Black holes in the quadratic-order extended vector-tensor theories

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We investigate the static and spherically black hole solutions in the quadratic-order extended vector-tensor theories without suffering from the Ostrogradsky instabilities, which include the quartic-order (beyond-)generalized Proca theories as the subclass. We start from the most general action of the vector-tensor theories constructed with up to the quadratic-order terms of the first-order covariant derivatives of the vector field, and derive the Euler-Lagrange equations for the metric and vector field variables in the static and spherically symmetric backgrounds. We then substitute the spacetime metric functions of the Schwarzschild, Schwarzschild-de Sitter/ anti-de Sitter, Reissner-Nordström-type, and Reissner-Nordström-de Sitter/ anti-de Sitter-type solutions and the vector field with the constant spacetime norm into the Euler-Lagrange equations, and obtain the conditions for the existence of these black hole solutions. These solutions are classified into the two cases 1) the solutions with the vanishing vector field strength; the stealth Schwarzschild and the Schwarzschild de Sitter/ anti-de Sitter solutions, and 2) those with the nonvanishing vector field strength; the charged stealth Schwarzschild and the charged Schwarzschild de Sitter/ anti-de Sitter solutions, in the case that the tuning relation among the coupling functions is satisfied. In the latter case, if this tuning relation is violated, the solution becomes the Reissner-Nordström-type solution. We show that the conditions for the existence of these solutions are compatible with the degeneracy conditions for the Class-A theories, and recover the black hole solutions in the generalized Proca theories as the particular cases.

I. INTRODUCTION

A. Black holes in modified theories of gravitation

Black holes are the most fundamental objects not only in general relativity but also in many theories of gravitation, and hence the properties of black hole solutions will provide us opportunities to test modified theories of gravitation from the theoretical and observational viewpoints. It is well known that many theories of gravitation share the same black hole solutions with general relativity (GR) [1, 2]. In vacuum GR, there is the uniqueness theorem stating that the asymptotically flat and stationary black holes solutions are described only by three parameters, i.e., mass, electric charge, and angular momentum [3–6]. The no-hair BH theorem is valid for the canonical scalar field minimally coupled to gravity [7, 8], and also holds for the scalar-tensor theories with the direct coupling of the scalar field to the Ricci scalar [9–11], the noncanonical kinetic terms of the scalar field [12, 13] and the higher-order derivative interactions of the scalar field [14, 15]. These no-hair theorems state that the Schwarzschild or Kerr solution with the vanishing scalar or vector field is the unique vacuum black hole solution under the given symmetries of the spacetime. On the other hand, the black hole solutions with the nontrivial profiles of the scalar and vector fields have also been found in the scalar-tensor and vector-tensor theories which are free from some of the assumptions for the no-hair theorems, for example, via the conformal coupling to the Ricci scalar with the singular behavior of the scalar field at the event horizon [16, 17], the nonminimal coupling to the Gauss-Bonnet term [18–23], the linear time dependence of the scalar field in the shift-symmetric scalar-tensor theories [24–35], the asymptotically anti-de Sitter spacetimes [36–39], and the self-derivative couplings and the nonminimal couplings of the vector field to the spacetime curvature [40–46] (see also Ref. [47] for a review).

B. The quadratic-order extended vector-tensor theories

In this paper, we will investigate the static and spherically black hole solutions with the constant norm of the vector field in the quadratic-order extended vector-tensor theories [48], which are currently recognized as the most general single-field vector-tensor theories without the Ostrogradsky instabilities [49]. The construction of the theories first considers the most general vector-tensor theories, which are constructed with up to the quadratic-order terms of the first-order covariant derivatives of the vector field, and imposes the degeneracy conditions to avoid the appearance of the Ostrogradsky instabilities (see below). The quadratic-order extended vector-theories also correspond to the extension of the quadratic-order degenerate higher-order scalar-tensor (DHOST) theories [50–53], which are known as the most general single-field scalar-tensor theories without suffering from the Ostrogradsky instabilities [49], to the vector-tensor theories. The extension is similar to that from the Horndesky theories [54–56] to the generalized Proca theories [57–60].

The most general vector-tensor theories which are constructed with up to the quadratic-order terms of the first-order covariant derivatives of the vector field $\nabla_\mu A_\nu$ are
given by
\[ S = \int d^4x \sqrt{-g} \left[ f_0(\mathcal{Y}) + f_2(\mathcal{Y}) R + C^{\mu\nu\rho\sigma} \nabla_\mu A_\nu \nabla_\rho A_\sigma \right], \]

where \( C^{\mu\nu\rho\sigma} \), \( \nabla_\mu \) represents the metric tensor, \( \nabla_\mu \) and \( R \) are the covariant derivative and Ricci scalar associated with \( g_{\mu\nu} \), respectively, \( A_\mu \) is the vector field,
\[ \mathcal{Y} := g^{\mu\nu} A_\mu A_\nu, \]
is the spacetime norm of the vector field \( A_\mu \), and \( f_0(\mathcal{Y}) \) and \( f_2(\mathcal{Y}) \) are the free functions of \( \mathcal{Y} \). We also define the symmetric rank-4 tensor constructed with the inverse metric and the vector field \([48]\):
\[ C^{\mu\nu\rho\sigma} := \alpha_1(\mathcal{Y}) g^{\mu(\rho} g^{\sigma)\nu} + \alpha_2(\mathcal{Y}) g^{\mu\nu} g^{\rho\sigma} \]
\[ + \frac{\alpha_3(\mathcal{Y})}{2} (A^\mu A^\nu g^{\rho\sigma} + A^\rho A^\sigma g^{\mu\nu}) \]
\[ + \frac{\alpha_4(\mathcal{Y})}{2} (A^\mu A^\nu g^{(\rho} g^{\sigma)\nu} + A^\nu A^{(\rho} g^{\sigma)\mu} \]
\[ + \alpha_5(\mathcal{Y}) A^\mu A^\nu A^\rho A^\sigma + \alpha_6(\mathcal{Y}) g^{\rho(\nu} g^{\mu)\sigma} \]
\[ + \frac{\alpha_7(\mathcal{Y})}{4} (A^\mu A^\nu g^{\rho\sigma} - A^\nu A^\rho g^{\mu\sigma}), \]

where \( \alpha_i(\mathcal{Y}) \) (the Latin induces run \( i = 1, 2, \ldots, 7, 8 \)) are also the free functions of \( \mathcal{Y} \). By introducing the symmetric and antisymmetric parts of the first-order covariant derivative of the vector field \( \nabla_\mu A_\nu \), respectively,
\[ S_{\mu\nu} := \nabla_\mu A_\nu + \nabla_\nu A_\mu, \]
\[ F_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu \]
the \( C^{\mu\nu\rho\sigma} \) term in the action (1) with Eq. (3) can be rewritten as
\[ 4C^{\mu\nu\rho\sigma} \nabla_\mu A_\nu \nabla_\rho A_\sigma \]
\[ = \alpha_1(\mathcal{Y}) S_{\mu\nu} S_{\rho\sigma} + \alpha_2(\mathcal{Y}) (S^\mu_{\nu\rho})^2 \]
\[ + \frac{\alpha_3(\mathcal{Y})}{2} A^\mu A^\nu S_{\rho\sigma} + \alpha_4(\mathcal{Y}) A^\mu A^\nu S_{\rho\sigma} + \alpha_5(\mathcal{Y}) A^\mu A^\nu F_{\mu\nu} F_{\rho\sigma} \]
\[ + \alpha_6(\mathcal{Y}) A^\mu A^\nu F_{\mu\rho} F_{\nu\sigma} + \alpha_7(\mathcal{Y}) A^\mu A^\nu F_{\mu\rho} S_{\nu\sigma} + \alpha_8(\mathcal{Y}) A^\mu A^\nu F_{\mu\rho} S_{\nu\sigma}. \]

In general, the vector-tensor theories (1) with Eq. (3) are not free from the Ostrogradsky instabilities [49]. The key idea to avoid the appearance of the Ostrogradsky instabilities is to impose the degeneracy conditions among the highest-derivative equations of motion.

Here, we demonstrate how the degeneracy condition can reduce the higher-derivative system to the second-order system with a simple example in the context of analytical mechanics [50, 53, 61, 62]. We consider a particle following the trajectory \((x(T), y(T))\) that minimizes the action \( S_p = \int dT L_p \):
\[ L_p = \frac{a_1}{2} \dot{x}^2 + a_2 \dot{x} \dot{y} + \frac{a_3}{2} \dot{y}^2 + \frac{1}{2} \dot{z}^2 - V(x, y), \]
where \( T \) represents the time coordinate \( t \), ‘dot’ means the derivative with respect to \( T \), \( a_j \) \((j = 1, 2, 3)\) are constants, and \( V(x, y) \) represents the potential term. The Euler-Lagrange equations for \( x \) and \( y \) are given by the fourth-order and second-order differential equations, and hence the theory generically contains the three degrees of freedom, one of which corresponds to the Ostrogradsky ghost. By eliminating \( y \) with \( y = z - (a_2/a_3) \dot{x} \), the above Lagrangian reduces to
\[ L_p = \frac{1}{2} \left( a_1 - \frac{a_2^2}{a_3} \right) \dot{x}^2 + \frac{a_3}{2} \dot{z}^2 + \frac{1}{2} \dot{z}^2 - V(x, y). \]

Hence, by imposing \( a_2^2 - a_1 a_3 = 0 \), which is called the degeneracy condition, the highest derivative term for \( x \) can be eliminated, and the system reduces to the second-order system of \((x, z)\) and contains only two physical degrees of freedom, namely, the Ostrogradsky ghost is eliminated. In Appendix A, we will review the theory (6) in terms of the Hamiltonian analysis, and that the degeneracy condition makes the Hamiltonian bounded from below. We note that more general class of the degenerate higher-derivative theories in analytical mechanics, has been investigated in Refs. [62–66].

The degenerate theories have been extended from analytical mechanics to scalar-tensor theories in Refs. [50–53] and to vector-tensor theories in Ref. [48]. In the case of the vector-tensor theories, after the Arnowitt-Deser-Misner (ADM) decomposition [67] of the theory (1) with Eq. (3), the degeneracy conditions yield the three solutions, namely, the three different classes of the degenerate theories [48], which are briefly summarized as follows (see Sec. IV for details) \(^{32}\):

- Class A: \( a_1 + a_2 = 0 \) and \( f_2 \neq 0 \),
- Class B: \( a_1 + a_2 = 0 \) and \( f_2 = 0 \),
- Class C: \( f_2 = 0 \).

Class A includes the quadratic- and quartic-order (beyond-)generalized Proca theories as the particular subclasses and can be mapped from them via the vector disformal transformation, while Classes B and C cannot be related to the (beyond-)generalized Proca theories via the same transformation. Within Class A, there are the subclasses from Class A1 to Class A4 [48], which will be explained in subsection IV A. In Class B, there are the subclasses from Class B1 to Class B6 [48], which we omit to show explicitly. We exclude Class C from our analysis, since there is no Einstein-Hilbert term in the

\(^{1}\) In order to avoid any confusion, we would like to distinguish the time coordinate \( T \) from that in the static and spherically symmetric spacetime \( t \). (see Eq. (8a)). \( T \) here represents the time coordinate in the mechanical system for a point particle following the trajectory \((x(T), y(T)), \)

\(^{2}\) In this paper, we call ‘Case A-C’ in Ref. [48] ‘Class A-C’, respectively.
gravitational action and hence no way to compare with the results in general relativity. Thus, in the rest of this paper, we focus on Class A and Class B.

As in the case of the quadratic-order DHOST theories [31–35], while the degeneracy conditions themselves are not relevant for the derivation of the static black hole solutions in the vector-tensor theories (see Secs. II and III), we will check the compatibility of our conditions with the above three classes in the quadratic-order extended vector-tensor theories (see Sec. IV). It should be noted that the quadratic-order extended vector-tensor theories reduce to the quadratic-order DHOST theories with the shift symmetry in the limit of \( A_\mu \to \partial_\mu \phi \), i.e., the field strength of the vector field vanishes, where \( \phi \) corresponds to the scalar field [48] (see Appendix B).

C. The static and spherically symmetric black hole solutions

In the vector-tensor theories without the \( U(1) \) gauge symmetry, the hairy black hole solutions have been investigated in Refs. [40–45]. One of the interesting solutions among them is of stealth type, which describes the Schwarzschild or Kerr black hole solutions in the scalar-tensor or vector-tensor theories where the scalar or vector field with the nontrivial profile does not backreact on the spacetime geometry. The stealth black hole solutions have been explored in Refs. [24, 25, 36, 40, 68]. Since the metric functions do not explicitly depend on the model parameters, these black hole solutions cannot be distinguished from those in GR, unless the scalar or vector field has the direct coupling to the matter sector. The stealth Schwarzschild solution has been obtained at the first time in the class of the generalized Proca theories with the nonminimal coupling to the Einstein tensor \( G^\mu\nu A_\mu A_\nu \), where \( G^\mu\nu \) represents the Einstein tensor [40]. The Schwarzschild de Sitter/anti-de Sitter and Reissner-Nordström de Sitter/anti-de Sitter solutions have been obtained in the class of the generalized Proca theories with the nonminimal coupling \( G^\mu\nu A_\mu A_\nu \) and the mass term \( m^2 g^{\mu\nu} A_\mu A_\nu \) in Ref. [41], where the mass parameter \( m \) plays the role of the effective cosmological constant which is different from the bare cosmological constant \( \Lambda_0 \). These analytic solutions have been found in the other classes of the generalized Proca theories [42–45]. While these analytic black hole solutions have been obtained under the assumption that the vector field has a constant spacetime norm \( \mathcal{Y} = \text{const} \), in the case that \( \mathcal{Y} \) is not a constant black hole solutions with the nontrivial vector field profile have been investigated numerically in Ref. [42–44]. These works have been extended to the case of the beyond-generalized Proca theories in Ref. [46]. The analysis in the present paper will analyze the static and spherically symmetric solutions with the constant spacetime norm of the vector field \( \mathcal{Y} = \text{const} \) in the quadratic-order extended vector-tensor theories (1).

Needless to say, our work corresponds to the direct extension of the above previous works [40, 41, 45]. Our black hole solutions will also be the extension of the works on hairy static and spherically symmetric black hole solutions in the shift-symmetric quadratic-order DHOST theories [31–34] to the quadratic-order extended vector-tensor theories. The conditions for the existence of the stealth Schwarzschild and Schwarzschild de Sitter/anti-de Sitter solutions in the quadratic-order DHOST theories will be reviewed in Appendix B. As seen in Appendix B, the several classes of the solutions obtained in Secs. II and III with the vanishing electric field strength will reproduce those for the static and spherically symmetric hairy black hole solutions in the shift-symmetric quadratic-order DHOST theories in the limit \( A_\mu \to \partial_\mu \phi \). In other words, the investigation of the black hole solutions in the quadratic-order extended vector-tensor theories will be along these previous studies, and should enrich our knowledge on the black hole solutions in modified theories of gravitation and clarify the relationship between different theories in terms of black hole physics. This is the main motivation of our work.

Along Refs. [40, 41, 43–45], we investigate the exact, static and spherically symmetric vacuum solutions in the quadratic-order extended vector-tensor theories. In order to obtain the exact black hole solutions, we follow the same strategy as that in the case of the quadratic-order DHOST theories [31–33]. First, we restrict our attention on the static and spherically symmetric solutions with the following ansatz:

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{h(r)} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \tag{8a}
\]

\[
A_\mu dx^\mu = A_t(r)dt + A_r(r)dr, \tag{8b}
\]

with

\[
\mathcal{Y} = -\frac{A_t(r)^2}{f(r)} + A_r(r)^2h(r), \tag{9}
\]

from Eq. (2), where \( r, t, \) and \( (\theta, \varphi) \) represent the time, radial, and angular coordinates, respectively. \( f(r), h(r), A_t(r), \) and \( A_r(r) \) are the functions of \( r \). Substituting Eq. (8) into the action (1) with Eq. (3) and varying it with respect to \( f(r), h(r), A_t(r), \) and \( A_r(r) \), we obtain the Euler-Lagrange equations for each of them, respectively. Our approach is different from the standard one that first one derives the covariant equations of motion by varying the covariant action and then substitute the ansatz for the metric and vector field in the static and spherically symmetric spacetime (8), and has been employed for the investigation of the black hole solutions in modified theories of gravitation [32–35, 43, 44, 46]. In Appendix C, we demonstrate that our approach correctly reproduces the Reissner-Nordström solution in the simplest Einstein-Maxwell theory.

We will explicitly specify the metric functions \( f(r) \) and \( h(r) \) to be those of the Schwarzschild, Schwarzschild-de Sitter/anti-de Sitter, Reissner-Nordström-type, and
Reissner-Nordström de Sitter/ anti-de Sitter-type solutions, respectively, given by

- **Schwarzschild solution**

\[ f(r) = h(r) = 1 - \frac{2M}{r}, \]

where \( M \) is the constant mass parameter of the black hole measured at the spatial infinity.

- **Schwarzschild-de Sitter/ anti-de Sitter solution**

\[ f(r) = h(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2, \]

where \( M \) and \( \Lambda \) are constant mass parameter of the black hole and the effective cosmological constant, respectively.

- **Reissner-Nordström-type solution**

\[ f(r) = h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \]

where the constant parameter \( Q \) is related to the electric charge \( Q \) (see Eq. (20)), depending on the class of the theories. For example, in the Einstein-Maxwell theory (see Eq. (C9)),

\[ Q = \frac{Q}{\sqrt{2M}}. \]

where \( M_p \) represents the reduced Planck mass (C1).

- **Reissner-Nordström-de Sitter/ anti-de Sitter-type solution**

\[ f(r) = h(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 + \frac{Q^2}{r^2}, \]

where again the constant parameter \( Q \) is related to the electric charge \( Q \).

Next, along Refs. [40, 41, 43–45], we will consider the vector field with the constant spacetime norm

\[ Y = Y_0 := \text{const}, \]

which yields

\[ A_r(r) = \pm \sqrt{\frac{Y_0 + A_r(r)^2/f(r)}{h(r)}}. \]

Assuming that \( A_t(r) \) is regular at the black hole event horizon as one approaches the event horizon \( r = r_g \) characterized by

\[ f(r \to r_g) \to 0, \]

\[ h(r \to r_g) \to 0, \]

\[ \frac{f(r \to r_g)}{h(r \to r_g)} \to \text{constant}, \]

and hence

\[ A_{\mu}dx^\mu \approx A_t(r) (dt \pm dr_*) = A_t(r) \times \begin{cases} du, & \text{if } \pm 1 \end{cases} \]

where \( dr_* := dr/\sqrt{h} \) represents the tortoise coordinate, and \( u := t+r_* \) and \( v := t-r_* \) are the null coordinates regular at the future and past event horizons, respectively. Hence, the vector field is regular at either the future or past event horizon. For instance, \( r_g = 2M \) in the case of the Schwarzschild solution (10). The similar discussion can also be applied to the cosmological horizon, if exists [41].

More specifically, we consider the following form of \( A_t \):

- **Constant** \( A_t \):

\[ A_t(r) = q, \]

where \( q \) is constant. Since \( F_{tr} = -\partial_t A_t = 0 \), the black hole solutions do not possess the electric charge. As argued in Appendix B, the solutions obtained in this subsection can be mapped to those in the quadratic-order DHOST theories with the scalar field \( \phi \), via the substitution \( A_{\mu} \to \partial_{\mu} \phi \) and the integration.

- **Coulomb form of \( A_t \):**

\[ A_t(r) = q + \frac{Q}{r}, \]

where \( Q \) is constant. Since \( F_{tr} = -\partial_t A_t = Q/r^2 \), the constant \( Q \) corresponds to the electric charge of the vector field.

Having the above assumptions for the metric and vector field, first we substitute the metric functions (either (10), (11), (12), or (14)), the temporal component of the vector field (either (19) or (20)), and the radial component of the vector field (16) into each component of the Euler-Lagrange equations, and then to reduce them to a set of the algebraic equations (see e.g., Refs. [30, 32–34, 40, 41, 43–45]). In the limit of \( r \gg r_g \), we expand each component of the Euler-Lagrange equations in terms of \( 1/r \). At each order of the \( 1/r \) expansion, we require that the coefficient vanishes, and obtain the condition among \( f_0(Y), f_2(Y), \alpha_i(Y) \) \( (i = 1, 2, \cdots, 7, 8) \) and their first-order derivatives \( Y \) evaluated at \( Y = Y_0 \). We then go to the next order of the \( 1/r \) expansion and impose the similar conditions.

We repeat this manipulation until all the coefficients of the \( 1/r \) expansion of the Euler-Lagrange equations automatically vanish. When the Euler-Lagrange equations are satisfied at all orders of the \( 1/r \) expansion, the series expansion of the Euler-Lagrange equations covers the entire domain from the spatial infinity toward the black hole event horizon \( r_g < r < \infty \) [30, 32–34]. We emphasize that the conditions for the existence of the black hole
solutions are independent of the choice of the reference point with respect to which the Euler-Lagrange equations are expanded, since in the end the series expansion covers the entire domain after all the conditions for the existence of the solutions are determined. Thus, the same tuning relations among \( f_0(\mathcal{Y}), f_2(\mathcal{Y}), \alpha_i(\mathcal{Y}) \) \( (i = 1, 2, \ldots, 7, 8) \) and their first-order derivatives \( \mathcal{Y} \) evaluated at \( \mathcal{Y} = \mathcal{Y}_0 \) can be obtained, even if we expand the Euler-Lagrange equations with respect to the black hole event horizon \( r = r_g \). Moreover, if there exists the cosmological horizon in the case of the asymptotically de Sitter solution, the same conditions can be obtained, when the Euler-Lagrange equations are expanded with respect to the cosmological horizon. We note that the solutions of Eqs. (10) and (11) correspond to the special cases of the solutions of Eqs. (12) and (14) with \( Q = 0 \), which can be obtained when the tuning relations among \( f_0(\mathcal{Y}), f_2(\mathcal{Y}), \alpha_i(\mathcal{Y}) \) \( (i = 1, 2, \ldots, 7, 8) \) evaluated at \( \mathcal{Y} = \mathcal{Y}_0 \) hold even for \( Q \neq 0 \) [40].

We will classify our exact black hole solutions into the two cases in terms of the spacetime norm of the vector field:

- Case-1-N \((N = 1, 2)\); \( \mathcal{Y}_0 = -q^2 \).

  The subcases of \( N = 1 \) and \( 2 \) correspond to the cases with \( A_t(r) \) given by Eqs. (19) and (20), respectively. “\( A \)” is also attached in the case that the solution is asymptotically de Sitter or anti-de Sitter.

- Case-2-N \((N = 1, 2)\); \( \mathcal{Y}_0 \neq -q^2 \).

  The subcases of \( N = 1 \) and \( 2 \) correspond to the cases with \( A_t(r) \) given by Eqs. (19) and (20), respectively. “\( A \)” is also attached in the case that the solution is asymptotically de Sitter or anti-de Sitter.

Finally, we check the compatibility of these conditions with the degeneracy conditions of the quadratic-order extended vector-tensor theories. As we will see later, in the cases both for the stealth Schwarzschild and Schwarzschild-de Sitter/anti-de Sitter solutions, in the most cases our conditions are compatible with the degeneracy conditions, and certainly recover the previous results in the case of the generalized Proca theories.

D. The construction of this paper

The construction of this paper is as follows; in Sec. II and Sec. III, we obtain the conditions for the existence of the stealth Schwarzschild and Schwarzschild-de Sitter/anti-de Sitter solutions in the quadratic-order extended vector-tensor theories, respectively. In Sec. IV, we check the compatibility of these conditions with the degeneracy conditions. In Sec. V, we discuss the limit of our results to the case of the generalized Proca theories. Sec. VI is devoted to giving the summary and conclusions.

In Appendix A, we review the Hamiltonian analysis of the theory (6) and how the degeneracy condition removes the Ostrogradsky instabilities. In Appendix B, we review the conditions for the existence of the stealth Schwarzschild and Schwarzschild-de Sitter solutions in the quadratic-order DHOST theories, and the check whether they coincide with the scalar-tensor limits of the conditions in the extended quadratic-order vector-tensor theories. In Appendix C, we illustrate that our approach reproduces the Reissner-Nordström solution in the Einstein-Maxwell theory.

II. THE SCHWARZSCHILD SOLUTIONS

In this section, we consider the case of the Schwarzschild solution given by Eq. (10). The black hole event horizon is located at \( r = r_g := 2M \). All the Schwarzschild solutions discussed below are of stealth type, in the sense that the mass parameter \( M \) is independent of the model parameters in the coupling functions \( f_0(\mathcal{Y}), f_2(\mathcal{Y}), \) and \( \alpha_i(\mathcal{Y}) \) \( (i = 1, 2, \ldots, 7, 8) \) [40].

A. The stealth Schwarzschild solution

First, we consider the case that \( A_t \) is given by Eq. (19). As noted in the subsection IC, there are the two branches of the stealth Schwarzschild solutions given by

- Case 1

  \[
  f_0 = f_0, \mathcal{Y} = 0, \tag{21a}
  \]

  \[
  \alpha_2 = -\alpha_1, \tag{21b}
  \]

  \[
  \alpha_2, \mathcal{Y} = -\alpha_1, \mathcal{Y}. \tag{21c}
  \]

- Case 2

  \[
  f_0 = f_0, \mathcal{Y} = 0, \tag{22a}
  \]

  \[
  \alpha_1 = \alpha_2 = 0, \tag{22b}
  \]

  \[
  \alpha_2, \mathcal{Y} = -\alpha_1, \mathcal{Y}, \tag{22c}
  \]

  \[
  \alpha_3 = -2\alpha_1, \mathcal{Y}. \tag{22d}
  \]

\( f_0, \mathcal{Y}, \) \( f_2, \mathcal{Y}, \) and \( \alpha_i(\mathcal{Y}) \) \( (i = 1, 2, \ldots, 7, 8) \) denote the derivatives of \( f_0(\mathcal{Y}), f_2(\mathcal{Y}), \) and \( \alpha_i(\mathcal{Y}) \), respectively.

It has been shown that Class A of the quadratic-order extended vector-tensor theories can be mapped from the quartic-order generalized Proca theories via the vector disformal transformation [48]. In Ref. [69], the disformal transformation of the static and stationary black hole solutions in the vector-tensor theories has been discussed. As shown in Ref. [69], via the vector disformal transformation

\[
\tilde{g}_{\mu \nu} = g_{\mu \nu} + Q(\mathcal{Y}) A_\mu A_\nu, \tag{23}
\]

the stealth Schwarzschild solution with the mass \( M \) (see Eq. (10)), \( A_t = q \) (see Eq. (19)), and the constant norm
\[ \mathcal{V} = -q^2 \] in a class of the generalized Proca theory is disformally mapped to the stealth Schwarzschild solution with the rescaled mass

\[ \tilde{M} = \frac{M}{1 - Q(-q^2)q^2}, \quad (24) \]

where we assume \( 1 - Q(-q^2)q^2 > 0 \), which in part explains why the stealth Schwarzschild solution also exists in the other (disformally related) quadratic-order extended vector-tensor theories satisfying Eq. (21). It is also interesting to note that as the special case of Eq. (24) the Minkowski solution with \( \tilde{M} = 0 \) and \( q \neq 0 \) is also mapped to the Minkowski solution with \( \tilde{M} = 0 \) and \( q \neq 0 \). Thus, the disformal transformation does not modify the vacuum structure with the nonzero vector field.

**B. The charged stealth Schwarzschild solution**

Second, we consider the case given by Eq. (20). Although the black holes are electrically charged, its contribution does not appear in the spacetime metric (10), and hence we call the solution obtained in this subsection the charged stealth Schwarzschild solution, which was originally obtained in the context of the generalized Proca theories in Refs. [40, 41].

We also obtain the two branches of the charged stealth Schwarzschild solutions given by

- **Case 1-2**

\[
\begin{align*}
  f_0 &= f_{0,Y} = 0, \quad (25a) \\
  \alpha_2 &= -\alpha_1, \quad (25b) \\
  \alpha_{2,Y} &= -\alpha_{1,Y}, \quad (25c) \\
  \alpha_6 &= 3\alpha_1 + \frac{q^2}{4}[2\alpha_3 - \alpha_4 + \alpha_7 \\
  &\quad + 6\alpha_{1,Y} - 2\alpha_{6,Y} + q^2(\alpha_{4,Y} + \alpha_{7,Y} + \alpha_{8,Y})], \quad (25d) \\
  \alpha_8 &= \frac{1}{2}[2\alpha_3 - 3\alpha_4 - \alpha_7 + 6\alpha_{1,Y} - 2\alpha_{6,Y} \\
  &\quad + q^2(\alpha_{4,Y} + \alpha_{7,Y} + \alpha_{8,Y})], \quad (25e) \\
  \alpha_1, \alpha_4, \alpha_6, \alpha_7, \text{ and } \alpha_8 \text{ satisfy the relation} \\
  -6\alpha_1 + 2\alpha_6 + \mathcal{V}(\alpha_4 + \alpha_7 + \alpha_8) &= 0. \quad (26)
\end{align*}
\]

- **Case 2-2**

\[
\begin{align*}
  f_0 &= f_{0,Y} = 0, \quad (27a) \\
  \alpha_1 &= \alpha_2 = 0, \quad (27b) \\
  \alpha_{2,Y} &= -\alpha_{1,Y}, \quad (27c) \\
  \alpha_3 &= -2\alpha_{1,Y}, \quad (27d) \\
  \alpha_6 &= \frac{\mathcal{V}_0}{4}[\alpha_4 - \alpha_7 - 2\alpha_{1,Y} + 2\alpha_{6,Y} \\
  &\quad + \mathcal{V}_0(\alpha_{4,Y} + \alpha_{7,Y} + \alpha_{8,Y})], \quad (27e) \\
  \alpha_8 &= -\frac{1}{2}[3\alpha_4 + \alpha_7 - 2\alpha_{1,Y} + 2\alpha_{6,Y} \\
  &\quad + \mathcal{V}_0(\alpha_{4,Y} + \alpha_{7,Y} + \alpha_{8,Y})], \quad (27f) \\
  \alpha_4, \alpha_6, \alpha_7, \text{ and } \alpha_8 \text{ satisfy the relation} \\
  2\alpha_6 + \mathcal{V}_0(\alpha_4 + \alpha_7 + \alpha_8) &= 0. \quad (28)
\end{align*}
\]

In Case 1-2, when the tuning relation (25d) is not satisfied, the spacetime metric effectively reduces to the form of the Reissner-Nordström-type solution given by Eq. (12) with

\[
\mathcal{Q} = \mathcal{Q}_1 := \frac{Q}{\sqrt{2}[8f_2 + 6q^2\alpha_1 + q^4(\alpha_3 + 4\alpha_{1,Y})]} \\
\times \left[12\alpha_1 + 2q^2\alpha_3 - q^2\alpha_4 - 4\alpha_6 \\
+ q^2\alpha_7 + 6q^2\alpha_{1,Y} - 2q^2\alpha_{6,Y} \\
+ q^4(\alpha_{4,Y} + \alpha_{7,Y} + \alpha_{8,Y})\right]^{\frac{1}{2}}. \quad (29)
\]

While Eqs. (25a)-(25c) remain the same, as the consequence of \( \mathcal{Q}_1 \neq 0 \), Eq. (25e) is modified.

In Case 2-2, when the tuning relation (27e) is not satisfied, the spacetime metric effectively reduces to the form of the Reissner-Nordström-type solution (12) with

\[
\mathcal{Q}_2 := \frac{Q}{2\sqrt{2f_2 + \mathcal{V}_0^2\alpha_1}} \left[\mathcal{Q}_0(\alpha_4 - 4\alpha_6) \\
+ \mathcal{V}_0(-\alpha_7 - 2\alpha_{1,Y} + 2\alpha_{6,Y} \\
+ \mathcal{V}_0(\alpha_{4,Y} + \alpha_{7,Y} + \alpha_{8,Y}))\right]^{\frac{1}{2}}. \quad (30)
\]

While Eqs. (27a)-(27d) remain the same, as the consequence of \( \mathcal{Q}_2 \neq 0 \), Eq. (27f) is modified.

### III. THE SCHWARZSCHILD-DE SITTER/ANTI-DE SITTER SOLUTIONS

Second, we consider the Schwarzschild-de Sitter / anti-de Sitter solutions (11). The black hole event horizon is located at \( r = r_g \), which corresponds to the smallest positive root of the equation \( f = h = 0 \). We will not call the Schwarzschild de Sitter/anti-de Sitter solutions obtained obtained in this section stealth solutions, since the effective cosmological constant \( \Lambda \) depends on the coupling functions \( f_0(\mathcal{V}), f_2(\mathcal{V}) \), and \( \alpha_i(\mathcal{V}) (i = 1, 2, \cdots, 7, 8) \).
A. The Schwarzschild-de Sitter/anti-de Sitter solution

First, we consider the case Eq. (19). As noted in the subsection IC, the solutions discussed in this subsection can be mapped to those in the quadratic-order DHOST theories with the scalar field (see Appendix B).

We find the two branches of the Schwarzschild-de Sitter/anti-de Sitter solutions, given by Eq. (11):

- Case 1-1-Λ

\[
f_0 = -2\Lambda (f_2 + q^2\alpha_1),
\]

(31a)

\[
f_{0,y} = \Lambda \left[ \alpha_1 + \frac{3}{2}q^2\alpha_3 - 4f_{2,y} + 2q^2\alpha_{1,y} \right],
\]

(31b)

\[
\alpha_2 = -\alpha_1,
\]

(31c)

\[
\alpha_{2,y} = -\alpha_{1,y}.
\]

(31d)

- Case 2-1-Λ

\[
f_0 = -2\Lambda f_2,
\]

(32a)

\[
f_{0,y} = \Lambda (-4f_{2,y} + \mathcal{Y}_0\alpha_{1,y}),
\]

(32b)

\[
\alpha_1 = \alpha_2 = 0,
\]

(32c)

\[
\alpha_{2,y} = -\alpha_{1,y},
\]

(32d)

\[
\alpha_3 = -2\alpha_{1,y}.
\]

(32e)

B. The charged Schwarzschild-de Sitter/anti-de Sitter solution

Second, we consider the case (20), where we find the two branches of the charged Schwarzschild-de Sitter/anti-de Sitter solutions, given by

- Case 1-2-Λ

\[
f_0 = -2\Lambda (f_2 + q^2\alpha_1),
\]

(33a)

\[
f_{0,y} = \frac{\Lambda}{2} \left[ 2\alpha_1 - 8f_{2,y} + q^2(3\alpha_3 + 4\alpha_{1,y}) \right],
\]

(33b)

\[
\alpha_2 = -\alpha_1,
\]

(33c)

\[
\alpha_{2,y} = -\alpha_{1,y},
\]

(33d)

\[
\alpha_6 = 3\alpha_1 + \frac{q^2}{4} \left[ 2\alpha_3 - \alpha_4 + \alpha_7 + 6\alpha_{1,y} - 2\alpha_{6,y} + q^2(\alpha_{4,y} + \alpha_{7,y} + \alpha_{8,y}) \right],
\]

(33e)

\[
\alpha_8 = \frac{1}{2} \left[ 2\alpha_3 - 3\alpha_4 - \alpha_7 + 6\alpha_{1,y} - 2\alpha_{6,y} + q^2(\alpha_{4,y} + \alpha_{7,y} + \alpha_{8,y}) \right],
\]

(33f)

which satisfies Eq. (26).

- Case 2-2-Λ

\[
f_0 = -2\Lambda f_2,
\]

(34a)

\[
f_{0,y} = \Lambda \left( -4f_{2,y} + \mathcal{Y}_0\alpha_{1,y} \right),
\]

(34b)

\[
\alpha_1 = 0,
\]

(34c)

\[
\alpha_2 = 0,
\]

(34d)

\[
\alpha_{2,y} = -\alpha_{1,y},
\]

(34e)

\[
\alpha_3 = -2\alpha_{1,y},
\]

(34f)

\[
\alpha_6 = \frac{\mathcal{Y}_0}{4} \left[ 4\alpha_4 - \alpha_7 - 2\alpha_{1,y} + 2\alpha_{6,y} + 2\mathcal{Y}_0(\alpha_{4,y} + \alpha_{7,y} + \alpha_{8,y}) \right],
\]

(34g)

\[
\alpha_8 = -\frac{1}{2} \left[ 3\alpha_4 + \alpha_7 - 2\alpha_{1,y} + 2\alpha_{6,y} + 2\mathcal{Y}_0(\alpha_{4,y} + \alpha_{7,y} + \alpha_{8,y}) \right],
\]

(34h)

which satisfy Eq. (28).

In Case 1-2-Λ, when the tuning relation (33e) is not satisfied, the spacetime metric effectively reduces to the form of the Reissner-Nordström-de Sitter/anti-de Sitter type solution given by (14) with \( Q = Q_1 \) as in Eq. (29) while \( \Lambda \) satisfies Eqs. (33a)-(33b) and Eqs. (33c)-(33d) remain the same, as the consequence of \( Q_1 \neq 0 \), Eq. (33f) is modified.

In Case 2-2-Λ, when the tuning relation (34g) is not satisfied, the spacetime metric effectively reduces to the form of the Reissner-Nordström-de Sitter/anti-de Sitter solution Eq. (14) with \( Q = Q_2 \) as in Eq. (30). While \( \Lambda \) satisfies Eqs. (34a)-(34b) and Eqs. (34a)-(34f) remain the same, as the consequence of \( Q_2 \neq 0 \), Eq. (34h) is modified.

In the above examples, \( \Lambda \) is determined not by the bare value of the cosmological constant but by the parameters in the coupling functions. Such a feature would be crucial for the realization of the self-tuning of the cosmological constant, which was originally suggested in the context of the Horndeski theory [25].

IV. THE COMPATIBILITY WITH THE DEGENERACY CONDITIONS

A. The degeneracy conditions

The degeneracy conditions for the quadratic-order extended vector-tensor theory (1) with Eq. (3) given in Ref. [48] were summarized in subsection IB. There are the following subclasses in Class A:

- Class A1

\[
\alpha_1 = -\alpha_2 = \frac{f_2}{\mathcal{Y}},
\]

(35a)

\[
\alpha_3 = \frac{2(f_2 - f_{2,y}\mathcal{Y})}{\mathcal{Y}^2},
\]

(35b)

which requires \( \mathcal{Y} \neq 0 \).
\begin{itemize}
  \item Class A2
  \begin{align}
  \alpha_1 & = -\alpha_2 = \frac{f_2}{\mathcal{Y}}, \\
  \alpha_4 & = \frac{6f_2 + \mathcal{Y}}{\mathcal{Y}^2} - \alpha_8,
  \end{align}
\end{itemize}

where

\begin{equation}
\beta := -2\alpha_6 - \alpha_7 \mathcal{Y},
\end{equation}

which requires \( \mathcal{Y} \neq 0 \).

\begin{itemize}
  \item Class A3
  \begin{align}
  \alpha_1 & = -\alpha_2 = \frac{(\alpha_4 + \alpha_8)\mathcal{Y} - \beta}{2}, \\
  \alpha_3 & = \frac{1}{4f_2} \left((\alpha_4 + \alpha_8)\mathcal{Y} - \beta\right) \\
  & \times \left((-2\beta + 8f_2\mathcal{Y} + \alpha_8\mathcal{Y}) - 2(2\alpha_4 + \alpha_8)f_2\right),
  \end{align}
\end{itemize}

which requires \( f_2 \neq 0 \).

\begin{itemize}
  \item Class A4
  \begin{align}
  \alpha_1 & = -\alpha_2, \\
  \alpha_4 & = \frac{3(2\alpha_2 + 4f_2\mathcal{Y})}{8(f_2 + \alpha_2\mathcal{Y})} - \alpha_3 - \alpha_5 \mathcal{Y}, \\
  \alpha_8 & = \frac{-W_2 \pm \sqrt{W_2^2 - 2W_1 W_3}}{2W_1},
  \end{align}
\end{itemize}

where for the definition of \( W_k \ (k = 1, 2, 3) \) see (3.28)-(3.30) of Ref. [48]. From (3.29a) in Ref. [48], \( W_1 \neq 0 \) requires that \( \mathcal{Y} \neq 0 \) and \( f_2 + \alpha_2 \mathcal{Y} \neq 0 \).

In Class A2, the second relation in Eq. (36) can be rewritten as

\begin{equation}
-6\alpha_1 + 2\alpha_6 + (\alpha_4 + \alpha_7 + \alpha_8) \mathcal{Y} = 0.
\end{equation}

In the case that the general theory (1) with Eq. (3) admits a solution with \( \alpha_1 = \alpha_2 = 0 \).

In Class A3, the relation (38a) can be rewritten as

\begin{equation}
2(\alpha_1 + \alpha_6) + \mathcal{Y}(\alpha_4 + \alpha_7 + \alpha_8) = 0.
\end{equation}

In the case that the general theory (1) with Eq. (3) admits a solution with \( \alpha_1 = \alpha_2 = 0 \), on this background Eq. (38a) suggests

\begin{equation}
2\alpha_6 + \mathcal{Y}(\alpha_4 + \alpha_7 + \alpha_8) = 0.
\end{equation}

\begin{table}[h]
\centering
\begin{tabular}{|c|cccccc|}
\hline
Class \( \setminus \) Case & 1-1(-\( \Lambda \)) & 1-2(-\( \Lambda \)) & 2-1(-\( \Lambda \)) & 2-2(-\( \Lambda \)) \\
\hline A1 & ✓ & ✓ & ✓ & ✓ & ✓ \\
A2 & ✓ & ✓ & ✓ & ✓ & ✓ \\
A3 & ✓\textsuperscript{a} & ✓\textsuperscript{a} & ✓\textsuperscript{a} & ✓\textsuperscript{a} & ✓\textsuperscript{a} \\
A4 & ✓\textsuperscript{b} & ✓\textsuperscript{b} & ✓\textsuperscript{b} & ✓\textsuperscript{b} & ✓\textsuperscript{b} \\
B & ✓ & ✓ & ✓ & ✓ & ✓ \\
\hline
\end{tabular}
\caption{The compatibility with the degeneracy conditions. “Class” represents the classes of the quadratic-order extended vector-tensor theories (see Secs. I B and IV A), which are the solutions of the degeneracy conditions, while “Case” represents the conditions for obtaining the exact black hole solutions (see Secs. II and IIII). Footnotes “a” and “b” represent the conditions under which the compatibility of the conditions for the existence of the exact black hole solutions with the degeneracy conditions hold.
\textsuperscript{a} \( f_2 \neq 0 \) evaluated at \( \mathcal{Y} = \mathcal{Y}_0 \).
\textsuperscript{b} \( \mathcal{Y} \neq 0 \) and \( f_2 + \alpha_2 \mathcal{Y} \neq 0 \) evaluated at \( \mathcal{Y} = \mathcal{Y}_0 \).}
\end{table}

B. The compatibility with the degeneracy conditions

The compatibility of the conditions for the existence of the solutions with the degeneracy conditions are summarized in Table I. The columns with ✓ correspond to the cases which are consistent with the degeneracy conditions under the assumptions shown in the footnotes, while those with × correspond to the cases which cannot be consistent.

All the solutions of Case 1-1, Case 1-2, Case 2-1, Case 2-2, Case 1-1-\( \Lambda \), Case 1-2-\( \Lambda \), Case 2-1-\( \Lambda \), and Case 2-2-\( \Lambda \) (Eqs. (21), (25), (22), (27), (31), (33), (32), and (34), respectively) are not compatible with Class B, since \( \alpha_1 + \alpha_2 \neq 0 \) at \( \mathcal{Y} = \mathcal{Y}_0 \). On the other hand, all the solutions are compatible with Class A.

C. The correspondence to the scalar-tensor theories

The vector-tensor theories (1) with Eq. (3) reduce to the shift-symmetric scalar-tensor theories (B1) with Eq. (B2), in the limit of \( A_\mu \rightarrow \partial_\mu \phi \), \( \mathcal{Y} \rightarrow \mathcal{X} \) (see Eq. (B3) for the definition of the canonical kinetic term of the scalar field), and \( \alpha_4 = 0 \ (i = 6, 7, 8) \). The vector field with the constant \( A_\mu \), Eq. (8b) with Eq. (19), then reduces to

\begin{equation}
\phi = \xi(t) + \psi(r), \quad \psi(r) := \int dr A_r(r).
\end{equation}

We confirm that the conditions for the existence of the black hole solutions with the vanishing electric field, Case 1-1, Case 2-1, Case 1-1-\( \Lambda \), and Case 2-1-\( \Lambda \) (Eqs. (21), (22), (31), and (32)), reduce to those of Case 1, Case 2, Case 1-\( \Lambda \), and Case 2-\( \Lambda \) obtained in Refs. [32] (see also Refs. [31, 68]), which will be reviewed in Appendix A.
D. On the black hole perturbations and the strong coupling problem

As shown in the previous subsections, the stealth Schwarzschild, the charged stealth Schwarzschild, the Schwarzschild de Sitter/anti-de Sitter, and the charged Schwarzschild de Sitter/anti-de Sitter solutions exist in the Class A theories. The limit of the other theories the generalized Proca theories will be discussed in Sec. V. Although at the background level the spacetime geometry remains that of the Schwarzschild or Schwarzschild de Sitter/anti-de Sitter solution, at the level of the linear perturbations the vector field degrees of freedom would propagate, which results in the existence of the extra polarization modes of gravitational waves. The existence of the extra polarization modes would be tested by the current and future observations of gravitational waves [70–72].

Regarding the perturbation analysis and the stability, the possible issues are raised by referring to the recent studies on the stability of black hole solutions in the quadratic-order DHOST theories [73–75]. In the case of the quadratic-order DHOST theories, it has been shown that the even-parity perturbations on the stealth Schwarzschild and Schwarzschild de Sitter solutions suffer from the strong coupling problem [74], which spoils the predictability of the perturbation theory. The similar strong coupling problem may exist for the black hole solutions in the quadratic-order extended vector-tensor theories reported in this paper.

In the case of the quadratic-order DHOST theories, a resolution to this problem by the controllable violation of the degeneracy conditions was argued in Ref. [76]. It has been shown that the effective field theories about the stealth Minkowski and de Sitter solutions predict the universal dispersion relation $\omega/M = c(k/M)^2$, where $c$ is the dimensionless constant and $M$ represents the scale characterizing the background scalar field, and are weakly coupled at the energy scales up to $M$ for $c = O(1)$. On the other hand, in the degenerate theories where the perturbations are forced to obey the second-order equations of motion, $c = 0$, indicating the appearance of the strong coupling problem [76]. Thus, the controllable violation of the degeneracy conditions makes the perturbations weakly coupled.

The similar resolution to the strong coupling problem may also be able to be applied to the stealth solutions in the quadratic-order extended vector-tensor theories. The conditions for the existence of the (charged) stealth Schwarzschild solutions and the (charged) Schwarzschild-de Sitter/anti-de Sitter solutions obtained in this paper should include the case where the degeneracy conditions are controllably violated within the framework of the general vector-tensor theories (1) with Eq. (3).

V. THE SPECIAL CASES IN THE GENERALIZED PROCA THEORIES

A. The generalized Proca theories

In this section, we apply the results in the previous sections to the quadratic- and quartic-order generalized Proca theories

$$S = \int d^4x \sqrt{-g} \left[ G_2(V) + G_4(V) R - \frac{1}{4} F_{\mu \nu} F_{\mu \nu} ight. \left. - 2G_4(V) \left( \nabla^\mu A_\mu \right)^2 - \nabla^\mu A^\nu \nabla_\nu A_\mu \right],$$

where $G_2$ and $G_4$ are the functions of $V$, which corresponds to the following choice of the quadratic-order extended vector-tensor theories (1) with Eq. (3)

$$f_0 = G_2, \quad f_2 = G_4,$$
$$\alpha_1 = -\alpha_2 = 2G_4V, \quad \alpha_6 = -2G_4V - 1, \quad \alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = 0.$$

Since $W_2 = W_3 = 0$ in Eq. (39), the generalized Proca theories (44) belong to the Class A4 theories.

The conditions for the stealth Schwarzschild and Schwarzschild-de Sitter/anti-de Sitter solutions obtained in the previous sections reduce as follows:

- Case 1-1
  The conditions (21) reduce to
  $$G_2 = G_{2,V} = 0.$$  \hspace{1cm} (46)

- Case 2-1
  The conditions (22) reduce to
  $$G_2 = G_{2,V} = G_{4,V} = G_{4,VV} = 0.$$  \hspace{1cm} (47)

- Case 1-2
  The conditions (25) reduce to
  $$1 + 8G_{4,V} = 0, \quad G_2 = G_{2,V} = G_{4,VV} = 0.$$  \hspace{1cm} (48a, 48b)

- Case 1-1-Λ
  The conditions (31) reduce to
  $$G_2 + 2\Lambda \left( G_4 + 2q^2 G_{4,V} \right) = 0, \quad G_{2,V} + 2\Lambda \left( G_{4,V} - 2q^2 G_{4,VV} \right) = 0.$$  \hspace{1cm} (49a, 49b)

- Case 2-1-Λ
  The conditions (32) reduce to
  $$G_2 + 2\Delta G_4 = 0, \quad G_{2,V} = G_{4,V} = G_{4,VV} = 0.$$  \hspace{1cm} (50a, 50b)
• Case 1-2-Λ
The conditions (33) reduce to
\[ G_2 - 2\Lambda (G_{4,y} + 2qG_{4,y}) = 0, \]
\[ G_{2,y} + 2\Lambda G_{4,y} = 0, \]
\[ 1 + 8G_{4,y} = 0, \]
\[ G_{4,yy} = 0. \]

• Case 2-2 and Case 2-2-Λ
The conditions (27) and (34) are not consistent in the generalized Proca theories (44).

B. The nonminimal coupling to the Einstein tensor

Furthermore, we consider the special case
\[ G_2 = -m^2\mathcal{Y} - M_p^2V_0, \]
\[ G_4 = \frac{M_p^2}{2} - \frac{\beta}{2}\mathcal{Y}, \]
where \( M_p \) and \( V_0 \) represent the reduced Planck mass and the cosmological constant, and \( m \) and \( \beta \) are the mass and the coupling constant of the vector field \( A_\mu \), respectively, which, up to the total derivative terms, gives rise to the generalized Proca theory with the nonminimal coupling to the Einstein tensor [57–59]
\[ S = \int d^4x\sqrt{-g} \left[ \frac{M_p^2}{2} (R - 2V_0) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - (m^2g^{\mu\nu} - \beta G^{\mu\nu})A_\mu A_\nu \right]. \]

Hence, \( \beta \) corresponds to the nonminimal coupling constant to the Einstein tensor.

• Case 1-1
The conditions (46) reduce to
\[ m = V_0 = 0, \]
which reproduced the stealth Schwarzschild solution with the vanishing electric field in the generalized Proca theory (53) discussed in Refs. [40, 41].

• Case 2-1
The conditions (47) reduce to
\[ m = \beta = V_0 = 0, \]
which means the Schwarzschild solution in general relativity without the cosmological constant and reproduces the stealth Schwarzschild solution discussed in Refs. [40, 41]. Since the \( U(1) \) gauge symmetry is restored, the solution describes the Schwarzschild solution with the vanishing electric field in the Einstein-Mawell theory.

• Case 1-2
The conditions (48) reduce to
\[ m = V_0 = 0, \]
\[ \beta = \frac{1}{4}, \]
which reproduces the charged stealth Schwarzschild solution discussed in Refs. [40, 41].

• Case 1-1-Λ
The conditions (49) reduce to
\[ \Lambda = -\frac{m^2}{\beta}, \]
\[ q = \pm \frac{M_p}{\sqrt{2}} \sqrt{1 + \frac{V_0}{m^2}}, \]
where we have to impose \( 1/\beta + V_0/m^2 \geq 0 \) which reproduces the Schwarzschild-(anti-) de Sitter solution discussed in Refs. [41].

• Case 2-1-Λ
The conditions (50) reduce to
\[ m = \beta = 0, \quad \Lambda = V_0. \]

Since the \( U(1) \) gauge symmetry is restored, the solution describes the Schwarzschild-de Sitter/ anti-de Sitter solution with the vanishing electric field in the Einstein-Mawell theory with the cosmological constant \( V_0\).

• Case 1-2-Λ
The conditions (51) reduce to
\[ \beta = \frac{1}{4}, \]
\[ \Lambda = -4m^2, \]
\[ q = \pm \frac{M_p}{\sqrt{2}} \sqrt{4 + \frac{V_0}{m^2}}, \]
where we have to impose \( 4m^2 + V_0 \geq 0 \) and \( m \neq 0 \), which reproduces the charged Schwarzschild-anti-de Sitter solutions discussed in Ref. [41].

VI. CONCLUSIONS

We have investigated the static and spherically symmetric black hole solutions in the quadratic-order extended vector-tensor theories without the Ostrogradsky instabilities, which include the generalized Proca theories as the particular subclass. The theories are given by Eq. (1) with Eq. (3), where the free functions of the spacetime norm of the vector field, \( \mathcal{Y} \) defined in Eq. (2),
satisfy the degeneracy conditions summarized in Sec IV.
The most general vector-tensor theories (I) with Eq. (3) constructed with up to the quadratic-order terms of the first-order covariant derivatives of the vector field are not free from the Ostrogradsky instabilities, unless the certain degeneracy conditions are imposed [48].

We have started from the most general action of the vector-tensor theories (I) with Eq. (3), and derived the Euler-Lagrange equations for the metric and vector field variables in the static and spherically symmetric backgrounds. We then substituted the metric functions for the Schwarzschild and Schwarzschild-de Sitter/ anti-de Sitter spacetimes and the vector field with the constant spacetime norm \( \mathcal{Y} = \mathcal{Y}_0 \) into the Euler-Lagrange equations. Under our ansatz (8) and assumptions, the vector field which is regular at either the future or past event horizon, depending on the choice of the branch.

The series expansion analysis of the Euler-Lagrange equations after the substitutions of the above ansatz yielded the conditions for the existence of the Schwarzschild and Schwarzschild-de Sitter/ anti-de Sitter solutions on the free functions of the spacetime norm of the vector field \( \mathcal{Y} \), evaluated at the constant value of \( \mathcal{Y} = \mathcal{Y}_0 \). As in the previous cases on the black hole solutions in the generalized Proca theories [40–44], we have derived the conditions for the existence of the stealth Schwarzschild, the charged stealth Schwarzschild, the Schwarzschild-de Sitter/ anti-de Sitter solutions, and the charged Schwarzschild-de Sitter/ anti-de Sitter solutions, where the metric functions do not depend on the electric charge \( Q \). When the tuning relation on the function \( \alpha_0 \) evaluated at \( \mathcal{Y} = \mathcal{Y}_0 \) is broken, then the solution takes the form of the Reissner-Nordström-type solution and Reissner-Nordström-de Sitter/ anti-de Sitter-type solution with the effective charge \( Q \) (see Eqs. (29) and (30)).

Second, we have compared the conditions for the existence of the black hole solutions of the black hole solutions obtained in the present paper with the degeneracy conditions. We have shown that the conditions for the existence the stealth Schwarzschild, the charged stealth Schwarzschild, the Schwarzschild-de Sitter/ anti-de Sitter solutions, and the charged Schwarzschild-de Sitter/ anti-de Sitter solutions are compatible with the degeneracy conditions for the Class A theories, while they are not compatible with the degeneracy conditions for the Class B theories (see Table 1). We have also recovered the stealth Schwarzschild, charged stealth Schwarzschild, and Schwarzschild-de Sitter solution in the limit to the generalized Proca theories.

Although at the background level the spacetime geometry remains that of the Schwarzschild and Schwarzschild de Sitter/ anti-de Sitter solutions, at the level of the linear perturbations the extra polarization modes of gravitational waves would also propagate, which would be tested by the current and future observations of gravitational waves [70–72]. As in the case of the static and spherically symmetric black hole solutions in the quadratic-order DHOST theories [74], for the solutions satisfying the degeneracy conditions the strong coupling problem may exist in the even parity sector of the black hole perturbations, which spoils the predictability of the linear perturbation theory. The strong coupling problem may be able to be resolved by the controllable violation of the degeneracy conditions as suggested in Ref. [76]. The conditions for the existence of the (charged) stealth Schwarzschild solutions and the (charged) Schwarzschild-de Sitter/ anti-de Sitter solutions obtained in this paper should admit the solutions which would not suffer from the strong coupling problem within the framework of the theory (I) with Eq. (3).

Finally, it will be very interesting to construct the black hole solutions with the nonconstant spacetime norm \( \mathcal{Y} = \mathcal{Y}(r) \) and the solutions of relativistic stars in the case of the static and spherically symmetric spacetime. As the next step, it will also be important to explore the black hole solutions beyond the spherical symmetry, e.g., the stationary and axisymmetric black hole solutions. We hope to come back to these issues in our future publication.

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Appendix A: The Ostrogradsky instabilities and degenerate theories in analytical mechanics

We analyze the analytical toy model (6) in the Hamiltonian analysis. More specifically, we consider the potential composed of the quadratic-order terms

\[
V(x, y) = \frac{g_1}{2} x^2 + g_2 xy + \frac{g_3}{2} y^2,
\]

(A1)

with \( g_1 \), \( g_2 \), and \( g_3 \) being constants. The theory equivalent to Eq. (6) can be obtained by introducing the auxiliary variable \( Q \)

\[
L_{p,2} = \frac{a_1}{2} \dot{Q}^2 + a_2 \dot{Q} \dot{y} + \frac{a_3}{2} y^2 + \frac{1}{2} x^2 - V(x, y) + \lambda (\dot{x} - Q).
\]

(A2)
We define their conjugate momenta to $x$, $Q$, and $y$, respectively, by
\[
P_Q := \frac{\partial L_{p,2}}{\partial \dot{Q}} = a_1 \dot{Q} + a_2 \dot{y}, \tag{A3}
\]
\[
P_y := \frac{\partial L_{p,2}}{\partial \dot{y}} = a_3 \dot{y} + a_2 \dot{Q}, \tag{A4}
\]
\[
P_x := \frac{\partial L_{p,2}}{\partial \dot{x}} = \lambda. \tag{A5}
\]

1. The nondegenerate case

First, we consider the nondegenerate case;
\[
\frac{\partial^2 L_{p,2}}{\partial Q^2} \frac{\partial^2 L_{p,2}}{\partial y^2} - \left( \frac{\partial^2 L_{p,2}}{\partial Q \partial y} \right)^2 = a_1 a_3 - a_2^2 \neq 0. \tag{A6}
\]
By rewriting $\dot{Q}$ and $\dot{y}$ in terms of $P_Q$ and $P_y$, we obtain the Hamiltonian
\[
H_p := P_Q \dot{Q} + P_x \dot{x} + P_y \dot{y} - L_{p,2} \\
= \frac{1}{2(a_1 a_3 - a_2^2)} (a_3 P_y^2 + 2 a_2 P_Q P_y + a_1 P_y^2) \\
+ P_x Q - \frac{1}{2} Q^2 + V(x, y), \tag{A7}
\]
where the dependence on $P_x$ appears only in the linear term $P_x Q$. Since there is no constraint which relates $P_x$ with the other canonical variables, the Hamiltonian Eq. (A7) is not bounded from below, which indicates the appearance of the Ostrogradsky instabilities. We note that the above discussion cannot be applied to the case of $a_1 a_3 - a_2^2 = 0$, which needs to be considered separately.

2. The degenerate case

Second, we consider the degenerate case
\[
\frac{\partial^2 L_{p,2}}{\partial Q^2} \frac{\partial^2 L_{p,2}}{\partial y^2} - \left( \frac{\partial^2 L_{p,2}}{\partial Q \partial y} \right)^2 = a_1 a_3 - a_2^2 = 0. \tag{A8}
\]
under which $P_Q$ and $P_y$ satisfy
\[
P_y - \frac{a_2}{a_1} P_Q = 0. \tag{A9}
\]
Regarding
\[
X_1 := P_y - \frac{a_2}{a_1} P_Q \approx 0, \tag{A10}
\]
as the primary constraint, the total Hamiltonian can be defined as
\[
\tilde{H}_p := H_p + \mu X_1 \\
= \frac{a_1 P_y^2}{2 a_2^2} + P_x Q - \frac{1}{2} Q^2 + V(x, y) + \mu X_1. \tag{A11}
\]
The time evolution of the primary constraint $X_1$ then generates the secondary constraint
\[
X_2 := \dot{X}_1 = \{X_1, \tilde{H}_p\} \\
= - g_2 x - g_3 y + \frac{a_2}{a_1} (P_x - Q) \approx 0, \tag{A12}
\]
where we define the Poisson bracket
\[
\{U_1, U_2\} := \left( \frac{\partial U_1}{\partial x} \frac{\partial U_2}{\partial P_x} - \frac{\partial U_1}{\partial P_x} \frac{\partial U_2}{\partial x} \right) \\
+ \left( \frac{\partial U_1}{\partial y} \frac{\partial U_2}{\partial P_y} - \frac{\partial U_1}{\partial P_y} \frac{\partial U_2}{\partial y} \right) \\
+ \left( \frac{\partial U_1}{\partial Q} \frac{\partial U_2}{\partial P_Q} - \frac{\partial U_1}{\partial P_Q} \frac{\partial U_2}{\partial Q} \right). \tag{A13}
\]
The secondary constraint (A12) relates $P_x$ to the other phase space variables and all the terms linear in the momentum are eliminated from the total Hamiltonian (A11). In other words, the Hamiltonian can be bounded from below. Since
\[
\{X_1, X_2\} = g_3 - \frac{a_2^2}{a_1^2}, \tag{A14}
\]
the time evolution of the secondary constraint $X_2$, $\dot{X}_2 := \{X_2, \tilde{H}_p\} \approx 0$, fixes $\mu$ and generates no more constraint for $g_3 \neq \frac{a_2^2}{a_1^2}$. From Eq. (A14), we note that the constraints $X_1 \approx 0$ and $X_2 \approx 0$, Eqs. (A10) and (A12), are of the second-class. Thus, starting from the 6-dimensional phase space $(x, y, Q, P_x, P_y, P_Q)$, the 2 second-class constraints reduce the dimensionality of the phase space to be $4(= 2 \times 2)$, namely, and hence the correct 2 degrees of freedom are recovered and the Ostrogradsky ghosts are eliminated.

Appendix B: The limit to the quadratic-order DHOST theories

In the limit of the scalar-tensor theories, $A_{\mu} \rightarrow \nabla_{\nu} \phi$, the quadratic-order extended vector-tensor theories (44) with Eq. (3) reduce to the quadratic-order DHOST theories [50, 51],
\[
S = \int d^4 x \sqrt{-\theta} \left[ f_0 (\mathcal{X}) + f_2 (\mathcal{X}) R \right. \\
+ C^{\mu\nu\rho\sigma} (\nabla_{\mu} \nabla_{\nu} \phi) (\nabla_{\rho} \nabla_{\sigma} \phi)], \tag{B1}
\]
with
\[
C^{\mu\nu\rho\sigma} (\nabla_{\mu} \nabla_{\nu} \phi) (\nabla_{\rho} \nabla_{\sigma} \phi) \\
= a_1 (\mathcal{X}) (\nabla_{\mu} \nabla_{\nu} \phi) (\nabla_{\rho} \nabla_{\sigma} \phi) + a_2 (\mathcal{X}) (\Box \phi)^2 \\
+ a_3 (\mathcal{X}) (\nabla_{\mu} \phi) (\nabla_{\nu} \phi) (\nabla_{\rho} \nabla_{\sigma} \phi) \\
+ a_4 (\mathcal{X}) (\nabla_{\mu} \phi) (\nabla_{\nu} \phi) (\nabla_{\rho} \nabla_{\sigma} \phi) \\
+ a_5 (\mathcal{X}) [(\nabla_{\mu} \phi) (\nabla_{\nu} \phi) (\nabla_{\rho} \nabla_{\sigma} \phi)^2], \tag{B2}
\]
where we have defined
\[
\mathcal{X} := g^{\mu\nu} \nabla_{\mu} \phi \nabla_{\nu} \phi, \quad \Box \phi := g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \phi. \tag{B3}
\]
and $\nabla_\mu$ denotes the covariant derivative associated with the metric $g_{\mu\nu}$. The relevant degeneracy conditions were obtained in Refs. [50-53].

We classify our exact black hole solutions into the two cases in terms of the constant value of $\lambda$;

- **Case 1**: $\lambda_0 = -q^2$.
  “$\Lambda$” is also attached in the case that the solution is asymptotically de Sitter or anti-de Sitter.

- **Case 2**: $\lambda_0 \neq -q^2$.
  “$\Lambda$” is also attached in the case that the solution is asymptotically de Sitter or anti-de Sitter.

Accordingly, in the limit of the scalar-tensor theories, the conditions Case 1-1, Case 2-1, Case 1-Λ, Case 2-1-Λ reduce to

- **Case 1**
  \[
  f_0 = f_{0,\lambda} = 0, \quad \alpha_2 = -\alpha_1, \quad \alpha_{2,\lambda} = -\alpha_{1,\lambda}.
  \]  
  \[
  f_0 = f_{0,\lambda} = 0, \quad \alpha_2 = -\alpha_1, \quad \alpha_{2,\lambda} = -\alpha_{1,\lambda}.
  \]

- **Case 2**
  \[
  f_0 = f_{0,\lambda} = 0, \quad \alpha_2 = 0, \quad \alpha_{2,\lambda} = -\alpha_{1,\lambda}, \quad \alpha_3 = -2\alpha_{1,\lambda}.
  \]

- **Case 1-Λ**
  \[
  f_0 = -2\Lambda f_2, \quad f_{0,\lambda} = \Lambda (\alpha_2, \alpha_{1,\lambda} + \alpha_{1,\lambda}), \quad \alpha_1 = \alpha_2 = 0, \quad \alpha_{2,\lambda} = -\alpha_1, \quad \alpha_3 = -2\alpha_{1,\lambda}.
  \]

These conditions exactly agree with those in “Case 1”, “Case 2”, “Case 1-Λ”, and “Case 2-Λ” obtained in Ref. [32] (see also Refs. [31, 68]), respectively.

**Appendix C: The Reissner-Nordström solution in the Einstein-Maxwell theory**

In this Appendix, we illustrate the derivation of the black hole solutions in the Einstein-Maxwell theory

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M^2}{3} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right].
\]

We assume the ansatz for the metric and vector field (8) where without loss of generality we may set $A_r = 0$. Substituting Eq. (8) into the Einstein-Maxwell action (C1),

\[
S = \int dt dr \frac{\pi}{r^2} \sqrt{f} \left\{ M_p^2 r^2 h f^2 - 4 M_p^2 h^2 (-1 + h + h') + r f \left[ -M_p^2 r f h' + 2 h (r A_r^2 - M_p^2 (2 f' + r f'')) \right] \right\}.
\]

where $Q$ is an integration constant. Substituting it into Eq. (C3a), we obtain

\[
Q^2 + 2 M_p^2 (r h' + h - 1) = 0,
\]

which can be integrated as

\[
h(r) = 1 - \frac{2 M}{r} + \frac{Q^2}{2 M_p^2 r^2},
\]

Varying the action with respect to $f$, $h$, and $A_t$, we obtain

\[
-2 h A_t^2 + 2 M_p^2 f (-1 + h + h') = 0,
\]

\[
2 M_p^2 f (-1 + h) + r h (r A_t^2 + 2 M_p^2 f') = 0,
\]

\[
r h A_t f' - f [r A_t^2 h' + 2 h (r A_t^2 + 2 A_t)] = 0.
\]

By integrating Eq. (C3a),

\[
A_t(r) = -\frac{Q}{r^2} \sqrt{\frac{f(r)}{h(r)}},
\]

\[
f_0 = -2\Lambda f_2, \quad f_{0,\lambda} = \Lambda (\alpha_2, \alpha_{1,\lambda} + \alpha_{1,\lambda}), \quad \alpha_1 = \alpha_2 = 0, \quad \alpha_{2,\lambda} = -\alpha_1, \quad \alpha_3 = -2\alpha_{1,\lambda}.
\]

(B7a)
where $M$ is an integration constant. Finally, substituting Eqs. (C4) and (C6) into Eq. (C3b),

\[
2 \left( Q^2 - 2M M_p^2 r \right) f(r) + r \left( Q^2 + 2M_p^2 (r - 2M) \right) f'(r) = 0,
\]

(C7)

which can be integrated as

\[
f(r) = c_1 h(r),
\]

(C8)

where $c$ denotes an integrating constant corresponding to the rescaling of the time coordinate. Without loss of generality, we can set $c_1 = 1$ and then reproduce the Reissner-Nordström solution

\[
f(r) = h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{2M_p^2 r^2},
\]

(C9a)

\[
A_i(r) = -\frac{Q}{r^2},
\]

(C9b)

where the integration constants $M$ and $Q$ physically represent the mass and electric charge of the black hole.

[1] D. Psaltis, D. Perrotin, K. R. Dienes, and I. Mocioiu, Phys.Rev.Lett. 100, 091101 (2008), arXiv:0710.4564 [astro-ph].
[2] H. Motohashi and M. Minamitsuji, Phys. Lett. B781, 728 (2018), arXiv:1804.01731 [gr-qc].
[3] W. Israel, Phys. Rev. 164, 1767 (1967).
[4] R. Ruffini and J. A. Wheeler, Phys. Today 24, 30 (1971).
[5] J. Bekenstein, Phys. Rev. D 5, 2403 (1972).
[6] S. Hawking, Commun. Math. Phys. 25, 167 (1972).
[7] J. Chase, Commun.Math.Phys. 19, 276 (1970).
[8] J. Bekenstein, Phys. Rev. Lett. 28, 452 (1972).
[9] S. Hawking, Commun. Math. Phys. 25, 152 (1972).
[10] J. Bekenstein, Phys. Rev. D 51, 6608 (1995).
[11] T. P. Sotiriou and V. Faraoni, Phys. Rev. Lett. 108, 081103 (2012), arXiv:1109.6524 [gr-qc].
[12] A. A. H. Graham and R. Jha, Phys. Rev. D 89, 064040 (2014), [Erratum: Phys.Rev.D 92, 069901 (2015)], arXiv:1401.8203 [gr-qc].
[13] A. A. H. Graham and R. Jha, Phys. Rev. D 90, 041501 (2014), arXiv:1407.6573 [gr-qc].
[14] L. Hui and A. Nicolis, Phys. Rev. Lett. 110, 241104 (2013), arXiv:1202.1296 [hep-th].
[15] E. Babichev, C. Charmousis, and A. Lehebel, JCAP 04, 027 (2017), arXiv:1702.01938 [gr-qc].
[16] N. Bocharova, K. Bronnikov, and V. Melnikov, Vestn. Mosk. Univ. Ser. III Fiz. Astronom., 706 (1970).
[17] J. Bekenstein, Annals Phys. 91, 75 (1975).
[18] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, Phys. Rev. D54, 5049 (1996), arXiv:hep-th/9511071 [hep-th].
[19] D. Ayzenberg and N. Yunes, Phys. Rev. D90, 044066 (2014), [Erratum: Phys.Rev.D91,no.6,069905(2015)], arXiv:1405.2133 [gr-qc].
[20] S. O. Alexeev and M. V. Ponomanov, Phys. Rev. D55, 2110 (1997), arXiv:hep-th/9605106 [hep-th].
[21] T. Torii, H. Yajima, and K.-i. Maeda, Phys. Rev. D55, 739 (1997), arXiv:gr-qc/9606034 [gr-qc].
[22] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. Lett. 112, 251102 (2014), arXiv:1312.3622 [gr-qc].
[23] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. D90, 124063 (2014), arXiv:1408.1698 [gr-qc].
[24] S. Mukohyama, Phys. Rev. D 71, 104019 (2005), arXiv:hep-th/0502189.
[25] E. Babichev and C. Charmousis, JHEP 1408, 106 (2014), arXiv:1312.3204 [gr-qc].
[26] S. Appleby, JCAP 05, 009 (2015), arXiv:1503.06768 [gr-qc].
[27] E. Babichev, C. Charmousis, and A. Lehebel, Class. Quant. Grav. 33, 154002 (2016), arXiv:1604.06402 [gr-qc].
[28] E. Babichev, C. Charmousis, and A. Lehebel, JCAP 04, 027 (2017), arXiv:1702.01938 [gr-qc].
[29] E. Babichev, C. Charmousis, G. Esposito-Farese, and A. Lehebel, Phys. Rev. Lett. 120, 241101 (2018), arXiv:1712.04398 [gr-qc].
[30] E. Babichev and G. Esposito-Farese, Phys. Rev. D 95, 024020 (2017), arXiv:1609.09798 [gr-qc].
[31] J. Ben Achour and H. Liu, Phys. Rev. D 99, 064042 (2019), arXiv:1811.05369 [gr-qc].
[32] H. Motohashi and M. Minamitsuji, Phys. Rev. D 99, 064040 (2019), arXiv:1901.04658 [gr-qc].
[33] M. Minamitsuji and J. Edholm, Phys. Rev. D 100, 044053 (2019), arXiv:1907.02072 [gr-qc].
[34] J. Ben Achour, H. Liu, and S. Mukohyama, JCAP 02, 023 (2020), arXiv:1910.11017 [gr-qc].
[35] M. Minamitsuji and J. Edholm, Phys. Rev. D 101, 044034 (2020), arXiv:1912.01744 [gr-qc].
[36] A. E. Ayon-Beato, C. Martinez, and J. Zanelli, Gen. Rel. Grav. 38, 145 (2006), arXiv:hep-th/0403228.
[37] M. Rinaldi, Phys. Rev. D86, 084048 (2012), arXiv:1208.0103 [gr-qc].
[38] A. Anabalon, A. Cisterna, and J. Oliva, Phys. Rev. D89, 084050 (2014), arXiv:1312.3597 [gr-qc].
[39] M. Minamitsuji, Phys. Rev. D89, 064017 (2014), arXiv:1312.3759 [gr-qc].
[40] J. Chagoya, G. Niz, and G. Tasinato, Class. Quant. Grav. 33, 175007 (2016), arXiv:1602.08697 [hep-th].
[41] M. Minamitsuji, Phys. Rev. D 94, 084039 (2016), arXiv:1607.06278 [gr-qc].
[42] E. Babichev, C. Charmousis, and M. Hassaine,
15

JHEP 05, 114 (2017), arXiv:1703.07676 [gr-qc].

[43] L. Heisenberg, R. Kase, M. Minamitsuji, and S. Tsujikawa, Phys. Rev. D 96, 084049 (2017), arXiv:1705.09662 [gr-qc].

[44] L. Heisenberg, R. Kase, M. Minamitsuji, and S. Tsujikawa, JCAP 08, 024 (2017), arXiv:1706.05115 [gr-qc].

[45] M. Minamitsuji, Gen. Rel. Grav. 49, 86 (2017).

[46] R. Kase, M. Minamitsuji, and S. Tsujikawa, Phys. Lett. B 782, 541 (2018), arXiv:1803.06335 [gr-qc].

[47] C. A. R. Herdeiro and E. Radu, Proceedings, 7th Black Holes Workshop 2014, Int. J. Mod. Phys. D24, 1542014 (2015), arXiv:1504.08209 [gr-qc].

[48] R. Kimura, A. Naruko, and D. Yoshida, JCAP 1701, 002 (2017), arXiv:1608.07066 [gr-qc].

[49] R. P. Woodard, Scholarpedia 10, 32243 (2015), arXiv:1506.02210 [hep-th].

[50] D. Langlois and K. Noui, JCAP 1602, 034 (2016), arXiv:1510.06930 [gr-qc].

[51] J. Ben Achour, D. Langlois, and K. Noui, Phys. Rev. D93, 124005 (2016), arXiv:1602.08398 [gr-qc].

[52] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, JHEP 12, 100 (2016), arXiv:1608.08135 [hep-th].

[53] D. Langlois, Int. J. Mod. Phys. D 28, 1942006 (2019), arXiv:1811.06271 [gr-qc].

[54] G. W. Horndeski, Int. J. Theor. Phys. 10, 363 (1974).

[55] C. Deffayet, S. Deser, and G. Esposito-Farèse, Phys. Rev. D80, 064015 (2009), arXiv:0906.1967 [gr-qc].

[56] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Prog. Theor. Phys. 126, 511 (2011), arXiv:1105.5723 [hep-th].

[57] G. Tasinato, JHEP 04, 067 (2014), arXiv:1402.6450 [hep-th].

[58] L. Heisenberg, JCAP 05, 015 (2014), arXiv:1402.7026 [hep-th].

[59] A. De Felice, L. Heisenberg, R. Kase, S. Tsujikawa, Y.-J. Zhang, and G.-B. Zhao, Phys.Rev.D 93, 104016 (2016), arXiv:1602.00371 [gr-qc].

[60] L. Heisenberg, Phys. Rept. 796, 1 (2019), arXiv:1807.01725 [gr-qc].

[61] T. Kobayashi, Rept. Prog. Phys. 82, 086901 (2019), arXiv:1901.07183 [gr-qc].

[62] H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi, and D. Langlois, JCAP 07, 033 (2016), arXiv:1603.09355 [hep-th].

[63] H. Motohashi and T. Suyama, Phys. Rev. D91, 085009 (2015), arXiv:1411.3721 [physics.class-ph].

[64] R. Klein and D. Roest, JHEP 07, 130 (2016), arXiv:1604.01719 [hep-th].

[65] H. Motohashi, T. Suyama, and M. Yamaguchi, J. Phys. Soc. Jap. 87, 063401 (2018), arXiv:1711.08125 [hep-th].

[66] H. Motohashi, T. Suyama, and M. Yamaguchi, JHEP 06, 133 (2018), arXiv:1804.07990 [hep-th].

[67] R. L. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 116, 1322 (1959).

[68] K. Takahashi and H. Motohashi, JCAP 06, 034 (2020), arXiv:2004.03883 [gr-qc].

[69] M. Minamitsuji, Phys. Rev. D 102, 124017 (2020), arXiv:2012.13526 [gr-qc].

[70] S. J. Chamberlin and X. Siemens, Phys. Rev. D 85, 082001 (2012), arXiv:1111.5661 [astro-ph.HE].

[71] Y. Hagiwara, N. Era, D. Iikawa, A. Nishizawa, and H. Asada, Phys. Rev. D 100, 064010 (2019), arXiv:1904.08198 [gr-qc].

[72] L. Barack et al., Class. Quant. Grav. 36, 143001 (2019), arXiv:1806.05195 [gr-qc].

[73] K. Takahashi, H. Motohashi, and M. Minamitsuji, Phys. Rev. D 100, 024041 (2019), arXiv:1904.03554 [gr-qc].

[74] C. de Rham and J. Zhang, Phys. Rev. D 88, 124023 (2019), arXiv:1907.00699 [hep-th].

[75] C. Charmousis, M. Crisostomi, D. Langlois, and K. Noui, Class. Quant. Grav. 36, 235008 (2019), arXiv:1907.02924 [gr-qc].

[76] H. Motohashi and S. Mukohyama, JCAP 01, 030 (2020), arXiv:1912.00378 [gr-qc].