Extensions, deformation and categorification of AssDer pairs

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Abstract

In this paper, we consider associative algebras equipped with derivations. Such a pair of an associative algebra with a derivation is called an AssDer pair. Using the Hochschild cohomology for associative algebras, we define cohomology for an AssDer pair with coefficients in a representation. We study central extensions and abelian extensions of AssDer pairs. Moreover, we consider extensions of a pair of derivations in central extensions of associative algebras. Next, we study formal one-parameter deformations of AssDer pair by deforming both the associative product and the derivation. They are governed by the cohomology of the AssDer pair with representation in itself. In the next part, we study 2-term $A_{\infty}$-algebras with homotopy derivations considered by Loday and Doubek-Lada. Finally, we introduce 2-derivations on associative 2-algebras and show that the category of associative 2-algebras with 2-derivations are equivalent to the category of 2-term $A_{\infty}$-algebras with homotopy derivations.

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1. Introduction

Algebraic structures, such as Lie algebras, associative algebras are important in various area of mathematics and physics. Algebras are also useful via their derivations. One can construct a homotopy Lie algebra out of a graded Lie algebra with a special derivation [28]. In [7] the authors use noncommuting derivations in an associative algebra to construct deformation formulas. Derivations are useful in control theory and gauge theories in quantum field theory [1,2]. Algebras with derivations are also studied from operadic point of view [11,20]. Recently, the authors in [27] considered Lie algebras equipped with derivations (also called LieDer pairs). More precisely, they study central extensions and deformations of LieDer pair from the cohomological point of view.

In this paper, we consider a pair of an associative algebra $A$ together with a distinguished derivation $\phi_A$. Such a pair $(A, \phi_A)$ of an associative algebra with a derivation is called an AssDer pair. If the algebra $A$ is polynomial algebra in $n$ variables, some special type of derivations (locally nilpotent, locally finite etc.) are studied extensively in the literature. For instance, to consider polynomial flows and also in connection to fourteenth problem of Hilbert as well (see in [24] and references). It is also known as a differential algebra in the literature and is itself a significant object of study in areas such as differential Galois theory [22]. Recently, the authors in [6] defined Hochschild homology for AssDer pairs motivated from noncommutative geometry.

Here we construct a cohomology for an AssDer pair, study their central and abelian extensions, and formal one-parameter deformations. Its Lie algebraic counterpart follows the results of [27]. We relate the cohomology of an AssDer pair with the cohomology of the corresponding commutator.
LieDer pair. This cohomology together with the homology defined in [6] might be starting point to study cyclic theory for AssDer pairs. Additionally, we define and study categorified derivation on a categorified associative algebra and relate them with homotopy derivations of 2-term $A_{\infty}$-algebras.

In section 2, we study representations and cohomology of AssDer pairs. Let $(A, \phi_A)$ be an AssDer pair. A representation of it consists of an $A$-bimodule $M$ together with a linear map $\phi_M$ which is compatible with the left and right actions of $A$ on $M$. It turns out that any AssDer pair is a representation of itself. Given a representation $(M, \phi_M)$, the pair $(M^*, -\phi_M^*)$ is also a representation, where $M^*$ is equipped with the $A$-bimodule structure dual to $M$ (cf. Proposition 2). Given a representation of an AssDer pair, one can construct a semi-direct product AssDer pair (cf. Proposition 3). Next, we study cohomology of an AssDer pair with coefficients in a representation. This cohomology is follow up by the Hochschild cohomology of the associative structure modified by the fixed derivation. Like Hochschild cohomology, we show that the cohomology of an AssDer pair with coefficients in itself carries a degree $-1$ graded Lie bracket (cf. Proposition 5).

It is known that the commutator bracket of an associative algebra gives rise to a Lie algebra structure. It turns out to be a functor $(\_)_c : \text{AssDer} \to \text{LieDer}$ from the category of AssDer pairs to the category of LieDer pairs. We construct a functor $U : \text{LieDer} \to \text{AssDer}$ using the universal enveloping algebra of a Lie algebra. This functor is left adjoint to $(\_)_c$ (cf. Proposition 6). We also show that there is a morphism from the cohomology of AssDer pair to the cohomology of the corresponding commutator LieDer pair (cf. Proposition 9).

In Sections 3 and 4 we study various questions about extensions of AssDer pairs. First, we study extensions of an AssDer pair by a trivial AssDer pair, called central extensions. We show that isomorphism classes of central extensions are classified by the second cohomology of the AssDer pair with coefficients in the trivial representation (cf. Theorem 1). Next, we study extensions of a pair of derivations in a central extension of associative algebras. Given a central extension of associative algebras $0 \to M \xrightarrow{j} \hat{A} \xrightarrow{p} A \to 0$ and a pair of derivations $(\phi_A, \phi_M) \in \text{Der}(A) \times \text{Der}(M)$, we associate a second cohomology class in the Hochschild cohomology of $A$ with trivial representation $M$ (cf. Proposition 11), called the obstruction class. When this cohomology class is null, the pair of derivations $(\phi_A, \phi_M)$ is extensible to a derivation $\hat{\phi}_A \in \text{Der}(\hat{A})$ which makes the above sequence into an exact sequence of AssDer pairs (cf. Theorem 2). Finally, we consider abelian extensions of an AssDer pair by an arbitrary representation and show that isomorphism classes of such extensions are classified by the second cohomology group of the AssDer pair (cf. Theorem 4).

In Section 5, we study formal one-parameter deformation of AssDer pair following the classical approach of Gerstenhaber for associative algebras [15] and Nijenhuis-Richardson for Lie algebras [23]. For this, we deform both the associative product as well as the given derivation. We remark that simultaneous deformation theory and the governing structure is studied in [13]. The results in this section are analogous to classical cases. The vanishing of the second cohomology of the AssDer pair with coefficients in itself implies that the structure is rigid (cf. Theorem 7, Remark 5). Given a finite order deformation of an AssDer pair, we associate a third cohomology class, called the obstruction class of the deformation (cf. Proposition 13). If the class is trivial, then the deformations extend to deformation of next order (cf. Theorem 8).

Homotopy associative algebras ($A_{\infty}$-algebras) were introduced by Stasheff to recognize loop spaces [26]. In Section 6, we consider homotopy derivations on $A_{\infty}$-algebras whose underlying graded vector space is concentrated in degrees 0 and 1 [11,20]. We denote the category of 2-term $A_{\infty}$-algebras with homotopy derivations by $2A_{\text{ssDer}}$. Homotopy derivations on skeletal $A_{\infty}$-algebras are characterized by third cocycles of AssDer pairs (cf. Proposition 14) and strict homotopy derivations on strict $A_{\infty}$-algebras are characterized by crossed modules of AssDer pairs (cf. Proposition 15).

In [4] Baez and Crans introduced Lie 2-algebras as a categorification of Lie algebras. They also showed that the category of 2-term $L_{\infty}$-algebras and the category of Lie 2-algebras are equivalent.
This result has been extended to various other algebraic structures, including groups, Leibniz algebras and twisted associative algebras [3,9,25]. In section 7, we introduce a categorification of AssDer pair. More precisely, we study AssDer pair on 2-vector space. We call such an object an AssDer 2-pair. The category of AssDer 2-pairs and morphisms between them is denoted by \( \text{AssDer}_2 \). Finally, we show that the category \( 2\text{AssDer}_\infty \) and \( \text{AssDer}_2 \) are equivalent (cf. Theorem 10).

In the whole paper we assume that \( K \) is a fixed field, all the vector spaces are over the field \( K \) and maps are \( K \)-linear maps unless otherwise stated.

2. AssDer pairs

In this section, we introduce representations and cohomology of AssDer pairs. We also relate this cohomology with the cohomology of LieDer pairs as introduced in [27].

Let \( A \) be an associative algebra. A bimodule over it consists of a vector space \( M \) together with two linear maps \( l : A \otimes M \to M \) and \( r : M \otimes A \to M \) satisfying

\[
l(ab, m) = l(a, l(b, m)), \quad r(l(a, m), b) = l(a, r(m, b)) \quad \text{and} \quad r(r(m, a), b) = r(m, ab),
\]

for all \( a, b \in A \) and \( m \in M \). It follows that \( A \) is a bimodule over the associative algebra \( A \) itself with the left and right actions given by the algebra multiplication. This is called the adjoint bimodule structure.

A derivation on \( A \) with values in the bimodule \( M \) is given by a linear map \( \phi : A \to M \) that satisfies

\[
\phi(ab) = r(\phi(a), b) + l(a, \phi(b)), \quad \text{for } a, b \in A.
\]

Derivations are 1-cocycles in the Hochschild complex of the associative algebra \( A \) with coefficients in \( M \) [16]. See also [19] for more details. Our main objective in this paper is a pair \((A, \phi_A)\) in which \( A \) is an associative algebra and \( \phi_A : A \to A \) is a derivation on \( A \) with values in the adjoint representation. Thus, \( \phi_A \) satisfies

\[
\phi_A(ab) = \phi_A(a)b + a\phi_A(b), \quad \text{for } a, b \in A.
\]

Such a pair \( (A, \phi_A) \) is called an AssDer pair. Here we give few examples of AssDer pairs.

Example 1. (i) The notion of derivations are generalization of the usual derivative of functions. For instance, if \( A = K[x_1, \ldots, x_n] \) is the polynomial algebra in \( n \) variables, then for \( 1 \leq i \leq n \), the partial derivatives \( \phi_i = \frac{\partial}{\partial x_i} \) are derivations on \( A \). In fact, the space of derivations on \( A \) is linearly spanned by \( \{ \phi_i \}_{1 \leq i \leq n} \).

(ii) Any derivation on the space \( C^\infty(M) \) of smooth functions on a manifold \( M \) is given by a vector field \( X \). Therefore \( (C^\infty(M), X) \) is an AssDer pair, for any \( X \in \mathfrak{X}(M) \)- denotes the space of vector fields on \( M \).

(iii) A (non-commutative) Poisson algebra is an associative algebra \( P \) together with a Lie bracket \( \{ \cdot, \cdot \} \) on it which is a derivation on each entry with respect to the associative product. It follows that if \( (P, \{ \cdot, \cdot \}) \) is a (non-commutative) Poisson algebra, then for any \( a \in P \), the linear map \( \phi_a = \{ a, \cdot \} \) is a derivation for associative product. Hence \( (P, \phi_a) \) is an AssDer pair.

(iv) Let \( V \) be a vector space and \( d : V \to V \) be a linear map. Consider the tensor algebra \( T(V) = \oplus_{n \geq 0} V \otimes^n \) (resp. reduced tensor algebra \( \overline{T}(V) = \oplus_{n \geq 1} V \otimes^n \)) with the concatenation product. The linear map \( d \) induces a linear map \( \overline{d} : T(V) \to T(V) \) (resp. \( \overline{d} : \overline{T}(V) \to \overline{T}(V) \))
as follows
\[
\overline{d}(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes dv_i \otimes \cdots \otimes v_n.
\]

It is easy to verify the \(\overline{d}\) is a derivation on \(T(V)\) (resp. \(\overline{T}(V)\)). Hence \((T(V), d)\) (resp. \((\overline{T}(V), \overline{d})\)) is an unital (resp. non-unital) AssDer pair. In fact, any derivation on \(T(V)\) (resp. \(\overline{T}(V)\)) arises in this way.

**Remark 1.** Let \(A\) be an unital, commutative associative algebra and \(\phi_A\) be a derivation on \(A\). Then \(\phi_A\) induces a Lie bracket on \(A\) given by
\[
[a, b]_{\phi_A} := a\phi_A(b) - \phi_A(a)b, \quad \text{for } a, b \in A.
\]
We denote this Lie algebra by \(A_{\phi_A}\). This Lie bracket additionally satisfies the following Leibniz rule
\[
[a, fb]_{\phi_A} = f[a, b]_{\phi_A} + \rho(a)(f)b, \quad \text{for } a, b, f \in A,
\]
where \(\rho : (A, [\ , \ ]_{\phi_A}) \to (\text{Der}(A), [\ , \ ]_{\phi_A})\) is the \(A\)-linear Lie algebra morphism given by \(\rho(a) = a\phi_A\). In other words, the triple \((A, A_{\phi_A}, \rho)\) is a Lie-Rinehart algebra [17]. Conversely, any Lie-Rinehart algebra structure on \((A, \overline{\phi})\) arises by a derivation in the above way. Thus, to better understand Lie-Rinehart algebra structures on \((A, \overline{\phi})\) one needs to know derivations on \(A\). In Remark 4, we observe that this association is compatible with deformation theories of AssDer pairs and Lie-Rinehart algebras.

**Definition 1.** Let \((A, \phi_A)\) and \((B, \phi_B)\) be two AssDer pairs. A morphism between them consists of an algebra map \(f : A \to B\) that commute with derivations, i.e.
\[
f \circ \phi_A = \phi_B \circ f.
\]
We denote the category of AssDer pairs together with morphisms between them by \(\text{AssDer}\).

Let \((V, d)\) be a vector space together with a linear map. The free AssDer pair over \((V, d)\) is an AssDer pair \((\mathcal{F}(V), \phi_{\mathcal{F}(V)})\) equipped with a linear map \(i : V \to \mathcal{F}(V)\) that satisfies \(\phi_{\mathcal{F}(V)} \circ i = i \circ d\) and the following universal condition holds:

For any AssDer pair \((A, \phi_A)\) and a linear map \(f : V \to A\) satisfying \(\phi_A \circ f = f \circ d\), there exists an unique AssDer pair morphism \(\widetilde{f} : (\mathcal{F}(V), \phi_{\mathcal{F}(V)}) \to (A, \phi_A)\) such that \(\widetilde{f} \circ i = f\).

It follows that the free AssDer pair over \((V, d)\) is well-defined up to a unique isomorphism.

**Proposition 1.** Let \((V, d)\) be a vector space together with a linear map. Then \((T(V), \overline{d})\) (resp. \((\overline{T}(V), \overline{d})\)) equipped with the inclusion map \(i\) is free unital (resp. non-unital) AssDer pair over \((V, d)\).

**2.1 Representation and cohomology of AssDer pair**

**Definition 2.** Let \((A, \phi_A)\) be an AssDer pair. A left module over it consists of a pair \((M, \phi_M)\) in which \(M\) is a left \(A\)-module and \(\phi_M : M \to M\) is a linear map satisfying
\[
\phi_M(am) = \phi_A(a)m + a\phi_M(m).
\]

Lie-Rinehart algebras are algebraic analogue of Lie algebroids [21]. More precisely, a Lie-Rinehart algebra consists of a triple \((A, \overline{\phi}, \rho)\) where \(A\) is an associative algebra, \(\overline{\phi}\) a Lie algebra and \(\overline{\phi}\)-module and an \(A\)-module map \(\rho : L \to \text{Der}(A)\) which is a morphism of Lie algebras and the Leibniz identity (1) holds. Thus a Lie algebra \(L\) is a Lie-Rinehart algebra in which \(A = \mathbb{K}\).
Similarly, a right module over \((A, \phi_A)\) is a pair \((M, \phi_M)\) in which \(M\) is a right \(A\)-module and \(\phi_M : M \rightarrow M\) is a linear map satisfying
\[
\phi_M(ma) = \phi_M(m)a + m\phi_A(a). \quad (3)
\]
A bimodule (representation) over \((A, \phi_A)\) is a pair \((M, \phi_M)\) which is both a left module and a right module over \((A, \phi_A)\) and \(M\) is an \(A\)-bimodule, i.e. \((am)b = a(mb)\), for all \(a, b \in A\) and \(m \in M\).

It follows that the AssDer pair \((A, \phi_A)\) is a representation of itself for the adjoint bimodule structure on \(A\).

**Proposition 2.** Let \((M, \phi_M)\) be a representation of the AssDer pair \((A, \phi_A)\). Then \((M^*, -\phi^*_M)\) is also a representation of \((A, \phi_A)\) where the \(A\)-bimodule structure on \(M^*\) is given by
\[
(a^* f)(m) = f(ma) \quad \text{for} \quad a \in A, f \in M^* \text{ and } m \in M.
\]

**Proof.** The fact that \(M^*\) is an \(A\)-bimodule is standard [19]. To verify that \((M^*, -\phi^*_M)\) is a representation of the AssDer pair, we observe that
\[
\langle -\phi^*_M(a^* f), m \rangle = \langle a^* f, -\phi_M(m) \rangle = f(-\phi_M(m)a) = f(m\phi_A(a)) - f(\phi_M(ma)) \quad \text{by (3)}
\]
\[
= \langle \phi_A(a)f, m \rangle - \langle a\phi_M(f), m \rangle.
\]
This shows that \(-\phi^*_M(a^* f) = \phi_A(a)f + a(-\phi^*_M(f))\). Similarly, we have
\[
\langle -\phi^*_M(f^* a), m \rangle = \langle f^* a, -\phi_M(m) \rangle = f(-a\phi_M(m)) = f(\phi_A(a)m) - f(\phi_M(am)) \quad \text{by (2)}
\]
\[
= \langle f\phi_A(a), m \rangle - \langle \phi^*_M(f)a, m \rangle.
\]
This shows that \(-\phi^*_M(f^* a) = -\phi^*_M(f)a + f\phi_A(a)\). Hence the result follows.

Thus by above Proposition, \((A^*, -\phi^*_A)\) is a representation of the AssDer pair \((A, \phi_A)\). It is called the coadjoint representation of the AssDer pair \((A, \phi_A)\).

Given an associative algebra and a bimodule over it, one can construct a semi-direct product associative algebra [19]. This result can be extended to AssDer pairs.

**Proposition 3.** Let \((A, \phi_A)\) be an AssDer pair and \((M, \phi_M)\) be a representation of it. Then \((A \oplus M, \phi_A \oplus \phi_M)\) is an AssDer pair where the associative structure on \(A \oplus M\) is given by the semi-direct product
\[
(a, m) \cdot (b, n) = (ab, an + mb).
\]

**Proof.** It is known that \(A \oplus M\) with the above product is an associative algebra. Thus, it is enough
to show that $\phi_A \oplus \phi_M : A \oplus M \to A \oplus M$ is a derivation. Observe that

$$(\phi_A \oplus \phi_M)((a, m) \cdot (b, n))$$

$$= (\phi_A(ab), \phi_M(mn))$$

$$= (\phi_A(a)b + a\phi_A(b), \phi_A(a)n + a\phi_M(n) + \phi_M(m)b + m\phi_A(b))$$

$$= (\phi_A(a)b, \phi_A(a)n + \phi_M(m)b) + (a\phi_A(b), a\phi_M(n) + m\phi_A(b))$$

$$= (\phi_A(a), \phi_M(m)) \cdot (b, n) + (a, m) \cdot (\phi_A(b) \oplus \phi_M(n))$$

$$= (\phi_A \oplus \phi_M)(a, m) \cdot (b, n) + (a, m) \cdot (\phi_A \oplus \phi_M)(b, n).$$

Hence the proof. \[ \square \]

Let $A$ be an associative algebra and $M$ be an $A$-bimodule. Then the classical Hochschild cohomology of $A$ with coefficients in $M$ is given as follows [16]. The $n$-th cochain group $C^n(A, M)$ is given by $C^n(A, M) := \text{Hom}(A^{\otimes n}, M)$ for $n \geq 0$ and the coboundary map $\delta_{\text{Hoch}} : C^n(A, M) \to C^{n+1}(A, M)$ is given by

$$\delta_{\text{Hoch}}(f)(a_1, \ldots, a_{n+1})$$

$$= a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_{i-1}, a_i a_{i+1}, \ldots, a_{n+1})$$

$$+ (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}. \tag{4}$$

The corresponding cohomology groups are denoted by $H^n_{\text{Hoch}}(A, M)$ for $n \geq 0$ and they are called the Hochschild cohomology groups of $A$ with coefficients in $M$.

It has been observed by Gerstenhaber [14] that the graded vector space of Hochschild cochains $C^*(A, A) = \oplus_n C^n(A, A)$ carries a degree $-1$ graded Lie bracket (called the Gerstenhaber bracket) given by

$$[f, g] := f \circ g - (-1)^{(m-1)(n-1)} g \circ f, \tag{6}$$

where $(f \circ g)(a_1, \ldots, a_{m+n-1})$ is defined as

$$\sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), \ldots, a_{m+n-1}),$$

for $f \in C^m(A, A)$ and $g \in C^n(A, A)$.

Let $\mu : A^{\otimes 2} \to A$ denote the associative multiplication on $A$. With the above notations, the Hochschild coboundary map $\delta_{\text{Hoch}}$ with coefficients in itself is given by

$$\delta_{\text{Hoch}} f = (-1)^{n-1} [\mu, f], \quad \text{for } f \in C^n(A, A).$$

Next, we use the above Hochschild cohomology for associative algebras to construct cohomology for an AssDer pair. In sections 3, 4 and 5, we will show that such cohomology (with appropriate co-efficients) is related to extensions of AssDer pairs and governs the one-parameter formal deformation of AssDer pairs.

Let $(A, \phi_A)$ be an AssDer pair and $(M, \phi_M)$ be a representation of it. We aim to define the cohomology of the AssDer pair $(A, \phi_A)$ with coefficients in $(M, \phi_M)$. Define the space $C^n_{\text{AssDer}}(A, M)$ of $0$-cochains to be $0$ and the space $C^1_{\text{AssDer}}(A, M)$ of $1$-cochains to be $\text{Hom}(A, M)$. For $n \geq 2$, the space $C^n_{\text{AssDer}}(A, M)$ of $n$-cochains to be defined by

$$C^n_{\text{AssDer}}(A, M) = C^n(A, M) \oplus C^{n-1}(A, M).$$
Before we introduce the coboundary operator, we define a new map

\[ \delta : C^n(A, M) \to C^n(A, M) \]

given by

\[
\delta f = \sum_{i=1}^n f \circ (\text{id} \otimes \cdots \otimes \phi_A \otimes \cdots \otimes \text{id}) - \phi_M \circ f.
\]

When \((M, \phi_M) = (A, \phi_A)\), the map \(\delta\) can be seen in terms of Gerstenhaber bracket as

\[ \delta f = -[\phi_A, f]. \]

Finally, we define the coboundary map \(\partial : C^n_{\text{AssDer}}(A, M) \to C^{n+1}_{\text{AssDer}}(A, M)\) as

\[
\begin{cases}
\partial f = (\delta_{\text{Hoch}} f, -\delta f), & \text{for } f \in C^1_{\text{AssDer}}(A, M) = \text{Hom}(A, M), \\
\partial (f_n, \overline{f}_n) = (\delta_{\text{Hoch}} f_n, \delta_{\text{Hoch}} \overline{f}_n + (-1)^n \delta f_n), & \text{for } (f_n, \overline{f}_n) \in C^n_{\text{AssDer}}(A, M).
\end{cases}
\]

(7)

To prove that \(\partial^2 = 0\) we use the following lemma whose proof will be given after Proposition 5.

**Lemma 1.** \(\delta_{\text{Hoch}} \circ \delta = \delta \circ \delta_{\text{Hoch}}\).

**Proposition 4.** The map \(\partial\) is a coboundary map, i.e. \(\partial^2 = 0\).

**Proof.** For \(f \in C^1_{\text{AssDer}}(A, M)\), we have

\[
\partial^2 f = \partial(\delta_{\text{Hoch}} f, -\delta f) = (\delta_{\text{Hoch}}^2 f, -\delta_{\text{Hoch}} \delta f + \delta \delta_{\text{Hoch}} f) = 0.
\]

For \((f_n, \overline{f}_n) \in C^n_{\text{AssDer}}(A, M)\), we have

\[
\begin{align*}
\partial^2 (f_n, \overline{f}_n) &= \partial(\delta_{\text{Hoch}} f_n, \delta_{\text{Hoch}} \overline{f}_n + (-1)^n \delta f_n) \\
&= (\delta_{\text{Hoch}}^2 f_n, \delta_{\text{Hoch}}^2 \overline{f}_n + (-1)^n \delta_{\text{Hoch}} \delta f_n + (-1)^{n+1} \delta \delta_{\text{Hoch}} f_n) = 0.
\end{align*}
\]

Hence the result follows. \(\square\)

We denote the corresponding cohomology groups by \(H^n_{\text{AssDer}}(A, M)\), for \(n \geq 1\). It is known that the Hochschild cohomology of an associative algebra \(A\) with coefficients in itself carries a degree \(-1\) graded Lie bracket (6). We prove a similar result for the cohomology of an AssDer pair using the bracket (6).

**Proposition 5.** The bracket

\[\llbracket \ , \rrbracket : C^n_{\text{AssDer}}(A, A) \times C^n_{\text{AssDer}}(A, A) \to C^{n+1}_{\text{AssDer}}(A, A)\]

\[\llbracket (f, \overline{f}), (g, \overline{g}) \rrbracket := ([f, g], (-1)^{m+1}[f, \overline{g}] + [\overline{f}, g])\]

is a degree \(-1\) graded Lie bracket on \(\bigoplus_n C^n_{\text{AssDer}}(A, A)\).

**Proof.** First note that, since \(\llbracket \ , \rrbracket\) is a degree \(-1\) graded Lie bracket, we have

\[\llbracket [f, g], h \rrbracket = [[f, g], h] + (-1)^{(m-1)(n-1)}[g, [f, h]],\]

(8)

for \(f \in \text{Hom}(A^\otimes m, A)\), \(g \in \text{Hom}(A^\otimes n, A)\) and \(h \in \text{Hom}(A^\otimes p, A)\). Now, for any \((f, \overline{f}) \in C^n_{\text{AssDer}}(A, A)\),
\((g, \overline{g}) \in C^n_{\text{AssDer}}(A, A)\) and \((h, \overline{h}) \in C^n_{\text{AssDer}}(A, A)\), we have

\[
\begin{align*}
[[f, \overline{f}], [(g, \overline{g}), (h, \overline{h})]] &= [[f, \overline{f}], ([g, h], (-1)^{n+1}[g, \overline{h}] + [\overline{g}, h])] \\
&= [[f, \overline{f}], ([g, h], (-1)^{n+1}[f, \overline{g}] + [\overline{f}, h])]
\end{align*}
\]

On the other hand,

\[
\begin{align*}
[[f, \overline{f}], [(g, \overline{g}), (h, \overline{h})]] + (-1)^{(m-1)(n-1)} [[g, \overline{g}], [(f, \overline{f}), (h, \overline{h})]] \\
&= (\langle f, [g, h], (-1)^{m+1}[g, \overline{h}] + (-1)^{m+1}[f, \overline{g}] + [\overline{f}, h] \rangle) + (-1)^{m+1} [\overline{g}, [f, h]]
\end{align*}
\]

It follows from (8) that

\[
[[f, \overline{f}], [(g, \overline{g}), (h, \overline{h})]] = [[f, \overline{f}], [(g, \overline{g}), (h, \overline{h})]] + (-1)^{(m-1)(n-1)} [[g, \overline{g}], [(f, \overline{f}), (h, \overline{h})]].
\]

Hence the result follows.

It follows from the above Proposition that the shifted graded vector space \(\bigoplus_n C_{\text{AssDer}}^{n+1}(A, A)\) carries a graded Lie bracket. This result is true for an arbitrary vector space \(A\) (not necessarily an associative algebra) and \(C_{\text{AssDer}}^n(A, A) = \text{Hom}(A^\otimes n, A) \oplus \text{Hom}(A^\otimes n^{-1}, A)\), for all \(n\).

Let \(A\) be a vector space, \(\mu : A^\otimes 2 \rightarrow A\) and \(\phi_A : A \rightarrow A\) be two linear maps. Consider the pair \((\mu, \phi_A) \in C_2^{2\text{AssDer}}(A, A)\). Then \(\mu\) defines an associative product on \(A\) and \(\phi_A\) is a derivation for the associative product if and only if \((\mu, \phi_A) \in C_2^{2\text{AssDer}}(A, A)\) is a Maurer-Cartan element in the graded Lie algebra \((\bigoplus_n C_{\text{AssDer}}^{n+1}(A, A), \llbracket, \rrbracket), \llbracket, \rrbracket\), i.e. \([\mu, \phi_A], (\mu, \phi_A) = 0\). With this notations, the differential (7) of the AssDer pair \((A, \phi_A)\) with coefficients in itself is given by

\[
\partial(f, \overline{f}) = (-1)^{n-1} [[\mu, \phi_A], (f, \overline{f})], \ \text{for } (f, \overline{f}) \in C^n_{\text{AssDer}}(A, A).
\]

As a consequence, we get that the graded space of cohomology \(\bigoplus_n H^{n+1}_{\text{AssDer}}(A, A)\) of the AssDer pair \((A, \phi_A)\) with coefficients in itself carries a graded Lie bracket.

**Proof. (of Lemma 1)** We first prove this result for an AssDer pair with coefficients in itself. Then using the semi-direct product AssDer pair, we conclude the same for any coefficients.

When \((M, \phi_M) = (A, \phi_A)\), we have \(\delta_{\text{Hoch}}(f) = (-1)^{n-1}[\mu, f]\) and \(\delta f = [-\phi_A, f], \ \text{for } f \in C^n(A, A)\). Hence

\[
\delta_{\text{Hoch}} \circ \delta(f) = -\delta_{\text{Hoch}}[\phi_A, f] = (-1)^n [\mu, [\phi_A, f]] = (-1)^n [[\mu, \phi_A], f] + (-1)^n [\phi_A, [\mu, f]] = \delta \circ \delta_{\text{Hoch}}(f).
\]

For any coefficient \((M, \phi_M)\), we consider the semi-direct product AssDer pair \((A \oplus M, \phi_A \oplus \phi_M)\). We use the same notation \(\delta_{\text{Hoch}}\) to denote the Hochschild cohomology of \(A\) with coefficients in \(M\), as well as the Hochschild cohomology of the semi-direct product algebra \(A \oplus M\). Similarly, we use the same notation \(\delta\) for the operator \(\delta\).

For any \(f \in C^n(A, M)\) can be extended to a map \(\tilde{f} \in C^n(A \oplus M, A \oplus M)\) by

\[
\tilde{f}(a_1, m_1, \ldots, a_n, m_n) = (0, f(a_1, \ldots, a_n)).
\]
The map \( f \) can be obtained from \( \tilde{f} \) just by restricting it to \( A^\otimes n \). Moreover, \( \tilde{f} = 0 \) implies that \( f = 0 \). Observe that \( \delta_{\text{Hoch}}(f) = \delta_{\text{Hoch}}(\tilde{f}) \) and \( \tilde{\delta} \circ \tilde{f} = \delta \tilde{f} \). Hence we have

\[
\delta_{\text{Hoch}} \circ \delta(f) = \delta_{\text{Hoch}}(\tilde{f}) = \delta_{\text{Hoch}} \circ \delta(\tilde{f}) = \delta \circ \delta_{\text{Hoch}}(\tilde{f}) = \delta \circ \tilde{\delta}_{\text{Hoch}}(f).
\]

This implies that \( \delta_{\text{Hoch}} \circ \delta = \delta \circ \delta_{\text{Hoch}} \).

### 2.2 Relation with LieDer pair

Let \((A, \phi_A)\) be an AssDer pair. Consider the Lie algebra structure on \( A \) with the commutator bracket \([a, b]_c = ab - ba\). We denote this Lie algebra by \( A_c \). Then \( \phi_A \) is still a derivation for the commutator Lie algebra \( A_c \) as

\[
\phi_A[a, b]_c = \phi_A(ab - ba) = \phi_A(a)b + a\phi_A(b) - \phi_A(b)a - b\phi_A(a) = [\phi_A(a), b]_c + [a, \phi_A(b)]_c.
\]

We get a functor \( (\ )_c : \text{AssDer} \to \text{LieDer} \). In the following, we construct a functor left adjoint to \( (\ )_c \), using the universal enveloping algebra of a Lie algebra.

Let \((\mathfrak{g}, \phi_\mathfrak{g})\) be a LieDer pair. Consider the tensor algebra \( T(\mathfrak{g}) \) of \( \mathfrak{g} \). The universal enveloping algebra \( U(\mathfrak{g}) \) is an associative algebra which is the quotient of \( T(\mathfrak{g}) \) by the two-sided ideal generated by the elements of the form \( x \otimes y - y \otimes x - [x, y] \), for \( x, y \in \mathfrak{g} \). Note that the linear map \( \overline{\phi}_\mathfrak{g} : T(\mathfrak{g}) \to T(\mathfrak{g}) \) (see Example 1 (iv)) induces a map \( \phi_{U(\mathfrak{g})} : U(\mathfrak{g}) \to U(\mathfrak{g}) \) as

\[
\overline{\phi}_\mathfrak{g}(x \otimes y - y \otimes x - [x, y]) = \overline{\phi}_\mathfrak{g}(x) \otimes y + x \otimes \overline{\phi}_\mathfrak{g}(y) - \overline{\phi}_\mathfrak{g}(y) \otimes x - y \otimes \overline{\phi}_\mathfrak{g}(x) - [\overline{\phi}_\mathfrak{g}(x), y] - [x, \overline{\phi}_\mathfrak{g}(y)]
\]

Moreover, \( \phi_{U(\mathfrak{g})} \) is a derivation on \( U(\mathfrak{g}) \). Thus, a LieDer pair \((\mathfrak{g}, \phi_\mathfrak{g})\) induces an AssDer pair \((U(\mathfrak{g}), \phi_{U(\mathfrak{g})})\) on the universal enveloping algebra \( U(\mathfrak{g}) \).

**Proposition 6.** The functor \( U : \text{LieDer} \to \text{AssDer} \) is left adjoint to the functor \( (\ )_c : \text{AssDer} \to \text{LieDer} \). In other words, there is an isomorphism

\[
\text{Hom}_{\text{AssDer}}(U(\mathfrak{g}), A) \cong \text{Hom}_{\text{LieDer}}(\mathfrak{g}, A_c),
\]

for any AssDer pair \((A, \phi_A)\) and any LieDer pair \((\mathfrak{g}, \phi_\mathfrak{g})\).

**Proof.** For any AssDer pair morphism \( f : U(\mathfrak{g}) \to A \), we consider its restriction to \( \mathfrak{g} \), which is a Lie algebra morphism \( \mathfrak{g} \to A_c \) and commute with derivations. Hence it is a morphism of LieDer pairs.

Conversely, for any LieDer pair morphism \( h : \mathfrak{g} \to A_c \), we consider the unique extension of \( h \) as an associative algebra morphism \( h : T(\mathfrak{g}) \to A \). This is indeed a morphism of AssDer pairs. It induces a map of AssDer pairs \( \tilde{h} : U(\mathfrak{g}) \to A \) as \( h \) is a LieDer pair morphism. Finally, the above two correspondences are inverses to each other.

Let \((\mathfrak{g}, \phi_\mathfrak{g})\) be a LieDer pair. A module over it consists of a \( \mathfrak{g} \)-module \( M \) together with a linear map \( \phi_M : M \to M \) satisfying

\[
\phi_M[x, m] = [\phi_\mathfrak{g}(x), m] + [x, \phi_M(m)], \quad \text{for all } x \in \mathfrak{g}, m \in M.
\]

**Proposition 7.** Let \((\mathfrak{g}, \phi_\mathfrak{g})\) be a LieDer pair. A \((\mathfrak{g}, \phi_\mathfrak{g})\)-module is equivalent to a left \((U(\mathfrak{g}), \phi_{U(\mathfrak{g})})\)-module.
Proof. It is known that any left $U(\mathfrak{g})$-module is equivalent to a $\mathfrak{g}$-module [19]. More precisely, let $M$ be a left $U(\mathfrak{g})$-module, then the $\mathfrak{g}$-module structure on $M$ is given by $[x, m] = xm$, for $x \in \mathfrak{g}$ and $m \in M$.

Next, take $(M, \phi_M)$ be a left module over the AssDer pair $(U(\mathfrak{g}), \phi_U(\mathfrak{g}))$. Then the condition $\phi_M(xm) = \phi_U(\mathfrak{g})(x)m + x\phi_M(m)$ is equivalent to $\phi_M[x, m] = [\phi_\mathfrak{g}(x), m] + [x, \phi_M(m)]$, for all $x \in \mathfrak{g}$ and $m \in M$. Hence the result follows.

Let $(M, \phi_M)$ be a representation of the AssDer pair $(A, M)$. Then $M$ can be considered as an $A_c$-module via $[\ ,
\ ] : A_c \times M \to M$, $[a, m] = am - ma$. Further $(M, \phi_M)$ is a representation of the LieDer pair $(A_c, \phi_A)$ as

$$
\phi_M[a, m] = \phi_M(am - ma) = \phi_A(a)m + a\phi_M(m) - \phi_M(m)a - m\phi_A(a)
= \phi_A(a)m - m\phi_A(a) + a\phi_M(m) - \phi_M(m)a
= [\phi_A(a), m] + [a, \phi_M(m)].
$$

Before we relate the cohomology of an AssDer pair with that of the corresponding commutator LieDer pair, we recall the following standard result.

**Proposition 8.** The collection of maps

$$
T_n : \text{Hom}(A^{\otimes n}, M) \to \text{Hom}(\wedge^n A_c, M), \ n \geq 0,
$$
defined by

$$
T_n(f)(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})
$$
is a morphism from the Hochschild complex of $A$ with coefficients in the $A$-bimodule $M$ to the Chevalley-Eilenberg cohomology of the commutator Lie algebra $A_c$ with coefficients in the module $M$.

This is standard and can be proved in a various way. First, it can be checked that $\{T_n\}$ maps Gerstenhaber bracket to the Nijenhuis-Richardson bracket. Note that an associative structure on $A$ and bimodule $M$ can be described by a Maurer-Cartan element in Gerstenhaber Lie bracket while a Lie algebra $\mathfrak{g}$ and representation on $M$ can be described by a Maurer-Cartan element in the Nijenhuis-Richardson bracket. The result follows as the differentials in both cases are induced by respective Maurer-Cartan elements and $T$ maps the Maurer-Cartan element for the associative structure to the Maurer-Cartan element for the corresponding Lie algebra structure.

In [27] the authors introduce a cohomology for a LieDer pair with coefficients in a representation. Let $(\mathfrak{g}, \phi_\mathfrak{g})$ be a LieDer pair and $(M, \phi_M)$ be a representation of it. We denote by $\delta_{\text{CE}} : \text{Hom}(\wedge^n \mathfrak{g}, M) \to \text{Hom}(\wedge^{n+1} \mathfrak{g}, M)$ the coboundary operator for the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coefficients in $M$. Define the 0-th cochain group of the LieDer pair $(\mathfrak{g}, \phi_\mathfrak{g})$ with coefficients in $(M, \phi_M)$ to be 0, and the higher cochain groups are defined by $C^n_{\text{LieDer}}(\mathfrak{g}, M) = \text{Hom}(\mathfrak{g}, M) \oplus \text{Hom}(\mathfrak{g}, M) \oplus \text{Hom}(\mathfrak{g}, M)$, for $n \geq 2$. The coboundary map $\partial : C^n_{\text{LieDer}}(\mathfrak{g}, M) \to C^{n+1}_{\text{LieDer}}(\mathfrak{g}, M)$ is given by

$$
\partial f = (\delta_{\text{CE}} f, -\delta f) \text{ and } \partial(f_n, \overline{f}_n) = (\delta_{\text{CE}} f_n, \delta_{\text{CE}} \overline{f}_n + (-1)^n \delta f_n),
$$
where $\delta : \text{Hom}(\wedge^n \mathfrak{g}, M) \to \text{Hom}(\wedge^n \mathfrak{g}, M)$ is

$$
\delta f_n = \sum_{i=1}^n f_n \circ (\text{id} \otimes \cdots \otimes \phi_\mathfrak{g} \otimes \cdots \otimes \text{id}) - \phi_M \circ f_n
$$

for $f \in C^n_{\text{LieDer}}(\mathfrak{g}, M)$ and $(f_n, \overline{f}_n) \in C^n_{\text{LieDer}}(\mathfrak{g}, M)$. When one considers the cohomology of the LieDer pair $(\mathfrak{g}, \phi_\mathfrak{g})$ with coefficients in itself, the cochain groups $\bigoplus_n C^n_{\text{LieDer}}(\mathfrak{g}, \mathfrak{g})$ carries a degree
−1 graded Lie bracket \[ [\cdot, \cdot] \] given by
\[
[(f, \overline{f}), (g, \overline{g})] = ([f, g], (-1)^{m+1}[f, \overline{g}] + [\overline{f}, g]),
\]
for \((f, \overline{f}) \in C^{m}_{\text{LieDer}}(\mathfrak{g}, \mathfrak{g})\), \((g, \overline{g}) \in C^{n}_{\text{LieDer}}(\mathfrak{g}, \mathfrak{g})\), where \([ \cdot, \cdot]\) is the Nijenhuis-Richardson bracket on the space of skew-symmetric multilinear maps on \(\mathfrak{g}\) given by
\[
[f, g](x_1, \ldots, x_{m+n-1}) = \sum_{\sigma \in \text{Sh}(n,m-1)} (-1)^{\sigma} f(g(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), x_{\sigma(n+1)}, \ldots, x_{\sigma(m+n-1)})
\]
\[- (-1)^{(m-1)(n-1)} \sum_{\sigma \in \text{Sh}(m,n-1)} (-1)^{\sigma} g(f(x_{\sigma(1)}, \ldots, x_{\sigma(m)}), x_{\sigma(m+1)}, \ldots, x_{\sigma(m+n-1)}).
\]
If the Lie bracket on \(\mathfrak{g}\) is given by a map \(\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\), then \((\omega, \phi_{\mathfrak{g}}) \in C^{2}_{\text{LieDer}}(\mathfrak{g}, \mathfrak{g})\) satisfies
\[
[[\omega, \phi_{\mathfrak{g}}], (f, \overline{f})] = 0
\]
and the differential if given by
\[
\partial(f, \overline{f}) = (-1)^{n-1}[[\omega, \phi_{\mathfrak{g}}], (f, \overline{f})].
\]

Proposition 9. Let \((A, \phi_{A})\) be an AssDer pair and \((M, \phi_{M})\) be a representation over it. Then the maps
\[
(T_n, T_{n-1}) : \text{Hom}(A \otimes^n M) \oplus \text{Hom}(A \otimes^{n-1} M) \to \text{Hom}(\wedge^n A_c, M) \oplus \text{Hom}(\wedge^{n-1} A_c, M)
\]
defines a morphism from the cohomology of the AssDer pair \((A, \phi_{A})\) with the cohomology of the corresponding LieDer pair \((A_c, \phi_{A})\).

3. Central extensions of AssDer pairs

In this section, we study central extensions of an AssDer pair, i.e., extensions of an AssDer pair by a trivial AssDer pair. We show that isomorphic classes of central extensions are classified by the second cohomology of the AssDer pair with coefficients in the trivial representation.

Let \((A, \phi_{A})\) be an AssDer pair and \((M, \phi_{M})\) a trivial AssDer pair. That is, the associative structure on \(M\) is trivial.

Definition 3. A central extension of \((A, \phi_{A})\) by \((M, \phi_{M})\) consists of an exact sequence of AssDer pairs
\[
0 \to (M, \phi_{M}) \xrightarrow{i} (\hat{A}, \phi_{\hat{A}}) \xrightarrow{p} (A, \phi_{A}) \to 0
\]
satisfying \(i(m) \cdot \hat{a} = \hat{a} \cdot i(m)\), for all \(m \in M\) and \(\hat{a} \in \hat{A}\).

We identify \(M\) with the corresponding subalgebra of \(\hat{A}\) and with this identification \(\phi_M = \phi_{\hat{A}}|M\).

Definition 4. Let \((\hat{A}_1, \phi_{\hat{A}_1})\) and \((\hat{A}_2, \phi_{\hat{A}_2})\) be two central extensions of the AssDer pair \((A, \phi_{A})\) by \((M, \phi_{M})\). These two central extensions are said to be isomorphic if there exists an AssDer pair
isomorphism \( \eta : (\hat{A}_1, \phi_{\hat{A}_1}) \to (\hat{A}_2, \phi_{\hat{A}_2}) \) such that the following diagram commute

\[
\begin{array}{ccc}
0 & \xrightarrow{s_1} & (M, \phi_M) \\
\downarrow s_2 & & \downarrow \eta \\
(\hat{A}_1, \phi_{\hat{A}_1}) & \xrightarrow{p_1} & (A, \phi_A) \to 0 \\
\downarrow p_2 & & \\
(\hat{A}_2, \phi_{\hat{A}_2}) & \xrightarrow{s_1} & (M, \phi_M)
\end{array}
\] (10)

Let \( 0 \to (M, \phi_M) \xrightarrow{\tilde{1}} (\hat{A}, \phi_{\hat{A}}) \xrightarrow{p} (A, \phi_A) \to 0 \) be a central extension of the AssDer pair \((A, \phi_A)\) by \((M, \phi_M)\). A section of it is given by a linear map \( s : A \to \hat{A} \) such that \( p \circ s = \text{id}_A \).

Let \( s \) be a section. Define two maps \( \psi : A \otimes^2 \to M \) and \( \chi : A \to M \) by

\[
\psi(a, b) = s(a) \cdot s(b) - s(ab), \quad \chi(a) = \phi_{\hat{A}}(s(a)) - s(\phi_A(a)), \quad \text{for } a, b \in A.
\]

Since the vector space \( \hat{A} \) is isomorphic to \( A \oplus M \) (via the section \( s \)), we may transfer the AssDer structure of \( \hat{A} \) to \( A \oplus M \). This will certainly depends on the section \( s \). The product and the linear map on \( A \oplus M \) are respectively given by

\[
(a \oplus m) \cdot (b \oplus n) = ab \oplus \psi(a, b) \quad \text{and} \quad \phi_{A\oplus M}(a \oplus m) = \phi_A(a) \oplus \phi_M(m) + \chi(a).
\]

With this notations, we have the following.

**Proposition 10.** The pair \((A \oplus M, \phi_{A\oplus M})\) is an AssDer pair if and only if \((\psi, \chi)\) is a 2-cocycle in the cohomology of the AssDer pair \((A, \phi_A)\) with coefficients in the trivial representation \((M = (M, l = 0, r = 0), \phi_M)\).

**Proof.** Note that \((\psi, \chi)\) is a 2-cocycle if and only if \( \delta_{\text{Hoch}} \psi = 0 \) and \( \delta_{\text{Hoch}} \chi + \delta \psi = 0 \), or equivalently,

\[
\begin{align}
\psi(ab, c) - \psi(a, bc) &= 0, \\
- \chi(ab) + \psi(\phi_A a, b) + \psi(a, \phi_A b) - \phi_M(\psi(a, b)) &= 0.
\end{align}
\] (11) (12)

First observe that \((A \oplus M, \phi_{A\oplus M})\) is an AssDer pair if and only if

\[
\begin{align}
((a \oplus m) \cdot (b \oplus n)) \cdot (c \oplus p) &= (a \oplus m) \cdot ((b \oplus n) \cdot (c \oplus p)), \\
\phi_{A\oplus M}((a \oplus m) \cdot (b \oplus n)) &= \phi_{A\oplus M}(a \oplus m) \cdot (b \oplus n) + (a \oplus m) \cdot \phi_{A\oplus M}(b \oplus n).
\end{align}
\] (13) (14)

The identity (13) is equivalent to (11) and the identity (14) is equivalent to (12). Hence the result follows. \( \square \)

**Theorem 1.** Let \((A, \phi_A)\) be an AssDer pair and \((M, \phi_M)\) be an abelian AssDer pair. Then the isomorphism classes of central extensions of \((A, \phi_A)\) by \((M, \phi_M)\) are classified by the second cohomology group \( H^2_{\text{AssDer}}(A, M) \) of the AssDer pair with coefficients in the trivial representation \((M = (M, l = 0, r = 0), \phi_M)\).

**Proof.** First we show that the cohomology class of the 2-cocycle \((\psi, \chi)\) does not depend on the choice of the section \( s \). Let \( s_1 \) and \( s_2 \) be two sections of (9). Define a map \( \phi : A \to M \) by
\( \phi(a) = s_1(a) - s_2(a) \). Then we have

\[
\psi_1(a, b) = s_1(a) \cdot s_1(b) - s_1(ab)
= (s_2(a) + \phi(a)) \cdot (s_2(b) + \phi(b)) - s_2(ab) - \phi(ab)
= \psi_2(a, b) - \phi(ab) \quad \text{(as } \phi(a), \phi(b) \in M) \]

and

\[
\chi_1(a) = \phi_{\hat{A}}(s_1(a)) - s_1(\phi_A(a))
= \phi_{\hat{A}}(s_2(a) + \phi(a)) - s_2(\phi_A(a)) - \phi(\phi_A(a))
= \phi_{\hat{A}}(s_2(a)) - s_2(\phi_A(a)) + \phi_M(\phi(a)) - \phi(\phi_A(a))
= \chi_2(a) + \phi_M(\phi(a)) - \phi(\phi_A(a)).
\]

This shows that \((\psi_1, \chi_1) = (\psi_2, \chi_2) + \partial \phi\). Hence \((\psi_1, \chi_1)\) and \((\psi_2, \chi_2)\) are representative of the same cohomology class.

Let \((\hat{A}_1, \phi_{\hat{A}_1})\) and \((\hat{A}_2, \phi_{\hat{A}_2})\) be two isomorphic central extensions of \((A, \phi_A)\) by \((M, \phi_M)\), and the isomorphism is given by \(\eta\). Let \(s_1 : A \to \hat{A}_1\) be a section of the first central extension. Then we have

\[
p_2 \circ (\eta \circ s_1) = (p_2 \circ \eta) \circ s_1 = p_1 \circ s = \text{id}_A.
\]

This shows that \(s_2 := \eta \circ s_1\) is a section for the second central extension. Since \(\eta\) is a morphism of AssDer pairs, we have \(\eta_M = \text{id}_M\). Thus, we have

\[
\psi_2(a, b) = s_2(a) \cdot s_2(b) - s_2(ab) = \eta(s_1(a) \cdot s_1(b) - s_1(ab)) = \psi_1(a, b)
\]

and

\[
\chi_2(a) = \phi_{\hat{A}_2}(s_2(a)) - s_2(\phi_A(a)) = \eta(\phi_{\hat{A}_1}(s_1(a)) - s_1(\phi_A(a))) = \chi_1(a).
\]

This shows that isomorphic central extensions give rise to same 2-cocycle, hence, same element in \(H^2_{\text{AssDer}}(A; M)\).

Conversely, let \((\psi_1, \chi_1)\) and \((\psi_2, \chi_2)\) be two cohomologous 2-cocycles. Therefore, there exists a linear map \(\phi : A \to M\) such that \((\psi_1, \chi_1) - (\psi_2, \chi_2) = \partial \phi\). Consider the corresponding AssDer pairs \((A \oplus M, \phi_{\hat{A} \oplus M}^1)\) and \((A \oplus M, \phi_{\hat{A} \oplus M}^2)\) given in Proposition 10. They are isomorphic as AssDer pairs via the map \(\eta : A \oplus M \to A \oplus M\) given by \(\eta(m) = a \oplus m + \phi(a)\). In fact, \(\eta\) defines an isomorphism between central extensions. Hence the proof.

### 3.1 Extensions of a pair of derivations

In this subsection, we study extensions of a pair of derivations in a central extension of associative algebras. The results of the present section are analogous to the results appeared for LieDer pair [27].

Let

\[
\begin{array}{c}
0 \longrightarrow & M & \overset{i}{\longrightarrow} & \hat{A} & \overset{p}{\longrightarrow} & A & \longrightarrow & 0
\end{array}
\]

be a fixed central extension of associative algebras.

**Definition 5.** A pair of derivations \((\phi_A, \phi_M) \in \text{Der}(A) \times \text{Der}(M)\) is said to be extensible if there
exists a derivation $\phi_M \in \text{Der}(\hat{A})$ such that

\[ 0 \longrightarrow (M, \phi_M) \longrightarrow (\hat{A}, \phi_M) \longrightarrow (A, \phi_A) \longrightarrow 0 \]  

is an exact sequence of AssDer pairs. In other words, $(\hat{A}, \phi_M)$ is a central extension of $(A, \phi_A)$ by $(M, \phi_M)$.

Let $s : A \to \hat{A}$ be a section of the central extension (15). Define a map $\psi : A^{\otimes 2} \to M$ by

\[
\psi(a, b) := s(a) \cdot s(b) - s(ab).
\]

For any pair of derivations $(\phi_A, \phi_M) \in \text{Der}(A) \times \text{Der}(M)$, we define a map $\text{Ob}^\hat{A}_{(\phi_A, \phi_M)} : A^{\otimes 2} \to M$ by

\[
\text{Ob}^\hat{A}_{(\phi_A, \phi_M)}(a, b) := \phi_M(\psi(a, b)) - \psi(\phi_A(a), b) - \psi(a, \phi_A(b)).
\]

**Proposition 11.** The map $\text{Ob}^\hat{A}_{(\phi_A, \phi_M)} : A^{\otimes 2} \to M$ is a 2-cocycle in the Hochschild cohomology of $A$ with coefficients in the trivial representation $M = (M, l = 0, r = 0)$. Moreover, the cohomology class $[\text{Ob}^\hat{A}_{(\phi_A, \phi_M)}] \in H^2_{\text{Hoch}}(A, M)$ does not depend on the choice of sections.

**Proof.** Note that $\psi$ is a 2-cocycle on $A$ with coefficients in the trivial $A$-bimodule $M$, i.e.

\[
\psi(ab, c) - \psi(a, bc) = 0.
\]

Observe that

\[
(\delta_{\text{Hoch}} \text{Ob}^\hat{A}_{(\phi_A, \phi_M)})(a, b, c) = - \text{Ob}^\hat{A}_{(\phi_A, \phi_M)}(ab, c) + \text{Ob}^\hat{A}_{(\phi_A, \phi_M)}(a, bc)
\]

\[
= - \phi_M(\psi(ab, c)) + \psi(\phi_A(ab), c) + \psi(ab, \phi_A(c)) + \phi_M(\psi(a, bc)) - \psi(\phi_A(a), bc) - \phi(a, \phi_A(bc))
\]

\[
= \psi(a\phi_A(b), c) + \psi(\phi_A(a)b, c) + \psi(ab, \phi_A(c)) - \psi(\phi_A(a), bc) - \psi(a, b\phi_A(c)) - \psi(a, \phi_A(b)c)
\]

\[
= 0 \quad (\text{by (17)}).
\]

This proves the first part. To prove the second part, take $s_1, s_2$ be two sections of (15). Define a map $\phi : A \to M$ by $\phi = s_1 - s_2$. Then we get

\[
\psi_1(a, b) = s_1(a) \cdot s_1(b) - s_1(ab)
\]

\[
= (s_2(a) + \phi(a)) \cdot (s_2(b) + \phi(b)) - s_2(ab) - \phi(ab)
\]

\[
= s_2(a) \cdot s_2(b) - s_2(ab) - \phi(ab) = \psi_2(a, b) - \phi(ab).
\]

If the 2-cocycles corresponding to $s_1$ and $s_2$ are respectively denoted by $1 \text{Ob}^\hat{A}_{(\phi_A, \phi_M)}$ and $2 \text{Ob}^\hat{A}_{(\phi_A, \phi_M)}$, then we get

\[
1 \text{Ob}^\hat{A}_{(\phi_A, \phi_M)}(a, b) = \phi_M(\psi_1(a, b)) - \psi_1(\phi_A(a), b) - \psi_1(a, \phi_A(b))
\]

\[
= \phi_M(\psi_2(a, b)) - \phi_M(\phi(ab)) - \psi_2(\phi_A(a), b) - \phi(\phi_A(a)b) - \phi(a\phi_A(b))
\]

\[
= 2 \text{Ob}^\hat{A}_{(\phi_A, \phi_M)}(a, b) + \delta_{\text{Hoch}}(\phi_M \circ \phi - \phi \circ \phi_A)(a, b).
\]

This shows that the 2-cocycles $1 \text{Ob}^\hat{A}_{(\phi_A, \phi_M)}$ and $2 \text{Ob}^\hat{A}_{(\phi_A, \phi_M)}$ are cohomologous, hence, they correspond to same cohomology class in $H^2_{\text{Hoch}}(A, M)$. \(\square\)

The cohomology class considered above $[\text{Ob}^\hat{A}_{(\phi_A, \phi_M)}] \in H^2_{\text{Hoch}}(A, M)$ is called the obstruction class to extend the pair of derivations $(\phi_A, \phi_M)$. 

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Theorem 2. A pair of derivations \((\phi_A, \phi_M) \in \text{Der}(A) \times \text{Der}(M)\) is extensible if and only if the obstruction class \([\text{Ob}^A_{\phi_A, \phi_M}] \in H^2_{\text{Hoch}}(A, M)\) is trivial.

Proof. Suppose that the pair \((\phi_A, \phi_M)\) is extensible. That is, there exists a derivation \(\phi_\hat{A} \in \text{Der}(\hat{A})\) such that (16) is an exact sequence of AssDer pairs. Define a map \(\lambda : A \rightarrow M\) by

\[
\lambda(a) = \phi_\hat{A}(s(a)) - s(\phi_A(a)).
\]

Note that the right hand side of the above equation lies in \(M\) as \(p(\phi_\hat{A}(s(a)) - s(\phi_A(a))) = 0\). Hence \(\phi_\hat{A}(s(a)) - s(\phi_A(a)) \in \ker(p) = \text{im}(i)\).

For any \(s(a) + m \in \hat{A}\), we observe that

\[
\phi_\hat{A}(s(a) + m) = \phi_\hat{A}(s(a)) + \phi_M(m)
\]

\[
= \phi_\hat{A}(s(a)) - s(\phi_A(a)) + s(\phi_A(a)) + \phi_M(m)
\]

\[
= s(\phi_A(a)) + \lambda(a) + \phi_M(m).
\]

Hence, for any \(s(a) + m, s(b) + n \in \hat{A}\), we have

\[
\phi_\hat{A}((s(a) + m) \cdot (s(b) + n)) = \phi_\hat{A}(s(a) \cdot s(b))
\]

\[
= \phi_\hat{A}(s(ab) + s(a) \cdot s(b) - s(ab))
\]

\[
= \phi_\hat{A}(s(ab) + \psi(a, b))
\]

\[
= s(\phi_A(ab)) + \lambda(ab) + \phi_M(\psi(a, b)). \hspace{1cm} (18)
\]

On the other hand,

\[
\phi_\hat{A}(s(a) + m) \cdot (s(b) + n) + (s(a) + m) \cdot \phi_\hat{A}(s(b) + n)
\]

\[
= (s(\phi_A(a)) + \lambda(a) + \phi_M(m)) \cdot (s(b) + n) + (s(a) + m) \cdot (s(\phi_A(b)) + \lambda(b) + \phi_M(n))
\]

\[
= (s(\phi_A(a)) + \lambda(a) + \phi_M(m)) \cdot (s(b) + n) + (s(a) + m) \cdot (s(\phi_A(b)) + \lambda(b) + \phi_M(n))
\]

\[
= s(\phi_A(a)) \cdot b + s(\phi_A(a)) \cdot s(b) - s(\phi_A(a) \cdot b) + s(a \cdot \phi_A(b)) + s(a) \cdot s(\phi_A(b)) - s(a \cdot \phi_A(b))
\]

\[
= s(\phi_A(a)) \cdot b + \psi(\phi_A(a), b) + s(a \cdot \phi_A(b)) + \psi(a, \phi_A(b)). \hspace{1cm} (19)
\]

Since \(\phi_\hat{A}\) is a derivation, it follows from (18) and (19) that

\[
\phi_M(\psi(a, b)) - \psi(\phi_A(a), b) - \psi(a, \phi_A(b)) = -\lambda(ab). \hspace{1cm} (20)
\]

This implies that the obstruction class \([\text{Ob}^A_{\phi_A, \phi_M}] = \partial\lambda\) is given by a coboundary. Hence \([\text{Ob}^A_{\phi_A, \phi_M}]\) is trivial.

Conversely, if the obstruction class is given by a coboundary, say \([\text{Ob}^A_{\phi_A, \phi_M}] = \partial\lambda\), for some \(\lambda : A \rightarrow M\). We define a map \(\phi_\hat{A}\) on \(\hat{A}\) by

\[
\phi_\hat{A}(s(a) + m) = s(\phi_A(a)) + \lambda(a) + \phi_M(m).
\]

It follows from (20) that (16) is an exact sequence of AssDer pairs. Hence the pair \((\phi_A, \phi_M)\) is extensible.

As a consequence, we have the following.

Corollary 1. If \(H^2_{\text{Hoch}}(A, M) = 0\) then any pair of derivations \((\phi_A, \phi_M) \in \text{Der}(A) \times \text{Der}(M)\) is extensible.

Let \(A\) be an associative algebra and \(M = (M, l = 0, r = 0)\) be a trivial bimodule. In the following,
we give conditions on a pair of derivations \((\phi_A, \phi_M) \in \text{Der}(A) \times \text{Der}(M)\) such that it is extensible in every central extensions of associative algebras.

Define a linear map \(\Theta : \text{Der}(A) \times \text{Der}(M) \to \mathfrak{gl}(H^2_{\text{Hoch}}(A, M))\) by
\[
\Theta(\phi_A, \phi_M)([\psi]) := [\phi_M \circ \psi - \psi \circ (\phi_A \otimes \text{id}) - \psi \circ (\text{id} \otimes \phi_A)].
\]

**Theorem 3.** A pair of derivations \((\phi_A, \phi_M) \in \text{Der}(A) \times \text{Der}(M)\) is extensible in every central extensions of \(A\) by \(M\) if and only if \(\Theta(\phi_A, \phi_M) = 0\).

**Proof.** Let \(0 \to M \xrightarrow{i} \hat{A} \xrightarrow{p} A \to 0\) be any central extension of \(A\) by \(M\). For any section \(s : A \to \hat{A}\), the map \(\psi : A^{\otimes 2} \to M, \psi(a, b) = s(a) \cdot s(b) - s(ab)\) is a 2-cocycle in the cohomology of \(A\) with coefficients in \(M\). If \(\Theta(\phi_A, \phi_M) = 0\) then we have
\[
[\text{Ob}^A_{(\phi_A, \phi_M)}] = [\phi_M \circ \psi - \psi \circ (\phi_A \otimes \text{id}) - \psi \circ (\text{id} \otimes \phi_A)] = \Theta(\phi_A, \phi_M)([\psi]) = 0.
\]

Hence by Theorem 2 the pair \((\phi_A, \phi_M)\) is extensible.

Conversely, let \((\phi_A, \phi_M)\) is extensible in every central extensions of \(A\) by \(M\). Take any class \([\psi] \in H^2_{\text{Hoch}}(A, M)\). This induces a central extension of \(A\) by \(M\):
\[
0 \to M \xrightarrow{i} A \oplus M \xrightarrow{p} A \to 0,
\]
where the associative product on \(A \oplus M\) is given by
\[
(a \oplus m) \cdot (b \oplus n) = ab \oplus \psi(a, b).
\]
Since \((\phi_A, \phi_M)\) is extensible in the central extension (21), by Theorem 16 we have
\[
\Theta(\phi_A, \phi_M)([\psi]) = [\phi_M \circ \psi - \psi \circ (\phi_A \otimes \text{id}) - \psi \circ (\text{id} \otimes \phi_A)] = [\text{Ob}^A_{(\phi_A, \phi_M)}] = 0.
\]
This shows that \(\Theta(\phi_A, \phi_M) = 0\). Hence the proof. \(\square\)

**4. Abelian extensions of AssDer pairs**

In this section, we study abelian extensions of AssDer pairs and show that equivalence classes of abelian extensions are classified by the second cohomology of AssDer pairs.

Let \((A, \phi_A)\) be an AssDer pair and \((M, \phi_M)\) be a vector space equipped with a linear map. Note that \(M\) can be considered as an associative algebra with trivial multiplication and \(\phi_M\) is a derivation on it.

**Definition 6.** An abelian extension of \((A, \phi_A)\) by \((M, \phi_M)\) is an exact sequence of AssDer pairs
\[
0 \to (M, \phi_M) \xrightarrow{i} (E, \phi_E) \xrightarrow{p} (A, \phi_A) \to 0
\]
(22) together with a \(K\)-splitting (given by \(s\)).

An abelian extension induces a representation of \((A, \phi_A)\) on \((M, \phi_M)\) by \(am = \mu_E(s(a), i(m))\) and \(ma = \mu_E(i(m), s(a))\), for \(a \in A, m \in M\), where \(\mu_E\) denotes the associative multiplication on \(E\). One can easily verify that this action is independent of the choice of \(s\).

**Definition 7.** Two abelian extensions \((E, \phi_E)\) and \((E', \phi_{E'})\) are said to be equivalent if there is a
morphism \( \Psi : (E, \phi_E) \to (E', \phi_{E'}) \) of AssDer pairs making the following diagram commutative

\[
\begin{array}{ccc}
0 & \xrightarrow{i} & (M, \phi_M) \\
& \searrow \Psi & \swarrow p' \\
& (E', \phi_{E'}) & \downarrow p \\
& \swarrow i' & (A, \phi_A) \xrightarrow{\mu} 0.
\end{array}
\]

Suppose \( (M, \phi_M) \) is a given \( (A, \phi_A) \)-representation. We denote by \( \text{Ext}(A, M) \) the equivalence classes of abelian extensions of \( (A, \phi_A) \) by \( (M, \phi_M) \) for which the induced \( (A, \phi_M) \)-representation is the given one.

**Theorem 4.** There is a bijective correspondence \( H^2_{\text{AssDer}}(A, M) \cong \text{Ext}(A, M) \).

**Proof.** Let \( (f, \overline{f}) \in C^2_{\text{AssDer}}(A, M) = \text{Hom}(A^{\otimes 2}, M) \oplus \text{Hom}(A, M) \) be a 2-cocycle. Then it follows that \( f \) is a Hochschild 2-cocycle and \( \delta_{\text{Hoch}}(\overline{f}) + \overline{f} = 0 \). Consider the direct sum \( E = A \oplus M \) with the following structure maps

\[
\mu_E((a, m), (b, n)) = (ab, an + mb + f(a, b)) \quad \text{and} \quad \phi_E(a, m) = (\phi_A(a), \phi_M(m) + \overline{f}(a)).
\]

(Observe that when the 2-cocycle is zero, this is the semi-direct product). Since \( f \) is a Hochschild 2-cocycle, it is easy to see that \( \mu_E \) defines an associative multiplication on \( E \). Moreover, \( \phi_E \) is a derivation on \( (E, \phi_E) \).

Therefore, \( 0 \to (M, \phi_M) \xrightarrow{i} (E, \phi_E) \xrightarrow{\mu} (A, \phi_A) \to 0 \) is an abelian extension with the obvious splitting.

Let \( (E' = A \oplus M, \mu_{E'}, \phi_{E'}) \) be the AssDer pair corresponding to the cohomologous 2-cocycle \( (f, \overline{f}) - \partial h = (f - \delta_{\text{Hoch}} h, \overline{f} + \delta h) \), for some \( h \in C^1_{\text{AssDer}}(A, M) = \text{Hom}(A, M) \). Then \( \Psi : E \to E', (a, m) \mapsto (a, h(a)) \) defines an equivalence between abelian extensions \( E \) and \( E' \).

Hence the map \( H^2_{\text{AssDer}}(A, M) \to \text{Ext}(A, M) \) is well defined.

Conversely, given any abelian extension (22) with splitting \( s \), the vector space \( E \) is isomorphic to \( A \oplus M \) and \( s \) is the map \( s(a) = (a, 0) \). The map \( i \) and \( p \) are the obvious ones with respect to the above splitting. Since \( p \) is an algebra map, we have \( p \circ \mu_E((a, 0), (b, 0)) = ab \). This implies that \( \mu_E((a, 0), (b, 0)) = (ab, f(a, b)) \), for some \( f \in \text{Hom}(A^{\otimes 2}, M) \). The associativity of \( \mu_E \) implies that \( f \) is a 2-cocycle in the Hochschild complex. Moreover, since \( p \circ \phi_E = \phi_A \circ p \), we have \( \phi_E(a, 0) = (\phi_A(a) + \overline{f}(a)) \), for some \( \overline{f} \in \text{Hom}(A, M) \). Finally, \( \phi_E \) being a derivation implies that \( \delta_{\text{Hoch}}(\overline{f}) + \overline{f} = 0 \). Therefore, \( (f, \overline{f}) \in C^2_{\text{AssDer}}(A, M) \) defines a 2-cocycle.

Similarly, one can show that any two equivalent extensions \( (E, \phi_E) \) and \( (E', \phi_{E'}) \) are related by a map \( E = A \oplus M \xrightarrow{\Psi} A \oplus M = E' \) defined by \( (a, m) \mapsto (a, m + h(a)) \), for some \( h \in \text{Hom}(A, M) \). By the fact that \( \Psi \) is a morphism of AssDer pairs, we can deduce that \( (f, \overline{f}) - (f', \overline{f'}) = \partial h \). Here \( (f', \overline{f'}) \in C^2_{\text{AssDer}}(A, M) \) is the 2-cocycle induced from the extension \( (E', \phi_{E'}) \). In other words, the map \( \text{Ext}(A, M) \to H^2_{\text{AssDer}}(A, M) \) is well defined. Moreover, the two maps are inverses to each other. Hence the proof. \( \square \)

**Remark 2.** In the classical case of associative algebras, it is known that the third Hochschild cohomology classifies 2-fold crossed extensions (see [19]). It is interesting to prove a similar result for the third cohomology of AssDer pairs. This will justify more the cohomology of AssDer pairs.
5. Deformations

In this section, we study formal deformations of an AssDer pair using deformations of both the algebra multiplication and the derivation.

Let \((A, \phi_A)\) be an AssDer pair. We denote the associative multiplication on \(A\) by \(\mu\). Consider the space \(A[[t]]\) of formal power series in \(t\) with coefficients from \(A\). Then \(A[[t]]\) is a \(\mathbb{K}[[t]]\)-module.

A formal 1-parameter deformation of the AssDer pair \((A, \phi_A)\) consists of two formal power series

\[
\mu_t = \sum_{i \geq 0} t^i \mu_i, \quad \mu_i \in \text{Hom}(A^\otimes i, A) \text{ with } \mu_0 = \mu,
\]

\[
\phi_t = \sum_{i \geq 0} t^i \phi_i, \quad \phi_i \in \text{Hom}(A, A) \text{ with } \phi_0 = \phi_A
\]
such that the \(\mathbb{K}[[t]]\)-module \(A[[t]]\) together with the multiplication \(\mu_t\) forms an associative algebra and \(\phi_t : A[[t]] \to A[[t]]\) is a derivation on it. In other words, \(A[[t]]\) with the associative multiplication \(\mu_t\) and the derivation \(\phi_t\) forms an AssDer pair over \(\mathbb{K}[[t]]\).

It is clear from the definition that \(\mu_t = \sum_{i \geq 0} t^i \mu_i\) defines a deformation of the associative structure on \(A\) in the sense of Gerstenhaber [15].

Let \((\mu_t, \phi_t)\) defines a deformation of the AssDer pair \((A, \phi_A)\). Since \(\mu_t\) defines an associative product on \(A[[t]]\) and \(\phi_t\) defines a derivation on the associative algebra \((A[[t]], \mu_t)\), we have

\[
\mu_t(\mu_t(a, b), c) = \mu_t(a, \mu_t(b, c)) \quad \text{and} \quad \phi_t(\mu_t(a, b)) = \mu_t(\phi_t(a, b) + \mu_t(a, \phi_t(b)));
\]

for all \(a, b, c \in A\). Expanding both the equations as power series in \(t\) and equating coefficients of \(t^n\) in both the equations, we get for \(n \geq 0\),

\[
\sum_{i+j=n} \mu_i(\mu_j(a, b), c) = \sum_{i+j=n} \mu_i(a, \mu_j(b, c)), \quad (23)
\]

\[
\sum_{i+j=n} \phi_i(\mu_j(a, b)) = \sum_{i+j=n} \mu_i(\phi_j(a, b) + \mu_i(a, \phi_j(b))). \quad (24)
\]

For \(n = 0\), the identity (23) and (24) both holds automatically. For \(n = 1\), we obtain

\[
\mu_1(ab, c) + \mu_1(a, b)c = a\mu_1(b, c) + \mu_1(a, bc), \quad (25)
\]

\[
\phi(\mu_1(a, b)) + \phi_1(ab) = \phi_1(a)b + \mu_1(\phi(a, b) + a\phi_1(b) + \mu_1(a, \phi(b)). \quad (26)
\]

The identity (25) is equivalent \(\delta_{\text{Hoch}}(\mu_1) = 0\) while the identity (26) is equivalent to \(\delta_{\text{Hoch}}(\phi_1) + \delta_{\mu_1} = 0\). It follows from (7) that \(\partial(\mu_1, \phi_1) = 0\).

Hence we have the following.

**Proposition 12.** Let \((\mu_t, \phi_t)\) be a formal deformation of an AssDer pair \((A, \phi_A)\). Then the linear term \((\mu_1, \phi_1)\) is a 2-cocycle in the cohomology of the AssDer pair \((A, \phi_A)\) with coefficients in itself.

The 2-cocycle \((\mu_1, \phi_1)\) is called the infinitesimal of the formal deformation \((\mu_t, \phi_t)\).

**Remark 3.** Let \((A, \phi_A)\) be an AssDer pair where the associative multiplication on \(A\) is denoted by \(\mu\). Consider a deformation \(\mu_t = \sum_{i \geq 0} t^i \mu_i\) of the associative product on \(A\). Then it follows that the skew-symmetrization of the linear term

\[
\{a, b\} := \mu_1(a, b) - \mu_1(b, a)
\]
defines a Poisson bracket on \((A, \mu)\) [18].

Next, we suppose that \(\phi_A : A[[t]] \to A[[t]]\) is still a derivation for the deformed algebra \((A[[t]], \mu_t)\),
Theorem 6. With this definition, we have the following.

An infinitesimal deformation of an AssDer pair \((A, \phi)\) is a pair \((\mu, \phi')\) of \(\mu \in \text{AssDer}^2(A, \mu, \{ , \})\) such that \(\phi'\) is a derivation for the associative product \(\mu\) and the Poisson bracket \(\{ , \}\) is obtained from the skew-symmetrization of the linear term \(\mu_1\). This question puts more restriction than the deformation quantization of Poisson algebras.

Remark 4. Let \((A, \phi_A)\) be an AssDer pair in which \(A\) is unital and commutative. Consider the Lie-Rinehart algebra \((A, A_\phi, \rho)\) as defined in Remark 1. Any deformation \((\mu_t, \phi_t)\) of the AssDer pair \((A, \phi_A)\) induces a deformation of the Lie-Rinehart algebra \((A, A_\phi, \rho)\) given by \(\mu_t, [a, b]_t = a\phi_t(b) - \phi_t(a)b, \rho_t = a\phi_t\).

Next, we discuss an equivalence between two formal deformations of an AssDer pair.

Definition 8. Let \((\mu_t, \phi_t)\) and \((\mu'_t, \phi'_t)\) be two formal deformations of an AssDer pair \((A, \phi_A)\). They are said to be equivalent if there exists a formal isomorphism \(\Phi_t = \sum_{i \geq 0} t^i \Phi_i : A[[t]] \rightarrow A[[t]]\) with \(\Phi_0 = \text{id}_A\), such that

\[
\Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \otimes \Phi_t) \quad \text{and} \quad \Phi_t \circ \phi_t = \phi'_t \circ \Phi_t.
\]

It follows that the following identities must hold (by equating coefficients of \(t^n\) from both sides)

\[
\sum_{i+j=n} \Phi_i \circ \mu_j = \sum_{i+j+k=n} \mu'_i \circ (\Phi_j \otimes \Phi_k) \quad \text{and} \quad \sum_{i+j=n} \Phi_i \circ \phi_j = \sum_{i+j=n} \phi'_i \circ \Phi_j.
\]

For \(n = 0\), both the identities hold as \(\Phi_0 = \text{id}_A\). For \(n = 1\), we obtain

\[
\mu_1 + \Phi_1 \circ \mu = \mu'_1 + \mu \circ (\Phi_1 \otimes \text{id}) + \mu \circ (\text{id} \otimes \Phi_1),
\]

\[
\phi_1 + \Phi_1 \circ \phi_A = \phi'_1 + \phi_A \circ \Phi.
\]

This implies that \((\mu_1, \phi_1) - (\mu'_1, \phi'_1) = \partial(\Phi_1)\). Thus we have the following.

Theorem 5. The infinitesimals corresponding to equivalent deformations of an AssDer pair \((A, \phi_A)\) are cohomologous. Therefore, they correspond to the same cohomology class.

To obtain a one-to-one correspondence between the second cohomology group \(H^2_{\text{AssDer}}(A, A)\) and equivalence classes of certain type deformations, we use the truncated version of formal deformations.

Definition 9. An infinitesimal deformation of an AssDer pair \((A, \phi_A)\) is a deformation of \((A, \phi_A)\) over \(\mathbb{K}[[t]]/(t^2)\) (the local Artinian ring of dual numbers).

Thus, an infinitesimal deformation of \((A, \phi_A)\) consists of a pair \((\mu_t, \phi_t)\) in which \(\mu_t = \mu + t\mu_1\) and \(\phi_t = \phi_A + t\phi_1\), such that \((\mu_1, \phi_1)\) is a 2-cocycle in the cohomology of the AssDer pair \((A, \phi_A)\).

With this definition, we have the following.

Theorem 6. There is a one-to-one correspondence between the space of equivalence classes of infinitesimal deformations of the AssDer pair \((A, \phi_A)\) and the second cohomology group \(H^2_{\text{AssDer}}(A, A)\).

A linear map \(D : A \rightarrow A\) in a Poisson algebra \((A, \mu, \{ , \})\) is called a Poisson derivation if \(D\) is a derivation for both the product \(\mu\) and the Lie bracket \(\{ , \}\).
that the linear term \((\mu, \phi)\) such that the cohomology class is trivial.

In this subsection, we consider deformations of order \(n\) of an AssDer pair \((A, \phi_A)\). To any deformation of order \(n\), we associate a 3-cocycle in the cohomology of the AssDer pair with coefficients in itself. We let \((\mu, \phi)\) be any formal 1-parameter deformation of \((A, \phi)\). Setting \(\Phi_t = \text{id}_A + t\mu\) and \(\phi_t = \phi_A + t\phi_1\) for some \(t \in \mathbb{K}[[t]]\). Therefore, the inverse map is also well defined.

Definition 10. A formal deformation \((\mu, \phi)\) of an AssDer pair \((A, \phi_A)\) is said to be trivial if it is equivalent to \((\mu' = \mu, \phi'_1 = \phi_A)\).

Theorem 7. If \(H^2_{\text{AssDer}}(A, A) = 0\) then every formal deformation of the AssDer pair \((A, \phi_A)\) is trivial.

Proof. Let \((\mu_t, \phi_t)\) be any formal 1-parameter deformation of \((A, \phi_A)\). It follows from Proposition 12 that the linear term \((\mu_1, \phi_1)\) is a 2-cocycle. From the given hypothesis, there exists a 1-cochain \(\Phi_1 \in C^1_{\text{AssDer}}(A, A) = \text{Hom}(A, A)\) such that \(\mu_1, \phi_1 = \partial \Phi_1\).

Let \((\mu_t, \phi_t)\) be any formal 1-parameter deformation of \((A, \phi)\). Setting \(\Phi_t = \text{id}_A + t\mu_t : A[[t]] \to A[[t]]\) and define

\[
\mu'_t = \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t), \quad \phi'_t = \Phi_t^{-1} \circ \phi_t \circ \Phi_t.
\]  

Then \((\mu'_t, \phi'_t)\) is equivalent to \((\mu_t, \phi_t)\). Moreover, it follows from (27) that \(\mu'_t\) and \(\phi'_t\) are of the form

\[
\mu'_t = \mu + t^2 \mu_2 + \cdots \quad \text{and} \quad \phi'_t = \phi_A + t^2 \phi_2 + \cdots.
\]

In other words, the linear terms of \(\mu'_t\) and \(\phi'_t\) vanish. By repeating this argument, one can show that \((\mu_t, \phi_t)\) is equivalent to \((\mu, \phi)\). Hence the proof.

Remark 5. An AssDer pair \((A, \phi_A)\) is said to be rigid if every formal deformation of it is equivalent to \((\mu, \phi)\). Thus the vanishing of the second cohomology is a sufficient condition for the rigidity of the AssDer pair \((A, \phi_A)\).

5.1 Extensions of finite order deformation

In this subsection, we consider deformations of order \(n\) of an AssDer pair. To any deformation of order \(n\), we associate a 3-cocycle in the cohomology of the AssDer pair with coefficients in itself. We show that such a deformation extends to deformation of order \((n+1)\) if and only if the corresponding cohomology class is trivial.

Let \((A, \phi_A)\) be an AssDer pair. Consider the \(\mathbb{K}[[t]]/(t^{n+1})\)-module \(A[[t]]/(t^{n+1})\). A deformation of order \(n\) of the AssDer \((A, \phi_A)\) consists of a pair \((\mu_t, \phi_t)\) where \(\mu_t = \sum_{i=0}^n t^i \mu_i\) and \(\phi_t = \sum_{i=0}^n t^i \phi_i\) such that \(\mu_t\) defines an associative product on \(A[[t]]/(t^{n+1})\) and \(\phi_t\) defines a derivation on it.

Thus, in a deformation of order \(n\), the following identities must hold

\[
\sum_{i+j=k} \mu_i(\mu_j(a, b), c) = \sum_{i+j=k} \mu_i(a, \mu_j(b, c)),
\]

\[
\sum_{i+j=k} \phi_i(\mu_j(a, b)) = \sum_{i+j=k} \mu_i(\phi_j(a), b) + \mu_i(a, \phi_j(b)),
\]

20
for $k = 0,1,\ldots,n$. In other words,
\[
\frac{1}{2} \sum_{i+j=k, i,j>0} [\mu_i, \mu_j] = - [\mu, \mu_k] \quad \text{and} \quad (28)
\]
\[
\sum_{i+j=k, i,j>0} [\phi_i, \mu_j] = - [\phi_A, \mu_k] + [\mu, \phi_k]. \quad (29)
\]

Let $(\mu_{n+1}, \phi_{n+1}) \in C^2_{\text{AssDer}}(A,A)$ be such that $(\mu'_t = \sum_{i=0}^n t^i \mu_i + t^{n+1} \mu_{n+1}, \phi'_t = \sum_{i=0}^n t^i \phi_i + t^{n+1} \phi_{n+1})$ defines a deformation of order $n + 1$. Then the deformation $(\mu_t = \sum_{i=0}^n t^i \mu_i, \phi_t = \sum_{i=0}^n t^i \phi_i)$ is said to be extensible. In such a case, two more deformation equations need to be satisfied, namely,
\[
\sum_{i+j=n+1} \mu_i(\mu_j(a, b), c) = \sum_{i+j=n+1} \mu_i(a, \mu_j(b, c)) \quad \text{and} \quad (30)
\]
\[
\sum_{i+j=n+1} \phi_i(\mu_j(a, b)) = \sum_{i+j=n+1} \mu_i(\phi_j(a), b) + \mu_i(a, \phi_j(b)). \quad (31)
\]

The identities (30) and (31) can be written as
\[
\delta_{\text{Hoch}}(\mu_{n+1})(a, b, c) = \sum_{i+j=n+1, i,j>0} \mu_i(\mu_j(a, b), c) - \mu_i(a, \mu_j(b, c)) = \text{Ob}^3(a, b, c) \quad \text{(say)}
\]
\[
(\delta_{\text{Hoch}}\phi_{n+1} + \delta_{\mu_{n+1}})(a, b) = \sum_{i+j=n+1, i,j>0} \phi_i(\mu_j(a, b)) - \mu_i(\phi_j(a), b) - \mu_i(a, \phi_j(b)) = \text{Ob}^2(a, b) \quad \text{(say)}
\]

Note that, in terms of the Gerstenhaber bracket (6), we have
\[
\text{Ob}^3 = \frac{1}{2} \sum_{i+j=n+1, i,j>0} [\mu_i, \mu_j] \quad \text{and} \quad \text{Ob}^2 = \sum_{i+j=n+1, i,j>0} [\phi_i, \mu_j].
\]

**Proposition 13.** The pair $(\text{Ob}^3, \text{Ob}^2)$ is a 3-cocycle in the cohomology of the AssDer pair $(A, \phi_A)$ with coefficients in itself.

**Proof.** It is known from the finite order deformations of associative algebras [15], the obstruction
Ob\(^3\) is a Hochschild 3-cocycle in the cohomology of \(A\), i.e. \(\delta_{\text{Hoch}}(\text{Ob}^3) = 0\). Moreover, we have

\[
\delta_{\text{Hoch}}(\text{Ob}^2) + (-1)^3 \delta(\text{Ob}^3)
\]

\[
= -[\mu, \text{Ob}^2] + [\phi_A, \text{Ob}^3]
\]

\[
= - \sum_{i+j=n+1, i,j>0} \left(\left[[\mu, \phi_i], \mu_j\right] + \left[[\phi_i, \mu_j], \mu\right]\right) + \frac{1}{2} \sum_{i+j=n+1, i,j>0} \left(\left[[\phi_A, \mu_i], \mu_j\right] + \left[[\mu_i, \phi_A, \mu_j]\right\}
\]

\[
= - \sum_{i+j=n+1, i,j>0} \left(\left[[\mu, \phi_i], \mu_j\right] + \left[[\phi_i, \mu_j], \mu\right]\right) + \frac{1}{2} \sum_{i+j=n+1, i,j>0} \left(\left[[\phi_A, \mu_i], \mu_j\right] + \left[[\mu_i, \phi_A, \mu_j]\right\}
\]

\[
+ \frac{1}{2} \sum_{i+j+\phi^{n+1}, i,j>0} \left(\left[[\phi', \mu'], \mu_j\right] + \left[[\mu', \phi', \mu_j]\right\}
\]

\[
= - \sum_{\phi', \mu', \mu_j} \left(\left[[\phi', \mu'], \mu_j\right] + \sum_{\phi', \mu', \mu_j} \left(\left[[\phi', \mu'], \mu_j\right]\right\}
\]

Thus, \(\partial(\text{Ob}^3, \text{Ob}^2) = (\delta_{\text{Hoch}}(\text{Ob}^3), \delta_{\text{Hoch}}(\text{Ob}^2) + (-1)^3 \delta(\text{Ob}^3)) = 0\).

Therefore, \((\text{Ob}^3, \text{Ob}^2)\) defines a cohomology class in \(H^3_{\text{AssDer}}(A, A)\). If this cohomology class vanishes, i.e \((\text{Ob}^3, \text{Ob}^2)\) is a coboundary, then we have

\[
\partial(\mu_{n+1}, \phi_{n+1}) = (\text{Ob}^3, \text{Ob}^2),
\]

for some \((\mu_{n+1}, \phi_{n+1}) \in C^2_{\text{AssDer}}(A, A)\). In such a case \((\mu'_t = \mu_t + t^{n+1}\mu_{n+1}, \phi'_t = \phi_t + t^{n+1}\phi_{n+1})\) defines a deformation of order \(n+1\). Therefore, the deformation \((\mu_t, \phi_t)\) becomes extensible. On the other hand, if \((\mu_t, \phi_t)\) is extensible, there exists \((\mu_{n+1}, \phi_{n+1}) \in C^2_{\text{AssDer}}(A, A)\) such that \((\mu'_t = \mu_t + t^{n+1}\mu_{n+1}, \phi'_t = \phi_t + t^{n+1}\phi_{n+1})\) is a deformation of order \(n+1\). Hence the obstruction \((\text{Ob}^3, \text{Ob}^2)\) is given by the coboundary \(\partial(\mu_{n+1}, \phi_{n+1})\). Thus the corresponding cohomology class is null. Therefore, we obtain the following.

**Theorem 8.** Let \((\mu_t, \phi_t)\) be a deformation of order \(n\) of the AssDer pair \((A, \phi_A)\). It is extensible if and only if the obstruction class \([\text{Ob}^3, \text{Ob}^2]\) vanishes.

**Theorem 9.** If \(H^3_{\text{AssDer}}(A, A) = 0\) then every finite order deformation of the AssDer pair \((A, \phi_A)\) extends to a deformation of next order.

**Corollary 2.** If \(H^3_{\text{AssDer}}(A, A) = 0\) then every 2-cocycle is the infinitesimal of a formal deformation of \((A, \phi_A)\).

### 6. Homotopy derivations on \(A_\infty\)-algebras

In this section, we are interested in homotopy associative algebras \((A_\infty\)-algebras [26]) with homotopy derivations. Note that homotopy derivation on \(A_\infty\)-algebras were studied by Loday [20] and further developed by Doubek-Lada [11]. However, we will be most interested in \(A_\infty\)-algebras whose underlying graded vector space is concentrated in degrees 0 and 1. We classify homotopy derivations on skeletal and strict \(A_\infty\)-algebras.
**Definition 11.** A 2-term $A_\infty$-algebra consists of a chain complex $A := (A_1 \overset{d}{\rightarrow} A_0)$ together with maps $\mu_2 : A_i \otimes A_j \rightarrow A_{i+j}$, for $0 \leq i, j, i+j \leq 1$ and a map $\mu_3 : A_0 \otimes A_0 \otimes A_0 \rightarrow A_1$ satisfying the followings: for any $a, b, c, e \in A_0$ and $m, n \in A_1$,

(a) $d\mu_2(a, m) = \mu_2(a, dm),$

(b) $d\mu_2(m, a) = \mu_2(dm, a),$

(c) $\mu_2(dm, n) = \mu_2(m, dn),$

(d) $d\mu_3(a, b, c) = \mu_2(\mu_2(a, b), c) - \mu_2(a, \mu_2(b, c)),$

(e) $\mu_3(a, b, dm) = \mu_2(\mu_2(a, b), m) - \mu_2(a, \mu_2(b, m)),$

(e) $\mu_3(a, dm, c) = \mu_2(\mu_2(a, m), c) - \mu_2(a, \mu_2(m, c)),$

(f) $\mu_3(\mu_2(a, b), c, e) - \mu_3(a, \mu_2(b, c), e) + \mu_3(a, b, \mu_2(c, e)) + \mu_3(a, b, c, e) = \mu_2(\mu_2(a, b), c, e) + \mu_2(a, \mu_2(b, c), e).$

A 2-term $A_\infty$-algebra as above may be denoted by $(B_1 \overset{d}{\rightarrow} B_0, \mu_2, \mu_3)$. When $A_1 = 0$, one simply get an associative algebra structure on $A_0$ with the multiplication given by $\mu_2 : A_0 \otimes A_0 \rightarrow A_0$.

A 2-term $A_\infty$-algebra $(B_1 \overset{d}{\rightarrow} B_0, \mu_2, \mu_3)$ is said to be skeletal if the differential $d = 0$. Skeletal algebras are related to Hochschild 3-cocycles of associative algebras. There is a one-to-one correspondence between skeletal algebras and triples $(A, M, \theta)$, where $A$ is an associative algebra, $M$ is an $A$-bimodule and $\theta$ is a Hochschild 3-cocycle of $A$ with coefficients in $M$ [9]. More precisely, let $(B_1 \overset{d}{\rightarrow} B_0, \mu_2, \mu_3)$ be a skeletal algebra. Then $(A_0, \mu_2)$ is an associative algebra, $A_1$ is an $A_0$-bimodule by $l(a, m) = \mu_2(a, m)$ and $r(m, a) = \mu_2(m, a)$; the map $\mu_3 : A_0 \otimes A_0 \otimes A_0 \rightarrow A_1$ defines a 3-cocycle on $A_0$ with coefficients in $A_1$.

**Definition 12.** Let $(A_1 \overset{d}{\rightarrow} A_0, \mu_2, \mu_3)$ and $(A'_1 \overset{d'}{\rightarrow} A'_0, \mu'_2, \mu'_3)$ be 2-term $A_\infty$-algebras. A morphism between them consists of a chain map $f : A \rightarrow A'$ (which consists of linear maps $f_0 : A_0 \rightarrow A'_0$ and $f_1 : A_1 \rightarrow A'_1$ with $f_0 \circ d = d' \circ f_1$) and a bilinear map $f_2 : A_0 \otimes A_0 \rightarrow A'_1$ such that for any $a, b, c \in A_0$ and $m, n \in A_1$, the following conditions hold

(a) $d'f_2(a, b) = f_0(\mu_2(a, b)) - \mu'_2(f_0(a), f_0(b)),$

(b) $f_2(a, dm) = f_1(\mu_2(a, m)) - \mu'_2(f_0(a), f_1(m)),$

(c) $f_2(dm, a) = f_1(\mu_2(m, a)) - \mu'_2(f_1(m), f_0(a)),$

(d) $f_2(\mu_2(a, b), c) - f_2(a, \mu_2(b, c)) - \mu'_2(f_2(a, b), f_0(c)) + \mu'_2(f_0(a), f_2(b, c))$

$= f_1(\mu_3(a, b, c)) - \mu'_3(f_0(a), f_0(b), f_0(c)).$

We denote the category of 2-term $A_\infty$-algebras and morphisms between them by $2A_\infty$.

**Definition 13.** Let $(A_1 \overset{d}{\rightarrow} A_0, \mu_2, \mu_3)$ be a 2-term $A_\infty$-algebra. A homotopy derivation of degree 0 on it consists of a chain map $\theta : A \rightarrow A$ (which consists of linear maps $\theta_i : A_i \rightarrow A_i$, for $i = 0, 1$ satisfying $\theta_0 \circ d = d \circ \theta_1$) and $\theta_2 : A_0 \otimes A_0 \rightarrow A_1$ satisfying the followings: for any $a, b, c \in A_0$ and $m, n \in A_1$,

(a) $d \circ \theta_2(a, b) = \mu_2(\theta_0 a, b) + \mu_2(a, \theta_0 b) - \theta_0(\mu_2(a, b)),$

(b) $\theta_2(a, dm) = \mu_2(\theta_0 a, m) + \mu_2(a, \theta_1 m) - \theta_1(\mu_2(a, m)),$

(c) $\theta_2(dm, a) = \mu_2(\theta_1 m, a) + \mu_2(m, \theta_0 a) - \theta_1(\mu_2(m, a)).$
(d) \( \theta_1 \circ \mu_3(a, b, c) = \theta_2(a, \mu_2(b, c)) - \theta_2(\mu_2(a, b), c) + \mu_2(a, \theta_2(b, c)) - \mu_2(\theta_2(a, b), c) + \mu_3(\theta_0a, b, c) + \\
\mu_3(a, \theta_0b, c) + \mu_3(a, b, \theta_0c). \)

We call a 2-term \( A_\infty \)-algebra with a homotopy derivation a 2-term \( \text{AssDer}_\infty \)-pair. We denote such a pair by \((A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)\). An \( \text{AssDer}_\infty \)-pair is said to be skeletal if the underlying 2-term \( A_\infty \)-algebra is skeletal, i.e. \( d = 0 \).

**Proposition 14.** There is a one-to-one correspondence between skeletal \( \text{AssDer}_\infty \)-pairs and triples \((A, \phi_A), (M, \phi_M), (\theta, \overline{\theta})\), where \((A, \phi_A)\) is an \( \text{AssDer} \) pair, \((M, \phi_M)\) is a representation and \((\theta, \overline{\theta}) \in C^3_{\text{AssDer}}(A, M)\) is a 3-cocycle in the cohomology of the \( \text{AssDer} \) pair with coefficients in \((M, \phi_M)\).

**Proof.** Let \((A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)\) be a skeletal \( \text{AssDer}_\infty \)-pair. Then it follows from Definition 13(a) that \( \theta_0 \) is a derivation for the associative algebra \((A_0, \theta_0)\). Moreover, the conditions (b) and (c) says that \((A_1, \theta_1)\) is a representation of the \( \text{AssDer} \) pair \((A_0, \theta_0)\). Finally, the condition (d) implies that \( \delta_\text{Hoch} \theta_2 + \delta \mu_3 = 0 \). Therefore, \((\mu_3, - \theta_2) \in C^3_{\text{AssDer}}(A_0, A_1)\) is a 3-cocycle in the cohomology of the \( \text{AssDer} \) pair with coefficients in \((A_1, \theta_1)\).

Conversely, let \((\mu_3, \theta_2) \in C^3_{\text{AssDer}}(A_0, A_1)\) be such a triple. Then it can be easily verify that \((M \xrightarrow{d} A, \mu_2 = (\mu_A, l_r), \theta, \phi_A, \phi_M, - \overline{\theta})\) is a skeletal \( \text{AssDer}_\infty \)-pair. The above correspondences are inverses to each other. \(\square\)

**Definition 14.** A 2-term \( \text{AssDer}_\infty \)-pair \((A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)\) is called strict if \( \mu_3 = 0 \) and \( \theta_2 = 0 \).

**Example 2.** Let \((A, \phi_A)\) be an \( \text{AssDer} \) pair. Take \( A_0 = A_1 = A, d = \text{id}, \mu_2 = \mu \) (the associative multiplication on \( A \)), \( \theta_0 = \theta_1 = \phi_A \). Then \((A_1 \xrightarrow{d} A_0, \mu_2, \mu_3 = 0, \theta_0, \theta_1, \theta_2 = 0)\) is strict \( \text{AssDer}_\infty \)-pair.

Next, we introduce crossed module of \( \text{AssDer} \) pairs and show that strict \( \text{AssDer}_\infty \)-pairs correspond to crossed module of \( \text{AssDer} \) pairs.

**Definition 15.** A crossed module of \( \text{AssDer} \) pairs consist of a tuple \((A, \phi_A), (B, \phi_B), d, \phi)\) in which \((A, \phi_A), (B, \phi_B)\) are both \( \text{AssDer} \) pairs, \( d : A \rightarrow B \) is a morphism of \( \text{AssDer} \) pairs and

\[
\phi : B \otimes A \rightarrow A \quad \phi : A \otimes B \rightarrow A
\]
defines an \( \text{AssDer} \) pair bimodule on \((A, \phi_A)\) satisfying the following conditions: for all \( b \in B \) and \( m, n \in A \),

(i) \( dt(\phi(b, m)) = \mu_B(b, dt(m)), \quad dt(\phi(m, b)) = \mu_B(dt(m), b), \)

(ii) \( \phi(dt(m), n) = \mu_A(m, n), \quad \phi(m, dt(n)) = \mu_A(m, n), \)

(iii) \( \phi(b, \mu_A(m, n)) = \mu_A(\phi(b, m), n), \quad \phi(\mu_A(m, n), b) = \mu_A(m, \phi(n, b)), \)

(iv) \( \phi_A(\phi(b, m)) = \phi(\phi_B(b), m) + \phi(b, \phi_A(m)), \quad \phi_A(\phi(m, b)) = \phi(\phi_A(m), b) + \phi(\phi_B(b)). \)

**Proposition 15.** There is a one-to-one correspondence between strict \( \text{AssDer}_\infty \)-pairs and crossed module of \( \text{AssDer} \) pairs.

**Proof.** It is already known that strict \( A_\infty \)-algebras are in one-to-one correspondence with crossed module of associative algebras [9]. More precisely, \((A_1 \xrightarrow{d} A_0, \mu_2, \mu_3 = 0)\) is a strict \( A_\infty \)-algebra if and only if \((A_1, A_0, d, \mu_2)\) is a crossed module of associative algebras. Note that the associative products
on $A_1$ and $A_0$ are respectively given by $\mu_{A_1}(m, n) := \mu_2(dm, n) = \mu_2(m, dn)$ and $\mu_{A_0}(a, b) = \mu_2(a, b)$, for $m, n \in A_1$ and $a, b \in A_0$. It follows from (a) and (b) of Definition 13 that $\theta_1$ is a derivation on $A_1$ and $\theta_0$ is a derivation on $A_0$. Hence $(A_1, \theta_1)$ and $(A_0, \theta_0)$ are AssDer pairs. Since $\theta_0 \circ d = d \circ \theta_1$, we have $dt = d : A_1 \to A_0$ is a morphism of AssDer pairs. Moreover, the conditions (b) and (c) of Definition 13 are equivalent to the last condition of Definition 15.

The crossed module corresponding to the strict AssDer$_\infty$-pair of Example 2 is given by $((A, \phi_A), (A, \phi_A), \text{id}, \mu_A)$.

**Definition 16.** Let $(A_1 \xrightarrow{d'} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$ and $(A'_1 \xrightarrow{d'} A'_0, \mu'_2, \mu'_3, \theta'_0, \theta'_1, \theta'_2)$ be two 2-term AssDer$_\infty$-pairs. A morphism between them consists of a morphism $(f_0, f_1, f_2)$ between the underlying 2-term $A_\infty$-algebra together with a map $\mathcal{B} : A_0 \to A'_1$ such that the following conditions hold:

(i) $\theta'_0(f_0(a)) - f_0(\theta_0(a)) = d'(\mathcal{B}(a))$,

(ii) $\theta'_1(f_1(m)) - f_1(\theta_1(m)) = \mathcal{B}(dm)$,

(iii) $f_1(\theta_2(a, b)) + f_2(\theta_0(a, b)) + f_2(a, \theta_0(b)) - \theta'_1(f_2(a, b)) - \theta'_2(f_0(a), f_0(b))$

$= \mu'_2(\mathcal{B}a, f_0(b)) + \mu'_2(f_0(a), \mathcal{B}b) - \mathcal{B}(f_2(a, b))$.

Let $A = (A_1 \xrightarrow{d} A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$ and $A' = (A'_1 \xrightarrow{d'} A'_0, \mu'_2, \mu'_3, \theta'_0, \theta'_1, \theta'_2)$ be two 2-term AssDer$_\infty$-pairs and $f = (f_0, f_1, f_2, \mathcal{B})$ be a morphism between them. Let $A'' = (A''_1 \xrightarrow{d''} A''_0, \mu''_2, \mu''_3, \theta''_0, \theta''_1, \theta''_2)$ be another 2-term AssDer$_\infty$-pair and $g = (g_0, g_1, g_2, \mathcal{C})$ be a morphism from $A'$ to $A''$. Their composition is a morphism $g \circ f : A \to A''$ of AssDer$_\infty$-pairs whose components are given by $(g \circ f)_0 = g_0 \circ f_0$, $(g \circ f)_1 = g_1 \circ f_1$, and

$$(g \circ f)_2(a, b) = g_2(f_0(a), f_0(b)) + g_1(f_2(a, b))$$

$\mathcal{D} = g_1 \circ \mathcal{B} + \mathcal{C} \circ f_0 : A_0 \to A''_1$.

For any 2-term AssDer$_\infty$-pair $A$, the identity morphism $\text{id}_A$ is given by the identity chain map $A \to A$ together with $(\text{id}_A)_2 = 0$, $(\text{id}_A)_3 = 0$ and $\mathcal{B} = 0$. The collection of 2-term AssDer$_\infty$-pairs and morphisms between them form a category. We denote this category by $\text{2AssDer}_\infty$.

7. **Categorification of AssDer pairs**

Categorification of Lie algebras was first studied by Baez and Crans [4]. The categorified Lie algebras are called Lie 2-algebras and they are related to 2-term homotopy Lie algebras ($L_\infty$-algebras).

In this section, we study categorification of AssDer pairs, which we call AssDer 2-pair. We show that the category of AssDer 2-pairs and the category $\text{2AssDer}_\infty$ are equivalent.

A 2-vector space $C$ is a category with vector space of objects $C_0$ and the vector space of arrows $C_1$ such that all structure maps in the category $C$ are linear. A morphism of 2-vector spaces is a functor $F = (F_0, F_1)$ which is linear in the space of objects and arrows. We denote the category of 2-vector spaces by $\text{2Vect}$. Given a 2-vector space $C = (C_1 \rightrightarrows C_0)$, we have a 2-term complex $\ker(s) \to C_0$. A morphism between 2-vector spaces induces a morphism between 2-term complexes. Conversely, any 2-term complex $A_1 \xrightarrow{d} A_0$ gives rise to a 2-vector space $V = (A_0 \oplus A_1 \rightrightarrows A_0)$ in which the set of objects is $A_0$ and the set of morphisms is $A_0 \oplus A_1$. The structure maps are given by $s(a, m) = a$, $t(a, m) = a + dm$ and $i(a) = (a, 0)$. A morphism between 2-term chain complexes induces a morphism between corresponding 2-vector spaces. We denote the category of 2-term complexes of real vector spaces by $\text{2Term}$. Then there is an equivalence of categories $\text{2Term} \simeq \text{2Vect}$.

**Definition 17.** An associative 2-algebra is a 2-vector space $C$ equipped with a bilinear functor $\mu : C \otimes C \to C$ and a trilinear natural isomorphism, called the associator

$$\mathcal{A}_{\xi, \eta, \zeta} : \mu(\mu(\xi, \eta), \zeta) \to \mu(\xi, \mu(\eta, \zeta))$$
satisfying the following identity represented by a pentagon.

\[
\begin{array}{c}
\mu(\mu(\mu(\xi,\eta),\zeta),\lambda) \\
A_{\xi,\eta,\zeta} \\
\mu(\mu(\xi,\mu(\eta,\zeta)),\lambda) \\
A_{\xi,\mu(\eta,\zeta),\lambda} \\
\mu(\xi,\mu(\mu(\eta,\zeta),\lambda)) \\
A_{\xi,\mu(\eta,\zeta),\lambda} \\
\mu(\mu(\xi,\eta),\mu(\zeta,\lambda)) \\
A_{\mu(\xi,\eta),\zeta,\lambda} \\
\mu(\xi,\mu(\xi,\eta,\zeta,\lambda)).
\end{array}
\]

**Definition 18.** A morphism between associative 2-algebras \((C,\mu,A)\) and \((C',\mu',A')\) consists of a functor \(F = (F_0, F_1)\) from the underlying 2-vector space \(C\) to \(C'\) and a bilinear natural transformation \(\Phi\) such that the following diagram commutes

\[
F_2(\xi,\eta) : \mu'(F_0(\xi),F_0(\eta)) \to F_0(\mu(\xi,\eta))
\]

such that the following diagram commutes

\[
\begin{array}{c}
\mu'(\mu'(F_0(\xi),F_0(\eta)),F_0(\zeta)) \\
A'_{F_0(\xi),F_0(\eta),F_0(\zeta)} \\
\mu'(F_0(\xi),\mu'(F_0(\eta),F_0(\zeta))) \\
F_2(\eta,\zeta) \\
\mu'(F_0(\xi),F_0(\mu(\eta,\zeta))) \\
F_2(\xi,\mu(\eta,\zeta)) \\
F_0(\mu(\xi,\eta),\zeta) \\
F_0(\mu(\xi,\mu(\eta,\zeta))).
\end{array}
\]

Composition of two associative 2-algebra morphisms is again a associative 2-algebra morphism. More precisely, let \(C, C'\) and \(C''\) be three associative 2-algebras and \(F : C \to C', G : C' \to C''\) be associative 2-algebra morphisms. Then their composition \(G \circ F : C \to C''\) is an associative 2-algebra morphism given by \((G \circ F)_0 = G_0 \circ F_0\), \((G \circ F)_1 = G_1 \circ F_1\) and \((G \circ F)_2\) is given by the composition

\[
\mu''(G_0 \circ F_0(\xi),G_0 \circ F_0(\eta)) \Rightarrow G_0(\mu'(F_0(\xi),F_0(\eta))) \Rightarrow (G_0 \circ F_0)(\mu(\xi,\eta)).
\]

Finally, for any associative 2-algebra \(C\), the identity morphism \(\text{id}_C : C \to C\) is given by the identity functor as its linear functor and the identity natural transformation as \((\text{id}_C)_2\). We denote the category of associative 2-algebras and morphisms between them by \(\text{Ass}_2\).

**Definition 19.** Let \((C,\mu,A)\) be an associative 2-algebra. A 2-derivation on it consists of a linear functor \(D : C \to C\) and a bilinear natural isomorphism, called the derivator

\[
D_{a,b} : D(\mu(a,b)) \to \mu(Da,b) + \mu(a,Db)
\]
For any $A$ defined on the 2-vector space $T$ of $A$, satisfying the following

\begin{align*}
    (a \cdot b) \cdot D(c) + D(a \cdot b) &\cdot c \\
    a \cdot D(b \cdot c) + D(a) \cdot (b \cdot c)
\end{align*}

\begin{align*}
    D((a \cdot b) \cdot c) \\
    (a \cdot b) \cdot D(c) + D(a) \cdot (b \cdot c)
\end{align*}

In the above diagram, we use the notation for the bifunctor $\mu$ as $\mu(a, b) = a \cdot b$. We call an associative 2-algebra together with a 2-derivation by an $AssDer$-pair.

**Definition 20.** Let $(C, \mu, A, D, D)$ and $(C', \mu', A', D', D')$ be two $AssDer$-pairs. A morphism between them consists of an associative 2-algebra morphism $(F = (F_0, F_1), F_2)$ and a natural isomorphism

$$\Phi_a : D' \circ F_0(a) \to F_0 \circ D(a)$$

such that the following diagram commute

\begin{align*}
    D'(F_0(a) \cdot F_0(b)) &\xrightarrow{\Phi} F_0(D(a) \cdot b + a \cdot D(b)) \\
    D'(F_0(a) \cdot F_0(b)) &\xrightarrow{\Phi} F_0(D(a) \cdot b + a \cdot D(b))
\end{align*}

Here we use the bifunctors $\mu$ and $\mu'$ as $\cdot$ and $'$ respectively. We denote the category of $AssDer$-pairs together with morphisms between them by $AssDer$.$\Rightarrow$.

It is known that the categories $2A_{\infty}$ and $Ass2$ are equivalent. See for example [9]. A functor $T : 2A_{\infty} \to Ass2$ is given as follows.

Let $A = (A_1 \xrightarrow{d} A_0, \mu_2, \mu_3)$ be a 2-term $A_{\infty}$-algebra. The corresponding associative 2-algebra is defined on the 2-vector space $A_0 \oplus A_1 \Rightarrow A_0$. The bifunctor $\mu$ and the associator $A$ is given by

\begin{align*}
    \mu((a, m), (b, n)) &= (\mu_2(a, b), \mu_2(a, n) + \mu_2(m, b) + \mu_2(dm, n)), \\
    A_{a,b,c} &= ((ab)c, \mu_3(a, b, c)).
\end{align*}

For any $A_{\infty}$-algebra morphism $(f_0, f_1, f_2)$ from $A$ to $A'$, the associative 2-algebra morphism from
$T(A)$ to $T(A')$ is given by

$$F_0 = f_0 \quad F_1 = f_1 \quad \text{and} \quad F_2(a, b) = (\mu'(f_0(a), f_0(b)), f_2(a, b)).$$

On the other hand, a functor $S : \text{Ass} \to 2\text{A}_{\infty}$ is given as follows. Given an associative 2-algebra $C = (C_1 \rightrightarrows C_0, \mu, A)$, the corresponding 2-term $A_{\infty}$-algebra is defined on the complex $A_1 = \ker d_{|\ker} \to C_0 = A_0$. Define $\mu_2 : A_i \otimes A_j \to A_{i+j}$ and $\mu_3 : A_0 \otimes A_0 \otimes A_0 \to A_1$ by

$$\mu_2(a, b) = \mu(a, b), \quad \mu_2(a, m) = \mu(i(a), m), \quad \mu_2(m, a) = \mu_2(m, i(a)), \quad \mu_2(m, n) = 0,$$

and $\mu_3(a, b, c) = A_{a,b,c} - i(s(A_{a,b,c}))$. 

For any associative 2-algebra morphism $(F_0, F_1, F_2) : C \to C'$, the corresponding $A_{\infty}$-algebra morphism from $S(C)$ to $S(C')$ is given by

$$f_0 = F_0 \quad f_1 = F_1|_{\ker} \quad f_2(a, b) = F_2(a, b) - i(sF_2(a, b)).$$

It was shown in [9] that these two functors provide the equivalence between $2\text{A}_{\infty}$ and $\text{Ass}$. This equivalence can be extended to respective categories equipped with derivations. More precisely, we have the following.

**Theorem 10.** The categories $2\text{AssDer}_{\infty}$ and $\text{AssDer}_2$ are equivalent.

**Proof.** Given a 2-term $\text{AssDer}_{\infty}$-pair $(A_1 \rightrightarrows A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$, consider the corresponding associative 2-algebra $T(A)$. A functor $D : T(A) \to T(A)$ and the derivator $\mathcal{D}$ is given by

$$D((a, m)) = (\theta_0(a), \theta_1(m)) \quad \mathcal{D}_{a,b} = (ab, \theta_2(a, b)).$$

It can be checked that if $(f_0, f_1, f_2, B)$ is a morphism of $\text{AssDer}_{\infty}$-pairs, then $(F_0, F_1, F_2, \Phi)$ is a morphism of corresponding $\text{AssDer}_2$-pairs where $\Phi(a) = B(a)$.

Conversely, for any $\text{AssDer}_2$-pair $(C, \mu, A, D, \mathcal{D})$, consider the 2-term $A_{\infty}$-algebra $S(C)$. A homotopy derivation on $S(C)$ is given by $\theta_0 = D(i(a)), \theta_1(m) = D|_{\ker}(m)$ and $\theta_2(a, b) = D_{a,b} - i(s'p(a, b))$. If $(f_0, f_1, f_2, \Phi)$ is a morphism of $\text{AssDer}_2$-pairs, then $(f_0, f_1, f_2, B)$ is a morphism between corresponding 2-term $\text{AssDer}_{\infty}$-pairs where $B(a) = \Phi(a)$.

Thus, it remains to prove that the composite $T \circ S$ is naturally isomorphic to the identity functor $1_{\text{AssDer}_2}$ and $S \circ T$ is naturally isomorphic to $1_{\text{AssDer}_{\infty}}$. For any $\text{AssDer}_2$-pair $(C, \mu, A, D, \mathcal{D})$, the $\text{AssDer}_2$-pair structure on $T \circ S(C)$ is defined on the 2-vector space $A_0 \oplus A_1 \cong A_0$, where $A_0 = C_0$ and $A_1 = \ker$. Define $\theta : T \circ S \to 1_{\text{AssDer}_2}$ by $\theta_C : T \circ S(C) \to 1_{\text{AssDer}_2}(C)$ with $(\theta_C)_0(a) = a, \quad (\theta_C)_1(a, m) = i(a) + m$. Then $\theta_C$ is an isomorphism of $\text{AssDer}_2$-pairs. It is also a natural isomorphism.

For any 2-term $\text{AssDer}_{\infty}$-pair $A = (A_1 \rightrightarrows A_0, \mu_2, \mu_3, \theta_0, \theta_1, \theta_2)$, the $\text{AssDer}_{\infty}$-pair structure on $S \circ T(A)$ is defined on the same complex $A_1 \rightrightarrows A_0$. In fact we get back the same 2-term $\text{AssDer}_{\infty}$-pair. Therefore, the natural isomorphism $\theta : S \circ T \to 1_{2\text{AssDer}_{\infty}}$ is given by the identity. Hence the proof.

**Conclusions.**

In this paper, we mainly concentrate on a pair of an associative algebra and a derivation on it. We call such a pair an AssDer pair. Among other things, we study extensions and deformations of an AssDer pair by extending the classical extensions and deformations of associative algebras. For this, we define a cohomology theory for AssDer pairs. This cohomology is based on the Hochschild cohomology of associative algebras.

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In [10] the author study deformation of multiplications in an operad which generalizes the deformation of associative algebras. As applications, the author formulates deformation of various Loday-type (e.g. dendriform, tridendriform, dialgebra, quadri) algebras. See [10,29] for cohomology of Loday-type algebras. Given an operad \( \mathcal{O} \) with a multiplication \( m \in \mathcal{O}(2) \) (i.e. \( m \) satisfies \( m \circ_1 m = m \circ_2 m \)), an element \( \phi \in \mathcal{O}(1) \) is called a derivation for \( m \) if \( \phi \) satisfies

\[
\phi \circ m = m \circ_1 \phi + m \circ_2 \phi.
\]

By the method of the present paper, one may study deformations of a pair \((m, \phi)\) where \( m \) is a multiplication on \( \mathcal{O} \) and \( \phi \) is a derivation for \( m \). Therefore, one may deduce deformations of Loday-type algebras equipped with derivations.

In [5] Balavoine studied deformations of algebras over quadratic operads. Derivations on an algebra over a quadratic operad are studied in [11,20]. Given a quadratic operad \( \mathcal{P} \), they construct a new operad \( \mathcal{DP} \) whose algebras are \( \mathcal{P} \)-algebras equipped with derivations. Therefore, deformations of algebras over \( \mathcal{DP} \) are deformations of \( \mathcal{P} \)-algebras with derivations. Motivated from this, in a subsequent paper, we aim to construct an explicit cohomology and deformation theory for \( \mathcal{P} \)-algebras equipped with derivations.

The results of the present paper can be dualized to study deformations of coalgebras with coderivations. Since an \( A_\infty \)-algebra can be described by a square-zero coderivation on the tensor coalgebra of a graded vector space, one can explore formal deformations of \( A_\infty \)-algebras and compare with the results of [12].

In [8] the authors consider the deformation problem of a Lie algebroid \( A \). Such deformation is governed by a (shifted) graded Lie algebra on the space of multiderivations on \( A \). An 1-cocycle of the corresponding complex is given by a Lie algebroid derivation on \( A \). A derivation of the underlying vector bundle \( A \) is a Lie algebroid derivation if it is also a derivation for the Lie bracket on \( A \). For any \( X \in \Gamma A \), the map \( \Phi_X : \Gamma A \rightarrow \Gamma A \) is a Lie algebroid derivation. Lie algebroid derivations are worth interesting as their flows give rise to Lie algebroid automorphisms. It would be interesting to extend the deformation theory of LieDer pair [27] in the context of Lie algebroid equipped with a Lie algebroid derivation.

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