W-algebra for solving problems with fuzzy parameters

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Abstract. A method of solving the problems with fuzzy parameters by means of a special algebraic structure is proposed. The structure defines its operations through operations on real numbers, which simplifies its use. It avoids deficiencies limiting applicability of the other known structures. Examples for solution of a quadratic equation, a system of linear equations and a network planning problem are given.

1. Introduction
The fuzzy set theory introduced by Zadeh [1] has been successfully used to represent the uncertainty caused by the use of qualitative scales in the measurement of parameters and variables of the objects. In particular, this uncertainty is typical for many control and decision-making tasks. Mathematical models of such problems include the parameters and variables that are defined as fuzzy numbers or fuzzy variables. It allows to receive the numerical and analytic solutions for control and decision-making problems under uncertainty. However, in practice, these options are not used in full due to the problems with implementation of fuzzy information processing. Classical approaches for handling the fuzzy numbers are based on the Zadeh's extension principle [2]. Representation of fuzzy numbers in the form of \( LR \)-numbers with a convex membership functions allows to significantly simplify the computational process. On this basis a wide variety of algebraic approaches to the implementation of fuzzy calculations developed in [3], [4], [5], [6], [7], [8], [9]. Nevertheless, some characteristics of algebraic and computational operations with fuzzy numbers restrain their practical applications. We point out the following negative characteristics:

- operations with fuzzy numbers can be impractical, because they may lead to an unjustified extension of the result uncertainty;
- fuzzy results of solving problems aren't always able to be interpreted properly due to distortion of the form of a fuzzy numerical result or (and) the distortion of natural properties and relations of classical models (e.g., fuzzy parameters do not retain the identity of the equation after substituting the solution). This implies a decrease, and sometimes a loss of the models adequacy;
- there are a number of software tools to solve various mathematical tasks of control and decision-making, which designed to work with the real numbers. They cannot be used in the fuzzy data processing due to implementation of arithmetic operations with fuzzy numbers and problem of comparing of the fuzzy numbers.

Overcoming these problems is possible by constructing a suitable algebraic structure of fuzzy information processing, which is the purpose of this work. The article consists of eight sections. We
look at the theoretical foundations of fuzzy theory and fuzzy numbers in Section 3. We construct the algebra of the single component numbers in section 4. Section 5 introduces $W$-algebra and shows some of its properties. Section 6 is devoted to the numerical examples showing some characteristics of compared approaches. Section 7 contains results and discussion of our work. Section 8 sets out the basic conclusions.

2. Materials and methods
Fuzzy set theory and the methods of algebraic structure theory used to develop and study the algebraic structure of fuzzy information processing that overcomes the previously mentioned characteristics. Linear programming methods used to find the critical path on a project’s network graph in the numerical examples section. Microsoft Office Excel© used to solve equations and find optimal solutions.

3. Fuzzy numbers
A classical (crisp) set is a collection of elements or objects $x \in X$ that can be finite, countable, or over-countable. Each single element can either belong to or not belong to a set $A$, $x \in X$. Such a classical set can be described using the characteristic function, in which 1 indicates membership and 0 non-membership. For a fuzzy set, the characteristic function allows various degrees of membership for the elements of a given set.

Definition 1. If $X$ is a collection of objects denoted generically by $x$, then a fuzzy set $A$ in $X$ is a set of ordered pairs:

$$A = \{x, \mu_a(x) \mid x \in X\}$$

$\mu_a(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of $x$ in $A$ that maps $X$ to the membership space $M$.

Definition 2. The support of a fuzzy set $A$, $S(A)$, is the crisp set of all $x \in X$ such that $\mu_a(x) > 0$.

Definition 3. The (crisp) set of elements that belong to the fuzzy set $A$ at least to the degree $r$ is called the $r$-level set [10]:

$$A_r = \{x \in X \mid \mu_a(x) \geq r\}.$$  \hspace{1cm} (2)

Definition 4. Fuzzy number $A$ is a fuzzy set $\mu_a(x) : X \rightarrow [0,1]$ which satisfies:

- $\mu$ is upper semicontinuous;
- $\mu(x) = 0$ outside some interval $[c, d]$;
- there are real numbers $a, b$: $c \leq a \leq b \leq d$ for which $\mu(x)$ is monotonic increasing on $[c, a]$, $\mu(x)$ is monotonic decreasing on $[b, d]$, and $\mu(x) = 1$ at $a \leq x \leq b$.

Definition 5. A fuzzy number $A$ in parametric form is a pair $(g(r), \overline{a}(r))$, $r \in [0,1]$, which satisfies the following requirements [11]:

- $g(r)$ is a bounded left continuous monotonic increasing function over $[0,1]$;
- $\overline{a}(r)$ is a bounded left continuous monotonic decreasing function over $[0,1]$;
- $g(r) \leq \overline{a}(r)$, $r \in [0,1]$.

Triangular fuzzy numbers are often used when there is not enough information to use more complex nonlinear membership functions. Also triangular fuzzy numbers help to reduce computational complexity and achieve sufficient accuracy.

Definition 6. Following [12] a triangular fuzzy number defined in parametric form can be presented as:

$$A(r) = (g(r); \overline{a}(r)) = (a + br; \overline{a} - b_r), \quad r \in [0,1],$$

3
where $a(r)$ is the left boundary and $\tilde{a}(r)$ is the right boundary of the corresponding $r$-interval, $b$ are non-negative coefficients of fuzziness.

The theoretical basis of arithmetic operations with fuzzy numbers is Zadeh's extension principle [2].

**Definition 7.** If $y = f(x_1, x_2, \ldots, x_n)$ is function of the fuzzy arguments where fuzzy numbers can be denoted as:

$$y = \bigcup_{r\in[0,1]} (y_r, \tilde{y}_r), x_i = \bigcup_{r\in[0,1]} (x_{ir}, \tilde{x}_{ir}), i = 1, \ldots, n,$$

then the function is calculated according to the expressions for any $r$-level:

$$y_r = \inf(f(x_{1r}, x_{2r}, \ldots, x_{nr})), \quad \tilde{y}_r = \sup(f(x_{1r}, x_{2r}, \ldots, x_{nr})),
$$

where $x_{ir} \in [\tilde{x}_{ir}, \tilde{x}_{ir}], i = 1, \ldots, n.$

The work aims at overcoming these shortcomings. It contains an original approach to the implementation of operations on fuzzy numbers and can overcome negative shortcomings of the extension principle.

4. **Algebra of single-component numbers (X-algebra)**

The concept of a single component number is used in the construction of the $W$-algebra.

**Definition 8.** One-component number $X$ (hereinafter the $X$-number) is called the structure represented in the form $X = x(r) = (a + br), r \in [0,1]$.

![Figure 1. X-number.](image)

It is known that an algebraic structure consists of a carrier set $K$, collective operations (signatures) $S$ over the set elements, and collective relations $R$ over the set $K$. Let the set be defined as a set of fuzzy $L$- or $R$-numbers: $K = \{x(r)\}; x(r) = a + br; r \in [0;1]$.

If we want a substitution of a fuzzy solution in the equation with fuzzy parameters to guarantee identical equivalency of the left and right parts, then it is necessary to introduce a non-fuzzy equivalency relation $R$ on the set $K$ as follows:

**Definition 9.** Two fuzzy numbers $x_1(r) = a_1 + b_1r$ and $x_2(r) = a_2 + b_2r$, $x_2(r)$ are equivalent, i.e. $x_1(r) R x_2(r)$ if and only if $a_1 = a_2$ and $b_1 = b_2$. Here $R$ is the equivalency relation.

Let us define a task of constructing an algebra $P = \{K; +, \ast\}$ on the set $K = \{x(r)\}; x(r) = a + br; r \in [0;1]$ of fuzzy $L$- and $R$-numbers satisfying the following conditions:

- limitation of an unjustifiable extension of uncertainty of the calculation result;
preservation of natural relations in the usual mathematical model of selection, including fuzzy parameters; for example, identity of an equation after substitution of its solution in the sense of exact equality of the corresponding inclusion functions;

- a guarantee that the carrier set $K$ is closed in relation to the algebra operations.

Let us introduce a binary summation operation on the set $K$:

$$ x_i(r) + x_2(r) = r_i(r) = a_i + a_r + (b_i + b_r) r \in K. \quad (6) $$

Let us define a single null element $O = (0 + 0 r) \in K$, such that

$$ \forall x(r) \in K: x(r) + O = a + br + 0 + 0r = x(r). $$

Let us introduce a binary multiplication operation on the set $K$, such that it guarantees preservation of the result in the form of a triangular number of type $L$ or $R$:

$$ x_i(r) \cdot x_2(r) = r_i(r) = a_i a_r + (a_i b_r + a_r b_i + b_r b_i) r; \ r_i(r) \in K \quad (7) $$

Let us define a single number $I = 1 + 0r \in K$ such that

$$ \forall x(r) \in K: \ I \cdot x(r) = (1 + 0r)(a + br) = x(r). $$

For all elements $x(r) = a + br \in K$ let us define a single reverse element $-x(r) = -a - br$, such that

$$ x(r) + (-x(r)) = a + br - a - br = O. $$

Then, the subtraction operation becomes:

$$ x_i(r) - x_2(r) = x_i(r) + (-x(r)). \quad (8) $$

In order to define the reverse element $x^{-1}(r) \in K$, such that the condition $x(r) \cdot x^{-1}(r) = 1$ is fulfilled, the following form is used:

$$ x^{-1}(r) = \frac{1}{a} - \frac{b}{a(a + b)} r, \ a \neq 0, \ a + b \neq 0 \quad (9) $$

Then, the division operation becomes:

$$ x_i(r) / x_2(r) = x_i(r) \cdot x_2^{-1}(r). \quad (10) $$

Limitations on the parameters of a fuzzy number in equation (9) should be accounted for during the formalization of an expert statement in the form of a fuzzy number.

Obtained reverse elements are used to define the subtraction and division operations of $L$ and $R$-numbers as summation and multiplication operations with the corresponding opposite and reverse numbers.

**Example 1.** Addition of two $X$-numbers $A = (4 + 2r)$ and $B = (1 - 3r)$:

$$ A + B = (4 + 1 + (2 - 3) r) = (5 - r). $$

**Example 2.** Subtraction of two $X$-numbers $A = (4 + 2r)$ and $B = (1 - 3r)$:

$$ A - B = (4 + 2r) + (-1 + 3r) = (3 + 5r), \text{ where } A - A = O. $$

**Example 3.** Multiplication of two $X$-numbers $A = (4 + 2r)$ and $B = (1 - 3r)$:

$$ A \cdot B = (4 + 1 \cdot (4 \cdot (-3) + 1 + 2 + 2 \cdot (-3)) r) = (4 + (-16)) r. $$

**Example 4.** Division of two $X$-numbers $A = (4 + 2r)$ and $B = (1 - 3r)$:

$$ A / B = (4 + 2r) \cdot \frac{1}{1 - \frac{3}{1 - (1 - 3)} r} = (4 + 2r) \cdot (1 - 1,5 r) = (4 - 7r), \text{ where } A / A = I. $$

It can easily be verified that the introduced algebra satisfies predefined conditions. However, implementation of operations (6,7,8,10) differs from the implementation of analogous operations on real numbers, which does not allow the use of standard computational software and justifies the need for construction of a more useful isomorphic algebra. Besides, known operations for comparison of
fuzzy numbers dictate only a partial order, which is insufficient for implementation of many algorithms of selection.

Let us denote the obtained algebra as \( P = \langle K_1; S_1 \rangle \).

**Definition 10.** Isomorphism of the algebra \( P = \langle K_1; S_1 \rangle \) on to the algebra \( G = \langle K_2; S_2 \rangle \) is a mapping \( \Gamma : K_1 \rightarrow K_2 \) satisfying the following conditions:

\[
\Gamma(S_1(k_1)) = S_2(\Gamma(k_1)), \quad k_1 \in K_1. \tag{11}
\]

\[
S_1(\Gamma^{-1}(k_2)) = \Gamma^{-1}S_2(k_2)), \quad k_2 \in K_2. \tag{12}
\]

It is necessary to find isomorphism \( \Gamma \), such that the implementation of algebra \( G \) uses operations on real numbers and allows the application of the standard software packages for solving problems with fuzzy numbers.

Let the elements of a set \( K_2 \) be represented as \( x(r) = x(0) + (x(1) - x(0))r \), where \( x(0), x(1) \) are real numbers. Signature \( S_2 \) is defined by

\[
x_1(r) \ast x_2(r) = rx(1) + (1 - r)x(0) = x(0) + (x(1) - x(0))r, \tag{13}
\]

where \( x(0) = x_1(0) \ast x_2(0); \quad x(1) = x_1(1) \ast x_2(1); \quad \ast \in \{+, -, \cdot, /\} \) and all operations satisfy the rules of operations with real numbers.

**Theorem 1.** Algebras \( P \) and \( G \) are isomorphic.

**Proof.** Consider elements of the sets \( K_2 \) and \( K_1 \). Obviously, any element of the set \( K_1 - x(r) = a + br \) is obtained from the elements of \( K_1 \) by using a simple substitution \( a = x(0), b = x(1) - x(0) \), in this case \( K_1 = K_2 \), and the mapping \( \Gamma \) is a mapping onto itself. The set \( K_1 \) is closed with respect to operations (6,7,8,10), so to prove homomorphism (11) and the reverse homomorphism (12) it is sufficient to show correctness of the equality \( S_1(k_1) = S_2(k_2) \). Since the result of operations in both algebras is a linear function of the parameter \( r \), it is sufficient to show that these results are similar in two points, for instance at \( r = 0 \) and \( r = 1 \). Similarity of the results is easy to check by performing operations using rules (6,7,8,10) and (13) for two fuzzy numbers \( a + br \) and \( a' + b'r \) [13].

5. Algebra of double-component numbers (\( W \)-algebra)

\( W \)-algebra is constructed using the \( X \)-numbers and \( X \)-algebra.

**Definition 10.** Two-component number \( W \) (hereinafter the \( W \)-number) is called the structure shown by the vector \( (v^l(r); v^h(r)) \), \( v^l(r) \in X \), \( v^h(r) \in X \), for which \( v^l(1) = v^h(1) \).

Triangular fuzzy numbers can be represented as \( W \)-numbers, where \( v^l(r) = a^l + br \), \( a^l \geq 0 \) and \( v^h(r) = a^h + b^h r \), \( a^h \leq 0 \).

Algebra of \( W \)-numbers (\( W \)-algebra) involves the independent performing of the operations on the left component with recording result to the left component of the final number and on the right component with the corresponding entry in the right component of the final number.

**Definition 11.** Let the left or right components of two \( W \)-numbers be represented as \( x(r) = x(0) + (x(1) - x(0))r \), where \( x(0), x(1) \) are crisp numbers. Then, as a result of the operation, the left or right component of the resulting \( W \)-number will be: \( x_1(r) \ast x_2(r) = x(0) + (x(1) - x(0))r \), where \( x(0) = x_1(0) \ast x_2(0), x(1) = x_1(1) \ast x_2(1), \ast \in \{+, -, \cdot, /\} \) and all operations are performed with crisp numbers.

Operations are linear with respect to \( r \), the set of all \( X \)-numbers is closed for all operations, that is, \( x_1(r), x_2(r) \in X \), then \( x(r) = x_1(r) \ast x_2(r) \in X, \ast \in \{+, -, \cdot, /\} \).
Definition 12. Let \( W_1 = (v_1^L(r); v_1^R(r)) \) and \( W_2 = (v_2^L(r); v_2^R(r)) \) be two \( W \)-numbers. The operations between these numbers will be made in accordance with the following rule: 
\[
(v_1^L, v_1^R) \ast (v_2^L, v_2^R) = \left( [v_1^L \ast v_2^L]; [v_1^R \ast v_2^R] \right) = (v^L, v^R), \text{ where } \ast \in \{ +, - , \ldots \}.
\]

Definition 13. Let \( W_1 = (a_1, b_1, c_1) \) and \( W_2 = (a_2, b_2, c_2) \) be two \( W \)-numbers. The operations between these numbers will be made in accordance with the following rule: 
\[
(a_1, b_1, c_1) \ast (a_2, b_2, c_2) = (a_1 \ast a_2, b_1 \ast b_2, c_1 \ast c_2), \text{ where } \ast \in \{ +, - , \ldots \}.
\]

In this case, the \( W \)-algebra provides a complete overcoming of all these negative shortcomings of arithmetic, based on Zadeh’s extension principle.

Theorem 2. \( W \)-number will be always the result of the operations on the \( W \)-numbers, i.e., closure condition always will be fulfilled.

Proof. We consider two \( W \)-numbers \( W_1 = [a_1 + b_1 r, a_2 + b_2 r] \) and \( W_2 = [c_1 + d_1 r, c_2 + d_2 r] \).

Since left and right components of \( W \)-numbers have the same value when \( r = 1 \) (hereinafter modal value), we have:
\[
a_1 + b_1 = a_2 + b_2 c_1 + d_1 \Rightarrow a_i = a_2 - b_i + b_2 c_i = c_2 - d_i + d_2 \tag{14}
\]

Let us prove the equality of modal values for each of the operations (addition, subtraction, multiplication, division).

Addition:
\[
\begin{align*}
v^L &= a_1 + b_1 r + c_1 + d_1 r \\
v^R &= a_2 + b_2 r + c_2 + d_2 r
\end{align*}
\tag{15}
\]

According to equation (14) for \( r = 1 \), we have:
\[
\begin{align*}
v^L &= a_1 + b_1 + c_1 + d_1 \\
v^R &= a_2 + b_2 + c_2 + d_2
\end{align*}
\tag{16}
\]

The proof is similar for the remaining operations.

Performing operations on \( W \)-numbers does not require special software, and as will be shown below, overcomes the shortcomings of the other arithmetic.

6. Numerical examples

We consider the results obtained by different approaches in the numerical examples.

Example 5. Subtraction of a triangular number from itself. Let us consider an arbitrary fuzzy number \( A = (a, b, c) \).
\[
\begin{align*}
(A - A)^L &= r(b - a) + a - (r(b - a) + a) = 0 \\
(A - A)^R &= c - r(c - b) - (c - r(c - b)) = 0
\end{align*}
\tag{17}
\]

In general, it is shown that the operation’s result of \( W \)-algebra equals zero.

Example 6. Division of a triangular number by itself. Let us consider an arbitrary fuzzy number \( A = (a, b, c) \).
\[
\begin{align*}
(A / A)^L &= r \left( \frac{b - a}{b - a} \right) + a = 1 \\
(A / A)^R &= r \left( \frac{b - c}{b - c} \right) + c = 1
\end{align*}
\tag{18}
\]

In general, it is shown that the operation’s result of \( W \)-algebra equals \((1,1,1)\).
Example 7. Solution of a linear equation. Let us consider the linear equation $Ax + 2 = 0$, where $A$ is a fuzzy number and $A = (3, 5, 6)$.

The solution of the linear equation:

$$
\begin{align*}
    x^L &= \frac{4r - 2}{15r - 3}, \\
    x^R &= \frac{-r - 1}{15r - 3}
\end{align*}
$$

The result of substitution of the solutions in the equation:

$$
\begin{align*}
    v^L &= 0, \\
    v^R &= 0
\end{align*}
$$

Example 8. Solution of a quadratic equation. Let us consider the quadratic equation $x^2 + Bx + 2 = 0$, where $B$ is a fuzzy number and $B = (3, 4, 6)$.

The solution of the quadratic equation:

$$
\begin{align*}
    x_{1,L} &= \left(\sqrt{2} - 1\right)r - 1, \\
    x_{1,R} &= \left(1 + \sqrt{2} - \sqrt{7}\right)r + \sqrt{7} - 3, \\
    x_{2,L} &= -\sqrt{2}r - 2, \\
    x_{2,R} &= \left(1 - \sqrt{2} + \sqrt{7}\right)r - \sqrt{7} - 3
\end{align*}
$$

The result of substitution in the equation:

$$
\begin{align*}
    \left(\frac{\sqrt{2} - 1}{\sqrt{2} - 1}\right)^2 + (r + 3)\left(\frac{\sqrt{2} - 1}{\sqrt{2} - 1}\right) + 2 &= 0, \\
    \left(1 + \sqrt{2} - \sqrt{7}\right)^2 + (6 - 2r)\left(1 + \sqrt{2} - \sqrt{7}\right) + 2 &= 0, \\
    \left(-\sqrt{2}r - 2\right)^2 + (r + 3)\left(-\sqrt{2}r - 2\right) + 2 &= 0, \\
    \left(1 - \sqrt{2} + \sqrt{7}\right)^2 + (6 - 2r)\left(1 - \sqrt{2} + \sqrt{7}\right) + 2 &= 0
\end{align*}
$$

Example 9. Solution of a system of linear equations.

Let us consider the system of linear equations:

$$
\begin{align*}
    a_1x_1 + b_1x_2 &= c_1, \\
    a_2x_1 + b_2x_2 &= c_2
\end{align*}
$$

where $a_1 = (1, 2, 4)$, $b_1 = (-8, -5, -3)$, $c_1 = (1, 4, 5)$, $a_2 = (2, 6, 7)$, $b_2 = (1, 3, 4)$, $c_2 = (1, 2, 5)$ are the fuzzy numbers.

The solution of the system of linear equations:

$$
\begin{align*}
    x_{1,L} &= \frac{25r + 9}{306 + 17}, \\
    x_{1,R} &= \frac{-6r + 1}{37 + 166}, \\
    x_{2,L} &= \frac{-153 - 17}{50r + 15}, \\
    x_{2,R} &= \frac{-333 - 37}{36}
\end{align*}
$$
The result of substitution of $x_{1,l}$, $x_{2,l}$, $x_{1,r}$ and $x_{2,r}$ in the system of linear equations:

$$\begin{align*}
-3r + \left(\frac{25r}{306} + \frac{9}{17}\right)(r+1) + \left(\frac{76r}{153} - \frac{1}{17}\right)(3r-8) - 1 &= 0; \\
-(r+1) + \left[\frac{76r}{153} - \frac{1}{17}\right](2r+1) + \left[\frac{25r}{306} + \frac{9}{17}\right](4r+2) &= 0; \\
-(5-r) + \left[\frac{50r}{333} - \frac{15}{37}\right](2-3r) + \left[\frac{35}{37} - \frac{223r}{666}\right](4-2r) &= 0; \\
\left(\frac{50r}{333} - \frac{15}{37}\right)(4-r) + \left[\frac{35}{37} - \frac{223r}{666}\right](7-r) - (5-3r) &= 0.
\end{align*}$$

(25)

**Example 10.** Cauchy problem for ordinary differential equation.

Let us consider the problem:

$$\begin{align*}
\frac{dy}{dr} &= Ay; \\
y(t = 0) &= 1;
\end{align*}$$

(26)

where $A = (-2, 0, 1)$.

The solution is $y(t) = (e^{-2t}, 1, e^t)$. Substitution leads to the equality $(-2e^{-2t}, 0, e^t) = (-2e^{-2t}, 0, e^t)$.

**Example 11.** A network project planning problem under fuzzy initial conditions of work duration.

Let us define a project in table 1:

| Table 1. Project. | Operation | Preceding operations | Fuzzy operation duration, $\tau$ |
|-------------------|-----------|----------------------|---------------------------------|
|                   | A         | -                    | $\tau^l$ $\tau^m$ $\tau'$     |
|                   | B         | -                    | 1 2 5                           |
|                   | C         | A                    | 3 7 9                           |
|                   | D         | B                    | 4 6 9                           |
|                   | E         | B                    | 9 10 12                         |
|                   | F         | C                    | 4 5 6                           |
|                   | G         | D, E                 | 1 5 6                           |
|                   | H         | F, G                 | 2 4 7                           |

Related network graph is shown in figure 2.

**Figure 2.** A network graph of the project.
It is required to determine the critical project operations and the overall fuzzy time for the project completion. Let us define a day as the unit of time measurement.

The necessary solution can be obtained by solving a linear programming problem with fuzzy parameters:

\[ \begin{align*}
    t_n - t_0 & \rightarrow \min; \\
    t_{\mu} - t_{\mu} \geq \tau_s, \quad \forall s = 1, \ldots, 8; \\
    \tau & = \sum_{s=1}^{8} \tau_s; \\
\end{align*} \]  

(27)

where \( t_0, t_n \) are the beginning and end times of the project; \( t_{\mu}, t_{\mu} \) are the beginning and end times of an \( s \)-operation; \( \tau_s \) is the fuzzy duration of an \( s \)-operation; \( \tau_{st} \) is the fuzzy duration of the critical path, or in other words, those operations for which the solution of the problem satisfies the equality \( t_{\mu st} - t_{\mu st} = \tau_{st} \) (in the sense of equivalency relation \( R \)). Such constraints in linear programming problems are called deficit constraints; \( \tau \) is the project duration.

The system of inequalities for (27) with respect to the data in table 1 takes the form:

\[ \begin{align*}
    t_1 - t_0 & \geq \tau_A; \\
    t_2 - t_0 & \geq \tau_B; \\
    t_3 - t_1 & \geq \tau_C; \\
    t_4 - t_2 & \geq \tau_D; \\
    t_5 - t_3 & \geq \tau_E; \\
    t_6 - t_4 & \geq \tau_F; \\
    t_7 - t_5 & \geq \tau_G; \\
    t_8 - t_6 & \geq \tau_H. \\
\end{align*} \]  

(28)

We point out that the proposed methodology eliminates the problem of comparing fuzzy numbers in the system of inequalities presented above, replacing it with a corresponding system of comparing non-fuzzy numbers.

Optimal operation start times are:

\[ \begin{align*}
    t_0 & = (0,0,0); \\
    t_1 & = (1,2,5); \\
    t_2 & = (2,4,5); \\
    t_3 & = (11,14,17); \\
    t_4 & = (11,14,17); \\
    t_5 & = (4,9,14); \\
    t_6 & = (12,19,23); \\
    t_7 & = (14,23,30). \\
\end{align*} \]  

(29)

The project's critical path contains operations B; E; G; H.

The results of solving the problem (27) have a natural interpretation:

- the sum of fuzzy critical path's operation durations equals (in the sense of equality of membership functions) the fuzzy time moment of the project's end, i.e., it is possible that the project is completed within the interval of \( \tau = (14,23,30) \) days;
- the total time reserve for operations A, C, F, for example, is determined by the expression \( (t_6 - t_0) - (\tau_A + \tau_C + \tau_F) = (12,19,23) - (8,14,20) = (4,5,3) \). Critical operations do not have a time reserve; for instance, the total time reserve for operation B equals to \( (t_2 - t_0) - \tau_B = (2,4,5) - (2,4,5) = (0,0,0) \).

The fulfillment of such natural relations proves the obtained solution is trustworthy.
7. Results and discussion
We have constructed a methodology for solving problems with non-fuzzy mathematical models and fuzzy parameters. This approach preserves qualities of the initial non-fuzzy mathematical model.

Future development of the proposed approach, in our opinion, extends beyond just static problems. By introducing the notion of a fuzzy function and its derivative, we are planning to switch to solving problems of optimal control of dynamic objects with a fuzzy parametric uncertainty. Another development path of the proposed approach is to investigate the problem of stability of solutions under conditions of fuzzy initial data.

8. Conclusion

W-algebra overcomes the shortcomings of the arithmetic based on Zadeh's extension principle.

Natural relations $A - A = 0$ and $A / A = 1$ are always provided in the $W$-algebra and a crisp identity is provided by substitution of the solution in the equation. These relations are not provided by Zadeh's extension principle.

The application of the $W$-algebra allows us to solve problems with fuzzy parameters and variables and use all the existing methods, algorithms and software tools designed to work with crisp numbers. The solution of the problem with crisp variables at three points and the construction of the corresponding result in the form of $W$-numbers are sufficient to obtain a solution. Consequently, the problem of the fuzzy numbers' comparing does not occur in the process of solutions. Zadeh's extension principle uses special algorithms and software and has greater computational complexity than the $W$-algebra for solving problems with fuzzy parameters and variables.

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