The local asymptotic estimation for the supremum of a random walk with generalized strong subexponential summands

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Abstract

In this paper, the local asymptotic estimation for the supremum of a random walk is presented, where the summands of the random walk has common long-tailed and generalized strong subexponential distribution. The generalized strong subexponential distribution class and corresponding generalized local subexponential distribution class are two new distribution classes with some good properties. Further, some long-tailed distributions with intuitive and concrete forms are found, showing that the generalized strong subexponential distribution class and the generalized locally subexponential distribution class properly contain the strong subexponential distribution class and the locally subexponential distribution class, respectively.

Keywords: random walks; supremum; local asymptotic estimation; generalized strong subexponential distribution; generalized locally subexponential distribution

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1 Introduction

In this paper, we primarily study the local asymptotics for the supremum of a random walk generated by summands with common long-tailed and generalized strong subexponential distribution. The generalized strong subexponential distribution class and corresponding generalized local subexponential distribution class are two new distribution classes with some good properties. Therefore, we introduce some related concepts and notations in this section.

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Unless otherwise stated, we always assume that a random variable (r.v.) $X$ has a proper and heavy-tailed distribution $F$ supported on $[0, \infty)$, that is its tailed distribution $F(x) = P(X > x)$, $x \in (-\infty, \infty)$ is always positive. By definition, a distribution $F$ is said to be heavy-tailed, if for all $\alpha > 0$,
\[
\int_0^{\infty} e^{\alpha y} dF(y) = \infty.
\]
As is known to all, heavy-tailed distributions have important applications in various fields of applied probability, such as risk theory, queuing system, warehousing management, branching theory, communication net and infinite divisible distribution theory, and so on. So they attract much interest of the researchers. However, the heavy-tailed distribution class is too large and it contains some distributions which cannot be “dominated”, so some subclasses of the heavy-tailed distribution class with good properties were introduced. Details about the heavy-tailed distributions can be found in Embrechts et al. (1997) and Foss et al. (2013), and so on. Here we first recall some existing subclasses of the heavy-tailed distribution class, and then introduce some new ones.

Hereafter, for any distribution $F$, and any $n \geq 0$, we use $F^{*n}$ to denote the $n$-fold convolution distribution of $F$ with itself, with the convention that $F^{*1} = F$ and $F^{*0}$ is the degenerated distribution at 0. We assume that all the limit relationships are for $x \to \infty$ unless otherwise stated. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$, if $\lim a(x)/b(x) = 1$; $a(x) = o(b(x))$, if $\lim a(x)/b(x) = 0$; $a(x) = O(b(x))$, if $\limsup a(x)/b(x) < \infty$ and $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leq 1$; $a(x) \approx b(x)$, if $a(x) = O(b(x))$ and $b(x) = O(a(x))$.

A most important heavy-tailed distribution subclass is defined as follows. We say that a distribution $F$ is subexponential, denoted by $F \in \mathcal{S}$, if
\[
F^{*2}(x) \sim 2F(x).
\]
It is known that the subexponential distribution class was introduced by Chistyakov (1964) in the study of the branching process, where it was proved that the subexponential distribution class belonged to the following heavy-tailed distribution subclass. We say that a distribution $F$ belongs to the long-tailed distribution class, denoted by $F \in \mathcal{L}$, if for any $y \in (-\infty, \infty)$,
\[
F(x + y) \sim F(x).
\]

For the convenience of studying, we introduce some quantity indexes of a distribution $F$. We write
\[
C_*(F) =: \liminf \frac{F^{*2}(x)}{F(x)} \quad \text{and} \quad C^*(F) =: \limsup \frac{F^{*2}(x)}{F(x)}.
\]
It was obtained by Theorem 1 of Foss and Korshunov (2007) that for any heavy-tailed distribution $F$, $C_*(F) = 2$. And it is obvious that $F \in \mathcal{S}$ if and only if $C_*(F) = C^*(F) = 2$, which means that for a subexponential distribution $F$, compared with other heavy-tailed distributions, the fluctuate of the ratios $\frac{F^{*2}(x)}{F(x)}$ is minimal as $x \to \infty$. So to some extent, we may regard such distribution $F$ as “optimal”. Also, we may say a distribution $F$ is “controllable” if
\[
2 \leq C^*(F) < \infty.
\]
And when $C^*(F) = \infty$, we say that the distribution $F$ is “uncontrollable”.

In a probabilistic model with heavy-tailed distributions, if we may choose distributions freely, then the first choice is of course the subexponential distributions, since they are “optimal” in the above sense. However, due to the complexity of the real world, the distributions usually are not decided by us. So it is necessary to study “controllable” distributions and “uncontrollable” distributions.

In fact, the “controllability” reflects the “principle of a single big jump”, since

$$P(\max\{X, Y\} > x) \sim 2F(x),$$

where $Y$ is a r.v with distribution $F$ and is independent to the r.v. $X$.

In the history of the distribution theory, Klüppelberg (1990) first considered the “controllable” distributions and called them “weak idempotents”. Shimura and Watanabe (2005) called the “controllable” distributions generalized subexponential and denoted the class of such distributions by $\mathcal{OS}$. In the terminology of Bingham et al. (1987), the distributions from the class $\mathcal{OS}$ are called O-regularly varying. Here we continue to use the notation $\mathcal{OS}$.

The distribution class $\mathcal{OS}$ is a rather large class which contains many heavy-tailed and light-tailed distributions. For research on $\mathcal{OS}$, besides the above-mentioned literature, we refer the reader to Klüppelberg and Villasenor (1991), Watanabe and Yamamura (2010), Lin and Wang (2012), Cheng and Wang (2012), Cheng et al. (2012), Yu and Wang (2013), Gao et al. (2013), Beck et al. (2013) and so on.

Present paper is interested in some subclasses of the class $\mathcal{OS}$, which respectively correspond to the number of subclasses of the class $\mathcal{S}$ as follows.

We say that a distribution $F$ is strong subexponential, denoted by $F \in \mathcal{S}^*$, if $0 < EX < \infty$ and

$$\int_0^x F(x-y)F(y)dy \sim 2EXF(x).$$

Here we point out that, in the above formulas, if the distribution $F$ is supported on $(-\infty, \infty)$, then $EX$ is replaced by $EX^+$, where $X^+ = \mathbf{1}(X > 0)$.

We say that a distribution $F$ is locally long-tailed, denoted by $F \in \mathcal{L}_{\Delta_T}$, where $T$ is some positive constant or $\infty$, if for some constant $x_0 > 0$, $F(x+\Delta_T) =: P(X \in x + \Delta_T) > 0$ for all $x \geq x_0$ and the relationship

$$F(x+y+\Delta_T) \sim F(x+\Delta_T)$$

holds uniformly for all $y \in (0, T)$, where $\Delta_T = (0, T)$, $x + \Delta_T = (x, x+T]$ when $T < \infty$, and $\Delta_T = (0, \infty)$, $x + \Delta_T = (x, \infty)$ when $T = \infty$. Further, we say that a distribution $F$ is locally subexponential, denoted by $F \in \mathcal{S}_{\Delta_T}$, if for some $0 < T \leq \infty$, $F \in \mathcal{L}_{\Delta_T}$ and

$$F^{*2}(x+\Delta_T) \sim 2F(x+\Delta_T).$$

It is known that the strong subexponential distribution and the locally subexponential distribution were introduced by Klüppelberg (1988) and Asmussen et al. (2003), respectively, which play an important role in various fields of applied probability. Inspired by the distribution classes $\mathcal{S}^*$, $\mathcal{S}_{\Delta_T}$, and $\mathcal{OS}$, we introduce the following two new distribution classes which are the main object of study of present paper.
Definition 1.1. We say that a distribution $F$ is generalized strong subexponential, denoted by $F \in \mathcal{OS}^*$, if 
\[
\int_0^x F(x - y) F(y) dy = O(F(x)).
\]

Definition 1.2. We say that a distribution $F$ is generalized locally subexponential for some $0 < T \leq \infty$, denoted by $F \in \mathcal{OS}_{\Delta T}$, if for some constant $x_0 > 0$, $F(x + \Delta T) > 0$ for all $x \geq x_0$ and 
\[
F^{*2}(x + \Delta T) = O(F(x + \Delta T)).
\]

Obviously, like the class $\mathcal{OS}$, the class $\mathcal{OS}^*$ and $\mathcal{OS}_{\Delta T}$ contain many heavy-tailed distributions and light-tailed distributions. Moreover, the classes $\mathcal{OS}^*$ and $\mathcal{OS}_{\Delta T}$ have also certain "controllability".

For the convenience of later research, similar to $C^*_s(F)$ and $C^*_T(F)$, we give some quantity indexes for the classes $\mathcal{OS}^*$ and $\mathcal{OS}_{\Delta T}$ for some $0 < T < \infty$, respectively. We write 

\[
C_{\otimes}(F) = \liminf \frac{\int_0^x F(x - y) F(y) dy}{F(x)} \leq C^\otimes(F) = \limsup \frac{\int_0^x F(x - y) F(y) dy}{F(x)}
\]

and 

\[
C_T(F) = \liminf \frac{F^{*2}(x + \Delta T)}{F(x + \Delta T)} \leq C^T(F) = \limsup \frac{F^{*2}(x + \Delta T)}{F(x + \Delta T)}.
\]

For a heavy-tailed distribution $F$, apart from proving the fact that $C^*_s(F) = 2$, Foss and Korshunov (2007) also proved that $C_{\otimes}(F) = 2EX$. However, for a light-tailed distribution $F$, the equalities $C_{\otimes}(F) = 2EX$ do not necessarily hold, see Foss and Korshunov (2007). Similarly, for a locally heavy-tailed distribution $F$, the equality $C_T(F) = 2$ for some $0 < T < \infty$ also does not necessarily hold, but if $F \in \mathcal{L}_{\Delta T}$ for some $0 < T < \infty$, then $C_T(F) = 2$, see Proposition 4.1 and Remark 4.1 of Chen et al. (2013).

In addition, inspired by Beck et al. (2013), we try to give a "physical" interpretation of the distribution class $\mathcal{OS}^*$. If $F \in \mathcal{OS}^*$, then for any $K > 0$, 

\[
\liminf \frac{P(\max\{X, Y\} > x - K, X + Y > x)}{\int_0^x F(x - y) F(y) dy} \geq \frac{2}{C_{\otimes}(F)} > 0.
\]

Further, if $F \in \mathcal{L} \cap \mathcal{OS}^*$, then for any $K > 0$, 

\[
0 < \frac{2}{C_{\otimes}(F)} = \liminf \frac{P(\max\{X, Y\} > x - K, X + Y > x)}{\int_0^x F(x - y) F(y) dy} \leq \limsup \frac{P(\max\{X, Y\} > x - K, X + Y > x)}{\int_0^x F(x - y) F(y) dy} = \frac{2}{C_{\otimes}(F)} = (EX)^{-1} < \infty.
\]

Particularly, if $F \in \mathcal{S}^*$, then for any $K > 0$, 

\[
P(\max\{X, Y\} > x - K, X + Y > x) \sim (EX)^{-1} \int_0^x F(x - y) F(y) dy.
\]
The remainder of this paper consists of three sections. In Section 2, the relationships among the two new distribution classes and some existing related ones are discussed. Some examples of long-tailed distribution show that the generalized strong subexponential distribution class and the generalized locally subexponential distribution class properly contain the strong subexponential distribution class and the locally subexponential distribution class, respectively. And proofs of this examples are given in Section 4. In Section 3, the local asymptotic estimation for the supremum of a random walk is presented, where the summands of the random walk has common long-tailed and generalized strong subexponential distribution.

The more distributions found in the classes $\mathcal{L} \cap \mathcal{OS} \setminus \mathcal{S}$, $(\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}^*$ and $(\mathcal{L}_\Delta \cap \mathcal{OS}_{\Delta'}) \setminus \mathcal{S}_{\Delta'}$ for some $0 < T < \infty$ and their applications explain the significance of further research on the class $\mathcal{L} \cap \mathcal{OS}$ as well as its subsets $\mathcal{L} \cap \mathcal{OS}^*$ and $\mathcal{L}_\Delta \cap \mathcal{OS}_{\Delta'}$ for some $0 < T < \infty$. It should be said that the method of construction of these distributions are not trivial, but these distributions are not particularly weird, especially their integral tail distributions are much more normal.

## 2 The relationships among the distribution classes

### 2.1 The relation between the classes $\mathcal{L} \cap \mathcal{OS}^*$ and $\mathcal{S}^*$

**Proposition 2.1.** The inclusion relation $\mathcal{S}^* \subset \mathcal{L} \cap \mathcal{OS}^*$ is proper.

**Proof.** Obviously, the inclusion relation $\mathcal{S}^* \subset \mathcal{L} \cap \mathcal{OS}^*$ holds. So, we just proves that the relationship is proper through the following two types of distributions.

**Example 2.1.** Let $m \geq 1$ be an integer. Choose any constant $\alpha \in (m^{-1}, 1 + m^{-1})$ and any constant $x_1 > 4^{m(\alpha - 1)}$. For all integers $n \geq 1$, let $x_{n+1} = x_n^{2^{-\theta(m\alpha - 1)}}$. Clearly, $x_{n+1} > 4x_n$ and $x_n \to \infty$ as $n \to \infty$. Now, define the distribution $F$ as follows:

\[
F(x) = \begin{cases} 
1(x < 0) + (x_1^{-1}(x_1^{-\alpha} - 1)x + 1)1(0 \leq x < x_1) \\
+ \sum_{n=1}^{\infty} ((x_n^{-\alpha} + (x_n^{-2\alpha - 1 + m^{-1}} - x_n^{-\alpha - 1}) (x - x_n))1(x_n \leq x < 2x_n) \\
+ x_n^{-2\alpha + m^{-1}}1(2x_n \leq x < x_{n+1}).
\end{cases} 
\]

Further, let

\[
G_m(x) =: (F(x))^m =: F^m(x), \ x \in (-\infty, \infty).
\]

Then $G_m \in (\mathcal{S} \cap \mathcal{OS}^*) \setminus \mathcal{S}^*$ for all $m \geq 1$.

Next, we give other some distributions in the class $(\mathcal{L} \cap \mathcal{OS}^*) \setminus \mathcal{S}$.

**Example 2.2.** Let $m \geq 1$ be an integer. Choose any constant $\alpha \in (2 + 2m^{-1}, \infty)$ and any constant $x_1 > 4^{\alpha}$. And, for all integers $n \geq 1$, let $x_{n+1} = x_n^{1+\alpha^{-1}}$. Clearly, $x_{n+1} > 4x_n$ and $x_n \to \infty$ as $n \to \infty$. Now, define the distribution $F$ as follows.

\[
F(x) = \begin{cases} 
1(x < 0) + (x_1^{-1}(x_1^{-\alpha} - 1)\sqrt{x} + 1)1(0 \leq x < x_1^2) \\
+ \sum_{n=1}^{\infty} ((x_n^{-\alpha} + (x_n^{-\alpha - 2} - x_n^{-\alpha - 1})(\sqrt{x} - x_n))1(x_n^2 \leq x < 4x_n^2) \\
+ x_n^{-\alpha - 1}1(4x_n^2 \leq x < x_{n+1}^2)).
\end{cases} 
\]
Further, let $G_m$ be the same as in Example 2.1, then $G_m \in (\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$, thus $G_m \notin \mathcal{S}^*$. Thus, the proposition is proved.

\[ \square \]

2.2 The relation between the classes $\mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T}$ and $\mathcal{S}_{\Delta T}$

Proposition 2.2. For all $0 < T \leq \infty$, the inclusion relation $\mathcal{S}_{\Delta T} \subset \mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T} \subset \mathcal{L}_{\Delta T} \cap \mathcal{OS}$ is proper.

Proof. When $T = \infty$, the corresponding counterexamples showing that there exist some distributions belonging to the class $\mathcal{L} \cap \mathcal{OS} \setminus \mathcal{S}$ may be found in Leslie (1989), Lin and Wang (2012), Example 2.2 and Example 2.5 below. So we only prove the result in the case that $0 < T < \infty$. First, we prove a simple fact that $\mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T} \subset \mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T} \subset \mathcal{L}_{\Delta T} \cap \mathcal{OS}$. Let $V \in \mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T}$ for some $0 < T < \infty$, we have

\[
\int_0^x \nabla(x - y)dV(y) \leq \sum_{k=0}^{[xT^{-1}]} \int_{kT}^{(k+1)T} \nabla(x - y)dV(y)
\]

\[
\leq \sum_{k=0}^{[xT^{-1}]} \sum_{l=0}^{\infty} V(x + lT - (k + 1)T + \Delta_T) V(kT + \Delta_T)
\]

\[
= O \left( \sum_{l=0}^{\infty} \sum_{k=0}^{[xT^{-1}]} \int_{kT}^{(k+1)T} V(x + lT - y + \Delta_T)dV(y) \right)
\]

\[
= O \left( \sum_{l=0}^{\infty} \int_0^x V(x + lT - y + \Delta_T)dV(y) \right)
\]

\[
= O(V(x)),
\]

thus $V \in \mathcal{OS}$.

The following Example 2.5 note that the distribution class $\mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T}$ is properly included in the distribution class $\mathcal{L}_{\Delta T} \cap \mathcal{OS}$. Now, we give the three counterexamples to show that the inclusion relationship $\mathcal{S}_{\Delta T} \subset \mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T}$ is also proper. To this end, we first introduce a concept of distribution. For some distribution $F$ with a finite and positive mean $EX$, we say that the distribution $F^I$ defined by

\[
F^I(x) := (EX)^{-1} \int_0^x F(y)dy 1(x > 0), \quad x \in (-\infty, \infty)
\]

is the integrated tail distribution of the distribution $F$.

Example 2.3. For any $m \geq 1$, let $G_m$ be the same as in Example 2.1 or Example 2.2. Then for all $0 < T < \infty$, $G_m^I \in (\mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T}) \setminus \mathcal{S}_{\Delta T}$.

In order to give the third counterexample, we first introduce some relevant notions and notations. We say that a distribution $F$ belongs to the exponential distribution class with the index $\gamma \geq 0$, denoted by $F \in \mathcal{L}(\gamma)$, if for all $t \in (-\infty, \infty)$,

\[
F(x + t) \sim e^{-\gamma t}F(x).
\]
We say that a distribution F belongs to the convolution equivalent distribution class with the index $\gamma \geq 0$, denoted by $F \in S(\gamma)$, if $F \in L(\gamma)$, $M_\gamma(F) =: \int_0^\infty e^{\gamma y}dF(y) < \infty$ and
\[
F*2(x) \sim 2M_\gamma(F)F(x).
\]
Obviously, when $\gamma = 0$, $L(0) = L$ and $S(0) = S$; when $\gamma > 0$, the distributions in $L(\gamma)$ are light-tailed.

Further, for a distribution $F$, if $M_\gamma(F) < \infty$ for some $\gamma > 0$, we may define a new distribution as follows.
\[
F_\gamma(x) =: (M_\gamma(F))^{-1}\int_0^x e^{\gamma y}dF(y), \ x \in (-\infty, \infty),
\]
which is called the $\gamma$-transform or the Escher transform of the distribution $F$. Similarly, we can define the $-\gamma$-transform of a distribution $F$ for any $\gamma > 0$.

**Example 2.4.** Klüppelberg and Villasenor (1991) found two distributions $F_i \in S(\gamma)$ for some $\gamma > 0$, $i = 1, 2$, but $F = F_1 * F_2 \in L(\gamma) \setminus S(\gamma)$. Then for all $0 < T < \infty$, $F_\gamma \in (L_{\Delta_T} \cap OS_{\Delta_T}) \setminus S_{\Delta_T}$.

In view of the three distributions in Examples 2.3 and 2.4, the proof of Proposition 2.2 is completed. $\blacksquare$

**Remark 2.1.** We point out that the distribution $F_\gamma \in L_{\Delta_T} \setminus S_{\Delta_T}$ for all $0 < T < \infty$ in Example 2.4 was firstly introduced by Proposition 2.1 of Chen et al. (2013). In addition, there Example 2.3 and Example 2.4 give two new ways to find more distributions in the class $(L \cap OS) \setminus S$ and its subclasses.

### 2.3 The relation between the classes $L \cap OS$ and $L \cap OS^*$.

**Proposition 2.3.** The inclusion relation $L \cap OS^* \subset L \cap OS$ is proper.

**Proof.** We first prove a simple fact that $L \cap OS^* \subset L \cap OS$. Let a distribution $V \in L \cap OS^*$, then
\[
\int_0^x \overline{V}(x-y)dV(y) \leq \sum_{k=0}^x \int_k^{x+1} \overline{V}(x-y)dV(y)
\]
\[
\leq \sum_{k=0}^x \overline{V}(x-k-1)\left(\overline{V}(k) - \overline{V}(k+1)\right)
\]
\[
\leq \sum_{k=0}^x \overline{V}(k)(\overline{V}(k+2))^{-1}\int_k^{x+2} \overline{V}(x-y)\overline{V}(y)dy
\]
\[
= O \left(\left(\int_0^x + \int_x^{x+2}\right) \overline{V}(x-y)\overline{V}(y)dy\right)
\]
\[
= O(\overline{V}(x)),
\]
thus $V \in OS$. 


Next, we prove that the above inclusion relation is proper by using the following example. First, we recall a distribution in the distribution class \( L \cap OS \setminus S \), which was found by Lin and Wang (2012).

**Example 2.5.** Let \( x_1 > 1 \) be any given number, and let \( x_{n+1} = (2x_n)^2, n \geq 1 \). For any \( \alpha \in (0, 1) \), define

\[
\overline{F}(x) = 1(x < 0) + (x_1^{-1}(x_1^{-\alpha} - 1)x + 1)1(0 \leq x < x_1) \\
+ \sum_{n=1}^{\infty} ((x_n^{-\alpha} + (2^{-2\alpha}x_n^{-2\alpha-1} - x_n^{-\alpha-1})(x - x_n))1(x_n \leq x < 2x_n) \\
+ (2x_n)^{-2\alpha}1(2x_n \leq x < x_{n+1})).
\] (2.3)

For any positive number \( m \in (\alpha^{-1}, 2\alpha^{-1}) \), let \( G_m(x) = \overline{F}^m(x), x \in (-\infty, \infty) \). It is obvious that the distribution \( G_m \) has a finite mean. Lin and Wang (2012) has proved that \( G_m \in (L \cap OS) \setminus S \). Further, we show that \( G_m \notin OS^*, G_m^I \in (L \cap OS) \setminus S \) and \( G_m^I \in L_{\Delta T} \setminus OS_{\Delta T} \) for all \( 0 < T < \infty \).

Thus the proposition is proved.

**Remark 2.2.** Through the proof of Proposition 2.3, we find that the condition \( V \in L \cap OS^* \) can be replaced by the weaker conditions that \( V \in OS^* \) and \( \overline{V}(k) = O(\overline{V}(k+1)) \) for all \( k \geq 1 \).

### 3 Local asymptotic estimations

In this section, we try to deliver local asymptotic estimations for the supremum of a random walk, where the distributions of the summands of the random walk belong to the class \( L \cap OS^* \), or equivalently, the integrated tail distributions of the summands belong to the class \( L_{\Delta T} \cap OS_{\Delta T} \) for any \( 0 < T < \infty \), see Lemma 3.3 below. This explains that distributions from the classes \( L \cap OS^* \) and \( L_{\Delta T} \cap OS_{\Delta T} \) possesses good properties, thus they have important value in applications.

In the following, we first introduce some concepts of a random walk and the main result of this paper. Then we give some lemmas in the second subsection. The proof of the main result will be presented at last.

#### 3.1 Related concepts and main result

Let \( \{X_i : i \geq 1\} \) be a sequence of independent, identically distributed r.v.s with a common non-degenerate distribution \( F \) supported on \(( -\infty, \infty) \). Denote the random walk by \( \{S_n =: \sum_{i=1}^{n} X_i : n \geq 0\} \), where \( S_0 = 0 \), and the supremum of the random walk by \( M =: \sup_{n \geq 0} S_n \) with a distribution \( W \) supported on \([0, \infty) \). Assume that

\[-\infty < -\mu =: EX_1 < 0,\]

then we know that \( S_n \) drifts to \(-\infty\) and \( W \) is a proper distribution.

Further, let \( \tau_+ =: \inf \{n \geq 1 : S_n > 0\} \) be the first ascending ladder-epoch and \( S_{\tau_+} \) the first ascending ladder height with a defective distribution \( F_+ \) supported on \([0, \infty) \), i.e.,
0 < p =: F_+ (∞) < 1. Denote \( G(x) =: p^{-1}F_+(x), x \in (-\infty, \infty) \), then \( G \) is a proper distribution supported on \([0, \infty)\). It is well known that, for any \( 0 < T \leq \infty \) and \( x \geq 0 \),

\[
W(x + \Delta_T) = (1 - p) \sum_{n=1}^{\infty} p^n G^n(x + \Delta_T).
\] (3.1)

For the random walk \( \{S_n : n \geq 0\} \), if \( F \in \mathcal{S}^* \), then Asmussen et al. (2002) delivered a local asymptotic estimation for \( W(x + \Delta_T) \) that

\[
W(x + \Delta_T) \sim \mu^{-1}T F(x),
\] (3.2)

for any \( 0 < T < \infty \). Naturally, one hopes to know that if \( F \in \mathcal{L} \cap \mathcal{OS}^* \), then how to estimate \( W(x + \Delta_T) \)? For this new stochastic model with long-tailed and generalized strong subexponential distribution, our answer is as follows.

**Theorem 3.1.** For the random walk \( \{S_n : n \geq 0\} \), if \( F \in \mathcal{L} \), then for any \( 0 < T < \infty \),

\[
\liminf \frac{W(x + \Delta_T)}{F(x)} = \mu^{-1}T.
\] (3.3)

Further, if \( F \in \mathcal{L} \cap \mathcal{OS}^* \) and

\[
C^\circ (F) < \mu + 2EX_1^+,
\] (3.4)

then for any \( 0 < T < \infty \),

\[
\limsup \frac{W(x + \Delta_T)}{F(x)} \leq \mu^{-1}T \left( 1 - \mu^{-1}(C^\circ (F) - 2EX_1^+) \right)^{-1},
\] (3.5)

and \( W \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T} \). In particular, if \( C^\circ (F) = 2EX_1^+ \), namely \( F \in \mathcal{S}^* \), then (3.4) holds. In addition, if \( F \in (\mathcal{L} \cap \mathcal{OS}^*) \setminus \mathcal{S}^* \), then \( W \in (\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T} \).

Here we remark that the theorem gives also us a new way to find more distributions in the class \((\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T}\) for any \( 0 < T < \infty \). Of course, we are more concerned about the condition (3.4). First, if the condition (3.4) is not satisfied, then the number \( \limsup \frac{W(x + \Delta_T)}{F(x)} \) can not have a upper bound as \( \mu^{-1}T \left( 1 - \mu^{-1}(C^\circ (F) - 2EX_1^+) \right)^{-1} \). In fact, if \( C^\circ (F) > \mu + 2EX_1^+ \), then (3.5) is not holds obviously. If \( C^\circ (F) = \mu + 2EX_1^+ \) and (3.5) holds for any \( 0 < T < \infty \), then by (3.3), (3.2) holds. Thus, according to Theorem 2(b) of Foss and Zachary (2003), we know that \( F \in \mathcal{S}^* \), which is contradictory to the fact that \( C^\circ (F) > 2EX_1^+ \). Secondly, we give the following results to illustrate the condition (3.4) from another angle.

**Proposition 3.4.** There exist a long-tailed and generalized strong subexponential distribution \( F \) supported on \((-\infty, \infty)\) satisfying the condition (3.4).
3.2 Some lemmas

In this section, we prepare more lemmas on the local distributions, which will be used in the proof of Theorem 3.1 and also have their own independent value.

First, we recall a known fact. If a distribution \( V \in \mathcal{L}_{\Delta T} \) for some \( 0 < T \leq \infty \), then

\[
\mathcal{H}_{\Delta T}(V) =: \{ h \text{ on } [0, \infty) : h(x) \uparrow \infty, h(x) = o(x) \text{ and } V(x + t + \Delta T) \sim V(x + \Delta T) \text{ holds uniformly for all } |t| \leq h(x) \} \neq \emptyset.
\]

And if \( h \in \mathcal{H}_{\Delta T}(V) \) and \( h(x) \geq h_1(x) \uparrow \infty \), then \( h_1 \in \mathcal{H}_{\Delta T}(V) \) too. Particularly, when \( T = \infty \), we denote \( \mathcal{H}_{\infty}(V) \) by \( \mathcal{H}(V) \).

**Lemma 3.1.** (i) If \( V \in \mathcal{L}_{\Delta T} \) for some \( 0 < T \leq \infty \), and for some \( h \in \mathcal{H}_{\Delta T}(V) \),

\[
\int_{h(x)}^{x-h(x)} V(x - y + \Delta T)dV(y) = O(V(x + \Delta T)), \quad (3.6)
\]

then \( V \in \mathcal{OS}_{\Delta T} \).

(ii) If \( V \in \mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T} \), then for all \( h \in \mathcal{H}_{\Delta T}(V) \), \((3.6)\) holds and

\[
\limsup \int_{h(x)}^{x-h(x)} \frac{V(x - y + \Delta T)}{V(x + \Delta T)}dV(y) = C_T^*(V) - 2. \quad (3.7)
\]

**Proof.** By \( V \in \mathcal{L}_{\Delta T} \) and a standard method, we have

\[
V^{*2}(x + \Delta T) \sim 2V(x + \Delta T) + \int_{h(x)}^{x-h(x)} V(x - y + \Delta T)dV(y), \quad (3.8)
\]

thus \( V \in \mathcal{OS}_{\Delta T} \) follows immediately from \((3.8)\) and \((3.6)\).

On the other hand, if \( V \in \mathcal{OS}_{\Delta T} \), then \((3.6)\) and \((3.7)\) follow directly from \((3.8)\). \( \square \)

**Lemma 3.2.** If \( V \in \mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T} \) for some \( 0 < T \leq \infty \), then for all \( n \geq 1 \),

\[
\limsup \frac{V^{*n}(x + \Delta T)}{V(x + \Delta T)} \leq \sum_{k=0}^{n-1} (C_T^*(V) - 1)^{n-1-k}. \quad (3.9)
\]

**Proof.** Apparently, \((3.9)\) holds for \( n = 1, 2 \). We assume that \((3.9)\) holds for \( n = m \) and aim to show that it holds for \( n = m + 1 \) too. For \( n \geq 1 \), denote

\[
C_n^T(V) =: \limsup \frac{V^{*n}(x + \Delta T)}{V(x + \Delta T)}.
\]

For any \( h \in \mathcal{H}_{\Delta T}(V) \cap \mathcal{H}_{\Delta T}(V^{*m}) \), by a standard method, we obtain

\[
V^{*(m+1)}(x + \Delta T) \sim V(x + \Delta T) + V^{*m}(x + \Delta T) + I(x), \quad (3.10)
\]

where

\[
I(x) \leq \int_{h(x)-T}^{x-h(x)+T} V^{*m}(x - y + \Delta T)dV(y) \\
\lesssim C_m^T(V) \int_{h(x)-T}^{x-h(x)+T} V(x - y + \Delta T)dV(y). \quad (3.11)
\]
It follows from (3.9)-(3.11) and Lemma 3.1 that
\[ C_{m+1}^T(V) \leq C_m^T(C^T(V) - 1) + 1 \leq \sum_{k=0}^{m} (C^T(V) - 1)^{m-k}, \]
namely (3.9) holds for \( n = m + 1 \).

**Lemma 3.3.** Let \( V \) be a proper distribution on \((-\infty, \infty)\). If \( V \in \mathcal{L}_{\Delta_T} \cap OS_{\Delta_T} \) for some \( 0 < T \leq \infty \), then for arbitrary \( \varepsilon > 0 \), there exists \( x_1 > 0 \) and \( K = K(\varepsilon, x_1) > 0 \) such that for all \( x \geq x_1 \) and \( n \geq 1 \),
\[ V^n(x + \Delta) \leq K(C^T(V) - 1 + \varepsilon)^n V(x + \Delta_T). \] \hspace{1cm} (3.12)

**Proof.** For any \( h \in \mathcal{H}_{\Delta_T}(V) \), by the condition \( V \in \mathcal{L}_{\Delta_T} \cap OS_{\Delta_T} \), we know that for any \( \varepsilon \in (0, 1) \), there exists a positive number \( x_1 \) large enough such that when \( x \geq x_1 \), \( h(x) > 2T \),
\[ \left| \frac{V(x - y + \Delta_T)}{V(x + \Delta_T)} - 1 \right| < \frac{1}{8} \varepsilon \quad \text{uniformly for} \quad |y| \leq h(x) \] \hspace{1cm} (3.13)
and
\[ \int_{h(x) - T}^{x - h(x) + T} V(x - y + \Delta_T) dV(y) < \left( C^T(V) - 2 + \frac{1}{8} \varepsilon \right) V(x + \Delta_T). \] \hspace{1cm} (3.14)

We now prove the lemma by induction. When \( n = 1 \), (3.12) is obvious. Assume that (3.12) holds for a fixed integer \( n \geq 1 \), we now show that it holds for \( n + 1 \). Denote
\[ A_n =: \sup_{x \geq x_1} \frac{V^n(x + \Delta_T)}{V(x + \Delta_T)}, \quad n \geq 1. \]

We have
\[ V^{n+1}(x + \Delta_T) = \int_{0}^{h(x)} V^n(x - y + \Delta_T) dV(y) + \int_{0}^{h(x)} V(x - y + \Delta_T) dV^n(y) + P(S_{n+1} \in x + \Delta_T, S_n > h(x), X_{n+1} > h(x)) =: I_1(x) + I_2(x) + I_3(x). \] \hspace{1cm} (3.15)

When \( x \geq x_1 \), by (3.13), we have
\[ \frac{I_1(x)}{V(x + \Delta_T)} = \int_{0}^{h(x)} \frac{V^n(x - y + \Delta_T)}{V(x + \Delta_T)} dV(y) \leq A_n \int_{0}^{h(x)} \frac{V(x - y + \Delta_T)}{V(x + \Delta_T)} dV(y) \leq A_n \left( 1 + \frac{1}{8} \varepsilon \right). \] \hspace{1cm} (3.16)
Similarly, we have
\[
\frac{I_2(x)}{V(x + \Delta_T)} \leq 1 + \frac{1}{8}\varepsilon.
\]
(3.17)

Finally, by (3.14),
\[
\frac{I_3(x)}{V(x + \Delta_T)} \leq \frac{P(S_{n+1} \in x + \Delta_T, h(x) - T \leq X_{n+1} < x - h(x) + T)}{V(x + \Delta_T)}
\leq \int_{h(x) - T}^{x-h(x)+T} \frac{V(x - y + \Delta_T)}{V(x + \Delta_T)} dV(y)
\leq A_n \int_{h(x) - T}^{x-h(x)+T} \frac{V(x - y + \Delta_T)}{V(x + \Delta_T)} dV(y)
\leq A_n \left(C^T(V) - 2 + \frac{1}{8}\varepsilon\right).
\]
(3.18)

So when \(x \geq x_1\), it follows from (3.15)-(3.18) that
\[
A_{n+1} \leq \sup_{x \geq x_1} \frac{V^{\ast n}(x + \Delta_T)}{V(x + \Delta_T)}
\leq 1 + \frac{1}{8}\varepsilon + A_n \left(C^T(V) - 1 + \frac{1}{4}\varepsilon\right).
\]
(3.19)

Taking \(K = K(\varepsilon) =: \frac{8}{3\varepsilon}\) and using (3.19), we get
\[
A_{n+1} \leq 2 + K \left(C^T(V) - 1 + \frac{3}{4}\varepsilon\right)
\leq K \left(C^T(V) - 1 + \varepsilon\right)^{n+1}.
\]

This completes the proof of Lemma 3.3. \(\square\)

For a r.v. \(\xi\) with a distribution \(V\) supported on \([0, \infty)\) and a positive and finite mean \(E\xi\), denote
\[
V_1(x) =: \min\{1, V^T(x)E\xi\}; \ x \in (-\infty, \infty),
\]
which may be a defective distribution.

Lemma 3.4. Let two distributions \(V\) and \(V_1\) be described as above. If \(V \in \mathcal{L}\), then \(V_1 \in \mathcal{L}_{\Delta_T}\) for all \(0 < T < \infty\); on the contrary, if \(V_1 \in \mathcal{L}_{\Delta_T}\) for some \(0 < T < \infty\), then \(V \in \mathcal{L}\). And both are able to derive the following asymptotic equivalence formula:
\[
V_1(x + \Delta_T) \sim V(x)T.
\]
(3.20)

Proof. Since the former is an obvious fact, so we need only to prove the latter. From the following fact that for any \(y > 0\) and \(x\) large enough,
\[
\overline{V}(x - y)T \leq V_1(x - y - T + \Delta_T) \sim V_1(x + \Delta_T) \leq \overline{V}(x)T,
\]
we know that \(V \in \mathcal{L}\) and (3.20) holds. \(\square\)
In the following, we give some new versions of Pitman’s theorem. When $T = \infty$, the result is due to Pitman (1980). To this end, we first recall a known fact. Yu and Wang (2013) pointed out that, if a distribution $V \in \mathcal{L} \cap \mathcal{OS}$, then for any $h \in \mathcal{H}(V)$,

$$\limsup_{h(x)} \int_{h(x)}^{x-h(x)} \frac{V(x-y)}{V(x)} dV(y) = C^*(V) - 2.$$  

Similarly, if a distribution $V \in \mathcal{L}_T \cap \mathcal{OS}_T$, for some $0 < T \leq \infty$, then for any $h \in \mathcal{H}_T(V)$,

$$\limsup_{h(x)} \int_{h(x)}^{x-h(x)} \frac{V(x-y+\Delta_T)}{V(x+\Delta_T)} dV(y) = C^T(V) - 2. \tag{3.21}$$

And if a distribution $V \in \mathcal{L} \cap \mathcal{OS}^*$, then for any $h \in \mathcal{H}(V)$,

$$\limsup_{h(x)} \int_{h(x)}^{x-h(x)} \frac{V(x-y)\overline{V}(y)}{V(x)} dy = C^\otimes(V) - 2E\xi. \tag{3.22}$$

**Lemma 3.5.** Let $V$ and $U$ be two distributions, then the following assertions hold.

(i) If $V \in \mathcal{L}_T \cap \mathcal{OS}_T$, for some $0 < T \leq \infty$ and there exist two constants $c_1$ and $c_2$ such that

$$0 < c_1 = \liminf_{x} \frac{U(x+\Delta_T)}{V(x+\Delta_T)} \leq \limsup_{x} \frac{U(x+\Delta_T)}{V(x+\Delta_T)} = c_2 < \infty, \tag{3.23}$$

then $U \in \mathcal{OS}_T$ and

$$C^T(U) - 2c_1^{-1}c_2 \leq c_1^{-1}c_2(C^T(V) - 2). \tag{3.24}$$

Particularly, if $c_1 = c_2 = c_0$, then $U \in \mathcal{L}_T \cap \mathcal{OS}_T$ and

$$C^T(U) - 2 = c_0(C^T(V) - 2). \tag{3.25}$$

(ii) If $V \in \mathcal{L} \cap \mathcal{OS}^*$ and there exist two constants $c_1$ and $c_2$ such that

$$-\infty < c_1 = \liminf_{x} \frac{U(x+\Delta_T)}{V(x)} \leq \limsup_{x} \frac{U(x+\Delta_T)}{V(x)} = c_2 < \infty, \tag{3.26}$$

for some $0 < T < \infty$, then $U \in \mathcal{OS}_T$ and

$$\frac{c_1^2}{c_2T}(C^\otimes(V) - 2E\xi) \leq C_T(U) - 2 \leq C^T(U) - 2 \leq \frac{c_2^2}{c_1T}(C^\otimes(V) - 2E\xi). \tag{3.27}$$

Particularly, if $c_1 = c_2 = c_0$, then both are able to derive the following equations:

$$C^T(U) - 2 = \frac{c_0}{T}(C^\otimes(V) - 2E\xi) \text{ and } C_T(U) - 2 = \frac{c_0}{T}(C^\otimes(V) - 2E\xi). \tag{3.28}$$

Thus, $V \in \mathcal{L} \cap \mathcal{OS}^*$ if and only if $U \in \mathcal{L}_T \cap \mathcal{OS}_T$ for all $0 < T < \infty$. Further, we have

$$C^T(V_1) - 2 = C^\otimes(V) - 2E\xi \text{ and } C_T(V_1) - 2 = \frac{c_0}{T}(C^\otimes(V) - 2E\xi). \tag{3.29}$$
Proof. (i) In (3.15), we take $n = 1$ and $V = U$, then for any $h \in \mathcal{H}_{\Delta T}(V)$,

$$
\limsup \frac{I_i(x)}{U(x + \Delta T)} \leq c_1^{-1}c_2, \ i = 1, 2.
$$

(3.30)

Let r.v.s $\xi_1$ and $\xi_2$ have distributions $V$ and $U$ respectively. Then by (3.23), we have

$$
\limsup \frac{I_3(x)}{U(x + \Delta T)} \leq c_2 \limsup \int_{x-h(x)-T}^{x-h(x)+T} \frac{V(x-y + \Delta T)}{U(x + \Delta T)} dU(y)
$$

$$
= c_2 \limsup \frac{P(\xi_1 + \xi_2 \in x + \Delta T, h(x) - T < \xi_2 \leq x - h(x) + T)}{U(x + \Delta T)}
$$

$$
\leq c_2 \limsup \frac{P(\xi_1 + \xi_2 \in x + \Delta T, h(x) - 2T < \xi_1 \leq x - h(x) + 2T)}{U(x + \Delta T)}
$$

$$
= c_2 \limsup \int_{x-h(x)-2T}^{x-h(x)+2T} \frac{U(x-y + \Delta T)}{U(x + \Delta T)} dV(y)
$$

$$
\leq c_1^{-1}c_2 \limsup \int_{x-h(x)-2T}^{x-h(x)+2T} \frac{V(x-y + \Delta T)}{V(x + \Delta T)} dV(y)
$$

$$
= c_1^{-1}c_2(C^T(V) - 2).
$$

Thus, by (3.8) of Lemma 3.1, (3.30) and (3.21), we know that $U \in \mathcal{OS}_{\Delta T}$ and (3.24) holds.

Particularly, if $c_1 = c_2 = c_0$, then

$$
C^T(V) - 2 \leq c_0^{-1}(C^T(U) - 2).
$$

Combining the above inequality and (3.24) yields the equality (3.25).

(ii) When $V \in \mathcal{L} \cap \mathcal{OS}^*$, similarly to (i), for some $h \in \mathcal{H}_{\Delta T}(U)$, we have

$$
\limsup \frac{I_i(x)}{U(x + \Delta T)} \leq c_1^{-1}c_2, \ i = 1, 2.
$$

(3.31)

For $I_3(x)$, without loss of generality, we may assume that $l_1(x) := (x - 2h_1(x))T^{-1}$ is an integer for $x$ large enough, where $h_1(x) = h(x) - T$. Thus, by (3.23), we have

$$
\limsup \frac{I_3(x)}{U(x + \Delta T)} \leq c_2 \limsup \int_{h_1(x)}^{x-h_1(x)} \frac{V(x-y)}{U(x + \Delta T)} dU(y)
$$

$$
= c_2 \limsup \sum_{k=1}^{l_1(x)} \int_{h_1(x) + (k-1)T}^{h_1(x) + kT} \frac{V(x-y)}{U(x + \Delta T)} dU(y)
$$

$$
\leq c_1^{-1}c_2 \sum_{k=1}^{l_1(x)} \frac{V(x-h_1(x) - kT)V(h_1(x) + (k-1)T)}{V(x)}
$$

$$
= c_1^{-1}c_2 T^{-1} \limsup \int_{h_1(x)}^{x-h_1(x)} \frac{V(x-y)V(y)}{V(x)} dy
$$

$$
= c_1^{-1}c_2 T^{-1}(C^\circ(V) - 2E\xi).
Thus, \( U \in \mathcal{OS}_{\Delta T} \).

On the other hand, we have

\[
\liminf \frac{I_i(x)}{U(x + \Delta_T)} \geq c_1c_2^{-1}, \quad i = 1, 2.
\] (3.32)

And without loss of generality, we may assume that \( l(x) = (x - 2h(x))T^{-1} \) is an integer for \( x \) large enough, then

\[
\liminf \frac{I_3(x)}{U(x + \Delta_T)} \geq c_1 \liminf \frac{\int_{h(x)}^{x-h(x)} \nabla(x-y) dU(y)}{U(x + \Delta_T)}
\]

\[
= c_1 \liminf \sum_{k=1}^{l(x)} \frac{\int_{h(x)+(k-1)T}^{h(x)+kT} \nabla(x-y) dU(y)}{U(x + \Delta_T)}
\]

\[
\geq c_1^2c_2^{-1} \liminf \sum_{k=1}^{l(x)} \frac{\nabla(x-h(x)-(k-1)T)\nabla(h_1(x)+(k-1)T)}{\nabla(x)}
\]

\[
= c_1^2c_2^{-1}T^{-1} \liminf \int_{h(x)}^{x-h(x)} \frac{\nabla(x-y)\nabla(y)}{\nabla(x)} dy
\]

\[
= c_1^2c_2^{-1}T^{-1}(C_\gamma - 2E\xi).
\]

By the above four inequalities, we know that (3.27) holds.

From these results, the final result can be obtained immediately. \( \square \)

Now, we discuss the relationship between a distribution \( V \) and its \( \gamma \)-transformation \( V_\gamma \).

**Lemma 3.6.** For some \( \gamma > 0 \), \( V \in \mathcal{L}(\gamma) \cap \mathcal{OS} \) if and only if \( V_\gamma \in \mathcal{L}_{\Delta T} \cap \mathcal{OS}_{\Delta T} \) for any \( 0 < T < \infty \). And both of them are able to derive the following asymptotic equivalence formula:

\[
C^T(V_\gamma) - 2 = (M_\gamma(V))^{-1} \gamma(C_\gamma(V) - 2E\xi) = (M_\gamma(V))^{-1}C^*(V) - 2.
\] (3.33)

**Proof.** According to Proposition 2.1 of Wang and Wang (2011), we know that \( V \in \mathcal{L}(\gamma) \) for some \( \gamma > 0 \) if and only if \( V_\gamma \in \mathcal{L}_{\Delta T} \) for any \( 0 < T < \infty \). And both of them are able to derive the following asymptotic equivalence formula:

\[
V_\gamma(x + \Delta_T) \sim (M_\gamma(V))^{-1} \gamma T e^{\gamma x} \nabla(x).
\] (3.34)

Select \( h \) and \( l \) as in the proof of Lemma 3.5. By (3.34), we know that

\[
\int_{h(x)}^{x-h(x)} \frac{\nabla(x-y)\nabla(y)}{\nabla(x)} dy \sim \frac{M_\gamma(V)}{\gamma T} \sum_{k=1}^{l(x)} \int_{h(x)+(k-1)T}^{h(x)+kT} \frac{V_\gamma(x-y+\Delta_T) V_\gamma(y+\Delta_T)}{V_\gamma(x+\Delta_T)} dy
\]

\[
\sim \frac{M_\gamma(V)}{\gamma} \sum_{k=1}^{l(x)} \frac{V_\gamma(x-h(x)-kT+\Delta_T) V_\gamma(h(x)+kT+\Delta_T)}{V_\gamma(x+\Delta_T)}
\]

\[
\sim M_\gamma(V) \gamma^{-1} \int_{h(x)}^{x-h(x)} \frac{V_\gamma(x-y+\Delta_T)}{V_\gamma(x+\Delta_T)} dV_\gamma(y)
\]

\[
= \gamma^{-1} \int_{h(x)}^{x-h(x)} \frac{\nabla(x-y)}{\nabla(x)} dV(y).
\]
Thus, $V \in OS$ if and only if $V_{\gamma} \in OS_{\Delta T}$ for any $0 < T < \infty$, and both of them are able to derive the (3.33) holds. \hfill \Box

Finally, we introduce an existing result which is due to Corollary 2.1 of Yu and Wang (2013).

**Lemma 3.7.** Let $V \in L(\gamma) \cap OS$ for some $\gamma \geq 0$ and $U = \sum_{n=0}^{\infty} p_n V^* n$, where $\{p_n : n \geq 0\}$ is a sequence of nonnegative numbers satisfying $\sum_{n=0}^{\infty} p_n = 1$. Suppose that there exists some $\varepsilon_0 > 0$ such that
\[
\sum_{n=0}^{\infty} p_n (C_V^* - \hat{V}(\gamma) + \varepsilon_0)^n < \infty ,
\] (3.35)
then $U \in L(\gamma) \cap OS$.

### 3.3 Proofs of Theorem 3.1 and Proposition 3.4

In this section, we prove Theorem 3.1 and Proposition 3.4, respectively.

**Proof of Theorem 3.1** We first prove (3.3). By $F \in L$, Corollary 3.1 of Wang and Wang (2006) and Lemma 3.4, we know that
\[
G(x + \Delta T) \sim (1 - p)p^{-1}\mu^{-1}F_1(x + \Delta T) \sim (1 - p)p^{-1}\mu^{-1}T F(x) ,
\] (3.36)
hence $G \in L_{\Delta T}$. Thus by Corollary 1 of Asmussen et al. (2003) and Theorem 3.1 of Yu et al. (2010), we know that
\[
\lim \inf \frac{W(x + \Delta T)}{G(x + \Delta T)} = p(1 - p)^{-1} .
\] (3.37)

So, (3.3) follows from (3.36) and (3.37).

Next, we prove (3.5). We know from (3.36), $F \in L \cap OS^*$, Lemmas 3.4 and 3.5 that $F_1 \in \mathcal{L}_{\Delta T} \cap OS_{\Delta T}$ and $G \in \mathcal{L}_{\Delta T} \cap OS_{\Delta T}$ for all $0 < T < \infty$. By condition (3.4) and Lemma 3.5, we have
\[
p(C_T^*(G) - 1) = p((1 - p)p^{-1}\mu^{-1}(C^\otimes(F) - 2) + 1) < 1 .
\] (3.38)

It follows from (3.1), (3.38), Lemmas 3.3 and 3.2 and the dominated convergence theorem that
\[
\limsup \frac{W(x + \Delta T)}{G(x + \Delta T)} \leq (1 - p) \sum_{n=1}^{\infty} p_n \limsup \frac{G^{*n}(x + \Delta T)}{G(x + \Delta T)} \\
\leq (1 - p) \sum_{n=1}^{\infty} p_n \sum_{k=0}^{n-1} (C_T^*(G) - 1)^{n-1-k} \\
= (1 - p) \sum_{n=1}^{\infty} p_n \frac{(C_T^*(G) - 1)^n - 1}{C_T^*(G) - 2} ,
\] (3.39)
where, if $C^T(G) = 2$, then we define $\frac{(C^T(G) - 1)^n - 1}{C^T(G) - 2} = n$ by continuity. By (3.38) and (3.39), we obtain that

$$\limsup \frac{W(x + \Delta T)}{G(x + \Delta T)} \leq (1 - p)(C^T(G) - 2)^{-1} \left( \frac{p(C^T(G) - 1)}{1 - p(C^T(G) - 1)} - \frac{p}{1 - p} \right) = p(1 - p(C^T(G) - 1))^{-1},$$

thus (3.35) follows from (3.36), (3.38) and (3.29).

Now, we show that $W \in L_{\Delta_T} \cap OS_{\Delta_T}$ for any $0 < T < \infty$. By (3.3), (3.4) and Lemma 3.5, we immediately get $W \in OS_{\Delta_T}$. Next, we prove $W \in L_{\Delta_T}$ for any $0 < T < \infty$. To this end, we denote the $-\gamma$-transform of $W$ and $G$ by $U = W_{-\gamma}$ and $V = G_{-\gamma}$, respectively. From (3.1), we know that

$$M_{-\gamma}(W) = (1 - p)(1 - p_1)^{-1} > 1, \quad (3.40)$$

thus by (3.40), we have

$$M_{\gamma}(V) = (M_{-\gamma}(G))^{-1} < 1 \quad (3.41)$$

and

$$U(x) = (1 - p_1) \sum_{n=1}^{\infty} p^n V^{*n}(x), \quad (3.42)$$

where $0 < p_1 = pM_{-\gamma}(G) < 1$ for some $\gamma$ large enough. By condition (3.36), Lemma 3.5 (ii) and (3.4), we have

$$C^T(G) - 2 = (1 - p)p^{-1}\mu^{-1}(C^\otimes(F) - 2EX^*_1) < (1 - p)p^{-1}. \quad (3.43)$$

From (3.43) and Lemma 3.6, we know that

$$p_1(C^\otimes V - M_{\gamma}(V)) < 1.$$

Thus, by (3.42) and Lemma 3.7, $W_{-\gamma} \in L(\gamma)$. According to Proposition 2.1 of Wang and Wang (2011), we know that $W \in L_{\Delta_T}$ for any $0 < T < \infty$.

Finally, if $F \in (L \cap OS^*) \setminus S^*$, then by Corollary 3.2 of Wang et al. (2007), we have $W \in (L_{\Delta_T} \cap OS_{\Delta_T}) \setminus S_{\Delta_T}$ for any $0 < T < \infty$. □

**Proof of Proposition 3.4.** Let $Y$ be a random variable with distribution $G \in L \cap OS^*$ supported on $(-\infty, \infty)$ and finite mean $EY = -\lambda < 0$. If $C^\otimes(G) < \lambda + 2EY^+$, we take distribution $F = G$ and $\mu = \lambda$, then distribution $F$ satisfy the condition (3.4). Otherwise, if $C^\otimes(G) \geq \mu + 2EY^+$, we set random variable $X = Y - a$ with distribution $F$ and mean $\mu = -EX$ for some $a$ large enough such that $C^\otimes(G) - 2EY^+ < \lambda + a$. It is easy to find that for any $h \in \mathcal{H}(F) = \mathcal{H}(G)$,

$$C^\otimes(F) - 2EX^+ = \limsup_{h(x)} \int_{h(x)}^{x-h(x)} \frac{F(x-y)F(y)dy}{F(x)} = \limsup_{h(x)} \int_{h(x)}^{x-h(x)} \frac{G(x-y)G(y)dy}{G(x)} = C^\otimes(G) - 2EY^+ < \lambda + a = \mu,$$

that is condition (3.4) holds. □
4 Proofs of Examples

In this section, we prove the conclusions in Examples 2.1-2.5, respectively.

**Proof of** $G_m \in (S \cap OS^*) \setminus S^*$ **in Example 2.1.**

It is not hard to see that for all $m \geq 1$ and $x \geq x_1$, one has

\begin{equation}
-x^{-2\alpha + m - 1} \leq F(x) \leq 2^\alpha x^{-\alpha}.
\end{equation}

Since $\alpha \in (m^{-1}, 1 + m^{-1})$, by (2.1), we know that the distribution $G_m$ has a finite mean $m(G_m) =: \mu$. Denote

$f(x) = \sum_{n=1}^{\infty} (x^{n-\alpha - 1} - x^{n-2\alpha + m - 1}) 1(x_n < x < 2x_n)$.

We first prove that $G_m \notin S^*$. To this end, for all $x \geq 0$, denote

\[ H(x) = (G_m(x))^{-1} \int_0^x G_m(x - y)G_m(y)dy = 2(G_m(x))^{-1} \int_{\frac{x}{2}}^x G_m(x - y)G_m(y)dy. \]

By (2.1), one has

\begin{align*}
H(2x_n) &= 2(F_m(2x_n))^{-1} \int_{x_n}^{2x_n} F_m(2x_n - y)F_m(y)dy \\
&= 2 \int_0^{x_n} F_m(y)(1 + (x_n^{\alpha - m - 1} - x_n^{-1}))y^m dy \\
&= 2 \int_0^{x_n} F_m(y)dy + 2 \int_0^{x_n} F_m(y) \left(1 + (x_n^{\alpha - m - 1} - x_n^{-1})y^m - 1\right) dy.
\end{align*}

By (4.1) and (2.1),

\[ \lim_{n \to \infty} x_n^{\alpha - m - 1} \int_0^{x_n} F_m(y)y^m dy = \lim_{n \to \infty} x_n^{\alpha - m - 1} \int_{2x_n - 1}^{x_n} F_m(y)y^m dy = (m + 1)^{-1}. \]

And for all $t = 1, \cdots, m - 1$,

\[ \lim_{n \to \infty} x_n^{(\alpha - m - 1)t} \int_0^{x_n} F_m(y)y^t dy = 0. \]

By (4.3)-(4.5),

\[ \lim_{n \to \infty} H(2x_n) = 2\mu + 2(m + 1)^{-1}, \]

thus $G_m \notin S^*$. 

Next, we prove $G_m \in \mathcal{OS}^*$. We estimate $H(x)$ in the cases $x_n \leq x < \frac{3}{2}x_n$, $\frac{3}{2}x_n \leq x < 2x_n$ and $2x_n \leq x < x_{n+1}$, $n \geq 1$, respectively. When $x \in [x_n, \frac{3}{2}x_n)$, by (4.1) and (2.1),

$$H(x) \leq 2^{m_1 \alpha + 1} \left( \frac{3}{2} x_n \right)^{1 - \alpha} \int_{x_n}^{x} F(x-y) y^{-\alpha} dy$$

$$\leq 2^{m_1 \alpha + 1 + m} \int_0^{x/2} F_m(y) dy \leq 2^{m_1 \alpha + 1 + m} \mu.$$

(4.7)

When $x \in \left[ \frac{3}{2}x_n, 2x_n \right)$, by (2.1), (4.3) and (4.6),

$$H(x) = 2 \left( \frac{F_m(x)}{x} \right)^{-1} \left( \int_{x}^{x_n} + \int_{x_n}^{x} \right) F(x-y) F_m(y) dy$$

$$= 2 \left( \frac{F_m(x)}{x} \right)^{-1} \left( x_n^{-\alpha} \int_{x-x_n}^{x} F_m(y) dy + \int_{x-x_n}^{x} F(x-y) F_m(y) dy \right)$$

$$\leq 2 \left( \frac{F_m(2x_n)}{x} \right)^{-1} x_n^{-\alpha} \int_{x-x_n}^{x} F_m(y) dy$$

$$+ 2 \int_0^{x-x_n} F_m(y) \left( 1 + \left( x_n \left( 1 - x_n^{-\alpha-1} \right) - (x-x_n)^{-1} \right)^{m} \right) dy$$

$$\leq 1 + 2 \int_0^{x-x_n} F_m(y) \left( 1 + \left( x_n \left( 1 - x_n^{-\alpha-1} \right) - x_n^{-1} \right)^{m} \right) dy$$

$$= 1 + H(2x_n)$$

$$\to 1 + 2\mu + 2(m + 1)^{-1}, \quad n \to \infty.$$

(4.8)

When $x \in [2x_n, x_{n+1})$, by (2.1) and (4.3),

$$H(x) \leq 2 \left( \frac{F_m(x)}{x} \right)^{-1} \left( \int_{x}^{x_{2x_n}} + \int_{x_{2x_n}}^{x} \right) F(x-y) F_m(y) dy$$

$$\leq H(2x_n) + 2 \int_0^{x-2x_n} F_m(y) dy$$

$$\to 4\mu + 2(m + 1)^{-1}, \quad n \to \infty.$$

(4.9)

It follows from (4.7)-(4.9) that $G_m \in \mathcal{OS}^*$.

Finally, we prove $G_m \in \mathcal{S}$. By (4.1), we have

$$\bar{F}_m^{2m} \left( \frac{x}{2} \right) \left( F_m(x) \right)^{-1} \leq 2^{4m_2 x} \to 0.$$

(4.10)

By (4.10) and

$$\bar{G}_m(x) = 2G_m(x) - \bar{G}_m^{2} \left( \frac{x}{2} \right) + 2 \int_{x}^{x} \bar{G}_m(x-y) dG_m(y),$$
we know that in order to prove \( G_m \in \mathcal{S} \), it suffices to prove

\[
T(x) = 2(F^m(x))^{-1} \int_x^\infty F^n(x-y)d\left(1-F^m(x)\right)
\]

\[
= 2m(F^m(x))^{-1} \int_x^\infty F^n(x-y)F^{m-1}(y)f(y)dy \to 0.
\quad (4.11)
\]

Clearly, \( T(x_n) = 0, n \geq 1 \). By (2.1) and (4.5), we have

\[
T(2x_n) = 2mF^n(2x_n)\int_0^{x_n} F^n(y)F^{m-1}(2x_n-y)f(2x_n-y)dy
\]

\[
= 2m(\alpha^{m-1}_n - x_n^{-1})\int_0^{x_n} F^n(y)\left(1 + (\alpha^{m-1}_n - x_n^{-1})y\right)^{m-1}dy
\]

\[
\leq 2m\alpha^{m-1}_n \int_0^{x_n} F^n(y)\left(1 + \alpha^{m-1}_n y\right)^{m-1}dy
\]

\[
= 2m\alpha^{m-1}_n \left(\int_0^{x_n} F^n(y)dy + \int_0^{x_n} F^n(y)\left(1 + \alpha^{m-1}_n y\right)^{m-1}dy\right)
\to 0, \quad n \to \infty.
\quad (4.12)
\]

In the following, we prove (4.11) in the cases \( x_n \leq x < 2x_n \) and \( 2x_n \leq x < x_{n+1}, n \geq 1 \), respectively. When \( x \in [x_n, 2x_n) \), by (2.1) and (4.12),

\[
T(x) = 2m(F^m(x))^{-1} \int_{x_n}^x F^n(x-y)F^{m-1}(y)f(y)dy
\]

\[
= 2m(F^m(x))^{-1} \int_{x_n}^{x-x_n} F^n(y)F^{m-1}(x-y)f(x-y)dy
\]

\[
\leq 2m(\alpha^{m-1}_n - x_n^{-1}) \int_0^{x-x_n} F^n(y)\left(1 + \left(x_n\left(1 - \alpha^{m-1}_n y\right)^{-1} - (x-x_n)^{-1}\right)\right)^{m-1}dy
\]

\[
\leq T(2x_n) \to 0, \quad n \to \infty.
\quad (4.13)
\]

When \( x \in [2x_n, x_{n+1}) \), by (2.1) and (4.12),

\[
T(x) \leq 2m(F^n(2x_n))^{-1} \int_{x_n}^{2x_n} F(2x_n-y)F^{m-1}(y)f(y)dy
\]

\[
= T(2x_n) \to 0, \quad n \to \infty.
\quad (4.14)
\]

According to (4.13) and (4.14), (4.11) holds, thus \( G_m \in \mathcal{S} \).

In summary, we have \( G_m \in (\mathcal{L} \cap \mathcal{O}^S) \setminus \mathcal{S}^* \).

\[\square\]

**Proof of** \( G_m \in (\mathcal{L} \cap \mathcal{O}^S) \setminus \mathcal{S} \) **in Example 2.2**

Still let \( f \) be the density of \( F \), when \( x \geq x_1 \), it is easily seen that

\[
x^{-\frac{m}{\alpha}} \leq F(x) \leq 2^\alpha x^{-\frac{m}{\alpha}} \quad \text{and} \quad f(x) \leq 2^{\alpha+1} x^{-\frac{m+1}{\alpha}}
\quad (4.15)

Moreover, one can easily find that the distribution \( G_m \) has a finite mean for \( m \geq 1 \), and we still denote it by \( \mu \).
Firstly, by (4.15) we have, 
\[(\overline{F}(x))^{-1}f(x) \leq 2^{\alpha+1}x^{-\frac{1}{2}} \to 0,\]
thus \(F \in \mathcal{L}\), so \(G_m \in \mathcal{L}\). Next, we prove that \(G_m \in \mathcal{OS}^*\). Let \(H(x), x \geq 0\) be the same as in (4.2), it is easily seen that

\[
H(4x_n^2) = 2(\overline{F}^{\alpha}(4x_n^2))^{-1}\int_0^{2x_n^2} \overline{F}^{\alpha}(y)\overline{F}^{\alpha}(4x_n^2 - y)dy
\]

\[
= 2 \int_0^{2x_n^2} \overline{F}^{\alpha}(y)(1 + (1 - x_n^{-1})(\sqrt{4x_n^2 - y} + 2x_n)^{-1})^m dy
\]

\[
\leq 2 \int_0^{2x_n^2} \overline{F}^{\alpha}(y)(1 + (2x_n)^{-1})^m dy
\]

\[
\leq 2x_n^2(1 + x_n^2)^m + 2^{\alpha+1}(2^{-1}m\alpha - m - 1)^{-1}x_n^{2m-\alpha}(1 + x_1^2)^m
\]

< \infty.

Just as in Example 2.1 we deal with \(H(x)\) in three cases \(x_n^2 \leq x < 2x_n^2, 2x_n^2 \leq x < 4x_n^2\) and \(4x_n^2 \leq x < x_{n+1}^2, n \geq 1\), respectively. When \(x \in [x_n^2, 2x_n^2]\), just as (4.7), by (2.2) and variable substitution, we have

\[
H(x) = 2(\overline{F}^{\alpha}(x))^{-1}\int_{\frac{x}{2}}^{x} \overline{F}^{\alpha}(x - y)\overline{F}^{\alpha}(y)dy
\]

\[
\leq 2(\overline{F}^{\alpha}(2x_n^2))^{-1}\int_{\frac{x}{2}}^{x} \overline{F}^{\alpha}(x - y)x_n^{-\alpha m}dy
\]

\[
\leq 2(2 - \sqrt{2})^{-m}\int_0^{x_n^2} \overline{F}^{\alpha}(y)dy
\]

\[
\leq 2(2 - \sqrt{2})^{-m} < \infty.
\]

(4.16)

When \(x \in [2x_n^2, 4x_n^2]\), just as (4.8), by (2.2), we have

\[
H(x) = 2(\overline{F}^{\alpha}(x))^{-1}\int_0^{\frac{x}{2}} \overline{F}^{\alpha}(y)\overline{F}^{\alpha}(x - y)dy
\]

\[
= 2\int_0^{\frac{x}{2}} \overline{F}^{\alpha}(y)(\overline{F}(x) + (x_n^{-\alpha - 1} - x_n^{-\alpha - 2})(\sqrt{x} + \sqrt{x - y})^{-1})^m dy
\]

\[
\leq 2\int_0^{\frac{x}{2}} \overline{F}^{\alpha}(y)(1 + (x_n^{-\alpha - 1} - x_n^{-\alpha - 2})(\overline{F}(\sqrt{x} + \sqrt{x - y})^{-1})^m dy
\]

\[
\leq 2\int_0^{\frac{x}{2}} \overline{F}^{\alpha}(y)(1 + (\sqrt{2}x_n)^{-1})^m dy
\]

\[
\leq 2x_n^2(1 + (\sqrt{2}x_n)^{-1}x_n^2)^m + 2^{\alpha+1}\int_{\frac{x}{2}}^{\infty} y^{m-\alpha}(\sqrt{2}x_n)^{-1}m dy
\]

\[
\leq 2x_n^2(1 + x_1^2)^m + 2^{\alpha+1}(2^{-1}m\alpha - m - 1)^{-1}x_n^{2m-\alpha}(1 + x_1^2)^m
\]

< \infty.

(4.17)
When \( x \in [4x_n^2, x_{n+1}^2] \), just as (4.9), by (2.2), we have

\[
H(x) \leq 2(F_n^n(x) - 1) \left( \int_{2x_n^2}^{4x_n^2} + \int_{4x_n^2}^{x} \right) F_n^n(x - y) F_n^n(y) dy
\]

\[
\leq H(4x_n^2) + 2 \int_{0}^{x - 4x_n^2} F_n^n(y) dy
\]

\[
\leq H(4x_n^2) + 2\mu < \infty.
\]

(4.18)

According to (4.16) and (4.17), (4.18) holds, that is \( G_m \in \mathcal{OS}^* \).

Finally, we prove \( G_m \notin \mathcal{S} \). Since

\[
\frac{1}{G_m^2} \left( 2x_n^2 \right) (G_m(4x_n^2))^{-1} = x_n^{m-\alpha} \left( (2 - \sqrt{2}) + (\sqrt{2} - 1)x_n^{-1} \right)^{2m}
\]

\[
\leq x_n^{m-\alpha} \rightarrow 0, \quad n \rightarrow \infty.
\]

Therefore, in order to prove \( G_m \notin \mathcal{S} \), we only need to prove

\[
\lim \inf \frac{T(x) = \lim \inf \left( \frac{G_m(x)}{x} \right) G_m(dy) > 0. \quad (4.19)
\]

In fact,

\[
T(4x_n^2) = m(F_n^n(4x_n^2))^{-1} \int_{2x_n^2}^{4x_n^2} F_n^n(4x_n^2 - y) F_n^n(y) f(y) dy
\]

\[
= 2^{-1}m(1 - x_n^{-1}) \int_{0}^{2x_n^2} F_n^n(y) (1 + (1 - x_n^{-1})(\sqrt{4x_n^2 - y} + 2x_n)^{y} - y^{m-1}y^{-2} dy
\]

\[
\geq 2^{-1}m(1 - x_n^{-1}) \int_{x_n^2}^{2x_n^2} F_n^n(y) y^{1/2} dy
\]

\[
\geq m(\alpha + m - 1)^{-1}(1 - x_n^{-1}) \left[ x_1^{1-\alpha-m} - (2x_n^2)^{-\frac{\alpha+m-1}{2}} \right]
\]

\[
\rightarrow m(\alpha + m - 1)^{-1}(1 - x_1^{-1}) \left[ x_1^{1-\alpha-m} > 0, \quad n \rightarrow \infty, \right.
\]

that is, (4.19) holds, that is \( G_m \notin \mathcal{S} \).

In summary, we have \( G_m \in (\mathcal{L} \cap \mathcal{OS}^*) \setminus \mathcal{S} \). \( \square \)

**Proof of \( G_m^I \in (\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T} \) in Example 2.3**

This conclusion follows directly from \( G_m \in (\mathcal{S} \cap \mathcal{OS}^*) \setminus \mathcal{S}^* \), Lemmas 3.4, 3.5 and Lemma 4.2 of Wang et al. (2007). \( \square \)

**Proof of \( G_m^I \in \mathcal{OS}_{\Delta_T} \setminus \mathcal{S}_{\Delta_T} \) for all \( 0 < T < \infty \) in Example 2.4**

By Proposition 2.1 of Chen et al. (2013), we have \( F_\gamma \in \mathcal{L}_{\Delta_T} \setminus \mathcal{S}_{\Delta_T} \) for all \( T > 0 \). We now prove \( F_\gamma \in \mathcal{OS}_{\Delta_T} \). Since \( F_i \in \mathcal{S}(\gamma) \subset \mathcal{OS}, \quad i = 1, 2 \), so \( F = F_1 \ast F_2 \in \mathcal{OS} \) by Proposition 6.1 of Yu and Wang (2013). By \( F \in \mathcal{OS} \) and (2.1) of Chen et al. (2013), we have

\[
\lim \sup F_\gamma(x + \Delta_T)(F_\gamma(x + \Delta_T))^{-1} = (M_\gamma(F))^{-1} \lim \sup F^{\gamma_2}(x)(F(x))^{-1} < \infty,
\]

thus \( F_\gamma \in \mathcal{OS}_{\Delta_T} \). \( \square \)
Proofs of $G_m \notin \mathcal{OS}^*$, $G_m^I \in (\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$ and $G_m^I \in \mathcal{L}_{\Delta T} \setminus \mathcal{OS}_{\Delta T}$ for all $0 < T < \infty$ in Example 2.5.

To show that $G_m \notin \mathcal{OS}^*$, we denote the density of $F$ by $f$. Consider the following quantity

$$I(x) =: \left( \frac{G_m(2x)}{G_m(x)} \right)^{-1} \int_{x_n}^{2x_n-2x_n-1} \frac{G_m(2x_n-y)G_m(y)dy}{G_m(2x_n)}.$$

Since $x_n \leq y \leq 2x_n - 2x_n - 1 \leq 2x_n$, we have $2x_n - 1 \leq 2x_n - y \leq x_n$, so by (2.3),

$$I(x) = 2^{2m \alpha} x_n^{ma} \int_{x_n}^{x_n-2x_n-1} F_m(y)dy \geq 2^{2m \alpha} (m+1)^{-1} x_n^{ma} \left( \sum_{i=0}^{m} x_n^{-\alpha i} (x_n^\alpha - f(x_n)(x_n - 2x_n - 1))^m \right) \rightarrow \infty, \ n \rightarrow \infty,$$

thus $G_m \notin \mathcal{OS}^*$.

Since $G_m \in \mathcal{L} \setminus \mathcal{OS}^*$, we immediately get $G_m^I \in \mathcal{L}_{\Delta T} \setminus \mathcal{OS}_{\Delta T}$ by Lemmas 3.4 and 3.5. Now we show that $G_m^I \notin \mathcal{S}$. By (2.3), it is obvious that

$$x^{-2\alpha} \leq F(x) \leq x^{-\alpha}. \quad (4.20)$$

By (4.20), we have

$$\left( \frac{G_m^I(x_n)}{G_m(2x_n)} \right)^{-1} \geq \mu^{-1} \left( \int_{x_n}^{2x_n} F_m(y)dy \right) \left( \int_{x_n}^{2x_n} F_m(y)dy \right)^{-1} \geq \mu^{-1} (m+1)^{-2} 2^{2m \alpha - 2} (1 - m^{-1} \alpha^{-1}) > 0,$$

thus by (2.4) of Murphree (1989), we complete the proof.

To prove $G_m^I \in \mathcal{OS}$, we consider

$$\int_{\frac{x}{2}}^{x} G_m^I(x - y) dG_m^I(y) = \mu^{-1} \int_{\frac{x}{2}}^{x} F_m(y) \left( \int_{x-y}^{\infty} F_m(z)dz \right) dy \geq \mu^{-1} \left( \int_{\frac{x}{2}}^{x} F_m(y) \left( \int_{x-y}^{\infty} F_m(z)dz \right) dy \right)^2 + \left( \int_{\frac{x}{2}}^{x} F_m(y) \int_{x-y}^{\infty} F_m(z)dz dy \right)^2 \geq \mu^{-1} (I_1(x) + I_2(x) + I_3(x)). \quad (4.21)$$
We first estimate $I_1(x)$ in the cases $x_n \leq x < 4x_n$ and $4x_n \leq x < x_{n+1}$, $n \geq 1$, respectively. When $x \in [x_n, 4x_n)$, by (2.3) and (4.20), we have

$$
\left( \int_{x}^{\infty} F^m(y)dy \right)^{-1} I_1(x) \leq \left( \int_{4x_n}^{x_{n+1}} F^m(y)dy \right)^{-1} \int_{x_n}^{x_{n+1}} \frac{x}{y} F^m(z)dz \int_{x-y}^{x} \frac{z}{y} F^m(z)dz dy
$$

\begin{align*}
&\leq 2^{2m-2}(x_n^{2m} - x_n^{1-m})^{-1} \int_{x_n}^{x_{n+1}} \int_{y}^{\infty} F^m(z)dz dy \\
&\leq 2^{2m-1}(x_n^{2m} - x_n^{1-m})^{-1} \left( \int_{x_n}^{x} \int_{y}^{x} z^{-m} dz dy + \int_{x_n}^{x_{n+1}} \int_{x}^{y} z^{-m} dz dy + \int_{x_n}^{x_{n+1}} \int_{y}^{x_{n+1}} dz dy \right) \\
&\leq 2^{m+1}(m-1)^{-1}(2-m)^{-1} + 2^{2m-1}(x_n^{2m} - x_n^{1-m})^{-1}(m-1)^{-1}x_n^{2m} + x_n^2 < \infty.
\end{align*}

Similarly, when $x \in [4x_n, x_{n+1})$, we have

$$
\left( \int_{x}^{\infty} F^m(y)dy \right)^{-1} I_1(x) \leq \left( \int_{4x_n}^{x_{n+1}} F^m(y)dy \right)^{-1} \int_{x_n}^{x_{n+1}} \frac{x}{y} F^m(z)dz \int_{x-y}^{x} \frac{z}{y} F^m(z)dz dy
$$

\begin{align*}
&\leq (2x_n)^{-2}(m+1) \left( \int_{x_n}^{x_{n+1}} \int_{y}^{x} z^{-m} dz dy + \int_{x_n}^{x_{n+1}} \int_{x}^{y} z^{-m} dz dy + \int_{x_n}^{x_{n+1}} \int_{y}^{x_{n+1}} dz dy \right) \\
&\leq (m+1)((m-1)^{-1}(2-m)^{-1}x_n^{2m} + x_n^2) < \infty.
\end{align*}

Next, we estimate $I_2(x)$. When $x \in [x_n, x_{n+1})$, by (4.20), we have

$$
\left( \int_{x}^{\infty} F^m(y)dy \right)^{-1} \left( \int_{x}^{x_{n+1}} F^m(y)dy \right)^2 \leq \left( \int_{x_n}^{x_{n+1}} F^m(y)dy \right)^{-1} \left( \int_{x_n}^{x_{n+1}} F^m(y)dy \right)^2
$$

\begin{align*}
&\leq (m+1)(2^{-4+2ma} + 3x_n^{1-m} + (2x_n)^{2-2m}) < \infty.
\end{align*}

Finally, for $I_3(x)$, it is obvious that

$$
\left( \int_{x}^{\infty} F^m(y)dy \right)^{-1} I_3(x) \leq \int_{x}^{x_{n+1}} F^m(y)dy \to 0.
$$

By (4.21)-(4.25), we get $G_m \in OS$. □

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