On Frequency LTL in Probabilistic Systems

Vojtěch Forejt
Department of Computer Science
University of Oxford, UK

Jan Krčál
Saarland University – Computer Science,
Saarbrücken, Germany

Abstract—We study frequency linear-time temporal logic (fLTL) which extends the linear-time temporal logic (LTL) with a path operator $G^p$ allowing to express that on a path, certain formula holds with at least a given frequency $p$, thus relaxing the semantics of the usual $G$ operator of LTL. Such logic is particularly useful in probabilistic systems, where some undesirable events such as random failures may occur and are acceptable if they are rare enough.

Frequency-related extensions of LTL have been previously studied by several authors, where mostly the logic is equipped with an extended “until” and “globally” operator, leading to undecidability of most interesting problems.

For the variant we study, we are able to establish fundamental decidability results. We show that for Markov chains, the problem of computing the probability with which a given fLTL formula holds has the same complexity as the analogous problem for LTL. We also show that for Markov decision processes the problem becomes more delicate, but that when restricting the frequency bound $p$ to be 1 and negations not to be outside any $G^p$ operator, we can compute the maximum probability of satisfying the fLTL formula. This can be again performed with the same time complexity as for the ordinary LTL formulas.

I. INTRODUCTION

Probabilistic verification is a vibrant area of research that aims to formally check properties of stochastic systems. Among the most prominent formalisms are Markov chains and Markov decision processes (MDPs). Markov chains are apt for modelling systems that contain stochastic behaviour, for example random failures, while MDPs can in addition also express nondeterminism, most commonly present as decisions of a controller or dually as adversarial events in the system. Applications of Markov chains and MDPs include, for example modelling network security protocols [1] or randomised algorithms [2].

More technically, MDP is a process that moves in discrete steps within a finite state space (labelled by sets of atomic propositions). Its evolution starts in a given initial state $s_0$. In each step a controller chooses an action $a_i$ from a finite set $A(s_i)$ of actions available in the current state $s_i$. The next state $s_{i+1}$ is then chosen randomly according to a fixed probability distribution $\Delta(s_i, a_i)$. The controller may base its choice on the previous evolution $s_0a_0\ldots a_{i-1}s_i$ and may also choose the action randomly. A Markov chain is an MDP where the set $A(s)$ is a singleton for each state $s$. Hence, there is no real choice, yielding a purely stochastic process.

The desired properties of systems modelled as Markov chains or MDPs are often expressed using a formula of a suitable linear-time or branching-time temporal logic. Probably the most wide-spread linear-time logic used in the verification community is the Linear Temporal Logic (LTL), which allows to express properties such as “whenever a signal arrives to the system, the system eventually switches off”.

Although LTL is suitable in many scenarios, it does not allow to capture some important properties, for example that a given event takes place sufficiently often. This issue becomes even more apparent in stochastic systems, in which the probabilities are often used to introduce random failures. It is then natural to express, in the objective, that such failures are infrequent, while still having the power of the LTL to talk about more complex behaviour.

A natural solution to the above problem is to extend LTL with operators that allow us to talk about frequencies of events. This needs to be done with care, since adding such operators can easily lead to undecidability, due to the fact that their presence often allows one to encode values of potentially infinite counters [3], [4]. In both the above papers this is caused by introduction of a variant of a “frequency until” operator that talks about the ratio of the number of given events happening along a finite path. The undecidability results from [3], [4] carry over to the stochastic setting easily, and so, to avoid undecidability, care needs to be taken.

In this paper, we take an approach similar to [5] and in addition to usual operators of LTL such as $X$, $U$, $G$ or $F$ we only allow frequency globally formulae $G^p\varphi$ that require the formula $\varphi$ to hold on $p$-fraction of suffixes of an infinite path, or more formally, $G^p\varphi$ is true on an infinite path $s_0a_0s_1\ldots$ of an MDP if and only if

$$\liminf_{n \to \infty} \frac{1}{n} \cdot \left\{ i \mid i < n \text{ and } s_i a_i s_{i+1} a_{i+1} \ldots \text{ satisfies } \varphi \right\}$$

This logic, which we call frequency LTL (fLTL), is still a significant extension to LTL, and because the operators can be nested, it allows to express much larger class of properties.

The problem studied in this paper asks, given a Markov chain and an fLTL formula, to compute the probability with which the formula is satisfied in the Markov chain when starting in the initial state. Analogously, for MDPs we study the controller synthesis problem which asks to compute the maximal probability of satisfying the given formula, over all controllers. Let us now explain the logic and its possible applications on two examples.

Example 1 (fLTL for Markov chains). We assume a model of a network service that can be started by an initialization command. When started, it accepts queries by sending an
acknowledgement back immediately, then it processes the request either on server 1 or on server 2 and eventually sends a response back. This property can be expressed in the standard LTL as

$$i \rightarrow \left( G(q \rightarrow a) \land G(a \rightarrow (p_1 \lor p_2 \lor r)) \right).$$

When the model is a Markov chain, we can model failures by probabilities. We may then require that behaviour specified by the formula has probability, e.g., at least 0.95. However, in order to keep the formula satisfied, the probabilities may only influence whether the system gets initialized or how long it takes to process a query. In particular, a response must be always sent almost surely. When using an fLTL formula

$$i \rightarrow \left( G(q \rightarrow a) \land G^{0.99}(a \rightarrow (p_1 \lor p_2 \lor r)) \right),$$

we allow (infrequent) failures causing a query not to be responded and we may then include such failures in the Markov chain model. Similarly as before, we can require the formula to be satisfied with probability at least 0.95. The formula may also contain several frequency operators, even nested. For instance, we may relax the requirements by

$$i \rightarrow \left( G(q \rightarrow a) \land G^{0.99}((a \rightarrow (p_1 \lor p_2 \lor r)) \lor \neg G^{0.02}a) \right).$$

Example 2 (fLTL for MDPs). When modelling a similar network service as an MDP we may leave some parts of the behaviour unspecified. The aim is to synthesise a control strategy that meets with a given probability the requirements on the system. Let us assume an fLTL formula

$$G F m \land G^1(q \rightarrow X r).$$

Here, we also consider maintenance of the service and require that it occurs infinitely often. The requirements can be satisfied in the following MDP model (where each action is depicted by one outgoing arrow, possibly splitting according to the probability distribution $\Delta$, unique actions are not labelled)

```
  m
 /      \
 w ----> m
  \      / \\
  |      |  \\
 r ----> q ----> q,m
  |      |  |    |
  0.5   0.5 0.5
```

In most states, a new query comes in the next step with probability 0.5. In the waiting state, the system chooses either to wait further (action w), or to start a maintenance (action m) which takes two steps to finish. During maintenance, the system is unable to respond. In order to push the frequency of correctly handled request towards 1, the controller needs to choose to perform the maintenance less and less frequently during operation.

A. Related work

Controller synthesis problem for ordinary LTL is a well studied problem that can be solved in time polynomial in the size of the model and doubly exponential in the size of the LTL formula [5]. Usually, the LTL formula is transformed to an equivalent Rabin automaton, and then a product of MDP (or Markov Chain) with the automaton is constructed and the probability of reaching certain subgraphs is computed. This approach cannot be extended to our setting, since as we mention below it is not clear what class of automata would express fLTL properties.

Significant attention has been given to the study of quantitative objectives. [7] introduces lexicographically ordered mean-payoff objectives in non-stochastic parity games, while [8] adds mean-payoff objectives to temporal logics, but only as atomic propositions and not allowing more complex properties to be quantified. [9] extends LTL with another form of quantitative operators, allowing accumulated weight constraint expressed using automata, again not allowing quantification over complex formulas.

A logic similar to our fLTL was studied in [5], where the LTL is extended with a mean-payoff reward constraints in which the reward structures are determined by validity of given subformulas. The authors show that any formula can be converted to a variant of nondeterministic Büchi automata, called multi-threshold mean-payoff Büchi automata, for which the emptiness problem is decidable, thus yielding decidability for model-checking and satisfiability problems of labelled transition systems. The approach of [5] cannot be extended to probabilistic systems, since in probabilistic systems one needs to work with deterministic automata, e.g. Rabin automata. As the authors themselves point out in [5, Section 4, Footnote 4] their approach heavily relies on non-determinism, since to determine a reward one needs to know the complete future, and so the class of “multi-threshold mean-payoff Rabin automata” is strictly less expressible than the logic.

Another variant of frequency LTL was studied in [3], [4], in which also a modified until operator is introduced. The work [3] maintains boolean semantics of the logic, while in [4] the value of a formula is a number between 0 and 1. Both of the works obtain undecidability results for their logics, and [3] also yields decidability when the nesting of the frequency until operator is restricted. Another logic that speaks about frequencies on a finite interval was introduced in [10].

The work [11] gives a polynomial time algorithm for almost-sure winning in MDPs with mean-payoff and parity objectives. These objectives do not allow to attach mean-payoff (i.e. frequencies) to properties more complex than atomic propositions. The solution to the problem requires infinite-memory strategy which at high level has a form similar to the form of strategies we construct for MDPs. Similar strategies also occur in [12], [13], [14] although each of these works deals with a fundamentally different problem.

In branching-time logics, CSL is sometimes equipped with a “steady-state” operator whose semantics is similar to our $G^p$
II. PRELIMINARIES

We now proceed with introducing basic notions we use throughout this paper.

A probability distribution over a finite or countable set $X$ is a function $d : X \rightarrow [0, 1]$ such that $\sum_{x \in X} d(x) = 1$.

A Markov decision process and Markov chains

A Markov decision process (MDP) is a tuple $M = (S, A, \Delta)$ where $S$ is a finite set of states, $A$ is a finite set of actions, and $\Delta : S \times A \rightarrow \mathcal{D}(S)$ is a partial probabilistic transition function. A Markov chain (MC) is an MDP in which for every $s \in S$ there is exactly one $a$ with $\Delta(s, a)$ being defined. To simplify the notation, we often omit actions completely when we speak about Markov chains and no confusion can arise.

An infinite path, also called run, in $M$ is a sequence $\omega = s_0a_1s_1 \ldots$ of states and actions such that $\Delta(s_i, a_i)(s_{i+1}) > 0$ for all $i$, and we denote $\omega(i)$ the suffix $s_ia_is_{i+1}a_{i+1} \ldots$. A finite path $h$, also called history, is a prefix of an infinite path ending in a state, and we use last($h$) for the last state of $h$ and $|h|$ for the number of states in $h$ (sometimes called length of $h$). The set of paths starting with a prefix $h$ is denoted by $Cyl(h)$, or simply by $h$ if it leads to no confusion. Analogously, we use a set of paths $H$ instead of the set of runs $\bigcup_{h \in H} Cyl(h)$ that start with some path in $H$. Given a finite path $h = s_0a_1s_1 \ldots s_i$ and a finite or infinite path $h' = s_ia_is_{i+1}a_{i+1} \ldots$ we use $h \cdot h'$ to denote the path $s_0a_1s_1 \ldots$.

A strategy is a function $\sigma$ that to every finite path $h$ assigns a probability distribution over actions such that if an action $a$ is assigned a nonzero probability, then $\Delta($last$(h), a)$ is defined. A strategy $\sigma$ is deterministic if it assigns Dirac distribution to any history, and randomised otherwise. Further, it is memoryless if its choice only depends on the last state of the history, and finite-memory if there is a finite automaton such that $\sigma$ only makes its choice based on the state the automaton ends in after reading the history.

An MDP $N$, a strategy $\sigma$ and an initial state $s_{in}$ give rise to a probability space $\mathbb{P}_n^{s_{in}}$ defined in a standard way [18]. Expectation of a random variable $X$ in this probability space is likewise denoted by $\mathbb{E}_n^{s_{in}}(X)$. Given a measurable set of runs $U$ and a finite path $h$, we put $\mathbb{P}_n^{s_{in}}(U) = \mathbb{P}_n^{s_{in}}(\{h : \omega \in U \ | \ h\}$). Similarly, for a random variable $X$, $\mathbb{E}_n^{s_{in}}(X)$ denotes $\mathbb{E}_n^{s_{in}}(X' \ | \ h)$ where $X'$ is defined by $X'(h : \omega) = X(\omega)$ and $X'(h : \omega) = 0$, elsewhere. We say a path $h$ is positive (subject to a strategy $\sigma$) if $h$ has nonzero probability under $\sigma$.

A bottom strongly connected component (bscc) of a Markov chain is a set of its states $S'$ such that for all $s \in S'$ the set of states reachable from $s$ is exactly $S'$.

B. Frequency LTL

Before introducing frequency LTL (fLTL), we first formalise the notion of frequency. Given an infinite sequence $\lambda = x_1x_2 \ldots$ of numbers, we define $\text{freq}(\lambda) = \lim_{i \to \infty} \frac{1}{i} \sum_{j=1}^{i} x_i$.

The syntax of frequency LTL (fLTL) is defined by the following equation:

$$\varphi ::= \alpha | \neg \varphi | \varphi \lor \varphi | X \varphi | \varphi U \varphi | G^p \varphi$$

where $\alpha$ ranges over a set $AP$ of atomic propositions. The logic LTL is obtained by omitting the rule for $G^p \varphi$. Given a
valuation $\nu : S \to 2^{AP}$, the semantics of fLTL is defined over a path $\omega$ of an MDP as follows.

$$
\begin{align*}
\omega \models \alpha & \iff \alpha \in \nu(s_0) \text{ where } \omega = s_0a_0s_1 \ldots \\
\omega \models \neg \varphi & \iff \omega \not\models \varphi \\
\omega \models \varphi_1 \lor \varphi_2 & \iff \omega \models \varphi_1 \text{ or } \omega \models \varphi_2 \\
\omega \models X\varphi & \iff \omega(1) \models \varphi \\
\omega \models \varphi_1 U \varphi_2 & \iff \exists k : \omega(k) \models \varphi_2 \\
\omega \models G\varphi & \iff \text{freq}(1_{\varphi,0}1_{\varphi,1} \ldots) \geq p
\end{align*}
$$

where $1_{\varphi,i}$ is 1 for $\omega$ iff $\omega(i) \models \varphi$, and 0 otherwise. We use $P_{\sigma}(\varphi)$ as a shorthand for $P_{\sigma}(\{\omega \mid \omega \models \varphi\})$. We also assume that for every $s \in S$ there is an atomic proposition of the same name, and given $S' \subseteq S$ we write just $S'$ instead of $\bigvee_{s \in S'} s$.

We define true $\equiv \alpha \lor \neg \alpha$ for some $\alpha \in AP$ and introduce standard operators $F$, and $G$ in the usual way by putting $F \varphi \equiv \text{true} U \varphi$, and $G \varphi \equiv \neg F \neg \varphi$. We also use $\land$ and $\rightarrow$ with their usual definitions.

**Definition 1** (Controller synthesis). The controller synthesis problem asks to decide, given an MDP $\mathcal{M}$, an initial state $s_{in}$, an fLTL formula $\varphi$ and a probability bound $\rho$, whether $P_{\rho}(\varphi) \geq 1$ for some strategy $\sigma$.

As an alternative to the above problem, we can ask to compute the maximal possible $x$ for which the answer is true. In the case of Markov chains, we speak about Satisfaction problem since there is no strategy to synthesise.

### C. Rabin automata

A (deterministic) Rabin automaton is a tuple $R = (Q, q_{in}, \Sigma, \delta, F)$ where $Q$ is a finite set of states, $\Sigma$ is an input alphabet, $\delta : Q \times \Sigma \to Q$ is a transition function, and $F \subseteq Q \times Q$ is an accepting condition. We sometimes use $\text{init}(R)$ to refer to the initial state $q_{in}$. A computation of $R$ on an infinite word $\lambda = a_0a_1 \ldots$ over the alphabet $\Sigma$ is the infinite sequence $R[\lambda] = q_0q_1 \ldots$ with $q_0 = q_{in}$ and $\delta(q_i, a_i) = q_{i+1}$. A computation is accepting (or “$R$ accepts $\lambda$”) if there is $F = (E, F) \in F$ such that all states of $E$ occur only finitely many times in the computation, and some state of $F$ occurs in it infinitely many times.

By $\inf_{\varphi}(\omega)$ (resp. $\text{occ}_{\varphi}(\omega)$) we denote the set of states of $R$ that occur infinitely many times in $R[\omega]$ (resp. that occur at least once in $R[\omega]$) when the Rabin automaton is started from state $q$ instead of $q_{in}$.

Given a Rabin automaton $R$ and a prefix $w$ of a word that the automaton can read, we use $R(w)$ to denote the state in which $R$ ends reading $w$.

For a run $\omega = s_0a_0s_1a_1 \ldots$ and a valuation $\nu$, we use $\nu(\omega)$ for the sequence $\nu(s_0)\nu(s_1) \ldots$ of sets of atomic propositions. We note that it is not clear whether the definition of Rabin automata and Lemma 1 can be extended to work with fLTL in a way that would be useful for our goals. The reason for this is, as pointed out in [5] Section 4, Footnote 4, that the frequencies in fLTL heavily depend on the future of a run, and so require non-determinism, which is not desirable in stochastic verification.

### III. Satisfaction problem for Markov chains

We now give an algorithm that computes the probability that an fLTL formula $\psi$ is satisfied in a Markov chain. Let $\mathcal{N}$ be a Markov chain, $s_{in}$ an initial state and $\psi$ an fLTL formula with $k$ subformulae of the form $G\varphi$. We show how to construct a formula $\psi'$ with only $k-1$ subformulae of the form $G\varphi$ such that $P_{\epsilon}(\psi) = P_{\epsilon}(\psi')$. By iterating this approach $k$ times we get a formula which is in fact an LTL formula, and so we can compute $P_{\epsilon}(\psi')$ using standard means [6].

Let $G\varphi$ be a subformula of $\psi$ such that $\varphi$ is an LTL formula. For every bssc $B$ of $\mathcal{N}$ and a state $s$ contained in $B$, let $x_s$ be the steady-state frequency of $s$ within $B$, i.e. the number $E(\text{freq}(1_{s,0}1_{s,1} \ldots))$; this number can easily be computed in polynomial time [19]. We create a fresh atomic proposition $a_{s,p}$ which is true in the states of $B$ iff $\sum_{s \in B} x_s \cdot P_{\sigma}(\varphi) \geq p$, and construct $\psi'$ from $\psi$ by replacing any occurrence of $G\varphi$ with $F a_{s,p}$. The correctness of the construction is obtained by applying the following proposition.

**Proposition 1.** We have $P_{\epsilon}(G\varphi) = P_{\epsilon}(F a_{s,p})$.

Note that although the above proposition might seem to be obviously true, the proof is not trivial. The main obstacle is that satisfaction $\varphi$ on $\omega(i)$ and $\omega(j)$ are not independent events in general: for example if $\varphi \equiv F \alpha$, and $i < j$, then $\omega(j) \models \varphi$ implies $\omega(i) \models \varphi$. For this reason it is not possible to directly use the Strong law of large numbers (SLLN) for independent random variables or Ergodic theorem for Markov chains [19] Theorem 1.10.1 and 1.10.2], which would otherwise be obvious candidates. Nevertheless, we can make use of the following variant of SLLN for correlated events.

**Lemma 2.** Let $Y_1, Y_2 \ldots$ be a sequence of random variables which only take values 0 or 1 and have expectation $\mu$. Assume there are $0 < r, c < 1$ such that for all $i, j \in \mathbb{N}$ we have $E((Y_i - \mu)(Y_j - \mu)) \leq r|i-j|$. Then $\lim_{n \to \infty} \sum_{i=0}^{n} Y_i/n = \mu$ almost surely.

**Proof:** By simple application of [20] Corollary 4.

We now show how Proposition 1 can be proved using the above lemma. Let $s$ be a state satisfying $a_{s,p}$, and let $t$ be an arbitrary state of a bssc containing $s$. We define a sequence of random variables $X_1^t, X_2^t \ldots$ by letting every variable $X_i^t$ take value 1 (resp. 0) on $\omega = s_0s_1 \ldots$ if the suffix of $\omega$ starting from the $i$-th visit to $t$ for satisfies (resp. does not satisfy) $\varphi$. If the state $t$ is visited less than $i$ times, we set $X_i^t$ to 0. We prove that Lemma 2 can be applied to the random variables $X_1^t, X_2^t \ldots$ above. Fix arbitrary $i$ and $j$ and w.l.o.g. suppose $i \leq j$. Consider the Markov chain which is a product of $\mathcal{N}$ and a
Rabin automaton $R$ equivalent to $\varphi$ (for the sake of evaluation of one particular $X_j^i$). Every word $s_0 s_1 \ldots$ corresponds to a path $(s_0, q_0)(s_1, q_1) \ldots$ in the product such that $q_0 q_1 \ldots$ is a computation of $R$ on $s_0 s_1 \ldots$. Moreover, for almost all such $s_0 s_1 \ldots$ we have that once $(s_j, q_j) \in$ a bsc of the product, then either almost every run of $Cy_l(s_0 \ldots s_j)$ satisfies $\varphi$, or almost every run of $Cy_l(s_0 \ldots s_j)$ does not satisfy $\varphi$. If this is the case, we call the path $s_0 \ldots s_j$ positively determined (resp. negatively determined).

Further, the probability that a run of the product enters a bsc within $k$ steps is at least $(1 - p)^{\lfloor k/M \rfloor}$ where $p$ is the minimum probability that occurs in the product, and $M$ is the number of states of the product. We denote $\Omega$ the set of all runs and $A$ (resp. $B$) the set of runs for which the path starting in $i$-th and ending in $j$-th visit to $t$ is positively (resp. negatively) determined.

$$
\mathbb{E}^n((X_i^j - \mu)(X_j^j - \mu)) = \mathbb{P}(A) \cdot \mathbb{E}^n((1 - \mu)(X_i^j - \mu) | A) + \mathbb{P}(B) \cdot \mathbb{E}^n((-\mu)(X_j^j - \mu) | B) + \mathbb{P}(\Omega \setminus (A \cup B)) \cdot \mathbb{E}^n((X_i^j - \mu)(X_j^j - \mu) | \Omega \setminus (A \cup B))
$$

$$
\leq (1 - p)^{\lfloor (i-j)/M \rfloor}
$$

Thus, Lemma 2 applies to the random variables $X_i^j$. Now let bsc$(s)$ contain the states from bsc of $s$, and let $X_i^j$ be the random variable returning the number of occurrences of $t$ in the first $n$ states of a run. We get that from $s$, almost surely the following is true:

$$
\lim_{n \to \infty} \frac{\sum_{i=0}^{n} 1_{\varphi,i}}{n} = \sum_{t \in \text{bsc}(s)} x_t \mathbb{P}^s(\varphi)
$$

This finishes proof of Proposition 1 and we get the following theorem.

**Theorem 1.** The satisfaction problem for Markov chains and fLTL is solvable in time polynomial in the size of the model, and doubly exponential in the size of the formula.

**IV. CONTROLLER SYNTHESIS FOR MDPs**

We now proceed with the controller synthesis problem for MDPs. For this section, we restrict to the fragment of fLTL in which the negations can only occur immediately preceding atomic propositions, and for any $G^p$ operators occurring in the formula we have $p = 1$ (in addition to the basic LTL operators we also allow $\land$, $\lor$ and $G$). We call this fragment 1-fLTL. Although the above might seem as a very strong restriction, a direct consequence of the following theorem is that the controller-synthesis problem for 1-fLTL is not equivalent to checking the LTL formula where $G^p$ is replaced with $G$.

**Theorem 2.** There is a 1-fLTL formula $\psi$ and a Markov decision process $\mathcal{M}$ such that the answer to the controller synthesis problem is "yes", but there is no finite-memory strategy witnessing this.

**Proof:** Consider the following MDP together with a formula $\psi = G^1(s_1 U s_2)$.

First, we argue that there is a satisfying strategy: Consider $\sigma$ which gives rise to the path

$(s_1 a)^i b s_2 c s_3 d(s_1 a)^i b s_2 c s_3 \ldots (s_1 a)^i b s_2 c s_3 d(s_1 a)^{i+1} \ldots$

Note that any suffix starting in $s_1$ satisfies $s_1 U s_2$, and the frequency of such states is 1.

On the other hand, any finite-memory strategy will yield a Markov chain in which any bsc either (i) does not contain $s_2$ and so $s_1 U s_2$ cannot be satisfied on any path within that bsc, or (ii) contains $s_3$, in which case $s_1 U s_2$ is satisfied with frequency lower than 1.

The above result suggests that it is not possible to easily re-use verification algorithms for ordinary LTL. Nevertheless, our results allow us to establish the following theorem.

**Theorem 3.** The controller-synthesis problem for 1-fLTL for MDPs is solvable in time polynomial in the size of the model and doubly exponential in the size of the formula.

For the rest of this section, in which we prove Theorem 3, we fix an MDP $\mathcal{M}$, an initial state $s_1$, and a 1-fLTL formula $\psi$. Without loss of generality suppose that the formula $\psi$ does not contain $G^1$ as the outermost operator, and that it contains $n$ subformulae of the form $G^1 \varphi$. Denote these subformulae $G^1 \varphi_1, \ldots, G^1 \varphi_n$. The first step of our construction is to convert these formulae $\psi, \varphi_1, \ldots, \varphi_n$ to Rabin automata. However, this is not directly possible since each of the formulae might contain a $G^1$ operator, and thus not be expressible using a Rabin automaton (and as pointed out by [5] there is a fundamental obstacle preventing us from extending Rabin automata to capture this). To overcome this, we replace all subformulae of the form $G^1 \varphi_i$ in such formulae by either true or false, to capture that the subformula is or is not true on a run. Of course, there are multiple possible replacements, and so we need to create multiple Rabin automata as follows.

For a formula $\xi$, let $sf(\xi) \subseteq \{1, \ldots, n\}$ contain $i$ whenever $G^1 \varphi_i$ occurs as a subformula of $\xi$ not guarded by any $G^1$. For a formula $\xi \in \{\psi, \varphi_1, \ldots, \varphi_n\}$ and any $J \subseteq \{1, \ldots, n\}$, let $R_{\xi, J}$ be a Rabin automaton for the formula $\xi^J$ obtained from $\xi$ by replacing $G^1 \varphi_j$ with true or false depending on whether $j \in J \cap sf(\xi)$ or not. We also use $Q$ for a disjoint union of the state spaces of these distinct Rabin automata, and for $q \in Q$ denote $R[\eta]q$ the automaton $R_{\xi,J}$ to whose state space $q$ belongs. We also call the automaton $R_{\psi,J}$ main for any $J$.

**Example 3.** The formula

$$
\psi = i \rightarrow (G(q \rightarrow a) \land G^1(p_1 U r \lor G^1 a))
$$

is decomposed into $\varphi_1 = (p_1 U r \lor G^1 a)$ and $\varphi_2 = a$. We then
have \( sf(\psi) = \{1\} \), \( sf(\varphi_1) = \{2\} \), and \( sf(\varphi_2) = \emptyset \). Then, e.g.,

\[
\psi^{(1)} = i \rightarrow (G(q \rightarrow a) \land \text{true}) = \psi^{(1,2)}, \quad \varphi_1 = (p_1 \lor r \lor \text{false})
\]

After decomposing \( \psi \) into standard LTL formulas, we prove as a further step the following proposition.

**Proposition 2.** The following two conditions are equivalent:

1) There is a strategy \( \sigma \) with \( \mathbb{P}_\sigma(\psi) = x \).

2) There is a set \( \Upsilon \subseteq S \times Q \times 2^{\{1,\ldots,n\}} \) such that the following two conditions hold:

   a) There is a strategy \( \sigma' \) with \( \mathbb{P}_\sigma(\varnothing \Upsilon) = x \) where \( \varnothing \Upsilon \) is the set of runs such that \( s_0a_0s_1a_1 \ldots \in \varnothing \Upsilon \) if and only if there is \( (s^*, q^*, I^*) \in \Upsilon \) and a position \( k \) satisfying \( s_k = s^* \) and the main automaton \( R_{\psi, I^*} \) ends in \( q^* \) after reading \( \nu(s_0a_0s_1 \ldots s_{k-1}) \).

   b) For all \( (s^*, q^*, I^*) \in \Upsilon \), we have that \( q^* \) is a state of \( R_{\psi, I^*} \) and there is a strategy \( \sigma_{s^*, q^*, I^*} \) such that for almost every run \( \omega \) initiated in \( s^* \) we have that

   \[
   \nu(\omega) \text{ is accepted by } R_{\psi, I^*} \text{ when started in } q^*, \quad \omega \models G^1\varphi_1 \text{ is true whenever } i \in I^*.
   \]

Note that here we split the problem into “reaching” a good set \( \Upsilon \) with probability \( x \) (condition 2a) and winning “from” \( \Upsilon \) with probability \( 1 \) (condition 2b). The states \( s^* \) and \( q^* \) are the states of the MDP and the main automaton, respectively, when \( \Upsilon \) is reached (provided all frequency formulae in \( I^* \) are satisfied) and hence in [23], the execution needs to continue from these states.

We now prove the above proposition. In the direction \( \Leftarrow \), it suffices to define \( \sigma \) so that it behaves as \( \sigma' \) initially, up to the position \( k \) defined in item 2a) above. Suppose that the history at position \( k \) is \( hs^* \) and the condition of item 2b) is satisfied for \( (s^*, q^*, I^*) \in \Upsilon \). The strategy \( \sigma \) then starts behaving on \( hs^* \cdot h' \) as \( \sigma_{s^*, q^*, I^*} \cdot h' \). The following lemma is then a trivial consequence of the construction.

**Lemma 3.** The strategy \( \sigma \) defined above satisfies \( \mathbb{P}_\sigma(\psi) = x \).

Let us now continue with the direction \( \Rightarrow \) of Proposition 2 which is significantly more difficult. In the proof, we will need to eliminate some unlikely events, and for this we will require that their probability is small to start with. For this purpose, we fix a very small positive number \( \varepsilon_1 \) to avoid cluttering of notation, we do not give a precise value of \( \varepsilon_1 \), but instead point out that it needs (and can) be chosen such that any numbers that depend on it in the following text have the required properties (i.e. are sufficiently small or big). We also fix \( \ell = |S_0| + 1 \) and \( \varepsilon_2 = 3 \cdot |\ell|^2 \cdot 2^n \cdot \varepsilon_1 \). The purpose of these numbers will become clearer later in the proof, but we should point out that \( \varepsilon_1 \) and \( \varepsilon_2 \) are influencing neither the size of representation of our strategy nor the complexity of our algorithm and are only required to make the proof work.

We mark every run \( \omega \) satisfying \( \psi \) with a set \( I_n \subseteq \{1, \ldots, n\} \) such that \( i \in I_n \) iff the formula \( G^1\varphi_i \) holds on the run. By the following variant of Lévy’s Zero-One Law, there is a prefix-free set \( \Gamma \) of histories such that \( \mathbb{P}_\sigma(\Gamma) = x \) and for every \( h^* \in \Gamma \) there is \( I_n \subseteq \{1, \ldots, n\} \) such that \( \mathbb{P}_\sigma((\omega \in \omega \models \psi \land \omega \uparrow h^*) \mid h^*) \geq 1 - \varepsilon_1 \).

**Lemma 4 (Lévy’s Zero-One Law [21]).** Let \( \sigma \) be a strategy and \( X \) a measurable set of runs. Then for almost every run \( \omega \) such that all its prefixes are positive, we have

\[
\lim_{n \to \infty} \mathbb{P}_\sigma(X \mid h_n) = 1^{X(\omega)}
\]

where each \( h_n \) denotes the prefix of \( \omega \) with \( n \) states.

This powerful lemma (seemingly on the edge of triviality) will be also used at several further places in the proof.

Now we show how to construct a witnessing strategy for item 2b) of Proposition 2. For the rest of the section, fix \( (s^*, q^*, I^*) \) from \( \Upsilon \) and a witnessing history \( h^* \in \Gamma \). Slightly rewriting the definition above, we have

\[
\mathbb{P}_\sigma(\psi^{(1)} \land \bigwedge_{i \in I^*} G^1\varphi_i^{(1)} \mid h^*) \geq 1 - \varepsilon_1 \quad (1)
\]

As we have shown in Theorem 2 strategies might require infinite memory, and this needs to be taken into consideration when constructing \( \sigma_{s^*, q^*, I^*} \). We define the strategy so that it cycles through two “phases”, called accumulating and reaching. In the accumulating phase, the strategy will be passing through states which satisfy (or, more precisely, paths from which almost surely satisfy) all formulae \( \varphi_i \) for \( i \in I^* \) that are required to be satisfied with frequency 1. Since each of the \( \varphi_i \) can place a requirement on more complex behaviour which cannot be satisfied in the accumulating phase, we repeatedly enter into a reaching phase which executes such behaviour. The formulae \( \varphi_i \) are not guaranteed to hold on suffixes starting in a reaching phase, and so we need to make sure we make the accumulating phases progressively longer so that in the long run they take place with frequency 1. The major obstacle in constructing \( \sigma_{s^*, q^*, I^*} \) is that \( \sigma \) might have a very complex behaviour, for example it might not be the case that any \( \varphi_i \) is ever satisfied almost surely from any state.

**Example 4.** We illustrate the set \( \Upsilon \) and accumulating and reaching phases on the formula \( \psi = X s_2 \land G^1(s_4 \lor s_5) \) that can be satisfied on the following MDP with probability 0.5.

On the left we show a Rabin automaton for the LTL formula \( \psi^{(1)} = X s_2 \land \text{true} \) and on the right for \( \varphi_1^{(1)} = s_4 \lor s_5 \).

\[\text{Diagram of the MDP and automaton for Example 4.}\]
Let us assume some strategy $\sigma$ that satisfies $\psi$ with probability $0.5$. We set $\Upsilon$ so that it contains the triple $(s_4, q_3, \{1\})$. The strategy $\sigma_{s_4, q_3, \{1\}}$ we would like to obtain

- “accumulates” arbitrarily many instances of the Rabin automaton $R_{\varphi_1,\{1\}}$ (all being in state $q_5$) by repeating action $a_4$ in $s_4$, and then
- “reaches” with all the Rabin automata accumulated in the previous phase their accepting state $q_0$ by taking action $a_2$ and in two further steps returns back to $s_4$ where the next accumulation phase can start.

When the strategy $\sigma_{s_4, q_3, \{1\}}$ makes successive accumulation phases longer and longer, it satisfies Proposition 2. However, it is not easy to extract this simple behaviour from the strategy $\sigma$ if the strategy for instance takes action $a_6$ with probability $1/2^i$ in the $i$-th visit to $s_4$ for every $i$. (Note that this strategy still satisfies $\mathbb{P}_\alpha(\psi) = 1/2$.)

Let us now explain the idea behind construction of accumulating and reaching phases in more detail. To satisfy the required conditions, the strategy $\sigma_{s_4, q_3, \{1\}}$ will need to make sure that the automata $R_{\varphi_1,\{1\}}$ for any $i \in I^*$ accept almost surely when started in any state of a accumulating phase. To facilitate this, we use a tailor-made product construction $M_\otimes$ which we define below. In the state of the product, we will store current states of Rabin automata started in previous accumulating phases that have read the history since then. By this strategy, the Rabin automaton eventually commits to an accepting condition or to visit a state of $R[in]$. When the strategy $\sigma$ still satisfies $\mathbb{P}_\alpha(\psi)$, we say $\sigma$ is legal in $(s, C)$ if:

- $C_a$ is not fulfilled and:
  - For every $(q, \star) \in C_a$, there is $(q, \star) \in C_a$.
  - For every $(q, \ast) \in C_a$, there is $(q, \ast) \in C_a$ or $(q, (E, F)_o) \in C_a$ for some accepting condition $(E, F)$ of $R[q]$.
  - For every $(q, (E, F)_x) \in C_a$, there is $(q, (E, F)_x) \in C_a$.

For all states $(s, C_a)$, all actions $(a, C_a)$ legal in $(s, C_a)$, and all $(t, C_t)$ such that $C_a \rightarrow C_t$, we set $\Delta_\otimes((s, C_a), (a, \bot))(t, C_t) = \Delta(s, a)(t)$. To avoid deadlocks, we also set $\Delta_\otimes((s, C_a), (a, \bot))(t, \bot) = \Delta(a)(t)$ for any $s, t \in S_\otimes$, $a \in A_\otimes$, and $C_a \in 2^E \cup \{\bot\}$.

**Example 5.** Figure 7 shows part of the product $M_\otimes$ for the MDP and the Rabin automata from Example 4. States with fulfilled second component are marked by double line around them, and the dotted part of the product is the one in which a strategy can “accumulate” arbitrarily and from which it can leave to visit a state with a fulfilled second component and subsequently return.

The following lemma relates the product $M_\otimes$ back to $M$. It shows an important property of every deterministic strategy in $M_\otimes$ that fulfills the second component infinitely often. Whenever the strategy “promises” to satisfy a formula by adding the initial state $(q_{\text{init}}, \star)$ of its Rabin automaton, the Rabin automaton actually accepts in $M$ with probability one.
Lemma 5. Let $\sigma_\otimes$ be a deterministic strategy in $M_\otimes$ and $(s_0, C_0)$ a state such that
\[ P_{\sigma_\otimes}^{s_0, C_0} (\text{fulfilled states visited infinitely often}) = 1. \]
There is a strategy $\sigma_\otimes^{\text{proj}}$ in $M$ such that for any positive path $(s_0, C_0) \cdots (a_n, D_n)(s_{n+1}, C_{n+1})$ and any $(q, \star) \in D_n$,
\[ P_{\sigma_\otimes^{\text{proj}}}_{s_0, C_0} ((\omega \mid R[q] \text{ accepts } \omega)) = 1. \]

Proof: For any finite path $h = s_0 a_0 \cdots a_{k-1} s_k$ in $M$ there is at most one path $(s_0, C_0) (a_0, D_0) \cdots (a_{k-1}, D_{k-1}) (s_k, C_k)$, denoted $h_{\otimes}$ with $C_0$ fixed above, all $D_i$ chosen with probability 1 by the deterministic strategy $\sigma_\otimes$ and all $C_i$ given uniquely by the definition of $M_\otimes$. Note that $h_{\otimes}$ may be undefined if $(s, \bot)$ is reached instead for some $s$.

We define the strategy $\sigma_\otimes^{\text{proj}}$ by $\sigma_\otimes^{\text{proj}}(h) = \sigma_\otimes(h_{\otimes})$ for all $h$ for which $h_{\otimes}$ is defined, and defining $\sigma_\otimes^{\text{proj}}(h)$ arbitrarily otherwise. Let $h_0 = (s_0, C_0) \cdots (a_n, D_n)(s_{n+1}, C_{n+1})$ be a positive path. Since $\sigma_\otimes$, when starting after $h_0$, almost surely fulfills infinitely often, it also never reaches $(s, \bot)$ for any $s$. Hence, $\sigma_\otimes^{\text{proj}}$ almost surely takes the same decisions as $\sigma_\otimes$. From the definition of $M_\otimes$, it is easy to see that fulfilling infinitely often implies that $R[q]$ accepts $\omega$.

The following proposition gives a crucial insight into the idea behind the proof of the main theorem, establishing the connection of our controller-synthesis problem to the product $M_\otimes$ defined above.

Proposition 3. If $\sigma$ satisfies $P_\sigma(q^* \wedge \bigwedge_{i \in I^*} C_i \downarrow \sigma^* \mid h^*) \geq 1 - \varepsilon_1$, there is $(M, N)$ with $M \subseteq S_\otimes$, $N \subseteq A_\otimes$ such that:
1) For some $(s^*, C) \in M$ we have $(q^*, \star) \in C$;
2) For any $(a, C) \in N$ and $i \in I^*$, $\bigwedge \{R_{\varphi_i, i}, \star\} \in C$;
3) For any $(s, C) \in M$ there is a finite-memory "accumulating" strategy $\sigma_{s,C}$ that, when starting in $(s, C)$, never leaves $M$ and never uses actions outside $N$;
4) For any $(s, C) \in M$ there is a finite-memory "reaching" strategy $\zeta_{s,C}$ that, when starting in $(s, C)$, almost surely reaches a state with fulfilled second component (possibly leaving $M$) and afterwards reaches $M$.

The proof of Proposition 3 is involved and is given in Section IV.B. Now we describe how the tuple $(M, N)$ given by Proposition 3 is used to finish the proof of Proposition 2, i.e. how the strategy $\sigma_{s,q,I}$ is constructed. We first construct a strategy $\sigma_\otimes$ for $M_\otimes$ that satisfies the conditions of Lemma 5 for the initial state $(s^*, C)$ given by item 1 above. The desired strategy $\sigma_{s,q,I}$ is then obtained by Lemma 5, by taking $\sigma_\otimes^{\text{proj}}$.

Inductively, for path $h$ in $M_\otimes$, we say that its $i$-th accumulating phase starts in the first step, $i$-th accumulating phase takes $i$ steps, and the $i+1$-th accumulating phase starts when the set $M$ is reached through a state with fulfilled second component after the $i$-th accumulating phase ended.

Suppose the current state is $(t, C)$, and the $i$-th accumulating phase is starting. The strategy $\sigma_\otimes$ is defined to play as $\sigma_{t,C}$ for $i$ steps. Suppose the current state is $(t', C')$ and an accumulating phase has just ended. The strategy then acts as $\zeta'_{C'}$ up to the point when it reaches a state $(t'', C'')$ of $M$ after visiting a fulfilled state.

Theorem 4. The strategy $\sigma_{s,q,I}$ defined above satisfies the conditions of Proposition 2 item 2b).

Proof: The lemma can be proved by using Ergodic theorem for Markov chains [19] and by Lemma 5.
A. The algorithm

To conclude the proof of Theorem 4, we need to give a procedure for computing the optimal probability of satisfying \( \psi \). It works in the following steps.

1) Construct the automata \( R_{\xi,I} \) for all \( \xi \in \{ \psi, \varphi_1, \ldots, \varphi_n \} \) and \( I \subseteq \{1, \ldots, n\} \).
2) Initialize \( \Upsilon := \emptyset \).
3) Repeat the following for every \( I \). Find possible candidates for \( (M,N) \) satisfying the conditions 2-4 of Proposition 4. It can be done as follows:
   - Let \( \text{trunc}(M,N) \) denote the tuple \( (M',N') \) that contains maximal subsets of \( M \) and \( N \) satisfying that for any \( s \in M \) there is \( a \in N \) such that \( \Delta(s,a) \) is defined and for any \( s' \) contained in the support of \( \Delta(s,a) \) we have \( s' \in M \). (Easily obtained by iteratively pruning actions and states violating the conditions.)
   - We start with \( M = S_0 \) and \( N \) containing all actions satisfying item 2 of Proposition 4. Then we apply the following steps until a fixpoint is reached:
     (a) \( (M,N) := \text{trunc}(M,N) \);
     (b) Remove from \( M \) all states that do not satisfy item 3 or item 4 of Proposition 4. (Easily achieved by qualitative safety and reachability analysis in \( M_\phi \).)
   - This yields a set \( (M,N) \), and we add to \( \Upsilon \) all triples \( (s,q,I) \) such that \( (q,\star) \in C \) for some \( C \) such that \( (s,C) \in M \).
4) Compute an optimal strategy \( \sigma' \) for “reaching” \( \Upsilon \) (defined in Proposition 4).

We explained all the steps of the algorithm except for step 4. This can be done as follows. Let \( M_\phi \) denote the “naive” product of \( M \) with all Rabin automata \( R_{\xi,I} \) for all \( J \subseteq \mathit{sf}(\psi) \). Formally, fixing \( I_0,\ldots,I_m \) an enumeration of subsets of \( \mathit{sf}(\psi) \), the state space \( S_\phi \) of \( M_\phi \) contains tuples \( (s,q_0^{I_0},\ldots,q_m^{I_m}) \) where \( q_i^{I_i} \) is a state of \( R_{\xi,I} \). The set of actions is \( A_\phi = A \), and the transition function \( \Delta_\phi \) is given by

\[
\Delta_\phi((s,q_0^{I_0},\ldots,q_m^{I_m}),a) = (s,q_0^{I_0'},\ldots,q_m^{I_m'})
\]

when \( q_i \rightarrow q_i' \). Furthermore, let \( \Upsilon_0 \subseteq S_\phi \) be the set of all \( (s,q_0^{I_0},\ldots,q_m^{I_m}) \) s.t. there is \( i \) and \( s \) with \( (s,q_i^{I_i},I_i) \in \Upsilon \).

Lemma 7. For any \( \sigma \) in \( M \) there is \( \sigma_\phi \) in \( M_\phi \), and also for any \( \sigma_\phi \) there is \( \sigma \) such that \( \mathbb{P}_\sigma(\Upsilon_0 \Upsilon) = \mathbb{P}_{\sigma_\phi}(F \Upsilon_\phi) \).

Proof: For any finite or infinite run \( \omega = s_0a_0s_1a_1\ldots \) in \( M \) initiated in \( s_0 \), there is a unique run \( \omega_\phi = (s_0,q_0^{I_0})a_0(s_1,q_1^{I_1})a_1\ldots \) with \( s_i = s_i' \) and \( a_i = a_i' \) for all \( i \). For a fixed \( \sigma_\phi \) we define \( \sigma \) by \( \sigma(h) = \sigma_\phi(h_\phi) \) for any \( h \). Similarly, for a fixed \( \sigma \), we define \( \sigma_\phi \) by \( \sigma_\phi(h_\phi) = \sigma(h) \) for any \( h \). The equality easily follows from the definitions.

Lemma 7 allows us to compute the strategy \( \sigma' \) using ordinary reachability algorithms.

Let us now analyse the complexity of the algorithm. Each of the Rabin automata in step 1) above can be computed in time \( 2^\mathit{sf}(\psi) \), and since there is exponentially many such automata (in \( |\varphi| \)), step 1) takes time \( 2^\mathit{sf}(\psi) \). In step 3), for a fixed \( I, M \) and \( N \) the result of \( \text{trunc}(M,N) \) can be computed in polynomial time in the size of \( M \) and \( N \); the same holds for satisfaction of the conditions in 2b. The size of \( M \subseteq S_\phi \) and \( N \subseteq A_\phi \) is \( \text{poly}(S) \cdot 2^\mathit{sf}(\psi) \), and for a fixed \( I \) the fixpoint is reached in at most \( |\Upsilon_\phi| \cdot |A_\phi| \) iterations. Moreover, there is at most \( 2^{|\varphi|} \) different \( I \). Hence, step 3) can be performed in time \( \text{poly}(S) \cdot 2^{|\varphi|} \). Finally, in step 4) we are computing reachability probability in a MDP \( M_\phi \) which is of size \( \text{poly}(S) \cdot 2^{|\varphi|} \). and so also this step can be done in time \( \text{poly}(S) \cdot 2^{|\varphi|} \). This completes proof of Theorem 5.

B. Proof of Proposition 4

We will now prove Proposition 4. As before, fix \( s^* \in S, q^* \in Q, I^* \subseteq \{1,\ldots,n\} \) and a finite path \( h^* \).

The following definition and lemma will help us identify (possible) recurring behaviour of \( \sigma \). We need to identify long enough parts of runs where all the frequency formulae \( \varphi_i^* \) are satisfied with probability very close to 1. Based on the behaviour of \( \sigma \) within these parts, we later define the “accumulating” strategy.

We say that a finite path \( h \) extending \( h^* \) is good if

\[
\mathbb{E}_\sigma\left(\sum_{k=0}^{\ell-1} Y | h_{|k+k} > \ell \cdot (1 - \varepsilon_1)\right.
\]

where \( Y_j(\omega) = 1_{\bar{\psi}_i(\omega) \cap (\omega_j | h_{|j})} \) is the indicator function that the suffix of \( \omega \) starting at \( j \)-th position satisfies all \( \varphi_i^* \).

Lemma 8. Almost every \( \omega \) satisfying \( \bigwedge_{i \in I} G^1 \varphi_i^* \) has infinitely many good prefixes.

By heavily relying on existence of good prefixes, we define labellings of histories of \( M \) that will help us establish a connection to \( M_\phi \). Namely, the labellings (1) identify what is the current state in \( M_\phi \) and (2) resolve the additional choices w.r.t. the second component of \( \Upsilon_\phi \).

We introduce functions \( \theta_a \) and \( \theta_i \) that label histories starting with \( h^* \) with elements of \( 2^I \cup \{\bot\} \) and define the current state and the current action to pick in \( M_\phi \) in the given history, respectively. Inductively, together with defining the labellings, we also assign one of two distinct tags to these histories, pseudo-accumulating or pseudo-reaching. We will then speak about pseudo-reaching and pseudo-accumulating phases which are maximal consecutive ranges within histories labelled so far such that all prefixes in this range are tagged as pseudo-reaching or pseudo-accumulating, respectively. A pseudo-accumulating phase is fulfilled if it contains a prefix \( h \) in its range such that \( \theta_a(h) \) is fulfilled.

Initially, we tag \( h^* \) as pseudo-reaching and set \( \theta_a(h^*) = \emptyset \) and \( \theta_i(h^*) = \{(q^*,\star)\} \).

Suppose that \( \theta_a(h) \) and \( \theta_i(h) \) has already been defined and the tag of \( h \) determined.

First, we tag any hat as pseudo-accumulating if (i) \( h \) is tagged as pseudo-accumulating and the length of the current pseudo-accumulating phase is less than \( \ell \) so far; or (ii) \( h \) is
in a fulfilled pseudo-reaching phase and $h$ is good. Otherwise we tag hat as pseudo-reaching.

Second, we define $\theta_a(h)$ and $\theta_h(h)$. Let $\theta_a(h) = \bot$ if $\theta_a(h) = \bot$ and let $\theta_h(h)$ be the set with $\theta_a(h) \supseteq \theta_h(h)$, otherwise, where $s = \text{last}(h)$. Similarly, we set $\theta_q(h) = \bot$ if $\theta_q(h) = \bot$ or if $(q, (E, F), s) \in \theta_a(h)$ with $q \in E$.

- If $h$ is in a pseudo-accumulating phase, then $(\text{init}(R_{\phi, i}), \star) \in \theta_a(h)$ for all $i \in I$.
- If $\theta_a(h)$ is not fulfilled:
  - For any $(q, \star) \in \theta_a(h)$ we put $(q, \star) \in \theta_a(h)$.
  - For any $(q, m) \in \theta_a(h)$ such that for some $(E, F)$, $P^\sigma_\theta(\{\omega \mid \text{occ}_q(\omega) \cap E = \emptyset, \text{inf}_q(\omega) \cap F \neq \emptyset\}) > 1 - \varepsilon_1$
    we put $(q, (E, F)\star) \in \theta_a(h)$, and otherwise we put $(q, m) \in \theta_a(h)$. In the case there are several $(E, F)$ satisfying the condition above, we pick the least one w.r.t. an arbitrary but arbitrarily fixed total order.
  - For any $(q, (E, F)\star) \in \theta_a(h)$ we put $(q, (E, F)\star) \in \theta_a(h)$.
- If $\theta_h(h)$ is fulfilled:
  - For every $(q, \star) \in \theta_h(h)$ we put $(q, \star) \in \theta_h(h)$.
  - For every $(q, (E, F)\star) \in \theta_h(h)$ we put $(q, (E, F)\star) \in \theta_h(h)$.

Finally, any (finite or infinite) path $\omega = s_0a_0s_1a_1 \ldots$ in $\mathcal{M}$ initiated in $h^*$ corresponds to a path $\omega_\ominus = (s_0, \theta_a(s_0))(a_0, \theta_a(s_0))(s_1, \theta_a(s_0a_0s_1))(a_1, \theta_a(s_0a_0s_1))$ in $\mathcal{M}_\ominus$. Similarly, the strategy $\sigma$ gives rise to a strategy $\sigma_\ominus$ defined, for any $h$, by $\sigma_\ominus(h)(a, \theta_a(s_0)) = \sigma(h)(a)$. The connection between the labellings and the MDP $\mathcal{M}_\ominus$ is completed by the following lemma that can be proven immediately from the definitions.

**Lemma 9.** For any set $T$, $P^h_\theta(T) = P^\sigma_\theta(\{\omega \mid \omega \in T\})$.

Note that the strategy $\sigma_\ominus$ in Lemma 9 is possibly still very complex in its structure and in particular can reach states of the form $(s, \bot)$. We however show that within a certain finite horizon that this happens with a small probability.

Let $\text{depth}(h)$ be the number of pseudo-accumulating phases along the path $h$. Let $T$ be the set of runs that have $\text{depth}(\omega) \leq \ell$, and for which no prefix $h$ with $\text{depth}(h) \leq \ell$ has $\theta_\alpha(h) = \bot$. We will show below that the probability of runs in $T$ is very large.

**Lemma 10.** $P_\sigma(T \cup h^*) \geq 1 - 3 \cdot \varepsilon^2$.

**Proof:** First, we start with the set of runs

$$U = \{\psi^r \wedge \mathcal{G}^1_{\phi_i^r} \} \cap h^*$$

with $P_\sigma(\Omega \setminus U \mid h^*) < \varepsilon_1$ as given by the assumption of Proposition 3 (here $\Omega$ denotes the set of all runs).

Furthermore, let $\mathcal{V} \subseteq \mathcal{U}$ be the set of runs where all the “accumulated” Rabin automata accept, i.e. runs $\omega$ such that for any $i \in I^*$ and for any prefix $h_0$ in an at most $\ell$-th pseudo-accumulating phase, we have that $R_{\phi_i^r, 1}$ accepts $\omega$′ such that $\omega = h_0 \cdot \omega'$. For a fixed accumulating phase which starts at some good history $h$, we have (denoting $\sum_{k=0}^{\ell} h_{\ell+k}$ by $Y_{1ph}$)

$$P(1 - \varepsilon_1) < P_\sigma(Y_{1ph} \mid h \wedge (\ell - 1) \cdot P_\sigma(Y_{1ph} < \ell \mid h)),$$

yielding $P_\sigma(Y_{1ph} \mid h) > 1 - \varepsilon_1$. Thus for $Y_{1ph}$ denoting the number of Rabin automata accepting in all $\ell$ accumulating phases, we easily obtain $P_\sigma(Y_{1ph} = \ell \mid h) > 1 - \ell^2 \cdot \varepsilon_1$ and thus $P_\sigma(U \setminus V) < \ell^2 \cdot \varepsilon_1$.

For any $i \in I$, we say that starting after $h$, the history $h'$ decides for an accepting condition $(E, F)$ of $R_{\phi_i^r, 1'}$, if

- $h$ is in pseudo-accumulating phase,
- $h'$ is the shortest history such that for some $(E', F')$
  $$P^h_{\phi_i^r, 1'}(\{\omega \mid \text{occ}_q(\omega) \cap E = \emptyset, \text{inf}_q(\omega) \cap F \neq \emptyset\}) > 1 - \varepsilon_1$$
  where $q = R_{\phi_i^r, 1'}(h')$,
- $(E, F)$ is the minimal one among such acceptance conditions $(E', F')$ (w.r.t. the above fixed order).

We define a set $W \subseteq V$ of runs where this “decision” turns out to be correct for all automata started in the first $\ell$ accumulating phases. Technically, $\omega \in W$ if for every $i \in I^*$ and every splitting $\omega = h \cdot h' \cdot \omega_2$ such that $h$ is in at most $\ell$-th pseudo-accumulating phace we have the following. If starting after $h$, $h'$ decides for some $(E, F)$, we have $\text{occ}_q(\omega_2) \cap E = \emptyset$ and $\text{inf}_q(\omega_2) \cap F \neq \emptyset$.

When starting after a single $h$, $h'$ decides for some $(E, F)$, the probability of not sticking to this decision is by definition at most $\varepsilon_1$ (conditioned by $h \cdot h'$). Similarly as before, there are at most $\ell^2 \times |I|$ decisions to take, yielding the overall probability at most $P_\sigma(V \setminus W) < |I| \cdot \varepsilon_1 \cdot \ell^2$ of runs that do not stick to decisions up to $\ell$.

For almost every run $\omega \in W$ we have that $\omega \in T$ if $\omega \in W$. Indeed, inductively, for any its prefix $h$ such that $h$ is in at most $\ell$-th pseudo-accumulating or pseudo-reaching phase and $\theta_a(h) \neq \bot$, we have $\theta_a(h) \neq \bot$ because no forbidden state $q$ of a previously decided automaton is visited along any $\omega$ of $W$. Furthermore, every label $(q, (E, F)\star)$ is eventually replaced by $(q, (E, F)\star)$ because $\omega \in V$; and every $(q, \star)$ is eventually replaced by some $(q, (E, F)\star)$ (for almost every $\omega \in V$) due to Lemma 11 given below. Thus, the set of labels along $\omega$ becomes at least $\ell$ times fulfilled.

Summing up $P^h_\sigma(\Omega \setminus U \cup V)$, $P^h_\sigma(U \setminus V)$ and $P^h_\sigma(V \setminus W)$, we obtain the statement of the lemma.

One important step in the previous proof was that on almost every accepting path there is a prefix where the Rabin automaton “decides” for one accepting condition with high probability. The proof is again based on Levy’s Zero-One Law.

**Lemma 11.** Let $R$ be a Rabin automaton, $h$ be a path, $V = \{h \cdot \omega \mid R \text{ accepts } \omega\}$, and $P_\sigma(V) > 0$. For almost all $h \cdot \omega' \in$
there is a prefix $h'$ of $\omega'$ and an acceptance pair $(E, F)$ of $R$ such that for $q = R(h')$ we have

$$P_{\sigma}^{h'}(\{\omega \mid \text{occ}_{q}(\omega) \cap E = \emptyset, \text{inf}_{q}(\omega) \cap F \neq \emptyset\}) > 1 - \varepsilon_1$$

Before constructing the accumulating and reaching strategies, we state the following lemmas that we will need. It says that if some events are unlikely in an MDP, they can be completely avoided.

We can achieve a certain event with a large enough probability in an MDP, then we can achieve it with probability 1. The proof follows from the fact that there are optimal deterministic strategies with memory of size 2.

**Lemma 12.** Let $M$ be an MDP with state space $S$, $p$ the minimal probability occurring in it, and $G_1$ and $G_2$ two sets of states. The following statements hold true:

1. If $\sup_{\sigma} P_{\sigma}(F (G_1 \land F G_2)) > 1 - p^{2|S|}$, then $P_{\sigma}(F (G_1 \land F G_2)) = 1$ for some $\sigma$.
2. If $\sup_{\sigma} P_{\sigma}(\{\omega = s_0 a_0 s_1 a_1 \ldots \mid \forall i \leq |S| : s_i \in G_1\}) > 1 - p^{2|S|}$, then $P_{\sigma}(G G_1)) = 1$ for some $\sigma$.

From now on, we will consider the strategy $\sigma_{\omega}$ in $M_{\omega}$ instead of $\sigma$. We transfer the labelling with a pseudo-reaching and pseudo-accumulating phase to runs of $M_{\omega}$ in the straightforward way.

Let $W_{\ell}$ be the set of histories that are in $i$-th pseudo-accumulating phase and whose predecessors are in $i$-th pseudo-accumulating phase. In order to define accumulating and reaching strategies, we need to select subsets of these histories that are “connected” with high probability. We thus select nonempty sets $W_{\ell} \subseteq W_{\ell}$ which in addition satisfy

$$P_{\sigma_{\omega}}(W_{\ell}) \geq 1 - 4 \varepsilon_2^{1/2^\ell - i}$$

for all $1 \leq i \leq \ell$, and

$$P_{\sigma_{\omega}}(W_{i+1} \mid h) \geq 1 - 4 \varepsilon_2^{1/2^\ell - i}$$

for all $h \in W_i$ and $1 \leq i \leq \ell$.

This is still not enough, we would like to get sets of histories that are “connected” with high probability from anywhere within the accumulating phase. For any $i$ and any $h \in W_i$ we apply the following lemma and obtain a prefix-free set of paths $Z_h$ such that $P_{\sigma_{\omega}}(Z_h) \geq 1 - 4 \varepsilon_2^{1/2^\ell}$ and for any prefix $h'$ of any path in $Z_h$ we have $P_{\sigma_{\omega}}(W_{i+1} \mid h') \geq 1 - 2 \varepsilon_2^{1/2^\ell}$.

**Lemma 14.** Let $W$ be a set of runs such that

$$P_{\sigma_{\omega}}(W) > 1 - \varepsilon$$

then there is a prefix-free set $V$ of finite paths of length $\ell$ such that $P_{\sigma_{\omega}}(V) \geq 1 - 2 \cdot 2^{1/2^\ell}$ and for any prefix $h'$ of a path in $V$ we have

$$P_{\sigma_{\omega}}(W \mid h') > 1 - \sqrt{\varepsilon}.$$

Finally, we are ready to obtain the accumulating and reaching strategies.

**Lemma 15.** For $i \leq \ell$, $h \in W_i$ and $h' \in Z_h$, there is a strategy $\omega_i$ that from last($h'$) almost surely reaches \{last($h''$) $|$ $h'' \in W_{i+1}\} after passing through a fulfilled state.

**Proof:** Such strategy always exists because of Lemma 12 and because $P_{\sigma_{\omega}}(W_{i+1} \mid w) \geq 1 - 2 \varepsilon_2^{1/2^\ell}$ by properties of elements of $W_i$ and $Z_w$.

The following lemma can be easily obtained from Lemma 12.

**Lemma 16.** For any $w \in W_i$, there is a memoryless deterministic strategy $\omega_w$ (in $M_{\omega}$) which, when started in last($w$), only ever reaches states and uses actions that occur on some history of $Z_w$.

In addition, denote by $\sigma_{s,C}$ a strategy $\omega_w$ for $w$ belonging to $W_i$ for $i = \min\{j \mid \exists \omega' \in W_j, \text{last}($w$') = (s,C)\}$. By $(M_{s,C}, N_{s,C})$ we denote the tuple of sets of states and actions that $\sigma_{s,C}$ visits when started in $(s,C)$.

Let $\zeta_{s,C}$ be a strategy $\omega_w$ where $w \in \bigcup_{i \leq \ell} W_i Z_w$ for $i = \min\{j \mid \exists \omega' \in W_j, Z_w \land \text{last}($w$') = (s,C)\}$. Let $\text{rank}((s,C)) = i$.

We inductively build $(M, N)$ as follows. Initially, $M = \{\text{last}($w$) \mid w \in W_1\}$ and $N = \emptyset$. We then keep adding to $M$ and $N$, until a fixpoint is reached, (i) the states and actions of $(M_{s,C}, N_{s,C})$ for any $(s,C) \in M$, and (ii) last states of histories $W_{i+1}$ for $i$ such that there is $(s,C) \in M$ with $\text{rank}((s,C)) = i$. We claim that this procedure is well-defined in the sense that the sets $W_{i+1}$ in step (ii) above were always defined, i.e. that $i < \ell$ in every case. For this, we need to show that whenever $H_{s,C}$ is taken in the definition, then $\text{rank}((s,C)) \leq \ell - 1 = |S_{\omega}|$. Letting $\text{rank}(M) = \max\{\text{rank}((s,C)) \mid (s,C) \in M\}$, we can argue that initially $\text{rank}(M) = 1$ and with every iteration of steps (i) and (ii) the rank increases at most by 1. Since only $|S_{\omega}|$ elements can be added to $M$ before a fixpoint is reached, we get that the bound on $\text{rank}(M)$ is $|S_{\omega}|$.

Now we claim that the Proposition 3 is satisfied. For item 2, note that we were only adding states to $N$ if they were last states of a history in a pseudo-accumulating phase, and by definition of $\theta_a$ we have $\{\text{init}(R_{\theta_a}), \emptyset\}$ in the second component of such states. For item 3, a witnessing strategy for an element $(s,C)$ is the strategy $\sigma_{s',C'}$ such that $(s,C) \in M_{s',C'}$. For 4, we gave the witnessing strategies above.
Example 6. To support reader’s intuition, Figure 2 highlights basic structure of the above proof. The pseudo-brewing phases are as marked, with pseudo-accumulating phases in between. For each \( w \in W_k \), there are histories determined by \( Z_w \) (marked in dark gray) from which \( \sigma \) is guaranteed to reach \( W_{k+1} \) with high probability, along paths marked in lighter gray. The strategies \( \zeta(s,C) \) and \( \sigma(s,C) \) strive to stay in these light and dark grayed areas, respectively, to ensure that the required properties can be guaranteed. In addition, to ensure that \((M,N)\) can be correctly defined, we need to define \( \zeta_w \) and \( \sigma_w \) such that \( w \) is (intuitively) as short as possible. For example, suppose that \( \text{last}(w_4) = \text{last}(w_0) = (s,C) \); then \( \sigma_{s,C} = \bar{\sigma}_{w_0} \).

V. CONCLUSIONS

In this paper we have given algorithms for controller synthesis of the logic LTL extended with an operator expressing that frequencies of some events exceeds a given bound. In the case of Markov chains we gave an algorithm working with the complete logic, and in the case of MDPs we require the formula to be from a certain fragment.

The obvious next step is extending the MDP results to the whole FLTL. This will require new insights. Our product formula to be from a certain fragment. This is no longer true when the frequency bound is lower than 1. In such cases different histories may require different probability of satisfying \( \varphi \). However, both authors strongly believe that even for this problem is decidable.

REFERENCES

[1] V. Shmatikov, “Probabilistic model checking of an anonymity system,” Journal of Computer Security, vol. 12, no. 3/4, pp. 355–377, 2004.
[2] M. Kwiatkowska, G. Norman, and D. Parker, “Probabilistic verification of herman’s self-stabilisation algorithm,” Formal Aspects of Computing, vol. 24, no. 4, pp. 661–670, 2012.
[3] B. Bollig, N. Decker, and M. Leucker, “Frequency linear-time temporal logic,” in TASE’12. Beijing, China: IEEE Computer Society Press, Jul. 2012, pp. 85–92.
[4] P. Bouyer, N. Markey, and R. M. Mateeplackel, “Averaging in LTL,” in CONCUR 2014, ser. LNCS, P. Baldan and D. Gorla, Eds., vol. 8704. Springer, 2014, pp. 266–280.
[5] T. Tomita, S. Hiura, S. Hagiwara, and N. Yonezaki, “A temporal logic with mean-payoff constraints,” in Formal Methods and Software Engineering, ser. LNCS, T. Aoki and K. Taguchi, Eds. Springer Berlin Heidelberg, 2012, vol. 7635, pp. 249–265.
[6] C. Baier and J.-P. Katoen, Principles of model checking. MIT Press, 2008.
[7] R. Bloem, K. Chatterjee, T. A. Henzinger, and B. Jobstmann, “Better quality in synthesis through quantitative objectives,” in Computer Aided Verification. Springer, 2009, pp. 140–156.
[8] U. Boker, K. Chatterjee, T. A. Henzinger, and O. Kupferman, “Temporal specifications with accumulative values,” in LICS 2011. IEEE, 2011, pp. 43–52.
[9] C. Baier, J. Klein, S. Klüppelholz, and S. Wunderlich, “Weight monitoring with linear temporal logic: Complexity and decidability,” in CSL-LICS. ACM, 2014, p. 11.
[10] T. Tomita, S. Hagiwara, and N. Yonezaki, “A probabilistic temporal logic with frequency operators and its model checking,” in INFINITY 2011, ser. EPTCS, F. Yu and C. Wang, Eds., vol. 73, 2011, pp. 79–93.
[11] K. Chatterjee and L. Doyen, “Energy and mean-payoff payoff Markov decision processes,” in MFCS 2011. Springer, 2011, pp. 206–218.
[12] K. Chatterjee, T. A. Henzinger, and M. Jurdzinski, “Mean-payoff parity games,” in LICS 2005. IEEE, 2005, pp. 178–187.
[13] K. Chatterjee and L. Doyen, “Games and markov decision processes with mean-payoff parity and energy parity objectives,” in MEMICS. Springer, 2012, pp. 37–46.
[14] T. Brázdil, V. Forejt, and A. Kučera, “Controller synthesis and verification for markov decision processes with qualitative branching time objectives,” in ICALP 2008, ser. LNCS, vol. 5126. Springer, 2008, pp. 148–159.
[15] C. Baier, B. Haverkort, H. Hermanns, and J.-P. Katoen, “Model checking continuous-time Markov chains by transient analysis,” in Computer Aided Verification, ser. LNCS. Springer Berlin Heidelberg, 2000, vol. 1855, pp. 358–372.
[16] A. Kučera and O. Stražovský, “On the controller synthesis for finite-state Markov decision processes,” in FSTTCS 2005. Springer, 2005, pp. 541–552.
[17] L. De Alfaro, “How to specify and verify the long-run average behaviour of probabilistic systems,” in LICS 1998. IEEE, 1998, pp. 454–465.
[18] J. Kemeny, J. Snell, and A. Knapp, Denumerable Markov Chains, 2nd ed. Springer-Verlag, 1976.
[19] R. Lyons, Markov chains. Cambridge University Press, 1998.
[20] R. Lyons, “Strong laws of large numbers for weakly correlated random variables.” Michigan Math. J., vol. 35, no. 3, pp. 353–359, 1988.
[21] K.-A. Chung, A Course in Probability Theory, 3rd ed. Academic Press, 2001.
A. Details for proof for Markov chains

Lemma 2. Let $Y_1, Y_2, \ldots$ be a sequence of random variables which only take values 0 or 1 and have expectation $\mu$. Assume there are $0 < r, c < 1$ such that for all $i, j \in \mathbb{N}$ we have $E((Y_i - \mu)(Y_j - \mu)) \leq r^{i|j-i|}$. Then $\lim_{n \to \infty} \sum_{i=0}^{n} Y_i/n = \mu$ almost surely.

Proof: We can use [20 Corollary 4] applied to random variables $Z_i = Y_i - \mu$ (we cannot use the result directly for $Y_i$ since [20] requires the random variables to have expectation value equal to 0). Clearly if $\lim_{n \to \infty} \sum_{i=0}^{n} Z_i/n = 0$, then $\lim_{n \to \infty} \sum_{i=0}^{n} Y_i/n = \lim_{n \to \infty} \sum_{i=0}^{n} (Z_i + \mu)/n = \mu$. Finally, the corollary of [20] indeed applies since $\sum_{k=0}^{\infty} r^k c^k \leq 1/(1 - r^2) < \infty$

The following is a more detailed computation for properties of the random variables $X_i^j$:

\[
E^s((X_i^j - \mu)(X_j^i - \mu)) = P(A) \cdot E^s((1 - \mu)(X_j^i - \mu) \mid A) + P(B) \cdot E^s((-\mu)(X_j^i - \mu) \mid B) + P(\Omega \setminus (A \cup B)) \cdot E^s((X_j^i - \mu)(X_j^i - \mu) \mid \Omega \setminus (A \cup B))
\]

and because $E^s(X_j^i - \mu) \mid A) = E^s(X_j^i - \mu) \mid B) = 0$

\[
= P(\Omega \setminus (A \cup B)) \cdot E^s((X_j^i - \mu)(X_j^i - \mu) \mid \Omega \setminus (A \cup B)) \\
\leq (1 - P(A \cup B)) \cdot 1 \\
\leq (1 - p)^{(i-j)/M}
\]

The following is the final computation for the proof of Proposition 1

\[
\lim_{n \to \infty} \sum_{i=0}^{n} 1_{\varphi,i}/n = \lim_{n \to \infty} \sum_{t \in \text{bscc}(s)} \sum_{i=0}^{N_t} X_i^t/n
\]

\[
= \sum_{t \in \text{bscc}(s)} \lim_{n \to \infty} \sum_{i=0}^{N_t} X_i^t/n
\]

\[
= \sum_{t \in \text{bscc}(s)} \lim_{n \to \infty} \sum_{i=0}^{n} X_i^t/(n/x_t)
\]

\[
= \sum_{t \in \text{bscc}(s)} x_t \lim_{n \to \infty} \sum_{i=0}^{n} X_i^t/n
\]

\[
= \sum_{t \in \text{bscc}(s)} x_t \mathbb{P}^s(\varphi)
\]

B. Details for proof for Markov decision processes

Lemma 17. Let $X$ be a set of runs, and let $J_i = \{h \mid \|h\| = i \text{ and } P(X \mid h) \geq \beta\}$ then $\lim_{i \to \infty} P(J_i) = P(X)$.

Proof: We show a stronger statement, namely that for every $\omega$ there is $i$ such that for all $i' > i$ we have $\omega \in J_{i'}$ iff $\omega \in X$.

If $\omega \notin X$, then by Lemma 4 there is $i$ such that for any $i' > i$ we have $P(X \mid h) \geq \beta$ where $h$ is the prefix of $\omega$ of length $i'$. Then $i$ is the required number. If $\omega \in X$, then again by Lemma 4 there is $i$ such that for any $i' > i$ we have $P(X \mid h) < \beta$ where $h$ is the prefix of $\omega$ of length $i'$. Then again we pick $i$.

The following lemma allows us to simplify the notation and only deal with one frequency-globally formula $\varphi := \bigwedge_{i \in I} \varphi_{i,1^i}$.

Lemma 18. Let $\xi_1, \ldots, \xi_n$ be LTL formulae, and $\omega$ a run. We have $\omega \models \bigwedge_{i=1}^{n} G^1 \xi_i$ if and only if $\omega \models G^1 \bigwedge_{i=1}^{n} \xi_i$. 


Lemma 8: Almost every \( \omega \) satisfying \( \bigwedge_{i \in I} G^1 \varphi_i' \) has infinitely many good prefixes.

Proof: By contradiction. Employing \([18]\) we can slightly simplify the problem and consider runs satisfying \( G^1 \varphi \) for \( \varphi \equiv \bigwedge_{i \in I} \varphi_i' \). Suppose that there is a set \( X' \) with \( \mathbb{P}_\sigma(X') > 0 \) such that all \( \omega \in X' \) satisfy \( G^1 \varphi \) and have only finitely many good prefixes.

Further, let \( m_\omega \) for a run \( \omega \in X' \) denote the smallest number such that for any \( m' \geq m_\omega \) and the prefix \( h \) of \( \omega \) of length \( m' \) is not good. We can pick \( m \) and \( X \subseteq X' \) satisfying that \( \mathbb{P}_\sigma(X) > 0 \), and every \( \omega \in X \) satisfies that \( m_\omega \leq m \). Note that such choice is possible, as with increasing \( m \) the set \( X \) tends monotonically to \( X' \).

Note that we have

\[
\mathbb{E}_\sigma(\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} Y_i \mid X) = 1 \tag{2}
\]

Furthermore, by Fatou’s Lemma, by linearity of expectation, and by taking a subsequence of averages of chunks of length \( \ell \), we have

\[
\frac{1}{\ell} \sum_{i=1}^{\ell} \mathbb{E}_\sigma(Y_{\ell j+i} \mid X)
\]

\[
= \frac{1}{\ell} \sum_{|h|=\ell j} \mathbb{P}_\sigma(h \mid X) \sum_{i=1}^{\ell} \mathbb{E}_\sigma(Y_{\ell j+i} \mid h \cap X)
\]

\[
= \frac{1}{\ell} \sum_{|h|=\ell j} \mathbb{P}_\sigma(h \mid X) \sum_{x \in \{0,1\}} \sum_{i=1}^{\ell} x \cdot \mathbb{P}_\sigma(Y_{\ell j+i} = 1 \mid h \cap X)
\]

\[
= \frac{1}{\ell} \sum_{|h|=\ell j} \frac{\mathbb{P}_\sigma(h \cap X)}{\mathbb{P}_\sigma(X)} \sum_{i=1}^{\ell} \frac{\mathbb{P}_\sigma(Y_{\ell j+i} = 1 \cap h \cap X)}{\mathbb{P}_\sigma(X \cap h)}
\]

\[
= \frac{1}{\ell} \frac{1}{\mathbb{P}_\sigma(X)} \left( \sum_{h \in J_{\ell j}} \sum_{i=1}^{\ell} \left( \mathbb{P}_\sigma(Y_{\ell j+i} = 1 \cap h) - \mathbb{P}_\sigma(Y_{\ell j+i} = 1 \cap (h \setminus X)) \right) 
\right.
\]

\[
+ \sum_{h \in co-J_{\ell j}} \sum_{i=1}^{\ell} \mathbb{P}_\sigma(Y_{\ell j+i} = 1 \cap h \cap X)
\]

\[
\leq \frac{1}{\ell} \frac{1}{\mathbb{P}_\sigma(X)} \left( \sum_{h \in J_{\ell j}} \sum_{i=1}^{\ell} \mathbb{P}_\sigma(Y_{\ell j+i} = 1 \cap h) + \sum_{h \in co-J_{\ell j}} \sum_{i=1}^{\ell} \mathbb{P}_\sigma(h \cap X) \right)
\]

\[
= \frac{1}{\ell} \frac{1}{\mathbb{P}_\sigma(X)} \left( \sum_{h \in J_{\ell j}} \sum_{i=1}^{\ell} \frac{\mathbb{P}_\sigma(Y_{\ell j+i} = 1 \cap h)}{\mathbb{P}_\sigma(h)} + \sum_{h \in co-J_{\ell j}} \ell \cdot \mathbb{P}_\sigma(h \cap X) \right)
\]

\[
= \frac{1}{\ell} \frac{1}{\mathbb{P}_\sigma(X)} \left( \sum_{h \in J_{\ell j}} \mathbb{P}_\sigma(h) \sum_{i=1}^{\ell} \mathbb{P}_\sigma(Y_{\ell j+i} = 1 \mid h) + \ell \cdot \mathbb{P}_\sigma(co-J_{\ell j} \cap X) \right)
\]

\[
\leq \frac{1}{\ell} \frac{1}{\mathbb{P}_\sigma(X)} \left( \mathbb{P}_\sigma(J_{\ell j}) \cdot (1 - \varepsilon) + \ell \mathbb{P}_\sigma(co-J_{\ell j} \cap X) \right)
\]
and hence

\[ (** \leq \liminf_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \left( \frac{\mathbb{P}_\sigma(J_{\ell j}) \cdot (1 - \varepsilon)}{\mathbb{P}_\sigma(X)} + \frac{\mathbb{P}_\sigma(\text{co}\text{-}J_{\ell j} \cap X)}{\mathbb{P}_\sigma(X)} \right) \]

and since \( \lim_{\ell j \to \infty} \mathbb{P}_\sigma(J_{\ell j}) = \mathbb{P}_\sigma(X) \), we get

\[ = (1 - \varepsilon) \]

which is a contradiction with \( \text{2} \). ■

**Lemma 11** Let \( R \) be a Rabin automaton, \( h \) be a path, \( V = \{ h \cdot \omega \mid R \text{ accepts } \omega \} \), and \( \mathbb{P}_\sigma(V) > 0 \). For almost all \( h \cdot \omega' \in V \) there is a prefix \( h' \) of \( \omega' \) and an acceptance pair \( (E, F) \) of \( R \) such that for \( q = R(h') \) we have

\[ \mathbb{P}_\sigma^{h \cdot h'}(\{ \omega \mid \text{occ}_q(\omega) \cap E = \emptyset, \text{inf}_q(\omega) \cap F \neq \emptyset \}) > 1 - \varepsilon_1 \]

**Proof:** Let the acceptance conditions of \( R \) be \( (E_i, F_i)_{1 \leq i \leq n} \) and its initial state be \( q_0 \). For each \( i \), let \( R_i \) be the set

\[ \{ h \cdot \omega' \in V \mid \text{inf}_{q_0}(\omega') \cap E_i = \emptyset, \text{inf}_q(\omega) \cap F_i \neq \emptyset \} \]

and \( 1_{R_i} \) be its indicator function. As for each \( h \cdot \omega' \) in some \( R_i \), \( 1_{R_i}(h \cdot \omega') = 1 \), we also have from Lemma 9 (Levy’s zero one law) that \( \lim_{k \to \infty} \mathbb{P}_\sigma(R_i \mid h_k) = 1 \) where \( h_k \) are the prefixes of \( h \cdot \omega' \) of length \( k \). Hence, there is \( k \) such that all prefixes \( h_k' \) for \( k' \geq k \) satisfy:

\[ \mathbb{P}_\sigma(R_i \mid h_k) > 1 - \frac{\varepsilon_1}{2}. \quad (3) \]

Let us fix an arbitrary partition of \( V \) into disjoint sets \( R'_1, \ldots, R'_n \) such that for any \( 1 \leq i \leq n, R'_i \subseteq R_i \). For each \( i \) and run \( h \cdot \omega' = s_0a_0 \cdots \) let

\[ \text{lastSim}_i(\omega') = \sup(\{0 \cup \{ n \mid s_n \in E_i \} \}). \]

Let \( h \cdot \omega' \in R'_i \). As we have \( 1_{\{\text{lastSim}_i > \text{lastSim}_i(h \cdot \omega')\}}(h \cdot \omega') = 0 \), we also have from Lemma 9 that \( \lim_{k \to \infty} \mathbb{P}_\sigma(\{ \text{lastSim}_i > \text{lastSim}_i(h \cdot \omega') \} \mid h_k) = 0 \). Hence, there is \( k \in \mathbb{N} \) such that \( k > \text{lastSim}_i(h \cdot \omega') \) and all prefixes \( h_{k'} \) of length \( k' \geq k \) satisfy

\[ \mathbb{P}_\sigma(\{ \text{lastSim}_i > k \} \mid h_k) < \frac{\varepsilon_1}{2}. \quad (4) \]

In total, we obtain from 3 and 4 the desired statement. ■

**Lemma 12** Let \( M \) be an MDP with state space \( S \), \( p \) the minimal probability occurring in it, and \( G_1 \) and \( G_2 \) two sets of states. The following statements hold true:

1) If \( \sup_\sigma \mathbb{P}_\sigma(F(G_1 \land F(G_2))) > 1 - p^{2|S|} \), then \( \mathbb{P}_\sigma(F(G_1 \land F(G_2))) = 1 \) for some \( \sigma \).

2) If \( \sup_\sigma \mathbb{P}_\sigma(\{ \omega = s_0a_0s_1a_1 \cdots \mid \forall i \leq |S| : s_i \in G_1 \}) > 1 - p^{|S|} \), then \( \mathbb{P}_\sigma(G(G_1)) = 1 \) for some \( \sigma \).

**Proof:** Let us analyse the second case which is slightly more technical. The set \( \{ \omega = s_0a_0s_1a_1 \cdots \mid \forall i \leq |S| : a_i \in G_1 \} \) can be captured using an LTL property and so the supremum is realised by some deterministic strategy \( \sigma' \). Suppose it is lower than 1. Then, since \( \sigma' \) is deterministic, there must be a positive history \( s_0a_0s_1a_1 \cdots s_i \) for \( i \leq |S| \) such that \( s_i \notin G_1 \) and \( \mathbb{P}_{\sigma'}(s_0a_0s_1a_1 \cdots s_i) \geq p^i \), which is a contradiction.

The first case can be proved similarly, we only need to consider that deterministic strategies with memory of size 2 are sufficient to achieve the supremum. ■

**Lemma 13** Let \( U \) be a set of runs such that \( \mathbb{P}(U) \geq 1 - \varepsilon \), and let \( H \) be the prefix-free set of finite paths such that \( \mathbb{P}(H) \geq 1 - \varepsilon \). There is a set \( V \subseteq H \) with \( \mathbb{P}(V) \geq 1 - 2\sqrt{\varepsilon} \) and \( \mathbb{P}(U \mid w) > 1 - \sqrt{\varepsilon} \) for all \( w \in V \).

**Proof:** Set \( V := \{ y \in H \mid \mathbb{P}(U \mid y) > 1 - \sqrt{\varepsilon} \} \). We have

\[ 1 - \varepsilon \leq \mathbb{P}(H) \cdot \mathbb{P}(U \mid H) + \mathbb{P}(V) \cdot \mathbb{P}(U \mid V) + \mathbb{P}(H \setminus V) \cdot \mathbb{P}(U \mid H \setminus V) \]

and since \( \mathbb{P}(U \mid H) \leq 1, \mathbb{P}(H) < \varepsilon, \mathbb{P}(U \mid V) \leq 1, \mathbb{P}(H \setminus V) \leq \mathbb{P}(V), \) and \( \mathbb{P}(U \mid H \setminus V) \leq 1 - \sqrt{\varepsilon} \), we obtain

\[ 1 - 2\varepsilon \leq \mathbb{P}(V) + \mathbb{P}(V) \cdot (1 - \sqrt{\varepsilon}) \]

\[ 1 - 2\varepsilon \leq \mathbb{P}(V) + (1 - \mathbb{P}(V)) \cdot (1 - \sqrt{\varepsilon}) \]

\[ \sqrt{\varepsilon} - 2\varepsilon \leq \mathbb{P}(V) \sqrt{\varepsilon} \]

and so \( \mathbb{P}(V) \geq 1 - 2\sqrt{\varepsilon} \). ■
Lemma 14. Let $W$ be a set of runs such that

$$P_{\sigma \otimes} (W) > 1 - \varepsilon$$

then there is a prefix-free set $V$ of finite paths of length $\ell$ such that $P_{\sigma \otimes} (V) \geq 1 - 2 \cdot \ell \cdot \sqrt{\varepsilon}$ and for any prefix $h'$ of a path in $V$ we have

$$P_{\sigma \otimes} (W \mid h') > 1 - \sqrt{\varepsilon}$$

Proof: For any $k$, we can find a set $V_k$ of paths of length $k$ such that $P_{\sigma \otimes} (V_k) \geq 1 - 2 \sqrt{\varepsilon}$ and $P_{\sigma \otimes} (W \mid h') > 1 - \sqrt{\varepsilon}$ for any $h' \in V_k$; this is possible by Lemma 13 where for $H$ we take all paths of length $\ell$. The set $V$ is then obtain $V = \bigcap_{i=1}^{\ell} V_i$ (note that this is indeed a set of paths).