Abstract

Fractional calculus has gained enormous visibility among the scientific community in last few decades and yet there are still many controversial and open theoretical discussions. In this work we want to elucidate in a simple way that it is possible to calculate the fractional integrals and derivatives of order $\alpha$ (using the Riemann-Liouville formulation) of power functions $(t - s)^\beta$ with $\beta$ being any real value, so long as one pays attention to the proper choosing of the lower and upper limits according to the original function’s domain. This is specially oriented as a review to those newly inducted on the process fractionally integrodifferentiating under the (in view of the nowadays multiples definitions) “classical” Riemann-Liouville setting. We, therefore, obtain valid expressions that are described in terms of function series of the type $(t - s)^{\alpha + k}$ and also observe that they are related to the famous hypergeometric functions of the Mathematical-Physics.

1 Introduction

The non-integer order calculus, popularly known as fractional calculus (FC) was born in 1695. Only after nearly 250 years did the first event dedicated exclusively to the theme [1, 2, 3]. Today, after more than 40 years, FC has gained enormous visibility both from a theoretical point of view and in applications.

In the theoretical point of view we mention, in addition to the classical formulations of the derivative (Riemann-Liouville, Caputo and Grünwald-Letnikov, just to cite a few) [4] a recent one due to Caputo-Fabrizio [5] where the kernel is non singular and whose properties were studied by Losada-Nieto [6] and as an application by Atangana [7] in the study of Fischer’s reaction-diffusion equation. As a generalization of these derivatives we cite the $\Psi$-Hilfer derivative [8] and the general Hilfer-Hadamard derivative [9]. On the other hand, also recent, there is a wide class of new derivatives that recover the classical (definition) derivative in the Newton/Leibniz sense and, even if fractional call, they are local derivatives [10].

As applications we can cite a wide range where the FC acts. Some of them are: Yang et al. [11] discuss anomalous diffusion models with general fractional derivatives within the kernels of the extended Mittag-Leffler functions; Yang [12] propose a fractional derivative of constant and variable orders applied to anomalous relaxation models in heat-transfer problems; Atangana and Baleann [13] introduce a new fractional derivative with non-local and non-singular kernel to discuss a particular heat transfer model; Yang et al. [14] present a new family of the local fractional partial differential equations; Yang et al. [15] discuss a LC-electric circuit modeled by a local fractional calculus; Gao and Yang [16] using local fractional Euler’s method, discuss the steady heat-conduction problem; Gómez and Capelas de Oliveira [17] propose and discuss a nonlinear partial differential equation variational iteration method; Costa et al. [18] present a nonlinear fractional Harry Dym equation whose solution is written in terms of the Fox’s $H$-function; Garrappa et al. [19] discuss models of dielectric relaxation.
based on completely monotone functions; Rosa and Capelas de Oliveira \cite{Rosa20}, discuss some particular fractional differential equation involving dielectrics and discuss the complete monotonicity and Capelas de Oliveira et al. \cite{Capelas21} discuss analytic components for the hadronic total cross-section using Mellin transform.

When it comes to “Calculus and Analysis” concepts such as limits, infinitesimals and continuity of functions are among the first that, at one step after another, leads to the definitions of the operations of differentiation and integration. Since these operations acts on a certain suitable class of functions, it is due to their simplicity and well understood behavior that in any introductory course to (Standard) Calculus the first class of functions to be “concretely” computed are the power functions, that is, functions of the type \( f(t) = (t - d)^\beta \), with \( d, \beta \in \mathbb{R} \). It is therefore logical that in the theory of the so-called Fractional Calculus (FC) the rules for operating these type of functions must also be carefully analyzed and established, but there is more to it than this simple argument. Power functions defines a very important class of functions and there are many natural (and artificial) phenomena that behaves according to power law distributions (e.g., Newtonian laws of gravity and electrostatics, Kepler’s third law about the orbital period of planets, the square-cube law relating the ratio between surface area and volume of an object, the Stefan-Boltzman law describing the power radiated from a black body in terms of its temperature, etc...), particularly, there are an increasing number of works showing the interesting link between FC models and physical phenomena described in terms of power laws. Just to mention a few examples, in \cite{Capelas22} it is presented a new fractional Laplacian time-space model to describe the frequency-dependent attenuation obeying empirical power law distribution. In \cite{Capelas23} it is studied the duality of Hamiltonian dynamics of a system of particles with power-like interactions with the solution of certain fractional differential equations. In \cite{Capelas24} is presented a fractional generalization of the Kelvin-Voigt rheology in order to better simulate the power law stress-strain relation of some biological media. In \cite{Capelas25} the authors study the link between multiple relaxation models, power law attenuation and fractional wave equations, providing some physically based evidence for the use of FC in the modelling processes. In \cite{Capelas26} it is proposed a fractional model to describe a power law relation between pain transmission processes in the human body and analgesia measurements, and the lists may go on. So all these works suggest that having a clear picture of the theoretical behavior and modus operandis of FC operators acting on power functions (and more generally to functions described in terms of power series such as analytic functions) may provide us with insights to several distinct power law related phenomena or, at the very least, a better and concise mathematical tool available to be used in the same way as already happens with the standard calculus.

In fact, in this work we are concerned exactly to this matter, because although the main literature provide some basic rules for fractionally operating power functions, we believe that there are still some fundamental aspects that are not fully cleared and understood and the matter still require more attention. To clarify this argument, we recall the following FC result: Let \( f(t) = (t - d)^\beta \), with \( d, \beta \in \mathbb{R} \). It is known that whenever \( \beta > -1 \), we can compute expressions for the Riemann-Liouville fractional integral (RLFI) and Riemann-Liouville fractional derivative (RLFD) of these power functions \cite{Capelas27}:

\[
\begin{align*}
\left[ d I_t^\alpha (x - d)^\beta \right] (t) &= \frac{\Gamma (\beta + 1)}{\Gamma (\beta + \alpha + 1)} (t - d)^{\beta + \alpha} , \\
\left[ d D_t^\alpha (x - d)^\beta \right] (t) &= \frac{\Gamma (\beta + 1)}{\Gamma (\beta - \alpha + 1)} (t - d)^{\beta - \alpha} ,
\end{align*}
\]

for \( \text{Re} (\alpha) \geq 0 \).

The proofs for the expressions in Eq.\,(1) and Eq.\,(2) are very straightforward and can be found, e.g. in \cite{Capelas28} and they make use of the integral representation for the Euler’s beta function

\[
B(\eta, \xi) = \int_0^1 x^{\eta - 1} (1 - x)^{\xi - 1} dx ,
\]

\[
= \frac{\Gamma (\eta) \Gamma (\xi)}{\Gamma (\eta + \xi)} , \quad \text{Re}[\eta], \text{Re}[\xi] > 0 .
\]
But here, we are particularly interested in evaluating possible expressions for the RLFI and the RLFD of order \( \alpha \in \mathbb{R} \) for the power functions stated above without restricting the value of the index \( \beta \in \mathbb{R} \), just as it happens when we’re dealing with the integer order calculus. Obviously, as one might expect, we won’t be able to use the same direct kind of Euler’s beta function approach, because of the restriction of the integral representation of the Euler’s beta function to the positive half complex-plane. We also point out that while the standard literature provide expressions for the RLFI and RLFD with the lower limit \( a \) of the operators coinciding with the shifting factor \( d \) in the argument of the power function (i.e., when \( a = d \) we call this situation, “centered”), we are also concerned with a more general setting when \( a \) and \( d \) are not necessarily equal (in such cases, we call this situation “displaced”). This is particularly important, because such “displaced” expressions frequently occurs when trying to solve fractional delay differential equations [29].

So in this work, we show that when one takes careful consideration on the choices for the lower and upper limits of these operations, it is possible to compute expressions for the RLFI and RLFD of any power function (regardless the index of the power) in terms of series that can be related to the famous hypergeometric functions [30] so important and commonly found in many problems of the Mathematical-Physics. It’s worth mentioning that in [31] the authors have provided two alternative definitions for fractional derivatives of power functions of any order, but they approached the problem in a very distinct way as they have not used the Riemann-Liouville formulation. Also, while there are many distinct formulations for a fractional differential operator [1, 5, 32, 33], it is our hope that with this work we can, not only provide helpful expressions for the aforementioned calculations of RLFI and RLFD of power functions to be used on analytical or numerical related problems, but also set some ground for a future discussion on the theoretical aspects of computing a fractional definite integral versus knowing its fractional primitives whenever such definitions are meaningful.

We recall some basic concepts in the preliminary section, in the second and third ones we present the main results stated as theorems, we then provide a summary of the results for convenience compiling the main formulas obtained. Finally we draw our conclusions and expose some further topics for researches. This work also contain an appendix where it is shown some calculations where we point out that our expressions can be identified with some hypergeometric functions.

2 Preliminaries

We recall that the RLFD is defined in terms of the RLFI which is, by the way, defined in terms of a “ordinary” definite integral, (see Def.1 and Def.2). So by construction, not only this makes the operator non-local, but also address the matter of fractional integrodifferentiability of a function to be dependant on the (upper and lower) limits of integration and the domain in which we want to operate, just as it happens in the classical integer order theory and we know that we can calculate the (classic integer order) indefinite integral of power functions \( f(t) = (t - d)^\beta \) regardless the values of \( \beta \in \mathbb{R} \). This means that any power function has a “classical” primitive and, as a consequence, one can calculate the \( n \)-fold (indefinite) integral (or \( n \)-fold primitive) of any order \( n \in \mathbb{N} \). The situation is similar when calculating the \( nth \)-order derivatives. Since the domain of the resulting power functions after the operations does eventually change and depend on the index \( \beta \) as well as the order \( n \) of the operators, then being (or not) able to calculate the definite integrals of these functions actually depends on where we perform the operations. After all in order to concretely calculate any “definite” results we must use the expressions for the functions obtained by the action of the formal operators: \( n \)-fold indefinite integrals and the \( nth \)-order derivatives. So, even though we can perform these integer order operators on power functions of any order, it is not necessarily true that one can calculate any definite integrals or derivative at a point for these functions at any arbitrary interval of \( \mathbb{R} \). And we point out that the situation for the fractional case should be similar.
Definition 1 Let $\Omega = [a, b] \subset \mathbb{R}$, $f \in L_1(\Omega)$ and $\alpha \in \mathbb{R}_+$. The expression for $[aJ_t^\alpha f(x)](t)$,

$$[aJ_t^\alpha f(x)](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx,$$

(5)
defines (leftwise) the Riemann-Liouville fractional integral of order $\alpha$ of the function $f$.

Definition 2 Let $\Omega = [a, b] \subset \mathbb{R}$, $n = [\alpha] + 1$ where $[\alpha]$ is the integer part of $\alpha \in \mathbb{R}_+$ and $f \in AC^n(\Omega)$. The expression for $[aD_t^n f(x)](t)$,

$$[aD_t^n f(x)](t) = \frac{d^n}{dt^n} [aJ_t^{n-\alpha} f(x)](t)$$

$$= \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f(x) dx,$$

(6)
defines (leftwise) the Riemann-Liouville fractional derivative of order $\alpha$ of the function $f$.

One can similarly define the (rightwise) versions of the RLFI and RLFD [34, 35].

In the definitions for the RLFI (Eq.(5)) and the RLFD (Eq.(6)), if $\alpha = 0$, we define both operators to be the identity operator $I$, while if one chooses $\alpha = n \in \mathbb{N}$ both operators reduces, respectively, to their integer order counterparts, that is, as $\alpha \to n$, $[aJ_t^\alpha f(x)](t) \to [aJ_t^n f(x)](t)$ $n$-fold integral and $[aD_t^n f(x)](t) \to [aD_t^n f(x)](t) = \frac{d^n}{dt^n} f(t)$ the $n$th-order derivative [28, 36].

In both definitions, the $\Gamma(\ast)$ symbol refers to the gamma function and we will be using some of its properties related to the ascending and descending Pochhammer symbols defined, respectively, as:

$$(z)_k = \begin{cases} 
1, & \text{if } k = 0, \\
z(z+1) \cdots (z+k-1), & \text{if } k \in \mathbb{N}.
\end{cases}$$

(7)

$$(z)_{-k} = \begin{cases} 
1, & \text{if } k = 0, \\
z(z-1) \cdots (z-k+1), & \text{if } k \in \mathbb{N},
\end{cases}$$

(8)

which can be rewritten in the following form

$$ (z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}, $$

(9)

$$ (z)_{-k} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)}, $$

(10)

and are valid the relations below

$$ (-z)_k = (-1)^k (z)_{-k}, $$

(11)

$$ (-z)_{-k} = (-1)^k (z)_k. $$

(12)

We also recall that the gamma function is uniquely determined as the function satisfying the functional relation

$$ \Gamma(1) = 1, $$

(13)

$$ z\Gamma(z) = \Gamma(z+1), \quad \text{Re}(z) > 0, $$

(14)

but the relation on Eq.(13) can be used to extend it analytically to all complex values, except on $N_0 = \{0, -1, -2, \ldots\}$, where $\Gamma(-n) \to \pm\infty$, $n \in \mathbb{N}$. Yet, the relation on Eq.(14) is valid for all complex values and when dealing with the elements of $N_0$ one should consider

$$ z^\pm \Gamma(z^\pm) = \Gamma(z^\pm + 1). $$

(15)
Although the gamma function is not defined for negative integers, the ratio of gamma functions of negative integers are defined \[36\]
\[
\frac{\Gamma(-n)}{\Gamma(-m)} = (-1)^{m-n} \frac{m!}{n!}, \quad m, n \in \mathbb{N}, \tag{16}
\]
and we point out that Eq. (16) is also valid when choosing \(m\) or \(n\) to be zero, with
\[
\frac{\Gamma(0)}{\Gamma(-m)} = (-1)^m \frac{m!}{0!} = (-1)^m m!, \tag{17}
\]
\[
\frac{\Gamma(-n)}{\Gamma(0)} = (-1)^{-n} \frac{0!}{n!} = \frac{(-1)^n}{n!}, \tag{18}
\]
\[
\frac{\Gamma(0)}{\Gamma(0)} = 1. \tag{19}
\]

Finally, due to the definition and properties of the (analytically extended) gamma function and its relation with the Pochhammer symbols above, we are allowed to generalize the binomial coefficients to non-integer values
\[
\left( \begin{array}{c} \beta \\ k \end{array} \right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-k+1)\Gamma(k+1)}, \quad \beta \in \mathbb{R} \text{ and } k \in \mathbb{N}_0, \tag{20}
\]
where we denote \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\).

In this work, we will restrict ourselves to \(0 < \alpha < 1\). This will simplify the main analysis without loosing great generality, since for any arbitrary order \(\alpha \in \mathbb{R}\), we have \(\alpha = [\alpha] + \{\alpha\}\), where \([\alpha]\) is the integer part of \(\alpha\) and \(0 < \{\alpha\} < 1\) its fractional part and the main feature of FC is related exactly to the non-integer part. Hence we are considering the evaluation of these expressions:

\[
\left[ aJ_\alpha^t (x-d)^\beta \right](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} (x-d)^\beta \, dx \tag{21}
\]
\[
\left[ aD_\alpha^t (x-d)^\beta \right](t) = \frac{d}{dt} \left[ aJ_\alpha^{1-\alpha} (x-d)^\beta \right](t)
= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-x)^{-\alpha} (x-d)^\beta \, dx. \tag{22}
\]

But before we begin with the explicit calculations, lets call attention to the domain in \(\mathbb{R}\) of \(f(t) = (t-d)^\beta\), which essentially depend on the values of \(d\) and \(\beta\) as it plays an important role on (definite) integrability and differentiability (at a point).

We have the following possibilities:

**Case 1:**
\[
\text{Dom}(f) = \begin{cases} \mathbb{R}, & \text{if } \beta \in \mathbb{N}_0, \\ \mathbb{R} \setminus \{d\}, & \text{if } \beta \in \mathbb{Z} \setminus \mathbb{N}_0. \end{cases} \tag{23}
\]

**Case 2:** If \(\beta \in \mathbb{Q} \setminus \mathbb{Z}\), that is, if \(\beta\) is a proper rational fraction, then we can assume without loss of generality that \(\beta = \frac{p}{q}\) with \(p \in \mathbb{Z}\) and \(q \in \mathbb{N}\) and under this hypothesis, we know that
\[
f(t) = (t-d)^\beta = \sqrt[q]{(t-d)^p}, \tag{24}
\]
and since the operation \(\sqrt[q]{(x)}\), \(q \in \mathbb{N}\) is well defined in \(\mathbb{R}_+^0 = [0, +\infty)\), we conclude that:
\[
\text{Dom}(f) = \begin{cases} \mathbb{R}, & \text{if } p \text{ is an even positive integer;} \\ [d, +\infty), & \text{if } p \text{ is an odd positive integer;} \\ (d, +\infty), & \text{if } p \text{ is a negative integer (even or odd).} \end{cases} \tag{25}
\]
Case 3: Now if $\beta \in \mathbb{R} \setminus \mathbb{Q}$, that is, if $\beta$ is irrational, then we can use the following identity:

$$
(t - d)^{\beta} = e^{\ln(t-d)^{\beta}} = e^{\beta \ln(t-d)},
$$

(26)

and since the domain in $\mathbb{R}$ of the logarithm is $\mathbb{R}_+$ while the domain in $\mathbb{R}$ of the exponential is $\mathbb{R}$ itself, then we conclude that in such case $\text{Dom}(f) = (d, +\infty)$.

3 Riemann-Liouville Integration of Power Functions

We start with the following theorem.

**Theorem 3** Let $f(t) = (t - d)^{\beta}$, $d, \beta \in \mathbb{R}$ and suppose $t \in \Omega \subset \mathbb{R}$, where $\Omega$ is an interval where $f$ is properly defined as a real valued function. Then

(1)

$$
\left[ aJ_t^\alpha (x - d)^{\beta} \right](t) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta + 1)(a - d)^{\beta-k}(t-a)^{\alpha+k}}{\Gamma(\beta - k + 1)\Gamma(\alpha + k + 1)}, \quad t \in \left[a, a + \epsilon\right),
$$

(27)

(II)

$$
\left[ d+J_t^\alpha (x - d)^{\beta} \right](t) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta + 1)\epsilon^{\beta-k}(t - d^+)^{\alpha+k}}{\Gamma(\beta - k + 1)\Gamma(\alpha + k + 1)}, \quad t \in \left[d^+, d^+ + \epsilon\right).
$$

(28)

Particularly, if $\beta = m \in \mathbb{N}_0$, then

$$
\left[ aJ_t^\alpha (x - d)^m \right](t) = \sum_{k=0}^{m} \frac{\Gamma(m + 1)(a - d)^{m-k}(t-a)^{\alpha+k}}{\Gamma(m - k + 1)\Gamma(\alpha + k + 1)}, \quad a, t \in \mathbb{R},
$$

(29)

while if $\beta = -m$ with $m \in \mathbb{N}$, then we can use alternatively the following expressions as well

$$
\left[ aJ_t^\alpha (x - d)^{-m} \right](t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(m + k)(a - d)^{-(m+k)}(t-a)^{\alpha+k}}{\Gamma(m)\Gamma(\alpha + k + 1)},
$$

(30)

$$
\left[ d+J_t^\alpha (x - d)^{-m} \right](t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(m + k)\epsilon^{-(m+k)}(t - d^+)^{\alpha+k}}{\Gamma(m)\Gamma(\alpha + k + 1)},
$$

(31)

with $t \in \left[a, a + \frac{\epsilon}{2}\right)$ and $t \in \left[d^+, d^+ + \epsilon\right)$, respectively and where in all cases, $\epsilon = |d - a| > 0$.

**Proof.** Initially, let $\beta \in \mathbb{R} \setminus \mathbb{Z}$. Using the definition of the RLFI (Eq.(15)) and integrating by parts a total of $p$ times, we get

$$
\left[ aJ_t^\alpha (x - d)^{\beta} \right](t) = \sum_{k=0}^{p-1} \frac{(\beta)_{-k}(a - d)^{\beta-k}(t-a)^{\alpha+k}}{\Gamma(\alpha + k + 1)} + \mathcal{R}_p,
$$

(32)

where

$$
\mathcal{R}_p = \frac{(\beta)_{-p}}{\Gamma(\alpha + p)} \int_a^t (x - d)^{\beta-p}(t-x)^{\alpha+p-1} dx.
$$

(33)

Now, using the identity in Eq.(10), we can write

$$
\left[ aJ_t^\alpha (x - d)^{\beta} \right](t) = \sum_{k=0}^{p-1} \frac{\Gamma(\beta + 1)(a - d)^{\beta-k}(t-a)^{\alpha+k}}{\Gamma(\beta - k + 1)\Gamma(\alpha + k + 1)} + \mathcal{R}_p,
$$

(34)
with
\[ R_p = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - p + 1) \Gamma(\alpha + p)} \int_a^t (x-d)^{\beta-p} (t-x)^{\alpha+p-1} dx. \] (35)

We now estimate the remainder \( R_p \)
\[
|R_p| \leq \frac{\Gamma(\beta + 1)}{\Gamma(\beta - p + 1) \Gamma(\alpha + p)} \int_a^t |x-d|^{\beta-p} |t-x|^{\alpha+p-1} dx
\]
\[
\leq \frac{\Gamma(\beta + 1)}{\Gamma(\beta - p + 1) \Gamma(\alpha + p)} |t-a|^{\alpha+p-1} \int_a^t |x-d|^{\beta-p} dx
\]
\[
= \frac{\Gamma(\beta + 1)}{\Gamma(\beta - p + 1) \Gamma(\alpha + p)} |t-a|^{\alpha+p-1} \left\{ \frac{|t-d|^{\beta-p+1} - |a-d|^{\beta-p+1}}{\beta-p+1} \right\}
\]
\[
= \frac{\Gamma(\beta+1)}{\Gamma(\beta-p+2) \Gamma(\alpha+p)} \left\{ \frac{|t-a|^\alpha |t-a|^{p-1}}{|t-d|^{\beta-p}} - \frac{|t-a|^\alpha |t-a|^{p-1}}{|a-d|^{\beta-p}} \right\}. \] (36)

While for the second factor in Eq. (36) it is straightforward that
\[
\lim_{p \to \infty} \left\{ \frac{|t-a|^\alpha |t-a|^{p-1}}{|t-d|^{\beta-p}} - \frac{|t-a|^\alpha |t-a|^{p-1}}{|a-d|^{\beta-p}} \right\} = 0,
\] (37)

whenever \( t \in \left[ a, a + \frac{|a-d|}{2} \right] \) if \( a \leq t < d \) or \( t \in [a, a + |a-d|) \) if \( d < a \leq t \), for the first factor, we need some further analysis. Using the well known identity [37]
\[
\Gamma(z-n) = \frac{(-1)^n \pi}{\sin(\pi z) \Gamma(n+1-z)}, \] (38)

we can write
\[
\left| \frac{\Gamma(\beta + 1)}{\Gamma(\beta - p + 2) \Gamma(\alpha + p)} \right| = \left| \frac{\Gamma(\beta + 1) \sin[\pi(\beta + 2)] \Gamma(p - \beta - 1)}{\Gamma(\beta - p + 1) \Gamma(\alpha + p)} \right|
\]
\[
\leq \left| \frac{\Gamma(\beta + 1) \sin[\pi(\beta + 2)]}{\Gamma(\alpha + p)} \right| \left| \frac{\Gamma(p - \beta - 1)}{\Gamma(\alpha + p)} \right|
\]
\[
= \frac{\Gamma(\beta + 1) \sin(\pi \beta)}{\pi} \left| \frac{\Gamma(p - \beta - 1)}{\Gamma(\alpha + p)} \right|. \] (39)

For our choice of \( \beta \), it is secured that \( \frac{\Gamma(\beta+1) \sin(\pi \beta)}{\pi} = M \) is always finite regardless of \( p \), while from [38] we have the asymptotic behavior of a ratio of gamma functions
\[
\lim_{p \to \infty} \left| \frac{\Gamma(p - \beta - 1)}{\Gamma(\alpha + p)} \right| = p^{-\alpha-\beta-1}, \] (40)

with convergence depending on \( -\alpha - 1 \leq \beta \). However, we are actually interested on the behavior of the product of the terms in Eq. (37) and Eq. (40). Since
\[
\left| \frac{t-a}{t-d} \right|^{p-1} = e^{-p \gamma}, \] (41)
\[
\left| \frac{t-a}{a-d} \right|^{p-1} = e^{-\eta p + \eta}, \] (42)
with $0 < \gamma = -\ln \left| \frac{t-a}{t-d} \right|$ and $0 < \eta = -\ln \left| \frac{t-a}{a-d} \right|$ (for the proper neighborhood as describe above) and the exponential decay (or growth) rate is always stronger (in the limit) than any power like rate of growth (decay), that is

$$\lim_{p \to \infty} \frac{p^{\alpha}}{e^{p}} = 0, \ \forall \alpha \in \mathbb{R},$$

we can conclude that

$$\lim_{p \to \infty} \left| R_p \right| \leq M \lim_{p \to \infty} p^{-\alpha - 1} \left\{ e^{-\gamma p} + e^{-\eta p} \right\} = 0,$$

proving the results in Eq. (27) and Eq. (28) for $\beta \in \mathbb{R} \setminus \mathbb{Z}$.

We now investigate the case $\beta = -m$, $m \in \mathbb{N}$. Initially, we will exclude the case $m = 1$, since they lead to logarithms. Recall that for $\beta = -2, -3, \ldots$ the domain of $f$ is

$$\text{Dom}(f) = \mathbb{R} \setminus \{d\} = (-\infty, d) \cup (d, \infty).$$

Therefore, when calculating $\left[ a J^\alpha_t (x - d)^{-m} \right]$, we need to take care of choosing the lower limit of integration $a$ in one of the following intervals: (I) $(-\infty, d)$ or (II) $(d, \infty)$.

(I) So let $a \leq t < d$. Then,

$$\left[ a J^\alpha_t (x - d)^{-m} \right] (t) = \frac{1}{\Gamma(\alpha)} \int_a^t (x - d)^{-m} (t - x)^{\alpha-1} dx. \tag{45}$$

If we integrate by parts the expression in the right side of Eq. (45) a total of $p$ times we get the following result

$$\left[ a J^\alpha_t (x - d)^{-m} \right] (t) = \sum_{k=0}^{p-1} \frac{(-1)^k (m)_k (a - d)^{-(m+k)} (t - a)^{\alpha+k}}{\Gamma(\alpha+k+1)} + R_p, \tag{46}$$

where

$$R_p = \frac{(-1)^p (m)_p}{\Gamma(\alpha+p)} \int_a^t (x - d)^{-(m+p)} (t - x)^{\alpha+p-1} dx. \tag{47}$$

Using the identity in Eq. (9), the expressions in Eq. (46) and Eq. (47) can be rewritten as

$$\left[ a J^\alpha_t (x - d)^{-m} \right] (t) = \sum_{k=0}^{p-1} \frac{(-1)^k \Gamma(m+k) (a - d)^{-(m+k)} (t - a)^{\alpha+k}}{\Gamma(m) \Gamma(\alpha+k+1)} + R_p, \tag{48}$$

where

$$R_p = \frac{(-1)^p \Gamma(m+p)}{\Gamma(m) \Gamma(\alpha+p)} \int_a^t (x - d)^{-(m+p)} (t - x)^{\alpha+p-1} dx. \tag{49}$$

Observe that the remainder $R_p$ can be estimated by the inequalities

$$\left| R_p \right| \leq \frac{(-1)^p \Gamma(m+p)}{\Gamma(m) \Gamma(\alpha+p)} \int_a^t |(x - d)|^{-(m+p)} |(t - x)|^{\alpha+p-1} dx$$

$$\leq \frac{\Gamma(m+p)}{\Gamma(m) \Gamma(\alpha+p)} \left| t - a \right|^{\alpha+p-1} \int_a^t |(x - d)|^{-(m+p)} dx$$

$$\leq \frac{\Gamma(m+p)}{\Gamma(m) \Gamma(\alpha+p)} \left| t - a \right|^{\alpha+p} \int_a^t |(t - d)|^{-(m+p)} dx$$

$$= \frac{\Gamma(m+p)}{\Gamma(m) \Gamma(\alpha+p)} \left| t - a \right|^\alpha \left\{ \frac{t - a}{t - d} \right\}^p \left( \frac{t - a}{t - d} \right)^{m+p}.$$
Now, in the last equality of Eq. (50), we have that for each fixed value of $t$ satisfying $a \leq t < d$ the fraction $|\frac{t-a}{t-d}|^\alpha$ has a fixed finite value. While

$$
\lim_{\mathbf{p} \to \infty} \left| \frac{t-a}{t-d} \right|^{\mathbf{p}} = 0,
$$

as long as $\left| \frac{t-a}{t-d} \right| < 1$, which is guaranteed for $t \in [a, a + \frac{\epsilon}{2})$, $\epsilon = |d - a|$. On the other hand, we have that

$$
\lim_{\mathbf{p} \to \infty} \left| \frac{\Gamma (m + \mathbf{p})}{\Gamma (m) \Gamma (\alpha + \mathbf{p})} \right| = \frac{p^{m-\alpha}}{\Gamma (m)} ,
$$

with convergence depending on $m - \alpha \leq 0$. However, again we are basically concerned with the limit of the product between Eq. (51) and Eq. (52) and since we can identify

$$
\left| \frac{t-a}{t-d} \right|^{\mathbf{p}} = e^{-\gamma \mathbf{p}},
$$

where $0 < \gamma = -\ln \left| \frac{t-a}{t-d} \right|$ (for $t \in [a, a + \frac{\epsilon}{2})$, $\epsilon = |d - a|$), and the exponential decay rate is faster (in the limit) than any power like rate, then we can conclude that

$$
\lim_{p \to \infty} |R_p| \leq \frac{|t-a|^\alpha}{\Gamma (m) |t-d|^m} \lim_{p \to \infty} \frac{p^{m-\alpha}}{e^{\gamma p}} = 0,
$$

therefore, as $\mathbf{p} \to \infty$, we conclude that Eq. (53) reduces to

$$
\left[ aJ_p (x-d)^{-m} \right] (t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma (m + k) (a-d)^{-\alpha k} (t-a)^k}{\Gamma (m) \Gamma (\alpha + k + 1)} ,
$$

for $t \in [a, a + \frac{\epsilon}{2})$, $\epsilon = |d - a|$, giving the expression claimed in Eq. (30). A very similar analysis but with some minor adjustment on the neighborhood would prove Eq. (31) as well.

It remains to explore the case where $m = 1$, that is when $\beta = -1$. For that, recall from integer order calculus that integration of functions of the type $(x - d)^{-1}$ lead to logarithms. Specifically we have

$$
\int_a^t (x-d)^{-1} \, dx = \ln \left( \frac{t-d}{a-d} \right) , \quad a > d.
$$

But we can also calculate the above integral in the following way: We consider the power series representation of $f(t) = (t - d)^{-1}$ centered at $t = a$ (for $a > d$),

$$
(t-d)^{-1} = -\sum_{k=0}^{\infty} \frac{(t-a)^k}{(a-d)^{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k (t-a)^k}{(a-d)^{k+1}}
$$

and then integrate it (inside its radius of convergence) to get

$$
\int_a^t (x-d)^{-1} \, dx = \int_a^t -\sum_{k=0}^{\infty} \frac{(t-a)^k}{(a-d)^{k+1}} \, dx
$$

$$
= -\sum_{k=0}^{\infty} \frac{1}{(a-d)^{k+1}} (t-a)^{k+1}
$$

$$
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \left( \frac{t-a}{a-d} \right)^{k+1} ,
$$
where either of the last two series are valid representations for \( \ln \left( \frac{t-d}{a-d} \right) \).

Now let's consider the explicit fractional case \( 0 < \alpha < 1 \). By definition, we have

\[
\left[ \alpha J_\alpha^t (x - d)^{-1} \right] (t) = \frac{1}{\Gamma(\alpha)} \int_a^t (x - d)^{-1} (t - x)^{\alpha - 1} \, dx,
\]

proceeding with integration by parts in the same way as the previous cases, one gets the result

\[
\left[ \alpha J_\alpha^t (x - d)^{-1} \right] (t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1+k) \epsilon^{-(1+k)} (t - d^+)^{\alpha+k}}{\Gamma(\alpha + k + 1)},
\]

for \( t \in [d^+, d^+ + \epsilon) \), which is nothing else but the formula in Eq.\( (31) \) when we set \( m = 1 \).

To recover the original expressions in Eq.\( (27) \) and Eq.\( (28) \) we can simply verify that since \( \beta = -m \), then

\[
\frac{(-1)^k \Gamma(m + k)}{\Gamma(m)} = \frac{(-1)^k \Gamma(-\beta + k)}{\Gamma(-\beta)}
\]

and by Eq.\( (11) \) together with the identities in Eq.\( (9) \) and Eq.\( (10) \), we have that

\[
\frac{(-1)^k \Gamma(-\beta + k)}{\Gamma(-\beta)} = (-1)^k (\beta)_k = (\beta)_{-k} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - k + 1)}.
\]

Finally, for the simplest case \( \beta = m \in \mathbb{N}, \, \text{Dom}(f) = \mathbb{R} \), so it really doesn’t matter where we take the lower limit \( t = a \), so

\[
\left[ \alpha J_\alpha^t (x - d)^m \right] (t) = \frac{1}{\Gamma(\alpha)} \int_a^t (x - d)^m (t - x)^{\alpha - 1} \, dx
\]

\[
= \text{after } m \text{ integration by parts}
\]

\[
= \sum_{k=0}^{m} \frac{(m)_k (a - d)^{m-k} (t - a)^{\alpha+k}}{\Gamma(\alpha + k + 1)}
\]

\[
= \sum_{k=0}^{m} \frac{\Gamma(m+1) (a - d)^{m-k} (t - a)^{\alpha+k}}{\Gamma(m-k+1) \Gamma(\alpha + k + 1)}, \, t \in \mathbb{R},
\]

where we have made use of the descending Pochhammer symbol Eq.\( (8) \) and the identity in Eq.\( (10) \).

**Corollary 4** Consider the hypothesis of Theorem 3. If the lower limit \( t = a \in U \subset \Omega \), where \( U \) is an interval where the power function is analytic, then calculating its RLFI can be done by operating on its Taylor series expansions.

**Proof.** Just consider the Taylor series expansion of \( f(t) = (t - d)^\beta \) at \( t = a \) and operate \( \alpha J_\alpha^t \) on it. Due to the uniform convergence, we can integrate term by term obtaining the expressions listed on Theorem 3.

**Remark 5** It is easily verifiable, that each of the expressions in Eq.\( (27) \) - Eq.\( (31) \), reduces to the expected integer order expressions when choosing \( \alpha = 0 \) or \( \alpha = 1 \) and, as an example, we will show this for Eq.\( (27) \) (the others are similar). Indeed, setting \( \alpha = 0 \) on Eq.\( (27) \) we directly get the Taylor series expansion of \( f(t) = (t - d)^\beta \) centered on \( t = a \), while setting \( \alpha = 1 \) on Eq.\( (27) \) we get

\[
\left[ \alpha J_\alpha^t (x - d)^\beta \right] (t) = \sum_{k=0}^{\infty} \frac{\Gamma(\beta + 1) (a - d)^{\beta-k} (t - a)^{k+1}}{\Gamma(\beta - k + 1) \Gamma(k+2)}.
\]
On the other hand, since the power function is analytic in the interval in consideration, then

\[
\frac{(t-d)^{\beta+1} - (a-d)^{\beta+1}}{\beta + 1} = \int_a^t (x-d)^\beta \, dx
\]

\[
= \int_a^t \sum_{k=0}^{\infty} \binom{\beta}{k} (a-d)^{\beta-k} (x-a)^k \, dx
\]

\[
= \sum_{k=0}^{\infty} \binom{\beta}{k} (a-d)^{\beta-k} \int_a^t (x-a)^k \, dx
\]

\[
= \sum_{k=0}^{\infty} \binom{\beta}{k} (a-d)^{\beta-k} \frac{(t-a)^{k+1}}{k+1},
\]

and clearly, Eq. (63) equals Eq. (61) since \((k + 1) \Gamma (k + 1) = \Gamma (k + 2)\).

Before ending this section, there’s one final observation that we want to call attention. All expressions listed on Theorem 3 for the power functions \(f(t) = (t-d)^\beta\) are only valid when we choose the lower limit \(a \neq d\), which guarantees the convergence of the series in their respective intervals of definition \([a, a + \frac{\epsilon}{2}]\) or \([d^+, d^+ + \epsilon]\), with \(\epsilon = |d-a| > 0\). The only exception to this is the case when \(\beta = m \in \mathbb{N}_0\), where if we set \(a = d\) (that means \(\epsilon = 0\)) in Eq. (29) it reduces to

\[
[aJ_t^\alpha (x-a)^m](t) = \frac{\Gamma (m + 1) (t-a)^{\alpha+m}}{\Gamma (\alpha + m + 1)}, \quad a, t \in \mathbb{R},
\]

which agrees with the usual formula in Eq. (1). But this is to be expected. In one hand, Eq. (29) is a finite sum and its convergence doesn’t depend on \(\epsilon\), on the other hand we known that the only class of power functions that are analytic everywhere in its domain of definition (which in such case includes the point \(t = d\)) are the polynomials. For all other values of the index \(\beta\), these functions are not analytic at \(t = d\) even if \(t = d\) belongs to its domain (e.g., consider \(f(t) = (t-d)^{\frac{1}{2}}\)).

4 Riemann-Liouville Differentiation of Power Functions

The result for the RLFD comes as a corollary of Theorem 3

\textbf{Corollary 6} Let \(f(t) = (t-d)^\beta\), \(d, \beta \in \mathbb{R}\) and suppose \(t \in \Omega \subset \mathbb{R}\), where \(\Omega\) is an interval where \(f\) is properly defined as real valued function. Then

(\text{I}) \hspace{1cm}
[aD_t^\alpha (x-d)^\beta](t) = \sum_{k=0}^{\infty} \frac{\Gamma (\beta + 1) (a-d)^{\beta-k} (t-a)^{-\alpha+k}}{\Gamma (\beta-k+1) \Gamma (-\alpha + k + 1)} , \quad t \in \left[a, a + \frac{\epsilon}{2}\right]

(\text{II}) \hspace{1cm}
[a^*D_t^\alpha (x-d)^\beta](t) = \sum_{k=0}^{\infty} \frac{\Gamma (\beta + 1) \epsilon^{\beta-k} (t-d^+)^{-\alpha+k}}{\Gamma (\beta-k+1) \Gamma (-\alpha + k + 1)} , \quad t \in \left[d^+, d^+ + \epsilon\right].

Particularly, if \(\beta = m \in \mathbb{N}_0\), then

\[
[aD_t^\alpha (x-d)^m](t) = \sum_{k=0}^{m} \frac{\Gamma (m + 1) (a-d)^{m-k} (t-a)^{k-\alpha}}{\Gamma (m-k+1) \Gamma (k+1-\alpha)} , \quad a, t \in \mathbb{R},
\]
while if \( \beta = -m \) with \( m \in \mathbb{N} \), then we can use alternatively the following expressions as well

\[
\begin{align*}
\left[ a D_t^\alpha (x - d)^{-m} \right] (t) &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma (m + k) (a - d)^{-m-k} (t-a)^{-\alpha+k}}{\Gamma (m) \Gamma (-\alpha + k + 1)}, \quad \text{and} \\
\left[ a J_t^\alpha (x - d)^{-m} \right] (t) &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma (m + k) \epsilon^{-m-k} (t-d^+)^{-\alpha+k}}{\Gamma (m) \Gamma (-\alpha + k + 1)},
\end{align*}
\]

with \( t \in \left[ a, a + \frac{\epsilon}{2} \right] \) and \( t \in [d^+, d^+ + \epsilon) \), respectively and where in all cases, \( \epsilon = |d - a| > 0 \).

**Proof.** We can use a similar tedious analysis as in Theorem 3 or simply realize that by the definition of the RLFD for \( 0 < \alpha < 1 \) (refer to Eq. (22)) we have

\[
\left[ a D_t^\alpha (x - d)^{\beta} \right] (t) = \frac{d}{dt} \left[ a J_t^{1-\alpha} (x - d)^{\beta} \right] (t),
\]

then it suffice to differentiate the corresponding expressions obtained for \( \left[ a J_t^{1-\alpha} (x - d)^{\beta} \right] (t) \) using the results of Theorem 3. □

**Corollary 7** Consider the hypothesis of Corollary 4. If the lower limit \( t = a \in U \subset \Omega \), where \( U \) is an interval where the power function is analytic, then calculating its RLFD can be done by operating on its Taylor series expansions.

**Proof.** Same as Corollary 4. □

**Remark 8** The expressions in Eq. (66) - Eq. (67) also reduce to the expected integer order formulas when setting \( \alpha = 0 \) and \( \alpha = 1 \) and the proofs can be done in a similar way as in Remark 4.

## 5 Summarizing the Results

We have then the following expressions for the RLFI of order \( 0 < \alpha < 1 \) for power functions of any order.

\[
\begin{align*}
\left[ a J_t^\alpha (x - d)^m \right] (t) &= \sum_{k=0}^{m} \frac{\Gamma (m + 1) (a - d)^{m-k} (t-a)^{\alpha+k}}{\Gamma (m - k + 1) \Gamma (\alpha + k + 1)}, \quad m \in \mathbb{N}_0, \ t \in \mathbb{R}, \\
\left[ a J_t^\alpha (x - d)^\beta \right] (t) &= \sum_{k=0}^{\infty} \frac{\Gamma (\beta + 1) (a - d)^{\beta-k} (t-a)^{\alpha+k}}{\Gamma (\beta - k + 1) \Gamma (\alpha + k + 1)}, \quad \beta \in \mathbb{R} \setminus \mathbb{N}_0, \ t \in \Omega_1, \\
\left[ a J_t^\alpha (x - d)^\beta \right] (t) &= \sum_{k=0}^{\infty} \frac{\Gamma (\beta + 1) \epsilon^{\beta-k} (t-d^+)^{\alpha+k}}{\Gamma (\beta - k + 1) \Gamma (\alpha + k + 1)}, \quad \beta \in \mathbb{R} \setminus \mathbb{N}_0, \ t \in \Omega_2,
\end{align*}
\]

where \( \Omega_1 = \left[ a, a + \frac{|d-a|}{2} \right] \) and \( \Omega_2 = [d^+, d^+ + \epsilon) \). Particularly, if \( \beta = -m \) with \( m \in \mathbb{N} \), then we can use alternatively these expressions as well

\[
\begin{align*}
\left[ a J_t^\alpha (x - d)^{-m} \right] (t) &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma (m + k) (a - d)^{-(m+k)} (t-a)^{\alpha+k}}{\Gamma (m) \Gamma (\alpha + k + 1)}, \quad t \in \Omega_1, \\
\left[ a J_t^\alpha (x - d)^{-m} \right] (t) &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma (m + k) \epsilon^{-(m+k)} (t-d^+)^{\alpha+k}}{\Gamma (m) \Gamma (\alpha + k + 1)}, \quad t \in \Omega_2.
\end{align*}
\]
Now for the RLFD of order 0 < α < 1 for power functions of any order, we have

\[ [aD^\alpha_t (x - d)^m] (t) = \sum_{k=0}^{m} \frac{\Gamma (m + 1) (a - d)^{m-k} (t - a)^{k-\alpha}}{\Gamma (m - k + 1) \Gamma (k + 1 - \alpha)}, m \in \mathbb{N}_0, t \in \mathbb{R}, \]

\[ [aD^\alpha_t (x - d)^\beta] (t) = \sum_{k=0}^{\infty} \frac{\Gamma (\beta + 1) (a - d)^{\beta-k} (t - a)^{-\alpha+k}}{\Gamma (\beta - k + 1) \Gamma (-\alpha + k + 1)}, \beta \in \mathbb{R} \setminus \mathbb{N}_0, t \in \Omega_1, \]

\[ [d^+ D^\alpha_t (x - d)^\beta] (t) = \sum_{k=0}^{\infty} \frac{\Gamma (\beta + 1) \epsilon^{\beta-k} (t - d^+)^{-\alpha+k}}{\Gamma (\beta - k + 1) \Gamma (-\alpha + k + 1)}, \beta \in \mathbb{R} \setminus \mathbb{N}_0, t \in \Omega_2, \]

particularly, if \( \beta = -m \) with \( m \in \mathbb{N} \), then we can use alternatively these expressions as well

\[ [aD^\alpha_t (x - d)^{-m}] (t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma (m + k) (a - d)^{-k} (t - a)^{-\alpha+k}}{\Gamma (m) \Gamma (-\alpha + k + 1)}, t \in \Omega_1, \]

\[ [d^+ D^\alpha_t (x - d)^\beta] (t) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma (m + k) \epsilon^{-k} (t - d^+)^{-\alpha+k}}{\Gamma (m) \Gamma (-\alpha + k + 1)}, t \in \Omega_2. \]

Finally, we point out that in the appendix we have related the expressions for \([aJ^\alpha_t (x - d)^\beta] (t), [d^+ J^\alpha_t (x - d)^\beta] (t), [aD^\alpha_t (x - d)^\beta] (t)\) and \([d^+ D^\alpha_t (x - d)^\beta] (t)\) in terms of the hypergeometric functions.

6 Conclusion

In this work, we have shown that it is possible to calculate the RLFI and RLFD of order 0 < α < 1 of power functions \((t - \ast)^\beta\) with \( \beta \) being any real value and we are able to express the results in terms of function series of the type \((t - \ast)^{\pm\alpha+k}\) and that such expressions can be related to the famous hypergeometric functions of the Mathematical-Physics. We have also observed that since the Riemann-Liouville formulations of the fractional integral and differential operators are actually defined in terms of the notion of a definite integral, then the series obtained are convergent on a proper neighborhood of the lower limit, therefore careful attention must be taken to check if the lower limit belongs (or not) to the original function’s domain. In fact, the whole problem of “not being able to integrodifferentiate the power functions of index strictly less than −1” is not really distinct than being (or not) able to integrate ordinarily power functions with a singularity in the lower limit. Recall that in ordinary calculus (integer orders), the integral’s final result of achieving or not a valid expression actually depends on “how strong is the singularity in the lower limit” versus how large is the order of the \(n\)-fold integral, so for the sake of instigating future discussions and works, this strongly suggest further investigation when setting the order \(\alpha\) with values greater than unity. It is also interesting to look if one can obtain a better formulation of the notions of an \(\alpha\)-primitive and therefore something as a fractional indefinite integral of order \(\alpha\).

7 Acknowledgment

F. G. Rodrigues would like to acknowledge the support from CNPq under grant 200832/2015-8.

A Appendix

We will show that following some algebraic manipulations and use of identities we can write the expressions found for the RLFI and RLFD also in terms of the hypergeometric functions. For starters, we will describe the steps for Eq. (28) and the others are done similarly.
So we have that

\[ [d^+ J_d^\alpha (x - d)^\beta] (t) = \sum_{k=0}^{\infty} \frac{\Gamma (\beta + 1) e^{\beta-k} (t - d^+)^{\alpha+k}}{\Gamma (\beta - k + 1) \Gamma (\alpha + k + 1)} \]

\[ = \Gamma (\beta + 1) e^{\alpha+\beta} \left( \frac{t - d}{e} - 1 \right) \sum_{k=0}^{\infty} \frac{(\frac{t-d}{e} - 1)^k}{\Gamma (\beta - k + 1) \Gamma (\alpha + k + 1)}. \]

To simplify the notation we introduce \( z = \frac{t-d}{e} > 0 \) and \( [d^+ J_d^\alpha (x - d)^\beta] (t) = J_d^\alpha (t) \). Hence,

\[ J_d^\alpha (t) = \Gamma (\beta + 1) e^{\alpha+\beta} (z - 1)^\alpha \sum_{k=0}^{\infty} \frac{(z - 1)^k}{\Gamma (\beta - k + 1) \Gamma (\alpha + k + 1)}. \]  

(72)

Then using the Pochhammer symbols notation and the identity

\[ \frac{(-\beta)_k}{(\alpha+1)_k} = (-1)^k \frac{\Gamma (\beta + 1) \Gamma (\alpha + 1)}{\Gamma (\beta - k + 1) \Gamma (\alpha + k + 1)}, \]

which implies

\[ \frac{1}{\Gamma (\beta - k + 1) \Gamma (\alpha + k + 1)} = \frac{(-1)^k}{\Gamma (\beta + 1) \Gamma (\alpha + 1) \Gamma (\alpha + 1)_k}, \]  

(73)

we have, after substituting Eq. (73) in Eq. (72) that

\[ J_d^\alpha (t) = \frac{e^{\alpha+\beta} (z - 1)^\alpha}{\Gamma (\alpha + 1)} \sum_{k=0}^{\infty} \frac{(-1)^k (-\beta)_k}{(\alpha + 1)_k} (z - 1)^k, \]

and since \( (1)_k = k! \) we have

\[ J_d^\alpha (t) = \frac{e^{\alpha+\beta} (z - 1)^\alpha}{\Gamma (\alpha + 1)} \sum_{k=0}^{\infty} \frac{(1)_k (-\beta)_k (1 - z)^k}{(\alpha + 1)_k k!}. \]  

(74)

Now, we recall that the hypergeometric function is defined by the series

\[ {}_2F_1 (a, b; c; \xi) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} \xi^k, \]

thus Eq. (74) can be written in the form

\[ J_d^\alpha (t) = \frac{e^{\alpha+\beta} (z - 1)^\alpha}{\Gamma (\alpha + 1)} {}_2F_1 (1, -\beta; \alpha + 1; 1 - z). \]  

(75)

We can express this hypergeometric function (conveniently) as a series if, and only if, \( \alpha + \beta \neq \pm m \), \( m \in \mathbb{N} \) and \( |\arg(z)| < \pi \) and we can rewrite this hypergeometric function as [37]:

\[ {}_2F_1 (1, -\beta; \alpha + 1; 1 - z) = \frac{\Gamma (\alpha + 1) \Gamma (\alpha + \beta)}{\Gamma (\alpha) \Gamma (\alpha + \beta + 1)} {}_2F_1 (1, -\beta; 1 - \alpha - \beta; z) + \]

\[ \frac{\Gamma (\alpha + 1) \Gamma (-\alpha - \beta)}{\Gamma (-\beta)} z^{\alpha+\beta} {}_2F_1 (\alpha, \alpha + \beta + 1; \alpha + \beta + 1; z). \]  

(76)

Substituting Eq. (76) in Eq. (75) we have

\[ J_d^\alpha (t) = \frac{e^{\alpha+\beta} (z - 1)^\alpha \Gamma (\alpha + \beta)}{\Gamma (\alpha) \Gamma (\alpha + \beta + 1)} {}_2F_1 (1, -\beta; 1 - \alpha - \beta; z) + \]

\[ + \frac{e^{\alpha+\beta} (z - 1)^\alpha \Gamma (-\alpha - \beta) z^{\alpha+\beta}}{\Gamma (-\beta)} {}_2F_1 (\alpha, \alpha + \beta + 1; \alpha + \beta + 1; z). \]  

(77)
We can continue simplifying this last expression, using some identities for the hypergeometric functions. First, for the second hypergeometric function in Eq. (77), we have:

\[
2F_1 (\alpha, \alpha + \beta + 1; \alpha + \beta + 1; z) = \sum_{k=0}^{\infty} (\alpha)_k z^k \frac{\Gamma (\alpha + k)}{\Gamma (\alpha) k!}
\]

\[
= \sum_{k=0}^{\infty} \frac{\Gamma (\alpha + k)}{\Gamma (\alpha) k!} z^k
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha + k - 1}{k} (-z)^k
\]

\[
= (1 - z)^{-\alpha}, \quad |z| < 1.
\]

(78)

While for the first hypergeometric function in Eq. (77) we use the relation known as Euler transformation

\[
2F_1 (\alpha, \beta; \alpha + \beta; z) = (1 - z)^{-\alpha - \beta} 2F_1 (\alpha + \beta; \alpha + \beta; z),
\]

and identify that

\[
c - a = 1 - \alpha - \beta - 1 = -\alpha - \beta,
\]

\[
c - b = 1 - \alpha - \beta + \beta = 1 - \alpha,
\]

\[
c - a - b = 1 - \alpha - \beta - 1 + \beta = -\alpha.
\]

Therefore,

\[
2F_1 (1, -\beta; 1 - \alpha - \beta; z) = (1 - z)^{-\alpha} 2F_1 (-\alpha - \beta; 1 - \alpha, 1 - \alpha - \beta; z),
\]

and it follows that Eq. (77) assume the following form

\[
J_{\alpha}^\beta (t) = \frac{(-1)^\alpha \Gamma (-\alpha - \beta)}{\Gamma (-\beta)} (t - d)^{\alpha + \beta}
\]

\[
+ \frac{(-1)^\alpha \Gamma (\alpha + \beta) \epsilon^{\alpha + \beta}}{\Gamma (\alpha) \Gamma (\alpha + \beta + 1)} 2F_1 (-\alpha - \beta, 1 - \alpha; 1 - \alpha - \beta; \frac{t - d}{\epsilon}).
\]

(79)

In a similar fashion for Eq. (77), calling \[\alpha J_{\delta}^\beta (x - d)^3\] \(t) = J_{\delta}^\beta (t)\) we have

\[
J_{\alpha}^\beta (t) = \frac{(-1)^\alpha \Gamma (-\alpha - \beta)}{\Gamma (-\beta)} (t - d)^{\alpha + \beta}
\]

\[
+ \frac{(-1)^\alpha \Gamma (\alpha + \beta) (a - d)^{\alpha + \beta}}{\Gamma (\alpha) \Gamma (\alpha + \beta + 1)} 2F_1 (-\alpha - \beta, 1 - \alpha; 1 - \alpha - \beta; \frac{t - d}{\epsilon}).
\]

(80)

In a similar fashion for Eq. (77), calling \[\delta D_{\alpha}^\beta (x - d)^3\] \(t) = D_{\alpha}^\beta (t)\) we have

\[
D_{\alpha}^\beta (t) = \frac{(-1)^{-\alpha} \Gamma (\alpha - \beta)}{\Gamma (-\beta)} (t - d)^{-\alpha + \beta}
\]

\[
+ \frac{(-1)^{-\alpha} \Gamma (-\alpha + \beta) \epsilon^{-\alpha + \beta}}{\Gamma (-\alpha) \Gamma (-\alpha + \beta + 1)} 2F_1 (\alpha - \beta, 1 + \alpha; 1 + \alpha - \beta; \frac{t - d}{\epsilon}).
\]

(81)

In a similar fashion for Eq. (77), calling \[\alpha D_{\alpha}^\beta (x - d)^3\] \(t) = D_{\alpha}^\beta (t)\) we have

\[
D_{\alpha}^\beta (t) = \frac{(-1)^{-\alpha} \Gamma (\alpha - \beta)}{\Gamma (-\beta)} (t - d)^{-\alpha + \beta}
\]

\[
+ \frac{(-1)^{-\alpha} \Gamma (-\alpha + \beta) \epsilon^{-\alpha + \beta}}{\Gamma (-\alpha) \Gamma (-\alpha + \beta + 1)} 2F_1 (\alpha - \beta, 1 + \alpha; 1 + \alpha - \beta; \frac{t - d}{\epsilon}).
\]

(82)
References

[1] B. Ross, *Fractional calculus*, Math. Mag., 50, 115-122, (1970).

[2] B. Ross, (Ed.), *Fractional Calculus and its Applications*: Proceedings of the International Conference, New Haven, June 1974, Springer Verlag, New York, (1974).

[3] B. Ross, *A brief history and exposition of the fundamental theory of fractional calculus*, Lecture Notes in Mathematics, 457, 1-36, (1975).

[4] E. Capelas de Oliveira and J. A. Tenreiro Machado, *A review of definitions for fractional derivatives and integral*, Math. Prob. Ing., 2014, ID 238459, (2014).

[5] M. Caputo and M. Fabrizio, *New definition of fractional derivative without singular kernel*, Prog. Fract. Diff. Appl., 1, 73-85 (2015).

[6] J. Losada and J. J. Nieto, *Properties of a new fractional derivative without singular kernel*, Prog. Fract. Diff. Appl., 1, 87-92 (2015).

[7] A. Atangana, *On the new fractional derivative and application to nonlinear Fischer’s reaction-diffusion equation*, Appl. Math. Comp., 273, 948-956 (2016).

[8] J. Vanterler C. Sousa and E. Capelas de Oliveira, *On the $Ψ$-Hilfer fractional derivative*, submitted (2017).

[9] D. S. de Oliveira and E. Capelas de Oliveira, *Hilfer-Katugampola fractional derivatives*, Submitted (2017).

[10] J. Vanterler C. Sousa and E. Capelas de Oliveira, *Mittag-Leffler functions and the truncated $Ψ$-fractional derivative*, Submitted (2017).

[11] X-J. Yang, J. A. Tenreiro Machado and D. Baleanu, *Anomalous diffusion models with general fractional derivatives within the kernels of the extended Mittag-Leffler functions*, Romanian Report in Physics, 69, 1-19 (2017).

[12] X-J. Yang, *Fractional derivative of constant and variable orders applied to anomalous relaxation models in heat-transfer problems*, Thermal Science, 21, 1161-1171 (2017).

[13] A. Atangana and D. Baleanu, *New fractional derivative with non-local and non-singular kernel: theory and application to heat transfer model*, Thermal Science, 20, 763-769 (2016).

[14] X-J. Yang, J. A. Tenreiro Machado and J. J. Nieto, *A new family of the local fractional partial differential equations*, Fundamenta Informaticae, 151, 63-75 (2017).

[15] X-J. Yang, J. A. Tenreiro Machado, C. Cattani and F. Gao, *On a fractal LC-electric circuit modeled by local fractional calculus*, Comm. Nonl. Sci. NUm. Simulat., 47, 200-206 (2016).

[16] F. Gao and X-J. Yang, *Local fractional Euler’s method for the steady heat-conduction problem*, Thermal Science, 20, S735-S738 (2016).

[17] A. R. Gómez Plata and E. Capelas de Oliveira, *Variational iteration method in the fractional Burgers equation*, Accepted: J. Applied Nonlinear Dynamic (2017).

[18] F. S. Costa, J. C. A. Soares, A. R. Gómez Plata and E. Capelas de Oliveira, *On the fractional Harry Dym equation*, Accepted: Computational and Applied Mathematics (2017).

[19] R. Garrappa, F. Mainardi, G. Maione, *Models of dielectric relaxation based on completely monotone functions*, Fract. Cal. & Appl. Anal., 19, 1105-1160 (2016).
[20] E. C. A. F. Rosa and E. Capelas de Oliveira, *Complete Monotonicity of Fractional Kinetic Functions*, to be submitted (2017).

[21] E. Capelas de Oliveira, M. J. Menon and P. V. R. G. Silva, *Analytic components for the hadronic total cross-section: Fractional calculus and Mellin transform*, to be submitted (2017).

[22] W. Chen and S. Holm, Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency, *J. Acoust. Soc. Am.* 115, (2004), 1424–1430.

[23] N. Korabel, G. M. Zaslavsky and V. E. Tarasov, Coupled oscillators with power-law interaction and their fractional dynamics analogues, *Commun. Nonlinear Sci. Numer. Simulat.* 12 (8), (2007), 1405-1417.

[24] M. Caputo, J. M. Carcione, F. Cavallini, Wave simulation in biologic media based on the Kelvin-Voigt fractional-derivative stress-strain relation, *Ultrasound Med Biol.*, (2011) Jun;37(6):996-1004. doi: 10.1016/j.ultrasmedbio.2011.03.009.

[25] S. P. Nasholm and S. Holm, Linking multiple relaxation, power-law attenuation, and fractional wave equations, *J. Acoust. Soc. Am.* 130 (5), (2011), 3038-3045.

[26] C. M. Ionescu and F. D. Ionescu, Power law and fractional derivative models can measure analgesia, *IEEE International Conference on Automation, Quality and Testing, Robotics* (2014). DOI: 10.1109/AQTR.2014.6857908.

[27] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B. V., Amsterdam, 2006.

[28] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.

[29] M. L. Morgado, N. J. Neville and P. M. Lima, Analysis and numerical methods for fractional differential equations with delay, *J. Comput. Appl. Math.* 252, (2013), 159-168.

[30] J. L. Lavoie, T. J. Osler and R. Tremblay, Fractional derivatives and special functions, *SIAM Review* 18, Issue 2, (1976), 240-268.

[31] R. Andriambololona, R. Hanitriarivo, T. Ranaivoson AND R. Raboanary, Two definitions of fractional derivatives of power functions, *Pure Appl. Math. J.*, 2, No.1, (2013), 10-19. doi: 10.11648/j.pamj.20130201.12.

[32] R. Khalil, et. al, A new definition of fractional derivative, *J. Comput. Appl. Math.* 264, (2014), 65-70.

[33] M. D. Ortigueira and J. A. Tenreiro Machado, What is a fractional derivative?, *J. Comput. Phys.*, 293 (2015), 4-13. http://dx.doi.org/10.1016/j.jcp.2014.07.019.

[34] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[35] F. G. Rodrigues, and E. Capelas de Oliveira, Confluent hypergeometric equation via fractional calculus, *J. Phys. Math.*, 6 (2), (2015), 1-4. http://dx.doi.org/10.4172/2090-0902.1000147

[36] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover Publications Inc., New York, 2002.

[37] W. Magnus, F. Oberhettinger and R. P. soui, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Edition, Springer-Verlag, New York, 1966.

[38] F. Tricomi and A. Ederlyi, The asymptotic expansion of a ratio of gamma functions, *Pacific J. Math.*, 1, (1951), 133-142.