Abuse notation of improper integrals

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Abstract. In the definition of \( \int_a^b f(x) \, dx \), it was assumed that the interval \([a, b]\) was finite. However, in many application in physics, engineering, economics, and probability we wish to allow \( a \) or \( b \) (or both) to be infinite or for the other case when \( f \) has an infinite discontinuity in \([a, b]\). In either case the integral is called improper integral. Definition of improper integral is the limit of a definite integral as an endpoint of the interval of integration approaches either a specified real number or \( \infty \) or \( -\infty \) or, in some cases, as both endpoints approach limits. Many of the first-year students in ITERA learning about improper integrals make an abuse notation like written improper integral just like standard definite integral. Based on this situation, we tried to change the habits of the students in the writing of improper integrals by providing sufficient understanding and considerable exercises. It turns out this way gives good result, based on the percentage of the value of the midterms to the problem of improper integrals give a fairly good percentage value.

1. Introduction

In addition to Physics and Chemistry, Calculus is one of the compulsory subjects to be taken by first year (TPB) Students of Institut Teknologi Sumatera (ITERA). Calculus can form a systematic, critical, logical, meticulous and consistent thinking pattern, and demands creative and innovative thinking.

Calculus has a big role in the advancement of science and technology because it has advantages such as language and rules in Calculus is well-designed and consistent, the structure of the network of mathematical information is very strong with a pattern of reasoning and systematic, Calculus acts as a way of approach to study Science and technology, Calculus is a tool in solving other field problems and through Calculus a problem can be seen in a compact, short and solid model \cite{1}. Calculus is used in many fields, as in Newton's law expressed by the rate of change referring to the derivative and the second law of the Newton’s law said force is the product of mass and acceleration while the acceleration itself is a derivative of the velocity. Another example, the function of vector has a big effect on the existence of a location in terms of the moving place. So, a vehicle can be known where and where is the location of the destination.

In the TPB, Calculus is divided into two parts that is Calculus I for the first semester and Calculus II for the second semester. Calculus I was discussed about Real Number System, Definitions and Various Functions, Definition of Limit and Continuity, The Derivatives and The Definite Integral and its application. Calculus II was discussed about techniques of integration, indeterminate form and improper integrals, infinite series, geometry in space and vector, derivative for function of two more
variables and multiple integral. All these materials are the basic that should be given to TPB students as a base for them when they return to their study program.

In this paper we will discuss about the errors that often occur in improper integrals problem. Many of the students solve improper integral problems without writing notation of the improper integral correctly. Yet in Calculus the writing of the improper integral definition correctly is very important. For example, when a student does not write a correct definition of it will get an unusual result from the calculation.

\[\int_{0}^{1} 1 + \ln x \, dx,\]

In this paper also performed simple data analysis of some sample of student value in the problem of improper integral.

2. Improper Integrals

In Calculus I, students defined the integral \(\int_{a}^{b} f(x) \, dx\) over a finite \([a, b]\). The function \(f\) was assumed to be continuous, at least bounded, otherwise the integral was not guaranteed to exist. However, in many application in physics, economics, and probability we wish to allow \(a\) or \(b\) (or both) to be \(\infty\) or \(-\infty\). We must therefore find a way to give meaning to symbols like

\[\int_{1}^{\infty} \frac{1}{1 + x^3} \, dx, \quad \int_{-\infty}^{-1} \frac{1}{1 + x^3} \, dx, \quad \int_{-\infty}^{\infty} \frac{1}{1 + x^3} \, dx\]  

The integrals are called improper integrals [2].

2.1 Definition (Improper Integral)

An integral is an improper integral if either the interval of integration is not finite (improper integral of type 1) or if the function to integrate is not continuous (not bounded) in the interval of integration (improper integral of type 2). We will give some example for each type of integral [3].

- \(\int_{1}^{\infty} e^x \, dx\)
  is an improper integral of type 1 since the upper limit of integration is infinite.

- \(\int_{0}^{1} \frac{1}{x^3} \, dx\)
  is an improper integral of type 2 because \(\frac{1}{x^3}\) is not continuous at 0.

- \(\int_{-\infty}^{-1} \frac{1}{x^2 - 1} \, dx\)
  is an improper integral of type 2 because \(\frac{1}{x^2 - 1}\) is not continuous at \(-1\) and 1.

- \(\int_{-\infty}^{\infty} \frac{1}{1 - x} \, dx\)
  is an improper integral of type 1 and also integral of type 2 because the lower limit of integration is infinite and also \(\frac{1}{1 - x}\) is not continuous at 1 and 1 is in the interval of integration.

Look at the interval of integration. If either the lower limit of integration, the upper limit of integration or both are not finite, it will be an improper integral of type 1.

2.1.1 Definition (improper integral of type 1)

1.

\[\int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx\]  

(2)
\[
\int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx
\]

If the limits on the right exist and have finite values, then we say that corresponding improper integrals converge and have those values. Otherwise, the integrals are said to diverge.

2. If both \( \int_c^\infty f(x) \, dx \) and \( \int_c^{-\infty} f(x) \, dx \) converge, then we define
\[
\int f(x) \, dx = \int f(x) \, dx + \int f(x) \, dx
\]

The integrals on the right are evaluated as shown in 1.

2.1.2 Definition (improper integral of type 2)

Improper integral of type 2 are evaluated as follows;

1. If \( f \) is continuous on \([a, b)\) and not continuous at \( b \) then we define
\[
\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx
\]

Provided the limit exists as a finite number. In this case, \( \int_a^b f(x) \, dx \) is said to be convergent (or to converge). Otherwise, \( \int_a^b f(x) \, dx \) is said to be divergent (or to diverge).

2. If \( f \) is continuous on \([a, b)\) and not continuous at \( a \) then we define
\[
\int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx
\]

provided the limit exists as a finite number. In this case, \( \int_a^b f(x) \, dx \) is said to be convergent (or to converge). Otherwise, \( \int_a^b f(x) \, dx \) is said to be divergent (or to diverge).

3. if \( f \) is not continuous at \( c \) where \( a < c < b \) and both \( \int_a^c f(x) \, dx \) and \( \int_c^b f(x) \, dx \) converge then we define
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

The integrals on the right are evaluated as shown in 1, and 2.

Evaluating an improper integral is exactly about limit problem and integral problem. Make sure that each integral is improper at only one place, that place should be either the lower limit of integration or the upper limit of integration.

- \( \int_1^\infty \frac{dx}{x^2} \)

This is an improper integral of type 1. We evaluate it by finding \( \lim_{t \to \infty} \int_1^t \frac{dx}{x^2} \).
\[
\int_1^\infty \frac{dx}{x^2} = \lim_{t \to \infty} \int_1^t \frac{dx}{x^2} = \lim_{t \to \infty} \left( 1 - \frac{1}{t} \right) = 1.
\]

Many of student solve that’s problem
\[
\int_1^\infty \frac{dx}{x^2} = \left( 1 - \frac{1}{\infty} \right) = 1
\]

Although the answer was correct, didn’t write improper integrals as a limit of definite integral is one of the abuse notation.
\[ \int_{0}^{\pi} \sec^2 x \, dx \]
This is an improper integral of type 2, because \( \sec^2 x \) is not continuous at \( \frac{\pi}{2} \). Thus,
\[
\int_{0}^{\pi} \sec^2 x \, dx = \int_{0}^{t} \sec^2 x \, dx + \int_{t}^{\pi/2} \sec^2 x \, dx
\]
\[
= \lim_{t \to \pi/2^-} \int_{0}^{t} \sec^2 x \, dx + \lim_{t \to \pi/2^+} \int_{t}^{\pi/2} \sec^2 x \, dx = \tan t + \infty
\]
Therefore \( \int_{0}^{\pi} \sec^2 x \, dx \) diverges.

If we had failed to see that the above integral is improper, and evaluated it without using definition of improper integral, we could have obtained a completely different (and wrong) answer
\[
\int_{0}^{\pi} \sec^2 x \, dx = \tan \pi - \tan 0 = 0 \quad \text{(this is not correct)}
\]
\[ \int_{0}^{1} 1 + \ln x \, dx \]
If we didn’t use right definition for the above integral we could obtained
\[
\int_{0}^{1} 1 + \ln x \, dx = (x + x \ln x - x)\bigg|_{t=0}^{t=1} = \infty
\]
the answer is not correct, since this is an improper integral of type 2, because \( \ln x \) is not continuous at 0. Thus,
\[
\int_{0}^{1} 1 + \ln x \, dx = \lim_{t \to 0^+} \left( \int_{0}^{t} dx + \int_{t}^{1} \ln x \, dx \right)
\]
\[
= \lim_{t \to 0^+} \left( (x + x \ln x - x)\bigg|_{t=0}^{t=1} \right)
\]
\[
= \lim_{t \to 0^+} (1 \ln 1 - t \ln t)
\]
\[
= 0 - \lim_{t \to 0^+} t \ln t = 0
\]

3. Result
We take a sample of mid test score from several students and will be tested by using hypothesis test of one sample proportion.

We claims that 50% of mid semester test results for the improper integral problem gets a good result. A random sample of 100 students has 61 students get a good result. We will test this claim at the 0.05 level of significance.

- State Hypotheses: \( H_0: p = 0.5 \)
  \( H_1: p > 0.5 \)
- Test Statistic: \( z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \)
  \[
  z = \frac{61/100 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{100}}} = 2.2
  \]
- \( \alpha \): 0.05
- Critical Value: 1.645
- Decision: Because \( z > 1.645 \), so we decided to Reject \( H_0 \)
- This test give conclusion that for 0.05 significance level more than 50% students get a good result for improper integrals problem.
From this simple analysis data we can say that after providing sufficient understanding and considerable exercise, more than 50% students can solve the improper integral problem correctly.

4. Conclusion
When given the problem of integrals we must know whether the integral is a definite integral or improper integrals. Incorrectly determining the type of integral will give very different results and possibly wrong answer. Be able to write an improper integral as the limit of definite integral is very important. Many of the students have been careless or unwise in this regard. For that reason, we provide a sufficient understanding of the integrals especially the improper integrals with adequate exercises on improper integrals problem. It turns out this way gives good result, based on the students mid test results.

5. Reference
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