Lepton decay constants of light mesons

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Abstract
A theory of lepton decay constants based on the path-integral formalism is given for chiral and vector mesons. Decay constants of the pseudoscalar and vector mesons are calculated and compared to other existing results.

1 Introduction

The decay constants $f_n$ in many cases may be directly measured in experiment and are important characteristics of mesons, where different theoretical approaches may be compared and their accuracy be estimated. The $f_n$ of light mesons have been studied in potential models [1, 2, 3, 4, 5, 6, 7, 8, 9] in the QCD sum rule method [10, 11, 12, 13, 14, 15, 16, 17] in chiral perturbation theory [18, 19], as well as in lattice simulations [20, 21, 22, 23, 24, 25, 26, 27] and in experiment [28, 29, 30, 31].

The important role of $f_n$ in theory and experiment is well illustrated by $f_\pi$ - the pion decay constant - which is the basic element and the natural scale of the chiral perturbation theory [32]. In the latter case $f_\pi$ is taken from experiment, and the computation of $f_\pi$ from the first principles is a serious challenge for the theory. On the lattice side a reasonable accuracy in computing $f_n$ was achieved recently [20, 21, 33], analytic methods include earlier attempts in the instanton vacuum [34] and within the Field Correlator Method (FCM) [35, 36, 37].

The present article is devoted to the systematic derivation of meson Green’s functions and decay constants $f_n$ for channels with arbitrary quantum numbers.
This paper is an update and extension of the earlier papers [38], devoted to the heavy-light pseudoscalar and light vector mesons and [39], devoted to heavy-light mesons. Those papers appeared before the systematic formulation of FCM and in particular of the path-integral Hamiltonian based on FCM [40], therefore some steps in [38, 39] required corrections. In particular, the rigorous derivation of the path-integral expression for the Green’s function in [40] has allowed to obtain the improved expressions for decay constants, which we exploit in what follows. Moreover Chiral Symmetry Breaking (CSB) was not incorporated in [38, 39]. In the present paper we present the consistent and general treatment of the meson Green’s functions and its spectral properties also for pseudoscalars accounting for CSB in the lowest states (π, K). The main problem which one encounters, when addressing the spectral properties in QCD, is the necessity of the quantitative nonperturbative methods, which describe the main dynamical phenomena: confinement and CSB.

The familiar (relativistic) potential model lacks the latter, while other models like instanton vacuum model, lacks the former QCD phenomenon.

In what follows we are using the Field Correlator Method, which was introduced in [35, 36, 37] has acquired the full form in [40] as a main tool to study and explain confinement. With respect to the QCD spectrum one derives in the FCM the effective Hamiltonian, which comprises confinement and relativistic effects, and contains only universal quantities: string tension σ, strong coupling αs and current quark masses m_q. The simple local form of the Hamiltonian which will be called the path-integral Hamiltonian (PIH) occurs for objects with temporal scales larger than the gluon correlation length \( \lambda \approx 0.2 \) fm, i.e. it is applicable for all QCD bound states except possibly toponium.

Explicit calculations of masses and wave functions using PIH have been done recently for light mesons [41], heavy quarkonia [42] and heavy-light mesons [43], and demonstrate good agreement with experimental masses.

In the present paper we devote a special attention to the chiral mesons and treat the chiral symmetry breaking (CSB) phenomenon within the formalism of [44], where CSB is the consequence of the confinement and the necessary relations can be derived from the basic parameters of QCD: string tension σ and vacuum correlation length \( \lambda \), so that a fundamental quantity entering \( f_\pi, f_k \) and \( m_\pi, m_k \) is \( M(0) \approx \sigma \lambda \) [44].

The paper is organized as follows: in section 2 the general path integral form of the meson Green’s function is presented, while in Appendices 1-5 the
details of derivation are given. In section 3 the obtained expression for $f_n$ is analyzed. In section 4 light pseudoscalar mesons are considered. The vector meson decay constants are studied in section 5. Section 6 contains summary and concluding remarks.

2 The meson Green’s function in the path integral formalism

We start with the one-body Green’s function.

The path-integral representation for $S_i$ is \[36, 37\]

$$S_i(x, y) = (m_i - \hat{D}^{(i)}) \int_0^\infty ds_i(Dz)_{xy} e^{-K_i \Phi^{(i)}(x, y)}$$

$$\equiv (m_i - \hat{D}^{(i)}) G_i(x, y), \quad (1)$$

where

$$K_i = m_i^2 s_i + \frac{1}{4} \int_0^{s_i} d\tau_i \left( \frac{dz^{(i)}_\mu}{d\tau_i} \right)^2 \quad (2)$$

$$\Phi^{(i)}(x, y) = P_A P_F \exp \left( ig \int_y^x A_\mu dz^{(i)}_\mu \right) \times$$

$$\times \exp \left( \int_0^{s_i} d\tau_i \sigma_{\mu\nu}(gF_{\mu\nu}) \right). \quad (3)$$

Here $F_{\mu\nu}$ is a gluon field tensor, $P_A, P_F$ are ordering operators, $\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$. Eqs. (1) hold for the quark, $i = 1$, while for the antiquark one should reverse the signs of $g$. In explicit form one writes

$$\sigma_{\mu\nu} F_{\mu\nu} = \begin{pmatrix} \sigma H & \sigma E \\ \sigma E & \sigma H \end{pmatrix}. \quad (4)$$

The two-body $q_1\bar{q}_2$ Green’s function can be written as \[20, 24\]

$$G_{q_1\bar{q}_2}(x, y) = \int_0^\infty ds_1 \int_0^\infty ds_2 (Dz^{(1)})_{xy} (Dz^{(2)})_{xy} \langle \hat{T} W_\sigma(A) \rangle A e^{-K_1 - K_2}, \quad (5)$$

where

$$\hat{T} = tr(\Gamma_1(m_1 - \hat{D}_1)\Gamma_2(m_2 - \hat{D}_2)), \quad (6)$$

$$3$$
“tr” is the trace over Dirac and color indices acting on all terms. Here $\langle W_\sigma(A) \rangle$ is the closed Wilson loop with the spin insertions and one should have in mind, that color spin insertions in general do not commute, which should be taken into account when computing spin-dependent part of interaction, see [45], in (5) this fact was disregarded.

$$W_\sigma(A) = P_F P_F \exp \left[ ig \oint A_\mu dz_\mu + g \int_0^{s_1} \sigma^{(1)}_{\mu\nu} F_{\mu\nu} d\tau_1 - g \int_0^{s_2} \sigma^{(2)}_{\mu\nu} F_{\mu\nu} d\tau_2 \right]. \quad (7)$$

As a result of the correlator averaging [36, 37], and neglecting the spin-dependent terms, one obtains

$$\langle \langle W \rangle \rangle = Z_W \exp \left( - \int_0^T \left[ V_0(r(t_E)) \right] dt_E \right), \quad (8)$$

where $r(t_E) = |z_1(t_E) - z_2(t_E)|$, and

$$V_0(r) = V_{\text{conf}}(r) + V_{\text{OGE}}(r), \quad (9)$$

$$V_{\text{conf}}(r) = 2r \int_0^r d\lambda \int_0^\infty d\nu D(\lambda, \nu) \rightarrow \sigma r, \quad (10)$$

$$\sigma = 2 \int_0^\infty d\nu \int_0^\infty d\lambda D(\nu, \lambda), \quad (11)$$

$$V_{\text{OGE}} = \int_0^r \lambda d\lambda \int_0^\infty d\nu D_{\text{pert}}(\lambda, \nu) = -\frac{4}{3} \frac{\alpha_s}{r} \quad (12)$$

At this point it is useful to introduce as in [40] the virtual quark (antiquark) energies $\omega_1(\omega_2)$ instead of proper times: $s_i = \frac{T}{2\omega_i}$, where $T$ is the Euclidean time interval, $T = x_4 - y_4$. As a result one obtains

$$\left( \frac{1}{(m_1^2 - D_1^2)(m_2^2 - D_2^2)} \right)_{xy} = \frac{T}{8\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} \left( D^2 z_1 \right)_{xy} \left( D^2 z_2 \right)_{xy} e^{-A(\omega_1, \omega_2, z_1, z_2)}, \quad (13)$$

where $A \equiv K_1(\omega_1) + K_2(\omega_2) + \int V_0(r(t_E)) dt_E$, and

$$K_i(\omega_i) = \frac{m_i^2 + \omega_i^2}{2\omega_i} T + \int_0^T dt_E \frac{\omega_i}{2} \left( \frac{dz^{(i)}}{dt_E} \right)^2$$

We can also introduce here the two-body 3d Hamiltonian $H(\omega_1, \omega_2, p_1, p_2)$ and rewrite (13) as
\[
\left( \frac{1}{(m_1^2 - \hat{D}_1^2)(m_2^2 - \hat{D}_2^2)} \right)_{xy} = \frac{T}{8\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} \langle x|e^{-H(\omega_1,\omega_2,P_1,P_2)}T|y \rangle,
\]

where \( H \) is obtained in a standard way from the action \( A(\omega_1,\omega_2,z_1,z_2) \) (we omit all e.m. fields except for external magnetic fields \( B \))

\[
H = \sum_{i=1}^2 \left( \frac{p^{(i)}_\mu}{2\omega_i} + \frac{m_i^2 + \omega_i^2}{\omega_i} \right) + V_0(r) + V_{ss} + \Delta M_{SE},
\]

and \( V_0 \) is given in (10). The spin-dependent part of \( H, V_{ss} \) and \( V_{LS} \) are obtained perturbatively from \( \sigma_{\mu\nu}F_{\mu\nu} \) terms in (27), and is calculated also in the presence of m.f. in [45]. It is considered as a perturbative correction and is a relativistic generalization of the standard hyperfine interaction,

\[
V_{ss}(r) = \frac{1}{4\omega_1\omega_2} \int \langle \sigma_{\mu\nu}F_{\mu\nu}(x)\sigma_{\rho\lambda}F_{\rho\lambda}(y) \rangle d(x_4 - y_4).
\]

Its explicit form is given in [45]. Finally, the correction \( \frac{\omega_i^k F(x)\sigma^{(i)}F(y)}{4\omega_i^k} \), where \( i \) refers to the same quark (antiquark) yields the spin-independent self-energy correction \( \Delta M_{SE} \) which was calculated earlier [16] and for zero mass quarks and no m.f. is

\[
\Delta M_{SE} = -\frac{3\sigma}{2\pi\omega_1} - \frac{3\sigma}{2\pi\omega_2}.
\]

For the case of nonzero m.f. the resulting \( \Delta M_{SE} \) is given in [45]. We can now write the total Green’s function of the \( q_1\bar{q}_2 \) system, denoting by \( Y \) the product of projection operators \( Y = \Gamma(m_1 - \hat{D}_1)\Gamma(m_2 - \hat{D}_2) \),

\[
m_1 - \hat{D}_1 = m_1 - i\hat{p}_1 = m_1 + \omega_1\gamma_4 - ip\gamma, \quad m_2 - \hat{D}_2 = m_2 - \omega_2\gamma_4 - ip\gamma,
\]

where \( \mathbf{p} \) is the quark 3 momentum in the c.m. system.

As a result one has

\[
\int d^3(x - y)G(x,y) = \int d^3(x - y)tr \left( \frac{4Y_{\Gamma}}{(m_1^2 - \hat{D}_1^2)(m_2^2 - \hat{D}_2^2)} \right)_{xy} = \frac{T}{2\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} \langle Y_{\Gamma} \rangle \langle x|e^{-H(\omega_1,\omega_2,P_1,P_2)T}|y \rangle,
\]

where
We have used in (18) the relations
\[ \langle Y \rangle = \text{tr}_\Gamma (m_1 - i \hat{p}_1) \Gamma (m_2 - i \hat{p}_2), \]
and neglect spin dependent terms in \( H \); we have taken into account, that \( D_\mu \) acting on Wilson line, i.e. \( D_\mu \exp (ig \int^x A_\mu dz_\mu) \Lambda \) yields \( \exp (ig \int^x A_\mu dz_\mu) \partial_\mu \Lambda \).

The c.m. projection of the Green’s function yields
\[ \int d^3(x - y) \langle x | e^{-H(\omega_1, \omega_2; p_1, p_2)} | y \rangle = \sum_n \varphi_n^2(0) e^{-M_n(\omega_1, \omega_2) T}. \] (19)

Here \( M_n(\omega_1, \omega_2) \) is the eigenvalue of \( H(\omega_1, \omega_2, p_1, p_2) \) in the c.m. system, where \( P = p_1 + p_2 = 0; \ p_1 = p = -p_2. \)

The integrals over \( d\omega_1, d\omega_2 \) for \( T \to \infty \) can be performed by the stationary point method, namely one has
\[ \int G(x, y) d^3(x - y) = \frac{T}{2\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} \sum_n e^{-M_n(\omega_1, \omega_2) T} \varphi_n^2(0) \langle Y \rangle \]
\[ = \sum_n \frac{e^{-M_n(\omega_1^{(0)}, \omega_2^{(0)}) T} \varphi_n^2(0) \langle Y \rangle}{\omega_1^{(0)} \omega_2^{(0)} \sqrt{(\omega_1^{(0)} M_n^{(1)})(\omega_2^{(0)} M_n^{(2)})}}, \] (20)
where
\[ \frac{\partial M_n(\omega_1, \omega_2)}{\partial \omega_i} \bigg|_{\omega_i = \omega_i^{(0)}} = 0, \quad M_n^{(i)} = \left. \frac{\partial M_n(\omega_1, \omega_2)}{\partial \omega_i^2} \right|_{\omega_i = \omega_i^{(0)}}. \] (21)

and we have neglected the mixed terms \( \frac{\partial^2 M_n}{\partial \omega_1 \partial \omega_2} \) for simplicity, however should keep them in concrete calculations: see exact result in Appendix 1. Comparing the results (19), (20) with the definitions of quark decay constants \( f_\Gamma^n \),
\[ \int G_\Gamma(x) d^3x = \sum_n \int d^3x \langle 0 | j_\Gamma | n \rangle \langle n | j_\Gamma | 0 \rangle e^{i \mathbf{P} \cdot \mathbf{x} - M_n T} \frac{d^3 \mathbf{P}}{2 M_n (2\pi)^3} \]
\[ = \sum_n \varepsilon_\Gamma \otimes \varepsilon_\Gamma \frac{(M_n f_\Gamma^n)^2}{2 M_n} e^{-M_n T}, \] (22)
where for \( \Gamma = \gamma_\mu, \gamma_\mu \gamma_5 \)
\[ \sum_{k=1,2,3} \varepsilon^{(k)}(q) \varepsilon^{(k)}(q) = \delta_{\mu \nu} - \frac{q_\mu q_\nu}{q^2}, \] (23)
and \( \varepsilon_\Gamma = 1 \) for \( \Gamma = 1, \gamma_5, \) one obtains the expression for \( f_\Gamma^n \) (to lowest order in \( V_{ss} \)).
\[(f_1^n)^2 = \frac{N_c(Y_Γ)|φ_n(0)|^2}{ω_i^{(0)}ω_j^{(0)}M_nξ_n}, \quad ξ_n \equiv \sqrt{(ω_1^{(0)}M_n(1))(ω_2^{(0)}M_n(2))}, \quad (24)\]

This expression coincides with the previously derived in [33, 34], when ξ_n = 1/2. In what follows we show, that ξ_n is close to that value, but different for light, heavy-light and heavy-heavy mesons.

3 Analysis of the obtained expressions

First calculate \(M_n, φ_n\) from the equation

\[Hφ_n = M_nφ_n, \quad (25)\]

treating \(V_{LS}, V_{SS}\) and \(V_{SE}\) in (9) as perturbation, \(H = H^{(0)} + V_{LS} + V_{SS} + \Delta M_{SE} = M_n^{(0)} + \Delta M_n\). One simplify the procedure introducing the relative coordinate in the c.m. system, \(η = r_1 - r_2, p = \frac{∂}{∂η}\), so that without magnetic field

\[H_0 = \frac{p^2}{2\tilde{ω}} + \sum \frac{m_i^2 + ω_i^2}{2ω_i} + V_0(r) + \Delta M_{SE}, \quad (26)\]

\[H = H_0 + V_{SS} + V_{LS}. \]

(Note, that this is not nonrelativistic expansion!) and finding stationary values of \(M_n^{(0)}, H_0φ_n = M_n^{(0)}φ_n\) with respect to \(ω_i, ω_i = ω_i^{(0)}\) from the equation

\[\frac{∂M_n^{(0)}(ω)}{∂ω_i} \bigg|_{ω_i=ω_i^{(0)}} = 0. \quad (27)\]

This is the basic approach in the string Hamiltonian formalism [47] and it was checked that the accuracy of the replacement \(\tilde{ω}_i = ω_i^{(0)}\) for lowest states is around 5% [47]. However the values \(φ_n(0)\) are more sensitive to the replacement (23), and one should use original Hamiltonian (15) to calculate \(φ_n(0)\) [38, 39], see Table 4 of [38] for comparisons.

It is essential, that we are using \(V_{SS} + V_{LS}\) as perturbation terms to compute the final hadron masses and hence \(M_n\) in different parts of (24) is inserted as computed from \(H_0\) (26), not containing \(V_{SS} + V_{LS}\).

ii) As it was argued above, the factor \(Y_Γ\) can be computed in terms of momenta of quark and antiquark, or in the c.m.system in terms of relative momentum \(p\), with the result.
\[ Y_V = m_1 m_2 + \bar{\omega}_1 \bar{\omega}_2 + \frac{1}{3} p^2, \quad (28) \]

\[ Y_S = -m_1 m_2 + \bar{\omega}_1 \bar{\omega}_2 | + p^2, \quad (29) \]

\[ \hat{Y}_{A_i} = -m_1 m_2 + \bar{\omega}_1 \bar{\omega}_2 + \frac{p^2}{3}, \quad (30) \]

\[ Y_{A_4} = m_1 m_2 + \bar{\omega}_1 \bar{\omega}_2 - p^2, \quad (31) \]

\[ Y_P = (m_1 m_2) + \bar{\omega}_1 \bar{\omega}_2 - p^2, \quad (32) \]

Here we used notations:

\[ \hat{Y}_V = \sum_i^3 \frac{1}{3} tr[(m_1 - \hat{D}_1)\gamma_i (m_2 - \hat{D}_2)\gamma_i], \quad (33) \]

\[ \hat{Y}_{A_i} = -\frac{1}{3} \sum_i tr[(m_1 - \hat{D}_1)\gamma_i \gamma_5 (m_2 - \hat{D}_2)\gamma_i \gamma_5], \quad (34) \]

\[ \hat{Y}_{A_4} = -[tr(m_1 - \hat{D}_1)\gamma_4 \gamma_5 (m_2 - \hat{D}_2)\gamma_4 \gamma_5]. \quad (35) \]

In case of the pseudoscalar channel in (31), (32) in the chiral limit \( m_1, m_2 \rightarrow 0 \) there appear additional mass terms due to CSB, which are computed through field correlators and are given in [44]. As it is shown in Appendix 2, the proper account of CSB leads to the fact, that Eq. (24) for \( f_P^2 \) retains its form for \( \pi, K \) mesons, but the expression for chiral mesons e.g. \( Y_{A_4} \) should be replaced by more general one,

\[ Y_{A_4}^{(chiral)} = (m_1 + M_1(0)) (m_2 + M_2(0)) + \bar{\omega}_1 \bar{\omega}_2 - p^2 \quad (36) \]

In the chiral limit \( m_1 = m_2 = 0 \) it was found in [44] that \( M_1(0) = M_2(0) \cong 0.15 \text{ GeV} \) and it was computed through the field correlators, \( M(0) = \sigma \lambda \), where \( \lambda \) is the vacuum correlation length, \( \lambda \approx 1 \text{ GeV}^{-1} \) [48].

In the nonrelativistic limit, \( m_i \gg \sqrt{\sigma} \), one can easily find that \( \bar{\omega}_i \approx m_i \), while \( \langle p^2 \rangle \sim O(\sigma) \), and therefore one has

\[ \langle Y_V \rangle_{NR} \approx 2m_1 m_2 + 0(\sigma), \quad \langle Y_S \rangle_{NR} \approx 0(\sigma), \quad (37) \]

\[ \langle Y_{A_i} \rangle_{NR} \approx 0(\sigma); \quad \langle Y_{A_4} \rangle_{NR} = 2m_1 m_2 + 0(\sigma), \quad (38) \]
\( (Y_P)_{NR} = 2m_1m_2 + 0(\sigma). \) \hspace{1cm} (39)

Therefore in the nonrelativistic limit \( m_1 \gg \sqrt{\sigma}, m_2 \gg \sqrt{\sigma} \), for \( f_n \) for V and P channels one obtains

\[
\begin{align*}
\left< f^2_n \right>_{NR} &= \frac{4N_c}{M_n} \left| \varphi_n(0) \right|^2 \\
\end{align*}
\] \hspace{1cm} (40)

as was found earlier [1].

As a final step one needs to compute the radiative corrections to \( f_n \), which come from the short-distance (large momentum) perturbative gluon contributions. Neglecting interference terms they can be written as in [23, 24, 32],

\[
\left< W_\sigma \right> = \left< W_{OGE} \right> \left< W_{nonpert} \right> \hspace{1cm} (41)
\]

and

\[
\left< W_{OGE} \right> = Z_m \exp \left( -\frac{4}{3\pi} \int \int \frac{dz dz' \alpha_s(z - z')}{(z - z')^2} \right), \hspace{1cm} (42)
\]

where \( Z_m \) is a regularization factor. After separating the Coulomb interaction in \( \hat{H} \) in this way, one gets the correction to \( \left< W_\sigma \right> \), and \( f^2_\Gamma \) can be written as

\[
f^2_\Gamma \to \xi_\Gamma f^2_\Gamma, \quad \xi_\Gamma = 1 + c_\Gamma \alpha_s + O(\alpha_s^2). \hspace{1cm} (43)
\]

but this correction is small and will be neglected below.

Another important contribution from perturbative gluon exchanges (GE) is the account of the running coupling constant in (40) which is especially important for \( \varphi_n(0) \) in the \( S \)-wave channels. Introducing the asymptotic freedom factor \( P_{AF} \) (we follow notations from [38])

\[
\rho_{AF} = \left| \frac{\varphi_n^{(AF)}(0)}{\varphi_n(0)} \right|^2, \hspace{1cm} (44)
\]

one should multiply \( f^2_n \) with this factor and finally gets

\[
f^2_\Gamma = f^2_\Gamma \rho_{AF}. \hspace{1cm} (45)
\]

We conclude this section with the discussion of input parameters in the approach described above. The set of parameters includes \( m_i, \alpha_s \) and \( \sigma \) in the first approximation and \( m_i, \alpha_s(r), \sigma \), when asymptotic freedom is taken into account.
Here $m_i$ are pole masses which are connected to the Lagrangian (current) masses in $\overline{MS}$ scheme as (see [28] for review and [39] for additional references).

$$m_i = \overline{m}_{\overline{MS}}(\overline{m}_{\overline{MS}}) \left\{ 1 + \frac{4}{3} \frac{\alpha_s(\overline{m}_{\overline{MS}})}{\pi} + \eta_2 \left( \frac{\alpha_s}{\pi} \right)^2 + O(\alpha_s^3) \right\} . \quad (46)$$

4 Light pseudoscalar mesons and current correlators in PS channels

The formalism of the previous section is of general character and be applied to any channel $\Gamma$. However to save space we shall consider below only PS and vector mesons. Pseudoscalar mesons appear both in $A_4$ and P channels. Their connection to the A channel is given by the chiral anomaly term:

$$\langle 0 | j_{\mu}^{(A)} | q, n \rangle = i f_P(n) q_{\mu} . \quad (47)$$

Exploiting this definition one obtains the same expression as before for $f_P(n)$, considering the c.m. system $P = 0$, namely

$$f_P^2(n) = \frac{N_c \langle Y_{A_4}^{(\text{chiral})} \rangle |\varphi_n(0)|^2}{\bar{\omega}_1 \bar{\omega}_2 M_n \xi_n} . \quad (48)$$

One should have in mind however that our formalism above in this paper till now did not take into account Chiral Symmetry Breaking (CSB) and therefore cannot be applied to the Nambu-Goldstone mesons $\pi, K, \eta$. For the latter one should use the technic suggested and exploited in [44], where $f_\pi$ was computed through the masses $m_n, \varphi_n(0)$ as in (24) but in addition there appears an effective mass parameter $M(0)$, see Appendix 2. The resulting equation for $\langle Y_{A_4} \rangle$ Eq. (36) can be written in the limit $m_i \to 0$ as

$$\langle Y_{A_4} \rangle = M^2(0) + \langle \sqrt{p^2 + m_1^2} \rangle \langle \sqrt{p^2 + m_2^2} \rangle - \langle p^2 \rangle \to M^2(0) . \quad (49)$$

In the chiral limit, $m_1 = m_2 = 0$, and taking into account that $\bar{\omega}_1 = \bar{\omega}_2 = \bar{\omega}$, and $\langle p^2 \rangle = \bar{\omega}^2$, one has

$$f_P^2(n) = \frac{N_c M^2(0) |\varphi_n(0)|^2}{\bar{\omega}^2 M_n \xi_n} . \quad (50)$$
For $\varphi_n(0)$ and $M_n$ one takes neglecting hyperfine interaction the same values, as for $\rho$-meson, i.e. $M_n = \bar{M}(n = 0) = 0.65 \text{ GeV}, |\varphi_n(0)|^2 = \frac{0.109 \text{ GeV}^3}{4\pi}, \bar{\omega} = 0.352 \text{ GeV}$. 

Taking now $\xi_n^{-1} = 2.45$ from A(12) and $M(0) = 0.15 \text{ GeV}$ one obtains $f_\pi^2 = 0.01782 \text{ GeV}^2$, $f_\pi = 133 \text{ MeV}$. This should be compared with the experimental value, which in the normalization of Eq. (50) is equal to $f_\pi^{ex} \approx \sqrt{2} \cdot 0.093 \text{ GeV} = 0.131 \text{ GeV}$. One see that agreement within 2%.

One should stress, that in absence of CSB, when $M(0) \equiv 0$, and $\langle Y_A \rangle \to m_1 m_2 \to 0$, also $f_\pi$ vanishes, implying that $f_\pi$ plays the role of the CSB order parameter (together with $\langle q\bar{q} \rangle$, which is also proportional to $M(0)$).

We now turn to the case of $K$ meson. Doing calculations in the same way as for pion above, and taking $m_s = 0.15 \text{ GeV}, \sigma = 0.18 \text{ GeV}^2$, one has for $K$-meson:

$$\omega_u(K) = 0.36 \text{ GeV}, \quad \omega_s(K) = 0.39 \text{ GeV}, \quad M_K^{(0)} = 0.84 \text{ GeV}.$$ 

The latter number is obtained without Coulomb and hyperfine interaction, which shift $M_k$ by $\Delta m_{Coul} = 0.05 \text{ GeV}$ and $\Delta m_{Hf} = 0.3 \text{ GeV}$, resulting in $m_K = M_K^{(0)} - \Delta m_{Coul} - \Delta m_{Hf} \approx 0.49 \text{ GeV}$. 

Using Appendix 1, Eq. (A2), one obtains $\xi_K^{-1} = 2.29$ so that $f_K^2$ is

$$f_K^2 = \frac{2.29 \cdot N_c \langle Y_K \rangle \varphi_K^2(0)}{\omega_u \omega_s \bar{M}_K^{(0)}}, \quad (51)$$

where $\langle Y_K \rangle = (M(0) + m_u)(M(0) + m_s) + \omega_u \omega_s - \langle p^2 \rangle = 0.06 \text{ GeV}^2$ and as a result $f_K = 0.165 \text{ GeV}$.

To compare $f_K$ and $f_\pi$ we write down for both mesons without hyperfine interaction and using (49)

$$\frac{f_K^2}{f_\pi^2} = 1.6, \quad (52)$$

$$\frac{f_K}{f_\pi} = 1.24$$

The result $\frac{f_K}{f_\pi} = 1.24$ is in agreement with the experimental values

$$f_\pi^{(exp)} = 130.7 \pm 0.1 \pm 0.36 \text{ MeV}, \quad (53)$$
Table 1: Decay constants of chiral mesons and its excitations

|     | $\pi(nS)$ |     | $K(nS)$ |     |     |     |
|-----|-----------|-----|---------|-----|-----|-----|
| n   | 1         | 2   | 3       | 1   | 2   | 3   |
| $f_{\pi n}$(GeV) | 0.138 | 0.069 | 0.048  | 0.165 | 0.104 | 0.085 |
| $f_{\pi n}/f_{\pi^+}$ | 1    | 0.5  | 0.35    | 1   | 0.63 | 0.515 |

$f_{K^+}^{(\text{exp})} = 159.8 \pm 1.4 \pm 0.44$ MeV,
which yields

$$\frac{f_{K^+}^{(\text{exp})}}{f_{\pi^+}^{(\text{exp})}} = 1.22 \pm 0.02,$$  (54)

while lattice calculation [20] yield for this ratio $1.195 \pm 0.006$.

We now turn to the radial excitations of the chiral mesons. In this case of high excitations there appear decay channels, which play a role of intermediate channels in the meson Green’s function. Therefore neglecting these channels, we shall make only rough upper limit estimates of decay constants.

To this end we exploit the fact (see Appendix 1) that $\xi_n$ does not depend on $n$, and $\varphi_n^2(0)$ can be estimated to the lowest order as $\frac{\sigma_n}{\pi} \left| \frac{\chi_n(x^2)}{\chi_n(0)} \right|$ and $\bar{M}_n \approx 2\omega_n$ (with the accuracy of $(10 \div 15)\%$). As a result one obtains

$$f_{\pi n}(\text{GeV}) \approx \frac{0.105}{\bar{M}_n(\text{GeV})}, \quad n > 1$$  (55)

where $M_n$ are the mass values before the chiral shift [49], and as a result one has the values listed in the Table 1.

One can see, that $f_{\pi n}$ and $f_{Kn}$ are slowly decreasing for growing $n$, implying a substantial leptonic decay contribution to the list of decay modes.

### 5 Decay constants of vector mesons

We now turn to vector mesons, where we take the same value for $\bar{M}_n$ and $\xi_n$ [49], not affected by the $hf$ splitting, which we take afterwards as a perturbation. Hence we take $\xi_{K^+}^{-1} = 2.29$ for $K^*$ meson and $\xi_{\rho}^{-1} = 2.45$ for $\rho$ and $\omega$ mesons, while for the $\phi$ meson one obtains $\xi_{\phi}^{-1} = 2.095$, and $\omega_s = 0.424$ GeV, $\Delta M_{SE} = -0.282$ GeV and $M_\phi = 1.040$ GeV. ($M_\phi(\text{exp}) = 1.020$ MeV).
| $V_i$ | $f_{i\rho}^V$ (GeV) | $f_{i\omega}^V$ (GeV) | $f_{i\phi}^V$ |
|-------|--------------------|--------------------|----------------|
| $i=\rho$ | 0.254 | 0.0846 | 0.096 |
| $i=\omega$ | 0.255 | 0.0756 | 0.107 |
| $i=\phi$ | \(7.04 \pm 0.06\) | \(0.60 \pm 0.02\) | \(1.26 \pm 0.02\) |

Table 2: Decay constants of vector mesons

Since vector mesons are connected to the electromagnetic current, the corresponding decay constants contain the effective charge \(\bar{e}_q^2(i) = \frac{1}{2}, \frac{1}{18}, \frac{1}{9}\) for \(i = \rho, \omega, \phi\) and vector decay constants have the form

\[
f_n^V(i) = \frac{\bar{e}_q^2(i) N_c \varphi_n^2(0) \langle Y_V \rangle}{\omega_1(0) \omega_2(0) M_n(i) \xi_n(i)}, \quad i = \rho, \omega, \phi,
\]

while the dielectron width is connected to \(f_n^V(i)\) as

\[
\Gamma(V \rightarrow e^+ e^-) = \frac{4\pi e^2}{3M_V(i)} (f_n^V(i))^2 (1 - \frac{16}{3\pi} \alpha_s).
\]

Keeping the value of the last factor in (57) to be equal \(0.32 (\alpha_s = 0.4)\) for \(\rho, \omega\) and \(\alpha_s = 0.3\) for \(\phi\), as a result one obtains the values of \(f^V(\text{exp})\), given in the Table 2. Lattice data [22] yield \(f_{\rho} = 239(18)\) in a satisfactory agreement with our result in Table 2.

6 Summary and conclusions

We have presented the theory of lepton decay constants for light mesons based on the path integral formalism. It essentially exploits the path integral Hamiltonian (PIH) depending on virtual \(q, \bar{q}\) energies \(\omega_1, \omega_2\) and the final form obtains after the stationary point analysis with respect to \(\omega_1, \omega_2\). The same approach has given a large number of observables (masses and wave functions, Regge trajectories etc.) in good agreement with experiment both for light and heavy quarks [41, 42, 43]. The lepton decay constants for heavy-light meson have been calculated in the same method in an approximate form in [38, 39] also in good agreement with lattice and experimental data.

In this paper we specifically considered light mesons and paid a special attention to the chiral mesons and their radial excitations. We have also exploited an improved form of the path integral from [40], which allows to...
obtain much better accuracy. Another important ingredient of the present paper is a new treatment of chiral mesons, which exploits the fundamental quantity - the scalar chiral mass parameter $M(0) \approx \sigma \lambda$ (the corresponding chiral correlation length is $1/\sigma \lambda$), which as shown before in [44] and here in Appendix 2, enters additively with the current quark mass $m_q$ and disappears at large $m_q$.

The resulting values of $f_\pi$ and $f_K$ are in good agreement with experimental data, however the only parameters of our theory are $m_q$, $\alpha_s$ and string tension $\sigma$. We have also calculated decay constants of vector mesons $\rho, \omega, \phi$ and found a satisfactory agreement with experiment.

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Appendix 1

The correction coefficient $\xi_n$

As it was shown in [40] (see also appendix of [40]), $\xi_n$ in [24] is defined as follows:

$$\xi_n = \sqrt{\omega_1^{(0)} \omega_2^{(0)} \Omega_n},$$

(A1.1)

with

$$\Omega_n = \left( \alpha \beta - \frac{\gamma^2}{4} \right),$$

(A1.2)

and

$$\alpha \equiv \frac{1}{2} \frac{\partial^2 M_n}{\partial \omega_1^2}, \quad \beta = \frac{1}{2} \frac{\partial^2 M_n}{\partial \omega_2^2}, \quad \gamma = \frac{\partial^2 M_n}{\partial \omega_1 \partial \omega_2},$$

(A1.3)

$M_n$ is defined as

$$M_n = \sum_{i=1,2} \frac{\omega_i^2 + m_i^2}{2 \omega_i} + \varepsilon_n (\tilde{\omega}^{-1}); \quad \tilde{\omega}^{-1} = \frac{\omega_1 + \omega_2}{\omega_1 \omega_2},$$

(A1.4)
and $\varepsilon_n$ is the eigenvalue of the equation

$$\left(\frac{p^2}{2\omega} + V(r)\right)\varphi_n(r) = \varepsilon_n\varphi_n(r), \quad (A1.5)$$

where $V(r)$ includes confinement $V_c(r)$ and gluon exchange interaction $V_g(r)$, but not hyperfine interaction, which is taken into account as a first order correction to the total mass, and hence is not to be present in (A1.3). Hence finally $\bar{M}_n$, entering in $f_n^2$, is the hyperfine averaged eigenvalue $M_n$ with the selfenergy term $\Delta_{SE}$ taken into account

$$\bar{M}_n = \sum \frac{(\omega_i^{(0)})^2 + m_i^2}{2\omega_i^{(0)}} + \varepsilon_n((\tilde{\omega}^{(0)})^{-1}) + \Delta_{SE} \quad (A1.6)$$

For $\alpha, \beta, \gamma$ one obtains

$$2\alpha, 2\beta = \frac{m_i^2}{\omega_i^3} + \frac{2}{\omega_i^3} \varepsilon_n' + \frac{1}{\omega_i^3} \varepsilon_n'', i = 1, 2. \quad (A1.7)$$

$$\gamma = \frac{\varepsilon_n''}{\omega_1^2 \omega_2^2} \quad (A1.8)$$

Here $\varepsilon_n' = \frac{\partial \varepsilon_n}{\partial \tilde{\omega}}, \quad \varepsilon_n'' = \frac{\partial^2 \varepsilon_n}{\partial (\tilde{\omega})^2}$.

For $m_1 = m_2 = m$ and hence $\omega_1 = \omega_2 \equiv \omega$ one has $\alpha = \beta$ and $\Omega_n$ is

$$\Omega_n = \alpha^2 - \frac{\gamma^2}{4} = \frac{\varepsilon_n''(\varepsilon_n' + m_1^2)}{2} + \omega(\varepsilon_n' + m_1^2)^2 \quad (A1.9)$$

In the nonrelativistic limit $\varepsilon_n \ll m, \omega \approx m$, one has

$$\Omega_n \approx \frac{1}{4m^2}, \xi_n = \frac{1}{2} \quad (A1.10)$$

Consider now light quarks and put $m_1 = m_2 = 0$, and first neglect the OGE interaction. Then

$$\varepsilon_n = 2^{-1/3}(\tilde{\omega}^{-1})^{1/3}\sigma^{2/3}a(n), \quad (A1.11)$$

with $a_0 = 2.338$ and other $a(n)$ given in Table 2 of [47].

Using (A1.7), (A1.8), one obtains

$$\xi_n(\alpha_s = 0, m_i = 0) = \frac{3}{\sqrt{5}4} = 0.408; \quad \xi_n^{-1} = 2.45. \quad (A1.12)$$
The resulting $f_2^2$ is
\[ f_2^2 = \frac{2.45 N_c \langle Y_V \rangle \phi_n^2(0)}{\omega_0^2 M_n}, \] (A1.13)
where $\phi_n^2(0) = \frac{\omega_0 \sigma}{4\pi}, \langle Y_V \rangle = \frac{4}{3}\omega_0^2, M_0 = 4\omega_0 + \Delta_{SE} \approx 2\omega_0$.

The inclusion of OGE interaction yields \[ \varepsilon_n^{(g)} = \varepsilon_n \frac{a(\lambda, L, n)}{a(n)} = 2^{-1/3}(\bar{\omega}^{-1})^{1/3} \sigma^{2/3} a(\lambda, L, n), \] (A1.14)
where $\lambda = \frac{4\alpha_s}{3} \left( \frac{2\omega_0}{\sqrt{\sigma}} \right)^{2/3}$ and $a(0, L, n) = a(L, n)$.

The values of $a(\lambda, 0, 0), \frac{\partial a}{\partial \lambda}(\lambda, 0, 0)$ are given in the Table 4 of [38], and one has an estimate of $\frac{\partial^2 a}{\partial \lambda^2} \approx 0.2$ for $\lambda < 0.9$.

The most important change due to OGE is in $\phi_n^2(0)$ which is now for $L = 0$
\[ \phi_n^2(0, \alpha_s) = \frac{\omega}{4\pi}(\sigma + \frac{4}{3}\alpha \langle r^{-2} \rangle_n) = \frac{\sigma \omega}{4\pi} \left| \frac{\chi(0)}{\chi(0)} \right|^2 \] (A1.15)

The values of $\phi_n^2(0, \alpha_s)$ are given in the Table 4 of [38], together with $\left| \frac{\chi(0)}{\chi(0)} \right|^2$.

**Appendix 2**

*Chiral correction length in the confining vacuum*

It was shown in [44], that nonzero field correlator $\langle FF \rangle \equiv \langle tr(F(u)\phi F(v)\phi) \rangle$, generating the kernel $J(x, y) \sim \int du \int dv \langle FF \rangle$, leads to the appearance in the quark Green’s function $S(x, y)$ the nonperturbative nonlocal mass operator $M(x, y), M(x, y) \sim JS$, satisfying equation
\[ (\hat{\partial} + m_q)S(x, y) + \int M(x, y) S(z, y) d^4z = \delta^4(x - y). \] (A2.1)
This general property can be further analyzed separating in $M$ scalar-isoscalar part $M_s$ and pseudoscalar-isovector pieces $\hat{U}$ which can be conveniently written as
\[ \mathcal{M} \rightarrow \mathcal{M}_S \hat{U}, \quad \hat{U} = \exp(i\gamma_5 \hat{\phi}) \quad (A2.2) \]

As a consequence the scalar nonlocal mass \( \mathcal{M}_S(x, y) \) enters into the quark Greens function \( S \) as a scalar piece together with the current quark mass \( m_q \),

\[ iS(x, y) = (\hat{\partial} + m_q + \mathcal{M}_S)^{-1}_{xy}. \quad (A2.3) \]

Note, that at large \( m_q \), the magnitude of \( \mathcal{M}_S \) is fast decreasing

\[ \mathcal{M}_S \sim O(1/m_q), \quad m_q \rightarrow \infty. \quad (A2.4) \]

In the current correlator \( \langle x|\bar{\Gamma}_S q \Gamma_S q|y\rangle \), \( \mathcal{M}_S \) enters in the local vertex form \( \mathcal{M}_S(x, x) = \mathcal{M}_S(0) \), and in the framework of the chiral approach \[44\] based on (A2.1) and (A2.2), one derives the relation for \( f_\pi^2 \) in the chiral limit, \( m_q \rightarrow 0 \),

\[ f_\pi^2 = N_c \mathcal{M}_S^2(0) \sum_{n=0} |\varphi_n(0)|^2 \frac{M_n^3}{M_0^3} \quad (A2.5) \]

where \( n \) refers to the radial excited \( q\bar{q} \) states with mass \( M_n \), and the sum over \( n \) is cut off by the factor \( \exp(-M_n \lambda) \), \( \lambda = 0(1 \text{ GeV}^{-1}) \) is the vacuum correlation length, calculated via gluelump masses \[48\], \( M(0) \) was calculated in the Appendix 4 of the third reference \[44\],

\[ M(0) \approx \frac{2\sigma \lambda}{\sqrt{\pi}} \approx 0.15 \text{ GeV}. \quad (A2.6) \]

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