Quantum Algorithms for Identifying Hidden Strings with Applications to Matroid Problems

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Abstract

In this paper, we explore quantum speedups for the problem, inspired by matroid theory, of identifying a pair of $n$-bit binary strings that are promised to have the same number of 1s and differ in exactly two bits, by using the max inner product oracle and the sub-set oracle. More specifically, given two string $s, s' \in \{0, 1\}^n$ satisfying the above constraints, for any $x \in \{0, 1\}^n$ the max inner product oracle $O_{\text{max}}(x)$ returns the max value between $s \cdot x$ and $s' \cdot x$, and the sub-set oracle $O_{\text{sub}}(x)$ indicates whether the index set of the 1s in $x$ is a subset of that in $s$ or $s'$. We present a quantum algorithm consuming $O(1)$ queries to the max inner product oracle for identifying the pair $\{s, s'\}$, and prove that any classical algorithm requires $\Omega(n/\log_2 n)$ queries. Also, we present a quantum algorithm consuming $n^{\frac{5}{2}} + O(\sqrt{n})$ queries to the subset oracle, and prove that any classical algorithm requires at least $n + \Omega(1)$ queries. Therefore, quantum speedups are revealed in the two oracle models. Furthermore, the above results are applied to the problem in matroid theory of finding all the bases of a 2-bases matroid, where a matroid is called $k$-bases if it has $k$ bases.

1. Introduction

The study of the string problem is important in theoretical computer science, and has a wide range of applications in many fields such as bioinformatics, text processing, artificial intelligence, and data mining, etc. Since the
1960s, the string problem has received much attention and been extensively studied in classical computing with a series of efficient algorithms proposed, including string matching\cite{1, 2}, string finding\cite{3, 4}, string learning\cite{5, 6} and so on. Compared to conventional classical computation, a new computing paradigm called quantum computation was proposed in the 1980s \cite{7, 8} and has developed rapidly \cite{9, 10, 11, 12, 13, 14, 15}. Finding more problems that admit quantum advantage has become one of the focus issues in the field of quantum computing, naturally including the string problem. In fact, the study of quantum algorithms for string problems started very early. As a notable example, the Bernstein-Vazirani algorithm proposed in 1997 \cite{12} is one of the most outstanding works, which can learn an $n$-bit hidden string using only one query to the inner product oracle, while the best deterministic classical algorithm needs $n$ queries to solve this problem. To date, a series of contributions have been made to solving string problems in the quantum setting \cite{16, 17, 18, 19, 20}.

String learning is one of the well-known research directions in string problems, which has important applications in bioinformatics \cite{21}, data mining \cite{22} and network security \cite{23}. In general, string learning is such a problem that the goal is to reconstruct a secret string hidden by an oracle through querying the oracle as few as possible. Considerable effort has been devoted to string learning in both classical and quantum computing over the past decades, involving different oracles and application scenarios. Skiena and Sundaram \cite{6} showed that $(\alpha - 1)n + \Theta(\alpha \sqrt{n})$ queries to the sub-string oracle are sufficient to reconstruct an unknown string, with $\alpha$ being the alphabet size and $n$ being the length of the string, and applied their algorithm to reconstruct DNA sequences. Du and Hwang \cite{24} considered the subset OR oracle in combinatorial group testing. Cleve et al. \cite{20} gave a quantum algorithm using $\frac{3}{4}n + o(n)$ queries to the sub-string oracle to reconstruct an unknown string of size $n$. Dam \cite{25} gave a quantum algorithm framework to identify a secret string of size $n$ using $\frac{n}{2} + O(\sqrt{n})$ queries to the binary oracle. Also, quantum algorithms associated with the balance oracle and the subset OR oracle were proposed for the quantum counterfeit coin problem \cite{26} and combinatorial group testing \cite{27}, respectively. Recently, Xu et al. \cite{28} gave a quantum algorithm using $\lfloor n/2 \rfloor$ queries to the longest common prefix oracle to learn a secret string of size of $n$, which has a double speedup over classical counterparts. Another interesting work is to play Mastermind on quantum computers. Li et al. \cite{29} studied quantum strategies for playing Mastermind with $n$ positions and $k$ colors using the black(-white)-peg oracle. Note these
String learning problems are all about learning a single hidden string, and a natural generalization is to think about learning multiple hidden strings for revealing possible quantum advantages.

In this paper, we investigate quantum algorithms to learn a pair of binary strings which are almost the same, that is, they have the same number of 1s and differ only in two bits. More generally, these two constraints are ubiquitous in a mathematical concept called matroid [30]. For example, any matroid with more than 1 bases has such a pair of bases that satisfy the above condition. Therefore, an effective quantum algorithm which learns a pair of binary strings that are almost the same implies that it can find the bases of a matroid with two bases. In the following, we first introduce some related background knowledge for convenience, and then introduce our main contributions of this paper.

1.1. Background Knowledge

Notations. \([n]\) denotes the set \(\{1, 2, \cdots, n\}\). For \(j \in [n], x \in \{0, 1\}^n, x_j\) denotes the \(j\)-th bit of \(x\). \(|x| = \sum_{i=1}^{n} x_i\) is the number of 1s in \(x\), i.e., the Hamming weight of \(x\). \(e_j\) represents a string such that its \(j\)-th bit is 1 and the other bits are 0. The symbols \(\land, \lor\) and \(\oplus\) represent the logical operation AND, OR, and XOR (modulo-2 addition), and would act as corresponding bitwise operations when applied to two bit strings. The inner product of \(x, y \in \{0, 1\}^n\) is \(x \cdot y = \sum_{j=1}^{n} x_jy_j\).

Definition 1 (Max Inner Product Oracle). The max inner product oracle associated with a string subset \(S \subseteq \{0, 1\}^n\) is a function \(O_{\text{max}}^S : \{0, 1\}^n \rightarrow \mathbb{Z}\) defined by

\[
O_{\text{max}}^S(x) = \max\{x \cdot s : s \in S\},
\]

for any \(x \in \{0, 1\}^n\).

Definition 2 (Sub-set Oracle). A sub-set oracle associated with a string subset \(S \subseteq \{0, 1\}^n\) is a function \(O_{\text{sub}}^S : \{0, 1\}^n \rightarrow \{0, 1\}\) defined by

\[
O_{\text{sub}}^S(x) = \begin{cases} 
1, & \text{if } \exists s \in S \text{ s.t. } \text{Idx}(x) \subseteq \text{Idx}(s), \\
0, & \text{otherwise},
\end{cases}
\]

for any \(x \in \{0, 1\}^n\), where the function \(\text{Idx}(z) = \{j \in [n] : z_j = 1\}\) represents the index set of the 1s in any string \(z \in \{0, 1\}^n\). Then \(O_{\text{sub}}^S(x)\) indicates whether the index set determined by \(x\) is a subset of any index set determined by \(s \in S\).
Now a natural problem is: Given the oracle $O^S_{\text{max}}$ or $O^S_{\text{sub}}$, how can we identify $S$ by using as few queries to the oracle as possible? In this paper, we will consider a particular case of this problem with a promise inspired by matroid theory as follows.

**Hidden String Problem (HSP):** Given the oracle $O^S_{\text{max}}$ in Eq. (1) or $O^S_{\text{sub}}$ in Eq. (2) with the promise that $S$ consists of two $n$-bit strings $s, s' \in \{0, 1\}^n$ satisfying $|s| = |s'|$ and $|s \oplus s'| = 2$, the goal is to identify $S = \{s, s'\}$ by using as few queries to the oracle as possible. In the following, we will omit the superscript $S$ in $O^S_{\text{max}}$ and $O^S_{\text{sub}}$ for this well-defined HSP.

Since HSP for $n = 2$ is trivial such that $S = \{01, 10\} = \{10, 01\}$, in this paper we focus on the non-trivial problem with $n \geq 3$. As mentioned before, the string learning problem has attracted much attention in both classical and quantum computing, and we hope to extend this widely studied problem for developing more applications, especially for solving some problems in matroid theory. Later, we will reveal the link between HSP and matroid theory with detailed discussions in Section 5.

### 1.2. Our Contributions

We will show that quantum computing can speed up the solution of HSP by constructing two quantum algorithms with query complexities lower than the classical lower bounds. More specifically, we have the following theorems.

**Theorem 1.** There is a quantum algorithm using $O(1)$ queries to the max inner product oracle to solve HSP with high probability.\(^2\)

**Theorem 2.** There is a quantum algorithm using $\frac{1}{2}n + O(\sqrt{n})$ queries to the sub-set oracle to solve HSP with high probability.

**Theorem 3.** Any classical algorithm for solving HSP requires $\Omega(n/\log_2 n)$ queries to the max inner product oracle or $n + \Omega(1)$ queries to the sub-set oracle.

The rest of the paper is organised as follows. In Section 2, we give a quantum algorithm with the max inner product oracle to prove Theorem 1.

\(^2\)In this paper, "with high probability" means with any constant probability greater than 1/2.
In Section 3, we give a quantum algorithm with the sub-set oracle to prove Theorem 2. In Section 4, we prove Theorem 3 with information theory. In Section 5, we present an application of our theorems about HSP to matroids. Finally, in Section 6 we raise some conclusions.

2. Quantum Algorithms with the Max Inner Product Oracle

In this section, we will prove Theorem 1 in two steps. First, we propose a procedure named Algorithm 1 that can extract useful information about the hidden strings \(\{s, s'\}\) (see Lemma 1). Next, based on this subroutine, we further propose Algorithm 2 to identify the hidden \(\{s, s'\}\) with high probability.

2.1. Information Extraction with the Max Inner Product Oracle

To identify \(\{s, s'\}\) by using as few queries to the max inner product oracle as possible, we need to extract as much information as possible with each query. In the quantum case, we reveal that one query to the max inner product can obtain certain effective information about \(\{s, s'\}\) as Lemma 1.

**Lemma 1.** There is a quantum algorithm using one query to the max inner product oracle \(O_{\text{max}}\) for the hidden strings \(\{s, s'\}\) in HSP to obtain each one of the four results \(\{s, s', s \land s', s \lor s'\}\) with probability \(\frac{1}{4}\).

**Algorithm 1** Extracting useful information with the max inner product oracle.

**Input:** The quantum max inner product oracle \(O_{\text{max}}\) in Eq. (4) with \(r(x)\) defined on \(s\) and \(s'\) in Eq. (3).

**Output:** A string in \(\{s, s', s \land s', s \lor s'\}\).

1: Initialize the \(n + m\) qubits to \(|0^n\rangle|0^{m-1}\rangle\), with \(m = \lceil \log_2(n) \rceil\).
2: Apply the unitary transformation \(H^{\otimes (n+m)}\).
3: Apply the quantum oracle \(O_{\text{max}}\).
4: Apply the unitary transformation \(H^{\otimes (n+m)}\).
5: Measure the first \(n\) qubits and obtain a string \(\tau \in \{s, s', s \land s', s \lor s'\}\).
6: **return** \(\tau\).
Proof. For simplicity, we denote the value of $O_{\text{max}}(x)$ for HSP by $r(x)$ as

$$r(x) = \max\{x \cdot s, x \cdot s'\} \text{ s.t. } |s| = |s'|, |s \oplus s'| = 2$$

for $x \in \{0, 1\}^n$ such that $r(x) \in [0, n-1]$, and we also use $O_{\text{max}}$ as a quantum oracle that acts on the computational basis as:

$$|x\rangle|y\rangle \xrightarrow{O_{\text{max}}} |x\rangle|y + r(x) \mod 2^m\rangle,$$

where the first register $|x\rangle$ is an $n$-qubit query register, and the second register $|y\rangle$ is the answer register containing $m = \lceil \log_2(n) \rceil$ qubits to encode a decimal integer $y$. Based on Eq. (4), we can propose Algorithm 1 to satisfy Lemma 1, which evolves as:

$$|0^n\rangle|0^{m-1}1\rangle \xrightarrow{H \otimes (n+m)} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \frac{1}{\sqrt{2^m}} \sum_{y=0}^{2^m-1} (-1)^y|y\rangle \xrightarrow{O_{\text{max}}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \frac{1}{\sqrt{2^m}} \sum_{y=0}^{2^m-1} (-1)^y|y + r(x) \mod 2^m\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{-r(x)}|x\rangle \frac{1}{\sqrt{2^m}} \sum_{y=0}^{2^m-1} (-1)^{y+r(x)}|y + r(x) \mod 2^m\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{r(x)}|x\rangle \frac{1}{\sqrt{2^m}} \sum_{y'=0}^{2^m-1} (-1)^{y'}|y'\rangle$$

$$\xrightarrow{H \otimes (n+m)} \sum_{\tau \in \{0,1\}^n} \left( \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{r(x)+\tau \cdot x}\right)|\tau\rangle|0^{m-1}1\rangle. \quad (5)$$

More generally, the process of Eq. (5) tells us how to extract information about a target oracle (e.g. $r(x)$) in the coefficient of each basis state $|\tau\rangle$ of the output state by querying the oracle only once. Note the well-known Bernstein-Vazirani algorithm [12] and its extended version [31] have used similar techniques to identify a target hidden string. In our case here, we give some technical treatment for calculating the probability of a measurement outcome.

Without loss of generality, we assume that $s$ and $s'$ are different in the $i$-th and $j$-th bits with $i < j$. For all $n$-bit strings $x \in \{0,1\}^n$, we divide
them into four categories as \(X_{00}, X_{01}, X_{10}\) and \(X_{11}\), according to the values of the \(i\)-th and \(j\)-th bits. That is, any string \(x \in X_{k_1k_2}\) for \(k_1k_2 \in \{0, 1\}^2\) has \(x_i = k_1\) and \(x_j = k_2\), leading to \(|X_{00}| = |X_{10}| = |X_{01}| = |X_{11}| = 2^{n-2}\). Note the string \(s_0 = s \land s'\) belongs to \(X_{00}\) and differs from \(s\) or \(s'\) in one bit, and we have

\[
r(x) = \begin{cases} 
  s_0 \cdot x, & x \in X_{00}, \\
  s_0 \cdot x + 1, & \text{otherwise},
\end{cases}
\]

by considering the definition of \(r(x)\) in Eq. (3). Based on Eq. (6), we can give the probability of getting result \(\tau = s\) upon measuring the output state in Eq. (5) as

\[
\Pr(\tau = s) = \left| \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} (-1)^{r(x) + s \cdot x} \right|^2
\]

\[
= \left| \frac{1}{2^n} \left( \sum_{x \in X_{00}} (-1)^{s_0 \cdot x + s \cdot x} + \sum_{x \in X_{10}} (-1)^{s_0 \cdot x + 1 + s \cdot x} + \sum_{x \in X_{01}} (-1)^{s_0 \cdot x + 1 + s \cdot x} + \sum_{x \in X_{11}} (-1)^{s_0 \cdot x + 1 + s \cdot x} \right) \right|^2
\]

\[
= \left| \frac{1}{2^n} \left( \sum_{x \in X_{00}} (-1)^{s_0 \cdot x} + \sum_{x \in X_{10}} (-1)^{s_0 \cdot x + 1} + \sum_{x \in X_{01}} (-1)^{s_0 \cdot x + 1} + \sum_{x \in X_{11}} (-1)^{s_0 \cdot x + 1} \right) \right|^2
\]

\[
= \left| \frac{1}{2^n} \left( \sum_{x \in X_{00}} 1 + \sum_{x \in X_{10}} 1 + \sum_{x \in X_{01}} 1 + \sum_{x \in X_{11}} 1 \right) \right|^2
\]

\[
= \left| \frac{1}{2^n} \left( 2^{n-2} + 2^{n-2} - 2^{n-2} + 2^{n-2} \right) \right|^2
\]

\[
= \frac{1}{4}.
\]

In a similar way, we can also compute \(\Pr(\tau = s') = \Pr(\tau = s \land s') = \Pr(\tau = s \lor s') = \frac{1}{4}\).

\[\square\]

2.2. Proof of Theorem 1

In this section, we prove Theorem 1 by designing Algorithm 2 based on previous Algorithm 1. For convenience, we restate Theorem 1 here.
Theorem 1. There is a quantum algorithm using $O(1)$ queries to the max inner product oracle to solve HSP with high probability.

Algorithm 2 Quantum algorithm for learning hidden strings with the max inner product oracle.

**Input:** The quantum max inner product oracle $O_{\text{max}}$ in Eq. (4) with $r(x)$ defined on $s$ and $s'$ in Eq. (3).

**Output:** $\{s, s'\}$ with success probability 0.96899.

1: $\omega \leftarrow 7$.
2: for $i \leftarrow 1$ to $\omega$ do
3: Run Algorithm 1 and obtain a string denoted $s_i$.
4: end for
5: if $\exists i \neq j$ such that $s_i \neq s_j$ and $|s_i| = |s_j|$ then
6: $s \leftarrow s_i, s' \leftarrow s_j$.
7: else if $\exists i, j$ such that $|s_i| - |s_j| = 2$ then
8: Find two indices $l, l'$ such that $s_i[l] \neq s_j[l]$ and $s_i[l'] \neq s_j[l']$.
9: $s \leftarrow s_j \lor e_l, s' \leftarrow s_j \lor e_{l'}$.
10: else
11: return NULL.
12: end if
13: return $\{s, s'\}$.

Proof. We will show that Algorithm 2 uses 7 queries to the max inner product oracle $O_{\text{max}}$ in Eq. (4) to obtain the hidden string pair $\{s, s'\}$ in Eq. (3) with a success probability higher than 0.95.

When we run Algorithm 1 $\omega$ ($\omega \geq 2$) times to get $\omega$ string results denoted $T = \{s_1, \ldots, s_{\omega}\}$, there are two cases that we can definitely identify $\{s, s'\}$:

(C1) If there exist two distinct $s_i$ and $s_j$ in $T$ such that $|s_i| = |s_j|$, then we can immediately identify the hidden strings as $s_i$ and $s_j$.

(C2) If there are two distinct $s_i$ and $s_j$ in $T$ such that $|s_i| - |s_j| = 2$ with $s_i[l] \neq s_j[l]$ and $s_i[l'] \neq s_j[l']$, it means that the target $s$ and $s'$ differ in two bits $l$ and $l'$ and $s_j = s \land s'$. Therefore, we can identify $\{s, s'\}$ as $\{s_j \lor e_l, s_j \lor e_{l'}\}$. 

8
We now discuss the success probability of identifying \( \{s, s'\} \) after running Algorithm 1 \( \omega \) times. The notation \( \Pr(a, b, c, d) \) is used to denote the probability that \( s, s', s \land s' \) and \( s \lor s' \) appear \( a, b, c \) and \( d \) times respectively in the \( \omega \) results \( T = \{s_1, \ldots, s_\omega\} \). By multinomial distribution and Lemma 1, we have
\[
\Pr(a, b, c, d) = \frac{\omega!}{a! \cdot b! \cdot c! \cdot d!} \left(\frac{1}{4}\right)^\omega. \tag{7}
\]
Then the probability of identifying \( \{s, s'\} \) is given by summing up the probabilities of all events satisfying (C1) or (C2) as
\[
\Pr(\text{Identify}\{s, s'\}) = \sum_{(C1) \text{ or (C2)}} \Pr(a, b, c, d), \tag{8}
\]
where the case (C1) or (C2) corresponds to:
\[
\begin{cases}
  a, b, c, d \text{ are non-negative integers;} \\
  a + b + c + d = \omega; \\
  ab > 0 \text{ or } cd > 0.
\end{cases} \tag{9}
\]
By combining Eqs. (7), (8) and (9), we can solve the inequality \( \Pr(\text{Identify}\{s, s'\}) \geq 0.95 \) and obtain \( \omega \geq 7 \). As a result, the above analysis with a parameter \( \omega = 7 \) is formulated as Algorithm 2, where we can identify \( \{s, s'\} \) of any size \( n \) with a success probability 0.96899. Thus, we prove Theorem 1 such that a constant number of queries to the max inner product oracle can solve HSP with high probability.

3. Quantum Algorithms with the Sub-set Oracle

In this section, we will prove Theorem 2 in two steps. First we use the technique proposed by Dam [25] to give a quantum algorithm based on the sub-set oracle to output \( s \lor s' \) with high probability (see Lemma 2). Then we use Grover’s algorithm to find \( s \) and \( s' \) in the set consisting of the largest proper subset of \( s \lor s' \).

3.1. Information Extraction with the Sub-set Oracle

Inspired by Ref. [25], we can use \( \frac{\omega}{2} + O(\sqrt{n}) \) queries to the general sub-set oracle \( O_{\text{sub}} \) in Eq. (2) to learn \( t = \bigvee_{s \in S}s \), i.e., the bitwise OR of the strings in \( S \) with arbitrarily small error probability, which would be useful for further identifying the case of \( S = \{s, s'\} \).
Lemma 2. There is a quantum algorithm using $\frac{n}{2} + O(\sqrt{n})$ queries to $O_{\text{sub}}$ in Eq. (2) to learn $t = \bigvee_{s \in S} s$, i.e., the bitwise OR of the strings in $S$, with an arbitrarily small error probability.

Algorithm 3 Extracting useful information with the sub-set oracle.

**Input:** A sub-set oracle $O_{\text{sub}}$ in Eq. (2); a parameter $k = n/2 + O(\sqrt{n})$.

**Output:** The bitwise OR of strings in $S$ with arbitrarily small probability.

1: Initialize the $(n + 1)$ registers to $|0^n\rangle|1\rangle$.
2: Apply the unitary transformation $U_k \otimes H$ with $U_k$ in Eq. (15).
3: Apply the unitary transformation $A_k$ in Eq. (11).
4: Apply the unitary transformation $H^\otimes n \otimes H$.
5: Measure the first $n$-qubit register and obtain $\tau \in \{0, 1\}^n$.
6: return $\tau$.

**Proof.** We will show that Algorithm 3 can obtain the bitwise OR of the strings in $S$ with arbitrarily small error probability by consuming $\frac{n}{2} + O(\sqrt{n})$ queries to $O_{\text{sub}}$. Let $t$ be the bitwise OR of strings in $S$.

At first, we introduce how to obtain the inner product of $t$ and $x \in \{0, 1\}^n$ by querying the oracle $O_{\text{sub}}$. It is easy to see that for any $j$, $O_{\text{sub}}(e_j) = 1$ if and only if $t_j = 1$. Thus, we have

$$x \cdot t = \sum_{j \in [n]} x_j t_j = \sum_{j \in [n]} x_j O_{\text{sub}}(e_j),$$

which indicates that we can use $|x|$ queries to $O_{\text{sub}}$ to calculate the value $x \cdot t$ for any $x \in \{0, 1\}^n$ and $t = \bigvee_{s \in S} s$.

Next, we construct an unitary transformation $A_k$ for a given threshold number $k$ such that

$$A_k|x\rangle|b\rangle = \begin{cases} |x\rangle|b \oplus (x \cdot t \mod 2)\rangle & \text{if } |x| \leq k, \\ |x\rangle|b\rangle & \text{if } |x| > k \end{cases}$$

as shown in Figure 1, which consists of a series of operators as:

1. The unitary operation $U$ acts on a qubit state $|a\rangle$ and an $n$-qubit register $|x\rangle = |x_1\rangle \ldots |x_n\rangle$, and tests whether the Hamming weight $|x|$ is less than or equal to $k$, that is

$$U|a\rangle|x\rangle = |a \oplus h_k(x)\rangle|x\rangle,$$
where \( h_k(x) = 1 \) if \(|x| \leq k\), otherwise \( h_k(x) = 0 \). For \(|x| \leq k\), the boxed module in the middle of Figure 1 is executed.

(2) The unitary operation \( V \) copies up to the first \( k \) 1s in \( x = x_1 \cdots x_n \) to \( y_1 \) to \( y_k \), and store the corresponding position where 1 appears in \( z_1 \) to \( z_k \) in turn.

(3) The unitary operation controlled-\( O \) acts on the control qubit \(|y_i\rangle\) and target qubits in \(|z_i\rangle\) and \(|b\rangle\) with \( i = 1, 2, \ldots, k \), such that \( O \) has the effect
\[
O|z_i\rangle|b\rangle = |z_i\rangle|b \oplus O_{\text{sub}}(e_{z_i})\rangle.
\] (13)

Finally, for \( k \) we define an integer
\[
M_k = \sum_{i=0}^{k} \binom{n}{i}
\] (14)
and an \( n \)-qubit unitary transformation \( U_k \) independent of \( O_{\text{sub}} \) such that
\[
U_k|0\rangle = \frac{1}{\sqrt{M_k}} \sum_{x \in \{0,1\}^n, |x| \leq k} |x\rangle.
\] (15)
Based on the introduced Eqs. (11) and (15), the Algorithm 3 evolves as:

\[
\begin{align*}
|0^n\rangle|1\rangle & \xrightarrow{U_k \otimes H} \frac{1}{\sqrt{M_k}} \sum_{x \in \{0,1\}^n : |x| \leq k} |x\rangle|\rangle \\
& \xrightarrow{A_k} \frac{1}{\sqrt{M_k}} \sum_{x \in \{0,1\}^n : |x| \leq k} (-1)^{x \cdot t} |x\rangle|\rangle \\
& \xrightarrow{H^\otimes n \otimes H} \sum_{\tau \in \{0,1\}^n} \left( \frac{1}{\sqrt{M_k} 2^n} \sum_{x \in \{0,1\}^n : |x| \leq k} (-1)^{x \cdot \tau} \right) |\tau\rangle|1\rangle,
\end{align*}
\]

for any value of \(k\), and thus the probability of getting result \(\tau = t\) upon measuring the output state is

\[
\Pr(\tau = t) = \left| \frac{1}{\sqrt{M_k} 2^n} \sum_{x \in \{0,1\}^n : |x| \leq k} (-1)^{x \cdot \tau} \right|^2 = M_k / 2^n.
\]

(16)

Now we consider the value range of \(k\) that enables the probability in Eq. (16) to exceed 0.95. Let \(X_1, \ldots, X_n\) be \(n\) random variables such that each \(X_j (j \in [n])\) represents the value of the \(j\)th bit in a random string \(x \in \{0,1\}^n\). Obviously, \(X_1, \ldots, X_n\) are independent and they are the \(0 - 1\) distribution with probability 1/2. Let \(X = \sum_{i=1}^n X_i\). By a simple calculation we have

\[
\Pr(X \leq k) = M_k / 2^n = \Pr(\tau = t).
\]

(17)

Therefore, we can get \(\Pr(\tau = t)\) by estimating \(\Pr(X \leq k)\). Also note that the expectation and variance of \(X\) are \(\mathbb{E}[X] = \frac{1}{2} n\) and \(D(X) = \frac{1}{4} n\), respectively. By De Moiver-Laplace Theorem we have

\[
\lim_{n \to \infty} \Pr \left( X \leq \frac{n}{2} + \frac{q \sqrt{n}}{2} \right) = \lim_{n \to \infty} \Pr \left( \frac{X - \mathbb{E}[X]}{\sqrt{D(X)}} \leq q \right) = \int_{-\infty}^q \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt
\]

(18)

for any \(q \in \mathbb{R}\). Obviously Eq. (18) monotonically increases with \(q\). By letting Eq. (18) > 0.95, we obtain \(q > 1.6449\). Thus for quite large \(n\), the
integer parameter taken as \( k = \lfloor n/2 + \sqrt{n} \rfloor \) in Algorithm 3 corresponds to \( q \approx 2 > 1.6449 \) in Eq. (18), such that the probability of obtaining \( t = \sqrt{s \in S} s \) is \( \Pr(\tau = t) = \Pr(X \leq k) > 0.95 \).

We also examine how Algorithm 3 works with small and moderate \( n \). Our numerical calculation shows that the probability curve \( \Pr(\tau = t) \) in Eq. (16) oscillates between 0.9648 and 1 for \( 3 \leq n \leq 1000 \) with \( k = \lfloor n/2 + \sqrt{n} \rfloor \), and seems to approach the case \( q = 2 \) and Eq. (18)=0.9773 as \( n \) increases (e.g. \( \Pr(\tau = t) = 0.9770, 0.9786, 0.9769 \) for \( n = 998, 999, 1000 \)).

Actually the error probability can be made arbitrarily small by letting \( k = n/2 + \lambda \sqrt{n} \), which is explained below. The complementary unit Gaussian distribution function is defined as

\[
Q(x) = 1 - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.
\]  

(19)

For \( x \geq 0 \), it is known that

\[
Q(x) \leq \frac{1}{2} e^{-x^2/2}
\]

(20)

with equality holding only at \( x = 0 \). Let \( k = n/2 + \lambda \sqrt{n} \). By Eq. (18), Eq. (19) and Eq. (20), for a large \( n \), we have

\[
\Pr(X > k) = \Pr(X > n/2 + \lambda \sqrt{n}) = Q(2\lambda) \leq \frac{1}{2} e^{-2\lambda^2}
\]

(21)

This means that the error probability \( \Pr(\tau \neq t) \) can be made arbitrarily small with \( n/2 + O(\sqrt{n}) \) queries to the sub-set oracle.

\[\square\]

3.2. Proof of Theorem 2

For convenience, we restate Theorem 2 here.

**Theorem 2.** There is a quantum algorithm using \( \frac{1}{2} n + O(\sqrt{n}) \) queries to the sub-set oracle to solve \( HSP \) with high probability.

**Proof.** First, by Lemma 2, we can use Algorithm 3 to obtain \( t = s \lor s' \) with an arbitrarily small error probability, which requires \( \frac{1}{2} n + O(\sqrt{n}) \) queries to the sub-set oracle.

Let \( \Delta \) be the set of strings obtained by replacing exactly one 1 in \( t \) by 0. Then there is \( |\Delta| = |t| - 1 \). Then we can use Grover’s algorithm to find the pair \( s, s' \in \Delta \) satisfying \( O_{\text{sub}}(s) = 1 \) and \( O_{\text{sub}}(s') = 1 \). This needs \( O(\sqrt{n}) \) queries to the sub-set oracle.

Therefore, Algorithm 4 consumes \( \frac{1}{2} n + O(\sqrt{n}) = \frac{1}{2} n + O(\sqrt{n}) \) queries to the sub-set oracle to obtain \( \{s, s'\} \) with high probability. \[\square\]
Algorithm 4 Quantum algorithm for learning hidden strings with the sub-set oracle.

**Input:** The sub-set oracle $O_{sub}$ in Eq. (2) with the hidden $S = \{s, s'\}$ satisfying $|s| = |s'|$ and $|s \oplus s'| = 2$.

**Output:** $\{s, s'\}$ with high probability.

1. $t \leftarrow$ Algorithm 3.
2. Apply Grover’s algorithm to find the pair $s, s'$ in $\Delta_t$ satisfying $O_{sub}(s) = 1$ and $O_{sub}(s') = 1$.
3. **return** $\{s, s'\}$.

4. Classical Lower Bounds

In this section, we give the information-theoretic lower bound on HSP to prove Theorem 3. Compared with Theorem 3, our quantum algorithms show substantial speedups over classical counterparts. First we restate Theorem 3 in the following.

**Theorem 3.** Any classical algorithm for solving HSP requires $\Omega(n/\log_2 n)$ queries to the max inner product oracle or $n + \Omega(1)$ queries to the sub-set oracle.

**Proof.** Let $N$ be the number of all possible solutions $\{s, s'\}$ to HSP satisfying the promise $|s \oplus s'| = 2$ and $|s| = |s'|$. Then there is

$$N = \binom{n}{2} \cdot 2^{n-2} = n(n-1) \cdot 2^{n-3}. \quad (22)$$

By noting that $\{s, s'\}$ and $\{s', s\}$ are the same string pair. Considering the range of oracle functions $O_{max}(x)$ and $O_{sub}(x)$ defined in Eq. (1) and Eq. (2) restricted to $S = \{s, s'\}$, respectively, a deterministic classical algorithm for HSP with the max inner product oracle or the sub-set oracle can be described as an $n$-ary or binary tree with $N$ leaf nodes. Therefore, the query complexity of a classical algorithm is the height of the tree, and the lower bound for the query complexity of HSP is the minimum height of a tree with $N$ leaf nodes.

Let $C_i(n)$ and $C_s(n)$ be the lower bound of HSP with the max inner
product oracle and the sub-set oracle, respectively. Then we can derive that
\[ C_i(n) = \lceil \log_2(n(N)) \rceil = \Omega(n/\log_2(n)), \]  
(23)
\[ C_s(n) = \lceil \log_2(n) \rceil = n + \Omega(1). \]  
(24)

5. Application to Matroids

With the rapid development of quantum computing, finding more problems that can take advantage of quantum speedup has become one of the focus issues in the field of quantum computing. Solving matroid problems is a potential application of quantum computing. Huang et al. [32] showed that quadratic quantum speedup is possible for the calculation problem of finding the girth or the number of circuits (bases, flats, hyperplanes) of a matroid, and for the decision problem of deciding whether a matroid is uniform, Eulerian or paving.

The problems related to bases of a matroid is very important in matroid theory, including finding a base (with maximum/minimum weigh), enumerating all the bases, counting the number of base and so on. These problems are of practical significance, for example, finding a base of a vector space or a (maximum/minimum weigh) spanning tree of a graph are the special case of finding a base of a matroid [33]. Counting the number of base has also been extensively studied [34, 35, 36, 37, 38], and determining the reliability of a graph is a special case of it. Enumerating the base of a matroid is interesting and also extensively studied. There are many works on how to efficiently enumerate the bases of a matroid or the spanning trees of a graph [39, 40, 41, 42, 43, 44, 45]. Due to the importance of the base problems and the potential speedup of quantum algorithms, we consider the following problem.

**Identify Matroids’ Bases.** Given a 2-bases matroid which has 2 bases and can be accessed by a matroid oracle (independence oracle or rank oracle), how many oracle queries are required to identify its bases?

In this section, we will show how to transform the above problem to the hidden string problem, so that the previous obtained quantum algorithms can be applied.
5.1. Matroid

Here we give some basic definitions and concepts on matroids. Matroid theory was established as a generalization of linear algebra and graph theory. Some concepts are similar to those of linear algebra or graphs. One can refer to [46] or [47] for more details about matroid theory.

**Definition 3 (Matroid).** A matroid is a combinational object defined by the tuple $M = (E, \mathcal{I})$ on the finite ground set $E$ and $\mathcal{I} \subseteq 2^E$ such that the following properties hold:

10. $\emptyset \in \mathcal{I}$;

11. If $A' \subseteq A$ and $A \in \mathcal{I}$, then $A' \in \mathcal{I}$;

12. For any two sets $A, B \in \mathcal{I}$ with $|A| < |B|$, there exists an element $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}$.

The members of $\mathcal{I}$ are the independent sets of $M$. A subset of $E$ not belonging to $\mathcal{I}$ is called dependent. A base of $M$ is a maximal independent subset of $E$, and the collection of bases is denoted by $\mathcal{B}(M)$. For a positive integer $k$, we call a matroid $M$ is a $k$-bases matroid (on $E$) if $|\mathcal{B}(M)| = k$.

**Definition 4 (Rank).** The rank function of matroid $M = (E, \mathcal{I})$ is a function $r : 2^E \rightarrow \mathbb{Z}$ defined by

$$r(A) = \max\{|X| : X \subseteq A, X \in \mathcal{I}\} \quad (A \subseteq E)$$

The rank of $M$ sometimes denoted by $r(M)$ is $r(E)$.

**Definition 5 (Matroid Oracles).** Given a matroid $M = (E, \mathcal{I})$, assume we can only access it by querying the independence oracle $O_i$ or the rank oracle $O_r$. For a subset $S \subseteq E$, $O_i(S) = 1$ if $S$ is an independent set of $M$, otherwise $O_i(S) = 0$ and $O_r(S) = r(S)$.

5.2. Identify 2-bases Matroids’ Bases

One way of representing a matroid is to represent its independent set with a binary indicator vector. Specifically, we first list the elements of $E$ as $\{e_1, \ldots, e_n\}$, and we also use $e_j$ to denote an $n$-bit 0-1 string, where the $j$-th bit is 1 and the other bits are 0. Then for any $A \subseteq E$, we use the $n$-bit
string $\bigoplus_{e \in A} e$ as the indicator vector of $A$. We define $\chi : 2^E \rightarrow \{0, 1\}^n$ as the mapping function of the set to its indicator vector by

$$\chi(A) = \bigoplus_{e \in A} e \quad (A \subseteq E).$$

**Max Inner Product Oracle and Rank Oracle.** From the definition of matroid, we know that for any independent set $J \in \mathcal{I}(M)$, there must be some base $B \in \mathcal{B}(M)$ that contains $J$. Then the rank function of matroid $M$ can be restate as

$$r(A) = \max\{|A \cap B| : B \in \mathcal{B}(M)\} \quad (A \subseteq E).$$

If the subsets of $E$ are represented by binary indicator vectors, the cardinality of the intersection of two sets is equal to the inner product of their corresponding indicator vectors. Thus, the rank function of matroid $M$ can also be described as

$$r(A) = \max\{\chi(A) \cdot \chi(B) : B \in \mathcal{B}(M)\} \quad (A \subseteq E).$$

Comparing the definitions of the max inner product oracle and the matroid rank oracle, we can see that they are the same:

$$O_r(A) = r(A) = O_{\max}(\chi(A)) \quad (A \subseteq E).$$

**Sub-set Oracle and Independence Oracle.** By the definition of matroid, we can restate the independence oracle as

$$O_i(A) = \begin{cases} 1 & \text{if } \exists B \in \mathcal{B}(M) \text{ s.t. } A \subseteq B, \\ 0 & \text{otherwise}. \end{cases}$$

From this definition, we can see that the independence oracle $O_i$ is determined by the set of the bases of a matroid. In other words, the independence oracle $O_i$ hides the bases of a matroid. Comparing the definitions of the sub-set oracle and the independence oracle, we can also see that they are the same.

**Identify Matroid’s Bases and Hidden Strings Problem.** Given a 2-bases matroid $M$, let $B, B' \in \mathcal{B}(M)$ be its two different bases. By the definition of matroid (Definition 3), it can be seen that $|B| = |B'|$ and $|(B - B') \cup (B' - B)| = 2$. Correspondingly, we have $|\chi(B)| = |\chi(B')|$ and $|\chi(B) \oplus \chi(B')| = 2$. Therefore, identifying the two bases $B, B'$ is equivalent to HSP with $S = \{\chi(B), \chi(B')\}$. Therefore, Theorem 1 and Theorem 2 can be applied to identify the bases of a 2-bases matroid.
6. Conclusion

In this paper we consider quantum algorithms for HSP with the max inner product oracle and the sub-set oracle. We present quantum algorithms using $O(1)$ queries to the max inner product oracle and $\frac{1}{2}n + O(\sqrt{n})$ queries to the sub-set oracle to solve HSP, respectively. Furthermore, our quantum algorithms are applied to identifying the bases of 2-bases matroids. In the further work, one may consider how to extend HSP to the case with more strings and apply it to more matroid problems or other problems.

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References

[1] D. E. Knuth, J. H. M. Jr., V. R. Pratt, Fast pattern matching in strings, SIAM J. Comput. 6 (2) (1977) 323–350. doi:10.1137/0206024. URL https://doi.org/10.1137/0206024

[2] L. Colussi, Fastest pattern matching in strings, J. Algorithms 16 (2) (1994) 163–189. doi:10.1006/jagm.1994.1008. URL https://doi.org/10.1006/jagm.1994.1008

[3] R. S. Boyer, J. S. Moore, A fast string searching algorithm, Commun. ACM 20 (10) (1977) 762–772. doi:10.1145/359842.359859. URL https://doi.org/10.1145/359842.359859

[4] P. Møller-Nielsen, J. Staunstrup, Experiments with a fast string searching algorithm, Inf. Process. Lett. 18 (3) (1984) 129–135. doi:10.1016/0020-0190(84)90015-2. URL https://doi.org/10.1016/0020-0190(84)90015-2

[5] D. Margaritis, S. Skiena, Reconstructing strings from substrings in rounds, in: 36th Annual Symposium on Foundations of Computer Science, Milwaukee, Wisconsin, USA, 23-25 October 1995, IEEE Computer Society, 1995, pp. 613–620. doi:10.1109/SFCS.1995.492591. URL https://doi.org/10.1109/SFCS.1995.492591
[6] S. Skiena, G. Sundaram, Reconstructing strings from substrings, J. Comput. Biol. 2 (2) (1995) 333–353. doi:10.1089/cmb.1995.2.333. URL https://doi.org/10.1089/cmb.1995.2.333

[7] Y. Manin, Computable and uncomputable, Sovetskoye Radio, Moscow 128 (1980).

[8] R. P. Feynman, Simulating physics with computers, International Journal of Theoretical Physics 21 (6) (1982) 467–488. doi:10.1007/BF02650179. URL https://doi.org/10.1007/BF02650179

[9] D. Deutsch, Quantum theory, the church-turing principle and the universal quantum computer, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 400 (1818) (1985) 97–117. doi:10.1098/rspa.1985.0070. URL https://doi.org/10.1098/rspa.1985.0070

[10] D. Deutsch, R. Jozsa, Rapid solution of problems by quantum computation, Proceedings of the Royal Society of London Series A 439 (1907) (1992) 553–558. doi:10.1098/rspa.1992.0167. URL https://doi.org/10.1098/rspa.1992.0167

[11] D. R. Simon, On the power of quantum computation, in: 35th Annual Symposium on Foundations of Computer Science, Santa Fe, New Mexico, USA, 20-22 November 1994, IEEE Computer Society, 1994, pp. 116–123. doi:10.1109/SFCS.1994.365701. URL https://doi.org/10.1109/SFCS.1994.365701

[12] E. Bernstein, U. V. Vazirani, Quantum complexity theory, SIAM J. Comput. 26 (5) (1997) 1411–1473. doi:10.1137/S0097539796300921. URL https://doi.org/10.1137/S0097539796300921

[13] P. W. Shor, Algorithms for quantum computation: Discrete logarithms and factoring, in: 35th Annual Symposium on Foundations of Computer Science, Santa Fe, New Mexico, USA, 20-22 November 1994, IEEE Computer Society, 1994, pp. 124–134. doi:10.1109/SFCS.1994.365700. URL https://doi.org/10.1109/SFCS.1994.365700

[14] L. K. Grover, A fast quantum mechanical algorithm for database search, in: G. L. Miller (Ed.), Proceedings of the Twenty-Eighth Annual ACM
[15] S. Zhang, L. Li, *A brief introduction to quantum algorithms*, CCF Transactions on High Performance Computing 4 (1) (2022) 53–62. doi:10.1007/s42514-022-00090-3. URL https://doi.org/10.1007/s42514-022-00090-3

[16] R. Hariharan, V. Vinay, *String matching in $\tilde{O}(\sqrt{n}+\sqrt{m})$ quantum time*, J. Discrete Algorithms 1 (1) (2003) 103–110. doi:10.1016/S1570-8667(03)00010-8. URL https://doi.org/10.1016/S1570-8667(03)00010-8

[17] A. Montanaro, *Quantum pattern matching fast on average*, Algorithmica 77 (1) (2017) 16–39. doi:10.1007/s00453-015-0060-4. URL https://doi.org/10.1007/s00453-015-0060-4

[18] F. L. Gall, S. Seddighin, *Quantum meets fine-grained complexity: Sublinear time quantum algorithms for string problems*, in: M. Braverman (Ed.), 13th Innovations in Theoretical Computer Science Conference, ITCS 2022, January 31 - February 3, 2022, Berkeley, CA, USA, Vol. 215 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022, pp. 97:1–97:23. doi:10.4230/LIPIcs.ITCS.2022.97. URL https://doi.org/10.4230/LIPIcs.ITCS.2022.97

[19] S. Akmal, C. Jin, *Near-optimal quantum algorithms for string problems*, in: J. S. Naor, N. Buchbinder (Eds.), Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022, SIAM, 2022, pp. 2791–2832. doi:10.1137/1.9781611977073.109. URL https://doi.org/10.1137/1.9781611977073.109

[20] R. Cleve, K. Iwama, F. L. Gall, H. Nishimura, S. Tani, J. Teruyama, S. Yamashita, *Reconstructing strings from substrings with quantum queries*, in: F. V. Fomin, P. Kaski (Eds.), Algorithm Theory - SWAT 2012 - 13th Scandinavian Symposium and Workshops, Helsinki, Finland, July 4-6, 2012. Proceedings, Vol. 7357 of Lecture Notes in Computer Science, Springer, 2012, pp. 388–397. doi:10.1007/978-3-642-31155-0\
[21] A. S. Motahari, G. Bresler, D. N. C. Tse, Information theory of DNA shotgun sequencing, IEEE Transactions on Information Theory 59 (10) (2013) 6273–6289.

[22] J. Dhaliwal, S. J. Puglisi, A. Turpin, Practical efficient string mining, IEEE Transactions on Knowledge and Data Engineering 24 (4) (2012) 735–744.

[23] Z. Li, B. Cai, H. Sun, H. Liu, L. Wan, S. Qin, Q. Wen, F. Gao, Novel quantum circuit implementation of advanced encryption standard with low costs, Science China: Physics, Mechanics and Astronomy 65 (9) (2022) 290311.

[24] D. Ding-Zhu, F. K. Hwang, Combinatorial group testing and its applications, World Scientific, 2000. doi:10.1142/1936.

[25] W. van Dam, Quantum oracle interrogation: Getting all information for almost half the price, in: 39th Annual Symposium on Foundations of Computer Science, FOCS ’98, November 8-11, 1998, Palo Alto, California, USA, IEEE Computer Society, 1998, pp. 362–367. doi:10.1109/SFCS.1998.743486.

[26] K. Iwama, H. Nishimura, R. Raymond, J. Teruyama, Quantum counterfeit coin problems, Theor. Comput. Sci. 456 (2012) 51–64. doi:10.1016/j.tcs.2012.05.039.

[27] A. Ambainis, A. Montanaro, Quantum algorithms for search with wildcards and combinatorial group testing, Quantum Inf. Comput. 14 (5-6) (2014) 439–453. doi:10.26421/QIC14.5-6-4.

[28] Y. Xu, S. Zhang, L. Li, Quantum algorithm for learning secret strings and its experimental demonstration, arXiv preprint arXiv:2206.11221 (2022). doi:110.48550/arXiv.2206.11221.

URL https://arxiv.org/abs/2206.11221v1
[29] L. Li, J. Luo, Y. Xu, Winning mastermind overwhelmingly on quantum
computers, arXiv preprint arXiv:2207.09356 (2022).

[30] H. Whitney, On the abstract properties of linear dependence, American
Journal of Mathematics 57 (3) (1935) 509–533. doi:10.2307/2371182.
URL http://www.jstor.org/stable/2371182

[31] M. Hunziker, D. A. Meyer, Quantum algorithms for highly structured
search problems, Quantum Inf. Process. 1 (3) (2002) 145–154. doi:
10.1023/A\%3A1019868924061.
URL https://doi.org/10.1023/A%3A1019868924061

[32] X. Huang, J. Luo, L. Li, Quantum speedup and limitations on matroid
properties, arXiv preprint arXiv:2111.12900 (2021).
URL https://arxiv.org/abs/2111.12900

[33] J. Edmonds, Matroids and the greedy algorithm, Mathematical pro-
gramming 1 (1) (1971) 127–136. doi:10.1007/BF01584082.
URL https://doi.org/10.1007/BF01584082

[34] T. Feder, M. Mihail, Balanced matroids, in: Proceedings of the twenty-
fourth annual ACM symposium on Theory of computing, 1992, pp. 26–
38. doi:10.1145/129712.129716.
URL https://doi.org/10.1145/129712.129716

[35] Y. Azar, A. Z. Broder, A. M. Frieze, On the problem of approximating
the number of bases of a matroid, Inf. Process. Lett. 50 (1) (1994) 9–11.
doi:10.1016/0020-0190(94)90037-X.
URL https://doi.org/10.1016/0020-0190(94)90037-X

[36] N. Anari, S. O. Gharan, C. Vinzant, Log-concave polynomials, entropy,
and a deterministic approximation algorithm for counting bases of ma-
troids, in: 2018 IEEE 59th Annual Symposium on Foundations of Com-
puter Science (FOCS), IEEE, 2018, pp. 35–46. doi:10.1109/FOCS.
2018.00013.
URL https://doi.org/10.1109/FOCS.2018.00013

[37] N. Anari, K. Liu, S. O. Gharan, C. Vinzant, Log-concave polynomials ii:
high-dimensional walks and an fpras for counting bases of a matroid, in:
Proceedings of the 51st Annual ACM SIGACT Symposium on Theory
[38] N. Anari, M. Dereziński, Isotropy and log-concave polynomials: Accelerated sampling and high-precision counting of matroid bases, in: 2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS), IEEE, 2020, pp. 1331–1344. doi:10.1109/FOCS46700.2020.00126. URL https://doi.org/10.1109/FOCS46700.2020.00126

[39] D. Avis, K. Fukuda, Reverse search for enumeration, Discret. Appl. Math. 65 (1-3) (1996) 21–46. doi:10.1016/0166-218X(95)00026-N. URL https://doi.org/10.1016/0166-218X(95)00026-N

[40] N. A. Neudauer, A. M. Meyers, B. Stevens, Enumeration of the bases of the bicircular matroid on a complete bipartite graph, Ars Comb. 66 (2003).

[41] L. G. Khachiyan, E. Boros, K. M. Elbassioni, V. Gurvich, K. Makino, On the complexity of some enumeration problems for matroids, SIAM J. Discret. Math. 19 (4) (2005) 966–984. doi:10.1137/S0895480103428338. URL https://doi.org/10.1137/S0895480103428338

[42] L. Khachiyan, E. Boros, K. Borys, K. M. Elbassioni, V. Gurvich, K. Makino, Enumerating spanning and connected subsets in graphs and matroids, in: Y. Azar, T. Erlebach (Eds.), Algorithms - ESA 2006, 14th Annual European Symposium, Zurich, Switzerland, September 11-13, 2006, Proceedings, Vol. 4168 of Lecture Notes in Computer Science, Springer, 2006, pp. 444–455. doi:10.1007/11841036_41. URL https://doi.org/10.1007/11841036_41

[43] M. Maxwell, Enumerating bases of self-dual matroids, J. Comb. Theory, Ser. A 116 (2) (2009) 351–378. doi:10.1016/j.jcta.2008.06.007. URL https://doi.org/10.1016/j.jcta.2008.06.007

[44] J. Cardinal, A. I. Merino, T. Mütze, Efficient generation of elimination trees and graph associahedra, in: J. S. Naor, N. Buchbinder (Eds.), Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12,
[45] A. I. Merino, T. Mütze, A. Williams, *All your bases are belong to us: Listing all bases of a matroid by greedy exchanges*, in: P. Fraigniaud, Y. Uno (Eds.), 11th International Conference on Fun with Algorithms, FUN 2022, May 30 to June 3, 2022, Island of Favignana, Sicily, Italy, Vol. 226 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022, pp. 22:1–22:28. doi:10.4230/LIPIcs.FUN.2022.22. URL https://doi.org/10.4230/LIPIcs.FUN.2022.22

[46] J. Oxley, *Matroid theory*, 2nd Edition, Vol. 21 of Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford, 2011. doi:10.1093/acprof:oso/9780198566946.001.0001. URL https://doi.org/10.1093/acprof:oso/9780198566946.001.0001

[47] D. J. A. Welsh, *Matroid theory*, L. M. S. Monographs, No. 8, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976.