A Class of Diffusion Algorithms with Logarithmic Cost over Adaptive Sparse Volterra Network

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Abstract—In this Letter, we present a novel class of diffusion algorithms that can be used to estimate the coefficients of sparse Volterra network. The development of the algorithms is based on the logarithmic cost and $l_p$-norm constraint. To further overcome the limitations of the impulsive noise environments from different nodes, a $p$-power error is selected via a new entropy matching estimator (EME). Specifically, our approach does not require a priori knowledge according to the noise characteristics. Simulations for Gaussian and impulsive scenarios are conducted to demonstrate the superior performance of the proposed algorithms as compared with the existing algorithms.

Index Terms—Diffusion adaptation; Volterra filter; Sparse; $l_p$-norm; $\alpha$-stable noise.

I. INTRODUCTION

DISTRIBUTED algorithms are the method for estimating parameters over adaptive networks, whose nodes can collect noisy observations related to a certain parameter of interest [1]. Attention in distributed adaptation has grown significantly over the past years due to its comprehensive applications in many areas [2-5]. In previous studies, two kinds of strategies of cooperation for adaptive networks have been studied extensively, including the incremental [6,7], and diffusion methods [8-10].

The diffusion method, uses the subset of neighbours to communicate, and therefore introduces low computational complexity and a stable behaviour in real-time adaptation [11]. Recently, some diffusion algorithms have been proposed. These algorithms include the diffusion least mean square (dLMS) algorithms [12-14], diffusion recursive least squares (dRLS) algorithms [15], etc. Additionally, similar algorithms were proposed by many researchers, such as Ni, Wen and so on [16,17], to improve the filtering performance in impulse interference scenarios.

Because of the limited nonlinear adaptive estimation capability, system designers are under pressure to achieve the satisfactory performance. To this end, many nonlinear filtering techniques were proposed, including [18-21]. Among these works, the Volterra filter has been widely used as a nonlinear system modelling tool with considerable success [22-24]. However, such a filter becomes very computationally expensive when a large number of coefficients are required. A costly second order Volterra (SOV) filter was developed to cope with the enormous amount of computations needed to obtain acceptable errors [25-26].

Motivated by these considerations, in this paper, we proposed a new diffusion algorithm for nonlinear network. The development of the algorithm is based on an innovative approach: the algorithms is introduced based on the minimization of cost functions with logarithmic dependence on the adaptation error, instead of minimizing the $p$th power error. Moreover, these algorithms with $l_p$-norm constraint are proposed in order to enhance the sparse Volterra network identification. Since nodes with different processes of impulsive noise are quite often in practical application areas, there is a need for an effective estimator to select $p$ value online. To accomplish this goal, we proposed a new estimator, inspired by the approach in [27]. Such estimator relies on matching the entropy of the generalized Gaussian density (GGD) modeled distribution, without resorting to a priori knowledge of the noise.

Notation: Throughout this paper, we use $\{\}$ to represent a set, $|\cdot|$ denotes absolute value of a scalar, $(\cdot)^T$ denotes the transposition, $E(\cdot)$ denotes the expectation, $tr\{\}$ denotes the trace operator, $\|\cdot\|_p$ denotes the $l_p$-norm, operator $@$ denotes the Kronecker product, and $\{\}^T$ is the real value of parameter. Besides, we use boldface and normal letters to denote the random quantities and deterministic quantities, respectively.

II. PROPOSED ALGORITHMS

A. Derivation of the proposed algorithms

We consider the problem of estimating Volterra coefficients from a network with $N$ sensor nodes. In each iteration $i$, each sensor node $k \in \{1,2,\ldots,N\}$ has access to the realization $\{d_k(i),u_{i,k}\}$ of some zero-mean random process $\{d_k(i),u_{i,k}\}$, where $u_{i,k}$ is an regression vector with length $M$, and $d_k(i)$ is a scalar. Suppose the measurements arising from the model

$$d_k(i) = \mathbf{u}_k^\perp w' + v_k(i)$$  \hfill (1)

where $w' \in \mathbb{C}^{M+1}$ is the coefficients of the Volterra series.
model, and $v_i(t)$ is the measurement noise. Here, we assume that $u_{k,i}$ and $v_i(t)$ are spatially independent and independent identically distributed (i.i.d.), and $v_i(t)$ is independent of $u_{k,i}$. The output of the SOV network can be expressed as

$$y_i(t) = w^T u_{k,i} + \sum_{m_i=0}^{\infty} h_{k,i}(m_i) v_i(t-m_i)$$

where $x_k(i)$ is the input data from at node $k$, $h_r$ is the $r$th order Volterra kernel at node $k$, $P$ is the length of Volterra system, $P = P(P+1)/2 = M$, and $u_{k,i}$ and the expanded coefficients vector $w$ of SOV system are expressed by

$$u_{k,i} = [x_k(i), \ldots, x_k(i-P+1), x_i^2(i), \ldots, x_i^2(i-P+1)]^T$$

$$w = [h_{k,i}(0), \ldots, h_{k,i}(P-1), h_{k,i}(0), \ldots, h_{k,i}(P-1), \ldots, h_{k,i}(P-1), \ldots, h_{k,i}(P-1)]^T.$$ (3)

To obtain an improved performance, we adopted the local cost function that has logarithmic dependence on the error [28], i.e.,

$$J^w(\omega) = \sum_{l \in N_k} a_{l,i} E[F(e_{l,i}) - \frac{1}{\delta} \log(\delta F(e_{l,i}))]$$

where $\delta > 0$, $N_k$ is the set of nodes with which node $k$ shares information (including $k$ itself), the weighting coefficients $\{a_{l,i}\}$ are real, non-negative, and satisfies $\sum_{l} a_{l,i} = 1$. The function $F(e_{l,i})$ denotes a function of the error signal. Using the steepest descent algorithm, we hold the steepest descent adaptation as

$$w_{k,i} = w_{k,i-1} - \mu \sum_{l \in N_k} a_{l,i} \frac{\partial \{F(e_{l,i}) - \frac{1}{\delta} \log(\delta F(e_{l,i}))\}}{\partial w}$$

where $\mu$ is the step-size. Under the linear combination assumption [12], let us define the linear combination $w_{k,i}$ at node $k$ as

$$w_{k,i} = \sum_{l \in N_k} a_{l,i} \phi_{l,i-1},$$

where $\phi_{l,i}$ is the local estimates at node $k$. Then, we can obtain an iterative formula for the intermediate estimate

$$\begin{align*}
\phi_{l,i} &= w_{k,i-1} - \mu \sum_{l \in N_k} a_{l,i} \frac{\partial F(e_{l,i})}{\partial \delta F(e_{l,i})} (\text{adaptation}) \\
&= \sum_{l \in N_k} a_{l,i} \phi_{l,i-1} (\text{combination}).
\end{align*}$$

In the adaptation step, $\phi_{l,i-1}$ is replaced by linear combination $w_{k,i-1}$. Such substitution is reasonable, since the linear combination contains more data information from neighbor nodes than $\phi_{l,i-1}$ [12].

**The diffusion least mean logarithmic square (dLMLS) algorithm**

For $F(e_{l,i}) = E(e_{l,i}^2)$, this simplifies into the dLMS algorithm:

$$\phi_{l,i} = w_{k,i-1} + \mu \sum_{l \in N_k} c_{l,i} \frac{\delta u_{l,i} e_{l,i}}{1 + \delta e_{l,i}^2}$$

where $\{c_{l,i}\}$ is the non-negative weighting coefficients, satisfying the condition $c_{l,i} = a_{l,i} = 0$ if $l \notin N_k$.

**The diffusion least logarithmic absolute difference (dLLAD) algorithm**

The dLLAD algorithm is derived by substituting $F(e_{l,i}) = E(|e_{l,i}|)$ in the general formula given by (7). Its update is given by

$$\phi_{l,i} = w_{k,i-1} + \mu \sum_{l \in N_k} c_{l,i} \frac{\delta u_{l,i} e_{l,i}}{1 + \delta |e_{l,i}|}.$$ (9)

**Remark 2.1:** The proposed dLMS algorithm is based on fourth-statistics of the error [28], and can therefore achieve a smaller steady-state kernel error. The dLLAD algorithm intrinsically combines the $l_2$-norm and $l_1$-norm. In the impulsive noise environments, it may be expected to converge faster than the dLMS algorithm.

**Remark 2.2:** The dLLMP algorithm resembles the algorithm that we proposed in [29]. Because the logarithmic-order class includes $\alpha$-stable noise process [30], it can be used estimate the coefficient with reduced negative effects of outliers. When $p_i = 1$, it reduces to the dLLLAD algorithm. When $p_i = 2$, it becomes the dLMS algorithm.

**Remark 2.3:** For the Eq. (8-10) above, the diffusion algorithms obtain solutions via Adapt-then-Combine (ATC) step. The Combine-then-Adapt (CTA) [12] algorithms can be easily derived by exchanging the order of these two steps in (7).

**B. l_0-norm-based algorithms**

To introduce sparsity, we can minimize the following penalized local cost function with $l_0$-norm:

$$J^w_0(\omega) = \sum_{l \in N_k} a_{l,i} E[F(e_{l,i}) - \frac{1}{\delta} \log(\delta F(e_{l,i}))] + \lambda \| w \|_0$$ (11)

where $\lambda$ is a controller to balance the new penalty and the estimation error. Similarly, the adaptation of logarithmic cost with $l_0$-norm can be given as

$$\phi_{l,i} = w_{k,i-1} - \mu \sum_{l \in N_k} a_{l,i} \frac{\partial F(e_{l,i})}{\partial \delta F(e_{l,i})} - \rho \| w_{k,i-1} \|_0$$

where $\rho = \lambda \mu$. Since the minimization of the $l_0$-norm is a Non-Polynomial (NP) hard problem, $\| w_{k,i-1} \|_0$ $l_0$-norm is usually approximated by a popular continuous function [31,32]

$$\| w_{k,i-1} \|_0 = \begin{cases} -\beta^2 w_{i} - \beta, & -1/\beta \leq w_i < 0 \\ -\beta^2 w_{i} + \beta, & 0 < w_i \leq 1/\beta \\ 0, & \text{elsewhere} \end{cases}$$

where $\beta$ is the parameter $1 < \beta < 2$ can be absorbed into the step size $\mu$ in (10).

\[1\] The parameter $1 < \beta < 2$ can be absorbed into the step size $\mu$ in (10).
where $\beta \in [5,20]$ is the positive constant.

C. Proposed entropy matching estimator (EME)

The main bottleneck of the dLMLP algorithm is the
requirement of quite precise a priori information to select
parameter $\rho$ in the presence of $\alpha$-stable noise. To avoid
this limitation, we proposed a truly on-line estimation based on
entropy matching estimator (EME) [27]. A new possible
estimator for the assumed distribution with entropy, denoted by
$H(x)$, is given by
\[
H(x) = h \cdot \text{sigmoid}(x) 
\begin{pmatrix}
\log_2 \left( \frac{2\Gamma(1/x)^{1/2}}{x\Gamma(3/x)^{1/2}} \right) + \frac{1}{x\ln 2}
\end{pmatrix}
\] (14)
where $h=1/3$ is the scalar factor, $\Gamma()$ defines the gamma
function, and $\text{sigmoid}(\cdot)$ denotes the sigmoidal function
\[
\text{sigmoid}(x) = \frac{1}{1 + \theta_1 e^{-\theta_2 x}}
\] (15)
where $\theta_1=1.8$ and $\theta_2=1.2$. Using the following formula, we can
obtain an estimation of $\hat{p}_{i,j}$
\[
\hat{p}_{i,j} = H^{-1} \left( \min \left( \frac{(1/L) \sum_{n=m-1}^{n-1} e_{i,j}^2}{(1/L) \sum_{n=m-1}^{n-1} e_{i,j}^2}, e_{i,j,\text{upper}} \right) \right). 
\] (16)
Note that $H^{-1}(\cdot)$ is the inverse function of $H(\cdot), e_{i,j,\text{upper}}=0.6$ is
the upper bound of $e_{i,j}$, and $L$ stands for sliding data length.

III. MEAN STABILITY ANALYSIS

In this section, we analyze the convergence property of the
proposed algorithm, the analysis relies on the assumption that
$\nu_j(i)$ is some temporally and spatially white noise,
and independent of $\{d_i(j), u_{i,j}\}$ for all $i,j$.

Then, we introduce the following global quantities
$q_i = \{\phi_{i,1}, \ldots, \phi_{i,k}\}, w_i=\{w_{i,1}, \ldots, w_{i,k}\}, d_i=\{d_{i,1}, \ldots, d_{i,k}\}, w^*=\{w_{i,1}^*, \ldots, w_{i,k}^*\}, U_i=\text{diag}(u_{i,1}, \ldots, u_{i,k}), V_i=\text{col}(v_{i,1}, \ldots, v_{i,k})\}, B=\text{diag}(\mu_{A,B}, \ldots, \mu_{A,B}), N_i=\text{diag}(N_{i,1}, \ldots, N_{i,k})\}. After defining these quantities, (12) can be rewritten as
\[
q_i = A\phi_{i-1} + BU_i^T(d_i - U_i A\phi_{i-1})F(e) + \delta f(e) + \zeta(\phi_{i-1})
\] (17)
where $A=X \otimes I_{\mu}$ is an $M \times N$ matrix, $\ell(e)$ is the function of $e^2$, $X$ is the
$N \times N$ matrix with individual entry $c_{i,j}$, and
$F(e) = E(f(e))$.

Let $\bar{\phi}_i$ be the mismatch between $w^*$ and $q_i$, given by
\[
\bar{\phi}_i = w^* - q_i.
\] (18)
After subtracting both sides of (17) from $w^*$, the following equation is obtained.
\[
\tilde{\phi}_i = (I_{\mu} - BU_i^T U_i \xi(e))A\phi_{i-1} - BU_i^T \xi(e) V_i + \zeta(\phi_{i-1})
\] (19)
where $\tilde{\phi}_i = (I_{\mu} - BU_i^T U_i \xi(e))A\phi_{i-1} - BU_i^T \xi(e) V_i + \zeta(\phi_{i-1})$
and $\zeta(\phi_{i-1}) = \rho N \| A\phi_{i-1} \|_0$.

According to the assumption in (1), the expectation of (19) can be taken as
\[
E(\bar{\phi}_i) = (I_{\mu} - BE(U_i^T U_i) \xi(e))A\phi_{i-1} + E(\zeta(\phi_{i-1}))
\] (20)
Since $E(\zeta(\phi_{i-1}))$ is a small bounded vector, therefore, the
algorithm can converge if
\[
\lambda_{\text{max}}(A) \leq \| X \|_2 \lambda_{\text{max}}(A) \leq \lambda_{\text{max}}(A)
\] (21)
where $\Phi = I_{\mu} - BE(U_i^T U_i) \xi(e)$, and $\lambda_{\text{max}}$ denotes the
maximal eigenvalue of a matrix. Thus, to ensure the stability of
proposed algorithms, the range of $\mu$ is obtained as
\[
0 < \mu < \frac{2}{\lambda_{\text{max}}(E(U_i^T U_i)) \xi(e)}.
\] (22)

IV. SIMULATION RESULTS

The newly developed algorithms based on ATC/CTA
schemes are evaluated for robust performances in this section.
The simulations are based on a network of 20 nodes (See Fig. 1).
The sparse SOVs have size $M=14(P=4)$, as shown in Fig. 2 [33].
The Gaussian signal with zero mean and unit variance is
employed as the exciting input. In the simulation study, the
effectiveness is assessed in terms of network mean-square
deviation (NMSD), which is defined as
\[
\text{NMSD} = \frac{1}{N} \sum_{k=1}^{N} \| w_{k} - w^* \|_2^2.
\] (23)
The noise is modeled by standard symmetric $\alpha$-stable (SaS)
distribution [22]:
\[
\eta(\ell) \equiv \exp \{- |\ell|^{\alpha} \}
\] (24)
where \( \alpha \) is a characteristic exponent. In all our simulations, the curves are drawn from the average of 25 independent runs.

A. Case 1

In this example, the noise signal is generated with \( \alpha=2 \), which corresponds to the Gaussian distribution \([33]\). Fig. 3 illustrates the NMSD curves for algorithms. It can be observed that the proposed family of logarithmic cost algorithms outperform the dLMS algorithm. In dLLAD, dLLMP, dLLAD \( l_0 \) and dLLMP \( \mu \) algorithms, the only sacrifice to make are the small convergence rate at the initial stage.

In Fig. 4, we demonstrate the steady-state NMSD of the dLLMP algorithm with CTA/ATC strategy. As the results indicate, the CTA-based algorithms achieve comparable performance as compared with ATC-based algorithms. For \( \mu \geq 0.3 \), it is interesting to note that the CTA algorithms outperform the ATC-Volterra algorithms by about 0.02dB.

B. Case 2

In second example, the SstS noises of all nodes are generated by random, as shown in Fig. 5. The learning curves for the algorithms are plotted in Fig. 6, where \( p \) in dLLMP is chosen by random. We can clearly see that the dLLMP \( l_0 \) method is very robust to the impulsive noise and has smaller kernel estimation error than the other algorithms. The other algorithms, in contrast, have large fluctuations during the adaptation. In Fig. 7, we illustrate the effectiveness of the dLLMP algorithm by using the proposed EME method, as compared with Bergamasco’s method \([34]\) and dLLMP algorithm (without using EME method). Obviously, the performance of the dLLMP with EME method is the best in terms of convergence rate and steady-state behavior, because \( p \) is close to \( \alpha \) (See Fig. 5).
An adaptive sparse Volterra network using a class of logarithmic cost algorithms has been proposed and investigated in detail through simulations. In Gaussian noise environments, the proposed algorithms enjoy smaller kernel misalignment as compared with the dLSM algorithm. If the network is corrupted by $\alpha$-stable noise, the proposed dLLMP algorithm will provide overwhelmingly better stability in comparison with the existing algorithms. Furthermore, selection of $\alpha$-power of the dLLMP algorithm has been addressed by using a new EME. Extension of the idea used in this work to other nonlinear networks is a topic for further exploration. Mean-Square analysis of the proposed algorithms is also a technically demanding issue to be attempted in the future.

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