The Von Zeipel–Lidov–Kozai Effect inside Mean Motion Resonances with Applications to Trans-Neptunian Objects

Hanlun Lei1,2, Jian Li1,2, Xiumin Huang1,2, and Muzi Li3

1 School of Astronomy and Space Science, Nanjing University, Nanjing 210023, People’s Republic of China; leihl@nju.edu.cn
2 Key Laboratory of Modern Astronomy and Astrophysics in Ministry of Education, Nanjing University, Nanjing 210023, People’s Republic of China
3 Shanghai Aerospace Control Technology Institute, Shanghai 201109, People’s Republic of China

Received 2022 April 21; revised 2022 June 22; accepted 2022 June 26; published 2022 August 2

Abstract

Secular dynamics inside mean motion resonances (MMRs) plays an essential role in governing the dynamical structure of the trans-Neptunian region and sculpting the orbital distribution of trans-Neptunian objects (TNOs). In this study, semianalytical developments are made to explore the von Zeipel–Lidov–Kozai resonance inside MMRs. To this end, a semi-secular model is formulated from averaging theory and then a single-degree-of-freedom integrable model is achieved based on the adiabatic invariance approximation. In particular, we introduce a modified adiabatic invariant, which is continuous around the separatrices of MMRs. During long-term evolution, both the resonant Hamiltonian and the adiabatic invariant remain unchanged, thus phase portraits can be produced by plotting level curves of the adiabatic invariant with a given Hamiltonian. The phase portraits provide global pictures to predict long-term behaviors of the eccentricity, inclination, and argument of pericenter. Applications to some representative TNOs inside MMRs (2018 VO137, 2005 SD276, 2015 PD312, Pluto, 2004 HA79, 1996 TR66, and 2014 SR373) show good agreements between the numerically propagated trajectories under the full N-body model and the level curves arising in phase portraits. Interestingly, 2018 VO137 and 2005 SD276 exhibit switching behaviors during their long-term evolution and currently they are inside 2:5 MMR with Neptune.

Unified Astronomy Thesaurus concepts: Celestial mechanics (211); Perturbation methods (1215); N-body problem (1082); Resonant Kuiper belt objects (1396); Trans-Neptunian objects (1705); Classical Kuiper belt objects (250)

1. Introduction

According to the independent works performed by von Zeipel (1910), Lidov (1962), and Kozai (1962), the coupled oscillations between eccentricity and inclination of test particles in long-term evolution are attributed to the von Zeipel–Lidov–Kozai (ZLK) effect (Ito & Ohtsuka 2019). In the Kuiper Belt, it is known that the ZLK effect inside mean motion resonances (MMRs) is important for understanding the orbital distribution of trans-Neptunian objects (TNOs; Gomes et al. 2005; Gomes 2011; Gallardo et al. 2012; Saillennest 2020). To understand the ZLK effect of TNOs inside MMRs with Neptune, it is of significance to formulate secular models (Saillennest et al. 2016). After averaging out the short-period terms from the Hamiltonian, the resulting semi-secular model has two degrees of freedom. One degree of freedom is associated with the MMR and the other with the long-term evolution (Saillennest 2020). To study secular dynamics inside MMRs analytically, it is required to further remove one degree of freedom from the semi-secular model. There are various methods to do this.

The simplest approach is to assume the critical argument associated with the MMR at the libration center (Kozai 1985; Yoshikawa 1989; Nesvorny et al. 2002; Wan & Huang 2007; Saillennest et al. 2017; Li et al. 2021; Pons & Gallardo 2022). By fixing the resonant angle or its amplitude to zero (this assumption corresponds to an adiabatic invariant equal to zero, as discussed later), the degree of freedom associated with MMR disappears and the dynamical model immediately reduces to a one-degree-of-freedom integrable system. In the resulting reduced model, the motion of TNOs inside MMRs happens on the isolines of the energy integral (Morbelli 2002). Thus, phase portraits can be used to estimate the ranges of variations of orbital elements. Such an approximate model could provide a reasonable approximation for those particles located deep inside MMRs. For a better approximation, some researchers assumed that the resonant angle associated with MMR evolves sinusoidally with constant center, frequency, and amplitude (Gomes et al. 2005; Gomes 2011; Gallardo et al. 2012; Brasil et al. 2014; Huang et al. 2018). Under this assumption, the degree of freedom associated with MMR is decoupled from the other degree of freedom. Thus, it becomes possible to average the Hamiltonian over the period of the resonant angle and thus the degree of freedom associated with MMR is eliminated. However, in the original system the libration center, frequency, and amplitudes associated with MMR are changed during the long-term evolution (Saillennest et al. 2016).

In order to understand the chaotic and quasi-periodic libration zones on the representative plane (Wisdom 1982, 1983; Murray & Fox 1984), Wisdom (1985) developed a semianalytical perturbation theory for the secular dynamics near the 3:1 MMR within the framework of the planar elliptic restricted three-body problem (ERTBP). Averaging the original system over the period of the resonant angle leads to the evolutionary equations, which can be used to describe the long-term behaviors of the slow variable. Following the same idea, Henrard & Lemaître (1987) extended Wisdom’s perturbation theory to the 2:1 Jovian resonance, and Yokoyama & Balthazar (1992) applied Wisdom’s perturbative method to asteroids inside 5:2 and 7:3 Jovian resonances within the framework of
the planar ERTBP. A generalization of Wisdom’s perturbative method can be found in Yokoyama (1996).

Based on the adiabatic invariance approximation, Henrard & Lemaitre (1986) developed a seminumerical perturbation method and proposed a general numerical procedure allowing description of the dynamics associated with a two- or more-degree-of-freedom separable Hamiltonian. In particular, Henrard (1990) presented a numerical description for the action–angle variables of the separable Hamiltonian system. The Hamiltonian and the adiabatic invariant remain unchanged in the long-term evolution, meaning that the motion of resonant objects happens on the isolines of the Hamiltonian and adiabatic invariant. As a result, two types of phase portraits revealing global structures in the phase space can be produced:
(a) plotting level curves of the Hamiltonian with given adiabatic invariant (Morbidelli 2002; Saillenfest et al. 2016; Saillenfest 2020) and (b) plotting level curves of the adiabatic invariant with given Hamiltonian (Wisdom 1985; Henrard & Caranicolas 1989; Henrard 1990; Sidorenko et al. 2014). More discussions about perturbative treatments can be found in Saillenfest et al. (2016), Saillenfest (2020), Efimov & Sidorenko (2020), and references therein.

The perturbation method based on the adiabatic invariance approximation has been widely used in different contexts. Saillenfest et al. (2016) formulated semianalytical one-degree-of-freedom integrable models for secular evolutions of minor bodies inside MMRs with Neptune. In this model, the precise variation of the resonant angle is taken into account to calculate the adiabatic invariant, and the secular evolutions of TNOs are represented by level curves of the Hamiltonian with given adiabatic invariant. Saillenfest et al. (2017) applied such a one-degree-of-freedom secular model to the distant trans-Neptunian region, showing pathways to high-perihelion distances and a “trapping mechanism” that maintained the objects on distant orbits for billions of years. Reviews of the long-term dynamics of resonant TNOs can be found in Gallardo et al. (2012) and Saillenfest (2020).

In this work, we adopt perturbative treatments based on the adiabatic invariant approximation to deal with the ZLK resonance inside MMRs. Traditionally, the adiabatic invariant is defined as the oriented area enclosed by the isoline of the Hamiltonian; see Morbidelli (2002) for example. As shown by Saillenfest et al. (2016; see Section 5.2 in their work for a detailed discussion), the adiabatic invariant is not continuous at a separatrix crossing. Thus, for those trajectories with switching behaviors, different phase portraits (level curves of the Hamiltonian with given adiabatic invariant) should be matched in order to predict long-term behaviors. For a given TNO, it is difficult to provide the magnitudes of the adiabatic invariant when it is inside and outside MMRs. To overcome the difficulty, we define an adiabatic invariant that is an extension of the standard definition and is continuous inside and outside MMRs. Then, phase portraits are produced by plotting level curves of the adiabatic invariant with given Hamiltonian. Finally, the analytical developments are applied to known TNOs inside MMRs. The analytical models are validated by comparing analytical results with those of numerical integrations made under the full N-body model.

The remainder of this article is organized as follows. In Section 2, the basic dynamical models are briefly introduced, including the full N-body model, the simplified N-body model, and the semi-secular model. The one-degree-of-freedom integrable model is developed based on the adiabatic invariance approximation and then it is validated by comparing the analytical results with those of numerical integrations made in Section 3. Applications to real TNOs inside MMRs are provided in Section 4 and conclusions of this work are summarized in Section 5.

2. Dynamical Models

Let us consider the model of the outer solar system (the full N-body model), where the masses of terrestrial planets are added to that of the Sun and the mutual gravitational attractions between giant planets (Jupiter, Saturn, Uranus, and Neptune) are taken into account. The trajectories propagated under such a full N-body model are taken as references to validate semi-secular and secular models formulated in this work.

Considering the fact that the eccentricities of the giant planets are generally small and all the giant planets hold very low inclinations relative to the invariable plane, it is reasonable to assume that the four giant planets are moving on circular and coplanar orbits around the Sun, i.e., the giant planets move in the invariable plane of the outer solar system. Under such an assumption, the Hamiltonian function that governs the evolution of small bodies involved in the system can be written as (Morbidelli & Moons 1993)

$$\mathcal{H} = -\frac{\mu_0}{2a} + \sum_{i=5}^{8} \frac{\lambda_i}{\ell^{2i}} - \sum_{i=5}^{8} R_i,$$

where \(\mu_0 = \bar{G}m_0\) is the gravitational parameter of the Sun, \(a\) is the semimajor axis of the test particle, \(\lambda_i\) is the time derivative of mean longitude for the \(i\)th planet, and \(\ell_i\) is the momentum conjugated to the mean longitude of the \(i\)th planet \(\lambda_i\). The subscripts 5, 6, 7, and 8 stand for Jupiter, Saturn, Uranus and Neptune, respectively. The planetary disturbing function \(R_i\) is given by

$$R_i = \mu_i \left( \frac{1}{|r_i - r|} - r_i \cdot r \right), \quad i = 5, 6, 7, 8.$$



Due to the assumption that the orbits of the giant planets are known and fixed, the dynamical model represented by Equation (1) is a simplified N-body model that has three degrees of freedom.

The orbits of the test particle and the \(i\)th giant planet around the Sun are described by orbital elements: the semimajor axis \(a(e_i)\), the eccentricity \(e(e_i)\), the inclination \(I(l_i)\), the longitude of the ascending node \(\Omega(\Omega_i)\), the argument of pericenter \(\omega(\omega_i)\), and the mean anomaly \(M(M_i)\). Usually, the longitude of pericenter \(\varpi(\varpi_i)\) and mean longitude \(\lambda(\lambda_i)\) are used for low-eccentricity and/or low-inclination cases. These variables are described in the Sun-centered right-handed inertial coordinate system with the invariable plane of the outer solar system as the fundamental plane and the total angular momentum vector as the direction of the z-axis. In addition, the time and space variables are normalized by taking the mean semimajor axis between the Sun and Neptune (over \(2 \times 10^7\) yr) as the unit of length, total mass of the Sun and Neptune as the unit of mass, and the two-body orbital period of Neptune divided by \(2\pi\) as the unit of time. The physical parameters adopted in this work are given in Table 1.
Table 1
Physical Parameters Adopted in This Study

| System of Units | Magnitude |
|------------------|-----------|
| Unit of time (yr) | 26.2948188764 |
| Unit of length (au) | 30.1094546076 |
| Unit of mass (m_{J}) | 332.965.1922595720 |

| Dimensionless Parameters | Magnitude |
|--------------------------|-----------|
| μ_0                      | 9.9994848907E-1 |
| μ_5                      | 9.5473704974E-4 |
| μ_6                      | 2.8586954577E-4 |
| μ_7                      | 4.3659930385E-5 |
| μ_8                      | 1.0 - μ_0 |
| a_5                      | 0.1727862090 |
| a_6                      | 0.3173329830 |
| a_7                      | 0.6382793627 |
| a_8                      | 1.0 |
| λ_{8}                    | 1.0027387100 |
| T_{8} (yr)               | 164.7639788626 |

Note. The Julian year (yr) consists of 365.25 Julian days and m_{J} is the mass of the Earth. The variables a_i (i = 5, 6, 7, 8) are the mean semimajor axes of Jupiter, Saturn, Uranus, and Neptune. See the text for the computation of a_i (i = 5, 6, 7, 8) and λ_{8}.

To formulate the Hamiltonian model, we introduce the modified Delaunay variables as follows (Morbidelli 2002):

\[ \Lambda = \sqrt{\mu_0 a}, \quad \lambda = M + \varpi, \]
\[ P = \sqrt{\mu_0 a}(1 - \sqrt{1 - e^2}), \quad p = - \varpi, \]
\[ Q = \sqrt{\mu_0 a}(1 - e^2)(1 - \cos i), \quad q = - \Omega, \]
\[ \Lambda_i, \quad \lambda_i. \]

Using Delaunay variables, the Hamiltonian of the dynamical system can be written as follows:

\[ \mathcal{H} = -\frac{\mu_0^2}{2\Lambda^2} + \sum_{i=5}^{8} \lambda_i \Lambda_i - \sum_{i=5}^{8} \mathcal{R}_i. \]  

For an object located inside k_p, k resonance with Neptune, the resonant angle is usually given by

\[ \sigma = k \lambda - k_p \lambda_p + (k_p - k) \varpi, \]

which is called the eccentricity-type resonant argument (Murray & Dermott 1999). Usually, the integer \( |k - k_p| \) is called the resonance order.

To formulate the semi-secular model, the following canonical variables are introduced:

\[ \Sigma = \frac{1}{k} \Lambda, \quad \sigma = k \lambda - k_p \lambda_p - (k_p - k) p, \]
\[ U = -P - \frac{k_p - k}{k} \Lambda, \quad u = q - p, \]
\[ V = -P - \frac{k_p - k}{k} \Lambda, \quad v = -q, \]
\[ W = \Lambda_8 + \frac{k_p}{k} \Lambda, \quad w = \lambda_8 \]

with the generating function

\[ S = k\Sigma - \mu(1 + k_p \Sigma - k \Sigma) + q(U - V) + \lambda_8(W - k_p \Sigma). \]

In terms of the classical orbital elements, the canonical variables given by Equation (5) can be expressed as

\[ \Sigma = \frac{1}{k} \sqrt{\mu_0 a}, \quad \sigma = k \lambda - k_p \lambda_p + (k_p - k) \varpi, \]
\[ U = \sqrt{\mu_0 a}\left(1 - e^2 - \frac{k_p}{k}\right), \quad u = \omega, \]
\[ V = \sqrt{\mu_0 a}\left(1 - e^2 \cos I - \frac{k_p}{k}\right), \quad v = \Omega, \]
\[ W = \Lambda_8 + \frac{k_p}{k} \sqrt{\mu_0 a}, \quad w = \lambda_8. \]

Among the angular variables shown in the Hamiltonian, it is not difficult to see that the mean longitudes (λ and λ_{8}) are short-period variables, σ is a semi-secular periodic variable, and ω and Ω are long-period variables (Saillenfest 2020). Based on this fact, those terms related to the short-period variables produce short-period influences upon the secular dynamics, thus they can be averaged out from the Hamiltonian by means of perturbation theory in order to study long-term evolutions (Hori 1966; Depret 1969).

Concerning the disturbing functions \( \mathcal{R}_{5,6,7} \) (nonresonant configurations), there are two short-period variables: λ and λ_{i} (i = 5, 6, 7). Thus, it is necessary to perform double averages over orbital periods of the test particle and giant planets to remove the short-period influences (Morbidelli & Moons 1993; Thomas & Morbidelli 1996):

\[ \mathcal{R}^* = \frac{\mu_i}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{|r - r_i|} \frac{r \cdot r_i}{|r_i|^3} d\lambda d\lambda_i. \]

In practical simulations, the double-averaged disturbing functions given by Equation (7) can be truncated at the fourth order in semimajor axis ratio a_i/a and they can be explicitly expressed as (Saillenfest 2020)

\[ \mathcal{R}^*_i = \frac{\mu_i}{a} + \frac{1}{8} \frac{a_i}{a} \left( \frac{a_i}{a} \right)^2 \frac{1}{(1 - e^2)^{3/2}} (3 \cos^2 I - 1) \]
\[ + \frac{9}{1024} \frac{a_i}{a} \left( \frac{a_i}{a} \right)^4 \frac{1}{(1 - e^2)^{3/2}} \mathcal{F}(e, I, \omega), \quad i = 5, 6, 7 \]

where

\[ \mathcal{F}(e, I, \omega) = (3 - 30 \cos^2 I + 35 \cos^4 I)(2 + 3e^2) + 10(7 \cos^2 I - 1)e^2 \sin^2 I \cos(2\omega). \]

About the disturbing function \( \mathcal{R}_8 \) (resonant configuration), there is only one short-period variable, λ_{8}. Thus, it is necessary to perform a single average for \( \mathcal{R}_8 \) over k times Neptune’s orbital period as follows (Gallardo 2006):

\[ \mathcal{R}^*_8 = \frac{\mu_8}{2k\pi} \int_0^{2\pi} \frac{1}{|r - r_8|} \frac{r \cdot r_8}{|r_8|^3} d\lambda_8. \]
which can be calculated by direct numerical quadrature (Schubart 1968). Evaluating Equation (9) by numerical quadrature is applicable for test particles with arbitrary semimajor axes, eccentricities, and inclinations.

To be consistent, the orbital elements appearing in \( R_i^k (i = 5, 6, 7, 8) \) should be replaced by the canonical variables \((\sigma, \Sigma, u, U)\). Removing those constant terms associated with \( \lambda_{5,6,7} \) from the Hamiltonian (without changing the Hamiltonian dynamics), we can obtain the averaged Hamiltonian function without short-period terms, given by

\[
\mathcal{H}^* = -\frac{\mu_0^2}{2k(S\Sigma)^2} - k_p \lambda_8 \Sigma - \sum_{i=5}^{8} R_i^k (\sigma, \Sigma, u, U). \tag{10}
\]

In order to produce the value of \( \lambda_8 = M_8 + \varepsilon_8 \), we numerically integrate the equations of motion under the full \( N \)-body model (the outer solar system model or the OSS model for short) over \( 2 \times 10^7 \) yr. Then, we identify the frequency of Neptune’s mean longitude \( \lambda_8 \) by means of linear fitting for the time series function \( \lambda_8(t) \) and determine the mean semimajor axes of giant planets by numerically averaging the time series function \( a_i(t) \). It should be mentioned that under the full \( N \)-body model the time evolution of the mean longitude of Neptune is not equal to mean \( n_8 \) (Neptune’s mean motion) because there is a contribution from the time evolution of \( \varepsilon_8 \). Under the full \( N \)-body model, the mean semimajor axis \( a_8 \) of Neptune will not satisfy the Keplerian relationship with mean \( n_8 \).

The practical values of \( a_i \) (\( i = 5, 6, 7, 8 \)) and \( \lambda_8 \) adopted in this study are provided in Table 1. In this work, the frequency of Neptune’s mean longitude is \( \lambda_8 = 1.002738718 \) in normalized units, which is very close to the value given by Murray & Dermott (1999); which is \( \lambda_8 = 1.002572368 \), as shown by Table A.3 in their textbook). The minor difference of \( \lambda_8 \) in magnitude is due to the difference in dynamical models.

To make the magnitude of the Hamiltonian be close to unity, we add some constant terms to the Hamiltonian without changing the Hamiltonian dynamics and then normalize it by the gravitational parameter of Neptune in the following manner (for convenience, we still use \( \mathcal{H} \) to stand for the normalized Hamiltonian and it is called the resonant Hamiltonian from now on):

\[
\mathcal{H} = \frac{1}{\mu_8} \left[ \mathcal{H}^* + \frac{\mu_0^2}{2(kS\Sigma)^2} + \lambda_8 k_p \Sigma_0 + \sum_{i=5}^{7} \frac{\mu_i}{a_0} \right], \tag{11}
\]

where \( a_0 \) and \( \Sigma_0 \) are the reference values of \( a \) and \( \Sigma \), given by \( (k_p/k)^2 a_0^3 = \mu_0 \) and \( \Sigma_0 = 1/k \sqrt{\mu_0 a_0} \). It is noted that the dynamical models represented by Equations (10) and (11) remain the same except that the time unit is magnified \( 1/\mu_8 \) times in the latter model.

Evidently, the semi-secular model represented by Equation (11) has two degrees of freedom with \( \sigma \) and \( u \) as the angular coordinates. Since the angular variable \( \nu = \Omega \) disappears from the resonant Hamiltonian, its conjugate momentum becomes an integral of motion, given by

\[
V = \sqrt{\mu_0 a} \sqrt{1 - e^2 \cos I - \frac{k_p}{k} \frac{k}{k}} = \text{const}. \tag{12}
\]

Assuming the semimajor axis at the nominal location of resonance, i.e., \( a = a_c = \left( \frac{\mu k}{\mu k^2} \right)^{1/3} \), the integral of motion can be expressed as

\[
V = \sqrt{\mu_0 a} \sqrt{1 - \tilde{e}^2 \cos \tilde{I} - \frac{k_p}{k}} = \text{const}, \tag{13}
\]

where \( \tilde{e} \) and \( \tilde{I} \) are called the equivalent eccentricity and inclination.

Furthermore, we assume the equivalent eccentricity \( \tilde{e} \) to be zero and then it is possible to get the maximum inclination \( I_{\text{max}} \) standing for the magnitude of \( V \) in the following manner:

\[
V = \sqrt{\mu_0 a} \left( \cos I_{\text{max}} - \frac{k_p}{k} \right) = \text{const}. \tag{14}
\]

The parameter \( I_{\text{max}} \) is used to specify the motion integral \( V \) and it is similar to the Kozai parameter \( i_0 \) (Kozai 1962). Up to now, a one-to-one correspondence between \( V \) and \( I_{\text{max}} \) has been made. As a result, the resonant Hamiltonian can be expressed as \( V(\lambda_{\text{max}}; \sigma, \Sigma, u, U) \), which determines a two-degree-of-freedom dynamical model, depending on the motion integral \( V \) (or the equivalent parameter \( I_{\text{max}} \)).

The Hamiltonian canonical relations lead to the equations of motion for trans-Neptunian objects inside MMRs as follows:

\[
\begin{align*}
\dot{\sigma} &= \partial \mathcal{H} / \partial \Sigma, \quad \dot{\Sigma} = -\partial \mathcal{H} / \partial \sigma, \\
\dot{u} &= \partial \mathcal{H} / \partial U, \quad \dot{U} = -\partial \mathcal{H} / \partial u,
\end{align*} \tag{15}
\]

which are the equations of motion of the semi-secular model. The degree of freedom associated with the MMR is represented by \( \sigma, \Sigma \) and the one associated with long-term orbital evolution is represented by \( (u, U) \). In the second degree of freedom, the angular variable is defined by \( u = \omega \) and the action variable \( U \) can be expressed in terms of the equivalent eccentricity as \( U = \sqrt{\mu_0 a} \sqrt{1 - \tilde{e}^2 - \frac{k_p}{k}} \) where \( a_i \) is the nominal location of the \( k_p \)-K MMR. Thus, the second degree of freedom can be represented by \( (\omega, \tilde{e}) \). Naturally, the resonant Hamiltonian can be denoted by \( V(\lambda_{\text{max}}; \sigma, \Sigma, \omega, \tilde{e}) \).

To validate the semi-secular model represented by Equation (15), we take 2018 VO137 as an example. It is currently located inside 2:5 resonance with Neptune, and its resonant argument is defined by \( \sigma = 5\lambda - 2\lambda_8 - 3\pi \). We propagate the orbits of asteroid 2018 VO137 under both the full \( N \)-body model and the semi-secular model over \( 4.0 \times 10^7 \) yr. The time histories of the resonant argument \( \sigma \) and argument of perihelion \( \omega \) are reported in Figure 1.

From Figure 1, it is observed that, under the full \( N \)-body model, 2018 VO137 transits between libration regions and circulation regions periodically and it is currently inside ZLK resonance with \( \omega \) librating around 90°.

Comparing the evolutions in different models, we can see that the angles \( \sigma \) and \( \omega \) produced under the semi-secular model are in good agreement with the ones produced under the full \( N \)-body model. In the following sections, we will make analytical developments based on the semi-secular model.

### 3. Secular Models for Resonant Objects

As stated in the previous section, the semi-secular dynamical model for describing the long-term dynamics inside or around MMRs has two degrees of freedom. The frequencies between the two degrees of freedom separate faraway from each other,
i.e., $|\dot{\sigma}| \gg |\dot{\omega}|$, meaning that such a dynamical model can be divided into “fast” and “slow” subsystems (Saillenfest et al. 2016; Saillenfest 2020). Perturbation theory is powerful for dealing with such a kind of separable Hamiltonian model (Wisdom 1985; Henrard 1990).

3.1. Semianalytical Developments

Considering the separability of frequencies between the two degrees of freedom, let us freeze the variables of the “slow” subsystem during the timescale of the “fast” subsystem associated with $(\sigma, \Sigma)$ and denote the associated Hamiltonian by

$$
\mathcal{H}(I_{\text{max}}, \omega, \tilde{e}; \sigma, \Sigma)
$$

where the slow variables $(\omega, \tilde{e})$ are treated as parameters (Saillenfest et al. 2016). Obviously, the Hamiltonian model represented by Equation (16) has a single degree of freedom. According to the usual perturbative treatments, Arnold action–angle variables can be introduced for the “fast” degree of freedom (Henrard 1990; Morbidelli 2002):

$$
\psi_\sigma = \frac{2\pi}{T_\sigma} t, \quad J_\sigma = \frac{1}{2\pi} \oint \Sigma \, d\sigma,
$$

where $T_\sigma$ is the libration period of $\sigma$. In geometry, $2\pi J_\sigma$ defined by the path integration in Equation (17) stands for the signed area enclosed by the solution curve of the “fast” subsystem. See Figure 4.2 in Morbidelli (2002) for the geometrical definition of $2\pi J_\sigma$. Using the action–angle variables, the Hamiltonian can be denoted by

$$
\mathcal{H}(I_{\text{max}}, \omega, \tilde{e}; J_\sigma),
$$

which determines a single-degree-of-freedom dynamical model for the “slow” subsystem with $J_\sigma$ as the motion integral. This approximation is an “adiabatic” one, and the motion integral $J_\sigma$ is referred to as an adiabatic invariant (Neishtadt 1987).

In particular, when the motion integral $J_\sigma$ is provided, the trajectories in the dynamical model determined by Equation (18) are level curves of a resonant Hamiltonian in the space $(\omega, \tilde{e})$. Thus, the conventional phase portraits can be produced by plotting level curves of a resonant Hamiltonian in the $(\omega, \tilde{e})$ space with given $J_\sigma$ (Morbidelli 2002). By analyzing the structures arising in the phase portraits, it is possible to know the global dynamics in the phase space. Saillenfest et al. (2016) pointed out that the computation of the Hamiltonian as well as its partial derivatives is time-consuming.

There are several difficulties for the problem at hand. At first, the motion integral $J_\sigma$ is not continuous when particles switch between libration and circulation regions, so that the phase portraits need to be considered separately and then matched for phase spaces inside and outside MMRs (see for instance Saillenfest et al. 2016 for more details). Thus, it is inconvenient to describe those particles with switching behaviors between circulation and libration. Second, it is not an easy task to inversely solve the state $(\sigma, \Sigma, \omega, \tilde{e})$ for a given motion integral $J_\sigma$ because $J_\sigma$ is not an explicit function of the variables $(\sigma, \Sigma, \omega, \tilde{e})$. Neishtadt & Sidorenko (2004) and Henrard & Lemaître (1986) have made some attempts to deal with the first point.

To make the adiabatic invariant be continuous around separatrices of the “fast” subsystem, we introduce the absolute of the area bounded by the isolines of the resonant Hamiltonian

$$
S(I_{\text{max}}, \mathcal{H}; \omega, \tilde{e}) = \left| \oint \Sigma \, d\sigma \right|
$$

inside the libration zone and

$$
S(I_{\text{max}}, \mathcal{H}; \omega, \tilde{e}) = \int_0^{2\pi} (\Sigma_{\text{up}} - \Sigma_{\text{down}}) d\sigma
$$

inside the circulation zone as a new adiabatic invariant. In Equation (20), $\Sigma_{\text{up}}$ is evaluated at the top isoline of the Hamiltonian and $\Sigma_{\text{down}}$ at the bottom isoline. Using the new adiabatic invariant $S$, the Hamiltonian can be written as

$$
\mathcal{H}(I_{\text{max}}, S; \omega, \tilde{e}),
$$

which is a single-degree-of-freedom dynamical model with $S$ as the motion integral (adiabatic invariant).
Inside the libration zone, it is known that the isoline of the resonant Hamiltonian is enclosed and isomorphic to a circle and thus in this case the area $S$ is exactly equivalent to the conventional motion integral (in magnitude, $S = 2\pi |I|$). However, in the circulation zone, there are two isolines of the resonant Hamiltonian in the phase space of MMR (see Figures 2 and 3 for details): one is located above the libration island and the other is below it. The definition given by Equation (20) shows that $S$ stands for the area bounded by the top and bottom isolines. Evidently, the definition of $S$ in the circulation case is different from the conventional one.

In summary, for the two-degree-of-freedom dynamical model at hand, there are two conserved quantities—the resonant Hamiltonian $H$ and the area $S$ enclosed by its isolines. These two integrals provide two nonlinear constraints embedded in the full four-dimensional phase space $(\sigma, \Sigma, \omega, \epsilon)$. Thus, when $H$ and $S$ are given, the motion of TNOs in the long-term evolution happens in a two-dimensional manifold. Such a two-dimensional manifold can be graphically illustrated by means of phase portraits (Henrard & Caramel-Colas 1989; Henrard 1990), Poincaré surfaces of section (Yokoyama 1996), and a set of guiding trajectories (Wisdom 1985).

According to Lei (2019) and Gallardo (2020), the location of the libration center is dependent on the eccentricity, inclination, and argument of pericenter. In particular, the 1:1-type MMRs have two stable libration centers in the phase space (meaning that tadpole and horseshoe trajectories are possible), while the non-1:1-type MMRs have only one stable libration center. In the following, the cases of 1:1 and non-1:1 MMRs are discussed separately.

In Figure 2, we show the geometrical definition of $S$ for the resonances of non-1:1 type. In practice, we take 2:5 resonance with Neptune as an example. The dynamical separatrices arising in the phase portraits of MMR are shown by red dashed lines. In the case of 2:5 resonance, the region bounded by the isoline of the given Hamiltonian is marked by the shaded area and, in particular, the left panel of Figure 2 shows the definition of $S$ inside the libration zone of MMR and the right panel shows the definition inside the circulation zone.

According to the magnitude of the resonant Hamiltonian, there are two special cases. The first case is that, when $H$ is equal to that of the resonant center, the magnitude of $S$ becomes zero, meaning that the small body is initially placed at the resonant center. As discussed in the introduction, such a special case with $S = 0$ has been adopted as an assumption in formulating secular models (Kozai 1985; Yoshikawa 1989; Nesvorný et al. 2002; Wan & Huang 2007; Saillenfest et al. 2017; Li et al. 2021; Pons & Gallardo 2022). The second case is that, when $H$ is equal to that of the saddle point (or the dynamical separatrix), $S$ is equal to the area bounded by the dynamical separatrices, and in this case the assumption that the two degrees of freedom are separable in frequencies fails, leading to the fact that $S$ is no longer an adiabatic invariant (Wisdom 1985; Tennyson et al. 1986; Neishtadt 1987; Neishtadt & Sidorenko 2004). In the following, we denote the curve on which the Hamiltonian $H$ is equal to that of the saddle point $H_U$; as the critical curve. From the phase portrait shown in Figure 2, we can see that the region with $H > H_U$ corresponds to libration and the region with $H < H_U$ corresponds to circulation. The phase-space regions around the critical curve are called zones of uncertainty, and the crossing of uncertainty zones is a generator of chaos (Wisdom 1985).

In Figure 3, the geometrical definition of $S$ is illustrated for resonances of 1:n type. In practice, we take 1:2 resonance with Neptune as an example. In the phase portraits of MMR, the dynamical separatrices corresponding to level curves of $H = H^{(1)}_U$ (inner separatrix) and $H = H^{(2)}_U$ (outer separatrix) are shown by red dashed lines. In Figure 3, the regions enclosed by level curves of a given resonant Hamiltonian are marked by shaded zones, and the area of these zones corresponds to the adiabatic invariant $S$. According to the magnitude of the resonant Hamiltonian, there are three cases:

**Case I: Inside the leading or trailing island.** When the resonant Hamiltonian $H$ is greater than $H^{(1)}_U$, the particle is located inside the asymmetric libration islands (the leading island is on the left and the trailing island is on the right). In this case, there are two isolines of $H$, one is inside the leading island and the other is inside the trailing island. Inside the asymmetric libration islands, the trajectories are of tadpole type. See Figures 3(a) and (b).
Case II: Inside the symmetric islands. When the resonant Hamiltonian $\mathcal{H}$ is smaller than $\mathcal{H}_{1}^{(2)}$ but greater than $\mathcal{H}_{2}^{(2)}$, the particle is located inside the symmetric libration island bounded by the inner and outer separatrices. In this case, both the asymmetric islands are bounded by the isoline of the resonant Hamiltonian and the trajectories in the symmetric libration island are of horseshoe type. See Figure 3 (c).

Case III: Outside the resonant regions. When the resonant Hamiltonian $\mathcal{H}$ is smaller than $\mathcal{H}_{2}^{(2)}$, test particles are located outside the resonant zones. See Figure 3 (d).

In the long-term evolution of TNOs under the considered two-degree-of-freedom dynamical model, both the resonant Hamiltonian $\mathcal{H}$ and the adiabatic invariant $S$ remain unchanged. Thus, it is not difficult to conclude that, when the parameters $I_{\text{max}}$ and $\mathcal{H}$ are provided, the motion of TNOs takes place on the isolines of the adiabatic invariant $S$ in the phase space spanned by $(\sigma, \Sigma)$. Based on this fact, it is possible to produce phase portraits for secular evolutions by plotting level curves of the adiabatic invariant $S$ with given $\mathcal{H}$ (Wisdom 1985; Henrard & Caranicolas 1989; Henrard 1990; Neishtadt & Sidorenko 2004) or by plotting level curves of the Hamiltonian $\mathcal{H}$ with given $S$ (Morbidelli 2002; Saillenfest et al. 2016; Saillenfest 2020). By analyzing phase portraits, it is possible to investigate long-term dynamics for TNOs inside or around MMRs. This is the semianalytical secular model formulated in this study.

To clearly illustrate our semianalytical model, we take 2:5 resonance with Neptune as an example to plot level curves of the adiabatic invariant $S$ in the space $(\sigma, \Sigma)$ by taking $I_{\text{max}} = 30^\circ$ and $\mathcal{H} = -0.614$. The results are shown in the left panel of Figure 4. The red dotted line represent the critical curve specified by $\mathcal{H} = 0.614$. The region below the critical curve corresponds to circulation (because in this region the Hamiltonian is smaller than that of the separatrix) and the region above the critical curve corresponds to libration (because in this region the Hamiltonian is greater than that of the separatrix). From the phase portrait shown in the left panel of Figure 4, we can see that the level curves of $S$ in the libration and circulation regions are continuous, as desired. In the long-term evolution, the test particle moves along a certain level curve shown in the phase portrait. As an example, the level curve of $S = 0.001555$ is explicitly marked by a blue line in the phase portrait and three typical points on the level curve are picked out and marked by pink stars. These points are numbered 1, 2, and 3. It is observed that points 1 and 3 are

![Figure 3](image-url)
located inside the libration regions (i.e., above the critical curve) and point 2 is located inside the circulation region (i.e., below the critical curve). In the right panel of Figure 4, geometrical definitions of the adiabatic invariant $S$ corresponding to these three points are illustrated in the space $(\sigma, \Sigma)$. Moving from point 1 to point 2, the particle transits from libration region to circulation region, and it then returns to the libration region when moving from point 2 to point 3. Theoretically speaking, and according to this model, a particle moving along this blue line will switch between libration and circulation periodically.

From the right panel of Figure 4, it is observed that, in the long-term evolution, the resonant angle $\sigma$ presents a significant variation in terms of both the resonant center and libration amplitude. This indicates that some artificial assumptions for the variation of $\sigma$ in previous works (e.g., zero or constant amplitude of $\sigma$) may lead to unreliable results. This problem was noted by Brasil et al. (2014) and Saillenfest et al. (2016).

From the phase portrait presented in the left panel of Figure 4, we could find some interesting dynamical structures: (a) both the arguments $\sigma$ and $\omega$ circulate in the regions below the critical curve, (b) ZLK resonances can be found in the region above the critical curve, (c) the ZLK islands centered at $\omega = 90^\circ$ and $\omega = 270^\circ$ are larger than the ones centered at $\omega = 0$ and $\omega = \pi$, (d) the ZLK resonance centered at $\omega = 0$ or $\omega = \pi$ occupies a higher-eccentricity space, and (e) the ZLK resonance disappears in the region with eccentricities greater than $0.41$.

When the particle is crossing the “zone of uncertainty,” the adiabatic invariant approximation is invalidated and chaos may occur (Wisdom 1985). However, for the current dynamical model, the zone of uncertainty is very narrow, meaning that the change of adiabatic invariant $S$ is negligible and the value of $S$ is still predictable for each possible transition. Thus, whether the particles are inside or around a certain MMR, the level curves of the adiabatic invariant $S$ are available to predict secular behaviors. See Saillenfest et al. (2016) for a detailed discussion of this problem.

### 3.2. Validation of the Semianalytical Developments

Before applications to real TNOs, we need to validate the semianalytical model by comparing the trajectories of particles predicted by phase portraits with the ones numerically propagated under the semi-secular model. For convenience, the resonances of non-$1:n$ type and those of $1:n$ type are discussed separately.

For the resonances of non-$1:n$ type, we take 2:5 resonance with Neptune as an example. Following the discussion presented in the previous subsection, we produce the phase portrait by plotting the level curves of $S$ with the parameters $I_{\text{max}} = 30^\circ$ and $\mathcal{H} = -0.614$, as shown in the upper left panel of Figure 5. In the phase portrait, the critical curve is marked by a red line. The region below the critical curve corresponds to circulation and the region above it to libration. In addition, with the same $I_{\text{max}}$ and $\mathcal{H}$, three trajectories are numerically propagated under the semi-secular model and the resulting numerical trajectories are numbered 1, 2, and 3, as shown by green lines. To show the detailed behaviors of evolution, the time histories of $\sigma$ and $\omega$ for these numerically propagated trajectories are reported in the remaining three panels of Figure 5.

A good agreement can be observed between the level curves shown in the phase portrait and the trajectories numerically propagated under the semi-secular model, as shown in the first panel of Figure 5. This shows that the adiabatic invariance approximation works very well for the problem at hand.

The remaining three panels of Figure 5 indicate that (a) following trajectory 1 the particle switches between libration and circulation in terms of $\sigma$ and it is located outside the ZLK resonance, (b) following trajectory 2 the particle also switches between libration and circulation in terms of $\sigma$ but it is located inside the ZLK resonance with $\omega$ librating around $90^\circ$, and (c) following trajectory 3 the particle is located inside both mean motion resonance and the ZLK resonance (both $\sigma$ and $\omega$ are librating around $180^\circ$).

As for the resonances of $1:n$ type, we take 1:2 resonance with Neptune as an example. To produce the phase portrait, we take the following parameters: $I_{\text{max}} = 30^\circ$ and $\mathcal{H} = -0.8$. The
level curves of the adiabatic invariant $S$ shown in the space $(\omega, \dot{\omega})$ are reported in the upper left panels of Figures 6 and 7. In the phase portrait, the critical curves are marked by red dashed lines. The critical curves divide the entire phase space into three subregions: the first one with circulating trajectories, the second one with horseshoe-type trajectories, and the last one with tadpole-type trajectories. The adiabatic invariant $S$ is evaluated inside the leading island in the phase portrait shown in Figure 6 and inside the trailing island for the phase portrait shown in Figure 7.

Under the semi-secular model, four trajectories are numerically propagated and they are denoted by letters A, B, C, and D, respectively. Their time histories of $\sigma$ and $\omega$ are reported in the last three panels of Figure 6 and in the right panel of Figure 7.

Also, an excellent agreement is observed between level curves in the phase portraits and the corresponding numerically propagated trajectories (see the first panels of Figures 6 and 7). This further indicates that the semi-analytical secular model formulated in this study is applicable for predicting long-term evolution for TNOs inside or around MMRs.

From Figures 6 and 7, we can see that (a) following trajectory A the particle transits between the libration region with horseshoe-type trajectories and the circulation region, (b) following trajectory B the particle is inside the libration region with horseshoe-type trajectories, (c) following trajectory C the particle is inside the leading island, and (d) following trajectory D the particle is inside the trailing island. The numerical behaviors are in quite good agreement with the ones predicted by phase portraits.

4. Applications to Real Trans-Neptunian Objects

In this section, we apply the semianalytical secular model formulated in Section 3 to real TNOs inside MMRs with Neptune. In practical applications, we take representative members from several groups of resonant TNOs as examples. It should be mentioned that the secular model developed in this work is also applicable to other resonant objects. According to the average defined by Equation (9), averaged elements of TNOs are required in the semi-secular dynamical model. To produce averaged elements of TNOs, we numerically integrate the equations of motion under the full $N$-body model over $k$ times the orbital period of Neptune to obtain the osculating elements, and then we identify the averaged elements of TNOs by numerically averaging the time series of osculating elements over the duration of integration. Table 2 provides the averaged elements, the maximum inclination $I_{\text{max}}$, and resonant Hamiltonian $\mathcal{H}$ for the TNOs chosen in this study. It should be noted that, in the process of evaluating the averaged elements, the initial epoch is taken on 2020 December 17.

In Figure 8, we take into account the secular dynamics of 2018 VO137, which is currently inside 2:5 resonance. For this TNO, its maximum inclination is $I_{\text{max}} = 40^\circ$ and the resonant Hamiltonian $\mathcal{H} = -0.614$ is marked by a red line. It is observed that the numerically propagated trajectories follow closely along the level curves in the phase portrait.

4 https://minorplanetcenter.net, retrieved 9 April 2021.
Hamiltonian is $\mathcal{H} = -0.58$ (see Table 2). With the given parameters $I_{\text{max}}$ and $\mathcal{H}$, the associated phase portrait is presented in the left panel of Figure 8. For convenience, the critical curve is marked by a red dashed line. Recall that the critical curve is defined as the curve on which the resonant Hamiltonian is equal to that of the dynamical separatrix. As stated before, the phase-space regions below the critical curve are of circulation and those above it are of libration. The trajectories are numerically propagated under both the semi-secular model and the full $N$-body model. The trajectory
VO137 is currently located inside the ZLK island centered at \( \text{2018 VO137} \) \( 2:5 \) \( 55.43896 \) \( 0.187185 \) \( 38.99327 \) \( 42.69996 \) \( 135.01319 \) \( 114.97992 \) \( \text{deg} \), showing that in the long-term evolution the resonant angle \( \omega \) is \( 90 \)°. The parameters for this TNO are provided in Table 2. In the left panel, the trajectory propagated from the full \( N \)-body model is in good agreement with that propagated under the semi-secular model, the one propagated in the full \( N \)-body model is shown by a blue line and the one propagated in the semi-secular model by a green line. The critical curve is marked by red dots (the region above the curve corresponds to libration and that below to circulation). Evidently, the trajectory of 2018 VO137 intersects the critical curve twice in a ZLK period, showing that in the long-term evolution the resonant angle \( \sigma \) switches between libration and circulation periodically. In the considered time interval, 2018 VO137 is currently located inside the ZLK island centered at \( \omega = 90 \)°.

Another two TNOs inside 2:5 resonance with Neptune are described by osculating elements (with short-period oscillations).

Table 2

| TNO     | \( k \cdot k \) | \( a \) (au) | \( e \) | \( I \) (deg) | \( \Omega \) (deg) | \( \omega \) (deg) | \( \sigma \) (deg) | \( \mathcal{H} \) | \( I_{\text{max}} \) (deg) | Figure |
|---------|----------------|-------------|-------|-------------|-----------------|-----------------|----------------|----------------|----------------|--------|
| \( \text{2018 VO137} \) | 2:5 | 55.43896 | 0.187185 | 38.99327 | 42.69996 | 135.01319 | 114.97992 | \( -0.584742 \) | 40.20489 | Figure 8 |
| \( \text{2005 SD278} \) | 2:5 | 55.58254 | 0.28142 | 17.89926 | 152.53972 | 218.34871 | 188.25311 | \( -0.62790 \) | 23.91183 | Figure 9(a) |
| \( \text{2015 PD112} \) | 2:5 | 55.48197 | 0.23409 | 23.01589 | 154.78276 | 226.08094 | 105.88611 | \( -0.60659 \) | 31.34801 | Figure 9(b) |
| Pluto   | 2:3 | 39.56190 | 0.24941 | 17.14126 | 110.25738 | 113.79771 | 235.18742 | \( -0.92683 \) | 22.18451 | Figure 10(a) |
| \( \text{2004 HA903} \) | 2:3 | 39.37383 | 0.24501 | 22.69980 | 203.20489 | 262.65667 | 161.56184 | \( -0.84521 \) | 26.56732 | Figure 10(b) |
| \( \text{1996 TR66} \) | 1:2 | 47.89136 | 0.38324 | 12.42382 | 343.04016 | 309.39781 | 58.14003 | \( -0.68111 \) | 26.28515 | Figure 11(a) |
| \( \text{2014 SR373} \) | 1:3 | 62.55550 | 0.38324 | 35.58437 | 165.74151 | 214.52424 | 80.01628 | \( -0.49380 \) | 41.29682 | Figure 11(b) |

Note. The averaged elements including semimajor axis \( a \), eccentricity \( e \), inclination \( I \), longitude of ascending node \( \Omega \), argument of perihelion \( \omega \), and the resonant argument \( \sigma \). To evaluate the averaged elements, the initial epoch is taken on 2020 December 17.

Figure 8. Level curves of the adiabatic invariant \( S \) (left panel) and time histories of the angles \( \sigma \) and \( \omega \) for the trajectory propagated under the full \( N \)-body model (right panel) for 2018 VO137, which is currently inside 2:5 resonance with Neptune. The parameters for this TNO are provided in Table 2. In the left panel, the trajectory propagated in the full \( N \)-body model is shown by a blue line and the one propagated in the semi-secular model by a green line. The critical curve is marked by red dots (the region above the curve corresponds to libration and that below to circulation). Evidently, the trajectory of 2018 VO137 intersects the critical curve twice in a ZLK period, showing that in the long-term evolution the resonant angle \( \sigma \) switches between libration and circulation periodically. In the considered time interval, 2018 VO137 is currently located inside the ZLK island centered at \( \omega = 90 \)°.
Naturally, coupled evolution between eccentricity and inclination indicates that the inclination should also have a large variation. For 2015 PD$_{312}$ (see the right panel of Figure 9), there are four ZLK islands centered at $\omega = 0^\circ$, $\omega = 90^\circ$, $\omega = 180^\circ$, and $\omega = 270^\circ$. The islands centered at $\omega = 0^\circ$ and $\omega = 180^\circ$ are located above the critical curve, meaning these two islands of ZLK resonance are inside the libration zones of MMR. In particular, 2015 PD$_{312}$ is currently inside the ZLK island centered at $\omega = 180^\circ$. This means that 2015 PD$_{312}$ is located inside the MMR and inside the ZLK resonance.

Two TNOs inside 2:3 resonance with Neptune (Pluto and 2004 HA$_{79}$) are considered in Figure 10, where the phase portraits and numerically propagated trajectories under the full $N$-body model are presented. It is observed that (a) the numerical trajectories are in agreement with the level curves arising in the phase portraits, and (b) Pluto is located inside the ZLK resonance with $\omega$ librating around $90^\circ$ and 2004 HA$_{79}$ is located inside the ZLK resonance with $\omega$ librating around $270^\circ$. For both TNOs considered here, the real location of libration center is coincident with the location predicted by the phase portrait. Gladman et al. (2012) showed that 2004 HA$_{79}$ is currently located inside ZLK resonance at $270^\circ$ with an amplitude of $30^\circ$, which is in agreement with our result. For TNOs trapped inside 2:3 MMR with Neptune, Wan & Huang (2007) analytically formulated a secular dynamical model by fixing the critical argument of MMR at the libration center of $180^\circ$. An application to Pluto shows that the real libration center under the full $N$-body model is higher than the analytical prediction (see Figure 4 in their work), which is mainly due to the assumption adopted for the critical argument of MMR.

Regarding the 1:$n$-type resonances with Neptune, we take the 1:2 and 1:3 resonances as examples. In particular, 1996 TR$_{66}$ inside 1:2 resonance and 2014 SR$_{373}$ inside 1:3 resonance are taken into account. Their phase portraits and trajectories numerically propagated under the full $N$-body model are reported in Figure 11. It is observed from Figure 11 that (a) the numerical trajectories are in quite good agreement with the level curves appearing in the phase portraits and (b) 1996 TR$_{66}$ is inside the ZLK resonance while 2014 SR$_{373}$ is outside it. In the left panel of Figure 11, it is observed that there are asymmetric centers of the ZLK islands. For the 1:$n$-type resonances, the centers of the ZLK islands are no longer at $\omega = 0^\circ$, $90^\circ$, $180^\circ$, and $270^\circ$. The asymmetric center of the ZLK
resonance inside 1:1-type MMRs has been known in previous works (Kozai 1985; Gallardo et al. 2012; Saillenfest et al. 2016).

5. Summary and Discussion

In this work, a semianalytical one-degree-of-freedom model was formulated in order to explore the ZLK resonance of TNOs inside MMRs with Neptune. First, we introduced a modified adiabatic invariant, denoted by $S$, which is equal to the absolute of the area enclosed by isolines of the resonant Hamiltonian in the space $(\sigma, \Sigma)$. Compared to the traditional version, the adiabatic invariant adopted in this work is continuous around the dynamical separatrix. The continuous characteristic is very useful to describe those TNOs with switching behaviors between libration and circulation, because it does not need to match different phase portraits to describe a single TNO moving in different regions. Second, phase portraits are produced by plotting level curves of the adiabatic invariant with given Hamiltonian, which can be used to predict long-term behaviors of TNOs. Compared to the conventional version of phase portraits (i.e., level curves of the Hamiltonian with given adiabatic invariant), there is a smaller computational burden to produce the new versions of phase portraits because the resonant Hamiltonian is an explicit function while the adiabatic invariant is an implicit function of the state variables $(\sigma, \Sigma, \omega, \varpi)$. Third, in the new analytical model, it is possible to produce critical curves (the curves on which the Hamiltonian is equal to that of the dynamical separatrix), which divide the entire phase space into different domains (with librations or circulations). Distributions of the critical curve are very useful for us to predict dynamical behaviors of TNOs in the long-term evolution.

Analytical developments are applied to real TNOs inside MMRs with Neptune. In particular, three representative TNOs inside 2:5 resonance (2018 VO137, 2005 SD278, and 2015 PD132), two representative TNOs inside 2:3 resonances (Pluto and 2004 HA79), and one TNO inside 1:2 resonance (1996 TR66), and one TNO inside 1:3 resonance (2014 SR373) are taken as examples. For each TNO considered, the trajectory is numerically propagated under the full $N$-body model and compared to the level curve arising in phase portraits. Good agreement is observed between numerical trajectories under the full $N$-body model and level curves arising in phase portraits, showing that our analytical developments are applicable for predicting long-term behaviors. In particular, level curves shown in phase portraits provide phase-space paths of eccentricity excitation, which is helpful in understanding the origin and evolution of high-eccentricity TNOs. In phase portraits, different numbers of ZLK islands can be found. If there exist four ZLK islands, we can see that the islands centered at $180^\circ$ or $360^\circ$ occupy higher-eccentricity zones than the ones centered at $90^\circ$ or $270^\circ$. Regarding the non-1:1 type of resonance, the ZLK centers are usually at symmetric locations of $\omega = 90^\circ$, $180^\circ$, $270^\circ$, and $360^\circ$, while for the 1:1-type resonances the ZLK centers are no longer at the symmetric locations, which is in agreement with previous works (Kozai 1985; Gallardo et al. 2012; Saillenfest et al. 2016).

Finally, some remarks are made here to explain the slight differences between numerical results under the full $N$-body model and level curves in phase portraits (see Figures 8–11). From the dynamical viewpoint, there are several possible reasons leading to the difference. First, the numerical trajectories propagated under the full $N$-body model are presented in osculating elements, so short-period oscillations can be observed in the long-term evolution. However, the level curves (or guiding trajectories in the secular model) represent the evolution of averaged elements. Second, in the full $N$-body model, the secular resonances associated with planets’ precession rates as well as the ZLK resonance may together influence the long-term evolution of TNOs. However, in the semi-secular or secular models, there are no other secular resonances besides the ZLK resonance. Finally, in the full $N$-body model, the osculating orbits of the giant planets change with time, and their inclinations and eccentricities are not equal to zero (although they are small). In the semi-secular or secular dynamical models, the orbits of the giant planets are fixed and they are assumed to be coplanar and circular.

The authors thank an anonymous reviewer for helpful suggestions that significantly improved the quality of the manuscript. This work is supported by the National Natural Science Foundation of China (Nos. 12073011, 11973027, 11933001), the National Key R&D Program of China (No. 2019YFA0706601), the Manned Space Project Foundation of China (No. ZYZZ12-QW04), and Shanghai Sailing Program.
This research has made use of data and/or services provided by the International Astronomical Union’s Minor Planet Center.

**ORCID iDs**

Hanlun Lei @ https://orcid.org/0000-0002-2747-5116  
Xiumin Huang @ https://orcid.org/0000-0002-2546-2012

**References**

Brasil, P., Gomes, R., & Soares, J. 2014, *A&A*, 564, A44  
Deprit, A. 1969, *CeMec*, 1, 12  
Efimov, S. S., & Sidorenko, V. V. 2020, *CeMDA*, 132, 1  
Gallardo, T. 2006, *Icar*, 184, 29  
Gallardo, T. 2020, *CeMDA*, 132, 1  
Gallardo, T., Hugo, G., & Pais, P. 2012, *Icar*, 220, 392  
Gladman, B., Lawler, S., Petit, J.-M., et al. 2012, *AJ*, 144, 23  
Gomes, R. S. 2011, *Icar*, 215, 661  
Gomes, R. S., Gallardo, T., Fernández, J. A., & Brunini, A. 2005, *CeMDA*, 91, 109  
Henrard, J. 1990, *CeMDA*, 49, 45  
Henrard, J., & Caranicolas, N. 1989, *CeMDA*, 47, 99  
Henrard, J., & Lemaître, A. 1986, *CeMec*, 39, 213  
Henrard, J., & Lemaître, A. 1987, *Icar*, 69, 266  
Hori, G.-i 1966, *PASI*, 18, 287  
Huang, Y., Li, M., Li, J., & Gong, S. 2018, *MNRAS*, 481, 5401  
Ito, T., & Ohtsuka, K. 2019, *MEEP*, 7, 1  
Kozai, Y. 1962, *AJ*, 67, 591  
Kozai, Y. 1985, *CeMec*, 36, 47  
Lei, H. 2019, *MNRAS*, 487, 2097  
Li, J., Lei, H., & Xia, Z. J. 2021, *MNRAS*, 505, 1730  
Lidov, M. 1962, *P&SS*, 9, 719  
Morbidelli, A. 2002, Modern Celestial Mechanics: Aspects of Solar System Dynamics (London: Taylor & Francis)  
Morbidelli, A., & Moons, M. 1993, *Icar*, 102, 316  
Murray, C., & Fox, K. 1984, *Icar*, 59, 221  
Murray, C. D., & Dermott, S. F. 1999, Solar System Dynamics (Cambridge: Cambridge Univ. Press)  
Neishtadt, A. 1987, *JApMM*, 51, 586  
Neishtadt, A. I., & Sidorenko, V. V. 2004, *CeMDA*, 90, 307  
Nesvorný, D., Thomas, F., Ferraz-Mello, S., & Morbidelli, A. 2002, *CeMDA*, 82, 323  
Pons, J., & Gallardo, T. 2022, *MNRAS*, 511, 1153  
Saillenfest, M. 2020, *CeMDA*, 132, 1  
Saillenfest, M., Fouchard, M., Tommei, G., & Valsecchi, G. B. 2016, *CeMDA*, 126, 369  
Saillenfest, M., Fouchard, M., Tommei, G., & Valsecchi, G. B. 2017, *CeMDA*, 127, 477  
Schubart, J. 1968, *AJ*, 73, 99  
Sidorenko, V. V., Neishtadt, A. I., Artemyev, A. V., & Zelenyi, L. M. 2014, *CeMDA*, 120, 131  
Tennyson, J., Cary, J. R., & Escande, D. 1986, *PhRvL*, 56, 2117  
Thomas, F., & Morbidelli, A. 1996, *CeMDA*, 64, 209  
von Zeipel, H. 1910, *AN*, 183, 345  
Wan, X.-S., & Huang, T.-Y. 2007, *MNRAS*, 377, 133  
Wisdom, J. 1982, *AJ*, 87, 577  
Wisdom, J. 1983, *Icar*, 56, 51  
Wisdom, J. 1985, *Icar*, 63, 272  
Yokoyama, T. 1996, *CeMDA*, 64, 243  
Yokoyama, T., & Balthazar, J. 1992, *Icar*, 99, 175  
Yoshikawa, M. 1989, *A&A*, 213, 436