Segal–Bargmann transform for unitary groups in the large-$N$ limit

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Mathematical Topics in Quantization
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September 2018
Shameless self-promotion

- “Quantum Theory for Mathematicians” and “Lie Groups, Lie Algebras, and Representations” (both Springer)
OUTLINE:

- Part 1: Segal–Bargmann transform for a compact Lie group
- Part 2: The large-$N$ limit
- Part 3: Application to random matrix theory
Part 1: Segal–Bargmann transform for a compact Lie group
SBT: The set-up

- $K$: connected compact Lie group
- $\mathfrak{k}$: Lie algebra
- $\langle \cdot , \cdot \rangle$: Ad-invariant inner product on $\mathfrak{k}$
- $K_{\mathbb{C}}$: complexification of $K$
- E.g., if $K = SU(2)$, $K_{\mathbb{C}} = SL(2; \mathbb{C})$; if $K = U(N)$, $K_{\mathbb{C}} = GL(N; \mathbb{C})$
\[ \Delta = \text{Laplacian on } K \]
\[ e^{t\Delta/2} : \text{heat operator on } K \]
\[ C_t : L^2(K) \rightarrow \mathcal{H}(K_C) \text{ defined by} \]
\[ C_t f = (e^{t\Delta/2} f)_C, \quad t > 0. \]

\[ (\cdot)_C \] denotes analytic continuation from \( K \) to \( K_C \)
Theorem (H, 1994)

There exists a measure \( \nu_t \) on \( K_C \) for which

\[
C_t : L^2(K) \rightarrow \mathcal{H}L^2(K_C, \nu_t)
\]

is a unitary map.

- \( \nu_t \) is a **heat kernel measure** on \( K_C \)
- Idea: Replace all Gaussians in the classical SBT with heat kernels
Same construction gives classical SBT on $\mathbb{R}^n/\mathbb{C}^n$

$K_{\mathbb{C}}$ identified with $T^*(K)$ ("phase space")

$\nu_t$ and $C_t$ can also be constructed using geometric quantization (H, 2002; Florentino–Mourão–Nunes, 2006)

Related to quantization of $(1+1)$-dimensional Yang–Mills theory on cylinder (Driver–H, 1999)

Gives rise to "coherent states" that have been used in quantum gravity (esp. Thomas Thiemann and collaborators)
Part 2: The large-$N$ limit
Take $K = U(N)$, $K_{\mathbb{C}} = GL(N; \mathbb{C})$

Take $\langle X, Y \rangle_N = N \text{ trace}(X^* Y)$

Thus $\Delta_N = \frac{1}{N} \Delta_{\text{tr}}(X^* Y)$

If $f(U) = U_{ij}$ then

$$\Delta_N(U_{ij}) = \underbrace{(-1)}_{\text{indep. of } N} U_{ij}.$$
Use “B-version” SBT:

\[ B_t : L^2(K, \rho_t) \to \mathcal{H}L^2(K_C, \mu_t) \]

where \( \rho_t \) and \( \mu_t \) are heat kernel measures

- \( B_t \) given by same formula as \( C_t \); only inner products have changed
- Still a unitary map!
- Extend to matrix-valued function “entrywise”
Define

\[ f(U) = U^2, \quad U \in U(N) \]

Can show:

\[
B_t^N(U^2) = e^{-t} \left[ \cosh \left( \frac{t}{N} \right) Z^2 - t \frac{\sinh \left( \frac{t}{N} \right)}{t/N} Z \operatorname{tr}(Z) \right]
\]

\[
(Z \in GL(N; \mathbb{C}))
\]

where \( \operatorname{tr}(\cdot) \) is the normalized trace.

You can all help me take the limit \( N \to \infty \)!
Large-\(N\) limit: Example

- For large \(N\)
  \[
  B_t^N(U^2) \approx e^{-t} \left[ Z^2 - tZ \operatorname{tr}(Z) \right].
  \]

- One more step: As \(N \to \infty\), the measure \(\mu^N_t\) concentrates onto the set where \(\operatorname{tr}(Z) = 1\)

- Thus
  \[
  B_t^N(U^2) \approx e^{-t} \left[ Z^2 - tZ \right]
  \]
  where closeness is measured in \(\mathcal{H}L^2(GL(N; \mathbb{C}), \mu^N_t)\)

- In the limit, only powers of \(Z\), no traces!
Theorem (Driver–H–Kemp, inspired by Biane)

For each polynomial $p$, there exists a polynomial $q_t$ such that

$$B_t^N(p(U)) \approx q_t(Z), \quad Z \in GL(N; \mathbb{C}).$$

- If $p(u) = u^2$, then $q_t(z) = e^{-t}(z^2 - tz)$
- Notation: $q_t = G_t(p)$
- Have explicit generating function for the polynomials
- Proof based on large-$N$ behavior of Laplacian on “trace polynomials”
Part 3: Application to random matrix theory
Random matrices: Biane’s measure $\gamma_t$

- Probability measure $\gamma_t$ on $S^1$
- Describes large-$N$ eigenvalue distribution for random matrices from $(U(N), \rho_t)$
- $\gamma_t(E) = \text{limiting fraction of eigenvalues in } E \subset S^1$
- Fact: for $t < 4$, have

$$\text{supp}(\gamma_t) \not\subset S^1$$
Random matrices: Biane’s measure $\gamma_t$
Biane showed that $G_t$ extends to a map

$$G_t : L^2(S^1, \gamma_t) \rightarrow \mathcal{H}(\Sigma_t)$$

where $\Sigma_t$ is a certain domain in the plane

Map satisfies

$$\|f\|_{L^2(S^1, \gamma_t)}^2 = \lim_{N \rightarrow \infty} \int_{U(N)} \|f(u)\|_{HS}^2 \, d\rho_t(u)$$

$$= \lim_{N \rightarrow \infty} \int_{GL(N)} \|G_t f(Z)\|_{HS}^2 \, d\mu_t(Z)$$

First equality is by the definition of $\gamma_t$ and second by unitarity of $B_t$
Define

\[ f_t(\lambda) = \lambda \exp \left\{ \frac{t \lambda + 1}{2 \lambda - 1} \right\} \]

If \(|\lambda| = 1\) then \(|f_t(\lambda)| = 1\)

But there are other points where \(|f_t(\lambda)| = 1\)

Then

\[ \partial \Sigma_t = \{ \lambda \in \mathbb{C} | \ |f_t(\lambda)| = 1 \text{ and } |\lambda| \neq 1 \} \]
Random matrices: properties of $\Sigma_t$

- Simply connected for $t \leq 4$, doubly connected for $t > 4$ (unit circle in black)
Random matrices: properties of $\Sigma_t$

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Figure: $t = 4.01$
Random matrices: properties of $\Sigma_t$

- **Key point:** $f_t$ maps $\mathbb{C} \setminus \tilde{\Sigma}_t$ conformally onto $\mathbb{C} \setminus \text{supp}(\gamma_t)$
- You will remember this, right?
Random matrices: Theorem

**Theorem (H–Kemp)**

Eigenvalues of random matrices from \((GL(N; \mathbb{C}), \mu_t)\) cluster into \(\bar{\Sigma}_t\) as \(N \to \infty\)

- Interpretation:

  \[ f_t(\text{resolvent set for } GL(N; \mathbb{C})) = \text{resolvent set for } U(N) \]

  (for large \(N\)).
Simulations of eigenvalues for $N = 2,000$ and $t = 3.0, 4.0$

$t = 3$

$t = 4$
If \( \lambda \notin \bar{\Sigma}_t \) then

\[
G_t^{-1}(((z - \lambda)^{-1}) = \frac{f_t(\lambda)}{\lambda} \frac{1}{u - f_t(\lambda)}, \quad u \in \text{supp}(\gamma_t) \subset S^1.
\]

Since \( \lambda \notin \bar{\Sigma}_t \) then \( f_t(\lambda) \notin \text{supp}(\gamma_t) \), so RHS is well-defined in \( L^2(S^1, \gamma_t) \).

Hence for \( \lambda \notin \bar{\Sigma}_t \), have

\[
\lim_{N \to \infty} \int_{\text{GL}(N)} \|(Z - \lambda)^{-1}\|_{HS}^2 \ d\mu_t(Z) = \|\ast\|^2 < \infty
\]
But

\[ \| (Z - \lambda)^{-1} \|_{HS}^2 \]

blows up as \( \lambda \) approaches spectrum of \( Z \)

Thus, previous result is possible only if \( \lambda \) is (with high probability as \( N \to \infty \)) not an eigenvalue of \( Z \in (GL(N; \mathbb{C}), \mu_t) \)
The heat kernel measure $\mu^N_t$ is distribution of Brownian motion $b^N_t$ in $GL(N; \mathbb{C})$

Then $b^N_t$ converges to a “free multiplicative Brownian motion” $b_t$ as $N \to \infty$ (element of an operator algebra)

Then $b_t$ has a Brown measure, which is a sort of “eigenvalue distribution” for the operator $b_t$

Our theorem: Support of Brown measure is in closure of $\Sigma_t$

Proof is mostly as indicated, but really need to let $N \to \infty$ first
• Joint work with Driver and Kemp: preliminary results and conjectures about actual distribution of eigenvalues in $\Sigma_t$.
• More precisely: computing the Brown measure of $b_t$ (not just its support)
• Expect: limiting eigenvalue distribution for matrices in $GL(N; \mathbb{C})$ will coincide with the Brown measure
THANK YOU FOR YOUR ATTENTION!