Multiloop calculations in HQET

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Recently, algorithms for calculation of 3-loop propagator diagrams in HQET and on-shell QCD with a heavy quark have been constructed and implemented. These algorithms (based on integration by parts recurrence relations) reduce an arbitrary diagram to a combination of a finite number of basis integrals. Here I discuss various ways to calculate non-trivial bases integrals, either exactly or as expansions in \( \varepsilon \). Some integrals of these two classes are related to each other by inversion, which provides a useful cross-check.

I presented a review talk about multiloop calculations in HQET at this conference in Pisa in 1995 [1]. Methods of calculation of two-loop propagator diagrams in HQET [2] and on-shell massive QCD [3,4], based on integration by parts [5], were discussed there. Recently, three-loop HQET [6] and on-shell [7] algorithms have been constructed. Here I discuss this substantial progress.

1. Three-loop massless diagrams

First, I briefly remind you the classic method of calculation of 3-loop massless propagator diagrams. There are 3 generic topologies of such diagrams. They can be reduces, using integration by parts, to 6 basis integrals [5]. This algorithm is implemented in the package Mincer [5] (first written in SCHOONSCHIP [6] and later rewritten in FORM [10]), and in the package Slicer [11] written in REDUCE [12,13].

Four basis integrals are trivial. One is a two-loop diagram with a non-integer power of the middle line. It can be found as a particular case of a more general expression [5,3] for the the two-loop diagram with three non-integer powers via a hypergeometric \( _3F_2 \) function of the unit argument, with indices tending to integers at \( \varepsilon \to 0 \). There is a rather straightforward algorithm for expanding such functions in \( \varepsilon \), with coefficients expressed via multiple \( \zeta \)-values. I have implemented it in REDUCE in the summer of 2000, some results produced by this program are published in [16]. It is clearly presented as Algorithm A in [17]; this paper also contains other, more complicated, algorithms. The algorithms of [17] are implemented in the C++ library nestedsums [18] based on the computer-algebra library GiNaC [19]. This implementation is very convenient; unfortunately, it requires one to install an outdated version of GiNaC. The Algorithm A seems to be also implemented in FORM [20], but I could not understand how to use it. Using my REDUCE procedure or nestedsums [18], one can quickly find as many terms of expansion of this basis integral in \( \varepsilon \) as needed, in terms of multiple \( \zeta \)-values. They can be expressed, up to weight 9, via a minimum set of independent \( \zeta \)-values, using the results of [22,23].

The two-loop diagram with a non-integer power of the middle line can also be expressed [21] via an \( _3F_2 \) function of the argument \(-1\). Expanding this expression in \( \varepsilon \) (say, using nestedsums [18]), we encounter more general Euler–Zagier sums, which were also considered in [22]. Reducing them to the minimal basis, we obtain, of course, the same \( \varepsilon \)-expansion of our basis integral.

Using this expansion and integration-by-parts relations, it is easy to recover the well-known result for the 3-loop ladder diagram, which is finite \( \varepsilon = 0 \): \( 20\zeta(5) + \mathcal{O}(\varepsilon) \). The last and most difficult basis diagram is non-planar. It is also finite at \( \varepsilon = 0 \). Using gluing of its external vertices [5], one can easily understand that it has the same value \( 20\zeta(5) \) at \( \varepsilon = 0 \). There is no easy way to find further terms of its \( \varepsilon \)-expansion.
2. Three-loop HQET diagrams

There are 10 generic topologies of 3-loop HQET propagator diagrams. They can be reduced, using integration by parts, to 8 basis integrals [4]. This algorithm is implemented in the REDUCE package Grinder [4], available at http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp00/ttp00-01. Five basis integrals are trivial. Two can be expressed via $_3F_2$ hypergeometric functions of the unit argument [23,6]. Their expansions in $\varepsilon$ can be obtained in the same way as in the massless case, the results are presented in [16]. The last and most difficult basis integral was found in [16] up to the finite term in $\varepsilon$, using direct integration in the coordinate space. More terms of its $\varepsilon$-expansion were recently obtained in [24] using inversion, as explained in the next Section.

3. Three-loop on-shell diagrams

Calculations of on-shell diagrams with massive quarks in QCD are necessary for obtaining coefficients in the HQET Lagrangian and $1/m$ HQET expansions of QCD operators by matching. There are 2 generic topologies of 2-loop on-shell propagator diagrams. They can be reduced, using integration by parts, to 3 basis integrals. This algorithm is implemented in the REDUCE package RECURSOR [3] and the FORM package SHELL2 [4]. Two basis integrals are trivial, and the third one is expressed via two $_3F_2$ hypergeometric functions of the unit argument. However, some of their indices tend to half-integers at $\varepsilon \to 0$, and the algorithm of expansion in $\varepsilon$ discussed in Sect. 4 is not applicable. This approach was used for QCD/HQET matching of heavy-light quark currents [25] and chromomagnetic interaction [26].

The case when there is another non-zero mass was systematically studied in [27]. There are 4 basis integrals, 2 of them trivial, and 2 are expressed via $_3F_2$ hypergeometric functions of the mass ratio squared. Finite parts at $\varepsilon \to 0$ are expressed via dilogarithms. More terms of expansions of the general results [27] in $\varepsilon$ were recently obtained [28]. The REDUCE package [27] is available at http://wwwthep.physik.uni-mainz.de/Publications/progdata/mzth9838/Mm.red.

There are 11 generic topologies of 3-loop on-shell propagator diagrams with a single non-zero mass (10 of them are the same as in HQET, and one involves a heavy-quark loop). They can be reduced, using integration by parts, to 18 basis integrals [7]. This algorithm is implemented as the FORM package SHELL3 [7]. The basis integrals are mostly known from QED [29]. Some on-shell diagrams are related to HQET ones by inversion of Euclidean integration momenta. One- and two-loop relations were pre-
presented in \[1\]. Three-loop relations are shown in \[2\]. The second of them was used in \[24\] to relate the convergent ladder HQET diagram at $\varepsilon = 0$ to the known on-shell ladder diagram.

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