PERIODIC MEASURES ARE DENSE IN INVARIANT MEASURES FOR RESIDUALLY FINITE AMENABLE GROUP ACTIONS WITH SPECIFICATION

XIANKUN REN
School of Mathematical Sciences
Peking University
Beijing 100871, China
and
Department of Mathematics
SUNY at Buffalo
Buffalo, NY 14260-2900, USA

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Abstract. We prove that for actions of a discrete countable residually finite amenable group on a compact metric space with specification property, periodic measures are dense in the set of invariant measures. We also prove that certain expansive actions of a countable discrete group by automorphisms of compact abelian groups have specification property.

1. Introduction. Let $G$ be a discrete countable residually finite amenable group acting on a compact metric space $X$. Denote by $\mathcal{M}(X,G)$ the set of $G$-invariant measures and $\mathcal{M}_e(X,G)$ the set of ergodic $G$-invariant measures. For a point $x \in X$, we call $x$ a periodic point if $|\text{orb}(x)| < \infty$. Define the periodic measure $\mu_x$ as a probability measure with mass $|\text{orb}(x)|^{-1}$ at each point of $\text{orb}(x)$ and we denote by $\mathcal{M}_P(X)$ the set of all such periodic measures.

Specification property, introduced by Bowen [3] in 70’s for $\mathbb{Z} -$actions, is a basic property used in smooth and topological dynamical systems to obtain maximal entropy measure, exponential growth of periodic orbits, density of periodic or ergodic measures, multifractal analysis etc. Specification property seems a very strong property, but there are many examples of dynamical systems satisfying this property, including subshifts of finite type, sofic shifts, the restriction of an Axiom A diffeomorphism to its non-wondering set, expanding differential maps and geodesic flows on manifold with negative curvature. Readers may refer [9, Chapter 21] for more details of specification. For non-uniformly hyperbolic dynamical systems, several versions of specification-like were introduced, including [2, 11, 13, 16]. Pfister and Sullivan [18] also introduced a weak specification property called $g$-almost product property, which was renamed as the almost specification by Tompson [22]. In [19], Ruelle introduced the notion of weak specification for $\mathbb{Z}^d$ actions and called the definition in [3] as strong specification. Recently, Chung and Li [6] generalized

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specification to general countable group actions. We will give the details in next section.

For smooth dynamical systems, the problem of density of periodic measures is well studied. For instance, Sigmund [21] proved that for uniformly hyperbolic diffeomorphisms with specification property, each invariant measure can be approximated by periodic measures. Hirayama [11] proved that each invariant measure supported by the closure of a Pesin set of a topological mixing measure is approximated by periodic measures. Liang, Liu and Sun [13] improved Hirayama’s result by weakening the assumption of mixing measure to that of hyperbolic ergodic measure.

Our main results are as follows.

**Theorem 1.1.** Let $G$ be a discrete countable residually finite amenable group acting on a compact metric space $X$ with specification property. Then $\mathcal{M}_P(X, G)$ is dense in $\mathcal{M}(X, G)$ in the weak* topology. Moreover $\mathcal{M}_e(X, G)$ is residual in $\mathcal{M}(X, G)$.

**Theorem 1.2.** Let $\Gamma$ be a countable discrete group and $f$ an element of $\mathbb{Z}\Gamma$ invertible in $l^1(\Gamma, \mathbb{R})$. Then the action of $\Gamma$ on $X_f$ which is the Pontryagin dual of $\mathbb{Z}\Gamma/\mathbb{Z}f$ has specification property.

2. Preliminary. In this section, we will recall some notions and basic facts about amenable groups and residually finite groups. Also we will give the definition of specification.

2.1. Amenable group. A countable group $G$ is amenable if there exists a sequence of nonempty finite subsets $\{F_n\}_{n \in \mathbb{N}}$ of $G$ satisfying

$$\lim_{n \to \infty} \frac{|F_n \triangle K F_n|}{|F_n|} = 0, \forall \text{ nonempty } K \in \mathcal{F}(G),$$

where $\mathcal{F}(G)$ is the collection of all finite subsets of $G$. Such sequences are called Følner sequences.

The quasi-tiling-theory is a useful tool for actions of amenable groups which is set up by Ornstein and Weiss in [17].

Subsets $A_1, A_2, \cdots, A_k \in \mathcal{F}(G)$ are $\varepsilon$-disjoint if there exists $\{B_1, B_2, \cdots, B_k\} \subset \mathcal{F}(G)$ such that

1. $B_i \subset A_i, i = 1, 2, \cdots, k,$
2. $B_i \cap B_j = \emptyset, 1 \leq i \neq j \leq k,$
3. $|B_i| > 1 - \varepsilon, i = 1, 2, \cdots, k.$

For $\alpha \in (0, 1]$, we say $\{A_1, A_2, \cdots, A_k\}$ $\alpha$-covers $A \in \mathcal{F}(G)$ if

$$\frac{A \cap (\bigcup_{i=1}^k A_i)}{|A|} \geq \alpha.$$

We say that $\{A_1, A_2, \cdots, A_k\} \subset \mathcal{F}(G)$ $\varepsilon$-quasi-tile $A \in \mathcal{F}(G)$ if there exists $\{C_1, C_2, \cdots, C_k\} \subset \mathcal{F}(G)$ satisfying

1. $A_i \subset C_i$ and $\{A_i c | c \in C_i\}$ forms an $\varepsilon$-disjoint family for $i = 1, 2, \cdots, k$,
2. $A_i \cap A_j C_j \neq \emptyset, 1 \leq i \neq j \leq k,$
3. $\{A_i C_i | i = 1, 2, \cdots, k\}$ forms a $(1 - \varepsilon)$-cover of $A$.

The subsets $C_1, C_2, \cdots, C_k$ are called the tiling centres.

The next proposition is [7, Lemma 9.4.14].

**Proposition 1.** Let $\{F_n\}_{n \in \mathbb{N}}$ and $\{F_n'\}_{n \in \mathbb{N}}$ be two Følner sequences of $G$. Then for any $\varepsilon \in (0, \frac{1}{4})$ and $N \in \mathbb{N}$ there exist integers $n_1, n_2, \cdots, n_k$ with $n_k > n_{k-1} > \cdots > n_1 > N$ such that $F_{n_1}, F_{n_2}, \cdots, F_{n_k}$ $\varepsilon$-quasi-tile $F_m$ when $m$ is large enough.
For our proof, we also need the Mean Ergodic Theorem for amenable group actions.

**Lemma 2.1 (Mean Ergodic Theorem).** Let $G$ be an amenable group acting on a probability measure space $(X,\mathcal{B},\mu)$ by measure preserving transformation, and let $\{F_n\}_{n\in \mathbb{N}}$ be a Følner sequence. For any $f \in L^2(\mu)$, set $A_n(f)(x) = \frac{1}{|F_n|} \sum_{g \in F_n} f(gx)$.

Then for any $f \in L^2(\mu)$, there is a $G$-invariant $f^* \in L^2(\mu)$ such that

$$\lim_{n \to \infty} A_n(f) = f^* \text{ in } L^2.$$ 

Moreover $\int f(x) d\mu = \int f^*(x) d\mu$.

For more information about ergodic theorem, readers may refer [14], [15] or [10, Chapter 8].

**2.2. Residually finite group.** A group is **residually finite** if the intersection of all its normal subgroups of finite index is trivial. Examples of groups that are residually finite are finite groups, free groups, finitely generated nilpotent groups, polycyclic-by-finite groups, finitely generated linear groups and fundamental groups of 3-manifolds. For more information about residually finite groups, readers can refer [5, Chapter 2].

Let $(G_n, n \geq 1)$ be a sequence of finite index normal subgroups in $G$. We say

$$\lim_{n \to \infty} G_n = \{e_G\}$$

if we can find, for any nonempty finite subset $K$ of $G$, an $N > 1$ with $G_n \cap (K^{-1}K) = \{e_G\}$ for every $n \geq N$. Clearly such sequence exists if $G$ is countable and residually finite.

If $G' \subset G$ is a subgroup with finite index, we say that $Q \subset G$ is a **fundamental domain** of the right coset space $G'/G$, i.e. a finite subset such that $\{G's | s \in Q\}$ is a partition of $G$.

The following proposition is [8, Corollary 5.6] and we will use it to control the periodic orbits we get. The original proof is a version of the Ornstein-Weiss quasi-tiling lemma. Also there is an algebraic proof from [1, Theorem 6].

**Proposition 2.** Let $G$ be a countable discrete residually finite amenable group and let $(G_n, n \geq 1)$ be a sequence of finite index normal subgroup with $\lim_{n \to \infty} G_n = \{e_G\}$. Then there exists a Følner sequence $(Q_n, n \geq 1)$ such that $Q_n$ is a fundamental domain of $G/G_n$ for every $n \geq 1$.

**2.3. Specification.** In this subsection, we will recall specification property of general group actions, which is from [6, Section 6].

Let $\alpha$ be a continuous $G$-action on a compact metric space $X$ with metric $\rho$. The action has specification property if there exist, for every $\varepsilon > 0$, a nonempty finite subset $F = F(\varepsilon)$ of $G$ with the following property:

for any finite collection of finite subsets $F_1, F_2, \ldots, F_m$ of $G$ satisfying

$$FF_i \cap F_j = \emptyset \quad 1 \leq i \neq j \leq m,$$

and for any subgroup $G'$ of $G$ with

$$FF_i \cap F_j(G' \setminus \{e_G\}) = \emptyset \quad \text{for} \quad 1 \leq i, j \leq m,$$

and for any collection of points $x^1, x^2, \ldots, x^m \in X$, there is a point $y \in X$ satisfying

$$\rho(sx^i, sy) \leq \varepsilon \quad \text{for all} \quad s \in F_i, \quad 1 \leq i \leq m,$$
and \( sy = y \) for all \( s \in G' \).

In [6], this property is called strong specification. We will call it specification since there is no misunderstanding.

3. **Proof of Theorem 1.1.** Let \((G_n, n \geq 1)\) be a sequence of finite index normal subgroups with \( \lim_{n \to \infty} G_n = \{e_G\} \) and \((Q_n, n \geq 1)\) be a Følner sequence such that \( Q_n \) is a fundamental domain of \( G/G_n \) as described in Proposition 2.

Let \( \nu \in \mathcal{M}(X, G), \varepsilon > 0 \) and \( W \) a finite subset of \( C(X) \), where \( C(X) \) is the set of all the continuous real valued functions on \( X \). Uniformly continuity of the elements of \( W \) implies that there is \( \delta \in (0, \varepsilon) \) such that \( |\xi(x) - \xi(y)| < \frac{\varepsilon}{5} \) for all \( x, y \in X \) with \( d(x, y) < \delta \) and all \( \xi \in W \). By Mean Ergodic theorem, \( A_n(\xi) \) converges to \( \xi^* \) in \( L^2 \) for all \( \xi \in W \), so we can choose a subsequence \( \{A_{n_k}\}_{k \in \mathbb{N}} \) such that \( A_{n_k}(\xi) \) converges to \( \xi^* \) \( \nu \)-a.e. for all \( \xi \in W \). For convenience, we will write the subsequence \((Q_{n_k}, k \geq 1)\) as \((Q_n, n \geq 1)\).

Set
\[
Q(G) = \{x \in X : \lim_{n \to \infty} \frac{1}{|Q_n|} \sum_{g \in Q_n} \xi(gx) \text{ exists for all } \xi \in C(X)\}.
\]

We know \( \nu(Q(G)) = 1 \). Denote by \( \xi^*(x) \) the limit for each \( x \in Q(G) \).

Next we will construct a finite partition of \( X \) as following:

Set \( D = \max_{\xi \in W} \|\xi\|_{\infty} \).

For \( j = 1, 2, \ldots, \left\lceil \frac{16|D|}{\varepsilon} \right\rceil + 1 \), set
\[
Q_j(\xi) = \{x \in Q(G) \mid -D + \frac{j-1}{8}\varepsilon \leq \xi^*(x) < -D + \frac{j}{8}\varepsilon\}.
\]

Since \( W \) is finite,
\[
\eta := \bigvee_{\xi \in W} \{Q_1(\xi), \ldots, Q_{\left\lceil \frac{16|D|}{\varepsilon} \right\rceil+1}(\xi)\}
\]
is a finite partition of \( Q(G) \), where \( \alpha \vee \beta = \{A_i \cap B_j : A_i \in \alpha, B_j \in \beta\} \) for partitions \( \alpha = \{A_i\} \) and \( \beta = \{B_j\} \).

Next we will construct \( F_1, F_2, \ldots, F_l \) and \( G_m \) satisfying (1) and (2) in specification property. The idea is from [25, Theorem 1.3] but with very minor changes. Suppose \( \eta = \{A_1, A_2, \ldots, A_l\} \). Let \( a_i = \nu(A_i) \) for \( i = 1, 2, \ldots, l \) and \( a = \min\{a_i : i = 1, 2, \ldots, l\} \). By Egorov’s Theorem, there exist a Borel subset \( X' \subset X \) with \( \nu(X') > 1 - \frac{1}{4}a \) such that \( \frac{1}{|Q_n|} \sum_{g \in Q_n} \xi(gx) \) converges to \( \xi^*(x) \) uniformly on \( X' \).

Take \( 0 < \gamma < \min\left(\frac{\delta}{4}, \frac{\delta}{4m}, \frac{\delta}{4|m|}, \frac{\delta}{4m}, \frac{\delta}{4m}\right) \) and \( F = F(\gamma) \) as in specification property. Take \( N_1 \in \mathbb{N} \) such that for all \( x \in X' \), \( \frac{1}{|Q_n|} \sum_{g \in Q_n} \xi(gx) - \xi^*(x) \mid < \frac{1}{16}\varepsilon \), for all \( n \geq N_1 \). Take \( N_2 > N_1 \) large enough s.t. \( \frac{|gQ_n \Delta Q_n|}{|Q_n|} < \frac{\gamma}{|F|}\varepsilon \) for all \( g \in F \), \( n \geq N_2 \). By Proposition 1, there exist \( n_k > n_{k-1} > \cdots > n_1 \geq N_2 \) and \( N_3 \in \mathbb{N} \) s.t. \( Q_m \) can be \( \frac{16m^2}{\varepsilon} \)-quasi-tiled by \( Q_{n_1}, Q_{n_2}, \ldots, Q_{n_k} \) when \( m > N_3 \). Also \( N_3 \) will be large enough such that the family of all the translations
\[
\mathcal{F} = \{Q_{n_jc_j} : 1 \leq j \leq k, c_j \in C_j\}
\]
can be partitioned into \( l \) subfamilies \( \mathcal{F}_l \), \( 1 \leq i \leq l \) satisfying
\[
\frac{1}{|Q_m|} \sum_{g \in Q_m} \xi(gx) - a_i < \frac{\gamma}{l},
\]
where \( \bigcup \mathcal{F} = \bigcup_{Q_n,j \in \mathcal{F}} \{Q_n, c_j\} \). Moreover, we can choose the elements in \( \mathcal{F} \) to be pairwise \( \frac{\gamma}{2|\mathcal{F}|} \)-disjoint. We can choose \( \{ T_{n_j}(c_j) \subset Q_n, c_j \in \mathcal{F} \} \) are pairwise disjoint and \( \frac{|T_{n_j}(c_j)|}{|Q_n|} > 1 - \frac{\gamma}{2|\mathcal{F}|} \). Denote

\[
T_{n_j}(c_j) = \{ s \in T_{n_j}(c_j) | Fs \subset Q_n \}
\]

and

\[
S_{n_j}(c_j) = T_{n_j}(c_j) \cap (\cap_{g \in F} g^{-1} T_{n_j}(c_j)).
\]

By the definition, we know \( \frac{|Q_n \setminus T_{n_j}(c_j)|}{|Q_n|} < \frac{\gamma}{2|\mathcal{F}|} \) and \( \frac{|Q_n \setminus S_{n_j}(c_j)|}{|Q_n|} < \gamma \). Denote \( \tilde{\mathcal{F}} = \{ S_{n_j}(c_j) | S_{n_j}(c_j) \subset Q_n, c_j \in \tilde{\mathcal{F}} \} \) and \( \tilde{\mathcal{F}}_i = \{ S_{n_j}(c_j) | Q_n, c_j \in \mathcal{F}_i \} \).

**Claim.** \( \{ S_{n_j}(c_j) | S_{n_j}(c_j) \subset \tilde{\mathcal{F}} \} \) and \( G_m \) satisfy the conditions in specification property.

For different \( S_{n_j}(c_j) c_j \) and \( S_{n_j}(c_j) c_j \), we have

\[
(S_{n_j}(c_j) c_j \cap FS_{n_j}(c_j) c_j) \subset (S_{n_j}(c_j) c_j \cap T_{n_j}(c_j) c_j)
\]

\[
\subset (T_{n_j}(c_j) c_j \cap T_{n_j}(c_j) c_j) = \emptyset.
\]

By Proposition 2, we know

\[
G = \bigcup_{g \in Q_m} G_m
\]

\[
= \bigcup_{g \in Q_m} gG_m
\]

\[
= \bigcup_{g \in Q_m} (g \bigcup (G_m \setminus \{c_G\})).
\]

For different \( S_{n_j}(c_j) c_j \) and \( S_{n_j}(c_j) c_j \), since \( FS_{n_j}(c_j) c_j \cap S_{n_j}(c_j) c_j = \emptyset \), by (5), \( FS_{n_j}(c_j) c_j \cap S_{n_j}(c_j) c_j G_m = \emptyset \); for the same \( S_{n_j}(c_j) c_j \) and \( S_{n_j}(c_j) c_j \), by (6), we get \( FS_{n_j}(c_j) c_j \cap S_{n_j}(c_j) c_j G_m \setminus \{c_G\} = \emptyset \). We get that \( \{ S_{n_j}(c_j) c_j \} \) and \( G_m \) satisfy the conditions (1) and (2) in specification property.

For \( S_n(c_j) c_j \in \tilde{\mathcal{F}}_i \), pick \( x_j(c_j) \) satisfying \( x_j(c_j) \in c_j^{-1} A_i \cap X' \). Using specification property, there is some \( y \in \mathcal{X} \) such that \( \rho(g x_j(c_j), y) < \gamma < \delta \), for all \( g \in S_n(c_j) c_j \). Denote by \( \mu_y \) the periodic measure supported on \( orb(y) \).

**Claim.** \( |\int \xi d\nu - \int \xi d\mu_y| < \varepsilon \), for all \( \xi \in \mathcal{W} \).

\[
|\int \xi d\nu - \int \xi d\mu_y| = \left| \int \xi^*(x) d\nu - \int \xi(x) d\mu_y \right|
\]

\[
\leq \left| \int \xi^*(x) d\nu - \frac{1}{|\cup \tilde{\mathcal{F}}|} \sum_{i=1}^l \sum_{S_{n_j}(c_j) c_j \in \tilde{\mathcal{F}}_i} \sum_{g \in S_{n_j}(c_j) c_j} \xi(gy) \right|
\]

\[
+ \left| \frac{1}{|\cup \tilde{\mathcal{F}}|} \sum_{i=1}^l \sum_{S_{n_j}(c_j) c_j \in \tilde{\mathcal{F}}_i} \sum_{g \in S_{n_j}(c_j) c_j} \xi(gy) - \frac{1}{|Q_m|} \sum_{g \in Q_m} \xi(gy) \right|
\]
\begin{align*}
&\leq \left| \int \xi^*(x) d\nu - \frac{1}{|\cup F|} \sum_{i=1}^l \sum_{n_j(c_j) \in \tilde{F}_i} \sum_{g \in S_{n_j}(c_j)c_j} \xi(g x_{n_j}(c_j)) \right| \\
&\quad + \frac{\varepsilon}{4}. \tag{7}
\end{align*}

Pick \( x_i \in A_i \cap X' \), then \( |\xi^*(x) - \xi^*(x_i)| < \frac{\varepsilon}{8} \) for all \( x \in A_i \) by the construction of \( \eta \). Then
\begin{align*}
&\left| \frac{1}{|Q_{n_j}|} \sum_{g \in Q_{n_j}} \xi(g c_j x_j(c_j)) - \frac{1}{|Q_{n_j}|} \sum_{g \in Q_{n_j}} \xi(g x_i) \right| \leq |\xi^*(c_j x_j(c_j)) - \xi^*(x_i)| + \frac{\varepsilon}{8} \\
&\quad \leq \frac{1}{4} \varepsilon. \tag{8}
\end{align*}

Also
\begin{align*}
&\left| \frac{1}{|S_{n_j}(c_j)|} \sum_{g \in S_{n_j}(c_j)} \xi(g c_j x_j(c_j)) - \frac{1}{|Q_{n_j}|} \sum_{g \in Q_{n_j}} \xi(g c_j x_j(c_j)) \right| \\
&\quad \leq \left| \frac{1}{|S_{n_j}(c_j)|} \sum_{g \in S_{n_j}(c_j)} \xi(g c_j x_j(c_j)) - \frac{1}{|S_{n_j}(c_j)|} \sum_{g \in S_{n_j}(c_j)} \xi(g c_j x_j(c_j)) \right| \\
&\quad + \left| \frac{1}{|S_{n_j}(c_j)|} \sum_{g \in S_{n_j}(c_j)} \xi(g c_j x_j(c_j)) - \frac{1}{|S_{n_j}(c_j)|} \sum_{g \in S_{n_j}(c_j)} \xi(g c_j x_j(c_j)) \right| \\
&\quad \leq \frac{1}{|S_{n_j}(c_j)|} D|Q_{n_j} \setminus S_{n_j}(c_j)| + \frac{|Q_{n_j}| D}{|S_{n_j}(c_j)||Q_{n_j}|} (|Q_{n_j}| - |S_{n_j}(c_j)|) \\
&\quad \leq \frac{2 D \gamma}{1 - \gamma} \\
&\quad \leq \frac{1}{4} \varepsilon. \tag{9}
\end{align*}

Using (7)-(9), we have,
\begin{align*}
&\left| \int \xi d\nu - \int \xi d\mu_y \right| \leq \left| \int \xi^* d\nu - \sum_{i=1}^l \sum_{n_j(c_j) \in \tilde{F}_i} \frac{|S_{n_j}(c_j)|}{|Q_{n_j}|} \sum_{g \in Q_{n_j}} \xi(g c_j x_j(c_j)) \right| \\
&\quad + \frac{3}{8} \varepsilon \\
&\quad \leq \left| \int \xi^*(x) d\nu - \sum_{i=1}^l \sum_{n_j(c_j) \in \tilde{F}_i} \frac{|S_{n_j}(c_j)|}{|\cup F|} \xi^*(c_j x_j(c_j)) \right| + \frac{1}{2} \varepsilon \\
&\quad \leq \left| \int \xi^*(x) d\nu - \sum_{i=1}^l a_i \xi^*(x_i) \right| + \frac{3}{4} \varepsilon \\
&\quad \leq \sum_{i=1}^l \int_{A_i} \xi^*(x) - \xi^*(x_i) d\nu + \frac{7}{8} \varepsilon \\
&\quad \leq \varepsilon. \tag{10}
\end{align*}
To finish the proof, we also need the following claim.

**Claim.** \((K,\rho)\) is a convex compact metric set and denote by \(\text{ext}(K)\) the set of extreme points of \(K\). Then \(\text{ext}(K)\) is a \(G_δ\) subset of \(K\).

Let

\[ K_n = \{ x \in K : \text{ there exist } y, z \in K \text{ such that } x = \frac{1}{2}(y + z) \text{ and } d(y, z) \geq \frac{1}{n} \}. \]

Obviously, \(K_n\) is closed. Set \(K_0 = \bigcup_{n \geq 1} K_n\). It is obvious \(K_0\) is a \(F_σ\) subset. Easy to check

\[ x \notin \text{ext}(K) \iff x \in K_0. \]

As a result, \(\text{ext}(K) = K \setminus K_0\) is a \(G_δ\) subset of \(K\).

We know that \(\mathcal{M}_e(X, G)\) is the set of extreme points of \(\mathcal{M}(X, G)\). So by the claim, \(\mathcal{M}_e(X, G)\) is a \(G_δ\) set. Thus we finish the proof. \(\square\)

4. **Proof of Theorem 1.2.** For a countable group \(\Gamma\) and an element \(f = \sum f_s s\) in the integral group ring \(\mathbb{Z}\Gamma\), where \(\mathbb{Z}\Gamma\) is the set of finitely supported \(\mathbb{Z}\)-valued functions on \(\Gamma\), consider the quotient \(\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f\) of \(\mathbb{Z}\Gamma\) by the left ideal \(\mathbb{Z}\Gamma f\) generated by \(f\). It is a discrete abelian group with a left \(\Gamma\)-action by multiplication. The Pontryagin dual of \(\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f\) which is denoted by \(X_f\) is a compact metrizable abelian group with a left action of \(\Gamma\) by continuous group automorphisms and denote by \(\rho\) some compliable metric on \(X_f\). Denote \(X = (\mathbb{R}/\mathbb{Z})^\Gamma\). We denote by \(\rho_1\) the canonical metric on \(\mathbb{R}/\mathbb{Z}\) defined by

\[ \rho_1(t + \mathbb{Z}, s + \mathbb{Z}) := \min_{m \in \mathbb{Z}} |t - s - m|. \]

The left and right actions \(l\) and \(r\) on \(X\) are defined by

\[(l^s x)_t = x_{s-1t} \text{ and } (r^s x)_t = x_{ts}\]

for every \(s, t \in \Gamma\) and \(x \in X\). We can extend those actions of \(\Gamma\) to commuting actions \(l\) and \(r\) of \(\mathbb{Z}\Gamma\) on \(X\) by setting

\[ l_f x = \sum_{s \in \Gamma} f_s l^s x \text{ and } r_f x = \sum_{s \in \Gamma} f_s r^s x. \]

It is easy to check \(X_f = \{ x \in X : r_f x = xf^* = 0 \}\), where \(f^* = \sum f_s s^{-1}\). Denote by

\[ \alpha_f = l|_{X_f}, \]

the restriction to \(X_f\) of the \(\Gamma\)-action \(l\) on \(X\).

From [8, Theorem 3.2], the action \(\alpha_f\) is expansive if and only if \(f\) is invertible in \(l^1(\Gamma)\). Denote by \(P\) the canonical map \(l^\infty(\Gamma, \mathbb{R}) \to X\). Let \(\xi = P \circ r_f^{-1} : l^\infty(\Gamma, \mathbb{Z}) \to X_f\). By [8, Proposition 4.2], \(\xi\) is a surjective group homomorphism.

The following lemma is [8, Lemma 4.5].

**Lemma 4.1.** For every \(x \in X_f\) there exists an element \(v \in l^\infty(\Gamma, \mathbb{Z})\) with \(\xi(v) = x\) and \(\|v\|_\infty \leq \frac{\|f\|_1}{2}\).

**Proof.** The proof is really easy. We choose \(w \in P^{-1}(X_f) \subset l^\infty(\Gamma, \mathbb{R})\) with \(P(w) = x\) and \(-\frac{1}{2} \leq w_s < \frac{1}{2}\) for every \(s \in \Gamma\). Then \(v = r_f(w)\) is what we need. \(\square\)

Let \(e_\Gamma\) and \(e_{X_f}\) be the unit elements of \(\Gamma\) and \(X_f\) respectively. Set \(W = \{ e_\Gamma \} \cup \text{ support}(f^*) = (\{ e_\Gamma \} \cup \text{ support}(f))^{-1}\). Let \(\varepsilon > 0\). Then we can find a nonempty finite subset \(W_1\) of \(\Gamma\) and \(\varepsilon_1 \in (0, \|f\|_1^{-1})\) such that if \(x, y \in X_f\) satisfy max\(s \in W_1 | x_s - y_s | ≤ \varepsilon_1\) then \(\xi(v) = x\).
\( \|v\|_\infty \leq \frac{\|f\|_1}{2} \) and \( \xi(v^i) = x^i \) for \( i = 1, 2, \ldots, m \). Let \( v \in l^\infty(\Gamma, \mathbb{Z}) \) be a point with \( \|v\|_\infty \leq \frac{\|f\|_1}{2} \) and \( v_s = v_s^i \) for \( s \in F^{-1}_iW_1W_2 \). \( v \) is well defined since \( F^{-1}_iW_1W_2 \cap F^{-1}_jW_1W_2 = \emptyset \) for \( 1 \leq i \neq j \leq m \).

Let \( \tilde{w} \in X \) be defined by

\[
\tilde{w}_s = \begin{cases} (f^{-1})_s & s \in W_2, \\ 0 & \text{otherwise.} \end{cases}
\]

**Claim 1.** \( y = \xi(v) \) satisfies \( \rho(sx^i, sy) < \varepsilon \) \( \forall s \in F_i \), \( i = 1, 2, \ldots, m \).

By the choice of \( W_1 \), we just need to show \( \rho_1((sx^i)_l, (sy)_l) < 2\varepsilon_1 \) \( \forall s \in F_i \) and \( t \in W_1 \) i.e. \( \rho_1(x^i_s, y_s) < 2\varepsilon_1 \) for all \( s \in F^{-1}_iW_1 \).

For any \( s \in F^{-1}_iW_1 \),

\[
\rho_1(x^i_s, y_s) = \rho_1(P \circ r_{f-1}(v^i)_s, P \circ r_{f-1}(v)_s) \\
\leq \rho_1(P \circ r_{f-1}(v^i)_s, P \circ r_{\tilde{w}}(v^i)_s) + \rho_1(P \circ r_{\tilde{w}}(v^i)_s, P \circ r_{f-1}(v^i)_s).
\]

For \( s \in F^{-1}_iW_1 \), we get the following calculations,

\[
| r_{f-1}(v^i)_s - r_{\tilde{w}}(v^i)_s | = \left| \sum_{t \in \Gamma} f^{i-1}_t v^i_{st} - \tilde{w}_t v^i_{st} \right| \\
\leq \left| \sum_{t \in W_2} f^{i-1}_t v^i_{st} - \tilde{w}_t v^i_{st} \right| + \left| \sum_{t \in \Gamma \setminus W_2} f^{i-1}_t v^i_{st} - \tilde{w}_t v^i_{st} \right| \\
\leq \frac{\varepsilon_1}{2 \|f\|_1} \cdot \frac{\|f\|_1}{2} = \frac{\varepsilon_1}{4}, \tag{11}
\]

and

\[
| r_{\tilde{w}}(v^i)_s - r_{\tilde{w}}(v)_s | = \left| \sum_{t \in \Gamma} \tilde{w}_t (v^i)_st - \tilde{w}_t v_{st} \right| \\
= \left| \sum_{t \in W_2} \tilde{w}_t (v^i)_st - \tilde{w}_t v_{st} \right| = 0, \tag{12}
\]

and

\[
| r_{\tilde{w}}(v)_s - r_{f-1}(v)_s | = \left| \sum_{t \in \Gamma} \tilde{w}_t v_{st} - f^{i-1}_t v_{st} \right| \\
\leq \left| \sum_{t \in W_2} \tilde{w}_t v_{st} - f^{i-1}_t v_{st} \right| + \left| \sum_{t \in \Gamma \setminus W_2} \tilde{w}_t v_{st} - f^{i-1}_t v_{st} \right| \\
\leq \left| \sum_{t \in W_2} f^{i-1}_t v_{st} \right| \\
\leq \frac{\varepsilon_1}{4}. \tag{13}
\]

By (11)–(13), we get

\[
\rho(sx^i, sy) < \varepsilon \text{ for all } s \in F_i, \ i = 1, 2, \ldots, m.
\]
The following lemma is a version of [4, Lemma 1]. The same argument also appeared in the proof of [6, Lemma 6.2].

**Lemma 4.2.** Let \( \alpha \) be an expansive continuous action of \( \Gamma \) on a compact metric space \((X, \rho)\). Let \( d > 0 \) such that if \( x, y \in X \) satisfy \( \sup_{s \in \Gamma} \rho(sx, sy) \leq d \), then \( x = y \). Let \( x, y \) satisfy \( \rho(sx, sy) \leq d \) for all but finitely many \( s \in \Gamma \). Then \( \rho(sx, sy) \to 0 \) as \( \Gamma \ni s \to \infty \).

A point \( x \in X_f \) is said to be homoclinic if \( sx \to e \) for any \( s \to \infty \). The set of all homoclinic points, denoted by \( \Delta(X_f) \), is a \( \Gamma \)-invariant normal subgroup of \( X_f \).

**Claim 2.** Let \( d \) be the expansive constant of \((X_f, \alpha_f)\) i.e. if \( x, y \in X_f \) with \( \rho(sx, sy) \leq d \) for all \( s \in \Gamma \) then \( x = y \). For any \( \varepsilon \in (0, d) \), \( F = W_1 W_2 (W_1 W_2)^{-1} \), any finite subset \( F_1 \) of \( \Gamma \) and \( x \in X_f \), there exists \( y \in \Delta(X_f) \) s.t. \( \max_{s \in F_1} \rho(sx, sy) \leq \varepsilon \) and \( \sup_{s \in \Gamma \setminus F_1} \rho(se_{X_f}, sy) \leq \varepsilon \).

To prove the above claim, we may assume \( F = F^{-1} \) otherwise we can replace \( F \) by \( F \cup F^{-1} \). Let \( F_1 \) be a finite subset of \( \Gamma \) and \( x \in X_f \). For each finite set \( F_2 \subset \Gamma \setminus F_1 \), from Claim 1, we can find \( y_{F_2} \in X_f \) such that \( \rho(sx, sy_{F_2}) \leq \varepsilon \) for all \( s \in F_1 \) and \( \rho(se_{X_f}, sy_{F_2}) \leq \varepsilon \) for all \( s \in F_2 \). Note that the collection of the finite subsets of \( \Gamma \setminus F_1 \) has a partial order. Take a limit point \( y \in X_f \) of \( \{y_{F_2}\} \). Then \( \rho(sx, sy) \leq \varepsilon \) for all \( s \in F_1 \) and \( \rho(se_{X_f}, sy) \leq \varepsilon \) for all \( \Gamma \setminus F_1 \). By Lemma 4.2, we know \( y \in \Delta(X_f) \).

Take some \( \delta > 0 \) such that \( \rho(x, y) < \delta \) implies \( |x_{e \Gamma} - y_{e \Gamma}| < \varepsilon_1 \), for all \( x, y \in X_f \).

Take \( W_3 = F(\delta) \) as described in Claim 2, such that for any finite subsets \( F_1 \) of \( \Gamma \) and \( x \in X_f \) there exists \( y \in \Delta(X_f) \) with \( \rho(sx, sy) < \delta \) for all \( s \in F_1 \) and \( \rho(sy, e_{X_f}) \leq \varepsilon_1 \) for all \( s \in \Gamma \setminus W_3 F_1 \) which implies \( |x_s - y_s| < \varepsilon_1 \) for all \( s \in F_1^{-1} \) and \( |y_s| < \varepsilon_1 \) for all \( s \in \Gamma \setminus (W_3 F_1)^{-1} \) by the choice of \( \delta \).

Set \( F = W_1 W_2 W_3^{-1} W(W_1 W_2 W_3^{-1} W)^{-1} \).

For any finite collection of finite subsets \( F_1, F_2, \ldots, F_m \) of \( \Gamma \) and subgroup \( \Gamma' \) satisfy Condition 5 and 6, by the form of \( F \), we will rewrite the conditions as

\[
F_i^{-1} W_1 W_2 W_3^{-1} W \cap F_j^{-1} W_1 W_2 W_3^{-1} W = \emptyset, \quad 1 \leq i \neq j \leq m, \quad (14)
\]

and

\[
F_i^{-1} W_1 W_2 W_3^{-1} W \cap (\Gamma' \setminus \{e\}) F_j^{-1} W_1 W_2 W_3^{-1} W = \emptyset, \quad 1 \leq i, j \leq m. \quad (15)
\]

For any collection of points \( x^1, x^2, \ldots, x^m \) in \( X_f \), we can pick \( y^1, y^2, \ldots, y^m \in \Delta(X_f) \) with

\[
\max_{s \in (W_1 W_2)^{-1} F_i} \rho(sx^i, sy^i) \leq \delta \quad \text{and} \quad \sup_{s \in \Gamma \setminus (W_3 (W_1 W_2)^{-1} F_i)} \rho(sy^i, e_{X_f}) \leq \delta
\]

which implies

\[
\max_{s \in (W_1 W_2)^{-1} F_i} \rho_1(x^i_s, y^i_s) \leq \varepsilon_1 \quad \text{and} \quad \sup_{s \in \Gamma \setminus (F_i^{-1} W_1 W_2 W_3^{-1})} \rho_1(y^i_s, 0) \leq \varepsilon_1 \quad (16)
\]

for \( i = 1, 2, \ldots, m \). Take \( \tilde{y}^i \in [-\frac{1}{2}, \frac{1}{2}]^\Gamma \) and \( \tilde{x}^i \in [-1, 1]^\Gamma \) with \( P(\tilde{y}^i) = y^i \) and \( P(\tilde{x}^i) = x^i \) respectively such that

\[
|\tilde{x}^i_s - \tilde{y}^i_s| \leq \varepsilon_1 \quad \forall s \in F_i^{-1} W_1 W_2 \quad \text{and} \quad |\hat{y}^i_s| \leq \varepsilon_1 \quad \forall s \in \Gamma \setminus (F_i^{-1} W_1 W_2 W_3^{-1}).
\]

For any \( s \in \Gamma \setminus (F_i^{-1} W_1 W_2 W_3^{-1}) \), one has

\[
|\langle P(F_s \tilde{y}) \rangle| = \left| \sum_{t \in \Gamma} f_t \tilde{y}_{st} \right| = \left| \sum_{t \in W^{-1}} f_t \tilde{y}_{st} \right| \leq \|f\|_1 \sup_{s \in \Gamma \setminus (F_i^{-1} W_1 W_2 W_3^{-1})} |\tilde{y}| < \|f\|_1 \varepsilon_1 < 1.
\]
We note that \( r_f y_t^i \in l^\infty(\Gamma, \mathbb{Z}) \), thus \( \text{support}(r_f y_t^i) \subset F_i^{-1}W_1W_2W_3^{-1}W \). By (14) and (15), we get the elements \( sr_f y_t^i \in l^\infty(\Gamma, \mathbb{Z}) \) for \( s \in \Gamma' \) and \( 1 \leq i \leq m \) have pairwise disjoint supports. Set \( \tilde{z} = \sum_{i \in \Gamma'} \sum_{i=1}^m sr_f y_t^i \). By the definition, one gets \( \tilde{z} \in l^\infty(\Gamma, \mathbb{Z}) \) and

\[
\|\tilde{z}\|_{\infty} = \max_{1 \leq i \leq m} \|r_f y_t^i\| \leq \|f\|_1 \max_{1 \leq i \leq m} \|y_t^i\|_{\infty} \leq \frac{\|f\|_1}{2}.
\]

Set \( \tilde{y} = r_f^{-1}(\tilde{z}) \) and \( y = P(\tilde{y}) = \xi(\tilde{z}) \).

**Claim 3.** \( y \) satisfies the conditions in specification property.

From the definition of \( \xi \), one gets \( y \in X_f \). For each \( s \in \Gamma' \), we have \( s\tilde{z} = z \) and hence \( sy = y \). For \( x', y' \in l^\infty(\Gamma, \mathbb{R}) \) satisfying \( \|x'\|_{\infty}, \|y'\|_{\infty} \leq \|f\|_1 \) and \( x', y' \) are equal on \( sW \) for some \( s \in \Gamma \), we have

\[
|(r_{f^{-1}}x')_s - (r_{f^{-1}}y')_s| = \left| \sum_{t \in \Gamma} (f_t^{-1}x_{st} - f_t^{-1}y_{st}) \right| = \left| \sum_{t \in \Gamma \setminus W_2} (f_t^{-1}x_{st} - f_t^{-1}y_{st}) \right| \leq \frac{\varepsilon_1}{2\|f\|_1} \cdot 2\|f\|_1 = \varepsilon_1.
\]

For any \( t \in W_1 \) and \( s \in F_i \),

\[
\rho_1((sx')_t, (sy)_t) \leq \rho_1(x_{s-t}^i, y_{s-t}^i) + \rho_1(y_{s-t}^i, y_{s-t}^i) \leq 2\varepsilon_1,
\]

for \( i = 1, 2, \ldots, m \) and the last inequality comes from (16) and (17).

By the choice of \( W_1 \), we finish the proof.

Combining Theorem 1.1 and Theorem 1.2, we can get the following corollary.

**Corollary 1.** Let \( \Gamma \) be a countable discrete residually finite amenable group and \( f \in ZT \) invertible in \( l^\infty(\Gamma, \mathbb{R}) \). Then for the system \((X_f, \alpha_f), \mathcal{M}_P(X_f)\) is dense in \( \mathcal{M}(X_f, \alpha_f) \).

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PERIODIC MEASURES ARE DENSE IN INVARIANT MEASURES

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E-mail address: renxiankun@pku.edu.cn