Exactly solvable discrete BCS - type Hamiltonians and the Six-Vertex model.

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Abstract

We propose the new family of the exactly solvable discrete state BCS - type Hamiltonians based on its relationship to the six-vertex model in the quasiclassical limit both in the rational and the trigonometric cases. We establish the relation of the BCS Hamiltonian and its eigenfunctions to the form of the monodromy matrix in the F-basis. Using the Algebraic Bethe Ansatz method for the standard BCS model with equal coupling the expression for the general scalar product and the determinant expressions for the physically interesting correlation functions for the finite number of sites which can be used in the numerical and analytical computations are obtained. We also compare the correlators with the results obtained by means of the variational method.

1. Introduction.

At present time the exactly solvable discrete-state Bardeen, Cooper and Schrieffer (BCS) model for the superconductivity [1] attracts much attention in connection with the problems in different areas of physics such as superconductivity, nuclear physics, physics of ultrasmall metallic grains and color superconductivity in QCD. The exact solution of the discrete-state BCS model is especially important for the study of the superconducting correlations in the atomic nuclei and the ultrasmall metallic grains since due to the finite number of particles the description in terms of the grand canonical ensemble [1] (in contrast to the microcanonical ensemble) is obviously not correct. For all of the above mentioned problems and also for the many-body problems of fermions with the long-range interaction, it is desirable to find the integrable BCS-type Hamiltonians with the attraction of the Cooper pairs, depending on the momentum (on the indices of sites in the discrete - state model) and on the occupation numbers on the other sites. Note that at present time the study of both the continuum and the discrete BCS model in the case of the equal spacing of the energy levels is not completed since the possibility of varying the number of pairs with the filling of the part of the energy levels by the single- electron states was not considered. Another important problem which motivates the study of different modifications of the BCS Hamiltonian is to find the realistic integrable BCS-type models which apart from the interaction of pairs, include the interactions describing
the disintegration of pairs i.e. the hopping of two single electrons (fermions) to another energy levels independently of whether they form a Cooper pair or not. In this case the determinant expressions for the correlators obtained in the present paper could be useful as well. The studying of the excitations of different kind in the case of the microcanonical ensemble for the BCS model is also an interesting problem. For the case of the continuum limit of the BCS model as well as for the case of the microcanonical ensemble the analytical and the numerical calculations for various correlation functions are important.

The eigenvectors and the eigenvalues of the discrete BCS Hamiltonian where first constructed by Richardson [2],[3] in the context of nuclear physics. Later the model was applied also to the case of the Bose gas [4]. The norm and the simplest correlation functions have been studied in ref. [3], [4]. Recently the integrability of the BCS model was proved by Cambiaggio, Rivas and Saraceno [5]. The set of the commuting operators which have much in common with the Gaudin magnets [6] was constructed. The connection of the Gaudin magnets with the six-vertex model was pointed out by Sklyanin [7]. The continuum limit of the BCS model for the equally spaced distribution of the energy levels was first considered by Gaudin [8] and later by Richardson [9], who also performed the numerical calculations [9], [10], and was found in agreement with the original (variational) BCS treatment [1]. Recently the solution of the model based on the off-shell Algebraic Bethe Ansatz construction [11] was given by Amico, Falci and Fazio [12]. The numerical calculation of the correlation functions for the finite systems was presented by Amico and Osterloh [13] using the method of calculation of the scalar products, based on the generating function, proposed by Sklyanin [14]. Nevertheless the calculations are quite involved and restricted to the systems with the small number of sites $N$. Therefore the explicit determinant expressions for the correlation functions of the model are highly desirable. Recently the connection of the BCS model with the twisted inhomogeneous six-vertex model was elaborated in ref’s [15] and [16], where the possibility of computation of the correlators with the help of the Algebraic Bethe Ansatz method was pointed out. However, the suitable determinant formulas for the correlators have not been obtained. The new multiparameter families of the BCS -type models connected with the trigonometric six-vertex model where proposed in ref’s [17], [18] and studied in more detail in ref. [19]. The connection of the BCS model with the conformal field theory and the modified Knizhnik-Zamolodchikov equations [20] have been studied by Sierra (for example, see [21]). The discrete BCS Hamiltonian was generalized to the case of arbitrary degeneracy of different energy levels of the Cooper pairs [6] and to the case of the Dicke model [22] (for example, see [23]).

The main goal of the present paper is the calculation of the correlators in the BCS -type models with the help of the methods developed in the context of the Algebraic Bethe Ansatz approach to the six-vertex model. We present the new determinant expressions for the basic correlation functions of the BCS model. We also review and propose the new approaches to the construction of various integrable BCS -type Hamiltonians.

In the first part of the paper we briefly review some of the recent results on the BCS-type
models, in particular, the derivation of the generalized BCS models from the six-vertex model both in the rational and the trigonometric cases in a way which is similar to that of ref’s [15] - [18]. We present the new way of construction of the BCS - type models with the interaction depending on the lattice sites by means of considering the limit of the transfer matrix spectral parameter $t \to \infty$. The several examples of the Hamiltonians with the position - dependent interaction are presented explicitly. We establish the relation of the BCS Hamiltonian and its eigenfunctions to the form of the monodromy matrix in the F-basis which also leads to the generalization of the BCS model to the case of the interaction between pairs depending on the occupation numbers on the other sites. Let us stress, that the formalism of the F- basis is used not to obtain the known formulas for the scalar products for the six-vertex model, but to develop the new formalism for the construction of various BCS- like integrable models.

In the second part of the paper we present the results on the calculation of various correlation functions for the BCS model using the Algebraic Bethe Ansatz method for the six-vertex model. First, we obtain the simple expressions for the scalar products and the formfactors of the model taking the quasiclassical limit in the corresponding expressions for the six-vertex model. Then, to obtain the expressions for the correlators, we use the commutational relations for the operators directly in the quasiclassical limit in order to reduce the problem to the calculation of the scalar products. We obtain the new determinant expressions for the different correlation functions which are useful for the numerical and the analytical calculation of the correlators. Let us stress, that our results for the correlators are different from the results obtained previously in ref’s [13], [16], and allow one the much more simple numerical evaluation of the correlators, since in all cases the correlators are represented as the determinants or a finite sum of the determinants.

The content of the paper is as follows. In Section 2 we propose the new family of the exactly solvable discrete BCS - type Hamiltonians based on its relationship to the six-vertex model in the quasiclassical limit both in the rational and the trigonometric cases. We present the examples of the Hamiltonians with the interaction between the pairs depending on the energy levels. We also review the results [17] on the BCS - type integrable models with the double set of parameters. In Section 3 we establish the relation of the BCS Hamiltonian and its eigenfunctions to the form of the monodromy matrix in the F-basis which also leads to the generalization of the BCS model to the case of the interaction between pairs depending on the occupation numbers on the other sites. The expression for the general scalar product and the determinant expressions for the physically interesting correlation functions for the finite number of sites which can be used in numerical computations are obtained in Section 4. The comparison with the expressions for the correlation functions obtained with the help of variation on the parameters is presented in the Appendix B. Thus we show that no special technique of the type proposed in ref.[14] for the calculation of correlators for the BCS model and for the Gaudin magnets is required. The results for the correlation functions for the Gaudin magnets are also interesting from the theoretical point of view. The results obtained can be useful for studying the correlation functions for the XXZ quantum spin chain. For
completeness some of the results on the diagonalization of the BCS Hamiltonian are also presented in the Appendix A. As an additional application of the solution of the BCS model we present the solution of the modified Knizhnik-Zamolodchikov equations [20] for the correlators of the conformal field theory (SU(2) WZW- model) in the Appendix C. We present in the Appendix D the brief review of the Gaudin’s solution of the BCS model in the thermodynamic limit.

2. BCS Hamiltonian and the Six-Vertex model.

The BCS Hamiltonian with an arbitrary parameters \( \epsilon_i \) has the form:

\[
H = \sum_{i=1}^{N} \epsilon_i n_i - g S^+ S^-,
\]

where \( n_i = b_i^+ b_i \) is the number of the hard-core bosons (electron pairs), \( S^+ = \sum_i b_i^+ \), \( S^- = \sum_i b_i \) and the coupling constant \( g \) is positive, which corresponds to the attraction.

To solve the Schrodinger equation for the Hamiltonian (1) and its generalizations (see below), instead of the usual transfer matrix of the six-vertex model with twisted boundary conditions, we consider the following transfer matrix for the twisted inhomogeneous six-vertex model

\[
Z(t, \{ \xi \}) = \text{Tr}_0 \left( \tilde{S}_{i0}(\xi_1, t) \tilde{S}_{20}(\xi_2, t) \ldots \tilde{S}_{N0}(\xi_N, t) \right),
\]

which can be equivalently represented as the trace of the monodromy matrix in the auxiliary space 0,

\[
Z(t, \{ \xi \}) = \text{Tr}_0 (T_0(t)), \quad T_0(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix},
\]

where \( \xi_i \) are the inhomogeneity parameters. The matrices \( \tilde{S}_{i0}(\xi_i, t) \) are equal to

\[
\tilde{S}_{i0}(\xi_i, t) = K_0 S_{i0}(\xi_i, t)
\]

where \( K_0 \) is the twist matrix and \( S_{i0} \) is the usual S-matrix of the six-vertex model obeying the Yang-Baxter equation \( S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12} \):

\[
K_0 = \begin{pmatrix} e^{\eta/2gN} & 0 \\ 0 & e^{-\eta/2gN} \end{pmatrix}, \quad S_{12}(t_1, t_2) = t_1 - t_2 + \frac{\eta}{2} (\sigma_1 \sigma_2)
\]

(we denote \( (\sigma_1 \sigma_2) = \sum_a \sigma_a^x \sigma_a^z \); \( a = x, y, z \)) for the rational case. Due to the well known property of the matrix \( R_{00'} = P_{00'} S_{00'} \) (\( P_{12} = \frac{1}{2} (1 + (\sigma_1 \sigma_2)) \) - is the permutation operator) \([K_0 K_0'; R_{00'}] = 0\) (it does not matter if the standard twist angle is imaginary or real) the matrices \( \tilde{S}_{i0}(\xi_i, t) \) obey the usual Yang-Baxter equation \( R_{00'} \tilde{S}_{i0} \tilde{S}_{i0'} = \tilde{S}_{i0'} \tilde{S}_{i0} R_{00'} \) and the transfer matrices (2) commute at different values of the spectral parameters \([Z(t); Z(t')] = 0\). The
Algebraic Bethe Ansatz method (for example, see [24]) can be readily applied to the monodromy matrix defined in (2) to obtain the same results as for the transfer matrix with the usual twisted boundary conditions

\[ Z(t) = \text{Tr}_0(K_0 T_0(t)) \]

where \( T_0(t) \) is the usual monodromy matrix constructed from the matrices \( S_{i0}(\xi_i - t) \) and \( K_0 \) is the diagonal matrix corresponding to the total twist angle \( \eta/2g \). However the construction presented above allows for another generalization which will be considered later. Considering the limit \( \eta \to 0 \) we get up to a factor \( (\xi_i - t) \):

\[
\tilde{S}_{i0} = 1 + (\eta/2gN)\sigma_0^z + (\eta/2(\xi_i - t))(\sigma_0 \sigma_i) + O(\eta^2),
\]

and retaining the terms of order \( O(\eta^2) \), we obtain the following Hamiltonian depending on the parameter \( t \):

\[
H(t) = -\frac{1}{2g} \sum_{i=1}^{N} \frac{1}{(t - \xi_i)} \sigma_i^z + \frac{1}{2} \sum_{i<j} \frac{1}{(t - \xi_i)(t - \xi_j)} (\sigma_i \sigma_j),
\]

where \( \sigma_i^z \) can be substituted by the number of the hard-core bosons \( \sigma_i^z = 2n_i - 1 \). Since the operators \( H(t) \) commute at different values of the parameter \( t \), one can obtain the set of the commuting operators, which generalize the Gaudin magnets [6], taking the limit \( t \to \xi_i \):

\[
H_i = -\frac{1}{g} n_i + \frac{1}{2} \sum_{l \neq i} \frac{\sigma_i \sigma_l}{(\xi_l - \xi_i)}, \quad [H_i; H_j] = 0 \quad [H; H_i] = 0.
\]

Note that since the commuting operators can be defined up to an additive constant, we choose the operators (5), omitting the constant term in the operator \( \sigma_i^z = 2n_i - 1 \). Note also that in these equations the operator \( (\sigma_i \sigma_j) \) can be represented through the hard-core boson operators \( b_i^+, b_i \) as

\[
(\sigma_i \sigma_j) = 2(b_i^+ b_j + b_j^+ b_i) + (2n_i - 1)(2n_j - 1).
\]

These operators also commute with the Hamiltonian (1) which was first found in ref.[5]. In fact, considering the limit \( t \to \infty \) in eq.(4) and retaining the terms of order \( \sim 1/t^2 \), we obtain exactly the Hamiltonian (1) with \( \epsilon_i = \xi_i \) (in what follows we omit the normalization factors and the additive constants depending on the total number of the hard-core bosons). Alternatively one can consider the following linear combination of the operators \( H_i \) to obtain the same Hamiltonian:

\[
H = -g \sum_{i=1}^{N} \xi_i H_i = \sum_{i=1}^{N} \xi_i n_i - gS^+ S^-,
\]

which coincides with the expression (1) up to an additive constant, depending on the number of bosons \( M \). The following correspondence between the spin and the hard-core boson operators is used: \( S^\pm = \sum_i S_i^\pm, \quad S_i^a = \frac{1}{2} \sigma_i^a, \quad S_i^z = b_i^+ (b_i) = \frac{1}{2} (\sigma_i^z \pm i \sigma_i^y) \). One can also consider an arbitrary linear combinations of the operators \( H_i \) with the coefficients \( \epsilon_i \neq \xi_i \). Then the following generalization of the BCS Hamiltonian appears:

\[
H = \sum_{i=1}^{N} \epsilon_i n_i - \frac{g}{2} \sum_{i<j} \frac{(\epsilon_i - \epsilon_j)}{(\xi_i - \xi_j)} (\sigma_i \sigma_j).
\]
Let us note also that using the different notations the Hamiltonian (4) can be represented in the form:

$$H = \frac{1}{g} \sum_{i=1}^{N} \epsilon_{ni} + \sum_{i<j} \epsilon_{i} \epsilon_{j} (\sigma_{i} \sigma_{j}).$$  \hspace{1cm} (8)$$

Thus, in general, one can construct the Hamiltonians with the interaction between the Cooper pairs depending on the momentum of the pairs. Another Hamiltonian of this type can be obtained by considering the derivative of the transfer matrix over the spectral parameter $t$. In this way the Hamiltonian takes the form similar to (8) with an extra factor $\epsilon^{-1/2} + \epsilon^{-1/2}$ contained in the sum of the second term. In general, since $H(t)$ eq.(4) depends on the additional parameter $t$ and commute at different $t$, $[H(t); H(t')] = 0$, one can take the derivatives over $t$ and consider the linear combination of the operators $H(n)(t)$ at arbitrary $t$ ($H = \sum_{n} C_{n} H^{(n)}(t)$). Thus, we obtain the new Hamiltonians of the BCS type (8) with the infinite number of additional parameters $C_{n}$. The other way to obtain new Hamiltonians is to consider the decomposition of $H(t)$ in the powers of $1/t^n$ at large $t$. In this way the new models with the dependence on the additional set of parameters can be obtained. Clearly, the similar procedures can be applied for the case of the trigonometric (hyperbolic) six-vertex model.

To obtain the eigenvalues of the BCS Hamiltonian one can start with the well known procedure of diagonalization of the transfer matrix (2)- the Algebraic Bethe Ansatz method (for example, see [24], [25]). The eigenstates are represented as

$$|\phi(t)\rangle = B(t_{1})B(t_{2}) \ldots B(t_{M})|0\rangle,$$

where the parameters $t_{1} \ldots t_{M}$ obey the system of Bethe Ansatz equations which do not depend on the spectral parameter $t$:

$$e^{-\eta/g} \prod_{\alpha=1}^{N} \left( \frac{t_{i} - \xi_{\alpha} - \eta/2}{t_{i} - \xi_{\alpha} + \eta/2} \right) = \prod_{\alpha=1}^{M} \left( \frac{t_{i} - t_{\alpha} - \eta}{t_{i} - t_{\alpha} + \eta} \right).$$  \hspace{1cm} (10)$$

Decomposing eq.(10) to the first order in the small parameter $\eta$, we obtain the Richardson’s equations

$$\sum_{\alpha=1}^{N} \frac{1}{t_{i} - \xi_{\alpha}} - 2 \sum_{\alpha \neq i}^{M} \frac{1}{t_{i} - t_{\alpha}} = -\frac{1}{g}.$$  \hspace{1cm} (11)$$

The corresponding eigenvalue for the Hamiltonian (1) is obtained as the limit at $t \to \infty$ of the eigenvalue of the transfer matrix $Z(t)$:

$$E = \sum_{i=1}^{M} t_{i}.$$  \hspace{1cm} (12)$$

The eigenvalues of the conserved operators $H_{i}$ (5) are easily evaluated from the eigenvalue of the transfer matrix (for the rational case)

$$\Lambda(t) = e^{-\eta/2g} \prod_{\alpha} \frac{\xi_{\alpha} - t - \eta/2}{\xi_{\alpha} - t} \prod_{i} \frac{t_{i} - t + \eta}{t_{i} - t} + e^{\eta/2g} \prod_{\alpha} \frac{\xi_{\alpha} - t + \eta/2}{\xi_{\alpha} - t} \prod_{i} \frac{t - t_{i} + \eta}{t - t_{i}}.$$  \hspace{1cm} (13)$$
which in the limit $\eta \to 0$ up to the terms of order $\eta^2$ and $t \to \xi_i$ produces the eigenvalues of the operators $H_i$ (5):

$$E_i = \frac{1}{2} \sum_{\alpha \neq i} \frac{1}{(\xi_i - \xi_{\alpha})} + \sum_j \frac{1}{(t_j - \xi_i)}$$

(here we take into account the definition of the constant term corresponding to the operators $H_i$ (5)). In the quasiclassical limit the operators $B(t)$, $C(t)$ reduce to the operators $\Sigma^\pm(t)$ obeying the commutational relations which can be easily obtained from the basic commutational relations for the elements of the monodromy matrix (Yang-Baxter equation):

$$B(t) \to \Sigma^+(t), \quad C(t) \to \Sigma^-(t), \quad \Sigma^\pm_z(t) = \sum_{i=1}^N \frac{\sigma_i^\pm, z}{t - \xi_i},$$

$$[\Sigma^-(t); \Sigma^+(t')] = \frac{2\Sigma^z(t) - \Sigma^z(t')}{t - t'}, \quad [\Sigma^z(t); \Sigma^\pm(t')] = \mp \frac{\Sigma^\pm(t) - \Sigma^\pm(t')}{t - t'}$$

Note that one could build up the eigenstates of the Hamiltonian (1) directly in terms of the operators $\Sigma^\pm(t)$ [12] (see also the Appendix A). Thus we see that instead of the off-shell Bethe Ansatz approach [11] used in ref.[12] one can use the usual on-shell Bethe Ansatz equations for the six-vertex model.

Let us consider the trigonometric S-matrix. Repeating the steps leading to the Hamiltonian (4) in the rational case (i.e. first taking the limit $\eta \to 0$) we obtain in the trigonometric case the following Hamiltonian:

$$H(t) = -\frac{1}{2g} \sum_{i=1}^N \frac{1}{n_i} \frac{1}{\sin(t - \xi_i)} \frac{1}{\sin(t - \xi_j)} \hat{Y}_{ij} + \sum_{i,j} \frac{1}{\sin(t - \xi_i - \xi_j)} \hat{A}_{ij},$$

where the following notations are used:

$$\hat{Y}_{ij} = b_i^+ b_j + b_j^+ b_i, \quad \hat{A}_{ij} = n_i n_j + (1 - n_i)(1 - n_j).$$

That is the one - parameter generalization of the Hamiltonian (4). First, one can proceed in the following way. Taking the limit $t \to \xi_i$, one readily obtains the trigonometric analogs of the commuting operators $H_i$ (5)

$$H_i = -\frac{1}{2g} n_i + \sum_{i \neq j} \frac{1}{\sin(\xi_i - \xi_j)} \hat{Y}_{ij} + \frac{1}{\t g(\xi_i - \xi_j)} \hat{A}_{ij},$$

and considering the linear combination $\sum_i \epsilon_i H_i$, we get the Hamiltonian

$$H = -\frac{1}{2g} \sum_{i=1}^N \epsilon_i n_i + \sum_{i \neq j} \frac{\epsilon_i - \epsilon_j}{\sin(\xi_i - \xi_j)} \hat{Y}_{ij} + \sum_{i \neq j} \frac{\epsilon_i - \epsilon_j}{\t g(\xi_i - \xi_j)} \hat{A}_{ij}.$$  

The second possibility is to consider the limit $t \to \infty$ in the hyperbolic version of eq.(14). Then we get the following integrable Hamiltonian ($\gamma_i = e^{\xi_i}$):

$$H = -\frac{1}{2g} \sum_{i=1}^N \gamma_i^2 n_i + \sum_{i \neq j} \gamma_i \gamma_j \hat{Y}_{ij}.$$  

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The omitted terms vanish in the limit considered.

Using the different notations the last Hamiltonian can be represented in the form:

\[ H = -\frac{1}{2g} \sum_{i=1}^{N} \epsilon_i n_i + \sum_{i \neq j} \sqrt{\epsilon_i} \sqrt{\epsilon_j} \left( b_i^+ b_j + b_j^+ b_i \right). \]

Note that the last expression is different from the Hamiltonian (8) obtained in the case of the rational six-vertex model. As in the rational case the equation (14) is more general than eq.(16) and taking the derivatives over \( t \) one also can obtain the various generalizations of the Hamiltonian (17). For the hyperbolic case which is obtained by considering \( \xi_i \) as an imaginary parameters the Hamiltonian (16) in the limit \( g \to \infty \) can be considered as the Hamiltonian of the open spin chain with the long-range interaction. Let us mention also that the other possibility to construct the exactly - solvable BCS- like Hamiltonians is to consider the various bilinear combinations of the form \( H = \sum_{i,j} f_{ij} H_i H_j \) with different coefficients \( f_{ij} \).

Let us comment on the form of the eigenstates and the equations for the eigenvalues for the trigonometric case. The analogs of the operators \( \Sigma^\pm(t) \) in this case are

\[ \Sigma^\pm(t) = \sum_i \frac{1}{\sin(t - \xi_i)} \sigma_i^\pm, \]

and the analogs of Richardson’s equations in the trigonometric case, which can be obtained from the trigonometric version of the equations (10), are

\[ \sum_{\alpha=1}^{N} \frac{1}{\tan(t_i - \xi_\alpha)} - 2 \sum_{\alpha \neq i}^{M} \frac{1}{\tan(t_i - t_\alpha)} = -\frac{1}{g}, \]

which represent the conditions for the common eigenvectors for the operators (15) and the Hamiltonian depending on the double set of parameters (17). Considering the quasiclassical limit of the eigenvalue \( \Lambda(t) \), we obtain, omitting the factor \( \prod_\alpha \sin(\xi_\alpha - t) \), the expression

\[ \Lambda(t) = e^{-\eta/2g} \prod_{i=1}^{M} \frac{\sin(t_i - t + \eta)}{\sin(t_i - t)} + e^{\eta/2g} \prod_{i=1}^{M} \frac{\sin(t_i - t - \eta)}{\sin(t_i - t)} \prod_{\alpha=1}^{N} \frac{\sin(\xi_\alpha - t + \eta)}{\sin(\xi_\alpha - t)}, \]

where the definition of the parameter \( t \) differs from the definition in the rational case by the shift \( t \to t + \eta/2 \) and considering the limit \( t \to \xi_i \), we get

\[ E_i = 1/2g - \sum_j \cot(t_j - \xi_i). \]

Here the difference with the expressions (12) is due to the different definitions of the operators \( H_i \) (note that \( P_{ij} = \hat{A}_{ij} + \hat{Y}_{ij} = (1/2)(1 + (\sigma_i \sigma_j)) \)).

Let us mention also the generalization of the construction proposed above. Consider the monodromy matrix constructed from the S- matrices \( \tilde{S}_{i0}(\xi_i, t) = K_0^{(i)} s_{i0}(\xi_i, t) \) where the twist matrix \( K_0^{(i)} \) now depends on the site \( i \):

\[ K_0^{(i)} = \begin{pmatrix} e^{\eta c_i/2gN} & 0 \\ 0 & e^{-\eta c_i/2gN} \end{pmatrix}. \]
where \(c_i\) are an arbitrary parameters. Clearly in this case the Algebraic Bethe Ansatz method can be used in the usual way. Let us denote \(c_0 = \sum c_i\). Then, first taking the limit \(\eta \to 0\), we will obtain the same Hamiltonian with the coupling constant depending on the constant \(c_0\) only. However, considering this limit in the case of \(c_0 = 1\) and the parameters \(c_i\) are of order \(c_i \sim N / \eta\), then, to obtain the Hamiltonian of the type (1), one should commute all the matrices \(K_0^{(i)}\) to the left, which effectively leads to the gauge-like transformation for the operators \(b_i^+, b_i\), which enter the operators \(S^\pm\) in the Hamiltonian (1).

The discrete BCS model (1) can be generalized to the case of arbitrary degeneracy of different energy levels corresponding to the energies of the Cooper pairs which is equivalent to the case of an arbitrary spin \(s_i\) assigned to the site \(i\) with the energy \(\epsilon_i\). Previously, in Section 2, the case \(s_i = 1/2\) was considered. One can show that the limit \(s_i \to \infty\) for the number of sites \(\epsilon_i\) corresponds to the special generalization of the Dicke model. Thus, using the general six-vertex model, the eigenfunctions and the eigenvalues of the Hamiltonian can be obtained. Clearly, instead of the elementary \(S\)-matrix one can take the following Lax operator in the equation (2):

\[
L_{i0}(\xi_i - t) = \xi_i - t + \eta(\sigma S_i),
\]

where \(S^a_i, \ a = x, y, z\) are the spin operators with the value of the spin \(s_i\), \((S_i)^2 = s_i(s_i + 1)\). Considering the quasiclassical limit of the transfer matrix (2) we obtain instead of (1) the Hamiltonian

\[
H = \sum_{i=1}^{N} \epsilon_i S^z_i - g S^+ S^-,
\]

where \(S^\pm = \sum_{i=1}^{N} S^\pm_i\). Clearly, in terms of the initial BCS model the integer parameters \(2s_i\) correspond to the degeneracies \(2s_i\) of \(i\)-th level with the energy \(\epsilon_i\). The construction of the eigenstates is the same as in Section 2 and the equations (11) become

\[
\sum_{\alpha=1}^{N} \frac{2s_i}{t_i - \xi_\alpha} - 2 \sum_{\alpha \neq i}^{M} \frac{1}{t_i - t_\alpha} = -\frac{1}{g},
\]

while the eigenvalues of the Hamiltonian have the same form \(E = \sum_{i=1}^{N} \epsilon_i\). The eigenstates can be constructed also with the help of the Gaudin operators \(\Sigma^\pm(t) = \sum_{i=1}^{N} \frac{S_i^\pm}{t - \xi_i}\), with \(\xi_i = \epsilon_i\). For the BCS model with the equal degeneracies \(2s\) at each site the equations (19) take the form

\[
(2s) \sum_{\alpha=1}^{N} \frac{1}{t_i - \xi_\alpha} - 2 \sum_{\alpha \neq i}^{M} \frac{1}{t_i - t_\alpha} = -\frac{1}{g},
\]

Considering the limit \(s \to \infty\) of the spin at the single site, and using the expressions of the spin operators at this site through the Holstein-Primakoff bosons, \([\phi; \phi^+] = 1, S^z = \phi^+ \phi - s, S^+ = \phi^+ (2s - \phi^+ \phi)^{1/2}, S^- = (2s - \phi^+ \phi)^{1/2}\phi\), we get after rescaling the parameters the Dicke Hamiltonian [22]:

\[
H = \omega \phi^+ \phi + \sum_{i=1}^{N} \epsilon_i n_i - g \left( S^+ \phi + S^- \phi^+ \right).
\]
Thus the spectrum and the eigenstates of this Hamiltonian can also be obtained using the appropriate limiting procedure of that of the transfer matrix of the six-vertex model. Remarkably, this model can be generalized to the case of several species of the oscillators $\phi_i^+, \phi_i$ with different frequencies $\omega_i$.

3. Connection with the monodromy matrix in the $F$-basis.

Let us show how the above results can be obtained using the operator expression of the monodromy matrix in the $F$-basis, the basis obtained with the help of the factorizing operator $F$ introduced in ref.[26]. One can construct the operator $F = F_{1\ldots N}$ which diagonalizes the operator $A(t)$ [26],[27],[28] ($A^F(t) = F^{-1}A(t)F$). Using the notations

$$\tilde{c}(t) = \frac{\phi(t)}{\phi(t + \eta)}, \quad \tilde{b}(t) = \frac{\phi(\eta)}{\phi(t + \eta)},$$

where $\phi(t) = t$ for the rational case and $\phi(t) = \sin(t)$ for the trigonometric case, the diagonal operator $A^F(t)$ has the following form:

$$A^F(t) = \prod_{i=1}^{N} (\tilde{c}(\xi_i - t)(1 - n_i) + n_i). \quad (22)$$

Let us briefly mention some of the properties of the operator $F$. The explicit form of the operator $F$ is

$$F_{12\ldots N} = \hat{F}_1 \hat{F}_2 \ldots \hat{F}_N, \quad \hat{F}_i = (1 - \hat{n}_i) + \hat{T}_i \hat{n}_i, \quad (23)$$

where $\hat{n}_i$ is the operator of the number of particles (spin up) at the site $i$ and the operator $T_n$ is given by the equation

$$T_n = S_{n+1,n}S_{n+2,n} \ldots S_{Nn}.$$ 

One can obtain the following formulas for the matrix elements of the operator $F$ [28] in the following form:

$$F_{\{m\}\{n\}} = \langle \{m\} | B(\xi_{n_1})B(\xi_{n_2}) \ldots B(\xi_{n_M})|0\rangle,$$

where the sets of coordinates $\{m\}$ and $\{n\}$ label the positions of the occupied sites. The similar expression can be obtained for the inverse operator $F^{-1}$. Apart from diagonalizing the operator $A(t)$, the operator $F$ is the factorizing operator [26] in the following sense. For any permutation of indices $\sigma \in S_N$ ($S_N$ - is the group of permutations) we have the equation $F = F^\sigma R^\sigma$, where $F^\sigma_{12\ldots N} = F_{\sigma_1\sigma_2\ldots\sigma N}^\sigma$ (including the permutation of the inhomogeneity parameters $\xi_i$) and $R^\sigma_{1\ldots N}$ is the operator constructed from the $S$-matrices defined in such a way that for the permutation of the monodromy matrix $T_0^\sigma = T_{0,\sigma_1\sigma_2\ldots\sigma N}$ we have $T_0^\sigma = (R^\sigma)^{-1}T_0 R^\sigma$. For the particular permutation $\sigma(\{n\})$ such that $\sigma 1 = n_1, \ldots, \sigma M = n_M$ ($n_1 < n_2 < \ldots < n_M$) the factorization condition is represented as $F(F^\sigma(\{n\}))^{-1} = T_{n_1} \ldots T_{n_M}$. To prove the factorizing property of the operator (23) it is sufficient to consider only one particular permutation, for example, the
permutation \((i, i + 1)\), since all the other can be obtained as a superposition of these ones for different \(i\). One can show, that \(F = S_{i+1,i} F^{(i,i+1)}\), which evidently proves the factorization property.

The matrix elements of the operators \(B(t)\) and \(C(t)\) in the \(F\)-basis: \(B^F(t) = F^{-1} B(t) F\) (and the same for \(C(t)\)) have the following form

\[
B^F(t) = \sum_i \sigma^+_i \hat{b}(\xi_i - t) \prod_{k \neq i} \left( \hat{c}(\xi_k - t)(\hat{c}(\xi_k - \xi_i))^{-1}(1 - n_k) + n_k \right).
\]

\[\text{(24)}\]

\[
C^F(t) = \sum_i \sigma^-_i \hat{b}(\xi_i - t) \prod_{k \neq i} \left( \hat{c}(\xi_k - t)(1 - n_k) + \hat{c}(\xi_i - \xi_k)^{-1} n_k \right).
\]

\[\text{(25)}\]

The operators (24) and (25) are quasilocal i.e. they describe flipping of the spin on a single site with the amplitude depending on the positions of the up-spins on the other sites of the chain. It is easily seen that in the limit \(\eta \to 0\) these operators reduce to the operators \(\Sigma^\pm(t)\) (13). The operator \(D^F(t)\) can be found, for example, using the quantum determinant relation and has a (quasi)bilocal form. In fact, the following operator identity for the elements of the monodromy matrix (2) can be derived:

\[
D(t)A(t - \eta) - B(t)C(t - \eta) = \prod_\alpha ( (t - \xi_\alpha + \eta/2)(t - \xi_\alpha - \eta/2) ),
\]

which is readily transformed to the \(F\)-basis. From this relation the explicit form of the operator \(D^F(t)\) can be obtained. However perhaps the simplest way to obtain \(D^F(t)\) is to use the well known basic commutational relation following from the Yang-Baxter equation:

\[
[B(t); C(q)] = \frac{\hat{b}(q - t)}{\hat{c}(q - t)} (D(t)A(q) - D(q)A(t)).
\]

Considering this equation in the \(F\)-basis in the limit \(q \to \infty\), we obtain the following expression for the operator \(D^F(t)\):

\[
D^F(t) = A^F(t) + [B^F(t); C^-],
\]

where the operators \(A^F\), \(B^F\) are given in (22), (24) and the operator \(C^- = \lim_{q \to \infty} (q/\eta) C^F(q)\) equals:

\[
C^- = - \sum_i b_i \prod_{k \neq i} \left( (1 - n_k) + (\hat{c}(\xi_i - \xi_k))^{-1} n_k \right).
\]

To find the physically interesting BCS-type models it is not necessary to consider the quasi-classical limit. For example, considering the limit \(t \to \infty\) for \(Z^F(t)\) we obtain (omitting an overall factor \(\eta/t\) and an additive constant depending on \(M\)) the Hamiltonian of the form

\[
H = [B^+; C^-],
\]

where the operator \(B^+\) equals

\[
B^+ = - \sum_i b^+_i \prod_{k \neq i} \left( n_k + (\hat{c}(\xi_k - \xi_i))^{-1}(1 - n_k) \right),
\]

\[\text{11}\]
and the operator $C^-$ was defined above. This Hamiltonian has a (quasi)bilocal form and, apart from the set of the parameters $\{\xi\}$, depends on the additional parameter $\eta$.

So the transfer matrix in the F-basis $Z^F(t) = A^F(t) + D^F(t)$ represents the Hamiltonian of the BCS-type model with the varying interaction of Cooper pairs (depending on the occupation numbers on the other sites) even without taking the quasiclassical limit. It is easily seen that in the limit $\eta \to 0$ the results of the previous section are reproduced in such a way that the second term in eq.(4) corresponds to the term $D^F(t)$. Note also that since the pseudovacuum state is invariant with respect to the action of the factorizing operator $F|0\rangle = |0\rangle$, the eigenfunctions of $Z^F(t)$ have the form $\prod_i B^F(t_i)|0\rangle$. As it was mentioned above, in the quasiclassical limit the operators $B^F(t)$ and $C^F(t)$ eqs.(24), (25) reduce to the operators $\Sigma^+(t)$ and $\Sigma^-(t)$ defined in eq.(13). Concluding this section, let us stress once more that the Hamiltonian $Z^F(t)$, generalized by including the twist angle, contains the terms of the form $\sum_i \epsilon_i n_i$ and the terms of the bilocal form $\sim b^+_i b_j$ with the amplitude depending on the occupation numbers on the other sites, and reduces to the BCS Hamiltonian (1) in the quasiclassical limit with the corresponding twist angle.

4. Correlation functions.

Here we derive the analytical expressions for the simplest physically interesting correlation functions: $\langle 0|n_i|0\rangle$, $\langle 0|b^+_i b_j|0\rangle$, $\langle 0|S^+ S^-|0\rangle$ and $\langle 0|\sigma_i \sigma_j|0\rangle$. For simplicity we restrict ourselves to the rational case although the similar formulas can be easily obtained for the general trigonometric (hyperbolic) models. First, we consider the following scalar product:

$$S_M(\{\lambda\}, \{t\}) = \langle 0|C(\lambda_1)C(\lambda_2)\ldots C(\lambda_M)B(t_1)B(t_2)\ldots B(t_M)|0\rangle,$$

where $\{\lambda\}$ and $\{t\}$ are the two sets of parameters, the set $\{t\}$ satisfies the Bethe Ansatz equations and $\{\lambda\}$ is an arbitrary set of parameters. According to the connection with the six-vertex model revealed in Section 2, to obtain the expression for the scalar product in the case when the set of the parameters $\{t\}$ satisfies the equations (11) one can use as a first step the known formula for the six - vertex model in the case when the parameters $\{t\}$ satisfy the usual Bethe Ansatz equations (10) [28], [29], [30], [31] (see also the direct proof in ref. [27] and in a different way in ref.[32]), and then decompose it in powers of $\eta$ (extract the leading power $\eta^{2M}$). The formulas for the XXX- spin chain are

$$S_M(\{\lambda\}, \{t\}) = \frac{1}{\prod_{i<j}(t_i - t_j) \prod_{j<i}(\lambda_i - \lambda_j)} \det_{ij}(M_{ij}(t, \lambda)),$$

where the matrix $M_{ij}$ equals:

$$M_{ij}(t, \lambda) = \frac{\eta}{(t_i - \lambda_j)} \left( \frac{a(\lambda_j)f^+(\lambda_j)}{t_i - \lambda_j + \eta} - \frac{f^-(\lambda_j)}{t_i - \lambda_j - \eta} \right),$$

or
where the following notations are used:

\[ f^\pm(\lambda) = \prod_{\alpha=1}^{M} (t_\alpha - \lambda \pm \eta), \]

and \( a(\lambda) = \prod_\alpha \tilde{c}(\xi_\alpha - \lambda) \). Note that in eq.(26) the operators \( B(t), C(t) \) are normalized in such a way that their quasiclassical limit is given by the operators (13). From the equation (28) taking the limit \( \lambda_i \to t_i \) one can easily obtain the formula for the norm of the Bethe eigenvector:

\[ N_M(t) = \eta^M \frac{\prod_{i \neq j} (t_i - t_j + \eta)}{\prod_{i \neq j} (t_i - t_j)} \det [N_{ij}(t)], \]

where the matrix \( N_{ij} \) can be represented in the form:

\[ N_{ij} = \frac{2\eta}{(t_{ij} + \eta)(t_{ij} - \eta)}, \quad (i \neq j), \quad N_{ii} = -\frac{\partial}{\partial t_i} \ln (a(t_i)) - \sum_{\alpha \neq i} \frac{2\eta}{(t_{ai} + \eta)(t_{ai} - \eta)}, \]

where \( t_{ij} = t_i - t_j \) and \( a(t) = e^{2\eta/g} \prod_\alpha \tilde{c}(\xi_\alpha - t) \). \( a(t) \to 1 + 2\eta/g + \eta \sum_\alpha (t - \xi_\alpha)^{-1} \) at \( \eta \to 0 \). Extracting from the last expression the term of order \( \eta^2 \), we get the norm of the eigenstate:

\[ \langle \phi(t)|\phi(t)\rangle = \det [N_{ij}(t)], \]

(here \( \langle \phi(t)|\phi(t)\rangle = S_M(t, t) \)) where the matrix \( N_{ij} \) is given by

\[ N_{ii} = \sum_\alpha \frac{1}{(t_i - \xi_\alpha)^2} - \sum_{\alpha \neq i} \frac{2}{(t_{\alpha i})^2}, \quad N_{ij} = \frac{2}{(t_{ij})^2}, \quad i \neq j. \quad (29) \]

The formula for the norm was first derived by Richardson [3]. Note that the norm does not depend on \( \epsilon_i \) explicitly. Similarly the general scalar product (26) is given by the formula (27) with the matrix \( M_{ij} \) of the following form:

\[ M_{ij}(\lambda, t) = \frac{1}{(t_i - \lambda_j)^2} \left( \prod_\alpha (t_\alpha - \lambda_j) \right) \left[ \sum_\alpha \frac{1}{(\lambda_j - \xi_\alpha)} - 2 \sum_{k \neq i} \frac{1}{(\lambda_j - t_k)} + \frac{1}{g} \right]. \quad (30) \]

Thus the scalar product (26) with one Bethe eigenstate is obtained in the form given by the equations (27), (30). Equivalently, since the product in the matrix (30) depends only on the number of the column \( j \) this scalar product can be also represented in the form

\[ S_M(\{\lambda\}, \{t\}) = \frac{\prod_{i \neq j} (t_i - \lambda_j)}{\prod_{i < j} (t_i - t_j) \prod_{i \neq j} (\lambda_i - \lambda_j)} \det (\tilde{M}_{ij}(\lambda, t)), \]

where the matrix \( \tilde{M}_{ij}(\lambda, t) \) is given by

\[ \tilde{M}_{ij}(\lambda, t) = \frac{1}{(t_i - \lambda_j)^2} \left[ \sum_\alpha \frac{1}{(\lambda_j - \xi_\alpha)} - 2 \sum_{k \neq i} \frac{1}{(\lambda_j - t_k)} + \frac{1}{g} \right]. \]
However for the calculation of the correlators in order to consider the limit $\lambda_i \to t_i$ for some of $\lambda_i$, it is more convenient to use the formulas (27), (30). The formula for the norm can be obtained directly by taking the limit $\lambda_i \to t_i$ in this expression for $S_M(\lambda, t)$ for the BCS model. Since the set \( \{ t \} \) satisfies the Richardson Bethe Ansatz equations (11) we obtain from (30) in this limit the matrix elements

$$M_{ii} \to \prod_{\alpha \neq i} (t_\alpha - t_i) \left( \sum_\alpha \frac{1}{(t_i - \xi_\alpha)^2} - 2 \sum_{\alpha \neq i} \frac{1}{(t_i^\alpha)^2} \right), \quad M_{ij} \to \prod_{\alpha \neq j} (t_\alpha - t_j) \frac{1}{(t_i - t_j)^2}, \quad i \neq j.$$

Again, since the product which depend only on the number of the column $j$ can be written in front of determinant, taking into account the products in front of the determinant (27), we obtain exactly the norm (29). For completeness, let us present another equivalent expression for the scalar product with one Bethe eigenstate which can also be used for the derivation of the norm and can be useful for taking different limits in the process of the calculation:

$$S_M(\lambda, t) = \frac{\prod_{\alpha \neq j} (t_i - \lambda_j)}{\prod_{i < j} (t_i - t_j) \prod_{j < i} (t_i - \lambda_j)} \det_{ij} \left( \tilde{M}_{ij}(\lambda, t) \right),$$

where the new matrix $\tilde{M}_{ij}$ is equal to

$$\tilde{M}_{ij}(\lambda, t) = \frac{(t_j - \lambda_j)}{(t_i - \lambda_j)^2} \left( \sum_\alpha \frac{1}{\lambda_j - \xi_\alpha} - 2 \sum_{k \neq i} \frac{1}{\lambda_j - t_k} + \frac{1}{g} \right),$$

where the first term in the numerator also depends only on the index $j$.

Now one can calculate the physically interesting correlation functions using the Algebraic Bethe Ansatz method. Although the expressions for some of the correlators can be obtained by the method of variations over the parameters [3], [6], which we present in the Appendix B, the determinant expressions obtained with the help of the Algebraic Bethe Ansatz method are different in the form (although equivalent to the variational ones). At the same time the direct Bethe Ansatz method allows for the computations of some of the correlation functions (such as $\langle b_i^+ b_j \rangle$, for example) that are not accessible by the variational method and can be useful for the computation of the correlators in the different BCS-like models.

Let us proceed with the calculation of the simplest correlation function $\langle n_i \rangle$ along the lines of ref.[27] and using the technique developed above. First, we consider the action of the operator $A(\xi_i)$ at the state $|\phi(\lambda)\rangle$ (9) where $\lambda_i$ are not necessarily satisfy the Bethe Ansatz equations. Due to the symmetry of the problem, one can consider the site $\xi_i$. One can use the formulas for the scalar product with Bethe eigenstate (27), (28) taking subsequently the quasiclassical limit. Note that due to the symmetry of the Hamiltonian (1) here it is not necessary to use the general formulae of the Quantum Inverse Scattering method for the six-vertex model [33]. However, the most simple way is to represent the eigenstate directly in terms of the operators (13),

$$|\phi(t)\rangle = \Sigma^+(t_1)\Sigma^+(t_2)\ldots\Sigma^+(t_M)|0\rangle (31)$$
and use the formula for the scalar product (30). To calculate the expectation value \( \langle \phi(t) | n_1 | \phi(t) \rangle \) we act by the operator \( n_1 \) on the state (31) using the following relation:

\[
n_1 \Sigma^+(t) = \Sigma^+(t)n_1 + \frac{1}{(t - \xi_1)} b_1^+, \quad n_1 | 0 \rangle = 0.
\]

The result can be represented in the form which allows one to apply the expression (30) for the scalar product:

\[
n_1 | \phi(t) \rangle = \sum_i \frac{1}{(t_i - \xi_1)} \lim_{\zeta \to \xi_1} (\zeta - \xi_1) \left( \Sigma^+(t_1) \ldots \Sigma^+(\zeta) \ldots \Sigma^+(t_M) \right) | 0 \rangle,
\]

where the operator \( \Sigma^+(\zeta) \) replace the operator \( \Sigma^+(t_i) \) in eq.(31). The factor \( (\zeta - \xi_1) \to 0 \) in this formula is cancelled by the corresponding term in the denominator in the first sum in the expression (30) for the matrix \( M_{ij} \) at \( \lambda_i = \zeta \). Next, one can use the following theorem. Consider the determinant of the sum of the two matrices \( \det_{ij} (M_{ij} + a_{ij}) \), where the second matrix \( a_{ij} = c_j \phi_i \) - is the matrix of rank 1. Then we have:

\[
\det_{ij} (M_{ij} + a_{ij}) = \det_{ij} (M_{ij}) + \sum_{k=1}^M c_k \det_{ij} (M_{ij}^{(k)}),
\]

where the matrices \( M_{ij}^{(k)} \) differ from the initial matrix \( M_{ij} \) only by the substitution of its \( k \)-th column by \( \phi_i \):

\[
M_{ij}^{(k)} = (1 - \delta_{jk}) M_{ij} + \delta_{jk} \phi_i.
\]

Applying this theorem to the case of the average \( \langle \phi(\lambda) | n_1 | \phi(t) \rangle \), the determinant in eq.(30) transforms into the sum of the determinants in (32) with

\[
\phi_i = M_{ik}^{(k)} = \frac{1}{(t_i - \xi_1)^2 \prod_\alpha (t_\alpha - \xi_1)}, \quad c_k = \prod_{\alpha \neq k} \left( \frac{\lambda_\alpha - \lambda_k}{\lambda_\alpha - \xi_1} \right) \frac{1}{(\lambda_k - \xi_1)}.
\]

Thus, we get the expression

\[
\langle \phi(\lambda) | n_1 | \phi(t) \rangle = \frac{1}{\prod_{i<j} (t_i - t_j) \prod_{j<i} (\lambda_i - \lambda_j)} \left( \det_{ij} (M_{ij} + H_{ij}) - \det_{ij} (M_{ij}) \right),
\]

where the matrix \( H_{ij} = c_j \phi_i \) in the limit \( \lambda_i \to t_i \) equals

\[
H_{ij}(\lambda \to t) = \prod_{\alpha \neq j} (t_\alpha - t_j) \frac{1}{(t_i - \xi_1)^2},
\]

and the matrix \( M_{ij} \) is given by the equation (30). Taking the limit \( \lambda_i \to t_i \) in the whole expression, we finally obtain the formula

\[
\langle \phi(t) | n_1 | \phi(t) \rangle = \det (N_{ij} + H_{ij}) - \det (N_{ij}), \quad (33)
\]
where $N_{ij}(t)$ is the matrix of the norm (29) and the rank-one matrix $H_{ij}$ equals

$$H_{ij} = \frac{1}{(t_i - \xi_1)^2}.$$ 

To find the expectation value one should divide this expression by the norm of the eigenstate $|\phi(t)\rangle$ which is given by $\det(N_{ij})$. Thus the expectation value $\langle n_i \rangle$ is represented as a ratio of the determinants. This formula can be used in the numerical evaluation of the occupation number in the case of finite system (for finite $N$).

The same method can be used to obtain the determinant representation for the two-point correlation function $\langle b_i^+ b_j \rangle$. Note that for this correlator the final expression was not obtained in ref.[3]. Due to the symmetry of the problem it is sufficient to calculate the average $\langle \phi(t)|b_2^+ b_1|\phi(t)\rangle$. First, we act by the operator $b_1$ on the state $|\phi(t)\rangle$ using the formula

$$b_1 \Sigma^+(t) = \Sigma^+(t)b_1 + \frac{1}{(t - \xi_1)}(1 - 2n_1).$$

Then the action of the operator $b_1$ produce the state (see also the Appendix A):

$$b_1|\phi(t)\rangle = \sum_i \frac{1}{(t_i - \xi_1)} \left( \Sigma^+(t_1) \ldots (i) \ldots \Sigma^+(t_M) \right) |0\rangle$$

$$- 2 \sum_{i<j} \frac{1}{(t_i - \xi_1)(t_j - \xi_1)} \left( \Sigma^+(t_1) \ldots (i) \ldots (j) \ldots \Sigma^+(t_M) \right) b_i^+ |0\rangle,$$

where we denote by $(i)$, $(j)$ the absence of the operators $\Sigma^+(t_i), \Sigma^+(t_i)$ in the last product. Clearly, the action of the operator $b_2^+ b_1$ produces the state with two terms (see the above formulas and the Appendix A for more details) where the first term is equal to

$$b_2^+ b_1|\phi(t)\rangle^{(1)} = \sum_i \frac{1}{(t_i - \xi_1)} \lim_{\zeta \to \xi_2} (\zeta - \xi_2) \left( \Sigma^+(t_1) \ldots \Sigma^+(\zeta) \ldots \Sigma^+(t_M) \right) |0\rangle,$$

where the operator $\Sigma^+(\zeta)$ is substituted instead of $i$-th operator $\Sigma^+(t_i)$. Proceeding in the same way as for the average $\langle n_i \rangle$ we get the similar expression for the first term

$$\langle \phi(t)|b_2^+ b_1|\phi(t)\rangle^{(1)} = \det(N_{ij} + \tilde{H}_{ij}^{(1)}) - \det(N_{ij}),$$

where the rank-one matrix $\tilde{H}_{ij}^{(1)}$ is equal to

$$\tilde{H}_{ij}^{(1)} = \frac{1}{(t_i - \xi_2)^2(t_j - \xi_1)^2}.$$
which means that the two columns should be replaced in the resulting determinant. Namely, repeating the above procedure and taking into account the form of the matrix $M_{ij}$ (30) we come to the expression

$$\langle \phi(t)|b_2^+b_1|\phi(t)\rangle^{(2)} = -2 \sum_{k<l} \frac{1}{(t_k - \xi_1)(t_l - \xi_1)} \left( \frac{(t_k - \xi_2)(t_l - \xi_1)}{(t_k - t_l)(\xi_2 - \xi_1)} \right) \det \left( N_{ij}^{(k,l)} \right),$$

where the expression in the parenthesis comes from the factor which can be written in front of the determinant in eq.(30) and the matrix $N_{ij}^{(k,l)}$ equals $N_{ij}$ with the exception of the two columns $k, l$, which are equal:

$$N_{ik}^{(k,l)} = \frac{(t_k - \xi_1)}{(t_i - \xi_1)^2}, \quad N_{il}^{(k,l)} = \frac{(t_l - \xi_2)}{(t_i - \xi_2)^2}.$$

In the equivalent form the last expression for the second term is

$$\langle \phi(t)|b_2^+b_1|\phi(t)\rangle^{(2)} = -2 \sum_{k<l} \left( \frac{1}{(t_k - t_l)(\xi_2 - \xi_1)} \right) \det \left( \tilde{N}_{ij}^{(k,l)} \right),$$

where the columns $k, l$ of the new matrix $\tilde{N}_{ij}^{(k,l)}$ are

$$\tilde{N}_{ik}^{(k,l)} = \phi_i^{(k)} = \frac{(t_k - \xi_2)}{(t_i - \xi_1)^2}, \quad \tilde{N}_{il}^{(k,l)} = \phi_i^{(l)} = \frac{(t_l - \xi_2)}{(t_i - \xi_2)^2}.$$ 

Note that in the particular case of the average $\langle b_1^+b_1 \rangle$ due to the first term we immediately get the expression for $\langle n_1 \rangle$, since the determinant in the second term is equal to zero due to the presence of two identical columns in the matrix $\tilde{N}_{ij}^{(k,l)}$. Thus, the remarkably simple expression as a sum of determinants is obtained.

Let us perform the similar calculations for the average $\langle n_i n_j \rangle$. Considering the action of the operator $n_2n_1$ on the state $|\phi(t)\rangle$, we get the expression similar to the second term for $b_2^+b_1|\phi(t)\rangle$:

$$n_2n_1|\phi(t)\rangle = \sum_{i\neq j} \frac{1}{(t_i - \xi_1)(t_j - \xi_2)} \lim_{\zeta_1 \rightarrow \xi_1, \zeta_2 \rightarrow \xi_2} (\zeta_1 - \xi_1)(\zeta_2 - \xi_2) \left( \Sigma^+(t_1)\ldots\Sigma^+(\zeta_1)\ldots\Sigma^+(\zeta_2)\ldots \right) |0\rangle.$$

Repeating the calculations performed for the average $\langle b_i^+b_j \rangle$, we obtain the following sum of the determinants

$$\langle \phi(t)|n_2n_1|\phi(t)\rangle = \sum_{k \neq l} \left( \frac{(t_k - \xi_2)(t_l - \xi_1)}{(t_k - t_l)(\xi_2 - \xi_1)} \right) \det \left( \tilde{N}_{ij}^{(k,l)} \right),$$

where the columns $k, l$ does not depend on the indices $k$ and $l$:

$$\tilde{N}_{ik}^{(k,l)} = \phi_i^{(k)} = \frac{1}{(t_i - \xi_1)^2}, \quad \tilde{N}_{il}^{(k,l)} = \phi_i^{(l)} = \frac{1}{(t_i - \xi_2)^2}.$$
The last expression can also be represented in the equivalent form since, taking into account
the factors in the parenthesis, one can modify the the columns of the matrix \( \hat{N}_{ij}^{(k,l)} \). Namely,
we have

\[
\langle \phi(t)|n_2n_1|\phi(t)\rangle = \sum_{k \neq l} \frac{1}{(t_k - t_l)(\xi_2 - \xi_1)} \det \left( \hat{N}_{ij}^{(k,l)} \right),
\]
where the two columns of the new matrix \( \hat{N}_{ij}^{(k,l)} \) are

\[
\hat{N}_{ik}^{(k,l)} = \phi_i^{(k)} = \frac{(t_k - \xi_2)}{(t_i - \xi_1)^2}, \quad \hat{N}_{il}^{(k,l)} = \phi_i^{(l)} = \frac{(t_l - \xi_1)}{(t_i - \xi_2)^2}.
\]

Note that the above expression for \( \langle n_in_j \rangle \) is obtained for \( i \neq j \). Clearly, at \( i = j \) the
average coincides with \( \langle n_i \rangle \). Since both averages \( \langle b_i^+b_j \rangle \) and \( \langle n_in_j \rangle \) are found, the correlator
\( \langle \sigma_i\sigma_j \rangle \) can also be calculated. It is straightforward but lengthy calculation to show that
the above expressions lead to the expression for \( \langle \sigma_i\sigma_j \rangle \) found by the variational method and
presented in the Appendix B. Thus, the remarkably simple determinant expressions for the
pair correlators of the model suitable for the numerical calculations even at the sufficiently
large \( N \) are obtained.

As for another physically interesting correlator - the expectation value \( \langle S^+S^- \rangle \), it could
be obtained either using the correlator \( \langle b_i^+b_j \rangle \) obtained above or directly with the help of the
representation of the average as a certain limit of the scalar product. However in fact it is
sufficient to calculate the average \( \langle \sum_i \xi_in_i \rangle \) using the formula (33) which leads to the final result

\[
\langle \phi(t)|\sum_i \xi_in_i|\phi(t)\rangle = \det(N_{ij} + \hat{H}_{ij}) - \det(N_{ij}), \quad (35)
\]
where \( \hat{H}_{ij} \) is again the rank-one matrix of the form

\[
\hat{H}_{ij} = \sum_{\alpha} \frac{\xi_{\alpha}}{(t_i - \xi_{\alpha})^2}.
\]

Clearly, due to the form of the Hamiltonian (1), from the equation (35) one can obtain the
expression for the average \( \langle S^+S^- \rangle \). We will show below that, in fact, the sum of eq.(35) and
the expression for \( \langle S^+S^- \rangle \) obtained by the different method gives the total energy \( E \).

Let us note that since the average should be divided by the norm of the state in all of the
above mentioned cases the expectation values of the operators can be represented in the form

\[
\det_{ij} \left( \delta_{ij} + (N^{-1}H)_{ij} \right),
\]

where \( \delta_{ij} \) is the Kronecker symbol and the matrix \( H_{ij} \) stands for one of the three matrices
introduced above (33), (34), (35). Since in the case of the averages \( \langle n_i \rangle \) and \( \langle S^+S^- \rangle \) (or,
equivalently, \( \langle \sum_i \xi_in_i \rangle \)) the rank-one matrices \( H_{ij}, \hat{H}_{ij} \) depend only on the first index \( i \), the
last expression can be represented in a different form with the help of the identity \( \det(\delta_{ij} + \chi_i) = 1 + \sum_i \chi_i \). Namely for the first average we get

\[
\langle n_i \rangle = \sum_{i,j} (N^{-1})_{ij} \phi_j, \quad \phi_i = \frac{1}{(t_i - \xi_i)^2}. \quad (36)
\]
From the equation (36) one readily obtains for the average \( \langle S^+ S^- \rangle = \sum_i \xi_i n_i/g - E/g \)

\[
\langle \sum_i \xi_i n_i \rangle = \sum_{i,j} \left( N^{-1} \right)_{ij} \phi_j, \quad \phi_i = \sum_\alpha \frac{\xi_\alpha}{(t_i - \xi_\alpha)^2},
\]

(37)
in agreement with the expression (35) obtained with the help of the different method. From the equation (36) one can check that the total number of particles is \( \sum_i \langle n_i \rangle = M \) which can be easily seen from the following property of the matrix \( N_{ij} \):

\[
\sum_\alpha N_{i\alpha} = f(t_i), \quad f(t_i) = \sum_\alpha \frac{1}{(t_i - \xi_\alpha)^2},
\]

see eq.(29), or equivalently \( 1_i = \sum_j (N^{-1})_{ij} f(t_j) \).

Now let us turn to the calculation of the average \( \langle S^+ S^- \rangle \) in a direct way. With the help of the formula \( S^+ = \lim_{\zeta \to \infty} \zeta \Sigma^+(\zeta) \) we get using the equations (11) that the operator \( S^+ S^- \) acts on the state \( |\phi(t)\rangle \) as

\[
S^+ S^- |\phi(t)\rangle = \frac{1}{g} \sum_i \lim_{\zeta \to \infty} \zeta \left( \Sigma^+(t_1) ... \Sigma^+(\zeta)(i) ... \Sigma^+(t_M) \right) |0\rangle,
\]

where \( \Sigma^+(\zeta) \) stands on the \( i \)-th place (see equation (42) in the Appendix A). Here the factor \( 1/g \) appears due to the equations (11). Then taking the limit \( \zeta \to \infty \) we finally obtain

\[
\langle \phi(t) | S^+ S^- | \phi(t) \rangle = \det(N_{ij} + H_{ij}) - \det(N_{ij}),
\]

with

\[
H_{ij} = \frac{1}{g^2},
\]

where the second factor \( 1/g \) is due to the last term in the matrix (30). The matrix \( H_{ij} \) does not depend on the indices at all. Dividing this expression by the norm and repeating the steps described above, we obtain the following formula for the average \( \langle S^+ S^- \rangle \):

\[
\langle S^+ S^- \rangle = \frac{1}{g^2} \sum_{i,j} \left( N^{-1} \right)_{ij},
\]

(38)
in agreement with the result based on the variational method. This form is analogous to the expressions (36), (37), for the other correlators. With these three expressions one can check the consistency of the above formulas. Namely the sum of (37) and (38) should be equal to the total energy of the system \( E \). To prove it we use the following equation for the matrix \( N_{ij} \):

\[
\sum_\alpha N_{i\alpha} t_\alpha = \sum_\alpha \frac{\xi_\alpha}{(t_i - \xi_\alpha)^2} - \frac{1}{g},
\]

which can be easily obtained using the equations (11). Multiplying both sides of this equation by the matrix \( N^{-1} \), we immediately get the energy \( E \) as a sum of (37) and (38).
In the cases, when the expressions for the correlation functions can be obtained with the help of the variation over the parameters of the total energy and the integrals of motions [3], [6], the results coincide with that obtained above as the ratio of the determinants. That can be easily proved for the averages $\langle n_i \rangle$, $\langle (\sigma_i \sigma_j) \rangle$, $\langle S^+ S^- \rangle$ using the formula for the matrix inverse to the norm matrix $N_{ij}$ (29). We discuss the correspondence of this two approaches in more details in the Appendix B.

Conclusion.

In conclusion, for the applications to the realistic fermionic systems, it would be interesting to find the realistic discrete-state integrable Hamiltonians with the interaction of fermions containing the terms describing the breaking of the Cooper pairs. It is possible that the models proposed in the present paper can be generalized to this case. Let us mention that even in the framework of the BCS model (1) one can include the terms describing the interaction of the single-electron states in the following way. Namely, consider the lattice Hamiltonian constructed from the fermionic operators $c^\dagger_{i\sigma} (c_{i\sigma})$ with $\sigma = \uparrow, \downarrow = 1, 2$:

$$H = \sum_i \epsilon_i (n_{1i} + n_{2i}) + \sum_{i<j} V_{ij} (S_i S_j) - g \sum_{i<j} \left( b^+_i b_j + b^+_j b_i \right),$$

where $n_{1\sigma} = c^\dagger_{i\sigma} c_{i\sigma}$, $b^+_i = c^\dagger_{i1} c_{i2}$ and $S^a_i = \frac{1}{2} (c^\dagger_i \sigma^a_i c_i)$ (the sum over the spin indices 1, 2 is implied). Since for this Hamiltonian the number of double-occupied sites is conserved the eigenstates are given by the superposition of the eigenstates of the BCS Hamiltonian and the eigenstates corresponding to the second term $\sum_{i<j} V_{ij} (S_i S_j)$. Here the coefficients $V_{ij}$ are not necessarily the constant but for an integrable model can correspond to any of the integrable quantum spin chains (for example, the XXX-spin chain or the Haldane-Shastry spin chain in its trigonometric [34] or hyperbolic [35] versions). Note also that using the method presented the Hamiltonians of the discrete-state BCS-like models related to the exactly solvable $t-J$-models of different symmetry [36] and the models based on the different Lie algebras both in the rational and the trigonometric cases can be constructed. The main shortcoming of these models from the point of view of the applications to the description of electrons close to the Fermi-surface is the presence of the single-electron hopping terms. In this context the study of Gaudin magnets for the different Lie algebras can be useful.

We have shown that for the calculation of the correlators for the Gaudin magnets and the BCS model no special technique for the calculation of the scalar products [14] is required. Using the determinant expressions for the correlation functions obtained in the present paper, one could hope to find the analytical results for the correlators in the thermodynamic limit for the systems with different density of energy levels. The determinant expressions for the correlation functions can also be useful for the numerical evaluation of correlators both in the case of the BCS model with the fixed number of pairs and the general BCS, which takes into account the existence of the single-occupied energy levels for the excited states. In
conclusion, let us note, that the similar determinant expressions can be obtained both for the BCS and the Gaudin magnets models based on the different Lie algebras, which is an interesting problem from the theoretical point of view.

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Appendix A.

Here we present the most simple and beautiful procedure of diagonalization of the Hamiltonian (see for example [6]) based on the construction directly in terms of the operators (13). We look for the eigenstate in the form:

$$|\phi(t)\rangle = \sum_{+}^{\infty} (t_1) \sum_{+}(t_2) \ldots \sum_{+}(t_M)|0\rangle$$

(39)

We use the following commutational relations for the operator $\Sigma^+(t)$ which can be easily proved directly or derived from the general formulas (13):

$$n_1 \Sigma^+(t) = \Sigma^+(t)n_1 + \frac{1}{(t-\xi_1)} b_1^+,$$

$$b_1 \Sigma^+(t) = \Sigma^+(t)b_1 + \frac{1}{(t-\xi_1)} (1-2n_1).$$

(40)

First, using the first relation, we obtain the formula for the action of the operator $n_\alpha$ to the state (39) $|\phi(t)\rangle = |t\rangle$:

$$n_\alpha |t\rangle = \sum_{i} \frac{1}{(t_i-\xi_\alpha)} \Sigma^+(t_1) \ldots (i) \ldots \Sigma^+(t_M) b_\alpha^+ |0\rangle,$$

where the sign $(i)$ means the absence of the operator $\Sigma^+(t_i)$ in the product. Then considering the sum $\sum_\alpha \xi_\alpha n_\alpha$ we obtain the following expression for the first part of the Hamiltonian:

$$\sum_\alpha \xi_\alpha n_\alpha |t\rangle = -S^+ \sum_i |t(i)\rangle + \left(\sum_i t_i\right) |t\rangle,$$

(41)

where we denote by the sign $|t(i)\rangle$ the state (39) without the single operator $\Sigma^+(t_i)$. Next, consider the action of the term $-gS^+S^-$ on the state (39) using the second commutational relation (40). First, considering the action of the operator $b_1$, we obtain the formulas:

$$b_1 |t\rangle = \sum_i \frac{1}{(t_i-\xi_1)} \left(\Sigma^+(t_1) \ldots (1-2n_i) \ldots \Sigma^+(t_M)\right) |0\rangle =$$

$$\sum_i \frac{1}{(t_i-\xi_1)} |t(i)\rangle - 2 \sum_{i<j} \frac{1}{(t_i-\xi_1)(t_j-\xi_1)} \left(\Sigma^+(t_1) \ldots (i)(j) \ldots \Sigma^+(t_M)\right) b_1^+ |0\rangle.$$
After the simple algebraic transformations, using the definition of the operator \( \Sigma^+ (t) \), we obtain the following expression:

\[
S^- |t\rangle = \sum_i \left( \sum_\alpha \frac{1}{t_i - \xi_\alpha} \right) |t(i)\rangle + 2 \sum_{i<j} \frac{1}{t_i - t_j} (|t(j)\rangle - |t(i)\rangle),
\]

which finally leads to the result:

\[
S^+ S^- |t\rangle = \sum_i \left( \sum_\alpha \frac{1}{t_i - \xi_\alpha} - 2 \sum_{\alpha \neq \lambda} \frac{1}{t_i - t_\alpha} \right) S^+ |t(i)\rangle.
\] (42)

Thus combining the equations (41) and (42) we get

\[
H |\phi(t)\rangle = E |\phi(t)\rangle - g \sum_i \left( \sum_\alpha \frac{1}{t_i - \xi_\alpha} - 2 \sum_{\alpha \neq \lambda} \frac{1}{t_i - t_\alpha} + \frac{1}{g} \right) S^+ |t(i)\rangle,
\]

and obtain the eigenvalue \( E = \sum t_i \) and the condition of the cancelation of the “unwanted” terms \( S^+ |t(i)\rangle \) - the equations (11):

\[
\sum_\alpha \frac{1}{t_i - \xi_\alpha} - 2 \sum_{\alpha \neq \lambda} \frac{1}{t_i - t_\alpha} = - \frac{1}{g}.
\]

It is straightforward to find also the eigenvalues of the conserved operators (5) using this method. In fact, performing the similar calculations, we finally obtain the formula for the action of the operators \( H_i \) (5) to the state \( |\phi(t)\rangle \) (39). For the operator

\[
H_1 = -\frac{1}{g} n_1 + \frac{1}{2} \sum_{j \neq 1} \frac{\sigma_1 \sigma_j}{(\xi_1 - \xi_j)}
\]

we get the expression

\[
H_1 |t\rangle = \left( \frac{1}{2} \sum_\alpha \frac{1}{(\xi_1 - \xi_\alpha)} + \sum_i \frac{1}{(t_i - \xi_1)} \right) |t\rangle - \sum_i \left[ \sum_\alpha \frac{1}{t_i - \xi_\alpha} - 2 \sum_{k \neq \lambda} \frac{1}{t_i - t_k} + \frac{1}{g} \right] \frac{1}{t_i - \xi_1} b_1^+ |t(i)\rangle.
\]

The condition of cancelation of the “unwanted” terms \( b_1^+ |t(i)\rangle \) is equivalent to the equations (11) and the eigenvalues of the operators \( H_i \) are given exactly by the equation (12), however, the last expression for \( H_1 |t\rangle \) is valid for the state of the form (39) for an arbitrary sets of the parameters \( \{t\} \) and \( \{\xi\} \).

Note that the similar expression for \( H_i |t\rangle \) can be easily obtained also for the trigonometric case even without the detailed calculations. In fact, it is clear, that the analog of the last formula has the form

\[
H_1 |t\rangle = E_1 (t, \xi) |t\rangle + \sum_i \frac{f(t_i)}{\sin(t_i - \xi_1)} b_1^+ |t(i)\rangle,
\]
where $E_1(t, \xi)$ is the eigenvalue of the operator $H_1$ and the function $f(t_i)$ equals

$$f(t_i) = \sum_{\alpha} \text{ctg}(t_i - \xi_{\alpha}) - 2 \sum_{\alpha} \text{ctg}(t_i - t_{\alpha}) + 1/g,$$

so that the trigonometric Richardson’s equations have the form $f(t_i) = 0$. The coefficient in front of the second term in the formula for $H_1|t\rangle$ can be fixed from the known expression for the action of the operator $n_1$ contained in $H_1$.

**Appendix B.**

Here we compare the results for the correlators obtained with the help of the Algebraic Bethe Ansatz method with the results obtained by means of the variation over the parameters [3], [6]. Variation over the parameters $\xi_i$ and $g$ gives the equation:

$$\sum_{\alpha} \frac{\delta t_i}{(t_i - \xi_{\alpha})^2} - 2 \sum_{\alpha \neq i} \frac{\delta t_i - \delta t_{\alpha}}{(t_i - t_{\alpha})^2} = \sum_{\beta} \frac{\delta \xi_{\beta}}{(t_i - \xi_{\beta})^2} + \frac{\delta g}{g^2}.$$  

For the case of the averages $\langle n_i \rangle, \langle S^+ S^- \rangle$ i.e. varying over the parameters $\xi_i, g$ this equation leads to

$$\sum_j N_{ij} \delta t_j = \sum_\beta \phi_{i\beta} \delta \xi_{\beta} + \delta g/g^2,$$

where $N_{ij}$ is the norm matrix (29) and the matrix $\phi_{i\beta} = 1/(t_i - \xi_{\beta})^2$. The solution of this system of equations gives the variation of the energy $\delta E = \sum_i \delta t_i$ which determines the above averages. One can easily see that for the correlators $\langle n_i \rangle, \langle S^+ S^- \rangle$ and also $\langle \sum_i \xi_i n_i \rangle$ the expressions obtained in Section 4 starting from the determinant expressions are reproduced.

In order to calculate the average $\langle (\sigma_i \sigma_j) \rangle$, we consider the variations of the eigenvalues (12) of the commuting operators (5) over the parameters $\xi_i$:

$$\frac{\delta H_i}{\delta \xi_j} = \frac{1}{2} \frac{1}{(\xi_i - \xi_j)^2} (\sigma_i \sigma_j).$$

We get for the eigenvalues

$$\frac{\delta E_i}{\delta \xi_j} = \frac{1}{2} \frac{1}{(\xi_i - \xi_j)^2} - \sum_l \frac{1}{(t_l - \xi_j)^2} \delta t_l,$$

where according to the formulas of Section 4 the matrix $\delta t_l/\delta \xi_{\beta}$ equals:

$$\frac{\delta t_l}{\delta \xi_{\beta}} = \sum_{j,\beta} (N^{-1})_{lj} \frac{1}{(t_j - \xi_{\beta})^2}.$$

Using these formulas we easily obtain for the average the expression

$$\langle (\sigma_i \sigma_j) \rangle = 1 - 2 (\xi_i - \xi_j)^2 \sum_{l,k} \frac{1}{(t_l - \xi_i)^2} (N^{-1})_{lk} \frac{1}{(t_k - \xi_j)^2}.$$
which can be shown to be in agreement with the determinant expression given in ref. [6]. In fact it is straightforward to represent this average as a ratio of the determinants:

\[ \langle (\sigma_i \sigma_j) \rangle = 1 - 2 (\xi_i - \xi_j)^2 \frac{\det(R_{ij})}{\det(N_{ij})}, \]

where the matrix \( R_{ij} \) is \((M + 1) \times (M + 1)\)-matrix with the indices \( i, j = 0, 1, \ldots M \) and with the matrix elements equal to \( R_{00} = 0 \) and

\[ R_{kl} = N_{kl}, \quad R_{k0} = \frac{1}{(t_k - \xi_j)^2}, \quad R_{0l} = \frac{1}{(t_l - \xi_i)^2} \]

for \( k, l = 1 \ldots M \). Thus the determinant expression is obtained.

**Appendix C**

Using the formulae obtained above, we present here the solution of the modified Knizhnik-Zamolodchikov equations for the vector- valued function \( |\Phi(\xi)\rangle \) of \( N \) variables \( \xi_1 \ldots \xi_N \) corresponding to the \( SL(2) \) algebra:

\[ \left( \gamma \frac{\partial}{\partial \xi_i} - \frac{1}{g} n_i + \frac{1}{2} \sum_{i \neq i} \frac{\sigma_i \sigma_i}{(\xi_i - \xi_j)} \right) |\Phi(\xi)\rangle = 0. \] (43)

The modification corresponds to the term \( \sim n_i \) and at \( g \) equal to infinity and the additional parameter \( \gamma = 1 \) we obtain the usual Knizhnik-Zamolodchikov equation [20] for the correlators of the conformal field theory (WZW- model). We consider the rational case although the same procedure can be performed for the trigonometric case. Our presentation follows the solution [11] and make use of the eigenstates of the BCS- model (or twisted six-vertex model in the quasiclassical limit). However, we present here the solution which is different (in the form) from the approach [11], and based on the off-shell Bethe Ansatz equations for the model (1), or, equivalently, on the equations for the six-vertex model after the quasiclassical limit was already performed. For simplicity consider the case \( \gamma = 1 \).

Following ref. [11] let us seek for the solution of the equation (43) in the form:

\[ |\Phi(\xi)\rangle = \oint dt \chi(t, \xi) |\phi(t)\rangle, \]

where \( |\phi(t)\rangle \) (39) is the eigenstate of the Hamiltonian (1), considered as a vector- valued function of the variables \( t_i \) and \( \xi_j \), and \( \chi(t, \xi) \) is some function to be specified below. We denote by the symbol \( \oint dt \) the integration over the variables \( t_1 \ldots t_M \) in the complex plane over some closed contours \( C_1 \ldots C_M \). The particular solution of the equation depends on the choice of the contours. Taking the derivative of this vector- valued function over the variable \( \xi_1 \) gives

\[ \frac{\partial}{\partial \xi_1} |\Phi(\xi)\rangle = \oint dt \left( \frac{\partial}{\partial \xi_1} \chi(t, \xi) \right) |\phi(t)\rangle + \oint dt \chi(t, \xi) \frac{\partial}{\partial \xi_1} |\phi(t)\rangle. \] (44)
Differentiating the eigenstate $|\phi(t)\rangle$ we obtain

$$\frac{\partial}{\partial \xi_1} |\phi(t)\rangle = \sum_i \frac{1}{(t_i - \xi)^2} b_i^+ |t(i)\rangle,$$

where the state $|t(i)\rangle$ is the state (39) without the single operator $\Sigma^+(t_i)$. Since we assume the closed contours of integration over $t_i$ and the integration by parts can be applied, the second term in the equation (44) can be represented as

$$\sum_i \int dt \left( \frac{\partial}{\partial t_i} \chi(t, \xi) \right) \frac{1}{(t_i - \xi)^2} b_i^+ |t(i)\rangle.$$

To rewrite this term we make use of the obvious similarity between this formula and the expression for $H_1|\phi(t)\rangle$, given by the last formula in the Appendix A, which is valid for an arbitrary parameters $t_i$ and $\xi_\alpha$. In fact, one can represent this equation in the equivalent form as

$$\sum_i f(t_i) \frac{1}{(t_i - \xi)^2} b_i^+ |t(i)\rangle = (E_1(t, \xi) - H_1) |t\rangle,$$

where

$$f(t_i) = \sum_\alpha \frac{1}{t_i - \xi_\alpha} - 2 \sum_{k \neq i} \frac{1}{t_i - t_k} + \frac{1}{g}.$$

If one can choose the function $\chi(t, \xi)$ in such a way that $(\partial/\partial t_i) \chi(t, \xi) = f(t_i) \chi(t, \xi)$ and the last term takes the form

$$\int dt \chi(t, \xi) (E_1(t, \xi) - H_1) |\phi(t)\rangle,$$

where $E_1(t, \xi)$ is the eigenvalue of the operator $H_1$ (12), and the operator $H_1$ is obviously does not depends on $t_i$, then we immediately find that the function $|\Phi(\xi)\rangle$ is the solution of the equation

$$\left( \frac{\partial}{\partial \xi_1} + H_1 \right) |\Phi(\xi)\rangle = 0,$$

provided the function $\chi(t, \xi)$ satisfies the equation $(\partial/\partial \xi_1) \chi(t, \xi) = E_1(t, \xi) \chi(t, \xi)$. Thus we get the following system of the equations for the function $\chi(t, \xi)$:

$$\frac{\partial}{\partial \xi_i} \chi(t, \xi) = \left( \sum_j \frac{1}{t_j - \xi_i} + \frac{1}{2} \sum_{\alpha \neq i} \frac{1}{\xi_i - \xi_\alpha} \right) \chi(t, \xi),$$

$$\frac{\partial}{\partial t_i} \chi(t, \xi) = \left( \sum_\alpha \frac{1}{t_i - \xi_\alpha} - 2 \sum_{k \neq i} \frac{1}{t_i - t_k} + \frac{1}{g} \right) \chi(t, \xi).$$

One can see that this two sets of the equations are compatible and the function $\chi(t, \xi)$ is given by

$$\chi(t, \xi) = e^{\sum_i t_i/g} \prod_{i < j} (t_i - t_j)^{-2} \prod_{\alpha < \beta} (\xi_\alpha - \xi_\beta)^{1/2} \prod_{i, \alpha} (t_i - \xi_\alpha).$$
Remarkably, the variation of this function over the variables $t_i$ gives the equations (11) if one assumes that $t_i$ are at the stationary points corresponding to the function $\chi(t, \xi)$ ($(\partial/\partial t_i)\chi(t, \xi) = 0$). For the parameter $\gamma \neq 1$ the similar expression for the function $\chi(t, \xi)$ can be obtained (the powers of the factors in the products should be divided by $\gamma$). In the similar way the solution of the equations based on the different Lie algebras can be obtained.

In the trigonometric case the solutions of the $SU(2)$ KZ - equations have the similar form:

$$\frac{\partial}{\partial \xi_i}\chi(t, \xi) = E_i(t, \xi)\chi(t, \xi), \quad \frac{\partial}{\partial t_i}\chi(t, \xi) = f(t, t, \xi)\chi(t, \xi),$$

where the expressions for $E_i(t, \xi)$ and $f(t, t, \xi)$ should be substituted by their trigonometric analogs presented above:

$$E_i = 1/2g - \sum_j \text{ctg}(t_j - \xi_i), \quad f(t) = \sum_\alpha \text{ctg}(t_i - \xi_\alpha) - 2\sum_\alpha \text{ctg}(t_i - t_\alpha) + 1/g.$$

From these equations the explicit solution for the function $\chi(t, \xi)$ in the trigonometric case can be obtained.

Let us mention that in the rational case, except the commuting differential operators $\partial/\partial \xi_i - H_i$, in the $SU(2)$ case there is an extra commuting differential operator $g^2\partial/\partial g - H$, where $H$ is the Richardson Hamiltonian (1), (6) and $g$ is the corresponding coupling constant:

$$\left[ g^2\frac{\partial}{\partial g} - H ; \frac{\partial}{\partial \xi_i} - H_i \right] = 0$$

(for example, see [37], [38]). In conclusion, let us mention that the integration contours $C_1, \ldots C_M$, which define the solution of the modified KZ - equations, are not necessarily the closed contours, which do not intersect the branch cuts of the integrand in the complex plane. The only condition is that the integral over the total derivative of the form

$$\sum_i \oint dt \frac{\partial}{\partial t_i} \left( \chi(t, \xi) \frac{1}{(t_i - \xi_\alpha)} b^+_\alpha |t(i)\rangle \right) = 0$$

for each $\alpha = 1, \ldots N$, which follows from the derivation presented above.

**Appendix D**

Here we present the analytical solution of the BCS model in the continuum limit (i.e. in the limit $N \to \infty$) for the case of the equal- spacing distribution of the energy levels $\xi_i$, or for the density of energy levels $\rho(\xi)$ equal to unity at some interval which can be chosen as $\xi \in (-1, 1)$. Although the solution of the equations (11) for the continuum limit has been considered previously [6], [9], with the help of the electrostatic analogy, and the final solution in agreement with the result of the variational BCS treatment (1) was obtained, some questions remained obscure. For instance, the choice of the ansatz for the complex electric field with a branch cut along the line is not well understood. Thus the additional arguments
and approaches for this problem are highly desirable. First of all, note that the equations (11) can be represented as the conditions of the minimum of the “energy” functional $\frac{\partial}{\partial t_i} \Phi(t) = 0$, where the roots $t_i$ are considered as a positions of charges at the two-dimensional complex plane interacting through the Coulomb potential and subjected to the homogeneous electric field of the strength $-1/g$, directed along the real axis,

$$\Phi(t) = \sum_{i,\alpha} \ln |t_i - \xi_\alpha| - 2 \sum_{i<j} \ln |t_i - t_j| + \frac{1}{g} \sum_i \text{Re}t_i.$$ 

The potential between the like charges $t_i$ of the value $+1$ corresponds to the repulsion, while the their interaction with the points $\xi_i$ with the charges $-1/2$ is repulsive.

Let us denote by $h(z)$ the holomorphic (which depends only on the coordinate $z$ at the part of the complex plane without the charges) complex electric field defined in such a way that the integral of $h(z)$ over some closed contour $C$ in the complex plane is equal $\oint_C dz h(z) = 2\pi i q$, where $q$ is the total electric charge enclosed by the contour $C$. Clearly, the unit charge, centered at the origin, gives the complex electric field equal to $1/z$.

First, let us suppose that in the continuum limit the roots $t_i$ form some curve $\Gamma$, symmetric with respect to the real axis, with the continuous complex density of roots along the curve $R(t)$ (we assume here that all roots $t_i$ appears in a complex pairs). Let us denote by $a$ and $b = a^*$ the endpoints of the curve $\Gamma$, and by $C^-$ the contour enclosing the curve $\Gamma$. We also assume that the curve $\Gamma$ does not intersect the domain of the charges $\xi$ at the real axis. The discontinuity of $h(z)$ at $\Gamma$ is equal to the density $R(t)$, $\Delta h(z) = 2\pi i R(z)$, which by the Cauchy theorem means that $h(z)$ is the analytic continuation of $R(z)$ from the curve $\Gamma$. In particular that means that the total number of particles is $\oint_C dz h(z) = 2\pi i M$ and the total energy is $\oint_C dz z h(z) = 2\pi i E$

The Gaudin’s assumption is that the field $h(z)$ has the branch cut along the line $(a, b)$ of the form

$$h(z) = \sqrt{(z-a)(z-b)} \int \frac{d\xi}{\xi - z} \phi(\xi),$$

where the function $\phi(\xi)$ can be fixed from the condition that the residues of the field at the points $\xi$ should be equal to $-1/2$:

$$\phi(\xi) = (1/2) \left( (\xi - a)(\xi - b) \right)^{-1/2}.$$

Since at infinity $|z| \to \infty$ the field should be equal to $-1/g$, we have the equation

$$\int d\xi \phi(\xi) = \frac{1}{2g}.$$ 

The conservation of the total number of particles gives the equation $\int_{\Gamma} dt R(t) = M$, and the total energy equals $E = \int_{\Gamma} dt t R(t)$.

Alternatively, one can write down the equations (11) in the continuum limit in the following form:

$$\int d\xi \frac{1}{t - \xi} - 2 \int_{\Gamma} dz \frac{R(z)}{t - z} = -\frac{1}{g},$$

27
where the integral over the curve $\Gamma$ can be transformed to the integral over the contour $C$ as
\[
\int \frac{d\xi}{t-\xi} - 2 \oint_C \frac{dz}{2\pi i} \frac{h(z)}{t-z} = -\frac{1}{g}.
\]
Substituting the ansatz for $h(z)$, we obtain
\[
\int \frac{d\xi}{t-\xi} - 2 \int \frac{d\xi \phi(\xi)}{t-\xi} \oint_C \frac{dz}{2\pi i} \sqrt{(z-a)(z-b)/(t-z)(\xi-z)} = -\frac{1}{g},
\]
and transforming the integration contour $C$ into the two contours, the first one is around the domain of $\xi$, and the second one is at the circle at the infinity, we get
\[
\int \frac{d\xi}{t-\xi} - 2 \left[ \int \frac{d\xi \phi(\xi)}{t-\xi} \sqrt{(\xi-a)(\xi-b)} + \int d\xi \phi(\xi) \right] = -\frac{1}{g}.
\]
This equation leads to the two equations, the first one is the equation for the function $\phi(\xi)$ and the second one is for the integral $\int d\xi \phi(\xi) = 1/2g$. Applying the same transformation of the integration contour to the integrals $M = \oint_C (dz/2\pi i)h(z)$ and $E = \oint_C (dz/2\pi i)zh(z)$, and substituting the value $a(b) = \pm i\Delta$, we obtain the gap equation, the equation for the particle number and the energy in agreement with the BCS theory. To investigate the $1/N$ corrections Richardson [9] has derived the closed Riccati type integro-differential equation for the electric field
\[
h(z) = \sum_i \frac{1}{z-t_i} - \frac{1}{2} \sum_\alpha \frac{1}{z-\xi_\alpha} - \frac{1}{g}
\]
in the continuum limit, using the same operation with the contour integration in the complex plane as above. However, at present time, the solution of this equation (without using the ansatz for the field $h(z)$) is absent. The form of the curve $\Gamma$ can be obtained from the condition that the component of the electric field along the curve should be equal to zero for the point at the curve $\Gamma$. One can imagine the curve $\Gamma$ as a metallic plate of the special form with the endpoints $a, b$. At each point near this plate the vector of the electric field has a direction perpendicular to the plate. In particular that means that the electric field at the points $a, b$ is equal to zero, which is fulfilled for the ansatz for $h(z)$. In other words, the curve $\Gamma$ can be found from the condition that it should be the equipotential curve for the electric field $h(z)$. One should stress that the form of the branch cut between the points $a$ and $b$ can be chosen in an arbitrary way, and for the ansatz for $h(z)$ it is assumed that the branch cut coincides with the curve $\Gamma$. Note that one can calculate the density of charges $\left| R(t) \right|$ along the curve and obtain the form $\sim ((t-a)(t-b))/f(t)$, where $f(t)$ is some smooth function, characteristic for the matrix models. Since one can imagine the conformal mapping of $\Gamma$ onto the line $(a,b)$, which reduce the problem to the solution of the matrix model with some potential, which should exhibit the density of states of the same form, this can be considered as a justification of the ansatz.
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