Quantum Error Correction and Fault-Tolerance

Daniel Gottesman

Abstract

I give an overview of the basic concepts behind quantum error correction and quantum fault tolerance. This includes the quantum error correction conditions, stabilizer codes, CSS codes, transversal gates, fault-tolerant error correction, and the threshold theorem.

1 Quantum Error Correction

Building a quantum computer or a quantum communications device in the real world means having to deal with errors. Any qubit stored unprotected or one transmitted through a communications channel will inevitably come out at least slightly changed. The theory of quantum error-correcting codes has been developed to counteract noise introduced in this way. By adding extra qubits and carefully encoding the quantum state we wish to protect, a quantum system can be insulated to great extent against errors.

To build a quantum computer, we face an even more daunting task: If our quantum gates are imperfect, everything we do will add to the error. The theory of fault-tolerant quantum computation tells us how to perform operations on states encoded in a quantum error-correcting code without compromising the code’s ability to protect against errors.

In general, a quantum error-correcting code is a subspace of a Hilbert space designed so that any of a set of possible errors can be corrected by an appropriate quantum operation. Specifically:

Definition 1 Let \( \mathcal{H}_n \) be a \( 2^n \)-dimensional Hilbert space (\( n \) qubits), and let \( C \) be a \( K \)-dimensional subspace of \( \mathcal{H}_n \). Then \( C \) is an \((n, K)\) (binary) quantum error-correcting code (QECC) correcting the set of errors \( \mathcal{E} = \{ E_a \} \) iff there exists a quantum operation \( \mathcal{R} \) s.t. \( \mathcal{R} \) is a quantum operation and \( (\mathcal{R} \circ E_a)(|\psi\rangle) = |\psi\rangle \) for all \( E_a \in \mathcal{E} \), \( |\psi\rangle \in C \).

\( \mathcal{R} \) is called the recovery or decoding operation and serves to actually perform the correction of the state. The decoder is sometimes also taken to map \( \mathcal{H}_n \) into an unencoded Hilbert space \( \mathcal{H}_{\log K} \) isomorphic to \( C \). This should be distinguished from the encoding operation which maps \( \mathcal{H}_{\log K} \) into \( \mathcal{H}_n \), determining the imbedding of \( C \). The computational complexity of the encoder is frequently a great deal lower than that of the decoder. In particular, the task of determining what error has occurred can be computationally difficult (NP-hard, in fact), and designing codes with efficient decoding algorithms is an important task in quantum error correction, as in classical error correction.

This article will cover only binary quantum codes, built with qubits as registers, but all of the techniques discussed here can be generalized to higher-dimensional registers, or qudits.

To determine whether a given subspace is able to correct a given set of errors, we can apply the quantum error-correction conditions [2, 7]:

Theorem 1 A QECC \( C \) corrects the set of errors \( \mathcal{E} \) iff

\[
\langle \psi_i | E_d^\dagger E_b | \psi_j \rangle = C_{ab} \delta_{ij},
\]

where \( E_a, E_b \in \mathcal{E} \) and \( \{ |\psi_i\rangle \} \) form an orthonormal basis for \( C \).

The salient point in these error-correction conditions is that the matrix element \( C_{ab} \) does not depend on the encoded basis states \( i \) and \( j \), which roughly speaking indicates that neither the environment nor the decoding operation learns any information about the encoded state. We can imagine the various possible errors taking the subspace \( C \) into other subspaces...
of $H_n$, and we want those subspaces to be isomorphic to $C$, and to be distinguishable from each other by an appropriate measurement. For instance, if $C_{ab} = \delta_{ab}$, then the various erroneous subspaces are orthogonal to each other.

Because of the linearity of quantum mechanics, we can always take the set of errors $E$ to be a linear space: If a QECC corrects $E_a$ and $E_b$, it will also correct $\alpha E_a + \beta E_b$ using the same recovery operation. In addition, if we write any superoperator $S$ in terms of its operator-sum representation $S(\rho)$ $\mapsto$ $\sum A_k \rho A_k^\dagger$, a QECC that corrects the set of errors $\{A_k\}$ automatically corrects $S$ as well. Thus, it is sufficient in general to check that the error-correction conditions hold for a basis of errors.

Frequently, we are interested in codes that correct any error affecting $t$ or fewer physical qubits. In that case, let us consider tensor products of the Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

Define the Pauli group $P_n$ as the group consisting of tensor products of $I$, $X$, $Y$, and $Z$ on $n$ qubits, with an overall phase of $\pm 1$ or $\pm i$. The weight $wt(P)$ of a Pauli operator $P \in P_n$ is the number of qubits on which it acts as $X$, $Y$, or $Z$ (i.e., not as the identity). Then the Pauli operators of weight $t$ or less form a basis for the set of all errors acting on $t$ or fewer qubits, so a QECC which corrects these Pauli operators corrects all errors acting on up to $t$ qubits. If we have a channel which causes errors independently with probability $O(\epsilon)$ on each qubit in the QECC, then the code will allow us to decode a correct state except with probability $O(\epsilon^{t+1})$, which is the probability of having more than $t$ errors. We get a similar result in the case where the noise is a general quantum operation on each qubit which differs from the identity by something of size $O(\epsilon)$.

**Definition 2** The distance $d$ of an $((n,K))$ QECC is the smallest weight of a nontrivial Pauli operator $E \in P_n$ s.t. the equation

$$\langle \psi_i | E | \psi_j \rangle = C(E) \delta_{ij} \quad (3)$$

fails.

We use the notation $((n,K,d))$ to refer to an $((n,K))$ QECC with distance $d$. Note that for $P,Q \in P_n$, $wt(PQ) \leq wt(P) + wt(Q)$. Then by comparing the definition of distance with the quantum error-correction conditions, we immediately see that a QECC corrects $t$ general errors iff its distance $d > 2t$. If we are instead interested in erasure errors, when the location of the error is known but not its precise nature, a distance $d$ code corrects $d-1$ erasure errors. If we only wish to detect errors, a distance $d$ code can detect errors on up to $d - 1$ qubits.

One of the central problems in the theory of quantum error correction is to find codes which maximize the ratios $(\log K)/n$ and $d/n$, so they can encode as many qubits as possible and correct as many errors as possible. Conversely, we are also interested in the problem of setting upper bounds on achievable values of $(\log K)/n$ and $d/n$. The quantum Singleton bound (or Knill-Laflamme bound [7]) states that any $((n,K,d))$ QECC must satisfy

$$n - \log K \geq 2d - 2. \quad (4)$$

We can set a lower bound on the existence of QECCs using the quantum Gilbert-Varshamov bound, which states that, for large $n$, an $((n,2^k,d))$ QECC exists provided that

$$k/n \leq 1 - (d/n) \log 3 - h(d/n), \quad (5)$$

where $h(x) = -x \log x - (1-x) \log (1-x)$ is the binary Hamming entropy. Note that the Gilbert-Varshamov bound simply states that codes at least this good exist; it does not suggest that better codes cannot exist.

## 2 Stabilizer Codes

In order to better manipulate and discover quantum error-correcting codes, it is helpful to have a more detailed mathematical structure to work with. The most widely-used structure gives a class of codes known as stabilizer codes [3, 4]. They are less general than arbitrary quantum codes, but have a number of
useful properties that make them easier to work with than the general QECC.

**Definition 3** Let $S \subset \mathcal{P}_n$ be an Abelian subgroup of the Pauli group that does not contain $-1$ or $\pm i$, and let $C(S) = \{ |\psi\rangle \text{ s.t. } P|\psi\rangle = |\psi\rangle \ \forall P \in S \}$. Then $C(S)$ is a stabilizer code and $S$ is its stabilizer.

Because of the simple structure of the Pauli group, any Abelian subgroup has order $2^{n-k}$ for some $k$ and can easily be specified by giving a set of $n-k$ commuting generators.

The codewords of the QECC are by definition in the +1-eigenspace of all elements of the stabilizer, but an error $E$ acting on a codeword will move the state into the $-1$-eigenspace of any stabilizer element $M$ which anticommutes with $E$:

$$M(E|\psi\rangle) = -EM|\psi\rangle = -E|\psi\rangle.$$  

Thus, measuring the eigenvalues of the generators of $S$ tells us information about the error that has occurred. The set of such eigenvalues can be represented as an $(n-k)$-dimensional binary vector known as the error syndrome. Note that the error syndrome does not tell us anything about the encoded state, only about the error that has occurred.

**Theorem 2** Let $S$ be a stabilizer with $n-k$ generators, and let $S^\perp = \{ E \in \mathcal{P}_n, \text{ s.t. } [E, M] = 0 \ \forall M \in S \}$. Then $S$ encodes $k$ qubits and has distance $d$, where $d$ is the smallest weight of an operator in $S^\perp \setminus S$.

We use the notation $[[n,k,d]]$ to refer to such a stabilizer code. Note that the square brackets specify that the code is a stabilizer code, and that the middle term $k$ refers to the number of encoded qubits, and not the dimension $2^k$ of the encoded subspace, as for the general QECC (whose dimension might not be a power of 2).

$S^\perp$ is the set of Pauli operators that commute with all elements of the stabilizer. They would therefore appear to be those errors which cannot be detected by the code. However, the theorem specifies the distance of the code by considering $S^\perp \setminus S$. A Pauli operator $P \in S$ cannot be detected by the code, but there is in fact no need to detect it, since all codewords remain fixed under $P$, making it equivalent to the identity operation. A distance $d$ stabilizer code which has nontrivial $P \in S$ with $\text{wt}(P) < d$ is called degenerate, whereas one which does not is non-degenerate. The phenomenon of degeneracy has no analogue for classical error correcting codes, and makes the study of quantum codes substantially more difficult than the study of classical error correction. For instance, a standard bound on classical error correction is the Hamming bound (or sphere-packing bound), but the analogous quantum Hamming bound

$$k/n \leq 1 - (t/n) \log 3 - h(t/n)$$

for $[[n,k,2t+1]]$ codes (when $n$ is large) is only known to apply to non-degenerate quantum codes (though in fact we do not know of any degenerate QECCs that violate the quantum Hamming bound).

An example of a stabilizer code is the 5-qubit code, a $[[5,1,3]]$ code whose stabilizer can be generated by

$$X \otimes Z \otimes Z \otimes X \otimes I,$$

$$I \otimes X \otimes Z \otimes Z \otimes X,$$

$$X \otimes I \otimes X \otimes Z \otimes Z,$$

$$Z \otimes X \otimes I \otimes X \otimes Z.$$

The 5-qubit code is a non-degenerate code, and is the smallest possible QECC which corrects 1 error (as one can see from the quantum Singleton bound).

It is frequently useful to consider other representations of stabilizer codes. For instance, $P \in \mathcal{P}_n$ can be represented by a pair of $n$-bit binary vectors $(p_X|p_Z)$ where $p_X$ is 1 for any location where $P$ has an $X$ or $Y$ tensor factor and is 0 elsewhere, and $p_Z$ is 1 for any location where $P$ has a $Y$ or $Z$ tensor factor. Two Pauli operators $P = (p_X|p_Z)$ and $Q = (q_X|q_Z)$ commute if $p_X \cdot q_Z + p_Z \cdot q_X = 0$. Then the stabilizer for a code becomes a pair of $(n-k) \times n$ binary matrices, and most interesting properties can be determined by an appropriate linear algebra exercise. Another useful representation is to map the single-qubit Pauli operators $I$, $X$, $Y$, $Z$ to the finite field $\mathbb{GF}(4)$, which sets up a connection between stabilizer codes and a subset of classical codes on 4-dimensional registers.
3 CSS codes

CSS codes are a very useful class of stabilizer codes invented by Calderbank and Shor, and by Steane [4, 10]. The construction takes two binary classical linear codes and produces a quantum code, and can therefore take advantage of much existing knowledge from classical coding theory. In addition, CSS codes have some very useful properties which make them excellent choices for fault-tolerant quantum computation.

A classical \([n, k, d]\) linear code (\(n\) physical bits, \(k\) logical bits, classical distance \(d\)) can be defined in terms of an \((n-k) \times n\) binary parity check matrix \(H\) — every classical codeword \(v\) must satisfy \(Hv = 0\).

Each row of the parity check matrix can be converted into a Pauli operator by replacing each 0 with an \(I\) operator and each 1 with a \(Z\) operator. Then the stabilizer code generated by these operators is precisely a quantum version of the classical error-correcting code given by \(H\). If the classical distance \(d = 2t + 1\), the quantum code can correct \(t\) bit flip (\(X\)) errors, just as could the classical code.

If we want to make a QECC that can also correct phase (\(Z\)) errors, we should choose two classical codes \(C_1\) and \(C_2\), with parity check matrices \(H_1\) and \(H_2\). Let \(C_1\) be an \([n, k_1, d_1]\) code and let \(C_2\) be an \([n, k_2, d_2]\) code. We convert \(H_1\) into stabilizer generators as above, replacing each \(0\) with an \(I\) operator and each \(1\) with a \(Z\) operator. Then for \(H_2\), we perform the same procedure, but each \(1\) is instead replaced by \(X\). The code will be able to correct bit flip (\(X\)) errors as if it had a distance \(d_1\) and to correct phase (\(Z\)) errors as if it had a distance \(d_2\). Since these two operations are completely separate, it can also correct \(Y\) errors as both a bit flip and a phase error. Thus, the distance of the quantum code is at least \(\min(d_1, d_2)\), but might be higher because of the possibility of degeneracy.

However, in order to have a stabilizer code at all, the generators produced by the above procedure must commute. Define the dual \(C_1^\perp\) of a classical code \(C\) as the set of vectors \(w\) s.t. \(w \cdot v = 0\) for all \(v \in C\). Then the \(Z\) generators from \(H_1\) will all commute with the \(X\) generators from \(H_2\) iff \(C_2^\perp \subseteq C_1\) (or equivalently, \(C_1^\perp \subseteq C_2\)). When this is true, \(C_1\) and \(C_2\) define an \([n, k_1 + k_2 - n, d]\) stabilizer code, where \(d \geq \min(d_1, d_2)\).

The smallest distance-3 CSS code is the 7-qubit code, a \([7, 1, 3]\) QECC created from the classical Hamming code (consisting of all sums of classical strings 1111000, 1100110, 1010101, and 1111111). The encoded \(\langle 0 \rangle\) for this code consists of the superposition of all even-weight classical codewords and the encoded \(\langle 1 \rangle\) is the superposition of all odd-weight classical codewords. The 7-qubit code is much studied because its properties make it particularly well-suited to fault-tolerant quantum computation.

4 Fault-Tolerance

Given a QECC, we can attempt to supplement it with protocols for performing fault-tolerant operations. The basic design principle of a fault-tolerant protocol is that an error in a single location — either a faulty gate or noise on a quiescent qubit — should not be able to alter more than a single qubit in each block of the quantum error-correcting code. If this condition is satisfied, \(t\) separate single-qubit or single-gate failures are required for a distance \(2t + 1\) code to fail.

Particular caution is necessary, as computational gates can cause errors to propagate from their original location onto qubits that were previously correct. In general, a gate coupling pairs of qubits allows errors to spread in both directions across the coupling.

The solution is to use transversal gates whenever possible [9]. A transversal operation is one in which the \(i\)th qubit in each block of a QECC interacts only with the \(i\)th qubit of other blocks of the code or of special ancilla states. An operation consisting only of single-qubit gates is automatically transversal. A transversal operation has the virtue that an error occurring on the 3rd qubit in a block, say, can only ever propagate to the 3rd qubit of other blocks of the code, no matter what other sequence of gates we perform before a complete error-correction procedure.

In the case of certain codes, such as the 7-qubit code, a number of different gates can be performed transversally. Unfortunately, it does not appear to be possible to perform universal quantum computations using just transversal gates. We therefore have to re-
sort to more complicated techniques. First we create special encoded ancilla states in a non-fault-tolerant way, but perform some sort of check on them (in addition to error correction) to make sure they are not too far off from the goal. Then we interact the ancilla with the encoded data qubits using gates from our stock of transversal gates and perform a fault-tolerant measurement. Then we complete the operation with a further transversal gate which depends on the outcome of the measurement.

5 Fault-Tolerant Gates

We will focus on stabilizer codes. Universal fault-tolerance is known to be possible for any stabilizer code, but in most cases the more complicated type of construction is needed for all but a few gates. The Pauli group $\mathcal{P}_k$, however, can be performed transversally on any stabilizer code. Indeed, the set $S^\perp \setminus S$ of undetectable errors is a boon in this case, as it allows us to perform these gates. In particular, each coset $S^\perp \setminus S$ corresponds to a different logical Pauli operator (with $S$ itself corresponding to the identity). On a stabilizer code, therefore, logical Pauli operations can be performed via a transversal Pauli operation on the physical qubits.

Stabilizer codes have a special relationship to a finite subgroup $\mathcal{C}_n$ of the unitary group $U(2^n)$ frequently called the Clifford group. The Clifford group on $n$ qubits is defined as the set of unitary operations which conjugate the Pauli group $\mathcal{P}_n$ into itself; $\mathcal{C}_n$ can be generated by the Hadamard transform, the CNOT, and the single-qubit $\pi/4$ phase rotation $\text{diag}(1, i)$. The set of stabilizer codes is exactly the set of codes which can be created by a Clifford group encoder circuit using $|0\rangle$ ancilla states.

Some stabilizer codes have interesting symmetries under the action of certain Clifford group elements, and these symmetries result in transversal gate operations. A particularly useful fact is that a transversal CNOT gate (i.e., CNOT acting between the $i$th qubit of one block of the QECC and the $i$th qubit of a second block for all $i$) acts as a logical CNOT gate on the encoded qubits for any CSS code. Furthermore, for the 7-qubit code, transversal Hadamard performs a logical Hadamard, and the transversal $\pi/4$ rotation performs a logical $-\pi/4$ rotation. Thus, for the 7-qubit code, the full logical Clifford group is accessible via transversal operations.

Unfortunately, the Clifford group by itself does not have much computational power: it can be efficiently simulated on a classical computer. We need to add some additional gate outside the Clifford group to allow universal quantum computation; a single gate will suffice, such as the single-qubit $\pi/8$ phase rotation $\text{diag}(1, \exp(i\pi/4))$. Note that this gives us a finite generating set of gates. However, by taking appropriate products, we get an infinite set of gates, one that is dense in the unitary group $U(2^n)$, allowing universal quantum computation.

The following circuit performs a $\pi/8$ rotation, given an ancilla state $|\psi_{\pi/8}\rangle = |0\rangle + \exp(i\pi/4)|1\rangle$:

Here $P$ is the $\pi/4$ phase rotation $\text{diag}(1, i)$, and $X$ is the bit flip. The product is in the Clifford group, and is only performed if the measurement outcome is 1. Therefore, given the ability to perform fault-tolerant Clifford group operations, fault-tolerant measurements, and to prepare the encoded $|\psi_{\pi/8}\rangle$ state, we have universal fault-tolerant quantum computation. A slight generalization of the fault-tolerant measurement procedure below can be used to fault-tolerantly verify the $|\psi_{\pi/8}\rangle$ state, which is a +1 eigenstate of $PX$. Using this or another verification procedure, we can check a non-fault-tolerant construction.
of them share some basic features: they involve creation and verification of specialized ancilla states, and use transversal gates which interact the data block with the ancilla state.

The simplest method, due to Shor, is very general but also requires the most overhead and is frequently the most susceptible to noise. Note that the following procedure can be used to measure (non-fault-tolerantly) the eigenvalue of any (possibly multi-qubit) Pauli operator $M$: Produce an ancilla qubit in the state $|+\rangle = |0\rangle + |1\rangle$. Perform a controlled-$M$ operation from the ancilla to the state being measured. In the case where $M$ is a multi-qubit Pauli operator, this can be broken down into a sequence of controlled-$X$, controlled-$Y$, and controlled-$Z$ operations. Then measure the ancilla in the basis of $|+\rangle$ and $|−\rangle = |0\rangle − |1\rangle$. If the state is a $+1$ eigenvector of $M$, the ancilla will be $|+\rangle$, and if the state is a $−1$ eigenvector, the ancilla will be $|−\rangle$.

The advantage of this procedure is that it measures just $M$ and nothing more. The disadvantage is that it is not transversal, and thus not fault-tolerant. Instead of the unencoded $|+\rangle$ state, we must use a more complex ancilla state $|00\ldots0\rangle + |11\ldots1\rangle$ known as a “cat” state. The cat state contains as many qubits as the operator $M$ to be measured, and we perform the controlled-$X$, -$Y$, or -$Z$ operations transversally from the appropriate qubits of the cat state to the appropriate qubits in the data block. Since, assuming the cat state is correct, all of its qubits are either $|0\rangle$ or $|1\rangle$, the procedure either leaves the data state alone or performs $M$ on it uniformly. A $+1$ eigenstate in the data therefore leaves us with $|00\ldots0\rangle + |11\ldots1\rangle$ in the ancilla and a $−1$ eigenstate leaves us with $|00\ldots0\rangle − |11\ldots1\rangle$. In either case, the final state still tells us nothing about the data beyond the eigenvalue of $M$. If we perform a Hadamard transform and then measure each qubit in the ancilla, we get either a random even weight string (for eigenvalue $+1$) or an odd weight string (for eigenvalue $−1$).

The procedure is transversal, so an error on a single qubit in the initial cat state or in a single gate during the interaction will only produce one error in the data. However, the initial construction of the cat state is not fault-tolerant, so a single gate error then could eventually produce two errors in the data block. Therefore, we must be careful and use some sort of technique to verify the cat state, for instance by checking if random pairs of qubits are the same. Also, note that a single phase error in the cat state will cause the final measurement outcome to be wrong (even and odd switch places), so we should repeat the measurement procedure multiple times for greater reliability.

We can then make a full fault-tolerant error correction procedure by performing the above measurement technique for each generator of the stabilizer. Each measurement gives us one bit of the error syndrome, which we then decipher classically to determine the actual error.

More sophisticated techniques for fault-tolerant error correction involve less interaction with the data but at the cost of more complicated ancilla states. A procedure due to Steane uses (for CSS codes) one ancilla in a logical $|0\rangle$ state of the same code and one ancilla in a logical $|0\rangle + |1\rangle$ state. A procedure due to Knill (for any stabilizer code) teleports the data qubit through an ancilla consisting of two blocks of the QECC containing an encoded Bell state $|00\rangle + |11\rangle$. Because the ancillas in Steane and Knill error correction are more complicated than the cat state, it is especially important to verify the ancillas before using them.

7 The Threshold for Fault-Tolerance

In an unencoded protocol, even one error can destroy the computation, but a fully fault-tolerant protocol will give the right answer unless multiple errors occur before they can be corrected. On the other hand, the fault-tolerant protocol is larger, requiring more qubits and more time to do each operation, and therefore providing more opportunities for errors. If errors occur on the physical qubits independently at random with probability $p$ per gate or timestep, the fault-tolerant protocol has probability of logical error for a single logical gate or timestep at most $Cp^2$, where $C$ is a constant that depends on the design of the fault-tolerant circuitry (assume the quantum error-
correcting code has distance 3, as for the 7-qubit code). When \( p < p_t = 1/C \), the fault-tolerance helps, decreasing the logical error rate. \( p_t \) is the threshold for fault-tolerant quantum computation. If the error rate is higher than the threshold, the extra overhead means that errors will occur faster than they can be reliably corrected, and we are better off with an unencoded system.

To further lower the logical error rate, we turn to a family of codes known as concatenated codes. Given a codeword of a particular \([n, 1]\) QECC, we can take each physical qubit and again encode it using the same code, producing an \([n^2, 1]\) QECC. We could repeat this procedure to get an \(n^3\)-qubit code, and so forth. The fault-tolerant procedures concatenate as well, and after \(L\) levels of concatenation, the effective logical error rate is \(p_t(p/p_t)^{2L}\) (for a base code correcting 1 error). Therefore, if \(p\) is below the threshold \(p_t\), we can achieve an arbitrarily good error rate \(\epsilon\) per logical gate or timestep using only \(\text{poly}(\log \epsilon)\) resources, which is excellent theoretical scaling.

Unfortunately, the practical requirements for this result are not nearly so good. The best rigorous proofs of the threshold to date show that the threshold is at least \(2 \times 10^{-5}\) (meaning one error per 50,000 operations). Optimized simulations of fault-tolerant protocols suggest the true threshold may be as high as 5%, but to tolerate this much error, existing protocols require enormous overhead, perhaps increasing the number of gates and qubits by a factor of a million or more for typical computations. For lower physical error rates, overhead requirements are more modest, particularly if we only attempt to optimize for calculations of a given size, but are still larger than one would like.

Furthermore, these calculations make a number of assumptions about the physical properties of the computer. The errors are assumed to be independent and uncorrelated between qubits except when a gate connects them. It is assumed that measurements and classical computations can be performed quickly and reliably, and that quantum gates can be performed between arbitrary pairs of qubits in the computer, irrespective of their physical proximity. Of these, only the assumption of independent errors is at all necessary, and that can be considerably relaxed to allow short-range correlations and certain kinds of non-Markovian environments. However, the effects of relaxing these assumptions on the threshold value and overhead requirements have not been well-studied.

References

[1] D. Aharonov and M. Ben-Or, “Fault-tolerant quantum computation with constant error rate,” quant-ph/9906129.
[2] C. Bennett, D. DiVincenzo, J. Smolin, and W. Wootters, “Mixed state entanglement and quantum error correction,” Phys. Rev. A 54 (1996), 3824–3851; quant-ph/9604024.
[3] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane, “Quantum error correction via codes over GF(4),” IEEE Trans. Inform. Theory 44 (1998), 1369–1387; quant-ph/9605005.
[4] A. R. Calderbank and P. W. Shor, “Good quantum error-correcting codes exist,” Phys. Rev. A 54 (1996), 1098–1105; quant-ph/9512032.
[5] D. Gottesman, “Class of quantum error-correcting codes saturating the quantum Hamming bound,” Phys. Rev. A 54 (1996), 1862–1868; quant-ph/9604038.
[6] A. Y. Kitaev, “Quantum error correction with imperfect gates,” Quantum Communication, Computing, and Measurement (Proc. 3rd Int. Conf. of Quantum Communication and Measurement) (Plenum Press, New York, 1997), p. 181–188.
[7] E. Knill and R. Laflamme, “A theory of quantum error-correcting codes,” Phys. Rev. A 55 (1997), 900–911; quant-ph/9604034.
[8] E. Knill, R. Laflamme, and W. H. Zurek, “Resilient quantum computation,” Science 279 (1998), 342–345.
[9] P. W. Shor, “Fault-tolerant quantum computation,” Proc. 35th Ann. Symp. on Fundamentals of Computer Science (IEEE Press, Los Alamitos, 1996), pp. 56–65; quant-ph/9605011.

[10] A. M. Steane, “Multiple particle interference and quantum error correction,” Proc. Roy. Soc. London A 452 (1996), 2551–2577; quant-ph/9601029.