2D-ZERNIKE POLYNOMIALS AND COHERENT STATE QUANTIZATION OF THE UNIT DISC

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Abstract. Using the orthonormality of the 2D-Zernike polynomials reproducing kernels, reproducing kernel Hilbert spaces and ensuing coherent states are attained. With the aid of the so obtained coherent states the complex unit disc is quantized. Associated upper symbols, lower symbols and related generalized Berezin transforms are also obtained. Along the way, necessary summation formulas for the 2D-Zernike polynomials are proved.

1. Introduction

Quantization is commonly understood as the transition from classical to quantum mechanics. One may also say, to a certain extent, quantization relates to a larger discipline than just restricting to specific domains of physics. In physics, the quantization is a procedure that associates with an algebra $A_{cl}$ of classical observables an algebra $A_q$ of quantum observables. The algebra $A_{cl}$ is usually realized as a commutative Poisson algebra of derivable functions on a symplectic (or phase) space $X$. The algebra $A_q$ is, however, noncommutative in general and the quantization procedure must provide a correspondence $A_{cl} \mapsto A_q : f \mapsto A_f$. Most physical quantum theories may be obtained as the result of a canonical quantization procedure. However, among the various quantization procedures available in the literature, the coherent state quantization (CS quantization) appear quite arbitrary because the only structure that a space $X$ must possess is a measure. Once a family of CS or frame labelled by a measure space $X$ is given one can quantize the measure space $X$. Various quantization schemes and their advantages and drawbacks are discussed in detail, for example, in [5, 13, 18, 2].

In CS quantization, a correspondence between classical and quantum observables is usually provided through a suitable generalization of the standard CS. The CS quantization is judged as equivalent to canonical quantization on a physical level. However, in [8] a family of CS were obtained with the complex Hermite polynomials $H_{m,n}(z, \overline{z})$ and it was then utilized to quantize the complex plane and these results were also further investigated and refined in [3]. The results so obtained were departed from the results of the standard canonical quantization. Further, in [13] the CS obtained with the complex hermite polynomials $H_{n}(z)$ were used to quantize the so-called noncommutative plane. A similar approach is also used in [6] to quantize cylindrical phase spaces.
Following these recent developments, in this note, in section 2 we revisit the general scheme of CS quantization, section 3 briefly describe the physical relevance of the unit disc and the standard CS on the disc. In section 4 we discuss 2D-Zernike polynomials as needed here and we prove some summation formulas for 2D-Zernike polynomials which are the essential tools in obtaining reproducing kernels, CS, lower symbols and associated Berezin transforms. In section 5, we obtain a class of reproducing kernels and related reproducing kernel Hilbert spaces and also acquire their direct sum as an $L^2$ space. Alongside we also discuss some projection operators and their integral Schwartz kernels. In section 6, with the aid of 2D-Zernike polynomials we implement the CS quantization of the unit disc (also referred to as Lobachevsky plane) and then obtain the associated upper symbols, lower symbols and a generalized Berezin transform. Section 7 discusses connections between the ladder operators of 2D-Zernike polynomials and the upper symbols of section 6. We end with a conclusion.

2. COHERENT STATE QUANTIZATION: GENERAL SCHEME

Let $(X, \mu)$ be a measure space and

$$L^2(X, \mu) = \left\{ f : X \longrightarrow \mathbb{C} \mid \int_X |f(x)|^2 d\mu(x) < \infty \right\}.$$  

The Klauder-Berezin or anti-Wick or Toeplitz or coherent state quantization, as used by various authors in the literature, associates a classical observable that is a function $f(x)$ on $X$ to an operator valued integral. We continue with the general procedure described in [13] and applied, for example, in [8, 14, 6].

Choose a countable orthonormal basis $O = \{ \phi_n \mid n \in \mathbb{N} \}$ in $L^2(X, \mu)$,

\begin{equation}
\langle \phi_n | \phi_m \rangle = \int_X \overline{\phi_m(x)} \phi_n(x) d\mu(x) = \delta_{mn}
\end{equation}

and assume that

\begin{equation}
0 < \sum_{n=0}^{\infty} |\phi_n(x)|^2 := N(x) < \infty \quad a.e.
\end{equation}

holds. Let $\mathcal{H}$ be a separable complex Hilbert space with orthonormal basis $\{ |e_n\rangle \mid n \in \mathbb{N} \}$ in 1-1 correspondence with $O$. In particular $\mathcal{H}$ can be taken as $K_0 = \text{span}O$ in $L^2(X, \mu)$.

Then the family $\mathcal{F}_H = \{|x\rangle \mid x \in X \}$ with

\begin{equation}
|x\rangle = N(x)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \overline{\phi_n(x)} |e_n\rangle \in \mathcal{H}
\end{equation}

forms a set of coherent states (CS). From (2.1) and (2.2) we have

\begin{equation}
\langle x | x \rangle = 1
\end{equation}

\begin{equation}
\int_X N(x) |x\rangle \langle x| d\mu(x) = I_{\mathcal{H}}.
\end{equation}

We call the set $\mathcal{F}_H$ a set of CS only for satisfying the normalization and a resolution of the identity. Equation (2.5) allows us to implement CS or frame quantization of the set of parameters $X$ by associating a function

$$X \ni x \mapsto f(x)$$
that satisfies appropriate conditions the following operator in $\mathcal{H}$

$$f(x) \mapsto A_f = \int_X N(x)f(x)|x|d\mu(x).$$

The matrix elements of $A_f$ with respect to the basis $\{|e_n\rangle\}$ are given by

$$(A_f)_{mn} = \langle e_m | A_f | e_n \rangle = \int_X f(x)\phi_m(x)\phi_n(x)d\mu(x).$$

The operator $A_f$ is

(a) symmetric if $f(x)$ is real valued.
(b) bounded if $f(x)$ is bounded.
(c) self-adjoint if $f(x)$ is real semi-bounded (through Friedrich’s extension).

In order to view the upper symbol $f$ of $A_f$ as a quantizable object (with respect to the family $\mathcal{F}_\mathcal{H}$) a reasonable requirement is that the so-called lower symbol of $A_f$ defined as

$$\tilde{f}(x) = \langle x | A_f | x \rangle = \int_X N(x')f(x')|\langle x | x' \rangle|^2d\mu(x')$$

be a smooth function on $X$ with respect to some topology assigned to the set $X$. Associating to the classical observable $f(x)$ the mean value $\langle x | A_f | x \rangle$ one can also get the so-called Berezin transform $B[f]$ with $B[f](x) = \langle x | A_f | x \rangle$, for example, see [17] for details.

3. UNIT DISC AS A PHASE SPACE

In order to make a comparison of our results to the existing theory, we briefly extract the material associated with CS on the unit disc from [13] as needed here. For an enhanced explanation one could view [13, 18, 5].

There exist many situations in physics where the unit disc, $D = \{z \in \mathbb{C} \mid |z| < 1\}$, is involved as a fundamental model or at least is used as a pedagogical toy. For instance, it is a model of phase space for the motion of a material particle on a one sheeted two-dimensional hyperboloid viewed as a (1+1)-dimensional space-time with negative constant curvature, namely, the two dimensional anti de Sitter space-time. The unit disc equipped with a Kählerian potential,

$$K_D(z, \overline{z}) = \frac{1}{4\pi} \left(1 - |z|^2\right)^2,$$

has the structure of a two-dimensional Kählerian manifold. Any Kählerian manifold is symplectic and so can be given a sense of phase space for some mechanical system.

3.1. Standard CS on the unit disc. Let $\eta > 1/2$ be a real parameter and let us equip the unit disc with a measure

$$\mu_\eta(d^2z) = \frac{2\eta - 1}{\pi} \frac{d^2z}{(1 - |z|^2)^2}.$$

Consider now the Hilbert space $L^2_\eta(D, \mu_\eta)$ of all functions $f(z, \overline{z})$ on $D$ that are square integrable with respect to $\mu_\eta$. Let

$$B = \left\{ \phi_n(z, \overline{z}) = \sqrt{\frac{(2\eta)_n}{n!}}(1 - |z|^2)^n \overline{z}^n \mid n \in \mathbb{N} \right\},$$
where \((2\eta)_n = \Gamma(2\eta + n)/\Gamma(2\eta)\) is the Pochhammer symbol, and \(K_+ = \text{span} B\). Then \(B\) is an orthonormal basis of \(K_+\) and it also satisfies

\[
\sum_{n=0}^{\infty} |\phi_n(z, \overline{z})|^2 = 1.
\]

Thereby we can readily write a set of CS

\[
|z, \eta\rangle = \sum_{n=0}^{\infty} \phi_n(z, \overline{z}) |e_n\rangle = (1 - |z|^2)^{\eta} \sum_{n=0}^{\infty} \sqrt{(2\eta)_n n!} z^n |e_n\rangle,
\]

where \(\{|e_n\rangle \mid n \in \mathbb{N}\}\) is an orthonormal basis of a separable Hilbert space \(\mathfrak{H}\). By construction

\[
\langle z, \eta | z, \eta \rangle = 1; \quad \int_{D} \mu_{\eta} (d^2 z) |z, \eta\rangle \langle z, \eta| = I_{\mathfrak{H}}.
\]

\(K_+\) is also a reproducing kernel Hilbert space, with reproducing kernel

\[
K(z, z') = (1 - |z|^2)^{\eta}(1 - |z'|^2)^{-2\eta}(1 - |z| z')^{-2\eta},
\]

and it is a Fock-Bargmann space. As it was introduced by Perelomov \[18\], the \(SU(1,1)\) CS are the CS in (3.5). Now, for a given \(\eta > 1\), introduce the Fock-Bargmann Hilbert space \(\mathcal{FB}_{\eta}\) of all analytic functions \(f(z)\) on \(D\) that are square integrable with respect to the scalar product

\[
\langle f_1 | f_2 \rangle = \frac{2\eta - 1}{2\pi} \int_{D} f_1(z) f_2(z) (1 - |z|^2)^{2\eta - 2} d^2 z.
\]

Then

\[
B_\eta = \left\{ p_n(z) = \sqrt{(2\eta)_n n!} z^n \mid n \in \mathbb{N}\right\}
\]

is an orthonormal basis of \(\mathcal{FB}_{\eta}\). On \(\mathcal{FB}_{\eta}\) define the differential operators,

\[
\hat{R}_0 = z \frac{d}{dz} + \eta
\]

\[
\hat{R}_1 = \frac{i}{2} (1 - |z|^2) \frac{d}{dz} + i\eta z
\]

\[
\hat{R}_2 = \frac{1}{2} (1 + |z|^2) \frac{d}{dz} + \eta z,
\]

then they obey

\[
[\hat{R}_0, \hat{R}_1] = i\hat{R}_2, \quad [\hat{R}_0, \hat{R}_2] = -i\hat{R}_1, \quad [\hat{R}_1, \hat{R}_2] = -i\hat{R}_0.
\]

Further \(\hat{R}_0\) acts on the orthonormal basis as

\[
\hat{R}_0 |p_n\rangle = (\eta + n) |p_n\rangle.
\]

Define new operators

\[
\hat{R}_\pm = \mp i(\hat{R}_1 \pm i\hat{R}_2) = \hat{R}_2 \mp i\hat{R}_1, \quad [\hat{R}_+, \hat{R}_-] = -2\hat{R}_0,
\]

and as differential operators they read

\[
\hat{R}_+ = z^2 \frac{d}{dz} + 2\eta z, \quad \hat{R}_- = \frac{d}{dz}.
\]

\(\hat{R}_+\) and \(\hat{R}_-\) are adjoint to each other and they act on the orthonormal basis as

\[
\hat{R}_+ |p_n\rangle = \sqrt{(n + 1)(2\eta + n)} |p_{n+1}\rangle, \quad \hat{R}_- |p_n\rangle = \sqrt{n(2\eta + n - 1)} |p_{n-1}\rangle.
\]
where $\mathcal{F}|p_0\rangle = 0$, also

$$
|p_n\rangle = \sqrt{\frac{\Gamma(2\eta)}{\Gamma(2\eta+n)n!}}(\mathcal{F}^+)^n|p_0\rangle.
$$

Now if we view the CS in (4.3) as vectors in $\mathcal{F}B_\eta$, we see

$$
|z, \eta\rangle = (1 - |z|^2)^\eta \sum_{n=0}^{\infty} \frac{(2\eta)_n}{n!} z^n |p_n\rangle.
$$

4. 2D-ZERNIKE POLYNOMIALS ON THE UNIT DISC

Let $z = x + iy = r e^{i\theta}$, $\bar{z} = x - iy = r e^{-i\theta}$, and $\mathcal{D} = \{z \in \mathbb{C} \mid |z|^2 = z\bar{z} < 1\}$ be the complex unit disc. Let $\alpha > -1$ be a continuous parameter. For $z \in \mathcal{D}$ the 2D-Zernike polynomials are defined as [22, 21]

$$
P^\alpha_{m,n}(z, \bar{z}) = \frac{(-1)^{m+n}\alpha!}{(m+n+\alpha)!} \frac{1}{(1-z\bar{z})^\alpha} \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} (1-z\bar{z})^{m+n+\alpha}
$$

$$
= \frac{m!n!\alpha!}{(m+\alpha)!(n+\alpha)!} \sum_{j=0}^{\min\{m,n\}} \frac{(-1)^j(m+n+\alpha-j)!}{j!(m-j)!(n-j)!} \frac{1}{z^m\bar{z}^n} z^{-j}\bar{z}^{-j}
$$

$$
= \sum_{k=0}^{\min\{m,n\}} \frac{(-1)^k m! n! \alpha!}{k!(m-k)!(n-k)!(k+\alpha)!(m+k-n-k)!} (1-z\bar{z})^{k} z^{-m-k}\bar{z}^{-n-k}
$$

$$
= z^m \bar{z}^n 2F_1(-m,-n;\alpha + 1;1 - \frac{1}{z\bar{z}}); \quad m, n = 0, 1, 2, ...,
$$

where $2F_1$ is the Gauss hypergeometric function and $\{m, n\}$ stands for the minimum of $m$ and $n$, $P^\alpha_{m,n}(z, \bar{z})$ are polynomials of $(z, \bar{z})$ of degree $m+n$ and they ensure the following properties,

$$
P^\alpha_{m,n}(-z, -\bar{z}) = (-1)^{m+n} P^\alpha_{m,n}(z, \bar{z}); \quad \overline{P^\alpha_{m,n}(z, \bar{z})} = P^\alpha_{m,n}(z, \bar{z}) = P^\alpha_{n,m}(z, \bar{z}).
$$

The area element of the plane is

$$
\frac{i}{2} dz \wedge d\bar{z} = rdr \wedge \phi = dx \wedge dy.
$$

We have

$$
\int_{z\bar{z} \leq 1} \frac{i}{2} dz \wedge d\bar{z} (-z\bar{z})^\alpha P^\alpha_{k,l}(z, \bar{z}) P^\alpha_{m,n}(z, \bar{z}) = A_\alpha(m, n) \delta_{km} \delta_{ln},
$$

where

$$
A_\alpha(m, n) = \frac{\pi m! n! \alpha!}{(m+n+\alpha+1)(m+\alpha)!(n+\alpha)!}.
$$

Introduce

$$
P^\alpha_{m,n}(z, \bar{z}) = A_\alpha(m, n) \frac{1}{2} (1-z\bar{z})^\alpha P^\alpha_{m,n}(z, \bar{z}),
$$

then

$$
\langle p^\alpha_{k,l} | p^\alpha_{m,n} \rangle = \int_{z\bar{z} \leq 1} \frac{i}{2} dz \wedge d\bar{z} p^\alpha_{k,l}(z, \bar{z}) p^\alpha_{m,n}(z, \bar{z}) = \delta_{km} \delta_{ln}.
$$
defines an inner product and the set $B_\alpha = \{ p_{m,n}^\alpha(z,\overline{z}) \mid m, n = 0, 1, 2, \ldots; \alpha \text{ fixed} \}$ is complete in $L^2_{\alpha}(D, \frac{i}{2} dz \wedge d\overline{z})$. The completeness relation is given by

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n}^\alpha(z,\overline{z})\overline{p_{m,n}^\alpha(z',\overline{z}')} = \delta(z - z', \overline{z} - \overline{z}'); \quad |z|, |z'| < 1.$$  

The lowering operator for $P_{m,n}^\alpha(z,\overline{z})$ is given by

$$(m\overline{z} + (1 - \overline{z}) \frac{\partial}{\partial z}) P_{m,n}^\alpha(z,\overline{z}) = m P_{m-1,n}^\alpha(z,\overline{z})$$

and raising operator is

$$(m + 1 + \alpha)z + (1 - z\overline{z}) \frac{\partial}{\partial z} P_{m,n}^\alpha(z,\overline{z}) = (m + 1 + \alpha) P_{m+1,n}^\alpha(z,\overline{z}).$$

Let

$$K_- = \sqrt{\frac{m + n + \alpha}{m + n + 1 + \alpha}} \left( (m + \frac{\alpha}{2})\overline{z} + (1 - z\overline{z}) \frac{\partial}{\partial z} \right)$$

$$K_+ = \sqrt{\frac{m + n + 2 + \alpha}{m + n + 1 + \alpha}} \left( (m + \frac{\alpha}{2})z - \frac{\partial}{\partial \overline{z}}(1 - z\overline{z}) \right).$$

Then we have

$$K_- p_{m,n}^\alpha(z,\overline{z}) = \sqrt{m(m + \alpha)} p_{m-1,n}^\alpha(z,\overline{z})$$

$$K_+ p_{m,n}^\alpha(z,\overline{z}) = \sqrt{(m+1)(m+1+\alpha)} p_{m+1,n}^\alpha(z,\overline{z}).$$

Further

$$K_+^\dagger = K_-$$

and

$$[K_-, K_+] p_{m,n}^\alpha(z,\overline{z}) = 2 \left( m + \frac{1+\alpha}{2} \right) p_{m,n}^\alpha(z,\overline{z}) = 2 K_0 p_{m,n}^\alpha(z,\overline{z}).$$

From [22] we also have the recurrence relation

$$(m + n + \alpha + 1)z P_{m,n}^\alpha(z,\overline{z}) = (m + 1 + \alpha) P_{m+1,n}^\alpha(z,\overline{z}) + n P_{m,n-1}^\alpha(z,\overline{z}).$$

The following lemma is essential in obtaining a reproducing kernel Hilbert space and, thereby, CS and to the computations of lower symbols and in so doing Berezin transforms. To the best of our knowledge, the following summation formulas for the 2D-Zernike polynomials have not yet been worked out. We shall prove it in the following lemma.
Lemma 4.1. For $\alpha > -1$, $z \bar{z} < 1$ and $w \bar{w} < 1$, we have

\[
\sum_{m=0}^{\infty} \frac{P_{m,n}^\alpha(z, \bar{z}) P_{m,n}^\alpha(w, \bar{w})}{A_\alpha(m, n)} = \left[ \frac{n \cdot (\alpha + 1)n}{\pi \cdot n!} \cdot (\bar{z} \cdot w)^n \right] \frac{(1 - \frac{\bar{w}}{w})^n \cdot (1 - \frac{z}{\bar{z}})^n}{(1 - z \cdot w)^{2n+1+\alpha}}
\]

\[
\times 2 F_1 \left( -n, -n; 1 + \alpha; \frac{z \bar{w}}{1 - \frac{1}{w \bar{w}}} \left( \frac{1 - \frac{1}{z \bar{z}}}{1 - \frac{1}{w \bar{w}}} \right) \right)
\]

\[
+ \left[ \frac{\alpha + 1 \cdot (\alpha + 1)n}{\pi \cdot n! \cdot (1 - z \bar{w})^{n+\alpha+2}} \cdot (\bar{z} \cdot w)^n \right] \left( 1 - \frac{\bar{w}}{w} \right)^n \sum_{k=0}^{n} \frac{(-n)_k \cdot (\alpha + 2)_k}{k! \cdot (\alpha + 1)_k}
\]

\[
\times \left( 1 - \frac{1}{w \bar{w}} \right)^k \cdot \left( \frac{z \bar{w}}{z \bar{w} - 1} \right)^k F_1 \left( -n, -k; -1 + \alpha; \frac{1 - \frac{1}{z \bar{z}}}{1 - \frac{1}{w \bar{w}}} \right), \left( \frac{1 - \frac{1}{z \bar{z}}}{1 - \frac{1}{w \bar{w}}} \right) \right)
\]

\[
= E_n^\alpha (z, \bar{w}) \quad \text{(call it so),}
\]

\[\text{(4.14)}\]

where $F_1$ is the Appell function of two variables [11]. As special cases, we also have

(a) \[\sum_{m=0}^{\infty} \frac{P_{m,0}^\alpha(z, \bar{z}) P_{m,0}^\alpha(w, \bar{w})}{A_\alpha(m, 0)} = \frac{(\alpha + 1)}{\pi (1 - w z)^{\alpha+2}}\]

(b) \[\sum_{m=0}^{\infty} \frac{|P_{m,n}^\alpha(z, \bar{z})|^2}{A_\alpha(m, n)} = \frac{(2n + \alpha + 1)}{\pi (1 - z \bar{z})^{\alpha+2}}; \quad n = 0, 1, 2 \ldots \]

Proof. See the appendix. \hfill \Box

Remark 4.2. In Equation (4.14), if we set $n = 0$ and $\alpha = 0$, we get the usual Bergman kernel for the open unit disc $\mathcal{D}$, see for example [16],

\[B(z, w) = \frac{1}{\pi (1 - z \bar{w})^2}; \quad z, w \in \mathcal{D}\]

and if we only set $n = 0$ we get the weighted Bergman kernel [12]

\[B_\alpha(z, w) = \frac{\alpha + 1}{\pi (1 - z \bar{w})^{\alpha+2}}; \quad z, w \in \mathcal{D}\]

of the weighted Bergman space $A^2_\alpha(\mathcal{D})$.

5. Reproducing kernel Hilbert spaces

For each fixed $\alpha > -1$, let $d\mu(z, \bar{z}) = \frac{i}{2} (1 - z \bar{z})^\alpha dz \wedge d\bar{z}$ and denote the corresponding complex valued $\mu$-square integrable functions on $\mathcal{D}$ by $L^2_\mu(\mathcal{D}, d\mu(z, \bar{z}))$ and for each $n$, let

\[B^\alpha_n = \left\{ \frac{P_{m,n}^\alpha(z, \bar{z})}{\sqrt{A_\alpha(m, n)}} \mid m = 0, 1, 2, \ldots \right\}.
\]
Let $\Lambda_0^\alpha(D) = \text{span}B_0^\alpha$. Then $B_0^\alpha$ is a basis of $\Lambda_0^\alpha(D)$ and it is a reproducing kernel Hilbert space with the kernel $E_0^\alpha(z, w)$ (see Lemma 4.1), we also need the following in the sequel

\begin{equation}
N_n(z, \overline{z}) = E_n^\alpha(z, \overline{z}) = \frac{(2n + \alpha + 1)}{\pi(1 - z\overline{z})^{\alpha + 2}}, \quad n = 0, 1, 2, \ldots.
\end{equation}

Further one can write

\begin{equation}
L_\alpha^2(D, d\mu(z, \overline{z})) = \bigoplus_{n=0}^{\infty} \Lambda_n^\alpha(D).
\end{equation}

For each $\alpha > -1$

$$B_0^\alpha = \left\{ \frac{z^m}{A_\alpha(m, 0)} \mid m = 0, 1, 2, \ldots \right\}.$$}

is the basis of the space $\Lambda_0^\alpha(D)$ and it is the classical Bargmann space of anti-holomorphic functions on the unit disc with reproducing kernel

$$E_0^\alpha(z, \overline{w}) = \frac{\alpha + 1}{\pi(1 - \overline{w}z)^{\alpha + 2}}.$$}

The closure of the space spanned by the set

$$\left\{ \frac{z^m}{\sqrt{A_\alpha(m, 0)}} \mid m = 0, 1, 2, \ldots \right\}$$

is the set of all holomorphic functions. Now for a given integral linear operator of the form

$$(Af)(x) = \int K_A(x, y)f(y)dy$$

the function $K_A(x, y)$ is called its Schwartz kernel [10]. As it was done in [15], here since by constructions $B_0^\alpha$ is an orthonormal basis for $\Lambda_0^\alpha(D)$ and are pairwise orthogonal in the Hilbert space $L_\alpha^2(D, d\mu(z, \overline{z}))$, for the projection operator

$$P_n^\alpha : L_\alpha^2(D, d\mu(z, \overline{z})) \to \Lambda_n^\alpha(D)$$

the integral Schwartz kernel $K_n^\alpha(z, w)$ can be obtained as

$$[P_n^\alpha f](z) = \sum_{m=0}^{\infty} \left\langle f, \frac{P^\alpha_{m, n}}{\sqrt{A_\alpha(m, n)}} \right\rangle \frac{P^\alpha_{m, n}(z, \overline{w})}{\sqrt{A_\alpha(m, n)}}$$

$$= \int_D f(w) \sum_{m=0}^{\infty} \frac{P_{m, n}(w, \overline{w})P^\alpha_{m, n}(z, \overline{w})}{A_\alpha(m, n)} d\mu(w).$$

Thereby the integral Schwartz kernel of the projection operator, $P_n^\alpha$, is given by

\begin{equation}
K_n^\alpha(z, w) = \sum_{m=0}^{\infty} \frac{P_{m, n}(w, \overline{w})P^\alpha_{m, n}(z, \overline{w})}{A_\alpha(m, n)} = E_n^\alpha(z, \overline{w}).
\end{equation}
6. QUANTIZATION OF THE COMPLEX UNIT DISC

In order to adapt to the general construction let us formulate for the disc as follows:

(a) Let \( d\mu(z, \overline{z}) = \frac{i}{2}(1 - z\overline{z})^\alpha dz \wedge d\overline{z} \).

(b) \((D, d\mu(z, \overline{z}))\) is a measure space.

(c) \( L_2^\alpha(D, d\mu(z, \overline{z})) \) is the corresponding Hilbert space of complex valued square integrable functions.

(d) \( \left\{ \frac{P_{m,n}(z, \overline{z})}{\sqrt{A_\alpha(m,n)}} \right\} \) is an orthonormal set in \( L_2^\alpha(D, d\mu(z, \overline{z})) \).

(e) \( \{ |e_m\rangle \mid m = 0, 1, 2, ... \} \) be an orthonormal basis of an abstract Hilbert space \( \mathcal{H} \).

With the above let us consider the vectors

\[
|z, \alpha, n\rangle = N_n(z, \overline{z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{P_{m,n}(z, \overline{z})}{\sqrt{A_\alpha(m,n)}} |e_m\rangle.
\]

From (4.5) and lemma (4.1)-(b), the vectors in (6.1) forms a set of CS in the sense that
- they are normalized, \( \langle z, \alpha, n | z, \alpha, n \rangle = 1 \) and
- satisfy a resolution of the identity

\[
\int_D N_n(z, \overline{z}) |z, \alpha, n\rangle \langle z, \alpha, n| d\mu(z, \overline{z}) = I_{\mathcal{H}}.
\]

Equation (6.2) allows us to implement CS quantization of the disc \( D \) by associating a function \( D \ni z \mapsto f(z, \overline{z}) \).

For this define the operator on \( \mathcal{H} \)

\[
f(z, \overline{z}) \mapsto A_f = \int_D N_n(z, \overline{z}) f(z, \overline{z}) |z, \alpha, n\rangle \langle z, \alpha, n| d\mu(z, \overline{z}).
\]

That is

\[
A_f = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} |e_m\rangle \langle e_l| \sqrt{\frac{A_\alpha(m,n)}{A_\alpha(l,n)}} \int_D f(z, \overline{z}) P_{m,n}(z, \overline{z}) P_{l,n}(z, \overline{z}) d\mu(z, \overline{z}).
\]

From the orthogonality relation (4.3), for \( f(z) = 1 \), we immediately get \( A_1 = I_{\mathcal{H}} \). Further using the recursion relation (4.13) and the orthogonality relation (4.3), it can easily be seen that

\[
A_z = \sum_{m=0}^{\infty} C(m, n, \alpha) |e_m\rangle \langle e_{m+1}|
\]

\[
A_{\overline{z}} = \sum_{m=0}^{\infty} C(m, n, \alpha) |e_{m+1}\rangle \langle e_m|
\]

where

\[
C(m, n, \alpha) = \sqrt{\frac{(m + 1)(m + \alpha + 1)}{(m + n + \alpha + 2)(m + n + \alpha + 1)}}.
\]

Thereby, we have

\[
A_z |e_0\rangle = 0, A_z |e_m\rangle = C(m - 1, n, \alpha) |e_{m-1}\rangle; m = 1, 2, \cdots
\]

\[
A_{\overline{z}} |e_m\rangle = C(m, n, \alpha) |e_{m+1}\rangle; m = 0, 1, 2, \cdots.
\]
That is \( A_z, A_\bar{z} \) are annihilation and creation operators respectively. Their commutators take the form

\[
(6.7) \quad [A_z, A_\bar{z}] = C^2(0, n, \alpha)|e_0\rangle\langle e_0| + \sum_{m=1}^{\infty} \left[ C^2(m, n, \alpha) - C^2(m - 1, n, \alpha) \right] |e_m\rangle\langle e_m|.
\]

It may be interesting to note that for \( n = 0 \) and \( \alpha = 0 \) it becomes

\[
[A_z, A_\bar{z}] = \sum_{m=0}^{\infty} \frac{|e_m\rangle\langle e_m|}{(m+1)(m+2)}.
\]

In the basis \( \{|e_m| \mid m = 0, 1, 2, \cdots \} \), the matrix elements read

\[
(A_z)_{k,l} = \langle e_k | A_z | e_l \rangle = \begin{cases} 
0 & \text{if } l \neq k + 1 \\
C(k+1, n, \alpha) & \text{if } l = k + 1
\end{cases}
\]

and

\[
(A_\bar{z})_{k,l} = \langle e_k | A_\bar{z} | e_l \rangle = \begin{cases} 
0 & \text{if } k \neq l + 1 \\
C(k-1, n, \alpha) & \text{if } k = l + 1
\end{cases}
\]

The position, \( Q \) and the momentum, \( P \) operators are easily obtained by using the quantized version of the phase conjugate coordinates \( q = \frac{z + z}{\sqrt{2}} \) and \( p = \frac{z - z}{i\sqrt{2}} \) as

\[
(6.8) \quad Q = A_q = \frac{1}{\sqrt{2}}(A_z + A_\bar{z}) = \frac{1}{\sqrt{2}} \sum_{m=0}^{\infty} C(m, n, \alpha) \left[ |e_m\rangle\langle e_{m+1}| + |e_{m+1}\rangle\langle e_m| \right]
\]

\[
(6.9) \quad P = A_p = \frac{1}{i\sqrt{2}}(A_z - A_\bar{z}) = \frac{1}{i\sqrt{2}} \sum_{m=0}^{\infty} C(m, n, \alpha) \left[ |e_m\rangle\langle e_{m+1}| - |e_{m+1}\rangle\langle e_m| \right]
\]

and their commutators become \([Q, P] = i[A_z, A_\bar{z}]\) and the quantized version of the classical Hamiltonian \( H = \frac{q^2 + p^2}{2} = |z|^2 \) takes the form

\[
(6.10) \quad H = \frac{Q^2 + P^2}{2} = \frac{1}{2} [A_z A_\bar{z} + A_\bar{z} A_z].
\]

**Remark 6.1.** In [3], by adjusting the parameters in \( A_z \) and \( A_\bar{z} \) the authors were able to recover the usual canonical commutation relations. However, in our case, by adjusting \( n \) and \( \alpha \) in \( A_z, A_\bar{z} \) we cannot recover [3.13] this is due to the fact that the basis \( B \) in [3.3] or \( B_\alpha \) in [3.9] cannot be obtained from the basis \( B_n^\alpha \) of [5.1] by adjusting the parameters \( n \) and \( \alpha \).

6.1. **Overlap of the CS, lower symbols and Berezin transform.** Using lemma [4.1] we can compute the overlap of two CS as

\[
(6.11) \quad \langle z, \alpha, n|w, \alpha, \alpha \rangle = \frac{1}{\sqrt{N_n(z, \overline{z})N_n(w, \overline{w})}} \sum_{m=0}^{\infty} \frac{P^\alpha_{m,n}(z, \overline{z}) P^\alpha_{m,n}(w, \overline{w})}{A_{\alpha}(m, n)}
\]

\[
\quad = \frac{\pi}{(2n + \alpha + 1)} \left[ (1 - z\overline{z})(1 - w\overline{w}) \right]^{\alpha + \frac{1}{2}} E_n^\alpha(\overline{z}, w).
\]

Now, from (2.3) and (6.11), the lower symbol of \( A_f \) can be computed as

\[
\hat{f} = \langle z, n, \alpha | A_f | z, n, \alpha \rangle = \int_{D} N_n(w, \overline{w}) f(w) |\langle z, n, \alpha | w, n, \alpha \rangle|^2 d\mu(w)
\]

\[
\quad = \frac{\pi}{(2n + \alpha + 1)} \int_{D} (1 - z\overline{z})^{\alpha + 2} \left[ E_n^\alpha(\overline{z}, w) \right]^2 f(w) d\mu(w).
\]
Thereby the Berezin transform can be written as

\[(6.12) \quad B_n^\alpha[f](z) = \langle z, n, \alpha | A_f | z, n, \alpha \rangle = \frac{\pi(1 - z\overline{z})^{\alpha+2}}{2n + \alpha + 1} \int_D |E_n^\alpha(z, w)|^2 f(w) d\mu(w).\]

Equation (6.12) can be considered as a more generalized Berezin transform for the unit disc in a sense that \(n = 0\), that is

\[E_0^\alpha(z, w) = \frac{(\alpha + 1)}{\pi(1 - w\overline{z})^{\alpha+2}},\]

leads to the Berezin transform of the weighted Bergman space, \(A_0^\alpha(D)\), (14)

\[(6.13) \quad B_n^\alpha[f](z) = \langle z, 0, \alpha | A_f | z, 0, \alpha \rangle = \frac{(\alpha + 1)}{\pi} \int_D \frac{(1 - z\overline{z})^{\alpha+2}}{(1 - w\overline{z})^{4+2\alpha}} f(w) d\mu(w)\]

and the values \(n = 0\) and \(\alpha = 0\), leads to the standard Berezin transform of the unit disc (see [1], pp-68), namely

\[(6.14) \quad B[f](z) = \frac{1}{\pi} (1 - |z|^2)^2 \int_D \frac{f(w)}{|1 - w\overline{z}|^4} d\mu(w).\]

7. Upper symbols as differential operators

From (1.3) and (1.6) \(B_\alpha = \{p_{m,n}^\alpha(z, \overline{z}) \mid m, n = 0, 1, 2; \alpha \text{ fixed}\}\) is an orthonormal basis of \(L_2^\alpha(D, \frac{i}{\pi} dz \wedge d\overline{z})\). Let \(B_\alpha^n = \{p_{m,n}^\alpha(z, \overline{z}) \mid m = 0, 1, 2, \ldots\}\) and

\[H_\alpha^n = \text{span} B_\alpha^n,\]

then \(B_\alpha^n\) is an orthonormal basis of \(H_\alpha^n\). Now consider the CS

\[|z, n, \alpha\rangle = \mathcal{N}(z, \overline{z})^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{P_{m,n}^\alpha(z, \overline{z})}{\mathcal{A}_\alpha(m, n)} p_{m,n}^\alpha,\]

that is in equation (6.1) replace the basis \(\{|e_m\rangle \mid m = 0, 1, 2, \ldots\}\) by the basis \(B_\alpha^n\). On the Hilbert space \(H_\alpha^n\) using equation (1.9) we can write

\[\sqrt{\frac{m+n+\alpha}{m+n+1+\alpha}} \left( \frac{m + \alpha}{2} \overline{z} + (1 - z\overline{z}) \frac{\partial}{\partial z} \right) p_{m,n}^\alpha(z, \overline{z}) = \sqrt{m(m + \alpha)} p_{m-1,n}^\alpha(z, \overline{z})\]

thereby, as in section 3,

\[\left( \frac{m + \alpha}{2} \overline{z} + (1 - z\overline{z}) \frac{\partial}{\partial z} \right) p_{m,n}^\alpha(z, \overline{z}) = \frac{\sqrt{m(m + \alpha)(m + n + \alpha + 1)}}{m + n + \alpha} p_{m-1,n}^\alpha(z, \overline{z})\]

\[= (m + n + 1 + \alpha) \sqrt{\frac{m(m + \alpha)}{(m + n + \alpha + 1)(m + n + \alpha)}} p_{m-1,n}^\alpha(z, \overline{z})\]

\[= (m + n + 1 + \alpha) C(m - 1, n, \alpha) p_{m-1,n}^\alpha(z, \overline{z})\]

\[= (m + n + 1 + \alpha) A_\alpha p_{m,n}^\alpha(z, \overline{z}).\]

Thus on \(H_\alpha^n\) we have

\[(7.1) \quad A_\alpha = \frac{1}{(m + n + 1 + \alpha)} \left( \frac{m + \alpha}{2} \overline{z} + (1 - z\overline{z}) \frac{\partial}{\partial z} \right)\]
or

\[ A_z = \frac{1}{\sqrt{(m + n + \alpha + 1)(m + n + \alpha)}} K_. \]

Similarly on \( H^\alpha_n \) we can obtain

\[
A_z = \frac{1}{(m + n + 1 + \alpha)} \left( (m + \frac{\alpha}{2})z - \frac{\partial}{\partial z} (1 - z\overline{z}) \right) \\
= \frac{1}{(m + n + 1 + \alpha)} \left( (m + 1 + \frac{\alpha}{2})z - (1 - z\overline{z}) \frac{\partial}{\partial z} \right) \\
= \frac{1}{\sqrt{(m + n + \alpha + 1)(m + n + \alpha + 2)}} K_+
\]

and also using (4.12) we can write

\[
[A_z, A_\overline{z}] = \frac{2}{(m + n + \alpha + 1)\sqrt{(m + n + \alpha)(m + n + \alpha + 2)}} K_0.
\]

8. CONCLUSION

Using the 2D-Zernike polynomials we have quantized the complex unit disc. The quantization provided in this note is quite different from the one obtained with the standard \( SU(1,1) \) CS. The difference occurred due to the basis and thereby the reproducing kernel Hilbert space used in this note, that is the typical discrete series basis of \( SU(1,1) \) cannot be obtained by adjusting \( n \) and \( \alpha \) in \( P_{\alpha,m,n}(z,\overline{z}) \sqrt{A_{\alpha}(m,n)} \). However, instead of directly using the 2D-Zernike polynomials if one uses a variation of it, as the one considered in [21], one may able to go back to the \( SU(1,1) \) case.

Using the CS constructed in this note and the 2D-Zernike polynomials one may consider studying modular structures on \( \mathcal{D} \) as it was done with the complex Hermite polynomials in [3]. Further as it was done in [15] one can study the spectral properties of Cauchy transform on \( L^2_0(\mathcal{D}, \frac{1}{2}(dz \wedge d\overline{z})) \) because the structure used in [15] is comparable to the one we have developed in section 5. In \( P_{\alpha,n,m}(z,\overline{z}) \) by setting \( n \geq m \) and \( n = m + s \) we can obtain a similar family of reproducing kernel Hilbert subspaces of \( L^2_0(\mathcal{D}, \frac{1}{2}(1 - z\overline{z})^\alpha dz \wedge d\overline{z}) \) as given in [3] for \( \mathbb{C} \) and make a study along the lines of [3] for the disc \( \mathcal{D} \). We shall tackle some of these issues in our future work.

9. APPENDIX

In this section, we prove Lemma 4.1 For the case \( n = 0 \) we have, using (4.2),

\[
\sum_{m=0}^{\infty} \frac{P_{\alpha,m,0}(z,\overline{z}) P_{\alpha,m,0}(w,\overline{w})}{A_{\alpha}(m,0)} = \frac{(\alpha + 1)}{\pi} \sum_{m=0}^{\infty} \frac{(\alpha + 2)_m}{m!} (\overline{w} z)_m = \frac{(\alpha + 1)}{\pi (1 - \overline{w} z)^{\alpha + 2}}.
\]

by means of

\[
(1 + \alpha + m) \cdot (1 + \alpha)_m = (1 + \alpha) \cdot (2 + \alpha)_m
\]

and

\[
\sum_{m=0}^{\infty} \frac{(\alpha + 2)_m}{m!} (\overline{w} z)_m = \frac{1}{\Gamma_0(2 + \alpha; -; \overline{w} z)} = (1 - \overline{w} z)^{-2 - \alpha}.
\]
By means of \((4.1)\), we have
\[
\sum_{m=0}^{\infty} \frac{P_{m,n}^\alpha(z,\bar{z})P_{m,n}^\alpha(w,\bar{w})}{A_\alpha(m,n)} = \frac{(\alpha+1)_n}{\pi \cdot n!} (\bar{z} \cdot w)^n \sum_{m=0}^{\infty} \frac{(m+n+\alpha+1) \cdot (\alpha+1)_m}{\Gamma(m+1)} \cdot (z \cdot \bar{w})^m
\]
\[
\times 2F_1(-m,-n;\alpha+1;1-\frac{1}{zz}) \cdot 2F_1(-m,-n;\alpha+1;1-\frac{1}{ww})
\]
\[
= n \frac{(\alpha+1)_n}{\pi \cdot n!} (\bar{z} \cdot w)^n \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{\Gamma(m+1)} \cdot (z \cdot \bar{w})^m
\]
\[
\times 2F_1(-m,-n;\alpha+1;1-\frac{1}{zz}) \cdot 2F_1(-m,-n;\alpha+1;1-\frac{1}{ww})
\]
\[
+ \frac{(\alpha+1) \cdot (\alpha+1)_n}{\pi \cdot n!} (\bar{z} \cdot w)^n \sum_{m=0}^{\infty} \frac{(\alpha+2)_m}{\Gamma(m+1)} \cdot (z \cdot \bar{w})^m
\]
\[
\times 2F_1(-m,-n;\alpha+1;1-\frac{1}{zz}) \cdot 2F_1(-m,-n;\alpha+1;1-\frac{1}{ww})
\]
\[
= n \frac{(\alpha+1)_n}{\pi \cdot n!} (\bar{z} \cdot w)^n \cdot S_1 + \frac{(\alpha+1) \cdot (\alpha+1)_n}{\pi \cdot n!} (\bar{z} \cdot w)^n \cdot S_2.
\]
(9.3)

For the first sum \(S_1\), we use the following identity (note we corrected here the typo of the formula as given in Reference [9], formula (3)),
\[
\sum_{m=0}^{\infty} \frac{(1+\alpha)_m \cdot z^m}{m!} \cdot 2F_1(-m,\beta;1+\alpha;\nu) \cdot 2F_1(-m,b;1+\alpha;u)
\]
\[
= (1-z)^{\beta+b-\alpha-1} (1-\nu(1-z+uz))^{-\beta} (1-z+uz)^{-b}
\]
\[
\times 2F_1 \left( \beta,b;1+\alpha;\frac{z\nu}{1-z+uz}(1-z+uz) \right)
\]
where \(|u| < 1, |\nu| < 1\) and \(|z| < 1\). We have
\[
S_1 = \frac{(1-\frac{w}{\bar{z}})^n}{(1-z \cdot \bar{w})^{n+1+\alpha}} \cdot 2F_1 \left( -n,-n;1+\alpha;\frac{z\bar{w}(1-\frac{1}{ww})}{1-\frac{1}{\bar{w}}} \cdot \frac{1-\frac{1}{z\bar{w}}}{1-\frac{1}{w}} \right)
\]
(9.5)

We can easily show, using the terminated series representation of the Gauss hypergeometric function \(2F_1(-n,\beta;\gamma;z)\) and collecting the terms, that
\[
\lim_{(w,\bar{w}) \to (z,\bar{z})} \left\{ \frac{n \cdot (\alpha+1)_n}{\pi \cdot n!} \cdot (\bar{z} \cdot w)^n \right\} \cdot S_1 = \frac{n}{\pi} (1-z\bar{w})^{-1-\alpha}.
\]
(9.6)

For the second sum \(S_2\), we use the following identity ([19], formula 2.6)
\[
\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \cdot 2F_1(-m,\mu;\alpha+1;x) \cdot 2F_1(-m,\nu;\beta+1;y) \cdot z^m
\]
\[
= (1-z)^{\mu-\lambda} (1-z+xz)^{-\mu} \sum_{m=0}^{\infty} \frac{(\lambda)_m \cdot (\nu)_m}{m! \cdot (\beta+1)_m} \left( \frac{yz}{z-1} \right)^m
\]
\[
\times 2F_1 \left( \mu,-m,\alpha-\lambda+1;\alpha+1;\frac{x}{1-z+xz},\frac{xz}{1-z+xz} \right)
\]
(9.7)
where $F_1$ is the Appell series of two variables $[11]$. Thus, we have

\[ S_2 = (1 - zw)^{-\alpha - 2} \left(1 - \frac{w}{z}\right)^n \sum_{k=0}^{n} \frac{(-n)_k \cdot (\alpha + 2)_k}{k! \cdot (\alpha + 1)_k} \cdot \left(1 - \frac{1}{wz}\right)^k \cdot \left(\frac{zw}{zw - 1}\right)^k \]

\[ \times F_1 \left(-n, -k, -1; 1 + \alpha; \frac{1 - \frac{1}{zw}}{1 - \frac{w}{z}}, \frac{1 - \frac{1}{zw}}{1 - \frac{w}{z}}\right) \]

(9.8)

where we used the fact that $(-n)_k = 0$ for $k > n$. Now, using the double series representation of the Appell function and collecting the terms, we can show that

\[ \lim_{(w, z) \to (z, w)} \left[ \frac{(\alpha + 1) \cdot (\alpha + 1)_n (\overline{z} \cdot w)^n}{\pi \cdot n!} \right] \cdot S_2 = \frac{(n + 1 + \alpha + nzw)}{\pi} \cdot (1 - zw)^{-\alpha - 2}. \]

(9.9)

Consequently, from (9.6) and (9.9), we have for (9.3) in the case of $z = w$ that

\[ \sum_{m=0}^{\infty} \frac{P_{m,n}^\alpha (z, \overline{z}) P_{m,n}^\alpha (w, \overline{w})}{A_\alpha (m, n)} = \frac{1}{\pi} \cdot (2n + 1 + \alpha) \cdot (1 - z \overline{z})^{-\alpha - 2} \]

(9.10)

In general, for $z \neq w$ (so $z \neq \overline{w}$), we have

\[ \sum_{m=0}^{\infty} \frac{P_{m,n}^\alpha (z, \overline{z}) P_{m,n}^\alpha (w, \overline{w})}{A_\alpha (m, n)} = \left[ \frac{n \cdot (\alpha + 1)_n}{\pi \cdot n!} \cdot (\overline{z} \cdot w)^n \right] \frac{(1 - \frac{w}{z})^n \cdot (1 - \frac{z}{w})^n}{(1 - z \overline{w})^{2n+1+\alpha}} \]

\[ \times F_1 \left(-n, -n; 1 + \alpha; \frac{z \overline{w}}{1 - \frac{w}{z}}, \frac{1 - \frac{1}{zw}}{1 - \frac{w}{z}}\right) \]

\[ + \left[ \frac{(\alpha + 1) \cdot (\alpha + 1)_n}{\pi \cdot n!} \cdot (\overline{z} \cdot w)^n \right] \left(1 - \frac{w}{z}\right)^n \cdot \sum_{k=0}^{n} \frac{(-n)_k \cdot (\alpha + 2)_k}{k! \cdot (\alpha + 1)_k} \]

\[ \times \left(1 - \frac{1}{wz}\right)^k \cdot \left(\frac{zw}{zw - 1}\right)^k F_1 \left(-n, -k, -1; 1 + \alpha; \frac{1 - \frac{1}{zw}}{1 - \frac{w}{z}}, \frac{1 - \frac{1}{zw}}{1 - \frac{w}{z}}\right) \]

(9.11)
This expression can be further simplified using for the Appell function

\[
F_1 \left( -n, -k, -1; 1 + \alpha; \frac{1 - \frac{1}{z\bar{z}}}{1 - \frac{w}{\bar{z}}}, \frac{1 - \frac{1}{z\bar{z}}}{1 - \frac{w}{\bar{z}}} \right)
\]

\[
= 2F_1 \left( -n, -k; 1 + \alpha; \frac{1 - \frac{1}{z\bar{z}}}{1 - \frac{w}{\bar{z}}} \right) + \frac{n \cdot \left( 1 - \frac{1}{z\bar{z}} \right) \cdot (z\bar{w})}{(1 + \alpha) \cdot \left( 1 - \frac{w}{\bar{z}} \right)} \cdot 2F_1 \left( -n + 1, -k; 2 + \alpha; \frac{1 - \frac{1}{z\bar{z}}}{1 - \frac{w}{\bar{z}}} \right)
\]

(9.12)

and the Chu-Vandermonde's theorem ([7], pp. 2-3)

\[
2F_1(-k, -1; 1 + \alpha; 1) = \frac{(2 + \alpha)_k}{(1 + \alpha)_k}
\]

then using the identity [9.7].

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