CONTINUOUS AND DISCRETE NEUMANN SYSTEMS ON STIEFEL VARIETIES AS MATRIX GENERALIZATIONS OF THE JACOBI–MUMFORD SYSTEMS

YURI FEDOROV
Department of Mathematics
Polytechnic university of Catalonia
C. Pau Gargallo, 14, 08028 Barcelona, Spain

BOŽIDAR JOVANOVIC*
Mathematical Institute
Serbian Academy of Sciences and Arts
Kneza Mihaila 36, 11000 Belgrade, Serbia

(Communicated by Álvaro Pelayo)

ABSTRACT. We study geometric and algebraic geometric properties of the continuous and discrete Neumann systems on cotangent bundles of Stiefel varieties $V_{n,r}$. The systems are integrable in the non-commutative sense, and by applying a $2r \times 2r$--Lax representation, we show that generic complex invariant manifolds are open subsets of affine Prym varieties on which the complex flow is linear. The characteristics of the varieties and the direction of the flow are calculated explicitly. Next, we construct a family of multi-valued integrable discretizations of the Neumann systems and describe them as translations on the Prym varieties, which are written explicitly in terms of divisors of points on the spectral curve.

1. Introduction. The classical integrable Neumann system on an $n$-dimensional sphere is known to be a toy model for the Jacobi–Mumford systems introduced in [45] and describing translationally invariant flows on the Jacobians of hyperelliptic curves and represented by $2 \times 2$ Lax pairs.

During the last three decades various generalizations of the Jacobi–Mumford systems associated with families of $r$-gonal and generic curves were constructed (see [51, 55, 10, 49, 34]). The paper [26] also introduced such systems linearized on special types of Prym varieties isomorphic to Jacobians. All such generalizations are closely related to integrable flows on coadjoint orbits of the loop algebra $\widehat{gl}(r)$, with Lax matrices being rank $r$ perturbations of a constant matrix. The complete algebraic geometric description of such flows in the generic case was given in the series of papers [18, 1, 2, 4].

There have also been several important results on discretizations (Bäcklund transformations) of the above systems, which are described by translations on the
Jacobians of hyperelliptic, $r$-gonal, and generic curves, and which can be related to addition theorems for meromorphic functions on such Jacobians (see [58, 39, 22]).

The present paper contributes to these fields of interest in both ways: by giving a complete algebraic geometric description and linearization of a continuous generalized Neumann problem and then performing its discretization.

First, as a further generalization of the Jacobi–Mumford systems, we consider natural analogs of the Neumann system on the Stiefel variety $V_{n,r}$ ($r < n$), the set of $n \times r$ matrices $X$ satisfying the constraints $X^T X = I_r$, where $I_r$ denotes the $r \times r$ unit matrix. These analogs were first introduced by Reiman and Semenov by presenting a “big” $(n \times n)$ Lax pair [53]. It appears that for $r > 1$ the big Lax pair itself does not define yet the system uniquely: there exists a family of noncommutatively integrable Neumann flows with different $SO(n)$-invariant metrics $ds_n^2$ on $V_{n,r}$ that possess the same big Lax representation and share the same foliation on invariant isotropic tori of the cotangent bundle $T^*V_{n,r}$ (see [24]).

The Neumann systems on $V_{n,r}$ also admit “small” $(2r \times 2r)$ Lax representations which, modulo the action of a discrete group of reflections $\mathbb{Z}_2^n$, define these systems uniquely (see [35, 24]). They have the symplectic block form

$$L(\lambda) = [L(\lambda), N_n(\lambda)],$$

$$L(\lambda) = \begin{pmatrix}
X^T(\lambda I_n - A)^{-1} P & X^T(\lambda I_n - A)^{-1} X \\
P - P^T X^{-1} P & -P^T(\lambda I_n - A)^{-1} X
\end{pmatrix},$$

where $A = \text{diag}(a_1, \ldots, a_n)$, $P$ is the $n \times r$-momentum satisfying the constraint $X^T P + P^T X = 0$, $\lambda$ is a rational spectral parameter, and the matrix $N_n(\lambda)$, which is shown below in the text, depends on the metric $ds_n^2$ (see Section 3).

The matrix $L(\lambda)$ can be regarded as a generalization of the $2 \times 2$ Lax matrix for the classical Neumann system (see [45, 6, 34]) and formally fits to the already studied class of rank $d$ perturbations of the constant matrix $A$. The latter are described by the Lax matrices of the form

$$Y + \sum_{i=1}^n \frac{N_i}{\lambda - a_i}, \quad Y, N_i \in \mathfrak{gl}(d),$$

where $Y$ is constant and $N_i$ depend on the variables of the corresponding dynamical system. However our $L(\lambda)$ belongs to the symplectic loop subalgebra $\mathfrak{sp}(2r) \subset \mathfrak{gl}(2r)$ and has a quite specific structure, moreover, $Y$ is not diagonalizable, it contains a Jordan block. As a result, the spectral curve $S$ of $L(\lambda)$ has extra strong singularities at the infinity (see Section 4). So, the previous results of [1, 2, 4] concerning calculation of the genus of $S$, number of its infinite points, etc, need to be modified in order to treat this case.

Calculating the order of singularity and the genus of the regularized curve $S'$ (Theorem 4.1), taking into account the involution $\sigma : S' \to S'$, we then demonstrate that, up to the action of $\mathbb{Z}_2^n$, generic complex invariant manifolds $\mathcal{I}_h$ of the Neumann systems on $T^*V_{n,r}$ are open subsets of affine Prym varieties $\text{Prym}(S', \sigma)$ of the same dimension (Theorem 5.3). The latter are algebraic noncompact subgroups of generalized Jacobian varieties $\text{Jac}(S', \infty)$, the extensions of $\text{Jac}(S')$ by $\mathbb{C}^{[r/2]}$. We show that the trajectories of the Neumann systems are straight lines on $\text{Prym}(S', \sigma)$ and calculate their direction (Theorem 5.4).

It should be noted that affine Prym varieties appear as complex invariant manifolds in several classical integrable systems on Lie groups, before the reduction of
the systems to the Lie algebras, for example, the spatial rigid body motion in the Clebsch case of the Kirchoff equations (see e.g., [8]). However, to our knowledge, such affine Prym varieties had never been discussed explicitly in the literature, neither in algebraic geometry, nor in connection with algebraic integrable systems.

Next, the Marsden–Weinstein reductions of the Neumann systems on $T^*V_{n,r}$ by the group $SO(r)$ are also integrable, and their complex invariant tori (again, modulo $\mathbb{Z}_2^n$–action), are shown to be usual Prym varieties $Prym(S',\sigma) \subset \text{Jac}(S')$ (Proposition 5). In particular, the reduction for the zero value of the momentum mapping is the Neumann system on the Grassmann variety $G_{n,r}$, which is integrable in the usual commutative (Liouville) sense (see Theorem 5.5).

Second, we construct a family of (multi-valued) integrable discretizations (Bäcklund transformations) $\mathcal{B}_r : (X, P) \rightarrow (\tilde{X}, \tilde{P})$ of the Neumann system on $T^*V_{n,r}$ (Section 6). The family is parameterized by $\lambda_s \in \mathbb{C}$ and described by the $2r \times 2r$ intertwining relation (discrete Lax pair)

$$\tilde{L}(\lambda)M(\lambda, \lambda_s) = M(\lambda, \lambda_s)L(\lambda), \quad M(\lambda, \lambda_s) = \begin{pmatrix} -\Gamma(\lambda_s) & I_r \\ (\lambda - \lambda_s)I_r + \Gamma^2(\lambda_s) & -\Gamma(\lambda_s) \end{pmatrix},$$

where $\tilde{L}(\lambda)$ depends on $\tilde{X}, \tilde{P}$ in the same way as $L(\lambda)$ depends on $X, P$, whereas $\Gamma(\lambda_s)$ depends on $X, \tilde{X}$ in a symmetric way (Theorem 6.1). The above relation is a matrix generalization of the $2 \times 2$ discrete Lax pair describing a Bäcklund transformation of the classical Neumann system (see [33, 39]).

Each discretization map $\mathcal{B}_r$ preserves the first integrals of the continuous system and geometrically is described by translations $T$ on $\text{Prym}(S', \sigma)$ (Theorems 6.2, 6.4). The translations are written explicitly in terms of $r$ non-involution points on the spectral curve $S'$ over the coordinate $\lambda_s$ (i.e., eigenvalues of $L(\lambda_s)$).

By adopting a classical result on solutions of quadratic matrix equations [50], we show that a choice of such points (partition) fixes the branch of the map and the corresponding translation $T$. Iterations of all possible translations generate a lattice of rank $2^{r-1}$ on the universal covering of $\text{Prym}(S', \sigma)$.

In Conclusion we discuss further possible generalizations.

Long technical proofs of some theorems are given in Appendix.

2. The Jacobi–Mumford and the Neumann systems. For completeness of the exposition, here we recall some basic definitions and properties of the Jacobi–Mumford systems, as well as the discretizations of the Neumann problem on the sphere.

2.1. The standard (odd order) Jacobi–Mumford systems. Let $\Gamma = \Gamma(R)$ be a smooth hyperelliptic curve of genus $g$, whose affine part is given by the equation $\mu^2 = R(\lambda)$, where $R(\lambda)$ is a monic polynomial of degree $2g + 1$. We regard $\Gamma$ as a compact Riemann surface having one infinite point $\infty$. Consider a generic divisor of $g$ points $P_1 = (\lambda_1, \mu_1), \ldots, P_g = (\lambda_g, \mu_g)$ on $\Gamma$. Following Mumford [45], the curve and the divisor can be associated to three polynomials $U(\lambda), V(\lambda)$, and $W(\lambda)$, $\lambda \in \mathbb{C}$ such that

$$U(\lambda_i) = 0, \quad V(\lambda_i) = \mu_i, \quad i = 1, \ldots, g, \quad (2.1)$$

$$W(\lambda)U(\lambda) + V^2(\lambda) = R(\lambda). \quad (2.2)$$

The degrees of $U(\lambda)$ and $V(\lambda)$ are at least $g$ and $g - 1$ respectively. For a fixed curve $\Gamma$ given by $R(\lambda)$, the polynomial $W(\lambda)$ is uniquely determined from the
condition (2.2). Then, if \(U(\lambda)\) is monic of degree \(g\), then \(W(\lambda)\) is monic and has degree \(g + 1\). That is,

\[
U(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_g) = \lambda^g + u_1 \lambda^{g-1} + \cdots + u_g,
\]

\[
V(\lambda) = \sum_{k=1}^{g} \mu_k \prod_{j \neq k} \frac{\lambda - \lambda_k}{\lambda_k - \lambda_j} = v_1 \lambda^{g-1} + \cdots + v_g,
\]

and

\[
W(\lambda) = \lambda^{g+1} + w_0 \lambda^g + w_1 \lambda^{g-1} + \cdots + w_g = (\lambda - \nu_1) \cdots (\lambda - \nu_{g+1}).
\]

Now let \(\mathcal{E}_g\) be a variety of all non-constant coefficients of \(U(\lambda), V(\lambda), W(\lambda)\). As shown in [45], for a fixed \(R(\lambda)\), formula (2.2) defines a set of equations on the coefficients which provides a purely algebraic description of an affine part of \(\text{Jac}(\Gamma)\), the Jacobian variety of the curve \(\Gamma\). The coefficients themselves are meromorphic functions on the Jacobian.

The variety \(\mathcal{E}_g\) itself can be completed to the fiber bundle \(\tilde{\mathcal{E}}_g \to \mathcal{R}\) over the \(2g+1\)-dimensional space \(\mathcal{R}\) spanned by the coefficients of \(R(\lambda)\) and parameterizing all odd order smooth hyperelliptic curves \(\Gamma = \Gamma(R)\) of genus \(g\) (the base), with fibers of \(\tilde{\mathcal{E}}_g\) being the Jacobians of the curves.

Any translationally invariant vector flow on \(\text{Jac}(\Gamma)\) can be extended (in different ways) to a vector flow on the whole space \(\tilde{\mathcal{E}}_g\), which leaves the fibers invariant. The latter flow is described as an integrable system of differential equations on the coefficients of the three polynomials, which is referred to as a Jacobi–Mumford or an (odd order) master system (see e.g., [45, 57, 27]).

Let \(R \in \mathcal{R}\), \(\Gamma = \Gamma(R)\) be the associated hyperelliptic curve and let \(\mathcal{A} : \Gamma \to \text{Jac}(\Gamma)\) be the Abel mapping given by the following integral with the basepoint \(\infty\)

\[
\mathcal{A}(P) = \int_{\infty}^{P} (\omega_1, \ldots, \omega_g)^T, \quad P \in \Gamma,
\]

where \(\omega_1, \ldots, \omega_g\) are independent holomorphic differentials on the curve \(\Gamma\).

Consider a translationally invariant flow on \(\text{Jac}(\Gamma)\) which is tangent to the image of the curve \(\mathcal{A}(\Gamma) \subset \text{Jac}(\Gamma)\) at a finite point \(\mathcal{A}(\lambda_*, \mu_*)\). Then, fixing \(\lambda_* \in \mathbb{C}\), but not \(R \in \mathcal{R}\), we get a vector flow on \(\mathcal{E}_g\) or on \(\tilde{\mathcal{E}}_g\).

The flow can be represented in the following Lax form with \(\lambda\) as a rational parameter (see [57]):

\[
\mathbf{L}(\lambda) = [\mathbf{L}(\lambda), N(\lambda)],
\]

where

\[
\mathbf{L}(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix},
\]

\[
N(\lambda) = \frac{1}{2(\lambda - \lambda^*)} \begin{pmatrix} -V(\lambda^*) & -U(\lambda^*) \\ -W(\lambda^*) - (\lambda - \lambda^*)U(\lambda^*) & V(\lambda^*) \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \mathbf{L}(\lambda^*) & 0 \\ -U(\lambda^*) & 0 \end{pmatrix}.
\]

Concerning the Lax equation (2.4), the coefficients of the spectral curve

\[
\det(\mathbf{L}(\lambda) - \mu \mathbb{1}_2) = \mu^2 - V^2(\lambda) - W(\lambda)U(\lambda) = \mu^2 - R(\lambda) = 0
\]

are the first integrals of the system, and we recover the family of hyperelliptic curves \(\Gamma = \Gamma(R)\). When \(\lambda_*\) tends to \(\infty\), after an appropriate time rescaling, the limit flow
is described by (2.4) with
\[ N(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + (w_0 - u_1) & 0 \end{pmatrix}. \] (2.5)

**Remark 1.** To a fixed curve \( \Gamma \) and the given divisor \( P_1 + \cdots + P_g \) on it, one can associate the dual divisor \( R_1 + \cdots + R_{g+1} \) given by the zeros of the meromorphic function \( f(\lambda, \mu) = (\mu + V(\lambda))/U(\lambda) \) (e.g., see [39, 40]). In view of (2.1) and (2.3), \( f(\lambda, \mu) \) have simple poles at the original divisor and \( \infty \), which implies
\[ A(P_1) + \cdots + A(P_g) + A(\infty) = A(R_1) + \cdots + A(R_g) + A(R_{g+1}). \] (2.6)

As follows from (2.2), one can also write \( f = W(\lambda)/(\mu - V(\lambda)) \). Hence, the coordinates \( \nu_k = \lambda(R_k), k = 1, \ldots, g + 1 \) are the roots of \( W(\lambda) \).

### 2.2. The Neumann system on \( S^{n-1} \)

The best known example of an integrable problem associated to the Jacobi–Mumford system is the Neumann system describing the motion of a point on the unit sphere \( S^{n-1} = \{ (q, q) = 1 \} \subset \mathbb{R}^n \), with the quadratic potential
\[ V(q) = -\frac{1}{2} \langle q, Aq \rangle, \quad A = \text{diag}(a_1, \ldots, a_n) \]
(see [37, 43, 47]). Upon introducing momentum \( p = \dot{q} \) and imposing the constraint \( \langle p, q \rangle = 0 \), the motion is described by the equations
\[ \dot{q} = p, \quad \dot{p} = Aq + \nu q, \quad \nu = -\langle p, p \rangle - \langle q, Aq \rangle, \] (2.7)
which possesses the \( 2 \times 2 \) Lax representation
\begin{align*}
\dot{L}(\lambda) &= [L(\lambda), N(\lambda)], \\
L(\lambda) &= \begin{pmatrix} \sum_{i=1}^{n} \frac{q_i p_i}{\lambda - a_i} & \sum_{i=1}^{n} \frac{q_i^2}{\lambda - a_i} \\ 1 - \sum_{i=1}^{n} \frac{q_i^2}{\lambda - a_i} & -\sum_{i=1}^{n} \frac{q_i p_i}{\lambda - a_i} \end{pmatrix}, \\
N(\lambda) &= \begin{pmatrix} 0 & 1 \\ \lambda + \nu(p, q) & 0 \end{pmatrix}. \quad (2.8)
\end{align*}

Let \( a(\lambda) = (\lambda - a_1) \cdots (\lambda - a_n) \). For the polynomial Lax matrix \( L(\lambda) = a(\lambda)L(\lambda) \) the characteristic equation \( |L(\lambda) - \mu I_2| = 0 \) has the form
\[ \mu^2 = -a^2(\lambda) \left( \sum_{i < j}^{n} \frac{(q_i p_j - q_j p_i)^2}{(\lambda - a_i)(\lambda - a_j)} - \sum_{i=1}^{n} \frac{q_i^2}{\lambda - a_i} \right), \] (2.10)
and gives the odd order hyperelliptic curve \( \Gamma \) of genus \( g = n - 1 \)
\[ \mu^2 = R(\lambda) = a(\lambda)(\lambda - c_1) \cdots (\lambda - c_{n-1}). \] (2.11)

For real \( (q, p) \), the zeros \( c_1, \ldots, c_{n-1} \) of \( R(\lambda) \) are real (see Lemma 4.5 in Chapter III of [45]) and they represent commuting integrals of the Neumann system.

One can then identify the \( 2 \times 2 \) polynomial matrices \( L(\lambda) = a(\lambda)L(\lambda) \) and (2.4) by setting \( g = n - 1 \), which gives
\begin{align*}
U(\lambda) &= a(\lambda) \sum_{i=1}^{n} \frac{q_i^2}{\lambda - a_i} = \lambda^g + (-\text{tr} A + \langle q, Aq \rangle)\lambda^{g-1} + \cdots, \\
V(\lambda) &= a(\lambda) \sum_{i=1}^{n} \frac{q_i p_i}{\lambda - a_i} = \langle p, Aq \rangle \lambda^{g-1} + \cdots, \\
W(\lambda) &= a(\lambda) \left( 1 - \sum_{i=1}^{n} \frac{p_i^2}{\lambda - a_i} \right) = \lambda^{g+1} - (\text{tr} A + \langle p, p \rangle)\lambda^g + \cdots,
\end{align*}
which, in particular, implies
\[ u_1 = -\text{tr} A + \langle q, Aq \rangle, \quad v_1 = \langle p, Aq \rangle, \quad w_0 = -\text{tr} A - \langle p, p \rangle. \]  
(2.12)

Then the second matrix (2.9) coincides with (2.5). Hence the Neumann flow is linearized on the Jacobian of \( \Gamma \), on which it is tangent to \( \mathcal{A}(\Gamma) \subset \text{Jac}(\Gamma) \) at its infinite point.

The above relations also give the parametrization of \( q_i^2 \) in terms of \( \lambda_1, \ldots, \lambda_g \), the \( \lambda \)-coordinates of the points \( P_1, \ldots, P_g \) on \( \Gamma \), which now play the role of the spherocental coordinates on \( S^{n-1} \):
\[ q_i^2 = \frac{(a_i - \lambda_1) \cdots (a_i - \lambda_{n-1})}{\prod_{j \neq i} (a_i - a_j)}, \quad i = 1, \ldots, n \]

In addition, \( p_i^2 \) can be expressed in terms of \( \lambda \)-coordinates of the dual divisor \( R_1, \ldots, R_{g+1} \):
\[ p_i^2 = \frac{(a_i - \nu_1) \cdots (a_i - \nu_n)}{\prod_{j \neq i} (a_i - a_j)}, \quad i = 1, \ldots, n. \]

Since \( p_i^2, q_i^2, p_iq_i \) are linear functions of the coefficients of \( U, V, W \), they are meromorphic function on \( \text{Jac}(\Gamma) \). However, the coordinates \( p_i, q_i \) themselves do not have this property. According to [45], the following theorem holds

**Theorem 2.1.**

1) Complex invariant manifolds of the Neumann system with the constants of motion \( c_i \) factorized by the action of the discrete group \( \mathbb{Z}_2^n \) generated by reflections
\[ (q_i, p_i) \mapsto (-q_i, -p_i), \quad i = 1, \ldots, n \]
are open subsets of the Jacobian \( \text{Jac}(\Gamma) \) of the spectral curve (2.11).

2) The complex invariant manifolds themselves are open subsets of unramified coverings of \( \text{Jac}(\Gamma) \) obtained by doubling some of the period vectors of \( \text{Jac}(\Gamma) \).

### 2.3. Discretizations of the Neumann system on \( T^*S^{n-1} \)

The first integrable discretization of the Neumann system (2.7) was found in [58] by using the approach of Lagrange correspondences and the idea of factorization of Lux operators (see [44] for the details). Combining the results of [39, 56], below we consider the discretization map \( \mathfrak{D} : T^*S^{n-1}(q, p) \rightarrow T^*S^{n-1}(\tilde{q}, \tilde{p}) \) containing an extra parameter \( \lambda_* \in \mathbb{C} \) written in the implicit form
\[ p = A^{1/2}(\lambda_*)\tilde{q} - \gamma q, \quad \tilde{p} = -A^{1/2}(\lambda_*)q + \gamma \tilde{q}, \]  
(2.13)
or, in the form
\[ \tilde{q} = A^{-1/2}(\lambda_*)(\gamma q + p), \quad \tilde{p} = -A^{1/2}(\lambda_*)q + A^{-1/2}(\lambda_*)(\gamma^2 q + \gamma p), \]  
(2.14)
where \( A(\lambda_*) = \lambda_*I_n - A = \text{diag}(\lambda_* - a_1, \ldots, \lambda_* - a_n) \), and \( \gamma \) is a multiplier.

The equations (2.14) were obtained in [33] in the context of Bäcklund transformations. As was shown in [33], up to the action of the group of reflections \( (p_i, q_i) \mapsto (-p_i, -q_i) \), the above map is equivalent to the following intertwining relation (discrete Lax pair):
\[ \tilde{L}(\lambda)M(\lambda|\lambda_*) = M(\lambda|\lambda_*)L(\lambda), \]  
(2.15)

\[ ^{1}\text{The above notation slightly differs from that of Chap. 21.6 of [56], where one should replace } h^2 \text{ and } \Omega \text{ by } 1/\lambda^* \text{ and } -A, \text{ respectively.} \]
where \( L(\lambda), \tilde{L}(\lambda) \) have the same structure as in (2.8) and depend on the “old” variables \((p, q)\) and the “new” ones \((\tilde{q}, \tilde{p})\) respectively, and
\[
M(\lambda|\lambda_a) = \begin{pmatrix} -\gamma & 1 \\ \lambda - \lambda_a + \gamma^2 & -\gamma \end{pmatrix}.
\]
Indeed, setting in (2.15) subsequently \( \lambda = a_1, \ldots, a_n \) and calculating the matrix
\[
M(\lambda|\lambda_a) L(\lambda) M^{-1}(\lambda|\lambda_a)|_{\lambda=a_i},
\]
one obtains
\[
\tilde{q}_i = \pm \frac{\gamma q_i + p_i}{\sqrt{\lambda_a - a_i}}, \quad \tilde{p}_i = \pm \frac{(\gamma^2 + a_i - \lambda_a)q_i + \gamma p_i}{\sqrt{\lambda_a - a_i}}, \quad i = 1, \ldots, n,
\]
which is equivalent to (2.14) if we fix sign “+” above.

To find the multiplier \( \gamma \) as a function of \((q, p)\), one applies the condition \((\tilde{q}, \tilde{p}) = 1\) to the first equation of (2.14) and obtains the quadratic equation
\[
\langle q, A^{-1}(\lambda_a)q \rangle \gamma^2 + 2(p, A^{-1}(\lambda_a)q) \gamma + \langle p, A^{-1}(\lambda_a)p \rangle - 1 = 0,
\]
which gives
\[
\gamma = \frac{-\langle p, A^{-1}(\lambda_a)q \rangle \pm \sqrt{(\langle p, A^{-1}(\lambda_a)q \rangle)^2 - \langle p, A^{-1}(\lambda_a)q \rangle (\langle p, A^{-1}(\lambda_a)p \rangle - 1)}}{\langle q, A^{-1}(\lambda_a)q \rangle},
\]
or, in view of (2.10), (2.11),
\[
\gamma = \frac{-\langle p, A^{-1}(\lambda_a)q \rangle + \mu(\lambda_a)/a(\lambda_a)}{\langle q, A^{-1}(\lambda_a)q \rangle} = \frac{-V(\lambda_a) + \mu(\lambda_a)}{U(\lambda_a)},
\]
\( \mu(\lambda_a) \) being the coordinate of one of the two points on the curve (2.11) over \( \lambda = \lambda_a \).

As a result, the map \( \mathfrak{B} \) preserves the same first integrals as the continuous Neumann system for any \( \lambda^* \), and, in view of Theorem 2.1, its complex invariant varieties are open subsets of unramified coverings of the Jacobian of the curve \( \Gamma \) given by (2.11). As also follows from (2.19), the map \( \mathfrak{B} \), as well as its inverse \( \mathfrak{B}^{-1} \), is generally 2-valued.

Similar discrete systems on pseudo-spheres and light-like cones in pseudo-Euclidean spaces are studied in [32]. Another discretization of the Neumann system as an even order Jacobi–Mumford systems, which has different first integrals in comparison with the classical case, was obtained by Ragnisco [52].

**Remark 2.** Obviously, equations (2.14) describe a real map when \( \mu(\lambda_a) \) is real, i.e., \( R(\lambda_a) > 0 \). In particular, \( R(\lambda_a) > 0 \) when \( \lambda_a > a_1, \ldots, a_n, c_1, \ldots, c_{n-1} \) (see (2.11)).

**Remark 3.** Using the above formulas, one can prove that the composition of the two branches of \( \mathfrak{B} \), corresponding to different choices of sign of \( \mu(\lambda_a) \), gives the map \((q, p) \mapsto (-q, -p)\). In Section 6 an analog of this property for the discrete system on \( V_{n,r} \) will be considered.

Algebraic geometrical interpretation of the map (2.14), was given in [58] (see also [44]) and, in the context of discretization of the Jacobi-Mumford systems, in [39]. Namely, let \( \mathfrak{B} \) be the reduction of \( \mathfrak{B} \) under the action of the reflection group \( \mathbb{Z}_2 \). Hence, following Theorem 2.1, its generic invariant varieties are open subsets of \( \text{Jac}(\Gamma) \). Then, according to the above remark, the two branches of \( \mathfrak{B} \) just represent a “shift back” and a “shift forward” maps.

Let, as above, \( \mathcal{A} : \Gamma \rightarrow \text{Jac}(\Gamma) \) be the Abel map. In [58] and [39] the following theorem was proven.
**Proposition 1.** The restriction of $\mathcal{B}$ onto $\text{Jac}(\Gamma)$ is described by translation by the vector

$$T = A(P) - A(\infty) = A(P), \quad P = (\lambda_*, \mu_*).$$

(2.20)

Notice that, although the translation vector does not depend explicitly on the constants of motion, it depends on them via the moduli of the curve $\Gamma$. The above formula allows to write explicit solution for the trajectory $\{(q_k, p_k), k \in \mathbb{Z}\}$, in terms of hyperelliptic theta-functions associated to $\Gamma$, whose arguments depend linearly on the discrete time $k \in \mathbb{Z}$. As follows from Proposition 1, the two branches of $\mathcal{B}$ corresponding to $P = (\lambda_*, \mu_*)$, $(\lambda_*, -\mu_*)$ are just shifts by opposite vectors $\pm T$. This observation is consistent with Remark 3 above.

2.4. **Special cases** $\lambda_* = a_j$. According to formula (2.20), when $(\lambda_*, \mu_*) = (a_j, 0)$ the translation $T$ is just a half-period in $\text{Jac}(\Gamma)$ and, in view of (2.19), the map $\mathcal{B}$ is single-valued. As expected, double iteration $\mathcal{B}^2$ gives the same point in the Jacobian. On the other hand, $(\tilde{q}, \tilde{p}) = \mathcal{B}^2(q, p) = (-q, -p).

This can be checked directly: let us set $\lambda_* = a_j$ in (2.17). Since the denominator in $j$-th equations vanishes, for $\tilde{q}_j$ to be finite, one has to set $\gamma = -p_j/q_j$. Then the second equation in (2.17) gives $\tilde{p}_j = \gamma \tilde{q}_j$, i.e., $\tilde{p}_j/\tilde{q}_j = \gamma$, which, under the next iteration of $\mathcal{B}$, implies $\gamma = -\gamma$. Using this relation and iterating (2.14) directly we get $\tilde{q} = -q$, $\tilde{p} = -p$ for any $p, q$. In this sense, the original map $\mathcal{B}$ with $\lambda_* = a_j$ represents the “imaginary unit map”.

2.5. **Continuous Limit.** As also follows from (2.20), the vector $T$ tends to zero when $\lambda_* \to \infty$, which must give the continuous limit of the map (2.14). We shall describe this limit in details in Section 6, jointly with the discrete Neumann system on the Stiefel variety $V_{n,r}$.

3. The Neumann systems on the Stiefel variety $V_{n,r}$.

3.1. **Notation.** The Stiefel variety $V_{n,r}$ is the variety of ordered sets of $r$ orthogonal unit vectors $e_1, \ldots, e_r$ in the Euclidean space $\mathbb{R}^n$, or, the set of $n \times r$ matrices $X = (e_1 \cdots e_r)$ satisfying the constraints $X^T X = I_r$, where, as above, $I_r$ denotes the $r \times r$ unit matrix. $V_{n,r}$ is a smooth $(rn - r(r + 1)/2)$-dimensional submanifold in the space of $n \times r$ real matrices $M_{n,r} = \mathbb{R}^{nr}$. Also, since the left $SO(n)$-action on $V_{n,r}$ is transitive, it is a homogeneous space $SO(n)/SO(n - r)$.

The cotangent bundle $T^*V_{n,r}$ can be realized as the set of pairs of $n \times r$ matrices $(X, P)$, $P = (p_1 \cdots p_r)$, that satisfy the constraints

$$X^T X = I_r, \quad X^T P + P^T X = 0.$$  

(3.1)

The canonical symplectic structure $\omega$ on $T^*V_{n,r}$ is the restriction of the 2-form $\omega_0 = \sum_{i=1}^n \sum_{s=1}^r dp_i^s \wedge de_i^s$ from the ambient space $T^*M_{n,r}$. It is convenient to work with the redundant variables $(X, P)$ and the corresponding Dirac–Poisson structure $\{\cdot, \cdot\}$ (see [16, 43]) on $T^*V_{n,r}$ is described in [24].

The Lie groups $SO(n)$ and $SO(r)$ naturally act on $T^*V_{n,r}$ by left and right multiplications respectively, with the equivariant momentum mappings given by

$$\Phi : T^*V_{n,r} \to so(n), \quad \Phi = PX^T - XP^T = p_1 \wedge e_1 + \cdots + p_r \wedge e_r,$$

$$\Psi : T^*V_{n,r} \to so(r), \quad \Psi = X^TP - P^TX.$$

Note that the actions of $SO(n)$ and $SO(r)$ commute and, in particular, the components of the momentum mapping, $\Phi_{ij}$ and $\Psi_{ij} = \langle e_i, p_j \rangle - \langle e_j, p_i \rangle$ are $SO(r)$
and $SO(n)$-invariant functions, respectively. Here we identified $so(n) \cong so(n)^*$ and $so(r) \cong so(r)^*$ by the use of the invariant metrics on $so(n)$ and $so(r)$ defined by $\langle \eta_1, \eta_2 \rangle = -\frac{1}{2} \text{tr}(\eta_1 \eta_2)$.

### 3.2. The Neumann systems.

By analogy with the system (2.7) on the sphere $S^{n-1}$, one defines a Neumann on the Stiefel variety $V_{n,r}$ as a natural mechanical system with an $SO(n)$-invariant kinetic energy and the quadratic potential function

$$V = -\frac{1}{2} \text{tr}(X^TAX) = -\frac{1}{2} \sum_{i=1}^{r} \langle e_i, A e_i \rangle, \quad A = \text{diag}(a_1, \ldots, a_n).$$

In this paper it is assumed that $a_i \neq a_j, i \neq j$.

Following [24], we consider a family $SO(n) \times SO(r)$-invariant metrics $ds^2_\kappa$ defined by the kinetic energy

$$T_\kappa(X, P) = \frac{1}{2} \langle \Phi, \Phi \rangle + \frac{1}{2} \kappa \langle \Psi, \Psi \rangle = \frac{1}{2} \text{tr}(P^T P) - \left( \frac{1}{2} + \kappa \right) \text{tr}((X^T P)^2),$$

$\kappa$ being a parameter ($\kappa > -1$). In the class of the metrics $ds^2_\kappa$ we have the normal metric induced from a bi-invariant metric on $SO(n)$ ($\kappa = 0$) and the Euclidean metric induced from the Euclidean metric of the ambient space $M_{n,r}$ ($\kappa = -1/2$).

The corresponding Hamiltonian flows are given by

$$\dot{X} = P - (1 + 2\kappa)XP^T X,$$

$$\dot{P} = AX + (1 + 2\kappa)PX^T P + X \Lambda,$$

where the Lagrange multiplier matrix does not depend on $\kappa$:

$$\Lambda = -X^TAX - P^T P. \quad (3.3)$$

Note that, due to the $SO(r)$-symmetry, the momentum mapping $\Psi$ is the integral of the system (3.2). The difference between the two vector fields in (3.2) with different $\kappa$ is proportional to $X' = -XP^T X, P' = PP^T X$. Since $X^T P = -P^T X = \frac{1}{2} \Psi$, the latter describes permanent (steady state) rotations of the vectors $e_1, \ldots, e_r$ in $\mathbb{R}^r = \text{span}(e_1, \ldots, e_r)$, similarly for the vectors $p_1, \ldots, p_r$.

The geometry of Riemannian spaces $(V_{n,r}, ds^2_\kappa)$ is studied in [29]. It appears that the class of the metrics $ds^2_\kappa$ is suitable for studying other natural mechanical problems on $V_{n,r}$ (see [25]).

### 3.3. Lax representations.

As was mentioned in [24], for all $\kappa$, the equations (3.2) imply the “big” $n \times n$-Lax representation with a rational spectral parameter $\lambda$,

$$\frac{d}{dt} \mathcal{L}(\lambda) = [\Phi + \lambda A, \mathcal{L}(\lambda)], \quad \mathcal{L}(\lambda) = \lambda \Phi + XX^T + \lambda^2 A. \quad (3.4)$$

Obviously, (3.4) is not equivalent to (3.2) if $r > 1$.

On the other hand, up to the action of the discrete group $\mathbb{Z}_n^2$ generated by $n$ reflections

$$(e_1^i, \ldots, e_r^i, p_1^i, \ldots, p_r^i) \mapsto (-e_1^i, \ldots, -e_r^i, -p_1^i, \ldots, -p_r^i), \quad i = 1, \ldots, n \quad (3.5)$$

the Neumann flows (3.2) are equivalent to the following “small” $2r \times 2r$ matrix Lax representations with a rational spectral parameter $\lambda$

$$\frac{d}{dt} L(\lambda) = [L(\lambda), N_\kappa(\lambda)], \quad (3.6)$$

$$L(\lambda) = \begin{pmatrix} X^T(\lambda I_n - A)^{-1} P & X^T(\lambda I_n - A)^{-1} X \\ I_r - P^T(\lambda I_n - A)^{-1} P & -P^T(\lambda I_n - A)^{-1} X \end{pmatrix}, \quad (3.7)$$
\[ N_\kappa(\lambda) = \begin{pmatrix} (1 + 2\kappa) XX^T P & I_r \\ \lambda I_r + \Lambda & -(1 + 2\kappa) P^T X \end{pmatrix} \text{,} \] (3.8)

the matrix \( r \times r \) factor \( \Lambda \) being already defined in (3.3).

The Lax representation (3.4) is closely related to that for the integrable Clebsch–Perelomov rigid body system [48] and for \( r = 1 \) it was given by Moser in [43]. It belongs to the class of the Lax matrix representations related to symmetric pairs decompositions of Lie algebras [53]. The small Lax pair (3.6)–(3.8) is a direct generalization of the \( 2 \times 2 \) Lax pair (2.8), (2.9), and it was first given in unpublished manuscript [35].

Note that, by the Wiesnstein–Aronszjn formula (see [3, 43]), the spectral curves associated to the Lax representations (3.4) and (3.6) are birationally equivalent and define the same invariant manifolds of the system.

3.4. Non-commutative integrability. From the Lax matrix \( \mathcal{L}(\lambda) \) we obtain the set of commuting integrals \( \{ f_{k,i}(X, P) \} \) defined by the expansions

\[ \text{tr}(\lambda(PX^T - XP^T) + XX^T + \lambda^2 A)^k = \sum f_{k,i} \lambda^i, \quad k = 1, \ldots, n. \] (3.9)

Apart from these integrals, which are \( SO(r) \)-invariant, the Neumann flows also possess the non-commutative algebra of integrals \( \Psi_{ij} \). As was shown in [24], the above systems are integrable in the non-commutative sense \([46, 41, 12]\). To describe the geometric structure and the dimension of the invariant tori, consider the Poisson reduced space \( T^* V_{n,r}/SO(r) \).

The integrals \( \text{tr}(\Psi^{2k}) \) induce Casimir functions \( J_k, k = 1, \ldots, [r/2] \) and a generic symplectic leaf \( \mathcal{U}_c = \{ J_k = c_k \} \subset T^* V_{n,r}/SO(r) \) has the dimension

\[ 2l = 2r(n - r) + \frac{r(r - 1)}{2} - \left[ \frac{r}{2} \right]. \] (3.10)

In [24] we proved that if all the eigenvalues of \( A \) are distinct, then (3.9) is a complete commutative set on \( \mathcal{U}_c \). As a result, there are \( l + r(r - 1)/2 \) independent functions within the algebra of integrals \( \{ f_{k,l}, \psi_{ij} \} \) and its center has \( l + [r/2] \) independent functions. Note that the invariants \( \text{tr}(\Psi^{2k}) \) functionally depend on \( \{ f_{k,l} \} \), so the center is generated by the integrals (3.9).

Since \( \dim T^* V_{n,r} = (l + r(r - 1)/2 + (l + [r/2]) \), by the theorem of the non-commutative integrability (see e.g., [12]), the Neumann flows (3.2) are completely integrable in the non-commutative sense. The generic motions of the systems, with the momentum \( \Psi \) of the maximal rank, are quasi-periodic over the isotropic tori of dimension

\[ \delta = l + \left[ \frac{r}{2} \right] = \frac{1}{2}(2r(n - r) + \frac{r(r - 1)}{2} - \left[ \frac{r}{2} \right]) + \left[ \frac{r}{2} \right]. \] (3.11)

There is also an alternative path to the reduced system. Namely, the symplectic leaves of the Poisson reduced space \( T^* V_{n,r}/SO(r) \) are the Marsden–Weinstein reduced spaces \( \Psi^{-1}(h)/SO(r)_h \), where \( SO(r)_h = \{ Q \in SO(r) \mid \text{Ad}_Q(h) = h \} \) is the isotropy subgroup of \( h \in \text{so}(r) \). For a regular \( h \), \( SO(r)_h \) is the maximal torus \( T^{[r/2]} \). The factorization of \( \Psi^{-1}(h) \) by \( SO(r)_h \cong T^{[r/2]} \) coincides with the corresponding leaf \( \mathcal{U}_c \subset T^* V_{n,r}/SO(r) \) with \( J_k = c_k, c_k = \text{tr}(h^{2k}), k = 1, \ldots, [r/2] \).

In this way we obtain the reduction of the restricted Neumann flows on \( \Psi^{-1}(h) \) to the symplectic manifolds \( \mathcal{U}_c \). The systems on \( \mathcal{U}_c \) are integrable in the usual commutative sense: the invariant isotropic tori \( T^\delta \) laying on \( \Psi^{-1}(h) \) reduce to the \( l \)-dimensional Lagrangian tori \( T^\delta \) laying on \( \mathcal{U}_c \). These observations can be summarized
The spectral curve $S$ has extra strong singularities. So, the previous results concerning its genus, number of infinite points, etc, should be adopted to our case. For this reason below we give our proper self-contained description.

### 3.5. The case $\Psi = 0$ and reduction to the Grassmann variety

The oriented Grassmannian $G_{n,r}$ is the variety of $r$-dimensional oriented planes passing through the origin in $\mathbb{R}^n$. It is a quotient space of the Stiefel manifold $V_{n,r}$ with respect to the right $SO(r)$–action via submersion $\pi(e_1 e_2 \cdots e_r) = e_1 \wedge e_2 \wedge \cdots \wedge e_r$, and its cotangent bundle $T^*G_{n,r}$ is symplectomorphic to the reduced space $U_0 = \Psi^{-1}(0)/SO(r)$. Note that $(X, P)$ belongs to $\Psi^{-1}(0)$ if and only if $X^T P = 0$, and all reduced systems have the same kinetic energy given by a normal metric on $G_{n,r}$. In [24] we proved that the integrals $\{f_{k,i}(X, P)\}$ induce a complete commutative set on $U_0$. Therefore, the reduced Neumann flow on $T^*G_{n,r}$ is completely integrable in the usual Liouville sense.

### 4. The spectral curve

The Lax matrix (3.7) is a particular case of so-called rank $\rho$ perturbations of the constant matrix $A = \text{diag}(a_1, \ldots, a_n)$, which generate Lax pairs of a great variety of integrable systems and have the form

$$L(\lambda) = Y + \sum_{i=1}^{n} \frac{N_i}{\lambda - a_i}, \quad Y, N_i \in \mathfrak{gl}(\rho),$$

(4.1)

where $Y$ is constant and $N_i$ depend on the variables of the corresponding dynamical system. In our case $\rho = 2r$, $N_i$ have rank 1, namely

$$N_i = (e_1^i \cdots e_r^i - p_1^i \cdots p_r^i)^T (p_1^i \cdots p_r^i e_1^i \cdots e_r^i),$$

and

$$Y = \begin{pmatrix} 0 & 0 \\ I_r & 0 \end{pmatrix}.$$  

(4.2)

General cases of rank $\rho$ perturbations corresponding to $Y$ with a simple spectrum and the properties of the corresponding spectral curves were studied in detail in [18, 51, 2, 4]. The matrix (3.7) however has a quite special structure, and its spectral curve $S$ has extra strong singularities. So, the previous results concerning its genus, number of infinite points, etc, should be adopted to our case. For this reason below we give our proper self-contained description.

Namely, multiplying $L(\lambda)$ in (3.7) by $a(\lambda) = (\lambda - a_1) \cdots (\lambda - a_n)$, we obtain the polynomial Lax matrix

$$L(\lambda) = a(\lambda)L(\lambda),$$

(4.3)

defining the spectral curve

$$S \subset \mathbb{C}^2{\lambda, w} : \quad F(\lambda, w) = \det(L(\lambda) - wI_{2r}) = 0.$$  

(4.4)

By expanding $F(\lambda, w)$ in $w$ we get

$$F(\lambda, w) = w^{2r} + w^{2r-2}a(\lambda)I_2(\lambda) + \cdots + w^2 a^{2r-3}(\lambda)I_{2r-2}(\lambda) + a^{2r-1}(\lambda)I_{2r}(\lambda),$$

in the following diagram, where the vertical arrows represent factorizations and the horizontal arrows the corresponding inclusions:

$$
\begin{array}{ccc}
T^*V_{r,n} & \xleftarrow{\Psi = h} & \Psi^{-1}(h) \supset T^h \\
\downarrow /SO(r) & & \downarrow /T^{[r/2]} \\
T^*V_{r,n}/SO(r) & \xleftarrow{\mathcal{L}_1 = e_1, \ldots, \mathcal{L}_{[r/2]} = e_{[r/2]}} & \mathcal{U}_c \supset T^l
\end{array}
$$

(3.12)
where $\mathcal{I}_{2l}(\lambda)$ is a polynomial of degree $n - l$ in $\lambda$ with the leading coefficient $C_l^r = r!c(r-l)$. Their explicit expressions are given in [24].

Due to the symplectic block structure of $L(\lambda)$, the coefficients at odd powers of $w$ in $F(\lambda, w)$ are zero, so the spectral curve has the involution

$$\sigma: (\lambda, w) \mapsto (\lambda, -w).$$

(4.5)

Note that, although the Lax matrix $L(\lambda)$ is not invariant under the right $SO(r)$-action, the spectral curve and therefore all the integrals $\mathcal{I}_{2l}(\lambda)$ are invariant. As we already mentioned, the spectral curve (4.4) is birationally equivalent to the spectral curve of the Lax matrix $L$, and the integrals $\{\mathcal{I}_{2l}(\lambda)\}$ are equivalent to the commuting integrals $\{f_{k,i}\}$.

4.1. Genus of the regularized spectral curve. From now on we consider the spectral curve $S$ in (4.4) as its projective closure in $\mathbb{P}^2(\xi: \eta: \zeta)$ such that $\lambda = \xi/\zeta$, $w = \eta/\zeta$. First, note that in its finite part, $S$ has singular points

$$S_i = (\lambda = a_i, w = 0), \quad i = 1, \ldots, n,$$

where all the branches of $w(\lambda)$ meet and all the partial derivative of $F(\lambda, w)$ vanish up to order $2r - 2$.

Let $S'$ be a complete regularization of $S$. To regularize $S$ at $S_i$, we observe that the eigenvectors of the polynomial matrix $L(\lambda)$ for $\lambda = a_i$ are proportional to those of $N_i$ in (4.1). As was shown in [4], if rank($N_i$) = $k_i$, then $L(a_i)$ has $2r - k_i$ independent eigenvectors. Hence, each $S_i \in S$ has rank 1 and $2r - 1$ independent eigenvectors. In our case any $N_i$ has rank 1 and $2r - 1$ different points on $S'$, one of them being ordinary (second order) branch point $P_{a_i}$ of the covering $S' \rightarrow \mathbb{P}^1$.

Following the theory of singularities (see e.g., [36]), this implies that at $S_i$, the curve $S$ has singularity ($\delta$-invariant) of order

$$\delta_i = (2r - 1)(r - 1), \quad i = 1, \ldots, n.$$

4.2. Singularity at the infinity. The curve $S \subset \mathbb{P}^2(\xi: \eta: \zeta)$ has only one infinite point $(0 : 1 : 0)$. Indeed, the structure of $F(\lambda, w)$ in (4.4) implies that $w \cong O(\lambda^{2n-1}/2)$ as $\lambda \rightarrow \infty$. Then, in the neighborhood of $(0 : 1 : 0)$, it is convenient to use local coordinates $t, w$ such that

$$\lambda = \frac{1}{t}, \quad w = \frac{w}{t^n} \quad \text{and, therefore,} \quad \xi = \frac{t^{n-1}}{w}, \quad \zeta = \frac{t^n}{w}.$$  

(4.6)

Substituting this into (4.4) and multiplying by $t^{2nr}$, we obtain the equation

$$(w^2 - t)^r + \text{higher order terms} = 0.$$  

This shows that $S$ has a strong singularity at the infinity. To regularize $S$ there, we construct the Puiseaux expansions of $w$ in powers of $t$. For this purpose, consider first the expansion of the polynomial Lax matrix (4.3)

$$L(\lambda) = \begin{pmatrix} \mathcal{V}(\lambda) & \mathcal{U}(\lambda) \\ \mathcal{W}(\lambda) & -\mathcal{T}(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^{n-1} \mathcal{V}_0 + \lambda^{n-2} \mathcal{V}_1 + \cdots \lambda^{-1} \mathcal{I}_0 + \lambda^{-2} \mathcal{U}_1 + \cdots \\ \lambda^n \mathcal{I}_0 + \lambda^{n-1} \mathcal{V}_0 + \lambda^{n-2} \mathcal{W}_1 + \cdots \end{pmatrix}$$

(4.7)

\footnote{One should replace $\lambda$ and $A$ by $-\lambda$ and $-A$, respectively, to relate the small Lax matrix $L(\lambda)$ and integrals $\mathcal{I}_{2l}(\lambda)$ used here with the matrix $L_{neum}(\lambda)$ and integrals $\mathcal{I}_{2l}(\lambda)$ used in [24].}
with the leading $r \times r$ matrix coefficients

$$V_0 = X^T P, \quad V_1 = X^T A P, \quad U_t = -\text{tr} A L + X^T A X, \quad W_0 = -\text{tr} A L - P^T P,$$

which are $r \times r$ matrix generalizations of the coefficients (2.12). Recall that the entries of $V_0 = X^T P$ are first integrals of the system: $V_0 = \frac{1}{2} \Psi$. Substituting (4.6) into the eigenvector equation

$$L(\lambda)\psi = w\psi, \quad \psi \in \mathbb{P}^{2r-1},$$

we get the expansion

$$\left( V_0 t + V_1 t^2 + \cdots \right) \psi(t) = w(t)\psi(t). \quad (4.8)$$

As $t \to 0$, the eigenvectors of $L(\lambda)$ tend to those of $Y$ in (4.2), which has $r$ different eigenvectors with zero eigenvalue. As a result, the regularized curve $S'$ must have $r$ different points over $\lambda = \infty$, which are all ordinary branch points of the covering $S' \to \mathbb{P}(\lambda)$, and will be denoted as $\infty_1, \ldots, \infty_r$. For a local coordinate near each of these points we take $\tau = \sqrt{t}$, so that $\lambda = 1/\tau^2$.

The structure of (4.8) allows to evaluate the expansions of $w$, $\psi$ in the neighborhood of each $\infty_s$, which will be used in the next sections.

**Proposition 2.**

1) Let rank of $V_0 = X^T P$ be maximal and $r$ be odd. Then let

$$\{\nu_1, \ldots, \nu_{[r/2]}, -\nu_1, \ldots, -\nu_{[r/2]}, 0\}$$

be the set of the eigenvalues of the matrix coefficient $V_0 \in \text{so}(r)$ and

$$\{v_1, \ldots, v_{[r/2]}, \bar{v}_1, \ldots, \bar{v}_{[r/2]}, v_0\}$$

be the corresponding eigenvectors normalized by

$$\langle \beta, v_s \rangle = 1, \quad \langle \beta, \bar{v}_s \rangle = 1, \quad \langle \beta, v_0 \rangle = 1,$$

where $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{C}^r$ is any non-zero vector, not orthogonal to any eigenvector of $V_0$. Then in the neighborhood of $\infty_1, \ldots, \infty_r$, the expansions $w(\tau)$ are, respectively,

$$w_s(\tau) = \tau + \nu_s \tau^2 + b_s \tau^3 + \cdots,$$

$$w_{[r/2]+s}(\tau) = \tau - \nu_s \tau^2 + b_s \tau^3 - \cdots, \quad s = 1, \ldots, [r/2], \quad (4.9)$$

$$w_r(\tau) = \tau + b_0 \tau^3 + B_0 \tau^5 + \cdots,$$

and the corresponding expansions of the eigenvector $\psi(\tau) = (\psi^1, \ldots, \psi^{2r})^T$ normalized by

$$\beta_1 \psi^{r+1} + \cdots + \beta_r \psi^{2r} = 1 \quad (4.10)$$

are

$$\psi_s(\tau) = \left( v_s \tau + h_s \tau^2 + O(\tau^3) \right),$$

$$\psi_{[r/2]+s}(\tau) = \left( \bar{v}_s \tau + h_s \tau^2 + O(\tau^3) \right),$$

$$\psi_r(\tau) = \left( v_0 \tau + h_0 \tau^2 + O(\tau^3) \right),$$

(4.11)
Here \( b_s, b_0, B_0 \) are certain constants, and \( \mathbf{h}_s, \mathbf{h}_l, \mathbf{h}_0 \) are constant vectors. The last expansions \( \mathbf{w}_s(\tau), \psi_s(\tau) \) contain only odd powers of \( \tau \).

2) If rank of \( \mathcal{V}_0 \) is maximal and \( r \) is even, then item 1) holds without the last expansions \( \mathbf{w}_s(\tau), \psi_s(\tau) \).

3) If \( \mathcal{V}_0 = 0 \), then all the expansions contain only odd powers of \( \tau \).\(^4\)

\textbf{Proof.} It is sufficient to substitute the above expansions into (4.8) and compare few leading terms in both sides.

The involution \( \sigma(\lambda, w) = (\lambda, -w) \) on \( S \) can be naturally continued to the involution on the regularized curve \( S' \) by means of the mapping

\[ \tau \mapsto -\tau. \] (4.12)

Namely, if rank(\( \mathcal{V}_0 \)) = \( 2[r/2] \), it takes the series \( \mathbf{w}_s(\tau), s = 1, \ldots, [r/2] \) in (4.9) to \(-\mathbf{w}_{[r/2]+s}(\tau)\) and vice versa (if \( r \) is odd, \( \mathbf{w}_r(\tau) = -\mathbf{w}_r(\tau) \)). Then, since \( \lambda(\tau) = 1/\tau^2 \) and \( w(\tau) = \mathbf{w}(\tau)/\tau^{2n} \), the mapping (4.12) on \( S' \setminus \{\infty_1, \ldots, \infty_r\} \) coincides with the action (4.5) on \( S \setminus \{(0 : 1 : 0)\} \).

Next, the infinite points of \( S' \) themselves admit the division into the subsets

\[ \{\infty_1, \ldots, \infty_{r/2}\}, \{\infty_1^+, \ldots, \infty_{r/2}^+\} \quad \text{if } r \text{ is even}, \]
\[ \{\infty_1, \ldots, \infty_{r/2}\}, \{\infty_1^+, \ldots, \infty_{r/2}^+\}, \{\infty_0\} \quad \text{if } r \text{ is odd}, \] (4.13)

such that the extension of \( \sigma \) onto \( S' \) is defined by

\[ \sigma(\infty_j^+) = \infty_{j+}, \quad j = 1, \ldots, [r/2], \quad \sigma(\infty_0) = \infty_0, \] (4.14)

and (4.12) maps the neighborhoods of \( \infty_j^-, \infty_j^+, \infty_0 \) to the neighborhoods of \( \infty_j^+, \infty_j^-, \infty_0 \), respectively.

Note that in the special case \( \mathcal{V}_0 = 0 \), due to item 3) of Proposition 2, all the infinite points are invariant with respect to the involution:

\[ \sigma(\infty_j) = \infty_j, \quad j = 1, \ldots, r. \] (4.15)

Using the expansions \( \mathbf{w}_s(t) \) and the second pair of relations (4.6), one calculates the set of the Puiseux expansions of the coordinate \( \zeta \) in powers of \( \xi \) in the neighborhood of \((0 : 1 : 0)\). Then, applying the theory of singularities of algebraic curves (see e.g., [13, 36]), we obtain the order \( \delta_\infty \) of singularity of \( S \) at \((0 : 1 : 0)\)\(^5\):

\[ \delta_\infty = 2nr(nr - 2r - 1) + 2r(r + 1) \quad \text{if } \text{rank}(X^TP) = 2[r/2], \]
\[ \delta_\infty = 2nr(nr - 2r - 1) + r^2 + \frac{3r(r + 1)}{2} \quad \text{if } X^TP = 0. \] (4.16)

Now, summing the singularity orders \( \delta_\infty \) and \( \delta_1, \ldots, \delta_n \) and taking into account that the degree of the curve \( S \) equals

\[ \deg(S) = n(2r - 1) + (n - r) = 2nr - r, \]
we use the Plücker formula to obtain the following result.

\textbf{Theorem 4.1.} For generic values of commuting integrals (3.9), the geometric genus of \( S' \) equals

\[ \text{gen}(S') = \frac{(\deg(S) - 1)(\deg(S) - 2)}{2} - \sum_{i=1}^{n} \delta_i - \delta_\infty = 2nr - n - \frac{3r^2}{2} - \frac{1}{2}r + 1. \]

\(^4\)Here we assume that the integrals \((I_{2\theta}(\lambda))\) take appropriate generic values, which ensures complete integrability of the Neumann system on \( G_{n,s} \).

\(^5\)These long calculations are due to Maria Alberich (UPC)
If \( \Psi = 0 \) we have \( \text{gen}(S') = 2r(n - r) - n + 1 \).

4.3. The “small” curve \( \mathcal{C} = S/\sigma \) and its genus. In view of the involution \( (4.5) \), the curve \( S \) is a 2-fold ramified covering of the curve \( C \subset \mathbb{C}^2(u, \lambda), u = w^2 \), given by the equation

\[
C: \quad u^r + u^{r-1}a(\lambda)I_2(\lambda) + \cdots + u \cdot a^{2r-3}(\lambda)I_{2r-2}(\lambda) + a^{2r-1}(\lambda)I_{2r}(\lambda) = 0
\]

Making the birational transformation

\[
(\lambda, u) \mapsto (\lambda, \mu)
\]

with \( u = a(\lambda) \mu \), we see that the equation of \( C \) gives the lines \( \{ \lambda = a_i \} \) and the curve

\[
\tilde{C}: \quad \mu^r + \mu^{r-1}I_2(\lambda) + \cdots + \mu a^{r-2}(\lambda)I_{2r-2}(\lambda) + a^{r-1}(\lambda)I_{2r}(\lambda) = 0. \tag{4.17}
\]

The latter is singular over \( \lambda = a_1, \ldots, a_n \) and at its infinite part.

Let \( \mathcal{C}' \) be a complete regularization of \( \tilde{C} \). Then the regularized curve \( S' \) is a 2-fold covering of \( \mathcal{C}' \). The covering

\[
\pi: \quad S' \longrightarrow \mathcal{C}' = S'/\sigma
\]

is branched when \( \mu = 0 \), that is, when \( \lambda = a_i \) or \( \lambda \) is a simple root of the last polynomial \( I_{2r}(\lambda) \) of degree \( n - r \) in \( (4.17) \). As mentioned above, the projection \( S' \rightarrow \mathbb{P}\{\lambda\} \) has only one ordinary branch point \( P_{a_i} \) over \( \lambda = a_i \), hence \( P_{a_i} \) is also the only branch point of \( (4.18) \) over \( \lambda = a_i \). (This means that the projection \( \mathcal{C}' \rightarrow \mathbb{P}\{\lambda\} \) is not ramified over \( \lambda = a_i \).) The roots of \( I_{2r}(\lambda) \) give other \( n - r \) ordinary branch points of \( \pi \). Therefore, \( \pi \) always has \( 2n - r \) finite ordinary branch points.

Next, as follows from \( (4.14) \) if \( \text{rank}(\Psi) = 2[r/2] \) and \( r \) is odd, the covering \( (4.18) \) is also ramified at \( \infty_0 \), and it is not ramified over infinity when \( r \) is even.

If \( \Psi = 0 \), \( (4.15) \) implies that \( \infty_1, \ldots, \infty_r \) are ordinary branch points of \( \pi \).

As a result, the covering \( (4.18) \) has in total \( B = 2n - 2[r/2] \) ordinary branch points if \( \text{rank}(\Psi) = 2[r/2] \), and \( B = 2n \) ordinary branch points if \( \Psi = 0 \). Then, in view of the Riemann–Hurwitz formula

\[
g = \text{gen}(S') = 2(\text{gen}(\mathcal{C}') - 1) + \frac{B}{2} + 1 \tag{4.19}
\]

and Theorem 4.1, the following result holds.

**Proposition 3.** If \( \text{rank}(\Psi) = 2[r/2] \),

\[
g_0 = \text{gen}(\mathcal{C}') = \begin{cases} n(r - 1) - 3(r^2 - 1)/4 & \text{if } r \text{ is odd}, \\ n(r - 1) - 3r^2/4 + 1 & \text{if } r \text{ is even} \end{cases}
\]

and, if \( \Psi = 0 \), \( g_0 = r(n - r) + n - 1 \).

In the next section the genera of \( S', \mathcal{C}' \) will be used to calculate the dimension of the Abelian subvarieties of the Jacobian of \( S' \), which are related to complex invariant tori of the system.

4.4. The case \( r = 2 \). In this simplest case it is possible to calculate the genera \( g_0, g \) without long calculations based on the above Puisaux expansions. Namely, the curve \( \tilde{C} \) in \( (4.17) \) takes the simple form

\[
\tilde{C}: \quad \mu^2 + \mu I_2(\lambda) + a(\lambda)I_4(\lambda) = 0.
\]
which, under the birational change $\mu = -I_2(\lambda)/2 + y/2$, gives the Weierstrass hyperelliptic form

$$y^2 = I_2^2(\lambda) - 4a(\lambda) I_4(\lambda).$$

(4.20)

To find its genus, note that here $I_2(\lambda), I_4(\lambda)$ are polynomials of degrees $n - 1, n - 2$ respectively, hence one might expect the right hand side of (4.20) to be a polynomial of degree $2n - 2$. However, in the general case (when $\Psi \neq 0$), the degree is

\[\text{deg}(\tilde{C}) = 2n - 3 = 2(n - 2) + 1\]

(in view of the expressions for $I_2(\lambda), I_4(\lambda)$, the coefficient at the leading power $\lambda^{2n-2}$ vanishes). Then the regularization $C'$ of $\tilde{C}$ has genus $g_0 = n - 2$ and one infinite point.

Next, as was counted above, for $r = 2$ and $\Psi \neq 0$, the covering (4.18) has in total $B = 2n - 2$ ordinary ramification points. Hence, by the Riemann–Hurwitz formula (4.19), the genus of the “big” curve $S'$ is

$$g = 2(n - 2 - 1) + (2n - 2)/2 + 1 = 3n - 6,$$

and $S'$ has 2 infinite points over $\lambda = \infty$, which pass to each other under $\sigma$.

In the case $\Psi = 0$ the right hand side of (4.20) becomes a polynomial of degree $2n - 4$, the curve $C'$ has now 2 infinite points, its genus drops to $g_0 - 3$, whereas the covering $\pi$ has $2n$ ordinary ramifications. All this yields $g = 3n - 7$.

5. Complexified toric foliations and linearization of the flows. In this section we study algebraic geometrical aspects of the complexified toric foliations presented at the diagram (3.12) and show that the invariant varieties of the non-reduced system, modulo $\mathbb{Z}_2^2$–action, are affine (i.e., non-compact) Prym subvarieties of generalized Jacobians of the spectral curve $S'$, while the complex invariant tori of the reduced system are the usual Prym varieties $\text{Prym}(S', \sigma) \subset \text{Jac}(S')$.

5.1. The Prym variety. From the curve $S'$ we pass to its Jacobian variety, $\text{Jac}(S')$, defined as the additive group of degree zero divisors on $S'$ modulo divisors of meromorphic functions. Equivalently, in some cases we will consider effective divisors $D$ as elements of $\text{Jac}(S')$ by choosing a basepoint $P_0 \in S'$ and associating to $D$ the degree zero divisor $D - NP_0$, where $N$ is the degree of $D$.

The involution $\sigma : S' \to S'$, $\sigma(\lambda, w) = (\lambda, -w)$ extends to the Jacobian of $S'$. Hence $\text{Jac}(S')$ contains two Abelian subvarieties: the Jacobian of the “small” underlying curve $C'$ of dimension $g_0 = \text{gen}(C')$ and $\text{Prym}(S', \sigma)$ of dimension

$$\text{dim}(\text{Prym}(S', \sigma)) = \text{gen}(S') - \text{gen}(C').$$

The vectors of $\text{Jac}(C') \subset \text{Jac}(S')$ are invariant with respect to $\sigma$, whereas the vectors of $\text{Prym}(S', \sigma) \subset \text{Jac}(S')$ are anti-invariant.

Algebraically, $\text{Prym}(S', \sigma)$ is defined as the set of degree zero divisors $D$ on $S'$ such that $D + \sigma D$ is the divisor of a meromorphic function on $S'$. For a given degree zero divisor $H$, a translate of $\text{Prym}(S', \sigma)$ in $\text{Jac}(S')$ is the set of degree zero divisors $D$ satisfying

$$D + \sigma D \equiv H,$$

where $\equiv$ denotes the equivalence modulo divisors of meromorphic functions on $S'$.

The variety $\text{Jac}(S')$ is isogeneous to $\text{Jac}(C') \times \text{Prym}(S', \sigma)$. A detailed algebraic description of Prym varieties in the case of a general double covering $S' \to C'$ ramified at $N > 2$ points is given in [20] (Chapter V). Using this description and
of degree zero divisors.

Both kinds include the simplest case when the singularized curve has only one double point. When several points of a smooth curve are glued to the same point, as described in [10, 54, 27].

our previous calculations, one can show that Prym($S', \sigma$) is an Abelian variety with polarization

$$\begin{align*}
(1, \ldots, 1, 2 \ldots, 2), & \quad (1, \ldots, 1, 2 \ldots, 2), \\
\underset{g_0}{n-\lfloor r/2 \rfloor - 1}, & \quad \underset{g_0}{n-1}
\end{align*}$$

for $\Psi$ of maximal rank and $\Psi = 0$, respectively. We will not use the polarization data in the sequel.

From Theorem 4.1 and Proposition 3 we immediately obtain

**Proposition 4.**

1) The dimension of the Abelian subvariety Prym($S', \sigma$), for generic constants of motion, equals

$$l = \frac{1}{2} \left(2r(n-r) + \frac{r(r-1)}{2} - \left\lfloor \frac{r}{2} \right\rfloor \right),$$

which coincides with one half of the dimension of a generic symplectic leaf $\mathcal{U}_c$ in $T^*V_{n,r}/SO(r)$.

2) In the special case $\Psi = 2X^TP = 0$ the dimension of Prym($S', \sigma$) is $r(n-r)$, which coincides with the dimension of $G_{n,r}$.

Note that in the simplest case $r = 2$ (and only in this case), the dimension of Prym($S', \sigma$) equals $2(n-2)$, i.e., the dimension of the Grassmanian $G_{n,2}$, regardless to whether $\Psi = 0$ or not.

5.2. The generalized Jacobian and the affine Prym subvariety. Assuming that the rank of the momentum $\Psi$ is maximal, we also introduce the generalized Jacobian of the singularized curve $S''$ obtained from $S'$ by gluing pairwise the involutive infinite points $\infty_{1,}, \infty_{2,}^\pm, s = 1, \ldots, \lfloor r/2 \rfloor$ described in (4.13), (4.14). Namely

$$S'' = S_0 \cup \{\infty_1, \ldots, \infty_{\lfloor r/2 \rfloor}\}, \quad S_0 = S' \setminus \{\infty_1^+, \infty_1^-, \ldots, \infty_{\lfloor r/2 \rfloor}^+, \infty_{\lfloor r/2 \rfloor}^-\}$$

so that $\infty_1, \ldots, \infty_{\lfloor r/2 \rfloor}$ are ordinary double points of $S''$. Then a meromorphic function $f$ on $S'$ will be also meromorphic on $S''$ if $f(\infty_{s}^-) = f(\infty_{s}^+)$. The generalized Jacobian of $S''$, denoted as Jac($S', \infty$) is then defined as the set of degree zero divisors $\mathcal{P}$ on the affine part $S_0$ considered modulo $\infty$-equivalence: $\mathcal{P} \equiv_{\infty} \mathcal{Q}$ if there exists a meromorphic function $f$ on the smooth curve $S'_{\infty}$ such that

1) $(f) = P - Q$, 2) $f(\infty_{s}^-) = f(\infty_{s}^+) \neq 0, \infty, \quad s = 1, \ldots, \lfloor r/2 \rfloor$

(note that the values $f(\infty_{s}^+)$, $f(\infty_{j}^+)$ for $i \neq j$ can be different). 6

Thus, there is an exact sequence of algebraic groups

$$
0 \xrightarrow{\exp} (\mathbb{C}^*)^{[r/2]} \xrightarrow{v} \text{Jac}(S', \infty) \xrightarrow{\phi} \text{Jac}(S') \xrightarrow{0}
$$

(5.1)

where, for $\rho = (\rho_1, \ldots, \rho_{\lfloor r/2 \rfloor}) \in \mathbb{C}^*[r/2]$, the image $v(\rho)$ is the divisor of any meromorphic function $f(p)$ on $S'$ satisfying

$$f(\infty_{s}^+)/f(\infty_{s}^-) = \rho_s, \quad s = 1, \ldots, \lfloor r/2 \rfloor,
$$

(5.2)

and for any degree zero divisor $\mathcal{P}$, $\phi(\mathcal{P})$ gives a point in Jac($S'$).

The analytical description of Jac($S', \infty$) is following (see, e.g., [14, 21]): let $\omega_1, \ldots, \omega_g$ be $g$ independent holomorphic differentials on $S'$ and $\Omega_j$ be meromorphic differential of the 3rd kind having a pair of simple poles at $\infty_{j}^+, \infty_{j}^-$ respectively.

---

6The above definition should be distinguished from the other kind of generalized Jacobians, when several points of a smooth curve are glued to the same point, as described in [10, 54, 27]. Both kinds include the simplest case when the singularized curve has only one double point.
One can always normalize the meromorphic differentials in such a way that they will be anti-invariant under the involution:

\[
\sigma^* \Omega_j = -\Omega_j, \quad j = 1, \ldots, [r/2].
\]

Next, let \( \Lambda \subset \mathbb{C}^g(z_1, \ldots, z_g) \) be the lattice generated by vectors of periods of \((\omega_1, \ldots, \omega_g)^T\) with respect to a basis in \(H_1(S', \mathbb{Z})\). Respectively, let \( \tilde{\Lambda} \subset \mathbb{C}^{g+[r/2]}(z_1, \ldots, z_g, Z_1, \ldots, Z_{[r/2]}\) be the lattice generated by vectors of periods of \((\omega_1, \ldots, \omega_g, \Omega_1, \ldots, \Omega_{[r/2]})^T\) with respect to a basis in \(H_1(S_0, \mathbb{Z})\) formed by \(2g\) canonical cycles on \(S'\) and the homology zero cycles embracing \(\infty_1^+, \ldots, \infty_{[r/2]}^+\). The lattice \( \Lambda \) has rank \(2g + [r/2]\), and the generalized Jacobian \( \tilde{\text{Jac}}(S', \infty) \) is the factor \( \mathbb{C}^{g+[r/2]}/\tilde{\Lambda}, \) which is a non-compact algebraic group. The map \( \phi \) in (5.1) acts as projection:

\[
\phi(z_1, \ldots, z_g, Z_1, \ldots, Z_{[r/2]}) = (z_1, \ldots, z_g).
\]

The relation between the above analytical and algebraic descriptions of \( \tilde{\text{Jac}}(S', \infty) \) is given by the generalized Abel map with a basepoint \( P_0 \in S_0 \)

\[
\tilde{\Delta}(P) = \int_{P_0}^P (\omega_1, \ldots, \omega_g, \Omega_1, \ldots, \Omega_{[r/2]})^T \in \mathbb{C}^{g+[r/2]}, \quad P \in S_0.
\]

A general point in \( \tilde{\text{Jac}}(S', \infty) \) is the Abel image of an effective divisor of degree \( g + [r/2] \) on \( S_0 \). The inversion of the generalized Abel map in terms of generalized theta-functions is described in [14, 21].

A generalization of the Abel theorem says that two effective divisors \( \mathcal{P} = P_1 + \cdots + P_N, \quad \mathcal{Q} = Q_1 + \cdots + Q_N \) on \( S_0 \) are \( \infty \)-equivalent \( (\mathcal{P} \equiv \mathcal{Q}) \) if and only if

\[
\tilde{\Delta}(P_1) + \cdots + \tilde{\Delta}(P_N) = \tilde{\Delta}(Q_1) + \cdots + \tilde{\Delta}(Q_N) \quad \text{modulo} \quad \tilde{\Lambda}.
\]

Note that two divisors \( D_1, D_2 \) on \( S_0 \) can correspond to the same point on \( \text{Jac}(S) \), but to different points on \( \tilde{\text{Jac}}(S', \infty) \).

5.3. The affine Prym variety. The involution (4.5) extends to \( \tilde{\text{Jac}}(S', \infty) \), which then contains two algebraic subgroups: one is invariant with respect to \( \sigma \) and the other is anti-invariant. The first is just the usual Jacobian \( \text{Jac}(C') \) of dimension \( g_0 \). The second is the affine Prym subvariety \( \tilde{\text{Prym}}(S', \sigma) \subset \tilde{\text{Jac}}(S', \infty) \), which can be defined as the set of degree zero divisors \( \mathcal{P} \) on \( S_0 \) such that

\[
\mathcal{P} + \sigma \mathcal{P} = \{ \text{a divisor of a meromorphic function } f \}
\]

with \( f(\infty_1^+) = f(\infty_s^-) \neq 0, \infty, \quad s = 1, \ldots, [r/2] \} \)

= the origin in \( \tilde{\text{Jac}}(S', \infty) \).

In view of the definition of the map \( \upsilon : (\mathbb{C}^*)^{[r/2]} \mapsto \tilde{\text{Jac}}(S', \infty) \), the action of \( \sigma \) on the set \( \upsilon \) \( (\mathbb{C}^*)^{[r/2]} \) is given by

\[
\sigma(\upsilon(\rho_1, \ldots, \rho_{[r/2]})) = \upsilon(1/\rho_1, \ldots, 1/\rho_{[r/2]}).
\]

Then we obtain

\footnote{We do not use term “generalized Prym variety”, quite natural in this situation, since it is related to a completely different construction considered in [9].}
Lemma 5.1. The set \( (\mathbb{C}^*)^{[r/2]} \) belongs to \( \widetilde{\text{Prym}}(S', \sigma) \). The involution \( \sigma \) on it has \( 2^{[r/2]} \) fixed points \( v(\pm 1, \ldots, \pm 1) \), which are also half-periods in \( \text{Jac}(S', \infty) \) and in \( \text{Prym}(S', \sigma) \).

Proof. Let \( \mathcal{P} \) be the divisor of any meromorphic function \( f(p) \) on \( S' \) satisfying (5.2), and let \( f_\sigma(p) = f(\sigma(p)) \) so that \( f_\sigma(\infty_+^+/\infty_-) = 1/\rho_\sigma \). Then
\[
(f \cdot f_\sigma) = \mathcal{P} + \sigma \mathcal{P}, \quad f(\infty_+^+) f_\sigma(\infty_+) = f(\infty_-) f_\sigma(\infty_-),
\]
hence, by the definition, \( \mathcal{P} \) belongs to \( \widetilde{\text{Prym}}(S', \sigma) \).

Next, if \( \mathcal{P} \) is the divisor of a meromorphic function \( f(p) \) satisfying \( f(\infty_+^+) = \pm f(\infty_-^+) \), then \( (f^2) = 2\mathcal{P} \) and \( f^2(\infty_+^+) = f^2(\infty_-^+) \). That is, \( 2\mathcal{P} \) corresponds to the origin in \( \text{Jac}(S', \infty) \), and \( \mathcal{P} \) is a half-period of the generalized Jacobian and of the affine Prym subvariety.

By the above lemma, the exact sequence (5.1) implies the following sequence of groups
\[
0 \to \text{exp} \to (\mathbb{C}^*)^{[r/2]} \to \widetilde{\text{Prym}}(S', \sigma) \to \text{Prym}(S, \sigma) \to 0.
\]
Since \( \phi(v((\mathbb{C}^*)^{[r/2]})) = 0 \), the sequence is also exact.

It follows that, like \( \text{Jac}(S', \infty) \), the variety \( \widetilde{\text{Prym}}(S', \sigma) \) is non-compact and has dimension \( g + [r/2] - g_0 = l + [r/2] \), which, in view of Proposition 4, equals \( \delta \), the dimension of generic invariant tori in \( T^*V_{n,r} \).

Let us note that the generalized Jacobian \( \text{Jac}(S', \infty) \) is isogeneous to the product \( \text{Jac}(\mathcal{C}) \times \widetilde{\text{Prym}}(S', \sigma) \).

5.4. The eigenvector map. To proceed further, we need to recall some basic notions relating Lax matrices and Abelian varieties, and we follow the description of [18, 38, 4, 10, 53].

Let \( L(\lambda) \) be a \( d \times d \) polynomial Lax matrix and \( S \) be its spectral curve (with its infinite points), which is assumed to be regular or appropriately regularized, of geometric genus \( g \). \( L(\lambda) \) defines the eigenvector bundle \( \varepsilon : S \to \mathbb{P}^{d-1} \) given by the eigenvectors \( \psi(P) = (\psi^1(P), \ldots, \psi^d(P))^T \), \( P \in S \) of \( L(\lambda) \).

Imposing a normalization \( \langle \alpha, \psi(P) \rangle = 1 \), where \( \alpha = (\alpha_1, \ldots, \alpha_{2r})^T \in \mathbb{P}^{2r-1} \) is an arbitrary constant vector, and consider the minimal effective divisor \( \mathcal{D}_\alpha \) on \( S \) such that
\[
(\psi^l(P)) = \text{zeros of } \psi^l(P) - \text{poles of } \psi^l(P) \geq -\mathcal{D}_\alpha, \quad l = 1, \ldots, d,
\]

that is, each of the normalized components \( \psi^l(P) \) can have poles at most at the points of \( \mathcal{D}_\alpha \). These points can be found as common zeros of certain polynomials of \( \lambda, w \) (see, e.g., [4]). Also
\[
\deg(\mathcal{D}_\alpha) = g + d - 1 = N. \tag{5.5}
\]
(One can always choose such a normalization \( \alpha \) that \( d - 1 \) points of \( \mathcal{D}_\alpha \) will be fixed in the infinite part of \( S \).)

It is known that two divisors \( \mathcal{D}_\alpha, \mathcal{D}_\alpha' \) corresponding to different normalizations \( \alpha, \alpha' \) are linearly equivalent. Therefore, for a basepoint \( P_0 \in S \), the degree zero divisors \( \mathcal{D}_\alpha - NP_0, \mathcal{D}_\alpha' - NP_0 \) give the same point in \( \text{Jac}(S) \), more precisely, in the open subset \( \text{Jac}(S) \setminus \Theta \), with \( \Theta \in \text{Jac}(S) \) being a translate of the theta-divisor, the Abel image of all the special divisors on \( S \).
Let now $I_S$ be the isospectral manifold: the set of all the above matrices $L(\lambda)$ having the same spectral curve $S$. Thus we get the eigenvector map

$$\mathcal{E}: I_S \rightarrow \text{Jac}(S).$$

If the equivalence class $\{D\}$ is the image of a matrix $L(\lambda) \in I_S$, the latter can be reconstructed up to a conjugation by an element of the group $\mathbb{PGL}(d, \mathbb{C})$ (not depending of $\lambda$). The main steps of this are as follows: Let $L(D)$ be the vector space of meromorphic functions $f(P)$ on $S$ with $(f) \geq D$ (it includes $f \equiv 1$).

According to the Riemann–Roch theorem, for a non-special degree $N$ divisor $D$, $\dim (L(D)) = N - g + 1$, hence, in the considered case, $\dim (L(D))$ is precisely $d$.

There exists a basis $\{f_1(P), \ldots, f_d(P)\}$ in $L(D)$ such that $f(P) = (f_1(P), \ldots, f_d(P))^T$, $P = (\lambda, \mu) \in S$ is an eigenvector$^8$ of $L(\lambda)$. To reconstruct $L(\lambda)$, choose $\lambda \in \mathbb{C}$ with distinct eigenvalues $\mu_1, \ldots, \mu_d$ and consider the $d \times d$ matrix

$$F(\lambda) = (f_1(P_1) \cdots f_d(P_d)), \quad P_j = (\lambda, \mu_j).$$

Then the matrix

$$\mathcal{X}(\lambda) = F(\lambda) \text{ diag}(\mu_1, \ldots, \mu_d) F^{-1}(\lambda) \quad (5.6)$$

is independent of the order of $\mu_1, \ldots, \mu_d$, is, in fact, polynomial in $\lambda$, and has the prescribed spectral curve $S$. It reconstructs the matrix $L(\lambda)$ up to a conjugation by a constant matrix (see e.g., [18, 2]).

Clearly, two matrices $L(\lambda), \tilde{L}(\lambda) \in I_S$ that are conjugated by a constant element of $\mathbb{PGL}(d, \mathbb{C})$ have equivalent divisors $D, \tilde{D}$. The inverse is also true: $D \equiv \tilde{D}$ means the existence of a meromorphic function $g(P)$ on $S$ with $(g) = D - \tilde{D}$. Let the components of $\tilde{f}(P), \bar{f}(P)$ form bases of $L(D), L(\tilde{D})$ respectively. Then the components of $g(P) \tilde{f}(P)$ form a basis of $L(D)$, and there is a non-degenerate matrix $R \in GL(d, \mathbb{C})$ independent of $P$ such that $f(P) = g(P)R\tilde{f}(P)$.

It follows that the induced map

$$\mathcal{M}: \{L(\lambda) \in I_S \text{ up to conjugation by matrices of } \mathbb{PGL}(d, \mathbb{C})\} \mapsto \text{Jac}(S) \setminus \Theta$$

is injective.

When the leading coefficient of $L(\lambda)$ is a constant matrix $J$ and there are no constraints on the other coefficients, it is natural to replace the isospectral manifold $I_S$ by $I_S^J$, the set of the matrices $L(\lambda)$ having the same spectral curve $S$ and leading coefficient $J$. In the general important case, when $J$ is diagonalizable, its stabilizer is the product $(\mathbb{C}^*)^{d-1}$. Then, as was shown in many publications (see [18, 10, 53, 2]), if the curve is smooth, the induced eigenvector map $I_S^J / (\mathbb{C}^*)^{d-1} \rightarrow \text{Jac}(S) \setminus \Theta$ is an isomorphism. The inverse map is described explicitly by means of theta-functions associated to the curve $S$.

If $J$ is not diagonalizable and/or $L(\lambda)$ belongs to a subgroup of $\mathfrak{gl}(d, \mathbb{C})$, the above result should be refined (one must consider the equivalence by conjugations by elements of a subgroup of $\mathbb{PGL}(d, \mathbb{C})$). Some cases of that have been considered in [28, 60, 17].

In addition, some general properties of the eigenvector map in our case, when $L(\lambda)$ belongs to the loop subalgebra $\mathfrak{sp}(2r, \mathbb{C})$, were already studied in [2] (Section 5, case ii).

---

$^8$It is always supposed that $f$ is normalized, i.e., its components do no have a common zero.
5.5. The complex manifolds $\mathcal{I}, \mathcal{I}_{\text{red}}, \mathcal{I}_h$. Now consider the Neumann system on the complexified cotangent bundle $T^*V_{n,r}(\mathbb{C})$ defined by the constrains (3.1) in the complex domain. Let us fix generic values of all commuting integrals $f_{k,i}$ described in Section 3. This also fixes the values of the invariants $\text{tr}(\Psi^{2j})$ and the spectral curve $S'$ of the “small” $2r \times 2r$ Lax matrix (3.7), (4.3), which belongs to the loop subalgebra $\mathfrak{sp}(2r, \mathbb{C})$.

Since there are $l + [r/2]$ independent commuting integrals, the corresponding invariant manifold

$$\mathcal{I} = \{(X, P) \mid f_{k,i} = c_{k,i}, \text{tr}(\Psi^{2j}) = c_j \} \subset T^*V_{n,r}(\mathbb{C})$$

has the (complex) dimension

$$\delta + r(r - 1)/2 - [r/2] = l + \dim SO(r)$$

with $\delta$ given by (3.11).

The manifold $\mathcal{I}$ and the curve $S'$ are invariant with respect to the right action of the complex group $SO(r, \mathbb{C})$ on $(X, P)$, as well as to the action of the group $\mathbb{Z}_2^n$ of reflections (3.5). These two actions commute between themselves and the reduced manifold

$$\mathcal{I}_{\text{red}} = \mathcal{I}/SO(r, \mathbb{C})/\mathbb{Z}_2^n$$

is the complex extension of the corresponding $l$-dimensional invariant isotropic torus of the reduced Neumann system on $T^*V_{n,r}/SO(r)/\mathbb{Z}_2^n$.

Next, $\mathcal{I}$ itself is foliated by complexified $\delta$-dimensional invariant isotropic tori

$$\mathcal{I}_h = \mathcal{I} \cap \Psi^{-1}(h) = \{(X, P) \in \mathcal{I} \mid \Psi = h\},$$

the joint level varieties of the extra non-commuting integrals of the set $\{\Psi_{ij}\}$\footnote{In the real domain, $\mathcal{I}$ is a coisotropic invariant manifold of our system.}. Here $h \in \text{so}(r, \mathbb{C})$ belongs to the adjoint orbit $\{\text{tr}(\Psi^{2j}) = c_j\}$. As follows from the above, the factor of each $\mathcal{I}_h$ by the direct product $(\mathbb{C}^*)^{[r/2]}$, which is the complex stabilizer of the momentum $h \in \text{so}(r, \mathbb{C})$ in $SO(r, \mathbb{C})$, and by the group of reflections $\mathbb{Z}_2^n$ coincides with $\mathcal{I}_{\text{red}}$. The relations between the manifolds can be described by the following diagram showing the factorizations and the inclusion

$$\mathcal{I}_h \xrightarrow{\Psi_{ij} = h_{ij}} \mathcal{I} \xrightarrow{(\mathbb{C}^*)^{[r/2]}/\mathbb{Z}_2^n} \mathcal{I}_{\text{red}} /SO(r, \mathbb{C})/\mathbb{Z}_2^n$$

This can also be seen as follows. Note that $\mathcal{I}/\mathbb{Z}_2^n$ can be identified with the isospectral manifold\footnote{In the simplest non-trivial case $r = 2$ the manifold $\mathcal{I}_h$ coincides with $\mathcal{I}$.} by assigning to each point $(X, P)$ the Lax matrix $\mathbf{L}(\lambda)$ in (4.7). The conjugations that preserve $\mathcal{I}/\mathbb{Z}_2^n$ are induced by the above mentioned right action

$$X \mapsto Xg, \quad P \mapsto Pg, \quad g \in SO(r, \mathbb{C}),$$

\footnote{Actually $\mathcal{I}/\mathbb{Z}_2^n$ is a subset of the isospectral manifold $\mathcal{I}_{S'}$ of the curve $S'$ in a sense of a general definition given in the subsection 5.4 (the $\text{PGL}(2r, \mathbb{C})$-action does not leave it invariant). However, as in the case when the leading term $J$ of the Lax matrix is fixed where one use $\mathcal{I}_{S'}$ instead $\mathcal{I}_S$ (see the subsection 5.4), in our case it is natural to take $\mathcal{I}/\mathbb{Z}_2^n$ instead $\mathcal{I}_{S'}$.}
Proposition 5. \[ \text{translated Prym subvariety.} \]

\[ \text{divisors on} \ V \ \text{which yields} \]

\[ \text{2580 YURI FEDOROV AND BOŽIDAR JOVANOVIĆ} \]

I

\[ \text{ifold} \] \langle \]

Note that, in view of Proposition 4, the dimensions of \( I_\infty \) where \( B \)

\[ \text{Prym} \]

\[ \text{infinite points} \]

\[ \text{with a fixed basepoint} \]

The reduced invariant manifold \( \text{Proposition 6.} \)

Since \( \psi \) an eigenvector with the opposite eigenvalue:

\[ \text{Proof of Proposition 5.} \]

Here we use the technique already applied in [11, 44, 18].

\[ \text{Prym} \]

\[ \text{eigenvector of} \]

\[ \text{be an eigenvector of} \]

\[ \text{L} \]

\[ \text{ψ} \]

\[ \text{the vector} \]

\[ \text{an eigenvector with the opposite eigenvalue:} \]

\[ \text{L}(\lambda) \psi(P) = -w\psi(P). \]

Introduce also the vector \( \psi^*(P) = (-\xi, \bar{\chi})^T \), which, due to the structure of \( L(\lambda) \) in (4.7), is an eigenvector of \( L^T(\lambda) \) with the eigenvalue \( w \):

\[ L^T(\lambda)\psi^*(P) = w\psi^*(P). \]

Indeed,

\[ L^T \left( \begin{array}{c} -\xi \\ \bar{\chi} \end{array} \right) = \left( \begin{array}{cc} \chi^T & \xi \\ \bar{U} & -\bar{V} \end{array} \right) \left( \begin{array}{c} -\xi \\ \bar{\chi} \end{array} \right) = \left( \begin{array}{c} -\chi^T\bar{\xi} + \xi\bar{\chi} \\ -\bar{U}\xi - \bar{V}\bar{\chi} \end{array} \right) = \left( \begin{array}{c} -w\xi \\ w\bar{\chi} \end{array} \right) = w \left( \begin{array}{c} -\xi \\ \bar{\chi} \end{array} \right). \]

\[ 12 \text{The injectivity of the map follows from injectivity of the eigenvector map on } I_{r}/\text{PGL}(2r, \mathbb{C}) \text{ described in the subsection 5.4.} \]
Now consider the function

\[ F(P) = (\psi^*(P), \psi(P)) = \psi^T(P)\psi^*(P) = \chi^\xi - \chi^\xi. \]  \hspace{1cm} (5.11)

The rest of the proof is based on item 1) of the following lemma that is proved in the Appendix.

**Lemma 5.2.**

1) The function \( F(P) \) has simple zeros only at the branch points of \( w \) as the function of \( \lambda \) and has the poles only at \( D \) and \( \sigma D \).

2) \( F(P) \) is anti-symmetric with respect to the involution \( \sigma : S' \rightarrow S' \).

Since \( F(P) \) is meromorphic on \( S' \), the divisors of its zeros and poles are equivalent, which leads to the relation (5.9) and Proposition 5. \[ \square \]

Since for an appropriate normalization of the eigenvector \( \psi(P) \) the pole divisor \( D \) is finite, the eigenvector map \( \tilde{M} : I_{\text{red}} \rightarrow \text{Jac}(S') \) can be extended to the map

\[ \tilde{M} : I_h/\mathbb{Z}_2^n \rightarrow \text{Jac}(S', \infty) \]

**Proposition 7.** The map \( \tilde{M} \) is injective.

**Proof.** The idea is borrowed from [27]. Namely, let \( \psi(P) \), \( P \in S' \) be the eigenvector bundle of a Lax matrix \( L(\lambda) \in I_h/\mathbb{Z}_2^n \) with the normalization (4.10), and \( D \) be the corresponding effective divisor on \( S' \) defining a point \( D = NP_0 \) in \( \text{Jac}(S', \infty) \). We will show that, given the \( \infty \)-equivalence class of \( D \), the matrix \( L(\lambda) \) can be reconstructed uniquely.

Let \( \tilde{D} \) be another effective divisor giving the same point in \( \tilde{\text{Jac}}(S', \infty) \). Then, by the generalized Abel theorem, there exists a meromorphic function \( G(P) \) on \( S' \) such that

\[ (G) = D - \tilde{D}, \quad G(\infty) = G(\infty) \neq 0, \quad s = 1, \ldots, [r/2]. \]

The components of \( \psi(P) \) form a basis of \( L(D) \). Respectively, let \( \tilde{\psi}(P) \) be a vector bundle whose components form a basis of \( L(\tilde{D}) \).

Then the components of \( G(P)\tilde{\psi}(P) \) form a basis of \( L(D) \), and there is a non-degenerate matrix \( R \in \text{GL}(2r, \mathbb{C}) \) independent of \( P \) such that \( \psi(P) = G(P)R\tilde{\psi}(P) \). The bundles \( \psi(P), \tilde{\psi}(P) \) define the same matrix \( L(\lambda) \) up to conjugation by \( R \). By (5.7), the conjugation preserves \( L(\lambda) \in I_h/\mathbb{Z}_2^n \) if and only if \( R = \text{diag}(g, g) \), where \( g \in SO(r, \mathbb{C}) \) is a stabilizer of \( V_0 \in \text{so}(r, \mathbb{C}) \).

On the other hand, the above relation between \( \psi, \tilde{\psi} \) implies

\[ \psi(\infty_k) = G(\infty_k)R\tilde{\psi}(\infty_k), \quad k = 1, \ldots, r. \]

Due to the chosen normalization, near \( \infty_k \) the components of \( \psi(\infty_k), \tilde{\psi}(\infty_k) \) are finite and given by the expansions (4.11) involving the eigenvectors \( v_s, \bar{v}_s, v_0 \) of \( V_0 \). This implies

\[ v_s | G(\infty)g v_s, \quad \bar{v}_s | G(\infty)g \bar{v}_s, \quad s = 1, \ldots, [r/2], \quad v_0 | G(\infty)g v_0 \]

hence all \( v_s, \bar{v}_s, v_0 \) must be also eigenvectors of \( g \). Since \( g \in SO(r, \mathbb{C}) \), we conclude that \( g \) and \( R \) are the unit matrices, therefore the divisors \( D, \tilde{D} \) correspond to the same matrix \( L(\lambda) \in I_h/\mathbb{Z}_2^n \). \[ \square \]

\[ ^{13} \text{Note that these components, in general, take different values at } \infty^- \text{ and } \infty^+. \]
Proposition 8. The image \( \tilde{\mathcal{M}}(I_h/\mathbb{Z}_2^n) \) belongs to a translate of \( \tilde{\text{Prym}}(S', \sigma) \) in \( \tilde{\text{Jac}}(S') \).

Proof. The zeros and poles of the meromorphic function \( F(P) \) given by (5.11) imply the equivalence (5.9) on \( \text{Jac}(S') \), namely \( D + \sigma D \equiv B \), which cannot be extended to the equivalence \( D + \sigma D \equiv \infty \), because \( F(\infty^+) = 0 \) and the divisor \( B \) contains all the infinite points:

\[
B = B_0 + \infty_1 + \cdots + \infty_r,
\]

\( B_0 \) being the finite part of \( B \). This can be repaired by introducing the function \( G(P) = F(P)^2 \lambda \), where \( \lambda \) is the coordinate on the curve \( S' \subset \mathbb{P}^2(\lambda, w) \) and \( (\lambda) = O - 2(\infty_1 + \cdots + \infty_r) \), \( O \) being the preimage of \( \lambda = 0 \) on \( S' \). Then

\[
(G) = 2B - 2(D + \sigma D) + O - 2(\infty_1 + \cdots + \infty_r) = 2B_0 + O - 2(D + \sigma D).
\]

In view of Lemma 5.2 and the symmetry \( \lambda(\sigma P) = \lambda(P) \),

\[
G(\infty^{-s}) = G(\infty^{+s}) \neq 0, \quad s = 1, \ldots, [r/2].
\]

Hence the expression for \( (G) \) yields

\[
2(D + \sigma D) \equiv \infty 2B_0 + O,
\]

(5.12)

The latter implies that \( D + \sigma D \) is a fixed divisor defined up to translations by half-periods \( \mathcal{P} \) of \( \tilde{\text{Jac}}(S', \infty) \). Then, for a certain translate \( \Sigma \) of \( \tilde{\text{Prym}}(S', \sigma) \subset \tilde{\text{Jac}}(S', \infty) \), the divisor \( D - NP_0 \) belongs to a union of translates of \( \Sigma \) by the half-periods.

On the other hand, comparing (5.12) with (5.9), one sees that any half-period \( \mathcal{P} \) must satisfy \( \phi(\mathcal{P}) = 0 \), where \( \phi \) is specified in (5.1). Therefore, \( \mathcal{P} \) includes only the half-periods \( v(\pm 1, \ldots, \pm 1) \), which, according to Lemma 5.1, are half-periods in \( \tilde{\text{Prym}}(S', \sigma) \). Hence, the translates of \( \Sigma \) by \( \mathcal{P} \) coincide with \( \Sigma \), which proves the proposition.

Since \( I_h \) and \( \tilde{\text{Prym}}(S', \sigma) \) have the same dimension \( \delta = l + [r/2] \) and the extended eigenvector map \( \tilde{\mathcal{M}}: I_h/\mathbb{Z}_2^n \to \tilde{\text{Jac}}(S', \infty) \) is injective, we arrive at the main theorem of the section.

Theorem 5.3. If the rank of the momentum \( \Psi \) is maximal, then, after factorization by the reflection group \( \mathbb{Z}_2^n \), a generic \( \delta \)-dimensional complex invariant manifold \( I_h \) of the Neumann system on \( T^*V_{n,r}(\mathbb{C}) \) is \( \tilde{\text{Prym}}_0(S', \sigma) \), an open subset of a translate of the affine Prym variety \( \text{Prym}(S', \sigma) \subset \text{Jac}(S', \infty) \).

This theorem together with Proposition 6 can be summarized with the following commutative diagram

\[
\begin{array}{ccc}
I_h/\mathbb{Z}_2^n & \xrightarrow{\tilde{\mathcal{M}}} & \tilde{\text{Prym}}_0(S', \sigma) \\
(I^*)^{[r/2]} \downarrow & & \downarrow \phi \\
I_{\text{red}} & \xrightarrow{\mathcal{M}} & \text{Prym}_0(S', \sigma)
\end{array}
\]
5.6. Linearization of the flows. We now show that under the eigenvector maps 
\[ \mathcal{M} : \mathcal{I}_n/\mathbb{Z}_2^2 \rightarrow \text{Jac}(S', \infty) \] 
and \[ \mathcal{M} : \mathcal{I}_{\text{red}} \rightarrow \text{Jac}(S') \], the trajectories \((X(t), P(t))\) of 
the complex Neumann system induce straight line trajectories on the (generalized) 
Jacobian variety.

Let, as above, \( \psi(P, t) \) be eigenvector of \( \mathbf{L}(\lambda) = a(\lambda) \mathbf{L}(\lambda) \) with a normalization \( \langle \alpha, \psi(P) \rangle = 1 \). As follows from the Lax representation (3.6), \( \psi + N_\kappa \psi \) is also an 
eigenvector of \( \mathbf{L}(\lambda) \). Therefore, it is proportional to \( \psi(P) \) with a meromorphic 
multiplier \( f \):

\[
\psi(P) + N_\kappa(\lambda)\psi(P) = f(P, t) \psi(P).
\]

Since \( \frac{d}{dt} \langle \alpha, \psi(P) \rangle = 0 \), the above implies \( f(P, t) = \langle \alpha, N_\kappa \psi \rangle \).

Now assume that \( \psi(P) \) is normalized as in Proposition 2, so its behavior near 
the infinite points \( \infty_1, \ldots, \infty_r \) is given by the expansions (4.11). Then, near \( \infty_s \) 
with the local coordinate \( \tau = 1/\tau^2 \), the following expansions hold

\[
N_\kappa \psi = \begin{pmatrix}
\kappa V_0 & I_r \\
\tau^{-2} I_r + \Lambda & \kappa V_0
\end{pmatrix}
\begin{pmatrix}
\nu_s + h_s \tau^2 + O(\tau^3) \\
\nu_s + h_s \tau + O(\tau^2)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\nu_s + (\kappa V_s + h_s) \tau + O(\tau^2) \\
\tau^{-1} v_s + \kappa V_s + h_s + O(\tau)
\end{pmatrix},
\]

where for \( j = 1, \ldots, \lfloor r/2 \rfloor \), \( v_{j+[r/2]} = \nu_j \), \( h_{j+[r/2]} = h_j \), \( v_{[r/2]+1} = -v_j \), and, if \( r \) is 
odd, \( v_r = v_0 \), \( h_r = h_0 \), \( v_r = 0 \). The chosen normalization (4.10) implies

\[
\langle \beta, v_s + h_s \tau + O(\tau^2) \rangle = 1,
\]

\[
f(\tau) = \langle \alpha, N_\kappa \psi \rangle = \langle \beta, \tau^{-1} [v_s + h_s \tau] + \kappa V_s + h_s + O(\tau) \rangle.
\]

As a result, near \( \infty_1, \ldots, \infty_r \), the function \( f(P, t) \) has the expansions

\[
\infty_s^+ : f = \frac{1}{\tau} + \kappa v_s + O(\tau),
\]

\[
\infty_s^- : f = \frac{1}{\tau} - \kappa v_s + O(\tau), \quad s = 1, \ldots, \lfloor r/2 \rfloor,
\]

\[
\infty_0^: f = \frac{1}{\tau} + O(\tau) \quad \text{(if } r \text{ is odd)},
\]

regardless to the choice of \( \beta \).

Let \( \mathcal{D}_t \) be the divisor of poles of \( \psi(P, t) \) and \( \tilde{A}(\mathcal{D}_t) \) be its image under the 
generalized Abel map (5.4) with holomorphic differentials \( \omega_1, \ldots, \omega_j \) and meromorphic 
differentials of the 3rd kind \( \Omega_j, j = 1, \ldots, \lfloor r/2 \rfloor \). Assume now that the latter are 
normalized: they have simple poles with the residues \( \pm 1 \) at \( \infty_j^+, \infty_j^- \) respectively, 
and no poles elsewhere.

**Theorem 5.4.** For generic values of the first integrals, the map \( \tilde{M} \) linearizes the 
complex Neumann flow on the generalized Jacobian \( \text{Jac}(S', \infty) \) as follows

\[
\tilde{A}(\mathcal{D}_t) - \tilde{A}(\mathcal{D}_0) = -t \sum_{k=1}^{r} \text{Res}_{\infty_k}(f \omega_1, \ldots, f \omega_g, f \Omega_1, \ldots, f \Omega_{\lfloor r/2 \rfloor})^T
\]

\[
- t(0, \ldots, 0, 2\kappa \nu_1, \ldots, 2\kappa \nu_{\lfloor r/2 \rfloor}) \quad (\text{mod } \tilde{A}).
\]

In particular, under the customary eigenvector map \( \mathcal{M} \), the complex Neumann 
flow gives the following flow on \( \text{Jac}(S') \)

\[
A(\mathcal{D}_t) - A(\mathcal{D}_0) = -t \sum_{s=1}^{r} \text{Res}_{\infty_s}(f \omega_1, \ldots, f \omega_g)^T \quad (\text{mod } A).
\]
Observe that as the residues of \( f \) at \( \infty \) do not depend on \( t \), the flows (5.16), (5.17) are indeed linear ones.

Note also that in the classical case \( r = 1 \) (the Neumann system on \( S^{n-1} \) linearized on the Jacobian of the hyperelliptic curve \( \Gamma \)) (5.17) gives a correct direction vector on \( \text{Jac}(\Gamma) \), whose components are the leading terms of the expansion of \( g = n - 1 \) holomorphic differentials near the infinite point of \( \Gamma \) (see e.g., [45]).

The proof of Theorem 5.4 is an extension of that of Theorem 6.39 in [6] describing linearization of equations admitting polynomial Lax representations, and it is given in Appendix.

The direction of the flow (5.16) can be written more specifically if we choose a basis of \( H^1(S', \mathbb{Z}) \) in the form \( (\omega^-_1, \ldots, \omega^-_g, \omega^+_1, \ldots, \omega^+_g) \), where \( \omega_i^-, \omega_i^+ \) are anti-symmetric and symmetric holomorphic differentials respectively, and, as above, \( l \) is the dimension of \( \text{Prym}(S', \sigma) \). Since the involution \( \sigma \) flips sign of the local parameter \( \tau \) (see (4.12), (4.14)), near the infinite points the differentials admit expansions

\[
\begin{align*}
\infty^+ & : \quad \omega^-_i = (\chi_{i,s} + O(\tau))d\tau, \quad \omega^+_j = (\xi_{j,s} + O(\tau))d\tau, \\
\infty^- & : \quad \omega^-_i = (\chi_{i,s} + O(\tau))d\tau, \quad \omega^+_j = - (\xi_{j,s} + O(\tau))d\tau, \\
\infty_0 & : \quad \omega^-_i = (\chi_{i,r} + O(\tau))d\tau, \quad \omega^+_j = O(\tau)d\tau,
\end{align*}
\]

where \( \chi_{i,s}, \xi_{j,s}, s = 1, \ldots, [r/2], i = 1, \ldots, l, j = 1, \ldots, n - l \) are some constants. Also, choose the meromorphic differentials satisfying

\[\sigma^* \Omega_k = - \Omega_k, \quad k = 1, \ldots, [r/2].\]

Again, according to (4.12), (4.14), near \( \infty^+_s, \infty^-_s \) we have

\[\Omega_k = \left( \frac{\delta_{ks}}{\tau} + \phi_{k,s} + O(\tau) \right)d\tau, \quad \Omega_k = \left( - \frac{\delta_{ks}}{\tau} + \phi_{k,s} + O(\tau) \right)d\tau, \quad (5.18)\]

where \( \delta_{ks} \) is the Kronecker symbol and \( \phi_{k,s}, k, s = 1, \ldots, [r/2] \) are some constants.

Theorem 5.4 implies that in the above basis of differentials the direction of the flow on \( \text{Jac}(S', \infty) \) is

\[
- \left( \chi_1, \ldots, \chi_l, 0, \ldots, 0, 4k \nu_1 + \phi_1, \ldots, 4k \nu_{[r/2]} + \phi_{[r/2]} \right)^T, \quad (5.19)
\]

\[\chi_i = \chi_{i,r} + 2 \sum_{s=1}^{[r/2]} \chi_{i,s}, \quad \phi_k = 2 \sum_{s=1}^{[r/2]} \phi_{k,s}.\]

Consequently, the direction of the reduced flow on \( \text{Jac}(S') \) is

\[
- \left( \chi_1, \ldots, \chi_l, 0, \ldots, 0 \right)^T, \quad (5.20)
\]

which shows that the flow goes along \( \text{Prym}(S', \sigma) \subset \text{Jac}(S') \), as expected.

**Remark 4.** The reduced Neumann flow on \( T^*V_{n,r}/SO(r) \) is described by the flow (5.17) on \( \text{Prym}(S', \sigma) \) whose direction (5.20) does not depend on the parameter \( \kappa \) in the metric on \( V_{n,r} \). This agrees with the fact that the metric term \( \kappa \langle \Psi, \Psi \rangle \) leads to a trivial vector field on \( T^*V_{n,r}/SO(r) \). On the contrary, the original, non-reduced problem on \( T^*V_{n,r} \) is linearized on \( \text{Prym}(S', \sigma) \), and the direction of the flow (5.19) depends on \( \kappa \).
5.7. Linearization of the Neumann system on the Grassmannian variety. The Neumann system on the Grassmannian $G_{n,r}$ is completely integrable in the Liouville sense. Since the Neumann system on $G_{n,r}$ can be seen as the $SO(r)$-reduction of a Neumann system on $V_{n,r}$ for the zero value of the momentum mapping $\Psi$, we have the natural identification:

$$I_{G_{n,r}}/\mathbb{Z}_{2}^{n} = I_{\text{red}} = I/\text{SO}(r,\mathbb{C})/\mathbb{Z}_{2}^{n},$$

where $I = I_{0}$ is a $(r(n-r)+r(r-1)/2)$-dimensional invariant manifold of the Neumann flow on the Stiefel variety in the special case $\Psi = 0$ and $I_{G_{n,r}}$ is the associated $r(n-r)$-dimensional complex level set of commuting integrals of the Neumann flow on the Grassmannian $G_{n,r}$.

From Theorem 4.1 and item 2) of Proposition 4, we have

$$\text{gen}(S') = \dim \text{Jac}(S') = 2r(n-r) - n + 1, \quad \dim \text{Prym}(S',\sigma) = r(n-r).$$

Besides, like in the general case, Proposition 5 implies that the map $\mathcal{M}$ realized a bijection between $I_{\text{red}}$ and an open subset of Prym$(S',\sigma)$.

The above considerations can be summarized in the following statement.

**Theorem 5.5.** For generic values of the first integrals of the complex Neumann system on the Grassmannian, the eigenvector map

$$\mathcal{M}: I_{G_{n,r}}/\mathbb{Z}_{2}^{n} \rightarrow \text{Jac}(S')$$

linearizes the flow on an open subset of a translate of the Prym variety Prym$(S',\sigma)$.

**Remark 5.** It can be proved that generic trajectories of the Neumann flows on the unreduced space with the zero momentum $\Psi$ filled up tori of dimension $r(n-r)$, while the level sets of the integrals $\{f_{k,i}, \Psi_{ij}\}$ have the dimension $r(n-r)+r(r-1)/2$. That is, over each torus in the factor $\Psi^{-1}(0)/\text{SO}(r)$ there is $\text{SO}(r)$-parametric family of tori in $\Psi^{-1}(0)$. On the algebraic-geometric side this can be seen as follows. In the special case $\Psi = 0$ all $r$ infinite points of the regularized curve $S'$ are invariant with respect to the involution $\sigma$ (see (4.15)). Such a situation has been studied in [9]: one considers the generalized Jacobian $\text{Jac}(S',\infty)$ of the singularized curve, obtained from $S'$ by gluing $\infty_{1},\ldots,\infty_{r}$ to one point. In contrast to the general case, now the part of $\text{Jac}(S',\infty)$, which is anti-invariant with respect to $\sigma$, is just the compact subvariety Prym$(S',\sigma)$. Then the corresponding extended eigenvector map $\tilde{\mathcal{M}}: I_{0}/\mathbb{Z}_{2}^{n} \rightarrow \tilde{\text{Jac}}(S',\infty)$ linearise a complex trajectory over a translate of Prym$(S',\sigma)$.

6. Integrable discretization.

6.1. The map and its basic properties. Using the notation of Section 3, consider the following matrix generalization of the map (2.13) (discrete Neumann system) with a real parameter $\lambda_{s}$

$$P = A^{1/2}(\lambda_{s}) \dot{X} - X \Gamma(\lambda_{s}),$$

$$\dot{P} = -A^{1/2}(\lambda_{s}) X + \dot{X} \Gamma(\lambda_{s}), \quad (6.1)$$

where, as above, $A(\lambda) = \lambda I_{n} - A$ and $\Gamma$ is a symmetric $r \times r$ matrix, which is found from the condition $X^{T}P + P^{T}X = 0$:

$$\Gamma(\lambda_{s}) = \frac{1}{2}(\dot{X}^{T}A^{1/2}(\lambda_{s})X + X^{T}A^{1/2}(\lambda_{s})\dot{X}). \quad (6.2)$$
The first matrix equation in (6.1) gives the definition of the discrete momentum $P$, whereas the second equation provides the discrete dynamics in an implicit form. In view of (6.2), the structure of (6.1) ensures preservation of the $SO(r)$ momentum:

$$\tilde{X}^T \tilde{P} = X^T P.$$  

Equations (6.1), (6.2) define a multi-valued map (a correspondence)

$$\mathcal{B}_r : T^* V_{n,r} \longrightarrow T^* V_{n,r}, \quad \mathcal{B}_r(X, P) = (\tilde{X}, \tilde{P}),$$

which can be regarded as a discrete Neumann system on $T^* V_{n,r}$.

To evaluate the map, we rewrite (6.1) in the form

$$\tilde{X} = A^{-1/2}(\lambda_*) (P + X \Gamma(\lambda_*)), \quad \tilde{P} = -A^{1/2}(\lambda_*) X + A^{-1/2}(\lambda_*) (P + X \Gamma(\lambda_*)) \Gamma(\lambda_*). \quad (6.3)$$

Then, applying the condition $\tilde{X}^T \tilde{X} = I_r$, we obtain the following quadratic matrix equation for $\Gamma$, which defines it implicitly as a function of $X, P$ and which generalizes the scalar equation (2.18),

$$\Gamma U \Gamma + \Gamma V + \bar{V}^T \Gamma - W = 0, \quad (6.4)$$

where

$$U = X^T A^{-1}(\lambda_*) X, \quad V = X^T A^{-1}(\lambda_*) P, \quad W = I_r - P^T A^{-1}(\lambda_*) P. \quad (6.5)$$

This is a system of $r(r+1)/2$ scalar quadratic equations for the $r(r+1)/2$ components of $\Gamma$. However, due to the structure of the matrix coefficients, the number of its solutions is less than that predicted by the Besout theorem. The matrix equation (6.4) is known in the literature in connection with stationary solutions of the matrix Riccati differential equation, optimum automatic control theory, and its complete solution was presented by Potter [50].

In order to describe it, we first observe that the coefficients (6.5) coincide with the $r \times r$ blocks of the Lax matrix $L(\lambda_*)$ in (3.7). Since the latter is symplectic, its eigenvalues $w_1, \ldots, w_{2r}$ are divided into $r$ pairs $(w_i, -w_i)$. Let $\psi_1, \ldots, \psi_r \in \mathbb{C}^{2r}$ be the eigenvectors of $L(\lambda_*)$ with distinct eigenvalues $w_1, \ldots, w_r$ such that $w_i \neq -w_j$. The corresponding matrix

$$\Psi = (\psi_1 \cdots \psi_r) \quad (6.6)$$

will be called a non-special eigenmatrix.

The following proposition is a direct consequence of the results of [50].

**Proposition 9.** Any symmetric solution of the matrix quadratic equation (6.4) has the form

$$\Gamma = \Upsilon \Xi^{-1},$$

where $\Xi, \Upsilon$ are upper and lower $r \times r$ halves of a non-special eigenmatrix (6.6).

**Remark 6.** The original paper [50] solves the equation

$$\Gamma U \Gamma + \Gamma V + \bar{V}^T \Gamma - W = 0, \quad (6.7)$$

where $U, V, W$ are arbitrary $n \times n$ complex matrices. As above, consider the matrix

$$K = \begin{pmatrix} V & U \\ W & -\bar{V}^T \end{pmatrix}.$$
According to [50], if $K$ has a diagonal Jordan canonical form with distinct eigenvalues $w_1, \ldots, w_r$ such that $w_i \neq w_j$, then the hermitian solutions ($\Gamma^T = \Gamma$) of (6.7) are exactly those described in Proposition 9: $\Gamma = \Upsilon \Xi^{-1}$, where $\Xi, \Upsilon$ are upper and lower $r \times r$ halves of the corresponding eigenmatrix of $K$. Proposition 9 dealing with symmetric matrices can be easily proved following the lines of [50].

Obviously, the product $\Upsilon \Xi^{-1}$ is invariant under any gouge transformation
$$\Psi \mapsto \Psi g, \quad g \in GL(r, C).$$
As was also shown in [50], if $\Psi$ is special (that is, contains eigenvectors $\psi_i, \psi_j$ with $w_i = -w_j$) or its columns are linear combinations of more than $r$ eigenvectors $\psi_i$, then the product $\Upsilon \Xi^{-1}$ also satisfies (6.4), but it is not a symmetric matrix.

Since for generic finite $\lambda_\sigma \neq a_i$ there are $2^r$ possible partitions
$$\mathbf{w} = \{w_1, \ldots, w_r \mid -w_1, \ldots, -w_r\},$$
we conclude that the equation (6.4) has precisely $2^r$ complex solutions. They admit a natural decomposition $\{\Gamma\} = \{\Gamma_-\} \cap \{\Gamma_+\}$ such that for any solution $\Gamma^* \in \{\Gamma_-\}$ corresponding to a partition $\mathbf{w}_- = \{w_1, \ldots, w_r \mid -w_1, \ldots, -w_r\}$ there is a unique $\Gamma^{**} \in \{\Gamma_+\}$ corresponding to the “opposite” partition $\mathbf{w}_+ = \{-w_1, \ldots, -w_r \mid w_1, \ldots, w_r\}$.

Since the spectral curve $S'$ has ordinary branch points $(\lambda = a_i, w = 0)$, in the special case $\lambda_\sigma = a_i$ there are only $r-1$ distinct non-zero eigenvalues $w_1, \ldots, w_{r-1}$ such that $w_i \neq -w_j$, and the number of symmetric solutions of (6.4) drops to $2^{r-1}$.

Once a solution for $\Gamma$ is chosen, $\bar{X}, \bar{P}$ are found uniquely from (6.1) or (6.3). Since, for generic $\lambda_\sigma$, different solutions $\Gamma$ lead to different images $\bar{X}, \bar{P}$, we conclude that the complex map
$$\mathfrak{B}_r: T^* V_{n, r}(\mathbb{C}) \rightarrow T^* V_{n, r}(\mathbb{C})$$
is $2^r$-valued.

Note that a similar situation where the choice of a partition of certain roots define a branch of the mapping appears in the integrable discrete Frahm–Manakov top on $SO(n)$, where the spectral curve is regular [44].

6.2. Continuous limit. Like in the case $r = 1$, the continuous limit of the map $\mathfrak{B}_r$ is obtained by letting $\lambda_\sigma \rightarrow \infty$. Namely, we set $\lambda_\sigma = 1/\epsilon^2$ and consider the expansions

$$A^{-1}(\lambda_\sigma) = \epsilon^2 \left( I_n + \epsilon^2 A + O(\epsilon^4) \right),$$
$$\bar{X} = X + \epsilon \bar{X} + O(\epsilon^2), \quad \bar{P} = P + \epsilon \bar{P} + O(\epsilon^2).$$

This gives
$$\Gamma = \frac{1}{\epsilon} I_r - \frac{\epsilon}{2} \left( X^T A X + P^T P \right) + O(\epsilon^2). \quad (6.8)$$

Indeed, substituting the above into the matrix quadratic equation (6.4) and using the expansions

$$U = \epsilon^2 (I_r + \epsilon^2 X^T A X + O(\epsilon^3)),$$
$$V = \epsilon^2 (X^T P + \epsilon^2 X^T A P + O(\epsilon^3)),$$
$$W = I_r - \epsilon^2 (P^T P + \epsilon^2 P^T A P + O(\epsilon^3)),$$

we find that the coefficients at $\epsilon^0, \epsilon, \epsilon^2$ vanish, which justifies (6.8).
On the other hand, from (6.9) and the condition
\begin{equation}
A^{-1/2}(\lambda_s) = \epsilon(\mathbf{I}_n + \frac{\epsilon^2}{2} A + O(\epsilon^4)), \quad A^{1/2}(\lambda_s) = \frac{1}{\epsilon}(\mathbf{I}_n - \frac{\epsilon^2}{2} A + O(\epsilon^4)),
\end{equation}
we conclude that \( \hat{X}, \hat{P} \) coincide with the right hand sides of the continuous Neumann system with the Euclidean metric (3.2), (3.3), \( \kappa = -1/2 \), modulo \( O(\epsilon) \).

6.3. Lagrangian description. The discrete Neumann system (6.1) can be considered as a discrete variational problem on the Stiefel variety as well. Namely, let \( \{(X_k, P_k), k \in \mathbb{Z}\} \) be a trajectory of the discrete Neumann system and \( \{\Gamma_k\} \) be the corresponding sequence of matrix multipliers. Comparing the first and the second equation in (6.1) with \( k \) replaced by \( k - 1 \), one obtains the equations
\begin{equation}
X_{k-1} + X_{k+1} = A^{-1/2}(\lambda_s)X_kB_k, \quad B_k = \Gamma_{k-1} + \Gamma_k.
\end{equation}
On the other hand, from (6.9) and the condition \( X_{k+1}^T X_{k+1} = \mathbf{I}_r \), we get the matrix equation
\begin{equation}
B_k(X_k^T A^{-1}(\lambda_s)X_k)B_k - (X_{k-1}^T A^{-1/2}(\lambda_s)X_k)B_k - B_k(X_k^T A^{-1/2}(\lambda_s)X_{k-1}) = 0,
\end{equation}
which determines \( B_k \) as a function of \( X_{k-1} \) and \( X_k \) only.

Equations (6.9) coincides with the discrete Euler-Lagrange equations on \( V_{n,r} \)
\begin{equation}
\frac{\partial L(X_k, X_{k+1})}{\partial X_k} + \frac{\partial L(X_{k-1}, X_k)}{\partial X_k} = X_kB_k, \quad k \in \mathbb{Z}
\end{equation}
of the functional \( S = \sum_{k \in \mathbb{Z}} L(X_k, X_{k+1}) \) with the Lagrangian
\begin{equation}
L : V_{n,r} \times V_{n,r} \rightarrow \mathbb{R}, \quad L(X, \tilde{X}) = \text{tr}(X^T A^{1/2}(\lambda_s)\tilde{X}),
\end{equation}
first derived by Moser and Veselov in [44]. In this sense, the expression for the momentum \( P \) in the system (6.1) is actually the discrete Legendre transformation:
\begin{equation}
\mathcal{L} : V_{n,r} \times V_{n,r} \rightarrow T^*V_{n,r}, \quad P = \frac{\partial L(X, \tilde{X})}{\partial X}, \quad \frac{\partial L(X, \tilde{X})}{\partial X} = XT, \quad \mathcal{L} = X\Gamma,
\end{equation}
where, as above, the multiplier \( \Gamma \) is given by (6.2). In particular, the correspondence \( \mathcal{L}_r \) is symplectic (e.g., see [32, 44, 59]).

6.4. The Lax representation. As we have seen above, the solutions of the matrix quadratic equation (6.4) are closely related to the Lax matrix of the continuous Neumann systems on \( V_{n,r} \). It appears that this matrix also forms a Lax representation of the discrete system.

**Theorem 6.1.** Up to the action of the group \( \mathbb{Z}_2 \) of reflections (3.5), the discrete Neumann system (6.1), (6.2) is equivalent to the intertwining matrix relation (discrete Lax pair)
\begin{equation}
\hat{L}(\lambda)M(\lambda, \lambda_s) = M(\lambda, \lambda_s)L(\lambda),
\end{equation}
where
\begin{align*}
L(\lambda) &= \begin{pmatrix}
X^T(\lambda\mathbf{I}_n - A)^{-1}P & X^T(\lambda\mathbf{I}_n - A)^{-1}X \\
\mathbf{I}_r - P^T(\lambda\mathbf{I}_n - A)^{-1}P & -P^T(\lambda\mathbf{I}_n - A)^{-1} \mathbf{I}_r
\end{pmatrix}, \\
M(\lambda, \lambda_s) &= \begin{pmatrix}
-\Gamma(\lambda_s) & \mathbf{I}_r \\
(\lambda - \lambda_s)\mathbf{I}_r + \Gamma^2(\lambda_s) & -\Gamma(\lambda_s)
\end{pmatrix},
\end{align*}
where \( \hat{L}(\lambda) \) depends on \( \tilde{X}, \tilde{P} \) in the same way as \( L(\lambda) \) depends on \( X, P \) and \( M(\lambda, \lambda_s) \) depends on \( X, \tilde{X} \) in a symmetric way via (6.2).
The proof is a direct computation (although quite a long one), it uses the constraints (3.1), the matrix identity
\[ A(\lambda I_n - A)^{-1} = (\lambda I_n - A)^{-1}A = \lambda(\lambda I_n - A)^{-1} - I_n, \] (6.14)
and \( SO(r) \)-momentum preservation \( \dot{X}^T \dot{P} = X^TP \).

Note that the \( 2r \times 2r \) matrix \( M \) is a direct generalization of (2.16).

It follows that, regardless to the branch of the map \( \mathcal{B}_r \), it preserves all the first integrals of the continuous Neumann systems on \( V_{n,r} \), including the non-commutative set given by the components of the \( SO(r) \)-momentum \( \Psi \). Thus, when the rank of \( \Psi \) is maximal, generic invariant manifolds of \( \mathcal{B}_r \) are \( \delta \)-dimensional isotropic tori.

**Theorem 6.2.** The discrete Neumann system (6.1) is completely integrable in the noncommutative sense with the set of integrals given by (3.9) and by the components of the \( SO(r) \)-momentum mapping \( \Psi_{ij} \).

The description of noncommutatively integrable symplectic correspondence is the same as that of a Liouville integrable correspondence (see [59]), with the only difference that the Lagrangian tori are replaced by the isotropic ones (see [31]).

6.5. **Double iterations.** Although the map \( \mathcal{B}_r \) is multi-valued, it has the following return property, which generalizes that mentioned in Remark 3.

**Theorem 6.3.** Let \( \mathcal{B}_r^- \), \( \mathcal{B}_r^+ \) be the branches of the map \( \mathcal{B}_r \) corresponding to some arbitrary opposite partitions \( w_- \), \( w_+ \). Then, for any initial \( (X, P) \in T^*V_{n,r}(\mathbb{C}) \),
\[ \mathcal{B}_r^- \circ \mathcal{B}_r^+(X, P) = (-X, -P), \]
i.e., double iterations of \( \mathcal{B}_r \) with opposite partitions multiply \( (X, P) \) by \(-1\).

The proof is given in Appendix.

6.6. **Real trajectories.** In view of (6.1), for the map \( \mathcal{B}_r \) to be real-valued it is necessary that \( \lambda_* > a_1, \ldots, a_n \), and even in this case not for any real pair \( (X, P) \in T^*V_{n,r} \) the matrix equation (6.4) has a real solution \( \Gamma \): as follows from the discrete Legendre transformation (6.12), the real momentum \( P \) must belong to the compact set
\[ \mathcal{L}(V_{n,r} \times V_{n,r}) = \left\{ A^{1/2}(\lambda_*) \dot{X} - \frac{1}{2}X \left( \dot{X}^T A^{1/2}(\lambda_*)X + X^T A^{1/2}(\lambda_*)\dot{X} \right) \right\}. \] (6.15)

**Proposition 10.** Assume \( \lambda_* > a_1, \ldots, a_n \), and let the momentum \( P_0 \) of the initial real point \( (X_0, P_0) \in T^*V_{n,r} \) belong to the set (6.15). Let \( L(\lambda_*) \) be evaluated at \( (X_0, P_0) \). Choose the branch of the map \( \mathcal{B}_r \) corresponding to a partition
\[ w = \{w_1, \ldots, w_r \mid -w_1, \ldots, -w_r \}, \]
such that \( w_1, \ldots, w_r \) are different, \( w_i + w_j \neq 0 \), \( w_i + \bar{w}_j \neq 0 \), \( i, j = 1, \ldots, r \). Then the trajectory of the discrete Neumann system (6.3) obtained from \( (X_0, P_0) \) by iterating the above branch is real.

**Proof.** Since \( w_1, \ldots, w_r \) are different and \( w_i + w_j \neq 0 \), the corresponding eigenmatrix (6.6) is non-special. Then, according to Proposition 9, the solution \( \Gamma = \Upsilon \Xi^{-1} \) of equation (6.4) with real matrix coefficients (6.5) corresponding to \( (X_0, P_0) \) is symmetric. On the other hand, due to the conditions \( w_i + \bar{w}_j \neq 0 \), after setting \( \Upsilon = U, \Xi = V, \Xi = W \), the equation (6.7) has the corresponding hermitian solution \( \Gamma \) obtained by the same eigenmatrix \( \Psi \) (see Remark 6). Since the matrix \( \Gamma \) is both symmetric and hermitian, it is a real symmetric matrix. \( \square \)
Remark 7. For the case $r = 1$, the eigenvalues $w, -w$ of $L(\lambda^*)$ can be pure real or imaginary. The above condition gives the real trajectory for real $w$, as in Remark 2. For the case $r > 1$ and real $\lambda^*$, generically the matrix $L(\lambda^*)$ has a set of four different eigenvalues invariant with respect to the inversion $w \mapsto -w$ and complex conjugation $w \mapsto \bar{w}$. For example, consider the case $r = 2$ and eigenvalues $w, -w, \bar{w}, -\bar{w}$. The non-special eigenmatrix $\Psi$ associated to the partition $w = \{w_1 = w, w_2 = \bar{w} \mid -w_1, -w_2\}$ defines the real branch of $\mathcal{B}_2$.

Note that with the same initial condition $(X_0, P_0)$ as one described in Proposition 10, we can have a complex solution of (6.4). That is why, it is natural to consider all objects complexified. In the next subsection we shall give an algebraic geometrical description of the mapping $\mathcal{B}_r$, which generalize Proposition 1.

The analogous discretization of the geodesic flow of the Euclidean metric (the case $A = 0$) has been treated in [30].

6.7. Algebraic geometric description of $\mathcal{B}_r$. Now we describe the correspondence (6.3) under the eigenvector map

$$\mathcal{M}: \mathcal{I}_{\text{red}} = \mathcal{I}/SO(r, \mathbb{C})/\mathbb{Z}_2^h \longrightarrow \text{Prym}(S', \sigma) \subset \text{Jac}(S')$$

introduced in (5.8). As above, for a fixed arbitrary $\lambda_\ast \neq a_i$ choose a partition of eigenvalues $w = \{w_1, \ldots, w_r \mid -w_1, \ldots, -w_r\}$ of $L(\lambda_\ast)$.

Note that the eigenvalues $w_i$ of $L(\lambda_\ast)$ and $\hat{w}_i$ of $L(\lambda_\ast) = a(\lambda_\ast)L(\lambda_\ast)$ are related as $\hat{w}_i = a(\lambda_\ast)w_i$, whereas the corresponding eigenvectors can be chosen the same. Then let

$$Q_1 = (\lambda_\ast, a(\lambda_\ast)w_1), \ldots, Q_r = (\lambda_\ast, a(\lambda_\ast)w_r)$$

be the corresponding points over $\lambda = \lambda_\ast$ on the regularized spectral curve $S'$ and

$$\Psi = \left(\begin{array}{c}
\Xi \\
\Upsilon
\end{array}\right) = (\psi(Q_1) \cdots \psi(Q_r))$$

be the corresponding non-special eigenmatrix. Let also $\Gamma = \Upsilon \Xi^{-1}$ be the corresponding solution of the matrix quadratic equation (6.4), which fixed the branch of the map $\mathcal{B}_r$.

Theorem 6.4. Under the eigenvector map $\mathcal{M}$, the above branch of $\mathcal{B}_r$ is the translation on $\text{Prym}(S', \sigma)$ by the vector given by the degree zero divisor

$$\mathcal{T} = \infty_1 + \cdots + \infty_r - Q_1 - \cdots - Q_r.$$  \hspace{1cm} (6.16)

The proof can be found in Appendix. Note that

$$\mathcal{T} + \sigma \mathcal{T} = 2(\infty_1 + \cdots + \infty_r) - Q_1 - \cdots - Q_r - \sigma Q_1 - \cdots - \sigma Q_r$$

which is the divisor of the meromorphic function $1/((\lambda - \lambda_\ast)$ on $S'$. Hence $\mathcal{T} + \sigma \mathcal{T} \equiv 0$, and $\mathcal{T}$ indeed belongs to $\text{Prym}(S', \sigma)$.

We also note that the expression (6.16) is a direct generalization of the one-point translation described in Proposition 1 for the case $r = 1$. In the continuous limit $\lambda_\ast \rightarrow \infty$ we have $Q_1 + \cdots + Q_r \mapsto \infty_1 + \cdots + \infty_r$, hence the shift vector $\mathcal{T} = \mathcal{A}(\mathcal{T})$ tends to zero.
6.8. Growth of the map. Theorem 6.4 says that the shift $\mathcal{T}$ on Prym$(S', \sigma)$ does not depend on the step of iteration of $\mathfrak{B}_r$, but only on the choice of partition $w = \{w_1, \ldots, w_r \mid -w_1, \ldots, -w_r\}$. The opposite partitions $w_-, w_+$ produce opposite shifts $\mathcal{T}_-, \mathcal{T}_+$ such that $\mathcal{T}_- + \mathcal{T}_+ \equiv 0$. Hence the $2^r$ branches of the map $\mathfrak{B}_r$ and of the composition $\mathcal{M} \circ \mathfrak{B}_r$ can conditionally be divided into $2^{r-1}$ “forward” and $2^{r-1}$ “backward” branches. Clearly, double iterations of $\mathcal{M} \circ \mathfrak{B}_r$ with opposite partitions give the original value of $\mathcal{M}(X, P)$. This property also holds for the map $\mathfrak{B}_r$ itself, but not completely: due to Theorem 6.3, the corresponding double iterations of $\mathfrak{B}_r$ give $(-X, -P)$ and not $(X, P)$.

In $C^1$, the universal covering of Prym$(S', \sigma)$, all the iterations of $\mathcal{M} \circ \mathfrak{B}_r$ form a lattice $\Xi$ of rank $\leq 2^{r-1}$. Due to the definition of the vectors $\mathcal{A}(\mathcal{T})$, for $r > 2$ not all of them are linearly independent, hence the lattice fits into a linear subspace of dimension less than $2^{r-1}$. For example, let $r = 3$ and denote $\infty = \infty_1 + \infty_2 + \infty_3$. The lattice $\Xi$ is generated by 4 shift vectors $\mathcal{A}(\mathcal{T}_1), \ldots, \mathcal{A}(\mathcal{T}_4)$ with

$$
\begin{align*}
\mathcal{T}_1 &= -Q_1 - Q_2 - Q_3 + \infty, \\
\mathcal{T}_2 &= -\sigma(Q_1) - Q_2 - Q_3 + \infty, \\
\mathcal{T}_3 &= -Q_1 - \sigma(Q_2) - Q_3 + \infty, \\
\mathcal{T}_4 &= -Q_1 - Q_2 - \sigma(Q_3) + \infty,
\end{align*}
$$

and one can see that $\mathcal{A}(\mathcal{T}_1) + \mathcal{A}(\mathcal{T}_2) + \mathcal{A}(\mathcal{T}_3) + \mathcal{A}(\mathcal{T}_4) = 0$.

The above implies that iterations of the “forward” branches of $\mathcal{M} \circ \mathfrak{B}_r$ give a linear growth of the images of the map.

7. Conclusion. We have seen that the Neumann systems on $V_{n,r}$ (continuous and discrete) inherit or naturally generalize the basic properties of the classical Neumann system on $S^{n-1}$ and, therefore, of the (odd) Jacobi–Mumford systems: the structure of the Lax matrices (symplectic), the spectral curve (with involution), the equations of motion, linearization on Abelian varieties, and, in the discrete case, the formula for the translation $\mathcal{T}$ on them.

For this reason, we believe that the Hamiltonian systems on $T^* V_{n,r}$ we consider represent one of the most natural matrix generalizations of the odd Jacobi–Mumford systems.

It worths also mentioning that the solutions of such systems can provide solutions of the matrix KdV equation, like the solutions of the classical Neumann system on $S^n$ are related to finite-gap solutions of the KdV equation.

On the other hand, the considered generalizations are quite specific since, by construction, the corresponding matrix residues $\mathcal{N}_i$ in (4.1) have rank 1. So, it is also natural to consider extensions of the Neumann systems whose Lax matrices $L(\lambda)$ remain to be symplectic, but with residues of a higher rank at $\lambda = \alpha_i$. To reach a full generality, one can introduce the space $E$ of three $r \times r$ matrix polynomials

$$
\begin{align*}
U(\lambda) &= \lambda^n I_r + U_1 \lambda^{n-1} + \cdots + U_n, \\
V(\lambda) &= V_0 \lambda^n + V_1 \lambda^{n-1} + \cdots + V_n, \\
W(\lambda) &= \lambda^{n+1} I_r + W_0 \lambda^n + W_1 \lambda^{n-1} + \cdots + W_n,
\end{align*}
$$
where $U_i, W_j$ are symmetric and $V_0$ is skew-symmetric arbitrary coefficients, and consider a hierarchy of flows on $\mathcal{E}$ given by the Lax pairs

$$\frac{d}{dt} L(\lambda) = [L(\lambda), N(\lambda)], \quad L(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V^T(\lambda) \end{pmatrix}. $$

In particular, $N(\lambda)$ can be chosen in the form similar to (3.8)

$$N(\lambda) = \begin{pmatrix} V_0 \\ \Lambda \end{pmatrix}, \quad \Lambda = W_0 - U_1,$$

which implies that $V_0$ is a matrix first integral of the flow. One can prove that in the case $V_0 = 0$ the matrix $\Lambda$ satisfies a stationary reduction of one of the equations of the $r \times r$ symmetric matrix KdV hierarchy considered, in particular, in [7].

The spectral curve $\mathcal{S} = \{[L(\lambda) - wI_{2r}] = 0\}$ has the involution $\sigma: (\lambda, w) \mapsto (\lambda, -w)$ and, as above, its complete regularization $\mathcal{S}'$ has $r$ infinite points. It is then natural to conjecture that generic complex invariant manifolds of the flows on $\mathcal{E}$ are open subsets of non-compact extensions of the Prym varieties Prym($\mathcal{S}'$, $\sigma$) $\subset$ Jac($\mathcal{S}'$), and if $V_0 = 0$, such manifolds are the Prym varieties themselves.

Similarly, one can introduce a family of multi-valued maps

$$B_{\lambda*} : \mathcal{E} \to \mathcal{E}, \quad (U_i, W_j, V_k) \to (\tilde{U}_i, \tilde{V}_j, \tilde{W}_k), \quad \lambda* \in \mathbb{C}$$

defined by the intertwining relation

$$L(\lambda)M(\lambda, \lambda*) = M(\lambda, \lambda*)L(\lambda), \quad (7.1)$$

$$L(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V^T(\lambda) \end{pmatrix}, \quad M(\lambda, \lambda*) = \begin{pmatrix} -\Gamma \\ (\lambda - \lambda*)I_r + \Gamma^2 \end{pmatrix},$$

where $\tilde{L}(\lambda)$ depends on the polynomials $\tilde{U}(\lambda), \tilde{V}(\lambda), \tilde{W}(\lambda)$ in the same way as $L(\lambda)$ depends on $U(\lambda), \ldots, W(\lambda)$, and $\Gamma$ is an $r \times r$ symmetric matrix determined from the compatibility of the left- and right hand sides of (7.1). This condition leads to the following $r \times r$ matrix quadratic equation

$$\Gamma U(\lambda*)\Gamma + \Gamma V(\lambda*) + V^T(\lambda*)\Gamma - W(\lambda*) = 0. \quad (7.2)$$

Once one of the $2^r$ solutions $\Gamma$ of (7.2) is fixed by using Proposition 9, the map $B_{\lambda*}$ is defined uniquely. Like in the continuous case, one finds that $\tilde{V}_0 = -\tilde{V}_0^T$, which implies preservation of the skew-symmetric matrix $\tilde{V}_0$ under any branch of $B_{\lambda*}$. It is expected that the branches of $B_{\lambda*}$ are given by translations on Jac($\mathcal{S}'$) described by Theorem 6.4.

**Appendix. Technical proofs.**

**Proof of Lemma 5.2.** 1) Take two distinct eigenvalues $w, w'$ of $L(\lambda)$ and the corresponding eigenvectors $\psi(\lambda, w), \psi^*(\lambda, w')$. Then, on the one hand,

$$(\psi^*(\lambda, w'))^T L(\lambda) \psi(\lambda, w) = w (\psi^*(\lambda, w'))^T \psi(\lambda, w).$$

On the other hand, in view of (5.10), the same product equals

$$(\psi^*(\lambda, w'))^T L(\lambda) \psi(\lambda, w) = (L^T(\lambda)\psi^*(\lambda, w'))^T \psi(\lambda, w) = w' (\psi^*(\lambda, w'))^T \psi(\lambda, w).$$

Since $w' \neq w$, the difference of the above two equations yields

$$(\psi^*(\lambda, w'))^T \psi(\lambda, w) = 0$$
for any \((\lambda, w) \neq (\lambda, w')\) and, by continuity, \((\psi^*(\lambda, w))^T \psi(\lambda, w) = 0\) when \((\lambda, w) \in B\). Hence, \(F(P)\) vanishes at all the branch points \(B\).

Next, since the components \(\chi(P), \xi(P)\) of \(\psi(P)\) have poles only at \(D\) and, by their definition, the components of \(\psi^*(P)\) have poles only at \(\sigma D\), the function \(F(P)\) has poles only at \(D + \sigma D\). Note that the latter points are all finite, that is, distinct from \(\infty_1, \ldots, \infty_r \in B\), if we impose an appropriate normalization, for example \(\psi^1(P) + \cdots + \psi^r(P) = 1\).

Finally, note that, apart from simple zeros at \(B\), the function \(F(P)\) cannot have other zeros on \(S'\), because, due to the Riemann–Hurwitz formula, the genus of the curve \(S'\), as a 2\(r\)-fold covering of \(\mathbb{P}\), equals

\[
g = \text{gen}(S') = 2r(\text{gen}(P) - 1) + \text{deg}(B)/2 + 1 = -2r + \text{deg}(B)/2 + 1,
\]

then we have \(\text{deg}(B) = 2(g + 2r - 1)\), and, in view of (5.5),

\[
\text{deg}(D + \sigma D) = 2\text{deg}(D) = 2(g + 2r - 1) = \text{deg}(B).
\]

2) Since \(\psi(\sigma P) = \tilde{\psi}(P) = (\tilde{\chi}, \tilde{\xi})^T\) and \(\psi^*(\sigma P)\) satisfies

\[
L^T(\lambda)\psi^*(\sigma P) = -w\psi^*(\sigma P),
\]

we have \(\psi^*(\sigma P) = (-\xi, \chi)^T\). As a result,

\[
F(\sigma P) = (\psi^*(\sigma P))^T \psi(\sigma P) = -\tilde{\chi}\xi + \tilde{\chi}\tilde{\xi} = -F(P),
\]

which completes the proof of the lemma. 

Proof of Theorem 5.4. At the first step, developing the idea of proof of Theorem 6.39 in [6], we show that for any differential \(\Omega\) from the generalized Abel map (5.4),

\[
\frac{d}{dt} \bigg|_{t=0} \int_{D_0}^{D_t} \Omega = -\lim_{t \to 0} \sum_{s=1}^{r} \frac{d}{dt} \int_{Q_s}^{P_s(t)} \Omega,
\]

(A.1)

where \(P_1(t), \ldots, P_r(t) \in S'\) are the solutions of the equation \(f(P, 0) = 1/t\) for small \(t\) and \(Q_1, \ldots, Q_r\) are any finite fixed points.

Namely, let \((U_0, U_\infty)\) denote an open cover of \(S'\) such that \(U_\infty \setminus U_0\) is a neighborhood of infinite points \(\infty_1, \ldots, \infty_r\), while \(U_0 \setminus U_\infty\) is a neighborhood of the support of \(D_0\). Apart from the divisor \(D_t\), for a small \(t \neq 0\) consider the divisor \(D'_t\) defined by the relation

\[
(1 - t f(P, 0))P_0 = D'_t - D_0,
\]

where \((1 - t f(P, 0))P_0\) is a part of the divisor \((1 - t f(P, 0))\) with a support belonging to \(U_0\). If \(P_1(t), \ldots, P_r(t) \in U_\infty \setminus U_0\) are the solutions of \(f(P, 0) = 1/t\) such that \(\lim_{t \to 0} P_s(t) = \infty_s\), then

\[
(1 - t f(P, 0)) = D'_t - D_0 + P_1(t) + \cdots + P_r(t) - (\infty_1 + \cdots + \infty_r).
\]

Next, introduce the function \(g(P, t) = (1 - t f(P, 0))^2/\lambda\) satisfying \(g(\infty_s^+) = g(\infty_s^-) = t^2 \neq 0, \infty, s = 1, \ldots, [r/2]\). Then, the divisor of \(g\) gives zero in \(\text{Jac}(S', \infty)\):

\[
\langle g \rangle = 2(1 - t f(P, 0)) - (\lambda) = 2(1 - t f(P, 0)) + 2(\infty_1 + \cdots + \infty_r) - \mathcal{O} = 2D'_t - 2D_0 + 2(P_1(t) + \cdots + P_r(t)) - \mathcal{O} \equiv_{\infty} 0,
\]
where, as above, the divisor \( \mathcal{O} = O_1 + \cdots + O_{2r} \) is the preimage of \( \lambda = 0 \) on \( S' \).
That is, for any \( t, A(2D'_t - 2D_0 + 2(P_1 + \cdots + P_r) - \mathcal{O}) \in A \) and, therefore,
\[
2 \lim_{t \to 0} \frac{d}{dt} \int_{D_0}^{D'_t} \Omega = - \sum_{s=1}^{r} \lim_{t \to 0} \frac{d}{dt} \left( \int_{O_s}^{P_s(t)} \Omega + \int_{O_{2s}}^{P_s(t)} \Omega \right),
\]
where \( \Omega \) is a holomorphic or a meromorphic differential of 3rd kind with the poles at the infinite points.
Since \( \mathcal{O} \) does not depend on \( t \), the above relation implies
\[
\lim_{t \to 0} \frac{d}{dt} \int_{D_0}^{D'_t} \Omega = - \sum_{s=1}^{r} \lim_{t \to 0} \frac{d}{dt} \int_{Q_s}^{P_s(t)} \Omega,
\]
with any fixed points \( Q_1, \ldots, Q_r \in U_\infty \setminus U_0 \) close to \( \infty_1, \ldots, \infty_r \), respectively.
Further, as in Proposition 6.37 of [6], for small \( t \) we have
\[
\int_{D_t}^{D'_t} \Omega = O(t^2).
\]
Indeed, all the arguments of that proposition can be applied in our situation. Although in [6] the property (A.3) was proved for holomorphic differentials, the proof uses only their restriction to \( U_0 \), so it is applicable for meromorphic differentials \( \Omega_k \) as well.
As a result, from (A.2), (A.3) we get the relation (A.1).
At the second step we use the following property.

**Lemma A 1.** If \( \Omega \) is a holomorphic near \( \infty_s \), then
\[
\lim_{t \to 0} \frac{d}{dt} \int_{Q_s}^{P_s(t)} \Omega = \text{Res}_{\infty_s} f \Omega.
\]
On the other hand, if \( \Omega = \Omega_k \) is a normalized anti–invariant meromorphic differential with the residues \( \pm 1 \) at \( \infty_k^+, \infty_k^- \), then
\[
\lim_{t \to 0} \frac{d}{dt} \left( \int_{Q_k}^{P_k(t)} \Omega_k + \int_{Q_{k+[r/2]}}^{P_{k+[r/2]}(t)} \Omega_k \right) = \text{Res}_{\infty_k^+} f \Omega_k + \text{Res}_{\infty_k^-} f \Omega_k + 2\kappa \nu_k,
\]
for any \( k = 1, \ldots, [r/2] \).
Since the residues \( \text{Res}_{\infty_s} f \Omega \) do not depend on \( t \), applying the above Lemma to (A.1) with \( \Omega = \omega_1, \ldots, \omega_g, \Omega_{1}, \ldots, \Omega_{[r/2]} \), we prove the relation (5.16) for the case of anti-invariant meromorphic differentials \( \Omega_k \). However, a general meromorphic differential with simple poles at \( \infty_k^+, \infty_k^- \) is a linear combination of \( \Omega_k \) and \( \omega_1, \ldots, \omega_g \), hence Theorem 5.4 holds for the general case as well.

**Proof of Lemma A 1.** The solutions \( P_1(t), \ldots, P_r(t) \) of the equation \( f(P, 0) = 1/t \) correspond to the following expansions of their local coordinates \( \tau \)
\[
P_k(t) : \quad \tau = t + \kappa \nu_k t^2 + O(t^3),
\]
\[
P_{[r/2]+k}(t) : \quad \tau = t - \kappa \nu_k t^2 + O(t^3), \quad k = 1, \ldots, [r/2],
\]
\[
P_r(t) : \quad \tau = t + O(t^3) \quad (\text{if } r \text{ is odd}).
\]
This can be proved by substituting the expansions (5.15) to \( f(P, 0) = 1/t \) and using the implicit function theorem.
Then, for a holomorphic differential $\Omega$ with the expansion $(\varphi_s + O(\tau))d\tau$ near $\infty_s$, we have

$$\lim_{t \to 0} \frac{d}{dt} \int_{Q_s} P_s(t) \Omega = \lim_{t \to 0} \left( \varphi_s + O(\tau(t)) \right) \dot{\tau}(t) = \varphi_s = \text{Res}_{\infty_s} f \Omega.$$

For a meromorphic differential $\Omega_k$ with the behavior (5.18) near $\infty^+_k$ and $\infty^-_k$, the expansions (A.4) imply $\dot{\tau}(t) = 1 + \kappa \nu_k O(t^2)$ and

$$\lim_{t \to 0} \frac{d}{dt} \left( \int_{Q_k} P_k(t) \Omega_k + \int_{Q_k + i/2} P_k(t) \Omega_k \right)$$

$$= \lim_{t \to 0} \left( \left( \frac{1}{\tau(t)} + \phi_{k,k} + O(\tau) \right) \dot{\tau}(t) + \left( - \frac{1}{\tau(t)} + \phi_{k,k} + O(\tau) \right) \dot{\tau}(t) \right)$$

$$= 2\phi_{k,k} + 4k\nu_k = \text{Res}_{\infty^+_k} f \Omega_k + \text{Res}_{\infty^-_k} f \Omega_k + 2k\nu_k.$$

\[ \square \]

**Proof of Theorem 6.3.** Expressing $X$ from the second matrix equation in (6.1) and using the condition $X^T X = I_r$, we obtain the following alternative equation for the matrix multiplier $\Gamma$,

$$\Gamma \dot{\Gamma} - \dot{\Gamma}^T \Gamma - \dot{W} = 0, \quad (A.5)$$

where $\ddot{\Gamma}$ is the nonspecial eigenmatrix of $\Gamma$ that corresponds to the opposite partition $\{w_1, \ldots, w_r, -w_1, \ldots, -w_r\}$, and $\ddot{\Gamma} \ddot{\Xi}^{-1} \ddot{\Upsilon}$ is a nonspecial eigenmatrix of the above matrix $\ddot{\Gamma}$ corresponding to the opposite partition $\{w_1, \ldots, w_r \mid w_1, \ldots, w_r\}$. Indeed,

$$\begin{pmatrix} \dddot{V} & \dddot{U} \\ \dddot{W} & \dddot{V}^T \end{pmatrix} \begin{pmatrix} \dddot{\Xi} \\ \dddot{\Upsilon} \end{pmatrix} = w_s \begin{pmatrix} \dddot{\Xi} \\ \dddot{\Upsilon} \end{pmatrix}$$

implies

$$\begin{pmatrix} -\dddot{V} & \dddot{U} \\ \dddot{W} & \dddot{V}^T \end{pmatrix} \begin{pmatrix} -\dddot{\Xi} \\ \dddot{\Upsilon} \end{pmatrix} = -w_s \begin{pmatrix} -\dddot{\Xi} \\ \dddot{\Upsilon} \end{pmatrix},$$

for $s = 1, \ldots, r$. Hence the set $\{\dddot{\Gamma}\}$ contains the solution $-\Gamma^*$. To complete the proof, consider the tilded version of the equations (6.1),

$$\dddot{P}^* = A_{1/2}(\lambda_s) \dddot{X} - \dddot{X}^* \dddot{\Gamma}(\lambda_s), \quad \dddot{P} = -A_{1/2}(\lambda_s) \dddot{X}^* + \dddot{X} \dddot{\Gamma}(\lambda_s). \quad (A.6)$$
Setting here $\tilde{\Gamma} = -\Gamma^*$ and assuming that $\tilde{\mathcal{D}} = -\mathcal{D}$, $\tilde{X} = -X$, we see that the first (second) equation of (A.6) becomes the second (first) equation in (6.1). Hence, the equations (A.6) for $\tilde{X}, \tilde{\mathcal{D}}$ have the solution $(-X, -P)$.

**Proof of Theorem 6.4.** As follows from the intertwining relation (6.13), if $\psi(P)$ is an eigenvector of $L(\lambda)$, then

$$
\hat{\psi}(P) = M(\lambda, \lambda_s)\psi(P) = \left( \begin{array}{cc} -\Gamma & \mathbf{I}_r \\ (\lambda - \lambda_s)\mathbf{I}_r + (\Gamma)^2 & -\Gamma \end{array} \right) \psi(P)
$$

is an eigenvector of $\tilde{L}(\lambda)$ with the same eigenvalue. Note that whereas $\psi(P)$ is normalized, the above $\hat{\psi}(P)$ is not, so we consider normalized eigenvector $\hat{\psi}(P) = f^{-1}(P)\hat{\psi}(P)$, $f = \langle \alpha, \hat{\psi}(P) \rangle$ for a normalization $\alpha \in \mathbb{P}^{2r-1}$.

We compare the divisors $\mathcal{D}, \mathcal{D}$ of poles of $\psi(P), \hat{\psi}(P)$ for the chosen $\Gamma$. First, note that det $M(\lambda, \lambda_s) = (\lambda - \lambda_s)^r$. Hence, apart from the points of $\mathcal{S}'$ over $\lambda = \lambda_s$ and $\lambda = \infty$, $M(\lambda, \lambda_s)$ is non-degenerate. Therefore, apart from these points, the divisors of poles of $\psi(P)$ and $\hat{\psi}(P)$ coincide.

Without loss of generality, assume that $\hat{\psi}(P)$ is normalized in the same way as in Proposition 4.1. Then in the neighborhood of the infinite points $\infty_1, \ldots, \infty_r$, these components have the expansions (4.11) and, near each $\infty_s$ with the local coordinate $\tau = 1/\sqrt{s}$, the following expansion holds

$$
\hat{\psi}(P) = \left( \begin{array}{c} \tau^{-2}\mathbf{I}_r + (\Gamma)^2 + O(1)\mathbf{I}_r \\ \mathbf{v}_s + O(\tau) \end{array} \right),
$$

where, as in (5.14), $\mathbf{v}_{j + [r/2]} = \mathbf{v}_j, j = 1, \ldots, [r/2]$, and, in the case when $r$ is odd, $\mathbf{v}_r = \mathbf{v}_0$.

Therefore, in contrast to $\psi(P)$, some components of $\hat{\psi}(P)$ have a first order pole at $\infty_s$. Next, observe that the eigenvectors $\psi(Q_1), \ldots, \psi(Q_r)$ that form the eigenmatrix $\Psi$, span the kernel of $M(\lambda_s, \lambda_s)$. Indeed, in view of the relation $\Gamma = \Upsilon \Xi^{-1}$,

$$
M(\lambda_s, \lambda_s) \Psi = \left( \begin{array}{cc} -\Gamma & \mathbf{I}_r \\ (\Gamma)^2 & -\Gamma \end{array} \right) \left( \begin{array}{c} \Xi \\ \Upsilon \end{array} \right) = \left( \begin{array}{cc} -\Gamma & \mathbf{I}_r \\ (\Gamma)^2 & -\Gamma \end{array} \right) \mathbf{I}_r \Upsilon = \mathbf{0}.
$$

The products of $M(\lambda_s, \lambda_s)$ with the other eigenvectors $\psi(\sigma Q_1), \ldots, \psi(\sigma Q_r)$ are non-zero. It follows that the divisor of the above normalizing factor $f(P)$ is

$$
(f) = Q_1 + \cdots + Q_r + \mathcal{R} - \mathcal{D} - \infty_1 - \cdots - \infty_r
$$

for a certain effective divisor $\mathcal{R}$. Then the divisor of poles $\mathcal{D}$ of $\hat{\psi}(P) = \hat{\psi}(P)/f(P)$ equals $\mathcal{R}$. Indeed, the zeros of $\hat{\psi}(P)$ and $f(P)$ at $Q_1 + \cdots + Q_r$ as well as their poles at $\mathcal{D} + \infty_1 + \cdots + \infty_r$ cancel each other. Since $f(P)$ is meromorphic on $\mathcal{S}'$, we conclude that $\mathcal{D}$ is equivalent to

$$
\mathcal{D} + \infty_1 + \cdots + \infty_r - Q_1 - \cdots - Q_r,
$$

which implies that the images of $\mathcal{D}, \tilde{\mathcal{D}}$ in $\text{Jac}(\mathcal{S}')$ differ by the translation $\mathcal{T}$ in (6.16).
Acknowledgments. The authors are grateful to M. Alberich for having done independently a series of hard calculations related to the singularities of the spectral curve $S'$ and of its genus. We also thank L. Gavrilov for useful comments on the generalized Jacobians and the anonymous referees for their suggestions.

The participation of Yu. F. was partially funded by the Spanish MINECO-FEDER grants MTM2016-80276-P, PGC2018-098676-B-I00/AEI/FEDER/UE, and the provincial grant 2017SGR1049. Research of B. J. was supported by the Serbian Ministry of Education, Science and Technical Development through the Mathematical Institute of the Serbian Academy of Sciences and Arts.

REFERENCES

[1] M. R. Adams, J. Harnad and E. Previato, Isospectral Hamiltonian flows in finite and infinite dimensions. I. Generalized Moser systems and moment maps into loop algebras, Comm. Math. Phys., 117 (1988), 451–500.

[2] M. R. Adams, J. Harnad and J. Hurtubise, Isospectral Hamiltonian flows in finite and infinite dimensions II. Integration of flows, Comm. Math. Phys., 134 (1990), 555–585.

[3] M. R. Adams, J. Harnad and J. Hurtubise, Dual moment maps into loop algebras, Lett. Math. Phys., 20 (1990), 299–308.

[4] M. R. Adams, J. Harnad and J. Hurtubise, Darboux coordinates on coadjoint orbits of Lie algebras, Lett. Math. Phys., 40 (1997), 41–57.

[5] M. Adler and P. van Moerbeke, Birkhoff strata, Backlund transformations and regularization of isospectral operators, Advances in Math., 108 (1994), 140–204.

[6] M. Adler, P. van Moerbeke and P. Vanhecke, Algebraic Integrability, Painleve Geometry and Lie Algebras, Springer, 2004.

[7] C. Athorne and A. Fordy, Generalized KdV and mKdV equations associated with symmetric spaces, J. Phys. A., 20 (1987), 1377–1386.

[8] E. D. Belokolos, A. I. Bobenko, V. Z. Enol’sii, A. R. Its and V. B. Matveev, Algebro-Geometric Approach to Nonlinear Integrable Equations, Springer Series in Nonlinear Dynamics. Springer–Verlag, 1994.

[9] A. Beauville, Prym varieties and the Schottky problem, Invent. Math., 41 (1977), 149–196.

[10] A. Beauville, Jacobiennes des courbes spectrales et systemes hamiltoniens completement integrables, Acta Math., 164 (1990), 211–235.

[11] O. I. Bogoyavlenski, New integrable problem of classical mechanics, Comm. Math. Phys., 94 (1984), 255–269.

[12] A. V. Bolsinov and B. Jovanović, Noncommutative integrability, moment map and geodesic flows, Annals of Global Analysis and Geometry, 23 (2003), 305–322, arXiv:math-ph/0109031.

[13] E. Casas-Alvero, Singularities of Plane Curves, London Math. Soc. Lecture Notes Series. 276 Cambridge University Press, 2000.

[14] A. Clebsch and P. Gordan, Theorie Der Abelschen Funktionen, Teubner, Leipzig, 1866.

[15] J. Eilbeck, V. Enol’ski, V. Kuznetsov and A. Tsiganov, Linear R-matrix algebra for classical separable systems, J. Phys. A: Math. Gen., 27 (1994), 567–578.

[16] P. A. Dirac, On generalized Hamiltonian dynamics, Can. J. Math., 2 (1950), 129–148.

[17] V. Dragović and B. Gajić, The Lagrange bitop on so(4) × so(4) and geometry of the Prym varieties, American J. of Math., 126 (2004), 981–1004, arXiv:math-ph/0201036.

[18] B. A. Dubrovin, Completely integrable Hamiltonian systems associated with matrix operators and Abelian varieties, Funkts. Anal. Prilozh., 11 (1977), 28–41 (Russian); English transl.: Funct. Anal. Appl., 11 (1977), 265–277.

[19] B. A. Dubrovin, S. P. Novikov and I. M. Krichever, Integrable Systems. I, in Itogi Nauki i Tekhniki. Sovr. Proble. Mat. Fund. Naprav., 4 (1985), 179–284. English transl.: Encyclopaedia of Math. Sciences, Vol. 4, Springer-Verlag, Berlin, 1989.

[20] J. Fay, Theta-functions on Riemann Surfaces, Springer Lecture Notes, 352, Springer-Verlag, 1973.

[21] Yu. Fedorov, Classical integrable systems related to generalized Jacobians, Acta Appl. Math., 55 (1999), 251–301.

[22] Yu. Fedorov, Bäcklund transformations on coadjoint orbits of the loop algebra $\tilde{\mathfrak{gl}}(r)$, Recent advances in integrable systems (Kowloon, Hong Kong, 2000). J. Nonlinear Math. Phys, 9 (2002), suppl. 1, 29–46.
Yu. Fedorov and B. Jovanović, Geodesic flows and Neumann systems on Stiefel varieties: Geometry and integrability, Math. Z., 270 (2012), 659–698, arXiv:1011.1835.

Yu Fedorov and B. Jovanović, Three natural mechanical systems on Stiefel varieties, J. Phys. A., 45 (2012), 165204, (15pp), arXiv:1202.1660.

R. L. Fernandes and P. Vanhaecke, Hyperelliptic Prym varieties and integrable systems, Comm. Math. Phys., 221 (2001), 169–196, arXiv:math-ph/0011051.

L. Gavrilov, Generalized Jacobians of spectral curves and completely integrable systems, Math. Z., 230 (1999), 487–508.

L. Gavrilov, Jacobians of singularized spectral curves and completely integrable systems, in The Kovalevski Property (Leeds, 2000), 59–68, CRM Proc. Lecture Notes, 32, Amer. Math. Soc., Providence, RI, 2002, arXiv:math/0111235.

G. Jensen, Einstein metrics on principal fiber bundles, J. Diff. Geom., 8 (1973), 599–614.

B. Jovanović and Yu. Fedorov, Discrete geodesic flows on Stiefel manifolds, Tr. Mat. Inst. Steklova, 310 (2020) 176–188, (Russian); English transl.: Proceedings of the Steklov Institute of Mathematics, 310 (2020), 163–174.

B. Jovanović and V. Jovanović, Virtual billiards in pseudo–Euclidean spaces: Discrete Hamiltonian and contact integrability, Discrete and Continuous Dynamical Systems–Series A, 37 (2017), 5163–5190, arXiv:1510.04037.

B. Jovanović and V. Jovanović, Heisenberg model in pseudo-Euclidean spaces II, Regular and Chaotic Dynamics, 23 (2018), 418–437, arXiv:1808.10783.

A. N. Hone, V. B. Kuznetsov and O. Ragnisco, Bäcklund transformations for many-body systems related to KdV, J. Phys. A, 32 (1999), L299–L306, arXiv:solv-int/9904003.

R. Inoue, Y. Konishi and T. Yamazaki, Jacobian variety and integrable system—after Mumford, Beauville and Vanhaecke, J. Geom. Phys., 57 (2007), 815–831.

S. Kapustin, The Neumann system on Stiefel varieties, Preprint, 1992 (Russian).

F. Kirwan, Complex Algebraic Curves, London Mathematical Society Student Texts, 23. Cambridge University Press, Cambridge, 1992.

H. Knörrer, Geodesics on quadrics and a mechanical problem of C. Neumann, J. Reine Angew. Math., 334 (1982), 69–78.

I. Krichiver, Methods of algebraic geometry in the theory of non-linear equations, Uspekhi Mat. Nauk., 32 (1977), 183–208 (Russian); English translation: Russ. Math. Surv., 32 (1977), 185–213.

V. Kuznetsov and P. Vanhaecke, Bäcklund transformations for finite-dimensional integrable systems: A geometric approach, J. Geom. Phys., 44 (2002), 1–40, arXiv:nlin/0004003.

H. P. McKean, Variation on a theme of Jacobi, Comm. Pure Appl. Math., 38 (1985), 669–678.

A. S. Mishchenko and A. T. Fomenko, Generalized Liouville method of integration of Hamiltonian systems, Funkts. Anal. Prilozh., 12 (1978), 46–56 (Russian); English translation: Funct. Anal. Appl., 12 (1978), 113–121.

P. van Moerbeke and D. Mumford, The spectrum of difference operators and algebraic curves, Acta Math., 143 (1979), 93–154.

J. Moser, Geometry of quadratic and spectral theory, in: Chern Symposium 1979, Berlin–Heidelberg, New York, 1980, 147–188.

J. Moser and A. Veselov, Discrete versions of some classical integrable systems and factorization of matrix polynomials, Comm. Math. Phys., 139 (1991), 217–243.

D. Mumford, Tata Lectures on Theta II, Progress in Math., Birkhäuser, 1984.

N. N. Nekhoroshev, Action-angle variables and their generalization, Tr. Mosk. Mat. O.-va., 26 (1972), 181–198, (Russian); English translation: Trans. Mosc. Math. Soc., 26 (1972), 180–198.

C. Neumann, De probleme quodam mechanico, quod ad primam integralium ultra-ellipticoram classsem revocatum, J. Reine Angew. Math., 56 (1859), 46–63.

A. M. Perelomov, Some remarks on the integrability of the equations of motion of a rigid body in an ideal fluid, Funkt. Anal. Prilozh., 15 (1981), 83–85 (Russian); English translation: Funct. Anal. Appl., 15 (1981), 144–146.

M. Pedroni and P. Vanhaecke, A Lie algebraic generalization of the Mumford system, its symmetries and its multi-Hamiltonian structure, J. Moser at (Russian). Regul. Chaotic Dyn., 3 (1998), 132–160.
[50] J. Potter, Matrix quadratic solutions, J. SIAM Appl. Math., 14 (1966), 496–501.
[51] E. Previato, Flows on r-gonal Jacobians, in: The legacy of Sonya Kovalevskaya (Cambridge, Mass., and Amherst, Mass., 1985), Contemp. Math., 64, Amer. Math. Soc., Providence, RI, 1987, 153–180.
[52] O. Ragnisco, A discrete Neumann system, Phys.Lett.A., 167 (1992), 165–171.
[53] A. G. Reyman and M. A. Semenov-Tian-Shanski, Group theoretical methods in the theory of finite dimensional integrable systems, in: Integrable Systems. VII, Itogi Nauki i Tekhniki. Sovr.Probl.Mat. Fund.Naprav, 16, VINITI, Moscow 1987, 116–225 (Russian). English transl.: Encyclopaedia of Math SCIENCES, 16, Dynamical systems VII, Springer 1994. https://www.springer.com/gp/book/9783540181767.
[54] J. P. Serre, Groupes Algébriques et Corps De Classes, Hermann, Paris, 1959.
[55] R. J. Schilling, Generalizations of the Neumann system. A curve theoretical approach. II, Comm. Pure Appl. Math., 42 (1989), 409–442.
[56] Yu. B. Suris, The Problem of Integrable Discretization: Hamiltonian Approach, Progress in Mathematics, 219. Birkhauser Verlag, Basel, 2003.
[57] P. Vanhaecke, Integrable Systems in the Realm of Algebraic Geometry, Springer Lecture Notes, 1638, 1996.
[58] A. P. Veselov, Integrable discrete-time systems and difference operators, Funkt. Anal. Prilozh., 22 (1988), 1–13 (Russian); English translation: Funct. An. and Appl., 22 (1988), 83–93.
[59] A. P. Veselov, Integrable maps, Uspekhi Mat. Nauk, 46 (1991), 3–45 (Russian); English translation: Russ. Math. Surv., 46 (1991), 1–51.
[60] O. Vivolo, Jacobians of singular spectral curves and completely integrable systems, Proc. Edinburg Math. Soc., 43 (2000), 605–623.

Received February 2020; revised September 2020.

E-mail address: Yuri.Fedorov@upc.edu
E-mail address: bozaj@mi.sanu.ac.rs