Article

Event-Triggered Adaptive Neural Network Tracking Control with Dynamic Gain and Prespecified Tracking Accuracy for a Class of Pure-Feedback Systems

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Abstract: This paper studies the event-triggered adaptive tracking control problem of a class of pure-feedback systems. Via the backstepping method and the neural network approximation with the central symmetric distribution, an event-triggered adaptive neural network controller is designed. In particular, a dynamic gain driven by the tracking error is introduced into the event-triggering mechanism. Then, by using the Lyapunov stability theory, the boundedness of all the closed-loop signals is proved, and the tracking error falls into a prespecified $\epsilon$-neighbourhood of zero. Meanwhile, the Zeno behaviour is avoided. Finally, two simulations verify the effectiveness of the proposed control scheme.

Keywords: event-triggered control; pure-feedback system; neural network; dynamic gain

1. Introduction

In the last several years, the event-triggered control (ETC) of nonlinear systems [1–6] has received extensive attention from scholars. For example, the output-feedback event-triggered (ET) controllers are designed for nonlinear plant models [1]. The Zeno behaviour is solved by combining ETC and the time-triggered control. In [2], the authors systematically co-design the adaptive controller and the event-triggering mechanism (ETM) and remove the input-to-state stability assumption on the measurement and requirement of the global Lipschitz continuousness of the nonlinearities. In [3], a periodic ETC policy is utilised to investigate the minimum bit rate conditions for stabilising a scalar nonlinear system under bounded network delay, where the sensor and the controller are updated asynchronously without acknowledgement. By adding different dwell times, two ETC strategies are proposed with static and dynamic ETMs, respectively [4]. Under a fixed/relative threshold strategy, the ETC problem is investigated for p-norm uncertain nonlinear systems to guarantee all the signals of the closed-loop systems converge to an arbitrarily small set [5]. A relative threshold event-triggered strategy is also investigated for large-scale high-order uncertain nonlinear systems [6]. Besides, an event-triggered (ET) stabilisation issue is investigated for IT2 fuzzy systems [7]. In [8], an ET strategy is proposed to save communication resources, and the triggering condition has an adaptive threshold. It is proved that the closed-loop system is semi-globally uniformly ultimately bounded (SGUUB). In [9–11], the ETC problems of strict-feedback systems with external disturbances are studied. In [12], the authors rewrite the closed-loop error system as a linear system with nonlinear perturbation and novelly construct an event-based dynamic surface control law under convex optimisation. Dynamic gains are used in [13,14] to design ET controllers for strict-feedback systems. In [13], based on the dynamic gain, an adaptive ET controller is designed to achieve the desired tracking target and avoid the Zeno...
behaviour. Obviously, most of the above-mentioned literature focuses on the ETC problems of strict-feedback systems.

Compared with strict-feedback systems, pure-feedback systems represent a more general class of lower-triangular systems. Many systems actually fall into this category, such as Duffing oscillator [15], biochemical processes [16], aircraft flight control systems [17] and mechanical systems [18]. Therefore, research on the ETC problems of pure-feedback systems is of great significance. In [19], a fixed-time controller under an event-triggering mechanism (ETM) is designed for a class of pure-feedback systems with non-differentiable and non-affine functions. The ETC problems of pure-feedback systems with constraints are studied in [20–23].

Usually, neural network (NN) and fuzzy logic systems (FLS) are used to approximate the unknown nonlinear smooth function or virtual controller [24,25]. In [26], NN is used to approximate the unknown nonlinear continuous function in the design process. At the same time, an ET switching threshold strategy is proposed to reduce the communication burden in the controller-actuator channel. Through the analysis, it is obtained that the closed-loop system is SGUUB, and the tracking error converges to some compact sets of zero. In [27], a control scheme is proposed, which updates the controller and NN weights only when the event is triggered. In order to solve the problem of discontinuous ET error caused by ETM, the system is described as a nonlinear impulsive dynamic system, and the stability of the system is proved. In [28], FLS is introduced to deal with the system’s unknown nonlinear functions. A new fixed-time performance function is established so that the tracking error can converge near the origin in fixed time, and the ET strategy is adopted to reduce the waste of communication resources. In addition, the ET adaptive NN/FLS control issues are studied for non-strict-feedback systems [29,30] and non-affine pure-feedback nonlinear multi-agent systems [31,32], respectively.

Based on the above analysis, it is of great significance to study the ET adaptive NN control (ANNC) problem of pure-feedback systems [33,34]. Therefore, in this paper, the dynamic gain in the sensor-controller channel and the ETM in the controller-actuator channel are considered together to solve the tracking control problem of a class of pure-feedback systems. In order to realise the tracking performance of the system’s output signal to the reference signal under the prespecified accuracy, an ET ANNC scheme involving a dynamic gain is proposed. The contributions of this paper are summarised as follows:

1. This is the first time that the dynamic gain driven by tracking error is applied to the controller design of the pure-feedback system. Compared with [26,35], the advantage is that when the tracking error is greater than the prespecified accuracy, the dynamic gain will be adjusted to change the size of the control input, so that the tracking error is within the prespecified accuracy. At this time, the tracking target is achieved and the dynamic gain no longer changes.

2. The ETM related to the bounded weights and the dynamic gain is designed in the controller-actuator channel. The weight estimates are updated only at the ET instants, which greatly saves communication resources.

The rest of this paper is organized as follows. The NN approximation and problem statement are given in Section 2. The ETM and adaptive NN controller are given in Section 3. In Section 4, two examples are provided to verify the results obtained in Section 3. Section 5 draws the conclusions.

2. NN Approximation and Problem Statement
   2.1. NN Approximation

Radial Basis Function (RBF) NNs are widely used for adaptive neural control. For any smooth function \( f_i(\rho) : \Omega_{\rho_i} \rightarrow \mathbb{R} \), there is the following approximation:

\[
f_i(\rho) = W_i^{*T} S_i(\rho) + \delta_i(\rho),
\]
where $\Omega_{\Omega}$ is a compact set; $\mathbb{R}$ denotes the set of real numbers; ideal weight vector $W^* := \arg \min_{W \in \mathbb{R}^1} \left\{ \sup_{\rho \in \Omega_{\Omega}} \left| W^T S_i(\rho) - f_i(\rho) \right| \right\}$ with $\mathbb{R}^l$ denoting $l$-dimension vector space; $S_i(\rho) = [s_1(\rho), \ldots, s_{l_i}(\rho)]^T$ is the basis function vector; $l_i$ is the number of neuron nodes and $\delta_i(\rho)$ is the approximation error. In this paper, Gaussian functions $s_j(\rho) = \exp \left\{ - \frac{\|\rho - \xi_j\|^2}{\rho_j^2} \right\} (j = 1, \ldots, l_i)$ are used as the basis functions, where $\xi_j \in \Omega_{\rho}$ and $\eta_j > 0$ are the center and width of the Gaussian function, respectively.

**Assumption 1** ([24]). The NN approximation error is bounded by $\theta$, which is an unknown positive constant, that is, $|\delta_i(\rho)| \leq \theta$.

2.2. Problem Statement

Consider the following pure-feedback system:

$$
\begin{align*}
\dot{x}_i &= f_i(x_i, x_{i+1}), \quad i = 1, \ldots, n - 1, \\
\dot{x}_n &= f_n(x_n) + g_n(\hat{x}_n)u, \\
y &= x_1,
\end{align*}
$$

(1)

where $x_i(i = 1, \ldots, n)$ are the system states with $x = [x_1, \ldots, x_i]^T \in \mathbb{R}^l; u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system input and output, respectively; $f_i(\cdot)(i = 1, \ldots, n)$ and $g_n(\cdot)$ are unknown smooth nonlinear functions.

The purpose of this paper is to design an ET adaptive NN controller to meet the following objectives:

(a) All the signals in the closed-loop systems are bounded on $[0, +\infty)$;

(b) Tracking error $y - y_r$ falls into a prespecified $\epsilon-$neighbourhood of zero;

(c) The infinite ET phenomenon will not happen, that is, Zeno behaviour is avoided.

Define $g_i(x_i, x_{i+1}) = \frac{\partial f_i(x_i, x_{i+1})}{\partial x_{i+1}}, i = 1, 2, \ldots, n - 1$. Note that $g_i(x_i, x_{i+1})$ is still an unknown nonlinear function. The following assumptions are necessary.

**Assumption 2** ([13]). Reference signal $y_r$ and its derivative $\dot{y}_r$ are known and bounded.

**Assumption 3** ([35]). The signs of $g_i(\cdot), i = 1, 2, \ldots, n$ are known and $g_i(\cdot), i = 1, 2, \ldots, n$ are bounded. Without loss of generality, suppose $0 \leq g_{im} \leq g_i(\cdot) \leq g_{im}, i = 1, 2, \ldots, n$, where both $g_{im}$ and $g_{im}$ are positive constants.

**Assumption 4** ([24]). There exists a constant $g_{is} > 0$ such that $|g_i(\cdot)| \leq g_{is}, \forall x_n \in \Omega_{\rho} \subset \mathbb{R}^n$, where $\Omega_{\rho}$ is a compact set, $i = 1, 2, \ldots, n$.

3. ET Tracking Controller and Stability Analysis

3.1. Adaptive Backstepping and ET NN Controller

In this section, an ET ANNNC scheme is proposed through the adaptive backstepping design, where a dynamic gain is introduced into the adaptive NN controller.

In order to show the idea of this paper more intuitively, a frame diagram of the ETC architecture is shown in Figure 1, where ZOH means zero-order holder. It can clearly be seen from Figure 1 that both the dynamic gain in the sensor-controller channel and the ETM in the controller-actuator channel are considered comprehensively, where the ETM is integrated into the controller to determine whether the communication occurs in the controller-actuator channel.
Figure 1. ETC architecture.

**Step 1.** Define tracking error $z_1 = x_1 - y_r$, whose derivative is:

$$z_1 = f_1(x_1, x_2) - y_r.$$  \hspace{1cm} (2)

Define:

$$\omega_1 = -\dot{y}_r + c_1 z_1,$$  \hspace{1cm} (3)

where $c_1$ is a positive constant. Obviously, $\omega_1$ is just a function of $y_r$ and $x_1$, so $\frac{\partial \omega_1}{\partial x_2} = 0$. Taking Assumption 3 into consideration, $\frac{\partial f_1(x_1, x_2)}{\partial x_2} > g_1 > 0$ holds for all $(x_1, x_2) \in \mathbb{R}^2$. Thus, $\frac{\partial f_1(x_1, x_2)}{\partial x_2} + \omega_1 > g_1 > 0$. According to the process in [24], there is an ideal virtual control input $\alpha_1^* (x_1, \omega_1)$ for each $x_1$ and $\omega_1$ such that:

$$f_1(x_1, x_2) = f_1(x_1, \alpha_1^*) + \bar{g}_1(x_2 - \alpha_1^*),$$  \hspace{1cm} (4)

where $\bar{g}_1 = g_1(x_1, x_2)$ with $x_2 = \mu_1 x_2 + (1 - \mu_1) \alpha_1^*$ and $0 < \mu_1 < 1$. It can be deduced that Assumption 3 made for $g_1(x_1, x_2)$ still applies to $\bar{g}_1$. Because $\alpha_1^*$ is a function of $x_1$ and $y_r$, $\bar{g}_1$ is a function of $x_1, x_2$ and $y_r$. Then one has:

$$\dot{\bar{g}}_1 = 2 \sum_{j=1}^2 \frac{\partial \bar{g}_1}{\partial x_j} \dot{x}_j + \frac{\partial \bar{g}_1}{\partial y_r} \dot{y}_r$$

$$= 2 \sum_{j=1}^2 \frac{\partial \bar{g}_1}{\partial x_j} f_1(x_{j+1}) + \frac{\partial \bar{g}_1}{\partial y_r} \dot{y}_r.$$  \hspace{1cm} (5)

According to Assumption 4, $|\dot{\bar{g}}_1| \leq \bar{g}_1$ still holds.

Combining (2)–(4), one can obtain:

$$z_1 = -c_1 z_1 + \bar{g}_1(x_2 - \alpha_1^*).$$  \hspace{1cm} (6)

RBF NN is used to approximate $\alpha_1^*$, i.e.,

$$\alpha_1^* = W_1^T S_1(\gamma_1) + \delta_1,$$  \hspace{1cm} (7)

where $\gamma_1 = (x_1, y_r)^T \in \Omega_1 \subset \mathbb{R}^2$.

Define $z_2 = x_2 - \alpha_1$, and design virtual controller:

$$\alpha_1 = -b_1 z_1 - \frac{G}{2} z_1 + W_1^T S_1(\gamma_1),$$  \hspace{1cm} (8)
where \( \hat{W}_1 \) is the estimate of \( W_1^* \); \( b_1 \) is a positive design constant and \( G \) is the dynamic gain defined as:

\[
G = \max \{ |y - y_r| - \epsilon, 0 \}, \quad G(0) = 1,
\]

with \( \epsilon \) being a prespecified tracking accuracy.

Then, Equation (6) becomes:

\[
\dot{z}_1 = -c_1 z_1 + \hat{g}_1 (z_2 + \alpha_1 - \alpha_1^1)
\]

\[
= -c_1 z_1 + \hat{g}_1 [z_2 - b_1 z_1 - \frac{G}{2} z_1 + \hat{W}_1^T S_1(\gamma_1) - \delta_1],
\]

where \( \hat{W}_1 = \hat{W}_1 - W_1^* \). Hereinafter, define \( (\cdot)^* = (\cdot)^{-1} \).

Select the Lyapunov function:

\[
V_1 = \frac{1}{2} \hat{g}_1 z_1^2 + \frac{1}{2} \hat{W}_1^T \Gamma_1^{-1} \hat{W}_1,
\]

where \( \Gamma_1 \) is a positive definite matrix with proper dimension and hereinafter, \( \Gamma_i \) throughout the paper has the same meaning. Its derivative is:

\[
\dot{V}_1 = \dot{z}_1 \dot{z}_1 - \frac{\hat{g}_1}{\hat{g}_1} \dot{z}_2^2 + \hat{W}_1^T \Gamma_1^{-1} \hat{W}_1
\]

\[
= -\frac{\hat{g}_1}{\hat{g}_1} z_1^2 + z_1 z_2 - b_1 z_1^2 - \frac{G}{2} z_1^2 - \frac{\hat{g}_1}{\hat{g}_1} z_1^2
\]

\[
+ \hat{W}_1^T S_1(\gamma_1) z_1 - z_1 \delta_1 + \hat{W}_1^T \Gamma_1^{-1} \hat{W}_1.
\]

According to the Young’s Inequality, it is obtained as:

\[
-z_1 \delta_1 \leq |z_1||\theta|
\]

\[
\leq \frac{G}{2} z_1^2 + \frac{\theta^2}{2G}.
\]

Design the adaptive law of \( \hat{W}_1 \) as:

\[
\dot{\hat{W}}_1 = \Gamma_1 [-S_1(\gamma_1) z_1 - \sigma_1 \hat{W}_1],
\]

where \( \sigma_1 \) is a small positive constant. By choosing \( b_1 = b_{10} + b_{11} \) with \( b_{10} > 0 \) and \( b_{11} > 0 \), Equation (12) becomes:

\[
V_1 \leq z_1 z_2 - (b_{10} + \frac{\hat{g}_1}{\hat{g}_1} - \frac{\theta^2}{2G}) z_1^2 - b_{11} z_1^2
\]

\[
+ \frac{\theta^2}{2G} - \sigma_1 \hat{W}_1^T \hat{W}_1.
\]

According to the Young’s Inequality, the following inequality holds:

\[
-\sigma_1 \hat{W}_1^T \hat{W}_1 = -\sigma_1 \hat{W}_1^T (\hat{W}_1 + W_1^*)
\]

\[
\leq -\frac{\sigma_1 ||\hat{W}_1||^2}{2} + \frac{\sigma_1 ||W_1^*||^2}{2}.
\]
Because \(- (b_{10} + \frac{\delta_1}{2s_1}) z_1^2 \leq -(b_{10} - \frac{\delta_1}{2s_{1w}}) z_1^2\), by choosing \(b_{10} = b_{10} - \frac{\delta_1}{2s_{1w}} > 0\), Equation (15) becomes:

\[
\dot{V}_1 < z_1z_2 - b_{10}z_1^2 - \frac{\sigma_1||\tilde{W}_1||^2}{2} + \frac{\theta^2}{2G} + \frac{\sigma_1\beta}{2G},
\]

(17)

where \(||W_1||^2 \leq \frac{\theta}{C} \leq \frac{\theta}{C} \) with \(\beta > 0\).

**Remark 1.** According to (9), it can be seen that \(G\) will gradually increase until \(\dot{G} = 0\), which implies that the control goal is achieved. Obviously, \(G\) is a non-decreasing function greater than 1.

**Remark 2.** The ideal weight is bounded, i.e., there is a positive constant \(\beta > 0\) satisfying \(||W_i^*||^2 \leq \frac{\theta}{C} \leq \frac{\theta}{C}\), where constant \(G\) is an upper bound of \(G\) and the boundedness of \(G\) will be proved later.

**Step i** (\(2 \leq i \leq n - 1\)). The derivative of \(z_i = x_i - a_{i-1}\) is:

\[
\dot{z}_i = f(x_r, x_{i+1}) - a_{i-1}.
\]

(18)

Similar to Step 1, there exists \(\mu_i (0 < \mu_i < 1)\) satisfying:

\[
\dot{z}_i = -c_i z_i + g_i(x_{i+1} - a_i^*),
\]

(19)

where \(g_i = g_i(x_r, x_{i+1})\) with \(x_{i+1} = \mu_i x_{i+1} + (1 - \mu_i)a_i^*\). Assumption 3 is still valid for \(g_i\). Owing to that \(a_{i-1}\) is a function of \(y_r, x_{i-1}\) and \(\tilde{W}_i, \ldots, \tilde{W}_{i-1}, a_{i-1}\) is:

\[
\dot{a}_{i-1} = \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial x_j} f_j(x_{j+1}) + D_{i-1},
\]

(20)

where

\[
D_{i-1} = \frac{\partial a_{i-1}}{\partial y_r} y_r + \sum_{j=1}^{i-1} \frac{\partial a_{i-1}}{\partial W_j} \dot{W}_j.
\]

(21)

An RBF NN is used to approximate \(a_i^*(x_i, \omega_i)\), i.e.,

\[
a_i^* = W_i^{*T} S_i(\gamma_i) + \delta_i,
\]

(22)

where \(\gamma_i = (x_r, \frac{\partial a_{i-1}}{\partial x_1}, \ldots, \frac{\partial a_{i-1}}{\partial x_n}, D_{i-1}, z_i)^T \in \Omega_i \subset \mathbb{R}^{2i+1}\).

Define \(z_{i+1} = x_{i+1} - a_i\) and design virtual controller:

\[
a_i = -z_{i-1} - b_i z_i - \frac{G}{2} z_i + \tilde{W}_i^{*T} S_i(\gamma_i),
\]

(23)

where \(\tilde{W}_i\) is the estimate of \(W_i^*\) and \(b_i\) is a positive constant.

Then, \(z_i\) becomes:

\[
z_i = -c_i z_i + \tilde{g}_i(z_{i+1} + a_i - a_i^*)
\]

\[= -c_i z_i + \tilde{g}_i(z_{i+1} - z_{i-1} - b_i z_i - \frac{G}{2} z_i + \tilde{W}_i^{*T} S_i(\gamma_i) - \delta_i),
\]

(24)
Choose Lyapunov function:

\[ V_i = V_{i-1} + \frac{1}{2} \hat{z}_i^2 + \frac{1}{2} \hat{W}_i^T \Gamma_i^{-1} \hat{W}_i. \]  

(25)

Its derivative is:

\[ \dot{V}_i = \dot{V}_{i-1} + \frac{z_i \hat{z}_i}{\hat{S}_i} - \frac{\hat{g}_i}{2 \hat{S}_i} \hat{z}_i^2 + \hat{W}_i^T \Gamma_i^{-1} \hat{\dot{W}}_i \]

\[ = \dot{V}_{i-1} - \frac{c_i}{\hat{S}_i} \hat{z}_i^2 - z_{i-1} \hat{z}_i + z_i \hat{z}_{i+1} - b_i \hat{z}_i^2 - \frac{G}{2} \hat{z}_i^2 \]

\[ - \frac{\hat{g}_i}{2 \hat{S}_i} \hat{z}_i^2 + \hat{W}_i^T \hat{S}_i(\gamma_i) \hat{z}_i - z_i \delta_i + \hat{W}_i^T \Gamma_i^{-1} \hat{\dot{W}}_i. \]  

(26)

Similar to Step 1, it holds:

\[-z_i \delta_i \leq \frac{G}{2} \hat{z}_i^2 + \frac{\theta^2}{2G}. \]  

(27)

The adaptation law of \( \hat{W}_i \) is designed:

\[ \hat{W}_i = \Gamma_i [-S_i(\gamma_i) \hat{z}_i - \sigma_i \hat{\dot{W}}_i], \]  

(28)

where \( \sigma_i \) is a small positive constant. By choosing \( b_i = b_{i0} + b_{i1} \) with \( b_{i0} > 0 \) and \( b_{i1} > 0 \), Equation (26) becomes:

\[ V_i \leq V_{i-1} - z_{i-1} \hat{z}_i + z_i \hat{z}_{i+1} - (b_{i0} + \frac{\hat{g}_i}{2 \hat{S}_i}) \hat{z}_i^2 \]

\[ - b_{i1} \hat{z}_i^2 + \frac{\theta^2}{2G} - \sigma_i \hat{W}_i^T \hat{\dot{W}}_i. \]  

(29)

Similar to Step 1, it holds:

\[-\sigma_i \hat{W}_i^T \hat{\dot{W}}_i \leq - \frac{\sigma_i ||W_i||^2}{2} + \frac{\sigma_i ||W_i^*||^2}{2}. \]  

(30)

Since \(- (b_{i0} + \frac{\hat{g}_i}{2 \hat{S}_i}) \hat{z}_i^2 \leq -(b_{i0} - \frac{k_{i0}}{2 \hat{S}_i}) \hat{z}_i^2\), by choosing \( b_{i0} \) large enough such that \( b_{i0} = b_{i0} - \frac{k_{i0}}{2 \hat{S}_i} > 0 \), Equation (29) becomes:

\[ \dot{V}_i \leq z_i \hat{z}_{i+1} - \sum_{j=1}^i b_{j0} \hat{z}_j^2 - \sum_{j=1}^i \sigma_j ||\hat{W}_j||^2 \]

\[ + \frac{i \theta^2}{2G} + \sum_{j=1}^i \sigma_j \beta \frac{2G}{2G}. \]  

(31)

**Step n.** The derivative of \( z_n = x_n - \alpha_{n-1} \) is:

\[ z_n = f_n(x_n) + g_n(x_n) u - \alpha_{n-1}. \]  

(32)

There exists the ideal virtual controller:

\[ \alpha_n^* = -z_{n-1} - b_n z_n - \frac{[f_n(x_n) - \alpha_{n-1}]}{g_n(x_n)}, \]  

(33)

where \( b_n \) is a positive constant.
On account of that \(\alpha_n - 1\) is a function of \(y_r, x_{n - 1}\) and \(\hat{W}_1, \ldots, \hat{W}_{n-1}\),

\[
\dot{\alpha}_{n - 1} = \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_j} f_j(x_{j+1}) + D_{n-1}, \tag{34}
\]

where,

\[
D_{n-1} = \frac{\partial \alpha_{n-1}}{\partial y_r} \dot{y}_r + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{W}_j} [\Gamma_j (-S_j(\gamma_j)z_j - \sigma_j \hat{W}_j)]. \tag{35}
\]

The unknown part of \(\hat{\alpha}_n^*\) is approximated by using an RBF NN, i.e.,

\[
- \left[ f_n(\hat{x}_n) - \hat{\alpha}_{n-1} \right] = W_n^T S_n(\alpha_n) + \delta_n, \tag{36}
\]

where \(\gamma_n = (\hat{x}_n, \frac{\partial \alpha_{n-1}}{\partial x_1}, \ldots, \frac{\partial \alpha_{n-1}}{\partial x_n}, D_{n-1})^T \in \Omega_n \subset \mathbb{R}^{2n}.\) Thus,

\[
\hat{\alpha}_n^* = -z_{n-1} - b_n z_n + W_n^T S_n(\gamma_n) + \delta_n. \tag{37}
\]

According to (37), design the adaptive NN controller and the weight update law as:

\[
\alpha_n = -z_{n-1} - b_n z_n - G z_n + \hat{W}_n^T S_n(\gamma_n), \tag{38}
\]

\[
\dot{\hat{W}}_n = \Gamma_n [ - S_n(\gamma_n) z_n - \sigma_n \hat{W}_n ], \tag{39}
\]

where \(\sigma_n\) is a small positive constant. Let \(b_n = b_{n0} + b_{n1}\) with \(b_{n0} \) and \(b_{n1} \) > 0.

In order to decrease the controller update frequency, an ETM is designed to decide the data transmission from controller to actuator. Now the ETM and the ET controller are given as follows:

\[
u(t) = \alpha_n(t_k), \quad t \in [t_k, t_{k+1}), \tag{40}
\]

\[
t_{k+1} = \inf\{t > t_k | |u(t) - \alpha_n(t)| \geq A\}, \tag{41}
\]

where \(k = 0, 1, \ldots, t_0 = 0; A\) is a positive constant.

Now select the Lyapunov function:

\[
V_n = V_{n-1} + \frac{1}{2} \frac{\dot{z}_n^2}{\tilde{g}_n(\hat{x}_n)} + \frac{1}{2} \hat{W}_n^T \Gamma_n^{-1} \hat{W}_n, \tag{42}
\]

and it is obtained as:

\[
\dot{V}_n = \dot{V}_{n-1} + \frac{z_n \dot{z}_n}{\tilde{g}_n(\hat{x}_n)} - \frac{\tilde{g}_n(\hat{x}_n)}{2 \tilde{g}_n^2(\hat{x}_n)} \dot{z}_n^2 + \hat{W}_n^T \Gamma_n^{-1} \hat{W}_n. \tag{43}
\]

Combining (32) and (43) gets:

\[
\dot{V}_n = \dot{V}_{n-1} + z_n u + z_n f_n(\hat{x}_n) - \frac{\tilde{g}_n(\hat{x}_n)}{2 \tilde{g}_n^2(\hat{x}_n)} \dot{z}_n^2 + \dot{\hat{W}}_n^T \Gamma_n^{-1} \dot{\hat{W}}_n. \tag{44}
\]

Substituting (32), (36), (38) and (39) into (44), one obtains:

\[
\dot{V}_n = V_{n-1} - z_{n-1} z_n - b_{n1} z_n^2 - G z_n^2 + z_n (u - \alpha_n) - z_n \delta_n - (b_{n0} + \frac{\tilde{g}_n(\hat{x}_n)}{2 \tilde{g}_n^2(\hat{x}_n)}) \dot{z}_n^2 - \sigma_n \hat{W}_n \dot{\hat{W}}_n. \tag{45}
\]

The following inequalities hold:

\[
z_n (u - \alpha_n) \leq \frac{G z_n^2}{2} + \frac{A^2}{2G}. \tag{46}
\]
\[-z_n\delta_n \leq \frac{G}{2}z_n^2 + \frac{\theta^2}{2G},\]  

(47)

\[-\sigma_n \dot{W}_n^T \dot{W}_n \leq - \frac{\sigma_n ||\dot{W}_n||^2}{2} + \frac{\sigma_n ||\dot{W}_n^*||^2}{2} \leq - \frac{\sigma_n ||\dot{W}_n||^2}{2} + \frac{\sigma_n \beta}{2G},\]  

(48)

Because \(-b_n^* = b_n - \frac{\delta_n}{2\delta_n} > 0\), by choosing \(b_n^*\) large enough such that \(b_n^* = b_n - \frac{\delta_n}{2\delta_n} > 0\), Equation (45) becomes:

\[V_n < - \sum_{j=1}^{n} b_n^* z_j^2 - \sum_{j=1}^{n} \frac{\sigma_j ||\dot{W}_j||^2}{2} + \frac{n\theta^2}{2G} + \sum_{j=1}^{n} \frac{\sigma_j \beta}{2G} + \frac{A^2}{2G}.\]  

(49)

3.2. Stability Analysis

**Theorem 1.** Under Assumptions 1–4, the closed-loop system consisting of pure-feedback system (1), dynamic gain (9), ETM (40)–(41) satisfies the following conclusions.

(i) All the signals are bounded on \([0, +\infty);\)

(ii) \(\exists T_0 > 0, \text{ when } t > T_0, |y - y_r| < \epsilon, \text{ where } \epsilon \text{ is the prespecified tracking accuracy};\)

(iii) Zeno behaviour is avoided, i.e., \(\inf\{t_k+1 - t_k\} > 0\).

**Proof.** (i) Equation (49) can be written as:

\[V_n < - \sum_{j=1}^{n} b_n^* z_j^2 - \sum_{j=1}^{n} \frac{\sigma_j ||\dot{W}_j||^2}{2} + \frac{\Delta}{G}, \forall t \in [0, +\infty),\]  

(50)

where \(\Delta = \frac{1}{2\delta_n^2} \sum_{j=1}^{n} \sigma_j \beta + n\theta^2 + A^2\) is a positive constant.

By choosing \(b_n^* \geq \frac{1}{2\delta_n^2}\), i.e., \(b_n^* \geq \frac{1}{2\delta_n^2} + \frac{\delta_n^2}{2\delta_n^2}\), and choosing \(\sigma_j\) and \(\Gamma_j\) to satisfy \(\sigma_j \geq \lambda_{\max}\{\Gamma_j^{-1}\}, j = 1, \ldots, n\), Equation (50) becomes:

\[V_n \leq - \sum_{j=1}^{n} \frac{1}{2\delta_n^2} z_j^2 - \sum_{j=1}^{n} \frac{\dot{W}_j^T \Gamma_j^{-1} \dot{W}_j}{2} + \frac{\Delta}{G} \leq - \sum_{j=1}^{n} \frac{1}{2\delta_n^2} z_j^2 + \sum_{j=1}^{n} \frac{\dot{W}_j^T \Gamma_j^{-1} \dot{W}_j}{2} + \frac{\Delta}{G} \leq - V_n + \frac{\Delta}{G}, \forall t \in [0, +\infty).\]  

(51)

From (51), one can obtain:

\[V_n(t) \leq V_n(0)e^{-t} + \int_{0}^{t} e^{-(t-\tau)} \frac{\Delta}{G(\tau)}d\tau, \forall t \in [0, +\infty),\]  

(52)

Because of \(G \geq 1\) and \(e^{-t} < 1 (\forall t > 0)\), there is:

\[V_n(t) \leq V_n(0) + \Delta, \forall t \in [0, +\infty).\]  

(53)
Based on the above analysis and the definition of $V_n$, the boundedness of $z_i$ and $\tilde{W}_i$ can be obtained directly. Since $W_i$ is bounded, the boundedness of $\hat{W}_i$ can obviously be obtained.

(ii) Because $G$ is a monotonically non-decreasing function, there exists finite time $T_1$ such that:

$$G \geq \frac{8\Delta}{\epsilon^2}, \quad \forall t > T_1. \tag{54}$$

Recalling (51), one obtains:

$$\dot{V}_n \leq -V_n + \frac{\epsilon^2}{8}, \quad \forall t > T_1, \tag{55}$$

and thus,

$$V_n(t) \leq V_n(T_1)e^{-(t-T_1)} + \frac{\epsilon^2}{8}(1 - e^{-(t-T_1)}). \tag{56}$$

According to the definition of $V_n$ and (56), there is sufficiently large time $T_2$ such that:

$$|z_1| \leq \sqrt{2V_n} \leq \sqrt{\frac{\epsilon^2}{4}} = \frac{\epsilon}{2} < \epsilon. \tag{57}$$

By (9), we have $\hat{G} = 0, \forall t > T_2$, which implies $\sup_{t \geq 0} G(t) = G(T_2) < +\infty$. Therefore, the boundedness of $G$ on $[0, +\infty)$ is obtained.

Moreover, because $y_r$ and $z_i$ are bounded on $[0, +\infty)$ and $z_i = x_i - \alpha_{i-1}$, it can be recursively derived that $x_i$ and $\alpha_i$ are bounded on $[0, +\infty)$. Based on the forms of $a_n$ and $u$, the boundedness of $u$ on $[0, +\infty)$ is obtained.

Thus, all the signals in the closed-loop system are bounded on $[0, +\infty)$.

Since $G$ is bounded on $[0, +\infty)$, it holds:

$$\lim_{t \to +\infty} \int_0^t \dot{G}(\tau)d\tau = G(\infty) - G(0) < +\infty. \tag{58}$$

Because $z_i$ is bounded and Lipschitz continuous, $\hat{G}$ is uniformly continuous in $t$ on $[0, +\infty)$. Taking Barbalat Lemma into consideration, $\lim_{t \to +\infty} \hat{G} = 0$ is obtained. Combining with (9), one has:

$$|y - y_r| = |z_1| < \epsilon, \forall t \geq T_0, \tag{59}$$

with finite time $T_0 > 0$.

(iii) From (40) and (41), it holds:

$$|\alpha(t_{k+1}) - \alpha(t_k)| \geq A. \tag{60}$$

Taking (38) into consideration, we have:

$$\dot{a}_n = -z_n - b_n z_n - \hat{G} z_n - \hat{G}^2 z_n + \hat{W}_n^T \dot{s}_n(\gamma_n). \tag{61}$$
The items in the right-hand side of (61) are continuous, so $\dot{\alpha}_n$ is continuous. Thus, $\exists E > 0, \forall t \in [t_k, t_{k+1}]$. Then one can have:

$$A \leq |a(t_{k+1}) - a(t_k)| = |\int_{t_k}^{t_{k+1}} \dot{\alpha}_n(\tau) d\tau| \leq \int_{t_k}^{t_{k+1}} |\dot{\alpha}_n(\tau)| d\tau \leq \int_{t_k}^{t_{k+1}} E d\tau = E(t_{k+1} - t_k).$$

Eventually, it is obtained that:

$$t_{k+1} - t_k \geq \frac{A}{E} > 0. \quad (63)$$

Therefore, no Zeno behaviour occurs.

The proof is completed. \(\square\)

4. Simulations

4.1. Numerical Simulation

We give the following numerical example to illustrate the effectiveness of the proposed scheme.

$$\begin{align*}
\dot{x}_1 &= x_1 + x_2 + 0.2x_3^2, \\
\dot{x}_2 &= x_1x_2 + u, \\
y &= x_1.
\end{align*} \quad (64)$$

There are two NNs used in the simulation. Thus, 25 and 125 nodes were chosen for NNs: $\hat{W}_1^T S_1$, $\hat{W}_2^T S_2$, whose centers were evenly spaced in $[-1, 1] \times [-1, 1]$ and $[-1, 1] \times [-1, 1] \times [-1, 1]$, respectively, with the same width $\psi = 2$. The parameters were selected as follows: $b_1 = 35, b_2 = 30, \sigma_1 = 5, \sigma_2 = 10, A = 0.3, \epsilon = 0.03, \Gamma_1 = \text{diag}[1]_{25}, \Gamma_2 = \text{diag}[1]_{125}$, where $\text{diag}[x]_l$ represents $l$-dimension diagonal matrix with elements $x$, and the system state at the initial moment is $[0.2, 0.8]^T$.

The reference trajectory is chosen as $y_d = 0.5 \cos(0.5t) + 0.6 \sin t$. System output trajectory $y$ and reference trajectory $y_d$ are shown in Figure 2. In Figure 3, it can be clearly seen that the tracking error trajectory is between $-0.03$ and $0.03$ after $0.4\text{vs}$, which implies that the control target is achieved. In Figure 4, the growth of dynamic gain $G$ is clear at a glance, and it remains unchanged after $0.4\text{vs}$, which corresponds to the fact that the tracking error in Figure 3 converges with the pre-specified accuracy after $0.4\text{vs}$. That is, the control target is achieved. The change of control $u$ can be clearly seen in Figure 5. Figure 6 is the time interval diagram between the consecutive triggered moments.
Figure 2. Output signal $y$ and reference signal $y_d$ for (64).

Figure 3. Tracking error $y - y_d$ for (64).

Figure 4. Dynamic gain $G$ for (64).
Theoretically, the value of $A$ is the fixed triggering threshold, which can be an arbitrary predetermined positive constant. On the premise of achieving the control objectives, different values of $A$ are taken for simulation and the triggered times of events under different values of $A$ are shown in Table 1. It can be seen from Table 1 that when $A$ is 0.3, the number of the triggered events is the least.

**Remark 3.** Indeed, from Table 1, it also can be seen that too small or too big $A$ values can create a large number of events. Obviously, the smaller the triggering threshold, the more the events are triggered. That is, more network resources are wasted. Hence, it seems that the value of $A$ should be chosen large enough. However, if $A$ is chosen too large, the error between the controller and the actuator may deteriorate the control performance and cause big fluctuation of the controller; thus, more events may be triggered. In practical applications, the triggering threshold need to be selected eclectically to trade off the system performance and the communication frequency, which seems to be a dilemmatic choice. Therefore, in the future, the dynamic threshold method also should be considered in our research.

**Table 1.** Triggering times with different $A$ for (64).

| $A$  | 0.01 | 0.02 | 0.05 | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  |
|------|------|------|------|------|------|------|------|------|
| Times | 559  | 375  | 260  | 261  | 245  | 205  | 226  | 229  |
4.2. Simulation of One-Link Robot

In this part, the effectiveness of the proposed scheme is further verified by the practical application of the one-link robot. The equation of the one-link robot system is as follows [25]:

\[
\begin{aligned}
R \ddot{p} + F \dot{p} + N \sin(p) &= \phi, \\
M \dot{\phi} &= u + 0.5 \sin(u) - L_m \dot{p}, \\
\end{aligned}
\]  
(65)

where \( p, \dot{p} \) and \( \ddot{p} \) represent the position, velocity and acceleration of the link, respectively. \( \phi \) and \( \dot{\phi} \) are the motor shaft angle and velocity. \( u \) denotes the control input of motor torque.

Let \( x_1 = p, x_2 = \dot{p}, x_3 = \phi \), and then system (65) can be rewritten as:

\[
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\frac{N \sin(x_1)}{R} - \frac{F x_2}{R} + x_3, \\
\dot{x}_3 &= -\frac{L_m x_2}{RM} - \frac{Q x_3}{M} + \frac{u + 0.5 \sin(u)}{RM}, \\
y &= x_1. \\
\end{aligned}
\]  
(66)

In the simulation, the system parameters are chosen as \( R = 1, F = 1, N = 10, L_m = 10, Q = 0.5, M = 0.05 \) and reference trajectory \( y_d \) is generated by the van der Pol oscillator system [25].

Here, 27, 243 and 2187 nodes were chosen for NNs: \( \hat{W}_T S_1, \hat{W}_T S_2 \) and \( \hat{W}_T S_3 \), whose centers are evenly spaced in \([-1, 1] \times [-1, 1] \times [-1, 1], [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1], \) and \([-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1], \) respectively, with the same width \( \psi = 2 \). The simulation parameters are selected as follows: \( b_1 = 50, b_2 = 20, b_3 = 5, c_1 = 5, c_2 = 10, c_3 = 10, \) \( A = 0.4, \epsilon = 0.05, \Gamma_1 = \text{diag}\{1\}_{27}, \Gamma_2 = \text{diag}\{1\}_{243}, \Gamma_2 = \text{diag}\{1\}_{2187} \). The system state vector at the initial moment is set as \([0.3, 0, 0]^T\), and the initial state vector of reference signal \( y_d \) is \([1.4, 0.8]^T\). The trajectories of the system output \( y \) and reference signal \( y_d \) are shown in Figure 7.

In Figure 8, it can be clearly seen that the tracking error trajectory is between \(-0.05\) and \(0.05\) after 0.5 s, which implies that the control target is achieved. In Figure 9, dynamic gain \( G \) grows at the beginning and remains unchanged after 0.5 s, which corresponds to the fact that the tracking error in Figure 8 converges with the prespecified accuracy after 0.4 s. That is, the control target has been achieved. The change of control \( u \) can be clearly seen in Figure 10. Figure 11 shows the time interval diagram between the consecutive triggered moments.

![Figure 7](image-url)  
**Figure 7.** Output signal \( y \) and reference signal \( y_d \) for (66).
Figure 8. Tracking error $y - y_d$ for (66).

Figure 9. Dynamic gain $G$ for (66).

Figure 10. Control $u$ for system (66).
As mentioned in Remark 3, different values of $A$ are also selected for simulation. The results are shown in Table 2. On the premise of achieving the control goal, when $A = 0.4$, the number of events is the least; that is, the saving of communication resources is the greatest.

Table 2. Triggering times with different $A$ for (66).

| $A$  | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|
| Triggering times | 670 | 460 | 401 | 369 | 371 | 379 | 394 | 397 |

5. Conclusions

In this paper, a new ETC scheme is proposed for a class of pure-feedback systems. The introduction of dynamic gain ensures that the output signal can track the reference signal effectively with the prespecified accuracy. Through theoretical analysis, it is proved that all signals are ultimately bounded, and Zeno behaviour is ruled out. Two simulations show the effectiveness of the designed scheme.

The threshold of the ETM in this paper is a given constant. In the future research, dynamic threshold [36] and controller fault-tolerance [37] will be further considered. The method proposed in this paper will also be expanded to the multi-agent systems with the symmetric communication topology.

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