On the Quantum Cohomology Rings of General Type Projective Hypersurfaces and Generalized Mirror Transformation

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Abstract

In this paper, we study the structure of the quantum cohomology ring of a projective hypersurface with non-positive 1st Chern class. We prove a theorem which suggests that the mirror transformation of the quantum cohomology of a projective Calabi-Yau hypersurface has a close relation with the ring of symmetric functions, or with Schur polynomials. With this result in mind, we propose a generalized mirror transformation on the quantum cohomology of a hypersurface with negative first Chern class and construct an explicit prediction formula for three point Gromov-Witten invariants up to cubic rational curves. We also construct a projective space resolution of the moduli space of polynomial maps, which is in a good correspondence with the terms that appear in the generalized mirror transformation.  

1 Introduction

This paper is a continuation of [3]. Here, we are mainly interested in the quantum cohomology ring of a projective hypersurface with non-positive 1st Chern class.

Up to now, many beautiful results on the quantum cohomology rings of Kähler manifolds with non-negative first Chern classes are known. In particular, A.B. Givental [5], B. Kim [10], and B. Lian-K. Liu-S. T. Yau [14] revealed a deep relation between the quantum cohomology of complete intersections in homogeneous spaces with non-negative first Chern class and the hypergeometric series. Then, what happens in the quantum cohomology of manifolds with negative first Chern class? Before we turn into this topic, we look back at the quantum cohomology of Calabi-Yau manifolds. Roughly speaking, the mirror calculation of the quantum cohomology of Calabi-Yau manifolds consists of two parts. The first part is the evaluation of the hypergeometric series obtained from the B-model of its mirror manifold and the second part is a translation of the hypergeometric data (B-model) into the geometrical one (A-model) by a coordinate change of the deformation parameter, i.e., the mirror map. In [5], Givental revealed the fact that in the case of projective Fano hypersurfaces with first Chern class $c \cdot H$, $(c \geq 2)$, we can do without the second part of the mirror calculation. In [3], [16], [14], it was argued that the second part has a close relation with a difference between the non-linear sigma model and the gauged linear sigma model, or, in other
words, a difference between the moduli space of polynomial maps and its toric compactification. An easy dimensional counting leads us to the conclusion that we have the same problem in treating the quantum cohomology of a projective hypersurface with negative first Chern class.

In this paper, we give some results arguing that there exists a generalization of two processes in the mirror calculation in the case of the quantum cohomology of the general type projective hypersurface. Our objects that correspond to the hypergeometric series in the case of a hypersurface with first Chern class \( c \cdot H \) (\( c \geq 2 \)). These recursive formulas express the structure constants for degree \( k \) hypersurface \( (M^k_N) \) in \( CP^{N-1} \) in terms of those of \( M^k_{N+1} \). By these formulas, we can reduce the dimension of the hypersurface conserving its degree. We constructed them up to mapping degree 4 in \( 3 \) (we later constructed the recursive formulas for rational curves of mapping degree 5). We observed that the virtual structure constants of the Calabi-Yau hypersurface \( M^k \) reproduce the coefficients of the hypergeometric series used in the mirror calculation up to mapping degree 5, and we expect here that these constants produce objects like the hypergeometric series, even in the case of the general type hypersurfaces. For the second part of the mirror calculation, we propose an idea for construction of the generalized mirror transformation. In this paper, we argue that this transformation can be effectively constructed from the one for Calabi-Yau hypersurfaces. In \( 3 \), we constructed an implicit formula that translates the virtual structure constants into the true ones in the case of the Calabi-Yau hypersurface, and here, we prove a theorem that implies a close relation between this mirror transformation and Schur polynomials, or partitions of integers (Young diagrams). Then we conjecture that the structure revealed in our theorem is still conserved in the case of the hypersurface with negative first Chern class. Indeed, we construct an explicit form of the generalized mirror transformation in this case up to mapping degree 3 (We also obtained partial results for mapping degree 4) with some technical assumption. Our formula can also be regarded as a prediction formula for the 3-point (with one insertion of Kähler form) Gromov-Witten invariant of the general type projective hypersurfaces.

Our idea of generalization of the mirror transformation has a geometrical origin. We construct a set-theoretic projective space resolution of the moduli space \( M^d_{CP^{N-1}} \), that is the moduli space of polynomial maps from \( CP^1 \) to \( CP^{N-1} \) of mapping degree \( d \). The resolution diagram consists of the direct product of \( CP^n \) (\( n \) varies) and these spaces are labeled by partitions of nonnegative integers, or mapping degree. We observe that there is one-to-one correspondence between the spaces in the resolution diagram and the terms in the generalized mirror transformation. On the other hand, as we mentioned in \( 1 \), dimensional count leads us to the conclusion that we can not see the resolution structure in the case of a Fano hypersurface with first Chern class \( c \cdot H \) (\( c \geq 2 \)). In sum, we expect a deep relation between this resolution and the ring of symmetric functions \( R \). We think this point of view should be pursued further.

This paper is organized as follows.

In section 2, we construct a projective space resolution of the moduli space \( M^d_{CP^{N-1}} \) of polynomial maps from \( CP^1 \) to \( CP^{N-1} \) of mapping degree \( d \) and explain its connection with partitions of integers.

In section 3, we briefly review the results on the Kähler sub-ring of the quantum cohomology ring of a projective hypersurface with nonnegative first Chern class. We also introduce here the virtual structure constants.

In section 4, we construct recursive formulas that represent the structure constants of the quantum cohomology of the hypersurface of degree \( k \) in \( CP^{N-1} \) in terms of the ones of the hypersurface of the same degree in \( CP^N \) in the case of non-positive first Chern class and construct the generalized mirror transformation up to mapping degree 2. Next, we state a theorem on the mirror transformation of the Calabi-Yau hypersurfaces and explain its relation with Schur polynomials and with the projective space resolution in section 2.

In section 5, which is the main section of this paper, we make a general conjecture on the form of the generalized mirror transformation using the results in section 4. Then we construct an explicit
formula of the transformation in the case of mapping degree 3, (we also obtained partial results for mapping degree 4) using our conjecture.

In section 6, we show some explicit examples of the quantum cohomology of general type hypersurface $M^k_N$ for lower $N$ and $k$.

2 Projective Space Resolution of Moduli Space

Topic of this section is not logically connected to the derivation of the generalized mirror transformation, which is the main topic of this paper, but it gives us a geometrical background of the conjecture that will be proposed in section 5.

Here, we propose a set-theoretic projective space resolution of the moduli space $M^{CP^{N-1}}_d$ of polynomial maps of mapping degree $d$ from $CP^1$ to $CP^{N-1}$. Note that this moduli space is non-compact and different from the moduli space of stable maps from $CP^1$ to $CP^{N-1}$. This point of view is already discussed by many authors and is called “Gauged Linear Sigma Model”.

Let us define that the sequence of maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is set-theoretically exact if $g$ is a bijective map between $B - \text{Im}(f)$ and $\text{Im}(g) \subset C$. If $B - \text{Im}(f)$ is the disjoint union of the sets $B_j$, $(j = 1, 2, \cdots, m)$ and if the maps $g_j : B_j \rightarrow C_j$, $(j = 1, 2, \cdots, m)$ are bijection between $B_j$ and $\text{Im}(g_j)$, we say that the branched sequence of maps $f : A \rightarrow B$ and $g_j : B_j \rightarrow C_j$ are set-theoretically exact. Then our resolution diagrams consist of the branched sequences of maps that are set-theoretically exact. For some lower degrees, the resolution diagrams are given in figure 1. Let us explain the inductive construction of these diagrams. First, take an

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Projective Space Resolution of $M^{CP^{N-1}}_d$ for $d = 1, 2, 3$}
\end{figure}
Here, we extend the definition of \( \iota \) from \( \mathcal{M}_d^{CP^{N-1}} \) to \( CP^{(d+1)N-1} \) in figure 2. The map \( \iota \) is defined as follows.

\[
\iota((\sum_{j=0}^{d} a_j^1 s^j t^{d-j}) : \ldots : (\sum_{j=0}^{d} a_j^N s^j t^{d-j})) = (a_0^1 : \ldots : a_d^1 : a_0^2 : \ldots : a_d^2 : \ldots : a_0^N : \ldots : a_d^N) \quad (2.2)
\]

Note that homogeneous polynomials of degree \( d \) \( \sum_{j=0}^{d} a_j^k s^j t^{d-j} \) \((k = 1, \ldots, N)\) must not have common divisor of positive degree, because otherwise they correspond to maps of lower degree. Combining \( \iota \) and \((\eta^d_j)^{-1}\), we obtain a set-theoretic short exact sequence in figure 3.

Since \( \mathcal{M}_d^{CP^{N-1}} \) is isomorphic to \( CP^{N-1} \), resolution of \( \mathcal{M}_d^{CP^{N-1}} \) is given as:

\[
\begin{align*}
\mathcal{M}_d^{CP^{N-1}} & \longrightarrow CP^{(d+1)N-1} \\
& \longrightarrow CP^{d} \times \mathcal{M}_0^{CP^{N-1}} \\
& \quad \vdots \\
& \quad \vdots \\
& \quad CP^{d} \times \mathcal{M}_1^{CP^{N-1}} \\
& \quad \vdots \\
& \quad \vdots \\
& \quad CP^{d} \times \mathcal{M}_{d-1}^{CP^{N-1}} \\
\end{align*}
\]

Figure 3: Set-Theoretic Short Exact Sequence

\[
\mathcal{M}_d^{CP^{N-1}} \rightarrow CP^{2N-1} \rightarrow CP^{1} \times CP^{N-1}. \quad (2.4)
\]

Now that we have the diagram of figure 3, what we have to do to construct projective space resolution of \( \mathcal{M}_d^{CP^{N-1}} \) is to construct projective space resolution of \( CP^{j} \times \mathcal{M}_d^{CP^{N-1}} \), \((j = 1, 2, \ldots, d)\). This can be done by applying \( CP^{j} \times \) to each space in the resolution diagram of \( \mathcal{M}_d^{CP^{N-1}} \). Then we can construct the resolution diagram of \( \mathcal{M}_d^{CP^{N-1}} \) inductively.

From figure 1, we can see that each space \( CP^{d_1} \times CP^{d_2} \times \ldots \times CP^{d_m} \times CP^{N(d'+1)-1} \) induced from a sequence \((d_1, d_2, \ldots, d_m; d')\) \((d_j \geq 1, d' \geq 0, \sum_{j=1}^{m} d_j + d' = d)\) appear at least once and only once in the resolution diagram of \( \mathcal{M}_d^{CP^{N-1}} \). Including the null sequence \((0; d)\) in the case of \( CP^{(d+1)N-1} \), total number of such integer sequences is \( 1 + \sum_{j=1}^{d} 2j-1 = 2d \).

We also associate a pair \( CP^{d_1} \times CP^{d_2} \times \ldots \times CP^{d_m} \times CP^{N(d'+1)-1} \rightarrow CP^{d_1} \times CP^{d_2} \times \ldots \times CP^{d_m} \times CP^{d'} \times CP^{N-1} \) with an integer sequence \((d_1, d_2, \ldots, d_m)\), \((d_j \geq 1, \sum_{j=1}^{m} d_j = k \quad (k = 1, \ldots, d - \ldots)\)
We denote the multiplicity of \( \sigma \) as the label of these pairs. Note that this label has multiplicity or, in other words, not one to one. These pairs appear in the resolution diagram once and only once. If we formally associate the order of the integer sequence, we can take a partition \( \sigma_k : k = d_1 + d_2 + \cdots + d_m \ (0 \leq k \leq d - 1) \) as the label of these pairs. Note that this label has multiplicity or, in other words, not one to one. We denote the multiplicity of \( \sigma_k \) by \( N(\sigma_k) \) and multiplicity of \( j \), \( (1 \leq j \leq k) \), appearing in \( \sigma_k \) by \( \text{mul}(j, \sigma_k) \). Then we have

\[
N(\sigma_k) = \frac{m!}{\prod_{j=1}^{k} \text{mul}(j, \sigma_k)!}.
\]

If we denote a set of partitions of integer \( k \) by \( P_k \), we see

\[
\sum_{\sigma_k \in P_k} N(\sigma_k) = 2^{k-1} \ (k \geq 1),
\]

\[
\sum_{k=0}^{d-1} \sum_{\sigma_k \in P_k} N(\sigma_k) = 2^{d-1},
\]

which are easy exercises of elementary combinatorics.

As we mentioned in our previous paper [8], these resolution structures do not occur in the quantum cohomology of degree \( k \) hypersurfaces in \( CP^{N-1} \) with \( N - k \geq 2 \) as we can see by dimensional counting, but in the case of \( N - k \leq 1 \) and especially in the case of \( N - k \leq 0 \), we will see the above structure appears. In these cases, we expect that a topological invariant obtained from integrating out the closed forms on \( M_d^{CP^{N-1}} \) can be represented as an alternate sum of the contributions from each pair in the resolution diagram (We denote length of partition \( \sigma_m \) by \( l(\sigma_m) \)):

\[
\text{(top. inv. on } M_d^{CP^{N-1}}) = \sum_{m=0}^{d-1} \sum_{\sigma_m \in P_m} (-1)^{l(\sigma_m)} N(\sigma_m) \text{(contribution from one pair labeled by } \sigma_m).\]

Here, we assumed that the contributions from each pair labeled by \( \sigma_m \) are the same. In section 4, we will see the formula that can be regarded as an example of (2.7).

3 Quantum Kähler Sub-Ring of Projective Hypersurfaces

3.1 Notation

In this section, we introduce the quantum Kähler sub-ring of the quantum cohomology ring of a degree \( k \) hypersurface in \( CP^{N-1} \). Let \( M^d_k \) be a hypersurface of degree \( k \) in \( CP^{N-1} \). We denote by \( QH^*_e(M^d_k) \) the sub-ring of the quantum cohomology ring \( QH^*(M^d_k) \) generated by \( O_e \) induced from the Kähler form \( e \) (or, equivalently the intersection \( H \cap M^d_k \) between a hyperplane class \( H \) of \( CP^{N-1} \) and \( M^d_k \)). The multiplication rule of \( QH^*_e(M^d_k) \) is determined by the Gromov-Witten invariant of genus 0 \( \langle O_e O_{e^{N-2-m}} O_{e^{m-1-(k-N)d}} \rangle_{d,M^d_k} \) and it is given as follows:

\[
L_{m}^{N,k,d} := \frac{1}{k} \langle O_e O_{e^{N-2-m}} O_{e^{m-1-(k-N)d}} \rangle,
\]

\[
O_e \cdot 1 = O_e,
\]

\[
O_e \cdot O_{e^{N-2-m}} = O_{e^{N-1-m}} + \sum_{d=1}^{\infty} L_{m}^{N,k,d} q^d O_{e^{N-1-m+(k-N)d}},
\]

\[
q := e^t.
\]

**Definition 1** We call \( L_{m}^{N,k,d} \) the structure constants of weighted degree \( d \).
Since $M_N^k$ is a complex $N - 2$ dimensional manifold, we see that the structure constants $L_{m,k,d}^{N,k,d}$ is non-zero only if the following condition is satisfied:

$$L_{m,k,d}^{N,k,d} \neq 0 \quad \iff \quad 1 \leq N - 2 - m \leq N - 2, 1 \leq m - 1 + (N - k)d \leq N - 2,$$

$$\iff \quad \max\{0, 2 - (N - k)d\} \leq m \leq \min\{N - 3, N - 1 - (N - k)d\}. \quad (3.9)$$

We rewrite (3.9) into

$$L_{m,k,d}^{N,k,d} \neq 0 \quad \iff \quad 0 \leq m \leq (N - 1) - (N - k)d \quad (N - k \geq 2),$$

$$\quad \iff \quad 1 \leq m \leq N - 3 \quad (N - k = 1, d = 1),$$

$$\quad \iff \quad 0 \leq m \leq N - 1 - (N - k)d \quad (N - k = 1, d \geq 2),$$

$$\quad \iff \quad 2 + (k - N)d \leq m \leq N - 3 \quad (N - k \leq 0). \quad (3.10)$$

From (3.10), we easily see that the number of the non-zero structure constants $L_{m,k,d}^{N,k,d}$ is finite except for the case of $N = k$. Moreover, if $N \geq 2k$, the non-zero structure constants come only from the $d = 1$ part and the number of them is $k$, which is independent of $N$. The $N > 2k$ region is studied by Beauville [1], and his result plays the role of an initial condition of our discussion later. In the case of $N = k$, the multiplication rule of $QH^*_c(M_k^k)$ is given as follows:

$$O_e \cdot 1 = O_e,$$

$$O_e \cdot O_{eN-2-m} = (1 + \sum_{d=1}^{\infty} q^d L_{m,k,d}^{N,k,d} O_{eN-1-m}) \quad (m = 2, 3, \cdots, N - 3),$$

$$O_e \cdot O_{eN-3} = O_{eN-2}. \quad (3.11)$$

We introduce here the generating function of the structure constants of the Calabi-Yau hypersurface $M_k^k$:

$$L_{m,k}^{k,k,d}(e^t) := 1 + \sum_{d=1}^{\infty} L_{m,k,d}^{k,k,d} e^{dt} \quad (m = 2, \cdots, k - 3). \quad (3.12)$$

In this paper, we assume that $L_{m,k,d}^{N,k,d}$ is geometrically given by the following formula:

$$L_{m,k,d}^{N,k,d} = \frac{1}{k} (O_e O_{eN-2-m} O_{eN-1-(k-N)d})_{d,M_k^k}$$

$$= \frac{d}{k} \int_{\mathcal{M}^{CP^{N-1}}_{0,d,2}} c_T(\pi_2^*(R^0\pi_1^*(\phi_1^*(k \cdot H)))) \land \phi_1^*(c_1^{N-2-n}(H)) \land \phi_2^*(c_1^{N-1-(k-N)d}(H)). \quad (3.13)$$

Here, $\mathcal{M}^{CP^{N-1}}_{0,d,n}$ is the moduli space of the stable maps from $CP^1$ with $n$ points to $CP^{N-1}$ of mapping degree $d$. $\pi_j : \mathcal{M}^{CP^{N-1}}_{0,d,j} \to \mathcal{M}^{CP^{N-1}}_{0,d,j-1}$ is the forgetful map and $\tilde{\pi}_m$ is given by $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_m : \mathcal{M}^{CP^{N-1}}_{0,d,m} \to \mathcal{M}^{CP^{N-1}}_{0,d,0}$. $\phi_1 : \mathcal{M}^{CP^{N-1}}_{0,d,m} \to CP^{N-1}$ $(j = 1, 2, \cdots, m)$ is the evaluation map at the $j$-th puncture. The direct image sheaf $R^0\pi_1^*(\phi_1^*(k \cdot H))$ is considered in the setting of the fibration $CP^1 \to \mathcal{M}^{CP^{N-1}}_{0,d,1} \to \mathcal{M}^{CP^{N-1}}_{0,d,n}$. The r.h.s. of (3.13) can be evaluated by the toroidal calculation method of Kontsevich [12], [3] and we will use this data to check numerically our prediction formula in section 5.

### 3.2 Review of Results for Fano and Calabi-Yau Hypersurfaces and Virtual Structure Constants

Let us summarize the results of [3]. In [3], we showed that the structure constants $L_{m,k,d}^{N,k,d}$ of $QH^*_c(M_k^k), \quad (N - k \geq 2)$, can be obtained by applying the recursive formulas in Appendix B,
Moreover we can obtain the mirror map $t$ structure constants of the Calabi-Yau hypersurface into the real ones as follows:

With these conjectures, we can construct the mirror transformation that transforms the virtual $L$ manifold of $M$ mirror manifold of

Then we conjectured $\tilde{L}^{N,k,d}$ for a Calabi-Yau hypersurface do not obey the recursive formulas in appendix B. We introduce here virtual structure constants $\tilde{L}^{N,k,d}$ as follows.

**Definition 2** Let $\tilde{L}^{N,k,d}$ be the rational number obtained by applying the recursion relations of Fano hypersurfaces in Appendix B for arbitrary $N$ and $k$ with the initial condition $L^{N,k,1}_n$, $(N \geq 2k)$, and $L^{N,k,d}_n = 0$, $(d \geq 2, \ N \geq 2k)$.

**Remark 1** In the $N - k \geq 2$ region, $\tilde{L}^{N,k,d} = L^{N,k,d}_n$.

We define the generating function of the virtual structure constants of the Calabi-Yau hypersurface $M^k$ as follows:

$$\tilde{L}_n^{k,k}(e^x) := 1 + \sum_{d=1}^{\infty} \tilde{L}_n^{k,k,d} e^{dx},$$

$$(n = 0, 1, \ldots, k - 1). \quad (3.17)$$

Then we conjectured that $\tilde{L}_n^{k,k}(e^x)$ gives us the information of the B-model of the mirror manifold of $M^k$. More explicitly, we conjectured

$$\tilde{L}_0^{k,k}(e^x) = \sum_{d=0}^{\infty} \frac{(kd)!}{(dl)!} e^{dx}, \quad (3.18)$$

where the r.h.s. of $(3.18)$ is the power series solution of the ODE for the period integral of the mirror manifold of $M^k$,

$$((\frac{d}{dx})^{k-1} - k^d e^x (\frac{d}{dx} + \frac{1}{k}) (\frac{d}{dx} + \frac{2}{k}) \cdots (\frac{d}{dx} + \frac{k-1}{k}))w(x) = 0. \quad (3.19)$$

Moreover we can obtain the mirror map $t = t(x)$ without using the mirror conjecture:

$$t(x) = x + \int_{-\infty}^{x} dt' (\tilde{L}_1^{k,k}(e^{t'}) - 1) = x + \sum_{d=1}^{\infty} \frac{\tilde{L}_1^{k,k,d}}{d} e^{dx}. \quad (3.20)$$

With these conjectures, we can construct the mirror transformation that transforms the virtual structure constants of the Calabi-Yau hypersurface into the real ones as follows:

$$L_n^{k,k}(e^t) = \frac{\tilde{L}_n^{k,k}(e^{t(x)})}{\tilde{L}_1^{k,k}(e^{t(x)})}. \quad (3.21)$$
Here we will discuss whether we can obtain useful informations from the virtual structure constants \( \tilde{L}_{m,k,d} \) in the \( N < k \) region. Note that \( \tilde{L}_{m,k,d} \) is non-zero only if \( 0 \leq m \leq N - 1 + (k - N)d \). So a hypersurface with non-positive first Chern class has infinite number of \( \tilde{L}_{m,k,d} \), though only a finite number of \( L_{n,k,d} \) appear in \( QH^*_\chi(M^k_N) \), \( (N < k) \). In this case, the recursive formulas in Appendix B restricted to \( \tilde{L}_{0,k,d} \) yield the following recursive relations:

\[
\frac{d!}{d^d} \tilde{L}_{0,k,d} = \frac{d!}{d^d} \tilde{L}_{N+1,k,d}.
\]

Then we obtain from (3.18)

\[
1 + \sum_{d=1}^{\infty} \tilde{L}_{0,k,d} e^d = \sum_{d=0}^{\infty} \frac{(kd)!}{(dl)^N} e^d.
\]

The r.h.s. of (3.23) resembles the power series solution of the ODE

\[
((\frac{d}{dx})^{N-1} - k^k e^x (\frac{d}{dx} + \frac{1}{k})(\frac{d}{dx} + \frac{2}{k}) \cdots (\frac{d}{dx} + \frac{k-1}{k})) w(x) = 0,
\]

given by

\[
w(x) = \sum_{d=0}^{\infty} \frac{(kd)!}{(dl)^N} e^d.
\]

These formulas suggest the existence of a mirror manifold of \( M^k_N \), \( (N < k) \). So we proceed further expecting that there exists formulas that transforms the virtual structure constants into the real structure constants like the case of \( M^k_k \). However, these information are less important in the relation with the structure constants of the real quantum cohomology ring \( L_{m,k,d} \) if we reconsider analogy with the case of Calabi-Yau hypersurfaces. In the case of Calabi-Yau hypersurfaces, the information of \( L_{m,k,d} \), \( (m = 1, 2, \cdots, k - 2) \), is used to construct \( L_{m,k,d} \), \( (m = 2, \cdots, k - 3) \), and \( \tilde{L}_{k,k,d} = \tilde{L}_{k-1,k,d} \) are thrown away. That corresponds to the “trivialization of line bundle factor”.

One of the reasons why we need \( \tilde{L}_{1,k,k,d} = \tilde{L}_{k-2,k,d} \) comes from the realization of the “flat metric condition”, or more formally, we need cancellation of the formulas that express \( L_{m,k,d} \) in terms of \( L_{1,k,k,d}, (d' \leq d) \), if we set \( m = 1 \) or \( m = k - 2 \). This condition assures us that the identity operator in \( QH^*_\chi(M^k_k) \) doesn’t receive quantum correction. Similarly, we require the formulas that express \( L_{m,k,d} \), \( (N < k) \), in terms of \( \tilde{L}_{N,k,d} \), \( (d' \leq d) \), to cancel if we formally set \( m = 1 + (k - N)d \) or \( m = N - 2 \). Moreover, we assume that what we need in constructing \( L_{m,k,d} \), which is non-zero only if \( 2 + (k - N)d \leq m \leq N - 3 \), is \( \tilde{L}_{N,k,d} \), \( (d' \leq d, 1 + (k - N) \leq m \leq N - 2) \). In other words, we throw away infinite number of \( \tilde{L}_{N,k,d} \) in the \( N < k \) case except for \( \tilde{L}_{N,k,d} \) \( (1 + (k - N) \leq m \leq N - 2) \). Later on, we call the formulas that express \( L_{m,k,d} \) in terms of \( \tilde{L}_{m,k,d} \), \( (d' \leq d) \), the “generalized mirror transformation”.

We introduce the following notation.

**Definition 3** Let \( V_{N,k,d} \) (resp. \( \hat{V}_{N,k,d} \)) be the vector space of weighted homogeneous polynomial of \( L_{n,k,d} \) \( (d' \leq d) \) (resp. \( \tilde{L}_{n,k,d} \) \( (d' \leq d) \)) of degree \( d \).

Then consider the commutative diagram

\[
\begin{array}{cccccc}
\cdots & \phi_{N+2} & \phi_{N+1} & \phi_N & \phi_{N-1} & \cdots \\
\tilde{V}_{N+2,k,d} & \tilde{V}_{N+1,k,d} & \tilde{V}_{N,k,d} & \tilde{V}_{N-1,k,d} & \phi_{N+2} & \phi_{N+1} \\
\cdots & \phi_{N+2} & \phi_{N+1} & \phi_N & \phi_{N-1} & \cdots \\
\end{array}
\]

The map \( \phi_N \) is the recursive formula between \( L_{N+1,k,d} \) and \( L_{N,k,d} \), which is difficult to calculate for the \( d \geq 3 \) cases (we will construct them for the \( d = 1, 2 \) cases in the next section) , and \( \phi_N \),
which is the recursive formula between $\tilde{L}^{N+1,k,d}_n$ and $\tilde{L}^{N,k,d}_n$, is obvious by definition. We denote by $m_N^k$ the generalized mirror transformation (of course, $m_N^k = \text{id}$ if $N - k \geq 2$). Then if we can determine $m_N^k$, we can construct $\phi_N$ from equation $\phi_N = (m^k_{N+1})^{-1} \circ \tilde{\phi}_N \circ (m^k_N)$. Conversely, we can construct $m_N^k$ if we can construct $\phi_N$. In the next section, we will give some results as some clues for constructing the generalized mirror transformation.

4 Some Preparatory Results

4.1 Recursive Formula for the $d = 1, 2$ cases

In this subsection, we prove a theorem on the structure constants of $QH^*_e(M_N^k)$, $(k > N)$, coming from the quantum correction of the $d = 1, 2$ part. Unfortunately, our method used in the proof of the theorem is effective only in these cases for the general type hypersurfaces. However, the result gives us a hint for the construction of the generalized mirror transformation proposed in section 5.

**Theorem 1** In the case of $k - N \geq 0$, the following formulas expressing $L^{N,k,d}_n$ in terms of $L^{N+1,k,d'}_{n'}$ ($d' \leq d$) for the $d = 1, 2$ cases hold:

$$
L^{N,k,1}_n = L^{N+1,k,1}_n - L^{N+1,k,1}_{1+(k-N)},
$$

$$
L^{N,k,2}_n = \frac{1}{2}(L^{N+1,k,2}_{n-1} + L^{N+1,k,2}_{n-1} + 2L^{N+1,k,1}_{n-1}L^{N+1,k,1}_{n-1} + \frac{1}{2}(L^{N+1,k,2}_{1+(k-N)} + L^{N+1,k,2}_{1+(k-N)} + 2L^{N+1,k,1}_{1+(k-N)}L^{N+1,k,1}_{1+(k-N)}) - 2L^{N+1,k,1}_{1+(k-N)}\sum_{j=0}^{k-N}(L^{N+1,k,1}_{j} - L^{N+1,k,1}_{1+(k-N)-j})).
$$

(4.26)

**proof** We roughly show a sketch of the proof along the line of [3]. Set

$$
[A_{a_1}, A_{a_2}, \cdots, A_{a_n}; N, k, d] := \langle O_{c_1} O_{c_2} \cdots O_{c_n} \rangle_{d,M_N^k}
$$

(4.27)

where $A_{a_j}$ is a linear subspace of codimension $a_j$ in $CP^{N-1}$, and we assume $A_{a_j}$’s are in general position. Let

$$
G[A_{a_1} \cap H, A_{a_1} \cap H, \cdots, A_{a_{k+\ell}} \cap H; N, k, d] = [A_{a_1}, A_{a_2}, \cdots, A_{a_{d+1}}; N, k, d] + R
$$

(4.28)

be the correlation function in special position, where $H$ is fixed hyperplane in $CP^{N-1}$, so the linear subspaces $A_{a_1} \cap H, \cdots, A_{a_{k+\ell}} \cap H$ are not in general position and $\cap_{j=1}^{k}(A_{a_j} \cap H) = (\cap_{j=1}^{k}A_{a_j}) \cap H$. Then $M_N^k$ is embedded in $M_{N+1}^{k+1}$ as $M_{N+1}^{k+1} \cap H$. Note that an irreducible rational curve of degree $d$ intersecting fixed hyperplane $H$ with $d + 1$ points must lie on $H$. Then, we have the equation

$$
G[A_{a_1} \cap H, A_{a_2} \cap H, \cdots, A_{a_{d+1}} \cap H; N + 1, k, d] = [A_{a_1}, A_{a_2}, \cdots, A_{a_{d+1}}; N, k, d] + R
$$

(4.29)

where $R$ consists of the contribution from the reduced curves with one component lying on $M_N^k$ and the other components lying on $M_{N+1}^{k+1}$, and of the contribution from some excess intersection that appear in the $d \geq 3$ case that is hard to determine geometrically. Obviously, $R$ is 0 if $d$ equals 1, and in the case of $d = 2$, the contribution from the reduced curves are given by the formula

$$
R = \frac{1}{k}[A_{N-2-(k-N)}; N + 1, k, 1] \cdot [A_{1+(k-N)}, A_{N-2-n}, A_{N-1-2(k-N)}, A_1; N, k, 1],
$$

(4.30)

The correlation function $G[A_{a_1} \cap H, A_{a_2} \cap H, \cdots, A_{a_{d+1}} \cap H; N + 1, k, d]$ can be evaluated by the specialization formula

$$
G[A_{a_1} \cap H, \cdots, A_{a_n} \cap H, A_{a_{n+1}+1}, A_{a_{n+2}+1}, \cdots, A_{a_{n+\ell}+1}; d, N + 1, k]
$$

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\[ G[A_{a_1} \cap H, \ldots, A_{a_s} \cap H, A_{a_{s+1}} \cap H, A_{a_{s+2}} \cap H, \ldots, A_{a_{s+t+1}}; d, N + 1, k] + \sum_{j=1}^{s} G[A_{a_1} \cap H, \ldots, A_{a_{j-1}} \cap H, A_{a_{j}+a_{s+1}} \cap H, A_{a_{j+1}} \cap H, \ldots, A_{a_{s}} \cap H, A_{a_{s+2}} \cap H, \ldots, A_{a_{s+t+1}}; d, N + 1, k]. \]

We apply the specialization formula to \( G[A_{N-2-n} \cap H, A_{n-1-(k-N)} \cap H; 1, N+1, k] \) and to \( G[A_{N-2-n} \cap H, A_{n-1-2(k-N)} \cap H, A_{1} \cap H; 2, N + 1, k] \), and obtain

\[
\begin{align*}
[A_{N-2-n}, A_{n-1-(k-N)}; 1, N + 1, k] \\
= G[A_{N-2-n} \cap H, A_{n-1-(k-N)} \cap H; 1, N + 1, k] \\
= [A_{N-1-n}, A_{n-1-(k-N)}; 1, N + 1, k] - [A_{N-2-(k-N)}; 1, N + 1, k] \\
[A_{N-2-n}, A_{n-1-2(k-N)}, A_2; 2, N + 1, k] \\
= G[A_{N-2-n} \cap H, A_{n-1-2(k-N)} \cap H, A_1 \cap H; 2, N + 1, k] \\
= [A_{N-1-n}, A_{n-1-2(k-N)}, A_2; 2, N + 1, k] \\
- [A_{N-1-n}, A_{n-2(k-N)}; 2, N + 1, k] - [A_{N-1-n}, A_{n+1-2(k-N)}; 2, N + 1, k] \\
- [A_{N-2-2(k-N)}, A_2; 2, N + 1, k] + 2[A_{N-1-2(k-N)}; 2, N + 1, k] \\
- [A_{N-2-2(k-N)}, A_2; 2, N + 1, k] + [A_{N-1-2(k-N)}; 2, N + 1, k].
\end{align*}
\]

Then using the microscopic version of the associativity equation \([4.29, 4.30]\), we rewrite the Gromov-Witten invariant of \( M_{N+1} \) in terms of \( L_{N}^{N+1,k_1} \) and \( L_{n}^{N+1,k_2} \)

\[
L_{n}^{N,k,1} = L_{n}^{N+1,k,1} - L_{1+(k-N)}^{N+1,k,1},
\]

\[
L_{n}^{N,k,2} + \frac{1}{k} R = \frac{1}{2} (L_{n}^{N+1,k,2} + L_{n}^{N+1,k,2} - L_{1+2(k-N)}^{N+1,k,2} - L_{2(k-N)}^{N+1,k,2})
+ L_{n}^{N+1,k,1}(L_{n}^{N+1,k,1} - L_{1+(k-N)}^{N+1,k,1})
- L_{1+(k-N)}^{N+1,k,1}(\sum_{j=1}^{k-N} (L_{n+j-1-(k-N)}^{N+1,k,1} - L_{n+j}^{N+1,k,1})).
\]

The first line of (4.33) directly leads to the theorem for the \( d = 1 \) case. By the associativity formula and the first line of (4.33), the second line of (4.33) turns out to be the desired formula for the \( d = 2 \) case. Q.E.D.

**Corollary 1** The generalized mirror transformation for \( d = 1, 2 \) is given by:

\[
L_{n}^{N,k,1} = \tilde{L}_{n}^{N,k,1} - \tilde{L}_{1+(k-N)}^{N,k,1},
\]

\[
L_{n}^{N,k,2} = \tilde{L}_{n}^{N,k,2} - \tilde{L}_{1+2(k-N)}^{N,k,2} - 2\tilde{L}_{1+(k-N)}^{N,k,1}(\sum_{j=0}^{k-N} (L_{n-j}^{N,k,1} - \tilde{L}_{1+2(k-N)-j}^{N,k,1})).
\]

### 4.2 Mirror Transformation of Calabi-Yau Hypersurfaces and Schur Polynomials

Here, we show some explicit results on the mirror transformation on the Calabi-Yau hypersurfaces as another clue for constructing the generalized mirror transformation. In the next section, we propose some conjectures on the structure of the generalized mirror transformation on the quantum cohomology of the general type \((N-k < 0)\) hypersurfaces. First, we expand \([4.21]\) into a power series in \( e^{t} \) and obtain the following formulas up to degree 4.

\[
L_{n}^{k,k,1} = \tilde{L}_{n}^{k,k,1} - \tilde{L}_{1}^{k,k,1},
\]

\[
L_{n}^{k,k,2} = \tilde{L}_{n}^{k,k,2} - \tilde{L}_{1}^{k,k,2} - 2\tilde{L}_{1}^{k,k,1}(\tilde{L}_{n}^{k,k,1} - \tilde{L}_{1}^{k,k,1}),
\]

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\[ L_n^{k,k,3} = \tilde{L}_n^{k,k,3} - \tilde{L}_1^{k,k,1}(\tilde{L}_n^{k,k,2} - \tilde{L}_1^{k,k,2}) + \left(\frac{9}{2}(\tilde{L}_1^{k,k,1})^2 - \frac{3}{2}\tilde{L}_1^{k,k,2}\right)(\tilde{L}_n^{k,k,1} - \tilde{L}_1^{k,k,1}), \]
\[ L_n^{k,k,4} = \tilde{L}_n^{k,k,4} - \tilde{L}_1^{k,k,1}(\tilde{L}_n^{k,k,3} - \tilde{L}_1^{k,k,3}) + \left(8(\tilde{L}_1^{k,k,1})^2 - 2\tilde{L}_1^{k,k,2}\right)(\tilde{L}_n^{k,k,2} - \tilde{L}_1^{k,k,2}) \]
\[ + \left(-\frac{32}{3}(\tilde{L}_1^{k,k,1})^3 + 8\tilde{L}_1^{k,k,2}\tilde{L}_1^{k,k,1} - \frac{4}{3}\tilde{L}_1^{k,k,3}\right)(\tilde{L}_n^{k,k,1} - \tilde{L}_1^{k,k,1}). \] (4.35)

Looking at (4.34) and (4.35), we speculate a close resemblance between these two formulas. We can imagine that (4.34) can be obtained from (4.35) by substituting \( \tilde{L}_n^{N,k,d} \) into \( \tilde{L}_1^{k,d} \) and \( \sum_{j=0}^{(d-d')(k-N)}(\tilde{L}_n^{N,k,d'} - \tilde{L}_1^{N,k,d'}) \) into \( \tilde{L}_k^{k,d'} - \tilde{L}_1^{k,d'} \). This speculation is not exactly correct, but not far from the truth. Before going ahead, we introduce a more general result on (3.21).

**Proposition 1** Let \( P_m \) be the set of partitions of \( m \) into positive integers and \( \sigma_m \) be an element of \( P_m \). We also denote the length of a partition \( \sigma_m \) by \( l(\sigma_m) \) (i.e., \( \sigma_m: m = d_1 + d_2 + \cdots + d_l(\sigma_m), \ d_1 \geq d_2 \geq \cdots \geq d_l(\sigma_m) \geq 1 \)). Then the above mirror transformation takes the form
\[ L_n^{k,k,d} = \sum_{m=0}^{d-1} \sum_{\sigma_m \in P_m} (C_{\sigma_m}^d) \cdot \prod_{i=1}^{\ell(\sigma_m)} (\tilde{L}_1^{k,k,d_i}) \cdot (\tilde{L}_n^{k,k,d-m} - \tilde{L}_1^{k,k,d-m}), \] (4.36)
where \( C_{\sigma_m}^d \) is a constant coefficient.

**proof** For brevity, we introduce the notation \( \tilde{L}_n^{k,k}(z) = \sum_{d=0}^{\infty} a_d z^d, \tilde{L}_1^{k,k}(z) = \sum_{d=0}^{\infty} b_d z^d \) and \( 1/(\tilde{L}_1^{k,k}(z)) = \sum_{d=0}^{\infty} c_d z^d \). Then we can easily see the following identity,
\[ \frac{\tilde{L}_n^{k,k}(z)}{\tilde{L}_1^{k,k}(z)} = 1 + \sum_{d=1}^{\infty} \left( \sum_{j=1}^{d} c_{d-j}(a_{j} - b_{j}) \right) z^d. \] (4.37)

Proposition immediately follows from this identity. Q.E.D.

We propose here the following theorem, which suggests a close relation between the mirror transformation for Calabi-Yau hypersurfaces (3.21) and the elementary Schur polynomials associated with the partitions of integers, or Young diagrams.

**Theorem 2** The mirror transformation is explicitly written as
\[ L_n^{k,k,d} = \sum_{m=0}^{d-1} \text{Res}_{z=0}(z^{-m-1} \exp(-d \cdot \sum_{j=1}^{\infty} \tilde{L}_1^{k,k,j}(z^j)) \cdot (\tilde{L}_n^{k,k,d-m} - \tilde{L}_1^{k,k,d-m}), \] (4.38)
The proof will be given in Appendix A.

**Corollary 2**
\[ C_{\sigma_m}^d = (-1)^l(\sigma_m) \frac{d^l(\sigma_m)}{\prod_{j=1}^{l(\sigma_m)} d_j} = (-1)^l(\sigma_m) \frac{d^l(\sigma_m)}{\prod_{j=1}^{l(\sigma_m)} d_j} \cdot N(\sigma_m), \] (4.39)
where \( \text{mul}(i, \sigma_m) \) is the multiplicity of \( i \) in the partition \( \sigma_m \in P_m \) and \( N(\sigma_m) \) is the number of the distinct elements in the orbit of the natural action of symmetric group \( S_l(\sigma_m) \) on the ordered sequence \( (d_1, d_2, \cdots, d_l(\sigma_m)) \).

Note that the multiplicity \( N(\sigma_m) \) of the partition \( \sigma_m \) in section 2 appears in \( C_{\sigma_m}^d \). (4.39) and (4.33) can be regarded as an example of (2.7). More explicitly, we expect the following correspondence:
\[ \text{(the contribution from one pair labeled by} \sigma_m \text{in the resolution diagram of} \mathcal{M}_{d}^{C,P_{N-1}}) \]
\[ = \frac{d^l(\sigma_m)}{(\prod_{j=1}^{l(\sigma_m)} d_j)^l(\sigma_m)} \left( \prod_{j=1}^{l(\sigma_m)} \tilde{L}_1^{N,k,d_j}(\tilde{L}_n^{N,k,d-m} - \tilde{L}_1^{N,k,d-m}) \right). \] (4.40)
We also introduce here a notation on partitions of integers, which is to be used in the next section and in the proof of Theorem 2.

**Definition 4** For two partitions $\sigma_m \in P_m$ and $\sigma_n \in P_n$, we define $\sigma_m \cup \sigma_n \in P_{m+n}$ to be the partition whose parts are those of $\sigma_m$ and $\sigma_n$, arranged in descending order.

## 5 Search for Generalized Mirror Transformation

### 5.1 General Conjecture

Now, we propose a conjecture on the generalized mirror transformation, bearing in mind the speculation in the previous subsection.

**Conjecture 1** The map $m'_{\mathcal{N}}$ is given as

$$L_{N,k,d}^{N,k,d} = \sum_{m=0}^{d-1} \sum_{\sigma_m \in P_m} (C_{m}^{d}(\sigma_m) \cdot \prod_{i=1}^{l(\sigma_m)} (\tilde{L}_{i+1}^{N,k,d} + \tilde{\sigma})) \cdot G_{d-m}^{N,k,d}(n;\sigma_m),$$

where $G_{d-m}^{N,k,d}(n;\sigma_m)$ is a degree $d-m$ weighted homogeneous polynomial of $\tilde{L}_{N,k,d}$.

Here $G_{d-m}^{N,k,d}(n;\sigma_m)$ is yet to be determined. $G_{d-m}^{N,k,d}(n;\sigma_m)$ must satisfy the following conditions.

(i) flat metric condition

$$G_{d-m}^{N,k,d}(1 + (k - N)d;\sigma_m) = G_{d-m}^{N,k,d}(N - 2;\sigma_m) = 0.$$  \hspace{1cm} (5.42)

(ii) symmetry

$$G_{d-m}^{N,k,d}(n;\sigma_m) = G_{d-m}^{N,k,d}(N - 1 + (k - N)d - n;\sigma_m).$$  \hspace{1cm} (5.43)

Moreover, we assume,

(iii) $G_{d-m}^{N,k,d}(n;\sigma_m)$ consists of $\tilde{L}_{i}^{N,k,d}$'s that satisfy the condition,

$$\tilde{L}_{i}^{N,k,d} - (1 \leq i \leq d), \quad (0 \leq j \leq (k - N)(d - d')),$$

$$\tilde{L}_{i}^{N,k,d} - (1 \leq i \leq d), \quad (0 \leq j \leq (k - N)(d - d')).$$  \hspace{1cm} (5.44)

Under these assumptions, we found the following numerical results for some lower $k$'s, using (3.13).

$$L_{5,k-1,k,3}^{k-1,k,3} = \tilde{L}_5^{k-1,k,3} - \tilde{L}_4^{k-1,k,3} - 3\tilde{L}_2^{k-1,k,1}(\tilde{L}_5^{k-1,k,2} - \tilde{L}_3^{k-1,k,2})$$

$$+ \frac{3}{2}\tilde{L}_4^{k-1,k,1} - \tilde{L}_2^{k-1,k,1},$$

$$+ \frac{9}{2}(\tilde{L}_5^{k-1,k,1})^2(\tilde{L}_5^{k-1,k,1} + \tilde{L}_4^{k-1,k,1} - \tilde{L}_3^{k-1,k,1} - \tilde{L}_2^{k-1,k,1}).$$  \hspace{1cm} (5.45)

$$L_{6,k-1,k,3}^{k-1,k,3} = \tilde{L}_6^{k-1,k,3} - \tilde{L}_5^{k-1,k,3} - 3\tilde{L}_2^{k-1,k,1}(\tilde{L}_6^{k-1,k,2} + \tilde{L}_5^{k-1,k,2} - \tilde{L}_4^{k-1,k,2} - \tilde{L}_3^{k-1,k,2})$$

$$+ \frac{3}{2}\tilde{L}_5^{k-1,k,1} - \tilde{L}_4^{k-1,k,1} - \tilde{L}_2^{k-1,k,1},$$

$$+ \frac{9}{2}(\tilde{L}_5^{k-1,k,1})^2(\tilde{L}_5^{k-1,k,1} + \tilde{L}_4^{k-1,k,1} - \tilde{L}_3^{k-1,k,1} - \tilde{L}_2^{k-1,k,1}),$$  \hspace{1cm} (5.46)
Looking at these formulas, we assume three more characteristics of $G_{d-m}^{N,k,d}(n;\sigma_m)$. 

(iv) 

$$G_{d-m}^{N,k,d}(n;0) = \tilde{L}_{n}^{N,k,d} - \tilde{L}_{1+d(k-N)}^{N,k,d}. \quad (5.48)$$

(v) 

$$G_{d-m}^{N,k,d}(2 + (k-N)(d+f);\sigma_m) = G_{d-m}^{N,k,d}(2 + (k-N)(d+f);\sigma_m \cup (f)). \quad (5.49)$$

(vi) 

$$G_{d-m}^{N,k,d}(n;\sigma_m) = \sum_{j=0}^{(k-N)(m)} (\tilde{L}_{n-j}^{N,k,d-m} - \tilde{L}_{1+d(k-N)-j}^{N,k,d-m})$$

+ (homogeneous polynomials of degree $d-m$ that consists of $\tilde{L}_{i}^{N,k,m'}$ ($m' < d-m$)). \quad (5.50)

In view of the resolution diagram of $\mathcal{M}^{(P^{N-1})}_{d}$ in section 2, $G_{d}^{N,k,d}(n;0)$ corresponds to the pair $CP^{N(d+1)-1} \to CP^d \times CP^{N-1}$, that is the element with the largest dimension in the diagram. The condition (v) suggests that $G_{d-m}^{N,k,d}(n;\sigma_m)$ inherits some combinatorial characteristics of the partitions of integers.

### 5.2 Solving the Ansatz (v) and Explicit Formula for the $d = 3$ case

In the first part of this section, we give a recipe (though it’s not complete) of the inductive determination of the factor $G_{d-m}^{N,k,d}(n;\sigma_m)$. We begin with the definition,

**Definition 5**

$$\tilde{G}_{d-m}^{N,k,d+f}(n;\sigma_m \cup (f)) := \sum_{j=0}^{(k-N)f} G_{d-m}^{N,k,d}(n-j;\sigma_m) - \sum_{j=0}^{(k-N)f} G_{d-m}^{N,k,d}(1+(k-N)(d+f) - j;\sigma_m). \quad (5.51)$$

Then we are led to the following proposition,

**Proposition 2** $\tilde{G}_{d-m}^{N,k,d+f}(n;\sigma_m \cup (f))$ satisfy the ansatz (v):

$$\tilde{G}_{d-m}^{N,k,d+f}(2 + (k-N)(d+f);\sigma_m \cup (f)) = G_{d-m}^{N,k,d}(2 + (k-N)(d+f);\sigma_m), \quad (5.52)$$

and the ansatz (i), (ii), and (iii).

**proof** The fact that $\tilde{G}_{d-m}^{N,k,d+f}(n;\sigma_m \cup (f))$ satisfy the ansatz (i), (iii) is rather obvious, and we first prove the ansatz (ii). It suffices to consider the part $\sum_{j=0}^{(k-N)f} G_{d-m}^{N,k,d}(n-j;\sigma_m)$:

$$\sum_{j=0}^{(k-N)f} G_{d-m}^{N,k,d}(N - 1 + (k-N)(d+f) - n - j;\sigma_m)$$
Next, we turn to the formula (5.52). By definition and the condition \( G_{d-m}^{N,k,d}(1+(k-N)d;\sigma_m) = 0 \), we obtain,

\[
\begin{align*}
\tilde{G}_{d-m}^{N,k,d+f}(2+(k-N)(d+f);\sigma_m \cup (f)) &= \sum_{j=0}^{(k-N)f} G_{d-m}^{N,k,d}(2+(k-N)(d+f) - j;\sigma_m) - \sum_{j=0}^{(k-N)f} G_{d-m}^{N,k,d}(1+(k-N)(d+f) - j;\sigma_m) \\
&= G_{d-m}^{N,k,d}(2+(k-N)(d+f);\sigma_m). 
\end{align*}
\]

Q.E.D.

**Definition 6**

\[
C_{d-m}^{N,k,d+f}(n;\sigma_m \cup (f)) := G_{d-m}^{N,k,d+f}(n;\sigma_m \cup (f)) - \tilde{G}_{d-m}^{N,k,d+f}(n;\sigma_m \cup (f))
\]

(5.55)

**Proposition 3** \( C_{d-m}^{N,k,d+f}(n;\sigma_m \cup (f)) \) satisfies the condition:

\[
C_{d-m}^{N,k,d+f}(2+(k-N)(d+f);\sigma_m \cup (f)) = C_{d-m}^{N,k,d+f}(1+(k-N)(d+f);\sigma_m \cup (f)) = 0,
\]

and the ansatz (i), (ii), (iii).

**Proof** Immediate. Q.E.D.

We can easily see that \( C_{d-m}^{N,k,d+f}(n;\sigma_m \cup (f)) \) consists of monomials of degree \( d-m \) only in \( \tilde{L}_j^{N,k,d'} \) \((m' < d-m)\), and we are led to the corollary:

**Corollary 3**

\[
G_{d-m}^{N,k,d}(n;\sigma_m) = \sum_{j_1=0}^{d_1(k-N)} \sum_{j_2=0}^{d_2(k-N)} \cdots \sum_{j_{l(\sigma_m)}=0}^{d_{l(\sigma_m)}(k-N)} \tilde{L}_j^{N,k,d-m} - \sum_{j_1=0}^{d_1(k-N)} \sum_{j_2=0}^{d_2(k-N)} \cdots \sum_{j_{l(\sigma_m)}=0}^{d_{l(\sigma_m)}(k-N)} \tilde{L}_j^{N,k,d-m} \\
+ \text{homogeneous polynomials of degree } d-m \text{ that consist of } \tilde{L}_j^{N,k,d'} \text{ (} m' < d-m \text{)}.
\]

(5.57)

\( C_{d-1}^{N,k,d}(n;\sigma_{d-1} \cup (f)) \) must be zero just because there are no positive integers less than 1. Hence we obtain,

**Corollary 4**

\[
G_{d-1}^{N,k,d}(n;\sigma_{d-1}) = \sum_{j_1=0}^{d_1(k-N)} \sum_{j_2=0}^{d_2(k-N)} \cdots \sum_{j_{l(\sigma_{d-1})}=0}^{d_{l(\sigma_{d-1})}(k-N)} \tilde{L}_j^{N,k,1} - \sum_{j_1=0}^{d_1(k-N)} \sum_{j_2=0}^{d_2(k-N)} \cdots \sum_{j_{l(\sigma_{d-1})}=0}^{d_{l(\sigma_{d-1})}(k-N)} \tilde{L}_j^{N,k,1}. 
\]

(5.58)
We further impose the symmetry condition and the condition \( C_{d-1,m} \) and the partition \( \sigma_m \). After all, what remains to be determined is the “hidden intersections” \( C_{d-1,m}^{d+f}(n; \sigma_m \cup (f)) \) that appear in the inductive process. We would like to give the general treatment of them in the future work, but in the \( d = 3 \) case, we managed to determine them using the condition (5.56) and the numerical results as will be discussed in the latter half of this section.

Now, we apply the discussion so far to the case of \( d = 3 \). In this case, we have to determine the following four factors:

\[
G_3^{N,k,3}(n; (0)), \quad G_2^{N,k,3}(n; (1))
\]
\[
G_1^{N,k,3}(n; (1) + (1)), \quad G_1^{N,k,3}(n; (2))
\]

(5.59)

Straightforward application of the results so far leads us to the following conjecture.

**Conjecture 2** The mirror transformation for \( d = 3 \) is given by

\[
L_n^{N,k,3} = \tilde{L}_n^{N,k,3} - \tilde{L}_n^{N,k,3} - 3\tilde{L}_n^{N,k,1} \left( \sum_{j=0}^{k-N} (\tilde{L}_{n-j}^{N,k,2} - \tilde{L}_{n+j}^{N,k,2}) \right) + C_{1,1}^{N,k,3}(n)
\]

\[
= \frac{3}{2} \tilde{L}_n^{N,k,2} \left( \sum_{j=0}^{2(k-N)} (\tilde{L}_{n-j}^{N,k,1} - \tilde{L}_{n+j}^{N,k,1}) \right)
\]

\[
+ \frac{9}{2} (\tilde{L}_n^{N,k,1})^2 \left( \sum_{j=0}^{2(k-N)} A_j (\tilde{L}_{n-j}^{N,k,1} - \tilde{L}_{n+j}^{N,k,1}) \right),
\]

(5.60)

where

\[
A_j := j + 1, \text{ if } (0 \leq j < k - N), \quad A_j := 1 + 2(k - N) - j, \text{ if } (k - N \leq j \leq 2(k - N)).
\]

(5.61)

and \( C_{1,1}^{N,k,3}(n) \) is some degree 2 polynomial of \( \tilde{L}_{n-j}^{N,k,1} \), \( 0 \leq j \leq 2(k - N) \), and of \( \tilde{L}_{1+3(k-N)-j}^{N,k,1} \), \( 0 \leq j \leq 2(k - N) \), that satisfy

\[
C_{1,1}^{N,k,3}(n) = C_{1,1}^{N,k,3}(N - 1 + 3(k - N) - n),
\]
\[
C_{1,1}^{N,k,3}(1 + 3(k - N)) = C_{1,1}^{N,k,3}(2 + 3(k - N)) = 0.
\]

(5.62)

As we mentioned before, the hidden intersection \( C_{1,1}^{N,k,3}(n) \) is yet to be determined. For simplicity, we discuss the case of \( k - N = 1 \) in detail.

By the flat metric condition and the condition (iii), \( C_{1,1}^{k-1,k,3}(n) \) must have the form

\[
\sum_{i=0}^{2} \sum_{j=0}^{2} A_{ij} \tilde{L}_{n-i}^{k-1,k,1} \tilde{L}_{n-j}^{k-1,k,1} + \sum_{i=0}^{2} \sum_{j=0}^{2} B_{ij} \tilde{L}_{n-i}^{k-1,k,1} \tilde{L}_{n+j}^{k-1,k,1} - \left( \sum_{i=0}^{2} \sum_{j=0}^{2} A_{ij} \tilde{L}_{n-i}^{k-1,k,1} \tilde{L}_{n-j}^{k-1,k,1} + \sum_{i=0}^{2} \sum_{j=0}^{2} B_{ij} \tilde{L}_{n-i}^{k-1,k,1} \tilde{L}_{n-j}^{k-1,k,1} \right).
\]

(5.63)

We further impose the symmetry condition and the condition \( C_{1,1}^{k-1,k,3}(5) = 0 \). Then we obtain

\[
C_{1,1}^{k-1,k,3}(n) = C_1(\tilde{L}_{n-1}^{k-1,k,1} \tilde{L}_{n-2}^{k-1,k,1} - \tilde{L}_{3}^{k-1,k,1} (\tilde{L}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-1}^{k-1,k,1} \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{3}^{k-1,k,1} \tilde{L}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} \tilde{L}_{n-1}^{k-1,k,1}))
\]

\[
+ C_2(\tilde{L}_{n-1}^{k-1,k,1} \tilde{L}_{n-2}^{k-1,k,1} - \tilde{L}_{3}^{k-1,k,1} (\tilde{L}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{3}^{k-1,k,1} \tilde{L}_{n-2}^{k-1,k,1})
\]

\[
+ \tilde{L}_{3}^{k-1,k,1} \tilde{L}_{n-1}^{k-1,k,1} - \tilde{L}_{3}^{k-1,k,1} \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{3}^{k-1,k,1} \tilde{L}_{n-2}^{k-1,k,1} \tilde{L}_{n-1}^{k-1,k,1} \tilde{L}_{n-2}^{k-1,k,1})
\]

(5.64)
If we compare (5.64) with (5.46), we can see $C_1 = 1, C_2 = 0$. Similarly, we determined $C_{1,1}^{N,k,3}(n)$ for the $k - N = 1, 2$ case using numerical results for lower $N$ and $k$.

$$C_{1,1}^{k-1,k,3}(n) = \tilde{I}_n^{k-1,k,1} \tilde{L}_n^{k-1,k,1} - \tilde{L}_n^{k-1,k,1}(\tilde{I}_n^{k-1,k,1} + \tilde{L}_n^{k-1,k,1}) + \tilde{L}_n^{k-1,k,1}(\tilde{I}_n^{k-1,k,1})$$

$$C_{1,1}^{k-2,k,3}(n) = \tilde{I}_n^{k-2,k,1} \tilde{L}_n^{k-2,k,1} - \tilde{L}_n^{k-2,k,1}(\tilde{I}_n^{k-2,k,1} + \tilde{L}_n^{k-2,k,1}) + \tilde{L}_n^{k-2,k,1}(\tilde{I}_n^{k-2,k,1})$$

From (5.65), we can speculate the form of $C_{1,1}^{N,k,3}(n)$ as

$$C_{1,1}^{N,k,3}(n) = \sum_{j=0}^{(k-N)-1} \left( \sum_{m=0}^{(k-N)-j-1} \tilde{I}_n^{N,k,1} \tilde{L}_n^{N,k,1} \right) - \tilde{I}_n^{N,k,1}(\sum_{m=0}^{2(k-N)-j-1} \tilde{I}_n^{N,k,1}) + \tilde{L}_n^{N,k,1}(\sum_{m=0}^{2(k-N)-j-1} \tilde{I}_n^{N,k,1})$$

We want to show that (5.64) and (5.66) correctly predicts $I_{N,k}^{k,3}$ for all $N$ and $k$. We did numerical check for some lower $N$ and $k$ by the method of toroidal calculation considered by Kontsevich [12] and no contradiction occurred.

5.3 Partial Results for the $d = 4$ case

In the $d = 4$ case, we can partially determine the form of the generalized mirror transformation, using our results so far, as follows:

$$L_n^{N,k,4} = \tilde{I}_n^{N,k,4} - \tilde{L}_n^{N,k,4} \tilde{I}_n^{1+4(k-N)}$$

$$-4 \tilde{L}_n^{N,k,4}(\sum_{j=0}^{(k-N)-1} (\tilde{I}_n^{N,k,3} - \tilde{L}_n^{N,k,3} \tilde{I}_n^{1+4(k-N)-j}) + C_{2,1}^{N,k,4}(n) + C_{1,1}^{N,k,4}(n)$$

$$-2 \tilde{L}_n^{N,k,2}(\sum_{j=0}^{2(k-N)-1} (\tilde{I}_n^{N,k,2} - \tilde{L}_n^{N,k,2} \tilde{I}_n^{1+2(k-N)-j}) + C_{1,1}^{N,k,4}(n)$$

$$-4 \tilde{I}_n^{N,k,3}(\sum_{j=0}^{3(k-N)-1} (\tilde{I}_n^{N,k,1} - \tilde{L}_n^{N,k,1} \tilde{I}_n^{1+4(k-N)-j}))$$
If we set
\[ A_{j} = \sum_{j=0}^{k-N} (E_{1,1}^{N,k,3}(n-j) - E_{1,1}^{N,k,3}(1 + 4(k-N) - j)) + D_{1,1}^{N,k,4}(n) \]

then
\[ +8 \bar{L}_{1+2(k-N)}^{N,k,2} \sum_{j=0}^{3(k-N)} B_{j}(\bar{L}_{n-j}^{N,k,1} - \bar{L}_{1+4(k-N)-j}^{N,k,1}) \]

\[ -\frac{32}{3} \bar{L}_{1+(k-N)}^{N,k,1} \sum_{j=0}^{3(k-N)} C_{j}(\bar{L}_{n-j}^{N,k,1} - \bar{L}_{1+4(k-N)-j}^{N,k,1}), \]  

where \( A_{j} \) is the same as defined in Conjecture 2 and

\[ B_{j} = \begin{cases} 
0 & (0 \leq j \leq k-N), \\
n & (k-N \leq j \leq 2(k-N)), \\
3(k-N) + 1 - j & (2(k-N) \leq j \leq 3(k-N)), \\
\end{cases} \]

\[ C_{j} = \begin{cases} 
(j+1)(j+2) & (0 \leq j \leq k-N), \\
-j^{2} + 3(k-N)j - \frac{3}{2}(k-N)(k-N-1) + 1 & (k-N \leq j \leq 2(k-N)), \\
(3(k-N) - j + 1)(3(k-N) - j + 2) & (2(k-N) \leq j \leq 3(k-N)), \\
\end{cases} \]

and

\[ C_{2,1}^{N,k,4}(n), C_{1,1,1}^{N,k,4}(n), C_{1,1}^{N,k,4}(n) \] and \( D_{1,1}^{N,k,4}(n) \) are “hidden intersections” and are some weighted homogeneous polynomials that respectively consist of \( \bar{L}_{i}^{N,k,2} \bar{L}_{j}^{N,k,1}, \bar{L}_{j}^{N,k,1} \bar{L}_{i}^{N,k,1}, \bar{L}_{i}^{N,k,1} \bar{L}_{j}^{N,k,1} \) and \( \bar{L}_{i}^{N,k,1} \bar{L}_{j}^{N,k,1} \), which satisfy

\[ C_{2,1}^{N,k,4}(1 + 4(k-N)) = C_{1,1,1}^{N,k,4}(1 + 4(k-N)) = C_{1,1}^{N,k,4}(1 + 4(k-N)) \]

\[ = C_{2,1}^{N,k,4}(2 + 4(k-N)) = C_{1,1,1}^{N,k,4}(2 + 4(k-N)) = C_{1,1}^{N,k,4}(2 + 4(k-N)) \]

\[ = D_{1,1}^{N,k,4}(1 + 4(k-N)) = D_{1,1,1}^{N,k,4}(2 + 4(k-N)) = 0, \]

\[ C_{2,1}^{N,k,4}(n) = C_{1,1,1}^{N,k,4}(N - 1 + 4(k-N) - n), \]

\[ C_{1,1,1}^{N,k,4}(n) = C_{1,1}^{N,k,4}(N - 1 + 4(k-N) - n), \]

\[ C_{1,1}^{N,k,4}(n) = C_{1,1}^{N,k,4}(N - 1 + 4(k-N) - n), \]

\[ D_{1,1}^{N,k,4}(n) = D_{1,1,1}^{N,k,4}(N - 1 + 4(k-N) - n). \]  

(5.69)

If we set \( n = 2 + 4(k-N) \), the formula \([5.67]\) includes no ambiguous part and we obtain the following proposition.

**Proposition 4** In the case of \( n = 2 + 4(k-N) \), the following formula predicts \( \bar{L}_{2+4(k-N)}^{N,k,4} \) for arbitrary \( N \) and \( k \).

\[ \bar{L}_{2+4(k-N)}^{N,k,4} = \bar{L}_{2+4(k-N)}^{N,k,4} - \bar{L}_{1+4(k-N)}^{N,k,4} \]  

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where

\[ A \]

Multiplication Rule

\[
\begin{align*}
\mathcal{O}_e \cdot 1 &= \mathcal{O}_e, \\
\mathcal{O}_e \cdot \mathcal{O}_c &= \mathcal{O}_{e2} + A\mathcal{O}_{e3}q, \\
\mathcal{O}_e \cdot \mathcal{O}_{e2} &= \mathcal{O}_{e3}, \\
\mathcal{O}_e \cdot \mathcal{O}_{e3} &= \mathcal{O}_e, \\
\mathcal{O}_e \cdot \mathcal{O}_{e4} &= 0.
\end{align*}
\]

Remark 1

From numerical computations for lower \( N \) and \( k \), we determined \( C_{1,1}^{N,k,4}(n) \) using the fact that \(-4\tilde{L}_{1+(k-N)}(C_{1,1}^{N,k,4}(n) + C_{1,1}^{N,k,4}(n)) + 8(\tilde{L}_{1+(k-N)} C_{2,1}^{N,k,4}(n)) \) is divisible by \( \tilde{L}_{1+(k-N)} \).

\[
C_{1,1}^{N,k,4}(n) = \sum_{j=0}^{2(k-N)-1} \left( \sum_{m=0}^{j} \tilde{L}_{n-m}^{N,k,1} \tilde{L}_{n-3(k-N)+j-m}^{N,k,1} - \tilde{L}_{(k-N)+2+j}^{N,k,1} \left( \sum_{m=0}^{3(k-N)} \tilde{L}_{n-m}^{N,k,1} \right) \right)
\]

\[
- \sum_{j=0}^{2(k-N)-1} \left( \sum_{m=0}^{j} \tilde{L}_{1+(k-N)}^{N,k,1} \tilde{L}_{1+(k-N)+j-m}^{N,k,1} - \tilde{L}_{(k-N)+2+j}^{N,k,1} \left( \sum_{m=0}^{3(k-N)} \tilde{L}_{1+(k-N)-m}^{N,k,1} \right) \right)
\]

\[
+ \tilde{L}_{1+(k-N)}^{N,k,1} \left( \sum_{j=0}^{k-N-1} \sum_{m=j+1}^{3(k-N)-j-1} \left( \tilde{L}_{n-m}^{N,k,1} - \tilde{L}_{1+(k-N)-m}^{N,k,1} \right) \right).
\]

This formula satisfies the condition \( C_{1,1}^{N,k,4}(2 + 5(k - N)) = C_{1,1}^{N,k,4}(2 + 5(k - N)) \), that is imposed by the ansatz (v).

6 Some Examples

In this section, we show some explicit examples of the quantum cohomology of general type hypersurfaces and the generalized mirror transformation. We put the real structure constants on the l.h.s. of the generalized mirror transformation. On the r.h.s. of the equalities, we put the virtual structure constants obtained by applying the recursive formulas in Appendix B with the initial conditions listed in Table 1. Some other examples of the generalized mirror transformation are listed in Table 2 and Table 3.

\[ M^4_{1,1} \] model

Multiplication Rule

\[
99715 = 300167 - 200452
\]
Generator Representation

\[
\begin{align*}
\mathcal{O}_e &= \mathcal{O}_e, \\
\mathcal{O}_{e^2} &= \frac{(\mathcal{O}_e)^2}{1 + A\mathcal{O}_e q}, \\
\mathcal{O}_{e^3} &= \frac{(\mathcal{O}_e)^3}{1 + A\mathcal{O}_e q}, \\
\mathcal{O}_{e^4} &= \frac{(\mathcal{O}_e)^4}{1 + A\mathcal{O}_e q}.
\end{align*}
\] (6.74)

Relation

\[
\frac{(\mathcal{O}_e)^5}{1 + A\mathcal{O}_e q} = 0.
\] (6.75)

\textit{M}_8^8 \text{ model}

Multiplication Rule

\[
\begin{align*}
\mathcal{O}_e \cdot 1 &= \mathcal{O}_e, \\
\mathcal{O}_e \cdot \mathcal{O}_e &= \mathcal{O}_{e^2} + A\mathcal{O}_e q + B\mathcal{O}_e q^2, \\
\mathcal{O}_e \cdot \mathcal{O}_{e^2} &= \mathcal{O}_{e^3} + A\mathcal{O}_e q, \\
\mathcal{O}_e \cdot \mathcal{O}_{e^3} &= \mathcal{O}_{e^4}, \\
\mathcal{O}_e \cdot \mathcal{O}_{e^4} &= \mathcal{O}_{e^5}, \\
\mathcal{O}_e \cdot \mathcal{O}_{e^5} &= 0.
\end{align*}
\] (6.76)

where \( A = 2689792, B = 21553860841856. \)

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\[
\begin{align*}
2689792 &= 5241984 - 2552192, \\
21553860841856 &= 9117348674880 - 55889894658816 \\
&\quad - 2 \times 2552192 \times (5241984 + 5241984 - 5241984 - 2552192). \quad (6.77)
\end{align*}
\]

Generator Representation

\[
\begin{align*}
\mathcal{O}_e &= \mathcal{O}_e, \\
\mathcal{O}_{e^2} &= \frac{(1 + A\mathcal{O}_e q)(\mathcal{O}_e)^2}{1 + 2A\mathcal{O}_e q + B(\mathcal{O}_e)^2 q^2}, \\
\mathcal{O}_{e^3} &= \frac{(\mathcal{O}_e)^3}{1 + 2A\mathcal{O}_e q + B(\mathcal{O}_e)^2 q^2}, \\
\mathcal{O}_{e^4} &= \frac{(\mathcal{O}_e)^4}{1 + 2A\mathcal{O}_e q + B(\mathcal{O}_e)^2 q^2}, \\
\mathcal{O}_{e^5} &= \frac{(\mathcal{O}_e)^5}{1 + 2A\mathcal{O}_e q + B(\mathcal{O}_e)^2 q^2}.
\end{align*}
\] (6.78)

Relation

\[
\frac{(\mathcal{O}_e)^6}{1 + 2A\mathcal{O}_e q + B(\mathcal{O}_e)^2 q^2} = 0.
\] (6.79)

\textit{M}_8^8 \text{ model}

Multiplication Rule

\[
\begin{align*}
\mathcal{O}_e \cdot 1 &= \mathcal{O}_e,
\end{align*}
\]
\[ O_c \cdot O_c = O_{c^2} + AO_{c^3}q + BO_{c^3}q^2 + CO_{c^3}q^3, \]
\[ O_c \cdot O_{c^2} = O_{c^3} + DO_{c^3}q + BO_{c^3}q^2, \]
\[ O_c \cdot O_{c^3} = O_{c^4} + AO_{c^3}q, \]
\[ O_c \cdot O_{c^4} = O_{c^5}, \]
\[ O_c \cdot O_{c^5} = O_{c^6}, \]
\[ O_c \cdot O_{c^6} = 0. \quad (6.80) \]

where \( A = 56718144, B = 35512880615374365/2, C = 403755597553238694522553, D = 90617373. \)

**Generalized Mirror Transformation**

\[
\begin{align*}
56718144 &= 90857052 - 34138908, \\
90617373 &= 124756281 - 34138908, \\
35512880615374365/2 &= 81506931029963973/2 - 16809868887197436 \\
&\quad - 2 \cdot 34138908 \cdot (124756281 + 90857052 - 90857052 - 34138908), \\
403755597553238694522553 &= 1889246549939149055742585 - 11447799161101387518646386 \\
&\quad - 3 \cdot 34138908 \cdot (81506931029963973/2 - 16809868887197436) \\
&\quad - (3/2) \cdot 16809868887197436 \cdot (90857052 - 34138908) \\
&\quad + (9/2) \cdot 34138908 \cdot 34138908 \cdot (124756281 - 34138908). \quad (6.81)
\end{align*}
\]

**Generator Representation**

\[
\begin{align*}
O_c &= O_c, \\
O_{c^2} &= \frac{(1 + (A + D)O_{c}q + B(\mathcal{O}_c)^2q^2)(\mathcal{O}_c)}{1 + (2A + D)\mathcal{O}_cq + (2B + A^2)(\mathcal{O}_c)^2q^2 + C(\mathcal{O}_c)^3q^4}, \\
O_{c^3} &= \frac{1 + (2A + D)\mathcal{O}_cq + (2B + A^2)(\mathcal{O}_c)^2q^2 + C(\mathcal{O}_c)^3q^4}{(\mathcal{O}_c)^4}, \\
O_{c^4} &= \frac{1 + (2A + D)\mathcal{O}_cq + (2B + A^2)(\mathcal{O}_c)^2q^2 + C(\mathcal{O}_c)^3q^4}{(\mathcal{O}_c)^5}, \\
O_{c^5} &= \frac{1 + (2A + D)\mathcal{O}_cq + (2B + A^2)(\mathcal{O}_c)^2q^2 + C(\mathcal{O}_c)^3q^4}{(\mathcal{O}_c)^6}, \\
O_{c^6} &= \frac{1 + (2A + D)\mathcal{O}_cq + (2B + A^2)(\mathcal{O}_c)^2q^2 + C(\mathcal{O}_c)^3q^4}{(\mathcal{O}_c)^7}. \quad (6.82)
\end{align*}
\]

**Relation**

\[
\frac{(\mathcal{O}_c)^7}{1 + (2A + D)\mathcal{O}_cq + (2B + A^2)(\mathcal{O}_c)^2q^2 + C(\mathcal{O}_c)^3q^4} = 0. \quad (6.83)
\]

**7 Conclusion**

In this paper, we have discussed a generalization of the mirror calculation of Calabi-Yau hypersurfaces to the case of general type hypersurfaces. However, our determination of the factor \( \mathcal{O}_c(N,k,d)(n;\sigma_m) \) relies on numerical results and depends on power of computers. That’s why we stopped up to the partial results for the \( d = 4 \) rational curves. So we have to search for other characteristics of \( \mathcal{O}_c(N,k,d)(n;\sigma_m) \) that is enough for determination of them. Geometrical derivation of our formula from the point of view of the projective space resolution in section 2 should be pursued further. Heuristically, we can expect that the pair

\[
CP^{d_{*}(1)} \times \ldots \times CP^{d_{*}((\sigma_m))} \times CP^{N(d-m+1)-1} \rightarrow CP^{d_{*}(1)} \times \ldots \times CP^{d_{*}((\sigma_m))} \times CP^{d-m} \times CP^{N-1},
\]

\[
(7.84)
\]
(\pi \in S_\ell(\sigma_m))\), in the resolution diagram corresponds to

\[
\frac{d_\ell(\sigma_m)}{(\prod_{j=1}^{l(\sigma_m)} d_j)! l(\sigma_m)!} \prod_{j=1}^{l(\sigma_m)} \tilde{L}_{1+\ell (k-N)d_j}^{N,k,d_j} G_{d-m}^{N,k,d}(n;\sigma_m)
\]  \hspace{1cm} \text{(7.85)}

in the generalized mirror transformation. Moreover it seems that \(CP^d\) and the pair \(CP^{N(d-m+1)-1} \to CP^{d-m} \times CP^{N-1}\) produce \(\tilde{L}_{1+\ell (k-N)d_j}^{N,k,d_j}\) and \(G_{d-m}^{N,k,d}(n;\sigma_m)\) respectively. Geometrical derivation of the factor \(\frac{d_\ell(\sigma_m)}{(\prod_{j=1}^{l(\sigma_m)} d_j)! l(\sigma_m)!}\) should also be considered deeply. These approaches will reveal the interaction between the geometrical structure of the moduli space of the holomorphic maps from \(CP^1\) to the Kähler manifold of general type and the structure of the oscillator algebra of the free bosons (or the ring of the symmetric functions). We hope our approach will contribute to deeper understanding of the mirror phenomena, which is now generalized to various types of manifolds, not only in the case of Calabi-Yau manifolds.

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Appendix A: Proof of Theorem 2

In this section, we give the proof of Theorem 2. Here we keep the notation used in the proof of Proposition 1.

**Lemma 1** For any formal power series \( f(z) = \sum_{j=1}^{\infty} u_j z^j \), the following formal power series identity holds.

\[
\sum_{d=1}^{\infty} \frac{1}{d} \text{Res}_{z=0}(z^{-d-1} \exp(-df(z))) \cdot z^d \exp(df(z)) = -f(z)
\]  

\[(7.86)\]

**proof** First, let \( S(\sigma_d) \) be defined by

\[
\exp(\sum_{j=1}^{\infty} u_j z^j) = \sum_{d=0}^{\infty} (\sum_{\sigma_d \in P_d} \prod_{j=1}^{l(\sigma_d)} u_{d_j}) z^d.
\]  

\[(7.87)\]

Then the above identity is equivalent to the combinatorial relations

\[
\sum_{f=1}^{d} \sum_{\sigma_f \in P_f, \sigma_d-f} (-1)^{l(\sigma_f)} f^{l(\sigma_d)-1} S(\sigma_f) S(\sigma_d-f) = -1, \quad (l(\sigma_d) = 1),
\]  

\[(7.88)\]

They are further unified into the relation,

\[
\sum_{f=0}^{d} \sum_{\sigma_f \in P_f, \sigma_d-f} (-1)^{l(\sigma_f)} f^{l(\sigma_d)-1} S(\sigma_f) S(\sigma_d-f) = 0, \quad (\text{otherwise}).
\]  

\[(7.89)\]

Hence it suffices to show

\[
\sum_{f=0}^{d} \sum_{\sigma_f \in P_f, \sigma_d-f} (-1)^{l(\sigma_f)} f^m S(\sigma_f) S(\sigma_d-f) = 0, \quad (m \leq l(\sigma_d) - 1).
\]  

\[(7.90)\]

The l.h.s. of \[(7.90)\] is the coefficient of \((\prod_{j=1}^{l(\sigma_d)} u_{d_j}) z^d\) in

\[
((z z^d) \frac{d}{dz})^m \exp(-\sum_{j=1}^{\infty} u_j z^j)) \cdot \exp(\sum_{j=1}^{\infty} u_j z^j)).
\]  

\[(7.91)\]

But \[(7.91)\] obviously contains no monomial \((\prod_{j=1}^{l(\sigma_d)} u_{d_j}) z^d\) with length \(l(\sigma_d) \geq m+1\) and this proves \[(7.90)\]. Q.E.D

From Lemma 1, we can explicitly construct the mirror map,

\[
x(t) = t + \sum_{d=1}^{\infty} \frac{u_d}{d} e^{dt};
\]  

\[
u_d = \text{Res}_{z=0}(z^{-d-1} \exp(-d(\sum_{j=1}^{\infty} \frac{b_j}{j} z^j))));
\]  

\[(7.92)\]
and we also have
\[ b_d = \text{Res}_{z=0}(z^{-d-1} \exp(-d(\sum_{j=1}^{\infty} \frac{t_j}{j}))). \] (7.93)

**Lemma 2**
\[ e^{dx(t)} = e^{dt}(1 + d(\sum_{f=1}^{\infty} e^{ft} \sum_{\sigma_f \in P_f} (-1)^{l(\sigma_f)}(d + f)^{l(\sigma_f)} - 1) S(\sigma_f) \prod_{j=1}^{l(\sigma_f)} \frac{b_d}{d_j}). \] (7.94)

**proof**

We prove more generally that the identity
\[ e^{\alpha x(t)} = e^{\alpha t}(1 + \alpha(\sum_{f=1}^{\infty} e^{ft} \sum_{\sigma_f \in P_f} (-1)^{l(\sigma_f)}(\alpha + f)^{l(\sigma_f)} - 1) S(\sigma_f) \prod_{j=1}^{l(\sigma_f)} \frac{b_d}{d_j}). \] (7.95)
holds for arbitrary \( \alpha \in C \). At first, we rewrite (7.95) into the form,
\[ \frac{1}{\alpha}(\exp(\alpha(x(t) - t)) - 1) = \sum_{f=1}^{\infty} e^{ft} \sum_{\sigma_f \in P_f} (-1)^{l(\sigma_f)}(\alpha + f)^{l(\sigma_f)} - 1) S(\sigma_f) \prod_{j=1}^{l(\sigma_f)} \frac{b_d}{d_j}. \] (7.96)

Then using the fact that \( t = t(x) \) and \( x(t(x)) = x \), we deduce from (7.96),
\[ \frac{1}{\alpha}(\exp(\alpha(x(t(x)) - t(x))) - 1) = \sum_{f=1}^{\infty} e^{fx} \exp(f(\sum_{j=1}^{\infty} \frac{b_j}{j} e^{jx})) \sum_{\sigma_f \in P_f} (-1)^{l(\sigma_f)}(\alpha + f)^{l(\sigma_f)} - 1) S(\sigma_f) \prod_{j=1}^{l(\sigma_f)} \frac{b_d}{d_j}. \] (7.97)

By expanding \( \exp(*) \) in (7.97), we obtain the following combinatorial relation among \( S(\sigma_m) \) that is equivalent to the statement of the lemma:
\[ (-1)^{l(\sigma_d)}\alpha^{l(\sigma_d)} - 1) S(\sigma_d) = \sum_{f=0}^{d} \sum_{\sigma_f \in P_f, \sigma_d-f \in P_{d-f}} (-1)^{l(\sigma_f)} f^{l(\sigma_d-f)}(\alpha + f)^{l(\sigma_f)} - 1) S(\sigma_f) S(\sigma_{d-f}) = \sum_{j=0}^{l(\sigma_d) - 1} \alpha^j \sum_{f=0}^{d} \sum_{\sigma_f \in P_f, \sigma_d-f \in P_{d-f}} (-1)^{l(\sigma_f)} f^{l(\sigma_d-f) - 1-j} l(\sigma_f) - 1) C_j S(\sigma_f) S(\sigma_{d-f}). \] (7.98)

So we have only to prove,
\[ \sum_{f=0}^{d} \sum_{\sigma_f \in P_f, \sigma_d-f \in P_{d-f}} (-1)^{l(\sigma_f)} l(\sigma_f) - 1) C(l(\sigma_d) - 1) S(\sigma_f) S(\sigma_{d-f}) = (-1)^{l(\sigma_d)} S(\sigma_d) \] (7.99)
and
\[ \frac{1}{j!} \sum_{f=0}^{d} \sum_{\sigma_f \in P_f, \sigma_d-f \in P_{d-f}} (-1)^{l(\sigma_f)} f^{l(\sigma_d) - 1-j} \prod_{k=1}^{j} (l(\sigma_f) - k) S(\sigma_f) S(\sigma_{d-f}) = 0, \] (7.100)

(0 \leq j \leq l(\sigma_d) - 2).
As we did in proving Lemma 1, we can unify (7.99) and (7.100) into a general statement:

\[
\sum_{\sigma_f \in \mathcal{P}_f} (-1)^{(\sigma_f)} \sum_{\sigma_{d-f} \in \mathcal{P}_{d-f}} (-1)^{(\sigma_{d-f})} (m \prod_{k=1}^{j} (l(\sigma_f) - k)) S(\sigma_f) S(\sigma_{d-f}) = 0,
\]

\[
0 \leq m + j \leq l(\sigma_d) - 1.
\]

The l.h.s. of (7.101) is coefficient of \( (\prod_{l=1}^{l(\sigma_d)} s_d^l) z^d \) of generating function,

\[
((z \frac{dz}{dz})^m (\frac{dz}{de})^j (\frac{1}{e} \exp(-\epsilon(\sum_{j=1}^{\infty} s_j z^j)))|_{\epsilon=1} \exp(\sum_{j=1}^{\infty} s_j z^j),
\]

(7.102)

but this does not contain monomials with length \( l(\sigma_d) \geq m + j + 1 \). So (7.101) holds. Q.E.D.

Using Lemma 2, we will now prove Theorem 2. From (4.37) and (7.94), we can see that Theorem 2 is equivalent to the combinatorial relation

\[
c_{d-j} + \sum_{g=j}^{d-1} g \cdot c_{g-j} \left( \sum_{\sigma_{d-g} \in \mathcal{P}_{d-g}} (-1)^{(\sigma_{d-g})} d^{l(\sigma_{d-g})-1} S(\sigma_{d-g}) \prod_{i=1}^{l(\sigma_{d-g})} \frac{b_{d-i}}{d_i} \right)
\]

\[
= \sum_{\sigma_{d-j} \in \mathcal{P}_{d-j}} (-1)^{(\sigma_{d-j})} d^{l(\sigma_{d-j})} S(\sigma_{d-j}) \prod_{i=1}^{l(\sigma_{d-j})} \frac{b_{d-i}}{d_i}.
\]

(7.103)

Let us denote as

\[
\exp(-d(\sum_{j=1}^{\infty} b_j z^j)) = \sum_{m=0}^{\infty} a_{m,-d} z^m.
\]

(7.104)

Then we can rewrite (7.103) as

\[
c_{d-j} + \sum_{g=j}^{d-1} g \cdot c_{g-j} \cdot a_{d-g,-d} = a_{d-j,-d}.
\]

(7.105)

We apply \( z \frac{dz}{dz} \) to the both sides of (7.104):

\[
(-d(\sum_{i=1}^{\infty} b_i z^i)) \exp(-d(\sum_{j=1}^{\infty} b_j z^j)) = \sum_{m=0}^{\infty} m \cdot a_{m,-d} z^m.
\]

(7.106)

So, we can easily derive

\[
\sum_{m=0}^{\infty} a_{m,-d} \cdot z^m = \sum_{m=0}^{\infty} c_m z^m + \sum_{m=1}^{\infty} \sum_{l=1}^{m} \frac{d-l}{d} \cdot \alpha_{l,-d} \cdot c_{m-l} \cdot z^m
\]

(7.107)

Substituting \( l = d - g \), we see that (7.107) is nothing but (7.107). Q.E.D.
Appendix B: Recursive Formulas for Fano Hypersurfaces
with $c_1(M_N^k) \geq 2$

\[
L_{m}^{N,k,1} = L_{m}^{N+1,k,1} := L_{m}^{k} \tag{7.108}
\]

\[
L_{m}^{N,k,2} = \frac{1}{2}(L_{m-1}^{N+1,k,2} + L_{m}^{N+1,k,2} + 2L_{m}^{N+1,k,1} \cdot L_{m+N-(N-k)}^{N+1,k,1}) \tag{7.109}
\]

\[
L_{m}^{N,k,3} = \frac{1}{18}(4L_{m-2}^{N+1,k,3} + 10L_{m-1}^{N+1,k,3} + 4L_{m}^{N+1,k,3}
+ 12L_{m-1}^{N+1,k,2} \cdot L_{m+2(N-k)}^{N+1,k,1}
+ 6L_{m-1}^{N+1,k,2} \cdot L_{m+1+2(N-k)}^{N+1,k,1}
+ 6L_{m-1}^{N+1,k,2} \cdot L_{m-1+(N-k)}^{N+1,k,1}
+ 12L_{m}^{N+1,k,1} \cdot L_{m+N-(N-k)}^{N+1,k,1}
+ 18L_{m}^{N+1,k,1} \cdot L_{m+(N-k)}^{N+1,k,1} \cdot L_{m+2(N-k)}^{N+1,k,1}) \tag{7.110}
\]

In the following, we omit $N + 1, k$ of $L_{n}^{N+1,k,4}$ in the r.h.s. for brevity.

\[
L_{n}^{N,k,4} = \frac{1}{32}(3L_{n-3}^{4} + 13L_{n-2}^{4} + 13L_{n-1}^{4} + 3L_{n}^{4})
+ \frac{1}{72}(9L_{n-2}^{4}L_{n-2}^{3} + 12L_{n-2}^{3}L_{n-2} + 16L_{n-2}^{3} + 6L_{n-2}^{3}L_{n-2+N-k} + 16L_{n-2}^{3}L_{n-2+N-k}
+ 36L_{n-1}^{4}L_{n-1}^{3} + 44L_{n-1}^{3}L_{n-1} + 27L_{n}^{3}L_{n+N-k})
+ \frac{1}{10}(3L_{n-1}^{2}L_{n-1}^{2}L_{n-1+2(N-k)} + 6L_{n-1}^{2}L_{n-1+2(N-k)} + 4L_{n-1}^{2}L_{n-1+2(N-k)}
+ 10L_{n}^{2}L_{n+2(N-k)} + 6L_{n}^{2}L_{n+2(N-k)} + 3L_{n}^{2}L_{n+2(N-k)})
+ \frac{1}{72}(27L_{n-2}^{3}L_{n-2}^{1} + 44L_{n-2}^{3}L_{n-2} + 16L_{n-2}^{3}L_{n-2}
+ 36L_{n-1}^{3}L_{n-1}^{1} + 12L_{n-1}^{3}L_{n-1} + 9L_{n}^{3}L_{n})
+ \frac{1}{12}(3L_{n-1}^{1}L_{n-1+N-k}L_{n-1+2(N-k)} + 4L_{n-1}^{1}L_{n-1+N-k}L_{n-1+2(N-k)}
+ 6L_{n}^{1}L_{n+N-k}L_{n-1+2(N-k)} + 9L_{n}^{1}L_{n+N-k}L_{n-1+2(N-k)}
+ \frac{1}{6}(3L_{n-1}^{2}L_{n-1+N-k}L_{n-1} + 4L_{n-1}^{2}L_{n-1+N-k}L_{n-1}
+ 4L_{n}^{2}L_{n+N-k}L_{n-1} + 3L_{n}^{2}L_{n+N-k}L_{n-1})
+ \frac{1}{12}(9L_{n-1}^{1}L_{n-1+N-k}L_{n-1} + 6L_{n}^{1}L_{n+N-k}L_{n-1} + 3L_{n}^{1}L_{n+N-k}L_{n-1}
+ L_{n}^{1}L_{n+N-k}L_{n-1}) \tag{7.111}
\]
\[ L_n^{N,k,5} = \]
\[ \frac{24}{625} L_n^{5} - \frac{154}{625} L_n^{5} - \frac{154}{625} L_n^{5} - 2 + \frac{269}{625} L_n^{5} - 1 + \frac{24}{625} L_n^{5} + \]
\[ + 6 \frac{L_n^{3}}{125} + 3 \frac{L_n^{1}}{50} L_n^{2} - N - 3 + N - k + \frac{3}{40} L_n^{1} L_n^{1} + \]
\[ + \frac{3}{32} L_n^{1} L_n^{1} - 3 + N - k + \frac{37}{125} L_n^{1} L_n^{1} - 2 + N - k + \frac{71}{200} L_n^{1} L_n^{1} - 2 + N - k + \]
\[ + \frac{17}{40} L_n^{1} L_n^{1} - 2 + N - k + \frac{58}{125} L_n^{1} L_n^{1} - 1 + N - k + \frac{393}{800} L_n^{1} L_n^{1} - 1 + N - k + \]
\[ + \frac{24}{125} L_n^{1} L_n^{1} - N + N - k + \]
\[ + \frac{8}{125} L_n^{3} L_n^{3} - 2 + 2(N - k) + \frac{8}{75} L_n^{3} L_n^{3} - 2 + 2(N - k) + \frac{4}{45} L_n^{3} L_n^{3} - 2 + 2(N - k) + \]
\[ + \frac{1}{9} L_n^{3} L_n^{3} - N - 2 + 2(N - k) + \frac{46}{125} L_n^{3} L_n^{3} - 1 + 2(N - k) + \frac{122}{225} L_n^{3} L_n^{3} - 1 + 2(N - k) + \]
\[ + \frac{29}{90} L_n^{3} L_n^{3} - N - 1 + 2(N - k) + \frac{59}{125} L_n^{3} L_n^{3} - N - 1 + 2(N - k) + \frac{6}{25} L_n^{3} L_n^{3} + 2(N - k) + \]
\[ + \frac{12}{125} L_n^{3} L_n^{3} - N + 1 + 2(N - k) + \]
\[ + \frac{12}{125} L_n^{3} L_n^{3} - N - 1 + 3(N - k) + \frac{6}{25} L_n^{3} L_n^{3} - N - 1 + 3(N - k) + \frac{9}{90} L_n^{3} L_n^{3} - N - 1 + 3(N - k) + \]
\[ + \frac{1}{9} L_n^{3} L_n^{3} - N - 1 + 3(N - k) + \frac{59}{125} L_n^{3} L_n^{3} - N - 2 + 3(N - k) + \frac{122}{225} L_n^{3} L_n^{3} - N - 1 + 3(N - k) + \]
\[ + \frac{8}{15} L_n^{3} L_n^{3} - N + 1 + 3(N - k) + \frac{46}{125} L_n^{3} L_n^{3} - N + 1 + 3(N - k) + \frac{8}{75} L_n^{3} L_n^{3} - N + 1 + 3(N - k) + \]
\[ + \frac{8}{25} L_n^{3} L_n^{3} - N + 1 + 3(N - k) + \]
\[ + \frac{24}{125} L_n^{3} L_n^{3} - N + 1 + 3(N - k) + \frac{393}{800} L_n^{3} L_n^{3} - N + 1 + 3(N - k) + \frac{17}{40} L_n^{3} L_n^{3} - N + 1 + 3(N - k) + \]
\[ + \frac{3}{32} L_n^{3} L_n^{3} - N + 1 + 4(N - k) + \frac{58}{125} L_n^{3} L_n^{3} - N + 1 + 4(N - k) + \frac{71}{200} L_n^{3} L_n^{3} - N + 1 + 4(N - k) + \]
\[ + \frac{3}{40} L_n^{3} L_n^{3} - N + 1 + 4(N - k) + \frac{37}{125} L_n^{3} L_n^{3} - N + 1 + 4(N - k) + \frac{3}{50} L_n^{3} L_n^{3} - N + 1 + 4(N - k) + \]
\[ + \frac{6}{125} L_n^{3} L_n^{3} - N + 1 + 4(N - k) + \]
\[ + \frac{2}{25} L_n^{3} L_n^{3} - N + 2 + N - k L_n^{3} L_n^{3} - N + 2 + 2(N - k) + \frac{1}{10} L_n^{3} L_n^{3} - N + 2 + N - k L_n^{3} L_n^{3} - N + 2 + 2(N - k) + \]
\[ + \frac{1}{8} L_n^{3} L_n^{3} - N + 2 + N - k L_n^{3} L_n^{3} - N + 2 + 2(N - k) + \frac{2}{15} L_n^{3} L_n^{3} - N + 1 + N - k L_n^{3} L_n^{3} - N + 2 + 2(N - k) + \]
\[ + \frac{1}{6} L_n^{3} L_n^{3} - N + 1 + N - k L_n^{3} L_n^{3} - N + 2 + 2(N - k) + \frac{2}{9} L_n^{3} L_n^{3} - N + 1 + N - k L_n^{3} L_n^{3} - N + 2 + 2(N - k) + \]
\[ + \frac{11}{25} L_n^{3} L_n^{3} - N + 1 + N - k L_n^{3} L_n^{3} - N + 1 + 2(N - k) + \frac{21}{40} L_n^{3} L_n^{3} - N + 1 + N - k L_n^{3} L_n^{3} - N + 1 + 2(N - k) + \]
\[ + \frac{29}{45} L_n^{3} L_n^{3} - N + 1 + N - k L_n^{3} L_n^{3} - N + 1 + 2(N - k) + \frac{12}{25} L_n^{3} L_n^{3} - N + 1 + N - k L_n^{3} L_n^{3} - N + 2(N - k) + \]
\[ + \frac{6}{25} L_n^{3} L_n^{3} - N + 2 + N - k L_n^{3} L_n^{3} - N + 4(N - k) + \frac{3}{10} L_n^{3} L_n^{3} - N + 2 + N - k L_n^{3} L_n^{3} - N + 4(N - k) + \]
\[ + \frac{3}{8} L_n^{3} L_n^{3} - N + 2 + N - k L_n^{3} L_n^{3} - N + 4(N - k) + \frac{23}{40} L_n^{3} L_n^{3} - N + 1 + N - k L_n^{3} L_n^{3} - N + 4(N - k) + \]
+\frac{2}{3}L_1L_{n-1+N-k}^3L_{n+4(N-k)} + \frac{3}{8}L_1^3L_{n+N-k}L_{n+4(N-k)}
+ \frac{13}{25}L_1^3L_{n+1-N-k}^3L_{n+4(N-k)} + \frac{23}{40}L_1^3L_{n+1-N-k}L_{n+1+4(N-k)}
+ \frac{3}{10}L_3^1L_{n+N-k}L_{n+1+4(N-k)} + \frac{6}{25}L_1^3L_{n+N-k}L_{n+2+4(N-k)}
+ \frac{12}{25}L_3^1L_{n-2}L_{n+3(N-k)}^1L_{n+4(N-k)} + \frac{29}{45}L_3^1L_{n-1}L_{n+3(N-k)}L_{n+4(N-k)}
+ \frac{1}{6}L_3^1L_{n}L_{n+1+3(N-k)}L_{n+4(N-k)} + \frac{8}{10}L_3^1L_{n+2+3(N-k)}L_{n+4(N-k)}
+ \frac{11}{25}L_3^1L_{n-1-N-k}^1L_{n+1+4(N-k)} + \frac{2}{15}L_3^1L_{n+1+3(N-k)}L_{n+1+4(N-k)}
+ \frac{1}{10}L_3^1L_{n+2+3(N-k)}L_{n+1+4(N-k)} + \frac{2}{25}L_3^1L_{n+2+3(N-k)}L_{n+2+4(N-k)}
+ \frac{3}{25}L_3^1L_{n-2-N-k}^2L_{n-1+3(N-k)} + \frac{3}{20}L_3^1L_{n-1}L_{n-2+N-k}L_{n-1+3(N-k)}
+ \frac{3}{16}L_3^1L_{n-2-N-k}L_{n-1+3(N-k)} + \frac{3}{10}L_3^1L_{n-1}L_{n-2+N-k}L_{n-1+3(N-k)}
+ \frac{3}{8}L_1^1L_{n-1-N-k}L_{n-1+3(N-k)} + \frac{1}{6}L_1^1L_{n+2+N-k}L_{n-1+3(N-k)}
+ \frac{14}{25}L_1^1L_{n-1-N-k}L_{n+3(N-k)} + \frac{53}{80}L_1^1L_{n-1-N-k}L_{n+3(N-k)}
+ \frac{8}{15}L_1^1L_{n-N-k}L_{n+3(N-k)} + \frac{8}{25}L_1^1L_{n-N-k}L_{n+3(N-k)}
+ \frac{4}{25}L_1^1L_{n-2-N-k}L_{n-1+3(N-k)} + \frac{4}{15}L_1^1L_{n-1-N-k}L_{n-1+2(N-k)}L_{n-1+3(N-k)}
+ \frac{1}{6}L_1^1L_{n-1+2(N-k)}L_{n-1+3(N-k)} + \frac{2}{5}L_1^1L_{n-1}L_{n+2(N-k)}L_{n-1+3(N-k)}
+ \frac{1}{4}L_1^1L_{n+2(N-k)}L_{n-1+3(N-k)} + \frac{1}{6}L_1^1L_{n+1+2(N-k)}L_{n-1+3(N-k)}
+ \frac{1}{4}L_1^1L_{n+1+2(N-k)}L_{n+3(N-k)} + \frac{4}{25}L_1^1L_{n+1+2(N-k)}L_{n+1+3(N-k)}
+ \frac{8}{25}L_1^1L_{n-2-N-k}L_{n+4(N-k)} + \frac{8}{15}L_1^1L_{n-1-N-k}L_{n+4(N-k)}
+ \frac{1}{3}L_1^1L_{n-1+2(N-k)}L_{n+4(N-k)} + \frac{53}{80}L_1^1L_{n-1-N-k}L_{n+4(N-k)}
+ \frac{3}{8}L_1^1L_{n+2(N-k)}L_{n+4(N-k)} + \frac{3}{16}L_1^1L_{n+1+2(N-k)}L_{n+4(N-k)}
+ \frac{14}{25}L_1^1L_{n-1-N-k}L_{n+1+4(N-k)} + \frac{3}{10}L_1^1L_{n+1+2(N-k)}L_{n+1+4(N-k)}
+ \frac{3}{20}L_1^1L_{n+1+2(N-k)}L_{n+1+4(N-k)} + \frac{3}{25}L_1^1L_{n+1+2(N-k)}L_{n+2+4(N-k)}
+ \frac{1}{5}L_3^1L_{n-1-N-k}L_{n-1+2(N-k)}L_{n-1+3(N-k)} + \frac{1}{4}L_3^1L_{n-1-N-k}L_{n-1+2(N-k)}L_{n-1+3(N-k)}
+ \frac{1}{3}L_3^1L_{n+1+2(N-k)}L_{n-1+3(N-k)} + \frac{1}{2}L_3^1L_{n+1+2(N-k)}L_{n-1+3(N-k)}
+ \frac{4}{5}L_3^1L_{n+1+2(N-k)}L_{n+3(N-k)} + \frac{2}{5}L_3^1L_{n-1-N-k}L_{n-1+2(N-k)}L_{n+4(N-k)}
\[+
\frac{1}{2}L_n^1L_{n-1}^1 + N-kL_{n-1}^2L_{n+1}^1 + 2(N-k)L_{n+1}^1L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[\frac{1}{4}L_n^1L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[\frac{3}{5}L_n^1L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[\frac{1}{2}L_n^1L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[\frac{2}{3}L_n^1L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[\frac{2}{5}L_n^1L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[\frac{1}{2}L_n^2L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[\frac{1}{3}L_n^2L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[\frac{1}{4}L_n^2L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]
\[L_n^1L_{n+1}^1 + N-kL_{n+1}^2L_{n+2}(N-k)L_{n+4}(N-k)
\]

(7.112)
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| k   | $L_{0,5,1}^{10,5,1}$ | $L_{0,5,1}^{10,5,1}$ | $L_{2,6,1}^{16,8,1}$ |
|-----|---------------------|---------------------|---------------------|
| 5   | 120                 | 770                 | 120                 |
| 6   | 720                 | 6264                | 16344               |
| 7   | 5040                | 56196               | 200452              |
| 8   | 40320               | 554112              | 2552192             |
| 9   | 362880              | 5973264             | 34138908            |
| 10  | 362880              | 699984000           | 482076000           |
| 11  | 39916800            | 886897440           | 719666969           |
| 12  | 479001600           | 120892953600        | 113548220928        |
| 13  | 6227020800          | 176484597120        | 189132239462         |
| 14  | 87178291200         | 2748022986240       | 3320520905318        |
| 15  | 1307674368000       | 4547329504000       | 613390541616000      |

(All the $L_{n}^{2k}\ell$ $(d \geq 2)$'s are zero.)
Table 2: Examples of the Generalized Mirror Transformation for the $d = 3, 4$ Cases

$L^{10,11,3}_5 = 760111401641235526556038149776315/3$
$= 39321013161929775850199104322696996/9 -
11051810355155318663381224366483160/9 -
3 * 7196676696 * (40216130393485580917383/2 - 2153815059753018506790) +
(9/2) * 7196676696 * 7196676696 * (83223447879 + 64088868338 -
28831752092 - 7196676696) -
(3/2) * 2153815059753018506790 * (83223447879 - 7196676696) -

$L^{10,11,3}_6 = 13783958731158999754651957610916334/3$
$= 61012348943229750362670547134032423/9 -
11051810355155318663381224366483160/9 -
3 * 7196676696 * (40216130393485580917383/2 + 40216130393485580917383/2 -
9451710952055403714441 - 2153815059753018506790 +
64088868338 * 64088868338 - 64088868338 * 28831752092 - 83223447879 - 28831752092 +
83223447879 * 7196676696 - 64088868338 * 7196676696 + 28831752092 * 7196676696 -
(9/2) * 7196676696 * 7196676696 * (64088868338 + 2 * 83223447879 -
2 * 28831752092 - 7196676696) -
(3/2) * 2153815059753018506790 * (64088868338 + 83223447879 -
28831752092 - 7196676696) -

$L^{10,11,4}_6 = 17309436345972797762158384063096760890661018005223/216$
$= 538190721372443294157468729700660158617785657/288 -
73190750568118350991831897085289272131127171095/96 -
4 * 7196676696 * (61012348943229750362670547134032423/9 -
11051810355155318663381224366483160/9) -
2 * 2153815059753018506790 * (40216130393485580917383/2 -
2153815059753018506790) -
(4/3) * 11051810355155318663381224366483160/9 * (64088868338 - 7196676696) +
8 * 7196676696 * 7196676696 * (40216130393485580917383/2 +
40216130393485580917383/2 -
9451710952055403714441 - 2153815059753018506790 +
64088868338 * 64088868338 - 64088868338 * 28831752092 - 83223447879 - 28831752092 +
83223447879 * 7196676696 - 64088868338 * 7196676696 + 28831752092 * 7196676696 -
8 * 7196676696 * 2153815059753018506790 * (64088868338 + 83223447879 -
28831752092 - 7196676696) -
32/3 * 7196676696 * 7196676696 * 7196676696 * (64088868338 + 2 * 83223447879 -
2 * 28831752092 - 7196676696) -

Table 3: More Example of the Generalized Mirror Transformation for the $d = 3$ Case

$L_{9}^{13,15,3} = 14356269698724856586166649497084995544762232083984375$

$= 23334416034364889092865122924300384023499300787109375 -$

$734983453001771990744124070334204964362034189837500 -$

$3 \times 4360309637094000 \times (14863292734375058940601059038015625/4 +$ 

$97609182646575948239192133869015625/2 - 16577238305553301159536028880531250 -$ 

$4454602732448692602148965129468750 + 50264090344359000 \times 50264090344359000 -$ 

$1852734278048000 \times (50264090344359000 + 90331361620677000 + 109607337529527375 +$ 

$90331361620677000 + 50264090344359000) +$ 

$4360309637094000 \times (90331361620677000 + 109607337529527375 + 90331361620677000) +$ 

$1852734278048000 \times (90331361620677000 + 50264090344359000 + 1852734278048000 +$ 

$4360309637094000) -$ 

$4360309637094000 \times (90331361620677000 + 50264090344359000 + 1852734278048000) +$ 

$50264090344359000 \times (90331361620677000 + 90331361620677000 + 50264090344359000 -$ 

$50264090344359000 \times (50264090344359000 + 90331361620677000 +$ 

$109607337529527375 + 90331361620677000 + 50264090344359000) +$ 

$4360309637094000 \times (109607337529527375 -$ 

$109607337529527375 \times 1852734278048000 - 90331361620677000 \times 360309637094000 +$ 

$50264090344359000 \times (109607337529527375 + 90331361620677000 + 50264090344359000 +$ 

$1852734278048000 + 4360309637094000) -$ 

$4360309637094000 \times 90331361620677000 -$ 

$2 \times 90331361620677000 + 3 \times 109607337529527375 +$ 

$2 \times 90331361620677000 + 50264090344359000 - 109607337529527375 -$ 

$2 \times 90331361620677000 - 3 \times 50264090344359000 - 2 \times 1852734278048000 -$ 

$4360309637094000) -$ 

$(3/2) \times 4454602732448692602148965129468750 \times (50264090344359000 +$ 

$90331361620677000 - 1852734278048000 - 4360309637094000) -$