TOPOLOGY OF A NONTOPOLOGICAL MAGNETIC MONOPOLE

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ABSTRACT

Certain nontopological magnetic monopoles, recently found by Lee and Weinberg, are reinterpreted as topological solitons of a non-Abelian gauged Higgs model. Our study makes the nature of the Lee-Weinberg monopoles more transparent, especially with regard to their singularity structure.

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When Dirac [1] founded the theory of magnetic monopoles in 1931, the monopole was not something that people could not live without. Things changed a great deal in the seventies when 't Hooft and Polyakov [2] showed that magnetic monopoles inevitably occur as solitons of spontaneously broken non-Abelian gauge theories; such as all grand unified theories where an internal semi-simple gauge symmetry is spontaneously broken to $U(1)$. Their existence is understood in terms of the nontrivial topology of the vacuum manifold, and as such the non-Abelian nature of the original gauge group plays a crucial role. In particular, the Dirac quantization rule is naturally enforced by the underlying non-Abelian structure.

Recently, however, Lee and Weinberg [3] constructed a new class of finite-energy magnetic monopoles in the context of a purely Abelian gauge theory. Amazingly enough, the corresponding $U(1)$ potential is simply that of a point Dirac monopole (with the monopole strength satisfying the Dirac quantization rule), yet the total energy is rendered finite by introducing a charged vector field of positive gyromagnetic ratio and by fine-tuning a quartic self-interaction thereof. For more general values of the couplings, this theory together with Einstein gravity was found to produce new magnetically charged black hole solutions with hair [3].

One might be tempted to conclude from this that the existence of the magnetic monopoles does not require an underlying non-Abelian structure, let alone a nontrivial topology of the vacuum manifold. We believe this is a bit premature, and is misleading as far as this particular model is concerned. As pointed out by Lee and Weinberg [3], their $U(1)$ theory may be regarded as a gauge-fixed version of the usual $SO(3)$ Higgs model at some special values of the couplings. An important question to ask here is whether there exists such a hidden structure at other values of the couplings as well. In this letter, we will show that there is indeed a hidden non-Abelian gauge symmetry for general values of the couplings, and that subsequently the integer-charged monopoles of Lee and Weinberg may be regarded as topological solitons associated with certain nonrenormalizable deformations of the $SO(3)$ Higgs model. Adopting the radial gauge rather than the unitary gauge, we found that the apparent Dirac string disappears as usual, while the singularity at the origin still needs to be examined. An important byproduct of our study is a topological understanding of the Dirac quantization rule for the integer-charged Lee-Weinberg
monopoles.

The Abelian model of Ref.\[3\] consists of a $U(1)$ electromagnetic potential $A_\mu$, a charged vector field $W_\mu$, and a real scalar $\phi$. The Lagrange density was chosen to have the form

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} |\bar{D}_\mu W_\nu - \bar{D}_\nu W_\mu|^2 + \frac{g}{4} H_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{4} H_{\mu\nu} H^{\mu\nu}$$

$$- \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m^2(\phi) |W_\mu|^2 - V(\phi),$$  \hspace{1cm} (1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\bar{D}_\mu W_\nu = \partial_\mu W_\nu + ie A_\mu W_\nu$. The magnetic moment $H_{\mu\nu}$ is given by the antisymmetric product $ie (W^*_\mu W_\nu - W^*_\nu W_\mu)$ so that $g$ is the gyromagnetic ratio of the charged vector. Here, $g$ and $\lambda$ are positive constants, while $m(\phi)$ vanishes at $\phi = 0$ but is equal to $m_W \neq 0$ when $\phi$ is at its (nontrivial) vacuum value. As mentioned above, when $g = 2$, $\lambda = 1$ and $m(\phi) = e\phi$, this is nothing but the unitary gauge version of the spontaneously broken $su(2)$ gauge theory of Ref.\[2\], and thus renormalizable. But for generic values of $g$ and $\lambda$, the theory is nonrenormalizable.

The ansatz for the unit-charged, spherically symmetric, Lee-Weinberg monopole can be written most succinctly in term of the differential forms $A \equiv A_\mu dx^\mu$ and $W \equiv W_\mu dx^\mu$:

$$\phi = \phi(r), \quad A = \frac{1}{e} (\cos \theta - 1) d\varphi, \quad W = \frac{i u(r)}{e \sqrt{2}} [\exp i \varphi] [d\theta + i \sin \theta d\varphi].$$  \hspace{1cm} (2)

$u$ and $\phi$ are radial functions to be determined by the field equations and appropriate boundary conditions. Note that the electromagnetic potential is simply that of a point-like Dirac monopole and the field configuration appears singular at origin, in addition to the presumably harmless Dirac string along the negative z-axis. A key observation in Ref.\[3\] is that, if the relationship $4\lambda = g^2$ holds, the total energy of the resulting solution can be actually finite despite this singular behaviour at origin. In addition to such a unit-charged solution, the Dirac quantization rule allows all integral and half-integral magnetic charges.

Let us turn to the question of the hidden non-Abelian structure. Consider the following nonrenormalizable $SO(3)$ gauge theory with the gauge connection 1-form $B = (B^a_\mu dx^\mu) T_a$ and a triplet Higgs $\Phi = \Phi^a T_a$.

$$L' = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{g - 2}{4} F_{\mu\nu} H^{\mu\nu} - \frac{\lambda - 1}{4} H_{\mu\nu} H^{\mu\nu}$$

$$- \frac{1}{2} (D_\mu \Phi)^a (D^\mu \Phi)^a - \frac{1}{e^2} (m^2(|\Phi|) - e^2 \Phi^a \Phi^a) (D_\mu \Phi)^a (D^\mu \Phi)^a - V(|\Phi|)$$  \hspace{1cm} (3)
\( G \) is the non-Abelian gauge field strength associated with \( B \), while 
\[ D_\mu \Phi^a \equiv \partial_\mu \Phi^a + \epsilon^{abc} B^b_\mu \Phi^c \] and 
\[ \hat{\Phi}^a \equiv \Phi^a / |\Phi| \]. In addition, we defined two gauge-invariant antisymmetric

tensor \( F \) and \( H \) as follows,

\[ F_{\mu \nu} \equiv \epsilon_{abc} \hat{\Phi}^a (D_\mu \hat{\Phi})^b (D_\nu \hat{\Phi})^c. \]  

(4)

Now in the unitary gauge \( \hat{\Phi}^a = \delta^a \), we may identify \( \phi \) with \( |\Phi| \), \( A_\mu \) with \( B_\mu^3 \), and \( W_\mu \) with \( (B^3_\mu + i B^2_\mu) / \sqrt{2} \). This results in the following reduction formulae:

\[ F_{\mu \nu} \Rightarrow F_{\mu \nu}, \]
\[ H_{\mu \nu} \Rightarrow H_{\mu \nu}, \]
\[ G_\mu^3 \Rightarrow F_{\mu \nu} - H_{\mu \nu}, \]
\[ G_\mu^1 + i G_\mu^2 \Rightarrow \sqrt{2} (\bar{D}_\mu W_\nu - \bar{D}_\nu W_\mu), \]
\[ (D_\mu \hat{\Phi})^a (D^\mu \hat{\Phi})^a \Rightarrow 2e^2 W_\mu^a W^\mu, \]
\[ (D_\mu \Phi)^a (D^\mu \Phi)^a \Rightarrow \partial_\mu \phi \partial_\mu \phi + 2e^2 \phi^2 W_\mu^a W^\mu. \]

Expanding the \( G^a_\mu G^{a \mu} \) term, one easily notices that the \( SO(3) \) invariant Lagrangian \( L' \) reduces to \( L \) of the apparently Abelian theory of Lee and Weinberg.

According to the standard argument \([4]\), all configurations with a regular and finite asymptotic behaviour may be classified in terms of a topological winding number:

\[ n = \frac{1}{8\pi} \oint dS^i \epsilon_{ijk} \epsilon_{abc} \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c, \]  

(5)

where the integral is over the asymptotic two-sphere. Note that \( n \) is an integer in general, and the magnetic monopole strength is related to this number by \( g_{\text{magnetic}} = -4\pi n / e \) \([5]\). The unit-charged case \((n = 1)\) is of special interest, because the corresponding solution has the spherical symmetry. For example, if \( g = 2 \), \( \lambda = 1 \) and \( m = e\phi \), the \( n = 1 \) solution is given by the usual 't Hooft-Polyakov monopole.

Actually, the ansatz \([2]\) for the spherically symmetric Lee-Weinberg monopole is exactly the same as that of the 't Hooft-Polyakov monopole but written in the unitary gauge. Written in the so-called radial gauge, the ansatz for the 't Hooft-Polyakov monopole has the following well-known hedgehog configuration:

\[ \Phi^a = \hat{x}^a \phi(r), \quad B^a = -\frac{\epsilon_{abc} x^b dx^c}{e r^2} [1 - u(r)]. \]  

(6)

\( ^3F \) was previously used by 't Hooft \([2]\) to represent physical electromagnetic fields.
Gauge transforming to the unitary gauge,

\[
\Omega^{-1} \partial^a T_a \Omega = T_3 \quad \text{with} \quad \Omega \equiv e^{-i\varphi T_3} e^{-i\theta T_2} e^{i\varphi T_3},
\]  

(7)
a lengthy but straightforward calculation shows that the hedgehog configuration (6) is indeed gauge-equivalent to the ansatz (2). This clearly demonstrates that the spherically symmetric unit-charged Lee-Weinberg monopole has the topological winding number \( n = 1 \), and that the Dirac string apparent in (2) is superfluous. Similar procedures for \( n > 1 \) non-Abelian configurations should also lead to all integer-charged Lee-Weinberg monopoles automatically, although, due to the lack of the spherical symmetry, the explicit forms of the necessary gauge transformations are more difficult to find.

Now that the Dirac string disappeared by virtue of the radial gauge, let us concentrate on the possible singular structure at origin \( r = 0 \). Generalizing to include dyons \([6]\) while maintaining \( n = 1 \), we may modify the hedgehog configuration (8) to have nontrivial \( B^a \).

\[
\Phi^a = \hat{x}^a \phi(r), \quad B^a = -\epsilon_{abc} \hat{x}^b \frac{dx^c}{e r^2} \left[ 1 - u(r) \right] - \hat{x}^a v(r) dt.
\]  

(8)

On such configurations, the total energy is given by the following functional of \( u \), \( \phi \), and \( v \).

\[
\mathcal{E} = \int dx^3 \left\{ \frac{1}{e^2 r^2} \left( \frac{du}{dr} \right)^2 + \frac{1}{2} \left( \frac{dv}{dr} \right)^2 + \frac{1}{2} \left( \frac{d\phi}{dr} \right)^2 + \frac{1}{2 e^2 r^4} \left[ \lambda u^4 - gu^2 + 1 \right] + \frac{u^2}{e^2 r^2} \left[ m^2 + e^2 v^2 \right] + V(\phi) \right\}.
\]  

(9)

For the integrand, we evaluated \( T_{00} \), the energy-density associated with \( \mathcal{L}' \), on the configuration (8) using such radial gauge results as \( F_{ij} = -\epsilon_{ija} x^a / er^3 \) and \( H_{ij} = u(r)^2 F_{ij} \).

Clearly, for the configuration (8) to be regular at origin, we need to have \( u(0) = 1 \), \( v(0) = 0 \), and \( \phi(0) = 0 \), while, for a finite total energy, we must also require

\[
\lambda u^4(0) - g u^2(0) + 1 = 0.
\]  

(10)

Only if this last condition is met, the integrand of the energy functional \( \mathcal{E} \) does not have any \( r^{-4} \) singularity near origin,

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4 What about half-integer-charged monopoles of Lee and Weinberg \([3]\)? The topological quantization \([3]\) clearly dictates that it is not possible to remove the Dirac string of such \( U(1) \) monopoles by lifting the configuration to the underlying \( SO(3) \) and by performing a non-Abelian gauge transformation. Instead, the lifted Dirac string will be carrying a \( \mathbb{Z}_2 \) flux that is again undetectable. Here, \( \mathbb{Z}_2 \) corresponds to the fundamental group of \( SO(3) \).
Does there exist an everywhere regular finite energy solution of the form (8)? The answer is no in general, such a solution being possible only when $g = 2$ and $\lambda = 1$. This can be easily seen as follows. To be a static solution to the field equations, the configuration must extremize the energy-functional. Suppose $\bar{u}(r)$ is such a regular finite energy solution, and consider a small variation $\delta u(r)$ around it such that $\delta u(0) = 0$. Varying $\mathcal{E}$, we now observe that the only contributions that may lead to a small-$r$ singularity of $r^{-3}$ are given by the following variation,

$$A \equiv \int dx^3 \frac{1}{2e^2 r^4} [\lambda u^4 - gu^2 + 1] \Rightarrow \delta A = \int dx^3 \frac{1}{2e^2 r^4} [4\lambda u^3 - 2gu] \delta u + \cdots. \quad (11)$$

Choosing $\delta u(r) = r + O(r^2)$, we find $4\lambda \bar{u}^3(0) - 2g \bar{u}(0) = 4\lambda - 2g = 0$ as a necessary condition for $\bar{u}(r)$ to satisfy the equation of motion. But the finite energy condition (10) requires $g = h + 1$ for such a regular solution, which is not compatible with $4\lambda - 2g = 0$ unless $g = 2$ and $\lambda = 1$. Thus the existence of a regular finite-energy monopole solution demands the special values of vector couplings, $g = 2$ and $\lambda = 1$, appropriate for the renormalizable field theory.

However, we do obtain a finite-energy solution if we choose not to insist on the everywhere regularity of the configuration (8). The finite-energy monopole of Lee and Weinberg belongs to this class. Instead of demanding $u(0) = 1$, let $u(r)$ approach a finite constant $C$ as $r \to 0$ but still let $v(0) = \phi(0) = 0$. With such a singular boundary condition, the finite-energy requirement and the stationary argument above translate into the following two conditions:

$$\lambda C^4 - g C^2 + 1 = 0, \quad 4\lambda C^3 - 2g C = 0. \quad (12)$$

This in turn leads to the Lee-Weinberg condition $4\lambda = g^2$ with $g > 0$ as well as to the boundary condition $u^2(0) = g/2\lambda = 2/g \quad (8)$. More careful study of such a singular finite-energy monopole may be carried out with the help of the following equations of motion for these radial functions,

$$\frac{d^2 u}{dr^2} + \left(\frac{g}{2} - \lambda u^2\right) \frac{u}{r^2} = (m^2 + e^2 v^2) u$$

$$\frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - \frac{u^2}{e^2 r^2} \frac{d m^2}{d\phi} = \frac{dV}{d\phi}$$

$$\frac{d^2 v}{dr^2} + \frac{2}{r} \frac{dv}{dr} - \frac{2u^2}{r^2} v = 0$$
For instance, a linearization of this nonlinear system near origin predicts that the solutions are in general non-analytic at $r = 0$. The same technique reproduces the well-known analytic behaviour, $u(r) = 1 + O(r^2)$ and $\phi = 1 + O(r)$, of the ’t Hooft-Polyakov monopoles when the couplings are those of the renormalizable theory.

The condition $4\lambda = g^2$ that is necessary for a finite-energy solution to exist can be also recognized on the basis of the energy-functional $E$ itself. The only part of $E$ that is not necessarily positive definite is the quantity $A$ of (12). But the integrand of $A$ is quadratic in $u^2(r)$ with coefficients $\lambda$ and $g$, and subsequently the energy functional $E$ is bounded from below if and only if $4\lambda \geq g^2$. In particular when $4\lambda = g^2$, it is easy to see that there exists a finite energy configuration of type (8) (but not necessarily a solution yet). In that case, we may secure a local minimum of $E$, thus a finite-energy solution, by suitably adjusting $u(r)$, $\phi(r)$ and $v(r)$. If $4\lambda > g^2$ ($4\lambda < g^2$) instead, monopole solutions are of infinite positive (negative) energy, but nevertheless physically significant if coupled to gravity [3].

In this letter, we presented a new interpretation of the integer-charged Lee-Weinberg monopoles in terms of a topological structure arising in a nonrenormalizable extension of the ’t Hooft-Polyakov monopole model. In the unitary gauge where the topological nature is obscured, the monopoles are riddled with Dirac strings, albeit harmless, and the singularity structures of the underlying non-Abelian fields are less transparent. We reexamined the core of the unit-charged monopole from the radial gauge viewpoint and reproduced some of the results in Ref.[3].

An obvious follow-up question to ask is: would it always be possible to provide a non-Abelian gauge theoretic interpretation for Lee-Weinberg-type monopoles, when the theory involves several charged vector fields? We find this unlikely. For example, consider the following generalization of Lee-Weinberg model with the $N$ charged vector fields $W^{(n)}$, $n = 1, \ldots, N$ and the corresponding magnetic moment tensors $H^{(n)}_{\mu\nu}$:

$$
\mathcal{L}_N = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \sum_n |D_\mu W^{(n)}_\nu - D_\nu W^{(n)}_\mu|^2 \\
+ \frac{1}{4} \sum_n g_n H^{(n)}_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \sum_{n,m} \lambda_{nm} H^{(n)}_{\mu\nu} H^{(m)\mu\nu} + \cdots.
$$

(13)

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5We are grateful to E. Weinberg for suggesting this.
As long as the relationship $4\lambda_{nm} = g_n g_m$ holds, an ansatz, entirely analogous to (4) of the $N = 1$ case, leads to a spherically symmetric finite-energy monopole with the boundary condition satisfying $\sum_{n=1}^{N} g_n [u^{(n)}(0)]^2 = 2$. But it is rather difficult to imagine why there should exist non-Abelian interpretations for all such generalized theories, specifically for all $N$ and for all values of the vector couplings. On the other hand, what one may expect to be true is that the resulting nontopological monopoles are generically singular at origin despite the finite total energy and that, just as we have observed in $N = 1$ case above, the complete regularity is recovered only when the relevant vector coupling structure is renormalizable, possibly corresponding to a spontaneously broken non-Abelian gauge theory.

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