On the distribution of winners’ scores in a round-robin tournament

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Abstract

In a classical chess round-robin tournament, each of $n$ players wins, draws, or loses a game against each of the other $n-1$ players. A win rewards a player with 1 points, a draw with $1/2$ point, and a loss with 0 points. We are interested in the distribution of the scores associated with ranks of $n$ players after $\binom{n}{2}$ games, i.e. the distribution of the maximal score, second maximum, and so on. The exact distribution for a general $n$ seems impossible to obtain; we obtain a limit distribution.

Keywords: Complete graph, extremes, negative correlation, Poisson approximation, total variation distance
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1 Introduction

In a classical chess round-robin tournament, each of $n$ players wins, draws, or loses a game against each of the other $n-1$ players. A win rewards a player with 1 points, a draw with $1/2$ point, and a loss with 0 points. Denoting by $X_{ij}$ the score of the player $i$ after the game with the player $j, j \neq i$, in this article, we consider the following model:

Model M:

For $i \neq j$, $X_{ij} + X_{ji} = 1$, $X_{ij} \in \{0, 1/2, 1\}$; we assume that all players are equally strong, i.e. $P(X_{ij} = 1) = P(X_{ji} = 1)$, and that the probability of a draw is the same.

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for all games, denoted by \( p = P(X_{ij} = 1/2) \). We also assume that all \( \binom{n}{2} \) pairs of scores 
\((X_{12}, X_{21}), \ldots, (X_{1n}, X_{n1}), \ldots, (X_{n-1,n}, X_{n,n-1})\) are independent.

Let \( s_i = \sum_{j=1, j \neq i}^n X_{ij} \) be a score of the player \( i (i = 1, \ldots, n) \) after playing with \( n - 1 \) opponents. We use a standard notation and denote by \( s^{(1)} \leq s^{(2)} \leq \cdots \leq s^{(n)} \) the order statistics of the random variables \( s_1, s_2, \ldots, s_n \), and further denote normalized scores (zero expectation and unit variance) by \( s^*_1, s^*_2, \ldots, s^*_n \) with the corresponding order statistics \( s^*_1 \leq s^*_2 \leq \cdots \leq s^*_n \).

For the case where there are no draws, i.e. \( X_{ij} \in \{0, 1\}, X_{ij} + X_{ji} = 1, p_{ij} = P(X_{ij} = 1) = \frac{1}{2} \), Huber (1963) proved that

\[
s^*_n - \sqrt{2 \log(n - 1)} \to 0
\]

in probability as \( n \to \infty \) (see also Moon (2013)), where \( \log(x) \) is the logarithm of \( x \), to base \( e \). The main step in his proof was establishing the following inequality (Lemma 1 in Huber (1963)):

\[
P (s_1 < k_1, \ldots, s_m < k_m) \leq P (s_1 < k_1) \cdots P (s_m < k_m)
\]

for any probability matrix \((p_{ij})\) and any numbers \((k_1, \ldots, k_m)\), \( m \leq n \).

Ross (2022) studied a binomial tournament model \((X_{ij} \sim Bin(n_{ij}, p_{ij}))\), and proved that Huber’s type lemma holds for that model. He established bounds for \( P(s_i > \max_{j \neq i} s_j) \) and for the number of wins for the winning team using the stochastic ordering property, which required the knowledge of certain negative dependence structures of the scores.

Malinovsky and Moon (2021) extended Huber’s lemma to a large class of discrete distributions of \( X_{ij} \) and showed that for generalizations of round-robin tournaments, this extension implies convergence in probability of the normalized maximal score. Model M is a particular case of such generalizations.

In this work, we are interested in the marginal distribution of the scores associated with the ranks of \( n \) players after \( \binom{n}{2} \) games under Model M, where rank 1 is the winner’s rank, rank 2 is the second best, and so on. This means that we are interested in finding the marginal distribution of \( s_{(i)} \). The exact distribution for a general \( n \) seems impossible to obtain; we obtain a limit distribution, and demonstrate it with the three best scores in Model M. Recently Malinovsky and Rinott (2023) proved that \( s_1, \ldots, s_n \) are negatively
associated (see Joag-Dev and Proschan (1983) for the definition). It simplifies the proof of the main result and allows all values of \( p \) in the interval \([0, 1)\) to be considered.

## 2 Main Result

Under Model M, we have the following properties of the scores \( s_1, s_2, \ldots, s_n \) that satisfy \( s_1 + s_2 + \cdots + s_n = n(n - 1)/2 \):

(a) \( E_n = E(s_1) = (n - 1)/2, \) \( \sigma_n = \sigma(s_1) = \sqrt{(n - 1)(1 - p)/4}, \)

(b) \( \rho_n = \text{corr}(s_1, s_2) = -1/(n - 1), \)

(c) The random variables \( s_1, s_2, \ldots, s_n \) are exchangeable for the fixed \( n \).

The normalized scores \( s_1^*, s_2^*, \ldots, s_n^* \) are exchangeable random variables for the fixed \( n \), i.e., \( n \)-exchangeable or finite exchangeable. Their distribution depends on \( n \), and their correlation is a function of \( n \). Therefore, if they are a segment of the infinite sequence \( s_1^*, s_2^*, \ldots \), then they are not exchangeable, i.e., not infinite exchangeable.

Let \( I_j^{(n)} = I(s_j^* > x_n(t)) \), where we choose \( x_n(t) = a_n t + b_n \), where

\[
a_n = (2 \log n)^{-\frac{1}{4}}, \quad b_n = (2 \log n)^{\frac{1}{2}} - \frac{1}{2}(2 \log n)^{-\frac{1}{2}} (\log \log n + \log 4\pi).
\]

Set \( W_n = I_1^{(n)} + I_2^{(n)} + \cdots + I_n^{(n)} \).

We prove the following result.

**Theorem 1.** For \( p \in [0, 1) \), a fixed value of \( k \), and a fixed real \( t \),

\[
\lim_{n \to \infty} P(W_n = k) = e^{-\lambda(t)} \frac{\lambda(t)^k}{k!}, \quad \lambda(t) = e^{-t}.
\]

**Proof.** (Theorem 1) The result follows from Assertions presented below. Set

\[
\pi_i^{(n)} = P(I_i^{(n)} = 1), \quad W_n = \sum_{i=1}^{n} I_i^{(n)}, \quad \lambda_n = E(W_n) = \sum_{i=1}^{n} \pi_i^{(n)}.
\]

**Assertion 1.**

\[
d_{TV}(L(W_n), \text{Poi}(\lambda_n)) \leq \frac{1 - e^{\lambda_n}}{\lambda_n} \left( \lambda_n - \text{Var}(W_n) \right) = \frac{1 - e^{\lambda_n}}{\lambda_n} \left( \sum_{i=1}^{n} \left( \pi_i^{(n)} \right)^2 - \sum_{i \neq j} \text{Cov} \left( I_i^{(n)}, I_j^{(n)} \right) \right),
\]

(A1)
where \( d_{TV}(L(W_n), Poi(\lambda_n)) \) is the total variation distance between distributions of \( W_n \) and Poisson distribution with mean \( \lambda_n \).

**Assertion 2.**

\[
\pi_1^{(n)} = P(s_1^* > x_n(t)) \sim 1 - \Phi(x_n(t)),
\]

where \( c_n \sim k_n \) means \( \lim_{n \to \infty} c_n/k_n = 1 \).

**Assertion 3.**

\[
\lim_{n \to \infty} n \pi_1^{(n)} = \lim_{n \to \infty} n P(s_1^* > x_n(t)) = \lambda(t) = e^{-t}.
\]

**Assertion 4.**

\[
\lim_{n \to \infty} n^2 P(s_1^* > x_n(t), s_2^* > x_n(t))) = \lambda(t)^2 = e^{-2t}.
\]

In our case, since \( s_1^*, \ldots, s_n^* \) are identically distributed, \( \sum_{i=1}^{n} \left( \pi_i^{(n)} \right)^2 = n P(s_1^* > x_n) P(s_1^* > x_n) \), and \( \sum_{i \neq j} Cov \left( I_i^{(n)}, I_j^{(n)} \right) = n(n - 1) [P(s_1^* > x_n(t), s_2^* > x_n(t)) - P(s_1^* > x_n(t)) P(s_2^* > x_n(t))] \).

Hence, from (A2) and (A3) it follows that

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( \pi_i^{(n)} \right)^2 = 0.
\]

and from (A3) and (A4) it follows that

\[
\lim_{n \to \infty} \sum_{i \neq j} Cov \left( I_i^{(n)}, I_j^{(n)} \right) = 0.
\]

Then, from (F1) and (F2) it follows that \( \lim_{n \to \infty} d_{TV}(L(W_n), Poi(\lambda_n)) = 0 \), and this completes the proof of Theorem 1.

**Proof. (Assertion 1).** Malinovsky and Rinott (2023) proved that \( s_1, \ldots, s_n \) are negatively associated (see Joag-Dev and Proschan (1983) for the definition). For any \( j = 1, \ldots, n \), the indicator \( I_j^{(n)} \) is an increasing function of \( s_j \). Hence, by Property 6 in Joag-Dev and Proschan (1983), the indicators \( I_1^{(n)}, \ldots, I_n^{(n)} \) are negatively associated. Combining Theorem 2.1 (Barbour et al., 1992) and the Corollary 2.C.2 (Barbour et al., 1992), we obtain (A1).

\[\square\]
Proof. (Assertion 2). Follows from Feller [1971] (p. 552-553, Theorem 2 or 3).

Proof. (Assertion 3). Follows from Assertion 2 combined with Cramér [1946] result on page 374 of his book.

Proof. (Assertion 4). Recall that 
\[ s_1 = X_{12} + X_{13} + \cdots + X_{1n}, \quad s_2 = X_{21} + X_{23} + \cdots + X_{2n} \]
Hence, condition on the event \( X_{12} = k, k \in \{0, 1/2, 1\} \), \( s_1 \) and \( s_2 \) are independent. Let \( s_1' = X_{13} + \cdots + X_{1n}, \ s_2' = X_{23} + \cdots + X_{2n} \) and denote by \( s_1^*, s_2^* \) the corresponding normalized scores (zero expectation and unit variance). We have,
\[
P(s_1^* > x_n(t), s_2^* > x_n(t) \mid X_{12} = k) = P(s_1^* > x_n(t) \mid X_{12} = k)P(s_2^* > x_n(t) \mid X_{12} = k)
= P\left(s_1^* > x_{n-1}(t)\frac{x_n(t)}{x_{n-1}(t)} \sqrt{\frac{n-1}{n-2}} - \frac{\sqrt{2(k - 1/2)}}{\sqrt{n-2}}\right)\]
\[
P\left(s_2^* > x_{n-1}(t)\frac{x_n(t)}{x_{n-1}(t)} \sqrt{\frac{n-1}{n-2}} - \frac{\sqrt{2((1 - k) - 1/2)}}{\sqrt{n-2}}\right)
\sim P\left(s_1^* > x_{n-1}(t)\right) P\left(s_2^* > x_{n-1}(t)\right).
\text{(F3)}
\]
Combining (F3) with the formula of total probability we obtain
\[
P(s_1^* > x_n(t), s_2^* > x_n(t)) \sim P\left(s_1^* > x_{n-1}(t)\right) P\left(s_2^* > x_{n-1}(t)\right),
\]
and combining it with Assertion 3 we obtain (A4).

Remark 1. It remains an open problem if Theorem [4] holds also for \( p \in (0, 1/3) \).

3 Asymptotic distribution of the order statistics of the normalized scores

An immediate consequence of Theorem [1] is given below and describes the asymptotic distribution of the ordered normalized scores.

Result 1. Suppose \( p \) is fixed, \( p \in [0, 1) \). Then, for a fixed \( j \) and a fixed real number \( t \),
\[
\lim_{n \to \infty} P\left(s_{(n-j)}^* \leq a_n t + b_n\right) = G(t) \sum_{k=0}^{j} e^{-\frac{k}{t}},
\]
where \( a_n \) and \( b_n \) are defined in (1) and \( G(t) = e^{-e^{-t}} \) ("Gumbel" distribution function).
Proof. Result [1] follows from Theorem [1] since

\[ P\left( s_{(n-j)}^* \leq x_n(t) \right) = P(W_n \leq j), \]

and therefore

\[
\lim_{n \to \infty} P\left( s_{(n-j)}^* \leq x_n(t) \right) = \lim_{n \to \infty} P(W_n \leq j) = e^{-e^{-t}} \sum_{k=0}^{j} \frac{e^{-tk}}{k!}.
\]

We demonstrate our results with the three best scores in Model M.

### 3.1 Maximal Score

For \( p \in [0, 1) \), we obtain from Result [1] the following corollary.

**Corollary 1.**

\[
E\left( s_{(n)} \right) \sim \frac{n-1}{2} + \sqrt{\frac{(n-1) \log(n)(1-p)}{2}} + \sqrt{\frac{(n-1)(1-p)}{2 \ln(n)}} \left\{ \frac{\gamma}{2} - \frac{1}{4} \left( \log \log(n) + \log(4\pi) \right) \right\} \equiv \hat{E}(n),
\]

\[
\sigma \left( s_{(n)} \right) \sim \frac{\pi}{4 \sqrt{3}} \sqrt{\frac{(n-1)(1-p)}{2 \log(n)}} \equiv \hat{\sigma}(n),
\]

where \( \gamma = 0.5772156649 \ldots \) is the Euler constant.

**Proof.** The moments under the distribution function \( G \) can be obtained based on the following consideration. If \( Y_1, \ldots, Y_n \) are independent \( \text{exp}(1) \) random variables, then straightforward calculation shows (see for example Grimmer and Stirzaker (2020)):

\[
\lim_{n \to \infty} P\left( Y_{(n)} - \log(n) \leq t \right) = G(t),
\]

(2)

and for \( r=0, 1, 2, \ldots, n, \)

\[
(n + 1 - r) \left( Y_{(r)} - Y_{(r-1)} \right)
\]

are independent exponential random variables with rate parameter 1, where \( Y_{(0)} \) is defined as zero. Since

\[
Y_{(k)} = Y_{(1)} + (Y_{(2)} - Y_{(1)}) + \cdots + (Y_{(k)} - Y_{(k-1)}),
\]

we obtain that

\[
E(Y_{(n)}) = \sum_{j=1}^{n} \frac{1}{j}, \quad Var(Y_{(n)}) = \sum_{j=1}^{n} \frac{1}{j^2}.
\]
From \( \lim_{n \to \infty} \left\{ \sum_{j=1}^{n} \frac{1}{j} - \log n \right\} = \gamma \), \( \lim_{n \to \infty} \sum_{j=1}^{n} \frac{1}{j^2} = \frac{\pi^2}{6} \) (see for example Courant and Robbins (1996)), and (2), we obtain the expectation and variance under the distribution function \( G \) as \( E_G = \gamma, \) \( \text{Var}_G = \frac{\pi^2}{6} \). Combining this with Result 1, we have

\[
E(s^*_n) \sim \gamma b_n + a_n, \quad \sigma(s^*_n) \sim \sqrt{\frac{\pi^2}{6} b_n}.
\] (4)

Then, upon substituting \( s^*_n = (s_n - E_n)/\sigma_n \), Corollary 1 follows.

In the following table, we compare \( E(s_n) \) with \( \hat{E}_n \) and \( \sigma(s_n) \) with \( \hat{\sigma}_n \) in this manner: We fix \( p = 2/3 \) and for \( n=10, 20, 50, 100, 1000, \) and \( 10000 \) we evaluate \( E(s_n) \) and \( \sigma(s_n) \) using Monte-Carlo (MC) simulation. Values of \( \hat{E}_n \) and \( \hat{\sigma}_n \) obtained based on Corollary 1.

| \( n \) | \( E(s_n) \) | \( \hat{E}_n \) | \( |\hat{E}_n/E(s_n) - 1| \times 100 \% \) | \( \sigma(s_n) \) | \( \hat{\sigma}_n \) | \( |\hat{\sigma}_n/\sigma(s_n) - 1| \times 100 \% \) |
|---|---|---|---|---|---|---|
| 10 | 5.833 | 5.912 | 1.360 | 0.469 | 0.518 | 10.454 |
| 20 | 11.89 | 11.944 | 0.456 | 0.627 | 0.659 | 5.189 |
| 50 | 29.08 | 29.162 | 0.283 | 0.912 | 0.927 | 1.563 |
| 100 | 56.73 | 56.843 | 0.199 | 1.219 | 1.214 | 0.426 |
| 1,000 | 529.12 | 529.352 | 0.044 | 3.259 | 3.148 | 3.529 |
| 10,000 | 5110.23 | 5111.295 | 0.0212 | 8.949 | 8.626 | 3.742 |

Table 1: The number of Monte-Carlo repetitions is 100,000 for \( n=10, 20, 50, 100; 10,000 \) for \( n=1000; \) and 500 for \( n=10,000. \)

### 3.2 Second and third largest scores

For \( p \in [0,1) \), we also obtain from Result 1 the following corollary.

**Corollary 2.**

\[
E(s^*_{(n-1)}) \sim \gamma b_n + a_n - b_n, \quad \sigma(s^*_{(n-1)}) \sim \sqrt{\left(\frac{\pi^2}{6} - 1\right)} b_n, \quad \text{Var}(s^*_{(n-1)}) = \frac{\pi^2}{6} \]
(5)

\[
E(s^*_{(n-2)}) \sim \gamma b_n + a_n - 3/2b_n, \quad \sigma(s^*_{(n-2)}) \sim \sqrt{\left(\frac{\pi^2}{6} - 1.25\right)} b_n
\] (6)
Proof. From Theorem 2.2.2 in Leadbetter et al. (1983), we obtain the following result: if $Y_1, \ldots, Y_n$ are independent $\exp(1)$ random variables, then for $j = 1, 2$

$$\lim_{n \to \infty} P \left( Y_{(n-j)} - \log(n) \leq t \right) = G(t) \left( 1 + e^{-t}/1! + \cdots + e^{-jt}/j! \right).$$

(7)

The rest of the proof is similar to the proof of Corollary 1.

Substituting $s^*_j = (s_j - E_n)/\sigma_n$ for $j = n-1, n-2$, we obtain the values $E(s_j), \sigma(s_j), \tilde{E}(j), \tilde{\sigma}(j)$, which are similar to the corresponding values obtained in Corollary 1 for the case $j = n$. In the case where $p = 2/3$, we provide numerical comparisons for the second and third largest scores in a similar manner as was done in Table 1.

| n   | $E(s_{n-1})(\sigma(s_{n-1}))$ | $\tilde{E}_{(n-1)}(\tilde{\sigma}_{(n-1)})$ | $E(s_{n-2})(\sigma(s_{n-2}))$ | $\tilde{E}_{(n-2)}(\tilde{\sigma}_{(n-2)})$ | $r_{(n-1)}$ | $r_{(n-2)}$ |
|-----|--------------------------------|---------------------------------|--------------------------------|---------------------------------|-------------|-------------|
| 10  | 5.400(0.338)                  | 5.509(0.324)                    | 5.093(0.273)                  | 5.307(0.254)                    | 2.009       | 4.195       |
| 20  | 11.305(0.446)                 | 11.43(0.413)                    | 10.95(0.374)                  | 11.173(0.323)                   | 1.106       | 2.037       |
| 50  | 28.277(0.649)                 | 28.44(0.580)                    | 27.816(0.541)                 | 28.079(0.454)                   | 0.576       | 0.946       |
| 100 | 55.695(0.858)                 | 55.896(0.760)                   | 55.113(0.712)                 | 55.423(0.595)                   | 0.361       | 0.563       |
| 1,000 | 526.48(2.154)              | 526.9(1.971)                    | 525.05(1.764)                 | 525.67(1.543)                   | 0.080       | 0.118       |
| 10,000 | 5103.2(5.866)            | 5104.6(5.401)                   | 5099.5(4.672)                 | 5101.2(4.227)                   | 0.027       | 0.033       |

Table 2: The number of Monte-Carlo repetitions is 100,000 for $n=10, 20, 50, 100; 10,000$ for $n=1000$; and 500 for $n=10,000$; $r_{(j)} = |\tilde{E}(s_{(j)})/E(s_{(j)}) - 1| \times 100\%$, $j = n-1, n-2$.

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References

Barbour, A. D., Holst, L., Janson, S. (1992). Poisson approximation. Oxford Studies in Probability. The Clarendon Press, Oxford, New York.
Cramér, H. (1946). Mathematical Methods of Statistics. Princeton University Press.

Courant, R., and Robbins, H. (1996). What Is Mathematics? An Elementary Approach to Ideas and Methods. Oxford University Press 2nd edition reviewed by Ian Stewart.

Feller, W. (1971). An introduction to probability theory and its applications. Vol. II. Second edition. New York-London-Sydney: Wiley.

Grimmett, G. R. and Stirzaker, D R. (2020). Probability and random processes. 4th Edition. Oxford University Press.

Huber, P. J. (1963). A remark on a paper of Trawinski and David entitled: Selection of the best treatment in a paired comparison experiment. Ann.Math.Statist. 34, 92–94.

Joag-Dev, K., Proschan, F. (1983). Negative association of random variables, with applications. Ann. Statist. 11, 286–295.

Leadbetter, M. R., Lindgren, G., Rootzén, H. (1983). Extremes and related properties of random sequences and processes. Springer Series in Statistics. Springer-Verlag, New York-Berlin.

Malinovsky, Y. (2022a). On the distribution of winners’ scores in a round-robin tournament. Prob. in Eng. and Inf. Sciences 36, 1098–1102.

Malinovsky, Y. (2022b). Correction to ”On the distribution of winners’ scores in a round-robin tournament.” Prob. in Eng. and Inf. Sciences. 37, 737–739.

Malinovsky, Y., Rinott, Y. (2023). On tournaments and negative dependence. J. Appl. Probab. 60, 945–954.

Malinovsky, Y., Moon, J. W. (2021). On the negative dependence inequalities and maximal score in round-robin tournament. https://arxiv.org/abs/2104.01450.

Moon, J. W. (2013). Topics on Tournaments. [Publicaly available on website of Project Gutenberg https://www.gutenberg.org/ebooks/42833].

Ross, S. M. (2022). Team’s seasonal win probabilities. Prob. in Eng. and Inf. Sciences. 36, 988–998.