2D $\mathcal{N} = (4, 4)$ superspace supergravity and bi-projective superfields

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Abstract

We propose a new superspace formulation for $\mathcal{N} = (4, 4)$ conformal supergravity in two dimensions. This is based on a geometry where the structure group of the curved superspace is chosen to be $\text{SO}(1,1) \times \text{SU}(2)_L \times \text{SU}(2)_R$. The off-shell supergravity multiplet possesses super-Weyl transformations generated by an unconstrained real scalar superfield. The new supergravity formulation turns out to be an extension of the minimal multiplet introduced in 1988 by Gates et. al. and it allows the existence of various off-shell matter supermultiplets. Covariant twisted-II and twisted-I multiplets respectively describe the field strength of an Abelian vector multiplet and its prepotential. Moreover, we introduce covariant bi-projective superfields. These define a large class of matter multiplets coupled to 2D $\mathcal{N} = (4, 4)$ conformal supergravity. They are the analogue of the covariant projective superfields recently introduced for 4D and 5D matter-coupled supergravity but they differ by the fact that bi-projective superfields are defined with the use of two $\mathbb{C}P^1$ instead of one. We conclude by giving a manifestly locally supersymmetric and super-Weyl invariant action principle in bi-projective superspace.
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1 Introduction

In revealing the off-shell structures of supersymmetric field theories a most natural framework is provided by superspace. This can offer a formalism to build general supersymmetric models with covariance fully guaranteed that is especially important, for example, in studying supergravity. An adequate superspace formalism is also powerful in the analysis of the quantum behavior of globally and locally supersymmetric field theories and it proves to be a unique tool in understanding the target space geometry of supersymmetric non-linear sigma-models. These statements have probably their best and simplest explanations in the case of 4D $\mathcal{N} = 1$ supersymmetry (see [1, 2, 3] for reviews). In the case of extended supersymmetry off-shell formulations using superspace, when possible, are less simple and complete prescriptions are, in our opinion, still to be found.

Exemplary is the case of supersymmetry with eight real supercharges in its most studied form: 4D $\mathcal{N} = 2$ supersymmetry. In this case, the study of supersymmetric multiplets and invariants naturally leads to the introduction of extended superspace coordinates related to the SU(2) automorphism group of the supersymmetry algebra [4, 5, 6]. Then, invariant sub-superspaces emerge and one can treat general multiplets including off-shell charged hypermultiplets. These, to close off-shell supersymmetry without central charges [7, 8], have an infinite number of auxiliary fields [5, 9, 6, 10, 11].

In the literature, two superspace formalism have been introduced to study 4D $\mathcal{N} = 2$ supersymmetric field theories. They go under the names of harmonic superspace (HS) [5, 9] and projective superspace (PS) [6, 10]. The methods make use of the two equivalent superspaces $\mathbb{R}^{4|8} \times S^2$ and $\mathbb{R}^{4|8} \times \mathbb{C}P^1$ respectively, however, they differ in the structure of the off-shell supermultiplets used and the supersymmetric action principle chosen. Due to their differences, the two approaches often prove to be complementary to each other.

In this paper we will focus on projective superspace.

Projective superspace was first introduced to study globally supersymmetric non-linear sigma-models providing, since then, a powerful generating formalism to build new hyper-Kähler metrics [6, 14, 15, 10]. PS has proved to be a useful approach in studying supersymmetric field theories also in 5D [17] and 6D [18, 19]. Superconformal field theories in PS have been described by Kuzenko in four and five dimensions [20, 21] providing a

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1 For global supersymmetry, the relationship between the harmonic and projective superspace has been described in [12]. See also [13] for a recent discussion.

2 For a review on this subject, and a partial list of references, see [16] where a nice discussion of the relationship between twistor spaces and projective superspace is given.
starting point for curved extensions. In the supergravity case, we recently proposed a PS formalism first in five \cite{23, 24} and then in four dimensions \cite{25, 26} (see \cite{27, 28, 29, 30} for recent developments and applications). This PS approach is conceptually based on two main ingredients: (i) a constrained superspace geometric formulation of the Weyl multiplet of conformal supergravity, which is based on “standard” Wess-Zumino superspace \cite{31} techniques; (ii) the existence of covariant projective multiplets and a locally supersymmetric and super-Weyl invariant action principle in PS that are consistently defined on the curved geometry of point (i). These ingredients allow a covariant off-shell setting for general 5D $\mathcal{N} = 1$ and 4D $\mathcal{N} = 2$ supergravity-matter systems similar to the Wess-Zumino superspace approach to 4D $\mathcal{N} = 1$ supergravity. It is worth mentioning that a prepotential formulation for 4D $\mathcal{N} = 2$ conformal supergravity was given in harmonic superspace twenty years ago \cite{36}. However, its relationship to the standard, curved superspace geometrical methods has not yet been elaborated in detail. A synthesis of HS and PS, could possibly provide a coherent superspace description of 4D $\mathcal{N} = 2$ supergravity, similar to the famous Gates-Siegel prepotential approach to the 4D $\mathcal{N} = 1$ case \cite{37}.

Projective superspace has been introduced also for two-dimensional $\mathcal{N} = (4, 4)$ supersymmetry \cite{14, 38, 39, 40}. The 2D case is interesting and peculiar for many reasons. First of all, once the number of supercharges is fixed, decreasing the space-time dimensions the number of inequivalent multiplets can increase due to “twisting” phenomena. In this regard, the 2D case is exemplar (see \cite{41} for a discussion of the 2D $\mathcal{N} = (4, 4)$ case). For instance, in 2D the PS approach makes the explicit use of two $\mathbb{C}P^1$ coordinates leading to a richer class of multiplets and sigma-models than in 4D. Moreover, 2D supersymmetry clearly has an important role in the classification of both superconformal field theories \cite{42} and string theory \cite{43}. Based on the seminal paper \cite{14}, the recent observation that generalized complex geometry (GCG) \cite{14} arises as the target space geometry of 2D supersymmetric non-linear sigma-models, has also renewed the interest on 2D superspace techniques especially for the $\mathcal{N} = (2, 2)$ cases \cite{14, 45, 46, 47}. A main interest in GCG is due to its importance in string theory compactification with fluxes (see \cite{48} for a review). The 2D $\mathcal{N} = (4, 4)$ case has been less explored but it could be interesting, for example, to generate new bi-Hermitian and generalized hyper-Kähler geometries (besides the works

\footnote{Building on the superconformal projective multiplets of \cite{20, 21}, for a curved geometry projective superfields were first introduced in studying field theory in 5D $\mathcal{N} = 1$ anti-de Sitter superspace \cite{22}.}

\footnote{For the Weyl multiplet of 5D $\mathcal{N} = 1$ conformal supergravity \cite{32} we gave a superspace formulation in \cite{24}. In the 4D $\mathcal{N} = 2$ conformal case we made use in \cite{25} of the Grimm’s superspace geometry \cite{33} while in \cite{26} we considered the Howe’s formulation \cite{34}; the supergravity of \cite{33} is obtained by a partial gauge–fixing of the geometry of \cite{34} but they both describe the Weyl multiplet \cite{35}.}
see [49] for a recent analysis in 2D $\mathcal{N} = (2, 2)$ superspace).

With the previous observations in mind, this work is focused on the development of new superspace techniques for 2D $\mathcal{N} = (4, 4)$ supergravity. In particular there are two main goals of the paper:

(a) a new superspace formulation for 2D $\mathcal{N} = (4, 4)$ conformal supergravity, and;

(b) a two-dimensional generalization of the four and five-dimensional curved projective superspace approach of [23, 24, 25, 26].

In building up the 2D $\mathcal{N} = (4, 4)$ curved PS techniques we follow the same principles used in 4D and 5D: (i) we identify a Wess-Zumino superspace constrained geometric formulation of 2D conformal supergravity; (ii) we introduce covariant supermultiplets and a locally supersymmetric and super-Weyl invariant action principle in 2D PS. Generalizing the flat case of [38, 39, 40], 2D curved PS depends on extra $\mathbb{C}P^1 \times \mathbb{C}P^1$ coordinates; for this reason we call the superspace and supermultiplets bi-projective.

Twenty years ago Gates et. al. presented a superspace formulation for 2D $\mathcal{N} = (4, 4)$ minimal supergravity [52] (see [53] for earlier components results). This was based on the gauging of a $\text{SO}(1,1) \times \text{SU}(2)_V$ tangent space group and a suitable irreducible set of constraints on the torsion of the curved superspace. Another important feature was the realization of the superconformal group. The super-Weyl transformations, that preserve the minimal torsion constraints of [52], are generated by two real superfields $S, S_{ij} = S_{ji}$ through the following infinitesimal variation of the spinor covariant derivative

$$\tilde{\delta} \nabla_{\alpha i} = \frac{1}{2} S \nabla_{\alpha i} + (\gamma^3)_{\alpha}^{\beta} S_{ij} \nabla_{\beta j} + (\gamma^3)_{\alpha}^{\gamma} (\nabla_{\gamma i} S) \mathcal{M} + (\nabla_{\alpha}^{k} S) \mathcal{V}_{ik}.$$  \hspace{1cm} (1.1)

From the previous equation, it turns out that the lowest component of $S$, $S| := S(x, \theta)|_{\theta = 0}$, generates local scale transformations. The second term in (1.1) induces a chiral $\text{SU}(2)_C$ transformation generated by $S_{ij}|$. Special supersymmetry transformations are generated by $\nabla_{\alpha i} S|$ and so on [52]. Moreover, looking at (1.1) it is clear that the $E_{\alpha i}^{A}$ supervielbein does not transform homogeneously under super-Weyl transformations. A natural question

5For 2D $\mathcal{N} = (4, 4)$ supersymmetry HS has been introduced in [50]. In analogy to bi-projective superspace [38], the 2D HS makes use of two sets of harmonics. A prepotential formulation for 2D $\mathcal{N} = (4, 4)$ conformal supergravity has been given in bi-harmonic superspace [51]. A detailed analysis of the relationship between 2D PS and HS superspaces, both in the flat and curved cases, would be useful.

6Here $\nabla_{\alpha i}$ are the complex spinor covariant derivatives, $\mathcal{M}$ and $\mathcal{V}_{kl}$ are respectively the Lorentz and $\text{SU}(2)_V$ generators. The $S, S_{ij}$ superfields are not independent satisfying the differential constraint $\nabla_{\alpha i} S_{kl} = -\frac{1}{2} (\gamma^3)_{\alpha}^{\beta} C_{(i(\kappa} \nabla_{\beta k)} S$. More discussions are in the body of the paper.
which arise from the previous observations is: does a 2D $\mathcal{N} = (4, 4)$ geometry with “homogeneous” super-Weyl transformations exists? The answer is an easy yes. We find one by enlarging the minimal tangent space group with the inclusion of $\text{SU}(2)_C$ transformations.\footnote{For reasons that will become clear in section 2, we will refer to the structure group of the new extended supergravity formulation as $\text{SO}(1,1) \times \text{SU}(2)_L \times \text{SU}(2)_R$.} Then, it is easy to find a set of torsion constraints for the curved superspace that are preserved by super-Weyl transformations generated by a single real unconstrained scalar superfield. Of course the new supergravity is reducible and, upon gauge fixing, gives the minimal multiplet of $[52]$.

The important point is that by using the extended supergravity formulation one can easily couple the geometry to many matter multiplets. Here we present covariant twisted-I (TM-I) [54, 14] and twisted-II (TM-II) [55] matter multiplets. Moreover, covariant bi-projective superfields are consistently defined; this is one of the main results of the paper. An advantage of the new supergravity is that all the matter multiplets considered possess homogeneous super-Weyl transformations in the extended geometry. In the minimal case, the matter multiplet’s super-Weyl transformations, that we will prove to be in-homogeneous and thus a bit more “tricky”, are easily spelled out by the details of the reduction from the extended geometry to the minimal one. The results contained in this paper then explain and extend the analysis of [52] and, more importantly, give a covariant prescription to study general 2D $\mathcal{N} = (4, 4)$ conformal supergravity-matter systems.

This paper is organized as follows. In section 2 we describe the geometry of the new extended $\text{SO}(1,1) \times \text{SU}(2)_L \times \text{SU}(2)_R$ curved superspace. We include the finite super-Weyl transformations and comment about the gauge fixing to the minimal supergravity multiplet. Section 3 contains the coupling to a vector multiplet. We observe that the geometry allows a coupling to an irreducible multiplet which has field strengths describing a covariant TM-II multiplet. We then describe a useful solution of the covariant TM-II constraints and observe how a covariant TM-I matter multiplet emerges as a prepotential for the TM-II. We then discuss again on the gauge fixing to minimal supergravity. Section 4 is devoted to the definition of 2D curved bi-projective superspace. We define a large class of covariant bi-projective superfields and formulate a locally supersymmetric and super-Weyl invariant action principle. Section 5 contains some concluding observations. This paper is accompanied by three technical appendices. In appendix A we collect our 2D conventions. Appendix B summarizes the solution of the Bianchi identities for the supergravity geometry of subsection 2.1. Appendix C contains a derivation of eq. (4.16) which is crucial for the analysis of section 4.
2D $\mathcal{N} = (4, 4)$ conformal supergravity geometries

In this section we present a new covariant superspace description of 2D $\mathcal{N} = (4, 4)$ conformal supergravity based on an extension of the minimal multiplet of Gates et. al. [52]. There are two main differences between the two formulations. The first is the choice of the supergravity structure group. In the minimal case this was $\text{SO}(1,1) \times \text{SU}(2)_V$, in the present case we make use of $\text{SO}(1,1) \times \text{SU}(2)_L \times \text{SU}(2)_R$. The second difference are the super-Weyl transformations. In the minimal case these are generated by a twisted-II multiplet [52] while in the present, extended formulation a real unconstrained scalar superfield is the transformation parameter. As we will see, the minimal multiplet can be obtained by a partial gauge fixing of the super-Weyl and SU(2) transformations.

2.1 New $\text{SO}(1,1) \times \text{SU}(2)_L \times \text{SU}(2)_R$ superspace geometry

Consider a curved 2D $\mathcal{N} = (4, 4)$ superspace, which we will denote by $\mathcal{M}^{2|4,4}$. This is locally parametrized by coordinates $z^M = (x^m, \theta^\mu \theta^i)$ where $m = 0, 1, \mu = +, -$ and $i = 1, 2$. In the light-cone coordinates the superspace is locally parametrized by $z^M = (x^{+\mu}, x_{-\mu}, \bar{\theta}^{+i}, \bar{\theta}^{-i})$ where $x^{+\mu} = \frac{1}{2}(x^1 + x^0)$ and $x_{-\mu} = \frac{1}{2}(x^1 - x^0)$. The Grassmann variables are related one to each other by the complex conjugation rule $(\bar{\theta}^\mu)^* = \bar{\theta}^\mu$ (see appendix A for our 2D conventions). We choose the supergravity structure group to be $\text{SO}(1,1) \times \text{SU}(2)_L \times \text{SU}(2)_R$ and let $\mathcal{M}$, $L_{ij}$, $R_{ij}$ be the corresponding Lorentz, SU(2)$_L$ and SU(2)$_R$ generators. The label $L$ or $R$ are associated to a SU(2) group that respectively acts non-trivially only on the left or right light-cone sectors.

The covariant derivatives $\nabla_A = (\nabla_a, \nabla_{\alpha i}, \bar{\nabla}_{\bar{\alpha} i})$ (or $\nabla_A = (\nabla_{++}, \nabla_{--}, \nabla_{+i}, \nabla_{-i}, \nabla_{++})$) are

\[
\nabla_A = E_A + \Omega_A \mathcal{M} + (\Phi_L)_A^{kl} L_{kl} + (\Phi_R)_A^{kl} R_{kl} , \quad (2.1a)
\]

\[
\nabla_A = E_A + \Omega_A \mathcal{M} + (\Phi_V)_A^{kl} V_{kl} + (\Phi_C)_A^{kl} C_{kl} . \quad (2.1b)
\]

Here $E_A = E_A^M(z) \partial_M$ is the supervielbein, with $\partial_M = \partial/\partial z^M$, $\Omega_A(z)$ is the Lorentz connection, $(\Phi_L)_A^{kl}(z)$ and $(\Phi_R)_A^{kl}(z)$ are the SU(2)$_L$ and SU(2)$_R$ connections, respectively. We have also introduced the generators $V_{kl}$ and $C_{kl}$ defined by

\[
V_{kl} = L_{kl} + R_{kl} , \quad C_{kl} = L_{kl} - R_{kl} , \quad L_{kl} = \frac{1}{2}(V_{kl} + C_{kl}) , \quad R_{kl} = \frac{1}{2}(V_{kl} - C_{kl}) , \quad (2.2)
\]

with $(\Phi_V)_A^{kl}(z)$ and $(\Phi_C)_A^{kl}(z)$ their connections.
The action of the Lorentz generator on the covariant derivatives is as follow

\[ [\mathcal{M}, \nabla_{a\dot{I}}] = \frac{1}{2} (\gamma^3)_{\alpha \beta} \nabla_{\beta i}, \quad [\mathcal{M}, \bar{\nabla}_{\dot{I}}] = \frac{1}{2} (\gamma^3)_{\alpha \beta} \bar{\nabla}^i_{\dot{I}} , \quad [\mathcal{M}, \nabla_{a}] = \epsilon_{ab} \nabla^b , \quad (2.3a) \]

\[ [\mathcal{M}, \nabla_{\pm}] = \pm \frac{1}{2} \nabla_{\pm} , \quad [\mathcal{M}, \bar{\nabla}_{\pm}] = \pm \frac{1}{2} \bar{\nabla}_{\pm} , \quad [\mathcal{M}, \nabla_{\mp}] = \pm \nabla_{\mp} . \quad (2.3b) \]

The action of the SU(2)\(_L\) and SU(2)\(_R\) generators on the covariant derivatives is

\[ [L_{kl}, \nabla_{+i}] = \frac{1}{2} C_{i(k} \nabla_{+l)} , \quad [L_{kl}, \bar{\nabla}^i_{+}] = -\frac{1}{2} \delta^i_{(k} \bar{\nabla}^l_{+)} , \quad [L_{kl}, \nabla_{-i}] = [L_{kl}, \bar{\nabla}^i_{-}] = 0 , \quad (2.4a) \]

\[ [R_{kl}, \nabla_{-i}] = \frac{1}{2} C_{i(k} \nabla_{-l)} , \quad [R_{kl}, \bar{\nabla}^i_{-}] = -\frac{1}{2} \delta^i_{(k} \bar{\nabla}^l_{-)} , \quad [R_{kl}, \nabla_{+i}] = [R_{kl}, \bar{\nabla}^i_{+}] = 0 , \quad (2.4b) \]

and \( \mathcal{V}_{kl} \) and \( C_{kl} \) satisfy

\[ [\mathcal{V}_{kl}, \nabla_{a\dot{I}}] = \frac{1}{2} C_{i(k} \nabla_{a\dot{l})} , \quad [\mathcal{V}_{kl}, \bar{\nabla}_{\dot{I}}] = -\frac{1}{2} \delta^i_{(k} \bar{\nabla}_{a\dot{l})} , \quad (2.5a) \]

\[ [C_{kl}, \nabla_{a\dot{I}}] = \frac{1}{2} (\gamma^3)_{\alpha \beta} C_{i(k} \nabla_{a\beta l)} , \quad [C_{kl}, \bar{\nabla}_{\dot{I}}] = -\frac{1}{2} (\gamma^3)_{\alpha \beta} \delta^i_{(k} \bar{\nabla}_{a\beta l)} . \quad (2.5b) \]

Moreover, it holds \( [L_{kl}, \nabla_{a}] = [R_{kl}, \nabla_{a}] = [\mathcal{V}_{kl}, \nabla_{a}] = [C_{kl}, \nabla_{a}] = 0 \). From the previous equations it is clear that the operator \( \mathcal{V}_{kl} \) generates a diagonal SU(2)\(_V\) subgroup inside SU(2)\(_L\)×SU(2)\(_R\) and \( C_{kl} \) generates chiral SU(2)\(_C\) transformations. The algebra of commutators of the structure group generators is given in appendix A eq. (A,15a) – (A,16a).

The supergravity gauge group is given by local general coordinate and tangent space transformations of the form

\[ \delta_{\mathcal{K}} \nabla_A = [\mathcal{K}, \nabla_A] , \quad (2.6a) \]

\[ \mathcal{K} = K^C \nabla_C + K^M + (K^L)_{kl} L_{kl} + (K^R)_{kl} R_{kl} , \quad (2.6b) \]

\[ \mathcal{K} = K^C \nabla_C + K^M + (K^V)_k \nabla_{k l} + (K^C)_k C_{k l} , \quad (2.6c) \]

with the gauge parameters obeying natural reality conditions, but otherwise arbitrary superfields. Given a tensor superfield \( \mathcal{U}(z) \), with its indices suppressed, it transforms as:

\[ \delta_{\mathcal{K}} \mathcal{U} = \mathcal{K} \mathcal{U} . \quad (2.7) \]

The covariant derivatives algebra has the form

\[ [\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + R_{AB} M + (R_L)_{AB} L_{kl} + (R_R)_{AB} R_{kl} , \quad (2.8a) \]

\[ [\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + R_{AB} M + (R_V)_{AB} V_{kl} + (R_C)_{AB} C_{kl} , \quad (2.8b) \]

where \( T_{AB}^C \) is the torsion, \( R_{AB} \) is the Lorentz curvature, \( (R_L)_{AB} \), \( (R_R)_{AB} \) are the SU(2)\(_L\)×SU(2)\(_R\) curvatures that have been recombined in the second line as \( (R_V)_{AB} \), \( (R_C)_{AB} \) in terms of the generators \( V_{kl}, C_{kl} \).
For the remainder of this section we will always use the $V_{kl}$, $C_{kl}$ parametrization of the SU(2)$_L \times$SU(2)$_R$ group. In that basis it will be trivial to see the reduction of our supergravity multiplet to the minimal one of $[52]$ where the structure group of the curved superspace was chosen to be SO(1,1)$\times$SU(2)$_Y$.

We impose the following constraints on the torsion
\[
T_{\alpha i}{}^j{}_{\beta} = 2i\delta_i^j (\gamma^c)_{\alpha\beta} , \quad T_{\alpha i}{}^j{}_{\beta} = 0 , \quad \text{(dimension 0)} \quad (2.9a)
\]
\[
T_{\alpha i}{}^j{}_{\beta} = T_{\alpha i}{}^j{}_{\gamma} = T_{\alpha i}{}^c{}_{\beta} = 0 , \quad \text{(dimension 1/2)} \quad (2.9b)
\]
\[
\delta^\beta T_{\alpha \beta(jk)} = \delta_{\gamma} T_{\alpha \beta(jk)} = T_{\alpha i}{}^c{}_{\beta} = 0 , \quad \text{(dimension 1)} \quad (2.9c)
\]
along with their complex conjugates.

The solution of the Bianchi identities based on the constraints $\{2.9a\} - \{2.9c\}$ is given for the interested reader in appendix [3]. Here we collect the main results.

The algebra of covariant derivatives based on $\{2.9a\} - \{2.9c\}$ results to be
\[
\{\nabla_{\alpha i}, \nabla_{\beta j}\} = -4i \left(C_{ij} C_{\alpha \beta} N - (\gamma^3)_{\alpha \beta} Y_{ij}\right) M + 4i \left((\gamma^3)_{\alpha \beta} N - (\gamma^a)_{\alpha \beta} A_a \right) V_{ij} + 2i C_{ij} C_{\alpha \beta} Y^{kl} V_{kl} + 2i (\gamma^3)_{\alpha \beta} Y_{(i} C_{j)k} - 4i (\gamma^a)_{\alpha \beta} \varepsilon_{ab} A_b C_{ij} , \quad (2.10a)
\]
\[
\{\bar{\nabla}_a, \bar{\nabla}_b\} = 4i \left(C^{ij} C_{\alpha \beta} \bar{N} - (\gamma^3)_{\alpha \beta} \bar{Y}^{ij}\right) M - 4i \left((\gamma^3)_{\alpha \beta} \bar{N} + (\gamma^a)_{\alpha \beta} \bar{A}_a\right) V_{ij} - 2i C^{ij} C_{\alpha \beta} Y^{lk} V_{kl} + 2i (\gamma^3)_{\alpha \beta} Y_{(i} C^{j)k} - 4i (\gamma^a)_{\alpha \beta} \varepsilon_{ab} \bar{A}_a C^{ij} , \quad (2.10b)
\]
\[
\{\nabla_{\alpha i}, \bar{\nabla}_j\} = 2i \delta_i^j (\gamma^a)_{\alpha \beta} \nabla_a - 4i \left(C_{\alpha \beta} \left(\delta_i^j T + i T_i^j\right) - (\gamma^3)_{\alpha \beta} \left(i \delta_i^j S + S_i^j\right) \right) M + 4i \left(i (\gamma^3)_{\alpha \beta} \bar{T} + (\gamma^a)_{\alpha \beta} B_a \right) V_{ij} + 2i \delta_i^j \left((\gamma^3)_{\alpha \beta} T^{kl} + i C_{\alpha \beta} S^{kl}\right) V_{kl} + 2i \bar{C}_{\alpha \beta} T_i^k C_i^j + 2i (\gamma^3)_{\alpha \beta} S_i^k C_i^j + 2i (\gamma^3)_{\alpha \beta} S^{jk} C_{ik} + 4(\gamma^a)_{\alpha \beta} \varepsilon_{ab} B^k C_{ij} , \quad (2.10c)
\]
\[
[\nabla_a, \nabla_b] = \left(\gamma_a\right)_\gamma (i \delta_i^j S + S_i^j) + \varepsilon_{ab} (\gamma^b)_\beta (\delta_i^j T + i T_i^j) + i \delta_i^j \delta^k B_a + i \delta^k (\gamma^3)_\gamma \varepsilon_{ab} B^k \right) \nabla_k + \left(\gamma_a\right)_{\beta \gamma} (i \delta_i^j S + S_i^j) + \varepsilon_{ab} (\gamma^b)_\beta (\delta_i^j T + i T_i^j) + i \delta_i^j \delta^k B_a + i \delta^k (\gamma^3)_\gamma \varepsilon_{ab} B^k \right) \nabla_k + \left(\gamma_a\right)_{\beta \gamma} \nabla_{\gamma j} N + \frac{2}{3} \varepsilon_{ab} (\gamma^b)_\beta \nabla_{\gamma k} S_{jk} + \frac{2i}{3} (\gamma_a)_\beta \nabla_{\gamma k} T_{jk} - \frac{i}{3} \varepsilon_{ab} (\gamma^b)_\beta \nabla_{\gamma k} Y_{jk} \right) M + \left(\gamma_a\right)_{\beta} \nabla_{\gamma j} N + \frac{1}{6} (\gamma_a)_\beta \delta_i^j \nabla_{\gamma p} S^{lj} p + \frac{1}{6} \varepsilon_{ab} (\gamma^b)_\beta \delta_i^j \nabla_{\gamma p} T^{lj} p - \frac{1}{12} (\gamma_a)_\beta \nabla_{\gamma j} Y_{ljp} C_{pj} - \frac{1}{2} \delta_i^j \left(\bar{A}_a\right)_\beta \right) V_{kl} + \left(\gamma_a\right)_{\beta} \nabla_{\gamma j} N + \frac{1}{6} (\gamma_a)_\beta \delta_i^j \nabla_{\gamma p} S^{lj} p + \frac{1}{6} \varepsilon_{ab} (\gamma^b)_\beta \delta_i^j \nabla_{\gamma p} T^{lj} p + \frac{1}{12} \varepsilon_{ab} (\gamma^b)_\beta \delta_i^j \nabla_{\gamma p} Y_{ljp} - \frac{1}{12} \varepsilon_{ab} (\gamma^b)_\beta \delta_i^j \nabla_{\gamma p} Y_{ljp} , \quad (2.10d)
\]
\[
[\nabla_a, \nabla^j_b] = \left( (\gamma_a)_\beta^\gamma (i\delta^2_a \mathcal{S} + \mathcal{S}^j_k) - \varepsilon_{ab}(\gamma^b_\beta^\gamma (\delta^2_i \mathcal{T} + i\mathcal{T}^j_k) - i\delta^2_b \epsilon_k \mathcal{B}_a - i\delta^2_b (\gamma^2_\beta^\gamma \varepsilon_{ab} \mathcal{B}^b) \right) \nabla^k_{\gamma}
\]
\[
+ \left( C^{jk}_\beta \bar{\nabla}^j_a + C^{jk}(\gamma^2_\beta^\gamma \varepsilon_{ab} \mathcal{A}^b - C^{jk}_\beta \varepsilon_{ab}(\gamma^2_\beta^\gamma \bar{\nabla} \mathcal{T}^j_k + \frac{1}{3} \varepsilon_{ab}(\gamma^2_\beta^\gamma \bar{\nabla} \mathcal{Y}^j_k) \mathcal{Y}^j_k) \nabla_{\gamma k} \right.
\]
\[
+ \left( \frac{1}{2} C^{j(k}_\beta \varepsilon_{ab}(\gamma^2_\beta^\gamma \bar{\nabla} \mathcal{Y}^j_k - \frac{1}{6} C^{j(k}_\beta (\gamma^2_\beta^\gamma \bar{\nabla} \mathcal{Y}^j_k + T^j_k + \frac{1}{3} \varepsilon_{ab}(\gamma^2_\beta^\gamma \bar{\nabla} \mathcal{Y}^j_k) \mathcal{Y}^j_k) \mathcal{Y}^j_k \right)
\]
\[
+ \left( \frac{1}{6} C^{j(k}_\beta \varepsilon_{ab}(\gamma^2_\beta^\gamma \bar{\nabla} \mathcal{Y}^j_k) \mathcal{Y}^j_k - \frac{1}{6} C^{j(k}_\beta (\gamma^2_\beta^\gamma \bar{\nabla} \mathcal{Y}^j_k + T^j_k + \frac{1}{3} \varepsilon_{ab}(\gamma^2_\beta^\gamma \bar{\nabla} \mathcal{Y}^j_k) \mathcal{Y}^j_k) \mathcal{Y}^j_k \right)
\]
\[
\left. \mathcal{C}_{kl} \right), \quad (2.10e)
\]
\[
[\nabla_a, \nabla_b] = -\frac{1}{2} \varepsilon_{ab} \left( i\nabla^k \bar{\nabla} \mathcal{Y}^j_k + \frac{2}{3} (\gamma^3)^\gamma \delta \nabla_{\delta b} \mathcal{S}^l k + \frac{2}{3} \nabla^k \mathcal{Y}^l k + \frac{i}{3} (\gamma^3)^\gamma \delta \nabla_{\delta b} \mathcal{Y}^l k \right) \nabla_{\gamma k}
\]
\[
+ \left( i\nabla^k \bar{\nabla} \mathcal{Y}^l k + \frac{2}{3} (\gamma^3)^\gamma \delta \nabla_{\delta b} \mathcal{S}^l k - \frac{2}{3} \nabla^k \mathcal{Y}^l k + \frac{i}{3} (\gamma^3)^\gamma \delta \nabla_{\delta b} \mathcal{Y}^l k \right) \nabla_{\gamma k}
\]
\[
+ \left( \frac{i}{4} (\gamma^3)^\alpha \beta [\bar{\nabla}_{\alpha k}, \nabla^k_{\beta l}] \bar{\nabla}^l \mathcal{Y}^l k - \frac{1}{4} (\gamma^3)^\alpha \beta [\bar{\nabla}_{\alpha k}, \nabla^k_{\beta l}] \bar{\nabla}^l \mathcal{Y}^l k - \frac{i}{12} [\bar{\nabla}_{\alpha k}, \nabla^k_{\beta l}] \mathcal{T}^l k + 4 \mathcal{T}^l k \bar{\nabla}^l \mathcal{T}^l k + 4 \mathcal{T}^l k \bar{\nabla}^l \mathcal{T}^l k + 4 \mathcal{T}^l k \bar{\nabla}^l \mathcal{T}^l k \right) \mathcal{T}^l k
\]
\[
+ \left( \frac{i}{16} [\bar{\nabla}_{\alpha k}, \nabla^k_{\beta l}] \bar{\nabla}^l \mathcal{T}^l k - \frac{i}{16} [\bar{\nabla}_{\alpha k}, \nabla^k_{\beta l}] \bar{\nabla}^l \mathcal{T}^l k - \frac{i}{16} (\gamma^3)^\alpha \beta [\bar{\nabla}_{\alpha p}, \nabla^p_{\beta l}] \mathcal{Y}^l k + 8 \mathcal{S}^l k \mathcal{T}^l k + 8 i \mathcal{T}^l k \mathcal{T}^l k \right) \mathcal{T}^l k
\]
\[
+ \left( \frac{i}{48} [\bar{\nabla}_{\alpha k}, \nabla^k_{\beta l}] \mathcal{Y}^l k - \frac{i}{48} [\bar{\nabla}_{\alpha k}, \nabla^k_{\beta l}] \mathcal{Y}^l k - 4 \mathcal{Y}^l k \mathcal{T}^l k \right) \mathcal{T}^l k \right). \quad (2.10f)
\]
Here the dimension-1 components of the torsion obey the symmetry relations
\[
Y_{ij} = Y_{ji} , \quad T_{ij} = T_{ji} , \quad S_{ij} = S_{ji} , \quad (2.11)
\]
and the reality conditions
\[
(N)^* = \bar{N} , \quad (T)^* = T , \quad (S)^* = S , \quad (A_a)^* = \bar{A}_a , \quad (B_a)^* = B_a , \quad (2.12a)
\]
\[
(Y^{ij})^* = \bar{Y}^{ij} , \quad (T^{ij})^* = T_{ij} , \quad (S^{ij})^* = S_{ij} . \quad (2.12b)
\]
All of the previous superfields are Lorentz scalars except the vectors $A_a$ and $B_a$. All of the superfields except $Y_{ij}, S_{ij}, T_{ij}$ are invariant under the action of the SU(2)$_V$ generator.
algebra and constraints reduce, up to trivial field redefinitions, to the minimal supergravity of those ones (see appendix B.3).

previous dimension-3/2 differential constraints. The dimension-2 Bianchi identities are omitted relations that can be easily obtained by complex conjugation of (2.15a)–(2.15g).

\[ \nabla_\alpha Y^{jk} = (\gamma^3)_\alpha^\beta C^{ij} \gamma_\beta^k N, \]

\[ \nabla_\alpha A_\beta = -\varepsilon_{bc}(\gamma^\gamma)_\alpha^\beta \nabla_\beta N, \]

\[ \nabla_\beta Y^{kl} = -2 \nabla_\beta S^{kl} = -2i(\gamma^3)^\gamma \nabla_\gamma T^{kl}, \]

\[ \nabla_\alpha S = \frac{1}{2} \gamma_\alpha^\beta \nabla_\beta N + \frac{1}{3}(\gamma^3)_\alpha^\beta \nabla_\beta \nabla^{kl} T^{ki} - \frac{i}{6} \nabla_\alpha T^{ki}, \]

\[ \nabla_\alpha T = -\frac{1}{2} \nabla_\alpha N + \frac{1}{3}(\gamma^3)_\alpha^\beta \nabla_\beta S^{ki} + \frac{1}{6}(\gamma^3)_\alpha^\beta \nabla_\beta Y^{ki}, \]

\[ \nabla_\beta A_\alpha = A_\alpha^\beta - \frac{1}{3}(\gamma_a)_\beta^\gamma \nabla_\gamma S^{kj} + \frac{i}{3} \varepsilon_{ab}(\gamma^a)_\beta^\gamma \nabla_\gamma T^{kj}, \quad (\gamma^a)_\alpha^\beta A_\alpha^\beta = 0, \]

\[ \nabla_\beta B_\alpha = \frac{1}{6}(\gamma_a)_\beta^\gamma \nabla_\gamma S^{jk} - \frac{1}{6} \varepsilon_{ab}(\gamma^a)_\beta^\gamma \nabla_\gamma T^{jk} + \frac{i}{6}(\gamma_a)_\beta^\gamma \nabla_\gamma Y^{jk} + \frac{i}{2} A_\alpha^\beta, \]

where the dimension-3/2 superfield \( A_\alpha^\beta \) has been introduced as the gamma-traceless part of \( \nabla_\beta A_\alpha \) according to (B.33). In the list of dimension-3/2 Bianchi identities we have omitted relations that can be easily obtained by complex conjugation of (2.15a)–(2.15g).

It is worth noting that the supergravity multiplet is completely determined by the previous dimension-3/2 differential constraints. The dimension-2 Bianchi identities are solved by making use of (2.15a)–(2.15g) and differential equations which are consequences of those ones (see appendix B.3).

We conclude by noting that if one imposes \( Y_{ij} = S_{ij} = T_{ij} = A_\alpha = B_\alpha = 0 \) the algebra and constraints reduce, up to trivial field redefinitions, to the minimal supergravity

\[ V_{kl} Y_{ij} = \frac{1}{2}(C_i(k Y_{lj} + C_j(k Y_{li})), \]

with \( S_{ij}, T_{ij} \) enjoying the same SU(2) transformation properties of \( Y_{ij} \). The transformation rules of the dimension-1 components of the torsion under the action of the \( C_{kl} \) operator are less trivial. It holds\(^8\)

\[ C_{kl} T = -S_{kl}, \quad C_{kl} S = -S_{kl}, \quad C_{kl} C_{ij} = -C_i(k C_{lj}) T, \quad C_{kl} N = -Y_{kl}, \quad C_{kl} Y_{ij} = -C_i(k C_{lj}) N, \quad C_{kl} N = -Y_{kl}, \quad C_{kl} Y_{ij} = -C_i(k C_{lj}) N, \]

\[ C_{kl} A_\alpha = C_{kl} A_\alpha = C_{kl} B_\alpha = 0. \]

\(^8\)These relations can be obtained for example by computing \([C_{kl}, \nabla_{\alpha i}], \nabla_{\beta j} \)], \([C_{kl}, \nabla_{\alpha i}, \nabla_{\beta j}] \) and \([C_{kl}, \nabla_{\alpha i}, \nabla_{\beta j}] \) and by using the equations (2.10a)–(2.10c) together with the commutation relations of the structure group operators given in appendix A eq. (A.16a)–(A.16c).
multiplet of Gates et al. [52]. We will discuss more about the connection between our supergravity formulation and the minimal one later in subsection 2.3 and 3.3.

2.2 Super-Weyl transformations

Here we consider super-Weyl transformations in analogy to the analysis of Howe and Tucker [56]. By direct computation, it can be shown that the constraints (2.9a)–(2.9c) are invariant under the finite super-Weyl transformations of the form:

$$\nabla'_{ai} = e^{\frac{2S}{i}} \left( \nabla_{ai} + (\gamma^3)_{a}^{\gamma}(\nabla_{\gamma i}S)\mathcal{M} - (\nabla_{ak}S)\nabla_i^k - (\gamma^3)_{a}^{\gamma}(\nabla_{\gamma k}S)\mathcal{C}_i^k \right), \quad (2.16a)$$

$$\nabla'^i_\alpha = e^{\frac{2S}{i}} \left( \nabla^i_\alpha + (\gamma^3)_{a}^{\gamma}(\nabla^i_{\gamma}S)\mathcal{M} + (\nabla^k_\alpha S)\nabla^i_k + (\gamma^3)_{a}^{\gamma}(\nabla^k_{\gamma}S)\mathcal{C}_i^k \right), \quad (2.16b)$$

$$\nabla'_a = e^{S} \left( \nabla_a + \frac{i}{2}(\gamma_a)^{\gamma}\nabla^k_{\gamma k} - \frac{i}{2}(\gamma_a)^{\gamma}(\nabla^k_{\gamma k}S)\nabla^k - \varepsilon_{ab}(\nabla^b S)\mathcal{M} - \frac{i}{8}(\gamma_a)^{\gamma}(\nabla^k_{\gamma k}S)\nabla^k - \frac{i}{8}\varepsilon_{ab}(\gamma^b)^{\gamma}(\nabla^k_{\gamma k}S)\mathcal{C}_{kl} \right). \quad (2.16c)$$

Here the parameter $S(z)$ is a real unconstrained superfield $(S)^* = S$ (not to be confused with the torsion component $S$). To ensure the invariance of the algebra under the super-Weyl transformations, the dimension-1 components of the torsion have to transform as

$$N' = e^S \left( N + \frac{i}{8}(\gamma^3)^{\gamma}(\nabla_{\gamma k}S)\nabla^k \right), \quad (2.17a)$$

$$\mathcal{T}' = e^S \left( \mathcal{T} + \frac{i}{16}(\gamma^3)^{\gamma}(\nabla_{\gamma k}S)\nabla^k \right), \quad (2.17b)$$

$$S' = e^S \left( S + \frac{1}{16}(\nabla_{\gamma k}S)\nabla^k \right), \quad (2.17c)$$

$$Y'_{ij} = e^S \left( Y_{ij} + \frac{i}{8}(\nabla_{\gamma i}S)\nabla^j \right), \quad (2.17d)$$

$$\mathcal{T}'_{ij} = e^S \left( \mathcal{T}_{ij} + \frac{1}{16}(\gamma^3)^{\gamma}(\nabla_{\gamma i}S)\nabla^j \right), \quad (2.17e)$$

$$S'_{ij} = e^S \left( S_{ij} + \frac{1}{16}(\nabla_{\gamma i}S)\nabla^j \right), \quad (2.17f)$$

$$A'_a = e^S \left( A_a - \frac{i}{8}(\gamma_a)^{\gamma}(\nabla_{\gamma k}S)\nabla^k - \frac{3i}{8}(\gamma_a)^{\gamma}(\nabla_{\gamma k}S)(\nabla^k S) \right), \quad (2.17g)$$

$$B'_a = e^S \left( B_a - \frac{i}{16}(\gamma_a)^{\gamma}(\nabla_{\gamma k}S)\nabla^k - \frac{3}{8}(\gamma_a)^{\gamma}(\nabla_{\gamma k}S)(\nabla^k S) \right), \quad (2.17h)$$

together with their complex conjugates. The proof that the covariant derivatives algebra of subsection 2.1 is invariant under the previous super-Weyl transformations is quite long but straightforward and it is left as a useful exercise to the interested reader.
For later use, we rewrite the super-Weyl transformations of the spinor covariant derivatives (2.16a) and (2.16b) in a form where the left/right Lorentz spin or indices are explicit

\[ \nabla'_{+i} = e^{\frac{i}{2}S} \left( \nabla_{+i} + (\nabla_{+i}S)M + 2(\nabla_{+i}S)L_{ik} \right) , \]  
(2.18a)

\[ \nabla''_{+i} = e^{\frac{i}{2}S} \left( \nabla_{+i} + (\nabla_{+i}S)M - 2(\nabla_{+i}S)L^{ik} \right) , \]  
(2.18b)

\[ \nabla'_{-i} = e^{\frac{i}{2}S} \left( \nabla_{-i} - (\nabla_{-i}S)M + 2(\nabla_{-i}S)R_{ik} \right) , \]  
(2.18c)

\[ \nabla''_{-i} = e^{\frac{i}{2}S} \left( \nabla_{-i} - (\nabla_{-i}S)M - 2(\nabla_{-i}S)R^{ik} \right) . \]  
(2.18d)

In this form one observes that, under super-Weyl transformations, only the SU(2)_{L} connections of the left covariant spinor derivatives transform non-homogeneously and, similarly, only the SU(2)_{R} connections of the right spinor derivatives transform non-homogeneously.

Observing (2.17a)–(2.17h), it is clear that one can gauge away all the theta independent dimension-1 components of the torsion. In particular using both super-Weyl and the supergravity gauge transformations one could choose a Wess-Zumino gauge in which the remaining fields are those of the Weyl multiplet of conformal supergravity which, in particular, does not contain auxiliary fields [52]. One easy way to prove this statement is by noting that under gauge fixing, the extended supergravity multiplet reduces to the minimal one and then follow the discussion of [52]. More on the fields content and the Wess-Zumino superspace reduction of the SO(1,1)×SU(2)_{L}×SU(2)_{R} extended geometry is planned to be the subject of a separate analysis and is beyond the scope of this paper.

### 2.3 On the minimal supergravity multiplet

So far in this section we have introduced a new superspace formulation for an extended supergravity multiplet having the structure group SO(1,1)×SU(2)_{L}×SU(2)_{R}. Its super-Weyl transformations, generated by an unconstrained real scalar superfield, induce homogeneous transformations on the inverse supervielbein in the spinor derivatives (2.16a), (2.16b). We have already mentioned that the extended multiplet can be gauged fixed to the minimal supergravity multiplet. For most applications, the minimal formulation is more convenient to work with even if, as explicitly described in the following, the super-Weyl transformations are more tricky. Let us consider here in greater detail the implications of the minimal gauge fixing.

First, we impose the following gauge condition in the supergravity multiplet

\[ S_{ij} = T_{ij} = Y_{ij} = A_{a} = B_{a} = 0 . \]  
(2.19)
It can be proved that the superfields $S_{ij}$, $T_{ij}$, $Y_{ij}$, $A_a$ and $B_a$ are pure gauge degrees of freedom under super-Weyl transformations; we will come back to this important point in subsection 3.3.

One readily observes that under (2.19) all the $R_C$ curvatures are identically zero and we can choose

$$(\Phi_C)_A{}^{kl} = 0 ,$$

in the covariant derivatives (2.1b). The resulting constraints on the surviving superfields $N, S, T$ are

$$\nabla_\beta^i N = 0 , \quad \nabla_\alpha^i S = \frac{i}{2}(\gamma_3)^\alpha_\beta \nabla_\beta^i N , \quad \nabla_\alpha^i T = -\frac{1}{2} \bar{\nabla}_\alpha^i N .$$

These, up to field redefinitions, are the constraints that characterize the dimension-1 torsion components of the minimal supergravity of [52]. In particular they describe a covariant extension of the dimension-1/2 differential constraints of the twisted-I multiplet [54, 14, 41].

The structure group of the resulting minimal multiplet now has a remaining local SO(1,1)$\times$SU(2)$_V$ symmetry. Moreover, the gauge choice (2.19) still has residual super-Weyl transformations (2.17a)–(2.17h). For simplicity, we restrict ourselves to infinitesimal transformations; the finite transformations can be easily derived along the same lines. To distinguish between the super-Weyl transformations of the extended and minimal geometry, we redefine in the minimal case the real superfield $S$ with $S$. Let us look again at the transformation (2.16a), which in the infinitesimal limit is

$$\delta \nabla_\alpha^i = \frac{1}{2} S \nabla_\alpha^i + (\gamma_3)^\alpha_\gamma (\nabla_\gamma S) M - (\nabla_\alpha S) V^k_i - (\gamma_3)^\alpha_\gamma (\nabla_\gamma S) C_i^k .$$

The last term in (2.22) tells us that the super-Weyl transformations alone break the gauge $(\Phi_C)_A{}^{kl} = 0$. This can be fixed by adding a compensating SU(2)$_C$ transformation to cancel the induced $(\Phi_C)_{\alpha_i{}^{kl}}$ spinor connection in (2.22).

An infinitesimal SU(2)$_C$ transformation of the spinor covariant derivatives, with real parameter $S_{ij} = (S^{ij})^*$, is

$$\delta_C \nabla_\alpha^i = [S^{kl} C_{kl}, \nabla_\alpha^i] = -(\gamma_3)^\alpha_\beta S_{ij} \nabla_{\beta j} - (\nabla_\alpha S_{kl}) C_i^k .$$

Imposing the following differential constraint between $S$ and $S_{ij}$

$$(\nabla_\alpha S_{kl}) = -\frac{1}{2}(\gamma_3)^\alpha_\beta C_{i(k}(\nabla_{\beta k})^i S) ,$$

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one obtains the modified super-Weyl transformation that preserves the gauge \((\Phi_c)_{A^{kl}} = 0\). This is given by

\[
\tilde{\delta} \nabla_{\alpha} = \frac{1}{2} S \nabla_{\alpha} + (\gamma^3)_\alpha^\beta S_i^j \nabla_{\beta j} + (\gamma^3)_\alpha^\gamma (\nabla_{\gamma i} S) M - (\nabla_{\alpha k} S) V_i^k .
\] (2.25)

Note that due to the compensating \(SU(2)_c\) transformation, the supervielbein in (2.25) does not transform homogeneously anymore. Equation (2.25) was first derived in [52].

Note that eq. (2.24) is the dimension-1/2 differential constraint of a twisted-II multiplet [55, 41]. It implies the following dimension-1 differential constraint on \(S\) and \(S_{ij}\)

\[
\nabla_{\alpha i} \nabla_{\beta j} S = -4i C_{\alpha \beta} NS_{ij} + \frac{1}{6} C_{ij} (\gamma^3)_\alpha^\beta (\nabla_{\gamma k} \nabla_{\gamma l} S_{kl}) ,
\] (2.26a)

\[
[\nabla_{\alpha}^i, \nabla_{\beta}^j] S = -\frac{1}{6} C_{ij} C_{\alpha \beta} (\gamma^3)^\gamma [\nabla_{\gamma k}, \nabla_{\alpha l}] S_{kl} + \frac{1}{6} C_{ij} (\gamma^3)_\alpha^\beta [\nabla_{\delta k}, \nabla_{\delta l}] S_{kl}
+ 4i \varepsilon^{ab} (\gamma_a)_{\alpha \beta} \nabla_b S_{ij} - 8 ((\gamma^3)_\alpha^\beta S + i C_{\alpha \beta} T) S_{ij} .
\] (2.26b)

By using the previous two results and (2.14a)–(2.14c), it can be explicitly observed that (2.19) are preserved by the \(\tilde{\delta}\) transformation and that the dimension-1 torsion components of the minimal multiplet transform according to the following rules

\[
\tilde{\delta} N = SN + \frac{i}{8} (\gamma^3)^\gamma \delta (\nabla_{\gamma k} \nabla_{\delta l} S) ,
\] (2.27a)

\[
\tilde{\delta} T = ST + \frac{i}{16} (\gamma^3)^\gamma [\nabla_{\gamma k}, \nabla_{\delta l}] S ,
\] (2.27b)

\[
\tilde{\delta} S = SS + \frac{1}{16} ([\nabla_{\gamma k}, \nabla_{\gamma l}] S) .
\] (2.27c)

The transformations of the \(\nabla_{\alpha}^i\) covariant derivative can be trivially obtained by complex conjugation of (2.25). We conclude by observing that for the vector covariant derivative it holds that

\[
\tilde{\delta} \nabla_a = S \nabla_a + \frac{1}{2} (\gamma_a)^{\gamma \delta} (\nabla_{\gamma k} S) \nabla_{\delta k}^i + \frac{1}{2} (\gamma_a)^{\gamma \delta} (\nabla_{\gamma k} S) \nabla_{\delta k}
+ \varepsilon_{ab} (\nabla^b S) M - \varepsilon_{ab} (\nabla^b S_{kl}) V_{kl} ,
\] (2.28)

where (2.26b) has been used.

3 Coupling to an Abelian vector multiplet

Let us couple the extended conformal supergravity multiplet to an off-shell vector multiplet. We describe here in detail the case of a single Abelian vector multiplet, which
will be interpreted as a real central charge. The resulting multiplet is of particular importance since it plays the role of a conformal compensator for supergravity. The covariant vector multiplet has field strength described by a scalar twisted-II multiplet. The covariant coupling with the algebra is useful because the structure group and super-Weyl transformations will be easily indicated by consistency of the geometry.

### 3.1 Twisted-II vector multiplet

The coupling of the supergravity geometry to an Abelian vector multiplet is achieved by modifying the covariant derivatives as follows

$$\nabla_A = \nabla_A + V_A Z,$$

with $V_A(z)$ the U(1)$_Z$ gauge connection. The gauge transformations of the covariant derivatives are

$$\delta Z \nabla_A = [\tau Z, \nabla_A],$$

with $\tau(z)$ the parameter of the U(1)$_Z$ transformations. The operator $Z$ is conveniently interpreted as a real central charge $(Z)^* = Z$. The multiplet introduced in this way is reducible. One can then impose appropriate covariant constraints on some components of the gauge-invariant field strength $F_{AB}$ which appears in the algebra of gauge-covariant derivatives

$$[\nabla_A, \nabla_B] = T_{ABC}^C \nabla_C + R_{AB}^D M + (R_V)_{AB}^{kl} V_{kl} + (R_C)_{AB}^{kl} C_{kl} + F_{AB} Z.$$

For consistency the field strength $F_{AB}$ has to satisfy the Bianchi identities

$$\sum_{[ABC]} \left( \nabla_A F_{BC} - T_{AB}^D F_{DC} \right) = 0.$$

Here a graded cyclic sum was assumed. The torsion $T_{AB}^C$ and curvatures $R_{AB}^D$, $(R_V)_{AB}^{kl}$ and $(R_C)_{AB}^{kl}$ are the ones appearing in (2.10a)–(2.10f). Note that in (3.4) we used the $\nabla_A$ derivatives instead of $\nabla_A$ since the field strength is neutral with respect to the central charge $Z$. Since the torsion and curvatures are also neutral, we will always use $\nabla_A$ in the Bianchi identities.
In the limit of flat superspace one can easily find two distinct irreducible representations for the vector multiplet field strength \[57\] . The first is described by the constraints

\[
F_{\alpha i\beta j} = -2C_{\alpha\beta}C_{ij}\bar{W}, \quad F^{ij}_{\alpha\beta} = -2C_{\alpha\beta}C^{ij}W, \quad F^{j}_{\alpha\beta} = 2i\delta^j_i(C_{\alpha\beta}P + i\gamma^3_{\alpha\beta}Q),
\]

\[(3.5a)\]

\[
F_{a\beta j} = -\frac{i}{2}(\gamma_a)_{\beta}\bar{D}_{\gamma j}\bar{W}, \quad F^{j}_{a\beta} = \frac{i}{2}(\gamma_a)_{\beta}D^j_{\gamma}W,
\]

\[(3.5b)\]

\[
F_{ab} = -\frac{1}{16}\varepsilon_{ab}\left((\gamma^3)^{\gamma\delta}D_{\gamma k}D^k_{\delta}W + (\gamma^3)^{\gamma\delta}D_{\gamma k}\bar{D}^k_{\delta}\bar{W}\right),
\]

\[(3.5c)\]

where \(D_A\) are the flat superspace covariant derivatives. The complex superfield \(W\) (\(\bar{W} = (W)^*\)) and the real superfields \(P, Q\) ((\(P\))^* = \(P\), (\(Q\))^* = \(Q\)) satisfy the differential constraints of a twisted-I multiplet (TM-I) \[54, 14, 41\]

\[
D_{\alpha i}\bar{W} = 0, \quad \bar{D}_{i\alpha}W = 0, \quad D_{\alpha i}P = -\frac{i}{2}\bar{D}_{\alpha i}\bar{W}, \quad D_{\alpha i}Q = \frac{1}{2}(\gamma^3)_{\alpha\beta}\bar{D}_{\beta i}\bar{W}.
\]

\[(3.6)\]

Note that the previous vector multiplet can be easily obtained by dimensionally reducing from 4D to 2D the well known 4D, \(N = 2\) vector multiplet constraints \[60\].

A second irreducible set of constraints for the vector multiplet field strength can be proven to be

\[
F_{\alpha i\beta j} = \left((\gamma^3)_{\alpha\beta}W_{ij} + \frac{1}{2}C_{\alpha\beta}C_{ij}F\right), \quad F^{ij}_{\alpha\beta} = \left((\gamma^3)_{\alpha\beta}W^{ij} + \frac{1}{2}C_{\alpha\beta}C^{ij}F\right),
\]

\[(3.7a)\]

\[
F^{j}_{\alpha\beta} = 0, \quad F_{a\beta j} = \frac{i}{4}(\gamma_a)_{\beta}\bar{D}_{\gamma j}\bar{F}, \quad F^{j}_{a\beta} = -\frac{i}{4}(\gamma_a)_{\beta}D^j_{\gamma}F,
\]

\[(3.7b)\]

\[
F_{ab} = -\frac{1}{48}\varepsilon_{ab}\left(D_{\gamma}^{k}D^{\gamma l}W_{kl} + \bar{D}_{\gamma}^{k}\bar{D}^{\gamma l}\bar{W}_{kl}\right),
\]

\[(3.7c)\]

provided that the real superfields \(W_{ij}, F\) satisfy the constraints

\[
D_{\alpha i}W_{jk} + \frac{1}{2}C_{ij}(\gamma^3)_{\alpha\beta}D_{\beta k}F = 0, \quad (W_{ij})^* = W^{ij}, \quad (F)^* = F.
\]

\[(3.8)\]

Then one sees that \(W_{ij}\) and \(F\) describe a twisted-II multiplet \[55, 41\].

It is interesting to note that the previous flat vector multiplet constraints can not be both consistently lifted to a coupling with the supergravity of subsection 2.1. The point is that once the vector multiplet is coupled to supergravity by using eq. \(3.1\) and \(3.3\), the structure group and super-Weyl transformation properties of the vector multiplet field strength \(F_{AB}\) are fixed by the geometry. In particular, by considering the commutator \([C_{kl}, \{\nabla_{\alpha i}, \nabla_{\beta j}\}]\) and eq. \(3.3\) together with the constraints \(3.5a\) one observes that the TM-I type of constraints on the field strength is inconsistent with the \(C_{kl}\) transformations.

\[9\] In 2D \(\mathcal{N} = (2, 2)\), dual formulations of minimal vector multiplets are also known, e. g. \[58, 59\].
Therefore, the constraints (3.5a) could not be extended to our supergravity case without fixing the SU(2)\textsubscript{C} group\textsuperscript{10}

On the other hand, by using the same arguments, it follows that the constraints (3.7a) are consistent with the \( C_{kl} \) transformations provided that the real superfields \( W_{ij}, F \) satisfy in the curved case

\[
C_{kl}F = 2W_{kl}, \quad C_{kl}W_{ij} = \frac{1}{2}C_{i(k}C_{l)j}F.
\]

Then, one can check that the constraints (3.7a)–(3.7c), in the curved geometry of subsection 2.1, become

\[
F_{\alpha i\beta j} = -\frac{1}{4} \epsilon_{\alpha\beta} \gamma^k \nabla_k W_{ij} + \bar{\gamma}^k \nabla_k \bar{W}_{ij} + 24i \left( \bar{N} - N \right) F + 24i \left( Y_{kl} - \bar{Y}^{kl} \right) W_{kl}.
\]

Here the superfields \( W_{ij}, F \) enjoy the covariant extension of the TM-II differential constraints

\[
\nabla_{\alpha i} W_{jk} = -\frac{1}{2} C_{i(j} \beta \gamma^{\alpha \beta} \nabla_{\beta k}) F, \quad \left( W_{ij} \right)^* = W^{ij}, \quad \left( F \right)^* = F,
\]

along with a complex conjugate constraint. The \( W_{ij}, F \) superfields are Lorentz scalars. Under SU(2)\textsubscript{V} transformations it holds \( \mathcal{V}_{kl} F = 0 \) and \( \mathcal{V}_{kl} W_{ij} = \frac{1}{2} \left( C_{i(k} W_{l)j} + C_{j(k} W_{l)i} \right) \).

The SU(2)\textsubscript{C} transformations are given in (3.9). By direct, but not short, computations one can prove that the Bianchi identities (3.4) are then identically satisfied.

As a final remark we observe that the consistency of (3.3) requires the superfields \( W_{ij}, F \) to transform homogeneously under the super-Weyl transformations (2.16a)–(2.16c), i. e.

\[
W'_{ij} = e^S W_{ij}, \quad F' = e^S F.
\]

Note that the TM-II differential constraint in eq. (3.11), is then invariant under super-Weyl transformations (3.12).

If one reduces the curved geometry to the one of the minimal supergravity multiplet, according to the discussion in subsection 2.3 the consistent infinitesimal super-Weyl transformations are

\[
\tilde{\delta} W_{ij} = S W_{ij} + S_{ij} F, \quad \tilde{\delta} F = S F - 2S_{kl} W_{kl}.
\]

\textsuperscript{10}In the case of minimal supergravity [52] one can prove that the constraints (3.5a) can be consistently coupled to the algebra. However, such coupling results to be inconsistent with the super-Weyl transformations of the minimal multiplet [61].
3.2 Chiral prepotential of TM-II and covariant matter TM-I

Here we want to prove the following statement: given a chiral super field $W$ invariant under structure group and super-Weyl transformations

$$\mathcal{M} W = \mathcal{V}_{kl} W = C_{kl} W = 0 , \quad W' = W ,$$

(3.14)

and subject to the conditions

$$\nabla^i W = 0 , \quad \nabla_{\alpha i} \bar{W} = 0 , \quad (W)^* = \bar{W} ,$$

(3.15a)

$$\nabla_{\alpha(i} \nabla_{\beta j)} W = \bar{\nabla}_{\alpha(i} \bar{\nabla}_{\beta j)} \bar{W} , \quad (\gamma^3)^{\alpha\beta} \nabla_{\alpha i} \nabla_{\beta i} W = (\gamma^3)^{\alpha\beta} \bar{\nabla}_{\alpha i} \bar{\nabla}_{\beta i} \bar{W} ,$$

(3.15b)

then the real descendant operators defined by

$$\Sigma_{ij} = \frac{1}{4} \nabla_{\alpha i} \nabla_{\beta j} W = \frac{1}{4} \bar{\nabla}_{\alpha i} \bar{\nabla}_{\beta j} \bar{W} = (\Sigma^{ij})^* ,$$

(3.16a)

$$\Sigma = -\frac{1}{4} (\gamma^3)^{\alpha\beta} \nabla_{\alpha i} \nabla_{\beta i} W = -\frac{1}{4} (\gamma^3)^{\alpha\beta} \bar{\nabla}_{\alpha i} \bar{\nabla}_{\beta i} \bar{W} = (\Sigma)^* ,$$

(3.16b)

define a covariant TM-II satisfying all the conditions (3.9), (3.11) and (3.12) with the identifications $W_{ij} = \Sigma_{ij}$ and $F = \Sigma$. Alternatively, this states that given a covariant TM-II, a constrained prepotential $\Sigma$ is given by a superfield $W$ satisfying (3.14)–(3.15b).

The proof of the previous statement involves some easy but instructive computations. Using (3.16a)–(3.16b), (3.14), (2.3a)–(2.5b) one obtains $\mathcal{M} \Sigma = \mathcal{M} \Sigma_{ij} = \mathcal{V}_{kl} \Sigma = 0$ and

$$\mathcal{V}_{kl} \Sigma_{ij} = \frac{1}{2} (C_{ik} \Sigma_{lj} + C_{lj} \Sigma_{ik}) , \quad C_{kl} \Sigma = 2 \Sigma_{kl} , \quad C_{kl} \Sigma_{ij} = \frac{1}{2} C_{i(k} \Sigma_{l)j} \Sigma .$$

(3.17)

Some $\nabla$-algebra gives

$$\nabla_{\alpha(i} \Sigma_{jk)} = \bar{\nabla}_{\alpha(i} \Sigma_{jk)} = 0 ,$$

(3.18a)

$$\nabla_{\alpha} \Sigma_{ij} = \frac{3i}{2} (\gamma^a)_{\alpha}^\beta \nabla_{\alpha} \bar{\nabla}_{\beta i} \bar{W} , \quad \nabla_{\alpha i} \Sigma = i \epsilon^{ab} (\gamma^a)_{\alpha}^\beta \nabla_{\beta i} \bar{\nabla} \bar{W} .$$

(3.18b)

The equations (3.18a)–(3.18b) then imply the TM-II differential constraint (3.11)

$$\nabla_{\alpha i} \Sigma_{jk} = -\frac{1}{2} C_{i(j} (\gamma^3)^{\alpha}^\beta \nabla_{\beta k)} \Sigma .$$

(3.19)

To conclude the proof that $\Sigma_{ij}$, $\Sigma$ describe a TM-II according to (3.9), (3.11) and (3.12), one has to prove that under super-Weyl transformations it holds $(\Sigma_{ij})' = e^S \Sigma_{ij}$ and

\[11\] It is worth noting that in the flat case a more complete analysis of TM-II constraints in terms of prepotentials has been described in [57]. This partly involved the use of a form of bi-projective superspace. Within the scope of the present paper the constrained prepotentials given in this subsection are enough.
\[ \Sigma' = e^\delta \Sigma. \] This can be easily seen by using the equations \((3.14)-(3.16)\) and the super-Weyl transformations of the covariant derivatives \((2.16a)\) and \((2.16b)\).

An irreducible realization for the superfield \(W\) is given by the chiral component of a covariant twisted-I multiplet. This is described by the superfields \(W, P, Q\). They are consistently chosen to be invariant under all the \(\text{SO}(1,1) \times \text{SU}(2)_L \times \text{SU}(2)_R\) and super-Weyl transformations and enjoy the following constraints\(^{12}\)

\[
\bar{\nabla}_\dot{\alpha}^i W = 0, \quad \nabla_\gamma k Q = \frac{1}{2} (\gamma^3)_{\dot{\gamma}\dot{\delta}} \delta_\delta k \bar{W}, \quad \nabla_\alpha i P = -\frac{i}{2} \bar{\nabla}_\alpha i \bar{W}, \quad (W)^* = \bar{W}, \quad (P)^* = P, \quad (Q)^* = Q. \tag{3.20a}
\]

In (3.20a) we have omitted some constraints that can be obtained by complex conjugation. Using (3.20a), it is easy to prove the relation

\[
\nabla_\alpha i \nabla_\beta j W = \nabla_\alpha i \nabla_\beta j \bar{W} - 4 C_{ij}^{a} (\gamma^a)_{\alpha\beta} \nabla_a P, \tag{3.21}
\]

which implies (3.15a). It is worth to mention that in [70], where the interested reader is referred, we present the solution of the covariant TM-I constraints in the language of the bi-projective superspace of section [4].

We conclude this subsection by remarking that there is a crucial difference between the TM-I prepotential introduced here and the supergravity multiplet in the minimal gauge \((2.19)-(2.20)\) described by the torsion components \(N, S, T\). The superfields \((W, P, Q)\) are invariant under super-Weyl transformations while \((N, S, T)\) are not and transform inhomogeneously according to \((2.27a)-(2.27c)\). This difference emphasizes that, even if both the sets of superfields consistently satisfy the covariant extensions of the dimension-1/2 TM-I differential constraints, \((W, P, Q)\) are matter superfields while \((N, S, T)\) are supergravity torsion components.

### 3.3 On the minimal supergravity multiplet: II

In subsection [2.3] we have described the relation between the extended \(SU(2)_L \times SU(2)_R\) supergravity formulation and the minimal \(SU(2)_V\) multiplet of [52]. Here, by making use of the covariant TM-II multiplet, we follow an analogue of the Howe’s procedure for 4D \(\mathcal{N} = 2\) [34] to introduce the minimal multiplet. The analysis goes along the same lines

---

\(^{12}\)The invariance of \(W, P\) and \(Q\) under the structure group and super-Weyl transformations clearly tells us that this version of the covariant TM-I can not be embedded in the field strengths of a vector multiplet differently to the covariant TM-II considered in this subsection.
of the 4D $\mathcal{N} = 2$ case described in [25, 26]. The SU(2)$_L \times$ SU(2)$_R$ supergravity multiplet plays the role of the U(2)-Howe formulation of the 4D $\mathcal{N} = 2$ Weyl multiplet [34], while the SU(2)-Grimm formulation [33] is the analogue of the 2D $\mathcal{N} = (4, 4)$ SU(2)$_V$ minimal supergravity of [52].

Suppose to have coupled the 2D $\mathcal{N} = (4, 4)$ extended supergravity geometry of subsection 2.1 to a TM-II Abelian vector multiplet such that at each point of the superspace: (i) $F \neq 0$ and, (ii) $W_{ij} = 0$. The second condition can always be achieved by the aid of a local SU(2)$_L \times$ SU(2)$_R$ transformation. Note that the previous condition is left invariant by SU(2)$_V$ transformations but breaks SU(2)$_C$. Under a super-Weyl transformation (3.12) with parameter $S = -\log F$ we can then impose the gauge

$$F = 1, \quad W_{ij} = 0, \quad (3.22)$$

which completely fixes the super-Weyl and local SU(2)$_C$ transformations.

The previous gauge implies various conditions. First, the covariant constraint (3.11)
\[
\nabla_{\alpha i} W_{jk} = -\frac{1}{2} C_{ij(\gamma^3)\alpha^\beta \nabla_{\gamma k)} F, \quad \text{in the limit (3.22), is}
\]

$$- (\Phi_C)_{\alpha i j k} = 0, \quad (3.23)$$

and therefore the spinor SU(2)$_C$ connections are zero

$$\Phi_C^{i^j k l} = (\Phi_C)^{i^j k l} = 0. \quad (3.24)$$

Note that due to the previous equations, it follows that in the gauge (3.22) it also holds the covariantly constant conditions $\nabla_{\alpha i} F = \nabla_{\alpha i} F = \nabla_{\alpha i} W_{jk} = \nabla_{\alpha i} W_{jk} = 0$.

The constraint (3.11) implies in general the following equations

\[
[\nabla_{\alpha i}, \nabla_{\beta j}] F = -\frac{1}{6} C_{ij(\gamma^3)\alpha^\beta \nabla_{\gamma k)} W_{kl} + \frac{1}{6} C_{ij(\gamma^3)\alpha^\beta \nabla_{\gamma k)} W_{kl} + 4i\varepsilon^{ab}(\gamma^3)_{\alpha^\beta} \nabla_{\gamma k)} W_{ij} - 8i C_{\alpha^\beta} \mathcal{T} W_{ij} - 8(\gamma^3)_{\alpha^\beta} \mathcal{S} W_{ij} - 4(\gamma^3)_{\alpha^\beta} \mathcal{S} W_{ij} + 4 C_{ij(\gamma^3)\alpha^\beta} B_{\alpha} F, \quad (3.25a)
\]

\[
\nabla_{\alpha i} \nabla_{\beta j} W_{jk} = -\frac{1}{4} C_{ij(\gamma^3)\alpha^\beta} \nabla_{\gamma k)} W_{kl} + 6i(\gamma^3)_{\alpha^\beta} N W_{ij} - 6i(\gamma^3)_{\alpha^\beta} A_{\alpha} W_{ij} + 3i(\gamma^3)_{\alpha^\beta} Y_{ij} F - 3i C_{\alpha^\beta} Y_{ijp} W_{ijp} - 3i C_{ij(\gamma^3)\alpha^\beta} e^{ab} A_{\alpha} F. \quad (3.25b)
\]

\footnote{By using (3.9) one observes that the SU(2)$_C$ transformation with gauge parameter $(K_C)_{kl} = (1/F) W_{kl}$ cancels $W_{ij}$ at the linearized level; it is not difficult to compute the finite analogue of this result.}
Equations (3.25a) and (3.25b) in the gauge (3.22) reduce to

\[ 0 = -4i\varepsilon^{ab}(\gamma_a)_{\alpha\beta}(\Phi_C)_{bij} - 4(\gamma^3)_{\alpha\beta}T_{ij} - 4C_{\alpha\beta}S_{ij} + 4C_{ij}(\gamma^a)_{\alpha\beta}B_a, \]  
(3.26a)

\[ 0 = 3i(\gamma^3)_{\alpha\beta}Y_{ij} - 3iC_{ij}(\gamma_a)_{\alpha\beta}\varepsilon^{ab}A_b, \]  
(3.26b)

that imply

\[ (\Phi_C)^{akl} = 0, \]  
(3.27a)

\[ S_{ij} = T_{ij} = Y_{ij} = A_a = B_a = 0. \]  
(3.27b)

It is clear that the gauge (3.22) reduces the extended supergravity of section 2.1 to the minimal multiplet [52] of subsection 2.3 coupled to a real constant central charge.

Now, recall the super-Weyl transformations of \( S_{ij} \), \( T_{ij} \), \( Y_{ij} \), \( A_a \) and \( B_a \) eqs. (2.17d)–(2.17h). The fact that there exists a gauge in which \( S'_{ij} = T'_{ij} = Y'_{ij} = A'_a = B'_a = 0 \) is equivalent to setting the left hand side of eqs. (2.17d)–(2.17h) to zero. This implies that one can solve the differential constraints of the \( S_{ij} \), \( T_{ij} \), \( Y_{ij} \), \( A_a \) and \( B_a \) superfields in terms of some real scalar superfields through the right hand side of eqs. (2.17d)–(2.17h). Then, we can reinterpret the derivation of eq. (3.27b) in the gauge (3.22), as a proof that \( S_{ij} \), \( T_{ij} \), \( Y_{ij} \), \( A_a \) and \( B_a \) are pure gauge degrees of freedom where the vector multiplet plays the role of a useful technical tool. Therefore, the \( S_{ij} \), \( T_{ij} \), \( Y_{ij} \), \( A_a \) and \( B_a \) superfields, in the general case of subsection 2.1, can be gauged away by a super-Weyl transformation. The previous analysis justify the gauge condition (2.19) and the results of subsection 2.3.

Note that the previous discussion is similar to the proof we gave in [26] that for the Howe’s formulation of 4D \( \mathcal{N} = 2 \) supergravity the \( G_{a}^{jk} \) superfield is a pure gauge degree of freedom.

4 \( \mathcal{N} = (4, 4) \) curved bi-projective superspace

In five [23, 24] and four [25, 26] dimensions, matter couplings in supergravity has been described in terms of covariant projective supermultiplets. In this section, we introduce the concept of covariant bi-projective supermultiplets for 2D \( \mathcal{N} = (4, 4) \) conformal supergravity, and then we present a locally supersymmetric and super-Weyl invariant action. The covariant bi-projective multiplets are a curved extensions of the multiplets introduced in the case of 2D \( \mathcal{N} = (4, 4) \) flat superspace [38, 14, 39, 40]. First, let us consider again the TM-II.
Before turning to the details let us make a note for the reader about our notations in this section. In the sections 2 and 3 we have always made use of SU(2)-indices denoted by lower-case letters like $i$, $j$. In this section we often make use of lower-case and capital SU(2)-indices, like $i$ and $I$, to distinguish between indices transforming respectively only under the SU(2)$_L$ and SU(2)$_R$ group. For example, according to such distinction, in this section we denote the left covariant derivatives as $(\nabla_+, \bar{\nabla}_i^j)$ and the right covariant derivatives as $(\nabla_-, \bar{\nabla}_I^L)$. In using, as in sections 2 and 3, the SU(2)$_V \times$SU(2)$_C$ parametrization of SU(2)$_L \times$SU(2)$_R$, this index difference is not natural but it turns out to be useful in working with the light-cone coordinates.

### 4.1 Rewriting the twisted-II multiplet

Here we want to give an equivalent description of the twisted-II multiplet in terms of a single superfield $T_{iI}$ satisfying a set of analyticity-like differential constraints. This description results to be a covariant extension of the TM-II as introduced for the first time in the flat superspace case in [55].

First let us rewrite the TM-II differential constraints (3.11) as

$$\nabla_+ W_{jk} = -\frac{1}{2} C_{i(j} \nabla_+ k) F, \quad \nabla_- W_{jk} = \frac{1}{2} C_{i(j} \nabla_- k) F,$$

where we have explicitly distinguished the left and right Lorentz spinor indices. We then define the real superfield $T_{iI}$ in terms of $W_{ij}$ and $F$ as

$$T_{iI} := W_{ij} + \frac{1}{2} C_{iI} F, \quad (T_{iI})^* = T_{iI}.$$

With the previous definition the TM-II differential constraints (4.1) are equivalent to the analyticity like constraints

$$\nabla_+ (i T_j)^I = \bar{\nabla}_+ (i T_j)^I = 0, \quad \nabla_- (i T_{[i|j]}^I = \bar{\nabla}_- (i T_{[i|j]}^I = 0.$$

The Lorentz scalar superfield $T_{iI}$ has transformations under the SU(2) groups defined by the one of $W_{ij}$, $F$. One finds

$$L_{kI} T_{iI} = \frac{1}{2} C_{i(k} T_{l)l} \quad R_{Kl} T_{iI} = \frac{1}{2} C_{T_{l(k} T_{l)}I}.$$

Then, it is clear that the index $i$ transforms only under the SU(2)$_L$ and the index $I$ under the SU(2)$_R$. The super-Weyl transformations of $T_{iI}$ are clearly

$$(T_{iI})' = e^S T_{iI}.$$
To conclude note that, in terms of the chiral prepotential $W$ introduced in subsection 3.2 equations (3.16) and (3.17), the superfield $T_{iI}$ can be expressed in the following form

$$T_{iI} = \frac{i}{4} [\nabla_{+i}, \nabla_{-J}] W = \frac{i}{4} [\bar{\nabla}_{+i}, \bar{\nabla}_{-J}] \bar{W} = (T^{iI})^* .$$

(4.6)

### 4.2 2D $\mathcal{N} = (4, 4)$ covariant bi-projective superfields

In subsection 4.1 we have rewritten the TM-II constraints in terms of analyticity like conditions on the left and right sectors of 2D $\mathcal{N} = (4, 4)$ supergravity. Here we want to introduce a large class of analytic multiplets living in, what we call, curved bi-projective superspace.

In defining curved bi-projective multiplets we follow the procedure recently developed in the cases of 5D $\mathcal{N} = 1$ supergravity [23, 24] and 4D $\mathcal{N} = 2$ supergravity [25, 26]. We then introduce isotwistor variables $u^i_\oplus \in \mathbb{C}^2 \setminus \{0\}$ and $v^I_\ominus \in \mathbb{C}^2 \setminus \{0\}$ defined to be inert under the action of the structure group. In the present 2D $\mathcal{N} = (4, 4)$ case the difference compared with [23, 24, 25, 26] is the use of two sets of isotwistor variables instead of one. This possibility is related to the fact that in (4.3) we have two independent set of analyticity like constraints. Note that the construction is based on and extends the flat case of [38, 14, 39, 40] and has clear similarities with the bi-harmonic superspace approach of [50, 51].

Using the $u, v$ isotwistors we define the covariant derivatives

$$\nabla_+^\oplus := u^i_\oplus \nabla_i^+ , \quad \nabla_+^\ominus := v^I_\ominus \nabla_I^+ ,$$

$$\nabla_-^\ominus := v^I_\ominus \nabla_I^- , \quad \nabla_-^\oplus := u^i_\oplus \nabla_i^- .$$

(4.7a)

(4.7b)

We are now ready to introduce a third equivalent definition of the covariant TM-II. By contracting the $u, v$ isotwistors with $T_{iI}$ the superfield $T^{\oplus \ominus}(z, u, v)$ is defined according to the following equation

$$T^{\oplus \ominus}(u, v) := u^i_\oplus v^I_\ominus T_{iI} .$$

(4.8)

The constraints (4.3) are then equivalent to the analyticity like conditions

$$\nabla_+^\oplus T^{\oplus \ominus} = \nabla_+^\ominus T^{\oplus \ominus} = 0 , \quad \nabla_-^\ominus T^{\oplus \ominus} = \nabla_-^\oplus T^{\oplus \ominus} = 0 .$$

(4.9)

It is important to note that the superfield $T^{\oplus \ominus}(u, v)$ is homogeneous of degree-(1,1) in the variables $u$ and $v$

$$T^{\oplus \ominus}(c_L u, v) = c_L T^{\oplus \ominus}(u, v) , \quad T^{\oplus \ominus}(u, c_R v) = c_R T^{\oplus \ominus}(u, v) , \quad c_L, c_R \in \mathbb{C} \setminus \{0\} .$$

(4.10)
In particular, $T^{\oplus \oplus}$ describes an holomorphic tensor field on the product of two complex projective spaces $\mathbb{C}P^1 \times \mathbb{C}P^1$. The transformation rules of $T^{\oplus \oplus}$ under $L_{kl}, R_{KL}$, that follow from (4.4), can be written as

$$L_{kl}T^{\oplus \oplus}(u^\oplus, v^\oplus) = -\frac{1}{2(u^\oplus u^\ominus)} \left( u^\ominus_k u^\oplus_l D^{\ominus \oplus} - u^\ominus_k u^\oplus_l \right) T^{\oplus \oplus}(u^\oplus, v^\ominus), \quad (4.11a)$$

$$R_{KL}T^{\oplus \oplus}(u^\oplus, v^\ominus) = -\frac{1}{2(v^\ominus v^\ominus)} \left( v^\ominus_k v^\ominus_l D^{\ominus \ominus} - v^\ominus_k v^\ominus_l \right) T^{\ominus \ominus}(u^\oplus, v^\ominus), \quad (4.11b)$$

where we have introduced

$$D^{\ominus \ominus} = u^\ominus_i \frac{\partial}{\partial u^\ominus_i}, \quad D^{\ominus \oplus} = v^\ominus I \frac{\partial}{\partial v^\ominus I}, \quad (4.12a)$$

$$(u^\ominus u^\oplus) := u^\ominus_i u^\oplus_i \neq 0, \quad (v^\ominus v^\ominus) := v^\ominus I v^\ominus I \neq 0. \quad (4.12b)$$

The equations (4.11a) and (4.11b) involve two new isotwistors $u^\ominus_i$ and $v^\ominus I$ which are subject to the only conditions (4.12b) and are otherwise completely arbitrary. The following relations also hold

$$C_{ij} = \frac{1}{(u^\ominus u^\ominus)} \left( u^\ominus_j u^\ominus_i - u^\ominus_i u^\ominus_j \right), \quad C_{IJ} = \frac{1}{(v^\ominus v^\ominus)} \left( v^\ominus_J v^\ominus_i - v^\ominus_i v^\ominus_J \right). \quad (4.13)$$

The TM-II, in the form of $T^{\ominus \ominus}$ just introduced, is the simplest example of a large class of multiplets living on $\mathcal{M}^{2|4.4} \times \mathbb{C}P^1 \times \mathbb{C}P^1$. We call these bi-isotwistor superfields.

A weight-$(m,n)$ bi-isotwistor superfield $U^{(m,n)}(z, u^\oplus, v^\ominus)$ is holomorphic on an open domain of $\{\mathbb{C}^2 \setminus \{0\} \times \{\mathbb{C}^2 \setminus \{0\}\}$ with respect to the homogeneous coordinates $(u^\oplus_i, v^\ominus_I)$ for $\mathbb{C}P^1 \times \mathbb{C}P^1$, and is characterized by the conditions:

(i) it is a homogeneous function of $(u^\oplus, v^\ominus)$ of degree $(m, n)$, that is,

$$U^{(m,n)}(z, c_L u^\oplus, v^\ominus) = (c_L)^m U^{(m,n)}(z, u^\oplus, v^\ominus), \quad c_L \in \mathbb{C} \setminus \{0\}, \quad (4.14a)$$

$$U^{(m,n)}(z, u^\oplus, c_R v^\ominus) = (c_R)^n U^{(m,n)}(z, u^\oplus, v^\ominus), \quad c_R \in \mathbb{C} \setminus \{0\}; \quad (4.14b)$$

(ii) the supergravity gauge transformations act on $U^{(m,n)}$ as follows:

$$\delta_K U^{(m,n)} = \left( K^C \nabla_C + KM + (K_L)_{kl} L_{kl} + (K_R)_{KL} R_{KL} \right) U^{(m,n)}, \quad (4.15a)$$

$$L_{kl} U^{(m,n)} = -\frac{1}{2(u^\oplus u^\ominus)} \left( u^\ominus_k u^\oplus_l D^{\ominus \ominus} - m u^\ominus_k u^\oplus_l \right) U^{(m,n)}, \quad (4.15b)$$

$$R_{KL} U^{(m,n)} = -\frac{1}{2(v^\ominus v^\ominus)} \left( v^\ominus_k v^\ominus_l D^{\ominus \ominus} - n v^\ominus_k v^\ominus_l \right) U^{(m,n)}, \quad (4.15c)$$

$$\mathcal{M} U^{(m,n)} = \frac{m - n}{2} U^{(m,n)}. \quad (4.15d)$$

---

\(^{14}\)See [23, 24, 25, 26] for the introduction and examples of isotwistor superfields in 4D and 5D supergravities.
Note that, due to (4.14a), the superfield \((L_{kl}U^{(m,n)})\) is independent of \(u^i\) even if the transformations in (4.15b) explicitly depend on it; similarly \((R_{KL}U^{(m,n)})\) is independent of \(v_I\). We refer the reader to [25] for a more detailed discussion on the SU(2) transformations of isotwistor-like superfields.

The most important property of 2D bi-isotwistor superfields is that the anticommutator among any of the covariant derivatives \(\nabla_+^\oplus, \nabla_+^\ominus, \nabla_+^\forall, \nabla_+^\exists\) is zero when acting on \(U^{(m,n)}\). Explicitly, it holds

\[
0 = \{\nabla_+^\oplus, \nabla_+^\ominus\} U^{(m,n)} = \{\nabla_+^\forall, \nabla_+^\exists\} U^{(m,n)} = \{\nabla_+^\exists, \nabla_+^\forall\} U^{(m,n)} = \cdots. \tag{4.16}
\]

We present a proof of this statement in appendix C. It is worth mentioning that the Lorentz transformations of \(U^{(m,n)}\) are uniquely fixed by requiring (4.16) with (4.15b) and (4.15c) assumed. In the case in which \((m - n)\) is odd we will generically consider \(U^{(m,n)}\) to be a fermionic superfield even if for the aim of the present discussion this is irrelevant.

With the definitions (i) and (ii) assumed, the set of bi-isotwistor superfields is closed under the product of superfields and the action of the \(\nabla_+^\oplus, \nabla_+^\ominus, \nabla_+^\forall, \nabla_+^\exists\) derivatives. More precisely, given a weight-\((m,n)\) \(U^{(m,n)}\) and a weight-\((p,q)\) \(U^{(p,q)}\) bi-isotwistor superfields the superfield \((U^{(m,n)}U^{(p,q)})\) is a weight-\((m+p,n+q)\) bi-isotwistor superfield. Moreover, the superfields \((\nabla_+^\oplus U^{(m,n)})\), \((\nabla_+^\forall U^{(m,n)})\) and \((\nabla_+^\exists U^{(m,n)})\), \((\nabla_+^\forall U^{(m,n)})\) are respectively weight-\((m+1,n)\) and weight-\((m,n+1)\) bi-isotwistor superfields.

If we consider the set of bi-isotwistor superfields transforming homogeneously under super-Weyl transformations \((U^{(m,n)})' = e^{wS}U^{(m,n)}\), it is natural to impose

\[
(U^{(m,n)})' = e^{\frac{m+n}{2}}S U^{(m,n)}. \tag{4.17}
\]

The conformal weight \(w\) in the previous relation is fixed by the requirement that the superfields \(\nabla_+^\oplus U^{(m,n)}, \nabla_+^\forall U^{(m,n)}, \nabla_+^\exists U^{(m,n)}, \nabla_+^\forall U^{(m,n)}\) also transform homogeneously. For example, it holds

\[
(\nabla_+^\forall U^{(m,n)})' = e^{\frac{m+n+1}{2}}S \nabla_+^\forall U^{(m,n)}. \tag{4.18}
\]

To prove the last relation one needs to use the equations (2.18a), (4.17), (4.15d) and the relation

\[
u^\ominus \mathbf{L}_{kl}U^{(m,n)} = \frac{m}{2}u^kU^{(m,n)}, \tag{4.19}
\]

which follows from eq. (4.15b). Analogously one can prove that also the superfields \(\nabla_+^\forall U^{(m,n)}, \nabla_+^\exists U^{(m,n)}\) and \(\nabla_+^\forall U^{(m,n)}\) have conformal weight \(w = (m + n + 1)/2\) if eq. (4.17) is assumed.

25
We are now ready to introduce 2D $\mathcal{N} = (4, 4)$ covariant bi-projective superfields. We define a weight-$(m, n)$ covariant bi-projective supermultiplet $Q^{(m, n)}(z, u^\oplus, v^\ominus)$ to be a bi-isotwistor superfield satisfying (i), (ii), (4.14a)–(4.15d) and to be constrained by the analyticity conditions

$$\nabla^\oplus_+ Q^{(m, n)} = \bar{\nabla}^\oplus_+ Q^{(m, n)} = 0, \quad \nabla^\ominus_- Q^{(m, n)} = \bar{\nabla}^\ominus_- Q^{(m, n)} = 0. \quad (4.20)$$

Note that the consistency of the previous constraints is guaranteed by eq. (4.16). This now takes the form of an integrability condition for the analyticity constraints.

If we ask $Q^{(m, n)}$ to have homogeneous super-Weyl transformations, it is clear by the previous discussion on the super-Weyl transformations of bi-isotwistor superfields, that the transformations

$$(Q^{(m, n)})' = e^{\frac{m+n}{2}S} Q^{(m, n)}, \quad (4.21)$$

preserve the analyticity conditions (4.20).

Given a bi-projective multiplet $Q^{(m, n)}(z, u^\oplus, v^\ominus)$, its complex conjugate is not covariantly analytic. However, one can introduce a generalized, analyticity-preserving conjugation, $Q^{(m, n)} \rightarrow \tilde{Q}^{(m, n)}$, defined as

$$\tilde{Q}^{(m, n)}(u^\oplus, v^\ominus) \equiv Q^{(m, n)}(\overline{u^\oplus}, \overline{v^\ominus}), \quad (4.22a)$$

$$\tilde{u}^\oplus = i \sigma_2 u^\oplus, \quad \tilde{v}^\ominus = i \sigma_2 v^\ominus, \quad (4.22b)$$

with $Q^{(m, n)}(u^\oplus, v^\ominus)$ the complex conjugate of $Q^{(m, n)}$ and $\overline{u^\oplus}, \overline{v^\ominus}$ the complex conjugates of $u^\oplus, v^\ominus$. It is easy to check that $\tilde{Q}^{(m, n)}(z, u^\oplus, v^\ominus)$ is a weight-$(m, n)$ bi-projective multiplet. One can see that $\tilde{Q}^{(m, n)} = (-1)^{m+n} Q^{(m, n)}$, and therefore real supermultiplets can be consistently defined when $(m + n)$ is even. The superfield $\tilde{Q}^{(m, n)}$ is called the smile-conjugate of $Q^{(m, n)}$. Geometrically, this conjugation is complex conjugation composed with the antipodal map on the two projective spaces $\mathbb{CP}^1 \times \mathbb{CP}^1$. The simplest example of real bi-projective superfield is again the TM-II. The reality condition $(T_{IJ})^* = T^{IJ}$ is equivalent to $\tilde{T}^{\oplus \ominus} = T^{\oplus \ominus}$.

Note that, by definition, the TM-II superfield $T^{\oplus \ominus}$ describes a regular holomorphic tensor field on the whole product of the two complex projective spaces $\mathbb{CP}^1 \times \mathbb{CP}^1$. Other simple examples of bi-projective superfields that are regular holomorphic tensor field on the whole $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be given by what we call $O(m, n)$ multiplets $(m, n > 0)$. They are described by a bi-projective superfield $O^{(m, n)}(z, u, v) := u^\oplus_{i_1} \cdots u^\ominus_{i_m} v^\ominus_{j_1} \cdots v^\oplus_{j_n} O^{i_1 \cdots i_m j_1 \cdots j_n}(z), \quad 26$
where the isotensor superfield \( O^{i_1 \cdots i_m | J_1 \cdots J_n} = \frac{1}{m! n!} O^{(i_1 \cdots i_m)(J_1 \cdots J_n)} \) is such that

\[
\mathcal{M} O^{i_1 \cdots i_m | J_1 \cdots J_n} = \frac{m-n}{2} O^{i_1 \cdots i_m | J_1 \cdots J_n},
\]

\[
L_{kl} O^{i_1 \cdots i_m | J_1 \cdots J_n} = \frac{1}{2} \frac{1}{(m-1)!} \delta^{(i_1 \cdots i_m)}_{(k \, O_l \, i_2 \cdots i_m)} J_1 \cdots J_n,
\]

\[
R_{K L} O^{i_1 \cdots i_m | J_1 \cdots J_n} = \frac{1}{2} \frac{1}{(n-1)!} \delta^{(i_1 \cdots i_m \mid L \, J_1 \cdots J_n)}_{1} ,
\]

\[
\nabla_+^{(k} O^{i_1 \cdots i_m \mid J_1 \cdots J_n} = \nabla_+^{(k} O^{i_1 \cdots i_m \mid J_1 \cdots J_n} = 0,
\]

\[
\nabla_-^{(K} O^{i_1 \cdots i_m \mid J_1 \cdots J_n} = \nabla_-^{(K} O^{i_1 \cdots i_m \mid J_1 \cdots J_n} = 0.
\]

Note that \( O^{(m,n)}(u,v) \) is polynomial in the isotwistor variables \( u, v \). More general bi-projective multiplets have poles and more complicated analytic properties on \( \mathbb{C}P^1 \times \mathbb{C}P^1 \). Then, in general 2D covariant bi-projective superfields possess an infinite number of standard superfields in a way completely analogous to the more studied 4D-5D curved cases \([23, 24, 25, 26]\). A more detailed classification of covariant bi-projective superfields will be considered elsewhere.

One can represent a bi-projective superfield \( Q^{(m,n)} \) in the form

\[
Q^{(m,n)} = -\frac{1}{4} \nabla_+^\oplus \nabla_-^\oplus \nabla_+^\ominus \nabla_-^\ominus U^{(m-2,n-2)} = \frac{1}{4} \nabla_+^\ominus \nabla_-^\ominus \nabla_+^\oplus \nabla_-^\oplus U^{(m-2,n-2)} = \cdots, \tag{4.24}
\]

for some bi-isotwistor superfield \( U^{(m-2,n-2)} \) satisfying \((4.14a)–(4.15d)\) and \((4.17)\). In \((4.24)\), thanks to \((4.16)\) and the defining properties of bi-isotwistor superfields, one can take any graded permutation of the covariant derivatives \( \nabla_+^\ominus, \nabla_+^\oplus, \nabla_-^\ominus, \nabla_-^\oplus \). Therefore it is trivial to prove that \((4.20)\) is identically satisfied. We will call a \( U^{(m-2,n-2)} \), such that \((4.24)\) holds, a bi-isotwistor prepotential of \( Q^{(m,n)} \).

To conclude we observe that all the results presented in this subsection remain true, up to few minor differences, if one reduces the supergravity geometry to the minimal multiplet. Due to the de-gauging from the \( SU(2)_L \times SU(2)_R \) group to \( SU(2)_V \), in the minimal case, the supergravity gauge transformations of bi-isotwistor and bi-projective superfields are modified from eq. \((4.15a)\) to

\[
\delta_k U^{(m,n)} = \left( K^C \nabla_C + K M + (K_V)^{kl} \nabla_{kl} \right) U^{(m,n)},
\]

Note that eqs. \((4.15b)–(4.15d)\) remain the same. A second modification that occurs regards the super-Weyl transformations. As explained in subsection 2.3, to preserve the gauge \((2.19)\), in the minimal case the super-Weyl transformations are generated by a TM-II \((S, S_{ij})\) couple of superfields through \( \delta = (\delta - \delta_C) \) infinitesimal transformations. For bi-isotwistor and bi-projective superfields this means that the infinitesimal super-Weyl transformations are modified to

\[
\tilde{\delta} U^{(m,n)} = \left( \frac{m+n}{2} S - S^{kl} C_{kl} \right) U^{(m,n)}
\]

that includes a compensating \( SU(2)_C \) transformation.
4.3 Action principle

Here we give a bi-projective superfield action principle invariant under the supergravity gauge group and super-Weyl transformations and such that in the flat limit it reduces to the one introduced in [38, 40].

Let $\mathcal{L}^{(0,0)}$ be a real bi-projective superfield of weight-$(0,0)$. In particular, according to (4.17), $\mathcal{L}^{(0,0)}$ is invariant under super-Weyl transformations. Moreover, we consider a TM-II described by $T^{\oplus \otimes}$ with $W, (\bar{W})$ a chiral prepotential. Associated with $\mathcal{L}^{(0,0)}$ we introduce the action principle

$$S = \frac{1}{4\pi^2} \oint (u^{\oplus} du^{\oplus}) \oint (v^{\otimes} dv^{\otimes}) \int d^2 x d^8 \theta E \frac{W \bar{W}}{(T^{\oplus \otimes})^2} \mathcal{L}^{(0,0)}, \quad E^{-1} = \text{Ber}(E_A^M).$$ (4.25)

By construction, the functional is invariant under the re-scaling $u_i^{\oplus}(t) \rightarrow c_L(t) u_i^{\oplus}(t)$, for an arbitrary function $c_L(t) \in \mathbb{C} \setminus \{0\}$, where $t$ denotes the evolution parameter along the first closed integration contour. Similarly, (4.25) is invariant under re-scalings $v_I^{\otimes}(s) \rightarrow c_R(s) v_I^{\otimes}(s)$, for an arbitrary function $c_R(s) \in \mathbb{C} \setminus \{0\}$, where $s$ denotes the evolution parameter along the second closed integration contour. Note that (4.25) has clear similarities with the action principles in four and five-dimensional curved projective superspace [23, 24, 25, 26].

By using that under super-Weyl transformations $E$ transforms like

$$E' = e^{2S} E,$$ (4.26)

and the transformations $(T^{\oplus \otimes})' = e^{S} T^{\oplus \otimes}$ and $W' = W$, one sees that $S$ is super-Weyl invariant. The action (4.25) is also invariant under arbitrary local supergravity gauge transformations (2.6a)–(2.7). The invariance under general coordinates and Lorentz transformations is trivial. The invariance under the two SU(2) transformations can be proved similarly to [23]. It is instructive to review this in the 2D case.

The proof of SU(2) invariance goes as follows. Under infinitesimal SU(2)$_L$ transformations the action varies like

$$\delta_L S = \frac{1}{4\pi^2} \oint (u^{\oplus} du^{\oplus}) \oint (v^{\otimes} dv^{\otimes}) \int d^2 x d^8 \theta E W \bar{W} (K_L)^{kl} L_{kl} \left( \frac{\mathcal{L}^{(0,0)}}{(T^{\oplus \otimes})^2} \right),$$ (4.27)

15For simplicity in this paper we consider the two contour integrals to be closed. Depending on the explicit form of the Lagrangian $\mathcal{L}^{(0,0)}$, one could consider different cases [14, 38, 39, 40] with line integrals not necessarily closed.
where we have used the invariance of \(E\) and \(W\) under SU(2) transformations. For weight-(-2,-2) bi-projective superfields, like \(Q^{(-2,-2)} := (\mathcal{L}^{(0,0)})/(T^{\oplus\oplus})^2\), it holds
\[
(K_L)^{kl} L_{kl} Q^{(-2,-2)} = -\frac{1}{(u^{\oplus\oplus} u^{\oplus\oplus})} D^{\oplus\oplus} \left( (K_L)^{\oplus\oplus} Q^{(-2,-2)} \right). 
\] (4.28)

Next, note that the \((u^{\oplus\oplus} du^{\oplus})\) integration measure, written in terms of the evolution parameter \(t\) of the closed contour, is equal to
\[
(u^{\oplus\oplus} du^{\oplus}) = -(u^{\oplus\oplus} u^{\oplus\oplus}) dt, \quad f := df(t)/dt. 
\] (4.29)

Then, being \((K_L)^{\oplus\oplus} Q^{(-2,-2)}\) homogeneous of degree zero in \(u^{\oplus\oplus}\) it is easy to note that it holds
\[
(u^{\oplus\oplus} du^{\oplus}) (K_L)^{kl} L_{kl} Q^{(-2,-2)} = -dt \frac{d}{dt} \left( (K_L)^{\oplus\oplus} Q^{(-2,-2)} \right). 
\] (4.30)

Since the integration contour is closed, eq. (4.27) is zero and the SU(2)_L-part of the supergravity transformations does not contribute to the variation of the action (4.25).

The proof of the invariance under the SU(2)_R transformations goes along the same lines.

If, according to eq. (4.24), we represent \(\mathcal{L}^{(0,0)}\) in terms of a bi-isotwistor prepotential \(\mathcal{U}^{(-2,-2)}\), then the action (4.25) can be rewritten as
\[
S = \frac{1}{4\pi^2} \oint (u^{\oplus\oplus} du^{\oplus}) \oint (v^{\oplus\oplus} dv^{\oplus}) \int d^2 x d^8 \theta E \mathcal{U}^{(-2,-2)}. 
\] (4.31)

Here we have used the relations \(T^{\oplus\oplus} = (i/4)[\nabla^{\oplus\oplus}, \nabla^{\oplus\oplus}] W = (i/4)[\nabla^{\oplus\oplus}, \nabla^{\oplus\oplus}] \bar{W}\) that follow from (4.6), and
\[
\nabla^{\oplus\oplus} \nabla^{\oplus\oplus} \nabla^{\oplus\oplus} (WW) = -4(T^{\oplus\oplus})^2. 
\] (4.32)

After integrating by parts, one obtains eq. (4.31) from (4.25). The equation (4.31) leads to an important result: if \(\mathcal{L}^{(0,0)}\) is a function of some supermultiplets to which the TM-II compensator does not belong, then the action \(S\) is independent of the superfields \(T^{\oplus\oplus}\), \(W\) and \(\bar{W}\) chosen.

Let us take the flat limit of the action principle (4.25). This is
\[
S_{\text{flat}} = \frac{1}{4\pi^2} \oint \frac{(u^{\oplus\oplus} du^{\oplus})}{(u^{\oplus\oplus} u^{\oplus\oplus})^2} \oint \frac{(v^{\oplus\oplus} dv^{\oplus})}{(v^{\oplus\oplus} v^{\oplus\oplus})^2} \int d^2 x D^{\oplus\oplus}_+ \bar{D}^{\oplus\oplus}_+ D^{\oplus\oplus}_+ \bar{D}^{\oplus\oplus}_+ D^{\oplus\oplus}_- \bar{D}^{\oplus\oplus}_- \frac{W \bar{W}}{(T^{\oplus\oplus})^2} L^{(0,0)}, 
\] (4.33)
where \(D^{\oplus\oplus}_{\alpha i}, \bar{D}^{\oplus\oplus}_{\bar{\alpha} i}\) are the flat covariant derivatives, \(L^{(0,0)}\) is the lagrangian in the flat case and \(D^{\oplus\oplus}_+ = u^{\oplus\oplus} D^{\oplus}_i, \bar{D}^{\oplus\oplus}_+ = u^{\oplus\oplus} \bar{D}^{\oplus}_i, D^{\oplus\oplus}_- = v^{\oplus\oplus} D^{\oplus}_i\) and \(\bar{D}^{\oplus\oplus}_- = v^{\oplus\oplus} \bar{D}^{\oplus}_i\). Using analyticity of \(T^{\oplus\oplus}\) and of the Lagrangian \(L^{(0,0)}\), and the relation (4.32) in the flat limit, we obtain
\[
S_{\text{flat}} = \frac{1}{\pi^2} \oint \frac{(u^{\oplus\oplus} du^{\oplus})}{(u^{\oplus\oplus} u^{\oplus\oplus})^2} \oint \frac{(v^{\oplus\oplus} dv^{\oplus})}{(v^{\oplus\oplus} v^{\oplus\oplus})^2} \int d^2 x D^{\oplus\oplus}_+ D^{\oplus\oplus}_- D^{\oplus\oplus}_- L^{(0,0)}. 
\] (4.34)

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In the north chart of both the $\mathbb{C}P^1$ one can obtain the flat action principle written in terms of inhomogeneous coordinates for $\mathbb{C}P^1 \times \mathbb{C}P^1$. This coincides with the action principle given in [38, 40]. The action (4.34) is a 2D analogue of the 4D case of [6, 10] and, being written in homogeneous coordinates for the projective spaces, it is closer in form to the one in [62, 57]. The action (4.34) is also invariant under arbitrary “projective” transformations of the form:

\[
(u_i^\oplus, u_i^\ominus) \rightarrow (u_i^\oplus, u_i^\ominus) P_L, \quad P_L = \begin{pmatrix} a_L & 0 \\ b_L & c_L \end{pmatrix} \in \text{GL}(2, \mathbb{C}), \quad (4.35a)
\]

\[
(v_I^\ominus, v_I^\oplus) \rightarrow (v_I^\ominus, v_I^\oplus) P_R, \quad P_R = \begin{pmatrix} a_R & 0 \\ b_R & c_R \end{pmatrix} \in \text{GL}(2, \mathbb{C}). \quad (4.35b)
\]

Projective transformations express the homogeneity of the formalism with respect to $u^\oplus, v^\ominus$ and the independence on $u^\ominus, v^\oplus$. This invariance results a powerful tool in superspace theories with eight supercharges. For example, in 5D $\mathcal{N} = 1$ [23] and 4D $\mathcal{N} = 2$ [30] supergravity it has been used to reduce the projective action principle to components. Along the same lines, one could approach the 2D $\mathcal{N} = (4, 4)$ case and continue the analysis of component reduction of 2D $\mathcal{N} = (4, 4)$ superspace action principles of [63, 64].

We conclude by noting that the action (4.25) has the same form if one considers the supergravity geometry reduced to the minimal multiplet. In such case (4.25) is invariant under arbitrary supergravity gauge transformations, with $\text{SO}(1,1) \times \text{SU}(2)^V$ as tangent space group. It is clearly invariant also under the $\bar{\delta}$ variation of subsection 2.3 generated by $S, S_{ij}$ which includes super-Weyl and compensating $\text{SU}(2)_C$ transformations.

We believe that the action (4.25) is suitable to describe general 2D $\mathcal{N} = (4, 4)$ superconformal matter systems, such as WZNW, Liouville systems and non-linear sigma models, covariantly coupled to supergravity. The investigation of that subjects and a more detailed study of bi-projective multiplets is left for future research.

5 Conclusion

In this paper we presented new results in the study of 2D $\mathcal{N} = (4, 4)$ supergravity using superspace techniques. We proposed a new superspace formulation for $\mathcal{N} = (4, 4)$ conformal supergravity in two dimensions which proves to be an extension of the minimal multiplet of [52]. We then described the covariant coupling of supergravity to a large class of multiplets. We begun by coupling the extended supergravity to an Abelian vector
multiplet described by a twisted-II multiplet. We have then introduced so called covariant bi-projective supermultiplets and presented a manifestly locally supersymmetric and super-Weyl invariant action principle in bi-projective superspace.

The formalism we have introduced should be suitable to study general classes of matter couplings in 2D $\mathcal{N} = (4, 4)$ supergravity. In the superspace supergravity framework presented here, possible subjects for future investigations would be the formulation of 2D $\mathcal{N} = (4, 4)$ super-conformal matter systems such as WZNW/Liouville-type systems, non-linear sigma models and (4, 4) non-critical strings.

We also believe that there are still open questions purely related to 2D $\mathcal{N} = (4, 4)$ supergravity in superspace. One first question is the existence of variant minimal formulations. The multiplet we presented in the paper is an extension of the minimal multiplet of [52] but our analysis indicate that the latter is the only minimal de-gauging of the supergravity of subsection 2.1. It is natural to believe that there exists another minimal formulation having TM-II torsion components and TM-I conformal compensator. Such new minimal multiplet would turn out to be dual to the one of [52] in a manner similar to the $\mathcal{N} = (2, 2)$ case of [65, 52, 66, 67]. One way to find it could be by dimensional reduction of 4D $\mathcal{N} = 2$ superspace supergravity. An alternative approach could be the study of a non-minimal supergravity in which the structure group of the curved superspace is the full automorphism group of $\mathcal{N} = (4, 4)$ supersymmetry [57]: $\text{SO}(1,1) \times \text{SO}(4)_{L} \times \text{SO}(4)_{R}$. In this paper, we didn’t considered the extra $\text{SU}(2)_{L} \times \text{SU}(2)_{R}$, which, for example, transform $\nabla_{+i}$ into $\bar{\nabla}_{+i}$. It would be useful to rewrite our results in a basis of derivatives $(\nabla_{+i}, \nabla_{-J})$ where the $\text{SO}(4)_{L} \times \text{SO}(4)_{R}$ structure is manifest; the new minimal multiplet could probably be identical to the one of [52] but with simply the non-underlined groups and indices changed with the underlined ones. This covariant derivative basis would help to compare in details our discussion with the bi-harmonic superspace results of [50, 51].

Clearly the solution of the constraints of the 2D $\mathcal{N} = (4, 4)$ supergravity multiplets would be interesting. Note that a first, but uncompleted, effort to solve the minimal constraints in terms of prepotentials was given in [68] along the lines of the 2D $\mathcal{N} = (2, 2)$ case of [66]. A formulation of 2D $\mathcal{N} = (4, 4)$ conformal supergravity purely based on prepotentials lies in the bi-harmonic superspace approach of [51]. A complete solution of the Wess-Zumino like constraints for the minimal, or non-minimal, multiplet could clarify the connection between the bi-projective and bi-harmonic superspace approaches and provide an understanding of 2D $\mathcal{N} = (4, 4)$ supergravity in the spirit of the Gates-Siegel prepotential approach to the 4D $\mathcal{N} = 1$ case [37]. To this regard the 2D case would be of example for the higher dimensional cases where the structure of the supergravity

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multiplets is more involved.

The reduction to subsuperspaces is another topic in which the 2D $\mathcal{N} = (4,4)$ case could be fruitful to clarify more involved higher-dimensional cases. A detailed analysis of bi-projective superfields and the action principle reduced to 2D $\mathcal{N} = (2,2)$ superspace would be very interesting.

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A 2D Conventions

In this section we collect the two dimensional conventions used in the paper. These are consistent with \[1, 64\]. The Minkowski metric, Levi-Civita tensor and $\gamma$-matrices in two dimensions are defined according to the equations

\[
\eta_{ab} = (1, -1) , \quad \varepsilon_{ab} \varepsilon^{cd} = -\delta_{[a}^{c} \delta_{b]}^{d} , \quad \varepsilon^{01} = +1 , \quad (A.1a)
\]

\[
(\gamma^a)_{\alpha}^{\gamma} (\gamma^b)_{\gamma}^{\beta} = \eta^{ab} \delta_{\alpha}^{\beta} - \varepsilon^{ab} (\gamma^3)_{\alpha}^{\beta} , \quad (A.1b)
\]

where the Lorentz spinor indices take values $\alpha = +, -$. It is important to remark that in this paper, the complete (anti)symmetrization of $n$ indices does not involve any $(1/n!)$ factor.\[16\] Equation (A.1b) imply

\[
(\gamma^a)_{\alpha}^{\gamma} (\gamma^a)_{\gamma}^{\beta} = 2\delta_{\alpha}^{\beta} , \quad (\gamma^3)_{\alpha}^{\gamma} (\gamma^a)_{\gamma}^{\beta} = -\varepsilon^{ab} (\gamma^b)_{\alpha}^{\beta} , \quad (A.2a)
\]

\[
(\gamma^3)_{\alpha}^{\beta} (\gamma^3)_{\beta}^{\alpha} = 0 , \quad (\gamma^3)_{\alpha}^{\gamma} (\gamma^3)_{\gamma}^{\beta} = \delta_{\alpha}^{\beta} . \quad (A.2b)
\]

\[16\]For example, our conventions tell that $\psi(\alpha\chi,\beta) = (\psi(\alpha\chi,\beta) + \psi(\beta\chi,\alpha))$ and $\psi|a\chi,\beta] = (\psi(\alpha\chi,\beta) - \psi(\beta\chi,\alpha))$. 

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Some Fierz identities used in the paper are:

\[
C_{\alpha\beta} C^{\gamma\delta} = \delta_{[\alpha}^{\gamma} \delta_{\beta]}^{\delta}, \\
(\gamma^a)_{\alpha\beta} (\gamma_a)^{\gamma\delta} + (\gamma^3)_{\alpha\beta} (\gamma^3)^{\gamma\delta} = -\delta_{(\alpha}^{\gamma} \delta_{\beta]}^{\delta}, \\
(\gamma^a)_{(\alpha} (\gamma_a)_{\beta)}^{\gamma\delta} + (\gamma^3)_{(\alpha} (\gamma^3)_{\beta)}^{\gamma\delta} = \delta_{(\alpha}^{\gamma} \delta_{\beta)}^{\delta}, \\
(\gamma^a)_{(\alpha} (\gamma_a)_{\beta)}^{\gamma\delta} = -2(\gamma^3)_{\alpha\beta} (\gamma^3)^{\gamma\delta}, \\
2(\gamma^a)_{\alpha\beta} (\gamma_a)^{\gamma\delta} + (\gamma^3)_{\alpha\beta} (\gamma^3)^{\gamma\delta} = -\delta_{(\alpha}^{\gamma} \delta_{\beta)}^{\delta}, \\
(\gamma_a)^{\alpha\delta} (\gamma_a)^{\gamma\beta} + (\gamma^3)_{\alpha\beta} (\gamma^3)^{\gamma\beta} = (\gamma^3)_{\alpha\beta} (\gamma^3)^{\gamma\delta}, \\
(\gamma^3)_{\alpha\beta} (\gamma^3)^{\gamma\delta} = - (\gamma_a)^{\alpha\delta}, \\
(\gamma^c)_{\alpha\beta} (\gamma^3)_{c\delta} = C_{\alpha\delta} (\gamma^3)^{\delta\beta} + (\gamma^3)_{\alpha\delta} C^{\delta\beta}.
\]  

(A.3a)

(A.3b)

(A.3c)

(A.3d)

(A.3e)

(A.3f)

(A.3g)

(A.3h)

In some of the previous relations, given for example a spinor \( \psi^\alpha(x) \), we have raised and lowered the spinor indices according to the rule

\[
\psi^\alpha(x) = C^{\alpha\beta} \psi_\beta(x), \quad \psi_\alpha(x) = \psi^\beta(x) C_{\beta\alpha}.
\]

(A.4)

In terms of an explicit representation, we can define the 2D \( \gamma \)-matrices by using the usual Pauli matrices according to

\[
(\gamma^0)_{\alpha\beta} \equiv (\sigma^2)_{\alpha\beta}, \quad (\gamma^1)_{\alpha\beta} \equiv -i(\sigma^1)_{\alpha\beta}, \quad (\gamma^3)_{\alpha\beta} \equiv (\sigma^3)_{\alpha\beta}.
\]

(A.5)

The spinor metric \( C_{\alpha\beta} \) and its inverse \( C^{\alpha\beta} \) can be defined by

\[
C_{\alpha\beta} \equiv (\sigma^2)_{\alpha\beta}, \quad C^{\alpha\beta} \equiv - (\sigma^2)^{\alpha\beta}.
\]

(A.6)

Using this explicit representation, it is easy to show the following symmetry properties

\[
(\gamma^a)_{\alpha\beta} = (\gamma^a)_{\beta\alpha}, \quad (\gamma^3)_{\alpha\beta} = (\gamma^3)_{\beta\alpha}, \quad C_{\alpha\beta} = - C_{\beta\alpha}.
\]

(A.7)

and similarly for the matrices with both up indices. The following complex conjugation properties can be derived

\[
((\gamma^a)_{\alpha\beta})^* = -(\gamma^a)_{\alpha\beta}, \quad ((\gamma^3)_{\alpha\beta})^* = (\gamma^3)_{\alpha\beta}, \quad (C_{\alpha\beta})^* = - C_{\alpha\beta},
\]

(A.8a)

\[
((\gamma^a)_{\alpha\beta})^* = (\gamma^a)_{\alpha\beta}, \quad ((\gamma^3)_{\alpha\beta})^* = -(\gamma^3)_{\alpha\beta},
\]

(A.8b)

and the same for the matrices with both indices raised. The choice of gamma matrices is in a Majorana representation and the simplest spinor one can choose is real \( \psi^\alpha(x) \),

\[
(\psi^\alpha(x))^* = \psi^\alpha(x), \quad (\psi_\alpha(x))^* = - \psi_\alpha(x).
\]

(A.9)
Clearly it is also possible to introduce complex spinors.

The SU(2) indices $i = 1, 2$ possess conventions similar to the one used for the Lorentz spinor indices. The SU(2) metric $C_{ij}$ and its inverse $C^{ij}$ satisfy

\[ C_{ij} \equiv (\sigma^2)_{ij}, \quad C^{ij} \equiv -(\sigma^2)^{ij}, \quad (A.10a) \]
\[ C_{ij} = -C_{ji}, \quad C^{ij} = -C^{ji}, \quad C_{ij}C^{kl} = \delta^k_i\delta^l_j. \quad (A.10b) \]

We raise and lower SU(2) indices according to

\[ \psi^i(x) = C^{ij}\psi_j(x), \quad \psi_i(x) = \psi^j(x)C_{ji}. \quad (A.11) \]

Note that for the SU(2) invariant it holds

\[ (C^{ij})^* = C_{ij}, \quad (C_{ij})^* = C^{ij}. \quad (A.12) \]

With the previous complex conjugation conventions we have that the local Grassmannian superspace coordinates $(\theta^{\mu\dot{i}}, \bar{\theta}^{\mu\dot{i}})$ are related one to each other by the rule

\[ (\theta^{\mu\dot{i}})^* = \bar{\theta}^{\dot{i}\mu}. \quad (A.13) \]

Accordingly, given a general complex superfield $A$ ($\bar{A} := (A)^*$, with Grassmann parity $\varepsilon(A)$, the complex conjugate of the spinor covariant derivatives of $A$ satisfies

\[ (\nabla_{\alpha i} A)^* = -(-)^{\varepsilon(A)}\nabla^i \bar{A}. \quad (A.14) \]

To conclude, let us give the commutation algebra of the SO(1,1) $\times$ SU(2)$_L \times$ SU(2)$_R$ generators $\mathcal{M}$, $\mathbf{L}_{kl}$ and $\mathbf{R}_{kl}$ which can be derived by using (2.3a)-(2.4b)

\[ [\mathcal{M}, \mathcal{M}] = [\mathcal{M}, \mathbf{L}_{kl}] = [\mathcal{M}, \mathbf{R}_{kl}] = [\mathbf{L}_{ij}, \mathbf{R}_{kl}] = [\mathbf{L}_{ij}, \mathbf{R}_{kl}] = 0, \quad (A.15a) \]
\[ [\mathbf{L}_{ij}, \mathbf{L}_{kl}] = \frac{1}{4}(C_{k(i}\mathbf{L}_{j)l} + C_{l(i}\mathbf{L}_{j)k}), \quad (A.15b) \]
\[ [\mathbf{R}_{ij}, \mathbf{R}_{kl}] = \frac{1}{4}(C_{k(i}\mathbf{R}_{j)l} + C_{l(i}\mathbf{R}_{j)k}), \quad (A.15c) \]

The commutation algebra for the operators $\mathcal{V}_{kl} = (\mathbf{L}_{kl} + \mathbf{R}_{kl})$ and $\mathcal{C}_{kl} = (\mathbf{L}_{kl} - \mathbf{R}_{kl})$ is

\[ [\mathcal{V}_{ij}, \mathcal{V}_{kl}] = \frac{1}{4}(C_{k(i}\mathbf{V}_{j)l} + C_{l(i}\mathbf{V}_{j)k}), \quad (A.16a) \]
\[ [\mathcal{V}_{ij}, \mathcal{C}_{kl}] = \frac{1}{4}(C_{k(i}\mathbf{C}_{j)l} + C_{l(i}\mathbf{C}_{j)k}), \quad (A.16b) \]
\[ [\mathcal{C}_{ij}, \mathcal{C}_{kl}] = \frac{1}{4}(C_{k(i}\mathbf{V}_{j)l} + C_{l(i}\mathbf{V}_{j)k}). \quad (A.16c) \]
B Solution of the supergravity Bianchi identities

In this appendix we want to give a description of the solution of the Bianchi identities for the 2D $\mathcal{N} = (4, 4)$ supergravity of subsection 2.1 based on the torsion constraints (2.9a)–(2.9c). In a standard and useful way the analysis is organized in accordance with the increasing mass dimension of the Bianchi identities involved.

The super-Jacobi identities for the covariant derivatives

$$\sum_{\{ABC\}} [\nabla_A, [\nabla_B, \nabla_C]] = 0,$$  \hspace{1cm} (B.1)

with the graded cyclic sum assumed, are equivalent to the following Bianchi identities for the torsion and curvature of the geometry

$$0 = \sum_{\{ABC\}} \left( R_{ABC}^D - \nabla_A T_{BC}^D + T_{AB}^E T_{EC}^D \right),$$  \hspace{1cm} (B.2a)

$$0 = \sum_{\{ABC\}} \left( \nabla_A (R_V)_{BC}^{kl} - T_{AB}^D (R_V)_{DC}^{kl} \right),$$  \hspace{1cm} (B.2b)

$$0 = \sum_{\{ABC\}} \left( \nabla_A (R_C)_{BC}^{kl} - T_{AB}^D (R_C)_{DC}^{kl} \right),$$  \hspace{1cm} (B.2c)

$$0 = \sum_{\{ABC\}} \left( \nabla_A R_{BC} - T_{AB}^D R_{DC} \right),$$  \hspace{1cm} (B.2d)

where

$$R_{ABC}^D \equiv (R_V)_{AB}^{kl} (V_{kl})_C^D + (R_C)_{AB}^{kl} (C_{kl})_C^D + R_{AB} (M)_C^D,$$  \hspace{1cm} (B.3a)

$$(V_{kl})_A^B \nabla_B \equiv [V_{kl}, \nabla_A], \quad (C_{kl})_A^B \nabla_B \equiv [C_{kl}, \nabla_A], \quad (\mathcal{M})_A^B \nabla_B \equiv [\mathcal{M}, \nabla_A], \hspace{1cm} (B.3b)$$

$$(V_{kl})_{\dot{\alpha}}^{\dot{\beta}} = \frac{1}{2} \delta_3^\beta \delta_i (C_{kl})^i_{\dot{\alpha}}, \quad (V_{kl})_{\dot{\beta}}^{\dot{\alpha}} = \frac{1}{2} \delta_3^\alpha \delta_i (C_{kl})^i_{\dot{\beta}},$$  \hspace{1cm} (B.3c)

$$(C_{kl})_{\dot{\alpha}}^{\dot{\beta}} = \frac{1}{2} \delta_3^\beta \delta_i (C_{kl})^i_{\dot{\alpha}}, \quad (C_{kl})_{\dot{\beta}}^{\dot{\alpha}} = \frac{1}{2} \delta_3^\alpha \delta_i (C_{kl})^i_{\dot{\beta}},$$  \hspace{1cm} (B.3d)

$$(\mathcal{M})_{\dot{\alpha}}^{\dot{\beta}} = \frac{1}{2} \delta_3^\beta \delta_i (\gamma^3)^{\alpha}_{\beta}, \quad (\mathcal{M})_{\dot{\beta}}^{\dot{\alpha}} = \frac{1}{2} \delta_3^\alpha \delta_i (\gamma^3)^{\beta}_{\alpha}, \quad (\mathcal{M})_a^b = \varepsilon_a^b,$$  \hspace{1cm} (B.3e)

with the other components of $(V_{kl})_C^D$, $(C_{kl})_C^D$ and $(\mathcal{M})_C^D$ being equal to zero.

In solving the Bianchi identities it is important to remember that, due to Dragon’s second theorem [69], it is sufficient to analyze only eq. (B.2a); all the equations (B.2b)–(B.2d) are identically satisfied, provided that (B.2a) holds. This gives a great reduction in the number of equations that have to be studied.

\textsuperscript{17}In this appendix we often use the condensed notation $A_{\dot{\alpha}} \equiv A_{\dot{\alpha}i}$ and $B_{\dot{\alpha}} \equiv B^i_{\dot{\alpha}}$; for instance we have $\nabla_{\dot{\alpha}} = \nabla_{\dot{\alpha}i}$ and $\nabla^i_{\dot{\alpha}} = \nabla^i_{\dot{\alpha}}$.
To distinguish the curvatures and unambiguously raise and lower indices, we introduce the notation

$$R_{\alpha i \beta j} := R_\alpha^{\bar{\beta},}$$

$$\hat{R}_{\alpha \beta} := R_\alpha^{\bar{\beta}}$$

$$\check{R}_{\alpha i \beta j} := \check{R}_\alpha^{\bar{\beta}}$$

$$\hat{R}_{\alpha \beta} := \hat{R}_\alpha^{\bar{\beta}}$$

and similarly for the SU(2) curvatures. Analogously, for the torsion it is useful to define different objects to freely raise and lower indices

$$T_{\alpha \beta j \gamma k} := T_{\alpha \beta}^{\gamma}$$

$$\hat{T}_{\alpha \beta j \gamma k} := \hat{T}_{\alpha \beta}^{\gamma}$$

$$\check{T}_{\alpha \beta j \gamma k} := \check{T}_{\alpha \beta}^{\gamma}$$

$$\hat{T}_{\alpha \beta j \gamma k} := \hat{T}_{\alpha \beta}^{\gamma}$$

At dimension-1/2, due to the choice of the torsion constraints (2.9a)–(2.9c), the Bianchi identities are identically satisfied. The non-trivial analysis begins at dimension-1.

### B.1 dimension-1

At dimension-1 there are many Bianchi identities that originate from eq (B.2a). In fact, (B.2a) gives the following set of equations: with ($A = a$, $B = \beta$, $C = \gamma$, $D = d$)

$$0 = R_{\beta j \gamma k a}^d + 2iT_{a \beta j \rho}^q \delta_q^d (\gamma)^\rho + 2iT_{a \gamma k q}^j \delta_j^d (\gamma)^\beta$$

with ($A = a$, $B = \beta$, $C = \dot{\gamma}$, $D = \dot{d}$)

$$0 = \check{R}_{\beta j \gamma k a}^d + 2iT_{a \beta j \rho}^q \delta_q^d (\gamma)^\rho + 2iT_{a \gamma k q}^j \delta_j^d (\gamma)^\beta$$

with ($A = \alpha$, $B = \beta$, $C = \dot{\gamma}$, $D = \dot{\delta}$)

$$0 = (R^\gamma)_{\alpha \beta j}^k \delta_l^\gamma + (R^\gamma)_{\alpha \beta j}^k (\gamma^3)^\gamma + \frac{1}{2}R_{\alpha \beta j}^k (\gamma^3)^\gamma \delta_l^k$$

$$+ 2i\delta_j^k (\gamma^e)_{\beta j} T_{ea \gamma}^\delta + 2i\delta_j^k (\gamma^e)_{\alpha j} T_{e \beta \gamma}^\delta ,$$

with ($A = \alpha$, $B = \dot{\beta}$, $C = \dot{\gamma}$, $D = \dot{\delta}$)

$$0 = (\hat{R}^\gamma)_{\alpha \beta j}^k \delta_l^\gamma + (\hat{R}^\gamma)_{\alpha \beta j}^k (\gamma^3)^\gamma + \frac{1}{2}R_{\alpha \beta j}^k (\gamma^3)^\gamma \delta_l^k$$

$$+ (\hat{R}^\gamma)_{\alpha \beta j}^k \delta_l^\gamma + (\hat{R}^\gamma)_{\alpha \beta j}^k (\gamma^3)^\gamma + \frac{1}{2}R_{\alpha \beta j}^k (\gamma^3)^\gamma \delta_l^k$$

$$+ 2i\delta_j^k (\gamma^e)_{\alpha \beta} \hat{T}_{e \gamma}^l + 2i\delta_j^k (\gamma^e)_{\alpha j} \hat{T}_{e \beta}^l ,$$
and with $(A = \alpha, \ B = \beta, \ C = \gamma, \ D = \delta)$

$$0 = (R_Y)_{\alpha i j k}^{\delta} \delta_{\gamma}^{\delta} + (R_C)_{\alpha i j k}^{\delta} (\gamma^3)^{\delta} - \frac{1}{2} R_{\alpha i j k}^{\delta} (\gamma^3)^{\delta} \delta_{\gamma}^{\delta}$$

$$+ (R_Y)_{\beta j k}^{\delta} \delta_{\alpha}^{\delta} + (R_C)_{\beta j k}^{\delta} (\gamma^3)^{\delta} - \frac{1}{2} R_{\beta j k}^{\delta} (\gamma^3)^{\delta} \delta_{\alpha}^{\delta}$$

$$+ (R_Y)_{\gamma \kappa A i j}^{\delta} \delta_{\beta}^{\delta} + (R_C)_{\gamma \kappa A i j}^{\delta} (\gamma^3)^{\delta} - \frac{1}{2} R_{\gamma \kappa A i j}^{\delta} (\gamma^3)^{\delta} \delta_{\beta}^{\delta}. \quad (B.6e)$$

Here we have omitted identities that follow by complex conjugating the previous ones.

Equation $[B.6a]$ gives the relation

$$R_{\beta j k}^{\gamma} = i\varepsilon^{a b} T_{a \beta j k}^{\rho} (\gamma b)_{\gamma \rho} + i\varepsilon^{a b} T_{a \gamma k}^{\rho} (\gamma b)_{\beta \rho}, \quad (B.7)$$

and also the following constraints to the torsion $T_{a \beta j k}^{\gamma}$

$$0 = T_{(a \beta k}^{j} (\gamma b)_{\gamma)} + T_{(a \gamma k}^{j} (\gamma b)_{\beta)} . \quad (B.8)$$

These equations set to zero some irreducible components of the torsion and imply

$$T_{a \beta j}^{\gamma k} = C_{\beta j}^{\gamma k} A_n + \delta_{j}^{k} (\gamma^3)^{\beta} C_{a} + \delta_{j}^{k} \varepsilon_{a b} (\gamma^3)^{\beta} N + (\gamma a)_{\beta \gamma} Y_{j}^{k}, \quad (B.9a)$$

$$R_{a i j}^{\beta} = -4iC_{i j} C_{a \beta} N + 4iY_{i j}^{(3)} (\gamma^3)^{a \beta}, \quad (B.9b)$$

where the complex superfield $Y_{i j}$ is symmetric $Y_{i j} = Y_{j i}$.

Now, consider equation $[B.6c]$ which, once used $[B.9a]$ and $[B.9b]$, turns out to be equivalent to

$$(R_Y)_{\alpha i j k l} C_{\gamma \delta} + (R_C)_{\alpha i j k l} (\gamma^3)^{\gamma \delta} = -2iC_{i j} C_{k l} C_{a \beta} (\gamma^3)^{\gamma \delta} N - 2iC_{j k} C_{i l} (\gamma^3)^{\beta \gamma} N + 2iC_{j k} (\gamma^3)^{\beta \gamma} Y_{i}^{l}$$

$$- 2iC_{i k} C_{j l} (\gamma^3)^{a \beta} N + 2iC_{i k} (\gamma^3)^{a \beta} Y_{l j} - 2iC_{i k} (\gamma^3)^{a \beta} C_{a b} C_{i l} A_{a} - 2iC_{i k} (\gamma^3)^{a \beta} C_{i l} A_{a}$$

$$- 2iC_{i k} (\gamma^3)^{a \beta} C_{i l} (\gamma^3)^{a \beta} C_{a} . \quad (B.10)$$

Taking the trace of the previous equation with $(\gamma^3)^{\gamma \delta}$ one finds

$$(R_C)_{\alpha i j}^{k l} = i(\gamma^3)^{a \beta} (\delta_{i}^{k} (Y_{j}^{l}) + \delta_{j}^{k} (Y_{i}^{l})) - i(\gamma a)_{k \beta} \delta_{l}^{k} (\delta_{i}^{j} (\varepsilon_{a b} A_{b} + C_{a})). \quad (B.11)$$

Taking the trace of equation $[B.10]$ with $C^{\gamma \delta}$ the following equation follows

$$(R_Y)_{\alpha i j}^{k l} = 2i(\delta_{i}^{k} \delta_{j}^{l}) (\gamma^3)^{a \beta} N + 2iC_{i j} C_{a \beta} Y_{k l} - i\delta_{i}^{l} (\delta_{j}^{k} (\gamma^3)^{a \beta} (A_{a} + \varepsilon_{a b} C_{b})). \quad (B.12)$$

Then, from the trace of equation $[B.10]$ with $(\gamma^3)^{\gamma \delta}$ one finds the constraint

$$C_{a} = \varepsilon_{a b} A_{b}. \quad (B.13)$$
Summarizing the results obtained so far, it holds

\[ T_{a\beta j\gamma} = \delta^k_j C_{\beta\gamma} A_a + \delta^k_j (\gamma^3)_{\beta\gamma} \varepsilon_{ab} A^b + \delta^k_j \varepsilon_{ab} (\gamma^b)_{\beta\gamma} N + (\gamma_a)_{\beta\gamma} Y_j^k , \quad (B.14a) \]

\[ R_{a\alpha i\beta j} = -4 i C_{ij} C_{\alpha\beta} N + 4 i Y_{ij} (\gamma^3)_{\alpha\beta} , \quad (B.14b) \]

\[ (R_\psi)_{a\alpha i\beta j}^{kl} = 2 i \delta^k_i (\delta^l_j)_{\gamma} \alpha\beta N + 2 i C_{ij} C_{\alpha\beta} Y^{kl} - 2 i \delta^k_i (\delta^l_j) (\gamma^a)_{\alpha\beta} A_a , \quad (B.14c) \]

\[ (R_c)_{a\alpha i\beta j}^{kl} = i (\gamma^3)_{\alpha\beta} (\delta^k_i Y_j^l + \delta^l_j Y_i^k) - 2 i \delta^k_i (\delta^l_j) (\gamma_a)_{\alpha\beta} \varepsilon_{ab} A_b . \quad (B.14d) \]

Let us now consider the Bianchi identity ([B.6d]) which is equivalent to

\[ \hat{R}_{\beta j\gamma k} \varepsilon_{ab} = 2 i \hat{T}_{a\beta j\rho k} (\gamma_h)_{\gamma} \rho^a + 2 i \hat{T}_{a\gamma k\rho j} (\gamma_h)_{\rho} . \quad (B.15) \]

Note that, by considering a real superfield \((A)^* = A , \varepsilon(A) = 0 , (\nabla_a i A)^* = - \nabla^i A\), being

\[ ([\nabla_a , \nabla_a i] A)^* = -(T_{a\alpha i})^* \nabla^i A + \cdots , \quad ([\nabla_a , \nabla_a i] A)^* = -T_{a\alpha i} (\gamma_a)_{\beta} \nabla^j A + \cdots , \quad (B.16) \]

we obtain the complex conjugation relation between \(T_{a\beta j}^\gamma\) and \(T_{a\beta j}^{-\gamma} k\):

\[ \hat{T}_{a\beta j}^\gamma = (T_{a\beta j}^{-\gamma})^* . \quad (B.17) \]

Using the torsion constraint ([2.9c]) together with ([B.17]), the symmetric part in \(a, b\) of eq. ([B.15]) implies that the torsion \(T_{a\alpha i} \beta j\) is

\[ T_{a\beta j}^\gamma = (\gamma_a)_{\beta} \gamma (i \delta^k_j S + S_j^k) + \varepsilon_{ab} (\gamma^b)_{\beta} \gamma (\delta^k_j T + i T_j^k) + i \delta^k_j (\delta^l_j B_a + i (\gamma^3)_{\beta} \rho^a \delta^k_j C_a , \quad (B.18) \]

where

\[ (S)^* = S , \quad (T)^* = T , \quad (B_a)^* = B_a , \quad (C_a)^* = C_a \quad (B.19a) \]

\[ (S^{ij})^* = S_{ij} , \quad (T^{ij})^* = T_{ij} , \quad S_{ij} = S_{ji} , \quad T_{ij} = T_{ji} . \quad (B.19b) \]

The antisymmetric part in \(a, b\) of ([B.15]) is solved by

\[ \hat{R}_{a\alpha i}^j = -4 i C_{\alpha\beta} (\delta^k_j T + i T_j^k) + 4 i (\gamma^3)_{\alpha\beta} (i \delta^j_i S + S_j^i) . \quad (B.20) \]

Let us now turn our attention to eq. ([B.6d]) which is equivalently written as

\[ (\hat{R}_\psi)_{a\beta}^{i j k l} C_{\gamma \delta} + (\hat{R}_c)_{a\beta}^{i j k l} (\gamma^3)_{\gamma \delta} + (\hat{R}_\psi)_{a\gamma}^{i k j l} C_{\beta \delta} + (\hat{R}_c)_{a\gamma}^{i k j l} (\gamma^3)_{\beta \delta} = \frac{1}{2} \hat{R}_{a\beta}^{i j} (\gamma^3)_{\gamma \delta} C^{k l} + \frac{1}{2} \hat{R}_{a\gamma}^{i k j l} (\gamma^3)_{\beta \delta} C_{\gamma \delta} - 2 i C_{ij} (\gamma^e)_{\alpha\beta} T_{e \gamma \delta}^{k l} - 2 i C_{ik} (\gamma^e)_{\alpha \gamma} T_{e \delta \beta}^{j l} . \quad (B.21) \]

The right hand side of the previous equation can be expressed in terms of the torsion components \(S, T, B_a, C_a, S_{ij}\) and \(T_{ij}\) by making use of the equations ([B.18], [B.17]) and
The solution of \( \text{eq. (B.21)} \) can then be approached by considering the trace of \( \text{eq. (B.21)} \) with \( C^{\gamma \delta} \), \( (\gamma^3)^{\gamma \delta} \), and \( (\gamma^a)^{\gamma \delta} \) respectively. Solving the three resulting equations one finds a new constraint for the torsion components

\[
C_a = \varepsilon_{ab} B^b, \quad (B.22)
\]

and also the following expressions for the remaining SU(2) curvatures

\[
(\hat{R}_V)_{a_{ij}^{kl}} = -2i(\gamma^3)_{a_{ij}^{kl}C^{a}l} - 2C_{a_{ij}^{kl}C^{a}l} S + 2(\gamma^3)_{a_{ij}^{kl}S^{a}l} + 2iC_{a_{ij}^{kl}S^{a}l} B^b,
\]

\[
(\hat{R}_C)_{a_{ij}^{kl}} = C_{a_{ij}^{kl}}(\delta^i_j C^{l}T + C^{l}T_i) + (\gamma^3)_{a_{ij}^{kl}}(i\delta^i_j S^{l}) + iC^{l}T_i S^b,
\]

\[
\begin{align*}
\hat{T}_{a_{ij}^{kl}} &= 0 = \nabla_{a_{ij}^{kl}} - \nabla_{a_{ij}^{kl}} \nabla_{a_{ij}^{kl}}. \quad (B.25)
\end{align*}
\]

By making use of the expressions obtained for the curvatures \( \text{eq. (B.14b)–(B.14d)} \), the last Bianchi identity to be checked, \( \text{eq. (B.6e)} \), turns out to be identically satisfied. This concludes the analysis of the dimension-1 Bianchi identities since other Bianchi identities, not explicitly studied, are identically satisfied by taking into account complex conjugation.

### B.2 dimension-3/2

We begin the analysis of the dimension-3/2 Bianchi identities by considering eq. \( \text{eq. (B.2a)} \) with \( (A = a, \ B = \beta, \ C = \gamma, \ D = \delta) \)

\[
0 = -\nabla_{\beta j} T_{a_{ij}^{kl}} - \nabla_{\gamma k} T_{a_{ij}^{kl}}. \quad (B.25)
\]

From the previous equation, a set of dimension-3/2 differential constraints on the torsion components \( N, Y_{ij} \) and \( A_a \) arises:

\[
\nabla_{\alpha} Y_{ij} = (\gamma^3)_{\alpha} C^{\alpha}l \nabla_{\beta} N, \quad (B.26a)
\]

\[
\nabla_{\beta} A_a = -\varepsilon_{ab} (\gamma^b)_{\beta} \nabla_{\delta} N. \quad (B.26b)
\]

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To continue, let us consider the Bianchi identity given by \([B.2a]\) with indices \((A = a, \; B = b, \; C = \gamma, \; D = d)\):

\[
0 = -R_{b\gamma k}\varepsilon^a_d + R_{a\gamma k}\varepsilon^b_d - 2i\tilde{T}_{ab\gamma}\delta^p_k (\gamma^d)_{\gamma\rho} .
\]  
(B.27)

This is equivalent to the following equation for the dimension-3/2 Lorentz curvature

\[
R_{a\beta j} = i\varepsilon^{bce}\tilde{T}_{be j}(\gamma_a)_{\gamma\beta} .
\]  
(B.28)

The Bianchi identity \([B.2a]\) with indices \((A = a, \; B = b, \; C = \gamma, \; D = \delta)\) is:

\[
0 = -\delta^\gamma_\delta (R_V)_{a\beta j} - (\gamma^3)_\beta \delta^\gamma_\delta (R_C)_{a\beta j} - \frac{1}{2}\delta^\gamma_\delta R_{a\beta j} (\gamma^3)_{\gamma\delta} - 2i\delta^\gamma_\delta (\gamma^3)_{\beta\gamma} T_{eal} \nonumber
\]  
(B.29)

By taking the trace of \([B.29]\) with \((\gamma^3)_{\delta\gamma}\) and solving the resulting equation one finds a set of constraints on the torsion components

\[
\nabla^j_{\beta} Y^{kl}_\gamma = -2\nabla^j_{\beta} S^{kl} = -2i(\gamma^3)_\beta \gamma \nabla^j T^{kl} ,
\]  
(B.30a)

\[
\nabla^j_{\beta} S = \frac{1}{2}(\gamma^3)_\beta \delta \nabla^j S^{kl} + \frac{1}{3}(\gamma^3)_\beta \gamma \nabla^j T^{jk} - \frac{i}{6}\nabla^j T^{jk} ,
\]  
(B.30b)

\[
\nabla^j_{\beta} T = -\frac{1}{2}\nabla^j_s N + \frac{1}{3}(\gamma^3)_\beta \delta \nabla^j S^{jk} + \frac{1}{6}(\gamma^3)_\beta \gamma \nabla^j T^{jk} ,
\]  
(B.30c)

\[
(\gamma^a)_\beta \gamma \nabla^j_{\gamma} A_a = -\frac{2}{3}\nabla^j_{\beta} S^{jk} + \frac{2i}{3}(\gamma^3)_\beta \delta \nabla^j T^{jk} ,
\]  
(B.30d)

and the following expression for the dimension-3/2 torsion

\[
\tilde{T}_{ab\gamma} = -\frac{1}{2}\varepsilon^a_{bc} \left( i\nabla^k N - \frac{2i}{3}(\gamma^3)_{\gamma} \delta \nabla^l S^{kl} + \frac{2}{3}\nabla^l T^{kl} + \frac{i}{3}(\gamma^3)_{\gamma} \delta \nabla^l Y^{kl} \right) .
\]  
(B.31)

It is also useful to decompose \(\nabla^k A_a\) in its irreducible gamma and gamma-traceless parts

\[
\nabla^j_{\beta} A_a = A^j_{a\beta} + \frac{1}{2}(\gamma_a)_{\beta} (\gamma^b)_{\gamma} \delta \nabla^j A_b , \quad (\gamma^a)_\gamma A_{a\gamma} = 0 ,
\]  
(B.32)

which by using \([B.30a]\) gives

\[
\nabla^k_{\gamma} A_a = A^k_{a\gamma} - \frac{1}{3}(\gamma_a)_{\gamma} \delta \nabla^k S^{kl} + \frac{i}{3}\varepsilon^a_{bc} (\gamma^b)_{\gamma} \delta \nabla^k T^{kl} .
\]  
(B.33)

By taking the trace of eq. \([B.29]\) with \(\delta^\gamma_\delta\) and by using eqs. \([B.30a]\)–\([B.33]\) we obtain the constraint

\[
\nabla^j_{\beta} B_a = \frac{i}{6}(\gamma_a)_{\beta} \gamma \nabla^p S^{jp} - \frac{1}{6}\varepsilon^a_{bc} (\gamma^b)_{\gamma} \gamma \nabla^p T^{jp} + \frac{i}{6}(\gamma_a)_{\beta} \gamma \nabla^p Y^{jp} + \frac{i}{2} A^j_{a\beta} ,
\]  
(B.34)
and the following expression for the dimension-3/2 SU(2)$_V$ curvature

$$(R_V)_{a\beta j}^{kl} = -\frac{1}{2} \delta_j^k \varepsilon_{ab} (\gamma^b)_{\beta} \gamma \overline{\nabla}_{\gamma p} S^l p + \frac{1}{6} \delta_j^k (\gamma_a)_{\beta} \gamma \overline{\nabla}_{\gamma p} T^l p + \frac{1}{6} \delta_j^k \varepsilon_{ab} (\gamma^b)_{\beta} \gamma \overline{\nabla}_{\gamma p} T^l p$$

To complete the analysis of the Bianchi identity (B.39) it is necessary to analyze its trace with $(\gamma^3)_{a\beta}$. Solving the resulting equation, with the help of the results obtained so far, the following expression for the dimension-3/2 SU(2)$_C$ curvature arises

$$(R_C)_{a\beta j}^{kl} = -\frac{1}{6} \delta_j^k \varepsilon_{ab} (\gamma^b)_{\beta} \gamma \overline{\nabla}_{\gamma p} S^l p - \frac{1}{6} \delta_j^k (\gamma_a)_{\beta} \delta \overline{\nabla}_{\delta p} Y^l p + \frac{1}{6} \delta_j^k \varepsilon_{ab} (\gamma^b)_{\beta} \delta \overline{\nabla}_{\delta p} Y^l p$$

It remains to consider the Bianchi identity (B.39a) with $(A = a, B = \beta, C = \gamma, D = \delta)$:

$$0 = (R_V)_{a\beta j}^{kl} \delta^k \gamma \overline{\nabla}_{\gamma p} S^l p + (R_V)_{a\gamma jk}^{(\gamma^3)_{A\beta}} + (R_C)_{a\gamma jk}^{(\gamma^3)_{A\beta}} + (\gamma^3)_{a\beta j}^{\gamma \overline{\nabla}_{\gamma p} Y^l p}$$

This turns out to be identically satisfied once used the results previously obtained. The rest of the dimension-3/2 Bianchi identities, not explicitly written here, are satisfied by taking into account complex conjugation.

### B.3 dimension-2

At dimension-2 the Bianchi identity (B.2a) with $(A = a, B = b, C = \gamma, D = \delta)$ gives

$$0 = \varepsilon^{ab} (R_V)_{ab k}^{i \delta \gamma} + \varepsilon^{ab} (R_C)_{ab k}^{l (\gamma^3)_{\gamma}} - \frac{1}{2} \varepsilon^{ab} R_{ab} (\gamma^3)_{\gamma} \delta \gamma \delta k + \nabla_{\gamma k} \varepsilon^{ab} T_{ab \delta k} + 2 \varepsilon^{ab} \nabla_a T_{b \gamma k \delta} \delta \gamma - 2 \varepsilon^{ab} T_{a \gamma k i}^{\alpha \gamma i} T_{b \alpha \delta}.$$

One can first consider the trace of the previous equation with $(\gamma^3)_{a\beta}$. Such equation turns out to be identically satisfied. To prove it one may use the following identities

$$i (\gamma_a)^{\alpha \beta} [\nabla_{ai}, \nabla^i_{\beta}] N = \frac{-2i}{3} \varepsilon_{ab} (\gamma^b)^{\alpha \beta} [\nabla_{ai}, \nabla_{bj}] S^{ij} + \frac{2}{3} (\gamma_a)^{\alpha \beta} [\nabla_{ai}, \nabla_{bj}] T^{ij}$$

$$+ 16 \nabla^a T + 16i \varepsilon_{ab} \nabla^b S - 16 A_a N,$$

$$i (\gamma_a)^{\alpha \beta} [\nabla_{ai}, \nabla^i_{\beta}] N = \frac{-2i}{3} \varepsilon_{ab} (\gamma^b)^{\alpha \beta} [\nabla_{ai}, \nabla_{bj}] S^{ij} - \frac{2}{3} (\gamma_a)^{\alpha \beta} [\nabla_{ai}, \nabla_{bj}] T^{ij}$$

$$+ 16 \nabla^a T - 16i \varepsilon_{ab} \nabla^b S - 16 A_a N,$$

$$(\gamma^3)_{a\beta} \nabla_{\delta} T^{ij} = 32i \nabla^a T^{ij} + 2i \varepsilon_{ab} (\gamma^b)^{\alpha \beta} \nabla_{\delta} (\nabla_{ai} Y^{j})^{p} + 64 B^a T^{ij} - 48 \varepsilon_{ab} A_b Y^{ij},$$

$$(\gamma^3)_{a\beta} \nabla_{\delta} (\nabla_{ai} Y^{j})^{p} = 32i \nabla^a S^{ij} - 2 (\gamma^3)^{\alpha \beta} \nabla_{\alpha} (\nabla_{\beta} Y^{j})^{p} + 64 B^a S^{ij} - 48i A^a Y^{ij}.$$
where clearly $\tilde{N} = (N)^*$ and $\tilde{Y}_{ij} = (Y^{ij})^*$. These equations are consequences of the dimension-3/2 Bianchi identities (2.15a)–(2.15g). One can obtain (B.39a)–(B.39d) in two steps: first derive a set of dimension-2 differential equations by applying a spinor covariant derivative to eqs. (2.15a)–(2.15g); then manipulate the resulting equations by taking into account the dimension-1 covariant derivatives algebra (2.10a)–(2.10b) and the structure group transformation properties of the dimension-1 torsion components (2.14a)–(2.14d).

Now, consider eq. (B.38) contracted with $\delta^\gamma_{\alpha} \delta^k_l$. The resulting equation is identically satisfied by making use of

$$(\gamma^3)^{\alpha\gamma}[\bar{\nabla}_{\alpha i}, \bar{\nabla}_{\gamma j}] S^{ij} - i[\nabla_{\alpha i}, \nabla^\alpha_{\gamma j}] T^{ij} + 24\varepsilon^{ab}_{\alpha} \nabla_a B_b = 0 \quad (B.40)$$

which is again a dimension-2 consequence of (2.15a)–(2.15g).

Contracting equation (B.38) with $(\gamma^3)^{\alpha\gamma} \delta^k_l$ the dimension-2 Lorentz curvature $R_{ab}$ can be computed. With the aid of the equations

$$0 = [\nabla_{\alpha i}, \nabla^\alpha_{\gamma j}] Y^{ij} + [\bar{\nabla}_{\alpha i}, \nabla^\alpha_{\gamma j}] Y^{ij} - 24\nabla^\alpha B_{\alpha} \quad (B.41a) ,$$

$$0 = i(\gamma^3)^{\alpha\beta}[\bar{\nabla}_{\alpha i}, \bar{\nabla}_{\beta j}] N + i(\gamma^3)^{\alpha\beta}[\nabla_{\alpha i}, \bar{\nabla}^\alpha_{\beta j}] \tilde{N} - \frac{i}{3}[\nabla_{\alpha i}, \nabla^\alpha_{\gamma j}] Y^{ij} - \frac{i}{12}[\nabla_{\alpha i}, \nabla^\alpha_{\gamma j}] \tilde{Y}^{ij} \quad (B.41b) ,$$

we find the expression

$$R_{ab} = -\frac{1}{2} \varepsilon^{ab}_{\alpha}( \frac{1}{4}(\gamma^3)^{\alpha\beta}[\bar{\nabla}_{\alpha i}, \bar{\nabla}^\beta_{\gamma j}] N + \frac{1}{4}(\gamma^3)^{\alpha\beta}[\nabla_{\alpha i}, \bar{\nabla}^\beta_{\gamma j}] \tilde{N} + \frac{1}{12}[\nabla_{\alpha i}, \nabla^\alpha_{\gamma j}] Y^{ij} - \frac{1}{12}[\nabla_{\alpha i}, \nabla^\alpha_{\gamma j}] \tilde{Y}^{ij} + \frac{i}{6}[\nabla_{\alpha i}, \nabla^\alpha_{\gamma j}] S^{ij} - \frac{1}{6}(\gamma^3)^{\alpha\beta}[\bar{\nabla}_{\alpha i}, \bar{\nabla}^\beta_{\gamma j}] \tilde{T}^{ij} + 8T^2 + 8\tilde{N} N + 8S^2 + 4S^2 S_{ij} + 4T^2 T_{ij} + 4\tilde{Y}^{ij} Y_{ij} \) \quad (B.42) .$$

Note that the relations (B.41a) and (B.41b) again derive from (2.15a)–(2.15g).

As a next step, contract equation (B.38) with $\delta^\gamma_{\alpha}$ and take the traceless part in the SU(2) indices $k, l$. From the resulting equation, by also using the relations

$$0 = [\nabla_{\alpha i}, \nabla^\alpha_{\gamma j}] N + [\bar{\nabla}_{\alpha i}, \bar{\nabla}^\alpha_{\gamma j}] \tilde{N} + (\gamma^3)^{\alpha\beta}[\nabla_{\alpha p}, \nabla^\alpha_{\beta j}] \tilde{Y}^{ij} + (\gamma^3)^{\alpha\beta}[\bar{\nabla}_{\alpha p}, \bar{\nabla}^\alpha_{\beta j}] \tilde{Y}^{ij} \quad (B.43a) ,$$

$$\bar{\nabla}_{\alpha p} \nabla^\alpha_{\beta j} T_{ij} = \frac{i}{8}(\gamma^3)^{\alpha\beta}[\bar{\nabla}_{\alpha p}, \bar{\nabla}^\beta_{\gamma j}] \tilde{Y}^{ij} - \frac{i}{8}(\gamma^3)^{\alpha\beta}[\nabla_{\alpha p}, \nabla^\beta_{\gamma j}] \tilde{Y}^{ij} - 4iS_{(i}^{\gamma k} \bar{T}_{jk)} \quad (B.43b) ,$$

$$\bar{\nabla}_{\alpha p} \nabla^\beta_{\gamma j} S_{ij} = \frac{i}{8}(\gamma^3)^{\alpha\beta}[\bar{\nabla}_{\alpha p}, \bar{\nabla}^\alpha_{\gamma j}] \tilde{Y}^{ij} - \frac{i}{8}(\gamma^3)^{\alpha\beta}[\nabla_{\alpha p}, \nabla^\alpha_{\gamma j}] \tilde{Y}^{ij} + 4S_{(i}^{\gamma} \bar{T}_{jk)} \quad (B.43c) ,$$

that derive from (2.15a)–(2.15g), we find the dimension-2 SU(2)$_V$ curvature

$$(R_Y)_{ab}^{kl} = -\frac{1}{2} \varepsilon_{ab}( \frac{i}{16}[\nabla_{\alpha i}, \nabla^\alpha_{\beta j}] N - \frac{i}{16}[\bar{\nabla}_{\alpha i}, \bar{\nabla}^\alpha_{\beta j}] \tilde{N} - \frac{i}{16}(\gamma^3)^{\alpha\beta}[\nabla_{\alpha p}, \nabla^\beta_{\gamma j}] Y^{kl} + \frac{i}{16}(\gamma^3)^{\alpha\beta}[\bar{\nabla}_{\alpha p}, \bar{\nabla}^\beta_{\gamma j}] \tilde{Y}^{kl} + 8S^{kl} T + 8iS_{(k}^{\gamma i} \bar{T}_{lp)} \) \quad (B.44) .$$
To conclude the solution of eq. (B.38), we take its trace with \((\gamma^3)_{\delta}^{\gamma}\) and consider the traceless part in the SU(2) indices \(k, l\). By using (2.15a)–(2.15g), that imply the relations

\[
\tilde{\nabla}_{\delta}^{\gamma} \nabla_{\alpha}^{\beta} S_{ij} = \frac{1}{16} [\tilde{\nabla}_{\alpha(k}, \nabla_{\alpha}^{k}) Y_{j)]_{k} + \frac{1}{16} [\nabla_{\alpha(i}, \nabla_{\alpha}^{k}) \tilde{Y}_{j)]_{k} - 2i \tilde{Y}(i Y_{j})_{k} - 8S S_{ij} + 8T T_{ij},
\]

(B.45a)

\[
(\gamma^{3})^{i\beta} \tilde{\nabla}_{\delta p}^{\alpha} \nabla_{\alpha}^{\alpha} T_{ij} = -\frac{i}{16} [\nabla_{\alpha(i}, \nabla_{\alpha}^{k}) Y_{j)]_{k} - \frac{i}{16} [\nabla_{\alpha(i}, \nabla_{\alpha}^{k}) \tilde{Y}_{j)]_{k} - 2 \tilde{Y}(i Y_{j})_{k} + 8i S S_{ij} - 8i T T_{ij},
\]

(B.45b)

we obtain the following expression for the dimension-2 SU(2) curvature

\[
(R_C)_{ab}^{kl} = -\frac{1}{2} \varepsilon_{ab} \left( \frac{i}{48} [\tilde{\nabla}_{\alpha(k}, \nabla_{\alpha}^{k}) Y^{l)]_{p} - \frac{i}{48} [\nabla_{\alpha(k}, \nabla_{\alpha}^{k}) \tilde{Y}^{l)}_{p} - 4 \tilde{Y}(p Y^{l)}_{j}) \right).
\]

(B.46)

The Bianchi identity (B.2a) with \((A = a, B = b, C = \gamma, D = \hat{\delta})\) is equivalent to the equation

\[
0 = \nabla_{\gamma k} \varepsilon^{ab} T_{ab}^{\delta} + 2\varepsilon^{ab} \nabla_{a} T_{b;ki}^{\delta} - 2\varepsilon^{ab} T_{a;ki}^{\alpha i} T_{b;ai}^{\delta} - 2\varepsilon^{ab} T_{a;ki}^{\alpha i} T_{ai}^{\delta}. \quad (B.47)
\]

This is identically satisfied by using (2.15a)–(2.15g), (2.10a)–(2.10b) and (2.14a)–(2.14c). The rest of the dimension-2 Bianchi identities, not explicitly written here, are satisfied by taking into account complex conjugation.

### C Derivation of eq. (4.16)

In this appendix we give a derivation of eq. (4.16) which is crucial for the analysis of section 4. Consider a general weight-(m,n) bi-isotwistor superfield \(U^{(m,n)}\) as defined in subsection 4.2. First, we analyze the pure left sector of (4.16). Being

\[
(\gamma^0)_{++} = (\gamma^1)_{++} = (\gamma^0)_{--} = 1, \quad (\gamma^1)_{--} = -1, \quad (\gamma^0)_{+-} = (\gamma^1)_{+-} = 0, \quad \text{(C.1)}
\]

by using (2.10a), one finds the \(\{\nabla^\oplus, \nabla^\oplus\}\) spinor derivatives anticommutator

\[
\{\nabla^\oplus_{\gamma}, \nabla^\oplus_{\nu}\} = -8i A_{++} L^\oplus_{\gamma}, \quad \text{(C.2)}
\]

where \(A_{++} = (\gamma^a)_{++} A_a = (A_0 + A_1)\) and \(L^\oplus_{\gamma} = u^\oplus_i u^\oplus_j L_{ij}\). From equation (4.15b) it is easy to observe that it holds

\[
L^\oplus_{\gamma} U^{(m,n)} = 0, \quad \Rightarrow \quad \{\nabla^\oplus_{\gamma}, \nabla^\oplus_{\nu}\} U^{(m,n)} = 0. \quad \text{(C.3)}
\]
In complete similarity, from (2.10c), (2.10b) and\( L^{\otimes \otimes}U^{(m,n)} = 0 \) one finds

\[
\{\nabla_+, \nabla_+\} = 8B_{++}L^{\otimes \otimes}, \quad \{\nabla_-, \nabla_-\} = -8iA_{++}L^{\otimes \otimes}, \quad (\text{C.4a})
\]

\[
\{\nabla_+, \nabla_+\}U^{(m,n)} = \{\nabla_-, \nabla_-\}U^{(m,n)} = 0. \quad (\text{C.4b})
\]

The same works in the right light-cone sector of the algebra. In fact, with \( A_- = (\gamma^a)_-A_a = (A_0 - A_1) \) and \( R^{\otimes \otimes} = v_i^j v_i^jR^{IJ} \) it holds

\[
\{\nabla_-, \nabla_-\} = -8iA_-R^{\otimes \otimes}, \quad \{\nabla_-, \nabla_-\} = 8B_-R^{\otimes \otimes}, \quad \{\nabla_-, \nabla_-\} = -8iA_-R^{\otimes \otimes}, \quad (\text{C.5})
\]

and, being \( R^{\otimes \otimes}U^{(m,n)} = 0 \), one finds

\[
\{\nabla_-, \nabla_-\}U^{(m,n)} = \{\nabla_-, \nabla_-\}U^{(m,n)} = \{\nabla_-, \nabla_-\}U^{(m,n)} = 0. \quad (\text{C.6})
\]

Consider now the mixed left-right sector which results a bit less trivial. Using

\[
C_{+-} = -C_{-+} = -i, \quad (\gamma^3)_{+-} = (\gamma^3)_{-+} = -i \quad (\text{C.7})
\]

and (2.10a), one finds

\[
\{\nabla_+, \nabla_-\} = 4((u^\otimes v^\otimes)N + Y^\otimes \otimes - 2(u^\otimes v^\otimes)Y^{kl}V_{kl} + 2u^\otimes v^\otimes Y_{iC}, I J, , \quad (\text{C.8})
\]

where, given any \( A^{ij} \) with two \( \text{SU}(2) \) indices, we use \( A^{\otimes \otimes} = u^i v^j A^{ij} \) as a contraction rule. The following relations hold

\[
u^\otimes v^\otimes L_{IJ}U^{(m,n)} = -\frac{m}{2}(u^\otimes v^\otimes)U^{(m,n)}, \quad u^\otimes v^\otimes R_{IJ}U^{(m,n)} = \frac{n}{2}(u^\otimes v^\otimes)U^{(m,n)}, \quad (\text{C.9a})
\]

\[
Y^{\otimes \otimes}U^{(m,n)} = \frac{n-m}{2}(u^\otimes v^\otimes)U^{(m,n)}, \quad C^{\otimes \otimes}U^{(m,n)} = -\frac{m+n}{2}(u^\otimes v^\otimes)U^{(m,n)}. \quad (\text{C.9b})
\]

Moreover, one finds that the combination \((u^\otimes v^\otimes)A^{kl}V_{kl} - u^\otimes v^\otimes A^{ij}_{iC}, I J, \), when acting on a bi-isotwistor superfield, satisfy the simple relation

\[
((u^\otimes v^\otimes)A^{kl}V_{kl} - u^\otimes v^\otimes A^{ij}_{iC}, I J, )U^{(m,n)} = (m-n)A^{\otimes \otimes}U^{(m,n)}. \quad (\text{C.10})
\]

Using the last results and \( MU^{(m,n)} = \frac{(m-n)}{2}U^{(m,n)} \), one easily obtains

\[
\{\nabla_+, \nabla_-\}U^{(m,n)} = 0. \quad (\text{C.11})
\]

Now, we consider the anticommutator \( \{\nabla_+, \nabla_-\} \) which can be easily seen to be

\[
\{\nabla_+, \nabla_-\} = 4(u^\otimes v^\otimes)T\{, M + 4T^{\otimes \otimes} - 4i(u^\otimes v^\otimes)S\{, M - 4iS^{\otimes \otimes} - 4i\bar{T}\{, \hat{M} + 2i(u^\otimes v^\otimes)\bar{T}^{kl}V_{kl} - 2iu^\otimes v^\otimes \bar{T}_{ij}C, I J, \} + 4S^{\otimes \otimes} - 2(u^\otimes v^\otimes)S^{kl}V_{kl} + 2u^\otimes v^\otimes S_{ij}C, I J, \}. \quad (\text{C.12})
\]
Therefore, by using (C.9a)-(C.10), one finds
\[
\{\nabla_+, \nabla_-\} U^{(m,n)} = 0 .
\] (C.13)

Along the same lines, it can be proved that
\[
\{\nabla^{\dagger}, \nabla_\dagger\} U^{(m,n)} = 0, \quad \{\nabla^+, \nabla^-\} U^{(m,n)} = 0 .
\] (C.14)

This concludes the derivation of the equations (4.16).

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