SIP-potentials and self-similar potentials of Shabat and Spiridonov: space asymmetric deformation

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Abstract
An appropriateness of a space asymmetry of shape invariant potentials with scaling of parameters and potentials of Shabat and Spiridonov in calculation of their forms, wave functions and discrete energy spectra has proved and has demonstrated on a simple example. Parameters, defined space asymmetry, have found. A new type of a hierarchy, in which superpotentials with neighboring numbers are connected by space rotation relatively a point of origin of space coordinates, has proposed.

1 Formalism of shape invariant potentials
Methods of SUSY QM increase essentially a class of exactly solvable potentials, which can be obtained by methods of direct and inverse approaches of quantum mechanics also. If to assume, that there is an additional interdependence between potentials-partners besides of interdependence of SUSY algebra, expressed by Darboux transformations (for example, see p. 275–277 in [1], p. 4–7 in [2], [3]) or transformations of non-linear supersymmetry [4], then in a number of cases it allows to find a form of such potentials and their spectral characteristics. If the additional interdependence points out to a similarity between shapes of these potentials, then the potentials are named as shape invariant [5]. One can find for them exactly energy spectrum [5] and wave functions [6].

We shall name as shape invariant (or SIP-potentials) the potentials $V_1(x)$ and $V_2(x)$, if

$$V_2(x, a_1) = V_1(x, a_2) + R(a_1), \quad R(a_1) = \text{const}(a_1),$$

(1)

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where \( a_1 \) is set of parameters, \( a_2 \) is function of \( a_1 \) (see p. 289-290 in \[1\]). Such SIP-potentials have been studied best of all, for which the dependence of parameters \( a_2 \) on \( a_1 \) has a form of translation (see \[7, 8\])

\[ a_2 = a_1 + \text{const}, \]  

(2)

or scaling (see \[9, 10\])

\[ a_2 = a_1 q^n, \quad 0 < q < 1. \]  

(3)

In this paper we analyze a possibility of a space asymmetry (at change \( x \to -x \)) of the SIP-potentials with scaling at \( n = 1 \) (which were introduced at the first time in \[10\]). We shall assume, that for a superpotential \( W(x, a_1) \) and constant \( R(a_1) \) there are representations in a form of convergent series (see p. 299-300 in \[1\], (7)–(8) in \[10\])

\[
W(x, a_1) = +\sum_{j=0}^{\infty} g_j(x) a_1^j, \quad R(a_1) = +\sum_{j=0}^{\infty} R_j a_1^j.
\]

(4)

2 A general solution for the superpotential

Using the definition \([1]\) of SIP-invariancy, Darboux transformations \(^1\), the representations \([4]\) and writing down expressions for variable \( a_1 \) with the same powers, one can make up equations for obtaining unknown \( g_n(x) \):

\[
2 \frac{dg_0(x)}{dx} = R_0, \quad \frac{dg_n(x)}{dx} + g_n(x) f_n(x) = h_n(x),
\]

(5)

\[
f_n(x) = 2d_n g_0(x), \quad h_n(x) = d_n r_n - d_n \sum_{j=1}^{n-1} g_j(x) g_{n-j}(x),
\]

(6)

where

\[
r_n = \frac{R_n}{1 - q^n}, \quad d_n = \frac{1 - q^n}{1 + q^n}, \quad n = 1, 2, 3 \ldots
\]

(7)

A solution of the function \( g_n(x) \) was found earlier in an integral form to a constant of integration (for example, see p. 299–300 in \[1\], in that time compiled a number of papers at this theme). However, a further construction of the SIP-potentials was fulfilled usually at zero values of these constants (also see (12) in \[10\]). It turns out, that “zero” choice of such constants of integration reduces the found potentials to their space symmetric solutions relatively point \( x = 0 \) of origin of space coordinates. One can conclude, that the SIP-potentials with scaling are essentially symmetric (see (12) and (13) in \[10\], p. 301 in \[1\]). Self-similar potentials, studied by Shabat \([11]\) and Spiridonov \([12]\), represent a subset in the set of such SIP-potentials (see p. 300 in \[1\], p. 4 (5) and p. 6 in \[10\]) and should be symmetric also.

\(^1\)In this paper we do not use the nonlinear supersymmetry of SIP-potentials \([4]\).
However, it turns out, that the non-zero account of the constants of integration brings a space asymmetry into the SIP-potentials. Changing these constants, one can deform asymmetrically the potentials. Thus, the constants of integration are such unique parameters, which defines the space asymmetry of the SIP-potentials. We do not find any asymmetric forms of the SIP-potentials with scaling in other papers.

Let’s consider a general solution of the function \( g_n(x) \) (from (5)):

\[
g_n(x) = \left( \int h_n(x)e^{\int \frac{f_n(x)dx}{dx} + C_n} \right) - \int f_n(x)dx. \tag{8}
\]

For the first numbers \( n \) we obtain:

\[
g_0(x) = -\frac{R_0x^2}{2} + C_0,
\]

\[
g_1(x) = \left( \int d_1r_1 e^{\int \frac{d_1R_0x^2}{2} + 2d_1C_0x + C_1} \right) e^{\frac{d_1R_0x^2}{2}} - 2d_1C_0x,
\]

\[
g_n(x) = \left( \int d_n \left( \sum_{j=1}^{n-1} g_j(x)g_{n-j}(x) \right) e^{\int \frac{d_nR_0x^2}{2} + 2d_nC_0x + C_n} \right) e^{\frac{d_nR_0x^2}{2}} - 2d_nC_0x,
\]

where \( C_n \) are the constants of integration.

We see, that the found functions \( g_n(x) \) at non-zero \( C_n \) are space asymmetric relatively point \( x = 0 \) and give the asymmetric solutions for the superpotential and potentials-partners, connected with it. The solution (9) is the general one and it can be considered as the asymmetric generalization of the known solutions (160) in [1], (12) in [10] (which in a context of these papers are antisymmetric).

One can see, that the definition of the SIP-potentials determines the function \( g_n(x) \) to \( n \) arbitrary constants \( C_i \) \( (i = 1 \ldots n) \). For arbitrary values of the constants \( C_n \) the function \( g_n(x) \) in the form (9) is the solution of the system (5). Therefore, the possibility of difference of the constants \( C_n \) on their zero values is rightful, the possibility of the asymmetry of the SIP-potentials is rightful.

We see, that an approach for calculation of the discrete component of the energy spectrum of the SIP-potentials with scaling of the parameters (see (173), p. 302 in [1]; solutions (171) in [1]) has an arbitrariness in selection of the values for the constants \( C_n \). Therefore, the possibility of difference of the constants \( C_n \) on their zero values in calculation of energy spectra is rightful.
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3 Are the potentials of Shabat and Spiridonov symmetric?

The potentials, studied by Shabat and Spiridonov, belong to a set of the SIP-potentials with scaling of the parameters and, because of this, they can be space asymmetric also. However, usually an independent formalism is used for their description (see [12, 11, 9]). Further, we shall show, that on the basis of such a formalism one can obtain the asymmetry of these potentials also.

Let’s define superpotentials of Shabat and Spiridonov by such way (see (8) in [12]; (2.8) in [9]):

\[ W_n(x) = q^n W(q^n x). \]  

(10)

We shall use a restriction \(0 < q < 1\), though, according to [12], it is not obligatory (as against [11]). Then using interdependence between the superpotentials with neighboring numbers of an arbitrary hierarchy (for example, see (2.4) in [9]):

\[ W_n^2(x) + \frac{dW_n(x)}{dx} = W_{n+1}^2(x) - \frac{dW_{n+1}(x)}{dx} + k_{n+1}, \quad n = 0, 1, 2... \]  

(11)

(constants \(k_n\) are defined by distances between neighboring levels \(E_n^{(0)}\) and \(E_n^{(0)}\) of the first potential \(V_0(x)\)) and using expansion of the function \(W(x)\) in series near to zero point \(x = 0\):

\[ W(x) = \sum_{m=0}^{+\infty} b_m x^m, \quad b_m = \text{const}, \]  

(12)

one can find relations between \(b_m\) and \(k_n\) (\(n = 0, 1, 2...\)):

\[ q^{2n} (b_1 (1 + q^2) + b_0^2 (1 - q^2)) = k_{n+1}, \]  

\[ b_{m+1} = -\frac{1 - q^{m+2}}{(m+1) (1 + q^{m+2})} \sum_{i=0}^{m} b_i b_{m-i}. \]  

(13)

Let’s calculate the first coefficients \(b_n\):

\[ b_1 = \frac{k_1 - b_0^2 (1 - q^2)}{1 + q^2}, \]
\[ b_2 = -\frac{1 - q^3}{1 + q^3} b_0 b_1 = -\frac{1 - q^3}{(1 + q^4)(1 + q^2)} b_0 \left( k_1 - b_0^2 (1 - q^2) \right), \]
\[ b_3 = -\frac{1 - q^2}{3(1 + q^4)} \left( k_1 - b_0^2 (1 - q^2) \right) \left( -2b_0 \frac{1 - q^2}{1 + q^4} + \frac{k_1 - b_0^2 (1 - q^2)}{1 + q^2} \right). \]  

(14)

One can see, that at \(b_0 = 0\) all coefficients \(b_n\) with even numbers \(n\) are equal to zero, the function \(W(x)\) are antysymmetric and coincides with the known solution (2.17) in [9]. Changing the values of the coefficients \(b_0\) and \(k_1\), one can deform the shape of the function \(W(x)\), obtaining new asymmetric solutions for the potentials.
Example. In Fig. 1 a displacement of the potential shape with changing of \(b_0\) has shown \((k_1 = 1.0, q = 0.5); \) for calculations of the function \(W(x)\) a partial sum \(W^{(n)}(x)\) at \(n = 75\) is used. For estimation of applicability of the method, let’s find a region of convergence, using a definition of the partial sum:

\[
W^{(n)}(x) = \sum_{j=0}^{n} b_j x^j.
\]  

(15)

Values of these sums in selected points \(x\) and at selected \(n\) are presented in the table (at \(b_0 = 0\)).:

| \(n=1\) | \(x = 1.0\) | \(x = 1.5\) | \(x = 2.0\) |
|---|---|---|---|
| \(n=5\) | 0.800009999940000 | 1.20000999991000 | 1.60000 |
| \(n=10\) | 0.670153606255581 | 1.00805575100451 | 1.96237 |
| \(n=20\) | 0.657567324973884 | 0.914274086501970 | 2.53156 |
| \(n=50\) | 0.656219813282069 | 0.809852235822209 | -3.76793 |
| \(n=100\) | 0.656223635811415 | 0.824633977282796 | 116.677 |
| \(n=200\) | 0.656223635811327 | 0.824569105154386 | -24219.3 |
| \(n=500\) | 0.656223635811327 | 0.824569105154386 | -8.89752 \(10^{22}\) |

One can see, that in the region \(|x| < 1.5\) the partial sums \(W_n(x)\) converge to the function \(W(x)\) easily enough. However, at values of \(|x| more then 1.5 (approximately) it is not possible to achieve such a convergence. With change of the parameters \(b_0, q\) and \(k_1\) (up to 1) the convergence of the function \(W(x)\) is kept. Therefore, one can accept the region \(|x| < 1.5\) as the region of the convergence of the representations \((12)\) for the function \(W(x), within which limits one can accept the shape of the function \(W(x)\) and its asymmetric deformation as rightful.

4 A hierarhy with rotations

Interesting type of deformation of the superpotential \(W_n(x)\) is its rotation in a plane \((x, W_n(x))\) around point of origin to a given angle.
4.1 Operator of rotation

Let’s consider a point \( A \) of a function \( f(x) \) with coordinates \( x_A \) and \( f_A = f(x_A) \). We introduce operator of rotation: \( T_{\Delta \varphi} \) rotates any point of the function \( f(x) \) in a plane \((x, f(x))\) around point of origin to an angle \( \Delta \varphi \):

\[
T_{\Delta \varphi} f_{\text{old}}(x_{\text{old}}) = f_{\text{new}}(x_{\text{new}}).
\]

(16)

Introducing a complex function \( z(x) \) by such a way:

\[
\text{Re}(z) = x; \quad \text{Im}(z) = f(x),
\]

(17)

one can describe the rotation of the function \( f(x) \):

\[
z_{\text{old}} = \rho e^{i\varphi_{\text{old}}}, \quad z_{\text{new}} = \rho e^{i\varphi_{\text{new}}} = \rho e^{i(\varphi_{\text{old}} + \Delta \varphi)},
\]

(18)

and also one can find a view of the operator of the rotation:

\[
T_{\Delta \varphi} = e^{i\Delta \varphi}, \quad \Delta \varphi = \varphi_{\text{new}} - \varphi_{\text{old}}.
\]

(19)

Then we obtain transformations of coordinates in the rotation:

\[
x_{\text{new}} = \text{Re}(z_{\text{new}}) = \text{Re}(\rho e^{i(\varphi_{\text{old}} + \Delta \varphi)}) = x_{\text{old}} \cos \Delta \varphi - f_{\text{old}} \sin \Delta \varphi,
\]

\[
f_{\text{new}} = \text{Im}(z_{\text{new}}) = \text{Im}(\rho e^{i(\varphi_{\text{old}} + \Delta \varphi)}) = f_{\text{old}} \cos \Delta \varphi + x_{\text{old}} \sin \Delta \varphi.
\]

(20)

4.2 Construction of a hierarchy of superpotentials with use of the rotations

Let’s use the function \( f_n(x) \) with number \( n \) for description of the superpotential \( W_n(x) \) with number \( n \). Then a sequence of the functions \( f_0(x), f_1(x), ... \) represents a hierarchy of the superpotentials \( W_n(x) \), which can be transformed one to another by use of such rotations.

We assume, that one can write the superpotential with the number \( n \) in some vicinity of point \( x = x_n \) in a form of the power series:

\[
W_n(x) = \sum_{m=0}^{+\infty} a_m(n) x^m.
\]

(21)

For this superpotential at point \( x = x_n \) we find:

\[
\left. \frac{dW_n(x)}{dx} \right|_{x=x_n} = \sum_{m=0}^{+\infty} (m+1)a_{m+1}(n)x_n^m,
\]

\[
W^2_n(x_n) = \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} a_m(n) a_k(n) x_n^{m+k} = \sum_{m=0}^{+\infty} \tilde{a}_m(n) x_n^m.
\]

(22)

Taking into account the coordinate transformations by rotation and representation \( \tilde{a}_m(n) \), we obtain:

\[
x_{n+1} = \cos \Delta \varphi x_n - \sin \Delta \varphi \sum_{m=0}^{+\infty} a_m(n) x_n^m,
\]

\[
x_{n-1} = x_n - \Delta x_n = (2 - \cos \Delta \varphi) x_n + \sin \Delta \varphi \sum_{m=0}^{+\infty} a_m(n) x_n^m = \sum_{m=0}^{+\infty} \tilde{a}_m(n) x_n^m,
\]

(23)
where \( \Delta x_n = x_{n+1} - x_n \).

From here, one can calculate the superpotential \( W_{n+1}(x) \) at point \( x_n \):

\[
W_{n+1}(x_n) = \cos \Delta \varphi \sum_{m=0}^{\infty} a_m^{(n)} x_{n-1}^m + \sin (\Delta \varphi) x_{n-1} \equiv \sum_{m=0}^{\infty} \tilde{a}_m^{(n+1)} x_n^m.
\]

(24)

And we find:

\[
\frac{dW_{n+1}(x)}{dx} \bigg|_{x=x_n} = \sum_{m=0}^{\infty} (m+1) \tilde{a}_m^{(n+1)} x_n^m,
\]

\[
W_{n+1}^2(x_n) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_m^{(n+1)} \tilde{a}_k^{(n+1)} x_{n+k}^m = \sum_{m=0}^{\infty} \tilde{a}_m^{(n+1)} x_n^m.
\]

(25)

Using (22) and (25), and also expressions (11) of the interdependence between the superpotentials with the neighboring numbers, we obtain the following system of equations for calculation of unknown coefficients \( a_m^{(n)} \):

\[
a_0^{(n)} + a_0^{(n+1)} - \tilde{a}_0^{(n+1)} = 0,
\]

\[
a_1^{(n)} + a_1^{(n+1)} - \tilde{a}_1^{(n+1)} = k_{n+1},
\]

\[
a_{m}^{(n)} + (m+1)a_{m+1}^{(n+1)} - \tilde{a}_{m+1}^{(n+1)} = 0, \quad m \geq 1.
\]

(26)

Resolving the system (26) relatively the coefficients \( a_m^{(n)} \), from (21) we obtain the shape of the superpotential \( W_n(x) \) with the number \( n \) in the vicinity of point \( x = x_n \). If to construct the new superpotential \( W_{n+1}(x) \) with the next number \( n+1 \) by use of the rotation of this superpotential with the number \( n \), then they should be connected by the SUSY transformations (11).

By such a way one can construct whole hierarchy of the superpotentials with rotations. However, here it needs else to use requirement, that the arbitrary consequence of the rotations of a superpotential with a given number \( n \) gives a new superpotential, which must be determined uniquely on the whole region of \( x \). For example, one can use a convergent geometric series for such a consequence of rotations.

5 Conclusions

In finishing we note main conclusions and new results.

- An appropriateness of a space asymmetry of shape invariant potentials with scaling of parameters in calculation of their forms, wave functions and discrete energy spectra has proved.

- The coefficients \( C_n \) are such unique parameters, which determine the space asymmetry of the SIP-potentials (without displacement of levels in spectra).

- It has shown, that the definition of the potentials of Shabat and Spiridonov in the form (11) allows their space asymmetric deformation.
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- The general solution for the function $W(x)$ and the superpotential $W_n(x)$ of Shabat and Spiridonov with number $n$ are defined by three parameters: $q$, $k_1$, and $b_0$. The parameter $q$ allows to deform a shape of the potentials, not influencing to their asymmetry. Energy spectrum is determined uniquely by two parameters $q$ and $k_1$ and is not depended on $b_0$. The parameter $b_0$ is only one parameter, determining the asymmetry of the potentials.

- A new approach for construction of the hierarchy, in which the superpotentials with the neighboring numbers are connected by their rotation relatively point of origin, has proposed.

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