Particle-Hole Symmetry Breaking and the 5/2 Fractional Quantum Hall Effect

Hao Wang\textsuperscript{1}, D. N. Sheng\textsuperscript{1}, and F. D. M. Haldane\textsuperscript{2}

\textsuperscript{1}Department of Physics and Astronomy, California State University, Northridge, California 91330, USA
\textsuperscript{2}Department of Physics, Princeton University, New Jersey 08544, USA

We report on the study of the fractional quantum Hall effect at the filling factor 5/2 using exact diagonalization method with torus geometry. The particle-hole symmetry breaking effect is considered using an additional three-body interaction. Both Pfaffian and anti-Pfaffian states can be the ground state depending on the sign of the three-body interaction. The results of the low-energy spectrum, the wave function overlap, and the particle-hole parity evolution, have shown the clear evidence of a direct sharp transition (possibly first-order) from the Pfaffian to the anti-Pfaffian state at the Coulomb point. A quantum phase diagram is established, where one finds further transitions from the Pfaffian or anti-Pfaffian state to the nearby compressible phases induced by a change of the pseudopotential.

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The fractional quantum Hall effect (FQHE) at the filling factor $\nu = 5/2$ has recently drawn intensive attentions in theoretical and experimental studies\textsuperscript{1,2,3,4,5,6,7,8,9,10,11,12}. The Moore-Read Pfaffian (Pf) state has been proposed as a successful candidate to describe the 5/2 FQHE\textsuperscript{2,3,4}. The Pf state as a non-Abelian topological phase suggests a potential application towards the quantum computing\textsuperscript{9}. However, the Pf state breaks particle-hole (PH) symmetry and its PH conjugate state, known as the anti-Pfaffian (APf) state, has been suggested to be another candidate state\textsuperscript{10,11}. In the limit of vanishing Landau-level (LL) mixing, the Pf and APf model wave functions have the same energy for the 5/2 FQHE system with the pure Coulomb interaction as the system has the PH symmetry. Numerically the ground state (GS) of such a system appears as a superposition of the Pf and APf states\textsuperscript{4,5}. However, the PH symmetry of the Hamiltonian in real systems can be broken by many factors, such as LL mixing. This raises a new challenge regarding the nature of the GS realized in the experimental system\textsuperscript{10,11}, which can be the Pf, APf, or superposition of them. To address this issue, we add a PH nonsymmetric three-body (3b) interaction together with the Coulomb interaction as the model Hamiltonian to reexamine the nature of the GS and to study the quantum phase transitions (QPT) between distinct quantum phases.

In this letter, we investigate the low-energy states of the 5/2 FQHE system using Lanczos method for finite-size systems with up to $N_e = 14$ electrons. The GS of a Coulomb system is found to have a transition into the Pf or APf state with the turn-on of the 3b interaction. The Pf and APf states are robust in a range of the 3b interaction, insensitive to the detailed form of the 3b interaction as long as the PH symmetry is broken. The calculations on the GS wave function overlap, energy and PH parity evolutions show the strong evidence that the phase transition between the Pf and the APf state appears to be first-order, occurring exactly at the Coulomb point. Under an extra short-range pseudopotential\textsuperscript{13}, Pf and APf states can have transition to the stripe phase or composite fermion liquid (CFL) phase.

We consider a two-dimensional electron system under a perpendicular magnetic field. Periodic boundary conditions for magnetic translational operators are imposed with a quantized flux $N_\phi$ through a rectangular unit cell $L_1 \times L_2$. The magnetic length $\ell$ is taken as the unit of the length and the energy is in units of $e^2/4\pi\epsilon\ell$. To reduce the size of the Hilbert space, we carry out our calculation at every pseudomomentum $K = (K_1, K_2)$\textsuperscript{13}, where $K_1(K_2)$ is in unit of $2\pi/L_1(2\pi/L_2)$ and the even number of electrons is used for $N_e$. The magnetic field is assumed to be strong enough so that the spin degeneracy of the LLs is lifted\textsuperscript{3,4,8}. One can thus project the system Hamiltonian into the topmost, half filled, $N = 1$ LL\textsuperscript{8}. The projected Hamiltonian for the Coulomb interaction has the form:

$$H_c = \frac{2}{N_\phi} \sum_{i<j} \sum_{\mathbf{q}} e^{-q^2/2} e^{i\mathbf{q}(\mathbf{r}_i-\mathbf{r}_j)} \sum_{m=0}^\infty V_m L_m(q^2),$$

where $V_m$ is the Haldane’s pseudopotential of the Coulomb interaction at $N = 1$ LL and $L_m(x)$ is the Laguerre polynomial. The momentum $\mathbf{q}$ is taken discrete values suitable for the unit cell lattice. $\mathbf{r}_i$ is the guiding center coordinate of the $i$-th electron.

The Pf state on torus can be obtained as the zero-energy GS of a repulsive 3b potential given by \textsuperscript{4}:

$$H_3 = -\sum_{i<j<k} S_{i,j,k} \left[ \nabla_i^2 \nabla_j^2 \delta^2(\mathbf{r}_i - \mathbf{r}_j) \delta^2(\mathbf{r}_j - \mathbf{r}_k) \right],$$

where $S_{i,j,k}$ is a symmetrizer. Besides the center-of-mass degeneracy, the Pf state appears at three particular pseudomomenta of $(0, N_e/2), (N_e/2, 0), (N_e/2, N_e/2)$\textsuperscript{12} as degenerate ground states. To study the effect of the PH symmetry breaking, we set up a model Hamiltonian with the form $H = H_c + V_{3b} \cdot H_3$, where the parameter $V_{3b}$ can change its sign and magnitude.
Fig. 1(a) and Fig. 1(b) for a pure Coulomb system with superposition states of Pf and APf. By adjusting the aspect ratio of the unit cell at the aspect ratios of $L_1/L_2 = 0.99$, we obtain low-energy states at the pseudomomentum $(0,6)$ are nearly degenerate. The plots of $(6,0)$ and $(6,6)$, respectively. The insets in (c) and (d) are the zoom-in spectra for $V_{3b}$ region, the projections on the two states quickly decohere from the other excited states. The wave function overlaps with Pf and APf states demonstrate that these six states are closely related to the superposition states of Pf and APf. By adjusting the aspect ratio of the unit cell from $L_1/L_2 = 0.99$ to 0.97, we obtain low-energy states at the pseudomomentum $(0,6)$ change from well separated to nearly degenerate (see also the inset of Fig. 1(c) and (d)). This is consistent with the fact that the tunneling between Pf and APf in a finite-size Coulomb system is sensitive to the specific geometry of the system. The fine tuning of the aspect ratio reduces the finite-size effect by reducing the energy gap between the lowest two states at the particular pseudomomentum.

Fig. 1(c) exhibits the low-lying excitation spectrum as a function of $V_{3b}$ for the $N_e = 12$ model system with the additional 3b interaction $H_3$. The unit cell is the same as the one described in Fig. 1(b). At the Coulomb point ($V_{3b} = 0$), each GS in the GS triplet and its first excited state have the opposite PH parity. In the region of $V_{3b} > 0$, with $H_3$ strength getting stronger, the degeneracy of the GS triplet improves and the energy gap between the GS triplet and other states increases. In the region of $V_{3b} < 0$, with the 3b strength $|V_{3b}| < 0.1$ the near degeneracy of the GS triplet persists. With more negative $V_{3b}$, the energy width of the lowest triplet increases and these energy levels cross with higher energy states, indicating a phase transition induced by $-H_3$.

We further check if the obtained results depend on the precise form of the 3b interaction. Noting that $H_3$ has the PH symmetric and anti-symmetric properties and we extrapolate the PH anti-symmetric component of $H_3$, which has the form of $H_3^\ast = (H_3 - \tilde{H}_3)/2$ with $\tilde{H}_3$ as the PH conjugate of $H_3$, as the other type of additional 3b interactions. This way, $H = H_e + V_{3b} \cdot H_3^\ast$ has the generic form of both symmetric and anti-symmetric parts. The obtained low-energy spectrum is shown in Fig. 1(d). Due to the PH anti-symmetry of the $H_3^\ast$, the spectrum is symmetric around the Coulomb point. Clearly, the three lowest energy states are well separated from the excited states, indicating the establishing of Pf or APf states.

To explore the nature of the GS under the additional 3b interaction, in Fig. 2(a) we plot the wave function overlap (squared) between the GS triplet and a Pf or an APf state as a function of $V_{3b}$ for the $N_e = 12$ model system with the unit cell as in Fig. 1(b). In the positive $V_{3b}$ region, with the $H_3$ strength growing, the GS projection on the Pf state monotonously increases towards unitary while the projection on the APf state decreases and drops to a small value, indicating that the GS is in the same class as the Pf state. In the negative $V_{3b}$ region, with the strength of $-H_3$ increasing, the projection on the Pf state quickly decreases and tends to be zero. On the other hand, the GS projection on the APf state, up to the strength $|V_{3b}| \sim 0.1$, increases and remains at a finite value between 0.6 and 0.7, indicating that the GS is associated with the APf in this region. When the strength increases further, the GS projection on the APf state continuously drops and eventually vanishes. The results for the model system with $H_3^\ast$ and the same unit cell have been shown in Fig. 2(b). In the positive $V_{3b}$ region, the projections on the APf state tend to vanish as the 3b strength grows while the projections on the Pf state increase to some finite value between 0.6 and 0.7 with the strength up to 0.1, indicating the GS is associated with the Pf state. In the negative $V_{3b}$ region, the GS has large overlap with the APf phase. The above characteristics from the wave function overlap are in agreement with the PH symmetric and anti-symmetric parts.
The projection of the system GS on the Pf and APf states as a function of $V_{3b}$. The aspect ratio $L_1/L_2$ are 0.94, 0.97 and 0.70 for the system with $N_e = 10, 12$ and 14 electrons, respectively. The curves with solid (open) symbols stand for the projection on the Pf (APf) state. States of $N_e = 12$ system with the additional 3b interaction of (a) $H_3$ and (b) $H_3^*$ at pseudomomenta (0,6), (6,0) and (6,6) are represented by the curves with circle, square and triangle symbols, respectively. (c) States of $N_e = 10$ system with $H_3$ at pseudomomentum (5,5), $N_e = 12$ at (0,6) and $N_e = 14$ at (7,7) are represented by the curves with diamond, square and triangle symbols, respectively.

The low-lying energy spectrum and the GS wave function overlaps agree with the spectrum feature shown in Fig. 1(c) and (d). We also notice that the projection curves of the local GS at the pseudomomentum (0,6), which has the nearly degenerate first excited state, exhibit the sharpest transition in a small region of $V_{3b}$ crossing the Coulomb point, indicating the strongest QPT signal. In the following discussion we target such particular GS from the GS triplet to investigate the phase transition between the Pf and the APf state. In Fig. 2(c), the results of the wave function overlap for different sizes of the system with the additional 3b interaction $H_3$ have been demonstrates. For all systems considered, the Pf state dominates on the $V_{3b} > 0$ side while the APf states dominates on the $V_{3b} < 0$ side as long as the 3b interaction strength is smaller than 0.1 and a sharp transition occurs at the Coulomb point.

The low-lying energy spectrum and the GS wave function projections provide us with a convincing picture for the existence of a QPT towards two different states, the Pf and APf states, with the turn-on of the additional 3b interaction. To gain the further understanding for this QPT, we study several physical quantities for a model system near the Coulomb point. In Fig. 3 we show the evolutions of the fidelity function $|F|$, energy and PH parity of the chosen GS for different system sizes when the strength of the 3b interaction $H_3$ varies in the range $|V_{3b}| < 0.1$ with the increment $\delta V = 0.01$. The pseudomomenta of the chosen GSs and the unit cells for $N_e = 10, 12, 14$ systems are the same as those in Fig. 2(c). The GS fidelity function, which we use in the plot of Fig. 3(a) to probe the response of the wave function to the variation of the parameter $V_{3b}$, has the form of $F(V_{3b}) = -\frac{\ln(||\Psi(0,0) - 4V_0\Psi(0,4)||^2)}{|\Psi(0,0) - 4V_0\Psi(0,4)|^2}$. The GS wave function is found insensitive to the change of the 3b interaction when its strength is larger than 0.02. However, within the strength range $|V_{3b}| < 0.02$, the value of the fidelity function abruptly increases, indicating a QPT occurs. The peak around $V_{3b} = 0$ becomes sharper if we improve the degeneracy between the chosen GS and its first excited state by carefully tuning the parameters of the unit cell. With the sharp peak located at $V_{3b} = 0$ for all different system sizes, we can identify the Coulomb point as the transition critical point. We can also trace the GS energy to probe a QPT. In Fig. 3(b) we plot the second derivative of the GS energy as a function of $V_{3b}$. For all the system sizes considered, the curves exhibit the singularity-like behavior at the Coulomb point, signaling a first-order QPT. Another useful protocol for us to understand the QPT crossing $V_{3b} = 0$ is the PH parity of the GS as shown in Fig. 3(c). Within a narrow range of $|V_{3b}| < 0.03$, the PH parity of the GS quickly collapses from the Coulomb point, where it is either unitary or
with a positive (negative) sign, the boundary point $\delta V_1^L$ locates where the GS projection on the CFL state begins to exceed the one on the Pf (APf) state with the square unit cell considered. As shown in the Fig. 4(a), on the $\delta V_1 > 0$ side both the Pf and the APf phase tend to expand their phase boundaries with the CFL phase when the 3b interaction strength grows. On the $\delta V_1 < 0$ side, the Pf phase extends its boundary with the stripe phase while the APf phase shrinks. The nonsymmetric behaviors of the Pf and APf phase result from the PH symmetric component of $H_3$. The phase diagram of the system with 3b interaction $H^3_3$, as shown in Fig. 4(b), is symmetric around the phase line $V_{3b} = 0$.

In summary, the PH nonsymmetric 3b term modeling a realistic 5/2 FQHE system can bring either the Pf or the APf state as the ground state depending on its sign. The pure Coulomb system is at the critical point for a possible first-order transition between these two states as one changes the sign of the 3b interaction. Our results suggest that the APf state is indeed a valid candidate[10, 11] for the experimental observed 5/2 FQHE.

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[1] R. L. Willett et al., Phys. Rev. Lett. 59, 1776 (1987); W. Pan et al., Phys. rev. Lett. 83, 3530 (1999); J. S. Xia et al., ibid. 93, 176809 (2004); J. P. Eisenstein et al., ibid. 88, 076801 (2002).
[2] G. Moore and N. Read, Nucl. Phys. B360, 362 (1991).
[3] R. H. Morf, Phys. Rev. Lett. 80, 1505 (1998).
[4] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. 84, 4685 (2000).
[5] M. R. Peterson, K. Park, and S. Das Sarma, Phys. Rev. Lett. 101, 156803 (2008).
[6] M. Greiter, X. G. Wen, and F. Wilczek, Phys. Rev. Lett. 66, 3205 (1991).
[7] Xin Wan, Zi-Xiang Hu, E. H. Rezayi, and Kun Yang, Phys. Rev. B 77, 165316 (2008).
[8] A. E. Feiguin, E. Rezayi, Kun Yang, C. Nayak, and S. Das Sarma, Phys. Rev. B 79, 115322 (2009).
[9] S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. 94, 166802 (2005).
[10] M. Levin, B. I. Halperin, and B. Rosenow, Phys. Rev. Lett. 99, 236806 (2007).
[11] S.-S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, Phys. Rev. Lett. 99, 236807 (2007).
[12] M. R. Peterson, Th. Jolicoeur, and S. Das Sarma, Phys. Rev. Lett. 101, 016807 (2008).
[13] F. D. M. Haldane, E. H. Rezayi, and K. Yang, Phys. Rev. Lett. 85, 5396 (2000).

[14] A. Hamma, W. Zhang, S. Haas, and D. A. Lidar, Phys. Rev. B 77, 155111 (2008).