Gradient modification of Newtonian gravity

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A second gradient generalization of Newtonian gravity is presented within the framework of gradient field theory. Weak nonlocality is introduced via first and second gradients of the gravitational field strength in the Lagrangian density. Gradient generalizations of the Poisson equation of Newtonian gravitation for the gravitational potential and of the generalized Gauss law for the gravitational field strength are presented. Such a gradient modification of Newtonian gravity provides a straightforward regularization of Newtonian gravity removing the classical Newtonian singularities. Finite gradient modifications of the gravitational potential energy and of the gravitational force law are constructed, with a possible connection to Yukawa interaction, and as suitable candidates for experimental tests of Newton’s inverse-square law at short-distances. In addition, nonlocal gravity of exponential type is investigated and its relation to gradient gravity theory is given.

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I. INTRODUCTION

During the last years, cosmological observations have shown that 70% of all the mass and energy of the Universe is related to a mysterious “dark energy” leading to a repulsive gravitational effect (see also [1, 2]). Only 5% of the mass of the Universe is in the form of baryons, while 25% is in the form of “dark matter”. The “dark energy” and “dark matter” play an important role in the evolution of the Universe. New physics phenomena due to the “dark energy”, “dark matter” and extra dimensions of M-theory could occur below the length scale associated with dark energy \( \lambda_d \approx 85 \mu m \) and may modify the gravitational inverse-square law.

Moreover, the singularities in Newtonian gravity indicate the limits of applicability of the theory. For this reason, they represent an important motivation to study generalized theories of gravitation (e.g. nonlocal gravity, gradient gravity, higher derivative quantum gravity) to make the theory free of singularities. The characteristic singularities appearing for a point mass in classical Newtonian gravity are 1/\( R \) and 1/\( R^2 \)-singularities in the Newtonian potential and Newtonian force, respectively.

Therefore, there is a strong interest for studying generalizations of and deviations from Newtonian gravity at small scales (see, e.g., [3–5]). It should be emphasized that we are not concerned with the known deviations from Newtonian gravity in strong gravitational fields or in relativistic systems at cosmic scales, which are described by Einstein’s general relativity and its generalizations. Our focus will be exclusively on possible derivations that arise in systems that should be described by Newtonian physics at short-distances.

From the theoretical point of view, there are interest-
Thus, the current experimental tests by Lee et al. [16] and Tan et al. [17] of Newton’s inverse-square law are in agreement with Newtonian gravity down to 52 µm and 48 µm, respectively. However, Newton’s inverse-square law of the gravitational force possesses a 1/R^2-singularity which should be modified at short-distances towards a singularity-free gravitational force expression. As always in theoretical physics, the range where the singularity becomes dominant shows that such theory is not valid at this range and must be modified or at least regularized. Therefore, the classical Newtonian singularities present in Newtonian gravity indicate the limits of the applicability of Newton’s theory of gravity at short-distances. Non-singular versions of Newtonian gravity may be nonlocal and gradient modifications of Newtonian gravity delivering easy-to-use singularity-free analytical expressions for the gravitational force (modified Newtonian force) and the modified Newtonian potential depending on characteristic length scale parameters, which might be used and determined in experiments for fitting data in the search and test of a modified Newtonian force or a modified Newtonian potential at short-distances. The appearing characteristic length scale parameters determine the range of the modification in the near field, and from the mathematical point of view they have the meaning of regularization parameters. Gradient modification of Newtonian gravity is in full agreement with Newtonian gravity in the far field. In the near field, gradient modification of Newtonian gravity provides a straightforward regularization based on higher order partial differential equations. Hence, a gradient modification of Newtonian gravity provides a regularization of Newtonian gravity at short-distances similar to the Pauli-Villars regularization in quantum electrodynamics. The regularized versions of the gravitational force and gravitational potential given in this paper can be used and tested in short-range gravity experiments.

The aim of the present work is to derive a modified Newtonian gravity based on gradient field theory which is a singularity-free generalized continuum theory valid at short-distances. Such a gradient modification of Newtonian gravity is nothing but a straightforward regularized version of Newtonian gravity. In particular, using such a gradient gravity we find that gravity weakens at short-distances, so that no singularity is created. Moreover, we want to find the gradient modifications of the gravitational potential energy and of the gravitational force law, with a possible connection to a Yukawa interaction.

The outline of this paper is as follows. In Section II, the theory of second gradient modification of Newtonian gravity including the generalized Gauss law for gravity is presented. In Section III, we give the relevant Green function and its first gradient. In Section IV, the non-singular gravitational fields of a point mass, the modified gravitational potential energy and modified gravitational force law are computed in the framework of second gradient gravity. The limit of those gravitational fields to the first gradient gravity and to the classical Newtonian gravity are given in Section V and Section VI, respectively. In Section VII, a nonlocal modification of exponential type of Newtonian gravity is investigated and the relation to gradient modifications of Newtonian gravity is given. The conclusions are given in Section VIII.

II. SECOND GRADIENT MODIFICATION OF NEWTONIAN GRAVITY

In this Section, we provide the field-theoretical framework of second gradient modification of Newtonian gravity. The Lagrangian density for second gradient modification of Newtonian gravity is given by

\[ \mathcal{L} = \frac{1}{8\pi G} \left( g \cdot g + \ell_1^2 \nabla g : \nabla g + \ell_2^4 \nabla \nabla g : \nabla \nabla g \right) + \rho \Phi \]

(3)

with the notation: \( g \cdot g = \rho g_0 \), \( \nabla g : \nabla g = \partial_j g_0 \partial_j g_i \), and \( \nabla \nabla g : \nabla \nabla g = \partial_k \partial_j g_0 \partial_k \partial_j g_i \). Here \( \rho \) denotes the gravitational field strength vector, \( \Phi \) is the gravitational potential, and \( \rho \) is the mass density. Moreover, \( \ell_1 \) and \( \ell_2 \) are two (real) internal characteristic length scale parameters of second gradient modification of Newtonian gravity and \( \nabla \) is the del operator. In addition to the classical term, first and second spatial derivatives of the gravitational field strength \( g \) multiplied by the characteristic lengths \( \ell_1 \) and \( \ell_2 \) appear in Eq. (3) which describe a weak nonlocality in space. In general, gradient gravity is a local theory with a finite number of derivatives of the gravitational field strength.

The gravitational field strength \( g \) can be written as gradient of a scalar potential, called the gravitational potential:

\[ g = -\nabla \Phi . \]

(4)

Therefore, the gravitational field strength is irrotational and satisfies the gravitational Bianchi identity

\[ \nabla \times g = 0 . \]

(5)

The Euler-Lagrange equation derived from the Lagrangian density (3) reads

\[ \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \Phi} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \Phi)} + \nabla : \frac{\partial \mathcal{L}}{\partial (\nabla \nabla \Phi)} - \nabla \nabla : \frac{\partial \mathcal{L}}{\partial (\nabla \nabla \nabla \Phi)} = 0 \]

(6)

and gives the inhomogeneous field equation of second gradient modification of Newtonian gravity

\[ L(\Delta) \nabla \cdot g = -4\pi G \rho , \]

(7)

where the differential operator of fourth order is given by

\[ L(\Delta) = 1 - \ell_1^2 \Delta + \ell_2^4 \Delta^2 . \]

(8)
Here $\Delta$ denotes the Laplacian. Eq. (7) represents the
generalized Gauss law for gravity.

By substituting Eq. (4) into Eq. (7), a modified Poisson
equation follows for the gravitational potential $\Phi$:

$$L(\Delta) \Delta \Phi = 4\pi G \rho,$$

which is a partial differential equation of sixth order.

Using the factorization of differential operators of
higher order into a product of differential operators of
lower order, the differential operator of fourth order (8)
can be factorized into a product of two differential op-

erators of second order, namely two Helmholtz operators
with characteristic length scale parameters $a_1$ and $a_2$:

$$L(\Delta) = (1 - a_1^2 \Delta)(1 - a_2^2 \Delta)$$

with

$$\ell_i^2 = a_i^2 + a_2^2, \quad i = 1, 2,$$

and

$$a_{1,2}^2 = \frac{\ell_1^4}{2} \left(1 \pm \sqrt{1 - 4 \frac{\ell_2^4}{\ell_1^4}}\right).$$

The differential operator (10) may be called bi-Helmholtz
operator.

The two length scales $a_1$ and $a_2$ might be real or com-
plex. The condition for the character, real or complex, of
the two lengths $a_1$ and $a_2$ is the condition for the dis-

criminant in Eq. (13), $1 - 4 \ell_2^4/\ell_1^4$, to be positive or negative.

Depending on the character of the two length scales $a_1$
and $a_2$, it can be distinguished between three cases:

1. $\ell_1^4 > 4 \ell_2^4$: real case

The two length scales $a_1$ and $a_2$ are real and distinct
and they read

$$a_{1,2} = \ell_1 \sqrt{\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4 \left(\frac{\ell_2}{\ell_1}\right)^4}},$$

satisfying the condition $a_1 > a_2$. The limit from second
gradient theory to first gradient theory is given by:
$\ell_2^4 \to 0$.

2. $\ell_1^4 = 4 \ell_2^4$: real degenerate case

The length scales $a_1$ and $a_2$ are real and equal

$$a_1 = a_2 = \frac{\ell_1}{\sqrt{2}} \equiv \ell_2.$$

There is no limit to first gradient theory.

3. $\ell_1^4 < 4 \ell_2^4$: complex conjugate case

The two length scales $a_1$ and $a_2$ are complex con-
jugate

$$a_{1,2} = A \pm iB,$$

with

$$A = \ell_2 \sqrt{\frac{1}{2} + \frac{\ell_2^2}{4 \ell_1^2}}, \quad B = \ell_2 \sqrt{\frac{1}{2} - \frac{\ell_2^2}{4 \ell_1^2}},$$

where $A > 0$ and $B > 0$. There is no limit to first
gradient theory.

III. GREEN FUNCTION IN SECOND GRADIENT MODIFICATION OF NEWTONIAN GRAVITY

In second gradient modification of Newtonian gravity,
the necessary Green function $G^{L\Delta}$ of the sixth order dif-

erential operator $L(\Delta)$ is defined by

$$L(\Delta) G^{L\Delta}(\mathbf{R}) = \delta(\mathbf{R}),$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\delta$ is the Dirac delta-function. The
Green function of Eq. (18) is given by (see, e.g., [18, 19])

$$G^{L\Delta}(\mathbf{R}) = \frac{1}{4\pi R} f_0(\mathbf{R}, a_1, a_2),$$

where $f_0(\mathbf{R}, a_1, a_2)$ is a characteristic auxiliary function.

For the case (1), the auxiliary function reads

$$f_0(\mathbf{R}, a_1, a_2) = 1 - \frac{1}{a_1^2 - a_2^2} \left[ a_1^2 e^{-R/a_1} - a_2^2 e^{-R/a_2} \right].$$

The series expansion (near field) of the auxiliary func-
tion (20) reads as

$$f_0(\mathbf{R}, a_1, a_2) = \frac{1}{(a_1 + a_2)} R + \mathcal{O}(R^3).$$

Therefore, the function $f_0(\mathbf{R}, a_1, a_2)$ regularizes up to a
$1/R$-singularity towards a non-singular field expression.
Indeed, the Green function (19) is non-singular and finite
at $R = 0$, namely

$$G^{L\Delta}(0) = -\frac{1}{4\pi(a_1 + a_2)}. (22)$$

For the case (2), the auxiliary function (20) becomes

$$f_0(\mathbf{R}, a_1, a_1) = 1 - \left[ 1 + \frac{R}{2a_1} \right] e^{-R/a_1},$$

with the near field behaviour

$$f_0(\mathbf{R}, a_1, a_1) = \frac{1}{2a_1} R + \mathcal{O}(R^3).$$

For the case (3), the auxiliary functions (20) reduces to

$$f_0(\mathbf{R}, a_1, a_2) = 1 - \left[ \cos(b R) + \frac{A^2 - B^2}{2AB} \sin(b R) \right] e^{-a R},$$

(25)
which is a real quantity with \( a = A/\ell_2^2 \) and \( b = B/\ell_2^2 \), and the near field behaviour reads as

\[
f_0(R, a_1, a_2) = \frac{1}{2A} R + \mathcal{O}(R^3). \tag{26}
\]

Therefore, the Green function \( G^{L,\Delta}(R) \) is a real quantity and finite at \( R = 0 \) for the three cases (1), (2) and (3).

The first gradient of the Green function (19) reads

\[
\nabla G^{L,\Delta}(R) = \frac{1}{4\pi} \frac{R}{R^3} f_1(R, a_1, a_2) \tag{27}
\]

with the auxiliary function for the case (1)

\[
f_1(R, a_1, a_2) = 1 - \frac{1}{a_1^2 - a_2^2} \left[ a_1^2 e^{-R/a_1} - a_2^2 e^{-R/a_2} \right] - \frac{R}{a_1^2 - a_2^2} \left[ a_1 e^{-R/a_1} - a_2 e^{-R/a_2} \right]. \tag{28}
\]

The series expansion (near field) of the auxiliary function (28) reads as

\[
f_1(R, a_1, a_2) = \frac{1}{3a_1a_2(a_1 + a_2)} R^3 + \mathcal{O}(R^4). \tag{29}
\]

One can see that the function \( f_1(R, a_1, a_2) \) regularizes up to a \( 1/R^3 \)-singularity towards a non-singular field expression.

For the case (2), the auxiliary function (28) becomes

\[
f_1(R, a_1, a_1) = 1 - \left[ 1 + \frac{R}{a_1} + \frac{R^2}{2a_1^2} \right] e^{-R/a_1} \tag{30}
\]

with near field

\[
f_1(R, a_1, a_1) = \frac{1}{6a_1} R^3 + \mathcal{O}(R^4). \tag{31}
\]

For the case (3), the auxiliary function (28) reduces to

\[
f_1(R, a_1, a_2) = 1 - \left[ 1 + \frac{R}{2A} \right] \cos(bR) + \left( \frac{A^2 - B^2}{2AB} + \frac{R}{2B} \right) \sin(bR) e^{-\alpha R}, \tag{32}
\]

which is a real quantity with the near field

\[
f_1(R, a_1, a_2) = \frac{1}{6A(A^2 + B^2)} R^3 + \mathcal{O}(R^4). \tag{33}
\]

Therefore, \( \nabla G^{L,\Delta} \) is real, non-singular and is zero at \( R = 0 \) for the three cases (1), (2) and (3). A more detailed study of the three cases (1)–(3) can be found in [18] in the framework of gradient elasticity of bi-Helmholtz type.

The main feature of the case (3) is the presence of oscillating terms. Depending on which quantity is greater in the pair \( (a, b) \), the oscillating terms might be relevant in the Green function \( G^{L,\Delta} \) and its gradient. The characteristic length of the Yukawa potential is \( 1/a \) and the period of oscillating terms is \( 2\pi/b \). For \( a > b \), the oscillations can be smooth and not dominant. For \( a < b \), the period of oscillations is smaller than the range of the Yukawa factor and an oscillation can give a relevant contribution; but this case changes \( \ell_1^2 \) to \( -\ell_1^2 \) in the Lagrangian density (3) and in the differential operator (8).

It is interesting to mention, that a similar observation was done in the so-called sixth order gravity model with real and complex massive poles [20] (see also [21, 22]).

\[\text{IV. GRAVITATIONAL FIELDS IN SECOND GRADIENT MODIFICATION OF NEWTONIAN GRAVITY}\]

In second gradient modification of Newtonian gravity, the gravitational potential is the solution of the modified Poisson equation (9) for a given mass density

\[
\Phi = 4\pi G G^{L,\Delta} * \rho, \tag{34}
\]

where the Green function is given in Eq. (19) and \( * \) denotes the convolution in space. The gravitational field strength vector is nothing but the (negative) gradient of Eq. (34) and it reads

\[
g = -4\pi G \nabla G^{L,\Delta} * \rho, \tag{35}
\]

where the gradient of the Green function is given in Eq. (27).

\[\text{A. Gravitational fields of a point mass}\]

The mass density of a point mass located at the position \( r' \) is given by

\[
\rho = M \delta(r - r'), \tag{36}
\]

where \( M \) denotes the mass. \( r' \) is the position vector of the point charge and \( r \) is the field vector.

On the one hand, substituting Eq. (36) into Eq. (34) and performing the convolution, the gravitational potential of a point mass \( M \) reduces to

\[
\Phi = 4\pi GM G^{L,\Delta}. \tag{37}
\]

If we insert the Green function (19) into Eq. (37), the explicit expression for the modified gravitational potential of a point mass reads in terms of the auxiliary function (20)

\[
\Phi = -\frac{GM}{R} f_0(R, a_1, a_2). \tag{38}
\]

Using the near field of \( f_0 \), Eq. (21), one can see that the gravitational potential (38) is finite at \( R = 0 \), namely...
In second gradient modification of Newtonian gravity, the gravitational potential (38) of a point mass is a superposition of a (long-range) Newtonian potential $-\frac{GM}{R}$ and a (short-range) bi-Yukawa-type potential $\frac{GM}{(a_1^2-a_2^2)R}[a_2 e^{-R/a_1} - a_2 e^{-R/a_2}]$ leading to a non-singular short-distance modification of the classical Newtonian potential of a mass $M$ (see also Fig. 1). The cancellation or regularization of the $1/R$-Newtonian singularity in the gravitational potential is due to the opposite signs of the Newtonian potential and the bi-Yukawa-type potential. Therefore, Eq. (38) is the singularity-free modified Newtonian potential in second gradient modification of Newtonian gravity.

Moreover, it is interesting to note that the modified Newtonian potential (38) with two free parameters is a particular version of the modified Newtonian potential with four free parameters obtained by Accioly et al. [20] in sixth order gravity.

On the other hand, the gravitational field strength (35) of a point mass (36) reduces to

$$g = -4\pi G \nabla G^{L\Delta}.$$  \hspace{1cm} (40)

Using Eq. (27), Eq. (40) becomes

$$g = -\frac{GM}{R^2} f_1(R, a_1, a_2).$$  \hspace{1cm} (41)

Using the near field of $f_1$, Eq. (29), one can see that the gravitational field strength (41) is zero at $R = 0$. Due to its spherical symmetry, the gravitational field strength (41) of a point mass reduces to

$$g = -\frac{GM}{R^2} f_1(R, a_1, a_2) e_R,$$  \hspace{1cm} (42)

where $e_R = R/R$ is the radial unit vector. In general, the gravitational field strength (42) of a point mass is non-singular and possesses an extremum value near the origin.

### B. Modified gravitational potential energy

The gravitational field energy (potential energy) is given by

$$U_r = -\frac{1}{4\pi G} \int_{\mathbb{R}^3} (g \cdot g + \ell_1^2 \nabla g : \nabla g + \ell_2^4 \nabla \nabla g : \nabla \nabla g) dV$$

$$= -\frac{1}{4\pi G} \int_{\mathbb{R}^3} L(\Delta) g \cdot g dV$$

$$= -\frac{1}{4\pi G} \int_{\mathbb{R}^3} L(\Delta) \nabla \cdot g \Phi dV$$

$$= \int_{\mathbb{R}^3} \rho \Phi dV,$$  \hspace{1cm} (43)

where we have used integration by parts, Eq. (7), and that the surface terms vanish at infinity. Substituting Eq. (36) into Eq. (43), the modified Newtonian gravitational interaction energy of the two point masses $M$ and $M'$ becomes

$$U_{MM'} = M\Phi_{M'}$$

$$= -\frac{GM'M'}{R} f_0(R, a_1, a_2),$$  \hspace{1cm} (44)

which is finite due to the non-singular potential (38) (see also Fig. 1).

### C. Modified gravitational force law

We proceed to the determination of the short-distance behavior of the modified force law.

In second gradient modification of Newtonian gravity, after a straightforward calculation the gravitational force reads as

$$F = \int_{\mathbb{R}^3} \rho g dV,$$  \hspace{1cm} (45)

which is the force acting on a mass distribution of density $\rho$ in presence of a gravitational field strength $g$. Substituting Eqs. (36) and (41) into Eq. (45), the gradient modification of Newton’s inverse-square law of gravity is found as

$$F_{MM'} = Mg_{M'}$$

$$= -\frac{GM'M'}{R^3} f_1(R, a_1, a_2).$$  \hspace{1cm} (46)
The force \( F_{MM'} = -\frac{GM'}{R'^2} f_1(R, a_1, a_2) e_R \). (47)

The force (47) is plotted in Fig. 2. Eq. (47) represents the attractive central conservative force acting on the point mass \( M \) at \( r \) due to the presence of the point mass \( M' \) at \( r' \). It is interesting to note that the force (47) is a linear superposition of the attractive (long-range) Newtonian force \( -\frac{GM'}{R'^2} \) and a repulsive (short-range) bi-Yukawa-type force with spatial decay lengths \( a_1 \) and \( a_2 \). The cancellation or regularization of the \( 1/R^2 \)-Newtonian singularity in the gravitational force is due to the opposite signs of the classical Newtonian force and the bi-Yukawa-type force. The exponential decay in the bi-Yukawa term originates from the fading of spatial memory or weak nonlocality significant at short-distances. At small scales \( (R < 4a_1) \), Newton’s inverse-square law of gravity is strongly modified due to the bi-Yukawa-type force. Thus, objects separated by less than this distance \( (R < 4a_1) \) would feel a reduced gravitational attraction, and the force vanishes at \( R = 0 \) (see Fig. 2). May this repulsive force be related with the mysterious “dark energy”? On the other hand, on the scales where \( R > 4a_1 \) the bi-Yukawa-type force can be neglected and the force of gravity is then essentially Newtonian. The modified gravitational force law (47) might be used for the investigation of a violation of Newton’s inverse-square law at short-distances. When a test particle is approaching \( R = 0 \), the gravitational force (47) applied to it tends to zero because the repulsive bi-Yukawa force cancels the attractive Newtonian force (see Fig. 2).

Last but not least, it is interesting to note that the short-distance modification of Newton’s force law based on Eq. (47) (see Fig. 2) possesses a similar form as the modified force law in Sundrum’s “fat graviton” model [23].

\section{V. GRAVITATIONAL FIELDS IN FIRST GRADIENT MODIFICATION OF NEWTONIAN GRAVITY}

Now, we perform the limit from second gradient modification of Newtonian gravity to first gradient modification of Newtonian gravity which is given by: \( \ell_1^2 \rightarrow 0 \), and therefore \( a_1 \rightarrow \ell_1, a_2 \rightarrow 0 \). In this limit, the auxiliary functions (20) and (28) reduce to

\begin{align*}
  f_0(R, \ell_1) &= 1 - e^{-R/\ell_1}, \quad (48) \\
  f_1(R, \ell_1) &= 1 - \left[ 1 + \frac{R}{\ell_1} \right] e^{-R/\ell_1}. \quad (49)
\end{align*}

The near fields of the auxiliary functions (48) and (49) read

\begin{align*}
  f_0(R, \ell_1) &= \frac{1}{\ell_1} R - \frac{1}{2\ell_1^2} R^2 + \mathcal{O}(R^3), \quad (50) \\
  f_1(R, \ell_1) &= \frac{1}{2\ell_1} R^2 - \frac{1}{3\ell_1^2} R^3 + \mathcal{O}(R^4). \quad (51)
\end{align*}

In Eqs. (50) and (51), it can be seen that the function \( f_0(R, \ell_1) \) regularizes up to a \( 1/R \)-singularity towards a non-singular field expression, and the function \( f_1(R, \ell_1) \) regularizes up to a \( 1/R^2 \)-singularity towards a non-singular field expression.

In this limit, the modified gravitational potential (38) reduces to

\[ \Phi = -\frac{GM}{R} f_0(R, \ell_1), \quad (52) \]

which is finite at \( R = 0 \), namely (see Fig. 1)

\[ \Phi(0) = -\frac{GM}{\ell_1}. \quad (53) \]

In first gradient modification of Newtonian gravity, the gravitational potential (52) of a point mass is a superposition of the (long-range) Newtonian term \( -\frac{GM}{R} \) and a (short-range) Yukawa term \( \frac{GM}{R} e^{-R/\ell_1} \) leading to a non-singular short-distance modification of the classical Newtonian potential (see Fig. 1). Moreover, it is noted that Eq. (52) is in agreement with the expression given by Treder [6] obtained in his unified field theory of gravitation with long- and short-range interactions, which is a particular version of the so-called fourth order gravity.

Note that the modified Newtonian potential of Bopp-Podolsky type (52) with one free parameter is a particular version of the finite “Stelle”-potential which is the modified Newtonian potential with two free parameters in fourth order gravity (see, e.g., [7, 11, 24]).
Moreover, the gravitational field strength vector (42) simplifies to
\[ g = \frac{GM}{R^2} f_1(R, \ell_1) e_R. \]  
(54)

Using Eq. (51), one can see that the gravitational field strength vector of a point mass is finite at the origin, namely
\[ g(R) = -\frac{GM}{2\ell_1} e_R + \mathcal{O}(R). \]  
(55)

The modified gravitational potential energy (44) for two point masses \( M \) and \( M' \) reduces to
\[ U_{MM'} = -\frac{GMM'}{R} f_0(R, \ell_1). \]  
(56)

The modified gravitational potential (56) is a linear superposition of the (long-range) Newtonian potential term \(-\frac{GMM'}{R}\), and a (short-range) Yukawa potential term \(\frac{GMM'}{R} e^{-R/\ell_1}\) with spatial decay length \(\ell_1\). It is interesting to note that if we compare (56) with (1), we find that the modified gravitational potential (56) is the only singularity-free version of Eq. (1) with \(\alpha = -1\) and \(\lambda = \ell_1\).

The modified force law (47) becomes
\[ F_{MM'} = -\frac{GMM'}{R^2} f_1(R, \ell_1) e_R. \]  
(57)

The force (57) is plotted in Fig. 2. Note that the force (57) is finite at \( R = 0 \). The modified force (57) is a linear superposition of the attractive (long-range) Newtonian force \(-\frac{GMM'}{R}\) and a repulsive (short-range) Yukawa-type force \(\frac{GMM'}{R^2} [1 + \frac{R}{\ell_1}] e^{-R/\ell_1}\). In the near field, the modified force (57) is stronger than the modified force (47) which becomes zero at \( R = 0 \). Note that the modified force law (57) is the singularity-free version of Eq. (2) with \(\alpha = -1\) and \(\lambda = \ell_1\). Therefore, in first gradient modification of Newtonian gravity, the strength of the Yukawa term in Eqs. (1) and (2) is fixed to be \(\alpha = -1\) to have singularity-free gravitational fields.

VI. GRAVITATIONAL FIELDS IN CLASSICAL NEWTONIAN GRAVITY

The limit from first gradient modification of Newtonian gravity to classical Newtonian gravity is given by \(\ell_1 \rightarrow 0\).

From Eqs. (52) and (54), the classical gravitational potential and classical gravitational field strength of a point mass are recovered
\[ \Phi = -\frac{GM}{R}, \]  
(58)

and
\[ g = \frac{GM}{R^2} e_R. \]  
(59)

VII. NONLOCAL GRAVITY OF EXponential TYPE

In this Section, we consider a nonlocal gravity of exponential type and its relation to gradient modification of gravity. In particular, the nonlocal modification of Newtonian gravity is considered.

For a nonlocal modification of Newtonian gravity, the Lagrangian density reads as
\[ \mathcal{L} = \frac{1}{8\pi G} \int_{\mathbb{R}^3} K(x - y) g(x) \cdot g(y) \, dy + \rho \Phi, \]  
(60)

where \(K(x - y)\) denotes the so-called nonlocal kernel function or form factor. Following Efimov (e.g. [25]), we consider in the static case a form factor obtained by acting an entire function of the Laplacian on the Dirac delta-function
\[ K(x - y) = L(\Delta) \delta(x - y). \]  
(61)

Using the form factor (61), the Lagrangian density (60) reduces to
\[ \mathcal{L} = \frac{1}{8\pi G} g \cdot (L(\Delta) g) + \rho \Phi. \]  
(62)

In nonlocal gravity of exponential type, an exponential operator is used (see, e.g., [13, 26, 27])
\[ L(\Delta) = e^{-\ell^2 \Delta}, \]  
(63)

where \(\ell\) is the characteristic length scale of the nonlocal theory. This differential operator is of infinite order, characteristic for a nonlocal theory. Eventually, the exponential operator may be represented as an infinite series in power of \(\Delta\):
\[ L(\Delta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \ell^{2n} \Delta^n. \]  
(64)

For the exponential differential operator (63), the solution of Eq. (9) reads for the modified Newton potential [13, 26, 27]
\[ \Phi = -\frac{GM}{R} \text{erf} \left(\frac{R}{2\ell}\right), \]  
(65)

where \(\text{erf}\) denotes the error function (also called the Gaussian error function). The modified Newton potential (65) is finite at \( R = 0 \), namely (see Fig. 3)
\[ \Phi(0) = -\frac{GM}{\sqrt{\pi} \ell}. \]  
(66)

The corresponding modified gravitational field strength reads as
\[ g = -\frac{GM}{R^2} \left(\text{erf} \left(\frac{R}{2\ell}\right) - \frac{R}{\sqrt{\pi} \ell} e^{-R^2/(4\ell^2)}\right) e_R, \]  
(67)
FIG. 3: Gravitational potential $\Phi$ of a point mass as a function of $R/\ell$ in nonlocal gravity of exponential type (Exponential), second gradient Newtonian gravity (2nd gradient) for $\ell_2^4 = 2\ell_1^4$, first gradient Newtonian gravity (1st gradient) and classical Newtonian gravity (Newton).

FIG. 4: Gravitational force $F$ between two point masses as a function of $R/\ell$ in nonlocal gravity of exponential type (Exponential), second gradient Newtonian gravity (2nd gradient) for $\ell_1^4 = 2\ell_2^4$, first gradient Newtonian gravity (1st gradient) and classical Newtonian gravity (Newton).

which is non-singular, and in particular it is zero at $R = 0$. Moreover, the modified force law of nonlocal gravity of exponential type becomes

$$F_{MM'} = -\frac{GMM'}{R^2} \left( \text{erf} \left( \frac{R}{2\ell} \right) - \frac{R}{\sqrt{\pi} \ell} e^{-R^2/(4\ell^2)} \right) e_R,$$

which is plotted in Fig. 4.

The series expansion of the exponential operator (63) up to the order $n = 2$ in Eq. (64) is given by

$$L(\Delta) = 1 - \ell^2 \Delta + \frac{1}{2} \ell^4 \Delta^2 + O(\Delta^3).$$

In Eq. (69), it can be seen that the differential operator (8) in second gradient modification of Newtonian gravity can be considered as truncated Taylor expansion of the exponential operator (63). If we compare Eq. (8) and Eq. (69), we find the relation for the length scales

$$\ell_1^2 = \ell^2, \quad \ell_2^4 = \frac{1}{2} \ell^4$$

and therefore, it yields

$$\ell_1^4 = 2\ell_2^4,$$

which belongs to the case (3) with complex conjugate length scales of the two Helmholtz operators in second gradient modification of Newtonian gravity.

Therefore, classical Newton gravity is the expansion of order $n = 0$ of nonlocal gravity of exponential type. First gradient modification of Newtonian gravity is the expansion of order $n = 1$ of nonlocal gravity of exponential type. Furthermore, second gradient modification of Newtonian gravity (case (3) with the relation for the two length scales (71)) is the expansion of order $n = 2$ of nonlocal gravity of exponential type. In this manner, second gradient modification of Newtonian gravity with complex conjugate length scales satisfying the condition (71) can be understood as a proper approximation of nonlocal gravity of exponential type (see Figs. 3 and 4).

VIII. CONCLUSIONS

Starting from first principles of field theory, a gradient modification of second order of Newton’s theory of gravitation has been derived. In such a gradient theory, the gravitational potential field is local, but satisfies a partial differential equation of sixth order including internal characteristic length scales determining the range of modification of Newtonian gravity. Moreover, the modified gravitational fields obtained in gradient theory are singularity-free and possess a modification at short-distance and in the far field they converge to the Newtonian gravitational fields. The cancellation of the Newtonian singularities are due to the opposite signs of a Newtonian term and a bi-Yukawa term. The analytical expressions for the modified gravitational potential energy and the modified gravitational force law obtained in the framework of gradient field theory can be used for testing Newton’s inverse-square law at short-distances and for searching effects of non-Newtonian gravity in the order of micrometers or even smaller. The modified gravitational potentials obtained in second gradient modification and in first gradient modification of Newtonian gravity correspond to the modified Newtonian potentials of particular versions of sixth order and fourth order gravity, respectively. Therefore, second gradient modification of Newtonian gravity is the Newtonian limit of a particular version of sixth order gravity representing a simple version of quantum gravity which is local, superrenormalizable and, in the case of complex massive poles, can be unitary.

In addition, nonlocal modification of exponential type of Newtonian gravity and the corresponding gravitational
fields, which are singularity-free, have been studied. The connection between such nonlocal gravity of exponential type and gradient modification of Newtonian gravity has been given. It turned out that second gradient modification of Newtonian gravity with complex conjugate length scales represents a very good approximation of nonlocal gravity of exponential type.

Due to the comparison of second gradient modification of Newtonian gravity with sixth order gravity and nonlocal gravity of exponential type, it can be concluded that second gradient modification of Newtonian gravity with complex conjugate length scales seems to be the most interesting case for a gradient modification of Newtonian gravity. An experimental test of such a modification of Newtonian gravity would be challenging.

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