Many Models for Water Waves

A unified theoretical approach

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Caveat lector  This document is an announcement and preview of a memoir whose full version is available on the Open Math Notes repository of the American Mathematical Society. In this memoir, I try to provide a fairly comprehensive picture of (mostly shallow water) asymptotic models for water waves. The work and presentation is heavily inspired by the book of D. Lannes [263], yet extends the discussion into several directions, notably high order and fully dispersive models, and internal/interfacial waves.

I plan to update this memoir from time to time when novel material fitting in the picture will arise. Please do not hesitate to contact me when you notice typos or mistakes, or if you have any question, comment or query, using the email address provided on the front page.

In the memoir one derives, discusses, and justifies as much as possible a large class of models describing in an approximate manner the propagation of waves at the surface of water, at the interface between two homogeneous fluids, or in the bulk of a continuously density-stratified fluid. In the considered idealized frameworks, these waves propagate from an initial perturbation of the rest state under the influence of gravity forces. Let me unveil a little bit of the material contained in the memoir in order to warn the potentially disappointed reader.

• The motivation is theoretical, in the sense that practical direct use of the results is not the main objective. The problem of the propagation of water waves is one example of partial differential equations which may be written under a compact formulation but forecasts a fascinating variety of phenomena, while enjoying a rich mathematical structure. It is hence a formidable toy on which one can apply advanced tools of modelization. Yet it is impossible not to have in mind that practical applications are just a few steps away, and many choices in the modelization procedure are grounded on applicative views, for instance robustness of the models or easy numerical implementation.

• The “master” equations, that is the system of equations from which all subsequent simplified models are derived, already incorporates many idealizations. To name a few, earth curvature, the Coriolis force, wind forcing, any dissipative effect and—most of the time—surface tension are neglected. In the “water waves” case one considers homogeneous fluids and potential flows. Moreover, the analysis is restricted to laminar (i.e. regular) rather than turbulent flows.

• While the equations at stake are of dispersive nature, little or none of the advanced tools on dispersive equations is used, and we barely report on the latest mathematical developments involving paradifferential calculus, normal forms, KAM theory, etc. The main mathematical tools that are put to use are rather old but robust: on one hand the elliptic theory to derive models from approximate solutions to a Laplace problem; and on the other hand the energy method to justify rigorously the resulting evolution equations (being of quasilinear hyperbolic nature). The heart of the matter consists in using these tools in a sufficiently refined manner so as to offer error estimates uniform with respect to the relevant parameters at stake.

• Given their number and diversity, it is impossible to present all relevant models based on the water waves system, even restricting to a specific asymptotic regime (the shallow water regime in our case). The memoir focuses on models which preserve as much as possible the structure (and in particular the Hamiltonian formulation) of the master equations, as well as mathematical properties (typically the well-posedness of the initial-value problem). That such models are often historical and among the most studied is not, to my opinion, a coincidence. Hence most of this work is dedicated to fairly standard models in oceanography. The aim of this document is to present such models together with more recent ones in a unified framework, and to address the state of their rigorous justification.

1The present document contains the prologue, slightly edited forewords of all chapters, and the bibliography.
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Y’a tant de vagues, et tant d’idées qu’on n’arrive plus à décider le faux du vrai

— Michel Berger, *Le paradis blanc*

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### Foreword

In this monograph we aim at describing the evolution in time of a body of fluid—typically water. Of course the features of the dynamics depend greatly on the framework, and in particular on the scales involved. As a rule of thumb, we will be motived by the description of the motion of the surface of water as seen by a human eye. These are often referred to as surface gravity waves, or simply *water waves*. As any wanderer knows, despite the restrictive framework, water waves are still remarkably diverse, and this is what makes them a fascinating subject of study. In order to get a grasp at the behavior of water waves in a given situation, one typically uses simplified models. Below we give examples of a few such models which appeared in the early literature, with the aim at emphasizing the diversity of possible waves and the hope of giving an insight at the possible mechanisms involved in the full picture. The models described further on in this work are refinements of such models.

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2To quote Feynman during his well-known Lectures on Physics (Vol. I, Ch. 51: Waves): “Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have.”

3We do not attempt at exhaustiveness. The relentless reader will find more in the present document and much more in the literature, using for instance [289, 263, 367, 64, 264] as starting points.

4The interested reader will find in [132] a detailed historical account on the early studies on water waves.
There are many ways to formally derive the models presented below. Considering the Saint-Venant system for instance, a typical way consists in integrating the horizontal velocity over the fluid layer and invoking a closure formula, based on physical principles such as energy conservation. One can also use some ad hoc hypotheses, such as columnar motion and hydrostatic approximation. Or a loose assumption that derivatives of a function are smaller than the function itself. Our strategy, called asymptotic modeling, is akin to the latter one, and provides a justification of the former ones, with quantitative estimates of the inaccuracies. We start with the so-called full Euler system (or more precisely, for models in this Prologue, the water waves system) whose solutions are regarded as “exact” (although, admittedly, the derivation of the equations relies on many oversimplifications). Using the typical scales of the flow, we can extract dimensionless parameters describing the strength of the main mechanisms involved. The asymptotic models are obtained through a description of the operators involved in the water waves system using assumptions on the size of these parameters, which will be called the asymptotic regime.

The complete rigorous justification of models in a given asymptotic regime typically proceeds in two steps. First we prove that sufficiently regular solutions to the water waves system satisfy the equations of the model—or the other way around—up to a small remainder term, measured by the size of the dimensionless parameters and data in a prescribed metric space; this is called consistency. Anticipating with future notations and results, we find that the water waves system is consistent with the acoustic wave equation (i) with precision $O(\mu + \varepsilon)$, with the linearized (Airy) equations (ii) with precision $O(\varepsilon)$, with the Saint-Venant system (iii) with precision $O(\mu)$, with all the Boussinesq systems (vii) with precision $O(\mu^2 + \mu\varepsilon)$, etc. This is however not sufficient, and there remains to prove that for a large class of sufficiently regular initial data (typically a neighborhood of the rest state in the aforementioned metric space), there exist unique solutions to both the water waves system and the asymptotic model, and that the two remain close on the relevant timescale. Following Lannes [263], we call the former property (uniform) well-posedness, and the latter convergence.

An important portion of this monograph is dedicated to the rigorous justification in the above sense—together with the study of a few basic properties—of standard and less-standard models for the propagation of surface, interfacial and internal gravity waves.
Arguably the simplest (partial differential) equation describing the motion of water waves, already put forward by Lagrange [260], is the following:

$$\partial_t^2 \zeta = g d \Delta_x \zeta. \quad (i)$$

Here $\zeta$ represents the deformation of the free surface, in the sense that the surface of the body of water at time $t$ is parameterized as $\Gamma_{\text{top}} = \{(x, z) \in \mathbb{R}^{d+1} : z = \zeta(t, x)\}$. Hence the function $\zeta$ depends on time, $t$, and horizontal space variable, $x$. For simplicity we assume that the horizontal variable lies in the full space $\mathbb{R}^d$. The constant $g$ denotes the gravity acceleration and $d$ is the depth of the layer. Equation (i) is called the linear acoustic wave equation as it governs the propagation of infinitesimally small acoustic waves through a material medium. It is only a coincidence that it also describes—very roughly, remember Feynman’s quote—water waves. In fact the above equation describes infinitely small and infinitely long water waves.

In the special case of horizontal dimension $d = 1$, the solution of the initial-value problem is easily found as

$$\zeta(t, x) = \frac{1}{2} \left( \zeta(0, x + c_0 t) + \zeta(0, x - c_0 t) \right) + \frac{1}{2 c_0} \int_{x-c_0 t}^{x+c_0 t} \partial_t \zeta(0, y) \, dy.$$

with $c_0 = \sqrt{g d}$. Hence the wave decomposes into the superposition of a right-going and a left-going components, both translating with velocity $c_0$. This is shown in Figure i where the evolution of the surface deformation when taken initially as Gaussians (with zero initial velocities) according to eq. (i) and eq. (ii) in dimension $d = 1$ is represented.

In dimension $d = 2$, the solution is less explicit, but a formula can still be written—at least for sufficiently regular initial data—with the use of Green’s function (we could also use Fourier representation as in the next section):

$$\zeta(t, x) = \frac{1}{2\pi c_0} \int_{|y-x| \leq c_0 t} \frac{\partial_t \zeta(0, y)}{\sqrt{(c_0 t)^2 - |y-x|^2}} \, dy + \frac{1}{2\pi c_0} \int_{|y-x| \leq c_0 t} \frac{\zeta(0, y)}{\sqrt{(c_0 t)^2 - |y-x|^2}} \, dy.$$

The one-dimensional framework $d = 1$ is relevant for instance for waves propagating along a narrow channel.
We can observe that the solution satisfies causality (but not Huygens’ principle): waves must be given enough time to propagate between two specified points. Again, $c_0$ is a good measure of the (scalar) velocity of waves according to eq. (i). Less obvious is the fact that for sufficiently smooth and decaying initial data, the amplitude of the solution decays for large time as $(c_0 t)^{-1/2}$. Figure ii represents the evolution of the surface deformation when taken initially as Gaussians (with zero initial velocities) according to eq. (i) and eq. (ii), in dimension $d = 2$.

Figure ii: Disintegration of Gaussian initial data, with zero initial velocities. $g = 9.81 \text{ m.s}^{-2}$, $d = 1 \text{ m}$. The bottom plot represents the solution on $\{ (x, y) : y = 0 \}$.

ii The linearized (Airy) water waves equations

The following equations describe the propagation of infinitesimally small waves without the long wave assumption of the previous section: it is the linearized system about the rest-state solution to the water waves equations, whose solutions shall be considered as “exact”, and which is introduced in Chapter A. Consider the linearized water waves equations as

$$\begin{align*}
\partial_t \zeta - \mathcal{G}_0 \psi &= 0, \\
\partial_t \psi + g \zeta &= 0
\end{align*}$$

where $\mathcal{G}_0 = |D| \tanh(d|D|)$ is the Fourier multiplier operator defined on sufficiently regular solutions by

$$\forall \xi \in \mathbb{R}^d, \quad \widehat{\mathcal{G}_0 \psi}(\xi) = |\xi| \tanh(d|\xi|) \widehat{\psi}(\xi).$$

Here, $g$, $d$ and $\zeta$ are as above and $\psi$ represents the trace of the velocity potential at the surface. Equation (ii) is a system of linear constant-coefficient equations of the form

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = L(D) \begin{pmatrix} \zeta \\ \psi \end{pmatrix}$$

where $L(D)$ is a matrix with Fourier multiplier coefficients.

Formally taking the limit $d \to 0$, we may replace $\tanh(d|\xi|)$ with $d|\xi|$ in $\mathcal{G}_0$, and then we recover the acoustic wave equation, eq. (i). In fact using eq. (ii) instead of eq. (i) in the left side of Figure i and Figure ii yields a very similar outcome; such is not the case for the narrower initial data used for right sides.
Modal analysis  Plane waves of the form \((\zeta, \psi) = (\zeta_0 e^{i(\xi \cdot x - \omega t)}, \psi_0 e^{i(\xi \cdot x - \omega t)})\) are solutions to eq. (ii) provided that \(i\omega \psi_0 = g\zeta_0\) and the dispersion relation holds \([245, 7]\):
\[
\omega(\xi)^2 = g|\xi| \tanh(d|\xi|).
\]
In other words, we can explicitly solve the equation in the Fourier space:
\[
\left( \hat{\zeta}(t, \xi) \hat{\psi}(t, \xi) \right) = \exp(L(\xi)t) \left( \hat{\zeta}(0, \xi) \hat{\psi}(0, \xi) \right) = \left( \begin{array}{c} \cos(\omega(\xi)|t|) \frac{\omega(\xi)}{g} \sin(|\omega(\xi)|t) \\ -\frac{\omega(\xi)}{g} \cos(|\omega(\xi)|t) \cos(\omega(\xi)|t|) \end{array} \right) \left( \begin{array}{c} \hat{\zeta}(0, \xi) \\ \hat{\psi}(0, \xi) \end{array} \right).
\]
For such plane wave solutions, \(\omega\) is called the (angular) frequency, \(\xi\) the (angular) wave vector (wavenumber if \(d = 1\)), and \(|\xi|\) the (angular) wavenumber. Phase velocities describe the velocity in a given direction of a plane wave with wave vector \(\xi\), and satisfy
\[
c_p \cdot \xi = \omega(\xi).
\]
The group velocity represents the traveling velocity of a wave packet about wave vector \(\xi\), and is given by
\[
c_g = \nabla_{\xi}(\omega(\xi)).
\]
Misusing these definitions, we shall also refer to
\[
c_p = \frac{\omega(\xi)}{|\xi|} = \sqrt{gd} \left( \frac{\tanh(d|\xi|)}{d|\xi|} \right)^{1/2}
\]
as the phase velocity, and to
\[
c_g = |c_g| = \sqrt{gd} \left( \frac{1}{2} \left( \frac{\tanh(d|\xi|)}{d|\xi|} \right)^{1/2} + \frac{1}{2} \left( \frac{d|\xi|}{\tanh(d|\xi|)} \right)^{1/2} \right)
\]
as the group velocity. They are represented in Figure iii. That the phase velocity is different (and greater) than the group velocity manifests the essential feature of the (linearized) water waves equations as being dispersive. Notice however that for small-normed wave vectors, \(d|\xi| \ll 1\), both velocities converge to \(c_0 = \sqrt{gd}\), the velocity of (non-dispersive) infinitely long waves. In the opposite direction, for \(d|\xi| \gg 1\), we have \(c_g \sim \frac{1}{2} c_p \sim \frac{\sqrt{g}}{2|\xi|^{1/2}}\).

Figure iii: Phase and group velocities of the linearized water waves system.
Large-time behavior  We can infer the large-time behavior of the solution, at least in dimension \(d = 1\), through the stationary phase theorem on oscillatory integrals; see e.g. \([384]\). Indeed, for any \(c \in \mathbb{R}\), and initial data such that \((\tilde{\zeta}(0, \cdot), |\omega(\cdot)|\tilde{\psi}(0, \cdot)) \in L^1(\mathbb{R})^2\), we have from the above

\[
\zeta(t, ct) = \frac{1}{4\pi} \int_{\mathbb{R}} e^{i(c\xi - \omega(\xi))t} \left( \tilde{\zeta}(0, \xi) + i\frac{\omega(\xi)}{g} \tilde{\psi}(0, \xi) \right) + e^{i((-c\xi + \omega(\xi))t/2)} \left( \tilde{\zeta}(0, \xi) - i\frac{\omega(\xi)}{g} \tilde{\psi}(0, \xi) \right) d\xi
\]

where we denote \(\omega(\xi) = \text{sgn}(\xi)\frac{|\xi|}{\tanh(d|\xi|)}\), and use a standard convention for the Fourier transform. We deduce that the following holds for sufficiently decaying and regular initial data.

i. For any \(c \in (-\infty, -\sqrt{gd}) \cup (\sqrt{gd}, +\infty)\), one has for any \(n \in \mathbb{N}\),

\[|\zeta|(t, ct) = O(t^{-n}).\]

ii. For any \(c \in (-\sqrt{gd}, \sqrt{gd}) \setminus \{0\}\), one has

\[|\zeta|(t, ct) \sim_{t \to \infty} \frac{1}{(2!)^{1/2}\Gamma(3/2)}|A(\xi_c)||\zeta(\xi_c)| (-\zeta(0, \xi_c) + \text{sgn}(c)i\frac{\omega(\xi_c)}{g} \tilde{\psi}(0, \xi_c));\]

where \(\xi_c\) is defined by the relation \(c = \omega'(\xi_c)\) and \(A(\xi_c) \equiv \tilde{\zeta}(0, \xi_c) + \text{sgn}(c)i\frac{\omega(\xi_c)}{g} \tilde{\psi}(0, \xi_c)\); unless \(A(\xi_c) = 0\) in which case the decay is at least \(O(t^{-1})\).

iii. If \(c \in \{-\sqrt{gd}, \sqrt{gd}\}\), one has

\[|\zeta|(t, ct) \sim_{t \to \infty} \frac{1}{(3!)^{1/2}\Gamma(1/2)}|A(0)||d^2\sqrt{gd}\ t\frac{1}{2} \approx a((d^2/\lambda^2)\sqrt{gd}/\lambda t)^{-1/2},\]

with \(A(0) \equiv \lim_{\xi \to 0} A(\xi)\) (notice we require regularity only on \(\xi \tilde{\psi}(0, \xi)\)); unless \(A(0) = 0\), in which case the decay is at least \(O(t^{-2})\). The last approximation is meant in a loose sense, where we set \(A(0) \approx a\lambda\). This allows to hint at the timescale for which dispersive mechanisms have a bearing on the behavior of the flow, which is large compared with the time period of long waves, \(T \equiv \frac{\lambda}{\sqrt{gd}}\), when \(d^2/\lambda^2 \ll 1\).

Above, \(\Gamma\) is the Euler Gamma function: \(\Gamma(s) \equiv \int_0^{+\infty} \tau^{s-1} e^{-\tau} d\tau\). A loose interpretation of the above is that for large time, the dominant part of the wave which will remain visible is the large wavelength component, traveling at velocity \(|c| \approx c_0 = \sqrt{gd}\).
iii The Saint-Venant system

Our first nonlinear model is the so-called shallow water, or Saint-Venant system [364]:

\[
\begin{aligned}
\partial_t \zeta + \nabla \cdot (hu) &= 0, \\
\partial_t u + g \nabla \zeta + (u \cdot \nabla) u &= 0,
\end{aligned}
\]

where \( h \) represents the water depth, and \( u \) a horizontal velocity. Indeed, denoting \( h = d + \zeta \), the first equation (mass conservation) in eq. (iii) can be seen as the Saint-Venant system.

System (iii) is hence a prototype of quasilinear hyperbolic systems. Hyperbolicity amounts to a basis of Riemann invariants. The Riemann invariants are explicit in this case: setting \( \zeta = \frac{u}{\sqrt{g h}} \), we find

\[
\zeta = \frac{u}{\sqrt{g h}},
\]

and the eigenvalues of the associated symbol (see e.g. [306]) are

\[
u \cdot \xi \quad \text{and} \quad \nu \cdot \xi \pm \sqrt{gh} |\xi| \cdot
\]

Notice here again the “sound speed” of long surface gravity waves as being \( c_0 = \sqrt{gh} \).

In dimension \( d = 1 \), as any quasilinear system of two scalar balance laws, eq. (iii) enjoys a basis of Riemann invariants. The Riemann invariants are explicit in this case: setting \( r_\pm = u \pm 2 \sqrt{gh} \), the system (iii) is equivalent to

\[
\begin{aligned}
\partial_t \zeta + \frac{3r_+ + r_-}{4} \partial_r r_+ &= 0, \\
\partial_r r_- + \frac{3r_+ + r_-}{4} \partial_r r_- &= 0.
\end{aligned}
\]

Notice that \( \frac{3r_+ + r_-}{4} = u + \sqrt{gh} \) and \( \frac{3r_+ + r_-}{4} = u - \sqrt{gh} \), consistently with the hyperbolicity discussion. The diagonal formulation, eq. (iv), allows to construct simple waves, i.e. solutions of the form

\[
(r_+, r_-) = R(\theta(t, x))
\]

where \( \theta \) is a scalar function. For instance, any sufficiently regular solution to eq. (iv) with initial data satisfying \( u|_{t=0} = 2 \sqrt{gh} |_{t=0} - 2 \sqrt{gd} \), the second equation yields \( r_- = -2 \sqrt{gd} \) for all times, from which we deduce \( r_+ = 2 \sqrt{gd} + 2u \), where \( u(t, x) \) satisfies the inviscid Burgers (or Hopf) equation

\[
\partial_t u + (\sqrt{gd} + \frac{3}{2} u) \partial_u u = 0.
\]

Conversely, any solution to the above equation provides a particular solution to eq. (iv) by setting \( (r_-, r_+) = (2 \sqrt{gd} + 2u, -2 \sqrt{gd}) \), or equivalently a solution to eq. (iii) with \( \zeta = g^{-1}(\sqrt{gd} u + \frac{1}{4} u^2) \).

Equation (v) may be solved by the hodograph transform, or the characteristics method, and exhibits a new phenomenon with respect to the linear equations discussed in previous sections: finite-time singularity formation. Assume \( u \) is a Lipschitz solution to eq. (v) and define, for any \( x_0 \in \mathbb{R} \), \( v_{x_0}(t) \) such that \( x_{x_0}(t) \) is defined by the initial condition \( x_{x_0}(t) = x_0 \) and the ordinary differential equation \( x_{x_0}'(t) = \sqrt{gd} + \frac{3}{2} u(t, x_{x_0}(t)) \). Chain rule and eq. (v) yields \( v_{x_0}'(t) = 0 \).

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6Sufficiently regular solutions with initial data in the hyperbolicity domain cannot leave the hyperbolicity domain due to first equation (mass conservation) in eq. (iii). Indeed, denoting \( h_{x_0}(t) = h(t, x_{x_0}(t)) \) where \( x_{x_0}(t) \) is defined by the final condition \( x_{x_0}(t) = x_0 \) and the ordinary differential equation \( x_{x_0}'(t) = u(t, x_{x_0}(t)) \) for \( t \in [0, t_*] \), we find \( h(t, x_0) = h_{x_0}(t) = h(0, x_{x_0}(0)) \exp(-
\int_0^t (|\nabla u|)(t, x_{x_0}(t)) \) \( t \).
and hence \( v_{x_0}(t) = u(0, x_0) \) and finally \( x_{x_0}(t) = x_0 + (\sqrt{g d} + \frac{1}{2} u(0, x_0)) t \). In other words, the solution is constant along the characteristics defined by \( x_{x_0}(t) \), for any \( x_0 \in \mathbb{R} \), and the characteristics are straight lines. This allows to define and describe solutions as long as two characteristics do not cross, i.e. as long as for any \( t \in (0, T) \), there does not exists \( x_0 \neq x_1 \in \mathbb{R} \) with

\[
x_0 + (\sqrt{g d} + \frac{1}{2} u(0, x_0)) t = x_1 + (\sqrt{g d} + \frac{1}{2} u(0, x_1)) t \iff \frac{u(0, x_1) - u(0, x_0)}{x_1 - x_0} = -\frac{2}{3T}.
\]

Hence we see that for any Lipschitz initial data \( u(t = 0, \cdot) = u_0 \in W^{1,\infty}(\mathbb{R}) \), the solution described above (which is unique) exists on the time domain \([0, T^*]\) where \( T^* = -\frac{2}{3} (\inf_{\mathbb{R}} u_0')^{-1} \) with the convention \( T^* = \infty \) if \( \inf_{\mathbb{R}} u_0' \geq 0 \). In the situation where \( \inf_{\mathbb{R}} u_0' < 0 \) (in particular for any non-trivial \( u_0 \) such that \( u_0 \to 0 \) as \(|x| \to \infty|\)), there exists indeed a singularity formation as \( t \to T^* \): since the solution remains bounded but \( \inf_{\mathbb{R}} \partial_x u(t, \cdot) \to -\infty \) as \( t \nearrow T^* \), we say that a shock, or a wavebreaking, occurs. We represent this situation in Figure iv.

![Overlapped initial and final time snapshots](image)

(a) Evolution in time  
(b) Characteristics

Figure iv: Wavebreaking of a simple wave according to eq. (v). The initial data for \( \zeta \) is the Gaussian \( \zeta(t = 0, x) = 0.5 \exp(-(0.1 x)^2) \) and corresponding velocity. \( g = 9.81 \text{ m.s}^{-2}, d = 1 \text{ m}. \)

Going back to the system case, eq. (iv), each of the Riemann invariants, \( r_\pm \), is constant along characteristics curves defined by

\[
x_{\pm, x_0}(t) = x_0, \quad x'_{\pm, x_0}(t) = \frac{1}{4} (3r_{\pm} + r_\mp)(t, x_{\pm, x_0}(t)).
\]

However the characteristics curves are no longer straight lines in general. Still we can infer the behavior of solutions for instance if we assume that initial data \( (\zeta(t = 0, \cdot), u(t = 0, \cdot)) \equiv (\zeta_0, u_0) \) have compact support, say in \((-\lambda, \lambda)\), and are are sufficiently small so that there exists \( c \in (0, c_0) \) with

\[
r_{+, 0} \equiv u_0 + 2\sqrt{d + \zeta_0} \in (2c_0 - c, 2c_0 + c) \quad \text{and} \quad r_{-, 0} \equiv u_0 - 2\sqrt{d + \zeta_0} \in (-2c_0 - c, -2c_0 + c).
\]

Because the Riemann invariants are constant along characteristics, we have, as long as the solution remains regular, \( \frac{3r_{+, c}}{c} \in (c_0 - c, c_0 + c) \) and \( \frac{3r_{-, c}}{c} \in (-c_0 - c, -c_0 + c) \), and as a consequence

\[
r_{+, t}(x, t) \equiv 2c_0 \text{ if } x \leq -\lambda + (c_0 - c)t \quad \text{and} \quad r_{-, t}(x, t) \equiv -2 \sqrt{\lambda - (c_0 - c)t}.
\]

If the initial data is sufficiently small in order to ensure that no shock formation occurs before \( T^* = \frac{1}{c_0 - c} \), we can afterwards decompose the flow as the superposition of two simple waves described by Hopf equations, and in particular a shock inevitably occurs after sufficiently large time.
iv Boussinesq systems

In his celebrated manuscript [59], Boussinesq introduced the first models for the propagation of surface gravity waves taking into account both (first order) nonlinear and dispersive effects. While restricted in the original work to unidirectional waves, models with similar flavor were later on obtained for general waves. Eventually, one may obtain a full family of systems [290, 51], often called (abcd) Boussinesq systems, of the form

\[
\begin{cases}
\partial_t (\zeta - b d^2 \Delta \zeta) + \nabla \cdot (hu + a d^2 \nabla \cdot u) = 0, \\
\partial_t (u - d d^2 \nabla \cdot u) + g \nabla (\zeta + c d^2 \Delta \zeta) + (u \cdot \nabla) u = 0,
\end{cases}
\tag{vii}
\]

where \( p = (a, b, c, d) \in \mathbb{R}^4 \) is such that (when neglecting surface tension) \( a + b + c + d = \frac{1}{2} \). In eq. (vii) the precise meaning of the velocity variable depends on the choice of the parameters. The freedom in the choice of \( (a, b, c, d) \in p \) is at the same time a blessing—for instance one may tune parameters so as to enhance the accuracy of the dispersion relation—and a curse, since important properties of the system will typically depend on the choice of \( (a, b, c, d) \in p \). In particular, the initial-value problem of a subfamily of eq. (vii) is strongly ill-posed, as can be seen from modal instabilities of the linearized equations about the rest state: the dispersion relation being

\[
\omega(\xi)^2 = gd|\xi|^2\left(1 - a|d\xi|^2\right)\left(1 - c|d\xi|^2\right)
\]

with right-hand side taking arbitrarily large negative values at large wavenumbers, \(|\xi|\), for ill-chosen \((a, b, c, d) \in p\). Incidentally, this is also the case for the original “bad” Boussinesq equation, eq. (vi). This is a useful reminder that consistency is not the only property to look for in a model.

In the other way, it is expected that for “good” choices of \((a, b, c, d) \in p\), dispersive properties of the Boussinesq systems prevent the wavebreaking scenario in the Saint-Venant model, eq. (iii). As a matter of fact, for several families of parameters, \((a, b, c, d) \in p\), global-in-time existence and uniqueness of solutions have been proved (see [369, Remark 1.1]) and—to the author’s knowledge—the emergence of finite-time singularity has not been proved or numerically witnessed on any of the models, at least for solutions maintaining positive layer depth. In the situation of long waves and relatively large amplitude, the solution typically generates a zone of rapid oscillations (or modulations) often called dispersive shock wave, in place of the shock predicted by the Saint-Venant system. Properties of these dispersive shock waves will typically depend on the choice of \((a, b, c, d) \in p\), and is not expected to accurately describe the real-life phenomenon.

An important property of nonlinear and dispersive equations such as eq. (vi) is that they allow the existence of traveling waves, that is solutions that maintain their shape while propagating at a constant velocity, including solitary waves which in addition bear finite energy. Once again the reader will find in [132] the fascinating and tumultuous story of the discovery and progressive acceptance of these waves. Existence and properties of traveling waves again typically depend on the choice of \((a, b, c, d) \in p\). We however expect that they exist at least for small supercritical velocities, \(0 < c - c_0 \ll 1\), and grow in amplitude with the velocity parameter; see e.g. [140]. We show examples in Figure v.

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3The transport term \((u \cdot \nabla) u\) is often replaced with \(\frac{1}{2} \nabla (|u|^2)\), trading the direct comparison with the Saint-Venant system, eq. (iii), with conservative form. The change is immaterial in dimension \(d = 1\), or when \(\text{rot } u = 0\). Similar systems can be derived using momentum-type variables instead of velocity variables, thus slightly altering the nonlinear/dispersive interplay; see [184]. These systems, sometimes called Abbott systems [2, 3], have conservative form. Other \(ad \ hoc\) transformations can be performed, for instance to improve the mathematical properties of the system; see [53]. Finally, the models can also be written as second order scalar equations similar to eq. (i), as in the original work of Boussinesq [59, (26), p. 75]:

\[
\partial_t^2 \zeta = gd \Delta (\zeta + \frac{1}{2c} \zeta^2 + \frac{d^2}{4} \Delta \zeta).
\tag{vi}
\]
Overlapped initial and final time snapshots

(a) Disintegration of Gaussian initial data

(b) Traveling waves

Figure v: Left: Disintegration of the Gaussian $\zeta(t = 0, x) = 0.25 \exp(-0.1x^2)$, with zero initial velocity. Right: Solitary wave solutions with velocities $c = 1.05c_0$ and $c = 1.01c_0$.

Both according to system (vii) with $-a = b = d = \frac{1}{3}, c = 0, g = 9.81 \text{ m.s}^{-2}, d = 1 \text{ m}.$

It would be impossible to review all known results on Boussinesq systems and closely related (symmetric, Abbott, etc.) variants. Let me lazily refer to [149, 264, 369]—in addition to previous references—and references therein, and conclude with a last warning. The Boussinesq systems typically lose important properties of the original water waves equations and in particular its Hamiltonian structure. Hence unless the parameters $(a, b, c, d)$ are well-chosen, we do not expect energy conservation, or Galilean invariance, etc.

v The Korteweg–de Vries and Whitham equations

It was mentioned in the previous section that Boussinesq’s original motivation was the study of unidirectional waves, and in particular solitary waves. Using such assumption one may derive\(^8\) (as did Boussinesq) simplified scalar equations, of which the most famous is the Korteweg–de Vries equation [61, 257] for right-going waves in dimension $d = 1$:

$$\partial_t \zeta + \sqrt{gd} \partial_x (\zeta + \frac{3}{4d} \zeta^2 + \frac{1}{6} d^2 \partial_x^2 \zeta) = 0.$$  \hspace{1cm} (viii)

One of the many reasons for the importance of the Korteweg–de Vries equation is the family of explicit solitary wave solutions\(^9\)

$$\zeta(t, x) = \zeta_c(x - ct), \quad \zeta_c(x) \overset{\text{def}}{=} 2d(\frac{c}{c_0} - 1) \sech^2 \left( \sqrt{\frac{1}{2d}} (\frac{c}{c_0} - 1) x \right)$$

where the velocity variable, $c$, may take any value $c > c_0 = \sqrt{gd}$.

\(^8\)We will not discuss in this document the interesting question of justifying such scalar equations from aforementioned systems of equations. Let me just mention that this justification is relatively straightforward for well-prepared initial data accounting for the assumption of unidirectional propagation, and much more involved for general initial data where we want to express that the flow can be decomposed at first order as the superposition of two counter-propagating unidirectional waves. Let me also refer—once again—to [263] and references therein (see also [36] for a recent development) for all details concerning the Korteweg–de Vries equation and to [176] for the Whitham equation.

\(^9\)Far from being simply some entertaining special solutions, solitary waves play a very important role as they allow to describe the large-time dynamics of generic solutions; a phenomenon designated as \textit{soliton resolution}. We will not discuss further on this feature as it relies on the integrability of the Korteweg–de Vries equation, a property which is not shared by other models in this document.
The existence of traveling waves with arbitrarily large amplitude and arbitrarily large velocity may found undesirable as nonphysical [387]. Such is the case also for the global-in-time well-posedness properties, preventing the aforedescribed wavebreaking scenario. With this in mind, Whitham [406] introduced the following equation which is now called *Whitham equation*:

$$
\partial_t \zeta + \sqrt{gd} \partial_x \left( \sqrt{\tanh(d|D|)} \zeta + \frac{3}{4g} \zeta^2 \right) = 0 .
$$

arguing that the fact that its linear dispersion relation reproduces exactly one branch of the dispersion relation of eq. (ii) would authorize wavebreaking and peaked traveling waves of extreme height. This prediction turned out to be valid, as recently shown in [216, 172, 396, 368]. A numerical comparison of solitary wave solutions to the Korteweg–de Vries and Whitam equations is shown in Figure vi.

![Figure vi: Solitary waves of unidirectional models with velocity $c = 1.05c_0$ (blue, smaller) and $c = 1.2290408c_0$ (red, larger). $g = 9.81 \text{ m.s}^{-2}$, $d = 1 \text{ m}$, $c_0 = \sqrt{gd}$.](image_url)

---

He also proposed [407, §13.14]

$$
\partial_t \zeta + \sqrt{g \tanh(d|D|)} \partial_x \zeta + (3\sqrt{g(d + \zeta)} - 3\sqrt{gd}) \partial_x \zeta = 0 ,
$$

where the advection term fits the decomposition in Riemann invariants of the Saint-Venant system.
CHAPTER A

The “master” equations

Le problème de l’établissement [...] des équations différentielles du mouvement, et ensuite de leur intégration approchée, aura encore sa difficulté souvent grande. Mais il ne présentera plus, envisagé ainsi, cette désespérante énigme contre laquelle des esprits distingués se sont heurtés en vain.

— Adhémar Barré de Saint-Venant, Comptes rendus des séances de l’Académie des sciences, séance du 18 mars 1872

Foreword

In this chapter, we introduce and provide a preliminary study of the systems of equations from which asymptotic models are derived in subsequent chapters. The presentation, as well as most of the notations, are borrowed from Lannes’ book [263]. However concision has been pursued and I cannot encourage enough a thorough reading of the book for a detailed account.

We first write down the most general system of equations which is considered in this work, that is Euler equations for a layer of (non-necessarily homogeneous) incompressible ideal fluid, coupled with boundary conditions accounting for the impermeable bottom and the free surface. The only external force acting on the system will be the gravity force, assumed constant and vertical. We refer to the system we obtain as the full Euler system. Then we focus on particular settings.

The homogeneous and irrotational framework is particularly rewarding, as it allows to rewrite the whole system as two evolution equations for unknown functions of time and horizontal space variables only. This system is referred to as the water waves equations.

Prominently important in the water waves equations is the Dirichlet-to-Neumann operator, which is defined after solving a Laplace problem on the fluid domain with Dirichlet and Neumann boundary conditions. Its study, and in particular the asymptotic expansions which allow to derive asymptotic models, are briefly reviewed.

Meanwhile we make a small step outside the world of homogeneous and potential flows to consider interfacial waves between two layers of homogeneous fluids with irrotational velocities. Additional Dirichlet/Neumann operators appearing in this framework are tackled.
“Begin at the beginning,” the King said gravely, “and go on till you come to the end: then stop.”

— Lewis Carroll, Alice in Wonderland

Figure B: Models in Chapter B (in green) and some filiations.
Foreword

We start our journey towards asymptotic models with ones among the oldest and simplest-looking. The so-called hydrostatic models can be formally derived from the “master” full Euler equations by using the hydrostatic assumption, that is approximating the pressure terms using an explicit formula stemming from neglecting the velocity advection terms in the horizontal momentum conservation equation, specifically

\[-\partial_z P = \rho g,\]

which we can integrate using the known pressure at the free surface. Additionally, one often adds the assumption of columnar motion, stating that the horizontal velocity (approximately) does not depend on the vertical variable. When both assumptions are made, then we quickly obtain models with the rewarding properties that the vertical space variable has disappeared from the equations and only (first order) differential operators are involved.

Yet we shall not assume a priori—but indirectly prove—the hydrostatic assumption nor the columnar motion and will rather justify models asymptotically—with quantitative error estimates—in the shallow water regime, as \( \mu \ll 1 \); and using the irrotationality assumption in lieu of columnar motion.

Our first model is derived from the water waves system, that is assuming that the density is homogeneous and the flow potential (as this allows to discard the vertical variable except in the Dirichlet-to-Neumann operator). We then obtain the well-known and much-studied Saint-Venant system, already introduced in Section iii. Its derivation and rigorous justification, together with a very short description of some of its properties, is the subject of a first section.

Then we move to the bilayer framework, with two layers of homogeneous potential flows. The situation is slightly messier as models differ whether we use the free-surface framework or the rigid-lid framework, and in the latter one often uses the so-called Boussinesq approximation. It turns out the rigid-lid assumption and Boussinesq approximation both follow from the same assumption of weak density contrast.

Finally we quickly extend the analysis to \( N \geq 2 \) layers as above. While this multilayer framework may appear artificial, it is expected to approximate (as \( N \gg 1 \)) the setting of continuously stratified flows, in view of withdrawing the assumptions of homogeneous density and potential flows while keeping the hydrostatic approximation in the shallow water regime.

Hydrostatic equations for continuously stratified flows are also discussed. It turns out very little is known on these equations, despite the fact that they are at the core of the primitive equations which are widely used in studies and numerical simulations of geophysical flows. This offers stimulating mathematical challenges.
Weakly dispersive models

parce que, [les Anciens] s'étant élevés jusqu'à un certain degré où ils nous ont portés, le moindre effort nous fait monter plus haut, et avec moins de peine et moins de gloire nous nous trouvons au-dessus d'eux. C'est de là que nous pouvons découvrir des choses qu'il leur était impossible d'apercevoir.

— Blaise Pascal, traité du vide

Figure C: Models in Chapter C (in green) and some filiations.
Foreword

This chapter is devoted to the derivation and analysis of weakly dispersive models. These models refine the hydrostatic equations studied in Chapter B (in fact, more precisely, the Saint-Venant equations since here we restrict the analysis to the water waves framework; see Chapter E for an extension to the bilayer framework) by introducing dispersive effects at first order. More refined models are presented in Chapter D.

The first section of this chapter has been meant as a showcase for a thorough study of water waves models. Here we introduce and analyze the so-called (Serre–)Green–Naghdi model. Firstly the model is quickly derived as an asymptotic model from the expansion of the Dirichlet-to-Neumann operator obtained in Chapter A. Yet the result which follows (namely the consistency of the model) is far from being sufficient to validate the Green–Naghdi equations as a good model for water waves. Firstly, its rigorous justification must be completed by well-posedness, stability and convergence results. They follow from careful energy estimates in suitable functional spaces. In a looser way, we also expect “good” models to retain important properties of the master equations (here the water waves system). Here we focus mostly on the variational structure of the equations: we observe that the Green–Naghdi equations not only preserve Zakharov’s canonical Hamiltonian structure of the water waves system, but it also enjoys a deeper Lagrangian formalism which embeds the system inside a natural family of conservation laws, which can be interpreted as equations for compressible fluids with inertia effects. Hence the structure of the equations becomes richer as we simplify the equations from the water waves system to the Green–Naghdi equations (and then from the Green–Naghdi equations to the Saint-Venant system). This explains in my opinion why the Green–Naghdi equations, among many other loosely equivalent models, has attracted so much attention from diverse communities. We review some basic properties of the Green–Naghdi equations: preserved quantities, modal analysis and dispersion relation, solitary wave solutions. Finally, some open questions are discussed.

Of course I do not claim that the Green–Naghdi model is perfect! One of its main drawback is certainly that numerically approximating the equations turns out to be quite costly. In a second section we explore some equations which have been proposed by Favrie and Gavrilyuk [180] to circumvent this issue. The equations are constructed using the aforementioned Lagrangian formalism, using a strategy akin to relaxation limits. Hence the system contains additional unknowns as well as a (large) parameter which is expected to measure the precision of solutions to the augmented equations with respect to solutions to the original Green–Naghdi equations, at least when initial data are well-prepared. The rigorous study of this singular limit is based on [156]. Again, the section is concluded by perspectives and open questions.

In a third section we introduce a fully dispersive analogue of the Green–Naghdi system, which we name Whitham–Green–Naghdi. When linearized about trivial equilibrium solutions, fully dispersive models coincide with the corresponding (Airy) linearized water waves equations, as introduced in Section ii. Interest in such fully dispersive models in the context of long water waves started with the work of Whitham, which proposed eq. (x) and eq. (ix) as suitable modifications of the standard Korteweg-de Vries equation, eq. (viii), in view of reproducing at least qualitatively important features of water waves such as wavebreaking and peaked traveling waves. Much more recently, the interest was renewed as Whitham’s predictions were proved to be valid [216, 172, 396, 368]. Yet the question of validating fully dispersive models as asymptotic models with improved accuracy with respect to their standard counterparts was mostly left aside. The precision of the Whitham–Green–Naghdi model (respectively Whitham–Boussinesq) introduced in this section significantly improves the precision of the Green–Naghdi (respectively Boussinesq) model for weak nonlinearities and mild bottom variations with the important price to pay that nonlocal operators (Fourier multipliers) are involved. These models also allow to rigorously justify the Whitham equations and observe a similar improvement with respect to the Korteweg–de Vries equation [176].
Jésus a dit : « Que celui qui cherche ne cesse pas de chercher, jusqu’à ce qu’il trouve. Et quand il aura trouvé, il sera troublé ; quand il sera troublé, il sera émerveillé, et il régnera sur le Tout. »

— Thomas l’apôtre, évangile selon Thomas

Figure D: Models in Chapter D (in green) and some filiations.
Foreword

In this chapter we introduce and discuss higher order models for the water waves system, building upon the Saint-Venant system (Chapter B) and the Green–Naghdi system (Chapter C). These are hierarchies of models, that is families of system depending on a parameter—always denoted $N$—which we call the rank of the model, of which the Saint-Venant and/or the Green–Naghdi system are typically the first rank elements. The Saint-Venant (resp. Green–Naghdi) system has been rigorously justified as a shallow water model for the water waves system, in—roughly speaking—the following way: the size of the difference between solutions to the dimensionless water waves system and the corresponding solutions to the model equations grows proportionally to the size of the initial data with a prefactor bounded as $C \mu t$ (resp. $C \mu^2 t$) over a relevant time interval (being of size inversely proportional to the size of the initial data), where $\mu$ is the shallow water parameter, and $C$ depends on an upper bound on the size of the admissible initial data (together with a lower bound on the minimal depth of the layer, an upper bound on admissible values for $\mu$, and the norms measuring the size of the data). In good cases we expect that a similar result holds for all elements in a hierarchy of models, with different prefactors $C_N \mu^{\alpha N} t$. There are typically two situations:

i. the order as a shallow water model increases with $N$, that is $\alpha N \to \infty$ as $N \to \infty$;

ii. the accuracy of the model improves with $N$, that is $C_N \to 0$ as $N \to \infty$.

In the latter but not in the former we can hope that the hierarchy provides a robust tool for the approximation of any (sufficiently regular) solution to the water waves system, and can be useful for instance to devise strategies for their numerical integration.

This chapter is decomposed into three sections, corresponding to three different strategies, each producing a variety of families of higher order models.\footnote{The list is by no means complete. In particular it lacks spectral methods based on expansions with respect to the steepness parameter, $\epsilon \equiv \sqrt{\mu}$, initiated in [146, 405, 128] (see e.g. [271, 370, 408, 338] for a detailed account and comparisons). Among them the strategy brought to light by Craig and Sulem in [128], consisting in expanding the Dirichlet-to-Neumann operator, $G^{\mu}[\xi \zeta]$, along the variable $\xi \zeta$, is particularly elegant and effective. Contrarily to the models introduced in this chapter, the family of models involve Fourier multipliers in addition to differential operators; see [95] for discussion, references and the explicit display of the models up to fifth order. This method has been extended and successfully employed in many situations (see [205] and references therein), despite the claim—based on numerical experiments and formal arguments—in [17, 301] that the Cauchy problem associated with systems in the family are ill-posed in Sobolev spaces.}

In the first section, we use an expansion due to Boussinesq [60] and Rayleigh [357] (see [139, §4.1] for discussion and other relevant references) of the velocity potential—as a solution to the Laplace problem—as a series involving powers of the shallow water parameter, $\mu$. We are hence typically in the framework of the first aforementioned situation, and this section emphasizes its possible shortcomings. Among the different models which can be naturally constructed by this way—which we call Friedrichs-type systems in acknowledgment to his Appendix to [386]—we introduce explicitly two families of models: the high order shallow water models and the extended Green–Naghdi models. These models involve differential operators of increasing order as the rank of the model grows, which yields several complications. Firstly, half of these models suffer from very serious high frequency instabilities which prevent any hope as for the well-posedness of the Cauchy problem. But even in good cases, it is expected that for fixed initial data the solutions to the systems—if they exist—do not converge towards the corresponding solution to the water waves system, as $N \to \infty$.

This can be seen in particular when studying the dispersion relation of the models, which converge towards the dispersion relation of the water waves system only for wavenumbers in a finite-size neighborhood of the origin.

In the second section we set up a Galerkin dimension reduction strategy to a reformulation of the Laplace problem, to devise the approximate formula for the velocity potential—or, more precisely, the horizontal velocity. As a second step, the usual procedure consists in using this approximate formula in the Hamiltonian functional of the water waves system, and express the model as the
canonical Hamiltonian equations associated with the approximate Hamiltonian. This procedure produces a different model for any (reasonable) choice of subspace of real-valued functions of the fluid domain used in the Galerkin method. Natural examples of such spaces in the shallow water framework are functions of the form

$$\Phi(t, x, z) = \sum_{i=1}^{N} \phi_i(t, x) \Psi_i(x, z)$$

where $\phi_i(t, x)$ are variable unknowns of the resulting model, which is characterized by the choice of the vertical distribution, $\{\Psi_i\}_{i \in \{1, \ldots, N\}}$. We explore the outcome of vertical distribution defined, following the finite element method, as piecewise polynomials in the vertical variable, $z$. We particularly emphasize two families of models (respectively playing with the degrees of the polynomials and the number of elements in the vertical discretization): the augmented Green–Naghdi models and the “multilayer” Green–Naghdi models. In each case, the system consists in two evolution equations coupled with a system of differential equations of order two mimicking the Laplace problem. The first family is a higher order shallow water hierarchy comparable to the models of the preceding section, yet instead of involving high order differential operators, the size of the system of differential equations grows with the rank, $N$. The second family has different properties, akin to the second situation described above. The term “multilayer” stems from the fact that the models can be interpreted as resulting from the vertical discretization of the fluid layer in $N$ prescribed—typically proportional—sublayers.

In the third section we describe the strategy referred in [256, 255] as “variational” (see [349] for an overview of related earlier and subsequent works). Of course the preceding strategy was also variational in nature, and we argue that the two strategies in fact differ only by the choice of the variational formulation of the Laplace problem. Yet in the latter, we plug directly the decomposition into Luke’s Lagrangian action for the water waves system, and let Hamilton’s principle do all the work in one single step. The outcome is surprising at first, as we obtain an overdetermined/underdetermined composite system of $N$ evolution equations for the surface deformation, $\zeta$, and only one evolution equation for $(\phi_1, \ldots, \phi_N)$. Yet the systems can in fact be written—as in the above hierarchies—under a canonical Hamiltonian formulation of two evolution equations coupled with a system of differential equations of order two. Again each choice of the vertical distribution, $\{\Psi_i\}_{i \in \{1, \ldots, N\}}$, yields a different model. We only quickly mention the “multilayer” systems and instead focus on the shallow water system named the Isobe–Kakinuma models in reference to [228, 234]. Indeed the latter benefit from a rigorous justification theory, thanks to the work of Iguchi and collaborators, culminating with [223].
Non-hydrostatic models for interfacial waves

Toujours vouloir tout essayer, et recommencer
— Michel Berger, Le paradis blanc

Figure E: Models in Chapter E (in green) and some filiations.
Foreword

In Chapter A the equations extending the water waves system to the framework of interfacial waves between two layers of incompressible, homogeneous, inviscid and immiscible fluids with potential flows are introduced. The physical motivation for studying such systems is the reported (and ubiquitous) existence of coherent waves traveling at the sharp interface between, say, fresh and warm water above denser salted cold water. One can refer to e.g. [230, 210] for a small peek at the vast literature on the subject. The main features of these waves is that they have tremendously large amplitudes—sometimes of the order of magnitude of the layer itself—, very long wave length, and travel over very long distances. Hence the assumptions of the shallow water regime, and in particular the fact that we do not impose any smallness assumption on the amplitude of the wave, is perfectly suited to the study of such waves. It is therefore very tempting to introduce asymptotic models for interfacial waves which are analogous to the asymptotic models for the water waves system. This is done in the hydrostatic framework in Chapter B and non-hydrostatic models are the subject of this chapter.

In addition to the physical motivation, there are interesting new features and challenges when studying interfacial waves. First and foremost, three additional dimensionless parameters come into view, namely

\[ \alpha = \frac{a_{\text{top}}}{a_{\text{int}}}; \quad \delta = \frac{d_1}{d_2}; \quad \gamma = \frac{\rho_1}{\rho_2} \]

respectively the amplitude ratio of the free surface and interface, the depth ratio between the two layers, and the density ratio. Hence there are plethora of interesting limits to consider. We will focus here on the framework which is the most similar to the one-layer case\(^\text{12}\) and in particular we will assume that the two layers are of comparable depth, both small with respect to the typical horizontal wavelength of the flow. The relation between the limit of small density contrast, \(\gamma \to 1\) and the rigid-lid hypothesis, is discussed in details in the hydrostatic framework in Chapter B. Somewhat inconsistently, we will restrict henceforth to the rigid-lid situation\(^\text{13}\) without assuming the Boussinesq approximation, yet allowing \(\gamma\) to approach unity. To summarize, our results hold for parameters in the following set.

**Definition** (Shallow water/Shallow water asymptotic regime). Given \(\mu^*, \delta^*, \delta^* > 0\), we let

\[ p_{\text{SW}} = \{ (\mu, \varepsilon, \beta, \delta, \gamma) : \mu \in (0, \mu^*], \varepsilon \in [0, 1], \beta \in [0, 1], \delta \in [\delta^*, \delta^*], \gamma \in [0, 1) \}. \]

One of the main striking difference between the water waves system and the corresponding interfacial waves system is the emergence of Kelvin–Helmholtz instabilities in the latter. Recall that the provided modal analysis shows that large wavenumber modes are unstable, and that the exponential growth rate takes arbitrarily large values as the wavenumber goes to infinity. This explains why the initial-value problem associated with the nonlinear system is strongly ill-posed outside of the analytic framework. This appears to contradict the fact that, as we said, large interfacial waves do exist and appear remarkably stable! An answer to this paradox has been given by Lannes in [262], by introducing interfacial tension effects: it is shown that well-posedness is restored, and more importantly the time of existence of solutions grows as \(\mu \downarrow 0\), consistently with the fact that the hydrostatic equations for interfacial waves are well-posed. It should be emphasized however that interfacial tension is not physical at the interface between two miscible fluids such as fresh and salted water; here it plays the role of a regularizing operator acting mostly on the high (spatial) frequency component of the flow. The real physical explanation is that mixing occurs, yet on a very thin transition layer: the pycnocline. In the absence—to my knowledge—one of a simple expression revealing the effective influence of such mixing in the equations, we shall discard any effect when deriving asymptotic models.

\(^\text{12}\)see e.g. [100, 54] for related studies in other physically relevant asymptotic regimes.

\(^\text{13}\)see e.g. [98, 150, 151] for related studies in the free-surface framework.
It should be emphasized however that the models studied in this manuscript behave very differently regarding Kelvin–Helmholtz instabilities. Indeed, as the derivation focuses on the low frequency (large wavelength) component of the flow, the high frequency behavior can be very dissimilar between the different models, and hence with respect to the original interfacial waves system. A key revelation of the forthcoming study is the following.

- The Miyata–Choi–Camassa model, which is analogous to the Green–Naghdi system (see Chapter C), overestimates Kelvin–Helmholtz instabilities.

- This unfortunate behavior can be corrected through artificial—but harmless for the precision (in the sense of consistency) of the asymptotic model—modifications, which naturally yields fully dispersive systems named Whitham–Choi–Camassa; or regularized systems.

- The Kakinuma model, which extends the Isobe–Kakinuma model (see Chapter D) to the bilayer framework, inherently tames Kelvin–Helmholtz instabilities.

The latter model can be expected to be useful for understanding the propagation of long interfacial waves, focusing on the large-scale dynamics of the flow, and discarding small-scale effects as irrelevant. Once again, this should not blur the fact that mixing do occur, and may in some circumstances play an important role on the large-scale dynamics. Models with the aim of tracking these effects—at least at first order—should use the continuously stratified Euler equations as a starting point. Yet as mentioned in Chapter B, very little is known for this system in the shallow water regime. An important reference—in my opinion—dealing with long weakly dispersive internal (and not interfacial) waves is [137]. The Perspectives section in that reference supports and complements the present discussion.
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