Definition 4.2.** Let $\pi$ denote the projection of $\mathbb{R}^{m+1}$ onto $\mathbb{R}^m$. A map $f: N^n \to \mathbb{R}^m$ is called prim if there exists an immersion $f': N^n \to \mathbb{R}^{m+1}$ such that $\pi \circ f' = f$.

**Corollary 4.3.** If $F: W^4 \to \mathbb{R}^5$ is a generic prim map then $\Sigma(F) \subset W^4$ is an orientable surface.

**Proof.** Let $s$ denote the sixth coordinate function of $F'$, i.e., $F' = (F, s)$. Since $F'$ is non-singular, the function $s$ is non-degenerate along $ker(dF)$. Thus we can orient $ker(dF)$ so that the derivative of $s$ in the positive direction of $ker(dF)$ is positive. But the orientability of $ker(dF)$ implies the orientability of $\Sigma(F)$ by Lemma 4.1. \hfill \square

The following definition, motivated by Corollary 4.3, gives an explicit isomorphism $\Psi$ between $ker(dF)$ and $det(T\Sigma(F))$.

**Definition 4.4.** Let $W^4$ be a compact oriented manifold with possibly non-empty boundary and let $F: W^4 \to \mathbb{R}^5$ be a generic map. For $p \in \Sigma(F)$ choose a small neighborhood $U_p \subset W^4$ of $p$ in which $ker(dF)$ is orientable. Put $F_p = F|_{U_p}$ and choose an orientation $\omega_p$ of $ker(dF_p)$. Then there exists a smooth function

$$(t_i^a, v_i^a)|sk_1(M^3) \text{ is homotopic to } (t_i^b, v_i^b)|sk_1(M^3) \text{ as maps into } V_2(\mathbb{R}^3) \text{ for } i = 0, 1.$$
s: \mathbb{U}_p \to \mathbb{R} such that the derivative of s in the direction of \( \alpha_p \) is positive. (First construct s along \( \Sigma(F_p) \) near \( \Sigma^{1,1}(F_p) \) then extend it to a tubular neighborhood of \( \Sigma(F_p) \).

The map \( \mathcal{F}'_p = (F_p, s): \mathbb{U}_p \leftrightarrow \mathbb{R}^6 \) is an immersion. If \( \mathbb{U}_p \) is chosen sufficiently small then we can even suppose that \( \mathcal{F}'_p \) is an embedding. Denote by \( e_6 \) the sixth coordinate direction in \( \mathbb{R}^6 \) and let \( \nu_6: \mathbb{U}_p \to T\mathbb{R}^6 \) denote the vector field along \( \mathcal{F}'_p \) defined by the formula \( \nu_6(x) = e_6 \in T_{\mathcal{F}'_p(x)}\mathbb{R}^6 \) for \( x \in \mathbb{U}_p \). Projecting \( \nu_6 \) into the normal bundle of \( \mathcal{F}'_p \) we obtain a normal field \( \mu_6 \) along \( \mathcal{F}'_p \) that vanishes exactly at the points of \( \Sigma(F_p) \). Perturb \( \mathcal{F}'_p \) in the direction of \( \mu_6 \) to obtain an embedding \( \mathcal{F}''_p \). Then orient \( \Sigma(F_p) \) as the intersection of \( \mathcal{F}'_p \) and \( \mathcal{F}''_p \) in \( \mathbb{R}^6 \). Here \( \mathbb{R}^6 \) is considered with its standard orientation. This orientation of \( \Sigma(F_p) \) does not depend on the choice of the function s, since if \( s_1 \) and \( s_2 \) are two such functions then for \( 0 \leq t \leq 1 \) the convex combination \((1-t)s_1 + ts_2 \) also satisfies the conditions for s.

If we reverse the orientation of \( \ker(dF_p) \), i.e. if we orient it by \(-\alpha_p\), then we can choose \(-s\) instead of s. Thus we obtain the embedding \((F_p, -s)\), which is the reflection of \( \mathcal{F}'_p \) in the hyperplane \( \mathbb{R}^5 \). Denote this reflection by \( R: \mathbb{R}^6 \to \mathbb{R}^6 \) (i.e., 

\( R(x_1, \ldots, x_5, x_6) = (x_1, \ldots, x_5, -x_6) \)). Then \( (F_p, -s) = R \circ \mathcal{F}'_p \). The vector field

\[ dR \circ \nu_6 \text{ along } R \circ \mathcal{F}'_p \text{ points in the direction } -e_6 \text{ and } R \circ \mathcal{F}''_p \text{ is the perturbation of } R \circ \mathcal{F}'_p \text{ in the direction of } dR \circ \mu_6. \]

But in this case we should perturb \( R \circ \mathcal{F}'_p \) in the direction of \(-dR \circ \mu_6\). We obtain the same orientation if we look at the intersection \((R \circ \mathcal{F}'_p) \cap (R \circ \mathcal{F}''_p)\) instead. Since the intersection is 2-dimensional and \( \mathbb{U}_p \) is 4-dimensional we get that \((R \circ \mathcal{F}'_p) \cap (R \circ \mathcal{F}'_p) = (R \circ \mathcal{F}'_p) \cap (R \circ \mathcal{F}''_p) \) in the oriented sense. The orientations of \( \Sigma(F_p) \) defined by the intersections \( \mathcal{F}'_p \cap \mathcal{F}''_p \) and \((R \circ \mathcal{F}'_p) \cap (R \circ \mathcal{F}''_p)\) are opposite. This can be seen from the following argument: For \( 0 \leq t \leq 1 \) denote by \( R_t \) the rotation of the hyperplane \( \mathbb{R}^6 \) in \( \mathbb{R}^7 \) around \( \mathbb{R}^5 \) by the angle \( \pi t \). The orientation of \((R_t \circ \mathcal{F}'_p) \cap (R_t \circ \mathcal{F}''_p)\) in \( R_t(\mathbb{R}^6) \) changes continuously as \( t \) goes from 0 to 1. The orientations of the hyperplanes \( R_t(\mathbb{R}^6) \) and \( \mathbb{R}^6 \) are opposite, thus the reflection \( R \) changes the orientation of the intersection \( \mathcal{F}'_p \cap \mathcal{F}''_p \).

So we have defined an isomorphism \( \Psi_p \) between \( \ker(dF_p) \) and \( \det(T\Sigma(F_p)) \) for every \( p \in W^4 \) in a compatible way (i.e., \( \Psi_p((U_p \cap \mathbb{U}_q)) = \Psi_q((U_p \cap \mathbb{U}_q)) \) for \( p, q \in W^4 \)). These local isomorphisms define a global isomorphism \( \Psi \) between \( \ker(dF) \) and \( \det(T\Sigma(F)) \).

5. The invariant

In this section we will give a geometric formula for the 3-dimensional obstruction to the existence of a regular homotopy between two immersions of \( M^3 \) into \( \mathbb{R}^5 \). This generalizes the results of [10] to immersions with non-trivial normal bundle.

**Definition 5.1.** Let \( W^4 \) be a compact oriented manifold with boundary \( M^3 \) and \( F: \mathcal{W} \to \mathbb{R}^5 \) a generic map such that \( f = F|M^3 \) is an immersion. Recall that \( \Sigma(F) \) denotes the set of singular points of F. Let us denote by \( C(F) \subset M^3 \) the 1-dimensional submanifold \( \partial \Sigma(F) \). Choose a trivialization \( \tau \) of \( \ker(dF)|C(F) \) so that it points into the interior of \( W^4 \). This is possible since \( f \) is non-singular (and so \( \ker(dF) \) never lies in \( TM^3 \)). Then \( \tau \) is normal to \( \Sigma(F) \) because \( F \) is generic and thus \( \Sigma^{1,1}(F) \cap C(F) = \emptyset \). So if we project \( \tau \) into \( TM \) along \( \Sigma(F) \) we obtain a nowhere vanishing normal field \( \nu(F) \) in \( \nu(C(F) \subset M^3) \).

Let \( U \) denote a small collar neighborhood of \( C(F) \) in \( \Sigma(F) \). Then clearly \( U \) is orientable. Using the isomorphism \( \Psi \) of Definition [10] the trivialization \( \tau \) of
ker\((dF)\) induces an orientation of \(U\). Thus \(C(F) \subseteq \partial U\) is also oriented. So we have assigned a pair \((C(F), \nu(F))\) to \(F\) as in Notation 5.1. Let \(r(F) = r(C(F), \nu(F))\).

**Notation 5.2.** For \(\chi \in H^2(M^3; \mathbb{Z})\) let us denote by \(\text{Imm}(M^3, \mathbb{R}^5)_{\chi}\) the space of immersions with normal Euler class \(\chi\).

Fix a cohomology class \(\chi \in H^2(M^3; \mathbb{Z})\). Our aim is to define an invariant \(i: \pi_0(\text{Imm}(M^3, \mathbb{R}^5)_{\chi}) \to \mathbb{Z}_{2d(\chi)}\).

**Proposition 5.3.** Let \(f \in \text{Imm}(M^3, \mathbb{R}^5)_{\chi}\) and let \(F: W^4 \to \mathbb{R}^5\) be a generic map such that \(\partial F = f\). Then \([C(F)] = D\chi\).

**Proof.** Let \(\kappa\) denote an inner normal field of \(M^3\) in \(W^4\) that extends \(\tau\) (see Definition 5.1). Then \(dF \circ \kappa\) is a vector field along \(f\) that is tangent to \(f\) exactly at the points of \(C(F) = \partial \Sigma(F)\). (If \(p \in C(F)\) then the rank of \((dF)p\) is 3, moreover \(dF|_{(T_pM^3)} = df\) is non-degenerate. Thus \(dF(\kappa_p) \in df|_{(T_pM^3)}\).) So if we project \(dF \circ \kappa\) into the normal bundle of \(f\) we obtain a normal field of \(f\) that vanishes along \(C(F)\). To see that \(C(F)\) represents the normal Euler class of \(f\), we have to know that it is oriented suitably.

Using the notations of Definition 5.1 we choose a function \(s: W^4 \to \mathbb{R}\) such that the derivative of \(s\) in the direction of \(\kappa\) (and thus \(\tau\)) is positive and \(s|_{M^3} = 0\). Then there exists a collar neighborhood \(V\) of \(M^3\) in \(W^4\) such that \(F' = (F, s)|V\) is an immersion. Denote by \(\nu_F\) the normal bundle of \(F'\) in \(\mathbb{R}^6\) and by \(\nu_F\) the normal bundle of \(f\) in \(\mathbb{R}^5\). Then \(\nu_F|_{M^3} = \nu_f\) are oriented bundles, since \(s\) is increasing along \(\kappa\) (here \(\mathbb{R}^5\) and \(\mathbb{R}^6\) are considered with their standard orientations). By Definition 2.21 the surface of singular points \(U = \Sigma(F|V)\) is oriented as the self-intersection of \(F'\) in \(\mathbb{R}^6\), or more precisely, as the intersection of the zero section and a generic section of \(\nu_{F'}\). Moreover, \(C(F)\) is oriented as the boundary of \(U\). Thus \(C(F)\) is the self-intersection of the zero section of \(\nu_{F'}|_{M^3} = \nu_f\), so it is dual to the Euler class \(e(\nu_{F'}) = \chi\). (Here we used the naturality of the Euler class.) \(\Box\)

**Definition 5.4.** Let \(f \in \text{Imm}(M^3, \mathbb{R}^5)_{\chi}\) and let \(F: W^4 \to \mathbb{R}^5\) be generic such that \(\partial F = f\). Denote the algebraic number of cusps of \(F\) by \(#\Sigma^{1,1}(F)\) (for the definition see 3). Then let \(j(f) = 3\sigma(W^4) - 3\alpha(M^3) + #\Sigma^{1,1}(F) + r(F) \in \mathbb{Z}_{2d(\chi)}\).

Note that if \(\chi = 0\) and \(F\) is an immersion near \(\partial W^4\) then \(r(F) = 0\). Thus in this case \(j(f)\) agrees with the double of the invariant introduced in 10 (see Definition 2.7).

**Theorem 5.5.** \(j(f)\) is well defined, i.e., it does not depend on the choice of the generic map \(F\). Moreover, if \(f_0\) and \(f_1\) are regularly homotopic then \(j(f_0) = j(f_1)\).

**Proof.** For \(i \in \{0, 1\}\) let \(F_i: W^4_i \to \mathbb{R}^5\) be a generic map such that \(\partial F_i = f_i\). Choose a regular homotopy \(\{h_t: 0 \leq t \leq 1\}\) connecting \(f_0\) and \(f_1\). This defines an immersion \(H: M^3 \times I \to \mathbb{R}^5 \times I\) by the formula \(H(x, t) = (h_t(x), t)\). Also choose a closed collar neighborhood \(U_i\) of \(M^3\) in \(W^4_i\) and a diffeomorphism \(d_i: U_i \to M^3 \times [0, \varepsilon]\) for \(i = 0, 1\). Let \(p: M^3 \times [0, \varepsilon] \to [0, \varepsilon]\) denote the projection onto the second factor. If \(\varepsilon\) (i.e., \(U_i\)) is sufficiently small then \(p \circ d_i\) is non-degenerate along \(\ker(dF_i)\) for \(i = 0, 1\) since \(\ker(dF_i)\) never lies in \(TM^3\). Let \(s_i\) be an arbitrary smooth extension of \(p \circ d_i\) over \(W^4_i\). Now let \(F'_0 = (F_0, -s_0)\): \(-W^4_0 \to \mathbb{R}^6\) and \(F'_1 = (F_1, s_1 + 1)\): \(W^4_1 \to \mathbb{R}^6\). Then \(F'_i\) is an immersion on \(U_i\). Notice that \(H|(M^3 \times \{0\}) = F'_0(\partial W^4_0)\) and \(H|(M^3 \times \{1\}) = F'_1(\partial W^4_1)\).
Denote by $\kappa_i$ the inner normal field of $W_i^4$ along $M^3 = \partial W^4$ and by $\nu_6$ the sixth coordinate direction in $\mathbb{R}^6$. Then the inner product $\langle dF'_i(\kappa_i), \nu_6 \rangle < 0$ and $\langle dF'_i(\kappa_1), \nu_6 \rangle > 0$. Furthermore, if $\lambda_i$ denotes the inner normal field of $M^3 \times I$ along $M^3 \times \{i\}$ for $i = 0, 1$ then $\langle dH(\lambda_0), \nu_6 \rangle > 0$ and $\langle dH(\lambda_1), \nu_6 \rangle < 0$. So $dH(\lambda_2)$ is homotopic to $-dF'_i(\kappa_i)$ in the space of vector fields normal to $H(M^3 \times \{i\})$. Using Smale’s lemma there exists a regular homotopy of $H$ fixed on the boundary $M^3 \times \{0, 1\}$ that induces the above homotopy of normal fields. Denote by $H'$ the result of this regular homotopy of $H$. Then $F'_0$, $H'$ and $F'_1$ fit together to a smooth map $F'$ of $W^4 = W_0^4 \cup (M^3 \times I) \cup W_1^4$ into $\mathbb{R}^6$ that is an immersion on $M^3 \times I$. Let $\pi: \mathbb{R}^6 \to \mathbb{R}^5$ denote the projection map. Then by a small perturbation of $H'$ we can achieve that $F = \pi \circ F'$ is generic.

Since $G = F|(M^3 \times I) = \pi \circ H'$ is prim, the singular surface $\Sigma(G)$ is oriented and a trivialization $\tau$ of $\ker(dG)$ is given. If we project $\tau$ into $\nu(\Sigma(G) \subset M^3 \times I)$ we obtain a normal field $\nu$ along $\Sigma(G)$ that vanishes exactly at the cusps of $G$, i.e., where $\tau$ is tangent to $\Sigma(G)$. So $\#\Sigma^{1,1}(G)$ is equal to the algebraic number of zeroes of $\nu$, which in turn is congruent to $\text{rd}(C(F_0), \nu(F_0)), (C(F_1), \nu(F_1))) = r(F_0) - r(F_1)$ modulo $2d(\chi)$ by Definition 5.2.

Now using the result of Szücs [13] that $3\sigma(W^4) + \#\Sigma^{1,1}(F) = 0$ we get that

$$\langle 3\sigma(W^4) + \#\Sigma^{1,1}(F_0) + r(F_0) \rangle + \langle 3\sigma(W^4) + \#\Sigma^{1,1}(F_1) + r(F_1) \rangle = 3\sigma(W^4) + \#\Sigma^{1,1}(F_0 \cup G \cup F_1) = 0.$$  

(5.1)

In the special case $f_0 = f_1 = f$ this implies that $j(f)$ is well defined, and for $f_0$ and $f_1$ arbitrary (but regularly homotopic) we get that $j$ is a regular homotopy invariant.

\begin{proof}

Choose an immersion $f_i \in \text{Imm}(M^3, \mathbb{R}^5)_0$ and denote $f$ by $f_0$. Since in [10] it is proved that $j(f_1)$ is always even $j(f_i) = 2i(f_i)$ for $i$ as in Definition 2.7 it is sufficient to prove that $j(f_0) \equiv j(f_1) \mod 2$. Choose a singular Seifert surface $F_i: W_i \to \mathbb{R}^5$ for $f_i$ ($i = 0, 1$) and let $G: M^3 \times I \to \mathbb{R}^5$ be a generic map such that $-F_0 \cup G \cup F_1$ is a smooth map on $-W_1 \cup (M^3 \times I) \cup W_2$. Then by equation 5.1 above it is sufficient to prove that $r(F_0) - r(F_1) \equiv \#\Sigma^{1,1}(G) \mod 2$. Since $f_1$ has trivial normal bundle we may choose $F_1$ to be an immersion in a neighborhood of $\partial W^4_2$. So $G$ is an immersion in a neighborhood of $M^3 \times \{1\}$, moreover $r(F_1) = 0$. The difference between the present situation and the proof of Theorem 5.5 is that now $\ker(dG)$ might be non-orientable. Using Definition 5.8 we get that $r_2(F_0) - r_2(F_1) \equiv r(F_0) - r(F_1) \mod 2$. Let $\nu$ denote a generic normal field along $\Sigma(G)$ that extends both $\nu(F_0)$ and $\nu(F_1)$. By definition $r_2(F_0) - r_2(F_1)$ equals the mod 2 number of zeroes of $\nu$. Thus we only have to prove that $\nu^{-1}(0) \equiv \#\Sigma^{1,1}(G) \mod 2$.

From now on we will denote $\Sigma^{1,1}(G)$ by $K$ and the line bundle $\ker(dG) < T(M^3 \times I)|K$ by $l$. Then $l$ is tangent to $K$ exactly at the points of $\Sigma^{1,1}(G)$. For $\varepsilon > 0$ sufficiently small let $\tilde{K}$ denote the sphere bundle $S_l$. If $\varepsilon$ is sufficiently small then the exponential map of $M^3 \times I$ defines an immersion $s: \tilde{K} \to M^3 \times I$ so that the double points of $s$ correspond exactly to the points of $\Sigma^{1,1}(G)$. So we have to prove that $\lfloor D_2(s) \rfloor \equiv \nu^{-1}(0) \mod 2$. By Lemma 5.9 the surface $\tilde{K}$ is the orientation double cover of $K$, in particular $\tilde{K}$ is oriented and a sign can be given to each
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double point of $s$ (here we also use that $\dim(\tilde{K})$ is even). The sign of a double point of $s$ is the opposite of the sign of the corresponding cusp of $G$ (since the sign of a cusp is defined as the self intersection of $K$). Thus $\#D_2(s) = -\#\Sigma^{1,1}(G)$. Let $p: \tilde{K} \to K$ denote the covering map. Then $p^*\nu_K \approx \nu_s$, thus $p^*\nu$ defines a section $\tilde{\nu}$ of $\nu_s$. From the construction of $\tilde{\nu}$ it is clear that $\#\tilde{\nu}^{-1}(0) = 2\#\nu^{-1}(0)$.

If we perturb $s$ in the direction of $\tilde{\nu}$ we get a self intersection point of $s$ for each element of $\tilde{\nu}^{-1}(0)$ and two self intersection points for each double point of $s$. Thus

$$s \cap (s + \epsilon \tilde{\nu}) = \#\tilde{\nu}^{-1}(0) + 2D_2(s) = 2(\#\nu^{-1}(0) - \#\Sigma^{1,1}(G)).$$

So we only have to show that the left hand side is divisible by 4.

From now on we will work in a fixed tubular neighborhood $T$ of $C(F_0) \subset M^3$. Note that $\partial(s, \tilde{\nu}) = (C(F_0) + \epsilon \nu(F_0), \nu(F_0)) \cup (-C(F_0) - \epsilon \nu(F_0), \nu(F_0)) \subset T \times \{0\}$. Let $\nu$ be an oriented submanifold of $M^3$. We define an embedding $\epsilon: C(F_0) \times [-\epsilon, \epsilon] \to T$ by the formula $\epsilon(x, t) = x + tv(F_0)$. Then $E = \text{Im}(\epsilon)$ is a 2-dimensional oriented submanifold of $T$ with boundary $C$. Thus $r((C, \nu(F_0)) = E \cap (C + \nu(F_0)))$ where the right hand side is considered to be a generic intersection (each fiber of $E$ is parallel to $\nu(F_0)$). Let $n$ be a small non-zero vector field along $C(F_0)$ orthogonal to $\nu(F_0)$. Then $E \cap (C + \nu(F_0) + n) = \emptyset$ (this can be verified by inspecting each fiber of $T$), thus $r((C, \nu(F_0))) = 0$. So we get that $\#\nu^{-1}(0) = \#\Sigma^{1,1}(G)$, not just a mod 2 congruence.

Remark 5.7. A small improvement on the proof of Proposition 5.6 yields an interesting result: Let $G: M^3 \times [0, 1] \to \mathbb{R}^5$ be a generic map connecting the immersions $f_0$ and $f_1$. Let $K$ denote the singular surface of $G$ and let $\nu_i$ be a trivialization of $\ker(dG)(M^3 \times \{i\})$ for $i = 0, 1$. Then $\#\Sigma^{1,1}(G)$ is equal to the relative twisted normal Euler class $\epsilon(\nu_K; \nu_0, \nu_1)$.

Note that if $\partial K = C_0 \cup C_1$ then by definition

$$\text{rdl}((C_0, \nu_0), (C_1, \nu_1)) \equiv \epsilon(\nu_K; \nu_0, \nu_1) \mod 2.$$

If in particular $K$ is an oriented cobordism between $C_0$ and $C_1$ then

$$\text{rdl}((C_0, \nu_0), (C_1, \nu_1)) = \epsilon(\nu_K; \nu_0, \nu_1) = \#\Sigma^{1,1}(G);$$

here $C_i$ is oriented by $\nu_i$ using the isomorphism $\Psi$ (see Definition 5.4).

Thus $j$ may take only $d(\chi)$ different values if $d(\chi) > 0$. (Since $j$ is additive if a connected sum is taken with an immersion of a sphere (Lemma 5.1) and $j(g)$ can be any even number for $g: S^3 \to \mathbb{R}^5$ it follows that $j$ is an epimorphism onto $2\mathbb{Z}_{2d(\chi)}$.) But Theorem 2.4 implies that there are exactly $2d(\chi)$ regular homotopy classes with normal Euler class $\chi$. So $j$ describes the regular homotopy class of $f$ up to a $2:1$ ambiguity. To resolve this problem we will lift the invariant $j \in \mathbb{Z}_{2d(\chi)}$ to an invariant $I \in \mathbb{Z}_{4d(\chi)}$. It follows from Proposition 5.3 that $I$ is always an even element, thus it defines an invariant $I \in \mathbb{Z}_{2d(\chi)}$ by the embedding $\mathbb{Z}_{2d(\chi)} \hookrightarrow \mathbb{Z}_{4d(\chi)}$.

Notation 5.8. Let $f \in \text{Imm}(M^3, \mathbb{R}^5)_\chi$ and let $F: W^4 \to \mathbb{R}^5$ be a singular Seifert surface for $f$. If $\kappa$ denotes the inner normal field along $\partial W^4$ then $\tilde{\nu}(F) \in \Gamma(\nu_f)$ be the projection of $dF(\kappa)$ into $\nu_f$. If $\bar{I} \in \Gamma(\varepsilon_1M)$ denotes a trivialization of $\varepsilon_1M$ then we can consider $\bar{I}$ and $\tilde{\nu}(F)$ to be sections of $\nu_f \oplus \varepsilon_1M$. Let $\bar{\nu}(F) = \tilde{\nu}(F) + \bar{I} \in \Gamma(\nu_f \oplus \varepsilon_1M)$. 
From now on we will fix a spin structure $s_M \in \text{Spin}(M^3)$. If we consider $\mathbb{R}^5$ with its unique spin structure then for every immersion $f: M^3 \leftrightarrow \mathbb{R}^5$ a spin structure $s(f)$ is induced on $\nu_f$ by $s_M$. Then $s(f)$ is equivalent to a trivialization $\tau(f): \varepsilon_M^3|\text{sk}_2(M) \to \nu_f \oplus \varepsilon_M^3|\text{sk}_2(M^3)$ up to homotopy. Since $\pi_2(\text{SO}(3)) = 0$ the trivialization $\tau(f)$ extends to an isomorphism $\tau(f): \varepsilon_M^3 \to \nu_f \oplus \varepsilon_M^1$, but this extension is not unique because $\pi_2(\text{SO}(3)) \neq 0$.

**Definition 5.9.** Using the above notations let $t(f), v(F) \in \Gamma(\varepsilon_M^3)$ be defined by the formulas $t(f) = \tau(f)^{-1} \circ t$ and $v(F) = \tau(f)^{-1} \circ v(F)$. Denote by $a(F)$ the quadruple $(C(F), \nu(F), t(f), v(F)) \in N(M^3, \chi)$. Then define $R(F) \in \mathbb{Z}_{4d(\chi)}$ to be $R(a(F))$. Since the homotopy class of the map $(t(f), v(F))|\text{sk}_2(M^3): \text{sk}_2(M^3) \to V_2(\mathbb{R}^3)$ is independent of the choice of the extension of $\tau(f)$ to $\varepsilon_M^3$ Proposition 5.11 implies that $R(F)$ is also independent of $\tau(f)$ and depends only on $s_M$.

**Remark 5.10.** Proposition 5.9 implies that $r(F) \equiv R(F) \mod 2d(\chi)$.

Now we can finally define a complete regular homotopy invariant.

**Definition 5.11.** For $f \in \text{Imm}(M^3, \mathbb{R}^5)_\chi$ and a singular Seifert surface $F$ let $I(f) \in \mathbb{Z}_{4d(\chi)}$ be defined as $3r(W^4) - 3\alpha(M^3) + \#\Sigma^{1,1}(F) + R(F)$. (Recall that we have fixed a spin structure $s_M$ on $M^3$ for the definition of $R(F)$.) Remark 5.10 above implies that $j(f) \equiv I(F) \mod 2d(\chi)$. Thus by Proposition 5.6 we get that $I(F)$ is always an even element of $\mathbb{Z}_{4d(\chi)}$. Let us denote by $\frac{1}{2}$ the isomorphism from $2\mathbb{Z}_{4d(\chi)}$ to $\mathbb{Z}_{2d(\chi)}$. Then let $i(F) = \frac{1}{2}I(F)$.

Clearly $j(f) = 2i(f)$ for every $f \in \text{Imm}(M^3, \mathbb{R}^5)_\chi$.

**Theorem 5.12.** $I(f)$ is well defined, i.e., it does not depend on the choice of the generic map $F$. Moreover, if $f_0$ and $f_1$ are regularly homotopic then $I(f_0) = I(f_1)$.

**Proof.** Using the notations of the proof of Theorem 5.9 we only have to show that the surface $K = \Sigma(G) \subset M^3 \times I$ satisfies Definition 5.11. I.e., there exist generic sections $t$ and $v$ of $\varepsilon_{M \times I}^3$ that extend $t(f_i)$ and $v(F_i)$ for $i = 0, 1$ and are linearly dependent exactly over $K$. The regular homotopy between $f_0$ and $f_1$ defines the immersion $H: M^3 \times I \leftrightarrow \mathbb{R}^5 \times I$. For $i \in \{0, 1\}$ there is a canonical isomorphism $\varphi_i: \nu_H(M^3 \times \{i\}) \to \nu_{f_i}$. Let $\bar{w} \in \Gamma(\nu_H)$ denote the projection of the sixth coordinate vector $v_6 \in \mathbb{R}^6$ into $\nu_H$. Then $\varphi_i(\bar{w}|M^3 \times \{i\}) = \bar{w}(F_i)$. Moreover, $K = \bar{w}^{-1}(0)$ and the orientation of $K$ is defined as the self intersection of $H$ if perturbed in the direction of $\bar{w}$. Define $\tilde{t}$ to be a trivialization of the $\varepsilon_{M \times I}^1$ component of the bundle $\nu_H \oplus \varepsilon_{M \times I}^1$ and let $\bar{v} = \bar{w} + \tilde{t}$. Then $\tilde{t}$ and $\bar{v}$ are linearly dependent exactly at the points of $K$. Note that $t(f_i) = \tau(f_i)^{-1} \circ (\varphi_i \oplus \text{id}_{\nu_{f_i}})(\tilde{t}|M^3 \times \{i\})$ and $v(F_i) = \tau(f_i)^{-1} \circ (\varphi_i \oplus \text{id}_{\nu_{f_i}})(\bar{v}|M^3 \times \{i\})$. Thus we only have to define a trivialization $\tau: \varepsilon_{M \times I}^3 \to \nu_H \oplus \varepsilon_{M \times I}^1$ such that

$$\tau(M^3 \times \{i\}) = \varphi_i^{-1} \circ \tau(f_i) \text{ for } i = 0, 1.$$  

The spin structure $s_M \in \text{Spin}(M^3)$ and the unique spin structure on $I$ define a spin structure on $M^3 \times I$. Together with the unique spin structure of $\mathbb{R}^6$ we get a spin structure $s_{\bar{H}}$ on $\nu_H$. When $s_{\bar{H}}$ is restricted to $M^3 \times \{i\}$ we get back the spin structure $s_M$. Thus $s_{\bar{H}}$ defines a trivialization $\tau_{\bar{H}}: \varepsilon_{M \times I}^3|\text{sk}_2(M^3 \times I) \to (\nu_H \oplus \varepsilon_{M \times I}^1)|\text{sk}_2(M^3 \times I)$ satisfying equation 5.2 over the 2-skeleton of $M^3 \times \{0, 1\}$. Note that the trivialization $\tau(f_i)$ is only well defined over $\text{sk}_2(M^3)$ and that we can choose an arbitrary
extension over $M^3$ in order to define the rotation difference. Thus we only have to extend $\tau_H$ to a trivialization $\tau$ of $\nu_f \oplus \varepsilon_{M \times I}$ and then define $\tau(f_i)$ by formula $(\ref{eq:tau_i})$.

First we extend $\tau_H$ to $sk_3(M^3 \times I) \setminus sk_3(M^3 \times \{1\})$. This is possible since the obstruction to extending the trivialization over a 3-simplex from its boundary lies in $\pi_2(SO(3)) = 0$. If $\sigma^3$ is a 3-simplex of $M^3$ then we can extend $\tau_H$ to $\sigma^3 \times I$ since it is given only on $\partial(\sigma^3 \times I) \setminus (\sigma^3 \times \{1\})$. Thus we have obtained the required extension $\tau$ of $\tau_H$. 

6. Connected sums and completeness of the invariant $i$

Lemma 6.1. If $f \in \text{Imm}(M^3, \mathbb{R}^5)$ and $g \in \text{Imm}(S^3, \mathbb{R}^5)$ then $c(f \# g) = c(f)$, in particular $c(\nu_f \# g) = c(\nu_f)$. Moreover

\[ i(f \# g) = i(f) + (i(g) \mod 2d(\chi)) \in \mathbb{Z}_{2d(\chi)}. \]

Proof. Since $c(f)$ describes the regular homotopy class of $f(M^3)$ it is trivial that $c(f \# g) = c(f)$. Let $F$ be a singular Seifert surface of $f$ and $G$ of $g$ such that $G$ is an immersion near the boundary. Then the result follows by inspecting the boundary connected sum $F \# G$ and the fact that $C(G) = \emptyset$. 

Theorem 6.2. Suppose that the immersions $f_0, f_1 \in \text{Imm}(M^3, \mathbb{R}^5)$ are regularly homotopic on $M^3 \setminus D$, where $D \subset M^3$ is diffeomorphic to the closed disc $D^3$ (i.e., $c(f_0) = c(f_1)$). Then $i(f_0) = i(f_1)$ implies that $f_0$ is regularly homotopic to $f_1$.

Proof. The proof consists of two cases according to the value of $d(\chi)$.

If $d(\chi) > 0$ then $i$ takes values in $\mathbb{Z}_{2d(\chi)}$ which is a finite group. Theorem 2.4 implies that there are exactly $2d(\chi)$ regular homotopy classes with a fixed Wu invariant $c$. Thus we only have to show that the invariant $i$ restricted to immersions with Wu invariant $c$ is an epimorphism onto $\mathbb{Z}_{2d(\chi)}$. For this end choose an immersion $f \in \text{Imm}(M^3, \mathbb{R}^5)$ such that $c(f) = c$. In \cite{3} it is shown that $i : \text{Imm}(M^3, \mathbb{R}^5) \rightarrow \mathbb{Z}$ is a bijection. Thus Lemma 6.1 implies that $c(f \# g) = c(f) = c$ for every $g \in \text{Imm}(S^3, \mathbb{R}^5)$, moreover $i : \{ f \# g : g \in \text{Imm}(S^3, \mathbb{R}^5) \} \rightarrow \mathbb{Z}_{2d(\chi)}$ is surjective.

If $d(\chi) = 0$ then $i$ maps into $\mathbb{Z}$. Using Smale’s lemma we can suppose that $f_0(M^3 \setminus D) = f_1(M^3 \setminus D)$. The normal bundles of $f_0|D$ and $f_1|D$ in $\mathbb{R}^5$ are trivial, choose a trivialization for both of them. Let $\tau_0$ be a non-zero normal field along $f_0|D$. Then $\tau_0|\partial D$ considered in the trivialization of the normal bundle of $f_1|D$ is a map $(\tau_0|\partial D) : \partial D \rightarrow S^3$. Since $\partial D$ is homeomorphic to $S^2$ and $\pi_2(S^3) = 0$ the normal field $\tau_0|\partial D$ can be extended to a normal field $\tau_1$ of $f_1|D$. Thus $\tau_1$ is a normal field of $f_i|D$ for $i = 0, 1$ and $\tau_0|\partial D = \tau_1|\partial D$.

Next choose an oriented compact manifold $W_0^4$ with boundary $M^3$. We push $D$ into the interior of $W_0^4$ fixing the boundary $\partial D$ to obtain a 3-disc $D_1 \subset W_0^4$ so that $\partial D = \partial D_1$ and $M_1^3 = (M^3 \setminus D) \cup D_1$ is a smooth submanifold of $W_0^4$. If we throw out the domain bounded by $D$ and $D_1$ in $W_0^4$ we obtain a 4-dimensional submanifold $W_1^4$ of $W_0^4$ with boundary $M_1^3$. Clearly $W_0^4$ is diffeomorphic to $W_1^4$.

We can choose a generic map $F_0 : W_0^4 \rightarrow \mathbb{R}^5$ with the following three properties:

1. $F_0|M_0^3 = f_0$ and $F_0|M_1^3 = f_1$ (where $M_1^3$ is identified with $M^3$ by a diffeomorphism keeping $M^3 \setminus D$ fixed).
2. $F_0$ is an immersion in a neighborhood of $D$ and $D_1$.
3. If $\kappa_0$ denotes the inner normal field of $D$ in $W_0^4$ and $\kappa_1$ denotes the inner normal field of $D_1$ in $W_1^4$ then $dF_0 \circ \kappa_0 = \tau_0$ and $dF_1 \circ \kappa_1 = \tau_1$. 

□
Let $F_1 = F_0 W_1^4$. Then (2) implies that $C(F_0) = C(F_1) \subset M^3 \setminus D$, moreover $\nu(F_0) = \nu(F_1)$. In particular, the normal Euler class of $f_0$ and $f_1$ coincide. Thus $R(F_0) = R(F_1)$. Since $\sigma(W_0^4) = \sigma(W_1^4)$, we get that

$$0 = i(f_0) - i(f_1) = \# \Sigma_{1,1}(F_0 \setminus (W_0^4 \setminus W_1^4)).$$

Choose diffeomorphisms $d_0: S^3_i \rightarrow D$ and $d_1: S^3_i \rightarrow D_1$, where $S^3_i$ denotes the northern hemisphere of $S^3$. Then the immersion $F_0 \circ d_i$ can be extended to an immersion $f'_i : S^3 \hookrightarrow \mathbb{R}^5$ for $i = 0, 1$ so that $f'_0 | S^3_i = f'_i | S^3_i$. (This is possible since $j^1(f_0) \partial D = j^1(f_1) \partial D$.) Now repeating the same argument as above for $f'_0$ and $f'_1$, we obtain that

$$i(f'_0) - i(f'_1) = \# \Sigma_{1,1}(F_0 \setminus (W_0^4 \setminus W_1^4)).$$

(Note that $\tau_0$ and $\tau_1$ have a common extension over $S^3_i$.) Thus $i(f'_0) - i(f'_1) = 0$, so using \ref{3} we get that $f'_0$ and $f'_1$ are regularly homotopic. But this implies that there exists a regular homotopy between $f'_0$ and $f'_1$ that is fixed on $S^3_i$ (see \ref{7}, Lemma 3.33). So $f_0 | D$ and $f_1 | D$ are regularly homotopic keeping the 1-jets on the boundary fixed, which completes the proof that $f_0$ and $f_1$ are regularly homotopic. \hfill \square

**Corollary 6.3.** The map

$$(c, i): \text{Imm}[M^3, \mathbb{R}^5] \rightarrow \prod_{c \in H^2(M^3; \mathbb{Z})} \mathbb{Z}_{4d(c)}$$

is a bijection.

We get more structure on the set of regular homotopy classes of immersions of oriented 3-manifolds into $\mathbb{R}^5$ if we endow it with the connected sum operation. Let us introduce the notation

$$I(3, 5) = \{ [f] : [f] \in \text{Imm}[M^3, \mathbb{R}^5] \text{ for } M^3 \text{ oriented} \}. $$

Then $(I(3, 5), \#)$ is a semigroup whose structure is described in the following theorem.

**Theorem 6.4.** Let $M_1^3$ and $M_2^3$ be oriented 3-manifolds. Then

\begin{equation}
H^2(M_1^3 \# M_2^3; \mathbb{Z}) \approx H^2(M_1^3; \mathbb{Z}) \oplus H^2(M_2^3; \mathbb{Z}).
\end{equation}

If $f_i \in \text{Imm}(M_i^3, \mathbb{R}^5)$ for $i = 1, 2$, then

\begin{equation}
c(f_1 \# f_2) = c(f_1) \oplus c(f_2) \in H^2(M_1^3 \# M_2^3; \mathbb{Z}).
\end{equation}

Moreover, if $\chi_i$ denotes the normal euler class of $f_i$ and $\chi$ the normal euler class of $f_1 \# f_2$ then $d(\chi) = \gcd(d(\chi_1), d(\chi_2))$. Finally,

\begin{equation}
i(f_1 \# f_2) = (i(f_1) \mod 2d(\chi)) + (i(f_2) \mod 2d(\chi)) \in \mathbb{Z}_{2d(\chi)},
\end{equation}

where $i(f_i) \in \mathbb{Z}_{2d(\chi_i)}$ for $i = 1, 2$.

**Proof.** Equation \ref{6.1} follows from the fact that $H^2(M_1^3; \mathbb{Z}) \approx H^2(M_1^3 \setminus D^3; \mathbb{Z})$ (see the long exact sequence of the pair $(M_1^3, M_1^3 \setminus D^3)$) and the Mayer-Vietoris exact sequence for $M_1^3 \setminus M_2^3 = (M_1^3 \setminus D^3) \cup (M_2^3 \setminus D^3)$.

Equation \ref{6.2} can be seen from the description of $c(f_i)$ as the regular homotopy class of $f_i | sk_2(M_i^3)$. Since $\chi = \chi_1 \oplus \chi_2$ the statement about $d(\chi)$ is trivial.

Finally, equation \ref{6.3} is obtained by taking the boundary connected sum $F_1 \# F_2$ of singular Seifert surfaces $F_1$ and $F_2$ for $f_1$, respectively $f_2$. \hfill \square
7. Immersions of $M^3$ into $\mathbb{R}^6$ with a normal field

Let $\text{Imm}_1(M^3, \mathbb{R}^6)$ denote the space of immersions of $M^3$ into $\mathbb{R}^6$ with a normal field $\nu$. Moreover, let $\text{Imm}_1[M^3, \mathbb{R}^6] = \pi_0(\text{Imm}(M^3, \mathbb{R}^6))$ be the set of regular homotopy classes of such immersions with normal fields. If we fix a trivialization of $TM^3$ then Hirsch's theorem implies that the natural map $\text{Imm}_1(M^3, \mathbb{R}^6) \to C(M^3, V_4(\mathbb{R}^6))$ is a weak homotopy equivalence.

For $f \in \text{Imm}(M^3, \mathbb{R}^5)$ let $\iota(f) \in \text{Imm}_1(M^3, \mathbb{R}^6)$ be the immersion $f$ with the constant normal field defined by the sixth coordinate vector in $\mathbb{R}^6$. Thus $\iota$ is an embedding of $\text{Imm}(M^3, \mathbb{R}^5)$ into $\text{Imm}_1(M^3, \mathbb{R}^6)$. As a special case of Hirsch's compression theorem we have the following proposition.

**Proposition 7.1.** $\iota_* : \text{Imm}[M^3, \mathbb{R}^5] \to \text{Imm}_1[M^3, \mathbb{R}^6]$ is a bijection.

**Proof.** The embedding $\mathbb{R}^5 \hookrightarrow \mathbb{R}^6$ induces an embedding $V_3(\mathbb{R}^5) \hookrightarrow V_4(\mathbb{R}^6)$ and thus a map $\psi : [M^3, V_3(\mathbb{R}^5)] \to [M^3, V_4(\mathbb{R}^6)]$ that makes the following diagram commutative.

$$
\begin{array}{ccc}
\text{Imm}[M^3, \mathbb{R}^5] & \xrightarrow{\iota_*} & \text{Imm}_1[M^3, \mathbb{R}^6] \\
\downarrow & & \downarrow \\
[M^3, V_3(\mathbb{R}^5)] & \xrightarrow{\psi} & [M^3, V_4(\mathbb{R}^6)].
\end{array}
$$

By Hirsch’s theorem the vertical arrows are bijections, thus it is sufficient to prove that $\psi$ is also a bijection. To see this consider the fibration $V_3(\mathbb{R}^5) \to V_4(\mathbb{R}^6) \to S^5$. Then from the homotopy exact sequence of this fibration we get that the homomorphism $\pi_i(V_3(\mathbb{R}^5)) \to \pi_i(V_4(\mathbb{R}^6))$ is an isomorphism for $i \leq 3$ and this implies that $\psi$ is a bijection. \qed

The natural forgetful map $\varphi : \text{Imm}_1(M^3, \mathbb{R}^6) \to \text{Imm}(M^3, \mathbb{R}^6)$ is a Serre fibration.

**Proposition 7.2.** For any immersion $f : M^3 \hookrightarrow \mathbb{R}^6$ the normal bundle $\nu_f$ is trivial.

**Proof.** Since $M^3$ is spin the normal bundle $\nu_f$ is also spin, thus it is trivial over the 2-skeleton of $M^3$. Such a trivialization can be extended to the 3-simplices of $M^3$ because $\pi_2(SO(3)) = 0$. \qed

So $\varphi$ is surjective and the fiber of $\varphi$ is homotopy equivalent to $\Gamma(\nu_f) = C(M^3, S^2)$. Thus the end of the homotopy exact sequence of $\varphi$ looks like as follows:

$$
\pi_1(\text{Imm}(M^3, \mathbb{R}^6)) \to [M^3, S^2] \to \text{Imm}_1[M^3, \mathbb{R}^6] \xrightarrow{\varphi_*} \text{Imm}(M^3, \mathbb{R}^6) \to 0.
$$

By Hirsch’s theorem there is a bijection $\text{Imm}_1[M^3, \mathbb{R}^6] \approx [M^3, V_3(\mathbb{R}^6)]$. Since $V_3(\mathbb{R}^6)$ is 2-connected and $\pi_3(V_3(\mathbb{R}^6)) \approx \mathbb{Z}_2$ we get from obstruction theory that $[M^3, V_3(\mathbb{R}^6)] \approx H^3(M^3; \mathbb{Z}_2) \approx \mathbb{Z}_2$. It is well known that for $f \in \text{Imm}(M^3, \mathbb{R}^6)$ the regular homotopy class of $f$ is determined by the number of its double points $D(f)$ modulo 2. This gives a geometric interpretation of the map $\varphi_* : (f, \nu) \in \text{Imm}_1(M^3, \mathbb{R}^6)$ the regular homotopy invariant $\varphi_*(f, \nu)$ is equal to $D(f)$ modulo 2.

How can we determine the value of $\varphi_*(f, \nu)$ for a generic $g \in \text{Imm}(M^3, \mathbb{R}^6)$? This question was answered in [12], let us recall that result now. The self-intersection set $A(g)$ of $g$ is a closed 1-dimensional submanifold of $M^3$ and $g(A(g))$ is also a closed 1-dimensional submanifold of $\mathbb{R}^5$. We say that a component $C$ of $g(A(g))$ is non-trivial if the double cover $g|g^{-1}(C) : g^{-1}(C) \to C$ is non-trivial, i.e., if $g^{-1}(C)$
Remark 7.5. Suppose that \( f: M^3 \to \mathbb{R}^6 \) is a generic immersion and that \( \pi: \mathbb{R}^6 \to \mathbb{R}^5 \) is a projection such that \( g = \pi \circ f \) is also a generic immersion. Then
\[
D(f) \equiv \delta(g) \pmod{2}.
\]

Note that the immersions \( f \) and \( g \) above are regularly homotopic in \( \mathbb{R}^6 \). Thus for any generic \( g \) we have that \( \varphi_s \circ \iota_s = \delta \).

Now we are going to determine the group \( \pi_1(\text{Imm}(M^3, \mathbb{R}^6)) \). Using Hirsch’s theorem we get that it is isomorphic to \( \pi_1(C(M^3, V_5(\mathbb{R}^6))) = \pi_3 M^3, V_3(\mathbb{R}^6) \). Here \( SM^3 \) denotes the suspension of \( M^3 \) and is a 4-dimensional CW complex. The space \( V_3(\mathbb{R}^6) \) is 2-connected and \( \pi_3(V_3(\mathbb{R}^6)) \cong \mathbb{Z}_2 \). Moreover, \( \pi_4(V_3(\mathbb{R}^6)) \cong 0 \). This can be seen as follows: From the homotopy exact sequence of the fibration \( V_3(\mathbb{R}^6) \to V_4(\mathbb{R}^7) \to \mathbb{R}^6 \) we get that \( \pi_4(V_3(\mathbb{R}^6)) \cong \pi_4(V_4(\mathbb{R}^7)) \). It was shown by Paechter [8] that for \( k \geq 4 \) the isomorphism \( \pi_k(V_k(\mathbb{R}^{2k-1})) \cong 0 \) holds if \( k \equiv 0 \mod 4 \). Thus obstruction theory yields that \( \pi_3 M^3, V_3(\mathbb{R}^6) \cong H^3(SM^3; \mathbb{Z}_2) \cong H^2(M^3; \mathbb{Z}_2) \).

Putting together the above results we obtain the following theorem.

**Theorem 7.4.** The following sequence is exact:
\[
H^2(M^3; \mathbb{Z}_2) \to \mathbb{Z}_2 \to \text{Imm}(M^3, \mathbb{R}^5) \to \mathbb{Z}_2 \to 0.
\]

**Remark 7.5.** If we fix a trivialization of \( TM^3 \) then non-zero vector fields (or equivalently, oriented 2-plane fields) on \( M^3 \) correspond to maps \( M^3 \to S^2 \). Thus the set of homotopy classes of oriented 2-plane fields on \( M^3 \) is equal to \( [M^3, S^2] \) which was determined in Remark 6.3. A geometric classification of such oriented 2-plane fields, avoiding the use of a trivialization of \( TM^3 \), was carried out by Gompf (see [1], section 4). A complete set of homotopy invariants, similar to those introduced in our present paper, were obtained in [4]. Gompf’s result and the regular homotopy classification of immersions of \( M^3 \) into \( \mathbb{R}^5 \) are related by Theorem 7.4.

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