Hermite finite elements for convection-diffusion equations

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Abstract

This work addresses techniques to solve convection-diffusion problems based on Hermite interpolation. More specifically we extend to the case of these equations a Hermite finite element method providing flux continuity across inter-element boundaries, shown to be a well-adapted tool for simulating pure diffusion phenomena\textsuperscript{21}. We consider two methods that can be viewed as non trivial improved versions of the lowest order Raviart-Thomas mixed method\textsuperscript{18}, corresponding to its extensions to convection-diffusion problems proposed by Douglas and Roberts\textsuperscript{10}. A detailed convergence study is carried out for one of the methods, and numerical results illustrate the performance of both of them, as compared to each other and to the corresponding mixed methods.

1 Introduction

Historically Hermite finite elements have mostly been used to solve fourth order partial differential equations, because minimum continuity of solution derivatives across inter-element boundaries is required in this case. However the construction of such elements can be rather laborious, as shown in\textsuperscript{8}. It is noticeable in this respect that the recent technique of the virtual element led to feasible constructions of $C^k$ functions for $k \geq 1$ on meshes consisting of straight elements of arbitrary shape\textsuperscript{3}, though by means of polynomials of rather high degree.

On the other hand Hermite interpolation has been showing to be a good alternative to solve several kinds of field problems modeled by second order boundary value problems in many respects. An outstanding demonstration of such an assertion is provided by the isogeometric analysis (IGA) introduced about ten years ago (see e.g.\textsuperscript{12}). In this case advantage is taken from data satisfying high continuity requirements supplied by CAD, for a subsequent finite element analysis. However IGA in connection with triangular or tetrahedral meshes is incipient, in spite of the undeniable geometric flexibility of this type of partitions. This is a good reason to study Hermite finite elements methods defined upon triangles or tetrahedra to solve second order partial differential equations, which are low order and easy to implement at a time. That is what we do in this work, by focusing more particularly the representation of fluxes for the simulation of phenomena or processes of the convection-diffusion type.

In practice quantities directly depending on partial derivatives of the variable in terms of which an equation is expressed, i.e., the primal variable, are often more important than this unknown itself. Among them one might quote the flux in a porous medium flow or in heat flow. As far as methods
allowing to enforce the continuity of normal derivatives or normal fluxes across the boundaries of triangular or tetrahedral cells are concerned, both mixed finite elements and finite volumes have been playing a prominent role since long. However Hermite interpolation can also be a tool well adapted for this purpose, as shown in [21] for pure diffusion equations in highly heterogeneous media. In this work and in [20] two finite element methods of the Hermite type based on a quadratic interpolation were studied. Both have, either identical or better convergence properties than some of the classical mixed methods, such as the $RT_0$ method, i.e. the lowest order Raviart-Thomas’ [18], according to the norm under consideration, though at comparable implementation cost.

As far as the convection-diffusion equations are concerned, a rather great amount of numerical solution techniques are available today. Nevertheless the fact that these equations lie on the basis of the mathematical modeling of countless physical phenomena, keeps encouraging specialists in the search for efficient methodology to solve this class of problems. This is particularly true of convection dominated processes, in which the correct capture of sharp boundary layers often reveal demerits of widespread computational techniques, even when the problem to solve is linear.

The main purpose of this work is to carry out a complete mathematical study of the Hermite method to solve the convection-diffusion equations introduced in [22]. More specifically such a method is an extension to the convection-diffusion equations of the Hermite finite element studied in [21] that can be regarded as a variant of the $RT_0$ mixed element [20]. The method is described in Section 2, where we recall that it is uniformly stable with respect to a suitable working norm. In Section 3 we apply these results which immediately lead to a priori error estimates in the same norm. Naturally enough they are of the first order. However similarly to [20] we also prove that the error in the $L^2$-norm is in terms of the square of the mesh size, in contrast to the first order ones that hold for the mixed extension of the $RT_0$ method to convection-diffusion-reaction equations in non divergence form proposed in [10]. In Section 4 we further consider a variant of the method under study that can be viewed as the Hermite analog of the Douglas & Roberts mixed method [10], applying to the C-D equations in divergence form. This variant is slightly different from the one introduced in [23]. The convergence properties and the accuracy of both new methods are checked and compared by means of numerical experiments reported in Section 5. In Section 6 we conclude with some comments on the whole work.

Referring to [1], in the sequel we employ the following notations: $S$ being a bounded open set of $\mathbb{R}^N$, we denote the standard norm of Sobolev spaces $H^m(S)$ (resp. $W^{m,p}(S)$ for $p \geq 1$, $p \neq 2$), for any non negative integer $m$ by $|\cdot|_{m,S}$ (resp. $||\cdot||_{m,p,S}$), including $L^2(S) = H^0(S)$. The standard semi-norm of $H^m(\Omega)$ (resp. $W^{m,p}(\Omega)$ for $p \geq 1$, $p \neq 2$) is denoted by $|\cdot|_{m,S}$ (resp. $||\cdot||_{m,p,S}$). Finally $\forall S \subset \Omega$ and $\forall f,g \in L^2(S)$, $(f,g)_S := \int_S fg \, ds$ if $S \neq \Omega$ and $(f,g) := \int_\Omega fg \, dx$.

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, $N = 2, 3$, with boundary $\Gamma$, $f \in L^2(\Omega)$ be given a function $f \in L^2(\Omega)$, $K$ be a tensor assumed to be constant, symmetric and positive-definite and let $w \in [C^0(\Omega)]^N$ denote a velocity field. In this work we study as a model the following equation, assumed to have a unique solution:

\[
\text{Find } u \text{ such that } u = 0 \text{ on } \Gamma \text{ and } \\
- \nabla \cdot K \nabla u + w \cdot \nabla u = f \quad \text{in } \Omega.
\]

2 A Hermite solution method

Henceforth we assume that $\Omega$ is a polygon if $N = 2$ or a polyhedron if $N = 3$, and that we are given a finite element partition $\mathcal{T}_h$ of $\Omega$, consisting of triangles or tetrahedra according to the value of $N$, and belonging to a regular family of partitions (cf. [8]). $h$ denotes the maximum diameter of the elements of $\mathcal{T}_h$.

In the following we define two finite element spaces $U_h$ and $V_h$ associated with $\mathcal{T}_h$. Let $w_h$ be the constant field in each element of $T \in \mathcal{T}_h$ whose value in $T$ is $w(x_T)$, where $x_T$ is the position vector of the centroid of $T$, and $w_h^k$ be the standard continuous piecewise linear interpolator of $w$ at the vertices
of $\mathcal{T}_h$. We further introduce the operators $\Pi_T : L^2(T) \rightarrow L^2(T)$ given by $\Pi_T[v] := \int_T v dx/meas(T)$ for $T \in \mathcal{T}_h$, and $\Pi_0 : L^2(\Omega) \rightarrow L^2(\Omega)$ by $\Pi_0[v] = \Pi_T[v] \forall T \in \mathcal{T}_h$. Now throughout this work we will work with the following

Local algebraic structure of the Hermite finite element spaces:

Every function $v \in V_h$ (resp. $\in U_h$) is such that in each element $T \in \mathcal{T}_h$ it is expressed by

$$v|_T = \mathbf{x}^T[\mathcal{K}^{-1}[a_T \mathbf{x} + b_T]] + d, \quad (2)$$

where $\mathbf{x}$ represents the space variable, $b$ is a constant vector of $\mathbb{R}^N$ and $a$ and $d$ are two real coefficients. In every $N$-simplex $T$ we associate with a quadratic function $v$ of the form (2):

Sets $\mathcal{D}_T$ and $\mathcal{E}_T$ of local degrees of freedom for the Hermite finite element spaces:

$F$ being an edge if $N = 2$ or a face if $N = 3$ belonging to the boundary $\partial T$ of an $N$-simplex $T$, and $n_T$ being the unit normal vector on $F$ oriented in a given manner for each $F \subset \partial T$, we set:

$$\begin{align*}
\mathcal{D}_T := \{ \cup_F \mathcal{U}_F \} \cup \mathcal{U}_T & \quad \text{where} \\
\mathcal{U}_F(v) = \int_F \mathcal{K} \nabla v \cdot n_T ds/meas(F) & \\
\mathcal{U}_T(v) = \int_T v dx/meas(T) \quad (3) \\
\mathcal{E}_T := \{ \cup_F \mathcal{V}_F \} \cup \mathcal{V}_T & \quad \text{where} \\
\mathcal{V}_F(v) = \int_F (\mathcal{K} \nabla v + w_h \Pi_T[v]) \cdot n_T ds/meas(F) & \\
\mathcal{V}_T(v) = \int_T v dx/meas(T). \quad (4)
\end{align*}$$

The canonical basis functions associated with these sets of degrees of freedom are as follows. First we note that $v \in V_h$ or $\in U_h$, $\nabla v|_T$ for $T \in \mathcal{T}_h$ is expressed by $\mathcal{K}^{-1}[a_T \mathbf{x} + b_T]$ for certain $a_T \in \mathbb{R}$ and $b_T \in \mathbb{R}^N$. Then the flux variable $\mathcal{K} \nabla v|_T$ is of the form $a_T \mathbf{x} + b_T$, and from a well-known property of the lowest order Raviart-Thomas mixed element $a_T$ and $b_T$ can be uniquely determined for prescribed $\mathcal{V}_F(v)$ (resp $\mathcal{U}_F(v)$), $\forall F \subset \partial T$. Indeed by construction the flux variable for the Hermite element is locally defined by functions of the same form as for the lowest order Raviart-Thomas element. Once $a_T$ and $b_T$ are known, we determine the value of the additive constant $d_T$ to complete the expression of $v|_T$, by enforcing the condition $\Pi_T[v] = 0$. As for the basis function corresponding to the degree of freedom $\mathcal{V}_T$ (resp. $\mathcal{U}_T$), the values of $a_T$ and $b_T$ are obtained in a trivial manner as specified in (22). Then the value of $d_T$ is adjusted in such a way that the mean value of the corresponding quadratic function is one.

This should be enough to determine the $N + 2$ basis functions associated with a given $N$-simplex $T$, corresponding to the sets $\mathcal{U}_T$ and $\mathcal{V}_T$ of degrees of freedom, for spaces $U_h$ and $V_h$ respectively, since the $RT_0$ method is well-known (cf. [25]). However for the sake of clarity we exhibit them below.

$T$ being an element of $\mathcal{T}_h$ let $\mathbf{x}_i^T$ be the position vector of the $i$-th vertex $S_i^T$ of $T$, $F_i^T$ be the face of $T$ opposite to $S_i^T$ and $h_i^T$ be the length of the corresponding height of $T$, for $i = 1, \ldots, N + 1$. We have:

Local basis functions $\varphi_i^T$ for space $U_h$:

The local basis function $\varphi_i^T$ associated with the degree of freedom $\mathcal{U}_F$ and the basis function $\varphi_{N+2}^T$ associated with the degree of freedom $\mathcal{U}_T$ are given by:

$$\begin{align*}
\varphi_i^T = \mathbf{x}^T[\mathcal{K}^{-1}[a_i^T \mathbf{x} + b_i^T]] + d_i^T \quad & \text{for } i = 1, \ldots, N + 2, \text{ where} \\
\begin{cases}
a_i^T = [h_i^T]^{-1}, \\
b_i^T = - \mathbf{x}_i^T a_i^T, \\
d_i^T = - \int_T \mathbf{x}^T[\mathcal{K}^{-1}[a_i^T \mathbf{x} + b_i^T]] dx/meas(T), \\
a_{N+2}^T = 0, \\
b_{N+2}^T = (0, \ldots, 0)^T, \\
d_{N+2}^T = 1.
\end{cases}
\end{align*} \quad (5)$$

Local basis functions $\psi_i^T$ for space $V_h$:

Akin to the case of $U_h$, the $\psi_i^T$’s are functions of the form (2) for $i = 1, \ldots, N + 2$. Since by definition
the mean values of all the first $N + 1$ $\varphi^T_i$'s vanish, the local basis functions $\psi^T_i$ of $V_h$ associated with the degree of freedom $V^T_F$ for $i = 1, \ldots, N + 1$, together with its local basis function $\psi^T_{N+2}$ associated with the degree of freedom $V_T$ are given by:

$$\begin{align*}
\begin{cases}
\psi^T_i = \varphi^T_i & \text{for } i = 1, \ldots, N + 1 \\
\psi^T_{N+2} = |x_T - x|^4 K^{-1} w_h + 1.
\end{cases}
\end{align*}$$

(6)

Next we define,

Hermite finite element spaces $U_h$ and $V_h$:

Consider that for every interface $F$ (an inner edge for $N = 2$ and an inner face for $N = 3$) of two elements in $T_h$, $n_F$ is oriented in the same manner for both of them. Then every function in $v \in V_h$ (resp. $U_h$) is such that its restriction to every $T \in T_h$ is a $N + 2$ coefficient quadratic function of the form (4), whose degrees of freedom of the type $V_F$ (resp. $U_F$) coincide on both sides of every interface $F$ of a pair of elements in $T_h$.

We proceed by setting the discrete variational problem (7) below, aimed at approximating (1), whose bi-linear form $a_h$ and linear form $L_h$ are given by (8):

Find $u_h \in U_h$ such that for all $v \in V_h$

$$a_h(u_h, v) = L_h(v),$$

(7)

holds, where $\forall u \in U_h$ and $\forall v \in V_h$,

$$\begin{align*}
a_h(u, v) & := \sum_{T \in T_h} \left[(\nabla \cdot Ku - \alpha \nabla u) \cdot \nabla v_T[v]T + (\nabla u, K \nabla v + \alpha w_h \nabla v_T[v])T + (u, \nabla \cdot K \nabla v)T; \right] \\
L_h(v) & := -(f, \Pi_h[v]).
\end{align*}$$

(8)

Now let us consider the space

$$V := \{v | v \in H^1(\Omega), \nabla \cdot K \nabla v \in L^2(\Omega)\}.$$

Clearly $a_h$ can be extended to $(U_h + V) \times (V_h + V)$. Then we further introduce the functional $\| \cdot \|_h$:

$$U_h + V_h + V \to \mathbb{R} \text{ given by:}$$

$$\| v \|_h := \left[\| \Pi_h v, \Pi_h v \| + \sum_{T \in T_h} \left\{ (\nabla v, \nabla v_T) + (\nabla \cdot K \nabla v, \nabla \cdot K \nabla v)_T \right\} \right]^{1/2}.$$ 

(9)

The expression $\| \cdot \|_h$ obviously defines a norm over $V, U_h$ and $V_h$. In this manner, it is not difficult to establish the continuity of $a_h$ over $(U_h + V) \times (V_h + V)$ with a mesh independent constant $M$ (cf. the proof of Proposition 3.1 hereafter):

$$a_h(u, v) \leq M \| u \|_h \| v \|_h.$$ 

(10)

On the other hand there is no way for $a_h$ to be coercive. Hence we resort to an inf-sup condition for $a_h$ over $U_h \times V_h$ [22], which directly implies that (7) has a unique solution. More specifically the following stability result was proved in [22].

**Proposition 2.1** ([22]) *If $h$ is sufficiently small and $w \in [W^{1, \infty}(\Omega)]^N$, there exists a constant $\alpha > 0$ independent of $h$ such that*

$$\forall u \in U_h \sup_{v \in V_h \setminus \{0\}} \frac{a_h(u, v)}{\| v \|_h} \geq \alpha \| u \|_h.$$ 

(11)
In the next section we derive estimates for $\| u - u_h \|_h$ using a modified Strang Lemma for non coercive problems given in [13]. In this aim we have to consider the following auxiliary problem:

Find $u_h^* \in U_h$ such that for all $v \in V_h$

$$a_h^*(u_h^*, v) = L_h(v), \tag{12}$$

holds, where $\forall u \in U_h + V$ and $\forall v \in V_h + V$,

$$a_h^*(u, v) := \sum_{T \in T_h} \left( [(\nabla \cdot K\nabla u - w \cdot \nabla u, \Pi_T[v])_T + (\nabla u, K\nabla v + w, \Pi_T[v])_T + (u, \nabla \cdot K\nabla v)_T \right). \tag{13}$$

Similarly to the case of problem (7) (cf. [22]) we can prove,

**Theorem 2.2** Problem (12) has a unique solution and moreover there exists a constant $C^*$ independent of $h$ such that

$$\| u_h^* \|_h \leq C^* \| f \|_{0, \Omega}. \tag{14}$$

Before proving **Theorem 2.2** we establish a stability result for problem (12), namely,

**Proposition 2.3** If $h$ is sufficiently small and $w \in [W^{1, \infty}(\Omega)]^N$, there exists a constant $\alpha^* > 0$ independent of $h$ such that

$$\forall u \in U_h \quad \sup_{v \in V_h \setminus \{0\}} \frac{a_h^*(u, v)}{\| v \|_h} \geq \alpha^* \| u \|_h. \tag{15}$$

**Proof:** Given $u \in U_h$ define $v := v_1 + v_2 + v_3$, where $v_i \in V_h$ for $i = 1, 2, 3$ are defined as follows:

$v_1 = \theta_1 w_1,$ $\theta_1$ being a non negative constant to be specified, and $w_1$ being defined by $\Pi_h[w_1] = \Pi_h[u],$

$\Pi_h[w_3]$ is defined by applying Theorem 4 of [18]. According to it there exists a field $p \in Q_h := \{ q \mid \exists u \in U_h \text{ such that } q|_T = K\nabla u|_T \forall T \in T_h \}$, satisfying for a constant $\tilde{C}$ independent of $h$:

$$\nabla \cdot p = \Pi_h[u] \text{ in } \Omega; \quad \| p \|_{0, \Omega} \leq \tilde{C} \| \Pi_h[u] \|_{0, \Omega}. \tag{16}$$

Then recalling that the normal traces over the faces of the elements in $T_h$ of fields belonging to $Q_h$ are constant [18] $v_3$ is defined in such a way that $\forall T \in T_h, (K\nabla v_3 + w, \Pi_T[v_3]) \cdot n_T = p \cdot n_T \forall T \in T_h$ and $\Pi_h[v_3] = -\theta_1 \Pi_h[u]$.

It is clear that $\nabla \cdot K\nabla v_1 = \nabla \cdot K\nabla v$. Moreover by construction we have,

$$\int_{\partial T} K\nabla w_1 \cdot n_T w_1 \, dS - (\nabla \cdot K w_1, w_1)_T = (K\nabla w_1, \nabla w_1)_T = (K\nabla w - w, \Pi_T[u], \nabla w)_T \forall T \in T_h. \tag{17}$$

Then $\lambda$ and $\Lambda$ being the smallest and the largest eigenvalue of $K$, after straightforward manipulations it follows that

$$\| \nabla w_1 \|_{0, T} \leq \lambda^{-1}(\Lambda \| \nabla w \|_{0, T} + \| w \|_{0, \infty, \Omega}) \| \Pi_T[u] \|_{0, T}. \tag{18}$$

This implies that for $\tilde{C}_1 = \lambda^{-1}(\Lambda^2 + \| w \|_{0, \infty, \Omega}^2)^{1/2}, \sum_{T \in T_h} \| \nabla w_1 \|_{0, T}^2 \leq \tilde{C}_1^2 \| u \|_h^2$, which immediately yields,

$$\| v_1 \|_h \leq C_1 \| u \|_h, \text{ with } C_1 = \theta_1(1 + \tilde{C}_1)^{1/2}. \tag{19}$$

As for $v_2$ we readily have,

$$\| v_2 \|_h \leq C_2 \| u \|_h, \text{ with } C_2 = \theta_2. \tag{20}$$
On the other hand by construction $v_3$ fulfills $\nabla \cdot [K \nabla v_3]_T = \nabla \cdot p_T = \Pi_T[u]$, $\forall T \in T_h$, and hence,
\[
\lambda \| \nabla v_3 \|^2_{0,T} \leq (K \nabla v_3, \nabla v_3)_T = \int_{\partial T} (K \nabla v_3) \cdot n_T v_3 dS - (v_3, \nabla \cdot p)_T = (\theta_1 w_h \Pi_T[u] + p, \nabla v_3)_T
\]
It easily follows that
\[
\| v_3 \|_{L^\infty(T)} \leq C_3 \| \Pi_T[u] \|_{0,T} \leq C_3 \| u \|_{h}, \text{ where } C_3 = [(C + \theta_1 \| w \|_{0,\infty,\Omega})^2 \lambda^{-2} + \theta_1^2 + 1]^{1/2}. \tag{22}
\]
Now taking into account (16) and (17), after straightforward calculations we obtain,
\[
a^*_h(u, v_1) = \theta_1 \sum_{T \in T_h} \{ 2(\nabla \cdot K \nabla u, \Pi_T[u])_T + (K \nabla u, \nabla u)_T - (w \cdot \nabla u, \Pi_T[u])_T \}; \tag{23}
\]
\[
a^*_h(u, v_2) = \theta_2 \sum_{T \in T_h} \{ \| \nabla \cdot K \nabla u \|^2_{0,T} + \| w_h - w \|_{0,T} + \| \nabla \cdot K \nabla u \|_{0,T} \}; \tag{24}
\]
\[
a^*_h(u, v_3) = \sum_{T \in T_h} \{ \| \Pi_T[u] \|^2_{0,T} + \theta_1 [(w \cdot \nabla u, \Pi_T[u])_T - (\nabla \cdot K \nabla u, \Pi_T[u])_T] + (p, \nabla u)_T \}. \tag{25}
\]
Then, recalling the definition of $w_h$, there exists a mesh independent constant $C_W$ (cf. [8]) such that
\[
\| w - w_h \|_{0,\infty,T} \leq C_W h \| w \|_{1,\infty,T} \forall T \in T_h. \] Using this fact, together with (23), (24), (25), simple manipulations lead to:
\[
\begin{align*}
a^*_h(u, v) & \geq \| \Pi_T[u] \|^2_{0,T} + \sum_{T \in T_h} \left[ \theta_1 \| \nabla u \|^2_{0,T} + \frac{\theta_2}{2} \| \nabla \cdot K \nabla u \|^2_{0,T} + (\theta_2 + \frac{\theta_2}{2}) \| \nabla \cdot K \nabla u \|^2_{1,T} \right] - \| p \|_{0,T} \| \nabla u \|_{0,T} - \\
& - \sum_{T \in T_h} \left[ \theta_1 \| \nabla \cdot K \nabla u \|_{0,T} \| \Pi_T[u] \|_{0,T} \| \nabla u \|_{0,T}^2 / 2 \right] \tag{26}
\end{align*}
\]
Now if we assume that $h^2 \leq \beta(C_W \| w \|_{1,\infty,\Omega})^{-2}$ with $\beta \leq 4\lambda^2/[D + (D^2 + 8\lambda^2)^{1/2}]$ for $D = 1 + C^2$, we may choose $\theta_1 > 0$ satisfying $\theta_1 \lambda - \frac{C^2}{2} - \theta_2 \geq 1/4$ with $\theta_2 = 1/2 + \theta_1^2$. It follows from (26), (19), (20) and (22), that,
\[
a^*_h(u, v) \geq \| u \|^2_{h}/4; \| v \|_{h,T} \leq C \| u \|_{h} , \text{ with } C = [3(C_1^2 + C_2^2 + C_3^2)]^{1/2}. \tag{27}
\]
This immediately yields (15) with $\alpha^* = 1/(4C)$. \[
\]
**Proof of Theorem 2.2.** Since $V_h$ is a finite dimensional space, according to [2] the existence and uniqueness of a solution to (12) follows from (15). Moreover, combining (12) and (15) we easily obtain,
\[
\alpha^* \| u^*_h \|_{h,T} \leq \sup_{v \in V_h \setminus \{0\}} \frac{L_h(v)}{\| \Pi_T[v] \|_{0,\Omega}}. \tag{28}
\]
Since $L_h(v) = \int_{\Omega} f \Pi_T[v] dx$ from (28) we finally derive (14) with $C^* = [\alpha^*]^{-1}$. \[
\]
**3 Convergence results**

Henceforth we denote by $\nabla_h$ the operator from $V + U_h + V_h$ onto $L^2(\Omega)$ defined by
\[
[\nabla_h w]_T = \nabla [w]_T \forall T \in T_h, \forall w \in V + U_h + V_h.
\]
Taking into account (14), (30)-(34)-(35) readily yield (33), independent of $h$.

Proof: Assume that $w$ is no need to use the operator $\nabla_h$ in this case.

In order to study the convergence of $u_h$ to $u$ in appropriate norms we first note that from the properties of $V_h$ and equation (I) we easily infer that $u$ satisfies

$$a^*_h(u, v) = L_h(v) \forall v \in V_h$$  \hfill (29)

From the continuity of $a^*_h$ and the uniform stability result proved in Proposition 2.3, we may apply the generalized First and Second Strang’s inequality for the weakly coercive case, namely, inequality (32) of [13]. In the case under study this writes,

$$\| u - u_h \|_h \leq \frac{M^*}{\alpha^*} \inf_{w \in U_h} \| u - w \|_h + \frac{1}{\alpha} \sup_{v \in V_h \setminus \{0\}} \frac{[a^*_h - a_h](u_h^*, v)}{\| v \|_h}$$  \hfill (30)

where $M^*$ is a constant such that

$$a^*_h(u, v) \leq M^* \| u \|_h \| v \|_h \forall u \in V + U_h, \forall v \in V + V_h.$$  \hfill (31)

Proposition 3.1 There exists a constant $M^*$ independent of $h$ such that (31) holds.

Proof: Since $(u, \nabla \cdot K \nabla v)_T = (\Pi_T[u], \nabla \cdot K \nabla v)_T \forall T \in T_h$ and $\forall v \in V_h$, we trivially have,

$$a^*_h(u, v) \leq (\| \nabla \cdot K \nabla u \|_{0, \Omega} + \| w \|_{0, \infty, \Omega}) \| \nabla_h u \|_{0, \Omega} \Pi_h[v] \Pi_h[v] + \|
\| \nabla_h u \|_{0, \Omega} (\Lambda \| \nabla_h v \|_{0, \Omega} + \| w \|_{0, \infty, \Omega}) \Pi_h[v] \Pi_h[v] + \sum_{T \in T_h} \| \Pi_T[u] \|_{0, T} \| \nabla \cdot K \nabla v \|_{0, T}$$

(32)

immediately yields (31) with $M^* = 2 \max[1, 2, \| w \|_{0, \infty, \Omega, \Lambda}]$. \hfill (32)

Next we prove the validity of the following a priori error estimate for the method under study:

Theorem 3.2 Assume that $w \in [W^{1,\infty}(\Omega)]^N$ and $h$ is sufficiently small. Then if $u \in H^2(\Omega)$ and $f \in H^1(\Omega)$ there exists a mesh independent constant $C'$ such that,

$$\| u - u_h \|_h \leq C' h\| u \|_{2, \Omega} + \| f \|_{1, \Omega}$$  \hfill (33)

Proof: By standard results applying to the $RT_0$ method, and since $\| \Pi_h[u - u_h] \|_{0, \Omega}$ is obviously bounded above by a mesh independent constant times $h \| u \|_{1, \Omega}$, for a suitable constant $C_I$ independent of $h$ it holds,

$$\inf_{w \in V_h} \| u - w \|_h \leq C_I h\| u \|_{2, \Omega} + \| f \|_{1, \Omega}$$  \hfill (34)

On the other hand we have $\| [a^*_h - a_h](w_h^*, v) \| = \| (w - w_h^*) \cdot \nabla_h u_h^*, \Pi_h[v] \|$. Hence for a mesh independent constant $C_W$ such that $\| w - w_h^* \|_{0, \infty, \Omega} \leq C_W h\| w \|_{1, \infty, \Omega}$ we derive,

$$\| a^*_h - a_h \|_{0, \Omega, \Omega} \leq C_W h\| w \|_{1, \infty, \Omega} \| \nabla_h u_h^* \|_{0, \Omega} \| v \|_h.$$  \hfill (35)

Taking into account (14), (30), (34), (35) readily yield (33), $C'$ being a mesh independent constant. \hfill (35)

Next we give a fundamental result of this work:

Theorem 3.3 If $\Omega$ is convex, $w \in [W^{2, A}(\Omega)]^N$ and $h$ is sufficiently small, there exists a constant $C''$ independent of $h$ such that,

$$\| u - u_h \|_{0, \Omega} \leq C'' h^2 \| u \|_{2, \Omega} + \| f \|_{1, \Omega}.$$  \hfill (36)
Proof: The proof is a non-trivial extension of the proof of Theorem 2.3 in [20], where the quadratic convergence was shown for the pure diffusion case. We first observe that the continuity of the normal components of $a_h$ and $\Pi_h$, together with the continuity of the normal components of $K\nabla v_h + w_h\Pi_h[v_h]$ on $\partial T$ for $v_h \in V_h$, we easily obtain,

$$a_h(u - u_h, v_h) + \left( (w_h^1 - w_h) \cdot \nabla u, \Pi_h[v_h] \right) = 0,$$

(37)

for all $v_h \in V_h$. Similarly, owing to the continuity of the normal components of $K\nabla u_h$ on $\partial T$ (cf. [20]):

$$(u - u_h, \nabla \cdot K\nabla v) = a_h(u - u_h, v) + (\nabla \cdot K\nabla (u - u_h), v - \Pi_h[v]) + ((w_h^1 - w_h) \cdot \nabla_h (u - u_h), \Pi_h[v]),$$

(38)

for all $v \in \{v \in H^1_0(\Omega), \nabla \cdot K\nabla v \in L^2(\Omega)\}$. By using the Aubin-Nitsche trick and (38) we can write

$$\| u - u_h \|_{0, \Omega} = \sup_{v \in D(\Omega) \setminus \{0\}} \frac{(u - u_h, \nabla \cdot K\nabla v)}{\| \nabla \cdot K\nabla v \|_{0, \Omega}} = \sup_{v \in D(\Omega) \setminus \{0\}} \frac{a_h(u - u_h, v) + (\nabla \cdot K\nabla (u - u_h), v - \Pi_h[v]) + ((w_h^1 - w_h) \cdot \nabla_h (u - u_h), \Pi_h[v])}{\| \nabla \cdot K\nabla v \|_{0, \Omega}},$$

(39)

where $D(\Omega) := V \cap H^1_0(\Omega)$. Let now $u \neq u_h$ (if $u = u_h$ then (36) trivially holds) and $B_D(0, 1) := \{v \in D(\Omega), \| \nabla \cdot K\nabla v \|_{0, \Omega} = 1\}$. We know that there exists $v_0 \in B_D(0, 1)$ such that $\nabla \cdot K\nabla v_0 = u - u_h$ (cf. (33)). Thus it is easy to see that due to (39) it holds,

$$\| u - u_h \|_{0, \Omega} = a_h(u - u_h, v_0) + (\nabla \cdot K\nabla (u - u_h), v_0 - \Pi_h[v_0]) + ((w_h^1 - w_h) \cdot \nabla_h (u - u_h), \Pi_h[v_0]).$$

(40)

By combining (37) and (40) we further get for any $v_h \in V_h$

$$\| u - u_h \|_{0, \Omega} = a_h(u - u_h, v_0 - v_h) + (\nabla \cdot K\nabla (u - u_h), v_0 - \Pi_h[v_0])$$

$$+ (\left( (w_h^1 - w_h) \cdot \nabla_h (u - u_h), \Pi_h[v_0] \right) + ((w_h^1 - w_h) \cdot \nabla u, \Pi_h[v_0]).$$

(41)

Since $D(\Omega) \subset H^2(\Omega)$ in case $\Omega$ is convex (cf. (33)), we can define the standard interpolate $I_h v \in V_h$ of every $v \in D(\Omega)$, based on the degrees of freedom of $V_h$. Taking $v_h = I_h v_0$ in (41), we get,

$$\| u - u_h \|_{0, \Omega} = a_h(u - u_h, v_0 - I_h v_0) + (\nabla \cdot K\nabla (u - u_h), v_0 - \Pi_h[v_0])$$

$$+ ((w_h^1 - w_h) \cdot \nabla_h(u - u_h), \Pi_h[v_0]) + ((w_h^1 - w_h) \cdot \nabla u, \Pi_h[I_h v_0]).$$

(42)

By using the continuity (10) of $a_h$ and the definition of $v_0$, together with the approximation properties of $I_h$ for $h$ sufficiently small (see [20], p. 239 for details), we have for a suitable $h$-independent constant $C$,

$$a_h(u - u_h, v_0 - I_h v_0) \leq \frac{3}{4} \| u - u_h \|_{0, \Omega} + C \| u - u_h \|_{H^1(\Omega)} \| v_0 - I_h v_0 \|_{0, \Omega} + \| \nabla_h(v_0 - I_h v_0) \|_{0, \Omega}.$$

(43)

Applying to (42) and (43) standard results to estimate $\| v_0 - \Pi_h[v_0] \|_{0, \Omega}$ together with $\| v_0 - I_h v_0 \|_{0, \Omega} + \| \nabla_h(v_0 - I_h v_0) \|_{0, \Omega}$, recalling (43) we easily conclude that there exists another constant $C$ independent of $h$ such that,

$$\| u - u_h \|_{0, \Omega} \leq C h^2(\| |u| \|_{2, 2} + |f|_{1, 1} + 4[((w_h^1 - w_h) \cdot \nabla_h(u - u_h), \Pi_h[v_0]) + ((w_h^1 - w_h) \cdot \nabla u, \Pi_h[I_h v_0])].$$

(44)

We proceed by estimating the two terms in brackets on the right hand side of (44) denoted by $T_1$ and $T_2$. First we note that from the Sobolev Embedding Theorem and the convexity of $\Omega$ (cf. (14)), there exist constants $C_\infty$ and $C'_\infty$ depending only on $\Omega$ such that

$$\| \Pi_h[v_0] \|_{0, \infty, \Omega} \leq \| v_0 \|_{0, \infty, \Omega} \leq C_\infty \| v_0 \|_{2, \Omega} \leq C'_\infty.$$
Thus for a mesh independent constant $C_2$ we have:

$$T_1 \leq \| w_h - w \|_{0, \Omega} \| \nabla_h (u - u_h) \|_{0, \Omega} \| \Pi_h [v_0] \|_{0, \infty, \Omega} \leq C_2 h^2 |u|_{2, \Omega}.$$  \hspace{1cm} (45)

For deriving the estimate (45) we used the fact that both $w_h$ and $w$ are interpolates of $w$, the result (33) and the boundedness of $\| \Pi_h [v_0] \|_{0, \infty, \Omega}$. $C_2$ is the product of $C_\infty$ with another constant not depending on $h$ and the semi-norm of $w$ in $H^1(\Omega)^N$.

The term $T_2$ can be estimated in a similar way. Using now the fact that $w_h$ is a piecewise linear interpolate of $w$, the regularity of the solution $u$ and standard properties of $\Pi_h$ and $I_h$, there holds

$$T_2 \leq \| w_h - w \|_{0, \Omega} \| \nabla u \|_{0, \Omega} \| \Pi_h [I_h v_0] \|_{0, \Omega} \leq C_3 h^2 \| u \|_{2, \Omega},$$  \hspace{1cm} (46)

where $C_3$ equals an $h$-independent constant times $|w|_{2, \Omega}$. Putting together (44)-(45)-(46), we obtain the quadratic convergence (36). \hfill \Box

### 4 A variant for the equations in divergence form

Like in [10] it is possible to consider a variant of the method described in Section 2 applying to the case where the normal component of the total flux $-K \nabla u + \text{wu}$ is continuous across the element interfaces. In the case of the mixed formulation this corresponds to introducing the auxiliary variable $p$ given by the above expression, and write the C-D equation equation (1) in divergence form, namely

Find $u$ satisfying $u = 0$ on $\Gamma$ and $p$ such that

$$\begin{align*}
\nabla \cdot p - \nabla \cdot w &\ u = f \quad \text{in } \Omega, \\
p + K \nabla u - \text{wu} &\ = 0 \quad \text{in } \Omega.
\end{align*}$$  \hspace{1cm} (47)

Recalling the space $H(div; \Omega) := \{ q \mid q \in [L^2(\Omega)]^N, \nabla \cdot q \in L^2(\Omega) \}$, a natural weak (variational) formulation equivalent to system (47) is given in [10], that is,

Find $u \in L^2(\Omega)$ and $p \in H(div; \Omega)$ such that for all $v \in L^2(\Omega)$, and for all $q \in H(div; \Omega)$,

$$\begin{align*}
\langle \nabla \cdot p, v \rangle - \langle \nabla \cdot w \ u, v \rangle &= (f, v), \\
\langle \mathcal{K}^{-1} p, q \rangle - \langle u, \nabla \cdot q \rangle - \langle \mathcal{K}^{-1} w \ u, q \rangle &= 0.
\end{align*}$$  \hspace{1cm} (48)

The extension of $RT_0$ to the C-D equation considered in [10] consists of using the Raviart-Thomas interpolation of the lowest order to represent $p$ and $q$ - i.e. to approximate $H(div; \Omega)$ - , and the space of constant functions in each element of the partition $T_h$ to represent $u$ and $v$. In contrast, here we shall mimic (48) by resorting to the space $U_h$, after adding up both relations in (48). More specifically we take in each element $T \in \mathcal{T}_h$, $q|_T = \mathcal{K} \nabla v|_T$ for $v \in U_h$. Now $u_h$ will be searched for in a space $W_h$ defined hereafter. First we have to construct field $\tilde{w}_h$ to replace $w_h$ (cf. Section 2), in order to preserve optimality of the approximation of $u$. In this aim it suffices that $\tilde{w}_h$ be of the form $cx + d$ in each $T \in \mathcal{T}_h$ for suitable real number $c$ and real vector $d = [d_1, \ldots, d_N]^t$. This representation is compatible with the requirement that the normal component of the flux variable $p = -K \nabla + \text{wu}$ be continuous across the mesh edges at discrete level. The natural choice of $\tilde{w}_h$ is certainly the interpolate of $w$ in the Raviart-Thomas ($RT_0$) space. Now we define the

Hermite finite element space $W_h$:

$W_h$ is the space of functions $v$ of the form $x^T \{ \mathcal{K}^{-1} [a x/2 + b] \} + d$ in every $T \in \mathcal{T}_h$, such that the mean normal flux $\int_F (-K \nabla v + \tilde{w}_h \Pi_h [v]) \cdot n \, ds/\text{meas}(F)$ is continuous across all the inner edges or faces $F$ of the partition. This is about all that is needed to complete the definition of $W_h$. Indeed using a procedure very similar to the one in Section 2 (cf. (5)) it is possible to uniquely determine the $N + 2$ local basis functions $\eta_i^T$ for each $N$-simplex $T \in \mathcal{T}_h$ related to the above set of degrees of freedom completed
with the function mean value in $T$, defining $W_h$ locally. More precisely, setting $[\tilde{w}_h]|_T = a_w^T x + b_w^T$, since by definition $\Pi_T \eta^T_i = 0$ for $i = 1, \ldots, N + 1$ and $\Pi_T \eta^T_{N+2} = 1$, recalling (5) we have:

$$\begin{cases}
\eta^T_i = \varphi^T_i \text{ for } i = 1, \ldots, N + 1, \\
\eta^T_{N+2} = x^T \{K^{-1} [a_w^T x/2 + b_w^T] \} + 1 - \int_T x^T \{K^{-1} [a_w^T x/2 + b_w^T] \} dx/\text{meas}(T).
\end{cases} \quad (49)$$

Now we replace in (48):

- $u$ with $\Pi_h[u_h]$;
- $w$ with $\tilde{w}_h$;
- $p$ with $-\mathcal{K} \nabla_h u_h + \tilde{w}_h \Pi_h[u_h]$ (taking $u_h \in W_h$);
- $q$ with $-\mathcal{K} \nabla_h v$ (taking $v \in U_h$);
- $f$ with $\Pi_h[f]$.

This leads to the following equation:

$$\sum_{T \in \mathcal{T}_h} \left[ (\nabla \cdot [\mathcal{K} \nabla u_h - \tilde{w}_h \Pi_T[u_h]], v)_{\mathcal{T}} + (\nabla \cdot \tilde{w}_h \Pi_T[u_h], v)_{\mathcal{T}} + (\mathcal{K} \nabla u_h - \tilde{w}_h \Pi_T[u_h], \nabla v)_{\mathcal{T}} + (\Pi_T[u_h], \nabla \cdot \mathcal{K} \nabla v)_{\mathcal{T}} + \left( \tilde{w}_h \Pi_T[u_h], \nabla v \right)_T \right] = -(\Pi_h[f], v) \forall v \in U_h. \quad (50)$$

After straightforward simplifications, and taking into account that $(\Pi_h[f], v) = (f, \Pi_h[v])$, we come up with the following Hermite finite element counterpart of (1): Find $u_h \in W_h$ such that for all $v \in U_h$

$$\tilde{a}_h(u_h, v) = L_h(v), \quad (51)$$

holds true, where $\forall u \in V + W_h$ and $\forall v \in V + U_h$,

$$\begin{cases}
\tilde{a}_h(u, v) := \sum_{T \in \mathcal{T}_h} \left[ (\nabla \cdot \mathcal{K} \nabla u, v)_{\mathcal{T}} + (\nabla u, \mathcal{K} \nabla v)_{\mathcal{T}} + (u, \nabla \cdot \mathcal{K} \nabla v)_{\mathcal{T}} \right] \\
L_h(v) := -(f, \Pi_h[v]).
\end{cases} \quad (52)$$

At a first glance (52) seems to indicate that the velocity $w$ does not appear in formulation (51). Nonetheless $w$ remains implicit therein through the definition of space $W_h$.

The fact that problem (51) has a unique solution can be established quite similarly to problem (7). The convergence results that hold for this method can be proved very much like in the case of the method defined in Section 2. The main difference is that it is necessary to require a little more regularity of $\nabla \cdot w$, namely, that this function lies in $W^{1,\infty}(\Omega)$. Apart from this assumption, the results are qualitatively equivalent, in the sense that a priori error estimates completely analogous to those of Theorem 3.2 and Theorem 3.3 apply to problem (51) as well. As far as this work is concerned, resulting properties among others we have not formally established here, are illustrated by means of a numerical example given in the following section.

## 5 Numerical experiments

In this section we present some numerical results obtained with the methods described in Sections 2 and 4 for two test problems, which particularly highlight their behavior. The following nomenclature is used for the different numerical methods having been experimented:

- Method $A$ - Douglas & Roberts version in non divergence form of mixed method $RT_0$;
• Method $hA$ - Hermite analog of Method $A$ (cf. Section 2);
• Method $B$ - Douglas & Roberts version in divergence form of mixed method $RT_0$;
• Method $hB$ - Hermite analog of Method $B$ (cf. Section 4).

Test-problem 1: In these experiments $\Omega$ is the unit square and a manufactured solution $u$ is given by $u(x_1, x_2) = (x_1 - x_2^2)(x_2 - x_1^2)/4$. This together with the choice $K = I$ and $w = Pe[x_1^2, x_2^2]/\sqrt{2}$ where Pe is the Péclet number, produces a right hand side datum $f$. A sequence of uniform meshes was employed with $2L^2$ triangles, for $L = 8, 16, 32, 64$, constructed by first subdividing $\Omega$ into $L^2$ equal squares and then each one of these squares into two triangles by means of their diagonals parallel to the line $x_1 = x_2$. In Figure 1 we display the absolute errors in four different respects for increasing values of $L$, of the approximate solutions obtained with methods $A$, $hA$, $B$ and $hB$ for $Pe = 1$. The notations are self-explanatory, except for $Max|e_u|$ which refers to the maximum of the absolute errors at the centroids of the mesh triangles. Henceforth we call $Max|e_u|$ the pseudo maximum semi-norm. More precisely the absolute errors of $u$, $\nabla u$ and $\Delta u = \nabla \cdot K \nabla u$ measured in the norm of $L^2(\Omega)$ together with $Max|e_u|$, are shown in the sub-figures where indicated. In Figures 2 the same kind of results are displayed for $Pe = 100$.

From Figures 1 and 2 one can infer that:

• Methods $A$ and $hA$ are fairly equivalent to Methods $B$ and $hB$ in all respects for a low Péclet number.
• Methods $A$ and $hA$ are superior to Methods $B$ and $hB$ in all respects when the Péclet number is not low.
• The theoretical results of Section 3 for Method $hA$ were confirmed in the case of both a low and a moderate Péclet number.
• As the Péclet number increases the convergence rate of Method $hB$ in $L^2(\Omega)$ seems to be decreasing from ca. two for $Pe = 1$.
• The numerical convergence rate in the pseudo-maximum semi-norm $Max|e_u|$ is approximately two for all the four methods.
• For $Pe = 100$ the mixed methods are a little more accurate than their Hermite counterparts, as far as errors at the triangle centroids are concerned.
• For $Pe = 100$ both $\nabla u$ and $\Delta u$ tend to be equally approximated by a mixed method and its Hermite counterpart.
Figure 1: Results for Test-problem 1 with \( \text{Pé} = 1 \)
Figure 2: Results for Test-problem 1 with Pe = 100
The numerical results for Methods $B$ and $hB$ deteriorated substantially as we switched to higher Péclet numbers, which was partially the case of Method $hA$, while most of the results obtained with Method $A$ remained quite reasonable. Taking $L = 64$ we illustrate the behavior of Methods $A$ and $hA$ in Tables 1 and 2, respectively, for increasing Péclet numbers. More precisely we took $Pé = 10^k$ for $k = 0, 1, 2, 3$, for which we display the absolute errors of $u$, $∇u$, $Δu$ measured in the norm of $L^2(Ω)$ and the absolute error of $u$ measured in the pseudo maximum semi-norm $Max|e_u|$

From Tables 1 and 2 we conclude that (mixed) Method $A$ is more stable than its Hermite counterpart $hA$, as the Péclet number increases.

Test-problem 2: In order to observe the behavior of the four methods being checked, in the presence of a curved boundary, in this test-problem the domain is a disk with unit radius. The manufactured solution $u$ is given by $u(x_1, x_2) = (1 - x_1^2 - x_2^2)/4$. Taking again $K = I$, the right hand side function $f = 1 - (x_1^2 + x_2^2)/2$ corresponds to a convective velocity $w = Pé[x_1, x_2]^T$. For symmetry reasons the computational domain $Ω$ is only the quarter of disk given by $x_1 > 0$ and $x_2 > 0$. A sequence of quasi-uniform meshes with $2L^2$ triangles was employed for $L = 8, 16, 32, 64$, constructed by mapping the meshes of Test-problem 1 into the actual meshes of $Ω$ using the transformation of cartesian into polar coordinates in the way described in [19]. We denote by $Ω_k$ the approximation of $Ω$ consisting of the union of the triangles in $T_h$. In Figure 3 we display the absolute errors in four different respects for increasing values of $L$, of the approximate solutions obtained with methods $A$, $hA$, $B$ and $hB$ for $Pé = 1$. The displayed errors and corresponding notations are the same as in Figures 1 and 2. More precisely the absolute errors of $u$, $∇u$ and $Δu = ∇ · K∇u$ measured in the norm of $L^2(Ω_h)$ together with $Max|e_u|$, are shown in four subfigures.

From Figure 3 we infer that:

- Methods $A$ and $hA$ are superior to Methods $B$ and $hB$ in all respects, except in the approximation of (constant) $Δu$, which is almost exactly approximated by all the four methods.

- Methods $A$ and $hA$ do not seem to be affected by the curved boundary approximation by polygons, while this seems to be case of Methods $B$ and $hB$.

- Method $A$ and $hA$ approximate both $∇u$ and $Δu$ to machine precision; this is an expected behavior since both functions in this test-problem can be exactly represented by the same underlying incomplete linear and constant interpolation for both methods.

| $Pé$ | $||e_u||_{0,Ω}$ | $||e∇u||_{0,Ω}$ | $||eΔu||_{0,Ω}$ | $Max|e_u|$ |
|------|----------------|----------------|----------------|----------|
| 1    | 0.13723841E03  | 0.58218263E-01 | 0.15249263E-02 | 0.20428256E05 |
| 100  | 0.13724039E03  | 0.58595099E-03 | 0.25587661E-01 | 0.26841141E05 |
| 10000| 0.18370239E-02 | 0.51526514E+00 | 0.15093095E+03 | 0.52745011E-01 |
| 100000| 0.20738981E-03 | 0.57979017E-01 | 0.22150639E+02 | 0.12137294E-02 |

Table 1: Absolute errors for Test-problem 1 solved by Method $A$

| $Pé$ | $||e_u||_{0,Ω}$ | $||e∇u||_{0,Ω}$ | $||eΔu||_{0,Ω}$ | $Max|e_u|$ |
|------|----------------|----------------|----------------|----------|
| 1    | 0.28250216E-05 | 0.58219418E-03 | 0.15249297E-02 | 0.28130033E05 |
| 100  | 0.25386722E-05 | 0.59256341E-03 | 0.24402972E-01 | 0.37752993E05 |
| 10000| 0.1089341E04   | 0.26546637E+06 | 0.75214576E+07 | 0.57326794E04 |
| 100000| 0.13681695E+00 | 0.26569231E+02 | 0.66544379E+02 | 0.27070200E+01 |

Table 2: Absolute errors for Test-problem 1 solved by Method $hA$
• The approximations of \( u \) by Method \( hA \) converge as an \( O(h^2) \) in \( L^2(\Omega_h) \), while those computed by Method \( A \) converge as an \( O(h) \), i.e. the best we can hope for.

• The numerical convergence rate in the pseudo-maximum semi-norm \( \max |e_u| \) is approximately two for both Method \( A \) and Method \( hA \) with an advantage of the former over the latter in terms of accuracy.

Akin to Test-problem 1, we checked the behavior of Methods \( A \) and \( hA \) as the Péclet number increases. Here again we took \( L = 64 \) and \( \text{Péclet} = 10^{2k} \) for \( k = 0, 1, 2, 3 \). The resulting errors measured in the same manner as in Tables 1 and 2 are displayed in Table 3 for Method \( A \) and in Table 4 for Method \( hA \).

From Tables 3 and 4 we observe that both Method \( A \) and Method \( hA \) are accurate to machine precision, irrespective of the Péclet number, as far as the approximations of \( \nabla u \) and \( \Delta u \) are concerned. The approximations of \( u \) in \( L^2(\Omega_h) \) and at the triangle centroids do not seem to be affected by the Péclet number either in this test-problem for both methods. In the former sense Method \( hA \) is much more accurate than Method \( A \) as expected, while in the latter sense Method \( A \) is slightly more precise than Method \( hA \). Notice that this test-problem is a little peculiar, since the exact solution is a quadratic function, whose gradient can be exactly represented by the gradient of the underlying interpolating functions.
Actually this also happens to the approximation of the function itself by Method \( hA \), but in this case other sources of errors came into play, such as numerical integration (see also Remark 3 hereafter).

### 6 Concluding remarks

We conclude this work with a few remarks.

**Remark 1** By means of similar test-problems at low to moderate Péclet numbers, it was shown in [22] and [23] that the Hermite methods work as well as the corresponding Douglas and Roberts extensions of the \( RT_0 \) element, as far as the fluxes are concerned. However the former behave much better in terms of the error of the primal variable in \( L^2(\Omega) \), as expected.

**Remark 2** According to the theoretical results derived in this work, numerical convergence in case the Péclet number is high could only be observed if meshes much finer than those used in Test-problem 1 and in [22] and [23] were used. However running tests with such meshes may become unrealistic. Therefore the authors intend to study modifications of the variational formulations employed in this work, in order to obtain stable solutions within acceptable accuracy, even in the case of high Péclet numbers, without resorting to excessive mesh refinement. The work of Park and Kim [15] for \( RT_0 \) discretizations could be an inspiring one in this connection.

**Remark 3** The bilinear forms and the linear form \( L_h \) considered in this work do not really reduce to those in [20] in the case where \( w \equiv 0 \). This is because somehow we wanted to incorporate numerical quadrature to the variational formulations in use, which is mandatory if \( f \) is not easy to integrate. However in this case we should rather take \( L_h(v) := (f_h, v) \) for a suitable \( f_h \) defined through point values of \( f \) only. Assuming for instance that \( f \in H^2(\Omega) \), \( f_h \) can be chosen to be a piecewise linear interpolate of \( f \) in every \( T \in \mathcal{T}_h \). Second order convergence results in \( L^2(\Omega) \) can still be proven to hold for such a choice, using the well-known analysis of variational crimes [24]. The same qualitative results can also be obtained by using a suitable quadrature formula to compute the integral of the function \( g := f v \) in every element of the mesh. For more details we also refer to [24], or to many other text books on the finite element method.

**Remark 4** The Hermite methods studied in this work can be viewed as a technique to improve the accuracy of the primal variable computations with mixed element \( RT_0 \) without resorting to post-processing...
(see e.g. [7], [16]) or hybridization (see e.g. [4] for the diffusion equation and [17] for convection-diffusion problems). Incidentally the method proposed in [17] can be applied also to BDM elements to obtain optimal estimates [5] improving in this way the classical BDM method for convection-diffusion equations [9] or for stabilization purposes [17, 6] like in several previous work on the subject. Notice that our method allows to achieve better accuracy directly from the numerical solution procedure, at negligible additional cost. Thus it seems worthwhile searching for Hermite analogs of BDM methods as well in the future.

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