Investigation of some basic thermodynamic properties of Na-K alloy

Rajesh C. Malan1,*, Aditya M. Vora2
1Applied Science and Humanities Department, Government Engineering College, Valsad -396001, Gujarat, India
2Department of Physics, University School of Sciences, Gujarat University, Ahmedabad-380009, Gujarat, India
*E-mail: rcmgeev@gmail.com

ABSTRACT
Investigation of thermodynamic of liquid Na1-xKx binary alloys using pseudopotential theory is reported. The potential suggested by Fiolhais et al. with its individual parameters is used for entire calculation. A transferability of the potential from the solid to liquid medium is achieved for the presently reported binary alloy. The internal energy components, Helmholtz free energy, entropy and total energy at various proportions of the participating alkali metals are included in the study. The comparison with the other data has been shown in the present article. Exchange and correlation effect is also tested with the help of various local field correction functions.

1. Introduction
Novelty in materials is key demand of industrial development. The successful synthesis of any material may be useful for the formation of fabrication of parts of various machineries and devices. The choice of the material has direct influence on overall performance of the device. The choice of the material must be on the base of some scientifically obtained physical and chemical properties. As per the present demand of requirement, variety of properties cannot be availed by the conventional metallic materials. The compounds and alloys are the better replacement of the pure metallic material to have combined properties in a single material. Focusing on alloys, the preliminary stage of the formation also requires the exact knowledge of thermodynamical properties of the alloy. The variation of property with respect to a variation of the proportion of each metal used for the formation of alloy is to be obtained to know the required proportion for desired property. Lots of work has been carried out for the thermodynamical properties of the many solid materials [1-3]. But on the other hand, the liquid materials are studied theoretically from the thermodynamical point of
view very rarely [4-5]. The liquid alkali metals are widely used in the area of nuclear reactors. As having very fast chemically reactive nature even at room temperature, the study of the thermodynamical properties is of great significance. Binary alloy of alkali metal is able to provide the mix properties of each of the metals of formed alloy. In the present article the focus is made on the one of the binary alkali alloys i.e., Na\(\alpha_{1-x}\)K\(\alpha_x\) in liquid state.

2. Computational Methodology

The well-established theory of the pseudopotential is applied for the present calculation. The potential suggested by the Fiolhais et al. [6] is chosen for as the model potentials. The accuracy obtained in previous calculations [9-14] by this potential for various properties is the main reason of the selection of the potential. As the structure factor and the entropy both can easily be characterized by considering the hard sphere system, the well-known Percus-Yevick hard sphere reference system [10] is used. The value of the Hamiltonian of the actual system can be estimated through the perturbation theory. According to the inequality of Gibbs and Bogoliubov [11, 12], the actual Hamiltonian of the reference system is always greater than or equal to the Hamiltonian of the reference system plus the perturbation part. The variational approach along with the Gibbs-Bogoliubov inequality has already been used to approximate the study of thermodynamical study of the metals. The present work extends the computation to the system of binary alkali alloy Na\(\alpha_{1-x}\)K\(\alpha_x\). The electronic free energy of the binary alloys with atoms positions \(\{R_1\}\) and \(\{R_2\}\) can be written as [13, 14],

\[
F_{ele}(R_1, R_2) = F_{eg} + F_1 F_2[R_1, R_2]
\]  

(1)

Here, \(F_1\) and \(F_2\) are the first and second order perturbation energy terms, respectively and \(F_{eg}\) is the free energy of the electron gas that can be expressed as,

\[
F_{eg} = \left[ \frac{3}{10} k_F^2 - \frac{3}{4\pi} k_F + E_c - \frac{1}{2} \gamma_{eg} T^2 \right] \bar{\varepsilon}_n.
\]  

(2)

To calculate the Helmhöltz free energy \((F_h)\) of the alloy, the effective potential energy of the ions can be obtained by introducing the direct Coulomb interaction between the ions. The expectation value of this effective potential can be given by,

\[
F_{ps} = F_{eg} + F_1 + F_2[R_1, R_2] + F_M.
\]  

(3)

The Madelung contribution \((F_M)\) shown in the above equation (3) can be given as,

\[
F_M = \frac{1}{\pi} \int_0^\infty \left[ c_1^2 Z_1^2 (S_{11} - 1) + 2c_1 c_2 (S_{12} - 1) + c_2^2 Z_2^2 (S_{22} - 1) \right] dq.
\]  

(4)

The \(S_{ij}\) \((i = j; 1,2)\) is the partial structure factors computed from the well-known relations given by as used by Faber [11].

The first order perturbation term \((F_1)\) is obtained from averaged valence density \((\bar{\varepsilon})\) and the zeroth Fourier component \((\varepsilon(q))\) of pseudopotential from the following equations (5) and (6), respectively.

\[
F_1 = (c_1 \alpha_1 + c_2 \alpha_2) \bar{\varepsilon}_n.
\]  

(5)

\[
\alpha_i = \lim_{q \to 0} \frac{W_b(q) + 4Z_i^2 \varepsilon^2}{q_c^2}.
\]  

(6)

The band structure energy or second order energy term \((F_2)\) is obtained from,

\[
F_2 = \frac{1}{16\pi} \int_0^\infty \left[ c_1 c_2 (W_{b_1} - W_{b_2})^2 + c_1^2 W_{b_1}^2 S_{11} + c_2^2 W_{b_2}^2 S_{22} \right] \left\{ \frac{1}{\varepsilon(q)} - 1 \right\} dq - \bar{\varepsilon}_n \gamma_2 (T) T^2.
\]  

(7)

Here, \(W_b\) is the screened form factor. \(\varepsilon(q)\) is the dielectric screening function. \(\gamma_2 (T)\) is the second order correction to the usual \(\gamma\) factor describing the low temperature electronic specific heat and is given by,

\[
\gamma_2 (T) = \frac{2}{3\pi^2} \int_0^\infty \frac{x^2 f(x)}{x^3 - 1} dx \left[ c_1 c_2 (W_{1} - W_{2})^2 + c_1^2 W_{1}^2 S_{11} + 2c_1 c_2 W_{1} W_{2} S_{12} + c_2^2 W_{2}^2 S_{22} \right].
\]  

(8)

in which, \(W_i\) is the screened form factor. While, the function \(f(x)\) is be defined as,
\[ f(x) = \left( \frac{x^2 - 1}{4x} \ln \frac{1 + x}{1 - x} \right) + \frac{1}{2}. \]  

(9)

The free energy per particle \((F_{\text{mix}})\) can be expressed as,

\[ F_{\text{mix}} = c_1 \mu_1 + c_2 \mu_2 - \frac{P_{hs}}{n}. \]  

(10)

The chemical potential is obtained by following equation,

\[
\frac{\mu_i}{k_B T} = \ln \left[ n_i \frac{2\pi n^2}{m_i k_B T} \right]^{3 \over 2} - \ln(1 - \eta) \\
+ \ln \left[ \frac{3X \sigma_i}{1 - \eta} \right] + \frac{3}{2} \left[ \frac{3X^2}{(1 - \eta)^2} + 2Y \right] \sigma_i^2 \\
+ \pi P_{hs} \sigma_i^3 - \frac{6k_B T}{3} \sigma_i^3 \\
\]

(11)

and

\[
\frac{P_{hs}}{k_B T} \eta = \frac{n(1 + \eta + \eta^2) - \frac{1}{2} \pi n_1 n_2 (\sigma_1 - \sigma_2)^2 (\sigma_1 + \sigma_2 + \sigma_1 \sigma_2 X)}{(1 - \eta)^3} 
\]

(12)

where,

\[ X = \frac{1}{6} \pi (n_1 \sigma_1^2 + n_2 \sigma_2^2) \]  

(13)

and

\[ Y = \frac{1}{6} \pi (n_1 \sigma_1 + n_2 \sigma_2). \]  

(14)

The internal energy \((F_{\text{int}})\) can be given as,

\[ F_{\text{int}} = \frac{3}{2} k_B T + F_{eg} + F_1 + F_2 (R_1, R_2) + F_M. \]  

(15)

Using equations (10)-(15),

\[ F_{\text{mix}} = \frac{3}{2} k_B T - TS_{\text{mix}}, \]  

(16)

where, \(S_{\text{mix}}\) is the total entropy for alloy system and can be divided into following four parts,

\[ S_{\text{mix}} = S_{\text{gas}} + S_c + S_\eta + S_\sigma. \]  

(17)

where, \(S_c\) is the ideal entropy of mixing, \(S_{\text{gas}}\) represents the gas term, \(S_\eta\) corresponds to packing density \(\eta\) and \(S_\sigma\) arises due to the difference in diameters of the hard sphere of participating elements of the alloy, respectively.

The various contributions of the entropy can be given as,

\[ \frac{S_c}{k_B} = -(c_1 \ln c_1 + c_2 \ln c_2), \]  

(18)

\[ \frac{S_\eta}{k_B} = \ln(1 - \eta) + \frac{3}{2} \left[ 1 - \frac{1}{(1 - \eta)^2} \right], \]  

(19)

\[ \frac{S_\sigma}{k_B} = \frac{\pi c_1 c_2 n (\sigma_1 - \sigma_2)^2 [2(\sigma_1 + \sigma_2) - \pi n (c_1 \sigma_1^4 + c_2 \sigma_2^4)]}{24(1 - \eta)^2} \]  

(20)

and

\[ \frac{S_{\text{gas}}}{k_B} = \ln \left[ \frac{e^{(Emk_B T)^{3 \over 2}}}{n (2\pi n^2)^{1 \over 2}} \right]. \]  

(21)

From all above required contributions, the Helmholtz free energy \((F_h)\) can be obtained as,

\[ F_h = F_{ps} + F_{\text{mix}}. \]  

(22)

To estimate the screening effect over the bare-ion potential the Hartree local field correction function \((H)\) is used [15]. The other local field correction functions used over the bare-ion potential are the functions suggested by Hubbard-Sham (HS) [16, 17], Vashishtha-Shingwi (VS) [18], Taylor (T) [19], Sarkar et al. (S) [20], Ichimaru-Utsumi (IU) [21] and Farid et al. (F) [22] and Nagy (N) [23].

3. Results and Discussion

The individual parameters of the potential of Fiolhais et al. [6] and the constants are used as given in the Table 1. The potential parameters are directly adopted from the original work of the Fiolhais et al. [6]. Table 2 shows the various parts \((F_{eg}, F_1, F_M\) and \(F_{\text{mix}}\) that contributes to the internal energy \((F_{\text{int}})\) of the alloy. Rather than calculating only internal energy \((F_{\text{int}})\) and the Helmhotz free energy \((F_h)\), the present calculation focused on calculation of the microscopic distributions of these energies in further contribution. The present results also compared with the others available theoretical results [14] to validate our results. It can be seen from the Table 2 that the \(F_{eg}\) and \(F_M\) has excellent agreement with compared results [14].
A very good agreement is obtained with the results of Vora [14] for the results of $F_{\text{mix}}$. This means the results generated by the potential used in present calculation and by Vora [14] provides the almost same type of approximation for free electron energy, free energy per particle of the binary mixture and the Madlung static electrical energy. The contributions provided by $F_{\text{eg}}$, $F_M$ and $F_{\text{mix}}$ are negative whereas that due to $F_1$ is positive. Out of these four parts, the $F_1$ and $F_2$ depend upon model potential. $F_1$ is obtained from the zero limit of the potential.

The results of the second order perturbation energy ($F_2$) for eight different local field corrections functions [15-23] are shown in the Table 3. The comparision of the present result with the other theoretical available data [14] is also shown in the Table 3. From the Table 3, it can be observed that the results due to N-function [23] is highest for all concentration ($X$). The HS-function [16, 17] provides the minimum exchange and correlation effect with respect to H-function [15].

The total internal energy ($F_{\text{int}}$) of the alloys under study is shown in the Table 4. The minimum deviation from the experimental value [20] is obtained for the N-function [23] and maximum for H-function [15]. However no significant effect is change is observed for any of the local field correction functions. Hence, the suggested bare-ion potential is sufficiently efficient to provide the screening and exchange and correlation effect.

It also can be observed from the Table 4 that the variation of $F_{\text{int}}$ from the experimental value [13] is maximum at $X = 0$ is and minimum for $X = 1$. Thus, the potential provides the poor results when proportion of Na is more and better results when proportion of K is more.

Various entropy contributions and total entropy $S_{\text{mix}}$ are obtained a shown in the Figure 1. The entropy contributes to the Helmholtz free energy ($F_h$) as shown in the equation [3] and [4].

The calculated Helmholtz free energy ($F_h$) is shown in the Table 6. The results obtained for the $F_h$ are found in well agreement with the experimental data [13] for $X = 1$, better agreement for $X = 0.5$ and reasonable agreement for $X = 0$.

### Table 1. Input parameters and constants.

| Metal | $\Omega$ (au) | $k_f$ (au) | $\alpha$ (au) [6] | $R$ (au) [6] |
|-------|---------------|------------|-------------------|-------------|
| Na    | 227.76        | 0.4742     | 3.517             | 0.492       |
| K     | 528.67        | 0.3826     | 3.349             | 0.679       |

**Fig. 1:** Various Entropy Contributions.
Table 2: $F_{eg}$, $F_1$, $F_M$ and $F_{mix}$ (* $10^{-3}$ in au)

| $X$ | $F_{eg}$ | Others [14] | $F_1$ | $F_M$ | Others [14] | $F_{mix}$ | Others [14] |
|-----|---------|-------------|-------|-------|-------------|-----------|-------------|
| 0.0 | -81.61  | -81.63      | 79.47 | -213.7| -210.17     | -6.45     | -7.51       |
| 0.1 | -81.73  | -81.74      | 78.97 | -203.1| -210.37     | -6.71     | -8.12       |
| 0.2 | -81.73  | -81.72      | 83.30 | -194.5| -192.15     | -6.93     | -8.54       |
| 0.3 | -81.64  | -81.62      | 86.57 | -187.5| -184.55     | -7.13     | -8.87       |
| 0.4 | -81.48  | -81.45      | 89.40 | -182.1| -178.36     | -7.32     | -9.12       |
| 0.5 | -81.28  | -81.24      | 91.87 | -178.0| -173.40     | -7.49     | -9.32       |
| 0.6 | -81.05  | -81.00      | 94.05 | -175.0| -169.53     | -7.64     | -9.45       |
| 0.7 | -80.78  | -80.73      | 95.99 | -172.9| -166.61     | -7.79     | -9.53       |
| 0.8 | -80.50  | -80.44      | 97.71 | -171.8| -164.54     | -7.92     | -9.55       |
| 0.9 | -80.21  | -80.13      | 99.27 | -171.5| -163.20     | -8.05     | -9.48       |
| 1.0 | -79.90  | -79.82      | 100.67| -171.8| -162.53     | -8.17     | -9.23       |

Table 3. $F_2 * 10^{-3}$ in au

| $X$ | H     | HS    | VS    | T     | S     | IU    | F     | N     | Others [14] |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------------|
| 0.0 | -16.09| -15.45| -14.88| -14.63| -15.14| -14.35| -14.33| -13.27| -13.69      |
| 0.1 | -70.51| -68.68| -67.09| -66.38| -67.83| -65.68| -65.61| -61.83| -           |
| 0.2 | -74.60| -72.33| -70.23| -69.20| -71.18| -68.30| -68.18| -63.93| -           |
| 0.3 | -61.07| -58.81| -56.55| -55.33| -57.54| -54.39| -54.24| -50.60| -           |
| 0.4 | -43.40| -41.39| -39.23| -37.96| -40.14| -37.09| -36.92| -34.29| -           |
| 0.5 | -27.86| -26.20| -24.30| -23.11| -25.08| -22.37| -22.20| -20.57| -19.20      |
| 0.6 | -17.44| -16.15| -14.59| -13.56| -15.22| -12.98| -12.83| -11.89| -           |
| 0.7 | -12.38| -11.40| -10.19| -9.38 | -10.67| -8.94 | -8.82 | -8.17 | -           |
| 0.8 | -12.21| -11.46| -10.56| -9.98 | -10.93| -9.66 | -9.57 | -8.80 | -           |
| 0.9 | -14.43| -13.83| -13.18| -12.83| -13.46| -12.58| -12.53| -11.39| -           |
| 1.0 | -12.62| -12.22| -11.84| -11.71| -12.03| -11.54| -11.53| -10.30| -14.50      |
Table: $F_{int} * 10^{-3}$ in au

| $X$ | $H$  | HS  | VS  | T   | S   | IU  | F   | N   | Exp [13] |
|-----|------|-----|-----|-----|-----|-----|-----|-----|----------|
| 0.0 | -308.75 | -308.11 | -307.55 | -307.29 | -307.81 | -307.02 | -307.00 | -305.93 | -226.00 |
| 0.1 | -274.13 | -272.30 | -270.71 | -270.00 | -271.45 | -269.30 | -269.23 | -265.45 | -        |
| 0.2 | -265.74 | -263.48 | -261.38 | -260.34 | -262.32 | -259.45 | -259.33 | -255.08 | -        |
| 0.3 | -241.92 | -239.66 | -237.40 | -236.18 | -238.38 | -235.24 | -235.09 | -231.45 | -        |
| 0.4 | -215.82 | -213.80 | -211.64 | -210.37 | -212.55 | -209.50 | -209.33 | -206.71 | -        |
| 0.5 | -193.46 | -191.80 | -189.90 | -188.70 | -190.68 | -187.97 | -187.80 | -186.16 | -        |
| 0.6 | -177.62 | -176.33 | -174.77 | -173.73 | -175.39 | -173.16 | -173.00 | -172.07 | -        |
| 0.7 | -168.35 | -167.37 | -166.16 | -165.35 | -166.64 | -164.92 | -164.79 | -164.14 | -        |
| 0.8 | -165.04 | -164.29 | -163.39 | -162.82 | -163.76 | -162.50 | -162.41 | -161.63 | -        |
| 0.9 | -165.06 | -164.46 | -163.81 | -163.46 | -164.10 | -163.21 | -163.17 | -162.02 | -        |
| 1.0 | -161.87 | -161.48 | -161.10 | -160.96 | -161.28 | -160.79 | -160.79 | -159.55 | -190.00 |

Table 5: $F_{h} * 10^{-3}$ in au

| $X$ | $H$  | HS  | VS  | T   | S   | IU  | F   | N   | Exp [13] |
|-----|------|-----|-----|-----|-----|-----|-----|-----|----------|
| 0.0 | -369.62 | -368.99 | -368.42 | -368.16 | -368.68 | -367.89 | -367.87 | -366.80 | -250.33 |
| 0.1 | -323.36 | -321.53 | -319.94 | -319.24 | -320.68 | -318.54 | -318.46 | -314.68 | -        |
| 0.2 | -320.67 | -318.41 | -316.30 | -315.27 | -317.25 | -314.37 | -314.25 | -310.01 | -        |
| 0.3 | -306.24 | -303.98 | -301.72 | -300.50 | -302.70 | -299.56 | -299.41 | -295.77 | -        |
| 0.4 | -287.95 | -285.94 | -283.78 | -282.51 | -284.69 | -281.64 | -281.47 | -278.84 | -        |
| 0.5 | -269.63 | -267.97 | -266.06 | -264.87 | -266.84 | -264.14 | -263.96 | -262.33 | -235.90 |
| 0.6 | -253.39 | -252.09 | -250.53 | -249.50 | -251.16 | -248.93 | -248.77 | -247.84 | -        |
| 0.7 | -239.82 | -238.84 | -237.63 | -236.82 | -238.12 | -236.39 | -236.26 | -235.61 | -        |
| 0.8 | -229.41 | -228.66 | -227.76 | -227.18 | -228.13 | -226.86 | -226.77 | -226.00 | -        |
| 0.9 | -221.56 | -220.97 | -220.32 | -219.97 | -220.60 | -219.72 | -219.67 | -218.52 | -        |
| 1.0 | -214.13 | -213.74 | -213.36 | -213.23 | -213.54 | -213.06 | -213.05 | -211.81 | -218.51 |
4. Conclusion

The thermodynamical investigation of the liquid alkali Na$_{1-x}$K$_x$ alloy has been done. The potential of Fiolhais et al. [6] is found appropriate to describe the thermodynamical properties of the alloy under study. The results for the internal energy ($F_{int}$) and Helmholtz energy ($F_H$) are obtained and found in a good agreement at some concentration value. The present results are found to be deviated from the experimental data [13] at lower concentration.

References

[1] J. K. Baria, Analysis of thermodynamics of liquid d- and f-shell metals with the variational approach. Chinese physics letters 20(6) (2003) 894. https://doi.org/10.1088/0256-307X/20/6/333

[2] R.C. Gosh, A.Z. Ahmed, and G. M. Bhuian, Investigation of surface entropy for liquid less simple metals, The European Physical Journal B 56(3) (2007) 177-181. https://doi.org/10.1140/epjb/e2007-00104-9

[3] P. Kumar, N.K. Bhatt, P.R. Vyas, and V.B. Gohel, The study of anharmonic properties and spd hybridization in barium at extreme environment, Phase Transitions 90(12) (2017) 1167-1178. https://doi.org/10.1080/01411594.2017.1326601

[4] P.B. Thakor, V.N. Patel, B.Y. Thakore, P.N. Gajjar, and A.R. Jani, Thermodynamic properties of some simple metals in liquid phase by pseudopotential theory, Indian Journal of Pure and Applied Physics 42 (2004) 684.

[5] N.E. Dubinin, L.D. Son, N.A. Vatolin, Thermodynamic properties of liquid binary transition-metal alloys in the Bretonnet-Silbert model, In Defect and Diffusion Forum 263 (2007) 105-110, Trans Tech Publications. https://doi.org/10.4028/www.scientific.net/DDF.263.105

[6] C. Fiolhais, J.P. Perdew, S.Q. Armster, J.M. MacLaren, and M. Brajczewska, Dominant density parameters and local pseudopotentials for simple metals, Physical Review B 51(20) (1995) 14001. https://doi.org/10.1103/PhysRevB.51.14001

[7] R.C. Malan, and A.M. Vora, Thermodynamical investigation of liquid Li$_{1-x}$K$_x$ alloy, In AIP Conference Proceedings (Vol. 2009, No. 1, p. 020052), AIP Publishing, 2018. https://doi.org/10.1063/1.5052121

[8] R.C. Malan, and A.M. Vora, Electrical resistivity of liquid Na-alkali alloys, In AIP Conference Proceedings (Vol. 1953, No. 1, p. 140014), AIP Publishing, 2018. https://doi.org/10.1063/1.5033189

[9] R.C. Malan, and A.M. Vora, Thermodynamical Investigation of Liquid Alkali Metals with Gibbs-Bogoliubov Method, J Nano Electronic Phys. 10(3) (2018) 3020. https://doi.org/10.21272/jnep.10(3).03020

[10] J.K. Percus, and G.J. Yevick., Analysis of classical statistical mechanics by means of collective coordinates, Physical Review 110(1) (1958) 1. https://doi.org/10.1103/PhysRev.110.1

[11] T.E. Faber, Introduction to the Theory of Liquid Metals, Cambridge Uni. Press, London, 1972.

[12] M. Shimoji, Liquid Metals, An Introduction to the Physics and Chemistry of Metals in the Liquid State. Academic Press, London, 1977.

[13] I.H. Umar, A. Meyer, M. Watabe, and W.H. Young, Thermodynamic calculations for liquid alloys with an application to sodium-potassium, Journal of Physics F: Metal Physics 4(10) (1974)1691. https://doi.org/10.1088/0305-4608/4/10/016

[14] A.M. Vora, Study of thermodynamical properties of liquid binary alloys by a pseudopotential method, J. Theo. App. Phys. 3(4) (2010) 25.

[15] Y. Waseda, The structure of non-crystalline materials: liquids and amorphous solids. McGraw-Hill, New York, 1977.

[16] J. Hubbard, The description of collective motions in terms of many-body perturbation Theory. II. The correlation energy of a free-electron gas, Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences 243(1234) (1958) 336-352. https://doi.org/10.1098/rspa.1958.0003

[17] L.J. Sham, A calculation of the phonon frequencies in sodium, Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences 283(1392) (1959) 33-49. https://doi.org/10.1098/rspa.1965.0005

[18] P. Vashishta, K.S Singwi, Electron correlations at metallic densities, Physical Review B 6(3) (1972) 875. https://doi.org/10.1103/PhysRevB.6.875

[19] R. Taylor, A simple, useful analytical form of the static electron gas dielectric function, Journal of Physics F: Metal Physics 8(8) (1978) 1699. https://doi.org/10.1088/0305-4608/8/8/011

[20] A. Sarkar, D. Sen, S. Haldar, and D. Roy, Static local field factor for dielectric screening function of
electron gas at metallic and lower densities, Modern Physics Letters B 12(16) (1998) 639-648. https://doi.org/10.1142/S0217984998000755

[21] S. Ichimaru, and K. Utsumi, Analytic expression for the dielectric screening function of strongly coupled electron liquids at metallic and lower densities, Physical Review B 24(12) (1981) 7385. https://doi.org/10.1103/PhysRevB.24.7385

[22] B. Farid, V. Heine, G.E. Engel, I.J. Robertson, Extremal properties of the Harris-Foulkes functional and an improved screening calculation for the electron gas, Physical Review B 48(16) (1993) 11602. https://doi.org/10.1103/PhysRevB.48.11602

[23] I. Nagy, Analytic expression for the static local field correlation function, Journal of Physics C: Solid State Physics, 19(22) (1986) L481. https://doi.org/10.1088/0022-3719/19/22/002