Time and space, frequency and wavevector: 
or, what I talk about when I talk about propagation

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The existence of waves with negative frequency is a surprising and perhaps controversial claim which has recently been revisited in optics and for water waves. Here I explain a context within which to understand the meaning of the “negative frequency” conception, and why it appears in some cases.

I. OMNISCIENT: ALL TIME AND ALL SPACE

The omniscient position is taken when we claim all possible knowledge about how some wave has and will move through its environment. Here we might express our knowledge of some wave-field \( F \) by writing it with full space and time arguments: i.e. \( F(t,r) \). Most likely, this view is a result of us having an analytic solution to some specific case of the physical model we are interested in.

Given this fully known wave field solution \( F(t,r) \), we are at liberty to Fourier transform in either time or space (or both) as we see fit. In the resulting double spectrum \( F(\omega,k) \), we will quite naturally see not only positive and negative frequency components, but also forward and backward wavevector components along each spatial coordinate axes.

Since the spectrum of some function is closely related to the complex conjugate of that spectrum for negative frequencies, we know for the doubly transformed \( F(\omega,k) \) that \( F(\omega,k) = F(-\omega,-k) \). Notably, in this case there is no obvious distinction between a positive frequency disturbance evolving in one direction and a negative frequency disturbance evolving in the opposite direction. We therefore need not be troubled by the presence of negative frequencies, since we can reinterpret them as oppositely directed positive ones – we might, for example, refer to the idea that a positron is only an electron travelling backwards in time\(^1\).

It is also possible to consider a “limited omniscience” position, where we claim complete knowledge of the wave behaviour in some defined region of space and time. This might be from some (past) experimental results, or be a numerical or analytic solution valid over that limited extent. In such a case, both time and space can be Fourier transformed, and the remarks above about handling and/or interpreting negative frequencies still hold. However, whilst such after-the-fact

\(^1\) Of course the handling of most particles is complicated by their mass, whereas photons, being massless, can be their own antiparticle.
Given some wave field state $F_i(r)$ known over all space $r$ at some specified time $t = t_0$, we are at liberty to Fourier transform in space, but not time. The spatial transform will give us the wavevector or momentum-like properties of the wave field, in a spectrum $F_{i0}(k)$. This will contain both forward and backward wavevector components along each spatial coordinate axis, and we can calculate it not only at the current time $t_0$, but also all past times $t < t_0$.

For simplicity, imagine we are following some trajectory through time and space, while counting interesting events that occur locally. Time is always increasing but we might travel any direction through space, although it is easier to think of the case where our position is fixed. Naturally, our count of interesting events will only ever increase, thus any estimate of the time period $T$ between events would be a positive number; as would any frequency estimation $f$ in events-per-second. However, the events themselves may have different spatial characteristics: notably, we may see objects that pass us travelling left, or perhaps travelling right. The differing character of events, accessible to us through our spatial knowledge $F_i(r)$, allows us to impose a sign (or signs) when adjusting our counting total – perhaps +1 for left-going objects, and -1 for right-going ones.

Having discussed how a counting/timing argument that gives us a strictly positive frequency estimate can be converted into a signed count using spatial information, let us revisit the temporally propagated spatial spectrum $F_{i0}(k)$ of our notional wave field. As already noted, the spatial spectrum of the wave has both positive and negative wavevector components. Starting with the well known velocity relation for waves $v = \omega/k$, we can use this to claim that the positive wavevector part of the spectrum represents forward evolving waves (with $v > 0$), while the negative wavevector part represents backward evolving waves (with $v < 0$). However, we have to be careful: even a static field profile will have positive and negative wavevector components, since for real-valued $F_i(z)$, $F_i(-k) = F_i^*(k)$. It is the changing complex phase(s) of $F_i(k)$ that represents the shift in $F_i(z)$ profile from one position to another.

Note that I have so far only discussed frequency estimates. Strictly speaking, in this picture, any system state exists only at an instant in time, so of itself that state has no frequency content at all. Further, even given our knowledge of past states, we cannot calculate a true frequency dependence, because we do not have the entire wave history to hand: we have the past but not the future. The best we might do in getting an up-to-date idea of the spectrum is apply a Laplace transform over the known past information, converting $t$ into a Laplace-spectral $s$, and get a hybrid spectrum $F'_{i0}(s,k)$. Nevertheless,

\[ F'_{i0}(s,k) = \int_{t=0}^{t_0} F_{i0}(s-kx) e^{-st} \, dx, \]

2 However, if we were to decide to propagate backwards in time from $t = +\infty$, as is sometimes done when we have preferred final-time boundary conditions, note that those become the computational initial conditions, and that the computational past of any point in the propagation will be its future light cone. The interested reader might try drawing a suitable counterpart to fig.1 themselves.

3 Although any one-sided transform that seemed appropriate, if applied over past (known) data, would also be fine.
it may be possible to characterise the dominant frequency-like properties of the propagation as a function of the know wavevector; and so be able to use a reference frequency \( \Omega(k) \) as a basis on which to simplify the propagation – perhaps by assuming it to be unidirectional.

However, if we judge that omitting the future behaviour will have a negligible effect, or that it is irrelevant since we only care about an experiment or computation that has already finished, we can simply take data from the appropriate past time interval, and calculate the frequency spectrum of that. This after-the-fact analysis of past data returns us to a version of the omniscient position, albeit one limited in scope, so that the same view of negative frequencies as there can be applied. This is what is usually done (either implicitly or explicitly), which is of course the reason why so many frequency responses & spectra appear in the literature, textbooks, and on datasheets.

So in this traditional picture, we need not be overly troubled by negative frequencies. Either we are being very rigorous, and so deny that a true frequency spectrum exists at all, or we have a spectrum calculated from known historical data, which has negative frequencies interpretable as positive ones, just as in the omniscient picture. As a final note, and as a result of the considerations above, the spectroscopists habit of giving spatial (wavevector, or momentum-like) spectra rather than temporal (frequency, or energy-like) ones makes sense from a theoretical perspective, as well as from a practical experimental one.

### III. DIRECTED: SPATIAL PROPAGATION

The directed position is taken when we claim all knowledge about how the wave the wave has moved in a preferred direction along some chosen path through its environment. This picture is shown on fig. 1, where the path is along the \( z \) axis from \( z = -\infty \), through \( z = 0 \), and towards \( z = +\infty \). Here we might express our knowledge of the wave-field \( F \) by writing it at the current position (e.g. \( z = z_0 \)) with full time and transverse space arguments: i.e. \( F(z_0(t,x,y)) \); and where it is implied that we also knew \( F(z_0(t,x,y)) \) at all prior positions \( z < z_0 \) along that path also. This often seems a very natural thing to do, especially when considering unidirectional beam propagation in waveguides or optical fibres. In particular, this position is made most explicit in the PSSD (pseudospectral spatial domain) method for propagating electromagnetic pulses, but also in many others [5][10][13]. Here, the starting state at \( z = z_0 \) for some spatially propagated simulation – the “computational initial conditions” – could be written \( F(z_0(t,x,y)) \), and are enthusiastically not the same as the traditional physical initial conditions at a time \( t = t_0 \) (written as e.g. \( F_0(x,y,z) \)). Contrast, for example, the starting states on figs. 1 and 2.

Given some wave field state \( F_1(t,x,y) \), known at some specified path position \( z_0 \), we are at liberty to Fourier transform in \( x \) and \( y \), and over time \( t \), but not along the propagation axis \( z \). The result will be a mixed spectrum \( F_2(\omega,k_x,k_y) \) as calculable for the current position \( z_0 \), as well as prior locations \( z < z_0 \). The spatial transform of this will give us the transverse wavevector spectrum of the wave field, which will contain both positive and negative wavevector components along each transverse coordinate axes.

![FIG. 2: Spatial propagation of waves, where disturbances (or pulses) evolve either forward or backward in time. At any point in the propagation, we know the full time behaviour of our wave field – both history and future, as indicated by the pale pink horizontal lines. The light pink shaded triangles indicates the computational past of a wave element at particular points (black circles) along the path of the disturbance. Note that unlike the temporally propagated case shown in fig. 1 the computational past of a spatially propagated system is not the same as the causal past. A notional interface has been added to the diagram to show how reflections behave – i.e. in an unexpected way].

For simplicity, and in analogy to the comparable discussion for the traditional picture, imagine we are following some trajectory through space and time, while counting interesting events that occur locally along that path. In this picture, the spatial coordinate along our propagation axis (e.g. \( z \)) is always increasing but our trajectory can move forward or back along the other spatial axes (\( x, y \)), and even forward of back in time; although it is easier to think of the case where our \( x,y \) position is fixed, as is time \( t \). Naturally, our count of interesting events will only ever increase, but rather than measuring second between events (or events per second), it would be in meters between events (or events per meter) – not a temporal period (or frequency), but a spatial interval \( \lambda \) or spatial recurrence rate (wavevector) \( k = 2\pi/\lambda \). Consequentially, any \( \lambda \) or \( k \) estimation we might make would be a positive number. However, the events themselves may have different characteristics: notably, we may see objects that pass us travelling along a transverse spatial axis in either direction. Further, since \( F_1(t,x,y) \)
contains a full time history, this picture can even encode objects travelling forward or backward in time! The differing character of events, accessible to us through the spatial and temporal knowledge in \( F_i(t,x,y) \), allows us to impose a sign (or signs) when adjusting our counting total – perhaps +1 for future-going objects, and -1 for past-going ones.

Having discussed how a counting/distance argument that gives us a strictly positive wavevector estimate can be converted into a signed count using temporal information, let us revisit the spatially propagated frequency spectrum \( F_0(\omega) \) of our notional wave field. As already noted, the frequency spectrum of the wave has both positive and negative frequency components. Again using the well known velocity relation for waves \( v = \omega/k \), we can now claim that the positive frequency part of the spectrum represents forward (in time) evolving waves (with \( v > 0 \)), while the negative wavevector part represents backward (in time) evolving waves (with \( v < 0 \)). However, we have to be careful: even a static wave will have positive and negative frequency components, since for real-valued \( F(i) \), \( F(-\omega) = F^*(\omega) \). It is the changing complex phase of \( F(\omega) \) that represents the shift in \( F(i) \) profile from one time to another.

Note that I have so far only discussed estimates of the propagation axis wavevector \( k_z \). In this picture, any system state exists only at a specific location (e.g. \( z = z_0 \)) along its path, and so of itself that state provides no information about a \( k_z \) at all. Further, even given our knowledge of previous states on the path, we cannot calculate a true \( k_z \), because we only know about where we have been (\( z \leq z_0 \)), not where we are yet to go (\( z > z_0 \)). The best we might do is apply a Laplace transform over the data from behind us along our path, converting \( z \) into a Laplace-spectral \( q_z \), and get hybrid spectrum \( F_0(q_\omega,k_x,k_y,q_z) \). Nevertheless, it is usually extremely advantageous to characterise the dominant \( k_z \)-like properties of the propagation as a function of the known frequency and so be able to use a reference \( k_z(\omega) \) as a basis on which to simplify the propagation – perhaps by assuming it to be unidirectional.

So in the directed picture, we can directly obtain a true frequency spectrum at any point in our propagation, and that spectrum will have negative components. These negative frequencies have a precise and well-defined meaning, but that meaning results from the convenient (but physically approximate) decision to treat propagation as if it were along a spatial path, rather than forward in time.

IV. CAUSAL: THE PAST LIGHT CONE

The causal position taken when we claim all knowledge allowed by causal signalling about how the wave has moved through its environment. Here we might express our knowledge of some wave-field \( F \) by writing it \( \hat{F}(\tau, B) \); where \( \tau \) and \( B \) are the proper time interval into the past and \( B \) the vector “rapidity” needed for a signal to travel from the past to our current location. These \( \tau \) and \( B \) are constructed in a similar way to Rindler coordinates, and quite naturally respect the light cone. This has been discussed in more detail elsewhere.

FIG. 3: Causal propagation, where our knowledge only advances with the edge of a single point’s (observer’s) lightcone; here I show the point travelling at slightly less than the maximum speed (i.e. of that of light) for clarity. At any point in the propagation, we know only the behaviour in our past lightcone; we show three such past lightcones, one being inscribed with curves showing how the rapidity \( B \) varies at a selection of fixed proper time \( \tau \) intervals. The initial conditions are a single point, and the final state the lightcone border. A notional interface has been added to the diagram to show how reflections behave.

Since here our frequency-like quantity relates to the proper time \( \tau \) and not \( t \), and because our wavevector-like quantity relates to \( B \) and not \( r \), I leave any more systematic analysis to later work. However, note that \( \tau \) is like time \( t \) in the sense that it is one sided – we only know the past; and that \( B \) is like \( r \) in that spans all (allowed) points in space.

On fig. 3 I indicate in diagram form how a strictly causal simulation might proceed if we consider the knowledge of (an observer) at a single point e.g. one co-moving with a point on the wave profile. The starting state of this causal propagation matches what would we would know the instant we switched on some sensors – i.e. only that of the observer’s current location. Here, the starting state for some causally propagated simulation – the “computational initial conditions” – are a limited (single point) subset of the traditional physical initial conditions, which tend to assume a much greater knowledge of the environment. If desired, these causal initial conditions could be expanded to cover the past light cone of the initial point. This initial knowledge would then expand, because as we propagate, we would integrate our model of the system to add a “new layer” to the outside of our prior past light cone to get our new (updated) past light cone.

It is debatable whether this rather purist causal picture is of much use in a scientific setting, except perhaps as part of a thought experiment. Nevertheless, it could be invaluable when considering cause and effect or the dynamical responses in specific situations, such as a metamaterial element being driven in the ultrafast and nanoscopic regime. On a more colourful level, for those engaged in spacecraft combat at relativistic velocities – or more likely, those science fiction authors writing about such things – it is the only view of the
known environment that makes sense.

V. SUMMARY

The above discussion shows that negative frequencies can have a well grounded and physical basis – as long as we either are sufficiently omniscient and have a complete knowledge of the behaviour we consider relevant, or if we start from the premise that choosing spatial propagation is reasonable. In such cases we cannot object to the appearance of negative frequency components, but it is worth noting that that we are not omniscient, and that spatial propagation – however useful – is (in practice) an approximation of reality. This means that the concept of negative frequencies must be treated with some caution in any kind of dynamical situation.

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Appendix: Time response models

The time response of a propagation medium is handled rather differently in the temporal and spatial propagation approaches. Consider a set of material properties $n_i$ that follow some kind of dynamical (temporal) response models defined by a differential equation. The differential equations for each $n_i$ will depend on both the local material states and that of the
local field:

\[ \partial_t n_i = r_i(\{n_j\}, F). \]  

(5.1)

Typical examples might be nonlinear response delays [10], a free carrier density model [18], a Raman response [19, 20], or even just a Drude or Lorentz oscillator [15] giving rise to dispersion. The requirement here for the model to be properly causal is that the order of the time derivative \( k \) is greater than any other time derivative parts present in the response function (or operator) \( r_i \) [21].

- In the traditional time propagated picture, each spatial location needs to not only know its field state \( F(r) \), but also its material property state(s) \( n_i(r) \). In some situations, values of \( F \) or \( n_i \) from earlier times – i.e. values from the computational past of the simulation – may need to be stored for use in subsequent computations. Further, as the propagation proceeds step-by-step in time, both the field \( F(r) \) and current material properties \( n_i(r) \) need to be updated.

- In the (directed) space propagated picture, each currently held state of the field holds the time history of each point. This means that it is never necessary to store the computational past of the simulation, since the values for earlier times are already incorporated into the current computational state \( F(t,x,y) \). We can simply solve equations such as eqn. (5.1) by directly integrating them from the distant past up to the desired time, using only that current state information. And, as already mentioned, if the response model is linear and has a known frequency response, as in the case of typical dispersive behaviours, it can be applied quickly and efficiently as a simple phase shift applied to the frequency spectrum – a process requiring only two Fourier transforms and a multiplication.

**Appendix: Nonlinear optics and negative frequencies**

There are some specific issues relating to nonlinear optics (NLO) and negative frequencies which bear additional examination. Consider the nonlinear Schrödinger equation (NLSE) as derived from Maxwell’s equations in the 1+1D regime of \( (t,z) \), with some sources of dispersion and a perturbative third order (Kerr) nonlinearity. Generally in NLO it is written in the convenient (i.e. directed) spatially propagated picture, in a unidirectional approximation with fields evolving forward in time only, as the zNLSE [5]

\[ \partial_z E^+(t) = i K E^+(t) + \sum_j \beta_j \partial_t^2 |E^+(t)|^2 E^+(t), \]

(5.2)

but note that it is also possible to derive a temporally propagated version, for the case of a unidirectional field evolving forward in space only, as the tNLSE\(^4\)

\[ \partial_tD_+(z) = i\Omega D_+(z) + \sum_j \gamma_j \partial_t^2 D_+(z) + i\chi_j |D_+(z)|^2 D_+(z). \]  

(5.3)

Note that in the “directed” spatial zNLSE eqn. (5.2) all sources of dispersion – whether due to time-dependent material response or the geometric properties of a confining waveguide – are treated as if they were solely due to a time-dependent response (by means of the power series in time derivatives). In contrast, in the “traditional” temporal zNLSE eqn. (5.3), all sources of dispersion in the tNLSE are treated as if they were solely due to the local geometric properties of that material i.e. by spatial dispersion [22]; typically this is expressed as a power series in either spatial derivatives or wavevector. However, given that most dispersions are rather weak in applications NLSE propagation, either picture can plausibly be assumed to be able to treat dispersion accurately enough for such purposes (although see [9]).

In both these cases we can see that it would be useful to Fourier transform the field to get temporal and spatial spectra \( \tilde{E}^+(\omega) \) and \( \tilde{D}_+(k) \) respectively, because then the dispersion model can be directly applied as a simple polynomial phase shift, i.e. either of

\[ \beta(\omega) = \sum_j \beta_j (\omega - i\omega)^j \]

(5.4)

\[ \gamma(k) = \sum_j \gamma_j (ik)^j \]

(5.5)

It is often stated that a Kerr nonlinearity causes third harmonic generation, but we now also need to consider what this generation is a “third harmonic” of. In the usual zNLSE model of eqn. (5.2), an existing harmonic field \( E^+(t) \sim E_0 e^{-i\omega_0t} + c.c. \) leads to not only some self phase modulation (SPM) but also third harmonic frequency generation at \( \omega_3 = 3\omega_0 \). In the tNLSE model of eqn. (5.3), an existing harmonic field \( D_+(z) \sim D_0 e^{ikz} + c.c. \) leads to not only some self phase modulation (SPM) but also third harmonic wavevector generation at \( k_3 = 3k_1 \).

For clarity, let us write this out as carefully as possible, in the case where we split the real-valued fields \( E^+ \) or \( D_+ \) into complex conjugate halves, in the style of [16]. With \( E^+(t) = E'(t)e^{-i\omega_0t} + E''(t)e^{i\omega_0t} \), the zNLSE equation can be partitioned into two complex conjugate halves. We can also dispense with the exponential oscillation by setting \( \omega_0 = 0 \) – although it is often useful to include such carrier oscillations when the fields are narrow band, having them written explicitly rather than implicitly is not required. The two complex

\(^4\) Author’s derivation, unpublished
conjugate zNLSE equations are
\[ \partial_t E'(t)e^{-i\omega_0 t} = iK E'(t)e^{-i\omega_0 t} + \sum_j \beta_j \partial_j E'(t)e^{-i\omega_0 t} \]
\[ + i\chi_\omega [E'(t)^3 e^{-3i\omega_0 t} + 3E'(t)^2 [E'(t)]^* e^{-i\omega_0 t}], \]
(5.6)
\[ \partial_t E''(t)e^{+i\omega_0 t} = iK E''(t)e^{+i\omega_0 t} + \sum_j \beta_j \partial_j E''(t)e^{+i\omega_0 t} \]
\[ + i\chi_\omega [E''(t)^3 e^{+3i\omega_0 t} + 3E''(t)^2 [E''(t)]^* e^{+i\omega_0 t}], \]
(5.7)

The sum of these two equations is just eqn. (5.2). Further, since they are exact complex conjugates of one another, we can solve just one version, which automatically gives us a solution to the other, and hence the solution to the real valued field \( E^+(t) \).

Note in particular that the nonlinearity in this exact mathematical re-expression of the zNLSE equation drives only resonant or third-harmonic frequencies. There is no explicit coupling to negative frequencies, which can be seen most clearly when assuming finite \( \omega_0 \) and a constant \( E' \); then, eqn. (5.6) does not drive negative frequencies of itself. Nevertheless, although the complex \( E'(\omega) \) might start with content solely in positive frequencies, its frequency bandwidth is not restricted. Consequently after propagating some distance it may well have evolved into a field whose spectrum is very wide, perhaps even extending past the origin. In this state, multi-frequency interactions that drive the field at negative frequencies could indeed occur as a result of the nonlinearity – remember that if represented in the frequency domain, the cubic nonlinear term transforms into a double convolution over the entire frequency spectra.

However, we could, if we wanted, make such “negative frequency” driving terms appear explicitly by repartitioning the whole nonlinear term into different complex conjugate halves, e.g. by writing not eqn. (5.6) but
\[ \partial_t E'(t)e^{-i\omega_0 t} = iK E'(t)e^{-i\omega_0 t} + \sum_j \beta_j \partial_j E'(t)e^{-i\omega_0 t} \]
\[ + i\chi_\omega [E'(t)^3 e^{-3i\omega_0 t} + \frac{3}{2}E'(t)^2 [E'(t)]^* e^{-i\omega_0 t} \]
\[ + \frac{3}{2} [E'(t)]^2 E'(t)e^{+i\omega_0 t}], \]
(5.8)

and there is of course also a matching complex conjugate counterpart of eqn. (5.7), and the sum of both will be equal to the original zNLSE equation. Whether or not this representation containing explicit driving of negative frequencies might be useful is another matter, but it is certainly possible to (re)express the mathematical model in order to construct them. But, apart from specific numerical difficulties that might occur when solving these propagation equations, the solutions gained from either form should be identical: both are just different ways of representing the same physical model.

Naturally one can apply the same method to the tNLSE as well, setting \( D'(z) = D'(z)e^{ik_1 z} + D''(z)e^{-ik_1 z} \), and arriving at two complex conjugate equations
\[ \partial_t D'(z)e^{+ik_1 z} = iK D'(z)e^{+ik_1 z} + \sum_j \gamma_j \partial_j D'(z)e^{+ik_1 z} \]
\[ + i\chi_\omega [D'(z)^3 e^{+3ik_1 z} + 3D'(z)^2 [D'(z)]^* e^{+ik_1 z}], \]
(5.9)
\[ \partial_t D''(z)e^{-ik_1 z} = iK D''(z)e^{-ik_1 z} + \sum_j \gamma_j \partial_j D''(z)e^{-ik_1 z} \]
\[ + i\chi_\omega [D''(z)^3 e^{-3ik_1 z} + 3D''(z)^2 [D''(z)]^* e^{-ik_1 z}]. \]
(5.10)

As we should expect, these two equations sum to eqn. (5.3).

Also, here there again is no explicit coupling to opposite parts of the wavevector spectra \( D'(k) \) in each equation, although (as before) we might repartition the nonlinear term to construct it, if we so desired.