On the cardinality of $S(n)$-spaces

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Abstract

In this paper we continue to study of properties of $S(n)$-spaces. We establish bounded on the cardinality of $S(n)$-spaces. Also we constructed the example of $S(n)$ not $\theta^n$-Urysohn space for every $n \geq 2 \in \mathbb{N}$. This is the answer to the question of F.A. Basile, N. Carlson and J. Porter.

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1. Introduction

Velichko [26] introduced the notion of $\theta$-closedness. Dikranjan and Giuli [9] introduced a more general notion $\theta^n$-closure operator and developed a theory of $S(n)$-spaces, $S(n)$-closed and $S(n)$-$\theta$-closed spaces. Many topologists actively applied this a general notion when studying nonregular Hausdorff spaces [3, 4, 6, 8, 9, 15, 16, 17, 19, 21, 24, 25]. In this paper we continue to study of properties of $S(n)$-spaces by applying notions $\theta^n$-closure and $\theta^n_0$-closure operators.

In Section 3 we introduce new cardinal functions: $sL_{\theta(n)}(X)$, $\theta(n)$-quasi-Menger number $qM_{\theta(n)}(X)$ and $s(n)$-quasi-Menger number $qM_{s(n)}(X)$ in order to extend some known cardinality bounded for Hausdorff and Urysohn spaces in the case of $S(n)$-spaces. In particular we prove the following:

- For every $S(2n)$-space $X$, $|X| \leq 2^{sL_{\theta(n)}(X)}$ (Theorem 3.6). For $n = 1$ we have Theorem 1 in [1].
For every $S(2n)$-space $X$, $|X| \leq 2^{qM_{\theta(n)}(X)\kappa_{\theta(n)}(X)}$ (Theorem 3.9). For $n = 1$ we have Theorem 3 in [1].

For every $S(2n)$-space $X$, $|X| \leq 2^{qM_{s(n)}(X)\kappa_{\theta(n)}(X)}$ (Theorem 3.21).

In Section 4 we construct the example of $S(n)$ not $\theta^n$-Urysohn space for every $n \geq 2 \in \mathbb{N}$. This is the answer to the question of Basile, Carlson and Porter in [4]. Finally, we get the example of Lindelöf $S(n)$-closed not $S(n)$-$\theta$-closed space for every $n \in \mathbb{N}$. This is the answer to the question in [22].

2. Main definitions and notation

Definition 2.1. ([9]). Suppose that $X$ is a topological space, $M \subset X$, and $x \in X$. For each $n \in \mathbb{N}$, the $\theta^n$-closure operator is defined as follows: $x \notin cl_{\theta^n}M$ if there exists a set of open neighborhoods $U_1 \subset U_2 \subset \ldots U_n$ of the point $x$ such that $clU_i \subset U_{i+1}$ for $i = 1, 2, \ldots, n - 1$ and $clU_n \cap M = \emptyset$. For $n = 0$, we put $cl_{\theta^0}M = clM$.

For $n = 1$, this definition gives the $\theta$-closure operator defined by Velichko ([26]).

A set $M$ is said to be $\theta^n$-closed if $M = cl_{\theta^n}M$. Similarly the $\theta^n$-interior of $M$ is defined and denoted by $Int_{\theta^n}M$, so $Int_{\theta^n}M = X \setminus cl_{\theta^n}(X \setminus M)$.

For any $n \in \mathbb{N}$, a point $x \in X$ is $S(n)$-separated from a subset $M$ if $x \notin cl_{\theta^n}M$. For example, $x$ is $S(0)$-separated from $M$ if $x \notin \overline{M}$.

Definition 2.2. ([3]). Let $n$ be a positive integer and $X$ be a space.

- $X$ is an $S(n)$-space (or $X$ satisfies the $S(n)$ separation axiom) if any two different points in $X$ are $S(n)$-separated;
- an open cover $\{U_\alpha\}$ of $X$ is an $S(n)$-cover if every point of $X$ is in the $\theta^n$-interior of some $U_\alpha$.

Obviously, any $S(0)$-space is $T_0$, any $S(1)$-space is Hausdorff, and any $S(2)$-space is Urysohn. In the class of topological $S(n)$-spaces, $S(n)$-closed ($S(n)$-$\theta$-closed) spaces are defined as $S(n)$-spaces which are closed (respectively, $\theta$-closed) in any ambient $S(n)$-space.

Definition 2.3. ([18]). An open set $U$ is called an $n$-hull of a set $A$ if there exists a family of open sets $U_1, U_2, \ldots, U_n = U$ such that $A \subset U_1$ and $clU_i \subset U_{i+1}$ for $i = 1, \ldots, n - 1$. 

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By a closed $n$-hull of a set $A$ we mean the closure of any $n$-hull of $A$.

All necessary definitions in theory of $S(n)$-spaces can be founded in [9, 18, 20, 22, 24, 25].

3. On cardinality bounds for $S(n)$-spaces

In [2], the cardinal function $sL(X)$ was introduced as being the smallest cardinal $\kappa$ such that for every $A \subset X$ and every open collection $\mathcal{U}$, with $\overline{A} \subset \cup \mathcal{U}$, there exists $\mathcal{V} \subset \mathcal{U}$ satisfying $|\mathcal{V}| \leq \kappa$ and $A \subset \overline{\cup \mathcal{V}}$. It was also shown that for a Hausdorff space $X$, $|X| \leq 2^{sL(X)\chi(X)}$.

O.T. Alas and Lj.D.R. Kočinac introduced the following definition.

**Definition 3.1.** ([1]) For a space $X$, $sL_\theta(X)$ is the smallest cardinal $\kappa$ such that if $A \subset X$, $\mathcal{U}$ is an open collection and $\operatorname{cl}_\theta(A) \subset \cup \mathcal{U}$, there is $\mathcal{V} \subset \mathcal{U}$ with $|\mathcal{V}| \leq \kappa$ and $A \subset \overline{\cup \mathcal{V}}$.

It is immediate that $sL_\theta(X) \leq sL(X)$ for every space $X$.

**Definition 3.2.** ([1]) For a Hausdorff space $X$, let $\kappa(X)$ be the smallest cardinal $\kappa$ such that for each point $x \in X$, there is a collection $\mathcal{V}_x$ of closed neighborhoods of $x$ so that $|\mathcal{V}_x| \leq \kappa$ and if $W$ is a closed neighborhood of $x$, then $W$ contains a member of $\mathcal{V}_x$.

It was also shown that for a Urysohn space $X$, $|X| \leq 2^{sL_\theta(X)\kappa(X)}$ (Theorem 1 in [1]).

We introduce the following definitions.

**Definition 3.3.** For a space $X$ and $n \in \mathbb{N}$, $sL_{\theta(n)}(X)$ is the smallest cardinal $\kappa$ such that if $A \subset X$, $\mathcal{U}$ is an open collection and $\operatorname{cl}_{\theta^n}(A) \subset \cup \mathcal{U}$, there is $\mathcal{V} \subset \mathcal{U}$ with $|\mathcal{V}| \leq \kappa$ and $A \subset \overline{\cup \mathcal{V}}$.

It is immediate that $sL_{\theta(n)}(X) \leq sL_{\theta(n-1)}(X) \leq \ldots \leq sL_{\theta}(X) \leq sL(X)$ for every space $X$.

**Definition 3.4.** For a space $X$ and $n \in \mathbb{N}$, let $\kappa_{\theta(n)}(X)$ be the smallest cardinal $\kappa$ such that for each point $x \in X$, there is a collection $\mathcal{V}_x$ of closed $n$-hulls of $x$ so that $|\mathcal{V}_x| \leq \kappa$ and if $W$ is a closed $n$-hull of $x$, then $W$ contains a member of $\mathcal{V}_x$.

We need the following lemma which can be easily shown.
Lemma 3.5. For a subset $A$ of a $S(2n)$-space $X$, $|cl_{\theta}(A)| \leq |A|^{\kappa_{\theta}(X)}$.

Theorem 3.6. For every $S(2n)$-space $X$, $|X| \leq 2^{s_{\theta(n)}(X)\kappa_{\theta}(X)}$.

Proof. Applying the well-known method of Pol-Sapirovska-Arhangel’skii-Grizlov [2, 11, 23], let $\tau = s_{\theta(n)}(X)\kappa_{\theta(n)}(X)$ and for each $\xi \in X$ let $B_\xi$ be a collection of closed $n$-hulls of $\xi$ such that $|B_\xi| \leq \tau$ and every closed $n$-hull $W$ of $\xi$ contains a member of $B_\xi$.

We shall define an increasing sequence $\{A_\alpha : \alpha \in \tau^+\}$ of subsets of $X$ and a sequence $\{U_\alpha : \alpha \in \tau^+\}$ of collections of open subsets of $X$ such that:

1. $|A_\alpha| \leq 2^\tau$, $\forall \alpha < \tau^+$ and $A_\alpha \supset cl_{\theta}(\cup_{\alpha < \alpha} A_\beta)$, $\forall \alpha < \tau^+$;
2. $U_\alpha = \{Int(M) : M \in \cup\{B_\xi : \xi \in cl_{\theta}(\cup_{\alpha < \alpha} A_\beta)\}\}$, $\alpha < \tau^+$;
3. If $V \in [U_\alpha]^{\leq \tau}$ and $\overline{\cup V} \neq X$, then $A_{\alpha+1} \setminus \overline{\cup V} \neq \emptyset$, $\alpha < \tau^+$.

Suppose that the sets $A_\beta U_\beta$, satisfying (1)-(3), have been defined for all $\beta < \alpha < \tau^+$ and let us define $A_\alpha$ and $U_\alpha$.

By Lemma $|cl_{\theta}(\cup_{\alpha < \alpha} A_\beta)| \leq |(\cup_{\alpha < \alpha} A_\beta)^{\kappa_{\theta}(X)}|$ and, hence, $cl_{\theta}(\cup_{\alpha < \alpha} A_\beta)$ has cardinality $\leq 2^\tau$, according to (2) $U_\alpha$ has cardinality $\leq 2^\tau$. For each $V \in [U_\alpha]^{\leq \tau}$ such that $X \setminus \overline{\cup V} \neq \emptyset$, fix a point $x_v \in X \setminus \overline{\cup V}$ and let $A_\alpha = cl_{\theta}(\cup_{\alpha < \alpha} A_\beta \cup \{x_v\})$.

Finally $A = \cup\{A_\alpha : \alpha < \tau^+\}$. Then $cl_{\theta}(A) = A$. Indeed, let $y \in X$ so that the closure of each $n$-hull of $y$ intersects $A$; then for each $F \in B_y$ there is $\alpha_F \leq \tau^+$ so that $F \cap A_{\alpha_F} \neq \emptyset$. Since $|\{\alpha_F : F \in B_y\}| \leq \tau$, there is $\psi < \tau^+$, so that $\psi > \alpha_F$ for every $F \in B_y$ and $y \in cl_{\theta}(A_\psi) \cap A$.

Now it is enough to show that $A = X$. On the contrary, there is $y \in X \setminus A$, then there is $W \in B_y$ so that $W \cap A = \emptyset$. Since $W$ is a closed $n$-hull of $y$, $W$ contains 1-hull $W_1$ of $y$. For each $x \in A$ choose a closed $n$-hull $D_x \in B_x$ so that $D_x \subset X \setminus W_1$. Since $\{Int(D_x) : x \in A\}$ is an open cover of $A = cl_{\theta}(A)$, then there is $B \subset A$, so that $|B| \leq s_{\theta(n)}(X) \leq \tau$ and $A \subset \bigcup_{x \in B} Int(D_x)$.

Since $|B| \leq \tau$, there is $\beta < \tau^+$ so that $B \subset A_\beta$ and $V = \{Int(D_x) : x \in B\}$ is a convenient collection of open sets which appears at the step $\beta + 1$. Hence, $A_{\beta+1} \setminus \overline{\cup V} \neq \emptyset$ and we have a contradiction that $A \subset \bigcup_{x \in B} Int(D_x)$.

\[ \square \]

Corollary 3.7. (Theorem 1 in [1]) For every Urysohn space $X$, $|X| \leq 2^{s_{\theta}(X)\kappa_{\theta}(X)}$.

Definition 3.8. (see [1] for $n = 1$) For a space $X$, the $\theta(n)$-quasi-Menger number $qM_{\theta(n)}(X)$ is the smallest cardinal number $\kappa$ such that for every
closed subset $A$ of $X$ and every collection $\{U_\alpha : \alpha \leq \kappa\}$ of families of open subsets of $X$ with $A \subset \bigcup_{\alpha<\kappa}(U_\alpha)$, there are finite subfamilies $\mathcal{V}_\alpha$ of $U_\alpha$, $\alpha < \kappa$, such that $A \subset \bigcup_{\alpha<\kappa}cl_{\theta^n}(\mathcal{V}_\alpha)$.

It is immediate that $qM_{\theta(n)}(X) \leq qM_{\theta(n-1)}(X) \leq \ldots \leq qM_{\theta(1)}(X) = qM_\theta(X)$ for every space $X$.

**Theorem 3.9.** For every $S(2n)$-space $X$, $|X| \leq 2^{qM_{\theta(n)}(X)\kappa_{\theta(n)}(X)}$.

**Proof.** Let $qM_{\theta(n)}(X)\kappa_{\theta(n)}(X) = \kappa$ and let for each $x \in X \mathcal{B}_x$ be a collection of closed $n$-hulls of $x$ such that $|\mathcal{B}_x| \leq \kappa$ and every closed $n$-hull of $x$ contains a member of $\mathcal{B}_x$. We shall define an increasing sequence $\{F_\alpha : \alpha \in \kappa^+\}$ of subsets of $X$ and a sequence $\{U_\alpha : \alpha \in \kappa^+\}$ of collections of open subsets of $X$ satisfying the following conditions:

1. $|F_\alpha| \leq 2^\kappa$, for every $\alpha < \kappa^+$;
2. $\mathcal{U}_\alpha = \{Int(M) : M \in \mathcal{U}_x : x \in cl_{\theta^n}(\bigcup_{\beta<\alpha}F_\beta)\}, \alpha < \kappa^+$;
3. If $V \in [\mathcal{U}_x]^{<\kappa}$ and $\bigcup V \neq X$, then $F_\alpha \setminus \bigcup V \neq \emptyset$, $\alpha < \kappa^+$.

Suppose $\alpha < \kappa^+$ and the sets $F_\beta$ and $\mathcal{U}_\beta$ satisfying (1)-(3) are already defined for all $\beta < \alpha$. We are going to define $F_\alpha$ and $\mathcal{U}_\alpha$.

Put $M_\alpha = cl_{\theta^n}(\bigcup_{\beta<\alpha}F_\beta)$. By the lemma, $|M_\alpha| \leq 2^\kappa$, hence, $|\mathcal{U}_\alpha| \leq 2^\kappa$. For every $V \in [\mathcal{U}_x]^{<\kappa}$ such that $\bigcup V \neq X$ take a point $x_V \in X \setminus \bigcup V$ and define $F_\alpha$ to be the $\theta^n$-closure of the union of $\bigcup_{\beta<\alpha}F_\beta$ with the set of all these $x_V$. Then $|F_\alpha| \leq 2^\kappa$.

Let $F = \bigcup\{F_\alpha : \alpha < \kappa^+\}$. Then $|F| \leq 2^\kappa$ and the proof will be finished if we show that $cl_{\theta^n}(F) = X$. First, we show that $cl_{\theta^n}(F) = \bigcup_{\alpha<\kappa^+}cl_{\theta^n}(F_\alpha)$. Let $x \in cl_{\theta^n}(F)$. The closure of every $n$-hull of $x$ intersects $F$, so that for each $B \in \mathcal{B}_x$ one can find some $\alpha_B < \kappa^+$ for which $B \cap F_{\alpha_B} \neq \emptyset$. Since $\kappa^+$ is a regular cardinal and $|\{\alpha_B : B \in \mathcal{B}_x\}| \leq \kappa$, there exists $\beta < \kappa^+$ such that $\beta > \alpha_B$ for every $B \in \mathcal{B}_x$ and $x \in cl_{\theta^n}(F_\beta) \subseteq F$. Note that $cl_{\theta^n}(F) = F$.

Suppose now $y \in X \setminus cl_{\theta^n}(F)$. Let $\mathcal{B}_y = \{B_y(\alpha) : \alpha < \kappa\}$. For each $\alpha < \kappa$ let $\mathcal{W}_\alpha$ be the collection of all members $W \in \bigcup\{B_x : x \in cl_{\theta^n}(F)\}$ such that $B_y(\alpha) \cap W = \emptyset$. Since $X$ is a $S(2n)$-space, $cl_{\theta^n}(F) \subseteq \bigcup_{\alpha<\kappa}\bigcup\{int(W) : W \in \mathcal{W}_\alpha\}$. As $cl_{\theta^n}(F)$ is $\theta^n$-closed, one can choose $\mathcal{V}_\alpha \in [\mathcal{W}_\alpha]^{<\omega}$ for each $\alpha < \kappa$ such that $cl_{\theta^n}(F) \subseteq \bigcup_{\alpha<\kappa}cl_{\theta^n}(\bigcup\{int(V) : V \in \mathcal{V}_\alpha\})$. Clearly, for every $\alpha < \kappa$, $\bigcup\{int(V) : V \in \mathcal{V}_\alpha\} \cap B_y(\alpha) = \emptyset$, hence, $y \notin cl_{\theta^n}(\bigcup\{int(V) : V \in \mathcal{V}_\alpha\})$. This means $y \notin \bigcup_{\alpha<\kappa}cl_{\theta^n}(\bigcup\{int(V) : V \in \mathcal{V}_\alpha\})$. There is a $\beta < \kappa^+$ such that all $\mathcal{V}_\alpha$, $\alpha < \kappa$, are contained in $\mathcal{U}_\beta$. Then by (3), $F_{\beta+1} \setminus \bigcup_{\alpha<\kappa}cl_{\theta^n}(\bigcup\{int(V) : V \in \mathcal{V}_\alpha\}) \neq \emptyset$ which is a contradiction.

$\square$
Corollary 3.10. (Theorem 3 in [1]) For every Urysohn space $X$, $|X| \leq 2^{qM_{\theta}(X)\kappa_\theta(X)}$.

Remark 3.11. Similarly, we can define the $\theta^0_0$-closure operator and obtain similar cardinality bounds for $S(2n-1)$-spaces.

Definition 3.12. Suppose that $X$ is a topological space, $M \subset X$, and $x \in X$. For each $n \in \mathbb{N}$, the $\theta^0_0$-closure operator is defined as follows: $x \notin cl_{\theta^0_0}M$ if there exists a set of open neighborhoods $U_1 \subset U_2 \subset ... U_n$ of the point $x$ such that $clU_i \subset U_{i+1}$ for $i = 1, 2, ..., n-1$ and $U_n \cap M = \emptyset$.

Definition 3.13. For a space $X$ and $n \in \mathbb{N}$, $sL_{\theta^0_0}(n)$ is the smallest cardinal $\kappa$ such that if $A$ is a $\theta^0_0$-closed subset of $X$, $U$ is an open cover of $A$, there is $V \subset U$ with $|V| \leq \kappa$ and $A \subset \bigcup V$.

Definition 3.14. For a space $X$ and $n \in \mathbb{N}$, $\kappa_{\theta^0_0}(n)$ be the smallest cardinal $\kappa$ such that for each point $x \in X$, there is a collection $V_x$ of $n$-hulls of $x$ so that $|V_x| \leq \kappa$ and if $W$ is a $n$-hull of $x$, then $W$ contains a member of $V_x$.

Lemma 3.15. For a subset $A$ of a $S(2n-1)$-space $X$, $|cl_{\theta^0_0}A| \leq |A|^{\kappa_{\theta^0_0}(n)}$.

Theorem 3.16. If $X$ is a $S(2n-1)$-space, then $|X| \leq 2^{sL_{\theta^0_0}(n)\kappa_{\theta^0_0}(n)}$.

Definition 3.17. For a space $X$, the $\theta_0(n)$-quasi-Menger number $qM_{\theta_0(n)}(X)$ is the smallest cardinal number $\kappa$ such that for every closed subset $A$ of $X$ and every collection $\{U_\alpha : \alpha \leq \kappa\}$ of families of open subsets of $X$ with $A \subset \bigcup_{\alpha < \kappa} U_\alpha$, there are finite subfamilies $V_\alpha$ of $U_\alpha$, $\alpha < \kappa$, such that $A \subset \bigcup_{\alpha < \kappa} cl_{\theta_0}U_\alpha$.

Theorem 3.18. For every $S(2n-1)$-space $X$, $|X| \leq 2^{qM_{\theta_0(n)}(X)\kappa_{\theta_0(n)}}$.

In [25], L. Stramaccia defined the notion of $S(n)$-set.

Definition 3.19. (25) Let $X$ be a topological space, $M$ a subset of $X$.

- A cover $U = \{U_\alpha : \alpha \in \Lambda\}$ of $M$ by open sets of $X$, is an $S(n)$-cover with respect to $M$, if $M \subset \bigcup \{Int_{\theta_0}U_\alpha : \alpha \in \Lambda\}$.
- $M$ is an $S(n)$-set of $X$ if every $S(n)$-cover with respect to $M$ has a finite subcover.
Definition 3.20. For a space $X$ and $n \in \mathbb{N}$, $sL_{\theta(n)}(X)$ is the smallest cardinal $\kappa$ such that if $A$ is a $\theta^n$-closed subset of $X$, $\mathcal{U}$ is an $S(n)$-cover with respect to $A$, there is $V \subset \mathcal{U}$ with $|V| \leq \kappa$ and $A \subset \bigcup V$.

Note that $sL_{\theta(n)}(X) \leq sL_{\theta(n)}(X)$ for every $n \in \mathbb{N}$.

Theorem 3.21. For every $S(2n)$-space $X$, $|X| \leq 2^{sL_{\theta(n)}(X)}$.

Proof. Note that in the proof of Theorem 3.6, $\{\text{Int}(D_x) : x \in A\}$ is an $S(n)$-cover with respect to $A$ and $A = \text{cl}_{\theta(n)}(A)$, hence, in the same way as in Theorem 3.6, we obtain a complete proof. \qed

Theorem 3.22. (Stramaccia). Let $X$ be $S(n+1)$-$\theta$-closed, $n \in \mathbb{N}$, and let $M \subset X$. $M$ is an $S(n)$-set of $X$ whenever it is $\theta^n$-closed in $X$.

Corollary 3.23. If $X$ is a $S(n+1)$-$\theta$-closed $S(2n)$-space, then $|X| \leq 2^{sL_{\theta(n)}(X)}$.

Proof. Note that in the proof of Theorem 3.6, $A = \text{cl}_{\theta(n)}(A)$, hence, by Theorem 3.22, $A$ is $S(n)$-set of $X$ and every $S(n)$-cover with respect to $A$ has a finite subcover. By Theorem 3.6, we obtain a complete proof. \qed

Corollary 3.24. If $X$ is a $S(2)$-$\theta$-closed Urysohn space, then $|X| \leq 2^{sL_{\theta(n)}(X)}$.

4. Example

Definition 4.1. ([18]). A point $x$ of $X$ is called a $\theta^0(n)$-accumulation point (a $\theta(n)$-accumulation point) for an infinite set $F$ if $|F \cap U| = |F|$ (respectively, $|F \cap U| = |F|$) for any $n$-hull $U$ of the point $x$.

Note that, for $n = 1$, any $\theta^0(1)$-accumulation point is a complete accumulation point, and any $\theta(1)$-accumulation point is a $\theta$-accumulation point.

Definition 4.2. ([18]). A topological space $X$ is said to be weakly $S(n)$-$\theta$-closed (weakly $S(n)$-closed) if any infinite subset of $X$ with regular cardinality has a $\theta^0(n)$-accumulation (respectively, $\theta(n)$-accumulation) point.

Theorem 4.3. (Theorem 1 in [18]) Let $X$ be an $S(n)$-$\theta$-closed $S(n)$-space. Then $X$ is weakly $S(n)$-$\theta$-closed space.

Theorem 4.4. (Theorem 3.6 in [22]) Let $X$ be a Lindelöf weakly $S(n)$-closed $S(n)$-space. Then $X$ is a $S(n)$-closed space.
Definition 4.5. \([\square]\) Let \(X\) be a space. For \(n \in \mathbb{N}\), the \(n\)-\(\theta\)-closure of a subset \(A\) of \(X\) is \(\text{cl}_\theta^n = \text{cl}_\theta \text{cl}_\theta \ldots \text{cl}_\theta(A)\), \(n\)-times.

Definition 4.6. \([\square]\) A space \(X\) is a \(\theta^n\)-Urysohn, for every \(n \in \mathbb{N}\), if for every \(x, y \in X\) with \(x \neq y\), there exist open subsets \(U\) and \(V\) of \(X\) with \(x \in U\) and \(y \in V\) such that \(\text{cl}_\theta^n(U) \cap \text{cl}_\theta^n(V) = \emptyset\).

In \([\square]\), Basile, Carlson and Porter posed the question: Does there exist a \(S(n)\)-space not \(\theta^n\)-Urysohn space for every \(n \geq 2 \in \mathbb{N}\)?

In \([\square]\), Osipov posed the question: Does there exist a non \(S(n)\)-\(\theta\)-closed Lindelöf \(S(n)\)-closed space for every \(n \geq 2 \in \mathbb{N}\)?

The following example answers both of these questions.

Example 4.7. Fix \(n \in \mathbb{N}\). Let \(\mathbb{R} = \bigcup_{i=1}^{2n} A_i\) where \(A_i\)'s are pairwise disjoint, each \(A_i\) is dense in \(\mathbb{R}\), \(|A_i| = 2^n\) for \(i \neq 2\). Let \(A_{2n+1}\) be a copy of \(A_1\) and let \(X_{2n+1} = \bigcup_{i=1}^{2n+1} A_i\).

If \(a, b \in \mathbb{R}\) and \(a < b\), an open base for \(X_{2n+1}\) is generated by these families of sets:

- \((1)\) if \(i \in \mathbb{N}\) is odd and \(1 \leq i \leq 2n + 1\), \((a, b) \cap A_i\) is open,
- \((2)\) if \(i \in \mathbb{N}\) is even and \(2 \leq i \leq 2n\), \((a, b) \cap (A_{i-1} \cup A_i \cup A_{i+1})\) is open.

Let \(a, b, c, d \in \mathbb{R}\), \(U = (a, b) \cap A_1\). Then \(\text{cl}_\theta(U) = \text{cl}(U) \subseteq [a, b] \cap (A_1 \cup A_2)\).

Let \(V = (c, d) \cap (A_3 \cup A_4 \cup A_5)\). Then \(\text{cl}_\theta(V) = \text{cl}(V) = [c, d] \cap (A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6)\). It follows that \(\text{cl}_\theta^n(U) = [a, b] \cap (A_1 \cup A_2 \cup A_3 \cup A_4)\). By induction, \(\text{cl}_\theta^n(U) = [a, b] \cap (A_1 \cup A_2 \cup \ldots \cup A_{2n})\). Likewise, starting from the right-hand subspace \(A_{2n+1}\) with \(U = (a, b) \cap A_{2n+1}\), we have \(\text{cl}_\theta^n(U) = [a, b] \cap (A_{2n+1} \cup A_{2n} \cup \ldots \cup A_2)\).

We have the following consequences:

- \((a)\). \(X_{2n+1}\) is a \(S(n)\)-space. Every \(x, y \in X_{2n+1}\), \(x \neq y\), are \(\theta^n\)-separated.
  For \(x, y \in X_{2n+1}\), \(x \neq y\), there are \(n\)-hull \(V(x)\) of \(x\) and \(W(y)\) of \(y\) such that
    - if \(n\) is odd then \(V(x) \cap W(y) = \emptyset\), and
    - if \(n\) is even then \(V(x) \cap W(y) = \emptyset\).
- \((b)\). Every pair of points \(x, x'\) such that \(x \in A_1\), \(x' \in A_{2n+1}\) (\(x'\) is a copy of \(x\)) are not \(\theta^n\)-Urysohn separated. Let \(a, b, c, d \in \mathbb{R}\), \(x \in (a, b)\) and \(x' \in (c, d)\), \(U = (a, b) \cap (c, d)\). Then \(\text{cl}_\theta^n(U) = \overline{U} \cap (A_1 \cup A_2 \cup \ldots \cup A_{2n})\) and
\[ cl_\theta^n(U) = \overline{U} \cap (A_{2n+1} \cup A_{2n} \cup \ldots \cup A_2). \] Thus, \( cl_\theta^n(U) \cap cl_\theta^n(U') \neq \emptyset; \) \( X_{2n+1} \) is not \( \theta^n \)-Urysohn.

(c). Consider the subspace \( Z = [0,1] \cap (\bigcup_{i=1}^{2n} A_i) \) of \( X_{2n+1} \).

Then \( Z \) is a Lindelöf \( S(n) \)-closed space, but it is not \( S(n) \)-\( \theta \)-closed space.

1. Since \( [0,1] \cap A_2 \) is subspace of \( \mathbb{R} \) and \( \mathbb{R} \) is hereditarily Lindelöf, then \( [0,1] \cap A_2 \) is Lindelöf and, hence, \( Z \) is Lindelöf.

2. Let \( a \in [0,1] \cap A_1 \). Consider a sequence \( \{a_m : m \in \mathbb{N}\} \) such that \( a_m \in [0,1] \cap A_{2n} \) for every \( m \in \mathbb{N} \) and \( \{a_m\}_{m \in \mathbb{N}} \) converges to \( a \) \((m \mapsto \infty)\) in natural topology of \( [0,1] \). Then there is a \( n \)-hull \( U(a) \) of the point \( a \) such that \( U(a) \cap \{a_n : n \in \mathbb{N}\} = \emptyset \). It follows that the set \( \{a_n : n \in \mathbb{N}\} \) has not a \( \theta^0(n) \)-accumulation point. Hence, by Theorem 4.3 \( Z \) is not \( S(n) \)-\( \theta \)-closed space.

3. Note that \( Z \) is weakly \( S(n) \)-closed space. Then, by Theorem 4.4 \( Z \) is a \( S(n) \)-closed space.

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