A Halfspace Theorem for Mean Curvature $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract
We prove a vertical halfspace theorem for surfaces with constant mean curvature $H = \frac{1}{2}$, properly immersed in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane and $\mathbb{R}$ is the set of real numbers. The proof is a geometric application of the classical maximum principle for second order elliptic PDE, using the family of non compact rotational $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

1 Introduction

This is a revised version of the article that we submit before. There was a problem in the construction of graphical ends. We are presently working to fix it (replace the previous boundary with a planar boundary curve and use Perron method). The main geometric constructions will be maintained. Here we present the halfspace type theorem, that correspond to Section 4 of the previous article.

D. Hofmann e W. Meeks proved a beautiful theorem on minimal surfaces, the so-called "Halfspace Theorem" in [3]: there is no non planar, complete, minimal surface properly immersed in a halfspace of $\mathbb{R}^3$. We focus in this paper complete surfaces with constant mean curvature $H = \frac{1}{2}$ in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane and $\mathbb{R}$ is the set of real numbers. In the context of $H$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$, it is natural to investigate about halfspace type results.

Before stating our result we would like to emphasize that, in last years there has been work in $H$-surfaces in homogeneous 3-manifolds, in particular in the product space $\mathbb{H}^2 \times \mathbb{R}$: new examples were produced and many theoretical results as well.

Halfspace theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is false, in fact there are many vertically bounded complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [9]. On the contrary, we are able to prove the following result for $H = \frac{1}{2}$ surfaces.

Theorem 1.1 Let $S$ be a simply connected rotational surface with constant mean curvature $H = \frac{1}{2}$. Let $\Sigma$ be a complete surface with constant mean curvature $H = \frac{1}{2}$, different from a rotational simply connected one. Then, $\Sigma$ can not be properly immersed in the mean convex side of $S$. 
In [4], L. Hauswirth, H. Rosenberg and J. Spruck prove a halfspace type theorem for surfaces on one side of a horocylinder. The result in [4] is different in nature from our result because in [4], the "halfspace" is one side of a horocylinder, while for us, the "halfspace" is the mean convex side of the rotational simply connected surface. The proof of our result is a geometric application of the classical maximum principle to surfaces with constant mean curvature \( H = \frac{1}{2} \) in \( \mathbb{H}^2 \times \mathbb{R} \).

**Maximum Principle.** Let \( S_1 \) and \( S_2 \) be two connected surfaces of constant mean curvature \( H = \frac{1}{2} \). Let \( p \in S_1 \cap S_2 \) be a point such that \( S_1 \) and \( S_2 \) are tangent at \( p \), the mean curvature vectors of \( S_1 \) and \( S_2 \) at \( p \) point towards the same side and \( S_1 \) is on one side of \( S_2 \) in a neighborhood of \( p \). Then \( S_1 \) coincide with \( S_2 \) around \( p \). By analytic continuation, they coincide everywhere.

The proof of the Maximum Principle is based on the fact that a constant mean curvature surface in \( \mathbb{H}^2 \times \mathbb{R} \) locally satisfies a second order elliptic PDE (cf. [2], [1] where the author prove the Maximum Principle in \( \mathbb{R}^n \); the proof generalizes to space forms and to \( \mathbb{H}^2 \times \mathbb{R} \) as well).

We notice that our surfaces are not compact, while the classical maximum principle applies at a finite point. It will be clear in the proof of Theorem [11] that we that we are able to reduce the analysis to finite tangent points, because of the geometry of rotational surfaces of constant mean curvature \( H = \frac{1}{2} \).

Our halfspace Theorem leads to the following conjecture (strong halfspace theorem).

**Conjecture.** Let \( \Sigma_1, \Sigma_2 \) be two complete properly embedded surfaces with constant mean curvature \( H = \frac{1}{2} \), different from the rotational simply connected one. Then \( \Sigma_i \) can not lie in the mean convex side of \( \Sigma_j, i \neq j \).

For \( H > \frac{1}{\sqrt{2}} \) the conjecture is true and it is known as maximum principle at infinity (cf. [6]).

2 Vertical Halfspace Theorem

R. Sa Earp and E. Toubiana find explicit integral formulas for rotational surfaces of constant mean curvature \( H \in (0, \frac{1}{2}] \) in [8]. A careful description of the geometry of these surfaces is contained in the Appendix of [7]. Here we recall some properties of rotational surfaces of constant mean curvature \( H = \frac{1}{2} \).

For any \( \alpha \in \mathbb{R}_+ \), there exists a rotational surface \( \mathcal{H}_\alpha \) of constant mean curvature \( H = \frac{1}{2} \). For \( \alpha \neq 1 \), the surface \( \mathcal{H}_\alpha \) has two vertical ends (where a vertical end is a topological annulus, with no asymptotic point at finite height) that are graphs over the exterior of a disk \( D_\alpha \) of hyperbolic radius \( r_\alpha = |\ln \alpha| \).
By graph we mean the following: the graph of a function $u$ defined on a subset $\Omega$ of $\mathbb{H}^2$ is $\{(x, y, t) \in \Omega \times \mathbb{R} \mid t = u(x, y)\}$. When the graph has constant mean curvature $H$, $u$ satisfies the following second order elliptic PDE

$$\text{div}_\mathbb{H} \left( \frac{\nabla_\mathbb{H} u}{W_u} \right) = 2H \quad (1)$$

where $\text{div}_\mathbb{H}$, $\nabla_\mathbb{H}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_\mathbb{H} u|^2_\mathbb{H}}$, being $|\cdot|_\mathbb{H}$ the norm in $\mathbb{H}^2 \times \{0\}$.

Furthermore, up to vertical translation, one can assume that $\mathcal{H}_\alpha$ is symmetric with respect to the horizontal plane $t = 0$. For $\alpha = 1$, the surface $\mathcal{H}_1$ has only one end and it is a graph over $\mathbb{H}^2$ and it is denoted by $S$.

When $\alpha > 1$ the surface $\mathcal{H}_\alpha$ is not embedded. The self intersection set is a horizontal circle on the plane $t = 0$. For $\alpha < 1$ the surface $\mathcal{H}_\alpha$ is embedded.

For any $\alpha \in \mathbb{R}_+$, each end of the surface $\mathcal{H}_\alpha$ is the vertical graph of a function $u_\alpha$ over the exterior of a disk $D_\alpha$ of radius $r_\alpha$. The asymptotic behavior of $u_\alpha$ has the following form: $u_\alpha(\rho) \simeq \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{\sqrt{\alpha}}}, \rho \to \infty$, where $\rho$ is the hyperbolic distance from the origin. The positive number $\frac{1}{\sqrt{\alpha}} \in \mathbb{R}_+$ is called the growth of the end.

The function $u_\alpha$ is vertical along the boundary of $D_\alpha$. Furthermore the radius $r_\alpha$ is always greater or equal to zero, it is zero if and only if $\alpha = 1$ and tends to infinity as $\alpha \to 0$ or $\alpha \to \infty$. As we pointed out before, the function $u_1 = 2 \cosh (\frac{\xi}{2})$ is entire and its graph corresponds to the unique simply connected example $S$.

Notice that, any end of an immersed rotational surface ($\alpha > 1$) has growth smaller than the growth of $S$, while any end of an embedded rotational surface ($\alpha < 1$) has growth greater than the growth of $S$. 

Figure 1: $H = \frac{1}{2}$: the profile curve in the embedded and immersed case ($R = \tanh \rho$).
Theorem 1.1 is called "vertical" because the end of the surface Σ is vertical, as it is contained in the mean convex side of S.

Proof of Theorem 1.1. One can assume that the surface S is tangent to the slice $t = 0$ at the origin and it is contained in $\{t \geq 0\}$. Suppose, by contradiction, that Σ is contained in the mean convex side of S. Lift vertically S. If there is an interior contact point between Σ and the translation of S, one has a contradiction by the maximum principle. As Σ is properly immersed, Σ is asymptotic at infinity to a vertical translation of S. One can assume that the surface Σ is asymptotic to the S tangent to the slice $t = 0$ at the origin and contained in $\{t \geq 0\}$.

Let $h$ be the height of one lowest point of Σ. Denote by $S(h)$ the vertical lifting of S of ratio h. One has one of the following facts.

- $S(h)$ and Σ has a first finite contact point $p$ : this means that $S(h - \varepsilon)$ does not meet Σ at a finite point, for $\varepsilon > 0$ and then $S(h)$ and Σ are tangent at $p$ with mean curvature vector pointing in the same direction. In this case, by the maximum principle $S(h)$ and Σ should coincide. Contradiction.

- $S(h)$ and Σ meet at a point $p$, but $p$ is not a first contact point. Then, for $\varepsilon$ small enough, $S(h - \varepsilon)$ intersect Σ transversally.

Denote by $W$ the non compact subset of $\mathbb{H}^2 \times \mathbb{R}$ above S and below $S(h - \varepsilon)$. It follows from the maximum principle that there are no compact component of Σ contained in W. Denote by $\Sigma_1$ a non compact connected component of Σ contained in W. Note that the boundary of $\Sigma_1$ is contained in $S(h - \varepsilon)$. Consider the family of rotational non embedded surfaces $H_\alpha$, $\alpha > 1$. Translate each $H_\alpha$ vertically in order to have the waist on the plane $t = h - \varepsilon$. By abuse of notation, we continue to call the translation, $H_\alpha$. The surface $H_\alpha$ intersects the plane $t = h - \varepsilon$ in two circles. Denote by $\rho_\alpha$ the radius of the larger circle. Denote by $H_\alpha^+$, the part of the surface outside the cylinder of radius $\rho_\alpha$. Notice that $H_\alpha^+$ is embedded. By the geometry of the $H_\alpha^+$, when $\alpha$ is great enough, say $\alpha_0$, $H_\alpha^+$ is outside the mean convex side of S. Then, $H_\alpha^+$ does not intersect Σ. Furthermore, when $\alpha \rightarrow 1$, $H_\alpha^+$ converge to $S(h - \varepsilon)$. Now, start to decrease $\alpha$ from $\alpha_0$ to one. Before reaching $\alpha = 1$, the surface $H_\alpha^+$ first meets S and then touches $\Sigma_1$ tangentially at an interior finite point, with $\Sigma_1$ above $H_\alpha^+$. This depends on the following two facts.

- The boundary of $\Sigma_1$ lies on $S(h - \varepsilon)$ and the boundary of any of the $H_\alpha^+$ lies on the horizontal plane $t = h - \varepsilon$.

- The growth of any of the $H_\alpha^+$ is strictly smaller than the growth of S. Thus the end of $H_\alpha^+$ is outside the end of S.

The existence of an such interior tangency point is a contradiction by the maximum principle.
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