High sigma model of pulsar wind nebulae

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ABSTRACT

Pulsars and central engines of long gamma ray burst – collapsars – may produce highly magnetized (Poynting flux dominated) outflows expanding in a dense surrounding (interstellar medium or stellar material). For certain injection conditions, the magnetic flux of the wind cannot be accommodated within the cavity. In this case, ideal (non-dissipative) MHD models, similar to the Kennel and Coroniti (1984) model of the Crab nebular, break down (the so-called \( \sigma \) problem). This is typically taken to imply that the wind should become particle-dominated on scales much smaller than the size of the cavity. The wind is then slowed down by a fluid-type (low magnetization) reverse shock. Recent Fermi results, indicating that synchrotron spectrum of the Crab nebula extends well beyond the upper limit of the most efficient radiation reaction-limited acceleration, contradict the presence of a low sigma reverse shock.

We propose an alternative possibility, that the excessive magnetic flux is destroyed in a reconnection-like process in two regions: near the rotational axis and near the equator. We construct an example of such highly magnetized wind having two distinct reconnection regions and suggest that these reconnection cites are observed as tori and jets in pulsar wind nebulae. The model reproduces, qualitatively, the observed morphology of the Crab nebula. In parts of the nebular the dissipation occurs in a relativistically moving wind, alleviating the requirements on the acceleration rate.

1. Introduction

The interaction of strongly magnetized, relativistic outflows with a dense surrounding is a basic problem in pulsar physics, which recently became important for gamma ray bursts (GRBs) central engines. This seemingly straightforward problem turns out to be a complicated one, as is illustrated by what became known as the "\( \sigma \) problem" (Kennel & Coroniti 1984). Following the work of Rees & Gunn (1974), Kennel & Coroniti (1984) assumed that
the central source produces a wind which blows a cavity in the surrounding medium. The key assumption of the model is that in the bulk of the cavity the wind is supersonic, causally disconnected from the source. Kennel & Coroniti (1984) found that for a non-relativistic expansion of the bubble, \( v_{\text{out}} \ll c \), if the Poynting flux in the wind is comparable to or exceeds the particle flux, there is no acceptable solution for the structure of a resulting cavity. Introducing a parameter \( \sigma \) as the ratio of Poynting \( F_{\text{Poynting}} \) to particle \( F_{\text{p}} \) fluxes, \( \sigma = F_{\text{Poynting}}/F_{\text{p}} \), they found that solutions where wind is injected into cavity with relativistic velocities, while the cavity expands non-relativistically, exist only for \( \sigma \sim v_{\text{out}}/c \ll 1 \). Formally, for \( \sigma \geq 1 \) the reverse shock in the wind is relativistic with respect to the downstream flow, and since the flow matches to the slowly expanding cavity, the shock is driven back into the pulsar on a light crossing time. As a result, the initial set up of the problem, that of a central engine producing relativistic wind, breaks down. The requirement \( \sigma \ll 1 \) in the bulk of the cavity runs contrary to models of pulsar magnetospheres, which all predict generation of highly magnetized, \( \sigma \gg 1 \) wind (Goldreich & Julian 1970). These expectations are confirmed by recent numerical simulations (Spitkovsky 2006). Kennel & Coroniti (1984) concluded that \( \sigma \) must change from being much larger than unity at the light cylinder to much smaller than unity in the bulk of a PWN.

The conversion of high \( \sigma \) to low is very hard to achieve in ideal MHD (Heyvaerts & Norman 2003), though a number of recipes have been proposed (e.g. Vlahakis 2004). It is beyond the scope of this paper to discuss this mathematically very difficult problem, but simple, non-dissipative models do not achieve high to low \( \sigma \) conversion (the applicability of fluid approach has also been questioned Lyubarsky 2003).

Note, that the requirement of evolution of \( \sigma \) from high to low follows from a self-consistency of the model, in a sense that for \( \sigma \geq 1 \) the whole model, that of a central pulsar producing a supersonic wind which is shocked within a PWN, becomes inconsistent. Thus, a question still remains, if \( \sigma \) remains high, how would a flow behave?. This is the main question addressed in this paper.

Recently this problem became important for long GRBs. Detection of Type Ic supernovae nearly coincidence with long GRBs unambiguously linked them with deaths of massive stars (Stanek et al. 2003; Hjorth et al. 2003). Studies of the host galaxies of long GRBs, which turned out to be actively star-forming, further strengthens this association (Djorgovski 2001). The outflow resulting in a GRB can be driven by a "millisecond magnetar" (a fast rotating strongly magnetized protoneutron star, e.g. Usov 1992) or a black holes with an accretion disc (e.g. MacFadyen & Woosley 1999). In both cases the wind produced by the central engine may be Poynting flux dominated, \( \sigma \gg 1 \). The evolution of the resulting bubble inside the star then resembles the evolution of a PWN. Again, in order to achieve
a self-consistent model, it is required that $\sigma$ changes from high to low values (Barkov & Komissarov 2008; Bucciantini et al. 2009). (In passing we note that the electro-magnetic model of GRBs (Lyutikov & Blandford 2003; Lyutikov 2006) explores the possibility that $\sigma$ remains high after the flow breaks out from the star.)

2. Radiation reaction-limited acceleration of leptons and Fermi spectrum of Crab

There is an upper limit on the frequency of synchrotron emission by radiation reaction-limited acceleration of electrons. If the accelerating electric field is a fraction $\eta \leq 1$ of the magnetic field (this is equivalent to acceleration on time scale of inverse cyclotron frequency $1/(\eta \omega_{B,rel})$, where $\omega_{B,rel} = \gamma/\omega_B$ is relativistic cyclotron frequency of a particle), equating the acceleration rate and synchrotron energy losses,

$$\eta e B c = \frac{2}{3} \frac{e^2}{c} \gamma^2 \omega_B^2$$

the peak frequency of emission becomes

$$\omega = \eta \omega_{\text{max}}, \quad \omega_{\text{max}} = 0.44 \frac{mc^3}{e^2}$$

This corresponds to energies (see also de Jager et al. 1996).

$$\epsilon_{\text{max}} = \hbar \omega_{\text{max}} = 0.44 \hbar \frac{mc^3}{e^2} \approx 30 \text{ MeV}.$$  

Note, the upper limit (3) assumes non-stochastic, DC-type acceleration. The break energy observed by Fermi satellite in Crab is $\sim 100$ MeV (Fermi Collaboration & Pulsar Timing Consortium 2009), already three times higher than the limit (3). If the spectrum above the break requires presence of higher energy particles, those particles should be accelerated by an electric field much larger than the magnetic field. This is highly problematic under astrophysical conditions, where abundant supply of particles ensures that $E \leq B$.

There is a number of possible resolutions of this contradiction. First, magnetic field can be strongly inhomogeneous, spatially or temporarily, so that particles are accelerated in a low magnetic field regions, e.g. very close to the equatorial current sheet, and then emit when they enter a high field region. The possibility of spatially inhomogeneous magnetic field is also consistent with the results of recent numerical simulations of relativistic shocks, indicating that only unmagnetized shocks are efficient accelerators (Sironi & Spitkovsky 2009, another case of highly efficient acceleration, quasi-parallel shocks, is not applicable to
The problem with this scenario is that the Crab nebula is a very efficient emitter: about 20% of the spin-down luminosity is converted into radiation. If acceleration is limited to a narrow current layer containing only a small part of the spin-down luminosity, this puts an unreasonable demands on the efficiency of acceleration.

Temporal variations in the strength of magnetic field near the termination shock, seen in numerical simulations by Camus et al. (2009), remain a distinct possibility, with a clear prediction: the Crab synchrotron spectrum should be strongly variable on time scale of month, of the order of wisp variability, with the cut-off energy dropping well below 100 MeV. This can be tested with the ongoing observations by Fermi satellite. Previously, EGRET data did indicate a moderate level of the cut-off energy variability (de Jager et al. 1996).

Another possibility to explain a high acceleration rate, is that the flow is relativistic in the post-shock region. In this case the observed emission is beamed by the bulk Lorentz factor $\Gamma$, $\omega \sim \eta \Gamma \omega_{\text{max}}$. Thus, a high post-shock Lorentz factors $\Gamma$ may alleviate the problem of super-efficient acceleration. Relativistic post-shock flows are indeed expected within the Komissarov & Lyubarsky (2004) model of PWNe (see also Del Zanna et al. 2004). In this model, the reverse shock is generally oblique, so that the post-shock flow is moving relativistically. But the post-shock flow is only mildly relativistic, independently of the pre-shock Lorentz factor, $\Gamma \sim 1/\theta \sim \text{few}$, where $\theta \sim 1$, is the angle that the radially moving unshocked wind makes with the forward shock (this angle can be small near the axis, but very little energy is coming along those directions).

As we show below, the high sigma flows allow dissipation in a relativistically moving plasma, boosting the maximum observed synchrotron frequency to $\omega \sim \eta \Gamma \omega_{\text{max}}$ and alleviating the requirements on $\eta$, which can be of the order $\eta \sim 1/\Gamma \ll 1$.

3. The $\sigma$ problem

The $\sigma$ problem of Kennel & Coroniti (1984) implies that ideal, non-relativistic, homologous expansion of a bubble of strongly magnetized plasma injected with nearly a speed of light, cannot occur. The reason is that magnetic flux and energy are supplied to the inflating bubble by the rates that cannot be accommodated in the bubble. The rate of supply is determined by the processes inside the light cylinder of a compact object, while inflation of the bubble is controlled by the external gas density. Formally, the surface of the bubble will influence back on the source and would start to control the injection rate, consistent with the allowed rate of expansion. But even if the wind remains subsonic, it is unlikely that processes at the edge of the inflating bubble would influence the wind generation region near
the light cylinder due to large separation in scales, thousands in case of GRBs and billions in case of PWNs.

One way to approach the $\sigma$ problem, is to try to solve magnetohydrodynamical reverse shock conditions in the strongly magnetized wind, that match the slowly expanding PWN cavity downstream. This can only be done for low $\sigma$: for $\sigma \geq 1$ the reverse shock is driven back into the pulsar on light crossing time at which point the model becomes meaningless (Kennel & Coroniti 1984).

To illustrate the $\sigma$ problem from another viewpoint, assume that the central source injects a relativistic wind that carries electromagnetic power and magnetic flux in a form of toroidal magnetic field. The central source injects energy and magnetic flux into a spherical cavity, PWN, with the injection velocity $v_{in} = \beta_{in}c$ and the outer boundary of the cavity expands with velocity $v_{out} = c\beta_{out} \ll v_{in}$. The core assumption here is that the injection velocity is independent of the conditions in the nebular. We assume that the expansion velocity of the nebular is much smaller than both the injection velocity and the fast magnetosonic speed inside a bubble, so that the interior is in a state of a quasistatic equilibrium.

Let us assume that magnetic flux is injected into the Crab nebular at some rate $\dot{\Phi}$. This rate depends on the integrated polar angle-dependent luminosity $L(\theta)$ produced by the pulsar. Inside the nebular, where the velocity of expansion becomes strongly sub-fast magnetosonic, the magnetic field will relax to a minimum energy state consistent with a given magnetic flux. It may be shown that such a state corresponds to the current concentrated on the symmetry axis. Since such a current distribution would correspond to the infinite current density, we will assume that there is a current-carrying core of conical shape with opening angle $\theta_c \ll 1$. Equivalently, the cavity is in a state of dynamical balance and since the pressure forces are negligible, the magnetic field relaxes to a state of (quasi)static force-free equilibrium with no velocity shear and associated electric fields. The only possible equilibrium of a purely toroidal magnetic field corresponds to a line current (see, e.g., Eq. (13)), with infinite current density on the axis. This state should have the magnetic flux equal the total injected flux.

The central source may have anisotropic energy flux. As an exemplary case, consider power distribution produced by an aligned pulsar Michel (1973). In this case $L \propto \sin^2 \theta$, and normalizing magnetic field in the wind to the total luminosity $L = 2\pi r_{in}^2/\beta_{in}c \int B^2 d\theta$, the magnetic field at the inner boundary of the nebular is

$$B = \frac{\sin \theta}{r_{in}} \sqrt{\frac{3L}{2\beta_{in}c}}$$

The rate of the injection of the magnetic flux is $\dot{\Phi} = \beta_{in}c \int B r dr d\theta = 2\sqrt{3\pi} \beta_{in} L c$, so that
total flux stored in a cavity is
\[ \Phi_{\text{tot}} = \sqrt{6} \beta_{\text{in}} L c t \] (5)
(and total energy \( E = Lt \)). We assume for simplicity that the luminosity is constant in time.

Inside the nebular, magnetic field corresponds to a current-carrying core of conical shape with opening angle \( \theta_c \). Equating the total stored flux in such a configuration to the injected flux, the magnetic field inside a PWN is
\[ B_{\text{in}} = \frac{\sqrt{6} c \beta_{\text{in}} L t}{r \sin \theta R(t) \ln \tan(\theta_c/2)} \] (6)
where \( R(t) \) is the outer radius of a PWN. The ratio of the stored to the injected energy is
\[ \frac{\int B_{\text{in}}^2/(8\pi) dV}{L t} = \frac{3 \beta_{\text{in}}}{2 \beta_{\text{out}}} \frac{1}{\ln \tan(\theta_c/2)} \] (7)
where we assumed \( R(t) = \beta_{\text{out}} c t \). For injection velocity of the order of the speed of light, \( \beta_{\text{in}} \sim 1 \) and expansion velocity several thousand km/sec, \( \beta_{\text{out}} \sim 10^{-2} \), the angle \( \theta_c \) required to accommodate all the injected flux and energy is unphysically small. Thus, the assumption of the model, of a source injecting at a given rate magnetically dominated, ideal flow into slowly expanding cavity, lead to unphysical consequence.

Finally, in the simplest example of the paradox, consider a cavity of fixed volume, thus neglecting the expansion completely, The energy and the magnetic flux are injected into the cavity by the wind linearly in time. But since toroidal flux grows linearly in time, the stored magnetic energy grows quadratically, in clear contradiction to linear injection of energy (Rees & Gunn 1974).

3.1. Possible resolutions of \( \sigma \)-paradox

There are several possible resolutions of this "paradox". First, our assumption of high \( \sigma \) in the bulk of PWN can be wrong. This is the resolution chosen by Kennel & Coroniti (1984), who postulated that that \( \sigma \) in the wind decreases from \( \sigma \gg 1 \) to \( \sigma \ll 1 \).

What if \( \sigma \) remains high in the bulk of the flow? One possibility is that the flow would break out of the cavity: in laboratory devices, where the sizes of the magnetic field generator and the enclosing cavity are similar, and where magnetic field is limited by strength of cavity’s material, this sometimes leads to spectacular accidents. In case of GRBs this will occur at the time of jet break-out, but cannot occur in PWNe.

Another possibility is that the cavity affects the central engine, so that the rate of injection of the magnetic flux adjusts to the rate with which it can be accommodated. We
consider the possibility unlikely due to large separation of scales. It is unimaginable that a PWN boundary, located at distances of parsecs away from the central source, affects generation of pulsar winds at scales of hundreds of kilometers. In the case of GRB central engines, the separation of scale is somewhat smaller, only three-four orders of magnitude, from $10^7$ cm (typical size of light cylinder radius of millisecond neutron star or Schwarzschild radius of a black hole) to $10^{11} - 10^{12}$ sm (size of a precollapse type Ib/c progenitor), but still there are some three-four orders of magnitude. We assume that injection is independent of the expansion and investigate consequences of this assumption.

Another possible resolution of the $\sigma$-paradox requires destruction of magnetic flux, which in turn requires resistivity and reconnection (reconnection in its original sense, as a process destroying magnetic flux, not necessarily the magnetic energy). Thus, dissipation must become important in the flow. Typically, reconnection is most efficient in a limited regions of space associated with high electric current concentration, current sheets. Current sheets produced locally, through turbulent motion of plasma, won’t destroy the large scale magnetic flux. The destruction of the the large scale toroidal magnetic flux can be achieved in two generic locations: near the axis in an O-type reconnection, or near the equatorial plane, where magnetic field lines of opposite polarity approach each other (assuming, for simplicity, an aligned rotator).

Thus, a flow may be ideal in the bulk and strongly resistive in regions of small measure. What would a structure of the flow be in this case? As the magnetic flux and necessarily the magnetic energy (and, thus, magnetic pressure) are dissipated, this would create a force misbalance that would push the flow towards the dissipation regions. Thus, a flow structure bringing magnetic field toward dissipation regions - rotational axis and equator - is set up. This, in turn, leads to the pile-up of magnetic field near the dissipation regions and to faster radial expansion in the direction of those regions (akin to toothpaste tube effect Lyutikov & Blandford 2003). This effect of magnetic pile-up would be most pronounced near the axis and may be responsible, as we propose, for generation of GRB and pulsar jets.

Previously, Kirk & Skjæraasen (2003) discussed magnetic field dissipation in the pulsar wind. In their model, the reconnection occurs locally, between different magnetic polarities in a striped wind. In this case, the total injected magnetic flux is zero if averaged over the pulsar’s period. In contrast, we assume that the pulsar rotation axis is nearly aligned with the magnetic axis, so that the magnetic field has a given polarity in each hemispheres.
4. Magnetized flow within a spherical cavity

Next we consider an idealized problem of a high-$\sigma$ dissipative flow bounded by a cavity. We expect that a flow pattern is set up in which a magnetized wind is injected at small radii and is deflected at intermediate regions towards the dissipation regions. First, instead of calculating the form of the cavity self-consistently, we assume it to be fixed and then calculated a flow within it. As the simplest example, we consider a spherical cavity, but the following derivation may be repeated for any of the separable coordinates. Second, we consider a case of highly magnetized plasma $\sigma \gg 1$ and completely neglect inertial contribution, the so-called force-free approximation (Gruzinov 1999, note, that the relativistic force-free approximation is different from vacuum, since massless charges create electric currents and charge density that affect the flow dynamics). Third, we assume that dissipation is limited to an infinitely small volume (formally of measure zero) concentrated near the axis and the magnetic equator, while in the bulk the flow is ideal. Fourth, we assume that the flow is generated within a volume small compared to the size of the cavity, so that the magnetic field is dominated by axially symmetric toroidal field. Neglect of the poloidal magnetic field is, perhaps, mathematically the most important simplifying assumption, since it untangles the structure of the poloidal current from the structure of magnetic flux surfaces, the main complication in solving the Grad-Shafranov equation. Finally, we assume that the flow is in a steady state condition, which implies that the velocity of the flow are much larger than the velocity of the cavity expansion and that the pulsar is an aligned rotator.

Equations of ideal force-free electrodynamics (Gruzinov 1999) include Maxwell’s equations,

$$\frac{\partial E}{\partial t} = \nabla \times B - 4\pi j$$ \hspace{1cm} (8)
$$\frac{\partial B}{\partial t} = -\nabla \times E$$ \hspace{1cm} (9)

and force-balance

$$\rho E + j \times B = 0,$$ \hspace{1cm} (10)

which allows one to relate the current to the electro-magnetic fields

$$j = \frac{(E \times B) \nabla \cdot E + (B \cdot \nabla \times B - E \cdot \nabla \times E) B}{4\pi B^2}$$ \hspace{1cm} (11)

In the asymptotic domain, the only nonzero components of the fields are $B = B_\phi, E_r, E_\theta$ and equations (9) and (11) give

$$\partial_t B = -\frac{1}{r} \partial_r (r E_\theta) + \frac{1}{r} \partial_\theta E_r$$
\[
\partial_t E_\theta = -\frac{1}{r} \partial_r(rB) + \frac{E_r}{B} \left( \frac{1}{r \sin \theta} \partial_\theta(\sin \theta E_\theta) + \frac{1}{r^2} \partial_r(r^2 E_r) \right)
\]
\[
\partial_t E_r = \frac{1}{r \sin \theta} \partial_\theta(\sin \theta B) - \frac{E_\theta}{B} \left( \frac{1}{r \sin \theta} \partial_\theta(\sin \theta E_\theta) + \frac{1}{r^2} \partial_r(r^2 E_r) \right)
\]  
(12)

In this formulation, the electrical currents arise exclusively due to charge transport across magnetic field by electric drift, \( j_r = \rho E_\theta B \), \( j_\theta = -\rho E_r B \), where \( \rho = \nabla \cdot E / (4\pi) \) is charge density. Thus, they are not subject to resistive decay in the bulk (e.g. Lyutikov 2003). Toroidal axially symmetric magnetic field is \( B_\phi = 2I / (cr \sin \theta) \), where \( I \) is a current flowing through a magnetic loop at radius \( r \) and polar angle \( \theta \). In ideal, time-stationary case the electric field is poloidal and irrotational, \( E = -\nabla \Phi \). The force-free condition then gives

\[
2\nabla I^2 = r^2 \sin^2 \theta \Delta \Phi \nabla \Phi
\]  
(13)

The neglect of poloidal magnetic flux makes the problem under-determined, in a sense that for a given toroidal magnetic field we may chose various distributions of electric field, the only constraint being that the flow lines do not intersect the given form of the outer boundary. In case of axial symmetry, the force balance gives one scalar equation (13) for two functions, toroidal magnetic field and poloidal electric field, or, equivalently, electric current and charge density. (Note, that in absence of a shear, \( \Phi = 0 \), Eq. (13) implies \( I = \)constant, and magnetic field corresponding to a line current.)

Equation (13) should be solved for a given form of the cavity. If the boundary is impenetrable, the velocity of plasma,

\[
v = \frac{E \times B}{B^2} = \frac{r \sin \theta (e_\phi \times \nabla \Phi)}{2I}
\]  
(14)

should vanish on the boundary. Thus, for a fixed spherical cavity of size \( R \), it is required that

\[
\partial_\theta \Phi(r = R) = 0
\]  
(15)

Equation (13) is underdetermined since it involves two unknown functions. A particularly simple choice of electric potential \( \Phi \) and current \( I \) corresponds to vanishing charge and current densities in the bulk, so that \( \Delta \Phi = 0 \) and \( I = \)constant. (Another analytically tractable case is discussed in Appendix A.) Taking the first terms in a series of harmonic functions (Legendre polynomials with \( m = 0, 1 \)), the potential that satisfies boundary condition (15) is

\[
\Phi = \Phi_0 \left( 1 - \frac{R}{r} \right) \ln \frac{\tan \frac{\theta}{2}}{\tan \frac{\theta^*}{2}}
\]  
(16)
where constant $\theta^*$ is the polar angle that separates flow lines that end on the axis, for $\theta_0 < \theta^*$ and at equator, for $\theta_0 > \theta^*$, where $\theta_0$ is the polar angle at the injection radius $r = r_0$. The flow lines are then given by (see Fig. 1) $\partial_\theta r = -\partial_\theta \Phi / \partial_r \Phi$, which integrates to

$$r = \frac{\ln \tan \frac{\theta_0}{2} R r_0}{R \ln \tan \frac{\theta_0}{2} + r_0 \ln \tan \frac{\theta^*}{2}}$$

(17)

The flow lines converge either on the axis or at the equator. It is implicitly assumed that there the dissipation sets in and destroys the magnetic field at the end points of flow lines. On the axis all the flow lines asymptote to the point $\theta = 0$, $r = R$, they never cross the line $\theta = 0$ at $r < R$. Situations at the equator is very different, a flow line that starts at $r_0, \theta_0$ reaches equator at

$$r_e = \frac{r_0 R \ln \cot (\theta^*/2)}{r_0 \ln \cot (\theta_0/2) + R \ln \tan \frac{\theta_0}{2}}$$

(18)

Thus, magnetic energy must be dissipated continuously and effectively at the equator, while near the axis a toothpaste-like effect pushes the flow in radial direction. Thus, on the axis all the dissipation (formally) happens at a point $\theta = 0$, $r = R$.

The flow velocity corresponding to Eq. (16) is

$$v_r = \beta_0 \left( \frac{R}{r} - 1 \right)$$

$$v_\theta = \beta_0 \frac{R \sin \theta}{r} \ln \frac{\tan \frac{\theta}{2}}{\tan \frac{\theta^*}{2}}$$

(19)

where we introduced $\beta_0 = \Phi_0/(2I)$. The flow settles down on the axis, $v_\theta \to 0$ as $\theta \to 0$, and has a finite $\theta$-velocity at equator, $v_\theta \to -\beta_0 \ln \tan \frac{\theta_0}{2} (R/r)$ at $\theta \to \pi/2$.

The corresponding Poynting flux toward the equator is

$$F_r = \frac{2(R - r)}{r^3 \sin^2 \theta} I \Phi_0$$

$$F_\theta = \frac{2R}{\sin \theta r^3} \ln \frac{\tan \frac{\theta}{2}}{\tan \frac{\theta^*}{2}} I \Phi_0$$

(20)

The energy flux towards the reconnection regions is $\propto r \sin \theta F_\theta$. Near the axis, the energy flux diverges as $\ln \theta$; assuming that resistivity becomes important within a core defined by the polar angle $\theta_c$, the ratio of energy fluxes going to the axis and to the equator is $\xi = \ln \theta_c / \ln \cot (\theta_0/2)$. We expect that in case of PWNe, the observed relative brightness of
Fig. 1.— Flow lines given by eq. (17). The cavity is assumed to be spherical. Injection of the wind occurs at some small radius $r_0$, labeled NS. Flow lines that start at $\theta < \theta^*$ (45° in this example) terminate near the axis, while those that start at $\theta > \theta^*$ terminate on equator.

jet and tori depends on the relative energy fluxes to the corresponding reconnection regions, so that the parameter $\xi$ is a measure of the relative brightness within this model. At a present stage, it is expressed in terms of two formal parameters, the core angle $\theta_c$ and the separatrix angle $\theta^*$. 
To summarize, we found a flow of strongly magnetized plasma within a given spherical cavity. The particular model, chosen mostly for simplicity, has a separatrix angle $\theta^*$ that delineates the flow lines ending on the equator and on the axis. The ratio of energies dissipated depends also on the core angle $\theta_c$, where dissipation sets in near the axis. The overall value of the velocity scales with the parameter $\beta_0$.

### 4.1. Applicability

So far, the parameters $\beta_0$ and $\theta^*$ were arbitrary. We should now check the conditions that the resulting solutions correspond to a physically meaningful velocity

$$v = \beta_0 \sqrt{\frac{(1 - R/r)^2 + (R/r)^2 \ln^2 \frac{\tan \theta/2}{\tan \theta^*/2} \sin^2 \theta}{1 + \ln \cot \theta^*/2}} < 1$$

at least somewhere inside the sphere $r = R$. Demanding that $v < 1$ on the equator at $r = R$ requires that

$$\beta_0 \ln \cot \frac{\theta^*}{2} < 1$$

For parameters satisfying Eq. (22), the condition $v < 1$ is violated for sufficiently small radii, $r/R \leq \beta_0 \sqrt{1 + \ln^2 \cot \theta^*/2}$. Thus, the model is not applicable at very small radii, where mass loading should be important. This is an artifact of our complete neglect of inertia.

### 4.2. Surface brightness of a model PWN

To produce an example of the observed PWN in our model, we assume that (i) the emission is generate in the equatorial plane and along the rotation axis; (2) the observed intensity is proportional to $\delta^{2.5}$, where $\delta = 1/\gamma(1 - v \cdot n)$ is a Doppler factor, $\gamma$ is a Lorentz factor corresponding to velocity (19), $n$ is a unit vector along the line of sight towards the observer; (3) chose inclination angle inferred for Crab nebular, $\theta_{ob} = \pi/3$ (the same value as used by Komissarov & Lyubarsky (2004)); (iv) chose $\theta^* = \pi/3$ (smaller values of $\theta^*$ give large Doppler factors and, correspondingly, more pronounced front-back asymmetry; (v) parameter $\beta_0 = 0.25$ (the values of $\beta_0$ mostly controls the size of the inner region, where the force-free approximation is not applicable; smaller $\beta_0$ correspond to smaller size of that region); (vi) axis inclination of 135° with respect to north-south direction. Results of the calculations are presented in Fig. 2.

Chandra observation of Crab nebular show that the pulsar is located inside a low brightness bubble, outlined by bright wisps. In our model, the “sudden” onset of dissipation is an
Fig. 2.— A negative image of the Crab nebular in the high-σ model. Point in the center denotes the central neutron star. Opening angle of the jets is chosen arbitrarily, 0.1 radian, for illustration purposes. Equatorial and jet emission is proportional to local Doppler factor. Within the central cavity the model is not applicable; see text for more details.

effects of Doppler boosting. Dissipation occurs in a limited region near equator and along the axis. At small radii, when radial velocity is large, the emission is boosted away from the observer. Further out, the flow slows down, as the radial velocity decreases with radius
\( \propto \frac{R}{r} - 1 \), (19) emission from corresponding parts comes into view. The resulting structure shows qualitative agreement with observations.

5. Discussion

In this paper we outlined a model of a high-\( \sigma \) (Poynting flux dominated) flow confined by a medium so dense, that the rate of the expansion of the blown-up cavity is not sufficient to accommodate the injected toroidal magnetic flux. The model postulates that excessive magnetic flux is destroyed within the nebular at a rate dictated by the expansion of the nebular. We do not address the related dissipation of energy. Formally, destruction of the flux can proceed at an infinitely small resistivity (a fast dynamo model, Vaǐnshteǐn & Zel’dovich 1972, is an example of model with non-conserved magnetic flux, yet negligible magnetic energy dissipation).

The nebular flow in this case is drastically different from the more conventional MHD model of PWNe by Rees & Gunn (1974) and Kennel & Coroniti (1984). The assumption of the model is that if the injection rate of toroidal magnetic field exceeds the rate at which magnetic flux can be accommodated, a flow will find ways to destroy the excessive magnetic flux. We present an example of how a nebular can achieve this: a flow pattern is set up, where the magnetic energy is injected nearly radially at small distances and is deflected toward reconnection cites in the bulk. We give an idealized example of such flows, in which the electric current and electric charge density are confined to two regions: near the symmetry (rotational) axis and equator. In this particular solution the bulk of the flow is current and charge density-free. This solution is very simple and is meant to illustrate a possible character of the flow with the nebular. In spite of it simplicity it reproduces qualitatively two dissipation regions observed in PWNs, axes and tori, as a well as collimated jet-like flows observed in some PWNe.

The model has implications for dynamics of collapsar winds at early stages of GRB explosions. Even in the current simple version of the model, the flow shows a “toothpaste tube” effect: the flow is collimated towards and is pushed along the symmetry axis. Our assumption of a rigid spherical boundary precluded formation of a jet. Thus, dissipation on the axis leads to formation of the flow towards the axis and increased magnetic field. This provides an additional (to the ideal case) collimation mechanism.

In addition, we expect that in the equatorial plane the flow also expands faster: conversion of the magnetic field energy density into kinetic energy of particles removes the confining hoop stresses. Thus we expect that both on the axis and at the equator a PWNe expands
faster than at midlatitudes.

This simple model misses a number of important effects: the properties of the solution at small distances do not quite correspond to what is expected from the pulsar injection. In the analytical example we discussed the angular dependence of luminosity does not correspond to what is expected from pulsar models (e.g. Spitkovsky 2006) (the model has more energy near the axis than expected. Also, at small distances the flow is formally superluminal. Dissipation of magnetic field inside the reconnection regions should lead to high plasma pressure, invalidating the assumption of magnetically dominated plasma. In the current formulation, the energy just disappears inside the reconnection regions: this is partly can be attributed to radiative losses, but physically most of the energy just change form, from magnetic to internal and bulk kinetic, and this will affect dynamics close to reconnection regions; the shape of the cavity was assumed to be fixed and given: it should be found self-consistently. Finally, we assumed an aligned rotator, while Crab is, in fact, nearly orthogonal (Moffett & Hankins 1999). For orthogonal rotators the injected magnetic flux is zero if averaged over the period of rotation, so that the reconnection can proceed locally, not involving large scales flow discussed in the current model.

A possible observational distinction between shock acceleration an reconnection is a spectral index of accelerating particles $p$. Shock acceleration typically produces $p \sim 2$ (Blandford & Eichler 1987), where $dn/d\gamma \propto \gamma^{-p}$ (see, though Lyubarsky & Liverts 2008). Non-linear shock acceleration model can, in principle, produce harder spectra, upto $p = 1.5$ for adiabatic index of $4/3$. Observations of Vela pulsar (Kargaltsev & Pavlov 2004) indicate $p = 1.1$, much flatter than expected. In contrast, acceleration in relativistic reconnection is expected to generically produce $p = 1$ (Zenitani & Hoshino 2007).

How can such a system be simulated on a computer? Due to a relatively limited dynamic range of simulations, the outer boundaries will tend to affect strongly the injection region, so that the rate of injected magnetic flux would tend to adjust to the expansion rate. As we argued, this is physically unrealistic. In simulations the (anomalous) resistivity should be adjusted so that all the injected flux (or at least most of it) is destroyed before it is reflected from the outer boundary and affects injection.

Komissarov (2006) gives a numerical example of resistive force-free magnetosphere, which has many similarities to the proposed model. Due to strong resistivity in the equatorial current, the force-free wind converges to the equator, so that all flux surfaces formally become closed. The observed behavior seem to be independent of the resistivity, suggesting that a small numerical resistivity is sufficient to relax to a particular solution. The electromagnetic energy flows into the current sheet and disappears inside of it.
The inertial and kinetic effects will modify the structure of the equatorial current sheet and polar current. Conversion of magnetic energy into heat and plasma pressure will slow down the influx toward the reconnection regions. On the other hand, increased pressure will lead to faster radial expansion in those regions and will reduce this effect. Simulations of Komissarov (2006), do show that some flux surfaces close outside the light cylinder in resistive MHD simulations.

The original impetus for this work came from discussions with Roger Blandford. I would like to thank Elena Amato, Jon Arons, Oleg Kargaltsev, Yuri Lyubarskii, Christopher Reynolds and Mallory Roberts for comments and discussions.

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A. Separable solution of Eq. (13)

Eq. (13) also allows a separable solution with distributed currents and charge densities

\[ I, \Phi \propto \left( \frac{1}{r^3} - \frac{r^2}{R^3} \right) \sin^2 \theta \]  

(A1)

The radial component of the Poynting flux is \( \propto (r - R^5/r^4)^2 \cos \theta \sin^2 \theta \), with flow lines given by \( \frac{\sin^2 \theta}{\sin^2 \theta_0} = (R^5/r_0^3 - r_0^2)/(R^5/r^2 - r^3) \). All the flow lines in this solution end at the equator.