Conformal couplings of a scalar field to higher curvature terms

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Abstract

We present a simple way of constructing conformal couplings of a scalar field to higher order Euler densities. This is done by constructing a four-rank tensor involving the curvature and derivatives of the field, which transforms covariantly under local Weyl rescalings. The equation of motion for the field and its energy–momentum tensor are shown to be of second order. The field equations for the spherically symmetric ansatz are integrated, and for generic non-homogeneous couplings, the solution is given in terms of a polynomial equation, in close analogy with Lovelock theories.

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1. Introduction

Conformally invariant theories have gained a lot of recent interest in various areas of physics and mathematics. One particular area of research is gravitational physics, where it is interesting to study the conformal coupling of a scalar field with gravity in arbitrary dimensions. Such a coupling was studied by Bocharova et al [1] over forty years ago and later independently by Bekenstein [2, 3], who found an exact black hole solution of Einstein equations in four spacetime dimensions. This solution is static, spherically symmetric and asymptotically flat. Later this solution was generalized in arbitrary dimensions [4] where it was shown that it represents a black hole only in four dimensions. In $D$ dimensions, the resulting equation of motion of the scalar field $\phi$ is given by

$$\Box \phi - \frac{D - 2}{4(D - 1)} R \phi = 0. \quad (1)$$

The second-order operator on the left-hand side, also known as the conformal Laplacian or the Yamabe operator, transforms covariantly under conformal transformations: $g_{ab} \rightarrow e^{2\Omega} g_{ab}$ and $\phi \rightarrow e^{1-\frac{D}{2}} \phi$ and plays an important role in the Yamabe problem [5].

It is natural to wonder if there are higher curvature generalizations of the conformal coupling of a scalar field to gravity. After a moment’s thought one realizes that, for order
$k \geq 2$, there is at least one trivial way of constructing such conformally invariant couplings of a scalar field. This can be done by simply taking $k$ conformal tensors and contracting all the indices with each other and then multiplying it by the scalar field $\phi$ raised to an appropriate power depending on the dimensions $D$. However, unlike the usual conformal coupling (linear in curvature), in this case the field equations are of fourth order. It turns out that there are other conformally invariant scalar densities, at each order $k$, out of which there is a unique density that leads to second-order field equations (in dimensions $D > 2k$). These can be thought of as the conformal couplings to Lovelock gravity [6]. In this work, we give a simple way of constructing such scalar densities. We also obtain the corresponding energy–momentum tensor and the general static spherically symmetric solutions of $T^{(k)\mu\nu} = 0$, i.e. non-trivial solutions with vanishing energy–momentum tensor. This equation is conformally invariant and hence the solution is determined up to an arbitrary conformal factor. We shall show that these equations leave the scalar field undetermined. However, if one adds Lovelock terms as purely gravitational interactions in the action and looks for static spherically symmetric solutions, then the field equations are of fourth order. It turns out that there are other indices with each other and then multiplying it by the scalar field $\phi$ raised to an appropriate power depending on the dimensions $D$. However, unlike the usual conformal coupling (linear in curvature), in this case the field equations are of fourth order. 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where $m_k = \frac{2k(1-s-D)}{s}$. Note that this action reduces to the $k$th order Lovelock action for a constant scalar field $\phi$. Varying the action with respect to the metric, we obtain the following energy–momentum tensor:

$$T_{p}^{(k)q} = \frac{1}{2k+1} \sqrt{-g} \phi^{m_0} \delta^{(D-2k+1)a_1 \ldots a_k b_1 \ldots b_s} c_{a_1} d_1 \ldots c_{a_k} d_k.$$  \hfill (7)

Whereas varying the scalar field $\phi$, we obtain the following equation of motion:

$$m_k + 2k \frac{2}{2k} \sqrt{-g} \phi^{m_0} \delta^{(D-2k+1)a_1 \ldots a_k b_1 \ldots b_s} c_{a_1} d_1 \ldots c_{a_k} d_k = 0.$$ \hfill (8)

The equation of motion ensures that the trace of the energy–momentum tensor $T_{p}^{(k)q}$ vanishes on its solutions as should be the case for conformally invariant theories. Also, note that all the field equations are of second order in the metric and the field (see appendix B).

3. Spherically symmetric solutions

Let us now evaluate the field equation on a general spherically symmetric ansatz given by the following line element:

$$ds^2 = \tilde{g}_{ij}(x)dx^i dx^j + e^{2\gamma(x)}d\Sigma_2^2,$$ \hfill (9)

where $d\Sigma_2^2 = \tilde{g}_{\rho\sigma}(y)dy^\rho dy^\sigma$ is the line element of a $(D - 2)$-dimensional space of constant curvature $\gamma$. Let $\tilde{\nabla}$ be the Lévi–Civitá connection on the two-dimensional space orthogonal to the constant curvature space and $\tilde{R}$ be the corresponding scalar curvature. To evaluate the field equations it is more convenient to express the scalar field as

$$\phi = e^{-\gamma(x)}.$$ \hfill (10)

Then the energy–momentum tensor and the equation of motion can be respectively written as

$$T_{p}^{(k)q} = \frac{1}{2k+1} \sqrt{-g} \phi^{m_0} \delta^{(D-2k+1)a_1 \ldots a_k b_1 \ldots b_s} c_{a_1} d_1 \ldots c_{a_k} d_k Z^{a_1 \ldots a_k b_1 \ldots b_s}.$$ \hfill (11)

and

$$m_k + 2k \frac{2}{2k} \sqrt{-g} \phi^{m_0} \delta^{(D-2k+1)a_1 \ldots a_k b_1 \ldots b_s} c_{a_1} d_1 \ldots c_{a_k} d_k Z^{a_1 \ldots a_k b_1 \ldots b_s} = 0.$$ \hfill (12)

where

$$Z^{a_1 \ldots a_k b_1 \ldots b_s} = R^{a_1 \ldots a_k b_1 \ldots b_s} - 4\delta^{(a_1 \ldots a_k)}_{(b_1 \ldots b_s)} + 4\delta^{(a_1 \ldots a_k)}_{(b_1 \ldots b_s)} \delta^{(c_{\ldots})}_{(d_{\ldots})} - 2\delta^{(c_{\ldots})}_{(d_{\ldots})} \delta^{(\lambda_{\ldots})}_{(\lambda_{\ldots})}.$$ \hfill (13)

Then the nontrivial components of the Riemann curvature tensor and the tensor $Z^{a_1 \ldots a_k b_1 \ldots b_s}$ are given by

$$R_{ji}^{\mu \nu} = \frac{1}{2} \tilde{\nabla}_i \delta_{j}^{\mu \nu}, \quad Z_{ji}^{\mu \nu} = \frac{1}{2} \left( \tilde{R} - 2(\tilde{\nabla}_m \lambda)(\tilde{\nabla}^{m} \lambda) \right) \delta_{j}^{\mu \nu} - 4\delta_{j}^{(\mu} \delta^{\nu \lambda)}.$$ \hfill (14)

$$R_{ji}^{\mu \nu} = \tilde{B} \delta_{ji}^{\mu \nu}, \quad Z_{ji}^{\mu \nu} = \tilde{B} \delta_{ji}^{\mu \nu},$$ \hfill (15)

$$R_{ji}^{\mu \nu} = -\tilde{A}_j^{\mu \nu}, \quad Z_{ji}^{\mu \nu} = -\tilde{A}_j^{\mu \nu},$$ \hfill (16)

where

$$\tilde{B} = \gamma e^{-2\gamma} - (\tilde{\nabla}_m \lambda)(\tilde{\nabla}^{m} \lambda), \quad \tilde{B} = \tilde{B} - 2(\tilde{\nabla}_k \lambda)(\tilde{\nabla}^{k} \lambda) - (\tilde{\nabla}_k \lambda)(\tilde{\nabla}^{k} \lambda)$$

$$\tilde{A}_j = \tilde{\nabla}_j \tilde{\nabla}^\mu + (\tilde{\nabla}^\mu \lambda)\tilde{\nabla}_j \lambda, \quad \tilde{A}_j = \tilde{A}_j + \tilde{C}_j + \delta_j^\mu \tilde{\nabla}_j \tilde{\nabla}^\mu (\lambda + \nu),$$

and

$$\tilde{C}_j = \tilde{\nabla}_j \tilde{\nabla}^\mu \lambda - \tilde{\nabla}_j \tilde{\nabla}^\mu \lambda.$$
Then the non-vanishing components of the energy–momentum tensor evaluated on the ansatz (9) are

\[
T^{(ji)} = \frac{-g}{2(D-2k-1)!} B^{-k-1} [(D-2k-1) \delta_j^i - \cdots] \quad \text{(17)}
\]

\[
T^{(ki)\alpha} = \frac{-g}{2(D-2k-1)!} B^{-k-2} \delta^\alpha_i [(D-2k-1)(D-2k-2) \delta_j^i \cdots] + k \tilde{B} [\tilde{R} - 2 \tilde{\nabla}_m \tilde{\nabla} \lambda] - 2(D-2k-1) \tilde{A}_j^i + 2k(k-1) \delta^\alpha_i \tilde{A}_j^j \quad \text{(18)}
\]

and the equation of motion or the trace of the energy–momentum tensor is given by

\[
\frac{-g}{(D-2k-1)!} B^{-k-1} [(D-2k)(D-2k-1) \tilde{B}^2 + (D-2k)(D-2k-1) \tilde{B}^2] + k \tilde{B} [\tilde{R} - 2 \tilde{\nabla}_m \tilde{\nabla} \lambda - 2(D-2k) \tilde{A}_j^i + 2k(k-1) \delta^\alpha_i \tilde{A}_j^j] = 0 \quad \text{(19)}
\]

Obviously, for \( k > 2 \) there is a trivial solution to the equations (17)–(19) which is \( \tilde{B} = 0 \), i.e.

\[
y e^{-2v} - \tilde{\nabla}_m (v + \lambda) \tilde{\nabla}^m (v + \lambda) = 0 \quad \text{(20)}
\]

Note that since the field equations are invariant with respect to local Weyl rescalings, one can as well gauge away the warp factor in front of the constant curvature base manifold in (9) by a conformal transformation. This implies that we can choose \( v \) to be zero without any loss of generality. In this case, the previous equation becomes \( (\tilde{\nabla}_\alpha \lambda)(\tilde{\nabla}_\beta \lambda) = \gamma \). This is the Hamilton–Jacobi equation in curved spacetime. For \( \gamma > 0 \), the general solution of equation (20) is \( \lambda = f(u) \) or \( \lambda = g(v) \), where \( (u, v) \) are the null coordinates in the two-dimensional case. For \( \gamma < 0 \), there is no general solution known. We next analyze the non-trivial case \( \tilde{B} = 0 \). We first show that the field equations imply that the metric is static. To see this, contract the index \( j \) after multiplying the equation obtained from (17) by \( e^{\xi_j} \) and then symmetrize the indices \( (j, k) \) to obtain

\[
e^{\xi_j} \tilde{\nabla}_j (e^{\xi_k} \tilde{\nabla} e^{-\lambda}) = 0. \quad \text{(21)}
\]

In other words, the vector \( \xi_k = e^{\xi_k} \tilde{\nabla} e^{-\lambda} \) is a Killing vector.

- **Case I**: \( \xi^k \) is null, i.e. \( (\tilde{\nabla}^2 \lambda)(\tilde{\nabla}_\lambda \lambda) = 0 \).

  In this case, we also assume that \( \gamma \neq 0 \) since otherwise we are led back to the trivial case of \( \tilde{B} = 0 \). Now, taking the trace of \( T^{(ji)} = 0 \) and plugging it back we obtain \( \tilde{A}_j^i \delta_j^i = 2 \tilde{A}_j^j \).

- **Case II**: \( \xi^k \) is non-null, i.e. \( (\tilde{\nabla}^2 \lambda)(\tilde{\nabla}_\lambda \lambda) \neq 0 \).

  In this case, we introduce one of the coordinates \( R = e^{-\lambda} \) and the other \( t \) such that \( \xi^k = \left( \frac{dt}{dR}, \frac{dR}{dR} \right) \). We then use the following metric ansatz:

\[
ds^2 = \cdots - f(R) \, dt^2 + \frac{dR^2}{g(R)} + d\Sigma_y^2. \quad \text{(22)}
\]

In this case, the \((t, R)\) and \((R, t)\) components of the field equations (17) are trivially satisfied. The equations \( T^{(ji)} = T^{(kj)} = 0 \) imply \( f(R) = \kappa g(R) \), where \( \kappa \) is a constant and can be absorbed by redefining the time coordinate. Finally, the equation \( T^{(kj)} + T^{(kj)} = 0 \) can then be expressed as

\[
(D - 2k - 1) \tilde{B}^2 - k R \frac{d\tilde{B}}{dR} = 0 \quad \text{where} \quad \tilde{B}^2 = \left( \gamma - \frac{g(R)}{R^2} \right) \quad \text{(23)}
\]
which can be integrated to obtain
\[ f(R) = g(R) = \gamma R^2 - CR^{\frac{D-4}{2}}, \]
and the scalar field is then given by \( \phi = R' \). This solves equation (18) trivially. The metric can as well be given in terms of the scalar field \( \phi \) in the following way:
\[ ds^2 = -f(\phi) \, dt^2 + \frac{d\phi^2}{g(\phi)} + d\Sigma_j^2, \]
(25)
where
\[ f(\phi) = \phi^{2/\gamma} \left( \gamma - C\phi^{\frac{D-4}{2}} \right) \]
\[ g(\phi) = s^2 \phi^2 \left( \gamma - C\phi^{\frac{D-4}{2}} \right). \]
The scalar field \( \phi \) is then left completely arbitrary.

4. Non-homogeneous couplings

Now it is fairly easy to consider non-homogeneous conformal couplings of a scalar field to higher order Lovelock densities. Consider the action:
\[ I = \sum_k c_k \tau^{(k)} = \sum_k \frac{1}{2\kappa} \int \sqrt{-g} d^Dx \, c_k \phi^m \delta_{\gamma}^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \psi^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \cdots \psi^{\alpha_{b_k} \beta}, \]
in which case the field equations and the equation of motion of the scalar field are given by
\[ T_p^q = \sum_k c_k T_p^k = \sum_k \frac{c_k}{2k+1} \sqrt{-g} \phi^m \delta_{\gamma}^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \psi^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \cdots \psi^{\alpha_{b_k} \beta} - \frac{2k}{2k+1} \sqrt{-g} \phi^m \delta_{\gamma}^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \psi^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \cdots \psi^{\alpha_{b_k} \beta} = 0. \]
(27)
Evaluating these equations on the spherically symmetric ansatz (9), we obtain
\[ T_p^j = \sum_k \frac{\sqrt{-g} \phi^m \delta_{\gamma}^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \psi^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \cdots \psi^{\alpha_{b_k} \beta}}{2(D-1)} \hat{B}^{k-1} \left[ (D-2k-1) \hat{B}^b \delta^b_j - 2M^a \hat{A}^a_k \right] \]
(29)
\[ T_p^a = \sum_k \frac{\sqrt{-g} \phi^m \delta_{\gamma}^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \psi^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \cdots \psi^{\alpha_{b_k} \beta}}{2(D-1)(D-2)} \hat{B}^{k-2} \delta^b_p \left[ (D-2k-1)(D-2k-2) \hat{B}^2 + 2 M \hat{B} \left( \hat{R} - 2 \hat{V}_m \hat{V}^m \lambda \right) 
-2(D-2k-1) \hat{A}^b_e + 2k(D-1) \delta^b_p \hat{A}^e_k \right] \]
where \( \hat{c}_k = \frac{(D-1)!}{(D-2k-1)!} \).
(30)
whereas the equation of motion take the form
\[ \sum_k \frac{\sqrt{-g} \phi^m \delta_{\gamma}^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \psi^{\alpha_\beta \gamma_1 \cdots \alpha_{b_1}} \cdots \psi^{\alpha_{b_k} \beta}}{(D-1)s} \hat{B}^{k-2} \left[ (D-2k)(D-2k-1) \hat{B}^2 + 2(D-2k-1) \hat{A}^b_e + 2k(D-1) \delta^b_p \hat{A}^e_k \right] = 0. \]
(31)
Now, again using the conformal invariance of the system we fix the gauge \( \nu = 0 \) and proceeding as before we can show that \( \hat{c}_k = \epsilon \hat{c}_k \hat{V}^e \hat{e}^{-s} \) is a Killing vector, provided \( \sum_k k \hat{c}_k \hat{B}(\hat{e}^{-s})^{k-1} \neq 0 \).

- Case I: \( \hat{e}^k \) is null, i.e. \( \hat{V}^k(\hat{V}_k \lambda) = 0 \).

Again assuming \( \gamma \neq 0 \), in this case the remaining field equations imply both the field \( \lambda \)
and \( \hat{R} \) are constants.
Case II: \( \xi^k \) is non-null, i.e. \((\tilde{\nabla}^k \lambda)(\tilde{\nabla}_k \lambda) \neq 0 \).

Introducing the coordinates \( R = e^{-\lambda} \) and \( t \) such that \( \xi^k = (\frac{\partial}{\partial t})^k \), we evaluate the field equations on the ansatz (22). The \((t, R)\) and \((R, t)\) components of the field equations (29) are then trivially satisfied. Furthermore, the equation \( T^t_t - T^R_R = 0 \) imply \( f(R) = \kappa g(R) \), where \( \kappa \) is a constant and can be absorbed by redefining the time coordinate. Finally, the equation

\[
\sum_k \hat{c}_k R^{-(D-2k)}\left[(D - 2k - 1)\tilde{B}^k - kR\tilde{B}^{k-1}\frac{d\tilde{B}}{dR}\right] = 0
\]

where \( \tilde{B} = \left(\gamma - \frac{g(R)}{R^2}\right) \).

This is a polynomial equation in \( \tilde{B} \). We may solve this polynomial to obtain \( \tilde{B} = \tilde{B}(R) \), which in turn allows one to express the metric functions as

\[
g(R) = f(R) = R^2(\gamma - \tilde{B}(R)).
\]

This satisfies the \((\alpha, \beta)\) components of the field equations. Again note that the scalar field is then given by \( \phi = R^2 \). As before, one can write the metric in terms of the scalar field \( \phi \) in the following way:

\[
d\tilde{s}^2 = -f(\phi) dt^2 + \frac{d\phi^2}{g(\phi)} + d\Sigma^2,
\]

where

\[
f(\phi) = \phi^{2/3}(\gamma - C\tilde{B}(\phi^{1/3}))
g(\phi) = s^2\phi^2(\gamma - C\tilde{B}(\phi^{1/3})).
\]

The scalar field \( \phi \) is then left completely arbitrary.

However, when one adds a purely gravitational interaction in the action which is not conformally invariant then the field equations are no longer conformally invariant. In this case, one cannot gauge away the warp factor in front of the \((D-2)\)-dimensional constant curvature base manifold. Moreover, the scalar field \( \phi \) is then determined by the field equations.

5. Conclusions

Here, we have presented a novel construction of conformal couplings of a scalar field to arbitrary higher order Euler densities. This is done by first constructing a four-rank tensor linear in the curvature which transforms covariantly under conformal transformations and has the symmetries of the Riemann tensor (except the Bianchi identity). This tensor along with the generalized Kronecker delta is then used to construct conformal invariants of higher order in parallel with the construction of Euler densities. The resulting energy–momentum tensor is shown to be of second order. We further solve the equations of motion under spherically symmetric conditions.

Let us now briefly mention some of the potential future directions of study where this work could be of some relevance.
Firstly, as mentioned in the introduction, the usual conformally coupled scalar field was originally studied in the context of black hole no-hair theorems. The BBMB or Bekenstein black hole circumvents the no-hair theorem since the scalar field diverges at the horizon. However, it was shown that the higher dimensional generalization of this solution does not represent a black hole [4]. Thus, the Bekenstein black hole is the only known, asymptotically flat, static black hole with a conformal scalar hair\(^1\). So, it is natural to look for black hole with a conformal scalar hair in higher dimensions where the coupling involve higher curvature terms.

Secondly, the scalar fields discussed here falls into the class of Galileons which are scalars whose equations of motion depend only on second derivatives. Hence, in flat space these are invariant under constant shifts of the fields and their gradients. These fields have gained some attention in the community due to various intriguing properties and their applications in particle physics and cosmology (see e.g. [11–14]). It has also been shown that the Galileons can be obtained through a standard Kaluza–Klein reduction of higher order Lovelock gravity [15]. This naturally raises a question about the compactification such that the Galileons have a conformal invariance.

Thirdly, one may note that there is a one-to-one correspondence of the spherically symmetric solutions of Lovelock theories to those of \(T^{(kq)}_p = 0\). However, in general Lovelock theories in odd dimensions, there is an enhancement of symmetry when the coupling constants are tuned such that the theory has a unique vacuum. This happens when the corresponding polynomial satisfied by the unknown metric function in static coordinates has a unique solution. Specifically, in odd dimensions, it has been shown that the theory can then be written as a Chern–Simons gauge theory [16, 17]. So, it is natural to investigate the role of any such enhancement of symmetry when the polynomial (34) has a unique solution.

Finally, even though in this work we have considered conformal couplings to Euler densities only, one can also use tensor (3) to construct other interesting couplings.

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Appendix A. Examples

Here, we provide the explicit expressions for the conformal couplings to the Euler densities up to the first few orders. Let us choose the conformal weight to be \(s = 1 - D/2\). Then, the coupling to the cosmological constant is given by

\[
\sqrt{-g} \, \phi \overset{\text{de}}{=} ,
\]  

(A.1)

that to the Einstein–Hilbert term is

\[
\frac{1}{2} \sqrt{-g} \, \delta_{\alpha \beta} S^\phi_{\alpha \beta} = \sqrt{-g} \left[ R \phi^2 - \frac{4(D-1)}{D-2} \phi \Box \phi \right] ,
\]

(A.2)

\(^1\) Recently a spacetime belonging to the family of the Plebanski–Demianski metrics was found to be a solution of the system [9], which reduces to the C-metric in the nonrotating case (also found in [10]). It is interesting to note, that the usual conical singularity in the C-metric is removed by the presence of the scalar field.
and that to the Gauss–Bonnet term is
\[
\frac{1}{4} \sqrt{-g} \phi^{-\frac{1}{2}} \delta_{a_1 b_1 c_1 d_1}^{e_1 f_1 g_1 h_1} \delta_{a_2 b_2 c_2 d_2}^{e_2 f_2 g_2 h_2} \left( R_{a b}^{\, c d} R_{c d}^{\, a b} - 4 R_{i j} R^{i j} + R^2 \right) \phi^4 \\
+ 8 \frac{D - 3}{(D - 2)} \left( 2 \phi^2 R_{a l}^{\, b l} \left( \phi \nabla^a \nabla^b \phi - \frac{D}{D - 2} \nabla^a \phi \nabla^b \phi \right) \\
- \phi^2 R(\phi \square \phi - \frac{2}{D - 2} \nabla^a \phi \nabla^a \phi) - 2 \phi^2 \nabla^a \phi (\nabla^a \nabla^b \phi - g_{a b} \square \phi) \\
+ \frac{4}{D - 2} \phi \nabla^a \phi (D \nabla^a \phi \nabla^b \nabla^b \phi - \nabla^a \phi \square \phi) - \frac{2 D(D - 1)}{(D - 2)^2} (\nabla^a \phi \nabla^a \phi)^2 \right) .
\]
(A.3)

Appendix B. A useful identity

As noted previously, the energy–momentum tensor for the scalar field is of second order in the fields. This can be checked by using the following identity. First let us define a four-rank tensor:
\[
X_{a_1 b_1}^{\, c_1 d_1} = \delta_{a_1 b_1}^{e_1 f_1 g_1 h_1} \delta_{a_2 b_2}^{c_2 d_2} \cdots Z_{a_1 b_1}^{c_1 d_1} .
\]
(B.1)

Then
\[
\nabla_a X^{a b}_{c d} = -(D - 2 k + 1) X^{e b}_{c d} + 2 \phi^2 X^{a b}_{c d} \phi_{c}^{\, c} .
\]
(B.2)

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