WELL-POSEDNESS OF THE MARTINGALE PROBLEM FOR NON-LOCAL PERTURBATIONS OF LÉVY-TYPE GENERATORS

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ABSTRACT. Let $L$ be a Lévy-type generator whose Lévy measure is controlled from below by that of a non-degenerate $\alpha$-stable ($0 < \alpha < 2$) process. In this paper, we study the martingale problem for the operator $L_t = L + K_t$, with $K_t$ being a time-dependent non-local operator defined by

$K_t f(x) := \int_{\mathbb{R}^d \setminus \{0\}} [f(x + y) - f(x) - \mathbb{1}_{|y| \leq 1} y \cdot \nabla f(x)] M(t, x, dy),$

where $M(t, x, \cdot)$ is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. We show that if

$\sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |y|^\beta M(t, x, dy) < \infty$

for some $0 < \beta < \alpha$, then the martingale problem for $L_t$ is well-posed.

1. Introduction

As a generalization of the fractional Laplacian $\Delta^{\alpha/2} (0 < \alpha < 2)$, the anisotropic fractional Laplacian is defined by

$Af(x) = \int_{\mathbb{R}^d \setminus \{0\}} [f(x + y) - f(x) - \mathbb{1}_{|y| \leq 1} y \cdot \nabla f(x)] \nu(dy),$

where

$\nu(B) = \int_{\mathbb{S}^{d-1}} \mu(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}}, \forall B \in \mathcal{B}(\mathbb{R}^d),$

and $\mu$ is a finite measure on $\mathbb{S}^{d-1}$. We call $\nu$ the Lévy measure and $\mu$ the spectral measure of $A$. Clearly the behavior of the anisotropic fractional Laplacian is solely determined by its spectral measure. Since $\mu$ can be any finite measure on $\mathbb{S}^{d-1}$, this leads to some interesting properties of $A$ that the fractional Laplacian $\Delta^{\alpha/2}$ does not possess. As an example, the heat kernel of $A$ may have very different type of estimates compared to $\Delta^{\alpha/2}$, see [14].

The anisotropic fractional Laplacian $A$ corresponds to a Markov process, namely, it is the generator of an $\alpha$-stable process. It is natural to ask the following question of stability: if we add a small perturbation $B$ to $A$, does $A + B$ still correspond to a Markov process, or more precisely, is the martingale problem for $A + B$ well-posed? This problem has been well-studied when $1 < \alpha < 2$ and the perturbation operator $B$ is of drift-type $B = b(t, \cdot) \cdot \nabla$. Depending on the regularity of the

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spectral measure $\mu$, various classes of drifts $b$ have been introduced such that the martingale problem for $A + b(t, \cdot) \cdot \nabla$ is well-posed. If $\mu$ is the surface measure on $S^{d-1}$, drifts belonging to the Kato class $K^{d-1}_\alpha$ were considered in [5, 1]; for the case when $\mu$ is non-degenerate, drifts from some Hölder or $L^p$ spaces were treated in [10, 15, 4].

In addition to drift-type perturbations mentioned above, perturbations of $A$ including a lower order non-local term have also been investigated. This type of perturbation was first considered in [6]. There, the perturbation operator $B$ took the form

$$Bf(x) = 1_{n>1}b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - 1_{n>1}1_{|y| \leq 1}y \cdot \nabla f(x)] M(x, dy),$$

and, under some appropriate conditions on $\mu$, $b$ and $M$, uniqueness of the martingale problem for $A+B$ was obtained. As an essential step, some non-local estimates on the resolvent of $A$ were established in [6]. To obtain these estimates, relatively strong regularity conditions on the spectral measure $\mu$ were needed. More precisely, it was assumed in [6] that the spectral measure $\mu$ has the Radon-Nikodym density $m(y)$, $y \in S^{d-1}$, with respect to the surface measure on $S^{d-1}$, and $m(\cdot)$ is $d$-times continuously differentiable on $S^{d-1}$ and not identically 0. Afterwards, similar perturbations of stable-like operators were considered in [9, 8, 7, 2]; among many other things, well-posedness of the corresponding martingale problem was obtained in [7, 2]. We remark that in [2], the jump measures of the stable-like operator don’t need to have densities with respect to the Lebesgue measure and are merely assumed to be controlled from above and below, respectively, by two Lévy measures of non-degenerate $\alpha$-stable processes.

The anisotropic fractional Laplacian is a special Lévy-type generator. A general Lévy-type generator is given by

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(x) + b \cdot \nabla f(x)$$

$$+ \int_{\mathbb{R}^d \setminus \{0\}} [f(x+y) - f(x) - 1_{|y| \leq 1}y \cdot \nabla f(x)] \nu(dy), \quad (1.1)$$

where $(a_{ij})_{1 \leq i,j \leq d}$ is a positive semi-definite symmetric $d \times d$ matrix, $b \in \mathbb{R}^d$, and $\nu$ is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$. The tuple $((a_{ij})_{1 \leq i,j \leq d}, b, \nu)$ is called the Lévy triple of $L$. In this paper, we study the martingale problem for (time-dependent) non-local perturbations of a general Lévy-type generator whose Lévy measure is controlled from below by that of a non-degenerate anisotropic fractional Laplacian.

Our main result is the following:

**Theorem 1.1.** Let $L$ be as in (1.1) and assume that there exist some $\alpha \in (0,2)$ and a non-degenerate finite measure $\mu$ on $S^{d-1}$ such that

$$\nu(B) \geq \int_{S^{d-1}} \mu(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad \forall B \in \mathcal{B}(\mathbb{R}^d). \quad (1.2)$$
Define the operator $K_t$ by

$$K_t f(x) := \int_{\mathbb{R}^d \setminus \{0\}} [f(x + y) - f(x) - 1_{\{|y| \leq 1\}} y \cdot \nabla f(x)] M(t, x, dy),$$  \hspace{1cm} (1.3)

where $M$ is a measurable kernel from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ and $M(t, x, \cdot)$ is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. If there exists some $\beta \in (0, \alpha)$ such that

$$\sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |y|^\beta M(t, x, dy) < \infty,$$  \hspace{1cm} (1.4)

then the martingale problem for $L_t = L + K_t$ is well-posed.

Note that the matrix $(a_{i,j})_{1 \leq i,j \leq d}$ in (1.1) is not assumed to be non-degenerate in Theorem 1.1. Indeed, if $(a_{i,j})_{1 \leq i,j \leq d}$ is non-degenerate, then by the classical results of Stroock [12], the assumption (1.4) in Theorem 1.1 can be relaxed to $\sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |y|^2 M(t, x, dy) < \infty$. Here we are more interested in the case where $(a_{i,j})_{1 \leq i,j \leq d}$ is degenerate and the non-local part of $L$ acts as the leading term.

The novelty of our Theorem 1.1, compared to the results of [6, 7, 2] in this direction, lies firstly in the fact that the generator $L$ here contains a possibly degenerate diffusion part. As far as the author knows, non-local perturbations of this kind of Lévy-type generators have not yet been considered. Another point we would like to mention is that the Lévy measure $\nu$ of $L$ is only required to satisfy the lower bound condition (1.4), which is weaker than those assumed in the above mentioned works. As a compensation, our assumption (1.4) on the perturbing jump kernel $M(t, x, \cdot)$, which guarantees that $K_t$ is a lower order perturbation of $L$, is actually slightly stronger than those in [6, 7].

Our strategy to prove the asserted uniqueness is motivated by the method of Komatsu in [6]. We will derive some non-local estimates of the resolvent of $L$. Since our assumption on the Lévy measure $\nu$ is much weaker than that of [6], together with the presence of the possibly degenerate diffusion part of $L$ and the time-dependsnoency of the kernel $M(t, x, \cdot)$, our arguments are technically more involved. To obtain the existence, we will first consider smooth approximations $L_{n,t}$ of $L_t$ and then derive some Krylov’s estimates for the martingale solutions corresponding to $L_{n,t}$. It turns out that the limit point (under the topology of weak convergence for measures) of these martingale solutions exists and solves the martingale problem for $L_t$.

The rest of the paper is organized as follows. In Section 2 we give some notation and recall the definition of the martingale problem for non-local generators. In Section 3 we establish some estimates on the time-space resolvent of the Lévy process with generator $L$. In Section 4 we construct the time-space resolvent corresponding to $L_t$. Finally, we prove Theorem 1.1 in Section 5.

2. Preliminaries

The inner product of $x$ and $y$ in $\mathbb{R}^d$ is written as $x \cdot y$. We use $|v|$ to denote the Euclidean norm of a vector $v \in \mathbb{R}^m$, $m \in \mathbb{N}$. For a bounded function $g : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow$
\[ \mathbb{R}^m \] we write \( \|g\| := \sup_{(s, x) \in \mathbb{R} \times \mathbb{R}^d} |g(s, x)|. \) Let \( \mathbb{S}^{d-1} := \{ x \in \mathbb{R}^d : |x| = 1 \} \) be the unitary sphere.

Let \( C^2_c(\mathbb{R}^d) \) denote the class of \( C^2 \) functions such that the function and its first and second order partial derivatives are bounded. Note that \( C^2_c(\mathbb{R}^d) \) is a Banach space endowed with the norm
\[
\|f\|_{C^2_c(\mathbb{R}^d)} := \|f\| + \sum_{i=1}^d \|\partial_i f\| + \sum_{i,j=1}^d \|\partial^2_{ij} f\|, \quad f \in C^2_c(\mathbb{R}^d),
\]
where \( \partial_i f := \partial_{x_i} f(x) \) and \( \partial^2_{ij} f(x) := \partial^2_{x_i x_j} f(x) \) for \( x \in \mathbb{R}^d \). For \( k \in \mathbb{N} \) and \( k \geq 3 \), the space \( C^k(\mathbb{R}^d) \) and the norm on \( C^k(\mathbb{R}^d) \) are similarly defined.

Consider a Lévy-type generator
\[
Lf(x) = \sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(x) + b \cdot \nabla f(x) + \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x+y) - f(x) - 1_{\{|y| \leq 1\}} y \cdot \nabla f(x) \right] \nu(dy),
\]
defined for every \( f \in C^2_c(\mathbb{R}^d) \), where \((a_{ij})_{1 \leq i,j \leq d}\) is a positive semi-definite symmetric \( d \times d \) matrix, \( b \in \mathbb{R}^d \), and \( \nu \) is a Lévy measure on \( \mathbb{R}^d \setminus \{0\} \).

Throughout this paper, we assume that the generator \( L \) satisfies the following assumption.

**Assumption 2.1.** There exist \( \alpha \in (0,2) \) and a non-degenerate finite measure \( \mu \) on \( \mathbb{S}^{d-1} \) such that
\[
\nu(B) \geq \int_{\mathbb{S}^{d-1}} \mu(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad \forall B \in \mathcal{B}(\mathbb{R}^d).
\]

By non-degeneracy of \( \mu \) we mean that the support of \( \mu \) is not contained in a proper linear subspace of \( \mathbb{R}^d \).

**Remark 2.2.** Since we don’t assume additional conditions on \((a_{ij})_{1 \leq i,j \leq d}\), the matrix \((a_{ij})_{1 \leq i,j \leq d}\) can be degenerate.

Recall that \( K_t \) is given by
\[
K_t f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x+y) - f(x) - 1_{\{|y| \leq 1\}} y \cdot \nabla f(x) \right] M(t, x, dy),
\]
where \( M \) is a kernel from \( \mathbb{R}_+ \times \mathbb{R}^d \) to \( \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \) with \( M(t, x, \cdot) \) being a Lévy measure on \( \mathbb{R}^d \setminus \{0\} \) for each \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \). Without any further specification, we will always assume the following:

**Assumption 2.3.** There exists \( \beta \in (0, \alpha) \) such that
\[
\sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |y|^\beta M(t, x, dy) < \infty.
\]

Let
\[
L_t := L + K_t,
\]
where \( L \) and \( K_t \) are defined in (2.1) and (2.3), respectively.
Let $D = D([0, \infty))$, the set of paths in $\mathbb{R}^d$ that are right continuous with left limits, endowed with the Skorokhod topology. Set $X_t(\omega) = \omega(t)$ for $\omega \in D$ and let $\mathcal{D} = \sigma(X_t : 0 \leq t < \infty)$ and $\mathcal{F}_t := \sigma(X_r : 0 \leq r \leq t)$. A probability measure $\mathbf{P}$ on $(D, \mathcal{D})$ is called a solution to the martingale problem for $\mathcal{L}_t$ starting from $(s, x)$, if
\[
\mathbf{P}(X_t = x, \forall t \leq s) = 1
\]
and under the measure $\mathbf{P}$,
\[
f(X_t) - \int_s^t \mathcal{L}_u f(X_u) du, \quad t \geq s,
\]
is an $\mathcal{F}_t$-martingale after time $s$ for all $f \in C^2_0(\mathbb{R}^d)$.

3. Estimates on the time-space resolvent of the Lévy process with generator $L$

In this section we consider a $d$-dimensional Lévy process $S = (S_t)_{t \geq 0}$ with generator $L$ that is defined in (2.1). So $S$ has the Lévy triple $((a_{ij})_{1 \leq i,j \leq d}, b, \nu)$, namely,
\[
\mathbb{E}[e^{iS_u}] = e^{-\psi(u)}, \quad u \in \mathbb{R}^d,
\]
and
\[
\psi(u) = \sum_{i,j=1}^d a_{ij}u_iu_j - \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{iu \cdot y} - 1 - 1_{\{|y| \leq 2\}} iu \cdot y \right)\nu(dy) - ib \cdot u,
\]
where $(a_{ij})_{1 \leq i,j \leq d}$, $b$ and $\nu$ are the same as in (2.1).

Let $\alpha \in (0, 2)$ and $\mu$ be as in Assumption 2.1. Define
\[
\tilde{\nu}(B) = \int_{\mathbb{R}^d} \mu(dx) \int_0^\infty 1_B(r \xi) \frac{dr}{r^{d+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}^d),
\]
and
\[
\tilde{\psi}(u) = -\int_{\mathbb{R}^d \setminus \{0\}} \left(e^{iu \cdot y} - 1 - 1_{\{|y| \leq 2\}} iu \cdot y \right)\tilde{\nu}(dy), \quad u \in \mathbb{R}^d.
\]
Then $\tilde{\psi}$ is the characteristic exponent of an $\alpha$-stable process $\tilde{S} = (\tilde{S}_t)_{t \geq 0}$. Let $\psi := \psi - \tilde{\psi}$. So $\psi$ is the characteristic exponent of a Lévy process $\tilde{S} = (\tilde{S}_t)_{t \geq 0}$ with the Lévy triple $(A, b, \nu - \tilde{\nu})$. Without loss of generality, we assume that $S$, $\tilde{S}$ and $\tilde{S}$ are defined on the same probability space.

Define
\[
\gamma := \begin{cases} 
-\int_{\{|y| \leq 1\}} y\tilde{\nu}(dy), & 0 < \alpha < 1, \\
\int_{\mathbb{R}^d \setminus \{0\}} \xi \mu(dx), & \alpha = 1, \\
\int_{\{|y| > 1\}} y\tilde{\nu}(dy), & 1 < \alpha < 2.
\end{cases}
\]
Then for $\alpha \neq 1$, the function $\tilde{\psi}(u) + iu \cdot \gamma$ becomes a homogeneous function (with variable $u$) of index $\alpha$. As a result, for $\alpha \neq 1$, we obtain
\[
\tilde{\psi}(\rho u) + i(\rho u \cdot \gamma) = \rho^\alpha \left(\tilde{\psi}(u) + i(u \cdot \gamma)\right), \quad \forall \rho > 0.
\]
The case with $\alpha = 1$ is a little different. For $\alpha = 1$, according to [11, p. 84, (14.20)] and its complex conjugate, it holds that
Lemma 3.1. Let $t > 0$ be arbitrary. Then the densities $\tilde{p}_t \in C_b^\infty(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ for all $r \geq 1$.

Since

$$E[e^{iS_t \cdot u}] = e^{-t\tilde{\psi}(u)} = e^{-t\psi(u)}e^{-i\tilde{\psi}(u)} = E[e^{i\tilde{S}_t \cdot u}]E[e^{i\tilde{S}_t \cdot u}],$$

the law of $S_t$ has a density $p_t$ that is given by

$$p_t(x) := \int_{\mathbb{R}^d} \tilde{p}_t(x-y)\tilde{m}_t(dy), \quad x \in \mathbb{R}^d, \quad t > 0,$$

where $\tilde{m}_t$ denotes the law of $\tilde{S}_t$. It follows from Lemma 3.1 that $p_t \in C_b^\infty(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ for all $r \geq 1$.

For $0 < \delta < 1$, define the integro-differential operator $|\partial|^\delta$ by

$$|\partial|^\delta f(x) = c_3 \int_{\mathbb{R}^d \setminus \{0\}} [f(x+y) - f(x)] \cdot |y|^{-d-\delta} dy, \quad f \in C_b^2(\mathbb{R}^d),$$

where the constant $c_3$ is given by

$$c_3 := 2^\delta \pi^{-d/2} \Gamma \left( \frac{d+\delta}{2} \right) / \Gamma \left( \frac{-\delta}{2} \right).$$

Note that

$$c_3 \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{iu \cdot y} - 1 \right) |y|^{-d-\delta} dy = -|u|^\delta, \quad u \in \mathbb{R}^d. \quad (3.11)$$

Next, we give an estimate of the $L^r$-norm of $|\partial|^\delta p_t$. 

\[
\tilde{\psi}(u) = \int_{\mathbb{R}^{d-1}} \left( \frac{\pi}{2} |u \cdot \xi| + iu \cdot (\xi \log |u \cdot \xi|) - ic_1 u \cdot \xi \right) \mu(d\xi), \quad u \in \mathbb{R}^d,
\]

where $c_1 = \int_1^\infty r^2 \sin rdr + \int_1^1 r^{-2}(\sin r - r)dr$; in this case, we have

$$\tilde{\psi}(pu) = p\tilde{\psi}(u) + i(p \log p)u \cdot \gamma, \quad \forall p > 0, \quad u \in \mathbb{R}^d. \quad (3.6)$$

According to Assumption 2.1 and [11, Prop. 24.20], there exists some constant $c_2 > 0$ such that

$$|e^{-t\tilde{\psi}(u)}| \leq e^{-c_2 t|u|^\alpha}, \quad \forall u \in \mathbb{R}^d, \quad t > 0. \quad (3.7)$$

By the inversion formula of Fourier transform, the law of $\tilde{S}_t$ has a density $\tilde{p}_t \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ that is given by

$$\tilde{p}_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu \cdot x} e^{-i\tilde{\psi}(u)} du, \quad x \in \mathbb{R}^d, \quad t > 0. \quad (3.8)$$

Moreover, according to [14, p. 2856, (2.3)], we have the following scaling property for $\tilde{p}_t$: for $x \in \mathbb{R}^d$, $t > 0$,

$$\tilde{p}_t(x) = \begin{cases} t^{-d/\alpha} \tilde{p}_1(t^{-1/\alpha}x + (1 - t^{-1/\alpha})\gamma), & (\alpha \neq 1), \\ t^{-d} \tilde{p}_1(t^{-1}x - \gamma \log t), & (\alpha = 1), \end{cases} \quad (3.9)$$

where $\gamma$ is given in (3.4).

The following result is a slight extension of [10, Lemma 3.1]. For its proof the reader is referred to [4, Lemma 3.1].

Lemma 3.1. For $0 < \delta < 1$, define the integro-differential operator $|\partial|^\delta$ by

$$|\partial|^\delta f(x) = c_3 \int_{\mathbb{R}^d \setminus \{0\}} [f(x+y) - f(x)] \cdot |y|^{-d-\delta} dy, \quad f \in C_b^2(\mathbb{R}^d),$$

where the constant $c_3$ is given by

$$c_3 := 2^\delta \pi^{-d/2} \Gamma \left( \frac{d+\delta}{2} \right) / \Gamma \left( \frac{-\delta}{2} \right).$$

Note that

$$c_3 \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{iu \cdot y} - 1 \right) |y|^{-d-\delta} dy = -|u|^\delta, \quad u \in \mathbb{R}^d. \quad (3.11)$$

Next, we give an estimate of the $L^r$-norm of $|\partial|^\delta p_t$. 

Lemma 3.2. Let $0 < \delta < 1$ and $r \geq 1$. Then there exists a constant $c_4 > 0$ that depends on $\delta$ and $r$ such that

$$|||\partial^\delta p_t|||_{L^r(\mathbb{R}^d)} \leq c_4 t^{(d/r-\delta-d)/\alpha}, \quad \forall t > 0.$$ (3.12)

Proof. Since $|||\partial^\delta p_t(x) = \int_{\mathbb{R}^d} |||\partial^\delta p_t(x - y)\mu_t(dy), t > 0$, by Jensen’s inequality, it suffices to prove

$$|||\partial^\delta \tilde{p}_t|||_{L^r(\mathbb{R}^d)} \leq t^{(d/r-\delta-d)/\alpha} |||\partial^\delta \tilde{p}_1|||_{L^r(\mathbb{R}^d)} < \infty, \quad \forall t > 0.$$ (3.13)

By (3.7), (3.8) and Fubini’s theorem, we easily obtain that for each $t > 0,$

$$|||\partial^\delta \tilde{p}_t|||_{L^r(\mathbb{R}^d)} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |u|^\delta e^{-i\psi(u)} e^{-iu \cdot x} du, \quad x \in \mathbb{R}^d.$$ (3.14)

We first assume $\alpha \neq 1$. Using a change of variables $u = t^{-1/\alpha} u'$ and noting (3.5), we obtain

$$|||\partial^\delta \tilde{p}_t(x) = \frac{1}{t^{-d/\alpha}} \int_{\mathbb{R}^d} t^{-\delta/\alpha} |u|^\delta e^{-i(\tilde{\psi}(u') + it^{-1/\alpha} u' \cdot \gamma + it^{-1/\alpha} u' \cdot x) du' = t^{-(\delta+d)/\alpha} |||\partial^\delta \tilde{p}_1(t^{-1/\alpha} x - \gamma(t^{1-1/\alpha} - 1)).$$

So

$$|||\partial^\delta \tilde{p}_t|||_{L^r(\mathbb{R}^d)} \leq t^{-(\delta+d)/\alpha} \left( \int_{\mathbb{R}^d} \left(|||\partial^\delta \tilde{p}_1(t^{-1/\alpha} x)\right)^r dx \right)^{1/r} = t^{(d/r-\delta-d)/\alpha} |||\partial^\delta \tilde{p}_1|||_{L^r(\mathbb{R}^d)}.$$ (3.15)

For the case $\alpha = 1$, we can apply (3.6) and a similar argument as above to also obtain (3.14). So (3.14) is true for all $\alpha \in (0, 2)$.

It remains to show that $|||\partial^\delta \tilde{p}_1|||_{L^r(\mathbb{R}^d)} < \infty$, or equivalently,

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} |u|^\delta e^{-i\tilde{\psi}(u)} e^{-iu \cdot y} du \right|^r dy < \infty.$$ (3.16)

To prove this fact, we use the same idea as in the proof of [4, Lemma 3.4]. Firstly, note that the characteristic exponent $\tilde{\psi}$ can be written as the sum of $\tilde{\psi}_1$ and $\tilde{\psi}_2$, where

$$\tilde{\psi}_1(u) = -\int_{\{0 < |y| \leq 1\}} \left( e^{iu \cdot y} - 1 - iu \cdot y \right) \nu(dy), \quad \tilde{\psi}_2 = \tilde{\psi} - \tilde{\psi}_1.$$ (3.17)

We can easily check that $\tilde{\psi}_1 \in C^\infty(\mathbb{R}^d)$. Since (3.7) holds, we see that $\exp(-\tilde{\psi}_1)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

According to (3.11), we can write $|u|^\delta = \psi_{\delta,1}(u) + \psi_{\delta,2}(u) + \psi_{\delta,3}$, where

$$\psi_{\delta,1}(u) = -c_3 \int_{\{0 < |y| \leq 1\}} (e^{iu \cdot y} - 1) |y|^{-d-\delta} dy$$

and

$$\psi_{\delta,2}(u) = -c_3 \int_{\{|y| > 1\}} e^{iu \cdot y} |y|^{-d-\delta} dy, \quad \psi_{\delta,3} = c_3 \int_{\{|y| > 1\}} |y|^{-d-\delta} dy.$$ (3.18)
Lemma 3.4. Indeed, the proof of Lemma 3.2 can be easily adapted to work also for this case. We only treat the first term on the right-hand side of (3.16), since the other two terms are similar. With the same reason as for exp(−ψ) above, we have ψ exp(−ψ) ∈ S(R^d). It is also easy to see that exp(−ψ) is bounded and is the characteristic function of an infinitely divisible probability measure ρ on R^d. As a consequence, we are allowed to define h to be the inverse Fourier transform of the ψ exp(−ψ), i.e.,

\[ h(y) := \frac{1}{(2\pi)^d} \int_{R^d} \psi \hat{1}_e^{-\psi} e^{-\psi} e^{-iu \cdot y} du, \quad y \in \mathbb{R}^d. \]

Since the Fourier transform is a one-to-one map of S(R^d) onto itself, we can find f ∈ S(R^d) with \( \hat{f} = \psi \hat{1}_e \exp(−\psi) \), where \( \hat{f} \) denotes the Fourier transform of f. In particular, we have f ∈ L^r(R^d). Let f * ρ be the convolution of f and ρ. We have

\[ \hat{f} \ast \rho = \hat{f} \rho = \psi \hat{1}_e^{-\psi} \hat{1}_e = \psi \hat{1}_e^{-\psi} = \hat{h}, \]

which implies h = f * ρ. Thus h ∈ C^∞_b (R^d). By Young’s inequality, we get h ∈ L^r(R^d), i.e.,

\[ \int_{R^d} \left| \int_{R^d} \psi \hat{1}_e^{-\psi} (u) e^{-iu \cdot y} du \right|^r dy < \infty. \tag{3.17} \]

Similarly, by noting that −ψ^2 and exp(−ψ^2) are both characteristic functions of some finite measures on R^d, we can show that

\[ \int_{R^d} \left| \int_{R^d} e^{-\psi} (-\psi^2) e^{-iu \cdot y} du \right|^r dy < \infty \tag{3.18} \]

and

\[ \int_{R^d} \left| \int_{R^d} \psi^3 e^{-\psi} \psi^2 e^{-iu \cdot y} du \right|^r dy < \infty. \tag{3.19} \]

Now, the inequality (3.15) follows from (3.16), (3.17), (3.18) and (3.19).

Remark 3.3. If we understand |δ|^0 as the identity map, then Lemma 3.2 holds also for the case δ = 0, namely, for each r ≥ 1, there exists a constant c_4 > 0 depending on r such that

\[ \| p_t \|_{L^r(R^d)} \leq c_4 t^{(d/r - d) / \alpha}, \quad \forall t > 0. \tag{3.20} \]

Indeed, the proof of Lemma 3.2 can be easily adapted to work also for this case.

In the next lemma we deal with a non-local estimate on the gradient of p_t when 1 < α < 2. Since its proof is completely similar to that of Lemma 3.2, we omit it here.

Lemma 3.4. Let 1 < α < 2, 0 < δ < α - 1 and r ≥ 1. Then there exists a constant c_5 > 0 which depends on δ and r such that for each \( i = 1, \cdots, d \),

\[ \| \partial^\delta \partial_i p_t \|_{L^r(R^d)} \leq c_5 t^{(d/r - d - 1 - d) / \alpha}, \quad \forall t > 0. \]
For $\lambda > 0$, the time-space resolvent $R_\lambda$ of the Lévy process $S$ is defined by

$$R_\lambda f(t, x) := \int_0^\infty e^{-\lambda u} \int_{\mathbb{R}^d} p_u(y - x) f(t + u, y) dy du, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad (3.21)$$

where $f \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$.

Before we state the next lemma, we recall two equalities from [6, Lemma 2.1]: for each $0 < \delta < 1$, there exist constants $c_6, c_7 > 0$, which depend on $\delta$, such that

$$\int_{\mathbb{R}^d} |(|w + z|^\delta - |w|^\delta - d)| \, dw = c_6 |z|^\delta, \quad (3.22)$$

and

$$f(x + z) - f(x) = c_7 \int_{\mathbb{R}^d} (|w + z|^\delta - |w|^\delta - d)|\partial \delta f(x - w) dw, \quad (3.23)$$

where $f \in C_b^\infty(\mathbb{R}^d)$ is arbitrary.

**Lemma 3.5.** Assume $0 < \delta < \alpha \wedge 1$.

(i) If $\lambda > 0$ and $g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$, then $|\partial \delta (R_\lambda g(t, \cdot))$ is well-defined for each $t \geq 0$. Moreover, there exists a constant $C_\lambda > 0$, independent of $g$, such that

$$\|\partial \delta (R_\lambda g(t, \cdot)) (x)\| \leq C\lambda \|g\| \quad (3.24)$$

and

$$|R_\lambda g(t, x + z) - R_\lambda g(t, x)| \leq C\lambda |z|^\delta \|g\| \quad (3.25)$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $z \in \mathbb{R}^d$. The constant $C_\lambda$ goes to 0 as $\lambda \to \infty$.

(ii) Let $T > 0$ and $g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$ be such that $\text{supp} (g) \subset [0, T] \times \mathbb{R}^d$ and $g \in L^p([0, T]; \mathbb{R}^d)$ with $p, q > 0$ and $d/p + \alpha/q < \alpha - \delta$. Then for each $\lambda > 0$, there exists a constant $N_\lambda > 0$, independent of $g$ and $T$, such that

$$\|\partial \delta (R_\lambda g(t, \cdot)) (x)\| \leq N_\lambda \|g\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \quad (3.26)$$

and

$$|R_\lambda g(t, x + z) - R_\lambda g(t, x)| \leq N_\lambda z^\delta \|g\|_{L^q([0, T]; L^p(\mathbb{R}^d))} \quad (3.27)$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $z \in \mathbb{R}^d$. Moreover, the constant $N_\lambda$ goes to 0 as $\lambda \to \infty$.

**Proof.** (i) Assume $g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$. Let $\epsilon > 0$ be a constant such that $\delta < \delta + \epsilon < \alpha \wedge 1$. For $z \in \mathbb{R}^d$, we have

$$|p_u(y - x - z) - p_u(y - x)| \leq c_7 \int_{\mathbb{R}^d} (|w - z|^\delta + \epsilon - d - |w|^\delta + \epsilon - d)|\partial \delta + \epsilon p_u(y - x - w)| dw. \quad (3.28)$$

It follows from (3.28) and Young’s inequality that

$$\int_{\mathbb{R}^d} |p_u(y - x - z) - p_u(y - x)| dy \leq c_7 \int_{\mathbb{R}^d} (|w - z|^\delta + \epsilon - d - |w|^\delta + \epsilon - d)|\partial \delta + \epsilon p_u(y - x - w)| dw \leq c_7 c_6 |z|^{\delta + \epsilon} \rho \|\partial \delta + \epsilon p_u\|_{L^1(\mathbb{R}^d)} \leq c_7 c_6 |z|^{\delta + \epsilon} \rho \|\partial \delta + \epsilon p_u\|_{L^1(\mathbb{R}^d)} \leq c_4 c_6 c_7 u^{-(\delta + \epsilon)/\alpha} |z|^{\delta + \epsilon}. \quad (3.29)$$
So
\[ |R_\lambda g(t, x + z) - R_\lambda g(t, x)| \leq \|g\| \int_0^\infty e^{-\lambda u} \int_{\mathbb{R}^d} |p_u(y - x - z) - p_u(y - x)|dydu \]
\[ \leq c_4 c_6 c_7 |z|^{\delta + \epsilon} \|g\| \int_0^\infty e^{-\lambda u} u^{-(\delta + \epsilon)/\alpha} du. \] (3.30)

On the other hand, we have
\[ |R_\lambda g(t, x + z) - R_\lambda g(t, x)| \leq 2\|R_\lambda g\| \leq 2\lambda^{-1}\|g\|. \] (3.31)

By (3.30) and (3.31), we can find a constant \( c > 0 \) such that
\[ |R_\lambda g(t, x + z) - R_\lambda g(t, x)| \leq c (|z|^\delta \wedge 1), \quad \forall z \in \mathbb{R}^d, \]
which implies that \( |\mathcal{D}^\delta (R_\lambda g(t, \cdot)) (x) \) is well-defined.

By Fubini’s theorem, we obtain that for all \( t \geq 0 \) and \( x \in \mathbb{R}^d, \)
\[ |\mathcal{D}^\delta (R_\lambda g(t, \cdot)) (x) = \int_0^\infty e^{-\lambda u} \int_{\mathbb{R}^d} |\mathcal{D}^\delta (p_u(y - \cdot)) (x) g(t + u, y)dydu. \] (3.32)

So for all \( t \geq 0 \) and \( x \in \mathbb{R}^d, \)
\[ ||\mathcal{D}^\delta (R_\lambda g(t, \cdot)) (x)| \leq \|g\| \int_0^\infty e^{-\lambda u} \|\mathcal{D}^\delta p_u\|_{L^1(\mathbb{R}^d)} du \leq C_\lambda \|g\|, \] (3.33)
where
\[ C_\lambda := c_4 \int_0^\infty e^{-\lambda u} u^{-\delta/\alpha} du. \]

Hence (3.24) is true. It is clear that \( C_\lambda \downarrow 0 \) as \( \lambda \to \infty. \)

It follows from (3.23) that
\[ R_\lambda g(t, x + z) - R_\lambda g(t, x) \]
\[ = \int_0^\infty e^{-\lambda u} \int_{\mathbb{R}^d} [p_u(y - x - z) - p_u(y - x)]g(t + u, y)dydu \]
\[ = \int_0^\infty e^{-\lambda u} \int_{\mathbb{R}^d} (c_7 \int_{\mathbb{R}^d} (|w - z|^{\delta - d} - |w|^{\delta - d})|\mathcal{D}^\delta p_u(y - x - w)dw) \]
\[ \times g(t + u, y)dydu. \]

In view of (3.12), (3.22) and (3.32), we can apply Fubini’s theorem to obtain that for all \( t \geq 0, \ x, z \in \mathbb{R}^d, \)
\[ R_\lambda g(t, x + z) - R_\lambda g(t, x) = c_7 \int_{\mathbb{R}^d} (|w - z|^{\delta - d} - |w|^{\delta - d})|\mathcal{D}^\delta (R_\lambda g(t, \cdot)) (x - w)dw. \] (3.34)

Combining (3.34), (3.22) and (3.33) yields (3.25).

(ii) Since (3.27) follows easily from (3.22), (3.26) and (3.34), we only need to prove (3.26). Note that \( \text{supp}(g) \subset [0, T] \times \mathbb{R}^d. \) By (3.32) and Hölder’s inequality,
we get

\[
||\partial^\delta (R_\lambda g(t, \cdot)) (x)|| = \left| \int_0^\infty e^{-\lambda u} \int_{\mathbb{R}^d} |\partial^\delta (p_u(y - \cdot))(x)g(t + u, y)dydu \right|
\]

\[
\leq \int_0^\infty e^{-\lambda u} ||\partial^\delta p_u||_{L^p(\mathbb{R}^d)} ||g(t + u, \cdot)||_{L^p(\mathbb{R}^d)} du
\]

\[
= \int_0^T e^{-\lambda u} ||\partial^\delta p_u||_{L^p(\mathbb{R}^d)} ||g(t + u, \cdot)||_{L^p(\mathbb{R}^d)} du
\]

\[
\leq \left( \int_0^T e^{-q^* \lambda u} ||\partial^\delta p_u||_{L^p(\mathbb{R}^d)}^q \right)^{1/q^*} \|g\|_{L^q([0,T];L^p(\mathbb{R}^d))},
\]

where \(p^*, q^* > 0\) are such that \(1/p^* + 1/p = 1\) and \(1/q^* + 1/q = 1\). By (3.12), we see that the inequality (3.26) holds with

\[
N_\lambda := \left( c_4 \int_0^\infty e^{-q^* \lambda u} u^{-1/d/p^* - \delta - d} du \right)^{1/q^*},
\]

which is finite if \(q^* \alpha - 1/d/p^* - \delta - d > -1\), or equivalently, \(d/p + \alpha/q < \alpha - \delta\). By dominated convergence theorem, \(\lim_{\lambda \to \infty} N_\lambda = 0\).

\[\square\]

**Lemma 3.6.** Let \(1 < \alpha < 2\) and \(0 < \delta < \alpha - 1\).

(i) If \(\lambda > 0\) and \(g \in \mathcal{B}_0(\mathbb{R}_+ \times \mathbb{R}^d)\), then \(\partial^\delta (\partial_t R_\lambda g(t, \cdot))\) is well-defined for each \(t \geq 0\). Moreover, there exists a constant \(\tilde{C}_\lambda > 0\), independent of \(g\), such that for all \(g \in \mathcal{B}_0(\mathbb{R}_+ \times \mathbb{R}^d)\),

\[
||\partial^\delta (\partial_t R_\lambda g(t, \cdot)) (x)|| \leq \tilde{C}_\lambda ||g||
\]

and

\[
||\partial_i R_\lambda g(t, x + z) - \partial_i R_\lambda g(t, x)|| \leq \tilde{C}_\lambda ||z|| ||g||
\]

for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\), \(z \in \mathbb{R}^d\) and \(i = 1, \ldots, d\). The constant \(\tilde{C}_\lambda\) goes to 0 as \(\lambda \to \infty\).

(ii) Let \(T > 0\) and \(g \in \mathcal{B}_0(\mathbb{R}_+ \times \mathbb{R}^d)\) be such that \(\text{supp}(g) \subset [0, T] \times \mathbb{R}^d\) and \(g \in L^q([0,T];L^p(\mathbb{R}^d))\) with \(d/p + \alpha/q < \alpha - 1 - \delta\). Then for each \(\lambda > 0\), there exists a constant \(\tilde{N}_\lambda > 0\), independent of \(g\) and \(T\), such that

\[
||\partial^\delta (\partial_t R_\lambda g(t, \cdot)) (x)|| \leq \tilde{N}_\lambda ||g||_{L^q([0,T];L^p(\mathbb{R}^d))}
\]

and

\[
||\partial_i R_\lambda g(t, x + z) - \partial_i R_\lambda g(t, x)|| \leq \tilde{N}_\lambda ||z|| ||g||_{L^q([0,T];L^p(\mathbb{R}^d))}
\]

for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\), \(z \in \mathbb{R}^d\) and \(i = 1, \ldots, d\). Moreover, the constant \(\tilde{N}_\lambda\) goes to 0 as \(\lambda \to \infty\).

**Proof.** Let \(g \in \mathcal{B}_0(\mathbb{R}_+ \times \mathbb{R}^d)\) be arbitrary. It is easy to see that for each \(i = 1, \ldots, d\),

\[
\partial_i R_\lambda g(t, x) = - \int_0^\infty e^{-\lambda u} \int_{\mathbb{R}^d} \partial_i p_u(y - x)g(t + u, y)dydu.
\]

In view of Lemma 3.4, we can argue in the same way as in Lemma 3.5 to derive the statements. We omit the details. \(\square\)
4. Construction of the time-space resolvent corresponding to $L_t$

In this section we give a purely analytical construction of the time-space resolvent $G_\lambda$ that corresponds to the generator $L_t := L + K_t$. Not to be precise, we can write $G_\lambda = (\lambda - \partial_t - L_t)^{-1}$. The main aim of this section is to establish rigorously, at least for large enough $\lambda > 0$, that

$$G_\lambda g = \sum_{k=0}^{\infty} R_\lambda (KR_\lambda)^k g, \quad g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d),$$

where $R_\lambda$ is the time-space resolvent of the Lévy process $S$ and the operator $KR_\lambda$ is defined by

$$KR_\lambda g(t,x) := \int_{\mathbb{R}^d \setminus \{0\}} [R_\lambda g(t,x + z) - R_\lambda g(t,x) - 1_{\alpha > 1} 1_{\{|z| \leq 1\}} z \cdot \nabla R_\lambda g(t,x) M(t,x,dz), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \quad (4.1)$$

To see that $KR_\lambda g$ in (4.1) is well-defined for $g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we need the following proposition.

**Proposition 4.1.** For each $\lambda > 0$, define

$$k_\lambda := \begin{cases} (C_\lambda + 2\lambda^{-1}) \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |z|^{\beta} M(t,x,dz), & 0 < \alpha \leq 1, \\ (\tilde{C}_\lambda + 2\lambda^{-1}) \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |z|^{\beta} M(t,x,dz), & 1 < \alpha < 2, \end{cases} \quad (4.2)$$

where $C_\lambda$ and $\tilde{C}_\lambda$ are the constants from Lemma 3.5 and Lemma 3.6, respectively.

Then

$$\|KR_\lambda g\| \leq k_\lambda \|g\|, \quad \forall g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d). \quad (4.3)$$

**Proof.** Let $\beta \in (0, \alpha)$ be the constant in Assumption 2.3. We distinguish between the cases with $0 < \alpha \leq 1$ and $1 < \alpha < 2$.

**Case 1:** $0 < \alpha \leq 1$. According to Lemma 3.5, there exists a constant $C_\lambda > 0$ such that for all $g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$|R_\lambda g(t,x + z) - R_\lambda g(t,x)| \leq C_\lambda \|g\| |z|^\beta, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad z \in \mathbb{R}^d,$$

and $C_\lambda$ goes to 0 as $\lambda \uparrow \infty$.

Let $g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$ be arbitrary. Then
\[
\int_{\mathbb{R}^d \setminus \{0\}} |R_{\lambda}g(t, x + z) - R_{\lambda}g(t, x)| M(t, x, dz) \\
= \int_{\{0 < |z| \leq 1\}} |R_{\lambda}g(t, x + z) - R_{\lambda}g(t, x)| M(t, x, dz) \\
+ \int_{\{|z| > 1\}} |R_{\lambda}g(t, x + z) - R_{\lambda}g(t, x)| M(t, x, dz) \\
\leq C_\lambda \|g\| \int_{\{0 < |z| \leq 1\}} |z|^{\beta} M(t, x, dz) + 2 \|R_{\lambda}g\| \int_{\{|z| > 1\}} 1 M(t, x, dz) \\
\leq (C_\lambda + 2\lambda^{-1}) \|g\| \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |z|^{\beta} M(t, x, dz).
\]

So \(K R_{\lambda}g\) is well-defined and \(\|K R_{\lambda}g\| \leq k_\lambda \|g\|\).

**Case 2:** \(1 < \alpha < 2\). Let \(\delta \in (0, 1)\) be such that \(\beta < \delta + 1 < \alpha\). According to Lemma 3.6, there exists a constant \(\tilde{C}_\lambda > 0\) such that for all \(g \in B_6(\mathbb{R}_+ \times \mathbb{R}^d)\),

\[
|\nabla R_{\lambda}g(t, x + z) - \nabla R_{\lambda}g(t, x)| \leq \tilde{C}_\lambda \|g\| |z|^\delta, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \; z \in \mathbb{R}^d,
\]

and \(\tilde{C}_\lambda\) goes to 0 as \(\lambda \uparrow \infty\).

Let \(g \in B_6(\mathbb{R}_+ \times \mathbb{R}^d)\). For all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) and \(z \in \mathbb{R}^d\), we have

\[
|R_{\lambda}g(t, x + z) - R_{\lambda}g(t, x) - z \cdot \nabla R_{\lambda}g(t, x)| \\
= \left| \int_0^1 \nabla R_{\lambda}g(t, x + rz) \cdot z \, dr - z \cdot \nabla R_{\lambda}g(t, x) \right| \\
= \left| \int_0^1 \left[ \nabla R_{\lambda}g(t, x + rz) - \nabla R_{\lambda}g(t, x) \right] \cdot z \, dr \right| \\
\leq |z| \int_0^1 |\nabla R_{\lambda}g(t, x + rz) - \nabla R_{\lambda}g(t, x)| \, dr \overset{(4.5)}{\leq} \tilde{C}_\lambda \|g\| |z|^\delta + 1. 
\]

So we obtain

\[
\int_{\mathbb{R}^d \setminus \{0\}} |R_{\lambda}g(t, x + z) - R_{\lambda}g(t, x) - 1_{\{|z| \leq 1\}} z \cdot \nabla R_{\lambda}g(t, x)| M(t, x, dz) \\
\leq \int_{\{0 < |z| \leq 1\}} |R_{\lambda}g(t, x + z) - R_{\lambda}g(t, x) - z \cdot \nabla R_{\lambda}g(t, x)| M(t, x, dz) \\
+ \int_{\{|z| > 1\}} |R_{\lambda}g(t, x + z) - R_{\lambda}g(t, x)| M(t, x, dz) \\
\overset{(4.6)}{\leq} \tilde{C}_\lambda \|g\| \int_{\{0 < |z| \leq 1\}} |z|^{\delta + 1} M(t, x, dz) + 2 \|R_{\lambda}g\| \int_{\{|z| > 1\}} 1 M(t, x, dz) \\
\leq (\tilde{C}_\lambda + 2\lambda^{-1}) \|g\| \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |z|^{\beta} M(t, x, dz).
\]

Hence \(\|K R_{\lambda}g\| \leq k_\lambda \|g\|\) for all \(B_6(\mathbb{R}_+ \times \mathbb{R}^d)\).
There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, we have $k_\lambda < 1/2$ and
\[
\left\| \sum_{i=0}^{\infty} R_\lambda(KR_\lambda)^i g \right\| \leq \sum_{i=0}^{\infty} \lambda^{-1}(k_\lambda)^i \| g \| \leq 2\lambda^{-1} \| g \|, \quad g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d). \tag{4.8}
\]

According to Corollary 4.2, for each $\lambda \geq \lambda_0$, we can define
\[
G_\lambda g := \sum_{i=0}^{\infty} R_\lambda(KR_\lambda)^i g, \quad g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d). \tag{4.9}
\]

Remark 4.3. By (4.3), (4.8) and (3.25), we see that if $\lambda \geq \lambda_0$ and $g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$, then the function $\mathbb{R}^d \ni x \mapsto G_\lambda g(t, x)$ is bounded continuous for each $t \geq 0$.

We have the following estimate of Krylov’s type.

Proposition 4.4. Let $T > 0$ and $g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)$ be such that $\text{supp}(g) \subset [0, T] \times \mathbb{R}^d$ and $g \in L^p([0, T]; L^p(\mathbb{R}^d))$ with $p, q > 0$ and $d/p + \alpha/q < \alpha - \beta$, where $\beta \in (0, \alpha)$ is the constant in Assumption 2.3. Then for each $\lambda \geq \lambda_0$, there exists a constant $l_\lambda > 0$, independent of $g$ and $T$, such that
\[
\| G_\lambda g \| \leq l_\lambda \| g \|_{L^p([0, T]; L^p(\mathbb{R}^d))}. \tag{4.10}
\]
Moreover, the constant $l_\lambda$ goes to 0 as $\lambda \to \infty$.

Proof. By (3.20) and the same proof of [4, Proposition 3.9 (i)], we can find a constant $c_\lambda > 0$, independent of $g$ and $T$, such that
\[
\| R_\lambda g \| \leq c_\lambda \| g \|_{L^p([0, T]; L^p(\mathbb{R}^d))}, \tag{4.11}
\]
where $c_\lambda$ goes to 0 as $\lambda \to \infty$.

For $0 < \alpha \leq 1$, by (4.4), (4.11) and Lemma 3.5 (ii), we have
\[
\int_{\mathbb{R}^d \setminus \{0\}} | R_\lambda g(t, x + z) - R_\lambda g(t, x) | M(t, x, dz) \\
\leq N_\lambda \| g \|_{L^p([0, T]; L^p(\mathbb{R}^d))} \int_{\{0 < |z| \leq 1\}} |z|^\beta M(t, x, dz) \\
+ 2c_\lambda \| g \|_{L^q([0, T]; L^p(\mathbb{R}^d))} \int_{\{|z| > 1\}} M(t, x, dz) \\
\leq (N_\lambda + 2c_\lambda) \| g \|_{L^p([0, T]; L^p(\mathbb{R}^d))} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |z|^\beta M(t, x, dz). \tag{4.12}
\]

For $1 < \alpha < 2$, similarly to (4.7), we obtain
\[
\int_{\mathbb{R}^d \setminus \{0\}} | R_\lambda g(t, x + z) - R_\lambda g(t, x) - 1_{\{|z| \leq 1\}} z \cdot \nabla R_\lambda g(t, x) | M(t, x, dz) \\
\leq (N_\lambda + 2c_\lambda) \| g \|_{L^p([0, T]; L^p(\mathbb{R}^d))} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |z|^\beta M(t, x, dz), \tag{4.13}
\]
where $\tilde{N}_\lambda > 0$ is the constant from Lemma 3.6 (ii). Summarizing (4.12) and (4.13), we obtain that for all $\alpha \in (0, 2)$,

$$\|KR_\lambda g\| \leq \tilde{c}_\alpha \|g\|_{L^q([0,T];L^p(\mathbb{R}^d))},$$

(4.14)

where

$$\tilde{c}_\lambda := \begin{cases} (N_\lambda + 2c_\alpha) \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |z|^p M(t, x, dz), & 0 < \alpha \leq 1, \\ (\tilde{N}_\lambda + 2c_\alpha) \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |z|^p M(t, x, dz), & 1 < \alpha < 2. \end{cases}$$

(4.15)

By (4.11), (4.14) and Lemma 4.1, we obtain that for all $\lambda \geq \lambda_0$,

$$\|R_\lambda(K R_\lambda)^i g\| \leq c_\lambda (k_\lambda)^{i-1} \|KR_\lambda g\| \leq c_\lambda (k_\lambda)^{i-1} \tilde{c}_\lambda \|g\|_{L^q([0,T];L^p(\mathbb{R}^d))},$$

which implies that for $\lambda \geq \lambda_0$,

$$\|G_\lambda g\| \leq \sum_{i=0}^\infty \|R_\lambda(K R_\lambda)^i g\| \leq c_\lambda \left( 1 + \sum_{i=1}^\infty \tilde{c}_\lambda (k_\lambda)^{i-1} \right) \|g\|_{L^q([0,T];L^p(\mathbb{R}^d))} \leq c_\lambda (1 + 2\tilde{c}_\lambda) \|g\|_{L^q([0,T];L^p(\mathbb{R}^d))}.$$ 

So (4.10) holds with

$$l_\lambda := c_\lambda (1 + 2\tilde{c}_\lambda) > 0.$$  

(4.16)

Since $c_\lambda, N_\lambda$ and $\tilde{N}_\lambda$ all converge to 0 as $\lambda \to \infty$, we see that $\lim_{\lambda \to \infty} l_\lambda = 0$. \( \square \)

5. Well-posedness of the martingale problem for $L_t$

In this section we prove our main result, namely, the martingale problem for $L_t$ is well-posed. In view of (4.9), the uniqueness problem can be solved by standard perturbation arguments. To obtain existence, we will first consider smooth approximations of $L_t$ and then construct a solution to the martingale problem for $L_t$ by weak convergence of probability measures.

Let $\phi \in C_0^\infty (\mathbb{R}^d)$ be such that $0 \leq \phi \leq 1$, $\int_{\mathbb{R}^d} \phi(x)dx = 1$ and $\phi(x) = 0$ for $|x| \geq 1$. Define $\phi_n(x) := n^d \phi(nx), x \in \mathbb{R}^d$. Given $n \in \mathbb{N}$, define $M_n(t, x, \cdot)$ as the kernel obtained by mollifying $M(t, x, \cdot)$ through $\phi_n$, that is,

$$M_n(t, x, B) := \int_{\mathbb{R}^d} M(t, x - z, B)\phi_n(z)dz, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

So $M_n(t, x, \cdot)$ is a kernel from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ and $M_n(t, x, \cdot)$ is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. By Fubini’s theorem, we have that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |y|^\beta M_n(t, x, dy) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |y|^\beta M(t, x - z, dy) \right) \phi_n(z)dz$$

$$\leq \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |y|^\beta M(t, x, dy) < \infty.$$ \(5.1\)

Define

$$K_{n,t}f(x) := \int_{\mathbb{R}^d \setminus \{0\}} [f(x + y) - f(x) - 1_{\alpha > 1}(|y| \leq 1)y \cdot \nabla f(x)]M_n(t, x, dy).$$
Lemma 5.1. Let $f \in C^3_b(\mathbb{R}^d)$ be arbitrary. Then for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we have

$$|K_{n,t}f(x) - K_t f * \phi_n(x)| \leq 4n^{-1} \|f\|_{C^3_b(\mathbb{R}^d)} \sup_{t,x} \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |h|^3) M(t, x, dh).$$

Proof. First note that

$$\int_{\mathbb{R}^d} |y| \phi_n(y) dy = \int_{\{|y| \leq 1/n\}} |y| \phi_n(y) dy \leq n^{-1}. \quad (5.2)$$

Let

$$\Delta_{n,t} f(x) := K_{n,t}f(x) - K_t f * \phi_n(x).$$

(i) For the case $0 < \alpha \leq 1$, we have

$$\Delta_{n,t} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} [f(x + h) - f(x)] M(t, x - y, dh) \phi_n(y) dy$$

$$- \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} [f(x - y + h) - f(x - y)] M(t, x - y, dh) \phi_n(y) dy. \quad (5.3)$$

Since

$$|f(x + h) - f(x - y + h) - f(x) + f(x - y)|$$

$$= \left| \int_0^1 [\nabla f(x + h - y + ry) - \nabla f(x - y + ry)] \cdot y dr \right|$$

$$\leq 2|y| (1 \wedge |h|) \|f\|_{C^2_b(\mathbb{R}^d)},$$

it follows from (5.3) that

$$|\Delta_{n,t} f(x)| \leq 2 \|f\|_{C^2_b(\mathbb{R}^d)} \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} (1 \wedge |h|)|y| M(t, x - y, dh) \phi_n(y) dy$$

$$\leq 2 \|f\|_{C^2_b(\mathbb{R}^d)} \sup_{t,x} \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |h|^3) M(t, x, dh) \int_{\mathbb{R}^d} |y| \phi_n(y) dy$$

$$\leq 2n^{-1} \|f\|_{C^2_b(\mathbb{R}^d)} \sup_{t,x} \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |h|^3) M(t, x, dh). \quad (5.2)$$

(ii) For $1 < \alpha < 2$, we have

$$\Delta_{n,t} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} [f(x + h) - f(x) - 1_{\{|h| \leq 1\}} h \cdot \nabla f(x)] M(t, x - y, dh) \phi_n(y) dy$$

$$- \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} [f(x - y + h) - f(x - y) - 1_{\{|h| \leq 1\}} h \cdot \nabla f(x - y)]$$

$$\times M(t, x - y, dh) \phi_n(y) dy.$$
Lemma 5.2. For each $L$, there is for each $(s, x)$ from $(1, \infty)$, we have

$$\|\Delta_{n,t}f(x)\| \leq 4 \|f\|_{C^3_b(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 \wedge |h|^2)|y|M(t, x - y, dh)\phi_n(y)dy$$

$$\leq 4 \|f\|_{C^3_b(\mathbb{R}^d)} \sup_{t, x} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 \wedge |h|^2)M(t, x, dh)\int_{\mathbb{R}^d} |y|\phi_n(y)dy$$

$$\leq 4n^{-1} \|f\|_{C^3_b(\mathbb{R}^d)} \sup_{t, x} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 \wedge |h|^2)M(t, x, dh).$$

Proof. To prove the solvability of the martingale problem for $L_{n,t} = L + K_{n,t}$ starting from $(s, x)$.

Set $c_\delta(t, x) = 1_{\alpha > 1} \int_{|y| \leq 1} yM_{n,t}^\delta(t, x, dy)$. Since

$$c_\delta(t, x) = 1_{\alpha > 1} \int_{|y| \leq 1} y\varphi_\delta(y)M_n(t, x, dy)$$

$$= 1_{\alpha > 1} \int_{\mathbb{R}^d} \left( \int_{|y| \leq 1} y\varphi_\delta(y)M(t, x, dy) \right) \phi_n(z)dz$$

$$= 1_{\alpha > 1} \int_{\mathbb{R}^d} \left( \int_{|y| \leq 1} y\varphi_\delta(y)M(t, x, dy) \right) \phi_n(x - z)dz,$$

we see that $|\nabla_x c_\delta(t, x)|$ is bounded on $\mathbb{R}_+ \times \mathbb{R}^d$. Hence $c_\delta(t, x)$ is globally Lipschitz continuous in $x$. Define the differential operator $A_\delta^f$ by

$$A_\delta^f(x) := \sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(x) + b \cdot \nabla f(x) - c_\delta(t, x) \cdot \nabla f(x).$$

By the Lipschitz continuity (in the space variable $x$) of the coefficients of $A_\delta^f$, there is for each $(s, x)$ a unique solution $Q_{n,t}^{s,x}$ to the martingale problem for $A_\delta^f$ starting from $(s, x)$, see, e.g., [13, Theorem 5.1.1 and Corollary 5.1.3]. By [13, Theorem
5.1.4], the mapping \((s, x) \mapsto Q^x_s(E)\) is measurable for all \(E \in \mathcal{D}\). Note that
\[ A^\delta f(x) + \int_{\mathbb{R}^d \setminus \{0\}} [f(x + y) - f(x)] M^\delta_n(t, x, dy) = Lf(x) + K^\delta_n(t, x, dy), \]
where
\[ K^\delta_{n,t}(x) := \int_{\mathbb{R}^d \setminus \{0\}} [f(x + y) - f(x) - 1_{\alpha > 1} \cdot \nabla f(x)] M^\delta_n(t, x, dy). \]
It follows from [12, Theorem (2.1)] that the martingale problem for \(L + K^\delta_{n,t}\) is solvable. For \(f \in C^\infty_0(\mathbb{R}^d)\), we have
\[
|K^\delta_{n,t}f(x) - K_{n,t}f(x)| \leq \int_{\{|y| \leq \delta\}} |f(x + y) - f(x) - 1_{\alpha > 1} \cdot \nabla f(x)| M_n(t, x, dy)
\]
\[
\leq \|f\|_{C^2_b(\mathbb{R}^d)} \int_{\{|y| \leq \delta\}} (1_{\alpha \leq 1} \cdot |y| + 1_{\alpha > 1} \cdot |y|^2) M_n(t, x, dy)
\]
\[
\leq \|f\|_{C^2_b(\mathbb{R}^d)} \int_{\{|y| \leq \delta\}} |y|^\alpha M_n(t, x, dy)
\]
\[
\leq \delta^{\alpha - \beta} \|f\|_{C^2_b(\mathbb{R}^d)} \sup_{t \geq 0, x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} 1 \land |y|^\beta M(t, x, dy), \tag{5.1}
\]
which implies that \(K^\delta_{n,t} f \to K_{n,t} f\) uniformly as \(\delta \to 0\). The rest of the proof goes in the same way as in [12, Theorem (2.2)]. We omit the details. \(\square\)

Recall that \(\lambda_0 > 0\) is the constant given in Corollary 4.2.

**Lemma 5.3.** Let \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) and \(P_n^{s,x}\) be a solution to the martingale problem for \(L_{n,t} = L + K_{n,t}\) starting from \((s, x)\). Then for any \(\lambda \geq \lambda_0\) and \(g \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\), we have
\[
\mathbb{E}_n^{s,x}[\int_s^\infty e^{-\lambda(t-s)}g(t, X_t)dt] = \sum_{k=0}^{\infty} R_\lambda(K_n R_\lambda)^k g(s, x), \tag{5.4}
\]
where \(\mathbb{E}_n^{s,x}[\cdot]\) denotes the expectation with respect to the measure \(P_n^{s,x}\) and \(K_n R_\lambda\) is defined by
\[
K_n R_\lambda g(t, x) := \int_{\mathbb{R}^d \setminus \{0\}} [R_\lambda g(t, x + y) - R_\lambda g(t, x)
\]
\[
- 1_{\alpha > 1} \cdot \nabla R_\lambda g(t, x)] M_n(t, x, dy), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{5.5}
\]

**Proof.** For \(\lambda > 0\) and \(f \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)\), define
\[
V_\lambda^n f := \mathbb{E}_n^{s,x} \left[ \int_s^\infty e^{-\lambda(t-s)}f(t, X_t)dt \right].
\]
For \(f \in C^{1,2}_b(\mathbb{R}_+ \times \mathbb{R}^d)\), we know that
\[
f(t, X_t) - f(s, X_s) = \text{“Martingale”} + \int_s^t \left( \frac{\partial f}{\partial u} + L_{n,u} f \right)(u, X_u)du.
\]
Taking expectations of both sides of the above equality gives
\[
E_n^s,x [f(t, X_t)] - f(s, x) = E_n^s,x \left[ \int_s^t (\partial f/\partial u + \mathcal{L}_{n,u} f)(u, X_u) \, du \right].
\] (5.6)

Multiplying both sides of (5.6) by \(e^{-\lambda(t-s)}\), integrating with respect to \(t\) from 0 to \(\infty\) and then applying Fubini’s theorem, we get
\[
E_n^s,x \left[ \int_s^\infty e^{-\lambda(t-s)} f(t, X_t) \, dt \right]
= \frac{1}{\lambda} f(s, x) + E_n^s,x \left[ \int_s^\infty e^{-\lambda(t-s)} \left( \frac{\partial f}{\partial u} + \mathcal{L}_{n,u} f \right)(u, X_u) \, du \right]
- \frac{1}{\lambda} f(s, x) - \frac{1}{\lambda} E_n^s,x \left[ \int_s^\infty e^{-\lambda(t-s)} \left( \frac{\partial f}{\partial u} + \mathcal{L}_{n,u} f \right)(u, X_u) \, du \right].
\] (5.7)

Therefore, for \(f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)\),
\[
\lambda V_n^\lambda f = f(s, x) + V_n^\lambda \left( \frac{\partial f}{\partial t} + \mathcal{L}_{n,t} f \right).
\] (5.8)

If \(g \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)\), then \(f := R_\lambda g \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)\) and
\[
\lambda f(t, y) - Lf(t, y) - \frac{\partial}{\partial t} f(t, y) = g(t, y), \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}^d,
\] (5.9)
see, e.g., the proof of [4, Proposition 3.8]. Substituting this \(f\) in (5.8), we obtain
\[
V_n^\lambda g = R_\lambda g(s, x) + V_n^\lambda (K_n R_\lambda g)
\]
for \(g \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)\). If \(g \in C_0(\mathbb{R}_+ \times \mathbb{R}^d)\), namely, \(g\) is continuous with compact support, then there exist \(g_k \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)\) such that \(g_k \to g\) boundedly and uniformly as \(k \to \infty\). It follows from (4.3) that \(K_n R_\lambda g_k \to K R_\lambda g\) boundedly and pointwise as \(k \to \infty\). By the dominated convergence theorem, we have
\[
V_n^\lambda g = \lim_{k \to \infty} V_n^\lambda g_k = \lim_{k \to \infty} \left\{ R_\lambda g_k(s, x) + V_n^\lambda (K_n R_\lambda g_k) \right\}
= R_\lambda g(s, x) + V_n^\lambda (K_n R_\lambda g), \quad g \in C_0(\mathbb{R}_+ \times \mathbb{R}^d).
\] (5.10)

For \(g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)\), we thus have
\[
V_n^\lambda g = R_\lambda g(s, x) + V_n^\lambda (K_n R_\lambda g)
\]
(5.10)
\[
= R_\lambda g(s, x) + R_\lambda K_n R_\lambda g(s, x) + V_n^\lambda (K_n R_\lambda)^2 g
= \cdots = \sum_{k=0}^i R_\lambda (K_n R_\lambda)^k g(s, x) + V_n^\lambda (K_n R_\lambda)^{i+1} g.
\] (5.11)

Let \(k_\lambda > 0\) be as in (4.2). By (5.1) and Proposition 4.1, we have
\[
\|K_n R_\lambda g\| \leq k_\lambda \|g\|, \quad \forall n \in \mathbb{N}, \quad g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d).
\]
According to Corollary 4.2, we have \(k_\lambda < 1/2\) for \(\lambda \geq \lambda_0\). Therefore, for all \(i, n \in \mathbb{N}, \lambda \geq \lambda_0\) and \(g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)\),
\[
|V_n^\lambda (K_n R_\lambda)^i g| \leq \lambda^{-i} (k_\lambda)^i \|g\| \leq \lambda^{-1} 2^{-i} \|g\|
\]
and

\[ \|R_\lambda(K_n R_\lambda)^k g\| \leq \lambda^{-1} (k\lambda)^i \|g\| \leq \lambda^{-1} 2^{-i}\|g\|. \]

Letting \( i \to \infty \) in (5.11) gives (5.4). This completes the proof. \( \square \)

**Remark 5.4.** In view of (5.1), we can repeat the proof of Proposition 4.4 to obtain that for each \( \lambda \geq \lambda_0 \),

\[ \left\| \sum_{k=0}^{\infty} R_\lambda(K_n R_\lambda)^k g \right\| \leq l_\lambda \|g\|_{L^q([0,T];L^p(\mathbb{R}^d))}, \tag{5.12} \]

where \( d/p + \alpha/q < \alpha - \beta \) and \( g \in B_b([0, \infty) \times \mathbb{R}^d) \cap L^q([0,T];L^p(\mathbb{R}^d)) \) is an arbitrary function satisfying \( \text{supp}(g) \subset [0,T] \times \mathbb{R}^d \). Indeed, by (4.15) and (4.16), the constant \( l_\lambda > 0 \) here can be chosen to be the same as in (4.10). In particular, \( l_\lambda \) in (5.12) is independent of \( n \in \mathbb{N} \).

**Corollary 5.5.** Let \( P_n^{s,x} \) be as in Lemma 5.3. Let \( p > (d + \alpha)/\alpha - \beta \). For each \( T > s \), there exists a constant \( C_T > 0 \), which is independent of \( n \), such that

\[ E_n^{s,x} \int_s^T |f(t, X_t)|dt \leq C_T \|f\|_{L^p([0,T] \times \mathbb{R}^d)}, \quad \forall f \in L^p([0,T] \times \mathbb{R}^d). \]

**Proof.** Let \( f \in B_b([0,T] \times \mathbb{R}^d) \cap L^p([0,T] \times \mathbb{R}^d) \). Applying (5.12) with \( p = q > (d + \alpha)/\alpha - \beta \), we get

\[
\begin{align*}
E_n^{s,x} \int_s^T |f(t, X_t)|dt &\leq e^{\lambda_0(T-s)} E_n^{s,x} \left[ \int_s^\infty e^{-\lambda_0(t-s)} 1_{[0,T]}(t)|f(t, X_t)|dt \right] \\
&\leq e^{\lambda_0(T-s)} \sum_{k=0}^{\infty} \lambda_0 k (1_{[0,T]}(t)|f|)(s, x) \\
&\leq l_\lambda e^{\lambda_0(T-s)} \|f\|_{L^p([0,T] \times \mathbb{R}^d)}.
\end{align*}
\]

For a general \( f \in L^p([0,T] \times \mathbb{R}^d) \), the assertion follows from the monotone convergence theorem. \( \square \)

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** "Existence": Let \( (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \) be fixed. It follows from Lemma 5.2 that there exists a solution \( P_n^{s,x} \) to the martingale problem for \( \mathcal{L}_{n,t} = L + K_{n,t} \) starting from \( (s, x) \).

By (5.1) and [12, Theorem (A.1)], the family \( \{P_n^{s,x}, n \in \mathbb{N}\} \) is tight. Let \( P_n^{s,x} \) be a limit point of \( \{P_n^{s,x}, n \in \mathbb{N}\} \). Then there exists a subsequence of \( \{P_n^{s,x}\}_{n \in \mathbb{N}} \) which converges weekly to \( P^{s,x} \). For simplicity, we denote this subsequence still by \( \{P_n^{s,x}\}_{n \in \mathbb{N}} \).

We next show that \( P^{s,x} \) is a solution to the martingale problem for \( \mathcal{L}_t \) starting from \( (s, x) \). Let \( f \in C_0^\infty(\mathbb{R}^d) \) be arbitrary. By [12, Theorem (1.1)], it suffices to show that

\[ f(X_t) - \int_s^t \mathcal{L}_u f(X_u)du \]
is a $P^{s,x}$-martingale after time $s$. Suppose $s \leq t_1 \leq t_2$, $0 \leq r_1 \leq \cdots \leq r_l \leq t_1$ and $g_1, \cdots, g_l \in C_0(\mathbb{R}^d)$, where $l \in \mathbb{N}$. Set $Y = \prod_{j=1}^l g_j(X_{r_j})$. It reduces to show that

$$E^{s,x}\left[ Y(f(X_{t_2}) - f(X_{t_1}) - \int_{t_1}^{t_2} L_u f(X_u) du) \right] = 0. \tag{5.13}$$

We will complete the proof of (5.13) in four steps. Firstly, note that by [3, Chap. 3, Lemma 7.7], there exists a countable set $I \subset \mathbb{R}_+$ such that

$$P^{s,x}(X_{t_1} = X_t) = 1, \quad \forall t \in \mathbb{R}_+ \setminus I. \tag{5.14}$$

Since $\mathbb{R}_+ \setminus I$ is dense in $\mathbb{R}_+$ and $t \mapsto X_t(\omega)$, $\omega \in D$, is right-continuous, it is enough to show (5.13) by additionally assuming that

$$r_1, \cdots, r_l, t_1, t_2 \in \mathbb{R}_+ \setminus I. \tag{5.15}$$

So from now on, we assume that (5.15) is true.

"Step 1": We establish an estimate of Krylov’s type for $P^{s,x}$. Let $p > (d + \alpha)/\alpha - \beta$. By Corollary 5.5, for each $T > s$, there exists a constant $C_T > 0$ such that

$$\sup_{n \in \mathbb{N}} E^{n,x}_s \left[ \int_s^T |f(t, X_t)| dt \right] \leq C_T \|f\|_{L^p([0,T] \times \mathbb{R}^d)}, \quad \forall f \in L^p([0,T] \times \mathbb{R}^d). \tag{5.16}$$

It follows that for each $T > s$,

$$E^{s,x}_s \left[ \int_s^T |f(t, X_t)| dt \right] \leq C_T \|f\|_{L^p([0,T] \times \mathbb{R}^d)}, \quad \forall f \in L^p([0,T] \times \mathbb{R}^d). \tag{5.17}$$

Indeed, if $f \in C_0([0,T] \times \mathbb{R}^d)$, namely, $f$ is continuous on $[0,T] \times \mathbb{R}^d$ with compact support, then (5.17) follows from (5.16) and the weak convergence of $P^{n,x}$ to $P^{s,x}$. By a standard monotone class argument, we obtain (5.17) for all $f \in L^p([0,T] \times \mathbb{R}^d)$.

"Step 2": We show that

$$\lim_{n \to \infty} E^{n,x}_s \left[ Y(f(X_{t_2}) - f(X_{t_1}) - \int_{t_1}^{t_2} L_u f(X_u) du) \right] = E^{s,x}_s \left[ Y(f(X_{t_2}) - f(X_{t_1}) - \int_{t_1}^{t_2} L_u f(X_u) du) \right]. \tag{5.18}$$

By Skorokhod’s representation theorem, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ and random elements $\xi, \xi_1, \cdots, \xi_n, \cdots : \Omega \rightarrow D$ such that $P^{n,x}_s = \mathbb{Q} \circ \xi_n^{-1}$, $P^{s,x}_s = \mathbb{Q} \circ \xi^{-1}$ and $d(\xi_n, \xi) \rightarrow 0$ $\mathbb{Q}$-a.s. (n), where $d$ is the Skorokhod metric on $D$. It follows from (5.14) and [3, Chap. 3, Prop. 5.2] that

$$\lim_{n \to \infty} X_t(\xi_n) = X_t(\xi) \quad \mathbb{Q}\text{-a.s.}, \quad \forall t \in \mathbb{R}_+ \setminus I. \tag{5.19}$$

Let $E[\cdot]$ denote the expectation with respect to the measure $\mathbb{Q}$. By (5.15) and the dominated convergence theorem, we have

$$\lim_{n \to \infty} E \left[ Y(\xi_n) \left\{ f(X_{t_2}(\xi_n)) - f(X_{t_1}(\xi_n)) - \int_{t_1}^{t_2} L f(X_u(\xi_n)) du \right\} \right] \stackrel{(5.19)}{=} E \left[ Y(\xi) \left\{ f(X_{t_2}(\xi)) - f(X_{t_1}(\xi)) - \int_{t_1}^{t_2} L f(X_u(\xi)) du \right\} \right].$$
Note that $\chi$ implies (5.18).

"Step 3": We show that

$$\lim_{n \to \infty} E_{i,n}^{s,x} \left[ Y \int_{t_1}^{t_2} K_{n,u} f(X_u) du \right] = E_{i,0}^{s,x} \left[ Y \int_{t_1}^{t_2} K_u f(X_u) du \right]. \quad (5.20)$$

Note that $Y$ is bounded. Let $C_Y := \sup_{\omega \in D} |Y(\omega)| < \infty$. For $r > 0$ let $\chi_r$ be a continuous non-negative function on $\mathbb{R}^d$ with $\chi_r(x) = 1$ for $|x| \leq r$, $\chi_r(x) = 0$ for $|x| > r + 1$ and $0 \leq \chi_r(x) \leq 1$ for $r < |x| \leq r + 1$; moreover, we can choose $\chi_r$ such that $\chi_r$ is monotone in $r$, namely, $\chi_{r_1} \leq \chi_{r_2}$ if $r_1 \leq r_2$. Note that $|K_{n,u} f|$ and $|K_u f|$ are both bounded, say, by a positive constant $C_K$. For $i \in \mathbb{N}$, we have

$$\left| E_{i,n}^{s,x} \left[ Y \int_{t_1}^{t_2} K_{n,u} f(X_u) du \right] - E_{i,0}^{s,x} \left[ Y \int_{t_1}^{t_2} K_u f(X_u) du \right] \right| \leq C_Y E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} |K_{n,u} f - K_u f| (X_u) du \right]$$

$$\leq C_Y E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} (|K_{n,u} f - K_u f* \phi_n| + |K_u f* \phi_n - K_u f|) (X_u) du \right]$$

$$\leq C_Y E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} |K_{n,u} f - K_u f* \phi_n| (X_u) du \right] + 2C_Y C_K E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} (1 - \chi_r) (X_u) du \right]$$

$$+ C_Y E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} \chi_r(X_u) |K_u f* \phi_n - K_u f| (X_u) du \right] \quad (5.16)$$

$$\leq t_2 C_Y \|K_{n,u} f - K_u f* \phi_n\| + 2C_Y C_K \sup_{i \in \mathbb{N}} E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} (1 - \chi_r) (X_u) du \right]$$

$$+ C_Y t_2 \chi_r (K_u f* \phi_n - K_u f) \|_{L^p([0,t_2] \times \mathbb{R}^d)} \leq J_1 + J_2 + J_3.$$

For any given $\epsilon_1 > 0$, by dominated convergence theorem, we can find sufficiently large $r_0 > 0$ such that

$$E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} (1 - \chi_{r_0}) (X_u) du \right] < \epsilon_1. \quad (5.21)$$

By the weak convergence of $P_{i,n}^{s,x}$ to $P_{i,0}^{s,x}$, we have

$$\lim_{i \to \infty} E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} (1 - \chi_{r_0}) (X_u) du \right] = E_{i,0}^{s,x} \left[ \int_{t_1}^{t_2} (1 - \chi_{r_0}) (X_u) du \right].$$

So there exists $i_0 \in \mathbb{N}$ such that

$$\sup_{i > i_0} E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} (1 - \chi_{r_0}) (X_u) du \right] \leq 2\epsilon_1. \quad (5.22)$$

Similarly to (5.21), for $i = 1, 2, \cdots, i_0$, we can find $r_1 > r_0$ such that

$$\sup_{1 \leq i \leq i_0} E_{i,n}^{s,x} \left[ \int_{t_1}^{t_2} (1 - \chi_{r_1}) (X_u) du \right] \leq \epsilon_1. \quad (5.23)$$
Combining (5.22) and (5.23) and noting that $\chi_r$ is non-decreasing in $r$, we get
\[
\sup_{i \in \mathbb{N}} \mathbb{E}^{s,x}_{i} \left[ \int_{t_1}^{t_2} (1 - \chi_r)(X_u)du \right] < 3\epsilon_1, \quad r \geq r_1.
\]
Hence we have shown that $\lim_{r \to \infty} J_2 = 0$. By Lemma 5.1, we have $J_1 \to 0$ as $n \to \infty$. It is also easy to see that $J_3 \to 0$ as $n \to \infty$. With a simple \("\epsilon - \delta\"\) argument, we obtain
\[
\lim_{n \to \infty} \left| \mathbb{E}^{s,x}_{i} \left[ Y \int_{t_1}^{t_2} K_{n,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{i} \left[ Y \int_{t_1}^{t_2} K_u f(X_u)du \right] \right| = 0, \quad (5.24)
\]
and the convergence is uniform with respect to $i \in \mathbb{N}$.

Similarly to (5.24), we have
\[
\lim_{n \to \infty} \left| \mathbb{E}^{s,x}_{i} \left[ Y \int_{t_1}^{t_2} K_{n,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{i} \left[ Y \int_{t_1}^{t_2} K_u f(X_u)du \right] \right| = 0. \quad (5.25)
\]
By (5.24) and (5.25), for any given $\epsilon > 0$, we can find $n_1 \in \mathbb{N}$, which is independent of $i$, such that for all $n, m \geq n_1$ and $i \in \mathbb{N},$
\[
\left| \mathbb{E}^{s,x}_{i} \left[ Y \int_{t_1}^{t_2} K_{n,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{i} \left[ Y \int_{t_1}^{t_2} K_u f(X_u)du \right] \right| < \epsilon
\]
and
\[
\left| \mathbb{E}^{s,x}_{i} \left[ Y \int_{t_1}^{t_2} K_{n,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{i} \left[ Y \int_{t_1}^{t_2} K_{m,u} f(X_u)du \right] \right| < \epsilon.
\]
Similarly to (5.18), there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$,
\[
\left| \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n_1,u} f(X_u)du \right] \right| < \epsilon.
\]
If $n \geq \sup\{n_1, n_2\}$, then
\[
\left| \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n_1,u} f(X_u)du \right] \right| \leq \left| \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n_1,u} f(X_u)du \right] \right| \]
\[
+ \left| \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n_1,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n_1,u} f(X_u)du \right] \right| \]
\[
+ \left| \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{n_1,u}f(X_u)du \right] - \mathbb{E}^{s,x}_{n} \left[ Y \int_{t_1}^{t_2} K_{u} f(X_u)du \right] \right| \leq 3\epsilon.
\]
So (5.20) is true.

“Step 4”: We finally prove (5.13) under the condition (5.15). Since $\mathbb{P}^{s,x}_{n}$ solves the martingale problem for $\mathcal{L}_{n,t}$, it follows that
\[
\mathbb{E}^{s,x}_{n} \left[ Y \left( f(X_{t_2}) - f(X_{t_1}) - \int_{t_1}^{t_2} \mathcal{L}_u f(X_u)du \right) \right] = 0. \quad (5.26)
\]
So (5.13) follows from (5.26), (5.18) and (5.20).

This completes the proof of existence.
“Uniqueness”: Let \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) be arbitrary and \(\hat{P}^{s, x}\) be a solution to the martingale problem for \(\mathcal{L}_t\) starting from \((s, x)\). For each \(f \in C_{b}^{1, 2}(\mathbb{R}_+ \times \mathbb{R}^d)\),

\[
f(t, X_t) - f(s, X_s) - \int_s^t \left( \frac{\partial f}{\partial u} + \mathcal{L}_u f \right) (u, X_u) du
\]

is an \(\mathcal{F}_t\)-martingale after \(s\) with respect to the measure \(\hat{P}^{s, x}\). For any \(s \leq t_1 < t\), \(C \in \mathcal{F}_{t_1}\), we thus have

\[
\hat{E}^{s, x}[1_C f(t, X_t)] = \hat{E}^{s, x}[1_C f(t_1, X_{t_1})] + \hat{E}^{s, x}\left[1_C \int_{t_1}^t \left( \frac{\partial f}{\partial u} + \mathcal{L}_u f \right) (u, X_u) du \right].
\]

Similarly to (5.7), by multiplying both sides of (5.27) by \(\exp(-\lambda(t-t_1))\) and then integrating with respect to \(t\) from \(t_1\) to \(\infty\), we get

\[
\hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(t-t_1)} f(t, X_t) dt \right]
= \lambda^{-1} \hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(u-t_1)} f(u, X_u) du \right] + \hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(u-t_1)} \left( \frac{\partial f}{\partial u} + \mathcal{L}_u f \right) (u, X_u) du \right].
\]

Therefore,

\[
\hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(t-t_1)} f(t, X_t) dt | \mathcal{F}_{t_1} \right]
= \lambda^{-1} f(t_1, X_{t_1}) + \hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(t-t_1)} \left( \frac{\partial f}{\partial t} + \mathcal{L}_t f \right) (t, X_t) dt | \mathcal{F}_{t_1} \right].
\]

If \(g \in C_{b}^{1, 2}(\mathbb{R}_+ \times \mathbb{R}^d)\), then \(f := R_\lambda g \in C_{b}^{1, 2}(\mathbb{R}_+ \times \mathbb{R}^d)\) and (5.9) holds. Substituting this \(f\) in (5.28), we obtain

\[
\hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(t-t_1)} g(t, X_t) dt | \mathcal{F}_{t_1} \right]
= R_\lambda g(t_1, X_{t_1}) + \hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(t-t_1)} K R_\lambda g(t, X_t) dt | \mathcal{F}_{t_1} \right].
\]

With a similar argument as in the proof of (5.10), we see that (5.29) is true for all \(g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)\).

If \(g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)\), then \(K R_\lambda g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)\). By (5.29) and a simple iteration, we obtain for each \(k \in \mathbb{N}\),

\[
\hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(t-t_1)} g(t, X_t) dt | \mathcal{F}_{t_1} \right]
= \sum_{i=0}^k R_\lambda(K R_\lambda)^i g(t_1, X_{t_1}) + \hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(t-t_1)} (K R_\lambda)^{k+1} g(t, X_t) dt | \mathcal{F}_{t_1} \right].
\]

By Proposition 4.1 and Corollary 4.2, we see that

\[
\hat{E}^{s, x}\left[\int_{t_1}^\infty e^{-\lambda(t-t_1)} g(t, X_t) dt | \mathcal{F}_{t_1} \right] = \sum_{i=0}^\infty R_\lambda(K R_\lambda)^i g(t_1, X_{t_1}) = G_\lambda g(t_1, X_{t_1})
\]

for all \(\lambda \geq \lambda_0\) and \(g \in B_b(\mathbb{R}_+ \times \mathbb{R}^d)\).
Note that the choice of $t_1 \in [s, \infty)$ in (5.30) is arbitrary. It follows from (5.30), Remark 4.3 and [6, Lemma 3.1] that there exists at most one solution to the martingale problem for $\mathcal{L}_t$ starting from $(s, x)$. □

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