Erratum: More on the flavor dependence of $m_\rho/f_\pi$

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Tree-level correlators and decay constant

The $m_q/f_\pi$ ratio was computed in the chiral-continuum limit in SU(3) gauge theory coupled to various numbers of fermions in the fundamental representation via non-perturbative lattice simulations [1, 2]. None of the non-perturbative results are affected by this erratum.

The issue to be corrected here concerns the free theory which was used for illustration and comparison only. Clearly, in a free theory both $m_\varrho$ and $f_\pi$ first needs to be defined at finite fermion mass $m$ and the chiral limit should be taken only for the ratio. Naturally, $m_\varrho = m_\pi = 2m$ where $m$ is the fermion mass. In [3] the result $f_\pi = \sqrt{12}/L$ was obtained from lattice simulations extrapolated to the continuum, in finite volume $m_\pi L = 1$. The convention for the normalization of $f_\pi$ used in [3] was not specified and it turns out it corresponded to $130 \text{ MeV}$ in QCD, which differs from our convention by a factor $\sqrt{2}$. In any case, from the finite ratio $m_\varrho/f_\pi$ in finite volume $m_\pi L = 1$, an incorrect conclusion was drawn in [1, 2], namely that $m_\varrho/f_\pi$ is volume independent and the value obtained in [3] holds in infinite volume too. Furthermore, $m_\varrho/f_\pi$ was misquoted in [1, 2] by a factor $\sqrt{2}$, beyond the $\sqrt{2}$ difference in conventions.

For completeness let us compute $f_\pi$ directly in the continuum both in finite and infinite volume at tree level in Euclidean signature. It is enough to consider $N_c = 1$ and at the end restore the $N_c$-dependence by $f_\pi \rightarrow \sqrt{N_c}f_\pi$. The decay constant is defined from the large $t$ behavior of the correlator at zero momentum,

$$C(x) = \langle (\bar{\psi}\gamma_5\psi)(x)(\bar{\psi}\gamma_5\psi)(0) \rangle \int d^3x C(x,t) \sim \frac{m_\pi^3}{4m^2}f_\pi^2 e^{-m_\pi t} = 2mf_\pi^2 e^{-2mt} \quad \text{for } t \gg \frac{1}{m}. \quad (1)$$

This normalization corresponds to $f_\pi = 92 \text{ MeV}$ in QCD as in [1, 2]. Now using the scalar and fermionic Green’s functions,

$$G(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{m^2 + p^2}$$

$$S(x) = (\gamma_\mu \partial_\mu + m)G(x) \quad (2)$$

we obtain in Fourier space,

$$\tilde{C}(p) = \int d^4x Tr[S(x)\gamma_5S(-x)\gamma_5] e^{-ip\cdot x} = 2\int d^4x e^{-ip\cdot x}G^2 + 4G(0)$$

$$\tilde{C}(p = 0, p_4) = 2\int dt e^{-itp_4}\partial_t^2 \int d^3x G^2(x,t) + 4G(0), \quad (3)$$

hence we need $\int d^3x G^2(x,t)$, which follows simply from (2),

$$\int d^3x G^2(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-2t\sqrt{p^2+m^2}}}{4(p^2 + m^2)}, \quad (4)$$

leading to the rather compact expression for $t > 0$,

$$C(t) = \int d^3x C(x,t) = 2\int \frac{d^3p}{(2\pi)^3} e^{-2t\sqrt{p^2+m^2}}. \quad (5)$$
The last expression holds in infinite volume, but in finite $L^3$ volume the momentum integral simply needs to be replaced by a momentum sum. The fermion fields are assumed to be periodic in all spatial directions.

Hence, in infinite volume, and positive time separation $t > 0$ we obtain,

$$C(t) = \frac{m^3}{\pi^2} K_2(2tm) \frac{e^{-2tm}}{2tm} \sim \frac{m^3}{4\pi^{3/2}} \frac{e^{-2tm}}{tm^{3/2}} \left( 1 + O\left(\frac{1}{tm}\right) \right) \quad \text{for} \quad t \gg \frac{1}{m},$$

with the Bessel function $K_2$. While in finite volume, $mL$ fixed,

$$C(t) = \frac{2}{L^3} \sum_{n=(n_1,n_2,n_3)} e^{-2t\sqrt{\left(\frac{2\pi}{L}\right)^2 n^2 + m^2}} \sim \frac{2}{L^3} e^{-2mt} + \cdots \quad \text{for} \quad t \gg \frac{1}{m},$$

where $\cdots$ refers to terms suppressed exponentially relative to the leading term $e^{-2mt}$. It is clear from (6) that the amplitude vanishes for $t \to \infty$ hence in infinite volume $f_\pi = 0$ even at finite $m$. In finite volume (7) shows that the amplitude is finite for asymptotically large time separations and we get, using (1),

$$f_\pi = \frac{m}{(mL)^{3/2}},$$

which coincides with the continuum extrapolated result of [3] at the particular finite volume $mL = 1/2$ once it is multiplied by $\sqrt{3}$ since $N_c = 3$ and also by $\sqrt{2}$ to take into account the normalization conventions ($92$ MeV vs $130$ MeV). As $L \to \infty$ at finite $m$, clearly $f_\pi \to 0$, consistently with the analysis directly in infinite volume.

Hence the ratio $m_\rho/f_\pi$ is divergent in infinite volume at tree-level. The tree-level result is relevant at the upper end of the conformal window, $N_f = 11N_c/2$. Hence presumably the non-perturbative result $m_\rho/f_\pi = 7.85(14)$ with $N_c = 3$ from [1, 2] valid for $2 \leq N_f \leq 10$ increases towards $N_f = 16.5$ contrary to what was stated in [1, 2].

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References

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