A signature index for third order topological insulators

L. B. Drissi and E. H. Saidi

1 LPHE, Modeling & Simulations, Faculty of Science, Mohammed V University, Rabat, Morocco
2 CPM, Centre of Physics and Mathematics, Mohammed V University in Rabat, Morocco
3 Peter Grünberg Institut and Institute for Advanced Simulation, Forschungszentrum Jülich & JARA, D-52425 Jülich, Germany

E-mail: ldrissi@fsr.ac.ma b.drissi@academiesciences.ma

Received 11 March 2020, revised 23 April 2020
Accepted for publication 5 May 2020
Published 15 June 2020

Abstract
In this work, we develop an index signature characterising the third order topological phases in 3D systems. This index is an alternating sum of monomial signatures of Higgs triplet values at 3D corners. We extend our method to N-dimensional systems with open boundaries, and demonstrate that the topological invariant can be efficiently generalised to any space dimension including the second order topological insulators. Known results on lower dimensional systems are recovered and an interpretation in the Higgs space parameters is given.

Keywords: Benalcazar–Bernevig–Hughes lattice models, second and third order topological insulators, index theorem, Higgs space

(Some figures may appear in colour only in the online journal)

1. Introduction

Recently, a new family of topological insulators (HOTIs) going beyond the standard Altland–Zirnbauer (AZ) classification [1, 2] has been discovered by Benalcazar–Bernevig–Hughes (BBH) [3, 4]; see also [5, 6] for related works. Higher order topological insulators—HOTIs—essentially concern the set of 2D and 3D matter systems exhibiting gapless modes at sub-regions of dimensions less or equal to D – 2. In 2D systems, with polygonal shapes including the square and the rectangle to be revisited in this study, second order topological insulators (SOTIs) are characterized by gapless states existing only at corners while the surface and the edges are gapped [7–13]. For 3D matter however; one distinguishes two kinds of HOTIs, namely (i) a second topological order phase on the 1-dimensional edges associated with a periodic boundary condition in z-direction while x- and y- dimensions are open [5]; and (ii) TOTIs; a third order topological insulator for the point vertices associated with full open boundaries [14]; this third order topological phase concerns 3D poly-faces hosting gapless states at corners (intersections of three 2D faces); while the bulk and the boundary surface as well as their edge intersections are all of them gapped. So far, TOTIs were constructed on the breathing Pyrochlore lattice where each corner of the tetrahedron carries 1/4 fractional charge [16]. Hall conductance quantized in units of $e^2/h$ was reported for reflection symmetric second-order topological crystalline insulators where the existence of edge states is ensured as long as surface and bulk remain gapped [17]. Gapless corner states were observed experimentally in a two-dimensional quadrupole topological insulator implemented using perturbative mechanical metamaterials [18]. It was reported in [19] that a realization of TOTIs was explored in an anisotropic diamond-lattice acoustic metamaterial; the bulk acoustic band structure has nontrivial topology characterized by quantized Wannier centers. By acoustic measurement, gapless states were observed at two corners of a rhombohedron-like structure in agreement with non trivial topology characterized by quantized Wannier centers [20].

Topological invariants are so important for a more comprehensive classification of topological crystalline insulators [4, 21]. Remarkably, this notion encodes information on the boundary physics and provides access to natural quantities
and observables [22–24]. A noninteracting $\mathbb{Z}_2$ topological invariant was given in term of the Berry curvature for topological insulators (TIs) [25]. For interacting and disordered TI systems, topological index, that determines their phase diagrams, can be experimentally measured through the topological magneto-electric effect [26–28]. For suitable types of noise, the classification of mixed-state topology in one dimension reveals reaining of its topological properties [29]. Furthermore, topological Thouless pump is induced by Markovian reservoirs in open quantum chains [30]. The integer invariant describing the topology of 2D open systems captures the number difference of gapless edge modes and gapless edge blind bands [31]. Remarkably, in the limit of TIs, topological invariants are well investigated, however, they are not perfectly developed and are to be described for HOTIs.

In this paper, we derive an explicit formula for the topological invariant characterizing TOTIs in 3D parallellepped systems with the cube as a particular case. For that, we consider the topological DBI class of the AZ periodic table which has reflection symmetries, in addition to the usual $T$- $P$- $C$ invariance. These kind of systems have full open boundary conditions with topological dynamics remarkably described by the limits of the lattice Hamiltonian near the Dirac points. It happens that the resulting limits can be interpreted in terms of couplings between fermions and scalar fields as done in [34]; and to which we refer below as Fukui-coupling. This nice observation has in fact a deep origin since from the quantum field theory view (QFT), the coupling can be put in a formal correspondence with the well known Yukawa tri-coupling $\psi^\dagger \phi \psi$ giving masses to fermions through a non zero vacuum expectation value $\langle \phi \rangle$ of the Higgs field $\phi$; the fields $\psi$ and $\phi$ describes the fermionic states. Borrowing this idea and applying it to 3D lattice systems, we develop the topological picture for 3-dimensional BBH system and show amongst others that the underlying topological index $\text{Ind}(H_{3D})$ is given by

$$\text{Ind}(H_{3D}) = \frac{1}{8} \sum_{pq} \text{sgn}(A_p B_q C_r)$$

(1.1)

with $A_\pm, B_\pm, C_\pm$ are non vanishing constants whose meaning may be imagined in terms of vacuum expectation values of an O(3) scalar (Higgs) field triplet $(\phi_+, \phi_0, \phi_-)$. These constants are the values of Higgs field at space infinities; they will be discussed in details in the heart of the paper. We also show that the above index formula has a remarkable factorisation as in equation (4.38) showing in turns that $\text{Ind}(H_{3D})$ is just an element of a sequence with a generic term as follows

$$\text{Ind}(H_{3D}) = \prod_{j=1}^{N} \left[ \sum_{k \neq j} \frac{1}{2} \text{sgn}(A_k) \right]$$.  

(1.2)

By setting $N = 3$, we recover exactly the index formula (1.1); for $N = 1$, one obtains an index formula for the 1-dimensional Su–Schrieffer–Heeger models (SSH) model [33] which reads here as $\frac{1}{2} (\text{sgn} A_+ - \text{sgn} A_-)$; and by setting $N = 2$, we discover the Fukui formula for $\text{Ind}(H_{2D})$ describing the second order topological systems of 2D matter systems. This 2D index formula can be expressed like $\sum_{pq=\pm} \frac{1}{2} \text{sgn}(A_p B_q)$; it is also factorable like

$$\text{Ind}(H_{2D}) = \frac{1}{2} \left[ \text{sgn}(A_+) - \text{sgn}(A_-) \right] \times \frac{1}{2} \left[ \text{sgn}(B_+) - \text{sgn}(B_-) \right]$$

(1.3)

and remarkably descends for the 3-dimensional index formula $\text{Ind}(H_{3D})$ by fixing the $C_\pm$ signatures of the third component $\phi_3$ of the Higgs field triplet as $\text{sgn}(C_+) = +1$ and $\text{sgn}(C_-) = -1$.

The organisation of this paper is as follows: In section 2, we revisit some useful aspects of the 2D model with full open boundary conditions and compute the topological index $\text{Ind}(H_{2D})$ by using two different methods that will be commented at proper places; the one used by Fukui and an equivalent one using the power of differential forms. In section 3, we shed more light on the derivation of the Fukui formula by using a direct approach based on topological mappings. In section 4, we develop the construction to 3-dimensions and show that the topological index $\text{Ind}(H_{3D})$ for the third order topological phase in DBI class with reflection symmetries is given by equation (1.1). Section 5 is devoted to conclusion and comments. In the appendices A.1 and A.2, we report some technical details which have been omitted from the core of the paper in order to keep the chain of ideas forward to the index derivation.

2. Two dimensional BBH model revisited

Following [34], the index of the Hamiltonian—$\text{Ind}(H)$—of the two dimensional BBH lattice model with open boundary conditions can be determined by studying the properties of a 2 + 2 component fermion $\psi = (\Delta, \chi)$ near the four Dirac-like points $k_{\pi x}$ of the model which are equal to $(n, \pi, n, \pi)$ with $n_i = 0, 1$; see details given after equation (2.1) and further details in appendix A.1. Around each one of these $k_{\pi x}$ points, the fermion $\psi$ is coupled to an external 2D vector $\varphi_{n \pi}$ with constant components $\varphi_{n \pi}^0, \varphi_{n \pi}^1, \varphi_{n \pi}^2$, playing the role of a mass—i.e.: a gap energy between valence and conduction bands. We refer below to this $\varphi_{n \pi}$ as constant $O(2)$ Higgs field doublet; its two components $\varphi_{n \pi}^0, \varphi_{n \pi}^1$ depend on the Dirac point on which one rests; they are given by $\Delta_\pi + \cos(n_\pi \pi)$ where $\Delta_\pi$ and $\Delta_\gamma$ are hopping parameters of the model. For example, near the Dirac point $n_x = n_y = 0$, the Higgs components $\varphi_{0\pi}^0$ and $\varphi_{0\pi}^1$ read respectively like $\Delta_\chi + 1$ and $\Delta_\pi + 1$; while near another point, say $n_x = 0, n_y = \pi$, we have $(\varphi_{0\pi}^0, \varphi_{0\pi}^1)$ with $y$-component like $\Delta_\chi - 1$. So, given a Dirac point, the interacting dynamics between fermion $\psi$, its adjoint $\psi^\dagger$ and the Higgs field is described by the following typical Hamiltonian matrix

$$H = \Lambda^0 k_x + \Lambda^1 k_y + \Omega^0 \varphi_x + \Omega^1 \varphi_y$$

(2.1)
where the hermitian 4 × 4 matrices $\Lambda'$ and $\Omega'$ and their basic properties will be specified below. To fix ideas, the term $\Lambda'\sin k_x + \Lambda'\sin k_y$ may be viewed as the leading contribution coming from the expansion of $\Lambda' \sin k_x + \Lambda' \sin k_y$ near the Dirac point $(k_x, k_y) = (0, 0)$ while the terms $\Omega' \varphi_x$ and $\Omega' \varphi_y$ derive from the expansion of $\Omega' (\Delta_x + \cos k_x)$ and $\Omega' (\Delta_y + \cos k_y)$ respectively [3, 4]. In appendix A.1, we show that the above $H$ is in fact a representative of a set of four Hamiltonians $H_{\alpha=\pi,0,\pi}$ obtained by the expansions of an underlying lattice $H_{\alpha}$ near the points $(k_x, k_y) = (n_x, \pi, n_y, \pi)$ with $n_x, n_y = 0, 1 \mod 2$. Notice by the way that the denomination of the usual four matrices $\gamma^a$ of Dirac theory by the pairs $\Lambda'$ and $\Omega'$ is for convenience; it will be justified later on. Observe also that, in addition to the 2D momentum vector variables $(k_x, k_y)$, equation (2.1) depends also on the constant $(\varphi_x, \varphi_y)$ that can be interpreted at this level as moduli parameters and whose role they play will be explored during this study. Notice as well that $H$ has discrete symmetries given by the basic $T \cdot P \cdot C$ invariance of AZ classification [1, 2] and moreover by mirror symmetries going beyond AZ. The basic symmetries, respectively generated by time reversing symmetry (TRS) operator $T$, particle-hole $P$ and chiral operator $C$, act on $H$ as follows

$$TH(k_x, k_y) T^{-1} = -H(-k_x, -k_y)$$
$$PH(k_x, k_y) P^{-1} = -H(-k_x, -k_y)$$
$$CH(k_x, k_y) C^{-1} = -H(k_x, k_y).$$

The obey the characteristics of the DBI class and are realised like: $T = K$ with $K$ standing for the usual $(\pm)$ complex conjugation; $C = \gamma_5$ for chirality and $P$ given by the composed operator $\gamma_5 K$. Regarding the mirror symmetries, they are generated by the $M_x$ and $M_y$ reflections in the $x$- and $y$-directions acting as follows

$$M_x H(k_x, k_y) M_x^{-1} = H(-k_x, k_y)$$
$$M_y H(k_x, k_y) M_y^{-1} = H(k_x, -k_y)$$

with $M_y = M_x M_y$ for inversion. The two $M_j$ operators are realised by $iA_j \gamma_5$ and capture important properties [17]; they commute with $T$ but anticommutes with $\gamma_5$; the last property prevents a $\gamma_5$ term in $H$. So, by requiring these reflection symmetries, the above time reversing invariant Dirac Hamiltonian $H$ has inevitably chirality symmetry which is at the basis of our following calculations. Recall that the gap energy of (2.1) is induced by the constant $\varphi$; gapless states require then the vanishing of $\varphi$. In the remainder of the paper, we think of $H$ as in (2.1) and capture important properties [17]; the $\Lambda_j$ and $\Omega_j$ matrices square to identity and anticommute among themselves; for a generic description of these matrices; it is interesting to set $\Lambda_1 = \gamma_1, \Omega_1 = \gamma_2$ and $\Lambda_2 = \gamma_1, \Omega_2 = \gamma_4$; by using these $\gamma_\mu$'s, the underlying 4-dim Euclidian Clifford algebra reads as $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$ and the chiral operator is given by $\gamma_5 = -\gamma_1 \gamma_2 \gamma_3 \gamma_4$; it represents the $C$ in equation (2.2) and defines chirality of the states. As given below, chiral and antichiral fermions have the half degrees of freedom carried by $\psi$.

$$\gamma_5 \left( \begin{array}{c} \lambda \\ \chi \end{array} \right) = - \left( \begin{array}{c} \lambda \\ \chi \end{array} \right), \quad \gamma_5 \left( \begin{array}{c} 0 \\ \chi \end{array} \right) = \left( \begin{array}{c} 0 \\ \chi \end{array} \right).$$

(b) A chirality condition, killing the half of the components of $\psi$, is given by the equation

$$\gamma_5 \left( \begin{array}{c} \lambda \\ \chi \end{array} \right) = - \left( \begin{array}{c} \lambda \\ \chi \end{array} \right), \quad \gamma_5 \left( \begin{array}{c} 0 \\ \chi \end{array} \right) = \left( \begin{array}{c} 0 \\ \chi \end{array} \right).$$

The explicit form of the wave functions describing gapless states at each corner are obtained by solving the two following constraint relations:

(a) The eigenvalue equation $H\psi = E\psi$ with energy $E = 0$ and local Hamiltonian as

$$H = -i\Lambda_1 \frac{\partial}{\partial x^1} + \Omega^a \phi_a$$

where we have used $\lambda_j = -i \frac{\partial}{\partial x^j}$. Here, the matrices $\Lambda'$ and $\Omega'$ are realised in terms of two sets of Pauli matrices $\sigma$ and $\tau$ as follows

$$\Lambda_1 = \tau_1 \otimes \sigma_1, \quad \Lambda_2 = \tau_2 \otimes \sigma_3$$
$$\Omega_1 = \tau_2 \otimes \sigma_2, \quad \Omega_2 = \tau_1 \otimes \sigma_0$$

with the properties $T \Lambda_1 T^{-1} = -\Lambda_1$ and $T \Omega_1 T^{-1} = -\Omega_1$ while $C \Lambda_1 C^{-1} = -\Lambda_1$ and $C \Omega_1 C^{-1} = -\Omega_1$. The $\Lambda_j$ and $\Omega_j$ matrices square to identity and anticommute among themselves; for a generic description of these matrices; it is interesting to set $\Lambda_1 = \gamma_1, \Omega_1 = \gamma_2$ and $\Lambda_2 = \gamma_1, \Omega_2 = \gamma_4$; by using these $\gamma_\mu$'s, the underlying 4-dim Euclidian Clifford algebra reads as $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$ and the chiral operator is given by $\gamma_5 = -\gamma_1 \gamma_2 \gamma_3 \gamma_4$; it represents the $C$ in equation (2.2) and defines chirality of the states. As given below, chiral and antichiral fermions have the half degrees of freedom carried by $\psi$.
The chiral charge \( q_\gamma \) takes either the value +1 or −1; the exact value of this charge at corner is fixed by the normalisation condition of \( \psi (r) \). By taking \( \phi_a \) positive constants for instance, the \( \psi \) is given by a real exponential \( e^{-\kappa r} \) with momentum \( \kappa \) which is function of \( q_\gamma \); by demanding \( \kappa, r > 0 \), one ends with a constraint on the sign of the chiral charge; see [38] for details.

After this digression on \( \psi \) and the algebra of the \( A^t \) and \( \Omega^a \) matrices, we come now to the study of the index of Hamiltonian \( H \); this index can be expressed in different, but equivalent, ways. The standard definition is given by the integer

\[
\text{Ind} (H) = N_+ - N_- \tag{2.10}
\]

with \( N_\pm \) standing for the numbers of chiral and antichiral zero modes in the ground state. To compute the value of this index, it is helpful to take advantage of two interesting features: first, we think of this integer as the flux of some two dimensional current \( j^I = J^i (r) \) as follows

\[
\text{Ind} (H) = \oint_C \varphi \mathbf{C} \quad \text{with} \quad \varphi = \frac{1}{2} (J^i dy - J^j dx) \tag{2.11}
\]

with closed contour \( C \) belonging to the \( x-y \) plane. Below, we refer to this position plane as \( \mathbb{R}^2 \) and so the loop \( C \) sits in \( \mathbb{R}^2 \); the reason for this notation is that we will encounter below another closed curve \( \Gamma \) belonging to another plane \( \mathbb{R}^4 \) parameterised by the \( \phi = (\phi_\alpha, \phi_\beta) \) components of the Higgs field doublet. By using the antisymmetric Levi–Civita symbol \( \epsilon^{ij} \) and its inverse \( \delta_{ij} \) with \( \epsilon^{12} = \epsilon^{21} = 1 \), the above index reads in a generic form like

\[
\text{Ind} (H) = \frac{1}{2} \oint_C \bigl( \epsilon^{ij} dx^j \bigr) \tag{2.12}
\]

showing that \( \text{Ind}(H) \) is indeed the flux of the \( J_i \) vector. Second, we interpret the vector current \( j^I \) as the vacuum expectation value (VEV) of an axial vector current \( J^I \) involving the matrix operator \( \gamma_5 \) generating chirality. In other words, we have \( J^I = \langle j^I \rangle \) reading in terms of the Hamiltonian \( H \) as follows [34, 38–42],

\[
J^I (r) = \lim_{r \rightarrow \infty} \left[ \lim_{m \rightarrow 0} \left( \gamma_5 A^I \frac{1}{m^2 + iH} \right) \right] \delta^I (r - r') \tag{2.13}
\]

By calculating the trace over the \( 4 \times 4 \) matrix product in (2.13) and taking the vanishing mass limit, the above vector current can be expressed in terms of the Higgs field as follows

\[
J^I = \frac{1}{\pi |\phi|^2} \epsilon^{ij} \phi_a \phi_b \partial_j \phi_b \frac{e^{i\phi_a \partial_j \phi_b}}{\partial_j \phi_b} \tag{2.14}
\]

with \( |\phi|^2 = \phi_1^2 + \phi_2^2 \). The derivation of equation (2.14) is somehow technical; a sketch of its obtention is reported in appendix A.2. By substituting this current vector back into (2.12), we end up with the following integral formula

\[
\text{Ind} (H) = \oint_C \frac{1}{2|\phi|^2} \epsilon^{ij} \phi_a \partial_j \phi_b \partial_j \phi_b \nu D x^j \tag{2.15}
\]

This relation shows that \( \text{Ind}(H) \) is intimately related with the Higgs fields; thanks to fermion-Higgs coupling which allowed to extract the ground state data \( N_+ - N_- \).

### 2.2. Signature formula for \( \text{Ind}(H) \)

To calculate \( \text{Ind}(H) \), we must perform the integral in (2.15) which can be interpreted as a function \( F [C] \) with variable given by an extended object namely the loop \( C \). For that, we \textit{a priori} have to specify the closed curve \( C \) in order to get the numerical value of \( \text{Ind}(H) \). In fact, this is not necessary since \( \text{Ind}(H) \) is a topological invariant and so it is non-sensitive to the shape of \( C \); we will use this property later on in order to re-derive the result of [34]. Before that, let us follow the Fukui approach by thinking of the curve in equation (2.15) as a big closed \( C_\infty \) at the boundary of the \( x-y \) plane; and moreover imagine it as a rectangular loop \( C_\infty^{(4)} \) with four edges as depicted in the figure 1.

\[
C_\infty^{(4)} = C_1 \cup C_2 \cup C_3 \cup C_4 \tag{2.16}
\]

We also require that at the four corners of the rectangular curve \( C_\infty^{(4)} \), the Higgs components \( (\phi_\alpha, \phi_\beta) \) take constant values denoted below like \( (A_\pm, B_\pm) \). In other words, on the boundary edges of \( C_\infty^{(4)} \), the Higgs field components satisfy space boundary conditions of the type

\[
\lim_{x \rightarrow \pm \infty} \phi_\alpha (x, y) = \phi_\alpha (\pm \infty, y) = A_\pm \tag{2.17}
\]

and

\[
\lim_{y \rightarrow \pm \infty} \phi_\beta (y, x) = \phi_\beta (y, \pm \infty) = B_\pm \tag{2.18}
\]

where \( A_\pm \) and \( B_\pm \) are non vanishing constants. We also have \( \lim_{x \rightarrow \pm \infty} F (x) = A_\pm \) and \( \lim_{y \rightarrow \pm \infty} G (y) = B_\pm \). This implies that the value of the field components \( (\phi_\alpha, \phi_\beta) \) at the corners of the rectangle are precisely given by \( (A_\pm, B_\pm) \). With these
boundary conditions the vector current (2.14) on the curve \( C_\infty \) becomes

\[
J^x(x,y)\big|_{C_1} \, dy = \frac{e^{\mu}}{\pi \left( A_1^2 + G_1^2 \right)} G_1 \partial_1 G_1 \, dy \\
J^y(x,y)\big|_{C_2} \, dx = \frac{e^{\mu}}{\pi \left( B_1^2 + F_1^2 \right)} F_1 \partial_1 F_1 \, dx
\]

(2.19)

Using the relation \( \arctan \xi + \arctan \eta = \frac{\pi}{2} \, \text{sgn}(\xi) \), we can put the above relation into the following form

\[
\text{Ind}(H) = \frac{1}{4} \left[ \text{sgn} \left( A_+ B_+ \right) + \text{sgn} \left( A_- B_- \right) \right] - \frac{1}{4} \left[ \text{sgn} \left( A_+ B_- \right) + \text{sgn} \left( A_- B_+ \right) \right]
\]

(2.26)

which shows that \( \text{Ind}(H) \) takes indeed an integer value; this is the Fukui formula [34]. For the example where \( A_+, B_+ \) have the same sign, the index vanishes identically; it vanishes also for other cases like \( \text{sgn}(A_+) = 1 \) and \( \text{sgn}(B_-) = - \text{sgn}(B_-) \). However, for the cases \( A_+, B_+ \) positive (respectively negative) definite numbers and \( A_-, B_- \) negative (respectively positive) definite ones; the index of the Hamiltonian is equal to +1 (respectively −1). For the example \( A_- = -A_+ \) and \( B_- = -B_+ \), the index of \( H \) reduces to \( \text{sgn} \left( A_+ B_- \right) \) which can be either +1 or −1 depending on the signs of \( A_+ \) and \( B_- \); for instance this is the case of \( A_- = B_+ \) and \( A_+ = -B_- \).

3. More on \( \text{Ind}(H) \) in 2-dim DBI

In this section, we shed more light of the derivation of the Fukui formula by using a direct approach based on topological mapping between the curve \( C \) in the \( x-y \) position space \( \mathbb{R}^2 \) and a corresponding curve \( \Gamma \) in the \( \phi_- \phi_+ \) Higgs space \( \mathbb{R}^2 \).

We showed amongst others that for a non trivial index the \( \text{sgn}(\phi_-) \) and \( \text{sgn}(\phi_+) \) have to change their polarity under space reflections.

3.1. Working in Higgs plane

The index formula (2.26) obtained by using the curve choice (2.16) is remarkable and suggestive; it involves only \( \text{sgn}(X) \) functions in agreement with the topological aspect of the index; it is given by the sum of four terms; each contributing with \( \pm \frac{1}{2} \) and add exactly to an integer. This relation shows that the four terms in (2.26) are in one to one correspondence with the four \( (A_-, B_-) \) corners of a rectangular curve \( \Gamma^{(4)} \) in the Higgs plane parameterised by \( (\phi_- , \phi_+ ) \). For the case \( A_- = B_+ = \pm L \), the curve \( \Gamma^{(4)} \) reduces to a square in \( \mathbb{R}^2 \), which can be imagined as depicted in the figure 2. Below, we want to re-derive equation (2.26) by working directly in the Higgs space \( \mathbb{R}^2 \). This result, which relies on using some differential geometry tools and topological mappings between \( \mathbb{R}^2 \) and \( \mathbb{R}^2 \), will be also used later on when we study the third order topological index of the three dimensional BBH lattice model.

Substituting the vector current \( J'(r) \) given by equation (2.14) into the relation equation (2.12) of the Hamiltonian index in the \( x-y \) plane namely

\[
\text{Ind}(H) = \oint_C \varpi \quad \text{with} \quad \varpi = -\frac{1}{2} \xi J'^{(i)} \, dx^i
\]

(3.1)

we obtain a new formula for the index which is completely expressed in the \( \phi_- \phi_+ \) Higgs plane; it is given by
To deal with (3.2), which reads also like the mapping terms of closed curve singularity at $R$ orientation of the circle; one may also get an integer of this loop a priori depends. This figure has two interpretations depending on whether we are sitting in $R^2$ or in $R^2_0$. From the view of $R^2_0$, we have a square $C^0$ in the $x$-$y$ plane circumscribed in a circle $S^0_0$ in red color. Under the mapping $\Phi: r \rightarrow \phi(r)$, this figure can be also interpreted in terms of closed curve $\Gamma^a$ circumscribed in a circle $S^a_0$ in the Higgs plane $R^2$.

$$\text{Ind}(H) = \oint C \frac{1}{2\pi i r^2} e^{a b} d\phi_b$$ (3.2)

where now the loop $\Gamma$ belongs to the $\phi, \rho$ Higgs plane; it is the image of the closed curve $C$ under the mapping $\phi_a : (x, y) \rightarrow (\phi_x, \phi_y)$). The above equation shows that $\text{Ind}(H)$ has a pole singularity at $|\phi| = 0$ indicating that gapless states live there. To deal with (3.2), which reads also like

$$\text{Ind}(H) = \oint C d\mu = \oint d\mu = e^{a b} d\phi_b$$ (3.3)

with $B_\mu = \frac{\phi}{2\pi i r} \phi_b$, it is interesting to use new field variables $\varrho = \varrho(x, y)$ and $\vartheta = \vartheta(x, y)$ related to the orders as

$$\begin{align*}
\phi_x &= \varrho \cos \vartheta, \\
\phi_y &= \varrho \sin \vartheta, \\
\vartheta &= \arctan \frac{\phi_y}{\phi_x}.
\end{align*}$$ (3.4)

This Higgs field change allows to bring the above index relation into the simple form

$$\text{Ind}(H) = \oint d\vartheta$$ (3.5)

which is easy to deal with. To exhibit the pole singularity, one may also use the complex variables $w = \phi_x + i \phi_y = \rho e^{i \vartheta}$ and $\bar{w} = \phi_x - i \phi_y = \rho e^{-i \vartheta}$, one ends up with the typical relation $w^{n+1} \bar{w}^{n+1}$ having non zero values for contour $\Gamma$ surrounding the pole at $w = 0$.

3.2. $\text{Ind}(H)$ in Higgs plane

Clearly, the explicit computation of the integral equation (3.5) depends a priori on the shape of the closed curve $\Gamma$. If thinking of this loop $\Gamma$ as an oriented circle with perimeter $2\pi$, we end up with $\text{Ind}(H) = \pm 1$ depending on the sense of orientation of the circle; one may also get an integer $n$ for the case of an oriented circle with a winding number $n$ where the perimeter of $\Gamma$ is $2\pi n$. However, for a loop $\Gamma$ given by a rectangular—square—cycle made of four edges as in the figures 2 and 3; one expects to obtain a relation like in (2.26).

In this rectangular case, we have

$$\Gamma^a_{\infty} = \Gamma^a_1 \cup \Gamma^a_2 \cup \Gamma^a_1 \cup \Gamma^a_2$$ (3.6)

where the $\Gamma^a_i$’s are as

$$\begin{align*}
\Gamma^a_1 : B_- \leq \varphi_y \leq B_+; \varphi_x = A_+ \\
\Gamma^a_2 : A_- \leq \varphi_x \leq A_+; \varphi_y = B_+
\end{align*}$$ (3.7)

where, to fix the ideas, we have assumed $A_+ < A_-$ and $B_+ < B_-$. But $A_+$ and $B_-$ can be either positive or negative showing that one can distinguish several cases. From these $\Gamma^a_i$ sets, we learn that they can be also defined by using ratios like

$$\begin{align*}
\Gamma^a_1 : \frac{\varphi_x}{A_+} &\leq \varphi_y \leq \frac{B_+}{A_+} \\
\Gamma^a_2 : \frac{A_-}{B_+} &\leq \varphi_y \leq \frac{A_+}{B_+}
\end{align*}$$ (3.8)

teaching us that $\frac{\varphi_y}{\varphi_x} = \arctan \vartheta$ and its inverse $\frac{\varphi_x}{\varphi_y} = \arccot \vartheta$ are the natural variables to use for describing the square loop $\Gamma^{(a)}$. Setting $\vartheta_{pq} = \frac{\varphi_y}{\varphi_x}$ and $\vartheta_{pq} = \frac{\varphi_x}{\varphi_y}$, we then have

$$\begin{align*}
\Gamma^a_1 : \arctan \frac{\varphi_x}{A_+} &\leq \vartheta \leq \arctan \frac{B_+}{A_+} \\
\Gamma^a_2 : \arccot \frac{A_-}{B_+} &\leq \vartheta \leq \arccot \frac{A_+}{B_+}.
\end{align*}$$ (3.9)

Putting the decomposition (3.6) back into the index formula (3.5) and expanding, we obtain

$$\text{Ind}(H) = \int_{\Gamma^a_1} \frac{d\vartheta}{2\pi} + \int_{\Gamma^a_2} \frac{d\vartheta}{2\pi} + \int_{\Gamma^a_1} \frac{d\vartheta}{2\pi} + \int_{\Gamma^a_2} \frac{d\vartheta}{2\pi}.$$ (3.10)

Then, using (3.9), we end up with

$$\text{Ind}(H) = \frac{1}{2\pi} \left( \arctan \frac{B_+}{A_+} - \arctan \frac{B_-}{A_-} \right) + \frac{1}{2\pi} \left( \arccot \frac{A_+}{B_+} - \arccot \frac{A_-}{B_-} \right) - \frac{1}{2\pi} \left( \arctan \frac{B_+}{A_+} - \arctan \frac{B_-}{A_-} \right) - \frac{1}{2\pi} \left( \arccot \frac{A_+}{B_+} - \arccot \frac{A_-}{B_-} \right).$$ (3.11)

that coincides precisely with the Fukui formula (2.26); and which we rewrite in the following remarkable factorised form

---

6 This number may be exhibited in our calculations by considering the general parametrisation $\phi_x = \varrho \cos(n \vartheta)$ and $\phi_y = \varrho \sin(n \vartheta)$ instead of equation (3.4) corresponding to setting $n = 1$. 
of these points of the 3D model, the fermion \( \phi \) is studied by studying the groundstate properties of a 4 \( \times \) 4 three dimensional BBH model |

A rectangular loop \( \Gamma_\infty \) circumscribed into an ellipse in the Higgs plane. One can move continuously from the ellipse to the rectangle by deforming the red and blue arcs of circles into straight line segments.

\[
\text{Ind} (H) = \frac{1}{2} \left( \text{sgn} (A_+) - \text{sgn} (A_-) \right) \times \frac{1}{2} \left[ \text{sgn} (B+) - \text{sgn} (B-) \right]
\]

(3.12)

from which we directly learn the values of \( \text{Ind} (H) \). This is a topological index relation; it depends only on the signature of the values of the Higgs components at spatial infinities; a non zero index requires the non vanishing of each factor in (3.12) showing that \( \phi_\lambda (x, y) \) has to change the sign when we go from \( x \to -\infty \) to \( x \to +\infty \) and the same thing should hold for \( \phi_\lambda (x, y) \) when going from \( y \to -\infty \) to \( y \to +\infty \). This feature ensures that the closed curve contains inside the pole singularity \( |\phi| = 0 \) where lives gapless states.

### 4. Three dimensional BBH model

The topological index of the three dimensional BBH lattice model with full open boundary condition can be determined by studying the ground state properties of a 4 \( + \times 4 \) component fermion \( \psi = \psi (x, y, z) \) near the Dirac points. Around each one of these points of the 3D model, the fermion \( \psi \) is coupled to an external space dependent field doublet \( \phi^\prime = \phi' (x, y, z) \)—an \( O (3) \) Higgs field triplet—with interacting dynamics described by the typical matrix Hamiltonian

\[
H = \Lambda^i k_i + \Lambda^i k_j + \Lambda^i k_i + \Omega^i \phi_i + \Omega^i \phi_j + \Omega^i \phi_i + \Omega^i \phi_j. \quad (4.1)
\]

In this relation, the six hermitian matrices \( \Lambda^i \) and \( \Omega^i \) are \( 8 \times 8 \) Dirac matrices realised in terms of three sets of Pauli matrices \( \sigma, \tau, \rho \) as follows

\[
\begin{align*}
\Lambda_1 &= \rho_0 \otimes \tau_2 \otimes \sigma_1, \\
\Lambda_2 &= \rho_0 \otimes \tau_2 \otimes \sigma_2, \\
\Lambda_3 &= -\rho_2 \otimes \tau_3 \otimes \sigma_0, \\
\Lambda_4 &= \rho_0 \otimes \tau_1 \otimes \sigma_0. \\
\Omega_1 &= \rho_0 \otimes \tau_2 \otimes \sigma_2, \\
\Omega_2 &= \rho_0 \otimes \tau_1 \otimes \sigma_0, \\
\Omega_3 &= -\rho_2 \otimes \tau_3 \otimes \sigma_0.
\end{align*}
\]

(4.2)

From these anticommuting matrices that generate a Clifford algebra in 6D, one define as well a chiral operator \( \Gamma^7 \) by the product \( \frac{1}{2} \Lambda_1 \Omega_1 \Lambda_2 \Omega_2 \Lambda_3 \Omega_3 \) which reads in terms of the Pauli matrices as follows

\[
\Gamma^7 = \rho_3 \otimes \gamma_5 = -\rho_3 \otimes \tau_3 \otimes \sigma_0. \quad (4.3)
\]

This operator characterises the corner states which are described by chiral wave functions. From the eigenvalue \( H \psi = E \psi \), one learns that \( \psi \) has 4 \times 4 components that can be formulated in various ways; for example like the tensor product of three two component spinors \( \xi \otimes \zeta \otimes \eta \) as done in [38]; or simply like \( \psi = (\lambda, \chi)^T \) with components

\[
\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}. \quad (4.4)
\]

The corner states are determined by solving two constraint relations generalising the previous ones in 2D; these are the gapless eigenvalue equation \( H \psi = 0 \) with

\[
H = -\frac{3}{2} \sum_{j=1}^3 \Lambda_j \frac{\partial}{\partial x^j} + \Omega^a \phi_a \quad (4.5)
\]

and the chirality condition

\[
\Gamma^7 \psi = q \tau \psi \quad (4.6)
\]

with chiral charge \( q \tau \) taking either the value +1 or −1 and which is fixed by the normality condition of the corner state.

#### 4.1. Integral formula for \( \text{Ind}(H) \) in 3D

The Hamiltonian index of the 3D model (4.5) is given by the number of chiral zero modes \( N_+ - N_- \) in the ground state. To calculate it, we proceed in a quite similar manner as we have done in 2D model. First, we think about this integer number as the flux of a three dimensional current \( J^i (r) \) like

\[
\text{Ind} (H) = \int_S \vec{J} \cdot d\vec{S} = \int_S J^i dS_i \quad (4.7)
\]

where \( d\vec{S} \) is a vector surface element and \( S \) a closed surface in the \( x-y-z \) space \( B^4_x \) traversed by the flux of \( \vec{J} \). We can equivalently express the above relation as follows

\[
\text{Ind} (H) = \int_S \varpi \quad (4.8)
\]

where now \( \varpi \) is 2-form in the real 3D space

\[
\varpi = \frac{1}{2} \varepsilon_{ijk} dx^i \wedge dx^j \quad (4.9)
\]

related to the vector current \( J^i \) like \( \varpi = \varepsilon_{ijk} J^j \) with \( \varepsilon_{ijk} \) the completely antisymmetric Levi–Civita tensor in 3D. By substituting, we obtain

\[
\text{Ind} (H) = \frac{1}{2} \int_S \varepsilon_{ijk} J^j dx^i \wedge dx^j \quad (4.10)
\]

Having introduced the integral formula for \( \text{Ind}(H) \), we come now to describe the vector current \( J^i = \langle J^i \rangle \). It is given by the vacuum expectation value of an axial vector current \( J^i \) involving the \( \Gamma^7 \) matrix as shown on the following expression

\[
J^i (r) = \lim_{r \to r'} \left[ \lim_{m \to 0} \text{tr} \left( \Gamma^7 \Lambda^i \frac{1}{m + i\hbar} \delta_3 (r - r') \right) \right] \quad (4.11)
\]
that descends from an underlying quantum field theory description where \( J' = \psi^\dagger \Gamma_7 \Lambda^i \psi \). The expression of the above current \( J' (r) \) can be also motivated from the two dimensional analysis of section 2; in particular from the study done subsection 2.1. The \( J' (r) \) in equation (4.13) is nothing but the 3D-extension of the two dimensional current given by equation (2.13). By comparing the two expressions (4.11) and (2.13), one learns amongst others the two following indicators: First, the 2D- Dirac- delta function \( \delta_2 (r - r') \) in (2.13)—with \( r = (x, y) \)—has been promoted to the three dimensional \( \delta_3 (r - r') \) in (4.11)—with \( r = (x, y, z) \). Second, the chiral operator \( \gamma_5 \) in equation (2.13) has been also promoted to the chiral operator \( \Gamma_7 \) of 3D space. Recall that the usual vector space dimension \( d_0 \) and the spinor dimension \( d_s \) are related to each other like \( d_s = 2 d_0 \). So, we have in 2D, we have \( d_s = 2^2 = 4 \) and then \( \gamma_5 \) is a \( 4 \times 4 \) matrix realised in our study as \( \tau_3 \otimes \sigma_0 \). In three dimensions, the spinor dimension \( d_s \) is equal to \( 2^3 = 8 \); then \( \Gamma_7 \) is a \( 8 \times 8 \) matrix realised here as \( \rho_3 \otimes \gamma_5 \). By calculating the trace over the \( 8 \times 8 \) matrix product in (4.11) and taking the vanishing mass limit, the above vector current can be expressed in terms of the Higgs field triplet as

\[
J' = \frac{\varepsilon^{abc}}{8\pi |\phi|^4} \epsilon^{ij\alpha} \phi_i \partial_\alpha \phi_j \partial_\alpha \phi_k \tag{4.12}
\]

with \( |\phi|^2 = \phi_x^2 + \phi_y^2 + \phi_z^2 \). An explicit derivation of equation (4.12) is as done in the appendix for the 2D case by following the same steps described there and by using properties of the \( \mathbf{A} \) and \( \mathbf{W} \) matrices (4.2) which are induced by the usual features of Pauli matrices. Putting this 3D vector current \( J' \) back into the relation (4.10), we get the following index formula

\[
\text{Ind} (H) = \int_{\Sigma} \frac{1}{8\pi |\phi|^4} \epsilon^{ij\alpha} \phi_i \partial_\alpha \phi_j \partial_\alpha \phi_k e^{abc} \tag{4.13}
\]

which can be interpreted as the winding number of the three component Higgs vector \( \phi_a \) field around the closed surface \( \Sigma \).

### 4.2. Topological index from 3D Higgs space

To determine the \( \text{Ind} (H) \) value of the Hamiltonian (4.5), we have to perform the integral in equation (4.13). To that purpose, we will proceed as follows: as a front matter, we give some useful properties of the relation (4.13) including two examples of shapes of the surface \( \Sigma \) appearing in the integral formula of \( \text{Ind} (H) \); these \( \Sigma \) shapes are given by a 2-sphere \( S^2_\infty \) at infinity; and a parallelepiped surface \( S^{(8)} \) respectively as in equations (4.16) and (4.17). Then, we use results from differential geometry on \( \mathbb{R}^3 \) to map equation (4.13) into an equivalent formula for \( \text{Ind} (H) \) which is completely expressed in the Higgs space. This new formula is given by

\[
\text{Ind} (H) = \int_{\Sigma} \frac{1}{8\pi |\phi|^4} \epsilon^{ij\alpha} \phi_i \partial_\alpha \phi_j \wedge d\phi_k e^{abc} \tag{4.14}
\]

it involves a closed surface \( \Sigma \) to be introduced later on and has a singularity at \( |\phi| = 0 \). The above formula, to be derived in what follows, plays an important role in our forthcoming calculations.

### 4.2.1. Irregular boundary surface \( \Sigma \)

Roughly speaking, the \( \text{Ind} (H) \) defined by equation (4.13) is a function of \( \Sigma \); so the integral formula (4.13) can be defined as

\[
\text{Ind} (H) = \mathcal{F} [\Sigma] \tag{4.15}
\]

This expression shows that if we want to find the numerical value of \( \text{Ind} (H) \), we should specify \( \Sigma \); unless if the integral (4.13) is independent of the shape of \( \Sigma \). This is precisely what happens in the present case since the \( \text{Ind} (H) \) is a topological invariant meaning that \( \mathcal{F} [\Sigma] = \mathcal{F} [\Sigma'] \) for any \( \Sigma \) related to \( \Sigma' \) by a continuous transformation. In this topological change the \( \Sigma \) is deformed to \( \Sigma' \) without affecting the value of \( \text{Ind} (H) \). Nevertheless, let us think of the closed surface in (4.13) as given by a big \( S_\infty \) at the boundary of the \( x\text{-y-z} \) space \( \mathbb{R}^3_\infty \) as in the following figure. For concreteness, we consider below two \( S_\infty \) and \( S^{(8)} \) shapes as follows: (i) \( S_\infty \) given by a 2-sphere \( S^2_\infty \) with defining equation

\[
S^2_\infty : x^2 + y^2 + z^2 = r^2_\infty \tag{4.16}
\]

it is the boundary of the ball \( V_\infty \) with equation \( |r| < r_\infty \); so we have \( S^2_\infty = \partial V_\infty \); and (ii) \( S^{(8)} \) given by the boundary surface \( S^{(8)} \) of cube—or in general a regular parallelepiped—. This cubic shape is an interesting situation that concerns 3D matter with full open boundary conditions. In this case, \( S^{(8)} \) delimits a volume \( V^{(8)} \) and is given by

\[
S^{(8)} = (S^1_+ \cup S^-_1) \cup (S^2_+ \cup S^-_2) \cup (S^3_+ \cup S^-_3) \tag{4.17}
\]

Concretely \( S^{(8)} \) has \( 3 + 3 \) planar faces \( S^j_\pm \); and eight tops with coordinates \((x, y, z) = (\pm L, \pm L, \pm L) \). The planar \( S^j_\pm \) are normal to the \( x\text{-y-z} \) directions as depicted in the figure 4; they bi-intersect along the 12 following segments

\[
C^j_{\pm} \sim S^j_\pm \cap S^j_\pm, \quad p, q = \pm \tag{4.18}
\]

and tri-intersect at the eight tops

\[
T^{(p,q,s)} \sim S^j_\pm \cap S^j_\pm \cap S^j_\pm, \quad p, q, s = \pm \tag{4.19}
\]

This \( S^{(8)} \) can be also interpreted as describing the boundary of a cube—parallelepiped—circumscribed into a 2-sphere—ellipsoid—as depicted in the figure 5. With this picture, one clearly see that one can pass from \( S^{(8)} \) to the 2-sphere \( S^2_\infty \) by deforming the planar faces \( S^j_\pm \) into spherical calottes with a square (rectangular) section. The surface shape (4.17) will be used later on when studying the 3D extension of the construction done for the 2D model studied in the previous section.
whose homologue in the Higgs space is obtained by setting $\phi=\phi_\Sigma$ at eight tops. It corresponds to $S^{(8)}$ from the view of the position space $R^3_\Sigma$ and to the surface $\Sigma^{(8)}_\infty$ from the view of $R^3_\phi$. 

4.2.2. Deriving the signature equation (1.1). A nice way to determine the $\text{Ind}(H)$ given by equation (4.13) is to formulate it directly in the Higgs space $R^3_\Phi$ parameterised by $(\phi_x, \phi_y, \phi_z)$ and given by equation (4.14). For that, we substitute the two following relations,

$$dS^{(p)}_\Sigma = \sum_{i,j} \epsilon_{ijkl} \phi_{ij} \phi_k \phi_l \; dx^i \wedge dx^j$$
$$d\phi_\Sigma = \sum_{i,j} \epsilon_{ijkl} \phi_{ij} \; \partial_l \phi_k \; dx^i$$

back into equation (4.13) to end up exactly with (4.14). But, in this index relation, the $\Sigma$ is a closed surface in the Higgs space—$\Sigma$ is contained in $R^3_\Phi$—; it is given by the correspondence $\Phi: S \to \Sigma$ mapping the real surface $S$ of the $x$-$y$-$z$ space $R^3_\Sigma$ into the surface $\Sigma$ belonging $\phi_x^i, \phi_y^j, \phi_z^k$ space. Under this correspondence, each point $r = (x, y, z)$ in the 3D position space $R^3_\Sigma$ gets mapped into a point $\Phi = (\phi_x, \phi_y, \phi_z)$ in the 3D Higgs space $R^3_\Phi$. Notice that because of this mapping, several relations in $R^3_\Sigma$ can be also mapped into corresponding ones in $R^3_\Phi$. Below, we give two interesting examples respectively dealing with equation (4.7) and equations (4.16) and (4.17). The first example concerns the flux formula (4.7) whose homologue in the Higgs space is obtained by setting

$$d\sigma^{\alpha} = \frac{1}{2} \epsilon^{\alpha\beta\gamma} d\phi_\beta \wedge d\phi_\gamma$$

into equation (4.14); this substitution allows to bring the flux formula into the interesting form

$$\text{Ind}(H) = \int \sum B_a d\sigma^\alpha = \int \sum B_a d\tau$$

with vector field $B_a = \frac{\partial \phi_\Sigma}{\partial x_a}$. This index relation, which describes the flux of $B_a$ through $\Sigma$, is remarkable in the sense it can be also put in correspondence with the well known Gauss theorem concerning the electrostatic field of Coulomb theory. The second example concerns the surface $S$ and its image $\Sigma$; for $S_\infty$ given by the 2-sphere $S_\infty^2$ of equation (4.16), it is associated a 2-sphere $\Sigma_\infty^2$ given by

$$S_\infty^2 : \phi_x^2 + \phi_y^2 + \phi_z^2 = \phi_\Sigma^2.$$ 

Similarly, for the $S^{(8)}_\infty$ given by (4.17), we have the corresponding Higgs space surface

$$\Sigma^{(8)}_\infty = (\Sigma^+_1 \cup \Sigma^{-}_1) \cup (\Sigma^+_2 \cup \Sigma^{-}_2) \cup (\Sigma^+_3 \cup \Sigma^{-}_3)$$

it has 3 + 3 faces $\Sigma^{(8)}_\infty$ normal to the $\phi_x$-directions in the Higgs space; twelve edges

$$C^{(p,q)}_f \sim \epsilon_{ij} \phi^p_{ij} \cap \Sigma^q_f$$

and eight tops

$$T^{(p,q,r)} = \Sigma^p_1 \cap \Sigma^q_2 \cap \Sigma^r_3, \quad p, q, r = \pm$$

with coordinates $(\phi_x, \phi_y, \phi_z) = (A_{\pm}, B_{\pm}, C_{\pm})$. Here also $\Sigma^{(8)}_\infty$ has an interpretation in terms of a cube—parallelipiped—circumscribed into the 2-sphere $S_\infty^2$—ellipsoid—as depicted in the figure 5. In this realisation, the two parallel faces $\Sigma^\pm_1$ are normal to $\phi_x$-direction and are located at $\phi_\alpha = \pm \phi_\Sigma$; a similar thing is valid for $\Sigma^\pm_2$ and $\Sigma^\pm_3$ respectively normal to $\phi_y$- and $\phi_z$-directions. For later use notice that the planar faces $\Sigma^\pm_1$ can be also defined as cross products of edges. Denoting the 12 edges of the cube—parallelipiped—like

$$C^{(p,q)}_f : A_- \leq \phi_x \leq A_+, \quad (\phi_y, \phi_z) = (B_p, C_q)$$

and the union of the four upper (respectively lower) edges of the face $\Sigma^+_1$ (respectively $\Sigma^-_1$) as

$$T^{(p)}_{\Sigma_1} = C^{(p,q)}_1 \cup C^{(-p,q)}_1 \cup C^{(-p,-q)}_1 \cup C^{(p,-q)}_1$$

we can think of $\Sigma^{(8)}_\infty$ as the union $\Sigma_L \cup (\Sigma^+_1 \cup \Sigma^-_1)$ with $\Sigma_L$ standing for the lateral surface $(\Sigma^+_2 \cup \Sigma^-_2) \cup (\Sigma^+_3 \cup \Sigma^-_3)$ of a cylinder with rectangular cross section. Moreover, using the relation $\Sigma_L \sim C^+_1 \times T_{\Sigma_1}$, we end up with
\[ \Sigma_{\infty}^{(8)} \sim (\Sigma_{x} \times \Sigma_{y}) \cup (\Sigma_{x}^{+} \cup \Sigma_{y}^{+}) \]  \hspace{1cm} (4.29)

With these data on the surface \( \Sigma_{\infty} \) and its image \( \Sigma_{\infty}^{(8)} \) in the Higgs space, we are now in position to determine the value of \( \text{Ind}(H) \) by using the Higgs space formula \((4.22)\). For the spherical choice \( \Sigma_{\infty} = \Sigma_{0}^{2} \), it is interesting to use following change of field variable,

\[
\phi_{x} = \phi \sin \vartheta \cos \varphi \\
\phi_{y} = \phi \sin \vartheta \sin \varphi \\
\phi_{z} = \phi \cos \vartheta
\]

(4.30)

to perform the integral in \((4.22)\). From this change, we learn the associated quantities \( \vartheta^{2} = \phi_{x}^{2} + \phi_{y}^{2} \) and \( \varphi^{2} = \vartheta^{2} + \phi_{z}^{2} \) as well as

\[
\tan \varphi = \frac{\phi_{y}}{\phi_{x}}, \quad \cos \vartheta = \frac{\phi_{z}}{\phi} , \quad \tan^{2} \vartheta = \frac{\vartheta^{2}}{\phi^{2}}.
\]

(4.31)

Putting the field change \((4.30)\) back into equation \((4.22)\), we obtain

\[
\text{Ind}(H) = \pm \frac{1}{4\pi} \int_{\Sigma_{\infty}} \sin \vartheta d\vartheta d\varphi
\]

(4.32)
giving \( \pm 1 \); thanks to the radial symmetry of \( B_{\rho} \). For the Cartesian shape \( \Sigma_{\infty}^{(8)} \) given by \((4.24)\), it is interesting to still use the \( (\vartheta, \varphi) \) angles to compute the index \( \int_{\Sigma_{\infty}} B_{\rho} d\sigma^{\rho} \) which, as shown on \((4.29)\), decomposes as the sum of three terms like \( \mathcal{J}_{0} + \mathcal{J}_{+} + \mathcal{J}_{-} \) with \( \mathcal{J}_{0,\pm} \) respectively given by

\[
\mathcal{J}_{0} = \int_{\Sigma_{x} \times \Sigma_{y}} B_{\rho} d\sigma^{\rho}, \quad \mathcal{J}_{\pm} = \int_{\Sigma_{x}^{\pm}} B_{\rho} d\sigma^{\rho}.
\]

(4.33)

Straightforward calculations lead to

\[
\mathcal{J}_{0} = \frac{T_{\rho \rho}}{2} (\cos \vartheta_{+} - \cos \vartheta_{-}) \\
\mathcal{J}_{+} = \frac{T_{\rho \rho}}{2} (1 - \cos \vartheta_{+}) \\
\mathcal{J}_{-} = \frac{T_{\rho \rho}}{2} (\cos \vartheta_{-} + 1)
\]

(4.34)

with \( \vartheta_{\pm} = \arccos \frac{\phi_{x}}{\phi} \) and \( T_{\rho \rho} \) as follows

\[
T_{\rho \rho} = \frac{1}{4} \left[ \text{sgn} (A_{+}B_{+}) + \text{sgn} (A_{-}B_{-}) \right] - \frac{1}{4} \left[ \text{sgn} (A_{+}B_{-}) + \text{sgn} (A_{-}B_{+}) \right].
\]

(4.35)

The sum of the three relations of equation \((4.34)\) gives exactly \( T_{\rho \rho} \); it looks as it does not depend on \( \text{sgn}(C_{\rho}) \); and, according to \((3.12)\), it is an integer. A way to interpret the absence of \( \text{sgn}(C_{\rho}) \) is because of the choice we have used in our calculation namely \( \text{sgn}(C_{\rho}) = - \text{sgn}(C_{\rho}) = -1 \). To implement, the contribution of \( \text{sgn}(C_{\rho}) \), we think of the above index as given by

\[
\text{Ind}(H) = \frac{T_{\rho \rho}}{2} \left[ \text{sgn} (C_{+}) - \text{sgn} (C_{-}) \right]
\]

(4.36)

by setting \( \text{sgn} (C_{+}) = 1 \), one recovers \((4.35)\). This relation can be also motivated by its factorised form \((3.12)\) and cyclic symmetry properties allowing to determine \( \mathcal{I}_{\rho \rho} \) and \( \mathcal{I}_{\rho \rho} \) from \( \mathcal{I}_{\rho \rho} \).

Expanding the above generalisation, we get

\[
\text{Ind}(H) = \frac{1}{8} \left[ \text{sgn} (A_{+}B_{+}C_{+}) + \text{sgn} (A_{-}B_{-}C_{-}) \right] + \frac{1}{8} \left[ \text{sgn} (A_{+}B_{+}C_{-}) + \text{sgn} (A_{-}B_{-}C_{+}) \right] - \frac{1}{8} \left[ \text{sgn} (A_{+}B_{-}C_{+}) + \text{sgn} (A_{-}B_{+}C_{-}) \right] - \frac{1}{8} \left[ \text{sgn} (A_{+}B_{-}C_{-}) + \text{sgn} (A_{-}B_{+}C_{+}) \right]
\]

(4.37)

having eight contributions in one to one correspondence with the corners of the cube—parallelepiped—with tops \((A_{\pm}, B_{\pm}, C_{\pm}) \). In the end notice that, like for \((3.12)\) of the 2D model, this topological formula factorises as follows,

\[
\text{Ind}(H) = \frac{1}{2} \left[ \text{sgn} (A_{+}) - \text{sgn} (A_{-}) \right] \frac{1}{2} \left[ \text{sgn} (B_{+}) - \text{sgn} (B_{-}) \right] \times \frac{1}{2} \left[ \text{sgn} (C_{+}) - \text{sgn} (C_{-}) \right]
\]

(4.38)

and showing that here also that \( \text{sgn}(\phi_{\rho}) \) has to change its polarity when we go from minus infinity to plus infinity in order to have a non trivial value of the index. For a non trivial value of \( \text{Ind}(H) \), the surface \( \Sigma \) has to englobe the pole singularity at \( |\phi| = 0 \).

5. Conclusion and comments

In this paper, we studied the topological properties of the three dimensional BBH lattice model falling into the DBI class in the AZ periodic table with reflection symmetries. First, we revisited the 2D model and re-derived the topological index \( \text{Ind}(H_{DP}) \) of this theory by using topological mapping from the real \( x-y \) plane \( \mathbb{R}^{2} \) into the \( \phi_{x, y} \) Higgs plane \( \mathbb{R}^{2} \). Then, we investigated the topological \( \text{Ind}(H_{ND}) \) of the 3D theory and calculated its expression in terms of the limit values of the Higgs field triplet at space infinity. The topological index formula given by \((4.37)\) can remarkably factorise like in \((4.38)\). Our method revealed that the results obtained for 2D, given by \((2.26) \) and \((3.12) \); and their homologue derived for 3D, correspond to leading terms of a general DBI topological index formula given by

\[
\text{Ind}(H_{ND}) = \prod_{\rho = 1}^{N} \frac{1}{2} \left[ \text{sgn} (A_{+\rho}) - \text{sgn} (A_{-\rho}) \right]
\]

(5.1)

where \( (A_{\pm\rho})_{\rho \in \mathbb{N}} \)’s stand for the values at space infinities of an \( N \)-component Higgs field multiplet; i.e. \( \phi = (\phi_{\rho})_{\rho \in \mathbb{N}} \). Non zero topological index requires \( A_{+\rho} \) and \( A_{-\rho} \) to have opposite signs. The above relation depends only on the signatures of the \( A_{\rho} \) values with \( p_{\rho} = \pm 1 \); and can be also expressed in other different ways: for example like

\[
\text{Ind}(H_{ND}) = \prod_{\rho = 1}^{N} \sum_{p_{\rho} = \pm 1} \frac{1}{2} \text{sgn} (A_{\rho}).
\]

(5.2)
By setting $\text{sgn} \left( A_{\pm a} \right) = (-)^{\xi \pm a}$ with integer $\xi, a = 0, 1 \text{ mod } 2$, the above index formula can be re-expressed as follows
\[
\text{Ind}(H_{ND}) = \prod_{a=1}^{N} \frac{1}{2} \left( e^{i\xi + a} - e^{-i\xi - a} \right) \tag{5.3}
\]
which, up to a sign, reads like $1/2\prod_{a=1}^{N} (1 - e^{i\xi})$ with $\xi$ given by the difference $\xi \pm a$. Like for 2D and 3D, a non zero value of the Hamiltonian index in higher dimensions requires that the boundary hypersurface $\Sigma$ contains in its inside the pole singularity $\phi = 0$.

**Acknowledgment**

The authors would like to acknowledge ‘Académie Hassan II des Sciences et Techniques-Morocco’ for financial support. L B Drissi acknowledges the Alexander von Humboldt Foundation for financial support via the Georg Forster Research Fellowship for experienced scientists (Ref 3.4-MAR-1202992).

**Appendices A**

Here, we give two appendices A.1 and A.2 aiming some technical details, which for simplicity of the presentation and also for the chain of ideas, have been omitted in the heart of the paper. In appendix A.1, we give some useful information on the Hamiltonians (2.1)–(2.6) and make a comment regarding the vector field $(\hat{\phi}_x, \hat{\phi}_y, \hat{\phi}_z)$. In appendix A.2, we give explicit details regarding the derivation of the 2D topological current of equation (2.14) used in sections 3 and 4.

**A.1. More on Hamiltonian (2.1)**

The 3D- extension of the two dimensional BBH lattice Hamiltonian model with full open boundary condition reads in reciprocal space as follows
\[
H_{\text{lat}} = + t_x (\sin k_x) \Lambda^x + (\Delta_x + t_c \cos k_x) \Omega^x + t_y (\sin k_y) \Lambda^y + (\Delta_y + t_c \cos k_y) \Omega^y + t_z (\sin k_z) \Lambda^z + (\Delta_z + t_c \cos k_z) \Omega^z \tag{6.1}
\]
where $\Lambda^i$ and $\Omega^i$ are hermitian $8 \times 8$ matrices given by equation (4.2) and where $\Delta$ and $t_c$ are hopping parameters: intra and extra unit cells. For simplicity of the presentation given below, we set $t_x = t_y = t_z = 1$; the reduction down to 2D can be obtained by cutting the $z$-direction ($t_z = \Delta_z = 0$) and thinking of the remaining reduced $\Lambda^x$ and $\Omega^x$ as hermitian $4 \times 4$ matrices with realisation as in equation (2.7). Notice that by setting $\Theta_i = (\Delta_i + \cos k_i)$ with $i = x, y, z$, we can express the above lattice Hamiltonian $H_{\text{lat}}$ as a matrix function of the 3 momentum variables $(k_x, k_y, k_z)$ and the three $(\Theta_x, \Theta_y, \Theta_z)$; that is
\[
H_{\text{lat}} = H_{\text{lat}} \left( k_x, k_y, k_z; \Theta_x, \Theta_y, \Theta_z \right). \tag{6.2}
\]
However as $\Theta_i = \Theta(k_i)$, the lattice Hamiltonian is then a function $H_{\text{lat}} \left( k_x, k_y, k_z \right)$; so the symmetry constraints (2.2) and (2.3) apply as well to $H_{\text{lat}} = \sum_i (\Lambda^i \sin k_i + \Omega^i \Theta_i)$. To see the relationship between this $H_{\text{lat}}$ and (2.1); we calculate the eight energy eigenvalues of $H_{\text{lat}}$: they are nicely obtained by computing $H^2_{\text{lat}}$ which turns out to be proportional to the identity matrix $I_{8 \times 8}$; that is $H^2_{\text{lat}} = E^2_{\text{lat}} I_{8 \times 8}$. This feature leads to the two following states $E^2_{\text{lat}}$ energies with multiplicity of order 4.
\[
E^2_{\text{lat}} = \pm \sqrt{\sum_{i=x,y,z} \left[ \sin^2 k_i + (\Delta_i + \cos k_i)^2 \right]} \tag{6.3}
\]
The gap energy is determined by looking for the minimal value of $(E^2_{\text{lat}})_{\text{min}}$; it is obtained by solving the vanishing of two following sets of equations:

(I): $\sin k_i = 0$,  \hspace{1cm} (II): $\Delta_i + \cos k_i = 0$. \hspace{1cm} (6.4)

While the set (II) shows that for $|\Delta| > 1$, the system is gapped; the first set teaches us interesting information. First, it has eight solutions that can be expressed like $k^*_i = n_i \pi$ with $n_i = 1, 0 \text{ mod } 2$. Therefore, given a fix point $(k^*_x, k^*_y, k^*_z) = (n_x \pi, n_y \pi, n_z \pi)$ with some integers $(n_x, n_y, n_z)$, the momentum vector $k$ around the $k^*_i$ expands as follows
\[
k_i \approx k^*_i + k'_i = n_i \pi + k'_i \tag{6.5}
\]
where $k'_i$ is a small deviation; that is $|k'_i|/|k^*_i| \ll 1$. Second, putting this change back into equation (6.1), we can approximate the lattice Hamiltonian $H_{\text{lat}}(k)$ near $k^*$ like
\[
H_{\text{lat}, n_i \pi, n_i \pi} \left( k' \right) + O \left( k'^2 \right) \tag{6.6}
\]
with $\phi'_i = 1 + \Delta_i \cos (n_i \pi)$. For example, near the point $(k^*_x, k^*_y, k^*_z)$ given by the origin $(0, 0, 0)$, we have the Hamiltonian $H_{0,0,0} = \sum_i \left( k'_i \Lambda^i + \phi'_i \Omega^i \right)$ with $\phi'_i = (1 + \Delta_i)$. Quantum fluctuations are described by realising the momentum deviations $k'_i$ in terms of the operators $k'_i = -i \nabla_i$ and so on. Topological properties are described by promoting the above $\phi'_i$ space coordinate dependent Higgs $\phi(r')$ with $r' = (x', y', z')$. By dropping out the primes from $k'$ and $r'$; one recovers amongst others equations (2.1)–(2.6).

**A.2. Derivation of equation (2.14)**

Here, we give a rapid sketch of the derivation of the topological current $J^r(r)$ given by equation (2.14) by starting from the following definition of the axial current using Pauli–Villars regularisation parameter $M$,
\[
J^r(r) = \lim_{r' \to r} \lim_{m \to 0} \lim_{M \to \infty} \left[ g_s \Lambda^r \left( \frac{1}{iH + m} - \frac{1}{iH + M} \right) \delta_2 \times (r' - r) \right]. \tag{6.7}
\]
Strictly speaking, this relation involves four operations that we have to perform in order to put $J^r(r)$ into the remarkable form (2.14) used in the paper. These operations are given by the matrix trace $tr(\ldots)$ which we have to calculate; and more-
over three limits namely \( \lim_{M\to\infty} \) and \( \lim_{m=0} \) as well as \( \lim_{r\to \infty} \) that we have to perform as well. In our present situation, the limit \( \lim_{M\to\infty} \) is trivial and the term \( 1/(H + M) \) in (6.7) can be dropped out. So, we are left with the basic term that we express like

\[
J'(r) = \lim_{r\to \infty} \lim_{m=0} \left[ \gamma_5 A' \frac{m - iH}{H^2 + m^2} \delta_2 (r - r') \right]. 
\] (6.8)

To proceed, we first perform the limit \( r\to \infty \) by substituting the Dirac-delta function \( \delta_2 (r - r') \) by its expression as an integral over the plane waves \( e^{ik(r-r')} \) and think of \( H \) like a differential operator \( -i\Lambda \partial_j + \Omega^a \partial_a \); this leads to put \( J'(r) \) in the form \( \lim_{m=0} J'(r; m) \) with

\[
J'(r; m) = \int \frac{d^3k}{(2\pi)^3} \text{tr} \left[ \gamma_5 A' e^{-ikr} \left( \frac{m - iH}{H^2 + m^2} \right) e^{ikr} \right]. 
\] (6.9)

We will show later on that the \( \lim_{m=0} \) is also trivial here; but let us keep it for a moment and kill it at proper time. The remaining steps to do are technical and rely on some key ingredients; in particular the two following ones that we want to comment on as they concern crucial stages: (i) The way to deal with the inverse operator \( \frac{1}{I + \Theta} \) appearing in (6.9) as it hides a difficulty that we have to overcome in order to get a simple expression of the current. The point is that the quantity \( e^{-ik^r H^2 + m^2} \) which is equal to \( k^2 + \phi^2 - i\Lambda \text{\Omega}^a \partial_a \Omega^b \) with \( \Omega^a = -\nabla \phi - 2i(k, \nabla) \), involves the field matrix \( \text{\Omega}^a \partial_a \Omega^b \). After substituting, we can put \( e^{-ikr H^2 + m^2} \) as the product of \( (k^2 + \phi^2 + m^2)^{-1} \) with \( 1/(I + \Theta_m) \) where \( \Theta_m \) is roughly given by the number \( (k^2 + \phi^2 + m^2) \); that is

\[
\Theta_m = \frac{-i\Lambda \text{\Omega}^a \partial_a \phi + \Omega_2}{k^2 + \phi^2 + m^2}. 
\] (6.10)

In other words, the term \( (H^2 + m^2)^{-1} \) in (6.9) involves the fraction \( 1/(I + \Theta_m) \) having matrices in the denominator. But this dependence poses a problem when coming to the explicit calculation of the matrix trace in (6.9). Hopefully, one can use perturbation theory methods to replace \( 1/(I + \Theta_m) \) by its expansion in \( \Theta \) powers given by \( I - \Theta_m + \) higher powers. This demands however assuming \( \Theta \) small which is equivalent the condition \( \partial_j \phi \ll \phi \) requiring a slow variation of the field gradient \( \nabla \phi \) with respect to the variation of \( \phi \). (ii) The second key ingredient we want to come from the computation of matrix traces like \( \text{tr} [\gamma_5 A' X] \) where \( X \) is given by products type \( \prod_{l=1}^{m} \Lambda^l \epsilon^{a_1 \cdots a_l} \Omega^b \) and having in mind that \( \text{tr} [\Lambda^n] = \text{tr} (\Omega^n) = 0 \) and \( \text{tr} (I_i) = 4 \). However, traces of the form \( \text{tr} [\gamma_5 A' X] \) have non vanishing values except for \( X \equiv I_1 \) proportional to the product \( \partial_j \Lambda^l \times (\Omega^a \Omega^b \epsilon_{ab}) \). In this regards, recall that \( \gamma_5 = -\Lambda^a \Omega^b \Omega^c \Omega^d \) and so \( \text{tr} [\gamma_5 \Lambda^l \Omega^a \Omega^b \Omega^c \Omega^d] \). This means that the \( m^{-1+n} \) in (6.9) should contribute by \( X_1 \) given by the product of three matrices: one appropriate \( \Lambda \) and two \( \Omega \)’s. This feature rules out the term \( \frac{m}{m^2+n} \) in (6.9) leaving only \( m^{-1+n} \). This is because the monomials \( [\Theta_m]^n \) in the expansion of \( 1/(I + \Theta_m) \) produces terms like \( (\Lambda^a \Omega^b)^n \) while a non vanishing trace requires terms as \( \varepsilon_j \Lambda^l \times (\Omega^a \Omega^b \epsilon_{ab}) \). By taking the limit \( \lim_{m=0} \), we can put (6.9) into the following factorised form

\[
J'(r) = \text{tr} (\gamma_5 \Lambda^a \Omega^a \Omega^b) \times \phi_a \partial_j \phi_b \times \int d^3k \frac{1}{(2\pi)^2 (k^2 + \phi^2)^2} \] (6.11)

where we have dropped out irrelevant terms such as those involving the \( \Omega \)’s. By substituting the trace \( \text{tr} (\gamma_5 \Lambda^a \Omega^a \Omega^b) \) by its value \( 4\varepsilon_j \epsilon_{ab} \); then performing the change \( q = k/|\phi| \) and using the integral result \( \int_0^\infty \frac{\rho_d}{(1+r^2)} = \frac{1}{2} \), we end up with the following expression

\[
J'(r) = \frac{1}{2} \times \frac{1}{2\pi} \times 4 \frac{1}{|\phi|^2} \varepsilon_j \epsilon_{ab} \] (6.12)

which is precisely equation (2.14).

**ORCID IDs**

L B Drissi \( \text{https://orcid.org/0000-0002-1966-9025} \)

**References**

[1] Altland A and Zirnbauer M R 1997 Phys. Rev. B 55 1142

[2] Schnyder A P, Ryu S, Furusaki A and Ludwig A W W 2008 Phys. Rev. B 78 195125

[3] Benalcazar W A, Bernevig B A and Hughes T L 2017 Science 357 61

[4] Benalcazar W A, Bernevig B A and Hughes T L 2017 Phys. Rev. B 96 245115

[5] Schindler F, Cook A M, Vergniory M G, Wang Z, Parkin S S P, Bernevig B A and Neupert T 2018 Sci. Adv. 4 eaat0346

[6] Song Z, Fang Z and Fang C 2017 Phys. Rev. Lett. 119 246402

[7] Serra-Garcia M, Peri V, Süssstrunk R, Bilal O R, Larsen T, Villanueva L G and Huber S D 2018 Nature 555 342–5

[8] Peterson C W, Benalcazar W A, Hughes T L and Bahl G 2018 Nature 555 346–50

[9] Noh J, Benalcazar W A, Huang S, Collins M J, Chen K P, Hughes T L and Reichsmann M C 2018 Nat. Photon. 12 408–15

[10] Imhof S et al 2018 Nat. Phys. 14 925–9

[11] Xue H, Yang Y, Gao F, Chong Y and Zhang B 2019 Nat. Mater. 18 108–12

[12] Ni X, Weiner M, Alì A and Khanikaev A B 2019 Nat. Mater. 18 113–20

[13] Serra-Garcia M, Süssstrunk R and Huber S D 2019 Phys. Rev. B 99 200304(R)

[14] Schindler F et al 2018 Nat. Phys. 14 918–24

[15] Chen R, Chen C-Z, Gao J-H, Zhou B and Xu D-H 2020 Phys. Rev. Lett. 124 036803

[16] Ezawa M 2018 Phys. Rev. Lett. 120 026801

[17] Langbehn J, Peng Y, Trifunovic L, von Oppen F and Brouwer P 2017 Phys. Rev. Lett. 119 246401

[18] Serra-Garcia M, Peri V, Süssstrunk R, Bilal O R, Larsen T, Villanueva L G and Huber S D 2018 Nature 555 342–5

[19] Xue H, Yang Y, Liu G, Gao F, Chong Y and Zhang B 2019 Phys. Rev. Lett. 122 244301

[20] Xue H, Yang Y, Gao F, Chong Y and Zhang B 2019 Nat. Mater. 18 108–12

[21] Ono S and Watanabe H 2018 Phys. Rev. B 98 115150
[22] Schnyder A P, Ryu S, Furusaki A and Ludwig A W W 2008 Phys. Rev. B 78 195125
[23] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. 82 3045
[24] Kennedy R and Zirnbauer M R 2016 Commun. Math. Phys. 342 909
[25] Wang Z, Qi X-L and Zhang S-C 2010 Phys. Rev. Lett. 105 256803
[26] Karch A 2009 Phys. Rev. Lett. 103 171601
[27] Rosenberg G and Franz M 2010 Phys. Rev. B 82 035105
[28] Li H and Sun an K 2020 Phys. Rev. Lett. 124 036401
[29] van Nieuwenburg E P L and Huber S D 2014 Phys. Rev. B 90 075141
[30] Linzner D, Wawer L, Grusdt F and Fleischhauer M 2016 Phys. Rev. B 94 201105(R)
[31] Bardyn C-E, Wawer L, Altland A, Fleischhauer M and Diehl S 2018 Phys. Rev. X 8 011035
[32] Zheng J-H and Hofstett W 2018 Phys. Rev. B 97 195434
[33] Su W P, Schrieffer J R and Heeger A J 1979 Phys. Rev. Lett. 42 1698–701
[34] Fukui T 2019 Phys. Rev. B 99 165129
[35] Jackiw R and Rossi P 1981 Nucl. Phys. B 190 681–91
[36] Takane Y 2019 J. Phys. Soc. Jpn. 88 094712
[37] Okugawa R, Hayashi S and Nakanishi T 2019 Phys. Rev. B 100 235302
[38] B Drissi L et al 2020 A coupled fermion-Higgs model for third order topological insulators (in preparation).
[39] Fukui T and Fujiwara T 2010 J. Phys. Soc. Jpn. 79 033701
[40] Fujiwara T and Fukui T 2012 Phys. Rev. D 85 125034
[41] Shiozaki K, Fukui T and Fujimoto S 2012 Phys. Rev. B 86 125405
[42] Fukui T and Fujiwara T 2017 Phys. Rev. B 96 205404