Dark energy cosmology from higher-order string-inspired gravity and its reconstruction

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In this paper we investigate the cosmological effects of modified gravity with string curvature corrections added to Einstein-Hilbert action in the presence of a dynamically evolving scalar field coupled to Riemann invariants. The scenario exhibits several features of cosmological interest for late universe. It is shown that higher order stringy corrections can lead to a class of dark energy models consistent with recent observations. The model can give rise to quintessence without recourse to scalar field potential. The detailed treatment of reconstruction program for general scalar-Gauss-Bonnet gravity is presented for any given cosmology. The explicit examples of reconstructed scalar potentials are given for accelerated (quintessence, cosmological constant or phantom) universe. Finally, the relation with modified $F(G)$ gravity is established on classical level and is extended to include third order terms on curvature.

PACS numbers: 11.25.-w, 95.36.+x, 98.80.-k

I. INTRODUCTION

One of the most remarkable discoveries of our time is related to the late time acceleration of our universe which is supported by observations of high redshift type Ia supernovae treated as standardized candles and, more indirectly, by observations of the cosmic microwave background and galaxy clustering. The criticality of universe supported by CMB observations fixes the total energy budget of universe. The study of large scale structure reveals that nearly 30 percent of the total cosmic budget is contributed by dark matter. Then there is a deficit of almost 70 percent; in the standard paradigm, the missing component is an exotic form of energy with large negative pressure dubbed dark energy [1, 2, 3, 4, 5, 6, 7]. The recent observations on baryon oscillations provide yet another independent support to dark energy hypothesis.

The dynamics of our universe is described by the Einstein equations in which the contribution of energy content of universe is represented by energy momentum tensor appearing on RHS of these equations. The LHS represents pure geometry given by the curvature of space-time. Gravitational equations in their original form with energy-momentum tensor of normal matter can not lead to acceleration. There are then two ways to obtain accelerated expansion, either by supplementing energy-momentum tensor by dark energy component or by modifying the geometry itself.

Dark energy problem is one of the most important problems of modern cosmology and despite of the number of efforts (for a recent review, see [3, 4]), there is no consistent theory which may successfully describe the late-time acceleration of the universe. General Relativity with cosmological constant does not solve the problem because such theory is in conflict with radiation/matter domination eras. An alternative approach to dark energy is related to modified theory of gravity (for a review, see [3]) in which dark energy emerges from the modification of geometry.
of our universe. In this approach, there appears quite an interesting possibility to mimic dark energy cosmology by string theory. It was suggested in refs.\textsuperscript{10,11,12} that dark energy cosmology may result from string-inspired gravity. In fact, scalar-Gauss-Bonnet gravity from bosonic or Type II strings was studied in the late universe \textsuperscript{10,11} (for review of the applications of such theory in the early universe, see \textsuperscript{13}). It is also interesting such theories may solve the initial singularity problem of the standard big-bang model(see \textsuperscript{14} and refs. therein). Moreover, the easy account of next order (third order, Lovelock term) is also possible in this approach (for recent discussion of such gravity, see \textsuperscript{15}).

In this paper we examine string-inspired gravity with third order curvature corrections (scalar-Gauss-Bonnet term and scalar-Euler term) and explore the cosmological dynamics of the system attributing special attention to dark energy (non-phantom/phantom) solutions. We confront our result with the recent observations. We also outline the general program of reconstruction of scalar-Gauss-Bonnet gravity for any \textit{a priori} given cosmology following the method \textsuperscript{16} developed in the scalar-tensor theory.

The paper is organized as follows. In section two, we consider the cosmological dynamics in the presence of string curvature corrections to Einstein-Hilbert action. We analyze cosmological solutions in the FRW background; special attention is paid at dark energy which naturally arises in the model thanks to higher order curvature terms induced by string corrections. Brief discussion on the comparison of theoretical results with recent observations is included.

The stability of dark energy solution is investigated in detail.

Section three is devoted to the study of late-time cosmology for scalar-Gauss-Bonnet gravity motivated by string theory but with the arbitrary scalar potentials. It is explicitly shown how such theory (actually, its potentials) may be reconstructed for any given cosmology. Several explicit examples of dark energy cosmology with transition from deceleration to acceleration and (or) cosmic speed-up (quintessence, phantom or de Sitter) phase or with oscillating (currently accelerating) behavior of scale factor are given. The corresponding scalar potentials are reconstructed. It is shown how such theory may be transformed to modified Gauss-Bonnet gravity which turns out to be just specific parametrization of scalar-Gauss-Bonnet gravity on classical level. Finally, it is shown how to include third order curvature terms in the above construction. Summary and outlook are given in the last section.

II. DARK ENERGY FROM HIGHER ORDER STRING CURVATURE CORRECTIONS

In this section we shall consider higher order curvature corrections to Einstein-Hilbert action. To avoid technical complications we restrict the discussion up to third order Riemann invariants coupled to a dynamical field $\phi$. The cosmological dynamics of the system will be developed in detail and general features of the solutions will be discussed.

It is really interesting that the model can account for recent observations on dark energy.

A. General action

We begin from the following action

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} \phi'' \right] + V(\phi) + \mathcal{L}_c + \mathcal{L}_m, \quad (1)$$

where $\phi$ is a scalar field which, in particular case, could be a dilaton. $\mathcal{L}_m$ is the Lagrangian of perfect fluid with energy density $\rho_m$ and pressure $p_m$. Note that scalar potential coupled to curvature (non-minimal coupling) \textsuperscript{17} does not appear in string-inspired gravity in the frame under consideration.

The quantum corrections are encoded in the term

$$\mathcal{L}_c = \xi_1(\phi)\mathcal{L}_c^{(1)} + \xi_2(\phi)\mathcal{L}_c^{(2)} \quad (2)$$

where $\xi_1(\phi)$ and $\xi_2(\phi)$ are the couplings of the field $\phi$ with higher curvature invariants. $\mathcal{L}_c^{(1)}$ and $\mathcal{L}_c^{(2)}$ are given by

$$\mathcal{L}_c^{(1)} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \quad (3)$$

$$\mathcal{L}_c^{(2)} = E_3 + R^{\alpha\beta\rho\sigma}R_{\lambda\rho\sigma}R_{\lambda\beta}, \quad (4)$$

The third order Euler density $E_3$ is proportional to

$$E_3 \propto \epsilon^{\mu\nu\rho\sigma\tau\sigma'}\epsilon_{\mu'\nu'\rho'\sigma'\tau'\sigma''} R_{\mu\nu'} R_{\rho\sigma'} R_{\tau\tau'} R_{\sigma''}, \quad (5)$$
Since there does not exist $\epsilon^{\mu\nu\sigma\tau\eta}$ if the space time dimension $D$ is less than 6; $E_3$ should vanish when $D < 6$, especially in four dimensions. By using

$$
epsilon^{\mu\nu\sigma\tau\eta} = \delta^\mu_\nu \delta^\tau_\eta - \delta^\nu_\tau \delta^\mu_\eta \pm \text{(permutations)},$$

we can rewrite the expression \(E_3\) as

$$E_3 \propto 8 \left( R^3 - 12 R R_{\mu\nu} R_{\mu\nu} + 3 R R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + 16 R_{\mu\nu} R_{\rho\sigma} R_{\mu\nu}^\rho R_{\rho\sigma} + 24 R_{\mu\nu}^\rho R_{\rho\sigma} R_{\mu\nu}^\sigma R_{\nu\sigma}^\rho - 24 R_{\mu\nu} R_{\mu\nu}^\rho R_{\rho\sigma} R_{\nu\sigma}^\rho + 2 R_{\mu\nu} \rho^\sigma R_{\rho\sigma} R_{\mu\nu}^\tau R_{\tau\eta}^\nu - 8 R_{\mu\nu} R_{\rho\sigma} R_{\mu\nu}^\rho R_{\rho\sigma} R_{\nu\sigma}^\nu \right).$$

We should note in the r.h.s. of (8), there appears \(6! = 720\) terms, which correspond to the sum of the absolute values of the coefficients in each term in the RHS of (7).

In what follows we shall be interested in the cosmological applications of modified equations of motion and thus assume a flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -N^2(t)dt^2 + a^2(t) \sum_{i=1}^{d} (dx_i)^2,$$

where $N(t)$ is the lapse function. With the metric (9), the Riemann invariants read

$$\mathcal{L}_c^{(1)} = 24 H^2 \left( \frac{\dot{H} + H^2}{N^4} - \frac{\dot{N}}{N^5} H \right), \quad \mathcal{L}_c^{(2)} = 24 \frac{N}{N^6} (H^6 + I^3) - \frac{72 N}{N^7} H I^2$$

where $I = \dot{H} + H^2$ and $H = \dot{a}/a$. It is straightforward though cumbersome to verify explicitly that third order Euler density $E_3$ is identically zero in the FRW background. The non-vanishing contribution in Eq. (10) comes from the second term in (11). To enforce the check in a particularly case, we consider $D$ dimensional de-Sitter space, where Riemann curvature is given by

$$R_{\mu\nu}^{\rho\sigma} = H_0 \left( \delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho \right).$$

Here $H_0$ is a constant corresponding to the Hubble rate. In the de-Sitter background we have

$$E_3 \propto D(D-1)(D-2)(D-3)(D-4)(D-5),$$

which is obviously zero in case of $D < 6$. For simplicity we shall limit the discussion to a homogeneous scalar field $\phi(t)$. Then the spatial volume can be integrated out from the measure in equation (11), which we rewrite as

$$S = \int dt Na^3 \left[ \frac{R}{2\kappa^2} + \mathcal{L}_c + \mathcal{L}_\phi + \mathcal{L}_m \right].$$

where $\mathcal{L}_\phi = -\frac{1}{2} \omega(\phi)(\nabla \phi)^2 - V(\phi)$. Varying the action (13) with respect to the lapse function $N$ we obtain (11)

$$\frac{3 H^2}{\kappa^2} = \rho_m + \rho_\phi + \rho_c$$

where

$$\rho_\phi = \frac{1}{2} \omega(\phi)^2 + V(\phi)$$

In Eq. (14), the energy density $\rho_c$ is induced by quantum corrections and is given by the following expression

$$\rho_c = \left( 3 H^2 \frac{\partial \mathcal{L}_c}{\partial N} + \frac{d}{dt} \frac{\partial \mathcal{L}_c}{\partial \dot{N}} - \frac{\partial \mathcal{L}_c}{\partial N} - \mathcal{L}_c \right) \bigg|_{N=1}$$

It would be convenient to rewrite $\rho_c$ as

$$\rho_c = \xi_1(\phi)\rho_c^{(1)} + \xi_2(\phi)\rho_c^{(2)}$$
Using Eqs. \((10)\) \& \((10)\) we obtain the expressions of \(\rho_c^{(1)}\) and \(\rho_c^{(2)}\)

\[
\rho_c^{(1)} = -24H^3\xi_1
\]

\[
\rho_c^{(2)} = -72HI^2\xi_2 - 72\left((HI^2 + 2IH) - 216H^2I^2 + 120\left(H^6 + I^3\right)\right)
\]

where \(\xi_1 = \dot{\xi}_1/\xi_1\) and \(\xi_2 = \dot{\xi}_2/\xi_2\). It is interesting to note that the contribution of Gauss-Bonnet term (described by Eq. \((10)\)) cancels in equations of motion for fixed \(\phi\) as it should be; it contributes for dynamically evolving scalar field only. In case of the third order curvature corrections, the Euler density is identically zero and hence it does not contribute to the equation of motion in general. Secondly, \(\mathcal{L}_c^{(2)}\) contributes for fixed field as well as for dynamically evolving \(\phi\). It contains corrections of third order in curvature beyond the Euler density.

We should note that such higher-derivative terms in string-inspired gravity may lead to ghosts and related instabilities (for recent discussion in scalar-Gauss-Bonnet gravity, see [18]). However, the ghost spectrum of such (quantum) gravity (for the review, see [19]) is more relevant at the early universe where curvature is strong, but less relevant at late universe. Moreover, in accordance with modified gravity approach, the emerging theory is purely classical, effective theory which comes from some unknown gravity which has different faces at different epochs. (Actually, it could be that our universe currently enters to instable phase). For instance, in near future the currently sub-leading terms may dominate in the modified gravity action which then has totally different form! Hence, this is that (unknown) gravity, and not its classical limit given by Eq. \((11)\) relevant during specific epoch, whose spectrum should be studied.

The point is best illustrated by the example of Fermi theory of weak interactions whose quantization runs into well known problems. Finally, on the phenomenological grounds, it is really interesting to include higher order terms. At present the situation is remarkably tolerant in cosmology, many exotic constructions attract attention provided they can lead to a viable model of dark energy.

The equation of motion for the field \(\phi\) reads from \(13\)

\[
\omega(\dot{\phi} + 3H\phi) + V' - \xi_1\mathcal{L}_c^{(1)} + \xi_2 \mathcal{L}_c^{(2)} + \omega ' \dot{\phi} - \omega ' \dot{\phi}^2 = 0
\]

(20)

In addition we have standard continuity equation for the barotropic background fluid with energy density \(\rho_m\) and pressure \(p_m\)

\[
\rho_m' + 3H(\rho_m + p_m) = 0
\]

(21)

Equations \((13), (20)\), and \((21)\) are the basic equations for our system under consideration.

Let us note that in the string theory context with the dilaton field \(\phi\) we have

\[
V(\phi) = 0, \quad \xi_1 = c_1\alpha' e^{2\phi/\phi_0}, \quad \xi_2 = c_2\alpha'^2 e^{4\phi/\phi_0}
\]

(22)

where \((c_1, c_2) = (0, 0, 1/8), (1/8, 0, 1/8)\) for Type II, Heterotic, and Bosonic strings, respectively.

**B. Fixed field case: general features of solutions.**

We now look for de-Sitter solutions in case of \(\phi = \text{constant}\) and \(\rho_m = 0\). In this case the modified Hubble Eqs.\((14)\) gives rise to de-Sitter solution

\[
3 = 24\xi_2H^4 \quad \text{or} \quad H = \left(\frac{1}{8\xi_2}\right)^{1/4}
\]

(23)

where \(\xi_2 = \frac{1}{8}exp(-4\phi/\phi_0)\) for type II and Bosonic strings. Normalizing \(\xi_2\) to one, we find that \(H = 0.6\) (we have set \(\kappa^2 = 1\) for convenience). Below we shall discuss the stability of the solution. There exists no de-Sitter solution for Heterotic case. Actually, de-Sitter solutions were investigated in similar background in Ref.\([11]\) where higher order curvature corrections up to order four were included. Since, here we confine ourselves up to the third order and the fourth order terms are excluded from the expression of \(\rho_c\); these terms come with different signs. Thus it becomes necessary to check whether or not the stability property of de-Sitter solutions is preserved order by order.

We further note that the modified Hubble Eqs.\((14)\) admits the following solution in the high curvature regime in presence of the barotropic fluid with equation of state parameter \(w\)

\[
a(t) = a_0 t^{h_0}, \quad \text{or} \quad a(t) = a_0(t_s - t)^{h_0}
\]

(24)
phantom matter, the effective EoS being less than $-1$ is typical for Big Rip singularity. It is really not surprising that we have inflationary solution at early epochs in the presence of higher order curvature correction to Einstein Hilbert action; an early example of this phenomenon is provided by $R^2$-gravity.

C. Autonomous form of equations of motion

Let us now cast the equations of motion in the autonomous form. Introducing the following notation ($\kappa^2 = 1$)

$$x = H, \quad y = \dot{H}, \quad u = \phi, \quad v = \dot{\phi}, \quad \omega = \rho_m$$

We shall assume $\omega(\phi) = \nu = \text{const.}$ we obtain the system of equations

$$\dot{x} = y,$$
$$\dot{y} = \frac{1}{2} v^2 - 24 \xi_1 \xi_2 \frac{\xi_2}{144 I(x, y) x^3} - 72(y I^2 + 4I y x^2) - 216 x^2 I^2 + 120(x^6 + I^3) \right) - 3 x^2 + \frac{z}{144 I_2 x},$$
$$\dot{u} = v, \quad \dot{v} = \frac{-3 \nu x v + \xi_1 L_2^{(1)} + \xi_2 L_2^{(2)}}{\nu}, \quad \dot{z} = -3x(1 + w)z$$

We shall be first interested in the case of fixed field for which we have (assuming $\nu = 1$)

$$\dot{x} = y,$$
$$\dot{y} = \frac{-72(y I(x, y)^2 + 4I(x, y) y x^2) + 216 x^2 I^2(x, y) + 120(x^6 + I^3(x, y)) - 3 x^2 + z}{144 I(x, y) x},$$
$$\dot{z} = -3x(1 + w)z$$

where

$$I(x, y) = x^2 + y, \quad L_2^{(1)} = 24 x^2 (y + x^2), \quad L_2^{(2)} = 24 (x^6 + I^3(x, y))$$

In the presence of perfect fluid, the de-Sitter fixed point is characterized by

$$x_c = 0.71, \quad y_c = 0, \quad z_c = 0$$

Perturbing the system around the critical point and keeping the linear terms we obtain

$$\delta x = \delta y,$$
$$\delta y = \left( \frac{21}{3} x_c^2 + \frac{10}{3 x_c} + \frac{1}{48 x_c^2} \right) \delta x + \left( \frac{2}{3} x_c + \frac{5}{6 x_c} + \frac{1}{48 x_c^2} \right) \delta y + \frac{1}{144 x_c^2} \delta z,$$
$$\delta z = -3x_c(1 + w)\delta z$$

Stability of the fixed points depends upon the nature of eigenvalues of perturbation matrix

$$\lambda_{1,2} = \frac{1}{2} \left( a_{22} \pm \sqrt{4a_{21} + a_{22}^2} \right), \quad \lambda_3 = a_{33} = -3x_c(1 + w)$$

For the fixed point given by $\xi_2^{(0)}$, $\lambda_1$ is positive where as $\lambda_2$ is negative making the de-Sitter solution an unstable node. In fact, $\lambda_1$ remains positive for any $x_c > 0$ thereby making the conclusion independent of the choice of $\xi_2^{(0)}$ (see FIG. 11).
D. Dynamically evolving field $\phi$ and dark energy solutions

In what follows we shall be interested in looking for an exact solution of equations of motion (14) and (20) which of interest to us from the point of view of dark energy in absence of the background fluid. In this case let us look for the following solution

$$H = \frac{h_0}{t}, \quad \phi = \phi_0 \ln \frac{t}{t_1} \quad \text{(when } h_0 > 0), \quad H = \frac{h_0}{t_s - t}, \quad \phi = \phi_0 \ln \frac{t_s - t}{t_1} \quad \text{(when } h_0 < 0).$$

(35)

Substituting (35) in evolution Eqs. (14) and (20) yields (we again set $\kappa^2 = 1$)

$$\nu(1 - 3h_0)\phi_0^2 + \frac{48\xi^{(1)}_0}{t_1^2} h_0^3 (h_0 - 1) + \frac{96\xi^{(0)}_0}{t_1^4} (h_0^6 + I_0^3) = 0,$$

$$-3h_0^2 + \frac{\nu}{2} \phi_0^2 - \frac{48\xi^{(0)}_1}{t_1^2} h_0^3 + \frac{\xi^{(0)}_2}{t_1^4} J(h_0) = 0$$

(36)

where

$$J = \frac{1}{96} \left( -288h_0 I_0^2 + 72(-h_0 I_0^2 + 2I_0h_0\dot{I}_0) - 216h_0 I_0^2 + 120(h_0^6 + I_0^3) \right),$$

$$I_0 = h_0(h_0 - 1), \quad \dot{I}_0 = -2h_0(h_0 - 1)$$

(37)
Using Eqs. (36), we express the couplings through $h$ and $\phi_0$

$$\frac{48\xi_1^{(0)}}{t_1^2} = \left[ \frac{3h_0^2 - \frac{\phi_0^2}{2} + \nu(3h_0 - 1)\phi_0^2}{J(h_0)(h_0 - 1) + h_0^3(h_0 + (h_0 - 1)^3)} \right].$$

(38)

$$\frac{96\xi_2^{(0)}}{t_1^4} = \frac{1}{h_0^3} \left[ -3h_0^2 + \frac{\nu\phi_0^2}{2} + \left( \frac{3h_0^2 - \frac{\phi_0^2}{2} + \nu(3h_0 - 1)\phi_0^2}{J(h_0)(h_0 - 1) + h_0^3(h_0 + (h_0 - 1)^3)} \right) \right].$$

(39)

Let us note that the string couplings ($\xi_1(\phi) = \xi_1^{(0)} e^{\frac{\phi}{\phi_0}}, \xi_2(\phi) = \xi_2^{(0)} e^{\frac{\phi}{\phi_0}}$ with $m = n^2 = 4$) are generic to solution described by (35); for other couplings such a solution does not exist. We also note that Eqs. (38) & (39) reduce to the earlier obtained results in Ref. [10] (see Refs. [13, 20, 21] on the related theme) where similar investigations were carried out confining to only second order curvature invariants in the action (1).

There are several free parameters in the problem. In order to extract important information from Eqs. (38) and (39), we proceed in the following manner. We fix $h_0$ corresponding to the observed value of dark energy equation of state parameter $w_{DE}$ and impose the positivity condition on the couplings $\xi_1^{(0)}$ and $\xi_2^{(0)}$ leading to allowed values of the parameter $\phi_0^2$. In the absence of coupling $\xi_2(\phi)$, it was shown in Ref. [10] that for a given value of $h_0$ from the allowed interval, the parameter $\phi_0$ takes a fixed value. Our model incorporates higher order curvature corrections allowing a one parameter flexibility in the values of $\phi_0$. This gives rise to comfortable choice of the equation of state consistent with observations.

The three years WMAP data is analyzed in Ref. [22], which shows that the combined analysis of WMAP with supernova Legacy survey (SNLS) constrains the dark energy equation of state $w_{DE}$ pushing it towards the cosmological constant. The marginalized best fit values of the equation of state parameter at 68% confidence level are given by $-1.14 \leq w_{DE} \leq -0.93$. In case of a prior that universe is flat, the combined data gives $-1.06 \leq w_{DE} \leq -0.90$. Our model can easily accommodate these values of $w_{DE}$. For instance, in case of non-phantom (standard dark energy) we find a one parameter family of dark energy models with $h_0 \approx 40$ ($w_{DE} = -0.98$) corresponding to $\phi_0^2 > 11$. Likewise, in case of phantom dark energy, we find that for $h_0 \approx -33$ ($w_{DE} = -1.02$), the viable range of the parameter $\phi_0^2$ is given by $6 < \phi_0^2 < 31.5$. These values are consistent with recent WMAP release and SNLS findings.

We should mention that the observations quoted above do not incorporate the dark energy perturbations which might severely constrain the phantom dark energy cosmologies. The combined data (CMB+LSS+SNLS) then forces the dark energy equation of state to vary as, $-1.001 < w_{DE} < -0.875$ [22]. Our model can easily incorporate these numerical values of $w_{DE}$ by constraining $h_0$ and $\phi_0^2$ similar to the case of non-clustering dark energy. A word of caution, the evolution of dark energy perturbation across the phantom divide needs additional assumptions; a complete analysis should take into account the non-adiabatic perturbations which makes dark energy gravitationally stable [1].
### Observational constraint on $w_{DE}$ with flatness prior

| Dark energy | $h_0$ | $\phi_0^2$ | $w_{DE}$ | Observational constraint on $w_{DE}$ | Constraint on $w_{DE}$ with flatness prior |
|-------------|-------|-------------|----------|--------------------------------------|------------------------------------------|
| Non-phantom | 40    | $\phi_0^2 > 41$ | $-0.98$  | $-1.00^{+0.13}_{-0.08}$                   | $-0.97^{+0.07}_{-0.09}$                   |
| Phantom     | 33.33 | $6 < \phi_0^2 < 31.5$ | $-1.02$  |                                      |                                          |

**TABLE I:** Observational constraints on (non-clustering) dark energy equation of state $w_{DE}$ dictated by the combined analysis of WMAP+SNLS data\cite{22} and the numerical values of model parameters consistent with the observations.

### E. Stability of dark energy solution

In what follows we shall examine the stability of the dark energy solution \cite{35} induced by purely stringy corrections. In general the analytical treatment becomes intractable; simplification, however, occurs in the limit of large $h_0$ corresponding to $w_{eff} \simeq -1$.

Let us consider the following situation of interest to us

$$\rho_m = 0, \quad \omega = \nu = 1, \quad \xi_1(\phi) = \xi_1(0)e^{2\phi/\phi_0}, \quad \xi_2(\phi) = \xi_2(0)e^{4\phi/\phi_0}. \tag{40}$$

In order to investigate the stability around the dark energy solution defined by (35), we need a convenient set of variables to cast the evolution equations into autonomous form. We now define the variables which are suited to our problem.

$$\mathcal{X} = \frac{\dot{\phi}}{H}, \quad \mathcal{Y} = \left(\dot{H} + H^2\right)^2 \xi_2(\phi), \quad \mathcal{Z} = H^2 \xi_1(\phi), \quad \frac{d}{dN} \equiv \frac{1}{H} \frac{d}{dt}. \tag{41}$$

With this choice, the evolution equations acquire the autonomous form

$$\frac{d\mathcal{X}}{dN} = -2\mathcal{X} + \xi_1(0)^2 \mathcal{Z} \mathcal{Y} - 48 \frac{\mathcal{Y}}{\phi_0} \left(\frac{\mathcal{Y}^2}{\mathcal{Z}^2} + \frac{96}{\phi_0^2} \frac{\xi_2(0)^2}{\xi_1(0)} Z^2 + \frac{96\xi_1(0)}{\phi_0} \mathcal{Y} \right) \frac{1}{\xi_2(0)} \frac{1}{\xi_2(0)}, \tag{42}$$

$$\frac{d\mathcal{Y}}{dN} = -\frac{1}{24}\frac{\mathcal{X}}{\mathcal{Z}} - \frac{1}{2} \frac{\mathcal{Z}}{\mathcal{X}} \mathcal{X} - 2\mathcal{Y} + \frac{2\xi_1(0)\mathcal{Y}}{3\mathcal{Z}} \left(\frac{\mathcal{Y}^2}{\xi_2(0)} \right) \frac{1}{\xi_2(0)} + \frac{5\xi_2(0)}{3\xi_1(0)} Z^2, \tag{43}$$

$$\frac{d\mathcal{Z}}{dN} = \left(\frac{1}{2} - \frac{2}{\phi_0}\mathcal{X}\right) \mathcal{Z} + 2\xi_1(0) \mathcal{Y} \frac{1}{\xi_2(0)} . \tag{44}$$

We have used the field equation \cite{20} and Eq.\cite{20} for $\dot{H}$ in deriving the above autonomous form of equations. For our solution given by \cite{35}, we have

$$\mathcal{X} = X_0 \equiv \frac{\phi_0}{h_0}, \quad \mathcal{Z} = Z_0 \equiv \frac{h_0^2 \xi_1(0)}{t_1^2}, \quad \mathcal{Y} = Y_0 = \frac{(-h_0 + h_0^2)^2}{t_1^2} \xi_2(0). \tag{45}$$

It can be checked that $(\mathcal{X}_0, \mathcal{Y}_0, \mathcal{Z}_0)$ is a fixed point of (42). We then consider small perturbations around (45) or equivalently around the original solution \cite{35}

$$\mathcal{X} = \mathcal{X}_0 + \delta\mathcal{X}, \quad \mathcal{Y} = \mathcal{Y}_0 + \delta\mathcal{Y}, \quad \mathcal{Z} = \mathcal{Z}_0 + \delta\mathcal{Z}. \tag{46}$$

Substituting (45) in (42) and retaining the linear terms in perturbations, we find

$$\frac{d}{dN} \left( \begin{array}{c} \delta\mathcal{X} \\ \delta\mathcal{Y} \\ \delta\mathcal{Z} \end{array} \right) = M \left( \begin{array}{c} \delta\mathcal{X} \\ \delta\mathcal{Y} \\ \delta\mathcal{Z} \end{array} \right). \tag{47}$$

Here $M$ is $3 \times 3$-matrix perturbation matrix whose components are given by

$$M_{11} = -2 + \frac{-1 + h_0}{h_0} ,$$
Stability of the fixed point(s) depends upon the nature eigenvalues of the perturbation matrix $M$. If there is an eigenvalue whose real part is positive, the system becomes unstable. Here for simplicity, we only consider the case of $h_0 \to \pm \infty$, which corresponds to the limit of $w_{\text{eff}} \sim -1$. In the case, we find

$$\frac{\xi_1^{(0)}}{t_1^2} \to \frac{1}{4h_0^2}, \quad \frac{\xi_2^{(0)}}{t_1^2} \to -\frac{1}{32h_0},$$

and the eigenvalue equation is given by

$$0 = F(\lambda) \equiv -\lambda^2 - 6\lambda^2 - \frac{h_0^3}{40h_0} \lambda - \frac{7h_0^3}{40\phi_0}.$$  \hfill (47)

The values of $\lambda$ satisfying $F(\lambda) = 0$ give eigenvalues of $M$. The solutions of (48) is given by

$$\lambda = \lambda_+ \equiv \pm \frac{|h_0| \sqrt{-h_0}}{\phi_0 \sqrt[4]{40}} + \mathcal{O}(|h_0|), \quad \lambda = \lambda_- \equiv -7 + \mathcal{O}\left(|h_0|^{-1/2}\right).$$  \hfill (49)

When $h_0 < 0$, the mode corresponding to $\lambda_+$ ($\lambda_-$) becomes stable (unstable). Since $\lambda_\pm$ are pure imaginary when $h_0 > 0$, the corresponding modes become stable in this case. On the other hand, the mode corresponding to $\lambda_0$ is always stable. Thus, the non-phantom dark energy solution induced by string corrections to Einstein gravity is stable. Such a solution exists in presence of a dynamically evolving field $\phi$ with $V(\phi) = 0$ coupled to Riemann invariants with couplings dictated by string theory. Dark energy can be realized in a variety of scalar field models by appropriately choosing the field potential. It is really interesting that we can obtain dark energy solution in string model without recourse to a scalar field potential.

Let us compare the results with those obtained in Ref. [10], where $\xi_2 = 0$ but $V(\phi) \neq 0$. The dark energy solution studied in Ref. [10], was shown to be stable when $h_0 > 0$ but unstable for $h_0 > 0$. The present investigations include $\xi_2$ and $V = 0$ which makes our model different from Ref. [10]; it is therefore not surprising that our results differ from Ref. [10]. Since $h_0 > 0$ corresponds to the quintessence phase and $h < 0$ to the phantom, the solution in the model (with $\xi_0$ and $V = 0$) is stable in the quintessence phase but unstable in the phantom phase. We should notice that the approximation we used to check the stability works fine for any generic value of $h_0$. For instance, $5 < h_0 < -667$ which corresponds to the variation of $w_{DE}$ in case the dark energy perturbations are taken into account. We also carried out numerical verification of our results.

### III. The Late-Time Cosmology in Scalar-Gauss-Bonnet Gravity

A number of scalar field models have recently been investigated in connection with dark energy (see Ref. [4] for details). The cosmological viability of these constructs depends upon how well the Hubble parameter predicted by
them compares with observations. One could also follow the reverse route and construct the Lagrangian using the observational input; such a scheme might help in the search of best fit models of dark energy\textsuperscript{[4]}. In what follows we shall describe how the reconstruction program is implemented in presence of higher order string curvature corrections.

### A. The reconstruction of scalar-Gauss-Bonnet gravity

In this section it will be shown how scalar-Gauss-Bonnet gravity may be reconstructed for any requested cosmology using the method\textsuperscript{[10]} developed in the scalar-tensor theory. We limit here only by Gauss-Bonnet term order (by technical reasons) but there is no principal problem to include higher order terms studied in previous section. It is interesting that the principal possibility appears to reconstruct the scalar-Gauss-Bonnet gravity for any (quintessence, cosmological constant or phantom) dark energy universe. The last possibility seems to be quite attractive due to the fact\textsuperscript{[10]}, that the phantom universe could be realized in the scalar-Gauss-Bonnet gravity without introducing ghost scalar field. In this section, we show that in scalar-Gauss-Bonnet gravity, any cosmology, including phantom cosmology, could be realized by properly choosing the potential and the coupling to the Gauss-Bonnet invariant with the canonical scalar.

The starting action is

$$S = \int d^4x\sqrt{-g} \left[ \frac{R}{2\kappa^2} - 1 + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \xi(\phi)G \right].$$

(50)

Here $G$ is the Gauss-Bonnet invariant $G \equiv \mathcal{L}_B^{(1)}$\textsuperscript{[3]} and the scalar field $\phi$ is canonical in\textsuperscript{[50]}. As in previous section, it is natural to assume the FRW universe\textsuperscript{[4]} with $N(t) = 1$ and the scalar field $\phi$ only depending on $t$. The FRW equations look like:

$$0 = -\frac{3}{\kappa^2} H^2 + \frac{1}{2} \dot{\phi}^2 + V(\phi) + 24H^2 \frac{d\xi}{dt}(\phi(t)),$$

(51)

$$0 = \frac{1}{\kappa^2} \left(2 \dot{H} + 3H^2\right) + \frac{1}{2} \dot{\phi}^2 - V(\phi) - 8H^2 \frac{d^2\xi}{dt^2}(\phi(t)) - 16H \frac{d\xi}{dt}(\phi(t)) - 16 \frac{d^3\xi}{dt^3}(\phi(t)).$$

(52)

and scalar field equation:

$$0 = \ddot{\phi} + 3H \dot{\phi} + V'(\phi) + \xi(\phi)G.$$

(53)

Combining (51) and (52), one gets

$$0 = \frac{2}{\kappa^2} \dot{H} + \dot{\phi}^2 - 8H^2 \frac{d^2\xi}{dt^2}(\phi(t)) - 16H \frac{d\xi}{dt}(\phi(t)) + 8H^3 \frac{d^3\xi}{dt^3}(\phi(t)) = \frac{2}{\kappa^2} \dot{H} + \dot{\phi}^2 - 8a \frac{d}{dt} \left( \frac{H^2}{a} \frac{d\xi}{dt}(\phi(t)) \right).$$

(54)

Eq. (54) can be solved with respect to $\xi(\phi(t))$ as

$$\xi(\phi(t)) = \frac{1}{8} \int^t dt_1 \frac{a(t_1)}{H(t_1)^2} \int^t_1 \frac{dt_2}{a(t_2)} \left( \frac{2}{\kappa^2} \dot{H}(t_2) + \dot{\phi}(t_2)^2 \right).$$

(55)

Combining (51) and (55), the scalar potential $V(\phi(t))$ is:

$$V(\phi(t)) = \frac{3}{\kappa^2} H(t)^2 - \frac{1}{2} \dot{\phi}(t)^2 - 3a(t) H(t) \int^t dt_1 \frac{a(t_1)}{a(t)} \left( \frac{2}{\kappa^2} \dot{H}(t_1) + \dot{\phi}(t_1)^2 \right).$$

(56)

We now identify $t$ with $f(\phi)$ and $H$ with $g'(t)$ where $f$ and $g$ are some unknown functions in analogy with Ref.\textsuperscript{[10]} since we know this leads to the solution of the FRW equations subject to existence of such functions. Then we consider the model where $V(\phi)$ and $\xi(\phi)$ may be expressed in terms of two functions $f$ and $g$ as

$$V(\phi) = \frac{3}{\kappa^2} g'(f(\phi))^2 - \frac{1}{2 f'(f(\phi))^2} - 3 g'(f(\phi)) e^{g(f(\phi))} \int^\phi d\phi_1 f'(\phi_1) e^{-g(f(\phi_1))} \times \left( \frac{2}{\kappa^2} g''(f(\phi_1)) + \frac{1}{f'(f(\phi))^2} \right),$$

$$\xi(\phi) = \frac{1}{8} \int^\phi d\phi_1 \frac{f'(\phi_1) e^{g(f(\phi_1))}}{g'(f(\phi_1))^2} \int^\phi_1 d\phi_2 f'(\phi_2) e^{-g(f(\phi_2))} \left( \frac{2}{\kappa^2} g''(f(\phi_2)) + \frac{1}{f'(f(\phi))^2} \right).$$

(57)

By choosing $V(\phi)$ and $\xi(\phi)$ as (57), we can easily find the following solution for Eqs. (51) and (52):

$$\phi = f^{-1}(t) \quad (t = f(\phi)) \quad a = a_0 e^{g(t)} \quad (H = g'(t)).$$

(58)
We can straightforwardly check the solution (68) satisfies the field equation (53).

Hence, any cosmology expressed as $H = g(\phi)$ in the model (51) with (57) can be realized, including the model exhibiting the transition from non-phantom phase to phantom phase without introducing the scalar field with wrong sign kinetic term.

In the Einstein gravity, the FRW equations are given by
\[
0 = -\frac{3}{\kappa^2} H^2 + \rho , \quad 0 = \frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) + p .
\]

Here $\rho$ and $p$ are total energy density and pressure in the universe. By comparing (59) with (68) we find the effective energy density $\tilde{\rho}$ and the pressure $\tilde{p}$ are given
\[
\tilde{\rho} = \frac{3}{\kappa^2} g'(t)^2 , \quad \tilde{p} = \frac{3}{\kappa^2} g'(t)^2 - \frac{2}{\kappa^2} g''(t) .
\]

Since $t = g^{-1} \left( (\kappa) \sqrt{\rho/3} \right)$, we obtain the following effective equation of the state (EoS):
\[
\tilde{p} = -\frac{\tilde{\rho}}{} - \frac{2}{\kappa^2} g'' \left( g^{-1} \left( \kappa \sqrt{\frac{\rho}{3}} \right) \right) ,
\]

which contains all the cases where the EoS is given by $p = w(\rho) \rho$. Furthermore, since $g^{-1}$ could NOT be always a single-valued function, Eq. (61) contains more general EoS given by
\[
0 = F (\tilde{\rho}, \tilde{p}) .
\]

This shows the equivalence between scalar-tensor and ideal fluid descriptions.

Let us come back now to scalar-Gauss-Bonnet gravity. It is not difficult to extend the above formulation to include matter with constant EoS parameter $w_m \equiv p_m/\rho_m$. Here $\rho_m$ and $p_m$ are energy density and pressure of the matter. Then, instead of (61) and (62) the FRW equations are
\[
\dot{\rho}_m + 3H (\rho_m + p_m) = 0 ,
\]
gives
\[
\rho_m = \rho_{m0} a^{-3(1+w_m)} ,
\]
with a constant $\rho_{m0}$. Instead of (57), if we consider the model with
\[
V(\phi) = \frac{3}{\kappa^2} g'(f(\phi))^2 - \frac{1}{2(f'(\phi))^2} - 3g'(f(\phi)) e^{g(f(\phi))} \int_{\phi}^{\phi_1} f'(\phi_1) e^{-g(f(\phi_1))} \left( \frac{2}{\kappa^2} g''(f(\phi_1)) + \frac{1}{2f'(\phi_1)^2} \right) d\phi_1 \\
+ (1 + w_m) g_0 e^{-3(1+w_m)g(f(\phi_1))} ,
\]
\[
\xi_1(\phi) = \frac{1}{8} \int_{\phi}^{\phi_1} f'(\phi_1) e^{g(f(\phi_1))} \int_{\phi_1}^{\phi_2} d\phi_2 f'(\phi_2) e^{-g(f(\phi_2))} \left( \frac{2}{\kappa^2} g''(f(\phi_2)) \right) \\
+ \frac{1}{2f'(\phi_2)^2} \int_{\phi_1}^{\phi_2} (1 + w_m) g_0 e^{-3(1+w_m)g(f(\phi_2))} ,
\]
we re-obtain the solution (68) even if the matter is included. However, a constant $a_0$ is given by
\[
a_0 = \frac{g_0}{\rho_0} .
\]
One can consider some explicit examples \[ 16 \]

\[ t = f(\phi) = \frac{\phi}{\phi_0} , \quad g(t) = h_0 \ln \frac{t}{t_s - t} , \quad (69) \]

which gives

\[ H = h_0 \left( \frac{1}{t} + \frac{1}{t_s - t} \right) , \quad \dot{H} = \frac{h_0 t_s (2t - t_s)}{t^2 (t_s - t)} . \quad (70) \]

Then the universe is in non-phantom phase when \( t < t_s / 2 \) and in phantom phase when \( t > t_s / 2 \). There is also a Big Rip singularity at \( t = t_s \). Especially in case \( w_m = 0 \) (that is, matter is dust) and \( h_0 = 2 \), we reconstruct the scalar-Gauss-Bonnet gravity with following potentials:

\[
V(\phi) = \frac{6\phi_0 \phi_s}{\kappa^2 \phi_s (\phi_s - \phi)} - \frac{1}{2} \phi_0^2 - \frac{4\phi_0^2 \phi_s \phi}{(\phi_s - \phi)} \cdot \frac{4}{\kappa^2} \left( \frac{3 \phi_s^3 - \phi_s}{3 \phi^2} \right) - \frac{2\phi_s \ln \phi}{\phi_s} + \phi \nonumber
\]

\[
+ \frac{g_0^2}{\phi_0^2} \left[ - \frac{\phi_0^8}{5 \phi_s^6} + 4 \phi_0^7 - 28 \phi_0^6 \phi_s^3 + 14 \phi_0^5 \phi_s^3 + 70 \phi_0^4 \phi_s^3 + 28 \phi_0^3 \phi_s^3 - 28 \phi_0^2 \phi_s^3 - 8 \phi_0 \ln \phi \right] + c_1 \n, \quad (71) \nonumber
\]

Here \( \phi_s \equiv \phi_0 t_s \) and \( c_1, c_2 \) are constants of the integration.

Another example, without matter \( (g) \), \[ 23 \]

\[
g(t) = h_0 \left( t + \frac{\cos \theta_0}{\omega} \sin \omega t \right) , \quad f^{-1}(t) = \phi_0 \sin \frac{\omega t}{2} . \quad (72) \nonumber
\]

Here \( h_0, \theta_0, \omega, \) and \( \phi_0 \) are constants. This leads to reconstruction of scalar-Gauss-Bonnet gravity with

\[
V(\phi) = \frac{3h_0}{\kappa^2} \left( 1 + \cos \theta_0 - \frac{2 \cos \theta_0}{\phi_0^2} \phi \right) - \frac{\phi_0^3 \omega^2}{8} \left( 1 - \frac{\phi_0^2}{\phi_0^2} \right)^{1/2} , \nonumber
\]

\[
\xi_1(\phi) = - \frac{\omega_0}{32h_0^2} \int \phi \left( 1 - \frac{\phi^2}{\phi_0^2} \right)^{-1/2} \left( 1 + \cos \theta_0 - \frac{2 \cos \theta_0}{\phi_0^2} \phi \right)^{-2} \, d\phi_1 , \quad (73) \nonumber
\]

Then from Eq. \[ 23 \] we find

\[
H = h_0 (1 + \cos \theta_0 \cos \omega t) \geq 0 , \quad \dot{H} = -h_0 \omega \cos \theta_0 \sin \omega t , \quad (74) \nonumber
\]

Then the Hubble rate \( H \) is oscillating but since \( H \) is positive, the universe continues to expand and if \( h_0 \omega \cos \theta_0 > 0 \), the universe is in non-phantom (phantom) phase when \( 2n\pi < \omega t < (2n + 1)\pi \) \((2n - 1)\pi < \omega t < 2n\pi \) with integer \( n \). Thus, the oscillating late-time cosmology in string-inspired gravity may be easily constructed.

One more example is \[ 23 \]

\[
g(t) = H_0 t - \frac{H_1}{H_0} \ln \cosh H_0 t \quad . \quad (75) \nonumber
\]

Here we assume \( H_0 > H_1 > 0 \). Since

\[
H = g'(t) = H_0 - H_1 \tanh H_0 t , \quad \dot{H} = g''(t) = - \frac{H_0 H_1}{\cosh^2 H_0 t} < 0 \quad , \quad (76) \nonumber
\]

when \( t \to \pm \infty \), the universe becomes asymptotically deSitter space, where \( H \) becomes a constant \( H \to H_0 \neq H_1 \) and therefore the universe is accelerating. When \( t = 0 \), we find

\[
\frac{\ddot{a}}{a} = \dot{H} + H^2 = -H_1 H_0 + H_0^2 < 0 \quad , \quad (77) \nonumber
\]
therefore the universe is decelerating. Then the universe is accelerating at first, turns to be decelerating, and after that universe becomes accelerating again. As $\dot{H}$ is always negative, the universe is in non-phantom phase. Furthermore with the choice

$$w_m = 0, \quad t = f(\phi) = \frac{1}{H_0} \tan \left( \frac{\kappa H_0}{2\sqrt{2}H_1} \phi \right),$$

(78)

we find the corresponding scalar-Gauss-Bonnet gravity

$$V(\phi) = \frac{3}{\kappa^2} (H_0 - H_1 \tanh \varphi)^2 - \frac{H_1}{\sqrt{2}\kappa^2 \cosh^2 \varphi}$$

$$- \frac{12g_0}{H_0} (H_0 - H_1 \tanh \varphi) \left( 1 + e^{2\varphi} \right) \left[ 2\varphi - \ln (1 + e^{2\varphi}) + \frac{5}{6 (1 + e^{2\varphi})} + \frac{5}{6 (1 + e^{2\varphi})^2} + \frac{2}{6 (1 + e^{2\varphi})^3} \right],$$

(79)

$$\xi_1(\phi) = \frac{g_0}{2H_0} \int d\varphi' \frac{1 + e^{2\varphi'}}{(H_0 - H_1 \tanh \varphi')^2} \left[ 2\varphi' - \ln (1 + e^{2\varphi'}) + \frac{5}{6 (1 + e^{2\varphi'})} + \frac{5}{6 (1 + e^{2\varphi'})^2} + \frac{2}{6 (1 + e^{2\varphi'})^3} \right].$$

Here

$$\varphi \equiv \tan \left( \frac{\kappa H_0}{2\sqrt{2}H_1} \phi \right).$$

(80)

Although it is difficult to give the explicit forms of $V(\phi)$ and $\xi_1(\phi)$, we may also consider the following example \cite{10}:

$$g(t) = h_0 \left( \frac{t^4}{12} - \frac{t_1 + t_2}{6} t^3 + \frac{t_1 + t_2}{2} t^2 \right), \quad (3t_1 > t_2 > t_1 > 0, \quad h_0 > 0).$$

(81)

Here $h_0, t_1, t_2$ are constants. Hence, Hubble rate is

$$H(t) = h_0 \left( \frac{t^3}{3} - \frac{t_1 + t_2}{2} t^2 + t_1 t_2 t \right), \quad \dot{H}(t) = h_0 (t - t_1) (t - t_2).$$

(82)

Since $H > 0$ when $t > 0$ and $H < 0$ when $t < 0$, the radius of the universe $a = a_0 e^{H(t)}$ has a minimum when $t = 0$. From the expression of $\dot{H}$ in \cite{82}, the universe is in phantom phase ($\dot{H} > 0$) when $t < t_1$ or $t > t_2$, and in non-phantom phase ($\dot{H} < 0$) when $t_1 < t < t_2$ (for other string-inspired models with similar cosmology, see for instance \cite{24}). Then we may identify the period $0 < t < t_1$ could correspond to the inflation and the period $t > t_2$ to the present acceleration of the universe (this is similar in spirit to unification of the inflation with the acceleration suggested in other class of modified gravities in ref.\cite{25}). If we define effective EoS parameter $w_{\text{eff}}$ as

$$w_{\text{eff}} = \frac{p}{\rho} = -1 - \frac{2\dot{H}}{3H^2},$$

(83)

we find $w_{\text{eff}} \to -1$ in the limit $t \to +\infty$. Although it is difficult to find the explicit forms of $V(\phi)$ and $\xi_1(\phi)$, one might give the rough forms by using the numerical calculations. From the expression of $V(\phi)$ and $\xi_1(\phi)$ in \cite{77}, if $f(\phi)$ is properly given, say as $t = f(\phi) = \phi/\phi_0$ with constant $\phi_0$, there cannot happen any singularity in $V(\phi)$ and $\xi_1(\phi)$ even if $t = t_1$ or $t = t_2$, which corresponds to the transition between phantom and non-phantom phases. Then the model \cite{81} could exhibit the smooth transition between phantom and non-phantom phases.

The next example is

$$g(t) = h_0 \ln \frac{t}{t_0}, \quad t = f(\phi) = t_0 e^{\frac{\phi}{\phi_0}}.$$  

(84)

Since

$$H = \frac{h_0}{t},$$

(85)

we have a constant effective EoS parameter:

$$w_{\text{eff}} = -1 + \frac{2}{3h_0}.$$  

(86)
Eqs. (84) give
\[ V(\phi) = -\frac{1}{(h_0 + 1) t_0^2} \left( \frac{3h_0^2 (1 - h_0)}{\kappa^2} + \frac{\phi_0^2}{2} (1 - 5h_0) \right) e^{\frac{2w_m}{\kappa}} + \frac{3h_0 (1 + w_m)}{4(3 + 3w_m)h_0 - 1} e^{-\frac{3(1 + w_m)h_0}{\kappa}} , \]
\[ \xi_1(\phi) = -\frac{t_0^2}{16h_0^2 (h_0 + 1)} \left( -\frac{2h_0}{\kappa^2} + \frac{\phi_0^2}{2} e^{\frac{2w_m}{\kappa}} \right) + \frac{1}{8} \left\{ 3 (1 + w_m) h_0 - 4 \right\}^{-1} \left\{ (4 + 3w_m) h_0 - 1 \right\}^{-1} (1 + w_m) g_0^4 e^{-\frac{3(1 + w_m)h_0}{\kappa}} . \] (87)

Thus, there appear exponential functions, which are typical in string-inspired gravity.

As clear from (84), if \( h_0 > 1 \), the universe is in quintessence phase, which corresponds to \(-1/3 < w_{\text{eff}} < -1\) in (84). If \( h_0 < 0 \), the universe is in phantom phase with \( w_{\text{eff}} < -1 \). In phantom phase, we choose \( t_0 \) to be negative and our universe corresponds to negative \( t \), or if we shift the time coordinate \( t \) as \( t \rightarrow t - t_s \), with a constant \( t_s \), \( t \) should be less than \( t_s \).

The model (10) corresponds to \( g_0 = 0 \) in (84). In the notations of ref. (10), \( t_0 = t_1 \), \( V(\phi) = V_0 e^{-\frac{2w_m}{\kappa}} \), and \( f(\phi) = f_0 e^{\frac{2w_m}{\kappa}} = -\xi_1(\phi) \). Then from the expression (84), one gets
\[ V_0 = -\frac{1}{(h_0 + 1) t_0^2} \left( \frac{3h_0^2 (1 - h_0)}{\kappa^2} + \frac{\phi_0^2}{2} (1 - 5h_0) \right) , \quad f_0 = \frac{t_0^2}{16h_0^2 (h_0 + 1)} \left( -\frac{2h_0}{\kappa^2} + \frac{\phi_0^2}{2} e^{\frac{2w_m}{\kappa}} \right) , \] (88)
which is identical (after replacing \( t_0 \) with \( t_1 \)) with (16) in (11).

Thus, we demonstrated that arbitrary late-time cosmology (from specific quintessence or phantom to oscillating cosmology) may be produced by scalar-Gauss-Bonnet gravity with scalar potentials defined by such cosmology. The reconstruction of string-inspired gravity may be always done. Moreover, one can extend this formulation to include the higher order terms in low-energy string effective action.

\section*{B. The relation with modified Gauss-Bonnet gravity}

In this section we show that scalar-Gauss-Bonnet gravity may be transformed to another form of modified Gauss-Bonnet gravity where no scalars present. In addition, the formulation may be extended to include higher order terms too. Starting from (84), one may redefine the scalar field \( \phi \) by \( \phi = \epsilon \varphi \). The action takes the following form
\[ S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{\epsilon^2}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \tilde{V}(\varphi) - \tilde{\xi}_1(\varphi) G \right] . \] (89)

Here
\[ \tilde{V}(\varphi) \equiv V(\epsilon \varphi) , \quad \tilde{\xi}_1(\varphi) \equiv \xi_1(\epsilon \varphi) . \] (90)

If a proper limit of \( \epsilon \to 0 \) exists, the action (89) reduces to
\[ S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \tilde{V}(\varphi) - \tilde{\xi}_1(\varphi) G \right] . \] (91)

Then \( \varphi \) is an auxiliary field. By the variation of \( \varphi \), we find
\[ 0 = \tilde{V}'(\varphi) - \tilde{\xi}'_1(\varphi) G , \] (92)
which may be solved with respect to \( \varphi \) as
\[ \varphi = \Phi(G) . \] (93)

Substituting (91) into the action (91), the \( F(G) \)-gravity follows (26):
\[ S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - F(G) \right] , \quad F(G) \equiv \tilde{V}(\Phi(G)) - \tilde{\xi}_1(\Phi(G)) G . \] (94)
For example, in case of (104), in $\epsilon \to 0$ limit after redefining $\phi = \epsilon \varphi$ and $\phi_0 = \epsilon \varphi_0$, $V(\phi)$ and $\xi_1(\phi)$ reduce to

$$\tilde{V}(\varphi) = \frac{3h_0^2 (h_0 - 1)}{(h_0 + 1) t_0^2 \kappa^2 e^{\frac{\varphi}{\varphi_0}}} + 3h_0 (1 + w_m) g_0 \frac{e^{\frac{3(1 + w_m)h_0 \varphi}{\varphi_0}}}{(4 + 3w_m) h_0 - 1},$$

$$\tilde{\xi}_1(\varphi) = \frac{t_0^2}{8h_0 (h_0 + 1) \kappa^2} e^{\frac{2\varphi}{\varphi_0}} + \frac{1}{8} \left(3 (1 + w_m) h_0 - 4\right)^{-1} \left((4 + 3w_m) h_0 - 1\right)^{-1} (1 + w_m) g_0 t_0^4 e^{-\frac{3(1 + w_m)h_0 \varphi}{\varphi_0}}. \quad (95)$$

The solution corresponding to (84) is:

$$g(t) = h_0 \ln \frac{t}{t_0}, \quad \varphi = \varphi_0 \ln \frac{t}{t_0}. \quad (96)$$

If we further consider the case $g_0 = 0$, Eq. (92) gives

$$e^{-\frac{\varphi}{\varphi_0}} = \frac{t_0^4}{24h_0^3 (h_0 - 1) G} \quad (97).$$

Eq. (97) could have a meaning only when $h_0 > 1$ or $h_0 < 0$ if $G$ is positive. In this situation

$$F(G) = A_0 G^{1/2}, \quad A_0 = \frac{1}{2(1 + h_0) \kappa^2} \sqrt{\frac{3(h_0 - 1) h_0}{2}}. \quad (98)$$

The above model has been discussed in [26]. Actually, in [26] the following type of the action has been considered:

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + F(G) \right). \quad (99)$$

In case that $F(G)$ is given by [26], in terms of [26], $A_0 = f_0$. Hence, $A_0$ [26] coincides with the Eq. (26) of [26].

As a further generalization, we may also consider the string-inspired theory of second section where next order term is coupled with scalar field:

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) - \xi_1(\phi) G + \xi_2(\phi) L_c^{(2)} \right]. \quad (100)$$

As in [89], we may redefine the scalar field $\phi$ by $\phi = \epsilon \varphi$. If a proper limit of $\epsilon \to 0$ exists, the action (100) reduces to

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \tilde{V}(\varphi) - \tilde{\xi}_1(\varphi) G + \tilde{\xi}_2(\varphi) L_c^{(2)} \right]. \quad (101)$$

Here

$$\tilde{\xi}_2 = \lim_{\epsilon \to 0} \xi_2(\epsilon \varphi). \quad (102)$$

Then $\varphi$ could be regarded as an auxiliary field and one gets

$$0 = \tilde{V}'(\varphi) - \tilde{\xi}_1'(\varphi) G + \tilde{\xi}_2' L_c^{(2)}, \quad (103)$$

which may be solved with respect to $\varphi$ as

$$\varphi = \Psi \left( G, L_c^{(2)} \right). \quad (104)$$

Substituting (104) into the action (101), we obtain $F(G, L_c^{(2)})$-gravity theory:

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - F(G, L_c^{(2)}) \right], \quad F(G) \equiv \tilde{V} \left( \Phi(G, L_c^{(2)}) \right) - \tilde{\xi}_1 \left( \Phi(G, L_c^{(2)}) \right) G + \tilde{\xi}_2 \left( \Phi(G, L_c^{(2)}) \right) L_c^{(2)}. \quad (105)$$

In case of the string-inspired gravity:

$$V = V_0 e^{-\frac{2\varphi}{\varphi_0}}, \quad \xi_1 = \xi_0 e^{\frac{2\varphi}{\varphi_0}}, \quad \xi_2 = \eta_0 e^{\frac{\varphi}{\varphi_0}}. \quad (106)$$
Here $\phi_0$, $V_0$, $\xi_0$, and $\eta_0$ are constants. We may consider the limit of $\epsilon \to 0$ after redefining $\phi = \epsilon \varphi$ and $\phi_0 = \epsilon \varphi_0$. Thus, Eq. (103) gives

$$e^{\frac{2\phi}{\varphi_0}} = \Theta(G, \mathcal{L}_c^{(2)}) \equiv \frac{\xi_0 G}{2\eta_0 \mathcal{L}_c^{(2)}} + Y(G, \mathcal{L}_c^{(2)}) .$$

(107)

Here

$$Y(G, \mathcal{L}_c^{(2)}) = y_+ + y_- , \quad y_+ e^{\frac{\phi_0}{\varphi_0}} + y_- e^{\frac{-\phi_0}{\varphi_0}} , \quad y_+ e^{\frac{\phi_0}{\varphi_0}} + y_- e^{\frac{-\phi_0}{\varphi_0}}$$

and

$$y_\pm \equiv \left\{ \frac{V_0}{4\eta_0 \mathcal{L}_c^{(2)}} \pm \sqrt{\left(\frac{V_0}{4\eta_0 \mathcal{L}_c^{(2)}}\right)^2 - \left(\frac{\xi_0 G}{6\eta_0 \mathcal{L}_c^{(2)}}\right)^6} \right\}^{1/3} .$$

(109)

Hence, the action of the corresponding $F(G, \mathcal{L}_c^{(2)})$-theory is

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - F(G, \mathcal{L}_c^{(2)}) \right] ,$$

$$F(G, \mathcal{L}_c^{(2)}) = \frac{V_0}{\Theta(G, \mathcal{L}_c^{(2)})} - \xi_0 \Theta(G, \mathcal{L}_c^{(2)}) G + \eta_0 \Theta(G, \mathcal{L}_c^{(2)})^2 \mathcal{L}_c^{(2)} .$$

(110)

Instead of (50), one may consider the model with one more scalar field $\chi$ coupled with the Gauss-Bonnet invariant:

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\epsilon}{2} \partial_\mu \chi \partial^\mu \chi - V(\phi) - U(\chi) - (\xi_1(\phi) + \theta(\chi)) G \right] .$$

(111)

This kind of action often appears in the models inspired by the string theory [14]. In such models, one scalar $\phi$ may correspond to the dilaton and another scalar $\chi$ to modulus. We now consider the case that the derivative of $\chi$, $\partial_\mu \chi$, is small or $\epsilon$ is very small. Then we may neglect the kinetic term of $\chi$ and $\chi$ could be regarded as an auxiliary field. Repeating the process of (52), (53), we obtain the $F(G)$-gravity coupled with the scalar field $\phi$:

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \xi_1(\phi) G + F(G) \right] .$$

(112)

The relation between scalar-Gauss-Bonnet gravity and modified Gauss-Bonnet gravity (or two parameterizations of the same theory) is discussed in this section. It is shown that cosmological solutions obtained in one of such theories may be used (with different physical interpretation) in another theory. It is often turns out that it is easier to work with specific parametrization of the same theory. Of course, only comparison with observational data may select the truly physical theory in correct parametrization.

### IV. CONCLUSION

In this paper we have studied several aspects of (dilaton) gravity in the presence of string corrections up to third order in curvature. The second order term is Euler density of order two called, the Gauss-Bonnet term. The next-to-leading term contains higher order Euler density ($E_3$) plus a term of order three in curvature. The expression of $E_3$ is identical zero in space-time of dimension less than six; the term beyond the Euler density contributes to equation of motion even for a fixed field $\phi$. We have verified that the de-Sitter solution which exists in the case of Type II and Bosonic strings is an unstable node. It is shown that in the presence of a barotropic fluid (radiation/matter), inflationary solution exists in the high curvature regime for constant field.

For a dynamically evolving field $\phi$ canonical in nature, there exists an interesting dark energy solution [56] characterized by $H = h_0/t$, $\phi = \phi_0 \ln t/t_1$ for $h_0 > 0$ ($H = h_0/t_2 - t$, $\phi = \phi_0 \ln(t_{t_2} - t/t_1)$) when $h_0 < 0$). The three years WMAP data taken with the SNL survey [22] suggests that $w_{DE} = -1.06^{+0.13}_{-0.08}$. We have shown that choosing a range of parameter $\phi_0^2$ (which is amplified thanks to third order curvature term contribution) we can easily obtain the observed values of $w_{DE}$ for phantom as well as for non-phantom dark energy. We have demonstrated, in detail, the stability of dark energy solution. For non-phantom energy, in the large $h_0$ limit, we presented analytical solution.
which shows that one of the eigenvalues of the $3 \times 3$ perturbation matrix is real and negative whereas the other two are purely imaginary, thereby, establishing the stability of solution \(\text{(35)}\). We have verified numerically that stability holds for all smaller and generic values of $h_0$ in this case. The phantom dark energy solution corresponding to $h_0 < 0$ turns out to be unstable. It is remarkable that string curvature corrections can account for late time acceleration and dark energy can be realized without the introduction of a field potential.

It is shown how scalar-Gauss-Bonnet gravity may be reconstructed for any given cosmology. The corresponding scalar potentials for several dark energy cosmologies including quintessence, phantom, cosmological constant or oscillatory regimes are explicitly found. This shows that having the realistic scale factor evolution, the principal possibility appears to present string-inspired gravity where such evolution is realized. It is explained how to transform scalar-Gauss-Bonnet gravity (even with account of third order curvature term) to modified Gauss-Bonnet gravity \(\text{(26)}\) which seems to pass the Solar System tests.

Different forms of modified gravity are attempted recently (for a review, see \(\text{(9)}\)) to describe dark energy universe; these models provide a qualitatively simple resolution of dark energy/coincidence problems and deserve further consideration. It is quite likely that time has come to reconsider the basics of General Relativity at the late universe in the search of realistic modified gravity/dark energy theory.

We should also mention that in the present study we have tested the background model against observations. The study of perturbations in the scenario discussed here is quite complicated and challenging and in our opinion it deserves attention; we defer this investigation to our future work.

Acknowledgments

The research of SDO is supported in part by the project FIS2005-01181 (MEC, Spain), by LRSS project N4489.2006.02 and by RFBR grant 06-01-00609 (Russia). MS thanks S. Panda, I. Neupane and S. Tsujikawa and SDO thanks M. Sasaki for useful discussions.

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