The Well-posedness and the Regularity of Global Attractor for a Couple Stress Fluid Through Porous Layer with the Local Thermal Non-equilibrium Effect

Liang Li¹ · Lan Jia²

Received: 11 September 2021 / Accepted: 16 December 2022 / Published online: 3 January 2023
© The Author(s), under exclusive licence to Springer Nature B.V. 2023

Abstract
In the article, we aim to investigate the well-posedness of solution and the regularity of the global attractor for the couple stress fluid in saturated porous media with the local thermal non-equilibrium effect. To be more specific, we firstly show the existence and uniqueness of global weak solution to the model by making use of the standard Galerkin method. Second, relying on verifying the uniformly compact condition required, we prove the existence of the global attractor of the model in the space where the weak solution resides. Finally, we improve the regularity of global attractor by uniformly compact condition and obtain the $C^\infty$ attractor for the model.

Keywords Well-posedness · The global attractor · Regularity · Couple stress fluid · Non-equilibrium effect

Mathematics Subject Classification (2020) 35A01 · 35A09 · 35B41

1 Introduction

The dynamics of fluid through porous media has attracted some of applied mathematicians like Hill [1] and Straughan [2], and still is a vigorously active research area. The widely used momentum equation for Newtonian fluid through porous media is either the Brinkman equation or the Darcy equation derived from the Navier-Stokes equation by statistical averages and simplifications of the complicated microscopic flow picture. Couple stress fluid, developed by V.K. Stokes [3] and discussed in detail by himself in his monograph [4], is a
typical non-Newtonian fluid and allows for the polar effects. The momentum equation [5, 6] for couple stress fluid in a saturated porous layer reads

$$\frac{\rho_0}{\varepsilon} \frac{\partial \mathbf{v}}{\partial t} + \frac{\mu_f}{K} \mathbf{v} = \frac{\mu_c}{K} \Delta \mathbf{v} - \nabla p - \rho_0(1 - \beta(T_f - T_u)) \mathbf{g},$$  \hspace{1cm} (1.1)

where $\mathbf{v} = (v_1, v_2)$ is the velocity field, $T_f$ and $T_u$ denote the fluid and upper surface temperatures respectively, $p$ means the pressure field and $\mathbf{g} = (0, -g)$ is the gravitational acceleration. The other numbers in the model are as follows: the porosity $\varepsilon$, the permeability $K$, the viscosity $\mu_f$, the coupling stress viscosity $\mu_c$, the thermal expand coefficient $\beta$ and the density $\rho_0$ at the initial moment.

The study of thermal convection in porous media has important practical significance, such as exploration of geothermal resources, techniques for preventing the spread of pollution sources in underground aquifers, early warning and protection technology of coal seam fire. However, a lot of work on convection mainly focus on Newtonian fluid, see. e.g. [7, 8]. Now, there is increasingly number of literature considering the convection in couple stress fluid, see. e.g. [9, 10]. It is worth noting that the energy models among the above literatures about couple stress fluid are considered under the uniform temperature gradient. However, the local thermal non-equilibrium (LTNE) effect has to be taken into account for establishing the energy equations which is better suitable for the practical situations such as high-speed flows or large temperature differences between the fluid and solid phases. Continuum theories for the LTNE effect on the flow in porous materials appear to have started in the late 1990’s, cf. the work of [11] where a modified energy equation that can be solved for very early departures from local thermal equilibrium (LTE) conditions is presented while assuming the velocity field is known from the solution of continuity and momentum equation, and [12]. Thus, for describing the energy equation, the two-field model [13] (page 34) [14]

$$\begin{align*}
\varepsilon (\rho c)_f &\frac{\partial T_f}{\partial t} + (\rho c)_f (\mathbf{v} \cdot \nabla) T_f = \varepsilon \kappa_f \Delta T_f + \kappa(T_s - T_f), \\
(1 - \varepsilon) (\rho c)_s &\frac{\partial T_s}{\partial t} = (1 - \varepsilon) \kappa_s \Delta T_s - \kappa(T_s - T_f)
\end{align*}$$

is employed, where the energy equations are coupled by the terms that account for the heat lost to or gain form the other phase, $T_s$ denotes the solid temperature, $c$ is the specific heat, the heat conductivity $\kappa$ with subscript $f$ and $s$ meaning fluid and solid phase respectively and the inter-phase heat transfer coefficient $\kappa$.

There are many studies involved with couple stress fluid through porous layer using a LTNE model: M.S. Malashetty [5] obtained the condition for the onset of convection by the linear stability theory and also presented asymptotic analysis for different values of the inter-phase heat transfer; Sunli [15] showed the equivalence of nonlinear stability threshold and linear instability boundary. Recently, Quan W. [16] studied the stability and transition of the LTNE model of coupled stress fluid. For more information about related model, please refer to the references [17, 18]. However, there is lack of work on well-posedness, i.e. the existence and uniqueness of solution, of related model. We will consider the problem later.

To the best of our knowledge, there is little literature about the existence and regularity of the global attractor for the couple stress fluid in saturated porous layer with the LTNE effect. The study of global attractor [19] which is defined as the maximal compact invariant set or the minimal set which uniformly attracts all bounded set can provide deep insight in the long-time behavior of dynamics of the model investigated. In general, there are two ways to prove the existence of the global attractor: Condition (C) [20] (page 106) [21] and uniformly compact condition. Although the C-condition compared with the uniformly compact condition is easier to be verified because there is no need to perform any operations in
higher regularity space, it is more convenient to employ the uniformly compact condition to obtain our desire under the help of semigroup. There are a lot of work on the existence of attractor by condition (C), see e.g. [21–23].

Inspired by the work [24] (page 512) where Ma proved the existence of the \( C^\infty \)-attractor of the 2-dimension Boussinesq equation, this work aim to study the existence of the \( C^\infty \)-attractor for the couple stress fluid in saturated porous layer with the LTNE effect. The approach employed in present article is as follows. Firstly, we show the existence of the global weak solution by using the standard Galerkin method [25]; secondly, relying on verifying the uniformly compact condition required, we prove the existence of the global attractor of the model in the space where the weak solution resides; finally, instead of improving the regularity of the global weak solution by the interpolation theorem [26] used in [24], we improve the regularity by estimating the expression of solution directly, improve the regularity of the global attractor by iterative use of uniformly compact condition and obtain the \( C^\infty \) attractor for the model. For more literatures on the regularity of the attractor, see. e.g. [27, 28].

The rest of this article is arranged as follows. In Sect. 2, we introduce the mathematical model studied in this paper and its dimensionless form and give some mathematical settings. In Sect. 3, we study the well-posedness of the model solution, that is, we prove the existence of the unique global weak solution and the global solution respectively. Next, we investigate the existence of the model attractor in detail in Sect. 4 and Sect. 5. We first prove that there is a global attractor in the weak solution space by using the uniformly compact method. By using the expression of the weak solution and the iterative method, we then improve the regularity of the weak solution and obtain the classical solution of the model under certain conditions. At the same time, we prove that there is a global attractor in the classical solution space. Finally, we summarize the results of the work in the article in Sect. 6.

2 Mathematic Setting

2.1 Mathematical Model

We consider the 2-D incompressible couple stress fluid model of saturated porous media where the region is a rectangle with depth \( d \) and width \( ad \). The fluid is heated from below and cooled from above (see Fig. 1). The temperature of lower surface and of upper surface are held at \( T = T_l \) and \( T = T_u(< T_l) \), respectively. Combining (1.1), (1.2) and continuity equation, the basic governing equations are as follows:

\[
\begin{align*}
\frac{\rho_0}{\kappa_f} \frac{\partial v}{\partial t} + \frac{\mu_f}{K} \nabla p - \rho_0(1 - \beta(T_f - T_u)) \mathbf{g}, \\
\frac{\varepsilon (\rho c)_f}{\kappa_f} \frac{\partial T_f}{\partial t} + (\rho c)_f (v \cdot \nabla) T_f &= \varepsilon \kappa_f \Delta T_f + h(T_s - T_f), \\
(1 - \varepsilon)(\rho c)_s \frac{\partial T_s}{\partial t} &= (1 - \varepsilon) \kappa_s \Delta T_s - h(T_s - T_f), \\
\nabla \cdot v &= 0,
\end{align*}
\]

where the region of the fluid saturated porous media is assumed in \( \Omega_0 = (0, ad) \times (0, d) \) and the time variable \( t > 0 \).

Now by the following transformations to make the equation (2.1) dimensionless:

\[
(x, z) = d(x^*, z^*), \quad (v_1, v_2) = \frac{\varepsilon \kappa_f}{(\rho c)_f} (v_1^*, v_2^*), \quad p = \frac{\kappa_f \mu_f}{(\rho c)_f K} p^*,
\]

\[
T_f = (T_l - T_u) \theta, \quad T_s = (T_l - T_u) \phi, \quad t = \frac{(\rho c)_f}{\kappa_f} t^*.
\]

\( \diamond \) Springer
After taking the curl of the first dimensionless equation and ignoring the superscript, equation (2.1) is as follows:

\[
\frac{Da}{Pr} \frac{\partial \Delta \psi}{\partial t} = (C \Delta - 1) \Delta \psi + Ra \frac{\partial \theta}{\partial x},
\]

\[
\frac{\partial \theta}{\partial t} = \Delta \theta + \lambda (\phi - \theta) - J(\psi, \theta),
\]

(2.3)

where stream function \( \psi \) satisfies

\[
v_1 = -\frac{\partial \psi}{\partial z}, \quad v_2 = \frac{\partial \psi}{\partial x},
\]

(2.4)

and the nonlinear part reads

\[
J(\psi, \theta) = \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \theta}{\partial x}.
\]

The non-dimensionalization procedure gives rise to the following seven numbers: the Darcy-Rayleigh number \( Ra = \frac{\rho_0 g \beta (T_l - T_u) (\rho c) f K}{\epsilon \mu_f \kappa_f} \), the dimensionless heat transfer coefficient \( \lambda = \frac{h}{\epsilon \kappa_f} \), the modified conductivity \( \gamma = \frac{\epsilon \kappa_f}{(1 - \epsilon) \kappa_s} \), the diffusion ration \( \alpha = \frac{\rho_c}{\rho c_f} \), the Darcy number \( Da = K \), the Prandtl number \( Pr = \frac{\mu_f \epsilon (\rho c)_f}{\rho_0 \kappa_f} \) and the couple stress number \( C = \frac{\nu_c}{\mu_f} \).

Now, for the bounded set \( \Omega \subset \mathbb{R}^2 \), we give the boundary and initial conditions to the equation (2.3) as follows:

\[
\psi|_{\partial \Omega} = \Delta \psi|_{\partial \Omega} = 0, \\
\theta|_{\partial \Omega} = \phi|_{\partial \Omega} = 0,
\]

(2.6)

\[
V(0, x, z) = (\psi(0, x, z), \theta(0, x, z), \phi(0, x, z)).
\]

3 Well-posedness of Solution

3.1 Mathematical Setting

We denote \( \langle \cdot, \cdot \rangle \) as the inner product of \( L^2 \) and \( \| \cdot \| \) as \( L^2 \)-norm. Let \( V = (\psi, \theta, \phi) \) and introduce the following spaces

\[
X = \{ V \in C^\infty(\Omega) \times C^\infty(\Omega) \times C^\infty(\Omega) | V \text{ satisfies (2.6)} \}.
\]
The Well-posedness and the Regularity of Global Attractor...

\[ \mathbf{Y} = H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \]
\[ \mathbf{Y}_1 = \{ \mathbf{V} \in H^3(\Omega) \times H^1(\Omega) \times H^1(\Omega) \mid \mathbf{V} \text{ satisfies (2.6)} \}, \]
\[ \mathbf{Y}_2 = \{ \mathbf{V} \in H^4(\Omega) \times H^2(\Omega) \times H^2(\Omega) \mid \mathbf{V} \text{ satisfies (2.6)} \}, \]
\[ \mathbf{Y}_k = \{ \mathbf{V} \in H^k(\Omega) \times H^k(\Omega) \times H^k(\Omega) \mid \mathbf{V} \text{ satisfies (2.6)} \} \quad (k \geq 2). \]

Because \( X \subset \mathbf{Y}_1 \) is dense and \( \mathbf{Y}_1 \) is separable, there exists a set of orthonormal basis \( \{ \tilde{e}_m \} \times \{ e_m \} \times \{ e_m \} \) which satisfies the following conditions
\[ \Delta \tilde{e}_m = \mu_m \tilde{e}_m, \quad \Delta e_m = \mu_m e_m, \]
\[ \langle \tilde{e}_{m1}, \tilde{e}_{m2} \rangle = \langle e_{m1}, e_{m2} \rangle = \delta_{m1m2}. \] (3.2)

In addition, according to [29], when \( \psi \in H^2 \cap H^1_0 \), there exists a constant \( c > 0 \) (depends on \( \Omega \)) such that
\[ \| \psi \|_{H^2} \leq c \| \Delta \psi \|. \] (3.3)

then by Poincaré inequality, \( \| \nabla \Delta \psi \| \) is equivalent to \( \| \psi \|_{H^3} \) when \( \mathbf{V} \in \mathbf{Y}_1 \).

3.2 Existence and Uniqueness of the Global Solution

Before giving the global solution of the equation (2.3), we consider the existence and uniqueness of weak solution.

**Definition 3.1 (Weak solution)** We say \( \mathbf{V} = (\psi, \theta, \phi) \in L^\infty((0, T), \mathbf{Y}) \cap L^2((0, T), \mathbf{Y}_2) \) is the weak solution of the equation (2.3), if for any \( \mathbf{W} = (w_1, w_2, w_3) \in X \) and \( 0 < t < T \) \( (T > 0 \) is arbitrary fixed constant), we have
\[
\frac{Da}{Pr} \langle \Delta \psi(t), w_1 \rangle + \langle \theta(t), w_2 \rangle + \alpha \langle \phi(t), w_3 \rangle \\
= \int_0^t \left[ -C \langle \nabla \Delta \psi, \nabla w_1 \rangle - \langle \Delta \psi, w_1 \rangle + Ra \langle \frac{\partial \theta}{\partial x}, w_1 \rangle - \langle \nabla \theta, \nabla w_2 \rangle \\
+ \lambda \langle \phi - \theta, w_2 \rangle - \langle J(\psi, \theta, \phi), w_2 \rangle - \langle \nabla \phi, \nabla w_3 \rangle + \gamma \lambda \langle \theta - \phi, w_3 \rangle \right] \, d\tau \\
+ \frac{Da}{Pr} \langle \Delta \psi(0), w_1 \rangle + \langle \theta(0), w_2 \rangle + \alpha \langle \phi(0), w_3 \rangle.
\] (3.4)

According to the Galerkin method, we can get the following results.

**Theorem 3.2 (Existence of weak solution)** For any initial value \( \mathbf{V}(0) \in \mathbf{Y}_{1} \), the equation (2.3) has a weak solution \( \mathbf{V}(t) \in L^\infty_{\text{loc}}((0, \infty), \mathbf{Y}) \cap L^2_{\text{loc}}((0, \infty), \mathbf{Y}_2) \).

**Proof** This proof is divided into three steps.

Step 1: Construct an approximate solution. Let \( \mathbf{V}^m = (\psi^m, \theta^m, \phi^m)(m \in \mathbb{N}) \), where
\[ \psi^m = \sum_{i=1}^{m} \psi^m_i(t) \tilde{e}_i, \]
\[ \theta^m = \sum_{i=1}^{m} \theta_i^m(t) e_i, \]
\[ \phi^m = \sum_{i=1}^{m} \phi_i^m(t) e_i. \]

Setting \( V = V^m \) in the equation (2.3) and taking the inner product of equation (2.3) with \((\bar{e}_i, e_i, e_i) (i = 1, \ldots, m)\), we have the set of the following ordinary equations

\[ \frac{Da}{Pr} \frac{d\psi_m}{dt} = (C \bar{\mu}_i - 1) \psi_i^m(t) + Raf_i(t), \]
\[ \frac{d\theta_m}{dt} = (\mu_i - \lambda) \theta_m(t) + \lambda \phi_m(t) - f_{21}(t), \]
\[ \alpha \frac{d\phi_m}{dt} = (\mu_i - \gamma \lambda) \phi_m(t) + \gamma \lambda \theta_m(t), \]

where

\[ f_{1i} = \sum_{k=1}^{m} \frac{\theta_k^m(t)}{\mu_k} \left( \frac{\partial e_k}{\partial x}, \bar{e}_i \right), \]
\[ f_{21} = \sum_{k_1, k_2=1}^{m} \psi_{k_1}(t) \theta_{k_2}^m(t) \left( \frac{\partial \bar{e}_{k_1}}{\partial x} \frac{\partial e_{k_2}}{\partial z} - \frac{\partial \bar{e}_{k_1}}{\partial z} \frac{\partial e_{k_2}}{\partial x}, e_i \right). \]

Based on the fundamental theory of ordinary differential equations in the space of smooth functions, the equation (3.5) has a smooth local solution, by which we have

\[ \frac{Da}{Pr} \left( \Delta \psi^m, w_1 \right) + \left( \theta^m, w_2 \right) + \alpha \left( \phi^m, w_3 \right) = \int_0^t \left[ -C \left( \nabla \Delta \psi^m, \nabla w_1 \right) - \left( \Delta \psi^m, w_1 \right) + Ra \left( \frac{\partial \theta^m}{\partial x}, w_1 \right) - \left( \nabla \theta^m, \nabla w_2 \right) + \lambda \left( \phi^m - \theta^m, w_2 \right) - \left( J(\psi^m, \theta^m), w_2 \right) - \left( \nabla \phi^m, \nabla w_3 \right) + \gamma \lambda \left( \theta^m - \phi^m, w_3 \right) \right] d\tau \]

\[ + \frac{Da}{Pr} \left( \Delta \psi^m(0), w_1 \right) + \left( \theta^m(0), w_2 \right) + \alpha \left( \phi^m(0), w_3 \right) \]

for any \( 0 < t < T \).

**Step 2: Priori estimates.** Substituting \( V^m = (\Delta \psi^m, \theta^m, \phi^m) \) \((m \in \mathbb{N})\) into the equation (2.3) and taking the inner product of with \( V^m \), we can obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \frac{Da}{Pr} \| \Delta \psi^m \|^2 + \| \theta^m \|^2 + \alpha \| \phi^m \|^2 \right) \]
\[ = - \left( C \| \nabla \Delta \psi^m \|^2 + \| \nabla \theta^m \|^2 + \| \nabla \phi^m \|^2 \right) - \| \Delta \psi^m \|^2 - Ra \left( \theta, \frac{\partial \Delta \psi^m}{\partial x} \right) \]
\[ + \lambda \left( \phi^m - \theta^m, \theta^m \right) + \gamma \lambda \left( \theta^m - \phi^m, \phi^m \right). \]
Then, the Young inequality is used to make estimates

\[ |Ra \langle \theta^m, \frac{\partial \Delta \psi^m}{\partial x} \rangle| \leq \frac{C}{2} \|\nabla \Delta \psi^m\|^2 + \frac{Ra^2}{2C} \|\theta^m\|^2, \quad (3.8) \]

and

\[ |(\lambda + \gamma \lambda) \langle \phi^m, \theta^m \rangle| \leq (\gamma \lambda + \frac{\lambda}{4}) \|\phi^m\|^2 + (\frac{\gamma \lambda}{4} + \lambda) \|\theta^m\|^2. \quad (3.9) \]

In combination with (3.7)–(3.9), we have

\[ \frac{d}{dt} \left( \frac{Da}{Pr} \|\Delta \psi^m\|^2 + \|\theta^m\|^2 + \alpha \|\phi^m\|^2 \right) \leq -M_1 \|V^m\|_{Y^2} + M_2 \|V^m\|_Y, \quad (3.10) \]

where

\[ M_1 = 2 \min \left\{ \frac{Pr C}{2Da}, 1, \frac{1}{\alpha} \right\}, \quad M_2 = 2 \max \left\{ 1, \frac{Ra^2}{2C} + \frac{\lambda \gamma}{4} + \frac{\lambda}{4\alpha} \right\}. \]

According to **Gronwall inequality**, it yields that

\[ \|V^m\|_Y^2 \leq \|V^m(0)\|_Y^2 e^{M_2 t}, \]

and

\[ \int_0^t \|V^m\|_{Y^2}^2 d\tau \leq \frac{e^{M_2 t}}{M_1} \|V^m(0)\|_Y^2, \]

which shows the approximate solution

\[ V^m \in L^\infty((0, T), Y) \cap L^2((0, T), \nabla Y^2) \quad \text{for} \quad \forall \ m \in \mathbb{N}. \]

**Step 3: Getting a convergent sequence in some sense.** In order to take the limit of the equation (3.6), the approximate solution must be convergent in some sense. Firstly, we prove \{\psi^m_i(t)\}, \{\theta^m_i(t)\} and \{\phi^m_i(t)\} are equicontinuous for fixed \( i \) and \( m = 1, 2, \ldots \).

On one hand, we have

\[ |\psi^m_i(t + h) - \psi^m_i(t)| = \left| \int_t^{t+h} \frac{d\psi^m_i(\tau)}{d\tau} d\tau \right| \]

\[ = \int_t^{t+h} \left[ \frac{Pr}{Da} (C \Delta - 1) \Delta \psi^m, \tilde{e}_i \right] + \frac{Pr Ra}{Da} \left( \frac{\partial \theta^m}{\partial x}, \tilde{e}_i \right) d\tau \]

\[ \leq M_3 h^{\frac{1}{2}}, \]

where

\[ M_3 = 3 \max_{0 < i < T} \left\{ \frac{Pr C}{Da} \|\nabla \tilde{e}_i\| \left[ \int_t^{t+h} \|\nabla \Delta \psi^m\|^2 d\tau \right]^\frac{1}{2}, \frac{Pr}{Da} \|\tilde{e}_i\| \|\Delta \psi^m\|, \frac{Pr Ra}{Da} \|\nabla \tilde{e}_i\| \left[ \int_t^{t+h} \|\theta^m\|^2 d\tau \right]^\frac{1}{2} \right\}. \]
On other hand, we have

$$|\theta_i^m(t+h) - \theta_i^m(t)| \leq M_4 h^{\frac{1}{2}},$$

and

$$|\phi_i^m(t+h) - \phi_i^m(t)| \leq M_5 h^{\frac{1}{2}},$$

where

$$M_4 = 3 \max_{0 < t < T} \{ \| \nabla e_i \| [\int_t^{t+h} \| \nabla \theta^m \|^2 d\tau]^{\frac{1}{2}} + \lambda \| \phi^m \| + \| \theta^m \| \| e_i \| \},$$

$$M_5 = 2 \max_{0 < t < T} \left\{ \frac{1}{\alpha} \| \nabla e_i \| [\int_t^{t+h} \| \nabla \phi^m \|^2 d\tau]^{\frac{1}{2}} + \frac{\gamma \lambda}{\alpha} [\| \theta^m \| + \| \phi^m \| \| e_i \| ] \right\},$$

where $c$ is in (3.3) and $M_{so}$ is from Sobolev inequality.

Note that $\{\psi_i^m(t)\}, \{\theta_i^m(t)\}$ and $\{\phi_i^m(t)\}$ are uniformly bounded by the results given in Step 2. Following the Ascoli-Arzela theorem, there exists convergent subsequences of $\{\psi_i^m(t)\}, \{\theta_i^m(t)\}$ and $\{\phi_i^m(t)\}$, denote by $\{\psi_i^m(t)\}, \{\theta_i^m(t)\}$ and $\{\phi_i^m(t)\}$, such that

$$\psi_i^m \to \psi_i, \quad \theta_i^m(t) \to \theta_i(t), \quad \phi_i^m(t) \to \phi_i(t)$$

as $m \to \infty$ uniformly, where $\psi_i(t), \theta_i(t)$ and $\phi_i(t)$ are all continuous.

Denote

$$V_0 = \left( \sum_{i=1}^{+\infty} \psi_i(t) \tilde{e}_i, \sum_{i=1}^{+\infty} \theta_i(t) e_i, \sum_{i=1}^{+\infty} \phi_i(t) e_i \right),$$

then for any $\tilde{V} \in X$ we have

$$\lim_{m \to \infty} \sup_{0 \leq t \leq T} \langle V^m - V_0, \tilde{V} \rangle = 0,$$ (3.11)

and infer that

$$\lim_{m \to \infty} \int_0^T |\langle V^m - V_0, \tilde{V} \rangle|^2 d\tau = 0.$$ (3.12)

Thus, it means $\{V^m\}$ is uniformly weakly convergent to $V_0$ in $L^2((0, T), Y_2)$. From (Lemma 4.28 in [24]), we get

$$\begin{cases}
D^\alpha \psi_i^m \to D^\alpha \psi_0, & \forall |\alpha| \leq 2, \\
\theta_i^m \to \theta_0, \\
\phi_i^m \to \phi_0,
\end{cases}$$ (3.13)

in $L^2((0, T) \times \Omega)$ as $m \to \infty.$
For simplicity, we only focus on the nonlinear term. A simple calculation shows that

\[
\left| \int_0^t \left( \frac{\partial^2 \psi^m}{\partial x \partial z} - \frac{\partial^2 \psi_0}{\partial x \partial z} \right) \theta^m, w_2 \right| d\tau \leq \left| \int_0^t \left( \frac{\partial^2 \psi_0}{\partial x \partial z} (\theta^m - \theta_0), w_2 \right) + \left( \frac{\partial \psi_0}{\partial x} - \frac{\partial \psi_0}{\partial x} \right) \theta^m, \frac{\partial w_2}{\partial x} \right| d\tau \\
+ \left| \int_0^t \left( \frac{\partial^2 \psi_0}{\partial x \partial z} (\theta_0 - \theta_0 - \theta_0), w_2 \right) + \left( \frac{\partial \psi_0}{\partial x} (\theta_0 - \theta_0), \frac{\partial w_2}{\partial x} \right) \right| d\tau \\
\rightarrow 0 \quad (m \to \infty).
\]

Taking the limit on both sides of the equation (3.6), we obtain

\[
\frac{D_a}{Pr} \langle \Delta \psi_0, w_1 \rangle + \langle \theta_0, w_2 \rangle + \alpha \langle \phi_0, w_3 \rangle
= \int_0^t \left[ -C \langle \nabla \Delta \psi_0, \nabla w_1 \rangle - \langle \Delta \psi_0, w_1 \rangle + Ra \frac{\partial \theta_0}{\partial x}, w_1 \rangle - \langle \nabla \theta_0, \nabla w_2 \rangle \right. \\
+ \lambda \langle \phi_0 - \theta_0, w_2 \rangle - \langle J(\psi_0, \theta_0), w_2 \rangle - \langle \nabla \phi_0, \nabla w_3 \rangle + \gamma \lambda \langle \theta_0 - \phi_0, w_3 \rangle \rangle \quad (3.6)
\]

\[d\tau \]

which means \( V_0 \) is the weak solution of the equation (2.3). \( \square \)

Moreover, by the two-dimensional Sobolev embedding theorem [24] (page 21), we have the following theorem.

**Theorem 3.3 (Uniqueness)** The weak solution of the equation (2.3) is unique.

**Proof** Assume both \( V_1 = (\psi_1, \theta_1, \phi_1) \) and \( V_2 = (\psi_2, \theta_2, \phi_2) \) are weak solutions of the equation (2.3). Let \( \tilde{V} = (\tilde{\psi}, \tilde{\theta}, \tilde{\phi}) = V_1 - V_2 \), that is

\[
\tilde{\psi} = \psi_1 - \psi_2, \quad \tilde{\theta} = \theta_1 - \theta_2, \quad \tilde{\phi} = \phi_1 - \phi_2.
\]

By simple calculation, we know \( \tilde{V}(0) = 0 \) and

\[
J(\psi_1, \theta_1) - J(\psi_2, \theta_2) = J(\tilde{\psi}, \tilde{\theta}) + J(\psi_2, \tilde{\theta}) = 0.
\]

Putting \( W = \tilde{V} \) into the equation (3.4), we obtain that

\[
\frac{D_a}{Pr} \| \nabla \tilde{\psi} \|^2 + \| \tilde{\theta} \|^2 + \alpha \| \tilde{\phi} \|^2
= \int_0^t \left[ -(C \| \Delta \tilde{\psi} \|^2 + \| \nabla \tilde{\theta} \|^2 + \| \nabla \tilde{\phi} \|^2) - (\| \nabla \tilde{\psi} \|^2 + \lambda \| \tilde{\theta} \|^2 + \gamma \lambda \| \tilde{\phi} \|^2) \right] \quad (3.4)
\]
\[-Ra\langle \tilde{\theta}, \frac{\partial \tilde{\psi}}{\partial x} \rangle + (\lambda + \gamma \lambda)\langle \tilde{\phi}, \tilde{\theta} \rangle - \langle J(\tilde{\psi}, \theta_1), \tilde{\theta} \rangle]d\tau.\]

By Young inequalities, we receive the following estimation

\[|Ra\langle \tilde{\theta}, \frac{\partial \tilde{\psi}}{\partial x} \rangle| \leq \|\nabla \tilde{\psi}\|^2 + \frac{Ra^2}{4}\|\tilde{\theta}\|^2,\]

and

\[|(\lambda + \gamma \lambda)\langle \tilde{\phi}, \tilde{\theta} \rangle| \leq (\gamma \lambda + \frac{\lambda}{4})\|\tilde{\phi}\|^2 + (\frac{\gamma \lambda}{4} + \lambda)\|\tilde{\theta}\|^2.\]

Besides, on the basis of Sobolev embedding theorem, we have

\[2\int_{\Omega} |\nabla \tilde{\psi}| \|
abla \theta_1\| |\tilde{\phi}| dx dz \leq 2\|\tilde{\theta}\|_{L^\infty} \int_{\Omega} |\nabla \tilde{\psi}| |\nabla \theta_1| dx dz \leq \|\nabla \tilde{\theta}\|^2 + M_{so}^2 \|
abla \theta_1\|^2 \|\nabla \tilde{\psi}\|^2,\]

where \(M_{so}\) just depends on \(\Omega\).

To sum up, we come to the conclusion

\[\frac{Da}{Pr} \|\nabla \tilde{\psi}\|^2 + \|\tilde{\theta}\|^2 + \alpha \|\tilde{\phi}\|^2 \leq \int_0^t \alpha(\tau)(\frac{Da}{Pr} \|\nabla \tilde{\psi}\|^2 + \|\tilde{\theta}\|^2 + \alpha \|\tilde{\phi}\|^2)d\tau,\]

where nonnegative function \(\alpha(\tau) = \max\{\frac{M_{so}^2 \|
abla \theta_1\|^2}{Da}, \frac{Ra^2 + \gamma \lambda}{4}, \frac{\lambda}{4\alpha}\} \in L^1(0, T)\). Finally, according to Gronwall inequality, we get

\[\frac{Da}{Pr} \|\nabla \tilde{\psi}\|^2 + \|\tilde{\theta}\|^2 + \alpha \|\tilde{\phi}\|^2 \leq 0,\]

which means the weak solution is unique.

Next, we will improve the regularity of weak solution with respect to \(t\). Let us start with the definition of the global solution.

**Definition 3.4 (Global solution)** We say \(V\) is the global solution of the equation (2.3), if for any \(W = (w_1, w_2, w_3) \in X\), there is

\[\frac{Da}{Pr} \langle \frac{d \Delta \psi(t)}{dt}, w_1 \rangle + \langle \frac{d \theta(t)}{dt}, w_2 \rangle + \alpha \langle \frac{d \phi(t)}{dt}, w_3 \rangle = -C \langle \nabla \Delta \psi, \nabla w_1 \rangle - \langle \Delta \psi, w_1 \rangle + Ra \langle \frac{\partial \theta}{\partial x}, w_1 \rangle - \langle \nabla \theta, \nabla w_2 \rangle + \lambda \langle \phi - \theta, w_2 \rangle - \langle J(\psi, \theta), w_2 \rangle - \langle \nabla \phi, \nabla w_3 \rangle + \gamma \lambda \langle \theta - \phi, w_3 \rangle.\]  

For the equation (2.3), we proved the following result.

**Theorem 3.5 (Existence and uniqueness of global solution)** If the initial value \(V(0) \in Y_1\), the equation (2.3) has a unique global solution

\[V \in W_{loc}^{1,\infty}((0, \infty), Y) \cap W_{loc}^{1,2}((0, \infty), Y_1).\]
Proof  Firstly, we will estimate the approximate solution $V^m$ which we have got.

Obviously, $V^m = (\psi^m, \theta^m, \phi^m)$ satisfies the following equations

$$
\int_0^t ((\Delta \dot{\psi}^m, w_1) + (\dot{\theta}^m, w_2) + (\dot{\phi}^m, w_3)) d\tau
$$

$$
= \int_0^t [- \frac{PrC}{Da} (\nabla \Delta \dot{\psi}^m, \nabla w_1) - \frac{Pr}{Da} (\Delta \dot{\psi}^m, w_1) + \frac{Pr Ra}{Da} (\frac{\partial \dot{\theta}^m}{\partial x}, w_1) - (\nabla \dot{\theta}^m, \nabla w_2)
$$

$$
+ \lambda (\dot{\phi}^m - \dot{\theta}^m, w_2) - (J(\dot{\psi}^m, \theta^m) + J(\psi^m, \dot{\theta}^m), w_2) - \frac{1}{\alpha} (\nabla \dot{\phi}^m, \nabla w_3)
$$

$$
+ \frac{\gamma \lambda}{\alpha} (\dot{\theta}^m - \dot{\phi}^m, w_3)] d\tau.
$$

Let $(w_1, w_2, w_3) = (\Delta \dot{\psi}^m, \dot{\theta}^m, \dot{\phi}^m)$. Then we have

$$
\|\Delta \dot{\psi}^m\|^2 + \|\dot{\theta}^m\|^2 + \|\dot{\phi}^m\|^2
$$

$$
= - 2 \int_0^t [ - \frac{PrC}{Da} \| \nabla \Delta \dot{\psi}^m \|^2 + \frac{Pr}{Da} \| \Delta \dot{\psi}^m \|^2 - \frac{Pr Ra}{Da} (\frac{\partial \dot{\theta}^m}{\partial x}, \Delta \dot{\psi}^m) + \| \nabla \dot{\theta}^m \|^2
$$

$$
+ \lambda (\dot{\phi}^m - \dot{\phi}^m, \dot{\phi}^m) + (J(\dot{\psi}^m, \theta^m), \dot{\phi}^m) + \frac{1}{\alpha} \| \nabla \dot{\phi}^m \|^2 + \frac{\gamma \lambda}{\alpha} (\dot{\phi}^m - \dot{\theta}^m, \dot{\phi}^m)] d\tau
$$

$$
+ \| \Delta \dot{\psi}^m(0) \|^2 + \| \dot{\theta}^m(0) \|^2 + \| \dot{\phi}^m(0) \|^2.
$$

Because

$$
\frac{Da}{Pr} (\Delta \dot{\psi}^m, w_1) = ((C \Delta - 1) \Delta \psi^m, w_1) + Ra (\frac{\partial \theta^m}{\partial x}, w_1),
$$

$$
(\dot{\theta}^m, w_2) = (\Delta \dot{\theta}^m, w_2) + \lambda (\phi^m - \theta^m, w_2) - (J(\psi^m, \theta^m), w_2),
$$

$$
\alpha (\dot{\phi}^m, w_3) = (\Delta \phi^m, w_3) + \gamma \lambda (\theta^m - \phi^m, w_3),
$$

let $(w_1, w_2, w_3) = (\Delta \dot{\psi}^m, \dot{\theta}^m, \dot{\phi}^m)$ in (3.16), and by Young inequality. Then we get

$$
\|\Delta \dot{\psi}^m(0)\|^2 + \|\dot{\theta}^m(0)\|^2 + \|\dot{\phi}^m(0)\|^2 \leq g(\psi^m(0), \theta^m(0), \phi^m(0)),
$$

where

$$
g(\psi^m(0), \theta^m(0), \phi^m(0))
$$

$$
= \frac{3Pr^2C^2}{Da^2} \| \Delta^2 \psi^m(0) \|^2 + \frac{3Pr^2}{Da^2} \| \Delta \psi^m(0) \|^2 + 3 \| \Delta \theta^m(0) \|^2
$$

$$
+ \frac{2}{\alpha^2} \| \Delta \phi^m(0) \|^2 + 12 \| \nabla \psi^m(0) \| \| \nabla \theta^m(0) \| + \frac{3Pr^2Ra^2}{Da^2} \| \nabla \dot{\theta}^m(0) \|^2
$$

$$
+ (12 \lambda^2 + \frac{8 \gamma^2 \lambda^2}{\alpha^2}) (\| \theta^m(0) \|^2 + \| \phi^m(0) \|^2).
$$

Thus, we choose $V(0) \in Y_1$, then $\{\|V^m(0)\|_{Y_1}\}$ is bounded and $g(\psi^m(0), \theta^m(0), \phi^m(0))$ is bounded.
Secondly, we can obtain the solution by functional analysis. According to the Sobolev inequality and (3.3), we know

\[
-2 \int_{0}^{t} \langle J(\dot{\psi}^{m}, \theta^{m}), \dot{\theta}^{m} \rangle d\tau \leq 4 \int_{0}^{t} \|\nabla \theta^{m}\| \|\nabla \dot{\psi}^{m}\| d\tau \\
\leq 4cM_{so} \int_{0}^{t} \|\nabla \theta^{m}\| \|\dot{V}^{m}\|_{Y}^{2} d\tau,
\]

where $M_{so} > 0$ is from Sobolev inequality and $c > 0$ is in (3.3). And by Young inequality, we receive that

\[
-2 \int_{0}^{t} \langle \partial_{x} \dot{\theta}^{m}, \Delta \dot{\psi}^{m} \rangle d\tau \leq \frac{Pr Ra}{Da} \int_{0}^{t} \|\dot{\theta}^{m}\|^{2} d\tau + \frac{3Pr}{Da} \int_{0}^{t} \|\Delta \dot{\psi}^{m}\|^{2} d\tau,
\]

and

\[
-2\lambda \int_{0}^{t} \langle \dot{\theta}^{m} - \dot{\phi}^{m}, \dot{\theta}^{m} \rangle - \frac{2\gamma \lambda}{\alpha} \langle \dot{\phi}^{m} - \dot{\phi}^{m}, \dot{\phi}^{m} \rangle d\tau \leq (1 + \frac{\gamma}{\alpha})\lambda \int_{0}^{t} \|\dot{\phi}^{m}\|^{2} d\tau + (1 + \frac{\gamma}{\alpha})\lambda \int_{0}^{t} \|\dot{\theta}^{m}\|^{2} d\tau.
\]

Combine the ones above, we have

\[
\|\dot{V}^{m}\|_{Y}^{2} \leq -M_{6} \int_{0}^{t} \|\alpha(\tau)\dot{V}^{m}\|_{Y}^{2} d\tau + g(V^{m}(0)),
\]

where

\[
M_{6} = 2 \min\{\frac{Pr C}{Da}, 1, \frac{1}{\alpha}\}
\]

and $\alpha(\tau) = \max\{\frac{Pr}{Da}, (1 + \frac{\gamma}{\alpha})\lambda + \frac{Pr Ra^{2}}{3Da}, (1 + \frac{\gamma}{\alpha})\lambda + 4cM_{so}\|\nabla \theta^{m}\|\}$ is nonnegative and local integrable. Thus, by Gronwall inequality, we know

\[
\|\dot{V}^{m}\|_{Y}^{2} < \infty, \quad \int_{0}^{t} \|\dot{V}^{m}\|_{Y}^{2} d\tau < +\infty,
\]

which means $\{V^{m}\} \subset W^{1,2}(0, T), Y_{\frac{1}{2}} \cap W^{1,\infty}(0, T), Y)$ is bounded. According to functional analysis, we obtain that

\[
V^{m} \xrightarrow{u} V_{0} \in W^{1,2}(0, T), Y_{\frac{1}{2}} \cap W^{1,\infty}(0, T), Y),
\]

by the uniqueness of weak limit, we know $V_{0}$ is the global solution of the equation (2.3).

The uniqueness of $V_{0}$ comes from the uniqueness of weak solution according to Theorem 3.3. \qed

Now, we prove the following corollary which will be used in improving the regularity later.

**Corollary 1** The global solution $V(t)$ of the equation (2.3) belongs to $L^{\infty}(0, T), Y_{\frac{1}{2}}$.
Proof After taking the inner product of the equation (2.3) with $V = (\Delta \psi, \theta, \phi)$ and identical transformations, we have

$$
\| \nabla \Delta \psi \|^2 + \| \nabla \theta \|^2 + \| \nabla \phi \|^2
= - \frac{Da}{PrC} \left( \frac{\partial \Delta \psi}{\partial t}, \Delta \psi \right) - \left( \frac{\partial \theta}{\partial t}, \theta \right) - \alpha \left( \frac{\partial \phi}{\partial t}, \phi \right) - \frac{1}{C} \| \Delta \psi \|^2
- \frac{Ra}{C} \left( \Delta \psi, \frac{\partial \theta}{\partial x} \right) + (\lambda + \gamma \lambda) \langle \phi, \theta \rangle - \lambda \| \theta \|^2 - \gamma \lambda \| \phi \|^2.
$$

After some simple calculations, the following estimated inequalities can be obtained

$$
| \frac{Da}{PrC} \left( \frac{\partial \Delta \psi}{\partial t}, \Delta \psi \right) | \leq \frac{1}{2} \| \Delta \psi_t \|^2 + \frac{Da^2}{2Pr^2C^2} \| \Delta \psi \|^2,
$$

$$
| \left( \frac{\partial \theta}{\partial t}, \theta \right) | \leq \frac{1}{2} \| \theta_t \|^2 + \frac{1}{2} \| \theta \|^2,
$$

and

$$
| \alpha \left( \frac{\partial \phi}{\partial t}, \phi \right) | \leq \frac{1}{2} \| \phi_t \|^2 + \frac{\alpha^2}{2} \| \phi \|^2,
$$

$$
| \frac{Ra}{C} \left( \Delta \psi, \frac{\partial \theta}{\partial x} \right) | \leq \frac{1}{2} \| \nabla \theta \|^2 + \frac{Ra^2}{2C^2} \| \Delta \psi \|^2,
$$

as well as

$$
| (\lambda + \gamma \lambda) \langle \phi, \theta \rangle | \leq \left( \frac{\lambda}{4} + \gamma \lambda \right) \| \phi \|^2 + \left( \lambda + \frac{\gamma \lambda}{4} \right) \| \theta \|^2.
$$

Combined with the above inequalities, whereupon it is concluded that

$$
\| \nabla \Delta \psi \|^2 + \| \nabla \theta \|^2 + \| \nabla \phi \|^2
\leq (\| \Delta \psi_t \|^2 + \| \theta_t \|^2 + \| \phi_t \|^2) + \left( \frac{\lambda}{2} + \alpha^2 \right) \| \phi \|^2
+ \left( 1 + \frac{\gamma \lambda}{2} \right) \| \theta \|^2 + \left( \frac{Da^2}{2Pr^2C^2} + \frac{2}{C} + \frac{Ra^2}{C^2} \right) \| \nabla \psi \|^2.
$$

And we have proved that

$$
V(t) \in W^{1,\infty}((0, T), Y) \cap W^{1,2}((0, T), Y^1_2),
$$

then we can receive

$$
\| \nabla \Delta \psi \|^2 + \| \nabla \theta \|^2 + \| \nabla \phi \|^2 < \infty.
$$

This means that

$$
V(t) \in L^\infty((0, T), Y_1^1) ,
$$

□
4 Existence of Attractors in Y

According to the conclusion in the previous sections, one can see that the equation (2.3) generates a dynamical system \( S(t) : Y_1 \rightarrow Y_2 \). In this section, we aim to prove the existence of attractor of the equation in \( Y \): we firstly prove there exists a bounded absorbing set in \( Y \), and then we demonstrate that semigroup of operators \( S(t) \) is uniformly compact in \( Y_2 \).

**Theorem 4.1 (Existence of attractor in \( Y \))** The equation (2.3) has an attractor \( A \) in \( Y \), and \( A \) absorbs all the bounded sets in \( Y \).

**Proof** Step 1, we will illustrate the existence of bounded absorbing set in \( Y \).

By considering the second and third equation in the equation (2.3), we find that we can estimate the two terms \( \theta \) and \( \phi \) firstly. Thus, after taking the inner product of the second and third equation with \( (\theta, \phi) \), we have

\[
\frac{d}{dt} \left( \frac{1}{2\lambda} \|\theta\|^2 + \frac{\alpha}{2\gamma \lambda} \|\phi\|^2 \right) = -\frac{\|\nabla \theta\|^2}{\lambda} - \frac{\|\nabla \phi\|^2}{\gamma \lambda} - [\|\theta\|^2 + \|\phi\|^2 - 2 \langle \phi, \theta \rangle].
\]

On the basis of Poincaré inequality, we know that

\[
\|\theta\|^2 \leq M_P \|\nabla \theta\|^2, \quad \|\phi\|^2 \leq M_P \|\nabla \phi\|^2,
\]

where \( M_P > 0 \) is constant.

Hence, we get the following inequality

\[
\frac{d}{dt} \left( \frac{1}{2\lambda} \|\theta\|^2 + \frac{\alpha}{2\gamma \lambda} \|\phi\|^2 \right) + \tilde{c} \left( \frac{1}{2\lambda} \|\nabla \theta\|^2 + \frac{\alpha}{2\gamma \lambda} \|\nabla \phi\|^2 \right) \leq -M_\gamma \left( \frac{1}{2\lambda} \|\theta\|^2 + \frac{\alpha}{2\gamma \lambda} \|\phi\|^2 \right),
\]

where

\[
M_\gamma = \frac{1}{M_P} \min\left\{1, \frac{1}{\alpha} \right\} > 0, \quad 0 < \tilde{c} < 1 (0 < \tilde{c} < 1).
\]

Finally, we take advantage of Gronwall inequality to receive the result

\[
\|\theta\|^2 + \|\phi\|^2 \leq M_8 e^{-M_\gamma t} (\|\theta(0)\|^2 + \|\phi(0)\|^2),
\]

and

\[
\int_0^t \|\nabla \theta\|^2 + \|\nabla \phi\|^2 d\tau \leq M_9 [\|\theta(0)\|^2 + \|\phi(0)\|^2],
\]

where

\[
M_8 = \max\left\{\frac{1}{2\lambda}, \frac{\alpha}{2\gamma \lambda}\right\}, \quad M_9 = \min\left\{\frac{1}{2\lambda}, \frac{\alpha}{2\gamma \lambda}\right\}.
\]

which means there exists \( t_0 > 0 \) such that \( \|\theta\|^2 + \|\phi\|^2 \leq \rho_0^2 \) (\( \rho_0 \) is a fixed constant) as \( t > t_0 \), i.e. \( \|\theta\| \) and \( \|\phi\| \) are uniformly bounded.

Similarly, after taking the inner product of the first equation with \( \Delta \psi \), we make use of the Young inequality as \( t > t_0 \) to have

\[
\frac{Da}{2Pr} \frac{d}{dt} \|\Delta \psi\|^2 = -C \|\nabla \Delta \psi\|^2 - \|\Delta \psi\|^2 - Ra(\theta, \frac{\partial \Delta \psi}{\partial x}),
\]
\[
\begin{align*}
\leq - \|\Delta \psi\|^2 + \frac{Ra^2}{4C} \|\theta\|^2, \\
\leq - \|\Delta \psi\|^2 + \frac{Ra^2 \rho_0^2}{4C}.
\end{align*}
\]

Then, according to the **Gronwall inequality**, it is concluded that

\[
\|\Delta \psi\|^2 \leq (1 - e^{2Pr(t_0-t)} \frac{Ra^2 \rho_0^2}{4C}).
\]

Thus, there exists \(\rho_R > 0\) such that \(\|\Delta \psi\|^2 + \|\theta\|^2 + \|\phi\|^2 < \rho_R^2\) as \(t > t_0\), i.e. there exists a bounded absorbing set in \(Y\). Moreover, by taking the integral

\[
\frac{Da}{2Pr} \frac{d}{dt} \|\Delta \psi\|^2 = -C\|\nabla \Delta \psi\|^2 - \|\Delta \psi\|^2 - Ra(\theta, \frac{\partial \Delta \psi}{\partial x})
\]

from \(t_0\) to \(t\) with respect to \(t\) and making use of (4.2), we have

\[
\int_{t_0}^{t} \|\nabla \Delta \psi\|^2 dt \leq \frac{Da}{PrC} \|\Delta \psi(t_0)\|^2 + \frac{Ra^2 M_8}{C^2 M_7} [\|\theta(0)\|^2 + \|\phi(0)\|^2].
\]

**Step 2:** We will prove \(S(t)\) is uniformly compact, i.e. bounded absorbing set in \(Y_\frac{1}{2}\).

After taking the inner product of the equation (2.3) with \((\Delta^2 \psi, \Delta \theta, \Delta \phi)\), we have

\[
\frac{d}{dt} \left[ \frac{Da}{Pr} \|\nabla \Delta \psi\|^2 + \|\nabla \theta\|^2 + \alpha \|\nabla \phi\|^2 \right]
\]

\[
= - 2[C\|\Delta^2 \psi\|^2 + \|\nabla \Delta \psi\|^2 + \|\Delta \theta\|^2 + \|\Delta \phi\|^2 + \lambda \|\nabla \theta\|^2]
\]

\[
+ \gamma \lambda \|\nabla \phi\|^2 + Ra(\frac{\partial \theta}{\partial x}, \Delta^2 \psi) + \lambda \langle \phi, \Delta \theta \rangle + \gamma \lambda \langle \theta, \Delta \phi \rangle + \langle J(\psi, \theta), \Delta \theta \rangle.
\]

We use **Young inequality** to get inequalities

\[
|Ra(\frac{\partial \theta}{\partial x}, \Delta^2 \psi)| \leq C\|\Delta^2 \psi\|^2 + \frac{Ra^2}{4C} \|\nabla \theta\|^2,
\]

and

\[
\lambda |\langle \phi, \Delta \theta \rangle| \leq \|\nabla \theta\|^2 + \frac{\lambda^2}{4} \|\nabla \phi\|^2,
\]

as well as

\[
\gamma \lambda |\langle \theta, \Delta \phi \rangle| \leq \|\Delta \phi\|^2 + \frac{\gamma^2 \lambda^2}{4} \|\theta\|^2.
\]

In the meantime, by **Sobolev embedding theorem**, we have

\[
|\langle J(\psi, \theta), \Delta \theta \rangle| \leq \|\nabla \theta\|^2 \|\psi\|_{H^2}^2 + \|\Delta \theta\|^2,
\]

\[
\leq c M_{so} \|\nabla \theta\|^2 \|\Delta \psi\|^2 + \|\Delta \theta\|^2.
\]
Combined with inequality (4.5)-(4.9), the following results are obtained

\[
\frac{d}{dt}(\frac{Da}{Pr}\|\nabla \Delta \psi\|^2 + \|\nabla \theta\|^2 + \alpha\|\nabla \phi\|^2) \leq M_{10}(\frac{Da}{Pr}\|\nabla \Delta \psi\|^2 + \|\nabla \theta\|^2 + \alpha\|\nabla \phi\|^2) + \frac{(\lambda^2 + \gamma^2\lambda^2)\rho_R^2}{2},
\]

where

\[
M_{10} = \frac{Ra^2}{2C} + 2\lambda + 2 + 2cM_{so}\|\Delta \psi\|^2 + \frac{4\gamma\lambda + \lambda^2}{2\alpha}.
\]

By uniform Gronwall inequality [30] (page 90), we set

\[
y = \frac{Da}{Pr}\|\nabla \Delta \psi\|^2 + \|\nabla \theta\|^2 + \alpha\|\nabla \phi\|^2, \quad g = M_{10}, \quad h = \frac{\lambda^2 + \gamma^2\lambda^2)\rho_R^2}{2},
\]

then according to (4.3)-(4.4), we have

\[
a_1 = 4 \int_t^{t+r} g(\tau)d\tau \leq M_{11} r.
\]

Used formula (4.2) and (4.4), we obtain that

\[
a_3 = \int_t^{t+r} y(\tau)d\tau \leq M_{12} r,
\]

where

\[
M_{11} = 2cM_{so}\rho_R^2 + \frac{Ra^2}{2C} + \frac{4\gamma\lambda + \lambda^2}{2\alpha} + 2\lambda + 2,
\]

\[
M_{12} = \frac{Da^2}{Pr^2C}\|\Delta \psi(0)\|^2 + \left(\frac{Ra^2M_8Da}{C^2M_7Pr} + (\alpha)M_9)(\|\theta(0)\|^2 + \|\phi(0)\|^2\right)
\]

It is easy to know

\[
a_2 = \int_t^{t+r} h(\tau)d\tau = \frac{(\lambda^2 + \gamma^2\lambda^2)\rho_R^2 r}{2},
\]

and then we receive that

\[
\frac{Da}{Pr}\|\nabla \Delta \psi\|^2 + \|\nabla \theta\|^2 + \alpha\|\nabla \phi\|^2 \leq \left(\frac{a_3}{r} + a_2\right)e^{a_1},
\]

which means there exists a bounded absorbing set in \(Y^{1/2}\).

Thus, the proof is completed.

\[\square\]

5 Existence of \(C^\infty\)-attractor

In this section, we aim to improve the regularity of the attractor. Thus, we need to improve the regularity of the global solution of the equation (2.3) and also prove there exists
a bounded absorbing set in $Y_k$ ($k \geq 2$). In order to improve the regularity, we now perform $\Delta^{-1}$ on both sides of the first equation of the equation (2.3) and rewrite it into abstract form as follows:

$$\frac{dV}{dt} = LV + G(V), \quad (5.1)$$

where

$$LV = \begin{pmatrix} \frac{pC}{Da} (\Delta^{-1}) \Delta^2 \psi \\ \Delta \theta \\ \frac{1}{a} \Delta \phi \end{pmatrix}, \quad G(V) = \begin{pmatrix} -\frac{pC}{Da} \psi + \frac{PrRa}{Da} (\Delta^{-1}) \frac{\partial \theta}{\partial x} \\ \lambda (\phi - \theta) - J (\psi, \theta) \\ \frac{\beta}{a} (\theta - \phi) \end{pmatrix}, \quad V = (\psi, \theta, \phi).$$

**Remark 1** Because of $\Delta : H^2(\Omega) \cap H^1_0(\Omega) \rightarrow L^2(\Omega)$ is isomorphism, $\Delta^{-1}$ is bounded.

**Remark 2** Under the condition (2.6), we can obtain $\Delta^{-1} \Delta \psi = \psi$ and $(\Delta^{-1})^{\Delta} \psi$, which means $(\Delta^{-1})^{\Delta} \psi$ is an infinitesimal generator of an analytic semigroup. Thus, operator $L : Y_1 \rightarrow Y$ is infinitesimal generator of the analytic semigroup denoted by $T(t)$.

Based on the fact that the solution $V \in L^\infty((0, T), Y_2)$, it is easy to know that $G(V) \in L^1((0, T), Y)$. Hence, according to Theorem 4.18 in [24] or theorem in page 259 [31], we have the following lemma:

**Lemma 5.1** The solution of the equation (5.1) $V(t, V(0))$ can be expressed as follow:

$$V(t, V(0)) = T(t)V(0) + \int_0^t T(t - \tau)G(V(\tau, V(0)))d\tau. \quad (5.2)$$

Lemma 5.1 is very useful for us to improve the regularity of $V(t)$. And a property of the analytic semigroup $T(t)$ will be used several times as follows:

$$\|L^\beta T(t)\| \leq M_\beta t^{-\beta} e^{-\delta t} \quad (\delta > 0, \ \beta \in R^1), \quad (5.3)$$

where $L^\beta$ is the fractional operator generated by $L$ and $M_\beta > 0$ is constant depending on $\beta$, and the norm $\|V\|_{Y_\beta} = \|L^\beta Y\|$.

In order to get the $C^\infty$-attractor, we improve the regularity of the solution of the equation (2.3). Although the boundary of a rectangular region is almost everywhere smooth, it can be approximated by a sequence of smooth regions $\{\Omega_n\}$, where $\Omega_n$ is a region enclosed by eight curves $\Gamma_j$ ($j = 1, \ldots, 8$) as follows:

$$\left\{ \begin{array}{l}
\Gamma_1 = \left[ \frac{1}{n}, a - \frac{1}{n} \right] \times \{ y = 0 \},
\Gamma_2 = \{ x = a \} \times \left[ \frac{1}{n}, 1 - \frac{1}{n} \right],
\Gamma_3 = \left[ \frac{1}{n}, a - \frac{1}{n} \right] \times \{ y = 1 \},
\Gamma_4 = \{ x = 0 \} \times \left[ \frac{1}{n}, 1 - \frac{1}{n} \right],
\Gamma_5 = \{ (x, y) \} \{ (x - \frac{1}{n})^2 + (y - \frac{1}{n})^2 = \frac{1}{n^2} \},
\Gamma_6 = \{ (x, y) \} \{ (x - a + \frac{1}{n})^2 + (y - \frac{1}{n})^2 = \frac{1}{n^2} \},
\Gamma_7 = \{ (x, y) \} \{ (x - a + \frac{1}{n})^2 + (y - 1 + \frac{1}{n})^2 = \frac{1}{n^2} \},
\Gamma_8 = \{ (x, y) \} \{ (x - \frac{1}{n})^2 + (y - 1 + \frac{1}{n})^2 = \frac{1}{n^2} \}.
\end{array} \right.$$
Theorem 5.2 The equation (5.1) has a unique solution

\[ V(t) \in C^\infty((0, T), Y_{k/2})(k \geq 2) \]

for arbitrary \( V(0) \in X \).

Proof Obviously, \( \Omega_n \to \Omega \) as \( n \to \infty \). All integrals on \( \Omega_n \) go to integrals on \( \Omega \) as \( n \to \infty \) because of the boundness. We take every integral on \( \Omega \) immediately rather than on \( \Omega_n \).

We still have three steps to show the results.

Step 1: we will improve the regularity of \( V(t) \) with respect to space variables by iteration. For the solution \( V(t) \in Y_{1/2} \), we obviously have

\[
\| J(\psi, \theta) \|_2 \leq 4 \int_\Omega |\nabla \psi|^2 |\nabla \theta|^2 \, dx \, dz \\
\leq 4 \| \nabla \psi \|_{L^\infty(\Omega)} \| \nabla \theta \|_2 \leq 4M_m (\| \nabla \psi \|_2 + \| \Delta \psi \|_2) |\nabla \theta|^2,
\]

which means \( G : Y_{1/2} \to Y \) is continuous and bounded. And by Lemma 5.1, we can obtain that

\[
\| V(t) \|_{Y_\beta} \leq \| T(t) V(0) \|_{Y_\beta} + \int_0^t \| L^{\beta} T(t - \tau) \| \| G(V) \| d\tau \\
\leq \| T(t) V(0) \|_{Y_\beta} + M_{\beta} \int_0^t t^{-\beta} e^{-\delta(t - \tau)} d\tau \| G(V) \| < \infty,
\]

for any \( \beta \in \left[ \frac{1}{2}, 1 \right) \).

It reveals that the solution of the equation (5.1)

\[ V(t) \in L^\infty((0, T), Y_{\beta}). \]

Then there exists a \( \theta(\beta) > 0 \) as \( \frac{1}{2} < \beta < 1 \) such that

\[ G : Y_\beta \to Y_{\theta(\beta)} \text{ continuous and bounded.} \]

According to Lemma 5.1, we have

\[
\| V(t) \|_{Y_1} \leq \| T(t) V(0) \|_{Y_1} + \int_0^t \| L^{1-\theta(\beta)} T(t - \tau) \| \| L^{\theta(\beta)} G(V) \| d\tau < \infty.
\]

That is to say the solution \( V(t) \in L^\infty((0, T), Y_{\beta}) \) \( \forall \frac{1}{2} \leq \beta \leq 1 \).

Step 2: we will prove the map \( G : Y_{1/2} \to Y_{k-1/2}(k \geq 2) \) is continuous and bounded.

It is easy to know

\[
J(\psi, \theta) = \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \theta}{\partial x} \in H^{k-1}(5.5)
\]

for \( \psi \in H^{k+2} \) and \( \theta \in H^k \), which means \( G : Y_{1/2} \to Y_{k-1/2} \) \( (k \geq 2) \) is a continuous and bounded map.
Above all, using an iterative approach, we have
\[ V(t) \in L^\infty((0, T), Y_{\frac{k}{2}})(k \geq 2). \] (5.6)

**Step 3:** we need to improve the regularity with respect to time variable. We have
\[
\frac{dV(t)}{dt} \|_{Y_{\frac{k}{2}}} \leq \|LT(t)V(0)\|_{Y_{\frac{k}{2}}} + \|G(V)\|_{Y_{\frac{k}{2}}} + \int_0^t \|L^\frac{1}{2}T(t-\tau)\| \|L^\frac{k+1}{2}G(V)\| d\tau
\]
\[
\leq M_{k+2} t^{-\frac{k+1}{2}} e^{-\delta t} \|V(0)\| + \|G(V)\|_{Y_{\frac{k}{2}}} + \int_0^t \|L^\frac{1}{2}T(t-\tau)\| \|L^\frac{k+1}{2}G(V)\| d\tau
\]
< \infty
for 0 < t < T, which means \(V(t) \in C^1((0, T), Y_{\frac{k}{2}})\).

Thanks to the iterative method again, for any \(k \geq 2\), we know the solution
\[ V(t) \in C^\infty((0, T), Y_{\frac{k}{2}}). \]

Finally, we introduce the regularity results of attractors, that is to say the existence of \(C^\infty\)-attractor.

**Theorem 5.3** For any \(k \geq 1\), there exists an attractor \(A \subset Y_{\frac{k}{2}}\) absorbs all the bounded sets in \(Y_{\frac{k}{2}}\).

**Proof** We have previously proved that there exists a bounded absorbing set \(B \subset Y_{\frac{1}{2}}\). And expression (5.6) combine with Lemma 5.1 illustrate that there exists a semigroup
\[ T(t) : Y_{\frac{k}{2}} \rightarrow Y_{\frac{k}{2}}(k \geq 2). \]
To complete the proof, then we just need to state that there is a bounded absorbing set \(B_{\frac{k}{2}} \subset Y_{\frac{k}{2}}\) for \(k \geq 2\).

For arbitrary initial value \(V(0) \in U_{\frac{k}{2}} \subset Y_{\frac{k}{2}}\), which \(U_{\frac{k}{2}}\) is bounded set in \(Y_{\frac{k}{2}}\), \(U_{\frac{k}{2}} \subset Y_{\frac{1}{2}}\) is also bounded. Then, there exists \(T_0 > 0\) such that for any \(T > T_0\) we have
\[ V(T, V(0)) \in B. \] (5.7)

Besides, when the time \(T\) satisfies \(t > T > T_0\), we have
\[
V(t) = T(t - T)V(T, V(0)) + \int_T^t T(t - \tau)G(V)d\tau.
\] (5.8)

At the same time, we can also get
\[
\|T(t - T)V(T, V(0))\|_{Y_{\frac{k}{2}}} = \|L^\frac{k+1}{2}T(t - T)V(T, V(0))\|_{Y_{\frac{k}{2}}}
\]
\[
\leq M_{k+1} t^{-\frac{k+1}{2}} e^{-\delta t} \|V\|_{Y_{\frac{1}{2}}}
\]
\[
= C_t \rightarrow 0
\]
(5.9)
as \(t \rightarrow \infty\). And then we obtain \(\|G(V)\|_{Y_{\frac{k}{2} - 1}} < \infty\) by means of the relation (5.5).
Whereupon, the following inequality is true

\[
\|V\|_{Y^\frac{1}{2}} \leq \|T(t - \tau)V(T, V(0))\|_{Y^\frac{1}{2}} + \int_t^T \|L^\frac{1}{2}T(t - \tau)\|\|G(V)\|_{Y^\frac{1}{2}} d\tau \\
\leq C_t + M_1 \int_t^T \tau^{-\frac{1}{2}} e^{-\delta \tau} d\tau \|G(V)\|_{Y^\frac{1}{2}} \\
< M_{13} \quad (t \to \infty),
\]

(5.10)

where \(M_{13}\) is independent of \(V(0)\).

By the foregoing reason, we have shown there exists a bounded absorbing set \(B_{\frac{1}{2}}^{k} \subset Y^\frac{1}{2} \) \((k \geq 2)\). This proof is completed.

\section*{6 Conclusions}

In this article, we studied the existence of the \(C^\infty\)-attractor for a couple stress fluid in saturated porous media. For the target model, we made mathematical deal by the stream function for the 2-dimension incompressible flow. Afterwards, we obtain the weak solution by Galerkin method and the weak solution is proved to be unique. In order to get the global solution, the regularity of the weak solution is improved with respect to time variable. So far, we say the model can generate a dynamic system. Taking advantage of semigroup, the solution can be expressed in the form of integral. In order to prove the existence of \(C\)-attractor, the regularity of the global solution is improved to \(C^\infty\) by estimating the integral solution and the existence of the bounded absorbing set in higher regularity space is also proved. Finally, by iterative use of uniformly compact condition, the existence of \(C^\infty\)-attractor is proved.

However, there is a problem worth discussing further: in 2-dimension case, in addition to the advantage of stream function, some key inequalities, which might not be valid in 3-dimension or higher-dimension case, can be proved by Sobolev embedding theorem. Thus, in higher-dimension case, it is much difficult to prove the similar results. In the future work, we will focus on this point.

\textbf{Funding} The work was supported by the National Nature Science Foundation of China (11901408) and (11711306).

\textbf{Declarations}

\textbf{Competing Interests} The authors declare no competing interests.

\textbf{References}

1. Hill, A.A., Morad, M.R.: Convective stability of carbon sequestration in anisotropic porous media. Proc. R. Soc. A, Math. Phys. Eng. Sci. 470(2170), 20140373 (2014)
2. Straughan, B.: Global nonlinear stability in porous convection with a thermal non-equilibrium model. Proc. R. Soc. A, Math. Phys. Eng. Sci. 462(2066), 409–418 (2005)
3. Stokes, V.K.: Couple stresses in fluids. In: Theories of Fluids with Microstructure, pp. 34–80. Springer, Berlin (1984)
4. Stokes, V.K.: Theories of Fluids with Microstructure. Springer, Berlin (1984)
5. Malashetty, M.S., Shivakumara, I.S., Kulkarni, S.: The onset of convection in a couple stress fluid saturated porous layer using a thermal non-equilibrium model. Phys. Lett. A 373(7), 781–790 (2009)
6. Shivakumara, I.S.: Onset of convection in a couple-stress fluid-saturated porous medium: effects of non-uniform temperature gradients. Arch. Appl. Mech. 80(8), 949–957 (2009)
7. Nield, D.A., Bejan, A.: Convection in Porous Media, 3rd edn. Springer, Berlin (2006)
8. Vafai, K.: Handbook of Porous Media. Taylor & Francis, Boca Raton (2005)
9. Gaikwad, S.N., Malashetty, M.S., Prasad, K.R.: An analytical study of linear and non-linear double diffusive convection with Soret and Dufour effects in couple stress fluid. Int. J. Non-Linear Mech. 42(7), 903–913 (2007)
10. Sunil, Sharma, R.C., Pal, M.: On a couple-stress fluid heated from below in a porous medium in the presence of a magnetic field and rotation. J. Porous Media 5(2), 10 (2002)
11. Minkowycz, W.J., Haji-Sheikh, A., Vafai, K.: On departure from local thermal equilibrium in porous media due to a rapidly changing heat source: the sparrow number. Int. J. Heat Mass Transf. 42(18), 3373–3385 (1999)
12. Nield, D.A., Kuznetsov, A.V., Xiong, M.: Effect of local thermal non-equilibrium on thermally developing forced convection in a porous medium. Int. J. Heat Mass Transf. 45(25), 4949–4955 (2002)
13. Brian, S.: Convection with Local Thermal Non-equilibrium and Microfluidic Effects. Springer, Berlin (2017)
14. Sunil, Choudhary, S., Mahajan, A.: Stability analysis of a couple-stress fluid saturating a porous medium with temperature and pressure dependent viscosity using a thermal non-equilibrium model. Appl. Math. Comput. 340, 15–30 (2019)
15. Pan, Z., Jia, L., Mao, Y., Wang, Q.: Transitions and bifurcations in couple stress fluid saturated porous media using a thermal non-equilibrium model. Appl. Math. Comput. 415, 126727 (2022)
16. Sunil, Choudhary, S., Mahajan, A.: Conditional stability for thermal convection in a rotating couple-stress fluid saturating a porous medium with temperature- and pressure-dependent viscosity using a thermal non-equilibrium model. J. Non-Equilib. Thermodyn. 39(2), 61–78 (2014)
17. Kumar, P.: Conditional stability for thermal convection in a rotating couple-stress fluid saturating a porous medium with temperature- and pressure-dependent viscosity using a thermal non-equilibrium model. WSEAS Transactions on Heat and Mass Transfers (2021)
18. Robinson, J.C.: Infinite-Dimensional Dynamical Systems. Cambridge University Press, Cambridge (2009)
19. Ma, T.: Stability and Bifurcation of Nonlinear Evolution Equations. Science Press, Beijing (2007)
20. Ma, Q., Wang, S., Zhong, C.: Necessary and sufficient conditions for the existence of global attractors for semigroups and applications. Indiana Univ. Math. J. 51(6), 1541–1570 (2002)
21. Zhang, Y., Zhong, C., Wang, S.: Attractors in for a class of reaction–diffusion equations. Nonlinear Anal. 71(5–6), 1901–1908 (2009)
22. Luo, H.: Global attractor of atmospheric circulation equations with humidity effect. Abstr. Appl. Anal. 2012, 1 (2012)
23. Evans, L.: Theory and Method of Partial Differential Equations. Science Press, Beijing (2011)
24. Stein, E.M., Shakarchi, R.: Functional Analysis: Introduction to Further Topics in Analysis. Princeton University Press, Princeton (2011)
25. Zhang, Y., Song, L., Ma, T.: The existence of global attractors for 2D Navier-Stokes equations in $H^k$ spaces. Acta Math. Sin. Engl. Ser. 25(1), 51–58 (2008)
26. Song, L., Zhang, Y., Ma, T.: Global attractor of the Cahn–Hilliard equation in $H^k$ spaces. J. Math. Anal. Appl. 355(1), 53–62 (2009)
27. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. SIAM, Philadelphia (2011)
28. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York (2013)
29. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.