Magnetocapillary Instability of Non–Conducting Liquid Jets
Revisited : Plateau versus Chandrasekhar

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March 22, 2022

Abstract

The magnetostatic and magnetocapillary instability problems of isothermal incompressible and
inviscid non–conducting liquid jets in a uniform magnetic field, is considered. The equivalence
between static and dynamic approaches at the onset of instability and cut–off wavelength is
shown. It is established that in the absence of electric currents the stability of permeable jets
can be changed by the magnetic field. A new dispersion relation for magnetocapillary instability
in such jets is derived. This relation differs from that given by Chandrasekhar. The existence
of critical magnetic field which stabilizes jets with finite susceptibility is established. It is shown
that the jet is stabilized by the field irrespective of its being para– or diamagnetic, but the extent
of stabilization is different.

Key words: Plateau problem, Magnetocapillary instability, Non–conducting liquid jet.
1 Introduction

Joseph Antoine Ferdinand Plateau (1801–1883) was a Belgian physicist who is best remembered in mathematics for Plateau problem. He wrote his seminal book \[1\] being already blind for the last 40 years of his life. The Plateau problem, in its brief formulation, is to find a surface of minimum area \(S\) given its boundary \(\partial \Omega\). The problem relates to the principle of minimum free energy at equilibrium. In this context it may be required to find a surface with isotropic tension \(\sigma\) which provides a minimum surface free energy \(\sigma S\). Rayleigh \[2\] gave a theoretical explanation for the instability of liquid cylinders that are longer than their circumference. Further generalization is called for if the excess free energy \(W\) of the cylinder comprises different types of energy that reflect a more complex structure of the liquid (e.g. elasticity \[3\] etc.) and well as its capacity to interact with external fields. The threshold of static instability, which is associated with the change of sign of \(W\), when the disturbed configuration becomes more preferable, is defined by

\[
W(kR, p) = 0 \rightarrow k_s = k_s(p),
\]

where \(k\) is a wave number assigned to a small surface disturbance, \(R\) is a radius of cylinder, and the parameter \(p\) stands for the effect of an external interaction, which contributes to the jet evolution, e.g. fluid polarization in the presence of external fields. Note that the static cut–off wave number \(k_s\) is related to the static cut–off wavelength \(\Lambda_s\) by \(\Lambda_s(p) = 2\pi/k_s\).

Plateau instability being a static problem has also a dynamic aspect. Consider Rayleigh instability in liquid jets of radius \(R\) and its corresponding dispersion equation \(s = s(kR, p)\). The latter determines the exponential evolution in time \(\Theta(r, t) \sim e^{st}\) of all hydrodynamic functions \(\Theta(r, t)\) of the jet, with the growth rate \(s\). Rayleigh’s theory of capillary instability in liquid jets states that the maximum of the dispersion function \(s(kR, p)\) which corresponds to the wave number \(k_{\text{max}}(p)\), gives rise to evolution of the largest capillary instability. The range of wave numbers \(k\) which contribute to the evolution of the instability is given by \(s > 0\). Thus, the threshold of instability follows as

\[
s(kR, p) = 0 \rightarrow k_d = k_d(p).
\]

Note that the dynamic cut–off wave number \(k_d\) is related to the dynamic cut–off wavelength \(\Lambda_d\) by \(\Lambda_d(p) = 2\pi/k_d\). The value \(\Lambda_d(0)\), which corresponds to free jet evolution is equal \(2\pi R\) according to Rayleigh \[2\]. If there exist a critical parameter \(p = p_{\text{cr}}\) such that for all \(p \geq p_{\text{cr}}\) the expression \(s(kR, p) \leq 0\) holds, then for \(p \geq p_{\text{cr}}\) the jet is stable for all wavelengths. The latter means that the liquid jet preserves its initial shape which is unaffected by small perturbations irrespective to the jet velocity. Thus, this conclusion must hold also in the limiting case of motionless fluid, i.e.

\[
\Lambda_d(p) \equiv \Lambda_s(p).
\]
This identity reflects a deep equivalence between the static approach (the threshold of static instability concerned with an excess free energy) and the dynamic approach (the bifurcation of the first non-trivial steady state of the inviscid hydrodynamic system). Thus, we come to Plateau problem for liquid cylinder with free surface in an external field.

The capillary instability of magnetizable liquid jets in the presence of an axial magnetic field constitutes a classical example. Chandrasekhar [4] treated the hydrodynamics of magnetic liquid jets and derived their dispersion relation. This relation reads particularly simple for superconducting $s_{sc}(\varpi)$ and non-conducting $s_{nc}(\varpi)$ liquids (see [4], p. 545, formula (165) and p. 549, formula (205))

$$s_{nc}^2(\varpi) = \frac{\sigma}{R^3 \rho} \left\{ \frac{\varpi I_1(\varpi)}{I_0(\varpi)} (1 - \varpi^2) - \left( \frac{H}{H_s} \right)^2 \frac{\varpi}{I_0(\varpi) K_1(\varpi)} \right\}, \quad s_{nc}^2(\varpi) = \frac{\sigma}{R^3 \rho} \frac{\varpi I_1(\varpi)}{I_0(\varpi)} (1 - \varpi^2), \quad (4)$$

where $\varpi = kR$ and $H_s = \sqrt{\sigma/\mu_0(1 + \chi)} R$ is designated by Chandrasekhar as characteristic field. $\chi$, $\sigma, \rho$ stand for magnetic susceptibility, isotropic surface tension and density of the liquid, respectively, and $\mu_0$ denotes the permeability of free space. $I_m(x)$ and $K_m(x)$ are the modified Bessel functions of order $m$ of the 1st and 2nd kind respectively. In the superconducting limit the relation (4) discloses the existence of critical magnetic field $H_{cr} = H_s/\sqrt{2}$ beyond which the jet is stable. In the non-conducting limit, this dispersion relation concides with that of Rayleigh [2] and ‘in this limit the magnetic field has no effect on the capillary instability as should, indeed, be the case’ (quotation from [4], Chap. 12, §112, p. 549).

The latter result seems rather unexpected. Indeed, the absence of the electric currents in the presence of static magnetic field does not exclude coupling between the magnetic dipole moments of the liquid and the external field. Since the linear problem sets no constraints on the initial velocity of the jet, the above observation must be also correct in statics (Plateau problem) when the magnetic cylinder is destabilized. We proceed to verify our observation, first for the case of Plateau instability (see Sections 2, 3, 4) and then move on to the dynamic case. In Section 5 we give an accurate solution of the magnetocapillary instability problem of isothermal, incompressible, inviscid and non-conducting jets, in the presence of a uniform magnetic field. We show that the dispersion relation differs from the one (4) found by Chandrasekhar [4]. The new relation accounts for the effect of magnetic fields and conforms with the solution of Plateau instability for magnetizable liquids. The reason for the discrepancy in Chandrasekhar treatment arises due to incorrect boundary conditions which he applied to the magnetic field.

\footnote{We quote from [4] (Chap. 12, §112, p. 547): ‘the field $\mathbf{h}$ must be continuous across the boundary’. The correct boundary conditions require continuity of the tangential component of the magnetic field and of the normal component of the magnetic induction (see equation (17) in Section 3 and equation (48) in Section 5.1 of the present paper).}
2 Free energy of liquid cylinder in the presence of magnetic field.

The Plateau problem of static instability of a non–conducting liquid cylinder which is subjected to a uniform magnetic field appears, at first glance, to be a simple generalization of its counterpart in the absence of external fields. However, deeper consideration shows that the presence of the field complicates considerably the physical picture and computational procedure. A fundamental question arises concerning the correct definition of the excess free energy $W$ which must be minimized via variation of the cylinder shape.

Consider an isothermal liquid cylinder in a uniform magnetic field $H_0$ that is applied in free space along the cylinder axis. The magnetic susceptibility $\chi$ of the cylinder is assumed isotropic, independent of magnetic field, and satisfying the thermodynamic condition $\chi > -1$ [5]. When the liquid cylinder is undisturbed the total free energy $F^0$ of the system is given by

$$F^0 = E_s^0 - \frac{\chi \mu_0 H_0^2}{2} \cdot \int_{\Omega^0_{cyl}} dv ,$$

where the integral represents the volume $\pi R^2 L$, enclosed by the area $\partial \Omega^0_{cyl}$ which is occupied by undisturbed cylinder. The term $E_s^0 = \sigma \int_{\partial \Omega^0_{cyl}} ds = 2\pi \sigma RL$ stands for the surface free energy of the undisturbed cylinder, where $R, L$ and $\sigma$ denote its radius, length and surface tension respectively. Deformation of the cylinder shape changes the field $H(r)$ over all space $\mathbb{R}^3$, i.e. in both the internal domain $\Omega_{cyl}$ and its complement (the exterior domain) $\mathbb{R}^3 \setminus \Omega_{cyl}$. Following Plateau, we assume conservation of the cylinder volume

$$\int_{\Omega^0_{cyl}} dv = \int_{\Omega_{cyl}} dv .$$

The total free energy $F$ of the disturbed cylinder takes the following form,

$$F = E_s - (1 + \chi) \frac{\mu_0}{2} \cdot \int_{\Omega_{cyl}} \text{in} H^2(r) dv - \frac{\mu_0}{2} \cdot \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \text{ex} H^2(r) dv + \frac{\mu_0}{2} \cdot \int_{\mathbb{R}^3} H_0^2 dv$$

where $\text{in} H(r)$ and $\text{ex} H(r)$ denote the internal and external magnetic fields, and $E_s = \sigma \int_{\partial \Omega_{cyl}} ds$. The excess free energy $W$ of the system is defined as,

$$W = F - F^0 .$$

From the mathematical standpoint, the variational problem for minimization of $W$, supplemented with constraint (6) for all smooth surfaces $\partial \Omega_{cyl}$, is known as the isoperimetric problem.

The cylinder instability can be studied assuming small perturbation in its shape. In this case the Plateau problem becomes solvable in closed form.
Let the extent of deformation be characterized by a length \( \zeta_0 \), such that \( \zeta_0/R = \epsilon \ll 1 \). Then the following approximation holds

\[
\int_{\partial \Omega_{cyl}} ds - \int_{\partial \Omega^0_{cyl}} ds \simeq A \epsilon^2 R L ,
\]

where \( A \) is a dimensionless parameter dependent on the deformed geometry.

Here, the fields \( \text{in} \mathbf{H}(r) \) and \( \text{ex} \mathbf{H}(r) \) which must satisfy Maxwell equations can be represented as small perturbations of \( \mathbf{H}_0 \),

\[
\text{in} \mathbf{H}(r) = \mathbf{H}_0 + \text{in} \mathbf{H}^1(r) \quad \text{and} \quad \text{ex} \mathbf{H}(r) = \mathbf{H}_0 + \text{ex} \mathbf{H}^1(r) ,
\]

where according to the assumption \( \epsilon \ll 1 \) the following approximations apply (see Section 3)

\[
\{ \text{in} \mathbf{H}^1_r, \text{ex} \mathbf{H}^1_r, \text{in} \mathbf{H}^1_z, \text{ex} \mathbf{H}^1_z \} = \{ \text{in} \mathbf{h}^1_r, \text{ex} \mathbf{h}^1_r, \text{in} \mathbf{h}^1_z, \text{ex} \mathbf{h}^1_z \} \times \epsilon \chi \mathbf{H}_0 .
\]

The dimensionless fields \( \text{in,ex} \mathbf{h}^1_{r,z}(r, z) \) are dependent on the coordinates. Inserting (11) into (7), and performing the integration, we evaluate \( W \) given in (8) as,

\[
W = A \sigma \epsilon^2 R L - \frac{\mu_0}{2} U ,
\]

where use was made of (5) and \( U \) is given by (see Appendix A)

\[
U = (1 + \chi) \int_{\Omega_{cyl}} \left\{ \left( \text{in} H^1_z \right)^2 + \left( \text{in} H^1_r \right)^2 \right\} dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left\{ \left( \text{ex} H^1_z \right)^2 + \left( \text{ex} H^1_r \right)^2 \right\} dv + 2H_0 \left( (1 + \chi) \int_{\Omega_{cyl}} \text{in} H^1_z dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \text{ex} H^1_z dv \right) .
\]

We specify the commonly used harmonic deformation of the cylinder as \( r(z) = R + \zeta_0 \cos k z \), where \( k = 2\pi/\Lambda \), \( \Lambda \) being the disturbance wavelength. By virtue of translational invariance of the problem

\[
\text{in,ex} \mathbf{h}^1_{r,z}(r, z + \Lambda) = \text{in,ex} \mathbf{h}^1_{r,z}(r, z)
\]

we set \( L = \Lambda \) and evaluate the free energy per unit wave length. For this type of deformation the parameter \( A \) is known [1],

\[
\sigma \int_{\partial \Omega_{cyl}} ds - 2\pi \sigma RL = \frac{\pi \zeta_0^2}{2L} L (\varpi^2 - 1) \quad \rightarrow \quad A = \frac{\pi}{2} (\varpi^2 - 1) , \quad \varpi = kR .
\]

Next we proceed to solve the boundary problem and get the distribution of the magnetic fields.

### 3 Boundary problem and its solution

The magnetostatics of the disturbed liquid cylinder is governed by Maxwell equations for the internal \( \text{in} \mathbf{H}(r) \) and external \( \text{ex} \mathbf{H}(r) \) magnetic fields

\[
\text{rot} \text{in} \mathbf{H} = \text{rot} \text{ex} \mathbf{H} = 0 , \quad \text{div} \text{in} \mathbf{B} = \text{div} \text{ex} \mathbf{B} = 0 ,
\]

where \( \mathbf{B} = \mu_0 \mathbf{H} \).
where \( \text{in} \mathbf{B} = \mu_0(1 + \chi)^{\text{in}} \mathbf{H} \) and \( \text{ex} \mathbf{B} = \mu_0^{\text{ex}} \mathbf{H} \) denote internal and external magnetic inductions, respectively. Equations (16) must be supplemented with boundary conditions (BC) at the interface \( r = R \),

\[
\langle \text{in} \mathbf{H}, \mathbf{t} \rangle = \langle \text{ex} \mathbf{H}, \mathbf{t} \rangle, \quad \langle \text{in} \mathbf{B}, \mathbf{e} \rangle = \langle \text{ex} \mathbf{B}, \mathbf{e} \rangle, \quad \langle \mathbf{e}, \mathbf{t} \rangle = 0, \quad (17)
\]

where \( \mathbf{t} \) and \( \mathbf{e} \) stand for tangential and normal unit vectors to the surface, respectively. Since the surface deformation is small, linearization can be applied,

\[
t_r = -e_z = \partial \zeta / \partial z, \quad t_z = e_r = \sqrt{1 - (\partial \zeta / \partial z)^2} \sim 1. \quad (18)
\]

A standard way to solve the problem is to introduce the magnetic potentials \( \Phi_{\text{in}}(r) \) and \( \Phi_{\text{ex}}(r) \) which are defined as \( \text{in} \mathbf{H}^1(r) = -\nabla \Phi_{\text{in}}, \text{ex} \mathbf{H}^1(r) = -\nabla \Phi_{\text{ex}} \), where \( |\nabla \Phi_{\text{in}}|, |\nabla \Phi_{\text{ex}}| \ll H_0 \). These potentials satisfy the first two equations in (16). The last two equations in (16) yield,

\[
\frac{\partial^2 \Phi_{\text{in}}}{\partial z^2} + \Delta_2 \Phi_{\text{in}} = 0, \quad \frac{\partial^2 \Phi_{\text{ex}}}{\partial z^2} + \Delta_2 \Phi_{\text{ex}} = 0, \quad \Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}, \quad (19)
\]

where \( \Delta_2 \) is the two-dimensional Laplacian. Reformulation of the BC (17) for \( \Phi_{\text{ex}}(r), \Phi_{\text{in}}(r) \) gives,

\[
\frac{\partial \Phi_{\text{ex}}}{\partial z} = \frac{\partial \Phi_{\text{in}}}{\partial z}, \quad \frac{\partial \Phi_{\text{ex}}}{\partial r} - (1 + \chi) \frac{\partial \Phi_{\text{in}}}{\partial r} = \chi H \frac{\partial \zeta}{\partial z}. \quad (20)
\]

Using \( \Phi_{\text{in}}(r, z) = \phi_{\text{in}}(r) \sin k z \) and \( \Phi_{\text{ex}}(r, z) = \phi_{\text{ex}}(r) \sin k z \) we find

\[
(\Delta_2 - k^2) \phi_{\text{ex}} = 0, \quad (\Delta_2 - k^2) \phi_{\text{in}} = 0 \quad (21)
\]

with BC at \( r = R \)

\[
\phi_{\text{ex}} = \phi_{\text{in}}, \quad (1 + \chi) \frac{\partial \phi_{\text{in}}}{\partial r} - \frac{\partial \phi_{\text{ex}}}{\partial r} = \chi H k \zeta_0. \quad (22)
\]

The solutions of (21), which satisfy BC (22) and are finite at \( r = 0 \) and \( r = \infty \), are obtained as,

\[
\phi_{\text{in}}(r) = \zeta_0 H_0 \frac{b(\varpi, \chi)}{\varpi} I_0(kr), \quad \phi_{\text{ex}}(r) = \zeta_0 H_0 \frac{c(\varpi, \chi)}{\varpi} K_0(kr), \quad (23)
\]

where

\[
b(\varpi, \chi) = \varpi^2 \frac{K_0(\varpi)}{T(\varpi, \chi)}, \quad c(\varpi, \chi) = \varpi^2 \frac{I_0(\varpi)}{T(\varpi, \chi)}, \quad T(\varpi, \chi) = 1 + \chi \varpi I_1(\varpi) K_0(\varpi). \quad (24)
\]

Recalling the definition (11) of the dimensionless fields \( \text{in,ex} h_{r,z}^1(r, z) \), the following final expressions are obtained,

\[
in h_{r}^1 = -b(\varpi, \chi) I_0(kr) \cos k z, \quad \text{in} h_{z}^1 = -b(\varpi, \chi) I_1(kr) \sin k z,
\]

\[
ex h_{r}^1 = -c(\varpi, \chi) K_0(kr) \cos k z, \quad ex h_{z}^1 = c(\varpi, \chi) K_1(kr) \sin k z. \quad (25)
\]

Bearing in mind that the function \( Q(\varpi) = \varpi I_1(\varpi) K_0(\varpi) \) is monotone growing, at the positive half axis and is bounded \( 0 \leq Q(\varpi) < 1/2 \), we conclude that the fields \( \text{in,ex} h_{r,z}^1(r, z) \) are free of singularities in the thermodynamically relevant region \( \chi > -1 \).
4 Plateau instability

In this Section we calculate via equation (11) the excess free energy $W$ in terms of the dimensionless fields $h_{z,r}^{1,2}$. Inserting (25) into (13) we get (see Appendix A)

$$W = \frac{\pi L \sigma R^2}{2} e^2 \cdot f(\omega, \chi, H_0), \quad f(\omega, \chi, H_0) = \omega^2 - 1 + \chi^2 \omega \frac{\mu_0 R H_0^2 I_0(\omega) K_0(\omega)}{T(\omega, \chi)},$$

where the dimensionless excess free energy $f(\omega, \chi, H_0)$ can be defined by introducing the characteristic field $H_{Ch}$

$$f(\omega, \chi, H_0) = \omega^2 - 1 + \chi^2 \omega \frac{I_0(\omega) K_0(\omega)}{T(\omega, \chi)} \left( \frac{H_0}{H_{Ch}} \right)^2, \quad H_{Ch} = \sqrt{\frac{\sigma}{\mu_0 \chi R}}.$$  

(27)

Notice that $H_{Ch}$ differs from $H_s$ introduced by Chandrasekhar in (4). Formula (27) serves for both paramagnetic ($\chi > 0$) and diamagnetic ($\chi < 0$) liquids, since the latter case does not lead to an imaginary expression in (27) due to the term $\chi^2$ in (26).

Consider the field $H_{cr}(\omega, \chi)$ satisfying $f(\omega, \chi, H_{cr}) = 0$ and call it critical field. Beyond this field, $H_0 \geq H_{cr}$, the cylinder is stable. The expression for the critical field is

$$H_{cr}(\omega, \chi) = \frac{H_{Ch}}{\omega} \sqrt{\frac{1 - \omega^2}{\chi} T(\omega, \chi) \frac{I_0(\omega) K_0(\omega)}{H_{Ch}}},$$

(28)

Figures 1, 2 show plots of $h_{cr} = H_{cr}/H_{Ch}$ for strong ($\chi \gg 1$) and weak ($|\chi| \ll 1$) magnetic liquids.

![Figure 1: A plot of $h_{cr}$ versus $k_s R$ for large positive susceptibilities: $\chi = 1, 10, 10^2, 10^3, 10^6$, from right to left, respectively.](image)

The corresponding asymptotics for $H_{cr}(\omega, \chi)$, in the case of weak $\chi \ll 1$ and strong $\chi \gg 1$ magnetic susceptibilities and in a long wave limit $\omega \to 0$, which are derived in Appendix B, are presented below

$$\omega \to 0, \quad |\chi| \ll 1, \quad \frac{H_{cr}(\omega, \chi)}{H_{Ch}} = \frac{1}{\omega \sqrt{-\chi \ln \omega}} \left( 1 - \frac{\chi}{2} \omega^2 \ln \omega \right),$$

(29)

$$\omega \to 0, \quad \chi \to \infty, \quad \frac{H_{cr}(\omega, \chi)}{H_{Ch}} = \frac{1}{\sqrt{2}} \left( 1 - \frac{9}{16} \omega^2 \right),$$

(30)

$$\omega \to 1, \quad \frac{H_{cr}(\omega, \chi)}{H_{Ch}} = B_1 \sqrt{B_2 + \frac{1}{\chi} \sqrt{1 - \omega^2}},$$

(31)
where \( B_1 = 1/\sqrt{I_0(1)K_0(1)} \simeq 1.3697, B_2 = I_1(1)K_0(1) \simeq 0.2379. \)

To the end of this Section we present physical arguments which justify our conclusion regarding stabilization of the permeable jets for both paramagnetic and diamagnetic liquids. The orientation of the dipole moments along the applied magnetic field in the paramagnetic liquid enforces the rigidity of the jet under small disturbances. In the case of diamagnetic liquid the increase of the jets’ stabilization is due to orientation of the dipole moments in the opposite direction which also enforces the rigidity of the jet. The similar increase of the rigidity, and its corresponding influence on the stabilization, was found recently \[3\] in the liquid crystalline jet due to elasticity of the media.

Although the critical fields for paramagnetic \( H_{cr}(\varpi, \chi) \) and diamagnetic \( H_{cr}(\varpi, -\chi) \) liquids are different the following universal relation holds,

\[
\frac{H_{cr}^2(\varpi, \chi)}{H_{Ch}^2} - \frac{H_{cr}^2(\varpi, -\chi)}{H_{Ch}^2} = 2 \left( 1 - \frac{\varpi^2}{\chi} \right) \frac{I_1(\varpi)}{\varpi I_0(\varpi)}. \tag{32}
\]

5 Hydrodynamics of Non–Conducting Jet in a Magnetic Field

Consider an isothermal, incompressible, inviscid and non–conducting jet in the presence of a magnetic field \( \mathbf{H}_0 \) applied along its \( z \)-axis. The deviation from initial values of the pressure is defined as \( P_{1in} = P_{in} - P_{0in} \), where \( P_{0in} \) is the unperturbed pressure within the cylindrical jet. The deviations of the internal and external magnetic fields are defined as \( \mathbf{H}_{1in} = \mathbf{H} - \mathbf{H}_0 \) and \( \mathbf{H}_{1ex} = \mathbf{H} - \mathbf{H}_0 \), respectively. The governing equations of magnetohydrodynamics which are given by

\[
\text{div} \mathbf{V}^{in} = 0, \quad \rho \frac{\partial V^{in}}{\partial t} = -\frac{\partial T^{in}}{\partial x_k} : \quad \text{in} T_{jk} = \left( P_{in} + \mu \frac{\mathbf{H}^2}{2} \right) \delta_{jk} - \mu \text{ in } H_j \text{ in } H_k, \tag{33}
\]

\[
\text{div} \mathbf{H}^{\alpha} = 0, \quad \text{rot} \mathbf{H}^{\alpha} = \gamma_{\alpha} (\mathbf{E}_{\alpha} + \mu_{\alpha} [\mathbf{V} \times \mathbf{H}]), \quad \alpha = \text{in}, \text{ex}. \tag{34}
\]
can be simplified considerably by applying Maxwell equations (34) for non–conducting media \((\gamma_{in} = \gamma_{ex} = 0)\) to Navier–Stokes equation (33). \(\gamma_{in}\) and \(\gamma_{ex}\) denote conductivities of the jet’s interior and exterior, respectively, \(T_{jk}\) stands for magnetic stress tensor, and \(V^{in}\) is local fluid velocity. Finally the magnetohydrodynamic problem is decoupled into the hydrodynamic and magnetostatic parts:

\[
\text{div} V^{in} = 0, \quad \frac{\partial V^{in}}{\partial t} = -\frac{1}{\rho} \text{grad} P^{in}, \quad \text{and} \quad \text{div} \alpha H = 0, \quad \text{rot} \alpha H = 0, \quad \alpha = \text{in, ex} . \quad (35)
\]

A standard way to solve the boundary problem (35) is to introduce the Stokes stream function \(\Psi (r,z,t)\) and magnetic potentials \(\Phi_{in}(r,z,t), \Phi_{ex}(r,z,t)\)

\[
V_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad V_z = \frac{1}{r} \frac{\partial \Psi}{\partial r}, \quad \alpha H = H_0 + \alpha H^1, \quad \alpha H^1 = -\nabla \Phi_\alpha, \quad \alpha = \text{in, ex} . \quad (36)
\]

This gives the following governing equations

\[
\frac{1}{\rho} \frac{\partial P^{in}}{\partial z} + \frac{1}{r} \frac{\partial^2 \Psi}{\partial r \partial t} = 0, \quad \frac{1}{\rho} \frac{\partial P^{in}}{\partial r} - \frac{1}{r} \frac{\partial^2 \Psi}{\partial z \partial t} = 0, \quad \left(\Delta_2 + \frac{\partial^2}{\partial z^2}\right) \Phi_\alpha = 0, \quad \alpha = \text{in, ex} . \quad (37)
\]

These equations must be supplemented by four boundary conditions which are derived in the next Section.

### 5.1 Boundary conditions

It is necessary to apply boundary conditions (17), which are imposed on \(H\) and \(B\), as well as those for the hydrodynamic variables. First, the velocity \(V_r\) must be compatible, at \(r = R\), with the assumed form of the deformed boundary \(\partial \zeta / \partial t\). Second, at the free surface of a liquid jet the jump in stress must be balanced by Laplace pressure [5],

\[
[T_{r z}]^{in}_{ex} e_z + [T_{r r}]^{in}_{ex} e_r = 2\sigma H e_r, \quad [T_{z z}]^{in}_{ex} e_z + [T_{z r}]^{in}_{ex} e_r = 2\sigma H e_z , \quad (38)
\]

where \([T_{jk}]^{in}_{ex} = T^{in}_{jk} - T^{ex}_{jk}\) and \(H\) is mean surface curvature decomposed as

\[
H = H_0 + H_1, \quad H_0 = \frac{1}{2R}, \quad H_1 = -\frac{1}{2} \left( \frac{\zeta}{R^2} + \frac{\partial^2 \zeta}{\partial z^2} \right) \propto \epsilon = \frac{\zeta_0}{R} . \quad (39)
\]

By virtue of (18) we get

\[
[T_{r r}]^{in}_{ex} - 2\sigma H = [T_{r z}]^{in}_{ex} \frac{\partial \zeta}{\partial z}, \quad [T_{z z}]^{in}_{ex} = ([T_{z z}]^{in}_{ex} - 2\sigma H) \frac{\partial \zeta}{\partial z}, \quad \text{or}
\]

\[
[T_{r r}]^{in}_{ex} - 2\sigma H = ([T_{z z}]^{in}_{ex} - 2\sigma H) \left( \frac{\partial \zeta}{\partial z} \right)^2, \quad [T_{z r}]^{in}_{ex} = ([T_{z z}]^{in}_{ex} - 2\sigma H) \frac{\partial \zeta}{\partial z} , \quad (40)
\]

where

\[
T^{in}_{rr} = P^{in} + \frac{\mu}{2} \left\{ (in H_z)^2 - (in H_r)^2 \right\}, \quad T^{in}_{zz} = P^{in} + \frac{\mu}{2} \left\{ (in H_z)^2 - (in H_z)^2 \right\}, \quad T^{in}_{rz} = -\mu \ーケ \text{in} H_r \text{in} H_z, \\
T^{ex}_{rr} = \frac{\mu_0}{2} \left\{ (ex H_z)^2 - (ex H_r)^2 \right\}, \quad T^{ex}_{zz} = \frac{\mu_0}{2} \left\{ (ex H_z)^2 - (ex H_z)^2 \right\}, \quad T^{ex}_{rz} = -\mu_0 \text{ex} H_r \text{ex} H_z.
\]

\[9\]
Recalling that \( \partial z / \partial z \propto \epsilon \) and

\[
\begin{align*}
(\text{in} H_z - H_0 &= \text{in} H_1^1 \propto \epsilon, \quad (\text{ex} H_z - H_0 = \text{ex} H_1^1 \propto \epsilon, \quad (\text{in} H_r = \text{in} H_1^1 \propto \epsilon, \quad (\text{ex} H_r = \text{ex} H_1^1 \propto \epsilon, \quad (\text{in} H_z)^2 = H_0^1 + 2H_0 \text{in} H_1^1 + O(\epsilon^2), \quad (\text{ex} H_z)^2 = H_0^2 + 2H_0 \text{ex} H_1^1 + O(\epsilon^2), \quad (\text{in} H_r)^2 = (\text{ex} H_r)^2 = O(\epsilon^2)
\end{align*}
\]

we get within the 1st order approximation in \( \epsilon \)

\[
[T_{rr}]_{\text{ex}}^{2 \sigma H} = P_{\text{ex}}^{\text{in}} - 2\sigma H + \left\{ \frac{\mu}{2} \left( (\text{in} H_z)^2 - \frac{\mu_0}{2} (\text{ex} H_z)^2 \right) + \left\{ \frac{\mu_0}{2} (\text{ex} H_r)^2 - \frac{\mu}{2} (\text{in} H_r)^2 \right\} = \right.
\]

\[
P_{\text{ex}}^{\text{in}} - 2\sigma H + \frac{\mu_0 \chi}{2} H_0^2 + H_0 \left( \mu \text{in} H_1^1 - \mu_0 \text{ex} H_1^1 \right) + O(\epsilon^2)
\]

\[
[T_{zz}]_{\text{ex}}^{2 \sigma H} = P_{\text{ex}}^{\text{in}} - 2\sigma H + \left\{ \frac{\mu}{2} \left( (\text{in} H_r)^2 - \frac{\mu_0}{2} (\text{ex} H_r)^2 \right) + \left\{ \frac{\mu_0}{2} (\text{ex} H_z)^2 - \frac{\mu}{2} (\text{in} H_z)^2 \right\} = \right.
\]

\[
P_{\text{ex}}^{\text{in}} - 2\sigma H - \frac{\mu_0 \chi}{2} H_0^2 - H_0 \left( \mu \text{in} H_1^1 - \mu_0 \text{ex} H_1^1 \right) + O(\epsilon^2)
\]

\[
[T_{rz}]_{\text{ex}} = \mu_0 \text{ex} H_r \text{ex} H_z - \mu \text{in} H_r \text{in} H_z = H_0 \left( \mu_0 \text{ex} H_1^1 - \mu \text{in} H_1^1 \right) + O(\epsilon^2).
\]

Assuming \( P_{\text{ex}}^{\text{in}} \propto \epsilon \) and combining the first equation in (40) with (41) we obtain

\[
P_{\text{ex}}^{\text{in}} - 2\sigma H + \frac{\mu_0 \chi}{2} H_0^2 + H_0 \left( \mu \text{in} H_1^1 - \mu_0 \text{ex} H_1^1 \right) = 0 \rightarrow \left\{ \begin{array}{l}
P_{\text{ex}}^{\text{in}} - 2\sigma H_0 + \frac{1}{2} \mu_0 \chi H_0^2 = 0, \\
\mu_0 \text{ex} H_r - \mu \text{in} H_r = \left( P_{\text{ex}}^{\text{in}} - 2\sigma H_0 - \frac{\mu_0 \chi}{2} H_0^2 \right) \frac{\partial \zeta}{\partial z}.
\end{array} \right.
\]

This gives the unperturbed pressure within the cylindrical jet as,

\[
P_{\text{ex}}^{\text{in}} = \frac{\sigma}{R} - \chi \frac{\mu_0 H_0^2}{2}.
\]

Combining the second boundary condition in (40) with (42), gives

\[
\left( \mu_0 \text{ex} H_r - \mu \text{in} H_r \right) H_0 = \left( P_{\text{ex}}^{\text{in}} - 2\sigma H_0 - \frac{\mu_0 \chi}{2} H_0^2 \right) \frac{\partial \zeta}{\partial z}.
\]

Both equations (44) and (45) lead to the conclusion

\[
\mu_0 \text{ex} H_r - \mu \text{in} H_r = -\mu_0 \chi H_0 \frac{\partial \zeta}{\partial z},
\]

that coincides with the second static boundary conditions in (17). Thus we arrive at

\[
P_{\text{ex}}^{\text{in}} + \sigma \left( \frac{\zeta}{R^2} + \frac{\partial^2 \zeta}{\partial z^2} \right) + H_0 \left( \mu \text{in} H_1^1 - \mu_0 \text{ex} H_1^1 \right) = 0, \quad V_r = \frac{\partial \zeta}{\partial \xi},
\]

\[
(1 + \chi) \text{in} H_r - \text{ex} H_r = \chi H_0 \frac{\partial \zeta}{\partial z}, \quad \text{in} H_1^1 = \text{ex} H_1^1,
\]

or using the notations of (36)

\[
\frac{\partial \Phi_{\text{ex}}}{\partial r} - (1 + \chi) \frac{\partial \Phi_{\text{in}}}{\partial r} = \chi H_0 \frac{\partial \zeta}{\partial z}, \quad \frac{\partial \Phi_{\text{ex}}}{\partial z} = \frac{\partial \Phi_{\text{in}}}{\partial z}.
\]
Assuming that an axisymmetrical disturbance, characterized by a wavelength \(2\pi/k\), increases exponentially in time with the growth rate \(s\), gives,

\[
\{\Phi_{\text{in}}, \Phi_{\text{ex}}, \Psi, \zeta, P_1^{\text{in}}\} = \{i \phi_{\text{in}}(r), i \phi_{\text{ex}}(r), i \psi(r), \zeta(r), p(r)\} \times e^{st+ikz}.
\] (51)

Consequently, the following boundary conditions hold

\[
p + \sigma \left( \frac{1}{R^2} - k^2 \right) \zeta + kH_0 (\mu \phi_{\text{ex}} - \mu_0 \phi_{\text{in}}) = 0, \quad s\zeta = k \frac{\psi}{r},
\]

\[
\frac{\partial \phi_{\text{ex}}}{\partial r} - (1 + \chi) \frac{\partial \phi_{\text{in}}}{\partial r} = k\chi H_0 \zeta, \quad \phi_{\text{ex}} = \phi_{\text{in}}.
\] (52)

The presence of the field–dependent term, in the first equation of (52), indicates that the coupling of hydrodynamics and magnetostatics in the boundary conditions must break the Rayleigh dispersion relation (4) for non–conducting jet.

### 5.2 Dispersion relation

Inserting (51) into (37) results in the following amplitude equations,

\[
\frac{1}{k} \frac{\partial p}{\partial r} + sp \frac{\psi}{r} = 0, \quad kp + sp \frac{1}{r} \frac{\partial \psi}{\partial r} = 0, \quad (\Delta_{2c} - k^2)\phi_{\text{in}} = 0, \quad (\Delta_{2c} - k^2)\phi_{\text{ex}} = 0.
\] (53)

The latter have fundamental solutions that are finite at \(r = 0\) and \(r = \infty\), as follows

\[
\psi(r) = A_1 kr I_1(\kappa r), \quad p(r) = -A_1 s\rho k I_0(\kappa r), \quad \phi_{\text{in}}(r) = A_2 I_0(\kappa r), \quad \phi_{\text{ex}}(r) = A_3 K_0(\kappa r),
\] (54)

where \(A_i\) are three indeterminate coefficients. Substituting (54) into (52) we get the following dispersion relation

\[
s^2(\varpi) = \frac{\sigma}{R^3 \rho} \frac{\varpi I_1(\varpi)}{I_0(\varpi)} (1 - \varpi^2) - \chi^2 \varpi^2 \mu_0 H_0^2 \frac{I_1(\varpi) K_0(\varpi)}{R^2 \rho} \frac{T(\varpi, \chi)}{T(\varpi, \chi)}. \] (55)

(see Figure 3). The last expression clearly differs from Chandrasekhar’s result (4). Moreover, (55) can be recasted as

\[
s_{nc}^2(\varpi) = -\frac{\sigma}{R^3 \rho} \frac{\varpi I_1(\varpi)}{I_0(\varpi)} f(\varpi, \chi, H_0),
\] (56)

where \(f(\varpi, \chi, H_0)\) can be recognized as the dimensionless excess free energy found in (27). Expression (56) confirms the equivalence (3) between the static and dynamic approaches in the problem of cut–off wavelength since both non–trivial zeroes, \(\varpi_d\) and \(\varpi_s\), of \(s_{nc}(\varpi_d)\) and \(f(\varpi_s, \chi, H_0)\), respectively, coincide.

It is noteworthy that in the limit \(\chi \to \infty\),

\[
s_{nc}^2(\varpi) \simeq \frac{\sigma}{R^3 \rho} \left\{ \frac{\varpi I_1(\varpi)}{I_0(\varpi)} (1 - \varpi^2) - \varpi^2 \left( \frac{H}{H_{Ch}} \right)^2 \right\},
\] (57)

which coincides with the dispersion relation (4) for a superconducting jet in the longwave limit \(\varpi \to 0\).
6 Conclusion

We have revised the theory of capillary instability of isothermal incompressible inviscid and non-conducting liquid jets in a uniform magnetic field which was treated by Chandrasekhar [4] about 50 years ago. The main result of [4] states that for non–conducting jets the field has no effect on their stability and the dispersion relation is given by Rayleigh’s theory for free jets. This statement contradicts the free energy approach (Plateau problem) in the framework of static instability of a magnetizable liquid cylinder in a uniform field.

We have found a new dispersion relation for magnetocapillary instability in such jets. The relation differs from that given by Chandrasekhar and agrees with the result for the cut–off wavelength obtained from the static approach. This reflects a deep equivalence between the static approach (the threshold of static instability which is concerned with an excess free energy) and the dynamic approach (the bifurcation of the first non–trivial steady state of the inviscid hydrodynamic system). In this context, the existence of critical magnetic field which stabilizes jets with finite susceptibility is established. It is shown that the jet is stabilized by the field irrespective of its being para– or diamagnetic, but the extent of stabilization is different.

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A Contribution of the magnetic field to free energy

Evaluate the contribution $\mu_0 U/2$ of the magnetic field inside $\Omega_{cyl}$ and outside $\mathbb{R}^3 \setminus \Omega_{cyl}$ of the disturbed liquid cylinder to the excess free energy $W$ (see equation (12))

$$U = \chi \left[ \int_{\Omega_{cyl}} \ln H^2(r) dv - \int_{\Omega_{cyl}^0} \ln H^2_0 dv \right] + \int_{\Omega_{cyl}} \ln H^2(r) dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \text{ex} \ln H^2(r) dv - \int_{\mathbb{R}^3} H^2_0 dv$$

$$= \chi \left[ \int_{\Omega_{cyl}} \left\{ (H_0 + \text{in} H_1^1)^2 + (\text{in} H_1^r)^2 \right\} dv - \int_{\Omega_{cyl}^0} H^2_0 dv \right] + \int_{\Omega_{cyl}} \left\{ (H_0 + \text{in} H_1^1)^2 + (\text{in} H_1^r)^2 \right\} dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left\{ (H_0 + \text{ex} H_1^1)^2 + (\text{ex} H_1^r)^2 \right\} dv - \int_{\mathbb{R}^3} H^2_0 dv$$

$$= \chi \left[ H^2_0 \left( \int_{\Omega_{cyl}} dv - \int_{\Omega_{cyl}^0} dv \right) + \int_{\Omega_{cyl}} \left\{ 2H_0 \text{in} H_1^1 + (\text{in} H_1^r)^2 \right\} dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left\{ 2H_0 \text{ex} H_1^1 + (\text{ex} H_1^r)^2 \right\} dv \right]$$

$$= (1 + \chi) \int_{\Omega_{cyl}} \text{in} H_1^1 dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \text{ex} H_1^1 dv$$

Hence, we conclude that

$$U = (C_1 \epsilon^2 \chi^2 + 2C_2 \epsilon \chi) H^2_0 ,$$

where

$$\frac{1}{\pi L} C_1 = \frac{1 + \chi}{\pi L} \int_{\Omega_{cyl}^0} \left\{ (\text{in} h_1^r)^2 + (\text{in} h_1^r)^2 \right\} dv + \frac{1}{\pi L} \int_{\mathbb{R}^3 \setminus \Omega_{cyl}^0} \left\{ (\text{ex} h_1^r)^2 + (\text{ex} h_1^r)^2 \right\} dv , \quad (A2)$$

$$\frac{1}{\pi L} C_2 = \frac{1 + \chi}{\pi L} \int_{\Omega_{cyl}} \text{in} h_1^1 dv + \frac{1}{\pi L} \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \text{ex} h_1^1 dv . \quad (A3)$$

Recall the expression (12) for $W$

$$W = \frac{\pi}{2} \left( \omega^2 - 1 \right) \sigma \epsilon^2 RL - (C_1 \epsilon^2 \chi^2 + 2C_2 \epsilon \chi) \frac{\mu_0 H^2_0}{2} = \frac{\pi L \sigma R}{2} \left[ \epsilon^2 (\pi^2 - 1) - \left( \frac{\epsilon^2 C_1}{\pi L} \chi^2 + 2\epsilon C_2 \frac{\chi^2}{\pi L} \right) \frac{\mu_0 H^2_0}{\sigma R} \right] . \quad (A4)$$

Making use of the integration

$$\int_0^u \left[ I_0^2(v) + I_1^2(v) \right] dv = u I_0(u) I_1(u) , \quad \int_u^\infty \left[ K_0^2(v) + K_1^2(v) \right] dv = u K_0(u) K_1(u) ,$$

$$\int_0^u I_0(v)dv = u I_1(u) , \quad \int_u^\infty K_0(v)dv = u K_1(u) ,$$
and inserting (25) into (A2) and (A3) gives,

\[
\frac{1}{\pi L} \int_{\Omega_{cyl}} \left\{ (\cos h_z)^2 + (\sin h_z)^2 \right\} dv = \frac{b^2}{\pi L} \cdot \frac{L}{2} \cdot 2\pi \int_{0}^{\infty} K^2_0(kr) (kr) d(kr)
\]

\[
= \frac{b^2 R^2}{2 \pi} \int_{0}^{\infty} I_0(\omega R) I_1(\omega) = \omega^2 R^2 \left( \frac{I_0(\omega) K_0(\omega)}{T(\omega)} \right)^2 I_1(\omega) \frac{(\omega)}{I_0(\omega)} \]

\[
\frac{1}{\pi L} \int_{\Omega_{cyl}} (\cos h_z)^2 + (\sin h_z)^2 dv = \frac{c^2}{\pi L} \cdot \frac{L}{2} \cdot 2\pi \int_{0}^{\infty} K^2_0(kr) (kr) d(kr)
\]

\[
= \frac{c^2 R^2}{2 \pi} \int_{0}^{\infty} I_0(\omega R) K_1(\omega) = \omega^2 R^2 \left( \frac{I_0(\omega) K_0(\omega)}{T(\omega)} \right)^2 K_1(\omega) \frac{(\omega)}{K_0(\omega)}
\]

Expanding \( I_1(x + \epsilon) \) and \( K_1(x + \epsilon) \) in the vicinity of \( x = 0 \), up to the first order in \( \epsilon \), gives

\[
I_1(x + \epsilon) = I_1(x) + (I_0(x) + I_2(x)) \frac{\epsilon}{2}, \quad K_1(x + \epsilon) = K_1(x) - (K_0(x) + K_2(x)) \frac{\epsilon}{2}.
\]

Finally we arrive at

\[
\frac{1}{\pi L} \int_{\Omega_{cyl}} \cos h_z dv = -\frac{2b}{L} \cdot \frac{R}{\omega} \int_{0}^{L} dz \cos k z \cdot \left\{ RI_1(\omega) + \zeta_0 \cos k z \left[ I_1(\omega) + \omega \frac{I_0(\omega) + I_2(\omega)}{2} \right] \right\}
\]

\[
= -\frac{2b}{L} \frac{R}{\omega} \zeta_0 \left[ I_1(\omega) + \omega \frac{I_0(\omega) + I_2(\omega)}{2} \right] \int_{0}^{L} dz \cos^2 k z \cdot \omega = -\omega b(\varpi, \chi) R^2 I_0(\varpi),
\]

\[
\frac{1}{\pi L} \int_{\Omega_{cyl}} \sin h_z dv = -\frac{2c}{L} \cdot \frac{R}{\omega} \int_{0}^{L} dz \cos k z \cdot \left\{ RK_1(\omega) + \zeta_0 \cos k z \left[ K_1(\omega) + \omega \frac{K_0(\omega) + K_2(\omega)}{2} \right] \right\}
\]

\[
= -\frac{2c}{L} \frac{R}{\omega} \zeta_0 \left[ K_1(\omega) - \omega \frac{K_0(\omega) + K_2(\omega)}{2} \right] \int_{0}^{L} dz \cos^2 k z = \omega c(\varpi, \chi) R^2 K_0(\varpi).
\]

Thus, we get

\[
\frac{1}{\pi L} C_1 = \omega^2 R^2 \left( \frac{I_0(\omega) K_0(\omega)}{T(\omega, \chi)} \right)^2 \left\{ (1 + \chi) \frac{I_1(\omega)}{I_0(\omega)} + \frac{K_1(\omega)}{K_0(\omega)} \right\} = \omega^2 R^2 \frac{I_0(\omega) K_0(\omega)}{T(\omega, \chi)}, \quad (A5)
\]

\[
\frac{1}{\pi L} C_2 = -\epsilon \omega^2 R^2 \left[ (1 + \chi) \frac{I_0(\omega) K_0(\omega)}{T(\omega, \chi)} - \frac{I_0(\omega) K_0(\omega)}{T(\omega, \chi)} \right] = -\epsilon \omega^2 R^2 \frac{I_0(\omega) K_0(\omega)}{T(\omega, \chi)}, \quad (A6)
\]

that consequently gives,

\[
W = \pi L \sigma R \frac{\epsilon^2}{2} \cdot f(\varpi, \chi), \quad f(\varpi, \chi) = \omega^2 - 1 - \left( \frac{C_1}{\pi L} \chi + 2 \frac{C_2}{\epsilon \pi L} \right) \frac{\varpi_0 H_0^2}{\sigma R} \cdot (A7)
\]

Insertion (A5) and (A6) in the latter expression, we obtain

\[
f(\varpi, \chi) = \omega^2 - 1 + \chi^2 \omega^2 \frac{\varpi_0 R H_0^2}{\sigma} \frac{I_0(\omega) K_0(\omega)}{T(\omega, \chi)} \cdot (A8)
\]
B  Asymptotics of expressions

Consider the expressions obtained in Appendix A and evaluate its asymptotics.

- \( \varpi \to 0 \), \( |\chi| \ll 1 \).

\[
I_1(\varpi) \sim \frac{\varpi}{2} \left( 1 + \frac{\varpi^2}{8} \right), \quad I_0(\varpi) \sim 1 + \frac{\varpi^2}{4}, \quad K_0(\varpi) \sim \beta(\varpi) \left( 1 + \frac{\varpi^2}{4} \right), \quad \beta(\varpi) = -\ln \frac{\gamma \varpi}{2},
\]

where \( \gamma = 0.577216 \) is Euler’s constant. The corresponding asymptotics for the dimensionless excess free energy \( f(\varpi, \chi) \) and the critical field \( H_{cr}(\varpi, \chi) \) read

\[
f(\varpi, \chi) = \varpi^2 - 1 + \chi \varpi^2 \beta(\varpi) \left( 1 - \frac{\chi}{2} \varpi^2 \beta(\varpi) \right) \left( \frac{H_0}{H_{Ch}} \right)^2, \quad (B1)
\]

\[
H_{cr}(\varpi, \chi) = \frac{H_{Ch}}{\varpi} \sqrt{\frac{1}{-\chi \ln \varpi} \left( 1 - \frac{\chi}{2} \varpi^2 \ln \varpi \right)}, \quad (B2)
\]

- \( \varpi \to 0 \), \( \chi \to \infty \).

\[
f(\varpi, \chi) \sim \varpi^2 - 1 + 2 \left( 1 + \frac{\varpi^2}{8} \right) \left( \frac{H_0}{H_{Ch}} \right)^2 \rightarrow \quad H_{cr}(\varpi, \chi) = \frac{H_{Ch}}{\sqrt{2}} \left( 1 - \frac{9}{16} \varpi^2 \right), \quad (B3)
\]

- \( \varpi \to 1 \)

\[
f(\varpi, \chi) \sim \varpi^2 - 1 + \frac{\chi I_0(1) K_0(1)}{1 + \chi I_1(1) K_0(1)} \left( \frac{H_0}{H_{Ch}} \right)^2, \quad H_{cr}(\varpi, \chi) = B_1 H_{Ch} \sqrt{B_2 + \frac{1}{\chi} \sqrt{1 - \varpi^2}}, \quad (B4)
\]

where \( B_1 = 1/\sqrt{I_0(1) K_0(1)} \simeq 1.3697, \quad B_2 = I_1(1) K_0(1) \simeq 0.2379. \)
strong paramagnetism
weak para- and diamagnetism