MINIMIZING THE NUMBER OF TILES IN A TILED RECTANGLE

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key words: rectangle, tiling, trapped block.

Abstract. In this paper, we prove that if a finite number of rectangles, every of which has at least one integer side, perfectly tile a big rectangle then there exists a strategy which reduces the number of these tiles (rectangles) without violating the condition on the borders of the tiles. Consequently this strategy leads to yet another solution to the famous rectangle tiling theorem.

1. Introduction

In [1] Stan Wagon collects 14 proofs of the result that whenever a rectangle is tiled by rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side. Many other solution to the above fact can also be found in the literature (see for example, [2], [3], [4] etc).

We have another solution of the same problem as a consequence of the following theorem:
Theorem 1. If $n$ rectangles, every of which has at least one integer side, perfectly tile a big rectangle $R$ then there exists a strategy which reduces the number $n$ of tiles (rectangles) to $n - 1$ such that at least one side of each rectangle in the new tiling is still an integer.

Coincidentally our idea has some resemblance with one of the solution (9 Induction) given in [1]. The strategy in this paper enables us to locate a particular rectangle in a given semi-integer (see definition 1) tiling of a rectangle. The surgery is done at the particular tile to minimize the number of tiles avoiding maximum destruction to the tiling. The area of the given rectangle as well as the semi-integer property of the tiling is invariant under this strategy. Hence the structure of an arbitrary such tiling is completely described.

2. Definitions and Proofs

Note: The words rectangle and tile from now on have the same meaning.

Definition 1. We call a rectangle semi – integer if at least one of its sides has integer length.

Definition 2. A block of $k + 1$, $k = 1, 2, 3, ...$ tiles is called trapped block if a single side of the fixed $(k + 1)$th (roof tile) tile $t_{k+1}$ is shared by a side of each of the remaining $k$ tiles (floor tiles) $t_i, 1 \leq i \leq k$ in such a way that the sum of the lengths of these $k$ shared sides with the roof tile $t_{k+1}$ is equal to the length of the shared side of $t_{k+1}$ (see, for example, figure 1).
Let $h_i$ (respectively, $v_i$) denotes the side of a tile $t_i$, $1 \leq i \leq k$ in $k$ tiles of a trapped block, parallel (respectively, perpendicular) to the shared side of $t_{k+1}$. The corresponding length is denoted by $l(h_i)$ (respectively, $l(v_i)$).

**Definition 3.** A trapped block is called minimal trapped block if no tile $t_i$ in $k$ floor tiles of this block lies between two tiles such that $l(v_i)$ is less than the perpendicular length of the left as well as the right tile.

**Remark 1.** If the $k$ floor tiles in a minimal trapped block are numbered consecutively then either $l(v_1) = \min\{l(v_i), 1 \leq i \leq k\}$ or $l(v_k) = \min\{l(v_i), 1 \leq i \leq k\}$ (see, for example, figure 3).

We follow the following operation given in R. Kenyon (Theorem 6 [2])

**Operation C:** Coalescing of two tiles which share a common edge into one tile by removing that edge.
Obviously this operation preserves the semi-integer property of the tiling.

**Lemma 1.** If at least one side of each of the tile $t_i, 1 \leq i \leq k+1$ in a minimal trapped block is an integer, then there exists a strategy (a semi-integer property preserving) which decrease the number of tiles in this block at least by one.

*Proof.* Without loss of generality let us assume that $l(v_1) = \min\{l(v_i), 1 \leq i \leq k\}$ as shown in the following figure

If $l(v_1)$ is an integer then we extend the lower $h_1$ in the above figure across the remaining $k-1$ tiles up to right $v_k$ and apply operation $C$. This produce a situation where operation $C$ is applicable $k$ times
and the tile $t_1$ disappears by increasing $l(v_{k+1})$ by $l(v_1)$ and the result follows.

If $l(v_1)$ is an not an integer then $l(h_1)$ must be an integer and we extend the right $v_1$ up to the upper horizontal side of $t_{k+1}$ and apply operation $C$ on the upper $h_1$ which decrease $l(h_{k+1})$ by $l(h_1)$. We see that there is a new minimal trapped block where $k - 1$ tiles are attached with the new form of $t_{k+1}$. In the new minimal trapped block either $l(v_2) = \min\{l(v_i), 2 \leq i \leq k\}$ or $l(v_k) = \min\{l(v_i), 2 \leq i \leq k\}$. Repeating the same procedure either we find that $l(v_i)$ is an integer or we come to the final minimal trapped in which only two tiles are coincide and apply the same operation $C$ and we arrive to the required result. \hfill \Box

Proof of the Theorem 1: We have two cases:

Case 1. If operation $C$ is applicable we apply and finish the proof.

Case 2. If case 1 is not possible then to show the result it is enough to show the existence of a minimal trapped block in the given tiled rectangle $R$. Since the above mentioned operation is not applicable so consider a side $h$ of the given rectangle $R$ shared by at least two tiles. For convenience let us take $h$ as horizontal side. We choose a tile $r_1$ attached to lower $h$ having minimum vertical length among the other attached tiles with $h$. Such a tile $r_1$ exists among the tiles attached to $h$ because operation $C$ is not applicable in this case. This tile is a roof in another trapped block. If this trapped block (having roof $r_1$) is minimal then we are done otherwise there will be a tile $r_2$ other than $r_1$ in the above mentioned trapped block such that $r_2$ has minimum vertical length than the left as well as the right tile. Due to the tiling of $R$, $r_2$ is a roof for some other trapped block. If this trapped block (where roof is $r_2$) is minimal then we are done otherwise we repeat the above process. Since $n$ is finite and operation $C$ is not applicable
therefore this process is finite and must ends with a minimal trapped block.

\[ \text{Figure 4} \]

\textit{Remark 2.} Theorem 1 is also true if integer is replaced by algebraic.

\textbf{References}

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