BRAUER GROUP OF MODULI SPACES OF PGL(r)–BUNDLES OVER A CURVE

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ABSTRACT. We compute the Brauer group of the moduli stack of stable PGL(r)–bundles on a curve X over an algebraically closed field of characteristic zero. We also show that this Brauer group of such a moduli stack coincides with the Brauer group of the smooth locus of the corresponding coarse moduli space of stable PGL(r)–bundles on X.

1. Introduction

Let k be an algebraically closed field of characteristic zero. Let X be an irreducible smooth projective curve defined over k of genus g, with g ≥ 2. Fix an integer r ≥ 2. Let N(r) be the moduli stack of stable PGL(r, k) bundles on X. Let

\[ N(r) \to N(r) \]

be the coarse moduli space. Our aim is to study the Brauer group of N(r) and that of the smooth locus of N(r).

Throughout this paper we assume that if g = 2, then r is at least three.

The following proposition is proved in Section 2 (see Proposition 2.4):

Proposition 1.1. The morphism N(r) \to N(r) is an isomorphism outside a codimension three closed subset. In particular, this morphism identifies the Brauer group of the smooth locus of N(r) with the Brauer group \( Br(N(r)) \).

Let

\[ N^0(r) \subset N(r) \]

be the locus of stable projective bundles admitting no nontrivial automorphisms. Similarly, let \( N^0(r) \subset N(r) \) be the substack of stable projective bundles whose automorphism group is trivial. Then the natural morphism \( N^0(r) \to N^0(r) \) is an isomorphism.

Fix a line bundle ξ over X. Let \( M(r, ξ) \) denote the moduli stack of stable vector bundles \( E \to X \) of rank r and determinant ξ, meaning \( \bigwedge^r E \) is isomorphic to ξ. This moduli stack \( M(r, ξ) \) is an irreducible smooth Deligne–Mumford stack having a smooth quasiprojective coarse moduli space. This coarse moduli space, which is of dimension

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\((g - 1)(r^2 - 1)\), will be denoted by \(M(r, \xi)\). The Brauer group of \(M(r, \xi)\) has the following description: the natural morphism
\[
\mathcal{M}(r, \xi) \rightarrow M(r, \xi)
\]
is a \(\mathbb{G}_m\)-gerbe, and its class in \(Br(M(r, \xi))\) is a generator of order \(\gcd(r, \deg(\xi))\) \([2]\).

The connected components of both \(N(r)\) and \(N(r)\) are indexed by \(\mathbb{Z}/r\mathbb{Z}\)
\[
N(r) = \bigcup_{i \in \mathbb{Z}/r\mathbb{Z}} N(r)_i \quad \text{and} \quad N(r) = \bigcup_{i \in \mathbb{Z}/r\mathbb{Z}} N(r)_i .
\]
The component \(N(r)_i\) corresponds to the projective bundles associated to the vector bundles on \(X\) of degree \(i\). More precisely, if \(\deg(\xi) \equiv i \mod r\), then \(N(r)_i\) is the quotient of \(M(r, \xi)\) by the subgroup \(\Gamma \subset \text{Pic}(X)\) of \(r\)-torsion points; the action of any \(\zeta \in \Gamma\) sends any \(E\) to \(E \otimes \zeta\).

As before, fix a line bundle \(\xi \rightarrow X\). The image of degree(\(\xi\)) in \(\mathbb{Z}/r\mathbb{Z}\) will be denoted by \(i\). If \(g = 2\), then we assume that \(r \geq 3\). For any \(\zeta \in \Gamma\), let \(\widehat{\zeta} : M(r, \xi) \rightarrow M(r, \xi)\) be the automorphism defined by \(E \mapsto E \otimes \zeta\). Let
\[
\mathcal{L}_0 \in \text{Pic}(M(r, \xi)) = \mathbb{Z}
\]
be the ample generator. Let \(\tilde{\Gamma}_i\) be the group of pairs of the form \((\zeta, \sigma)\), where \(\zeta \in \Gamma\) and \(\sigma\) is an isomorphism of \(\mathcal{L}_0\) with \(\widehat{\zeta}^* \mathcal{L}_0\). So we have a central extension
\[
0 \rightarrow k^* \rightarrow \tilde{\Gamma}_i \rightarrow \Gamma \rightarrow 0 .
\]
Let \(\nu_i \in H^2(\Gamma, k^*)\) be the corresponding extension class.

We prove the following theorem; its proof is given in Section \[3\].

**Theorem 1.2.** There is a short exact sequence
\[
0 \rightarrow H^2(\Gamma, k^*)_H \rightarrow Br(N(r)_i) \rightarrow \mathbb{Z}/\delta \mathbb{Z} \rightarrow 0 ,
\]
where \(H \subset H^2(\Gamma, k^*)\) is the subgroup of order \(\delta := \text{g.c.d.}(r, i)\) generated by \(\nu_i\).

The class \(\nu_i\) is described in Remark \[3.3\]

The short exact sequence in Theorem \[1.2\] is constructed using the Leray spectral sequence for the quotient map \(M(r, \xi) \rightarrow M(r, \xi)/\Gamma = N(r)_i\).

The following theorem provides more information on the exact sequence in Theorem \[1.2\]

**Theorem 1.3.** For a point \(x \in X(k)\), let \(i_x : N(r) \rightarrow N(r) \times_k X\) be the constant section of the projection \(N(r) \times_k X \rightarrow N(r)\) passing through \(x\). Let \(\alpha \in Br(N(r))\) be the class of the projective bundle obtained by pulling back the universal \(\text{PGL}(r, k)\) bundle over \(N(r) \times_k X\) by \(i_x\).
Then the following two hold:

(i) The class $\alpha$ is independent of $x$, and $\tau(\alpha) = 1 \in \mathbb{Z}/\delta\mathbb{Z}$, where $\tau$ is the homomorphism in Theorem 1.2.

(ii) The order of $\alpha$ restricted to each component of $\mathcal{N}(r)$ is exactly $r$. Therefore, the short exact sequence in Theorem 1.2 is split if and only if $\delta = r$.

From the second part of Theorem 1.3 it follows that the principal $\text{PGL}(r, k)$–bundle over $N^0(r)$ obtained by restricting the projective bundle in Theorem 1.3 is stable (see Corollary 5.1).

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2. Fixed points of the moduli space

Throughout this section we fix a line bundle $\xi$ over $X$, and we also fix an integer $r \geq 2$. As mentioned in the introduction, if $g = 2$, then $r$ is taken to be at least three. Let $M(r, \xi)$ denote the moduli space of stable vector bundles $E \to X$ of rank $r$ with $\bigwedge^r E = \xi$. For convenience the moduli space $M(r, \xi)$ will also be denoted by $M$. Let

$$\Gamma := \text{Pic}^0(X)_r = \{ \eta \in \text{Pic}^0(X) \mid \eta^{\otimes r} \cong \mathcal{O}_X \}$$

be the subgroup of $r$–torsion points of the Picard group of $X$. For any $\eta \in \Gamma$, we have the automorphism

$$\phi_\eta : M \to M$$

that sends any $E$ to $E \otimes \eta$. Let

$$\phi : \Gamma \to \text{Aut}(M)$$

be the homomorphism defined by $\eta \mapsto \phi_\eta$.

Take any nontrivial line bundle

$$\eta \in \Gamma \setminus \{ \mathcal{O}_X \}.$$

Let

$$M^\eta \subset M$$

be the closed subvariety fixed by the automorphism $\phi_\eta$ in (2.2). We will describe $M^\eta$.

Let $m$ be the order of $\eta$. Fix a nonzero section

$$\sigma : X \to \eta^{\otimes m}.$$

Consider the morphism of varieties

$$f_\eta : \eta \to \eta^{\otimes m}$$

defined by $v \mapsto v^{\otimes m}$. Set

$$Y := f_\eta^{-1}(\sigma(X)) \subset \eta,$$
where $\sigma$ is the section in (2.5). Let

\[(2.7) \quad \gamma : Y \longrightarrow X\]

be the morphism obtained by restricting the natural projection of $\eta$ to $X$. So $\gamma$ is an étale Galois covering of degree $m$.

The curve $Y$ is irreducible. To prove this, let $Y_1$ be an irreducible component of $Y$. Let $m_1$ be the degree of the restriction $\gamma_1 := \gamma|_{Y_1}$, where $\gamma$ is defined in (2.7). We have a nonzero section of $\eta^{\otimes m_1}$ whose fiber over any point $x \in X$ is $\bigotimes_{\theta \in \gamma_1^{-1}(x)} \theta$ (recall that $\gamma_1^{-1}(x)$ is a subset of the fiber $\eta_x$). Therefore, the line bundle $\eta^{\otimes m_1}$ is trivial. Since the order of $\eta$ is precisely $m$, we have $Y_1 = Y$. Hence $Y$ is irreducible.

We note that the covering $Y$ is independent of the choice of the section $\sigma$. Indeed, if $\sigma$ is replaced by $c \cdot \sigma$, where $c \in k^*$, then the automorphism of $\eta$ defined by multiplication with $c$ takes $Y$ in (2.6) to the covering corresponding to the section $c \cdot \sigma$.

Lemma 2.1. There is a nonempty Zariski open subset $U \subset N_\gamma$ such that for any $F \in U$, $\gamma_* F \in M^\eta$.

Furthermore, the morphism

$$U \longrightarrow M^\eta$$

defined by $F \longmapsto \gamma_* F$ is surjective.

Proof. Note that the line bundle $\gamma^* \eta$ has a tautological trivialization because $Y$ is contained in the complement of the image of the zero section in the total space of $\eta$ over which the pullback of $\eta$ is tautologically trivial. Take any vector bundle $V$ over $Y$. The trivialization of $\gamma^* \eta$ yields an isomorphism

$$V = V \otimes \mathcal{O}_Y \longrightarrow V \otimes \gamma^* \eta.$$ 

This isomorphism gives an isomorphism

$$\gamma_* V \longrightarrow \gamma_*(V \otimes \gamma^* \eta) = (\gamma_* V) \otimes \eta$$

(the vector bundle $(\gamma_* V) \otimes \eta$ is identified with $\gamma_*(V \otimes \gamma^* \eta)$ using the projection formula).

Therefore, if $V \in N_\gamma$, and $\gamma_* V$ is stable, then $\gamma_* V \in M^\eta$.

Take any $E \in M^\eta$. Fix an isomorphism

$$\beta : E \longrightarrow E \otimes \eta.$$ 

Since $E$ is stable, it follows that $E$ is simple, which means that all global endomorphisms of $E$ are constant scalar multiplications. Therefore, any two isomorphisms between $E$ and $E \otimes \eta$ differ by multiplication with a constant scalar.
We re-scale the section $\beta$ by multiplying it with a nonzero scalar such that the $m$–fold composition

\[
\underbrace{\beta \circ \cdots \circ \beta}_{m\text{-times}} : E \rightarrow E \otimes \eta^\otimes m
\]

coincides with $\text{Id}_E \otimes \sigma$, where $\sigma$ is the section in (2.5).

Let

\[
\theta \in H^0(X, \text{End}(E) \otimes \eta)
\]

be the section defined by $\beta$. Consider the pullback

\[
\gamma^* \theta \in H^0(Y, \gamma^* \text{End}(E) \otimes \gamma^* \eta)
\]

of the section in (2.9). Using the canonical trivialization of $\gamma^* \eta$, it defines a section

\[
\theta_0 \in H^0(Y, \gamma^* \text{End}(E)) = H^0(Y, \text{End}(\gamma^* E))
\]

Since $Y$ is irreducible, the characteristic polynomial of $\theta_0(y)$ is independent of $y \in Y$. Therefore, the eigenvalues of $\theta_0(y)$, along with their multiplicities, do not change as $y$ moves over $Y$. Consequently, for each eigenvalue $\lambda$ of $\theta_0(y)$, we have the associated generalized eigenbundle

\[
\gamma^* E \supset E^\lambda \rightarrow Y
\]

whose fiber over any $y \in Y$ is the generalized eigenspace of $\theta_0(y) \in \text{End}((\gamma^* E)_y)$ for the fixed eigenvalue $\lambda$.

Since the composition in (2.8) coincides with $\text{Id}_E \otimes \sigma$, we have

\[
\underbrace{\theta_0 \circ \cdots \circ \theta_0}_{m\text{-times}} = \text{Id}_{\gamma^* E},
\]

where $\theta_0$ is constructed in (2.10) from $\beta$. Therefore, if $\lambda$ is an eigenvalue of $\theta_0(x)$, then $\lambda^m = 1$.

The Galois group $\text{Gal}(\gamma)$ for the covering $\gamma$ in (2.7) is identified with the group of all $m$–th roots of 1; the group of $m$–th roots of 1 will be denoted by $\mu_m$. The action of $\mu_m$ on $Y$ is obtained by restricting the multiplicative action of $\mathbb{G}_m$ on the line bundle $\eta$. Note that $\text{Gal}(\gamma)$ has a natural action on the pullback $\gamma^* E$ which is a lift of the action of $\text{Gal}(\gamma)$ on $Y$. Examining the construction of $\theta_0$ from $\beta$ it follows that the action of any $\rho \in \text{Gal}(\gamma) = \mu_m$

on $\gamma^* E$ takes the eigenbundle $E^\lambda$ (see (2.11)) to the eigenbundle $E^{\lambda \rho}$. This immediately implies that each element of $\mu_m$ is an eigenvalue of $\theta_0(y)$ (we noted earlier that the eigenvalues lie in $\mu_m$), and the multiplicity of each eigenvalue of $\theta_0(y)$ is $r/m$.

Consider the subbundle

\[
E^1 \rightarrow Y
\]
of \( \gamma^*E \), which is the eigenbundle for the eigenvalue \( 1 \in \mu_m \) (see (2.11)). Define
\[
\widetilde{E}^1 := \bigoplus_{\rho \in \text{Gal}(\gamma)} \rho^*E^1.
\]
There is a natural action of \( \text{Gal}(\gamma) = \mu_m \) on \( \widetilde{E}^1 \). Since the action of any \( \rho \in \mu_m \) on \( \gamma^*E \) takes the eigenbundle \( E^\lambda \) to the eigenbundle \( E^{\lambda\rho} \), it follows immediately that we have a \( \text{Gal}(\gamma) \)-equivariant identification
\[
(2.13) \quad \gamma^*E = \widetilde{E}^1 := \bigoplus_{\rho \in \text{Gal}(\gamma)} \rho^*E^1.
\]
In view of this \( \text{Gal}(\gamma) \)-equivariant isomorphism we conclude that the vector bundle \( \gamma^*E^1 \) is isomorphic to \( E \).

To complete the proof of the lemma it remains to show that the vector bundle \( E^1 \) is stable.

Take any vector bundle \( W \to Y \). Since the map \( \gamma \) is finite, we have
\[
H^i(Y, W) = H^i(X, \gamma_*W)
\]
for all \( i \). Let \( d_W \) (respectively, \( \overline{d} \)) be the degree of \( W \) (respectively, \( \gamma_*W \)), and let \( r_W \) be the rank of \( W \); so, \( \text{rank}(\gamma_*W) = mr_W \). From Riemann–Roch theorem and the Hurwitz’s formula
\[
(2.14) \quad \text{genus}(Y) = m(g - 1) + 1
\]
we have
\[
\chi(W) = d_W - r_W m(g - 1) = \overline{d} - m r_W (g - 1) = \chi(\gamma_*W).
\]
Hence
\[
(2.15) \quad d_W = \overline{d}.
\]
We now note that if \( W_1 \subset W \) is a nonzero algebraic subbundle such that
\[
\text{degree}(W_1)/\text{rank}(W_1) \geq \text{degree}(W)/\text{rank}(W),
\]
then from (2.15) we have
\[
\text{degree}(\gamma_*W_1)/\text{rank}(\gamma_*W_1) \geq \text{degree}(\gamma_*W)/\text{rank}(\gamma_*W).
\]
Hence \( W \) is stable if \( \gamma_*W \) is so. In particular, \( E^1 \) is stable because \( \gamma_*E^1 = E \) is stable. \( \square \)

Using (2.14), we have
\[
(2.16) \quad \dim N_\gamma = \frac{r^2}{m^2} (\text{genus}(Y) - 1) + 1 - g = (g - 1) \left( \frac{r^2}{m} - 1 \right).
\]
Hence \( \dim M - \dim N_\gamma > 3 \). Let
\[
(2.17) \quad Z := \bigcup_{L \in \Gamma \setminus \{O_X\}} M^L \subset M
\]
be the closed subscheme. We note that Lemma 2.1 has the following corollary.

**Corollary 2.2.** The codimension of the closed subscheme \( Z \subset M \) is at least three.
For a vector space $V$, by $\mathbb{P}(V)$ we will denote the projective space parametrizing all lines in $V$. Similarly, for a vector bundle $W$, by $\mathbb{P}(W)$ we denote the projective bundle parametrizing all lines in $W$.

**Lemma 2.3.** Take any $E \in M \setminus \mathcal{Z}$, where $\mathcal{Z}$ is constructed in (2.17). Consider the vector bundle $F = \phi_\eta(E) = E \otimes \eta$, where $\phi_\eta$ is the automorphism in (2.2). Then there is a unique isomorphism of the projective bundle

$\mathbb{P}(F) \rightarrow X$

with $\mathbb{P}(E) \rightarrow X$ over the identity map of $X$.

**Proof.** There is a natural isomorphism of $\mathbb{P}(E)$ with $\mathbb{P}(E \otimes \eta)$. If there are two distinct isomorphisms of $\mathbb{P}(E)$ with $\mathbb{P}(E \otimes L)$, we get a nontrivial automorphism of $\mathbb{P}(E)$.

Let $f : \mathbb{P}(E) \rightarrow \mathbb{P}(E)$ be a nontrivial automorphism. There is a line bundle $L_0$ and an isomorphism of vector bundles

$\tilde{f} : E \rightarrow E \otimes L_0$

such that $\tilde{f}$ induces $f$. Since $\tilde{f}$ induces an isomorphism of $\det E$ with $\det(E \otimes L_0) = (\det E) \otimes L_0^{\otimes r}$, we conclude that $L_0 \in \Gamma$. Next note that $L_0$ must be trivial because $E \notin \mathcal{Z}$.

The vector bundle $E$ is simple because it is stable. Hence all automorphisms of $E$ induce the identity map of $\mathbb{P}(E)$. Therefore, $\mathbb{P}(E)$ does not admit a nontrivial automorphism. This completes the proof of the lemma.

**Proposition 2.4.** The morphism $N(r) \rightarrow N(r)$ in (1.1) is an isomorphism in codimension three.

**Proof.** Let $i \in \mathbb{Z}/r\mathbb{Z}$ be the image of degree($\xi$). In the proof of Lemma 2.3 we saw that if a projective bundle $P \in N(r)_i$ admits a nontrivial automorphism, then $P$ lies in the image of the subscheme $\mathcal{Z}$ by the projection $M(r, \xi) \rightarrow N(r)_i$. Therefore, the proposition follows from Corollary 2.2.

Let

$$U := M \setminus \mathcal{Z},$$

be the Zariski open subset, where $\mathcal{Z}$ is constructed in (2.17). Let

$$P \rightarrow M \times X$$

be the universal projective bundle (see [2]).

Fix a point $x \in X$. Consider the restriction

$$P_x : = P|_{M \times \{x\}} \rightarrow M.$$

Let

$$P^0 : = P_x|_U \rightarrow U$$

be the restriction of $P_x$ to the open subset $U$ defined in (2.18).
The action of $\Gamma$ on $M$ (see (2.3)) clearly preserves $U$. The resulting action of $\Gamma$ on $U$ and the trivial action of $\Gamma$ on $X$ together define an action of $\Gamma$ on $U \times X$.

Lemma 2.3 has the following corollary:

**Corollary 2.5.** There is a unique lift of the action of $\Gamma$ on $U \times X$ to the projective bundle $P^U := P|_{U \times X} \to U \times X$, where $P$ is defined in (2.19). In particular, the projective bundle $P^0$ in (2.21) admits a canonical lift of the action of $\Gamma$ on $U$, which is obtained by restricting the action of $\Gamma$ on $P|_{U \times X}$.

**Remark 2.6.** Take any vector bundle $E \in M^\eta$ with $\eta \neq O_X$. Then the automorphism of $P(E)$ given by an isomorphism of $E$ with $E \otimes \eta$ is nontrivial. Therefore, a vector bundle $F$ lies in $U$ if and only if the projective bundle $P(F)$ does not admit any nontrivial automorphism.

### 3. Moduli space of PGL($r,k$–bundles

Define

\begin{equation}
U_P := U/\Gamma,
\end{equation}

where $U$ and $\Gamma$ are defined in (2.18) and (2.1) respectively. Note that $\Gamma$ acts freely on $U$ and thus $U_P$ is an open substack of the component $N$ of $N(r)$, where

\[\deg \xi \equiv i \mod r.\]

In fact, $U_P$ is also the moduli stack of stable PGL($r,k$–bundles over $X$ without automorphisms and with topological invariant degree($\xi$) $\in \mathbb{Z}/r\mathbb{Z}$ (see Remark 2.6).

We will now recall the descriptions of Pic($U_P$) and Pic($M(r,\xi)$). Let $\delta$ be the greatest common divisor of $r$ and degree($\xi$). Fix a semistable vector bundle

\[F \to X\]

such that rank($F$) = $r/\delta$ and

\[\text{degree}(F) = \frac{r(g-1) - \text{degree}(\xi)}{\delta}.
\]

Let $\tilde{N}$ be the moduli space of semistable vector bundles of rank $r^2/\delta$ and degree $r^2(g-1)/\delta$. This moduli space has the theta divisor $\Theta$ that parametrizes vector bundles $V$ with $h^0(V) \neq 0$ (from Riemann–Roch, $h^0(V) = h^1(V)$ for all $V \in \tilde{N}$). We have a morphism

\[\psi : M(r,\xi) \to \tilde{N}\]

defined by $E \mapsto E \otimes F$, where $F$ is the vector bundle fixed above.

The Picard group of $M(r,\xi)$ is isomorphic to $\mathbb{Z}$, and the ample generator of Pic($M(r,\xi)$) is the pull back

\begin{equation}
L_0 := \psi^*O_{\tilde{N}}(\Theta) \to M(r,\xi)
\end{equation}

(see [4]).
Remark 3.1. From Corollary 2.2 it follows that the inclusion of $\mathcal{U}$ in $M(r, \xi)$ (see (2.18) for $\mathcal{U}$) induces an isomorphism of Picard groups. Therefore, $\text{Pic}(\mathcal{U}) = \mathbb{Z}$, and it is generated by the restriction of $\mathcal{L}_0$.

Lemma 3.2. Let $\mathcal{U}_P$ be the quotient defined in (3.1). Then the image of the homomorphism

$$\text{Pic}(\mathcal{U}_P) \rightarrow \text{Pic}(\mathcal{U})$$

induced by the quotient map is generated by $\mathcal{L}_\delta$, where $\delta = \text{g.c.d.}(r, \text{degree}(\xi))$, and $\mathcal{L}_0$ is the line bundle in (3.2).

The above lemma follows from [3, p. 184, Theorem].

The quotient

(3.3) $\mathcal{P}^U/\Gamma \rightarrow \mathcal{U}_P \times X$

(see Corollary 2.5) is the universal projective bundle. Let

(3.4) $\mathcal{P}_P^0 := (\mathcal{P}^U/\Gamma)|_{\mathcal{U}_P \times \{x\}} \rightarrow \mathcal{U}_P$

be the projective bundle, where $x$ is a fixed point of $X$ as in (2.20). Note that

$$\mathcal{P}_P^0 = \mathcal{P}^0/\Gamma,$$

where $\mathcal{P}^0$ is constructed in (2.21), and the quotient is for the canonical action of $\Gamma$ constructed in Corollary 2.5.

Proof of Theorem 1.2. Fix a line bundle $\xi \rightarrow X$ such that $\text{deg}(\xi) \equiv i \mod r$. Let

$$\pi : \mathcal{U} \rightarrow \mathcal{U}/\Gamma$$

be the quotient map (see (3.1)). By Leray spectral sequence, and the identification

$$\text{H}^1(\Gamma, \text{Pic}(\mathcal{U})) = \text{H}^1(\Gamma, \mathbb{Z}) = 0,$$

we get the following exact sequence:

(3.5) $\text{Pic}(\mathcal{U}_P) \rightarrow \text{Pic}(\mathcal{U})^\Gamma \xrightarrow{\varphi} \text{H}^2(\Gamma, k^*) \rightarrow \text{Br}(\mathcal{U}_P) \rightarrow \text{Br}(\mathcal{U})^\Gamma$.

The action of $\Gamma$ on $\text{Pic}(\mathcal{U})$ is trivial because the action of $\Gamma$ preserves the ample generator of $\text{Pic}(\mathcal{M}(r, \xi)) = \text{Pic}(\mathcal{U})$ (see Remark 3.1). Thus by Lemma 3.2 the cokernel of the homomorphism

$$\text{Pic}(\mathcal{N}(r,i)) \rightarrow \text{Pic}(\mathcal{M}(r, \xi))^\Gamma$$

is isomorphic to $\mathbb{Z}/\delta$, where $\delta = \text{g.c.d.}(r, i)$.

From Corollary 2.2 it follows that the inclusion map of $\mathcal{U}$ in $M(r, \xi)$ induces an isomorphism of Brauer groups. The Brauer group of $M(r, \xi)$ is generated by the class of the projective bundle $\mathcal{P}^0$ constructed in (2.21) [2]. Therefore, by Corollary 2.5 the homomorphism

$$\text{Br}(\mathcal{U}_P) \rightarrow \text{Br}(\mathcal{U})$$

is surjective.
From Corollary 2.2 we know that the inclusion map of $U_P$ in $N(r)$ induces an isomorphism of Brauer groups (see also Proposition 2.4). Thus we get the following exact sequence

\[(3.6) \quad 0 \rightarrow H^2(\Gamma, k^*) \rightarrow Br(N(r)_i) \rightarrow \tau/\delta \rightarrow 0,\]

where $H \subset H^2(\Gamma, k^*)$ is the subgroup of order $\delta$. It remains to find the generator of $H$.

Consider the line bundle $L_0$ in (3.2). Let $\tilde{\Gamma}_i$ be the group of all pairs of the form $(\zeta, \sigma)$, where $\zeta \in \Gamma$, and $\sigma$ is an isomorphism $L_0 \rightarrow \phi^*_\zeta L_0$; the map $\phi\zeta$ is defined in (2.2). The group operation on $\tilde{\Gamma}_i$ is the following:

\[(\zeta_1, \sigma_1(\phi\zeta(z))) \cdot (\zeta, \sigma(z)) = (\zeta_1 \otimes \zeta, (\sigma_1 \circ \sigma)(z)),\]

where $z \in M(r, \xi)$. There is a natural projection $\tilde{\Gamma}_i \rightarrow \Gamma$ that sends any $(\zeta, \sigma)$ to $\zeta$. Consequently, we have the central extension

\[0 \rightarrow k^* \rightarrow \tilde{\Gamma}_i \rightarrow \Gamma \rightarrow 0.\]

Let

\[(3.7) \quad \nu_i \in H^2(\Gamma, k^*)\]

be the corresponding extension class. First note that

\[\nu_i = \varphi(L_0),\]

where $\varphi$ is the homomorphism in (3.5). Recall that $L_0$ is the generator of $\text{Pic}(U)$. Hence the subgroup $H$ is (3.6) is generated by $\nu_i$. This completes the proof of the theorem. \square

**Remark 3.3.** Let $\bigwedge^2 \Gamma$ be the quotient of $\Gamma \otimes \Gamma$ by the subgroup of elements of the form $x \otimes y - y \otimes x$, where $x, y \in \Gamma$. The space of all extensions of $\Gamma$ by $k^*$ is parametrized by $\text{Hom}(\bigwedge^2 \Gamma, k^*)$ (see [6, pp. 217–218, Theorem 4.4]). We have $\Gamma = H^1_{et}(X, \mu_r)$, and $\text{Hom}(\bigwedge^2 \Gamma, k^*) = \text{Hom}(\bigwedge^2 H^1_{et}(X, \mu_r), \mu_r)$, where $\mu_r$ is the group of $r$–th roots of 1. The cup product

\[H^1_{et}(X, \mu_r) \otimes H^1_{et}(X, \mu_r) \rightarrow H^2_{et}(X, \mu_r) = \mu_r\]

defines an element $\hat{\nu} \in \text{Hom}(\bigwedge^2 H^1_{et}(X, \mu_r), \mu_r)$. The element $\nu_i$ in (3.7) coincides with $(r/\delta) \cdot \hat{\nu}$.

### 4. Twisted Bundles on a $\mu_r$–Gerbe over a Curve

In this section we prove some results on twisted sheaves on a $\mu_r$–gerbe over a curve; these will be required in the proof of Theorem 1.3. We fix the following notation:

- $K$ is any field of characteristic zero, and $\overline{K}$ is an algebraic closure of $K$.
- $X/K$ is a smooth projective geometrically connected curve of genus $g \geq 2$, and having a $K$–point.
- $\mathcal{L}$ and $\Lambda$ are line bundles on $X$, with $\text{degree}(\mathcal{L}) = d$ and $\text{degree}(\Lambda) = 1$.
- $h : Y \rightarrow X$ is a $\mu_r$–gerbe; if $g = 2$, then we assume that $r \geq 3$. 

• $\mathcal{T}$ is the moduli stack of stable twisted sheaves on $X$ of rank $r$ and determinant $\mathcal{L}$, and $q : \mathcal{T} \to T$ is its coarse moduli space.

• $E \to T \times_K Y$ is the universal twisted bundle.

• $T := \mathcal{T} \times_K \overline{K}$, and $\overline{T} := T \times_K \overline{K}$.

Let $\xi$ be a line bundle on $X$ of degree $d$. As before, $\mathcal{M}(r, \xi)$ will denote the moduli stack of vector bundles over $X$ of rank $r$ and determinant $\xi$. Let $\pi : \mathcal{M}(r, \xi) \times X \to \mathcal{M}(r, \xi)$ denote the natural projection. Let

$$\delta = g.c.d.(r, d) = g.c.d.(r, \chi),$$

where $\chi = \chi(\xi) - (r - 1)(g - 1)$. So $\chi = \chi(E)$ for any vector bundle $E \to X$ parametrized by $\mathcal{M}(r, \xi)$. Let

$$\mathcal{E} \to \mathcal{M}(r, \xi) \times X$$

be the universal vector bundle. Define the line bundle

$$(4.1) \quad F := \det(\pi_* \mathcal{E}) \otimes \det(\mathcal{R}^1 \pi_* \mathcal{E})^* \to \mathcal{M}(r, \xi)$$

be the line bundle. Let $i_x : \mathcal{M}(r, \xi) \to \mathcal{M}(r, \xi) \times X$ be the constant section determined by a $K$–point $x \in X(K)$.

See [4] for the following theorem.

**Theorem 4.1.** The line bundle $(F^*)^{\otimes \frac{1}{\delta}} \otimes (i_x^* \det(\mathcal{E}))^{\otimes \frac{1}{\delta}}$ descends to a line bundle on the coarse moduli space $M(r, \xi)$, where $F$ is constructed in (4.1). Moreover, this descended line bundle is the ample generator of $\text{Pic}(M(r, \xi)) = \mathbb{Z}$.

**Proposition 4.2.** Using the above notation:

(i) $\overline{T}$ is non–canonically isomorphic to the moduli stack of stable vector bundles on $X \times_K \overline{K}$ of rank $r$ and determinant $\zeta$ for some fixed line bundle $\zeta$. Further, if

$$Y \times_K \overline{K} \to X \times_K \overline{K}$$

is a neutral $\mu_r$–gerbe, then $\text{degree}(\zeta) \equiv d \text{ mod } r$ for any such $\zeta$.

(ii) $\text{Pic}(T) = \text{Pic}(\overline{T}) = \mathbb{Z}$.

(iii) The natural homomorphism $\text{Br}(K) \to \text{Br}(T)$ is injective.

**Proof.** (i): This is proved in [5 3.1.2.1].

(ii): This follows from statement (i) and Theorem [4.1].

(iii): This follows immediately from statement (ii) and the exactness of the following sequence:

$$0 \to \text{Pic}(T) \to \text{Pic}(\overline{T})^{\text{Gal}(K)} \to \text{Br}(K) \to \text{Br}(T).$$

This completes the proof of the proposition. $\square$

**Proposition 4.3.** Assume the $\mu_r$–gerbe $Y \to X$ is a pull back of a $\mu_r$–gerbe on $\text{Spec}(K)$ which defines a class of order $r$ in $\text{Br}(K)$. For any $K$–point $x$ in $X$, let

$$i_x : T \to T \times_K X$$

denote the corresponding section. Then the following two statements hold.
(i) The class in $Br(T)$ defined by the $\mathbb{G}_m$-gerbe $T \rightarrow T$ has order exactly equal to $\delta = \text{g.c.d.}(r, d)$.

(ii) The order of $i^*_x(\text{End}(E))$ is exactly equal to $r$.

Proof. Let $\gamma$ denote the class of $T \rightarrow T$ in $Br(T)$. Let us first show that $\gamma$ is annihilated by $r$. Consider the Azumaya algebra $\text{End}(E)$ on $T \times_K X$. The class of $i^*_x(\text{End}(E))$ in $Br(T)$ is $\gamma + \beta$ where $\beta \in Br(T)$ is the image, by the natural homomorphism $Br(K) \rightarrow Br(T)$, of the class in $Br(K)$ defined by the restriction of $Y \rightarrow X$ to $x$. We have $r \cdot \beta = 0$ because $\beta$ is associated to a $\mu_r$-gerbe. Moreover, $r \cdot (\gamma + \beta) = 0$ since $\gamma + \beta$ is represented by an Azumaya algebra of rank $r^2$. Thus $r \cdot \gamma = 0$.

Proof of (i): Note that after base field extension to $\overline{K}$, the image of $\gamma$ in $Br(\overline{T})$ is precisely $\delta$. Thus it is sufficient to show $\delta \cdot \gamma = 0$. There exists an integer $m$ and a twisted vector bundle $F$ on $Y$ of rank $r^m$ and trivial determinant. Consider the vector bundle

$$\mathcal{H} = E \otimes F^*$$

on $T \times Y$. This vector bundle descends to a vector bundle on $T \times X$. Also note that the class of $i^*_x(\text{End}(\mathcal{H}))$ in $Br(T)$ is precisely $\gamma$. Let $\pi : T \times X \rightarrow T$ denote the natural projection. $\mathcal{H}$ can be thought of as a family of vector bundles on $X$ parametrized by $T$. Since this family is bounded, there exists an integer $m_0$ such that for all $m \geq m_0$,

$$\mathcal{H} \otimes \Lambda^m$$

has no higher cohomology on fibers of $\pi$. The vector bundle

$$\pi^*(\mathcal{H} \otimes \Lambda^m)$$

is a twisted vector bundle on $T$ of rank

$$d + rm + (r^m + r) \cdot (1 - g) .$$

Thus $\gamma$ is annihilated by $d + rm + (r^m + r) \cdot (1 - g)$. We already know that it is annihilated by $r$. This proves $\delta \cdot \gamma = 0$.

Proof of (ii): Let $a(\gamma + \beta) = 0$. We will show $a$ is divisible by $r$. Since the image of $\gamma$ in $Br(\overline{T})$ has order $\delta$ and image of $\beta$ in $Br(\overline{T})$ is zero, it follows that $a$ is divisible by $\delta$. Thus we get

$$a \cdot \beta = 0 .$$

By Proposition 4(ii), the order of the class $\beta$ is $r$. Hence (4.2) implies that $a$ is divisible by $r$. This completes the proof of the proposition. □

5. Proof of Theorem 1.3

Proof of (i): Let $E$ be the universal vector bundle over $\mathcal{M}(r, \xi) \times X$. For any point $x \in X$, let $E_x \rightarrow \mathcal{M}(r, \xi)$ be the vector bundle obtained by restricting $E$ to $\mathcal{M}(r, \xi) \times \{x\}$. Take two points $x, y \in X$. Note that the multiplicative action of $k^*$ on $E$ induces a trivial action of $k^*$ on $E_x^* \otimes E_y$. Therefore, $E_x^* \otimes E_y$ descends to $\mathcal{M}(r, \xi)$. This descended vector bundle on $\mathcal{M}(r, \xi)$ will also be denoted by $E_x^* \otimes E_y$. Next we note that the action of $\Gamma$ on
\( \mathcal{M}(r, \xi) \) lifts to an action of \( \Gamma \) on the vector bundle \( E^*_x \otimes E_y \). Let \( \mathcal{W} \) denote the vector bundle on \( \mathcal{U}_p \) defined by the \( \Gamma \)-equivariant vector bundle \( E^*_x \otimes E_y \). It is easy to see that the \( \text{PGL}(r^3, k) \)-bundle \( \mathcal{P}_p^0 \otimes \mathbb{P}(\mathcal{W}) \), where \( \mathcal{P}_p^0 \) is constructed in [3.4], is isomorphic to \( \mathcal{P}_p^{y,0} \otimes \mathbb{P}(\text{End}(E_x)) \), where \( \mathcal{P}_p^{y,0} \) is the projective bundle obtained by substituting \( x \) with \( y \) in the construction of \( \mathcal{P}_p^0 \). Consequently, the class \( \alpha \) in Theorem 1.3 is independent of \( x \). From [2] we know that \( \alpha \) maps to the generator of \( Br(\mathcal{M}(r, \xi)) \). Hence \( \alpha \) maps to \( 1 \in \mathbb{Z}/\delta \mathbb{Z} \) in Theorem 1.2.

Proof of (ii): Let \( \mathcal{N}(r)_i \) be a connected component of \( \mathcal{N}(r) \) for \( i \in \mathbb{Z}/r \). To prove the theorem it is enough to construct a \( k \)-scheme \( T \) together with an Azumaya algebra \( B \) on \( T \times_k X \) such that

(i) \( B \) gives a family of stable \( \text{PGL}(r, k) \) bundles on \( X \) lying in component \( \mathcal{N}(r)_i \), and

(ii) if \( i_x : T \to T \times_k X \) is the section given by the point \( x \), then \( i_x^* B \) has order precisely \( r \) in \( Br(T) \).

We carry out this construction below.

We first claim that there is a field extension \( K/k \) such that \( Br(K) \) contains an element \( \beta \) of order \( r \). To prove this, take the purely transcendental extension \( K = k(x, y) \), and define \( \beta \) to be the class of the cyclic algebra \( (x, y)_\zeta \), where \( \zeta \) is a primitive \( r \)-th root of unity. Note that \( (x, y)_\zeta \) is a division algebra.

In an earlier version, we had a very long argument for the existence of \( K \) and \( \beta \). The above short argument was provided by the referee.

Define \( X_K := X \times_k K \). Let \( \beta' \) denote the pullback of \( \beta \) in \( Br(X_K) \). Since \( \beta' \) is of order \( r \), there is a \( \mu_r \)-gerbe

\[ Y \to X_K \]

representing the class \( \beta' \). We fix the following notation:

- \( \mathcal{T} \) be the moduli stack of stable twisted rank \( r \) vector bundles on \( Y \) with determinant \( \mathcal{L} \), where \( \text{deg}(\mathcal{L}) \equiv i \mod r \).
- Let \( E \) be the universal bundle on \( \mathcal{T} \times_K Y \).
- Let \( \mathcal{T} \to T \) be the coarse moduli space of \( \mathcal{T} \).

Now \( \text{End}(E) \) descends to an Azumaya algebra on \( T \times_k X \). Since this is a stable family, we get a map \( T \to \mathcal{N}(r)_i \). The proof now follows from Proposition 4.3 since order of \( i_x^* \text{End}(E) \) in \( Br(T) \) is precisely \( r \). This completes the proof of the theorem. \( \square \)

Let \( S \) be a smooth variety defined over \( k \), with \( \text{dim } S \geq 1 \). Let

\[ \mathbb{P}_S \to S \]

be a projective bundle of relative dimension \( r - 1 \). So \( \mathbb{P}_S \) defines an algebraic principal \( \text{PGL}(r, k) \)-bundle

\[ E_{\text{PGL}(r, k)} \to S. \]
Let
\[ \beta \in Br(S) \]
be the class defined by \( \mathbb{P}_S \). If the order of \( \beta \) is \( r \), then \( E_{PGL(r,k)} \) does not admit any reduction of structure group to any proper parabolic subgroup of \( PGL(r,k) \) over any nonempty open subset of \( S \) (see [2, p. 267, Lemma 2.1]).

Suppose there is an irreducible normal projective variety \( \overline{S} \) over \( k \) such that \( S \) is the smooth locus of \( \overline{S} \). Fix a polarization on \( \overline{S} \). Assume that the order of \( \beta \) in (5.2) is \( r \). Since \( E_{PGL(r,k)} \) does not admit any reduction of structure group to any proper parabolic subgroup of \( PGL(r,k) \) over any nonempty open subset of \( S \), the principal \( PGL(r,k) \)–bundle in (5.1) is stable.

Consider \( N^0(r) \) defined in (1.2). Let
\[ F_{PGL(r,k)} \to N^0(r) \]
be the principal \( PGL(r,k) \)–bundle obtained by pulling back the universal projective bundle using the map \( i_x \) in Theorem 1.3 (recall that \( N^0(r) \) is isomorphic to \( N^0(r) \)). From the second part of Theorem 1.3 (combined with Proposition 1.1) we have the following corollary:

**Corollary 5.1.** The principal \( PGL(r,k) \)–bundle \( F_{PGL(r,k)} \to N^0(r) \) is stable.

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