EFFICIENT COMPUTATION OF SOME SPECIAL FUNCTIONS

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ABSTRACT. We introduce a new algorithm to efficiently compute the functions belonging to a suitable set \( \mathcal{F} \) defined as follows: \( f \in \mathcal{F} \) means that \( f(s,x), s \in A \subset \mathbb{R} \) being fixed and \( x > 0 \), has a power series expansion centred at \( x_0 = 1 \) with convergence radius greater or equal than 1; moreover, it satisfies a difference equation of step 1 and the Euler-Maclaurin summation formula can be applied to \( f \). Denoting Euler’s function as \( \Gamma \), we will show, for \( x > 0 \), that \( \log \Gamma(x) \), the digamma function \( \psi(x) \), the polygamma functions \( \psi^{(w)}(x), w \in \mathbb{N}, w \geq 1 \), and, for \( s > 1 \) being fixed, the Hurwitz \( \zeta(s,x) \)-function and its first partial derivative \( \frac{\partial}{\partial x} \zeta(s,x) \) are in \( \mathcal{F} \). In all these cases the coefficients of the involved power series will depend on the values of \( \zeta(u), u > 1 \), where \( \zeta \) is Riemann’s function. As a by-product, we will also show how to compute efficiently the Dirichlet \( L \)-functions \( L(s,\chi) \) and \( L'(s,\chi) \), \( s > 1 \), \( \chi \) being a primitive Dirichlet character, by inserting the reflection formulae of \( \zeta(s,x) \) and \( \frac{\partial}{\partial x} \zeta(s,x) \) into the first step of the Fast Fourier Transform algorithm. Moreover, we will obtain some new formulae and algorithms for the Dirichlet \( \beta \)-function and for the Catalan constant \( G \). Finally, we will study the case of the Bateman \( G \)-function. In the last section we will also describe some tests that show a performance gain with respect to a standard multiprecision implementation of \( \zeta(s,x) \) and \( \frac{\partial}{\partial x} \zeta(s,x), s > 1, x > 0 \).

1. Introduction

The goal of this paper is to show that for a suitable set of functions \( \mathcal{F} \) there exists an unified and efficient computational strategy, at least when the main variable is a positive real number. The set of functions we will work on is described in the following

Definition 1 (The set \( \mathcal{F} \)). We will say that a function \( f : A \times (0, +\infty) \to \mathbb{R} \), where \( A \subset \mathbb{R} \), is in the set \( \mathcal{F} \) if and only if it has the following properties:

i) for every fixed \( s \in A \) there exists a sequence of functions \( c_f(s,k) \in \mathbb{R} \) and \( \rho_f(s) \geq 1 \) such that \( \sum_{k=0}^{\infty} c_f(s,k)(1-z)^k \) is absolutely convergent to \( f(s,z) \) for every \( z \in (1-\rho_f(s),1+\rho_f(s)) \);

ii) for every fixed \( s \in A \), there exists \( k_f(s) \in \mathbb{N} \) such that \( |c_f(s,k)| \) is a decreasing sequence for every \( k \geq k_f(s) \);

iii) there exists a function \( g_f : A \times (0, +\infty) \to \mathbb{R} \) such that, for every fixed \( s \in A \), we have

\[
 f(s,z+1) = f(s,z) + g_f(s,z) \quad \text{for every } z \in (0, +\infty);
\]

iv) for every fixed \( s \in A \), the function \( g_f(s,\cdot) \) in (1) can be used in the Euler-Maclaurin summation formula; i.e., for every fixed \( s \in A \), \( g_f(s,\cdot) \) verifies the hypotheses of Lemma 2 below.

As we will see later, log \( \Gamma(x) \in \mathcal{F} \) for \( x > 0 \), (\( \Gamma \) is Euler’s function) together with the digamma function \( \psi(x) \), and the treatment here presented generalizes and improves on the one in [10]. These functions have the coefficients \( c_f(s,k) \) depending on the values of the Riemann \( \zeta \)-function at positive integers greater than 1; thus, to evaluate them for \( x \in (0,1) \), we will need to precompute a sufficiently large number of values of \( \zeta(j), j \in \mathbb{N}, j \geq 2 \). For \( j \) even, the Bernoulli numbers are hence needed. For \( x > 1 \) we will use the Euler-Maclaurin formula, and hence we will need to precompute a sufficiently large set of Bernoulli numbers; to be able to obtain a sufficiently good accuracy in this step, we will also need to exploit an

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2020 Mathematics Subject Classification. Primary 33F05; secondary 33B15, 65D20, 11M35, 11-04.

Key words and phrases. Euler’s Gamma, digamma, polygamma functions; Hurwitz zeta-function and Dirichlet \( L \)-functions. Dirichlet \( \beta \)-function, Catalan constant, Bateman \( G \)-function.

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“horizontal shift” trick, see Section 3.3.1 which will require to compute some values of the function \( g_f(s, x) \) used in point (iii) of Definition 1.

Another special function belonging to \( \mathcal{F} \) is the Hurwitz zeta-function \( \zeta(s, x) \), together with its first partial derivative \( \zeta'(s, x) := \frac{\partial \zeta}{\partial s}(s, x) \), for \( s > 1 \) being fixed and \( x > 0 \). In these cases the coefficients \( c_n(s, k), c_n''(s, k) \) will depend on \( \zeta(w), \zeta'(w), w > 1 \), and on some Euler’s B-function values. All the functions mentioned before will be involved in computing \( L(s, \chi) \) and \( L'(s, \chi), s > 1 \), where \( L \) denotes a Dirichlet \( L \)-function and \( \chi \) is a non-principal Dirichlet character modulo an odd prime number \( q \).

As a by-product, since the polygamma functions \( \psi^{(w)}(x), w \in \mathbb{N}, w \geq 1 \), can be written in terms of \( \zeta(w + 1, x) \), we will also obtain that \( \psi^{(w)}(x) \in \mathcal{F}, x > 0 \). Moreover, we will show some new formulae and algorithms for the Dirichlet beta-function \( \beta(s) \), together with \( \beta'(s) \) and \( \beta'/\beta(s), s > 1 \), too; we will also obtain a new fast convergent series for the Catalan constant \( G \). Finally, we will insert another example regarding the Bateman \( G \)-function to show that, even if \( G(\cdot) \notin \mathcal{F} \) because it is not the solution of a difference equation of the type in point (iii) of Definition 1 our strategy can be adapted to this case too.

1.1. Outline. In Section 2 we will collect some useful lemmas, and in the following one we will write the algorithm for \( f \in \mathcal{F} \); we will also study its computational cost. In Sections 3-7 we will show how to adapt the general algorithm to each one of the special functions mentioned before. In Section 8 we will then discuss how to use the reflection formulae available for these functions in the Fast Fourier Transform algorithm with the goal of computing the Dirichlet \( L \)-functions attached to primitive Dirichlet characters. In the same section we will also show the results on the Dirichlet \( \beta \)-function, the Catalan constant \( G \) and the Bateman \( G \)-function. Moreover, to show the good performances of this algorithm, in Section 9 we will describe some computational results obtained with programs developed using PARI/GP [13]: source codes and examples are available on the following page: http://www.math.unipd.it/~languasc/specialfunctions.html.

2. Lemmas

To estimate the error we have in approximating \( f \) with a finite sum in point (ii) of Definition 1 we will need the following lemma which is an adaptation of the ratio test.

**Lemma 1.** Let \( c_k \in \mathbb{C}, \sum_{k=0}^{+\infty} c_k \) be an absolutely convergent series, \( \mu \in (0, 1) \) and \( K \in \mathbb{N} \) such that \( |c_{k+1}/c_k| \leq \mu \) for every \( k \geq K \). Denoting \( \sum_{k=0}^{+\infty} c_k =: C \in \mathbb{C} \) and \( \sum_{k=0}^{m} c_k =: C_m \in \mathbb{C} \), we have that \( |C - C_m| \leq |c_{m+1}|/(1 - \mu) \) for every \( m \geq K - 1 \).

**Proof.** Let \( m+1 \geq K \). We have \( |c_{m+2}| \leq \mu |c_{m+1}| \) and \( |c_{m+3}| \leq \mu |c_{m+2}| \leq \mu^2 |c_{m+1}| \). Arguing by induction, we prove, for every \( p \in \mathbb{N}, p \geq 1 \), that \( |c_{m+p}| \leq \mu^p |c_{m+1}| \). Hence we get \( |C - C_m| = \left| \sum_{k=m+1}^{+\infty} c_k \right| \leq \sum_{k=m+1}^{+\infty} |c_k| = \sum_{p=1}^{+\infty} |c_{m+p}| \leq \sum_{p=1}^{+\infty} \mu^{p-1} |c_{m+1}| \leq \frac{|c_{m+1}|}{1 - \mu} \) using the well-know theorem about the geometric series.

A similar result holds true for the root test too. We will also need a statement about the Euler-Maclaurin formula; we will use the following one that can be obtained by combining the topics in Sections 3.3-3.4 of Stoer-Bulirsch [14], or referring to Cohen [3] Corollary 9.2.3(2) and Proposition 9.2.5(2).

**Lemma 2.** Let \( a \in \mathbb{R}, N \in \mathbb{N}, \) and \( m \in \mathbb{N}, m \geq 1 \). Assume that \( h \in C^{2m+4}([a, N + a]) \) and that both \( h^{(2m+2)}, h^{(2m+4)} \) have constant sign on \([a, a + N]\). We have \( \sum_{j=0}^{N} h(j + a) = \int_{a}^{N+a} h(w) \, dw + \frac{1}{2} (h(N + a) + h(a)) + \sum_{n=1}^{m} \frac{B_{2n}}{2n!} (h^{(2n-1)}(N + a) - h^{(2n-1)}(a)) \)
In the remaining case in which $\zeta(\cdot)$ is the Riemann zeta-function and $\Re(u) > 1$, we have that

$$
1 + \frac{1}{2^x} < \zeta(x) < 1 + \frac{1}{2^x} - 1
$$

and, for $x \geq 3$, also that

$$
-\frac{\log 2 + (2/3) \log 3}{2^x} < \zeta'(x) < -\frac{\log 2}{2^x}.
$$

**Proof.** The first inequality follows from Theorem 5 of Gordon [6]. The estimate on $\zeta(x)$, $x > 1$, can be obtained from the definition of the Riemann $\zeta$-function in $\Re(u) > 1$, and the integral test. Recalling that $\zeta'(u) = -\sum_{n=2}^{\infty} (\log n) n^{-u}$, $\Re(u) > 1$, we have $-\zeta'(x) > (\log 2)2^{-x}$ for every $x > 1$. Moreover, using that $(\log x)/x$ is a decreasing sequence for $x \geq e$, the last part of the lemma follows by remarking

$$
-\zeta'(x) = \frac{\log 2}{2^x} + \sum_{n=3}^{\infty} \frac{\log n}{n^x} < \frac{\log 2}{2^x} + \frac{\log 3}{3} \sum_{n=3}^{\infty} \frac{1}{n^{x-1}} = \frac{\log 2}{2^x} + \frac{\log 3}{3} \left( \zeta(x-1) - 1 - \frac{1}{2^{x-1}} \right)
$$

and using the inequality $\zeta(x) < 1 + 2^{1-x}$, $x \geq 3$, which follows from the one previously proved. \qed

#### 3. The Algorithm for $f \in \mathcal{F}$

Let $f \in \mathcal{F}$, $s \in A$ be fixed and $x > 0$. Denote by $\lfloor x \rfloor$ and $\{x\}$ the integral and fractional parts of $x$. If $\lfloor x \rfloor \geq 1$ and $\{x\} > 0$, we can use (1) to write

$$
f(s, x) = f(s, x - 1) + g_f(s, x - 1) = \cdots = f(s, \{x\}) + \sum_{j=0}^{\lfloor x \rfloor - 1} g_f(s, \{x\} + j).
$$

In the remaining case in which $x = \ell \in \mathbb{N}$, $\ell \geq 2$, formula (2) becomes

$$
f(s, \ell) = f(s, \ell - 1) + g_f(s, \ell - 1) = \cdots = f(s, 1) + \sum_{j=1}^{\ell-1} g_f(s, j).
$$

In this way we split the problem of computing $f(s, x)$ in two parts: the first is to evaluate $f(s, \{x\})$ and the second is to evaluate the tail, i.e, the difference $f(s, x) - f(s, \{x\})$ expressed as a finite sum of values of the $g_f$-function. Unfortunately the length of such a sum may be too large, so we will evaluate the tail using Lemma 2. We see now how to perform the computation of the quantities in (2); the ones in (3) can be handled analogously.
3.1. Computation of \( f(s, \{x\}), \{x\} > 0, s \in A \) being fixed. Recalling that \( s \in A \) is fixed, the starting point is the first part of Definition 1, namely

\[
f(s, z) = \sum_{k=0}^{\infty} c_f(s, k)(1 - z)^k,
\]

for every \( z \in (1 - \rho_f(s), 1 + \rho_f(s)) \). The series in (4) absolutely converges for \( z \in (1 - \rho_f(s), 1 + \rho_f(s)) \); since \( \rho_f(s) \geq 1 \), this fact and (1) let us obtain \( f(s, \{x\}) \) directly or as \( f(s, \{x\} + 1) - g_f(s, \{x\}) \). In this way we will be able to have a good estimate on the number of summands needed to approximate, up to a desired accuracy, the series in (4) with a finite sum.

Before being more precise, we need the following remark. Using point ii) of Definition 1, we have \( |c_{f(s, k+1)}| \leq 1 \) for \( k \geq k_f(s) \) and hence, using Lemma 1 on (4), we have \( \mu_f = |1 - z| \). Letting now \( a \in \mathbb{N} \), \( a \geq 2 \), we have that for every \( z \in (0, 2) \subset (1 - \rho_f(s), 1 + \rho_f(s)) \) there exists \( r = r_f(s, z, a) \geq k_f(s) \) such that

\[
\left| \sum_{k=r+1}^{\infty} c_f(s, k)(1 - z)^k \right| \leq |c_f(s, r + 1)| \frac{|1 - z|^{r+1}}{|1 - |1 - z||} \leq |c_f(s, k_f(s) + 1)| \frac{|1 - z|^{k_f(s) + 1}}{|1 - |1 - z||} < 2^{-a-1},
\]

since \( c_f(s, k) \) is a decreasing sequence for \( k \geq k_f(s) \). A straightforward computation reveals that in (5) we can choose

\[
r_f(s, z, a) = \max \left( \frac{(a + 1) \log 2 + |\log(1 - |1 - z||) + \log |c_f(s, k_f(s) + 1)||}{|\log |1 - z||} - 1; k_f(s) \right),
\]

where we denoted as \( \lceil u \rceil \) the least integer greater than or equal to \( u \in \mathbb{R} \).

3.1.1. Optimising the number of summands for \( f(s, \{x\}), s \in A \) being fixed. Clearly \( r_f(s, z, a) \) becomes larger as \( |1 - z| \) increases. So when \( 0 < z < 1/2 \) we will evaluate \( f \) at \( 1 + z \) exploiting (1). In this way we will always use the best convergence interval, \( z \in (1/2, 3/2) \), we have for the series in (4); we also remark that

\[
r_f(s, z, a) \leq r_f(s, 1/2, a) = r_f(s, 3/2, a) = \max \left( a + 1 + \frac{|\log |c_f(s, k_f(s) + 1)||}{\log 2}; k_f(s) \right)
\]

for every \( z \in (1/2, 3/2) \). Summarising, using (4)-(5), for \( \{x\} \in (1/2, 1) \) we have that there exists \( \theta = \theta(s, \{x\}) \in (-1/2, 1/2) \) such that

\[
f(s, \{x\}) = \sum_{k=0}^{\infty} c_f(s, k)(1 - \{x\})^k = \sum_{k=0}^{r_f(s, \{x\}, a)} c_f(s, k)(1 - \{x\})^k + |\theta|2^{-a}.
\]

We also remark that for \( \{x\} \in (1/2, 1) \), we have

\[
r_f(s, \{x\}, a) \leq \max \left( a + 1 + \frac{|\log |c_f(s, k_f(s) + 1)||}{\log 2}; k_f(s) \right).
\]

Moreover, using (1) and (4)-(5), for \( \{x\} \in (0, 1/2) \) we have that there exists \( \eta = \eta(s, \{x\}) \in (-1/2, 1/2) \) such that

\[
f(s, \{x\}) = -g_f(s, \{x\}) + f(s, \{x\} + 1) = -g_f(s, \{x\}) + \sum_{k=0}^{\infty} c_f(s, k)(-\{x\})^k
\]

\[
r_f'(s, \{x\}, a) = -g_f(s, \{x\}) + \sum_{k=0}^{r_f'(s, \{x\}, a)} c_f(s, k)(-\{x\})^k + |\eta|2^{-a},
\]

where

\[
r_f'(s, \{x\}, a) := r_f(s, 1 + \{x\}, a)
\]
which, as we will see in the next sections, we will have a small error term. Hence, by choosing
Assuming that the needed \(c_f(s, k)\)-values can be precomputed and stored with a sufficiently good accuracy, the formulae in (6) and (8) allow us to compute \(f(s, \{x\})\), with an accuracy of \(a\) binary digits using about \(a + 1\) summands. We will be more precise on this in the next subsection.

Remark 1. In fact, for \([x] \geq 1\) and \(\{x\} \in (0, 1/2)\), we can combine the first part of (8) with (2) thus obtaining

\[
f(s, x) = f(s, \{x\} + 1) + \sum_{j=1}^{[x]-1} g_f(s, \{x\} + j).
\]  
(10)

In this case it is faster to use (10) since it involves a shorter sum than the one in the right hand side of (2) and \(z = \{x\} + 1 \in (1, 3/2)\) is already in the best convergence interval for the series which defines \(f(s, \cdot)\).

3.2. Computational costs and error terms for \(f(s, x), x \in (0, 1), s \in A\) being fixed. We assume the value of \(f(s, 1/2)\) is known. The estimates on \(r_f(s, \{x\}, a), r'_f(s, \{x\}, a)\) in (7) and (9) for every \(\{x\} \in (1/2, 1)\) and, respectively, \(\{x\} \in (0, 1/2)\), imply that \(f(s, \{x\}), x \in (0, 1)\), can be obtained with an \(a\)-bit accuracy using essentially at most \(a + 1\) summands. The summation can be performed combining the “pairwise summation” (7) algorithm with Kahan’s (8) method (the minimal block for the pairwise summation algorithm is summed using Kahan’s procedure) to have a good compromise between accuracy, computational cost and execution speed. Since the needed powers of \(\{x\}\) and \(1 - \{x\}\) can be obtained using a repeated product strategy, the cost of computing \(f(s, \{x\})\), \(\{x\} \in (1/2, 1)\) is essentially \(\Theta(a)\) evaluations of \(c_f(s, k)\), \(\Theta(a)\) floating point products, \(\Theta(a)\) floating point summations with an accuracy of \(a\) binary digits; for \(\{x\} \in (0, 1/2)\) we have the same plus the cost of computing \(g_f(s, \{x\})\).

3.3. The tail. We explain here the general idea on how to evaluate the tail of the problem in (2), i.e, \(f(s, x) - f(s, \{x\}) = \sum_{j=0}^{[x]-1} g_f(s, \{x\} + j)\). Using Lemma 2 with \(h(\cdot) = g_f(s, \cdot), s \in A\) being fixed, the error terms will depend on the size of the derivatives of \(h\). Here we just make some general considerations, but the actual evaluation of the error term of this part will be directly performed for the examples we will see in the next sections. We just recall here that, in general, a well known characteristic of the Euler-Maclaurin formula, see, e.g., Stoer-Bulirsch [14] Sections 3.3-3.4], is that the error term might not converge to 0; but, in fact, it initially decreases until it reaches a minimal value. We will need then to evaluate if such a minimal value satisfies our desired accuracy; if this is not the case, it is possible to improve on it by using the following idea.

3.3.1. The horizontal shift. The tail-computation can be rearranged in the following way. Letting \(t \in \mathbb{N}, 3 \leq t \leq [x]\), and \(v_1 := \{x\} + t > t\). Moreover, letting \(u_1 = [x] - t \in \mathbb{N}\), we can write that

\[
f(s, \{x\} + [x]) = f(s, v_1 + u_1)\text{ and hence}
\]

\[
f(s, \{x\} + [x]) - f(s, \{x\}) = f(s, v_1 + u_1) - f(s, v_1) + f(s, v_1) - f(s, \{x\})
\]

\[
= f(s, v_1 + u_1) - f(s, v_1) + \sum_{j=0}^{t-1} g_f(s, \{x\} + j).
\]  
(11)

Now we can apply the Euler-Maclaurin formula to \(f(s, v_1 + u_1) - f(s, v_1)\) in a situation, \(v_1 > t \geq 3\), in which, as we will see in the next sections, we will have a small error term. Hence, by choosing \(t\) large enough, and at the cost of evaluating \(t\)-times the \(g_f\)-function, we can obtain a final better accuracy.
3.4. Reflection formulae for \( f(s, x), x \in (0, 1), s \in A \) being fixed. In the applications involving the use of the Fast Fourier Transform, see Section \[8.1.1\] we can speed up the global computation, and improve on the memory occupation, by using the values of \( f(s, x) \pm f(s, 1-x) \) for \( x \in (0, 1), s \in A \) being fixed, instead of the ones of \( f(s, x) \). A nice feature of using \([6]\) and \([8]\) for this goal is that the odd or the even summands of the series of such quantities will vanish thus reducing by a factor of 2 the computational effort of getting such values. We summarise the situation in the following

**Proposition 1.** Let \( f \in \mathcal{F}, s \in A \) being fixed, \( x \in (0, 1), x \neq 1/2, a \in \mathbb{N}, a \geq 2 \). Recalling \([7]\) and \([9]\), let further \( r_1(s, x, a) = r_f(s, x, a)/2 \) and \( r_2(s, x, a) = r'_f(s, x, a)/2 \). There exists \( \theta = \theta(s, x) \in (-1/2, 1/2) \) such that for \( 0 < x < 1/2 \) we have

\[
f(s, x) + f(s, 1-x) = -g_f(s, x) + 2 \sum_{\ell=0}^{\infty} c_f(s, 2\ell)x^{2\ell} + |\theta|2^{-a},
\]

and for \( 1/2 < x < 1 \) we have

\[
f(s, x) - f(s, 1-x) = -g_f(s, x) - 2 \sum_{\ell=0}^{\infty} c_f(s, 2\ell+1)x^{2\ell+1} + |\theta|2^{-a},
\]

Proof. Assume that \( 0 < x < 1/2 \); in this case we compute \( f(s, x) \) with \([8]\) and \( f(s, 1-x) \) with \([6]\). Since the involved series absolutely converge, their sum is the series having as summands the sum of their coefficients; they are \( c_f(s, k)(x^k + (-x)^k) = 2c_f(s, k)x^k \) when \( k \) is even and zero otherwise. Arguing as in Section \[8.1.1\] and remarking that \( r_1(s, x, a) = r_f(s, 1-x, a)/2 = r'_f(s, x, a)/2 \), we immediately have that \([12]\) holds. Assume that \( 1/2 < x < 1 \); in this case we compute \( f(s, x) \) with \([6]\) and \( f(s, 1-x) \) with \([8]\). Arguing as before and remarking that \( r_2(s, x, a) = r_f(s, x, a)/2 = r'_f(s, 1-x, a)/2 \), we immediately have that \([14]\) holds. In a similar way we can prove \([13]\) and \([15]\): the only difference is that now the coefficients are \( c_f(s, k)(x^k - (-x)^k) = 2c_f(s, k)x^k \) when \( k \) is odd and zero otherwise. This completes the proof. \( \square \)

We remark that in Proposition \[1\] we have \( r_2(s, x, a) = r_1(s, 1-x, a) \) for \( x \in (0, 1) \) and hence the right hand side of \([14]\) can be obtained from the right hand side of \([12]\) by replacing any occurrence of \( x \) with \( 1-x \) and vice versa. Something similar hold for \([13]\) and \([15]\) if we also perform a sign change.

We finally remark that the proof of Proposition \[1\] reveals that a similar statement holds for the infinite series of \( f(s, x) \pm f(s, 1-x) \) too.

4. The \( \log \Gamma \) function

We now show that \( \log \Gamma(x), x > 0, \) is in \( \mathcal{F} \). Recalling that the well known Euler formula (see, e.g., Lagarias [9 Section 3]) gives

\[
\log \Gamma(z) = \gamma(1-z) + \sum_{k=2}^{+\infty} \frac{\zeta(k)}{k} (1-z)^k,
\]

where \( \zeta(\cdot) \) is the Riemann zeta-function, \( \gamma \) is the Euler-Mascheroni constant and \( z > 0 \), by letting \( c_\Gamma(k) = \zeta(k)/k \) for \( k \geq 2 \), \( c_\Gamma(1) = \gamma \) and \( c_\Gamma(0) = 0 \), we have that point \[9\] of Definition \[1\] holds. By
Lemma it is easy to see that point of Definition also holds. Using the functional equation
\[ \log \Gamma(1 + z) = \log \Gamma(z) + \log z, \quad z > 0, \]
we can say that point of Definition holds with \( g_\Gamma(s, z) = \log z, \quad z > 0. \) The analysis on the number of summands that follows from Sections 3.1-3.2 gives the results already proved in Section 3 of [10]; the same happens for the reflection formulae we obtain by specialising Proposition. We now see how to handle the tail of this case.

4.1. The Euler-Maclaurin formula for the tail of \( \log \Gamma. \) We now see how to evaluate the tail of the problem in (2), with \( f(s, x) = \log \Gamma(x) \) and \( g_\Gamma(s, x) = \log x. \) First of all, equation (11) becomes
\[ \log \Gamma(\{x\} + [x]) - \log \Gamma(\{x\}) = \log \Gamma(v_1 + u_1) - \log \Gamma(v_1) + \sum_{j=0}^{t-1} \log(\{x\} + j), \tag{16} \]
where \( t \in \mathbb{N}, 3 \leq t \leq [x], \) \( u_1 = [x] - t \in \mathbb{N} \) and \( v_1 := \{x\} + t > t. \) Now we can apply the Euler-Maclaurin formula as in Lemma 2 to \( \log \Gamma(v_1 + u_1) - \log \Gamma(v_1) \) in a situation, \( v_1 > t \geq 3, \) in which we have a small error term.

Before applying Lemma 2 to \( h(w) = \log w, \) \( w > 0, \) we need a couple of definitions. For \( v > 0, \) \( u \in \mathbb{N}, \) \( u \geq 1, m \in \mathbb{N}, m \geq 1, \) we define
\[ d(u, v) := \left( 1 + \frac{u-1}{v} \right)^{-1}, \tag{17} \]
\[ J_{2m}(v, u) := \sum_{n=1}^{m} \frac{B_{2n}}{2n(2n-1)} \frac{d(u, v)^{2n-1} - 1}{v^{2n-1}}, \tag{18} \]
where \( B_{2n} \) are the even-index Bernoulli numbers (we recall that, with the unique exception of \( B_1 = -1/2, \) the odd-index Bernoulli numbers are equal to 0), and
\[ E_{2m}(v, u) := \frac{1}{2m} \int_v^{v+u-1} B_{2m}(\{t\} \{v\}) \frac{dt}{t^{2m}}. \tag{19} \]
We remark that \( J_{2m}(v, 1) = 0 \) and \( E_{2m}(v, 1) = 0. \) We also define \( E_{2m}(0, u) := 0 \) for every \( u \in \mathbb{N}, u \geq 1. \) We will use the following

**Proposition 1.** Let \( v > 0, u \in \mathbb{N}, u \geq 3, m \in \mathbb{N}, m \geq 1. \) Then
\[ \log \frac{\Gamma(v + u)}{\Gamma(v)} = u \log v - u + 1 - \left( v + u - 1 \right) \log d(u, v) + J_{2m}(v, u) + E_{2m}(v, u), \tag{20} \]
where \( \Gamma \) is Euler’s function, \( d(u, v), J_{2m}(v, u), E_{2m}(v, u) \) are respectively defined in (17)-(19). We further have that
\[ |E_{2m}(v, u)| < 2 \left( 1 + \frac{1}{4^{m+1}} \frac{2m + 3}{2m + 1} \right) \frac{(2m)!}{(2\pi)^{2m+2}} \frac{1 - d(u, v)^{2m+1}}{v^{2m+1}}. \tag{21} \]
Moreover, for \( v > 0, u \in \{1, 2\}, \) we have
\[ \log \frac{\Gamma(v + u)}{\Gamma(v)} = u \log v - (u - 1) \log d(u, v). \]

**Proof.** Let \( v > 0, u \in \mathbb{N}, u \geq 1. \) We use the Euler-Maclaurin summation formula on \( \sum_{j=0}^{u-1} \log(v + j). \) Applying the first part of Lemma 2 to \( h(w) = \log w, \) \( a = v \) and \( N = u - 1, \) we obtain that
\[ \sum_{j=0}^{u-1} \log(v + j) = \int_v^{v+u-1} \log w \, dw + \frac{1}{2} \left( \log(v + u - 1) + \log v \right). \]
since \((\log w)^{(\ell)} = (-1)^{\ell-1}w^{-\ell}(\ell - 1)\) for every \(w > 0\) and \(\ell \geq 1\). Recalling (17), that \(\int_{z_1}^{z_2} \log w \, dw = z_1 \log z_1 - z_1 - z_2 \log z_2 + z_2\) and performing the needed computations in (22), the first part of Proposition 2 follows. Using the second part of Lemma 2 we also have

\[
\left| \frac{1}{2m} \int_{v}^{v+u-1} B_{2m}(\{w - v\}) \frac{dw}{w_{2m}} \right| \leq \frac{|B_{2m+2}(2m)!|}{(2m+2)!} \left( \frac{1}{v + u - 1} \right)^{2m+1} - \frac{1}{\sqrt{v^{2m+1}}}.
\]

Recalling \(B_{2\ell} = 2(-1)^{\ell-1} (2\ell)! \zeta(2\ell)\), where \(\ell \in \mathbb{N}, \ell \geq 1\), by Lemma 3 one has \(\zeta(2\ell) < 1 + \frac{1}{4\pi^2} 2^{2\ell+1}\), so that

\[
|B_{2m+2}| < 2 \left( 1 + \frac{1}{4^{m+1}} \frac{2m+3}{2m+1} \right) \frac{(2m+2)!}{(2\pi)^{2m+2}}
\]

and the second part of Proposition 2 follows.

\(\Box\)

4.1.1. **Why we need the horizontal shift.** A well known characteristic of the Euler-Maclaurin summation formula, see, e.g., Stoer-Bulirsch [14, §3.3-3.4], is, for \(v, u > 0\) fixed, that \(E_{2m}(v, u)\) might not converge to 0 as \(m \to +\infty\).

In our application we have \(v = \{x\} < 1\) and it is clear that for \(v < 1\), \(E_{2m}(v, u)\), defined in (19), diverges. But, after having used the horizontal shift as in (16), it is still possible to use Proposition 2 to efficiently and accurately evaluate \(\log \Gamma(v_1 + u_1) - \log \Gamma(v_1)\); if it is sufficiently small for our goals we then proceed to evaluate the remaining part of (20). Moreover, it is easy to see that the order of magnitude of the right hand side of (21) depends weakly from large values of \(u\); hence, for \(v > 1\) fixed, we can find \(\widetilde{m}\) such that

\[
E_{2m}(v) := \left( 1 + \frac{1}{4^{m+1}} \frac{2m+3}{2m+1} \right) \frac{(2m)!}{(2\pi)^{2m+2}} \frac{2}{\sqrt{v^{2m+1}}}
\]

is minimal and then use \(J_{2\widetilde{m}}(v, u)\) and \(E_{2\widetilde{m}}(v, u)\) in (20), since \(|E_{2\widetilde{m}}(v, u)| \leq E_{2m}(v)\). In practice, this procedure for \(v > 1\) works very well and its accuracy improves as \(v\) becomes larger.

Hence, by choosing \(t\) large enough, and at the cost of evaluating \(t\) logarithms (or a logarithm of a product having \(t\) factors; a quantity that might be evaluated with a better computational cost if \(t\) is not too large), we can obtain a much smaller estimate for \(E_{2m}(v_1, u_1)\) than the one originally available for \(E_{2m}(v, u)\). For fixed precision computations we can use the parameters of Table 1 we experimentally computed; for multiprecision applications, such an \(m\)-choice has to be performed at runtime.

We finally remark that the maximal order of magnitude as \(x \to 0^+\) for \(\log \Gamma(x)\) is \(\log x\); so to get its accurate evaluation one needs to work with at least \([\log_2 |\log_2 |\log x||]\) binary digits, where \(\log_2(w)\) denotes the base-2 logarithm of \(w\). For \(x \to +\infty\), Stirling’s formula reveals that we need at most \([\log_2 x + \log_2 \log x]\) binary digits.

\[
\begin{array}{|c|c|c|c|}
\hline
 t & \widetilde{m} & B_{2m}(v) & \text{binary precision} \\
\hline
 4 & 12 & 1.947867 \cdot 10^{-12} < 2^{-33} & 32 \\
 8 & 25 & 1.664034 \cdot 10^{-23} < 2^{-65} & 64 \\
 10 & 31 & 5.192957 \cdot 10^{-29} < 2^{-81} & 80 \\
 15 & 47 & 9.626509 \cdot 10^{-43} < 2^{-129} & 128 \\
\hline
\end{array}
\]

**Table 1.** Horizontal shift optimal error evaluation for \(\log \Gamma\).
4.2. **Computational costs and error terms for** \( \log \Gamma(x), x > 0 \). Thanks to Section 3.2 we can say that the cost of computing \( \log \Gamma(\{x\}) \), \( \{x\} \in (1/2, 1) \) is \( \Theta(a) \) floating point products and \( \Theta(a) \) floating point summations with an accuracy of \( a \) binary digits; for \( \{x\} \in (0, 1/2) \) we have the same plus the cost of computing \( \log(\{x\}) \).

We now evaluate the computational cost of the Euler-Maclaurin summation formula. Let now \( \Delta \in (0, 1) \) be fixed and \( m := m(\Delta, v, u) \in \mathbb{N} \). We consider the sum \( J_m(v, u) \) as defined in (18). We can precompute and store the values of \( B_{2\ell}/(2\ell(2\ell - 1)) \); for example

\[
B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad \ldots
\]

Then for a fixed \( \ell \), to compute the summand in (18) we need four products and two sums since the powers of \( v \) and \( d(u, v) \) can be obtained exploiting a repeated product strategy. So the cost of computing \( J_m(v, u) \) is \( 4m \) products and \( 2m \) sums. The summations are performed as in the previous case. To obtain a final result having a \( \Delta \)-accuracy using (21), we need to first determine the smallest \( t \) and the optimal \( \tilde{m} \) such that \( E_{2\tilde{m}}(v) < \Delta \) using (23); then, we choose the smallest \( m \leq \tilde{m} \) such that \( E_{2m}(v) < \Delta \). To combine the previous analyses we have to choose \( \Delta = 2^{-a-1} \). We also have to consider the cost of the horizontal shift procedure which is just the cost of computing the sum of \( t \) logarithms, or, in alternative, the cost of a logarithm of a product having \( t \) factors.

5. **The digamma function**

The digamma function \( \psi(x), x > 0 \), is in \( \mathcal{F} \) too. Recalling that the well known Euler formula (see, e.g., Lagarias [9, section 3])

\[
\psi(z) = -\gamma - \sum_{k=1}^{\infty} \zeta(k+1)(1-z)^k,
\]

where \( \gamma \) is the Euler-Mascheroni constant and \( z > 0 \), by letting \( c_\psi(k) = \zeta(k+1) \) for \( k \geq 1 \) and \( c_\psi(0) = \gamma \), we have that point [I] of Definition 1 holds. By Lemma 3 it is easy to see that point [II] of Definition 1 also holds. Using the functional equation

\[
\psi(1+z) = \psi(z) + \frac{1}{z}, \quad z > 0,
\]

we can say that point [III] of Definition 1 holds with \( g_\psi(s, z) = 1/z, z > 0 \). The analysis on the number of summands that follows from Sections 3.1 and 3.2 gives the results already proved in Section 4 of [10]; the same happens for the reflection formulae we obtain by specialising Proposition 1. We now see how to handle the tail of this case.

5.1. **The Euler-Maclaurin formula for the tail of** \( \psi(x), x > 0 \). We now see how to evaluate the tail of the problem in (2), with \( f(s, x) = \psi(x) \) and \( g_\psi(s, x) = 1/x, x > 0 \). First of all, equation (11) becomes

\[
\psi(\{x\} + [x]) - \psi(\{x\}) = \psi(v_1 + u_1) - \psi(v_1) + \sum_{j=0}^{\lfloor x \rfloor - 1} \frac{1}{\{x\} + j},
\]

where \( t \in \mathbb{N}, 3 \leq t \leq \lfloor x \rfloor, u_1 = \lfloor x \rfloor - t \in \mathbb{N} \) and \( v_1 := \{x\} + t > t \). Now we can apply the Euler-Maclaurin formula to \( \psi(v_1 + u_1) - \psi(v_1) \) in a situation, \( v_1 > t \geq 3 \), in which we have a small error term.

The computation of \( \sum_{j=0}^{\lfloor x \rfloor - 1} (v_1 + j)^{-1} \) can be performed using Lemma 2 with \( g_f(s, z) = z^{-1} \), as we did for proving Proposition 2. Recalling (17), in this way we get that

\[
\psi(v_1 + u_1) - \psi(v_1) = -\log d(u_1, v_1) + \frac{1}{2v_1}(1 + d(u_1, v_1)) + \sum_{n=1}^{\lfloor x \rfloor} \frac{B_{2n}}{2n} \frac{1 - d(u_1, v_1)^{2n}}{v_1^{2n}}
\]
\[ - \int_{v_1}^{v_1 + u_1 - 1} B_{2m}(\{w - v_1\}) \frac{dw}{w^{2m+1}}, \quad (25) \]

and

\[ \left| \int_{v_1}^{v_1 + u_1 - 1} B_{2m}(\{w - v_1\}) \frac{dw}{w^{2m+1}} \right| < 2 \left( 1 + \frac{1}{4m+1} \frac{2m+3}{2m+1} \right) \frac{(2m+1)!}{(2\pi)^{2m+2}} \frac{1 - d(u_1, v_1)^{2m+2}}{v_1^{2m+2}}. \]

5.2. **Computational costs and error terms for** \( \psi(x), x > 0 \). Thanks to Section 3.2, we can say that the cost of computing \( \psi((x)), \{x\} \in (1/2, 1) \) is \( \Theta(a) \) floating point products and \( \Theta(a) \) floating point sums with an accuracy of a binary digits; for \( \{x\} \in (0, 1/2) \) we have the same plus the cost of computing \( 1/\{x\} \).

Let now \( \Delta \in (0, 1) \) be fixed and let \( m := m(\Delta, v, u) \in \mathbb{N} \). We consider the sum in (25). In this case too the key point is that we can store the values of \( B_{2l}/(2l) \). Then for a fixed \( n \), to compute the summand in (18) we need four products and two sums since the powers of \( v \) and \( d(u, v) \) can be obtained exploiting a repeated product strategy. So the cost of computing the sum in (25) is \( 4m \) products and \( 2m \) sums. To obtain a final result having a \( \Delta \)-accuracy using (21), we need to first determine the smallest \( v \) and the optimal \( \tilde{m}(v) \) such that

\[ E_{2m}^\psi(v) := \left( 1 + \frac{1}{4m+1} \frac{2m+3}{2m+1} \right) \frac{(2m+1)!}{(2\pi)^{2m+2}} \frac{2}{v^{2m+2}} \]

is minimal and less than \( \Delta \); then we choose the smallest \( m(v, \Delta) \leq \tilde{m}(v) \) such that \( E_{2m}^\psi(v) < \Delta \). In this procedure it might be necessary to use a suitable horizontal shift \( t \). To combine the previous analyses we have to choose \( \Delta = 2^{-\alpha - 1} \). We also have to consider the cost of the horizontal shift procedure which is just the cost of computing the sum of \( t \) fractions. For fixed precision computations we can use the parameters of Table 2 we experimentally computed; for multiprecision applications, such a choice has to be performed at runtime.

We finally remark that the maximal order of magnitude as \( x \to 0^+ \) for \( \psi(x) \) is \( 1/x \); so to get its accurate evaluation one needs to work with at least \( \lceil \log_2 x \rceil \) binary digits. For \( x \to +\infty \), the well known asymptotic properties of \( \psi(x) \) reveal that we need at most \( \lceil \log_2 \log x \rceil \) binary digits.

| \( t \) | \( \tilde{m} \) | \( E_{2m}^\psi(v) \) | binary precision |
|-----|-----|----------------|-----------------|
| 4   | 12  | \( \approx 1.21741 \cdot 10^{-11} < 2^{-33} \) | 32               |
| 8   | 24  | \( \approx 1.05109 \cdot 10^{-22} < 2^{-65} \) | 64               |
| 10  | 31  | \( \approx 3.27156 \cdot 10^{-28} < 2^{-81} \) | 80               |
| 15  | 46  | \( \approx 6.06445 \cdot 10^{-42} < 2^{-129} \) | 128              |

**Table 2.** Horizontal shift optimal error evaluation for \( \psi \).

6. **The Hurwitz zeta-function**

Letting \( s > 1, z \in (0, 2) \) and using \( \frac{\partial}{\partial z} \zeta(s, z) = -z \zeta(s+1, z) \), it is not hard to prove that

\[ \zeta(s, z) = \sum_{k=0}^{4 \infty} \frac{\Gamma(k+s)}{(k!)\Gamma(s)} \zeta(k+s)(1-z)^k. \quad (27) \]

For \( s > 1 \) fixed, unfortunately, \( \Gamma(k+s)/[(k!)\Gamma(s)] \) is not a decreasing sequence in \( k \). To overcome this problem, we isolate the power series at 1 of \( z^{-s} - 1 \) in (27) thus obtaining

\[ \zeta(s, z) = \zeta(s) - 1 + \frac{1}{z^s} + \sum_{k=1}^{4 \infty} \frac{\Gamma(k+s)}{(k!)\Gamma(s)} (\zeta(k+s) - 1)(1-z)^k. \quad (28) \]
The series in (28) absolutely converges for every \( z \in (-1, 3) \); this proves that it satisfies point \( i \) of Definition \( \[ \]\) with \( c_{\zeta_H}(s, 0) = 0 \),
\[
c_{\zeta_H}(s, k) = \frac{\Gamma(k + s)}{(k!)\Gamma(s)}(\zeta(k + s) - 1) = \frac{\zeta(k + s) - 1}{kB(s, k)}
\] (29)
for \( k \geq 1 \) and \( s > 1 \), where \( B(x, y) := \Gamma(x)\Gamma(y)/\Gamma(x + y) \) is Euler’s B-function. By Lemma \( \[ \]\) and using the functional equation \( \Gamma(w + 1) = w\Gamma(w), w > 0 \), we obtain
\[
\left| \frac{c_{\zeta_H}(s, k + 1)}{c_{\zeta_H}(s, k)} \right| = \frac{\zeta(k + s + 1) - 1}{\zeta(k + s) - 1} \frac{kB(s, k)}{(k + 1)B(s, k + 1)} < \frac{1}{2} \frac{k + s + 2}{k + s + 1} = \frac{1}{2} \frac{k + s + 2}{k + 1} \leq 1
\]
for \( k \geq s \). Hence point \( ii \) of Definition \( \[ \]\) holds with \( k_{\zeta_H}(s) = \lfloor s \rfloor \). For \( z \in (0, 2) \) we can now write that
\[
r_{\zeta_H}(s, z, a) = \max \left\{ \left( a + 1 \right) \log 2 + \left| \log(1 - |1 - z|) \right| + \left| \log c_{\zeta_H}(s, \lfloor s \rfloor + 1) \right| \right\} - \lfloor s \rfloor, (30)
\]
and that \( r'_{\zeta_H}(s, z, a) = r_{\zeta_H}(s, 1 + z, a) \). The difference equation for this function is
\[
\zeta(s, z + 1) = \zeta(s, z) - z^{-s},
\] (31)
for every fixed \( s > 1 \) and \( z > 0 \). Hence, \( \zeta(s, x), s > 1 \) being fixed, \( x > 0 \), is in \( \mathbb{F} \).

From (28) and (31) we can obtain \( \zeta(s, x), x \in (0, 1) \), in two different ways. Specialising (6) and (8), we obtain for \( x \in (1/2, 1) \) that there exists \( \theta = \theta(s, x) \in (-1/2, 1/2) \) such that
\[
\zeta(s, x) = \zeta(s) - 1 + \frac{1}{x^s} + \frac{r_{\zeta_H}(s, x, a)}{kB(s, k)}(1 - x)^k + |\theta|2^{-a}. (32)
\]
For \( x \in (0, 1/2) \) we have that there exists \( \eta = \eta(s, x) \in (-1/2, 1/2) \) such that
\[
\zeta(s, x) = \zeta(s) - 1 + \frac{1}{(1 + x)^s} + \frac{r'_{\zeta_H}(s, x, a)}{kB(s, k)}(-x)^k + |\eta|2^{-a}.
\]

We recall that \( \zeta(s, 1/2) = (2^s - 1)\zeta(s) \). The reflection formulae are collected in the following

**Proposition 3.** Let \( x \in (0, 1), x \neq 1/2, a \in \mathbb{N}, a \geq 2, s > 1, r_1(s, x, a) = r_{\zeta_H}(s, x, a)/2 \) and \( r_2(s, x, a) = r'_{\zeta_H}(s, x, a)/2. \) Recalling (29), there exists \( \theta = \theta(s, x) \in (-1/2, 1/2) \) such that for \( 0 < x < 1/2 \) we obtain
\[
\zeta(s, x) - \zeta(s, 1 - x) = \frac{1}{x^s} - \frac{1}{(1 - x)^s} + \frac{1}{(1 + x)^s} - 2 \sum_{\ell=1}^{r_1} \frac{\zeta(2\ell - 1 + s) - 1}{(2\ell - 1)B(s, 2\ell - 1)} x^{2\ell - 1} + |\theta|2^{-a}, (33)
\]
\[
\zeta(s, x) + \zeta(s, 1 - x) = 2\zeta(s) - 2 + \frac{1}{x^s} + \frac{1}{(1 - x)^s} + \frac{1}{(1 + x)^s} + 2 \sum_{\ell=1}^{r_1} \frac{\zeta(2\ell + s) - 1}{2\ell B(s, 2\ell)} x^{2\ell} + |\theta|2^{-a},
\]
and for \( 1/2 < x < 1 \) we have
\[
\zeta(s, x) - \zeta(s, 1 - x) = \frac{1}{x^s} - \frac{1}{(1 - x)^s} - \frac{1}{(2 - x)^s} + 2 \sum_{\ell=1}^{r_2} \frac{\zeta(2\ell - 1 + s) - 1}{(2\ell - 1)B(s, 2\ell - 1)} (1 - x)^{2\ell - 1} + |\theta|2^{-a},
\]
\[
\zeta(s, x) + \zeta(s, 1 - x) = 2\zeta(s) - 2 + \frac{1}{x^s} + \frac{1}{(1 - x)^s} + \frac{1}{(2 - x)^s} + 2 \sum_{\ell=1}^{r_2} \frac{\zeta(2\ell + s) - 1}{2\ell B(s, 2\ell)} (1 - x)^{2\ell} + |\theta|2^{-a}.
\]
Similar formulae hold for the infinite series too.
Moreover point \( iii \) of Definition \( \[ \]\) holds with \( g_{\zeta_H}(s, z) = -z^{-s} \) thanks to (31).
we can say that the cost of computing the sum in (32) is computed using a repeated product strategy since
\[
\Gamma = \prod_{j=0}^{k-1} \frac{(s + j)\Gamma(s)}{(k + 1)!\Gamma(s)} = \frac{s + k}{k + 1} b_{s,k} = \left(1 + \frac{s - 1}{k + 1}\right)b_{s,k},
\]
in which we used again that \(\Gamma(w + 1) = w\Gamma(w), w > 0\).

6.1. **The Euler-Maclaurin formula for the tail of \(\zeta(s,x), s > 1\ being\ fixed\**, \(x > 0\). We now see how to evaluate the tail of the problem in (2), with \(f(s,x) = \zeta(s,x)\) and \(g_{\zeta}(s,x) = -x^{-s}, s > 1\ being\ fixed\ and\ \(x > 0\). First of all, equation (11) becomes
\[
\zeta(s,\{x\} + [x]) - \zeta(s,\{x\}) = \zeta(s, v_1 + u_1) - \zeta(s, v_1) - \sum_{j=0}^{t-1} (\{x\} + j)^{-s},
\]
where \(t = \tau(s) \in \mathbb{N}, 3 \leq t \leq \lfloor x \rfloor\), \(u_1 = [x] - t \in \mathbb{N}\) and \(v_1 := \{x\} + t > t\). Now we can apply the Euler-Maclaurin formula as in Proposition 2 to \(\zeta(s, v_1 + u_1) - \zeta(s, v_1)\) in a situation, \(v_1 > t \geq 3\), in which, as we will see in the next subsection, we have a small error term. The computation of \(\Sigma_{j=0}^{m-1} (v_1 + j)^{-s}\) can be performed using Lemma 2 with \(g_{\zeta}(s, z) = -z^{-s}\), as we did for proving Proposition 2. Recalling (17), in this way we get that
\[
\zeta(s, v_1 + u_1) - \zeta(s, v_1) = -v_1^{-s} \left[\frac{v_1}{s - 1} (1 - d(u_1, v_1)^{s-1}) + \frac{1}{2} (1 + d(u_1, v_1)^s)\right] + \sum_{n=1}^{m} B_{2n} \frac{(2n-2)!}{(2n)!} \int_{v_1}^{v_1+u_1-1} B_{2m}\{w - v_1\}\frac{dw}{w^{2m+s}},
\]
and
\[
\int_{v_1}^{v_1+u_1-1} B_{2m}\{w - v_1\}\frac{dw}{w^{2m+s}} < \left(1 + \frac{1}{4m+1} \frac{2m + 3}{2m + 1}\right) \frac{\prod_{j=0}^{2m-1} (s + j)}{\pi(2\pi)^{2m+1}} \frac{1 - d(u_1, v_1)^{2m+1}}{v_1^{2m+s+1}}.
\]

6.2. **Computational costs and error terms for \(\zeta(s,x), s > 1\ being\ fixed\**, \(x > 0\). Thanks to Section 3.2 we can say that the cost of computing the sum in (32) is \(O\) (a) floating point products and \(O\) (a) floating point sums with an accuracy of a binary digits. To obtain the values \(\zeta(s,\{x\}), \{x\} \in (1/2, 1), s > 1\ being fixed\, we also need to further add the cost of computing \(\{x\}^{-s}\). For \(\{x\} \in (0,1/2)\) we have the same plus the cost of computing \((1 + \{x\})^{-s}\).

For estimating the computational cost of using the Euler-Maclaurin formula, we can repeat the same argument already used for \(\log\Gamma\) and \(\psi\). Let now \(\Delta \in (0,1)\) be fixed and \(m := m(s, \Delta, v, u) \in \mathbb{N}\). We consider the sum in (35). In this case too the key point is that we can store the values of \(B_{2r}/(2\ell)\). Hence, for a fixed \(n\), to compute the summand in (35) we need five products and three sums, since \(b_{s,n} = \prod_{j=0}^{n-1} (s + j) / (j + 1)\) can be obtained as in (34) and the powers of \(v\) and \(d(u, v)\) can be obtained exploiting a repeated product strategy. So the cost of computing the sum in (35) is \(5m\) products and \(3m\) sums. To obtain a final result having a \(\Delta\)-accuracy using (35), we need to first determine the smallest \(v\) and the optimal \(\bar{m}(v, s)\) such that
\[
E_{2m}(s, v) := \left(1 + \frac{1}{4m+1} \frac{2m + 3}{2m + 1}\right) \frac{\prod_{j=0}^{2m-1} (s + j)}{\pi(2\pi)^{2m+1}} \frac{1}{v_1^{2m+s+1}}
\]
is minimal and less than \(\Delta\); then, we choose the smallest \(m(v, \Delta, s) \leq \bar{m}(v, s)\) such that \(E_{2m}(s, v) < \Delta\). In this procedure it might be necessary to use a suitable horizontal shift \(t(s)\). To combine the previous...
analyses we have to choose $\Delta = 2^{-a-1}$. We also have to consider the cost of the horizontal shift procedure (which is the cost of computing the sum of $t$ powers with real exponents). The dependence from $s$ of the error term of $E_{2m}^\varepsilon (v, s)$ prevents us to a priori establish what is the optimal level for the horizontal shift parameter as we did for $\log \Gamma$ and $\psi$; hence this has to be computed at runtime for both the fixed precision and the multiprecision cases.

We finally remark that the maximal order of magnitude as $x \to 0^+$, $s > 1$ being fixed, for $\zeta(s, x)$ is $x^{-s}$; so to get its accurate evaluation one needs to work with at least $\lceil s \log_2 x \rceil$ binary digits. For $x \to +\infty$, $s > 1$ being fixed, the well known asymptotic properties of $\zeta(s, x)$ reveal that we need at least $\lceil (s - 1) \log_2 x \rceil$ binary digits.

### 7. The Hurwitz zeta-function: first partial derivative

For $\zeta'(s, z) := \frac{\partial \zeta}{\partial s}(s, z)$, $s > 1$ being fixed and $z > 0$, we proceed as follows. Differentiating (28) we obtain, for $s > 1, z \in (0, 2)$, that

$$
\zeta'(s, z) = \zeta'(s) - \frac{\log z}{z^s} + \sum_{k=1}^{+\infty} c_{\zeta_H}^z(s, k) (1 - z)^k,
$$

where

$$
c_{\zeta_H}^z(s, k) := \frac{\alpha(s, k)}{B(s, k)}, \quad \alpha(s, k) := \frac{(\zeta(k + s) - 1)(\psi(k + s) - \psi(s)) + \zeta'(k + s)}{k}
$$

and $B(s, k)$ is Euler’s $B$-function. By Lemma [5] for $k \geq s + 1, s > 1$, and $k \geq 3$ we have

$$
|\alpha(s, k)| < \frac{1}{k^{2k+s}} \left( \log(k + s) \frac{k + s + 1}{k + s - 1} + 1.43 \right) \leq \frac{1}{2^{k+s}} \left( \max \left( \frac{4 \cdot \log 5}{9}, \frac{\log(2s + 1)}{s} \right) + 1.43 \right) < \frac{1.64}{2^{k+s}}.
$$

Hence for $k \geq \max(s + 1; 3)$ and $s > 1$ we have $|c_{\zeta_H}^z(s, k)| < 1.64/(2^{k+s}B(s, k))$. Since

$$
\frac{2^{k+s}B(s, k)}{2^{k+s+1}B(s, k + 1)} = \frac{1 + s}{2k} = \frac{1}{2} + \frac{s}{2k} < 1
$$

for $k \geq s + 1$, for $r \geq \max(s; 2)$ and $s > 1$ we have

$$
\left| \sum_{k=r+1}^{+\infty} c_{\zeta_H}^z(s, k)(1 - z)^k \right| < 1.64 \sum_{k=r+1}^{+\infty} \frac{|1 - z|^k}{2^{k+s}B(s, k)} < \frac{0.82}{2^{s+r}B(s, r + 1)} \frac{|1 - z|^{r+1}}{1 - |1 - z|}
$$

where, in the last step, we used Lemma [1]. Let now $a \in \mathbb{N}$, $a \geq 2$. Arguing as in Section 3.1.1, for $z \in (1/2, 3/2)$ we obtain that $|\sum_{k=r+1}^{+\infty} c_{\zeta_H}^z(s, k)(1 - z)^k| < 2^{-a-1}$ for

$$
r_{\zeta_H}^z(s, z, a) = \max \left\{ \left[ \frac{(a + 1 + 2[s]) \log 2 + |\log(1 - |1 - z|)| + |\log B(s, [s] + 1)|}{\log|1 - z|} \right] - 1; [s]; 2 \right\},
$$

and $r_{\zeta_H}^z(s, z, a) = r_{\zeta_H}^z(s, 1 + z, a)$. Differentiating (31), we obtain that

$$
\zeta'(s, z + 1) = \zeta'(s, z) + \frac{\log z}{z^s},
$$

for every $s > 1$ and $z > 0$. Specialising (6) and (8), we obtain for $x \in (1/2, 1)$ that there exists $\theta = \theta(s, x) \in (-1/2, 1/2)$ such that

$$
\zeta'(s, x) = \zeta'(s) - \frac{\log x}{x^s} + \sum_{k=1} r_{\zeta_H}^z(s, x, a) c_{\zeta_H}^z(s, k)(1 - x)^k + |\theta|2^{-a}.
$$
For $x \in (0, 1/2)$ we have that there exists $\eta = \eta(s, x) \in (-1/2, 1/2)$ such that

$$
\zeta'(s, x) = \zeta'(s) - \frac{\log x}{x^s} - \frac{\log(1 + x)}{(1 + x)^s} + \frac{r'_\zeta(s, a)}{(s, a)} + \sum_{k=1}^{r'_{\zeta_H}(s, k)} c_{\zeta_H}(s, k)(-x)^k + |\eta|2^{-a}.
$$

It is easy to obtain that $\zeta'(s, 1/2) = 2^s(\log 2)\zeta(s) + (2^s - 1)\zeta'(s)$. The reflection formulae are collected in the following

**Proposition 4.** Let $x \in (0, 1)$, $x \neq 1/2$, $a \in \mathbb{N}$, $a \geq 2$, $s > 1$ being fixed, $r_1(s, x, a) = r'_{\zeta_H}(s, x, a)/2$ and $r_2(s, x, a) = r'_{\zeta_H}(s, x, a)/2$. Recalling the definition of $c_{\zeta_H}(k, s)$ in (37), there exists $\theta = \theta(s, x) \in (-1/2, 1/2)$ such that for $0 < x < 1/2$ we have

$$
\zeta'(s, x) + \zeta'(s, 1 - x) = 2\zeta'(s) - \frac{\log x}{x^s} - \frac{\log(1 - x)}{(1 - x)^s} - \frac{\log(1 + x)}{(1 + x)^s} + 2 \sum_{\ell=1}^{r_1} c_{\zeta_H}(s, 2\ell)x^{2\ell} + |\theta|2^{-a},
$$

$$
\zeta'(s, x) - \zeta'(s, 1 - x) = -\frac{\log x}{x^s} + \frac{\log(1 - x)}{(1 - x)^s} - \frac{\log(1 + x)}{(1 + x)^s} - 2 \sum_{\ell=1}^{r_1} c_{\zeta_H}(s, 2\ell - 1)x^{2\ell - 1} + |\theta|2^{-a},
$$

and for $1/2 < x < 1$ we have

$$
\zeta'(s, x) + \zeta'(s, 1 - x) = 2\zeta'(s) - \frac{\log x}{x^s} - \frac{\log(1 - x)}{(1 - x)^s} - \frac{\log(1 + x)}{(1 + x)^s} + 2 \sum_{\ell=1}^{r_2} c_{\zeta_H}(s, 2\ell - 1)x^{2\ell - 1} + |\theta|2^{-a},
$$

$$
\zeta'(s, x) - \zeta'(s, 1 - x) = -\frac{\log x}{x^s} + \frac{\log(1 - x)}{(1 - x)^s} + \frac{\log(1 + x)}{(1 + x)^s} + 2 \sum_{\ell=1}^{r_2} c_{\zeta_H}(s, 2\ell - 1)(1 - x)^{2\ell - 1} + |\theta|2^{-a}.
$$

Similar formulae hold for the infinite series too.

Moreover, thanks to (38), point iii) of Definition 1 holds with $g_{\zeta'(s, z)} = (\log z)z^{-s}$, $s > 1, z > 0$.

For this function the application of the Euler-Maclaurin formula is still possible but the formulae are much more complicated due to the form of the derivatives of $(\log x)x^{-s}$. Since in Section 8.1 we will just need the values of $\zeta''(s, x)$ for $x \in (0, 1)$, we do not insert this topic here.

We finally remark that the maximal order of magnitude for $\zeta''(s, x)$ is $(\log x)x^{-s}$ for $x \to 0^+$; so to get its accurate evaluation one needs at least $[s]\log_2 x]+ |\log_2 |\log x| |$ binary digits.

**Remark 3 (Coefficients computations).** In this case we have to precompute both $\zeta(k + s)$ and $\zeta'(k + s)$; moreover the coefficients $b_{s,k} := (kB(s, k))^{-1}$ can be computed as in (34). For $d_{s,k} := \psi(k + s) - \psi(s)$, using (24) we obtain $d_{s,0} = 0, d_{s,1} = s^{-1}, d_{s,2} = (s + 1)^{-1} + d_{s,1}$, and hence

$$
d_{s,k+1} = \psi(k + s + 1) - \psi(s) = \frac{1}{s + k} + \psi(k + s) - \psi(s) = \frac{1}{s + k} + d_{s,k}.
$$

8. **Some related topics**

8.1. **How to compute the Dirichlet $L$-functions at $s \geq 1$**. Assume that $q$ is an odd prime and let $\chi$ be a non-principal Dirichlet character mod $q$. For $s = 1$ we use the approach in [10], [11] and [12]. For $s > 1$ will use the well known formulae

$$
L(s, \chi) = q^{-s} \sum_{a=1}^{q-1} \chi(a)\zeta\left(s, \frac{a}{q}\right) \quad \text{and} \quad L'(s, \chi) = -(\log q) L(s, \chi) + q^{-s} \sum_{a=1}^{q-1} \chi(a)\zeta'\left(s, \frac{a}{q}\right).
$$

It is clear that we need the values of $\zeta(s, z)$ and $\zeta'(s, z)$ for $z = a/q \in (0, 1)$. To this goal we can respectively use the results described in Sections 3-7 in this application it is particularly efficient the fact
that, for $s > 1$ fixed, the coefficients $c_{\ell h}(s, k)$, $c_{\ell h}'(s, k)$ can be precomputed and stored since they do not depend on $z = a/b \in (0, 1)$.

If $q$ is not too small, in fact, the computational cost of performing the sums over $a$ in (40) becomes too large. But the trivial summation procedure can be replaced by the use of the Fast Fourier Transform; it is in this context that the reflection formulae of Propositions 34 will be useful.

8.1.1. The FFT setting. First of all we remark that, since $q$ is prime, it is enough to get $g$, a primitive root of $q$, and $\chi_1$, the Dirichlet character mod $q$ given by $\chi_1(g) = e^{2\pi i/(q-1)}$, to see that the set of the non-principal characters mod $q$ is $\{\chi_1': j = 1, \ldots, q-2\}$. Hence, if, for every $k \in \{0, \ldots, q-2\}$, we denote $g^k \equiv a_k \in \{1, \ldots, q-1\}$, every summation in (40) is of the type

$$\sum_{k=0}^{q-2} e\left(\frac{\sigma j k}{q-1}\right) F\left(\frac{a_k}{q}\right),$$

where $e(x) := \exp(2\pi i x)$, $j \in \{1, \ldots, q-2\}$, $\sigma = \pm 1$, and $F(\cdot)$ is a suitable function which assumes real values. As a consequence, such quantities are, depending on $\sigma$, the Discrete Fourier Transforms, or its inverse transformations, of the sequence $\{F(a_k/q) : k = 0, \ldots, q-2\}$. We can now insert the decimation in frequency strategy: assuming that in (41) one has to distinguish between the parity of $j$ (hence on the parity of the Dirichlet character), letting $\overline{q} = (q-1)/2$, for every $j = 0, 1, \ldots, q-2$ we have that

$$\sum_{k=0}^{q-2} e\left(\frac{\sigma j k}{q-1}\right) F\left(\frac{a_k}{q}\right) = \sum_{k=0}^{\overline{q}-1} e\left(\frac{\sigma j k}{q-1}\right) F\left(\frac{a_k}{q}\right) + \sum_{k=0}^{\overline{q}-1} e\left(\frac{\sigma j (k + \overline{q})}{q-1}\right) F\left(\frac{a_k + \overline{q}}{q}\right)$$

$$= \sum_{k=0}^{\overline{q}-1} e\left(\frac{\sigma j k}{q-1}\right) F\left(\frac{a_k}{q}\right) + (-1)^j F\left(\frac{a_k + \overline{q}}{q}\right).$$

Let now $j = 2t + \ell$, where $\ell \in \{0, 1\}$ and $t \in \mathbb{Z}$. Then, the previous equation becomes

$$\sum_{k=0}^{q-2} e\left(\frac{\sigma j k}{q-1}\right) F\left(\frac{a_k}{q}\right) = \sum_{k=0}^{\overline{q}-1} e\left(\frac{\sigma t k}{m}\right) e\left(\frac{\sigma \ell k}{q-1}\right) \left(F\left(\frac{a_k}{q}\right) + (-1)^\ell F\left(\frac{a_k + \overline{q}}{q}\right)\right)$$

$$= \begin{cases} 
\sum_{k=0}^{\overline{q}-1} e\left(\frac{\sigma t k}{m}\right) B_k & \text{if } \ell = 0 \\
\sum_{k=0}^{\overline{q}-1} e\left(\frac{\sigma t k}{m}\right) C_k & \text{if } \ell = 1,
\end{cases}$$

where $t = 0, \ldots, \overline{q} - 1$, $\sigma = \pm 1$,

$$B_k := F\left(\frac{a_k}{q}\right) + F\left(\frac{a_k + \overline{q}}{q}\right) \quad \text{and} \quad C_k := e\left(\frac{\sigma k}{q-1}\right) \left(F\left(\frac{a_k}{q}\right) - F\left(\frac{a_k + \overline{q}}{q}\right)\right).$$

Hence we can split the original problem in summing over the even, or odd, Dirichlet characters; in this way instead of computing an FFT transform of length $q - 1$ we can evaluate two FFTs of length $(q - 1)/2$ each, applied on suitably modified sequences according to (42). Clearly this represents a gain in both the speed and the memory usage in running the actual computer program. Moreover, if the values of $F(a_k/q)$ have to be precomputed and stored, this also means that the quantity of information we have to save during the precomputation and to recall for the FFT algorithm, is reduced by a factor of 2.

Now, since $\langle g \rangle = \mathbb{Z}_q^*$, it trivially follows that $g^q \equiv q - 1 \mod q$, where $\overline{q} = (q-1)/2$. Hence, recalling $a_k \equiv g^k \mod q$, we obtain $a_k + \overline{q} \equiv g^{k + \overline{q}} \equiv a_k(q-1) \equiv q - a_k \mod q$ and, as a consequence, we get

$$F\left(\frac{a_k + \overline{q}}{q}\right) = F\left(\frac{q - a_k}{q}\right) = F\left(1 - \frac{a_k}{q}\right).$$
we use the fact that
\[
\left| c_{\psi(w)}(k + 1) \right| \left| \frac{(w!)c_{\zeta_H}(w + 1, k + 1)}{(w!)c_{\zeta_H}(w + 1, k)} \right| = \left| c_{\zeta_H}(w + 1, k + 1) \right|
\]
and both sequences can be then computed using the reflection formulae for \( F(\cdot) \).

**Remark 4 (Further gain in the computational effort).** Since for the functions described in the previous sections the number of the non-trivial summands in their reflection formulae is reduced by a factor of 2, their use leads to reduce the computational effort of getting \( B_k \) and \( C_k \) by a factor of 4.

8.2. **Polygamma functions.** Let \( w \in \mathbb{N}, w \geq 1 \). The polygamma function \( \psi^{(w)}(z) \) is defined as the \( w \)-th derivative of the digamma function \( \psi(z), z > 0 \). Instead of writing an ad hoc treatment for such functions, we use the fact that
\[
\psi^{(w)}(z) = (-1)^{w-1}(w!)\zeta(w + 1, z),
\]
where \( \zeta(s, z) \) is the Hurwitz zeta-function. Hence
\[
\left| \frac{c_{\psi(w)}(k + 1)}{c_{\psi(w)}(k)} \right| = \left| \frac{(w!)c_{\zeta_H}(w + 1, k + 1)}{(w!)c_{\zeta_H}(w + 1, k)} \right| = \left| c_{\zeta_H}(w + 1, k + 1) \right|
\]
and we can argue about the computation for \( z \in (0, 1) \) as in Section 6 (it is enough to change any occurrence of \( c_{\zeta_H}(s, k) \) with \((-1)^{w-1}(w!)c_{\zeta_H}(w + 1, k)\)). This implies that an extra summand \( \log(w!) \) has to be inserted at the numerator of (30). For \( z > 1 \) the same remark applies too, but in the Euler-Maclaurin formula application of Section 6.1 we also need to control the extra \( w! \)-term by suitably increasing the value of the horizontal shift \( t \) or, if possible, the chosen value of \( m(w + 1, \Delta, v) \). We also recall that \( \psi^{(w)}(1/2) \) is \((-1)^{w-1}(w!)(2^{w+1} - 1)\zeta(w + 1) \) and \( \psi^{(w)}(1) \) is \((-1)^{w-1}(w!)\zeta(w + 1) \).

8.3. **The Dirichlet \( \beta \)-function.** The Dirichlet \( \beta \)-function is defined as
\[
\beta(s) := 4^{-s}(\zeta(s, 1/4) - \zeta(s, 3/4)) = \sum_{n=0}^{\infty} (-1)^n(2n + 1)^{-s} = L(s, \chi_{-4}),
\]
where \( \chi_{-4} \) is the quadratic Dirichlet character mod 4. From (28) and (31), for \( x \in (0, 1/2) \) and \( s > 1 \) we have
\[
\zeta(s, x) - \zeta(s, 1 - x) = \frac{1}{x^s} + \frac{1}{(1 + x)^s} - \frac{1}{(1 - x)^s} - 2 \sum_{\ell=1}^{\infty} \frac{\zeta(2\ell - 1 + s) - 1}{(2\ell - 1)B(s, 2\ell - 1)}x^{2\ell-1}
\]
and hence, for \( s > 1 \) we immediately obtain that
\[
\beta(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{2^{2s-3}} \sum_{\ell=1}^{\infty} \frac{\zeta(2\ell - 1 + s) - 1}{16^{\ell}(2\ell - 1)B(s, 2\ell - 1)}
\]
(43)
and a corresponding truncated formula can be obtained arguing as we did in Proposition 3 see, e.g., formula (33). Recalling (29) and (37), or differentiating both sides of formula (43), for \( s > 1 \) we can also get
\[
\beta'(s) = -2(\log 2)\beta(s) + 4^{-s}(\zeta'(s, 1/4) - \zeta'(s, 3/4))
\]
\[
= -2\log 2 \left( 1 + \frac{1}{3^s} - \frac{1}{2^{2s-3}} \sum_{\ell=1}^{\infty} c_{\zeta_H}(s, 2\ell - 1)16^{-\ell} \right)
\]
\[
+ 2\log 2 - \frac{\log(5/4)}{5^s} + \frac{\log(3/4)}{3^s} - \frac{1}{2^{2s-3}} \sum_{\ell=1}^{\infty} c_{\zeta_H}'(s, 2\ell - 1)16^{-\ell}
\]
We also remark that a truncated formula for (44) can be proved directly or using (33). From a computational point of view, the bottleneck in (44) is the large number of Riemann zeta-values at odd integers needed, for every \( \ell \cdot 64^{-\ell} \); much sharper than the one of Glaisher’s formula. We also remark that a truncated formula for (44) can be proved directly or using (33). From a computational point of view, the bottleneck in (44) is the large number of Riemann zeta-values at odd integers needed, since in this case they are just used once. We recall that Glaisher used his formula to compute \( G \) with an accuracy of 32 decimal digits; currently \( G \) is now known up to \( 10^{12} \) decimal digits.

\[
\frac{\beta'(s)}{\beta(s)} = -2 \log 2 + \frac{\zeta'(s, 1/4) - \zeta'(s, 3/4)}{\zeta(s, 1/4) - \zeta(s, 3/4)}
\]

where \( c_{\zeta}(s, k) \) and \( c_{\zeta'}(s, k) \) are respectively defined in (29) and (37).

8.4. Catalan’s constant. Using (43) for \( s = 2 \), we immediately obtain a formula similar to Glaisher’s result [5] on Catalan’s constant \( G \), namely

\[
G := \beta(2) = \frac{209}{225} - \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{\zeta(2\ell + 1) - 1}{16^\ell (2\ell - 1) B(2, 2\ell - 1)} = \frac{209}{225} - \sum_{\ell=1}^{\infty} \frac{\ell}{16^\ell} (\zeta(2\ell + 1) - 1)
\]

in which we also used \( d_{k,2} = (kB(2, k))^{-1} = k + 1 \), for \( k \geq 1 \). Moreover, thanks to Lemma [3] its convergence ratio is, roughly speaking, about \( \ell \cdot 64^{-\ell} \); much sharper than the one of Glaisher’s formula. We also remark that a truncated formula for (44) can be proved directly or using (33). From a computational point of view, the bottleneck in (44) is the large number of Riemann zeta-values at odd integers needed, since in this case they are just used once. We recall that Glaisher used his formula to compute \( G \) with an accuracy of 32 decimal digits; currently \( G \) is now known up to \( 10^{12} \) decimal digits.

8.5. The Bateman G-function. The Bateman G-function is defined as

\[
G(z) := \psi \left( \frac{z + 1}{2} \right) - \psi \left( \frac{z}{2} \right)
\]

for every \( z > 0 \). Useful special values are \( G(1) = 2 \log 2 \) and \( G(1/2) = \pi \). According to Beebe [11, p. 555], it is not a good idea to compute \( G(z) \) as a difference of \( \psi \)-values, and hence we use a different strategy. Unfortunately \( G \not\in \mathcal{F} \), since its recursion formula is

\[
G(1 + z) = \frac{2}{z} - G(z), \quad z > 0
\]

and hence point \( \text{iii} \) of Definition [1] does not hold. However, we can use part of the framework previously described in this case too. The starting point is that, using eq. (6) on page 46 of [4], we can write

\[
G(z) = 2 \log 2 + 2 \sum_{k=1}^{\infty} (1 - 2^{-k}) \zeta(k + 1)(1 + z)^k
\]

for every \( z \in (0, 2) \). Unfortunately, \( (1 - 2^{-k}) \zeta(k + 1) \) is not a decreasing sequence. However, subtracting from (46) the difference of the geometric series of ratios \( (1 - z) \) and \( (1 - z)/2 \), whose sum is \( \frac{1 - z}{1 - z} = \frac{1 - 2^{-k}}{z(1 + z)} \), it is easy to obtain

\[
G(z) = 2 \log 2 + \frac{2(1 - z)}{z(1 + z)} + 2 \sum_{k=1}^{\infty} (1 - 2^{-k})(\zeta(k + 1) - 1)(1 + z)^k
\]

(47)

Letting \( c_G(0) := 0 \) and

\[
c_G(k) := 2(1 - 2^{-k})(\zeta(k + 1) - 1)
\]

(48)
for \( k \geq 1 \), by Lemma 3 one gets

\[
\frac{|c_G(k + 1)|}{c_G(k)} = \frac{\zeta(k + 2) - 1}{\zeta(k + 1) - 1} \frac{1 - 2^{-k}}{1 - 2^{-k}} < \frac{1}{k} < 1
\]

for \( k \geq 2 \), and this implies that point (ii) of Definition 1 holds with \( k_G = 2 \). Moreover, since the series in (47) absolutely converges for \( z \in (-1, 3) \), we have that point (ii) of Definition 1 holds. Since \( c_G(k) \) is a decreasing sequence and \(|\log(c_G(2))| = |\log((3/2)(\zeta(3) - 1))| < 1.2\), we obtain that

\[
r_G(z, a) = \max \left\{ \left[ \log |1 - z| + 1.2 \right], 1 \right\},
\]

and that \( r'_G(z, a) = r_G(1 + z, a) \).

Hence we can compute \( G(x) \), \( x \in (0, 1) \), in the following way. For \( x \in (1/2, 1) \) we obtain that there exists \( \theta = \theta(x) \in (-1/2, 1/2) \) such that

\[
G(x) = 2 \log 2 + \frac{2(1 - x)}{x(1 + x)} + 2 \sum_{k=1}^{r_G(x,a)} (1 - 2^{-k})(\zeta(k + 1) - 1)(1 - x)^k + |\theta|2^{-a}. \tag{49}
\]

For \( x \in (0, 1/2) \) we have that there exists \( \eta = \eta(x) \in (-1/2, 1/2) \) such that

\[
G(x) = -2 \log 2 + \frac{2x}{x(2 + x)(1 + x)} - 2 \sum_{k=1}^{r'_G(x,a)} (1 - 2^{-k})(\zeta(k + 1) - 1)(-x)^k + |\eta|2^{-a}. \tag{50}
\]

The reflection formulae are collected in the following

**Proposition 5.** Let \( x \in (0, 1), x \neq 1/2, a \in \mathbb{N}, a \geq 2, r_1(x, a) = r_G(x, a)/2 \) and \( r_2(x, a) = r'_G(x, a)/2 \). Recalling (48), there exists \( \theta = \theta(x) \in (-1/2, 1/2) \) such that for \( 0 < x < 1/2 \) we have

\[
G(x) + G(1 - x) = \frac{2}{1 - x} + \frac{2(1 - x)}{(3 - x)(2 - x)} + \frac{2(1 - x)}{x(1 + x)} + 2 \sum_{\ell=1}^{r_1} c_G(2\ell - 1)x^{2\ell-1} + |\theta|2^{-a},
\]

\[
G(x) - G(1 - x) = -4 \log 2 + \frac{2}{1 - x} - \frac{2(1 - x)}{(3 - x)(2 - x)} - \frac{2x}{(2 - x)(1 - x)} - 2 \sum_{\ell=1}^{r_1} c_G(2\ell)x^{2\ell} + |\theta|2^{-a},
\]

and for \( 1/2 < x < 1 \) we have

\[
G(x) + G(1 - x) = \frac{2}{1 - x} + \frac{2(1 - x)}{(3 - x)(2 - x)} + \frac{2(1 - x)}{x(1 + x)} + 2 \sum_{\ell=1}^{r_1} c_G(2\ell - 1)(1 - x)^{2\ell-1} + |\theta|2^{-a},
\]

\[
G(x) - G(1 - x) = 4 \log 2 - \frac{2}{1 - x} - \frac{2(1 - x)}{(3 - x)(2 - x)} + \frac{2(1 - x)}{x(1 + x)} + 2 \sum_{\ell=1}^{r_1} c_G(2\ell)(1 - x)^{2\ell} + |\theta|2^{-a}.
\]

Similar formulae hold for the infinite series too.

**8.5.1. The tail of the \( G \)-function.** Due to (45) we cannot directly use the Euler-Maclaurin formula. But we can argue as follows. Using (24), for the special cases \( x = 2\ell \) or \( x = 2\ell + 1, \ell \in \mathbb{N} \), we respectively have that

\[
G(2\ell) = \psi \left( \frac{3}{2} \right) - \psi \left( 1 \right) + \sum_{j=1}^{\ell-1} \left( \frac{1}{j + 1/2} - \frac{1}{j} \right) = 2 - G(1) + \sum_{j=1}^{[\ell/2]-1} \left( \frac{1}{j + 1/2} - \frac{1}{j} \right), \quad (x = 2\ell), \tag{51}
\]

\[
G(2\ell + 1) = \frac{1}{\ell} - G(2\ell) = G(1) - 2 + \frac{1}{[\ell/2]} - \sum_{j=1}^{[\ell/2]-1} \left( \frac{1}{j + 1/2} - \frac{1}{j} \right), \quad (x = 2\ell + 1), \tag{52}
\]
in which we also used \( \Gamma \). Moreover, for \( x \in (2\ell, 2\ell + 1) \) we have \( \lfloor x/2 \rfloor = \ell, \{x/2\} \in (0, 1/2) \), \( \{x/2\} + 1/2 \in (1/2, 1) \), so that, using (24), we obtain

\[
G(x) = \psi \left( \frac{x}{2} + \ell + \frac{1}{2} \right) - \psi \left( \frac{x}{2} + \ell \right) = \psi \left( \frac{x}{2} + \frac{1}{2} \right) - \psi \left( \frac{x}{2} \right) + \sum_{j=0}^{\lfloor x/2 \rfloor - 1} \left( \frac{1}{\{x/2\} + j + 1/2} - \frac{1}{\{x/2\} + j} \right)
\]

\[
= G \left( 2 \left( \frac{x}{2} \right) \right) + \sum_{j=0}^{\lfloor x/2 \rfloor - 1} \left( \frac{1}{\{x/2\} + j + 1/2} - \frac{1}{\{x/2\} + j} \right), \quad x \in (2\ell, 2\ell + 1), \quad (53)
\]

and, since \( 2\{x/2\} \in (0, 1) \), we can use (49)-(50) to compute \( G(2\{x/2\}) \). For \( x \in (2\ell + 1, 2\ell + 2) \) we get \( \lfloor x/2 \rfloor = \ell, \lfloor (x+1)/2 \rfloor = \ell + 1, \{x/2\} \in (1/2, 1), \{x/2 + 1/2\} = \{x/2\} - 1/2 \in (0, 1/2) \), so that

\[
G(x) = \psi \left( \frac{x}{2} + \ell + \frac{1}{2} \right) - \psi \left( \frac{x}{2} + \ell \right) = \psi \left( \frac{x}{2} + \frac{1}{2} \right) - \psi \left( \frac{x}{2} \right) + \sum_{j=0}^{\lfloor x/2 \rfloor - 1} \left( \frac{1}{\{x/2\} + j + 1/2} - \frac{1}{\{x/2\} + j} \right)
\]

\[
= \psi \left( \frac{x}{2} - \frac{1}{2} \right) - \psi \left( \frac{x}{2} \right) + \frac{1}{\{x/2\} - 1/2} + \sum_{j=0}^{\lfloor x/2 \rfloor - 1} \left( \frac{1}{\{x/2\} + j + 1/2} - \frac{1}{\{x/2\} + j} \right)
\]

\[
= -G \left( 2 \left( \frac{x}{2} \right) - 1 \right) + \frac{2}{2\{x/2\} - 1} + \sum_{j=0}^{\lfloor x/2 \rfloor - 1} \left( \frac{1}{\{x/2\} + j + 1/2} - \frac{1}{\{x/2\} + j} \right), \quad x \in (2\ell + 1, 2\ell + 2), \quad (54)
\]

and, since \( 2\{x/2\} - 1 \in (0, 1) \), we can use (49)-(50) to compute \( G(2\{x/2\} - 1) \).

Moreover, it is now clear that to evaluate the tails in (51)-(54) we can use Lemma 2 with the function \( h(u) = (u(2u + 1))^{-1} \). As an alternative, it is enough to use twice equation (25); the final error terms for \( G(x), x > 0 \), will be bounded by \( 2E_{2m}^G(\{x/2\}) \), where \( E_{2m}^G(v) \) is defined in (26). Hence the value of \( m_G \) has to be suitably modified with respect to the one of \( m_\psi \), to be sure that the error term for \( G \) will be less of the desired accuracy. If this is not possible, we can use the formulae in this paragraph to build a suitable horizontal shift as we did for \( \psi \) in Section 5.1.

9. Some tests

We performed some tests by implementing in PARI/GP \(^\text{[13]}\) the algorithms here described\(^\text{[4]}\). The running time comparisons refer to the internal PARI/GP functions (however, such functions work for complex inputs too while ours work only for positive inputs) and are obtained on a Dell OptiPlex-3050, equipped with an Intel i5-7500 CPU, 3.40GHz, 16GB of RAM and running Ubuntu 20.04.3LTS. A professional implementation of our algorithm should be able to obtain better practical performances than the ones described below.

We decided to allow an accuracy of 32, 64, 80 or 128 bits in the implementations of these functions but it is easy to modify this to allow an arbitrarily large precision. The choice of the length of the sum, i.e., of the parameter \( m \), in the Euler-Maclaurin formula is performed at runtime, even if for \( \log \Gamma(x) \), \( \psi(x) \) and \( G(x) \) it is also possible to use Tables\(^\text{[12]}\). The precomputation of the needed Riemann zeta-function values gives us excellent performances; in particular, for \( \log \Gamma(x), \psi(x), G(x) \) this works nicely for \( 0 < x < 1 \) and in this case our algorithm is about twice times faster that the internal PARI/GP functions\(^\text{[2]}\). For \( x > 1 \)

\(^1\)The programs can be downloaded at the page: \text{http://www.math.unipd.it/~languasc/specialfunctions.html}

\(^2\)Recently (November 8th, 2021) K. Belabas \([2]\), maintainer and developer of PARI/GP, communicated me that some of the ideas used here and in \([10]\) to compute \( \Gamma(x) \) and \( \log \Gamma(x) \) for \( 0 < x < 1 \) have been used to improve their PARI/GP implementations (from version 2.9.0 on) and that further tunings of their code are about to be released. Analogous ideas for \( \psi(x) \) will be inserted in PARI/GP-2.15.
the computation of the tail of such functions requires too much time and the performances of our method become worse than the ones of PARI/GP.

We also tested the case of the Hurwitz zeta-function by computing 10000 times the same values with our algorithm and the internal PARI/GP functions (both with an accuracy of 128 bits). For integral values of \( s \), our script is about 5.5-times faster and for non-integral values of \( s \) the performances are even better; for example, we evaluated \( \zeta(8.3, 1345.1234) \) for 10000 times in less than 4.9 seconds with our algorithm while the internal functions of PARI/GP required about two minutes and 21 seconds of time (with a factor of about 30 as a performance gain). To test the implementation designed for the Fast Fourier Transform applications, see Section 8.1, we have chosen a small prime \( q = 305741 \) and we computed \( \zeta(s, a_k/q) \) and \( \zeta'(s, a_k/q) \) for some values of \( s \), where \( a_k \equiv g^k \mod q \), \( g \) is a primitive root of \( q \) and \( k \) runs from 0 to \( q - 2 \). In the worst case our script is twice times faster than the internal PARI/GP functions. Moreover, our algorithm seems to be particularly efficient for \( \zeta'(s, a_k/q) \): for example, with an accuracy of 128 bits, \( s = 8.3, q = 307541 \) and \( k \) running from 0 to \( q - 2 \), our implementation is about 32-times faster than the one that uses directly the internal PARI/GP functions. Moreover, this can be further improved using the script that computes at the same time both \( \zeta(s, a_k/q) \) and \( \zeta'(s, a_k/q) \). We also remark that computing \( \zeta(s, a_k/q) \pm \zeta(s, 1 - a_k/q) \) and \( \zeta'(s, a_k/q) \pm \zeta'(s, 1 - a_k/q) \) gives much better performances thanks to the use of the reflection formulae and Remark\[4\] with an accuracy of 128 bits, \( s = 8.3, q = 307541 \) and \( k \) running from 0 to \( q - 2 \), our implementation is about 60-times faster than the one that uses directly the internal PARI/GP functions. For \( q = 6766811 \) the gain factor in the speed of the last computation is about 72-times; in fact, we think that the larger \( q \) we use, the better the performance gain becomes. It is hence clear that in computing the values of the Dirichlet \( L \)-functions, see Section 8.1, the use of our algorithm gives excellent performances.

For the Dirichlet \( \beta \)-function we computed 10000 times the same values both with our algorithm and the internal PARI/GP functions (both with a precision of 128 bits). For integral values of \( s \), our script is about three times faster and for non-integral values of \( s \) the performances are even better; for example we evaluated \( \beta(8.3) \) for 10000 times in 535 milliseconds with our algorithm while the internal functions of PARI/GP required about seven seconds of time. A better performance gain is obtained for both \( \beta'(s) \) and \( \beta'/\beta(s) \); with \( s = 8.3 \), our algorithm is respectively faster by a factor of 39 and 52. Moreover, using our script for \( \beta'/\beta(s) \) is particularly convenient since is also gives the values of \( \beta'(s) \) and \( \beta(s) \).

From all the previous tests and examples, we can conclude that the algorithm here presented is particularly useful when it is possible to exploit the precomputation of the \( c_j(s, k) \)-coefficients of the series in point[4] of Definition[4] as for the applications that require the use of the Fast Fourier Transform algorithm.

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