Second order Lagrangian and symplectic current for gravitationally perturbed Dirac-Goto-Nambu strings and branes

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We extend a recent analysis of gravitational perturbations on Dirac-Goto-Nambu type strings, membranes and higher dimensional branes. In an arbitrary gauge, it is shown that the relevant first order equations governing the displacement vector of the worldsheet and metric perturbation are obtainable from a variational principle whose Lagrangian is constructed as a second order perturbation of the standard Dirac-Goto-Nambu action density. A symplectic current functional is obtained as a by-product that is potentially useful for the derivation of conservation laws in particular circumstances.

I. INTRODUCTION

The purpose of this work is to extend our recent analysis \cite{1} of the dynamics of small perturbations of the worldsheet for a simple, internally structureless Dirac-Goto-Nambu type string, membrane, or higher dimensional brane in a pseudo-Riemannian background of arbitrary dimension. The main motivation for this investigation is to prepare the way for a study radiation backreaction that has already been undertaken \cite{2,3} in the technically simpler, but physically more speculative context of scalar axion fields. In view of this ultimate objective, our analysis \cite{1} specifically allowed for the effect of perturbing gravitational waves, unlike earlier work that was restricted to effectively free perturbations of the worldsheet in a fixed flat \cite{5} or curved \cite{6,7,8} space-time background. The main result of this analysis to date has been the demonstration \cite{1} that the divergent part of the gravitational self interaction exactly cancels for a Nambu-Goto string in ordinary four-dimensional space-time, contrary to previous claims in the literature \cite{9,10}, but consistently with concurrent work \cite{11}.

Most simple treatments of the dynamics of branes use a formalism originally developed by Eisenhart \cite{7}, using the explicit details of a specifically chosen worldsheet reference scheme at each step. However, particularly when there are other physically independent fields (gravitational or otherwise) to be taken into account, it is efficient to use a more economical treatment \cite{5,6} in which the amount of auxiliary mathematical structure involved is reduced. The approach we have used \cite{5,6} reduces the mathematical paraphernalia to its strict minimum by working exclusively in terms of fields whose tensorial components are defined directly in terms of the background space time coordinates, that is, without any reference to special frames or internal coordinates. While particular coordinate systems are genuinely useful for specific applications, the excessive number of indices can lead to the mistakes which have already been discussed above.

Although it did not allow for gravitational wave perturbations, the first of the works discussed above \cite{5} went further than our more recent generalization \cite{1}. In both cases, the first-order perturbation equations governing the dynamical evolution of the relevant fields (a vectorial surface field $\xi^\mu$ in the case of ref. \cite{5} and the surface field plus the gravitational wave perturbation $h_{\mu\nu}$ for ref. \cite{1}) were deduced. But for the simpler case of just the surface field, it was shown that these dynamical equations could be derived via a variational principle, from a Lagrangian of quadratic order in $\xi^\mu$, which was constructed as a second order perturbation of the original action — in this case the Dirac-Goto-Nambu action. A by-product of this second-order approach is the construction of a bi-linear symplectic surface current satisfying a Noether type conservation law.

In the present work it is shown how to construct an equivalent second-order Lagrangian, and the corresponding bi-linear symplectic current in the more general case which includes gravitational wave perturbations. From this Lagrangian, we deduce that the same dynamical equations of motion already found in ref. \cite{1} and used in ref. \cite{5}. However, it will be seen that the gravitational waves perturbations act as a source term for the symplectic current, and hence in this more general case the relevant Noetherian surface current no longer satisfies a strict conservation law.

We will use a similar notation scheme as before to describe a p-brane with a $p+1$ dimensional worldsheet, in $n$ space-time dimensions. As emphasized already, the need to refer any internal worldsheet coordinates, $\sigma^a$ ($a=0,\ldots,p$) say, is avoided by basing the analysis on the first fundamental tensor, with components $\eta^{\mu\nu}$ defined with respect to ordinary background space-time coordinates $x^{\mu} = X^{\mu}(\sigma^a)$ ($\mu = 1,\ldots,n$). This tensor is determined by the background space-time metric $g_{\mu\nu}$ as the projection of the contravariant inverse of the induced metric on the
worldsheet, that is, in terms of the internal coordinates (using $\partial_a$ for partial differentiation), as $\gamma^{\mu\nu} = \gamma^{ab} \partial_a X^\mu \partial_b X^\nu$, where the induced metric is given by $\gamma_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$. Contracting any vector with this fundamental tensor has the effect of projecting it onto the part tangential to the worldsheet, while the normal part is obtainable by contraction with the orthogonal projection tensor $\perp^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$. We denote the operator of tangentially projected covariant differentiation — the only kind that is well defined for a tensor field with support confined to the worldsheet — by

$$\overline{\nabla}^\mu = \eta^{\mu\nu} \nabla_\nu,$$

where $\nabla_\nu$ is the usual operator of covariant differentiation as defined with respect to the Riemannian background connection, namely $\Gamma^\lambda_{\sigma\tau} = g^{\lambda\rho} \left( g_{\rho(\sigma,\tau)} - \frac{1}{2} g_{\sigma\tau,\rho} \right)$ using round brackets for index symmetrisation. The information characterizing the various (intrinsic and extrinsic) kinds of curvature associated with the worldsheet embedding is provided [12] by the second fundamental tensor, as defined in terms the first one by

$$K_{\rho}^{\mu\nu} = \eta^{\sigma\nu} \overline{\nabla}^\mu \eta_{\rho\sigma}.$$  

(2)

Since it is automatically surface tangential, as well as symmetric, with respect to its first two indices but surface orthogonal with respect to its last index, contraction with the former provides its only non-identically vanishing trace, namely the curvature vector

$$K^\rho = \eta^{\mu\nu} K_{\rho}^{\mu\nu}.$$  

(3)

The non-trivial dynamical requirement that this vector $K^\rho$ should actually vanish is the tensorially covariant expression of the equations of motion for the kind of strings and higher branes that we shall be considering, namely those characterised by the Dirac-Goto-Nambu action principle. In this case, the relevant action $\mathcal{I}$ is just proportional to the geometrically induced measure of the worldsheet, meaning that it is obtained by taking a fixed value for the Lagrangian density $L$ in the general expression for a worldsheet action, as given by

$$\mathcal{I} = \int L d\Sigma,$$

(4)

where $d\Sigma = ||\gamma||^{1/2} dp^{+1}\sigma$, in terms of internal coordinates.

II. RELATION BETWEEN LAGRANGIAN AND EULERIAN VARIATIONS

In order to proceed with the second-order analysis of the action principle, we need to define the relevant perturbations and in particular the difference between Eulerian and Lagrangian variations. In ref. [1] we only included the very much simpler first-order perturbations, while in ref. [5] second-order perturbations were included in a fixed background, that is, no Eulerian variation. In this section, we generalize these second-order variations to include Eulerian variations of the metric. The starting point for this is the finite action variation, which takes the form

$$\Delta \mathcal{I} = \delta \mathcal{I} + \frac{1}{2} \delta^2 \mathcal{I} + \mathcal{O}(\delta^3),$$

(5)

where $\delta$ is an infinitesimal variation operator normalised with respect to a variation parameter with magnitude proportional to that of the displacement vector field.

The Eulerian variation of the metric is given by

$$g_{\mu\nu} \Rightarrow g_{\mu\nu}^{\text{E}} = g_{\mu\nu} + h_{\mu\nu},$$

(6)

defined with respect to some predetermined position identification scheme, for example, harmonic coordinates. It is possible in the analysis of a smoothly distributed fluid to work entirely with Eulerian, that is, ‘fixed point’, variations. However, for the cases under consideration here (point particles, strings, and higher dimensional branes) where the relevant fields are confined to a lower dimensional support manifold that may be displaced by the perturbation, it is evident that the concept of an Eulerian variation will generically fail to be well defined, so that it becomes necessary to use an approach based on Lagrangian, that is, ‘comoving’, variations. A similar approach is often also used in dealing with smooth fluid media.

As a prerequisite for this, the relevant Eulerian background variation (5) must first be translated into Lagrangian form. It is well known that the Lagrangian variation
\[ \Delta_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu} , \]  
representing the difference between the comoving perturbed metric \( g_{\mu\nu} \) and the unperturbed value \( g_{\mu\nu} \) will deviate from its Eulerian analogue by a term that to first order in the displacement \( \xi^\mu \) will be given just by the corresponding Lie derivative of \( g_{\mu\nu} \) due to the worldsheet, and to the worldsheet motions may play a role in the intermediate steps of the calculations, they must cancel out (a condition on \( \eta^\mu \) for which only the values of \( \xi^\mu \) is defined. The natural way to do this [12] (as suggested, for example, by the Bunting identity [13], and the analysis of Boisseau and Letelier [14]) is to specify the displacement \( \xi^\mu \) in terms of the corresponding infinitesimal geodesic. On this basis, according to (12) of ref. [3], one finds that [14] that the corresponding Lagrangian value \( h_{\mu\nu} \) of the perturbed metric will be given in terms of its Eulerian analogue \( g_{\mu\nu} \) and the associated Riemannian differentiation operator \( \nabla^\ell \) and curvature \( R_{\mu\nu\rho\sigma} \) by

\[ g_{\mu\nu} - h_{\mu\nu} = 2\nabla^\rho (\xi^\mu) \nabla_\rho \xi_\nu - \xi_\rho \xi_\sigma R_{\mu\nu\rho\sigma} + O\{\delta^3\} . \]  
Subject to the understanding that the metric perturbation is at most of the same infinitesimal order as the displacement, \( h_{\mu\nu} = O\{\delta\} \), it then follows that the required Lagrangian variation of the metric will be expressible in terms of the unperturbed Riemannian differentiation operator \( \nabla^\ell \) and curvature \( R_{\mu\nu\rho\sigma} \) by the ubiquitously applicable formula

\[ \Delta_{\mu\nu} - h_{\mu\nu} = 2\nabla^\rho (\xi^\mu) + \xi^\rho \nabla_\rho h_{\mu\nu} + 2h_{\rho(\xi^\mu) \nabla_\rho} - \xi^\rho \xi^\sigma R_{\mu\nu\rho\sigma} + O\{\delta^3\} \]  
The linear order part given by the first term on the right in this formula is the well known Lie derivative term. The novelty here is inclusion of the quadratic order adjustment, whose potential importance is to be emphasised; it is likely to be useful in more general physical contexts, not just for treating the particular case of the Dirac-Goto-Nambu strings and higher branes that are considered here.

It is to be remarked that there is a lot of arbitrary gauge freedom in the way one chooses the displacement generating field \( \xi^\mu \) at positions off the worldsheet. However, although derivatives of the displacement field in directions orthogonal to the worldsheet may play a role in the intermediate steps of the calculations, they must cancel out (a condition which provides a useful check on the algebra) in the physical equations governing the motion of the worldsheet, for which only the values of \( \xi^\mu \) at positions actually on the worldsheet are relevant. In the particular case to be considered here, Dirac-Nambu-Goto branes, there is another kind of gauge freedom involved, related to the choice of the tangential component of the displacement fields. This is because moving along the worldsheet has no effect on its locus, which — in the absence of internal currents and fields — is the only physical structure that matters. Hence, in the Dirac-Goto-Nambu case there is no loss of physical generality if, for the purpose of obtaining well behaved hyperbolic dynamical equations, one chooses to eliminate this freedom by fixing the gauge on the worldsheet through imposing the condition of orthogonality, \( \eta^\mu \xi^\mu = 0 \). In the present article, we shall nevertheless impose no restriction on the gauge to be used, leaving open the possibility of including an arbitrary tangential component in the choice of \( \xi^\mu \).

III. SECOND ORDER VARIATION OF THE ACTION.

We are now in a position to vary the action [3] in the form [3]. It will be convenient to use an abbreviation scheme in which an acute accent indicates differentiation with respect to the relevant variation parameter, so that in the particular case of the action we are able to simply write the variation parameter, so that in the particular case of the action we are able to simply write \[ \delta I = \hat{I} \]  and \[ \delta^2 I = \hat{I} \]. We should note that, since the action is global, the distinction between Lagrangian and Eulerian variations does not arise at this stage.

It is evident from (3) that in this scheme the first order Lagrangian variation of the metric will be given by an expression of the familiar form

\[ \delta L g_{\mu\nu} = \hat{L}_{\mu\nu} + 2\nabla^\rho (\xi^\mu) \]  
while the corresponding second order Lagrangian variation of the metric will be given by

\[ \delta^2 L g_{\mu\nu} = \hat{L}_{\mu\nu} + 2\nabla^\rho (\xi^\mu) + 2\xi^\rho \nabla_\rho \hat{L}_{\mu\nu} + 4h_{\rho(\xi^\mu) \nabla_\rho} - \xi_\rho \xi_\sigma R_{\mu\nu\rho\sigma} \]  
This last expression can be slightly simplified by omission of the first term on the right if the perturbing field \( h_{\mu\nu} \) is prescribed to scale directly with the variation parameter, so that one simply has \( h_{\mu\nu} = \hat{h}_{\mu\nu} \) and \( \hat{h}_{\mu\nu} = 0 \). However, one can not analogously exclude the presence of higher order contributions in the expansion \( \xi^\mu = \xi^\mu + \frac{1}{2} \hat{\xi}^\mu + ... \) for
the a priori unknown, displacement field. At this stage we do not make any simplification, but subsequent sections we shall.

To allow for the effects of the perturbation on the surface measure, whose first order Lagrangian variation will be given \[ \delta \mathcal{L} \] in terms of the fundamental tensor \( \eta_{\mu\nu} \) by an expression of the familiar form

\[
\delta \mathcal{L} = \frac{1}{2} \eta_{\mu\nu} (\delta g_{\mu\nu}) d\Sigma,
\]

it is convenient to express the first and second order variations of any action integral of the form \( \mathcal{I} \) as

\[
\dot{\mathcal{I}} = \int (\hat{\delta} \mathcal{L}) d\Sigma, \quad \ddot{\mathcal{I}} = \int (\hat{\delta}^2 \mathcal{L}) d\Sigma.
\]

The measure weighted ‘diamond-differential’ operator \( \hat{\delta} \) is surface generalisation of the ‘boldface-differential’ operator already used by Friedman and Schutz \( \hat{\delta} \). When applied to the Lagrangian density it gives

\[
\hat{\delta} \mathcal{L} = \hat{\delta} \mathcal{L} + \frac{1}{2} \mathcal{L} \eta^{\mu\nu} \delta g_{\mu\nu},
\]

so that at second-order one obtains

\[
\hat{\delta}^2 \mathcal{L} = \hat{\delta} \mathcal{L} + \frac{1}{2} \mathcal{L} \eta^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{4} \mathcal{L} \eta^{\mu\nu} \eta^{\rho\sigma} - 2 \mathcal{L} \eta^{\mu\rho} \eta^{\nu\sigma}) (\delta g_{\mu\nu}) \delta g_{\rho\sigma} + \frac{1}{2} \mathcal{L} \delta^2 \eta^{\mu\nu} g_{\mu\nu}.
\]

Provided any internal fields that may be present satisfy the corresponding dynamical equations, which implies that they give no contribution to the first-order variation, and provided the only relevant external field is that of the background metric \( g_{\mu\nu} \), then the variation in \( \mathcal{I} \) will be expressible just in terms of the surface energy-momentum density tensor \( T_{\mu\nu} \) in the standard form

\[
\delta \mathcal{I} = \frac{1}{2} \int T_{\mu\nu} (\delta g_{\mu\nu}) d\Sigma,
\]

where

\[
T_{\mu\nu} = \frac{2}{m^{p+1}} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + \mathcal{L} \eta^{\mu\nu}.
\]

We shall now restrict our attention to the simple case of a Dirac-Goto-Nambu membrane or string, for which Lagrangian density in \( \mathcal{L} \) is just a constant,

\[
\mathcal{L} = -m^{p+1},
\]

where \( m \) is a fixed parameter which in natural units will have the dimensions of mass. For topological defect system, such as a cosmic string or domain wall, this Kibble mass parameter will have the same order of magnitude as the Higgs mass associated with the underlying spontaneous symmetry breaking. In this simple case, the Lagrangian variation of the Lagrangian density will be given trivially by \( \delta \mathcal{L} = 0 \) and the surface energy-momentum density tensor \( T_{\mu\nu} \) will simply be given in terms of the fundamental tensor by the proportionality relation

\[
T_{\mu\nu} = -m^{p+1} \eta^{\mu\nu}.
\]

Under these circumstances it can immediately be seen from the preceding formulae \( \mathcal{L} \) and \( \mathcal{I} \) that the integrand of the first order variation of the action will be given by

\[
- m^{-p+1} \hat{\delta} \mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \delta g_{\mu\nu} = \frac{1}{2} \eta^{\mu\nu} h_{\mu\nu} + \nabla_{\mu} \xi^{\mu}.
\]

The content of the variation principle is that the surface integral of this quantity must be made independent of the infinitesimal displacement vector field \( \xi^{\mu} \). In other words the dynamical equations are specified as the condition for all the displacement dependent terms on the right hand side of \( \mathcal{I} \) to be contained within a surface current divergence, that is, a current which is tangential to the worldsheet. The relevant term \( \nabla_{\mu} \xi^{\mu} \) would already have the form of a surface current divergence as it stands, if the displacement \( \xi^{\mu} \) satisfied the worldsheet tangentiality condition \( \xi^{\mu} = \eta^{\mu\nu} \xi_{\nu} \). More generally, in the non trivial case for which it has an orthogonal part, \( \perp_{\nu} \xi^{\mu} \neq 0 \), the formula \( \nabla_{\mu} \perp_{\nu} = -K_{\mu} \) can be used to expand the term in question as

\[
\nabla_{\mu} \xi^{\mu} = \nabla_{\mu} (\eta^{\mu\nu} \xi_{\nu}) - K_{\mu} \xi^{\mu},
\]

\[
\nabla_{\mu} \xi^{\mu} = \nabla_{\mu} (\eta^{\mu\nu} \xi_{\nu}) - K_{\mu} \xi^{\mu},
\]
where $K_{\mu}$ is the curvature vector, as defined by (3). Since the first term in (20) has the form of a surface current divergence, and as such is irrelevant from the point of view of the variation principle, we can replace (19) by the expression

$$- m^{-(p+1)} \delta L \equiv \frac{1}{2} \eta^{\mu\nu} \dot{h}_{\mu\nu} - K_{\mu} \dot{\xi}^\mu,$$

(21)

using the equivalence symbol $\equiv$ to indicate equality modulo divergences that are removable by Green’s theorem for a variation with compact support. At this first order level the metric variation $\dot{h}_{\mu\nu}$ cannot couple to the displacement. It thus becomes evident that the content of the ensuing Dirac Goto Nambu type field equation is simply the well known condition that this vector $K_{\mu}$ should vanish.

We are now ready to move onto new ground by using (11) and (15) to obtain the second-order analogue of (19) which works out as

$$- m^{-(p+1)} \delta L = \left( \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} - \frac{1}{2} \eta^{\mu\rho} \eta^{\nu\sigma} \right) \dot{h}_{\mu\nu} \dot{h}_{\rho\sigma} + \frac{1}{2} \rho_{\sigma} \left( 2 \dot{h}_{\mu} \nabla_{\nu} \dot{\xi}^\sigma + \eta^{\mu\nu} \dot{\xi}^\sigma \nabla_{\rho} \dot{h}_{\mu\nu} + \eta^{\rho\sigma} \nabla_{\nu} \left( \dot{h}_{\rho\sigma} \dot{\xi}^\nu \right) + 2 \left( \left( \nabla_{\mu} \dot{\xi}^\nu \right) \nabla_{\nu} \dot{\xi}^\rho + \frac{1}{2} \rho_{\sigma} \left( \nabla_{\nu} \dot{\xi}^\rho \nabla_{\sigma} \dot{\xi}^\nu \right) - \eta^{\mu\nu} R_{\rho\mu\sigma} \dot{\xi}^\rho \dot{\xi}^\sigma + \frac{1}{2} \eta^{\nu\sigma} \dot{h}_{\rho\sigma} + \nabla_{\nu} \dot{\xi}^\rho \right).$$

(22)

In the manner already demonstrated in the special case (3) for which the gravitational perturbation $\dot{h}_{\mu\nu}$ is absent, the second variational integrand (22) will be employable as the Lagrangian density of a variational principle for the perturbed dynamical equations at first-order. For this purpose, the only relevant terms are those having homogeneously quadratic dependence on first-order variations. The remaining terms, that is to say the last two terms which are homogeneously linear in second-order variations, are of the same form as the linear contribution (19) in the first-order contribution, which means that as consequence of the zero order field equations their contribution to the action integral will just be a constant. Since the present analysis is based on the treatment of the gravitational perturbation as a given background field, the the first term in (22) will also just be a constant as far as the variational principle is concerned.

IV. DERIVATION OF THE PERTURBED DYNAMICAL EQUATIONS

Therefore, we have argued that dropping the constant contributions (from the first term in (22) and from the second-order variations, which are irrelevant for the application of the variation principle, one can obtain a second-order action $I_{(2)}$ say, (differing from $\mathcal{L}$ in (3) only by the omitted constant) of the form

$$I_{(2)} = \int L_{(2)} \ d\Sigma,$$

(23)

where the relevant second-order Lagrangian is given by

$$- m^{-(p+1)} L_{(2)} = \left( \nabla_{\mu} \dot{\xi}^\nu \right) \nabla_{\nu} \dot{\xi}^\rho + \frac{1}{2} \rho_{\sigma} \left( \left( \nabla_{\nu} \dot{\xi}^\rho \right) \nabla_{\sigma} \dot{\xi}^\nu - \eta^{\mu\nu} R_{\rho\mu\sigma} \dot{\xi}^\rho \dot{\xi}^\sigma + \frac{1}{2} \eta^{\nu\sigma} \dot{h}_{\rho\sigma} + \nabla_{\nu} \dot{\xi}^\rho \right).$$

(24)

To apply the variation principle to $L_{(2)}$ we consider a process in which the linearized field variables themselves undergo independent infinitesimal ‘virtual’ variations $\dot{\xi}^\mu \rightarrow \dot{\xi}^\mu + \delta \dot{\xi}^\mu$ and $\dot{h}_{\mu\nu} \rightarrow \dot{h}_{\mu\nu} + \delta \dot{h}_{\mu\nu}$. Writing $\dot{\xi}^\mu = \delta \dot{\xi}^\mu$ and $\dot{h}_{\mu\nu} = \delta \dot{h}_{\mu\nu}$, that is, using a grave accent to indicate differentiation with respect to the ‘virtual’ variation parameter (as distinct from differentiation with respect to the ‘real’ or ‘physical’ variations for which the acute accent is used), then the ensuing action differential will have the form

$$\delta I_{(2)} = 2 \int \mathcal{L}_{(1,1)} \ d\Sigma,$$

(25)

where $\mathcal{L}_{(1,1)}$ is a symmetric bi-linear functional of the independent ‘real’ (acute accented) and ‘virtual’ (grave accented) variations, that is given by

$$- m^{-(p+1)} \mathcal{L}_{(1,1)} = \left( \nabla_{\mu} \dot{\xi}^\nu \right) \nabla_{\nu} \dot{\xi}^\rho + \frac{1}{2} \rho_{\sigma} \left( \left( \nabla_{\nu} \dot{\xi}^\rho \right) \nabla_{\sigma} \dot{\xi}^\nu - \eta^{\mu\nu} R_{\rho\mu\sigma} \dot{\xi}^\rho \dot{\xi}^\sigma + \frac{1}{2} \eta^{\nu\sigma} \dot{h}_{\rho\sigma} + \nabla_{\nu} \dot{\xi}^\rho \right).$$

(26)
At the expense of sacrificing the manifest symmetry between ‘real’ and ‘virtual’ contributions, the next step is to introduce linear functionals \( Q = Q\{\xi\} \) and \( F = F\{h\} \) depending respectively on the displacement vector \( \xi^\mu \) and the gravitational perturbation \( h_{\mu\nu} \), so as rewrite this in the more directly applicable form

\[
-m^{-({p+1})}\mathcal{I}_{(1,1)} = \dot{\xi}^\mu (\dot{F}_\mu - \dot{Q}_\mu) + \xi^\mu \dot{F}_\mu + \nabla_\mu \dot{J}^\mu ,
\]

where we have used the obvious notation

\[
\dot{Q}_\mu = Q_\mu(\xi) , \quad \dot{F}_\mu = F_\mu(h) , \quad \dot{F}_\mu = F_\mu(h) .
\]

The surface current vector \( \dot{J}^\mu \) is a symmetric bi-linear functional of the two variations, \( Q_\sigma \) is the dynamical functional for the surface perturbation, and \( F_\sigma \) is the equivalent functional for the gravitational perturbation. These expressions are given by,

\[
\dot{J}^\mu = \pm^\alpha \xi^\alpha \nabla^\mu \xi_\rho + \eta^\rho_{\sigma} (\xi^\alpha \nabla_\alpha \xi^\nu - \xi^\nu \nabla_\alpha \xi^\alpha \xi^\alpha \eta_\sigma - \eta^\rho_\sigma (\dot{\xi}_\rho \dot{\xi}_\sigma + \dot{\xi}_\sigma \dot{\xi}_\rho) ,
\]

\[
Q_\sigma(\xi) = \pm^\alpha \nabla_\sigma \nabla^\rho \xi_\rho - 2K_\sigma^\alpha \nabla_\alpha \xi^\rho + 2K_\sigma^\alpha \nabla_\alpha \xi^\nu + \pm^\alpha \eta^\rho_{\sigma} R^\nu_\rho \xi_\rho ,
\]

\[
F_\sigma(h) = \pm^\alpha (\frac{1}{2} \eta^\mu_\sigma \nabla_\mu h_{\rho\nu} - \nabla_\rho h_{\mu\nu}) + (K_\sigma^\rho_\sigma - \pm^\rho \eta^\rho_\nu K_\nu) h_{\mu\nu} .
\]

One can see that the last functional is automatically worldsheet orthogonal, that is, for an arbitrary perturbing field \( h_{\mu\nu} \)

\[
\eta^\rho_\sigma F_\sigma(h) = 0 ,
\]

is an identity.

The requirement of the variation principle is that the surface integral \( \mathcal{I}_{(2)} \) of \( \mathcal{I} \) should be independent of the virtual displacement \( \dot{\xi}^\mu \) for a variation with compact support. Since the term involving the current \( \dot{J}^\mu \) is a pure surface divergence, this requirement is evidently equivalent just to the condition that for a given ‘real’ gravitational perturbation field \( h_{\mu\nu} \) the corresponding ‘real displacement vector \( \dot{\xi}^\mu \) should satisfy a field equation of the form

\[
\dot{Q}_\sigma = \dot{F}_\sigma .
\]

The validity of this field equation can be immediately confirmed by checking that when the unperturbed, zero order dynamical equations \( K_\rho = 0 \) are satisfied, \( \mathcal{I}_{(2)} \) does indeed agree with the first-order perturbed dynamical equations given by equation (21) of our preceding work \[1\]. In particular, it can be verified that the backreaction terms \( F_1, F_2 \) that were defined there by equation (45) add up to a total that matches the gravitational forcing term on the right of \( \mathcal{I} \), that is, the linear functional defined here by \( \mathcal{I} \) is expressible in terms of our previous notation as \( F_\sigma = F_1 + F_2 \). According to (41) of ref. \[1\], the precise physical interpretation of this is that the quantity \( \mathcal{I} = -m^{(p+1)}F_\sigma \) represents the gravitational force density exerted on the brane by a perturbing field \( h_{\mu\nu} \).

**V. THE SYMPLECTIC CURRENT**

Having confirmed that the homogeneous quadratic functional \( \mathcal{I}_{(2)} \) given by \( \mathcal{I} \) does indeed act as a Lagrangian for the linearly perturbed field equations, we can exploit it for the purpose of derivation of various kinds of conservation law. In case where the gravitational perturbation \( h_{\mu\nu} \), it was shown that the current treatment yielded a Noether type identity. This expresses the fact that, although it is not manifest in the \( \mathcal{I} \), this bilinear variation term is by construction symmetric in the sense of being invariant under interchange of the two independent, mathematically equivalent variations involved. This symmetry property is equivalently expressible as the vanishing of the antisymmetric quantity obtained by taking the difference between \( \mathcal{I} \) and its analogue obtained by swapping the acute and grave accents. This identity which reduces, by the symmetry property of partial differentiation, to the form

\[
\nabla_\mu \dot{C}^\mu = \dot{\xi}^\mu \dot{Q}_\mu - \xi^\mu \dot{Q}_\mu ,
\]

in terms of a symplectic — meaning antisymmetric bilinear — surface current \( \dot{C}^\mu \) given by \( \dot{C} = \dot{C} = \dot{J}^\mu - J^\mu \) where \( J^\mu \) is obtained from \( \dot{J}^\mu \) by the interchange of ‘real’ and ‘virtual’ variations. This gives

\[
\dot{C}^\mu = (\eta^\mu_\rho \pm_\sigma + 2\eta^\rho_\sigma \eta_\rho^\nu) (\dot{\xi}^\nu \nabla_\nu \xi_\sigma - \xi^\nu \dot{\xi}^\nu \xi_\sigma) ,
\]
in which it is to be noticed that which the external gravitational perturbation field has cancelled out. Hence, the geometrically defined current functional $C^\mu$ thus retains exactly the same form, apart from the omission of the physical weighting factor $m^{\nu+1}$ in its definition, as the corresponding dimensionally weighted symplectic current $\tilde{C}^\mu$ that was obtained in ref. [1]. The current $\tilde{C}^\mu$ can be rewritten in the form

$$\tilde{C}^\mu = \frac{1}{2} \epsilon_{\rho\sigma} \left( \tilde{\xi}^\rho \nabla^\mu \tilde{\xi}^\sigma - \tilde{\xi}^\sigma \nabla^\mu \tilde{\xi}^\rho \right) + 2 \eta^\nu_{\rho} \nabla_\nu \left( \tilde{\xi}^\rho \xi^\sigma \right).$$  \hspace{1cm} (36)$$

from which it can be seen that, for the case of worldsheet conserving displacements, that is, when $\tilde{\xi}^\rho$ and $\tilde{\xi}^\sigma$ are both tangential to the brane, the first term will simply vanish while the second term will have the form of an exact surface divergence. This implies that that it will be trivially conserved as an identity, independently of any field equations that may be satisfied [7].

Although it retains the same form, the symplectic current now no longer obeys the simple surface current conservation law that was obtained [5] in the absence of gravitational perturbations. $\tilde{\xi}^\mu$ is postulated to satisfy a dynamical equation of the same form as the first order perturbation equation (33) that must be satisfied by $\xi^\mu$, allowing the dependence on the displacements to be eliminated from the right hand side of (34), and hence one is left with a relation of the form

$$\nabla_\mu \tilde{C}^\mu = \tilde{\xi}^\mu \tilde{F}_\mu - \xi^\mu \tilde{F}_\mu,$$ \hspace{1cm} (37)$$

in which, generically, there remains a source term on the right in cases where the forcing effect of gravitational wave perturbations is present.

By the orthogonality property (32), the trivial tangential parts of $\tilde{\xi}^\mu$ and $\xi^\mu$ make no contribution to the right hand side of (37). To see how they affect the left hand side, it is useful to rewrite the formula (34) for the current in the form

$$\tilde{C}^\mu = \frac{1}{2} \epsilon_{\rho\sigma} \left( \tilde{\xi}^\rho \nabla^\mu (\perp_\nu \tilde{\xi}^\nu) - \tilde{\xi}^\sigma \nabla^\mu (\perp_\nu \tilde{\xi}^\nu) \right) + 2 \eta^\nu_{\rho} \nabla_\nu \left( \eta^\rho \xi^\sigma \right) + 2 \eta^\nu_{\rho} \nabla_\nu \left( \eta^\rho \eta^{\sigma} \tilde{\xi}^\nu \xi^\sigma \right),$$ \hspace{1cm} (38)$$

in which the last term has the form of an exact surface divergence, so that independently of the field equations it will be trivially conserved as an identity. Provided the unperturbed field equations, $K_\mu = 0$ are satisfied, the second term simply drops out, so the ‘reduced’ symplectic current vector $\tilde{S}^\mu$ consisting just of the first term, namely

$$\tilde{S}^\mu = \frac{1}{2} \epsilon_{\rho\sigma} \left( \tilde{\xi}^\rho \nabla^\mu \perp_\nu \tilde{\xi}^\nu - \tilde{\xi}^\sigma \nabla^\mu \perp_\nu \tilde{\xi}^\nu \right),$$ \hspace{1cm} (39)$$

will satisfy an equation of the same form as the original symplectic current, that is,

$$\nabla_\mu \tilde{S}^\mu = \tilde{\xi}^\mu \tilde{F}_\mu - \xi^\mu \tilde{F}_\mu.$$ \hspace{1cm} (40)$$

Unlike the original current $\tilde{C}^\mu$, this ‘reduced’ current $\tilde{S}^\mu$ has the advantage of being gauge independent in the sense of being entirely independent of the tangentially projected parts of the vectors $\xi^\mu$ and $\tilde{\xi}^\mu$ on which it depends.

In the kind of practical application for which such a Noetherian identity is typically used, namely for exploiting underlying symmetries of various kinds, it is usually sufficient to consider only virtual displacements satisfying the unforced dynamical equations, which means taking $\tilde{\xi}^\mu = 0$, and thereby setting $\tilde{F}_\mu = 0$ in the right hand side of (37). A virtual displacement field satisfying this condition will be given automatically by setting $\xi^\mu = k^\mu$ where $k^\mu$ is any solution of the background Killing equations $\nabla_{\mu} k_{\nu} = 0$, since the action of such a translation on any given unperturbed solution evidently translates it onto a nearby solution that is geometrically identical and therefore also a solution. What this means – for any real physical solution $\xi^\mu$ of the displacement perturbation equation (32) – is that associated with the geometric symmetry generated by $k^\mu$ there will be a corresponding generalised momentum current

$$\tilde{S}^\mu \{ k \} = \frac{1}{2} \epsilon_{\rho\sigma} \left( k^\rho \nabla^\mu \perp_\nu \tilde{\xi}^\nu - \tilde{\xi}^\sigma \nabla^\mu \perp_\nu \tilde{\xi}^\nu \right),$$ \hspace{1cm} (41)$$

that can be seen from (37) to obey a divergence equation of the form

$$\nabla_\mu \tilde{S}^\mu \{ k \} = k^\mu \tilde{F}_\mu.$$ \hspace{1cm} (42)$$

In view of the orthogonality property (32) the source term on the right will evidently vanish in the case for which $k^\mu$ is tangent to the worldsheet, $\perp_\mu k^\nu = 0$, i.e. when the unperturbed solution is invariant under the action of the relevant symmetry, but in such a case the ensuing strict conservation law is vacuous, since the current $\tilde{S}^\mu \{ k \}$ will itself
be zero. In the case for which the unperturbed solution is invariant under the action of \( k^\mu \) one can however obtain a non trivial application of (40) by using the fact that in these circumstances if the first order dynamical equation (33) is satisfied by a “real” displacement field \( \xi^\mu \) for a given ‘real” gravitational perturbing field \( \hat{h}_{\rho\sigma} \), then the first order dynamical equation will also be satisfied by the Lie derivatives of these fields with respect to \( k^\mu \), so these Lie derivatives will be utilisable used as the “virtual” fields in (40), i.e. we shall be able to take 

\[
\xi^\mu = k^\nu \nabla_\nu \xi^\mu - \xi^\nu \nabla_\nu k^\nu \quad \text{and} \quad \hat{h}_{\rho\sigma} = k^\nu \nabla_\nu \hat{h}_{\rho\sigma} + 2 \hat{h}_{\nu(\sigma} \nabla_{\rho)} k^{\nu}).
\]

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