On a Type of Semi-Symmetric Non-Metric Connection on Riemannian Manifolds

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Abstract. The object of the present paper is to characterize a Riemannian manifold admitting a type of semi-symmetric non-metric connection.

1. Introduction

In 1924, Friedmann and Schouten [1] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\nabla$ on a differentiable manifold $(M^n, g)$ with Riemannian connection $\nabla$ is said to be a semi-symmetric connection if the torsion tensor $T$ of the connection $\nabla$ satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where $\eta$ is a 1-form and $\xi$ is a vector field defined by $\eta(X) = g(X,\xi)$, for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on $M^n$.

In 1932, Hayden [4] introduced the idea of semi-symmetric connections on a Riemannian manifold $(M^n, g)$. A semi-symmetric connection $\nabla$ is said to be a semi-symmetric metric connection if

$$\nabla g = 0.$$

The study of semi-symmetric metric connection was further developed by Yano [6], Amur and Pujar [7], Chaki and Konar [12], De [17] and many others.
After long gap the study of a semi-symmetric connection $\nabla$ satisfying

$$\nabla g \neq 0.$$ 

was initiated by Prvanović [9] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [14].

In 1992, Agashe and Chaffe [13] introduced and studied a semi-symmetric non-metric connection. The semi-symmetric non-metric connections was further developed by several authors such as De and Biswas [18], Biswas, De and Barua [16], De and Kamilya ([20], [21]) and many others.

In 1967, R.N.Sen and M.C.Chaki [15] studied certain curvature restrictions on a certain kind of conformally flat Riemannian space of class one and they obtained the following expression of the covariant derivative of the curvature tensor:

$$K_{ijkl} = 2\lambda_l K_{ijk}^h + \lambda_i K_{ljk}^h + \lambda_j K_{ilk}^h + \lambda_k K_{ijl}^h + \lambda^h K_{ijkl},$$

where $K_{ijkl}$ are the components of the curvature tensor $K$ with respect to the Levi-Civita connection,

$$K_{ijkl} = g_{hi} K_{ijkl}^h,$$

$\lambda_i$ is a non-zero covariant vector and "," denotes covariant differentiation with respect to the metric tensor $g_{ij}$.

Later in 1987, M.C.Chaki [10] called a manifold a pseudo symmetric manifold whose curvature tensor satisfies (1.2). In index free notation this can be stated as follows: A non-flat Riemannian manifold $(M^n, g)$, $n \geq 2$ is said to be a pseudo symmetric manifold [10] if its curvature tensor $K$ satisfies the condition

$$(D_X K)(Y, Z)W = 2A(X)K(Y, Z)W + A(Y)K(X, Z)W + A(Z)K(Y, X)W$$

$$+ A(W)K(Y, Z)X + g(K(Y, Z)W, X)U,$$

where $D$ denotes the operator of covariant differentiation with respect to the metric tensor $g$. The 1-form $A$ is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan [3]. An $n$-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. In this connection we can mention the notion of weakly symmetric manifold introduced by Tamássy and Binh [8]. Such a manifold was denoted by $(WS)_n$.

In a recent paper De and Gazi [19] introduced a type of non-flat Riemannian manifold $(M^n, g)$, $n \geq 2$ whose curvature tensor $K$ of type (1,3) satisfies the condition

$$(D_X K)(Y, Z)W = 2A(X)K(Y, Z)W + A(Y)K(X, Z)W + A(Z)K(Y, X)W$$

$$+ A(W)K(Y, Z)X + g(K(Y, Z)W, X)U,$$
On a Type of Semi-Symmetric Non-Metric

(1.3)

\[(D_X K)(Y, Z)W = [A(X) + B(X)] K(Y, Z)W + A(Y) K(X, Z)W + A(Z) K(Y, X)W + A(W) K(Y, Z)X + g(K(Y, Z)W, X)U,\]

where A, U and D have the meaning already mentioned and B is a non-zero 1-form, V is a vector field defined by B(X) = g(X, V), \(\forall X\).

Such a manifold was called an almost pseudo symmetric manifold and was denoted by \((APS)_n\).

If \(B = A\), then from the definitions it follows that \((APS)_n\) deduces to a \((PS)_n\).

In the same paper the authors constructed two non-trivial examples of \((APS)_n\). It may be mentioned that almost pseudo symmetric manifolds is not a particular case of weakly symmetric manifolds.

Let \((M^n, g)\), \((n > 3)\) be a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric, that is,

(1.4)

\[(\nabla_X T)(Y, Z) = [A(X) + B(X)] T(Y, Z) + A(Y) T(X, Z) + A(Z) T(Y, X) + g(T(Y, Z), X)U,\]

where A and B are defined earlier.

A non-flat Riemannian manifold \((M^n, g)\), \(n \geq 3\) is said to be a quasi-Einstein manifold [11] if its Ricci tensor \(\tilde{S}\) of the Levi-Civita connection is of the form

\[\tilde{S}(X, Y) = ag(X, Y) + bA(X)A(Y),\]

where \(a\) and \(b\) are smooth functions of the manifold.

In the present paper we consider a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric.

The paper is organized as follows:

After preliminaries in section 3, we first obtain the expressions of the curvature tensor and the Ricci tensor of the semi-symmetric non-metric connection. In this section we prove that if a Riemannian Manifold admits a semi-symmetric non-metric connection whose curvature tensor vanishes and the torsion tensor is almost pseudo symmetric with respect to the semi-symmetric non-metric connection, then the manifold becomes a quasi-Einstein manifold. Finally, we deal with a simply connected \((APS)_n\), \((n > 3)\) admitting such a semi-symmetric non-metric connection.
2. Preliminaries

Let $\tilde{r}$ denotes the scalar curvature of the manifold with respect to the Levi-Civita connection and $L$ denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, that is,

$$g(LX, Y) = \tilde{S}(X, Y),$$

for any vector field $X, Y$.

Contracting $Y$ in (1.3), it follows that

$$d\tilde{r}(X) = \left[ A(X) + B(X) \right] \tilde{r} + 4A(LX).$$

3. Riemannian Manifolds Admitting a Special Type of the Semi-Symmetric Non-Metric Connection

**Theorem 3.1.** If a Riemannian manifold admits a semi-symmetric non-metric connection whose curvature tensor $R$ vanishes and torsion tensor $T$ satisfies (1.4), then the manifold is a quasi-Einstein manifold.

**Proof.** Let $M$ be an $n$-dimensional Riemannian manifold with Riemannian metric $g$. If $\nabla$ is the semi-symmetric non-metric connection of a Riemannian manifold $M$, then $\nabla$ is given by [13]

$$\nabla_X Y = D_X Y + A(Y)X.$$ 

Let $R$ be the curvature tensor with respect to semi-symmetric non-metric connection. Then $R$ and $K$ are related by [13]

$$R(X, Y)Z = K(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X,$$

for all vector fields $X, Y, Z$ on $M$, where $\alpha$ is a $(0, 2)$ tensor given by

$$\alpha(X, Z) = (D_X A)(Z) - A(X)A(Z).$$

In this section we consider a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor $T$ satisfies (1.4).

From (1.1), contracting over $X$, we get

$$(C^1_i T)(Y) = (n - 1)A(Y).$$
From (3.4), it follows that
\[(\nabla_X C^1 T)(Y) = (n - 1)(\nabla_X A)(Y).\]

Contracting over \(Z\) in (1.4) and using (3.4), we obtain
\[(\nabla_X C^1 T)(Y) = (2n - 3)A(X)A(Y) + (n - 1)B(X)A(Y) + A(U)g(X, Y).\]

From (3.5) and (3.6) yields
\[(n - 1)(\nabla_X A)(Y) = (2n - 3)A(X)A(Y) + (n - 1)B(X)A(Y) + A(U)g(X, Y).\]

Using (3.1) and (3.3), it follows that
\[(\nabla_X A)(Y) = (D_X A)(Y) - A(X)A(Y) = (\alpha(X, Y)).\]

Therefore, from (3.7) and (3.8), we have
\[(\nabla_X A)(Y) = (2n - 3)A(X)A(Y) + (n - 1)B(X)A(Y) + 1\]
\[\alpha(X, Y) = \frac{n - 1}{n - 1} A(X)A(Y) + \frac{1}{n - 1} A(U)g(X, Y).\]

Using (3.9) in (3.2) yields
\[R(X, Y)Z = K(X, Y)Z + \frac{2n - 3}{n - 1} A(X)A(Z)Y + B(X)A(Z)Y + \frac{1}{n - 1} A(U)g(X, Z)Y\]
\[\quad - \frac{2n - 3}{n - 1} A(Y)A(Z)X - B(Y)A(Z)X - \frac{1}{n - 1} A(U)g(Y, Z)X.\]

From (3.10), we get
\[\tilde{R}(X, Y, Z, W) = \tilde{K}(X, Y, Z, W) + \frac{2n - 3}{n - 1} A(X)A(Z)g(Y, W) + B(X)A(Z)g(Y, W)\]
\[\quad + \frac{1}{n - 1} A(U)g(Y, Z)g(Y, W) - \frac{2n - 3}{n - 1} A(Y)A(Z)g(X, W)\]
\[\quad - \frac{1}{n - 1} A(U)g(Y, Z)g(X, W) - B(Y)A(Z)g(X, W),\]

where \(\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)\) and \(\tilde{K}(X, Y, Z, W) = g(K(X, Y)Z, W)\).

Putting \(X = W = e_i\) in (3.11) where \(\{e_i\}, 1 \leq i \leq n\) is an orthonormal basis of the tangent space at any point of the manifold \(M^n\) and then summing over \(i\), we
obtain
(3.12)
\[ S(Y, Z) = \tilde{S}(Y, Z) - (2n - 3)A(Y)A(Z) - (n - 1)B(Y)A(Z) - A(U)g(Y, Z), \]
where \( S \) be the Ricci tensor with respect to the semi-symmetric non-metric connection.

**Proposition 3.1.** If a Riemannian manifold admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric, then
(i) the curvature tensor of the semi-symmetric non-metric connection is given by (3.10),
(ii) the Ricci tensor of the semi-symmetric non-metric connection is given by (3.12),
(iii) the Ricci tensor \( S \) is symmetric if and only if \( B(Y)A(Z) = B(Z)A(Y) \).

Suppose \( R(X, Y)Z = 0 \).

Then from the above equation, we have
\[ S(Y, Z) = 0. \]

Hence the equation (3.12) reduces to
(3.13)
\[ \tilde{S}(Y, Z) = (2n - 3)A(Y)A(Z) + (n - 1)B(Y)A(Z) + A(U)g(Y, Z). \]

Since \( \tilde{S} \) is symmetric, therefore
Therefore,
(3.14)
\[ B(Y)A(Z) = B(Z)A(Y). \]

Putting \( Z = U \) in (3.14), it follows that
(3.15)
\[ B(Y) = fA(Y). \]

where \( f = \frac{B(U)}{A(U)} \).

Now using (3.17) in (3.15), we obtain
(3.16)
\[ \tilde{S}(Y, Z) = A(U)g(Y, Z) + [(2n - 3) + (n - 1)f]A(Y)A(Z). \]

Therefore , \( \tilde{S}(X, Y) = ag(X, Y) + bA(X)A(Y) \),
where \( a = A(U) \) and \( b = [(2n - 3) + (n - 1)f] \).

Hence the proof is completed. \( \Box \)

4. Special Conformally Flat \((APS)_{n}\) Admitting a Special Type of the Semi-Symmetric Non-Metric Connection
Theorem 4.1. If a \((APS)_n\) \((n > 3)\) admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes, then the manifold is a particular kind of a special conformally flat manifold, namely a subprojective manifold.

Proof. Chen and Yano [2] introduced the notion of a special conformally flat manifold which generalizes the notion of a subprojective manifold. A conformally flat manifold is called a special conformally flat manifold if the tensor \(H\) of type \((0, 2)\) defined by

\[
(4.1) \quad H(X,Y) = -\frac{1}{n-2} \tilde{S}(X,Y) + \frac{\tilde{r}}{2(n-1)(n-2)} g(X,Y),
\]

is expressible in the form

\[
H(X,Y) = -\frac{\alpha^2}{2} g(X,Y) + \beta (D_X\alpha)(D_Y\alpha),
\]

where \(\alpha\) and \(\beta\) are two scalars such that \(\alpha\) is positive. In particular, if \(\beta\) is a function of \(\alpha\) then the special conformally flat manifold is called a subprojective manifold [5].

Let us consider \((APS)_n\) admitting a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes.

Using (3.16) in (4.1), we get

\[
(4.2) \quad H(X,Y) = \frac{\tilde{r} - 2(n-1)A(U)}{2(n-1)(n-2)} g(X,Y) - \frac{2n-3+f(n-1)}{n-2} A(X)A(Y).
\]

Now, put

\[
(4.3) \quad \alpha^2 = -\frac{\tilde{r} - 2(n-1)A(U)}{(n-1)(n-2)}.
\]

From (3.16), we get

\[
(4.4) \quad \tilde{r} = (n-1)(3+f)A(U), \quad n \geq 3.
\]

Since \(\tilde{r} \neq 0\), it follows that \(\alpha^2\) will be positive provided that \(\tilde{r} < 0\).

From (3.16) and (2.1), it follows that

\[
(4.5) \quad LY = [2n-3+(n-1)f]A(Y)U + A(U)Y.
\]

From (4.5), we obtain

\[
(4.6) \quad A(LY) = (n-1)(2+f)A(U)A(Y).
\]

Using (4.6) and (3.15) in (2.3), we have

\[
(4.7) \quad d\tilde{r}(X) = [(1+f)\tilde{r} + 4(n-1)(2+f)A(U)]A(X).
\]
Let us take the covariant derivative of both side of (4.3) with respect to $X$ and using (4.7), we obtain

\[(4.8)\]
\[D_X \alpha = -\frac{(1 + f)\tilde{r} + 4(n - 1)(2 + f)A(U)}{2(n - 1)(n - 2)\alpha} A(X).\]

From (4.8), we have

\[(4.9)\]
\[A(X) = -\frac{2(n - 1)(n - 2)\alpha}{(1 + f)\tilde{r} + 4(n - 1)(2 + f)A(U)} D_X \alpha.\]

Thus, due to (4.3), (4.9) and (4.2) can be expressed in the form

\[H(X, Y) = -\frac{\alpha^2}{2} g(X, Y) + \beta(D_X \alpha)(D_Y \alpha),\]

where

\[(4.10)\]
\[\beta = -\frac{4(2n - 3) + (n - 1)f(n - 1)^2(n - 2)}{[(1 + f)\tilde{r} + 4(n - 1)(2 + f)A(U)]^2} \alpha^2.\]

In virtue of (4.10), we deduce that $\beta$ is a function of $\alpha$. Thus the theorem is proved.

**Corollary 4.1.** (2) Every simply connected subprojective space can be isometrically immersed in a Euclidean space as a hypersurface.

Moreover, using this Corollary, we can also state the following theorem:

**Theorem 4.2.** If a simply connected $(APS)_n$ $(n > 3)$ admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes, then the manifold can be isometrically immersed in a Euclidean space $E^{n+1}$ as a hypersurface.

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On a Type of Semi-Symmetric Non-Metric

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