Gravitational lensing of a charged Weyl black hole surrounded by plasma

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We study the light deflection as it passes a plasmic medium, surrounding a charged Weyl black hole. Considering two specific algebraic ansätze for the plasmic refractive index, we characterize the photon sphere for each of the cases. This will be used further to calculate the angular diameter of the corresponding black hole shadow. The results indicate how the complexity of the refractive index could result in the dependence of the light ray behavior on the spacetime parameters.

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\section{I. INTRODUCTION}

The recent spectacular black hole imaging [1, 2] is with no argument one of the most astonishing discoveries of the recent century. Ever since the advent of the vacuum solutions to general relativity and the possibility of gravitational collapse, rigid endeavors to find and observe these strange objects had commenced. Specially, since the last decades of the 20th century, scientists have been trying to simulate black holes by inspecting the gravitational lensing and the behavior of light in the spacetime they would define. This in fact has had a long history. The reader is encouraged to see for example Refs. [3–5] presenting studies on static black holes and Ref. [6] and references therein which provide more information on different types of lensing and their applications in astrophysics and cosmology. Note that one of the most spectacular black hole imaging has been done for the movie \textit{Interstellar} which exploits the lensing phenomena [7].

Strictly speaking, light propagation around black holes is restricted to some rules according to which, photons may or may not be able to escape to infinity and reach a distant observer. Synge had figured out such conditions in the Schwarzschild spacetime, by devising an escape cone [8] which was called later, the \textit{cone of avoidance} by Chandrasekhar [9]. This constructs the foundations of the study of the so-called \textit{black hole shadow} [10–14]. Gravitational lensing, together with the near-horizon light confinement on photon surfaces, constitute firm tools in figuring out that how black holes would look like.

The real observations of lensing effects are however done through intergalactic materials which are certainly refractive. A more realistic approach to gravitational lensing by black holes will therefore require the consideration of materials that surround these gravitating systems. This in fact have formed the foundations of optical gravity, in the sense that the exterior spacetime of a gravitating system could itself be considered as a refractive medium. Hence, the light deflection in vacuum spacetime, caused by curvature, is reduced to that caused by the refraction of a specific material [15] (see also Refs. [16, 17] for discussions on gravito-opto-mechanical analogies). This however is different from considering an extra refractive material around the gravitational source, because in this case, we need to take into account the material’s properties into the Hamiltonian system. One of the earlier discussions argued that lensing through (non-)uniform plasmic background can be thought of a tool in characterizing astrophysical objects [18, 19]. Since then, gravitational lensing in refractive (plasmic) medium has given more rigorous attention. The studies that have been published so far, cover discussions on higher order black hole imaging, as well as impacts of plasma on the shadow of (non-)static black holes [13, 20–23] (also see [24] for a good review). Having the mathematical tools in hand, we can also talk about the lensing effects generated by black holes defined in the context of alternative theories of gravity.

The inclination to such alternative theories is empowered by this belief among some scientists that the reference to the dark matter and dark energy scenarios stems from the inability of general relativity to explain unexpected cosmological phenomena, like flat galactic rotation curves [25], exorbitant gravitational lensing [26] and the accelerated expansion of the universe [27–29]. Becoming the most exciting and yet mysterious problems of the modern cosmology, the dark matter and dark energy scenarios are given attention in generalized (like $f(R)$-gravity [30]) and alternative theories [31] and these theories have been confronted with the cosmological observations [32].

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In particular, paying attention to an older higher order theory, named Weyl conformal gravity, which had been introduced by H. Weyl in 1918 [33] and revived by R.J. Riegert in 1984 [34], P.D. Mannheim and D. Kazanas were able to find a static spherically symmetric solution by solving its fourth order vacuum field equations. The solution contained some terms which could regenerate the flat galactic rotation curves as well as the vacuum energy, to support accelerated expansion of the universe [35]. This theory, since then, has been applied to various aspects of astrophysical and cosmology [36–62]. The Weyl theory of gravity, although is complicated in form, has shown interesting properties, mostly regarding its conformal invariance. We have therefore chosen this theory in our current study. This is done by studying a specific charged black hole solution obtained in Ref. [63] by confronting values from the observable universe with constant coefficients introduced in a general metric potential. This black hole has recently examined regarding the behavior of null geodesics passing its exterior geometry [64].

In this paper, we investigate light propagation around this charged Weyl black hole by immersing it into a refractive plasma, and analyze the resultant gravitational lensing caused by the geometry and the plasma. For this, we apply a canonical Hamiltonian formalism and use two different ansatzes for the plasma refractive index. This way, we determine the light deflection and the relevant photon sphere as it can be observed outside the black hole event horizon. To elaborate this, in Sec. II, we first bring some fundamentals on optical gravity which is followed by the determination of the optical paths in the plasmic surrounding, by means of a Hamiltonian formalism on the spacetime manifold’s cotangent bundle. In this section, we also build the contribution of the properties of the refractive media (the plasma) through which the light travels, and introduce the general formulation of the light’s deflection angle. In Sec. III, we propose two substantially different ansatzes for the plasma’s refractive index. Accordingly, we calculate the light deflection angle for each case and discuss their peculiarities. Furthermore, by calculating the radius of the photon spheres in either of the media with given refractive refractions, in Sec. IV, we stipulate to what extent the orbiting photons can approach the black hole without falling into its event horizon. In Sec. V we put a small gap in our discussion, to talk more about physical implications of the refractive plasma. To do this, we discuss the particle concentration in the region of casual connection outside the black hole event horizon, which is followed by making a comparison between the refractive media and a dark matter halo in the context of Navarro-Frenk-White density profile [65, 66]. The significance of photon spheres is used in Sec. VI to obtain the angular diameter of the black hole shadow in each of the mentioned plasmic media. The results also show some confinements on the impact parameter which is associated with the trajectories. Final notes are given in Sec. VII.

II. LIGHT PROPAGATION IN PLASMIC MEDIUM

A. Some backgrounds

Light propagation in medium is indeed described in the phase space, whose Hamiltonian dynamics gives the structure of the manifold’s cotangent bundle. Given the manifold \((M, g_{\alpha\beta})\) expressed in the chart \(x^\alpha\), the cotangent bundle \(T^*M\) provides the means to define the Hamiltonian \(H \equiv H(x^\alpha, p_\alpha)\) where \(p_\alpha\) is the momentum (wave) covector associated with the cotangent bundle. The Hamilton-Jacobi equation is therefore given in the form

\[
H(x^\alpha, p_\alpha) = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta = 0,
\]

in which \(g^{\alpha\beta}(x^\rho)\) is the metric describing \(T^*M\), and is called the optical metric. In this sense, the wave (co)vector \(p_\alpha\) is considered parallel to the tangential velocity 4-vector \(u^\alpha \equiv \dot{x}^{\alpha 1}\) of the light congruence, i.e. \(p_\alpha = g_{\alpha\beta} u^\beta\) and according to Eq. (1), the light propagates on null congruences with respect to the cotangent bundle. This however is not what an observer on \(M\) would measure, because \(p_\alpha \neq g_{\alpha\beta} u^\beta\) and \(g^{\alpha\beta} p_\alpha p_\beta \neq 0\). This means that light behaves like massive particles during its propagation in a medium. In general, such media are given the properties of dielectrics. In fact, the connection between the light propagation in dielectric media and that in the gravitational systems, was recognized in the early days of the advent of general relativity. According to Eddington, relativistic forms of light propagation near a massive object, can be emulated in an appropriate refractive medium [67]. In reverse, Gordon pointed out that light propagation in a medium with specific refractive properties, can be emulated in a curved spacetime background endowed with an optical metric inferred from the optical properties of that medium [68]. This connection was elaborated further in terms of the effect permittivity (\(\varepsilon\)) and permeability (\(\mu\)) of an arbitrary spacetime metric by

\[\text{Here, over-dots indicate } \partial_\tau, \text{ where } \tau \text{ is the congruence affine parameter.}\]
Plebanski [69] and for the first time, the Gordon’s optical metric was used by de Felice to construct (mathematically) a dielectric medium which could mimic a Schwarzschild black hole [15]. The Gordon’s optical metric is written as [70]

\[
g^{\alpha \beta} = g^{\alpha \beta} + (1 - n^2) v^\alpha v^\beta, \tag{2}\]

where \(n(x^\alpha) \equiv \sqrt{\varepsilon_{\mu
u}}\) and \(v^\alpha\) are respectively the scalar refractive index and the tangential velocity 4-vector of the dielectric in the comoving frame\(^2\). In order to include anisotropy, birefringence and magnetoelectric couplings, the notion of the optical metric has been given efforts to be generalized [71–74]. In the most covariant form, this metric is pseudo-Finslerian, according to the relation

\[
g^{\alpha \beta} = \frac{\partial^2 H}{\partial p^\alpha \partial p^\beta}. \tag{3}\]

In what follows, we consider that a spherically symmetric region (the exterior geometry of a black hole) is filled with a dielectric material, in the form of an inhomogeneous cold plasma with a scalar refractive index. We can therefore assume that the light follows the trajectories on the background described by Gordon’s optical metric (2).

**B. Light propagation in a spherically symmetric plasmic medium surrounding a Weyl black hole**

Weyl gravity is a theory of fourth order in the metric. The simplified form of its action reads as [35]

\[
I_W = -2\mathcal{K} \int d^4x \sqrt{-g} \left( R^{\alpha \beta} R_{\alpha \beta} - \frac{1}{3} R^2 \right), \tag{4}\]

where \(\mathcal{K}\) is coupling constant. Applying the principle of least action in the form \(\delta I_W / \delta g_{\alpha \beta} = 0\), leads to the Bach equation

\[
W_{\alpha \beta} = 0 \quad \text{in which the Bach tensor is defined as} \quad [75, 76]
\]

\[
W_{\alpha \beta} = \nabla^\sigma \nabla_\alpha R_{\beta \sigma} + \nabla^\sigma \nabla_\beta R_{\alpha \sigma} - \Box R_{\alpha \beta} - g_{\alpha \beta} \nabla^\sigma \nabla_\gamma R^{\sigma \gamma} - 2 R^\sigma_{\alpha \beta \sigma} + \frac{1}{2} g_{\alpha \beta} R^2 - 2 g_{\alpha \beta} \Box R - 2 R R_{\alpha \beta} + 1 \frac{1}{2} g_{\alpha \beta} R^2. \tag{5}\]

The spherically symmetric vacuum solution to the Bach equation, proposed by Mannheim and Kazanas was in the form [35]

\[
ds^2 = -B(r)dt^2 + B(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{6}\]

where the lapse function \(B(r)\) included dark energy and dark matter relevant terms. Here however, we consider a non-vacuum solution for the equation

\[
W_{\alpha \beta} = \frac{1}{4\mathcal{K}} \mathcal{T}_{\alpha \beta}, \tag{7}\]

in which

\[
\mathcal{T}_{\alpha \beta} = M_{\alpha \beta} + \chi_{\alpha \beta}, \tag{8}\]

is the energy-momentum tensor composed of the massive \((M_{\alpha \beta})\) and the electromagnetic \((\chi_{\alpha \beta})\) parts. Considering the completely static case, in which only the 00 element of the above tensors takes part, the exterior geometry of a charged black hole on a cosmological background has been obtained as [63]:

\[
B(r) = 1 - \frac{r^2}{\lambda^2} - \frac{Q^2}{4r^2}. \tag{9}\]

\(^2\) In fact, since the observer moves on a time-like curve on \(\mathcal{M}\), then in the \((-+++)\) sign convention, \(g_{\alpha \beta} v^\alpha v^\beta = -1\). In the same sense, the contraction \(v^\alpha p_{\alpha}\) should be normalized to a real value, which here is the energy of a photon of frequency \(\omega\) (\(E = \hbar \omega\)), evaluated by an observer, comoving with the plasma.
in which
\[
\frac{1}{\lambda^2} = \frac{3}{r^3} \left( \frac{\dot{m}}{\bar{r}^3} + \frac{2}{3} c_1 \right), \quad Q = \sqrt{2} \bar{q}, \quad (10)
\]

where \( \dot{m} \) and \( \bar{q} \) are respectively the mass and the charge distributed in a source of radius \( \bar{r} \). It is readily noted that this spacetime allows two horizons; an event horizon \( r_+ \) together with a cosmological horizon \( r_{++} \), placed at
\[
r_+ = \lambda \left[ \frac{1}{2} - \sqrt{\frac{1}{4} \left( 1 - \frac{Q^2}{\lambda^2} \right)} \right]^{1/2},
\]
\[
r_{++} = \lambda \left[ \frac{1}{2} + \sqrt{\frac{1}{4} \left( 1 - \frac{Q^2}{\lambda^2} \right)} \right]^{1/2},
\]
respectively. Thus, we can write Eq. (9) conveniently as
\[
B(r) = \frac{(r_{++}^2 - r^2)(r^2 - r_+^2)}{\lambda^2 r^2}.
\]

Accordingly, the extremal black hole with a unique horizon at \( r_{ex} = r_+ = r_{++} = \lambda/\sqrt{2} \) is obtained when \( \lambda = Q \), and the naked singularity appears when \( \lambda < Q \).

As mentioned before, we consider that this black hole is surrounded by an inhomogeneous non-magnetized, optically-thin plasmic shell. The index of refraction of such medium is given by the relation
\[
n^2(r) = 1 - \frac{\omega_p^2(r)}{\omega^2(r)},
\]
where \( \omega_p \) is the electron plasma frequency given by
\[
\omega_p^2(r) = K e N(r), \quad K = \frac{e^2}{\epsilon_0 m_e} = 3182.6 \text{[m}^3/\text{s}^2].
\]

Here \( N(r) \) is the electron concentration in plasma, \( e \) is the electric charge of the electron and \( m_e \) is the electron mass.

For the sake of simplicity, in what follows, we restrict our analysis to the equatorial plane (\( \vartheta = \pi/2 \)); hence, \( p_\vartheta = 0 \). Under such condition, applying the optical metric (2) to the Hamiltonian in Eq. (1) we get
\[
H = \frac{1}{2} \left[ g^{\alpha\beta} p_\alpha p_\beta + \hbar^2 \omega_p^2(r) \right]
\]
\[
= \frac{1}{2} \left( -\frac{p_t^2}{B(r)} + B(r)p_r^2 + \frac{\ell^2}{r^2} + \hbar^2 \omega_p^2(r) \right).
\]

Accordingly, the canonical Hamilton’s equations
\[
\dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha}, \quad \dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad (17)
\]
in the cyclic coordinates \( (t, \phi) \) yield
\[
\dot{p}_t = -\frac{\partial H}{\partial t} = 0 \Rightarrow p_t = -\hbar \omega_0 = \text{cte.}, \quad (18)
\]
\[
\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \Rightarrow p_\phi = \ell = \text{cte.}, \quad (19)
\]
regarding which, we can infer that \( \hbar \omega_0 \equiv \mathcal{E}_0 \) and \( \ell \) are constants of motion, associated with its temporal and rotational
invariance. The remaining equations read

\[ \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{\ell^2}{r^3} - \frac{d}{dr} \left[ \frac{\hbar^2 \omega_p^2(r)}{2} \right] - \frac{1}{2} \frac{dB(r)}{dr} \left[ p_r^2 + \frac{\hbar^2 \omega_p^2(r)}{B^2(r)} \right], \]

\[ \dot{\theta} = \frac{\partial H}{\partial \theta} = p_\theta = \frac{\hbar \omega_0}{B(r)}, \]

\[ \dot{\phi} = \frac{\partial H}{\partial \phi} = \frac{p_\phi}{r^2} = \frac{\ell}{r^2}, \]

\[ \dot{r} = \frac{\partial H}{\partial p_r} = B(r) p_r. \]

There is also an extra condition

\[ 0 = \frac{\ell^2}{r^2} + B(r) p_r^2 - \left[ \frac{\hbar \omega_0}{\sqrt{B(r)}} \right]^2 - \frac{\hbar^2 \omega_p^2(r)}{r^2}, \]

inferred from the Hamilton-Jacobi equation. Note that, the radial dependence of the photon’s frequency, measured by the comoving observer, is obtained by the redshift formula

\[ \omega(r) = \frac{\omega_0}{\sqrt{B(r)}}. \]

Therefore, it is no hard to see from Eqs. (24) and (25) that, in a given position \( r \), the photon frequency \( \omega(r) \) is bigger than the plasma frequency \( \omega_p(r) \), i.e.

\[ \omega(r) > \omega_p(r), \]

which is an empirical constraint for light propagation in plasma [77].

Now, turning to the subject in hand, we commence studying the light propagation in the above system. Using Eqs. (22) and (23), the general orbits are governed by

\[ \left( \frac{dr}{d\phi} \right)^2 = \frac{j^2}{\phi^2} = \mathcal{F}(r), \]

with

\[ \mathcal{F}(r) = r^2 B(r) \left[ \frac{h^2(r)}{h^2(\mathcal{R})} - 1 \right], \]

where

\[ h^2(r) = \frac{r^2 n^2(r)}{B(r)} = \frac{r^2}{B(r)} \left( 1 - \frac{\omega_p^2(r)}{\omega^2(r)} \right), \]

\[ h^2(\mathcal{R}) = \frac{\ell^2}{\hbar^2 \omega_0^2} = b^2. \]

Equation (30) relates the closest approach to the source, \( \mathcal{R} \), to the impact parameter \( b \) (another constant of motion). Therefore, exploiting Eqs. (27) to (30), the deflection angle for a light ray traveling from \( r_{++} \) to \( \mathcal{R} \) and goes again to \( r_{++} \), can be calculated as

\[ \hat{\alpha} = 2b \int_{r_{++}}^{\mathcal{R}} \frac{dr}{\sqrt{r^2 B(r) h^2(r) - b^2 r^2 B(r)}} - \pi \]

\[ = 2b \int_{r_{++}}^{\mathcal{R}} \frac{dr}{\sqrt{r^2 n^2(r) - b^2 r^2 B(r)}} - \pi. \]

The above deflection relates to the lensing effect caused by massive sources. This shows that how outermost objects can change their apparent position. The above deflection angle however could be determined specifically, once \( n(r) \) is given an appropriate algebraic expression, regarding the causality conditions. We deal with such expressions in the next section.
FIG. 1. The causal structure offered by a charged Weyl black hole. Events outside $r_{++}$ does not have casual connections with the observers residing inside it.

III. SPECIFIC CASES OF $n(r)$

The casual connection in the spacetime constructed by the charged Weyl black hole in Eq. (9), suggests that an observer inside the cosmological horizon cannot be aware of the events from the region covered by $r > r_{++}$ (see Fig. 1). For this reason, any algebraic assignment for the refractive index $n(r)$ should respect this kind of causality. This means that the refraction is well-defined only inside the boarders of the casual connection. Accordingly, we propose relevant algebraic forms, regarding the boundaries of the causality.

A. First ansatz

Taking into account a case in which $n(r_{++}) = n(r_+) = 0$, we propose the following ansatz:

$$n^2(r) = B(r) \left[ \frac{r_{++}^2}{r^2} + 1 \right],$$

(32)

which of course, has its maximum at $r_+ < r_{\text{max}} < r_{++}$. By means of Eq. (13), this can be rewritten as

$$n^2(r) = \frac{(r_{++}^4 - r^4)(r^2 - r_+^2)}{\lambda^2 r^4}.$$  

(33)

The integrand in Eq. (31) is $\frac{1}{\sqrt{\mathcal{P}(r)}}$, in which, according to the above definition, we have

$$\mathcal{P}(r) = \frac{(r_{++}^2 - r^2)(r^2 - r_+^2)}{\lambda^2 (r^2 - R^2)}.$$  

(34)

Here, $R = \sqrt{b^2 - r_{++}^2}$ is the closest approach as appeared in Eq. (30). This implies that $b > r_{++}$. Now, recasting

$$\mathcal{P}(r) = \frac{r_+^2 r_{++}^2 R^2}{\lambda^2} \left( \frac{1}{r^2} - \frac{1}{r_{++}^2} \right) \left( \frac{1}{r^2} - \frac{1}{r_+^2} \right) \left( \frac{1}{R^2} - \frac{1}{r^2} \right),$$

(35)

we can rewrite the deflection angle in Eq. (31) as

$$\delta \equiv \hat{\alpha} + \pi = \frac{b \lambda}{r_{++} + R} \int_0^{\xi_{++}} \frac{d\xi}{\sqrt{\xi (\xi_{++} + \xi)(\xi_+ + \xi)}},$$

(36)

for which, we have used the change of variable

$$\xi(r) = \frac{1}{R^2} - \frac{1}{r^2},$$

(37)
and have defined
\[ \xi_+ \equiv \xi(r_{++}), \]
\[ \xi_+ = \frac{1}{r_{++}^2} - \frac{1}{R^2}. \]
(38)

The integral in Eq. (36) is in fact an elliptic integral of the first kind. We therefore get
\[ \delta = \frac{b\lambda}{r_{++} + r_+} \tilde{g} K(k) \]
(40)
in which [78]
\[ \tilde{g} = \frac{2}{\sqrt{\xi_+ + \xi_+}} = \frac{2r_{++}r_+}{\sqrt{r_{++}^2 + r_+^2}}, \]
(41)
\[ K(k) = F(\varphi(\xi_+), k) \]
\[ = \int_0^\pi \frac{d\eta}{\sqrt{1 - k^2 \sin^2 \eta}}, \]
(42)
where the latter is the complete elliptic integral of the first kind, given
\[ \varphi(y) = \arcsin \left( \sqrt{\frac{\xi_+ + \xi_+}{y + \xi_+}} \right), \]
(43)
\[ k = \sqrt{\frac{\xi_+ + \xi_+}{\xi_+ + \xi_+}} = \frac{r_+}{R} \sqrt{\frac{r_{++}^2 + R^2}{r_{++}^2 - r_+^2}}. \]
(44)

Regarding the relation between \( b \) and \( R \), the deflection could be rewritten in terms of either of the above parameters as
\[ \delta(\lambda, b) = \frac{2b\lambda}{\sqrt{(b^2 - r_{++}^2)(r_{++}^2 - r_+^2)}} K(k(b)), \]
(45a)
\[ \delta(\lambda, R) = \frac{2\lambda}{R} \sqrt{\frac{R^2 + r_{++}}{r_{++}^2 - r_+^2}} K(k(R)). \]
(45b)

Note that, not all values of \( b \) are allowed for the light ray trajectories. Since \( b > r_{++} \) and \( k > 0 \), regarding Eq. (44), we have either \( \sqrt{\frac{3}{2}} r_{++} \leq b < \sqrt{2} r_{++} \) or \( r_{++} < b \leq \sqrt{\frac{3}{2}} r_{++} \). This has been shown in Fig. 2. Also, the behavior of \( \delta \) has been demonstrated in Fig. 3, distinctly for the above two categories. The plots show that the second kind of confinement for \( b \), results in more fast varying deflections.

**B. Second ansatz**

As the second guess, we consider a more complicated algebraic form, reading
\[ n^2(r) = \frac{B(r)}{r^2} \left[ b^2 + (r^2 + \sigma^2)^2 \left( r^2 - r_{++}^2 \left( 1 - \frac{\sigma^2}{r_+^2} \right) \right) \right], \]
(46)
in which \( \sigma \equiv \sigma(r_+, r_{++}) \) is a function whose value satisfies the condition \( 0 < \sigma < r_+ \). Exploiting this in the integrand, we get
\[ \mathfrak{g}(r) = \frac{1}{\lambda^2} \left[ (r^2 - r_{++}^2)(r_{++}^2 + r^2)(r^2 + \sigma^2)^2(r^2 - R^2) \right], \]
(47)
where the newly defined closest approach is \( R = \sqrt{r_{++}^2 \left( 1 - \frac{\sigma^2}{r_+^2} \right)} \). Upon recasting, the above polynomial becomes
\[ \mathfrak{g}(r) = \left( \frac{r^2 r_{++} + \sigma^2 R^2}{\lambda} \right) \left( \frac{1}{r_+^2} - \frac{1}{r^2} \right) \left( \frac{1}{r^2} - \frac{1}{r_{++}^2} \right) \left( \frac{1}{r^2 + \sigma^2} \right)^2 \left( \frac{1}{R^2} - \frac{1}{r^2} \right). \]
(48)
Applying the same change of variable as in Eq. (37), we get
\[
\delta = \frac{b \lambda}{r_+ r_{++} R \sigma} I_1,
\]
where
\[
I_1 = \int_0^{\xi_{++}} \frac{(\xi - \frac{1}{R^2})}{(\xi - \zeta)(\zeta - \xi)} \frac{d\xi}{\sqrt{\xi - \xi}},
\]
Here we have defined
\[
\zeta = \frac{1}{R^2} + \frac{1}{\sigma^2},
\]
and other definitions remain the same as in the previous case. The integral in Eq. (50) has an elliptic counterpart so that we can rewrite it as [78]
\[
I_1 = \frac{\bar{g}}{R^2 \xi} \int_0^{K(k)} \frac{1 - \beta_1^2 \text{sn}^2(\eta)}{1 - \beta^2 \text{sn}^2(\eta)} d\eta,
\]
in which
\[
\beta^2 = \frac{1}{\xi}(\xi_+ + \bar{\xi}) \beta_1^2 = \frac{\xi_+ + \bar{\xi}}{\xi_+ + \xi_++},
\]
and \(\text{sn}(\eta)\) is a Jacobi elliptic function, doubly periodic in \(\eta\), and is defined as [78]
\[
\text{sn}(\eta) = \sin(\varphi),
\]
with \(\varphi\) given in Eq. (43). Considering the above elliptic counterpart, we get
\[
I_1 = \frac{\bar{g}}{R^2 \beta^2 \mu} \left[ \beta_1^2 K(k) + (\beta^2 - \beta_1^2) \Pi(\beta^2, k) \right],
\]
FIG. 2. The region of allowed values for \(b\) for which the condition \(k > 0\) is satisfied. The plot has been done for \(Q = 0.1\). The considered range for \(b\) is from \(1.02 r_+\) to \(1.4 r_+\) for the given \(\lambda\) and \(Q\), so that it can cover the allowed values.
\( \lambda \) and \( Q \). So, for certain black holes, not all rays can provide imaging through gravitational lensing. In the plots of Fig. 4, light ray deflections are given in terms of changes of the parameter \( \sigma \).

In this section, we talked about two completely different possibilities of the radial dependence of the refractive index. This parameter tells us about how light can deviate during its travel inside the plasma and in our case, at the same time, how can be affected by the background geometry. The obtained deflection angles, corresponding to these specific cases of the refractive index, demonstrate the ability of the plasma to contribute in the usual spacetime

\[
\Pi(\beta^2, k) = \int_0^{\frac{\pi}{2}} \frac{d\eta}{1 - \beta^2 \sin^2 \eta} \frac{1}{\sqrt{1 - k^2 \sin^2 \eta}} \tag{56}
\]

is the complete elliptic integral of the third kind. With this in mind, and taking into account the definition in Eq. (41), we finally get

\[
\delta = \frac{2b\lambda}{\mathcal{R}\sigma(1 + \frac{R^2}{\sigma^2})} \sqrt{r_+^2 - r_+^2} \left( \frac{r_+^2 + \sigma^2}{\mathcal{R}^2 + \sigma^2} K(k) + \frac{\mathcal{R}^2 - r_+^2}{\mathcal{R}^2 + \sigma^2} \Pi(\beta^2, k) \right), \tag{57}
\]

which is compatible with

\[
\beta^2 = \frac{\mathcal{R}^2 + \sigma^2}{r_+^2 + \sigma^2} \beta_1 = \frac{(r_+^2 + \mathcal{R}^2)(r_+^2 + \sigma^2)}{(r_+^2 - r_+^2)(\mathcal{R}^2 + \sigma^2)}, \tag{58}
\]

and \( k = (r_+ / \mathcal{R})\beta_1 \). Note that, since \( b \) does not have any contribution in the parameter \( \mathcal{R} \), this angle does not put any restrictions on the impact parameter and the condition \( k > 0 \) is always satisfied. The behavior of the deflection in Eq. (57) has been plotted in Fig. 4 for some different impact parameter. The asymptotic behavior of the plots, stems in the elliptic functions included in the description of \( \delta \). Similar behavior was observed in Fig. 3. Physically, this means that light rays with definite impact parameters, can only contribute to the lensing process of black holes with definite physical properties (namely \( \lambda \) and \( Q \)). So, for certain black holes, not all rays can provide imaging through gravitational lensing. In the plots of Fig. 4, light ray deflections are given in terms of changes of the parameter \( \sigma \).
curvature caused by the black hole. However, once the deflection is so high, in a way that the light rays are confined to circulating on a surface around a black hole, they form a photon surface which constitutes the foundations of the so-called black hole shadow. In the next section, we exploit the recently assessed forms of \( n^2(r) \) to investigate the characteristics of the corresponding photon surfaces.

### IV. THE PHOTON SPHERE

Photon spheres are those hypersurfaces, on which light rays can stay on a stable circular path. The innermost photon sphere has the radius \( R \) introduced above. The photon surfaces however can be determined by analyzing purely angular light orbits. This condition requires \( \dot{r} = \ddot{r} = 0 \), that from Eq. (23) it follows that \( p_r = 0 \). We therefore can rewrite the Hamilton-Jacobi equation as

\[
\ell^2 = \hbar^2 r^2 \left[ \frac{\omega_0^2}{B(r)} - \frac{\omega_p^2(r)}{r} \right].
\]

Furthermore, differentiating Eq. (23) with respect to the affine parameter, results in

\[
\dot{p}_r = \frac{1}{B(r)} \left( \dot{r} - \frac{dB(r)}{dr} \dot{r} p_r \right),
\]

according to which, the zero radial velocity condition implies \( \dot{p}_r = 0 \). Hence, Eq. (20) can be recast as

\[
\ell^2 = \hbar^2 \frac{d}{dr} \frac{\omega_p^2(r)}{\omega_0^2} \left[ \frac{\omega_0^2}{B(r)} + \frac{dB(r)}{dr} \right].
\]

Subtracting the above equations and after some manipulations, we get the equation governing the radius of the circular light orbits

\[
\frac{d}{dr} h^2(r) = 0.
\]

Solutions to this equation determine the radius of photon spheres. Satisfaction of Eq. (62) is done by letting \( h^2(r) = c = \text{const} \). Applying this in Eq. (29) and taking into account the redshift in Eq. (25) we get

\[
\omega_p^2(r) = \frac{\omega_0^2}{B(r)} \left[ 1 - \frac{c B(r)}{r^2} \right].
\]

This demands the following condition for \( r > r_+ \):

\[
\frac{r^2}{B(r)} > c.
\]
Furthermore, considering Eq. (14) in Eq. (62) we get
\[
0 = \left(2B(r) - r \left(\frac{d}{dr} B(r)\right)\right) \left(1 - B(r) \frac{\omega_p^2(r)}{\omega_0^2}\right)
\]
\[-r B(r) \left[\frac{d}{dr} B(r) \frac{\omega_p^2(r)}{\omega_0^2} + 2B(r)\omega_p(r) \left(\frac{d}{dr} \omega_p(r)\right)\right].
\]
(65)

In the case of no plasmic surroundings, we have \(\omega_p(r) = 0\), yielding the following photon sphere radius in vacuum:
\[
r_{\text{ph}}^{(\text{vac})} = \frac{\sqrt{2} r_+ r_{++}}{r_+^2 + r_{++}^2}.
\]
(66)

From the values in Eqs. (11) and (12), this gives \(r_{\text{ph}}^{(\text{vac})} = Q/\sqrt{2}\), which is the same as the radius of the critical orbits, \(r_c\), obtained in Ref. [64] for the same black hole in vacuum\(^3\).

However, in the presence of plasma, this photon sphere is characterized by solving Eq. (65). Considering Eq. (13), this differential equation yields
\[
\omega_0^2(r) = \frac{\lambda^2 \omega_0^2 \left(r^2 (r_+^2 + r_{++}^2) - r_+^2 r_{++}^2\right)}{r^2 (r^2 - r_+^2) (r_+^2 - r_{++}^2)}.
\]
(67)

Note that, as long as the condition \(\lambda > Q\) is satisfied, the positivity of the right hand side of the above relation is guaranteed.

Given the frequency in Eq. (67), the radius \(r_{\text{ph}}\) now depends on one other characteristic of the plasmic medium, namely the refractive index. This can be seen through Eq. (14), providing \(\omega_0^2(r) = (\omega_0^2/B(r))(1 - n^2(r))\). This, together with Eq. (67), results in the following alternative for the refractive index:
\[
n^2(r) = 1 - \frac{r_+^2 + r_{++}^2}{r^2} + \left(\frac{r_+ r_{++}}{r^2}\right)^2.
\]
(68)

The determination of \(r_{\text{ph}}\) however, requires other definitions for \(n^2(r)\). To deal with this, we therefore recall the specific cases discussed in the previous section.

- For the first ansatz in Eq. (33) (plasma of the first kind (PKF)), Eq. (68) provides \(r_{\text{ph}} = r_+\). This means that the corresponding hypersurface, formed as the 3-dimensional (3D) closure of the 2D circles characterized by \(r = r_+\), is indeed a null surface. Although this result could seem unexpected, we here refer the reader to the fact that this photon surface is observed through a dispersive medium (plasma) that based on the geometric structure of the respected refractive index, could affect the photon surface to be located differently from that in the vacuum.

- For the case in Eq. (46) (plasma of the second kind (PSK)), we get
\[
r_{\text{ph}} = \frac{A^{1/4}}{\sqrt[4]{6} r_+} \left[2^4 (r_+ r_{++})^4 + 2(r_+ r_{++})^2 A^{1/4} + (-2A)^{3/4}\right.
\]
\[+ \sigma^2 \left(2^3 (r_+ r_{++})^2 (r_+^2 - r_{++}^2) - 2 (2r_+^2 + r_{++}^2) A^{1/4}\right)
\]
\[+ \sigma^4 \left(2^2 (r_+^4 + r_{++}^4) - 2^2 (r_+ r_{++})^2\right)\left]^{1/4},
\]
(69)

with
\[
A = \sqrt{B^2 - 4 \left((\sigma r_{++})^2 + r_+^2 (r_{++}^2 + \sigma^2)\right)^6 - B},
\]
(70)

where
\[
B = 27b^2 r_+^6 + 2(\sigma r_{++})^6 + 6 \sigma^4 \left(r_{++}^2 + \sigma^2\right) (r_+ r_{++})^2 \left[(r_{++}^2 + \sigma^2) r_+^2 - r_{++}^2\right]
\]
\[-r_+^6 (2r_{++}^2 - 27\lambda^2 + 6 \sigma^2 r_{++}^2 (r_{++}^2 + \sigma^2)) + 2\sigma^6\].
\]
(71)

---

\(^3\) Note that, the radius in Eq. (66) will never regain the famous Schwarzschild \(r = 3M\) photon sphere, by letting \(r_+ = r_{++} = 2M\). This is because the metric potential in Eq. (9) is totally different in structure, regarding the presence and the definition of the \(\lambda\) parameter.
FIG. 5. Confronting the radii of vacuum and plasmic photon spheres. The plots have been done for \( b = 10 \), \( Q = 7 \) and five different values of \( \sigma \) which have been selected according to \( \sigma < r_+ \). Changes in \( b \) do not have any effects on the form of the curves. Obviously, the value of \( r_{ph}^{(vac)} \) does not depend on \( \lambda \) and is therefore a constant in this regard. This is while the plasmic \( r_{ph} \) raises constantly for smaller values of \( \sigma \), whereas it drops fast for larger ones.

In Fig. 5, we have confronted the above radius for different values of \( \sigma \), with the radius of the photon sphere in the vacuum case. We have considered a fixed \( b \), because the curves with different values of \( b \) will coincide. The vacuum photon sphere exhibits a constant size, whereas the plasmic one can change its radius, depending on the value of \( \sigma \). It is observed that, increase in \( \lambda \) has different effects on \( r_{ph} \), depending on the corresponding \( \sigma \). This means that, the small–\( \sigma \) photon spheres expand as \( \lambda \) increases, whereas the large–\( \sigma \) ones would shrink.

In this section, the light rays were considered to travel on a circular path around the black hole and we discussed the outcome of the combination of the background geometry and plasma in confining a photon sphere. This sphere defines the boundary of the black hole’s shadow. Now, before going any further on this, let us examine the refractive plasmas under study, in a more physical context.

V. THE IMPLICATIONS FOR \( N(r) \)

Even though the spacetime effects are imposed on the description of the refractive index, nevertheless, the physical interpretation of the particle distribution inside the spacetime is given by the concentration function \( N(r) \). Applying the definition given in Eqs. (14) and (15), we get

\[
N(r) = \frac{\omega_0^2}{K_e B(r)} (1 - n^2(r)).
\]

(73)

In this section, paying attention to this quantity we go deeper into the physical implications of both kind plasmas.

The PFK generates

\[
N_1(r) = \frac{\omega_0^2}{K_e r^2} \left[ \frac{r^4 \lambda^2}{r^2 (r^2 - r_+^2)} - b^2 \right],
\]

(74)

the behavior of which has been illustrated in Fig. 6 inside the causal region. For the PSK, the concentration becomes

\[
N_2(r) = \frac{\omega_0^2}{K_e r^2} \left[ \frac{r^4 \lambda^2}{(r^2 - r_+^2)(r_++^2 - r^2)} - b^2 - (r^2 + \sigma^2)^2 \left( r^2 - r_+^2 \left( 1 - \frac{\sigma^2}{r_+^2} \right) \right) \right],
\]

(75)

which evolves as plotted in Fig. 7 for five different values of \( \sigma \), in the region \( r_+ < r < r_++ \). As it is expected, the concentration drops from its highest values at the vicinity of \( r_+ \), by moving toward \( r_++ \). As we can see from the plots of \( N_1(r) \) and \( N_2(r) \) (for definite values of \( \sigma \)), the electron concentration can tend to zero long before reaching the cosmological horizon (where the concentration should be indefinite). One important implication of this property, is that the effect of the plasma can be seen in regions outside its presence, because the refraction \( n(r) \) is available in all the region \( r_+ < r < r_++ \). This can be interpreted as a combination of electromagnetic effects and optical
FIG. 6. The behavior of particle concentration $N_1(r)$ for four values of $\omega_0$, in the region between the horizons. All four concentrations have a maximum at the same radial distance and at the horizons, $N_1$ is indefinite. It however tends to zero at the vicinity of both horizons. The plots have been done for $Q = 7$, $\lambda = 19.4$, $b = 9$ and we have absorbed $K_e$ into $\omega_0$ (all values are in arbitrary length units).

FIG. 7. The evolution of particle concentration $N_2(r)$ for five values of $\sigma$, in the region between the horizons. The concentration drops by moving toward $r_{++}$. Significantly, larger $\sigma$ results in a less steep decrease in the concentration. The plots have been done for $Q = 7$, $\lambda = 15$, $b = 10$ and $\omega_0 = 1.37$ (we have absorbed $K_e$ into $\omega_0$ and all values are in arbitrary length units). The above values have been chosen to obtain a good scale of observation and alternations in these values will just change the scale of the plots, not their form of behavior.

As a matter of interest, let us think of the PSK as a spherically symmetric halo, filling the region $r_+ < r < r_{++}$. Although electrons are not usually considered as dark matter candidates, however, it may be of interest to revisit their plasmic distribution in the cold dark matter realm. In this regard, we therefore compare the total masses obtained from the above particle concentration, and that given by the Navarro-Frenk-White (NFW) density profile. The NFW profile for a cold dark matter distribution is

$$\rho(r) = \frac{\rho_0}{\frac{r}{r_s} \left(1 + \frac{r}{r_s}\right)^2}, \quad (76)$$

in which the initial density $\rho_0$ and the scale radius $r_s$ depend on the characteristics of the halos. The integrated mass of the halo is obtained by integrating the above profile within the total volume. Considering a spherically symmetric gravity, manifesting themselves through the refractive index. For the second kind plasma, the fall in the value of $N_2(r)$ happens faster for smaller $\sigma$. However we should bear in mind that, through their relation to the horizons, every pair $(\lambda, Q)$ is related to a range for $\sigma$, which has to satisfy $0 < \sigma < r_+$. 

As a matter of interest, let us think of the PSK as a spherically symmetric halo, filling the region $r_+ < r < r_{++}$. Although electrons are not usually considered as dark matter candidates, however, it may be of interest to revisit their plasmic distribution in the cold dark matter realm. In this regard, we therefore compare the total masses obtained from the above particle concentration, and that given by the Navarro-Frenk-White (NFW) density profile. The NFW profile for a cold dark matter distribution is

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in which the initial density $\rho_0$ and the scale radius $r_s$ depend on the characteristics of the halos. The integrated mass of the halo is obtained by integrating the above profile within the total volume. Considering a spherically symmetric
FIG. 8. The numerical evaluation of the $M_P \leq M_{NFW}$ condition. The plot has been done for $b = 5.2$, $Q = 3$, $\omega_0 = 2$, $\rho_0 = 1$, $r_s = 0.6$ and $r_{\text{max}} = 10$. The solid blue line shows the possibility of having a plasmic electron distribution, which obeys the NFW cold dark matter density profile.

Solving the equation $M_P = M_{NFW}$ for either of $\sigma$ or $\lambda$, one can get an estimation criteria, in which the plasmic surrounding can behave as a cold dark matter halo in the context of NFW description. Solutions to this equation however, although achievable, are rather complicated and do not have algebraic values. We instead, demonstrate the above criteria in a plot as in Fig. 8. The figure indicates more possible similarity between the electron plasma and the NFW cold dark matter, for the lower limits of $\sigma$ and $\lambda$.
FIG. 9. For an observer located at $O$, the angular diameter ($\psi$) of the black hole depends on the closest approach to the black hole. When $R \to r_{ph}$, then $\psi$ indicates the angular diameter of the shadow (here $\psi_{sh}$).

VI. SHADOW OF THE BLACK HOLE

The deflecting trajectories, governed by the angular equation of motion in Eq. (27), can be divided into orbits of the first and second kind (respectively abbreviated as OFK and OSK). The former provides the well-known escape to infinity in terms of a definite deflection angle, whereas the latter results in falling onto the singularity [9]. The OSK therefore result in the darkness of the sky for an observer who is observing the black hole. Hence, this observer encounters a dark disk which is the black hole's shadow. This shadow is surrounded by the photon trajectories following OFK. For this reason, it can be noticed by the observer. In this regard, the photon sphere is in fact the boundary of the shadow because it is the final possible limit, at which the photons can lie. The photon sphere is therefore unstable with respect to perturbations. This is essential in the determination of the shadow.

To proceed, we calculate the angular diameter of the shadow, by considering an observer located outside the outermost photon sphere. Pursuing the method given in Ref. [77], let us consider the scheme in Fig. 9. The observer, located at the distance $r_{O}$, sends a light ray into the past at an angle $\psi$, which according to the line element in Eq. (6), is given by

$$\psi = \arccot \left( \sqrt{\frac{1}{r^2 B(r)}} \frac{dr}{d\phi} \right) \bigg|_{r=r_{O}}. \quad (79)$$

which by means of Eqs. (27) and (28), becomes

$$\psi = \arccot \left( \sqrt{\frac{h^2(r)}{h^2(R)} - 1} \right) \bigg|_{r=r_{O}}. \quad (80)$$

This can be recast as

$$\sin^2 \psi = \frac{h^2(R)}{h^2(r_{O})}. \quad (81)$$

Once the light rays have reached their final possible stable orbits at $r_{ph}$, they indicate the outermost boundary of the black hole. Hence, the shadow can be determined by letting $R \to r_{ph}$ (see Fig. 9). Accordingly, the corresponding angular diameter of the shadow is obtained as

$$\sin^2 \psi_{sh} = \frac{h^2(r_{ph})}{h^2(r_{O})}. \quad (82)$$

Applying Eq. (29) we can calculate the above angle for the shadow. In the absence of plasma (i.e. for $h^2(r) = r^2/B(r)$), applying the radius in Eq. (66), this angle becomes

$$\sin^2 \psi_{sh}^{(vac)} = \frac{4r_{+}^2 r_{ph}^2 (r_{ph}^2 - r_{+}^2) (r_{+}^2 - r_{O}^2)}{r_{O}^4 (r_{+}^2 - r_{+}^2)^2}. \quad (83)$$

For the PFK and PSK, discussed and analyzed in the previous sections, we get the following results:
From Eq. (32) we get
\[
\sin^2 \psi_{sh} = \frac{r_0^2 + r_+^+}{r_0^2 + r_+^+},
\]
for \( r_{ph} = r_+ \) \((b = \sqrt{r_0^2 + r_+^2})\).

From Eq. (46), the angle in Eq. (81) becomes
\[
\sin^2 \psi = \frac{b^2 r_+^2 (\sigma^2 + \mathcal{R}^2)^2 (r_{\mathcal{O}}^2 + \sigma^2 - r_+^2 (r_{\mathcal{O}}^2 - \mathcal{R}^2))}{b^2 r_0^2 (r_{\mathcal{O}}^2 + \sigma^2)^2 (r_+^2 - r_+^2 (r_0^2 - \mathcal{R}^2))}.
\]

Applying the condition in Eq. (82) and the radius in Eq. (69), the angular diameter of the shadow is obtain as
\[
\sin^2 \psi_{sh} = \frac{\lambda^2 r_+^2}{(r_{\mathcal{O}}^2 + \sigma^2)^2 (r_0^2 + (r_+^2 - \sigma^2) - r_0^2 r_+^2) - b^2 r_+^2}.
\]

Note that, not all values of \( b \) are permitted to be possessed by the photons. This means that only certain photons with allowed impact parameters can identify the shadow. Such photons are those which could escape the black hole by passing the nearest possible distance (the critical distance) from it. According to the above relation, the condition \( 0 < \sin^2 \psi_{sh} < 1 \) implies
\[
b^2 < b_{max}^2 - \lambda^2,
\]
in which
\[
b_{max}^2 = \frac{(r_{\mathcal{O}}^2 + \sigma^2)^2 (r_0^2 + (r_+^2 - \sigma^2) - r_0^2 r_+^2)}{r_+^2}.
\]

This means that for every triplet \((\lambda, Q, \sigma)\), only photons satisfying the condition in Eq. (87) can identify the shadow. In Fig. 10, a region has been plotted in which, the values of \( b \) satisfy the above condition. Accordingly, and in Fig. 11, the angular diameters of the shadow have been plotted respectively for the vacuum, the PFK and the PSK. For all cases, no extremal black holes are observable. However, shadow of the black hole surrounded by the PFK, achieves its maximum angular diameter for the lower values of \( \lambda \). This is while for the one corresponding to the PSK, \( \psi_{sh} \) tends to zero for same range of \( \lambda \). This means that, this model of plasmic surrounding prohibits the shadow to appear to the observer, when the cosmological term in Eq. (9) is dominant.

The discussion in this section, dealt with the way though which a charged Weyl black hole manifests itself to an observer residing in \( r_+ < r < r_+^+ \). To demonstrate the shadow, it is usual to define some celestial coordinates which are obtained by doing a frame transformation from the curved background spacetime to the frame of a local observer (see for example the method of obtaining the shadow for rotating black holes in Refs. [9, 79] in vacuum and Ref. [23] in the presence of plasma. The case of static vacuum spacetime has also been investigated for example in Ref. [80]). To do the above frame transformation however, the spacetime needs to be asymptotically flat which is not the case for the black hole under consideration. We therefore leave the discussion here and in the next section we bring the final notes and summarize the results.

**VII. CONCLUSION**

Light propagation in a plasmic medium, surrounding a charged Weyl black hole was the main aim of this paper. We calculated the equations of motion in connection with the plasma’s energy density and refraction. Then by proposing two different ansatzes for the refractive index, we obtained analytical expressions for the light deflection which is the significance of the gravitational lensing caused by the black hole in the media. The solutions were given in terms of the elliptic integrals and for both kinds of plasma, we discovered that, depending on their energy and angular momentum, not all rays can contribute in the lensing process.

Further, we demonstrated the particular way, through which, the radius of the photon sphere can be obtained. The photon sphere constitutes the closure of the final possible stable orbits around the black hole. In the first kind plasma, photons, regardless of their impact parameter, can form only one single photon sphere which depends only
FIG. 10. The allowed values of $b$ which satisfy the condition $0 < \sin^2 \psi_{sh} < 1$. The region has been plotted for $Q = 0.6$ and $r_{O} = 0.8$.

FIG. 11. The radial diameter given in terms of $\sin^2 \psi_{sh}$, for (a) vacuum, (b) PFK and (c) PSK. The plots have been done for $Q = 0.6$, $r_{O} = 0.78$ and the impact parameter for the plots (b) and (c) has been taken as $b = 0.8$ (arbitrary length units have been considered).

on black hole’s characteristics. In contrast, the formation of photon sphere in the plasma of the second kind, depends directly on the test particles’ energy and angular momentum and evolves in terms of the plasma’s refraction. We continued our discussion by comparing the mass relation derived from the second kind plasma with that obtained from the NFW dark matter halo and demonstrated the extent of black hole properties, to which, these two could be similar in value. As the last concept, we considered the black hole’s shadow and obtained its angular diameter in both cases of plasmic surrounding. The second kind plasma showed that not all photons can contribute in the formation
of the shadow. We demonstrated this by plotting the angular diameter. In conclusion, we highlight the importance of the investigation of light propagation in refractive media when one is interested in inspecting the appearance of black holes to distant observers. For the cases studied here, we found that the impacts of plasma can make strong changes in the way the black hole is seen. In the present study, this became apparent in the demonstrated evolution of the deflection angles and the photon spheres.

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