Analysis on Wiener Space and Applications

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Introduction

The aim of this book is to give a rigorous introduction for the graduate students to Analysis on Wiener space, a subject which has grown up very quickly these recent years under the new impulse of the Stochastic Calculus of Variations of Paul Malliavin (cf. [55]). A portion of the material exposed is our own research, in particular, with Moshe Zakai and Denis Feyel for the rest we have used the works listed in the bibliography.

The origin of this book goes back to a series of seminars that I had given in Bilkent University of Ankara in the summer of 1987 and also during the spring and some portion of the summer of 1993 at the Mathematics Institute of Oslo University and a graduate course dispensed at the University of Paris VI. An initial and rather naive version of these notes has been published in Lecture Notes in Mathematics series of Springer at 1995. Since then we have assisted to a very quick development and progress of the subject in several directions. In particular, its use has been remarked by mathematical economists. Consequently I have decided to write a more complete text with additional contemporary applications to illustrate the strength and the applicability of the subject. Several new results like the logarithmic Sobolev inequalities, applications of the capacity theory to the local and global differentiability of Wiener functionals, probabilistic notions of the convexity and log-concavity, the Monge and the Monge-Kantorovitch measure transportation problems in the infinite dimensional setting and the analysis on the path space of a compact Lie group are added.

Although some concepts are given in the first chapter, I assumed that the students had already acquired the notions of stochastic calculus with semimartingales, Brownian motion and some rudiments of the theory of Markov processes.

The second chapter deals with the definition of the (so-called) Gross-Sobolev derivative and the Ornstein-Uhlenbeck operator which are indispensable tools of the analysis on Wiener space. In the third chapter we begin the proof of the Meyer inequalities, for which the hypercontractivity property of the Ornstein-Uhlenbeck semi-group is needed. We expose this last topic in the fourth chapter and give the classical proof of the logarithmic Sobolev inequality of L. Gross for the Wiener measure. In chapter V, we complete the proof of Meyer inequalities and study the distribution spaces which are defined via the Ornstein-Uhlenbeck operator. In particular we show that the derivative and divergence operators extend continuously to distribution spaces. In the appendix we indicate how one can transfer all these results to arbitrary abstract Wiener spaces using the notion of time associated to a
Chapter VI begins with an extension of Clark’s formula to the distributions defined in the preceding chapter. This formula is applied to prove the classical $0 - 1$-law and as an application of the latter, we prove the positivity improving property of the Ornstein-Uhlenbeck semigroup. We then show that the functional composition of a non-degenerate Wiener functional with values in $\mathbb{R}^n$, (in the sense of Malliavin) with a real-valued smooth function on $\mathbb{R}^n$ can be extended when the latter is a tempered distribution if we look at to the result as a distribution on the Wiener space. This result contains the fact that the probability density of a non-degenerate functional is not only $C^\infty$ but also it is rapidly decreasing. This observation is then applied to prove the regularity of the solutions of Zakai equation of the nonlinear filtering and to an extension of the Ito formula to the space of tempered distributions with non-degenerate Ito processes. We complete this chapter with two non-standard applications of Clark’s formula, the first concerns the equivalence between the independence of two measurable sets and the orthogonality of the corresponding kernels of their Ito-Clark representation and the latter is another proof of the logarithmic Sobolev inequality via Clark’s formula.

Chapter VII begins with the characterization of positive (Meyer) distributions as Radon measures and an application of this result to local times. Using capacities defined with respect to the Ornstein-Uhlenbeck process, we prove also a stronger version of the $0 - 1$-law already exposed in Chapter VI: it says that any $H$-invariant subset of the Wiener space or its complement has zero $C_{r,1}$-capacity. This result is then used that the $H$- gauge functionals of measurable sets are finite quasi-everywhere instead of almost everywhere. We define also there the local Sobolev spaces, which is a useful notion when we study the problems where the integrability is not a priori obvious. We show how to patch them together to obtain global functionals. Finally we give a short section about the distribution spaces defined with the second quantization of a general “elliptic” operator, and as an example show that the action of a shift define a distribution in this sense.

In chapter eight we study the independence of some Wiener functionals with the previously developed tools.

The ninth chapter is devoted to some series of moment inequalities which are important in applications like large deviations, stochastic differential equations, etc. In the tenth chapter we expose the contractive version of Ramer’s theorem as another example of the applications of moment inequalities developed in the preceding chapter and as an application we show the validity of the logarithmic Sobolev inequality under this perturbated measures. Chapter XI deals with a rather new notion of convexity and concavity which is quite appropriate for the equivalence classes of Wiener functionals.
We believe that it will have important applications in the field of convex analysis and financial mathematics. Chapter XII can be regarded as an immediate application of Chapter XI, where we study the problem of G. Monge and its generalization, called the Monge-Kantorovich\textsuperscript{1} measure transportation problem \textbf{for general measures} with a singular quadratic cost function, namely the square of the Cameron-Martin norm. Later we study in detail when the initial measure is the Wiener measure.

The last chapter is devoted to construct a similar Sobolev analysis on the path space over a compact Lie group, which is the simplest non-linear situation. This problem has been studied in the more general case of compact Riemannian manifolds (cf. \cite{56}, \cite{57}), however, I think that the case of Lie groups, as an intermediate step to clarify the ideas, is quite useful.

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\textsuperscript{1}Another spelling is “Kantorovich”.

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Chapter 1

Introduction to Stochastic Analysis

This chapter is devoted to the basic results about the Wiener measure, Brownian motion, construction of the Ito stochastic integral, Cameron-Martin and Girsanov theorems, representation of the Wiener functionals with stochastic integrals and the Ito-Wiener chaos decomposition which results from it. The proofs are rather sketchy whenever they are given; for complete treatment of these results we refer the reader to the excellent references given in the bibliography.

1.1 The Brownian Motion and the Wiener Measure

Let $W = C_0([0,1])$, define $W_t$ as to be the coordinate functional, i.e., for $w \in W$ and $t \in [0,1]$, let $W_t(w) = w(t)$. If we note by $\mathcal{B}_t = \sigma\{W_s; s \leq t\}$, then, the following theorem is well-known (cf. for instance [81]):

**Theorem 1.1.1** There is one and only one measure $\mu$ on $W$ which satisfies the following properties:

1. $\mu\{w \in W : W_0(w) = 0\} = 1$,
2. For any $f \in C_0^\infty(\mathbb{R})$, the stochastic process process

\[
(t, w) \mapsto f(W_t(w)) - \frac{1}{2} \int_0^t \Delta f(W_s(w))ds
\]

is a $(\mathcal{B}_t, \mu)$-martingale, where $\Delta$ denotes the Laplace operator. $\mu$ is called the (standard) Wiener measure.
Brownian Motion

From Theorem 1.1.1, it follows that, for \( t > s \),

\[
E_\mu \left[ e^{i\alpha (W_t - W_s)} | \mathcal{B}_s \right] = \exp \left\{ -\frac{1}{2} \alpha^2 (t - s) \right\},
\]

hence \((t, w) \mapsto W_t(w)\) is a continuous additive process (i.e., a process with independent increments) and \((W_t; t \in [0, 1])\) is also a continuous martingale.

1.2 Stochastic Integration

The stochastic integration with respect to the Brownian motion is first defined on the adapted step processes and then extended to their completion by isometry. A mapping \( K : [0, 1] \times W \to \mathbb{R} \) is called a step process if it can be represented in the following form:

\[
K_t(w) = \sum_{i=1}^{n} a_i(w) \cdot 1_{[t_i, t_{i+1})}(t), \quad a_i(w) \in L^2(\mathcal{B}_{t_i}).
\]

For such a step process, we define its stochastic integral with respect to the Brownian motion, which is denoted by

\[
I(K) = \int_0^1 K_s dW_s(w)
\]
as to be

\[
\sum_{i=1}^{n} a_i(w) \left( W_{t_{i+1}}(w) - W_{t_i}(w) \right).
\]

Using the independence of the increments of \((W_t, t \in [0, 1])\), it is easy to see that

\[
E \left[ \left| \int_0^1 K_s dW_s \right|^2 \right] = E \int_0^1 |K_s|^2 \, ds,
\]
i.e., \( I \) is an isometry from the adapted step processes into \( L^2(\mu) \), hence it has a unique extension as an isometry from

\[
L^2([0, 1] \times W, \mathcal{A}, dt \times d\mu) \xrightarrow{I} L^2(\mu)
\]
where \( \mathcal{A} \) denotes the sigma algebra on \([0, 1] \times W\) generated by the adapted, left (or right) continuous processes. The extension of \( I(K) \) is called the stochastic integral of \( K \) and it is denoted as \( \int_0^1 K_s dW_s \). If we define

\[
I_t(K) = \int_0^t K_s dW_s
\]
as
\[ \int_{0}^{1} \mathbf{1}_{[0,1]}(s)K_s dW_s, \]

it follows from the Doob inequality that the stochastic process \( t \mapsto I_t(K) \) is a continuous, square integrable martingale. With some localization techniques using stopping times, \( I \) can be extended to any adapted process \( K \) such that \( \int_{0}^{1} K_s^2(s)ds < \infty \) a.s. In this case the process \( t \mapsto I_t(K) \) becomes a local martingale, i.e., there exists a sequence of stopping times increasing to one, say \( (T_n, n \in \mathbb{N}) \) such that the process \( t \mapsto I_{t \wedge T_n}(K) \) is a (square integrable) martingale. Vector (i.e. \( \mathbb{R}^n \)) valued Brownian motion is defined as a process whose components are independent, real-valued Brownian motions. A stochastic process \( (X_t, t \geq 0) \) with values in a finite dimensional Euclidean space is called an Ito process if it can be represented as
\[ X_t = X_0 + \int_{0}^{t} a_s dW_s + \int_{0}^{t} b_s ds, \]

where \( (W_t, t \geq 0) \) is a vector valued Brownian motion and \( a \) and \( b \) are respectively matrix and vector valued, adapted, measurable processes with \( \int_{0}^{t} (|a_s|^2 + |b_s|)ds < \infty \) almost surely for any \( t \geq 0 \). In the sequel the notation \( \int_{0}^{t} H_s dX_s \) will mean \( \int_{0}^{t} H_s a_s dW_s + \int_{0}^{t} H_s b_s ds \), we shall also denote by \( ([X,X]_t, t \geq 0) \) the Doob-Meyer process defined as
\[ [X,X]_t = \int_{0}^{t} \text{trace } (a_s a_s^\ast)ds. \]

This is the unique increasing process such that \( ([X]_t^2 - [X,X]_t, t \geq 0) \) is a (continuous) local martingale. It can be calculated as the limit of the sums
\[ \lim \sum_{i=1}^{n} \left( |X_{t_{i+1}}^2 - |X_{t_i}|^2 \right) = \lim \sum_{i=1}^{n} \left( |X_{t_{i+1}}| - |X_{t_i}| \right)^2 \]
\[ = \lim \sum_{i=1}^{n} \mathbb{E} \left[ (|X_{t_{i+1}}|^2 - |X_{t_i}|^2) \mathcal{F}_{t_i} \right], \]

where the limit is taken as the length of the partition \( \{t_1, \ldots, t_{n+1}\} \) of \([0,t]\), defined by \( \sup_i |t_{i+1} - t_i| \), tends to zero.

1.3 Ito formula

The following result is one of the most important applications of the stochastic integration:
Theorem 1.3.1 Let $f \in C^2(\mathbb{R})$ and let $(X_t, t \in [0,1])$ be an Ito process, i.e.,
\[ X_t = X_0 + \int_0^t K_s dW_s + \int_0^t U_r dr \]
where $X_0$ is $\mathcal{B}_0$-measurable, $K$ and $U$ are adapted processes with
\[ \int_0^1 \left[ |K_r|^2 + |U_r| \right] dr < \infty \]
(1.3.1)
almost surely. Then
\[ f(X_t) = f(X_0) + \int_0^t f'(X_s)K_s dW_s + \frac{1}{2} \int_0^t f''(X_s)K_s^2 ds \]
\[ + \int_0^t f'(X_r)U_r dr. \]

Remark 1.3.2 This formula is also valid in the several dimensional case. In fact, if $K$ is and $U$ are adapted processes with values in $\mathbb{R}^n \otimes \mathbb{R}^m$ and $\mathbb{R}^m$ respectively whose components are satisfying the condition (1.3.1), then we have, for any $f \in C^2(\mathbb{R}^m),$
\[ f(X_t) = f(X_0) + \int_0^t \partial_i f(X_s)K_{ij}(r) dW^j_s + \int_0^t \partial_i f(X_s)U_i(r) dr \]
\[ + \frac{1}{2} \int_0^t \partial_{ij} f(X_r)(K_rK_r^*)_{ij} dr \]
almost surely.

To prove the Ito formula we shall proceed by

Lemma 1.3.3 Let $X = (X_t, t \geq 0)$ and $Y = (Y_t, t \geq 0)$ be two Ito real-valued processes, then
\[ X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X,Y]_t, \]
(1.3.2)
amost surely, where $[X,Y]$ denotes the Doob-Meyer process. In particular
\[ X_t^2 = X_0^2 + 2 \int_0^t X_s dX_s + [X,X]_t. \]
(1.3.3)

Proof: Evidently it suffices to prove the relation (1.3.3), since we can obtain (1.3.2) via a polarization argument. Since $X$ has almost surely continuous trajectories, using a stopping time argument we can assume that $X$ is almost surely bounded. Assume now that $\{t_1, \ldots, t_n\}$ is a partition of $[0,t]$ and
Ito Formula

denote by \((M_t, t \geq 0)\) the local martingale part and by \((A_t, t \geq 0)\) the finite variation part of \(X\). We have

\[
X_t^2 - X_0^2 = \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2 + 2 \sum_{k=1}^{n} X_{t_{k-1}} (X_{t_k} - X_{t_{k-1}}) \quad (1.3.4)
\]

\[
= \sum_{k=1}^{n} (X_{t_k}^2 - X_{t_{k-1}}^2) + 2 \sum_{k=1}^{n} X_{t_{k-1}} (M_{t_k} - M_{t_{k-1}}) \quad (1.3.5)
\]

\[
+ 2 \sum_{k=1}^{n} X_{t_{k-1}} (A_{t_k} - A_{t_{k-1}}). \quad (1.3.6)
\]

Now, when \(\sup_k |t_k - t_{k-1}| \to 0\), then the first term at the right hand side of (1.3.5) converges to \([X, X]_t\) and the sum of the second term with (1.3.6) converges to \(2 \int_0^t X_s dX_s\) in probability.

**Proof of the Ito formula:**

Using a stopping argument we can assume that \(X\) takes its values in a bounded interval, say \([-K, K]\). The interest of this argument resides in the fact that we can approach a \(C^2\) function, as well as its first two derivatives uniformly by the polynomials on any compact interval. On the other hand, using Lemma 1.3.3, we see that the formula is valid for the polynomials. Let us denote by \((\Gamma f)_t\) the random variable

\[
f(X_t) - f(X_0) - \int_0^t f'(X_s) dX_s - \frac{1}{2} \int_0^t f''(X_s) d[X, X]_s.
\]

Assume moreover that \((p_n, n \geq 1)\) is a sequence of polynomials such that \((p_n, n \geq 1)\), \((p'_n, n \geq 1)\) and \((p''_n, n \geq 1)\) converge uniformly on \([-K, K]\) to \(f, f'\) and to \(f''\) respectively. Choosing a subsequence, if necessary, we may assume that

\[
\sup_{x \in [-K, K]} (|f(x) - p_n(x)| + |f'(x) - p'_n(x)| + |f''(x) - p''_n(x)|) \leq 1/n.
\]

Using the Doob and the Chebytchev inequalities, it is easy to see that \((\Gamma f)_t - (\Gamma p_n)_t\) converges to zero in probability, since \((\Gamma p_n)_t = 0\) almost surely, \((\Gamma f)_t\) should be also zero almost surely and this completes the proof of the Ito formula.

As an immediate corollary of the Ito formula we have

**Corollary 1.3.4** For any \(h \in L^2([0, 1])\), the process defined by

\[
\mathcal{E}_t(I(h)) = \exp \left( \int_0^t h_s dW_s - \frac{1}{2} \int_0^t h_s^2 ds \right)
\]

is a martingale.
Proof: Let us denote $E_t(I(h))$ by $M_t$, then from the Ito formula we have

$$M_t = 1 + \int_0^t M_s h_s dW_s,$$

hence $(M_t, t \in [0, 1])$ is a local martingale, moreover, since $I(h)$ is Gaussian, $M_1$ is in all the $L^p$-spaces, hence $(M_t, t \in [0, 1])$ is a square integrable martingale.

\[\square\]

1.4 Alternative constructions of the Wiener measure

A) Let us state first the celebrated theorem of Ito-Nisio about the convergence of the random series of independent, Banach space valued random variables (cf. [42]):

**Theorem 1.4.1 (Ito-Nisio Theorem)** Assume that $(X_n, n \in \mathbb{N})$ is a sequence of independent random variables with values in a separable Banach space $B$ whose continuous dual is denoted by $B^*$. The sequence $(S_n, n \in \mathbb{N})$ defined as

$$S_n = \sum_{i=1}^{n} X_i,$$

converges almost surely in the norm topology of $B$ if and only if there exists a probability measure $\nu$ on $B$ such that

$$\lim_{n} E\left[e^{i<\xi,S_n>}\right] = \int_{B} e^{i<\xi,y>} \nu(dy)$$

for any $\xi \in B^*$.

We can give another construction of the Brownian motion using Theorem 1.4.1 as follows: Let $(\gamma_i; i \in \mathbb{N})$ be an independent sequence of $N_1(0,1)$-Gaussian random variables. Let $(g_i)$ be a complete, orthonormal basis of $L^2([0,1])$. Then $W_t$ defined by

$$W_t(w) = \sum_{i=1}^{\infty} \gamma_i(w) \cdot \int_0^t g_i(s) ds$$

converges almost surely uniformly with respect to $t \in [0,1]$ and $(W_t, t \in [0,1])$ is a Brownian motion. In fact to see this it suffices to apply Theorem 1.4.1 to the sequence $(X_n, n \in \mathbb{N})$ defined by

$$X_n(w) = \gamma_n(w) \int_0^t g_n(s) ds.$$
Remark 1.4.2 In the sequel we shall denote by $H$ the so-called Cameron-Martin space $H([0, 1], \mathbb{R}^n)$ (in case $n = 1$ we shall again write simply $H$ or, in case of necessity $H([0, 1])$ i.e., the isometric image of $L^2([0, 1], \mathbb{R}^n)$ under the mapping
\[
g \mapsto \int_0^1 g(\tau) d\tau.
\]
Hence for any complete, orthonormal basis $(g_i, i \in \mathbb{N})$ of $L^2([0, 1], \mathbb{R}^n)$, $(\int_0^1 g_i(s) ds, i \in \mathbb{N})$ is a complete orthonormal basis of $H([0, 1])$. The use of the generic notation $H$ will be preferred as long as the results are dimension independent.

B) Let $(\Omega, \mathcal{F}, P)$ be any abstract probability space and let $H$ be any separable Hilbert space. If $L : H \rightarrow L^2(\Omega, \mathcal{F}, P)$ is a linear operator such that for any $h \in H$, $E[\exp iL(h)] = \exp -\frac{1}{2} |h|^2_H$, then there exists a Banach space with dense injection
\[H \hookrightarrow W\]
dense, hence
\[W^* \hookrightarrow H\]
is also dense and there exists a probability measure $\mu$ on $W$ such that
\[
\int_W \exp\langle w^*, w \rangle d\mu(w) = \exp -\frac{1}{2} |j^*(w^*)|^2_H
\]
and
\[
L(j^*(w^*))(w) = \langle w^*, w \rangle
\]
almost surely. $(W, H, \mu)$ is called an Abstract Wiener space and $\mu$ is the Wiener measure (cf. [37]). In the case $H$ is chosen to be
\[
H([0, 1]) = \left\{ h : h(t) = \int_0^t h(s) ds, |h|_H = |h|_{L^2([0, 1])} \right\}
\]
then $\mu$ is the classical Wiener measure and $W$ can be taken as $C_0([0, 1])$.

Remark 1.4.3 In the case of the classical Wiener space, any element $\lambda$ of $W^*$ is a signed measure on $[0, 1]$, and its image in $H = H([0, 1])$ can be represented as $j^*(\lambda)(t) = \int_0^t \lambda([s, 1]) ds$. In fact, we have for any $h \in H$
\[
(j^*(\lambda), h) = \langle \lambda, j(h) \rangle = \int_0^1 h(s) \lambda(ds) = h(1)\lambda([0, 1]) - \int_0^1 \lambda([0, s]) \dot{h}(s) ds
\]
\[
\begin{align*}
&= \int_0^1 (\lambda([0,1]) - \lambda([0,s])) \dot{h}(s) ds \\
&= \int_0^1 \lambda([s,1]) \dot{h}(s) ds.
\end{align*}
\]

## 1.5 Cameron-Martin and Girsanov Theorems

In the sequel we shall often need approximation of the Wiener functional with cylindrical smooth functions on the Wiener space. This kind of properties hold in every Wiener space since this is due to the analyticity of the characteristic function of the Wiener measure. However, they are very easy to explain in the case of classical Wiener space, that is why we have chosen to work in this frame. In particular the Cameron-Martin theorem which is explained in this section is absolutely indispensable for the development of the next chapters.

**Lemma 1.5.1** The set of random variables

\[
\left\{ f(W_{t_1}, \ldots, W_{t_n}); t_i \in [0,1], f \in \mathcal{S}(\mathbb{R}^n); n \in \mathbb{N} \right\}
\]

is dense in \( L^2(\mu) \), where \( \mathcal{S}(\mathbb{R}^n) \) denotes the space of infinitely differentiable, rapidly decreasing functions on \( \mathbb{R}^n \).

**Proof:** It follows from the martingale convergence theorem and the monotone class theorem.

**Lemma 1.5.2** The linear span of the set

\[
\Theta = \left\{ \exp \left[ \int_0^1 h_s dW_s - \frac{1}{2} \int_0^1 h_s^2 ds \right] : h \in L^2([0,1]) \right\}
\]

is dense in \( L^2(\mu) \).

**Proof:** It follows from Lemma 1.5.1 via the Fourier transform.

**Remark:** Although the set \( \Theta \) separates the points of \( L^2(\mu) \), it does not give any indication about the positivity.

**Lemma 1.5.3** The polynomials are dense in \( L^2(\mu) \).

**Proof:** The proof follows by the analyticity of the characteristic function of the Wiener measure, in fact, due to this property, the elements of the set in Lemma 1.5.2 can be approached by the polynomials.
Theorem 1.5.4 (Cameron-Martin Theorem) For any bounded Borel measurable function $F$ on $C_0([0, 1])$ and $h \in L^2([0, 1])$, we have

$$E_\mu \left[ F \left( w + \int_0^1 h_s ds \right) \exp \left\{ - \int_0^1 h_s dW_s - \frac{1}{2} \int_0^1 h_s^2 ds \right\} \right] = E_\mu[F].$$

This assertion implies in particular that the process $(t, w) \to W_t(w) + \int_0^t h_s ds$ is again a Brownian motion under the new probability measure

$$\exp \left\{ - \int_0^1 h_s dW_s - \frac{1}{2} \int_0^1 h_s^2 ds \right\} d\mu.$$

Proof: It is sufficient to show that the new probability has the same characteristic function as $\mu$: if $x^* \in W^*$, then $x^*$ is a measure on $[0, 1]$ and

$$W^* \langle x^*, w \rangle_W = \int_0^1 W_s(w) x^*(ds)$$

$$= W_t(w) \cdot x^*([0, t]) \mid_0^1 - \int_0^1 x^*([0, t]) dW_t(w)$$

$$= W_t x^*([0, 1]) - \int_0^1 x^*([0, t]) dW_t$$

$$= \int_0^1 x^*((t, 1]) dW_t.$$

Consequently

$$E \left[ \left\{ \exp i \int_0^1 x^*([t, 1]) dW_t \right\} \left( w + \int_0^1 h_s ds \right) \exp(-I(h)) \right]$$

$$= E \left[ \exp \left\{ i \int_0^1 x^*([t, 1]) dW_t + i \int_0^1 x^*([t, 1]) h_t dt - \int_0^1 h_t dW_t - \frac{1}{2} \int_0^1 h_t^2 dt \right\} \right]$$

$$= E \left[ \exp \left\{ i \int_0^1 (ix^*([t, 1]) - h_t) dW_t \right\} \exp \left\{ i \int_0^1 x^*([t, 1]) h_t dt - \frac{1}{2} \int_0^1 h_t^2 dt \right\} \right]$$

$$= \exp \left\{ \frac{1}{2} \int_0^1 (ix^*([t, 1]) - h_t)^2 dt + i \int_0^1 x^*([t, 1]) h_t dt - \frac{1}{2} \int_0^1 h_t^2 dt \right\}$$

$$= \exp \left\{ \frac{1}{2} \left( j(x^*) \right)^2 \right\}$$

and this achieves the proof. \qed

The following corollary is one of the most important results of the modern probability theory:
Corollary 1.5.5 (Paul Lévy’s Theorem) Suppose that \((M_t, t \in [0, 1])\) is a continuous martingale with \(M_0 = 0\) and that \((M^2_t - t, t \in [0, 1])\) is again a martingale. Then \((M_t, t \in [0, 1])\) is a Brownian motion.

Proof: From the Ito formula
\[
f(M_t) = f(0) + \int_0^t f^\prime(M_s) \cdot dM_s + \frac{1}{2} \int_0^t \Delta f(M_s) \, ds.
\]
Hence the law of \((M_t : t \in [0, 1])\) is \(\mu\).

As an application of Paul Lévy’s theorem we can prove easily the following result known as the Girsanov theorem which generalizes the Cameron-Martin theorem. This theorem is basic in several applications like the filtering of the random signals corrupted by a Brownian motion, or the problem of optimal control of Itô processes.

Theorem 1.5.6 (Girsanov Theorem) Assume that \(u : [0, 1] \times W \to \mathbb{R}^n\) is a measurable process adapted to the Brownian filtration satisfying
\[
\int_0^1 |u_s|^2 ds < \infty
\]
\(\mu\)-almost surely. Let
\[
\Lambda_t = \exp \left\{ - \int_0^t (u_s, dW_s) - 1/2 \int_0^t |u_s|^2 ds \right\}.
\]
Assume that
\[
E [\Lambda_1] = 1.
\]
Then the process \((t, w) \to W_t(w) + \int_0^t u_s(w) ds\) is a Brownian motion under the probability \(\Lambda_1 d\mu\).

Remark 1.5.7 The condition (1.5.7) is satisfied in particular if we have
\[
E \left[ \exp \frac{1}{2} \int_0^1 |u_s|^2 ds \right] < \infty.
\]
This is called the Novikov condition (cf. \[67, 101\]). There is another, slightly more general sufficient condition due to Kazamaki \[45\], which is
\[
E \left[ \exp \frac{1}{2} \int_0^1 u_s dW_s \right] < \infty.
\]
Note that the difference between the Cameron–Martin theorem and the Girsanov theorem is that in the former the mapping \(w \to w + \int_0^1 h(s) ds\) is an invertible transformation of the Wiener space \(W\) and in the latter the corresponding map \(w \to w + \int_0^1 u_s(w) ds\) is not necessarily invertible.
1.6 The Ito Representation Theorem

The following result is known as the Ito representation formula:

**Theorem 1.6.1** Any $\varphi \in L^2(\mu)$ can be represented as

$$\varphi = E[\varphi] + \int_0^1 K_s dW_s$$

where $K \in L^2([0,1] \times W)$ and it is adapted.

**Proof:** Since the Wick exponentials

$$E(I(h)) = \exp\left\{\int_0^1 h_s dW_s - 1/2 \int_0^1 h_s^2 ds\right\}$$

can be represented as claimed and since their finite linear combinations are dense in $L^2(\mu)$, the proof follows. □

**Remark 1.6.2** Let $\phi$ be an integrable real random variable on the Wiener space. We say that it belongs to the class $H^1$ if the martingale $M = (M_t, t \in [0,1])$ satisfies the property that

$$E[<M,M>_{1/2}] < \infty.$$ 

The Ito representation theorem extends via stopping techniques to the random variables of class $H^1$.

1.7 Ito-Wiener chaos representation

For any $h \in L^2([0,1])$, define $K_t = \int_0^t h_s dW_s, t \in [0,1]$. Then, from the Ito formula, we can write

$$K_t^p = p \int_0^1 K_s^{p-1} h_s dW_s + \frac{p(p-1)}{2} \int_0^1 K_s^{p-2} h_s^2 ds$$

$$= p \int_0^1 (p-1) \int_0^{t_1} K_t^{p-2} h_{t_2} dW_{t_2} + \frac{(p-1)(p-1)}{2} \int_0^{t_1} K_{t_2}^{p-3} h_{t_2}^2 dt_2 dW_{t_1}$$

$$+ \cdots$$

where $p$ is a positive integer. Iterating this procedure we see that $K_t^p$ can be written as the linear combination of the multiple integrals of deterministic integrands of the type

$$J_p = \int_{0<t_2<t_3<...<t_p<1} h_{t_1} h_{t_2} \cdots h_{t_p} dW_{t_1}^{i_1} \cdots dW_{t_p}^{i_p},$$
\[ i_j = 0 \text{ or } 1 \text{ with } dW^0_t = dt \text{ and } dW^1_t = dW_t. \] Hence we can express the polynomials as multiple Wiener-Ito integrals. Let us now combine this observation with the Ito representation:

Assume that \( \varphi \in L^2(\mu) \), then from the Ito representation theorem:

\[ \varphi = E[\varphi] + \int_0^1 K_s dW_s. \]

Iterating the same procedure for the integrand of the above stochastic integral:

\[ \varphi = E[\varphi] + \int_0^1 E[K_s] dW_s + \int_0^1 \int_0^{t_1} E[K_{t_1,t_2}] dW_{t_2} dW_{t_1} \]

\[ + \int_0^1 \int_0^{t_1} \int_0^{t_2} K_{t_1,t_2,t_3} dW_{t_3} dW_{t_2} dW_{t_1}. \]

After \( n \) iterations we end up with

\[ \varphi = \sum_{p=0}^{n} J_p(K_p) + \varphi_{n+1} \]

and each element of the sum is orthogonal to the other one. Hence \( (\varphi_n; n \in \mathbb{N}) \) is bounded in the Hilbert space \( L^2(\mu) \) and this means that it is weakly relatively compact. Let \( (\varphi_{n_k}) \) be a weakly convergent subsequence and \( \varphi_\infty = \lim_{k \to \infty} \varphi_{n_k} \). Then it is easy from the first part that \( \varphi_\infty \) is orthogonal to the polynomials, therefore \( \varphi_\infty = 0 \) and the weak limit

\[ w - \lim_{n \to \infty} \sum_{p=0}^{n} J_p(K_p) \]

exists and it is equal to \( \varphi \) almost surely. Let

\[ S_n = \sum_{p=0}^{n} J_p(K_p), \]

then, from the weak convergence, we have

\[ \lim_n E[|S_n|^2] = \lim_n E[S_n \varphi] = E[|\varphi|^2], \]

hence \( (S_n, n \geq 1) \) converges weakly to \( \varphi \) and its \( L^2 \)-norm converges to the \( L^2 \)-norm of \( \varphi \) and this implies that the series

\[ \sum_{p=1}^{\infty} J_p(K_p) \]
converges to $\varphi$ in the strong topology of $L^2(\mu)$. Let now $\tilde{K}_p$ be an element of $\tilde{L}^2[0,1]^p$ (i.e. symmetric), defined as $\tilde{K}_p = K_p$ on $C_p = \{t_1 < \cdots < t_p\}$. We define $I_p(\tilde{K}_p) = p! J_p(K_p)$ in such a way that

$$E[|I_p(\tilde{K}_p)|^2] = (p!)^2 \int_{C_p} K_p^2 dt_1 \cdots dt_p = p! \int_{[0,1]^p} |\tilde{K}_p|^2 dt_1 \cdots dt_p.$$ 

Let $\varphi_p = \frac{\tilde{K}_p}{p!}$, then we have proven

**Theorem 1.7.1** Any element $\varphi$ of $L^2(\mu)$, can be decomposed as an orthogonal sum of multiple Wiener-Ito integrals

$$\varphi = E[\varphi] + \sum_{p=1}^{\infty} I_p(\varphi_p)$$

where $\varphi_p$ is a symmetric element of $L^2[0,1]^p$. Moreover, this decomposition is unique.

**Remark:** In the following chapters we shall give an explicit representation of the kernels $\varphi_p$ using the Gross-Sobolev derivative.

**Notes and suggested reading**

The basic references for the stochastic calculus are the books of Dellacherie-Meyer [21] and of Stroock-Varadhan [81]. Especially in the former, the theory is established for the general semimartingales with jumps. For the construction of the Wiener measure on Banach spaces we refer the reader to [37] and especially to [49].
Chapter 2

Sobolev Derivative, Divergence and Ornstein-Uhlenbeck Operators

2.1 Introduction

Let $W = C_0([0, 1], \mathbb{R}^d)$ be the classical Wiener space equipped with $\mu$ the Wiener measure. We want to construct on $W$ a Sobolev type analysis in such a way that we can apply it to the random variables that we encounter in the applications. Mainly we want to construct a differentiation operator and to be able to apply it to practical examples. The Fréchet derivative is not satisfactory. In fact the most frequently encountered Wiener functionals, as the multiple (or single) Wiener integrals or the solutions of stochastic differential equations with smooth coefficients are not even continuous with respect to the Fréchet norm of the Wiener space. Therefore, what we need is in fact to define a derivative on the $L^p(\mu)$-spaces of random variables, but in general, to be able to do this, we need the following property which is essential: if $F, G \in L^p(\mu)$, and if we want to define their directional derivative, in the direction, say $\hat{w} \in W$, we write $\frac{d}{dt} F(w + t\hat{w})|_{t=0}$ and $\frac{d}{dt} G(w + t\hat{w})|_{t=0}$. If $F = G \mu$-a.s., it is natural to ask that their derivatives are also equal a.s. For this, the only way is to choose $\hat{w}$ in some specific subspace of $W$, namely, the Cameron-Martin space $H$:

$$H = \left\{ h : [0, 1] \to \mathbb{R}^d / h(t) = \int_0^t h(s)ds, \ |h|^2_H = \int_0^1 |h(s)|^2 ds \right\}.$$  

In fact, the theorem of Cameron-Martin says that for any $F \in L^p(\mu), p > 1,$
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\[ h \in H \]

\[ E_\mu \left[ F(w + h) \exp \left\{ - \int_0^1 \dot{h}(s) \cdot dW_s - \frac{1}{2} |h|^2_H \right\} \right] = E_\mu[F], \]

or equivalently

\[ E_\mu[F(w + h)] = E \left[ F(w) \cdot \exp \left\{ \int_0^1 \dot{h}(s) \cdot dW_s - \frac{1}{2} |h|^2_H \right\} \right]. \]

That is to say, if \( F = G \) a.s., then \( F(\cdot + h) = G(\cdot + h) \) a.s. for all \( h \in H \).

### 2.2 The Construction of \( \nabla \) and its properties

If \( F : W \to \mathbb{R} \) is a function of the following type (called cylindrical):

\[ F(w) = f(W_{t_1}(w), \ldots, W_{t_n}(w)), \quad f \in \mathcal{S}(\mathbb{R}^n), \]

we define, for \( h \in H \),

\[ \nabla_h F(w) = \frac{d}{d\lambda} F(w + \lambda h)|_{\lambda=0}. \]

Noting that \( W_t(w + h) = W_t(w) + h(t) \), we obtain

\[ \nabla_h F(w) = \sum_{i=1}^n \partial_i f(W_{t_1}(w), \ldots, W_{t_n}(w))h(t_i), \]

in particular

\[ \nabla_h W_t(w) = h(t) = \int_0^t \dot{h}(s)ds = \int_0^1 1_{[0,t]}(s) \dot{h}(s)ds. \]

If we denote by \( U_t \) the element of \( H \) defined as \( U_t(s) = \int_0^s 1_{[0,t]}(r)dr \), we have \( \nabla_h W_t(w) = (U_t, h)_H \). Looking at the linear map \( h \mapsto \nabla_h F(w) \) we see that it defines a random element with values in \( H^* \), since we have identified \( H \) with \( H^* \). \( \nabla F \) is an \( H \)-valued random variable. Now we can prove:

**Proposition 2.2.1** \( \nabla \) is a closable operator on any \( L^p(\mu) \) (\( p > 1 \)).

**Proof:** Closable means that if \( (F_n : n \in \mathbb{N}) \) are cylindrical functions on \( W \), such that \( F_n \to 0 \) in \( L^p(\mu) \) and if \( (\nabla F_n : n \in \mathbb{N}) \) is Cauchy in \( L^p(\mu, H) \), then its limit is zero. Hence suppose that \( \nabla F_n \to \xi \) in \( L^p(\mu; H) \). In order to prove \( \xi = 0 \) \( \mu \)-a.s., we use the Cameron-Martin theorem: Let \( \varphi \) be any cylindrical
function. Since such \( \varphi \)'s are dense in \( L^p(\mu) \), it is sufficient to prove that 
\[
E[(\xi, h)_H \cdot \varphi] = 0 \quad \text{for any } h \in H.
\]
This follows from
\[
E[\langle \nabla F_n, h \rangle \varphi] = \frac{d}{d\lambda} E[F_n(w + \lambda h) \cdot \varphi] |_{\lambda=0}
\]
\[
= \frac{d}{d\lambda} E \left[ F_n(w) \varphi(w - \lambda h) \exp \left( \lambda \int_0^1 \dot{h}(s) dW_s - \frac{\lambda^2}{2} \int_0^1 |\dot{h}_s|^2 ds \right) \right] |_{\lambda=0}
\]
\[
= E \left[ F_n(w) \left( -\nabla_h \varphi(w) + \varphi(w) \int_0^1 \dot{h}(s) dW_s \right) \right] \xrightarrow{n \to \infty} 0
\]
since \((F_n, n \in \mathbb{N})\) converges to zero in \( L^p(\mu) \). \( \square \)

Proposition 2.2.1 tells us that the operator \( \nabla \) can be extended to larger classes of Wiener functionals than the cylindrical ones. In fact we define first the extended \( L^p \)-domain of \( \nabla \), denoted by \( \text{Dom}_p(\nabla) \) as

**Definition 2.2.2** \( F \in \text{Dom}_p(\nabla) \) if and only if there exists a sequence \((F_n; n \in \mathbb{N})\) of cylindrical functions such that \( F_n \to F \) in \( L^p(\mu) \) and \( (\nabla F_n) \) is Cauchy in \( L^p(\mu, H) \). Then, for any \( F \in \text{Dom}_p(\nabla) \), we define
\[
\nabla F = \lim_{n \to \infty} \nabla F_n.
\]
The extended operator \( \nabla \) is called **Gross-Sobolev derivative**.

**Remark 2.2.3** Proposition 2.2.1 implies that the definition of \( \nabla F \) is independent of the choice of the approximating sequence \((F_n)\).

Now we are ready to define

**Definition 2.2.4** We will denote by \( \mathbb{D}_{p,1} \) the linear space \( \text{Dom}_p(\nabla) \) equipped with the norm \( \|F\|_{p,1} = \|F\|_p + \|\nabla F\|_{L^p(\mu, H)} \).

**Remark 2.2.5**

1. If \( \Xi \) is a separable Hilbert space we can define \( \mathbb{D}_{p,1}(\Xi) \) exactly in the same way as before, the only difference is that we take \( \mathcal{S}_\Xi \) instead of \( \mathcal{S} \), i.e., the rapidly decreasing functions with values in \( \Xi \). Then we leave to the reader to prove that the same closability result holds.

2. Hence we can define \( \mathbb{D}_{p,k} \) by iteration:
   i) We say that \( F \in \mathbb{D}_{p,2} \) if \( \nabla F \in \mathbb{D}_{p,1}(H) \), then write \( \nabla^2 F = \nabla(\nabla F) \).
   ii) \( F \in \mathbb{D}_{p,k} \) if \( \nabla^{k-1} F \in \mathbb{D}_{p,1}(H^{\otimes(k-1)}) \).
3. Note that, for $F \in \mathbb{D}_{p,k}$, $\nabla^k F$ is in fact with values $H^{\otimes k}$ (i.e. symmetric tensor product).

4. From the proof we have that if $F \in \mathbb{D}_{p,1}$, $h \in H$ and $\varphi$ is cylindrical, we have

$$E[\nabla_h F \cdot \varphi] = -E[F \cdot \nabla_h \varphi] + E[I(h) \cdot F \cdot \varphi],$$

where $I(h)$ is the first order Wiener integral of the (Lebesgue) density of $h$. If $\varphi \in \mathbb{D}_{q,1}$ ($q^{-1} + p^{-1} = 1$), by a limiting argument, the same relation holds again. Let us note that this limiting procedure shows in fact that if $\nabla F \in L^p(\mu, H)$ then $F.I(h) \in L^p(\mu)$, i.e., $F$ is more than $p$-integrable. This observation gives rise to the logarithmic Sobolev inequality.

2.3 Derivative of the Itô integral

Let $\varphi = f(W_{t_1}, \ldots, W_{t_n})$, $t_i \leq t$, $f$ smooth. Then we have

$$\nabla_h \varphi(w) = \sum_{i=1}^{n} \partial_i f(W_{t_1}, \ldots, W_{t_n}) h(t_i),$$

hence $\nabla \varphi$ is again a random variable which is $\mathcal{B}_t$-measurable. In fact this property is satisfied by a larger class of Wiener functionals:

**Proposition 2.3.1** Let $\varphi \in \mathbb{D}_{p,1}$, $p > 1$ and suppose that $\varphi$ is $\mathcal{B}_t$-measurable for a given $t \geq 0$. Then $\nabla \varphi$ is also $\mathcal{B}_t$-measurable and furthermore, for any $h \in H$, whose support is in $[t, 1]$, $\nabla_h \varphi = (\nabla \varphi, h)_H = 0$ a.s.

**Proof:** Let $(\varphi_n)$ be a sequence of cylindrical random variable converging to $\varphi$ in $\mathbb{D}_{p,1}$. If $\varphi_n$ is of the form $f(W_{t_1}, \ldots, W_{t_k})$, it is easy to see that, even if $\varphi_n$ is not $\mathcal{B}_t$-measurable, $E[\varphi_n | \mathcal{B}_t]$ is another cylindrical random variable, say $\theta_n(W_{t_1 \wedge t}, \ldots, W_{t_k \wedge t})$. In fact, suppose that $t_k > t$ and $t_1, \ldots, t_{k-1} \leq t$. We have

$$E[f(W_{t_1}, \ldots, W_{t_k} | \mathcal{B}_t] = E[f(W_{t_1}, \ldots, W_{t_{k-1}}, W_t - W_t + W_t) | \mathcal{B}_t]$$

$$= \int_{\mathbb{R}} f(W_{t_1}, \ldots, W_{t_{k-1}}, W_t + x)p_{t_k-t}(x) dx$$

$$= \theta(W_{t_1}, \ldots, W_{t_{k-1}}, W_t),$$

and $\theta \in \mathcal{S}$ if $f \in \mathcal{S}(\mathbb{R}^k)$, where $p_t$ denotes the heat kernel. Hence we can choose a sequence $(\varphi_n)$ converging to $\varphi$ in $\mathbb{D}_{p,1}$ such that $\nabla \varphi_n$ is $\mathcal{B}_t$-measurable for each $n \in \mathbb{N}$. Hence $\nabla \varphi$ is also $\mathcal{B}_t$-measurable. If $h \in H$ has
its support in $[t, 1]$, then, for each $n$, we have $\nabla_h \varphi_n = 0$ a.s., because $\nabla \varphi_n$ has its support in $[0, t]$ as one can see from the explicit calculation for $\nabla \varphi_n$. Taking an a.s. convergent subsequence, we see that $\nabla_h \varphi = 0$ a.s. also.

Let now $K$ be an adapted simple process:

$$K_t(w) = \sum_{i=1}^{n} a_i(w) 1_{(t_i, t_{i+1})}(t)$$

where $a_i \in \mathbb{D}_{p, 1}$ and $\mathcal{B}_{t_i}$-measurable for any $i$. Then we have

$$\int_0^1 K_s dW_s = \sum_{i=1}^{n} a_i(W_{t_{i+1}} - W_{t_i})$$

and

$$\nabla_h \int_0^1 K_s dW_s = \sum_{i=1}^{n} \nabla_h a_i(W_{t_{i+1}} - W_{t_i}) + \sum_{i=1}^{n} a_i(h(t_{i+1}) - h(t_i)) = \int_0^1 \nabla_h K_s dW_s + \int_0^1 K_s h(s) ds .$$

Hence

$$\left| \nabla \int_0^1 K_s dW_s \right|^2_H \leq 2 \left\{ \int_0^1 \nabla K_s dW_s \right|^2_H + \int_0^1 |K_s|^2 ds \right\}$$

and

$$E \left[ \left( \left| \nabla \int_0^1 K_s dW_s \right|^2_H \right)^{p/2} \right] \leq 2^p E \left[ \left( \left| \int_0^1 \nabla K_s dW_s \right|^p_H \right)^{p/2} \right] + \int_0^1 |K_s|^2 ds \right) \right] .$$

Using the Burkholder-Davis-Gundy inequality for the Hilbert space valued martingales, the above quantity is majorized by

$$2c_p E \left\{ \left( \int_0^1 \nabla K_s^2_H ds \right)^{p/2} + \left( \int_0^1 |K_s|^2 ds \right)^{p/2} \right\}$$

$$= \tilde{c}_p \left\| \nabla \tilde{K} \right\|^p_{L^p(\mu, H \otimes H)} + \left\| \tilde{K} \right\|^p_{L^p(\mu, H)} ,$$

where

$$\tilde{K} = \int_0^1 K_r dr .$$

Thanks to this majoration, we have proved:
Proposition 2.3.2 Let $\tilde{K} \in \mathbb{D}_{p,1}(H)$ such that $K_t = \frac{d\tilde{K}(t)}{dt}$ be $\mathcal{B}_t$-measurable for almost all $t$. Then we have

$$
\nabla \int_0^1 K_s dW_s = \int_0^1 \nabla K_s dW_s + \tilde{K}
$$

(2.3.1)

almost surely.

Remark 2.3.3 The relation (2.3.1) means that, for any $h \in H$, we have

$$
\nabla_h \int_0^1 K_s dW_s = \int_0^1 \nabla_h K_s dW_s + \int_0^1 K_s \dot{h}(s) ds.
$$

Corollary 2.3.4 If $\varphi = I_n(f_n)$, $f_n \in \dot{L}^2([0,1]^n)$, then we have, for $h \in H$,

$$
\nabla_h I_n(f_n) = n \int_{[0,1]^n} f(t_1, \ldots, t_n) \dot{h}(t_n) dW_{t_1} \ldots dW_{t_{n-1}} dt_n.
$$

Proof: Apply the above proposition $n$-times to the case in which, first $f_n$ is $C^\infty([0,1]^n)$, then pass to the limit in $L^2(\mu)$. 

The following result will be extended in the sequel to much larger classes of random variables:

Corollary 2.3.5 Let $\varphi : W \to \mathbb{R}$ be analytic in $H$-direction. Then we have

$$
\varphi = E[\varphi] + \sum_{n=1}^{\infty} \tilde{I}_n \left( \frac{E[\nabla^n \varphi]}{n!} \right),
$$

where $\tilde{I}_n(g)$, for a symmetric $g \in H^\otimes n$, denotes the multiple Wiener integral of

$$
\frac{\partial^n g}{\partial t_1 \ldots \partial t_n}(t_1, \ldots, t_n).
$$

In other words the kernel $\varphi_n \in \dot{L}^2[0,1]^n$ of the Wiener chaos decomposition of $\varphi$ is equal to

$$
\frac{\partial^n E[\nabla^n \varphi]}{\partial t_1 \ldots \partial t_n n!}.
$$

Proof: We have, on one hand, for any $h \in H$,

$$
E[\varphi(w + h)] = E[\varphi \exp \int_0^1 \dot{h}_s dW_s - \frac{1}{2} \int_0^1 \dot{h}_s^2 ds] = E[\varphi E(I(h))].
$$
On the other hand, from Taylor’s formula:

\[
E[\varphi(w + h)] = E[\varphi] + \sum_{n=1}^{\infty} \frac{1}{n!} E\left[ (\nabla^n \varphi(w), h^{\otimes n}) \right] \\
= E[\varphi] + \sum_{n=1}^{\infty} \frac{1}{n!} E\left[ (\hat{I}_n(E[\nabla^n \varphi]) \hat{I}_n(h^{\otimes n})) \right] \\
= E[\varphi] + \sum_{n=1}^{\infty} \frac{1}{n!} E\left[ \hat{I}_n(E[\nabla^n \varphi]) \hat{I}_n(h^{\otimes n}) \right]
\]

hence, from the symmetry, we have

\[
\hat{I}_n(\varphi_n) = \frac{1}{n!} \hat{I}_n(E[\nabla^n \varphi]),
\]

where we have used the notation \( \hat{I}_1(h) = \hat{I}(h) = \int_0^1 \dot{h}_s dW_s \) and

\[
\hat{I}_n(\varphi_n) = \int_{[0,1]^n} \frac{\partial^n \varphi_n}{\partial t_1 \cdots \partial t_n}(t_1, \ldots, t_n) dW_{t_1} \cdots dW_{t_n}.
\]

\[
\square
\]

### 2.4 The divergence operator

The divergence operator, which is the adjoint of the Sobolev derivative with respect to the Wiener measure, is one of the most important tools of the Stochastic Analysis. We begin with its formal definition:

**Definition 2.4.1** Let \( \xi : W \to H \) be a random variable. We say that \( \xi \in \text{Dom}_p(\delta) \), if for any \( \varphi \in \mathbb{D}_{q,1} \) (\( q^{-1} + p^{-1} = 1 \)), we have

\[
E[[(\nabla \varphi, \xi)_{H}]] \leq c_{p,q}(\xi) \| \varphi \|_q,
\]

and in this case we define \( \delta \xi \) by

\[
E[\{\delta \xi\} \varphi] = E[[(\xi, \nabla \varphi)_{H}]],
\]

i.e., \( \delta \xi = \nabla^* \xi \), where \( \nabla^* \) denotes the adjoint of \( \nabla \) with respect to the Wiener measure \( \mu \), it is called the divergence operator.
Remark: For the emergence of this operator cf. [47], [35] and the references there.

Let us give some properties of $\delta$:

1.) Let $a : W \to \mathbb{R}$ be “smooth”, $\xi \in \mathrm{Dom}_p(\delta)$. Then we have, for any $\varphi \in \mathbb{D}_{q,1}$,

$$E[\delta(a\xi)\varphi] = E[(a\xi, \nabla \varphi)_H] = E[(\xi, a\nabla \varphi)_H] = E[(\xi, \nabla (a\varphi) - \varphi \nabla a)_H] = E[(\delta \xi) a\varphi - \varphi (\nabla a, \xi)_H],$$

hence

$$\delta(a\xi) = a\delta \xi - (\nabla a, \xi)_H. \quad (2.4.2)$$

2.) Let $h \in H$, then we pretend that

$$\delta h = \int_0^1 \dot{h}(s)dW_s.$$ 

To see this, it is sufficient to test this relation on the exponential martingales: if $k \in H$, we have

$$E \left[ \exp \left\{ \int_0^1 \dot{k}_s dW_s - \frac{1}{2} \int_0^1 \dot{k}_s^2 ds \right\} \right] = E[(h, \nabla \mathcal{E}(I(k))_H)] = E[(h, k)_H \mathcal{E}(I(k))] = (h, k)_H.$$

On the other hand, supposing first $h \in W^*$,

$$E[I(h) \mathcal{E}(I(k))] = E[I(h)(w + k)] = E[I(h)] + (h, k)_H = (h, k)_H.$$

Hence in particular, if we denote by $\tilde{1}_{[s,t]}$ the element of $H$ such that $\tilde{1}_{[s,t]}(r) = \int_s^r 1_{[s,t]}(u)du$, we have that

$$\delta(\tilde{1}_{[s,t]}) = W_t - W_s. \quad (2.4.3)$$
3.) Let now \( K \) be an adapted, simple process
\[
K_t(w) = \sum_{i=1}^{n} a_i(w) \cdot 1_{[t_i, t_{i+1}]}(t),
\]
where \( a_i \in \mathbb{D}_{p,1} \) and \( \mathcal{B}_{t_i} \)-measurable for each \( i \). Let \( \hat{K} \) be \( \int_0^t K_s ds \). Then from the identity (2.4.3), we have
\[
\delta \hat{K} = \delta \left( \sum_{i=1}^{n} a_i \cdot \tilde{1}_{[t_i, t_{i+1}]} \right) = \sum_{i=1}^{n} \left\{ a_i \delta (\tilde{1}_{[t_i, t_{i+1}]} - (\nabla a_i, \tilde{1}_{[t_i, t_{i+1}]})) \right\}.
\]
From the relation (2.4.3), we have \( \delta (\tilde{1}_{[t_i, t_{i+1}]} = W_{t_{i+1}} - W_{t_i} \), furthermore, from the Proposition 2.3.1, the support of \( \nabla a_i \) is in \([0, t_i]\), consequently, we obtain
\[
\delta \hat{K} = \sum_{i=1}^{n} a_i (W_{t_{i+1}} - W_{t_i}) = \int_0^1 K_s dW_s.
\]
Hence we have the important result which says that

**Theorem 2.4.2** \( \text{Dom}_p(\delta) \ (p > 1) \) contains the set consisting of the primitives of adapted stochastic processes satisfying
\[
E \left[ \left( \int_0^1 K_s^2 ds \right)^{p/2} \right] < \infty.
\]
Moreover one has
\[
\delta \left\{ \int_0^1 K_s ds \right\} = \int_0^1 K_s dW_s.
\]

### 2.5 Local characters of \( \nabla \) and \( \delta \)

Before proceeding further, we shall prove the locality of the Gross-Sobolev derivative and the divergence operators in this section:

**Lemma 2.5.1** Let \( \phi \in \mathbb{D}_{p,1} \) for some \( p > 1 \), then we have, for any constant \( c \in \mathbb{R} \),
\[
\nabla \phi = 0 \ \text{on} \ \{ \phi = c \},
\]
almost surely.

**Proof:** Replacing \( \phi \) by \( \phi - c \), we may assume that \( c = 0 \). Let now \( f \) be a positive, smooth function of compact support on \( \mathbb{R} \) such that \( f(0) = 1 \).
Let \( f_\varepsilon(t) = f(t/\varepsilon) \) and let \( F_\varepsilon \) be its primitive. For any smooth, cylindrical, \( H \)-valued random variable \( u \), we have

\[
E[F_\varepsilon(\phi) \delta u] = E[(\nabla F_\varepsilon(\phi), u)_H] = E[f_\varepsilon(\phi)\nabla \phi, u)_H] \rightarrow E[1_{\{\phi=0\}}(\nabla \phi, u)_H]
\]

as \( \varepsilon \to 0 \). On the other hand \( |F_\varepsilon(\phi)| \leq \varepsilon \left\| f \right\|_{L^1(\mathbb{R},d\varepsilon)} \), hence it follows that

\[
E[1_{\{\phi=0\}}(\nabla \phi, u)_H] = 0,
\]

since such \( u \)'s are dense in \( L^q(\mu,H) \), the proof follows.

The divergence operator has an analogous property:

**Lemma 2.5.2** Assume that \( u \in \text{Dom}_p(\delta) \), \( p > 1 \), and that the operator norm of \( \nabla u \), denoted by \( \left\| \nabla u \right\|_{\text{op}} \) is in \( L^p(\mu) \). Then

\[
\delta u = 0 \quad \text{a.s. on } \{ w \in W : u(w) = 0 \}.
\]

**Proof:** Let \( f_\varepsilon \) be as in the proof of Lemma 2.5.1, then for any cylindrical \( \phi \), using the integration by parts formula:

\[
E \left[ f_\varepsilon \left( |u|^2_H \right) \delta u \phi \right] = E \left[ f'_\varepsilon \left( |u|^2_H \right) \left( u, \nabla |u|^2_H \right)_H \phi \right] + E \left[ f_\varepsilon \left( |u|^2_H \right) (u, \nabla \phi)_H \right]. \tag{2.5.4}
\]

Note that

\[
\left| f'_\varepsilon \left( |u|^2_H \right) (u, \nabla |u|^2_H)_H \right| \leq |u|^2_H \left| f'_\varepsilon \left( |u|^2_H \right) \right| \left\| \nabla u \right\|_{\text{op}} \leq \varepsilon \sup_{\mathbb{R}} |xf'(x)| \left\| \nabla u \right\|_{\text{op}}.
\]

Hence from the dominated convergence theorem, the first term at the right of (2.5.4) tends to zero with \( \varepsilon \). Evidently the second one also converges to zero and this completes the proof.

**Remark 2.5.3** Using the local character of the Sobolev derivative one can define the local Sobolev spaces as we shall see later.
2.6 The Ornstein-Uhlenbeck Operator

For a nice function $f$ on $W$, $t \geq 0$, we define

$$P_t f(x) = \int_W f \left( e^{-t}x + \sqrt{1-e^{-2t}}y \right) \mu(dy),$$

(2.6.5)

this expression for $P_t$ is called Mehler’s formula. Since $\mu(dx)\mu(dy)$ is invariant under the rotations of $W \times W$, i.e., $(\mu \times \mu)(dx,dy)$ is invariant under the transformation

$$T_t(x,y) = \left( xe^{-t} + y(1-e^{-2t})^{1/2}, x(1-e^{-2t})^{1/2} - ye^{-t} \right),$$

we have obviously

$$\|P_t f(x)\|_{L^p(\mu)} \leq \iint |(f \otimes 1)(T_t(x,y))|p \mu(dx)\mu(dy)$$

$$= \iint |(f \otimes 1)(x,y)|p \mu(dx)\mu(dy)$$

$$= \iint |f(x)|p \mu(dx),$$

for any $p \geq 1$, $\|P_t f\|_{L^p} \leq \|f\|_{L^p}$; hence also for $p = \infty$ by duality. A straightforward calculation gives that, for any $h \in H \cap W^* (= W^*)$,

$$P_t(\mathcal{E}(I(h))) = \mathcal{E}(e^{-t}I(h))$$

$$= \sum_{n=0}^{\infty} e^{-nt} I_n(h^{\otimes n}) \frac{n!}{n!}.$$

Hence, by homogeneity, we have

$$P_t(I_n(h^{\otimes n})) = e^{-nt} I_n(h^{\otimes n})$$

and by a density argument, we obtain

$$P_t I_n(f_n) = e^{-nt} I_n(f_n),$$

for any $f_n \in \hat{L}^2([0,1]^n)$. Consequently $P_s \circ P_t = P_{s+t}$, i.e., $(P_t)$ is a measure preserving Markov semi-group. Its infinitesimal generator is denoted by $-\mathcal{L}$ and is $\mathcal{L}$ is called the Ornstein-Uhlenbeck or the number operator. Evidently, we have

$$\mathcal{L}I_n(f_n) = n I_n(f_n)$$

(2.6.6)

and this relation means that the Wiener chaos are its eigenspaces. From the definition, it follows directly that (for $a_i$ being $\mathcal{F}_{t_i}$-measurable)

$$P_t \left( \sum a_i(W_{t_{i+1}} - W_{t_i}) \right) = e^{-t} \sum (P_t a_i)(W_{t_{i+1}} - W_{t_i}),$$
that is to say 
\[ P_t \int_0^1 H_s dW_s = e^{-t} \int_0^1 P_t H_s dW_s, \]
\[ \mathcal{L} \int_0^1 H_s dW_s = \int_0^1 (I + \mathcal{L}) H_s dW_s. \] 
(2.6.7)

Also we have 
\[ \nabla P_t \varphi = e^{-t} P_t \nabla \varphi. \] 
(2.6.8)

The following lemma is a consequence of the relation (2.6.6):

**Lemma 2.6.1** Assume that \( \phi \in L^2(\mu) \) with the Wiener chaos representation
\[ \phi = \sum_{n=0}^{\infty} I_n(\phi_n) \] satisfying 
\[ \sum_{n=1}^{\infty} n (n!) \| \phi_n \|^2_{H^n} < \infty. \]
Then 
\[ \delta \circ \nabla \phi = \mathcal{L} \phi, \]
where \( \delta \) is the divergence operator.

**Proof:** It is sufficient to prove for \( \varphi = \mathcal{E}(I(h)) \). In this case from the identity (2.4.2)
\[ (\delta \circ \nabla) \varphi = \delta(h \mathcal{E}(I(h))) = \left[ I(h) - |h|^2_H \right] \mathcal{E}(I(h)) = \mathcal{L} \mathcal{E}(I(h)). \]

**Remark 2.6.2** Let us define for the smooth functions \( \varphi \), a semi-norm
\[ \| \varphi \|_{p,k} = \| (I + \mathcal{L})^{k/2} \varphi \|_{L^p(\mu)} . \]
At first glance, these semi-norms (in fact norms), seem different from the one defined by \( \| \varphi \|_{p,k} = \sum_{j=0}^{k} \| \nabla^j \varphi \|_{L^p(\mu, H^\otimes j)} \). We will show in the next chapters that they are equivalent.

\[ ^1 \text{Sometimes, in the classical case, it is also called Hitsuda-Ramer-Skorohod integral.} \]
2.7 Exercises

These exercises are aimed to give some useful formulas about the iterated divergence operator and related commutation properties.

1. Prove that
   \[ \nabla P_t \phi = e^{-t} P_t \nabla \phi \quad (2.7.9) \]
   and
   \[ P_t \delta u = e^{-t} \delta P_t u \quad (2.7.10) \]
   for any \( \phi \in \mathbb{D}_{p,1} \) and \( u \in \mathbb{D}_{p,1}(H) \).

2. Assume that \( u : W \to H \) is a cylindrical random variable. Prove that
   \[ \delta u = \sum_{i=1}^{\infty} \{(u, e_i)_H \delta e_i - \nabla e_i (u, e_i)_H\}, \]
   for any complete, orthonormal basis \((e_i, i \in \mathbb{N})\) of \( H \). In particular, in the finite dimensional case we can write
   \[ \delta u(w) = \langle u(w), w \rangle - \text{trace } \nabla u(w), \]
   although in infinite dimensional case such an expression is meaningless in general. In case the trace \( \nabla u \) exists, the remaining part is called the Stratonovitch integral.

3. Assume that \( u : W \to H \) is a cylindrical random variable. Prove that
   \[ E[(\delta u)^2] = E[|u|^2_H] + E[\text{trace } (\nabla u \nabla u)]. \]

4. Let \( u \) be as above, prove the identity
   \[ \delta^2 u \otimes^2 = (\delta u)^2 - |u|^2_H - \text{trace } (\nabla u \nabla u) - 2\delta(\nabla u), \]
   where \( \delta^2 u \otimes^2 \) is defined by the integration by parts formula as
   \[ E[\delta^2 u \otimes^2 \phi] = E[(\nabla^2 \phi, u \otimes^2)_2], \]
   for any test function \( \phi \) and \((\cdot, \cdot)_2\) denotes the inner product of the space of Hilbert-Schmidt operators on \( H \). Prove that more generally one has
   \[ \delta \alpha \delta \beta = \delta^2 (\alpha \otimes \beta) + \text{trace } (\nabla \alpha \nabla \beta) \]
   \[ + \delta(\nabla_\alpha \beta + \nabla_\beta \alpha) + (\alpha, \beta)_H, \]
   where \( \alpha \) and \( \beta \) are two \( H \)-valued, cylindrical random variables.

5. With the same hypothesis as above, show that one has
   \[ \delta^{p+1} u \otimes^{p+1} = \delta u \delta^p u \otimes^p - \nabla_u (\delta^p u \otimes^p) - \delta(\nabla_u \otimes^p u). \]

6. For a \( u : W \to H \) as above, prove that
   \[ (\delta u)^p = \delta \left( u(\delta u)^{p-1} \right) \]
   \[ + (\delta u)^{p-2} \left[(p-1)|u|^2_H + (p-2) (\delta(\nabla u) + \text{trace } (\nabla u \nabla u)) \right] \]
   for any \( p \in \mathbb{N} \).
Notes and suggested reading

The notion of derivation in the setting of a Gaussian measure on an infinite dimensional setting can be found in the books of Quantum Field Theory, cf. [77] also [47] and the references there. It has also been studied in a little bit more restricted case under the name $H$-derivative by L. Gross, cf. also [49], [47]. However the full use of the quasi-invariance with respect to the translations from the Cameron-Martin space combined with the $L^p$-closure of it in the sense of Sobolev has become popular with the advent of the stochastic calculus of variations of Paul Malliavin: cf. [62], [76], [56].
Chapter 3
Meyer Inequalities

Meyer Inequalities and Distributions

Meyer inequalities are essential to control the Sobolev norms defined with the Sobolev derivative with the norms defined via the Ornstein-Uhlenbeck operator. They can be summarized as the equivalence of the two norms defined on the (real-valued) Wiener functionals as

\[ \|\phi\|_{p,k} = \sum_{i=0}^{k} \|\nabla^i \phi\|_{L^p(\mu, H^\otimes i)}, \]

and

\[ \|\phi\|_{p,k} = \|(I + \mathcal{L})^{k/2} \phi\|_{L^p(\mu)}, \]

for any \( p > 1 \) and \( k \in \mathbb{N} \). The key point is the continuity property of the Riesz transform on \( L^p([0, 2\pi], dx) \), i.e., from a totally analytic origin, although the original proof of P. A. Meyer was probabilistic (cf. [62]). Here we develop the proof suggested by [28].

3.1 Some Preparations

Let \( f \) be a function on \([0, 2\pi]\), extended to the whole \( \mathbb{R} \) by periodicity. We denote by \( \tilde{f}(x) \) the function defined by

\[ \tilde{f}(x) = \frac{1}{\pi} \text{p.v.} \int_0^\pi \frac{f(x + t) - f(x - t)}{2 \tan t/2} dt, \]  

where p.v. denotes the the principal value of the integral in (3.1.1). The famous theorem of M. Riesz, cf. [105], asserts that, for any \( f \in L^p[0, 2\pi], \)
\( \hat{f} \in L^p([0,2\pi]), \text{ for } 1 < p < \infty \text{ with} \)
\[
\|\hat{f}\|_p \leq A_p \|f\|_p,
\]
where \(A_p\) is a constant depending only on \(p\). Most of the classical functional analysis of the 20-th century has been devoted to extend this result to the case where the function \(f\) was taking its values in more abstract spaces than the real line. We will show that our problem also can be reduced to this one.

In fact, the main result that we are going to show will be that
\[
\|\nabla (I + \mathcal{L})^{-1/2} \varphi\|_p \approx \|\varphi\|_p
\]
by rewriting \(\nabla (I + \mathcal{L})^{-1/2}\) as an \(L^p(\mu,H)\)-valued Riesz transform. For this we need first, the following elementary

**Lemma 3.1.1** Let \(K\) be any function on \([0,2\pi]\) such that
\[
K(\theta) - \frac{1}{2} \cot \frac{\theta}{2} \in L^\infty([0,\pi]),
\]
then the operator \(f \rightarrow T_K f\) defined by
\[
T_K f(x) = \frac{1}{\pi} \text{p.v.} \int_0^\pi (f(x + t) - f(x - t))K(t)dt
\]
is again a bounded operator on \(L^p([0,2\pi])\) with
\[
\|T_K f\|_p \leq B_p \|f\|_p \text{ for any } p \in (1,\infty)
\]
where \(B_p\) depends only on \(p\).

**Proof:** In fact we have
\[
|T_K f - \hat{f}|(x) \leq \frac{1}{\pi} \int_0^\pi |f(x + t) - f(x - t)|\ |K(t) - \frac{1}{2} \cot \frac{\theta}{2}| \ dt
\]
\[
\leq c \|f\|_{L^p} \left\| K - \frac{1}{2} \cot \frac{\theta}{2} \right\|_{L^\infty}.
\]
Hence
\[
\|T_K f\|_p \leq \left( c \left\| K - \frac{1}{2} \cot \frac{\theta}{2} \right\|_{L^\infty} + A_p \right) \|f\|_p.
\]
Remark 3.1.2 If for some $a \neq 0$, $aK(\theta) - \frac{1}{2} \cot \frac{\theta}{2} \in L^\infty([0, 2\pi])$, then we have

$$
\|T_K f\|_p = \frac{1}{|a|} \|aT_K f\|_p \\
\leq \frac{1}{|a|} \left\{ \|aT_K f - \tilde{f}\|_p + \|\tilde{f}\|_p \right\} \\
\leq \frac{1}{|a|} \left\{ \|aK - \frac{1}{2} \cot \frac{\theta}{2}\|_{L^\infty} \|f\|_p + A_p \|f\|_p \right\} \\
\leq c_p \|f\|_p
$$

with another constant $c_p$.

Corollary 3.1.3 Let $K$ be a function on $[0, \pi]$ such that $K = 0$ on $\left[ \frac{\pi}{2}, \pi \right]$ and $K - \frac{1}{2} \cot \frac{\theta}{2} \in L^\infty\left( \left[ 0, \frac{\pi}{2} \right] \right)$. Then $T_K$ defined by

$$
T_K f(x) = \int_0^{\pi/2} [f(x + t) - f(x - t)] K(t) dt
$$

is continuous from $L^p([0, 2\pi])$ into itself for any $p \in [1, \infty)$.

Proof: We have

$$
c K(\theta) 1_{[0, \frac{\pi}{2}]} - \frac{1}{2} \cot \frac{\theta}{2} \in L^\infty([0, \pi])
$$

since on the interval $\left[ \frac{\pi}{2}, \pi \right]$, $\sin \frac{\theta}{2} \in \left[ \frac{\sqrt{2}}{2}, 1 \right]$, then the result follows from the Lemma 3.1.1.

3.2 $\nabla (I + L)^{-1/2}$ as the Riesz Transform

Let us denote by $R_\theta(x, y)$ the rotation on $W \times W$ defined by

$$
R_\theta(x, y) = \left( x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta \right).
$$

Note that $R_\theta \circ R_\phi = R_{\phi + \theta}$. We have also, putting $e^{-t} = \cos \theta$,

$$
P_t f(x) = \int_W f(e^{-t}x + \sqrt{1 - e^{-2t}} y) \mu(dy) \\
= \int_W (f \otimes 1)(R_\theta(x, y)) \mu(dy) \\
= P_{-\log \cos \theta} f(x).
$$
Let us now calculate \((I + \mathcal{L})^{-1/2} \varphi\) using this transformation:

\[
(I + \mathcal{L})^{-1/2} \varphi(x) = \int_0^\infty t^{-1/2} e^{-t} P_t \varphi(x) dt
\]

\[
= \int_0^{\pi/2} (- \log \cos \theta)^{-1/2} \cos \theta \cdot \int_W (\varphi \otimes 1)(R_\theta(x, y)) \mu(dy) \tan \theta d\theta
\]

\[
= \int_W \mu(dy) \left[ \int_0^{\pi/2} (- \log \cos \theta)^{-1/2} \sin \theta (\varphi \otimes 1)(R_\theta(x, y)) d\theta \right].
\]

On the other hand, we have, for \(h \in H\)

\[
\nabla_h P_t \varphi(x) = \frac{d}{d\lambda} P_t \varphi(x + \lambda h) |_{\lambda=0}
\]

\[
= \frac{d}{d\lambda} \int \varphi \left( e^{-t} x + \lambda h + \sqrt{1 - e^{-2t}} y \right) \mu(dy) |_{\lambda=0}
\]

\[
= \frac{d}{d\lambda} \int \varphi \left( e^{-t} x + \sqrt{1 - e^{-2t}} \left( y + \frac{\lambda e^{-t}}{\sqrt{1 - e^{-2t}}} h \right) \right) \mu(dy) |_{\lambda=0}
\]

\[
= \frac{d}{d\lambda} \int \varphi \left( e^{-t} x + \sqrt{1 - e^{-2t}} y \right) \mathcal{E} \left( \frac{\lambda e^{-t}}{\sqrt{1 - e^{-2t}}} I(h) \right)(y) \mu(dy) |_{\lambda=0}
\]

\[
= \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_W \varphi \left( e^{-t} x + \sqrt{1 - e^{-2t}} y \right) \delta h(y) \mu(dy). 
\]

Therefore

\[
\nabla_h (I + \mathcal{L})^{-1/2} \varphi(x)
\]

\[
= \int_0^\infty t^{-1/2} e^{-t} \nabla_h P_t \varphi(x) dt
\]

\[
= \int_0^\infty t^{-1/2} \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \int_W \delta h(y) \varphi \left( e^{-t} x + \sqrt{1 - e^{-2t}} y \right) \mu(dy) dt
\]

\[
= \int_0^{\pi/2} (- \log \cos \theta)^{-1/2} \cos \theta \sin \theta \tan \theta \int \delta h(y) \varphi \otimes 1 \left( R_\theta(x, y) \right) \mu(dy) d\theta
\]

\[
= \int_0^{\pi/2} (- \log \cos \theta)^{-1/2} \cos \theta \int_W \delta h(y) \cdot (\varphi \otimes 1)(R_\theta(x, y)) \mu(dy) d\theta
\]

Since \(\mu(dy)\) is invariant under the transformation \(y \mapsto -y\), we have

\[
\int \delta h(y)(\varphi \otimes 1)(R_\theta(x, y)) \mu(dy) = - \int \delta h(y)(\varphi \otimes 1)(R_{-\theta}(x, y)) \mu(dy),
\]

therefore:

\[
\nabla_h (I + \mathcal{L})^{-1/2} \varphi(x)
\]
Riesz Transform

\[ R(\theta) = \int_0^{\pi/2} (- \log \cos \theta)^{-1/2} \int W \delta_h(y) \left( (\varphi \otimes 1)(R_\theta(x,y)) - (\varphi \otimes 1)(R_{-\theta}(x,y)) \right) \mu(dy) d\theta \]

where \( K(\theta) = \frac{1}{2} \cos \theta (- \log \cos \theta)^{-1/2} \).

**Lemma 3.2.1** We have

\[ 2K(\theta) - \cot \frac{\theta}{2} \in L^\infty((0,\pi/2]). \]

**Proof:** The only problem is when \( \theta \to 0 \). To see this let us put \( e^{-t} = \cos \theta \), then

\[ \cot \frac{\theta}{2} = \frac{\sqrt{1 + e^{-t}}}{\sqrt{1 - e^{-t}}} \approx \frac{2}{\sqrt{t}} \]

and

\[ K(\theta) = \frac{e^{-t}}{\sqrt{t}} \approx \frac{1}{\sqrt{t}} \]

hence

\[ 2K(\theta) - \cot \frac{\theta}{2} \in L^\infty \left([0, \frac{\pi}{2}] \right). \]

Using Lemma 3.1.1, Remark 3.1.2 following it and Corollary 3.1.3, we see that the map \( f \mapsto p.v. \int_{\pi/2}^0 (f(x + \theta) - f(x - \theta)) K(\theta) d\theta \) is a bounded map from \( L^p[0, \pi] \) into itself. Moreover

**Lemma 3.2.2** Let \( F : W \times W \to \mathbb{R} \) be a measurable, bounded function. Define \( TF(x,y) \) as

\[ TF(x,y) = p.v. \int_0^{\pi/2} [F \circ R_\theta(x,y) - F \circ R_{-\theta}(x,y)] K(\theta) d\theta. \]

Then, for any \( p > 1 \), there exists some \( c_p > 0 \) such that

\[ \|TF\|_{L^p(\mu \times \mu)} \leq c_p \|F\|_{L^p(\mu \times \mu)}. \]

**Proof:** We have

\[ (TF)(R_\theta(x,y)) = p.v. \int_0^{\pi/2} (F(R_{\theta+\theta}(x,y)) - F(R_{\theta-\theta}(x,y))) K(\theta) d\theta, \]
this is the Riesz transform for fixed \((x, y) \in W \times W\), hence we have
\[
\int_0^{\pi/2} |TF(R_\beta(x, y))|^p d\beta \leq c_p \int_0^{\pi} |F(R_\beta(x, y))|^p d\beta ,
\]

taking the expectation with respect to \(\mu \times \mu\), which is invariant under \(R_\beta\), we have
\[
E_{\mu \times \mu} \int_0^{\pi} |TF(R_\beta(x, y))|^p d\beta = \frac{\pi}{2} E[|TF|^p] 
\]
\[
\leq c_p E \int_0^{\pi} |F(R_\beta(x, y))|^p d\beta 
\]
\[
= \pi c_p E[|F|^p] .
\]

\(\Box\)

We have

**Theorem 3.2.3** \(\nabla \circ (I + \mathcal{L})^{-1/2} : L^p(\mu) \to L^p(\mu, H)\) is a linear continuous operator for any \(p \in (1, \infty)\).

**Proof:** With the notations of Lemma 3.2.2, we have
\[
\nabla_h (I + \mathcal{L})^{-1/2} \varphi = \int_W \delta_h(y) T(\varphi \otimes 1)(x, y) \mu(dy) .
\]

From Hölder inequality:
\[
|\nabla_h (I + \mathcal{L})^{1/2} \varphi(x)| \leq \|\delta_h\|_q \left( \int_W |T(\varphi \otimes 1)(x, y)|^p \mu(dy) \right)^{1/p} 
\]
\[
\leq c_p |h|_H \left( \int_W |T(\varphi \otimes 1)(x, y)|^p \mu(dy) \right)^{1/p} ,
\]

where the last inequality follows from the fact that \(y \to \delta_h(y)\) is an \(N_1(0, |h|_H^2)\)–Gaussian random variable. Hence
\[
|\nabla (I + \mathcal{L})^{-1/2} \varphi(x)|_H \leq \left( \int_W |T(\varphi \otimes 1)(x, y)|^p \mu(dy) \right)^{1/p}
\]

consequently, from Lemma 3.2.2
\[
\|\nabla (I + \mathcal{L})^{-1/2} \varphi\|_p \leq \int_{W \times W} |T(\varphi \otimes 1)(x, y)|^p \mu(dx) \mu(dy) 
\]
\[
\leq \|\varphi \otimes 1\|_{L^p(\mu \times \mu)} 
\]
\[
= \|\varphi\|_p^n 
\]

and this completes the proof. \(\Box\)
Corollary 3.2.4 We have
\[ \| (I + \mathcal{L})^{-1/2} \delta \xi \|_p \leq c_p \| \xi \|_p, \]
for any \( \xi \in L^p(\mu; H) \) and for any \( p \in (1, \infty) \).

**Proof:** It suffices to take the adjoint of \( \nabla (I + \mathcal{L})^{-1/2} \).

**Corollary 3.2.5** The following identities are valid for any \( \varphi \in \mathbb{D} \):

1. \( \| \nabla \varphi \|_p \leq c_p \| (I + \mathcal{L})^{1/2} \varphi \|_p \)
2. \( \| (I + \mathcal{L})^{1/2} \varphi \|_p \leq \tilde{c}_p (\| \varphi \|_p + \| \nabla \varphi \|_p) \)

where \( c_p \) and \( \tilde{c}_p \) are two constants independent of \( \varphi \).

**Proof:** The first identity follows easily as
\[ \| \nabla \varphi \|_p = \| \nabla (I + \mathcal{L})^{-1/2}(I + \mathcal{L})^{1/2} \varphi \|_p \leq c_p \| (I + \mathcal{L})^{1/2} \varphi \|_p. \]

To prove the second we have
\[
\| (I + \mathcal{L})^{1/2} \varphi \|_p = \| (I + \mathcal{L})^{-1/2}(I + \mathcal{L})\varphi \|_p \\
= \| (I + \mathcal{L})^{-1/2}(I + \delta \nabla) \varphi \|_p \\
\leq \| (I + \mathcal{L})^{-1/2} \varphi \|_p + \| (I + \mathcal{L})^{-1/2} \delta \nabla \varphi \|_p \\
\leq \| \varphi \|_p + c_p \| \nabla \varphi \|_p,
\]
where the last inequality follows from Corollary 3.2.4.

**Notes and suggested reading**

The inequalities studied in this chapter are due to P. A. Meyer in his seminal paper \[62\]. He discusses at the last part of it already about the space of test functions defined by the Ornstein-Uhlenbeck operator and proves that this space is an algebra. Then the classical duality results give birth immediately to the space of the distributions on the Wiener space, and this is done in \[102\]. Later the proof of P. A. Meyer has been simplified by several people. Here we have followed an idea of D. Feyel, cf. \[28\].
Meyer Inequalities
Chapter 4

Hypercontractivity

Introduction

We know that the semi-group of Ornstein-Uhlenbeck is a bounded operator on $L^p(\mu)$, for any $p \in [1, \infty]$. In fact for $p \in (1, \infty)$, it is more than bounded. It increases the degree of integrability, this property is called hypercontractivity and it is used to show the continuity of linear operators on $L^p(\mu)$-spaces defined via the Wiener chaos decomposition or the spectral decomposition of the Ornstein-Uhlenbeck operator. We shall use it in the next chapter to complete the proof of the Meyer inequalities. Hypercontractivity has been first discovered by E. Nelson, here we follow the proof given by [66]. We complete the chapter by an analytic proof of the logarithmic Sobolev inequality of Leonard Gross (cf. [36], [22]) for which we shall give another proof in the fifth chapter.

4.1 Hypercontractivity

In the sequel we shall show that this result can be proved using the Ito formula. Let $(\Omega, A, P)$ be a probability space with $(B_t; t \in \mathbb{R}_+)$ being a filtration. We take two Brownian motions $(X_t; t \geq 0)$ and $(Y_t; t \geq 0)$ which are not necessarily independent, i.e., $X$ and $Y$ are two continuous, real martingales such that $(X_t^2 - t)$ and $(Y_t^2 - t)$ are again martingales (with respect to $(B_t)$) and that $X_t - X_s$ and $Y_t - Y_s$ are independent of $B_s$, for $t > s$. Moreover there exists $(\rho_t; t \in \mathbb{R}_+)$, progressively measurable with values in $[-1, 1]$ such that

$$(X_t Y_t - \int_0^t \rho_s ds, t \geq 0)$$
Hypercontractivity

is again a \((\mathcal{B}_t)\)-martingale. Let us denote by

\[\Xi_t = \sigma(X_s; s \leq t), \quad \mathcal{Y}_t = \sigma(Y_s; s \leq t)\]

i.e., the corresponding filtrations of \(X\) and \(Y\) and by \(\Xi\) and by \(\mathcal{Y}\) their respective supremum.

**Lemma 4.1.1**

1. For any \(\varphi \in L^1(\Omega, \Xi, P), t \geq 0\), we have

\[E[\varphi | \mathcal{B}_t] = E[\varphi | \Xi_t] \text{ a.s.}\]

2. For any \(\psi \in L^1(\Omega, \mathcal{Y}, P), t \geq 0\), we have

\[E[\psi | \mathcal{B}_t] = E[\psi | \mathcal{Y}_t] \text{ a.s.}\]

**Proof:** Since the two claims are similar, we shall prove only the first one. From Paul Lévy’s theorem, we have also that \((X_t)\) is an \((\Xi_t)\)-Brownian motion. Hence

\[\varphi = E[\varphi] + \int_0^\infty H_s dX_s\]

where \(H\) is \((\Xi_t)\)-adapted process. Hence

\[E[\varphi | \mathcal{B}_t] = E[\varphi] + \int_0^t H_s dX_s = E[\varphi | \Xi_t].\]

Let \(T\) be the operator \(T : L^1(\Omega, \Xi, P) \to L^1(\Omega, \mathcal{Y}, P)\) defined as the restriction of \(E[\cdot | \mathcal{Y}]\) to the space \(L^1(\Omega, \Xi, P)\). We know that \(T : L^p(\Xi) \to L^p(\mathcal{Y})\) is a contraction for any \(p \geq 1\). If we impose supplementary conditions to \(\rho\), then we have more:

**Proposition 4.1.2** If 
\[|\rho_t(w)| \leq r (dt \times dP \text{ a.s.}) \text{ for some } r \in [0, 1],\]

then \(T : L^p(\Xi) \to L^q(\mathcal{Y})\) is a bounded operator, where

\[p - 1 \geq r^2(q - 1).\]

**Proof:** \(p = 1\) is already known. So suppose \(p, q \in ]1, \infty[\). Since \(L^\infty(\Xi)\) is dense in \(L^p(\Xi)\), it is enough to prove that \(\|TF\|_q \leq \|F\|_p\) for any \(F \in L^\infty(\Xi)\). Moreover, since \(T\) is a positive operator, we have \(|T(F)| \leq T(|F|)\), hence we can work as well with \(F \in L^\infty(\Xi)\). Due to the duality between \(L^p\)-spaces, it suffices to show that

\[E[T(F)G] \leq \|F\|_p \|G\|_{q'}, \quad \left(\frac{1}{q'} + \frac{1}{q} = 1\right),\]
for any $F \in L^\infty_+(\Xi)$, $G \in L^\infty_+(\mathcal{Y})$. Since bounded and positive random variables are dense in all $L^p_+$ for any $p > 1$, we can suppose without loss of generality that $F, G \in [a, b]$ almost surely for some $0 < a < b < \infty$. Let

$$M_t = E[F^{|t|}]$$
$$N_t = E[G^{|t|}] .$$

Then, from the Ito representation theorem we have

$$M_t = M_0 + \int_0^t \phi_s dX_s$$
$$N_t = N_0 + \int_0^t \psi_s dY_s$$

where $\phi$ is $\Xi$-adapted, $\psi$ is $\mathcal{Y}$-adapted, $M_0 = E[F^{|t|}]$, $N_0 = E[G^{|t|}]$. From the Ito formula, we have

$$M_t^\alpha N_t^\beta = M_0^\alpha N_0^\beta + \int_0^t \alpha M_s^{\alpha-1} N_s^\beta dM_s + \beta \int_0^t M_s^{\alpha-1} N_s^{\beta-1} dN_s + \frac{1}{2} \int_0^t M_s^{\alpha-2} N_s^\beta A_s ds$$

where

$$A_t = \alpha(\alpha - 1) \left( \frac{\phi_t}{M_t} \right)^2 + 2\alpha \beta \frac{\phi_t \psi_t}{M_t N_t} \rho_t + \beta(\beta - 1) \left( \frac{\psi_t}{N_t} \right)^2$$

and $\alpha = \frac{1}{p}$, $\beta = \frac{1}{q'}$. To see this it suffices to use the Ito formula as

$$M_t^\alpha = M_0^\alpha + \alpha \int_0^t M_s^{\alpha-1} \phi_s dX_s + \frac{\alpha(\alpha - 1)}{2} \int_0^t M_s^{\alpha-2} \phi_s^2 ds$$
$$N_t^\beta = \ldots$$

and then as

$$M_t^\alpha N_t^\beta - M_0^\alpha N_0^\beta$$
$$= \int_0^t M_s^\alpha dN_s^\beta + \int_0^t N_s^\beta dM_s^\alpha + \alpha \beta \int_0^t M_s^{\alpha-1} N_s^{\beta-1} \phi_s \psi_s \rho_s ds$$
$$= \int_0^t M_s^\alpha \left( \beta N_s^{\beta-1} \psi_s dY_s + \frac{\beta(\beta - 1)}{2} N_s^{\beta-2} \psi_s^2 ds \right)$$
$$+ \int_0^t N_s^\beta \left( \alpha M_s^{\alpha-1} \phi_s dX_s + \frac{\alpha(\alpha - 1)}{2} M_s^{\alpha-2} \phi_s^2 ds \right)$$
$$+ \alpha \beta \int_0^t M_s^{\alpha-1} N_s^{\beta-1} \phi_s \psi_s \rho_s ds .$$
and finally to pick up together all the integrands integrated with respect to
the Lebesgue measure $ds$.
As everything is square integrable, it comes
\[
E[M_\infty^\alpha N_\infty^\beta] = E[E[F^\alpha|\Xi_\infty]^\alpha \cdot E[G^{q'}|\mathcal{Y}_\infty]^\beta]
\]
\[
= E[F \cdot G]
\]
\[
= \frac{1}{2} \int_0^\infty E[N_t^\beta M_t^\alpha A_t]dt + EM_0^\alpha N_0^\beta
\]
\[
= E[F^\alpha E[G^{q'}]^\beta + \frac{1}{2} \int_0^\infty E[M_t^\alpha N_t^\beta A_t]dt .
\]
Consequently
\[
E[FG] - \|X\|_p \|Y\|_{q'} = \frac{1}{2} \int_0^\infty E[M_t^\alpha N_t^\beta A_t] dt .
\]
Look now at $A_t$ as a polynomial of second degree with respect to $\frac{\rho}{\alpha}$. Then
\[
\frac{\delta}{4} = \alpha^2 \beta^2 \rho_t^2 - \alpha(\alpha - 1)\beta(\beta - 1).
\]
If $\delta \leq 0$ then the sign of $A_t$ is the same as the sign of $\alpha(\alpha - 1) \leq 0$, i.e., if
\[
\rho_t^2 \leq \frac{(\alpha - 1)(\beta - 1)}{\alpha \beta} = \left(1 - \frac{1}{\alpha}\right)\left(1 - \frac{1}{\beta}\right) = (p - 1)(q' - 1)
\]
a.s., then we obtain
\[
E[FG] = E[T(F)G] \leq \|F\|_p \|G\|_{q'}
\]
which achieves the proof. □

**Lemma 4.1.3** Let $(w, z) = W \times W$ be independent Brownian paths. For
$\rho \in [0, 1]$, define $x = \rho w + \sqrt{1 - \rho^2} z$, $\Xi_\infty$ the $\sigma$-algebra associated to the
paths $x$. Then we have
\[
E[F(w)|\Xi_\infty] = \int_W F\left(\rho x + \sqrt{1 - \rho^2} z\right) \mu(dz).
\]
**Proof:** For any $G \in L^\infty(\Xi_\infty)$, we have
\[
E[F(w) \cdot G(x)] = E\left[E[F(w)G\left(\rho w + \sqrt{1 - \rho^2} z\right)]\right]
\]
\[
= E\left[F\left(\rho w + \sqrt{1 - \rho^2} z\right) G(w)\right]
\]
\[
= \int \int F\left(\rho \bar{w} + \sqrt{1 - \rho^2} \bar{z}\right) G(\bar{w}) \cdot \mu(d\bar{w}) \mu(d\bar{z})
\]
\[
= E\left[G(x) \int F\left(\rho x + \sqrt{1 - \rho^2} z\right) \cdot \mu(dz)\right]
\]
where \( \bar{w}, \bar{z} \) represent the dummy variables of integration.

**Corollary 4.1.4** Under the hypothesis of the above lemma, we have

\[
\left\| \int_W F \left( \rho x + \sqrt{1 - \rho^2 \bar{z}} \right) \mu(d\bar{z}) \right\|_{L^q(\mu)} \leq \| F \|_{L^p(\mu)}
\]

for any

\[ (p - 1) \geq \rho^2(q - 1). \]

### 4.2 Logarithmic Sobolev Inequality

Let \((P_t, t \geq 0)\) be the Ornstein-Uhlenbeck semigroup. The commutation relation (cf. 2.7.9)

\[ \nabla P_t f = e^{-t} P_t \nabla f \]

is directly related to the logarithmic Sobolev inequality of L. Gross:

\[
E \left[ f^2 \log f^2 \right] - E[f^2] \log E[f^2] \leq 2E \left[ |\nabla f|^2_H \right].
\]

In fact, suppose that \( f \) is strictly positive and lower and upper bounded. We have

\[
E[f \log f] - E[f] \log E[f] = -\int_0^\infty E \left[ \frac{d}{dt} P_t f \log P_t f \right] dt
\]

\[ = \int_0^\infty E \left[ \mathcal{L} P_t f \log P_t f \right] dt
\]

\[ = \int_0^\infty E \left[ \frac{|\nabla P_t f|^2}{P_t f} \right] dt
\]

\[ = \int_0^\infty e^{-2t} E \left[ \frac{|P_t \nabla f|^2}{P_t f} \right] dt. \quad (4.2.1)
\]

Now insert in (4.2.1) the following,

\[
|P_t(\nabla f)|^2_H = \left| P_t \left( f^{1/2} \frac{\nabla f}{f^{1/2}} \right) \right|^2_H
\]

\[ \leq (P_t f) P_t \left( \frac{|\nabla f|^2_H}{f} \right)
\]
which is a consequence of the Hölder inequality, to obtain

\[
E[f \log f] - E[f] \log E[f] \leq \int_0^\infty e^{-2t} E\left[P_t \left( \frac{\left| \nabla f \right|^2}{f} \right) \right] dt
\]

\[
= \int_0^\infty e^{-2t} 4E\left[|\nabla \sqrt{f}|^2_H\right] dt
\]

\[
= 2E[\left| \nabla \sqrt{f} \right|^2_H],
\]

replacing \( f \) by \( f^2 \) completes the proof of the inequality.

**Remark 4.2.1** Here we have used the fact that if \( f > 0 \) almost surely, then \( P_t f > 0 \) also. In fact one can prove, using the Cameron Martin theorem, that, if \( \mu \{ g > 0 \} > 0 \), then \( P_t g > 0 \) almost surely. \( P_t \) is called a positivity improving semi-group (cf. Corollary 6.1.7).

**Notes and suggested reading**

The hypercontractivity property of the Ornstein-Uhlenbeck semigroup is due to E. Nelson. The proof given here follows the lines given by J. Neveu, cf. [66]. For the logarithmic Sobolev inequality and its relations to hypercontractivity cf. [22].

AND DISTRIBUTIONS
Chapter 5

$L^p$-Multipliers Theorem, Meyer Inequalities and Distributions

5.1 $L^p$-Multipliers Theorem

$L^p$-Multipliers Theorem gives us a tool to perform some sort of symbolic calculus to study the continuity of the operators defined via the Wiener chaos decomposition of the Wiener functionals. With the help of this calculus we will complete the proof of the Meyer’s inequalities.

Almost all of these results have been discovered by P. A. Meyer (cf. [62]) and they are consequences of the Nelson’s hypercontractivity theorem ([65]).

First let us give first the following simple and important result:

**Theorem 5.1.1** Let $F \in L^p(\mu)$, $p > 1$, denote by $I_n(F_n)$ the projection of $F$ on the $n$-th Wiener chaos, $n \geq 1$. Then the map $F \rightarrow I_n(F_n)$ is continuous on $L^p(\mu)$.

**Proof:** Suppose first $p > 2$. Let $t$ be such that $p = e^{2t} + 1$, then we have

$$\|P_tF\|_p \leq \|F\|_2.$$ 

Moreover

$$\|P_tF\|_p \leq \|I_n(F_n)\|_2 \leq \|F\|_2 \leq \|F\|_p$$

but $P_tF(I_n(F_n)) = e^{-nt}I_n(F_n)$, hence

$$\|I_n(F_n)\|_p \leq e^{nt}\|F\|_p.$$ 

For $1 < p < 2$ we use the duality: let $F \rightarrow I_n(F_n) = J_n(F)$. Then

$$\|I_n(F)\|_p = \sup_{\|G\|_q \leq 1} |\langle G, J_n(F) \rangle|$$
\[ = \sup |\langle J_n(G), F \rangle| \]
\[ = \sup |\langle J_n G, J_n F \rangle| \]
\[ \leq \sup e^{nt} \|G\|_q \|F\|_p \]
\[ = e^{nt} \|F\|_p. \]

**Proposition 5.1.2 (Meyer’s Multipliers theorem)** Let the function \( h \) be defined as
\[ h(x) = \sum_{k=0}^{\infty} a_k x^k \]
be analytic around the origin with
\[ \sum_{k=1}^{\infty} |a_k| \left( \frac{1}{|x|^n} \right)^k < \infty \]
for \( n \geq n_0 \), for some \( n_0 \in \mathbb{N} \). Let \( \phi(x) = h(x^{-\alpha}) \) and define \( T_\phi \) on \( L^p(\mu) \) as
\[ T_\phi F = \sum_{n=0}^{\infty} \phi(n) I_n(F_n). \]
Then the operator to \( T_\phi \) is bounded on \( L^p(\mu) \) for any \( p > 1 \).

**Proof:** Suppose first \( \alpha = 1 \). Let \( T_\phi = T_1 + T_2 \) where
\[ T_1 F = \sum_{n=0}^{n_0-1} \phi(n) I_n(F_n), \quad T_2 F = (I - T_1) F. \]
From the hypercontractivity, \( F \mapsto T_1 F \) is continuous on \( L^p(\mu) \). Let
\[ \Delta_{n_0} F = \sum_{n=n_0}^{\infty} I_n(F_n). \]
Since
\[ (I - \Delta_{n_0})(F) = \sum_{n=n_0}^{n_0-1} I_n(F_n), \]
\( \Delta_{n_0} : L^p \to L^p \) is continuous, hence \( P_t \Delta_{n_0} : L^p \to L^p \) is also continuous. Applying Riesz-Thorin interpolation theorem, which says that if \( P_t \Delta_{n_0} \) is \( L^q \to L^q \) and \( L^2 \to L^2 \) then it is \( L^p \to L^p \) for any \( p \) such that \( \frac{1}{p} \) is in the interval \( \left[ \frac{1}{q}, \frac{1}{2} \right] \), we obtain
\[ \|P_t \Delta_{n_0}\|_{p,p} \leq \|P_t \Delta_{n_0}\|_{2,2}^{\frac{\theta}{q}} \|P_t \Delta_{n_0}\|_{q,q}^{1-\frac{\theta}{q}} \leq \|P_t \Delta_{n_0}\|_{2,2}^{\frac{\theta}{q}} \|\Delta_{n_0}\|_{q,q}^{1-\frac{\theta}{q}}. \]
where $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q}$, $\theta \in (0, 1)$. Choose $q$ large enough such that $\theta \approx 1$ (if necessary). Hence we have

$$\|P_t \Delta_{n_0}\|_{p,p} \leq e^{-n_0 t \theta} K, \quad K = K(n_0, \theta).$$

A similar argument holds for $p \in (1, 2)$ by duality.

We then have

$$T_2(F) = \sum_{n \geq n_0} \phi(n) I_n(F_n)$$

$$= \sum_{n \geq n_0} \left( \sum_k a_k \left( \frac{1}{n} \right)^k \right) I_n(F_n)$$

$$= \sum_k a_k \sum_{n \geq n_0} \left( \frac{1}{n} \right)^k I_n(F_n)$$

$$= \sum_k a_k \mathcal{L}^{-k} I_n(F_n)$$

$$= \sum_k a_k \mathcal{L}^{-k} \Delta_{n_0} F.$$

We also have

$$\|\mathcal{L}^{-1} \Delta_{n_0} F\|_p = \left\| \int_0^\infty P_t \Delta_{n_0} F dt \right\|_p \leq K \int_0^\infty e^{-n_0 t \theta} \|F\|_p dt \leq K \cdot \frac{\|F\|_p}{n_0 \theta},$$

$$\|\mathcal{L}^{-2} \Delta_{n_0} F\|_p = \left\| \int_0^\infty \int_0^\infty P_{t+s} \Delta_{n_0} F ds dt \right\|_p \leq K \cdot \frac{\|F\|_p}{(n_0 \theta)^2},$$

$$\|\mathcal{L}^{-k} \Delta_{n_0} F\|_p \leq K \cdot \frac{1}{(n_0 \theta)^k} \cdot \frac{\|F\|_p}{(n_0 \theta)^k}.$$

Therefore

$$\|T_2(F)\|_p \leq \sum_k K \cdot \frac{1}{n_0 \theta} \cdot \frac{\|F\|_p}{n_0 \theta} \approx \sum K \cdot \frac{1}{n_0 \theta} \cdot \frac{\|F\|_p}{n_0 \theta}$$

by the hypothesis (take $n_0 + 1$ instead of $n_0$ if necessary).

For the case $\alpha \in (0, 1)$, let $\theta_t^{(\alpha)}(ds)$ be the measure on $\mathbb{R}_+$, defined by

$$\int_{\mathbb{R}_+} e^{-s \theta_t^{(\alpha)}}(ds) = e^{-t \lambda^\alpha}.$$

Define

$$Q^\alpha_t F = \sum_n e^{-n^\alpha} I_n(F_n) = \int_0^\infty P_s F \theta_t^{(\alpha)}(ds).$$

Then

$$\|Q^\alpha_t \Delta_{n_0} F\|_p \leq \|F\|_p \int_0^\infty e^{-n_0 s \theta_t^{(\alpha)}}(ds)$$

$$= \|F\|_p e^{-t(n_0 \theta)^\alpha}.$$. 
the rest of the proof goes as in the case $\alpha = 1$. \qed

**Examples:**

1) Let
\[
\phi(n) = \left( \frac{1 + \sqrt{n}}{\sqrt{1 + n}} \right)^s, \quad s \in (-\infty, \infty)
\]
\[
= h\left( \frac{1}{\sqrt{n}} \right), \quad h(x) = \left( \frac{1 + x}{\sqrt{1 + x^2}} \right)^s.
\]
Then $T_\phi : L^p \to L^p$ is bounded. Moreover $\phi^{-1}(n) = \frac{1}{\phi(n)} = h^{-1}\left( \frac{1}{\sqrt{n}} \right)$, $h^{-1}(x) = \frac{1}{h(x)}$ is also analytic near the origin, hence $T_{\phi^{-1}} : L^p \to L^p$ is also a bounded operator.

2) Let $\phi(n) = \frac{1 + \sqrt{n}}{\sqrt{2 + n}}$ then $h(x) = \sqrt{\frac{x}{2x+1}}$ satisfies also the above hypothesis.

3) As an application of (2), look at
\[
\|(I + \mathcal{L})^{1/2} \nabla \varphi\|_p = \|\nabla (2I + \mathcal{L})^{1/2} \varphi\|_p \\
\leq \|(I + \mathcal{L})^{1/2} (2I + \mathcal{L})^{1/2} \varphi\|_p \\
= \|(2I + \mathcal{L})^{1/2} (I + \mathcal{L})^{1/2} \varphi\|_p \\
= \|T_{\varphi}(I + \mathcal{L})^{1/2} (I + \mathcal{L})^{1/2} \varphi\|_p \\
\leq c_p \|(I + \mathcal{L}) \varphi\|_p.
\]
Continuing this way we can show that
\[
\|\nabla^k \varphi\|_{L^p(\mu, H^{\otimes k})} \leq c_{p,k} \|\varphi\|_{p,k} = \|(I + \mathcal{L})^{k/2} \varphi\|_p \\
\leq \tilde{c}_{p,k} (\|\varphi\|_p + \|\nabla^k \varphi\|_{L^p(\mu, H^{\otimes k})})
\]
and this completes the proof of the Meyer inequalities for the scalar-valued Wiener functionals. If $\Xi$ is a separable Hilbert space, we denote with $\mathbb{D}_{p,k}(\Xi)$ the completion of the $\Xi$-valued polynomials with respect to the norm
\[
\|\alpha\|_{\mathbb{D}_{p,k}(\Xi)} = \|(I + \mathcal{L})^{k/2}\|_{L^p(\mu, \Xi)}.
\]
We define as in the case $\Xi = \mathbb{R}$, the Sobolev derivative $\nabla$, the divergence $\delta$, etc. All we have said for the real case extend trivially to the vector case, including the Meyer inequalities. In fact, in the proof of these inequalities the main step is the Riesz inequality for the Hilbert transform. However this
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inequality is also true for any Hilbert space (in fact it holds also for a class of Banach spaces which contains Hilbert spaces, called UMD spaces). The rest is almost the transcription of the real case combined with the Khintchine inequalities. We leave hence this passage to the reader.

**Corollary 5.1.3** For every $p > 1$, $k \in \mathbb{R}$, $\nabla$ has a continuous extension as a map $\mathbb{D}_{p,k} \rightarrow \mathbb{D}_{p,k-1}(H)$.

**Proof:** We have

\[
\|\nabla \varphi\|_{p,k} = \|(I + \mathcal{L})^{k/2} \nabla \varphi\|_p \\
= \|\nabla (2I + \mathcal{L})^{k/2} \varphi\|_p \\
\leq c_p \|(1 + \mathcal{L})^{1/2} (2I + \mathcal{L})^{k/2} \varphi\|_p \\
\approx \|(I + \mathcal{L})^{(k+1)/2} \varphi\|_p \\
= \|\varphi\|_{p,k+1}.
\]

**Corollary 5.1.4** $\delta = \nabla^* : \mathbb{D}_{p,k}(H) \rightarrow \mathbb{D}_{p,k-1} \text{ is continuous for all } p > 1 \text{ and } k \in \mathbb{R}.$

**Proof:** The proof follows from the duality.

In particular we have:

**Corollary 5.1.5** The Sobolev derivative and its adjoint extend to distribution spaces as explained below:

- **Sobolev derivative operates as a continuous operator on**

\[
\nabla : \mathbb{D} = \bigcap_{p,k} \mathbb{D}_{p,k} \rightarrow D(H) = \bigcap_{p,k} \mathbb{D}_{p,k}(H)
\]

and it extends continuously as a map

\[
\nabla : \mathbb{D}' = \bigcup_{p,k} \mathbb{D}_{p,k} \rightarrow \mathbb{D}'(H) = \bigcup_{p,k} \mathbb{D}_{p,k}(H).
\]

The elements of the space $\mathbb{D}'$ are called Meyer-Watanabe distributions.
Consequently its adjoint has similar properties:

\[ \delta : \bigcap_{p,k} \mathcal{D}_{p,k}(H) = \mathcal{D}(H) \to \mathcal{D} \]

is continuous and this map has a continuous extension to

\[ \delta : \mathcal{D}'(H) \to \mathcal{D}' \]

**Proof:** Everything follows from the dualities

\[ (\mathcal{D})' = \mathcal{D}', (\mathcal{D}(H))' = \mathcal{D}'(H). \]

\[ \square \]

**Definition 5.1.6** For \( n \geq 1 \), we define \( \delta^n \) as \( (\nabla^n)^* \) with respect to \( \mu \).

Here is the generalization of Corollary 2.3.5 promised in Chapter 2:

**Proposition 5.1.7** For \( \varphi \in L^2(\mu) \), we have

\[ \varphi = E[\varphi] + \sum_{n \geq 1} \frac{1}{n!} \delta^n(E[\nabla^n \varphi]). \]  

(5.1.1)

**Proof:** If \( f \) is a symmetric element of \( H^{\otimes n} \), we shall denote by \( \tilde{I}_n(f) \) the \( n \)-th order multiple Ito-Wiener integral of the density of \( f \) with respect to the Lebesgue measure on \([0,1]^n\) (cf. also Corollary 2.3.5). With this notational convention, suppose that \( h \mapsto \varphi(w+h) \) is analytic for almost all \( w \). Then we have

\[ \varphi(w+h) = \varphi(w) + \sum_{n \geq 1} \frac{\nabla^n \varphi(w), h^{\otimes n}}{n!} \cdot H^{\otimes n} \]

Take the expectations:

\[ E[\varphi(w+h)] = E[\varphi \mathcal{E}(\delta h)] \]

\[ = E[\varphi] + \sum_n \frac{E[\nabla^n \varphi, h^{\otimes n}]}{n!} \]

\[ = E[\varphi] + \sum_{n \geq 1} E \left[ \tilde{I}_n(E[\nabla^n \varphi]) \mathcal{E}(\delta h) \right]. \]

Since the finite linear combinations of the elements of the set \( \{ \mathcal{E}(\delta h); h \in H \} \) is dense in any \( L^p(\mu) \), we obtain the identity

\[ \varphi(w) = E[\varphi] + \sum_{n \geq 1} \frac{\tilde{I}_n(E[\nabla^n \varphi])}{n!}. \]
Let $\psi \in \mathcal{D}$, then we have (with $E[\psi] = 0$),

$$
\langle \varphi, \psi \rangle = \sum_{n \geq 1} E [\hat{I}_n (\varphi_n) \hat{I}_n (\psi_n)]
$$

$$
= \sum_n E \left[ \frac{\hat{I}_n (E [\nabla^n \varphi])}{n!} : \hat{I}_n (\psi_n) \right]
$$

$$
= \sum_n (E [\nabla^n \varphi], \psi_n)
$$

$$
= \sum_n \frac{1}{n!} (E [\nabla^n \varphi], E [\nabla^n \psi])
$$

$$
= \sum_n \frac{1}{n!} E[ (E [\nabla^n \varphi], \nabla^n \psi) ]
$$

$$
= \sum_n \frac{1}{n!} E [\delta^n (E [\nabla^n \varphi]) \cdot \psi]
$$

hence we obtain that

$$
\varphi = \sum_n \frac{1}{n!} \delta^n E [\nabla^n \varphi].
$$

In particular it holds true that

$$
\delta^n \{ E [\nabla^n \varphi] \} = \hat{I}_n (E [\nabla^n \varphi]).
$$

Evidently this identity is valid not only for the expectation of the n-th derivative of a Wiener functional but for any symmetric element of $H^{\otimes n}$.

Remark 5.1.8 Although in the literature Proposition 5.1.7 is announced for the elements of $\mathcal{D}$, the proof given here shows its validity for the elements of $L^2(\mu)$. In fact, although $\nabla^n \varphi$ is a distribution, its expectation is an ordinary symmetric tensor of order $n$, hence the corresponding multiple Wiener integrals are well-defined. With a small extra work, we can show that in fact the formula (5.1.1) holds for any $\phi \in \cup_{k \in \mathbb{Z}} \mathcal{D}_{2,k}$.

Let us give another result important for the applications:

Proposition 5.1.9 Let $F$ be in some $L^p(\mu)$ with $p > 1$ and suppose that the distributional derivative $\nabla F$ of $F$, is in some $L^r(\mu, H)$, ($1 < r$). Then $F$ belongs to $\mathcal{D}_{r \wedge p, 1}$.

Proof: Without loss of generality, we can assume that $r \leq p$. Let $(e_i; i \in \mathbb{N})$ be a complete, orthonormal basis of the Cameron-Martin space $H$. Denote by $V_n$ the sigma-field generated by $\delta e_1, \ldots, \delta e_n$, and by $\pi_n$ the orthogonal
projection of $H$ onto the subspace spanned by $e_1, \ldots, e_n$, $n \in \mathbb{N}$. Let us define $F_n$ by

$$F_n = E[P_{1/n} F | V_n],$$

where $P_{1/n}$ is the Ornstein-Uhlenbeck semi-group at $t = 1/n$. Then $F_n$ belongs to $\mathbb{D}_{r,k}$ for any $k \in \mathbb{N}$ and converges to $F$ in $L^r(\mu)$. Moreover, from Doob’s lemma, $F_n$ is of the form

$$F_n(w) = \alpha(\delta e_1, \ldots, \delta e_n),$$

with $\alpha$ being a Borel function on $\mathbb{R}^n$, which is in the intersection of the Sobolev spaces $\cap_k W_{r,k}(\mathbb{R}^n, \mu_n)$ defined with the Ornstein-Uhlenbeck operator $L_n = -\Delta + x \cdot \nabla$ on $\mathbb{R}^n$. Since $L_n$ is elliptic, the Weyl lemma implies that $\alpha$ can be chosen as a $C^\infty$-function. Consequently, $\nabla F_n$ is again $V_n$-measurable and we find, using the very definition of conditional expectation and the Mehler formula, that

$$\nabla F_n = E[e^{-1/n} \pi_n P_{1/n} \nabla F | V_n].$$

Consequently, from the martingale convergence theorem and from the fact that $\pi_n \to I_H$ in the weak operators topology, it follows that

$$\nabla F_n \to \nabla F,$$

in $L^r(\mu, H)$, consequently $F$ belongs to $\mathbb{D}_{r,1}$. 

Appendix: Passing from the classical Wiener space to the Abstract Wiener Space (or vice-versa):

Let $(W, H, \mu)$ be an abstract Wiener space. Since, à priori, there is no notion of time, it seems that we can not define the notion of anticipation, non-anticipation, etc. This difficulty can be overcome in the following way:

Let $(p_\lambda; \lambda \in \Sigma), \Sigma \subset \mathbb{R}$, be a resolution of identity on the separable Hilbert space $H$, i.e., each $p_\lambda$ is an orthogonal projection, increasing to $I_H$, in the sense that $\lambda \mapsto (p_\lambda h, h)$ is an increasing function. Let us denote by $H_\lambda = \overline{p_\lambda(H)}$, where $p_\lambda(H)$ denotes the closure of $p_\lambda(H)$ in $H$.

**Definition 5.1.10** We will denote by $\mathcal{F}_\lambda$ the $\sigma$-algebra generated by the real polynomials $\varphi$ on $W$ such that $\nabla \varphi \in H_\lambda \mu$-almost surely.
Lemma 5.1.11 We have
\[ \bigvee_{\lambda \in \Sigma} F^\lambda = \mathcal{B}(W) \]
up to \( \mu \)-negligible sets.

Proof: We have already \( \bigvee F^\lambda \subset \mathcal{B}(W) \). Conversely, if \( h \in H \), then \( \nabla \delta h = h \). Since \( \bigcup_{\lambda \in \Sigma} H^\lambda \) is dense in \( H \), there exists \( (h_n) \subset \bigcup_{\lambda \in \Sigma} H^\lambda \) such that \( h_n \to h \) in \( H \). Hence \( \delta h_n \to \delta h \) in \( L^p(\mu) \), for all \( p \geq 1 \). Since each \( \delta h_n \) is \( \bigvee F^\lambda \)-measurable, so does \( \delta h \). Since \( \mathcal{B}(W) \) is generated by \( \{ \delta h; h \in H \} \) the proof is completed.

Definition 5.1.12 A random variable \( \xi : W \to H \) is called a simple, adapted vector field if it can be written as a finite sum:
\[ \xi = \sum_{i < \infty} F_i (p_{\lambda_i+1} h_i - p_{\lambda_i} h_i) \]
where \( h_i \in H \), \( F_i \) are \( \mathcal{F}^\lambda_i \)-measurable (and smooth for the time being) random variables.

Proposition 5.1.13 For each adapted simple vector field we have

i) \( \delta \xi = \sum_{i < \infty} F_i \delta (p_{\lambda_i+1} h_i - p_{\lambda_i} h_i) \)

ii) with Itô's isometry:
\[ E \left[ |\delta \xi|^2 \right] = E \left[ |\xi|^2 \right] \]

Proof: The first part follows from the usual identity
\[ \delta [F_i (p_{\lambda_i+1} - p_{\lambda_i}) h_i] = F_i \delta [(p_{\lambda_i+1} - p_{\lambda_i}) h_i] - \left( \nabla F_i, (p_{\lambda_i+1} - p_{\lambda_i}) h_i \right)_H \]
and from the fact that the second term is null since \( \nabla F_i \in H^\lambda \) almost surely. The verification of the second relation is left to the reader.

Remark 5.1.14 If we denote \( \Sigma F_i 1_{[\lambda_i, \lambda_i+1]}(\lambda) h_i \) by \( \dot{\xi}(\lambda) \), we have the following relations:
\[ \delta \xi = \delta \int_{\Sigma} \dot{\xi}(\lambda) dp_{\lambda} \quad \text{with} \quad ||\delta \xi||^2_2 = E \int_{\Sigma} d(\dot{\xi}, p_{\lambda} \dot{\xi}_\lambda) = ||\xi||^2_{L^2(\mu; H)} \]
which are significantly analogous to the relations that we have seen before.
Multipliers and Inequalities

The Ito representation theorem can be stated in this setting as follows: suppose that \((p_\lambda; \lambda \in \Sigma)\) is weakly continuous. We mean by this that the function

\[ \lambda \mapsto (p_\lambda h, k)_H \]

is continuous for any \(h, k \in H\). Then

**Theorem 5.1.15** Let us denote with \(\mathbb{D}^{2,0}_2(H)\) the completion of adapted simple vector fields with respect to the \(L^2(\mu, H)\)-norm. Then we have

\[ L_2(\mu) = \mathbb{R} + \{\delta \xi : \xi \in \mathbb{D}^{2,0}_2(H)\}, \]

i.e., any \(\varphi \in L_2(\mu)\) can be written as

\[ \varphi = E[\varphi] + \delta \xi \]

for some \(\xi \in \mathbb{D}^{2,0}_2(H)\). Moreover such \(\xi\) is unique up to \(L^2(\mu, H)\)-equivalence classes.

The following result explains the reason of the existence of the Brownian motion (cf. also [90]):

**Theorem 5.1.16** Suppose that there exists some \(\Omega_0 \in H\) such that the set \(\{p_\lambda \Omega_0; \lambda \in \Sigma\}\) has a dense span in \(H\) (i.e. the linear combinations from it is a dense set). Then the real-valued \((\mathcal{F}_\lambda)\)-martingale defined by

\[ b_\lambda = \delta p_\lambda \Omega_0 \]

is a Brownian motion with a deterministic time change and \((\mathcal{F}_\lambda; \lambda \in \Sigma)\) is its canonical filtration completed with the negligible sets.

**Example:** Let \(H = H_1([0, 1])\), define \(A\) as the operator defined by \(Ah(t) = \int_0^t sh(s)ds\). Then \(A\) is a self-adjoint operator on \(H\) with a continuous spectrum which is equal to \([0, 1]\). Moreover we have

\[ (p_\lambda h)(t) = \int_0^t 1_{[0, \lambda]}(s) h(s)ds \]

and \(\Omega_0(t) = \int_0^t 1_{[0, 1]}(s)ds\) satisfies the hypothesis of the above theorem. \(\Omega_0\) is called the vacuum vector (in physics).

This is the main example, since all the (separable) Hilbert spaces are isomorphic, we can carry this time structure to any abstract Hilbert-Wiener space as long as we do not need any particular structure of time.
5.2 Exercises

1. Give a detailed proof of Corollary 5.1.4 in particular explain the why of the existence of continuous extensions of $\delta$ and $\nabla$.

2. Prove the last claim of Remark 5.1.8

Notes and suggested reading

To complete the series of the Meyer inequalities, we have been obliged to use the hypercontractivity property of the Ornstein-Uhlenbeck semigroup as done in [62]. Once this is done the extensions of $\nabla$ and $\delta$ to the distributions are immediate via the duality techniques. Proposition 5.1.7 is due to Stroock, [80] with a different proof. The results of the appendix are essentially due to the author, cf. [90]. In [101] a stochastic calculus is constructed in more detail.
Chapter 6

Some Applications

Introduction

In this chapter we give some applications of the extended versions of the derivative and the divergence operators. First we give an extension of the Ito-Clark formula to the space of the scalar distributions. We refer the reader to [11] and [70] for the developments of this formula in the case of Sobolev differentiable Wiener functionals. Let us briefly explain the problem: although, we know from the Ito representation theorem, that each square integrable Wiener functional can be represented as the stochastic integral of an adapted process, without the use of the distributions, we can not calculate this process, since any square integrable random variable is not necessarily in $D_{2,1}$, hence it is not Sobolev differentiable in the ordinary sense. As it will be explained, this problem is completely solved using the differentiation in the sense of distributions. Afterwards we give a straightforward application of this result to prove a $0 - 1$ law for the Wiener measure. At the second section we construct the composition of the tempered distributions with non-degenerate Wiener functionals as Meyer-Watanabe distributions. This construction carries also the information that the probability density of a non-degenerate random variable is not only infinitely differentiable but also it is rapidly decreasing. The same idea is then applied to prove the regularity of the solutions of the Zakai equation for the filtering of non-linear diffusions.
6.1 Extension of the Ito-Clark formula

Let $F$ be any integrable random variable. Then we the celebrated Ito Representation Theorem 1.6.1 tells us that $F$ can be represented as

$$F = E[F] + \int_0^1 H_s dW_s,$$

where $(H_s; s \in [0, 1])$ is an adapted process such that, it is unique and

$$\int_0^1 H_s^2 ds < +\infty \text{ a.s.}$$

Moreover, if $F \in L^p (p > 1)$, then we also have

$$E\left[\left( \int_0^1 |H_s|^2 ds \right)^{p/2}\right] < \infty.$$

One question is how to calculate the process $H$. In fact, below we will extend the Ito representation and answer to the above question for any $F \in \mathbb{D}'$ (i.e., the Meyer-Watanabe distributions). We begin with

**Lemma 6.1.1** Let $\xi \in \mathbb{D}(H)$ be represented as $\xi(t) = \int_0^t \dot{\xi}_s ds$, then $\pi\xi$ defined by

$$\pi\xi(t) = \int_0^t E[\dot{\xi}_s | \mathcal{F}_s] ds$$

belongs again to $\mathbb{D}(H)$. In other words $\pi : \mathbb{D}(H) \to \mathbb{D}(H)$ is a linear continuous operator.

**Proof:** Let $(P_t, t \in \mathbb{R}_+)$ be the Ornstein-Uhlenbeck semigroup. Then it is easy to see that, for any $\tau \in [0, 1]$, if $\phi \in L^1(\mu)$ is $\mathcal{B}_\tau$-measurable, then so is also $P_t\phi$ for any $t \in \mathbb{R}_+$. This implies in particular that $\mathcal{L}\pi\xi = \pi\mathcal{L}\xi$. Therefore

$$\|\pi\xi\|_{p,k} = E\left[\left( \int_0^1 |(I + \mathcal{L})^{k/2} E[\dot{\xi}_s | \mathcal{F}_s]|^2 ds \right)^{p/2}\right] =
= E\left[\left( \int_0^1 |E[(I + \mathcal{L})^{k/2}\dot{\xi}_s | \mathcal{F}_s]|^2 ds \right)^{p/2}\right] \leq c_p E\left[\left( \int_0^1 |(I + \mathcal{L}^{k/2}\dot{\xi}_s|^2 ds \right)^{p/2}\right] (c_p \cong p)$$

where the last inequality follows from the convexity inequalities of the dual predictable projections (c.f. [21]).

\[ \square \]
Lemma 6.1.2 $\pi : \mathbb{D}(H) \to \mathbb{D}(H)$ extends as a continuous mapping to $\mathbb{D}'(H) \to \mathbb{D}'(H)$.

**Proof:** Let $\xi \in \mathbb{D}(H)$, then we have, for $k > 0$,

$$
\|\pi \xi\|_{p,-k} = \|(I + \mathcal{L})^{-k/2} \pi \xi\|_p \\
= \|\pi (I + \mathcal{L})^{-k/2} \xi\|_p \leq c_p \|(I + \mathcal{L})^{-k/2} \xi\|_p \\
\leq c_p \|\xi\|_{p,-k},
$$

then the proof follows since $\mathbb{D}(H)$ is dense in $\mathbb{D}'(H)$.

Before going further let us give a notation: if $F$ is in some $\mathbb{D}_{p,1}$ then its Gross-Sobolev derivative $\nabla F$ is an $H$-valued random variable. Hence $t \mapsto \nabla F(t)$ is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. We shall denote by $D_s F$ its Radon-Nikodym derivative with respect to the Lebesgue measure. Note that $D_s F$ is $ds \times d\mu$-almost everywhere well-defined.

Lemma 6.1.3 Let $\varphi \in \mathbb{D}$, then we have

$$
\varphi = E[\varphi] + \int_0^1 E[D_s \varphi | \mathcal{F}_s] dW_s \\
= E[\varphi] + \delta \pi \nabla \varphi.
$$

Moreover $\pi \nabla \varphi \in \mathbb{D}(H)$.

**Proof:** Let $u$ be an element of $L^2(\mu, H)$ such that $u(t) = \int_0^t \dot{u}_s ds$ with $(\dot{u}_t; t \in [0,1])$ being an adapted and bounded process. Then we have, from the Girsanov theorem,

$$
E \left[ \varphi(w + \lambda u(w)). \exp \left\{ -\lambda \int_0^1 \dot{u}_s dW_s - \frac{\lambda^2}{2} \int_0^1 \dot{u}_s ds \right\} \right] = E[\varphi].
$$

Differentiating both sides at $\lambda = 0$, we obtain:

$$
E[(\nabla \varphi(w), u) - \varphi \int_0^1 \dot{u}_s dW_s] = 0,
$$

i.e.,

$$
E[(\nabla \varphi, u)] = E[\varphi \int_0^1 \dot{u}_s dW_s].
$$

Furthermore

$$
E \left[ \int_0^1 D_s \varphi \dot{u}_s ds \right] = E \left[ \int_0^1 E[D_s \varphi | \mathcal{F}_s] \dot{u}_s ds \right] \\
= E[(\pi \nabla \varphi, u)_H] \\
= E \left[ \left( \int_0^1 E[D_s \varphi | \mathcal{F}_s] dW_s \right) \left( \int_0^1 \dot{u}_s dW_s \right) \right].
$$
Since the set of the stochastic integrals \( \int_0^1 \dot{u}_s dW_s \) of the processes \( \dot{u} \) as above is dense in \( L_0^2(\mu) = \{ F \in L^2(\mu) : E[F] = 0 \} \), we see that
\[
\varphi - E[\varphi] = \int_0^1 E[D_s \varphi | F_s] dW_s = \delta \pi \nabla \varphi.
\]
The rest is obvious from the Lemma 6.1.1.

Lemma 6.1.3 extends to \( \mathbb{D}' \) as:

**Theorem 6.1.4** For any \( T \in \mathbb{D}' \), we have
\[
T = \langle T, 1 \rangle + \delta \pi \nabla T.
\]

**Proof:** Let \( (\varphi_n) \subset \mathbb{D} \) such that \( \varphi_n \to T \) in \( D' \). Then we have
\[
T = \lim_n \varphi_n = \lim_n \{ E[\varphi_n] + \delta \pi \nabla \varphi_n \}
= \lim_n E[\varphi_n] + \lim_n \delta \pi \nabla \varphi_n = \lim_n \langle 1, \varphi_n \rangle + \lim_n \delta \pi \nabla \varphi_n = \langle 1, T \rangle + \delta \pi \nabla T
\]
since \( \nabla : \mathbb{D}' \to \mathbb{D}'(H), \pi : \mathbb{D}'(H) \to \mathbb{D}'(H) \) and \( \delta : \mathbb{D}'(H) \to \mathbb{D}' \) are all linear, continuous mappings.

Here is a nontrivial application of the Theorem 6.1.4:

**Theorem 6.1.5 (0–1 law)**

Let \( A \in \mathcal{B}(W) \) such that \( \nabla_h 1_A = 0 \), \( h \in H \), where the derivative is in the sense of the distributions. Then \( \mu(A) = 0 \) or \( 1 \).

**Remark:** In particular, the above hypothesis is satisfied when \( A + H \subset A \).

**Proof:** Let \( T_A = 1_A \), then Theorem 6.1.4 implies that
\[
T_A = \langle T_A, 1 \rangle = \mu(A),
\]
hence \( \mu(A)^2 = \mu(A) \). Another proof can be given as follows: let \( T_t \) be defined as \( P_t 1_A \), where \( (P_t, t \geq 0) \) is the Ornstein-Uhlenbeck semigroup. Then, from the hypothesis, \( \nabla T_t = e^{-t} P_t \nabla 1_A = 0 \), consequently \( T_t \) is almost surely a constant for any \( t > 0 \), this implies that \( \lim_{t \to 0} T_t = 1_A \) is also a constant.
Remark 6.1.6 From Doob-Burkholder inequalities, it follows via a duality technique, that

\[
E \left[ \left( \int_0^1 (I + \mathcal{L})^{k/2} E[D_s \phi | \mathcal{F}_s]^2 ds \right)^{p/2} \right] \\
\leq c_p \|(I + \mathcal{L})^{k/2} \phi\|_{L^p(\mu)}^p \\
\leq c_{p,k} \|\phi\|_{p,k}^p,
\]

for any \( p > 1 \) and \( k \in \mathbb{R} \). Consequently, for any \( \phi \in \mathbb{D}_{p,k}, \pi \nabla \phi \in \mathbb{D}_{p,k}(H) \) and the Ito integral is an isomorphism from the adapted elements of \( \mathbb{D}_{p,k}(H) \) onto \( \mathbb{D}_{p,k}^o = \{ \phi \in \mathbb{D}_{p,k} : \langle \phi, 1 \rangle = 0 \} \) (cf. [85] for further details).

Corollary 6.1.7 (Positivity improving) Let \( F \in L^p(\mu), p > 1 \) be a non-negative Wiener functional such that \( \mu\{F > 0\} > 0 \), denote by \( (P_\tau, \tau \in \mathbb{R}^+) \) the Ornstein-Uhlenbeck semi-group. Then, for any \( t > 0 \), the set \( A_t = \{ w : P_t F(w) > 0 \} \) has full \( \mu \)-measure, in fact we have

\[ A_t + H \subset A_t. \]

Proof: From the Mehler and Cameron-Martin formulae, we have

\[
P_t F(w + h) = \int_W F(e^{-t}(w + h) + \sqrt{1 - e^{-2t}}y)\mu(dy) \\
= \int_W F(e^{-t}w + \sqrt{1 - e^{-2t}}y)\rho(\alpha_t h(y))\mu(dy)
\]

where

\[ \alpha_t = \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \]

and

\[ \rho(\delta h) = \exp \left\{ \delta h - 1/2|h|_H^2 \right\} . \]

This proves the claim about the \( H \)-invariance of \( A_t \) and the proof follows from Theorem 6.1.5.

6.2 Lifting of \( S'(\mathbb{R}^d) \) with random variables

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^1_b \)-function, \( F \in \mathbb{D} \). Then we know that

\[ \nabla(f(F)) = f'(F) \nabla F. \]
Now suppose that $|\nabla F|^2_H^{-\frac{1}{2}} \in \bigcap L^p(\mu)$, then

$$f'(F) = \frac{(\nabla(f(F)), \nabla F)_H}{|\nabla F|^2_H^{-\frac{1}{2}}}$$

Even if $f$ is not $C^1$, the right hand side of this equality has a sense if we look at $\nabla(f(F))$ as an element of $\mathbb{D}'$. In the following we will develop this idea:

**Definition 6.2.1** Let $F : W \to \mathbb{R}^d$ be a random variable such that $F_i \in \mathbb{ID}$, for all $i = 1, \ldots, d$, and that

$$[\det(\nabla F_i, \nabla F_j)]^{-1} \in \bigcap_{p > 1} L^p(\mu).$$

Then we say that $F$ is a **non-degenerate** random variable.

**Lemma 6.2.2** Let us denote by $\sigma_{ij} = (\nabla F_i, \nabla F_j)_H$ and by $\gamma = \sigma^{-1}$ (as a matrix). Then $\gamma \in \mathbb{D}(\mathbb{R}^d \otimes \mathbb{R}^d)$, in particular $\det \gamma \in \mathbb{ID}$.

**Proof:** Formally, we have, using the relation $\sigma \cdot \gamma = \text{Id}$,

$$\nabla \gamma_{ij} = \sum_{k,l} \gamma_{ik} \gamma_{jl} \nabla \sigma_{kl}.$$  

To justify this we define first $\sigma_{ij}^\varepsilon = \sigma_{ij} + \varepsilon \delta_{ij}$, $\varepsilon > 0$. Then we can write $\gamma_{ij}^\varepsilon = f_{ij}(\sigma^\varepsilon)$, where $f : \mathbb{R}^d \otimes \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a smooth function of polynomial growth. Hence $\gamma_{ij}^\varepsilon \in \mathbb{ID}$. Then from the dominated convergence theorem we have $\gamma_{ij}^\varepsilon \to \gamma_{ij}$ in $L^p$ and $\nabla \gamma_{ij}^\varepsilon \to \nabla \gamma_{ij}$ in $L^p(\mu, H^{\otimes k})$ (the latter follows again from $\gamma^\varepsilon \cdot \sigma^\varepsilon = \text{Id}$).

**Lemma 6.2.3** Let $G \in \mathbb{ID}$. Then, for all $f \in \mathcal{S}(\mathbb{R}^d)$, the following identities are true:

1. 

$$E[\partial_i f(F) \cdot G] = E[f(F) \cdot l_i(G)],$$

where $G \mapsto l_i(G)$ is linear and for any $1 < r < q < \infty$,

$$\sup_{\|G\|_{\mathcal{L}^r,1} \leq 1} \|l_i(G)\|_r < +\infty.$$  

2. Similarly

$$E[\partial_{i_1 \cdots i_k} f \circ F \cdot G] = E[f(F) \cdot l_{i_1 \cdots i_k}(G)]$$

and

$$\sup_{\|G\|_{\mathcal{L}^r,1} \leq 1} \|l_{i_1 \cdots i_k}(G)\|_r < \infty.$$
Proof: We have \[
\nabla(f \circ F) = \sum_{i=1}^{d} \partial_i f(F) \nabla F_i
\]
hence \[
(\nabla(f \circ F), \nabla F_j)_H = \sum_{j=1}^{d} \sigma_{ij} \partial_i f(F).
\]
Since \(\sigma\) is invertible, we obtain:
\[
\partial_i f(F) = \sum_j \gamma_{ij} (\nabla(f \circ F), \nabla F_j)_H.
\]
Then
\[
E[\partial_i f(F).G] = \sum_j E[\gamma_{ij} (\nabla(f \circ F), \nabla F_j)_H G]
= \sum_j E[f \circ F \delta\{\nabla F_j \gamma_{ij} G\}],
\]
hence we see that \(l_i(G) = \sum_j \delta\{\nabla F_j \gamma_{ij} G\}\). Developing this expression gives
\[
l_i(G) = -\sum_j (\nabla(\gamma_{ij} G), \nabla F_j)_H - \gamma_{ij} G \mathcal{L} F_j
\]

Hence
\[
|l_i(G)| \leq \sum_j \left[ \sum_{k,l} |\gamma_{ik}| |\nabla \sigma_{kl}| |\nabla F_j| |G| \\
+ |\gamma_{ij}| |\nabla F_j| |\nabla G| + |\gamma_{ij}| |G| |\mathcal{L} F_j| \right].
\]
Choose \(p\) such that \(\frac{1}{p} = \frac{1}{r} + \frac{1}{q}\) and apply Hölder’s inequality:
\[
\|l_i(G)\|_r \leq \sum_{j=1}^{d} \left[ \sum_{k,l} \|G\|_q |\gamma_{ik} \gamma_{jl}| |\nabla \sigma_{kl}| |\nabla F_j|_H \right]_p \\
+ \|\gamma_{ij} |\nabla F_j|_H \|_p \|\nabla G\|_q + \|\gamma_{ij} \mathcal{L} F_j\|_p \|G\|_q \\
\leq \|G\|_{q,1} \left[ \sum_{j=1}^{d} \|\gamma_{ik} \gamma_{jl}| \nabla F_{kl}| |\nabla F_j|_p \\
+ \|\gamma_{ij} |\nabla F_j|_p \| + \|\gamma_{ij} \mathcal{L} F_j\|_p \right].
\]
Lifting of $S'(\mathbb{R}^d)$

To prove the last part we iterate this procedure for $i > 1$.

**Theorem 6.2.4** Let $F \in D(\mathbb{R}^d)$ be a non-degenerate random variable. Then we have for $f \in S(\mathbb{R}^d)$:

$$\|f \circ F\|_{p,-2k} \leq c_{p,k} \|f\|_{-2k}.$$  

**Proof:** Let $\psi = A^{-k}f \in S(\mathbb{R}^d)$. For $G \in D$, from Lemma 6.2.3, we know that there exists some $\eta_{2k}(G) \in D$ with $G \mapsto \eta_{2k}(G)$ being linear, such that

$$E[A^k \psi \circ F G] = E[\psi \circ F \eta_{2k}(G)],$$  

i.e.,

$$E[f \circ F G] = E[(A^{-k}f) \circ F \eta_{2k}(G)].$$

Hence

$$|E[f \circ F G]| \leq \|A^{-k}f\|_{\infty} \|\eta_{2k}(G)\|_{L^1}$$

and

$$\sup_{\|G\|_{q,2k} \leq 1} |E[f \circ F G]| \leq \|A^{-k}f\|_{\infty} \sup_{\|G\|_{q,2k} \leq 1} \|\eta_{2k}(G)\|_{L^1}$$

$$= K \|f\|_{-2k}.$$  

Consequently

$$\|f \circ F\|_{p,-2k} \leq K \|f\|_{-2k}.$$  

**Corollary 6.2.5** The linear map $f \mapsto f \circ F$ from $S(\mathbb{R}^d)$ into $D$ extends continuously to a map from $S'(\mathbb{R}^d)$ into $D'$ whenever $F \in D(\mathbb{R}^d)$ is non-degenerate.

As we have seen in Theorem 6.2.4 and Corollary 6.2.5, if $F : W \to \mathbb{R}^d$ is a non-degenerate random variable, then the map $f \mapsto f \circ F$ from $S(\mathbb{R}^d) \to D$ has a continuous extension to $S'(\mathbb{R}^d) \to D'$ which we shall denote by $T \mapsto T \circ F$. 
For $f \in S(\mathbb{R}^d)$, let us look at the following Pettis integral:

$$\int_{\mathbb{R}^d} f(x)\mathcal{E}_x dx,$$

where $\mathcal{E}_x$ denotes the Dirac measure at $x \in \mathbb{R}^d$. We have, for any $g \in S(\mathbb{R}^d)$,

$$\left\langle \int f(x)\mathcal{E}_x dx, g \right\rangle = \int \left\langle f(x)\mathcal{E}_x, g \right\rangle dx = \int f(x)\langle \mathcal{E}_x, g \rangle dx = \int f(x)g(x) dx = \langle f, g \rangle.$$

Hence we have proven:

**Lemma 6.2.6** The following representation holds in $S(\mathbb{R}^d)$:

$$f = \int_{\mathbb{R}^d} f(x)\mathcal{E}_x dx.$$

From Lemma 6.2.6 we have

**Lemma 6.2.7** We have

$$\int \langle \mathcal{E}_y(F), \varphi \rangle f(y) dy = E[f(F) \varphi],$$

for any $\varphi \in \mathbb{D}$, where $\langle \cdot, \cdot \rangle$ denotes the bilinear form of duality between $\mathbb{D}'$ and $\mathbb{D}$.

**Proof:** Let $\rho_\varepsilon$ be a mollifier. Then $\mathcal{E}_y * \rho_\varepsilon \rightarrow \mathcal{E}_y$ in $S'$ on the other hand

$$\int_{\mathbb{R}^d} (\mathcal{E}_y * \rho_\varepsilon)(F)f(y) dy = \int_{\mathbb{R}^d} \rho_\varepsilon(F + y) f(y) dy = \int_{\mathbb{R}^d} \rho_\varepsilon(y)f(y + F) dy \xrightarrow{\varepsilon \rightarrow 0} f(F).$$

On the other hand, for $\varphi \in \mathbb{D}$,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \langle (\mathcal{E}_y * \rho_\varepsilon)(F), \varphi \rangle f(y) dy = \int_{\mathbb{R}^d} \lim_{\varepsilon \rightarrow 0} \langle (\mathcal{E}_y * \rho_\varepsilon)(F), \varphi \rangle f(y) dy = \int_{\mathbb{R}^d} \mathcal{E}_y(F), \varphi \rangle f(y) dy = \langle f(F), \varphi \rangle = E[f(F) \varphi].$$
Corollary 6.2.8 We have
\[
\langle \mathcal{E}_x(F), 1 \rangle = \frac{d(F^*\mu)}{dx}(x) = p_F(x),
\]
moreover \( p_F \in \mathcal{S}(\mathbb{R}^d) \) (i.e., the probability density of \( F \) is not only \( C^\infty \) but it is also a rapidly decreasing function).

Proof: We know that, for any \( \varphi \in \mathcal{D} \), the map \( T \to \langle T(F), \varphi \rangle \) is continuous on \( \mathcal{S}'(\mathbb{R}^d) \) hence there exists some \( p_{F,\varphi} \in \mathcal{S}(\mathbb{R}^d) \) such that
\[
E[T(F) \cdot \varphi] = \langle p_{F,\varphi}, T \rangle_{\mathcal{S}'}.
\]
Let \( p_{F,1} = p_F \), then it follows from the Lemma 6.2.6 that
\[
E[f(F)] = \int \langle \mathcal{E}_y(F), 1 \rangle f(y)dy
= \int p_F(y)f(y)dy.
\]

Remark 6.2.9 From the disintegration of measures, we have
\[
E[f(F) \varphi] = \int_{\mathbb{R}^d} p_F(x)E[\varphi|F = x]f(x)dx
= \int_{\mathbb{R}^d} f(x)\langle \mathcal{E}_x(F), \varphi \rangle dx
\]
hence
\[
E[\varphi|F = x] = \frac{\langle \mathcal{E}_x(F), \varphi \rangle}{p_F(x)}
\]
d\( x \)-almost surely on the support of the law of \( F \). In fact the right hand side is an everywhere defined version of this conditional probability.

6.2.1 Extension of the Ito Formula

Let \( (x_t) \) be the solution of the following stochastic differential equation:
\[
dx_t(\omega) = b_t(x_t(\omega))dt + \sigma_t(x_t(\omega))dw^i_t
\]
\[
x_0 = x \text{ given},
\]
where \( b: \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma_t: \mathbb{R}^d \to \mathbb{R}^d \) are smooth vector fields with bounded derivatives. Let us denote by
\[
X_0 = \sum_{i=1}^d \frac{\partial b^i_0}{\partial x_i}, \quad X_j = \sum \sigma^i_j \frac{\partial}{\partial x_j}
\]
Filtering

where

\[ \tilde{b}^i(x) = b^i(x) - \frac{1}{2} \sum_{k,\alpha} \partial_k \sigma^i_{\alpha}(x) \sigma^k_{\alpha}(x). \]

If the Lie algebra of vector fields generated by \( \{X_0, X_1, \ldots, X_d\} \) has dimension equal to \( d \) at any \( x \in \mathbb{R}^d \), then \( x_t(w) \) is non-degenerate cf. [102]. In fact it is also uniformly non-degenerate in the following sense:

\[ E \int_s^t |\det(\nabla x^i_t, \nabla x^j_t)|^{-p} dr < \infty, \]

forall \( 0 < s < t \) and \( p > 1 \).

As a corollary of this result, combined with the lifting of \( \mathcal{S}' \) to \( \mathbb{D}' \), we can show the following:

**Theorem 6.2.10** For any \( T \in \mathcal{S}'(\mathbb{R}^d) \), one has the following:

\[ T(x_t) - T(x_s) = \int_s^t A T(x_s) ds + \int_s^t \sigma_{ij}(x_s) \cdot \partial_j T(x_s) dW^i_s, \]

where the Lebesgue integral is a Bochner integral, the stochastic integral is as defined at the first section of this chapter and we have used the following notation:

\[ A = \sum b^i \partial_i + \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad a(x) = (\sigma^*)_{ij}, \quad \sigma = [\sigma_1, \ldots, \sigma_d]. \]

### 6.2.2 Applications to the filtering of the diffusions

Suppose that we are given, for any \( t \geq 0 \),

\[ y_t = \int_0^t h(x_s) ds + B_t \]

where \( h \in C^\infty_0(\mathbb{R}^d) \otimes \mathbb{R}^d \), \( B \) is another Brownian motion independent of \( w \) above. The process \( (y_t; t \in [0, 1]) \) is called an (noisy) observation of \( (x_t, t \in \mathbb{R}_+) \). Let \( \mathcal{Y}_t = \sigma\{y_s; s \in [0, t]\} \) be the observed data till \( t \). The filtering problem consists of calculating the random measure \( f \mapsto E[f(x_t)|\mathcal{Y}_t] \). Let \( P^0 \) be the probability defined by

\[ dP^0 = Z_t^{-1} dP \]

where

\[ Z_t = \exp \left\{ \int_0^t (h(x_s), dy_s) - \frac{1}{2} \int_0^t |h(x_s)|^2 ds \right\}. \]
Then for any bounded, \( \mathcal{Y}_t \)-measurable random variable \( Y_t \), we have:

\[
E[f(x_t)Y_t] = E\left[\frac{Z_t}{Z_t} f(x_t)Y_t\right] \\
= E^0 [Z_t f(x_t)Y_t] \\
= E^0 \left[E^0[Z_t f(x_t)|\mathcal{Y}_t] \cdot Y_t\right] \\
= E \left[\frac{1}{E^0[Z_t |\mathcal{Y}_t]} E^0[Z_t f(x_t)|\mathcal{Y}_t] \cdot Y_t\right],
\]

hence

\[
E[f(x_t)|\mathcal{Y}_t] = \frac{E^0[Z_t f(x_t)|\mathcal{Y}_t]}{E^0[Z_t |\mathcal{Y}_t]}.
\]

If we want to study the smoothness of the measure \( f \mapsto E[f(x_t)|\mathcal{Y}_t] \), then from the above formula, we see that it is sufficient to study the smoothness of \( f \mapsto E^0[Z_t f(x_t)|\mathcal{Y}_t] \). The reason for the use of \( P^0 \) is that \( w \) and \((y_t; t \in [0, 1])\) are two independent Brownian motions \( \square \) under \( P^0 \).

**Remark 6.2.11** Let us note that the random distribution \( f \mapsto \nu_t(f) \) defined by

\[
\nu_t(f) = E^0[Z_t f(x_t)|\mathcal{Y}_t]
\]

satisfies the Zakai equation:

\[
\nu_t(f) = \nu_0(f) + \int_0^t \nu_s(Af) ds + \int_0^t \sum_i \nu_s(h_i f) dy_i,
\]

where \( A \) denotes the infinitesimal generator of the diffusion process \((x_t, t \in \mathbb{R}_+)^1\).

After this preliminaries, we can prove the following

**Theorem 6.2.12** Suppose that the map \( f \mapsto f(x_t) \) from \( \mathcal{S}(\mathbb{R}^d) \) into \( \mathbb{D} \) has a continuous extension as a map from \( \mathcal{S}'(\mathbb{R}^d) \) into \( \mathbb{D}' \). Then the measure \( f \mapsto E[f(x_t)|\mathcal{Y}_t] \) has a density in \( \mathcal{S}(\mathbb{R}^d) \).

**Proof:** As explained above, it is sufficient to prove that the (random) measure \( f \mapsto E^0[Z_t f(x_t)|\mathcal{Y}_t] \) has a density in \( \mathcal{S}(\mathbb{R}^d) \). Let \( \mathcal{L}_y \) be the Ornstein-Uhlenbeck operator on the space of the Brownian motion \((y_t; t \in [0, 1])\). Then we have

\[
\mathcal{L}_y Z_t = Z_t \left(-\int_0^t h(x_s) dy_s + \frac{1}{2} \int_0^t |h(x_s)|^2 ds\right) \in \cap_p L_p.
\]

\(^1\) This claim follows directly from Paul Lévy’s theorem of the characterization of the Brownian motion.
It is also easy to see that
\[ \mathcal{L}_w^k Z_t \in \bigcap_p L^p. \]

From these observations we draw the following conclusions:

- Hence \( Z_t(w, y) \in \mathcal{D}(w, y) \), where \( \mathcal{D}(w, y) \) denotes the space of test functions defined on the product Wiener space with respect to the laws of \( w \) and \( y \).

- The second point is that the operator \( E^0[\cdot|\mathcal{Y}_t] \) is a continuous mapping from \( \mathcal{D}_{p,k}(w, y) \) into \( \mathcal{D}_{p,k}^0(y) \), for any \( p \geq 1, k \in \mathbb{Z} \), since \( \mathcal{L}_y \) commutes with \( E^0[\cdot|\mathcal{Y}_t] \).

- Hence the map
  \[ T \mapsto E^0[T(x_t)Z_t|\mathcal{Y}_t] \]
  is continuous from \( \mathcal{S}'(\mathbb{R}^d) \to \mathcal{D}'(y) \). In particular, for fixed \( T \in \mathcal{S}' \), there exist \( p > 1 \) and \( k \in \mathbb{N} \) such that \( T(x_t) \in \mathcal{D}_{p,-k}(w) \). Since \( Z_t \in \mathcal{D}(w, y) \),

\[ Z_tT(x_t) \in \mathcal{D}_{p,-k}(w, y) \]

and

\[ T(x_t)(I + \mathcal{L}_y)^{k/2}Z_t \in \mathcal{D}_{p,-k}(w, y). \]

- Consequently

\[ E^0[T(x_t) \cdot (I + \mathcal{L}_y)^{k/2}Z_t|\mathcal{Y}_t] \in \mathcal{D}_{p,-k}(y). \]

- Finally it follows from the latter that

\[ (I + \mathcal{L})^{-k/2}E^0[T(x_t)(I + \mathcal{L}_y)^{k/2}Z_t|\mathcal{Y}_t] = E^0[T(x_t)Z_t|\mathcal{Y}_t] \]

belongs to \( L^p(y) \). Therefore we see that:

\[ T \mapsto E^0[T(x_t)Z_t|\mathcal{Y}_t] \]

defines a linear, continuous (use the closed graph theorem for instance) map from \( \mathcal{S}'(\mathbb{R}^d) \) into \( L^p(y) \).

Since \( \mathcal{S}'(\mathbb{R}^d) \) is a nuclear space, the map

\[ T \mapsto E^0[T(x_t)Z_t|\mathcal{Y}_t] \]

is a nuclear operator. This implies that \( \Theta \) can be represented as

\[ \Theta = \sum_{i=1}^{\infty} \lambda_i f_i \otimes \alpha_i \]
Applications of Ito Clark Formula

where \((\lambda_i) \in l^1\), \((f_i) \subset S(\mathbb{R}^d)\) and \((\alpha_i) \subset L^p(y)\) are bounded sequences. Define

\[
k_t(x,y) = \sum_{i=1}^{\infty} \lambda_i f_i(x) \alpha_i(y) \in S(\mathbb{R}^d) \otimes_1 L^p(y)
\]

where \(\otimes_1\) denotes the projective tensor product topology. It is easy now to see that, for \(g \in S(\mathbb{R}^d)\)

\[
\int_{\mathbb{R}^d} g(x) k_t(x,y) dx = E_0[g(x_t) \cdot Z_t|\mathcal{F}_t]
\]

and this completes the proof. \(\square\)

6.3 Some applications of the Clark formula

6.3.1 Case of non-differentiable functionals

In this example we use the Clark representation theorem for the elements of \(\mathbb{D}'\) and the composition of the tempered distributions with the non-degenerate Wiener functionals: Let \(w \mapsto \kappa(w)\) be the sign of the random variable \(w \mapsto W_1(w)\) where \(W_1\) denotes the value of the Wiener path \((W_t, t \in [0,1])\) at time \(t = 1\). We have, using Theorem 6.2.4

\[
E[D_t \kappa |\mathcal{F}_t] = 2 \exp \left\{ - \frac{W_t^2}{2(1-t)} \right\} \frac{1}{\sqrt{2\pi(1-t)}},
\]

dt \times d\mu-almost surely. Hence

\[
\kappa = 2 \int_0^1 \exp \left\{ - \frac{W_t^2}{2(1-t)} \right\} \frac{1}{\sqrt{2\pi(1-t)}} dW_t,
\]

\(\mu\)-almost surely. Note that, although \(\kappa\) is not strongly Sobolev differentiable, the integrand of the stochastic integral is an ordinary square integrable process. This phenomena can be explained by the fact that the conditional expectation tames the distribution, in such a way that the result becomes an ordinary random variable.

Here is another application of the Clark formula:

Proposition 6.3.1 Assume that \(A\) is a measurable subset of \(W\), then from Theorem 6.1.4, there exists an \(e_A \in L^2(\mu, H)\) which can be represented as \(e_A(t) = \int_0^t \dot{e}_A(\tau)d\tau, t \in [0,1]\), such that \(\dot{e}_A\) is adapted and

\[
1_A = \mu(A) + \delta e_A.
\]
If \( B \) is another measurable set, then \( A \) and \( B \) are independent if and only if
\[
E[(e_A, e_B)_H] = 0.
\]

**Proof:** It suffices to observe that
\[
\mu(A \cap B) = \mu(A)\mu(B) + E[(e_A, e_B)_H],
\]
hence \( A \) and \( B \) is independent if and only if the last term in (6.3.1) is null.

### 6.3.2 Logarithmic Sobolev Inequality

As another application of the Clark representation theorem, we shall give a quick proof of the logarithmic Sobolev inequality of L. Gross\(^2\) (cf. [36]).

**Theorem 6.3.2 (log-Sobolev inequality)** For any \( \phi \in \mathbb{D}_{2,1} \), we have
\[
E[\phi^2 \log \phi^2] \leq E[\phi^2] \log E[\phi^2] + 2E[|\nabla \phi|^2_H].
\]

**Proof:** Clearly it suffices to prove the following inequality
\[
E[f \log f] \leq \frac{1}{2} E \left[ \frac{1}{f} |\nabla f|^2_H \right],
\]
for any \( f \in \mathbb{D}_{2,1} \) which is strictly positive, lower bounded with some \( \varepsilon > 0 \) and with \( E[f] = 1 \). Using the Itô-Clark representation theorem, we can write
\[
f = \exp \left( \int_0^1 \frac{E[D_s f | \mathcal{F}_s]}{f_s} dW_s - \frac{1}{2} \int_0^1 \left( \frac{E[D_s f | \mathcal{F}_s]}{f_s} \right)^2 ds \right),
\]
where \( f_s = E[f | \mathcal{F}_s] \). It follows from the Itô formula that
\[
E[f \log f] = \frac{1}{2} E \left[ f \int_0^1 \left( \frac{E[D_s f | \mathcal{F}_s]}{f_s} \right)^2 ds \right].
\]

Let \( \nu \) be the probability defined by \( d\nu = f d\mu \). Then we have
\[
E[f \log f] = \frac{1}{2} E \left[ f \int_0^1 \left( \frac{E[D_s f | \mathcal{F}_s]}{f_s} \right)^2 ds \right].
\]

\(^2\)The proof which is given here is similar to that of B. Maurey.
\[
\begin{align*}
= & \frac{1}{2}E\nu \left[ \int_0^1 (E\nu[D_s \log f|\mathcal{F}_s])^2 ds \right] \\
\leq & \frac{1}{2}E\nu \int_0^1 (D_s \log f)^2 ds \\
= & \frac{1}{2}E[f |\nabla \log f|_{H}^2] \\
= & \frac{1}{2}E \left[ \frac{|\nabla f|_{H}^2}{f} \right],
\end{align*}
\]

Remark 6.3.3 We have given the proof in the frame of the classical Wiener space. However this result extends immediately to any abstract Wiener space by the use of the techniques explained in the Appendix of the fourth chapter.

Remark 6.3.4 A straightforward implication of the Clark representation, as we have seen in the sequel of the proof, is the Poincaré inequality which says that, for any \( F \in \mathbb{D}_{2,1} \), one has

\[
E[|F - E[F]|^2] \leq E[|\nabla F|_{H}^2].
\]

This inequality is the first step towards the logarithmic Sobolev inequality.

Exercises

1. Assume that \( F : W \rightarrow X \) is a measurable Wiener function, where \( X \) is a separable Hilbert space. Assume further that

\[
|F(w + h) - F(w + k)||_{X} \leq K|h - k|_{H}
\]

\( \mu \)-almost surely, for any \( h, k \in H \). Prove that there exists \( F' = F \) almost surely such that

\[
|F'(w + h) - F'(w + k)||_{X} \leq K|h - k|_{H}
\]

for any \( w \in W \) and \( h, k \in H \).

2. Deduce from this result that if \( A \) is a measurable subset of \( W \), such that \( A + h \subset A \) almost surely, then \( A \) has a modification, say \( A' \) such that \( A + H \subset A \).

Notes and suggested reading

Ito-Clark formula has been discovered first by Clark in the case of Fréchet differentiable Wiener functionals. Later its connections with the Girsanov theorem has been remarked by J.-M. Bismut, \cite{Bismut}. D. Ocone has extended it to the Wiener functionals in \( \mathbb{D}_{2,1} \), cf.
Its extension to the distributions is due to the author, cf. [85]. Later D. Ocone and I. Karatzas have also extended it to the functionals of $\mathcal{D}_{1,1}$.

Composition of the non-degenerate Wiener functionals with the elements of $\mathcal{S}'(\mathbb{R}^d)$ is due to Kuo [50]. Watanabe has generalized it to more general Wiener functionals, [102, 103]. Later it has been observed by the author that this implies automatically the fact that the density of the law of a non-degenerate Wiener functional is a rapidly decreasing $C^\infty$ function. This last result remains true for the conditional density of the non-linear filtering as it has been first proven in [89].
Chapter 7

Positive distributions and applications

7.1 Positive Meyer-Watanabe distributions

If $\theta$ is a positive distribution on $\mathbb{R}^d$, then a well-known theorem says that $\theta$ is a positive measure, finite on the compact sets. We will prove an analogous result for the Meyer-Watanabe distributions in this section, show that they are absolutely continuous with respect to the capacities defined with respect to the scale of the Sobolev spaces on the Wiener space and give an application to the construction of the local time of the Wiener process. We end the chapter by making some remarks about the Sobolev spaces constructed by the second quantization of an elliptic operator on the Cameron-Martin space.

We will work on the classical Wiener space $C_0([0, 1]) = W$. First we have the following:

**Proposition 7.1.1** Suppose $(T_n) \subset \mathbb{D}'$ and each $T_n$ is also a probability on $W$. If $T_n \to T$ in $\mathbb{D}'$, then $T$ is also a probability and $T_n \to T$ in the weak-star topology of measures on $W$.

For the proof of this proposition, we shall need the following:

**Lemma 7.1.2 (Garsia-Rademich-Ramsey lemma)** Let $p, \psi$ be two continuous, strictly increasing functions on $\mathbb{R}_+$ such that $\psi(0) = p(0) = 0$ and that $\lim_{t \to \infty} \psi(t) = \infty$. Let $T > 0$ and $f \in C([0, T], \mathbb{R}^d)$. If

$$\int_{[0,T]^2} \psi \left( \frac{|f(t) - f(s)|}{p(|t-s|)} \right) ds \, dt \leq B,$$

then...
then for any $0 \leq s \leq t \leq T$, we have

$$|f(t) - f(s)| \leq 8 \int_0^{t-s} \psi^{-1}\left(\frac{AB}{u^2}\right) p(du).$$

**Proof:** [of the Proposition] It is sufficient to prove that the sequence of probability measures $(\nu_n, n \geq 1)$ associated to $(T_n, n \geq 1)$, is tight. In fact, let $S = \mathbb{D} \cap C_b(W)$, if the tightness holds, then we would have, for $\nu = w - \lim \nu_n$ (taking a subsequence if necessary), where $w - \lim$ denotes the limit in the weak-star topology of measures,

$$\nu(\varphi) = T(\varphi) \text{ on } S.$$ 

Since the mapping $w \to e^{i \langle w, w^* \rangle}$ ($w^* \in W^*$) belongs to $S$, $S$ separates the probability measures on $(W, \mathcal{B}(W))$ and the proof would follow.

In order to realize this program, let $G : W \to \mathbb{R}$ be defined as

$$G(w) = \int_0^1 \int_0^1 \frac{|w(t) - w(s)|^8}{|t - s|^3} ds dt.$$ 

Then $G \in \mathbb{D}$ and $A_\lambda = \{G(w) \leq \lambda\}$ is a compact subset of $W$. In fact, from Garsia-Rademich-Rumsey lemma [81], the inequality $G(w) \leq \lambda$ implies the existence of a constant $K_\lambda$ such that

$$|w(s) - w(t)| \leq K_\lambda |t - s|^{15},$$

for $0 \leq s < t \leq 1$, hence $A_\lambda$ is equicontinuous, then the Arzela-Ascoli Theorem implies that the set $\{w : G(w) \leq \lambda\}$ is relatively compact in $W$, moreover it is a closed set since $G$ is a lower semi-continuous function by the Fatou Lemma. In particular, it is measurable with respect to the non-completed Borel sigma algebra of $W$. Moreover, we have $\bigcup_{\lambda \geq 0} A_\lambda = W$ almost surely. Let $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$; $\varphi(x) = 1$ for $x \geq 0$, $\varphi(x) = 0$ for $x \leq -1$. Let $\varphi_\lambda(x) = \varphi(x - \lambda)$. We have

$$\nu_n(A_\lambda^c) \leq \int_W \varphi_\lambda(G(w)) \nu_n(dw).$$

We claim that

$$\int_W \varphi_\lambda(G) d\nu_n = \langle \varphi_\lambda(G), T_n \rangle.$$ 

To see this, for $\varepsilon > 0$, write

$$G_\varepsilon(w) = \int_{[0,1]^2} \frac{|w(t) - w(s)|^8}{(\varepsilon + |t - s|)^3} ds dt.$$
Then \( \varphi_\lambda(G_\varepsilon) \in S \) (but not \( \varphi_\lambda(G) \), since \( G \) is not continuous on \( W \)) and we have
\[
\int \varphi_\lambda(G_\varepsilon) d\nu_n = \langle \varphi_\lambda(G_\varepsilon), T_n \rangle.
\]
Moreover \( \varphi_\lambda(G_\varepsilon) \to \varphi_\lambda(G) \) in \( \mathbb{D} \), hence
\[
\lim_{\varepsilon \to 0} \langle \varphi_\lambda(G_\varepsilon), T_n \rangle = \langle \varphi_\lambda(G), T_n \rangle.
\]
From the dominated convergence theorem, we have also
\[
\lim_{\varepsilon \to 0} \int \varphi_\lambda(G_\varepsilon) d\nu_n = \int \varphi_\lambda(G) d\nu_n.
\]
Since \( T_n \to T \) in \( \mathbb{D}' \), there exist some \( k > 0 \) and \( p > 1 \) such that \( T_n \to T \) in \( \mathbb{D}_{p,-k} \). Therefore
\[
\langle \varphi_\lambda(G), T_n \rangle = \langle (I + \mathcal{L})^{k/2} \varphi_\lambda(G), (I + \mathcal{L})^{-k/2} T_n \rangle 
\leq \left\| (I + \mathcal{L})^{k/2} \varphi_\lambda(G) \right\|_q \sup_n \left\| (I + \mathcal{L})^{-k/2} T_n \right\|_p.
\]
From the Meyer inequalities, we see that
\[
\lim_{\lambda \to \infty} \left\| (I + \mathcal{L})^{k/2} \varphi_\lambda(G) \right\|_q = 0,
\]
in fact, it is sufficient to see that \( \nabla^i(\varphi_\lambda(G)) \to 0 \) in \( L^p \) for all \( i \leq [k] + 1 \), but this is obvious from the choice of \( \varphi_\lambda \). We have proven that
\[
\lim_{\lambda \to \infty} \sup_n \nu_n(A^\lambda_n) 
\leq \sup_n \left\| (I + \mathcal{L})^{-k/2} T_n \right\|_p \lim_{\lambda \to \infty} \left\| (I + \mathcal{L})^{k/2} \varphi_\lambda(G) \right\|_p = 0,
\]
which implies the tightness and the proof is completed. \( \square \)

**Corollary 7.1.3** Let \( T \in \mathbb{D}' \) such that \( \langle T, \varphi \rangle \geq 0 \), for all positive \( \varphi \in \mathbb{D} \). Then \( T \) is a Radon measure on \( W \).

**Proof:** Let \( (h_i) \subset H \) be a complete, orthonormal basis of \( H \). Let \( V_n = \sigma\{\delta h_1, \ldots, \delta h_n\} \). Define \( T_n \) as \( T_n = E[P_{1/n}T[V_n]] \) where \( P_{1/n} \) is the Ornstein-Uhlenbeck semi-group on \( W \). Then \( T_n \geq 0 \) and it is a random variable in some \( L^p(\mu) \). Therefore it defines a measure on \( W \) (it is even absolutely continuous with respect to \( \mu \)). Moreover \( T_n \to T \) in \( \mathbb{D}' \), hence the proof follows from Proposition 7.1.1. \( \square \)
7.2 Capacities and positive Wiener functionals

We begin with the following definitions:

**Definition 7.2.1** Let $p \in [1, \infty)$ and $k > 0$. If $O \subset W$ is an open set, we define the $(p, k)$-capacity of $O$ as

$$C_{p,k}(O) = \inf \{\|\varphi\|_{p,k}^p : \varphi \in D_{p,k}, \varphi \geq 1 \mu - a.e. \text{ on } O\}.$$  

If $A \subset W$ is any set, define its $(p, k)$-capacity as

$$C_{p,k}(A) = \inf \{C_{p,k}(O) ; O \text{ is open } O \supset A\}.$$ 

- We say that some property takes place $(p, k)$-quasi everywhere if the set on which it does not hold has $(p, k)$-capacity zero.
- We say $N$ is a slim set if $C_{p,k}(N) = 0$, for all $p > 1$, $k > 0$.
- A function is called $(p, k)$-quasi continuous if for any $\varepsilon > 0$, there exists an open set $O_\varepsilon$ such that $C_{p,k}(O_\varepsilon) < \varepsilon$ and the function is continuous on $O_\varepsilon$.
- A function is called $\infty$-quasi continuous if it is $(p, k)$-quasi continuous for any $p > 1$, $k \in \mathbb{N}$.

The results contained in the next lemma are proved by Fukushima & Kaneko (cf. [33]):

**Lemma 7.2.2**

1. If $F \in D_{p,k}$, then there exists a $(p, k)$-quasi continuous function $\tilde{F}$ such that $F = \tilde{F}$ $\mu$-a.e. and $\tilde{F}$ is $(p, k)$-quasi everywhere defined, i.e. if $\tilde{G}$ is another such function, then $C_{p,k}(\{F \neq G\}) = 0$.

2. If $A \subset W$ is arbitrary, then

$$C_{p,k}(A) = \inf \{\|\varphi\|_{p,k} : \varphi \in D_{p,k}, \varphi \geq 1 (p, r) - q.e. \text{ on } A\}.$$  

3. There exists a unique element $U_A \in D_{p,k}$ such that $\tilde{U}_A \geq 1 (p, k)$-quasi everywhere on $A$ with $C_{p,k}(A) = \|U_A\|_{p,k}$, and $\tilde{U}_A \geq 0 (p, k)$-quasi everywhere. $U_A$ is called the $(p, k)$-equilibrium potential of $A$. 

**Theorem 7.2.3** Let \( T \in \mathcal{D}' \) be a positive distribution and suppose that \( T \in \mathcal{D}_{q-k} \) for some \( q > 1, k \geq 0 \). Then, if we denote by \( \nu_T \) the measure associated to \( T \), we have

\[
\bar{\nu}_T(A) \leq \|T\|_{q-k} (C_{p,k}(A))^{1/p},
\]

for any set \( A \subset W \), where \( \bar{\nu}_T \) denotes the outer measure with respect to \( \nu_T \). In particular \( \nu_T \) does not charge the slim sets.

**Proof:** Let \( V \) be an open set in \( W \) and let \( U_V \) be its equilibrium potential of order \((p, k)\). We have

\[
\langle P_{1/n}T, U_V \rangle = \int P_{1/n}TU_Vd\mu \\
\geq \int_V P_{1/n}TU_Vd\mu \\
\geq \int_V P_{1/n}Td\mu \\
= \nu_{P_{1/n}T}(V).
\]

Since \( V \) is open, we have, from the fact that \( \nu_{P_{1/n}T} \rightarrow \nu_T \) weakly,

\[
\liminf_{n \rightarrow \infty} \nu_{P_{1/n}T}(V) \geq \nu_T(V).
\]

On the other hand

\[
\lim_{n \rightarrow \infty} \langle P_{1/n}T, U_V \rangle = \langle T, U_V \rangle \\
\leq \|T\|_{q-k} \|U_V\|_{p,k} \\
= \|T\|_{q-k} C_{p,k}(V)^{1/p}.
\]

\( \square \)

### 7.3 Some Applications

Below we use the characterization of the positive distributions to give a different interpretation of the local times. Afterwards the 0–1 law is revisited via the capacities.

#### 7.3.1 Applications to Ito formula and local times

Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be a function from \( \mathcal{S}'(\mathbb{R}^d) \) and suppose that \((X_t, t \geq 0)\) is a hypoelliptic diffusion on \( \mathbb{R}^d \) which is constructed as the solution of the
following stochastic differential equation with smooth coefficients:

\[ dX_t = \sigma(X_t) dW_t + b(X_t) dt \quad (7.3.1) \]

\[ X_0 = x \in \mathbb{R}^d. \]

We denote by \( L \) the infinitesimal generator of the diffusion process \((X_t, t \geq 0)\). For any \( t > 0 \), \( X_t \) is a non-degenerate random variable in the sense of Definition 6.2.1. Consequently we have the extension of the Ito formula

\[ f(X_t) - f(X_u) = \int_u^t Lf(X_s) ds + \int_u^t \sigma_{ij}(X_s) \partial_i f(X_s) dW_s^j, \]

for \( 0 < u \leq t \leq 1 \). Note that, since we did not make any differentiability hypothesis about \( f \), the above integrals are to be regarded as the elements of \( \mathbb{D}' \).

Suppose that \( Lf \) is a bounded measure on \( \mathbb{R}^d \), from our result about the positive distributions, we see that \( \int_u^t Lf(X_s) ds \) is a measure on \( W \) which does not charge the slim sets. By difference, so does the term \( \int_u^t \sigma_{ij}(X_s) \partial_i f(X_s) dW_s^j. \)

As a particular case, we can take \( d = 1, L = \frac{1}{2}\Delta \) (i.e. \( \sigma = 1 \)), \( f(x) = |x| \) and this gives

\[ |W_t| - |W_u| = \frac{1}{2} \int_u^t \Delta|x|(W_s) ds + \int_u^t \frac{d}{dx}|x|(W_s) dW_s. \]

As \( \frac{d}{dx}|x| = \text{sign}(x) \), we have

\[ \int_u^t \frac{d}{dx}|x|(W_s) dW_s = \int_u^t \text{sign}(W_s) dW_s = M^u_t \]

is a measure absolutely continuous with respect to \( \mu \). Since \( \lim_{u \to 0} M^u_t = N_t \) exists in all \( L^p \), so does

\[ \lim_{u \to 0} \int_u^t \Delta|x|(W_s) ds \]

in \( L^p \) for any \( p \geq 1 \). Consequently \( \int_0^t \Delta|x|(W_s) ds \) is absolutely continuous with respect to \( \mu \), i.e., it is a random variable. It is easy to see that

\[ \Delta|x|(W_s) = 2\mathcal{E}_0(W_s), \]

where \( \mathcal{E}_0 \) denotes the Dirac measure at zero, hence we obtain

\[ \int_0^t 2\mathcal{E}_0(W_s) ds = \int_0^t \Delta|x|(W_s) ds = 2l^0_t \]

which is the local time of Tanaka. Note that, although \( \mathcal{E}_0(W_s) \) is singular with respect to \( \mu \), its Pettis integral is absolutely continuous with respect to \( \mu \).
Remark 7.3.1 If \( F : W \to \mathbb{R}^d \) is a non-degenerate random variable, then for any \( S \in \mathcal{S}'(\mathbb{R}^d) \) with \( S \geq 0 \) on \( \mathcal{S}_+(\mathbb{R}^d) \), \( S(F) \in \mathcal{D}' \) is a positive distribution, hence it is a positive Radon measure on \( W \). In particular \( \mathcal{E}_s(F) \) is a positive Radon measure.

7.3.2 Applications to \( 0 \rightarrow 1 \) law and to the gauge functionals of sets

In Theorem 6.1.5 we have seen that an \( H \)-invariant subset of \( W \) has measure which is equal either to zero or to one. In this section we shall refine this result using the capacities. Let us first begin by defining the gauge functional of a measurable subset of \( W \): if \( A \in \mathcal{B}(W) \), define

\[
q_A(w) = \inf[|h|_H : h \in (A - w) \cap H],
\]

where the infimum is defined as to be infinity on the empty set. We have

Lemma 7.3.2 For any \( A \in \mathcal{B}(W) \), the map \( q_A \) is measurable with respect to the \( \mu \)-completion of \( \mathcal{B}(W) \). Moreover

\[
|q_A(w + h) - q_A(w)| \leq |h|_H
\]

almost surely, for any \( h \in H \) and \( \mu\{q_A < \infty\} = 0 \) or 1.

Proof: Without loss of generality, we may assume that \( A \) is a compact subset of \( W \) with \( \mu(A) > 0 \). Then the set \( K(w) = (A - w) \cap H \neq \emptyset \) almost surely. Therefore \( w \to K(w) \) is a multivalued map with values in the non-empty subsets of \( H \) for almost all \( w \in W \). Let us denote by \( G(K) \) its graph, i.e.,

\[
G(K) = \{(h, w) : h \in K(w)\}.
\]

Since \( (h, w) \mapsto h + w \) is measurable from \( H \times W \) to \( W \) when the first space is equipped with the product sigma algebra, due to the continuity of the map \( (h, w) \to h + w \), it follows that \( G(K) \) is a measurable subset of \( H \times W \). From a theorem about the measurable multi-valued maps, it follows that \( w \to K(w) \) is measurable with respect to the \( \mu \)-completed sigma field \( \mathcal{B}(W) \) (cf. [16]). Hence there is a countable sequence of \( H \)-valued measurable selectors \( (u_i, i \in \mathbb{N}) \) of \( K \) (i.e., \( u_i : W \to H \) such that \( u_i(w) \in K(w) \) almost surely) such that \( (u_i(w), i \in \mathbb{N}) \) is dense in \( K(w) \) almost surely. To see the measurability, it suffices to remark that

\[
q_A(w) = \inf(|u_i(w)|_H : i \in \mathbb{N}).
\]
The relation 7.3.3 is evident from the definition of \( q_A \). To complete the proof it suffices to remark that the set \( Z = \{ w : q_A(w) < \infty \} \) is \( H \)-invariant, hence from Theorem 6.1.5 \( \mu(Z) = 0 \) or 1. Since \( Z \) contains \( A \) and \( \mu(A) > 0 \), \( \mu(Z) = 1 \). □

The following result refines the 0 − 1–law (cf. [29], [52]):

**Theorem 7.3.3** Assume that \( A \subset W \) is an \( H \)-invariant set of zero Wiener measure. Then

\[
C_{r,1}(A) = 0
\]

for any \( r > 1 \).

**Proof:** Choose a compact \( K \subset A^c \) with \( \mu(K) > 0 \). Denote by \( B_n \) the ball of radius \( n \) of \( H \) and define \( K_n = K + B_n \). It is easy to see that \( \cup_n K_n \) is an \( H \)-invariant set. Moreover

\[
(\cup_n K_n) \cap A = \emptyset,
\]

otherwise, due to the \( H \)-invariance of of \( A \), we would have \( A \cap K \neq \emptyset \). We also have \( \mu(K_n) \to 1 \). Let

\[
p_n(w) = q_{K_n}(w) \wedge 1.
\]

From Proposition 5.1.9, we see that \( p_n \in \cap_p \mathcal{D}_{p,1} \). Moreover \( p_n(w) = 1 \) on \( K_{n+1}^c \) (hence on \( A \)) by construction. Since \( p_n = 0 \) on \( K_n \), from Lemma 2.5.1 \( \nabla p_n = 0 \) almost surely on \( K_n \). Consequently

\[
C_{r,1}(A) \leq \int (|p_n|^r + |\nabla p_n|_H^r) d\mu = \int_{K_n^c} (|p_n|^r + |\nabla p_n|_H^r) d\mu \leq 2 \mu(K_n^c) \to 0
\]

as \( n \to \infty \). □

### 7.4 Local Sobolev spaces

In Chapter II we have observed the local character of the Sobolev derivative and the divergence operator. This permits us to define the local Sobolev spaces as follows:
Definition 7.4.1 We say that a Wiener functional $F$ with values in some separable Hilbert space $X$ belongs to $\mathbb{D}^{loc}_{p,1}(X)$, $p > 1$, if there exists a sequence $(\Omega_n, n \geq 1)$ of measurable subsets of $W$ whose union is equal to $W$ almost surely and

$$F = F_n \text{ a.s. on } \Omega_n,$$

where $F_n \in \mathbb{D}_{p,1}(X)$ for any $n \geq 1$. We call $((\Omega_n, F_n), n \geq 1)$ a localizing sequence for $F$.

Lemma 2.5.1 and Lemma 2.5.2 of Section 2.5 permit us to define the local Sobolev derivative and local divergence of the Wiener functionals. In fact, if $F \in \mathbb{D}^{loc}_{p,1}(X)$, then we define $\nabla^{loc} F$ as

$$\nabla^{loc} F = \nabla F_n \text{ on } \Omega_n.$$

Similarly, if $\xi \in \mathbb{D}^{loc}_{p,1}(X \otimes H)$, then we define

$$\delta^{loc} \xi = \delta \xi_n \text{ on } \Omega_n.$$

From the lemmas quoted above $\nabla^{loc} F$ and $\delta^{loc} \xi$ are independent of the choice of their localizing sequences.

Remark: Note that we can define also the spaces $\mathbb{D}^{loc}_{p,k}(X)$ similarly.

The most essential property of the Sobolev derivative and the divergence operator is the fact that the latter is the adjoint of the former under the Wiener measure. In other words they satisfy the integration by parts formula:

$$E[\langle \nabla \phi, \xi \rangle_H] = E[\phi \delta \xi].$$

In general this important formula is no longer valid when we replace $\nabla$ and $\delta$ with $\nabla^{loc}$ and $\delta^{loc}$ respectively. The theorem given below gives the exact condition when the local derivative or divergence of a vector field is in fact equal to the global one.

Theorem 7.4.2 Assume that $\phi \in \mathbb{D}^{loc}_{p,1}(X)$, and let $((\phi_n, \Omega_n), n \in \mathbb{N})$ be a localizing sequence of $\phi$. A necessary and sufficient condition for $\phi \in \mathbb{D}_{p,1}(X)$ and for $\nabla \phi = \nabla^{loc} \phi$ almost surely, is

$$\lim_{n \rightarrow \infty} C_{p,1}(\Omega^c_n) = 0. \quad (7.4.4)$$

Proof: The necessity is trivial since, from Lemma 7.2.2 To prove the sufficiency we can assume without loss of generality that $\phi$ is bounded. In fact, if the theorem is proved for the bounded functions, then to prove the general case, we can replace $\phi$ by

$$\phi_k = \left(1 + \frac{1}{k} \|\phi\|_X\right)^{-1} \phi,$$
which converges in $\mathbb{D}_{p,1}(X)$ as $k \to \infty$ due to the closedness of the Sobolev derivative. Hence we shall assume that $\phi$ is bounded. Let $\varepsilon > 0$ be arbitrary, since $C_{p,1}(\Omega_n^c) \to 0$, by Lemma [7.2.2] there exists some $F_n \in \mathbb{D}_{p,1}$ such that $F_n \geq 1$ on $\Omega_n^c$ quasi-everywhere and $\|F_n\|_{p,1} \leq C_{p,1}(\Omega_n^c) + \varepsilon 2^{-n}$, for any $n \in \mathbb{N}$. Evidently, the sequence $(F_n, n \in \mathbb{N})$ converges to zero in $\mathbb{D}_{p,1}$. Let $f : \mathbb{R} \to [0, 1]$ be a smooth function such that $f(t) = 0$ for $|t| \geq 3/4$ and $f(t) = 1$ for $|t| \leq 1/2$. Define $A_n = f \circ F_n$, then $A_n = 0$ on $\Omega_n^c$ quasi-everywhere and the sequence $(A_n, n \in \mathbb{N})$ converges to the constant 1 in $\mathbb{D}_{p,1}$. As a consequence of this observation $\phi A_n = \phi_n A_n$ almost surely and by the dominated convergence theorem, $(\phi_n A_n, n \in \mathbb{N})$ converges to $\phi$ in $L^p(\mu, X)$. Moreover

$$\nabla(\phi A_n) = \nabla(\phi_n A_n) = A_n \nabla^{loc} \phi + \phi \nabla A_n \to \nabla^{loc} \phi$$

in $L^p(\mu, X \otimes H)$ since $(A_n, n \in \mathbb{N})$ and $\phi$ are bounded. Consequently $\nabla(\phi_n A_n) \to \nabla^{loc} \phi$ in $L^p(\mu, X \otimes H)$, since $\nabla$ is a closed operator on $L^p(\mu, X)$ the convergence takes place also in $\mathbb{D}_{p,1}(X)$ and the proof is completed. 

We have also a similar result for the divergence operator:

**Theorem 7.4.3** Let $\xi$ be in $\mathbb{D}_{p,1}^{loc}(H)$ with a localizing sequence $((\xi_n, \Omega_n), n \in \mathbb{N})$ such that $\xi \in L^p(\mu, H)$ and $\delta^{loc} \xi \in L^p(\mu)$. Assume moreover

$$\lim_{n \to \infty} C_{q,1}(\Omega_n^c) = 0, \quad (7.4.5)$$

where $q = p/(p - 1)$. Then $\xi \in \text{Dom}_p(\delta)$ and $\delta^{loc} \xi = \delta \xi$ almost surely.

**Proof:** Due to the hypothesis $(7.4.5)$, we can construct a sequence $(A_n, n \in \mathbb{N})$ as in the proof of Theorem [7.4.2] which is bounded in $L^\infty(\mu)$, converging to the constant function 1 in $\mathbb{D}_{q,1}$ such that $A_n = 0$ on $\Omega_n^c$. Let $\gamma \in \mathbb{D}$ be bounded, with a bounded Sobolev derivative. We have

$$E \left[ A_n(\delta^{loc} \xi) \gamma \right] = E \left[ A_n \delta \xi_n \gamma \right] = E \left[ (\nabla A_n, \xi_n)_{H} \right] + E \left[ (\nabla A_n, \xi_n)_{H} \right] = E \left[ A_n(\xi, \nabla \gamma)_{H} \right] + E \left[ (\nabla A_n, \xi)_{H} \right] \to E \left[ (\xi, \nabla \gamma)_{H} \right].$$

Moreover, from the dominated convergence theorem we have

$$\lim_n E[A_n(\delta^{loc} \xi) \gamma] = E[(\delta^{loc} \xi) \gamma],$$
Since the set of functionals $\gamma$ with the above prescribed properties is dense in $\mathbb{D}_{q,1}$, the proof is completed. \[\square\]

7.5 Distributions associated to $\Gamma(A)$

It is sometimes useful to have a scale of distribution spaces which are defined with a more “elliptic” operator than the Ornstein-Uhlenbeck semigroup. In this way objects which are more singular than Meyer distributions can be interpreted as the elements of the dual space. This is important essentially for the constructive Quantum field theory, cf. [77]. We begin with an abstract Wiener space $(W, H, \mu)$. Let $A$ be a self-adjoint operator on $H$, we suppose that its spectrum lies in $(1, \infty)$, hence $A^{-1}$ is bounded and $\|A^{-1}\| < 1$. Let

$$H_\infty = \bigcap_n \text{Dom}(A^n),$$

hence $H_\infty$ is dense in $H$ and $\alpha \mapsto (A^\alpha h, h)_H$ is increasing. Denote by $H_\alpha$ the completion of $H_\infty$ with respect to the norm $|h|_\alpha^2 = (A^\alpha h, h); \alpha \in \mathbb{R}$. Evidently $H'_\alpha \cong H_{-\alpha}$ (isomorphism). If $\varphi : W \to \mathbb{R}$ is a nice Wiener functional with $\varphi = \sum_{n=0}^{\infty} I_n(\varphi_n)$, define the second quantization of $A$

$$\Gamma(A)\varphi = E[\varphi] + \sum_{n=1}^{\infty} I_n(A \otimes h^\alpha, \varphi_n).$$

**Definition 7.5.1** For $p > 1$, $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}$, we define $\mathbb{D}_{p,k}^\alpha$ as the completion of polynomials based on $H_\infty$, with respect to the norm:

$$\|\varphi\|_{p,k,\alpha} = \|(I + \mathcal{L})^{k/2}\Gamma(A^{\alpha/2}\varphi)\|_{L^p(\mu)},$$

where $\varphi(w) = p(\delta h_1, \ldots, \delta h_n)$, $p$ is a polynomial on $\mathbb{R}^n$ and $h_i \in H_\infty$. If $\Xi$ is a separable Hilbert space, $\mathbb{D}_{p,k}^\alpha(\Xi)$ is defined similarly except that $\varphi$ is taken as an $\Xi$-valued polynomial.

**Remark 7.5.2** If $\varphi = \exp(\delta h - \frac{1}{2}|h|^2)$ then we have

$$\Gamma(A)\varphi = \exp \left\{ \delta(A h) - \frac{1}{2}|Ah|^2 \right\}.$$

**Remark 7.5.3** $\mathbb{D}_{p,k}^\alpha$ is decreasing with respect to $\alpha$, $p$ and $k$. 
Theorem 7.5.4  Let \((W^\alpha, H_\alpha, \mu_\alpha)\) be the abstract Wiener space corresponding to the Cameron-Martin space \(H_\alpha\). Let us denote by \(D^{(\alpha)}_{p,k}\) the Sobolev space on \(W^\alpha\) defined by

\[
\|\varphi\|_{D^{(\alpha)}_{p,k}} = \|(I + \mathcal{L})^{k/2}\varphi\|_{L^p(\mu_\alpha, W^\alpha)}
\]

Then \(D^{(\alpha)}_{p,k}\) and \(D^{\alpha}_{p,k}\) are isomorphic.

Remark: This isomorphism is not algebraic, i.e., it does not commute with the point-wise multiplication.

Proof: We have

\[
E[e^{i\delta(A^\alpha/2)h}] = \exp\left(\frac{1}{2}|A^\alpha/2h|^2\right) = \exp\left(\frac{|h|^2}{2\alpha}\right)
\]

which is the characteristic function of \(\mu_\alpha\) on \(W^\alpha\). \(\square\)

Theorem 7.5.5  
1. For \(p > 2\), \(\alpha \in \mathbb{R}\), \(k \in \mathbb{Z}\), there exists some \(\beta > \frac{\alpha}{2}\) such that

\[
\|\varphi\|_{D^\alpha_{p,k}} \leq \|\varphi\|_{D^{\beta}_{2,0}}
\]

consequently

\[
\bigcap_{\alpha,k} D^\alpha_{2,0} = \bigcap_{\alpha,p,k} D^\alpha_{p,k}.
\]

2. Moreover, for some \(\beta > \alpha\) we have

\[
\|\varphi\|_{D^\alpha_{p,k}} \leq \|\varphi\|_{D^{\beta}_{2,0}},
\]

hence we have also

\[
\bigcap_{\alpha} D^\alpha_{2,0} = \bigcap_{\alpha,p,k} D^\alpha_{p,k}.
\]

Proof: 1) We have

\[
\|\varphi\|_{D^\alpha_{p,k}} = \|\sum_n (1 + n)^{k/2} I_n((A^\alpha/2)^\otimes_n \varphi_n)\|_{L^p} = \|\sum_n (1 + n)^{k/2} e^{nt} e^{- nt} I_n((A^\alpha/2)^\otimes_n \varphi_n)\|_{L^p}.
\]

From the hypercontractivity of \(P_t\), we can choose \(t\) such that \(p = e^{2t} + 1\) then

\[
\|\sum_n (1 + n)^{k/2} e^{nt} e^{- nt} I_n(\ldots)\|_p \leq \|\sum_n (1 + n)^{k/2} e^{nt} I_n(\ldots)\|_2.
\]
Choose $\beta > 0$ such that $\|A^{-\beta}\| \leq e^{-t}$, hence

$$\|\sum (1+n)^{k/2}e^{nt}I_n(\ldots)\|_2 \leq \|\sum (1+n)^{k/2}\Gamma(A^\beta)\Gamma(A^{-\beta})e^{nt}I_n((A^{\alpha/2})^{\otimes n}\varphi_n)\|_2 \leq \sum (1+n)^{k/2}\|I_n((A^{\beta+\alpha/2})^{\otimes n}\varphi_n))\|_2 = \|\varphi\|_{\mathbb{D}^2_{2k+\alpha}}.$$ 

2) If we choose $\|A^{-\beta}\| < e^{-t}$ then the difference suffices to absorb the action of the multiplicator $(1+n)^{k/2}$ which is of polynomial growth and the former gives an exponential decrease.

Corollary 7.5.6 We have similar relations for any separable Hilbert space valued functionals.

Proof: This statement follows easily from the Khintchine inequality.

As another corollary we have

Corollary 7.5.7 Let us denote by $\Phi(H_\infty)$ the space $\bigcap_\alpha \Phi(H_\alpha)$. Then

1. $\nabla : \Phi \to \Phi(H_\infty)$ and $\delta : \Phi(H_\infty) \to \Phi$ are linear continuous operators. Consequently $\nabla$ and $\delta$ have continuous extensions as linear operators $\nabla : \Phi' \to \Phi'(H_\infty)$ and $\delta : \Phi'(H_\infty) \to \Phi'$.

2. $\Phi$ is an algebra.

3. For any $T \in \Phi'$, there exists some $\zeta \in \Phi'(H_\infty)$ such that

$$T = \langle T, 1 \rangle + \delta \zeta.$$ 

Proof: The first claim follows from Theorems 7.5.4 and 7.5.5. To prove the second one it is sufficient to show that $\varphi^2 \in \Phi$ if $\varphi \in \Phi$. This follows from the multiplication formula of the multiple Wiener integrals. (cf. Lemma 8.1.1). To prove the last one let us observe that if $T \in \Phi'$, then there exists some $\alpha > 0$ such that $T \in \mathbb{D}_{2\alpha,0}$, i.e., $T$ under the isomorphism of Theorem 7.5.4 is in $L^2(\mu_\alpha, W^n)$ on which we have Itô representation (cf. Appendix to the Chapter IV).

Proposition 7.5.8 Suppose that $A^{-1}$ is $p$-nuclear, i.e., there exists some $p \geq 1$ such that $A^{-p}$ is nuclear. Then $\Phi$ is a nuclear Fréchet space.

Proof: This goes as in the classical white noise case, except that the eigenvectors of $\Gamma(A^{-1})$ are of the form $H_\alpha(\delta h_{\alpha_1}, \ldots, \delta h_n)$ with $h_{\alpha_1}$ are the eigenvectors of $A$. 


7.6 Applications to positive distributions

Let $T \in \Phi'$ be a positive distribution. Then, from the construction of the distribution spaces, there exists some $I D_{-\alpha, -k}$ such that $T \in I D_{-\alpha, -k}$ and $\langle T, \varphi \rangle \geq 0$ for any $\varphi \in D_{q,k}, \varphi \geq 0$. Hence $i_\alpha(T)$ is a positive functional on $D_{1,k}$ which is the Sobolev space on $W^\alpha$. Therefore $i_\alpha(T)$ is a Radon measure on $W^{-\alpha}$ and we find in fact that the support of $T$ is $W^{-\alpha}$ which is much smaller than $H_{-\infty}$. Let us give an example of such a positive distribution:

**Proposition 7.6.1** Assume that $u \in L^2(\mu, H)$ such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} E \left[ |u|^n_H \right]^{1/2} < \infty.$$  \hspace{1cm} (7.6.6)

Then the mapping defined by

$$\phi \rightarrow E[\phi(w + u(w))] = < L_u, \phi >$$

is a positive distribution and it can be expressed as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \delta^n u^\otimes_n.$$  

Moreover this sum is weakly uniformly convergent in $D_{-\alpha, 0}$, for any $\alpha > 0$ such that $\|A^{-1}\|^{2\alpha} < \frac{1}{2}$.

**Proof:** It follows trivially from the Taylor formula and from the definition of $\delta^n$ as the adjoint of $\nabla^n$ with respect to $\mu$, that

$$< L_u, \phi > = \sum_{n=0}^{\infty} \frac{1}{n!} E \left[ \phi \delta^n u^\otimes_n \right]$$

for any cylindrical, analytic function $\phi$. To complete the proof it suffices to show that $\phi \rightarrow < L_u, \phi >$ extends continuously to $\Phi$. If $\phi$ has the chaos decomposition

$$\phi = \sum_{k=0}^{\infty} I_k(\phi_k),$$

with $\phi_k \in H_\infty^{\otimes k}$, then

$$E \left[ \|\nabla^n \phi\|_{H^{\otimes k}}^2 \right] = \sum_{k \geq n} \frac{(k!)^2}{(k-n)!} \|\phi_k\|_{H^{\otimes k}}^2$$

$$\leq \sum_{k \geq n} e^{-ak} \frac{(k!)^2}{(k-n)!} \|\phi_k\|_{H^{\otimes k}}^2,$$
where $c^{-\alpha}$ is an upper bound for the norm of $A^{-\alpha/2}$. Hence we the following a priori bound:

$$| <L_u, \phi> | \leq \sum_{n=0}^{\infty} \frac{1}{n!} <\nabla^n \phi, u^{\otimes n}> |$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} E \left[ ||u||_H^n \right]^{1/2} \left( \sum_{k\geq n} c^{-\alpha k} \frac{(k!)^2}{(k-n)!} ||\phi_k||_{H^{k_0}}^2 \right)^{1/2}$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} E \left[ ||u||_H^n \right]^{1/2} \left( \sum_{k\geq n} (2c^{-\alpha})^k ||\phi_k||_{H^{k_0}}^2 \right)^{1/2} .$$

Choose now $\alpha$ such that $c^{-\alpha} < 1/2$, then the sum inside the square root is dominated by

$$\sum_{k=0}^{\infty} k! ||\phi_k||_{H^{k_0}}^2 = ||\phi||_{H^{k_0}}^2 .$$

Hence the sum is absolutely convergent provided that $u$ satisfies the condition (7.6.6).

7.7 Exercises

1. Let $K$ be a closed vector subspace of $H$ and denote by $P$ the orthogonal projection associated to it. Denote by $\mathcal{F}_K$ the sigma algebra generated by $\{ \delta k, k \in K \}$. Prove that

$$\Gamma(P)f = E[f|\mathcal{F}_K],$$

for any $f \in L^2(\mu)$.

2. Assume that $M$ and $N$ are two closed vector subspaces of the Cameron-Martin space $H$, denote by $P$ and $Q$ respectively the corresponding orthogonal projections. For any $f, g \in L^2(\mu)$ prove the following inequality:

$$|E[(f - E[f])(g - E[g])]| \leq ||PQ|| \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)},$$

where $||PQ||$ is the operator norm of $PQ$.

3. Prove that $\Gamma(e^{-tI_H}) = P_t, t \geq 0$, where $P_t$ denotes the Ornstein-Uhlenbeck semigroup.

4. Let $B$ be a bounded operator on $H$, define $d\Gamma(B)$ as

$$d\Gamma(B)f = \frac{d}{dt} \Gamma(e^{tB})f |_{t=0} .$$

Prove that

$$d\Gamma(B)f = \delta \{ B \nabla f \}$$

and that

$$d\Gamma(B)(fg) = f \, d\Gamma(B)g + g \, d\Gamma(B)f ,$$

(i.e., $d\Gamma(B)$ is a derivation) for any $f, g \in \mathcal{D}$ whenever $B$ is skew-symmetric.
Notes and suggested reading

The fact that a positive Meyer distribution defines a Radon measure on the Wiener space has been indicated for the first time in \[3\]. The notion of the capacity in an abstract frame has been studied by several people, cf. in particular \[12\], \[56\] and the references there. Application to the local times is original, the capacity version of 0 – 1–law is taken from \[52\]. Proposition 7.6.1 is taken from \[46\], for the more general distribution spaces we refer the reader to \[46\] \[48\] \[63\] and to the references there.
Chapter 8

Characterization of independence of some Wiener functionals

Introduction

In probability theory, probably the most important concept is the independence since it is the basic property which differentiates the probability theory from the abstract measure theory or from the functional analysis. Besides it is almost always difficult to verify the independence of random variables. In fact, even in the elementary probability, the tests required to verify the independence of three or more random variables get very quickly quite cumbersome. Hence it is very tempting to try to characterize the independence of random variables via the local operators as $\nabla$ or $\delta$ that we have studied in the preceding chapters.

Let us begin with two random variables: let $F, G \in \mathbb{D}_{p,1}$ for some $p > 1$. They are independent if and only if

$$E[e^{i\alpha F} e^{i\beta G}] = E[e^{i\alpha F}] E[e^{i\beta G}]$$

for any $\alpha, \beta \in \mathbb{R}$, which is equivalent to

$$E[a(F)b(G)] = E[a(F)] E[b(G)]$$

for any $a, b \in C_b(\mathbb{R})$.

Let us denote by $\tilde{a}(F) = a(F) - E[a(F)]$, then we have: $F$ and $G$ are independent if and only if

$$E[\tilde{a}(F) \cdot b(G)] = 0, \quad \forall a, b \in C_b(\mathbb{R}).$$
Since $e^{i\alpha x}$ can be approximated point-wise with smooth functions, we can suppose as well that $a, b \in C^1_b(\mathbb{R})$ (or $C^\infty_0(\mathbb{R})$). Since $L$ is invertible on the centered random variables, we have

\[ E[\tilde{a}(F)b(G)] = E[L^{-1}\tilde{a}(F) \cdot b(G)] 
= E[\delta \nabla L^{-1}\tilde{a}(F) \cdot b(G)] 
= E[(\nabla L^{-1}\tilde{a}(F), \nabla (b(G)))_H] 
= E[((I + L)^{-1}\tilde{a}(F) \cdot \nabla (b(G)))_H] 
= E[b'(G) \cdot ((I + L)^{-1}(a'(F) \nabla F), \nabla G)_H] 
= E[b'(G) \cdot E[((I + L)^{-1}(a'(F) \nabla F), \nabla G)_H | \sigma(G))] \]

In particular choosing $a = e^{i\alpha x}$, we find that

**Proposition 8.0.1** $F$ and $G$ (in $\mathbb{D}_{p,1}$) are independent if and only if

\[ E[((I + L)^{-1}(e^{i\alpha F} \nabla F), \nabla G)_H | \sigma(G))] = 0 \text{ a.s.} \]

**8.1 The case of multiple Wiener integrals**

Proposition 8.0.1 is not very useful, because of the non-localness property of the operator $L^{-1}$. Let us however look at the case of multiple Wiener integrals:

First recall the following multiplication formula of the multiple Wiener integrals:

**Lemma 8.1.1** Let $f \in \hat{L}^2([0,1]^p)$, $g \in \hat{L}^2([0,1]^q)$. Then we have

\[ I_p(f) I_q(g) = \sum_{m=0}^{p \wedge q} \frac{p! \cdot q!}{m!(p-m)!(q-m)!} I_{p+q-2m}(f \otimes_m g), \]

where $f \otimes_m g$ denotes the contraction of order $m$ of the tensor $f \otimes g$, i.e., the partial scalar product of $f$ and $g$ in $L^2([0,1]^m)$.

By the help of this lemma we will prove:

**Theorem 8.1.2** $I_p(f)$ and $I_q(g)$ are independent if and only if

\[ f \otimes_1 g = 0 \text{ a.s. on } [0,1]^{p+q-2}. \]
Proof: \((\Rightarrow)\): By independence, we have
\[
E[I_p^2 I_q^2] = p!\|f\|^2 q!\|g\|^2 = p!q!\|f \otimes g\|^2.
\]
On the other hand
\[
I_p(f)I_q(g) = \sum_0^{p \wedge q} m!C_m p!C_m q! I_{p+q-2m}(f \otimes_m g),
\]
hence
\[
E[(I_p(f)I_q(g))^2] = \sum_0^{p \wedge q} (m!C_m p!C_m q!)^2 (p + q - 2m)!\|f \otimes_m g\|^2 
\geq (p + q)!\|f \otimes g\|^2 \quad \text{(dropping the terms with } m \geq 1).\]

We have, by definition:
\[
\|f \otimes g\|^2 = \left\| \frac{1}{(p + q)!} \sum_{\sigma \in S_{p+q}} f(t_{\sigma(1)}, \ldots, t_{\sigma(p)})g(t_{\sigma(p+1)}, \ldots, t_{\sigma(p+q)}) \right\|^2 
= \frac{1}{((p + q)!)^2} \sum_{\sigma, \pi \in S_{p+q}} \lambda_{\sigma, \pi},
\]
where \(S_{p+q}\) denotes the group of permutations of order \(p + q\) and
\[
\lambda_{\sigma, \pi} = \int_{[0,1]^{p+q}} f(t_{\sigma(1)}, \ldots, t_{\sigma(p)})g(t_{\sigma(p+1)}, \ldots, t_{\sigma(p+q)}) 
\cdot f(t_{\pi(1)}, \ldots, t_{\pi(p)})g(t_{\pi(p+1)}, \ldots, t_{\pi(p+q)})dt_1 \ldots dt_{p+q}.
\]
Without loss of generality, we may suppose that \(p \leq q\). Suppose now that \((\sigma(1), \ldots, \sigma(p))\) and \((\pi(1), \ldots, \pi(p))\) has \(k \geq 0\) elements in common. If we use the block notations, then
\[
(t_{\sigma(1)}, \ldots, t_{\sigma(p)}) = (A_k, \bar{A})
(t_{\sigma(p+1)}, \ldots, t_{\sigma(p+q)}) = B
(t_{\pi(1)}, \ldots, t_{\pi(p)}) = (A_k, \bar{C})
(t_{\pi(p+1)}, \ldots, t_{\pi(p+q)}) = D
\]
where \(A_k\) is the sub-block containing elements common to \((t_{\pi(1)}, \ldots, t_{\pi(p)})\) and \((t_{\sigma(1)}, \ldots, t_{\sigma(p)})\). Then we have
\[
\lambda_{\sigma, \pi} = \int_{[0,1]^{p+q}} f(A_k, \bar{A})g(B) \cdot f(A_k, \bar{C})g(D)dt_1 \ldots dt_{p+q}.
\]
Note that \( A_k \cup \tilde{A} \cup B = A_k \cup \tilde{C} \cup D = \{ t_1, \ldots, t_{p+q} \} \), \( \tilde{A} \cap \tilde{C} = \emptyset \). Hence we have \( \tilde{A} \cup B = \tilde{C} \cup D \). Since \( \tilde{A} \cap \tilde{C} = \emptyset \), we have \( C \subset B \) and \( \tilde{A} \subset D \). From the fact that \((\tilde{A}, B)\) and \((\tilde{C}, D)\) are the partitions of the same set, we have \( D \setminus \tilde{A} = B \setminus \tilde{C} \). Hence we can write, with the obvious notations:

\[
\lambda_{\sigma, \pi} = \\
= \int_{[0,1]^{p+q}} f(A_k, \tilde{A}) g(\tilde{C}, B \setminus \tilde{C}) \cdot f(A_k, \tilde{C}) g(\tilde{A}, D \setminus \tilde{A}) dt_1 \ldots dt_{p+q}
\]

where we have used the relation \( D \setminus \tilde{A} = B \setminus \tilde{C} \) in the second line of the above equalities. Note that for \( k = p \) we have \( \lambda_{\sigma, \pi} = \| f \otimes g \|_{L^2}^2 \). Hence we have

\[
E[I_p^2(f) I_q^2(g)] = p!\| f \|^2 \cdot q!\| g \| ^2 \\
\geq (p + q)! \left[ \frac{1}{((p + q)!)^2} \left( \sum_{\sigma, \pi} \lambda_{\sigma, \pi}(k = p) + \sum_{\sigma, \pi} \lambda_{\sigma, \pi}(k \neq p) \right) \right].
\]

The number of \( \lambda_{\sigma, \pi} \) with \( (k = p) \) is exactly \( \binom{p+q}{p} (p!)^2 (q!)^2 \), hence we have

\[
p!q!\| f \| ^2 \| g \| ^2 \geq p!q!\| f \otimes g \| ^2 + \sum_{k=0}^{p-1} c_k \| f \otimes_{p-k} g \|_{L^2([0,1]^{q-p+2k})}^2
\]

with \( c_k > 0 \). For this relation to hold we should have

\[
\| f \otimes_{p-k} g \| = 0, \quad k = 0, \ldots, p - 1
\]

in particular for \( k = p - 1 \), we have

\[
\| f \otimes_1 g \| = 0.
\]

(\( \Leftarrow \)): From the Proposition 8.0.1 we see that it is sufficient to prove

\[
((I + L)^{-1} e^{i\alpha F} \nabla F, \nabla I_q(g)) = 0 \quad \text{a.s.}
\]

with \( F = I_p(f) \), under the hypothesis \( f \otimes_1 g = 0 \) a.s. Let us write

\[
e^{i\alpha I_p(f)} = \sum_{k=0}^{\infty} I_k(h_k),
\]
then
\[ e^{i\alpha I_p(f)} \nabla I_p(f) = p \sum_{k=0}^{\infty} I_k(h_k) \cdot I_{p-1}(f) \]
\[ = p \sum_{k=0}^{\infty} \sum_{r=0}^{k \wedge (p-1)} \alpha_{p,k,r} I_{p-1+k-2r}(h_k \otimes_r f) . \]

Hence
\[ (I + \mathcal{L})^{-1} e^{i\alpha F} \nabla F = p \sum_{k} \sum_{r=0}^{k \wedge (p-1)} (1 + p + k - 1 - 2r)^{-1} I_{p-1+k-2r}(h_k \otimes_p f) . \]

When we take the scalar product with \( \nabla I_q(g) \), we will have terms of the type:
\[ (I_{p-1+k-2r}(h_k \otimes_r f), I_{q-1}(g))_H = \]
\[ = \sum_{i=1}^{\infty} I_{p-1+k-2r}(h_k \otimes_r f(e_i)) I_{q-1}(g(e_i)). \]

If we use the multiplication formula to calculate each term, we find the terms as
\[ \sum_{i=1}^{\infty} \int (h_k \otimes_r f(e_i))(t_1, \ldots, t_{p+k-2r-1}) g(e_i)(t_1, \ldots, t_{q-1}) dt_1 dt_2 \ldots \]
\[ = \int_{\theta=0}^{1} \int f(\theta,t_1 \ldots,t_{p+k-2r-1}) g(\theta,t_1, \ldots, t_{q-1}) d\theta dt_1 \ldots \]

From the hypothesis we have
\[ \int_{0}^{1} f(\theta,t_1 \ldots ) g(\theta,s_1 \ldots ) d\theta = 0 \quad \text{a.s.} , \]
hence the Fubini theorem completes the proof. \( \blacksquare \)

**Remark:** For a more elementary proof of the sufficiency of Theorem 8.1.2 cf. [43].

**Remark 8.1.3** In the proof of the necessity we have used only the fact that \( I_p(f)^2 \) and \( I_q(g)^2 \) are independent. Hence, as a byproduct we obtain also the fact that \( I_p \) and \( I_q \) are independent if and only if their squares are independent.

**Corollary 8.1.4** Let \( f \) and \( g \) be symmetric \( L^2 \)-kernels respectively on \([0,1]^p\) and \([0,1]^q\). Let
\[ S_f = \text{span}\{ f \otimes_{p-1} h : h \in L^2([0,1])^{p-1} \} \]
Independence

and

\[ S_g = \text{span}\{g \otimes_{q-1} k; k \in L^2([0,1]^{q-1})\}. \]

Then the following are equivalent:

i) \( I_p(f) \) and \( I_q(g) \) are independent,

ii) \( I_p(f)^2 \) and \( I_q(g)^2 \) are independent,

iii) \( S_f \) and \( S_g \) are orthogonal in \( H \),

iv) the Gaussian-generated \( \sigma \)-fields \( \sigma\{I_1(k); k \in S_f\} \) and \( \sigma\{I_1(l); l \in S_g\} \) are independent.

Proof: As it is indicated in Remark 8.1.3, the independence of \( I_p \) and \( I_q \) is equivalent to the independence of their squares.

(i\( \Rightarrow \)iii): The hypothesis implies that 

\[ f \otimes_{1} g = 0 \text{ a.s.} \]

If \( a \in S_f, b \in S_g \) then they can be written as finite linear combinations of the vectors \( f \otimes_{p-1} h \) and \( g \otimes_{q-1} k \) respectively. Hence, it suffices to assume, by linearity, that 

\[ a = f \otimes_{p-1} h \text{ and } b = g \otimes_{q-1} k. \]

Then it follows from the Fubini theorem

\[(a, b) = (f \otimes_{p-1} h, g \otimes_{q-1} k) = (f \otimes_{1} g, h \otimes k)(L^2(\otimes_{p+q-2})) = 0.\]

(iii\( \Rightarrow \)i) If \( f \otimes_{1} g, h \otimes k \) = 0 for all \( h \in L^2([0,1]^{p-1}), k \in L^2([0,1]^{q-1}) \), then \( f \otimes_{1} g = 0 \text{ a.s.} \) since finite combinations of \( h \otimes k \) are dense in \( L^2([0,1]^{p+q-2}) \).

Finally, the equivalence of (iii) and (iv) is obvious.

Proposition 8.1.5 Suppose that \( I_p(f) \) is independent of \( I_q(g) \) and \( I_p(f) \) is independent of \( I_r(h) \). Then \( I_p(f) \) is independent of \( \{I_q(g), I_r(h)\} \).

Proof: We have 

\[ f \otimes_{1} g = f \otimes_{1} h = 0 \text{ a.s.} \]

This implies the independence of \( I_p(f) \) and \( \{I_q(g), I_r(h)\} \) from the calculations similar to those of the proof of sufficiency of the theorem.

In a similar way we have

Proposition 8.1.6 Let \( \{I_{p\alpha}(f_\alpha); \alpha \in J\} \) and \( \{I_{q\beta}(g_\beta); \beta \in K\} \) be two arbitrary families of multiple Wiener integrals. The two families are independent if and only if \( I_{p\alpha}(f_\alpha) \) is independent of \( I_{q\beta}(g_\beta) \) for all \( (\alpha, \beta) \in J \times K \).

Corollary 8.1.7 If \( I_p(f) \) and \( I_q(g) \) are independent, so are also \( I_p(f)(w + \hat{h}) \) and \( I_q(g)(w + k) \) for any \( \hat{h}, k \in H \).
Proof: Let us denote, respectively, by \( h \) and \( k \) the Lebesgue densities of \( \tilde{h} \) and \( \tilde{k} \). We have then
\[
I_p(f)(w + \tilde{h}) = \sum_{i=0}^{p} \binom{p}{i} (I_{p-i}(f), h^{\otimes i})_{H^{\otimes i}}.
\]
Let us define \( f[h^{\otimes i}] \in L^2[0, 1]^{p-i} \) by
\[
I_{p-i}(f[h^{\otimes i}]) = (I_{p-i}(f), h^{\otimes i}).
\]
If \( f \otimes_1 g = 0 \) then it is easy to see that
\[
f[h^{\otimes i}] \otimes_1 g[k^{\otimes j}] = 0,
\]
hence the corollary follows from Theorem 8.1.2.

From the corollary it follows

**Corollary 8.1.8** \( I_p(f) \) and \( I_q(g) \) are independent if and only if the germ \( \sigma \)-fields
\[
\sigma \{ I_p(f), \nabla I_p(f), \ldots, \nabla^{p-1} I_p(f) \}
\]
and
\[
\sigma \{ I_q(g), \ldots, \nabla^{q-1} I_q(g) \}
\]
are independent.

**Corollary 8.1.9** Let \( X, Y \in L^2(\mu) \), \( Y = \sum_0^{\infty} I_n(g_n) \). If
\[
\nabla X \otimes_1 g_n = 0 \quad \text{a.s.} \quad \forall n,
\]
then \( X \) and \( Y \) are independent.

**Proof:** This follows from Proposition 8.0.1.

**Corollary 8.1.10** In particular, if \( \tilde{h} \in H \), then \( \nabla \tilde{h} \varphi = 0 \) a.s. implies that \( \varphi \) and \( I_1(h) = \delta \tilde{h} \) are independent.

8.2 Exercises

1. Let \( f \in \hat{L}^2([0, 1]^p) \) and \( h \in L^2([0, 1]) \). Prove the product formula
\[
I_p(f) I_1(h) = I_{p+1}(f \otimes h) + p I_{p-1}(f \otimes_1 h).
\]

2. Prove by induction and with the help of (8.2.1), the general multiplication formula
\[
I_p(f) I_q(g) = \sum_{m=0}^{p \wedge q} \frac{p! q!}{m!(p-m)!(q-m)!} I_{p+q-2m}(f \otimes_m g),
\]
where \( f \in \hat{L}^2([0, 1]^p) \) and \( g \in L^2([0, 1]^q) \).
Notes and suggested reading

All the results of this chapter are taken from [93, 94], cf. also [43] for some simplification of the sufficiency of Theorem 8.1.2. Note that, in Theorem 8.1.2, we have used only the independence of \( I_p(f)^2 \) and \( I_q(g)^2 \). Hence two multiple Ito-Wiener integrals are independent if and only if their squares are independent.
Chapter 9

Moment inequalities for Wiener functionals

Introduction

In several applications, as limit theorems, large deviations, degree theory of Wiener maps, calculation of the Radon-Nikodym densities, etc., it is important to control the (exponential) moments of Wiener functionals by those of their derivatives. In this chapter we will give two results on this subject. The first one concerns the tail probabilities of the Wiener functionals with essentially bounded Gross-Sobolev derivatives. This result is a straightforward generalization of the celebrated Fernique’s lemma which says that the square of the supremum of the Brownian path on any bounded interval has an exponential moment provided that it is multiplied with a sufficiently small, positive constant. The second inequality says that for a Wiener functional $F \in \mathcal{D}_{p,1}$, we have

$$E_w \times E_z [U(\pi_1(\nabla F(w)) (z))] \leq E_w \times E_z \left[ U \left( \frac{\pi}{2} I_1(\nabla F(w)) \right) (z) \right], \quad (9.0.1)$$

where $w$ and $z$ represent two independent Wiener paths, $E_w$ and $E_z$ are the corresponding expectations, and $I_1(\nabla F(w))(z)$ is the first order Wiener integral with respect to $z$ of $\nabla F(w)$ and $U$ is any lower bounded, convex function on $\mathbb{R}$. Then combining these two inequalities we will obtain some interesting majorations.

In the next section we show that the log-Sobolev inequality implies the exponential integrability of the square of the Wiener functionals whose derivatives are essentially bounded. In this section we study with general measures which satisfy a logarithmic Sobolev inequality.
The next inequality is an interpolation inequality which says that the Sobolev norm of first order can be upper bounded by the product of the second order and of the zero-th order Sobolev norms.

In the last part we study the exponential integrability of the Wiener functionals in the divergence form, a problem which has gained considerable importance due to the degree theorem on the Wiener space as it is explained in more detail in the notes at the end of this chapter.

9.1 Exponential tightness

First we will show the following result which is a consequence of the Doob inequality:

**Theorem 9.1.1** Let $\varphi \in \mathcal{D}_{p,1}$ for some $p > 1$. Suppose that $\nabla \varphi \in L^\infty(\mu, H)$. Then we have

$$
\mu\{\|\varphi\| > c\} \leq 2 \exp\left\{-\frac{(c - E[\varphi])^2}{2\|\nabla \varphi\|^2_{L^\infty(\mu, H)}}\right\}
$$

for any $c \geq 0$.

**Proof:** Suppose that $E[\varphi] = 0$. Let $(e_i) \subset H$ be a complete, orthonormal basis of $H$. Define $V_n = \sigma\{\delta e_1, \ldots, \delta e_n\}$ and let $\varphi_n = E[P_{1/n}\varphi|\nabla_n]$, where $P_t$ denotes the Ornstein-Uhlenbeck semi-group on $W$. Then, from Doob’s Lemma,

$$
\varphi_n = f_n(\delta e_1, \ldots, \delta e_n).
$$

Note that, since $f_n \in \bigcap_{p,k} W_{p,k}(\mathbb{R}^n; \mu_n)$, the Sobolev embedding theorem implies that after a modification on a set of null Lebesgue measure, $f_n$ can be chosen in $C^\infty(\mathbb{R}^n)$. Let $(B_t; t \in [0, 1])$ be an $\mathbb{R}^n$-valued Brownian motion. Then

$$
\mu\{\|\varphi_n\| > c\} = P\{|f_n(B_1)| > c\}
\leq P\{\sup_{t \in [0,1]} |E[f_n(B_1)|B_t]| > c\}
= P\{\sup_{t \in [0,1]} |Q_{1-t}f_n(B_t)| > c\},
$$

where $P$ is the canonical Wiener measure on $C([0, 1], \mathbb{R}^n)$ and $Q_t$ is the heat kernel associated to $(B_t)$, i.e.

$$
Q_t(x, A) = P\{B_t + x \in A\}.
$$
From the Ito formula, we have

\[ Q_{1-t}f_n(B_t) = Q_{1}f_n(B_0) + \int_0^t (DQ_{1-s}f_n(B_s), dB_s). \]

By definition

\[ Q_{1}f_n(B_0) = Q_{1}f_n(0) = \int f_n(y) \cdot Q_1(0, dy) \]
\[ = \int_{\mathbb{R}^n} f_n(y) e^{-\frac{1}{2}|y|^2} \frac{dy}{(2\pi)^{n/2}} \]
\[ = E \left[ E[P_{1/n}\varphi | V_n] \right] \]
\[ = E \left[ P_{1/n}\varphi \right] \]
\[ = E[\varphi] \]
\[ = 0. \]

Moreover we have \( DQ_t f = Q_t Df \), hence

\[ Q_{1-t}f_n(B_t) = \int_0^t (Q_{1-s}Df_n(B_s), dB_s) = M^n_t. \]

The Doob-Meyer process \( \langle M^n, M^n \rangle_t, t \in \mathbb{R}_+ \) of the martingale \( M^n \) can be controlled as

\[ \langle M^n, M^n \rangle_t = \int_0^t |DQ_{1-s}f_n(B_s)|^2 ds \]
\[ \leq \int_0^t \|Df_n\|_C^2 ds = t\|Df_n\|_C^2 \]
\[ = t\|\nabla f_n\|_{L^\infty(\mu_n)}^2 \]
\[ \leq t\|\nabla \varphi\|_{L^\infty(\mu, H)}^2. \]

Hence from the exponential Doob inequality, we obtain

\[ P \left\{ \sup_{t \in [0,1]} |Q_{1-t}f_n(B_t)| > c \right\} \leq 2 \exp \left[ -\frac{c^2}{2\|\nabla \varphi\|_{L^\infty(\mu, H)}^2} \right]. \]

Consequently

\[ \mu\{|\varphi_n| > c\} \leq 2 \exp \left[ -\frac{c^2}{2\|\nabla \varphi\|_{L^\infty(\mu, H)}^2} \right]. \]

Since \( \varphi_n \to \varphi \) in probability the proof is completed. \( \square \)
Corollary 9.1.2 Under the hypothesis of the theorem, for any
\[ \lambda < \left[ 2 \| \nabla \varphi \|_{L^\infty(\mu, H)} \right]^{-1}, \]
we have
\[ E \left[ \exp \lambda |\varphi|^2 \right] < \infty. \]

Proof: The first part follows from the fact that, for \( F \geq 0 \) a.s.,
\[ E[F] = \int_0^\infty P\{F > t\} dt. \]

Remark: In the next sections we will give more precise estimate for \( E[\exp \lambda F^2]. \)

In the applications, we encounter random variables \( F \) satisfying
\[ |F(w + h) - F(w)| \leq c|h|_H, \]
almost surely, for any \( h \) in the Cameron-Martin space \( H \) and a fixed constant \( c > 0 \), without any hypothesis of integrability. For example, \( F(w) = \sup_{t \in [0,1]} |w(t)| \), defined on \( C_0[0,1] \) is such a functional. In fact the above hypothesis contains the integrability and Sobolev differentiability of \( F \). We begin first by proving that under the integrability hypothesis, such a functional is in the domain of \( \nabla \):

Lemma 9.1.3 Suppose that \( F : W \mapsto \mathbb{R} \) is a measurable random variable in \( \cup_{p>1} L^p(\mu) \), satisfying
\[ |F(w + h) - F(w)| \leq c|h|_H, \quad (9.1.2) \]
almost surely, for any \( h \in H \), where \( c > 0 \) is a fixed constant. Then \( F \) belongs to \( \mathbb{D}_{p,1} \) for any \( p > 1 \).

Remark: If in (9.1.2) the negligible set on which the inequality is satisfied is independent of \( h \in H \), then the functional \( F \) is called H-Lipschitz.

Proof: Since, for some \( p_0 > 1 \), \( F \in L^{p_0} \), the distributional derivative of \( F \), \( \nabla F \) exists. We have \( \nabla_k F \in \mathbb{D}' \) for any \( k \in H \). Moreover, for \( \phi \in \mathbb{D} \), from the integration by parts formula
\[ E[\nabla_k F \phi] = -E[F \nabla_k \phi] + E[F \delta k \phi] \]
\[ = - \frac{d}{dt} \bigg|_{t=0} E[F \phi(w + tk)] + E[F \delta k \phi] \]
\[ = - \frac{d}{dt} \bigg|_{t=0} E[F(w - tk) \phi \varepsilon(t\delta k)] + E[F \delta k \phi] \]
\[ = \lim_{t \to 0^-} -E \left[ \frac{F(w - tk) - F(w)}{t} \phi \right], \]
where $\varepsilon(\delta k)$ denotes the Wick exponential of the Gaussian random variable $\delta k$, i.e.,

$$
\varepsilon(\delta k) = \exp\left\{ \delta k - \frac{1}{2} |k|^2 \right\}.
$$

Consequently,

$$
|E[\nabla_k F \phi]| \leq c |k|_H E[|\phi|] \leq c |k|_H \|\phi\|_q,
$$

for any $q > 1$, i.e., $\nabla F$ belongs to $L^p(\mu, H)$ for any $p > 1$. Let now $(e_i; i \in \mathbb{N})$ be a complete, orthonormal basis of $H$, denote by $V_n$ the sigma-field generated by $\delta e_1, \ldots, \delta e_n$, $n \in \mathbb{N}$ and let $\pi_n$ be the orthogonal projection onto the the subspace of $H$ spanned by $e_1, \ldots, e_n$. Let us define

$$
F_n = E[P_{1/n}F|V_n],
$$

where $P_{1/n}$ is the Ornstein-Uhlenbeck semi-group at the instant $t = 1/n$. Then $F_n \in \cap_k D_{p_0,k}$ and it is immediate, from the martingale convergence theorem and from the fact that $\pi_n$ tends to the identity operator of $H$ pointwise, that

$$
\nabla F_n = E[e^{-1/n} \pi_n P_{1/n} \nabla F|V_n] \to \nabla F,
$$

in $L^p(\mu, H)$, for any $p > 1$, as $n$ tends to infinity. Since, by construction, $(F_n; n \in \mathbb{N})$ converges also to $F$ in $L^{p_0}(\mu)$, $F$ belongs to $D_{p_0,1}$. Hence we can apply the Corollary 9.1.2.

\[\square\]

**Lemma 9.1.4** Suppose that $F : W \mapsto \mathbb{R}$ is a measurable random variable satisfying

$$
|F(w + h) - F(w)| \leq c|h|_H,
$$

almost surely, for any $h \in H$, where $c > 0$ is a fixed constant. Then $F$ belongs to $D_{p,1}$ for any $p > 1$.

**Proof:** Let $F_n = |F| \wedge n$, $n \in \mathbb{N}$. A simple calculation shows that

$$
|F_n(w + h) - F_n(w)| \leq c|h|_H,
$$

hence $F_n \in D_{p,1}$ for any $p > 1$ and $|\nabla F_n| \leq c$ almost surely from Lemma 9.1.3. We have from the Ito-Clark formula (cf. Theorem 6.1.4),

$$
F_n = E[F_n] + \int_0^1 E[D_s F_n|F_s]dW_s.
$$
From the definition of the stochastic integral, we have
\[
E \left[ \left( \int_0^1 E[D_s F_n | F_s] dW_s \right)^2 \right] = \int_0^1 E[E[D_s F_n | F_s]^2] ds \\
\leq E \left[ \int_0^1 |D_s F_n|^2 ds \right] \\
= E[|\nabla F_n|^2] \\
\leq c^2.
\]
Since \( F_n \) converges to \(|F|\) in probability, and the stochastic integral is bounded in \( L^2(\mu) \), by taking the difference, we see that \( (E[F_n], n \in \mathbb{N}) \) is a sequence of (degenerate) random variables bounded in the space of random variables under the topology of convergence in probability, denoted by \( L^0(\mu) \). Therefore \( \sup_n \mu \{ E[F_n] > c \} \rightarrow 0 \) as \( c \rightarrow \infty \). Hence \( \lim_n E[F_n] = E[|F|] \) is finite. Now we apply the dominated convergence theorem to obtain that \( F \in L^2(\mu) \). Since the distributional derivative of \( F \) is a square integrable random variable, \( F \in D_{2,1} \). We can now apply the Lemma \ref{lem:moment_inequalities} which implies that \( F \in D_{p,1} \) for any \( p \).

**Remark:** Although we have used the classical Wiener space structure in the proof, the case of the Abstract Wiener space can be reduced to this case using the method explained in the appendix of Chapter IV.

**Corollary 9.1.5 (Fernique’s Lemma)** For any \( \lambda < \frac{1}{2} \), we have
\[
E[\exp \lambda \|w\|_W^2] < \infty,
\]
where \( \|w\| \) is the norm of the Wiener path \( w \in W \).

**Proof:** It suffices to remark that
\[
\|w + h\| - \|w\| \leq |h|_H
\]
for any \( h \in H \) and \( w \in W \).

\[
\square
\]

### 9.2 Coupling inequalities

We begin with the following elementary lemma (cf. \[71\]):
**Lemma 9.2.1** Let $X$ be a Gaussian random variable with values in $\mathbb{R}^d$. Then for any convex function $U$ on $\mathbb{R}$ and $C^1$-function $V : \mathbb{R}^d \to \mathbb{R}$, we have the following inequality:

$$E[U(V(X) - V(Y))] \leq E\left[U\left(\frac{\pi}{2}(V'(X), Y)|_{\mathbb{R}^d}\right)\right],$$

where $Y$ is an independent copy of $X$ and $E$ is the expectation with respect to the product measure.

**Proof:** Let $X_\theta = X \sin \theta + Y \cos \theta$. Then

$$V(X) - V(Y) = \int_{[0, \pi/2]} \frac{d}{d\theta} V(X_\theta) d\theta = \int_{[0, \pi/2]} \langle V'(X_\theta), X'_\theta \rangle_{\mathbb{R}^d} d\theta = \frac{\pi}{2} \int_{[0, \pi/2]} \langle V'(X_\theta), X'_\theta \rangle_{\mathbb{R}^d} d\tilde{\theta},$$

where $d\tilde{\theta} = \frac{d\theta}{\pi/2}$. Since $U$ is convex, we have

$$U(V(X) - V(Y)) \leq \int_0^{\pi/2} U\left(\frac{\pi}{2}(V'(X_\theta), X'_\theta)\right) d\tilde{\theta}.$$ 

Moreover $X_\theta$ and $X'_\theta$ are two independent Gaussian random variables with the same law as the one of $X$. Hence

$$E[U(V(X) - V(Y))] \leq \int_0^{\pi/2} E\left[U\left(\frac{\pi}{2}(V'(X_\theta), X'_\theta)\right)\right] d\tilde{\theta} = E\left[U\left(\frac{\pi}{2}(V'(X), Y)|_{\mathbb{R}^d}\right)\right].$$

Now we will extend this result to the Wiener space:

**Theorem 9.2.2** Suppose that $\varphi \in \mathcal{D}_{p,1}$, for some $p > 1$ and $U$ is a lower bounded, convex function (hence lower semi-continuous) on $\mathbb{R}$. We have

$$E[U(\varphi(w) - \varphi(z))] \leq E\left[U\left(\frac{\pi}{2} I_1(\nabla \varphi(w))|_{\mathbb{R}^d}\right)\right]$$

where $E$ is taken with respect to $\mu(dw) \times \mu(dz)$ on $W \times W$ and on the classical Wiener space, we have

$$I_1(\nabla \varphi(w))(z) = \int_0^1 \frac{d}{dt} \nabla \varphi(w, t)dz_t.$$
\textbf{Proof:} Suppose first that
\[ \varphi = f(\delta h_1(w), \ldots, \delta h_n(w)) \]
with \( f \) smooth on \( \mathbb{R}^n, h_i \in H, (h_i, h_j) = \delta_{ij}. \) We have
\[
I_1(\nabla \varphi(w))(z) = I_1 \left( \sum_{i=1}^{n} \partial_i f(\delta h_1(w), \ldots, \delta h_n(w))h_i \right)
= \sum_{i=1}^{n} \partial_i f(\delta h_1(w), \ldots, \delta h_n(w))I_1(h_i)(z)
= \left( f'(X), Y \right)_{\mathbb{R}^n}
\]
where \( X = (\delta h_1(w), \ldots, \delta h_n(w)) \) and \( Y = (\delta h_1(z), \ldots, \delta h_n(z)). \) Hence the inequality is trivially true in this case.

For general \( \varphi, \) let \( (h_i) \) be a complete, orthonormal basis in \( H, \)
\[ V_n = \sigma\{\delta h_1, \ldots, \delta h_n\} \]
and let
\[ \varphi_n = E[P_{1/n}\varphi|V_n], \]
where \( P_{1/n} \) is the Ornstein-Uhlenbeck semi-group on \( W. \) We have then
\[ E[U(\varphi_n(w) - \varphi_n(z))] \leq E \left[ U \left( \frac{\pi}{2} I_1(\nabla \varphi_n(w))(z) \right) \right]. \]

Let \( \pi_n \) be the orthogonal projection from \( H \) onto span \( \{h_1, \ldots, h_n\}. \) We have
\[
I_1(\nabla \varphi_n(w))(z) = I_1(\nabla w E_w[P_{1/n}\varphi|V_n])(z)
= I_1(E_w[e^{1/n} P_{1/n} \pi_n \nabla \varphi|V_n])(z)
= I_1(\pi_n E_w[e^{1/n} P_{1/n} \nabla \varphi|V_n])(z)
= E_{\tilde{V}_n}[I_1(\nabla w e^{-1/n} P_{1/n}^{\tilde{V}_n} \nabla \varphi|V_n)]\]
where \( \tilde{V}_n \) is the copy of \( V_n \) on the second Wiener space. Then
\[
E \left[ U \left( \frac{\pi}{2} I_1(\nabla \varphi_n(w))(z) \right) \right]
\leq E \left[ U \left( \frac{\pi}{2} I_1(E_w[e^{1/n} P_{1/n} \nabla \varphi|V_n])(z) \right) \right]
= E \left[ U \left( \frac{\pi}{2} e^{-1/n} E_w[I_1(P_{1/n} \nabla \varphi(w))(z)|V_n] \right) \right]
\leq E \left[ U \left( \frac{\pi}{2} e^{-1/n} I_1(P_{1/n} \nabla \varphi(w))(z) \right) \right]
= E \left[ U \left( \frac{\pi}{2} e^{-1/n} P_{1/n}^{\tilde{V}_n} I_1(\nabla \varphi(w))(z) \right) \right]
\]
Now Fatou’s lemma completes the proof.

Let us give some consequences of this result:

Theorem 9.2.3 The following Poincaré inequalities are valid:

i) \( E[\exp(\varphi - E[\varphi])] \leq E\left[\exp \left( \frac{\pi^2}{8} |\nabla \varphi|_H^2 \right) \right], \)

ii) \( E[|\varphi - E[\varphi]|] \leq \frac{\pi}{2} E[|\nabla \varphi|_H]. \)

iii) \( E[|\varphi - E[\varphi]|^{2k}] \leq \left( \frac{\pi}{2} \right)^{2k} \frac{(2k)!}{2^{2k} k!} E[|\nabla \varphi|_H^{2k}], \quad k \in \mathbb{N}. \)

Remark 9.2.4 Let us note that the result of (ii) can not be obtained with the classical methods, such as the Ito-Clark representation theorem, since the optional projection is not a continuous map in \( L^1 \)-setting. Moreover, using the Hölder inequality and the Stirling formula, we deduce the following set of inequalities:

\[
\|\varphi - E[\varphi]\|_p \leq p \frac{\pi}{2} \|\nabla \varphi\|_{L^p(\mu,H)},
\]

for any \( p \geq 1 \). To compare this result with those already known, let us recall that using first the Ito-Clark formula, then the Burkholder-Davis-Gundy inequality combined with the convexity inequalities for the dual projections and some duality techniques, we obtain, only for \( p > 1 \) the inequality

\[
\|\varphi - E[\varphi]\|_p \leq K p^{3/2} \|\nabla \varphi\|_{L^p(\mu,H)},
\]

where \( K \) is some positive constant.

Proof: Replacing the function \( U \) of Theorem 9.2 by the exponential function, we have

\[
E[\exp(\varphi - E[\varphi])] \leq E_w \times E_z[\exp(\varphi(w) - \varphi(z))] \leq E_w \left[ E_z \left[ \exp \frac{\pi^2}{2} I_1(\nabla \varphi(w))(z) \right] \right] = E \left[ \exp \frac{\pi^2}{8} |\nabla \varphi|_H^2 \right].
\]

(ii) and (iii) are similar provided that we take \( U(x) = |x|^k, \quad k \in \mathbb{N}. \)

\[\square\]
Theorem 9.2.5 Let $\varphi \in \mathbb{D}_{p,2}$ for some $p > 1$ and that $\nabla |\nabla \varphi|_H \in L^\infty(\mu, H)$. Then there exists some $\lambda > 0$ such that
\[
E[\exp \lambda |\varphi|] < \infty.
\]
In particular, this hypothesis is satisfied if $\|\nabla^2 \varphi\|_{\text{op}} \in L^\infty(\mu)$, where $\| \cdot \|_{\text{op}}$ denotes the operator norm.

Proof: From Theorem 9.2.3 (i), we know that
\[
E[\exp \lambda (|\varphi - E[\varphi]|)] \leq 2 E \left[ \exp \frac{\lambda^2}{8} |\nabla \varphi|^2 \right].
\]
Hence it is sufficient to prove that
\[
E \left[ \exp \lambda^2 |\nabla \varphi|^2 \right] < \infty
\]
for some $\lambda > 0$. However Theorem 9.1.1 applies since $|\nabla |\nabla \varphi|_H \in L^\infty(\mu, H)$. The last claim is obvious since $|\nabla |\nabla \varphi|_H \leq \|\nabla^2 \varphi\|_{\text{op}}$ almost surely.

Corollary 9.2.6 Let $F \in \mathbb{D}_{p,1}$ for some $p > 1$ such that $|\nabla F|_H \in L^\infty(\mu)$. We then have
\[
E[\exp \lambda F^2] \leq E \left[ \frac{1}{\sqrt{1 - \frac{\lambda \pi^2}{4} |\nabla F|^2}} \exp \left( \frac{\lambda E[F]^2}{1 - \frac{\lambda \pi^2}{4} |\nabla F|^2} \right) \right], \quad \text{(9.2.3)}
\]
for any $\lambda > 0$ such that $\| |\nabla F|_H \|^2_{L^\infty(\mu)} \frac{\lambda \pi^2}{4} < 1$.

Proof: Let $Y$ be an auxiliary, real-valued Gaussian random variable, living on a separate probability space $(\Omega, \mathcal{U}, P)$ with variance one and zero expectation. We have, using Theorem 9.2.3:
\[
E[\exp \lambda F^2] = E \otimes E_P[\exp \sqrt{2\lambda FY}]
\leq E \otimes E_P \left[ \exp \left\{ \sqrt{2\lambda E[F]Y + |\nabla F|^2 Y^2 \frac{\lambda \pi^2}{4}} \right\} \right]
\leq E \left[ \frac{1}{\sqrt{1 - \frac{\lambda \pi^2}{2} |\nabla F|^2}} \exp \left( \frac{\lambda E[F]^2}{1 - \frac{\lambda \pi^2}{2} |\nabla F|^2} \right) \right],
\]
where $E_P$ denotes the expectation with respect to the probability $P$.

Remark: In the next section we shall obtain a better estimate then the one given by (9.2.3).
9.3 Log-Sobolev inequality and exponential integrability

There is a close relationship between the probability measures satisfying the log-Sobolev inequality and the exponential integrability of the random variables having essentially bounded Sobolev derivatives. We shall explain this in the frame of the Wiener space: let $\nu$ be a probability measure on $(W, \mathcal{B}(W))$ such that the operator $\nabla$ is a closable operator on $L^2(\nu)$. Assume that we have

$$E_\nu[\mathcal{H}_\nu(f^2)] \leq K E_\nu[|\nabla f|_H^2]$$

for any cylindrical $f : W \to \mathbb{R}$, where $\mathcal{H}_\nu(f^2) = f^2(\log f^2 - \log E_\nu[f^2])$. Since $\nabla$ is a closable operator, of course this inequality extends immediately to the extended $L^2$-domain of it.

**Lemma 9.3.1** Assume now that $f$ is in the extended $L^2$-domain of $\nabla$ such that $|\nabla f|_H$ is $\nu$-essentially bounded by one. Then

$$E_\nu[e^{tf}] \leq \exp\left\{ tE_\nu[f] + \frac{Kt^2}{4} \right\}, \quad (9.3.4)$$

for any $t \in \mathbb{R}$.

**Proof:** Let $f_n = \min(|f|, n)$, then it is easy to see that $|\nabla f_n|_H \leq |\nabla f|_H$ $\nu$-almost surely. Let $t \in \mathbb{R}$ and define $g_n$ as to be $e^{tf_n}$. Denote by $\theta(t)$ the function $E[\exp(tf_n)]$. Then it follows from the above inequality that

$$t\theta'(t) - \theta(t) \log \theta(t) \leq \frac{Kt^2}{4} \theta(t). \quad (9.3.5)$$

If we write $\beta(t) = \frac{1}{t} \log \theta(t)$, then $\lim_{t \to 0} \beta(t) = E[f_n]$, and (9.3.5) implies that $\beta'(t) \leq K/4$, hence we have

$$\beta(t) \leq E_\nu[f_n] + \frac{Kt}{4},$$

therefore

$$\theta(t) \leq \exp\left(tE_\nu[f_n] + \frac{Kt^2}{4}\right). \quad (9.3.6)$$

It follows from the monotone convergence theorem that $E[\exp(tf)] < \infty$, for any $t \in \mathbb{R}$. Hence the function $\theta(t) = E[\exp(tf)]$ satisfies also the inequality (9.3.5) which implies the inequality (9.3.4).

Using now the inequality (9.3.4) and an auxiliary Gaussian random variable as in Corollary 9.2.6 we can show easily:
Proposition 9.3.2 Assume that $f \in L^p(\nu)$ has $\nu$-essentially bounded Sobolev derivative and that this bound is equal to one. Then we have, for any $\varepsilon > 0$,

$$E_\nu[e^{\varepsilon f^2}] \leq \frac{1}{\sqrt{1 - \varepsilon K}} \exp \left( \frac{2\varepsilon E_\nu[f]^2}{1 - \varepsilon K} \right),$$

provided $\varepsilon K < 1$.

9.4 An interpolation inequality

Another useful inequality for the Wiener functionals is the following interpolation inequality which helps to control the $L^p$- norm of $\nabla F$ with the help of the $L^p$-norms of $F$ and $\nabla^2 F$.

Theorem 9.4.1 For any $p > 1$, there exists a constant $C_p$, such that, for any $F \in D_{p,2}$, one has

$$\|\nabla F\|_p \leq C_p \left[ \|F\|_p + \|F\|^{1/2}_p \|\nabla^2 F\|^{1/2}_p \right].$$

Theorem 9.4.1 will be proven, thanks to the Meyer inequalities, if we can prove the following

Theorem 9.4.2 For any $p > 1$, we have

$$\|(I + \mathcal{L})^{1/2} F\|_p \leq \frac{4}{\Gamma(1/2)} \|F\|^{1/2}_p \|(I + \mathcal{L}) F\|^{1/2}_p.$$ 

Proof: Denote by $G$ the functional $(I + \mathcal{L}) F$. Then we have $F = (I + \mathcal{L})^{-1} G$. Therefore it suffices to show that

$$\|(I + \mathcal{L})^{-1/2} G\|_p \leq \frac{4}{\Gamma(1/2)} \|G\|^{1/2}_p \|(I + \mathcal{L})^{-1} G\|^{1/2}_p.$$ 

We have

$$(I + \mathcal{L})^{-1/2} G = \frac{\sqrt{2}}{\Gamma(1/2)} \int_0^\infty t^{-1/2} e^{-t} P_t G dt,$$

where $P_t$ denotes the semi-group of Ornstein-Uhlenbeck. For any $a > 0$, we can write

$$(I + \mathcal{L})^{-1/2} G = \frac{\sqrt{2}}{\Gamma(1/2)} \left[ \int_0^a t^{-1/2} e^{-t} P_t G dt + \int_a^\infty t^{-1/2} e^{-t} P_t G dt \right].$$

This result has been proven as an answer to a question posed by D. W. Stroock, cf. also [19].
Let us denote the two terms at the right hand side of the above equality, respectively, by \( I_a \) and \( II_a \). We have

\[
\|(I + \mathcal{L})^{-1/2}G\|_p \leq \frac{\sqrt{2}}{\Gamma(1/2)} \|I_a\|_p + \|II_a\|_p.
\]

The first term at the right hand side can be upper bounded as

\[
\|I_a\|_p \leq \int_0^a t^{-1/2}\|G\|_p dt = 2\sqrt{a}\|G\|_p.
\]

Let \( g = (I + \mathcal{L})^{-1}G \). Then

\[
\int_a^\infty t^{-1/2}e^{-t}P_tGdt = \int_a^\infty t^{-1/2}e^{-t}P_t(I + \mathcal{L})(I + \mathcal{L})^{-1}Gdt = \int_a^\infty t^{-1/2}e^{-t}P_t(I + \mathcal{L})gdt = \int_a^\infty t^{-1/2}\frac{d}{dt}(e^{-t}P_t)dt = -a^{-1/2}e^{-a}P_ag + \frac{1}{2}\int_a^\infty t^{-3/2}e^{-t}P_tgdt,
\]

where the third equality follows from the integration by parts formula. Therefore

\[
\|II_a\|_p \leq a^{-1/2}\|e^{-a}P_ag\|_p + \frac{1}{2}\int_a^\infty t^{-3/2}\|e^{-t}P_tg\|_p dt \leq a^{-1/2}\|g\|_p + \frac{1}{2}\int_a^\infty t^{-3/2}\|g\|_p dt = 2a^{-1/2}\|g\|_p = 2a^{-1/2}\|(I + \mathcal{L})^{-1}G\|_p.
\]

Finally we have

\[
\|(I + \mathcal{L})^{-1/2}G\|_p \leq \frac{2}{\Gamma(1/2)} \left[ a^{1/2}\|G\|_p + a^{-1/2}\|(I + \mathcal{L})^{-1}G\|_p \right].
\]

This expression attains its minimum when we take

\[
a = \frac{\|(I + \mathcal{L})^{-1}G\|_p}{\|G\|_p}.
\]

Combining Theorem 9.4.1 with Meyer inequalities, we have

**Corollary 9.4.3** Suppose that \((F_n, n \in \mathbb{N})\) converges to zero in \(\mathbb{D}_{p,k}, \ p > 1, k \in \mathbb{Z}\), and that it is bounded in \(\mathbb{D}_{p,k+2}\). Then the convergence takes place also in \(\mathbb{D}_{p,k+1}\).
9.5 Exponential integrability of the divergence

We begin with two lemmas which are of some interest:

**Lemma 9.5.1** Let $\phi \in L^p(\mu), p > 1$, then, for any $h \in H$, $t > 0$, we have

$$\nabla_h P_t \phi(x) = \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_W \phi(e^{-t}x + \sqrt{1 - e^{-2t}}y)\delta h(y)\mu(dy)$$

almost surely, where $\nabla_h P_t \phi$ represents $(\nabla P_t \phi, h)_H$.

**Proof:** From the Mehler formula (cf. [2.6.5]), we have

$$\nabla_h P_t \phi(x) = \frac{d}{d\lambda}|_{\lambda=0} \int_W \phi \left( e^{-t}(x + \lambda h) + \sqrt{1 - e^{-2t}}y \right) \mu(dy)$$

$$= \frac{d}{d\lambda}|_{\lambda=0} \int_W \phi \left( e^{-t}x + \sqrt{1 - e^{-2t}}\left( y + \frac{\lambda e^{-t}}{\sqrt{1 - e^{-2t}}} h \right) \right) \mu(dy)$$

$$= \frac{d}{d\lambda}|_{\lambda=0} \int_W \phi \left( e^{-t}x + \sqrt{1 - e^{-2t}}y \right) \varepsilon \left( \frac{\lambda e^{-t}}{\sqrt{1 - e^{-2t}}} \delta h \right)(y) \mu(dy)$$

$$= \int_W \phi \left( e^{-t}x + \sqrt{1 - e^{-2t}}y \right) \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \delta h(y) \mu(dy),$$

where $\varepsilon(\delta h)$ denotes $\exp(\delta h - 1/2|h|^2_H)$.

**Lemma 9.5.2** Let $\xi \in L^p(\mu, H), p > 1$ and for $(x, y) \in W \times W$, $t \geq 0$, define

$$R_t(x, y) = e^{-t}x + (1 - e^{-2t})^{1/2} y$$

and

$$S_t(x, y) = (1 - e^{-2t})^{1/2} x - e^{-t} y.$$ 

Then $S_t(x, y)$ and $R_t(x, y)$ are indepdendent, identically distributed Gaussian random variables on $(W \times W, \mu(dx) \times \mu(dy))$. Moreover the following identity holds true:

$$P_t \delta \xi(x) = \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_W I_1(\xi(R_t(x, y)))(S_t(x, y)) \mu(dy),$$

where

$$I_1(\xi(R_t(x, y)))(S_t(x, y))$$

denotes the first order Wiener integral of $\xi(R_t(x, y))$ with respect to the independent path $S_t(x, y)$ under the product measure $\mu(dx) \times \mu(dy)$. 


Proof: The first part of the lemma is a well-known property of the Gaussian random variables and left to the reader. In the proof of the second part, for the typographical facility, we shall denote in the sequel by $e(t)$ the function $(\exp -t)/(1 - \exp -2t)^{1/2}$. Let now $\phi$ be an element of $\mathbb{D}$, we have, via duality and using Lemma 9.5.1

$$< P_t \delta \xi, \phi > = < \xi, \nabla P_t \phi >$$

$$= \sum_{i=1}^{\infty} < \xi_i, \nabla_{h_i} P_t \phi >$$

$$= \sum_i e(t) E \left[ \int_W \xi_i(x) \phi (R_t(x,y)) \delta h_i(y) \mu(dy) \right]$$

where $(h_i; i \in \mathbb{N}) \subset W^*$ is a complete orthonormal basis of $H$, $\xi_i$ is the component of $\xi$ in the direction of $e_i$ and $< . , >$ represents the duality bracket corresponding to the dual pairs $(\mathbb{D}, \mathbb{D}')$ or $(\mathbb{D}(H), \mathbb{D}'(H))$. Let us make the following change of variables, which preserves $\mu \times \mu$:

$$x \mapsto e^{-t}x + \sqrt{1 - e^{-2t}}y$$

$$y \mapsto \sqrt{1 - e^{-2t}}x - e^{-t}y.$$}

We then obtain

$$< P_t (\delta \xi), \phi > = e(t) \int_W \phi(x) I_1 (\xi (R_t(x,y))) (S_t(x,y)) \mu(dx) \mu(dy),$$

for any $\phi \in \mathbb{D}$ and the lemma follows from the density of $\mathbb{D}$ in all $L^p$-spaces. \hfill \Box

We are now ready to prove the following

**Theorem 9.5.3** Let $\beta > 1/2$ and suppose that $\eta \in \mathbb{D}_{2,2\beta}(H)$. Then we have

$$E[\exp \delta \eta] \leq E \left[ \exp \left( \alpha |(2I + \mathcal{L})^\beta \eta|^2_H \right) \right],$$

for any $\alpha$ satisfying

$$\alpha \geq \frac{1}{2} \left[ \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^+} t^{\beta-1} e^{-2t} \sqrt{1 - e^{-2t}} dt \right]^{-2},$$

where $\mathcal{L}$ denotes the Ornstein-Uhlenbeck or the number operator on $W$.

**Proof:** Let $\xi = (2I + \mathcal{L})^\beta \eta$, then the above inequality is equivalent to

$$E \left[ \exp \left( (I + \mathcal{L})^{-\beta} \delta \xi \right) \right] \leq E \left[ \exp \alpha |\xi|^2_H \right],$$
where we have used the identity
\[(I + \mathcal{L})^{-\beta} \delta \xi = \delta \left( (2I + \mathcal{L})^{-\beta} \xi \right) .\]

We have from the resolvent identity and from the Lemma 9.5.2,
\[
(I + \mathcal{L})^{-\beta} \delta \xi = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^+} t^{\beta - 1} e^{-t} P_t \delta \xi \, dt
\]
\[
= \int_{\mathbb{R}^+ \times W} \frac{e^{-t}}{\Gamma(\beta) \sqrt{1 - e^{-2t}}} t^{\beta - 1} e^{-t} I_1(\xi(R_t(x, y)))(S_t(x, y)) \mu(dy) \, dt.
\]

Let
\[
\lambda_0 = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^+} \frac{t^{\beta - 1} e^{-2t}}{\sqrt{1 - e^{-2t}}} \, dt
\]
and
\[
\nu(dt) = \mathbf{1}_{\mathbb{R}^+}(t) \frac{1}{\lambda_0 \Gamma(\beta)} \frac{t^{\beta - 1} e^{-2t}}{\sqrt{1 - e^{-2t}}} \, dt.
\]

Then, from the Hölder inequality
\[
E \left[ \exp \left\{ (I + \mathcal{L})^{-\beta} \delta \xi \right\} \right]
\leq \int_{\mathbb{R}^+} \int_{W} \exp \left\{ \lambda_0 I_1(\xi(R_t(x, y)))(S_t(x, y)) \mu(dy) \nu(dt) \right\} \mu(dx) \mu(dy) \nu(dt)
\leq E \left[ \exp \left\{ \frac{\lambda_0^2}{2} \| \xi \|^2_H \right\} \right],
\]
which completes the proof.

In the applications, we need also to control the moments like \( E[\exp \| \nabla \eta \|^2_2] \) (cf. [101]), where \( \eta \) is an \( H \)-valued random variable and \( \| \cdot \|_2 \) denotes the Hilbert-Schmidt norm. The following result gives an answer to this question:

**Proposition 9.5.4** Suppose that \( \beta > 1/2 \) and that \( \eta \in \mathbb{D}_{2,2\beta}(H) \). Then we have
\[
E[\exp \| \nabla \eta \|^2_2] \leq E[\exp c(I + \mathcal{L})^2 \eta]_H, \]
for any
\[
c \geq c_0 = \left[ \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^+} t^{\beta - 1} e^{-2t}(1 - e^{-2t})^{-1/2} \, dt \right]^2.
\]
In particular, for \( \beta = 1 \) we have \( c \geq 1/4 \).
Proof: Setting $\xi = (I + \mathcal{L})^\beta \eta$, it is sufficient to show that

\[ E \left[ \exp \| \nabla (I + \mathcal{L})^{-\beta} \xi \|^2 \right] \leq E \left[ \exp c|\xi|^2_H \right]. \]

Let $(E_i, i \in \mathbb{N})$ be a complete, orthonormal basis of $H \otimes H$ which is the completion of the tensor product of $H$ with itself under the Hilbert-Schmidt topology. Then

\[ \| \nabla \eta \|^2 = \sum_i K_i (\nabla \eta, E_i)_2, \]

where $(.,.)_2$ is the scalar product in $H \otimes H$ and $K_i = (\nabla \eta, E_i)_2$. Let $\theta(t)$ be the function

\[ \frac{1}{\Gamma(\beta)} t^{\beta-1} e^{-2t(1 - e^{-2t})^{-1/2}} \]

and let $\gamma_0 = \int_0^\infty \theta(t) dt$. From Lemmas 9.5.1 and 9.5.2, we have

\[ \| \nabla \eta(x) \|^2 \leq \sum_i K_i(x) \int_{\mathbb{R}^+} \theta(t) \int_W (I_1(E_i)(y), \xi(R_t(x, y)))_H \mu(dy) dt \]

\[ \leq \int_{\mathbb{R}^+} \theta(t) \left( \int_W |I_1(\nabla \eta(x))(y)|^2_H \mu(dy) \right)^{1/2} \left( \int_W |\xi(R_t(x, y))|^2_H \mu(dy) \right)^{1/2} dt \]

\[ = \int_{\mathbb{R}^+} \theta(t) \| \nabla \eta(x) \|^2_2 \left( P_t(|\xi|^2_H) \right)^{1/2} dt, \]

where $I_1(\nabla \eta(x))(y)$ denotes the first order Wiener integral of $\nabla \eta(x)$ with respect to the independent path (or variable) $y$. Consequently we have the following inequality:

\[ \| \nabla \eta \|^2 \leq \int_{\mathbb{R}^+} \theta(t) \left( P_t(|\xi|^2_H) \right)^{1/2} dt. \]

Therefore

\[ E[\exp \| \nabla \eta \|^2_2] \leq E \left[ \exp \left( \int_{\mathbb{R}^+} \frac{\gamma_0^2 P_t(|\xi|^2_H)}{\gamma_0} \frac{\theta(t)}{\gamma_0} dt \right) \right] \]

\[ \leq E \int_{\mathbb{R}^+} \frac{\theta(t)}{\gamma_0} \exp \left\{ \gamma_0^2 P_t(|\xi|^2_H) \right\} dt \]

\[ \leq E \int_{\mathbb{R}^+} \frac{\theta(t)}{\gamma_0} \exp \left\{ \gamma_0^2 |\xi|^2_H \right\} dt \]

\[ = E \left[ \exp \gamma_0^2 |\xi|^2_H \right]. \]
As an example of application of these results let us give the following theorem of the degree theory of the Wiener maps (cf. [101]):

**Corollary 9.5.5** Suppose that \( \eta \in D_{2,2\beta}(H) \), \( \beta > 1/2 \), satisfies

\[
E \left[ \exp a \left( (2I + \mathcal{L})^\beta \eta \right)_H^2 \right] < \infty,
\]

for some \( a > 0 \). Then for any \( \lambda \leq \sqrt{\frac{a}{4c_0}} \) and \( h \in H \), we have

\[
E \left[ e^{i(\delta h + \lambda(h,\eta))} \Lambda \right] = \exp \left( -\frac{1}{2} |h|^2_H \right),
\]

where \( \Lambda \) is defined by

\[
\Lambda = \det(2(I_H + \lambda \nabla \eta)) \exp \left( -\lambda \delta h - \frac{\lambda^2}{2} |\eta|^2_H \right).
\]

In particular, if we deal with the classical Wiener space, the path defined by

\[
T_\lambda(w) = w + \lambda \eta(w),
\]

is a Brownian motion under the new probability measure \( E[\Lambda | \sigma(T_\lambda)] d\mu \), where \( \sigma(T_\lambda) \) denotes the sigma field generated by the mapping \( T_\lambda \).

**Proof:** This result follows from the degree theorem for Wiener maps (cf. [94]). In fact from the Theorem 3.2 of [94] (cf. also [101]), it follows that \( E[\Lambda] = 1 \). On the other hand, from the Theorem 3.1 of the same reference, we have

\[
E[F \circ T_\lambda \Lambda] = E[F] E[\Lambda].
\]

Hence the proof follows.

---

**Notes and suggested reading**

The results about the exponential tightness go back till to the celebrated Lemma of X. Fernique about the exponential integrability of the square of semi-norms (cf. [49]). It is also proven by B. Maurey in the finite dimensional case for the Lipschitz continuous maps with the same method that we have used here (cf. [71]). A similar result in the abstract Wiener space case has been given by S. Kusuoka under the hypothesis of \( H \)-continuity, i.e., \( h \to \phi(w + h) \) is continuous for any \( w \in W \). We have proven the actual result without this latter hypothesis. However, it has been proven later that the essential boundedness of
the Sobolev derivative implies the existence of a version which is $H$-continuous by Enchev and Stroock (cf. [24]). Later it has been discovered that the exponential integrability is implied by the logarithmic Sobolev inequality (cf. [2]). The derivation of the inequality (9.3.6) is attributed to Herbst (cf. [54]).

In any case the exponential integrability of the square of the Wiener functionals has found one of its most important applications in the analysis of non-linear Gaussian functionals. In fact in the proof of the Ramer theorem and its extensions this property plays an important role (cf. Chapter X, [97], [98] and [101]). Corollary 9.5.5 uses some results about the degree theory of the Wiener maps which are explained below:

**Theorem 9.5.6** Assume that $\gamma$ and $r$ be fixed strictly positive numbers such that $r > (1 + \gamma)\gamma - 1$. Let $u \in D_{r,2}(H)$ and assume that

1. $\Lambda u \in L^{1+\gamma}(\mu)$,
2. $\Lambda_u(I_H + \nabla u)^{-1}h \in L^{1+\gamma}(\mu, H)$ for any $h \in H$,

where

$$\Lambda_u = \det_2(I_H + \nabla u) \exp \left\{ -\delta u - \frac{1}{2} |u|_H^2 \right\}.$$

Then, for any $F \in C_b(W)$, we have

$$E[F(w + u(w))\Lambda_u] = E[\Lambda_u]E[F].$$

In particular, using a homotopy argument, one can show that, if

$$\exp \left\{ -\delta u + \frac{1}{2} + \varepsilon \frac{1}{2} \right\} \in L^{1+\alpha}(\mu),$$

for some $\alpha > 0, \varepsilon > 0$, then $E[\Lambda_u] = 1$. We refer the reader to [101] for further information about this topic.
Chapter 10

Introduction to the Theorem of Ramer

Introduction

The Girsanov theorem tells us that if \( u : W \to H \) is a Wiener functional such that \( \frac{du}{dt} = \dot{u}(t) \) is an adapted process such that

\[
E \left[ \exp \left\{ - \int_0^1 \dot{u}(s) dW_s - \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right\} \right] = 1,
\]

then under the new probability \( L d\mu \), where

\[
L = \exp \left\{ - \int_0^1 \dot{u}(s) dW_s - \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right\},
\]

\( w \to w + u(w) \) is a Brownian motion. The theorem of Ramer studies the same problem without hypothesis of adaptedness of the process \( u \). This problem has been initiated by Cameron and Martin. Their work has been extended by Gross and others. It was Ramer [74] who gave a main impulse to the problem by realizing that the ordinary determinant can be replaced by the modified Carleman-Fredholm determinant via defining a Gaussian divergence instead of the ordinary Lebesgue divergence. The problem has been further studied by Kusuoka [51] and the final solution in the case of (locally) differentiable shifts in the Cameron-Martin space direction has been given by Üstünel and Zakai [97]. In this chapter we will give a partial (however indispensable for the proof of the general) result.

To understand the problem, let us consider first the finite dimensional case: let \( W = \mathbb{R}^n \) and let \( \mu_n \) be the standard Gauss measure on \( \mathbb{R}^n \). If \( u : \mathbb{R}^n \to \mathbb{R}^n \) is a differentiable mapping such that \( I + u \) is a diffeomorphism
of $\mathbb{R}^n$, then the theorem of Jacobi tells us that, for any smooth function $F$ on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} F(x + u(x)) |\det(I + \partial u(x))| \exp \left\{ -u(x), x > -\frac{1}{2}|u|^2 \right\} \mu_n(dx)$$

$$= \int_{\mathbb{R}^n} F(x) \mu_n(dx),$$

where $\partial u$ denotes the derivative of $u$. The natural idea now is to pass to the infinite dimension. For this, note that, if we define $\det_2(I + \partial u)$ by

$$\det_2(I + \partial u(x)) = \det(I + \partial u(x)) e^{-\text{trace}[\partial u(x)]}$$

$$= \prod_i (1 + \lambda_i) \exp -\lambda_i,$$

where $(\lambda_i)$ are the eigenvalues of $\partial u(x)$ counted with respect to their multiplicity, then the density of the left hand side can be written as

$$\Lambda = |\det_2(I + \partial u(x))| \exp \left\{ -<u(x), x> + \text{trace} \partial u(x) - \frac{1}{2}|u|^2 \right\}$$

and let us remark that

$$<u(x), x> - \text{trace} \partial u(x) = \delta u(x),$$

where $\delta$ is the adjoint of the $\partial$ with respect to the Gaussian measure $\mu_n$. Hence, we can express the density $\Lambda$ as

$$\Lambda = |\det_2(I + \partial u(x))| \exp \left\{ -\delta u(x) - \frac{|u(x)|^2}{2} \right\}. $$

As remarked first by Ramer, cf. [74], this expression has two advantages: first $\det_2(I + \partial u)$, called Carleman-Fredholm determinant, can be defined for the mappings $u$ such that $\partial u(x)$ is with values in the space of Hilbert-Schmidt operators rather than nuclear operators (the latter is a smaller class than the former), secondly, as we have already seen, $\delta u$ is well-defined for a large class of mappings meanwhile $<u(x), x>$ is a highly singular object in the Wiener space.

### 10.1 Ramer’s Theorem

After these preliminaries, we can announce, using our standard notations, the main result of this chapter:
Theorem 10.1.1 Suppose that $u : W \mapsto H$ is a measurable map belonging to $\mathbb{D}_{p,1}(H)$ for some $p > 1$. Assume that there are constants $c$ and $d$ with $c < 1$ such that for almost all $w \in W$,

$$\|\nabla u\| \leq c < 1$$

and

$$\|\nabla u\|_2 \leq d < \infty,$$

where $\|\cdot\|$ denotes the operator norm and $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm for the linear operators on $H$. Then:

- Almost surely $w \mapsto T(w) = w + u(w)$ is bijective. The inverse of $T$, denoted by $S$ is of the form $S(w) = w + v(w)$, where $v$ belongs to $\mathbb{D}_{p,1}(H)$ for any $p > 1$, moreover,

$$\|\nabla v\| \leq \frac{c}{1-c} \text{ and } \|\nabla v\|_2 \leq \frac{d}{1-c},$$

$\mu$-almost surely.

- For all bounded and measurable $F$, we have

$$E[F(w)] = E[F(T(w)) \cdot |\Lambda_u(w)|]$$

and in particular

$$E|\Lambda_u| = 1,$$

where

$$\Lambda_u = |\det_2(I + \nabla u)| \exp -\delta u - \frac{1}{2} |u|_H^2,$$

and $\det_2(I + \nabla u)$ denotes the Carleman-Fredholm determinant of $I + \nabla u$.

- The measures $\mu$, $T^*\mu$ and $S^*\mu$ are mutually absolutely continuous, where $T^*\mu$ (respectively $S^*\mu$) denotes the image of $\mu$ under $T$ (respectively $S$). We have

$$\frac{dS^*\mu}{d\mu} = |\Lambda_u|,$$

$$\frac{dT^*\mu}{d\mu} = |\Lambda_v|,$$

where $\Lambda_v$ is defined similarly.
Remark 10.1.2 If \( \| \nabla u \| \leq 1 \) instead of \( \| \nabla u \| \leq c < 1 \), then taking \( u_\varepsilon = (1 - \varepsilon) u \) we see that the hypothesis of the theorem are satisfied for \( u_\varepsilon \). Hence using the Fatou lemma, we obtain

\[
E[F \circ T | \Lambda_u] \leq E[F]
\]

for any positive \( F \in C_b(W) \). Consequently, if \( \Lambda_u \neq 0 \) almost surely, then \( T^* \mu \) is absolutely continuous with respect to \( \mu \).

The proof of Theorem 10.1.1 will be done in several steps. As we have indicated above, the main idea is to pass to the limit from finite to infinite dimensions. The key point in this procedure will be the use of the Theorem 1 of the preceding chapter which will imply the uniform integrability of the finite dimensional densities. We shall first prove the same theorem in the cylindrical case:

Lemma 10.1.3 Let \( \xi : W \mapsto H \) be a shift of the following form:

\[
\xi(w) = \sum_{i=1}^{n} \alpha_i(\delta h_1, \ldots, \delta h_n)h_i,
\]

with \( \alpha_i \in C^\infty(\mathbb{R}^n) \) with bounded first derivative, \( h_i \in W^* \) are orthonormal \(^1\) in \( H \). Suppose furthermore that \( \| \nabla \xi \| \leq c < 1 \) and that \( \| \nabla \xi \|_2^2 \leq d \) as above. Then we have

- Almost surely \( w \mapsto U(w) = w + \xi(w) \) is bijective.
- The measures \( \mu \) and \( U^* \mu \) are mutually absolutely continuous.
- For all bounded and measurable \( F \), we have

\[
E[F(w)] = E[F(U(w))] \cdot |\Lambda_\xi(w)|
\]

for all bounded and measurable \( F \) and in particular

\[
E[|\Lambda_\xi|] = 1,
\]

where

\[
\Lambda_\xi = |\det_2(I + \nabla \xi)| \exp(-\delta \xi - \frac{1}{2} |\xi|^2_H).
\]

\(^1\)In fact \( h_i \in W^* \) should be distinguished from its image in \( H \), denoted by \( j(h) \). For notational simplicity, we denote both by \( h_i \), as long as there is no ambiguity.
The inverse of $U$, denoted by $V$ is of the form $V(w) = w + \eta(w)$, where

$$\eta(w) = \sum_{i=1}^{n} \beta_i(\delta h_1, \ldots, \delta h_n)h_i,$$

such that $\|\nabla \eta\| \leq \frac{c}{1-c}$ and $\|\nabla \eta\|_2 \leq \frac{d}{1-c}$.

**Proof:** Note first that due to the Corollary 9.1.2 of the Chapter VIII, $E[\exp \lambda |\xi|^2] < \infty$ for any $\lambda < \frac{1}{2c}$. We shall construct the inverse of $U$ by imitating the fixed point techniques: let $\eta_0(w) = 0$ and $\eta_{n+1}(w) = -\xi(w + \eta_n(w))$.

We have

$$|\eta_{n+1}(w) - \eta_n(w)|_H \leq c|\eta_n(w) - \eta_{n-1}(w)|_H \leq c^n|\xi(w)|_H.$$

Therefore $\eta(w) = \lim_{n \to \infty} \eta_n(w)$ exists and it is bounded by $\frac{1}{1-c}|\xi(w)|_H$. By the triangle inequality

$$|\eta_{n+1}(w + h) - \eta_{n+1}(w)|_H \leq |\xi(w + h + \eta_n(w + h)) - \xi(w + \eta_n(w))|_H \leq c|h|_H + c|\eta_n(w + h) - \eta_n(w)|_H.$$

Hence passing to the limit, we find

$$|\eta(w + h) - \eta(w)|_H \leq \frac{c}{1-c}|h|_H.$$

We also have

$$U(w + \eta(w)) = w + \eta(w) + \xi(w + \eta(w)) = w + \eta(w) - \eta(w) = w,$$

hence $U \circ (I_W + \eta) = I_W$, i.e., $U$ is an onto map. If $U(w) = U(w')$, then

$$|\xi(w) - \xi(w')|_H = |\xi(w + \xi(w') - \xi(w)) - \xi(w')|_H \leq c|\xi(w) - \xi(w')|_H,$$

which implies that $U$ is also injective. To show the Girsanov identity, let us complete the sequence $(h_i, i \leq n)$ to a complete orthonormal basis whose
elements are chosen from $W^*$. From a theorem of Ito-Nisio \[42\], we can express the Wiener path $w$ as

$$w = \sum_{i=1}^{\infty} \delta h_i(w) h_i,$$

where the sum converges almost surely in the norm topology of $W$. Let $F$ be a nice function on $W$, denote by $\mu_n$ the image of the Wiener measure $\mu$ under the map $w \mapsto \sum_{i \leq n} \delta h_i(w) h_i$ and by $\nu$ the image of $\mu$ under $w \mapsto \sum_{i > n} \delta h_i(w) h_i$. Evidently $\mu = \mu_n \times \nu$. Therefore

$$E[F \circ U | \Lambda_\xi] = \int_{\mathbb{R}^n} E_\nu \left[ F \left( w + \sum_{i \leq n} (x_i + \alpha_i(x_1, \ldots, x_n)) h_i \right) | \Lambda_\xi \right] \mu_{\mathbb{R}^n}(dx) = E[F],$$

where $\mu_{\mathbb{R}^n}(dx)$ denotes the standard Gaussian measure on $\mathbb{R}^n$ and the equality follows from the Fubini theorem. In fact by changing the order of integrals, we reduce the problem to a finite dimensional one and then the result is immediate from the theorem of Jacobi as explained above. From the construction of $V$, it is trivial to see that

$$\eta(w) = \sum_{i \leq n} \beta_i \delta h_i, \ldots, \delta h_n h_i,$$

for some vector field $(\beta_1, \ldots, \beta_n)$ which is a $C^\infty$ mapping from $\mathbb{R}^n$ into itself due to the finite dimensional inverse mapping theorem. Now it is routine to verify that

$$\nabla \eta = -(I + \nabla \eta)^* \nabla \xi \circ V,$$

hence

$$\| \nabla \eta \|_2 \leq \| I + \nabla \eta \| \| \nabla \xi \circ V \|_2 \\
\leq (1 + \| \nabla \eta \|) \| \nabla \xi \circ V \|_2 \\
\leq d \left( 1 + \frac{c}{1 - c} \right) \\
= \frac{d}{1 - c}.$$

\[\Box\]

**Lemma 10.1.4** With the notations and hypothesis of Lemma \[10.1.3\] we have

$$\delta \xi \circ V = -\delta \eta - |\eta|^2_H + \text{trace} [(\nabla \xi \circ V) \cdot \nabla \eta],$$

almost surely.
Proof: We have
\[
\delta \xi = \sum_{i=1}^{\infty} \{ (\xi, e_i)_H \delta e_i - \nabla e_i(\xi, e_i)_H \},
\]
where the sum converges in \(L^2\) and the result is independent of the choice of the orthonormal basis \((e_i; i \in \mathbb{N})\). Therefore we can choose as basis \(h_1, \ldots, h_n\) that we have already used in Lemma 10.1.3 completed with the elements of \(W^*\) to form an orthonormal basis of \(H\), denoted by \((h_i; i \in \mathbb{N})\). Hence
\[
\delta \xi = \sum_{i=1}^{n} \{ (\xi, h_i)_H \delta h_i - \nabla h_i(\xi, h_i)_H \}.
\]
From the Lemma 10.1.3 we have \(\xi \circ V = -\eta\) and since, \(h_i\) are originating from \(W^*\), it is immediate to see that \(\delta h_i \circ V = \delta h_i + (h_i, \eta)_H\). Moreover, from the preceding lemma we know that \(\nabla (\xi \circ V) = (I + \nabla \eta)^* \nabla \xi \circ V\). Consequently, applying all this, we obtain
\[
\delta \xi \circ V = \sum_{i=1}^{n} (\xi \circ V, h_i)_H (\delta h_i + (h_i, \eta)_H) - (\nabla h_i(\xi, h_i)_H) \circ V
\]
\[
= (\xi \circ V, \eta)_H + \delta (\xi \circ V) + \sum_{i=1}^{n} \nabla h_i(\xi \circ V, h_i)_H - \nabla h_i(\xi, h_i)_H \circ V
\]
\[
= -|\eta|^2_H - \delta \eta + \sum_{i=1}^{n} (\nabla \xi \circ V[h_i], \nabla \eta[h_i])_H
\]
\[
= -|\eta|^2_H - \delta \eta + \text{trace}(\nabla \xi \circ V \cdot \nabla \eta),
\]
where \(\nabla \xi[h]\) denotes the Hilbert-Schmidt operator \(\nabla \xi\) applied to the vector \(h \in H\). \(\square\)

Remark 10.1.5 Since \(\xi\) and \(\eta\) are symmetric, we have \(\eta \circ U = -\xi\) and consequently
\[
\delta \eta \circ U = -\delta \xi - |\xi|^2_H + \text{trace} \left[ (\nabla \eta \circ U) \cdot \nabla \xi \right].
\]

Corollary 10.1.6 For any cylindrical function \(F\) on \(W\), we have
\[
E[F \circ V] = E[F|\Lambda_\xi].
\]
\[
E[F \circ U] = E[F|\Lambda_\eta].
\]

Proof: The first part follows from the identity
\[
E[F|\Lambda_\xi] = E[F \circ V \circ U |\Lambda_\xi] = E[F \circ V].
\]
To see the second part, we have
\[
E[F \circ U] = E \left[ F \circ U \frac{1}{|\Lambda_\xi| \circ V} \circ U |\Lambda_\xi| \right]
= E \left[ F \frac{1}{|\Lambda_\xi|} \circ V \right].
\]

From Lemma \ref{lem:10.1.4}, it follows that
\[
\frac{1}{|\Lambda_\xi| \circ V} = \frac{1}{|\det_2(I + \nabla \xi) \circ V|} \exp \left\{ \delta \xi + 1/2|\xi|^2_H \right\} \circ V
= \frac{1}{|\det_2(I + \nabla \xi) \circ V|} \exp \left\{ -\delta \eta - 1/2|\eta|^2_H + \text{trace}((\nabla \xi \circ V) \cdot \nabla \eta) \right\}
= |\Lambda_\eta|,
\]
since, for general Hilbert-Schmidt maps \(A\) and \(B\), we have
\[
\det_2(I + A) \cdot \det_2(I + B) = \exp\{\text{trace}(AB)\} \cdot \det_2((I + A)(I + B)) \quad (10.1.1)
\]
and in our case we have
\[
(I + \nabla \xi \circ V) \cdot (I + \nabla \eta) = I.
\]

\[\square\]

**Remark:** In fact the equality \((10.1.1)\) follows from the multiplicative property of the ordinary determinants and from the formula (cf. [23], page 1106, Lemma 22):
\[
\det_2(I + A) = \prod_{i=1}^{\infty} (1 + \lambda_i) e^{-\lambda_i},
\]
where \((\lambda_i, i \in \mathbb{N})\) are the eigenvalues of \(A\) counted with respect to their multiplicity.

**Proof of Theorem \ref{thm:10.1.1}:** Let \((h_i, i \in \mathbb{N}) \subset W^*\) be a complete orthonormal basis of \(H\). For \(n \in \mathbb{N}\), let \(V_n\) be the sigma algebra on \(W\) generated by \(\delta h_1, \ldots, \delta h_n\), \(\pi_n\) be the orthogonal projection of \(H\) onto the subspace spanned by \(\{h_1, \ldots, h_n\}\). Define
\[
\xi_n = E \left[ \pi_n P_{1/n} u |V_n \right],
\]
where $P_{1/n}$ is the Ornstein-Uhlenbeck semi-group on $W$ with $t = 1/n$. Then $\xi_n \to \xi$ in $\mathbb{D}_{p,1}(H)$ for any $p > 1$ (cf., Lemma [9.1.4] of Chapter IX). Moreover $\xi_n$ has the following form:

$$\xi_n = \sum_{i=1}^{n} \alpha^n_i(\delta h_1, \ldots, \delta h_n)h_i,$$

where $\alpha^n_i$ are $C^\infty$-functions due to the finite dimensional Sobolev embedding theorem. We have

$$\nabla \xi_n = E \left[ \pi_n \otimes \pi_n e^{-1/n} P_{1/n} \nabla u | V_n \right],$$

hence

$$\|\nabla \xi_n\| \leq e^{-1/n} E \left[ P_{1/n} \|\nabla u\| | V_n \right],$$

and the same inequality holds also with the Hilbert-Schmidt norm. Consequently, we have

$$\|\nabla \xi_n\| \leq c, \quad \|\nabla \xi_n\|_2 \leq d,$$

$\mu$-almost surely. Hence, each $\xi_n$ satisfies the hypothesis of Lemma [10.1.3]. Let us denote by $\eta_n$ the shift corresponding to the inverse of $U_n = I + \xi_n$ and let $V_n = I + \eta_n$. Denote by $\Lambda_n$ and $L_n$ the densities corresponding, respectively, to $\xi_n$ and $\eta_n$, i.e., with the old notations

$$\Lambda_n = \Lambda_{\xi_n} \quad \text{and} \quad L_n = \Lambda_{\eta_n}.$$

We will prove that the sequences of densities

$$\{\Lambda_n : n \in \mathbb{N}\} \quad \text{and} \quad \{L_n : n \in \mathbb{N}\}$$

are uniformly integrable. In fact we will do this only for the first sequence since the proof for the second is very similar to the proof of the first case. To prove the uniform integrability, from the lemma of de la Vallé-Poussin, it suffices to show

$$\sup_n E \left[ |\Lambda_n| \log \Lambda_n \right] < \infty,$$

which amounts to show, from the Corollary [10.1.6] that

$$\sup_n E \left[ |\log \Lambda_n \circ V_n| \right] < \infty.$$

Hence we have to control

$$E \left[ |\log \det_2(I + \nabla \xi_n \circ V_n)| + |\delta \xi_n \circ V_n| + 1/2|\xi_n \circ V_n|^2 \right].$$
Theorem of Ramer

From the Lemma 10.1.4, we have
\[ \delta \xi_n \circ V_n = -\delta \eta_n - |\eta_n|^2_H + \text{trace}(\nabla \xi_n \circ V_n) \cdot \nabla \eta_n, \]

hence
\[
E[|\delta \xi_n \circ V_n|] \leq \|\delta \eta_n\|_{L^2(\mu)} + E[|\eta_n|^2] + E[\|\nabla \xi_n \circ V_n\|_2 \|\nabla \eta_n\|_2]
\leq \|\eta_n\|_{L^2(\mu, H)} + \|\eta_n\|_{L^2(\mu, H \otimes H)}^2 + d^2 \frac{d(1 + d)}{1 - c},
\]

where the second inequality follows from
\[ \|\delta \gamma\|_{L^2(\mu)} \leq \|\nabla \gamma\|_{L^2(\mu, H \otimes H)} + \|\gamma\|_{L^2(\mu, H)}.\]

From the Corollary 9.1.2 of Chapter IX, we have
\[ \sup_n E[\exp \alpha |\eta_n|^2_H] < \infty, \]

for any \( \alpha < \frac{(1-c)^2}{2d^2} \), hence
\[ \sup_n E[|\eta_n|^2] < \infty. \]

We have a well-known inequality (cf. [101], Appendix), which says that
\[ |\det_2(I + A)| \leq \exp \frac{1}{2} \|A\|_2^2 \]

for any Hilbert-Schmidt operator \( A \) on \( H \). Applying this inequality to our case, we obtain
\[ \sup_n |\log \det_2(I + \nabla \xi_n \circ V_n)| \leq \frac{d^2}{2} \]

and this proves the uniform integrability of \( (\Lambda_n, n \in \mathbb{N}) \). Therefore the sequence \( (\Lambda_n, n \in \mathbb{N}) \) converges to \( \Lambda_u \) in \( L^1(\mu) \) and we have
\[ E[F \circ T |\Lambda_u] = E[F], \]

for any \( F \in C_b(W) \), where \( T(w) = w + u(w) \).

To prove the existence of the inverse transformation we begin with
\[ |\eta_n - \eta_m|_H \leq |\xi_n \circ V_n - \xi_m \circ V_m|_H + |\xi_m \circ V_n - \xi_m \circ V_m|_H \leq |\xi_n \circ V_n - \xi_m \circ V_m|_H + c |\eta_n - \eta_m|_H, \]
since $c < 1$, we obtain:

$$(1 - c)|\eta_n - \eta_m|_H \leq |\xi_n \circ V_n - \xi_m \circ V_n|_H.$$ 

Consequently, for any $K > 0$,

$$\mu \left\{ |\eta_n - \eta_m|_H > K \right\} \leq \mu \left\{ |\xi_n \circ V_n - \xi_m \circ V_n|_H > (1 - c)K \right\}$$

$$= E \left[ |\Lambda_n| 1_{|\xi_n - \xi_m|_H > (1 - c)K} \right] \to 0,$$

as $n$ and $m$ go to infinity, by the uniform integrability of $(\Lambda_n; n \in \mathbb{N})$ and by the convergence in probability of $(\eta_n; n \in \mathbb{N})$. As the sequence $(\eta_n; n \in \mathbb{N})$ is bounded in all $L^p$ spaces, this result implies the existence of an $H$-valued random variable, say $v$ which is the limit of $(\eta_n; n \in \mathbb{N})$ in probability. By uniform integrability, the convergence takes place in $L^p(\mu, H)$ for any $p > 1$ and since the sequence $(\nabla \eta_n; n \in \mathbb{N})$ is bounded in $L^\infty(\mu, H \otimes H)$, also the convergence takes place in $D_{p,1}(H)$ for any $p > 1$. Consequently, we have

$$E[F(w + v(w)) | \Lambda_n] = E[F],$$

and

$$E[F(w + v(w))] = E[F | \Lambda_n],$$

for any $F \in C_b(W)$.

Let us show that $S : W \to W$, defined by $S(w) = w + v(w)$ is the inverse of $T$ : let $a > 0$ be any number, then

$$\mu \left\{ \|T \circ S - w\|_W > a \right\} = \mu \left\{ \|T \circ S - U_n \circ S\|_W > a/2 \right\}$$

$$+ \mu \left\{ \|U_n \circ S - U_n \circ V_n\|_W > a/2 \right\}$$

$$= E \left[ |\Lambda_u| 1_{\|T - U_n\|_W > a/2} \right]$$

$$+ \mu \left\{ |\xi_n(w + v(w)) - \xi_n(w + \eta_n(w))|_H > \frac{a}{2} \right\}$$

$$\leq E \left[ |\Lambda_u| 1_{|w - \eta_n|_H > a/2} \right]$$

$$+ \mu \left\{ |v - \eta_n|_H > \frac{a}{2c} \right\} \to 0,$$

as $n$ tends to infinity, hence $\mu$-almost surely $T \circ S(w) = w$. Moreover

$$\mu \left\{ \|S \circ T(w) - w\|_W > a \right\} = \mu \left\{ \|S \circ T - S \circ U_n\|_W > a/2 \right\}$$

$$+ \mu \left\{ \|S \circ U_n - V_n \circ U_n\|_W > a/2 \right\}$$

$$\leq \mu \left\{ |u - \xi_n|_H > \frac{a(1 - c)}{2c} \right\}$$

$$+ E \left[ |\Lambda_n| 1_{|v - \eta_n|_H > a/2} \right] \to 0,$$

by the uniform integrability of $(\Lambda_n; n \in \mathbb{N})$, therefore $\mu$-almost surely, we have $S \circ T(w) = w$. \qed
10.2 Applications

In the sequel we shall give two applications. The first one consists of a very simple case of the Ramer formula which is used in Physics litterature (cf. [20] for more details). The second one concerns the logarithmic Sobolev inequality for the measures \( T^*\mu \) for the shifts \( T \) studied in this chapter.

10.2.1 Van-Vleck formula

**Lemma 10.2.1** Let \( K \in L^2(H) \) be a symmetric Hilbert–Schmidt operator on \( H \) such that \(-1\) does not belong to its spectrum. Set \( T_K(w) = w + \delta K(w) \), then \( T_K : W \to W \) is almost surely invertible and

\[
T_K^{-1}(w) = w - \delta[(I + K)^{-1}K](w),
\]

almost surely.

**Proof:** By the properties of the divergence operator (cf. Lemma 10.1.4)

\[
T_K(w - \delta((I + K)^{-1}K))(w) = w - \delta((I + K)^{-1}K)(w) + \delta K(w) - \langle \delta((I + K)^{-1}K), K \rangle_H
\]

\[
= w - \delta((I + K)^{-1}K)(w) + \delta K(w) - K(\delta((I + K)^{-1}K)(w))
\]

\[
= w + \delta K(w) - (I + K)\delta((I + K)^{-1}K)(w)
\]

\[
= w,
\]

and this proves the lemma.

**Lemma 10.2.2** Let \( K \) be a symmetric Hilbert–Schmidt operator on \( H \). We have

\[
\|\delta K\|_H^2 = \delta^{(2)} K^2 + \text{trace } K^2,
\]

where \( \delta^{(2)} \) denotes the second order divergence, i.e., \( \delta^{(2)} = (\nabla^2)^* \) with respect to \( \mu \).

**Proof:** Let \( \{e_i, i \geq 0\} \) be the complete, orthonormal basis of \( H \) corresponding to the eigenfunctions of \( K \) and denote by \( \{\alpha_i, i \geq 0\} \) its eigenvalues. We can represent \( K \) as

\[
K = \sum_{i=0}^{\infty} \alpha_i e_i \otimes e_i
\]

and

\[
K^2 = \sum_{i=0}^{\infty} \alpha_i^2 e_i \otimes e_i.
\]
Since $\delta K = \sum_i \alpha_i \delta e_i e_i$, we have
\[
\|\delta K\|_H^2 = \sum_{i=0}^{\infty} \alpha_i^2 \delta e_i^2
\]
\[
= \sum_{i=0}^{\infty} \alpha_i^2 (\delta e_i^2 - 1) + \sum_{i=0}^{\infty} \alpha_i^2
\]
\[
= \sum_{i=0}^{\infty} \alpha_i^2 \delta(e_i, e_i) + \text{trace } K^2
\]
\[
= \delta(2) K^2 + \text{trace } K^2.
\]

\[\Box\]

**Theorem 10.2.3** Let $K \in L^2(H)$ be a symmetric Hilbert–Schmidt operator such that $(I + K)$ is invertible and let $h_1, \ldots, h_n$ be $n$ linearly independent elements of $H$. Denote by $\delta \vec{h}$ the random vector $(\delta h_1, \ldots, \delta h_n)$. Then we have, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$
\[
E\left[ \exp\left(-\delta(2) \left( K + \frac{1}{2} K^2 \right) \right) | \delta \vec{h} = x \right]
\]
\[
= \exp\left( \frac{1}{2} \text{trace } K^2 \right) |\det_2(I + K)|^{-1} q_K(x)
\]
where $q_0(x)$ and $q_K(x)$ denote respectively the densities of the laws of the Gaussian vectors $(\delta h_1, \ldots, \delta h_n)$ and
\[
(\delta(I + K)^{-1} h_1, \ldots, \delta(I + K)^{-1} h_n).
\]

**Proof:** By the Ramer formula (cf. Theorem 10.1.1), for any nice function $f$ on $\mathbb{R}^n$, we have
\[
E\left[ f(\delta \vec{h}) | \det_2(I + K)| \exp\left(-\delta(2) K - \frac{1}{2} \|\delta K\|_H^2 \right) \right]
\]
\[
= E\left[ f(\delta \vec{h}) \circ T_K^{-1}(w) \right].
\]
Hence
\[
\int_{\mathbb{R}^n} E\left[ \exp\left(-\delta(2) \left( K + \frac{1}{2} K^2 \right) \right) | \delta \vec{h} = x \right] f(x) q_0(x) \, dx
\]
\[
= \exp\left( \frac{1}{2} \text{trace } K^2 \right) |\det_2(I + K)|^{-1} \int_{\mathbb{R}^n} f(x) q_K(x) dx.
\]
\[\Box\]
Corollary 10.2.4 Suppose that $A$ is a symmetric Hilbert–Schmidt operator whose spectrum is included in $(-1/2,1/2)$. Let $h_1,\ldots,h_n$ be $n$ linearly independent elements of $H$ and define the symmetric, Hilbert-Schmidt operator $K$ as $K = (I + 2A)^{1/2} - I$. Then the following identity holds:

$$E\left[\exp(-\delta^{(2)}A) \mid \delta\vec{h} = x\right] = \frac{1}{\sqrt{\det_2(I + 2A)}} q_{K}(x),$$

(10.2.2)

for any $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$.

Proof: Since the spectrum of $A$ is included in $(-1/2,1/2)$, the operator $I + 2A$ is symmetric and definite. It is easy to see that the operator $K$ is Hilbert-Schmidt. We have $K + K^2/2 = A$, hence the result follows by Theorem 10.2.3.

10.2.2 Logarithmic Sobolev inequality

Recall that the logarithmic Sobolev inequality for the Wiener measure says

$$E\left[f^2 \log \frac{f^2}{E[f^2]}\right] \leq 2E[|\nabla f|_H^2],$$

(10.2.3)

for any $f \in \mathbb{D}_{2,1}$. We can extend this inequality easily to the measures $\nu = T^*\mu$, where $T = I_W + u$ satisfies the hypothesis of Theorem 10.1.1.

Theorem 10.2.5 Assume that $\nu$ is a measure given by $\nu = T^*\mu$, where $T = I_W + u$ satisfies the hypothesis of Theorem 10.1.1, in particular $\|\nabla u\| \leq c$ almost surely for some $c \in (0,1)$. Then, we have

$$E_{\nu}\left[f^2 \log \frac{f^2}{E[f^2]}\right] \leq 2\left(\frac{c}{1-c}\right)^2 E_{\nu}[|\nabla f|_H^2]$$

(10.2.4)

for any cylindrical Wiener functional $f$, where $E_{\nu}[\cdot]$ represents the expectation with respect to $\nu$.

Proof: Let us denote by $S = I_W + v$ the inverse of $T$ whose existence has been proven in Theorem 10.1.1. Apply now the inequality 10.2.3 to $f \circ T$:

$$E\left[(f \circ T)^2 \log \frac{(f \circ T)^2}{E[(f \circ T)^2]}\right] \leq 2E\left[|\nabla (f \circ T)|_H^2\right] \leq 2E\left[|\nabla f \circ T|_H^2\|I_H + \nabla u\|^2\right] = 2E\left[|\nabla f \circ T|_H^2\|I_H + \nabla u \circ S \circ T\|^2\right]$$
Log-Sobolev Inequality

\[ 2E_\nu \left[ |\nabla f|^2 H \right] \]

\[ = 2E_\nu \left[ |\nabla f|^2 \| (I_H + \nabla v)^{-1} \|^2 \right] \]

\[ \leq 2 \left( \frac{c}{1-c} \right)^2 E_\nu \left[ |\nabla f|^2 H \right] \]

and this completes the proof. \( \square \)

We have also the following:

**Theorem 10.2.6** The operator \( \nabla \) is closable in \( L^p(\nu) \) for any \( p > 1 \).

**Proof:** Assume that \((f_n, n \in \mathbb{N})\) is a sequence of cylindrical Wiener functionals, converging in \( L^p(\nu) \) to zero, and assume also that \((\nabla f_n, n \in \mathbb{N})\) is Cauchy in \( L^p(\nu, H) \), denote its limit by \( \xi \). Then, by definition, \((f_n \circ T, n \in \mathbb{N})\) converges to zero in \( L^p(\mu) \), hence \((\nabla(f_n \circ T), n \in \mathbb{N})\) converges to zero in \( ID_{p-1}(H) \). Moreover

\[ \nabla(f_n \circ T) = (I_H + \nabla u)^* \nabla f_n \circ T, \]

hence for any cylindrical \( \eta \in \mathbb{D}(H) \), we have

\[ \lim_n E[(\nabla(f_n \circ T), \eta)_H] = \lim_n E[(\nabla f_n \circ T, (I_H + \nabla u)\eta)_H] \]

\[ = E[(\xi \circ T, (I_H + \nabla u)\eta)_H] \]

\[ = 0. \]

Since \( T \) is invertible, the sigma algebra generated by \( T \) is equal to the Borel sigma algebra of \( W \) upto the negligible sets. Consequently, we have

\[ (I_H + \nabla u)^* \xi \circ T = 0 \]

\( \mu \)-almost surely. Since \( I_H + \nabla u \) is almost surely invertible, \( \mu \)-almost surely we have \( \xi \circ T = 0 \) and this amounts up to saying \( \xi = 0 \) \( \nu \)-almost surely. \( \square \)

**Notes and suggested reading**

The Ramer theorem has been proved, with some stronger hypothesis (Fréchet regularity of \( u \)) in [74], later some of its hypothesis have been relaxed in [51]. The version given here has been proved in [97]. We refer the reader to [101] for its further extensions and applications to the degree theory of Wiener maps (cf. [98] also). The Van-Vleck formula is well-known in Physics, however the general approach that we have used here as well as the logarithmic Sobolev inequalities with these new measures are original.
Theorem of Ramer
Chapter 11

Convexity on Wiener space

Introduction

On an infinite dimensional vector space $W$ the notion of convex or concave function is well-known. Assume now that this space is equipped with a probability measure. Suppose that there are two measurable functions on this vector space, say $F$ and $G$ such that $F = G$ almost surely. If $F$ is a convex function, then from the probabilistic point of view, we would like to say that $G$ is also convex. However this is false; since in general the underlying probability measure is not (quasi) invariant under the translations by the elements of the vector space. If $W$ contains a dense subspace $H$ such that $w \to w + h$ ($h \in H$) induces a measure which is equivalent to the initial measure or absolutely continuous with respect to it, then we can define a notion of “$H$–convexity” or “$H$–concavity in the direction of $H$ for the equivalence classes of real random variables. Hence these notions will be particularly useful for the probabilistic calculations.

The notion of $H$-convexity has been used in [101] to study the absolute continuity of the image of the Wiener measure under the monotone shifts. In this chapter we study further properties of such functions and some additional ones in the frame of an abstract Wiener space, namely $H$-convex, $H$-concave, log $H$-concave and log $H$-convex Wiener functions, where $H$ denotes the associated Cameron-Martin space. In particular we extend some finite dimensional results of [73] and [13] to this setting and prove that some finite dimensional convexity-concavity inequalities have their counterparts in infinite dimensions.
11.1 Preliminaries

In the sequel \((W,H,\mu)\) denotes an abstract Wiener space, i.e., \(H\) is a separable Hilbert space, called the Cameron-Martin space. It is identified with its continuous dual. \(W\) is a Banach or a Fréchet space into which \(H\) is injected continuously and densely. \(\mu\) is the standard cylindrical Gaussian measure on \(H\) which is concentrated in \(W\) as a Radon probability measure.

In the sequel we shall use the notion of second quantization of bounded operators on \(H\); although this is a well-known subject, we give a brief outline below for the reader’s convenience (cf. [8], [30], [77]). Assume that \(A : H \to H\) is a bounded, linear operator, then it has a unique, \(\mu\)-measurable (i.e., measurable with respect to the \(\mu\)-completion of \(B(W)\)) extension, denoted by \(\tilde{A}\), as a linear map on \(W\) (cf. [8, 30]). Assume in particular that \(\|A\| \leq 1\) and define \(S = (I_H - A^*A)^{1/2}\), \(T = (I_H - AA^*)^{1/2}\) and \(U : H \times H \to H \times H\) as \(U(h, k) = (Ah + Tk, -Sh + A^*k)\). \(U\) is then a unitary operator on \(H \times H\), hence its \(\mu \times \mu\)-measurable linear extension to \(W \times W\) preserves the Wiener measure \(\mu \times \mu\) (this is called the rotation associated to \(U\), cf. [101], Chapter VIII). Using this observation, one can define the second quantization of \(A\) via the generalized Mehler formula as

\[
\Gamma(A)f(w) = \int_W f(\tilde{A}^*w + \tilde{S}y)\mu(dy),
\]

which happens to be a Markovian contraction on \(L^p(\mu)\) for any \(p \geq 1\). \(\Gamma(A)\) can be calculated explicitly for the Wick exponentials as

\[
\Gamma(A) \exp \left\{ \delta h - 1/2|h|^2_H \right\} = \exp \left\{ \delta Ah - 1/2|Ah|^2_H \right\} \quad (h \in H).
\]

This identity implies that \(\Gamma(AB) = \Gamma(A)\Gamma(B)\) and that for any sequence \((A_n, n \in \mathbb{N})\) of operators whose norms are bounded by one, \(\Gamma(A_n)\) converges strongly to \(\Gamma(A)\) if \(\lim_n A_n = A\) in the strong operator topology. A particular case of interest is when we take \(A = e^{-t}I_H\), then \(\Gamma(e^{-t}I_H)\) equals to the Ornstein-Uhlenbeck semigroup \(P_t\). Also if \(\pi\) is the orthogonal projection of \(H\) onto a closed vector subspace \(K\), then \(\Gamma(\pi)\) is the conditional expectation with respect to the sigma field generated by \(\{\delta k, k \in K\}\).

11.2 \(H\)-convexity and its properties

Let us give the notion of \(H\)-convexity on the Wiener space \(W\):

**Definition 11.2.1** Let \(F : W \to \mathbb{R} \cup \{\infty\}\) be a measurable function. It is called \(H\)-convex if for any \(h, k \in H\), \(\alpha \in [0, 1]\)

\[
F(w + \alpha h + (1 - \alpha)k) \leq \alpha F(w + h) + (1 - \alpha)F(w + k) \quad (11.2.1)
\]
almost surely.

Remarks:

- This definition is more general than the one given in [99, 101] since $F$ may be infinite on a set of positive measure.

- Note that the negligible set on which the relation (11.2.1) fails may depend on the choice of $h, k$ and of $\alpha$.

- If $G : W \to \mathbb{R} \cup \{\infty\}$ is a measurable convex function, then it is necessarily $H$-convex.

- To conclude the $H$-convexity, it suffices to verify the relation (11.2.1) for $k = -h$ and $\alpha = 1/2$.

The following properties of $H$-convex Wiener functionals have been proved in [99, 100, 101]:

**Theorem 11.2.2**

1. If $(F_n, n \in \mathbb{N})$ is a sequence of $H$-convex functionals converging in probability, then the limit is also $H$-convex.

2. If $F \in L^p(\mu)$ ($p > 1$) is $H$-convex if and only if $\nabla^2 F$ is positive and symmetric Hilbert-Schmidt operator valued distribution on $W$.

3. If $F \in L^1(\mu)$ is $H$-convex, then $P_tF$ is also $H$-convex for any $t \geq 0$, where $P_t$ is the Ornstein-Uhlenbeck semi-group on $W$.

The following result is immediate from Theorem 11.2.2:

**Corollary 11.2.3**

$F \in \cup_{p>1} L^p(\mu)$ is $H$-convex if and only if

$$E \left[ \varphi \left( \nabla^2 F(w), h \otimes h \right) \right] \geq 0$$

for any $h \in H$ and $\varphi \in \mathbb{D}_+$, where $(\cdot, \cdot)_2$ denotes the scalar product for the Hilbert-Schmidt operators on $H$.

We have also

**Corollary 11.2.4**

If $F \in L^p(\mu), p > 1$, is $H$-convex and if $E[\nabla^2 F] = 0$, then $F$ is of the form

$$F = E[F] + \delta \left( E[\nabla F] \right).$$
Proof: Let \((P_t, t \geq 0)\) denote the Ornstein-Uhlenbeck semigroup, then \(P_tF\) is again \(H\)-convex and Sobolev differentiable. Moreover \(\nabla^2 P_tF = e^{-2t}P_t\nabla^2 F\). Hence \(E[\nabla^2 P_tF] = 0\), and the positivity of \(\nabla^2 P_tF\) implies that \(\nabla^2 P_tF = 0\) almost surely, hence \(\nabla^2 F = 0\). This implies that \(F\) is in the first two Wiener chaos. □

Remark: It may be worth-while to note that the random variable which represents the share price of the Black and Scholes model in financial mathematics is \(H\)-convex.

We shall need also the concept of \(C\)-convex functionals:

**Definition 11.2.5** Let \((e_i, i \in \mathbb{N}) \subset W^*\) be any complete, orthonormal basis of \(H\). For \(w \in W\), define \(w_n = \sum_{i=1}^{n} \delta_{e_i}(w)e_i\) and \(w_n^\perp = w - w_n\), then a Wiener functional \(f : W \to \mathbb{R}\) is called \(C\)-convex if, for any such basis \((e_i, i \in \mathbb{N})\), for almost all \(w_n^\perp\), the partial map

\[
  w_n \to f(w_n^\perp + w_n)
\]

has a modification which is convex on the space \(\text{span}\{e_1, \ldots, e_n\} \simeq \mathbb{R}^n\).

**Remark:** It follows from Corollary 11.2.3 that, if \(f\) is \(H\)-convex and in some \(L^p(\mu)\) \((p > 1)\), then it is \(C\)-convex. We shall prove that this is also true without any integrability hypothesis.

We begin with the following lemma whose proof is obvious:

**Lemma 11.2.6** If \(f\) is \(C\)-convex then it is \(H\)-convex.

In order to prove the validity of the converse of Lemma 11.2.6 we need some technical results from the harmonic analysis on finite dimensional Euclidean spaces that we shall state as separate lemmas:

**Lemma 11.2.7** Let \(B \in \mathcal{B}(\mathbb{R}^n)\) be a set of positive Lebesgue measure. Then \(B + B\) contains a non-empty open set.

**Proof:** Let \(\phi(x) = 1_B \ast 1_B(x)\), where “\(\ast\)” denotes the convolution of functions with respect to the Lebesgue measure. Then \(\phi\) is a non-negative, continuous function, hence the set \(O = \{x \in \mathbb{R}^n : \phi(x) > 0\}\) is an open set. Since \(B\) has positive measure, \(\phi\) can not be identically zero, hence \(O\) is non-empty. Besides, if \(x \in O\), then the set of \(y \in \mathbb{R}^n\) such that \(y \in B\) and \(x - y \in B\) has positive Lebesgue measure, otherwise \(\phi(x)\) would have been null. Consequently \(O \subset B + B\). □

The following lemma gives a more precise statement than Lemma 11.2.7.
Lemma 11.2.8 Let $B \in \mathcal{B}(\mathbb{R}^n)$ be a set of positive Lebesgue measure and assume that $A \subset \mathbb{R}^n \times \mathbb{R}^n$ with $B \times B = A$ almost surely with respect to the Lebesgue measure of $\mathbb{R}^n \times \mathbb{R}^n$. Then the set $\{x + y : (x, y) \in A\}$ contains almost surely an open subset of $\mathbb{R}^n$.

Proof: It follows from an obvious change of variables that

$$1_A(y, x - y) = 1_B(y)1_B(x - y)$$

almost surely, hence

$$\int_{\mathbb{R}^n} 1_A(y, x - y) dy = \phi(x)$$

almost surely, where $\phi(x) = 1_B \ast 1_B(x)$. Consequently, for almost all $x \in \mathbb{R}^n$ such that $\phi(x) > 0$, one has $(y, x - y) \in A$, this means that

$$\{x \in \mathbb{R}^n : \phi(x) > 0\} \subset \{u + v : (u, v) \in A\}$$

almost surely. □

The following lemma is particularly important for the sequel:

Lemma 11.2.9 Let $f : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ be a Borel function which is finite on a set of positive Lebesgue measure. Assume that, for any $u \in \mathbb{R}^n$,

$$f(x) \leq \frac{1}{2}[f(x + u) + f(x - u)] \quad (11.2.2)$$

almost surely (the negligeable set on which the inequality (11.2.2) fails may depend on u). Then there exists a non-empty, open convex subset $U$ of $\mathbb{R}^n$ such that $f$ is locally essentially bounded on $U$. Moreover let $D$ be the set consisting of $x \in \mathbb{R}^n$ such that any neighbourhood of $x \in D$ contains a Borel set of positive Lebesgue measure on which $f$ is finite, then $D \subset \overline{U}$, in particular $f = \infty$ almost surely on the complement of $\overline{U}$.

Proof: From the theorem of Fubini, the inequality (11.2.2) implies that

$$2f\left(\frac{x + y}{2}\right) \leq f(x) + f(y) \quad (11.2.3)$$

dx × dy-almost surely. Let $B \in \mathcal{B}(\mathbb{R}^n)$ be a set of positive Lebesgue measure on which $f$ is bounded by some constant $M > 0$. Then from Lemma 11.2.7, $B + B$ contains an open set $O$. Let $A$ be the set consisting of the elements of $B \times B$ for which the inequality (11.2.3) holds. Then $A = B \times B$ almost surely, hence from Lemma 11.2.8 the set $\Gamma = \{x + y : (x, y) \in A\}$ contains almost surely the open set $O$. Hence for almost all $z \in \frac{1}{2}O$, $2z$ belongs to...
the set $\Gamma$, consequently $z = \frac{1}{2}(x + y)$, with $(x, y) \in A$. This implies, from (11.2.3), that $f(z) \leq M$. Consequently $f$ is essentially bounded on the open set $\frac{1}{2}O$.

Let now $U$ be set of points which have neighbourhoods on which $f$ is essentially bounded. Clearly $U$ is open and non-empty by what we have shown above. Let $S$ and $T$ be two balls of radius $\rho$, on which $f$ is bounded by some $M > 0$. Assume that they are centered at the points $a$ and $b$ respectively. Let $u = \frac{1}{2}(b - a)$, then for almost all $x \in \frac{1}{2}(S + T)$, $x + u \in T$ and $x - u \in S$, hence, from the inequality (11.2.2) $f(x) \leq M$, which shows that $f$ is essentially bounded on the set $\frac{1}{2}(S + T)$ and this proves the convexity of $U$.

To prove the last claim, let $x$ be any element of $D$ and let $V$ be any neighbourhood of $x$; without loss of generality, we may assume that $V$ is convex. Then there exists a Borel set $B \subset V$ of positive measure on which $f$ is bounded, hence from the first part of the proof, there exists an open neighbourhood $O \subset B + B$ such that $f$ is essentially bounded on $\frac{1}{2}O \subset \frac{1}{2}(V + V) \subset V$, hence $\frac{1}{2}O \subset U$. Consequently $V \cap U \neq \emptyset$, and this implies that $x$ is in the closure of $U$, i.e. $D \subset \overline{U}$. The fact that $f = \infty$ almost surely on the complement of $U$ is obvious from the definition of $D$. 

\begin{proof}
Assume first that $g$ is positive, then with the notations of Lemma 11.2.9 define $g' = g$ on the open, convex set $U$ and as $g' = \infty$ on $U^c$. From the relation (11.2.4), $g'$ is a distribution on $U$ whose second derivative is positive, hence it is convex on $U$, hence it is convex on the whole space $\mathbb{R}^n$. Moreover we have $\{g' \neq \infty\} \subset \partial U$ and $\partial U$ has zero Lebesgue measure, consequently $g = g'$ almost surely. For general $g$, define $f_\varepsilon = e^{g_\varepsilon}$ ($\varepsilon > 0$), then, from what is proven above, $f_\varepsilon$ has a modification $f'_\varepsilon$ which is convex (with the same fixed open and convex set $U$), hence $\limsup_{\varepsilon \to 0} \frac{f'_{\varepsilon} - 1}{\varepsilon} = g'$ is also convex and $g = g'$ almost surely.
\end{proof}

\begin{theorem}
A Wiener functional $F : W \to \mathbb{R} \cup \{\infty\}$ is $H$-convex if and only if it is $C$-convex.
\end{theorem}
Proof: We have already proven the sufficiency. To prove the necessity, with the notations of Definition 11.2.5, $H$-convexity implies that $h \to F(w_n^+ + w_n + h)$ satisfies the hypothesis of Theorem 11.2.10 when $h$ runs in any $n$-dimensional Euclidean subspace of $H$, hence the partial mapping $w_n \to F(w_n^+ + w_n)$ has a modification which is convex on the vector space spanned by $\{e_1, \ldots, e_n\}$.

11.3 Log $H$-concave and $C$-log concave Wiener functionals

Definition 11.3.1 Let $F$ be a measurable mapping from $W$ into $\mathbb{R}_+$ with $\mu\{F > 0\} > 0$.

1. $F$ is called log $H$-concave, if for any $h, k \in H$, $\alpha \in [0, 1]$, one has

$$F(w + \alpha h + (1 - \alpha)k) \geq F(w + h)^\alpha F(w + k)^{1-\alpha} \quad (11.3.5)$$

almost surely, where the negligible set on which the relation (11.3.5) fails may depend on $h$, $k$ and on $\alpha$.

2. We shall say that $F$ is $C$-log concave, if for any complete, orthonormal basis $(e_i, i \in \mathbb{N}) \subset W^*$ of $H$, the partial map $w_n \to F(w_n^+ + w_n)$ is log-concave (cf. Definition 11.2.5 for the notation), up to a modification, on $\text{span}\{e_1, \ldots, e_n\} \simeq \mathbb{R}^n$.

Let us remark immediately that if $F = G$ almost surely then $G$ is also log $H$-concave. Moreover, any limit in probability of log $H$-concave random variables is again log $H$-concave. We shall prove below some less immediate properties. Let us begin with the following observation which is a direct consequence of Theorem 11.2.11:

Remark: $F$ is log $H$-concave if and only if $-\log F$ is $H$-convex (which may be infinity with a positive probability), hence if and only if $F$ is $C$-log concave.

Theorem 11.3.2 Suppose that $(W_i, H_i, \mu_i)$, $i = 1, 2$, are two abstract Wiener spaces. Consider $(W_1 \times W_2, H_1 \times H_1, \mu_1 \times \mu_2)$ as an abstract Wiener space. Assume that $F: W_1 \times W_2 \to \mathbb{R}_+$ is log $H_1 \times H_2$-concave. Then the map

$$w_2 \to \int_{W_1} F(w_1, w_2) \, d\mu_1(w_1)$$

is log $H_2$-concave.
**Proof:** If $F$ is log $H \times H$-concave, so is also $F \wedge c$ ($c \in \mathbb{R}_+$), hence we may suppose without loss of generality that $F$ is bounded. Let $(e_i, i \in \mathbb{N})$ be a complete, orthonormal basis in $H_2$. It suffices to prove that

$$E_1[F](w_2 + \alpha h + \beta l) \geq (E_1[F](w_2 + h))^\alpha (E_1[F](w_2 + l))^\beta$$

almost surely, for any $h, l \in \text{span}\{e_1, \ldots, e_k\}$, $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, where $E_1$ denotes the expectation with respect to $\mu_1$. Let $(P_n, n \in \mathbb{N})$ be a sequence of orthogonal projections of finite rank on $H_1$ increasing to the identity map of it. Denote by $\mu_1^n$ the image of $\mu_1$ under the map $w_1 \rightarrow \tilde{P}_n w_1$ and by $\mu_1^{n,\perp}$ the image of $\mu_1$ under $w_1 \rightarrow w_1 - \tilde{P}_n w_1$. We have, from the martingale convergence theorem,

$$\int_{W_1} F(w_1, w_2) \, d\mu_1(w_1) = \lim_n \int F(w_1^n, w_2^n) \, d\mu^n_1(w_1^n)$$

almost surely. Let $(Q_n, n \in \mathbb{N})$ be a sequence of orthogonal projections of finite rank on $H_2$ increasing to the identity, corresponding to the basis $(e_n, n \in \mathbb{N})$. Let $w_2^n = \tilde{Q}_n w_2$ and $w_2^{n,\perp} = w_2 - w_2^n$. Write

$$F(w_1, w_2) = F(w_1^n, w_2^n, w_2^{n,\perp}) = F_{w_1^n, w_2^{n,\perp}}(w_1, w_2).$$

From the hypothesis

$$(w_1^n, w_2^n) \rightarrow F_{w_1^n, w_2^{n,\perp}}(w_1^n, w_2^n)$$

has a log concave modification on the $(n + k)$-dimensional Euclidean space. From the theorem of Prékopa (cf. [73]), it follows that

$$w_2^n \rightarrow \int F_{w_1^n, w_2^{n,\perp}}(w_1^n, w_2^n) \, d\mu^n_1(w_1^n)$$

is log concave on $\mathbb{R}^k$ for any $k \in \mathbb{N}$ (upto a modification), hence

$$w_2 \rightarrow \int F(w_1^n, w_1^n, w_2) \, d\mu(w_1^n)$$

is log $H_2$-concave for any $n \in \mathbb{N}$, then the proof follows by passing to the limit with respect to $n$. \qed

**Theorem 11.3.3** Let $A : H \rightarrow H$ be a linear operator with $\|A\| \leq 1$, denote by $\Gamma(A)$ its second quantization as explained in the preliminaries. If $F : W \rightarrow \mathbb{R}_+$ is a log $H$-concave Wiener functional, then $\Gamma(A)F$ is also log $H$-concave.
**Proof:** Replacing $F$ by $F \wedge c = \min(F, c)$, $c > 0$, we may suppose that $F$ is bounded. It is easy to see that the mapping

$$(w, y) \mapsto F(\tilde{A}^w + \tilde{S}y)$$

is log $H \times H$-concave on $W \times W$. In fact, for any $\alpha + \beta = 1$, $h, k, u, v \in H$, one has

$$F(\tilde{A}^w + \tilde{S}y + \alpha(A^h + Sk) + \beta(A^u + Sv)) \geq F(\tilde{A}^w + \tilde{S}y + A^h + Sk)\alpha \cdot F(\tilde{A}^w + \tilde{S}y + A^u + Sv)\beta,$$

d$\mu \times d\mu$-almost surely. Let us recall that, since the image of $\mu \times \mu$ under the map $(w, y) \mapsto \tilde{A}^w + \tilde{S}y$ is $\mu$, the terms in the inequality (11.3.6) are defined without ambiguity. Hence

$$\Gamma(A)F(w) = \int_W F(\tilde{A}^w + \tilde{S}y)\mu(dy)$$

is log $H$-concave on $W$ from Theorem 11.3.2.

**Corollary 11.3.4** Let $F : W \to \mathbb{R}_+$ be a log $H$-concave functional. Assume that $K$ is any closed vector subspace of $H$ and denote by $V(K)$ the sigma algebra generated by $\{\delta k, k \in K\}$. Then the conditional expectation of $F$ with respect to $V(K)$, i.e., $E[F|V(K)]$ is again log $H$-concave.

**Proof:** The proof follows from Theorem 11.3.3 as soon as we remark that $\Gamma(\pi_K)F = E[F|V(K)]$, where $\pi_K$ denotes the orthogonal projection associated to $K$.

**Corollary 11.3.5** Let $F$ be log $H$-concave. If $P_t$ denotes the Ornstein-Uhlenbeck semigroup on $W$, then $w \mapsto P_tF(w)$ is log $H$-concave.

**Proof:** Since $P_t = \Gamma(e^{-t}I_H)$, the proof follows from Theorem 11.3.3.

Here is an important application of these results:

**Theorem 11.3.6** Assume that $F : W \to \mathbb{R} \cup \{\infty\}$ is an $H$-convex Wiener functional, then $F$ has a modification $F'$ which is a Borel measurable convex function on $W$. Any log $H$-concave functional $G$ has a modification $G'$ which is Borel measurable and log-concave on $W$. 
Proof: Assume first that $F$ is positive, let $G = \exp -F$, then $G$ is a positive, bounded $C$-log concave function. Define $G_n$ as

$$G_n = E[P_{1/n}G|V_n],$$

where $V_n$ is the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$, and $(\delta i, i \in \mathbb{N}) \subset W^*$ is a complete orthonormal basis of $H$. Since $P_{1/n}E[G|V_n] = E[P_{1/n}G|V_n]$, the positivity improving property of the Ornstein-Uhlenbeck semigroup implies that $G_n$ is almost surely strictly positive (even quasi-surely). As we have attained the finite dimensional case, $G_n$ has a modification $G'_n$ which is continuous on $W$ and, from Corollary 11.3.4 and Corollary 11.3.5, it satisfies

$$G'_n(w + ah + bk) \geq G'_n(w + h)^a G'_n(w + k)^b$$

almost surely, for any $h, k \in H$ and $a + b = 1$. The continuity of $G'_n$ implies that the relation (11.3.7) holds for any $h, k \in H$, $w \in W$ and $a \in [0, 1]$. Hence $G'_n$ is log-concave on $W$ and this implies that $-\log G'_n$ is convex on $W$. Define $F' = \limsup_n ( -\log G'_n)$, then $F'$ is convex and Borel measurable on $W$ and $F = F'$ almost surely.

For general $F$, define $f_\varepsilon = e^{\varepsilon F}$, then from above, there exists a modification of $f_\varepsilon$, say $f'_\varepsilon$ which is convex and Borel measurable on $W$. To complete the proof it suffices to define $F'$ as

$$F' = \limsup_{\varepsilon \to 0} f'_\varepsilon - \frac{1}{\varepsilon}.$$

The rest is now obvious. \qed

Under the light of Theorem 11.3.6, the following definition is natural:

**Definition 11.3.7** A Wiener functional $F : W \to \mathbb{R} \cup \{\infty\}$ will be called almost surely convex if it has a modification $F'$ which is convex and Borel measurable on $W$. Similarly, a non-negative functional $G$ will be called almost surely log-concave if it has a modification $G'$ which is log-concave on $W$.

The following proposition summarizes the main results of this section:

**Theorem 11.3.8** Assume that $F : W \to \mathbb{R} \cup \{\infty\}$ is a Wiener functional such that

$$\mu\{F < \infty\} > 0.$$

Then the following are equivalent:

1. $F$ is $H$-convex,
2. $F$ is $C$-convex,

3. $F$ is almost surely convex.

Similarly, for $G : W \to \mathbb{R}_+$, with $\mu\{G > 0\} > 0$, the following properties are equivalent:

1. $G$ is log $H$-concave,

2. $G$ is log $C$-concave,

3. $G$ is almost surely log-concave.

The notion of a convex set can be extended as

**Definition 11.3.9** Any measurable subset $A$ of $W$ will be called $H$-convex if its indicator function $1_A$ is log $H$-concave.

**Remark:** Evidently any measurable convex subset of $W$ is $H$-convex. Moreover, if $A = A'$ almost surely and if $A$ is $H$-convex, then $A'$ is also $H$-convex.

**Remark:** If $\phi$ is an $H$-convex Wiener functional, then the set

$$\{w \in W : \phi(w) \leq t\}$$

is $H$-convex for any $t \in \mathbb{R}$.

We have the following result about the characterization of the $H$-convex sets:

**Theorem 11.3.10** Assume that $A$ is an $H$-convex set, then there exists a convex set $A'$, which is Borel measurable such that $A = A'$ almost surely.

**Proof:** Since, by definition, $1_A$ is a log $H$-concave Wiener functional, from Theorem 11.3.6 there exists a log-concave Wiener functional $f_A$ such that $f_A = 1_A$ almost surely. It suffices to define $A'$ as the set

$$A' = \{w \in W : f_A(w) \geq 1\}.$$

**Example:** Assume that $A$ is an $H$-convex subset of $W$ of positive measure. Define $p_A$ as

$$p_A(w) = \inf \{\|h\|_H : h \in (A - w) \cap H\}.$$

Then $p_A$ is $H$-convex, hence almost surely convex (and $H$-Lipschitz c.f. [101]). Moreover, the $\{w : p_A(w) \leq \alpha\}$ is an $H$-convex set for any $\alpha \in \mathbb{R}_+$. 
11.4 Extensions and some applications

Definition 11.4.1 Let \((e_i, i \in \mathbb{N})\) be any complete orthonormal basis of \(H\). We shall denote, as before, by \(w_n = \sum_{i=1}^n \delta e_i(w) e_i\) and \(w_n^\perp = w - w_n\). Assume now that \(F : W \to \mathbb{R} \cup \{\infty\}\) is a measurable mapping with \(\mu\{F < \infty\} > 0\).

1. We say that it is \(a\)-convex \((a \in \mathbb{R})\), if the partial map
\[
\frac{a}{2}|w_n|^2 + F(w_n^\perp + w_n)
\]

is almost surely convex for any \(n \geq 1\), where \(|w_n|\) is the Euclidean norm of \(w_n\).

2. We call \(G\) a \(a\)-log-concave if
\[
\frac{a}{2}|w_n|^2 + F(w_n^\perp + w_n)
\]
is almost surely log-concave for any \(n \in \mathbb{N}\).

Remark: \(G\) is a \(a\)-log-concave if and only if \(-\log G\) is \(a\)-convex.

The following theorem gives a practical method to verify \(a\)-convexity or log-concavity:

Theorem 11.4.2 Let \(F : W \to \mathbb{R} \cup \{\infty\}\) be a measurable map such that \(\mu\{F < \infty\} > 0\). Define the map \(F_a\) on \(H \times W\) as
\[
F_a(h, w + h) = \frac{a}{2}|h|^2_H + F(w) + h).
\]
Then \(F\) is \(a\)-convex if and only if, for any \(h, k \in H\) and \(\alpha, \beta \in [0, 1]\) with \(\alpha + \beta = 1\), one has
\[
F_a(\alpha h + \beta k, w + \alpha h + \beta k) \leq \alpha F_a(h, w + h) + \beta F_a(k, w + k) \tag{11.4.8}
\]
\(\mu\)-almost surely, where the negligible set on which the inequality (11.4.8) fails may depend on the choice of \(h, k\) and of \(a\).

Similarly a measurable mapping \(G : W \to \mathbb{R}_+\) is a \(a\)-log-concave if and only if the map defined by
\[
G_a(h, w + h) = \exp \left\{ -\frac{a}{2}|h|^2_H \right\} G(w + h)
\]
satisfies the inequality
\[
G_a(\alpha h + \beta k, w + \alpha h + \beta k) \geq G_a(h, w + h)^\alpha G_a(k, w + k)^\beta, \tag{11.4.9}
\]
\(\mu\)-almost surely, where the negligible set on which the inequality (11.4.9) fails may depend on the choice of \(h, k\) and of \(a\).
Proof: Let us denote by \( h_n \) its projection on the vector space spanned by \( \{ e_1, \ldots, e_n \} \), i.e. \( h_n = \sum_{i \leq n} (h, e_i)_H e_i \). Then, from Theorem 11.3.8 \( F \) is \( a \)-convex if and only if the map

\[
    h_n \rightarrow \frac{\alpha}{2} \left[ |w_n|^2 + 2(w_n, h_n) + |h_n|^2 \right] + F(w + h_n)
\]

satisfies a convexity inequality like (11.4.8). Besides the term \( |w_n|^2 \) being kept constant in this operation, it can be removed from the both sides of the inequality. Similarly, since \( h_n \rightarrow (w_n, h_n) \) is being affine, it also cancels from the both sides of this inequality. Hence \( a \)-convexity is equivalent to

\[
    F_a(\alpha h_n + \beta k_n, w + \alpha h_n + \beta k_n) \leq \alpha F_a(h_n, w + h_n) + \beta F_a(k_n, w + k_n)
\]

where \( k_n \) is defined as \( h_n \) from a \( k \in H \).

The second part of the theorem is obvious since \( G \) is \( a \)-log-concave if and only if \(-\log G\) is \( a \)-convex.

Corollary 11.4.3

1. Let \( \hat{L}^0(\mu) \) be the space of the \( \mu \)-equivalence classes of \( \mathbb{R} \cup \{ \infty \} \)-valued random variables regarded as a topological semigroup under addition and convergence in probability. Then \( F \in \hat{L}^0(\mu) \) is \( \beta \)-convex if and only if the mapping

\[
    h \rightarrow \frac{\beta}{2} |h|^2_H + F(w + h)
\]

is a convex and continuous mapping from \( H \) into \( \hat{L}^0(\mu) \).

2. \( F \in L^p(\mu), p > 1 \) is \( \beta \)-convex if and only if

\[
    E \left[ \left( (\beta I_H + \nabla^2 F) h, h \right)_H \phi \right] \geq 0
\]

for any \( \phi \in \mathcal{D} \) positive and \( h \in H \), where \( \nabla^2 F \) is to be understood in the sense of the distributions \( \mathcal{D}' \).

Example: Note for instance that \( \sin \delta h \) with \( |h|^H = 1 \), is a 1-convex random variable and that \( \exp(\sin \delta h) \) is 1-log-concave.

The following result is a direct consequence of Prekopa’s theorem:

Proposition 11.4.4 Let \( G \) be an \( a \)-log concave Wiener functional, \( a \in [0, 1] \), and assume that \( V \) is any sigma algebra generated by the elements of the first Wiener chaos. Then \( E[G|V] \) is again \( a \)-log-concave.
Proof: From Corollary [11.4.3] it suffices to prove the case $V$ is generated by 
\{δe_1, \ldots, δe_k\}, where $(e_n, n \in \mathbb{N})$ is an orthonormal basis of $H$. Let

$$w_k = \sum_{i \leq k} δe_i(w)e_i$$

$$z_k = w - w_k$$

$$z_{k,n} = \sum_{i=k+1} δe_i(w)e_i$$

and let $z_{k,n}^⊥ = z_k - z_{k,n}$. Then we have

$$E[G|V] = \int G(z_k + w_k) d\mu(z_k)$$

$$= \lim_{n} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} G(z_{k,n}^⊥ + z_{k,n} + w_k) e^{-\frac{|z_{k,n}|^2}{2}} d(z_{k,n})$$

Since

$$(z_{k,n}, w_k) \to \exp\left\{-\frac{1}{2}(a|w_k|^2 + |z_{n,k}|^2)\right\} G(z_{k,n}^⊥ + z_{k,n} + w_k)$$

is almost surely log-concave, the proof follows from Prekopa’s theorem (cf. [73]).}

The following theorem extends Theorem [11.3.3]:

**Theorem 11.4.5** Let $G$ be an $α$-log-concave Wiener functional, where $α \in [0, 1]$. Then $Γ(A)G$ is a $α$-log-concave, where $A \in L(H, H)$ (i.e. the space of bounded linear operators on $H$) with $\|A\| \leq 1$. In particular $P_t G$ is a $α$-log-concave for any $t \geq 0$, where $(P_t, t \geq 0)$ denotes the Ornstein-Uhlenbeck semi-group on $W$.

**Proof:** Let $(e_i, i \in \mathbb{N})$ be a complete, orthonormal basis of $H$, denote by $π_n$ the orthogonal projection from $H$ onto the linear space spanned by \{e_1, \ldots, e_n\} and by $V_n$ the sigma algebra generated by \{δe_1, \ldots, δe_n\}. From Proposition [11.4.4] and from the fact that $Γ(π_n A π_n) → Γ(A)$ in the strong operator topology as $n$ tends to infinity, it suffices to prove the theorem when $W = \mathbb{R}^n$. We may then assume that $G$ is bounded and of compact support. Define $F$ as

$$G(x) = F(x) e^{\frac{1}{2}|x|^2}$$

$$= F(x) \int_{\mathbb{R}^n} e^{\sqrt{a}(x, ξ)} dμ(ξ) .$$
From the hypothesis, $F$ is almost surely log-concave. Then, using the notations explained in Section 2:

$$e^{-a|x|^2/2} \Gamma(A)G(x)$$

$$= \int \int F(A^*x + Sy) \exp \left\{-a \frac{|x|^2}{2} + \sqrt{a}(A^*x + Sy, \xi) \right\} d\mu(y) d\mu(\xi)$$

$$= (2\pi)^{-n} \int \int F(A^*x + Sy) \exp \left\{-\frac{\Theta(x, y, \xi)}{2} \right\} dy d\xi,$$

where

$$\Theta(x, y, \xi) = a|x|^2 - 2\sqrt{a}(A^*x + Sy, \xi) + |y|^2 + |\xi|^2$$

$$= |\sqrt{a}x - A\xi|^2 + |\sqrt{a}y - S\xi|^2 + (1 - a)|y|^2,$$

which is a convex function of $(x, y, \xi)$. Hence the proof follows from Prékopa’s theorem (cf. [73]). \qed

The following proposition extends a well-known finite dimensional inequality (cf. [41]):

**Proposition 11.4.6** Assume that $f$ and $g$ are $H$-convex Wiener functionals such that $f \in L^p(\mu)$ and $g \in L^q(\mu)$ with $p > 1$, $p^{-1} = 1 - q^{-1}$. Then

$$E[f g] \geq E[f]E[g] + (E[\nabla f], E[\nabla g])_H.$$  \hfill (11.4.10)

**Proof:** Define the smooth and convex functions $f_n$ and $g_n$ on $W$ by

$$P_{1/n} f = f_n$$

$$P_{1/n} g = g_n.$$

Using the fact that $P_t = e^{-tL}$, where $L$ is the number operator $L = \delta \circ \nabla$ and the commutation relation $\nabla P_t = e^{-tP_t} \nabla$, for any $0 \leq t \leq T$, we have

$$E[P_{T-t} f_n g_n] = E[P_T f_n g_n] + \int_0^t E[L P_{T-s} f_n g_n] \, ds$$

$$= E[P_T f_n g_n] + \int_0^t e^{-(T-s)} E[(P_{T-s} \nabla f_n, \nabla g_n)_H] \, ds$$

$$= E[P_T f_n g_n] + \int_0^t e^{-(T-s)} E[(P_T \nabla f_n, \nabla g_n)_H] \, ds$$

$$+ e^{-2T} \int_0^t \int_0^s e^{s+\tau} E\left[(P_{T-\tau} \nabla^2 f_n, \nabla^2 g_n)_H\right] \, d\tau \, ds$$

$$\geq E[P_T f_n g_n]$$

$$+ E[(P_T \nabla f_n, \nabla g_n)_H] e^{-T}(e^t - 1)$$ \hfill (11.4.11)
where $(\cdot, \cdot)_2$ denotes the Hilbert-Schmidt scalar product and the inequality (11.4.11) follows from the convexity of $f_n$ and $g_n$. In fact, their convexity implies that $P_t \nabla^2 f_n$ and $\nabla^2 g_n$ are positive operators, hence their Hilbert-Schmidt tensor product is positive. Letting $T = t$ in the above inequality we have

$$E[f_n g_n] \geq E[P_T f_n g_n] + (1 - e^{-T}) E[(P_T \nabla f_n, \nabla g_n)_H].$$

(11.4.12)

Letting $T \to \infty$ in (11.4.12), we obtain, by the ergodicity of $(P_t, t \geq 0)$, the claimed inequality for $f_n$ and $g_n$. It suffices then to take the limit of this inequality as $n$ tends to infinity.

**Proposition 11.4.7** Let $G$ be a (positive) $\gamma$-log-concave Wiener functional with $\gamma \in [0, 1]$. Then the map $h \to E[G(w + h)]$ is a log-concave mapping on $H$. In particular, if $G$ is symmetric, i.e., if $G(w) = G(-w)$, then

$$E[G(w + h)] \leq E[G].$$

**Proof:** Without loss of generality, we may suppose that $G$ is bounded. Using the usual notations, we have, for any $h$ in any finite dimensional subspace $L$ of $H$,

$$E[G(w + h)] = \lim_n \frac{1}{(2\pi)^{n/2}} \int_{W_n} G(w_n^+ + w_n + h) \exp \left\{ -\frac{|w_n|^2}{2} \right\} dw_n,$$

from the hypothesis, the integrand is almost surely log-concave on $W_n \times L$, from Prekopa's theorem, the integral is log-concave on $L$, hence the limit is also log-concave. Since $L$ is arbitrary, the first part of the proof follows. To prove the second part, let $g(h) = E[G(w + h)]$, then, from the log-concavity of $g$ and symmetry of $G$, we have

$$E[G] = g(0) = g(1/2(h) + 1/2(-h)) \geq g(h)^{1/2}g(-h)^{1/2} = g(h) = E[G(w + h)].$$

**Remark:** In fact, with a little bit more attention, we can see that the map $h \to \exp\left\{\frac{1}{2}(1 - \gamma)|h|^2_H\right\} E[G(w + h)]$ is log-concave on $H$.

We have the following immediate corollary:
Corollary 11.4.8 Assume that $A \subset W$ is an $H$-convex and symmetric set. Then we have
\[ \mu(A + h) \leq \mu(A), \]
for any $h \in H$.

Proof: Since $1_A$ is log $H$-concave, the proof follows from Proposition 11.4.7.

Proposition 11.4.9 Let $F \in L^p(\mu)$ be a positive log $H$-convex function. Then for any
\[ u \in \mathbb{D}_{q,2}(H), \]
we have
\[ E_F \left[ (\delta u - E_F[\delta u])^2 \right] \geq E_F \left[ |u|^2_H + 2\delta(\nabla u) + \text{trace} (\nabla u \cdot \nabla u) \right], \]
where $E_F$ denotes the mathematical expectation with respect to the probability defined as
\[ \frac{F}{E[F]} d\mu. \]

Proof: Let $F_\tau$ be $P_\tau F$, where $(P_\tau, \tau \in \mathbb{R}_+)$ denotes the Ornstein-Uhlenbeck semi-group. $F_\tau$ has a modification, denoted again by the same letter, such that the mapping $h \mapsto F_\tau(w + h)$ is real-analytic on $H$ for all $w \in W$ (cf. [101]). Suppose first also that $\|\nabla u\|_2 \in L^\infty(\mu, H \otimes H)$ where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm. Then, for any $r > 1$, there exists some $t_r > 0$ such that, for any $0 \leq t < t_r$, the image of the Wiener measure under $w \mapsto w + tu(w)$ is equivalent to $\mu$ with the Radon-Nikodym density $L_t \in L^r(\mu)$. Hence $w \mapsto F_\tau(w + tu(w))$ is a well-defined mapping on $W$ and it is in some $L^r(\mu)$ for small $t > 0$ (cf. [101], Chapter 3 and Lemma B.8.8). Besides $t \mapsto F(w + tu(w))$ is log convex on $\mathbb{R}$ since $F_\tau$ is log $H$-convex. Consequently $t \mapsto E[F_\tau(w + tu(w))]$ is log convex and strictly positive. Then the second derivative of its logarithm at $t = 0$ should be positive. This implies immediately the claimed inequality for $\nabla u$ bounded. We then pass to the limit with respect to $u$ in $\mathbb{D}_{q,2}(H)$ and then let $\tau \to 0$ to complete the proof.

11.5 Poincaré and logarithmic Sobolev inequalities

The following theorem extends the Poincaré- Brascamp-Lieb inequality:
Theorem 11.5.1 Assume that $F$ is a Wiener functional in $\cup_{p>1} D_{p,2}$ with $e^{-F} \in L^1(\mu)$ and assume also that there exists a constant $\varepsilon > 0$ such that
\[
\left( (I_H + \nabla^2 F) h, h \right)_H \geq \varepsilon |h|^2_H
\]
almost surely, for any $h \in H$, i.e. $F$ is $(1-\varepsilon)$-convex. Let us denote by $\nu_F$ the probability measure on $(W, \mathcal{B}(W))$ defined by
\[
d\nu_F = \exp \left\{ -F - \log E \left[ e^{-F} \right] \right\} d\mu.
\]
Then for any smooth cylindrical Wiener functional $\phi$, we have
\[
\int_W |\phi - E_{\nu_F}[\phi]|^2 d\nu_F \leq \int_W \left( (I_H + \nabla^2 F)^{-1} \nabla \phi, \nabla \phi \right)_H d\nu_F.
\]
(11.5.14)

In particular, if $F$ is an $H$-convex Wiener functional, then the condition $\varepsilon > 0$ is satisfied with $\varepsilon = 1$.

Proof: Assume first that $W = \mathbb{R}^n$ and that $F$ is a smooth function on $\mathbb{R}^n$ satisfying the inequality (11.5.13) in this setting. Assume also for the typographical facility that $E[e^{-F}] = 1$. For any smooth function $\phi$ on $\mathbb{R}^n$, we have
\[
\int_{\mathbb{R}^n} |\phi - E_{\nu_F}[\phi]|^2 d\nu_F = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-F(x) - |x|^2/2} |\phi(x) - E_F[\phi]|^2 dx.
\]
(11.5.15)
The function $G(x) = F(x) + \frac{1}{2} |x|^2$ is a strictly convex smooth function. Hence Brascamp-Lieb inequality (cf. [13]) implies that:
\[
\int_{\mathbb{R}^n} |\phi - E_{\nu_F}[\phi]|^2 d\nu_F \leq \int_{\mathbb{R}^n} \left( (\text{Hess } G(x))^{-1} \nabla \phi(x), \nabla \phi(x) \right)_{\mathbb{R}^n} d\nu_F(x)
\]
\[
= \int_{\mathbb{R}^n} \left( (I_{\mathbb{R}^n} + \nabla^2 F)^{-1} \nabla \phi, \nabla \phi \right)_{\mathbb{R}^n} d\nu_F.
\]
To prove the general case we proceed by approximation as before: indeed let $(e_i, i \in \mathbb{N})$ be a complete, orthonormal basis of $H$, denote by $V_n$ the sigma algebra generated by $\{\delta \varepsilon, \ldots, \delta \varepsilon_n\}$. Define $F_n$ as to be $E[P_{1/n} F|V_n]$, where $P_{1/n}$ is the Ornstein-Uhlenbeck semigroup at $t = 1/n$. Then from the martingale convergence theorem and the fact that $V_n$ is a smooth sigma algebra, the sequence $(F_n, n \in \mathbb{N})$ converges to $F$ in some $D_{p,2}$. Moreover $F_n$ satisfies the hypothesis (with a better constant in the inequality (11.5.13)) since $\nabla^2 F_n = e^{-2/n} E[Q_n^{1/2} \nabla^2 F|V_n]$, where $Q_n$ denotes the orthogonal projection onto the vector space spanned by $\{\varepsilon_1, \ldots, \varepsilon_n\}$. Besides $F_n$ can be represented as $F_n = \theta(\delta \varepsilon_1, \ldots, \delta \varepsilon_n)$, where $\theta$ is a smooth function on $\mathbb{R}^n$ satisfying
\[
((I_{\mathbb{R}^n} + \nabla^2 \theta(x)) y, y)_{\mathbb{R}^n} \geq \varepsilon |y|^2_{\mathbb{R}^n},
\]
for any $x, y \in \mathbb{R}^n$. Let $w_n = \tilde{Q}_n(w) = \sum_{i \leq n}(\delta e_i)e_i$, $W_n = \tilde{P}_n(W)$ and $W_n \perp = (I_W - \tilde{Q}_n)(W)$ as before. Let us denote by $\nu_n$ the probability measure corresponding to $F_n$. Let us also denote by $V_n$ the sigma algebra generated by $\{\delta e_k, k > n\}$. Using the finite dimensional result that we have derived, the Fubini theorem and the inequality $2|ab| \leq \kappa a^2 + \frac{1}{\kappa} b^2$, for any $\kappa > 0$, we obtain

$$E_{\nu_n}[|\phi - E_{\nu_n}[\phi]|^2]$$

$$= \int_{W_n \times V_n} e^{-F_n(w_n)}|\phi(w_n + w_n ^\perp) - E_{\nu_n}[\phi]|^2 d\mu_n(w_n) d\mu_n^+(w_n ^\perp)$$

$$\leq (1 + \kappa) \int_W e^{-F_n}|\phi - E[e^{-F_n}\phi]|V_n ^\perp|^2 d\mu$$

$$+ \left(1 + \frac{1}{\kappa}\right) \int_W e^{-F_n}|E[e^{-F_n}\phi]|V_n ^\perp| - E_{\nu_n}[\phi]|^2 d\mu$$

$$\leq (1 + \kappa) E_{\nu_n} \left[ ((I_H + \nabla^2 F_n)^{-1} \nabla \phi, \nabla \phi) \right]_H$$

$$+ \left(1 + \frac{1}{\kappa}\right) \int_W e^{-F_n}|E[e^{-F_n}\phi]|V_n ^\perp| - E_{\nu_n}[\phi]|^2 d\mu , \quad (11.5.16)$$

where $F'_n$ denotes $F_n - \log E[e^{-F_n}]$. Since $V_n$ and $V_n ^\perp$ are independent sigma algebras, we have

$$|E[e^{-F_n}\phi]|V_n ^\perp| = \frac{1}{E[e^{-F_n}]}|E[e^{-F_n}\phi]|V_n ^\perp|$$

$$\leq \frac{1}{E[e^{-F_n}]}E[e^{-F_n}|V_n ^\perp|]||\phi||_\infty$$

$$= ||\phi||_\infty ,$$

hence, using the triangle inequality and the dominated convergence theorem, we realize that the last term in (11.5.16) converges to zero as $n$ tends to infinity. Since the sequence of operator valued random variables $((I_H + \nabla^2 F_n)^{-1}, n \in \mathbb{N})$ is essentially bounded in the strong operator norm, we can pass to the limit on both sides and this gives the claimed inequality with a factor $1 + \kappa$, since $\kappa > 0$ is arbitrary, the proof is completed.

\[\square\]

**Remark:** Let $T : W \rightarrow W$ be a shift defined as $T(w) = w + u(w)$, where $u : W \rightarrow H$ is a measurable map satisfying $(u(w + h) - u(w), h)_H \geq -\varepsilon|h|^2$. In [99] and in [101], Chapter 6, we have studied such transformations, called $\varepsilon$-monotone shifts. Here the hypothesis of Theorem [11.5.1] says that the shift $T = I_W + \nabla F$ is $\varepsilon$-monotone.

The Sobolev regularity hypothesis can be omitted if we are after a Poincaré inequality with another constant:
Theorem 11.5.2 Assume that \( F \in \bigcup_{p>1} L^p(\mu) \) with \( E\left[e^{-F}\right] \) is finite and that, for some constant \( \varepsilon > 0 \),
\[
E \left[ \left( (I_H + \nabla^2 F)h, h \right)_H \psi \right] \geq \varepsilon |h|_H^2 E[\psi],
\]
for any \( h \in H \) and positive test function \( \psi \in \mathcal{D} \), where \( \nabla^2 F \) denotes the second order derivative in the sense of the distributions. Then we have
\[
E_{\nu_F} \left[ |\phi - E_F[\phi]|^2 \right] \leq \frac{1}{\varepsilon} E_{\nu_F} [ |\nabla \phi|_H^2 ]
\]
for any cylindrical Wiener functional \( \phi \). In particular, if \( F \) is \( H \)-convex, then we can take \( \varepsilon = 1 \).

Proof: Let \( F_t \) be defined as \( P_t F \), where \( P_t \) denotes the Ornstein-Uhlenbeck semigroup. Then \( F_t \) satisfies the hypothesis of Theorem 11.5.1, hence we have
\[
E_{\nu_{F_t}} \left[ |\phi - E_{F_t}[\phi]|^2 \right] \leq \frac{1}{\varepsilon} E_{\nu_{F_t}} [ |\nabla \phi|_H^2 ]
\]
for any \( t > 0 \). The claim follows when we take the limits of both sides as \( t \to 0 \). \( \square \)

Example: Let \( F(w) = \|w\| + \frac{1}{2} \sin(\delta h) \) with \( |h|_H \leq 1 \), where \( \| \cdot \| \) denotes the norm of the Banach space \( \tilde{W} \). Then in general \( F \) is not in \( \bigcup_{p>1} \mathbb{D}_{p,2} \), however the Poincaré inequality (11.5.17) holds with \( \varepsilon = 1/2 \).

Theorem 11.5.3 Assume that \( F \) is a Wiener functional in \( \bigcup_{p>1} \mathbb{D}_{p,2} \) with \( E[\exp -F] < \infty \). Assume that there exists a constant \( \varepsilon > 0 \) such that
\[
\left( (I_H + \nabla^2 F)h, h \right)_H \geq \varepsilon |h|_H^2 \tag{11.5.18}
\]
almost surely, for any \( h \in H \). Let us denote by \( \nu_F \) the probability measure on \( (W, \mathcal{B}(W)) \) defined by
\[
d\nu_F = \exp \left\{ -F - \log E\left[e^{-F}\right] \right\} d\mu.
\]
Then for any smooth cylindrical Wiener functional \( \phi \), we have
\[
E_{\nu_F} \left[ \phi^2 \left\{ \log \phi^2 - \log \|\phi\|_{L^2(\nu_F)}^2 \right\} \right] \leq \frac{2}{\varepsilon} E_{\nu_F} \left[ |\nabla \phi|_H^2 \right]. \tag{11.5.19}
\]
In particular, if \( F \) is an \( H \)-convex Wiener functional, then the condition (11.5.18) is satisfied with \( \varepsilon = 1 \).
Proof: We shall proceed as in the proof of Theorem 11.5.1. Assume then that $W = \mathbb{R}^n$ and that $F$ is a smooth function satisfying the inequality (11.5.18) in this frame. In this case it is immediate to see that function $G(x) = \frac{1}{2}|x|^2 + F(x)$ satisfies the Bakry-Emery condition (cf. [9], [23]), which is known as a sufficient condition for the inequality (11.5.19). For the infinite dimensional case we define as in the proof of Theorem 11.5.1, $F_n, \nu_n, V_n, V_n^\perp$. Then, denoting by $E_n$ the expectation with respect to the probability $\exp\{-F'_n\}d\mu$, where $F'_n = F_n - \log E[\exp(-F_n)]$, we have

\begin{align}
E_n \left[ \phi^2 \left\{ \log \phi^2 - \log \|\phi\|^2_{L^2(\nu_F)} \right\} \right] \\
= E_n \left[ \phi^2 \left\{ \log \phi^2 - \log E[\exp(-F'_n \phi^2)|V_n^\perp] \right\} \right] \\
+ E_n \left[ \phi^2 \left\{ \log E[\exp(-F'_n \phi^2)|V_n^\perp] - \log E_n[\phi^2] \right\} \right] \\
\leq \frac{2}{\varepsilon} E_n \left[ |\nabla \phi|^2_H \right] \\
+ E_n \left[ \phi^2 \left\{ \log E[\exp(-F'_n \phi^2)|V_n^\perp] - \log E_n[\phi^2] \right\} \right],
\end{align}

(11.5.20)

where we have used, as in the proof of Theorem 11.5.1, the finite dimensional log-Sobolev inequality to obtain the inequality (11.5.20). Since in the above inequalities everything is squared, we can assume that $\phi$ is positive, and adding a constant $\kappa > 0$, we can also replace $\phi$ with $\phi_\kappa = \phi + \kappa$. Again by the independance of $V_n$ and $V_n^\perp$, we can pass to the limit with respect to $n$ in the inequality (11.5.20) for $\phi = \phi_\kappa$ to obtain

\begin{align}
E_{\nu_F} \left[ \phi_\kappa^2 \left\{ \log \phi_\kappa^2 - \log \|\phi_\kappa\|^2_{L^2(\nu_F)} \right\} \right] &\leq \frac{2}{\varepsilon} E_{\nu_F} \left[ |\nabla \phi_\kappa|^2_H \right].
\end{align}

To complete the proof it suffices to pass to the limit as $\kappa \to 0$. \hfill \Box

The following theorem fully extends Theorem 11.5.3 and it is useful for the applications:

**Theorem 11.5.4** Assume that $G$ is a (positive) $\gamma$-log-concave Wiener functional for some $\gamma \in [0, 1)$ with $E[G] < \infty$. Let us denote by $E_G[\cdot]$ the expectation with respect to the probability measure defined by

\begin{align}
d\nu_G = \frac{G}{E[G]}d\mu.
\end{align}

Then we have

\begin{align}
E_G \left[ \phi^2 \left\{ \log \phi^2 - \log E_G[\phi^2] \right\} \right] &\leq \frac{2}{1 - \gamma} E_G[|\nabla \phi|^2_H],
\end{align}

(11.5.21)

for any cylindrical Wiener functional $\phi$. 
Proof: Since \( G \land c, c > 0, \) is again \( \gamma \)-log-concave, we may suppose without loss of generality that \( G \) is bounded. Let now \((e_i, i \in \mathbb{N})\) be a complete, orthonormal basis for \( H \), denote by \( V_n \) the sigma algebra generated by \( \{\delta e_1, \ldots, \delta e_n\} \). Define \( G_n \) as to be \( E[P_{1/n} G|V_n] \). From Proposition 11.4.4 and Theorem 11.4.5, \( G_n \) is again a \( \gamma \)-log-concave, strictly positive Wiener functional. It can be represented as

\[
G_n(w) = g_n(\delta e_1, \ldots, \delta e_n)
\]

and due to the Sobolev embedding theorem, after a modification on a set of zero Lebesgue measure, we can assume that \( g_n \) is a smooth function on \( \mathbb{R}^n \). Since it is strictly positive, it is of the form \( e^{-f_n} \), where \( f_n \) is a smooth, \( \gamma \)-convex function. It follows then from Theorem 11.5.3 that the inequality (11.5.21) holds when we replace \( G \) by \( G_n \), then the proof follows by taking the limits of both sides as \( n \to \infty \).

Example: Assume that \( A \) is a measurable subset of \( W \) and let \( H \) be a measurable Wiener functional with values in \( \mathbb{R} \cup \{\infty\} \). If \( G \) defined by \( G = 1_A H \) is \( \gamma \)-log-concave with \( \gamma \in [0, 1) \), then the hypothesis of Theorem 11.5.4 are satisfied.

Definition 11.5.5 Let \( T \in \mathcal{D}' \) be a positive distribution. We say that it is \( a \)-log-concave if \( P_t T \) is an \( a \)-log-concave Wiener functional. If \( a = 0 \), then we call \( T \) simply log-concave.

Remark: From Corollary 7.1.3, to any positive distribution on \( W \), it corresponds a positive Radon measure \( \nu_T \) such that

\[
<T, \phi> = \int_W \tilde{\phi}(w) d\nu_T(w)
\]

for any \( \phi \in \mathcal{D} \), where \( \tilde{\phi} \) represents a quasi-continuous version of \( \phi \).

Example: Let \((w_t, t \in [0, 1])\) be the one-dimensional Wiener process and denote by \( p_t \) the heat kernel on \( \mathbb{R} \). Then the distribution defined as \( \varepsilon_0(w_1) = \lim_{t \to 0} p_t(w_1) \) is log-concave, where \( \varepsilon_0 \) denotes the Dirac measure at zero.

The following result is a Corollary of Theorem 11.5.4.

Theorem 11.5.6 Assume that \( T \in \mathcal{D}' \) is a positive, \( \beta \)-log-concave distribution with \( \beta \in [0, 1) \). Let \( \gamma \) be the probability Radon measure defined by

\[
\gamma = \frac{\nu_T}{<T, 1>}
\]
Then we have
\[ E_\gamma \left[ \phi^2 \left\{ \log \phi^2 - \log E_\gamma[\phi^2] \right\} \right] \leq \frac{2}{1-\beta} E_\gamma[|\nabla \phi|^2_H], \] (11.5.22)
for any smooth cylindrical function \( \phi : W \to \mathbb{R} \).

Here is an application of this result:

**Proposition 11.5.7** Let \( F \) be a Wiener functional in \( \mathbb{D}_{r,2} \) for some \( r > 1 \). Suppose that it is \( p \)-non-degenerate in the sense that
\[ \delta \left\{ \frac{|\nabla F|^2}{|F|^2} \phi \right\} \in L^p(\mu) \] (11.5.23)
for any \( \phi \in \mathbb{D} \), for some \( p > 1 \). Assume furthermore that, for some \( x_0 \in \mathbb{R} \),
\[ (F - x_0)\nabla^2 F + \nabla F \otimes \nabla F \geq 0 \] (11.5.24)
almost surely. Then we have
\[ E \left[ \phi^2 \left\{ \log \phi^2 - \log E \left[ \phi^2 | F = x_0 \right] \right\} | F = x_0 \right] \leq 2 E \left[ |\nabla \phi|^2_H | F = x_0 \right] \]
for any smooth cylindrical \( \phi \).

**Proof:** Note that the non-degeneracy hypothesis (11.5.23) implies the existence of a continuous density of the law of \( F \) with respect to the Lebesgue measure (cf. [56] and the references there). Moreover it implies also the fact that
\[ \lim_{\tau \to 0} p_\tau(F - x_0) = \varepsilon_{x_0}(F), \]
in \( \mathbb{D}' \), where \( \varepsilon_{x_0} \) denotes the Dirac measure at \( x_0 \) and \( p_\tau \) is the heat kernel on \( \mathbb{R} \). The inequality (11.5.24) implies that the distribution defined by
\[ \phi \to E[\phi|F = x_0] = \frac{\langle \varepsilon_{x_0}(F), \phi \rangle}{\langle \varepsilon_{x_0}(F), 1 \rangle} \]
is log-concave, hence the conclusion follows from Theorem 11.5.6. \( \square \)

### 11.6 Change of variables formula and log-Sobolev inequality

In this section we shall derive a different kind of logarithmic Sobolev inequality using the change of variables formula for the monotone shifts studied in [99] and in more detail in [101]. An analogous approach to derive log-Sobolev-type inequalities using the Girsanov theorem has been employed in [92].
Theorem 11.6.1 Suppose that $F \in L^p(\mu)$, for some $p > 1$, is an $a$-convex Wiener functional, $a \in [0, 1)$ with $E[F] = 0$. Assume that
\[
E \left[ \exp \left\{ e \|\nabla^2 \mathcal{L}^{-1} F\|_2^2 \right\} \right] < \infty,
\]
for some
\[
e > \frac{2 + (1 - a)}{2(1 - a)},
\]
where $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm on $H \otimes H$ and $\mathcal{L}^{-1} F = \int_{\mathbb{R}^+} P_t F \, dt$. Denote by $\nu$ the probability measure defined by
\[
d\nu = \Lambda \, d\mu,
\]
where
\[
\Lambda = \det_2(I_H + \nabla^2 \mathcal{L}^{-1} F) \exp \left\{ -\frac{1}{2} \|\nabla \mathcal{L}^{-1} F\|_H^2 \right\}
\]
and $\det_2(I_H + \nabla^2 \mathcal{L}^{-1} F)$ denotes the modified Carleman-Fredholm determinant. Then we have
\[
E_\nu \left[ f^2 \log \left( \frac{f^2}{\|f\|^2_{L^2(\nu)}} \right) \right] \leq 2 E_\nu \left[ \| (I_H + \nabla^2 \mathcal{L}^{-1} F)^{-1} \nabla f \|_H^2 \right]
\]
and
\[
E_\nu \|[f - E_\nu[f]]^2\| \leq E_\nu \left[ \| (I_H + \nabla^2 \mathcal{L}^{-1} F)^{-1} \nabla f \|_H^2 \right]
\]
for any smooth, cylindrical $f$.

Proof: Let $F_n = E[P_{1/n} F|V_n]$, where $V_n$ is the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$ and let $(e_n, n \in \mathbb{N})$ be a complete, orthonormal basis of $H$. Define $\xi_n$ by $\nabla \mathcal{L}^{-1} F_n$, then $\xi_n$ is $(1 - a)$-strongly monotone (cf. [99] or [101]) and smooth. Consequently, the shift $T_n : W \to W$, defined by $T_n(w) = w + \xi_n(w)$ is a bijection of $W$ (cf. [101] Corollary 6.4.1), whose inverse is of the form $S_n = I_W + \eta_n$, where $\eta_n(w) = g_n(\delta e_1, \ldots, \delta e_n)$ such that $g_n : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. Moreover the images of $\mu$ under $T_n$ and $S_n$, denoted by $T_n^* \mu$ and $S_n^* \mu$ respectively, are equivalent to $\mu$ and we have
\[
\frac{dS_n^* \mu}{d\mu} = \Lambda_n \quad \frac{dT_n^* \mu}{d\mu} = L_n
\]
where
\[
\Lambda_n = \det_2(I_H + \nabla \xi_n) \exp \left\{ -\delta \xi_n - \frac{1}{2} |\xi_n|^2_H \right\}
\]
\[
L_n = \det_2(I_H + \nabla \eta_n) \exp \left\{ -\delta \eta_n - \frac{1}{2} |\eta_n|^2_H \right\}.
\]

The hypothesis (11.6.25) implies the uniform integrability of the densities \((\Lambda_n, n \geq 1)\) and \((L_n, n \geq 1)\) (cf. [100, 101]). For any probability \(P\) on \((W, \mathcal{B}(W))\) and any positive, measurable function \(f\), define \(\mathcal{H}_P(f)\) as
\[
\mathcal{H}_P(f) = f(\log f - \log E_P[f]).
\] (11.6.28)

Using the logarithmic Sobolev inequality of L. Gross for \(\mu\) (cf. [36]) and the relation \((I_H + \nabla \eta_n) \circ T_n = (I_H + \nabla \xi_n)^{-1}\), we have
\[
E[\Lambda_n \mathcal{H}_\mu(f^2)] = E[\mathcal{H}_\mu(f^2 \circ S_n)] \\
\leq 2E[|\nabla (f \circ S_n)|^2_H] \\
= 2E[|(I_H + \nabla \eta_n) \nabla f \circ S_n|^2_H] \\
= 2E[|\Lambda_n|(I_H + \nabla \xi_n)^{-1} \nabla f|^2_H].
\] (11.6.29)

It follows by the \(a\)-convexity of \(F\) that
\[
\|(I_H + \nabla \xi_n)^{-1}\| \leq \frac{1}{1 - a}
\]
almost surely for any \(n \geq 1\), where \(\|\cdot\|\) denotes the operator norm. Since the sequence \((\Lambda_n, n \in \mathbb{N})\) is uniformly integrable, the limit of (11.6.29) exists in \(L^1(\mu)\) and the proof of (11.6.26) follows. The proof of the inequality (11.6.27) is now trivial.

**Corollary 11.6.2** Assume that \(F\) satisfies the hypothesis of Theorem 11.6.1. Let \(Z\) be the functional defined by
\[
Z = \det_2(I_H + \nabla^2 \mathcal{L}^{-1} F) \exp \frac{1}{2} \left| \nabla \mathcal{L}^{-1} F \right|^2_H
\]
and assume that \(Z, Z^{-1} \in L^\infty(\mu)\). Then we have
\[
E\left[ e^{-F} f^2 \log \left( \frac{f^2}{E[e^{-F} f^2]} \right) \right] \leq 2KE\left[ e^{-F} \left| (I_H + \nabla^2 \mathcal{L}^{-1} F)^{-1} \nabla f \right|^2_H \right]
\] (11.6.30)
and
\[ E \left[ e^{-F} \left| f - E[e^{-F}f] \right|^2 \right] \leq KE \left[ e^{-F} \left| (I_H + \nabla^2 \mathcal{L}^{-1} F)^{-1} \nabla f \right|^2_H \right] \] (11.6.31)
for any smooth, cylindrical \( f \), where \( K = \|Z\|_{L^\infty(\mu)} \|Z^{-1}\|_{L^\infty(\mu)} \).

**Proof:** Using the identity remarked by Holley and Stroock (cf. [40], p.1183)
\[ E_P \left[ H_P(f^2) \right] = \inf_{x > 0} E_P \left[ f^2 \log \left( \frac{f^2}{x} \right) - (f^2 - x) \right] , \]
where \( P \) is an arbitrary probability measure, and \( H \) is defined by the relation (11.6.28), we see that the inequality (11.6.30) follows from Theorem 11.6.1 and the inequality (11.6.31) is trivial.

**Exercises**

1. Assume that \( A_1, \ldots, A_n \) are almost surely convex and symmetric sets. Prove the following inequality:
\[ \mu \left( \bigcap_{i=1}^{n} (A_i + h_i) \right) \leq \mu \left( \bigcap_{i=1}^{n} A_i \right) , \] (11.6.32)
for any \( h_1, \ldots, h_n \in H \).

2. Assume that \( F \) is a positive, symmetric, almost surely log-concave Wiener functional such that \( \mu\{F > 0\} > 0 \). Denote by \( \mu_F \) the probability defined by
\[ d\mu_F = \frac{F}{E[F]} d\mu . \]
Prove the inequality (11.6.32) when \( \mu \) is replaced by \( \mu_F \).

3. Let \( A \) and \( B \) be two almost surely convex sets. For \( \alpha \in [0,1] \), define the map \((\alpha, w) \to f(\alpha, w)\) as
\[ f(\alpha, w) = 1_{C_\alpha}(w) , \]
where \( C_\alpha = \alpha A + (1-\alpha) B \). Prove that \((\alpha, w) \to f(\alpha, w)\) is almost surely log-concave. Deduce from that and from Prékopa’s theorem the inequality:
\[ \mu(C_\alpha) \geq \mu(A)^\alpha \mu(B)^{1-\alpha} . \]

4. Let \( F \) and \( G \) be two almost surely convex, symmetric Wiener functionals from \( D_{2,2} \). Prove that
\[ E[(\nabla F, \nabla G)_H] \geq 0 . \]

5. Let \( W \) be the classical Wiener space \( C_0([0,1], \mathbb{R}) \) and let \( f \) and \( g \) be two \( H \)-convex functions in \( L^2(\mu) \). With the help of the Clark’s formula, prove that
\[ E[E[D_{t1}f|F_{t1}] E[D_{t1}g|F_{t1}]] \geq E[D_{t1}f] E[D_{t1}g] , \]
d\( t \)-almost surely.
Notes and references

The notion of convexity for the equivalence classes of Wiener random variables is a new subject. It has been studied for the first time in [31]. Even in the finite dimensional case it is not evident to find a result about the $H$-convexity.

The log-Sobolev inequalities given here are well-known in the finite dimensional case except the content of the last section. The fact that log-concavity is preserved under the action of certain semi-groups and especially its implications concerning log-concave distributions seem to be novel.
Change of Variables
Chapter 12

Monge-Kantorovitch Mass Transportation

12.1 Introduction

In 1781, Gaspard Monge has published his celebrated memoire about the most economical way of earth-moving [64]. The configurations of excavated earth and remblai were modelized as two measures of equal mass, say $\rho$ and $\nu$, that Monge had supposed absolutely continuous with respect to the volume measure. Later Ampère has studied an analogous question about the electricity current in a media with varying conductivity. In modern language of measure theory we can express the problem in the following terms: let $W$ be a Polish space on which are given two positive measures $\rho$ and $\nu$, of finite, equal mass. Let $c(x, y)$ be a cost function on $W \times W$, which is, usually, assumed positive. Does there exist a map $T : W \to W$ such that $T \rho = \nu$ and $T$ minimizes the integral

$$\int_W c(x, T(x))d\rho(x)$$

between all such maps? The problem has been further studied by Appell [6 7] and by Kantorovich [44]. Kantarovitch has succeeded to transform this highly nonlinear problem of Monge into a linear problem by replacing the search for $T$ with the search of a measure $\gamma$ on $W \times W$ with marginals $\rho$ and $\nu$ such that the integral

$$\int_{W \times W} c(x, y)d\gamma(x, y)$$

is the minimum of all the integrals

$$\int_{W \times W} c(x, y)d\beta(x, y)$$

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where $\beta$ runs in the set of measures on $W \times W$ whose marginals are $\rho$ and $\nu$. Since then the problem addressed above is called the Monge problem and the quest of the optimal measure is called the Monge-Kantorovitch problem.

In this chapter we study the Monge-Kantorovitch and the Monge problem in the frame of an abstract Wiener space with a singular cost. In other words, let $W$ be a separable Fréchet space with its Borel sigma algebra $\mathcal{B}(W)$ and assume that there is a separable Hilbert space $H$ which is injected densely and continuously into $W$, hence in general the topology of $H$ is stronger than the topology induced by $W$. The cost function $c : W \times W \to \mathbb{R}_+ \cup \{\infty\}$ is defined as

$$c(x, y) = |x - y|^2_H,$$

we suppose that $c(x, y) = \infty$ if $x - y$ does not belong to $H$. Clearly, this choice of the function $c$ is not arbitrary, in fact it is closely related to Ito Calculus, hence also to the problems originating from Physics, quantum chemistry, large deviations, etc. Since for all the interesting measures on $W$, the Cameron-Martin space is a negligible set, the cost function will be infinity very frequently. Let $\Sigma(\rho, \nu)$ denote the set of probability measures on $W \times W$ with given marginals $\rho$ and $\nu$. It is a convex, compact set under the weak topology $\sigma(\Sigma, C_b(W \times W))$. As explained above, the problem of Monge consists of finding a measurable map $T : W \to W$, called the optimal transport of $\rho$ to $\nu$, i.e., $T\rho = \nu$ which minimizes the cost

$$U \to \int_W |x - U(x)|^2_H d\rho(x),$$

between all the maps $U : W \to W$ such that $U\rho = \nu$. The Monge-Kantorovitch problem will consist of finding a measure on $W \times W$, which minimizes the function $\theta \to J(\theta)$, defined by

$$J(\theta) = \int_{W \times W} |x - y|^2_H d\theta(x, y),$$

(12.1.1)

where $\theta$ runs in $\Sigma(\rho, \nu)$. Note that $\inf\{J(\theta) : \theta \in \Sigma(\rho, \nu)\}$ is the square of Wasserstein metric $d_H(\rho, \nu)$ with respect to the Cameron-Martin space $H$.

Any solution $\gamma$ of the Monge-Kantorovitch problem will give a solution to the Monge problem provided that its support is included in the graph of a map. Hence our work consists of realizing this program. Although in the finite dimensional case this problem is well-studied in the path-breaking papers of Brenier [14] and McCann [59, 60] the things do not come up easily in our setting and the difficulty is due to the fact that the cost function is not continuous with respect to the Fréchet topology of $W$, for instance the

\footnote{We denote the push-forward of $\rho$ by $T$, i.e., the image of $\rho$ under $T$, by $T\rho$.}
weak convergence of the probability measures does not imply the convergence of the integrals of the cost function. In other words the function $|x - y|_H^2$ takes the value plus infinity “very often”. On the other hand the results we obtain seem to have important applications to several problems of stochastic analysis that we shall explain while enumerating the contents of this chapter.

Section 12.3 is devoted to the derivation of some inequalities which control the Wasserstein distance. In particular, with the help of the Girsanov theorem, we give a very simple proof of an inequality, initially discovered by Talagrand ([33]); this facility gives already an idea about the efficiency of the infinite dimensional techniques for the Monge-Kantorovitch problem. We indicate some simple consequences of this inequality to control the measures of subsets of the Wiener space with respect to second moments of their gauge functionals defined with the Cameron-Martin distance. These inequalities are quite useful in the theory of large deviations. Using a different representation of the target measure, namely by constructing a flow of diffeomorphisms of the Wiener space (cf. Chapter V of [101]) which maps the Wiener measure to the target measure, we obtain also a new control of the Kantorovitch-Rubinstein metric of order one. The method we employ for this inequality generalizes directly to a more general class of measures, namely those for which one can define a reasonable divergence operator.

In Section 12.4 we solve directly the original problem of Monge when the first measure is the Wiener measure and the second one is given with a density, in such a way that the Wasserstein distance between these two measures is finite. We prove the existence and the uniqueness of a transformation of $W$ of the form $T = I_W + \nabla \phi$, where $\phi$ is a 1-convex function in the Gaussian Sobolev space $D_{2,1}$ such that the measure $\gamma = (I_W \times T)\mu$ is the unique solution of the problem of Monge-Kantorovitch. This result gives a new insight to the question of representing an integrable, positive random variable whose expectation is unity, as the Radon-Nikodym derivative of the image of the Wiener measure under a map which is a perturbation of identity, a problem which has been studied by X. Fernique and by one of us with M. Zakai (cf., [26, 27, 101]). In [101], Chapter II, it is shown that such random variables are dense in $L^1_{1,+}(\mu)$ (the lower index 1 means that the expectations are equal to one), here we prove that this set of random variables contains the random variables who are at finite Wasserstein distance from the Wiener measure. In fact even if this distance is infinite, we show that there is a solution to this problem if we enlarge $W$ slightly by taking $\mathbb{N} \times W$.

Section 12.5 is devoted to the immediate implications of the existence and the uniqueness of the solutions of Monge-Kantorovitch and Monge prob-

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2In Section 12.7 we shall see another illustration of this phenomena.
lems constructed in Section 12.4. Indeed the uniqueness implies at once that the absolutely continuous transformations of the Wiener space, at finite (Wasserstein) distance, have a unique decomposition in the sense that they can be written as the composition of a measure preserving map in the form of the perturbation of identity with another one which is the perturbation of identity with the Sobolev derivative of a 1-convex function. This means in particular that the class of 1-convex functions is as basic as the class of adapted processes in the setting of Wiener space.

In Section 12.6 we prove the existence and the uniqueness of solutions of the Monge-Kantorovitch and Monge problems for the measures which are at finite Wasserstein distance from each other. The fundamental hypothesis we use is that the regular conditional probabilities which are obtained by the disintegration of one of the measures along the orthogonals of a sequence of regular, finite dimensional projections vanish on the sets of co-dimension one. In particular, this hypothesis is satisfied if the measure under question is absolutely continuous with respect to the Wiener measure. The method we use in this section is totally different from the one of Section 12.4; it is based on the notion of cyclic monotonicity of the supports of the regular conditional probabilities obtained through some specific disintegrations of the optimal measures. The importance of cyclic monotonicity has first been remarked by McCann and used abundantly in [59] and in [34] for the finite dimensional case. Here the things are much more complicated due to the singularity of the cost function, in particular, contrary to the finite dimensional case, the cyclic monotonicity is not compatible with the weak convergence of probability measures. A curious reader may ask why we did not treat first the general case and then attack the subject of Section 12.4. The answer is twofold: even if we had done so, we would have needed similar calculations as in Section 12.4 in order to show the Sobolev regularity of the transport map, hence concerning the volume, the order that we have chosen does not change anything. Secondly, the construction used in Section 12.4 has an interest by itself since it explains interesting relations between the transport map and its inverse and the optimal measure in a more detectable situation, in this sense this construction is rather complementary to the material of Section 12.6.

Section 12.7 studies the Monge-Ampère equation for the measures which are absolutely continuous with respect to the Wiener measure. First we briefly indicate the notion of second order Alexandroff derivative and the Alexandroff version of the Ornstein-Uhlenbeck operator applied to a 1-convex function in the finite dimensional case. With the help of these observations, we write the corresponding Jacobian using the modified Carleman-Fredholm determinant which is natural in the infinite dimensional case (cf., [101]).
Afterwards we attack the infinite dimensional case by proving that the absolutely continuous part of the Ornstein-Uhlenbeck operator applied to the finite rank conditional expectations of the transport function is a submartingale which converges almost surely. Hence the only difficulty lies in the calculation of the limit of the Carleman-Fredholm determinants. Here we have a major difficulty which originates from the pathology of the Radon-Nikodym derivatives of the vector measures with respect to a scalar measure as explained in [84]: in fact even if the second order Sobolev derivative of a Wiener function is a vector measure with values in the space of Hilbert-Schmidt operators, its absolutely continuous part has no reason to be Hilbert-Schmidt. Hence the Carleman-Fredholm determinant may not exist, however due to the 1-convexity, the determinants of the approximating sequence are all with values in the interval [0, 1]. Consequently we can construct the subsolutions with the help of the Fatou lemma.

Last but not the least, in section 12.7.1, we prove that all these difficulties can be overcome thanks to the natural renormalization of the Ito stochastic calculus. In fact using the Ito representation theorem and the Wiener space analysis extended to the distributions, we can give the explicit solution of the Monge-Ampère equation. This is a remarkable result in the sense that such techniques do not exist in the finite dimensional case.

12.2 Preliminaries and notations

Let $W$ be a separable Fréchet space equipped with a Gaussian measure $\mu$ of zero mean whose support is the whole space. The corresponding Cameron-Martin space is denoted by $H$. Recall that the injection $H \hookrightarrow W$ is compact and its adjoint is the natural injection $W^* \hookrightarrow H^* \subseteq L^2(\mu)$. The triple $(W, \mu, H)$ is called an abstract Wiener space. Recall that $W = H$ if and only if $W$ is finite dimensional. A subspace $F$ of $H$ is called regular if the corresponding orthogonal projection has a continuous extension to $W$, denoted again by the same letter. It is well-known that there exists an increasing sequence of regular subspaces $(F_n, n \geq 1)$, called total, such that $\bigcup_n F_n$ is dense in $H$ and in $W$. Let $\sigma(\pi_{F_n})$ be the $\sigma$-algebra generated by $\pi_{F_n}$, then for any $f \in L^p(\mu)$, the martingale sequence $(E[f|\sigma(\pi_{F_n})], n \geq 1)$ converges to $f$ (strongly if $p < \infty$) in $L^p(\mu)$. Observe that the function $f_n = E[f|\sigma(\pi_{F_n})]$ can be identified with a function on the finite dimensional abstract Wiener space $(F_n, \mu_n, F_n)$, where $\mu_n = \pi_n\mu$.

Let us recall some facts from the convex analysis. Let $K$ be a Hilbert space, a subset $S$ of $K \times K$ is called cyclically monotone if any finite subset

\footnote{For the notational simplicity, in the sequel we shall denote it by $\pi_{F_n}$.}
\{(x_1, y_1), \ldots, (x_N, y_N)\} of \(S\) satisfies the following algebraic condition:

\[
\langle y_1, x_2 - x_1 \rangle + \langle y_2, x_3 - x_2 \rangle + \cdots + \langle y_{N-1}, x_N - x_{N-1} \rangle + \langle y_N, x_1 - x_N \rangle \leq 0,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the inner product of \(K\). It turns out that \(S\) is cyclically monotone if and only if

\[
\sum_{i=1}^{N} (y_i, x_{\sigma(i)} - x_i) \leq 0,
\]

for any permutation \(\sigma\) of \(\{1, \ldots, N\}\) and for any finite subset \(\{(x_i, y_i) : i = 1, \ldots, N\}\) of \(S\). Note that \(S\) is cyclically monotone if and only if any translate of it is cyclically monotone. By a theorem of Rockafellar, any cyclically monotone set is contained in the graph of the subdifferential of a convex function in the sense of convex analysis ([75]) and even if the function may not be unique its subdifferential is unique.

Let now \((W, \mu, H)\) be an abstract Wiener space; a measurable function \(f : W \to \mathbb{R} \cup \{\infty\}\) is called 1-convex if the map

\[
h \mapsto f(x + h) + \frac{1}{2} |h|_H^2 = F(x, h)
\]

is convex on the Cameron-Martin space \(H\) with values in \(L^0(\mu)\). Note that this notion is compatible with the \(\mu\)-equivalence classes of random variables thanks to the Cameron-Martin theorem. It is proven in Chapter [1] that this definition is equivalent the following condition: Let \((\pi_n, n \geq 1)\) be a sequence of regular, finite dimensional, orthogonal projections of \(H\), increasing to the identity map \(I_H\). Denote also by \(\pi_n\) its continuous extension to \(W\) and define \(\pi_n^\perp = I_W - \pi_n\). For \(x \in W\), let \(x_n = \pi_n x\) and \(x_n^\perp = \pi_n^\perp x\). Then \(f\) is 1-convex if and only if

\[
x_n \rightarrow \frac{1}{2} |x_n|_H^2 + f(x_n + x_n^\perp)
\]

is \(\pi_n^\perp \mu\)-almost surely convex.

### 12.3 Some Inequalities

**Definition 12.3.1** Let \(\xi\) and \(\eta\) be two probabilities on \((W, \mathcal{B}(W))\). We say that a probability \(\gamma\) on \((W \times W, \mathcal{B}(W \times W))\) is a solution of the Monge-Kantorovitch problem associated to the couple \((\xi, \eta)\) if the first marginal of \(\gamma\) is \(\xi\), the second one is \(\eta\) and if

\[
J(\gamma) = \int_{W \times W} |x - y|_H^2 d\gamma(x, y) = \inf \left\{ \int_{W \times W} |x - y|_H^2 d\beta(x, y) : \beta \in \Sigma(\xi, \eta) \right\},
\]
where $\Sigma(\xi, \eta)$ denotes the set of all the probability measures on $W \times W$ whose first and second marginals are respectively $\xi$ and $\eta$. We shall denote the Wasserstein distance between $\xi$ and $\eta$, which is the positive square-root of this infimum, with $d_H(\xi, \eta)$.

**Remark:** Since the set of probability measures on $W \times W$ is weakly compact and since the integrand in the definition is lower semi-continuous and strictly convex, the infimum in the definition is always attained even if the functional $J$ is identically infinity.

The following result is an extension of an inequality due to Talagrand [83] and it gives a sufficient condition for the Wasserstein distance to be finite:

**Theorem 12.3.2** Let $L \in \mathbb{L} \log \mathbb{L}(\mu)$ be a positive random variable with $E[L] = 1$ and let $\nu$ be the measure $d\nu = Ld\mu$. We then have

$$d_H^2(\nu, \mu) \leq 2 E[L \log L].$$  \hspace{1cm} (12.3.2)

**Proof:** Without loss of generality, we may suppose that $W$ is equipped with a filtration of sigma algebras in such a way that it becomes a classical Wiener space as $W = C_0(\mathbb{R}_+, \mathbb{R}^d)$. Assume first that $L$ is a strictly positive and bounded random variable. We can represent it as

$$L = \exp \left[ - \int_0^\infty (\dot{u}_s, dW_s) - \frac{1}{2} |u|^2_H \right],$$

where $u = \int_0^t \dot{u}_s ds$ is an $H$-valued, adapted random variable. Define $\tau_n$ as

$$\tau_n(x) = \inf \left\{ t \in \mathbb{R}_+ : \int_0^t |\dot{u}_s(x)|^2 ds > n \right\}.$$

$\tau_n$ is a stopping time with respect to the canonical filtration $(\mathcal{F}_t, t \in \mathbb{R}_+)$ of the Wiener process $(W_t, t \in \mathbb{R}_+)$ and $\lim_n \tau_n = \infty$ almost surely. Define $u^n$ as

$$u^n(t, x) = \int_0^t \mathbf{1}_{[0, \tau_n(x)]}(s) \dot{u}_s(x) ds.$$

Let $U_n : W \rightarrow W$ be the map $U_n(x) = x + u^n(x)$, then the Girsanov theorem says that $(t, x) \rightarrow U_n(x)(t) = x(t) + \int_0^t \dot{u}_s^n ds$ is a Wiener process under the measure $L_n d\mu$, where $L_n = E[L|\mathcal{F}_{\tau_n}]$. Therefore

$$E[L_n \log L_n] = E \left[ L_n \left\{ - \int_0^\infty (\dot{u}_s^n, dW_s) - \frac{1}{2} |u^n|^2_H \right\} \right]$$

$$= \frac{1}{2} E[L_n |u^n|^2_H]$$

$$= \frac{1}{2} E[L |u^n|^2_H].$$
Define now the measure $\beta_n$ on $W \times W$ as
\[
\int_{W \times W} f(x, y) d\beta_n(x, y) = \int_W f(U_n(x), x) L_n(x) d\mu(x).
\]
Then the first marginal of $\beta_n$ is $\mu$ and the second one is $L_n.\mu$. Consequently
\[
\inf \left\{ \int_{W \times W} |x - y|^2 d\theta : \pi_1 \theta = \mu, \pi_2 \theta = L_n.\mu \right\}
\leq \int_W |U_n(x) - x|^2 L_n d\mu
= 2E[L_n \log L_n].
\]
Hence we obtain
\[
d^2_H(L_n.\mu, \mu) = J(\gamma_n) \leq 2E[L_n \log L_n],
\]
where $\gamma_n$ is a solution of the Monge-Kantorovitch problem in $\Sigma(L.\mu, \mu)$. Let now $\gamma$ be any cluster point of the sequence $(\gamma_n, n \geq 1)$, since $\gamma \to J(\gamma)$ is lower semi-continuous with respect to the weak topology of probability measures, we have
\[
J(\gamma) \leq \lim inf_n J(\gamma_n)
\leq \sup_n 2E[L_n \log L_n]
\leq 2E[L \log L],
\]
since $\gamma \in \Sigma(L.\mu, \mu)$, it follows that
\[
d^2_H(L.\mu, \mu) \leq 2E[L \log L].
\]
For the general case we stop the martingale $E[L_t|\mathcal{F}_t]$ appropriately to obtain a bounded density $L_n$, then replace it by $P_{1/n}L_n$ to improve the positivity, where $(P_t, t \geq 0)$ denotes the Ornstein-Uhlenbeck semigroup. Then, from the Jensen inequality,
\[
E[P_{1/n}L_n\log P_{1/n}L_n] \leq E[L \log L],
\]
therefore, using the same reasoning as above
\[
d^2_H(L.\mu, \mu) \leq \lim inf_n d^2_H(P_{1/n}L_n.\mu, \mu)
\leq 2E[L \log L],
\]
and this completes the proof. \hfill \square
Corollary 12.3.3 Assume that \( \nu_i (i = 1, 2) \) have Radon-Nikodym densities \( L_i (i = 1, 2) \) with respect to the Wiener measure \( \mu \) which are in \( \mathbb{I} \log \mathbb{I} \). Then
\[
d_H(\nu_1, \nu_2) < \infty.
\]

Proof: This is a simple consequence of the triangle inequality (cf. [10]):
\[
d_H(\nu_1, \nu_2) \leq d_H(\nu_1, \mu) + d_H(\nu_2, \mu).
\]

Let us give a simple application of the above result in the lines of [58]:

Corollary 12.3.4 Assume that \( A \in \mathcal{B}(W) \) is any set of positive Wiener measure. Define the \( H \)-gauge function of \( A \) as
\[
q_A(x) = \inf(|h|_H : h \in (A - x) \cap H).
\]
Then we have
\[
E[q_A^2] \leq 2 \log \frac{1}{\mu(A)},
\]
in other words
\[
\mu(A) \leq \exp \left( - \frac{E[q_A^2]}{2} \right).
\]
Similarly if \( A \) and \( B \) are \( H \)-separated, i.e., if \( A_\varepsilon \cap B = \emptyset \), for some \( \varepsilon > 0 \), where \( A_\varepsilon = \{ x \in W : q_A(x) \leq \varepsilon \} \), then
\[
\mu(A_\varepsilon) \leq \frac{1}{\mu(A)} e^{-\varepsilon^2/4}
\]
and consequently
\[
\mu(A) \mu(B) \leq \exp \left( - \frac{\varepsilon^2}{4} \right).
\]

Remark: We already know that, from the 0–1–law, \( q_A \) is almost surely finite, besides it satisfies \( |q_A(x + h) - q_A(x)| \leq |h|_H \), hence \( E[\exp \lambda q_A^2] < \infty \) for any \( \lambda < 1/2 \) (cf. [101]). In fact all these assertions can also be proved with the technique used below.

Proof: Let \( \nu_A \) be the measure defined by
\[
d\nu_A = \frac{1}{\mu(A)} 1_A d\mu.
\]
Let $\gamma_A$ be the solution of the Monge-Kantorovitch problem, it is easy to see that the support of $\gamma_A$ is included in $W \times A$, hence

$$|x - y|_H \geq \inf \{|x - z|_H : z \in A\} = q_A(x),$$

$\gamma_A$-almost surely. This implies in particular that $q_A$ is almost surely finite. It follows now from the inequality $E[q_A] \leq -2 \log \mu(A)$, hence the proof of the first inequality follows. For the second let $B = A^c$ and let $\gamma_{AB}$ be the solution of the Monge-Kantorovitch problem corresponding to $\nu_A, \nu_B$. Then we have from the Corollary 12.3.3,

$$d_H^2(\nu_A, \nu_B) \leq -4 \log \mu(A) \mu(B).$$

Besides the support of the measure $\gamma_{AB}$ is in $A \times B$, hence $\gamma_{AB}$-almost surely $|x - y|_H \geq \varepsilon$ and the proof follows.

For the distance defined by

$$d_1(\nu, \mu) = \inf \left\{ \int_{W \times W} |x - y|_H d\theta : \pi_1 \theta = \mu, \pi_2 \theta = \nu \right\},$$

we have the following control:

**Theorem 12.3.5** Let $L \in L^1_+(\mu)$ with $E[L] = 1$. Then we have

$$d_1(L, \mu, \mu) \leq E \left[ (|I + L|^{-1}) \nabla L \right].$$

(12.3.3)

**Proof:** To prove the theorem we shall use a technique developed in [18]. Using the conditioning with respect to the sigma algebra $V_n = \sigma\{\delta e_1, \ldots, \delta e_n\}$, where $(e_i, i \geq 1)$ is a complete, orthonormal basis of $H$, we reduce the problem to the finite dimensional case. Moreover, we can assume that $L$ is a smooth, strictly positive function on $\mathbb{R}^n$. Define now $\sigma = (I + L)^{-1} \nabla L$ and

$$\sigma_t(x) = \frac{\sigma(x)}{t + (1 - t)L},$$

for $t \in [0, 1]$. Let $(\phi_{s,t}(x), s \leq t \in [0, 1])$ be the flow of diffeomorphisms defined by the following differential equation:

$$\phi_{s,t}(x) = x - \int_s^t \sigma_r(\phi_{s,r}(x)) d\tau.$$
From the standard results (cf. [101], Chapter V), it follows that \( x \rightarrow \phi_{s,t}(x) \) is Gaussian under the probability \( \Lambda_{s,t} \mu \), where

\[
\Lambda_{s,t} = \exp \int_s^t (\delta \sigma_\tau)(\phi_{s,\tau}(x)) d\tau
\]

is the Radon-Nikodym density of \( \phi_{s,t}^{-1} \mu \) with respect to \( \mu \). Define

\[
H_s(t, x) = \Lambda_{s,t}(x) \left\{ t + (1 - t) L \circ \phi_{s,t}(x) \right\}.
\]

It is easy to see that

\[
\frac{d}{dt} H_s(t, x) = 0
\]

for \( t \in (s, 1) \). Hence the map \( t \rightarrow H_s(t, x) \) is a constant, this implies that

\[
\Lambda_{s,1}(x) = s + (1 - s)L(x).
\]

We have, as in the proof of Theorem 12.3.2,

\[
d_1(L_\mu, \mu) \leq E[|\phi_{0,1}(x) - x|_L L_{0,1}]
\]

\[
\leq E \left[ \Lambda_{0,1} \int_0^1 |\sigma_t(\phi_{0,t}(x))|_H dt \right]
\]

\[
= E \left[ \int_0^1 |\sigma_t(\phi_{0,t} \circ \phi_{0,1}^{-1})(x)|_H dt \right]
\]

\[
= E \left[ \int_0^1 |\sigma_t(\phi_{0,1}^{-1}(x))|_H dt \right]
\]

\[
= E \left[ \int_0^1 |\sigma_t(x)|_H L_{t,1} dt \right]
\]

\[
= E[|\sigma|_H],
\]

and the general case follows via the usual approximation procedure. \( \square \)

### 12.4 Construction of the transport map

In this section we give the construction of the transport map in the Gaussian case. We begin with the following lemma:

**Lemma 12.4.1** Let \((W, \mu, H)\) be an abstract Wiener space, assume that \( f : W \rightarrow \mathbb{R} \) is a measurable function such that it is Gâteaux differentiable in the direction of the Cameron-Martin space \( H \), i.e., there exists some \( \nabla f : W \rightarrow H \) such that

\[
f(x + h) = f(x) + \int_0^1 (\nabla f(x + \tau h), h)_H d\tau,
\]

where \( \nabla f(x) \in H \). Then the transport map \( \phi_{s,t}(x) \) is given by

\[
\phi_{s,t}(x) = \exp \left[ \int_s^t (\delta \sigma_\tau)(\phi_{s,\tau}(x)) d\tau \right]
\]

where \( \Lambda_{s,t} \mu \) is the Radon-Nikodym density of \( \phi_{s,t}^{-1} \mu \) with respect to \( \mu \).
\( \mu \)-almost surely, for any \( h \in H \). If \( |\nabla f|_H \in L^2(\mu) \), then \( f \) belongs to the Sobolev space \( D_{2,1} \).

**Proof:** Since \( |\nabla | |_{H} \leq |\nabla f|_H \), we can assume that \( f \) is positive. Moreover, for any \( n \in \mathbb{N} \), the function \( f_n = \min(f, n) \) has also a Gâteaux derivative such that \( |\nabla f_n|_H \leq |\nabla f|_H \) \( \mu \)-almost surely. It follows from the Poincaré inequality that the sequence \( (f_n - E[f_n], n \geq 1) \) is bounded in \( L^2(\mu) \), hence it is also bounded in \( L^0(\mu) \). Since \( f \) is almost surely finite, the sequence \( (f_n, n \geq 1) \) is bounded in \( L^0(\mu) \). This means that \( \sup_n E[f_n] < \infty \), hence the monotone convergence theorem implies that \( E[f] < \infty \) and the proof is completed.

**Theorem 12.4.2** Let \( \nu \) be the measure \( d\nu = Ld\mu \), where \( L \) is a positive random variable, with \( E[L] = 1 \). Assume that \( dH(\mu, \nu) < \infty \) (for instance \( L \in L \log L \)). Then there exists a 1-convex function \( \phi \in D_{2,1} \), unique up to a constant, such that the map \( T = I_W + \nabla \phi \) is the unique solution of the original problem of Monge. Moreover, its graph supports the unique solution of the Monge-Kantorovitch problem \( \gamma \). Consequently

\[
(I_W \times T)\mu = \gamma
\]

In particular \( T \) maps \( \mu \) to \( \nu \) and \( T \) is almost surely invertible, i.e., there exists some \( T^{-1} \) such that \( T^{-1}\nu = \mu \) and that

\[
1 = \mu \left\{ x : T^{-1} \circ T(x) = x \right\} = \nu \left\{ y \in W : T \circ T^{-1}(y) = y \right\}.
\]

**Proof:** Let \( (\pi_n, n \geq 1) \) be a sequence of regular, finite dimensional orthogonal projections of \( H \) increasing to \( I_H \). Denote their continuous extensions to \( W \) by the same letters. For \( x \in W \), we define \( \pi_n x =: x_n = x - \pi_n x \). Let \( \nu_n \) be the measure \( \pi_n \nu \). Since \( \nu \) is absolutely continuous with respect to \( \mu \), \( \nu_n \) is absolutely continuous with respect to \( \mu_n := \pi_n \mu \) and

\[
\frac{d\nu_n}{d\mu_n} \circ \pi_n = E[L|V_n] =: L_n,
\]

where \( V_n \) is the sigma algebra \( \sigma(\pi_n) \) and the conditional expectation is taken with respect to \( \mu \). On the space \( H_n \), the Monge-Kantorovitch problem, which consists of finding the probability measure which realizes the following infimum

\[
d_H^2(\mu_n, \nu_n) = \inf \{ J(\beta) : \beta \in M_1(H_n \times H_n), p_1 \beta = \mu_n, p_2 \beta = \nu_n \}
\]
where
\[ J(\beta) = \int_{H_n \times H_n} |x - y|^2 d\beta(x, y), \]
has a unique solution \( \gamma_n \), where \( p_i, i = 1, 2 \) denote the projections \( (x_1, x_2) \to x_i, i = 1, 2 \) from \( H_n \times H_n \) to \( H_n \) and \( M_1(H_n \times H_n) \) denotes the set of probability measures on \( H_n \times H_n \). The measure \( \gamma_n \) may be regarded as a measure on \( W \times W \), by taking its image under the injection \( H_n \times H_n \hookrightarrow W \times W \) which we shall denote again by \( \gamma_n \). It results from the finite dimensional results of Brenier and of McCann\([14], [59]\) that there are two convex continuous functions (hence almost everywhere differentiable) \( \Phi_n \) and \( \Psi_n \) on \( H_n \) such that
\[ \Phi_n(x) + \Psi_n(y) \geq (x, y) \]
for all \( x, y \in H_n \) and that
\[ \Phi_n(x) + \Psi_n(y) = (x, y) \]
\( \gamma_n \)-almost everywhere. Hence the support of \( \gamma_n \) is included in the graph of the derivative \( \nabla \Phi_n \) of \( \Phi_n \), hence \( \nabla \Phi_n \mu_n = \nu_n \) and the inverse of \( \nabla \Phi_n \) is equal to \( \nabla \Psi_n \). Let
\[ \phi_n(x) = \Phi_n(x) - \frac{1}{2} |x|_H^2 \]
\[ \psi_n(y) = \Psi_n(y) - \frac{1}{2} |y|_H^2 . \]
Then \( \phi_n \) and \( \psi_n \) are 1-convex functions and they satisfy the following relations:
\[ \phi_n(x) + \psi_n(y) + \frac{1}{2} |x - y|_H^2 \geq 0 , \quad (12.4.4) \]
for all \( x, y \in H_n \) and
\[ \phi_n(x) + \psi_n(y) + \frac{1}{2} |x - y|_H^2 = 0 , \quad (12.4.5) \]
\( \gamma_n \)-almost everywhere. From what we have said above, it follows that \( \gamma_n \)-almost surely \( y = x + \nabla \phi_n(x) \), consequently
\[ J(\gamma_n) = E[|\nabla \phi_n|^2_H] . \quad (12.4.6) \]
Let \( q_n : W \times W \to H_n \times H_n \) be defined as \( q_n(x, y) = (\pi_n x, \pi_n y) \). If \( \gamma \) is any solution of the Monge-Kantororvitch problem, then \( q_n \gamma \in \Sigma(\mu_n, \nu_n) \), hence
\[ J(\gamma_n) \leq J(q_n \gamma) \leq J(\gamma) = d_H^2(\mu, \nu) . \quad (12.4.7) \]
Combining the relation (12.4.6) with the inequality (12.4.7), we obtain the following bound

\[ \sup_n J(\gamma_n) = \sup_n d_H^2(\mu_n, \nu_n) = \sup_n E[|\nabla \phi_n|^2_H] \leq d_H^2(\mu, \nu) = J(\gamma). \] (12.4.8)

For \( m \leq n, q_m \gamma_n \in \Sigma(\mu_m, \nu_m) \), hence we should have

\[ J(\gamma_m) = \int_{W \times W} |\pi_m x - \pi_m y|^2_H d\gamma_m(x, y) \leq \int_{W \times W} |\pi_n x - \pi_n y|^2_H d\gamma_n(x, y) \leq \int_{W \times W} |x - y|^2_H d\gamma_n(x, y) = J(\gamma_n), \]

where the third equality follows from the fact that we have denoted the \( \gamma_n \) on \( H_n \times H_n \) and its image in \( W \times W \) by the same letter. Let now \( \gamma \) be a weak cluster point of the sequence of measures \( (\gamma_n, n \geq 1) \), where the word “weak”\(^4\) refers to the weak convergence of measures on \( W \times W \). Since \( (x, y) \rightarrow |x - y|_H \) is lower semi-continuous, we have

\[ J(\gamma) = \int_{W \times W} |x - y|^2_H d\gamma(x, y) \leq \liminf_n \int_{W \times W} |x - y|^2_H d\gamma_n(x, y) = \liminf_n J(\gamma_n) \leq \sup_n J(\gamma_n) \leq J(\gamma) = d_H^2(\mu, \nu), \]

from the relation (12.4.8). Consequently

\[ J(\gamma) = \lim_n J(\gamma_n). \] (12.4.9)

Again from (12.4.8), if we replace \( \phi_n \) with \( \phi_n - E[\phi_n] \) and \( \psi_n \) with \( \psi_n + E[\phi_n] \) we obtain a bounded sequence \( (\phi_n, n \geq 1) \) in \( D_{2,1} \), in particular it is bounded

---

\(^4\)To prevent the reader against the trivial errors let us emphasize that \( \gamma_n \) is not the projection of \( \gamma \) on \( W_n \times W_n \).
in the space $L^2(\gamma)$ if we inject it into latter by $\phi_n(x) \to \phi_n(x) \otimes 1(y)$. Consider now the sequence of the positive, lower semi-continuous functions $(F_n, n \geq 1)$ defined on $W \times W$ as

$$F_n(x, y) = \phi_n(x) + \psi_n(y) + \frac{1}{2}|x - y|_H^2.$$  

We have, from the relation (12.4.5)

$$\int_{W \times W} F_n(x, y) d\gamma(x, y) = \int_W \phi_n d\mu + \int_W \psi_n(y) d\nu + \frac{1}{2} J(\gamma)$$

$$= \frac{1}{2} (J(\gamma) - J(\gamma_n)) \to 0.$$  

Consequently the sequence $(F_n, n \geq 1)$ converges to zero in $L^1(\gamma)$, therefore it is uniformly integrable. Since $(\phi_n, n \geq 1)$ is uniformly integrable as explained above and since $|x - y|^2_H$ has a finite expectation with respect to $\gamma$, it follows that $(\psi_n, n \geq 1)$ is also uniformly integrable in $L^1(\gamma)$ hence also in $L^1(\nu)$.

Let $\phi'$ be a weak cluster point of $(\phi_n, n \geq 1)$, then there exists a sequence $(\phi'_n, n \geq 1)$ whose elements are the convex combinations of some elements of $(\phi_k, k \geq n)$ such that $(\phi'_n, n \geq 1)$ converges in the norm topology of $\mathcal{D}_{2,1}$ and $\mu$-almost everywhere. Therefore the sequence $(\psi'_n, n \geq 1)$, constructed from $(\psi_k, k \geq n)$, converges in $L^1(\nu)$ and $\nu$-almost surely. Define $\phi$ and $\psi$ as

$$\phi(x) = \limsup_n \phi'_n(x)$$
$$\psi(y) = \limsup_n \psi'_n(y),$$

hence we have

$$G(x, y) = \phi(x) + \psi(y) + \frac{1}{2}|x - y|_H^2 \geq 0$$

for all $(x, y) \in W \times W$, also the equality holds $\gamma$-almost everywhere. Let now $h$ be any element of $H$, since $x - y$ is in $H$ for $\gamma$-almost all $(x, y) \in W \times W$, we have

$$|x + h - y|_H^2 = |x - y|_H^2 + |h|_H^2 + 2(h, x - y)_H$$

$\gamma$-almost surely. Consequently

$$\phi(x + h) - \phi(x) \geq -(h, x - y)_H - \frac{1}{2}|h|_H^2$$

$\gamma$-almost surely and this implies that

$$y = x + \nabla \phi(x)$$
\(\gamma\)-almost everywhere. Define now the map \(T : W \rightarrow W\) as \(T(x) = x + \nabla \phi(x)\), then

\[
\int_{W \times W} f(x, y) d\gamma(x, y) = \int_{W \times W} f(x, T(x)) d\gamma(x, y) = \int_W f(x, T(x)) d\mu(x),
\]

for any \(f \in C_b(W \times W)\), consequently \((I_W \times T) \mu = \gamma\), in particular \(T \mu = \nu\).

Let us notice that any weak cluster point of \((\phi_n, n \geq 1)\), say \(\tilde{\phi}\), satisfies

\[
\nabla \tilde{\phi}(x) = y - x
\]

\(\gamma\)-almost surely, hence \(\mu\)-almost surely we have \(\tilde{\phi} = \phi\). This implies that \((\phi_n, n \geq 1)\) has a unique cluster point \(\phi\), consequently the sequence \((\phi_n, n \geq 1)\) converges weakly in \(\mathbb{D}_{2,1}\) to \(\phi\). Besides we have

\[
\lim_n \int_W |\nabla \phi_n|^2_H d\mu = \lim_n J(\gamma_n) = J(\gamma) = \int_{W \times W} |x - y|^2_H d\gamma(x, y) = \int_W |\nabla \phi|^2_H d\mu,
\]

hence \((\phi_n, n \geq 1)\) converges to \(\phi\) in the norm topology of \(\mathbb{D}_{2,1}\). Let us recapitulate what we have done till here: we have taken an arbitrary optimal \(\gamma \in \Sigma(\mu, \nu)\) and an arbitrary cluster point \(\phi\) of \((\phi_n, n \geq 1)\) and we have proved that \(\gamma\) is carried by the graph of \(T = I_W + \nabla \phi\). This implies that \(\gamma\) and \(\phi\) are unique and that the sequence \((\gamma_n, n \geq 1)\) has a unique cluster point \(\gamma\).

Certainly \((\psi_n, \geq 1)\) converges also in the norm topology of \(L^1(\nu)\). Moreover, from the finite dimensional situation, we have \(\nabla \phi_n(x) + \nabla \psi_n(y) = 0\) \(\gamma_n\)-almost everywhere. Hence

\[
E_\nu[|\nabla \psi_n|^2_H] = E[|\nabla \phi_n|^2_H]
\]

this implies the boundedness of \((\nabla \psi_n, n \geq 1)\) in \(L^2(\nu, H)\) (i.e., \(H\)-valued functions). To complete the proof we have to show that, for some measurable, \(H\)-valued map, say \(\eta\), it holds that \(x = y + \eta(y)\) \(\gamma\)-almost surely. For this let \(F\) be a finite dimensional, regular subspace of \(H\) and denote by \(\pi_F\) the projection operator onto \(F\) which is continuously extended to \(W\), put \(\nabla_\perp = I_W - \pi_F\). We have \(W = F \oplus F_\perp\), with \(F_\perp = \ker \pi_F = \pi_F^\perp(W)\). Define the measures \(\nu_F = \pi_F(\nu)\) and \(\nu_F^\perp = \pi_F^\perp(\nu)\). From the construction of \(\psi\), we know
that, for any \( v \in F^\perp \), the partial map \( u \to \psi(u + v) \) is 1-convex on \( F \). Let also \( A = \{ y \in W : \psi(y) < \infty \} \), then \( A \) is a Borel set with \( \nu(A) = 1 \) and it is easy to see that, for \( \nu_\perp F \)-almost all \( v \in F^\perp \), one has
\[
\nu(A|\pi_\perp F = v) > 0.
\]
It then follows from Lemma 3.4 of Chapter 11 and from the fact that the regular conditional probability \( \nu(\cdot |\pi_\perp F = v) \) is absolutely continuous with respect to the Lebesgue measure of \( F \), that \( u \to \psi(u + v) \) is \( \nu_\perp F \)-almost everywhere differentiable on \( F \) for \( \nu_\perp F \)-almost all \( v \in F^\perp \). It then follows that, \( \nu \)-almost surely, \( \psi \) is differentiable in the directions of \( F \), i.e., there exists \( \nabla_F \psi \in F \) \( \nu \)-almost surely. Since we also have
\[
\psi(y + k) - \psi(y) \geq (x - y, k)_H - \frac{1}{2}|k|^2_H,
\]
we obtain, \( \gamma \)-almost surely
\[
(\nabla_F \psi(y), k)_H = (x - y, k)_H,
\]
for any \( k \in F \). Consequently
\[
\nabla_F \psi(y) = \pi_F(x - y)
\]
\( \gamma \)-almost surely. Let now \( (F_n, n \geq 1) \) be a total, increasing sequence of regular subspaces of \( H \), we have a sequence \( (\nabla_n \psi, n \geq 1) \) bounded in \( L^2(\nu) \) hence also bounded in \( L^2(\gamma) \). Besides \( \nabla_n \psi(y) = \pi_n x - \pi_n y \) \( \gamma \)-almost surely. Since \( (\pi_n(x - y), n \geq 1) \) converges in \( L^2(\gamma, H) \), \( (\nabla_n \psi, n \geq 1) \) converges in the norm topology of \( L^2(\gamma, H) \). Let us denote this limit by \( \eta \), then we have \( x = y + \eta(y) \) \( \gamma \)-almost surely. Note that, since \( \pi_n \eta = \nabla_n \psi \), we can even write in a weak sense that \( \eta = \nabla \psi \). If we define \( T^{-1}(y) = y + \eta(y) \), we see that
\[
1 = \gamma \{(x, y) \in W \times W : T \circ T^{-1}(y) = y \} = \gamma \{(x, y) \in W \times W : T^{-1} \circ T(x) = x \},
\]
and this completes the proof of the theorem.

\begin{remark}
Assume that the operator \( \nabla \) is closable with respect to \( \nu \), then we have \( \eta = \nabla \psi \). In particular, if \( \nu \) and \( \mu \) are equivalent, then we have
\[
T^{-1} = I_W + \nabla \psi,
\]
where is \( \psi \) is a 1-convex function.
\end{remark}
Remark 12.4.4 Assume that $L \in L^1_+(\mu)$, with $E[L] = 1$ and let $(D_k, k \in \mathbb{N})$ be a measurable partition of $W$ such that on each $D_k$, $L$ is bounded. Define $d\nu = L \, d\mu$ and $\nu_k = \nu(\cdot|D_k)$. It follows from Theorem 12.3.2, that $d_H(\mu, \nu_k) < \infty$. Let then $T_k$ be the map constructed in Theorem 12.4.2 satisfying $T_k \mu = \nu_k$. Define $n(\cdot|k)$ as the probability distribution on $\mathbb{N}$ given by $n(\{k\}) = \nu(D_k)$, $k \in \mathbb{N}$. Then we have

$$\int_W f(y) \, d\nu(y) = \int_{W \times \mathbb{N}} f(T_k(x)) \mu(dx)n(\cdot|k).$$

A similar result is given in [27], the difference with that of above lies in the fact that we have a more precise information about the probability space on which $T$ is defined.

12.5 Polar factorization of the absolutely continuous transformations of the Wiener space

Assume that $V = I_W + v : W \to W$ be an absolutely continuous transformation and let $L \in L^1_+(\mu)$ be the Radon-Nikodym derivative of $V\mu$ with respect to $\mu$. Let $T = I_W + \nabla \phi$ be the transport map such that $T \mu = L \mu$. Then it is easy to see that the map $s = T^{-1} \circ V$ is a rotation, i.e., $s \mu = \mu$ (cf. [10]), and it can be represented as $s = I_W + \alpha$. In particular we have

$$\alpha + \nabla \phi \circ s = v. \quad (12.5.10)$$

Since $\phi$ is a 1-convex map, we have $h \to \frac{1}{2}|h|^2_H + \phi(x + h)$ is almost surely convex (cf. Chapter 11). Let $s' = I_W + \alpha'$ be another rotation with $\alpha' : W \to H$. By the 1-convexity of $\phi$, we have

$$\frac{1}{2}|\alpha'|^2_H + \phi \circ s' \geq \frac{1}{2}|\alpha|^2_H + \phi \circ s + (\alpha + \nabla \phi \circ s, \alpha' - \alpha)_H,$$

$\mu$-almost surely. Taking the expectation of both sides, using the fact that $s$ and $s'$ preserve the Wiener measure $\mu$ and the identity (12.5.10), we obtain

$$E\left[\frac{1}{2}|\alpha|^2_H - (v, \alpha)_H\right] \leq E\left[\frac{1}{2}|\alpha'|^2_H - (v, \alpha')_H\right].$$

Hence we have proven the existence part of the following

Proposition 12.5.1 Let $R_2$ denote the subset of $L^2(\mu, H)$ whose elements are defined by the property that $x \to x + \eta(x)$ is a rotation, i.e., it preserves
the Wiener measure. Then $\alpha$ is the unique element of $R_2$ which minimizes the functional

$$\eta \to M_v(\eta) = E\left[\frac{1}{2}|\eta|^2_H - (v, \eta)_H\right].$$

**Proof:** To show the uniqueness, assume that $\eta \in R_2$ be another map minimizing $J_v$. Let $\beta$ be the measure on $W \times W$, defined as

$$\int_{W \times W} f(x, y) d\beta(x, y) = \int_W f(x + \eta(x), V(x)) d\mu.$$ 

Then the first marginal of $\beta$ is $\mu$ and the second marginal is $L.\mu$. Since $\gamma = (I_W \times T)\mu$ is the unique solution of the Monge-Kantorovitch problem, we should have

$$\int |x - y|^2_H d\beta(x, y) > \int |x - y|^2_H d\gamma(x, y) = E[|\nabla \phi|^2_H].$$

However we have

$$\int_{W \times W} |x - y|^2_H d\beta(x, y) = E\left[|v - \eta|^2_H\right] = E\left[|v|^2_H + 2M_v(\eta)\right] = E\left[|v|^2_H + 2M_v(\alpha)\right] = E\left[|v - \alpha|^2_H\right] = E\left[|\nabla \phi \circ s|^2_H\right] = E\left[|\nabla \phi|^2_H\right] = \int_{W \times W} |x - y|^2_H d\gamma(x, y) = J(\gamma)$$

and this gives a contradiction to the uniqueness of $\gamma$. \hfill \Box

The following theorem, whose proof is rather easy, gives a better understanding of the structure of absolutely continuous transformations of the Wiener measure:

**Theorem 12.5.2** Assume that $U : W \to W$ be a measurable map and $L \in L \log L$ a positive random variable with $E[L] = 1$. Assume that the measure $\nu = L \cdot \mu$ is a Girsanov measure for $U$, i.e., that one has

$$E[f \circ U L] = E[f],$$

for any $f \in C_b(W)$. Then there exists a unique map $T = I_W + \nabla \phi$ with $\phi \in D_{2,1}$ is $1$-convex, and a measure preserving transformation $R : W \to W$ such that $U \circ T = R \mu$-almost surely and $U = R \circ T^{-1} \nu$-almost surely.
Proof: By Theorem 12.4.2 there is a unique map \( T = I_W + \nabla \phi \), with \( \phi \in \mathbb{D}_{2,1} \), 1-convex such that \( T \) transports \( \mu \) to \( \nu \). Since \( U\nu = \mu \), we have

\[
E[f \circ U L] = E[f \circ U \circ T] = E[f].
\]

Therefore \( x \rightarrow U \circ T(x) \) preserves the measure \( \mu \). The rest is obvious since \( T^{-1} \) exists \( \nu \)-almost surely. □

Another version of Theorem 12.5.2 can be announced as follows:

**Theorem 12.5.3** Assume that \( Z : W \rightarrow W \) is a measurable map such that \( Z\mu \ll \mu \), with \( d_H(Z\mu, \mu) < \infty \). Then \( Z \) can be decomposed as

\[
Z = T \circ s,
\]

where \( T \) is the unique transport map of the Monge-Kantorovitch problem for \( \Sigma(\mu, Z\mu) \) and \( s \) is a rotation.

**Proof:** Let \( L \) be the Radon-Nikodym derivative of \( Z\mu \) with respect to \( \mu \). We have, from Theorem 12.4.2

\[
E[f] = E[f \circ T^{-1} \circ T] = E[f \circ T^{-1} L] = E[f \circ T^{-1} \circ Z],
\]

for any \( f \in C_b(W) \). Hence \( T^{-1} \circ Z = s \) is a rotation. Since \( T \) is uniquely defined, \( s \) is also uniquely defined. □

Although the following result is a translation of the results of this section, it is interesting from the point of view of stochastic differential equations:

**Theorem 12.5.4** Let \((W, \mu, H)\) be the standard Wiener space on \( \mathbb{R}^d \), i.e., \( W = C(\mathbb{R}_+, \mathbb{R}^d) \). Assume that there exists a probability \( P \ll \mu \) which is the weak solution of the stochastic differential equation

\[
dy_t = dW_t + b(t, y)dt,
\]

such that \( d_H(P, \mu) < \infty \). Then there exists a process \((T_t, t \in \mathbb{R}_+)\) which is a pathwise solution of some stochastic differential equation whose law is equal to \( P \).
Proof: Let $T$ be the transport map constructed in Theorem 12.4.2 corresponding to $dP/d\mu$. Then it has an inverse $T^{-1}$ such that $\mu\{T^{-1} \circ T(x) = x\} = 1$. Let $\phi$ be the 1-convex function such that $T = I_W + \nabla \phi$ and denote by $(D_s \phi, s \in \mathbb{R}_+)$ the representation of $\nabla \phi$ in $L^2(\mathbb{R}_+, ds)$. Define $T_t(x)$ as the trajectory $T(x)$ evaluated at $t \in \mathbb{R}_+$. Then it is easy to see that $(T_t, t \in \mathbb{R}_+)$ satisfies the stochastic differential equation

$$T_t(x) = W_t(x) + \int_0^t l(s, T(x)) ds, \quad t \in \mathbb{R}_+,$$

where $W_t(x) = x(t)$ and $l(s, x) = D_s \phi \circ T^{-1}(x)$.

\[\square\]

12.6 Construction and uniqueness of the transport map in the general case

In this section we call optimal every probability measure $\gamma$ on $W \times W$ such that $J(\gamma) < \infty$ and that $J(\gamma) \leq J(\theta)$ for every other probability $\theta$ having the same marginals as those of $\gamma$. We recall that a finite dimensional subspace $F$ of $W$ is called regular if the corresponding projection is continuous. Similarly a finite dimensional projection of $H$ is called regular if it has a continuous extension to $W$.

We begin with the following lemma which answers all kind of questions of measurability that we may encounter in the sequel:

Lemma 12.6.1 Consider two uncountable Polish spaces $X$ and $T$. Let $t \mapsto \gamma_t$ be a Borel family of probabilities on $X$ and let $\mathcal{F}$ be a separable sub-$\sigma$-algebra of the Borel $\sigma$-algebra $\mathcal{B}$ of $X$. Then there exists a Borel kernel

$$N_t f(x) = \int_X f(y) N_t(x, dy),$$

such that, for any bounded Borel function $f$ on $X$, the following properties hold true:

i) $(t, x) \mapsto N_t f(x)$ is Borel measurable on $T \times X$.

ii) For any $t \in T$, $N_t f$ is an $\mathcal{F}$-measurable version of the conditional expectation $E_{\gamma_t}[f | \mathcal{F}]$.

\[5\text{In fact the results of this section are essentially true for the bounded, positive measures.}\]
Proof: Assume first that \( F \) is finite, hence it is generated by a finite partition \( \{A_1, \ldots, A_k\} \). In this case it suffices to take

\[
N_t f(x) = \sum_{i=1}^{k} \frac{1}{\gamma_t(A_i)} \left( \int_{A_i} f \, d\gamma_t \right) 1_{A_i}(x) \quad \text{with } 0 = 0 .
\]

For the general case, take an increasing sequence \( (F_n, n \geq 1) \) of finite sub-sigma-algebras whose union generates \( F \). Without loss of generality we can assume that \((X, \mathcal{B})\) is the Cantor set (Kuratowski Theorem, cf., [21]). Then for every clopen set (i.e., a set which is closed and open at the same time) \( G \) and any \( t \in T \), the sequence \( (N_t^n 1_G, n \geq 1) \) converges \( \gamma_t \)-almost everywhere. Define

\[
H_G(t, x) = \limsup_{m, n \to \infty} |N_t^n 1_G(x) - N_t^m 1_G(x)| .
\]

\( H_G \) is a Borel function on \( T \times X \) which vanishes \( \gamma_t \)-almost all \( x \in X \), besides, for any \( t \in T \), \( x \to H_G(t, x) \) is \( \mathcal{F} \)-measurable. As there exist only countably many clopen sets in \( X \), the function

\[
H(t, x) = \sup_G H_G(t, x)
\]

inherits all the measurability properties. Let \( \theta \) be any probability on \( X \), for any clopen \( G \), define

\[
N_t 1_G(x) = \begin{cases} 
\lim_n N_t^n 1_G(x) & \text{if } H(t, x) = 0 , \\
\theta(G) & \text{if } H(t, x) > 0 .
\end{cases}
\]

Hence, for any \( t \in T \), we get an additive measure on the Boolean algebra of clopen sets of \( X \). Since such a measure is \( \sigma \)-additive and extends uniquely as a \( \sigma \)-additive measure on \( \mathcal{B} \), the proof is completed.

Remark 12.6.2

1. This result holds in fact for the Lusin spaces since they are Borel isomorphic to the Cantor set. Besides it extends easily to countable spaces.

2. The particular case where \( T = M_1(X) \), i.e., the space of probability measures on \( X \) under the weak topology and \( t \to \gamma_t \) being the identity map, is particularly important for the sequel. In this case we obtain a kernel \( N \) such that \( (x, \gamma) \to N \gamma f(x) \) is measurable and \( N \gamma f \) is an \( \mathcal{F} \)-measurable version of \( E \gamma [f | \mathcal{F}] \).
Lemma 12.6.3 Let $\rho$ and $\nu$ be two probability measures on $W$ such that

$$d_H(\rho, \nu) < \infty$$

and let $\gamma \in \Sigma(\rho, \nu)$ be an optimal measure, i.e., $J(\gamma) = d_H^2(\rho, \nu)$, where $J$ is given by (12.1.4). Assume that $F$ is a regular finite dimensional subspace of $W$ with the corresponding projection $\pi_F$ from $W$ to $F$ and let $\pi_F^\perp = I_W - \pi_F$. Define $p_F$ as the projection from $W \times W$ onto $F$ with $p_F(x, y) = \pi_F x$ and let $p_F^\perp(x, y) = \pi_F^\perp x$. Consider the Borel disintegration

$$\gamma(\cdot) = \int_{F^\perp \times W} \gamma(\cdot | x^\perp)(dx^\perp)$$

$$= \int_{F^\perp} \gamma(\cdot | x^\perp)\rho^\perp(dx^\perp)$$

along the projection of $W \times W$ on $F^\perp$, where $\rho^\perp$ is the measure $\pi_F^\perp \rho$, $\gamma(\cdot | x^\perp)$ denotes the regular conditional probability $\gamma(\cdot | p_F^\perp = x^\perp)$ and $\gamma^\perp$ is the measure $p_F^\perp \gamma$. Then, $\rho^\perp$ and $\gamma^\perp$-almost surely $\gamma(\cdot | x^\perp)$ is optimal on $(x^\perp + F) \times W$.

**Proof:** Let $p_1, p_2$ be the projections of $W \times W$ defined as $p_1(x, y) = \pi_F(x)$ and $p_2(x, y) = \pi_F(y)$. Note first the following obvious identity:

$$p_1 \gamma(\cdot | x^\perp) = \rho(\cdot | x^\perp),$$

$\rho^\perp$ and $\gamma^\perp$-almost surely. Define the sets $B \subset F^\perp \times \mathcal{M}_1(F \times F)$ and $C$ as

$$B = \{ (x^\perp, \theta) : \theta \in \Sigma(p_1 \gamma(\cdot | x^\perp), p_2 \gamma(\cdot | x^\perp)) \}$$

$$C = \{ (x^\perp, \theta) \in B : J(\theta) < J(\gamma(\cdot | x^\perp)) \},$$

where $\mathcal{M}_1(F \times F)$ denotes the set of probability measures on $F \times F$. Let $K$ be the projection of $C$ on $F^\perp$. Since $B$ and $C$ are Borel measurable, $K$ is a Souslin set, hence it is $\rho^\perp$-measurable. The selection theorem (cf. [21]) implies the existence of a measurable map

$$x^\perp \to \theta_{x^\perp}$$

from $K$ to $\mathcal{M}_1(F \times F)$ such that, $\rho^\perp$-almost surely, $(x^\perp, \theta_{x^\perp}) \in C$. Define

$$\theta(\cdot) = \int_K \theta_{x^\perp}(\cdot) d\rho^\perp(x^\perp) + \int_{K^c} \gamma(\cdot | x^\perp) d\rho^\perp(x^\perp).$$

Then $\theta \in \Sigma(\rho, \nu)$ and we have

$$J(\theta) = \int_K J(\theta_{x^\perp}) d\rho^\perp(x^\perp) + \int_{K^c} J(\gamma(\cdot | x^\perp)) d\rho^\perp(x^\perp)$$

$$< \int_K J(\gamma(\cdot | x^\perp)) d\rho^\perp(x^\perp) + \int_{K^c} J(\gamma(\cdot | x^\perp)) d\rho^\perp(x^\perp)$$

$$= J(\gamma),$$
hence we obtain \( J(\theta) < J(\gamma) \) which is a contradiction to the optimality of \( \gamma \).

**Lemma 12.6.4** Assume that the hypothesis of Lemma 12.6.3 holds and let \( F \) be any regular finite dimensional subspace of \( W \). Denote by \( \pi_F \) the projection operator associated to it and let \( \pi_F^1 = I_W - \pi_F \). If \( \pi_F^1 \rho \)-almost surely, the regular conditional probability \( \rho(\cdot \mid \pi_F^1 = x^\perp) \) vanishes on the subsets of \( x^\perp + F \) whose Hausdorff dimension are at most equal to \( \dim(F) - 1 \), then there exists a map \( T_F : F \times F^\perp \to F \) such that

\[
\gamma \left( \left\{ (x, y) \in W \times W : \pi_F y = T_F(\pi_F x, \pi_F^1 x) \right\} \right) = 1, \]

**Proof:** Let \( C_{x^\perp} \) be the support of the regular conditional probability \( \gamma(\cdot \mid x^\perp) \) in \( (x^\perp + F) \times W \). We know from Lemma 12.6.3 that the measure \( \gamma(\cdot \mid x^\perp) \) is optimal in \( \Sigma(\pi_1 \gamma(\cdot \mid x^\perp), \pi_2 \gamma(\cdot \mid x^\perp)) \), with \( J(\gamma(\cdot \mid x^\perp)) < \infty \) for \( \rho^\perp \)-almost everywhere \( x^\perp \). From Theorem 2.3 of [34] and from [1], the set \( C_{x^\perp} \) is cyclically monotone, moreover, \( C_{x^\perp} \) is a subset of \( (x^\perp + F) \times H \), hence the cyclic monotonicity of it implies that the set \( K_{x^\perp} \subset F \times F \), defined as

\[
K_{x^\perp} = \{ (u, \pi_F v) \in F \times F : (x^\perp + u, v) \in C_{x^\perp} \}
\]

is cyclically monotone in \( F \times F \). Therefore \( K_{x^\perp} \) is included in the subdifferential of a convex function defined on \( F \). Since, by hypothesis, the first marginal of \( \gamma(\cdot \mid x^\perp) \), i.e., \( \rho(\cdot \mid x^\perp) \) vanishes on the subsets of \( x^\perp + F \) of codimension one, the subdifferential under question, denoted as \( U_F(u, x^\perp) \) is \( \rho(\cdot \mid x^\perp) \)-almost surely univalent (cf. [5, 59]). This implies that

\[
\gamma(\cdot \mid x^\perp) \left( \left\{ (u, v) \in C_{x^\perp} : \pi_F v = U_F(u, x^\perp) \right\} \right) = 1, \]

\( \rho^\perp \)-almost surely. Let

\[
K_{x^\perp, u} = \{ v \in W : (u, v) \in K_{x^\perp} \}.
\]

Then \( K_{x^\perp, u} \) consists of a single point for almost all \( u \) with respect to \( \rho(\cdot \mid x^\perp) \). Let

\[
N = \left\{ (u, x^\perp) \in F \times F^\perp : \text{Card}(K_{x^\perp, u}) > 1 \right\},
\]

note that \( N \) is a Souslin set, hence it is universally measurable. Let \( \sigma \) be the measure which is defined as the image of \( \rho \) under the projection \( x \to (\pi_F x, \pi_F^1 x) \). We then have

\[
\sigma(N) = \int_{F^\perp} \rho^\perp(dx^\perp) \int_F 1_N(u, x^\perp) \rho(du|x^\perp) = 0.
\]
Hence \((u, x^\perp) \mapsto K_{x^\perp,u} = \{y\}\) is \(\rho\) and \(\gamma\)-almost surely well-defined and it suffices to denote this map by \(T_F\) to achieve the proof.

\[\text{Theorem 12.6.5} \quad \text{Suppose that } \rho \text{ and } \nu \text{ are two probability measures on } W \text{ such that} \]
\[d_H(\rho, \nu) < \infty.\]

Let \((\pi_n, n \geq 1)\) be a total increasing sequence of regular projections (of \(H\), converging to the identity map of \(H\)). Suppose that, for any \(n \geq 1\), the regular conditional probabilities \(\rho(\cdot | \pi_n^{\perp} = x^\perp)\) vanish \(\pi_n^\perp \rho\)-almost surely on the subsets of \((\pi_n^\perp)^{-1}(W)\) with Hausdorff dimension \(n - 1\). Then there exists a unique solution of the Monge-Kantorovitch problem, denoted by \(\gamma \in \Sigma(\rho, \nu)\) and \(\gamma\) is supported by the graph of a Borel map \(T\) which is the solution of the Monge problem. \(T : W \to W\) is of the form \(T = I_W + \xi\), where \(\xi \in H\) almost surely. Besides we have
\[d^2_H(\rho, \nu) = \int_{W \times W} |T(x) - x|^2_H d\gamma(x, y)\]
\[= \int_W |T(x) - x|^2_H d\rho(x),\]
and for \(\pi_n^\perp \rho\)-almost almost all \(x_n^\perp\), the map \(u \mapsto u + \xi(u + x_n^\perp)\) is cyclically monotone on \((\pi_n^\perp)^{-1}\{x_n^\perp\}\), in the sense that
\[\sum_{i=1}^N \left( u_i + \xi(x_n^\perp + u_i), u_{i+1} - u_i \right)^2_H \leq 0\]
\(\pi_n^\perp \rho\)-almost surely, for any cyclic sequence \(\{u_1, \ldots, u_N, u_{N+1} = u_1\}\) from \(\pi_n(W)\). Finally, if, for any \(n \geq 1, \pi_n^\perp \nu\)-almost surely, \(\nu(\cdot | \pi_n^\perp = y^\perp)\) also vanishes on the \(n - 1\)-Hausdorff dimensional subsets of \((\pi_n^\perp)^{-1}(W)\), then \(T\) is invertible, i.e., there exists \(S : W \to W\) of the form \(S = I_W + \eta\) such that \(\eta \in H\) satisfies a similar cyclic monotonicity property as \(\xi\) and that
\[1 = \gamma \{ (x, y) \in W \times W : T \circ S(y) = y \}\]
\[= \gamma \{ (x, y) \in W \times W : S \circ T(x) = x \}.\]

In particular we have
\[d^2_H(\rho, \nu) = \int_{W \times W} |S(y) - y|^2_H d\gamma(x, y)\]
\[= \int_W |S(y) - y|^2_H d\nu(y).\]
Remark 12.6.6 In particular, for all the measures $\rho$ which are absolutely continuous with respect to the Wiener measure $\mu$, the second hypothesis is satisfied, i.e., the measure $\rho(\cdot | \pi_n^+ = x_n^+)$ vanishes on the sets of Hausdorff dimension $n-1$.

Proof: Let $(F_n, n \geq 1)$ be the increasing sequence of regular subspaces associated to $(\pi_n, n \geq 1)$, whose union is dense in $W$. From Lemma 12.6.4, for any $F_n$, there exists a map $T_n$, such that $\pi_n y = T_n(\pi_n x, \pi_n^+ x)$ for $\gamma$-almost all $(x, y)$, where $\pi_n^+ = I_W - \pi_n$. Write $T_n$ as $I_n + \xi_n$, where $I_n$ denotes the identity map on $F_n$. Then we have the following representation:

$$\pi_n y = \pi_n x + \xi_n(\pi_n x, \pi_n^+ x),$$

$\gamma$-almost surely. Since

$$\pi_n y - \pi_n x = \pi_n (y - x) = \xi_n(\pi_n x, \pi_n^+ x)$$

and since $y - x \in H$ $\gamma$-almost surely, $(\pi_n y - \pi_n x, n \geq 1)$ converges $\gamma$-almost surely. Consequently $(\xi_n, n \geq 1)$ converges $\gamma$, hence $\rho$ almost surely to a measurable $\xi$. Consequently we obtain

$$\gamma(\{(x, y) \in W \times W : y = x + \xi(x)\}) = 1.$$ 

Since $J(\gamma) < \infty$, $\xi$ takes its values almost surely in the Cameron-Martin space $H$. The cyclic monotonicity of $\xi$ is obvious. To prove the uniqueness, assume that we have two optimal solutions $\gamma_1$ and $\gamma_2$ with the same marginals and $J(\gamma_1) = J(\gamma_2)$. Since $\beta \to J(\beta)$ is linear, the measure defined as $\gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$ is also optimal and it has also the same marginals $\rho$ and $\nu$. Consequently, it is also supported by the graph of a map $T$. Note that $\gamma_1$ and $\gamma_2$ are absolutely continuous with respect to $\gamma$, let $L_1(x, y)$ be the Radon-Nikodym density of $\gamma_1$ with respect to $\gamma$. For any $f \in C_b(W)$, we then have

$$\int_W f \ d\rho = \int_{W \times W} f(x) d\gamma_1(x, y) = \int_{W \times W} f(x) L_1(x, y) d\gamma(x, y) = \int_W f(x) L_1(x, T(x)) d\rho(x).$$

Therefore we should have $\rho$-almost surely, $L_1(x, T(x)) = 1$, hence also $L_1 = 1$ almost everywhere $\gamma$ and this implies that $\gamma = \gamma_1 = \gamma_2$. The second part about the invertibility of $T$ is totally symmetric, hence its proof follows along the same lines as the proof for $T$. $\square$
Corollary 12.6.7 Assume that $\rho$ is equivalent to the Wiener measure $\mu$, then for any $h_1, \ldots, h_N \in H$ and for any permutation $\tau$ of $\{1, \ldots, N\}$, we have, with the notations of Theorem [12.6.5]

$$
\sum_{i=1}^{N} \left( h_i + \xi(x + h_i), h_{\tau(i)} - h_i \right)_H \leq 0
$$

$\rho$-almost surely.

**Proof:** Again with the notations of the theorem, $\rho^k$-almost surely, the graph of the map $x_k \rightarrow x_k + \xi_k(x_k, x_k^+)$ is cyclically monotone on $F_k$. Hence, for the case $h_i \in F_n$ for all $i = 1, \ldots, N$ and $n \leq k$, we have

$$
\sum_{i=1}^{N} \left( h_i + x_k + \xi_k(x_k + h_i, x_k^+), h_{\tau(i)} - h_i \right)_H \leq 0.
$$

Since $\sum_i (x_k, h_{\tau(i)} - h_i)_H = 0$, we also have

$$
\sum_{i=1}^{N} \left( h_i + \xi_k(x_k + h_i, x_k^+), h_{\tau(i)} - h_i \right)_H \leq 0.
$$

We know that $\xi_k(x_k + h_i, x_k^+)$ converges to $\xi(x + h_i)$ $\rho$-almost surely. Moreover $h \rightarrow \xi(x + h)$ is continuous from $H$ to $L^0(\rho)$ and the proof follows. 

---

### 12.7 The Monge-Ampère equation

Assume that $W = \mathbb{R}^n$ and take a density $L \in \mathbb{D} \log \mathbb{D}$. Let $\phi \in \mathbb{D}_{2,1}$ be the 1-convex function such that $T = I + \nabla \phi$ maps $\mu$ to $L \cdot \mu$. Let $S = I + \nabla \psi$ be its inverse with $\psi \in \mathbb{D}_{2,1}$. Let now $\nabla^2 \phi$ be the second Alexandrov derivative of $\phi$, i.e., the Radon-Nikodym derivative of the absolutely continuous part of the vector measure $\nabla^2 \phi$ with respect to the Gaussian measure $\mu$ on $\mathbb{R}^n$. Since $\phi$ is 1-convex, it follows that $\nabla^2 \phi \geq -I_{\mathbb{R}^n}$ in the sense of the distributions, consequently $\nabla^2 \phi \geq -I_{\mathbb{R}^n} \mu$-almost surely. Define also the Alexandrov version $\mathcal{L}_a \phi$ of $\mathcal{L} \phi$ as the Radon-Nikodym derivative of the absolutely continuous part of the distribution $\mathcal{L} \phi$. Since we are in finite dimensional situation, we have the explicit expression for $\mathcal{L}_a \phi$ as

$$
\mathcal{L}_a \phi(x) = (\nabla \phi(x), x)_{\mathbb{R}^n} - \text{trace} \left( \nabla^2 \phi \right).
$$

Let $\Lambda$ be the Gaussian Jacobian

$$
\Lambda = \text{det}_2 \left( I_{\mathbb{R}^n} + \nabla^2 \phi \right) \exp \left\{ -\mathcal{L}_a \phi - \frac{1}{2} |\nabla \phi|^2_{\mathbb{R}^n} \right\}.
$$
Remark 12.7.1 In this expression as well as in the sequel, the notation \( \det_2(I_H + A) \) denotes the modified Carleman-Fredholm determinant of the operator \( I_H + A \) on a Hilbert space \( H \). If \( A \) is an operator of finite rank, then it is defined as

\[
\det_2 (I_H + A) = \prod_{i=1}^{n} (1 + l_i) e^{-l_i},
\]

where \((l_i, i \leq n)\) denotes the eigenvalues of \( A \) counted with respect to their multiplicity. In fact this determinant has an analytic extension to the space of Hilbert-Schmidt operators on a separable Hilbert space, cf. [23] and Appendix A.2 of [101]. As explained in [101], the modified determinant exists for the Hilbert-Schmidt operators while the ordinary determinant does not, since the latter requires the existence of the trace of \( A \). Hence the modified Carleman-Fredholm determinant is particularly useful when one studies the absolute continuity properties of the image of a Gaussian measure under non-linear transformations in the setting of infinite dimensional Banach spaces (cf., [101] for further information).

It follows from the change of variables formula given in Corollary 4.3 of [60], that, for any \( f \in C_b(\mathbb{R}^n) \),

\[
E[f \circ T \Lambda] = E\left[f 1_{\partial \Phi(M)}\right],
\]

where \( M \) is the set of non-degeneracy of \( I_{\mathbb{R}^n} + \nabla_2^2 \phi \),

\[
\Phi(x) = \frac{1}{2} |x|^2 + \phi(x)
\]

and \( \partial \Phi \) denotes the subdifferential of the convex function \( \Phi \). Let us note that, in case \( L > 0 \) almost surely, \( T \) has a global inverse \( S \), i.e., \( S \circ T = T \circ S = I_{\mathbb{R}^n} \) \( \mu \)-almost surely and \( \mu(\partial \Phi(M)) = \mu(S^{-1}(M)) \). Assume now that \( \Lambda > 0 \) almost surely, i.e., that \( \mu(M) = 1 \). Then, for any \( f \in C_b(\mathbb{R}^n) \), we have

\[
E[f \circ T] = E\left[f \circ T \frac{\Lambda}{\Lambda \circ T^{-1} \circ T}\right] = E\left[f \frac{1}{\Lambda \circ T^{-1} \circ \partial \Phi(M)}\right] = E[f L],
\]

where \( T^{-1} \) denotes the left inverse of \( T \) whose existence is guaranteed by Theorem 12.4.2. Since \( T(x) \in \partial \Phi(M) \) almost surely, it follows from the above calculations

\[
\frac{1}{\Lambda} = L \circ T,
\]
almost surely. Take now any \( t \in [0,1) \), the map \( x \to \frac{1}{2}|x|_H^2 + t\phi(x) = \Phi_t(x) \) is strictly convex and a simple calculation implies that the mapping \( T_t = I + t\nabla \phi \) is \((1-t)\)-monotone (cf. [101], Chapter 6), consequently it has a left inverse denoted by \( S_t \). Let us denote by \( \Psi_t \) the Legendre transformation of \( \Phi_t \):

\[
\Psi_t(y) = \sup_{x \in \mathbb{R}^n} \{(x,y) - \Phi_t(x)\}.
\]

A simple calculation shows that

\[
\Psi_t(y) = \sup_x \left[ (1-t) \left\{ (x,y) - \frac{|x|^2}{2} \right\} + t \left\{ (x,y) - \frac{|x|^2}{2} - \phi(x) \right\} \right]
\]

\[
\leq (1-t)\frac{|y|^2}{2} + t\Psi_1(y).
\]

Since \( \Psi_1 \) is the Legendre transformation of \( \Phi_1(x) = |x|^2/2 + \phi(x) \) and since \( L \in \mathbb{I}_L \log \mathbb{I}_L \), it is finite on a convex set of full measure, hence it is finite everywhere. Consequently \( \Psi_t(y) < \infty \) for any \( y \in \mathbb{R}^n \). Since a finite, convex function is almost everywhere differentiable, \( \nabla \Psi_t \) exists almost everywhere on and it is equal almost everywhere on \( T_t(M_t) \) to the left inverse \( T_t^{-1} \), where \( M_t \) is the set of non-degeneracy of \( I_{\mathbb{R}^n} + t\nabla^2 \phi \). Note that \( \mu(M_t) = 1 \). The strict convexity implies that \( T_t^{-1} \) is Lipschitz with a Lipschitz constant \( \frac{1}{1-t} \).

Let now \( \Lambda_t \) be the Gaussian Jacobian

\[
\Lambda_t = \det_2 \left( I_{\mathbb{R}^n} + t\nabla^2 \phi \right) \exp \left\{ -tL_{a}\phi - \frac{t^2}{2} |\nabla \phi|^2_{\mathbb{R}^n} \right\}.
\]

Since the domain of \( \phi \) is the whole space \( \mathbb{R}^n \), \( \Lambda_t > 0 \) almost surely, hence, as we have explained above, it follows from the change of variables formula of [60] that \( T_t \mu \) is absolutely continuous with respect to \( \mu \) and that

\[
\frac{1}{\Lambda_t} = L_t \circ T_t,
\]

\( \mu \)-almost surely.

Let us come back to the infinite dimensional case: we first give an inequality which may be useful.

**Theorem 12.7.2** Assume that \( (W, \mu, H) \) is an abstract Wiener space, assume that \( K, L \in \mathbb{I}_L^1(\mu) \) with \( K > 0 \) almost surely and denote by \( T : W \to W \) the transfer map \( T = I_W + \nabla \phi \), which maps the measure \( Kd\mu \) to the measure \( Ld\mu \). Then the following inequality holds:

\[
\frac{1}{2} E[|\nabla \phi|^2_H] \leq E[- \log K + \log L \circ T].
\]
Proof: Let us define $k$ as $k = K \circ T^{-1}$, then for any $f \in C_b(W)$, we have
\[
\int_W f(y) L(y) d\mu(y) = \int_W f \circ T(x) K(x) d\mu(x) = \int_W f \circ T(x) k \circ T(x) d\mu(x),
\]
hence
\[
T\mu = \frac{L}{k} \mu.
\]
It then follows from the inequality 12.3.2 that
\[
\frac{1}{2} E \left[ |\nabla \phi|^2_H \right] \leq E \left[ \frac{L}{k} \log \frac{L}{k} \right] = E \left[ \log \frac{L \circ T}{k \circ T} \right] = E \left[ -\log K + \log L \circ T \right].
\]

Lemma 12.7.3 Let $\phi \in D_{2,1}$ be 1-convex Wiener functional. Let $V_n$ be the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$, where $(e_n, n \geq 1)$ is an orthonormal basis of the Cameron-Martin space $H$. Then $\phi_n = E[\phi | V_n]$ is again 1-convex (cf. Chapter 11), hence $\mathcal{L}\phi_n$ is a measure as it can be easily verified. However the sequence $(\mathcal{L}\phi_n, n \geq 1)$ converges to $\mathcal{L}\phi$ only in $D'$. Consequently, there is no reason for the limit $\mathcal{L}\phi$ to be a measure. In case this happens, we shall denote the Radon-Nikodym density with respect to $\mu$, of the absolutely continuous part of this measure by $\mathcal{L}_a\phi$.

Proof: Note that, due to the 1-convexity, we have $\mathcal{L}_a F_n \geq \mathcal{L} F_n$ for any $n \in \mathbb{N}$. Let $X_n = \mathcal{L}_a F_n$ and $f \in \mathcal{D}$ be a positive, $V_n$-measurable test function. Since $\mathcal{L} E[\phi | V_n] = E[\mathcal{L}\phi | V_n]$, we have
\[
E[X_{n+1} f] \geq \langle \mathcal{L} F_{n+1}, f \rangle = \langle \mathcal{L} F_n, f \rangle,
\]
where $\langle \cdot, \cdot \rangle$ denotes the duality bracket for the dual pair $(D', D)$. Consequently
\[
E[f E[X_{n+1} | V_n]] \geq \langle \mathcal{L} F_n, f \rangle,
\]
for any positive, $V_n$-measurable test function $f$, it follows that the absolutely continuous part of $\mathcal{L}F_n$ is also dominated by the same conditional expectation and this proves the submartingale property.

\[\square\]

**Lemma 12.7.4** Assume that $L \in \mathbb{I} \log \mathbb{I}$ is a positive random variable whose expectation is one. Assume further that it is lower bounded by a constant $a > 0$. Let $T = I_W + \nabla \phi$ be the transport map such that $T \mu = L \mu$ and let $T^{-1} = I_W + \nabla \psi$. Then $\mathcal{L} \psi$ is a Radon measure on $(W, B(W))$. If $L$ is upper bounded by $b > 0$, then $\mathcal{L} \psi$ is also a Radon measure on $(W, B(W))$.

**Proof:** Let $L_n = E[L|V_n]$, then $L_n \geq a$ almost surely. Let $T_n = I_W + \nabla \phi_n$ be the transport map which satisfies $T_n \mu = L_n \mu$ and let $T^{-1}_n = I_W + \nabla \psi_n$ be its inverse. We have

\[L_n = \det_2 \left( I_H + \nabla^2 \psi_n \right) \exp \left[ -\mathcal{L}_a \psi_n - \frac{1}{2} |\nabla \psi_n|^2_H \right].\]

By the hypothesis $-\log L_n \leq -\log a$. Since $\psi_n$ is 1-convex, it follows from the finite dimensional results that $\det_2 \left( I_H + \nabla^2 \psi_n \right) \in [0, 1]$ almost surely. Therefore we have

\[\mathcal{L}_a \psi_n \leq -\log a,\]

besides $\mathcal{L} \psi_n \leq \mathcal{L}_a \psi_n$ as distributions, consequently

\[\mathcal{L} \psi_n \leq -\log a\]

as distributions, for any $n \geq 1$. Since $\lim_n \mathcal{L} \psi_n = \mathcal{L} \psi$ in $\mathbb{D}'$, we obtain $\mathcal{L} \psi \leq -\log a$, hence $-\log a - \mathcal{L} \psi \geq 0$ as a distribution, hence $\mathcal{L} \psi$ is a Radon measure on $W$. This proves the first claim. Note that whenever $L$ is upper bounded, $\Lambda = 1/L \circ T$ is lower bounded, hence the proof of the second claim is similar to that of the first one. \[\square\]

**Theorem 12.7.5** Assume that $L$ is a strictly positive bounded random variable with $E[L] = 1$. Let $\phi \in \mathbb{D}_{2,1}$ be the 1-convex Wiener functional such that

\[T = I_W + \nabla \phi\]

is the transport map realizing the measure $L \mu$ and let $S = I_W + \nabla \psi$ be its inverse. Define $F_n = E[\phi|V_n]$, then the submartingale $(\mathcal{L}_a F_n, n \geq 1)$ converges almost surely to $\mathcal{L}_a \phi$. Let $\lambda(\phi)$ be the random variable defined as

\[\lambda(\phi) = \lim_{n \to \infty} \Lambda_n\]

\[= \left( \lim_{n} \inf \det_2 \left( I_H + \nabla^2 F_n \right) \right) \exp \left\{ -\mathcal{L}_a \phi - \frac{1}{2} |\nabla \phi|^2_H \right\}\]
where
\[ \Lambda_n = \det_2 \left( I_H + \nabla^2 F_n \right) \exp \left\{ -\mathcal{L}_a F_n - \frac{1}{2} |\nabla F_n|^2_H \right\}. \]

Then it holds true that
\[ E[f \circ T \lambda(\phi)] \leq E[f] \quad (12.7.12) \]
for any \( f \in C^+_W \), in particular \( \lambda(\phi) \leq \frac{1}{L \circ T} \) almost surely. If \( E[\lambda(\phi)] = 1 \), then the inequality in (12.7.12) becomes an equality and we also have
\[ \lambda(\phi) = \frac{1}{L \circ T}. \]

**Proof:** Let us remark that, due to the 1-convexity, \( 0 \leq \det_2 \left( I_H + \nabla^2 F_n \right) \leq 1 \), hence the lim inf exists. Now, Lemma 12.7.4 implies that \( \mathcal{L} \phi \) is a Radon measure. Let \( F_n = E[\phi|V_n] \), then we know from Lemma 12.7.3 that \( (\mathcal{L}_a F_n, n \geq 1) \) is a submartingale. Let \( \mathcal{L}^+ \phi \) denote the positive part of the measure \( \mathcal{L} \phi \). Since \( \mathcal{L}^+ \phi \geq \mathcal{L} \phi \), we have also \( E[\mathcal{L}^+ \phi|V_n] \geq E[\mathcal{L} \phi|V_n] = \mathcal{L} F_n \). This implies that \( E[\mathcal{L}^+ \phi|V_n] \geq \mathcal{L}^+_a F_n \). Hence we find that
\[ \sup_n E[\mathcal{L}^+_a F_n] < \infty \]
and this condition implies that the submartingale \( (\mathcal{L}_a F_n, n \geq 1) \) converges almost surely. We shall now identify the limit of this submartingale. Let \( \mathcal{L} sG \) be the singular part of the measure \( \mathcal{L} G \) for a Wiener function \( G \) such that \( \mathcal{L} G \) is a measure. We have
\[ E[\mathcal{L} \phi|V_n] = E[\mathcal{L}_a \phi|V_n] + E[\mathcal{L}_s \phi|V_n] \]
\[ = \mathcal{L}_a F_n + \mathcal{L}_s F_n, \]

hence
\[ \mathcal{L}_a F_n = E[\mathcal{L}_a \phi|V_n] + E[\mathcal{L}_s \phi|V_n]_a \]
almost surely, where \( E[\mathcal{L}_s \phi|V_n]_a \) denotes the absolutely continuous part of the measure \( E[\mathcal{L}_s \phi|V_n] \). Note that, from the Theorem of Jessen (cf., for example Theorem 1.2.1 of [101]), \( \lim_n E[\mathcal{L}_s^+ \phi|V_n]_a = 0 \) and \( \lim_n E[\mathcal{L}_s^- \phi|V_n]_a = 0 \) almost surely, hence we have
\[ \lim_n \mathcal{L}_a F_n = \mathcal{L}_a \phi, \]
\( \mu \)-almost surely. To complete the proof, an application of the Fatou lemma implies that
\[ E[f \circ T \lambda(\phi)] \leq E[f] \]
\[ = E \left[ f \circ T \frac{1}{L \circ T} \right], \]
for any $f \in C^+_b(W)$. Since $T$ is invertible, it follows that

$$\lambda(\phi) \leq \frac{1}{L \circ T}$$

almost surely. Therefore, in case $E[\lambda(\phi)] = 1$, we have

$$\lambda(\phi) = \frac{1}{L \circ T},$$

and this completes the proof. \hfill \Box

**Corollary 12.7.6** Assume that $K, L$ are two positive random variables with values in a bounded interval $[a, b] \subset (0, \infty)$ such that $E[K] = E[L] = 1$. Let $T = I_W + \nabla \phi$, $\phi \in \mathcal{D}_{2,1}$, be the transport map pushing $Kd\mu$ to $Ld\mu$, i.e., $T(Kd\mu) = Ld\mu$. We then have

$$L \circ T \lambda(\phi) \leq K,$$

$\mu$-almost surely. In particular, if $E[\lambda(\phi)] = 1$, then $T$ is the solution of the Monge-Ampère equation.

**Proof:** Since $a > 0$,

$$\frac{dT\mu}{d\mu} = \frac{L}{K \circ T} \leq \frac{b}{a}.$$ 

Hence, Theorem [12.7.12] implies that

$$E[f \circ T L \circ T \lambda(\phi)] \leq E[f L] = E[f \circ T K],$$

consequently

$$L \circ T \lambda(\phi) \leq K,$$

the rest of the claim is now obvious. \hfill \Box

For later use we give also the following result:

**Theorem 12.7.7** Assume that $L$ is a positive random variable of class $\mathcal{IL}_{\log \mathcal{IL}}$ such that $E[L] = 1$. Let $\phi \in \mathcal{D}_{2,1}$ be the 1-convex function corresponding to the transport map $T = I_W + \nabla \phi$. Define $T_t = I_W + t\nabla \phi$, where $t \in [0, 1]$. Then, for any $t \in [0, 1]$, $T_t\mu$ is absolutely continuous with respect to the Wiener measure $\mu$. 
Monge-Kantorovich

**Proof:** Let \( \phi_n \) be defined as the transport map corresponding to \( L_n = E[P_{1/n} L_n | V_n] \) and define \( T_n \) as \( I_W + \nabla \phi_n \). For \( t \in [0, 1) \), let \( T_{n,t} = I_W + t \nabla \phi_n \). It follows from the finite dimensional results which are summarized in the beginning of this section, that \( T_{n,t} \mu \) is absolutely continuous with respect to \( \mu \). Let \( L_{n,t} \) be the corresponding Radon-Nikodym density and define \( \Lambda_{n,t} \) as

\[
\Lambda_{n,t} = \det_2 \left( I_H + t \nabla^2 \phi_n \right) \exp \left\{ -t \mathcal{L}_a \phi_n - \frac{t^2}{2} |\nabla \phi_n|_H^2 \right\} .
\]

Besides, for any \( t \in [0, 1) \),

\[
\left( (I_H + t \nabla^2 \phi_n) h, h \right)_H > 0, \quad (12.7.13)
\]

\( \mu \)-almost surely for any \( 0 \neq h \in H \). Since \( \phi_n \) is of finite rank, (12.7.13) implies that \( \Lambda_{n,t} > 0 \) \( \mu \)-almost surely and we have shown at the beginning of this section

\[
\Lambda_{n,t} = \frac{1}{L_{n,t} \circ T_{n,t}}
\]

\( \mu \)-almost surely. An easy calculation shows that \( t \to \log \det_2 (I + t \nabla^2 \phi_n) \) is a non-increasing function. Since \( \mathcal{L}_a \phi_n \geq \mathcal{L} \phi_n \), we have \( E[\mathcal{L}_a \phi_n] \geq 0 \).

Consequently

\[
E \left[ L_{t,n} \log L_{t,n} \right] = E \left[ \log L_{n,t} \circ T_{n,t} \right]
\]

\[
= -E \left[ \log \Lambda_{t,n} \right]
\]

\[
= E \left[ -\log \det_2 \left( I_H + t \nabla^2 \phi_n \right) + t \mathcal{L}_a \phi_n + \frac{t^2}{2} |\nabla \phi_n|_H^2 \right]
\]

\[
\leq E \left[ -\log \det_2 \left( I_H + \nabla^2 \phi_n \right) + \mathcal{L}_a \phi_n + \frac{1}{2} |\nabla \phi_n|_H^2 \right]
\]

\[
= E \left[ L_n \log L_n \right]
\]

\[
\leq E[L \log L],
\]

by the Jensen inequality. Therefore

\[
\sup_n E[L_{n,t} \log L_{n,t}] < \infty
\]

and this implies that the sequence \((L_{n,t}, n \geq 1)\) is uniformly integrable for any \( t \in [0, 1] \). Consequently it has a subsequence which converges weakly in \( L^1(\mu) \) to some \( L_t \). Since, from Theorem 12.4.2, \( \lim_n \phi_n = \phi \) in \( D_{2,1} \), where \( \phi \) is the transport map associated to \( L \), for any \( f \in C_b(W) \), we have

\[
E[f \circ T_t] = \lim_k E[f \circ T_{n_k,t}]
\]

\[
= \lim_k E[f L_{n_k,t}]
\]

\[
= E[f L_t],
\]
hence the theorem is proved.

12.7.1 The solution of the Monge-Ampère equation via Ito-renormalization

We can interpret the Monge-Ampère equation as follows: given two probability densities $K$ and $L$, find a map $T : W \to W$ such that

$$L \circ T J(T) = K$$

almost surely, where $J(T)$ is a kind of Jacobian to be written in terms of $T$. In Corollary 12.7.6, we have shown the existence of some $\lambda(\phi)$ which gives an inequality instead of the equality. Although in the finite dimensional case there are some regularity results about the transport map (cf., [15]), in the infinite dimensional case such techniques do not work. All these difficulties can be circumvented using the miraculous renormalization of the Ito calculus. In fact assume that $K$ and $L$ satisfy the hypothesis of the corollary. First let us indicate that we can assume $W = C_0([0, 1], \mathbb{R})$ (cf., [10], Chapter II, to see how one can pass from an abstract Wiener space to the standard one) and in this case the Cameron-Martin space $H$ becomes $H^1([0, 1])$, which is the space of absolutely continuous functions on $[0, 1]$, with a square integrable Sobolev derivative. Let now

$$\Lambda = \frac{K}{L \circ T},$$

where $T$ is as constructed above. Then $\Lambda, \mu$ is a Girsanov measure for the map $T$. This means that the law of the stochastic process $(t, x) \to T_t(x)$ under $\Lambda, \mu$ is equal to the Wiener measure, where $T_t(x)$ is defined as the evaluation of the trajectory $T(x)$ at $t \in [0, 1]$. In other words the process $(t, x) \to T_t(x)$ is a Brownian motion under the probability $\Lambda, \mu$. Let $(\mathcal{F}_t^T, t \in [0, 1])$ be its filtration, the invertibility of $T$ implies that

$$\bigvee_{t \in [0, 1]} \mathcal{F}_t^T = \mathcal{B}(W).$$

$\Lambda$ is upper and lower bounded $\mu$-almost surely, hence also $\Lambda, \mu$-almost surely. The Ito representation theorem implies that it can be represented as

$$\Lambda = E[\Lambda^2] \exp \left\{ - \int_0^1 \dot{\alpha}_s dT_s - \frac{1}{2} \int_0^1 |\dot{\alpha}_s|^2 ds \right\},$$
where $\alpha(\cdot) = \int_0^\cdot \dot{\alpha}_s ds$ is an $H$-valued random variable. In fact $\alpha$ can be calculated explicitly using the Ito-Clark representation theorem, and it is given as

$$
\dot{\alpha}_t = \frac{E_\lambda[D_t\lambda|F_t^T]}{E_\lambda[\lambda|F_t^T]} \quad (12.7.14)
$$

$dt \times \Lambda d\mu$-almost surely, where $E_\lambda$ denotes the expectation operator with respect to $\Lambda, \mu$ and $D_t\lambda$ is the Lebesgue density of the absolutely continuous map $t \to \nabla \Lambda(t, x)$. From the relation (12.7.14), it follows that $\alpha$ is a function of $T$, hence we have obtained the strong solution of the Monge-Ampère equation. Let us announce all this as

**Theorem 12.7.8** Assume that $K$ and $L$ are upper and lower bounded densities, let $T$ be the transport map constructed in Theorem 12.6.5. Then $T$ is also the strong solution of the Monge-Ampère equation in the Ito sense, namely

$$
E[\lambda^2] L \circ T \exp \left\{ - \int_0^1 \dot{\alpha}_s dT_s - \frac{1}{2} \int_0^1 |\dot{\alpha}_s|^2 ds \right\} = K,
$$

$\mu$-almost surely, where $\alpha$ is given with (12.7.14).
Chapter 13

Stochastic Analysis on Lie Groups

Introduction

This chapter is a partial survey of the construction of Sobolev-type analysis on the path space of a Lie group. The word partial refers to the fact that we give some new results about the quasi-invariance of anticipative transformations and the corresponding measure theoretical degree theorems in the last section. Almost all the theory has been initiated by S. Albeverio and R. H.-Krohn ([4]), L. Gross ([38, 39]) and M. P. Malliavin and P. Malliavin ([57]). Although the study of the similar subjects has already begun in the case of manifolds (cf. [17]), we prefer to understand first the case of the Lie groups because of their relative simplicity and this will give a better idea of what is going on in former situation; since the frame of the Lie group-valued Brownian motion represents the simplest non-linear and non-trivial case in which we can construct a Sobolev type functional analysis on the space of the trajectories.

After some preliminaries in the second section we give the definitions of the basic tools in the third section, namely the left and right derivatives on the path space. The fourth section is devoted to the left divergence, in the next one we study the Ornstein-Uhlenbeck operator, Sobolev spaces and some applications like the zero-one law. Sixth section is a compilation of the formulas based essentially on the variation of the constants method of the ordinary linear differential equations which are to be used in the following sections. Section seven is devoted to the right derivative which is more technical and interesting than the left one; since it contains a rotation of the path in the sense of [96]. We also define there the skew-symmetric rotational
derivative and study some of its properties. Eighth section is devoted to the quasi-invariance at the left and at the right with respect to the multiplication of the path with deterministic paths of finite variation. Loop space case is also considered there.

Section nine deals with the absolute continuity of the path and loop measures under the transformation which consists of multiplying from the left the generic trajectory with some random, absolutely continuous and anticipative path. We prove a generalization of the Campbell-Baker-Hausdorff formula which is fundamental. To prove this we have been obliged to employ all the recent sophisticated techniques derived in the flat case. Afterwards, the extension of the Ramer and the degree theorems are immediate.

In this chapter we have focuse our attention to the probabilistic and functional analytic problems. For the more general case of Riemannian manifolds cf. [56] and the references therein.

13.1 Analytic tools on group valued paths

Let $G$ be a finite dimensional, connected, locally compact Lie group and $\mathcal{G}$ be its Lie algebra of left invariant vector fields which is isomorphic to the tangent space at identity of $G$, denoted by $T_e(G)$ which is supposed to be equipped with an inner product. $C = C_G$ denotes $C_c([0, 1], G)$ (i.e., $p(0) = e$ for $p \in C_G$). $C_G$ denotes $C_0([0, 1], \mathcal{G})$. Let

$$H = H_\mathcal{G} = \left\{ h \in C_\mathcal{G} : \int_0^1 |\dot{h}(t)|^2 dt = |h|^2 < \infty \right\}.$$ 

Our basic Wiener space is $(C_\mathcal{G}, H, \mu)$. We denote by $p(w)$ the solution of the following stochastic differential equation:

$$p_t = e + \int_0^t p_s(w) dW_s(w)$$

where the integral is in Stratonovitch sense and $W$ is the canonical Brownian motion on $C_\mathcal{G}$. In general this equation is to be understood as following: for any smooth function $f$ on $G$, we have

$$f(p_t) = f(e) + \int_0^t H_i f(p_s) dW_s^i,$$

where $(H_i)$ is a basis of $\mathcal{G}$ and $W_t^i = (H_i, W_t)$. Hence $w \mapsto p(w)$ defines a mapping from $C_\mathcal{G}$ into $C_G$ and we denote by $\nu$ the image of $\mu$ under this
mapping. Similarly, if $h \in H$ then we denote by $e(h)$ the solution of the following differential equation:

$$e_t(h) = e + \int_0^t e_s(h) \dot{h}_s ds.$$  \hfill (13.1.1)

**Theorem 13.1.1 (Campbell-Baker-Hausdorff Formula)** For any $h \in H$ the following identity is valid almost surely:

$$p(w + h) = e(\widetilde{\text{Ad}}p(w)h)p(w),$$  \hfill (13.1.2)

where $\widetilde{\text{Ad}}p(w)h$ is the $H$-valued random variable defined by

$$\left(\widetilde{\text{Ad}}p(w)h\right)(t) = \int_0^t \text{Ad}_p(s)\dot{h}(s)ds.$$  

**Remark:** In case we work with matrices, $\widetilde{\text{Ad}}p(w)h$ is defined as

$$\int_0^t p_s(w)\dot{h}(s)p^{-1}_s(w)ds.$$  

**Remark:** This theorem implies in particular that the $C_G$-valued random variable $w \to p(w)$ has a modification, denoted again by the same letter $p$, such that $h \to p(w + h)$ is a smooth function of $h \in H$ for any $w \in C_G$.

**Calculation of $\nabla (f(p_t(w)))$:** We have $f(p_t(w + \lambda h)) = f(e_t(\widetilde{\text{Ad}}p\lambda h)p_t)$ where $e_t(h), h \in H$ is defined by the equation (13.1.1). Let us write $g = p_t(w)$ and $F(x) = f(xg)$. Then

$$F(e_t(\lambda \widetilde{\text{Ad}}ph)) = F(e) + \lambda \int_0^t \text{Ad}_p \dot{h}_s F(e_s(\lambda \widetilde{\text{Ad}}ph))ds.$$  

Hence

$$\frac{d}{d\lambda}F(e_t(\lambda \widetilde{\text{Ad}}ph))|_{\lambda=0} = \int_0^t \text{Ad}_p \dot{h}_s F(e)ds.$$  

Now if $X$ is a left invariant vector field on $G$, then we have $XF(x) = X(f(xg)) = X(f(gg^{-1}xg)) = (\text{Ad}g^{-1}X)f(gx)$ by the left invariance of $X$. In particular, for $x = e$, we have $XF(e) = (\text{Ad}^{-1}X)f(g)$. Replacing $g$ with $p_t(w)$ above, we obtain

$$\nabla_h(f(p_t)) = \text{Ad}^{-1}_p \int_0^t \text{Ad}_p \dot{h}_s f(p_s)ds$$  \hfill (13.1.3)

$$= \left(\text{Ad}^{-1}_p \int_0^t \text{Ad}_p \dot{h}_s ds\right) f(p_t).$$  \hfill (13.1.4)
Notation: In the sequel, we shall denote the map $h \mapsto \int_0^1 \Ad_p \dot{h}(s) ds$ by $\theta_p h$ or by $\widetilde{\Ad} ph$ as before, depending on the notational convenience.

Definition 13.1.2 If $F : C_G \rightarrow \mathbb{R}$ is a cylindrical function, $h \in H$, we define

$$L_h F(p) = \frac{d}{d\lambda} F(e(\lambda h)p)|_{\lambda=0}$$

(13.1.5)

$$R_h F(p) = \frac{d}{d\lambda} F(pe(\lambda h))|_{\lambda=0}$$

(13.1.6)

where $p$ is a generic point of $C_G$. $L$ is called the left derivative and $R$ is called the right derivative.

A similar calculation as above gives us

$$L_h f(p_t) = \Ad_p^{-1} h_t f(p_t)$$

(13.1.7)

$$R_h f(p_t) = h_t f(p_t)$$

(13.1.8)

If $F(p) = f(p_{t_1}, \cdots, p_{t_n})$, then

$$L_h F(p) = \sum_{i=1}^n \Ad_{p_{t_i}}^{-1} h_{t_i} f(p_{t_1}, \cdots, p_{t_n})$$

(13.1.9)

$$R_h F(p) = \sum_{i=1}^n h_{t_i} f(p_{t_1}, \cdots, p_{t_n})$$

(13.1.10)

$$\nabla_h (F \circ p(w)) = \sum_{i=1}^n \Ad_{p_{t_i}}^{-1}(w) \theta_{p(w)} h_{t_i} f(p_{t_1}, \cdots, p_{t_n})(w)$$

(13.1.11)

Proposition 13.1.3 $L_h$ is a closable operator on $L^p(\nu)$ for any $p > 1$ and $h \in H$. Moreover, we have

$$(L_h F)(p(w)) = \nabla_{\theta_{p(w)}^{-1}(h)} (F(p(w))) .$$

Proof: Suppose that $(F_n)$ is a sequence of cylindrical functions on $C_G$ converging to zero in $L^p(\nu)$ and that $(L_h F_n)$ is Cauchy in $L^p(\nu)$. Then, from the formulas (7) and (9), we have

$$(L_h F_n)(p(w)) = \nabla_{\theta_{p(w)}^{-1}(h)} (F_n(p(w))) ,$$

since $\nabla$ is a closed operator on $L^p(\mu)$, we have necessarily $\lim_n L_h F_n = 0$ $\nu$-almost surely. \qed
Remark 13.1.4 On the cylindrical functions we have the identity
\[
R_h F(p(w)) = \nabla_{m(h)}(F(p(w)))
\]
where \(m(h)_t = \text{Ad}_p(w) \int_0^t \text{Ad}_p^{-1}h(s) \, ds\), but this process is not absolutely continuous with respect to \(t\), consequently, in general, the right derivative is not a closable operator without further hypothesis on the structure of \(G\), we will come back to this problem later.

Remark 13.1.5 While working with matrix groups (i.e., the linear case) we can also define all these in an alternative way (cf. also [38])
\[
L_h F(p) = \frac{d}{d\lambda} F(e^{\lambda h} p)|_{\lambda=0}
\]
\[
R_h F(p) = \frac{d}{d\lambda} F(p e^{\lambda h})|_{\lambda=0},
\]
where \(e^h\) is defined (pointwise) as \(e^h(t) = e^{h(t)}\). The advantage of this definition is that the right derivative commutes with the right multiplication (however, as we will see later the corresponding Radon-Nikodym derivative is more complicated):
\[
\frac{d}{d\lambda} F(p e^{\lambda h}) = R_h F(p e^{\lambda h}),
\]
almost surely. Let us also note the following identity which can be easily verified on the cylindrical functions:
\[
\frac{d}{d\lambda} F(p e^{\lambda h}) = R_{\theta e^{\lambda h}} F(p e^{\lambda h}),
\]
where \(\theta e^{\lambda h} k \in H\) is defined as
\[
\theta e^{\lambda h} k(t) = \int_0^t \text{Ade}_s(h) \dot{k}(s) \, ds.
\]

Remark 13.1.6 On the extended domain of \(L\), we have the identity
\[
L_h F \circ p(w) = \nabla_{\theta^{-1}_{p(w)}(h)}(F \circ p) \quad (13.1.12)
\]
\[
= (\theta^{-1}_{p(w)} \nabla (F \circ p), h) \quad (13.1.13)
\]
\[
= (\theta_{p(w)} \nabla (F \circ p), h) \quad (13.1.14)
\]
if we assume that the scalar product of \(\mathcal{G}\) is invariant with respect to the inner automorphisms, in which case \(G\) becomes of compact type, hence linear, i.e., a space of matrices and \(\theta_p\) becomes an isometry of \(H\).
Proposition 13.1.7 If $\eta : C_G \to H$ is a measurable random variable, then we have

$$(L_\eta F) \circ p = \nabla_{\theta_p^{-1}(\eta \circ p)} (F \circ p) = (\theta_p \nabla (F \circ p), \eta \circ p).$$

**Proof:** By definition, $F \in \text{Dom}(L)$ iff $F \circ p \in \text{Dom}(\nabla)$ and in this case $h \mapsto L_h F$ induces an $H$-valued random variable, denoted by $LF$. Then, for any complete orthonormal basis $(h_i, i \in \mathbb{N})$ of $H$

$$L_\eta F \circ p = \sum_i L_{h_i} F \circ p(\eta, h_i) \circ p = \sum_i \nabla_{\theta^{-1}h_i}(F \circ p)(\eta, h_i)_{H} \circ p = \sum_i \nabla_{\theta^{-1}h_i}(F \circ p)(\theta^{-1}\eta \circ p, \theta^{-1}h_i)_{H} = \nabla_{\theta_p^{-1}(\eta \circ p)}(F \circ p) = (\theta_p \nabla (F \circ p), \eta \circ p)_H.$$

13.2 The left divergence $L^*$

If $\eta : C_G \to H$ is a cylindrical random variable and if $F$ is a smooth function on $C_G$, we have

$$E_\nu[L_\eta F] = E_\mu[(L_\eta F) \circ p] = E_\mu[\nabla_{\theta^{-1}(\eta \circ p)} (F \circ p)] = E_\mu[F \circ p \delta(\theta^{-1}(\eta \circ p))].$$

Since $L$ is a closed operator, its adjoint with respect to $\nu$ is well-defined and we have

$$E_\nu[L_\eta F] = E_\eta[F L^*\eta] = E_\mu[F \circ p (L^*\eta) \circ p].$$

We have

Proposition 13.2.1 The following identity is true:

$$(L^*\eta) \circ p = \delta(\theta^{-1}(\eta \circ p)).$$
Proof: We have already tested this identity for cylindrical \( \eta \) and \( F \). To complete the proof it is sufficient to prove that the cylindrical \( F \) are dense in \( L^p(\nu) \). Then the proof will follow from the closability of \( L \). The density follows from the fact that \((p_t; t \in [0,1])\) and the Wiener process generate the same sigma algebra and from the monotone class theorem.

Lemma 13.2.2 Let \((H_t, t \in [0,1])\) be the filtration (eventually completed) of the process \((p_t, t \in [0,1])\) and \((F_t, t \in [0,1])\) be the filtration of the basic Wiener process. We have

\[
E_\nu[\phi | H_t] \circ p = E_\mu[\phi \circ p | F_t]
\]

\( \mu \)-almost surely.

Proof: Let \( f \) be a smooth function on \( \mathbb{R}^n \). Then

\[
E_\mu[\phi \circ f(p_{t_1}(w), \ldots, p_{t_n}(w))] = E_\nu[\phi f(p_{t_1}, \ldots, p_{t_n})]
\]

\[
= E_\nu[E_\nu[\phi | H_t] f(p_{t_1}, \ldots, p_{t_n})]
\]

\[
= E_\nu[E_\nu[\phi | H_t] \circ p f(p_{t_1}(w), \ldots, p_{t_n}(w))],
\]

since \( E_\nu[\phi | H_t] \circ p \) is \( \mathcal{F}_t \)-measurable, the proof follows.

If \( F \) is a nice random variable on \( C_G \) and denote by \( \pi \) the optional projection with respect to \( \mathcal{F}_t \). Using Ito-Clark representation theorem, we have

\[
F \circ p = E_\mu[F \circ p] + \delta [\pi \nabla(F \circ p)]
\]

\[
= E_\nu[F] + \delta [\theta_p^{-1} \pi \nabla(F \circ p)]
\]

\[
= E_\nu[F] + \delta [\theta_p^{-1} \pi_p \nabla(F \circ p)]
\]

\[
= E_\nu[F] + \delta [\theta_p^{-1}(\tilde{\pi}LF) \circ p]
\]

\[
= E_\nu[F] + (L^*(\tilde{\pi}LF)) \circ p
\]

\( \mu \)-almost surely, where \( \tilde{\pi} \) denotes the optional projection with respect to the filtration \((H_t)\). Consequently, we have proved the following

Theorem 13.2.3 Suppose that \( F \in L^p(\nu), p > 1 \) such that \( F \circ p \in D_{p,1} \).

Then we have

\[
F = E_\nu[F] + L^* \tilde{\pi}LF
\]

\( \nu \)-almost surely.
13.3 Ornstein-Uhlenbeck operator and the Wiener chaos

Let $F$ be a nice function on $C_G$, then

$$(L^*LF) \circ p = L^*(LF) \circ p \quad (13.3.15)$$

$$= \delta \left[ \theta_p^{-1} (LF \circ p) \right] \quad (13.3.16)$$

$$= \delta \left[ \theta^{-1} \theta (F \circ p) \right] \quad (13.3.17)$$

$$= \delta \nabla (F \circ p) \quad (13.3.18)$$

$$= \mathcal{L} (F \circ p), \quad (13.3.19)$$

where $\mathcal{L} = \delta \nabla$ is the Ornstein-Uhlenbeck operator on $W$.

**Definition 13.3.1** We denote by $\mathcal{K}$ the operator $L^*L$ and call it the Ornstein-Uhlenbeck operator on $C_G$.

Let $F$ be a cylindrical function on $G$, for $t \geq 0$, define $Q_tF(p)$ as

$$Q_tF(p(w)) = P_t(F \circ p)(w),$$

where $P_t$ is the Ornstein-Uhlenbeck semigroup on $C_G$, i.e.,

$$P_t f(w) = \int_{C_G} f(e^{-t}w + \sqrt{1 - e^{-2t}}y) \mu(dy).$$

Then it is easy to see that

$$\frac{d}{dt} Q_tF(p)|_{t=0} = -\mathcal{K}F(p).$$

Hence we can define the spaces of distributions, verify Meyer inequalities, etc. , as in the flat case (cf. [72]): Let $\phi$ be an equivalence class of random variables on $(C_G, \nu)$ with values in some separable Hilbert space $X$. For $q > 1$, $k \in \mathbb{N}$, we will say that $\phi$ is in $S_{q,k}(X)$, if there exists a sequence of cylindrical functions $(\phi_n)$ which converges to $\phi$ in $L^q(\nu, X)$ such that $(\phi_n \circ p)$ is Cauchy in $\mathbb{D}_{q,k}(X)$. For $X = \mathbb{R}$, we write simply $S_{q,k}$ instead of $S_{q,k}(\mathbb{R})$. We denote by $S(X)$ the projective limit of the spaces $(S_{q,k}; q > 1, k \in \mathbb{N})$.

Using Meyer inequalities and the fact that $w \mapsto p(w)$ is smooth in the Sobolev sense, we can show easily that, for $q > 1$, $k \in \mathbb{Z}$

1. the left derivative $L$ possesses a continuous extension from $S_{q,k}(X)$ into $S_{q,k-1}(X \otimes H)$, where

$$S_{q,k}^-(X) = \bigcup_{\epsilon > 0} S_{q,-\epsilon,k}(X).$$
2. $L^*$ has a continuous extension as a map from $S_{q,k}(X \otimes H)$ into $S_{q,k-1}(X)$.

3. Consequently $L$ maps $S(X)$ continuously into $S(X \otimes H)$ and $L^*$ maps $S(X \otimes H)$ continuously into $S(X)$.

4. By duality, $L$ and $L^*$ have continuous extensions, respectively, from $S'(X)$ to $S'(X \otimes H)$ and from $S'(X \otimes H)$ to $S'(X)$.

We can now state the $0 - 1$ law as a corollary:

**Proposition 13.3.2** Let $A \in \mathcal{B}(C_G)$ such that $A = e(h)A$ $\nu$-almost surely for any $h \in H$, then $\nu(A) = 0$ or 1.

**Proof:** It is easy to see that $L_h 1_A = 0$ (in the sense of the distributions) for any $h \in H$, hence, from Theorem 6.1.5, we obtain

$$1_A = \nu(A)$$

almost surely. $\square$

Using the calculations above we obtain

**Proposition 13.3.3** We have the following identity:

$$L^n(F \circ p) = (K^n F) \circ p$$

$\mu$-almost surely.

**Notation:** In the sequel we will denote by $\tau$ the operator $\theta_p(w)$ whenever $p(w)$ is replaced by the generic trajectory $p$ of $C_G$.

Let $F$ be a cylindrical function on $C_G$. We know that

$$F \circ p = E_\mu[F \circ p] + \sum_{i=1}^{\infty} \frac{1}{i!} \delta^n E_\mu[\nabla^n(F \circ p)].$$

On the other hand

$$\nabla(F \circ p) = \theta^{-1}(LF \circ p) = (\tau^{-1}LF) \circ p$$

$\mu$-almost surely. Iterating this identity, we obtain

$$\nabla^n(F \circ p) = ((\tau^{-1}L)^n F) \circ p.$$ 

Therefore

$$E_\mu[\nabla^n(F \circ p)] = E_\mu[((\tau^{-1}L)^n F) \circ p]$$

(13.3.20)

$$= E_\nu[(\tau^{-1}L)^n F].$$

(13.3.21)
On the other hand, for $K$ in $H^\otimes n$ (i.e., the symmetric tensor product), we have

$$E_\mu[\delta^n KH \circ p] = E_\mu[(K, \nabla^n (H \circ p))_n] = E_\mu[(K, (\tau^{-1} L)^n H) \circ p]_n = E_\nu[(K, (\tau^{-1} L)^n H)_n] = E_\nu[(L^* \tau)^n K H] = E_\mu[((L^* \tau)^n K) \circ p H \circ p],$$

for any cylindrical function $H$ on $C_G$, where $\langle \cdot, \cdot \rangle_n$ denotes the scalar product in $H^\otimes n$. We have proved the identity

$$\delta^n K = ((L^* \tau)^n K) \circ p,$$

consequently the following Wiener decomposition holds:

**Theorem 13.3.4** For any $F \in L^2(\nu)$, one has

$$F = E_\nu[F] + \sum_{n=1}^{\infty} \frac{1}{n!} (L^* \tau)^n \left( E_\nu[(\tau^{-1} L)^n F] \right)$$

where the sum converges in $L^2$.

The Ito-Clark representation theorem suggests us a second kind of Wiener chaos decomposition. First we need the following:

**Lemma 13.3.5** The set

$$\Psi = \left\{ \exp \left( L^* h - \frac{1}{2} |h|^2_H \right); \ h \in H \right\}$$

is dense in $L^p(\nu)$ for any $p \geq 1$.

**Proof:** We have

$$L^* h \circ p = \delta(\theta^{-1}_{p(w)} h) = \int_0^1 (\text{Ad}_{p(s)}^{-1} h(s), dW_s) = \int_0^1 (\dot{h}(s), \text{Ad}_{p(s)} dW_s).$$

By Paul Lévy’s theorem, $t \mapsto B_t = \int_0^t \text{Ad}_{p(s)} dW_s$ defines a Brownian motion. Hence, to prove the lemma, it suffices to show that $W$ and $B$ generate the
same filtration. To see this, note that the process \((p_t)\) satisfies the following stochastic differential equation:

\[
df(p_t) = H_i f(p_t) \, dW_t^i,
\]

\((f \in C^\infty(G))\), replacing \(dW_t^i\) by \(\text{Ad}_p dB_t^i\) we obtain

\[
df(p_t) = \text{Ad}_{p_t}^{-1} H_i f(p_t) \, dB_t^i.
\]

Since everything is smooth, we see that \(p(w)\) is measurable with respect to the filtration of \(B\). But we know that the filtrations of \(p\) and \(W\) are equal from the lemma 13.2.2.

**Remark 13.3.6** Using the Brownian motion \(B_t\) defined above we can also represent the Wiener functionals, this gives another Wiener chaos decomposition.

### 13.4 Some useful formulae

Let us first recall the variation of constant method for matrix-valued equations:

**Lemma 13.4.1** The solution of the equation

\[
\beta_t(h) = \Phi_t + \int_0^t \beta_s(h) \dot{h}(s) \, ds
\]

is given by

\[
\beta_t(h) = \Phi_0 + \left( \int_0^t \frac{d}{ds} \Phi_s e_s(h)^{-1} \, ds \right) e_t(h).
\]

**Corollary 13.4.2** We have

\[
\frac{d}{d\lambda} e_t(\lambda h) = \left( \int_0^t \text{Ade}_s(\lambda h) \dot{h}(s) \, ds \right) e_t(\lambda h)
\]

\[
= (\theta_{e(\lambda h)} h)(t) e_t(\lambda h).
\]

**Corollary 13.4.3** We have

\[
\frac{d}{d\lambda} \text{Ade}_t(\lambda h) \dot{k}_t = \left[ \int_0^t \text{Ade}_s(\lambda h) \dot{k}_s \, ds, \text{Ade}_t(\lambda h) \dot{k}_t \right].
\]
Corollary 13.4.4 We have
\[
\frac{d}{d\lambda} \operatorname{Ade}_t^{-1}(\lambda h) \dot{k}_t = -\operatorname{Ade}_t^{-1}(\lambda h) \left[ \int_0^t \operatorname{Ade}_s(\lambda h) \dot{h}_s ds, \dot{k}_t \right].
\]

Proof: Since \( \operatorname{Ade} \operatorname{Ade}^{-1} = I \), we have
\[
0 = \frac{d}{d\lambda} \operatorname{Ade}_t(\lambda h) \operatorname{Ade}_t^{-1}(\lambda h) \dot{k}_t
\]
\[
= \left( \frac{d}{d\lambda} \operatorname{Ade}_t(\lambda h) \right) \operatorname{Ade}_t^{-1}(\lambda h) \dot{k}_t + \operatorname{Ade}_t(\lambda h) \frac{d}{d\lambda} \operatorname{Ade}_t^{-1}(\lambda h) \dot{k}_t,
\]
hence
\[
\frac{d}{d\lambda} \operatorname{Ade}_t^{-1}(\lambda h) \dot{k}_t = -\operatorname{Ade}_t^{-1}(\lambda h) \left( \frac{d}{d\lambda} \operatorname{Ade}_t(\lambda h) \right) \operatorname{Ade}_t^{-1}(\lambda h) \dot{k}_t
\]
\[
= -\operatorname{Ade}_t^{-1}(\lambda h) \left[ \int_0^t \operatorname{Ade}_s(\lambda h) \dot{h}_s ds, \operatorname{Ade}_t(\lambda h) \operatorname{Ade}_t^{-1}(\lambda h) \dot{k}_t \right]
\]
\[
= -\operatorname{Ade}_t^{-1}(\lambda h) \left[ \int_0^t \operatorname{Ade}_s(\lambda h) \dot{h}_s ds, \dot{k}_t \right].
\]

In further calculations we shall need to control the terms like
\[
|\operatorname{Ade}_t^{-1}(\nu) \dot{h}_t - \operatorname{Ade}_t^{-1}(\alpha) \dot{h}_t|_G.
\]

For this, we have
\[
\operatorname{Ade}_t^{-1}(\nu) \dot{h}_t - \operatorname{Ade}_t^{-1}(\alpha) \dot{h}_t = \int_0^1 \frac{d}{d\lambda} \operatorname{Ade}_t^{-1}(\lambda(\nu + (1 - \lambda)\alpha)) \dot{h}_t d\lambda
\]
\[
= \int_0^1 \frac{d}{d\lambda} \operatorname{Ade}_t^{-1}(\lambda(\nu - \alpha) + \alpha) \dot{h}_t d\lambda.
\]

From the Corollary 6.3, we have
\[
\frac{d}{d\lambda} \operatorname{Ade}_t^{-1}(\lambda(\nu - \alpha) + \alpha) \dot{h}_t =
\]
\[
-\operatorname{Ade}_t^{-1}(\lambda(\nu - \alpha) + \alpha) \left[ \int_0^t \operatorname{Ade}_s(\lambda(\nu - \alpha) + \alpha)(\dot{\nu}_s - \dot{\alpha}_s) ds, \dot{h}_t \right].
\]
Therefore
\[
|\operatorname{Ade}_t^{-1}(\nu) \dot{h}_t - \operatorname{Ade}_t^{-1}(\alpha) \dot{h}_t|_G \leq
\]
\[
\int_0^1 \left| \left[ \int_0^t \operatorname{Ade}_s(\lambda(\nu - \alpha) + \alpha)(\dot{\nu}_s - \dot{\alpha}_s) ds, \dot{h}_t \right] \right|_G d\lambda.
\]
Now we need to control the $\mathcal{G}$-norm of the Lie brackets: for this we introduce some notations: let $(e_i)$ be a complete, orthonormal basis of $\mathcal{G}$. Since $[e_i, e_j] \in \mathcal{G}$ we should have

$$[e_i, e_j] = \sum_{k=1}^{n} \gamma^k_{ij} e_k.$$ 

For $h, k \in \mathcal{G},$

$$[h, k] = \left[ \sum_{i} h_i e_i, \sum_{i} k_i e_i \right] = \sum_{i,j} h_i k_j [e_i, e_j] = \sum_{i,j,k} h_i k_j \gamma^k_{ij}.$$ 

Consequently

$$|[h, k]|^2_{\mathcal{G}} = \sum_{l} \left[ \sum_{i,j} h_i k_j \gamma^l_{ij} \right]^2 
\leq \sum_{l} \left( \sum_{i,j} h_i^2 k_j^2 \right) \left( \sum_{i,j} (\gamma^l_{ij})^2 \right) 
= \sum_{l} |h|_G^2 |k|_G^2 |\gamma^l|_2^2 
= |h|_G^2 |k|_G^2 \sum_{l} |\gamma^l|_2^2,$$

where $| \cdot |_2$ refers to the Hilbert-Schmidt norm on $\mathcal{G}$. Although this is well-known, let us announce the above result as a lemma for later reference:

**Lemma 13.4.5** For any $h, k \in \mathcal{G}$, we have

$$|[h, k]|_{\mathcal{G}} \leq |h|_{\mathcal{G}} |k|_{\mathcal{G}} \left( \sum_{l} |\gamma^l|_2^2 \right)^{1/2}.$$ 

We have also the immediate consequence

**Lemma 13.4.6** For any $h, k \in H$

$$|\text{Ad}_{t^{-1}}(v) \dot{h}_t - \text{Ad}_{t^{-1}}(\alpha) \dot{h}_t|_{\mathcal{G}} \leq \|\gamma\|_2 |\dot{h}_t|_{\mathcal{G}} \int_{0}^{t} |\dot{v}_s - \dot{\alpha}_s|_{\mathcal{G}} ds,$$

where $\|\gamma\|_2^2 = \sum |\gamma^l|_2^2.$
Lemma 13.4.7 We have
\[ \frac{d}{d\lambda} \phi(e(\lambda h)p) = \left( L\phi(e(\lambda h)p), \tilde{A}de(\lambda h)h \right)_H. \]

Proof: We have
\[ e_t(ah)e_t(bh) = e_t(a\tilde{A}de^{-1}(bh)h + bh), \]
hence
\[ e_t(ah + bh) = e_t(a\tilde{A}de^{-1}(bh)h)e_t(bh) = e_t(b\tilde{A}de^{-1}(ah)h)e_t(ah), \]
therefore
\[ e_t((\lambda + \mu)h) = e_t(\mu\tilde{A}de(\lambda h)h)e_t(\lambda h), \]
which gives
\[ \frac{d}{d\lambda} \phi(e(\lambda h)p) = \left( L\phi(e(\lambda h)p), \tilde{A}de(\lambda h)h \right)_H. \]

13.5 Right derivative

Recall that we have defined
\[ R_h\phi(p) = \frac{d}{d\lambda} \phi(pe(\lambda h))|_{\lambda=0}. \]

Since \( G \) consists of left invariant vector fields, we have, using the global notations:
\[ R_hf(p_t) = (h_t f)(p_t), \]
where \( h_t f \) is the function obtained by applying the vector field \( h_t \) to the smooth function \( f \). The following is straightforward:

Lemma 13.5.1 We have
\[ p_t(u)e_t(h) = p_t \left( \int_0^t \tilde{A}de^{-1}(h)dW_s + h \right), \]
where \( e_t(h) \) for \( h \in H \) is defined in (13.1.1).
Lemma 13.5.2 We have

\[ E_\mu [R_h F \circ p] = E_\mu [F \circ p \int_0^1 h_s dW_s], \]

for any cylindrical function \( F \).

Proof: From the Lemma 13.5.1, \( p_t(w)e_t(\lambda h) = p_t(\lambda h + \int_0^t \text{Ade}_s^{-1}(\lambda h) dW_s) \).

Since \( \int_0^t \text{Ade}_s^{-1}(\lambda h) dW_s \) is a Brownian motion, it follows from the Girsanov theorem that

\[ E \left[ F(p(w)e(\lambda h)) \exp \left\{-\lambda \int_0^1 (\dot{h}_s, \text{Ade}_s^{-1}(\lambda h) dW_s) - \frac{\lambda^2}{2} |h|^2_H \right\} \right] = E[F], \]

differentiating at \( \lambda = 0 \) gives the result. \( \square \)

Definition 13.5.3 For \( h \in H \) and \( F \) smooth, define

- \( Q_h F(w) \) by

\[ Q_h F(w) = F \left( \int_0^1 \text{Ade}_s^{-1}(h) dW_s \right), \]

note that since \( \int_0^1 \text{Ade}_s^{-1}(h) dW_s \) is a Brownian motion, the composition of it with \( F \) is well-defined.

- And

\[ X_h F(w) = \frac{d}{d\lambda} Q_{\lambda h} F(w) \bigg|_{\lambda=0}. \]

Example 13.5.4 Let us see how the derivation operator \( X_h \) operates on the simple functional \( F = \exp \delta k, k \in H \): we have

\[ Q_{\lambda h} F = \exp \int_0^1 (\dot{k}_s, \text{Ade}_s^{-1}(\lambda h) dW_s) \]

\[ = \exp \int_0^1 (\text{Ade}_s(\lambda h) \dot{k}_s, dW_s), \]

hence

\[ X_h e^{\delta k} = e^{\delta k} \int_0^1 \left( [h(s), \dot{k}_s], dW_s \right). \]

Proposition 13.5.5 We have the following identity:

\[ (R_h F) \circ p = \nabla_h (F \circ p) + X_h (F \circ p), \]

for any \( F : C_G \to \mathbb{R} \) smooth. In particular, \( R_h \) and \( X_h \) are closable operators.
Remark 13.5.6 From the above definition, we see that

\[(R^2_h)^*1 = \delta^2 h \otimes^2 - \int_0^1 ([h_s, \dot{h}_s], dW_s) .\]

Hence \(R^n\) does not give the pure chaos but mixes them with those of lower order. Here enters the notion of universal enveloping algebra.

**Notation**: For \(h \in H\), we will denote by \(\tilde{ad}_h\) the linear operator on \(H\) defined as

\[
\tilde{ad}_h(k)(t) = \int_0^t [h_s, \dot{k}_s] \, ds .
\]

Remark 13.5.7 Suppose that \(R_h \delta k = 0\), i.e.,

\[
(h, k) + \int_0^1 [h_s, \dot{k}_s] \cdot dW_s = 0 .
\]

Then \((h, k) = 0\) and \([h(t), \dot{k}(t)] = 0\) \(dt\)-almost surely. Hence this gives more information than the independence of \(\delta h\) and \(\delta k\).

Remark 13.5.8 Suppose that \(R_h F = 0\) a.s. for any \(h \in H\). Then we have, denoting \(F = \sum I_n(f_n)\), \(R_h F = 0\) implies

\[
f_n(h) + d\Gamma(\tilde{ad}_h) f_{n-1} = 0 , \quad k \in H .
\]

Since \(f_1 = 0\) (this follows from \(E[R_h F] = E[\nabla_h F] = 0\)), we find that \(f_n(h) = 0\) for any \(h \in H\), hence \(f_n = 0\), and \(F\) is a constant.

Remark 13.5.9 If \(X_h F = 0\) for any \(h \in H\), we find that

\[
d\Gamma(\tilde{ad}_h) f_n = 0
\]

for any \(h \in H\) and for any \(n\). Therefore \(f_n\)'s take their values in the tensor spaces constructed from the center of \(G\).

Recall that in the case of an abstract Wiener space, if \(A\) is a deterministic operator on the Cameron-Martin space \(H\), then the operator \(d\Gamma(A)\) is defined on the Fock as

\[
d\Gamma(A)\phi = \frac{d}{dt} \Gamma(e^{tA})\phi|_{t=0}
\]

for any cylindrical Wiener functional \(\phi\). We will need the following result which is well-known in the Quantum Field Theory folklore:
Lemma 13.5.10 Suppose that $A$ is a skew-symmetric operator on $H$ (i.e., $A + A^* = 0$). Then we have

$$d \Gamma(A) \phi = \delta A \nabla \phi,$$

for any $\phi \in \cup_{p>1} D_{p,2}$.

Proof: By a density argument, it is sufficient to prove the identity for the functionals $\phi = \exp[\delta h - 1/2|h|^2_H]$, $h \in H$. In this case we have

$$\Gamma(e^{tA}) \phi = \exp\left\{ \delta e^{tA}h - \frac{1}{2}|e^{tA}h|^2_H \right\}$$

where the last equality follows from the fact that $e^{tA}$ is an isometry of $H$. Hence, by differentiation, we obtain

$$d \Gamma(A) \phi = \delta (Ah) \phi.$$

On the other hand

$$\delta A \nabla \phi = \delta \left[ Ah \ e^{\delta h - \frac{1}{2}|h|^2_H} \right]$$

$$= [\delta(Ah) - (Ah, h)_H] e^{\delta h - \frac{1}{2}|h|^2_H}$$

$$= \delta(Ah)e^{\delta h - \frac{1}{2}|h|^2_H},$$

since $(Ah, h)_H = 0$. 

As a corollary, we have

Corollary 13.5.11 For any cylindrical function $F$ on $(C_G, H, \mu)$, we have the following commutation relation:

$$[\nabla_h, X_k] F = -\nabla_{\text{ad}_k(h)} F,$$

where $h, k \in H$.

We have also

Proposition 13.5.12 Let $\phi$ be a cylindrical function on $(C_G, H, \mu)$ and $h \in H$. We have

$$E_{\mu}[(X_h \phi)^2] \leq ||\gamma||^2_H |h|^2_H E \left\{ |\nabla \phi|^2_H + ||\nabla^2 \phi||^2_2 \right\},$$

where $\gamma$ is the structure constant of $G$ and $|| \cdot ||_2$ denotes the Hilbert-Schmidt norm of $H \otimes H$. 

Proof: From Lemma 13.5.10 we have $X_h \phi = \delta \left( \tilde{\text{ad}}(\nabla \phi) \right)$. Hence
\[
E \left[ (X_h \phi)^2 \right] = E[|\tilde{\text{ad}} \nabla \phi|_H^2] + E \left[ \text{trace} \left( \nabla \tilde{\text{ad}} \nabla \phi \right)^2 \right].
\]

From Lemma 13.4.5 we have
\[
|\tilde{\text{ad}} \nabla \phi|_H^2 \leq \|\gamma\|_2^2 |h|_H^2 |\nabla \phi|_H^2
\]
and
\[
\text{trace} \left( \nabla \tilde{\text{ad}} \nabla \phi \right)^2 \leq \|\gamma\|_2^2 |h|_H^2 \|\nabla^2 \phi\|_2^2.
\]

Suppose that $u \in D(H)$ and define $X_u F$, where $F$ is a cylindrical function on $C_G$, as $\delta \text{ad} u \nabla F$. Then using similar calculations, we see that

Corollary 13.5.13 We have the following majoration:
\[
E[|X_u F|^2] \leq \|\gamma\|^2 E \left[ |u|_H^2 |\nabla F|_H^2 \right] + 2 \|\gamma\|^2 \left( |u|_H^2 \|\nabla^2 F\|_2^2 + \|\nabla u\|_2^2 |\nabla F|_H^2 \right).
\]

13.6 Quasi-invariance

Let $\gamma_t$ be a curve in $G$ such that $t \mapsto \gamma_t$ is absolutely continuous. We can write it as
\[
d\gamma_t = \dot{\gamma}_t dt = \gamma_t^{-1} \dot{\gamma}_t dt
\]
Hence $\gamma_t = e_t \left( \int_0^t \gamma_s^{-1} \dot{\gamma}_s ds \right)$ provided $\int_0^1 |\gamma_t^{-1} \dot{\gamma}_t|^2 dt < \infty$. Under these hypothesis, we have
\[
\gamma_t p_t(w) = p_t \left( w + \int_0^t \text{Ad} p^{-1}_s(w)(\gamma_s^{-1} \dot{\gamma}_s)(ds) \right).
\]
For any cylindrical $\phi : G \to \mathbb{R}$, we have
\[
E_{\nu} [\phi(\gamma p) J_\gamma] = E_{\nu} [\phi]
\]
where
\[
J_\gamma \circ p(w) = \exp \left\{ - \int_0^1 \left( \text{Ad} p^{-1}_s(\gamma_s^{-1} \dot{\gamma}_s), dW_s \right) - \frac{1}{2} \int_0^1 |\gamma_s^{-1} \dot{\gamma}_s|^2 ds \right\}.
\]
Similarly
\[ p_t(w)\gamma_t = p_t \left( \int_0^t \mathrm{Ad} \gamma_s^{-1} dW_s + \int_0^t \gamma_s^{-1} \dot{\gamma}_s ds \right), \]
hence
\[ E_\nu [\phi(p \gamma) K_\gamma] = E_\nu [\phi] \]
where
\[ K_\gamma \circ p(w) = \exp \left\{ - \int_0^1 (\gamma_s^{-1} \dot{\gamma}_s, \mathrm{Ad} \gamma_s^{-1} dW_s) - \frac{1}{2} \int_0^1 |\gamma_s^{-1} \dot{\gamma}_s|^2 ds \right\} \]
\[ = \exp \left\{ - \int_0^1 (\dot{\gamma}_s \gamma_s^{-1}, dW_s) - \frac{1}{2} \int_0^1 |\gamma_s^{-1} \dot{\gamma}_s|^2 ds \right\}. \quad (13.6.24) \]

As an application of these results, let us choose \( \gamma = e^h \) and denote by \( K_h \) the Radon-Nikodym density defined by
\[ E_\nu [F(p e^h)] = E[F K_h]. \]
Since \( \lambda \mapsto K_{\lambda h} \) is analytic, from Remark 13.1.5 for smooth, cylindrical \( F \), we have
\[ E[F(p e^{\lambda h})] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E[R_h^n F(p)] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E[F(p) R_h^n 1], \]
hence we have the identity
\[ K_{\lambda h} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} R_h^n 1. \]

Let us now choose \( F(p) \) of the form \( f(p_1) \), where \( f \) is a smooth function on \( G \). Then
\[ E[f(p_1 e^{\lambda h(1)})] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E[R_h^n f(p_1)] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E[h(1)^n f(p_1)]. \]
Let \( q(x) dx \) be the law of \( p_1 \) where \( dx \) is the right invariant Haar measure on \( G \). Then
\[ E[h(1)^n f(p_1)] = \int_G h(1)^n f(x) q(x) dx \]
\[ = (-1)^n \int_G f(x) \frac{h(1)^n q(x)}{q(x)} q(x) dx. \]
Hence we have proved
Proposition 13.6.1 We have the following identity:

\[ E_\nu[K_h | p_1 = x] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{h(1)^n q(x)}{q(x)} , \]

for all \( x \in G \). In particular, if \( h(1) = 0 \) then

\[ E_\nu[K_h | p_1 = x] = 1 \]

for all \( x \in G \).

Proof: The only claim to be justified is “all \( x \)” instead of almost all \( x \). This follows from the fact that \( x \mapsto E_\nu[K_h | p_1 = x] \) is continuous due to the non-degeneracy of the random variable \( p_1 \) in the sense of the Malliavin calculus.

Although the analogue of the following result is obvious in the flat case, in the case of the Lie groups, the proof requires more work:

Proposition 13.6.2 The span of \( \{K_h; h \in H\} \) is dense in \( L^r(\nu) \) for any \( r > 1 \).

Proof: Let us denote by \( \Theta \) the span of the set of the densities. Suppose that \( F \in L^s \) with \( E_\nu[F] = 0 \), where \( s \) is the conjugate of \( r \), is orthogonal to \( \Theta \). In the sequel we shall denote again by \( F \) the random variable defined as \( w \mapsto F \circ p(w) \). From the orthogonality hypothesis, we have \( E[R^n_h F] = 0 \) for any \( h \in H \) and \( n \in \mathbb{N} \) (we have not made any differentiability hypothesis about \( F \) since all these calculations are interpreted in the distributional sense). For \( n = 1 \), this gives

\[ 0 = E_\mu[\nabla_h F + X_h F] = E_\mu[\nabla_h F] , \]

since \( X_h + X^*_h = 0 \). For \( n = 2 \)

\[ 0 = E_\mu[R^2_h F] = E_\mu[\nabla_h^2 F + X_h \nabla_h F + X_h X_h F + X^2_h F] = E_\mu[\nabla_h^2 F] + E_\mu[\nabla_h X_h F] . \]

Also we have from the calculations of the first order

\[ E_\mu[\nabla_h X_h F] = E_\mu[X_h F \delta h] = -E_\mu[F \delta(\tilde{ad} h(h))] = -E_\mu[(\nabla F, \tilde{ad}(h))_H] = 0 . \]
By polarization, we deduce that, as a tensor in $H^2$, $E_\mu[\nabla^2 F] = 0$. Suppose now that $E_\mu[\nabla^i F] = 0$ for $i \leq n$. We have

$$E_\mu[R_{h+1}^n F] = E_\mu[\nabla_{h+1}^n F] + \text{supplementary terms}. $$

Between these supplementary terms, those who begin with $X_h$ or its powers have automatically zero expectation. We can show via induction hypothesis that the others are also null. For instance let us take the term $E_\mu[\nabla^h X_h^n F]$:

$$E_\mu[\nabla_h X_h^n F] = E_\mu[X_h^n F \delta h] = (-1)^n E_\mu[F \delta((\text{ad} h)^n h)] = 0,$$

the other terms can be treated similarly.

We shall apply these results to the loop measure by choosing a special form of $\gamma$. Let us first explain the strategy: replace in the above expression the random variable $\phi(p(w))$ by $\phi \circ p(w) f(p_1(w))$. Then we have

$$E_\mu[\phi(\gamma p(w)) f(\gamma p_1(w)) J_\gamma p(w)] = E_\mu[\phi \circ p(w) f(p_1(w))],$$

and

$$E_\mu[\phi(p(w) \gamma) f(p_1(\gamma(1))) K_\gamma \circ p(w)] = E_\mu[\phi \circ p(w) f(p_1(w))].$$

We shall proceed as follows: let $f : G \to \mathbb{R}$ be a smooth cylindrical function. Replace in the above expressions the map $\phi \circ p$ by $\phi \circ p f(p_1(w))$ where $f$ is a smooth function on $G$. Then we have on the one hand

$$E_\nu[\phi(\gamma p) f(\gamma(1)p_1) J_\gamma] = E_\nu[\phi(p) f(p_1)],$$

and on the other hand

$$E_\nu[\phi(p(\gamma) f(p_1(\gamma(1)))) K_\gamma] = E_\nu[\phi(p) f(p_1)] .$$

Choose $\gamma$ such that $\gamma(1) = e$ (i.e., the identity of $G$). Hence (13.6.25) becomes

$$E_\nu[\phi(\gamma p) f(p_1) J_\gamma] = E[\phi(p) f(p_1)],$$

therefore

$$\int_G E_\nu[\phi(\gamma p) J_\gamma(p)|p_1 = x] f(x) q_1(x) dx = \int_G E_\nu[\phi(p)|p_1 = x] f(x) q_1(x) dx .$$
where \( dx \) is the Haar measure on \( G \) and \( q_1 \) is the density of the law of \( p_1 \) with respect to Haar measure which is smooth and strictly positive. Consequently we obtain

\[
E_\nu[\phi(\gamma p)J_\gamma(p)|p_1 = x] = E_\nu[\phi(p)|p_1 = x].
\]

Since both sides are continuous with respect to \( x \), this equality holds everywhere. We obtain a similar result also for the right perturbation using the relation (13.6.26).

A natural candidate for \( \gamma \) for the loop measure based at \( e \), i.e., for the measure \( E_\nu[\cdot|p_1 = e] \) which we will denote by \( E_1 \), would be

\[
\gamma_t(h) = e_t(h)e_1^{-1}(th).
\]

From the calculations of the sixth section, we have

\[
\dot{\gamma}_t(h) = e_t(h)[h_t e_1^{-1}(th) - e_1^{-1}(th)(\theta_{e(th)}h)(1)].
\]

Hence

**Lemma 13.6.3** For \( \gamma_t(h) = e_t(h)e_1^{-1}(th) \), we have

\[
\gamma_t^{-1}(h)\dot{\gamma}_t(h) = \mathrm{Ad}_e(th)\dot{h}_t - (\theta_{e(th)}h)(1).
\]

In this case \( J_\gamma \) becomes

\[
J_\gamma \circ p = \exp - \int_0^1 \left( \mathrm{Ad} p_s^{-1}[\mathrm{Ad} e_1(sh)\dot{h}_s - (\theta_{e(sh)}h)(1)], dW_s \right) \exp - \frac{1}{2} \int_0^1 |\mathrm{Ad} e_1(sh)\dot{h}_s - (\theta_{e(sh)}h)(1)|^2 ds.
\]

For \( K_\gamma \) we have

\[
\mathrm{Ad} \gamma_t(\gamma_t^{-1}\dot{\gamma}_t) = \dot{\gamma}_t \gamma_t^{-1} = \mathrm{Ad}_e(th)\dot{h}_t - \mathrm{Ad}(e_t(h)e_1^{-1}(th))(\theta_{e(th)}h)(1).
\]

Since \( |\cdot| \) is \( \mathrm{Ad} \)-invariant, we have

\[
K_\gamma \circ p = \exp - \int_0^1 \left( \mathrm{Ad}_e(th)\dot{h}_t - \mathrm{Ad}(e_t(h)e_1^{-1}(th))(\theta_{e(th)}h)(1)), dW_t \right) \exp - \int_0^1 |\dot{h}_t - \mathrm{Ad}_1^{-1}(th)(\theta_{e(th)}h(1))|^2 dt.
\]

**Remark 13.6.4** Note that \( \gamma \) as chosen above satisfies the following differential equation:

\[
\dot{\gamma}_t = \gamma_t(h)[\mathrm{Ad}_e(th)\dot{h}_t - \theta_{e(th)}h(1)].
\]
Let us calculate
\[
\frac{d}{d\lambda} \phi(p_\gamma(\lambda h))|_{\lambda=0} \quad \text{and} \quad \frac{d}{d\lambda} \phi(\gamma(\lambda h)p)|_{\lambda=0}
\]
for cylindrical \( \phi \). Denote by \( P_0 : H \to H_0 \) the orthogonal projection defined by
\[
P_0 h(t) = h(t) - th(1).
\]
Then it is easy to see that
\[
\frac{d}{d\lambda} \phi(\gamma(\lambda h)p)|_{\lambda=0} = L_{P_0 h} \phi(p)
\]
and
\[
\frac{d}{d\lambda} \phi(p_\gamma(\lambda h))|_{\lambda=0} = R_{P_0 h} \phi(p).
\]
Moreover, we have
\[
\frac{d}{d\lambda} J_{\gamma(\lambda h)}(p(w))|_{\lambda=0} = - \int_0^1 \left( \text{Ad}_{p_\alpha}^{-1}(w) (\dot h_s - h(1)), dW_s \right)
\]
and
\[
\frac{d}{d\lambda} K_{\gamma(\lambda h)}(p)|_{\lambda=0} = - \int_0^1 (\dot h_s - h(1), dW_s)
\]
Consequently we have proven

**Theorem 13.6.5** For any cylindrical function \( \phi \) on the loop space of \( G \), we have
\[
E_1[L_{P_0 h} \phi] = E_1[\phi L^* P_0 h]
\]
and
\[
E_1[R_{P_0 h} \phi] = E_1[\phi \delta P_0 h]
\]
for any \( h \in H \). In particular, the operators \( L_{P_0 h} \) and \( R_{P_0 h} \) are closable on \( L^p(\nu_1) \) for any \( p > 1 \).

Before closing this section let us give a result of L. Gross (cf. [38]):

**Lemma 13.6.6** For \( \alpha < 1 \) the measure \( \nu(\cdot | p(1) = e) \) is equivalent to \( \nu \) on \( (C_G, \mathcal{H}_\alpha) \) and for any \( \mathcal{H}_\alpha \)-measurable random variable \( F \), we have
\[
E_\nu[F|p(1) = e] = E_\nu \left[ F \frac{q_{1-\alpha}(p_\alpha, e)}{q_1(e, e)} \right],
\]
where \( q_\alpha \) is the density of the law of \( p_\alpha \) with respect to the Haar measure.
Proof: Without loss of generality we can suppose that $F$ is a continuous and bounded function on $C_G$. Let $g$ be a nice function on $G$, from the Markov property, it follows that

$$E_\nu[F g(p(1))] = E\left[F \int_G q_1-\alpha(p_\alpha, y)g(y)dy\right].$$

On the other hand, from the disintegration of measures, we have

$$E_\nu[F g(p(1))] = \int_G E_\nu[F | p(1) = y]g(y)q_1(e, y)dy.$$ 

Equating both sides gives

$$E_\nu[F | p(1) = y] = \frac{1}{q_1(e, y)}E_\nu[F q_1-\alpha(p_\alpha, y)]$$

$dy$-almost surely. Since both sides are continuous in $y$ the result follows if we put $y = e$. \qed

Remark 13.6.7 Note that we have the following identity:

$$E_\nu[F(p)|p(1) = e] = E_\mu[F \circ p(w)|p_1(w) = e]$$

for any cylindrical function $F$ on $C_G$.

### 13.7 Anticipative transformations

In this section we shall study the absolute continuity of the measures which are defined as the image of $\nu$ under the mappings which are defined as the left multiplication of the path $p$ with the exponentials of anticipative $G$-valued processes. To be able to use the results of the flat case we need to extend the Campbell-Baker-Hausdorff formula to this case. We begin by recalling the following

**Definition 13.7.1** Let $(W, H, \mu)$ be an abstract Wiener space. A random variable $F$ defined on this space is said to be of class $R_{\alpha,k}^p$ if $F \in D_{q,r}$ for some $q > 1$, $r \geq 1$ and

$$\sup_{|h|_{H^r} \leq \alpha} |\nabla^k F(w + h)| \in L^p(\mu).$$

- If $p = 0$, we write $F \in R_{\alpha,k}^0$. 

We write $F \in R_{\infty,k}^p$ if the above condition holds for any $\alpha > 0$, and $F \in R_{\infty,\infty}^p$ if $F \in R_{\infty,k}^p$ for any $k \in \mathbb{N}$.

Finally, we say that $F \in R(\infty)$ if $F \in R_{\infty,\infty}^p$ for any $p > 1$.

Remark 13.7.2 The importance of this class is easy to realize: suppose that $u$ is an $H$-valued random variable, and let $F \in R_{0,\infty}^\infty$. If $(u_n)$ is a sequence of random variables of the form $\sum_{i<\infty} h_i 1_{A_i}$ converging in probability to $u$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, $h_i \in H$, then we can define $F(w + u_n(w))$ as $\sum F(w + h_i) 1_{A_i}(w)$ and evidently the sequence $(F \circ T_n)$ converges in probability, where $T_n = I_W + u_n$. Furthermore, the limit is independent of the particular choice of the elements of the equivalence class of $u$. Moreover, if we choose a sequence approximating $F$ as $F_n = E[P_{1/n}F|V_n]$, where $(h_n)$ is a complete basis of $H$, $V_n$ is the sigma algebra generated by $\delta h_1, \ldots, \delta h_n$ and $P_{1/n}$ is the Ornstein-Uhlenbeck semigroup at $t = 1/n$, then $\sup_n \sup_{|h| \leq \alpha} |\nabla^k F_n(w + h)| < \infty$ almost surely for any $\alpha > 0$, $k \in \mathbb{N}$, and we can show, using an equicontinuity argument (cf. [101]) that the limit of $(F \circ T_n)$ is measurable with respect to the sigma algebra of $T = I_W + u$.

Lemma 13.7.3 For any $t \geq 0$, the random variable $w \mapsto p_t(w)$ belongs to the class $R(\infty)$. Consequently, for any $H$-valued random variable $u$, the random variable $w \mapsto p_t(w + u(w))$ is well-defined and it is independent of the choice of the elements of the equivalence class of $u$.

Proof: In fact in [101], p.175, it has been proven that any diffusion with smooth coefficients of compact support belongs to $R_{0,\infty}^\infty$. In our particular case it is easy to see that

$$\sup_{|h| \leq \alpha} \|\nabla^k p_t(w + h)\| \in \cap p L^p(\mu)$$

for any $\alpha > 0$ and $k, n \in \mathbb{N}$, where $\|\cdot\|$ is the Euclidean norm on $M(\mathbb{R}^n) \otimes H^\otimes k$ and $M(\mathbb{R}^n)$ denotes the space of linear operators on $\mathbb{R}^n$. \hfill \Box

Lemma 13.7.4 Suppose that $\xi \in R_{\infty,\infty}^\infty(H) \cap D(H)$, and $\delta \xi \in R_{\infty,\infty}^\infty$ and that $u \in D(H)$ with $|u|_H \leq a \leq \alpha$ almost surely. Denote by $T$ the mapping $I_W + u$, then we have

$$(\delta \xi) \circ T = \delta(\xi \circ T) + (\xi \circ T, u)_H + \text{trace}(\nabla \xi \circ T \cdot \nabla u),$$

almost surely.
Proof: Let \((e_i)\) be a complete, orthonormal basis in \(H\), denote by \(V_k\) the sigma algebra generated by \(\{\delta e_1, \cdots, \delta e_k\}\), by \(\pi_k\) the orthogonal projection of \(H\) onto the vector space generated by \(\{e_1, \cdots, e_k\}\). Let \(u_n\) be defined as \(E[\pi_n P_{1/n} u|V_n]\), then \(|u_n|_H \leq a\) almost surely again. From the finite dimensional Sobolev injection theorem one can show that the map \(\phi \mapsto \phi \circ T_n\) is continuous from \(D\) into itself and we have

\[
\nabla(\phi \circ T_n) = (I + \nabla u_n) \cdot \nabla \phi \circ T_n
\]

(cf. [101]). For \(\xi\) as above, it is not difficult to show the claimed identity, beginning first with a cylindrical \(\xi\) then passing to the limit with the help of the continuity of the map \(\phi \mapsto \phi \circ T_n\). To pass to the limit with respect to \(n\), note that we have

\[
|\delta \xi \circ T_n - \delta \xi \circ T| \leq \sup_{|h|_H \leq a} \|\nabla \delta \xi(w + h)|_{H} u_n(w) - u(w)|_H,
\]

and, from the hypothesis, this sequence converges to zero in all the \(L^p\) spaces. For the other terms we proceed similarly.

Theorem 13.7.5 Let \(u\) be in \(D_{q,1}(H)\) for some \(q > 1\), then we have

\[
p_t \circ T(w) = e_t(\theta_p u)p_t(w),
\]

where \(e_t(\theta_p u)\) is the solution of the ordinary differential equation given by \(e_t = e_t \text{Ad} \theta_p u_t\).

Proof: Suppose first that \(u\) is also bounded. From Lemma 13.7.3, \(p_t\) belongs to \(R(\infty)\) hence the same thing is also true for the Stratonovitch integral \(\int_0^t p_s dW_s\). We can write the Stratonovitch integral as the sum of the Ito integral of \(p_s\) plus \(\frac{1}{2} \int_0^t C p_s ds\), where \(C\) denotes the Casimir operator (cf. [25]). Since \(\sup_{|h|_H \leq a} |e_t(\theta_p h)| \leq \exp t |h|_H\), \(t \mapsto p_t \circ T\) is almost surely continuous. Moreover, it is not difficult to see that \(\int_0^t C p_s ds\) is in \(R(\infty)\). Hence we can commute the Lebesgue integral with the composition with \(T\). Consequently we have, using Lemma 13.7.4

\[
\left(\int_0^t p_s dW_s\right) \circ T = \int_0^t p_s \circ T \delta W_s + \int_0^t \frac{1}{2} C p_s \circ T ds
\]

\[
+ \int_0^t p_s \circ T \dot{u}_s ds
\]

\[
+ \int_0^t \int_0^s (D_r p_s) \circ T D_s \dot{u}_r dr ds
\]
where $\delta W_s$ denotes the Skorohod integral and $D_s \phi$ is the notation for the Lebesgue density of the $H$-valued random variable $\nabla \phi$. We can write this expression simply as

$$
\left( \int_0^t p_s dW_s \right) \circ T = \int_0^t p_s \circ T d^c W_s + \int_0^t p_s \circ T \dot{u}_s ds,
$$

where $d^c W_s$ represents the anticipative Stratonovitch integral, i.e., we add the trace term to the divergence, whenever it is well-defined. Therefore we obtain the relation

$$
p_t \circ T = e + \int_0^t p_s \circ T d^c W_s + \int_0^t p_s \circ T \dot{u}_s ds.
$$

Let us now develop $e_t(\theta p_t u) p_t(w)$ using the Itô formula for anticipative processes (cf. [90]):

$$
e_t(\theta p_t u) p_t(w) = e + \int_0^t e_s(\theta p_t u) p_s(w) d^c W_s + \int_0^t e_s(\theta p_t u) Ad_p \dot{u}_s p_s ds
$$

$$= e + \int_0^t e_s(\theta p_t u) p_s(w) d^c W_s + \int_0^t e_s(\theta p_t u) p_s \dot{u}_s ds.
$$

Hence, both $p_t \circ T$ and $e_t(\theta p_t u) p_t$ satisfy the same anticipative stochastic differential equation with the obvious unique solution, therefore the proof is completed for the case where $u$ is bounded. To get rid of the boundedness hypothesis, let $(u_n)$ be a sequence in $D_{q,1}(H)$ converging to $u$ (with respect to $(q,1)$-Sobolev norm) such that $|u_n|_H \leq 2n+1$ and $u_n = u$ on the set $\{w : |u(w)|_H \leq n\}$. Then from the bounded case, we have $p_t(w + u_n(w)) = e_t(\theta p_t u_n)(w) p_t(w)$ almost surely. Moreover both sides of this equality converge in probability respectively to $p_t \circ T$ and $e_t(\theta p_t u) p_t$ and the proof is completed.

The following results now follow immediately from the flat case and Theorem 13.7.5 using the change of variable formula for the anticipative shifts on the abstract Wiener spaces (cf. [94]), we can prove

**Theorem 13.7.6** Suppose that $u : C_G \to H$ be a random variable such that

1. $\|Lu\|_{L^\infty(\nu, H\otimes H)} < \infty$,
2. $\|Lu\|_{op, L^\infty(\nu)} \leq c < 1$, where $c$ is a fixed constant.

Then we have

$$
E_\nu [F(e(\theta p_t u(p))p) | J_u] = E_\nu[F]
$$

for any $F \in C_b(C_G)$, where

$$
J_u = \det_2(I_H + \theta_p^{-1} Lu) \exp -L^*(\theta_p u) - \frac{1}{2} |u|^2.
$$
Proof: Let us denote by \( u'(w) \) the random variable \( u \circ p \) which is defined on \( W = C([0, 1], G) \). From Campbell-Baker-Hausdorff formula, we have
\[
p(w + u'(w)) = e(\theta_{p(w)}u'(w))p(w)
\]
(in fact here we are dealing with anticipative processes but the calculations go as if the things were adapted thanks to the Stratonovitch integral which defines the trajectory \( p \)). We know from [94] that
\[
E_\mu[F(p(w + u'(w)) | \Lambda_{u'}] = E_\mu[F(p(w))]
\]
where
\[
\Lambda_{u'} = \det_2(I_H + \nabla u'(w)) \exp -\delta u' - \frac{1}{2}|u'|^2.
\]
To complete the proof it suffices to remark that
\[
\nabla u'(w) = \nabla (u \circ p(w)) = \theta_p^{-1} Lu \circ p(w)
\]
\[
\delta u'(w) = \delta (u \circ p)(w) = L^*(\theta_p u) \circ p(w).
\]
\[\Box\]

We shall observe first the based loop space case. We need the following notations: if \( \gamma(t) \) is an absolutely continuous curve with values in \( G \), we denote by \( \kappa(\gamma) \) the curve with values in \( G \) defined by
\[
\kappa(\gamma)_t = \int_0^t \gamma_s^{-1} \dot{\gamma}_s ds,
\]
where we use, as before, the matrix notation.

**Theorem 13.7.7** Suppose that \( \gamma : [0, 1] \times C_G \to G \) be a random variable which is absolutely continuous with respect to \( dt \) and that \( \gamma(0) = \gamma(1) = e \), where \( e \) denotes the unit element of \( G \). Suppose moreover that

1. \( \|L\theta_p^{-1}\kappa(\gamma)\|_{L^\infty(\nu, H \otimes H)} < \infty \),
2. \( \|L\theta_p^{-1}\kappa(\gamma)\|_{op}\|_{L^\infty(\nu)} \leq c < 1 \),
3. \( J_{\gamma} \in S_{r,1} \) for some \( r > 1 \), where \( S_{r,1} \) is the Sobolev space on \( C_G \) which consists of the completion of the cylindrical functionals with respect to the norm \( \|\phi\|_{r,1} = \|\phi\|_{L^r(\nu)} + \|L\phi\|_{L^r(\nu, H)}. \)
Then we have

\[ E_1 [F(\gamma(p)p) | J_\gamma] = E_1[F] \]

for any \( F \in C_b(C_G) \), where

\[ J_\gamma = \det_2(I_H + \theta_p^{-1}L\theta_p^{-1}\kappa(\gamma)) \exp \left\{ -L^* \kappa(\gamma) - \frac{1}{2} |\kappa(\gamma)|^2 \right\}. \]

**Proof:** It is sufficient to take \( u = \theta_p^{-1}\kappa(\gamma) \) in the preceding theorem and then apply the usual conditioning trick to obtain

\[ E_\nu[F(\gamma(p)p)|J_\gamma|p(1) = y] = E_\nu[F|p(1) = y] \]

d\( y \)-almost surely. Note that by the hypothesis, there is some \( q > 1 \) such that \( J_\gamma \circ p \) belongs to the Sobolev space \( \mathbb{D}_{q,1} \) and \( \varepsilon_e \circ p(1) \) (\( \varepsilon_e \) denotes the Dirac measure at \( e \)) belongs to \( \bigcap_s \mathbb{D}_{s,-1} \) (cf. [103]), hence both sides of the above equality are continuous with respect to \( y \) and the proof follows. \( \square \)

### 13.7.1 Degree type results

In this section we will give some straight-forward applications of the measure theoretic degree theorem on the flat Wiener space to the path and loop spaces on the Lie group \( G \). The following theorem is a direct consequence of the results of the preceding section and the degree theory in the flat case (cf. [97, 98], [101] and Theorem 9.5.6):

**Theorem 13.7.8** Let \( \gamma : [0,1] \times C_G \to G \) be a random variable which is absolutely continuous with respect to \( dt \) and that \( \gamma(0) = e \). Suppose moreover that, for some \( a > 0 \),

1. \( J_\gamma \in L^{1+a}(\nu) \),
2. \( J_\gamma \left( I_H + \theta_p^{-1}L\theta_p^{-1}\kappa(\gamma) \right) \in L^{1+a}(\nu) \), for any \( h \in H \),
3. \( \kappa(\gamma) \in S_{r,2}(H) \), for some \( r > \frac{1+a}{a} \), where \( S_{r,2} \) is the Sobolev space of \( H \)-valued functionals as defined before.

Then we have

\[ E_\nu[F(\gamma(p)p)J_\gamma] = E_\nu[F]E_\nu[J_\gamma], \]

for any \( F \in C_b(C_G) \).

The following is a consequence of Theorem 3.2 of [98]:
**Proposition 13.7.9** Suppose that \( \kappa(\gamma) \in S_{q,1}(H) \) for some \( q > 1 \) and that
\[
\exp \left( -L^* \theta_p^{-1} \kappa(\gamma) + 1/2 \| L \theta_p^{-1} \kappa(\gamma) \|^2 \right) \in L^{1+b}(\nu),
\]
for some \( b > 1 \). Then \( E_{\nu}[J_\gamma] = 1 \).

Let us look at the loop space case:

**Proposition 13.7.10** Let \( \gamma \) be as in Theorem 13.7.8 with \( \gamma(1) = e \) and suppose moreover that \( J_\gamma \in S_{c,1} \), for some \( c > 1 \). Then
\[
E_1 [F(\gamma(p)p)J_\gamma] = E_1 [F] E_{\nu}[J_\gamma],
\]
for any smooth, cylindrical function \( F \).

**Proof:** Let \( f \) be a nice function on \( G \). From Theorem 9.4, we have
\[
E_{\nu}[F(\gamma(p)p)f(p_1)J_\gamma] = E_{\nu}[F(\gamma(p)p)f(\gamma_1(p)p_1)J_\gamma] = E_{\nu}[F(p)f(p_1)] E_{\nu}[J_\gamma]
\]
hence
\[
E_{\nu}[F(\gamma(p)p)J_\gamma|p_1 = y] = E_{\nu}[F(p)|p_1 = y] E_{\nu}[J_\gamma]
\]
d\( y \) almost surely. Since both sides are continuous with respect to \( y \), the equality remains true for every \( y \in G \). \( \square \)

**Remark 13.7.11** Note that the “degree” of \( \gamma \), namely \( E_\nu[J_\gamma] \) remains the same in both path and loop spaces.
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