Recurrence region of multiuser Aloha

Ghurumuruhan Ganesan *

New York University, Abu Dhabi

Abstract

In this paper, we provide upper and lower bounds for the region of positive recurrence for a general finite user Aloha network.

Key words: Recurrence region, multiuser Aloha.

AMS 2000 Subject Classification: Primary: 60J10, 60K35; Secondary: 60C05, 62E10, 90B15, 91D30.

1 Introduction

Consider a network of $M$ users assigned to access a common channel via random access. Much of previous literature has focused on finding bounds on the stability region for various channel models (see for example [3, 4, 2] and references therein). It is also important to determine the region where the overall network is positive recurrent. In this paper, we find bounds for the recurrence region for a general multiuser Aloha network.

Channel Model

The users are labelled $U_1, \ldots, U_M$. Time is divided into slots of unit length and in each time slot, packets arrive randomly at the queue of each user. For $1 \leq i \leq M$, let $A_i(n)$ be the random number of packets arriving at
user $U_i$ in time slot $n$. We assume that $\{A_i(n)\}_{n \geq 1}$ are independent and identically distributed (i.i.d) with

$$\lambda_i = \mathbb{E}A_i(n) \quad (1.1)$$

and satisfying

$$\mathbb{P}(A_i(n) = 1) > 0. \quad (1.2)$$

For $i \neq j$, we also assume that $\{A_i(n)\}_{n \geq 1}$ is independent of $\{A_j(n)\}_{n \geq 1}$. Depending on the queue length and the instantaneous channel conditions, user $U_i$ attempts to transmit packets. Formally, let $\{W_i(n)\}_{n \geq 1}$ be i.i.d non-negative integer valued random variables with mean

$$C_i = \mathbb{E}W_i(n) \quad (1.3)$$

and

$$\mathbb{P}(W_i(n) = 1) > 0. \quad (1.4)$$

Let $Q_i(n)$ be the queue length of user $U_i$ at time slot $n$. For time slot $n \geq 0$, the update equation for $Q_i(.)$ is

$$Q_i(n + 1) = Q_i(n) + A_i(n + 1)$$

$$- \min(Q_i(n), W_i(n + 1)) \mathbf{1} \left( B_i(n + 1) \cap \bigcap_{j \neq i} B_j^c(n + 1) \right) \quad (1.5)$$

where

$$B_i(n + 1) = \{Q_i(n) \geq 1\} \cap \{W_i(n + 1) \geq 1\} \quad (1.6)$$

is the intersection of the events that the queue of user $U_i$ is nonempty and the user $U_i$ attempts to transmit in time slot $n + 1$. Define

$$p_i := \mathbb{P}(W_i(n) \geq 1) \quad (1.7)$$

to be the attempt probability of user $U_i$.

Conditions (1.2) and (1.4) ensure that the Markov chain

$$Q(n) = (Q_1(n), \ldots, Q_M(n)), n \geq 1$$

is irreducible. The following result provides necessary and sufficient conditions for recurrence of $\{Q(n)\}$. 

2
Theorem 1. If \( W_i(n) \in \{0, 1\} \) for all \( n \) and
\[
\sum_{i=1}^{M} \frac{\lambda_i}{C_i \prod_{j \neq i} (1 - p_j)} < 1, \tag{1.8}
\]
then \( \{Q(n)\} \) is positive recurrent.

If \( \max_{1 \leq i \leq M} \mathbb{E}A_i^4(1) < \infty, \max_{1 \leq i \leq M} \mathbb{E}W_i^4(1) < \infty \) and
\[
\lambda_i > C_i \prod_{j \neq i} (1 - p_j), \tag{1.9}
\]
for all \( 1 \leq i \leq M \), then \( \{Q(n)\} \) is transient.

For the particular case when \( W_i(n) \in \{0, 1\} \), we obtain the usual single packet Aloha network with
\[
p_i := \mathbb{P}(W_i(n) = 1) = 1 - \mathbb{P}(W_i(n) = 0). \tag{1.10}
\]
For \( p = (p_1, \ldots, p_M) \), the Markov chain \( Q(n) = Q(p, n) \) and so define the recurrence region
\[
\mathcal{R} = \{ (\lambda_1, \ldots, \lambda_M) : \exists p \in (0, 1)^M \text{ such that} \}
\[
\{Q(p, n)\}_{n \geq 1} \text{ is positive recurrent} \} \tag{1.11}
\]
To obtain bounds for \( \mathcal{R} \) using Theorem 1, we have some definitions. For \( p = (p_1, \ldots, p_M) \in (0, 1)^M \), define
\[
C_1(p) := \left\{ (\lambda_1, \ldots, \lambda_M) \in (0, 1)^M : \sum_{i=1}^{M} \frac{\lambda_i}{p_i \prod_{j \neq i} (1 - p_j)} < 1 \right\} \tag{1.12}
\]
and let
\[
C_1 = \bigcup_{p \in (0, 1)^M} C_1(p). \tag{1.13}
\]
Similarly define
\[
C_2(p) := \left\{ (\lambda_1, \ldots, \lambda_M) \in (0, 1)^M : \lambda_i < p_i \prod_{j \neq i} (1 - p_j) \text{ for some } 1 \leq i \leq M \right\} \tag{1.14}
\]
and let
\[
C_2 = \bigcup_{p \in (0, 1)^M} C_2(p). \tag{1.15}
\]
Corollary 1. The recurrence region $R$ defined in (1.11) satisfies
\[ C_1 \subseteq R \subseteq \text{cl}(C_2) \] (1.16)
where cl($C_2$) is the closure of the set $C_2$.

The paper is organized as follows: In Section 1, we prove Theorem 1.

**Proof of Theorem 1**

**Lower bound**

From (1.8), there exists $\epsilon_0 > 0$ so that
\[ \sum_{i=1}^{M} \frac{\lambda_i}{v_i} \leq 1 - \epsilon_0, \] (1.17)
where $v_i := C_i \prod_{j \neq i} (1 - p_j)$.

For $1 \leq i \leq M$ we have from the definition of the event $B_i(.)$ in (1.6) that
\[ 1(B_j(n+1)) \leq 1(W_j(n+1) \geq 1) \]
for all $j \neq i$. Moreover if $Q_i(n) \geq 1$, then $\min(Q_i(n), W_i(n+1)) = W_i(n+1)$ since $W_i(n+1) \in \{0,1\}$ and so from (1.5),
\[ Q_i(n+1) \leq Q_i(n) + A_i(n+1) - W_i(n+1)1(Q_i(n) \geq 1)1(V_i(n+1)) \] (1.18)
where
\[ V_i(n+1) := \{W_i(n+1) \geq 1\} \cap \bigcap_{j \neq i} \{W_j(n+1) = 0\} \] (1.19)
is the event that only user $U_i$ attempts to transmit at time slot $n+1$. The term $W_i(n+1)1(V_i(n+1))$ is independent of $Q_i(n)$ and has mean
\[ \mathbb{E}W_i(n+1)1(V_i(n+1)) = C_i \prod_{j \neq i} (1 - p_j) = v_i, \] (1.20)
from (1.17).

Let $Q_i(0) = 0$ for all $1 \leq i \leq M$ and define
\[ T = \inf\{k \geq 1 : Q_i(k) = 0 \text{ for all } 1 \leq i \leq M\} \] (1.21)
to be the first time that the queues of all the users are simultaneously empty, again. Analogous to \([1]\), we show that the expected time to return to the origin \(ET\) is finite. Multiplying both sides of \((1.18)\) by \(1(T \geq n + 1)\) and using the fact that \(Q_i(n)1(T \geq n + 1) \leq Q_i(n)1(T \geq n)\), we get

\[
Q_i(n + 1)1(T \geq n + 1) \\
\leq Q_i(n)1(T \geq n) + A_i(n + 1)1(T \geq n + 1) \\
- 1(Q_i(n) \geq 1)1(T \geq n + 1)W_i(n + 1)1(V_i(n + 1)). \tag{1.22}
\]

Defining

\[
y_n(i) := \mathbb{E}Q_i(n)1(T \geq n) \tag{1.23}
\]

we get from \((1.22)\) that

\[
y_{n+1}(i) \leq y_n(i) + \mathbb{E}A_i(n + 1)1(T \geq n + 1) \\
- \mathbb{E}1(Q_i(n) \geq 1)1(T \geq n + 1)W_i(n + 1)1(V_i(n + 1)) \\
= y_n(i) + \mathbb{E}A_i(n + 1)\mathbb{E}1(T \geq n + 1) \\
- \mathbb{E}1(Q_i(n) \geq 1)1(T \geq n + 1)\mathbb{E}W_i(n + 1)1(V_i(n + 1)) \tag{1.24}
\]

\[
y_{n+1}(i) = y_n(i) + \lambda_i\mathbb{P}(T \geq n + 1) - v_i\mathbb{P}(\{Q_i(n) \geq 1\} \cap \{T \geq n + 1\}). \tag{1.25}
\]

The relation \((1.24)\) follows from the fact \(\{T \geq n + 1\} = \{T \leq n\}^c\) and that \(A_i(n + 1), W_i(n + 1)\) and \(V_i(n + 1)\) are independent of the process up to \(n\) time slots. The final estimate in \((1.25)\) is obtained using \((1.20)\).

Defining

\[
y_n := \sum_{i=1}^{M} \frac{y_n(i)}{v_i}, \tag{1.26}
\]

we get from \((1.25)\) that

\[
y_{n+1} \leq y_n + \left(\sum_{i=1}^{M} \frac{\lambda_i}{v_i}\right)\mathbb{P}(T \geq n + 1) - \Delta_n \tag{1.27}
\]
where
\[
\Delta_n := \sum_{i=1}^{M} \mathbb{P}\left(\{Q_i(n) \geq 1\} \cap \{T \geq n + 1\}\right)
\]
\[
\geq \mathbb{P}\left(\bigcup_{i=1}^{M} \{Q_i(n) \geq 1\} \cap \{T \geq n + 1\}\right)
\]
\[
= \mathbb{P}(T \geq n + 1),
\] (1.28)
since if $T \geq n + 1$, then at least one of the queues at time slot $n \geq 1$ is non empty.

Using (1.28) in (1.27) gives
\[
y_{n+1} \leq y_n + \left(\sum_{i=1}^{M} \frac{\lambda_i}{v_i} - 1\right) \mathbb{P}(T \geq n + 1)
\]
\[
\leq y_n - \epsilon_0 \mathbb{P}(T \geq n + 1)
\] by (1.17). Thus
\[
\mathbb{P}(T \geq n + 1) \leq \frac{1}{\epsilon_0} (y_n - y_{n+1})
\]
and adding telescopically gives for $J \geq 1$ that
\[
\sum_{k=1}^{J} \mathbb{P}(T \geq k + 1) \leq \frac{1}{\epsilon_0} (y_1 - y_{J+1}) \leq \frac{1}{\epsilon_0} y_1,
\] (1.29)
where
\[
y_1 = \sum_{i=1}^{M} \frac{y_1(i)}{v_i}
\]
using (1.26) and
\[
y_1(i) = \mathbb{E}Q_i(1)1(T \geq 1) \leq \mathbb{E}Q_i(1) \leq \mathbb{E}A_i(1) = \lambda_i < \infty
\]
using (1.23). This implies that $0 \leq y_1 < \infty$ and since $J$ is arbitrary, we get from (1.29) that $\mathbb{E}T < \infty$ and so the Markov chain $\{Q(n)\}$ is positive recurrent.
Upper bound

Let $K \geq 1$ be a large integer constant to be determined later and let $T_0 := 0$ and $T_1 := K$. We now observe the overall queue process from time slot $T_0 + 1 = 1$ to time slot $T_1$. Recall from (1.5) that $W_i(n+1)$ is the maximum number of packets transmitted by user $U_i$ in time slot $n+1$. If

$$Z_i(T_0, T_1) := \left\{ \sum_{n=T_0}^{T_1-1} W_i(n+1) < 2C_i(T_1 - T_0) \right\}$$

(1.30)

then

$$\mathbb{P}(Z_i^c(T_0, T_1)) = \mathbb{P}(S_i \geq C_i(T_1 - T_0)), \quad (1.31)$$

where $S_i = \sum_{n=T_0}^{T_1-1}(W_i(n+1) - C_i)$ is a sum of independent zero mean random variables and so

$$\mathbb{E}S_i^4 = \sum_n \mathbb{E}(W_i(n+1) - C_i)^4 + \sum_{n \neq m} \mathbb{E}(W_i(n+1) - C_i)^2 \mathbb{E}(W_i(m+1) - C_i)^2.$$  

(1.32)

Using the finite fourth moment condition of $W_i(n)$ (see statement prior to (1.9)), the first term in (1.32) is $\mathbb{E}(W_i(1) - C_i)^4(T_1 - T_0)$ and the second term in (1.32) is at most

$$(T_1 - T_0)^2 \mathbb{E}(W_i(1) - C_i)^2 \leq (T_1 - T_0)^2 \mathbb{E}(W_i(1) - C_i)^4.$$  

Combining,

$$\mathbb{E}S_i^4 \leq \alpha_1(i)(T_1 - T_0)^2 \quad (1.33)$$

for some constant $\alpha_1(i) > 0$, not depending on $T_1$ or $T_0$. From (1.31), (1.33) and Markov inequality, we get

$$\mathbb{P}(Z_i^c(T_0, T_1)) \leq \frac{\alpha_1(i)(T_1 - T_0)^2}{C_i^4(T_1 - T_0)^4} \leq \frac{\alpha_2}{(T_1 - T_0)^2} \quad (1.34)$$

where $\alpha_2 = \max_{1 \leq i \leq M} \frac{\alpha_1(i)}{C_i^4}$ is a constant. If

$$Z(T_0, T_1) := \bigcap_{i=1}^{M} Z_i(T_0, T_1),$$

(1.35)

then from (1.31),

$$\mathbb{P}(Z(T_0, T_1)) \geq 1 - \frac{\alpha_2 M}{(T_1 - T_0)^2}. \quad (1.36)$$

7
For $1 \leq i \leq M$, set the initial queue length

$$Q_i(T_0) = Q_i(0) = \delta(T_1 - T_0) =: L_1,$$

(1.37)

where $\delta = 3 \max_{1 \leq i \leq M} C_i$ and $T_1 = K$ is large so that $\delta(T_1 - T_0) = \delta K > 1$. If $Z(T_0, T_1)$ occurs, then at most $2C_i(T_1 - T_0)$ packets are transmitted from user $U_i$ and so the queue of user $U_i$ never becomes empty between time slots $T_0 + 1$ and $T_1$. From the queue update equation (1.5) we therefore get for $T_0 \leq n \leq T_1$ and $1 \leq i \leq M$ that

$$Q_i(n + 1) \geq Q_i(n) + A_i(n + 1) - W_i(n + 1)1(V_i(n + 1))$$

(1.38)

where $V_i(.)$ is as defined in (1.19). Adding telescopically,

$$Q_i(T_1) = Q_i(T_0) + R_i(T_0, T_1),$$

(1.39)

where

$$R_i(T_0, T_1) := \sum_{n=T_0}^{T_1-1} (A_i(n + 1) - W_i(n + 1)1(V_i(n + 1))).$$

(1.40)

From (1.9) we have that

$$\mathbb{E}(A_i(n + 1) - W_i(n + 1)1(V_i(n + 1))) = 2\epsilon_1(i) > 0$$

(1.41)

for all $n$. Moreover, the term $R_i(T_0, T_1)$ is also a sum of i.i.d zero mean random variables and so arguing as in (1.34), we get

$$\mathbb{P}(R_i(T_0, T_1) \geq \epsilon_1(i)(T_1 - T_0)) \geq 1 - \frac{\alpha_3(i)}{(T_1 - T_0)^2}$$

(1.42)

for some constant $\alpha_3(i) > 0$, not depending on $T_1$ or $T_0$.

Letting $\epsilon_1 = \min_{1 \leq i \leq M} \epsilon_1(i)$ and $\alpha_3 = \max_{1 \leq i \leq M} \alpha_3(i)$ and defining

$$X(T_0, T_1) := \bigcap_{i=1}^{M} \{R_i(T_0, T_1) \geq \epsilon_1(T_1 - T_0)\},$$

(1.43)

we get from (1.42) that

$$\mathbb{P}(X(T_0, T_1)) \geq 1 - \frac{\alpha_3 M}{(T_1 - T_0)^2}$$

(1.44)
and if
\[ Y(T_0, T_1) := Z(T_0, T_1) \cap X(T_0, T_1), \] (1.45)
then from (1.31) and (1.44), we get
\[ \mathbb{P}(Y(T_0, T_1)) \geq 1 - \frac{\alpha_4}{(T_1 - T_0)^2} \] (1.46)
for some constant \( \alpha_4 > 0 \), not depending on \( T_0 \) or \( T_1 \).

Suppose now that \( Y(T_0, T_1) \) occurs. Between time slots \( T_0 \) and \( T_1 \), none of the queues of the \( M \) users ever becomes empty and at time slot \( T_1 \), the queue length \( Q_i(T_1) \) is at least
\[ Q_i(T_0) + \epsilon_1(T_1 - T_0) = L_1 + \epsilon_1(T_1 - T_0) = (\delta + \epsilon_1)(T_1 - T_0) =: L_2, \] (1.47)
using (1.37) and (1.39). For \( j \geq 2 \), we now repeat the above procedure between time slots \( T_{j-1} + 1 \) and \( T_j \), where \( T_j \) is determined by the relation
\[ \delta(T_j - T_{j-1}) = L_j = (\delta + \epsilon_1)(T_{j-1} - T_{j-2}) \] (1.48)
and \( \delta > 0 \) is as in (1.37). Using the first and last relations in (1.48) iteratively, we get
\[ L_j = \delta(T_1 - T_0) \left( 1 + \frac{\epsilon_1}{\delta} \right)^{j-1} \geq (T_1 - T_0)(\delta + (j - 1)\epsilon_1) \] (1.49)
and so from (1.48),
\[ T_j - T_{j-1} \geq (T_1 - T_0) \left( 1 + (j - 1)\frac{\epsilon_1}{\delta} \right). \] (1.50)
Also analogous to (1.46), we have
\[ \mathbb{P}(Y(T_{j-1}, T_j)) \geq 1 - \frac{\alpha_4}{(T_j - T_{j-1})^2} \geq 1 - \frac{\alpha_5}{(\delta + (j - 1)\epsilon_1)^2} \] (1.51)
for all \( j \geq 2 \) and for some constant \( \alpha_5 > 0 \) not depending on \( j \).

If the event
\[ Y := \bigcap_{j \geq 1} Y(T_{j-1}, T_j) \] (1.52)
occurs, then none of the queues of any user ever becomes empty. Using (1.51) and the Markov property we also have
\[ \mathbb{P}(Y) \geq \prod_j \left( 1 - \frac{\alpha_6}{(\delta + (j - 1)\epsilon_1)^2} \right) > 0, \]
since \( \sum_j \frac{\alpha_j}{(\delta + (j-1)\epsilon_j)^2} < \infty \). Recall that the initial queue length of each user is \( \delta(T_1 - T_0) = \delta K \) (see (1.37)) and so starting from \( (\delta K, \ldots, \delta K) \), the above discussion implies that with positive probability, the Markov chain \( \{Q(n)\} \) never reaches the origin. Since \( \{Q(n)\} \) is irreducible, this implies that starting from the origin, the chain \( \{Q(n)\} \) never returns to the origin, with positive probability. Therefore \( \{Q(n)\} \) is transient.

References

[1] S. Asmussen. (2003). *Applied probability and queues*. Springer.

[2] W. Luo and A. Ephremides. (1999). Stability of \( N \) interacting queues in random-access systems. *IEEE Transactions on Information Theory*, **45**, 1579–1587.

[3] V. Naware, G. Mergen and L. Tong. (2005). Stability and delay of finite-user slotted Aloha with multipacket reception. *IEEE Transactions of Information Theory*, **51**, 2636–2656.

[4] W. Szpankowski. (1986). Stability conditions for multidimensional queueing systems and applications to analysis of computer systems. *Computer Science Technical Reports*, 86-601.