Gregory-Laflamme instability of BTZ black hole in new massive gravity

Taeyoon Moon* and Yun Soo Myung†,

Institute of Basic Sciences and Department of Computer Simulation, Inje University, Gimhae 621-749, Korea

Abstract

We find the Gregory-Laflamme s-mode instability of the non-rotating BTZ black hole in new massive gravity. This instability shows that the BTZ black hole could not exist as a stable static solution to the new massive gravity. For non-rotating BTZ black string in four dimensions, however, it is demonstrated that the BTZ black string can be stable against the metric perturbation.

PACS numbers:04.30.Nk, 04.70.Bw

* e-mail address: tymoon@sogang.ac.kr
†e-mail address: ysmyung@inje.ac.kr
1 Introduction

The dRGT gravity \cite{1,2,3} is considered as a promising massive gravity model which yields Einstein gravity in the massless limit. Recently, it was shown that the stability of the Schwarzschild black hole in the four dimensional dRGT gravity could be determined by the Gregory-Laflamme (GL) instability \cite{4,5} of a five-dimensional black string. The small Schwarzschild black hole with mass $M_5$ in the dRGT gravity and its bi-gravity extension \cite{6,7}, and fourth-order gravity \cite{8} is unstable against metric and Ricci tensor perturbations for $m' \leq \frac{C_2}{M_5}$ and $m_2 \leq \frac{1}{2M_5}$, respectively. These results may indicate that static black holes in massive gravity do not exist.

Interestingly, it turned out that in a massive theory of the Einstein-Weyl gravity, the linearized Einstein tensor perturbations exhibit unstable modes of the Schwarzschild-AdS black hole featuring the GL instability of five-dimensional AdS black string, in contrast to the stable Schwarzschild-AdS black hole in the Einstein gravity \cite{9}. The linearized Ricci tensor perturbations were employed to exhibit unstable modes of the Schwarzschild-Tangherlini (higher dimensional Schwarzschild) black hole in higher-dimensional fourth order gravity which features the GL instability of higher dimensional black strings \cite{10}, in comparison with the stable Schwarzschild-Tangherlini black holes in higher-dimensional Einstein gravity. These imply that the GL instability of the black holes in the massive gravity originates from the massiveness, but not a nature of the fourth-order gravity giving ghost states. Also, one could avoid the ghost problem arising from the metric perturbations in the fourth-order gravity when using the linearized Einstein and Ricci tensors because their linearized equations become the second-order tensor equations.

On the other hand, it was shown that the four-dimensional BTZ black string in Einstein gravity is stable against metric perturbations regardless of the horizon size, which is also supported by a thermodynamic argument of Gubser-Mitra conjecture \cite{11,12}. Later on, however, it was argued that the BTZ black string is not always stable against metric perturbations \cite{13}. In the literatures \cite{11,12}, there exists a threshold value for $\mu^2 > 0$ (we use a different notation $\mu^2$ to avoid a confusion $m^2$ here) which is related to the compactification of the extra dimension of the tensor perturbation. It was shown in \cite{13} that for $\mu^2 \geq 3/\ell^2$ with $\ell$ AdS$_3$ curvature radius, the BTZ black string is stable against s-mode metric perturbation, while for $\mu^2 < 3/\ell^2$ it is unstable. Therefore, it seems to be necessary to point out which one is correct.
The new massive gravity has been introduced as a fourth-order gravity with a healthy massive spin-2 mode and a massless spin-2 ghost mode, which is pure gauge only in three dimensions [14]. This parity-even gravity describes two modes of helicity $+2$ and $-2$ [2 degrees of freedom (DOF)] of a massive graviton, but it has a drawback of serving as a unitary model of massive gravity only in three dimensions [15]. For $m^2 > 1/2\ell^2$ with $m^2$ the mass of graviton, the three-dimensional BTZ black hole in the new massive gravity is shown to be stable against $s$-mode metric perturbation [16] when using the positivity of the potential outside the horizon. To this direction, it was recently reported that the stability of the BTZ black hole was mainly determined by the asymptotes of black hole spacetime: the condition of the $s$-mode stability is consistent with the generalized Breitenlohner-Freedman (BF) bound ($m^2 \geq -1/2\ell^2$) for metric perturbations on asymptotically AdS$_3$ spacetime [17]. This result may imply that the stability condition is extended simply from $m^2 > 1/2\ell^2$ to $m^2 \geq -1/2\ell^2$ if $m^2$ is allowed to be a negative quantity. However, one expects that two different type of instabilities appears for the BTZ black hole in new massive gravity: one is from the BF bound based on the tensor propagation on asymptotically AdS$_3$ spacetime, while the other is the GL instability of a massive graviton propagating on the BTZ black hole spacetime. This is similar to two instabilities of AdS black holes to trigger a holographic superconductor phase within the AdS/CFT correspondence [18]. In these models, the AdS$_d$ black hole becomes unstable to form non-trivial fields outside its horizon when being close to extremality whose near-horizon geometry is AdS$_2 \times M_{d-2}$. For a massive scalar with mass $m^2$ between $-(d-1)^2/4\ell^2$ and $-1/4\ell^2$, two AdS spacetimes are unstable [19].

Hence, it suggests strongly that the stability of BTZ black hole should be revisited in new massive gravity by observing the GL instability of four-dimensional black string. We will show that the instability of a massive graviton persists even in three-dimensional BTZ black hole. This establishes that the instability of the black holes in the $D \geq 3$-dimensional massive gravity originates from the massiveness, but not a nature of the fourth-order gravity giving ghost states. As a byproduct, we will show that the four-dimensional BTZ black string is stable against the $s$-mode metric perturbation because its mass squared is positive ($\mu^2 > 0$).
2 Linearized perturbation equation

We start with the three-dimensional fourth order gravity defined as

\[
S_{\text{lag}} = \int d^3x \sqrt{-g} \left[ R - \frac{2\lambda_S}{\kappa} + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right]
\]  

(2.1)

with \( \kappa \) the three-dimensional gravitational coupling constant. From the action (2.1), the Einstein equation is derived to be

\[
\frac{1}{\kappa} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda_S g_{\mu\nu} \right) + E_{\mu\nu} = 0,
\]

(2.2)

where \( E_{\mu\nu} \) takes the form

\[
E_{\mu\nu} = 2\beta \left( R_{\mu\rho\nu\sigma} R^{\rho\sigma} - \frac{1}{4} R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} \right) + 2\alpha R \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) + \beta \left( \nabla^2 R_{\mu\nu} + \frac{1}{2} \nabla^2 R g_{\mu\nu} - \nabla_\mu \nabla_\nu R \right) + 2\alpha \left( g_{\mu\nu} \nabla^2 R - \nabla_\mu \nabla_\nu R \right).
\]

(2.3)

For \( \Lambda = \lambda_S + 2\kappa(3\alpha + \beta)\Lambda^2 \) with \( \bar{R}_{\mu\nu} = 2\Lambda \bar{g}_{\mu\nu} \), the non-rotating BTZ black hole solution is given by

\[
ds_{\text{BTZ}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -V(r) dt^2 + \frac{dr^2}{V(r)} + r^2 d\phi^2,
\]

(2.4)

where the metric function takes the form

\[
V(r) = -\mathcal{M} + \frac{r^2}{\ell^2}
\]

(2.5)

with \( \ell^2 = -1/\Lambda \) and \( \mathcal{M} \) the ADM mass. From the condition of \( V(r_+) = 0 \), the horizon is located at \( r = r_+ \).

For a perturbation around the BTZ black hole

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},
\]

(2.6)

the linearized Einstein tensor, Ricci tensor, and Ricci scalar are given by

\[
\delta G_{\mu\nu}(h) = \delta R_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \delta R - 2\Lambda h_{\mu\nu},
\]

(2.7)

\[
\delta R_{\mu\nu}(h) = \frac{1}{2} \left( \tilde{\nabla}^\rho \tilde{\nabla}_\mu h_{\rho\nu} + \tilde{\nabla}^\rho \tilde{\nabla}_\nu h_{\rho\mu} - \tilde{\nabla}^2 h_{\mu\nu} - \tilde{\nabla}_\mu \tilde{\nabla}_\nu h \right),
\]

(2.8)

\[
\delta R(h) = \tilde{\nabla}_\alpha \tilde{\nabla}_\beta h^{\alpha\beta} - \tilde{\nabla}^2 h - 2\Lambda h,
\]

(2.9)
where the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - R g_{\mu\nu}/2 + \Lambda g_{\mu\nu}$. With the help of the above quantities, the linearized Einstein equation can be written as

$$\left[\frac{1}{\kappa} + (12\alpha + 2\beta)\Lambda\right]\delta G_{\mu\nu} + (2\alpha + \beta)\left[\bar{g}_{\mu\nu}\nabla^2 - \bar{\nabla}_\mu \nabla_\nu + 2\Lambda \bar{g}_{\mu\nu}\right]\delta R + \beta \left(\nabla^2 \delta G_{\mu\nu} - \Lambda \bar{g}_{\mu\nu}\delta R\right) = 0. \tag{2.10}$$

Taking the trace of (2.10) provides the linearized Ricci scalar equation

$$\left[(8\alpha + 3\beta)\nabla^2 - \left\{\frac{1}{\kappa} - 4(3\alpha + \beta)\Lambda\right\}\right]\delta R = 0, \tag{2.11}$$

which implies that for $8\alpha + 3\beta = 0$ (the new massive gravity), the d’Alembertian operator is removed. In the new massive gravity \cite{14}

$$S_{\text{NMG}} = \int d^3x \sqrt{-g} \left[\frac{R - 2\lambda s}{\kappa} + \beta \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2\right)\right], \tag{2.12}$$

$\delta R$ is constrained to vanish

$$\delta R = 0 \tag{2.13}$$

provided that $\kappa \beta \Lambda \neq -2$. This explains why we choose the new massive gravity. Plugging $\delta R = 0$ and $\alpha = -3\beta/8$ into (2.10) leads to the linearized Einstein tensor equation

$$\left(\nabla^2 - 2\Lambda - M^2\right)\delta G_{\mu\nu} = 0, \tag{2.14}$$

where the mass squared is given by

$$M^2 = -\frac{1}{\kappa \beta} + \frac{\Lambda}{2} \equiv m^2 - \frac{1}{2\ell^2}. \tag{2.15}$$

Choosing the transverse-traceless gauge

$$\nabla^\mu h_{\mu\nu} = 0, \quad h = 0, \tag{2.16}$$

Eq. (2.14) leads to the fourth-order equation for the metric perturbation $h_{\mu\nu}$

$$\left(\nabla^2 - 2\Lambda\right)\left(\nabla^2 - 2\Lambda - M^2\right)h_{\mu\nu} = 0, \tag{2.17}$$

which might imply the two second-order linearized equations

$$\left[\nabla^2 - 2\Lambda\right]h_{\mu\nu} = 0, \tag{2.18}$$

$$\left[\nabla^2 - 2\Lambda - M^2\right]h_{\mu\nu} = 0 \tag{2.19}$$
off critical point \((M^2 \neq 0, m^2 \neq 1/2\ell^2)\). Even though Eq. (2.19) may describe 2 DOF for a massive graviton in three dimensions, it might not be a correct equation because the ‘−’ sign disappears when splitting (2.17) into (2.18) and (2.19). In order to see this ghost problem explicitly, we consider the three-dimensional flat spacetime. In the case of \(\Lambda \to 0 (\lambda_S \to 0)\), the tree-level scattering amplitude \(A\) between two conserved sources \(T''_{\mu\nu}\) and \(T_{\mu\nu}\) is given by \([20]\)

\[
4A = -\frac{2}{\beta} T''_{\mu\nu} \frac{1}{p^2(p^2 + m^2)} T_{\mu\nu} + \frac{1}{\beta} T'' \frac{1}{p^2(p^2 + m^2)} T - \kappa T' \frac{1}{p^2} T \tag{2.20}
\]

with \(p^2 = -\partial^2\). After partial fractions, this leads to

\[
4A = 2\kappa T''_{\mu\nu} \left( \frac{1}{p^2} - \frac{1}{p^2 + m^2} \right) T_{\mu\nu} - \kappa T' \left( \frac{2}{p^2} - \frac{1}{p^2 + m^2} \right) T. \tag{2.21}
\]

In order not to have a tachyon, we have to choose \(\kappa \beta < 0 (m^2 > 0)\) with \(\beta > 0\). Also, we require \(\kappa < 0\) to have a healthy massive spin-2 without negative norm states (ghosts) and a massless spin-2 ghost. This is why one chooses a negative \(\kappa\) for a positive \(\beta\) in 3D flat spacetime. However, we might miss the ghost problem when splitting (2.17) into (2.18) and (2.19) without imposing the sign correction. Hence, it would be better to use the second-order equation (2.14) for the linearized Einstein tensor if it could describe 2 DOF. For this purpose, we have two constraints

\[
\delta G = -\delta R = 0, \quad \nabla^\mu \delta G_{\mu\nu} = 0, \tag{2.22}
\]

where the last one comes from contracting the linearized Bianchi identity \(\nabla_{[\tau} \delta R_{\mu\nu]} = 0\) with \(g^{\tau\rho} g^{\mu\nu}\). Then, we have 2 DOF because \(6 - 1 - 3 = 2\). Hence, we note that Eq. (2.14) is interpreted as a boosted-up version of Eq. (2.19) \([15]\).

In 3D flat spacetime, the mass squared should be positive \((m^2 > 0)\) because of the non-tachyonic condition but it is allowed to be negative in asymptotically AdS\(3\) spacetime. Accordingly, the BF bound for a tensor field could be read off from Eqs. (2.14) and (2.19) as

\[
M^2 \geq -\frac{1}{\ell^2} \Rightarrow m^2 \geq -\frac{1}{2\ell^2}. \tag{2.23}
\]

However, in the case of 4D BTZ black string, one requires \(\mu^2 > 0\) because it comes from the compactification of an extra dimension \([13]\). Hence, the BF bound does not require a further condition for the stability of the 4D BTZ black string.
3 Gregory-Laflamme s-mode instability

In order to investigate the GL instability of the massive graviton propagating on the BTZ black hole spacetime, we first start with (2.19) for convenience, because (2.14) and (2.19) are the same equation. For \( s(k = 0) \)-mode analysis, a metric perturbation takes the form with four components \( H_{tt}, \ H_{tr}, \ H_{rr}, \) and \( H_3 \) as

\[
h_{\mu
u}(t, \phi, r) = e^{i\Omega t}e^{ik\phi}|_{k=0} \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 \\
H_{tr}(r) & H_{rr}(r) & 0 \\
0 & 0 & H_3(r) \end{pmatrix}.
\]

Substituting Eq. (3.1) into Eq. (2.19) and after a tedious manipulation, we obtain the second-order equation for a single physical field \( H_{tr}(r) \) as

\[
\begin{aligned}
\{(m^2 - 1/2\ell^2)(r^2/\ell^2 - \mathcal{M}) + r^2/\ell^2 - 2\mathcal{M}/\ell^2 + \Omega^2\} H_{tt}'' + \left\{ \frac{7r^2/\ell^2 - \mathcal{M}}{r(r^2/\ell^2 - \mathcal{M})}\Omega^2 \\
+ \frac{5r^2/\ell^2 - \mathcal{M}}{r^2 - 1/2\ell^2} + \frac{5r^4/\ell^4 - 13\mathcal{M}r^2/\ell^2 + 2\mathcal{M}^2}{r^2(\ell^2 - \mathcal{M})} \right\} H_{tr}' + \left\{ \frac{6r^4/\ell^4 - \mathcal{M}^2}{r^2(\ell^2 - \mathcal{M})^2}\Omega^2 \\
+ \frac{2r^4/\ell^4 - 2\mathcal{M}r^2/\ell^2 - \mathcal{M}^2}{r^2(\ell^2 - \mathcal{M})} (m^2 - 1/2\ell^2) + \frac{3r^2/\ell^2 - 2\mathcal{M}}{r^2(\ell^2 - \mathcal{M})}\right\} H_{tr} = 0
\end{aligned}
\]

with \( \mathcal{M} = r_+^2/\ell^2 \). This implies that the \( s \)-mode perturbation is described by a single field \( H_{tr} \), even though we were starting with four components and the massive graviton has two DOF for \( k \neq 0 \).

It turns out that the second-order equation (3.2) can be reduced to two first-order equations with a constraint when using the perturbation equation (2.19) together with the TT gauge condition (2.10) [5]. The two coupled first-order equations are given by

\[
H' = \frac{\mathcal{M} - 3r^2/\ell^2}{rV} H - \frac{\Omega}{2V} (H_+ + H_-),
\]

\[
H'_- = \frac{\Omega}{\mathcal{M}} H + \left[ \frac{1}{2r} - \frac{(2m^2 + 1/\ell^2)r}{4\mathcal{M}} \right] H_+ + \left[ - \frac{3}{2r} + \frac{(2m^2 + 1/\ell^2)r}{4\mathcal{M}} + \frac{r\Omega^2}{MV} \right] H_-.
\]

A constraint equation can be written as

\[
r\Omega \left[ - 5\mathcal{M}/\ell^2 + r^2/\ell^4 + 2m^2 V + 4\Omega^2 \right] H_- - rV(2m^2 + 1/\ell^2)\Omega H_+ \\
+ 2V(2m^2 \mathcal{M} - \mathcal{M}/\ell^2 + 2\Omega^2) H = 0,
\]

\[7\]
where
\[ H \equiv -H_{tr}, \quad H_\pm \equiv \frac{H_\ell}{V(r)} \pm V(r)H_{rr}. \]  

(3.6)

At infinity of \( r \to \infty \), asymptotic solutions to Eqs. (3.3) and (3.4) are
\[
H^{(\infty)} = C_1^{(\infty)} r^{-2 - \sqrt{m^2 \ell^2 + 1}/2} + C_2^{(\infty)} r^{-2 + \sqrt{m^2 \ell^2 + 1}/2},
\]
\[
H_-^{(\infty)} = \tilde{C}_1^{(\infty)} r^{-1 - \sqrt{m^2 \ell^2 + 1}/2} + \tilde{C}_2^{(\infty)} r^{-1 + \sqrt{m^2 \ell^2 + 1}/2},
\]

(3.7)

where
\[
\tilde{C}_1^{(\infty)} = \frac{2m^2 - 1/\ell^2}{\Omega(2 + \sqrt{4m^2 \ell^2 + 2})} C_1^{(\infty)}, \quad \tilde{C}_2^{(\infty)} = \frac{2m^2 - 1/\ell^2}{\Omega(2 - \sqrt{4m^2 \ell^2 + 2})} C_2^{(\infty)}.
\]

(3.8)

At the horizon of \( r_+ = \ell \sqrt{M} \), their solutions are given by
\[
H^{(r_+)} = C_1^{(r_+)} (r - r_+)^{-1 - \Omega \ell/(2\sqrt{M})} + C_2^{(r_+)} (r - r_+)^{-1 + \Omega \ell/(2\sqrt{M})},
\]
\[
H_-^{(r_+)} = \tilde{C}_1^{(r_+)} (r - r_+)^{-\Omega \ell/(2\sqrt{M})} + \tilde{C}_2^{(r_+)} (r - r_+)^{\Omega \ell/(2\sqrt{M})},
\]

(3.9)

(3.10)

where \( \tilde{C}_1^{(r_+)} \) take the forms
\[
\tilde{C}_1^{(r_+)} = \frac{M(-2m^2\ell^2 + 1) - 2\Omega^2 \ell^2 + \Omega \ell(2m^2\ell^2 + 1)\sqrt{M}}{\Omega \ell(M - \Omega^2 \ell^2)} C_1^{(r_+)},
\]
\[
\tilde{C}_2^{(r_+)} = \frac{M(-2m^2\ell^2 + 1) - 2\Omega^2 \ell^2 - \Omega \ell(2m^2\ell^2 + 1)\sqrt{M}}{\Omega \ell(M - \Omega^2 \ell^2)} C_2^{(r_+)},
\]

(3.11)

Imposing two boundary conditions of the regular solutions at infinity and horizon correspond to choosing \( C_2^{(\infty)} = 0 \) and \( C_1^{(r_+)} = 0 \), respectively.

Eliminating \( H_+ \) in Eqs. (3.3) and (3.4) by using the constraint (3.5) leads to the two coupled equations with \( H \) and \( H_- \) only. For fixed \( m \) and various values of \( \Omega \), we solve these equations numerically\( ^1 \) which yields permitted values of \( \Omega \) as a function of \( m^2 \). As a result, this shows clearly that there exist unstable modes (see Fig.1).

\( ^1 \)Solving the first-order equations (3.3) and (3.4) numerically, we begin it by considering asymptotic solutions (3.9) and (3.10) at the horizon. For given \( m \), we find the consistent values of \( \Omega \) to have the asymptotic behavior of \( H^{(\infty)} \sim r^{-2 - \sqrt{m^2 \ell^2 + 1}/2} \). For a complete analysis, we have already checked the results given in Fig.3 of the literature [5].
Figure 1: Ω graphs are depicted as function of $m^2$ for three different horizon radii of $r_+ = 1, 2, 3$ from bottom to top with $\ell = 1$. The data range for $m^2$ is between $-1/2$ and $1/2$.

Comparing with higher dimensional black strings [4, 5], we would like to mention a couple of key points observed from Fig. 1. Firstly we can fix the AdS$_3$ curvature radius ($\ell = 1$) by taking into account the scaling symmetry given in Eqs. (3.3)-(3.5) as

$$r \to \alpha r, \quad m \to m/\alpha, \quad \Omega \to \Omega/\alpha, \quad \ell^2 \to \alpha^2 \ell^2$$

with an arbitrary constant $\alpha$. The second point is that the threshold mass for $r_+ = 1, 2, 3$ is given near $m^2 \approx 0.5$, which implies that GL instability exists for

$$m^2 < \frac{1}{2\ell^2}$$

(3.13)

when recovering the AdS$_3$ curvature radius $\ell$. This means that the BTZ black hole is unstable against the $s$-mode metric perturbation regardless of the horizon size. On the other hand, the threshold mass of $m^2 = 1/2\ell^2$ can be read off approximately by taking the limit of $\Omega \to 0$ in the constraint equation (3.5). Therefore, the stability condition is given by

$$m^2 > \frac{1}{2\ell^2}$$

(3.14)

which is exactly the same condition obtained from the positivity of the potential ($V_\Psi >$
The potential appears in the Schrödinger equation

\[
\frac{d^2 \Psi}{dr^2} + [\omega^2 - V(r)]\Psi = 0, \quad \omega = i\Omega
\]  

(3.15)

which was derived from the second-order equation (3.2) by introducing a new field \( \Psi \) defined by \( \Psi = H_{tt}/f(r) \). Combining it with the BF bound (2.23) for stability condition of tensor field at asymptotically AdS\(_3\) spacetime, we find that the instability of the BTZ black hole in new massive gravity is extended to

\[
-\frac{1}{2\ell^2} \leq m^2 < \frac{1}{2\ell^2}.
\]  

(3.16)

Up to now, we have made our instability analysis with (2.19) for the metric tensor. Since two equations (2.14) and (2.19) take the same form when replacing \( \delta G_{\mu\nu} \) by \( h_{\mu\nu} \) and they describe 2 DOF with (2.16) and (2.22), the instability analysis for \( h_{\mu\nu} \) persists in that of \( \delta G_{\mu\nu} \). One additional advantage when using (2.14) is to avoid the ghost problem.

Finally, considering \( M^2 = m^2 - 1/2\ell^2 \) (2.15), we rewrite the stability condition (3.14) as

\[
M^2 > 0
\]  

(3.17)

which is exactly the same condition for the 4D black string with positive mass squared \( \mu^2 > 0 \). This implies that the 4D black string is stable under the s-mode metric perturbation regardless of the horizon size [12].

4 Discussions

We have shown that the new massive gravity which is known to be a unitary gravity model in three dimensions, could not accommodate the BTZ black hole by observing the GL instability for the mass \( m^2 \) between \(-1/2\ell^2\) and \(1/2\ell^2\). The GL instability has nothing to do with the ghost issue arising from the fourth-order gravity of the new massive gravity because we have used two second-order equations (2.19) and (2.14) for \( h_{\mu\nu} \) and \( \delta G_{\mu\nu} \), respectively. Also, this instability could not be explained in terms of the Gubser-Mitra conjecture because the heat capacity of the BTZ black hole is always positive. This instability arises from the massiveness \( (M^2 = m^2 - 1/2\ell^2 \neq 0) \) of the new massive gravity. In the massless case of \( M^2 = 0 \), one has a massless graviton propagating on the BTZ black hole spacetime which
is gauge artefact in three dimensions. In this case, the stability issue of the BTZ black hole is meaningless.

This establishes that the instability of the static black holes in the $D \geq 3$-dimensional massive gravity originates from the massiveness, but not a nature of the fourth-order gravity giving ghost states.

Also, the unstable condition of (3.16) explains that choosing the graviton mass above the three-dimensional BF bound but below the GL instability bound has made the BTZ black hole unstable to a formation of massive graviton which is regular at horizon and infinity. This is similar to two instabilities of AdS black holes to realize a holographic superconductor phase in the bulk within the AdS/CFT correspondence \cite{18,19}.

Finally, we would like to mention that the 4D black string is stable under the $s$-mode metric perturbation in Einstein gravity \cite{12}, while the BTZ black hole is unstable in the new massive gravity. This shows a newly interesting feature of low dimensional black string and black hole when comparing with higher dimensional black string and black holes.

**Acknowledgement**

T.M. would like to thank Dr. Miok Park for useful discussions. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No.2012-R1A1A2A10040499). Y.M. was supported partly by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) through the Center for Quantum Spacetime (CQUeST) of Sogang University with grant number 2005-0049409.
References

[1] C. de Rham and G. Gabadadze, Phys. Rev. D 82, 044020 (2010) arXiv:1007.0443 [hep-th].

[2] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. 106, 231101 (2011) arXiv:1011.1232 [hep-th].

[3] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. 108, 041101 (2012) arXiv:1106.3344 [hep-th].

[4] R. Gregory and R. Laflamme, Phys. Rev. Lett. 70, 2837 (1993) hep-th/9301052.

[5] R. Gregory and R. Laflamme, Nucl. Phys. B 428 (1994) 399 hep-th/9404071.

[6] E. Babichev and A. Fabbri, Class. Quant. Grav. 30, 152001 (2013) arXiv:1304.5992 [gr-qc].

[7] R. Brito, V. Cardoso and P. Pani, Phys. Rev. D 88, 023514 (2013) arXiv:1304.6725 [gr-qc].

[8] Y. S. Myung, Phys. Rev. D 88, 024039 (2013) arXiv:1306.3725 [gr-qc].

[9] Y. S. Myung, arXiv:1308.1455 [gr-qc].

[10] Y. S. Myung, Phys. Rev. D 88, 084006 (2013) arXiv:1308.3907 [gr-qc].

[11] G. Kang, hep-th/0202147.

[12] G. Kang and Y. O. Lee, AIP Conf. Proc. 805, 358 (2006).

[13] L. -h. Liu and B. Wang, Phys. Rev. D 78, 064001 (2008) arXiv:0803.0455 [hep-th].

[14] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. 102, 201301 (2009) arXiv:0901.1766 [hep-th].

[15] E. Bergshoeff, M. Kovacevic, L. Parra and T. Zojer, PoS Corfu 2012, 053 (2013).

[16] Y. S. Myung, Y. -W. Kim, T. Moon and Y. -J. Park, Phys. Rev. D 84, 024044 (2011) arXiv:1105.4205 [hep-th].
[17] T. Moon and Y. S. Myung, Gen. Rel. Grav. (2013) [arXiv:1303.5893 [hep-th]].

[18] S. A. Hartnoll, Class. Quant. Grav. 26, 224002 (2009) [arXiv:0903.3246 [hep-th]].

[19] B. Hartmann, [arXiv:1310.0300 [gr-qc]].

[20] I. Gullu and B. Tekin, Phys. Rev. D 80, 064033 (2009) [arXiv:0906.0102 [hep-th]].