Defining the space in a general spacetime

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Abstract

A global vector field $v$ on a “spacetime” differentiable manifold $V$, of dimension $N+1$, defines a congruence of world lines: the maximal integral curves of $v$, or orbits. The associated global space $N_v$ is the set of these orbits. A “$v$-adapted” chart on $V$ is one for which the $\mathbb{R}^N$ vector $x \equiv (x^j) (j = 1, ..., N)$ of the “spatial” coordinates remains constant on any orbit $l$. We consider non-vanishing vector fields $v$ that have non-periodic orbits, each of which is a closed set. We prove transversality theorems relevant to such vector fields. Due to these results, it can be considered plausible that, for such a vector field, there exists in the neighborhood of any point $X \in V$ a chart $\chi$ that is $v$-adapted and “nice”, i.e., such that the mapping $\bar{\chi} : l \mapsto x$ is injective — unless $v$ has some “pathological” character. This leads us to define a notion of “normal” vector field. For any such vector field, the mappings $\bar{\chi}$ build an atlas of charts, thus providing $N_v$ with a canonical structure of differentiable manifold (when the topology defined on $N_v$ is Hausdorff, for which we give a sufficient condition met in important physical situations). Previously, a local space manifold $M_F$ had been associated with any “reference frame” $F$, defined as an equivalence class of charts. We show that, if $F$ is made of nice $v$-adapted charts, $M_F$ is naturally identified with an open subset of the global space manifold $N_v$.

Keywords: Physical space; global vector field; reference fluid; orbit space; adapted chart; differentiable manifold; Kruskal-Szekeres coordinates.
1 Introduction

1.1 Physical motivation

The theory of relativity says that space and time merge into “a kind of union of the two” (in Minkowski’s words): the spacetime. However, the notion of a physical space should be useful also in relativistic physics. In our opinion it is even needed, for the following two reasons. (i) In experimental/observational work, one of course needs to define the spatial position of the experimental apparatus and/or of the observed system, and this is true also if the relativistic effects have to be considered. (ii) In quantum mechanics, the “state” of a quantum-mechanical particle is a function $\psi$ of the position $x$ belonging to some 3-D “physical space” $M$, and taking values in $\mathbb{C}$ (or in a complex vector bundle). Note that defining such a space as a spacelike 3-D submanifold of the spacetime manifold (e.g. [1]) can work to define an initial condition for a field in space-time, but does not allow one to define a spatial position in the way that is needed in the two foregoing examples: in those, one needs to identify spatial points that exist at least for some open interval of time — e.g. to state that some objects maintain a fixed spatial position in some reference frame, or to define the stationary states of a quantum particle. In practice, the spatial position is taken to be the triplet of the spatial coordinates, $x \equiv (x^j) \ (j = 1, 2, 3)$. However, a priori, $x$ does not have a precise geometric meaning in a theory starting from a spacetime structure. Only a notion of “spatial tensors” has been defined for a general spacetime of relativistic gravity. This definition was based on the concept of “reference fluid” [2, 3, 4], also named “reference body” [5] — i.e., a three-dimensional congruence of world lines, whose the tangent vector is assumed to be a time-like vector field. The latter can be normed to become a four-velocity field $v$, i.e. $g(v,v) = 1$ where $g$ is the spacetime metric. The data of the four-velocity field $v$ allows one to define the spatial projection operator $\Pi_X$ (depending on the point $X$ in the spacetime manifold $V$) [2, 3, 4, 5, 6, 7]. A “spatial vector” is then defined as a spacetime vector which is equal to its spatial projection. A full algebra of “spatial tensors” can be defined in the same way, and also, once a relevant connection has been defined, a spatial tensor analysis [2, 3, 4, 5, 6].

However, it is possible in a general spacetime manifold $V$ to define a relevant physical space as a 3-D differentiable manifold, at least locally in $V$. 

To see this, consider a coordinate system or chart:

\[ \chi : U \to \mathbb{R}^4, \quad X \mapsto \chi(X) = \mathbf{X} \equiv (x^\mu) \quad (\mu = 0, \ldots, 3), \]

where \( U \) is an open subset of \( V \): the domain of the chart. Then one may define a set of world lines, each of which, \( l \), has constant spatial coordinates \( a^j \) in the chart \( \chi \):

\[ l_a = \{ X \in U; \ \chi(X) = (x^\mu) \text{ is such that } x^j = a^j \text{ for } j = 1, 2, 3 \}. \]

Let us suppose for a moment that the chart \( \chi \) is in fact a Cartesian coordinate system on the Minkowski spacetime. Then that chart defines an inertial reference frame. In that case, it is clear that, for any event \( X \), with \( \chi(X) = (x^\mu) \), the triplet \( \mathbf{x} \equiv (x^j) \ (j = 1, 2, 3) \) defines the spatial position associated in that chart with the event \( X \). Note that the data of \( \mathbf{x} \) is equivalent to specifying a unique world line in the “congruence \( (3) \)” [By this we mean the set of the world lines \( (3) \), when \( a \equiv (a^j) \) takes any value in \( \mathbb{R}^3 \) such that the corresponding world line \( (3) \) is not empty.] That world line is thus uniquely determined by the event \( X \) and may be noted \( l(X) \). Events \( X' \) that have different values of the time coordinate \( x^0 \), but that have the same values of the spatial coordinates \( x^j \ (j = 1, 2, 3) \), can be said to occur at the same spatial position in the inertial frame as does \( X \). Thus, the whole of \( l(X) \) is needed. However, each world line in the congruence \( (3) \) stays invariant if we change the coordinate system by a purely spatial coordinate change:

\[ x'^0 = x^0, \quad x'^k = \phi^k((x^j)). \]

It is clear that this transformation leaves us in the same inertial reference frame. With the new chart \( \chi' \), the new triplet \( \mathbf{x}' \equiv (x'^k) \equiv \phi(\mathbf{x}) \) corresponds to the same spatial position in the inertial frame as does \( \mathbf{x} \) with the first chart \( \chi \). And indeed, that world line of the congruence which is defined in the chart \( \chi' \) by the data of \( \mathbf{x}' \) is just the same as that world line of the congruence which is defined in the chart \( \chi \) by the data of \( \mathbf{x} \). The spatial position

\[ v^0 = \frac{1}{\sqrt{g_{00}}} \quad v^j = 0. \]

\[ 1 \] Note that, if we assume that \( V \) is endowed with a Lorentzian metric \( g \) whose component \( g_{00} \) in the chart \( \chi \) verifies \( g_{00} > 0 \) in \( U \), then each among the world lines \( l \) is time-like, because in the chart \( \chi \) the tangent vector to \( l \) has components \( \propto (1, 0, 0, 0) \), which may be normed to
of the event \( X \) in the inertial frame is therefore most precisely defined by the world line \( l(X) \) of the congruence which passes at \( X \).

Now note that very little in the foregoing paragraph actually depends on whether or not the chart \( \chi \) is a Cartesian coordinate system on the Minkowski spacetime: only the qualification of the reference frame as being inertial depends on that. It is just that we are accustomed to consider a spatial position in an inertial frame in a flat spacetime, and special relativity makes it natural to accept that it is actually the world line \( l(X) \) which best represents that spatial position. Hence, consider a general spacetime, and define a congruence of world lines from the data of a coordinate system as in Eq. (3). In the domain of the chart, we may then define the spatial position of an event \( X \) as the unique world line \( l(X) \) of the congruence (3) which passes at \( X \) — i.e., \( l(X) \) is the unique world line of the congruence (3), such that \( X \in l(X) \). Thus, the data of a coordinate system on the spacetime defines a three-dimensional space \( M \), of which the points (the elements of \( M \)) are the world lines of the congruence (3) associated with that coordinate system.

1.2 Local reference frame and local space manifold

The foregoing approach can be used to define precise notions of a reference frame and its unique associated space manifold \( \mathbb{R}^3 \). First, the invariance of the congruence (3) under the purely spatial coordinate changes (4) allows one to define a reference frame as being an equivalence class of charts related by a change (4). More exactly, the following is an equivalence relation between charts which are defined on a given open subspace \( U \) of the spacetime manifold \( V \):

\[
\chi' \mathcal{R}_U \chi' \iff \forall X \in \chi(U), \quad \phi^0(X) = x^0 \quad \text{and} \quad \frac{\partial \phi^k}{\partial x^0}(X) = 0 \quad (k = 1, \ldots, 3),
\]

where \( f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu) \) is the transition map, which is defined on \( \chi(U) \). Thus a reference frame \( F \) is a set of charts defined on the same open domain \( U \) and exchanging by a purely spatial coordinate change (4). More exactly, the following is an equivalence relation between charts which are defined on a given open subspace \( U \) of the spacetime manifold \( V \):

\[
\chi' \mathcal{R}_U \chi' \iff \forall X \in \chi(U), \quad \phi^0(X) = x^0 \quad \text{and} \quad \frac{\partial \phi^k}{\partial x^0}(X) = 0 \quad (k = 1, \ldots, 3),
\]

where \( f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu) \) is the transition map, which is defined on \( \chi(U) \). Thus a reference frame \( F \) is a set of charts defined on the same open domain \( U \) and exchanging by a purely spatial coordinate change (4). Then the space manifold \( M \) or \( M_F \) associated with the reference frame \( F \) is defined as the set of the world lines (3). In detail: let \( P_S : \mathbb{R}^4 \to \mathbb{R}^3, X \equiv (x^\mu) \mapsto x \equiv (x^j) \), be the spatial projection. A world line \( l \) is an element of \( M_F \) iff there is a chart \( \chi \in F \) and a triplet \( x \in P_S(\chi(U)) \), such that \( l \) is the set of all points \( X \) in
the domain $U$, whose spatial coordinates are $x$:

$$l \equiv \{ X \in U; \ P_S(\chi(X)) = x \}. \quad (6)$$

(Thus, $l$ is not necessarily a connected set.) It results easily from (4) that (6) holds true then in any chart $\chi' \in F$, of course with the transformed spatial projection triplet $x' = \phi(x) \equiv (\phi^i(x)) \[8]$. For a chart $\chi \in F$, one defines the “associated chart” as the mapping which associates, with a world line $l \in M$, the constant triplet of the spatial coordinates of the points $X \in l$:

$$\tilde{\chi}: M \to \mathbb{R}^3, \quad l \mapsto x$$

such that $\forall X \in l, \ P_S(\chi(X)) = x. \quad (7)$

One shows then that the set $\mathcal{T}$ of the subsets $\Omega \subset M$ such that,

$$\forall \chi \in F, \ \tilde{\chi}(\Omega) \text{ is an open set in } \mathbb{R}^3 \quad (8)$$

is a topology on $M$. Finally one shows that the set of the associated charts: $\tilde{F} \equiv \{ \tilde{\chi}; \ \chi \in F \}$, is an atlas on the topological space $(M, \mathcal{T})$, hence defines a structure of differentiable manifold on $M \[8]$. Thus the space manifold $M_F$ is browsed by precisely the triplet $x \equiv (x^i)$ made with the spatial projection of the spacetime coordinates $X \equiv (x^\mu) \equiv \chi(X)$ of a chart $\chi \in F$, see Eq. (7).

Using these results, one may define the space of quantum-mechanical states, for a given reference frame $F$ in a given spacetime $(V, g)$, as being the set $\mathcal{H}$ of the square-integrable functions defined on the corresponding space manifold $M \[9]$. One may also define the full algebra of spatial tensors: the pointwise algebra is defined simply as the tensor algebra of the tangent vector space $TM_x$ to the space manifold $M$ at some arbitrary point $x \in M \[10]$.

### 1.3 Goal and summary

Thus, by defining a reference frame as a set $F$ of charts that all have the same open domain $U$ and that exchange by a purely spatial coordinate change (4), one can then define the associated space manifold $M_F$ as the set of the world lines (6) \[8]. These definitions are relevant to physical applications \[9] \[10]. However, they apply to a parametrizable domain $U$ of the spacetime manifold $V$, i.e., to an open set $U$, such that at least one regular chart can be defined over the whole of $U$. Since the manifold $V$ itself as a whole is in general not
parametrizable, a reference frame is in general only a local one, and so the associated space manifold does not look “maximal”. The aim of the present work is to define \textit{global} reference fluids, to associate with any of them a \textit{global physical space}, and to link these concepts with the formerly defined \textit{local concepts}. As the “spacetime”, we consider a differentiable manifold \( V \) having dimension \( N + 1 \), thus \( N \) is the dimension of the “space” manifold to be defined. We define a global reference fluid by the data of a \textit{non-vanishing} global vector field \( v \) on \( V \). We do not need that \( N = 3 \), nor that \( V \) be endowed with a Lorentzian metric \( g \) for which \( v \) be a time-like vector field. This was already true for the former “local” work [8]. Note, however, that a time-like vector field on a Lorentzian manifold \( (V, g) \) is non-vanishing; and that, if a Lorentzian manifold \( (V, g) \) is time-oriented, which indeed is usually required for a spacetime, then by definition there exists at least one global time-like vector field on \( V \). We define the “global space” associated with \( v \) as the set \( N_v \) of the maximal integral curves (or “orbits”) of \( v \). To reach our goal, we take the following steps:

\( (a) \) Section 2 studies when a given vector field \( v \) on a differentiable manifold \( V \) is such that there locally exists charts of \( V \) which are \textit{adapted} \[2\] to the congruence associated with \( v \); i.e., charts in which the “spatial” position \( x \equiv (x^j) \) \( (j = 1, \ldots, N) \) is constant on any orbit \( l \) of \( v \); see \textbf{Definition 1}. We need also that the mapping \( l \mapsto x \) be injective. The desired situation is defined by \textbf{Proposition 2}. According to a \textbf{transversality argument} this situation should be attainable, in general, if \( v \) does not vanish and each of its orbits is non-periodic and is closed in \( V \). In Subsect. 2.4, two theorems of transversality and another theorem pertaining to differential topology allow us to formalize that argument in \textbf{Theorem 4}. This justifies us in introducing a notion of “normal” vector field by \textbf{Definition 2}, which ensures the local existence of \( v \)-adapted charts through \textbf{Theorem 5}.

\( (b) \) For any \( v \)-adapted chart \( \chi \), one may define the mapping \( \bar{\chi} \) which associates with any orbit \( l \) the constant spatial position \( x \). We show in Section 3 that, using the set \( \mathcal{A} \) of the injective mappings \( \bar{\chi} \), one can endow the global orbit set \( N_v \) with a topology \( \mathcal{T}' \) for which this set is an atlas of charts. See \textbf{Theorem 6}. Thus, when \( \mathcal{T}' \) is metrizable and separable, we do have a canonical structure of differentiable manifold on the orbit set \( N_v \), for which the mappings \( \bar{\chi} \) defined by Eq. (14) are local charts on the “space” manifold \( N_v \). I.e., if \( N = 3 \), also the global space \( N_v \) is browsed (locally) by precisely the
triplet $\mathbf{x} \equiv (x^j)$ made with the spatial projection of the spacetime coordinates $\mathbf{X} \equiv (x^\mu) \equiv \chi(X)$ of a $v$-adapted chart $\chi$.

(c) In Section 4 we establish the link with the previously defined space manifold $M_F$, associated with a given local reference frame $F$ — defined as an equivalence class of charts for the relation (5). We show in Theorem 7 that, when the charts belonging to $F$ are $v$-adapted and with the mapping $l \mapsto \mathbf{x}$ being injective, then $M_F$ is naturally identified with an open subset of $N_v$.

The definitions of a local reference frame $F$ and the corresponding “space” manifold $M_F$ do not need any metrical structure on the “spacetime” manifold $V$ [8]. Just the same can be said for the definition of the global “space” manifold $N_v$, beyond the very fact that $V$ should have a metrizable topology.

2 Local existence of adapted charts

2.1 Definitions

Let $V$ be an $(N + 1)$—dimensional real differentiable manifold, with $N \geq 1$. We consider a given global, smooth vector field $v$ on $V$. A continuously differentiable ($C^1$) mapping $C$ from an open interval $I$ of $\mathbb{R}$ into $V$ defines an integral curve of $v$ iff $\frac{dC}{ds} = v(C(s))$ for $s \in I$. For any $X \in V$, let $C_X$ be the solution of

$$\frac{dC}{ds} = v(C(s)), \quad C(0) = X$$

for which the open interval $I$ is maximal, and denote this maximal interval by $I_X$ [12]. That is, $I_X$ is an open interval defined as the union of all open intervals $I$, each containing 0, in which a solution of (9) is defined. The solution $C_X$ is defined on $I_X$ and is unique [12]. Let $s \in I_X$ and set $Y = C_X(s)$. These definitions imply easily [12] that

$$I_Y = I_X - s \quad \text{and} \quad \forall t \in I_Y, \quad C_Y(t) = C_X(s + t).$$

For any $X \in V$, call the range $l_X \equiv C_X(I_X) \subset V$ the “maximal integral curve at $X$”. From (10), “$l_X$ does not depend on the point $X \in l_X$”:

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2 We understand “differentiable manifold” as a topological space endowed with an atlas of compatible charts, hence with the corresponding equivalence class of compatible atlases — with the restriction that that space should be metrizable and separable [11].
then $l_Y = l_X$. We define the set of the maximal integral curves (or orbits) of $v$:

$$N_v \equiv \{l_X; \ X \in \mathcal{V}\}. \quad (11)$$

Once endowed with further structure, $N_v$ will be the global space manifold associated with the global vector field $v$ (when the latter is non-vanishing and obeys another assumption). Note that, if the set $U \subset \mathcal{V}$ is not empty, then the following subset of $N_v$:

$$D_U \equiv \{l \in N_v; \ l \cap U \neq \emptyset\} \quad (12)$$

is not empty. Indeed, for any $X \in U$, the world line $l \equiv l_X$ belongs to $D_U$. Let $P_S : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$, $(x^\mu) \mapsto (x^j)$ ($\mu = 0, ..., N; \ j = 1, ..., N$) be the “spatial” projection.

**Definition 1.** A mapping $\chi : U \rightarrow \mathbb{R}^{N+1}$ with $U \subset \mathcal{V}$ is said “$v$-adapted” iff for any $l \in D_U$, there exists $x \in \mathbb{R}^N$ such that

$$\forall Y \in l \cap U, \ P_S(\chi(Y)) = x. \quad (13)$$

If Eq. (13) is verified by some world line $l \in N_v$, then necessarily $l \in D_U$, and $x$ is obviously unique. Thus, for any $v$-adapted mapping $\chi$, the mapping

$$\bar{\chi} : D_U \rightarrow \mathbb{R}^N, \ l \mapsto x \text{ such that (13) is verified} \quad (14)$$

is well defined. In Section 3, we will endow the set $N_v$ with first a topology and then a structure of differentiable manifold, for which the charts (of $N_v$) will be mappings $\bar{\chi}$, where $\chi$ is a $v$-adapted chart of $\mathcal{V}$. Since any chart is in particular a one-to-one mapping, we need to restrict ourselves to $v$-adapted charts $\chi$ such that the associated mapping $\bar{\chi}$ is injective. Thus, we define that a $v$-adapted chart $\chi$ is “nice” iff $\bar{\chi}$ is injective on $D_U$, with $U \subset \mathcal{V}$ the (open) domain of $\chi$.

### 2.2 Straightening-out vs $v$-adapted charts

In the remainder of this section, we investigate whether there exist nice $v$-adapted charts in the neighborhood of any point $X_0 \in \mathcal{V}$. If the vector field $v$ does not vanish, a well-known theorem applies at any point $X_0 \in \mathcal{V}$:
Straightening-out theorem (e.g. [13]). Let $v$ be a vector field of class $C^\infty$ defined on $V$. Suppose that at $X_0 \in V$ we have $v(X_0) \neq 0$. There is a “straightening-out chart” $\chi$ defined on an open neighborhood $U$ of $X_0$, i.e. $\chi$ is such that:

- (i) $\chi(U) = I \times \Omega$, $I = ]-a, +a[$, $a \neq 0$, $\Omega$ open set in $\mathbb{R}^N$.
- (ii) For any $x \in \Omega$, $\chi^{-1}(I \times \{x\})$ is an integral curve of $v$.
- (iii) In $U$, we have $v = \partial_0$, where $(\partial_\mu)$ is the natural basis associated with the chart $\chi$.

However, the direct link with the notion of a $v$-adapted chart works in the wrong direction:

**Proposition 0.** Let $(\chi, U)$ be a $v$-adapted chart. (i) We have $v|_U = f \partial_0$ with $f : U \to \mathbb{R}$ a smooth function. (ii) Given any point $X \in U$ such that $v(X) \neq 0$, one may obtain a straightening-out chart $(\chi', U')$, with $U' \subset U$ being an open neighborhood of $X$, by changing merely the $y^0$ coordinate.

**Proof.** (i) To say that $(\chi, U)$ is $v$-adapted means, according to Definition 1, that for any given $X \in U$, we have for any $Y \in l_X \cap U$: $P_S(\chi(Y)) = P_S(\chi(X))$. In the coordinates $\chi(X) = (x^\mu) \equiv X$, $\chi(Y) = (y^\mu) \equiv Y$ ($\mu = 0, ..., N$), the latter rewrites as

$$y^j = x^j \quad (j = 1, ..., N). \quad (15)$$

On the other hand, remembering the definition of $l_X$ as a (maximal) integral curve of $v$, Eq. (9) and below, let $J$ be the connected component of 0 in $l_X \cap U \equiv \{s \in l_X; C_X(s) \in U\}$: $J$ is an open interval containing 0 and we have $l' = C_X(J) \subset l_X \cap U$. Let us denote this as $l' = \{Y(s); s \in J\}$, with

$$\frac{dy^\mu}{ds} = v^\mu(Y(s)) \quad (\mu = 0, ..., N), \quad Y(0) = X, \quad (16)$$

$v^\mu = v^\mu(Y)$ being the components of $v$ in the chart $\chi$. It follows from (15) and (16) that for $s \in J$ we have $v^j(Y(s)) = 0$ ($j = 1, ..., N$), i.e. $v(Y(s)) = v^0(Y(s)) \partial_0(Y(s))$, thus in particular $v(X) = v^0(X) \partial_0(X) \equiv f(X) \partial_0(X)$. Since this is true at any point $X \in U$, our statement (i) is proved.
(ii) If we leave the coordinates $y^j$ ($j = 1, ... , N$) unchanged: $\chi'(Y) \equiv Y' = (g(Y), (y^j))$ where $Y = (y^0, (y^j)) = \chi(Y)$ ($Y \in U' \subset U$), then the components of $v$ in the new chart $(\chi', U')$ are $v^j = v^j = 0$ ($j = 1, ... , N$) and $v^0(Y') = \frac{\partial y}{\partial y^0} = \frac{\partial y}{\partial y^0} f(Y)$. The latter must be equal to 1 for a straightening-out chart. Since $v|_U = f\partial_b$ and $v(X) \neq 0$, we have $f(Y) \neq 0$ when $Y$ is in some neighborhood $U'' \subset U$ of $X$. Hence, we get $v^0 = 1$ in $U'$ if we take $U' = \chi^{-1}(B)$ with $B = ]x^0 - r, x^0 + r[\times ... \times]x^N - r, x^N + r[ \subset \chi(U'')$ and define $y^0 \equiv g(Y) = \int_{x^0}^{y^0} du/f(u, (y^j))$ for $Y \in B$. Property (i) in the straightening-out theorem is then got by a mere shift $y^0 \mapsto y^0 + \delta$, and its Property (ii) is a straightforward consequence of Properties (i) and (iii). \[\square\]

Conversely, if $(\chi, U)$ is a straightening-out chart, i.e. it fulfills conditions (i) to (iii) of the theorem above, then let $x \in \Omega$ and set $l' \equiv \chi^{-1}(I \times \{x\})$. From (ii), $l'$ is an integral curve of $v$ and, since $l' \subset U$ by construction, we have

$$\forall Y \in l' \cap U, \quad P_S(\chi(Y)) = x. \quad (17)$$

Hence, at first sight, it might seem that $\chi$ is a $v$-adapted chart. However, let $X = \chi^{-1}(s, x) \in l'$ and let $l \equiv l_X \subset N_v$ be the maximal integral curve of $v$ passing at $X$. We have $l' \subset (l \cap U)$ since $l'$ is an integral curve of $v$ that is included in $U$ and that passes at $X$. But nothing guarantees that $l' \supset (l \cap U)$: the intersection $l \cap U$ may contain other arcs, say $l_1' \equiv \chi^{-1}(I \times \{x_1\})$ with $x_1 \in \Omega$ and $x_1 \neq x$. In such a case, the straightening-out chart $\chi$ is not $v$-adapted, since for $Y \in l \cap U$, $P_S(\chi(Y))$ may take different values $x, x_1, ...$

As we already noted, Property (ii) in the straightening-out theorem follows easily from Properties (i) and (iii). More generally, if a chart $(\chi, U)$ satisfies Property (iii), i.e. $v = \partial_b$ in $U$, and if $\chi(U)$ contains a set $I \times \{x\}$, with $x \in \mathbb{R}^N$ and $I$ an open interval, then Property (ii) applies for this $x \in \mathbb{R}^N$. Later on, we will need that the boundary of the open set $U$ be a smooth hypersurface, hence we must consider charts for which (iii) is true, but not (i).

**Proposition 1.** Let $(\chi, U)$ be a chart such that $v = \partial_b$ in $U$. Assume there is an open subset $\Omega \subset \mathbb{R}^N$ such that

$$\chi(U) = \bigcup_{x \in \Omega} I_x \times \{x\}, \quad (18)$$

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where $I_x (x \in \Omega)$ are open intervals.

(i) In order that the chart $(\chi, U)$ be $v$-adapted, it is necessary and sufficient that
\[ \forall X \in U, \ \chi(l_X \cap U) \text{ has the form } I_x \times \{x\} \text{ for some } x \in \Omega. \quad (19) \]

(ii) Moreover, if that is the case, then the $v$-adapted chart $(\chi, U)$ is nice.

**Proof.** (i) If Condition (19) is satisfied, let $l \in D_U$. Thus, $l \cap U \neq \emptyset$, so let $X \in l \cap U$, hence $l = l_X$. Let $x \in \Omega$ be given by (19) for precisely the maximal integral curve $l = l_X$. For any $Y \in l \cap U$, we have thus $\chi(Y) = (s, x)$ for some $s \in I_x$. Hence, we have (13). Therefore, according to Definition 1, $\chi$ is $v$-adapted. Conversely, assume that $(\chi, U)$ is $v$-adapted. Let $X \in U$ and set $l \equiv l_X$. Further, let $x$ be given by (13). That is, any point $Y \in \chi(l \cap U)$ has the form $(s, x)$ for some $s \in \mathbb{R}$. Since moreover $\chi(U)$ has the form (18), we have also $Y = (t, x')$ for some $x' \in \Omega$ and some $t \in I_{x'}$, so $x = x'$ and $s = t \in I_x$. Hence $\chi(l \cap U) \subseteq I_x \times \{x\}$. But also $I_x \times \{x\} \subseteq \chi(l \cap U)$. Indeed, setting $l' \equiv \chi^{-1}(I_x \times \{x\})$, we have $l' \subset (l \cap U)$ because, as noted before the statement of this Proposition 1, $l'$ is an integral curve of $v$ that is included in $U$ and that passes at $X$.

(ii) Assuming that the chart $(\chi, U)$ obeys Condition (19) [and hence, by (i), is a $v$-adapted chart], let us show that the mapping $\bar{\chi}$ defined by (14) is injective. Thus, let $l, l' \in \mathbb{R}^N$, assume that both intersect $U$, and let $x$ and $x'$ be the images of $l$ and $l'$ by $\bar{\chi}$. This means, according to the definition (14), that we have (13), and similarly
\[ \forall Y \in l' \cap U, \ \ P_S(\chi(Y)) = x'. \quad (20) \]

From Condition (19), we get thus:
\[ \chi(l \cap U) = I_x \times \{x\} \quad \text{and} \quad \chi(l' \cap U) = I_{x'} \times \{x'\}. \quad (21) \]

Therefore, if $x = x'$, it is clear that $l = l'$. The proof is complete. $\square$

The condition that the open set $A \equiv \chi(U) \subseteq \mathbb{R}^{N+1}$ have the form (18) is fulfilled, in particular, if $A$ is convex. So it is fulfilled if one restricts the chart $\chi$ to $\chi^{-1}(A)$ with $A$ a convex open subset of $\chi(U_0)$. Unfortunately, what we have in a rather general situation is the following result, which does not ensure the applicability of Proposition 1:
Theorem 0. Suppose that some maximal integral curve \( l \) of the \( C^\infty \) vector field \( v \) is closed in \( V \) and is not reduced to a point. Since \( v \) then does not vanish on \( l \), let \( \chi : U \to I \times \Omega \) be a straightening-out chart in an open neighborhood \( U \) of some point of \( l \). Then we have \( \chi(l \cap U) = I \times E \), where \( E \) is a closed countable subset of \( \Omega \), and any point \( X \in l \cap U \) is “transversally isolated”, i.e. \( x = P_S(\chi(X)) \) is isolated in \( E \).

(Note that the set \( E \) is closed in \( \Omega \), which is an open set in \( \mathbb{R}^N \). So \( E \) is not necessarily closed in \( \mathbb{R}^N \).) Assume that some straightening-out chart \( (\chi, U) \) is such that, for any \( X \in U \), the maximal integral curve \( l_X \) is closed in \( V \). From Point (i) in Proposition 1, we get that this is a \( v \)-adapted chart iff, in addition, for any point \( X \in U \), the countable closed subset \( E_X \) of \( \Omega \), whose existence is ensured by Theorem 0 for the curve \( l_X \), is actually reduced to a point. We can further characterize this desired situation:

Proposition 2. Let \( \chi : U_0 \to I \times \Omega_0 \) be a straightening-out chart for the \( C^\infty \) vector field \( v \). Let \( U \subset U_0 \) be an open set such that \( \chi(U) \) has the form (18). Assume that, for any \( X \in U \), the maximal integral curve \( l_X \) is closed in \( V \).

(i) For any given \( X \in U \), set \( \chi(X) = (s, x) \). The connected component \( \lambda'' \) of \( (s, x) \) in \( \lambda \equiv \chi(l_X \cap U) \) is equal to \( \lambda' \equiv I_x \times \{x\} \).

(ii) In order that \( (\chi, U) \) be a \( v \)-adapted chart, it is necessary and sufficient that, for any \( X \in U \), the intersection \( l_X \cap U \) be a connected set.

(iii) Let \( W \) be an open subset of \( U \) such that, for any \( X \in W \), \( l_X \cap U \) be a connected set. Then the restriction \( (\chi, W) \) is a nice \( v \)-adapted chart.

Proof. (i) Since \( I_x \) is an interval, the set \( \lambda' \equiv I_x \times \{x\} \) is connected. To show that \( \lambda'' = \lambda' \) is always true, we note first that, \( \chi(U) \) having the form (18), \( \chi(X) = (s, x) \) is such that \( s \in I_x \). Hence \( (s, x) \in \lambda' \), and since \( \lambda' \) is connected, we have \( \lambda' \subset \lambda'' \). To prove that in fact \( \lambda' = \lambda'' \), we will show that \( \lambda' \) is open and closed in \( \lambda'' \). We know from Theorem 0 that \( \lambda_0 \equiv \chi(l_X \cap U_0) \) has the form \( \lambda_0 = I \times E \), where \( E \) is a subset of \( \Omega_0 \) having only isolated points. Thus, \( x \) being isolated in \( E \), let \( r > 0 \) be such that \( B \cap E = \{x\} \), where \( B \equiv B(x, r) \) is the open ball of radius \( r \) in \( \mathbb{R}^N \), centered at \( x \). Hence, we have

\[(I_x \times B) \cap \lambda'' \subset (I_x \times B) \cap \lambda_0 = (I_x \times B) \cap (I \times E) = (I_x \cap I) \times (B \cap E) = I_x \times \{x\} \equiv \lambda'. \]

(22)
We have also \( \lambda' \subset (I \times B) \cap \lambda'' \), because \( \lambda' \subset I \times B \) and \( \lambda' \subset \lambda'' \). So \( \lambda' = (I \times B) \cap \lambda'' \) is an open subset of \( \lambda'' \). On the other hand, if a sequence of points of \( \lambda' \) tends towards a limit in \( \lambda'' \), say \((s_n, x) \to (s', y) \in \lambda''\), then \( y = x \) and, since \( \lambda'' \subset \chi(U) \) which is given by (18), we have \((s', x) \in (\mathbb{R} \times \{x\}) \cap \chi(U) = I_x \times \{x\} \equiv \lambda' \), hence the limit \((s', x) \) is in \( \lambda' \), so that \( \lambda' \) is a closed subset of \( \lambda'' \). Being non-empty and an open-and-closed subset of the connected set \( \lambda'' \), \( \lambda' \) is equal to \( \lambda'' \).

(ii) Since \( \chi : U_0 \to I \times \Omega_0 \) is a bicontinuous mapping, it is of course equivalent to say that \( l_X \cap U \) or \( \lambda_X \equiv \chi(l_X \cap U) \) is connected. Therefore, (ii) follows immediately from (i) and from Statement (i) in Proposition 1.

(iii) Let \( X \) be any point in \( W \) and set \( \chi(X) = (s, x) \). Since \( l_X \cap U \) is connected, we have \( \chi(l_X \cap U) = I_x \times \{x\} \) from (i). Thus, \( P_S(\chi(Y)) = x \) is true for any \( Y \in l_X \cap U \), hence a fortiori for any \( Y \in l_X \cap W \). Hence the chart \( (\chi, W) \) is \( v \)-adapted. In a similar way, it is easy to adapt the proof of Statement (ii) in Proposition 1 to conclude that the \( v \)-adapted chart \( (\chi, W) \) is nice. \( \square \)

**Proposition 3.** Let \( \chi : U_0 \to I \times \Omega_0 \) be a straightening-out chart for the \( C^\infty \) vector field \( v \) and assume that, for some \( X \in U_0 \), the maximal integral curve \( l_X \) is closed. Let \( E \) be the closed countable subset of \( \Omega_0 \) given by Theorem 1 for the maximal curve \( l \equiv l_X \), thus \( \chi(l_X \cap U_0) = I \times E \). Set \( x = P_S(\chi(X)) \) and, as ensured by Theorem 1, let \( \Omega \subset \Omega_0 \) be any open neighborhood of \( x \) such that \( E \cap \Omega = \{x\} \). Let \( U \subset U_0 \) be any open set such that \( \chi(U) \) has the form (18) with this set \( \Omega \). Then \( l_X \cap U \) is connected, \( \chi(l_X \cap U) = I_x \times \{x\} \).

**Proof.** Since \( \chi : U_0 \to I \times \Omega_0 \) is a bijection, we have

\[
\begin{align*}
\chi(l_X \cap U) &= \chi((l_X \cap U_0) \cup U) = \chi(l_X \cap U_0) \cap \chi(U) \\
&= (I \times E) \cap \left( \bigcup_{x' \in \Omega} I_{x'} \times \{x'\} \right) = \bigcup_{x' \in \Omega} (I \cap I_{x'}) \times (E \cap \{x'\}) \\
&= I_x \times \{x\}. \quad \square
\end{align*}
\]
2.3 Intersections of straight lines with inverse images under the flow

Recall that the flow of the global vector field $v$ on $V$ is the mapping $F : D \to V$, $(s, X) \mapsto F(s, X) \equiv C_X(s)$, where $D$ is the domain of the flow $F$:

$$D \equiv \bigcup_{X \in V} I_X \times \{X\} \subset \mathbb{R} \times V.$$  \hfill (24)

If $v$ is $C^q$ ($q \geq 1$, possibly $q = \infty$), $D$ is an open set in $\mathbb{R} \times V$, moreover $F$ is $C^q$ on $D$ [12, 13].

**Proposition 4.** Let $U$ be an open subset of $V$. (i) For any $X \in V$, we have

$$l_X \cap U = F_X(I'_{XU}) = F(I'_{XU} \times \{X\}),$$  \hfill (25)

where $F_X \equiv F(., X) = C_X$ is defined on $I_X \subset \mathbb{R}$, and where

$$I'_{XU} \equiv F_X^{-1}(U) = \{s \in I_X; F(s, X) \in U\}.$$  \hfill (26)

(ii) Further, we have for any $X \in V$:

$$I'_{XU} \times \{X\} = (\mathbb{R} \times \{X\}) \cap D_U = (I_X \times \{X\}) \cap D_U,$$  \hfill (27)

where

$$D_U \equiv F^{-1}(U) = \bigcup_{X \in V} I'_{XU} \times \{X\}.$$  \hfill (28)

(iii) Assume that $v$ is $C^\infty$ and that, for some $X \in V$, $l_X$ is closed in $V$ and not reduced to a point. Then (a) $l_X$ is a submanifold of $V$ and the mapping $F_X : I_X \to l_X$ is a local diffeomorphism at any point $s \in I_X$. (b) Assume moreover that $F_X$ is non-periodic. Then it is a (global) diffeomorphism of $I_X$ onto $l_X$. The connected components of $l_X \cap U$ are the images by $F_X$ of the connected components of $I'_{XU} \equiv F_X^{-1}(U) = F_X^{-1}(l_X \cap U)$, which are open intervals of $\mathbb{R}$. In particular, in order that $l_X \cap U$ be connected, it is necessary and sufficient that $I'_{XU}$ be an open interval.

**Proof.** Points (i) and (ii) follow immediately from the definitions. Let us prove Point (iii). (a) Since the maximal integral curve $l_X$ is closed in $V$, this is a submanifold of $V$ [13]. Since $l_X$ is not reduced to a point, the vector field $v$ does not vanish on $l_X$. [If $v(Y) = 0$ for some $Y \in l_X$, then we have
An argument of transversality. Assume that $v$ does not vanish and that each maximal integral curve is non-periodic and is closed in $V$. Given an arbitrary point $X \in V$, let $\chi : U_0 \to I \times \Omega_0$ be a straightening-out chart in the neighborhood of $X$, and let $E$ be the closed countable subset of $\Omega_0$, having only isolated points, such that $\chi(l_X \cap U_0) = I \times E$. As Proposition 3 states: by restricting $\chi$ to an open subset $U \subset U_0$ for which $\chi(U)$ has the form (18) with $\Omega \subset \Omega_0$ any open neighborhood of $x \equiv P_S(\chi(X))$ such that $E \cap \Omega = \{x\}$, we ensure that $l_X \cap U$ is connected. From Point (iii) of Proposition 2, it follows that we will obtain a nice $v$-adapted chart by restricting $\chi$ to an open neighborhood $W \subset U$ of $X$ — if it exists — such that, for any $Y \in W$, $l_Y \cap U$ be connected. To this purpose, we observe that, when the boundary of $U$ is a regular hypersurface, then the same should be true for the boundary of the open set $D_U \equiv F^{-1}(U) \subset \mathbb{R} \times V$: that boundary $\Sigma_U \equiv \text{Fr}(D_U)$ should “normally” be a hypersurface in $\mathbb{R} \times V$. I.e., $\Sigma_U$ should be a submanifold of codimension 1 of the $(N+2)$—dimensional manifold $\mathbb{R} \times V$. Then, “generically”, a straight line $\mathbb{R} \times \{X\} \subset \mathbb{R} \times V$ that intersects $\Sigma_U$ is nowhere tangent to it, i.e. it is transverse to it at each intersection point. Thus, when $Y$ is sufficiently close to $X$, the intersection points should be slightly displaced but should remain in the same number, hence the structure of $I'_Y \times \{Y\} = (\mathbb{R} \times \{Y\}) \cap D_U$ should be the same as for $X$. In particular, if $I'_X \times \{X\}$ is an interval, i.e. (according to Proposition 4) if $l_X \cap U$ is connected, then also $I'_Y \times \{Y\}$ should be an interval, i.e., also $l_Y \cap U$ should be connected.

To make this line of reasoning precise, we first introduce, for a general hypersurface $\Sigma$ in $\mathbb{R} \times V$, the set $S_\Sigma$ of the points $X$ in $V$ for which the straight
line $\mathbb{R} \times \{X\}$ is tangent to $\Sigma$ at one point at least. (Our aim is to use this when $\Sigma$ is the boundary hypersurface $\Sigma_U \equiv \text{Fr}(D_U)$ introduced above.) Note that the tangent space to $\mathbb{R} \times X$ at some point $(s, X)$ is $T_{(s,X)}(\mathbb{R} \times V) \simeq T_s \mathbb{R} \times T_X V \simeq \mathbb{R} \times T_X V$. The tangent space to $\mathbb{R} \times \{X\}$ at $(s, X)$ is $T_{(s,X)}\mathbb{R} \times T_X V \simeq \mathbb{R} \times T_X V$. Thus we define

$$S_\Sigma \equiv \{X \in V; \exists s \in \mathbb{R} : (s, X) \in \Sigma \text{ and } (1, 0_X) \in T_{(s,X)}\Sigma\}. \quad (29)$$

However, we note that $D_U \equiv F^{-1}(U)$ is in general not a bounded domain of $\mathbb{R} \times V$, even if the open set $U \subset V$ is bounded. Therefore, even if $X \in V$ is such that the straight line $\mathbb{R} \times \{X\}$ is not tangent to the boundary hypersurface $\Sigma_U$, i.e. $X \not\in S_{\Sigma_U}$, it may happen that $\mathbb{R} \times \{X\}$ is “tangent to $\Sigma_U$ at infinity”, in which case any line $\mathbb{R} \times \{Y\}$, however close $Y$ can be to $X$, may have “new” intersections with the hypersurface $\Sigma_U$. For a general hypersurface $\Sigma$ in $\mathbb{R} \times V$, in fact for any subset $\Sigma$ of $\mathbb{R} \times V$, we thus introduce:

$$S_{\Sigma_\infty} \equiv \{X \in V; \lim_{r \to \infty} \inf_{|s| \geq r} d(X, \Sigma_s) = 0\}, \quad (30)$$

where, for any subset $B$ of $\mathbb{R} \times V$, we define $B_s$ to be its slice in $V$ at $s \in \mathbb{R}$:

$$B_s \equiv \{X \in V; (s, X) \in B\} \subset V, \quad (31)$$

and where, given any distance $d$ that generates the (metrizable) topology of $V$, one defines for any subset $A$ of $V$ and any point $X \in V$: $d(X, A) \equiv \inf_{Z \in A} d(X, Z)$.

### 2.4 Relevant theorems

**Theorem 1.** Let $\Sigma$ be a hypersurface of $\mathbb{R} \times V$ that is closed in $\mathbb{R} \times V$. Let $K = [\alpha, \beta]$ be a compact interval of $\mathbb{R}$ ($\alpha < \beta$). Suppose that, for some $X_0 \in V$, the intersection $(K \times \{X_0\}) \cap \Sigma$ be a singleton $(s_0, X_0)$ with $\alpha < s_0 < \beta$ and that this intersection be transverse, i.e. $X_0 \not\in S_{\Sigma}$. Then there is a neighborhood $W$ of $X_0$ such that, for any $X \in W$, $(K \times \{X\}) \cap \Sigma$ is a singleton $\Phi(X)$ and this intersection is transverse. Moreover, the function

---

Note for instance, consider a constant vector field $v$ on $V \equiv \mathbb{R}^n$. Then we have simply $F(s, X) = X + sv$. Hence, for any $X_0 \in V$, its inverse image is the unbounded straight line $F^{-1}(X_0) = \{(s, X) \in \mathbb{R} \times V; s \in \mathbb{R}, X = X_0 - sv\}$. From this, one may deduce more general examples by applying a diffeomorphism.
\[ \Phi : \mathcal{W} \to \Sigma \text{ is smooth.} \]

**Proof.** Since \((s_0, X_0) \in \Sigma\), and since \(\Sigma\) is a submanifold of dimension \(N + 1\) of the \((N + 2)\)-dimensional differentiable manifold \(\mathbb{R} \times V\), there is a chart \((\Psi, \mathcal{U})\) on \(\mathbb{R} \times V\), with \((s_0, X_0) \in \mathcal{U}\), for which the function

\[ g : \mathcal{U} \to \mathbb{R}, (s, X) \mapsto g(s, X) \equiv \Psi^{N+2}(s, X) \]  

(32)

[the last component of \(\Psi(s, X)\)] is such that

\[ (s, X) \in \mathcal{U} \cap \Sigma \iff g(s, X) = 0 \]  

(33)

{e.g. [11], §(16.8.3)}. Then, if \((s, X) \in \mathcal{U} \cap \Sigma\), a vector \(\eta = (a, u) \in T(s, X) (\mathbb{R} \times V) \simeq \mathbb{R} \times T_X V\) is in the tangent space \(T(s, X) \Sigma\), iff \(dg(s, X)(\eta) = 0\).

Since the intersection \((K \times \{X_0\}) \cap \Sigma = \{(s_0, X_0)\}\) is transverse, we have

\[ dg(s_0, X_0)(\xi_0) \neq 0, \quad \xi_0 \equiv (1, 0, 0) \]  

(34)

Since \(K\) is a neighborhood of \(s_0\): replacing \(\mathcal{U}\) with a smaller neighborhood of \((s_0, X_0)\) if necessary, we may assume that, if \((s, X) \in \mathcal{U}\), then \(s \in K\). Consider a chart \((\chi, \mathcal{U})\) of \(V\) in a neighborhood \(\mathcal{U}\) of \(X_0\), thus \(\chi(X) = X = (x^\mu) \in \mathbb{R}^{N+1}\) for \(X \in \mathcal{U}\). The mapping \(\Xi : (s, X) \mapsto (s, \chi(X))\) is a local chart on \(\mathbb{R} \times V\) defined in the neighborhood \(\mathbb{R} \times \mathcal{U}\) of \((s_0, X_0)\), while \(\Psi\) is also a local chart of \(\mathbb{R} \times V\), defined in the neighborhood \(\mathcal{U}\) of \((s_0, X_0)\). Let

\[ f(s, X) \equiv g(\Xi^{-1}(s, X)) = g(s, \chi^{-1}(X)) \]  

(35)

be the local expression of \(g\) in the chart \(\Xi\). This function \(f\) is defined and \(C^\infty\) on the open subset \(\mathcal{O} \equiv \Xi((\mathbb{R} \times \mathcal{U}) \cap \mathcal{U})\) of \(\mathbb{R}^{N+2}\). Note that \(\mathcal{O}\) contains \((s_0, X_0)\), where \(X_0 \equiv \chi(X_0)\). With this local expression, we have for any vector \(\eta = (a, u) \in T(s, X) (\mathbb{R} \times V)\):

\[ dg(s, X)(\eta) = \frac{\partial f}{\partial s}(s, X) a + \frac{\partial f}{\partial x^\mu}(s, X) u^\mu, \]  

(36)

where \(X \equiv \chi(X)\), and where \((a, (u^\mu)) (\mu = 0, ..., N)\) are the components of \(\eta\) in the product chart \(\Xi\). From (33), we have

\[ f(s_0, X_0) = 0. \]  

(37)

From (34) and (36) we have

\[ dg(s_0, X_0)(\xi_0) = \frac{\partial f}{\partial s}(s_0, X_0) \neq 0. \]  

(38)
We can thus apply the implicit function theorem: there is an open neighborhood A of $X_0$ and a unique smooth function $\varphi : A \to \mathbb{R}$, such that $\varphi(X_0) = s_0$ and that, for any $X \in A$: 

$$ (\varphi(X), X) \in \mathcal{O}, \quad f(\varphi(X), X) = 0, \quad \frac{\partial f}{\partial s}(\varphi(X), X) \neq 0. \quad (39) $$

We define a smooth mapping $\Phi$ by setting for any $X \in W' \equiv \chi^{-1}(A) \subset V$:

$$ \Phi(X) \equiv (\varphi(\chi(X)), X). \quad (40) $$

Thus, for any $X \in W'$, with $X \equiv \chi(X) \in A$, we have $\Phi(X) = (\varphi(X), \chi^{-1}(X))$, hence $\Phi(X) \in \Xi^{-1}(\varphi(X), X) \in \Xi^{-1}(\mathcal{O}) \subset U$. In particular, $\Phi(X_0) = (s_0, X_0)$. For any $X \in W'$, we have from (33), (39) and (40): $g(\Phi(X)) = 0$, thus $\Phi(X) \in \Sigma \cap U$. As with (34) and (38), (39) means that the intersection $\Phi(X) \in (K \times \{X\}) \cap \Sigma$ is transverse. The definition (40) entails also that $\Pr_2 \circ \Phi = \text{Id}_{W'}$, where $\Pr_2 : \mathbb{R} \times V \to V$, $(s, X) \mapsto X$. Thus, the rank of $\Pr_2 \circ \Phi$ is dim $V = N + 1$, and since the rank of $\Pr_2 \circ \Phi$ is not larger than that of $\Phi$, this latter is also $N + 1 = \dim \Sigma$, i.e. $\Phi$ is a submersion. (All of this is true at any point $X \in W'$.) It follows that $W'' \equiv \Phi(W')$ is an open set in the manifold $\Sigma$ (II, §(16.7.5)), hence is a neighborhood of $(s_0, X_0) = \Phi(X_0)$ in $\Sigma$. We have

$$ \forall (s, X) \in W'', \quad (s, X) = \Phi(X). \quad (41) $$

We claim that there is some neighborhood $W \subset W'$ of $X_0$, such that

$$ \forall X \in W, \quad [s \in K \text{ and } (s, X) \in \Sigma] \implies [(s, X) = \Phi(X)]. \quad (42) $$

Note that, if $W \subset W'$, the reverse implication is also true for any $X \in W$, by the construction of the mapping $\Phi$. Thus, if (42) is true, $W$ is as stated by Theorem 1. We will reason ab absurdo. If it does not exist such a neighborhood $W$, then, for any integer $n > 0$, one can find $s_n \in K$ and $X_n \in W' \cap B(X_0, 1/n)$ such that $(s_n, X_n) \in \Sigma$ and $(s_n, X_n) \neq \Phi(X_n)$. Thus $X_n \to X_0$ and, by extraction in the compact set $K$, we may assume that $s_n$ has a limit $s \in K$, so that $(s_n, X_n) \to (s, X_0)$. If it happened that $s = s_0$, then, since $W''$ is a neighborhood of $(s_0, X_0)$ and $(s_n, X_n) \to (s, X_0)$, we would have $(s_n, X_n) \in W''$ for large enough $n$, hence $(s_n, X_n) = \Phi(X_n)$ from (41), which is a contradiction. Thus $s \neq s_0$. But since $\Sigma$ is closed, we have $(s, X_0) \in \Sigma$, which contradicts the assumption that $(K \times \{X_0\}) \cap \Sigma = \{(s_0, X_0)\}$. This completes the proof of Theorem 1. $\Box$
Theorem 2. Let $\mathcal{U}$ be an open domain in $\mathbb{R} \times V$, whose boundary $\Sigma$ be a hypersurface of $\mathbb{R} \times V$. Assume that, for some $X \in V$, all intersections of the straight line $\mathcal{L} \equiv \mathbb{R} \times \{X\}$ with $\Sigma$ are transverse. Then the boundary of $\mathcal{L} \cap \mathcal{U}$ in $\mathcal{L}$ is $\text{Fr}(\mathcal{L}(\mathcal{L} \cap \mathcal{U})) = \mathcal{L} \cap \Sigma$.

Proof. The boundary of a subset is of course relative to which containing set is considered. Here, the boundary $\Sigma$ of the open set $\mathcal{U} \subset \mathbb{R} \times V$ is relative to the whole manifold $\mathcal{V} \equiv \mathbb{R} \times V$, i.e. $\Sigma \equiv \overline{\mathcal{U}} \cap \mathcal{U}^c$, where the upper bar $\overline{\cdot}$ denotes the adherence in $\mathcal{V}$ and $\mathcal{U}^c$ the complementary set in $\mathcal{V}$. Thus

$$\Sigma = \overline{\mathcal{U}} \setminus \mathcal{U},$$

(43)
since $\mathcal{U}$ is open. The boundary of some subset $\mathcal{B} \subset \mathcal{L}$ in $\mathcal{L}$ (or relative to $\mathcal{L}$) is $\text{Fr}(\mathcal{L})(\mathcal{B}) \equiv \overline{\mathcal{B}}^c \cap \overline{\mathcal{L}} \mathcal{B}^c$, where $\overline{\mathcal{B}}^c$ denotes the adherence of $\mathcal{B}$ in $\mathcal{L}$, and where $\overline{\mathcal{L}} \mathcal{B} \equiv \mathcal{L} \setminus \mathcal{B}$ is the complementary set of $\mathcal{B}$ in $\mathcal{L}$. However, here $\mathcal{L} \equiv \mathbb{R} \times \{X\}$ is closed in $\mathcal{V} = \mathbb{R} \times V$, hence we have $\overline{\mathcal{B}}^c = \mathcal{B}$, the adherence in the whole set $\mathcal{V}$. Thus

$$\text{Fr}(\mathcal{L}(\mathcal{L} \cap \mathcal{U})) \equiv \overline{\mathcal{L} \cap \mathcal{U}}^c \cap \overline{\overline{\mathcal{L} \cap \mathcal{U}}} = \overline{\mathcal{L} \cap \mathcal{U}} \cap \overline{\overline{\mathcal{L} \cap \mathcal{U}}}.$$  

(44)

Again because $\mathcal{L}$ is closed in $\mathcal{V}$, we have $\overline{\mathcal{L} \cap \mathcal{U}} \subset \mathcal{L} \cap \overline{\mathcal{U}}$. We will show that we have exactly

$$\overline{\mathcal{L} \cap \mathcal{U}} = \mathcal{L} \cap \overline{\mathcal{U}}.$$  

(45)

We shall in fact show that

$$\mathcal{L} \cap \Sigma \subset \overline{\mathcal{L} \cap \mathcal{U}}.$$  

(46)

Since (43) implies that $(\mathcal{L} \cap \mathcal{U}) \setminus \overline{\mathcal{L} \cap \mathcal{U}} \subset (\mathcal{L} \cap \mathcal{U}) \setminus (\mathcal{L} \cap \mathcal{U}) = \mathcal{L} \cap \Sigma$, this will prove that $\mathcal{L} \cap \overline{\mathcal{U}} \subset \mathcal{L} \cap \overline{\mathcal{U}}$, whence (45).

To prove (46), consider an arbitrary point $p_0 = (s_0, X) \in \mathcal{L} \cap \Sigma$. As in the proof of [Theorem 1] let $(\Psi, \mathcal{W})$ be a chart on $\mathcal{V}$, with $p_0 \in \mathcal{W}$, such that $p \in \mathcal{W} \cap \Sigma \iff g(p) = 0$, where $g = \Psi^n$, with $n = \text{dim}(\mathcal{V}) = N + 2$. Since we assume that the intersection $p_0 \in \mathcal{L} \cap \Sigma$ is transverse, we have again (34). Hence, there is an interval $J = \)s_0 - r, s_0 + r[\text{ in which } s = s_0\text{ is the only zero of the smooth function } \varphi(s) \equiv g(s, X).$ (That is, $p_0$ is the only intersection of $J \times \{X\}$ with $\Sigma$.) Thus, we may assume that, say, $g(s, X) > 0 \text{ for } s_0 - r < s < s_0$, so that

$$g(s, X) < 0 \text{ for } s_0 - r < s < s_0.$$  

(47)
Replacing $\Psi(p)$ by $\Psi(p) - \Psi(p_0)$, we may assume that $\Psi(p_0) = 0_{\mathbb{R}^n}$. There is some open ball $W = B(0, r) \subset \Psi(W)$. Replacing $W$ by $\Psi^{-1}(W)$, we have that $W_+ \equiv \{ p \in W; x^n \equiv \Psi^n(p) > 0 \}$ is just $W_+ = \Psi^{-1}(W_+)$, where $W_+ \equiv \{ P \in W; x^n > 0 \}$, hence $W_+$ is non-empty and connected as is $W_+$. The same is true for $W_- \equiv \{ p \in W; \Psi(x^n(p)) < 0 \}$. Since $p_0 \in \Sigma \subset \mathcal{U}$, and since $W$ is a neighborhood of $p_0$, we have $U \cap W \neq \emptyset$, so let $p_0' \in U \cap W$. Because $W$ is the disjointed union $W = W_+ \cup W_- \cup (W \cap \Sigma)$, and because $p_0' \in U$ cannot belong to $\Sigma = \mathcal{U} \setminus U$, we have either $p_0' \in W_+$ or $p_0' \in W_-$. Let us assume that, for instance, $p_0' \in W_-$, so that $U \cap W_- \neq \emptyset$. It follows that $W_- \subset U \cap W$, for otherwise the connected set $W_-$ would intersect both $U$ and $\mathcal{U}$, hence would intersect the boundary $\Sigma$ — which is impossible, since we have $x^n = 0$ in $\Sigma$, not $x^n < 0$. Therefore, we get from (47) that $]s_0 - r, s_0[\times \{ X \} \subset U \cap W$, so that $p_0 \equiv (s_0, X) \in \mathcal{U} \cap \mathcal{U}$. This proves (49).

Combining (44) and (45) gives us:

\[
Fr_{(L)}(L \cap U) = L \cap \mathcal{U} \cap \overline{C_L U}.
\]  

(48)

But $C_L U = L \cap \mathcal{U}$ is closed in $\mathcal{V}$, for $L$ is closed and $U$ is open. Hence $L \cap \overline{C_L U} = L \cap \mathcal{U} = L \cap \mathcal{U} = L \cap \mathcal{U}$. Therefore, (48) rewrites as

\[
Fr_{(L)}(L \cap U) = L \cap \mathcal{U} \cap \overline{U} \equiv L \cap \Sigma,
\]  

(49)

which proves Theorem 2. □

**Remark 2.1.** With small and straightforward modifications, the foregoing proof shows the result in the much more general case that $\mathbb{R} \times \mathcal{V}$ (with $\mathcal{V}$ a differentiable manifold) is replaced by a general differentiable manifold $\mathcal{V}$ and $L$ is the range $L = C(J)$, assumed closed, of a smooth curve $C : J \rightarrow \mathcal{V}$ (with $J$ an interval of $\mathbb{R}$), such that all intersections $p \in L \cap \mathcal{V}$ are transverse. It is easy to see that the latter assumption is necessary.

**Remark 2.2.** In the course of the proof, the following intuitively obvious result was proved: Suppose that the line $L = C(J)$ intersects transversely at point $p = C(s_0)$ the boundary $\Sigma$, assumed to be a regular hypersurface, of some open domain $U \subset \mathcal{V}$. Then, among the two parts of $L$: $s < s_0$ and $s > s_0$, at least one is such that, when $s$ is close enough to $s_0$, we have $C(s) \in U$. 

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**Theorem 3.** Let $U$ be an open subset of $V$. (i) Assume that $F^{-1}(U) \subset \mathcal{D}$, with $\mathcal{D}$ the domain of the flow $F$. (This is true, in particular, if the flow is complete, i.e. $\mathcal{D} = \mathcal{V}$.) Then we have:

$$\text{Fr}(F^{-1}(U)) = F^{-1}(\text{Fr}(U)). \quad (50)$$

(ii) If $\text{Fr}(U)$ is a hypersurface of $V$, then $F^{-1}(\text{Fr}(U))$ is a hypersurface of $\mathcal{V} \equiv \mathbb{R} \times \mathcal{V}$.

**Proof of Point (i).** First, consider the more general context that $\mathcal{V}$ and $V$ are merely metric spaces and $F : \mathcal{D} \to \mathcal{V}$ is merely a continuous map, with $\mathcal{D}$ an open subset of $\mathcal{V}$. Then, if $F^{-1}(U) \subset \mathcal{D}$, we have

$$\text{Fr}(F^{-1}(U)) \subset F^{-1}(\text{Fr}(U)). \quad (51)$$

Indeed, since $U$ is open in $V$, we have $\text{Fr}(U) = \overline{U}^V \setminus U$, as with Eq. (13). Since $F$ is continuous on $\mathcal{D}$, $F^{-1}(U)$ is open in $\mathcal{D}$; hence, $\mathcal{D}$ being an open subset of $\mathcal{V}$, $F^{-1}(U)$ is open in $\mathcal{V}$. Therefore, we have similarly $\text{Fr}(F^{-1}(U)) = F^{-1}(\text{Fr}(U)) \setminus F^{-1}(U)$. Again because $F$ is continuous on $\mathcal{D}$, we have $F^{-1}(U)^\mathcal{D} \subset F^{-1}(\text{Fr}(U)) \setminus F^{-1}(U)$. But $F^{-1}(U)^\mathcal{D} = F^{-1}(U)$ since $F^{-1}(U) \subset \mathcal{D}$. Thus $\text{Fr}(F^{-1}(U)) \subset F^{-1}(\text{Fr}(U)) \setminus F^{-1}(U)$, whence (51).

To prove the reverse inclusion, we consider any $p = (s, X) \in F^{-1}(\text{Fr}(U))$, thus $Y \equiv F(s, X) \in \text{Fr}(U)$, and we will show that $p \in \text{Fr}(F^{-1}(U))$. There exists an open neighborhood $\mathcal{U}$ of $p$, having the form $\mathcal{U} = J \times W$, with $J$ an open interval containing $s$ and 0, and with $W$ an open neighborhood of $X$ in $V$, such that $\mathcal{U} \subset \mathcal{D}$ {Ref. [12], §18.2.5}). For all $t \in J$, $F_t \equiv F(t, \cdot)$ is a homeomorphism of $W$ onto $W_t \equiv F_t(W)$, moreover $F_{-t}$ is defined over $W_t$ and is the inverse homeomorphism of $F_t$ [these two points result from (10)] {Ref. [12], §18.2.8}). (We note for the proof of Point (ii) that $F_t$, as well as $F_{-t}$, is of class $\mathcal{C}^q$ if the vector field $v$ is itself $\mathcal{C}^q$ [12].) Now we consider any neighborhood $\mathcal{U}'$ of $p$ and we show that it intersects both $F^{-1}(U)$ and $\mathcal{C}F^{-1}(U)$. We may assume that $\mathcal{U}' \subset \mathcal{U}$ and has the form $\mathcal{U}' = J' \times W'$, with $J' \subset J$ an open interval containing $s$, and with $W' \subset W$ an open neighborhood of $X$ in $V$. Thus $F_s(W')$ is an open neighborhood of $Y = F_s(X)$. Since $Y \in \text{Fr}(U)$, both $F_s(W') \cap \text{Fr}(U)$ and $F_s(W') \cap \mathcal{C}U$ are non-empty, so let $Y' \in F_s(W') \cap U$ and $Y'' \in F_s(W') \cap \mathcal{C}U$. Therefore, $X' \equiv F_{-s}(Y') \in W'$. Thus, $F(s, X') = Y' \in U$, and also $(s, X') \in J' \times W'$, so that $(s, X') \in \mathcal{U}' \cap \mathcal{C}F^{-1}(U)$. In just the
same way, with \( X'' = F_{-s}(Y'') \), we get that \((s, X'') \in (J' \times W') \cap F^{-1}(\mathcal{U})\). Since \( F^{-1}(\mathcal{U}) = \mathcal{P}F^{-1}(U) \subset \mathcal{C}F^{-1}(U) \equiv \mathcal{V} \setminus F^{-1}(U) \), we have shown that any neighborhood \( U' \) of \( p \) intersects both \( F^{-1}(U) \) and \( \mathcal{C}F^{-1}(U) \), thus \( p \in \text{Fr}(F^{-1}(U)) \). Therefore, we have indeed \( F^{-1}(\text{Fr}(U)) \subset \text{Fr}(F^{-1}(U)) \). Together with (51), this proves (50).

**Proof of Point (ii).** Let \( p = (s, X) \in F^{-1}(\text{Fr}(U)) \subset \mathcal{V} \). Define \( J \) (with \( s \in J \)), \( W \subset \mathcal{V} \) (with \( X \in W \)), \( \mathcal{U} = J \times W \), and \( F_t \) \((t \in J)\) just as at the beginning of the foregoing paragraph. Since \( S \equiv \text{Fr}(U) \) is assumed to be a hypersurface of \( \mathcal{V} \), and since by hypothesis \( Y \equiv F(s, X) \in S \), let \( \chi : Y' \mapsto Y \equiv (y_1, ..., y_{n-1}) \) be a chart of \( \mathcal{V} \) (with \( n-1 = \dim(\mathcal{V}) - 1 = \dim(\mathcal{V}) \)), defined in the neighborhood of \( Y \), and such that

\[
Y' \in S \cap \text{Dom}(\chi) \iff y^{n-1} \equiv \chi^{n-1}(Y') = 0. \tag{52}
\]

We may assume that \( \text{Dom}(\chi) \), the domain of \( \chi \), is just \( W_s \equiv F_s(W) \). Then the mapping

\[
\Xi : \mathcal{U} = J \times W \rightarrow \mathbb{R}^n, \quad (t, X') \mapsto (t, \chi(F_s(X'))) = (t, Y) \tag{53}
\]

is a chart of \( \mathcal{V} \) in the neighborhood of \( p \). Consider the mapping

\[
\Psi : \mathcal{U}' \equiv F^{-1}(W_s) \rightarrow \mathbb{R}^n, \quad (t, X') \mapsto (t, \chi(F(t, X'))). \tag{54}
\]

(Note that \( \mathcal{U}' \) is an open neighborhood of \( p \).) We have (cf. Eq. (10)):

\[
F(t, X') = F(s + u, X') = F(u, F(s, X')) = F(u, F_s(X')), \quad u \equiv t - s. \tag{55}
\]

Hence, the local expression of \( \Psi \) in the chart \( \Xi \) is:

\[
G(t, Y) \equiv \Psi(\Xi^{-1}(t, Y)) = (t, Z(t, Y)) \tag{56}
\]

with

\[
Z(t, Y) = (z^1, ..., z^{n-1}) \equiv \chi(F(u, \chi^{-1}(Y))). \tag{57}
\]

Let \( v = (v^1, ..., v^{n-1}) = \chi(Y) \) be the local expression of \( v \) in the chart \( \chi \). Thus \( Z \) is the value at \( u \equiv t - s \) of the solution of \( \frac{dZ'}{dw} = v(Z'(w)), \quad Z'(w = 0) = Y \). Hence we have, uniformly w.r.t. \( (t, Y) \) in some neighborhood of \((s, Y_0) \equiv \Xi(p)\):

\[
Z(t, Y) = Y + (t - s)v(Y) + O(u^2). \tag{58}
\]
The Jacobian matrix of \( G(t, Y) = (t, Z(t, Y)) \) at point \((s, Y_0)\) is therefore:

\[
J = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
v^1(Y_0) & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
v^{n-1}(Y_0) & 0 & \ldots & 1
\end{pmatrix},
\]

(59)
a triangular matrix with 1 on the diagonal, so \( \det J = 1 \). It follows that \( \Psi \) also is a chart of \( V \) in some neighborhood \( U'' \subset U \cap U' \) of \( p \). When \((t, X') \in U''\), we have from (52) and (54):

\[
(t, X') \in F^{-1}(S) \iff z^{n-1} = \Psi^n(t, X') = \chi^{n-1}(F(t, X')) = 0. 
\]

(60)

Hence, \( F^{-1}(S) \) is a hypersurface of \( V \). The proof of Theorem 3 is complete. □

**Remark 3.1.** From (54) and (56), we have also \( Z(t, Y) = \chi(F(\Xi^{-1}(t, Y))) \), thus \( Z(t, Y) \) is the local expression of \( F \) in the charts \( \Xi \) on \( V \) and \( \chi \) on \( V \). Equation (59) shows also that the Jacobian matrix of \( Z \) at \((s, Y_0)\) \( \equiv \Xi(p) \) has rank \( n - 1 = \dim V \). [Relation (52), hence the fact that \( p \in F^{-1}(S) \), are not used to get this; hence this is true at any point \( p \in D = \text{Dom}(F) \).] Thus, \( F \) is a submersion. Hence, it is transversal to any submanifold of \( V \). Point (ii) of Theorem 3 follows also from this [16].

**Proposition 5.** Let \( \Sigma \) be a subset of \( \mathbb{R} \times V \). If \( X \in V \setminus S_{\Sigma, \infty} \) [cf. (50)], there is a neighborhood \( W \) of \( X \) and a real \( R > 0 \) such that

\[
(Y \in W \text{ and } |s| \geq R) \Rightarrow (s, Y) \not\in \Sigma. 
\]

(61)

**Proof.** Clearly, \( f(X) = \lim_{r \to \infty} \inf_{|s| \geq r} d(X, (\Sigma)_s) \) is well defined for any \( X \in V \) and verifies \( 0 \leq f(X) \leq +\infty \). Hence, if \( X \not\in S_{\Sigma, \infty} \) it means that \( f(X) > 0 \), or equivalently that there exists \( R > 0 \) and \( \delta > 0 \) such that

\[
|s| \geq R \Rightarrow d(X, \Sigma_s) \geq \delta,
\]

(62)
hence \( d(X, Z) \geq \delta \) is true for any \( Z \in \Sigma_s \) if \(|s| \geq R \). We deduce from this that, if \( Y \in V \) verifies \( d(X, Y) < \delta/2 \), and if \(|s| \geq R \), we have \( d(Y, Z) > \delta/2 \) for any \( Z \in \Sigma_s \), hence \( d(Y, \Sigma_s) \geq \delta/2 \). (Thus, \( V \setminus S_{\Sigma, \infty} \) is open in \( V \), in other words \( S_{\Sigma, \infty} \) is closed.) In particular, if \( d(X, Y) < \delta/2 \) and \(|s| \geq R \), then \( Y \not\in \Sigma_s \), i.e. \((s, Y) \not\in \Sigma \).

□

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Theorem 4. Let $U$ be an open subset of $V$ and set $\mathcal{D}_U \equiv F^{-1}(U)$ and $\Sigma_U \equiv \text{Fr}(\mathcal{D}_U)$. Assume that $\Sigma_U$ is a hypersurface of $V \equiv \mathbb{R} \times V$ which is closed in $V$, that for some $X \in U$, the open set $I_{XU}$ is a bounded interval, and that $X \not\in (S_{\Sigma_U} \cup S_{\Sigma_U} \infty)$.

(i) There is a neighborhood $W$ of $X$, $W \subset U$, such that for $Y \in W$, also $I_{YU}$ is a bounded interval. (ii) If $v$ does not vanish and all maximal integral curves are closed in $V$ and non-periodic, then $I_{V \cup U}$ is connected for any $Y \in W$. (iii) If moreover $\chi(U)$ has the form (18) and there is a straightening-out chart $(\chi, U_0)$ with $U_0 \supset U$, then the restriction of $\chi$ to $W$ is a nice $v$-adapted chart.

Proof. (i) Since $X \not\in S_{\Sigma_U} \infty$, by Proposition 4, there is a neighborhood $W_0$ of $X$ and a real $R > 0$ such that

$$Y \in W_0 \quad |s| \geq R \Rightarrow (s, Y) \not\in \Sigma_U.$$

(63)

Since $I_{XU}$ is assumed to be a bounded interval (and $I_{XU} \not= \emptyset$ since $0 \in I_{XU}$), let $I_{XU} = [a, b]$, with $a, b \in \mathbb{R}$, $a < b$. In (63) we may assume that $-R < a < b < R$. By Proposition 4, we have

$$I_{XU} \times \{X\} \equiv [a, b] \times \{X\} = (\mathbb{R} \times \{X\}) \cap \mathcal{D}_U.$$

(64)

Since by assumption $X \not\in S_{\Sigma_U}$, we get by Theorem 2 (setting $\mathcal{L} \equiv \mathbb{R} \times \{X\}$):

$$\{(a, X), (b, X)\} = \text{Fr}(\mathcal{L}) \cap \mathcal{L} \cap \Sigma_U.$$

(65)

Thus, considering the compact intervals $K_1 \equiv [-R, \frac{a+b}{2}]$ and $K_2 \equiv [\frac{a+b}{2}, R]$, we have

$$(K_1 \times \{X\}) \cap \Sigma_U = \{(a, X)\}, \quad (K_2 \times \{X\}) \cap \Sigma_U = \{(b, X)\}.$$

(66)

Therefore, by Theorem 1, there are neighborhoods $W_1$ and $W_2$ of $X$, and smooth functions $\Phi_j : W_j \to \Sigma_U$ ($j = 1, 2$), such that:

$$Y \in W_j \Rightarrow (K_j \times \{Y\}) \cap \Sigma_U = \{\Phi_j(Y)\} \quad (j = 1, 2),$$

(67)

with transverse intersection. We have $\Phi_j(Y) = (\varphi_j(Y), Y)$, with $\varphi_j : W_j \to \mathbb{R}$ a smooth function. From (66) and (67), we get: $\varphi_1(X) = a$ and $\varphi_2(X) = b$. Thus $\varphi_1(X) = a \neq b = \varphi_2(X)$; hence, by considering a small enough neighborhood $W$ of $X$, with $W \subset W_0 \cap W_1 \cap W_2$, we get $\varphi_1(Y) \neq \varphi_2(Y)$ for $Y \in W$. With (63) and (67), this implies that

$$Y \in W \Rightarrow (\mathbb{R} \times \{Y\}) \cap \Sigma_U = \{\Phi_1(Y), \Phi_2(Y)\}.$$

(68)
Another application of Theorem 2 proves that
\[ Fr(\mathbb{R} \times \{Y\}) \cap D_U = (\mathbb{R} \times \{Y\}) \cap \Sigma_U. \] (69)
Together with (68), this implies that \( I'_{Y \cap U} \), the open subset of \( \mathbb{R} \) such that \( I'_{Y \cap U} \cap \{Y\} = (\mathbb{R} \times \{Y\}) \cap \Sigma_U \), has boundary \( \{\varphi_1(Y), \varphi_2(Y)\} \). Therefore, \( I'_{Y \cap U} = \varphi_1(Y), \varphi_2(Y) \).

(ii) This follows from Point (iii) in Proposition 4.
(iii) This follows from Point (iii) in Proposition 2.

\[ \square \]

2.5 Adapted charts and “normal” vector fields

With Theorem 4, we formalized our transversality argument in Subsect. 2.3 to investigate the problem of the existence, in the neighborhood of any point \( X \in V \), of a nice \( v \)-adapted chart. Assuming that \( v \) does not vanish and that all maximal integral curves are closed in \( V \) and non-periodic, let us check if Theorem 4 applies. Due to Proposition 4, \( I_{X \cap U} = F^{-1}(l_X \cap U) \) is an interval iff \( l_X \cap U \) is connected, and \( I_{X \cap U} \) is bounded if \( U \) is relatively compact. As shown by Proposition 3, the assumption that \( l_X \cap U \) is connected may be fulfilled by starting from a straightening-out chart \( \chi : U_0 \to I \times \Omega_0 \) in the neighborhood of the arbitrary point \( X \in V \) and by restricting \( \chi \) to an open subset \( U \subset U_0 \) such that \( \chi(U) \) has the form (18) with \( \Omega \subset \Omega_0 \) a small enough open neighborhood of \( x = P_S(\chi(X)) \).

As shown by Theorem 3, the assumption that \( \Sigma_U \) is a hypersurface of \( V \) that is closed in \( V \) is fulfilled, in particular, if the boundary of the open set \( U \subset V \) is itself a hypersurface of \( V \) that is closed in \( V \), and if moreover \( F^{-1}(U) \subset D \). The latter inclusion is true, in particular, if \( D = V \), i.e., if the vector field \( v \) is complete (in other words, if every maximal integral curve of \( v \) is defined over the whole real line). Actually, this does not restrict in any way the set of the maximum integral curves, \( N_v \equiv \{l_X; X \in V\} \) (the “congruence of world lines”, in the physical context with \( N = 3 \)). Indeed, there always exists a smooth function \( \lambda : V \to \mathbb{R}_+ \), such that the vector field \( \lambda v \) is complete, moreover the mappings \( C_X \) corresponding to the maximal integral curves of \( \lambda v \) are mere reparameterizations of those of \( v \), so that the curves \( l_X \) themselves are unchanged [17]. Thus, the assumption that \( \Sigma_U \) is a hypersurface of \( V \) that is closed in \( V \) is not a restrictive one.
The assumption “$X \notin (S_{\Sigma U} \cup S_{\Sigma U \infty})$” means that the straight line $\mathbb{R} \times \{X\}$ is not tangent to the hypersurface $\Sigma_U$, and is not “tangent to it at infinity”. For a given hypersurface $\Sigma_U$, the points thus excluded form a kind of apparent contour (of that hypersurface $\Sigma_U$), having “normally” measure zero in $V$, hence this is true for a “generic” point $X$. However, here the hypersurface $\Sigma_U = F^{-1}(Fr(U))$ of $\mathbb{R} \times V$ depends on the selected neighborhood $U$ of the given point $X \in V$. There is much freedom in the choice of this neighborhood, since it is merely required to have a regular boundary and have the form (13) with $\Omega$ a small enough open neighborhood of $x \equiv P_S(\chi(X))$. If it turns out that $X \in (S_{\Sigma U} \cup S_{\Sigma U \infty})$ for some $U$ satisfying these requirements, then a slightly deformed neighborhood $U'$ does also satisfy them, but the boundary $\Sigma_{U'}$ is also deformed w.r.t. $\Sigma_U$. Hence, it seems plausible that, due to this freedom, there always exists $U$ satisfying these requirements and such that $X \notin (S_{\Sigma U} \cup S_{\Sigma U \infty})$ — unless $v$ has some “pathology” that we were not able to describe in a more explicit way.

Thus, for any point $X \in V$, the assumptions of Theorem 4 should be fulfilled in a suitable neighborhood $U$ of $X$ if the vector field $v$ does not vanish, has all maximal integral curves closed in $V$ and non-periodic, and does not suffer from the “pathology” alluded to. Therefore, we set the following definition, the word “normal” being justified by the foregoing discussion.

**Definition 2.** A non-vanishing $C^\infty$ vector field $v$ is called “normal” iff all maximal integral curves are closed in $V$ and, moreover, any point $X \in V$ has nested open neighborhoods $W \subset U \subset U_0$ such that: (i) There is a straightening-out chart $(\chi, U_0)$. (ii) $\chi(U)$ has the form (13). (iii) For any maximal integral curve $l$ of $v$ intersecting $W$, the line $l \cap U$ is connected.

The following result shows that this concept is relevant. Note that we do not need to assume that the maximal integral curves are non-periodic.

**Theorem 5.** Let $v$ be a non-vanishing global vector field, such that all maximal integral curves are closed in $V$. (i) In order that, for any point $X \in V$, there exist a nice $v$-adapted chart whose domain be an open neighborhood of $X$, it is necessary and sufficient that $v$ be normal. (ii) Also, in order that $v$ be normal, it is necessary and sufficient that any point $X \in V$ have an open neighborhood $W$ on which there is a straightening-out chart $(\chi, W)$, and such
that, for any maximal integral curve $l$ of $v$, the line $l \cap W$ is connected.

**Proof.** (i) The sufficiency is an immediate consequence of Point (iii) in Proposition 2. Conversely, if for any $X \in V$ there is a nice $v$-adapted chart $(\chi_1, U_1)$ with $X \in U_1$, then by Point (ii) of Proposition 0 we get a chart $(\chi, W)$, with an open set $W \subset U_1$ and $X \in W$, which (a) differs from $\chi_1$ merely by the time coordinate $y^0$, and (b) is a straightening-out chart. From (a), $(\chi, W)$ is also a $v$-adapted chart. Therefore, by Point (ii) of Proposition 2: for any $Y \in W$, the intersection $l_Y \cap W$ is a connected set. This implies that for any maximal integral curve $l$ of $v$, the line $l \cap W$ is a connected (possibly empty) set. By this and (b) above, and since $X \in V$ is arbitrary, the vector field $v$ is normal. (This is the case $W = U = U_0$ in the definition.)

(ii) For the reason just invoked, the condition is sufficient in order that $v$ be normal. Conversely, if $v$ is normal, consider any $X \in V$. We know that there is a nice $v$-adapted chart $(\chi_1, U_1)$ with $X \in U_1$, and the proof of the necessity at Point (i) shows that from it we deduce a a straightening-out chart $(\chi, W)$, with $X \in W$, and such that for any maximal integral curve $l$ of $v$, the line $l \cap W$ is a connected set. □

**Examples.** (i) Take $V = \mathbb{R}^{N+1}$ and consider any constant vector field $v(X) = v = \text{Constant} \neq 0$. The maximal integral curve at $X \in V$ is $l_X = \{Y = X + tv; t \in \mathbb{R}\}$. To define a straightening-out chart explicitly, take $u_1, \ldots, u_N$ such that the vectors $u_0 \equiv v, u_1, \ldots, u_N$ form a basis of $\mathbb{R}^{N+1}$, and define an invertible linear transformation $L$ of $\mathbb{R}^{N+1}$ by $L(x^\mu u_\mu) = x^\mu e_\mu$, where $(e_\mu) (\mu = 0, \ldots, N)$ is the canonical basis of $\mathbb{R}^{N+1}$. Now consider any open set of the form $U = L^{-1}(I \times \Omega)$ with $I = [-a, +a]$ and $\Omega$ an open subset of $\mathbb{R}^N$: we have explicitly $U = \{sv + x^j u_j; s \in I, x \equiv (x^j) \in \Omega\}$. The restriction $\chi$ of $L$ to $U$ defines a straightening-out chart, because $L(v) = e_0$ means that $v = \partial_0$ in that chart. Moreover, given $X = sv + x^j u_j \in U$ (thus $s \in I, x \equiv (x^j) \in \Omega$), a point $Y = X + tv$ of $l_X$ belongs to $U$, iff $s + t \in I$, so $l_X \cap U$ is connected. Hence, a constant vector field is normal. And indeed, we have for $Y \in l_X \cap U$: $\chi(Y) = L(X + tv) = \chi(X) + te_0$, hence $\chi$ is $v$-adapted. Moreover, let $l \in D_U$, where $D_U$ is defined in Eq. (12). Thus $l = l_X$, where $X = sv + x^j u_j \in U$, i.e. $s \in I, x \equiv (x^j) \in \Omega$. Then we have from the
definition (14): \( \tilde{\chi}(l) = x = L(x_0) \) with \( x_0 \equiv x^j u_j \). Hence, the mapping \( \tilde{\chi} : \mathbb{D}_U \to \mathbb{R}^N \), \( l = l_x = l_{sv+x_0} \to x = L(x_0) \) is injective, i.e., the \( v \)-adapted chart \( \chi \) is nice.

(ii) If \( \phi : V \to V' \) is a diffeomorphism, set \( v' \equiv \phi^* v : v'(X') \equiv D\phi_{\phi^{-1}(X')} (v(\phi^{-1}(X'))) \). The maximal integral curves of \( v' \), as well as the associate flow \( F' \), are just the images of their counterparts for \( v: F'(s, X') \equiv C'_{X'}(s) = \phi(C_{\phi^{-1}(X')}(s)) = \phi(F(s, \phi^{-1}(X'))) \). Moreover, if \( (\chi, U_0) \) is a straightening-out chart in the neighborhood of \( X \in V \), then so is \( (\chi \circ \phi^{-1}, \phi(U_0)) \) in the neighborhood of \( X' \equiv \phi(X) \in V' \). Therefore, if \( v \) is normal, so is \( v' \).

(iii) If we have a normal vector field \( v \) on some differentiable manifold \( V \), having non-periodic orbits, and if \( U \) is any open subset of \( V \), let us show that the restriction \( v' \equiv v|_U \) is a normal vector field on the differentiable manifold \( U \). Given any \( X \in U \), it is easy to check from the definitions that the maximal open interval \( I'_X \) defining the orbit (maximal integral curve) \( l'_X \) of \( v' \) at \( X \) is the connected component of 0 in \( I'_X \), the latter being defined in Eq. (26). (This result is true for any vector field.) It follows by Point (iii) in Proposition 4 that \( l'_X \) is the connected component of \( X \) in \( I_X \cap U \), where \( l_X \) is the orbit of \( v \) in \( V \) at \( X \). Therefore, since the orbits of \( v \) are closed subsets of \( V \), the orbits of \( v' \) are closed subsets of \( U \). If \( X \in U \), there exists by Point (ii) of Theorem 5 a straightening-out chart \( (\chi, W) \) of \( (V, v) \), with \( X \in W \) and \( \chi(W) = I \times \Omega \), such that for any \( Y \in W \), \( l_Y \cap W \) is connected. Let \( B = [x^0 - r, x^0 + r \times \ldots \times]x^N - r, x^N + r \times [ \) be a ball centered at \( \chi(X) = (x^0) \) and such that \( B \subset \chi(U) \), and set \( W' \equiv \chi^{-1}(B) \). Then the restriction \( \chi' \equiv \chi|_{W'} \) is a straightening-out chart for \( v' \) (up to a shift in \( y^0 \)). For \( Y \in W' \), set \( y \equiv P_S(\chi(Y)) \). Since \( l_Y \cap W \) is connected, we have \( l_Y \cap W = \chi^{-1}(I \times \{ y \}) \) by Point (i) of Proposition 2. Hence \( l_Y \cap W' = \chi^{-1}(I \times \{ y \}) \), thus a connected set \( \subset l_Y \cap U \). Since \( l'_Y \) is the connected component of \( Y \) in \( l_Y \), we have therefore \( l'_Y \cap W' = l_Y \cap W' = \chi^{-1}(I \times \{ y \}) \). The conclusion follows by Point (ii) of Theorem 5.

(iv) By combining the three former examples, we get that, if a manifold \( V \) is diffeomorphic to an open subset \( \Gamma \) of \( \mathbb{R}^{N+1} \) and \( \phi : \Gamma \to V \) is any diffeomorphism, then for any constant vector field \( v \neq 0 \) on \( \Gamma \), its pushforward vector field by \( \phi, v = \phi^* v \), is a normal vector field on \( V \). In the application to physics (for which \( N = 3 \), as far as we know), this describes already a wide
variety of spacetimes and vector fields, together with the associated reference fluids. Those are deformable in a very general way with respect to each other, by changing \(\phi\), i.e. by transforming the integral curves by any diffeomorphism of \(V\). Of course, we expect that much more general normal vector fields do exist, due to the discussion at the beginning of this subsection.

3 The set of orbits of \(v\) as a differentiable manifold

The set \(N_v\) of the maximal integral curves of \(v\) has been defined in Eq. (11). In this section, we will show that, when \(v\) is a normal vector field on the differentiable manifold \(V\), the set \(N_v\) can be endowed with a canonical structure of differentiable manifold.

Proposition 6. Let \(v\) be a normal vector field on \(V\). Define the set \(F_v\) made of all nice \(v\)-adapted charts on \(V\). For any chart \(\chi \in F_v\), with domain \(U \subset V\), let \(D_U\) be defined by \(I\), and, for any subset \(O \subset N_v\), define \(\bar{\chi}(O) \equiv \bar{\chi}(O \cap D_U)\), where \(\bar{\chi}\) is defined in Eq. \(I\) on \(\text{Dom}(\bar{\chi}) \equiv D_U\). Let \(\mathcal{T}'\) be the set of the subsets \(O \subset N_v\) such that

\[
\forall \chi \in F_v, \quad \bar{\chi}(O) \text{ is an open set in } \mathbb{R}^N.
\] (70)

The set \(\mathcal{T}'\) is a topology on \(N_v\).

Proof. This is an adaptation of the proof of Proposition C in Ref. \(S\), replacing \(M\) by \(N_v\), \(F\) by \(F_v\), \(\bar{\chi}\) by \(\chi\), and \(\mathbb{R}^3\) by \(\mathbb{R}^N\). In particular, the proof that the whole set \(N_v\) (instead of \(M\)) belongs to \(\mathcal{T}'\) is exactly identical. Also, by definition of a nice \(v\)-adapted chart, the mapping \(\bar{\chi} : D_U \rightarrow \mathbb{R}^N\) is injective. Therefore, if \(O_1 \in \mathcal{T}'\) and \(O_2 \in \mathcal{T}'\), we have

\[
\bar{\chi}(O_1 \cap O_2) \equiv \bar{\chi}((O_1 \cap O_2) \cap D_U) = \bar{\chi}((O_1 \cap D_U) \cap (O_2 \cap D_U)) = \bar{\chi}(O_1 \cap D_U) \cap \bar{\chi}(O_2 \cap D_U) \equiv \bar{\chi}(O_1) \cap \bar{\chi}(O_2),
\] (71)

which is thus an open set of \(\mathbb{R}^N\), for any \(\chi \in F_v\), so that \(O_1 \cap O_2 \in \mathcal{T}'\). It is also trivial to check that the union of any family of subsets \(O_i \in \mathcal{T}'\) is still an element of \(\mathcal{T}'\). \(\square\)
To prove that the mappings $\bar{\chi}$ are continuous for this topology, and to prove the compatibility of any two mappings $\bar{\chi}, \bar{\chi}'$ on $N_v$, associated with two nice $v$-adapted charts $\chi, \chi' \in \mathcal{F}_v$, the following difficulty arises: $\chi$ and $\chi'$ have in general different domains $U$ and $U'$. We may have $U \cap U' = \emptyset$, although there is some $l \in N_v$ with

$$l \cap U \neq \emptyset, \quad l \cap U' \neq \emptyset. \tag{72}$$

I.e., it may happen that the domains of the charts $\chi$ and $\chi'$ do not overlap, and that the domains of the mappings $\bar{\chi}$ and $\bar{\chi}'$ do. To overcome this difficulty, we use the flow $F$ of the vector field $v$ to associate smoothly with any point $Y$ in some neighborhood $W \subset U$ of a point $X$, a point $g(Y) \in U'$:

**Lemma.** Let $v$ be a $C^\infty$ vector field on $V$. Let $\chi, \chi' \in \mathcal{F}_v$, with domains $U$ and $U'$, be such that $D_U \cap D_{U'} \neq \emptyset$. Let $l \in D_U \cap D_{U'}$ and $X \in l \cap U$ and set $\chi(X) = (t, x)$, so that $\bar{\chi}(l) = x$. There is an open neighborhood $\Omega$ of $x$ in $\mathbb{R}^N$ and a $C^\infty$ mapping $g$ defined on an open neighborhood $W \subset U$ of $X$, such that for any $y \in \Omega$ we have $(t, y) \in \chi(W)$ (so that $y \in \bar{\chi}(D_U)$), $\bar{\chi}^{-1}(y) \in D_{U'}$, and

$$\forall y \in \Omega, \quad \bar{\chi}' \circ \bar{\chi}^{-1}(y) = P_S(\chi'(g(\chi^{-1}(t, y))))). \tag{73}$$

**Proof.** Since $l \in D_{U'}$, there is some point $X' \in l \cap U'$, and since $X \in l$ there is some $s \in I_X$ such that $X' = F(s, X)$. Thus, the domain $D$ of $F$ being open in $\mathbb{R} \times V$ and $F$ being continuous, there is an interval $I$ centered at $s$ and an open neighborhood $W \subset U$ of $X$ in $V$, such that $I \times W \subset D$ and $F(I \times W) \subset U'$. For $Y \in W$, set $g(Y) = F(s, Y)$. This defines a $C^\infty$-mapping $g : W \to U'$. Because $\chi(W)$ is open in $\mathbb{R}^{N+1}$ and $(t, x) \in \chi(W)$, there is an interval $J$ centered at $t$ and an open neighborhood $\Omega$ of $x$ in $\mathbb{R}^N$, with $J \times \Omega \subset \chi(W)$. Hence, if $y \in \Omega$, then indeed $(t, y) \in \chi(W)$, thus $Y \equiv \chi^{-1}(t, y) \in W \subset U$, which implies that $l_Y \in D_U$ and that

$$y = P_S(\chi(Y)) = \bar{\chi}(l_Y), \tag{74}$$

so $y \in \bar{\chi}(D_U)$. Moreover, we have $Y' \equiv g(Y) = F(s, Y) \in l_Y \cap U'$, so $l_Y = \bar{\chi}^{-1}(y) \in D_{U'}$ and

$$y' \equiv P_S(\chi'(Y')) = \bar{\chi}'(l_Y), \tag{75}$$

whence follows (73). \hfill \Box
Theorem 6. Let $v$ be a normal vector field on $V$. Let $\mathcal{A}$ be the set of all mappings $\bar{\chi}$, where $\chi \in \mathcal{F}_v$. This set $\mathcal{A}$ is an atlas on the topological space $(N_v, \mathcal{T'})$.

Proof. (i) Consider any $\chi' \in \mathcal{F}_v$, with domain $U$. Let us prove that $\bar{\chi}'$, which is defined on $D_U$, is continuous for the topology induced on $D_U$ by the topology $\mathcal{T'}$ on $N_v$. Thus, $A$ being any open set in $\mathbb{R}^N$, we must show that the set $\mathcal{O}_1 \equiv \bar{\chi}'^{-1}(A)$ has the form $\mathcal{O}_1 = \mathcal{O} \cap D_U$, where $\mathcal{O}$ is such that we have (70). We shall actually show that $\mathcal{O}_1 \in \mathcal{T'}$, i.e., that we have (70) with $\mathcal{O} \equiv \mathcal{O}_1 \subset D_U$. To prove this, we may assume that $\mathcal{O}_1 \neq \emptyset$. Moreover, let $\chi \in \mathcal{F}_v$, with domain $U$. We may also assume that $\bar{\chi}(\mathcal{O}_1) \neq \emptyset$, i.e., $\mathcal{O}_1 \cap D_U \neq \emptyset$, so let $x \in \bar{\chi}(\mathcal{O}_1)$. We have to find a neighborhood $B$ of $x$ in $\mathbb{R}^N$, such that $B \subset \bar{\chi}(\mathcal{O}_1)$, i.e., $B \subset \bar{\chi}(\bar{\chi}'^{-1}(A))$. Since $x \in \bar{\chi}(\mathcal{O}_1) \equiv \bar{\chi}(\mathcal{O}_1 \cap D_U)$, we have that $l \equiv \bar{\chi}'^{-1}(x) \in (\bar{\chi}'^{-1}(\mathcal{A})) \cap D_U \subset D_U \cap D_U$. In particular, there is some $X \in l \cap U$, with $\chi(X) = (t, x)$ for some $t \in \mathbb{R}$. Hence, we may apply the Lemma and get the corresponding open neighborhood $\Omega$ of $x$. Thus (73) shows that the mapping $f \equiv \bar{\chi}' \circ \bar{\chi}'^{-1}$ is well defined and continuous over $\Omega$. Hence, since $x \in \Omega$ and $x' \equiv f(x) = \bar{\chi}'(l) \in A$, which is open in $\mathbb{R}^N$, there is a neighborhood $B \subset \Omega$ of $x$ in $\mathbb{R}^N$, such that $f(B) \subset A$. Set $f' \equiv \bar{\chi}' \circ \bar{\chi}'^{-1} = f^{-1}$. For $y \in B$, we have thus $y' \equiv f(y) \in A$, that is, $y = f'(y') \in f'(A) = (\bar{\chi}' \circ \bar{\chi}'^{-1})(A)$. Thus, for any open set $A$ in $\mathbb{R}^N$, the set $\mathcal{O}_1 \equiv \bar{\chi}'^{-1}(A)$ belongs to $\mathcal{T'}$, as announced. This proves that $\bar{\chi}'$ is continuous for the topology induced on $D_U$ by the topology $\mathcal{T'}$ on $N_v$. Moreover, taking $A = \mathbb{R}^N$, we get that $D_U = \bar{\chi}'^{-1}(\mathbb{R}^N)$ is open in $N_v$, $D_U \in \mathcal{T'}$.

(ii) Given any $\chi \in \mathcal{F}_v$, with domain $U$, let us show that the mapping $\bar{\chi}^{-1} : \bar{\chi}(D_U) \rightarrow D_U \subset N_v$, is continuous. Since $D_U$ is open as seen at the end of (i), we have to show that, for any $\mathcal{O} \in \mathcal{T'}$ such that $\mathcal{O} \subset D_U$, the set $\Omega \equiv (\bar{\chi}^{-1})^{-1}(\mathcal{O})$ is open in $\mathbb{R}^N$. But since $\mathcal{O} \subset D_U = \text{Dom}(\bar{\chi})$, and since $\bar{\chi}$ is injective, we have $\Omega = \bar{\chi}(\mathcal{O})$. The fact that this is open in $\mathbb{R}^N$ follows from the very definition of the topology $\mathcal{T'}$ in Eq. (70). With (i), this means that, for any $\chi \in \mathcal{F}_v$, the mapping $\bar{\chi} : D_U \rightarrow \bar{\chi}(D_U) \subset \mathbb{R}^N$ is bicontinuous, thus is indeed a chart on the topological space $(N_v, \mathcal{T'})$.

(iii) Let us show that the domains of definition of the charts $\bar{\chi}$, for $\chi \in \mathcal{F}_v$, cover the whole set $N_v$. Given $l \in N_v$, let $X \in l$. Since $v$ is a normal vector field on $V$, we know from Theorem 5 that there is a nice $v$-adapted chart $\chi$ whose domain $U$ is a neighborhood of $X$. Thus $X \in l \cap U$, so $l \in D_U$, Q.E.D.
Finally, given any two nice \( v \)-adapted charts \( \chi, \chi' \in \mathcal{F}_v \), having domains \( U \) and \( U' \) respectively, let us show the compatibility of the two charts \( \bar{\chi}, \bar{\chi}' \) on \( N_v \), with domains \( D_U \) and \( D_{U'} \). The relevant case is when \( D_U \cap D_{U'} \neq \emptyset \), so that \( \text{Dom}(\bar{\chi}' \circ \bar{\chi}^{-1}) = \bar{\chi}(D_U \cap D_{U'}) \neq \emptyset \). Thus we may apply the Lemma. Its Eq. (73) shows that, given any \( x \in \bar{\chi}(D_U \cap D_{U'}) \), it has a neighborhood \( \Omega \) in which the function \( \bar{\chi}' \circ \bar{\chi}^{-1} \) is given as a composition of \( C^\infty \) functions, hence \( \bar{\chi}' \circ \bar{\chi}^{-1} \) is \( C^\infty \) on its domain. \( \square \)

Thus, we have endowed the set \( N_v \) with first the topology \( T' \) defined by (70), and then with a canonical atlas \( \mathcal{A} \) of compatible charts, which are simply the mappings \( \bar{\chi} \), where \( \chi \in \mathcal{F}_v \) is any nice \( v \)-adapted chart. To call this a differentiable manifold in the rather usual sense of Note 2 needs that the topological space \( (N_v, T') \) be metrizable and separable — hence, in particular, that it be Hausdorff. We do not have very general results on the latter point.

Proposition 7. Let \( v \) be a normal vector field on \( V \). (i) Any two points \( l \neq l' \) in the orbit space \( N_v \) are topologically distinguishable, i.e., there exists an open set \( O \in T' \) such that \( l \in O \) and \( l' \notin O \).
(ii) Suppose \( l \neq l' \) but \( l \) and \( l' \) both belong to the domain \( D_U \) of a chart \( \bar{\chi} \), where \( \chi \in \mathcal{F}_v \) with \( \text{Dom} \chi = U \). Then \( l \) and \( l' \) are separated by neighborhoods, i.e., there are two open sets \( O, O' \in T' \), such that \( l \in O \), \( l' \in O' \), and \( O \cap O' = \emptyset \).
(iii) Suppose there is a chart \( \chi \in \mathcal{F}_v \), such that any maximal integral curve \( l \in N_v \) intersects its domain \( U \), so that \( D_U = N_v \). Then the topological space \( (N_v, T') \) is metrizable and separable. [Hence, in particular, it has the Hausdorff property, i.e., any two distinct elements \( l, l' \in N_v \) are separated by neighborhoods — as follows also from Point (ii).]

Proof. (i) Let \( X \in l \). There is some open neighborhood \( U \) of \( X \), such that \( U \cap l' = \emptyset \): if that were not true, then, taking any distance on \( V \) that defines its topology, and considering \( U_n \equiv B(X, 1/n) \), we would get a sequence \( (X_n) \) with \( X_n \in l' \) and \( X_n \to X \), hence \( X \in l' \) since we defined that a normal vector field has all its maximal integral curves closed; but this implies that \( l = l' \), which is a contradiction. By Theorem 5 let \( \chi \in \mathcal{F}_v \) whose domain \( U_2 \) is an open neighborhood of \( X \), and set \( U_1 \equiv U \cap U_2 \). The restriction \( \chi_1 \) of \( \chi \) to \( U_1 \subset U \) is still a nice \( v \)-adapted chart: it is obviously \( v \)-adapted, we have
$D_{U_1} \subset D_U$, and the mapping $\bar{\chi}_1 : D_{U_1} \to \mathbb{R}^N$ is clearly the restriction of $\bar{\chi}$ to $D_U$, hence it also is injective. Therefore, $\mathcal{O} \equiv D_{U_1}$ is an open set, $\mathcal{O} \in \mathcal{T}'$. Since $X \in l \cap U_1$, we have $l \in \mathcal{O}$; and since $l' \cap U_1 \subset l' \cap U = \emptyset$, we have $l' \notin \mathcal{O}$.

(ii) Since $l \in D_U$ and $l' \in D_U$ with $l \neq l'$, and since $\bar{\chi}$ is defined on $D_U$ and injective, we have $\mathbf{x} \equiv \bar{\chi}(l) \neq \mathbf{x}' \equiv \bar{\chi}(l')$. (76)

Let $\Omega$ and $\Omega'$ be open neighborhoods in $\mathbb{R}^N$ of $\mathbf{x}$ and $\mathbf{x}'$ respectively, such that $\Omega \cap \Omega' = \emptyset$. Set $\mathcal{O} \equiv \bar{\chi}^{-1}(\Omega)$ and $\mathcal{O}' \equiv \bar{\chi}^{-1}(\Omega')$. These are open sets such that $l \in \mathcal{O}$ and $l' \in \mathcal{O}'$, and we have $\mathcal{O} \cap \mathcal{O}' = \bar{\chi}^{-1}(\Omega \cap \Omega') = \emptyset$.

(iii) This follows from the fact that $\bar{\chi}$ is a homeomorphism of its domain $D_U$ onto its range $\bar{\chi}(D_U) \subset \mathbb{R}^N$. □

Example. The assumption made for Point (iii) is fulfilled, in particular, in the following case, which occurs frequently in relativistic theories of gravitation. Assume a chart $(\chi, U)$ is defined on the whole of the manifold: $U = V$, which means that $\chi$ is a diffeomorphism of $V$ onto the open subset $\Gamma \equiv \chi(V)$ of $\mathbb{R}^{N+1}$. Then, the tangent vector field $\mathbf{v}$ to the world lines $l_a$ given by (3) for $N = 3$, with constant component vector $\mathbf{v}_0 = (1, 0, ..., 0)$ in the chart $\chi$, is the pushforward vector field of the constant vector field $\mathbf{v}(\mathbf{X}) = \mathbf{v}_0$ for $\mathbf{X} \in \Gamma$ by the diffeomorphism $\chi^{-1}$. Hence, by No. (iv) in the Examples above, $\mathbf{v}$ is a normal vector field on $V$. Due to No. (ii) in these examples, the orbits of $\mathbf{v}$ are the images of the orbits of $\mathbf{v}$ by $\chi^{-1}$, hence [by No. (iii)] are the connected components of the lines $l_a [a = (a') \in P_S(\Gamma)]$. Hence, the chart $\chi$ is $\mathbf{v}$-adapted, for $P_S(\chi(\mathbf{X})) = a$ if $\mathbf{X} \in l_a$. If actually all lines $l_a$ defined in (3) are connected (which happens iff the domain of the time coordinate $x^0$ is an interval for any such line), then $\chi$ is nice ($a = a' \Rightarrow l_a = l_{a'}$). Thus, in that case, $\chi \in \mathcal{F}_v$. Since the domain of $\chi$ is $U = V$, we have then $D_U = N_v$, so $N_v$ is metrizable and separable. This case includes of course standard situations, e.g. an inertial frame (e.g. with Cartesian coordinates) in Minkowski spacetime; a uniformly rotating frame (e.g. with “rotating Cartesian coordinates” [10]) in Minkowski spacetime [even though in that case the lines (3) are spacelike when $\rho \equiv \sqrt{x^2 + y^2} > c/\omega$]; harmonic coordinates in an asymptotically flat spacetime [18]; etc. It also includes known singular solutions of general relativity such as the singular Schwarzschild-Kruskal-Szekeres spacetime: the Kruskal-Szekeres coordinates $(T, \xi, \theta, \phi)$ [19, 20] cover the whole
of the “maximally extended” Schwarzschild manifold. Since the domain of the coordinates \( T, \xi \) is: \( \xi \in \mathbb{R}, T^2 - \xi^2 < 1 \), i.e. \( T \in ]-\sqrt{1+\xi^2}, +\sqrt{1+\xi^2}[ \), each line \( l_\alpha \) [with \( \alpha \equiv (\xi, \theta, \phi) \)] is connected. Thus this global chart on the Schwarzschild spacetime does define a global space manifold. Moreover, the tangent vector field \( v \) to these lines (3) is time-like.

**Proposition 8.** Assume that \( v \) is a normal vector field on \( V \). (i) There is a countable cover of \( V \) by open sets \( U_n \) such that, for any integer \( n \), there is a nice \( v \)-adapted chart, \( \chi_n \in \mathcal{F}_v \), having domain \( U_n \). (ii) Then, setting \( D_n \equiv D_{U_n} \), the sequence \( (D_n) \) is a countable cover of \( N_v \) by metrizable open subsets. Hence the topological space \( (N_v, T') \) is separable.

**Proof.** (i) By Theorem 5, for any \( X \in V \) there is a nice \( v \)-adapted chart \( \chi_X \in \mathcal{F}_v \), such that its domain \( U_X \) is an open neighborhood of \( X \). But, since \( V \) is metrizable and separable, there exists a countable basis \( (V_n)_{n\in\mathbb{N}} \) for the open sets of \( V \). Hence, for any \( X \in V \), there is some integer \( \tilde{n}(X) \) such that

\[
X \in V_{\tilde{n}(X)} \subset U_X.
\]  

This defines a mapping \( \tilde{n} : V \to \mathbb{N} \) and we have

\[
V = \bigcup_{n\in\tilde{n}(V)} V_n.
\]  

We may define a mapping \( \tilde{n}(V) \to V, n \mapsto X_n \), by choosing \( X_n \), for any \( n \in \tilde{n}(V) \), as being one of the points \( X \in V \) such that \( n = \tilde{n}(X) \). From (77), it follows then that, for any \( n \in \tilde{n}(V) \), we have

\[
V_n = V_{\tilde{n}(X_n)} \subset U_{X_n}.
\]  

For \( n \in \tilde{n}(V) \), define \( U_n \equiv U_{X_n} \) and \( \chi_n \equiv \chi_{X_n} \in \mathcal{F}_v \). From (78) and (79), it results that the countable family \( (U_n)_{n\in\tilde{n}(V)} \) is as in Statement (i).

(ii) Note first that, since \( \chi_n \in \mathcal{F}_v \), with domain \( U_n \), it follows from Theorem 6 that \( D_n \), the domain of the associated chart \( \bar{\chi}_n \) on \( N_v \), is open in \( N_v \). If \( l \in N_v \), let \( X \in l \) and, since \( (U_n) \) is a cover of \( V \), let \( n \) be such that \( X \in U_n \). We have thus \( l \cap U_n \neq \emptyset \), i.e. \( l \in D_n \). So \( (D_n) \) is a countable open cover of \( N_v \). Since \( \bar{\chi}_n \) is a homeomorphism of \( D_n \) onto \( \bar{\chi}_n(D_n) \subset \mathbb{R}^N \), it follows that \( D_n \) is a metrizable and separable space. Hence it is second-countable, i.e., there exists a countable basis \( (O_{nm})_{m\in\mathbb{N}} \) for the open subsets
of \(D_n\). Since any open subset \(O\) of \(N_v\) is the countable union of the open subsets \(O_n \equiv O \cap D_n\) of \(D_n\), we have that \((O_{nm})_{n,m \in \mathbb{N}}\) is a countable basis for the topology \(T'\) of \(N_v\). Thus also \(N_v\) is second-countable, hence it is separable. 

\[ \Box \]

4 The local manifold as an open subset of the global one

Let \(v\) be a normal vector field on \(V\). In addition, as in Subsect. 1.2, let \(F\) be a (local) reference frame, thus an equivalence class of charts for the relation (\(\mathbb{I}\)), in which \(U\) is a given open subset of \(V\). In Subsect. 1.2 the local space manifold \(M_F\) associated with \(F\) was defined as the set of the world lines (6). On the other hand, the orbit set \(N_v\) defined in Subsect. 2.1, the set of the maximal integral curves of \(v\), was endowed in Sect. 3 with a topology \(T'\) and an atlas \(\mathcal{A}\), which (assuming that \(T'\) is metrizable and separable) makes it a differentiable manifold. Thus, we have also a global space manifold: \(N_v\). When the charts \(\chi \in F\), all having domain \(U\), are \(v\)-adapted charts, i.e. belong to \(\mathcal{F}_v\), we have the following tight relation between \(M_F\) and \(N_v\):

**Theorem 7.** Assume that \(F \subset \mathcal{F}_v\). For any \(l \in M_F\), there is a unique maximal integral curve \(l' \in N_v\) such that, for any \(X \in l\), we have \(l' = l_X\). It holds \(l = l' \cap U\). The mapping \(I : l \mapsto l'\) is a diffeomorphism of \(M_F\) onto the open subset \(D_U\) of \(N_v\).

**Proof.** Let \(l \in M_F\). By the definition of \(M_F\) near Eq. (6), there is some chart \(\chi \in F\) and some \(x \in P_S(\chi(U)) \subset \mathbb{R}^N\), such that

\[ l = \{ X \in U; P_S(\chi(X)) = x \}. \tag{80} \]

Let \(X_1 \in l\) and \(X_2 \in l\). Denote the maximal integral curves of \(v\) at \(X_1\) and \(X_2\) as \(l_1' \equiv l_{X_1}, l_2' \equiv l_{X_2}\). Since \(\chi\) is \(v\)-adapted, there exist \(x_1, x_2 \in \mathbb{R}^N\), such that \(\forall X \in l_1' \cap U, P_S(\chi(X)) = x_1\); and \(\forall X \in l_2' \cap U, P_S(\chi(X)) = x_2\). In particular, since \(X_j \in l_j' \cap U\), we have \(P_S(\chi(X_j)) = x_j\) (\(j = 1, 2\)). But since

---

4 As in Sects. [2] and [3] the dimension of \(V\) is \(N + 1\), where \(N\) is any integer \(\geq 1\). All results summarized in Subsect. 1.2 hold if one substitutes any integer \(N \geq 1\) for the integer 3, and \(N + 1\) for 4 [8].
Thus, if \( F \subseteq F \), we can say that \( M = \bar{I} \). Therefore, we define a mapping \( I : M \rightarrow N \) by associating with any \( l \in M \), the unique maximal integral curve \( l' \in N \), such that for any \( X \in l \), we have \( l' = l_X \). Note that actually \( l' \in D \). Owing to the definitions (7) and (14), we have \( x = \tilde{\chi}(l) = \tilde{\chi}(l') \), and since here \( l \) is any element of \( M \), this shows that \( \tilde{\chi}(M) \subseteq \tilde{\chi}(D) \). Thus

\[
I(l) = \tilde{\chi}^{-1}(\tilde{\chi}(l)), \tag{81}
\]

so that we have simply \( I = \tilde{\chi}^{-1} \circ \tilde{\chi} \), for whatever chart \( \chi \in F \). Let us show that \( l = l' \cap U \). By definition, for any \( X \in l \), we have \( l' = l_X \), hence \( X \in l' \), and since \( l \subseteq U \) by the definition (81), we have \( l \subseteq l' \cap U \). Conversely, consider any \( \chi \in F \); this is by assumption a \( \nu \)-adapted chart and, as we showed before (81), we have \( \chi(l') = \chi(l) \equiv x \). Therefore, by the definition (13), we have for any \( X \in l' \cap U \): \( P_S(\chi(X)) = x \). Then (80) implies that \( X \subseteq l \), so \( l' \cap U \subseteq l \).

As we showed, the mapping \( I \) is defined on the whole of \( M \) and ranges into \( D \), which is an open subset of \( N \). Let us show that \( I(M) = D \). Let \( l' \subseteq D \), so there exists \( X \subseteq l' \cap U \). Let \( \chi \subseteq F \) and set \( x \equiv P_S(\chi(X)) \) and \( l \equiv \{ Y \subseteq U \mid P_S(\chi(Y)) = x \} \). Clearly \( l \subseteq M \) and \( X \subseteq l \). By the definition of \( I \), we have that, for any \( Y \subseteq l \), \( I_Y = I(l) \). In particular, \( I_X = I(l) \). But since \( X \subseteq l' \), we have \( l_X = l' \), hence \( l' = I(l) \); thus indeed \( D \subseteq I(M) \). Note that again here, from the definitions (7) and (14) we have \( x = \tilde{\chi}(l) = \tilde{\chi}(l') \), and since now \( l' \) is any element of \( D \), this shows that \( \tilde{\chi}(D) \subseteq \tilde{\chi}(M) \). Since the reverse inclusion has been proved before (81), we have \( \tilde{\chi}(D) = \tilde{\chi}(M) \).

As shown in Ref. [5], \( \tilde{\chi} \) is a global chart on the differentiable manifold \( M \), for any \( \chi \in F \). As shown in Theorem 6, \( \tilde{\chi} \) is a chart with domain \( D \) on the differentiable manifold \( N \), also for any \( \chi \in F \). Moreover, as we just saw, we have \( \tilde{\chi}(D) = \tilde{\chi}(M) \). It follows that the one-to-one mapping \( I = \tilde{\chi}^{-1} \circ \tilde{\chi} \), from \( \text{Dom}(\tilde{\chi}) = M \) onto \( I(M) = D = \text{Dom}(\tilde{\chi}) \), is a diffeomorphism. Therefore, \( I \) is an immersion of \( M \) into \( N \). Actually, recall that \( D \) is more specifically an open subset of \( N \). □

Thus, if \( F \subseteq F \), we may identify the local space \( M \) with the open subset \( I(M) = D \) of the global space \( N \). Since we proved that \( l = I(l) \cap U \), we can say that \( M \) is made of the intersections with the local domain \( U \) of the
maximal integral curves of $v$. Given that each world line $l \in \mathcal{M}_F$ is invariant under any exchange of the chart $\chi \in F$ for another chart $\chi' \in F$, to say that $F \subset \mathcal{F}_v$ is equivalent to say that one chart $\chi \in F$ is a nice $v$-adapted chart.

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