Instability of Chaplygin gas trajectories in unified dark matter models

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In the past few years, the Chaplygin gas (CG) has been considered as an appealing candidate for unifying dark matter and dark energy into a single substance. This picture had to face several problems when trying to fit its predictions with cosmological observations. We point out another potential problem of this model, showing that, when the CG is described through a classical scalar field, the corresponding trajectories are strongly unstable. This implies that an extreme fine tuning must apply to the initial values of the field in order to end up today with values allowed by observational data.

Since the observations of distant type Ia Supernovae pointed towards an accelerated expansion of the Universe \cite{1, 2}, a great effort has been done to provide a reasonable, physically motivated possible for this speeding up. Apparently, a dark energy component adds up to the dark matter, coming to dominate the total energy budget at very recent times. The remarkable new feature of dark energy is that it appears to violate the strong energy condition \cite{5}; understanding the nature of dark energy is probably one of the most important open problems in modern physics. Besides the possibility of a quintessence scalar field \cite{4}, modifications of gravity \cite{5}, or an uncanceled cosmological constant \cite{6}, an interesting alternative class of dark energy models is that of the so-called Chaplygin gas \cite{7, 10, 14}. In its simplest formulation, the Chaplygin gas is a perfect fluid with equation of state $p = -A/\rho$, $A$ being a positive constant with the dimensions of an energy squared. A generalized form of the Chaplygin gas, containing an additional free parameter, has also been studied in detail (see, e.g., \cite{10, 14}).

While first introduced in a hydrodynamics context \cite{24}, the Chaplygin gas has recently raised growing interest in particle physics, thanks to its connection with string theory \cite{8, 23, 20, 24, 28, 29, 30} and because it is the only fluid known to admit a supersymmetric generalization \cite{31}.

From a cosmological point of view, the striking feature of the Chaplygin Gas (hereafter, CG) is that it allows for a very elegant unification of dark energy (DE) and dark matter (DM), since its equation of state interpolates between a dust dominated phase at early times and a de Sitter phase at late times. Despite the appeal of these intriguing features, several fatal drawbacks turned out to affect CG models. There are, actually, two different approaches to the cosmological role of the CG: one can consider it as a unified dark matter component, so that $\Omega_{DM} = 0$ and the CG plays both the roles of dark matter and of a cosmological constant, at different epochs (see, e.g., Ref. \cite{32}); or one can look at the CG as a Dark Energy candidate, which adds to the standard DM component, as well as to baryons and radiation \cite{18, 19}. In both cases, the CG is unable to match several observational data.

In Ref. \cite{32} it was pointed out that perturbations in the CG should affect the formation of structures, producing oscillations or exponential blowups in the matter power-spectrum which are inconsistent with observations; the analysis ruled out the CG as a unified dark matter candidate ($\Omega_{DM} = 0$) at 99.99% level.

The impact on cosmology has also been investigated in the context of Supernovae data \cite{11, 13, 33, 34, 35, 36}, showing initially a certain degree of consistency with data for a class of CG models; Cosmic Microwave Background (CMB) measurements, though, provided stronger constraints to CG cosmologies \cite{18, 19, 21}. While Ref. \cite{18} found that the joint analysis of type Ia Supernovae and CMB data only allows for a CG gas which is today indistinguishable from a cosmological constant, the analysis of Ref. \cite{19} rules out the CG as a Dark Energy candidate at 99.99% confidence level, and also indicates that the CG as a unified model of dark matter and dark energy is strongly disfavoured by the latest CMB data.

In this paper, we point out another possible problem related to the dynamical behavior of the CG gas. We focus on the unified dark matter cosmological model and we analyze the trajectories of the
classical scalar field which is meant to correspond to the CG hydrodynamical representation, searching for attractor solutions. For a flat universe, the parameters of the CG can be easily related to the expansion rate today, $H_0$, and to the present cosmic equation of state, $w_0$. The latter quantity is constrained by the recent estimates of the present deceleration parameter $q_0$ [37], since $w_0 = \frac{2q_0 - 1}{3}$; we thus require $w_0 \sim -0.7$ today. We find that, in order to finish up at the present epoch with such an equation of state $w_0$ (different than $-1$) and a plausible value of the expansion rate, a strong fine-tuning must be applied on the initial conditions of the scalar field. In other words, the trajectories which reproduce the observed parameters are strongly unstable under variations of the initial conditions on the field. The only attractor of the dynamical system under study is indeed the de Sitter Universe, to which the model converges at late times. Our conclusions are complementary to the perturbation analysis of [32], showing that, already at the background level, the unified dark matter model shows serious instabilities.

We will undertake our analysis in a flat Friedmann-Robertson-Walker (FRW) cosmology; since we are mostly interested in the dynamics at recent times, radiation is not included in our treatment. From the energy conservation and the CG equation of state, it follows that

$$
\rho_{\text{Chap}} = \sqrt{\frac{A + B}{a^6}} \quad p_{\text{Chap}} = \frac{-A}{\sqrt{A + B a^6}} 
$$

\[ (1) \]

$a$ being the scale factor and $B$ an integration constant; the ratio $B/A$ characterizes the transition between the matter-like behavior and the cosmological constant.

In the unified dark matter model, the expansion rate and the cosmic equation of state today are, respectively, $H_0^2 = \sqrt{A + B}$ and $w_0 = -(1 + B/A)^{-1}$. We see that the constants $A$ and $B$ can be determined once $H_0$ and $w_0$ are known. Our aim is to provide a description of the Chaplygin gas through a minimally-coupled, classical scalar field. In [7,16] this description is simply built from the analogy of the energy density and pressure of the CG in Eq. (1) with the analogous quantitites for a scalar field $\phi$ evolving in a potential $V(\phi)$, i.e.

$$
\rho_\phi = \dot{\phi}^2/2 + V(\phi) \quad p_\phi = \dot{\phi}^2/2 - V(\phi) 
$$

\[ (2) \]

where a dot denotes differentiation w.r.t. the cosmic time. Namely, by requiring that $\rho_\phi = \rho_{\text{Chap}}$ and $p_\phi = p_{\text{Chap}}$, we can relate the cosmic scale factor to the kinetic and potential energy a scalar field should have in order to reproduce the Chaplygin fluid dynamics:

$$
\dot{\phi}^2 = \frac{B}{a^6 \sqrt{A + B}} \quad V(\phi(a)) = \frac{2a^6 A + B}{2a^6 \sqrt{A + B}}. 
$$

\[ (3) \]

By making use of the Friedmann equation and of Eq. (3), one can infer $d\phi/da$, which can be solved to give $a(\phi)$; The authors of Ref. [7] give

$$
a^6 = \frac{4B\exp(\phi)}{A(1 - \exp(6\phi))^2}. 
$$

\[ (4) \]

There is a crucial assumption in Eq. (4): the integration constant has been choosen so that the value of the scalar field at some (arbitrary) time $t_i$ is fixed; in particular, setting the initial conditions at $a_i$, one should have

$$
\phi_i = \frac{1}{6} \ln \left( \frac{\sqrt{\frac{1}{4}a_i^6 + 1} - 1}{\sqrt{\frac{1}{4}a_i^6 + 1} + 1} \right); 
$$

\[ (5) \]

The initial “velocity” $\dot{\phi}_i$ is fixed too, by Eq. (4). Substituting $a(\phi)$ in $V(a)$, one has

$$
V(\phi) = \frac{1}{2} \sqrt{2} \left( \cosh 3\phi + \frac{1}{\cosh 3\phi} \right). 
$$

\[ (6) \]

As already noticed in Ref. [22], a scalar field evolving in such a potential will produce a cosmological evolution coinciding with that of the Chaplygin gas only if the initial conditions are appropriately handled to satisfy the relation $\dot{\phi}_i^2 = 4(V^2(\phi_i) - A)$; indeed, the potential (6) has been built under this assumption. The evolution of a minimally-coupled scalar field follows the Klein-Gordon equation of motion:

$$
\ddot{\phi} = -3H\dot{\phi} - V, \phi. 
$$

\[ (7) \]
of the Chaplygin fluid; first, a unified model of dark matter needs at least an epoch when the equation of state \( w_{\text{CG}} = p_{\phi}/\rho_{\phi} \) is zero, so that the energy density drops like a matter component. This phase should end up in a de Sitter phase, when the energy density of the scalar field is constant (as for \( a >> 1 \) in \( 1 \)). Finally, since the cosmic equation of state today is kinematically related to the deceleration parameter \( q \equiv a/\dot{a}H^2 \), the most recent measurements give \( w_0 \sim 0.7 \) \( 37 \). It follows that the intermediate stage of the CG evolution should be characterized by a decreasing equation of state, shifting smoothly from 0 to \(-1\). The point we focus on here is how much fine-tuning, if any, is required to do that. In other words, is there any freedom in setting the initial values of the field if one has to reproduce the scalings in Eq. \( 1 \)? The question may be turned into the phase-space language, by asking whether the Chaplygin fluid trajectory is an attractor for the system \( 7 \). As we said, we can always provide a scalar-field description of the Chaplygin gas, by appropriately setting its initial conditions and building up the corresponding potential. Such a field will obey the Klein-Gordon equation, and, in addition, \( \rho_{\phi}, p_{\phi} \) will scale as in Eq. \( 1 \) at any time. The scalar field model obtained with this particular choice of initial conditions would perfectly reproduce the dynamics of the Chaplygin gas. We will refer to \( \phi_{\text{Chap}}(t) \) as the solution of Eq. \( 7 \) with potential \( 9 \), which satisfies the required initial conditions.

As for the late time behavior of \( 7 \), it is easily seen that, whatever initial conditions we choose, the system will end up in its global attractor, the de Sitter stable node (see \( 38 \)) with \( \dot{\phi} = 0, H = \sqrt{A} \) (note that the de Sitter node is the only critical point of the dynamical system under investigation). In particular, this is true for the solution \( \phi_{\text{Chap}}(t) \). However, we are interested in finding whether the trajectory \( \phi_{\text{Chap}} \) itself is an attractor for the problem on hand, in the sense that many possible solutions of the system converge to a common evolutionary track represented by the \( \phi_{\text{Chap}} \) trajectory. We find that this is not the case; while \( \phi_{\text{Chap}} \) requires fine-tuned initial values to reproduce the Chaplygin gas cosmology, slightly different initial conditions on \( \phi_{\text{Chap}} \) may result in completely different trajectories, which may even be unable to reproduce the matter behavior at all. The general solutions of this dynamical system turn out to behave much differently than the “tracking solutions” of \( 39 \). Following the notation of \( 39 \), we may anticipate our result, saying that the system does not admit tracking solutions because one of the two necessary conditions on \( \Gamma \equiv VV_{\phi\phi}/V_{\phi}^2 \) is violated, namely \( \Gamma = 1 \) for almost any plausible initial field value. One should be careful, however, in using the formalism of \( 39 \) in this case, in that the Chaplygin gas is supposed to be the only component of the cosmic fluid, and there is no other background component which may drive its trajectories (having neglected the role of radiation in our analysis). For this reason we found it useful to directly check for the stability of the \( \phi_{\text{Chap}} \) solutions with a numerical integration of the system.

Our plots refer to four different choices of initial conditions, set at redshift \( z \sim 10^3 \). The solid lines refer to the \( \phi_{\text{Chap}} \) trajectories. The present values of \( H_0 \) and \( w_0 \) unambiguously determine \( A \) and \( B \) for a flat universe and the present values of \( \phi_{\text{Chap}}, \dot{\phi}_{\text{Chap}} \). We set \( h = 0.7 \) and \( w_0 = -0.7 \). Together with the \( \phi_{\text{Chap}} \) trajectories, Figs. \( 1 \) and \( 2 \) show, respectively, the energy density of the scalar field and the corresponding equation of state, when the field starts with the following initial conditions:

a) \( \phi_1 = \phi_{\text{Chap},i} \times 1.1, \dot{\phi}_1 = \dot{\phi}_{\text{Chap},i} \) (dotted lines);

b) \( \phi_1 = \phi_{\text{Chap},i}, \dot{\phi}_1 = \dot{\phi}_{\text{Chap},i} \times 10^2 \) (dashed lines);

c) \( \phi_1 = \phi_{\text{Chap},i} \times 0.9, \dot{\phi}_1 = \dot{\phi}_{\text{Chap},i} \times 10 \) (dot-dashed lines).

Even a slight modification of the initial values with respect to \( \phi_{\text{Chap}} \) can make the equation of state today completely different from the required value, resulting in trajectories which never join the \( \phi_{\text{Chap}} \) curves, nor reproduce the CG behavior at any epoch. Furthermore, the conditions a), b), c) finish up at present with unacceptable values of the Hubble expansion rate. Starting with a bigger amount of kinetic energy, as in b) and c), an initial kination precedes the phase in which the field behaves like matter. In each of the plotted cases, there is a very sharp transition between the matter-like scaling \( (w = 0) \) and the \( \Lambda \)-like one \( (w = -1) \): because of this sharpness, values of the equation of state different than \( 0, -1 \) turn out to be unnatural. This behavior is very different than in tracking quintessence solutions, where the only parameters which need to be adjusted are the amplitude of the potential and the current equation of state: in that case, once these present values are fixed, there is no dependence on the initial values of the quintessence field.

Concluding, we have shown that, in order to have a unified dark matter model consistent with the most recent estimates of the cosmic equation of state, one is forced to introduce an extreme fine tuning on the initial values of the scalar field, because the problem does not admit tracking solutions; the de Sitter universe is the only global
FIG. 1: Energy densities for the $\phi_{\text{Chap}}$ solution and for different initial conditions, corresponding to the cases a), b), c) described in the text.

FIG. 2: Equation of state for the $\phi_{\text{Chap}}$ solution and for different initial conditions, corresponding to the cases a), b), c) described in the text.

attractor of the system. Although the potential (6) requires ad hoc initial conditions on $\phi$, which allowed us to define $\phi_{\text{Chap}}$, this is not a unique representation of the Chaplygin fluid. One may start with different values of $\phi_i, \dot{\phi}_i$ and, by a similar approach, build a potential $V(\phi)$ (generally, different from (6)) for each set of initial conditions, so to reproduce the behavior in this way, there would be an infinite class of potentials, mapping the behavior of the Chaplygin fluid. However, our conclusions apply to any on this potentials, due to the peculiar sharpness of the matter-Λ transition of the CG fluid. Therefore, the CG as a unified dark matter candidate, seems to be strongly disfavored from the point of view of its dynamics.

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