A FAMILY OF CRITERIA FOR IRRATIONALITY OF EULER’S CONSTANT

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Abstract. Following earlier results of Sondow, we propose another criterion of irrationality for Euler’s constant $\gamma$. It involves similar linear combinations of logarithm numbers $L_{n,m}$. To prove that $\gamma$ is irrational, it suffices to prove that, for some fixed $m$, the distance of $d_n L_{n,m}$ ($d_n$ is the least common multiple of the $n$ first integers) to the set of integers \( \mathbb{Z} \) does not converge to 0. A similar result is obtained by replacing logarithms numbers by rational numbers: it gives a sufficient condition involving only rational numbers. Unfortunately, the chaotic behavior of $d_n$ is an obstacle to verify this sufficient condition. All the proofs use in a large manner the theory of Padé approximation.

1. Introduction

In [Sondow 2003], the author, using Beukers’ integral [Beukers 1979], found a criterion for irrationality of Euler’s constant $\gamma$. It depends on the limit of the fractional part of the following expression

$$L_n = 2 \sum_{k=1}^{n} \sum_{i=0}^{k-1} \left( \frac{n}{i} \right)^2 \left( H_{n-i} - H_i \right) \ln(n+k)$$

where $H_n$ is the Harmonic number $H_n := \sum_{k=1}^{n} \frac{1}{k}$. In this paper, we establish the connexion between Sondow’s criteria and Padé approximant of the function $\frac{\ln u}{u-1}$. Moreover, following the same idea, we find a family of new criteria: for each integer $m \leq n$, let us set

$$L_{n,m} := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1), \quad (1.1)$$

$d_n := \text{LCM}(1,\ldots,n)$ and $\{x\} = x - \lfloor x \rfloor$ the fractional part of the real $x$. If, for some integer $m$, if the sequence $\{d_n(-1)^m L_{n,m}\}$ does not converge to 0 when $n$ tends to infinity, then $\gamma$ is irrational.

Using the property of the error term, a more precise criterion is proved here: if, for some integer $m$, the sequence $\{d_{2n}(-1)^m L_{2n,m}\}$ is asymptotically non decreasing, when $n$ tends to infinity, then $\gamma$ is irrational.

2000 Mathematics Subject Classification. Primary 11J72, Secondary 41A21.

Key words and phrases. Euler’s constant, irrationality, Padé approximations.
2. SONDOW’S CRITERION WITH PADÉ APPROXIMANT

Sondow considers the double integral (so-called Beukers’ integral)

\[ I_n = \int_0^1 \int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)\ln(xy)} \, dx \, dy. \]

Applying Taylor expansion of \(1/(1-xy)\) around 0, he proved the following identity

\[ I_n = \left( \begin{array}{c} 2n \\ n \end{array} \right) \gamma + L_n - A_n = O(2^{-4n}n^{-1/2}) \tag{2.1} \]

where \(A_n = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right)^2 H_{n+i} \). After multiplication by \(d_{2n}\), it arises

\[ d_{2n} \left( \begin{array}{c} 2n \\ n \end{array} \right) \gamma = d_{2n}(A_n - L_n) + o(1). \]

**Sondow’s criterion:**

Since \(d_{2n}A_n \in \mathbb{Z}\), if the sequence of fractional part \(\{d_{2n}L_n\}\) does not converge to 0 then \(\gamma \notin \mathbb{Q}\).

Sebah computed this sequence for \(1 \leq n \leq 2500\). Its cumulative average seems to converge 1/2, but the mathematical proof remains to establish.

In the following, we will show that the sequence involved in the paper by Sondow can be recovered by means of Padé approximation.

Let us consider the function \((\ln u)/(u-1)\) and its Padé Approximant \([n-1/n]\) of degree \((n-1/n)\) at the point \(u = 1\):

\[ \frac{\ln u}{u-1} = \frac{N_n(u)}{D_n(u)} + R_n(u) \tag{2.2} \]

where \(N_n\) and \(D_n\) are polynomials of respective degree \(n-1\) and \(n\), normalized by \(N_n(1) = D_n(1) = 1\), and \(R_n(u) = \mathcal{O}(u^{2n})\)  

From the theory of Padé approximation, it is well known that \(D_n\) is related with the shifted Legendre Polynomial orthogonal on the interval \([0,1]\) with respect to the Lebesgue weight function. Some of these expressions are

\[ P_n^*(t) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)^2 t^{n-k}(t-1)^k \tag{2.3} \]

\[ \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} n+k \\ k \end{array} \right) (-1)^{n+k}t^k \tag{2.4} \]
$D_n$ has the following expression in terms of $P_n^*$

$$D_n(u) = P_n^* \left( \frac{1}{1-u} \right) (1-u)^n \left( \frac{2n}{n} \right)^{-1}. \quad (2.5)$$

Replacing $P_n^*$ by its expressions (2.3, 2.4), formula (2.5) becomes

$$D_n(u) = \left( \frac{2n}{n} \right)^{-1} \sum_{k=0}^{n} \binom{n}{k}^2 u^k$$

$$= \left( \frac{2n}{n} \right)^{-1} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{n+k}{k} \right) (1-u)^{n-k} u^k.$$

The numerator $N_n(u)$ of $[n - 1/n]$, is related with the associated polynomial of the denominator:

$$N_n(u) = 2 \left( \frac{2n}{n} \right)^{-1} \sum_{k=1}^{n} \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i) u^{k-1}$$

$$= \left( \frac{2n}{n} \right)^{-1} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{n+k}{k} \right) \sum_{i=0}^{k-1} (u-1)^{n-k+i} (-1)^i \frac{i}{i+1}.$$

Now, what is the link between Padé Approximation and Sondow’s criterion?

The definition of $\gamma$ is, primarily,

$$\gamma = \lim_{n \to \infty} (H_n - \ln n).$$

An integral representation for Euler’s constant is

$$\gamma = \int_0^1 \left( \frac{1}{\ln u} + \frac{1}{1-u} \right) \, du. \quad (2.6)$$

Formula (2.2) can be rewritten as

$$\frac{N_n(u)}{\ln u} + \frac{D_n(u)}{1-u} = - \frac{R_n(u) D_n(u)}{\ln u}$$

and

$$\gamma = \int_0^1 \frac{1-u^n N_n(u)}{\ln u} \, du + \int_0^1 \frac{1-u^n D_n(u)}{1-u} \, du - \int_0^1 \frac{u^n R_n(u) D_n(u)}{\ln u} \, du.$$
By linearity, the second term is expanded as
\[
\int_0^1 \frac{1 - u^n D_n(u)}{1 - u} \, du = \left( \frac{2n}{n} \right)^{-1} \sum_{k=0}^{n} \binom{n}{k}^2 \int_0^1 \frac{u^{n+k} - 1}{u - 1} \, du \tag{2.7}
\]
\[
= \left( \frac{2n}{n} \right)^{-1} \sum_{k=0}^{n} \binom{n}{k}^2 H_{n+k} \tag{2.8}
\]
\[
= \left( \frac{2n}{n} \right)^{-1} A_n. \tag{2.9}
\]

The first integral can be computed as following:
\[
\int_0^1 \frac{1 - u^n N_n(u)}{\ln(u)} \, du = 2 \left( \frac{2n}{n} \right)^{-1} \sum_{k=1}^{n} \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i) \int_0^1 \frac{1 - u^{n+k-1}}{\ln(u)} \, du \tag{2.10}
\]
\[
= -2 \left( \frac{2n}{n} \right)^{-1} \sum_{k=1}^{n} \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i) \ln(n + k) \tag{2.11}
\]
\[
= - \left( \frac{2n}{n} \right)^{-1} L_n. \tag{2.12}
\]

From the theory of Padé approximation, in formula (2.2), the remainder term $R_n$ has an integral representation
\[
R_n(u) = \frac{(1 - u)^n}{D_n(u)} \int_0^1 \frac{t^n D_n(1 - 1/t)}{1 - (1 - u)t} \, dt.
\]

Thanks to formulas (2.9, 2.12), $\gamma$ satisfies
\[
\gamma = \frac{A_n - L_n}{\binom{2n}{n}} - \int_0^1 \frac{u^n R_n(u) D_n(u)}{\ln(u)} \, du.
\]

Thus another expression of the remainder term $I_n$ of Sondow is
\[
I_n = \left( \frac{2n}{n} \right) \gamma - A_n + L_n = - \int_0^1 \frac{u^n R_n(u) D_n(u)}{\ln(u)} \, du
\]
\[
= \int_0^1 \int_0^1 \frac{u^n (1 - u)^n}{\ln(u)} \frac{P_n^*(t)}{1 - (1 - u)t} \, dt \, du
\]
\[
= \int_0^1 \int_0^1 \frac{u^n (1 - u)^{2n}}{\ln(u)} \frac{t^n (1 - t)^n}{(1 - (1 - u)t)^{n+1}} \, dt \, du
\]
thanks to integration by parts and Rodrigues formula for orthogonal polynomials.

Thus the approximation for Euler’s constant $\gamma$ (2.1) is a consequence of the Padé approximation to the function $(\ln u)/(1 - u)$. 
In the same manner, Pilehrood [Pilehrood 2004] found irrationality criteria for generalized Euler’s constant. He defined the following linear form in logarithms

\[ L(n_1, n_2)(\alpha) = \sum_{m=1}^{n_1} \sum_{k=0}^{m-1} \binom{n_1}{k} \binom{n_2}{k} (H_{n_1-k} + H_{n_2-k} - 2H_k) \ln(m + n_1 + \alpha - 1) + \sum_{m=n_1+1}^{n_2} \sum_{k=m}^{n_2} (-1)^{k-1-n_1/k} \binom{n_2}{k}/\binom{k-1}{n_1} (\ln(m + n_1 + \alpha - 1) \]

Actually, following the same idea as for Sondow’s criterion, it is possible to prove that Pilehrood’ criterion comes from Padé approximations \([n_2-1, n_1] = R_{n_2-1}(u)/S_{n_1}(u)\) (normalized by \(R_{n_2-1}(1) = S_{n_1}(1) = 1\) to the function \(\ln(u)/(u-1)\) at the point \(u = 1\). The linear form \(L(n_1, n_2)(\alpha)\) satisfies:

\[ L(n_1, n_2)(\alpha) = \int_0^1 (1 - u^{n_1+\alpha-1}R_{n_2-1}(u)) \frac{1}{\ln(u)} du \]

3. Statement of the results

In order to simplify Sondow’s criterion, it is convenient to choose a more simple approximation. This method leads to the following theorems.

**Theorem 1.** For \(0 \leq m \leq n\), let us define

\[ J_{n,m} = \int_0^1 u^{n-m}P_n^*(u) \left( \frac{1}{\ln u} + \frac{1}{1-u} \right) du \]
\[ L_{n,m} = -\int_0^1 \frac{1-u^{n-m}P_n^*(u)}{\ln u} du \]
\[ A_{n,m} = \int_0^1 \frac{1-u^{n-m}P_n^*(u)}{1-u} du \]

then

\[ \gamma = A_{n,m} - L_{n,m} + J_{n,m} \] \hspace{1cm} (3.1)

\[ L_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n - m + k + 1) \] \hspace{1cm} (3.2)

\[ A_{n,m} = 2H_n \] \hspace{1cm} (3.3)

**Theorem 2.** The following are equivalent:

(a) The fractional part of \(d_nL_{n,m}\) is given by \(\{d_n(-1)^mL_{n,m}\} = d_n(-1)^mJ_{n,m}\) (*) for some \(n\) or \(m\).

(b) The formula (*) holds for some \(m\) and for all sufficiently large \(n\).

(c) Euler’s constant is a rational number.
A sufficient condition which involves $d_n L_{n,m}$ but not $J_{n,m}$ is the following

**Theorem 3.** If for some integer $m$, $\{d_n(-1)^{m} L_{n,m}\} \geq 0.707^n$ infinitely often, then $\gamma$ is irrational.

Computations (see Table 1) show that this condition is satisfied for $n \leq 1000$. Numerical results also suggest that for each $m$, $\{d_n(-1)^{m} L_{n,m}\}$ is dense in the interval $(0,1)$ and the cumulative average $n^{-1} \sum_{k=1}^{\infty} \{d_k(-1)^{m} L_{k,m}\}$ converges to 0.5 (see Figure 1 and 2).

To prove $\gamma$ irrational, it just suffices to show that $\{d_n(-1)^{m} L_{n,m}\}$ does not converge to 0.

In section 6, we will prove the asymptotic formula

$$\gamma = A_{n,m} - L_{n,m} + \mathcal{O}(4^{-n}). \quad (3.4)$$

Actually, we will prove that the error term $J_{n,m}$ is a totally monotone sequence (i.e. a sequence of moments with respect a positive measure), converging to 0 as $4^{-n}$.

By substituting in $L_{n,m}$, $\ln(n + 1 + k - m)$ by some suitable Padé approximants, a sufficient condition, involving only rational numbers is the following

**Corollary 1.** Let us define, for $n - m + 1 = 2^p, p \in \mathbb{Z}$,

$$\tilde{L}_{n,m} := p \left[ n/n \right]_{t=1} + \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k}[n/n]_{t=k/(n-m+1)} \quad (3.5)$$
where \([n/n]\) is the Padé approximant of \(\ln(1 + t)\) at \(t = 0\):

\[
[n/n]_t = \frac{t \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left( \sum_{i=0}^{k-1} \frac{t^{i-k+n(-1)^i}}{i+1} \right)}{\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} t^{n-k}}
\]

If for some integer \(m\), \(\{d_{2^p}(-1)^m \tilde{L}_{2^p+m-1,m}\}\) does not converge to 0 when \(p\) tends to infinity, then \(\gamma\) is irrational.

Another sufficient condition comes from the property of the error term in the asymptotic formula (3.4) and from the upper and lower bound of the LCM(1, \ldots, n):

**Corollary 2.** If for some \(m\), \(\{d_{2^p}(-1)^m \tilde{L}_{2^p,m}\}\) is asymptotically non decreasing, then \(\gamma\) is irrational.

4. Two Lemmas

**Lemma 1.** The function \(\frac{1}{\ln(1-u)} + \frac{1}{u}\) is a Markov-Stieltjes function. More precisely,

\[
\frac{1}{\ln(1-u)} + \frac{1}{u} = \int_0^1 \frac{1}{1-u} w(t) \, dt
\]  

\( (4.1) \)
Table 1.

| n   | \(d_n L_{n,0}\) | \(-d_n L_{n,1}\) | \(d_n L_{n,2}\) | \(-d_n L_{n,3}\) |
|-----|-----------------|-----------------|-----------------|-----------------|
| 1   | 1.38868         | 1.81209         | —               | —               |
| 2   | 0.56003         | 0.58439         | 0.56609         | —               |
| 3   | 0.61882         | 0.64252         | 0.63428         | 0.67030         |
| 4   | 2.97160         | 3.31151         | 3.23310         | 0.38225         |
| 5   | 0.44808         | 0.45886         | 0.45719         | 0.45913         |
| 6   | 0.31896         | 0.32064         | 0.32044         | 0.32061         |
| 7   | 0.14391         | 0.14467         | 0.14460         | 0.14465         |
| 8   | 0.41138         | 0.41543         | 0.41511         | 0.41528         |
| 9   | 0.09667         | 0.09689         | 0.09687         | 0.09688         |
| 10  | 0.06778         | 0.06781         | 0.06781         | 0.06781         |
| 11  | 0.03395         | 0.03398         | 0.03398         | 0.03398         |
| 12  | 0.02378         | 0.02379         | 0.02379         | 0.02379         |
| 13  | 0.01719         | 0.01721         | 0.01720         | 0.01720         |
| 14  | 0.01204         | 0.01204         | 0.01204         | 0.01204         |
| 15  | 0.00843         | 0.00843         | 0.00843         | 0.00843         |
| 16  | 0.02637         | 0.02637         | 0.02637         | 0.02637         |
| 17  | 0.01639         | 0.01639         | 0.01639         | 0.01639         |
| 18  | 0.01147         | 0.01147         | 0.01147         | 0.01147         |
| 19  | 0.00163         | 0.00163         | 0.00163         | 0.00163         |
| 20  | 0.001147        | 0.00114         | 0.00114         | 0.00114         |

where the weight function \(w\) is

\[
w(t) := \frac{1}{t \left( \ln^2 \left( \frac{1}{t} - 1 \right) + \pi^2 \right)}
\]

**Proof.** After a change of variable \((u \to 1 - u)\) and \(x = 1/t - 1\), formula (4.1) is equivalent to

\[
\ln(u) + \frac{1}{1-u} = \int_0^\infty \frac{1}{x+u} \frac{1}{\ln^2 x + \pi^2} \, dt
\]

(4.2)

The weight function \(w\) can be found with the Stieltjes inversion formula (see [Widder 1941]). Another way to prove formula (4.2) is to apply residue theorem to the function \(f(x) := \frac{1}{x+u \ln x + i\pi}\).

Taking the determination of \(\ln x\) on the complex plane cut along the positive real axis, the poles of \(f\) are \(x = -u\) and \(x = -1\).
Let us define \( \gamma_r \) a small semi-circle \( z = re^{i\theta}, -\pi/2 \leq \theta \leq \pi/2, r > 0 \). \( D_r^+ \) the line 
\( z = x + ir \), \( x \) running from 0 to \( R \), \( \Gamma_R \) the circle \( z = Re^{i\theta}, 0 \leq \theta \leq 2\pi \) and \( D_r^- \) the line 
\( z = x - ir \), for \( x \) from \( R \) to 0.

Now, we compute \( \int_C f(x) \, dx \) where \( C \) is the union of \( D_r^+, \Gamma_R, D_r^- \) and \( \gamma_r \), with the theorem of residue to obtain
\[
\int_0^\infty \frac{1}{x + u} \left( \frac{1}{\ln x + i\pi} + \frac{-1}{\ln x - i\pi} \right) \, dx = \int_0^\infty \frac{1}{x + u} \left( \frac{-2i\pi}{\ln^2 x + \pi^2} \right) \, dx = -2i\pi \left( \frac{1}{\ln u + 1 - u} \right)
\]

Now, we are in position to prove a new formula for the Euler’s constant \( \gamma \).

**Theorem 4.** The Euler’s constant \( \gamma \) satisfies
\[
\gamma = \int_{-\infty}^{\infty} \frac{\ln(1 + e^{-z})e^z}{z^2 + \pi^2} \, dz
\]

**Proof.** In the integral representation of \( \gamma \) (2.6), let us substitute the integrand by the expression (4.1). This leads to
\[
\gamma = \int_0^1 -\frac{\ln(1-t)}{t} \frac{1}{t(\ln^2(1/t) + \pi^2)} \, dt = \int_{-\infty}^{\infty} \frac{\ln(1 + e^{-z})e^z}{z^2 + \pi^2} \, dz
\]
with the change of variable \( t = (1 + e^z)^{-1} \).

**Lemma 2.** For each fixed integer \( m \), the sequence \( ((-1)^m J_{n,m})_n \) defined in Theorem 2 is totally monotonic. More precisely
\[
(-1)^m J_{n,m} = \int_0^{1/4} v^n \rho_m(v) \, dv
\]
where the weight function is
\[
\rho_m(v) = \int_{\frac{1 + \sqrt{1 + 4m}}{2}}^{\frac{1 - \sqrt{1 + 4m}}{2}} \left( \frac{u - u^2 - v}{u v} \right)^m \left( \frac{1}{u - u^2 - v} \right)^{\frac{1}{2}} \left( \frac{-u v}{u^2 - u + v} \right)^{\frac{1}{2}} \, du.
\]

**Proof.** \( J_{n,m} = \int_0^1 u^{n-m} P_n^*(u) \left( \frac{1}{\ln u} + \frac{1}{1 - u} \right) \, du \) appears as Legendre modified moments of the weight function \( \left( \frac{1}{\ln u} + \frac{1}{1 - u} \right) \).
For some particular cases of weight function, a sequence of modified moments can be itself a sequence of moments, with respect to a positive measure (see [Prévost 1994]). Using Rodrigues formula for orthogonal polynomials, Lemma 1, Fubini’s theorem and after \( n \) integrations by parts, it arises

\[
 J_{n,m} = \int_0^1 u^{n-m} \frac{(-1)^n}{n!} \frac{d^n}{du^n} (u^n (1-u)^n) \left( \frac{1}{\ln u} + \frac{1}{1-u} \right) du
 = \int_0^1 u^{n-m} \frac{(-1)^n}{n!} \frac{d^n}{du^n} (u^n (1-u)^n) \left( \frac{1}{1-(1-u)t} \right) w(t) dt
 = \int_0^1 \int_0^1 \frac{(-1)^n}{n!} u^n (1-u)^n \frac{d^n}{du^n} \left( \frac{u^{n-m}}{1-(1-u)t} \right) w(t) dt \cdot du.
\]

The computation of \( \frac{d^n}{du^n} \left( \frac{u^{n-m}}{1-(1-u)t} \right) \) needs the partial decomposition of the rational function \( \frac{u^{n-m}}{1-(1-u)t} = q(u) + \left( \frac{t-1}{t} \right)^{n-m} \frac{1}{1-(1-u)t} \), where \( q \) is polynomial of degree \( n - m - 1 \).

Another expression of \( J_{n,m} \) is then

\[
 J_{n,m} = \int_0^1 \int_0^1 u^n (1-u)^n \left( \frac{t-1}{t} \right)^{n-m} \frac{t^n}{(1-(1-u)t)^{n+1}} w(t) dt \cdot du.
\]

We do the following change of variable

\[
 v = \frac{u(1-u)(1-t)}{1-(1-u)t} \in [0, 1/4] \Leftrightarrow t = \phi(v) = \frac{u^2 - u + v}{(u-v)(u-1)} \in [0, 1].
\]

Let \( \phi_1(v) \) and \( \phi_2(v) \) denote the two roots of the quadratic equation \( v = u - u^2 \),

\[
 \phi_1(v) = \frac{1 + \sqrt{1 - 4v}}{2}, \quad \phi_2(v) = \frac{1 - \sqrt{1 - 4v}}{2}.
\]

\[
 J_{n,m} = \int_0^{1/4} v^n dv \int_{\phi_1(v)}^{\phi_2(v)} (-1)^m \left( \frac{\phi(v)}{\phi(v) - 1} \right)^m w(\phi(v)) \frac{u^2}{(u-1)(u-v)^2} \frac{(-1)^m}{1-(1-u)\phi(v)} du
\]

which proves the lemma.

\[\square\]

5. Proof of Theorem 1

We first prove the identity (3.1) linking Euler’s constant \( \gamma \), the linear combination of logarithms numbers \( L_{n,m} \), the rational numbers \( A_{n,m} \) and the integrals \( J_{n,m} \). From formula (2.6), one substitute the integrand \( \left( \frac{1}{\ln u} + \frac{1}{1-u} \right) \) by an approximation involving
Legendre Polynomials as follows:

\[
\gamma = \int_0^1 \left( \frac{1}{\ln u} + \frac{1}{1 - u} \right) \, du = \int_0^1 \left( \frac{1 - u^{n-m}P_n^*(u)}{\ln u} + \frac{1 - u^{n-m}P_n^*(u)}{1 - u} \right) \, du + \int_0^1 u^{n-m}P_n^*(u) \left( \frac{1}{\ln u} + \frac{1}{1 - u} \right) \, du.
\]

The expression (2.4) of \( P_n^* \) leads to analogous expressions \( L_{n,m} \).

By linearity

\[
L_{n,m} = - \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \int_0^1 \left( \frac{1 - u^{k+n-m}}{\ln u} \right) \, du = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n - m + k + 1)
\]

\( A_{n,m} \) is treated quite differently:

\( P_n^* \) satisfies the following orthogonality relation

\[
\int_0^1 P_n^*(u)q(u) \, du = 0, \text{ for all polynomial } q \text{ of degree less than } n.
\]

Thus, by taking \( q(u) = \frac{1 - u^{n-m}}{1 - u} \), another expression for \( A_{n,m} \) is

\[
A_{n,m} = \int_0^1 \frac{1 - P_n^*(u)}{1 - u} \, du = \int_0^1 \frac{P_n^*(1) - P_n^*(u)}{1 - u} \, du
\]

(5.1)

and so \( A_{n,m} \) is independent of \( 0 \leq m \leq n \).

Let us now compute the integral in (5.1).

Legendre polynomials satisfy a three term recurrence relation which is

\[
(n + 1)P_{n+1}^*(u) = (2n + 1)(2u - 1)P_n^*(u) - nP_{n-1}^*(u)
\]

\( P_0^*(u) = 1 \quad P_1^*(u) = 2u - 1 \)

Thus, \( A_{n,m} \) is also satisfy a similar recurrence relation

\[
(n + 1)A_{n+1,m} = (2n + 1)(2u - 1)A_{n,m}(u) - nA_{n-1,m}(u)
\]

(5.2)

\( A_0 = 0 \quad A_1 = 2 \)

(5.3)

With (5.2) and (5.3), it is not difficult to prove that

\[
A_{n,m} = 2H_n, \quad 0 \leq m \leq n
\]
6. Proof of Theorem 2 and Theorem 3

All the arguments are based on the formula (3.1):

\[ \gamma = A_{n,m} - L_{n,m} + J_{n,m} \iff d_n \gamma = d_n A_{n,m} - d_n L_{n,m} + d_n J_{n,m} \]

Thus,

\[ d_n \gamma \in \mathbb{Z} \iff d_n L_{n,m} - d_n J_{n,m} \in \mathbb{Z} \quad (6.1) \]

\[ \{d_n (-1)^m L_{n,m}\} = \{d_n (-1)^m J_{n,m}\} \quad (6.2) \]

since \( A_{n,m} = 2H_n \) and \((-1)^m J_{n,m}\) is positive.

On the other hand, Lemma 2 implies that the sequence \((J_{n,m})_n\) converges to 0 as \(4^{-n}\). The numbers \(d_n\) converges to infinity as \(e^n\). Thus \(d_n\gamma \in \mathbb{Z}\) if \((J_{n,m})_n\) converges to 0.

Thus, for all sufficiently large \(n\), \(d_n (-1)^m J_{n,m}\) is decreasing to 0. So, for all \(n \geq N\), \(d_n \gamma \in \mathbb{Z}\) for some \(N\) and \(d_n \gamma \in \mathbb{Z}\).

In (6.2), we substitute \(d_n\) by an upper bound: \(d_n \leq e^{1.039}\, 4^{-n}\) [Rosser et al. 1962]. Thus \(\forall n, d_n (-1)^m J_{n,m} \leq e^{1.039} \, 4^{-n} < 0.707^n\) and Theorem 3 is proved.

7. Proof of Corollaries

1) In the numerical computation of formula

\[ L_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1), \]

the problem is the evaluation of logarithmic functions.

A mean to avoid this drawback is the substitution of \(\ln(n-m+k+1)\) by some suitable approximations, enough good to keep the irrationality criteria. We will show now that Padé approximants satisfy this condition:

another expression of \(L_{n,m}\) is

\[ L_{n,m} = \ln(n-m+1) + \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln \left(1 + \frac{k}{n-m+1}\right). \]

The Padé error for the logarithmic function is

\[ \ln(1+x) - [n/n]_x = \frac{(-1)^n x^{n+1}}{P_n(-1/x)} \int_0^1 \frac{t^n (1-t)^n}{(1+x t)^{n+1}} \, dt \quad (7.1) \]

Let us set

\[ L'_{n,m} := \ln(n-m+1) + \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} [n/n]_{t=k/(n-m+1)} \]

We have to evaluate the difference \(\delta_{n,m} := L_{n,m} - L'_{n,m}\). For sake of simplicity, we set \(\zeta_k = \frac{k}{n-m+1}\).
\[ \delta_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} (\ln(1 + \zeta_k) - \lfloor n/n \rfloor t = \zeta_k) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{k} \frac{\zeta_k^{n+1}}{P_n^*(-\zeta_k^{-1})} \int_0^1 \frac{t^n (1-t)^n}{(1 + \zeta_k t)^{n+1}} \, dt \]

Since \( \zeta_k \in [0, 1] \) and \( P_n^* \) has all its roots in \([0, 1] \), \( \left| \frac{\zeta_k^n}{P_n^*(-\zeta_k^{-1})} \right| \leq \frac{1}{|P_n^*(-1)|} \). On the other hand, the integral

\[ \int_0^1 \frac{t^n (1-t)^n}{(1 + \zeta_k t)^{n+1}} \leq 4^{-n} \int_0^1 \frac{1}{(1 + \zeta_k t)^{n+1}} \leq 4^{-n} \frac{1}{n \zeta_k}. \]

So,

\[ |\delta_{n,m}| \leq \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left| \frac{\zeta_k^{n+1}}{P_n^*(-\zeta_k^{-1})} \right| \left| \int_0^1 \frac{t^n (1-t)^n}{(1 + \zeta_k t)^{n+1}} \, dt \right| \]

\[ \leq \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{\zeta_k}{|P_n^*(-1)|} 4^{-n} \frac{1}{n \zeta_k} \]

\[ \leq \frac{1}{|nP_n^*(-1)|} 4^{-n} \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} = \frac{1}{|nP_n^*(-1)|} 4^{-n} |P_n^*(-1)| = (n \cdot 4^n)^{-1} \]

The goal is partly reached since the error between \( L_{n,m} \) and its approximation is less than \( J_{n,m} \). Now, let us consider the approximation of \( \ln(n - m + 1) \). It is difficult to approximate this number (which tends to infinity) with an error less than \( 4^{-n} \). So, we consider sequences of integers \( n \), such that \( n - m + 1 \) is a power of \( 2 \): \( n - m + 1 = 2^p \). With this hypothesis, \( \ln(n - m + 1) = p \ln 2 \).

In (7.1), if \( x = 1 \), \( \ln 2 - \lfloor n/n \rfloor_x = \left( \frac{-1}{n} \right) \int_0^1 \frac{t^n (1-t)^n}{(1 + t)^{n+1}} \, dt \). The asymptotics for Legendre polynomials are well known

\( P_n(\alpha) \sim (\alpha + \sqrt{\alpha^2 - 1})^n \), for \( \alpha \in \mathbb{R} \setminus [-1, 1] \).

Thus shifted Legendre Polynomials satisfy

\( P_n^*(t) \sim ((2t - 1) + 2\sqrt{t^2 - t})^n \), for \( t \in \mathbb{R} \setminus [0, 1] \).
The maximum of the fraction \( \frac{t(1-t)}{1+t} \) for \( t \in [0, 1] \) is obtained for \( t = \sqrt{2} - 1 \), and its value is \((3 - 2\sqrt{2})\). Thus

\[
|\ln 2 - [n/n]_{x=1}| \leq \frac{(3 - 2\sqrt{2})^n}{(3 + 2\sqrt{2})^n} \ln 2
\]

For \( n - m + 1 = 2^p \), \( \ln(2^p) - p \, [n/n]_{x=1} \leq p \, (3 - 2\sqrt{2})^2 \) which is a \( o(4^{-n}/n) \). At last, the error \( |L_{n,m} - \tilde{L}_{n,m}| \) satisfies

\[
|L_{n,m} - \tilde{L}_{n,m}| \leq (4^{-n}/n)
\]

and the corollary 1 is proved.

2) For the proof of Corollary 2 we exploit the property of totally monotonic sequences (TMS).

A sequence \( u_n \) is called TMS if there exists a non negative measure \( d\mu \) with infinitely many points of increase such that

\[
\forall n \in \mathbb{N}, \quad u_n = \int_0^{\infty} x^n \, d\mu(x).
\]

If the support of the measure \( d\mu \) is the interval \([0, 1/R]\), then \( \forall n, u_{n+1}/u_n \leq R \) and \( \lim_{n} \frac{u_{n+1}}{u_n} = R \). If \( R = 1 \), it is equivalent to

\[
\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, \quad (-1)^k \Delta^k(u_n) > 0
\]

where \( \Delta^0(u_n) := u_n \) and \( \Delta^{k+1}u_n = \Delta^k u_{n+1} - \Delta^k u_n \). (see W\( \text{idder} \) 1941, p. 108).

The previous properties can be applied to the sequence \( J_{n,m} \) for which we prove some convergence properties. If they are not satisfied by \( \{d_n(-1)^m L_{n,m}\} \) then \( \gamma \) is irrational.

First we will prove that \( J_{n,m} \) satisfies \( d_{2n}(-1)^m J_{2n,m} < J_{n,m} \): the numbers \( d_n \) and \( J_{n,m} \) satisfy \( 2^n \leq d_n < e^{1.039 \, n} \) (see Tenenbaum 1990, p.12-13 for the lower bound and Rosser et al. 1962 for the upper one) \( \frac{J_{n+1,m}}{J_{n,m}} < 1/4 \) (property of totally monotonic sequence W\( \text{idder} \) 1941, p.135).

\[
\frac{d_n J_{n,m}}{d_{2n} J_{2n,m}} > \frac{2^n}{e^{1.039 \times 2n}} 4^n > 1.0014
\]

Thus, for all integer \( m \), \( (d_{2^p}(-1)^m J_{2^p,m})_{p \in \mathbb{N}} \) is a positive decreasing sequence, converging to 0. So, if \( \{d_{2^p}(-1)^m L_{2^p,m}\}_p \) is non decreasing for \( p \) greater than any integer, then \( \gamma \) is irrational.

I would thank my colleague S. Eliahou for reference (Pilehrood 2004)
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