The effect of cyclic loading on the nonlinear response of structural concrete members with arbitrary cross-sectional shapes

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Abstract. This paper presents a numerical methodology for analysis of the behaviour of structural concrete sections subjected to cyclic loading. Classical beam theory was utilised to calculate sectional deformations (strain and curvature) under normal force and biaxial moment, while a nonlinear sectional analysis is performed via computer software written in Python by discretising the section of any shape, that is a combination of polygons and circles, into infinitesimal triangles or quadrilaterals. The stress-strain relationship of concrete and steel materials under monotonic loading was also implemented to evaluate the concrete’s secant modulus of elasticity at any strain level. A concrete cyclic model representing stress-strain relationships during unloading is thus proposed in which the secant modulus of elasticity of each mesh element is modified based on the stress level attained at the start of the unloading stage. Cyclic behaviour in tension and compression was investigated, including the transition from compression to tension and vice versa. The cyclic model was then implemented in sectional analysis to allow computation of the behaviour of the section at different loading stages. This study also proposes a new technique to calculate residual deformation during each analysis step and thus to plot the hysteresis behaviour of the section. Software was thus developed to estimate the failure load and produce a biaxial moment-curvature diagram, interaction diagram, and stress-strain diagram across the depth of the section for each load case.

1. Introduction
Sectional analysis of structural concrete members is widely used to estimate section capacity, and most international standards adopt sectional analysis as a design method based on ultimate loads. Traditional hand-calculated versions of sectional analysis use estimated material behaviour, such as the Whitney stress block for concrete [1]; however, current technologies and computer power now permit accurate results to be obtained in less time using sophisticated analytical material models and computational algorithms.

Traditional design methods assume that concrete sections undergo static loading and thus design them based on material monotonic peak stress, with conservative factors added to account for cyclic loading degradation. The cyclic behaviour of concrete was first investigated by Sinha et al. [2] in 1964, and since then, several attempts have been made to study cyclic behaviour and residual damage under different loading regimes. Most studies and proposed models have been conducted at a material level, with the concrete elements studied being cubes and cylinders under either direct tension or compression.
Theoretical concrete sectional analysis under static loading shows good consistency with experimental data in general, and two main techniques of sectional analysis have emerged from the literature. The first technique [3] calculates sectional response (strain and curvature) at specific external loadings (axial force and biaxial moments), while the second technique [4, 5] evaluates the specific loading required to obtain a certain deformation. The first technique is less complicated and leads to fewer errors, while the second technique is more efficient in determining section capacity and the descending branch of the moment-curvature diagram [6]. However, cyclic sectional behaviour during unloading and reloading cannot be determined analytically using either method, as a concrete section suffers from damage at different rates throughout the section depending on the strain level attained. This paper thus proposes a new approach to determining the sectional behaviour of structural concrete members during the unloading phase. Stress-strain diagrams of static and unloading phases are extracted from models developed in the literature, and the development of new software to implement these models in the proposed sectional analysis approach is examined.

2. Static behaviour of concrete
Concrete is a non-homogeneous mixture of cement and aggregate; its strength thus varies according to the mixing rates and curing methods used. However, the monotonic behaviour of concrete has been studied intensively, and several mathematical models have been proposed to describe concrete’s monotonic stress-strain diagram under both compression and tension. The main parameters required to produce any model are the initial tangent modulus of elasticity, the peak stress, and the strain corresponding to the peak stress. Existing models of compression behaviour include those by Hognestad et al. [7], Collins et al. [8], Mazars and Pijaudier-Cabot [9], and Karpenko et al. [10], while tensile behaviour under monotonic loading has been studied by Hillerborg [11], Gopalaratman and Shah [12], Vecchio and Collins [13], and Yankelevsky and Reinhardt [14]. Karpenko et al. [10] also proposed a nonlinear relationship, as adopted in this paper, which can be used to simulate both concrete and steel under different loading regimes, including tensile and compression loading, based on adjusting a few parameters.

2.1. General Karpenko model
The stress-strain relationship \((\sigma - \varepsilon)\) of any material can be expressed using equation (1), in which \(E\) is the secant modulus of elasticity. For materials with linear behaviour, the secant modulus, \(E\), represents a constant value equalling the initial tangent modulus of elasticity at the origin point, \(E_o\). For materials with nonlinear behaviour, such as concrete, the secant modulus, \(E\), at a given strain magnitude is expressed by a percentage of the initial tangent modulus of elasticity known as the coefficient of change of secant modulus, \(\nu\), as shown in equation (2). Karpenko et al. [10] thus proposed equation (3) to describe the coefficient of change of secant modulus as a function of the attained strain.

\[
\sigma = E \times \varepsilon
\]  
\(E = \left\{ \begin{array}{ll}
E_o & \text{linear} \\
\nu \times E_o & \text{nonlinear}
\end{array} \right.\)  
\[
\nu = \nu_o + (\nu_o - \nu)\sqrt{1 - \omega \times \eta - (1 - \omega) \times \eta^2} \quad (1 \leq \nu \leq 0)
\]
\[
\eta = \frac{\sigma - \sigma_{el}}{\sigma - \sigma_{el}} \quad (0 \leq \eta \leq 1)
\]

where, \(\nu_o\) is the coefficient of change of the secant modulus at the start of the stress-strain diagram, \(\nu\) is the coefficient of change of the secant modulus at the peak of the stress-strain diagram, \(\omega\) is the curvature parameter of the stress-strain diagram, \(\eta\) is the relative stress level as expressed in equation (4), \(\sigma_{el}\) is the elastic stress limit, and \(\sigma\) is the stress at the peak point of the stress-strain diagram.
Equation (3) can be expressed in the quadratic form shown in equation (5) based on substitution in equation (1) and equation (4). Although equation (5) is visually a complicated formula, it can be easily solved with the use of computer programs.

\[ a \times \nu^2 + b \times \nu + c = 0 \]  

\[ a = (1 - \omega) \times (v_0 - \hat{\nu})^2 \times \hat{\varepsilon}^2 + (\hat{\nu} \times \varepsilon - \varepsilon_{el})^2 \]

\[ b = (v_0 - \hat{\nu})^2 \times \varepsilon \times [(\omega \times (\hat{\nu} \times \hat{\varepsilon} - \varepsilon_{el}) - 2 \times \varepsilon_{el} \times (1 - \omega))] - 2 \times \hat{\nu} \times (\hat{\nu} \times \hat{\varepsilon} - \varepsilon_{el})^2 \]

\[ c = (v_0 - \hat{\nu})^2 \times \varepsilon_{el} \times [(1 - \omega) \times \varepsilon_{el} - \omega \times (\hat{\nu} \times \hat{\varepsilon} - \varepsilon_{el})] + (\hat{\nu} \times \hat{\varepsilon} - \varepsilon_{el})^2 \times (2 \times \hat{\nu} - v_o) \times v_o \]

2.2. Concrete stress-strain diagram

The stress-strain diagram of concrete under compression or tension can be modelled in two branches: an ascending branch for strain magnitudes less than the strain corresponding to the peak stress, and a descending branch for strain magnitudes higher than the strain corresponding to the peak stress. Most available models ignore the elastic limit in compression and assume full nonlinear behaviour, including those developed by Hognestad et al. [7] and Collins et al. [8]. However, Mazars and Pijaudier-Cabot assumed an elastic limit in compression of up to 50% of the peak stress [9]. The elastic limit is ignored in this study as the coefficient of secant modulus gradually decreases at low stress levels. This allows the quadratic parameters of equation (5) to be simplified to equation (6) by substituting zero for \( \varepsilon_{el} \):

\[ a \times \nu^2 + b \times \nu + c = 0 \]

\[ a = (1 - \omega) \times (v_0 - \hat{\nu})^2 \times \hat{\varepsilon}^2 + \hat{\nu}^2 \times \varepsilon^2 \]

\[ b = (v_0 - \hat{\nu})^2 \times \omega \times \hat{\nu} \times \hat{\varepsilon} \times \varepsilon - 2 \times \hat{\nu} \times \varepsilon^3 \times \varepsilon^2 \]

\[ c = (2 \times \hat{\nu} - v_o) \times v_o \times \hat{\varepsilon}^2 \times \varepsilon^2 \]

The expression for the curvature parameter, \( \omega \), is presented in equation (7) for both ascending and descending branches, while the coefficient of change of the secant modulus at the start of the stress-strain diagram, \( v_o \), is presented in equation (8). The stress-strain diagram of concrete can be easily plotted if the peak stress points (compressive peak and tensile peak) and the initial tangent modulus of elasticity are known. Figure 1 demonstrates the monotonic stress-strain diagram under both compression and tension, while figure 2 illustrates the relationship between the coefficient of secant modulus and the strain attained.

![Figure 1](image1.png)  
**Figure 1.** Concrete stress-strain diagram under monotonic loading.

![Figure 2](image2.png)  
**Figure 2.** Relationship between the coefficient of secant modulus and strain magnitude.
\[
\omega = \begin{cases} 
2 - 2.5 \times \dot{\varepsilon} & \varepsilon \leq \dot{\varepsilon} \\
1.95 \times \dot{\varepsilon} - 0.138 & \varepsilon \geq \dot{\varepsilon}
\end{cases}
\] (7)

\[
v_0 = \begin{cases} 
1 & \varepsilon \leq \dot{\varepsilon} \\
2.05 \times \dot{\varepsilon} & \varepsilon \geq \dot{\varepsilon}
\end{cases}
\] (8)

3. Cyclic behaviour of concrete

Experimental data reported in the literature suggests that concrete follows a concave path during unloading from a particular strain level during typical cyclic tests under compression or tension loads. The unloading path begins with a high stiffness, which gradually declines until it becomes approximately flat at a low stress level. The magnitude of residual strain after complete unloading is depending mainly on the strain level at the unloading point. Existing models of concrete unloading curves in compression include the multilinear curve published by Yankelevsky and Reinhardt [15], the power-type equation created by Bahn and Hsu [16], the Ramberg-Osgood equation developed by Palermo and Vecchio [17], and the exponential-type equation offered by Sima et al. [18].

The cyclic behaviour of concrete in tension has also been experimentally investigated in several studies, including Reinhardt [19], Nouailletas et al. [20], and Chen et al. [21]. Reinhardt [19] conducted static and cyclic tensile tests on 100 plain concrete prisms. Despite concave tensile unloading curves being observed in several studies, a straight line was selected by Foster and Marti [22], Mansor and Hsu [23], Sima et al. [18], and Chen et al. [21] due to its insignificant impact on results compared with compression. Chen et al. [24] also proposed a model to describe the tensile cyclic behaviour during full and partial unloading and reloading phases. However, the current paper considers the unloading phase of the first cycle only in tension and compression loading, accompanied by the transition phase to the original monotonic stress-strain diagram, as subsequent cycles require a more comprehensive model that simulates the stress-strain diagram for each cycle under various repeated loading regimes.

3.1. Unloading in compression

This study adopted the exponential model developed by Sima et al. [18] for simulation of the concrete stress-strain relationship during the compressive unloading stage. The main parameters involved in the model are unload strain, plastic residual strain, and tangent modulus at the plastic residual point. The magnitudes of residual strain and tangent modulus at the plastic point were determined empirically by Sima et al. [18], based on experimental data from seven studies, though Breccolotti et al. [25] proposed a modified version of Sima et al.’s [18] equation parameters. The original parameters were adopted in this research, however, as they fit well with data from the existing literature, while the new parameters by Breccolotti el al. [25] achieve higher consistency with the boundary conditions at both low and high strain levels by assuming that plastic strain coincides with unloading strain when \( \nu_u \) equals zero, and plastic modulus coincides with initial modulus when \( \nu_u \) equals one. Equation (9) represents the stress-strain relationship along the compressive unload path, which has been mathematically revised to resemble equation (1). Equations (10) and (11) were similarly derived from Sima et al. [18].

\[
\sigma = \left( \frac{\sigma_u}{E_p \times (\varepsilon_u - \varepsilon_p)} \right) \times E_p \times (\varepsilon - \varepsilon_p) \quad (\varepsilon_p \leq \varepsilon \leq \varepsilon_u)
\] (9)

\[
E_p = E_o \times (0.215 \times \nu_u^2 + 0.27 \times \nu_u)
\] (10)

\[
\varepsilon_p = \frac{\varepsilon_u}{4.694 \times \nu_u + 1.063}
\] (11)
where, $\sigma_u, \varepsilon_u, \nu_u$ are stress, strain, and coefficient of change in secant modulus, respectively, at the unloading point of the monotonic diagram; $\varepsilon_p$ is the plastic residual strain at zero stress level; and $E_p$ is the tangent modulus of the unload path at $\varepsilon_p$.

### 3.2. Compression-tension transition

After complete unloading during compression testing, a crush relaxation stress is required to remove all residual strain and restore material stiffness in the opposite loading direction. The transition from residual state to tensile state is very linear, as the level of tensile stress required to relieve the concrete element from compression damage is relatively low. Tasnimi and Lavasani [26] developed a transition model of two opposite curves passing through a specific intermediate point determined based on the plastic residual point. Although their model adopts a nonlinear relationship, there is no noticeable nonlinearity in the full-scale stress-strain diagram; a linear relationship from the residual point to the intersection point with the stress-strain diagram was thus accepted in this work for simplicity. The intersection point suggested by Tasnimi and Lavasani [26] was selected, such that the stress at the intersection is 10% of the tensile peak stress of concrete, as shown in equation (12). Figure 3 illustrates the unloading and transition curves at different compressive unloading strain levels:

$$\sigma_{tr} = 0.1 \times \varepsilon_t$$  \hspace{1cm} (12)

![Figure 3. Unloading paths in compression at different strain levels.](image)

### 3.3. Unloading in tension

The model adopted for the tensile unloading stress-strain diagram was developed by Chen et al. [24]. This model implements the concept of non-classical strain, which represents the inelastic strain as calculated by equations (13) and (14).

$$\varepsilon = \varepsilon_e + \varepsilon_{nc} = \frac{\sigma}{E_o} + \varepsilon_{nc}$$  \hspace{1cm} (13)

$$\varepsilon_{nc} = \varepsilon - \frac{\sigma}{E_o}$$  \hspace{1cm} (14)

where $\varepsilon$ is the total strain used throughout the paper, $\varepsilon_e$ is the classical elastic strain calculated using Hook’s law, and $\varepsilon_{nc}$ is the non-classical strain. The tensile unloading stress-strain diagram follows a nonlinear concave shape, represented by equation (15):
\[ \sigma = \sigma_u \times \frac{X_u}{1 - a_u \times (1 - (X_u)^{b_u})} \]  

\[ X_u = \frac{\varepsilon_{nc} - \varepsilon_p}{\varepsilon_{nc,u} - \varepsilon_p} \quad (\varepsilon_p \leq \varepsilon_{nc} \leq \varepsilon_{nc,u}) \]

where, \( \sigma_u \) is the tensile stress at unloading point, \( X_u \) is the normalised non-classical strain of the unloading segment of the stress-strain diagram, \( a_u \) and \( b_u \) are the unloading segment curvature parameters, \( \varepsilon_{nc,u} \) is the non-classical strain at the unloading point, and \( \varepsilon_p \) is the plastic residual strain after full unload. \( \varepsilon_p \) is not subscripted with the nc abbreviation, as the total strain at the plastic residual point equals the non-classical strain due to the elastic part being equal to zero. The magnitude of plastic residual strain depends mainly on the strain of the unloading point; most existing empirical equations therefore use the unloading strain as a parameter. Chen et al. [24] proposed equation (17), with a power-type relationship between the residual strain and unloading strain; \( 10^6 \) is a conversion factor form microstrain unit, added to maintain consistency in the units used throughout this paper.

\[ \varepsilon_p = -0.245 \times (-\varepsilon_{nc,u} \times 10^6)^{1.139} \times 10^{-6} \]  

The curvature parameters of the unloading diagram, \( a_u \) and \( b_u \), were derived from the tangent modulus at the plastic residual point. The recommended relationships were developed for various loading rates and verified with experimental data by Chen et al. [24]. Equations (18) and (19) express the curvature parameters of the tensile unloading diagram.

\[ a_u = -0.00688 + \frac{0.597}{\sigma_u} + \frac{0.0753}{\sigma_u^2} - 0.0206 \times \log(\dot{\varepsilon}) \]  

\[ b_u = 0.114 \times \exp\left(-\frac{2.36}{\sigma_u - 0.376}\right) - 0.571 \times \log(\dot{\varepsilon}) \]

where \( \dot{\varepsilon} \) is the unloading strain rate in (sec\(^{-1}\)) unit. This study adopted a slow unloading rate, up to one microstrain per second, and the equations above were slightly modified such that the unloading stress and strain always had negative signs, to comply with the sign convention used for the tension zone. The output of equation (15) was then processed via equation (13) to obtain the coordinates of the unloading stress-strain diagram.

3.4. Tension-compression transition

The shape of the stress-strain diagram in the transition stage from tension to compression is convex. It begins at a steady rate from the residual plastic point and suddenly increases at the inflection point until it meets the original stress-strain diagram at compression. The intersection point with the original stress-strain diagram at compression represents the stress required to close the crack based on tensile damage level. The stress-strain diagram beyond the intersection point is not affected by tensile damage, a feature reported throughout the research literature. Yankelevsky and Reinhardt [27] proposed a focal point model to simulate the transition diagram, while Tasnimi and Lavasani [26] developed a transition model of two opposite curves passing through a specific intermediate point determined based on the plastic residual point. However, Sima et al. [18] adopted a linear relationship between the plastic residual point and the meeting point, and Chen et al. [24] proposed a similar approach to the unloading segment as that presented above and employed in the present study.

Yankelevsky and Reinhardt [27] assumed that the intersection point has three times the peak stress in tension, while Tasnimi and Lavasani [26], and Sima et al. [18] indicated that 10% of the peak compressive stress was adequate to close the crack opening completely. Chen et al. [24] provided an equation for the strain in the intersection point that predicted that the stress would be between 2.5 MPa to 3 MPa, based on the strain loading rate. However, this study implemented the suggestions of Tasnimi...
and Lavasani [26] and Sima et al. [18] to ensure a smooth transition curve and to avoid any jump at the intersection point. The stress-strain diagram of the tension-compression transition curve was thus calculated using equation (20).

\[
\sigma = \sigma_{tr} \times \frac{X_{tr}}{1 - a_{tr} \times (1 - (X_{tr})^{b_{tr}})}
\]

(20)

\[
\sigma_{tr} = 0.1 \times \delta_c
\]

(21)

\[
X_{tr} = \frac{\varepsilon_{nc} - \varepsilon_p}{\varepsilon_{nc,tr} - \varepsilon_p} (\varepsilon_p \leq \varepsilon_{nc} \leq \varepsilon_{nc,tr})
\]

(22)

where, \(\sigma_{tr}\) is the stress at the intersection between the transition curve and compression diagram, \(X_{tr}\) is the normalised non-classical strain of the transition curve, \(a_{tr}\) and \(b_{tr}\) are the curvature parameters of the transition curve, \(\varepsilon_{nc}\) is the non-classical strain of the intersection point corresponding to \(\sigma_{tr}\) as calculated numerically using equation (1) and equation (14), and \(\varepsilon_p\) is the plastic residual strain obtained from equation (17). The curvature parameters expressed in equation (23) and (24) were derived by assuming that the unload curve and transition curve have the same tangent modulus at the plastic residual point.

\[
a_{tr} = 1 - \frac{\sigma_{tr}}{\sigma_u} \times \frac{\varepsilon_{nc,u} - \varepsilon_p}{\varepsilon_{nc,tr} - \varepsilon_p} \times (1 - a_u)
\]

(23)

\[
b_{tr} = \exp \left(0.144 - 0.535 \frac{0.0667 \times \log(\dot{\varepsilon})}{a_u}\right)
\]

(24)

Finally, the coordinates of tension-compression transition curve were obtained from equation (20) after processing via equation (13). The output of unloading and transition curves were thus plotted for different unloading strain levels, with results as shown in figure 4.

![Figure 4. Unloading paths in tension at different strain levels.](image)

4. Traditional sectional analysis

The purpose of sectional analysis is to calculate the internal forces in a concrete beam section and to equate these with externally-applied forces. The internal forces at a given deformation can be calculated utilising classical beam theory; thus, internal deformations due to external forces can be calculated by
performing an iterative solution procedure. Sectional analysis employs Bernoulli-Navier’s assumption in which a plane section before deformation remains plane after deformation [28]. In addition, the section is treated as infinitesimal fibres undergoing uniaxial load and Poisson’s effect is ignored. Equation (25) represents the linear relationship of Bernoulli-Navier’s assumption used to calculate the strain magnitude at a given point:

$$\varepsilon = \varepsilon_o + \psi_x \times y + \psi_y \times x$$  \hspace{1cm} (25)

where, $\psi_x$ and $\psi_y$ are the bending curvatures about the x-axis and y-axis respectively, $x$ and $y$ are the coordinates of the point under consideration, and $\varepsilon_o$ is the strain magnitude at the origin point. The three equations of classical beam theory can therefore be rewritten in a matrix form, as shown in equation (26):

$$\begin{bmatrix} N \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} \int \sigma \, dA \\ \int \sigma \times y \, dA \\ \int \sigma \times x \, dA \end{bmatrix} \begin{bmatrix} \varepsilon_o \\ \psi_x \\ \psi_y \end{bmatrix}$$  \hspace{1cm} (26)

where $N$ is the normal force, $M_x$ and $M_y$ are the bending moments about the x-axis and y-axis respectively, $\sigma$ is the stress function, and $A$ is the area of the section. Equation (26) can also be rewritten in the form of equation (27) by multiplying the right-hand side by $(\varepsilon \times \varepsilon^{-1})$ and substituting equation (25). The integration can then be solved using numerical summation and substituting equation (1). Equation (28) thus represents the final formulation of the traditional sectional analysis matrix:

$$\begin{bmatrix} N \\ M_x \\ M_y \end{bmatrix} = \sum_m \begin{bmatrix} E_m \times \begin{bmatrix} 1 & y_m & x_m \\ y_m & y_m^2 & x_m \times y_m \\ x_m & x_m \times y_m & x_m^2 \end{bmatrix} \times A_m \end{bmatrix} \begin{bmatrix} \varepsilon_o \\ \psi_x \\ \psi_y \end{bmatrix}$$  \hspace{1cm} (28)

$$\{F\} = \sum_m \begin{bmatrix} E_m \times [C_m] \times A_m \end{bmatrix} \{\varepsilon\}$$  \hspace{1cm} (29)

where $x_m$ and $y_m$ are the coordinates of an elementary unit in the section relative to the origin point, $A_m$ is the area of that elementary unit, and $E_m$ is the secant modulus of elasticity of the elementary unit’s material at the given strain vector. Solving equation (28) for a given load vector requires the value of the secant modulus, which is a function of the strain vector. The strain vector can be calculated iteratively using the Newton-Raphson technique by calculating the secant modulus using the strain vector from the previous iteration and repeating this calculation until acceptable tolerance between the last two iterations is reached, as shown in equation (30):

$$\{\varepsilon\}_i = \left[ \sum_m \begin{bmatrix} E_m \{\varepsilon\}_{i-1} \times [C_m] \times A_m \end{bmatrix} \right]^{-1} \times \{F\}$$  \hspace{1cm} (30)

The solution thus involves discretising the section into infinitesimal cells and calculating the area and centroid of each cell, with the material properties and the load vector defined in advance. The secant modulus is calculated beginning with a zero-strain vector, and the stiffness matrix is determined by summing this over all mesh elements, including any embedded steel bars. A new strain vector is then obtained at the specified load vector using equation (30), and a convergence check is performed to determine whether an acceptable tolerance is achieved, as shown in equation (31). Finally, the strain vector is assessed against the material strain limit at section vertices to check failure at the section. The
accuracy of results depends mainly on the size of the mesh elements used to discretise the section, the accuracy of the stress-strain diagram in terms of describing the actual material, and the specified tolerance:

\[ |\varepsilon_i - \varepsilon_{i-1}| \leq \text{given tolerance} \]  

5. Proposed sectional analysis

Applying cyclic loading on a concrete beam changes the stress-strain diagram and results in residual deformation being stored in the section, with the stress-strain diagram no longer passing through the origin point (0,0). As a result, equation (30) cannot be solved at the zero strain vector as this would give a secant modulus of infinity. Moreover, cyclic analysis is mostly concerned with the determination of residual strain, while equation (30) requires a non-zero load vector to be solved mathematically, and if a zero-load vector is supplied, the solution is always a zero-strain vector. This means that the traditional sectional analysis approach cannot be directly used to solve sections under cyclic loading. Although the first issue can be solved by beginning analysis with a strain vector corresponding to the unloading load vector, the second issue requires equation (30) to be modified to allow resolution. The proposed solution in this work is to use the unloading step as a reference and to perform the analysis relative to that point. This allows equations (26) to (30) to be rewritten in the form of equations (32) to (36):

\[
\begin{align*}
\begin{pmatrix}
\Delta N \\
\Delta M_x \\
\Delta M_y
\end{pmatrix} &= \begin{bmatrix}
\int \Delta \sigma \, dA \\
\int \Delta \sigma \times y \, dA \\
\int \Delta \sigma \times x \, dA
\end{bmatrix}
\end{align*}
\]

\[
\begin{pmatrix}
N_2 - N_1 \\
M_{2,x} - M_{1,x} \\
M_{2,y} - M_{1,y}
\end{pmatrix} = \begin{bmatrix}
\frac{\sigma_2 - \sigma_1}{\varepsilon_2 - \varepsilon_1} & 1 & y \\
y & y^2 & x \times y \\
x & x \times y & x^2
\end{bmatrix} \times dA \begin{pmatrix}
\varepsilon_{2,0} - \varepsilon_{1,0} \\
\psi_{2,x} - \psi_{1,x} \\
\psi_{2,y} - \psi_{1,y}
\end{pmatrix} 
\]

\[
\begin{pmatrix}
N_2 - N_1 \\
M_{2,x} - M_{1,x} \\
M_{2,y} - M_{1,y}
\end{pmatrix} = \sum_m \begin{pmatrix}
\varepsilon_i \\
\psi_i \\
\varepsilon_{1,i}
\end{pmatrix} \times \begin{bmatrix}
1 & y_m & x_m \\
y_m & y_m^2 & x_m \times y_m \\
x_m & x_m \times y_m & x_m^2
\end{bmatrix} \times A_m \begin{pmatrix}
\varepsilon_{2,0} - \varepsilon_{1,0} \\
\psi_{2,x} - \psi_{1,x} \\
\psi_{2,y} - \psi_{1,y}
\end{pmatrix}
\]

\[
\{\Delta F\} = \sum_m (\bar{E}_m \times [C_m] \times A_m) \times \{\Delta \varepsilon\}
\]

\[
\{\varepsilon_{2,i}\} = \left[\sum_m (\bar{E}_m ((\varepsilon_{2,i-1}, \{\varepsilon_i\}) \times [C_m] \times A_m) \right]^{-1} \times \{\Delta F\} + \{\varepsilon_i\}
\]

where, \( F_1 \) and \( \varepsilon_1 \) are the reference step load and strain vectors as previously calculated using traditional sectional analysis, \( F_2 \) and \( \varepsilon_2 \) are the load and strain vectors of the step under consideration, and \( \bar{E}_m \) is the relative secant modulus. Note that substituting zero for \( F_1 \) and \( \varepsilon_1 \) turns equation (36) into equation (30), allowing the improved equation to solve both cases. In some cases, strain vectors \( \varepsilon_1 \) and \( \varepsilon_2 \), are the same, however, which will cause an error in calculating the relative secant modulus. In such cases, the tangent modulus can be used instead, as shown in equation (37):

\[
\bar{E}_m = \begin{cases}
\frac{E_{\text{tan},d} (\varepsilon_1)}{\sigma (\varepsilon_2) - \sigma (\varepsilon_1)} & \varepsilon_1 = \varepsilon_2 \\
\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 - \varepsilon_1} & \varepsilon_1 \neq \varepsilon_2
\end{cases}
\]
6. Program development
Oukaili [29] developed a program using Fortran to perform traditional sectional analysis of structural concrete members with a general cross-section of rectangular-shaped components undergoing static loads. The program could detect the failure load vector and the flexural crack depth across a section at different loading stages [30],[31]. In the current research, a new software package, written in Python [32], was developed, utilising the proposed sectional analysis algorithm, named SectionPy. It implements object-oriented programming concepts by distributing the program into reusable modules and classes. The software thus consists of eight modules: material, mesh, element, section, load, solver, postprocessor, and miscellaneous functions.

6.1. Material Module
The material module consists of an abstract material class that generates the stress-strain diagram from a specific nonlinear relationship by computing the corresponding stress of a massive number of strain ordinates with a small increment \(0.1 \mu \varepsilon\). The material properties required at given strain values are then calculated by linear interpolation. The technique used enhances the flexibility of the program, enabling it to be used with any material model or even with experimental data. The material module also includes a cyclic material class that can be used during cyclic analysis. This is a collector of abstract material instances, which thus has the same size as the number of cells in the mesh. The primary function of the cyclic material class is to give each cell element a specific stress-strain diagram and to calculate the required material properties in an array manner.

6.2. Mesh Module
The mesh module is used to discretise the section being analysed uses the built-in rectangular mesh generator or the Gmsh API [33], allowing SectionPy to analyse sections of arbitrary shapes such as polygons, circles, or any combination of both. Moreover, sections with voids can also be discretised using Boolean operations such as union, difference, rotation, and translation. The mesh module also calculates the area and the centroid of each cell, with calculation performed at array level using the NumPy library [34], which enhances performance as compared to traditional Python for-loop. Cell areas are thus computed using the shoelace technique, based on cell vertices, regardless of the shape of the mesh element. The mesh can then be discretised into one-dimensional layers, structured rectangles, right triangles, quadrilaterals, and non-structured triangles. Figure 5 shows various discretised concrete sections, including hollow core slab, AASHTO girder, rectangle, and trapezoidal sections.

Figure 5. Different mesh instances generated by the developed program.
6.3. Element Module
The element module includes the subsection and wire classes. A subsection is a 2-D solid element with specific material properties that takes a single mesh instance and material instance as input. The wire element represents a reinforcing bar or a prestressing strand, and thus area, location, and prestressing value are provided as input. The element module calculates the contribution of subsections and rebars to the stiffness matrix as well as ensuring that the maximum strain at the element is not reached during analysis.

6.4. Section Module
The section module is a collection of elements (subsections and wires) that consists of class methods to be used by the user to build the section and solve the matrix by summing all elements in the section. It also calculates the prestress load vector at each strand and adds this to the supplied load vector every time a solution is sought. The solve method thus outputs a step instance that stores the load vector, strain vector, stiffness matrix, and the results of the strain limit check.

6.5. Load Module
The load module is a small module used to calculate the resultant load vector for any load type. It consists of force, moment, and a combination of forces and moments, and it can perform most mathematical operations, such as addition, subtraction, multiplication, and division of load vectors.

6.6. Solver Module and Postprocessor Module
The solver module consists of the special analysis procedures used to calculate the required data by implementing the above modules. It consists of a stepping solver, failure solver, and cyclic solver. The stepping solver can be used to perform the analysis between two load vectors at a specific increment. It can be run in a multiprocessing approach that solves multiple load vectors simultaneously instead of the one-by-one traditional approach. The failure solver uses the bisection method to detect the ultimate load vector to be applied at the section in a specific load direction. Finally, the postprocessor is used to visualise the output steps from the solvers in the form of the desired plots, such as the stress-strain curve at a specific point or a biaxial moment-curvature diagram. The stress-strain curve along a specific line and the stress-strain map can also be generated in a video format that shows the behaviour of the beam under specific loading patterns.

7. Computational Example
A beam with a T-shaped section was adopted to demonstrate the output of the developed program. The general section was defined as two rectangles, of 150 mm × 225 mm for the web and 400 mm × 75 mm for the flange, using the input data outlined in table 1. A reference point located in the middle of the beam soffit was then adopted as the origin of the global coordinate system. A structured mesh of 12.5 mm in size was nominated, and the concrete material stress-strain diagram shown in figure 1 assigned for the concrete subsection instance. The rebars used were two Ø16 mm bars placed at the bottom and four Ø10 mm bars distributed at the top of the beam, placed such that 30 mm cover to the main rebar edge was achieved. The input data for the reinforcement bars, including their area and location relative to the reference point, are listed in table 2. A linear material stress-strain relationship was used for all rebars, with a yield stress of 500 MPa and modulus of elasticity of 200,000 MPa. Figure 6 shows the final section plot and mesh discretisation as generated by the software.

| Table 1. Concrete subsection input data. |
|-----------------------------------------|
| Item | Type   | X[mm] | Y[mm] | Width[mm] | Height[mm] |
| Web  | Rectangle | -75   | 0     | 150       | 225        |
| Flange | Rectangle | -200  | 225   | 400       | 75         |
Table 2. Reinforcement detailing input data.

| Item | X[mm] | Y[mm] | Area[mm²] |
|------|-------|-------|-----------|
| W1   | -37   | 38    | 200       |
| W2   | 37    | 38    | 200       |
| W1   | -165  | 265   | 78.5      |
| W2   | -40   | 265   | 78.5      |
| W3   | 40    | 265   | 78.5      |
| W4   | 165   | 265   | 78.5      |

Figure 6. T-beam section and mesh used in the analysis.

Figure 7. Moment curvature diagram of monotonic and unloading paths.

The applied load vector was then defined as a moment instance in the software, and a 10 kN.m moment about the x-axis used as an incremental load vector. The maximum bending moment about the x-axis before the tensile steel started to yield was expected to exceed 50 kN.m; moment-curvature beyond the yield point was thus ignored in this analysis, as the cyclic behaviour of steel bars was not under investigation at this stage. The unloading path was computed by solving the unload moment, with the material of each cell in the mesh being modified based on the strain level of the unload step.
Subsequent steps were solved via equation (36) at one kN.m intervals, based on the previously solved unload step, until the section was fully unloaded. Figure 7 demonstrates the moment-curvature diagram during monotonic loading accompanied by an unloading path at several unloading moments generated by the software. Although the concrete unloading model used was nonlinear, an almost linear unloading moment-curvature relationship can be observed, with only minor nonlinearity near the unloading point. In figure 8, the unloading moment versus residual curvature is plotted, showing that the curve can be categorised into three distinct regions: pre-cracking, cracking, and post cracking. The pre-cracking region starts with a sharp slope, after which the curve is almost flat during the cracking stage. Finally, the curve rises again with a less-steep gradient compared to that in the pre-cracking stage.

8. Conclusion
This paper presents an analytical methodology that can be used to analyse structural concrete sections under cyclic loading. The model developed by Karpenko et al. was used to simulate the monotonic stress-strain diagram, and two unloading relationships were employed, that of Sima et al. in compression, and that of Chen et al. in tension. The traditional technique of sectional analysis was explained in detail, and an improved version of the sectional analysis matrix proposed in conjunction with an advanced program used to run the analysis. The program can thus be used to discretise any configuration of polygons and circles into the desired mesh size. A T-shaped concrete section was then utilised to demonstrate the analysis output of the program, with the moment-curvature relationship during the unload stage is plotted for different unloading moments. In addition, the relationship between unloading moment and residual curvature was investigated. Based on the research results and supplementary computational example, the following remarks are offered:

- The new approach can determine sectional response incrementally based on a previously calculated step, which improves the calculation speed and converges on the solution with fewer iterations.
- Modelling each mesh cell as an independent element, with its own specific material properties and geometry, enables the program to determine sectional response during cyclic loading by modifying the material property of each mesh cell at the start of unloading and reloading steps based on its attained strain.
- The presented program can calculate the residual strain after full unloading or reloading (at zero load vector) by solving the matrix relative to the unload or reload step, as equation (36) is always solvable at the zero step.
- In the example, the observed unload response was almost linear despite a nonlinear material model being used.
- The results show a significant increase in residual curvature once the section is cracked.

The proposed method could be further developed by implementing the cyclic behaviour of reinforcing bars, allowing the section to be analysed at higher load amplitudes. Furthermore, unload and reload stress-strain diagrams of concrete beyond the first cycle could be utilised to run fatigue analysis at any stress level and for more than one cycle. Finally, more complex applications may be developed and investigated, such as those related to the strengthening of initially-damaged members and the damage induced during prestressing of prestressed girders.

9. References
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