Scattering of massless particles in one-dimensional chiral channel

Mikhail Pletyukhov\textsuperscript{1} and Vladimir Gritsev\textsuperscript{2,3}

\textsuperscript{1} Institute for Theory of Statistical Physics, RWTH Aachen, 52056 Aachen, Germany and JARA—Fundamentals of Future Information Technology
\textsuperscript{2} Physics Department, University of Fribourg, Chemin du Musee 3, 1700 Fribourg, Switzerland
E-mail: vladimir.gritsev@unifr.ch

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Abstract. We present a general formalism describing the propagation of an arbitrary multi-particle wave packet in a one-dimensional multimode chiral channel coupled to an ensemble of emitters which are distributed at arbitrary positions. The formalism is based on a direct and exact resummation of a diagrammatic series for the multi-particle scattering matrix. It is complementary to the Bethe Ansatz approach and to approaches based on equations of motion, and it reveals a simple and transparent structure of scattering states. In particular, we demonstrate how this formalism works on various examples, including the scattering of one- and two-photon states off two- and three-level emitters, off an array of emitters, as well as the scattering of coherent light. We argue that this formalism can be used constructively for the study of the scattering of an arbitrary initial photonic state off emitters with an arbitrary degree of complexity.

\textsuperscript{3} Author to whom any correspondence should be addressed.
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1. Introduction

Recent advances in fabricating quasi-one-dimensional (1D) nanostructures have stimulated and
motivated extensive theoretical work on propagating photons in reduced dimensions. One of
the goals of this research is to achieve considerable photonic nonlinearities for a single photon
pass through the structure which could become a fertile platform for a variety of applications
in future technology and quantum information [1]. The only way nonlinear effects may show
up in the photonic component of scattering is through an interaction with an ensemble of
emitters. This can be achieved by inserting emitters into the 1D channel or by side coupling to
evanescent modes of the channel. The scattering probability of an individual quantum particle
of wavelength $\lambda$ in the 1D channel is controlled by the ratio $\lambda^2/A$ where $A$ is the effective
scattering area of the channel. Therefore, reducing the effective scattering area makes nonlinear
effects feasible in the range of wavelengths close to $\lambda$. Several physical realizations of this
kind of scattering have been recently suggested: tapered optical fibres with an atomic ensemble
coupled to the evanescent field (currently tapered regions can have optical subwavelength
diameters down to 50 nm) [2]; hollow optical fibres, ‘stuffed’ with cold atoms [3] and photonic
crystals [4]. Besides conventional atomic ensembles, various artificial emitters and channels can
also be used. Thus, superconducting $q$-bits coupled to transmission lines have been engineered
to demonstrate two- and three-level emitters (see [5] for an extensive recent review). Moreover, photons can be replaced by surface plasmons coupled to quantum dots [6].

Bearing these and other physical realizations in mind, we consider the Hamiltonian model which describes the interaction of a massless, multi-mode, quantized bosonic field propagating unidirectionally in 1D geometry with an array of emitters. The coupling between bosons and emitters has a standard dipole-like form. The emitters are assumed to have a multi-level structure (in particular in this paper we focus on two- and three-level structures, although our formalism also allows for a straightforward generalization to more complicated cases). We do not impose any specific details on the spatial distribution of emitters in the 1D channel. However, we distinguish between the case of the Dicke arrangement (when all emitters are placed in a space region smaller than a typical wavelength) and the case of remotely distributed emitters. The problem we focus on is of the scattering type: an initial state consists of a direct product of the photonic state and the ground state of an atomic ensemble. Typically, this state is an eigenstate of the free (non-interacting) Hamiltonian. An interaction between light and matter is assumed to be switched on adiabatically in the infinite past and switched off adiabatically in the infinite future. The adiabaticity parameter controls the so-called on-shell condition which plays an important role in the scattering matrix approach (more precisely, the adiabaticity parameter defines an uncertainty in the energy conservation condition). The outgoing state is derived from the incoming state by an application of a multi-particle scattering matrix. Our main goal is to evaluate the scattering matrix.

We provide a generic solution to the problem (see equation (3) below) under the assumptions specified above. It can be further extended to the case of multiple emitters. The coupling constant, detuning and the level structure may vary from emitter to emitter. Arbitrary initial conditions can also be studied, however, we specifically focus here on the Fock and coherent states. We found that the multi-particle scattering matrix generically contains many-body bound states in the photonic sector, while in the atomic sector it has a projector-like structure. The result of the scattering consists in preparing the ensemble of scatterers in a specific linear combination of levels, which is nothing else but the dark state, i.e. the state which does not emit.

Previous studies in the 1980s [7–9], 1990s [10] and over the last 12 years [11–16] have already clarified a number of theoretical questions related to propagating of photons in 1D geometry. Many of these studies have been based on the property of integrability. Our analysis here extends these studies in various directions: we consider arrays of emitters with multi-level structures and arbitrary coupling constants, and treat the problem comprehensively using the methodology of the scattering formalism. Our approach does not require any a priori knowledge about the system’s integrability. Nevertheless, our results match those obtained by integrability methods, whenever the latter are available.

In the next section, we present our general result with an outline of its derivation. In the following sections, we demonstrate its application on specific examples of light scattering off two- and three-level systems. We consider separately the scattering of Fock states with a well-defined number of photons $N$ and the scattering of coherent light. The latter study serves two purposes: firstly, the scattering matrix in the coherent state basis describes a scattering problem of coherent light itself, and, secondly, it can be considered as a generating functional for scattering problems in sectors with a well-defined number of photons.
In turn, the latter amounts to the scattering matrix $S^{\nu}$. The non-trivial part of scattering is contained in the so-called $T$-matrix, which is related to the scattering matrix $S$ via the expression

$$S = 1 - 2\pi i \delta(E_{\text{in}} - E_{\text{out}}) T(E_{\text{in}}),$$

where $E_{\text{in}}$ and $E_{\text{out}}$ are the energies of incoming and outgoing states.

With the assumptions we stated above, we show that the $T$-matrix in the $N$-particle sector equals

$$T^{(N)}(\omega) = G_0^{-1} G(\hat{a}_{\nu_1} S_{\nu_1} G \cdots G(\hat{a}_{\nu_2} S_{\nu_2}) G G_0^{-1},$$

where the summation over the set of $\nu_i = \pm$ and the integration over the set of $\nu_i$ is implicitly assumed. The hats over the operators $a$ mean that they can only be contracted with external operators (that is, operators creating incoming and outgoing states). The operators $\hat{a}$ are effectively normal-ordered in (3), but it is still necessary to account for their commutation relationship with $a^\dagger a$ appearing in the bare $G_0^{-1} = \omega - H_0$ and dressed $G^{-1}$ Green’s functions. In turn, the latter amounts to

$$G^{-1}(\omega) = G_0^{-1}(\omega) - \Sigma = \omega - H_0 + i\pi S_+ S_-.$$
the projector onto the atomic states with zero broadening (dark states). Therefore the first task is to evaluate this building block of (3).

We also note that from the diagrammatic point of view, the class of models we study here does not allow for diagrams with an intersection and overlapping of photonic lines, what follows from the spectrum linearity and the causality imposed by the absence of backscattering for propagating chiral modes. It is possible to extend the diagrammatic approach beyond this class, but then the non-crossing approximation no longer provides an exact solution, and one should expect effects associated with vertex corrections to the vertices $V$.

2.2. Derivation of the main result

Below we derive our main expression (3). First we consider several basic facts about the scattering matrix approach, illustrating them on a simple example of scattering off the two-level system. Then, we outline the main steps of our derivation.

2.2.1. Scattering problem. The main goal of the scattering theory is to calculate the scattering matrix

$$S = T \exp \left[ -i \int_{-\infty}^{+\infty} V(t) dt \right],$$

(5)

where the interaction term $V(t)$ is evaluated in the interaction picture. The matrix elements of the scattering matrix $S_{n'n} = \langle n'|S|n \rangle$ are defined in the states of the non-interacting Hamiltonian, which in our case have the following form

$$|n\rangle = a^\dagger_{k_1} \cdots a^\dagger_{k_n} |0\rangle_b |\sigma\rangle, \quad \varepsilon_n = \sum_{i=1}^{n} k_i + \sigma \frac{\Omega}{2},$$

(6)

$$|n'\rangle = a^\dagger_{p_1} \cdots a^\dagger_{p_{n'}} |0\rangle_b |\sigma'\rangle, \quad \varepsilon_{n'} = \sum_{i=1}^{n'} p_i + \sigma' \frac{\Omega}{2},$$

(7)

where $|0\rangle_b$ is the photon vacuum state, and $\sigma = \pm$ labels the states of the two-level system, and $\Omega$ is the level splitting.

By definition, the $T$-matrix is

$$T(\omega) = V + V \hat{G}(\omega)V,$$

(8)

where the full Green’s function is defined as $\hat{G}(\omega) = (\omega - H + i\eta)^{-1}$. The parameter $\eta$ controls the adiabaticity of switching the interaction on and off in the far past and far future, respectively. The matrix elements of the $S$-matrix (5) and the $T$-matrix (8) are related to each other by the following equation, $\langle n'|S|n \rangle = \langle n'|n \rangle - 2\pi i \delta(\varepsilon_n - \varepsilon_{n'}) |n'|T(\varepsilon_n)|n \rangle$. Therefore, it is sufficient to calculate the matrix elements of the on-shell $T$-matrix (that is, at $\omega = \varepsilon_n$).

2.2.2. Calculation of the $T$-matrix. We expand $\hat{G}$ in (8) in a series of the interaction $V$

$$\hat{G}(\omega) = G_0(\omega) + G_0(\omega)V G_0(\omega) + G_0(\omega)V G_0(\omega)V G_0(\omega) + \cdots,$$

(9)

where

$$G_0(\omega) = \frac{1}{\omega - H_0 + i\eta} = \frac{P_+}{\omega - \frac{\Omega}{2} - H_b + i\eta} + \frac{P_-}{\omega + \frac{\Omega}{2} - H_b + i\eta}.$$

(10)
is the bare Green’s function. The projectors $P_{\pm} = \sigma_{\pm}\sigma_{\pm} = \frac{1}{2}(1 \pm \sigma_{z})$ map onto the spin states $|\pm\rangle$, respectively. We note that $P_{\pm}\sigma_{z} = \sigma_{\pm}P_{\pm} = 0$, and thus $\hat{P}_{\pm}\sigma_{z} = \sigma_{\pm}\hat{P}_{\pm}$, and therefore $T = V + VG_{0}V + VG_{0}VG_{0}V + \cdots$. As $V$ is linear in bosonic operators, we can omit the odd powers of $V$ in this expansion: they do not conserve the number of photons and will vanish in the calculation of matrix elements. Then $T = W + W G_{0}W + \cdots$, where $W = VG_{0}V$.

Using the properties of $P_{\pm}$, one can show that only the diagonal elements $T_{\pm\pm} = \langle \pm|T|\pm\rangle$ in spin space are non-zero

\[
T_{++}(\omega) = g^{2}a_{\nu'1}^{\dagger}\frac{1}{\omega + \frac{i}{2} - H_{b} + i\eta} - a_{\nu1}^{\dagger} + g^{4}\left(a_{\nu'1}^{\dagger}\frac{1}{\omega + \frac{i}{2} - H_{b} + i\eta} - a_{\nu1}^{\dagger}\right) \\
\times \frac{1}{\omega + \frac{i}{2} - H_{b} + i\eta} - a_{\nu1}^{\dagger} + g^{4}\left(a_{\nu2}^{\dagger}\frac{1}{\omega + \frac{i}{2} - H_{b} + i\eta} - a_{\nu1}^{\dagger}\right) + \cdots. \tag{11}
\]

\[
T_{--}(\omega) = g^{2}a_{\nu2}^{\dagger}\frac{1}{\omega - \frac{i}{2} - H_{b} + i\eta} - a_{\nu1}^{\dagger} + g^{4}\left(a_{\nu'1}^{\dagger}\frac{1}{\omega - \frac{i}{2} - H_{b} + i\eta} - a_{\nu1}^{\dagger}\right) \\
\times \frac{1}{\omega + \frac{i}{2} - H_{b} + i\eta} - a_{\nu1}^{\dagger} + g^{4}\left(a_{\nu2}^{\dagger}\frac{1}{\omega - \frac{i}{2} - H_{b} + i\eta} - a_{\nu1}^{\dagger}\right) + \cdots. \tag{12}
\]

Here the integration over the frequencies $\{\nu, \nu'\}$ is implicitly assumed.

Let us put $T_{++}$ and $T_{--}$ partially on-shell introducing $\omega = \omega_{b} + \frac{i}{2}$ in (11), and $\omega = \omega_{b} - \frac{i}{2}$ in (12), where $\omega_{b}$ is the energy of photons, $\omega_{b} = \sum_{i} k_{i} = \sum_{i} p_{i}$. Then, we can rewrite

\[
T_{++}(\omega_{b}) = g^{2}a_{\nu'1}^{\dagger}\frac{1}{\omega_{b} + \Omega - H_{b} + i\eta} - a_{\nu1}^{\dagger} + g^{4}\left(a_{\nu'1}^{\dagger}\frac{1}{\omega_{b} + \Omega - H_{b} + i\eta} - a_{\nu1}^{\dagger}\right) \\
\times \frac{1}{\omega_{b} + \Omega - H_{b} + i\eta} - a_{\nu1}^{\dagger} + g^{4}\left(a_{\nu2}^{\dagger}\frac{1}{\omega_{b} + \Omega - H_{b} + i\eta} - a_{\nu1}^{\dagger}\right) + \cdots. \tag{13}
\]

\[
T_{--}(\omega_{b}) = g^{2}a_{\nu2}^{\dagger}\frac{1}{\omega_{b} - \Omega - H_{b} + i\eta} - a_{\nu1}^{\dagger} + g^{4}\left(a_{\nu'1}^{\dagger}\frac{1}{\omega_{b} - \Omega - H_{b} + i\eta} - a_{\nu1}^{\dagger}\right) \\
\times \frac{1}{\omega_{b} - \Omega - H_{b} + i\eta} - a_{\nu1}^{\dagger} + g^{4}\left(a_{\nu2}^{\dagger}\frac{1}{\omega_{b} - \Omega - H_{b} + i\eta} - a_{\nu1}^{\dagger}\right) + \cdots. \tag{14}
\]

As we are interested in evaluating average values in the bosonic eigenstates $\langle a_{p_{n}}\cdots a_{p_{1}}T_{\pm\pm}a_{k_{1}}^{\dagger}\cdots a_{k_{m}}^{\dagger}\rangle$, it is logical to exploit the Wick’s theorem, implying $\langle a_{\nu'}a_{\nu}^{\dagger}\rangle = \delta(\nu - \nu')$ and $\langle a_{\nu}^{\dagger}a_{\nu'}\rangle = 0$. We also note the following intertwining properties of bosonic operators

\[
a_{\nu}^{\dagger}\frac{1}{\omega - H_{b} - \nu} = \frac{1}{\omega - H_{b}}a_{\nu}^{\dagger}, \quad a_{\nu}^{\dagger}\frac{1}{\omega - H_{b} + \nu'} = \frac{1}{\omega - H_{b}}a_{\nu}^{\dagger}. \tag{15}
\]

An application of the Wick’s theorem implies that we have to contract, pair-wise, the operators $a_{\nu'}$ and $a_{\nu}^{\dagger}$. In order to do that, it is necessary to move $a_{\nu'}$ to the right, commuting it by means of (15) with all propagators appearing in between its initial position and the position of $a_{\nu}^{\dagger}$. There are only two possibilities for such a contraction: (i) $a_{\nu}$ is contracted with an adjacent operator $a_{\nu'}^{\dagger}$ standing to the right from it; (ii) $a_{\nu'}$ is contracted with some external operator $a_{k_{i}}^{\dagger}$. In the diagrammatic representation this means that only diagrams with non-crossing lines of contraction are allowed.

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The reason for this restriction is that if \( a_{\nu} \) is reshuffled with more than one propagator, and after that it is contracted to some internal operator, we obtain an integral over \( \nu' \) with more than one pole lying in the same half-plane. Such an integral identically vanishes. The operator \( a_{\nu} \) can then be loosely reshuffled to the right end, which provides the second possibility. A contraction of the two adjacent operators yields

\[
\left( a_{\nu} \frac{1}{\omega - H_b + i\eta} a_{\nu}^{\dagger} \right) = \left( \int d\nu' d\nu' \frac{\delta (\nu - \nu')}{\omega - H_b - \nu' + i\eta} \right) = -i\pi.
\]

This generates the self-energy insertion \( \Sigma \).

Let us denote by \( a_{\mu'} \) and \( a_{\mu}^{\dagger} \) the operators which are contracted to the external operators. In the \( N \)-photon sector we have the number \( N \) of both species, moreover for the two-level system they alternate (due to \( \sigma_2^+ = \sigma_2^- = 0 \)), that is, after \( a_{\mu'} \) we must have \( a_{\mu}^{\dagger} \), not \( a_{\mu}^{\dagger} \). All self-energy insertions between such pairs can be resummed, and we get for \( T_{++} \) and \( T_{--} \) the following result in the \( N \)-particle sector

\[
T_{++}^{(N)}(\omega_b) = g^{2N} (\omega_b - H_b + i\eta) \frac{1}{\omega_b - H_b + i\eta g^2} \left( a_{\mu'} \frac{1}{\omega_b + \Omega - H_b + i\eta} a_{\mu}^{\dagger} \right) \cdot \frac{1}{\omega_b - H_b + i\eta g^2} \left( a_{\mu}^{\dagger} \frac{1}{\omega_b - \Omega - H_b + i\eta} a_{\mu'} \right) \frac{1}{\omega_b - H_b + i\eta g^2} \left( a_{\mu'} \frac{1}{\omega_b - H_b + i\eta} a_{\mu}^{\dagger} \right) \frac{1}{\omega_b - H_b + i\eta g^2} \left( a_{\mu}^{\dagger} \frac{1}{\omega_b - \Omega - H_b + i\eta} a_{\mu'} \right) \ldots
\]

\[
T_{--}^{(N)}(\omega_b) = g^{2N} a_{\mu}^{\dagger} \omega_b - \Omega - H_b + i\eta g^2 \left( a_{\mu}^{\dagger} \frac{1}{\omega_b - H_b + i\eta} a_{\mu}^{\dagger} \right) \ldots
\]

In these relationships, operators \( a \) and \( a^{\dagger} \) effectively commute with each other, and one has to take carefully into account permutations of \( a(a^{\dagger}) \) with the propagators using (15). Note that \( T_{++}(\omega_b) \) vanishes, when it is put on-shell, because of the factors \( (\omega_b - H_b + i\eta) \) at the beginning and at the end of the corresponding expression.

We now consider \( T_{--}^{(N)}(\omega_b) \). Moving all propagators to the left and putting them on-shell, we have

\[
T_{--}^{(N)} = g^{2N} \frac{1}{\mu_1 - \alpha} \Delta_{11'}^+ + \eta \frac{1}{\Delta_{11'} + \mu_2 - \alpha} \ldots \sum_{i=1}^{N-2} \Delta_{ii'} + \mu_{N-1} - \alpha \frac{1}{\sum_{i=1}^{N-1} \Delta_{ii'} + \eta} \frac{1}{\sum_{i=1}^{N-1} \Delta_{ii'} + \mu_N - \alpha} a_{\mu_1}^{\dagger} \ldots a_{\mu_N}^{\dagger} a_{\mu_N}^{\dagger} \ldots a_{\mu_1},
\]

where \( \Delta_{ii'} = \mu_i - \mu_{i'} \) and \( \alpha = \Omega - ig^2 \).

A direct generalization of this derivation to cases with a more complicated level structure leads to our main equation (3).
3. Examples

Here we consider several important examples and compute the scattering matrix for cases of a single emitter coupled to a 1D waveguide within and beyond the RWA. We focus on the two-level and three-level cases.

3.1. One emitter in the rotating wave approximation (RWA)

For pedagogical reasons, we repeat here the derivation of (19) from the general expression (3).

In the case of a single atom interacting with a 1D field in the RWA we define $\epsilon_k = \pm \frac{\Omega}{2}$, $P_\pm = \sigma_\pm \sigma_\mp$ and $S_\pm = g \sigma_\pm = g \frac{\sigma_+,\sigma_0}{2}$. We find then $\Sigma = -i \pi S_+ S_- = -i \pi g^2 P_+$ and $G(\omega) = P_+(\omega - \frac{\Omega}{2} - H_b + i \pi g^2)^{-1} + P_-(\omega + \frac{\Omega}{2} - H_b + i \eta)^{-1} = P_+ G_+ + P_- G_-$. Therefore, the main building blocks of the equation for the $T$-matrix are given by

$$G_0^{-1} G = \left(\omega - \frac{\Omega}{2} - H_b + i \eta\right) \frac{P_+}{\omega - \frac{\Omega}{2} - H_b + i \pi g^2} + P_-,$$

(20)

$$GG_0^{-1} = \frac{P_+}{\omega - \frac{\Omega}{2} - H_b + i \pi g^2} \left(\omega - \frac{\Omega}{2} - H_b + i \eta\right) + P_-.$$

(21)

Putting them on-shell we find

$$G_0^{-1} G|_{\text{los}} = GG_0^{-1}|_{\text{los}} = P_-,$$

(22)

and therefore

$$T(\omega) = g^{2N} P_-(\hat{\alpha}_{v_1i} \sigma_-) P_+ G_+ \hat{\alpha}_{v_1i} \sigma_+ \cdot \cdot \cdot P_- G_- \hat{\alpha}_{v_{2N-1}i} \sigma_-) P_+ G_+ \hat{\alpha}_{v_{2N-1}i} \sigma_+ P_-$$

$$= P_- g^{2N} \hat{\alpha}_{v_1i} G_+ \hat{\alpha}_{v_2i} G_- \cdot \cdot \cdot \hat{\alpha}_{v_{2N-1}i} G_+ \hat{\alpha}_{v_{2N}i}.$$  

(23)

Introducing $\omega = \omega_b - \frac{\Omega}{2}$, we can rewrite $G_\pm$ as $G_+ = (\omega_b - H_b - \Omega + i \pi g^2)^{-1}$ and $G_- = (\omega - H_b + i \eta)^{-1}$.

3.2. $M$ emitters in the RWA (Dicke case)

In the case of many emitters confined in a small region of space (compared to the typical wavelength) we have $\epsilon_m = m \Omega$, $P_\pm = |m\rangle \langle m|$, while $S_\pm \to g S_\pm$. The total effective spin is conserved, $S_z S_a = l(l + 1)$. If the system is initially in the ground state, that is $S_z = -M/2$, then it is sufficient to consider the representation with the largest weight $l = M/2$, to which the ground state belongs. In this case we find

$$\Sigma = -i \pi g^2 S_+ S_- = -i \pi g^2 (S_z^2 - S_\mp^2) = -i \pi g^2 \sum_{m=-l}^l P_m [l(l+1) - m(m-1)],$$

(24)

and therefore for the dressed Green’s function we find

$$G(\omega) = \sum_{m=-l}^l \frac{P_m}{\omega - H_b - m \Omega + i \pi g^2 (l + m)(l - m + 1)}.$$  

(25)

The main building block is therefore equal to

$$G_0^{-1} G|_{\text{los}} = GG_0^{-1}|_{\text{los}} = P_{-l} = | -l \rangle \langle -l |.$$  

(26)
In the one-photon sector we immediately get

\[
T^{(1)}(\omega) = g^2 P_{-l}(a_{v_1}^\dagger S_-) P_{-l+1} G_{-l+1}(a_{v_2} S_+) P_{-l} \\
= g^2 P_{-l}(|-l|S_-| - l + 1|)|^2 a_{v_1}^\dagger \frac{1}{\omega - H_b + (l - 1)\Omega + i2\pi g^2 l a_{v_2}}.
\]

(27)

Using the action of the collective spin operators on the states of emitters

\[
S_\pm|m\rangle = \sqrt{(l \mp m)(l \pm m + 1)}|m\pm 1\rangle,
\]

(28)

we find that \(|-l|S_-| - l + 1|)^2 = 2l. Introducing \(\omega = \omega_b - \Omega l = \omega_b - \Omega M/2\) as well as \(\alpha_M = \Omega - i\pi g^2 M\), and omitting \(P_{-l}\), we find \(T^{(1)}(\omega_b) = g^2 M a_{v_1}^\dagger (\omega_b - H_b - \alpha_M)^{-1} a_{v_2}\). Putting this expression on-shell results in

\[
T^{(1)} = g^2 M \frac{1}{\nu_1 - \alpha_M} a_{v_1}^\dagger a_{v_2}.
\]

(29)

In the two-photon sector

\[
T^{(2)}(\omega) = g^4 P_{-l}(a_{v_1}^\dagger S_-) P_{-l+1} G_{-l+1}(a_{v_2} S_+) P_{-l+1} G_{-l+1}(a_{v_4} S_+) P_{-l} \\
= g^4 P_{-l}(|-l|S_-| - l + 1|)|^2(-l + 1)\Omega S_{\dagger 2} S_{\dagger 1} (m_2 | S_{\dagger 2}, -l + 1) \\
\times a_{v_1}^\dagger \frac{1}{\omega - H_b + (l - 1)\Omega + i2\pi g^2 l a_{v_2}} \omega - H_b - m_2\Omega + i\pi g^2 (l + m_2)(l - m_2 + 1) \\
\times a_{\nu_3}^\dagger \frac{1}{\omega - H_b + (l - 1)\Omega + i2\pi g^2 l a_{v_4}}.
\]

(30)

Using (28) we get

\[
|(-l|S_-| - l + 1|)^2(-l + 1)\Omega S_{\dagger 2} S_{\dagger 1} (m_2 | S_{\dagger 2}, -l + 1) \\
= (2l)\delta_{\nu_3}\delta_{\nu_2} \delta_{m_2, -l} \sqrt{2l + \delta_{\nu_3}\delta_{m_2, -l+2} \sqrt{2(2l - 1)}},
\]

(31)

and

\[
T^{(2)}(\omega) = P_{-l} g^4 l^2 a_{v_1}^\dagger \frac{1}{\omega - H_b + (l - 1)\Omega + i2\pi g^2 l a_{v_2}} \frac{1}{\omega - H_b + l\Omega + i\eta} \\
\times a_{\nu_3}^\dagger \frac{1}{\omega - H_b + (l - 1)\Omega + i2\pi g^2 l a_{v_4}} + P_{-l} g^4 l^2 (2l - 1) a_{v_1}^\dagger \\
\times a_{\nu_3}^\dagger \frac{1}{\omega - H_b + (l - 1)\Omega + i2\pi g^2 l a_{v_4}} + P_{-l} g^4 l^2 (2l - 1) \times a_{v_1}^\dagger \\
\times a_{\nu_3}^\dagger \frac{1}{\omega - H_b + (l - 1)\Omega + i2\pi g^2 l a_{v_4}}.
\]

(32)

Introducing \(\alpha_{M-1} = \Omega - i\pi g^2 (M - 1)\) and omitting \(P_{-l}\), we arrive at

\[
T^{(2)}(\omega_b) = g^4 M^2 a_{v_1}^\dagger \frac{1}{\omega_b - H_b - \alpha_M} a_{v_2} \frac{1}{\omega_b - H_b + i\eta} a_{v_3}^\dagger \frac{1}{\omega_b - H_b - \alpha_M} a_{v_4} \\
+ g^4 M (M - 1) a_{v_1}^\dagger \frac{1}{\omega_b - H_b - \alpha_M} a_{v_2} \frac{1}{\omega_b - H_b + i\eta} a_{v_3}^\dagger \frac{1}{\omega_b - H_b - \alpha_M} a_{v_4}.
\]

(33)
Putting $T^{(2)}$ on-shell, we get (note relabelling $v_2 \leftrightarrow v_3$ in the first term which is allowed because of the symmetric nature of the photonic wavefunction)

$$T^{(2)} = g^4 M \frac{1}{v_1 - \alpha_M} \frac{1}{v_1 + v_2 - v_3 - \alpha_M} \left[ \frac{M}{v_1 - v_3 + i \eta} + \frac{M - 1}{\frac{1}{2} (v_1 + v_2) - \alpha_{M-1}} \right] a_{v_1}^\dagger a_{v_2}^\dagger a_{v_3} a_{v_4}. \quad (34)$$

In the one-photon sector we have

$$t_{p_{1k_1}} = \delta_{p_{1k_1}} (a_{p_1} T^{(1)} a_{k_1}^\dagger) = \delta_{p_{1k_1}} \frac{M g^2}{v_1 - \alpha_M} (a_{p_1} a_{v_1}^\dagger a_{v_2} a_{v_4}^\dagger) = \delta_{p_{1k_1}} \frac{M g^2}{p_1 - \alpha_M}, \quad (35)$$

and therefore the scattering matrix is

$$S_{p_{1k_1}} = \delta_{p_{1k_1}} - 2 \pi i t_{p_{1k_1}} = \delta_{p_{1k_1}} \left( 1 - \frac{2i \pi M g^2}{p_1 - \alpha_M} \right). \quad (36)$$

We now consider the scattering matrix in the two-photon sector. First, we consider

$$\delta_{p_1+p_2,k_1+k_2} (a_{p_1} a_{p_1} T^{(1)} a_{k_1}^\dagger a_{k_2}^\dagger) = \delta_{p_2 k_1} t_{p_{1k_1}} + \delta_{p_{1k_1}} t_{p_{2k_2}} + \delta_{p_{1k_1}} t_{p_{k_2}} + \delta_{p_{1k_2}} t_{p_{1k_1}}. \quad (37)$$

Then we consider,

$$\delta_{p_1+p_2,k_1+k_2} (a_{p_1} a_{p_1} T^{(2)} a_{k_1}^\dagger a_{k_2}^\dagger) = \delta_{p_1+p_2,k_1+k_2} g^4 M \frac{1}{v_1 - \alpha_M} \frac{1}{v_4 - \alpha_M}
\times \left[ \frac{M}{v_1 - v_3} - i M \pi \delta_{v_1 v_3} + \frac{M - 1}{\frac{1}{2} (v_1 + v_2) - \alpha_{M-1}} \right]
\times (\delta_{p_{1v_1}} \delta_{p_{2v_2}} + \delta_{p_{1v_2}} \delta_{p_{2v_1}}) (\delta_{k_{1v_4}} \delta_{k_{2v_3}} + \delta_{k_{1v_3}} \delta_{k_{2v_4}}), \quad (38)$$

and therefore

$$S_{p_{1p_2,k_1+k_2}} = S_{p_{1k_1}} S_{p_{2k_2}} + S_{p_{2k_1}} S_{p_{1k_2}} + i T^{(2)}_{p_{1p_2,k_1+k_2}}, \quad (39)$$

where

$$T^{(2)}_{p_{1p_2,k_1+k_2}} = -2 \pi \delta_{p_1+p_2,k_1+k_2} g^4 M \frac{1}{v_1 - \alpha_M} \frac{1}{v_4 - \alpha_M}
\times \left[ \frac{M}{v_1 - v_3} + \frac{M - 1}{\frac{1}{2} (v_1 + v_2) - \alpha_{M-1}} \right]
\times (\delta_{p_{1v_1}} \delta_{p_{2v_2}} + \delta_{p_{1v_2}} \delta_{p_{2v_1}}) (\delta_{k_{1v_4}} \delta_{k_{2v_3}} + \delta_{k_{1v_3}} \delta_{k_{2v_4}}). \quad (40)$$

Defining $E = p_1 + p_2 = k_1 + k_2$, $\Delta = \frac{1}{2} (k_1 - k_2)$ and $\Delta' = \frac{1}{2} (p_1 - p_2)$ we find that $k_{1,2} = \frac{E}{2} \pm \Delta$, $p_{1,2} = \frac{E}{2} \pm \Delta'$. Therefore the $T$-matrix is

$$T^{(2)}_{p_{1p_2,k_1+k_2}} = \frac{8 \pi \delta_{p_1+p_2,k_1+k_2} g^4 M \left( \frac{E}{2} - \alpha_M \right)}{\left( \frac{E}{2} - \alpha_M \right)^2 - \Delta^2} \left[ \frac{M - (M - 1) \left( \frac{E}{2} - \alpha_M \right)}{\left( \frac{E}{2} - \alpha_M \right)^2 - \Delta^2} \right]
\times \left[ \frac{M - (M - 1) \left( \frac{E}{2} - \alpha_M \right)}{\left( \frac{E}{2} - \alpha_M \right)^2 - \Delta^2} \right]
\times \left[ \frac{E}{2} - \alpha_M \right], \quad (41)$$

which agrees with the result of [16]. The emergence of the poles in (41) involving more than one individual photon’s energy reflects a formation of the photonic bound state.
The non-analyticity is hidden in the non-commutativity of the limits $\eta \to 0$ and therefore

$$\eta \to 0,$$

and therefore

$$G = \frac{P_1}{\omega - \epsilon_1 - H_b + i\eta} + \frac{P_2}{\omega - \epsilon_2 - H_b + i\eta} + \frac{P_3}{\omega - \epsilon_3 - H_b + i\pi g^2} = P_i G_i. \quad (42)$$

Therefore the building block is

$$G_{0^{-1}} G|_{\text{los}} = GG_{0^{-1}}|_{\text{los}} = P_1 + P_2. \quad (43)$$

The $T$-matrix in the single-photon sector is

$$T^{(1)}(\omega) = (P_1 + P_2)(\hat{a}_{v_1}^+, S_{-}) P_3 G_3(\hat{a}_{v_2} S_{+})(P_1 + P_2)$$

$$= \left\{ g_{31}^2 P_1 + g_{32}^2 P_2 + g_{31} g_{32} (|2\rangle \langle 1| + |1\rangle \langle 2|) \right\} \hat{a}_{v_1}^+ \frac{1}{\omega - \epsilon_3 - H_b + i\pi g^2} \hat{a}_{v_2}, \quad (44)$$

and therefore

$$S_{pk} = \delta_{pk} - 2\pi i g_{31}^2 \delta_{pk} \frac{P_1}{p + \epsilon_1 - \epsilon_3 + i\pi g^2} - 2\pi i g_{32}^2 \delta_{pk} \frac{P_2}{p + \epsilon_2 - \epsilon_3 + i\pi g^2}$$

$$- 2\pi i g_{31} g_{32} \left[ \delta_{k+1, p+2} \frac{|2\rangle \langle 1|}{p + \epsilon_2 - \epsilon_3 + i\pi g^2} + \delta_{p+1, k+2} \frac{|1\rangle \langle 2|}{p + \epsilon_1 - \epsilon_3 + i\pi g^2} \right]. \quad (45)$$
\[ |2\rangle \quad \begin{array}{c} g_{12} \\ g_{13} \end{array} \quad |3\rangle \quad |1\rangle \]

**Figure 2.** Three-level V-scheme.

In the two-photon sector of scattering we find (cf [13])

\[
T^{(2)}(\omega) = (P_1 + P_2)(\hat{\alpha}^+_v S_+) P_3 G_3(\hat{\alpha}_{v_2} S_z)(P_4 G_4 + P_4 G_5)(\hat{\alpha}^+_v S_-) P_3 G_3(\hat{\alpha}_{v_3} S_z)(P_4 + P_5) \\
= \left\{ g_{31}^2 P_1 + g_{32}^2 P_1 + g_{31} g_{32} (2|1\rangle + |1\rangle 2) \right\} \hat{\alpha}^+_v G_3 \hat{\alpha}_{v_2} (g_{31}^2 G_1 + g_{32}^2 G_2) \hat{\alpha}^+_v G_3 \hat{\alpha}_{v_3}. \tag{46}
\]

3.5. Three-level system, V-scheme

In the case of the three-level V-scheme (see figure 2) the \( S_\pm \) operators are \( S_+ = g_{31} |3\rangle \langle 1| + g_{21} |2\rangle \langle 1| \), and \( S_- = g_{31} |1\rangle \langle 3| + g_{21} |1\rangle \langle 2| \). The corresponding self-energy is therefore \( \Sigma = -i\pi g_{31}^2 P_3 - i\pi g_{32}^2 P_1 - i\pi g_{21} g_{31} (2|3\rangle + |3\rangle 2) \), while the dressed Green’s function is

\[
G = P_1 (\omega - \epsilon_1 - H_b + i\eta)^{-1} + \sum_{a=2,3} \tilde{P}_a (\omega - H_b - \lambda_a)^{-1},
\]

where

\[
\tilde{P}_2 = \xi_{22}^2 P_2 + \xi_{23}^2 P_3 + \xi_{22} \xi_{23} |2\rangle \langle 3| + \xi_{23} \xi_{22} |3\rangle \langle 2|, \\
\tilde{P}_3 = \xi_{33}^2 P_3 + \xi_{32}^2 P_2 + \xi_{32} \xi_{33} |2\rangle \langle 3| + \xi_{33} \xi_{32} |3\rangle \langle 2|,
\]

\[
\lambda_2 - \lambda_3 = \frac{\cos \phi}{2}, \quad \xi_{22} = \frac{\sin \phi}{2}, \\
\lambda_2 - \lambda_3 = \frac{\cos \phi}{2}, \quad \xi_{23} = \frac{\sin \phi}{2},
\]

\[
\frac{\lambda_2 + \lambda_3}{2} = \frac{\epsilon_2 + \epsilon_3}{2} - i\pi \frac{g_{21}^2 + g_{31}^2}{2}, \quad \lambda_2 \lambda_3 = (\epsilon_2 - i\pi g_{21}^2)(\epsilon_3 - i\pi g_{31}^2) + \pi^2 g_{21}^2 g_{31}^2,
\]

\[
(\lambda - \lambda_2)(\lambda - \lambda_3) = (\lambda - \epsilon_2 + i\pi g_{21}^2)(\lambda - \epsilon_3 + i\pi g_{31}^2) + \pi^2 g_{21}^2 g_{31}^2.
\]

Using these relationships, we observe that our basic building block \( G_0^{-1} G_{10} = G G_0^{-1} G_{10} = P_1 \), and therefore

\[
T^{(1)}(\omega) = P_1 (\hat{\alpha}^+_v S_-) G_a(\hat{\alpha}_{v_2} S_z) P_1 \\
= P_1 \hat{\alpha}^+_v \left[ \frac{1 + \cos \phi}{2} g_{21}^2 + \frac{1 - \cos \phi}{2} g_{31}^2 + g_{21} g_{31} \sin \phi \frac{\omega - H_b - \lambda_2}{2} + \frac{1 - \cos \phi}{2} g_{21}^2 + \frac{1 + \cos \phi}{2} g_{31}^2 - g_{21} g_{31} \sin \phi \frac{\omega - H_b - \lambda_3}{2} \right] \hat{\alpha}_{v_2}. \tag{48}
\]
Combining the terms in square brackets, for the single-particle scattering matrix, we get

\[ S_{pk} = \delta_{pk} \left[ 1 - 2 \frac{i \pi g_{21}^2 (p + \epsilon_1 - \epsilon_3) + i \pi g_{31}^2 (p + \epsilon_1 - \epsilon_2)}{(p + \epsilon_1 - \epsilon_2 + i \pi g_{21}^2)(p + \epsilon_1 - \epsilon_3 + i \pi g_{31}^2) + \pi^2 g_{21}^2 g_{31}^2} \right] \]

\[ = \delta_{pk} \frac{(p + \epsilon_1 - \epsilon_2 - i \pi g_{21}^2)(p + \epsilon_1 - \epsilon_3 - i \pi g_{31}^2) + \pi^2 g_{21}^2 g_{31}^2}{(p + \epsilon_1 - \epsilon_2 + i \pi g_{21}^2)(p + \epsilon_1 - \epsilon_3 + i \pi g_{31}^2) + \pi^2 g_{21}^2 g_{31}^2} \] (49)

3.6. Three-level system, \( \Sigma \)-scheme

In the case of the \( \Sigma \)-scheme (see figure 3) the role of \( S_{\pm} \) operators is played by \( S_+ = g_{32} |3\rangle \langle 2| + g_{21} |2\rangle \langle 1| \), and \( S_- = g_{32} |2\rangle \langle 3| + g_{21} |1\rangle \langle 2| \). The corresponding self-energy is \( \Sigma = -i \pi g_{32}^2 P_3 - i \pi g_{21}^2 P_2 \) and therefore \( G = P_i (\omega - \epsilon_1 - H_b + i \eta)^{-1} + P_2 (\omega - \epsilon_2 - H_b + i \pi g_{21}^2)^{-1} + P_3 (\omega - \epsilon_3 - H_b + i \pi g_{32}^2)^{-1} = P_i G_j \).

The building block is \( G_0^{-1} G|_{\text{los}} = GG_0^{-1}|_{\text{los}} = P_i \), and therefore

\[ T^{(1)}(\omega) = P_i (\tilde{a}_v^\dagger S_-) P_2 G_2 (\tilde{a}_v S_+) P_1 = P_1 g_{21}^2 \tilde{a}_v^\dagger \frac{1}{\omega - \epsilon_2 + i \pi g_{21}^2} \tilde{a}_v \] (50)

while

\[ S_{pk} = \delta_{pk} \left[ 1 - 2 \pi i \frac{g_{21}^2}{p + \epsilon_1 - \epsilon_2 + i \pi g_{21}^2} \right] = \delta_{pk} \frac{p + \epsilon_1 - \epsilon_2 - i \pi g_{21}^2}{p + \epsilon_1 - \epsilon_2 + i \pi g_{21}^2} \] (51)

We note that for the \( \Sigma \)-scheme the one-photon scattering is the same as in the two-level model, the third level being inefficient.

The results for the one-photon scattering of sections 3.4–3.6 agree with the ones derived in [12].

4. A model with several emitters

Here we apply the same formalism to the case of several emitters. We distinguish the case of distributed emitters from the case of concentrated system (located at the same point). Moreover, we focus mainly on two-level systems and discuss single- and two-photon scattering separately. We show that our main result works even when different emitters have different coupling constants. These calculations explicitly prove one of our main statements about the independent character of scattering of unidirectional photons in distributed systems.
4.1. The case of two atoms

To be specific we focus on the following Hamiltonian model with two emitters

\[ H = \int \mathrm{d}v \Gamma a^\dagger(v)a(v) + \frac{\Omega_1}{2} \sigma_z^{(1)} + \frac{\Omega_2}{2} \sigma_z^{(2)} + \int \mathrm{d}v (a^\dagger(v)S_-(v) + a(v)S_+(v)) \]

\[ \equiv H_0 + a_{av}S_{av}, \]  

(52)

where the following combinations now play the role of spin operators

\[ S_+(v) = g_1 \sigma_+^{(1)} e^{i\nu_1 r_1} + g_2 \sigma_+^{(2)} e^{i\nu_2 r_2}, \]

(53)

\[ S_-(v) = g_1 \sigma_-^{(1)} e^{-i\nu_1 r_1} + g_2 \sigma_-^{(2)} e^{-i\nu_2 r_2}. \]

(54)

It is important to notice that this form of coupling is effectively energy (momentum) dependent, and for \( r_1 \neq r_2 \) it does not fulfil one of the assumptions we used to derive (3). Therefore, an extension of (3) is required, which is done below.

It is convenient to introduce the following projection operators \( P_{ab} = P_{ab}^{(1)} P_{ab}^{(2)}, a, b = \pm, \) so that \( \sum_{a,b} P_{ab} = 1. \) We then find

\[ \frac{\Omega_1}{2} \sigma_z^{(1)} + \frac{\Omega_2}{2} \sigma_z^{(2)} = \frac{\Omega_1}{2} (P_+ - P_-)(P_+ + P_-) + \frac{\Omega_2}{2} (P_+ + P_-)(P_+ - P_-) \]

\[ = \frac{\Omega_1 + \Omega_2}{2} P_{++} + \frac{\Omega_1 - \Omega_2}{2} (P_{+-} - P_{-+}) - \frac{\Omega_1}{2} P_{--}. \]  

(55)

It follows that the bare Green’s function is

\[ G_0(\omega) = \frac{P_{++}}{\omega - H_b - \frac{\Omega_1 + \Omega_2}{2} + i\eta} + \frac{P_{+-}}{\omega - H_b - \frac{\Omega_1 - \Omega_2}{2} + i\eta} + \frac{P_{-+}}{\omega - H_b + \frac{\Omega_1 + \Omega_2}{2} + i\eta} \]

\[ + \frac{P_{--}}{\omega - H_b + \frac{\Omega_1 - \Omega_2}{2} + i\eta} = P_{ab} G_{ab}. \]  

(56)

4.2. Calculation of the self-energy and scattering matrices

Here is the only diagram allowed for the self-energy

\[ \Sigma^{(2)}(\omega) = a_{av} S_{av} P_{ab} G_{ab}(\omega) a_{av}^\dagger = S_{av} P_{ab} G_{ab}(\omega - \nu) S_{-v} \]

\[ = [g_1 \sigma_+^{(1)} e^{i\nu_1 r_1} + g_2 \sigma_+^{(2)} e^{i\nu_2 r_2}] P_{ab} G_{ab} [g_1 \sigma_-^{(1)} e^{-i\nu_1 r_1} + g_2 \sigma_-^{(2)} e^{-i\nu_2 r_2}] \]

\[ = [g_1 e^{i\nu_1 r_1} \sigma_+^{(1)} P_{ab} G_{-b} + g_2 e^{i\nu_2 r_2} P_{a}^{(2)} \sigma_+^{(2)} G_{a,-}] [g_1 \sigma_-^{(1)} e^{-i\nu_1 r_1} + g_2 \sigma_-^{(2)} e^{-i\nu_2 r_2}] \]

\[ = \int \mathrm{d}v [g_1^2 P_{ab} G_{-b} + g_1 g_2 e^{i(\nu_1 r_1 - \nu_2 r_2)} \sigma_+^{(1)} \sigma_-^{(2)} G_{-b} \]

\[ + g_1 g_2 e^{i(\nu_1 r_1 - \nu_2 r_2)} \sigma_+^{(1)} \sigma_-^{(2)} G_{-b} + g_2^2 P_{a}^{(2)} G_{a,-}] \]

\[ = -i\pi \left[ (g_1^2 + g_2^2) P_{++} + g_1^2 P_{+-} + g_2^2 P_{-+} + g_1 g_2 f(r_1, r_2) \sigma_+^{(1)} \sigma_-^{(2)} \right] \]

\[ + g_1 g_2 f(r_1, r_2) \sigma_+^{(1)} \sigma_-^{(2)}, \]  

(57)

where \( f(r_1, r_2; \omega_b - H_b) = f(r_1, r_2) = 2 \exp[i((\omega_b - H_b)(r_1 - r_2))] \Theta(r_1 - r_2), \) and \( \omega = \omega_b - \Omega, \) \( \omega_b \) being the energy of incoming (and outgoing) photons. The standard symbol for the Wick
contraction is also used. The dressed Green’s function is therefore
\[
G(\omega) = G(\omega_h - H_b) = \frac{P_{++}}{\omega_h - H_b - \Omega_1 - \Omega_2 + i\pi(g_1^2 + g_2^2)} + \frac{P_{--}}{\omega_h - H_b + i\eta} + \mathcal{M},
\]
where \( \mathcal{M} \) is the matrix to be specified below.

Now we consider the two cases separately: the case of a concentrated system (when positions of two emitters \( r_1 \) and \( r_2 \) coincide) and the case of a distributed system when the coordinates of emitters are different. In the former case the analysis based on equation (3) is sufficient, while the latter case requires its extension.

### 4.2.1. Concentrated case.

For coinciding positions of emitters \( r_1 = r_2 \) and we have \( f_{12} = f_{21} = 1 \) and
\[
\mathcal{M} = \left\{ (G_{++}^0)^{-1} + i\pi g_1^2 \right\} P_{+-} + \left\{ (G_{--}^0)^{-1} + i\pi g_2^2 \right\} P_{-+} + i\pi g_1 g_2 \frac{\sigma_{-}^{(1)} \sigma_{+}^{(2)} + \sigma_{+}^{(1)} \sigma_{-}^{(2)}}{2}
\]
\[
= \frac{1}{\det \mathcal{M}} \left\{ (G_{++}^0)^{-1} + i\pi g_1^2 \right\} P_{+-} + \left\{ (G_{--}^0)^{-1} + i\pi g_2^2 \right\} P_{-+} - i\pi g_1 g_2 \frac{\sigma_{-}^{(1)} \sigma_{+}^{(2)} + \sigma_{+}^{(1)} \sigma_{-}^{(2)}}{2}
\]
where
\[
\det \mathcal{M} = (G_{++}^0)^{-1} (G_{--}^0)^{-1} + i\pi \left[ g_2^2 (G_{++}^0)^{-1} + g_1^2 (G_{--}^0)^{-1} \right]
\]
\[
= \left( \omega_h - H_b - \frac{\Omega_1 + \Omega_2}{2} - \Lambda_+ \right) \left( \omega_h - H_b - \frac{\Omega_1 + \Omega_2}{2} - \Lambda_- \right),
\]
\[
\Lambda_{\pm} = -\frac{i\pi (g_1^2 + g_2^2)}{2} \pm \frac{1}{2} \sqrt{-\pi^2 (g_1^2 + g_2^2)^2 + (\Omega_1 - \Omega_2)^2 - 2i\pi (\Omega_1 - \Omega_2)(g_1^2 - g_2^2)}.
\]

In the limiting case \( \Omega_1 = \Omega_2 = \Omega \)
\[
\mathcal{M} = \frac{P_{++} + P_{--}}{2} \left[ \frac{1}{\omega_h - H_b - \Omega + i\pi (g_1^2 + g_2^2)} + \frac{1}{\omega_h - H_b - \Omega + i\eta} \right]
\]
\[
+ \frac{P_{+-} - P_{-+}}{2} \cos \theta \left[ \frac{1}{\omega_h - H_b - \Omega + i\pi (g_1^2 + g_2^2)} - \frac{1}{\omega_h - H_b - \Omega + i\eta} \right]
\]
\[
+ \frac{\sigma_{-}^{(1)} \sigma_{+}^{(2)} + \sigma_{+}^{(1)} \sigma_{-}^{(2)}}{2} \sin \theta \left[ \frac{1}{\omega_h - H_b - \Omega + i\pi (g_1^2 + g_2^2)} - \frac{1}{\omega_h - H_b - \Omega + i\eta} \right],
\]
where \( \tan \frac{\theta}{2} = \frac{g_2}{g_1} \). Then
\[
T^{(1)} = \left[ g_1 \sigma_{-}^{(1)} P_{-}^{(2)} + g_2 P_{-}^{(1)} \sigma_{+}^{(2)} \right] G(v_1) \left[ g_1 \sigma_{+}^{(1)} P_{+}^{(2)} + g_2 P_{+}^{(1)} \sigma_{+}^{(2)} \right] a_{v_1}^\dagger a_{v_2}
\]
\[
= a_{v_1}^\dagger a_{v_2} P_{--} \left( g_1^2 + g_2^2 \right) \frac{g_1^2 + g_2^2}{v_1 - \Omega + i\pi (g_1^2 + g_2^2)},
\]

and the corresponding scattering matrix is
\[
S_{pk} = \delta_{pk} \left( 1 - 2 \frac{i\pi (g_{1}^2 + g_{2}^2)}{p - \Omega + i\pi (g_{1}^2 + g_{2}^2)} \right) = \delta_{pk} \frac{p - \Omega - i\pi (g_{1}^2 + g_{2}^2)}{p - \Omega + i\pi (g_{1}^2 + g_{2}^2)}. \tag{64}
\]

4.2.2. Distributed system. For the distributed system \( r_1 \neq r_2 \) we get
\[
\mathcal{M} = \frac{P_{-+}}{\omega_b - \Omega_1 + i\pi g_{1}^2} + \frac{P_{+-}}{\omega_b - \Omega_2 + i\pi g_{2}^2} - i\pi g_{1}g_{2} \frac{f(r_2, r_1)\sigma_{-}^{(1)}\sigma_{+}^{(2)} + f(r_1, r_2)\sigma_{+}^{(1)}\sigma_{-}^{(2)}}{(\omega_b - \Omega_1 + i\pi g_{1}^2)(\omega_b - \Omega_2 + i\pi g_{2}^2)}.
\]

The one-photon \( T \)-matrix amounts to
\[
T^{(1)} = P_{-}a_i S_{-i} G a_{i'} S_{+i'} P_{+} = \left[ g_{1}\sigma_{-}^{(1)} P_{-}^{(2)} e^{-i\nu \tau_1} + g_{2} P_{-}^{(1)} \sigma_{-}^{(2)} e^{-i\nu \tau_2} \right] G(v_1) \\
\times \left[ g_{1}\sigma_{+}^{(1)} P_{+}^{(2)} e^{+i\nu \tau_1} + g_{2} P_{+}^{(1)} \sigma_{+}^{(2)} e^{+i\nu \tau_2} \right] a_{i'} a_{i'}
\]
\[
= a_{i'} a_{i'} P_{-} g_{1}^2 (v_1 - \Omega_2) + g_{2}^2 (v_1 - \Omega_1)
\]
\[
= a_{i'} a_{i'} P_{+} \frac{g_{1}^2 (v_1 - \Omega_2) + g_{2}^2 (v_1 - \Omega_1)}{(v_1 - \Omega_1 + i\pi g_{1}^2)(v_1 - \Omega_2 + i\pi g_{2}^2)}, \tag{66}
\]
and the corresponding scattering matrix reads
\[
S_{pk} = \delta_{pk} \left( 1 - 2 \frac{i\pi g_{1}^2 (p - \Omega_2) + i\pi g_{2}^2 (p - \Omega_1)}{(p - \Omega_1 + i\pi g_{1}^2)(p - \Omega_2 + i\pi g_{2}^2)} \right) = \delta_{pk} \prod_{i=1}^{2} \frac{p - \Omega_i - i\pi g_{i}^2}{p - \Omega_i + i\pi g_{i}^2}. \tag{67}
\]
We note that (67) can be represented as a convolution
\[
S_{pk} = \int dk' S_{2;pk} S_{1;k'k} \tag{68}
\]
of the scattering matrices \( S_1 \) and \( S_2 \) on the first and second emitters, respectively. This property is a consequence of the absent backscattering for chiral photons, and subsequently it will be generalized to the arbitrary \( N \)-photon sector.

We also make the two following observations: (i) in the absence of backscattering there is no interference between counter-propagating waves, and therefore the outgoing state contains no information about the positions of emitters, that is, there is no dependence on \( r_1 - r_2 \) in (67). (ii) The result (67) would smoothly cross over to (64) on a scale of the phononic wavelength. The latter quantity is of the order of an inverse bandwidth, which is effectively set to zero in our theory. Therefore, (64) and (67) are not analytically connected with each other in the confines of the vanishing distance \( r_1 - r_2 \) between the emitters.

4.3. General approach: one-photon scattering

Here we present a general approach aimed at calculating the scattering matrices for the distributed system. Therefore in the following we only consider the model with two emitters
located in different positions and having different $\Omega_i$ and $g_i$,

$$H = \int dv \, v a^\dagger(v)a(v) + \frac{\Omega_1}{2} \sigma^{(1)}_z + \frac{\Omega_2}{2} \sigma^{(2)}_z + V_1 + V_2, \quad (69)$$

where $V_i = v_i + v_i^\dagger$, where $i = 1, 2$. Here $v_i = g_i \sigma^i_\uparrow \int dv \, e^{ivr_i}a(v) \equiv g_i \sigma^i_\uparrow A_i$. We label the atoms in such a way that $r_1 > r_2$. Our aim is to calculate the $T$-matrix

$$T(\omega) = V + V \frac{1}{\omega - H + i\eta} V,$$  

where $V = V_1 + V_2$, in the ground state of the atomic system $|↓↓\rangle$. In the following we omit $i\eta$ assuming $\omega \to \omega + i\eta$.

The first term in (70) can be disregarded as it is off-diagonal in spin states. In calculating the second term $(V_1 + V_2)\frac{1}{\omega - H_0 - V_1 - V_2}(V_1 + V_2)$ we can only retain $(v_1^\dagger + v_2^\dagger)$ in the left factor and $(v_1 + v_2)$ in the right factor, respectively, since $\sigma^i_\uparrow |↓↓\rangle = 0$ and $\langle ↓↓|\sigma^i_\uparrow = 0$. Thus we have a sum of four terms

$$T = v_1^\dagger \frac{1}{\omega - H_0 - V_1 - V_2} v_1 \quad (71)$$

$$+ v_2^\dagger \frac{1}{\omega - H_0 - V_1 - V_2} v_2 \quad (72)$$

$$+ v_1^\dagger \frac{1}{\omega - H_0 - V_1 - V_2} v_2 \quad (73)$$

$$+ v_2^\dagger \frac{1}{\omega - H_0 - V_1 - V_2} v_1. \quad (74)$$

As a consequence of the RWA we derive the operators $A_i^\dagger \sim v_i^\dagger$ to the left of the resolvent, and the operators $A_i \sim v_i$ to the right of the resolvent in each term of this sum.

We now consider each term, expanding them first in $V_1$, and then in $V_2$. In the following we apply an important observation that $v_1$ can be only paired with the adjacent $v_1^\dagger$, while any contraction of $v_2$ and $v_1^\dagger$ always yields zero. More generally, the contraction

$$\langle v_i^\dagger v_j \rangle = 0, \quad i > j, \quad (75)$$

vanishes as it typically implies an integral of the kind

$$\int dv \, e^{i(vr_1 - r_j)} \frac{1}{(\cdots - v + i\eta) \cdots (\cdots - v + i\eta)} = 0. \quad (76)$$

This integral is zero as it has all poles in the upper half-plane, and the exponential function $e^{i(vr_1 - r_j)}$ decays quickly enough in the lower half-plane for $r_i < r_j$. Therefore we can close the integration contour in the lower half-plane and get zero.

We now consider the different components (71–74) of the $T$-matrix in more detail. In the contribution $T_{11}$ given by (71)

$$T_{11} = v_1^\dagger \left( \frac{1}{\omega - H_0 - V_2} + \frac{1}{\omega - H_0 - V_2} v_1 \frac{1}{\omega - H_0 - V_2} v_1^\dagger \frac{1}{\omega - H_0 - V_2} + \cdots \right) v_1, \quad (77)$$

we can omit the terms in parentheses which are odd in $V_1$, as they are off-diagonal in the spin states of the $i = 1$ emitter. Moreover, the operators $v_1$ and $v_1^\dagger$ must alternate.
If we are exclusively interested in the single-photon scattering, then we should only consider terms with the single creation (annihilation) operator on the left (right) side. For this reason we can neglect $V_2$ everywhere in (77); an expansion in $V_2$ cannot contain $v_2^\dagger$ in the left-most position because of the spin state of the second atom, and it cannot contain $v_2^\dagger$ because of the photon state. Note that this argument is implicitly based on the RWA.

Resumming the remaining series, we find in the one-photon sector

$$T_{11}^{(1)} = v_1^\dagger \frac{1}{\omega - H_0 - \Sigma_1} v_1 = g_1^2 A_1^\dagger \frac{1}{\omega - H_0 - \Omega_1 - \Sigma_1} A_1,$$

where $\Sigma_1 = -i\pi g_1^2 \equiv -i\Gamma_1$.

Applying similar arguments to $T_{22} (72)$, we find

$$T_{22}^{(1)} = v_2^\dagger \frac{1}{\omega - H_0 - \Sigma_2} v_2 = g_2^2 A_2^\dagger \frac{1}{\omega - H_0 - \Omega_2 - \Sigma_2} A_2,$$

where $\Sigma_2 = -i\pi g_2^2 \equiv -i\Gamma_2$.

Analogously we find for $T_{12} (73)$ that

$$T_{12} = v_1^\dagger \left( \frac{1}{\omega - H_0 - V_2} v_1 \frac{1}{\omega - H_0 - V_2} + \cdots \right) v_2.$$

The terms in parentheses which are even in $V_1$ are omitted; an expansion starts from $v_1$. Once a term containing $v_1^\dagger$ occurs (e.g. $\sim v_1 v_1^\dagger v_1$), a pairing of adjacent $v_1$ and $v_1^\dagger$ should be performed, as there is no other possibility for $v_1^\dagger$ to be paired in the one-photon sector (the latter implies that $v_1^\dagger$ cannot be paired to an external vertex). This leads to the expression

$$T_{12}^{(1)} = v_1^\dagger \frac{1}{\omega - H_0 - V_2 - \Sigma_1} v_1 \frac{1}{\omega - H_0 - V_2} v_2.$$  

In the first propagator one can neglect $V_2$ as the corresponding expansion can start neither from $v_2$ nor from $v_2^\dagger$. After expanding the second propagator in $V_2$, one keeps only the odd terms in $V_2$, the expansion starting from $v_2^\dagger$. Resumming the series containing the power of contraction between $v_2$ and $v_2^\dagger$, one transforms (81) into

$$T_{12}^{(1)} = v_1^\dagger \frac{1}{\omega - H_0 - \Sigma_1} v_1 \frac{1}{\omega - H_0} v_2^\dagger \frac{1}{\omega - H_0 - \Sigma_2} v_2.$$  

It now only remains to pair $v_1$ and $v_2^\dagger$ which results in

$$T_{12}^{(1)} = -2\pi i g_1^2 g_2^2 A_1^\dagger \frac{1}{\omega - H_0 - \Omega_1 - \Sigma_1} e^{i(\omega - H_0)(r_1 - r_2)} \frac{1}{\omega - H_0 - \Omega_2 - \Sigma_2} A_2,$$

where we have also used

$$\sigma \equiv \frac{1}{\omega - \frac{i}{2} \sigma_z} = \frac{1}{\omega - \frac{i}{2} \sigma_z - \Omega \sigma}. $$

Finally, we consider $T_{21} (74)$

$$T_{21} = v_2^\dagger \left( \frac{1}{\omega - H_0 - V_2} v_1^\dagger \frac{1}{\omega - H_0 - V_2} + \cdots \right) v_1. $$

The terms in parentheses which are even in $V_1$ are omitted; an expansion starts from $v_1^\dagger$. However, this operator cannot be paired to any operator standing to the left of it as it follows from (75). Therefore, $T_{21}$ identically vanishes in the one-photon sector, $T_{21}^{(1)} = 0$. 

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The first and the fourth vertices in these diagrams are not renormalized, which is a consequence of the algebra of spin operators. In between vertices, one can insert the dressed Green’s functions given by equations (58) and (65).

We now collect all the terms and calculate the scattering matrix in the one-photon sector

\[
S_{pk} = \delta_{pk} - 2i\delta_{pk} \left[ \frac{\Gamma_1}{p - \Omega_1 + r\Gamma_1} + \frac{\Gamma_2}{p - \Omega_2 + i\Gamma_2} - 2i\Gamma_1 \Gamma_2 e^{-ipr_1} \right. \\
\left. \times \frac{1}{p - \Omega_1 + i\Gamma_1} e^{ip(r_1 - r_2)} \frac{1}{p - \Omega_2 + i\Gamma_2} e^{ipr_2} \right]
\]

\[
= \delta_{pk} \left[ 1 - 2 \frac{i\Gamma_1 (p - \Omega_2) + i\Gamma_2 (p - \Omega_1)}{(p - \Omega_1 + i\Gamma_1)(p - \Omega_2 + i\Gamma_2)} \right]
\]

\[
= \delta_{pk} \frac{(p - \Omega_1 - i\Gamma_1)(p - \Omega_2 - i\Gamma_2)}{(p - \Omega_1 + i\Gamma_1)(p - \Omega_2 + i\Gamma_2)}, \tag{86}
\]

which coincides with the result (67) from the previous subsection.

4.4. Two-photon sector

In order to find an exact expression for a scattering matrix in the two-photon sector, it is necessary to classify all the possible arrangements of external vertices with their eventual renormalization which are allowed by the algebra of spin operators. In between vertices, one can insert the dressed Green’s functions given by equations (58) and (65).

The diagrams which do not vanish in the two-photon sector are shown in figures 4 and 5. The first and the fourth vertices in these diagrams are not renormalized, which is a consequence of the RWA. This allows us to evaluate the part which is the same for all diagrams

\[
a_{v_1}^\dagger (g_1 \sigma_1^{(1)} P_-^{(2)} e^{-i\nu r_1} + g_2 P_-^{(1)} \sigma_-^{(2)} e^{-i\nu r_2}) G \cdots G a_{v_2} (g_1 \sigma_1^{(1)} P_-^{(2)} e^{i\nu r_1} + g_2 P_-^{(1)} \sigma_-^{(2)} e^{i\nu r_2})
\]

\[
= a_{v_1}^\dagger (g_1 M_1 \sigma_1^{(1)} P_-^{(2)} e^{-i\nu r_1} + g_2 M_2 P_-^{(1)} \sigma_-^{(2)} e^{-i\nu r_2})
\]

\[
\times \cdots \times a_{v_1} (M_1 g_1 \sigma_1^{(1)} P_-^{(2)} e^{i\nu r_1} + M_2 g_2 P_-^{(1)} \sigma_-^{(2)} e^{i\nu r_2})
\]

\[
= a_{v_1}^\dagger (g_1 M_1 \sigma_1^{(1)} P_-^{(2)} e^{-i\nu r_1} + g_2 M_2 S_{12} P_-^{(1)} \sigma_-^{(2)} e^{-i\nu r_2})
\]

\[
\times \cdots \times a_{v_1} (M_1 g_1 S_{12} \sigma_1^{(1)} P_-^{(2)} e^{i\nu r_1} + M_2 g_2 P_-^{(1)} \sigma_-^{(2)} e^{i\nu r_2}), \tag{87}
\]

where \( S_{1,2} = 1 - 2\pi i g_{1,2}^2 M_{1,2} \), and \( M_{1,2} \) are defined in (65).
The first term in (88) reads
\[
P_{-}\sigma_{a\beta}^{*}(g_{1}^{2}M_{1}e^{-i(v_{1}-v_{2})\nu_{1}} + g_{2}^{2}M_{2}S_{1}e^{-i(v_{1}-v_{2})\nu_{2}})a_{v_{2}} \frac{1}{\omega_{b} - H_{b} + i\eta} \times a_{v_{1}}^{\dagger}(g_{1}^{2}M_{1}S_{2}e^{-i(v_{1}-v_{3})\nu_{1}} + g_{2}^{2}M_{2}e^{-i(v_{3}-v_{4})\nu_{2}})a_{v_{4}}
\]
\[
= P_{-} \left( g_{1}^{2}M_{1}(v_{4})S_{2}(v_{1})e^{i(v_{1}-v_{3})\nu_{1}} + g_{2}^{2}M_{2}(v_{1})S_{2}(v_{4})e^{i(v_{3}-v_{4})\nu_{2}} \right) a_{v_{1}}^{\dagger}a_{v_{2}}^{\dagger}a_{v_{3}}a_{v_{4}},
\]
\[
(88)
\]

The second term in (88) reads
\[
P_{-}\sigma_{a\beta}^{*}g_{1}g_{2}(M_{1}e^{-i(v_{1}-v_{2})\nu_{1}} + M_{2}S_{1}e^{-i(v_{1}-v_{2})\nu_{2}})a_{v_{2}}^{\dagger} \frac{1}{\omega_{b} - H_{b} - \alpha_{1} - \alpha_{2}} a_{v_{1}}g_{1}g_{2}
\times (M_{1}S_{1}e^{i\nu_{2}r_{1}+i\nu_{1}r_{2}} + M_{2}e^{i\nu_{2}r_{1}+i\nu_{1}r_{2}})a_{v_{4}}
\]
\[
= P_{-}g_{1}g_{2}^{2} \left( M_{1}(v_{1})e^{-i(v_{1}-v_{2})\nu_{1}} + M_{2}(v_{1})S_{1}(v_{2})e^{-i(v_{1}-v_{2})r_{1}} \right)
\times \frac{1}{E - \alpha_{1} - \alpha_{2}} (M_{1}(v_{1})S_{2}(v_{4})e^{i\nu_{2}r_{1}+i\nu_{1}r_{2}} + M_{2}(v_{4})e^{i\nu_{2}r_{1}+i\nu_{1}r_{2}})a_{v_{1}}^{\dagger}a_{v_{2}}^{\dagger}a_{v_{3}}a_{v_{4}}
\]
\[
= \frac{1}{E - \alpha_{1} - \alpha_{2}} \left( M_{1}(v_{1})M_{1}(v_{4})S_{2}(v_{4})e^{-i(v_{1}-v_{4})r_{1}+i\nu_{2}r_{1}} + M_{2}(v_{2})S_{1}(v_{1})M_{2}(v_{4})e^{i(v_{1}-v_{4})r_{1}+i\nu_{2}r_{1}} + M_{2}(v_{1})S_{1}(v_{1})M_{1}(v_{4})S_{2}(v_{4})e^{i(v_{1}-v_{4})r_{1}+i\nu_{2}r_{1}} + M_{1}(v_{1})M_{2}(v_{4})e^{-i(v_{1}-v_{4})r_{1}+i\nu_{2}r_{1}} \right)
\times \frac{P_{-}g_{1}g_{2}^{2}}{E - \alpha_{1} - \alpha_{2}} a_{v_{1}}^{\dagger}a_{v_{2}}^{\dagger}a_{v_{3}}a_{v_{4}},
\]
\[
(90)
\]

The first figure shows diagrams with renormalization of double vertices. Notations are the same as before.

We first evaluate the diagrams shown in figure 4. The diagram without vertex renormalization is represented by the expression
\[
P_{-}\sigma_{a\beta}^{*}S_{-\nu_{1}}Ga_{\nu_{2}}S_{-\nu_{2}}Ga_{\nu_{3}}S_{-\nu_{4}}P_{-} + P_{-}\sigma_{a\beta}^{*}S_{-\nu_{1}}Ga_{\nu_{2}}S_{-\nu_{2}}Ga_{\nu_{3}}S_{+\nu_{3}}Ga_{\nu_{4}}S_{+\nu_{4}}P_{-}.
\]

The first term in (88) reads
\[
(88)
\]

The second term in (88) reads
\[
(90)
\]

where \(\alpha_{1,2} = \Omega_{1,2} - i\pi g_{1,2}^{2}\) and \(E = v_{1} + v_{2} = v_{3} + v_{4}\).
In order to evaluate the diagram with renormalization of the second vertex we find the following vertex correction to the second vertex
\[
(g_1\sigma_+^{(1)} a_v e^{i\nu v}) G(g_1\sigma_+^{(1)} a_v^\dagger e^{-i\nu v}) G(g_2\sigma_-^{(2)} a_v^\dagger e^{-i\nu v})
\]
\[
= g_1^2 g_2^2 \int dv \frac{e^{i\nu_1 r_1 - \nu_2 r_2}}{(\omega_b - H_b - \nu + i\eta)(\omega_b - H_b - \nu + v_2 - \alpha_1)} a_v^\dagger.
\]

Such vertex correction can only occur in the diagram which has $v_1^\dagger$ in the first position (from the left). Therefore we can effectively replace $\omega_b - H_b \to \nu_1$. Evaluating the integral we get
\[
P_+^{(1)}(1) \sigma_+^{(2)} e^{-i\nu_2 r_2} g_2^2 (-2\pi i) e^{i\nu_1 (r_1 - r_2)} M_1(v_2) \left[1 - e^{i(v_2 - \nu_1)(r_2 - r_1)}\right] a_v^\dagger = \sigma_+^{(1)}(P_+^{(2)}) e^{i\nu_2 r_2} g_2^2 a_v \int dv \frac{e^{i\nu_1 r_1 - \nu_2 r_2}}{(\omega_b - H_b - \nu + v_3 - \alpha_2)(\omega_b - H_b - \nu + i\eta)}.
\]
\[
M_1(v_1) [S_1(v_2) - 1] \left[1 - e^{i(v_1 - \nu_3)(r_1 - r_2)}\right] (M_1(v_4) S_2(v_4) e^{i(v_1 - v_3)(r_1 - r_2)} + M_2(v_4) e^{i(v_1 - v_4)(r_1 - r_2)})
\]
\[
\times \frac{P_+^{(2)}}{E - \alpha_1 - \alpha_2} a_v^\dagger a_v^\dagger a_v^\dagger a_v^\dagger.
\]

In order to evaluate the diagram with renormalization of the third vertex, we use the following vertex correction to the third vertex
\[
(g_1\sigma_+^{(1)} a_v e^{i\nu v}) G(g_2\sigma_+^{(2)} a_v e^{i\nu v}) G(g_2\sigma_-^{(2)} a_v^\dagger e^{-i\nu v})
\]
\[
= g_1^2 g_2^2 \int dv \frac{e^{i\nu r} g_2^2 a_v (-2\pi i) e^{i\nu_1 (r_1 - r_2)} M_2(v_3) \left[1 - e^{i(v_3 - \nu_2)(r_1 - r_2)}\right]}{(\omega_b - H_b - v + v_3 - \alpha_2)(\omega_b - H_b - v + i\eta)}.
\]
\[
\sigma_+^{(1)}(P_+^{(2)}) e^{i\nu_3 r_3} g_1^2 g_2 a_v \int dv \frac{e^{i\nu_1 r_1 - \nu_3 r_3}}{(\omega_b - H_b - \nu + v_3 - \alpha_2)(\omega_b - H_b - \nu + i\eta)}.
\]
\[
M_1(v_1) e^{-i(v_1 - v_3)(r_1 - r_2)} M_2(v_1) S_1(v_1) e^{i(v_1 - v_3)(r_1 - r_2)} [S_2(v_3) - 1] \left[1 - e^{i(v_3 - \nu_2)(r_1 - r_2)}\right] M_2(v_3)
\]
\[
\times \frac{P_+^{(2)}}{E - \alpha_1 - \alpha_2} a_v^\dagger a_v^\dagger a_v^\dagger a_v^\dagger.
\]
Summarizing the results of equations (91), (94), (97) and (98), we find the following intermediate expression for $T^{(2)}$

\[-i\pi \delta_{\nu_\eta}(v_1) \left[ g_1^4 M_1(v_1) M_1(v_4) S_2(v_4) + g_2^4 S_1(v_1) M_2(v_1) M_2(v_4) \right] \]

\[-i\pi \delta_{\nu_\eta}(v_1) g_1^2 g_2^2 \left[ M_1(v_1) M_2(v_4) + M_1(v_4) S_2(v_4) M_2(v_1) S_1(v_1) \right] \]

\[+ \frac{1}{v_1 - v_3} \left[ g_1^4 M_1(v_1) M_1(v_4) S_2(v_4) + g_2^4 S_1(v_1) M_2(v_1) M_2(v_4) \right] \]

\[+ \frac{g_1^2 g_2^2}{v_1 - v_3} \left[ M_1(v_1) M_2(v_4) - M_1(v_3) S_2(v_3) M_2(v_2) S_1(v_2) \right] e^{-i(v_1 - v_3)(r_1 - r_2)} \]

\[+ \frac{g_1^2 g_2^2}{E - \alpha_1 - \alpha_2} e^{-i(v_1 - v_3)(r_1 - r_2)} \]

\[\times \left\{ M_1(v_1) + M_2(v_2) S_2(v_2) + M_1(v_2) [S_1(v_1) - 1] [1 - e^{i(v_1 - \alpha_1)(r_1 - r_2)}] \right\} \]

\[\times \left\{ M_2(v_4) + M_1(v_3) S_2(v_3) + M_2(v_3) [S_2(v_4) - 1] [1 - e^{i(v_2 - \alpha_2)(r_1 - r_2)}] \right\}, \tag{99} \]

which has to be convoluted with $a^\dagger_{v_1} a^\dagger_{v_2} a_{v_3} a_{v_4}$. The projector $P_{--}$ onto the ground state of the atomic system is also omitted.

We see that the dependence on atomic positions is still present in (99). Exchanging the dummy frequencies $v_1 \leftrightarrow v_2$ and $v_3 \leftrightarrow v_4$, when necessary, and exploiting the obvious identities

\[M_1(v_2) [S_1(v_1) - 1] = M_1(v_1) [S_1(v_2) - 1], \tag{100} \]

\[M_2(v_3) [S_2(v_4) - 1] = M_2(v_4) [S_2(v_3) - 1], \tag{101} \]

we can assign the position-dependent part of (99) to

\[\frac{g_1^2 g_2^2}{v_1 - v_3} M_1(v_1) M_2(v_4) [1 - S_1(v_2) S_2(v_3)] e^{-i(v_1 - v_3)(r_1 - r_2)} \]

\[-g_1^2 g_2^2 M_1(v_1) [S_1(v_2) - 1] M_1(v_3) M_2(v_4) S_2(v_3) e^{i(v_3 - \alpha_1)(r_1 - r_2)} \]

\[-g_1^2 g_2^2 M_2(v_4) [S_2(v_3) - 1] M_1(v_1) M_2(v_2) S_1(v_2) e^{i(v_2 - \alpha_2)(r_1 - r_2)} \]

\[+ \frac{g_1^2 g_2^2}{E - \alpha_1 - \alpha_2} M_1(v_1) M_2(v_4) [S_1(v_2) - 1] [S_2(v_3) - 1] e^{i(E - \alpha_1 - \alpha_2)(r_1 - r_2)}. \tag{102} \]

A cancellation of (102) is achieved by an account of diagrams containing double vertices shown in figure 5. The first possibility of generating a double vertex is provided by one photon covering the second and third vertices (upper panel of figure 5). The corresponding double vertex reads

\[(g_1 \sigma_+^{(1)} a_v e^{i v_1 r_1}) G \left( g_1 \sigma_-^{(i)} a^\dagger_{v_2} e^{-i v_2 r_2} \right) G \left( g_1 \sigma_+^{(i)} a_{v_3} e^{i v_3 r_3} \right) G \left( g_2 \sigma_-^{(2)} a^\dagger_v e^{-i v_4 r_4} \right) \]

\[= g_1^3 g_2^2 \sigma_+^{(1)} \sigma_-^{(2)} a_v G a^\dagger_{v_2} G a_{v_3} a^\dagger_v e^{i(v_3 - \alpha_1)(r_1 - r_2)} e^{-i(v_2 - v_3) r_1} \]

\[+ g_1 g_2^3 \sigma_+^{(1)} \sigma_-^{(2)} a_v G a^\dagger_{v_2} G a_{v_3} a^\dagger_v e^{i(v_2 - \alpha_2)(r_1 - r_2)} e^{-i(v_2 - v_3) r_2} \]

\[= -g_1 g_2^{(1)} \sigma_+^{(2)} a_v a^\dagger_{v_2} \left\{ g_1^2 e^{-i(v_1 - v_3) r_1} \int dv \frac{e^{i(v_1 - v_3) r_1}}{(v - v_1 - i\eta)(v - E + \alpha_1)(v - v_4 - i\eta)} \right\} \]

\[+ g_2^2 e^{-i(v_1 - v_3) r_2} \int dv \frac{e^{i(v_1 - v_3) r_2}}{(v - v_1 - i\eta)(v - E + \alpha_2)(v - v_4 - i\eta)} \]

\[\int_{v_1}^{v_4} dv \frac{e^{i(v_1 - v_3) r_1}}{(v - v_1 - i\eta)(v - v_4 - i\eta)}. \]
\[ 2\pi i g_2 \sigma_+^{(1)} \sigma_-^{(2)} a_v^\dagger a_v \left\{ g_1^2 e^{-i(v_2-v_3)r_1} \left[ M_1(v_2) e^{i(v_1-r_2)} - M_1(v_3) e^{i(v_1-r_2)} \right] \frac{M_1(v_2) e^{i(v_1-r_2)} - M_1(v_3) e^{i(v_1-r_2)}}{v_1 - v_4} \\
- M_1(v_2) M_1(v_3) e^{i(E-\alpha)(v_1-r_2)} \right] + g_2^2 e^{-i(v_2-v_3)r_2} \times \left[ \frac{M_2(v_2) e^{i(v_1-r_2)} - M_2(v_3) e^{i(v_2-r_2)}}{v_1 - v_4} - M_2(v_2) M_2(v_3) e^{i(E-\alpha)(v_1-r_2)} \right] \right\}. \]

The whole diagram results in
\[ g_1^2 g_2^2 a_v^\dagger a_v^\dagger a_v a_v \left\{ [S_1(v_2) - 1] M_2(v_4) \frac{M_1(v_1) e^{-i(v_1-v_3)(v_1-r_2)}}{v_1 - v_3} + [S_1(v_1) - 1] M_2(v_3) \frac{M_1(v_4)}{v_1 - v_3} \\
+ M_1(v_1) M_2(v_4) [S_1(v_2) - 1] M_1(v_4) e^{i(v_1-v_3)(v_1-r_2)} + M_1(v_2) [S_2(v_4) - 1] \frac{M_2(v_1)}{v_1 - v_3} \\
+ M_1(v_1) [S_2(v_3) - 1] \frac{M_2(v_4) e^{-i(v_1-v_3)(v_1-r_2)}}{v_1 - v_3} \\
+ M_1(v_1) M_2(v_4) M_2(v_2) [S_2(v_3) - 1] e^{i(v_2-v_3)(v_1-r_2)} \right\}. \] (103)

The second possibility of generating a double vertex is provided by two photon lines covering the second and the third vertices, respectively, and at the same time intersecting each other (lower panel of figure 5). The corresponding expression reads
\[ (g_1 \sigma_+^{(1)} \nu_v e^{i\gamma}) \ G \left( g_1 \sigma_-^{(1)} a_v^\dagger e^{-i\gamma_2} \right) G \left( g_1 \sigma_+^{(1)} a_\mu e^{i\gamma_1} \right) G \left( g_2 \sigma_-^{(2)} a_v^\dagger e^{-i\gamma_2} \right) \]
\[ \times G \left( g_2 \sigma_-^{(2)} a_\nu e^{i\gamma_2} \right) G \left( g_2 \sigma_+^{(2)} a_\mu e^{-i\gamma_2} \right) \]
\[ = g_1^3 g_2^3 a_v^\dagger \ G_+ a_v^\dagger G_+ a_\mu G_+ a_\nu G_+ a_v G_+ a_\mu a_v G_+ a_\nu a_v \ e^{i(v_1-r_2)} e^{i(v_2-r_2)} e^{-i(v_2+r_2)} \\
\times \int d\nu \int d\mu \frac{e^{i\gamma_1}}{(v - v_3 - i\eta)(v - E + \alpha_1)(v + \mu - E - i\eta)(\mu - E + \alpha_2)(\mu - v_4 - i\eta)}. \] (104)

First we evaluate the integral over \( \mu \)
\[ \int d\mu \frac{e^{i\gamma_1}}{(v + \mu - E - i\eta)(\mu - E + \alpha_2)(\mu - v_4 - i\eta)} \]
\[ = 2\pi i \left[ \frac{e^{i(E-v)(v_1-r_2)}}{(-v + v_2)(v_3 - v)} + \frac{e^{i(E-\alpha)(v_1-r_2)}}{(v - \alpha_2)(v_3 - \alpha_2)} + \frac{e^{i\gamma_1}}{(v_3 - \alpha_2)} \right] \]
\[ = 2\pi i e^{i(E-v)(v_1-r_2)} M_2(v_3) \left[ \frac{1 - e^{i(v_1-v_3)r_2}}{v - v_3} - \frac{1 - e^{i(v_2-v_3)r_2}}{v - \alpha_2} \right]. \] (105)
We now have to perform an integration over \( \nu \). Note that there is no pole at \( \nu = \nu_3 \) and \( \nu = \alpha_2 \). Collecting the contributions from the poles at \( \nu = \nu_1 + i\eta \) and \( \nu = E - \alpha_1 \), we assign (105) to

\[
(2\pi i)^2 g_1^3 g_2^3 e^{i\nu_1 - i\nu_2 - \sigma_4^{(1)}} \sigma_9^{(1)} a_{\nu_1}^\dagger a_{\nu_2} M_2(v_3) M_1(v_2) \times \left\{ \frac{1}{\nu_1 - \nu_3} \left[ \frac{1 - e^{i(\nu_1 - \nu_3)(\nu_1 - \nu_2)}}{E - \alpha_1 - \alpha_2} - \frac{1 - e^{i(\nu_1 - \alpha_2)(\nu_1 - \nu_2)}}{E - \alpha_1 - \alpha_2} \right] - M_2(v_1) \right\}.
\]  

(107)

The whole diagram with this vertex equals

\[
(2\pi i)^2 g_1^4 g_2^4 P \sum a_{\nu_1}^\dagger a_{\nu_2} a_{\nu_3} M_1(v_1) M_2(v_3) M_1(v_2) M_2(v_4) \times \left\{ \frac{1}{\nu_1 - \nu_3} \left[ \frac{1 - e^{i(\nu_1 - \nu_3)(\nu_1 - \nu_2)}}{E - \alpha_1 - \alpha_2} - \frac{1 - e^{i(\nu_1 - \alpha_2)(\nu_1 - \nu_2)}}{E - \alpha_1 - \alpha_2} \right] - M_2(v_1) \right\} = g_1^2 g_2^2 P \sum a_{\nu_1}^\dagger a_{\nu_2} a_{\nu_3} M_1(v_1) M_2(v_3) M_1(v_2) M_2(v_4) \times \left\{ - M_2(v_1) M_2(v_2)[S_1(v_1) - 1][S_2(v_3) - 1] \frac{1 - e^{i(\nu_1 - \nu_3)(\nu_1 - \nu_2)}}{E - \alpha_1 - \alpha_2} - M_2(v_1)[S_2(v_3) - 1][S_2(v_3) - 1] \frac{1 - e^{i(\nu_1 - \nu_3)(\nu_1 - \nu_2)}}{E - \alpha_1 - \alpha_2} \right\}.
\]  

(108)

Collecting all contributions to \( T^{(2)} \), namely (102), (104), (108), and the terms \( \sim g_1^4, g_2^4 \) from (99), we find

\[
-2\pi i T^{(2)} = -\frac{2\pi i}{\nu_1 - \nu_3} \left[ g_1^4 M_1(v_1) M_1(v_4) S_2(v_4) + g_2^4 S_1(v_1) M_2(v_1) M_2(v_4) \right] - 2\pi i g_1^2 g_2^2 \left\{ [S_1(v_1) - 1][S_2(v_3) - 1] \frac{M_2(v_4)}{\nu_1 - \nu_3} + M_1(v_2)[S_2(v_4) - 1] \frac{M_2(v_1)}{\nu_1 - \nu_3} \right\} - M_2(v_1) M_2(v_4)[S_1(v_1) - 1][S_2(v_3) - 1] \frac{1}{\nu_1 - \nu_3} + M_1(v_1)[S_2(v_3) - 1][S_2(v_3) - 1] \frac{M_2(v_4)}{\nu_1 - \nu_3} + M_1(v_2)[S_2(v_4) - 1][S_1(v_1) - 1] \frac{M_2(v_4)}{E - \alpha_1 - \alpha_2} - M_1(v_1)[S_2(v_3) - 1][S_2(v_3) - 1] M_2(v_4) M_2(v_2) - M_1(v_1) M_2(v_4)[S_1(v_1) - 1][S_2(v_3) - 1] M_1(v_3) \right\}.
\]  

(109)

One can observe that the dependence on emitter coordinates has disappeared. Transforming \( T^{(2)} \) further, we find

\[
-2\pi i T^{(2)} = \frac{E - 2\alpha_2}{4\pi i} S_1(v_1) S_1(v_2) M_2(v_1) M_2(v_2)[S_2(v_4) - 1][S_2(v_3) - 1] + \frac{E - 2\alpha_1}{4\pi i} M_1(v_4) M_1(v_3) [S_1(v_2) - 1][S_2(v_4) - 1] S_1(v_1) - \frac{1}{2\pi i} [S_1(v_1) - 1][S_2(v_3) - 1][S_1(v_2) - 1][S_2(v_4) - 1] \frac{1}{E - \alpha_1 - \alpha_2}.
\]  

(110)
This expression should be multiplied by $a_{q_k}^\dagger a_{p_k}^\dagger a_{p_1} a_{q_1}$. Then the irreducible part of $S^{(2)}$ equals

$$\delta_{p_1,p_2,k_1,k_2}^{(2)\text{irred}} = -2\pi i \delta_{p_1+p_2,k_1+k_2}^i (a^\dagger_{p_2} a_{p_1} T^{(2)} a^\dagger_{k_1} a^\dagger_{k_2}).$$

(111)

Note that if one of the emitters is decoupled (say, $g_2 = 0$), then $S_2 \equiv 1$, and one recovers from (110), the known result for the two-photon scattering matrix on a single emitter.

It now remains to evaluate a reducible contribution to the two-photon scattering. To this end we use an expression for $T^{(1)}$ and the $\delta$-part of (102) and find that the corresponding expression factorizes into products of the one-photon scattering matrices

$$S^{(2)\text{red}}_{p_1,p_2,k_1,k_2} = (\delta_{p_1,k_1} \delta_{p_2,k_2} + \delta_{p_1,k_2} \delta_{p_2,k_1}) \left\{ 1 - 2\pi i \left[ g_1^2 M_1(p_2) + g_2^2 M_2(p_2) - 2\pi i g_1^2 g_2^2 M_1(p_2) M_2(p_2) \right] \right. $$

$$+ \left. \frac{1}{2} (2\pi i)^2 \left[ g_1^2 M_1(p_1) M_1(p_2) S_2(p_2) + g_2^2 S_1(p_1) M_2(p_1) M_2(p_2) \right] \right. $$

$$+ \left. \frac{1}{2} (2\pi i)^2 g_2^2 \left[ M_1(p_1) M_2(p_2) + M_1(p_2) M_2(p_1) S_1(p_1) \right] \right. $$

$$+ \left. \frac{1}{2} (2\pi i)^2 \left[ g_1^2 M_1(p_2) M_1(p_1) S_2(p_1) + g_2^2 S_1(p_2) M_2(p_2) M_2(p_1) \right] \right. $$

$$+ \left. \frac{1}{2} (2\pi i)^2 g_1^2 \left[ M_1(p_2) M_2(p_1) + M_1(p_1) S_2(p_1) M_2(p_2) S_1(p_2) \right] \right\}$$

$$= \frac{1}{2} (\delta_{p_1,k_1} \delta_{p_2,k_2} + \delta_{p_1,k_2} \delta_{p_2,k_1}) \left[ 2 + 2(S_1(p_2) - 1) + 2(S_2(p_2) - 1) \right]$$

$$+ 2[S_1(p_2) - 1][S_2(p_2) - 1] + 2(S_1(p_1) - 1) + 2(S_2(p_1) - 1)$$

$$+ 2[S_1(p_1) - 1][S_2(p_1) - 1] + [S_1(p_1) - 1][S_1(p_2) - 1]$$

$$+ [S_1(p_1) - 1][S_2(p_2) - 1] + [S_1(p_1) - 1][S_1(p_2) - 1]$$

$$+ [S_1(p_2) - 1][S_2(p_1) - 1] + [S_1(p_2) - 1][S_2(p_1) - 1]$$

$$+ [S_1(p_1) - 1][S_2(p_1) - 1]$$

$$= (\delta_{p_1,k_1} \delta_{p_2,k_2} + \delta_{p_1,k_2} \delta_{p_2,k_1}) S_1(p_1) S_2(p_1) S_2(p_2) S_1(p_2) = \sum^{(1) \text{red}}_{p_1,k_1} S^{(1)}_{p_1,k_1} S^{(1)}_{p_2,k_2} S^{(1)}_{p_2,k_2}.$$  

(112)

We now prove that the two-photon scattering matrix on two emitters can be represented as a convolution of the two-photon scattering matrices on individual emitters, that is,

$$S^{(2)}_{p_1,p_2,k_1,k_2} = \frac{1}{2!} \int dk_1' dk_2' S^{(2)}_{2;p_1,p_2,k_1,k_2} S^{(2)}_{1;k_1,k_2,k_1,k_2}$$

$$= \frac{1}{2!} \int dk_1' dk_2' \left( S^{(1)}_{2;p_1,k_1} S^{(1)}_{2;p_2,k_2} + S^{(1)}_{2;p_2,k_2} S^{(1)}_{2;p_1,k_1} + i T^{(2)}_{2;p_1,p_2,k_1,k_2} \right)$$

$$\times \left( S^{(1)}_{1;k_1,k_1} S^{(1)}_{1;k_2,k_2} + S^{(1)}_{1;k_2,k_2} S^{(1)}_{1;k_1,k_1} + i T^{(2)}_{1;k_1,k_2,k_1,k_2} \right)$$

$$= (\delta_{p_1,k_1} \delta_{p_2,k_2} + \delta_{p_1,k_2} \delta_{p_2,k_1}) S_2(p_1) S_2(p_2) S_1(p_1) S_1(p_2) + S_2(p_1) S_2(p_2) i T_{1;p_1,p_2,k_1,k_2}$$

$$+ S_1(k_1) S_1(k_2) i T_{2;p_1,p_2,k_1,k_2} - \frac{1}{2} \int dk_1' dk_2' T^{(2)}_{2;p_1,p_2,k_1,k_2} T^{(2)}_{1;k_1,k_2,k_1,k_2}.$$  

(113)
The first term is obviously equal to the reducible contribution (112). We now show that the second and the third terms together yield $S^{(2)\text{irred}}_{p_1p_2,k_1k_2}$. We get

$$4\delta_{p_1+p_2,k_1+k_2} \left[ \frac{(2\pi i g_1^2)^2}{4\pi i} (E - 2\alpha_1) M_1(p_1) M_1(p_2) M_1(k_1) M_1(k_2) S_2(p_1) S_2(p_2) 
+ \frac{(2\pi i g_2^2)^2}{4\pi i} (E - 2\alpha_2) M_2(p_1) M_2(p_2) M_2(k_1) M_2(k_2) S_1(k_1) S_1(k_2) 
- \frac{1}{2} (4\pi)^2 S_1 g_1^2 S_2 g_2^2 \delta_{p_1+p_2,k_1+k_2} (E - 2\alpha_1)(E - 2\alpha_2) M_2(p_1) M_2(p_2) M_1(k_1) M_1(k_2) 
\times \int dk_1' M_2(k_1') M_2(E - k_1') M_1(k_1') M_1(E - k_1). \right]$$

Evaluating the last integral we find

$$\int dk_1' M_2(k_1') M_2(E - k_1') M_1(k_1') M_1(E - k_1') = - \frac{4\pi i}{(E - \alpha_1 - \alpha_2)(E - 2\alpha_1)(E - 2\alpha_2)}, \quad (115)$$

and therefore

$$4\delta_{p_1+p_2,k_1+k_2} \left[ \frac{(2\pi i g_1^2)^2}{4\pi i} (E - 2\alpha_1) M_1(p_1) M_1(p_2) M_1(k_1) M_1(k_2) S_2(p_1) S_2(p_2) 
+ \frac{(2\pi i g_2^2)^2}{4\pi i} (E - 2\alpha_2) M_2(p_1) M_2(p_2) M_2(k_1) M_2(k_2) S_1(k_1) S_1(k_2) 
- \frac{(2\pi i g_1^2)^2 (2\pi i g_2^2)^2}{2\pi i} \delta_{p_1+p_2,k_1+k_2} \frac{M_2(p_1) M_2(p_2) M_1(k_1) M_1(k_2)}{E - \alpha_1 - \alpha_2} \right] = S^{(2)\text{irred}}_{p_1p_2,k_1k_2}, \quad (116)$$

which reproduces the result contained in equations (110) and (111).

Note that the sequence of $S_1$ and $S_2$ in the convolution (113) corresponds to the order in which the right-moving photons encounter the atoms 1 and 2 along the line of propagation. The presence of the irreducible part $T^{(2)}$ in the two-photon scattering makes the convolution (113) non-commutative $S_2^{(2)} * S_1^{(2)} \neq S_1^{(2)} * S_2^{(2)}$, in contrast to the case of single-photon scattering (68), where the scattering is commutative, $S_2^{(1)} * S_1^{(1)} = S_1^{(1)} * S_2^{(1)}$.

The result (110)–(112) generalizes the corresponding expressions of [9] to arbitrary coupling strengths $g_1 \neq g_2$. Applying the convolution property, one can also write an expression for the two-photon scattering on $M$ emitters with different $g_i$’s.

The calculations outlined above allow us to make the following general statement: the two-particle scattering matrix on an array of distributed emitters is a momentum space convolution of scattering matrices on individual emitters. This property has a fundamental cause: the absence of back-scattering for unidirectionally propagating photons. Therefore, it is also expected to hold in the arbitrary $N$-photon sector of scattering.

5. Coherent light scattering

In this section we consider the scattering of coherent light off a single two-level emitter. The aim of this calculation is twofold: firstly, we describe the specific physical situation relevant to a discussion of the resonant fluorescence [17] in nanostructures, and, secondly, we provide a generating functional for a $N$-particle scattering matrix.
We start by introducing operators of photons propagating inside a waveguide of a finite length \( L \)

\[
c_{k} = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{L/2} \text{d}x a(x) e^{-ikx}, \quad c_{k}^{\dagger} = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{L/2} \text{d}x a^{\dagger}(x) e^{ikx},
\]

where \( a(x) \) and \( a^{\dagger}(x) \) are the Fourier transforms of the operators \( a_{k} \) and \( a_{k}^{\dagger} \)

\[
a(x) = \frac{1}{\sqrt{2\pi}} \int \text{d}k a_{k} e^{ikx}, \quad a^{\dagger}(x) = \frac{1}{\sqrt{2\pi}} \int \text{d}k a_{k}^{\dagger} e^{-ikx}.
\]

(117)

The construction (117) explicitly containing the finite length \( L \) is very useful as it allows for the treatment of different limits such as: a finite photon number limit \((\bar{N} \equiv |\alpha_{k}|^{2} = \text{const}, L \to \infty)\) and a finite photon density limit \((\bar{N}/L = \text{const at } \bar{N}, L \to \infty)\).

The operators (117) obey the commutation relations

\[
[c_{k}, c_{p}^{\dagger}] = [c_{k}, a_{p}^{\dagger}] = [a_{p}, c_{k}^{\dagger}] = \frac{1}{2\pi i} \frac{e^{i(k-p)L/2} - e^{-i(k-p)L/2}}{k-p} \equiv \delta_{\Delta}(k-p),
\]

(119)

\[
[a(x), c_{k}^{\dagger}] = \frac{e^{ikx}}{\sqrt{2\pi}} \Theta(-L/2 < x < L/2),
\]

(120)

where \( \delta_{\Delta}(k-p) \) is a delta function broadened by \( \Delta \equiv 2\pi/L \) and reaching the peak value \( 1/\Delta \) at \( k = p \). An expression of the operator \( c_{k} \) in terms of \( a_{k} \) is given by

\[
c_{k} = \int \text{d}p a_{p}\delta_{\Delta}(p-k).
\]

(121)

As an initial state we choose a coherent state in the mode \( k \)

\[
|\alpha_{k}\rangle = e^{-|\alpha_{k}|^{2}/2} \sum_{n_{k}=0}^{\infty} \left( \frac{2\pi}{L} \right)^{n_{k}/2} \frac{\alpha_{k}^{n_{k}} (c_{k}^{\dagger})^{n_{k}}}{n_{k}!} |0\rangle = e^{-|\alpha_{k}|^{2}/2} \sum_{n_{k}=0}^{\infty} \frac{\alpha_{k}^{n_{k}}}{\sqrt{n_{k}!}} |n_{k}\rangle,
\]

(122)

This state is normalized to unity, which is ensured by the finite waveguide’s length \( L \). It is a superposition of Fock states with different numbers of photons \( n_{k} \), and therefore we introduce the operator \( O \) which projects on-shell and accounts for the energy conservation in each photon’s number sector. An outgoing state, after the scattering off a two-level system, is thus obtained by applying the operator \( O \) followed by the action of the \( T \)-matrix, as prescribed by (19),

\[
|\beta_{k}\rangle = (1-2\pi i T O) |\alpha_{k}\rangle
\]

\[
|\beta_{k}\rangle = |\alpha_{k}\rangle - 2\pi i e^{-|\alpha_{k}|^{2}/2} \sum_{n_{k}=1}^{\infty} \left( \frac{2\pi}{L} \right)^{n_{k}/2} \alpha_{k}^{n_{k}} \sum_{i_{1}=1}^{n_{k}} \cdots \sum_{i_{1}=1}^{n_{k}} \int \left( \prod_{i=1}^{n_{k}} \text{d}q_{i} \text{d}p_{i} \right) \delta \left( \sum_{i=1}^{n_{k}} p_{i} - \sum_{i=1}^{n_{k}} q_{i} \right)
\]

\[
\times a_{p_{1}}^{\dagger} \frac{1}{kn_{k} - H_{b} - \alpha} a_{q_{1}}^{\dagger} \frac{1}{kn_{k} - H_{b} + i\eta} \cdots a_{p_{n_{k}}}^{\dagger} \frac{1}{kn_{k} - H_{b} - \alpha} a_{q_{n_{k}}}^{\dagger} \frac{1}{n_{k}!} |0\rangle
\]

\[
= |\alpha_{k}\rangle - 2\pi i e^{-|\alpha_{k}|^{2}/2} \sum_{n_{k}=1}^{\infty} \left( \frac{2\pi}{L} \right)^{n_{k}/2} \alpha_{k}^{n_{k}} \sum_{i_{1}=1}^{n_{k}} \cdots \sum_{i_{1}=1}^{n_{k}} \int \left( \prod_{i=1}^{n_{k}} \text{d}q_{i} \text{d}p_{i} \right) \delta \left( \sum_{i=1}^{n_{k}} p_{i} - \sum_{i=1}^{n_{k}} q_{i} \right)
\]

\[
\times a_{p_{1}}^{\dagger} \frac{1}{kn_{k} - H_{b} - \alpha} a_{q_{1}}^{\dagger} \frac{1}{kn_{k} - H_{b} + i\eta} \cdots a_{p_{n_{k}}}^{\dagger} \frac{1}{kn_{k} - H_{b} - \alpha} a_{q_{n_{k}}}^{\dagger} \frac{1}{n_{k}!} |0\rangle
\]

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Performing the inverse Fourier transformation (cf (118))

\[ a_p^\dagger = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x_i a_i^\dagger(x_i) e^{ipx_i}, \]

(126)

Exchanging the order of summations \( \sum_{n_k=1}^{\infty} \sum_{n_1=1}^{\infty} = \sum_{n_1=1}^{\infty} \sum_{n_k=1}^{\infty} \) and shifting \( n_k \to n_k + n \) we find

\[
|\beta_k\rangle = |\alpha_k\rangle - 2\pi i e^{-|\alpha_k|^2/2} \sum_{n_k} \sum_{n_1=0}^{\infty} \left( \frac{2\pi}{L} \right)^n \alpha_k^{n_k} \sum_{n_1=1}^{n_k} (c_i^\dagger)^{n_1-n} s^{2n_k} \alpha_k^{n_k} \int \left( \prod_{i=1}^{n_k} \mathrm{d}q_i \mathrm{d}p_i \right) \delta \left( \sum_{i=1}^{n_k} (p_i - q_i) \right) \left( \prod_{i=1}^{n} \delta(q_i - k) \right) \times a_p^\dagger \left[ q_1 + \sum_{i=1}^{n_k} (q_i - p_i) - \alpha \right] \left( q_n - p_n + \mathrm{i} \eta \right) a_{p_1}^\dagger \left( q_{n-1} + q_n - p_n - \alpha \right) |\alpha_k\rangle.
\]

(124)
and substituting it into (125), we get

\[
|\gamma_k^{(n)}\rangle = \frac{1}{(2\pi)^{n/2}} \int dx_1 \cdots dx_n \int dP_1 \cdots dP_n \int dq_1 \cdots dq_n \delta \left( \sum_{i=1}^{n} (p_i - q_i) \right) \left( \prod_{i=1}^{n} \delta_\Delta (q_i - k) \right) \\
\times \frac{1}{[q_1 + \sum_{i=2}^{n} (q_i - p_i) - \alpha] \cdot [\sum_{i=2}^{n} (q_i - p_i) + i\eta]} \cdots \\
\times \frac{1}{[q_{n-1} + q_n - p_n - \alpha] \cdot [q_n - p_n + i\eta]} \frac{1}{q_n - \alpha} e^{i\sum_{i=1}^{n} p_i \cdot x_i} a^\dagger(x_1) \cdots a^\dagger(x_n) |\alpha_k\rangle.
\]

(127)

Introducing new variables \( P_j = \sum_{i=j}^{n} p_i \) and \( Q_j = \sum_{i=j}^{n} q_i \), so that \( p_i = P_i - P_{i+1} \) and \( q_i = Q_i - Q_{i+1} \), we assign (127) to

\[
|\gamma_k^{(n)}\rangle = \frac{1}{(2\pi)^{n/2}} \int dx_1 \cdots dx_n \int dP_1 \cdots dP_n \int dq_1 \cdots dq_n \left( P_1 - Q_1 \right) \left( \prod_{i=1}^{n} \delta_\Delta (q_i - k) \right) \\
\times e^{i\sum_{i=1}^{n} P_i \cdot x_i} \frac{1}{[Q_1 - P_2 - \alpha] \cdot [Q_2 - P_2 + i\eta]} \cdots \frac{1}{[Q_{n-1} - P_n - \alpha] \cdot [Q_n - P_n + i\eta]} \\
\times e^{i\sum_{i=1}^{n} P_n \cdot x_n} \frac{1}{Q_n - \alpha} a^\dagger(x_1) \cdots a^\dagger(x_n) |\alpha_k\rangle \\
= \frac{1}{(2\pi)^{n/2}} \int dx_1 \cdots dx_n \int dq_1 \cdots dq_n \left( \prod_{i=1}^{n} \delta_\Delta (q_i - k) \right) e^{i\sum_{i=1}^{n} Q_1 \cdot x_i} \frac{1}{Q_n - \alpha} \\
\times \left( \int dP_2 e^{-i\sum_{i=1}^{n} P_2 \cdot x_2} \right) \cdots \\
\times \left( \int dP_n e^{-i\sum_{i=1}^{n} P_n \cdot x_n} \right) a^\dagger(x_1) \cdots a^\dagger(x_n) |\alpha_k\rangle.
\]

(128)

If one can close the contour of integration in the upper half-plane, each integral over \( dP_j \) does not vanish. It is only possible for \( \Delta x_j \equiv x_j - x_{j-1} > 0 \). Then we get

\[
\int dP_j \frac{e^{iP_j \Delta x_j}}{[Q_{j-1} - P_j - \alpha] \cdot [Q_j - P_j + i\eta]} = \Theta(x_j - x_{j-1}) 2\pi i e^{iQ_j \Delta x_j} \frac{e^{i(\alpha - Q_j) \Delta x_j} - 1}{Q_{j-1} - \alpha},
\]

and therefore

\[
|\beta_k\rangle = |\alpha_k\rangle + \sum_{n=1}^{\infty} (-2\pi i g^2 \alpha_k)^n \int dx_1 \cdots dx_n \int dq_1 \cdots dq_n \left( \prod_{i=1}^{n} \delta_\Delta (q_i - k) \right) \Theta(x_n > \cdots > x_1) \\
\times e^{i\sum_{i=1}^{n} q_i \cdot x_i} \frac{1}{q_n - \alpha} \prod_{j=2}^{n} \frac{1 - e^{i(\alpha - Q_j) \Delta x_j}}{q_{j-1} - \alpha} a^\dagger(x_1) \cdots a^\dagger(x_n) |\alpha_k\rangle
\]

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= |\alpha_k| + \sum_{n=1}^{\infty} (-2\pi i g^2 \tilde{\alpha}_k)^n \int dx_1 \cdots dx_n \Theta(x_n > \cdots > x_1) \\
\times \left( \int dq_j \frac{e^{i(q_j-x_j)} e^{i\Delta x_{j+1}} \delta(q_j-x_j)}{q_j - \alpha} \right) \left( \prod_{j=1}^{n-1} \int dq_j \frac{e^{i(q_j-x_j)} 1 - e^{i(q_j-x_j)\Delta x_{j+1}}}{q_j - \alpha} \delta(q_j-x_j) \right) \\
\times \alpha(x_1) \cdots \alpha(x_n)|\alpha_k|)
\tag{130}

where \(\tilde{\alpha}_k = \alpha_k/\sqrt{L}\).

We now need to evaluate the integrals
\[
\int dq_j \frac{e^{i(q_j-x_j)} e^{i\Delta x_{j+1}} \delta(q_j-x_j)}{q_j - \alpha} = \frac{e^{ikx_j}}{4\pi i} \int dq_j \frac{e^{i(q_j-x_j)\Delta x_{j+1}}}{q_j + k - \alpha} \left[ \frac{1}{q_j + i0^+} + \frac{1}{q_j - i0^+} \right]
\tag{131}
\]

and
\[
- \int dq_j \frac{e^{i(q_j-x_j)\Delta x_{j+1}} e^{i(q_j-x_j)\Delta x_{j+1}} \delta(q_j-x_j)}{q_j - \alpha} = - e^{-i\alpha \Delta x_{j+1}} \int dq_j \frac{e^{i(q_j-x_j)\Delta x_{j+1}}}{q_j - \alpha} \delta(q_j-x_j)
\tag{132}
\]

Summing up the both terms (131) and (132) gives
\[
\Theta(x_{j+1} - x_j) \int dq_j \frac{e^{i(q_j-x_j)[1 - e^{i(q_j-x_j)\Delta x_{j+1}}]}}{q_j - \alpha} \delta(q_j-x_j)
\]
\[
= \Theta(-L/2 < x_j < x_{j+1} < L/2) \frac{e^{ikx_j}}{k - \alpha} \left[ 1 - e^{-i(k-\alpha)\Delta x_{j+1}} \right] \\
+ \Theta(x_j < -L/2 < x_{j+1} < L/2) \frac{e^{ikx_j}}{k - \alpha} \left[ e^{-i(k-\alpha)(x_j+L/2)} - e^{i(k-\alpha)\Delta x_{j+1}} \right].
\tag{133}
\]

Note that there is no contribution to (133) when both \(x_j\) and \(x_{j+1}\) are smaller than \(-L/2\).
Thus, we finally get an expression for the state $|\beta_k\rangle$ which emerges after the scattering of the initially prepared coherent state $|\alpha_k\rangle$ off the two-level system

$$
|\beta_k\rangle = |\alpha_k\rangle + \sum_{n=1}^{\infty} \left( \frac{-2\pi ig^2\alpha_k}{k-\alpha} \right)^n \int dx_1 \cdots dx_n \ e^{ik(x_1+\cdots+x_n)}
$$

$$
\times \left[ \Theta(L/2 > x_n > \cdots > x_2 > x_1 > -L/2) \prod_{j=1}^{n} (1 - e^{i(k-\alpha)\Delta x_{j+1}}) 
\right.
$$

$$
\left. + \Theta(L/2 > x_n > \cdots > x_2 > -L/2 > x_1) e^{-i(k-\alpha)(x_1+L/2)} (1 - e^{i(k-\alpha)(x_2+L/2)}) \times \prod_{j=2}^{n} (1 - e^{i(k-\alpha)\Delta x_{j+1}}) \right] a^\dagger(x_1) \cdots a^\dagger(x_n)|\alpha_k\rangle,
$$

(134)

where $x_{n+1} \equiv L/2$.

The state (134) has a remarkable property: it vanishes when two coordinates in the integrand approach each other. This property lies in the origin of the anti-bunching of photons which is conventionally observed in the density–density correlation functions [8, 17]. An occurrence of the two contributions with $x_1 > -L/2$ and $x_1 < -L/2$ has already been observed in [8], both being important for a proper normalization of the state (134).

The explicit expression for the outgoing state (134) opens the possibility for a study of correlation functions as well as photons’ statistics in all parametric regimes, which will be the subject of a subsequent publication [18].

6. Conclusions

We have developed the scattering approach to problems of propagating bosons in one dimensional geometry. We have derived the general form of the $T$-matrix (the nontrivial part of the scattering matrix), equation (3), given the following assumptions about the spectrum of bosons: the spectrum is linear, chiral and infinite. Our formalism is complementary to the Bethe Ansatz solutions [7] and to traditional approaches based on equations of motion.

We have applied the developed formalism to several specific examples including emitters with two- and three-level structures. The emitters can be either distributed across the 1D channel or concentrated in a tiny region of space. We have shown that the scattering results in projecting the state of the emitter onto the specific—dark—state which does not emit.

We have also shown that the one- and two-particle scattering matrices off two emitters can be represented as a convolution of scattering matrices corresponding to individual emitters. Thus the microscopic properties of different emitters can vary (i.e. coupling constants to photons, detunings and level structure). We suggest that this property holds generically for multi-emitter arrays as well as for multi-particle scattering with $N > 2$ photons. We intend to elaborate further on these issues in future studies [18].

The developed approach can be applied to the arbitrary initial state, which can either conserve the particle number or not. In the case of the number-conserving initial state, we observe the formation of photonic bound states, which is reflected in the emergence of a pole of the $S$-matrix involving in its development the energy of more than one individual photon. In the
case of coherent light scattering, we clearly observe the fermionized behaviour typical of the Tonks–Girardeau gas [19] discussed in the photonic context in [20].

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