Abstract. Motivated by a fundamental geometrical object, the cut locus, we introduce and study a new combinatorial structure on graphs.

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1 Introduction

The motivation of this work comes from a basic notion in riemannian geometry, that we shortly present in the following. In this paper, by a surface we always mean a complete, compact and connected 2-dimensional riemannian manifold without boundary.

The cut locus $C(x)$ of the point $x$ in the surface $S$ is the set of all extremities (different from $x$) of maximal (with respect to inclusion) shortest paths (geodesic segments) starting at $x$; for basic properties and equivalent definitions refer, for example, to [13] or [16]. The notion was introduced by H. Poincaré [15] and gain, since then, an important place in global riemannian geometry.

For surfaces $S$ is known that $C(x)$, if not a single point, is a local tree (i.e., each of its points $z$ has a neighbourhood $V$ in $S$ such that the component $K_z(V)$ of $z$ in $C(x) \cap V$ is a tree), even a tree if $S$ is homeomorphic to the sphere. A tree is a set $T$ any two points of which can be joined by a unique Jordan arc included in $T$.

All our graphs are finite, connected, undirected, and may have multiple edges or loops.

S. B. Myers [14] established that the cut locus of a real analytic surface is (homeomorphic to) a graph, and M. Buchner [3] extended the result for manifolds of arbitrary dimension. For not analytic riemannian metrics on $S$, Brainard...
cut loci may be quite large sets, see the work of J. Hebda [6] and of the first author [9]. Other contributions to the study of this notion were brought, among others, by M. Buchner [2], [4], H. Gluck and D. Singer [5], J. Hebda [7], J. Itoh [8], K. Shiohama and M. Tanaka [17], T. Zamfirescu [18], [19], A. D. Weinstein [20].

We show in another paper [10] that for every graph $G$ there exists a surface $S_G$ and a point $x$ in $S$ whose cut locus $C(x)$ is isomorphic to $G$; rephrasing, every graph can be realized as a cut locus.

If $G$ has an odd number $q$ of generating cycles then any surface $S_G$ realizing $G$ is non-orientable, but if $q$ is even then one cannot generally distinguish, by simply looking to the graph $G$, whether $S_G$ is orientable or not: explicit examples show that both possibilities can occur [11]. In other words, seen as a graph, the cut locus does not encode the orientability of the ambient space.

This is our main motivation to endow graphs with a combinatorial structure – that of cut locus structure, or shortly CL-structure.

In this paper we treat combinatorial aspects of this new notion: in Section 2 we introduce and discuss this notion, in Section 3 we give two planar representations of CL-structures, and in the last section we enumerate all such structures on “small” graphs.

In a second paper [10] we show that every CL-structure actually corresponds to a cut locus on a surface, while in a subsequent one [11] we consider the orientability of the surfaces realizing CL-structures as cut loci. In particular, any graph endowed with a CL-structure does encode the orientability of the ambient space where is lives as a cut locus. An upper bound on the number of CL-structures on a graph is given in [12].

At the end of this section we recall a few notions from graph theory, in order to fix the notation.

Let $G$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by $B$ the set of all bridges in the graph $G$; i.e., edges whose removal disconnects $G$. Each non-vertex component of $G \setminus B$ is called a 2-connected component of $G$.

A $k$-graph is a graph all vertices of which have degree $k$.

The power set $\mathcal{E}$ of $E$ becomes a $\mathbb{Z}_2$-vector space over the two-element field $\mathbb{Z}_2$ if endowed with the symmetric difference as addition. $\mathcal{E}$ can be thought of as the space of all functions $E \to \mathbb{Z}_2$, and called the (binary) edge space of $G$. The (binary) cycle space is the subspace $\mathcal{Q}$ of $\mathcal{E}$ generated by
(the edge sets of) all simple cycles of $G$. If $G$ is seen as a simplicial complex, $Q$ is the space of 1-cycles of $G$ with mod 2 coefficients.

# 2 Cut locus structures

**Definition 2.1** A $G$-patch on the graph $G$ is a topological surface $P_G$ with boundary, containing (a graph isomorphic to) $G$ and contractible to it.

**Remark 2.2** Every boundary component of a patch is homeomorphic to a circle, as a 1-dimensional manifold without boundary.

**Definition 2.3** A $G$-strip (or a strip on $G$, or simply a strip, if the graph is clear from the context), is a $G$-patch with 1-component boundary; i.e., whose boundary is one topological circle; see Figure 1 (a).

The next remark gives the geometrical background for the notion of cut locus structure.

**Remark 2.4** Consider a point $x$ on a surface $S$, and a geodesic segment $\gamma : [0, l] \rightarrow S$ parameterized by arclength, with $\gamma(0) = x$ and $\gamma(l) \in C(x)$. For $\varepsilon > 0$ smaller than the injectivity radius at $x$, and hence smaller than $l$, the point $\gamma(l - \varepsilon)$ is well defined. Since $S \setminus C(x)$ is contractible to $x$ along geodesic segments, and thus homeomorphic to an open disk, the union over all $\gamma$s of those points $\gamma(l - \varepsilon)$ is homeomorphic to the unit circle, and therefore the set $\bigcup_\gamma \{ \gamma(l - \mu) : 0 \leq \mu \leq \varepsilon \}$ is a $C(x)$-strip.

**Definition 2.5** A cut locus structure (shortly, a CL-structure) on the graph $G$ is a strip on the cyclic part $G^{\text{cp}}$ of $G$.

**Remark 2.6** We show in another paper [10], with geometrical tools, the converse to Remark 2.4: every CL-structure can be obtained (with some suitable surface and point on the surface) as described in Remark 2.4.

**Remark 2.7** Each $G$-strip defines a circular order around each vertex of $G$, and thus a rotation system. Conversely, one can alternatively define a $G$-strip as the graph associated to a rotation system, together with a 2-cell embedding having precisely one face. We choose not to follow this way, and to keep in our presentation as much as possible of the geometrical intuition.
Figure 1: A strip and its elementary decomposition.
Definition 2.8 An elementary strip is an edge-strip (arc-strip) or a point-strip; i.e., a strip defined by the graph with precisely one edge (arc) of different extremities, respectively by the graph consisting of one single vertex.

Definition 2.9 An elementary decomposition of a $G$-patch $P_G$ is a decomposition of $P_G$ into elementary strips such that:
- each edge-strip corresponds to precisely one edge of $G$;
- each point-strip corresponds to precisely one vertex of $G$; see Figure 1 (b) and (c).

Remark 2.10 Our notion of “$G$-patch” is equivalent to that of “fibered surface” introduced by M. Bestvina and M. Handel: “a fibered surface is a compact surface $F$ with boundary which is decomposed into arcs and into polygons that are modeled on $k$-junctions, $k = 1, 2, 3, ...$. The components of the subsurface fibered by arcs are strips. Shrinking the decomposition elements to points produces a graph $G$, where vertices (of valence $k$) correspond to ($k$-) junctions and strips to edges. We can think of $G$ as being embedded in $F$, representing the spine of $F$” [1]. We choose the most (in our opinion) appropriate name for our purpose, and thus different from theirs.

In order to easier handle a CL-structure, we associate to it an object of combinatorial nature. To this goal, denote by $\mathcal{P}$ and $\mathcal{A}$ the set of all point-strips, respectively edge-strips, of a CL-structure $\mathcal{C}$ on the graph $G$.

Definition 2.11 Consider an elementary decomposition of the $G$-strip $P_G$ such that each elementary strip has a distinguished face, labeled $\bar{0}$. The face opposite to the distinguished face will be labeled $\bar{1}$. Here, $0$ and $1$ are the elements of the 2-element group $(\mathbb{Z}_2, \oplus)$.

To each pair $(v, e) \in V \times E$ consisting of a vertex $v$ and an edge $e$ incident to $v$, we associate the $\mathbb{Z}_2$-sum $\bar{s}(v, e)$ of the labels of the elementary strips $\nu \in \mathcal{P}$, $\varepsilon \in \mathcal{A}$ associated to $v$ and $e$; i.e., $\bar{s}(v, e) = \bar{0}$ if the distinguished faces of $\nu$ and $\varepsilon$ agree to each other, and $\bar{1}$ otherwise. Therefore, to any cut locus structure $\mathcal{C}$ we can associate a function $s_\mathcal{C} : E \to \{\bar{0}, \bar{1}\}$,

$$s_\mathcal{C}(e) = \bar{s}(v, e) \oplus \bar{s}(v', e), \quad (1)$$

where $v$ and $v'$ are the vertices of the edge $e \in E$.

We call the function $s_\mathcal{C}$ defined by (1) the companion function of $\mathcal{C}$.

The value $s_\mathcal{C}(e)$ above can be thought of as the switch of the edge $e$. 

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Figure 2: Equivalent CL-structures (a), (b) and (c), and schematic representation (d). The edge-strip at (a) corresponds to a rectangular band whose base is $\pi$-rotated “to the left” with respect to the top; the edge-strip at (b) corresponds to a rectangular band whose base is $\pi$-rotated “to the right” with respect to the top; the edge-strip at (c) corresponds to a rectangular band whose base is $(2k + 1)\pi$-rotated “to the left” with respect to the top.

**Definition 2.12** Assume first that the graph $G$ is 2-connected. Two CL-structures $C, C'$ on $G$ are called equivalent if their companion functions are equivalent: i.e., $s_C$ and $s_{C'}$ are equal, up to a simultaneous change of the distinguished face for all elementary strips in $G$ (i.e., either $s_C = s_{C'}$, or $s_C = \overline{1} \oplus s_{C'}$).

If $G$ is not 2-connected, the CL-structures $C, C'$ on $G$ are called equivalent if their companion functions are equivalent on every 2-connected component of $G$. See Figure 2.

**Definition 2.13** An edge-strip $P_e$ (or simply an edge $e$) in a CL-structure $C$ is called switched if $s_C(e) = \overline{1}$.

**Proposition 2.14** If two CL-structures on the same graph $G$ are equivalent then the corresponding $G$-strips are homeomorphic surfaces.

**Proof:** We may assume that $G$ is cyclic.

Assume, moreover, that we have two CL-structures on $G$, whose companion functions are equivalent on every 2-connected component of $G$. The desired homeomorphism can be constructed inductively, extending it with each new “gluing” of an elementary strip, see Figure 2.
3 Representations of CL-structures

We propose two ways to planary represent a CL-structure \( C \) on the graph \( G \).

**Definition 3.1** The graph representation of \( C \) starts with some planar representation of \( G \), and afterward points out the CL-structure, see Figure 3 (a).

**Definition 3.2** The natural representation of \( C \) starts by representing in the plane each vertex-strip such that its distinguished face is “up”, and afterward connects the vertex-strips by edge-strips. The idea is illustrated by Figure 3 (b) and (c).

**Remark 3.3** Consider the natural representation of a CL-structure on a cubic graph. We shall overwrite an “x” to the drawn image of an edge if its strip is switched, and an “=” to the drawn image of an edge if its strip is not-switched. See Figures 4 and 3.

**Remark 3.4** Neither the natural representation, nor the graph representation, of a CL-structure on a graph is unique.

**Proposition 3.5** For any planar cubic graph \( G \) and any CL-structure on \( G \), the natural representation and the graph representation coincide, up to planar homeomorphisms.

*Proof:* This follows from the definitions above. \( \square \)

**Example 3.6** If the 3-graph \( G \) is not planar, Proposition 3.5 is not true. An easy example, obtained from a flat torus of rectangular fundamental domain (see the procedure described in Remark 2.4), is illustrated by Figure 5.

Directly from the definitions we have the following.

**Lemma 3.7** In any natural representation of a strip, each cycle-patch contains at least one switched edge-strip.

We can give four open questions.
Figure 3: Representations of CL-structures. a) *Graph representation* of a strip. b) Intermediate step to obtain (c). c) *Natural representation* for the strip at (a). Additional points $x, y$ are indicated to make clear the transformation.
Figure 4: Schematic representation of the strip in Figure 1 (a).

Figure 5: CL-structure obtained from a flat torus of rectangular fundamental domain.
Figure 6: All 3-graphs with 2 generating cycles.

Question 3.8 Characterize the companion functions of CL-structures in the set $S = \{ s : E \to \{\bar{0}, \bar{1}\} \}$.

Question 3.9 A planar graph is, by definition, a graph which can be represented in the plane without crossings (self-intersections). As we have seen in Example 3.6, there are CL-structures on (not cubic) planar graphs whose natural representations in the plane necessarily produce crossings. What is the minimal number of such crossings which guarantees a planar natural representation?

The same question can be asked for non-planar graphs too, where the (minimal number of necessary) crossings of the graphs is a new parameter.

Question 3.10 How many CL-structures can coexist on the same graph?

Some (not sharp) upper bound will be given in [11].

Question 3.11 Which of the graphs with $q$ generating cycles has the largest number of different CL-structures?

We shall address in the following section the last two questions above, for graphs with two and three generating cycles.

4 CL-structures on small graphs

We present in this section all distinct cut locus structures on 3-graphs with $q = 2, 3$ generating cycles.

The following statement can be obtained by straightforward inductive constructions.
Lemma 4.1 There are precisely 2, respectively 6, distinct 3-graphs with 2, respectively 3, generating cycles, see Figures 6 and 7.

Theorem 4.2 a) There are precisely 3 non-equivalent CL-structures on the 3-graphs with 2 generating cycles, see Figures 8 and 9.

b) There are precisely 17 non-equivalent CL-structures on the 3-graphs with 3 generating cycles, see Figures 10 – 15.

Proof: We employ the natural representation of CL-structures. It is straightforward to generate all patches on the graphs in Figures 6 – 7, to keep only
the strips (by the use of Lemma 3.7, and to use Definition 2.12 and the symmetries of the graphs to identify equivalent CL-structures. □

Our last result shows that the case of CL-structures on 3-graphs is, in some sense, sufficient. For, define the degree of a graph as the maximal degree of its vertices.

**Theorem 4.3** Any CL-structures on a graph with \( q \) generating cycles and degree larger than 3 can be obtained from CL-structures on 3-graphs with \( q \) generating cycles, by contracting non-switched edge-strips.

**Proof:** Fix \( q \); we consider only graphs with \( q \) generating cycles, and proceed by induction over the number of vertices of degree larger that 3. Denote by \( D(G) \) this number for the graph \( G \).

Assume the cyclic graph \( G \) has \( D(G) \geq 1 \), and choose a vertex \( v \) in \( G \) with \( \text{deg}(v) = d > 3 \).

Let \( C \) be a CL-structure on \( G \), and denote by \( v_1, ..., v_d \) the neighbours of \( v \) in \( G \), and by \( T \) the subtree of \( G \) rooted at \( v \), with leaves \( v_1, ..., v_d \). Let \( G^- \) be the complement of \( T \) in \( G \), and \( C^- \) be the union of patches naturally induced by \( C \) on \( G^- \). Let \( s_C^- \) be the restriction of the companion function \( s_C \) of \( C \) to \( G^- \).

Replace \( T \) in \( G \) by a tree \( T_3 \) of leaves \( v_1, ..., v_d \), all of whose internal vertices have degree 3 (\( T_3 \) is generally not unique), and denote by \( G^u \) the
Figure 11: 4 CL-structures on the graph in Figure 7 (ii).

Figure 12: 4 CL-structures on the graph in Figure 7 (iii).

Figure 13: 3 CL-structures on the graph in Figure 7 (iv).
Figure 14: Unique CL-structure on the graph in Figure 7 (v).

Figure 15: 4 CL-structures on the graph in Figure 7 (vi).
new graph. Now complete $C^-$ to a CL-structure $C^v$ on $G^v$, by extending $s_{c^-}$ on the internal edges of $T_3$ with $\bar{0}$, and on the external edges of $T_3$ with the original values of $s_c$. Observe that $C^v$ is indeed a CL-structure on $G^v$, and $D(G^v) = D(G) - 1$, so the proof is complete. \hfill \Box

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