Collective excitations in a fermion-fermion mixture with different Fermi surfaces

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In this paper, collective excitations in a homogeneous fermion-fermion mixture with different Fermi surfaces are studied. In the Fermi liquid phase, the zero-sound velocity is found to be larger than the largest Fermi velocity. With attractive interactions, the superfluid phase appears below a critical temperature, and the phase mode is the low-energy collective excitation. The velocity of the phase mode is proportional to the geometric mean of the two Fermi velocities. The difference between the two velocities may serve as a tool to detect the superfluid phase.

I. INTRODUCTION

A lot of progress has been made in the area of low temperature Fermi gases in recent several years. Particularly, the quantum degenerate regime was reached \[1\] and Feshbach resonance was observed \[2\]. Currently, a lot of effort are concentrated on creating and studying the superfluid phase \[3\].

At low temperature, s-wave scattering is the dominant interaction between atoms. Interactions with higher angular momentum are ineffective for cooling. To take advantage of the s-wave interaction, the fermion system must contain more than one species. In most experiments so far, fermions are trapped in two different hyperfine-spin states. The two Fermi surfaces are usually different, which makes the system more complex.

Collective excitations are important properties of low-temperature systems. Landau predicted the existence of the zero sound in Fermi liquids. In \(^3\)He systems, sound modes are distinct signatures of exotic pairing phases. The collective excitations in Fermi gases have been extensively studied theoretically\[4, 5, 6, 7\]. However, so far, most studies assume the two species have the same Fermi surface, which is often not true in experiments.

In this paper, we study the collective excitations in the fermion-fermion mixture with different Fermi surfaces. For simplicity, we consider a homogeneous system. We find that the dispersion of collective excitations are affected by the Fermi velocities. The superfluid phase can be exclusively identified by its sound velocity as proposed in Ref.\[6, 7\].

The Hamiltonian describing the fermion-fermion mixture is given by

\[
H = \frac{\hbar^2}{2m_a} \nabla \psi_a^\dagger \cdot \nabla \psi_a + \frac{\hbar^2}{2m_b} \nabla \psi_b^\dagger \cdot \nabla \psi_b + g \psi_a^\dagger \psi_b^\dagger \psi_b \psi_a,
\]

where \(m_a\) and \(m_b\) are the masses of the \(a\)- and \(b\)-species. The single-particle dispersion is given by \(\epsilon_k^{a,b} = \frac{\hbar^2 k^2}{2m_{a,b}} - \mu_{a,b}\), where \(\mu_a\) and \(\mu_b\) are the chemical potentials of the two species. The coupling constant between the two species is given by \(g\). In this paper, only the s-wave scattering is considered, and the interaction between atoms of the same species is ignored.

To obtain the spectrum of collective excitations, we construct the kinetic equations,

\[
\frac{\partial n}{\partial t} = [n, H],
\]

where \(n\) is a 2 by 2 density matrix given by

\[
n_{kk'} = \begin{pmatrix} a_k^\dagger a_{k'} & a_k^\dagger b_{-k'} \\ b_{-k}^\dagger a_{k'} & b_{-k}^\dagger b_{-k'} \end{pmatrix}.
\]

In this paper, we consider the low-energy and low-temperature region, where the system is in the collisionless region and the collision integral can be ignored. The density fluctuation \(\delta n\) obeys the simple kinetic equation

\[
\omega \delta n_{kk'q} = \delta n_{kk'q}^0 - \epsilon_k^{a,b} \delta n_{kk'q} - \frac{\hbar^2 k^2}{2m_{a,b}} \delta n_{kk'q} + \frac{\hbar^2 k^2}{2m_{a,b}} \delta n_{kk'q},
\]

where \(n_{kk'}^0\) is the density in equilibrium, \(\epsilon_k^{a,b}\) is the mean-field energy, \(\delta \epsilon\) is the energy fluctuation, and they are all 2 by 2 matrices.

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In the low-frequency and long-wavelength limit, we apply gradient expansion to second order, and eq. (3) is reduced to

$$\omega \delta n_k = \left[ \delta n_k, \epsilon_k^0 \right] + \left[ \delta n_k, \frac{1}{2} q \cdot \nabla_k \epsilon_k^0 \right] + \left[ n_k^0, \delta \epsilon_k \right] - \left[ \delta \epsilon_k, \frac{1}{2} q \cdot \nabla_k n_k^0 \right] + \left[ \frac{1}{8} q \cdot q \frac{\partial^2 n_k^0}{\partial k_i \partial k_j}, \delta \epsilon_k \right],$$

(4)

where the second derivative of $\epsilon_k^0$ can be ignored because the fermi energies are usually much bigger than the pairing gap. The fluctuations in energy and density are also related through interaction,

$$\delta \epsilon_k^{(1),(2),(3)} = g \int \frac{d^3k'}{(2\pi)^3} \delta n_k^{(1),(2),(3)};$$

$$\delta \epsilon_k^{(0)} = -g \int \frac{d^3k'}{(2\pi)^3} \delta n_k^{(0)};$$

where $\delta \epsilon = \delta \epsilon^{(0)} \sigma_i$, $\delta n = \delta n^{(0)} \sigma_i$, $\sigma_i$'s are Pauli matrices, and $\sigma_0$ is the identity matrix. As a consequence of the s-wave scattering, $\delta \epsilon_k$ is independent of $k$. So in the following, we omit its subscript $k$.

**II. ZERO SOUND**

In the Fermi-liquid phase, the low-energy collective excitations come from density fluctuation. Since the pairing fluctuation is massive, the off-diagonal matrix elements $\epsilon^{(1),(2)}$ and $n^{(1),(2)}$ can be ignored. The mean-field energy and density are given by

$$\epsilon_k^0 = \begin{pmatrix} \epsilon_k^a & 0 \\ 0 & -\epsilon_k^b \end{pmatrix},$$

$$n_k^0 = \begin{pmatrix} f(\epsilon_k^a) & 0 \\ 0 & 1 - f(\epsilon_k^b) \end{pmatrix},$$

where $f(\epsilon) = 1/(1 + e^{\beta \epsilon})$ is the fermi function.

At low temperature, $\nabla_k f(\epsilon_k^{a,b}) \approx \delta(\epsilon_k^{a,b}) v_F^{a,b} \hat{k}$, the kinetic equations are approximately given by

$$(\omega - v_F^{a,b} \hat{k} \cdot q) \delta n_k^{a,b} = v_F^{a,b} \hat{k} \cdot q \delta(\epsilon_k^{a,b}) \delta \epsilon^{a,b},$$

(5)

where $\delta \epsilon^{a,b} = \delta \epsilon^{(3)} \pm \delta \epsilon^{(0)}$, $\delta n^{a,b} = \delta n^{(3)} \pm \delta n^{(0)}$, and $v_F^{a,b}$ are the Fermi velocities. Since the density fluctuation is always around the Fermi surfaces, it is a good approximation to assume $\delta n_k^{a,b} = \delta n_k^{a,b} v_k^{a,b}$, where $v_k^{a,b}$ are functions of $\hat{k}$, $q$, and $\omega$. The kinetic equations are now reduced to

$$(\omega - v_F^{a,b} \hat{k} \cdot q) v_k^{a,b} = v_F^{a,b} \hat{k} \cdot q \delta(\epsilon_k^{a,b}),$$

(6)

$$\delta \epsilon^{a,b} = g \int \frac{dk^3}{(2\pi)^3} v_k^{a,b} \delta \epsilon^{b,a}. $$

(7)

The above equations can be further simplified to the following form

$$1 = g^2 D_a(x_a) D_b(x_b),$$

(8)

where

$$D_{a,b}(x_a, b) = N_{a,b}(0) \int \frac{\hat{q} \cdot \hat{k} dk}{4\pi (x_{a,b} - \hat{q} \cdot \hat{k})},$$

$x_{a,b} = \omega/(v_F^{a,b} q)$, and $N_{a,b}(0)$ are the densities of states of the two species. The function $D_a(x_a)$ has an imaginary part when $x_a < 1$. Therefore Eq. (8) has an undamped solution only when

$$\omega > \max(v_F^{a,b} q).$$

(9)

In the weak coupling limit, the zero sound velocity is approximately given by the largest Fermi velocity,

$$\omega \approx \max(v_F^{a,b} q).$$

(10)

For stronger couplings, the difference between the sound velocity and the largest Fermi velocity becomes bigger.
III. PHASE MODE

With attractive interactions, \( g < 0 \), the system can go into a pairing phase below a critical temperature. The fermion excitations have a gap \( \Delta \) which can be obtained from the gap equation in the standard BCS formulism,

\[
1 = -\frac{g}{\mathcal{V}} \sum_{\mathbf{k}} \frac{\tanh \left[ \frac{\mathcal{E}_k}{2} / (E_k + \epsilon_k) \right] + \tanh \left[ \frac{\mathcal{E}_k}{2} / (E_k - \epsilon_k) \right]}{4E_k},
\]

where \( E_k = \sqrt{\epsilon_k^2 + \Delta^2} \), \( \epsilon_k^\pm = (\epsilon_k \pm \epsilon_k^2)/2 \), and \( \Delta \) is positive for simplicity. The fermion excitations in this phase have two branches with dispersions given by \( \pm \theta(\epsilon_k - |\epsilon_k|) \).

At zero temperature, there is no pairing contribution from the region where \( E_k < |\epsilon_k| \), as shown in the gap equation,

\[
1 = -\frac{g}{\mathcal{V}} \sum_{\mathbf{k}} \frac{\theta(E_k - |\epsilon_k|)}{2E_k}.
\]

When the two Fermi surfaces are different, there is no infra-red divergence on the right-hand side of Eq. (12) in the limit \( \Delta \to 0 \). The coupling constant has to be bigger than a critical value, \( g > g_c \), for the pairing phase to be stable. In contrast, when the two Fermi surfaces are identical, the critical coupling constant is zero, \( g_c = 0 \), and the pairing of fermions is stronger.

In the following, we consider the case \( g > g_c \) and at zero temperature for simplicity. The mean-field energy and density matrix are given by

\[
\epsilon_k^0 = \epsilon_k^I + \epsilon_k^\sigma_3 + \Delta \sigma_1, \\
n_k^0 = \frac{1}{2} I - \Delta \theta_k \sigma_1 - \phi_k \sigma_3,
\]

where \( \theta_k = \frac{1}{2\pi E_k} \) and \( \phi_k = \epsilon_k^+ \theta_k \). The kinetic equation Eq. (11) is now given by

\[
\Omega_k \delta n_k = M_k \delta \epsilon,
\]

where \( \Omega_k \) and \( M_k \) are 4 by 4 matrices given by

\[
\Omega_k = \begin{pmatrix}
\omega - \eta_k^- & 0 & 0 & -\eta_k^+
0 & \omega - \eta_k^- & -2i \epsilon_k^+ & 0
0 & -2i \epsilon_k^- & \omega - \eta_k^- & -2i \Delta
-\eta_k^+ & 0 & 2i \Delta & \omega - \eta_k^-
\end{pmatrix},
\]

\[
M_k = \begin{pmatrix}
0 & \eta_k^+ \theta_k^+ \Delta & 0 & \eta_k^+ \phi_k^+
\eta_k^- \theta_k^+ \Delta & 0 & 2i \phi_k^+ + i \eta_k^\sigma_2 \phi_k^\sigma_1 & 0
0 & -2i \phi_k - i \eta_k^\sigma_2 \phi_k & 0 & 2i \Delta \theta_k + \frac{1}{2} \eta_k^\sigma_2 \Delta \phi_k^\sigma_1
\eta_k^- \phi_k^+ & 0 & -2i \Delta \theta_k - \frac{i}{2} \eta_k^\sigma_2 \Delta \phi_k^\sigma_1 & 0
\end{pmatrix},
\]

where \( \eta_k^\pm = v_F \cdot \mathbf{k} \cdot \eta_k^\sigma_1 \), \( \phi_k^\sigma_1 = \frac{d \phi_k}{d x_k} \), \( \phi_k^\sigma_2 = \frac{d \phi_k}{d x_k} \), \( \theta_k^\sigma_1 \) and \( \theta_k^\sigma_2 \) are similarly defined. The kinetic energy \( \epsilon_k^+ \) is linearized around the place where \( \epsilon_k^+ = 0 \), \( \nabla_k \epsilon_k^\pm = v_F \cdot \mathbf{k} \). The curvature of \( \epsilon_k^\pm \) is negligible as long as the chemical potentials of the two species are much larger than the pairing gap. To a good approximation, \( v_F^\pm = (v_F^x \pm v_F^y)/2 \). The fluctuations in energy and density are related through interaction

\[
\delta \epsilon = \lambda g \int \frac{d^3k}{(2\pi)^3} \delta n_k,
\]

where \( \lambda \) is the 4 by 4 matrix given by

\[
\lambda = \begin{pmatrix}
-1 & 0 & 0 & 0
0 & 1 & 0 & 0
0 & 0 & 1 & 0
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Using Eq. (13) and Eq. (14), we obtain the equation for the dispersion of the collective mode
\[
\det |I - \lambda g \int \frac{d^3 k}{(2\pi)^3} \Omega_k^{-1} M_k| = 0. \tag{15}
\]
To the leading nontrivial order of \(\omega\) and \(q\), the above equation is reduced to
\[
\det \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -g N_+(0) & 0 & 0 \\
0 & 0 & \frac{g N_+(0)}{4\Delta^2} (\omega^2 - \frac{1}{3} v_a^2 v_b^2 q^2) & \frac{i g N_+(0)}{2\Delta} \omega \\
0 & 0 & -\frac{i g N_+(0)}{2\Delta} \omega & 1 + g N_+(0)
\end{bmatrix} = 0, \tag{16}
\]
where \(N_+(0)\) is the density of states of \(\epsilon_k\), approximately given by \(N_+(0) = 2N_a(0)N_b(0)/[N_a(0) + N_b(0)]\). As shown in Eq. (16), the spin fluctuation and the pairing amplitude fluctuation are decoupled from the rest fluctuations. The uniform density fluctuation is closely coupled to the pairing phase fluctuation.

The dispersion of the phase mode to the leading order of \(q\) is given by
\[
\omega = q \sqrt{\frac{1}{3} v_a^2 v_b^2 (1 + g N_+(0))}. \tag{17}
\]
In the case of weak coupling, the phase-mode velocity is noticeably smaller than the zero-sound velocity given by Eq. (10). This difference between the two velocities is large enough to be used in detecting the existence of the paring state, as propose in Refs. [6, 7]. Further work for the case of trapped systems are needed to compare with experiments.

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