SEMILINEAR HYPERBOLIC SYSTEMS VIOLATING THE NULL CONDITION

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Dedicated to the memory of Professor Rentaro Agemi

Abstract. We consider systems of semilinear wave equations in three space dimensions with quadratic nonlinear terms not satisfying the null condition. We prove small data global existence of the classical solution under a new structural condition related to the weak null condition. For two-component systems satisfying this condition, we also observe a new kind of asymptotic behavior: Only one component is dissipated and the other one behaves like a free solution in the large time.

1. Introduction

This paper is concerned with global existence and large time behavior of classical solutions to the Cauchy problem for systems of semilinear wave equations of the following type:

\[ \square u = F(\partial u) \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}^3, \quad (1.1) \]

\[ u(0, x) = \varepsilon f(x), \quad (\partial_t u)(0, x) = \varepsilon g(x) \quad \text{for } x \in \mathbb{R}^3, \quad (1.2) \]

where \( u = (u_1, \ldots, u_N)^T \) is an \( \mathbb{R}^N \)-valued unknown function of \((t, x) \in [0, \infty) \times \mathbb{R}^3, \quad \square = \partial_t^2 - \Delta_x = \partial_t^2 - \sum_{k=1}^3 \partial^2_{x_k}, \) and \( \partial u = (\partial_0 u, \partial_1 u, \partial_2 u, \partial_3 u) \) with the notation

\[ \partial_0 = \partial_t = \frac{\partial}{\partial t}, \quad \partial_k = \frac{\partial}{\partial x_k}, \quad k = 1, 2, 3. \]

Here \( B^T \) stands for the transpose of a matrix (or vector) \( B. \) For simplicity, we suppose that \( f, g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N), \) and that the nonlinear term \( F = (F_1, \ldots, F_N)^T \) has the form

\[ F_j(\partial u) = \sum_{k,l=1}^N \sum_{a,b=0}^3 c_{kl,ab}^{j}(\partial_au_k)(\partial_bu_l), \quad j = 1, \ldots, N \quad (1.3) \]

with some constants \( c_{kl,ab}^{j} \in \mathbb{R}. \) \( \varepsilon \) is a parameter which will be always assumed to be sufficiently small.

Let us briefly review known results concerning the global existence and the asymptotic behavior. In general, it is known that the solution to the Cauchy
problem (1.1)-(1.2) blows up in finite time no matter how small $\varepsilon$ is; for example, if $N = 1$ and $F(\partial u) = (\partial_i u)^2$, then the solution $u$ blows up in finite time for any $\varepsilon > 0$ unless $(f, g) \equiv (0, 0)$ in (1.2) (see John [9]). Therefore, we need some restriction on the nonlinearity to obtain global solutions even for small data. We say that we have small data global existence for the problem (1.1)-(1.2) if for any $f, g \in C^\infty_0(\mathbb{R}^3)$, there is a positive constant $\varepsilon_0$ such that the Cauchy problem (1.1)-(1.2) admits a global solution for any $\varepsilon \in (0, \varepsilon_0]$. Klainerman [16] introduced the notion of the weak null condition, and proved the small data global existence, known as the null condition (see also Christodoulou [3]): We say that the null condition is satisfied if we have

$$F^{\text{red}}(\omega, Y) = 0, \quad \omega = (\omega_1, \omega_2, \omega_3) \in S^2, \quad Y = (Y_1, \ldots, Y_N)^T \in \mathbb{R}^N,$$

where the reduced nonlinearity $F^{\text{red}}(\omega, Y) = (F^{\text{red}}_1(\omega, Y), \ldots, F^{\text{red}}_N(\omega, Y))^T$ is defined by

$$F^{\text{red}}_j(\omega, Y) := F_j(\omega_0 Y, \omega_1 Y, \omega_2 Y, \omega_3 Y) = \sum_{k,l=1}^{N} \sum_{a,b=0}^{3} c_{kl}^{ab} \omega_a \omega_{k} Y_{k} Y_{l}$$

for $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ and $Y = (Y_1, \ldots, Y_N)^T \in \mathbb{R}^N$ with $\omega_0 = -1$. Here the constants $c_{kl}^{ab}$ are from (1.3). In [3] and [16], it was proved that the null condition implies small data global existence. It is also easy to see that this global solution $u$ for small $\varepsilon$ is asymptotically free, that is to say that there is $(f^+, g^+) \in \dot{H}^1(\mathbb{R}^3; \mathbb{R}^N) \times L^2(\mathbb{R}^3; \mathbb{R}^N)$ such that we have

$$\lim_{t \to \infty} ||u(t) - u^+(t)||_E = 0$$

for the solution $u^+$ to the free wave equation $\Box u^+ = 0$ with initial data $(u^+, \partial_t u^+)(0) = (f^+, g^+)$. Here and in the sequel, $|| \cdot ||_E$ is the energy norm defined by

$$||\phi(t)||_E = \left(\frac{1}{2} \int_{\mathbb{R}^3} (|\partial_t \phi(t, x)|^2 + |\nabla_x \phi(t, x)|^2) \, dx \right)^{1/2},$$

and $\dot{H}^1(\mathbb{R}^3)$ denotes the completion of $C^\infty_0(\mathbb{R}^3)$ with respect to the norm given by $||\phi||_{\dot{H}^1} = ||\nabla_x \phi||_{L^2}$. Introducing the null forms

$$Q_0(\phi, \psi) := (\partial_t \phi)(\partial_t \psi) - (\nabla_x \phi) \cdot (\nabla_x \psi),$$

$$Q_{ab}(\phi, \psi) := (\partial_a \phi)(\partial_b \psi) - (\partial_b \phi)(\partial_a \psi), \quad a, b \in \{0, 1, 2, 3\},$$

we see that the nonlinearity $F$ of the form (1.3) satisfies the null condition if and only if each component $F_j$ can be written as a linear combination of the null forms $Q_0(u_k, u_l)$ and $Q_{ab}(u_k, u_l)$ with $k, l \in \{1, \ldots, N\}$ and $a, b \in \{0, \ldots, 3\}$.}

In connection with the Einstein equation that can be expressed in wave coordinates as a system of quasilinear wave equations, Lindblad-Rodnianski [22] introduced the notion of the weak null condition, and proved the small data global existence for the Einstein equation in wave coordinates (see [23] and
The small data global existence is also obtained for a closely related equation
\[ \Box u = \sum_{a,b=0}^{3} g^{ab} u(\partial_a \partial_b u), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3 \]
with constants \( g^{ab} \in \mathbb{R} \), which satisfies the weak null condition, but violates the null condition (see Alinhac [1], and Lindblad [20], [21]). These successful examples suggest that the weak null condition implies the small data global existence in general; however this is still an open problem even for semilinear systems.

Before we proceed to further discussion, we give the definition of the weak null condition for the semilinear case here: We say that the weak null condition is satisfied if the reduced system
\[ \partial_t V(t;\sigma,\omega) = -\frac{1}{2t} F_{\text{red}}(\omega, V(t;\sigma,\omega)), \quad t \geq 1, \sigma \in \mathbb{R}, \omega \in S^2 \]
(admits a global solution \( V \) with at most polynomial growth of small power in \( t \) for small data given at \( t = 1 \), where (1.8) is obtained as an asymptotic equation for \( V(t;\sigma,\omega) = (\partial_r - \partial_t)(ru(t,r\omega))/2 \) with \( \sigma = r - t \) (see (4.5) and (5.19) below; see also Hörmander [8]). The null condition (1.4) immediately implies the weak null condition. It is not easy to check whether or not the weak null condition is satisfied in general, because it depends on the global behavior of the reduced system (1.8).

In connection with the weak null condition, Alinhac [2] considered systems of semilinear wave equations and introduced an algebraic condition to ensure the small data global existence. His condition is stronger than the weak null condition, but still weaker than the null condition. His condition was slightly extended, and the asymptotic behavior of global solutions under this extended condition was studied in Katayama [10] (see also Katayama-Kubo [12]). Quite roughly speaking, the (extended) Alinhac condition says that the reduced system, through some change of unknowns, can be expressed as
\[ \partial_t V_j = \begin{cases} \frac{1}{2t} \sum_{k=M+1}^{N} \sum_{l=1}^{N} C_{kl}(\omega)V_k V_l, & 1 \leq j \leq M, \\ 0, & M + 1 \leq j \leq N \end{cases} \]
(1.9)
with smooth coefficients \( C_{kl}(\omega) \) and \( M \in \mathbb{N} \). If the reduced system is of the form (1.9), then we can easily check that the weak null condition is satisfied; however the null condition is violated unless all the coefficients \( C_{kl}(\omega) \) vanish identically. Here we give three typical examples satisfying the (extended) Alinhac condition (and thus the weak null condition), but violating the null condition, and their asymptotic behavior (see [10] for details): The first example is
\[ \begin{cases} \Box u_1 = (\partial_t u_2)(\partial_t u_1) + (\text{null forms}), \\ \Box u_2 = (\text{null forms}). \end{cases} \]
(1.10)
The second example is
\[
\begin{cases}
\Box u_1 = (\partial_t u_2)^2 + \text{(null forms)}, \\
\Box u_2 = \text{(null forms)}.
\end{cases}
\] (1.11)
We can choose \( f \) and \( g \) in (1.2) such that we have \( \|u(t)\|_E \geq C\varepsilon (1 + t)^C \) for (1.10), and \( \|u(t)\|_E \geq C\varepsilon (1 + \varepsilon \log(2 + t)) \) for (1.11), with a positive constant \( C \). In both cases, the energy grows up to infinity, and the global solution \( u \) is not asymptotically free for such data. The third example is
\[
\begin{cases}
\Box u_1 = -(\partial_t u_3)(\partial_t u_2) + \text{(null forms)}, \\
\Box u_2 = (\partial_t u_3)(\partial_t u_1) + \text{(null forms)}, \\
\Box u_3 = \text{(null forms)}.
\end{cases}
\] (1.12)
For this example, we can see that \( C^{-1}\varepsilon \leq \|u(t)\|_E \leq C\varepsilon \) for some positive constant \( C \) unless \( (f, g) \equiv (0, 0) \); nonetheless, for appropriately chosen \( f \) and \( g \), we can show that the global solution \( u \) is not asymptotically free.

Our aim in this paper is to obtain another kind of algebraic condition which implies the small data global existence (and the weak null condition). We will also show that, under this condition, we have the asymptotic behavior that is quite different from the known cases.

2. The main results

In what follows, we assume the following condition on the nonlinearity:

(H) There is an \( N \times N \)-matrix valued continuous function \( A = A(\omega) \) on \( S^2 \) such that \( A(\omega) \) is a positive-definite symmetric matrix for each \( \omega \in S^2 \), and that
\[
Y^T A(\omega) F^{\text{red}}(\omega, Y) = 0, \quad \omega \in S^2, \ Y \in \mathbb{R}^N.
\]

Concerning the global existence, our main result is the following:

**Theorem 2.1** (Global existence). Suppose that the condition (H) is satisfied. Then, for any \( f, g \in C^0_0(\mathbb{R}^3; \mathbb{R}^N) \), there exists \( \varepsilon_0 > 0 \) such that (1.1)–(1.2) admits a unique global \( C^\infty \)-solution \( u \) for \( (t, x) \in [0, \infty) \times \mathbb{R}^3 \) if \( \varepsilon \in (0, \varepsilon_0] \).

Since the local existence of the solution is well known, what we have to do for the proof of Theorem 2.1 is to get a suitable \textit{a priori} estimate for the solution to (1.1)–(1.2). This will be carried out in Section 3 after some preliminaries in Sections 3 and 4.

Under the condition (H), there is a positive constant \( M_0 \) such that
\[
M_0^{-1}|Y|^2 \leq Y^T A(\omega) Y \leq M_0|Y|^2, \quad \omega \in S^2.
\] (2.1)
Indeed, if we denote the eigenvalues of \( A(\omega) \) by \( \lambda_1(\omega), \ldots, \lambda_N(\omega) \) with each eigenvalue being counted up to its algebraic multiplicity, then we have
\[
\min_{1 \leq j \leq N} \lambda_j(\omega)|Y|^2 \leq Y^T A(\omega) Y \leq \max_{1 \leq j \leq N} \lambda_j(\omega)|Y|^2,
\]
which leads to (2.1) because we may assume \( \lambda_f(\omega) \) is positive and continuous in \( \omega \in S^2 \). Since (1.8) and (H) implies

\[
\partial_t (V^T A(\omega) V) = -\frac{1}{t} V^T A(\omega) F^\text{red}(\omega, V) = 0,
\]

we have an a priori bound for \( |V| \) in view of (2.1). Hence the condition (H) implies the weak null condition. If the null condition is satisfied, then the condition (H) is trivially satisfied with \( A(\omega) = I_N \), where \( I_N \) is the identity \( N \times N \)-matrix. To sum up, the condition (H) is stronger than the weak null condition, and weaker than the null condition.

There is no inclusion between the condition (H) and the (extended) Alinhac condition, though both of them are satisfied for (1.12); the examples (1.10) and (1.11) satisfy the Alinhac condition but not the condition (H); the next examples satisfy the condition (H) but not the Alinhac condition.

**Example 2.1.** Let \( N = 2 \) and

\[
\begin{aligned}
F_1(\partial u) &= -c_0 \sum_{a,b=0}^3 c_{ab}(\partial_a u_1)(\partial_b u_2) + N_1(\partial u), \\
F_2(\partial u) &= \sum_{a,b=0}^3 c_{ab}(\partial_a u_1)(\partial_b u_1) + N_2(\partial u),
\end{aligned}
\]

where \( c_0 \) is a positive constant and \( c_{ab} \) are real constants, while \( N_1 \) and \( N_2 \) are written as linear combinations of the null forms. If we put

\[
c(\omega) := \sum_{a,b=0}^3 c_{ab} \omega_a \omega_b, \quad \omega = (\omega_1, \omega_2, \omega_3) \in S^2 \text{ with } \omega_0 = -1,
\]

then we get \( F_1^{\text{red}}(\omega, Y) = -c_0 c(\omega) Y_1 Y_2 \) and \( F_2^{\text{red}}(\omega, Y) = c(\omega) Y_2^2 \). Hence we have

\[
Y^T A F^{\text{red}}(\omega, Y) = Y_1 F_1^{\text{red}}(\omega, Y) + c_0 Y_2 F_2^{\text{red}}(\omega, Y) = 0
\]

with \( A = \text{diag}(1, c_0) \), and we see that the condition (H) is satisfied. The null condition is not satisfied unless \( c(\omega) \equiv 0 \). Due to the result in [10], we can also see that the Alinhac condition is not satisfied unless \( c(\omega) \equiv 0 \), because the asymptotic behavior as seen in Theorem 2.2 below never happens under the Alinhac condition.

**Example 2.2.** Let \( N = 2 \) and

\[
\begin{aligned}
F_1(\partial u) &= -(\partial_t u_1)^2 - 4(\partial_t u_1)(\partial_t u_2) - 3(\partial_t u_2)^2 + (\partial_t u_1 + \partial_t u_2)^2, \\
F_2(\partial u) &= 3(\partial_t u_1)^2 + 4(\partial_t u_1)(\partial_t u_2) + (\partial_t u_2)^2 - (\partial_t u_1 + \partial_t u_2)^2.
\end{aligned}
\]

Then we can check that the condition (H) is satisfied with

\[
A(\omega) = \frac{1}{2} \begin{pmatrix} 3 - \omega_1^2 & 1 - \omega_1^2 \\ 1 - \omega_2^2 & 3 - \omega_2^2 \end{pmatrix}.
\]
Indeed,
\[ F^\text{red}(\omega, Y) = (Y_1 + Y_2) \left( (\omega_1^2 - 1)Y_1 + (\omega_1^2 - 3)Y_2 \right) \]
is perpendicular to \(A(\omega)Y\). Note that the eigenvalues of \(A(\omega)\) are 1 and \((2 - \omega_1^2)\), both of which are positive for all \(\omega \in \mathbb{S}^2\).

**Remark 2.1.** Let us consider \((1.1) - (1.2)\) with \(N = 2\) and
\[
\begin{cases}
F_1 = -c_1(\partial_t u_1)(\partial_t u_2), \\
F_2 = c_2(\partial_t u_1)^2.
\end{cases}
\]

We have small data global existence if \(c_1c_2 \geq 0\) because the condition \((H)\) is satisfied for the case \(c_1c_2 > 0\), while the case \(c_1c_2 = 0\) is trivial. On the other hand, if \(c_1c_2 < 0\), then there are \(f, g \in C_0^\infty(\mathbb{R}^3)\) such that the solution \(u\) blows up in finite time no matter how small \(\varepsilon\) is. Indeed, if we choose \(\phi, \psi \in C_0^\infty(\mathbb{R}^3)\) with \((\phi, \psi) \neq (0, 0)\), then the solution \(u = (u_1, u_2)^T\) with
\[
(f_1, g_1) = \frac{1}{\sqrt{-c_1c_2}}(\phi, \psi), \quad (f_2, g_2) = -\frac{1}{c_1}(\phi, \psi)
\]
can be written as \(u_1 = w/\sqrt{-c_1c_2}\) and \(u_2 = -w/c_1\), where \(w\) is the solution to \(\Box w = (\partial_tw)^2\) with \((w, \partial_tw)(0) = (\varepsilon\phi, \varepsilon\psi)\), which blows up in finite time by the result of John [9].

Now we give our main result for the asymptotic behavior of global solutions under the condition \((H)\). In order to keep the description not too complicated, we consider only the case of \((2.2)\) here. The case of general two-component systems satisfying \((H)\) will be outlined in Section 8. For simplicity of exposition, we put \(\mathcal{H}_0(\mathbb{R}^3) := H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) and
\[
\| (\phi, \psi) \|^2_{\mathcal{H}_0} := \frac{1}{2} \left( \| \phi \|^2_{H^1} + \| \psi \|^2_{L^2} \right), \quad (\phi, \psi) \in \mathcal{H}_0.
\]
Note that \(\| (\varphi(t), \partial_t\varphi(t)) \|_{\mathcal{H}_0} = \| \varphi(t) \|_E\).

**Theorem 2.2** (Asymptotic behavior). Let \(N = 2\), and assume that \(F\) is of the form \((2.2)\) with \(c_0 > 0\). Suppose that \(c(\omega) \neq 0\) on \(\mathbb{S}^2\), where \(c(\omega)\) is defined by \((2.3)\). Given \(f = (f_1, f_2)^T, g = (g_1, g_2)^T \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^2)\), let \(\varepsilon\) be sufficiently small, and \(u = (u_1, u_2)^T\) be the global solution for \((1.1) - (1.2)\) whose existence is guaranteed by Theorem 2.1. Then we have
\[
\lim_{t \to \infty} \| u_1(t) \|_E = 0,
\]
and there exists \((f_2^+, g_2^+) \in \mathcal{H}_0(\mathbb{R}^3)\) such that
\[
\lim_{t \to \infty} \| u_2(t) - u_2^+(t) \|_E = 0,
\]
where \(u_2^+ = u_2^+(t, x)\) solves \(\Box u_2^+ = 0\) with \((u_2^+, \partial_t u_2^+)(0) = (f_2^+, g_2^+)\). Moreover we have
\[
\| (f_2^+, g_2^+) \|_{\mathcal{H}_0} = \varepsilon \left( c_0^{-1} \| (f_1, g_1) \|_{\mathcal{H}_0}^2 + \| (f_2, g_2) \|_{\mathcal{H}_0}^2 \right)^{1/2} + O(\varepsilon^2)
\]
as \(\varepsilon \to +0\).
The proof of Theorem 2.2 will be given in Sections 6 and 7. Note that the null condition is violated in the assumption of Theorem 2.2 because we have assumed $c(\omega) \not\equiv 0$.

From (2.8), we see that $(f^+_2, g^+_2) \neq (0, 0)$ for small $\varepsilon$, unless the Cauchy data for the original problem vanish identically. Therefore Theorem 2.2 tells us that only $u_1$ is dissipated and $u_2$ behaves like a non-trivial free solution in the large time. As far as the authors know, there are no previous results on such decoupling in the context of nonlinear wave equations.

Under the Alinhac condition, the global solution (at least for some data) behaves differently from the free solution in the large time unless the null condition is satisfied. In contrast, we may say that the global solution $u$ under the assumption of Theorem 2.2 is asymptotically free by understanding (2.6) as

$$
\|u_1(t) - u^+_1(t)\|_E = 0 \text{ with } u^+_1 \equiv 0, \text{ which trivially satisfies } \Box u^+_1 = 0;
$$

however, this case should be strictly distinguished from the situation under the null condition for the following reason. As we have stated in the introduction, if $N = 2$ and the null condition is satisfied (that is $c(\omega) \equiv 0$ for (2.2)), then the solution $u = (u_1, u_2)^T$ tends to $u^+ = (u^+_1, u^+_2)^T$ in the energy norm, where $u^+$ is the solution to $\Box u^+ = 0$ with some data $(u^+_j, \partial_t u^+_j)(0) = (f^+_j, g^+_j)$. Moreover we can easily obtain

$$
\|(f^+_j, g^+_j) - \varepsilon (f_j, g_j)\|_{\mathcal{H}_0} = O(\varepsilon^2), \quad j = 1, 2 \quad (2.9)
$$

as $\varepsilon \to +0$, which shows that the effect of the nonlinearity is rather weak. (2.6) and (2.8) make a sharp contrast to (2.9), and they are the consequence of the strong effect of the nonlinearity.

Remark 2.2. Since the condition (H) is invariant under the change of variables $(t, x) \mapsto (-t, -x)$, we can also treat the backward Cauchy problem. Hence the existence of global $C^\infty$-solution for $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ under the condition (H) follows from Theorem 2.1 provided that $\varepsilon$ is small enough. Since the form (2.2) is also invariant, we can apply Theorem 2.2 to obtain

$$
\lim_{t \to -\infty} \|u_1(t)\|_E = 0
$$

and

$$
\lim_{t \to -\infty} \|u_2(t) - u^-_2(t)\|_E = 0
$$

for the solution $u^-_2$ to $\Box u^-_2 = 0$ with some data $(u^-_2, \partial_t u^-_2)(0) = (f^-_2, g^-_2) \in \mathcal{H}_0(\mathbb{R}^3)$, provided that the assumption of Theorem 2.2 is fulfilled. Hence $u_1$ is dissipated not only forward but also backward in time.

Remark 2.3. Here we mention some related topics:

• Nonlinear Klein-Gordon systems in two space dimensions with nonlinearity of type (2.2) was considered in Kawahara-Sunagawa [15] as an example violating the null condition for the Klein-Gordon systems (see [4], [14], [15], [26], and references cited therein for the null condition for Klein-Gordon systems).
A system of nonlinear Schrödinger equations related to (2.5) was considered in Hayashi-Li-Naumkin [3] and [6], where one needs some restriction on the final state (see also [18], [13]). This might correspond to the situation in Theorem 2.2 which suggests that the final state has the special form.

3. Commuting vector fields

In this section, we recall basic properties of the vector fields associated with the wave equation. In what follows, we denote several positive constants by $C$ which may vary from one line to another. For $y \in \mathbb{R}^d$ with a positive integer $d$, the notation $(y) = (1 + |y|^2)^{1/2}$ will be often used. Also we will use the following convention on implicit constants: The expression $f = \sum_{\alpha \in \Lambda} A_\alpha$ means that there exists a family $\{A_\lambda\}_{\lambda \in \Lambda}$ of constants such that $f = \sum_{\lambda \in \Lambda} A_\lambda g_\lambda$.

Let us introduce

$$S = t \partial_t + \sum_{j=1}^3 x_j \partial_j,$$

$$L_j = t \partial_j + x_j \partial_t, \quad j \in \{1, 2, 3\},$$

$$\Omega_{jk} = x_j \partial_k - x_k \partial_j, \quad j, k \in \{1, 2, 3\},$$

$$\partial = (\partial_{a})_{a=0,1,2,3} = (\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}),$$

and we set

$$\Gamma = (\Gamma_0, \Gamma_1, \ldots, \Gamma_{10}) = (S, L_1, L_2, L_3, \Omega_{23}, \Omega_{31}, \Omega_{12}, \partial_0, \partial_1, \partial_2, \partial_3).$$

For a multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{10})$, we write $\Gamma^\alpha = \Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_{10}^{\alpha_{10}}$ and $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_{10}$. We define

$$|\phi(t, x)|_k = \left( \sum_{|\alpha| \leq k} |\Gamma^\alpha \phi(t, x)|^2 \right)^{1/2}, \quad \|\phi(t, \cdot)\|_k = \left( \sum_{|\alpha| \leq k} \|\Gamma^\alpha \phi(t, \cdot)\|_{L^2}^2 \right)^{1/2}$$

for a non-negative integer $k$ and a smooth function $\phi = \phi(t, x)$. As is well known, these vector fields satisfy $[\Box, S] = 2\Box$ and $[\Box, L_j] = [\Box, \Omega_{jk}] = [\Box, \partial_a] = 0$, where $[A, B] = AB - BA$ for linear operators $A$ and $B$. From them it follows that

$$\Box \Gamma^\alpha \phi = \tilde{\Gamma}^\alpha \Box \phi,$$

where $\tilde{\Gamma}^\alpha = (\Gamma_0 + 2)^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_{10}^{\alpha_{10}}$. We also note that

$$[\Gamma_j, \Gamma_k] = \sum_{t=0}^{10} \Gamma_t, \quad [\Gamma_j, \partial_a] = \sum_{b=0}^{3} \partial_b.$$

Hence we can check that the estimates

$$|\Gamma^\alpha \Gamma^\beta \phi| \leq C|\phi|_{|\alpha|+|\beta|},$$

$$C^{-1}|\partial \phi|_s \leq \sum_{|\alpha| \leq s} |\partial \Gamma^\alpha \phi| \leq C|\partial \phi|_s$$

(3.2)
are valid for any multi-indices $\alpha$, $\beta$ and any non-negative integer $s$.

Next we set $r = |x|$, $\omega = x_j/r$, $\omega = \sum_{j=1}^3 \omega_j \partial_j$, and $\partial_\pm = \partial_t \pm \partial_r$. We write $\omega = (\omega_j)_{j=1,2,3}$. For simplicity of exposition, we also introduce

\[ D_\pm = \frac{1}{2} \partial_\pm = \frac{1}{2} (\partial_t \pm \partial_r). \]

We summarize several useful inequalities related to $\Gamma$.

**Lemma 3.1.** For a smooth function $\phi$ of $(t, x) \in [0, \infty) \times \mathbb{R}^3$, we have

\[ |D_+(r \phi(t, x))| \leq C|\phi(t, x)|_1, \quad (3.3) \]

\[ |r \partial_t \phi(t, x) + D_-(r \phi(t, x))| \leq C|\phi(t, x)|_1, \quad (3.4) \]

and

\[ |r \partial_j \phi(t, x) - \omega_j D_-(r \phi(t, x))| \leq C|\phi(t, x)|_1 \quad (3.5) \]

for $j = 1, 2, 3$.

**Proof.** (3.3) and (3.4) are direct consequences of the following relations:

\[ D_+(r \phi) = \frac{r}{2(r + t)} (S \phi + L_r \phi) + \frac{\phi}{2}, \]

\[ r \partial_t \phi = - D_-(r \phi) + D_+(r \phi), \]

where $L_r = r \partial_t + t \partial_r = \sum_{j=1}^3 \omega_j L_j$. (3.5) follows just from

\[ r(\partial_j - \omega_j \partial_r) \phi = \sum_{k=1}^3 \omega_k \Omega_{kj} \phi \quad (3.6) \]

and

\[ r \partial_r \phi = D_-(r \phi) + D_+(r \phi) - \phi, \]

if we use (3.3) to estimate $D_+ \phi$. \qed

**Lemma 3.2.** For a smooth function $\phi$ of $(t, x) \in [0, \infty) \times \mathbb{R}^3$ and a non-negative integer $s$, we have

\[ |\partial \phi(t, x)|_s \leq C(t - |x|)^{-1} |\phi(t, x)|_{s+1}. \]

This lemma is due to Lindblad [19], which comes from the identities

\[ (t - r) \partial_t \phi = \frac{1}{t + r} (tS - rL_r) \phi, \]

\[ (t - r) \partial_r \phi = \frac{1}{t + r} (tL_r - rS) \phi, \]

and $t \Omega_{kj} \phi = x_k L_j \phi - x_j L_k \phi$, as well as (3.6) (see [19] for the detail of the proof).

We close this section with the following decay estimate for solutions to inhomogeneous wave equations.
Lemma 3.3 (Hörmander’s $L^1$–$L^\infty$ estimate). Let $\phi$ be a smooth solution to
$$\square \phi = G, \quad (t, x) \in (0, T) \times \mathbb{R}^3$$
with $\phi(0, x) = \partial_t \phi(0, x) = 0$. It holds that
$$\langle t + |x| \rangle |\phi(t, x)| \leq C \sum_{|\alpha| \leq 2} \int_0^t \|\Gamma^\alpha G(\tau, \cdot)\|_{L^1(\mathbb{R}^3)} \frac{d\tau}{\langle \tau \rangle}$$
for $0 \leq t < T$. Here the constant $C$ is independent of $T$.

See [7] for the proof (see also Lemma 6.6.8 of [8], or Section 2.1 of [25]).

Remark 3.1. Various kinds of decay estimates for homogeneous wave equations are also available. Here we only mention the following one that is a simple corollary to Lemma 3.3 via the cut-off argument (see [21] for the proof): For $R > 0$, there is a positive constant $C_R$ such that we have
$$\langle t + |x| \rangle |\phi(t, x)| \leq C_R \|\partial \phi(0)\|^2$$
for a smooth solution $\square \phi(t, x) = 0$ for $(t, x) \in (0, \infty) \times \mathbb{R}^3$, provided that $\phi(0, x) = (\partial_t \phi)(0, x) = 0$ for $|x| \geq R$.

4. THE PROFILE EQUATION

Let $0 < T \leq \infty$, and let $u$ be the solution to (1.1)–(1.2) on $[0, T) \times \mathbb{R}^3$. We suppose that
$$\text{supp } f \cup \text{supp } g \subset B_R$$
for some $R > 0$, where $B_M = \{x \in \mathbb{R}^3; |x| \leq M\}$ for $M > 0$. Then, from the finite propagation property, we have
$$\text{supp } u(t, \cdot) \subset B_{t+R}, \quad 0 \leq t < T.$$  

Now we put $r = |x|, \omega = (\omega_1, \omega_2, \omega_3) = x/|x|$ and set
$$\Delta_{S^2} = \sum_{1 \leq j < k \leq 3} \Omega_{jk}^2,$$
so that
$$r \square \phi = \partial_+ \partial_-(r \phi) - \frac{1}{r} \Delta_{S^2} \phi.$$  

We define $U = (U_1, \ldots, U_N)^T$ by
$$U(t, x) := D_- (ru(t, x)), \quad (t, x) \in [0, T) \times (\mathbb{R}^3 \setminus \{0\})$$
for the solution $u$ of (1.1). In view of (3.4) and (3.5), the asymptotic profiles as $t \to \infty$ of $\partial_+ u$ and $\nabla_x u$ should be given by $-U/r$ and $\omega U/r$, respectively, because we can expect $|u(t, x)|_1 \to 0$ as $t \to \infty$. Also it follows from (4.3) that
$$\partial_+ U(t, x) = -\frac{1}{2t} F^{\text{red}}(\omega, U(t, x)) + H(t, x),$$
where $F^{\text{red}} = F^{\text{red}}(\omega, Y)$ is defined by (1.5), and $H = H(t, x)$ is given by

$$H = -\frac{1}{2} \left( r F(\partial u) - \frac{1}{t} F^{\text{red}}(\omega, U) \right) - \frac{1}{2r} \Delta_{S^2} u. \tag{4.6}$$

As we will see in Lemma 4.1 below, $H$ can be regarded as a remainder. For these reasons, we call (4.5) the profile equation associated with (1.1), which plays an important role in our analysis. Observe that the reduced system (1.8) is obtained by neglecting $H$ and changing variables in (4.5) (see (5.19) below).

We also need an analogous equation for $\Gamma^\alpha u$ with a multi-index $\alpha$. For this purpose, we put

$$U^{(\alpha)}(t, x) = (U_1^{(\alpha)}(t, x), \ldots, U_N^{(\alpha)}(t, x))^T := D_-(r \Gamma^\alpha u(t, x)). \tag{4.7}$$

Since $\square(\Gamma^\alpha u) = \tilde{\Gamma}^\alpha (F(\partial u))$, we deduce from (4.3) that

$$\partial_+ U^{(\alpha)} = -\frac{1}{2r} G_\alpha(\omega, U, U^{(\alpha)}) + H_\alpha \tag{4.8}$$

for $|\alpha| \geq 1$, where $G_\alpha = (G_{\alpha,j})_{1 \leq j \leq N}$ is given by

$$G_{\alpha,j}(\omega, U, U^{(\alpha)}) = \sum_{k=1}^N \frac{\partial F^{\text{red}}}{\partial Y_k}(\omega, U) U_k^{(\alpha)}$$

$$= \sum_{k,l=1}^N \sum_{a,b=0}^3 c_{j}^{kl,ab} \omega_a \omega_b \left( U_k U_l^{(\alpha)} + U_k^{(\alpha)} U_l \right) \tag{4.9}$$

with the constants $c_{j}^{kl,ab}$ appeared in (1.3), and $H_\alpha$ is given by

$$H_\alpha(t, x) = -\frac{1}{2} \left( r \tilde{\Gamma}^\alpha F(\partial u) - \frac{1}{t} G_\alpha(\omega, U, U^{(\alpha)}) \right) - \frac{1}{2r} \Delta_{S^2} \Gamma^\alpha u. \tag{4.10}$$

In the rest part of this section, we focus on preliminary estimates for $H$ and $H_\alpha$ in terms of the solution $u$ near the light cone. To be more specific, we put

$$A_{T,R} := \{ (t, x) \in [0, T) \times \mathbb{R}^3; 1 \leq t/2 \leq |x| \leq t + R \}.$$

Note that we have

$$(1 + t + |x|)^{-1} \leq |x|^{-1} \leq 2t^{-1} \leq 3(1 + t)^{-1} \leq 3(R + 2)(1 + t + |x|)^{-1}$$

for $(t, x) \in A_{T,R}$. In other words, the weights $(t + |x|)^{-1}, (1 + t)^{-1}, |x|^{-1}$, and $t^{-1}$ are equivalent to each other in $A_{T,R}$. For a non-negative integer $s$, we also introduce an auxiliary notation $| \cdot |_{t,s}$ by

$$|\phi(t, x)|_{s+1} := |\partial \phi(t, x)|_s + (t + |x|)^{-1} |\phi(t, x)|_{s+1}. \tag{4.11}$$

Lemma 4.1. We have

$$|H(t, x)| \leq C \left( |u(t, x)|_{1,0} |u(t, x)|_1 + t^{-1} |u(t, x)|_2 \right),$$
for \((t,x) \in \Lambda_{T,R}\). Here the constant \(C\) is independent of \(T\). Also, in the case of \(s \geq 1\), we have
\[
\sum_{|\alpha| = s} |H_\alpha(t,x)| \leq C_s(|u|_{2,s+1} + t|\partial u|_{s-1}^2 + t^{-1}|u|_{s+2})
\]
for \((t,x) \in \Lambda_{T,R}\), where \(C_s\) is a positive constant which does not depend on \(T\).

Proof. Let \((t,x) = (t,r\omega) \in \Lambda_{T,R}\) and \(|\alpha| = s \geq 0\). First we note that
\[
|U^{(\alpha)}(t,x)| \leq r|D_- \Gamma^\alpha u| + \frac{1}{2}|\Gamma^\alpha u|
\]
by the definition of \(|\cdot|_{s,s}\), and that
\[
(t-r)|U^{(\alpha)}(t,x)| \leq Ct \left( (t-r)|\partial u(t,x)|_s + \frac{(t-r)}{(t+r)}|u(t,x)|_s \right)
\]
\[
\leq Ct|u(t,x)|_{s+1}
\]
(4.13)
by Lemma 3.2.

Now we consider the estimate for \(H\). We decompose it as follows:
\[
H = \frac{1}{2r} \left( r^2 F(\partial u) - F^{\text{red}}(\omega, U) \right) - \frac{t-r}{2rt} F^{\text{red}}(\omega, U) - \frac{1}{2r} \Delta_{S^2} u.
\]
It is easy to see that the third term can be dominated by \(Ct^{-1}|u|_2\). As for the second term, we have
\[
\left| \frac{t-r}{rt} |F^{\text{red}}(\omega, U)| \right| \leq Ct^{-1} (t-r) |U| : t^{-1} |U| \leq C |u|_1 |u|_{L,0},
\]
because of (4.12) and (4.13) with \(s = 0\). To estimate the first term, noting that (3.3) and (3.5) imply
\[
|\partial u_k(\omega k)\omega_l - (\omega_k \omega_l)\partial u| \leq |\partial u_k - \omega_k U_k|, k \partial u_l - \omega_l U_l| \leq C(r |\partial u| + |U|)|u|_1
\]
with \(\omega_0 = -1\), and that (1.3) yields
\[
r^2 F_j(\partial u) - F^{\text{red}}_j(\omega, U) = \sum_{k,l=1}^{N} \sum_{a,b=0}^{3} c_{j,k,l}^{a,b} \left( (\partial a u_k)(\partial b u_l) - (\omega_k \omega_l)(\partial a u_l) \right),
\]
we obtain
\[
\frac{1}{2r} \left| r^2 F(\partial u) - F^{\text{red}}(\omega, U) \right| \leq C (|\partial u| + r^{-1} |U|) |u|_1 \leq C |u|_{L,0} |u|_1
\]
with the help of (4.12).

Next we turn to the estimate for \(H_\alpha\) with \(|\alpha| = s \geq 1\). For this purpose, we set 
\[
\tilde{F}_\alpha = (\tilde{F}_{\alpha,j})_{1 \leq j \leq N}^{T} \text{ with }
\]
\[
\tilde{F}_{\alpha,j} = \sum_{k,l=1}^{N} \sum_{a,b=0}^{3} c_{j,k,l}^{a,b} \left( (\partial a u_k)(\Gamma^\alpha \partial b u_l) + (\partial a u_k)(\partial b u_l) \right)
\]
to split $H_\alpha$ into the following form:

$$H_\alpha = -\frac{r}{2} \left( \tilde{\Gamma}^\alpha F(\partial u) - \tilde{F}_\alpha \right) - \frac{1}{2r} \left( r^2 \tilde{F}_\alpha - G_\alpha \right) - \frac{t - r}{2rt} G_\alpha - \frac{1}{2r} \Delta_{\mathbb{S}^2} \Gamma^\alpha u.$$ 

Since the first term consists of a linear combination of the terms in the form $r(\Gamma^\beta \partial a u_k)(\Gamma^\gamma \partial b u_l)$ with $|\beta|, |\gamma| \leq s-1$, $k, l \in \{1, \ldots, N\}$, and $a, b \in \{0, 1, 2, 3\}$, it can be estimated by $C t |\partial u|^2_{s-1}$. Other terms can be treated in the same way as in the previous case.

5. Proof of Theorem 2.1

Let $u(t, x)$ be a smooth solution to (1.1)–(1.2) on $[0, T_0) \times \mathbb{R}^3$ with some $T_0 \in (0, \infty]$. For $0 < T \leq T_0$, we put

$$e[u](T) = \sup_{(t,x) \in [0,T) \times \mathbb{R}^3} \left( (t + |x|)(t - |x|)^{1-\mu} |\partial u(t, x)| + (t + |x|)^{1-\nu}(t - |x|)^{1-\mu} |\partial u(t, x)|_k \right)$$

with some $\mu, \nu > 0$ and a positive integer $k$. We also put

$$e[u](0) = \lim_{T \to +0} e[u](T).$$

Observe that there is a positive constant $\varepsilon_1$ such that $0 < \varepsilon \leq \varepsilon_1$ implies

$$e[u](0) \leq \sqrt{\varepsilon}/2,$$

because we have $e[u](0) = O(\varepsilon)$.

The main step toward global existence is to show the following.

**Proposition 5.1 (A priori estimate).** Let $k \geq 3$, $0 < \mu < 1/2$, and $0 < 4(k+1)\nu \leq \mu$. There exist positive constants $\varepsilon_2$ and $m$, which depend only on $k$, $\mu$ and $\nu$, such that

$$e[u](T) \leq \sqrt{\varepsilon}$$

implies

$$e[u](T) \leq m\varepsilon,$$  \hspace{1cm} (5.2)

provided that $0 < \varepsilon \leq \varepsilon_2$ and $0 < T \leq T_0$.

Once the above proposition is obtained, we can show the small data global existence for (1.1)–(1.2) by the so-called continuity argument: Let $T^*$ be the lifespan of the classical solution for (1.1)–(1.2) and assume $T^* < \infty$. Then, it follows from the standard blow-up criterion (see e.g., [25]) that

$$\lim_{t \to T^*-0} \|\partial u(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} = \infty.$$  \hspace{1cm} (5.3)

On the other hand, by setting

$$T_* = \sup \{ T \in [0, T^*) ; e[u](T) \leq \sqrt{\varepsilon} \},$$

we can see that Proposition 5.1 yields $T_* = T^*$, provided that $\varepsilon$ is small enough. Indeed, if $T_* < T^*$, then we have $e[u](T_*) \leq \sqrt{\varepsilon}$, and Proposition 5.1 implies that

$$e[u](T_*) \leq m\varepsilon \leq \frac{\sqrt{\varepsilon}}{2}$$
for \( 0 < \varepsilon \leq \min\{\varepsilon_1, \varepsilon_2, 1/4m^2\} \) (note that we have \( T_* > 0 \) for \( \varepsilon \leq \varepsilon_1 \)). Then, by the continuity of \([0, T^*) \ni T \mapsto e[u](T)\), we can take \( \delta > 0 \) such that
\[
e[u](T_* + \delta) \leq \sqrt{\varepsilon},
\]
which contradicts the definition of \( T_* \), and we conclude that \( T_* = T^* \).

In particular, we have
\[
e[u](T^*) \leq \sqrt{\varepsilon}.
\]
This implies that (5.3) never occurs for small \( \varepsilon \). In other words, we must have \( T^* = \infty \), that is, the solution \( u \) exists globally for small data. We also note that
\[
e[u](\infty) \leq \sqrt{\varepsilon} \quad (5.4)
\]
holds for this global solution \( u \), and Proposition 5.1 again yields
\[
e[u](\infty) \leq m\varepsilon. \quad (5.5)
\]

Now we turn to the proof of Proposition 5.1. It will be divided into several steps.

Proof of Proposition 5.1. In what follows, we always suppose \( 0 \leq t < T \).

**Step 1: Rough bounds for \( |u(t, x)|_{k+2} \) and \( |\partial u(t, x)|_{k+1} \).**

First of all, we will establish the following energy estimates:
\[
\|\partial u(t)\|_l \leq C_\varepsilon(1 + t)^{C_*\sqrt{\varepsilon} + l\nu} \quad (5.6)
\]
for \( l \in \{0, 1, \ldots, 2k + 1\} \), where \( C_* \) is a positive constant to be fixed later.

In preparation for the proof of (5.6), we make some observations: Let \( 0 \leq l \leq 2k + 1 \). In what follows we neglect terms including \( |\partial u|_{l-1} \) or \( |\partial u|_{l-1} \) when \( l = 0 \). From (3.1), (3.2), and the standard energy inequality, we get
\[
\|\partial u(t)\|_l \leq C_{1,l} \left( \|\partial u(0)\|_l + \int_0^t \|F(\partial u(\tau))\|_l d\tau \right), \quad (5.7)
\]
where \( C_{1,l} \) is a positive constant depending only on \( l \). From (5.1) we have
\[
|\partial u(t, x)| \leq \sqrt{2\varepsilon}(1 + t)^{-1} \quad \text{and} \quad |\partial u(t, x)|_k \leq \sqrt{2\varepsilon}(1 + t)^{-1}, \quad \text{since} \quad \langle t + |x| \rangle^{-1} \leq \sqrt{2}(1 + t)^{-1}.
\]
Hence we get
\[
|F(\partial u)|_l \leq C_{2,l} \left( |\partial u| |\partial u| + |\partial u|_{l/2} |\partial u|_{l-1} \right)
\]
\[
\leq C_{2,l} \sqrt{2\varepsilon} \left( (1 + t)^{-1} |\partial u| + (1 + t)^{-1} |\partial u|_{l-1} \right)
\]
with a positive constant \( C_{2,l} \) depending only on \( l \), which leads to
\[
\|F(\partial u(t))\|_l \leq \sqrt{2}C_{2,l} \sqrt{\varepsilon} \left( (1 + t)^{-1} \|\partial u(t)\|_l + (1 + t)^{-1} \|\partial u(t)\|_{l-1} \right). \quad (5.8)
\]

Now we put \( C_* = \max_{0 \leq l \leq 2k+1} \sqrt{2}C_{1,l}C_{2,l} \), and we shall prove (5.6) by induction on \( l \). If \( l = 0 \), it follows from (5.7) and (5.8) that
\[
\|\partial u(t)\|_0 \leq C\varepsilon + C_*\sqrt{\varepsilon} \int_0^t (1 + \tau)^{-1} \|\partial u(\tau)\|_0 d\tau,
\]
whence the Gronwall lemma implies
\[
\|\partial u(t)\|_0 \leq C\varepsilon(1 + t)^{C_*\sqrt{\varepsilon}}.
\]
Next we assume that (5.6) holds for some \( l \in \{0, 1, \ldots, 2k\} \). Then it follows from (5.7) and (5.8) that
\[
\|\partial u(t)\|_{l+1} \leq C\varepsilon + C_\varepsilon \varepsilon \int_0^t \left((1 + \tau)^{-1}\|\partial u(\tau)\|_{l+1} + (1 + \tau)^{-1+\nu}\|\partial u(\tau)\|_l\right) d\tau \\
\leq C\varepsilon + C_\varepsilon \varepsilon \int_0^t (1 + \tau)^{-1}\|\partial u(\tau)\|_{l+1} d\tau \\
+ C\varepsilon^{3/2} \int_0^t (1 + \tau)^{-1+3\sqrt{\varepsilon^2 + (t+\tau)^{1+\nu}}} d\tau \\
\leq C\varepsilon + C_\varepsilon \varepsilon \int_0^t (1 + \tau)^{-1}\|\partial u(\tau)\|_{l+1} d\tau + C\varepsilon^{3/2}(1 + t)^{3\sqrt{\varepsilon^2 + (t+\tau)^{1+\nu}}},
\]
which yields
\[
\|\partial u(t)\|_{l+1} \leq C\varepsilon(1 + t)^{3\sqrt{\varepsilon^2 + (t+\tau)^{1+\nu}}} + C\varepsilon^{3/2}(1 + t)^{3\sqrt{\varepsilon^2 + (t+\tau)^{1+\nu}}} \leq C\varepsilon(1 + t)^{3\sqrt{\varepsilon^2 + (t+\tau)^{1+\nu}}}.
\]
This means that (5.6) remains true when \( l \) is replaced by \( l + 1 \), and (5.6) has been proved for all \( l \in \{0, 1, \ldots, 2k+1\} \).

From now on, we assume that \( \varepsilon \leq \nu^2 / C_\varepsilon^2 \). Then, since \( k \geq 3 \) and \( 2(k+1)\nu \leq \mu/2 \), it follows from (5.6) with \( l = 2k+1 \) that
\[
\|\partial u(t)\|_{k+4} \leq \|\partial u(t)\|_{2k+1} \leq C\varepsilon(t)^{2(k+1)\nu} \leq C\varepsilon(t)^{\mu/2}
\]
and
\[
\| |F(\partial u(t, \cdot))|_{k+4}\|_{L^1(\mathbb{R}^3)} \leq C\|\partial u(t)\|_{k+4} \leq C\varepsilon^2(t)\mu.
\]
Hence Lemma 3.3 and Remark 3.1 yield
\[
\langle t + |x| \rangle |u(t, x)|_{k+2} \leq C_\varepsilon \|\partial u(0)\|_{k+4} + C \int_0^t \frac{\| |F(\partial u(\tau))|_{k+4}\|_{L^1}}{\langle \tau \rangle} d\tau \\
\leq C\varepsilon + C\varepsilon^2 \int_0^t \langle \tau \rangle^{\mu-1} d\tau \leq C\varepsilon \langle t + |x| \rangle^\mu,
\]
that is,
\[
|u(t, x)|_{k+2} \leq C\varepsilon \langle t + |x| \rangle^{-1+\mu}
\]
for \( (t, x) \in [0, T) \times \mathbb{R}^3 \). By Lemma 3.2 we also have
\[
|\partial u(t, x)|_{k+1} \leq C\varepsilon \langle t + |x| \rangle^{-1+\mu} \langle t - |x| \rangle^{-1}
\]
for \( (t, x) \in [0, T) \times \mathbb{R}^3 \).

**Step 2: Estimates for \( |\partial u(t, x)|_k \) away from the light cone.**

Now we put \( \Lambda_{T, R} := ([0, T] \times \mathbb{R}^3) \setminus \Lambda_{T, R} \), where \( R \) is the constant appearing in (4.1). In the case of \( t/2 < 1 \) or \( |x| < t/2 \), we see that
\[
\langle t - |x| \rangle \leq \langle t + |x| \rangle \leq C\langle t - |x| \rangle.
\]
On the other hand, it follows from (1.2) that \( u(t, x) = 0 \) if \( |x| > t + R \). Hence (5.11) implies
\[
\sup_{(t, x) \in \Lambda_{T, R}} \langle t + |x| \rangle^{-1-\mu} |\partial u(t, x)|_k \leq C\varepsilon.
\]
Step 3: Estimates for $|\partial u(t, x)|$ near the light cone.

Let $(t, x) \in \Lambda_{T, R}$ throughout this step. Remember that $t^{-1}, |x|^{-1}, \langle t \rangle^{-1}$, and $(t + |x|)^{-1}$ are equivalent to each other in $\Lambda_{T, R}$. We define $U, U^{(\alpha)}, H, H_\alpha$ and $| \cdot |_\beta$ as in Section 4 (see (4.4), (4.6), (4.7), (4.10), and (4.11)). We see from (5.10) and (5.11) that

$$|u(t, x)|_{t, k} \leq C \varepsilon t^{\mu - 1} \langle t - |x| \rangle^{-1}.$$  

(5.13)

By (3.2), (3.4), (3.5), and (5.10), we have

$$t |\partial u(t, x)|_t \leq C \sum_{|\alpha| \leq l} |x| |\partial^\alpha u(t, x)|$$

$$\leq C \sum_{|\alpha| \leq l} |U^{(\alpha)}(t, x)| + C \varepsilon t^{\mu - 1}$$  

(5.14)

for $l \leq k$. Also, it follows from (5.10), (5.13), and Lemma 4.1 that

$$|H(t, x)| \leq C \left( \varepsilon^2 t^{2\mu - 2} \langle t - |x| \rangle^{-1} + \varepsilon t^{\mu - 2} \right) \leq C \varepsilon t^{2\mu - 2} \langle t - |x| \rangle^{-\mu}.$$  

(5.15)

Next we put

$$\Sigma = \{(t, x) \in \Lambda_{T, R}; \; t/2 = 1 \text{ or } t/2 = |x|\}$$

and we define $t_{0, \sigma} = \max\{2, -2\sigma\}$ for $\sigma \leq R$. What is important here is that the line segment $\{(t, (t + \sigma) \omega); 0 \leq t < T\}$ meets $\Sigma$ at the point $(t_{0, \sigma}, (t_{0, \sigma} + \sigma) \omega)$ for each fixed $(\sigma, \omega) \in (-\infty, R] \times S^2$. We also remark that

$$C^{-1}(\sigma) \leq t_{0, \sigma} \leq C(\sigma), \quad \sigma \leq R.$$  

(5.16)

When $(t, x) \in \Sigma$, we have $t^\mu \leq C \langle t - |x| \rangle^\mu$. So it follows from (4.12) and (5.13) that

$$\sum_{|\alpha| \leq k} |U^{(\alpha)}(t, x)| \leq C \varepsilon t^\mu \langle t - |x| \rangle^{-1} \leq C \varepsilon \langle t - |x| \rangle^{\mu - 1}, \quad (t, x) \in \Sigma.$$  

(5.17)

Now we define

$$V(t; \sigma, \omega) = (V_1(t; \sigma, \omega), \ldots, V_N(t; \sigma, \omega))^T = U(t, (t + \sigma) \omega)$$  

(5.18)

for $0 \leq t < T$ and $(\sigma, \omega) \in \mathbb{R} \times S^2$. In what follows, we fix $(\sigma, \omega) \in (-\infty, R] \times S^2$ and write $V(t)$ for $V(t; \sigma, \omega)$. Then, since the profile equation (4.5) is rewritten as

$$\frac{\partial V}{\partial t}(t) = (\partial_t U)(t, (t + \sigma) \omega) = -\frac{1}{2t} F^{\text{red}}(\omega, V(t)) + H(t, (t + \sigma) \omega)$$  

(5.19)
for \( t_{0,\sigma} < t < T \), it follows from the condition (H) that
\[
\frac{\partial}{\partial t} \left( V(t)^T A(\omega)V(t) \right) = 2V(t)^T A(\omega) \frac{\partial V}{\partial t}(t)
\]
\[
= 2V(t)^T A(\omega) \left( -\frac{1}{2t} F^{\text{red}}(\omega, V(t)) + H(t, (t + \sigma)\omega) \right)
\]
\[
= 2V(t)^T A(\omega) H(t, (t + \sigma)\omega)
\]
\[
\leq C|V(t)| |H(t, (t + \sigma)\omega)|
\]
\[
\leq C\sqrt{V(t)^T A(\omega)V(t)}|H(t, (t + \sigma)\omega)|
\]
(5.20)
for \( t_{0,\sigma} < t < T \), where we have used (2.1) to obtain the last line. We also note that (5.17) for \( k = 0 \) can be interpreted as
\[
|V(t_{0,\sigma})| = |V(t_{0,\sigma} + (t_{0,\sigma} + \sigma)\omega)| \leq C\varepsilon \langle \sigma \rangle^{\mu-1}.
\]
(5.21)
From (2.1), (5.15), (5.16), (5.20), and (5.21) we get
\[
|V(t)| \leq \sqrt{M_0 V(t)^T A(\omega)V(t)}
\]
\[
\leq C \left( \frac{\sqrt{V(t_{0,\sigma})^T A(\omega)V(t_{0,\sigma})}}{t_{0,\sigma}} + \int_{t_{0,\sigma}}^t |H(\tau, (\tau + \sigma)\omega)|d\tau \right)
\]
\[
\leq C\varepsilon \langle \sigma \rangle^{\mu-1} + C\varepsilon \langle \sigma \rangle^{-\mu} \int_{t_{0,\sigma}}^t \tau^{2\mu-2}d\tau
\]
\[
\leq C\varepsilon \left( \langle \sigma \rangle^{\mu-1} + \langle \sigma \rangle^{-\mu} t_{0,\sigma}^{2\mu-1} \right)
\]
\[
\leq C\varepsilon \langle \sigma \rangle^{\mu-1}
\]
(5.22)
for \( t \geq t_{0,\sigma} \), where \( C \) is independent of \( \varepsilon, \sigma, \) and \( \omega \).
(5.22) implies
\[
|U(t, x)| = |V(t; |x| - t, x/|x|)| \leq C\varepsilon (t - |x|)^{\mu-1}, \quad (t, x) \in \Lambda_{T,R}.
\]
Finally, in view of (5.14) with \( l = 0 \), we obtain
\[
\sup_{(t, x) \in \Lambda_{T,R}} (t + |x|)(t - |x|)^{1-\mu}\right|\partial u(t, x)\right| \leq C\varepsilon.
\]
(5.23)
We remark that the derivation of (5.20) is the only point where we make use of the condition (H) throughout this proof.

**Step 4: Estimates for \(|\partial u(t, x)|_k\) near the light cone.**
We assume \((t, x) \in \Lambda_{T,R}\) also in this step. For a non-negative integer \( s \), we set
\[
U^{(s)}(t, x) := \sum_{|\beta| \leq s} |U^{(\beta)}(t, x)|.
\]
Let \( 1 \leq |\alpha| \leq k \). By (5.14) we get
\[
|\partial u(t, x)|_{|\alpha|-1} \leq C \left( t^{-1} U^{(|\alpha|-1)}(t, x) + \varepsilon \mu^{-2} \right).
\]
(5.24)
It follows from (5.10), (5.13), (5.24), and Lemma 4.1 that

\[ |H_\alpha(t, x)| \leq C \left( \varepsilon t^{2^{-1} \mu - 2} - |x|^{-1} + \varepsilon t^{\mu - 2} + \varepsilon^2 t^{2\mu - 3} + t^{-1}U^{(|\alpha| - 1)}(t, x) \right)^2 \leq C \varepsilon t^{2\mu - 2} - |x|^{-\mu} + C t^{-1}U^{(|\alpha| - 1)}(t, x)^2. \] (5.25)

We put

\[ V^{(\alpha)}(t; \sigma, \omega) = U^{(\alpha)}(t, (t + \sigma)\omega) \]

for 0 \leq t < T and (\sigma, \omega) \in (-\infty, R] \times S^2. We fix (\sigma, \omega) \in (-\infty, R] \times S^2 and write \( V^{(\alpha)}(t) \) for \( V^{(\alpha)}(t; \sigma, \omega) \). Then (4.8) is rewritten as

\[ \frac{\partial V^{(\alpha)}}{\partial t}(t) = -\frac{1}{2t}G_\alpha(\omega, V(t), V^{(\alpha)}(t)) + H_\alpha(t, (t + \sigma)\omega) \]

for \( t_{0, \sigma} < t < T \). Hence by (5.22) and (5.25) we obtain

\[ \frac{\partial}{\partial t}|V^{(\alpha)}(t)|^2 \leq \frac{C}{t} |V(t)||V^{(\alpha)}(t)|^2 + 2|H_\alpha(t, (t + \sigma)\omega)| |V^{(\alpha)}(t)| \leq 2C^* \varepsilon |V^{(\alpha)}(t)|^2 + C \left( \varepsilon t^{2\mu - 2} - |\mu| + t^{-1}(U^{(|\alpha| - 1)}(t))^2 \right) |V^{(\alpha)}(t)|, \]

where

\[ V^{(\sigma)}(t) = V^{(\sigma)}(t; \sigma, \omega) := \sum_{|\beta| \leq \alpha} |V^{(\beta)}(t; \sigma, \omega)|, \]

and \( C^* \) is a positive constant independent of \( \alpha \). Therefore it follows from (5.16) and (5.17) that

\[ t^{-C^* \varepsilon} |V^{(\alpha)}(t)| \leq t_{0, \sigma}^{-C^* \varepsilon} |V^{(\alpha)}(t_{0, \sigma})| + C \varepsilon \langle \sigma \rangle^{-\mu} \int_{t_{0, \sigma}}^t \tau^{-C^* \varepsilon + 2\mu - 2} d\tau \]

\[ + C \int_{t_{0, \sigma}}^t \tau^{-C^* \varepsilon - 1}(U^{(|\alpha| - 1)}(\tau))^2 d\tau \]

\[ \leq C \varepsilon \langle \sigma \rangle^{-\mu - 1} + C \int_{t_{0, \sigma}}^t \tau^{-C^* \varepsilon - 1}(U^{(|\alpha| - 1)}(\tau))^2 d\tau. \]

To sum up with respect to \(|\alpha| \leq l\), we have

\[ t^{-C^* \varepsilon} \mathcal{V}^{(l)}(t) \leq C \varepsilon \langle \sigma \rangle^{-\mu - 1} + C \int_{t_{0, \sigma}}^t \tau^{-C^* \varepsilon - 1}(U^{(l - 1)}(\tau))^2 d\tau \]

for \( l \in \{1, \ldots, k\} \). Using this inequality, we can show inductively that

\[ \mathcal{V}^{(l)}(t) \leq C \varepsilon \langle \sigma \rangle^{-\mu - 1} t^{2l - 1} \]

(5.26)

for \( t_{0, \sigma} \leq t < T \) and \( l \in \{1, \ldots, k\} \). Indeed, we already know that

\[ \mathcal{V}^{(0)}(t) = |V(t)| \leq C \varepsilon \langle \sigma \rangle^{-\mu - 1} \]

by (5.22). Hence we have

\[ t^{-C^* \varepsilon} \mathcal{V}^{(1)}(t) \leq C \varepsilon \langle \sigma \rangle^{-\mu - 1} + C \varepsilon^2 \langle \sigma \rangle^{2\mu - 2} \int_{t_{0, \sigma}}^\infty \tau^{-C^* \varepsilon - 1} d\tau \leq C \varepsilon \langle \sigma \rangle^{-\mu - 1}, \]
which implies (5.26) for $l = 1$. Next we suppose that (5.26) is true for some $l \in \{1, \ldots, k - 1\}$. Then we have

\[
t^{-C^*} \mathcal{V}^{(l+1)}(t) \leq C \varepsilon \langle \sigma \rangle^{\mu-1} + C \varepsilon^2 \langle \sigma \rangle^{2\mu-2} \int_{2}^{t} \tau^{(2^l-1)C^*\varepsilon-1} d\tau
\]

\[
\leq C \varepsilon \langle \sigma \rangle^{\mu-1} t^{(2^l-1)C^*\varepsilon},
\]

which yields (5.26) with $l$ replaced by $l + 1$. Hence (5.26) for $l \in \{1, \ldots, k\}$ has been proved.

By (5.14) and (5.26) with $l = k$, we have

\[
|\partial u(t, x)|_{k} \leq C \varepsilon (t + |x|)^{-1+2^{-k-1}C^*\varepsilon} (t - |x|)^{-1+\mu}, \quad (t, x) \in \Lambda_{T,R}.
\]

Finally we take $\varepsilon \leq 2^{1-k}\nu/C^*$ to obtain

\[
\sup_{(t,x) \in \Lambda_{T,R}} (t + |x|)^{1-\nu} (t - |x|)^{1-\mu} |\partial u(t, x)|_{k} \leq C \varepsilon.
\]

(5.27)

**The final step.**

By (5.12), (5.23), and (5.27), we see that there exist two positive constants $\varepsilon_2$ and $m$ such that (5.2) holds for $0 < \varepsilon \leq \varepsilon_2$. This completes the proof of Proposition 5.1. □

### 6. Asymptotics for the Solution to the Profile Equation

This section is devoted to preliminaries for the proof of Theorem 2.2. We assume $N = 2$ and (2.2) with $c_0 > 0$ throughout this section. Let $u = (u_1, u_2)^T$ be the global solution to (1.1)–(1.2), whose existence is guaranteed by Theorem 2.1 for small $\varepsilon$, and let $U = (U_1, U_2)^T$ be given by (4.4). For simplicity of exposition, we introduce a complex-valued function

\[
U_c(t, x) := \sqrt{c_0} U_1(t, x) + ic_0 U_2(t, x),
\]

(6.1)

where $i = \sqrt{-1}$. Then it follows from (2.2) and (4.5) that

\[
\partial_+ U_c(t, x) = -\frac{i}{2t} c(\omega) (\text{Re} U_c(t, x)) U_c(t, x) + H_c(t, x)
\]

(6.2)

with $H_c = \sqrt{c_0} H_1 + ic_0 H_2$, where $c(\omega)$ is given by (2.3), and $H = (H_1, H_2)^T$ by (4.4).

Let $t_0 \geq 1$. Keeping the application to the profile equation (6.2) in mind, we consider the following ordinary differential equation for $t > t_0$:

\[
l \frac{dz}{dt}(t) = \frac{\Phi(z(t))}{t} z(t) + J(t),
\]

(6.3)

where $\Phi : \mathbb{C} \to \mathbb{R}$ satisfies

\[
|\Phi(z) - \Phi(w)| \leq C_0 |z - w| \quad \text{for } z, w \in \mathbb{C}
\]

(6.4)

with a positive constant $C_0$, and $J : [t_0, \infty) \to \mathbb{C}$ satisfies

\[
|J(t)| \leq E_0 t^{-1-\lambda}
\]

(6.5)
with positive constants $E_0$ and $\lambda$. The important structure here is that $\Phi$ is real-valued. Concerning the asymptotics for the solution $z(t)$ of (6.3), we have the following lemma.

**Lemma 6.1.** Let $z(t)$ be the global solution of (6.3), and suppose

$$C_0(E_0 t_0^{-\lambda} + |z(t_0)|\lambda) < \lambda^2.$$  

Then there is a $C^1$-function $p = p(s)$ on $[\log t_0, \infty)$ such that we have

$$|z(t) - p(\log t)| \leq \frac{E_0 \lambda}{\lambda^2 - C_0(E_0 t_0^{-\lambda} + |z(t_0)|\lambda)} t^\lambda, \quad t \geq t_0,$$  

and

$$i \frac{dp}{ds}(s) = \Phi(p(s)) p(s), \quad s \geq \log t_0.$$  

To prove Lemma 6.1, we introduce some sequences. For the solution $z(t)$ of (6.3), we define sequences $\{z_n(t)\}_{n=0}^{\infty}$, $\{\Theta_n(t)\}_{n=0}^{\infty}$, and $\{\zeta_n\}_{n=0}^{\infty}$ in the following way: We set $z_0(t) = z(t)$, and inductively define

$$\Theta_n(t) = \int_{t_0}^{t} \Phi(z_n(\tau)) d\tau, \quad t \geq t_0,$$  

$$\zeta_n = \lim_{\tau \to \infty} z_n(\tau) e^{i\Theta_n(\tau)},$$  

$$z_{n+1}(t) = \zeta_n e^{-i\Theta_n(t)}, \quad t \geq t_0$$

for $n \in \mathbb{N}_0$, where $\mathbb{N}_0$ denotes the set of non-negative integers. In order to see that this definition works well, we have only to check the convergence of $\lim_{\tau \to \infty} z_n(\tau) e^{i\Theta_n(\tau)}$ for each $n$.

**Lemma 6.2.** The above sequences $\{z_n(t)\}_{n=0}^{\infty}$, $\{\Theta_n(t)\}_{n=0}^{\infty}$, and $\{\zeta_n\}_{n=0}^{\infty}$ are well-defined. Moreover we have

$$\zeta_n = \left( z(t_0) - i \int_{t_0}^{\infty} J(\tau) e^{i\Theta_0(\tau)} d\tau \right)$$

$$\times \exp \left( i \int_{t_0}^{\infty} \{\Phi(z_n(\tau)) - \Phi(z_0(\tau))\} \frac{d\tau}{\tau} \right)$$

and

$$|z_{n+1}(t) - z_n(t)| \leq \frac{E_0}{\lambda t^\lambda} \left( C_0(E_0 t_0^{-\lambda} + |z(t_0)|\lambda) \right)^n$$

for $n \in \mathbb{N}_0$.

**Proof.** We prove Lemma 6.2 by the induction on $n$.

First we consider the case of $n = 0$. Since $z_0 = z$, it follows from (6.3) that

$$\left( z_0(t) e^{i\Theta_0(t)} \right)' = -i J(t) e^{i\Theta_0(t)},$$

which yields

$$z_0(t) e^{i\Theta_0(t)} = z(t_0) - i \int_{t_0}^{t} J(\tau) e^{i\Theta_0(\tau)} d\tau.$$
This shows that \( z_0(\tau) e^{i\Theta_0(\tau)} \) converges as \( \tau \to \infty \), and that (6.11) holds for \( n = 0 \), because (6.5) implies \( J(\cdot) e^{i\Theta_0(\cdot)} \in L^1(t_0, \infty) \). As for (6.12) with \( n = 0 \), we have

\[
(z_1(t) - z_0(t)) e^{i\Theta_0(t)} = \zeta_0 - z_0(t) e^{i\Theta_0(t)} = -i \int_t^\infty J(\tau) e^{i\Theta_0(\tau)} d\tau,
\]

whence

\[
|z_1(t) - z_0(t)| \leq \int_t^\infty |J(\tau)| d\tau \leq \frac{E_0}{\lambda t^\lambda}.
\]

Note that by (6.5) we have

\[
|\zeta_0| = |z(t_0) - i \int_{t_0}^\infty J(\tau) e^{i\Theta_0(\tau)} d\tau| \leq |z(t_0)| + \frac{E_0}{\lambda t_0^\lambda}. \tag{6.13}
\]

Next we consider the case of \( n = n_0 + 1 \) under the assumption that \( \zeta_n \) for \( n \leq n_0 \) are well-defined (thus \( z_n(t) \) and \( \Theta_n(t) \) for \( n \leq n_0 + 1 \) are also well-defined), and that (6.11) and (6.12) are true for \( n \leq n_0 \). We set \( K = C_0 E_0 t_0^{-\lambda} + |z(t_0)|/\lambda \). By (6.4) and (6.12) for \( n = n_0 \), we get

\[
|\Phi (z_{n_0+1}(t)) - \Phi (z_{n_0}(t))| \leq C_0 |z_{n_0+1}(t) - z_{n_0}(t)| \leq \frac{C_0 E_0}{\lambda t^\lambda} K^{n_0}. \tag{6.14}
\]

We put

\[
\theta_{n_0} = \int_{t_0}^\infty \left\{ \Phi (z_{n_0+1}(\tau)) - \Phi (z_{n_0}(\tau)) \right\} \frac{d\tau}{\tau},
\]

which has a finite value because of (6.11). It also follows from (6.12) that

\[
|\Theta_{n_0+1}(t) - \Theta_{n_0}(t) - \theta_{n_0}| \leq \int_{t_0}^\infty |\Phi (z_{n_0+1}(\tau)) - \Phi (z_{n_0}(\tau))| \frac{d\tau}{\tau} \leq C_0 E_0 \frac{K^{n_0}}{\lambda^2 t^\lambda}. \tag{6.15}
\]

Now we obtain from (6.10) for \( n = n_0 \) and (6.15) that

\[
\zeta_{n_0+1} = \lim_{\tau \to \infty} (z_{n_0+1}(\tau)) e^{i\Theta_{n_0+1}(\tau)} = \zeta_{n_0} \exp \left( i \lim_{\tau \to \infty} (\Theta_{n_0+1}(\tau) - \Theta_{n_0}(\tau)) \right)
\]

\[
= \zeta_{n_0} e^{i\theta_{n_0}},
\]

which immediately leads to (6.11) for \( n = n_0 + 1 \) if we replace \( \zeta_{n_0} \) by the right-hand side of (6.11) for \( n = n_0 \). Since \( |\zeta_{n_0}| = |\zeta_0| \), it follows from (6.10), (6.13), and (6.15) that

\[
|z_{n_0+2}(t) - z_{n_0+1}(t)| \leq |\zeta_{n_0} e^{i\theta_{n_0}} e^{-i\Theta_{n_0+1}(t)} - \zeta_{n_0} e^{-i\Theta_{n_0}(t)}| \]

\[
\leq |\zeta_{n_0}| |\theta_{n_0} - \Theta_{n_0+1}(t) + \Theta_{n_0}(t)| \]

\[
\leq \left( |z(t_0)| + \frac{E_0}{\lambda t_0^\lambda} \right) \frac{C_0 E_0}{\lambda^2 t_0^\lambda} K^{n_0+1},
\]

which is (6.12) for \( n = n_0 + 1 \). This completes the proof. \( \square \)
Now we are in a position to prove Lemma 6.1.

**Proof of Lemma 6.1.** Since \( z_0 \) is continuous on \([t_0, \infty)\), it follows from (6.8) and (6.11) that each \( z_n \) is also continuous on \([t_0, \infty)\). We put 
\[
K = C_0(E_0 t_0^{-\lambda} + |z(t_0)|\lambda)/\lambda^2
\]
as before. Then we have \( 0 < K < 1 \) from the assumption. By (6.12) we can easily show that \( \{ z_n(\cdot) \}_{n=0}^\infty \) is a uniform Cauchy sequence on \([t_0, \infty)\), and \( \{ z_n(\cdot) \}_{n=0}^\infty \) converges uniformly on \([t_0, \infty)\) as \( n \to \infty \). Hence if we put 
\[
p(s) := \lim_{n \to \infty} z_n(e^s), \quad s \geq \log t_0,
\]
\( p \) is continuous on \([\log t_0, \infty)\). Since we have \( p(\log t) = \lim_{n \to \infty} z_n(t) \) and \( 0 < K < 1 \), it follows from (6.12) that 
\[
|z(t) - p(\log t)| = \lim_{n \to \infty} |z_0(t) - z_n(t)|
\]
\[
\leq \sum_{n=0}^\infty |z_{n+1}(t) - z_n(t)| \leq \sum_{n=0}^\infty \frac{E_0}{\lambda^t t^n} K^n \leq \frac{E_0}{\lambda(1 - K)t^\lambda},
\]
which is (6.6).

To show (6.4), we set 
\[
\Theta_\infty(t) = \int_{t_0}^t \Phi(p(\log \tau)) \frac{d\tau}{\tau} = \int_{\log t_0}^{\log t} \Phi(p(\sigma)) d\sigma,
\]
which is well-defined because the integrands are continuous functions. Then it follows that 
\[
|\Theta_\infty(t) - \Theta_n(t)| \leq \int_{t_0}^t C_0 |p(\log \tau) - z_n(\tau)| \frac{d\tau}{\tau}
\]
\[
\leq \int_{t_0}^t C_0 \sum_{j=n}^\infty \frac{E_0}{\lambda^j t^\lambda} K^j \frac{d\tau}{\tau}
\]
\[
\leq \frac{C_0 E_0 K^n}{\lambda^2(1 - K)t_0^\lambda},
\]
whence \( \lim_{n \to \infty} \Theta_n(t) = \Theta_\infty(t) \). Similarly, we can show 
\[
\lim_{n \to \infty} \int_{t_0}^\infty \left\{ \Phi(z_n(\tau)) - \Phi(z_0(\tau)) \right\} \frac{d\tau}{\tau} = \int_{t_0}^\infty \left\{ \Phi(p(\log \tau)) - \Phi(z_0(\tau)) \right\} \frac{d\tau}{\tau},
\]
which implies that \( \{ \zeta_n \} \) converges as \( n \to \infty \) with the help of (6.11) (note that (6.6) shows the existence of the integral on the right-hand side of the identity above). Thus, by setting \( \zeta_\infty = \lim_{n \to \infty} \zeta_n \), we have 
\[
p(s) = \lim_{n \to \infty} \zeta_n \exp(-i \Theta_{n-1}(e^s)) = \zeta_\infty e^{-i \Theta_\infty(e^s)} = \zeta_\infty \exp \left( -i \int_{\log t_0}^s \Phi(p(\sigma)) d\sigma \right)
\]
By differentiation, we see that \( p(s) \) solves the desired equation (6.4).

In the remaining part of this section, we will apply Lemma 6.1 to the profile equation (6.2). We put 
\[
V_\varepsilon(t; \sigma, \omega) = U_\varepsilon(t, (t + \sigma)\omega)
\]
(6.16)
for $(\sigma, \omega) \in \mathbb{R} \times S^2$ and $t > \max\{0, -\sigma\}$. Note that we have $V_s(t; \sigma, \omega) = \sqrt{c_0} V_1(t; \sigma, \omega) + i c_0 V_2(t; \sigma, \omega)$, where $V = (V_1, V_2)^T$ is given by (5.18). Let $R$ be the constant appearing in (1.1). It follows from (6.2) that $V_e(t; \sigma, \omega)$ satisfies

\[i \partial_t V_e(t; \sigma, \omega) = \frac{c(\omega) \text{Re}(V_e(t; \sigma, \omega))}{2t} V_e(t; \sigma, \omega) + i H_e(t, (t + \sigma) \omega)\]  

(6.17)

for $t > t_{0, \sigma}$ and $\sigma \leq R$. Note that all the estimates obtained in the proof of Proposition 5.1 are valid with $T = \infty$, because we have already shown that (5.4) is valid. On the other hand, for $\sigma > R$, we have

\[\lim_{t \to \infty} V_e(t; \sigma, \omega) = \lim_{t \to \infty} 0 = 0\]

because of the finite propagation property (1.2).

As an application of Lemma 6.1 we have the following.

**Corollary 6.3.** Let $\varepsilon$ be sufficiently small. Suppose that $c(\omega) \neq 0$ on $S^2$. Then $\lim_{t \to \infty} V_e(t; \sigma, \omega)$ exists for each $(\sigma, \omega) \in \mathbb{R} \times S^2$. If we put

\[V_e^+(\sigma, \omega) := \lim_{t \to \infty} V_e(t; \sigma, \omega)\]

for each $(\sigma, \omega) \in \mathbb{R} \times S^2$, then we have

\[\text{Re} V_e^+(\sigma, \omega) = 0\]  

(6.18)

for almost all $(\sigma, \omega) \in \mathbb{R} \times S^2$. Moreover we have $V_e^+ \in L^2(\mathbb{R} \times S^2)$ and

\[\lim_{t \to \infty} \int_{\mathbb{R} \times S^2} |\chi_t(\sigma)V_e(t; \sigma, \omega) - V_e^+(\sigma, \omega)|^2 d\sigma dS_\omega = 0,\]  

(6.19)

where $\chi_t(\sigma) = 1$ for $\sigma > -t$, and $\chi_t(\sigma) = 0$ for $\sigma \leq -t$.

**Proof.** First we show the convergence of $V_e(t; \sigma, \omega)$ as $t \to \infty$, and (6.18). We have only to consider the case $\sigma \leq R$, because the opposite case is trivial. By (5.15) and (5.21), we can apply Lemma 6.1 to (6.17) with $z(t) = V_e(t; \sigma, \omega)$, $\Phi(z) = c(\omega) \text{Re} z/2$, $J(t) = i H_e(t, (t + \sigma) \omega)$, and $t_0 = t_{0, \sigma}$, provided that $\varepsilon$ is small enough, because we have

\[C_0(E_0 t_0^{-\lambda} + |z(t_0)| \lambda) \leq C_1 \varepsilon < \lambda^2\]

for $0 < \varepsilon < \lambda^2/C_1$, where we have taken $C_0 = \max_{\omega \in S^2} c(\omega)/2$, $E_0 = C \varepsilon(\sigma)^{-\mu}$, and $\lambda = 1 - 2\mu$, while $C_1$ is an appropriate positive constant independent of $\sigma$ and $\omega$. It follows from Lemma 6.1 that for any $(\sigma, \omega) \in (-\infty, R] \times S^2$, there is $p(s)$ satisfying

\[i \frac{dp}{ds}(s) = \frac{c(\omega) \text{Re}(p(s))}{2} p(s)\]

and

\[\lim_{t \to \infty} |V_e(t; \sigma, \omega) - p(\log t)| = 0.\]

So it is enough to show that $p(s)$ converges as $s \to \infty$, and that $\text{Re} p(s) \to 0$ as $s \to \infty$ for almost all $(\sigma, \omega) \in (-\infty, R] \times S^2$. If $c(\omega) = 0$, then $p(s)$ is independent of $s$ and the convergence of $p(s)$ as $s \to \infty$ is trivial. Since $c(\omega)$ is a polynomial of degree 2 in $\omega$, the set of $(\sigma, \omega) \in \mathbb{R} \times S^2$ with $c(\omega) = 0$ has
We observe that
\[ \lim_{c} \]
where the double sign depends on the signature of measure zero unless \( c(\omega) \) vanishes identically on \( S^2 \). Hence we may assume \( c(\omega) \neq 0 \) from now on, and we are going to show that \( p(s) \) converges to a pure imaginary number as \( s \to \infty \). For this purpose, we set \( X(s) = \text{Re} p(s)/2 \), \( Y(s) = \text{Im} p(s)/2 \) to rewrite the above equation as
\[
\frac{dX}{ds}(s) = c(\omega)X(s)Y(s), \quad \frac{dY}{ds}(s) = -c(\omega)X(s)^2. \tag{6.20}
\]
We observe that
\[
\frac{d}{ds}(X(s)^2 + Y(s)^2) = 0,
\]
which implies that \( X(s)^2 + Y(s)^2 \) is independent of \( s \). We denote this conserved quantity by \( \rho^2 \), where \( \rho \geq 0 \). The case \( \rho = 0 \) is trivial, because we have \( X(s) = Y(s) \equiv 0 \). Hence we assume \( \rho > 0 \) from now on. From the second equation of (6.20) we have
\[
\frac{dY}{ds}(s) = c(\omega) \left( Y(s)^2 - \rho^2 \right).
\]
This can be explicitly integrated as
\[
Y(s) = \frac{(\rho + \eta)e^{-c(\omega)\rho s} - (\rho - \eta)e^{c(\omega)\rho s}}{(\rho + \eta)e^{-c(\omega)\rho s} + (\rho - \eta)e^{c(\omega)\rho s}}
\]
with some real constant \( \eta \) satisfying \( |\eta| \leq \rho \). We can also see that
\[
X(s) = \frac{2\rho \xi}{(\rho + \eta)e^{-c(\omega)\rho s} + (\rho - \eta)e^{c(\omega)\rho s}}
\]
with some real constant \( \xi \) satisfying \( \xi^2 + \eta^2 = \rho^2 \). If \( \xi = 0 \), then we have \( X(s) \equiv 0 \), and \( Y(s) \equiv \pm \rho \). If \( \xi \neq 0 \), then \( \eta^2 < \rho^2 \). Especially we have \( \rho \neq \pm \eta \neq 0 \), and we get
\[
\lim_{s \to \infty} X(s) = \lim_{s \to \infty} \frac{2\rho \xi e^{-c(\omega)\rho s}}{(\rho \pm \eta)e^{-2c(\omega)\rho s} + (\rho \mp \eta)} = 0,
\]
\[
\lim_{s \to \infty} Y(s) = \rho \lim_{s \to \infty} \frac{\pm ((\rho \pm \eta)e^{-2c(\omega)\rho s} - (\rho \mp \eta))}{(\rho \pm \eta)e^{-2c(\omega)\rho s} + (\rho \mp \eta)} = \mp \rho.
\]
where the double sign depends on the signature of \( c(\omega) \). Now the existence of \( \lim_{t \to \infty} V_{c}(t; \sigma, \omega) \) and (6.13) have been established.

It follows from (5.5) and (5.10) that
\[
|U_{c}(t, r\omega)| \leq C|U(t, r\omega)| = C\left| D_{-}(ru(t, r\omega)) \right| \leq C\varepsilon(t - r)^{-1+\mu}
\]
for any \((t, r, \omega) \in [0, \infty) \times (0, \infty) \times S^2 \). Since \( V_{c}(t; \sigma, \omega) = U_{c}(t, (t + \sigma)\omega) \), we obtain
\[
|V_{c}(t; \sigma, \omega)| \leq C\varepsilon(\sigma)^{-1+\mu} \tag{6.21}
\]
for \((\sigma, \omega) \in \mathbb{R} \times S^2 \) and \( t > \max\{0, -\sigma\} \). Hence, by taking the limit of this inequality as \( t \to \infty \), we have
\[
|V_{c}^{+}(\sigma, \omega)| \leq C\varepsilon(\sigma)^{-1+\mu}, \quad (\sigma, \omega) \in \mathbb{R} \times S^2,
\]
which shows \( V^+_c \in L^2(\mathbb{R} \times S^2) \) since \( \mu < 1/2 \). Furthermore we have
\[
|\chi_t(\sigma)V_e(t; \sigma, \omega) - V^+_c(\sigma, \omega)|^2 \leq C\varepsilon^2(\sigma)^{-2\mu} \in L^1(\mathbb{R} \times S^2)
\]
for \( t \geq 0 \). Now, since \( \lim_{t \to -\infty} |\chi_t(\sigma)V_e(t; \sigma, \omega) - V^+_c(\sigma, \omega)|^2 = 0 \) for each \((\sigma, \omega) \in \mathbb{R} \times S^2\), Lebesgue’s convergence theorem implies \((6.19)\). This completes the proof.

\[\square\]

7. Proof of Theorem 2.2

In the following, we write
\[
\hat{\omega}(x) = \left(\hat{\omega}_a(x)\right)_{a=0,1,2,3} = (-1, x_1/|x|, x_2/|x|, x_3/|x|)
\]
for \( x \in \mathbb{R}^3 \setminus \{0\} \). For the proof of Theorem 2.2 we will use the following lemma:

**Lemma 7.1.** Let \( \phi \in C([0, \infty); \dot{H}^1(\mathbb{R}^3)) \cap C^1([0, \infty); L^2(\mathbb{R}^3)) \). The following assertions (i) and (ii) are equivalent:

(i) There exists \((\phi_0^+, \phi_1^+) \in \mathcal{H}_0(\mathbb{R}^3) = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) such that
\[
\lim_{t \to \infty} \|\phi(t) - \phi^+(t)\|_E = 0,
\]
where \( \phi^+ \in C([0, \infty); \dot{H}^1(\mathbb{R}^3)) \cap C^1([0, \infty); L^2(\mathbb{R}^3)) \) is the unique solution to \(\Box \phi^+ = 0\) with \(\phi^+, \partial_t \phi^+)(0) = (\phi_0^+, \phi_1^+)\).

(ii) There is a function \(P = P(\sigma, \omega) \in L^2(\mathbb{R} \times S^2)\) such that
\[
\lim_{t \to \infty} \|\partial \phi(t, \cdot) - \hat{\omega}(\cdot)P^x(t, \cdot)\|_{L^2(\mathbb{R}^3)} = 0,
\]
where \(P^x\) is given by
\[
P^x(t, x) = \frac{1}{|x|}P(|x| - t, |x|^{-1}x), \quad x \neq 0.
\]

See \([11]\) for the proof (see also \([10]\), where the above result was implicitly proved). We note that \((\phi_0^+, \phi_1^+)\) and \(P\) above are related by \(P = \mathcal{T}[\phi_0^+, \phi_1^+]\), where \(\mathcal{T}[\phi_0^+, \phi_1^+]\) is the so-called translation representation of \((\phi_0^+, \phi_1^+)\) introduced by Lax-Phillips \([17]\) Chapter IV. More precisely, \(\mathcal{T}\) is an isometric isomorphism from \(\mathcal{H}_0(\mathbb{R}^3)\) to \(L^2(\mathbb{R} \times S^2)\) which can be represented as
\[
\mathcal{T}[\phi_0, \phi_1](\sigma, \omega) = \frac{1}{4\pi}(-\partial_\sigma \mathcal{R}[\phi_0](\sigma, \omega) + \mathcal{R}[\phi_1](\sigma, \omega)), \quad (\sigma, \omega) \in \mathbb{R} \times S^2,
\]
for \((\phi_0, \phi_1) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)\), where \(\mathcal{R}[\psi]\) is the Radon transform of \(\psi\), given by
\[
\mathcal{R}[\psi](\sigma, \omega) = \int_{y \cdot \omega = \sigma} \psi(y) dS_y
\]
with the surface element \(dS_y\) on the plane \(\{y \in \mathbb{R}^3; y \cdot \omega = \sigma\}\).

**Proof of Theorem 2.2** Let \(u = (u_1, u_2)^T\) be the global solution to \((1.1)-(1.2)\) with \((2.2)\) for small \(\varepsilon\), and \(U = (U_1, U_2)^T\) and \(V = (V_1, V_2)^T\) be given by \((1.4)\) and \((5.18)\), respectively. Suppose that \(c_0 > 0\) and \(c(\omega) \neq 0\) on \(S^2\). Recall that all the estimates in the proof of Proposition 5.1 are valid in our present setting.
As in the previous section, we define $U_c = U_c(t, x)$ by (6.1), and $V_c = V_c(t; \sigma, \omega)$ by (6.10). We write $V_c^+(\sigma, \omega) = \lim_{t \to \infty} V_c(t; \sigma, \omega)$ whose existence is guaranteed by Corollary 6.3. If we put
\[
V_1^+(\sigma, \omega) = c_0^{-1/2} \text{Re} V_c^+(\sigma, \omega) \quad \text{and} \quad V_2^+(\sigma, \omega) = c_0^{-1} \text{Im} V_c^+(\sigma, \omega),
\]
then Corollary 6.3 implies that $V_1^+(\sigma, \omega) = 0$ almost everywhere and $V_2^+ \in L^2(\mathbb{R} \times \mathbb{S}^2)$. Hence, if we can prove
\[
\lim_{t \to \infty} \sum_{j=1}^2 \| \partial u_j(t, \cdot) - \hat{\omega}(\cdot) V_j^+(t, \cdot) \|_{L^2(\mathbb{R}^3)} = 0, \tag{7.1}
\]
then we obtain (2.6) immediately, and also (2.7) with the help of Lemma 7.1, where
\[
V_j^{+, t}(t, x) := \frac{1}{|x|} V_j^+(|x| - t, |x|^{-1} x), \quad x \neq 0
\]
for $j = 1, 2$. We define
\[
J_1(t) = \left( \sum_{j=1}^2 \int_{\mathbb{S}^2} \left( \int_0^\infty |r \partial u_j(t, r \omega) - \hat{\omega}(r \omega)V_j(t; t + r, \omega)|^2 dr \right) dS_\omega \right)^{1/2},
\]
\[
J_2(t) = \left( \sum_{j=1}^2 \int_{\mathbb{S}^2} \left( \int_0^\infty |\hat{\omega}(r \omega)V_j(t; r - t, \omega) - r \hat{\omega}(r \omega)V_j^{+, t}(t, r \omega)|^2 dr \right) dS_\omega \right)^{1/2}.
\]
It follows from (3.4), (3.5), and (5.10) that
\[
J_1(t)^2 \leq C \int_{\mathbb{S}^2} \left( \int_0^\infty |u(t, r \omega)|^2 dr \right) dS_\omega \leq C \varepsilon^2 \int_0^\infty (t + r)^{2\mu - 2} dr \leq C \varepsilon^2 (t)^{2\mu - 1} \to 0
\]
as $t \to \infty$. Therefore (7.1) follows from
\[
\lim_{t \to \infty} J_2(t) = 0, \tag{7.2}
\]
because we have $\sum_{j=1}^2 \| \partial u_j(t) - \hat{\omega} V_j^{+, t}(t) \|_{L^2} \leq J_1(t) + J_2(t)$. In order to prove (7.2), we introduce
\[
V_c^{+, t}(t, x) = \frac{1}{|x|} V_c^+(|x| - t, |x|^{-1} x), \quad x \neq 0.
\]
Let
\[
J_3(t) = \left( \int_{\mathbb{S}^2} \left( \int_0^\infty |\hat{\omega}(r \omega)V_c(t; t + r, \omega) - r \hat{\omega}(r \omega)V_c^{+, t}(t, r \omega)|^2 dr \right) dS_\omega \right)^{1/2}.
\]
By (6.19) we get
\[ J_3(t)^2 = 2 \int_{S^2} \left( \int_0^\infty |V_c(t - r, \omega) - V_c^+(r - t, \omega)|^2 dr \right) dS_\omega \]
\[ = 2 \int_{S^2} \left( \int_{-t}^{\infty} |\chi_t(\sigma)V_c(t; \sigma, \omega) - V_c^+(\sigma, \omega)|^2 d\sigma \right) dS_\omega \]
\[ \leq 2 \int_{S^2} \left( \int_{\mathbb{R}} |\chi_t(\sigma)V_c(t; \sigma, \omega) - V_c^+(\sigma, \omega)|^2 d\sigma \right) dS_\omega \rightarrow 0 \]
as \( t \to \infty \), because \( \chi_t(\sigma) = 1 \) for \( \sigma > -t \). Since \( J_2(t) \leq C J_3(t) \), we obtain (7.2) immediately.

It remains to prove (2.8). We set
\[ \|u(t)\|_E^2 := c_0^{-1}\|u_1(t)\|_E^2 + \|u_2(t)\|_E^2 \]
for \( u = (u_1, u_2)^T \). By the standard argument of the energy, we have
\[ \frac{d}{dt} (\|u(t)\|_E^2) = \int_{\mathbb{R}^3} F_1(\partial u(t, x)) \partial_t u_1(t, x) dx + \int_{\mathbb{R}^3} F_2(\partial u(t, x)) \partial_t u_2(t, x) dx. \]

Let \( R \) be the constant appearing in (4.1), and we put \( \Lambda_{\infty, R} = \{(t, x) \in [0, \infty) \times \mathbb{R}^3; 1 \leq t/2 \leq |x| \leq t + R\} \). We put \( \chi(t, x) = 1 \) if \( (t, x) \in \Lambda_{\infty, R} \) and \( \chi(t, x) = 0 \) otherwise. Since \((1 - \chi(t, x))|\partial u(t, x)| \leq C \varepsilon (t + r)^{\mu - 2} \) by (5.9), it follows from (5.9) that
\[ \sum_{j=1}^{2} \int_{\mathbb{R}^3} (1 - \chi(t, x)) \left| F_j(\partial u(t, x)) (\partial_t u_j)(t, x) \right| dx \leq C \varepsilon (1 + t)^{\mu - 2} \|\partial u(t)\|_{L^2}^2 \]
\[ \leq C \varepsilon^2 (1 + t)^{(3\mu/2) - 2} \|u(t)\|_E \]
for sufficiently small \( \varepsilon \). For \((t, x) \in \Lambda_{\infty, R} \), we obtain from (3.4) and (3.5) that
\[ |(\partial_t u_k)(\partial_t u_1)(\partial_t u_j) + \omega_0 D_- u_k(D_- u_1)(D_- u_j) \leq C (t + r)^{-1} u_1(t, x), \]
which leads to
\[ |(\partial_t u_k)(\partial_t u_1)(\partial_t u_j) + \omega_0 \omega_0 (D_- u_k)(D_- u_1)(D_- u_j) \leq C (1 + t)^{-1} |u_1| \partial u_1^2 \]
\[ \leq C (1 + t)^{-2} |\partial u|^2 \]
with the help of (5.10). As an immediate consequence, we obtain
\[ \frac{F_1(\partial u)}{c_0}(\partial_t u_1) + F_2(\partial u)(\partial_t u_2) = -\frac{F_1^{\text{red}}(\omega, D_- u)}{c_0}(D_- u_1) \]
\[ - F_2^{\text{red}}(\omega, D_- u)(D_- u_2) + O(\varepsilon(1 + t)^{\mu - 2}|\partial u|^2) \]
\[ = O(\varepsilon(1 + t)^{\mu - 2}|\partial u|^2) \]
for \((t, x) \in \Lambda_{T, R} \), because of the structure (2.2). Therefore we get
\[ \int_{\mathbb{R}^3} \chi(t, x) \left| \frac{F_1(\partial u)}{c_0}(\partial_t u_1) + F_2(\partial u)(\partial_t u_2) \right| dx \leq C \varepsilon (1 + t)^{\mu - 2} \|\partial u(t)\|_{L^2}^2 \]
\[ \leq C \varepsilon^2 (1 + t)^{(3\mu/2) - 2} \|u(t)\|_E, \]
provided that \( \varepsilon \) is small enough. To sum up, we obtain
\[
\left| \frac{d}{dt} \left( \|u(t)\|_E^2 \right) \right| \leq C\varepsilon^2 (1 + t)^{(3n/2)-2}\|u(t)\|_E,
\]
which yields
\[
\|u(t)\|_E - \|u(0)\|_E \leq C\varepsilon^2 \int_0^\infty (1 + \tau)^{(3n/2)-2}d\tau \leq C\varepsilon^2.
\]
Since we have \( \|u_2^+(t)\|_E = \|u_2^+(0)\|_E \), it follows that
\[
\|u(0)\|_E - \|u_2^+(0)\|_E \leq \|u(0)\|_E - \|u(t)\|_E + \|u(t)\|_E - \|u_2^+(t)\|_E \leq C \left( \varepsilon^2 + \|u_1(t)\|_E + \|u_2(t) - u_2^+(t)\|_E \right).
\]
By (2.6) and (2.7), taking the limit as \( t \to \infty \) in the inequality above, we obtain
\[
\|u(0)\|_E - \|u_2^+(0)\|_E \leq C\varepsilon^2,
\]
which immediately yields (2.8). This completes the proof. \( \square \)

8. ASYMPTOTIC BEHAVIOR FOR GENERAL TWO-COMPONENT SYSTEMS UNDER THE CONDITION (H)

In this section, we discuss the asymptotic behavior for general two-component systems which are not necessarily of the form (2.2). If the condition (H) is satisfied with some \( A(\omega) \), then the condition (H) with \( A(\omega) \) replaced by \( h(\omega)A(\omega) \) remains valid for an arbitrary continuous function \( h \) on \( S^2 \) with positive values. Therefore, without loss of generality, we may assume that \( A(\omega) \) has 1 and \( c_0(\omega) \) as its eigenvalues, where \( c_0 \) is a positive and continuous function on \( S^2 \).

Then we can take an orthogonal matrix \( \mathcal{P}(\omega) \) such that
\[
A(\omega) = \mathcal{P}(\omega)^T \begin{pmatrix} 1 & 0 \\ 0 & c_0(\omega) \end{pmatrix} \mathcal{P}(\omega).
\]
Since the condition (H) yields
\[
(\mathcal{P}(\omega)Y)^T \begin{pmatrix} 1 & 0 \\ 0 & c_0(\omega) \end{pmatrix} \mathcal{P}(\omega) F_{\text{red}}(\omega, Y) = 0,
\]
we see that \( \mathcal{P}(\omega) F_{\text{red}}(\omega, \mathcal{P}(\omega)^T \tilde{Y}) \) is perpendicular to \( (\tilde{Y}_1, c_0(\omega)\tilde{Y}_2)^T \) for all \( \tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)^T \in \mathbb{R}^2 \), by substituting \( Y = \mathcal{P}(\omega)^T \tilde{Y} \). Accordingly, we deduce that
\[
\mathcal{P}(\omega) F_{\text{red}}(\omega, \mathcal{P}(\omega)^T \tilde{Y}) = \left( \tilde{c}_1(\omega)\tilde{Y}_1 + \tilde{c}_2(\omega)\tilde{Y}_2 \right)^T \left( \frac{-c_0(\omega)\tilde{Y}_2}{\tilde{Y}_1} \right) \quad (8.1)
\]
with some \( \tilde{c}_1(\omega) \) and \( \tilde{c}_2(\omega) \). Here \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are bounded functions on \( S^2 \). In fact, substituting \( \tilde{Y} = (1, 0)^T \) in (8.1), we find that
\[
|\tilde{c}_1(\omega)| = \left| \mathcal{P}(\omega) F_{\text{red}}(\omega, \mathcal{P}(\omega)^T \tilde{Y}) \right| \leq \max_{\eta \in S^2, |\tilde{Y}| = 1} |F_{\text{red}}(\eta, Y)|, \quad \omega \in S^2,
\]
and a similar estimate for $\tilde{c}_2$ can be obtained by choosing $\tilde{Y} = (0,1)^T$. It is easy to see that the null condition is satisfied if and only if $\tilde{c}_1(\omega)^2 + \tilde{c}_2(\omega)^2 = 0$ for all $\omega \in S^2$. Moreover, the set of $\omega \in S^2$ with $\tilde{c}_1(\omega)^2 + \tilde{c}_2(\omega)^2 = 0$ is of surface measure zero when the null condition is violated. Indeed, for $\omega$ satisfying $\tilde{c}_1(\omega)^2 + \tilde{c}_2(\omega)^2 = 0$, we find $F^{\text{red}}(\omega, P(\omega)^T \tilde{Y}) = 0$ for all $\tilde{Y} \in \mathbb{R}^2$, and hence $F^{\text{red}}(\omega,Y) = 0$ for all $Y \in \mathbb{R}^2$; if the null condition is violated, then the set of such $\omega$ has surface measure zero, since the coefficients of $Y_tY_i$ with $k,l \in \{1,2\}$ in $F^{\text{red}}(\omega,Y)$ are polynomials of degree 2 in $\omega$.

Suppose that the condition (H) is satisfied, but the null condition is violated. Let $u = (u_1, u_2)^T$ be the global solution to \eqref{eq:11}–\eqref{eq:12}. Let $U = (U_1, U_2)^T$ and $V = (V_1, V_2)^T$ be given by \eqref{eq:4} and \eqref{eq:5}, respectively. We put

$$
\tilde{V}(t; \sigma, \omega) = (\tilde{V}_1(t; \sigma, \omega), \tilde{V}_2(t; \sigma, \omega))^T = P(\omega)V(t; \sigma, \omega),
$$

and

$$
\tilde{V}_c(t; \sigma, \omega) = \sqrt{c_0(\omega)}\tilde{V}_1(t; \sigma, \omega) + ic_0(\omega)\tilde{V}_2(t; \sigma, \omega) = C(\omega)^T \tilde{V}(t; \sigma, \omega)
$$

with $C(\omega) = \left(\sqrt{c_0(\omega)}, ic_0(\omega)\right)^T$. Multiplying \eqref{eq:5} by $C(\omega)^TP(\omega)$ from the left, and using \eqref{eq:8}, we get

$$
\partial_t \tilde{V}_c(t) = -\frac{1}{2t}C(\omega)^TP(\omega)F^{\text{red}}(\omega, P(\omega)^T \tilde{V}(t)) + \tilde{H}_e(t, (t+\sigma)\omega)
$$

where $\tilde{H}_e(t, x) = C(|x|^{-1}x)^T P(|x|^{-1}x) H(t, x)$ and

$$
\Psi(z) = \frac{1}{2} \left( \tilde{c}_1(\omega) (\Re z) + \frac{\tilde{c}_2(\omega)}{\sqrt{c_0(\omega)}} (\Im z) \right).
$$

In view of Lemma \ref{lem:6.1} we need to solve

$$
i \frac{d\tilde{p}}{ds}(s) = \Psi(\tilde{p}(s)) \tilde{p}(s)
$$

in order to specify the asymptotic profile of $\tilde{V}_c(t; \sigma, \omega)$ for fixed $(\sigma, \omega)$. As is done in Section \ref{sec:6.3} we introduce

$$
\begin{pmatrix}
\tilde{X}(s) \\
\tilde{Y}(s)
\end{pmatrix} = \frac{1}{2 \sqrt{c_0(\omega)}} \begin{pmatrix}
\sqrt{c_0(\omega)}\tilde{c}_1(\omega) & \tilde{c}_2(\omega) \\
\tilde{c}_1(\omega) & -\sqrt{c_0(\omega)}\tilde{c}_1(\omega)
\end{pmatrix} \begin{pmatrix}
\Re \tilde{p}(s) \\
\Im \tilde{p}(s)
\end{pmatrix}
$$

so that we can reduce \eqref{eq:8.3} to the simpler system

$$
\frac{d\tilde{X}}{ds}(s) = -\tilde{X}(s)\tilde{Y}(s), \quad \frac{d\tilde{Y}}{ds}(s) = \tilde{X}(s)^2.
$$

Now going similar lines to the proof of Corollary \ref{cor:6.3} we see that

$$
\lim_{t \to \infty} \tilde{V}_c(t; \sigma, \omega) = \tilde{V}_c^+(\sigma, \omega)
$$

(8.4)
for almost every \((\sigma, \omega) \in \mathbb{R} \times S^2\), where

\[
\tilde{V}^+(\sigma, \omega) = \frac{2\sqrt{c_0(\omega)}\left(\tilde{c}_2(\omega) - i\sqrt{c_0(\omega)}\tilde{c}_1(\omega)\right)}{c_0(\omega)\tilde{c}_1(\omega)^2 + \tilde{c}_2(\omega)^2}\rho(\sigma, \omega)
\]

with some function \(\rho = \rho(\sigma, \omega)\) when \(\tilde{c}_1(\omega)^2 + \tilde{c}_2(\omega)^2 \neq 0\). Since we have \(|\tilde{V}_c(t; \sigma, \omega)| \leq C\varepsilon(\sigma)^{-1+\mu}\) as in (8.21), we can prove \(\tilde{V}_c^+ \in L^2(\mathbb{R} \times S^2)\) and

\[
\lim_{t \to \infty} \int_{\mathbb{R} \times S^2} \left|\chi_t(\sigma)\tilde{V}_c(t; \sigma, \omega) - \tilde{V}_c^+(\sigma, \omega)\right|^2 d\sigma dS_\omega = 0 \tag{8.5}
\]
as before. Now, we put

\[
\tilde{V}^+(\sigma, \omega) = \frac{\left(\tilde{V}_1^+(\sigma, \omega), \tilde{V}_2^+(\sigma, \omega)\right)}{\left(\tilde{V}_1^+(\sigma, \omega), \tilde{V}_2^+(\sigma, \omega)\right)} = \frac{1}{c_0(\omega)} \left(\sqrt{c_0(\omega)}\Re \tilde{V}_c^+(\sigma, \omega)\right) + \frac{1}{c_0(\omega)} \left(\sqrt{c_0(\omega)}\Im \tilde{V}_c^+(\sigma, \omega)\right)
\]

\[
V^+(\sigma, \omega) = \left(V_1^+(\sigma, \omega), V_2^+(\sigma, \omega)\right) = \mathcal{P}(\omega)^T \tilde{V}^+(\sigma, \omega)
\]

Then, recalling that \(\mathcal{P}(\omega)\) is an orthogonal matrix, we have

\[
\int_{S^2} \left(\int_0^\infty \left|V(t; r-t, \omega) - rV^+(t, r\omega)\right|^2 dr\right) dS_\omega
\]

\[
= \int_{S^2} \left(\int_0^\infty \left|\tilde{V}(t; r-t, \omega) - r\tilde{V}^+(t, r\omega)\right|^2 dr\right) dS_\omega
\]

\[
\leq C \int_{\mathbb{R} \times S^2} \left|\chi_t(\sigma)\tilde{V}_c(t; \sigma, \omega) - \tilde{V}_c^+(\sigma, \omega)\right|^2 d\sigma dS_\omega \to 0
\]
as \(t \to \infty\), where \(V^+\) and \(\tilde{V}^+\) are defined from \(V^+\) and \(\tilde{V}^+\) as before. Finally, noting that (8.6) implies \(c_1(\omega)V_1^+(\sigma, \omega) + c_2(\omega)V_2^+(\sigma, \omega) = 0\) with

\[
\begin{pmatrix}
\tilde{c}_2(\omega) \\
\tilde{c}_1(\omega)
\end{pmatrix}
:= \mathcal{P}(\omega)^T \begin{pmatrix}
\tilde{c}_2(\omega) \\
\tilde{c}_1(\omega)
\end{pmatrix},
\]

we can modify the proof of Theorem 2.2 to obtain the following:

**Theorem 8.1.** Suppose that \(N = 2\) and the condition (H) is satisfied, but the null condition is violated. Let \(\varepsilon\) be sufficiently small, and \(u = (u_1, u_2)^T\) be the global solution to (1.1)–(1.2). Then there is \((f_j^+, g_j^+) \in H_0(\mathbb{R}^3)\) such that

\[
\lim_{t \to \infty} \|u_j(t) - u_j^+(t)\|_E = 0, \quad j = 1, 2,
\]

where \(u_j^+\) is the solution to the free wave equation \(\Box u_j^+ = 0\) with initial data \((u_j^+, \partial_t u_j^+)(0) = (f_j^+, g_j^+)\). Moreover, there are bounded functions \(c_1 = c_1(\omega)\) and \(c_2 = c_2(\omega)\) of \(\omega \in S^2\) such that \((c_1(\omega), c_2(\omega)) \neq (0, 0)\) and

\[
c_1(\omega)T[f_1^+, g_1^+](\sigma, \omega) + c_2(\omega)T[f_2^+, g_2^+](\sigma, \omega) = 0 \tag{8.7}
\]
for almost all \((\sigma,\omega) \in \mathbb{R} \times S^2\), where \(T\) is the translation representation. Here \(c_1\) and \(c_2\) depend only on the coefficients of the nonlinearity \(F\).

**Remark 8.1.** The result of Theorem 2.2 corresponds to the case where \(c_1(\omega) \equiv 1\) and \(c_2(\omega) \equiv 0\) in Theorem 8.1.

We conclude this paper with the following remark: From Theorem 8.1 we see that the global solution for small data to a two-component system satisfying the condition (H) and violating the null condition is asymptotically free, but there is a strong relationship \(S.7\) between the asymptotic profiles for the components \(u_1\) and \(u_2\). This is the special feature of the condition (H) with \(N = 2\). Since the solution for \(1.12\) is not always asymptotically free, Theorem 8.1 cannot be extended to the case \(N \geq 3\) directly; there might be a wider variety of asymptotic behavior.

**ACKNOWLEDGMENTS**

The authors would like to express their sincere gratitude to Professor Akitaka Matsumura for his comments on the earlier version of this work. The work of S. K. is supported by Grant-in-Aid for Scientific Research (C) (No. 23540241), JSPS. The work of H. S. is supported by Grant-in-Aid for Young Scientists (B) (No. 22740089) and Grant-in-Aid for Scientific Research (C) (No. 25400161), JSPS.

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