NOTES ON THE FIDELITY OF SYMPLECTIC QUANTUM ERROR-CORRECTING CODES

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Two observations are given on the fidelity of schemes for quantum information processing. In the first one, we show that the fidelity of a symplectic (stabilizer) code, if properly defined, exactly equals the ‘probability’ of the correctable errors for general quantum channels. The second observation states that for any coding rate below the quantum capacity, exponential convergence of the fidelity of some codes to unity is possible.

Keywords: codes, symplectic; fidelity; entanglement

1. Introduction

Two observations are given in this paper on the fidelity of schemes for quantum information processing, especially on that of quantum codes and entanglement distillation protocols. In the first one, we give a formula for the fidelity of symplectic (stabilizer) codes\textsuperscript{1,2,3}. While relating the fidelity of symplectic codes with the ‘probability’ of correctable errors for channels represented by trace-preserving completely positive (TPCP) maps was already done in the literature\textsuperscript{4,5,6}, this work shows that the fidelity, if properly defined, exactly equals the ‘probability’ of the correctable errors for general quantum channels. This formula is also useful for assessing the security of quantum key distribution (QKD) protocols\textsuperscript{7}. In fact, one of the motivations for analyzing the fidelity of symplectic codes was to prove the security of the Bennett-Brassard 1984 (BB84) QKD protocol\textsuperscript{8} or its analogs along the lines of Shor and Preskill\textsuperscript{9,5,7}.

The second observation is related to the problem of the quantum capacity of noisy quantum channels\textsuperscript{10,11}. It states that for any coding rate below the quantum capacity, exponential convergence of the fidelity of some codes to unity is possible.

This paper is organized as follows. In Section 2, several basic notions such as Weyl’s unitary basis are introduced. Section 3 contains the formula for the fidelity of symplectic codes, which is applied to entanglement distillation in Section 4. Sections 5 and 6, respectively, contain the observation on exponential convergence of fidelity and a known lemma to be used in the subsequent section, where the observation is proved. Sections 8 and 9 contain a remark and a summary, respectively. Two appendices are given to explicate the basics of symplectic codes and to give a
2. Basic Notions

2.1. Terminology and Notation

We will treat copies of a quantum system described with $H$, $d = \dim H < +\infty$. A composite system consisting of $n$ such copies is sometimes called an $n$-quantum-($d$-ary-)digit system. The set of all linear maps from a Hilbert space $H$ into itself is denoted by $L(H)$. Hereafter throughout, it is assumed that $H$ is a Hilbert space whose dimension $d$ is a prime number, though the results in this section are true for any integer $d \geq 2$. We assume this because the structure of vector spaces over the finite field $\mathbb{F}_d = \mathbb{Z}/d\mathbb{Z}$ will be exploited. For two subsets $A$ and $B$ of an additive group, $A + B$ denotes \{ $a + b$ | $a \in A, b \in B$ \}, and $a + B$ denotes \{ $a$ \} + $B$.

In this paper, the way to specify quantum codes varies according to the context. For most parts, a quantum code indicates a pair $(C, R)$ consisting of a code subspace $C$ of $H \otimes H$ and a recovery operator $R$; sometimes $C$ alone is called a quantum code. A more general definition allowing encoding maps will appear in a later section.

2.2. The Weyl Basis

A representation $U : G \ni x \mapsto U_x \in L(H)$ of a group $G$ usually indicates one with the property $U_{x+y} = U_x U_y$, $x, y \in G$. However, in quantum mechanics, vectors in $L(H)$ proportional to each other stand for a single quantum state, so that it is natural to weaken the stipulation $U_{x+y} = U_x U_y$ to that $U_{x+y} = \eta(x, y)U_x U_y$, $x, y \in G$, for some collection of complex numbers $\eta(x, y)$, $x, y \in G$. If $U$ satisfies the weaker assumption, it is called a ray (projective) representation.

Weyl$^{12}$ introduced two unitary operators, $X$ and $Z$, on $H$ satisfying the property

$$XZ = \omega ZX,$$

with $\omega$ being a primitive $d$-th root of unity to give a unitary ray representation, $N$, of $\mathcal{X} = \mathbb{F}_d^2$, the 2-dimensional numerical vector space. A concrete form of $N$ can be given as follows. Fix an orthonormal basis \{ $|0\rangle, \ldots, |d-1\rangle$ \} of $H$. Define $X$ and $Z$ by

$$X|a\rangle = |a-1\rangle, \quad Z|a\rangle = \omega^a|a\rangle, \quad a \in \mathbb{F}_d.$$  \hspace{1cm} (2)

We define $N$ by

$$N_{(a,b)} = \sqrt{-1}^{ab} X^a Z^b, \quad (a, b) \in \mathcal{X}$$  \hspace{1cm} (3)

for $d = 2$, and by

$$N_{(a,b)} = X^a Z^b, \quad (a, b) \in \mathcal{X}$$  \hspace{1cm} (4)

for $d > 2$. For $d = 2$, $N_{(a,b)}, (a, b) \neq (0,0)$, are the Pauli operators. Note that there are many systems of complex numbers $\zeta(a, b)$ of modulus 1 such that $\zeta(a, b)X^a Z^b$...
is a ray representation of $\mathcal{X}$. Using the factor $\sqrt{-1}ab$ in the case of $d = 2$ is for a technical reason (Appendix A, Section A.5). It is remarked that Weyl actually derived the concrete representation in (4) from (1) with more natural stipulations such as the irreducibility of $N$.

We identify $((x_1, z_1), \ldots, (x_n, z_n)) \in \mathcal{X}^n$ with $(x_1, z_1, \ldots, x_n, z_n) \in \mathbb{F}_d^{2n}$. To cope with composite quantum systems, we write $N_y = N_{y_1} \otimes \cdots \otimes N_{y_n}$, where $y = (y_1, \ldots, y_n) \in \mathcal{X}^n$, and $N_J = \{N_y \mid y \in J\}$, where $J \subseteq \mathbb{F}_d^{2n}$. We call the operators $N_y$ Weyl unitaries and the system $\{N_y\}_{y \in \mathbb{F}_d^{2n}}$ Weyl basis. An important property of the Weyl basis is the commutation relation

$$N_y N_{y'} = \omega^{(y, y')_{sp}} N_{y'} N_y,$$

(5)

where

$$(y, y')_{sp} = \sum_{i=1}^n x_i z_i' - z_i x_i'$$

(6)

for $y = (x_1, z_1, \ldots, x_n, z_n)$ and $y' = (x_1', z_1', \ldots, x_n', z_n') \in \mathbb{F}_d^{2n}$. The commutation relation (5) follows from $N(a, b)N(a', b') = \omega^{-ba'} N(a+a', b+b')$, $a, b, a', b' \in \mathbb{F}_d$, (7) which in turn follows from the primitive relation (1), and the map that sends $(y, y')$ to $(y, y')_{sp}$ in (6) is known as a symplectic bilinear form. The relation (5) implies that $(x, y)_{sp} = 0$ if and only if $N_x$ and $N_y$ commute.

We have a lemma \textsuperscript{13,14}.

**Lemma 1:** The vectors

$$|\Psi_y\rangle = \frac{1}{\sqrt{d^n}} \sum_{I \in \mathbb{F}_d^n} |I\rangle \otimes N_y |I\rangle, \quad y \in \mathbb{F}_d^{2n}$$

where $|(I_1, \ldots, I_n)\rangle = |I_1\rangle \otimes \cdots \otimes |I_n\rangle$, form an orthonormal basis of $\mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n$.

Note that putting $|\Psi\rangle = |\Psi_{0^{2n}}\rangle$ for the zero vector $0^{2n} \in \mathbb{F}_d^{2n}$, we can rewrite $|\Psi_y\rangle$ as $(I \otimes N_y)|\Psi\rangle$. The zero vector $0^n \in \mathbb{F}_d^n$ will be sometimes abbreviated as $0$ if there is no fear of confusion.

### 2.3. Choi’s Matrix

A simple but helpful tool in quantum information theory is the following one-to-one map of Choi\textsuperscript{15} between the CP maps on $L(\mathcal{H}^\otimes n)$ and the positive semi-definite operators in $L(\mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n)$:

$$M_n(A) = [I \otimes A](|\Psi\rangle\langle\Psi|),$$

(9)

where $I$ is the identity map on $L(\mathcal{H}^\otimes n)$. In fact, Choi introduced $d^n M_n(A)$ in the matrix form (with more flexibility on dimensionality) to yield fundamentals of CP maps.
According to Theorem 1 of Choi\textsuperscript{15}, if \( \rho_n = M_n(\mathcal{A}) \) is written as
\[
\rho_n = \sum_{y,z \in \mathbb{F}_d^2} \alpha_{y,z} |\Psi_y\rangle\langle \Psi_z|,
\] (10)
or equivalently as
\[
\rho_n = \frac{1}{d^n} \sum_{l,m \in \mathbb{F}_d^n} |l\rangle \langle m| \otimes \alpha_{y,z} N_y |l\rangle \langle m| N_z^\dagger,
\] then the CP map \( \mathcal{A} \) is represented as
\[
\mathcal{A} : \sigma \mapsto \sum_{y,z \in \mathbb{F}_d^2} \alpha_{y,z} N_y \sigma N_z^\dagger.
\] (11)

This immediately follows from the fact that Choi’s matrix, viz., the matrix of \( d^n M_n(\mathcal{A}) \) with respect to the basis \( \{|l\rangle \otimes |m\rangle \}_{l,m} \), is the \( d^n \times d^n \) block matrix whose \( (l,m) \)-entry is the \( d^n \times d^n \) matrix of \( \mathcal{A}(|l\rangle \langle m|) \).

2.4. Discrete Twirling

We begin with proving the following formula for discrete twirling (Appendix A of Ref. 16, Ref. 17): For an operator \( \rho_n \in \mathcal{L}(\mathcal{H}^\otimes n) \) in (10), we have
\[
\frac{1}{d^{2n}} \sum_{x \in \mathbb{F}_d^2} (\overline{N_x} \otimes N_x) \rho_n (\overline{N_x} \otimes N_x)^\dagger = \sum_{y \in \mathbb{F}_d^2} \alpha_{y,y} |\Psi_y\rangle\langle \Psi_y|
\] (12)
where \( \overline{U} \) is the complex conjugate of \( U \), viz., the element \( \langle |l\rangle \otimes |m\rangle |U| |m\rangle \) is the complex conjugate of \( \langle |l\rangle |U| |m\rangle \) for \( l, m \in \mathbb{F}_d^n \).

\textbf{Proof of (12).} Put
\[
\rho'_n = \frac{1}{d^{2n}} \sum_{x \in \mathbb{F}_d^2} (\overline{N_x} \otimes N_x) \rho_n (\overline{N_x} \otimes N_x)^\dagger.
\]
Then,
\[
\rho'_n = \frac{1}{d^{2n}} \sum_{x,y,z \in \mathbb{F}_d^2} \alpha_{y,z} (\overline{N_x} \otimes N_x N_y) |\Psi_y\rangle\langle \Psi_y| (\overline{N_x} \otimes N_x N_z^\dagger)
\] (13)
where we used the relation
\[
(A \otimes I)|\Psi\rangle = (I \otimes A^T)|\Psi\rangle
\] (14)
with \( A^T \) being the transpose of \( A \) with respect to \( \{|j\rangle\} \), which means that if \( A = \sum_{l,m} a_{l,m} |l\rangle \langle m| \), then \( A^T = \sum_{l,m} a_{m,l} |l\rangle \langle m| \). Using (5), we then have
\[
\rho'_n = \frac{1}{d^{2n}} \sum_{x,y,z \in \mathbb{F}_d^2} \alpha_{y,z} \omega(x,y-z) \delta (\overline{N_y}) |\Psi_y\rangle\langle \Psi_y| (I \otimes N_z)^\dagger.
\]
Since
\[ \sum_{x \in \mathbb{F}_2^n} \omega^{(x,y-z)_{sp}} = 0 \quad \text{whenever} \quad y \neq z, \]
which holds because \( f_{y-z} : x \mapsto \omega^{(x,y-z)_{sp}} \), where \( y \neq z \), is a character of \( \mathbb{F}_2^n \) such that \( f_{y-z}(x) \neq 0 \) for some \( x \in \mathbb{F}_2^n \) (e.g., Ref. 18 or Section III of Ref. 17), we obtain the formula (12), as desired.

### 2.5. Twirled Channel

Suppose a TPCP map \( A \) on \( L(\mathcal{H}^n) \) is given, and the twirling is applied to the corresponding state \( \rho_n = M_n(A) \). Then, the resulting state is given by (13), and this can be regarded as the mixture
\[ \rho'_n = \frac{1}{d^2n} \sum_{x \in \mathbb{F}_2^n} M_n(N_x A N_x^{-1}) \]
where \( N_x : \sigma \mapsto N_x \sigma N_x^\dagger \) and \( M \mathcal{L} \) denotes the composition that maps \( \sigma \) to \( M(\mathcal{L}(\sigma)) \), etc., on account of the representation of CP maps in (11) [and the block structure of Choi’s matrix mentioned below (11)]. In other words, the channel \( \tilde{A} \) that corresponds to the twirled state \( \rho'_n \) via \( \tilde{A} = M_n^{-1}(\rho'_n) \) is given by
\[ \tilde{A} = \frac{1}{d^2n} \sum_{x \in \mathbb{F}_2^n} N_x A N_x^{-1}. \] (15)

Since the matrix of \( M_n(\tilde{A}) \) is diagonal with respect to the basis \( \{ |\Psi_x \rangle \}_{x \in \mathbb{F}_2^n} \), the channel \( \tilde{A} \) can be expressed as
\[ \tilde{A} : \sigma \mapsto \sum_x P_A(x) N_x \sigma N_x^\dagger, \]
where \( P_A \) is the probability distribution on \( \mathbb{F}_2^n \) defined by
\[ P_A(x) = \langle \Psi_x | M_n(A) | \Psi_x \rangle, \quad x \in \mathcal{X}^n \] (16)
with the basis \( \{ |\Psi_x \rangle \} \) in Lemma 1.

### 3. Fidelity of Symplectic Codes

In this section, we present the formula for the fidelity of symplectic codes. A self-contained exposition of symplectic codes, as well as proofs of the lemmas in this section, can be found in Appendix A, which is a recast of Section III of Ref. 19 except the proof of Theorem 3.

Recall that a symplectic code is obtained from a subspace \( L \subseteq \mathbb{F}_2^n \) that is contained in the symplectic dual \( L^\perp \) of \( L \). Specifically, (a code subspace of) a symplectic code associated with \( L \) is a subspace of the form
\[ \{ \psi \in \mathcal{H}^n \mid N_x \psi = \tau(x) \psi, \ x \in L \} \]
where \( \tau(x), x \in L \), are some complex numbers. When \( \dim_d L = n - k \), we have \( d^{n-k} \) such subspaces, and the collection of these subspaces is also referred to as the symplectic code associated with \( L \). With a basis \((g_1, \ldots, g_{n-k})\) of \( L \) fixed, we have \( d^{n-k} \) cosets of \( L^\perp \) in \( F_d^{2n} \) of the form \( \{ x \in F_d^{2n} \mid (g_i, x)_s = s_i, \ i = 1, \ldots, n - k \} \), where \( s = (s_1, \ldots, s_{n-k}) \in F_d^{n-k} \). Thus, we can label the cosets of \( L^\perp \) by \( s \in F_d^{n-k} \).

It is known that there is a one-to-one correspondence between the set of these cosets and that of the code subspaces, \( C(s) \), \( s \in F_d^{n-k} \). For the specification of \( C(s) \), see Appendix A. If we choose a vector \( \tilde{x}(s) \) from each coset \( s \) of \( L^\perp \) in \( F_d^{2n} \), and denote the set of coset representatives \( \tilde{x}(s) \) by \( J_0 \), we have quantum codes \((C(s), R(s))\), \( s \in F_d^{n-k} \), where \( R(s) : L(H^{\otimes n}) \to L(H^{\otimes n}) \) is a recovery operator designed so that the code is \( N_J \)-correcting, \( J = J_0 + L \).

The recovery operator can be specified by Kraus operators,

\[
K_t^{(s)} = N_{\tilde{x}(t)}^\dagger \Pi_{\tilde{x}(s)}, \quad t \in F_d^{n-k},
\]

where \( \Pi_{\tilde{x}} \) is the projection onto the code subspace \( C(\tilde{x}) \), viz.,

\[
R(s)(\sigma) = \sum_{t \in F_d^{n-k}} K_t^{(s)} \sigma K_t^{(s)\dagger}.
\]

This operation is expressed as the measurement \( \{ \Pi_{\tilde{x}(t)} \}_t \) followed by the unitary \( N_{\tilde{x}(t)} \). The measurement result \( t \) represents the ‘relative syndrome’, so to speak, for the code \( C(s) \). We denote the trace-decreasing CP map \( \sigma \mapsto K_t^{(s)} \sigma K_t^{(s)\dagger} \) by \( R(s,t) \), so that \( R(s) = \sum_{t \in F_d^{n-k}} R(s,t) \).

Let \( \pi_C \) denote the projection operator onto \( C \) divided by \( \dim C \). The entanglement fidelity\(^{20} \) of the \( N_J \)-correcting code \( C \) used on a channel \( A : L(H^{\otimes n}) \to L(H^{\otimes n}) \), \( \sigma \mapsto \sum_{x \in F_d^{2n}} P_n(x) \sigma N_x N_x^\dagger \), where \( P_n \) is a probability distribution on \( X^n \), is given by

\[
F_e(\pi_C, R(s,t) A) = P_n(J) = \sum_{x \in J} P_n(x)
\]

for any \( s \in F_d^{n-k} \). This follows from a finer analysis on the entanglement fidelity for \( R(s,t) \), namely, from the next lemma, which is proved in Appendix A.

**Lemma 2:** Let a subspace \( L \subseteq F_d^{2n} \) which is self-orthogonal with respect to the symplectic form \((\cdot, \cdot)_s\) and \( \tilde{x}(t), t \in F_d^{n-k} \), be given as above. Then,

\[
F_e(\pi_C, R(s,t) A) = P_n(\tilde{x}(t) + L) = \sum_{x \in \tilde{x}(t) + L} P_n(x)
\]

for any \( s, t \in F_d^{n-k} \) and channel \( A : L(H^{\otimes n}) \to L(H^{\otimes n}) \), \( \sigma \mapsto \sum_{x \in F_d^{2n}} P_n(x) \sigma N_x N_x^\dagger \).

**Remark.** Throughout, \( F_e \) is to be understood as the unnormalized entanglement fidelity\(^{21} \).

The corresponding statement for general channels is given in the next theorem, which will be proved in Appendix A.
Theorem 3: Let a subspace \( L \subseteq F_d^{2n} \) and \( \tilde{\pi}(t) \), \( t \in F_d^{n-k} \), be given as above. Then, the symplectic codes \( (C(s), R(s)) = \sum_t R(s,t) \) associated with \( L \) satisfy

\[
\frac{1}{d^{n-k}} \sum_{s \in F_d^{n-k}} F_e(\tilde{\pi}_C(s), R(s,t)A) = \sum_{x \in \tilde{\pi}(t) + L} P_A(x),
\]

for any \( t \in F_d^{n-k} \) and TPCP map \( A : L(H^\otimes n) \to L(H^\otimes n) \), where \( P_A \) is associated with \( A \) by (16).

Corollary 4: For \( J = \bigcup_{x \in \tilde{\pi}(t) + L} \),

\[
\frac{1}{d^{n-k}} \sum_{s \in F_d^{n-k}} F_e(\tilde{\pi}_C(s), R(s,t)A) = \sum_{x \in J} P_A(x).
\]

Remark. That \( \sum_{x \in J} P_A(x) \) is a lower bound to the average fidelity in Corollary 4 easily follows from the observation of Gottesman and Preskill\(^5\) as remarked in Ref. 7.

4. Fidelity of Entanglement Distillation

4.1. One-Way Protocols

In this section, we will consider the problem of evaluating the fidelity of entanglement distillation schemes and see its close relation to quantum error-correcting codes. Shor and Preskill described their famous proof of the security of the BB84 protocol in terms of entanglement distillation. The entanglement distillation protocol they used is as follows, where as usual, the protocol is performed by Alice and Bob. First, imagine they are given a bipartite state \( M_n(A_n) = [\mathcal{I} \otimes A_n](\rho)\langle \Psi|\Psi \rangle \), where \( |\Psi \rangle = d^{-n/2} \sum_{s,u} |s,u \rangle \otimes |s,\bar{u} \rangle \), \( \{ |s,\bar{u} \rangle \}_u \) is an orthonormal basis of \( \mathcal{C}^{(s)} \) for each \( s \in F_d^{n-k} \), \( \{ |s,u \rangle \}_s \) is an orthonormal basis of \( H^\otimes n \), and \( \mathcal{C}^{(s)} \) is spanned by \( |s,\bar{u} \rangle \), \( u \in F_d^k \), for each \( s \in F_d^{n-k} \). Alice performs the local measurement \( \{ \Pi_s \} \) on the first half of the system, where \( \Pi_s \) denotes the projection onto the subspace \( \mathcal{C}^{(s)} \), and Bob performs the recovery operation for the \( N_f \)-correcting code \( C^{(s)} \) knowing that Alice’s measurement result is \( s \). Now recall the physical meaning of entanglement fidelity\(^20\): Suppose an ideal bipartite state \( |\Phi \rangle = |\Phi_s \rangle = d^{-k} \sum_u |s,u \rangle \otimes |s,\bar{u} \rangle \) is given, where \( \{ |s,u \rangle \}_u \) plays the role of an orthonormal basis of the ‘reference’ system\(^20\); then, \( F_e(\pi_C, B) = \langle \Phi|\mathcal{I} \otimes B|\Phi \rangle \langle \Phi|\Phi \rangle \). Since Alice obtains each measurement result \( s \) with the equal probabilities and the resulting state is \( [\mathcal{I} \otimes A_n](|\Phi_s \rangle \langle \Phi_s |) \) conditioned on this event, the fidelity of this distillation protocol for \( M_n(A_n) \) is exactly the same as the average entanglement fidelity of the code \( (C^{(s)}, R^{(s)}) \) in Corollary 4.

For the security proof, the above argument is enough\(^7\). For the purposes of entanglement distillation, however, we should start with \( M_n(A_n) = [\mathcal{I} \otimes A_n](\rho)\langle \Psi|\Psi \rangle \), rather than \( M_n(A_n) \), since in the standard setting the given bipartite states are of the form \( \rho^\otimes n \), which is written in (or reduced by twirling to) the form \( M_n(A^\otimes n) \). This problem is resolved upon noticing the relation \( \langle U \otimes U|\Psi \rangle = |\Psi \rangle \), which
holds for any unitary $U$ by (14), and the existence of the unitary $U$ that maps $| (s,u) \rangle$, to $| s,u \rangle$, $(s,u) \in \mathbb{F}_d^{n-k} \times \mathbb{F}_d^k \cong \mathbb{F}_d^n$. In fact, we can choose $U| (s,u) \rangle$ as $| s,u \rangle$, $(s,u) \in \mathbb{F}_d^{n-k} \times \mathbb{F}_d^k \cong \mathbb{F}_d^n$. Thus, we see the average entanglement fidelity given in Theorem 3 is the fidelity of the following one-way entanglement distillation protocol for the state $M_n(A_n)$ (or for any bipartite state $\rho_n \in \mathbb{L}(\mathbb{H}^{\otimes n} \otimes \mathbb{H}^\otimes n)$ if the participants of the distillation protocol perform the discrete twirling as a preprocessing).

Protocol. First, Alice performs the orthogonal measurement consisting of the projections onto $C'(s)$, where provided Alice’s measurement result is $s$, the resulting state is $\rho^{(s)} = [I \otimes A_n](|\Phi_s\rangle \langle \Phi_s|)$. Bob applies the recovery operator $R^{(s)}$ to his system. Alice and Bob, respectively, apply some unitaries $U_A$ and $U_B$ such that $U_A| s,u \rangle = | 0^{n-k}, u \rangle$ and $U_B| s,u \rangle = | 0^{n-k}, u \rangle$.

Protocols thus obtained will be sometimes called symplectic (entanglement) distillation protocols. This class of one-way protocols are also applicable to correlated states.$^{22,17}$

### 4.2. Two-Way Protocols

Theorem 3 is also useful for analyses of two-way entanglement distillation from multiple copies of a state $\rho$. In this case, the corresponding channel $A_n$ can be written as $A^{\otimes n}$ for some channel $A : \mathbb{L}(\mathbb{H}) \rightarrow \mathbb{L}(\mathbb{H})$. For example, consider Bennett et al.’s protocol$^{23}$, where Alice and Bob use the symplectic code associated with $\text{span} \{(0, 1, 0, 1), (0, 0, 0, 0), (0, 1, 0, 1)\}$, where $n = 2$ and $k = 1$. [This code is sometimes called $[[2, 1]]$ cat code and the core of this distillation protocol was originally described$^{23}$ in terms of quantum gates as a decoding network of the cat code was$^{24,25}$.] The protocol consists of several iterations of the two-way procedure using $\text{span} \{(0, 1, 0, 1)\}$ and a one-way entanglement distillation protocol. The two-way procedure using $\text{span} \{(0, 1, 0, 1)\}$ is not much different from the one-way symplectic distillation protocol using it: In each step, Alice and Bob pair up surviving states, and for each pair they do the same measurement and unitaries as described in Section 4.1, where in the second or further step, the basis $\{|l\rangle \otimes |m\rangle\}_{l,m \in \mathbb{F}_d^n}$ is to be understood as the basis $\{| 0^{n-k}, u \rangle \otimes | 0^{n-k}, u' \rangle\}_{u,u'}$ obtained newly in the previous step. In the present case of two-way distillation, however, they retain only states with result $t = 0$, or $t \in T$ for some fixed proper subset of $\mathbb{F}_d^{n-k}$ [recall $\mathcal{R}^{(s)} = \sum_t \mathcal{R}^{(s,t)}$], and discard the rest. Clearly, both the two-way subroutine and the final one-way procedure can be replaced by arbitrary ones based on symplectic codes that are described or exemplified above$^{26}$, though the problem of estimating the fidelity for such schemes is non-trivial for general states, which is solved by Theorem 3.
5. Exponential Convergence of Fidelity

Recently, a formula for the quantum capacity written with coherent information, which had been conjectured by several authors, was confirmed\textsuperscript{10,11}. Regarding this topic, from a viewpoint of information theory or large-deviation theory, we will consider the problem of finding attainable speeds of convergence (exponents) of the fidelity of quantum codes, or other similar schemes, to unity.

A memoryless quantum channel is a TPCP map

\[ A : \mathcal{L}(H_c) \to \mathcal{L}(H_o). \]

The term ‘memoryless’ refers to the property that \( A \) acts on a density operator \( \rho \) in \( \mathcal{L}(H_c^\otimes n) \) as \( A^\otimes n(\rho) \). A coding scheme or code for \( A^\otimes n \) is a triple \((C_n, \mathcal{E}_n, \mathcal{D}_n)\) that consists of a Hilbert space \( C_n \), and TPCP maps

\begin{align*}
\mathcal{E}_n : \mathcal{L}(C_n) &\to \mathcal{L}(H_c^\otimes n), \\
\mathcal{D}_n : \mathcal{L}(H_o^\otimes n) &\to \mathcal{L}(C_n).
\end{align*}

(21) (22)

**Definition 5:** A number \( R \) is said to be an achievable rate for \( A \) if there exists a sequence of codes \((C_n, \mathcal{E}_n, \mathcal{D}_n)\) for \( A^\otimes n \) such that

\[ \limsup_{n \to \infty} \frac{\log d \dim C_n}{n} \geq R \]

and

\[ \lim_{n \to \infty} F_e(\pi_{C_n}, \mathcal{D}_n A^\otimes n \mathcal{E}_n) = 1. \]

**Definition 6:** The supremum of achievable rates for a memoryless channel \( A \) is called the quantum capacity and denoted by \( Q(A) \).

**Remark.** This definition is essentially the same as the one using the subspace fidelity in Ref. 21, but we employ \( F_e(\pi_C, B) \) rather than the minimum pure-state fidelity. For the equivalence, see Appendix B or examine the arguments in Ref. 21.

**Definition 7:** A number \( E \) is said to be an attainable exponent for a channel \( A \) and a rate \( R \) if there exists a sequence of codes \((C_n, \mathcal{E}_n, \mathcal{D}_n)\) for \( A^\otimes n \) such that

\[ \liminf_{n \to \infty} \frac{\log d \dim C_n}{n} \geq R \]

and

\[ \liminf_{n \to \infty} \frac{\log d [1 - F_e(\pi_{C_n}, \mathcal{D}_n A^\otimes n \mathcal{E}_n)]}{n} \geq E. \]

We will prove the next theorem in what follows.

**Theorem 8:** For any memoryless channel \( A \), and any rate \( R \) smaller than \( Q(A) \), we have a positive attainable exponent.
6. Random Coding Bound for Symplectic Codes

A random coding argument shows the next lemma. In fact, the proof of the main result of Ref. 6 or Ref. 27 applies to this lemma if we replace $X$ thereof by $X^m$. Alternatively, the proof in Ref. 19 works if we assume the inner code of the concatenated code thereof to be the identity map.

**Lemma 9:** For any positive integer $m$, number $R$, $0 \leq R < 1$, and memoryless channel $B : L(H^\otimes m) \to L(H^\otimes m)$, there exists a sequence of symplectic codes $\{(C_\nu \subseteq H^\otimes m, R_\nu)\}_\nu$ such that $\log_d \dim C_\nu \geq m\nu R$, and

$$1 - \mathbb{E}E_{\nu}(\pi C_\nu, R_\nu B^\otimes \nu) \leq f(\nu) \exp_{\nu} \{-\nu E_m(R, P_B)\},$$

(23)

where

$$E_m(R, P_B) = \min_Q [D(Q||P_B)/m + \log_d \dim C_\nu \geq m\nu R, \text{and for any memoryless channel } B : L(H^\otimes m) \to L(H^\otimes m), (23) \text{ is satisfied}].$$

Remainder. The symplectic code $\{(C_\nu \subseteq H^\otimes m, R_\nu)\}_\nu$ is to be understood as the ensemble $\{(C_\nu^{(s)}(x), R_\nu^{(s)})\}_s$, where $s$ runs through all syndromes, and $E$ denotes the expectation operation to produce the ensemble average with respect to the uniform distribution over all syndromes, by which Corollary 4 is applicable. The statement can be strengthened to 'For any positive integer $m$, number $R$, $0 \leq R < 1$, there exists a sequence of symplectic codes $\{(C_\nu \subseteq H^\otimes m, R_\nu)\}_\nu$ such that $\log_d \dim C_\nu \geq m\nu R$ and for any memoryless channel $B : L(H^\otimes m) \to L(H^\otimes m)$, (23) is satisfied'. This means that we can find symplectic codes whose structures do not depend on the channel characteristics, especially on $P_B$. The proof of this refinement is essentially the same as that in Ref. 7. The proof uses the existence of a symplectic code whose 'type spectrum', which is a natural generalization of the weight spectrum (distribution) in coding theory, is 'well balanced', and the fact that the fidelity of any symplectic code on a memoryless channel is invariant under permutations of the coordinates (digits).

7. Proof of Theorem 8

Suppose a rate $r$ is achievable for $A$. Then, there exists a sequence of codes $\{(C_n, E_n = E, D_n = D)\}$ whose rate, as $n$ becomes large, approaches $r$, which may be arbitrarily close to $Q(A)$. We may assume $\dim C_n = d^m$ for some integer $m$ for every $n$ as argued in Appendix B (since $Q = Q_{e,d}$). We apply Lemma 9 setting $B = D A^\otimes \nu E$ and identifying $H^\otimes m$ with $C_n$. Namely, we use two-stage coding in which the $\nu m$-quantum-digits system is divided into $\nu$ blocks of length $n$, each block is coded with $(C_n, E_n = E, D_n = D)$, and $\nu$ blocks are coded with the code.
for $B''$ the existence of which is ensured in Lemma 9 (Fig. 1). The two-stage codes have overall rates not smaller than

$$\frac{m}{n} R.$$  

From (24), $E_m(R, P_B)$ is positive if $R < 1 - \frac{H(P_B)}{m}$, i.e., if

$$\frac{m}{n} R < \frac{m}{n} \left[ 1 - \frac{H(P_B)}{m} \right].$$  

The number $1 - \frac{H(P_B)}{m}$ can be bounded as

$$1 - \frac{H(P_B)}{m} \geq 1 - \frac{h(P_B(0^{2m})) + [1 - P_B(0^{2m})]2m}{m},$$

where $h$ is the binary entropy function. Note also by the definition of the entanglement fidelity, we have

$$F_e(\pi_{C_n}, DA^{\otimes n}E) = F_e(\pi_{C_n}, B) = \langle \Psi | M_m(B) | \Psi \rangle = P_B(0^{2m}),$$

where $|\Psi\rangle = |\Psi_{0^{2m}}\rangle$. Then, because $m/n$ and $F_e(\pi_{C_n}, DA^{\otimes n}E) = P_B(0^{2m})$ tend to $r$ and 1, respectively, as $n$ grows large, the number on the right-hand side of (25), for a large enough $n$, will be arbitrarily close to $r$, which in turn can be made close to $Q(A)$.

Thus, we have a sequence of codes of desired performance for $A^l$, $l = n, 2n, \ldots$. To interpolate a code for $A^l$ with $l = n\nu + i$, $0 < i < n$, into this sequence, we just past a trivial code of dimension one for the $i$-quantum-digit system to the large code for $n\nu$-quantum-digit system.

**Remark.** Instead of assuming $\dim C_n = d^m$ for some integer $m$ to use the argument in Appendix B, we can generalize Lemma 9 so that it applies to memoryless channels $B : L(H') \rightarrow L(H')$ with $\dim H'$ arbitrary but finite. To do this, write the number $\dim H'$ as the product of the prime factors $d_1 \cdots d_m$, and use the tensor product of code subspaces of symplectic codes for quantum-$d_i$-ary-digit systems.
8. Exponents for Entanglement Distillation

The above argument can be accommodated to the problem of entanglement distillation from multiple copies of a bipartite state \( \rho \otimes \rho \). In fact, the achievability of a rate, the capacity analog \( D_C(\rho) \) (sometimes called the distillable entanglement), and attainable error exponents for a bipartite state \( \rho \) can be similarly defined for a given class of distillation protocols \( C \). Assume that the participants of a protocol are allowed to apply a one-way symplectic distillation protocol to multiple copies of \( \mathcal{D}(\rho^\otimes n) \) in the class \( C \), where \( \mathcal{D} \) is another protocol in \( C \). Note that most of protocol classes discussed in the literature, e.g., \( C_1 \) through \( C_{17} \) of Ref. 29, possess this property. Then, since the symplectic quantum code in Lemma 9 can be used as a one-way symplectic distillation protocol, we conclude that for any state \( \rho \) of a bipartite system and any rate below \( D_C(\rho) \), we have a positive attainable exponent.

Clearly, this conclusion as well as its reasoning extends to the scenario of entanglement generation over memoryless quantum channels, where the sender Alice begins with an arbitrary initial bipartite state \( \rho_{in} \) in some prescribed class \( C_{in} \), sends the half of \( \rho_{in} \) to produce \( I \otimes A \otimes n(\rho_{in}) \), and then Alice and Bob apply some distillation protocol to \( I \otimes A \otimes n(\rho_{in}) \) allowed in a prescribed class \( C \). [The term ‘entanglement generation’ is from Ref. 11, where the allowed operations are those of local TPCP maps at the receiver’s end.]

9. Conclusion

In summary, based on Weyl’s ray representation of \( \mathbb{Z}/d\mathbb{Z} \), with which the standard symplectic form was associated naturally in considering the commutation relation for the representation, the fidelities of schemes for quantum information processing using the property of the symplectic geometry were evaluated.

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Appendix A. Basics of Symplectic Codes

A1. Symplectic Codes

In this section, the framework of symplectic codes is rebuilt on the theory of geometric algebra \(^{30,31}\). For a subspace \( L \subseteq F_2^n \), let \( L^\perp \) be defined by

\[
L^\perp = \{ y \in F_2^n \mid \forall x \in L, \ (x,y)_{sp} = 0 \}.
\]

By linear algebra, the matrices of commuting unitary operators are diagonal with respect to a common basis. A symplectic code is a collection of simultaneous eigenspaces of a set of commuting operators in the Weyl basis. By (5), if a set \( L \subseteq F_2^n \) has the property that the operators \( N_x, x \in L \), commute with each other,
then \( \text{span} L \) has the same property. Hence, it is enough to consider a subspace \( L \subseteq \mathbb{F}_d^{2n} \) such that
\[
\forall x, y \in L, \quad (x, y)_{\text{sp}} = 0,
\]
which is equivalent to \( L \subseteq L^\perp \). A subspace \( L \subseteq \mathbb{F}_d^{2n} \) is said to be self-orthogonal (with respect to the symplectic bilinear form) if \( L \subseteq L^\perp \).

The statement of the following lemma can be found in Ref. 32, Section 3.2, and Ref. 33. A proof based on the very basics of symplectic geometry\(^{30,31}\) has been given in Ref. 19.

**Proposition 1:** Let \( L \) be a self-orthogonal subspace with \( \dim L = n - k \) and \( L = \text{span} \{g_1, \ldots, g_{n-k}\} \). Then, we can find vectors \( g_{n-k+1}, \ldots, g_n \) and \( h_1, \ldots, h_n \) such that
\[
\begin{align*}
(g_i, h_j)_{\text{sp}} &= \delta_{ij}, \\
(g_i, g_j)_{\text{sp}} &= 0, \\
(h_i, h_j)_{\text{sp}} &= 0
\end{align*}
\]
for \( i, j = 1, \ldots, n \), where \( \delta_{ij} \) is the Kronecker delta.

A pair of linearly independent vectors \((g, h)\) with \((g, h)_{\text{sp}} = 1\) is called a hyperbolic pair, and it is known that a space with a nondegenerate symplectic form, such as the one defined by (6), can be decomposed into an orthogonal sum of the form
\[
\text{span} \{w_1, z_1\} \perp \ldots \perp \text{span} \{w_n, z_n\}
\]
in such a way that \((w_i, z_i), i = 1, \ldots, n,\) are hyperbolic pairs\(^{30}\). Following Artin\(^{30}\), we have referred to the direct sum of \( U_1, \ldots, U_n \) as the orthogonal sum of spaces \( U_1, \ldots, U_n \) if \( U_1, \ldots, U_n \) are orthogonal. The three equations in the above lemma say that \( \mathbb{F}_d^{2n} \) is the orthogonal sum of \( \text{span} \{g_i, h_i\}, i = 1, \ldots, n \). In the present case with the bilinear form in (6), the simplest example of such a decomposition of the space \( \mathbb{F}_d^{2n} \) is \( \text{span} \{e_1, e_2\} \perp \ldots \perp \text{span} \{e_{2n-1}, e_{2n}\} \), where \( \{e_i\}_{1 \leq i \leq 2n} \) is the standard basis of \( \mathbb{F}_d^{2n} \) that consists of \( e_i = (\delta_{ij})_{0 \leq j \leq 2n} \in \mathbb{F}_d^{2n} \), \( 1 \leq i \leq 2n \).

For the remainder of this appendix, we fix an arbitrary self-orthogonal subspace \( L \) with \( \dim L = n - k \) and such hyperbolic pairs \((g_1, h_1), \ldots, (g_n, h_n)\) as given in Proposition 1. Any vector \( x \in \mathbb{F}_d^{2n} \) can be expanded into
\[
x = \sum_{i=1}^{n} (w_i g_i + z_i h_i).
\]

Thus, the hyperbolic pairs \((g_1, h_1), \ldots, (g_n, h_n)\) determines the map that sends \( x \) to \((w_1, z_1, \ldots, w_n, z_n)\), which is clearly an isometry. For \( z = (z_1, \ldots, z_m) \in \mathbb{F}_d^{m} \), \( 1 \leq m \leq n \), we write
\[
X^z = \prod_{i=1}^{m} (N_{h_i})^{z_i}
\]
where the product on the right-hand side is unambiguous because \((N_{h_i})^{z_i}, i = 1, \ldots, m,\) commute with each other. Note that by (7), \( X^z \) and \( N_x \), where \( x =
\[
\sum_{i=1}^{m} z_i h_i, \text{ are the same up to a phase factor. Similarly, for } w = (w_1, \ldots, w_m) \in \mathbb{F}_d^m, \quad 1 \leq m \leq n, \text{ we write }
\]

\[
\mathcal{Z}^w = \prod_{i=1}^{m} (N_{g_i})^{w_i}. \quad (A.4)
\]

We have seen that any basis \( \{g_1, \ldots, g_{n-k}\} \) of a self-orthogonal space can be extended to \( \{g_1, \ldots, g_n\} \) in such a way that \( \text{span} \{g_1, \ldots, g_n\} \) is self-orthogonal. Since \( N_{g_i}, i = 1, \ldots, n, \) commute with each other, we can find a basis of \( L(H) \) on which \( N_{g_i} \) are simultaneously diagonalized in matrix forms. Hence, we can find an \( n \)-tuple of scalars \( \mu_i \) \( 1 \leq i \leq n \) for which the space consisting of \( \psi \) with \( N_{g_i} \psi = \mu_i \psi, \quad i = 1, \ldots, n, \) (A.5)

is not empty. We call a nonzero vector (respectively, the set of vectors) satisfying (A.5) an eigenvector (respectively, the eigenspace) of \( \{N_{g_i}\}_{1 \leq i \leq n} \) with eigenvalue list \( (\mu_i)_{1 \leq i \leq n} \). Take a normalized vector \( |0, \ldots, 0\rangle \) from this eigenspace, where the label \( (0, \ldots, 0) \) belongs to \( \mathbb{F}_d^n \). Applying an operator \( N_x \) to both sides of (A.5) from left and using (5) as well as the symplectic property \( (x,y)_{sp} = -(y,x)_{sp} \), we have

\[
N_x N_{g_i} \psi = \mu_i N_x \psi, \quad i = 1, \ldots, n, \quad (A.6)
\]

This means that \( N_x \psi \) is an eigenvector with eigenvalue list \( (\mu_i (g_i, x))_{sp} \) \( 1 \leq i \leq n \). If we expand \( x \) as in (A.2), then we have \( (g_i, x)_{sp} = z_i, i = 1, \ldots, n, \) and hence there are, at least, \( d^n \) possible eigenvalue lists for \( \{N_{g_i}\}_{1 \leq i \leq n} \). However, for any pair of distinct eigenvalue lists, the corresponding eigenspaces of \( \{N_{g_i}\}_{1 \leq i \leq n} \) are orthogonal, and hence there are no more eigenvalue lists. Thus, we have an orthonormal basis \( \{|s_1, \ldots, s_n\rangle\}_{(s_1, \ldots, s_n) \in \mathbb{F}_d^n} \) defined by

\[
|s_1, \ldots, s_n\rangle = X |0, \ldots, 0\rangle, \quad \text{where} \quad s = (s_1, \ldots, s_n). \quad (A.7)
\]

It is easy to check that \( (N_{(a,b)})^d \) is the identity operator, which implies eigenvalues of \( N_x, x \in \mathbb{F}_d^n \), are \( d \)-th roots of unity. Hence, we can take \( \mu_i, 1 \leq i \leq n, \) to be all one, which we will assume throughout. Note that the basis \( \{|s_1, \ldots, s_n\rangle\}_{(s_1, \ldots, s_n) \in \mathbb{F}_d^n} \) depends on \( (g_i, h_i), i = 1, \ldots, n \).

We expand \( x \) as in (A.2) and put

\[
z = (z_1, \ldots, z_n), \quad w = (w_1, \ldots, w_n). \quad (A.8)
\]
Define \([a, b] = (a_1, b_1, \ldots, a_n, b_n) \in \mathbb{F}_d^{2n}\), \(X^a\) as \(X^{a_1} \otimes \cdots \otimes X^{a_n}\) and \(Z^b\) as \(Z^{b_1} \otimes \cdots \otimes Z^{b_n}\) for \(a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{F}_d^n\). Then, \(N_{[a,b]} = X^aZ^b\),

\[X^a|l_1, \ldots, l_n\rangle = |l_1 - a_1, \ldots, l_n - a_n\rangle\]

and

\[Z^b|l_1, \ldots, l_n\rangle = \prod_{i=1}^{n} \omega^{b_i l_i}|l_1, \ldots, l_n\rangle,\]

\(a, b, (l_1, \ldots, l_n) \in \mathbb{F}_d^n\), by the definitions of \(N\), \(X\) and \(Z\). We notice that the actions of \(X^a\) and \(Z^w\), \(z, w \in \mathbb{F}_d^n\), on the new basis is quite similar to those of \(X^{-1}\) and \(Z\) on \(|l_1, \ldots, l_n\rangle\):

\[
X^a|l_1, \ldots, l_n\rangle = |l_1 + z_1, \ldots, l_n + z_n\rangle \quad \text{(A.9)}
\]

and

\[
Z^w|l_1, \ldots, l_n\rangle = \prod_{i=1}^{n} \omega^{w_i l_i}|l_1, \ldots, l_n\rangle, \quad \text{(A.10)}
\]

\(z, w, (l_1, \ldots, l_n) \in \mathbb{F}_d^n\). Eq. (A.9) holds by definition, and (A.10) can be checked as follows.

**Proof of (A.10).** Since \(N\) is a ray representation, \(X^a\) and \(Z^w\) can be written as

\[
X^l = \lambda N_{\Sigma, l, h_i}, Z^w = \lambda' N_{\Sigma, w, g_i},
\]

with some constants \(\lambda\) and \(\lambda'\), where \(l = (l_1, \ldots, l_n)\) and \(i\) runs through 1 to \(n\) in the summations. Then,

\[
Z^w|l\rangle = Z^w X^l|0^n\rangle
\]

\[
= \lambda' N_{\Sigma, w, g_i} N_{\Sigma, l, h_i} |0^n\rangle
\]

\[
\overset{(a)}{=} \lambda' \omega^{(\Sigma, w, g_i, \Sigma, l, h_i) w} N_{\Sigma, l, h_i} N_{\Sigma, w, g_i} |0^n\rangle
\]

\[
= \omega^{\Sigma, w, l} X^l Z^w|0^n\rangle
\]

\[
\overset{(b)}{=} \omega^{\Sigma, w, l} X^l|0^n\rangle = \omega^{\Sigma, w, l}|l\rangle,
\]

where the equalities (a) and (b) follow from (5) and (A.5) with the assumption \(\mu_i = 1\) for all \(i\), respectively. \(\square\)

Now we are ready to see the principle of symplectic codes.

**Proposition 2:** \(^{1,2,3}\) Let a subspace \(L \subseteq \mathbb{F}_d^{2n}\) satisfy

\[L \subseteq L^\perp \quad \text{and} \quad \dim L = n - k. \quad \text{(A.11)}\]

In addition, let \(J_0 \subseteq \mathbb{F}_d^{2n}\) be a set satisfying

\[\forall x, y \in J_0, [y - x \in L^\perp \Rightarrow x = y], \quad \text{(A.12)}\]

and put

\[J = J_0 + L = \{z + w \mid z \in J_0, w \in L\}.\]
Then, the \(d^k\)-dimensional subspaces of the form
\[
\{ \psi \in \mathbb{H}^\otimes n \mid \forall M \in N_L, \ M \psi = \tau(M)\psi \}, \tag{A.13}
\]
where \(\tau(M)\) are eigenvalues of \(M \in N_L\), are \(N_J\)-correcting codes.

In fact, the subspace
\[
C^{(s)} = \text{span} \{ [s_1, \ldots, s_{n-k}, s_{n-k+1}, \ldots, s_n] \mid (s_{n-k+1}, \ldots, s_n) \in \mathbb{F}_d^k \} \tag{A.14}
\]
with a fixed \((n-k)\)-tuple \(s = (s_1, \ldots, s_{n-k}) \in \mathbb{F}_d^{n-k}\) is such a quantum code. The equivalence of (A.13) and (A.14) follows from (7). Since there are \(d^{n-k}\) possible choices for \((s_1, \ldots, s_{n-k})\), we have \(d^{n-k}\) codes. The term codes is applied to both a self-orthogonal subspace \(L \subseteq \mathbb{F}_d^{2n}\), and quantum codes \(C^{(s)}\) associated with \(L\).

The collection of quantum codes \(C^{(s)}\) or one from the collection, each possibly accompanied by a recovery operator, is called a symplectic code associated with \(L\) or symplectic (stabilizer) code with stabilizer \(N_L\).

Since \(L^\perp\) is spanned by \(g_1, \ldots, g_n\) and \(h_{n-k+1}, \ldots, h_n\), any coset of \(L^\perp\) in \(\mathbb{F}_d^{2n}\) is of the form
\[
\left\{ \sum_{i=1}^{n} (w_ig_i + z_ih_i) \mid z_i = s_i, \ i = 1, \ldots, n-k \right\}
= \{ x \mid (g_i, x)_{sp} = s_i, \ i = 1, \ldots, n-k \} \tag{A.15}
\]
with some \((n-k)\)-tuple \(s = (s_1, \ldots, s_{n-k})\). The set of cosets of \(L^\perp\) and \(\{ N_xC^{(0)} \mid x \in J_0 \}\), where \(N_xC^{(0)}\) denotes \(\{ N_x\psi \mid \psi \in C^{(0)} \}\) with 0 being the abbreviation of \((0, \ldots, 0) \in \mathbb{F}_d^{n-k}\), are in a one-to-one correspondence when \(J_0\) is a transversal (a set of coset representatives such that each coset has exactly one representative in it), i.e., when \(|J_0| = d^{n-k}\). In fact, for any vector \(x\) in the coset in (A.15), we have, by (A.9) and (A.10) or Section A.3 below,
\[
C^{(s)} = N_xC^{(0)}. \tag{A.16}
\]
The \((n-k)\)-tuple \((s_i)_{1 \leq i \leq n-k}\) is called a syndrome on the analogy with classical linear codes.

To show that the subspace, say \(C\), in (A.13) or (A.14) is really \(N_J\)-correcting, we may use Theorem III.2 of Knill and Laflamme⁴. Alternatively, we can directly check the error-correcting capability using the recovery operator specified by (17) and (18) as will be done in Section A.3.

**A2. Coset Arrays**

In discussing symplectic codes, it is often useful to conceive a coset array of \(L\) which has the form

\[
y_0 + x_0 + L \quad y_0 + x_1 + L \cdots \quad y_0 + x_{K-1} + L \\
y_1 + x_0 + L \quad y_1 + x_1 + L \cdots \quad y_1 + x_{K-1} + L \\
\vdots \quad \vdots \quad \vdots \\
y_{M-1} + x_0 + L \quad y_{M-1} + x_1 + L \cdots \quad y_{M-1} + x_{K-1} + L \quad \tag{A.17}
\]
where $K = d^{2k}$, $M = d^{n-k}$, $\{x_i\}$ is a transversal of the cosets of $L$ in $L^\perp$, and $\{y_i\}$ is that of the cosets of $L^\perp$ in $F_d^{2n}$. Here, the integer index $i$ of $x_i$ is identified with $s \in F_d^{n-k}$, which can be viewed as a $d$-ary number, and that of $y_i$ is to be similarly understood. In the array, each entry is a coset of $L$ in $F_d^{2n}$, and each row form a coset of $L^\perp$ in $F_d^{2n}$. This array resembles standard arrays often used in classical coding theory\textsuperscript{34,35}, and there is an analogy between them. For example, if we choose one coset $y_s + x_u + L$ from each row, and denote the union of these cosets by $J$, then there are recovery operators such that the resulting symplectic codes are $N_J$-correcting, which was already mentioned in the previous section and will be proved in the next section. [Entries of a standard array of a classical linear code are not cosets but vectors, and if we choose a vector from each row, and denote the set of these vectors by $J$, then we can decode it in such a way that the resulting code is $J$-correcting.]

**A3. Proof of Lemma 2: Fidelity of Codes on Channels Subject to Probabilistic Weyl Unitaries**

To calculate the fidelity, we trace the action of $I \otimes N_x$ on the state $|\Phi_s\rangle \langle \Phi_s|$, where

$$
|\Phi_s\rangle = \frac{1}{d^{k/2}} \sum_{(l_1, \ldots, l_k) \in F_d^k} |l_1, \ldots, l_k\rangle \otimes |s_1, \ldots, s_{n-k}, l_1, \ldots, l_k\rangle
$$

is a purification of $\pi_{\mathcal{C}(s)}$, $s = (s_1, \ldots, s_{n-k})$.

Suppose an error $N_x$, $x \in F_d^{2n}$, has occurred on a state $\pi_{\mathcal{C}(s)}$. We decompose $x$ into

$$
x = \sum_{i=1}^{n-k} w_i g_i + \sum_{i=1}^{n-k} z_i h_i + \sum_{i=1}^{k} z_{i+n-k} h_{i+n-k} + \sum_{i=1}^{k} w_{i+n-k} g_{i+n-k}.
$$

Then, $N_x$ is the same as $U_3 U_2 U_1$ up to an irrelevant phase factor, where $U_1 = Z^u$, $v = (w_1, \ldots, w_{n-k})$, $U_2 = X^t$, $t = (z_1, \ldots, z_{n-k})$, and $U_3 = X^u Z^v$, $u = (0, \ldots, 0, z_{n-k+1}, \ldots, z_n)$, $u' = (0, \ldots, 0, w_{n-k+1}, \ldots, w_n)$. By (A.9) and (A.10), $I \otimes (U_2 U_1) |\Phi_s\rangle \langle \Phi_s| I \otimes (U_2 U_1) = I \otimes U_2 |\Phi_s\rangle \langle \Phi_s| I \otimes U_2 = |\Phi_{s+t}\rangle \langle \Phi_{s+t}|$. The final part $I \otimes U_3$ acts on the state $|\Phi_{s+t}\rangle \langle \Phi_{s+t}|$ as a Weyl unitary. [These actions may be visualized in terms of a coset array as follows. Assume for simplicity $s = 0^{n-k}$, recall $N_x C^{(0)} = C^{(t)}$, and write $C^{(t)}$ beside the $i$-th row of the array; $U_2$ does nothing, $U_2$ translates the half of the state $|\Phi_0\rangle$ along the vertical lines to $C^{(t)}$ and $U_3$ acts as the Weyl unitary specified by $(u, u')$ that corresponds to a horizontal index of the array in a one-to-one fashion.]

Now suppose $\hat{x}(t)$ is expanded as $x$ was to yield $\hat{v}, \hat{u}$ and $\hat{u}'$ in place of $v, u$ and $u'$. Then, only the effect of errors $N_x$ such that $u = \hat{u}$ and $u' = \hat{u}'$ is properly canceled out by applying $N_x^{\dagger} \hat{x}(t)$. In fact, by Lemma 1, the entanglement fidelity equals one if $x \in \hat{x}(t) + L$ and zero otherwise since the final states is $X^u Z^{u'} - \hat{u}' |\Phi_s\rangle$. Hence, we obtain the desired formula.
\section*{A4. Proof of Theorem 3}

Suppose the twirling is applied to \( \rho_n = M_n(\mathcal{A}) \) to yield \( M_n(\tilde{\mathcal{A}}) \). Since the matrix of \( M_n(\tilde{\mathcal{A}}) \) is diagonal with respect to the basis \( \{|\Psi_x\rangle\}_{x \in \mathbb{F}_d^{2n}} \), the mixed channel \( \tilde{\mathcal{A}} \) has the form \( \tilde{\mathcal{A}} : \sigma \mapsto \sum_x P_n(x)N_x\sigma N_x^\dagger \) with the probability distribution \( P_n = P_\mathcal{A} \) on \( \mathbb{F}_d^{2n} \) by Theorem 1 of Choi\textsuperscript{15} as argued in Section 2.5.

Now assume \( C \subseteq H^{\otimes n} \) is a code subspace, say \( C(0) \), of the symplectic code. Then, by Lemma 2

\[ P_\mathcal{A}(\tilde{x}(t) + L) = F_e(\pi_C, R^{(0,t)}_{\tilde{\mathcal{A}}}) \]

\[ = F_e(\pi_C, d^{-2n}R^{(0,t)}_{\tilde{\mathcal{A}}}) \sum_x N_x A N_x^{-1} \]

\[ = \frac{1}{d^{2n}} \sum_x F_e(N_x^{-1}(\pi_C), N_x^{-1}R^{(0,t)}_{\tilde{\mathcal{A}}}). \]

Since \( N_x^\dagger C = \{|N_x^\dagger \psi\rangle | \psi \in C\} \) ranges uniformly over the whole set of code subspaces of the symplectic code associated with \( L \) (Section A.3 of this appendix or Section III of Ref. 19) as \( x \) runs through \( \mathbb{F}_d^{2n} \) and \( N_x^{-1}R^{(0,t)}_{\tilde{\mathcal{A}}} = R^{(s,t)}_{\tilde{\mathcal{A}}} \) as can be checked easily, this means that the entanglement fidelity \( F_e(\pi_C, R^{(s,t)}_{\tilde{\mathcal{A}}}) \) of the symplectic code averaged over all code subspaces \( C(s), s \in \mathbb{F}_d^{2n-k} \), is given by \( P_\mathcal{A}(\tilde{x}(t) + L) \), as promised.

\section*{A5. Remark on Symplectic Stabilizer Codes}

If we define \( N \) by (4) for \( d = 2 \), most existing arguments on symplectic codes work. In this case, however, we cannot assume \( \mu_i, 1 \leq i \leq n \), to be all one in general. For example, recall the eigenvalues of \( XZ \).

\section*{Appendix B. Fidelities and Quantum Capacity}

In this appendix, only for a technical reason, we define three variants of \( Q \), which will appear as \( Q_{e,d} \), \( Q_p \), and \( Q_{p,d} \), and show that these are all equal to each other. This fact is used in the proof of Theorem 8 in Section 7.

In Definition 5, we could have used minimum pure state fidelity

\[ F_p(C, B) = \min_{|\varphi\rangle \in C, \|\varphi\| = 1} \langle \varphi | B(|\varphi\rangle \langle \varphi|) | \varphi \rangle \]

in place of entanglement fidelity. The \( Q_p \) is defined in the same way as \( Q \) with \( F_e \) replaced by \( F_p \). In Definition 5, we could also have restrict ourselves to codes \( \{(C_n, E_n, D_n)\} \) such that \( \dim C_n = d^m \) for some integer \( m \) for every \( n \). We can define the achievability with this restriction on codes, and provided the employed fidelity is \( F_e, [F_p] \), we denote the corresponding capacity by \( Q_{e,d} \) and \( Q_{p,d} \).

Now we will check the equalities among the four quantities. Put \( Q_e = Q \) for accordance with the other three. It is known\textsuperscript{21} that

\[ 1 - F_e(\pi_C, B) \leq (3/2)[1 - \]
For any TPCP map $\mathcal{B}$, hence, $Q_p \leq Q_e$, which is shorthand for $Q_p(A) \leq Q_e(A)$ for any memoryless channel $A$. From this fact and by definitions, we have

$$Q_{p,d} \leq Q_p \leq Q_e,$$

and

$$Q_{p,d} \leq Q_e, d \leq Q_e.$$

Then, all we have to show is $Q_{p,d} \geq Q_e$. This follows from that the entanglement fidelity $F_e(\pi_C, \mathcal{B})$ is not larger than the pure-state fidelity $(\dim C)^{-1} \sum_{\phi \in \mathcal{S}} \langle \phi | \mathcal{B}(|\phi\rangle \langle \phi|) | \phi \rangle$ averaged over $\mathcal{S}$, where $\mathcal{S}$ is an arbitrary orthonormal basis of $C^{20}$. In fact, we can reduce $C$ to a good subspace $C' \subseteq C$ of dimension $[d^{-1} \dim C]$ only with negligible loss of the fidelity as in the proof of Lemma 1 of Ref. 6 or as in Section V-A of Ref. 21. Specifically, $F_p(C'_n, \mathcal{B}_n) \rightarrow 1$ for a good choice of $C'_n \subseteq C_n$ provided $F_e(\pi_{C_n}, \mathcal{B}_n) \rightarrow 1$ as $n \rightarrow \infty$, where $\mathcal{B}_n = D_n \mathcal{A}^\otimes n \mathcal{E}_n$.

This implies $Q_{p,d} \geq Q_e$, completing the proof.

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