MORPHISMS ON INFINITE ALPHABETS, COUNTABLE STATES AUTOMATA AND REGULAR SEQUENCES

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Abstract. In this paper, we prove that a class of regular sequences can be viewed as projections of fixed points of uniform morphisms on a countable alphabet, and also can be generated by countable states automata. Moreover, we prove that the regularity of some regular sequences is invariant under some codings.

1. Introduction

Morphisms on a finite alphabet are widely studied in many fields, such as finite automata, symbolic dynamics, formal languages, number theory and also in physics in relation to quasi-crystals (see [4, 8, 15, 8, 11]). A morphism is a map \( \sigma : \Sigma^* \to \Sigma^* \) satisfying that \( \sigma(uv) = \sigma(u)\sigma(v) \) for all words \( u, v \in \Sigma^* \), where \( \Sigma^* \) is the free monoid generated by a finite alphabet \( \Sigma \) (with \( \epsilon \) as the neutral element). Naturally, the morphism can be extended to \( \Sigma^N \), which is the set of infinite sequences. The morphisms of constant length are called uniform morphisms and the sequence \( u = u(0)u(1)u(2) \cdots \in \Sigma^N \) satisfying that \( \sigma(u) = u \) is a fixed point of \( \sigma \).

In [12], Cobham showed that a sequence is a fixed point of a uniform morphism (under a coding) if and only if it is an automatic sequence. Recall that we call a sequence is automatic if it can be generated by a finite state automaton. Moreover, a sequence \( \{u(n)\}_{n \geq 0} \) is \( k \)-automatic if and only if its \( k \)-kernel is finite, where the \( k \)-kernel is defined by the set of subsequences,

\[
\{\{u(kn+j)\}_{n \geq 0} : 0 \leq j \leq k^i - 1\}.
\]

However, the range of automatic sequences is necessarily finite. To overcome this limit, Allouche and Shallit [2] introduced a more general class of regular sequences which take their values in a (possibly infinite) Noetherian ring \( R \). Formally, a sequence is \( k \)-regular if the module generated by its \( k \)-kernel is finitely generated.

Many regular sequences have been found and studied in [3, 5, 6, 20, 21, 22]. If a sequence \( \{u(n)\}_{n \geq 0} \) takes finitely many values, Allouche and Shallit showed that it is regular if and only if it is automatic in [2]. Hence, we always assume that regular sequences take infinitely many values. If a sequence \( \{u(n)\}_{n \geq 0} \) is an unbounded integer regular sequence, Allouche and Shallit [2] proved that there exists \( c_1 \geq 0 \) such that \( u(n) = O(n^{c_1}) \) for all \( n \) and Bell et al. [7] showed that there exists \( c_2 \geq 0 \) such that \( |u(n)| > c_2 \log n \) infinitely often. Recently, Charlier et al. characterized the regular sequences by counting the paths in the corresponding graph with finite vertices in [10].

Despite all this, there are no descriptions for regular sequences by automata. Note that automatic sequences can be generated by finite state automata, it is a natural question that can regular sequences be generated by automata with countable states, or morphisms on a countable alphabet?

Morphisms on infinite alphabets and countable states automata have been studied by many authors. In [22], Mauduit concerned the arithmetical and statistical properties of sequences generated by deterministic countable states automata or morphisms on a countable alphabet. Meanwhile, Ferenczi [14] studied morphism dynamical systems on infinite alphabets and Le Gonidec [17, 18, 19, 20] studied complexity function for some \( q^\infty \)-automatic sequences. More about morphisms on infinite alphabets and countable states automata, please see in [25, 16, 9].

In the present paper, we focus on morphisms on a countable alphabet and automata with countable states. We find a class of automata with countable states which can generate regular sequences. That is to say, a class of regular sequences can be generated by countable states automata.

This paper is organized as follows. In Section 2, we give some notations and definitions. In Section 3, we introduce a class of morphisms on infinite alphabets and countable states automata, which are called to be \( m \)-periodic \( k \)-uniform morphism and \( m \)-periodic \( k \)-DCAO, respectively. We prove that all the infinite sequences generated by them are \( k \)-regular. In Section 4, we consider the codings generated by the sequences satisfying a linear recurrence. Under some conditions, we show that the regularity is invariant under these codings. In the last section, we outline some generalizations.
2. Preliminary

Let \( \mathbb{N}^{\geq 2} \) be the set of integers greater than 2. For every integer \( b \in \mathbb{N}^{\geq 2} \), we define a nonempty alphabet \( \Sigma_b := \{0, 1, \ldots, b - 1\} \) and a countable alphabet \( \Sigma_\infty := \mathbb{N} = \{0, 1, 2, \ldots\} \). For \( b \in \mathbb{N}^{\geq 2} \cup \{\infty\} \), let \( \Sigma_b^v \) denote the set of all finite words on \( \Sigma_b \). If \( w \in \Sigma_b^v \), then its length is denoted by \( |w| \). If \( |w| = 0 \), then we call \( w \) is the empty word, denoted by \( \epsilon \). Let \( \Sigma^v_k \) denote the set of words of length \( k \) on \( \Sigma_b \), i.e., \( \Sigma^v_k := \bigcup_{n \geq k} \Sigma^v_n \). Let \( u = u(0)(1) \cdots u(n) \) and \( v = v(0)v(1) \cdots v(n) \in \Sigma^v_b \). The word \( uv := u(0)(1) \cdots u(n)v(0)v(1) \cdots v(n) \) denotes their concatenation. If \( |u| \geq 1 \) (resp. \( |v| \geq 1 \)), then \( u \) (resp. \( v \)) is a prefix (resp. suffix) of \( uv \). Clearly, the set \( \Sigma_b^v \) together with the concatenation forms a monoid, and the empty word \( \epsilon \) plays the role of the neutral element.

If \( b \in \mathbb{N}^{\geq 2} \), then every non-negative integer \( n \) has a unique representation of the form \( n = \sum_{i=0}^{l} n_i b^i \) with \( n_i \neq 0 \) and \( n_i \in \Sigma_b \). We call \( n \) its canonical representation in base \( b \), denoted by \( (n)_b \). If \( l \geq |(n)_b| \), denote \( (n)_b^l = 0^l(n)_b \) with \( i = l - |(n)_b| \). If \( (n)_b = n_b n_{b-1} \cdots n_0 \), then the base-\( b \) sum of digits function is denoted by \( s_b(n) := \sum_{i=0}^{l} n_i \). If \( b \in \mathbb{N}^{\geq 2} \) and \( w = w_1 w_{-1} \cdots w_0 \), then \( |w| := \sum_{i=0}^{l} w_i \cdot b^i \). We denote by \( \text{rem}_b(n) := r \) if \( n \equiv r \mod b \) (\( 0 \leq r < b - 1 \)).

In this paper, unless otherwise stated, all alphabets under consideration are countable.

2.1. Morphisms on countable alphabets. Let \( \Sigma \) and \( \Delta \) be two alphabets. A morphism (or substitution) is a map \( \sigma \) from \( \Sigma^* \) to \( \Delta^* \) satisfying that \( \sigma(uv) = \sigma(u)\sigma(v) \) for all words \( u, v \in \Sigma^* \). In the whole paper, we use the term “morphism”.

Note that \( \sigma(\epsilon) = \epsilon \). If \( \Sigma = \Delta \), then we can iterate the application of \( \sigma \). That is, \( \sigma^i(a) = \sigma((\sigma^{-1}(a))) \) for all \( i \geq 1 \) and \( \sigma^0(a) = a \).

Let \( \sigma \) be a morphism defined on \( \Sigma = \{q_0, q_1, \ldots, q_n, \ldots\} \). If \( \sigma(q_i) = q_i \sigma_0, q_i \sigma_1 \ldots, q_i \sigma_k \) with \( i = a_i, b_j \) and \( a_j, b_j \in \mathbb{Z} \), for every \( i \geq 0 \), then \( \sigma \) is called a linear morphism. If there exists some integer \( k \geq 1 \) such that \( \sigma(a) = k \) for all \( a \in \Sigma \), then \( \sigma \) is called a \( k \)-uniform morphism (or \( k \)-constant length morphism). A \( 1 \)-uniform morphism is called a coding. If there exists a finite or infinite word \( w \in \Sigma^v \) such that \( \sigma(w) = w \), then the word \( w \) is a fixed point of \( \sigma \). In fact, if \( \sigma(a) = a \) for some letter \( a \in \Sigma \) and nonempty \( w \in \Sigma^* \), then the sequence of words \( a, \sigma(a), \sigma^2(a), \cdots \) converges to the infinite word \( \sigma^\infty(a) := awa(1)\sigma^2(2)w \cdots \), where the limit is defined by the metric \( d(w, v) = 2^{-\min\{i:w(i) \neq v(i)\}} \) for \( u = u(0)u(1) \cdots \) and \( v = v(0)v(1) \cdots \). Clearly, \( \sigma^\infty(a) \) is a fixed point of \( \sigma \). Hence, for every morphism \( \sigma \) on the alphabet \( \Sigma \), we always assume that there exists a letter \( a \in \Sigma \) such that \( \sigma(a) = aw \) with a nonempty word \( w \in \Sigma^* \).

Example 1. Let \( \Sigma = \Sigma_\infty := \{0, 1, \ldots, n, \ldots\} \). Define a 2-uniform morphism \( \sigma(i) = i(i + 1) \) for all \( i \geq 0 \), then \( \sigma^\infty(i) = i(i + 1)(i + 1)(i + 2) \cdots \) is a fixed point of \( \sigma \). In particular, the fixed point \( \sigma^\infty(0) = 01121223 \cdots \) is the sequence of base-2 sum of digits function \( s_2(n) \) \( n \geq 0 \).

Example 2. (The drunken man morphism) Let \( \Sigma = \{i \} \cup \mathbb{Z} \). Define a 2-uniform morphism \( \sigma(i) = i + 1 \) and \( \sigma(i + 1) = (i - 1)(i + 1) \) for all \( i \in \mathbb{Z} \), then the infinite word \( \sigma^\infty(i) = 102(-1)113(-2)0020224 \cdots \) is the only non-empty fixed point of \( \sigma \).

Example 3. (Infinibonacci morphism) Let \( \Sigma = \mathbb{N} \). Define a 2-uniform morphism \( \sigma(i) = i(i + 1) \) for all \( i \geq 0 \), then \( \sigma^\infty(0) = 0102010301020104 \cdots \) is a fixed point of \( \sigma \).

2.2. Deterministic infinite states automata. A deterministic countable automaton (DCA) is a directed graph \( M = (Q, \Sigma, \delta, q_0, F) \), where \( Q \) is a countable set of states, \( q_0 \in Q \) is the initial state, \( \Sigma \) is the finite input alphabet, \( F \subseteq Q \) is the set of accepting states, \( \delta : Q \times \Sigma \rightarrow Q \) is the transition function. The transition function \( \delta \) can be extended to \( Q \times \Sigma^* \) by \( \delta(q, e) = q \) and \( \delta(q, wa) = \delta(\delta(q, w), a) \) for all \( q \in Q, a \in \Sigma \) and \( w \in \Sigma^* \).

Similarly, a deterministic countable states automaton with output (DCAO) is defined to be a 6-tuple \( M = (Q, \Sigma, \Delta, \delta, q_0, \tau) \), where \( Q, \Sigma, \Delta, \delta, q_0 \) are as in the definition of DCA as above, \( \Delta \) is the output alphabet and \( \tau : Q \rightarrow \Delta \) is the output function. In particular, if the input alphabet \( \Sigma = \Sigma_k \) for some \( k \in \mathbb{N}^{\geq 2} \), then the automaton \( M \) is always called to be a \( k \)-DCAO.

Let \( \{u(n)\}_{n \geq 0} = u(0)u(1)u(2) \cdots \) be a sequence on the alphabet \( \Delta \). The sequence \( \{u(n)\}_{n \geq 0} \) is called to be \( k \)-automatic, if the sequence can be generated by a \( k \)-DCAO, that is, there exists a \( k \)-DCAO \( M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau) \) such that \( u(n) = \tau(\delta(q_0, w)) \) for all \( n \geq 0, w \in \Sigma_k \) and \( |w|_k = n \).

By the choice of DCAO \( M \) satisfying that \( \delta(q_0, 0) = q_0 \), our machine \( M \) always computes the same \( u(n) \) even if the input one has leading zeroes. Hence, unless otherwise stated, all DCAOs satisfy \( \delta(q_0, 0) = q_0 \) and \( u(n) = \tau(\delta(q_0, (n)_k)) \) for all \( n \geq 0 \).
Example 4. Let $Q = \{q_0, q_1, q_2, \cdots\}$, $\Delta = \mathbb{N}, \delta(q_0, 0) = q_i, \delta(q_i, 1) = q_{i+1}$ and $\tau(q_i) = i$ for all $i \geq 0$. Then, the sequence of base-2 sum of digits function $\{s_2(n)\}_{n \geq 0}$ is 2-automatic. It can be generated by a 2-DCAO in Figure 1.

![Figure 1. DCAO generating the base-2 sum of digits function.](image)

Example 5. Let $Q = \{q_0\} \cup \mathbb{Z}, \Delta = \{i\} \cup \mathbb{Z}, \delta(q_0, 0) = q_0, \delta(q_0, 1) = 1, \delta(i, 0) = i - 1, \delta(i, 1) = i + 1, \tau(q_0) = i$ and $\tau(i) = i$ for all $i \in \mathbb{Z}$. Then, the sequence defined in Example 3 can be generated by a 2-DCAO in Figure 2.

![Figure 2. DCAO generating the sequence defined in Example 2](image)

Example 6. Let $Q = \{q_0, q_1, q_2, \cdots\}$, $\Delta = \mathbb{N}, \delta(q_0, 0) = q_{0}, \delta(q_i, 1) = q_{i+1}$ and $\tau(q_i) = i$ for all $i \geq 0$. Then, the sequence defined in Example 3 is 2-automatic. It can be generated by a 2-DCAO in Figure 3.

![Figure 3. DCAO generating the sequence defined in Example 3](image)

By the definitions of $k$-uniform morphism and $k$-DCAO, we note that each sequence $u = \{u(n)\}_{n \geq 0}$ can be generated by a $k$-uniform morphism or a $k$-DCAO for every $k \in \mathbb{N}^\geq 2$. In fact, let $\sigma : \mathbb{N} \rightarrow \mathbb{N}^*$ be a $k$-uniform morphism defined by $\sigma(i) = (ki)(ki + 1) \cdots (ki + k - 1)$ and $\rho$ be a coding by $\rho(i) = u(i)$. We have $u = \rho(\sigma^\infty(0))$. Similarly, it can be generated by a $k$-DCAO by choosing $\delta(q_0, (n)_k) = q_{n}$ and $\tau(q_n) = u(n)$ for all $n \geq 0$.

Hence, in the whole paper, we will consider the $k$-uniform morphisms and the $k$-DCAOs that the codings and the output functions satisfying some conditions. Assume that $\sigma$ is a morphism on the alphabet $\{q_0, q_1, q_2, \cdots\}$, we will consider the $k$-DCAO satisfying $\tau(q_i) = i$ in Section 3, $\tau(q_i) = L_i$ in Section 4 and some others, where $\{L_i\}_{i \geq 0}$ is a given sequence.

3. Regularity of the Index Sequences Generated by $m$-Periodic $k$-Uniform Morphisms

In this section, we first introduce a definition in the following.

Definition 1. If there exists a matrix $T = (t_{r,s})_{0 \leq r \leq m-1, 0 \leq s \leq k-1} \in \mathbb{N}^{m \times k}$ such that

$$\sigma(q_{mi+n}) = q_{mi+tn_0q_{mi+tn_1} \cdots q_{mi+tn_{k-1}}}$$

for every $0 \leq n \leq m-1$ and $i \geq 0$, then $\sigma$ is called to be a $m$-periodic $k$-uniform morphism on $\{q_0, q_1, q_2, \cdots\}$ and we always denote it by $\sigma_T$. The matrix $T$ is called to be the index matrix of $\sigma$. 

If there exists an integer $0 \leq n \leq m - 1$ such that $t_{n,0} = n$, then $\sigma_T^n(q_n)$ is a fixed point of $\sigma_T$. Unless otherwise stated, we assume $t_{0,0} = 0$, then $\sigma_T^n(q_0)$ is a fixed point of $\sigma_T$. By the definition of $m$-periodic $k$-uniform morphism, we have the following definition similarly.

**Definition 2.** If there exists a matrix $T = (t_{r,s})_{0 \leq r \leq m-1, 0 \leq s \leq k-1} \in \mathbb{N}^{m \times k}$ such that the $k$-DCAO $M = (Q, \Sigma_k, q_0, \delta, \Delta, \tau)$ satisfying that

$$\delta(q_{mi+n}, j) = q_{mi+t_{n,j}},$$

for every $0 \leq n \leq m - 1, 0 \leq j \leq k - 1$ and $i \geq 0$, then the automaton $M$ is called to be a $m$-periodic $k$-DCAO and we always denote it by $M_T$.

Clearly, the fixed point $\tau(\sigma_T^n(q_0))$ can be generated by the $k$-DCAO $M_T$. In the rest of this section, we will consider the coding $\tau : q_i \rightarrow i \ (i \geq 0)$ and its corresponding sequence $\{i(n)\}_{n \geq 0} := \tau(\sigma_T^n(q_0))$. Note that

$$\sigma_T^n(q_0) = \{q_i(n)\}_{n \geq 0}.$$ 

Hence, the sequence $\{i(n)\}_{n \geq 0}$ is called to be the *index sequence* of the morphism $\sigma_T$.

In fact, the index sequence $\{i(n)\}_{n \geq 0}$ can be generated by the morphism $\sigma'$ defined on $\mathbb{N}$ by

$$\sigma'(mi + n) = (mi + t_{n,0}) (mi + t_{n,1}) \cdots (mi + t_{n,k-1}),$$

for every $0 \leq n \leq m - 1$ and $i \geq 0$. Similarly, the corresponding transition function can be defined by $\delta(r, s) = t_{r,s} (0 \leq r \leq m-1, 0 \leq s \leq k-1)$ and $\delta(mi + j, a) = mi + \delta(j, a)$, for every $i \geq 0, j \in \Sigma_m$ and $a \in \Sigma_k$. It is easy to check that for every $n \geq 0$, $i(n) = \delta(0, (n)_k)$ and $\delta(n, w) = n + \delta(\text{rem}_m(n), w) - \text{rem}_m(n)$ for every $w \in \Sigma_k^*$.

**Lemma 1.** For every $t \geq 0$ and $0 \leq j \leq k^l - 1$, we have

$$i(k^l n + j) = i(n) + \delta(\text{rem}_m(i(n)), (j)_k) - \text{rem}_m(i(n)).$$

**Proof.** For every $t \geq 0$ and $0 \leq j \leq k^l - 1$, we denote $(j)_k := j_{t-1}j_{t-2} \cdots j_0$. Then,

$$i(k^l n + j) = \delta(0, (k^l n + j)_k) = \delta(0, (n)_k j_{t-1} j_{t-2} \cdots j_0)$$

$$= \delta(0, (n)_k), j_{t-1} j_{t-2} \cdots j_0) = \delta(i(n), j_{t-1} j_{t-2} \cdots j_0)$$

$$= i(n) + \delta(t, j_{t-1} j_{t-2} \cdots j_0) - t$$

where $t = \text{rem}_m(i(n))$. \qed

By Lemma 1 we will prove the following theorem.

**Theorem 1.** If $\sigma$ is a $m$-periodic $k$-uniform morphism, then the index sequence $\{i(n)\}_{n \geq 0}$ is $k$-regular.

**Proof.** For every $t \geq 0$ and $0 \leq j \leq k^l - 1$, let $(j)_k = j_{t-1} j_{t-2} \cdots j_0$. Define

$$V_{i,j} := \begin{pmatrix}
\delta(0, j_{t-1} j_{t-2} \cdots j_0) \\
\delta(1, j_{t-1} j_{t-2} \cdots j_0) - 1 \\
\delta(2, j_{t-1} j_{t-2} \cdots j_0) - 2 \\
\vdots \\
\delta(m - 1, j_{t-1} j_{t-2} \cdots j_0) - (m - 1)
\end{pmatrix}.$$ 

By Lemma 1 if $V_{i,j} = c_1 V_{i,j'} + c_2 V_{i,j''}$ ($l', l'' < l$), then $i(k^l n + j) = c_1 i(k^l n + j') + c_2 i(k^l n + j'') - (c_1 + c_2 - 1)i(n)$. Note from 3 that if a sequence satisfies linear recurrence relations, then it is regular. Hence, it suffices to prove that the vectors satisfy the linear recurrence relations.
By the definition of the vector $V_{i,j}$, we have

$$V_{i,j} = \begin{pmatrix} 
\delta(0, j_i 1 j_i 2 \cdots j_1, j_0) \\
\delta(1, j_i 1 j_i 2 \cdots j_1, j_0) - 1 \\
\delta(2, j_i 1 j_i 2 \cdots j_1, j_0) - 2 \\
\vdots \\
\delta(m - 1, j_i 1 j_i 2 \cdots j_1, j_0) - (m - 1) \\
\delta(0, j_i 1 j_i 2 \cdots j_1, j_0) + \delta(l_0, j_0) - l_0 \\
\delta(1, j_i 1 j_i 2 \cdots j_1) - 1 + \delta(l_1, j_0) - l_1 \\
\delta(2, j_i 1 j_i 2 \cdots j_1) - 2 + \delta(l_2, j_0) - l_2 \\
\vdots \\
\delta(m - 1, j_i 1 j_i 2 \cdots j_1) - (m - 1) + \delta(l_{m-1}, j_0) - l_{m-1}
\end{pmatrix},$$

where $l_i = \text{rem}_m(\delta(i, j_i 1 j_i 2 \cdots j_1))$ for $0 \leq i \leq m - 1$. Since $l_0, l_1, \ldots, l_{m-1} \in \Sigma_m$ and $j_0 \in \Sigma_k$, there are at most $m^m$ different vectors on the last equation. Hence, there exists an integer $L$ such that for all $l > L$ and $0 \leq j \leq k^l - 1$, there exist integers $l' \leq L$ and $0 \leq j' \leq k^{l'} - 1$ satisfying

$$V_{l,j} - V_{l-1,j} = V_{l',j'} - V_{l'-1,j'},$$

where $j_0, j'_0$ are the least digits of the $k$-ary expansion of $j$ and $j'$ respectively. Hence, each vector $V_{i,j}$ is a linear combination of the vectors $V_{l',j'}$ with $l' \leq L$ and $0 \leq j' \leq k^{l'} - 1$. It implies that each sequence \(\{i(k^l n + j)\}_{n \geq 0}\) is a linear combination of the sequences \(\{i(k^l n + j')\}_{n \geq 0}\) with $l' \leq L$ and $0 \leq j' \leq k^{l'} - 1$. Thus, the index sequence \(\{i(n)\}_{n \geq 0}\) is $k$-regular.

By Theorem 2.5 in [2], if the integer sequences \(\{u(n)\}_{n \geq 0}\) and \(\{v(n)\}_{n \geq 0}\) are both $k$-regular sequences, then \(\{u(n) + v(n)\}_{n \geq 0}\), \(\{au(n)\}_{n \geq 0}\) and \(\{u(n)v(n)\}_{n \geq 0}\) are also $k$-regular. Hence, we have the following proposition.

**Proposition 1.** Take $\tau : q_i \rightarrow f(i)$ for every $i \geq 0$, where $f(i)$ is a polynomial with integer coefficients. If $\sigma$ is a $m$-periodic $k$-uniform morphism, then the sequence $\tau(\sigma^\infty(q_0))$ is $k$-regular.

**Proof.** Since $\sigma^\infty(q_0) = \{q_{\sigma^\infty(n)}\}_{n \geq 0}$, we have

$$\tau(\sigma^\infty(q_0)) = \tau(\{q_{\sigma^\infty(n)}\}_{n \geq 0}) = \{f(i(n))\}_{n \geq 0}.$$  
Hence, by Theorem 4 and Theorem 2.5 in [2], both $\{i(n)\}_{n \geq 0}$ and $\{f(i(n))\}_{n \geq 0}$ are $k$-regular.

The following example shows that the periodic condition in Theorem 4 is necessary.

**Example 7.** The index sequence of $\sigma_1 : q_0 \rightarrow q_0 q_1$, $q_i \rightarrow q_{i-1} q_{i+1}$ $(i \geq 1)$ is not 2-regular. It can be generated by a DCAO in Figure 4.

![Figure 4. A DCAO generating the index sequence of $\sigma_1$.](image-url)

Assume that the index sequence $\{i(n)\}_{n \geq 0}$ of $\sigma_1$ is a 2-regular sequence, then $\{\text{rem}_2(i(n))\}_{n \geq 0}$ is 2-automatic, which implies that the set $\{\text{rem}_2(i(2^k n))\}_{n \geq 0} : k \geq 0\}$ is finite. Hence, there exist two integers $0 < k_1 < k_2$ such that $\text{rem}_2(i(2^{k_1} n)) = \text{rem}_2(i(2^{k_2} n))$ for all $n \geq 0$. However, taking $n = 2^{k_1 + 1} - 1$, we have $\text{rem}_2(i(2^{k_1 + 1} n)) = \text{rem}_2(i(2^{k_1 + 1} (2^{k_1 + 1} - 1))) = 1$, but $\text{rem}_2(i(2^{k_2} n)) = \text{rem}_2(i(2^{k_2} (2^{k_1 + 1} - 1))) = 0$, which is a contradiction.

In fact, for every $n \geq 0$,

$$i(n) = \begin{cases} 
2s_2(n) - |(n)_2|, & \text{if } 2s_2(n) > |(n)_2|, \\
0, & \text{otherwise}.
\end{cases}$$
That is, $i(n)$ equals 0 if the number of 0’s is more than 1’s in the $(n)_2$, otherwise it equals the difference between the number of 1’s and 0’s in the $(n)_2$.

Similarly, the index sequence of $\sigma_2 : q_0 \rightarrow q_0 q_1$, $q_1 \rightarrow q_{i+1} q_{i-1}$ ($i \geq 1$) is not 2-regular.

Let $b_j \in \mathbb{N}$ and $\sigma : q_1 \rightarrow q_1 q_1 + b_1 \cdots q_1 + b_k$, be a 1-periodic $k$-uniform morphism. By Theorem 1, its index sequence is $k$-regular. For every $i \geq 0$, we define

$$\sigma : q_i \rightarrow q_{a_1} q_{a_2} + b_1 \cdots q_{a_{k-1}} + b_{k-1}$$

where $a_j \in \mathbb{N}$. Then, we have the following proposition.

**Proposition 2.** The index sequence $\{i(n)\}_{n \geq 0}$ of $\sigma$ defined by (1) is $k$-regular.

**Proof.** By the definition of $\sigma$, we have $\sigma(q(n)) = q_{a_1} q_{a_2} + b_1 \cdots q_{a_{k-1}} + b_{k-1}$. Since $\sigma$ is $k$-uniform, we also have $\sigma(q_{a_2}) = q_{a_3} q_{a_4} + b_2 \cdots q_{a_{k-1}} + b_{k-2}$. Hence, for every $0 \leq j \leq k-1$, $i(n+j) = a_j i(n) + b_j$, for all $n \geq 0$, where $b_0 = 0$, which completes our proof.

For every $0 \leq n \leq m - 1$ and $i \geq 0$, we define a morphism to be $\sigma$ by

$$\sigma(q_{a_1} q_{a_2} + b_1 \cdots q_{a_{k-1}} + b_{k-1}) = q_{a_1} q_{a_2} + b_1 \cdots q_{a_{k-1}} + b_{k-1}$$

where $A = (a_{b_1} q_{a_2} + b_1 \cdots q_{a_{k-1}} + b_{k-1})$ and $T = (b_1, b_2, \ldots, b_{k-1}) \in \mathbb{N}^{k \times 1}$. Denote this morphism by $\sigma_{A, T}$ briefly. If all elements of $A$ are 1, then the morphism $\sigma_{A, T}$ is a $m$-periodic $k$-uniform morphism. Hence, by Theorem 1, its index sequence is $k$-regular. However, for a general matrix $A$, we do not know that whether the index sequence of this morphism $\sigma_{A, T}$ is $k$-regular or not.

Until now, all the morphisms we considered are linear. The following example gives a non-linear morphism and shows that it is not 2-regular.

**Example 8.** Let $\alpha > 1$ be a real number. The index sequence of $\sigma_2 : q_1 \rightarrow q_0 q_2$ ($i \geq 0$) is not 2-regular.

In fact, let $\{i(n)\}_{n \geq 0}$ be the index sequence of this morphism $\sigma_2$. Then $i(0) = 0$, $i(1) = 2$, $i(2n) = (i(n))^\alpha$ and $i(2n+1) = 2$ for all $n \geq 0$. Hence, $i(2^k) = 2^{\alpha^k}$ and

$$\frac{\log_2(i(2^k))}{\log_2(2^k)} = \frac{\alpha^k}{k} \rightarrow \infty \text{ as } (k \rightarrow \infty).$$

Thus, by Theorem 2.10 in [2], the sequence $\{i(n)\}_{n \geq 0}$ is not 2-regular.

**4. Regularity of the sequences generated by linear recurrence codings**

Given an integer sequence $L = \{L_n\}_{n \geq 0}$, we define a coding $\tau_L$ to be $\tau_L(q_i) = L_i$ for all $i \geq 0$. In particular, if $L = \{n\}_{n \geq 0}$, then $\tau_L$ is the coding under consideration in Section 3. If there exist integers $p \geq 0$, $C_0, C_1, C_2, \ldots, C_p$ satisfying that

$$L(n) = \sum_{i=1}^{p} C_i L(n-i) + C_0 \text{ (} n \geq p \text{),}$$

then we say the sequence $\{L_n\}_{n \geq 0}$ satisfies a linear recurrence. In this section, we will consider the coding $\tau_L$ satisfying a linear recurrence and its corresponding sequences.

Let $\{u(n)\}_{n \geq 0}$ be an integer sequence. If there exist integers $l \geq 0$ and $0 \leq j \leq k^l - 1$ such that for every $l' > l$ and $0 \leq j' \leq k^{l'} - 1$,

$$u(k^{l'} n + j') = u(k^l n + j) + c_{l',j'}$$

where $c_{l',j'}$ is a constant, depending on $l',j'$, then the sequence $\{u(n)\}_{n \geq 0}$ is called $(l',j')$-order recursive. If there exist integers $l \geq 0$ and $0 \leq j \leq k^l - 1$ satisfying that $\{u(k^l n + j) : n \geq 0\} \subseteq \mathbb{N}$, then the sequence $\{u(n)\}_{n \geq 0}$ is called $(l,j)$-order complete. For example, the sequence $\{S_2(n)\}_{n \geq 0}$ is both $(l,j)$-order recursive and $(l,j)$-order complete with $l = j = 0$, $c_{l,j} = s_2(j')$, since $s_2(2n) = s_2(n)$ and $s_2(2n+1) = s_2(n+1)$.

Note that, for every $l \geq 0$ and $0 \leq j \leq k^l - 1$, a $(l,j)$-order recursive integer sequence is $k$-regular. In particular, we have the following theorem.

**Theorem 2.** Let $k \geq 2$ be an integer and $\{u(n)\}_{n \geq 0}$ be a non-negative integer sequence. If $\{u(n)\}_{n \geq 0}$ is both $(0,0)$-order recursive and $(0,0)$-order complete, then $\{L_n\}_{n \geq 0}$ satisfies a linear recurrence if and only if $\{L_{u(n)}\}_{n \geq 0}$ is $k$-regular.

**Remark 1.** In fact, if there exist integers $l \geq 0$, $0 \leq j \leq k^l - 1$ satisfying that $\{u(n)\}_{n \geq 0}$ is $(l,j)$-order recursive and $\{L_n\}_{n \geq 0}$ satisfies a linear recurrence, then $\{L_{u(n)}\}_{n \geq 0}$ is also $k$-regular.
Remark 2. Theorem 3 tells us that the images of some regular sequences under linear recurrence codings are also regular.

To prove Theorem 3 we need the following two lemmas.

Lemma 2. If \( \{u(n)\}_{n \geq 0} \) is \((0,0)\)-order recursive, then for every \( l > 0 \) and \( 0 \leq j \leq k^l - 1 \),

\[
Cl_j = \sum_{s=0}^{l-1} c_{1,j_s}
\]

where \( (j)_k = j_{l-1} \cdots j_1 j_0 \).

Proof. This is an immediate consequence of the fact that \( u(kn+j) = u(n) + c_{1,j} \) for \( 0 \leq j \leq k - 1 \). \( \square \)

Lemma 3. If \( \{u(n)\}_{n \geq 0} \) is \((0,0)\)-order recursive and \((0,0)\)-complete, then max\( \{c_{1,j} : 0 \leq j \leq k - 1\} > 0 \). Moreover, if \( \{u(n)\}_{n \geq 0} \) is non-negative, then \( c_{1,j} \geq 0 \) for every \( l > 0 \) and \( 0 \leq j \leq k^l - 1 \).

Proof. If max\( \{c_{1,j} : 0 \leq j \leq k - 1\} \leq 0 \), then \( c_{1,j} \leq 0 \) for all \( 0 \leq j \leq k - 1 \). By Lemma 2, we have \( c_{1,j} \leq 0 \) for all \( l > 0 \) and \( 0 \leq j \leq k^l - 1 \). Hence, if \( (n)_k = j_{l-1} \cdots j_1 j_0 \), then \( u(n) = u(0) + \sum_{s=0}^{l-1} c_{1,j_s} \leq u(0) \), which contradicts the \((0,0)\)-completeness of \( \{u(n)\}_{n \geq 0} \).

If there exists an integer \( 0 \leq r \leq k - 1 \) such that \( c_{1,r} < 0 \), choosing a nature number \( l \) such that \( u(0) + l c_{1,r} < 0 \), then \( u(n) = u(0) + \sum_{s=0}^{l-1} c_{1,j_s} < 0 \) where \( (n)_k = r^l \). Since \( \{u(n)\}_{n \geq 0} \) is a non-negative sequence, it implies that \( c_{1,j} \geq 0 \) for every \( 0 \leq j \leq k - 1 \). By Lemma 2 \( c_{1,j} \geq 0 \), for every \( l > 0 \) and \( 0 \leq j \leq k^l - 1 \). \( \square \)

Now, we are going to prove Theorem 2.

Proof of Theorem 2. Assume that the sequence \( \{L_n\}_{n \geq 0} \) satisfies a linear recurrence, i.e., there exists an integer \( p \geq 1 \) such that \( L_n = \sum_{k=1}^{p} C_k L_{n-k} + C_0 (n \geq p) \), where \( C_k (0 \leq i \leq p) \) are constants. Then, for every integer \( c \geq 0 \), \( L_{n+c} = \sum_{k=1}^{p} C_k(c) L_{n-k} + C_0(c) (n \geq p) \), where \( C_k(c) (0 \leq i \leq p) \) are constants.

Hence, by Lemma 3 and the \((0,0)\)-order recursive relation, for \( l \geq 0 \) and \( 0 \leq j \leq k^l - 1 \), we have

\[
L_{u(k^n+j)} = L_{u(n)+c_{1,j}} = \sum_{i=1}^{p} C_i L_{u(n)+c_{1,j}-i} + C_0 = \sum_{i=1}^{p} C_i(l,j) L_{u(n)-i} + C_0(l,j),
\]

where \( C_i(l,j) (0 \leq i \leq p) \) are all constants, depending on \( l \) and \( j \). Thus, each sequence \( \{L_{u(k^n+j)}\}_{n \geq 0} \) is a combination of the sequences \( \{L_{u(n)}\}_{n \geq 0} \), \( \{L_{u(n)}\}_{n \geq 0} \), \( \cdots \), \( \{L_{u(n)-p}\}_{n \geq 0} \) and the constant sequence, which implies that the sequence \( \{L_{u(n)}\}_{n \geq 0} \) is \( k \)-regular.

Conversely, if \( \{L_{u(n)}\}_{n \geq 0} \) is a \( k \)-regular sequence, then the module generated by its \( k \)-kernel is generated by \( \{L_{u(k^n+r)}\}_{n \geq 0} \) for some \( 0 \leq r \leq k - 1 \). By Lemma 3 \( c_{1,r} > 0 \). Taking \( j = [r^l]k = r \cdot \frac{r - 1}{k^l - 1} \). Then, by Lemma 2 we have \( u(k^n+j) = u(n) + l c_{1,r} \). Hence, we have

\[
L_{u(n)+l c_{1,r}} = L_{u(k^n+j)} = \sum_{s=1}^{M} C_s(l,j) L_{u(k^n+j)_s} = \sum_{s=1}^{M} C_s(l,j) L_{u(n)+c_{1,j_s}},
\]

By the choice of \( l \), note that for all \( 1 \leq s \leq M \) and \( c_{1,r} > c_{1,j_s} \). Let \( m = u(n) + l c_{1,r} \). Then,

\[
L_m = \sum_{s=1}^{M} C_s(l,j) L_{m-(l c_{1,r} - c_{1,j_s})}.
\]

Since \( u(n) \) is \((0,0)\)-order complete, \( m \) ranges all natural numbers except finitely many terms. Hence, \( \{L_n\}_{n \geq 0} \) satisfies a linear recurrence. \( \square \)

Example 9. Let \( \sigma : q_i \rightarrow q_i q_{i+1} \) and \( \{F_n\}_{n \geq 0} \) be the Fibonacci numbers defined by \( F_0 = 1, F_1 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) for every \( n \geq 2 \). Note that the index sequence \( \{i(n)\}_{n \geq 0} \) satisfying \( \sigma \) is \((0,0)\)-order recursive and \((0,0)\)-order complete, \( \{F_n\}_{n \geq 0} \) satisfies a linear recurrence. Hence, the sequence \( \tau_F(\sigma^\infty(q_0)) = \{F_i(n)\}_{n \geq 0} \) is \( 2 \)-regular.

It is clearly that the index sequence \( \{i(n)\}_{n \geq 0} \) is a \( 2 \)-regular sequence which can be generated by \( i(2n) = i(n) \) and \( i(2n+1) = i(n) + 1 \) with \( i(0) = 0 \). Under the coding \( \rho : i \rightarrow F_i \), we obtain a new \( 2 \)-regular sequence \( \rho(i(n)) = \{F_i(n)\}_{n \geq 0} = \tau_F(\sigma^\infty(q_0)) \) which can be generated by the formulas:

\[
F_{i(0)} = F_{i(1)} = 1, F_{i(2n)} = F_{i(n)}, F_{i(4n+1)} = F_{i(2n+1)}, F_{i(4n+3)} = F_{i(2n+1)} + F_{i(n)}.
\]
Let $\sigma : q_i \to q_{i+t_0}q_{i+t_1}\cdots q_{i+t_{k-1}}$ be a 1-periodic $k$-uniform morphism with $t_0 = 0, t_i \in \mathbb{N}$ ($0 \leq i \leq k - 1$). Assume that \{\(i(n)\)\}_{n \geq 0} is the index sequence of $\sigma$, then \(i(kn+j) = i(n) + t_j\) for all $0 \leq j \leq k - 1$. Note that \{\(i(n)\)\}_{n \geq 0} is $(0,0)$-order recursive with $c_{1,j} = t_j$ and max\{$c_{1,j} : 0 \leq j \leq k - 1\} > 0$. Hence, by Theorem 2 we have the following corollary.

**Corollary 1.** If there exist two integers $1 \leq i, j \leq k - 1$ such that $(t_i, t_j) = 1$, then the sequence \(\{L_n\}_{n \geq 0}\) satisfies a linear recurrence if and only if the sequence $\tau_L(\sigma^{\infty}(q_0))$ is $k$-regular.

**Proof.** Since $\tau_L(\sigma^{\infty}(q_0)) = \{\tau_L(i(n))\}_{n \geq 0} = \{L_{i(n)}\}_{n \geq 0}$, by Theorem 2 it suffices to show that \{\(i(n)\)\}_{n \geq 0} is $(0,0)$-order complete. Assume that $0 < t_i = p < q = t_j$ satisfies $(p, q) = 1$ for some $0 \leq i, j \leq k - 1$. Now, we claim that for every large integer $m \geq 0$, there exist integers $c_1, c_2 \geq 0$ such that $m = c_1 p + c_2 q$.

Assume $m = pn + i$ with $0 \leq i \leq p - 1$, then $m = p(n - j) + (pj + i)$ for every $j \geq 0$. Since $(p, q) = 1$, \{\(pj + i : 0 \leq j \leq q - 1\)\} forms a complete system of incongruent residues $(\mod \ q)$. Hence, there exist integers $0 \leq j_1 \leq j_2$ and $j_2 \geq 2$ such that $pj_1 + i = qj_2$. Thus, $m = (n - j_1)p + jq_2$ which completes the claim.

For every $0 \leq i, j \leq k - 1$, let $n = [i^e j^f]_k$. Then by Lemma 2 \(i(n) = i(0) + c_1 t_i + c_2 t_j = i(0) + c_1 p + c_2 q\).

Hence, \{\(i(n)\)\}_{n \geq 0} is $(0,0)$-order complete except finitely many terms. \(\square\)

**Remark 3.** The condition of Corollary 1 can be weakened by the condition that the greatest common factor of all nonzero integers $t_i$ $(0 \leq i \leq k - 1)$ is 1.

Let $\sigma : q_i \to q_0 q_{i+t_0} q_{i+t_1} \cdots q_{i+t_{k-1}}$ $(i \geq 0)$ be a $k$-uniform morphism, where $t_i \geq 0$ $(1 \leq i \leq k - 1)$ are integers. Assume that \{\(i(n)\)\}_{n \geq 0} is the index sequence of this morphism, then $i(kn) = 0$ and $i(kn+j) = i(n) + t_j$ $(1 \leq j \leq k - 1)$ for every $n \geq 0$. Note that the sequence \{\(i(n)\)\}_{n \geq 0} is $(0,0)$-order recursive.

The following proposition shows that if \{\(u(n)\)\}_{n \geq 0} is not $(0,0)$-order recursive, then, Theorem 2 also holds.

**Proposition 3.** If for every two integers $1 \leq i, j \leq k - 1$ such that $(t_i, t_j) = 1$, then the sequence \(\{L_n\}_{n \geq 0}\) satisfies a linear recurrence if and only if the sequence $\tau_L(\sigma^{\infty}(q_0)) = \{L_{i(n)}\}_{n \geq 0}$ is $k$-regular.

**Proof.** For every $l \geq 0, 0 \leq j \leq k^l - 1$, let \(j_k^{i} = j_{i-1}j_{i-2} \cdots j_0\). If $j_k \neq 0$ for all $0 \leq i \leq k - 1$, then $i(k^l + j) = i(n) + \sum_{i=0}^{l-1} t_i$. Otherwise, assume that $s = \min\{0 \leq i \leq l - 1 : j_i = 0\}$, then $i(k^l + j) = \sum_{i=0}^{l-s} t_i$.

If the sequence \(\{L_n\}_{n \geq 0}\) satisfies a linear recurrence, i.e., there exists an integer $p \geq 1$ satisfying $L_n = \sum_{i=1}^{p} C_i L_{n-i} + C_0$ $(n \geq p)$, where $C_i$ $(0 \leq i \leq p)$ are constants. Then, for every integer $c \geq 0$, $L_{n+c}$ is a combination of $L_{n-i}$ with $1 \leq i \leq p$. Hence, for every $l \geq 0$ and $0 \leq j \leq k^l - 1$, the sequence \(\{L_{i(k^l+j)}\}_{n \geq 0}\) is a combination of \(\{L_{i(n)}\}_{n \geq 0}\) $(1 \leq i \leq p)$ and the constant sequence, which implies the sequence \(\{L_{i(n)}\}_{n \geq 0}\) is $k$-regular.

The other direction is the same as the proof of Theorem 2 and Corollary 1 so we omit it here. \(\square\)

Let $\sigma$ be the morphism defined by formula (1), i.e., $\sigma : q_i \to q_{a_i} q_{a_{i+1}} \cdots q_{a_{k-1}+a_{k-1}}$ with $a_i \in \mathbb{N}$. Corollary 1 and Proposition 3 have studied the cases $a_i \in \{0, 1\}$ for all $0 \leq i \leq k - 1$. If there exists $0 \leq i \leq k - 1$ such that $a_i \geq 2$, then the following example shows that Theorem 2 does not hold.

**Example 10.** For every $0 \geq 0$, let $\sigma : q_i \to q_{2-i} q_{i+1}$, $\tau : q_i \to F_i$, where \(\{F_n\}_{n \geq 0}\) is the Fibonacci numbers. Then the sequence $\tau(\sigma^{\infty}(q_0))$ is not 2-regular.

Let \{\(i(n)\)\}_{n \geq 0} be the index sequence of $\sigma$ in Example 8. Then $i(2n) = 2i(n)$ and $i(2n+1) = i(n) + 1$ with $i(0) = 0$. Note that $F_{2n} = F_{2n}^2 + 2F_n F_{n+1}$ and assume that $d_n = F_n(1)$, then,

\[
d_{2n} = F_{2i(2n)} = F_{2i(n)} = F_{i(n)}^2 + 2F_{i(n)} F_{i(n)+1} > F_{i(n)}^2 = d_n^2.
\]

Since $d_2 = 2$ and \[
\frac{\log_2(d_2 \cdot 2^k)}{\log_2(2 \cdot 2^k)} > \frac{2^k}{k+1} \frac{d_2}{k+1} \to \infty \quad (k \to \infty).
\]

By Theorem 2.10 in [2], the sequence $\tau(\sigma^{\infty}(q_0)) = \{F_{i(n)}\}_{n \geq 0} = \{d_n\}_{n \geq 0}$ is not 2-regular.

In Theorem 2 if the sequence \(\{L_n\}_{n \geq 0}\) takes only finitely many values, then, we have the following proposition.

**Proposition 4.** Let $k \geq 2$ be an integer and \{\(u(n)\)\}_{n \geq 0} be a non-negative integer sequence. If the sequence \{\(u(n)\)\}_{n \geq 0} is both $(0,0)$-order recursive and $(0,0)$-order complete, then the sequence \(\{L_{u(n)}\}_{n \geq 0}\) is k-automatic if and only if the sequence \(\{L_n\}_{n \geq 0}\) is ultimately periodic.
Proof. Note that if a sequence takes only finitely many values, then Everest et al. in [13] told us that it satisfies a linear recurrence if and only if it is ultimately periodic, and Allouche and Shallit in [2] showed that it is regular if and only if it is k-automatic. Then, by Theorem 2 we complete this proof.

Remark 4. Proposition 4 shows that we can obtain an automatic sequence from a ultimately periodic sequence by a “a(n)” index picking, where \{a(n)\} \(n \geq 0\) is both (0,0)-order recursive and (0,0)-order complete.

It is known that the Fibonacci sequence \(\{f_n\}_{n \geq 0} = \psi^\infty(0)\) is not ultimately periodic, where \(\psi : 0 \rightarrow 01, 1 \rightarrow 0\). Hence, if \(\sigma : q_i \mapsto q_i q_{i+1} \cdots q_{i+k-1}\) and \((t_i, t_j) = 1\) for some \(1 \leq i, j \leq k - 1\), then by Proposition 3 \(\tau(f(\sigma^\infty(q_0)))\) is not k-automatic.

5. SOME GENERALIZATIONS

Both the alphabet \(\Sigma\) and the state set \(Q\) in Section 3 and Section 4 are denoted by \(\{q_n : n \in \mathbb{N}\}\) and the index sequences take values in \(\mathbb{N}\). A possible extension of the present approach is to focus on the infinite set \(\{q_n : n \in \mathbb{Z}\}\) and the index sequences on \(\mathbb{Z}\). On the infinite set \(\{q_n : n \in \mathbb{Z}\}\), we can define morphisms and DCAs (DCAOs) similarly. Moreover, we generalize \(m\)-periodic \(k\)-uniform morphisms (or the \(m\)-periodic \(k\)-DCAOs) by \(T \in \mathbb{Z}^{m \times k}\).

Let \(\sigma\) be a \(m\)-periodic \(k\)-uniform morphism and \(\sigma^\infty(q_0)\) be a fixed point of \(\sigma\). If \(\tau(q_i) = i\), then the sequence \(\{t(n)\}_{n \geq 0} := \sigma^\infty(q_0))\) is called to be the index sequence of \(\sigma\). The following results are similar as Theorem 1 and Proposition 1.

Theorem 1. If \(\sigma\) is an \(m\)-periodic \(k\)-uniform morphism, then its index sequence \(\{t(n)\}_{n \geq 0}\) is \(k\)-regular.

Proposition 1. Taking \(\tau : q_i \rightarrow f(i)\) for every \(i \geq 0\), where \(f(i)\) is a polynomial with integer coefficients. If \(\sigma\) is a \(m\)-periodic \(k\)-uniform morphism, then the sequence \(\tau(\sigma^\infty(q_0))\) is \(k\)-regular.

We give the following 2-periodic 2-uniform morphism on \(\{q_n : n \in \mathbb{Z}\}\).

Example 11. Let \(\sigma\) be a morphism defined by \(q_{2i} \mapsto q_{2i+1}, q_{2i+1} \mapsto q_{2i+2}(i \in \mathbb{Z})\). Then, \(\sigma^\infty(q_0)\) is the unique fixed point of \(\sigma\). Let \(\tau(q_i) = i\). Then, the index sequence \(\{t(n)\}_{n \geq 0} = \tau(\sigma^\infty(q_0)) = 01(-1)2(-3)02(5)(-5)012314\cdots\). It also can be generated by a DCAO in Figure 5.

![Figure 5. A DCAO generating the sequence \(\{b_n\}_{n \geq 0}\).](image)

We end this section by two regular sequences which can be generated by countable state automata and also morphisms on a countable alphabet.

Example 12. Let \(a(n) = \lfloor \log_b (\alpha n + \beta) \rfloor\). Then, the integer sequence \(\{a(n)\}_{n \geq 0}\) has been studied by Allouche, Shallit, Bell, Moshe, Rowland and Zhang et al. in [3 23 24 27] respectively. Especially, if \(b = 2, \alpha = 1, \beta = 0\), we obtain a 2-periodic sequence \(\{\lfloor \log_2 n \rfloor\}_{n \geq 1}\). Let \(b_0 = 1, b_n = \lfloor \log_2 n \rfloor (n \geq 1)\). Then, it can be generated by formulas: \(b_0 = 1, b_1 = 0, b_{2n} = b_{2n+1} = b_n + 1 (n \geq 1)\). It also can be generated by a DCAO in Figure 6.

![Figure 6. A DCAO generating the sequence \(\{b_n\}_{n \geq 0}\).](image)
Example 13. Let \( f(n) \) be a polynomial with integer coefficients. Then, \( \{f(n)\}_{n \geq 0} \) is a \( k \)-regular sequence for every \( k \geq 2 \) in [2]. Especially, the sequence \( \{b_n\}_{n \geq 0} = \{n\}_{n \geq 0} \) is 2-regular, since it can be generated by formulas: \( b_{2n} = 2b_n \) and \( b_{2n+1} = 2b_n + 1 \). It also can be generated by 2-uniform morphism \( \sigma : i \rightarrow (2i) (2i + 1) (i \geq 0) \), i.e., \( \{b_n\}_{n \geq 0} = \sigma^\infty(0) \).

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References.

[1] J. P. Allouche, Automates finis en théorie des nombres, Exposition Math., 1987, 5: 239-266.
[2] J. P. Allouche, J. Shallit, The ring of \( k \)-regular sequences, Theoret. Comput. Sci., 1992, 98: 163-197.
[3] J. P. Allouche, J. Shallit, The ring of \( k \)-regular sequences, II, Theoret. Comput. Sci., 2003, 307: 3-29.
[4] J. P. Allouche, J. Shallit, Automatic sequences. Theory, Applications, Generalizations, Cambridge University Press, Cambridge, 2003.
[5] J. P. Allouche, K. Scheicher, R. F. Tichy, Regular maps in generalized number systems, Math. Slovaca., 2000, 50: 41-58.
[6] J. P. Bell, A generalization of Cobbach’s theorem for regular sequences, Sémin. Lothar. Combin., 2006, 54A, Art. B54Ap.
[7] J. P. Bell, M. Coons, K. G. Hare, The minimal growth of a \( k \)-regular sequence, Bull. Aust. Math. Soc., 2014, 90: 195-209.
[8] V. Berthé, M. Rigo, Combinatorics, Automata and Number Theory, Encyclopedia Math. Appl., vol. 135, Cambridge University Press, Cambridge, 2010.
[9] J. Cassaigne, M. Le Gonidec, Propriétés et limites de la reconnaissance d’ensembles d’entiers par automates dénombrables, Journal de Théorie des Nombres de Bordeaux., 2010, 22: 307-338.
[10] E. Charlier, N. Rampersad, J. Shallit, Enumeration and decidable properties of automatic sequences, International J. of Foundations of Comput. Sci., 2012, 5: 1035-1066.
[11] G. Christol, T. Kamae, M. Mendès France, G. Rauzy, Suites algébriques, automates et substitutions, Bull. Soc. Math. France., 1980, 108: 401-419.
[12] A. Cobham, Uniform tag sequences, Math. Systems Theory., 1972, 6: 164-192.
[13] G. Everest, A. van der Poorten, I. Shparlinski, T. Ward. Recurrence sequences, Math. Surveys Monogr., vol. 104, Amer. Math. Soc., Providence, RI, 2003.
[14] S. Ferenczi, Substitution dynamical systems on infinite alphabets, Ann. Inst. Fourier, Grenoble., 2006, 56(7): 2315-2343.
[15] N. P. Fogg, Substitutions in dynamics, arithmetics and combinatorics, Lecture Notes in Math., vol. 1794, Springer-Verlag, Berlin, 2002.
[16] M. G. Paquet, J. Shallit, Avoiding squares and overlaps over the natural numbers, Discrrete Math., 2009, 309: 6245-6254.
[17] M. Le Gonidec, Sur la complexité de mots infinis engendrés par des \( q \)-automates dénombrables, Ann. Inst. Fourier, Grenoble., 2006, 56(7): 2463-2491.
[18] M. Le Gonidec, Sur la complexité des mots \( q^\omega \)-automatiques, Ph.D. thesis, Université Aix-Marseille II, 2006.
[19] M. Le Gonidec, Drunken man infinite words complexity, Rairo-Theor. Inf. Appl., 2008, 42: 599-613.
[20] M. Le Gonidec, On complexity functions of infinite words associated with generalized Dyck languages, Theoret. Comput. Sci., 2008, 407: 117-133.
[21] J. H. Loxton, A. J. van der Poorten, Arithmetic properties of automata: regular sequences, J. Reine Angew. Math., 1988, 392: 57-69.
[22] C. Mauduit, Propriétés arithmétiques des substitutions et automates infinis, Ann. Inst. Fourier, Grenoble., 2006, 56(7): 2525-2549.
[23] Y. Moshe, On some questions regarding \( k \)-regular and \( k \)-context-free sequences, Theoret. Comput. Sci., 2008, 400: 62-69.
[24] E. Rowland, Non-regularity of \( \lfloor a + \log_d n \rfloor \), Integers., 2010, 10(1): 19-23.
[25] W. Thomas, A short introduction to infinite automata, In: Proc. 5th International Conference Developments in Language Theory, Springer LNCS., 2001, 2295: 134-144.
[26] W. Thomas, A short introduction to infinite automata, In: Proc. 5th International Conference in Language Theory, Springer LNCS., 2001, 2295: 134-144.
[27] Z. X. Wen, J. M. Zhang, W. Wu, On the regular sum-free sets, European J. Combin., 2015, 49: 42-56.
[28] S. Zhang, J. Y. Yao, Analytic functions over \( \mathbb{Z}_p \) and \( p \)-regular sequences, C. R. Acad. Sci. Paris, Ser. I., 2011, 349: 947-952.