THE BORSUK-ULAM THEOREM FOR CLOSED 3-DIMENSIONAL MANIFOLDS HAVING NIL GEOMETRY

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Abstract.
Let $M$ be a closed, connected 3-manifold which admits Nil geometry, we determine all free involutions $\tau$ on $M$ and the Borsuk-Ulam index of $(M, \tau)$.

1. Introduction

The theorem now known as the Borsuk-Ulam theorem seems to have first appeared in a paper by Lyusternik and Schnirel’man [17] in 1930, then in a paper by Borsuk [7] in 1933 (where a footnote mentions that the theorem was posed as a conjecture by S. Ulam). One of the most familiar statements (Borsuk’s Satz II) is that for any continuous map $f: S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. The theorem has many equivalent forms and generalizations, an obvious one being to replace $S^n$ and its antipodal involution $\tau(x) = -x$ by any finite dimensional CW-complex $X$ equipped with some fixed point free involution $\tau$, and ask whether $f(x) = f(\tau(x))$ must hold for some $x \in X$. The original theorem and its generalizations have many applications in topology, and also – since Lovász’s [16] and Bárany’s [1] pioneering work in 1978 – in combinatorics and graph theory. An excellent general reference is Matoušek’s book [18]. For more examples of new applications, see [13] or [20].

In recent years, the following generalization of the question raised by Ulam has been studied (e.g. [10], [12], [11], [21], [4], [8]) for many families of pairs $(M, \tau)$, where $\tau$ is a free involution on the space $M$:

Given $(M, \tau)$, determine all positive integers $n$ such that for every map $f: M \to \mathbb{R}^n$, there is an $x \in M$ for which $f(x) = f(\tau(x))$.

When $n$ belongs to this family, we say that the pair $(M, \tau)$ has the Borsuk-Ulam property with respect to maps into $\mathbb{R}^n$. If $M$ is a closed connected manifold, the greatest such integer $n$ – which is at least 1 ([18], [20]) and at most the dimension of $M$ ([12]) – is called the $\mathbb{Z}_2$-index of the pair $(M, \tau)$.

In this work, we compute this index when $M$ is any of the closed connected 3-dimensional manifolds having Nil geometry [23]. Our main result consists in a list of seven Propositions and six Theorems. The propositions give a complete list (up to a certain equivalence) of all double coverings $M \to N$ of each manifold $N$ with Nil geometry, and the theorems give the $\mathbb{Z}_2$-index of the corresponding pairs $(M, \tau)$.

This work contains four sections besides this one. In Section 2 we give some preliminaries and the list of manifolds with Nil geometry. In Section 3 we specify the notion of equivalence of two coverings and in seven propositions we list all equivalence classes of double coverings of manifolds with Nil geometry. In Section 4...
we recall the results about the Borsuk-Ulam property and apply them to manifolds with Nil geometry. In Section 5, in six theorems, we list (up to equivalence) all free involutions on manifolds with Nil geometry, and their $\mathbb{Z}_2$-index.

2. Manifolds with Nil geometry

The (3-dimensional, closed, connected) manifolds having Nil geometry are known (see for instance [9]). Any such manifold $M$ is orientable and is the total space of a Seifert bundle $M \to S$, which is unique once an orientation of $M$ is chosen ([23]) and is characterised ([24], [22]) by an invariant of the form:

$$(b, \varepsilon, g', a_1, b_1, \ldots, a_n, b_n)$$

where

- $b$ is an integer,
- $\varepsilon = +1$ if the surface $S$ is orientable and $\varepsilon = -1$ if it isn’t,
- $g'$ is the rank of $H_1(S)$ (i.e. $g' = \frac{\varepsilon + 3}{2} g$, $g \in \mathbb{N}$ being the genus of $S$, $g \geq 1$ if $\varepsilon = -1$),
- $n \in \mathbb{N}$ and for each $i \in \{1, \ldots, n\}$, the integers $a_i, b_i$ describing a critical fibre are coprime and $0 < b_i < a_i$.

Moreover ([23, p. 469]):

- the Euler orbifold characteristic $\chi(M)$ is zero:
  $$0 = \chi(M) := \chi(S) - \sum \left( 1 - \frac{1}{a_i} \right),$$
  i.e.
  $$\sum \left( 1 - \frac{1}{a_i} \right) = 2 - g';$$

- the circle bundle Euler number $e(M)$ is non-zero, hence strictly positive for one of the two orientations: $0 < e(M) := b + \sum \frac{b_i}{a_i}$, i.e.
  $$b \geq b_{\min} := - \left\lceil \sum \frac{b_i}{a_i} \right\rceil + 1.$$

Conversely, for any orientable Seifert manifold with invariants satisfying these two conditions, the geometry is Nil ([23, p. 441]).

These conditions leaving very few possibilities, we recover directly (in a different order) Dekimpe’s list of all (closed, connected) 3-manifolds having Nil geometry [9, Theorem 6.5.5, Chapter 6, p. 154]. For later use, we add to the table two columns computing the integers $c, d$ defined as follows:

**Definition 1.** For any orientable Seifert manifold $M$ with invariant

$$(b, \varepsilon, g', a_1, b_1, \ldots, a_n, b_n),$$

we define two integers $c, d$ by:

- $d$ is the number of indices $i$ such that $a_i$ is even;
- $c = e(M)a = ba + \sum \frac{b_i}{a_i}$, where $a$ denotes the least commun multiple of the $a_i$’s.
about orientable Seifert manifolds, and soon after, make it more precise for those
abelianization of the fundamental group. Let us first mention a general remark
of triples
Remark 1. In the Seifert invariant, the traditional ordered list (of ordered pairs)
\( V \) gives a presentation of the same group).
The first homology group (with integral coefficients) is easily computed, as the
\[ \pi_1(M) = \left\langle \frac{s_1, \ldots, s_n}{h} \mid s_i h^a, \quad 1 \leq i \leq n \right\rangle \]
where \( V = [v_1, v_2] \ldots [v_{2g-1}, v_{2g}] \) if \( \varepsilon = +1 \) and \( V = v_1^2 \ldots v_g^2 \) if \( \varepsilon = -1 \).

**Remark 1.** In the Seifert invariant, the traditional ordered list (of ordered pairs)
\(((a_1, b_1), \ldots, (a_n, b_n))\) should in fact – due to its geometrical origin – be considered
as a multiset \{ \((a_1, b_1), \ldots, (a_n, b_n)\) \} (one may check that any reordering of the list
of triples \((a_i, a_i, b_i)\)'s gives a presentation of the same group).

The first homology group (with integral coefficients) is easily computed, as the
abelianization of the fundamental group. Let us first mention a general remark
about orientable Seifert manifolds, and soon after, make it more precise for those
having Nil geometry. Homology with coefficients in \( \mathbb{Z}_2 \) would be easier to compute,
and sufficient for Propositions 5 to 11 but integral homology will be needed in
Corollary 14 of the next section.

**Proposition 1.** The first homology group \( H_1(M) \) of an orientable Seifert manifold
with invariant \((b, \varepsilon, g', \ldots)\) is the direct sum of its torsion subgroup and of the free
abelian subgroup \( \sum_{j=1}^{g'} v_j \mathbb{Z} \) if \( \varepsilon = +1 \), \( \sum_{j=1}^{g'-1} v_j \mathbb{Z} \) if \( \varepsilon = -1 \).

**Proof.** \( H_1(M) \) is the abelian group with (commuting) generators
\[ s_1, \ldots, s_n, v_1, \ldots, v_{g'}, h \]
satisfying certain relations.

- If \( \varepsilon = +1 \), the relations are \( a_i s_i = -b_i h \) (for \( 1 \leq i \leq n \)) and \( bh = \sum s_i \).
  They imply (for \( a \) and \( c \) as in Definition 1) \( abh = \sum a_i (-b_i h) \) hence \( ch = 0 \),
  which, in turn, entails \((ca_i) s_i = -b_i (ch) = 0\), hence the subgroup generated
  by \( h, s_1, \ldots, s_n \) is torsion.
Proposition 2. The first homology groups of manifolds with Nil geometry are the following.

- \( H_1(M_T(b)) = v_1 \mathbb{Z} \oplus v_2 \mathbb{Z} \oplus h\mathbb{Z}_c \) (\( c = b \)).
- \( H_1(M_K(b)) = \begin{cases} v_1 \mathbb{Z} \oplus (v_1 + v_2)\mathbb{Z}_4 & \text{if } b \text{ is odd} \\ v_1 \mathbb{Z} \oplus (v_1 + v_2)\mathbb{Z}_2 \oplus h\mathbb{Z}_2 & \text{if } b \text{ is even.} \end{cases} \)
- \( H_1(M_{22}(b)) = v_1\mathbb{Z}_4 \oplus s_1\mathbb{Z}_4 \) with \( h = -2s_1 \) and \( s_2 = -(2b + 1)s_1 - 2v_1 \)
- \( H_1(M_{222}(b)) = (s_2 - s_1)\mathbb{Z}_2 \oplus (s_1 - s_1)\mathbb{Z}_2 \oplus s_1\mathbb{Z}_c \) (\( c = 2b + 4 \))
  with \( h = -2s_1 \) and \( s_4 = -(2b + 1)s_1 - s_2 - s_3 \).
- \( H_1(M_{23b}(b, b_2, b_3))_{(b_2 \in (1, 2), b_3 \in (1, 5), b_5 \in (1, 5), b_5 \in (1, 5))} = (s_2 - s_1)\mathbb{Z}_6c \) (\( c = 6b + 3 + 2b_2 + b_3 \))
  with \( s_1 = 3(2b_2 - 3)s_2 - s_1, h = -2s_1 \) and \( s_3 = -(2b + 1)s_1 - s_2 \).
- \( H_1(M_{244}(b, b_2, b_3))_{(b, b_2, b_3) \in (1, 2, 1, 3)} = \mathbb{Z}_2 \oplus \mathbb{Z}_4c \) (\( c = 4b + 2 + b_2 + b_3 \))
  with \( s_1 = (1, 2b_2 - 4), s_2 = s_1 + (0, 1), h = -2s_1 \) and \( s_3 = -(2b + 1)s_1 - s_2 \).
- \( H_1(M_{333}(b, b_1, b_2, b_3))_{(b, b_1, b_2, b_3) \in (1, 2, 1, 3)} = \mathbb{Z}_3 \oplus \mathbb{Z}_c \) (\( c = 3b + b_1 + b_2 + b_3 \))
  with \( h = (0, 3), s_1 = (0, -b_1), s_2 = (-b_2, -b_2) \) and \( s_3 = bh - s_1 - s_2 \).

Proof. Elementary exercise (see e.g. \cite{15} chap. 6) for the standard method). \( \square \)

3. Double coverings of each manifold with Nil geometry

The family of (closed, connected) 3-manifolds having Nil geometry is closed by 2-quotients \cite[Theorem 2.1]{19}. This will allow us to compute the list of all pairs \((M, \tau)\), where \( M \) belongs to this family and \( \tau \) is a free involution on \( M \). The procedure (same as in \cite{2} and \cite{3}) is the following: starting with a manifold \( N \) of the table above, we shall determine all double coverings \( \tilde{N} \to N \) by a systematic use of Reidemeister-Schreier algorithm (\cite{5}), and find that \( \tilde{N} \) again belongs to the family. We obtain a list of double coverings \( M \to N \), indexed by \( N \). Reordering and reindexing it by \( N \), we get the list of all 2-quotients of members \( M \) of the family. This, in turn, determines all free involutions on these \( M \)'s (up to a natural equivalence relation).

Let us recall some standard results about \((M, \tau)\), where \( M \) is a manifold and \( \tau \) a free involution on \( M \). The projection \( p: M \to M/\tau \) is a non trivial principal \( \mathbb{Z}_2 \)-bundle (i.e. double covering). Isomorphism classes of double coverings of a fixed base \( B \) are in 1-to-1 correspondence with \( H^1(B, \mathbb{Z}_2) \). In the rest of the article, we use the isomorphism between \( H^1(B, \mathbb{Z}_2) \) and \( \text{Hom}(\pi_1(B, \mathbb{Z}_2)) \). Hence the characteristic class \( \varphi \in H^1(M/\tau; \mathbb{Z}_2) \) \( \setminus \{0\} \) of \( p: M \to M/\tau \) is also an epimorphism \( \varphi: \pi_1(M/\tau) \to \mathbb{Z}_2 \). It is determined by its kernel, and we have a short exact sequence of groups

\[
0 \to \pi_1 M \xrightarrow{p_1} \pi_1(M/\tau) \xrightarrow{\varphi} \mathbb{Z}_2 \to 0.
\]

Definition 2. We say that \((M_1, \tau_1)\) and \((M_2, \tau_2)\) are equivalent if the two bundles \( p_i: M_i \to M_i/\tau_i \) are isomorphic.

The obvious necessary condition for the two bundles to be isomorphic is in fact also sufficient:
Proposition 3. Two pairs $(M_i, \tau_i)$ with characteristic classes $\varphi_i : \pi_1(M_i/\tau_i) \to \mathbb{Z}_2$ $(i \in \{1, 2\})$ are equivalent if (and only if) there exists a homeomorphism $F : M_1/\tau_1 \to M_2/\tau_2$ such that $\varphi_2 \circ F = \varphi_1$, where $F : \pi_1(M_1/\tau_1) \to \pi_1(M_2/\tau_2)$ is the isomorphism induced by $F$ in homotopy.

Proof. Assume there exists a homeomorphism $F$ such that $\varphi_1 \circ F^{-1} = \varphi_2$. This means that the double coverings $F \circ p_1$ and $p_2$ of $M_2/\tau_2$ have the same characteristic class. Then, these two coverings are isomorphic over $\text{id}_{M_2/\tau_2}$, which amounts to say that $p_1$ and $p_2$ are isomorphic over $F$.

When convenient, we shall then say that the epimorphisms $\varphi_1$ and $\varphi_2$ are equivalent.

The only equivalences used in this paper will be the following:

Corollary 4. Let $N$ be a closed connected manifold with Nil geometry, with invariant $(b, c, g', a_1, b_1, \ldots, a_n, b_n)$. Two epimorphisms $\varphi_1, \varphi_2 : \pi_1(N) \to \mathbb{Z}_2$ are equivalent in any of the following situations (some of which may happen simultaneously):

1. $\varphi_1, \varphi_2$ coincide on $h, s_1, \ldots, s_n$ and $\varphi_i(h) = 1$;
2. For some $I \subset \{1, \ldots, n\}$ on which $i \mapsto (a_i, b_i)$ is constant and for some permutation $\sigma$ of $I$,
   $\varphi_2(s_i) = \varphi_1(s_{\sigma(i)})$ $(\forall i \in I)$, and $\varphi_1, \varphi_2$ coincide on the other generators of $\pi_1(N)$;
3. $N$ belongs to the class $T$, $\varphi_1(h) = \varphi_2(h)$, $(\varphi_1(v_1), \varphi_1(v_2)) = (1, 1)$ and $(\varphi_2(v_1), \varphi_2(v_2))$ equals either $(1, 0)$ or $(0, 1)$.
4. $N$ belongs to the class $K$, $\varphi_1(h) = \varphi_2(h)$, $(\varphi_1(v_1), \varphi_1(v_2)) = (1, 0)$ and $(\varphi_2(v_1), \varphi_2(v_2)) = (0, 1)$.
5. $N$ belongs to the class $22$ and $\varphi_1, \varphi_2$ both send $s_2$ to $1$.

Proof. Since $N$ is sufficiently large, in order to prove the equivalence, it suffices to construct an automorphism $\theta$ of $\pi_1(N)$ such that $\varphi_2 = \varphi_1 \circ \theta$ (2). In each of the five cases, we shall define $\theta$ on the canonical generators of $\pi_1(N)$ (the reader will easily check it is well defined and invertible):

1. We set $\theta(v_j) = v_j h$ for any $j$ such that $\varphi_2(v_j) \neq \varphi_1(v_j)$ and let $\theta$ fix the other generators.
2. We may assume that $\sigma$ is a transposition (by decomposition of the permutation) and that moreover, $I = \{1, 2\}$ (by Remark 1). We then set $\theta(s_1) = s_1 s_2 s_1^{-1}$, $\theta(s_2) = s_1$, and let $\theta$ fix the other generators.
3. We let $\theta$ fix $h$ and send $(v_1, v_2)$ to $(v_1, v_2 v_1)$ in the first case, and to $(v_1 v_2, v_2)$ in the second case.
4. We let $\theta$ fix $h$ and send $(v_1, v_2)$ to $(v_1^2 v_2 v_1^{-2}, v_1)$. 

(5) Note that $\varphi_1, \varphi_2$ coincide on $s_2$ (by hypothesis) but also on $h$ and $s_1$ (by Proposition 2): $\varphi_1(h) = 0$ and $\varphi_1(s_1) = 1$. We set $\theta(v_1) = v'_1 := s_2 v_1$, $\theta(s_2) = v'_1 s_2 v'_1$, and let $\theta$ fix $s_1$ and $h$. □

In the following Propositions 5 to 11, for each manifold $M$ in the table, starting from the description of all epimorphisms $\varphi : H_1(M) \to \mathbb{Z}_2$ given by Proposition 2, we shall compute the equivalence class of these epimorphisms (using the previous corollary) and for each class, the total space of the corresponding double covering (given by [5], which, using Reidemeister-Schreier algorithm, identifies ker $\varphi$ as the fundamental group of some Seifert manifold). For each of these seven propositions, we shall just indicate which parts of the previous corollary and of [5] are used.

**Proposition 5.** For any $b \in \mathbb{N}^*$, the three epimorphisms 
$$
\varphi : H_1(M_T(b)) = v_1 \mathbb{Z} \oplus v_2 \mathbb{Z} \oplus h \mathbb{Z}_b \to \mathbb{Z}_2
$$
such that $\varphi(h) = 0$ are equivalent. Their associated double covering is 
$$
M_T(2b) \to M_T(b).
$$
If $b$ is odd, these three epimorphisms are the only ones. If $b$ is even, the four other epimorphisms are equivalent. Their associated double covering is 
$$
M_T(b/2) \to M_T(b).
$$

**Proof.** Corollary 4 (3) and (1). [5] Proposition 12 and Theorem 1. □

**Proposition 6.** For any $b \in \mathbb{N}^*$, the two epimorphisms 
$$
\varphi : H_1(M_K(b)) = \begin{cases} 
v_1 \mathbb{Z} \oplus (v_1 + v_2) \mathbb{Z}_4 & \text{if } b \text{ is odd} \\
 v_1 \mathbb{Z} \oplus (v_1 + v_2) \mathbb{Z}_2 \oplus h \mathbb{Z}_2 & \text{if } b \text{ is even}
\end{cases} \to \mathbb{Z}_2
$$
such that $\varphi(h) = 0$ and $\varphi(v_1 + v_2) = 1$ are equivalent. The associated double covering is 
$$
M_K(2b) \to M_K(b).
$$
The double covering associated to the epimorphism such that $\varphi(h) = \varphi(v_1 + v_2) = 0$ is 
$$
M_T(2b) \to M_K(b).
$$
If $b$ is odd, these three epimorphisms are the only ones. If $b$ is even, the four other epimorphisms are equivalent. The associated double covering is 
$$
M_K(b/2) \to M_K(b).
$$

**Proof.** Corollary 4 (4) and (1). [5] Proposition 14 and Theorem 1. □

**Proposition 7.** For any $b \in \mathbb{N}$, the three epimorphisms 
$$
\varphi : H_1(M_{22}(b)) = v_1 \mathbb{Z}_4 \oplus s_1 \mathbb{Z}_4 \to \mathbb{Z}_2
$$
satisfy $\varphi(h) = 0$ and $\varphi(s_1) = \varphi(s_2)$. The two of them which send $s_2$ to 1 are equivalent. Their associated double covering is 
$$
M_K(2b + 2) \to M_{22}(b).
$$
The double covering associated to the epimorphism which sends \(s_2\) to 0 is
\[ M_{2222}(2b) \rightarrow M_{22}(b). \]

Proof. Corollary [3 (5). [5] Lemma 4 and Proposition 14]. \( \square \)

Proposition 8. For any integer \(b \geq -1\), the seven epimorphisms
\[ \varphi: H_1(M_{2222}(b)) = \mathbb{Z}_4(b+2)s_1 \oplus (s_2-s_1)\mathbb{Z}_2 \oplus (s_3-s_1)\mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \]
send \(h\) and \(\sum s_i\) to 0.
The six of them which send two of the \(s_i\)'s to 1 and the other two to 0 are equivalent.
Their associated double covering is
\[ M_{2222}(2b+2) \rightarrow M_{2222}(b). \]
The double covering associated to the epimorphism which sends all the \(s_i\)'s to 1 is
\[ M_T(2b+4) \rightarrow M_{2222}(b). \]

Proof. Corollary [3 (2). [5] Lemma 4]. \( \square \)

Proposition 9. For any integers \(b_2 \in \{1, 2\}, b_3 \in \{1, 5\}\) and \(b \geq b_{\text{min}}\), the only epimorphism
\[ \varphi: H_1(M_{236}(b, b_2, b_3)) = \mathbb{Z}_{4c} \rightarrow \mathbb{Z}_2 \]
sends \(s_2, h\) to 0 and \(s_1, s_3\) to 1.
The associated double covering is
\[ M_{333}(2b + (b_3 + 3)/4, b_2, b_3) \rightarrow M_{236}(b, b_2, b_3), \]
up to a reordering of \((b_2, b_2, (b_3 + 3)/4)\), i.e. replacement by \(( (b_3 + 3)/4, b_2, b_2)\) if
\((b_2, b_3) = (2, 1)\).

Proof. [5] Lemma 4]. \( \square \)

Proposition 10. For any integers \(b_2, b_3 \in \{1, 3\}\) and \(b \geq b_{\text{min}}\), the three epimorphisms
\[ \varphi: H_1(M_{2444}(b, b_2, b_3)) = \mathbb{Z}_2 \oplus \mathbb{Z}_{4c} \rightarrow \mathbb{Z}_2 \]
send \(s_1 + s_2 + s_3\) and \(h\) to 0.
The two of them which send \(s_1\) to 1 are equivalent if \(b_2 = b_3\).
The associated double coverings are:
- if \(\varphi(s_3) = 0\): \( M_{2444}(2b + (b_2+1)/2, b_3, b_3) \rightarrow M_{2444}(b_2, b_2, b_3); \)
- if \(\varphi(s_2) = 0\): \( M_{2444}(2b + (b_3+1)/2, b_2, b_2) \rightarrow M_{2444}(b_2, b_2, b_3); \)
- if \(\varphi(s_1) = 0\): \( M_{2222}(2b - 1 + (b_2 + b_3)/2) \rightarrow M_{2444}(b, b_2, b_3). \)

Proof. Corollary [3 (2). [5] Lemma 4]. \( \square \)

Proposition 11. For any integers \(b_1, b_2, b_3 \in \{1, 2\}\) and \(b \geq b_{\text{min}}\), there is an epimorphism
\[ \varphi: H_1(M_{333}(b, b_1, b_2, b_3)) = \mathbb{Z}_3 \oplus \mathbb{Z}_{4c} \rightarrow \mathbb{Z}_2 \]
only if \(b + b_1 + b_2 + b_3\) is even and then, \(\varphi\) sends \(h\) to 1 and \(s_i\) to \(b_i \mod 2\).
The associated double covering is
\[ M_{333}((b + b_1 + b_2 + b_3 - 6)/2, 3 - b_1, 3 - b_2, 3 - b_1) \rightarrow M_{333}(b_1, b_2, b_3). \]

Proof. [5] Theorem 1]. \( \square \)
4. The Borsuk-Ulam theorem for each double covering

**Definition 3.** Let $\tau$ be a free involution on $M$. We say that:

- the pair $(M, \tau)$ verifies the Borsuk-Ulam theorem for $\mathbb{R}^n$ if for any continuous map $f: M \to \mathbb{R}^n$ there is at least one point $x \in M$ such that $f(x) = f(\tau(x))$;
- the $\mathbb{Z}_2$-index of the pair $(M, \tau)$ – or of the double covering $M \to M/\tau$ – is the greatest integer $n$ such that $(M, \tau)$ verifies the Borsuk-Ulam theorem for $\mathbb{R}^n$.

**Notation 12.** The total space of a double covering of $N$ with characteristic class $\varphi$ will be denoted $N_\varphi$:

$$N_\varphi \to N.$$ 

For closed connected manifolds of dimension 3, the $\mathbb{Z}_2$-index may only take the values 1, 2 or 3, hence the free involutions will be completely classified if we isolate those of index 1 and those of index 3.

From [12] Theorem 3.1, we have:

**Theorem 13.** The $\mathbb{Z}_2$-index of a non trivial double covering $N_\varphi \to N$ of a closed, connected manifold $N$ equals 1 if and only if its characteristic class $\varphi: \pi_1(N) \to \mathbb{Z}_2$ factors through the projection $\mathbb{Z} \to \mathbb{Z}_2$.

From this theorem and our Proposition 2, we deduce:

**Corollary 14.** For closed, connected manifolds with Nil geometry, the only non trivial double coverings $N_\varphi \to N$ with $\mathbb{Z}_2$-index 1 are those such that:

- $N$ is of class $T$ and $\varphi(h) = 0$ (and $(\varphi(v_1), \varphi(v_2)) = (1,0), (0,1)$ or $(1,1)$);
- $N$ is of class $K$, $\varphi(h) = 0$ and $(\varphi(v_1), \varphi(v_2)) = (1,1)$.

From [12] Theorem 3.4, we have:

**Theorem 15.** For closed, connected manifolds of dimension $m$, the $\mathbb{Z}_2$-index of a non trivial double covering $N_\varphi \to N$ equals $m$ if and only if the $m$-th cup power $\varphi^m$ of its characteristic class $\varphi \in H^1(N; \mathbb{Z}_2)$ is non-zero.

From [4] or using [6], we have:

**Proposition 16.** Let $N$ be an orientable Seifert manifold with invariant $(b, \varepsilon, g', a_1, b_1, \ldots, a_n, b_n)$ and let $c, d$ be as in Definition 7

The cup-cube of an element $\varphi \in H^1(N; \mathbb{Z}_2)$ is non-zero if and only if

- either $d = 0$, $\varphi(h) = 1$ and
  - either $\varepsilon = +1$ and $c \equiv 2 \mod 4$,
  - or $\varepsilon = -1$ and $c + 2g' \equiv 2 \mod 4$;
- or $d > 0$ and $\sum \varphi(s_j) \frac{a_j}{2} = 1$.

From this proposition and our Propositions 3 to 11, we deduce:

**Corollary 17.** For a manifold $N$ with Nil geometry, the $\mathbb{Z}_2$-index of a non trivial double covering $N_\varphi \to N$ equals 3 if and only if:

- $N = M_T(b)$ or $M_K(b)$, $b \equiv 2 \mod 4$ and $\varphi(h) = 1$;
- $N = M_{333}(b, b_1, b_2, b_3)$ and $b \equiv 2 + \sum b_i \mod 4$;
- $N$ is of class $333$ and $\varphi(s_1) = 1$. 

5. Free involutions on manifolds with Nil geometry, and their $\mathbb{Z}_2$-index

This section is a mere synthesis of the two previous ones, with a sorting by $M$ (instead of $N$) of the double coverings $M \to N$. When we talk about the number of free involutions on $M$, it will be up to equivalence, i.e. up to conjugation by an automorphism of $M$.

**Theorem 18.** The manifolds $M_{22}(b), M_{244}(b,1,3)$ and $M_{236}(b,b_2,b_3)$ do not support any free involution.

**Notation 19.** A figure

$$M \xrightarrow{i} N$$

will mean that $M \to N$ is a double covering with $\mathbb{Z}_2$-index $i$.

**Theorem 20.**

- If $b$ is odd, $M_T(b)$ is the double covering of one manifold, with $\mathbb{Z}_2$-index 3:
  
  $$M_T(b), b \text{ odd} \quad 3 \quad M_T(2b).$$

- If $b$ is even, $M_T(b)$ is the double covering of four manifolds (two with $\mathbb{Z}_2$-index 1 and two with $\mathbb{Z}_2$-index 2):

  $$M_T(b), b \text{ even} \quad 2 \quad 1 \quad 2 \quad 1$$

  $$M_T(2b) \quad M_T(b/2) \quad M_{2222}(b/2 - 2) \quad M_K(b/2).$$

**Theorem 21.**

- If $b$ is odd, $M_K(b)$ is the double covering of one manifold, with $\mathbb{Z}_2$-index 3:

  $$M_K(b), b \text{ odd} \quad 3 \quad M_K(2b).$$

- If $b$ is even, $M_K(b)$ is the double covering of three manifolds, all with $\mathbb{Z}_2$-index 2:

  $$M_K(b), b \text{ even} \quad 2 \quad 2 \quad 2$$

  $$M_K(2b) \quad M_K(b/2) \quad M_{22}(b/2 - 1).$$

**Theorem 22.**

- If $b$ is odd, $M_{2222}(b)$ is the double covering of one manifold, with $\mathbb{Z}_2$-index 2:

  $$M_{2222}(b), b \text{ odd} \quad 2$$

  $$M_{244}((b - 1)/2, 1, 3).$$
• If $b$ is even, $M_{2222}(b)$ is the double covering of four manifolds, all with $\mathbb{Z}_2$-index 2:

$$M_{2222}(b), b \text{ even}$$

$$\begin{align*}
M_{2222}(b/2 - 1) & \rightarrow M_{22}(b/2) \\
M_{244}(b/2 - 1, 3, 3) & \rightarrow M_{244}(b/2, 1, 1).
\end{align*}$$

**Theorem 23.** $M_{244}(b, x, x)$ ($x \in \{1, 3\}$) is the double covering of one manifold, with $\mathbb{Z}_2$-index 3:

$$M_{244}(b, x, x), b \text{ odd}$$

$$\begin{align*}
M_{244}((b - 1)/2, 1, x) & \rightarrow M_{244}(b/2 - 1, x, 3).
\end{align*}$$

**Theorem 24.** Up to reordering if $(x, y) = (2, 1)$, i.e. replacement of $(2, 2, 1)$ by $(1, 2, 2)$ and of $(2, 1, 1)$ by $(1, 1, 2)$:

• if $b - y$ is odd, $M_{333}(b, x, x, y)$ is the double covering of one manifold, with $\mathbb{Z}_2$-index 3:

$$M_{333}(b, x, x, y), b - y \text{ odd}$$

$$M_{333}(2b + 2x + y - 3, 3 - y, 3 - x, 3 - x);$$

• if $b - y$ is even, $M_{333}(b, x, x, y)$ is the double covering of two manifolds, both with $\mathbb{Z}_2$-index 2:

$$M_{333}(b, x, x, y), b - y \text{ even}$$

$$\begin{align*}
M_{333}(2b + 2x + y - 3, 3 - y, 3 - x, 3 - x) & \rightarrow M_{236} ((b - y)/2, x, 4y - 3).
\end{align*}$$

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