D-brane field theory on compact spaces

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Abstract

We consider Dirichlet p-branes in type II string theory on a space which has been toroidally compactified in d dimensions. We give an explicit construction of the field theory description of this system by putting a countably infinite number of copies of each brane on the noncompact covering space, and modding out the resulting gauge theory by $\mathbb{Z}^d$. The resulting theory is a gauge theory with graded fields corresponding to strings winding around the torus an arbitrary number of times. In accordance with T-duality, this theory is equivalent to the gauge theory for the dual system of $(d+p)$-branes wrapped around the compact directions, where the winding number is exchanged for momentum in the compact direction.
D-branes\[1, 2\] are an important new tool for studying many aspects of string theory, including duality and black hole physics. The low energy dynamics of parallel Dirichlet $p$-branes in noncompact space are neatly described by the dimensional reduction of 10-dimensional supersymmetric gauge theory to $p + 1$ dimensions\[3\]. A similar field theory description of D-brane dynamics on general compact spaces which is globally valid has not yet been given, although a number of important aspects of such systems have been studied in the context of bound states and black holes\[1, 4, 5, 6, 7, 8, 9\].

Recently, the motion of D-branes on orbifold spaces has been studied by looking at the motion of multiple copies of each D-brane on the simply connected covering space and taking a quotient by a discrete orbifold group\[10, 11, 12, 13\]. In this note we carry out an analogous construction for D-branes moving on toroidally compactified space-times. We make an infinite number of copies of each D-brane which we allow to move on a flat noncompact space; modding out the resulting theory by the lattice $\mathbb{Z}^d$ defining a $d$-dimensional torus gives a field theory description of D-branes moving on the quotient space. The quotient theory thus described is precisely equivalent as a field theory to the usual gauge theory description of the T-dual system of D-branes wrapped around the torus. Thus, although the results of the construction described here agree with T-duality, the construction itself is independent of duality, and thus may be useful in studying systems whose T-duals are not already understood.

We now present the quotient construction explicitly in the simplest case of 0-branes moving on $(S^1 \times \mathbb{R}^8) \times \mathbb{R}$. Thereafter we will discuss briefly the generalization to higher dimensional branes and higher dimensional compactifications, which is straightforward.

Let us begin by recalling the gauge theory description of $k$ 0-branes in type IIA theory in flat space. The low-energy physics of such a system is completely described by dimensionally reducing the 10-dimensional super Yang-Mills action to $0 + 1$ dimensions. In units where $2\pi\alpha' = 1$, this gives an action

\[
S = \int dt \frac{1}{2g} \left[ \sum_i \text{Tr} \left( \dot{X}^i - i [A_0, X^i] \right)^2 + \sum_{i<j} \text{Tr} \left[ X^i, X^j \right]^2 - i \text{Tr} \bar{\psi} \Gamma^i D_i \psi \right]
\]  

(1)
where $X^i$ and $\psi^i$ are bosonic and fermionic fields written as $k \times k$ matrices. If we gauge fix $A_0 = 0$, the bosonic part of the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2g} \left[ \text{Tr} \dot{X}^i \dot{X}^i + \frac{1}{2} \text{Tr} [X^i, X^j]^2 \right].$$  (2)

Throughout this paper we will only concern ourselves with the bosonic part of the action. The fermionic components can be replaced at any point using the supersymmetry of the theory.

Since the matrices $X^i$ are hermitian, the classical equations of motion for the Lagrangian (2) are $[X^i, X^j] = 0$. When these equations are satisfied, the matrices are simultaneously diagonalizable. In this case, the diagonal elements correspond to well-defined positions of the 0-branes. Thus, the moduli space of classical configurations is just $(\mathbb{R}^9)^k/S_k$ where the quotient by the symmetric group on $k$ elements arises from the residual gauge invariance, and expresses the indistinguishability of the particles. When the matrices do not commute, the 0-branes are smeared out and cannot be thought of as having classical positions.

We will now compactify one of the spatial dimensions $X^1$ on a circle $S^1$ of radius $R$. The circle is just the quotient of an infinite line $\mathbb{R}$ by the discrete group $\Gamma = 2\pi R \mathbb{Z}$. Thus, we can describe the physics of 0-branes moving on $(S^1 \times \mathbb{R}^8) \times \mathbb{R}$ by making a copy of each 0-brane for each element of $\mathbb{Z}$, and then imposing the symmetry under $\Gamma$ (for a clear description of the analogous construction on an orbifold where the group $\Gamma$ is finite, see [13]). If we wish to describe $k$ 0-branes moving on the compact space, then the usual index $i$ labeling the branes will be replaced by a pair of indices $j, n$ where $1 \leq j \leq k$ and $n \in \mathbb{Z}$. For notational convenience we will write the resulting infinite dimensional matrices $X^i$ in terms of $k \times k$ blocks $X^i_{mn}$ which satisfy $X_{mn} = X^i_{nm}$. The Lagrangian for the infinite system of particles moving on the covering space now reads

$$\mathcal{L} = \frac{1}{2g} \left[ \text{Tr} \dot{X}^i_{mn} \dot{X}^i_{nm} + \frac{1}{2} \text{Tr} (X^i_{mq} X^j_{qn} - X^j_{mq} X^i_{qn}) (X^i_{nr} X^i_{rm} - X^i_{nr} X^i_{rm}) \right].$$  (3)

*Note: formally, this Lagrangian should be divided by the (infinite) order of the discrete group $\Gamma$. However, this factor will later be multiplied back in when we move to a system of reduced variables, so we simply drop this factor in all formulae.*
Now, in order to describe the physics of the particles moving on the quotient space, we impose the symmetry under $\Gamma$, which gives the constraints

\[
X_{mn}^i = X_{(m-1)(n-1)}^i, \quad i > 1 \\
X_{mn}^1 = X_{(m-1)(n-1)}^1, \quad m \neq n \\
X_{mn}^1 = 2\pi RI + X_{(n-1)(n-1)}^i
\]

The term proportional to the identity matrix in the last line expresses the condition that the $n$th copy of each 0-brane is displaced by a distance $2\pi Rn$ from the 0th copy in the compactified direction. As a result of these constraints, all the dynamical information in the system is contained in the matrices

\[
X_n^i = X_{0n}^i
\]

where $(X_n^i)^\dagger = X_{-n}^i$. We can now rewrite the action in terms of these reduced variables. This is a straightforward calculation; the only subtlety appears in the term in the potential corresponding to $\text{Tr} [X^1, X^j]^2$. Expressed in terms of the reduced variables, this term in the potential becomes

\[
\frac{1}{2g} \left[ \sum_{n=-\infty}^{\infty} \frac{(2\pi Rn)^2}{2} \text{Tr} X_n^j X_n^{-j} - \sum_{m+n+p=0} 2\pi R(n-p) X_m^1 X_n^j X_p^j \\
- \sum_{k+l+m+n=0} \text{Tr} (X_n^1 X_m^1 X_k^j X_l^j - X_n^1 X_m^1 X_k^j X_l^j) \right]
\]

Writing

\[
S_n^j = \sum_q \left( X_q^1 X_{n-q}^1 - X_q^j X_{n-q}^j \right) - 2\pi Rn X_n^j = \sum_q \left( [X_q^1, X_{n-q}^1] \right) - 2\pi Rn X_n^j
\]

and

\[
T_{nk}^{jk} = \sum_q [X_q^j, X_{n-q}^k],
\]

we can rewrite the Lagrangian in reduced variables as

\[
\mathcal{L} = \frac{1}{2g} \left[ \sum_{i=1}^{9} \text{Tr} X_n^i X_n^{-i} - \sum_{j=2}^{9} \text{Tr} S_n^j (S_n^j)^\dagger - \frac{1}{2} \sum_{j,k=2}^{9} \text{Tr} T_{nk}^{jk} (T_{nk}^{jk})^\dagger \right]
\]
This gives a description of 0-branes moving on the compactified space in terms of quantum mechanics on an infinite dimensional space.

The classical potential of this theory is minimized when

\[
X_n^i = 0, \quad n \neq 0
\]

\[
[X_0^i, X_0^j] = 0
\]

Under these conditions, the hermitian matrices \(X_0^i\) are simultaneously diagonalizable, and have eigenvalues corresponding to definite 0-brane positions. To check that the action corresponds to what we would expect of this physical interpretation, we can expand around such a classical configuration and compute the masses of the fields corresponding to strings stretched between distinct 0-branes. For example, the mass of the field \(X_n^i\) corresponding to a string stretching from a D-brane at position \(x^1\) to a D-brane at \(y^1\) (with all other coordinates equal) is correctly proportional to \(y^1 - x^1 - 2\pi R n\), due to the extra \(n\) times that the string wraps around the compact direction. The other interaction terms in (5) similarly correspond to tree-level open string interactions.

According to T-duality, the system we have just described should be equivalent to the T-dual field theory of \(k\) Dirichlet 1-branes wrapped around a compact \(S^1\) of radius \(R' = 1/(2\pi R)\). In fact, we can demonstrate this equivalence explicitly by writing a Fourier mode expansion of the dual theory. The 1-brane theory is the dimensional reduction to 1 + 1 dimensions of 10D SUSY Yang-Mills. In this theory, there is a gauge field \(A_1^i, i = 0, 1\) and there are matter fields \(Y^i, 2 \leq i \leq 9\). In the gauge \(A_0^0 = 0\), the bosonic part of the action is

\[
S = \int dt \frac{dx}{2\pi R' 2g} \left[ \text{Tr} \ Y^i \dot{Y}^i + \text{Tr} \ A_1^i \dot{A}_1^i - \text{Tr} \ (\partial_1 Y^i - i[A_1^i, Y^i])^2 + \frac{1}{2} \text{Tr} \ [Y^i, Y^j]^2 \right]
\]

(10)

Writing

\[
A_1^i = \sum_n e^{inx/R'} X_n^i
\]

\[
Y^i = \sum_n e^{inx/R'} X_n^i
\]

and explicitly performing the integral over \(x\), we find that (10) precisely reproduces the Lagrangian (8).
We now make several observations regarding the T-duality between these theories. Firstly, this equivalence of field theories gives an explicit description of how D-brane configurations corresponding to noncommuting position matrices can be directly converted to gauge field configurations in the dual theory. This description can be compared with other discussions of T-duality in the excitation spectra of these systems which have been carried out in a perturbative context, expanding around a fixed classical configuration associated with definite D-brane positions (see for example[11, 15]). Secondly, note that there is a residual symmetry in each of these theories. In the 0-brane theory the residual symmetry corresponds to the fact that our choice of which copy of each brane we chose as the 0th reference copy was arbitrary. When the matrices are noncommuting the expression of this residual symmetry is somewhat complicated. However, when the matrices do commute, the symmetry simply corresponds to the equivalence of the classical 0-brane configuration under a shift of each 0-brane’s coordinate by an independent multiple of $2\pi R$ in the compactified direction. In the dual 1-brane theory, this symmetry just corresponds to the symmetry under global gauge transformations which are not connected to the identity. Because the space of matrices which satisfy (9) and thus minimize the classical potential in the 0-brane theory must be modded out by this symmetry, the space of classical configurations of the 0-branes is given as expected by $(S^1 \times \mathbb{R}^8)^k/S_k$.

We have described here explicitly the case of 0-branes moving on a space which has been compactified in a single direction. The generalization to parallel Dirichlet $p$-branes of arbitrary dimension moving on a space which has been compactified in $d$ directions is straightforward. In the general case, the indices of each field become a $d$-tuple of integers, corresponding to winding/Fourier modes on the $d$-torus. The only other substantial difference in the construction is that the potential terms analogous to (9) becomes slightly more complicated. When two fields $X^i$ and $X^j$ both correspond to compactified directions, a term appears in (8) of the form $(-\text{Tr} Q_{nm}^{ij}(Q_{nm}^{ij})^\dagger)$ where

$$Q_{nm}^{ij} = \sum_{q,r} \left( [X^i_{q,r}, X^j_{n-q,m-r}] - 2\pi RnX^j_{n,m} - 2\pi RmX^i_{n,m} \right).$$

(11)

(All winding indices except the two of interest have been dropped, but of course generically
all fields will have \( d \) integer indices.) In the T-dual theory, such a term corresponds to a curvature term \( F_{ij} \) in a pair of compactified directions around which the dual D-branes are wound.

We conclude this note with a brief discussion of several situations in which the construction described here may be relevant. In the situation described here of parallel D-branes of the same dimension, the quotient field theory we constructed was precisely equivalent to the known field theory of the T-dual system. However, there are situations where this construction might be applied where no simple T-dual field theory is known. An example of such a situation is a system of interacting \( p \)-branes and \( p + 4 \)-branes where the latter objects are wrapped around a 4-dimensional compact manifold. Systems of interacting branes of this type have been used in recent studies of black holes in string theory[8, 16]. The spectrum of BPS bound states of these systems can be understood from duality arguments[5] or by quantizing the five-brane[9], and can be counted explicitly from the D-brane picture asymptotically[8, 16] and exactly in special cases[3, 17]. Furthermore, it has been suggested that the moduli space of classical configurations of such a system can be identified with the moduli space of instantons on the compact space[7]. This suggestion agrees with the observation[18, 19] that instantons in the \( p+4 \)-brane world-sheet theory carry \( p \)-brane charge. In [4] the matter fields in such a theory corresponding to ND strings connecting the \( p \)- and \( p + 4 \)-branes were described. However, because there is no general description of a global field theory for the system of interacting D-branes on a compact space, it has been difficult to prove this conjecture by explicitly constructing the moduli space. The situation is better in noncompact space, where it has been shown that if the gauge fields on the \( p + 4 \) branes are fixed, the moduli space of vacua has an explicit construction as a hyperkähler quotient[18, 11]. This hyperkähler quotient construction is precisely equivalent to the ADHM construction of instantons on \( \mathbb{R}^4 \), showing that in this case the moduli space of instantons arises directly from the D-brane field theory. There are several obstacles to repeating this argument in the compact case, including the fact that there is no finite dimensional analogue to the ADHM construction for instantons on compact 4-manifolds and the absence
of a global field theory formulation of the intersecting D-brane system. The approach described here, which provides a global description of a D-brane system on a compact space as a quotient of a noncompact system, may provide a tool for better understanding systems of interacting branes of this type and may shed light on the connection between such systems and instanton moduli space. Work in this direction is in progress.

Another possible application of the method described here is to 0-brane quantum mechanics on compact spaces. Recently, it was suggested that the 0-brane matrix quantum mechanics described by the action (1) is more than just a low energy description of 0-branes in type IIA string theory, and actually contains within it at least a large part, if not all, of the physics of 11-dimensional M-theory[20]. In particular, the fact that this matrix quantum mechanics contains naturally within it the 11-dimensional supermembrane[21, 22] gives strong support to this conjecture. It is clearly important to understand how membranes in this theory would behave upon compactification. Perhaps the approach outlined in this note will be helpful in elucidating this issue.

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