Homogenization of Evolutionary Incompressible Navier–Stokes System in Perforated Domains

Yong Lu and Peikang Yang

Communicated by E. Feireisl

This paper is dedicated to the memory of Antonín Novotný.

Abstract. In this paper, we consider the homogenization problems for evolutionary incompressible Navier–Stokes system in three dimensional domains perforated with a large number of small holes which are periodically located. We first establish certain uniform estimates for the weak solutions. To overcome the extra difficulties coming from the time derivative, we use the idea of Temam (Navier–Stokes equations, North-Holland, Amsterdam, 1979) and consider the equations by integrating in time variable. After suitably extending the weak solutions to the whole domain, we employ the generalized cell problem to study the limit process.

Keywords. Homogenization, Evolutionary incompressible Navier–Stokes equations, perforated domains.

1. Introduction

1.1. Background

In this paper we study the homogenization of homogeneous incompressible Navier–Stokes equations in a perforated domain in \( \mathbb{R}^3 \) under Dirichlet boundary condition. Our goal is to describe the limit behavior of the (weak) solutions as the number of holes goes to infinity and the size of holes goes to zero simultaneously.

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain of class \( C^{2,\beta} \) \((0 < \beta < 1)\). The holes in \( \Omega \) are denoted by \( T_{\varepsilon,k} \) which are assumed to satisfy

\[
B(\varepsilon x_k, \delta_0 a_{\varepsilon}) \subset T_{\varepsilon,k} = \varepsilon x_k + a_{\varepsilon} T_0 \subset \subset B(\varepsilon x_k, \delta_1 a_{\varepsilon}) \subset \subset B(\varepsilon x_k, \delta_2 a_{\varepsilon}) \subset B(\varepsilon x_k, \delta_3 \varepsilon) \subset \varepsilon Q_k, \quad (1.1)
\]

where the cube \( Q_k := \left( -\frac{1}{2}, \frac{1}{2} \right)^3 + k \) and \( x_k = x_0 + k \) with \( x_0 \in T_0 \), for each \( k \in \mathbb{Z}^3 \); \( T_0 \) is a model hole which is assumed to be a closed bounded and simply connected \( C^{2,\beta} \) domain; \( \delta_i \), \( i = 0, 1, 2, 3 \) are fixed positive numbers. The perforation parameters \( \varepsilon \) and \( a_{\varepsilon} \) are used to measure the mutual distance of holes and the size of holes, and \( \varepsilon x_k = \varepsilon x_0 + \varepsilon k \) present the locations of holes. Without loss of generality, we assume that \( x_0 = 0 \) and \( 0 < a_{\varepsilon} \leq \varepsilon \leq 1 \).

The perforated domain \( \Omega_{\varepsilon} \) under consideration is described as follows:

\[
\Omega_{\varepsilon} := \Omega \setminus \bigcup_{k \in K_{\varepsilon}} T_{\varepsilon,k}, \quad K_{\varepsilon} := \{k \in \mathbb{Z}^3 : \varepsilon Q_k \subset \Omega \}. \quad (1.2)
\]

The study of homogenization problems in fluid mechanics have gained a lot interest. In particular, the homogenization of Stokes system in perforated domains has been systematically studied. In 1980s,
Tartar [22] considered the case where the size of holes is proportional to the mutual distance of holes and derived Darcy’s law. In 1990s, Allaire [1,2] considered general size of holes and obtained complete results in periodic setting. By introducing a local problem and employing an abstract framework the idea of which goes back to [6], Allaire found that the homogenized limit equations are determined by the ratio \( \sigma \) given as

\[
\sigma_\varepsilon := \left( \frac{\varepsilon d}{d_\varepsilon - 2} \right)^{\frac{1}{2}}, \quad \varepsilon \geq 3; \quad \sigma_* := \varepsilon \left| \log \frac{a_\varepsilon}{\varepsilon} \right|^{\frac{1}{2}}, \quad d = 2,
\]

where \( d \) is the spatial dimension. More precisely, if \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = 0 \) corresponding to the case of large holes, the homogenized system is the Darcy’s law; if \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty \) corresponding to the case of small holes, the motion of the fluid does not change much in the homogenization process and in the limit there arise the same Stokes equations; if \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma_* \in (0, \infty) \) corresponding to the case of critical size of holes, the homogenized system is governed by the Brinkman’s law—a combination of the Darcy’s law and the Stokes equations.

The homogenization study is extended to more complicated models describing fluid flows: Mikelić [20] has studied the incompressible Navier–Stokes equations in a porous medium; Masmoudi [19] studied the compressible Navier–Stokes equations. In all these previous studies, only the case where the size of holes is proportional to the mutual distance of holes and the Darcy’s law is recovered in the limit. Recently, Feireisl, Novotný, Namlyeyeva, and Namlyeyeva [10] studied the case with critical size of holes for the incompressible Navier–Stokes equations and they derived Brinkman’s law; Feireisl et al. also considered the case of small holes for the compressible Navier–Stokes equations [7,9,18]. Bella and Oschmann [5] studied the homogenization of compressible Navier–Stokes equations with very small and randomly distributed holes. The low Mach number and homogenization limit of compressible Navier–Stokes equations in perforated domains were considered by Höfer, Kowalczyk, and Schwarzache [12] for the case of large holes, and by Bella and Oschmann [4] for the case of critical size of holes.

In [1,2], Allaire also gave a rather complete description concerning the homogenization of stationary incompressible Navier–Stokes equations and the results coincide with the Stokes equations: for the case of small holes, the equations remain unchanged; for the case of large holes, Darcy’s law is derived; for the case of critical size of holes, Brinkman type equations are recovered. While, for the evolutionary incompressible Navier–Stokes equations, the study is not complete with respect to the size of holes, even in periodic setting: in [20], Mikelić considered the case when the size of holes is proportional to the mutual distance of holes, and in [10] the critical size of holes is considered.

In this paper, we shall consider the homogenization of evolutionary incompressible Navier–Stokes equations in perforated domain \( \Omega_\varepsilon \) with Dirichlet boundary condition and our goal is to give a complete description for the homogenization process related to small and large sizes of holes in periodic setting. Let \( T > 0 \), the initial boundary problem in space-time cylinder \( \Omega_\varepsilon \times (0, T) \) under consideration is the following:

\[
\begin{aligned}
\partial_t u_\varepsilon + \text{div} (u_\varepsilon \otimes u_\varepsilon) - \mu \Delta u_\varepsilon + \nabla p_\varepsilon &= f_\varepsilon, \quad \text{in } \Omega_\varepsilon \times (0, T), \\
\text{div } u_\varepsilon &= 0, \quad \text{in } \Omega_\varepsilon \times (0, T), \\
u_\varepsilon |_{t=0} &= u_0 \in L^2(\Omega_\varepsilon; \mathbb{R}^3), \quad \text{on } \partial \Omega_\varepsilon \times (0, T), \\
\end{aligned}
\]

Here \( u_\varepsilon \) is the fluid velocity field in \( \mathbb{R}^3 \) and \( p_\varepsilon \) is the fluid pressure. The external force \( f_\varepsilon \) is assumed to be in \( L^2(\Omega_\varepsilon \times (0, T); \mathbb{R}^3) \).

We recall some notations. Let \( W^{1,q}_0(\Omega) \) be the collection of Sobolev functions in \( W^{1,q}(\Omega) \) with zero trace, and let \( W^{-1,q} \) be the dual space of \( W^{1,q}_0(\Omega) \). We set \( V^{1,q}(\Omega) := \{ v \in W^{1,q}_0(\Omega) \}, \quad \text{div } v = 0 \) with \( W^{1,q} \) norm and \( V^{-1,q}(\Omega) \) be the dual space of \( V^{1,q}(\Omega) \). Let \( L^q(\Omega) \) be the collection of \( L^q(\Omega) \) integrable functions that are of zero average. We sometimes use \( L^pL^q \) to denote the Bochner space \( L^p(0, T; L^q(\Omega)) \) or.

\[ Birkhäuser \]
\( L^r(0, T; L^s(\Omega); \mathbb{R}^3) \) for short. For a function \( g \) in \( \Omega \), we use the notation \( \tilde{g} \) to represent its zero extension in \( \Omega \):

\[
\tilde{g} = g \text{ in } \Omega_{\varepsilon}, \quad \tilde{g} = 0 \text{ in } \Omega \setminus \Omega_{\varepsilon} = \bigcup_{k \in K_\varepsilon} T_{\varepsilon,k}.
\]

We now recall the definition of (finite energy) weak solutions:

**Definition 1.1.** We call \( u_\varepsilon \) a weak solution of (1.4) in \( \Omega \times (0, T) \) provided:

- There holds:
  \[
  u_\varepsilon \in L^2(0, T; V^{1,2}(\Omega_{\varepsilon})) \cap C_{\text{weak}}([0, T], L^2(\Omega_{\varepsilon})) \cap C([0, T], L^q(\Omega_{\varepsilon})),
  \partial_t u_\varepsilon \in L^\frac{2}{q}(0, T; V^{-1,2}(\Omega_{\varepsilon})),
  \]
  for any \( 1 \leq q < 2 \).
- For any \( \varphi \in C_c^\infty(\Omega \times [0, T); \mathbb{R}^3) \) with \( \text{div} \varphi = 0 \), there holds
  \[
  \int_0^T \int_{\Omega_{\varepsilon}} -u_\varepsilon \cdot \partial_t \varphi - u_\varepsilon \otimes u_\varepsilon : \nabla \varphi + \mu \nabla u_\varepsilon : \nabla \varphi \, dx \, dt
  = \int_0^T \int_{\Omega_{\varepsilon}} f_\varepsilon \cdot \varphi \, dx \, dt + \int_{\Omega_{\varepsilon}} u^0_\varepsilon \cdot \varphi(0, 0) \, dx \, dt.
  \]

The pressure \( p_\varepsilon \) is determined by

\[
\langle \nabla p_\varepsilon, \psi \rangle = -\langle \partial_t u_\varepsilon + \text{div} (u_\varepsilon \otimes u_\varepsilon) - \mu \Delta u_\varepsilon - f_\varepsilon, \psi \rangle, \quad \forall \psi \in C_c^\infty(\Omega \times (0, T); \mathbb{R}^3).
\]

We further call \( u_\varepsilon \) a finite energy weak solution of (1.4) provided there holds in addition the energy inequality: for a.a. \( t \in (0, T) \),

\[
\frac{1}{2} \int_{\Omega_{\varepsilon}} |u_\varepsilon(x, t)|^2 \, dx + \int_0^t \int_{\Omega_{\varepsilon}} |\nabla u_\varepsilon(x, s)|^2 \, dx \, ds \leq \int_0^t \int_{\Omega_{\varepsilon}} |f_\varepsilon \cdot u_\varepsilon| \, dx \, ds + \frac{1}{2} \int \int_{\Omega_{\varepsilon}} |u^0_\varepsilon(x)|^2 \, dx.
\]

For each fixed \( \varepsilon \), the existence of a global finite energy weak solution \( u_\varepsilon \) is known, see for example Leray’s pioneer work [14] or the classical books [15,23]. We need to investigate the behavior of the solutions as \( \varepsilon \to 0 \).

### 1.2. Main Results

From (1.3), we see that the size of holes \( a_\varepsilon \) is typically chosen to be \( \varepsilon^\alpha \) (\( \alpha \geq 1 \)) in three or higher dimensional spaces, while \( a_\varepsilon \) is typically chosen to be \( e^{-\varepsilon^{-\alpha}} \) (\( \alpha > 0 \)) in two dimensional case. Here for the study of three dimensional case, we shall take

\[
a_\varepsilon = \varepsilon^\alpha \text{ with } \alpha \geq 1, \text{ which implies } \sigma_\varepsilon = \varepsilon^\frac{2-\alpha}{2}.
\]

Throughout the paper, we will assume the zero extension of the initial datum and the external force satisfy

\[
\tilde{u}^0_\varepsilon \to u^0 \text{ strongly in } L^2(\Omega), \quad \tilde{f}_\varepsilon \to f \text{ strongly in } L^2(\Omega \times (0, T)).
\]

Let \( u_\varepsilon \) be a finite energy weak solution of (1.4). We employ the idea of Mikelić [20] and Temam [23] and introduce for any \( t \in (0, T) \),

\[
U_\varepsilon(\cdot, t) := \int_0^t u_\varepsilon(\cdot, s) \, ds, \quad \Psi_\varepsilon(\cdot, t) := \int_0^t u_\varepsilon(\cdot, s) \otimes u_\varepsilon(\cdot, s) \, ds, \quad F_\varepsilon(\cdot, t) := \int_0^t f_\varepsilon(\cdot, s) \, ds.
\]
Then $U_\varepsilon \in C([0, T]; W^{1,2}_0(\Omega_\varepsilon))$, $\text{div } U_\varepsilon = 0$, $F_\varepsilon \in C([0, T]; L^2(\Omega_\varepsilon))$ and $\text{div } \Psi_\varepsilon \in C([0, T]; L^{\frac{2}{3}}(\Omega_\varepsilon))$. As shown in the proof of our theorems, instead of showing the limit behavior of $\mathbf{u}_\varepsilon$, we turn to study the limit behavior of $U_\varepsilon$ which has better regularity in time variable. Clearly

$$\lim_{\varepsilon \to 0} \mathbf{F}_\varepsilon = \mathbf{F} = \int_0^t f(s) \, ds \text{ strongly in } L^2(\Omega \times (0, T)). \quad (1.11)$$

The classical theory on Stokes equations implies that there exists $P_\varepsilon \in C([0, T]; L_0^2(\Omega_\varepsilon))$ (see Chapter 3 in [23]), such that for any $t \in (0, T)$,

$$\mathbf{u}_\varepsilon(t) - \mathbf{u}_0^\varepsilon + \text{div } \Psi_\varepsilon(t) - \mu \Delta \mathbf{U}_\varepsilon(t) + \nabla P_\varepsilon(t) = \mathbf{F}_\varepsilon(t), \quad \text{in } W^{-1,2}(\Omega_\varepsilon). \quad (1.12)$$

Now we state our results corresponding to different sizes of holes. The case of critical size of holes is considered by Feireisl–Namlyeyeva–Nečasová in [10], so we are focusing only on the case of small holes and large holes. Note that the limits are taken up to possible extractions of subsequences. The first result corresponds to the case of small holes:

**Theorem 1.1.** Let $\mathbf{u}_\varepsilon$ be a finite energy weak solution of the Navier–Stokes system (1.4) in the sense of Definition 1.1 with initial datum and external force satisfying (1.9). Let $\tilde{p}_\varepsilon$ be the extension of $p_\varepsilon$ defined by $\tilde{p}_\varepsilon = \partial_t \tilde{P}_\varepsilon$ where $\tilde{P}_\varepsilon$ is the extension of $P_\varepsilon$ defined in (2.8) and (2.11). If $\alpha > 3$, i.e. $\lim_{\varepsilon \to \infty} \sigma_\varepsilon = \infty$, then

$$\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ weakly(*) in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega)), \quad (1.13)$$

and

$$\tilde{p}_\varepsilon \to p \text{ weakly in } W^{-1,2}(0, T; L^2(\Omega)). \quad (1.14)$$

Moreover, $(\mathbf{u}, p)$ is a weak solution of the Navier–Stokes equations in homogeneous domain $\Omega$:

$$\begin{aligned}
\partial_t \mathbf{u} + (\mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla p = \mathbf{f}, & \quad \text{in } \Omega \times (0, T), \\
\text{div } \mathbf{u} = 0, & \quad \text{in } \Omega \times (0, T), \\
\mathbf{u} = 0, & \quad \text{on } \partial \Omega \times (0, T), \\
|\mathbf{u}|_{t=0} = \mathbf{u}^0.
\end{aligned} \quad (1.15)$$

For the case of large holes, we consider the time-scaled Navier–Stokes system:

$$\begin{aligned}
\sigma_\varepsilon^2 \partial_t \mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \mu \Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}, & \quad \text{in } \Omega_\varepsilon \times (0, T), \\
\text{div } \mathbf{u}_\varepsilon = 0, & \quad \text{in } \Omega_\varepsilon \times (0, T), \\
\mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0^\varepsilon \in L^2(\Omega_\varepsilon; \mathbb{R}^3).
\end{aligned} \quad (1.16)$$

The time scaling parameter $\sigma_\varepsilon^2$ in (1.16) is inspired by the study in [19]. This ensures the energy estimates (see (3.2)) more compatible.

For a solution $\mathbf{u}_\varepsilon$ to (1.16), we similarly introduce $U_\varepsilon, \Psi_\varepsilon, F_\varepsilon$ as in (1.10). There exists $P_\varepsilon \in C([0, T]; L_0^2(\Omega_\varepsilon))$, such that for any $t \in (0, T)$,

$$\sigma_\varepsilon^2 \mathbf{u}_\varepsilon(t) - \sigma_\varepsilon^2 \mathbf{u}_0^\varepsilon + \Phi_\varepsilon(t) - \mu \Delta \mathbf{U}_\varepsilon(t) + \nabla P_\varepsilon(t) = \mathbf{F}_\varepsilon(t), \quad \text{in } W^{-1,2}(\Omega_\varepsilon). \quad (1.17)$$

We have the following theorem concerning the case of large holes:

**Theorem 1.2.** Let $\mathbf{u}_\varepsilon$ be a finite energy weak solution of the time-scaled Navier–Stokes system (1.16) in the sense of Definition 1.1 with initial datum and external force satisfying (1.9). Let $\tilde{p}_\varepsilon$ be the extension of $p_\varepsilon$ defined by $\tilde{p}_\varepsilon = \partial_t \tilde{P}_\varepsilon$ where $\tilde{P}_\varepsilon$ is the extension of $P_\varepsilon$ defined in (3.14) and (3.15). If $1 < \alpha < 3$, i.e. $\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0$, then

$$734 \sigma_\varepsilon^{-2} \tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{R}^3)), \quad (1.18)$$
and
\[\tilde{p}_\varepsilon = \tilde{p}^{(1)}_\varepsilon + \sigma_{\varepsilon}^{1/2} \tilde{p}^{(2)}_\varepsilon + \sigma_{\varepsilon}^2 \tilde{p}^{(3)}_\varepsilon,\]  
(1.19)

with
\[\tilde{p}^{(1)}_\varepsilon \to p \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)),\]
\[\tilde{p}^{(2)}_\varepsilon \text{ bounded in } L^{4/3}(0, T; L^2_0(\Omega)),\]
\[\tilde{p}^{(3)}_\varepsilon \text{ bounded in } W^{-1,2}(0, T; L^2_0(\Omega)).\]  
(1.20)

Moreover \((u, p)\) is a weak solution of the Darcy’s law:
\[
\begin{aligned}
\mu u &= A(f - \nabla p), \quad \text{in } \Omega \times (0, T), \\
\text{div } u &= 0, \quad \text{in } \Omega \times (0, T), \\
u \cdot n &= 0, \quad \text{on } \partial \Omega \times (0, T),
\end{aligned}
\]  
(1.21)

where \(n\) is the unit normal vector on the boundary of \(\Omega\).

Here in (1.21), the permeability tensor \(A\) is a constant positive definite matrix (see [3]) given by:
\[
A_{i,j} := \lim_{\eta \to 0} c_{\eta}^{-2} \int_{Q_\eta} \nabla w^i_{\eta} : \nabla w^j_{\eta} \, dx = \lim_{\eta \to 0} \int_{Q_\eta} (w^j_{\eta})_i \, dx := (\bar{w}^j)_i,
\]
where \(w^i_{\eta}\) satisfies the generalized cell problem introduced in Sect. 1.3.

Remark 1.3. The case \(\alpha = 1\) is considered by Mikelić [20]. In this case the permeability tensor \(A_{i,j}\) is defined by the classical cell problem where \(\eta = 1\) (see also [22]) which is slightly different from the definition above.

Sections 2 and 3 are devoted to the Proofs of Theorems 1.1 and 1.2 respectively. In the sequel, \(C\) denotes a constant independent of \(\varepsilon\), while its value may differ from line to line.

1.3. Generalized Cell Problem

We introduce the idea of generalized cell problem [13,17,22] which is used to study the homogenization process. Near each single hole, after a scaling of size \(\varepsilon^{-1}\) such that the controlling cube becomes the \(O(1)\) size, one can consider the following modified cell problem:
\[
\begin{aligned}
-\Delta w^i_\eta + \nabla q^i_\eta &= c_{\eta}^2 \varepsilon^i, \quad \text{in } Q_\eta := Q_0 \setminus (\eta T_0), \\
\text{div } w^i_\eta &= 0, \quad \text{in } Q_\eta, \\
w^i_\eta &= 0, \quad \text{on } \partial \eta T_0, \\
(w^i_\eta, q^i_\eta) \text{ is } Q_0\text{-periodic.}
\end{aligned}
\]

Here \(Q_0 = (-\frac{1}{2}, \frac{1}{2})^d\), \(\eta := \frac{a \varepsilon}{\sigma_{\varepsilon}}\), \(c_{\eta} := \frac{\sigma_{\varepsilon}}{\varepsilon}\), and \(\{e^i\}_{i=1}^d\) is the standard Euclidean coordinate of \(\mathbb{R}^d\). Clearly \(c_{\eta} \to 0\) when \(\eta \to 0\). When \(a_{\varepsilon}\) is proportional to \(\varepsilon\), \(\eta\) becomes a positive constant independent of \(\varepsilon\) and \(Q_\eta\) becomes a fixed domain of type \(Q_0\setminus T_0\); this is the case considered by Tartar [22].

For each fixed \(\eta > 0\), the generalized cell problem admits a unique regular solution. We now recall two lemmas concerning the estimates of the cell problems. The proofs can be found in [17].

Lemma 1.4. The solution \((w^i_\eta, q^i_\eta)\) of the generalized cell problem has the estimates:
\[
\|\nabla w^i_\eta\|_{L^2(Q_\eta)} \leq C c_{\eta}, \quad \|w^i_\eta\|_{L^2(Q_\eta)} \leq C, \quad \|q^i_\eta\|_{L^2(Q_\eta)} \leq C c_{\eta},
\]  
(1.22)
Define the scaled cell solutions

\[ w^i_{\eta, \varepsilon}(\cdot) := w^i_{\eta}(\frac{\cdot}{\varepsilon}), \quad q^i_{\eta, \varepsilon}(\cdot) := q^i_{\eta}(\frac{\cdot}{\varepsilon}) \]

which solves

\[
\begin{cases}
-\varepsilon^2 \Delta w^i_{\eta, \varepsilon} + \varepsilon \nabla q^i_{\eta, \varepsilon} = c_\eta^2 \varepsilon^i, & \text{in } \varepsilon Q_0 \setminus (a_\varepsilon T_0), \\
\text{div } w^i_{\eta, \varepsilon} = 0, & \text{in } \varepsilon Q_\eta, \\
\frac{\partial w^i_{\eta, \varepsilon}}{\partial n} = 0, & \text{on } a_\varepsilon T_0, \\
(w^i_{\eta, \varepsilon}, q^i_{\eta, \varepsilon}) \text{ is } \varepsilon Q_0\text{-periodic.}
\end{cases}
\] (1.23)

Employing the estimates of \((w^i_{\eta}, q^i_{\eta})\) in (1.22) gives

**Lemma 1.5.** The scaled cell solution \((w^i_{\eta, \varepsilon}, q^i_{\eta, \varepsilon})\) has the estimates:

\[
\|w^i_{\eta, \varepsilon}\|_{L^2(\Omega)} \leq C\|w^i_{\eta}\|_{L^2(Q_0)} \leq C,
\]

\[
\|q^i_{\eta, \varepsilon}\|_{L^2(\Omega)} \leq C\|q^i_{\eta, \varepsilon}\|_{L^2(Q_0)} \leq C\varepsilon^{-1} c_\eta \leq C\sigma^{-1}_\varepsilon.
\] (1.24)

From these estimates we have

\[ w^i_{\eta, \varepsilon} \to \bar{w}^i \text{ weakly in } L^2(\Omega), \quad c^{-1}_\eta q^i_{\eta, \varepsilon} \to \bar{q}^i \text{ weakly in } L^2(\Omega). \] (1.25)

**2. Proof of Theorem 1.1**

This section is devoted to proving Theorem 1.1 concerning the case of small holes where \(a_\varepsilon = \varepsilon^\alpha\) with \(\alpha > 3\) and \(\sigma_\varepsilon = \varepsilon^{\frac{3-\alpha}{2}} \to \infty\) (see (1.8)).

**2.1. Estimates of Velocity**

In this case, the uniform estimates of \(u_\varepsilon\) follow directly from the energy inequality (1.7). Indeed, using Hölder’s inequality and Poincaré inequality gives

\[
\frac{1}{2} \int_{0}^{t} \int_{\Omega_\varepsilon} |u_\varepsilon(x, t)|^2 \, dx + \int_{0}^{t} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x, s)|^2 \, dx \, ds \\
\leq \int_{0}^{t} \int_{\Omega_\varepsilon} f_\varepsilon \cdot u_\varepsilon \, dx \, ds + \frac{1}{2} \int_{\Omega_\varepsilon} |u_\varepsilon^0(x)|^2 \, dx \\
\leq C \sup_{0 < \varepsilon \leq 1} \|f_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}^2 + \frac{1}{2}\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}^2 + \frac{1}{2} \sup_{0 < \varepsilon \leq 1} \|u_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}^2.
\]

Together with the assumption on the initial datum and the external force in (1.9), we deduce

\[
\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C, \quad \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C.
\] (2.1)

Since \(u_\varepsilon \in L^2(0,T;W^{1,2}_{0}(\Omega_\varepsilon))\) has zero trace on the boundary, its zero extension \(\tilde{u}_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}_{0}(\Omega))\) has the estimates:

\[
\|\tilde{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \|\tilde{u}_\varepsilon\|_{L^2(0,T;W^{1,2}_{0}(\Omega))} \leq C.
\] (2.2)

Thus, up to a subsequence, there holds the convergence

\[
\tilde{u}_\varepsilon \to u \text{ weakly(*) in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}_{0}(\Omega)),
\] (2.3)

which is exactly (1.13) in Theorem 1.1. Moreover, by the definition of \(U_\varepsilon\) in (1.10) we have

\[
\|\tilde{U}_\varepsilon\|_{W^{1,\infty}(0,T;L^2(\Omega))} \leq C, \quad \|\tilde{U}_\varepsilon\|_{W^{1,2}(0,T;W^{1,2}_{0}(\Omega))} \leq C, \quad \|\tilde{U}_\varepsilon\|_{C([0,T];W^{1,2}_{0}(\Omega))} \leq C.
\] (2.4)
2.2. Extension of Pressure

The extension of the pressure is given by using the dual formula and employing the so-called restriction operator due to Allaire [1,2] for general sizes of holes, and due to Tartar [22] for the case where the size of the holes is proportional to their mutual distance. A restriction operator $R_\varepsilon$ is a linear operator $R_\varepsilon : W^{1,2}_0(\Omega; \mathbb{R}^d) \rightarrow W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^d)$ such that:

$$
\begin{align*}
\mathbf{u} \in W^{1,2}_0(\Omega_\varepsilon; \mathbb{R}^d) & \implies R_\varepsilon(\mathbf{u}) = \mathbf{u} \text{ in } \Omega_\varepsilon, \quad \text{where } \tilde{u} := \begin{cases} 
\mathbf{u} & \text{in } \Omega_\varepsilon, \\
0 & \text{on } \Omega \setminus \Omega_\varepsilon,
\end{cases} \\
\mathbf{u} \in W^{1,2}_0(\Omega; \mathbb{R}^d), \text{ div } \mathbf{u} = 0 \text{ in } \Omega & \implies \text{div } R_\varepsilon(\mathbf{u}) = 0 \text{ in } \Omega_\varepsilon, \\
\mathbf{u} \in W^{1,2}_0(\Omega; \mathbb{R}^d) & \implies \|\nabla R_\varepsilon(\mathbf{u})\|_{L^2(\Omega_\varepsilon)} \leq C \left( \|\nabla \mathbf{u}\|_{L^2(\Omega)} + (1 + \sigma_\varepsilon^{-1}) \|\mathbf{u}\|_{L^2(\Omega)} \right).
\end{align*}
$$

(2.5)

For each $\varphi \in L^q(0,T; W^{1,2}_0(\Omega))$ with $1 < q < \infty$, the restriction $R_\varepsilon(\varphi)$ is taken only on spatial variable:

$$
R_\varepsilon(\varphi)(\cdot,t) = R_\varepsilon(\varphi(\cdot,t))(\cdot) \quad \text{for each } t \in (0,T).
$$

Clearly $R_\varepsilon$ maps $L^q(0,T; W^{1,2}_0(\Omega))$ onto $L^q(0,T; W^{1,2}_0(\Omega_\varepsilon))$ with the estimate:

$$
\|\nabla R_\varepsilon(\varphi)\|_{L^q(0,T; L^2(\Omega_\varepsilon))} \leq C \left( \|\nabla \varphi\|_{L^q(0,T; L^2(\Omega))} + (1 + \sigma_\varepsilon^{-1}) \|\varphi\|_{L^q(0,T; L^2(\Omega))} \right).
$$

(2.6)

**Lemma 2.1.** Let $1 < q < \infty$. Assume $H \in L^q(0,T; W^{-1,2}(\Omega; \mathbb{R}^3))$ satisfying

$$
\langle H, \varphi \rangle_{\Omega \times (0,T)} = 0, \quad \forall \varphi \in C^\infty_c(\Omega \times (0,T); \mathbb{R}^3), \quad \text{div } \varphi = 0.
$$

Then there exists a scalar function $P \in L^q(0,T; L^2_0(\Omega))$ such that:

$$
H = \nabla P, \quad \text{with } \|P\|_{L^q(0,T; L^2_0(\Omega))} \leq C \|H\|_{L^q(0,T; W^{-1,2}(\Omega))}.
$$

(2.7)

**Proof.** For each $\psi \in C^\infty_c(\Omega; \mathbb{R}^3)$ with $\text{div } \psi \equiv 0$ and $\phi \in C^\infty_c(0,T)$, there holds

$$
0 = \langle H, \phi(t)\psi(x) \rangle_{\Omega \times (0,T)} = \int_0^T \langle H(\cdot,t), \psi(\cdot) \rangle_{\Omega} \phi(t) dt.
$$

Thus we have for a.a. $t \in (0,T)$ that

$$
\langle H(\cdot,t), \psi(\cdot) \rangle_{\Omega} = 0, \quad \forall \psi \in C^\infty_c(\Omega; \mathbb{R}^3), \quad \text{div } \psi \equiv 0.
$$

Therefore, for a.a. $t \in (0,T)$, there exists $P(\cdot,t) \in L^2_0(\Omega)$ (see [11]), such that

$$
H(\cdot,t) = \nabla P(\cdot,t), \quad \text{with } \|P(\cdot,t)\|_{L^2_0(\Omega)} \leq \|H(\cdot,t)\|_{W^{-1,2}(\Omega)}.
$$

This implies immediately (2.7). \qed

Now we define a functional $\tilde{H}_\varepsilon$ in $\mathcal{D}'(\Omega \times (0,T))$ by the following dual formulation:

$$
\langle \tilde{H}_\varepsilon, \varphi \rangle_{\Omega \times (0,T)} = \langle \nabla P_\varepsilon, R(\varphi) \rangle_{\Omega \times (0,T)} , \quad \forall \varphi \in C^\infty_c(\Omega \times (0,T); \mathbb{R}^3),
$$

where $P_\varepsilon \in C([0,T], L^2_0(\Omega_\varepsilon))$ is given in (1.12). Then for any $\varphi \in C^\infty_c(\Omega \times (0,T); \mathbb{R}^3)$,

$$
\langle \tilde{H}_\varepsilon, \varphi \rangle_{\Omega \times (0,T)} = \langle \nabla P_\varepsilon(t), R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon \times (0,T)}
$$

$$
= \langle \mathbf{F}_\varepsilon(t) - \mathbf{u}_\varepsilon(t) + \mathbf{u}_\varepsilon^0 + \mu \Delta \mathbf{U}_\varepsilon(t) - \Phi_\varepsilon(t), R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon \times (0,T)}.
$$

(2.8)
Using the uniform estimates in (2.1)–(2.4) implies

\[
\begin{align*}
\langle \Delta U_\varepsilon(t), R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon \times (0,T)} & \leq \| \nabla U_\varepsilon(t) \|_{L^2} \| \nabla R_\varepsilon(\varphi) \|_{L^2} \leq C \| \varphi \|_{L^2 W^{1,2}_0}, \\
\langle \Phi_\varepsilon(t), R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon \times (0,T)} & = \| \langle \Psi_\varepsilon(t), \nabla R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon \times (0,T)} \| \leq \| \Psi_\varepsilon \|_{L^\infty} \| \nabla R_\varepsilon(\varphi) \|_{L^2} \\
& \leq C \| \psi_\varepsilon \otimes u_\varepsilon \|_{L^2} \| \nabla \varphi \|_{L^2 W^{1,2}_0} \leq C \| \varphi \|_{L^2 W^{1,2}_0}, \\
\langle u_\varepsilon, R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon \times (0,T)} & \leq C \| u_\varepsilon \|_{L^\infty(0,T; L^2(\Omega_\varepsilon))} \| R_\varepsilon(\varphi) \|_{L^2} \leq C \| \varphi \|_{L^2 W^{1,2}_0}, \\
\langle u_\varepsilon^0, R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon \times (0,T)} & \leq C \| u_\varepsilon^0 \|_{L^2} \| R_\varepsilon(\varphi) \|_{L^2} \leq C \| \varphi \|_{L^2 W^{1,2}_0}, \\
\langle F_\varepsilon, R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon \times (0,T)} & \leq C \| F_\varepsilon \|_{L^2} \| R_\varepsilon(\varphi) \|_{L^2} \leq C \| \varphi \|_{L^2 W^{1,2}_0}.
\end{align*}
\]

Estimates in (2.9) imply

\[
\langle \tilde{H}_\varepsilon, \varphi \rangle_{\Omega \times (0,T)} \leq C \| \varphi \|_{L^2 W^{1,2}_0}.
\]

Thus \( \tilde{H}_\varepsilon \) is bounded in \( L^2(0,T; W^{-1,2}(\Omega; \mathbb{R}^3)) \). Moreover, the second property of the restriction operator in (2.5) implies that

\[
\langle \tilde{H}_\varepsilon, \varphi \rangle_{\Omega \times (0,T)} = 0, \quad \forall \varphi \in C^\infty_c(\Omega \times (0,T; \mathbb{R}^3)), \quad \text{div} \varphi = 0.
\]

Thus, by (2.10), we can apply Lemma 2.1 and deduce that there exists \( \tilde{P}_\varepsilon \in L^2(0,T; L^2(\Omega)) \) such that

\[
\tilde{H}_\varepsilon = \nabla \tilde{P}_\varepsilon,
\]

and

\[
\| \tilde{P}_\varepsilon \|_{L^2(0,T; L^2(\Omega))} \leq C \| \tilde{H}_\varepsilon \|_{L^2(0,T; W^{-1,2}(\Omega))} \leq C.
\]

### 2.3. Momentum Equations in the Homogeneous Domain

In the case of small holes, we find that the extension \( \tilde{u}_\varepsilon \) satisfies the Navier–Stokes equations in \( \Omega \) up to a small remainder:

**Proposition 2.2.** Under the assumptions in Theorem 1.1, the extension \( \tilde{u}_\varepsilon \) satisfies the following equations in the sense of distribution:

\[
\partial_t \tilde{u}_\varepsilon + \text{div} (\tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon) - \mu \Delta \tilde{u}_\varepsilon + \nabla \tilde{p}_\varepsilon = \tilde{f}_\varepsilon + G_\varepsilon, \quad \text{div} \tilde{u}_\varepsilon = 0,
\]

where \( G_\varepsilon \in \mathcal{D}'(\Omega \times (0,T)) \) satisfying

\[
\| G_\varepsilon, \varphi \| \leq C^\sigma (\| \partial_t \varphi \|_{L^4 L^2} + \| \nabla \varphi \|_{L^4 L^{r_1}}), \quad \forall \varphi \in C^\infty_c(\Omega \times (0,T; \mathbb{R}^3)), \quad \text{div} \varphi = 0.
\]

Here \( \sigma := ((3-q)\alpha - 3)/q > 0 \) for some \( q > 2 \) close to 2, and \( 2 < r_1 < 3 \) given in (2.18).

**Proof.** Let \( \varphi \in C^\infty_c(\Omega \times (0,T; \mathbb{R}^3)) \) with \( \text{div} \varphi = 0 \). To extend the Navier–Stokes equations from \( \Omega \) to \( \Omega_\varepsilon \), an idea is to find a family of functions \( \{ g_\varepsilon \}_{\varepsilon > 0} \) vanishing on the holes and converges to 1 in some Sobolev space \( W^{1,3}(\Omega) \) and decompose \( \varphi \) as

\[
\varphi = g_\varepsilon \varphi + (1 - g_\varepsilon) \varphi.
\]

Then \( g_\varepsilon \varphi \) can be treated as a test function for the momentum equations in \( \Omega_\varepsilon \). While for the terms related the other part \( (1 - g_\varepsilon) \varphi \), we show that they are small and converge to zero. However, such a decomposition destroyed the divergence free property of \( \varphi \); \( \text{div} (g_\varepsilon \varphi) \neq 0 \). To overcome this trouble, we introduce the following Bogovskii type operator in perforated domain \( \Omega_\varepsilon \) (see Proposition 2.2 in [18] and Theorem 2.3 in [7]):

\[\text{Birkhäuser} \]
Lemma 2.3. Let $\Omega_\varepsilon$ defined as in (1.1) and (1.2) with $\alpha \geq 1$. Then for any $1 < q < \infty$, there exists a linear operator $B_\varepsilon : L^R_0(\Omega_\varepsilon) \to W^{1,q}_0(\Omega_\varepsilon; \mathbb{R}^3)$ such that for any $f \in L^R_0(\Omega_\varepsilon)$, there holds
\[
\text{div} B_\varepsilon(f) = f \text{ in } \Omega_\varepsilon, \quad \|B_\varepsilon(f)\|_{W^{1,q}_0(\Omega_\varepsilon; \mathbb{R}^3)} \leq C \left( 1 + \varepsilon \frac{(3-q)\alpha-3}{q} \right) \|f\|_{L^R(\Omega_\varepsilon)} \tag{2.15}
\]
for some constant $C$ independent of $\varepsilon$.

For any $r > 3/2$, the linear operator $B_\varepsilon$ can be extended as a linear operator from $\{\text{div} g : g \in L^r(\Omega_\varepsilon; \mathbb{R}^3), g \cdot n = 0 \text{ on } \partial \Omega_\varepsilon\}$ to $L^r(\Omega_\varepsilon; \mathbb{R}^3)$ satisfying
\[
\|B_\varepsilon(\text{div} g)\|_{L^r(\Omega_\varepsilon; \mathbb{R}^3)} \leq C\|g\|_{L^r(\Omega_\varepsilon; \mathbb{R}^3)}, \tag{2.16}
\]
for some constant $C$ independent of $\varepsilon$.

By the description of the holes in (1.1), there exists cut-off functions $\{g_\varepsilon\}_{\varepsilon > 0} \subset C^\infty(\mathbb{R}^3)$ such that $0 \leq g_\varepsilon \leq 1$ and
\[
g_\varepsilon = 0 \text{ on } \bigcup_{k \in K_\varepsilon} B(\varepsilon x_k, \delta_1 \varepsilon^\alpha), \quad g_\varepsilon = 1 \text{ on } \left( \bigcup_{k \in K_\varepsilon} B(\varepsilon x_k, \delta_2 \varepsilon^\alpha) \right)^c, \quad |\nabla g_\varepsilon| \leq C\varepsilon^{-\alpha}.
\]
Then for each $1 \leq q \leq \infty$ there holds
\[
\|g_\varepsilon - 1\|_{L^q(\Omega_\varepsilon)} \leq C\varepsilon^{3\alpha-3}, \quad \|\nabla g_\varepsilon\|_{L^q(\mathbb{R}^3)} \leq C\varepsilon^{3\alpha-3-\alpha}. \tag{2.17}
\]
Now we estimate
\[
I^\varepsilon := \int_0^T \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon \cdot \partial_t \varphi + \tilde{u}_\varepsilon \cdot \nabla \varphi - \nabla \tilde{u}_\varepsilon : \nabla \varphi + \tilde{f}_\varepsilon \cdot \varphi \, dx \, dt.
\]
Using the decomposition (2.14) we write
\[
I^\varepsilon = \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon \cdot \partial_t (g_\varepsilon \varphi) + u_\varepsilon \otimes u_\varepsilon : \nabla (g_\varepsilon \varphi) - \nabla u_\varepsilon : \nabla (g_\varepsilon \varphi) + f_\varepsilon (g_\varepsilon \varphi) \, dx \, dt + \sum_{j=1}^4 I_j
\]
with
\[
I_1 = \int_0^T \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon (1 - g_\varepsilon) \partial_t \varphi \, dx \, dt,
\]
\[
I_2 = \int_0^T \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon : (1 - g_\varepsilon) \nabla \varphi - \tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon : \nabla g_\varepsilon \otimes \varphi \, dx \, dt,
\]
\[
I_3 = \int_0^T \int_{\Omega_\varepsilon} \nabla \tilde{u}_\varepsilon : (1 - g_\varepsilon) \nabla \varphi - \nabla \tilde{u}_\varepsilon : \nabla g_\varepsilon \otimes \varphi \, dx \, dt,
\]
\[
I_4 = \int_0^T \int_{\Omega_\varepsilon} \tilde{f}_\varepsilon (g_\varepsilon - 1) \varphi \, dx \, dt.
\]
Observe that
\[
\int_{\Omega_\varepsilon} \varphi \cdot \nabla g_\varepsilon \, dx = \int_{\Omega_\varepsilon} \text{div} (\varphi g_\varepsilon) \, dx = 0.
\]
Thus, we can apply Lemma 2.3 and introduce
\[
\varphi_1 := \varphi - \varphi_2, \quad \varphi_2 := B_\varepsilon(\text{div} (\varphi g_\varepsilon)) = B_\varepsilon(\varphi \cdot \nabla g_\varepsilon).
\]
Then $\varphi_1 \in C^\infty_c((0,T); W^{1,2}_0(\Omega_\varepsilon))$ satisfying $\text{div} \varphi_1 = 0$. Using the weak formulation (1.6) gives
\[
I^\varepsilon = \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon \cdot \partial_t \varphi_1 + u_\varepsilon \otimes u_\varepsilon : \nabla \varphi_1 - \nabla u_\varepsilon : \nabla \varphi_1 + f_\varepsilon \cdot \varphi_1 \, dx \, dt + \sum_{j=1}^4 I_j + \sum_{j=5}^8 I_j
\]
\[
= \sum_{j=1}^4 I_j + \sum_{j=5}^8 I_j.
\]
where

\[ I_5 = \int_0^T \int_{\Omega_\varepsilon} u_x \partial_t \varphi_2 dx dt, \quad I_6 = \int_0^T \int_{\Omega_\varepsilon} u_x \otimes u_x : \nabla \varphi_2 dx dt, \]

\[ I_7 = \int_0^T \int_{\Omega_\varepsilon} -\nabla u_x : \nabla \varphi_2 dx dt, \quad I_8 = \int_0^T \int_{\Omega_\varepsilon} f_x \cdot \varphi_2 dx dt. \]

Now we estimate \( I_j \) term by term. Since \( \alpha > 3 \), there exists \( q \in (2, 3) \) close to 2 such that

\[ \sigma := \frac{(3 - q)\alpha - 3}{q} > 0. \]

Let \( q^* \) be the Sobolev conjugate component to \( q \) with \( \frac{1}{q^*} = \frac{1}{q} - \frac{1}{3} \). Clearly \( 6 < q^* < \infty \).

By the uniform estimates (2.2) and (2.17), together with interpolation and Sobolev embedding, we obtain

\[ |I_1| \leq \| \tilde{u}_x \|_{L^4 L^3} \| 1 - g_\varepsilon \|_{L^6} \| \partial_t \varphi \|_{L^\frac{4}{3} L^2} \leq \| \tilde{u}_x \|_{L^\infty L^2 \cap L^2 L^6} \| 1 - g_\varepsilon \|_{L^6} \| \partial_t \varphi \|_{L^\frac{4}{3} L^2} \leq C \varepsilon^\sigma \| \partial_t \varphi \|_{L^\frac{4}{3} L^2}. \]

Again by interpolation and using Sobolev embedding, we have

\[ |I_2| \leq \| \tilde{u}_x \otimes \tilde{u}_x \|_{L^\frac{4}{3} L^2} (\| g_\varepsilon - 1 \|_{L^q} \| \nabla \varphi \|_{L^4 L^2} + \| \nabla g_\varepsilon \|_{L^q} \| \varphi \|_{L^4 L^2}) \leq C \varepsilon^\sigma \| \nabla \varphi \|_{L^4 L^2}. \]

where

\[ \frac{1}{r_1} = \frac{1}{2} - \frac{1}{q^*} = \frac{5}{6} - \frac{1}{q} > \frac{1}{3}, \quad \frac{1}{r_2} = \frac{1}{2} - \frac{1}{q} = \frac{1}{r_1}. \quad (2.18) \]

Similarly,

\[ |I_3| \leq \| \tilde{u}_x \|_{L^2 L^2} (\| g_\varepsilon - 1 \|_{L^q} \| \nabla \varphi \|_{L^4 L^2} + \| \nabla g_\varepsilon \|_{L^q} \| \varphi \|_{L^4 L^2}) \leq C \varepsilon^\sigma \| \nabla \varphi \|_{L^2 L^2}. \]

For \( I_4 \),

\[ |I_4| \leq \| f_x \|_{L^2 L^2} \| g_\varepsilon - 1 \|_{L^q} \| \varphi \|_{L^2 L^2} \leq C \varepsilon^\sigma \| \varphi \|_{L^2 L^2}. \]

Next we estimate \( I_j, j = 5, 6, 7, 8 \) for which the estimates of the Bogovskii operator \( B_\varepsilon \) in (2.15) and (2.16) will be repeatedly used. Since the Bogovskii operator \( B_\varepsilon \) only applies on spatial variable, then

\[ \partial_t \varphi_2 = \partial_t B_\varepsilon (\nabla g_\varepsilon) = B_\varepsilon (\partial_t \varphi \cdot \nabla g_\varepsilon) = B_\varepsilon (\text{div} ((\partial_t \varphi) g_\varepsilon)) = B_\varepsilon (\text{div} ((\partial_t \varphi)(g_\varepsilon - 1))). \]

Thus, taking \( r_5 > 3/2 \) and using (2.16) gives

\[ |I_5| \leq \| u_x \|_{L^4 L^3} \| \partial_t \varphi_2 \|_{L^\frac{4}{3} L^{r_5}} \leq C \| \partial_t \varphi (g_\varepsilon - 1) \|_{L^\frac{4}{3} L^{r_5}} \leq C \| g_\varepsilon - 1 \|_{L^q} \| \partial_t \varphi \|_{L^\frac{4}{3} L^{r_5}} \leq C \varepsilon^\sigma \| \partial_t \varphi \|_{L^\frac{4}{3} L^{r_5}}, \]

where

\[ \frac{1}{r_6} = \frac{1}{r_5} - \frac{1}{q^*} = \frac{1}{r_5} - \frac{1}{q} + \frac{1}{3}. \]

Since \( q > 2 \), we can choose \( r_5 > 3/2 \) close to 3/2 such that

\[ \frac{1}{r_5} - \frac{1}{q} = \left( \frac{1}{r_5} - \frac{2}{3} \right) + \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{1}{6} = \frac{1}{6}. \]

With such a choice of \( r_5 \) we have \( r_6 = 2 \) and

\[ |I_5| \leq C \varepsilon^\sigma \| \partial_t \varphi \|_{L^\frac{4}{3} L^2}. \]

Using (2.15), by similar argument one has

\[ |I_6| + |I_7| + |I_8| \leq C \varepsilon^\sigma \| \nabla \varphi \|_{L^4 L^{r_1}}. \]

Summing up the above estimates for \( I_j \) gives

\[ |I^*| \leq C \varepsilon^\sigma (\| \partial_t \varphi \|_{L^\frac{4}{3} L^2} + \| \nabla \varphi \|_{L^4 L^{r_1}}). \]

This implies our desired result in (2.13). \( \square \)
2.4. Convergence of the Nonlinear Convective Term

Here we will show \( \tilde{u}_\varepsilon \) has certain compactness such that the convergence of the nonlinear convective term \( \tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon \) can be obtained. A key observation is that some uniform estimates related to time derivative can be deduced from Proposition 2.2. Indeed, By Proposition 2.2, for any \( \varphi \in C_c^\infty(\Omega \times (0,T);\mathbb{R}^3) \) with \( \text{div } \varphi = 0 \), we have

\[
|\langle \partial_t \tilde{u}_\varepsilon, \varphi \rangle| \leq \int_0^T \int_\Omega |\tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon : \nabla \varphi| + |\nabla \tilde{u}_\varepsilon : \nabla \varphi| + |f_\varepsilon \cdot \varphi| \, dxdt + |\langle G_\varepsilon, \varphi \rangle|.
\]

Recall that \( \sigma > 0 \) and \( r_1 \in (2,3) \). Thus, we have the following decomposition

\[
\tilde{u}_\varepsilon = \tilde{u}_\varepsilon^{(1)} + \varepsilon \tilde{u}_\varepsilon^{(2)},
\]

where \( \partial_t \tilde{u}_\varepsilon^{(1)} \) is uniformly bounded in \( L^{\frac{4}{3}}(0,T;V^{-1,r'_1}(\Omega)) \) and \( \tilde{u}_\varepsilon^{(2)} \) is uniformly bounded in \( L^4(0,T;L^2(\Omega)) \).

Since \( \tilde{u}_\varepsilon \) is uniformly bounded in \( L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^6(\Omega)) \), then \( \tilde{u}_\varepsilon^{(1)} = \tilde{u}_\varepsilon - \varepsilon \sigma \tilde{u}_\varepsilon^{(2)} \) is uniformly bounded in \( L^4(0,T;L^2(\Omega)) \). Thus, up to a subsequence,

\[
\tilde{u}_\varepsilon^{(1)} \rightarrow u \text{ weakly in } L^4(0,T;L^2(\Omega)).
\]

Recall \( \tilde{u}_\varepsilon \) is uniformly bounded in \( L^2(0,T;W_0^{1,2}(\Omega)) \). Then apply Aubin-Lions type argument (see Lemma 5.1 in [16]) gives

\[
\tilde{u}_\varepsilon^{(1)} \otimes \tilde{u}_\varepsilon \rightarrow u \otimes u \text{ in } \mathcal{D}'(\Omega \times (0,T)).
\]

Clearly

\[
\varepsilon \sigma \tilde{u}_\varepsilon^{(2)} \otimes \tilde{u}_\varepsilon \rightarrow 0 \text{ strongly in } L^{\frac{4}{3}}(0,T;L^2(\Omega)) \cap L^4(0,T;L^1(\Omega)).
\]

Thus,

\[
\tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon \rightarrow u \otimes u \text{ in } \mathcal{D}'(\Omega \times (0,T)). \tag{2.19}
\]

Together with the uniform estimates of \( \tilde{u}_\varepsilon \) in (2.2), we deduce from (2.19) that

\[
\tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon \rightarrow u \otimes u \text{ weakly in } L^\frac{4}{3}(0,T;L^2(\Omega)). \tag{2.20}
\]

2.5. Passing to the Limit

Now we are ready to pass \( \varepsilon \rightarrow 0 \) and prove the following result in \( U_\varepsilon \) from which we can deduce our desired limit equations in \( u_\varepsilon \).

**Proposition 2.4.** Let \( (U_\varepsilon, P_\varepsilon) \) be the solution of the equation (1.12) and \( \tilde{P}_\varepsilon \) be the extension in \( \Omega \) defined by (2.11). If \( \alpha > 3 \), i.e. \( \lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon = \infty \), then

\[
\tilde{U}_\varepsilon \rightarrow U \text{ weakly*(s) in } W^{1,\infty}((0,T);L^2(\Omega)) \cap W^{1,2}((0,T);W_0^{1,2}(\Omega)),
\]

\[
\tilde{P}_\varepsilon \rightarrow P \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{2.21}
\]

Moreover, \( (U, P) \) solves

\[
u - u^0 + \text{div } \Psi - \mu \Delta U + \nabla P = F, \quad \text{div } U = 0 \text{ in } \mathcal{D}'(\Omega \times (0,T)), \tag{2.22}
\]

with

\[
U(\cdot,t) = \int_0^t u(\cdot,s) \, ds, \quad \Psi(\cdot,t) := \int_0^t u(\cdot,s) \otimes u(\cdot,s) \, ds, \quad F(\cdot,t) = \int_0^t f(\cdot,s) \, ds, \tag{2.23}
\]

where \( u \) is the limit of \( \tilde{u}_\varepsilon \) given in (2.3), \( u^0 \) and \( f \) are given in (1.9).
Proof. The weak convergences in (2.21) follow immediately from the uniform estimates (2.4) and (2.12). The divergence free condition \( \text{div} \mathbf{U} = 0 \) follows from \( \text{div} \hat{\mathbf{U}}_\varepsilon = 0 \). Moreover, the weak convergence of \( \hat{\mathbf{u}}_\varepsilon \) in (2.3) implies for each \( \varphi \in C_c^\infty(\Omega \times (0, T); \mathbb{R}^3) \) that

\[
\int_0^T \int_\Omega \hat{\mathbf{u}}_\varepsilon \cdot \varphi(t, x) dx dt = \int_0^T \int_\Omega \hat{\mathbf{u}}_\varepsilon(s, x) ds \cdot \varphi(t, x) dx dt \\
= \int_0^T \int_0^t \int_\Omega \mathbf{u}(s, x) \cdot \varphi(t, x) dx ds dt \\
= \int_0^T \int_0^t \int_\Omega \mathbf{u}(s, x) \cdot \varphi(t, x) dx ds dt
\]

(2.24)

Thus, the uniqueness of weak limits implies

\[ \mathbf{U}(\cdot, t) = \int_0^t \mathbf{u}(\cdot, s) ds, \]

which is the first equality in (2.23).

Given any scalar test function \( \phi \in C_c^\infty(\Omega \times (0, T)) \), we choose \( w_{\eta, \varepsilon}^i \phi \) as a test function to (1.12) to obtain the weak formulation:

\[
\int_0^T \int_\Omega \hat{\mathbf{u}}_\varepsilon \cdot (w_{\eta, \varepsilon}^i \phi) - \hat{\mathbf{U}}_\varepsilon : \nabla (w_{\eta, \varepsilon}^i \phi) + \mu \nabla \mathbf{U}_\varepsilon : \nabla (w_{\eta, \varepsilon}^i \phi) - \hat{P}_\varepsilon \text{div} (w_{\eta, \varepsilon}^i \phi) dx dt \\
= \int_0^T \int_\Omega \mathbf{F}_\varepsilon \cdot (w_{\eta, \varepsilon}^i \phi) dx dt + \int_0^T \int_\Omega \mathbf{u}_\varepsilon^0 \cdot (w_{\eta, \varepsilon}^i \phi) dx dt.
\]

(2.25)

By (1.24), we know that \( \| \nabla w_{\eta, \varepsilon}^i \|_{L^2(\Omega)} \leq C\varepsilon^{-\frac{3}{2}} \rightarrow 0 \). Then \( w_{\eta, \varepsilon}^i \rightarrow \bar{w}^i \) strongly in \( L^2(\Omega) \) by Sobolev compact embedding. This implies

\[
\int_0^T \int_\Omega \hat{\mathbf{u}}_\varepsilon \cdot w_{\eta, \varepsilon}^i \phi dx dt \rightarrow \int_0^T \int_\Omega \mathbf{u} \cdot \bar{w}^i \phi dx dt,
\]

(2.26)

For the nonlinear convective term one has

\[
\int_0^T \int_\Omega \hat{\mathbf{U}}_\varepsilon : \nabla (w_{\eta, \varepsilon}^i \phi) dx dt = \int_0^T \int_\Omega \hat{\mathbf{U}}_\varepsilon : \nabla \phi \otimes w_{\eta, \varepsilon}^i dx dt,
\]

where the first term on the right-hand side satisfies

\[
\left| \int_0^T \int_\Omega \hat{\mathbf{U}}_\varepsilon : \nabla w_{\eta, \varepsilon}^i \phi dx dt \right| \leq C\| \hat{\mathbf{u}}_\varepsilon \|_{L^2 L^6}^2 \| \nabla w_{\eta, \varepsilon}^i \|_{L^2} \| \phi \|_{L^\infty L^6} \leq C\varepsilon^{-\frac{3}{2}} \rightarrow 0.
\]

For the second term, by the convergence of the convective term in (2.20) and the strong convergence of \( w_{\eta, \varepsilon}^i \), we have

\[
\int_0^T \int_\Omega \hat{\mathbf{U}}_\varepsilon : \nabla \phi \otimes w_{\eta, \varepsilon}^i dx dt \rightarrow \int_0^T \int_\Omega \mathbf{U} : \nabla \phi \otimes \bar{w}^i dx dt.
\]

So we obtain

\[
\int_0^T \int_\Omega \hat{\mathbf{U}}_\varepsilon : \nabla (w_{\eta, \varepsilon}^i \phi) dx dt \rightarrow \int_0^T \int_\Omega \mathbf{U} : \nabla (\bar{w}^i \phi) dx dt.
\]

(2.27)
Using $\nabla w_{n,\varepsilon}^i \to 0$ and the strong convergence of $w_{n,\varepsilon}^i$ again implies
\[
\int_0^T \int_\Omega \nabla \bar{U}_\varepsilon : \nabla (w_{n,\varepsilon}^i \varphi) \, dx \, dt = \int_0^T \int_\Omega \nabla \bar{U}_\varepsilon : \nabla w_{n,\varepsilon}^i \varphi \, dx \, dt + \int_0^T \int_\Omega \nabla \bar{U}_\varepsilon : \nabla \varphi \otimes w_{n,\varepsilon}^i \, dx \, dt
\]
\[
= \int_0^T \int_\Omega \nabla \bar{U}_\varepsilon : \nabla \varphi \otimes \bar{w}_i \, dx \, dt = \int_0^T \int_\Omega \nabla \bar{U}_\varepsilon : \nabla (\bar{w}_i \varphi) \, dx \, dt.
\]  
(2.28)

For the term related to the pressure, again by the strong convergence of $w_{n,\varepsilon}^i$, we have
\[
\int_0^T \int_\Omega \bar{P}_\varepsilon \div (w_{n,\varepsilon}^i \phi) \, dx \, dt = \int_0^T \int_\Omega \bar{P}_\varepsilon (w_{n,\varepsilon}^i \cdot \nabla \phi) \, dx \, dt
\]
\[
= \int_0^T \int_\Omega P(\bar{w}_i \cdot \phi) \, dx \, dt = \int_0^T \int_\Omega P(\bar{w}_i) \, dx \, dt.
\]  
(2.29)

For the external force term, by the strong convergence of $\bar{F}_\varepsilon$ in (1.11), we have
\[
\int_0^T \int_\Omega \bar{F}_\varepsilon (w_{n,\varepsilon}^i \phi) \, dx \, dt \to \int_0^T \int_\Omega F(\bar{w}_i \phi) \, dx \, dt.
\]  
(2.30)

Summarizing the convergences in (2.26)–(2.30) and passing $\varepsilon \to 0$ in (2.25) implies
\[
\int_0^T \int_\Omega \bar{u} \cdot (\bar{w}_i \phi) - \Psi : \nabla (\bar{w}_i \phi) + \mu \nabla \bar{U} : \nabla (\bar{w}_i \phi) - P \div (\bar{w}_i \phi) \, dx \, dt
\]
\[
= \int_0^T \int_\Omega F \cdot (\bar{w}_i \phi) \, dx \, dt + \int_0^T \int_\Omega \bar{u}_0 \cdot (\bar{w}_i \phi) \, dx \, dt.
\]  
(2.31)

Equation (2.31) is equivalent to
\[
\int_0^T \int_\Omega \bar{u} \cdot (A \varphi) - \Psi : \nabla (A \varphi) + \mu \nabla \bar{U} : \nabla (A \varphi) - P \div (A \varphi) \, dx \, dt
\]
\[
= \int_0^T \int_\Omega F \cdot (A \varphi) \, dx \, dt + \int_0^T \int_\Omega \bar{u}_0 \cdot (A \varphi) \, dx \, dt
\]
for any $\varphi \in C^\infty_c(\Omega \times (0, T); \mathbb{R}^3)$ with $A = (A_{i,j}) = (\bar{w}_i)_{i} = (\bar{w}_i)_{j}$. As shown in [3], $A$ is a positive definite matrix. This implies
\[
\bar{u} - \bar{u}_0 + \div \Psi - \mu \Delta \bar{U} + \nabla P - F = 0, \quad \text{in} \ D'(\Omega \times (0, T)).
\]

The Proof of Proposition 2.4 is completed. \hfill \Box

2.6. End of the Proof

From Proposition 2.4, we can deduce the limit equations in $\bar{u}$. Indeed, differentiating in $t$ to Eq. (2.22) implies
\[
\partial_t \bar{u} + \div (\bar{u} \otimes \bar{u}) - \mu \Delta \bar{u} + \nabla P = \bar{f}, \quad \text{in} \ D'(\Omega \times (0, T)),
\]
(2.32)

with $p = \partial_t P$. By (2.21), we know that
\[
\bar{p}_\varepsilon := \partial_t \bar{P}_\varepsilon \to p \ \text{weakly in} \ W^{-1,2}(0, T; L^2_0(\Omega)),
\]
which is exactly (1.14).

Clearly $\div \bar{u} = 0$ which follows from $\div \bar{u}_\varepsilon = 0$. Since $\bar{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega))$, using equation (2.32) implies
\[
\partial_t \bar{u} \in L^{\frac{4}{3}}(0, T; V^{-1,2}(\Omega)).
\]
Thus

\[ u \in C^\text{weak}(\lbrack 0, T \rbrack, L^2(\Omega)) \cap C(\lbrack 0, T \rbrack, L^q(\Omega)), \quad \text{for any } 1 \leq q < 2. \]  

(2.33)

We shall further show the attainment of initial datum for \( u \). With such continuity of \( u \) in time variable in (2.33), by density argument, we deduce from (2.32) that for each \( \varphi \in C^1_c(\Omega \times \lbrack 0, T \rbrack; \mathbb{R}^3) \), \( \nabla \varphi = 0 \),

\[
\int_0^T \int_{\Omega} -u \partial_t \varphi - u \otimes u : \nabla \varphi + \nabla u : \nabla \varphi \, dx \, dt = \int_0^T \int_{\Omega} f \cdot \varphi + \int_{\Omega} u(x, 0) \varphi(x, 0) \, dx. \]  

(2.34)

Following the Proof of Proposition 2.2, we can deduce for each \( \varphi \in C^1_c(\Omega \times \lbrack 0, T \rbrack; \mathbb{R}^3) \), \( \nabla \varphi = 0 \) that

\[
\int_0^T \int_{\Omega} -\tilde{u}_\varepsilon \partial_t \varphi - \tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon : \nabla \varphi + \nabla \tilde{u}_\varepsilon : \nabla \varphi \, dx \, dt = \int_0^T \int_{\Omega} \tilde{f}_\varepsilon \cdot \varphi + \int_{\Omega} \tilde{u}_\varepsilon^0(x) \varphi(x, 0) \, dx + H_\varepsilon(\varphi), \]  

(2.35)

where the remainder term \( H_\varepsilon(\varphi) \) satisfies

\[
|H_\varepsilon(\varphi)| \leq C \varepsilon^\sigma \left( \|\partial_t \varphi\|_{L^4_{\varepsilon}L^2} + \|\nabla \varphi\|_{L^4_{\varepsilon}L^4} + \|\varphi(\cdot, 0)\|_{L^r_{\varepsilon}L^1(\Omega)} \right),
\]

with \( \sigma > 0 \) and \( 2 < r_1 < 3 \) given in Proposition 2.2 and \( r_1^* \) the Sobolev conjugate number of \( r_1 \) such that

\[
\frac{1}{r_1^*} = \frac{1}{r_1} - \frac{1}{3}.
\]

By the convergence of the convective term in (2.20), passing \( \varepsilon \to 0 \) in (2.35) implies

\[
\int_0^T \int_{\Omega} -u \partial_t \varphi - u \otimes u : \nabla \varphi + \nabla u : \nabla \varphi \, dx \, dt = \int_0^T \int_{\Omega} f \cdot \varphi + \int_{\Omega} u^0(x) \varphi(x, 0) \, dx. \]  

(2.36)

Comparing (2.34) and (2.36) implies the attainment of initial datum:

\[
u |_{t=0} = u^0.
\]

We obtain system (1.15) and thus complete the Proof of Theorem 1.1.

### 3. Proof of Theorem 1.2

In this section we shall prove Theorem 1.2 where \( a_\varepsilon = \varepsilon^\alpha \) with \( 1 < \alpha < 3 \) and \( \sigma_\varepsilon = \varepsilon^{3-\alpha} \) concerning the case of large holes. In this case we consider the time-scaled Navier–Stokes system (1.16).

In the case of large holes, one can benefit from the zero boundary condition on the holes and obtain the following perforation version of Poincaré inequality (see Lemma 3.4.1 in [2]):

**Lemma 3.1.** Let \( \Omega_\varepsilon \) be the perforated domain defined by (1.1) and (1.2) with \( a_\varepsilon = \varepsilon^\alpha \), \( 1 < \alpha < 3 \). There exists a constant \( C \) independent of \( \varepsilon \) such that

\[
\|u\|_{L^2(\Omega_\varepsilon)} \leq C \sigma_\varepsilon \|
abla u\|_{L^2(\Omega_\varepsilon)}, \quad \text{for any } u \in W^{1,2}_0(\Omega_\varepsilon).
\]

(3.1)
3.1. Estimates of Velocity

From the energy inequality (1.7), using Poincaré inequality (3.1) and Hölder’s inequality gives

\[
\frac{1}{2} \sigma_\varepsilon^2 \int_{\Omega_\varepsilon} |u_\varepsilon(x,t)|^2 \, dx + \int_0^t \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x,s)|^2 \, dx \, ds = \frac{1}{2} \sigma_\varepsilon^2 \int_{\Omega_\varepsilon} |u_\varepsilon^0(x)|^2 \, dx \\
\leq \left( \|f_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))} \right)^2 \|u_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))} + \frac{1}{2} \sigma_\varepsilon^2 \sup_{0<s<1} \|u_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}^2 \\
\leq C \sigma_\varepsilon \|f_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))} \|\nabla u_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))} + \frac{1}{2} \sigma_\varepsilon^2 \sup_{0<s<1} \|u_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}^2 \\
\leq C \sigma_\varepsilon^2 \sup_{0<s<1} \left( \|f_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))}^2 + \frac{1}{2} \|\nabla u_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))}^2 \right) + \frac{1}{2} \sigma_\varepsilon^2 \sup_{0<s<1} \|u_\varepsilon^0\|_{L^2(\Omega_\varepsilon)}^2.
\]

This implies

\[
\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C, \quad \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \sigma_\varepsilon, \quad \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C \sigma_\varepsilon^2,
\]

where we used again (3.1). Therefore the extension satisfies

\[
\|\tilde{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \|\nabla \tilde{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq C \sigma_\varepsilon, \quad \|\tilde{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq C \sigma_\varepsilon^2.
\]

Then, up to a subsequence, we have the convergence

\[
\sigma_\varepsilon^{-2} \tilde{u}_\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;L^2(\Omega)),
\]

which is exactly (1.18).

By (1.10), it follows from (3.4) that

\[
\|\tilde{U}_\varepsilon\|_{W^{1,\infty}(0,T;L^2(\Omega))} \leq C, \quad \|\nabla \tilde{U}_\varepsilon\|_{W^{1,2}(0,T;L^2(\Omega))} \leq C \sigma_\varepsilon, \quad \|\tilde{U}_\varepsilon\|_{W^{1,2}(0,T;L^2(\Omega))} \leq C \sigma_\varepsilon^2.
\]

Then, up to a subsequence,

\[
\sigma_\varepsilon^{-2} \tilde{U}_\varepsilon \rightharpoonup U \text{ weakly in } W^{1,2}(0,T;L^2(\Omega)),
\]

As the argument in (2.24), the uniqueness of the weak limits implies that

\[
U(\cdot,t) = \int_0^t u(\cdot,s) \, ds.
\]

3.2. Extension of Pressure

Let \(P_\varepsilon\) be given in the Stokes equations (1.17). Recall the estimate of \(R_\varepsilon\) in (2.6):

\[
\|\nabla R_\varepsilon(\varphi)\|_{L^\tau(0,T;L^2(\Omega_\varepsilon))} \leq C \left( \|\nabla \varphi\|_{L^\tau(0,T;L^2(\Omega_\varepsilon))} + \sigma_\varepsilon^{-1} \|\varphi\|_{L^\tau(0,T;L^2(\Omega_\varepsilon))} \right)
\]

which will be repeatedly used in this section. Here \(\sigma_\varepsilon \to 0\) for the case of large holes. As in Section 2.2, we define a functional \(\tilde{H}_\varepsilon\) in \(\mathcal{D}'(\Omega \times (0,T);\mathbb{R}^3)\) by the following dual formulation: for each \(\varphi \in C_c^\infty(\Omega \times (0,T);\mathbb{R}^3)\),

\[
\left\langle \tilde{H}_\varepsilon, \varphi \right\rangle_{\Omega \times (0,T)} = \left\langle \nabla P_\varepsilon(t), R_\varepsilon(\varphi) \right\rangle_{\Omega_\varepsilon \times (0,T)} \\
= \left\langle F_\varepsilon(t) - \sigma_\varepsilon^2 u_\varepsilon(t), u_\varepsilon^0(t) + \mu \Delta U_\varepsilon(t) - \text{div} \Psi_\varepsilon(t), R_\varepsilon(\varphi) \right\rangle_{\Omega_\varepsilon \times (0,T)}.
\]
Therefore, this shows that

\[ \langle \partial_t \tilde{H}_\varepsilon, \varphi \rangle_{\Omega \times (0,T)} \leq C \| \varphi \|_{L^2(0,T;L^2(\Omega))} + \sigma_\varepsilon L^2(0,T,W^{-1,2}(\Omega)). \]  

This indicates \( \tilde{H}_\varepsilon \) is bounded in \( L^2(0,T,L^2(\Omega)) + \sigma_\varepsilon L^2(0,T,W^{-1,2}(\Omega)) \).

On the other hand,

\[
\langle \partial_t \tilde{H}_\varepsilon, \varphi \rangle_{\Omega \times (0,T)} = -\langle \tilde{H}_\varepsilon, \partial_t \varphi \rangle_{\Omega \times (0,T)} = -\langle &l_\varepsilon(t) - \varphi_\varepsilon(t) + 2\Delta u_\varepsilon(t) - \Phi_\varepsilon(t), R_\varepsilon(\partial_t \varphi) \rangle_{\Omega \times (0,T)} \\
-\langle \Phi_\varepsilon(t), R_\varepsilon(\partial_t \varphi) \rangle_{\Omega \times (0,T)} + \langle \mu \Delta u_\varepsilon, R_\varepsilon(\partial_t \varphi) \rangle_{\Omega \times (0,T)} + \langle \mu \Delta u_\varepsilon, R_\varepsilon(\varphi) \rangle_{\Omega \times (0,T)} - \langle \text{div}(u_\varepsilon \otimes u_\varepsilon), R_\varepsilon(\varphi) \rangle_{\Omega \times (0,T)}.
\]

By the uniform estimates in (3.3)–(3.6) and Lemma 3.1, we have

\[
\| (f_\varepsilon, R_\varepsilon(\varphi)) \| \leq C \| f_\varepsilon \|_{L^2(L^2)} \| R_\varepsilon(\varphi) \|_{L^2(L^2)} \leq C(\sigma_\varepsilon \| \varphi \|_{L^2(L^2)} + \| \varphi \|_{L^2(L^2)}),
\]

\[
\| (\varphi_\varepsilon, R_\varepsilon(\partial_t \varphi)) \| \leq C \| \varphi_\varepsilon \|_{L^2(L^2)} \| R_\varepsilon(\partial_t \varphi) \|_{L^2(L^2)} \leq C \sigma_\varepsilon \| \varphi \|_{L^2(L^2)} + \| \partial_t \varphi \|_{L^2(L^2)},
\]

\[
\| (\mu \Delta u_\varepsilon, R_\varepsilon(\varphi)) \| \leq C(\sigma_\varepsilon \| \varphi \|_{L^2(L^2)} + \| \varphi \|_{L^2(L^2)}),
\]

\[
\| \text{div}(u_\varepsilon \otimes u_\varepsilon), R_\varepsilon(\varphi) \| \leq C \| u_\varepsilon \|_{L^2(L^2)} \| R_\varepsilon(\varphi) \|_{L^2(L^2)} \leq C \sigma_\varepsilon \| \varphi \|_{L^2(L^2)} + \| \varphi \|_{L^2(L^2)}.
\]

This shows that

\[
\langle \partial_t \tilde{H}_\varepsilon, \varphi \rangle_{\Omega \times (0,T)} \leq C(\| \varphi \|_{L^2(L^2)} + \| \varphi \|_{L^2(L^2)} + \| \varphi \|_{L^2(L^2)} + \| \varphi \|_{L^2(L^2)} + \| \varphi \|_{L^2(L^2)}).
\]

Therefore, \( \tilde{H}_\varepsilon \) is uniformly bounded in \( W^{1,2}L^2 + \sigma_\varepsilon \| W^{1,2}L^2 + \sigma_\varepsilon \| L^2W^{-1,2} + \sigma_\varepsilon \| L^2W^{-1,2} \). Moreover, the second property of the restriction operator in (2.5) implies that

\[ \langle \tilde{H}_\varepsilon, \varphi \rangle_{\Omega \times (0,T)} = 0, \quad \forall \varphi \in C^{\infty}(\Omega \times (0,T); \mathbb{R}^3), \quad \text{div} \varphi = 0. \]
Thus, we can apply Lemma 2.1 and deduce that there exists \( \tilde{P}_\varepsilon \in W^{1,2}(W^{1,2} \cap L^2_0) + \sigma_\varepsilon \frac{1}{2} W^{1,\frac{4}{3}} L^2_0 + \sigma_\varepsilon^2 L^2 L^2_0 \) such that

\[
\tilde{H}_\varepsilon = \tilde{H}_\varepsilon^{(1)} + \sigma_\varepsilon \tilde{H}_\varepsilon^{(2)} + \sigma_\varepsilon^2 \tilde{H}_\varepsilon^{(3)}, \quad \tilde{P}_\varepsilon = \tilde{P}_\varepsilon^{(1)} + \sigma_\varepsilon \tilde{P}_\varepsilon^{(2)} + \sigma_\varepsilon^2 \tilde{P}_\varepsilon^{(3)}, \tag{3.14}
\]

with

\[
\tilde{H}_\varepsilon^{(1)} = \nabla \tilde{P}_\varepsilon^{(1)}, \quad \tilde{H}_\varepsilon^{(2)} = \nabla \tilde{P}_\varepsilon^{(2)}, \quad \tilde{H}_\varepsilon^{(3)} = \nabla \tilde{P}_\varepsilon^{(3)}, \tag{3.15}
\]

and

\[
\begin{align*}
\| \tilde{P}_\varepsilon^{(1)} \|_{W^{1,2}W^{1,2}} & \leq \| \tilde{H}_\varepsilon^{(1)} \|_{W^{1,2}L^2} \leq C, \\
\| \tilde{P}_\varepsilon^{(2)} \|_{W^{1,\frac{4}{3}} L^2_0} & \leq \| \tilde{H}_\varepsilon^{(2)} \|_{W^{1,\frac{4}{3}} W^{-1,2}} \leq C, \\
\| \tilde{P}_\varepsilon^{(3)} \|_{L^2 L^2_0} & \leq \| \tilde{H}_\varepsilon^{(3)} \|_{L^2 W^{-1,2}} \leq C.
\end{align*} \tag{3.16}
\]

### 3.3. Passing to the Limit

Instead of showing the limit equations in \( u \) directly, we prove the following results in \( U \):

**Proposition 3.2.** Let \((U_\varepsilon, P_\varepsilon)\) be the weak solution of equation (1.17) and \((\tilde{U}_\varepsilon, \tilde{P}_\varepsilon)\) be their extension in \( \Omega \) defined in (1.10), (3.9), (3.14) and (3.15). If \( \lim_{\varepsilon \to 0} \sigma_\varepsilon = 0 \) i.e. \( 1 < \alpha < 3 \), then

\[
\sigma_\varepsilon^{-2} \tilde{U}_\varepsilon \to U \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{3.17}
\]

and

\[
\tilde{P}_\varepsilon \to P \text{ weakly in } W^{1,2}(0,T;W^{1,2}(\Omega)). \tag{3.19}
\]

Moreover, \((U, P)\) solves

\[
\mu U = A(\mathbf{F} - \nabla P), \text{ in } \mathcal{D}'(\Omega \times (0,T)), \tag{3.20}
\]

with

\[
U(\cdot, t) = \int_0^t u(\cdot, s)ds, \quad F(\cdot, t) = \int_0^t f(\cdot, s)ds. \tag{3.21}
\]

**Proof.** The convergence of \( \tilde{U}_\varepsilon \) in (3.17) follows immediately from the uniform estimates (3.6) and was given already in (3.7). The weak convergence of \( \tilde{P}_\varepsilon \) in (3.19) follows from the uniform estimates (3.16). The relations in (3.21) follows from the uniqueness for weak limits and were given already in (1.11) and (3.8).

Given any scalar test function \( \phi \in C_\infty^\infty(\Omega \times (0,T)) \), we choose \( w_{i,\varepsilon}^i \phi \) as a test function to (1.17) and obtain

\[
\begin{align*}
&\int_0^T \int_\Omega \sigma_\varepsilon^2 \tilde{\mathbf{u}}_\varepsilon \cdot (w_{i,\varepsilon}^i \phi) - \tilde{\Psi}_\varepsilon : \nabla (w_{i,\varepsilon}^i \phi) + \mu \nabla U_\varepsilon : \nabla (w_{i,\varepsilon}^i \phi) - \tilde{P}_\varepsilon \text{div} (w_{i,\varepsilon}^i \phi) dxdt \\
= &\int_0^T \int_\Omega F_\varepsilon \cdot (w_{i,\varepsilon}^i \phi) dxdt + \sigma_\varepsilon^2 \int_0^T \int_\Omega \tilde{\mathbf{u}}_\varepsilon^0 \cdot (w_{i,\varepsilon}^i \phi) dxdt.
\end{align*} \tag{3.22}
\]
By the uniform estimates in (3.4) and (1.24), and the convergences in (1.25) and (1.11), we have

\[
\begin{align*}
\sigma^2 \int_0^T \int_\Omega \tilde{u}_\varepsilon \cdot (w^i_{\eta, \varepsilon} \phi) \, dx \, dt &\leq \sigma^2 \| \tilde{u}_\varepsilon \|_{L^2(L^6)} \| w^i_{\eta, \varepsilon} \|_{L^2(\Omega)} \| \phi \|_{L^2 L^3} \leq C \sigma^3 \to 0, \\
\sigma^2 \int_0^T \int_\Omega u^0_\varepsilon \cdot (w^i_{\eta, \varepsilon} \phi) \, dx \, dt &\leq C \sigma^2 \| u^0_\varepsilon \|_{L^2(\Omega)} \| w^i_{\eta, \varepsilon} \|_{L^2(\Omega)} \| \phi \|_{L^1 L^\infty} \leq C \sigma^2 \to 0 \to 0, \\
\int_0^T \int_\Omega \tilde{F}_\varepsilon \cdot (w^i_{\eta, \varepsilon} \phi) \, dx \, dt &\to \int_0^T \int_\Omega \tilde{F} \cdot (w^i \phi) \, dx \, dt.
\end{align*}
\]

For the convective term, by (1.24) and (3.4), we have

\[
\begin{align*}
\left| \int_0^T \int_\Omega \tilde{\psi}_\varepsilon : \nabla (w^i_{\eta, \varepsilon} \phi) \, dx \, dt \right| &\leq \| \tilde{\psi}_\varepsilon \|_{L^\infty L^2} \| w^i_{\eta, \varepsilon} \|_{W^{1,2}} \| \phi \|_{W^{1,\infty}} \\
&\leq \| \tilde{\psi}_\varepsilon \otimes \tilde{u}_\varepsilon \|_{L^1 L^2} \| w^i_{\eta, \varepsilon} \|_{W^{1,2}} \| \phi \|_{W^{1,\infty}} \\
&\leq C \| \tilde{\psi}_\varepsilon \|_{L^2 L^3} \| w^i_{\eta, \varepsilon} \|_{W^{1,2}} \leq C \sigma_\varepsilon \to 0.
\end{align*}
\]

Using Eq. (1.23) in \((w^i_{\eta, \varepsilon}, q^i_{\eta, \varepsilon})\) in Section 1.3 gives us

\[
\begin{align*}
\int_0^T \int_\Omega \nabla \tilde{U}_\varepsilon : \nabla (w^i_{\eta, \varepsilon} \phi) \, dx \, dt &= \int_0^T \int_\Omega \nabla \tilde{U}_\varepsilon : \nabla \phi \otimes w^i_{\eta, \varepsilon} + \nabla \tilde{U}_\varepsilon : \nabla w^i_{\eta, \varepsilon} \phi \, dx \, dt \\
&= \int_0^T \int_\Omega \nabla \tilde{U}_\varepsilon : \nabla \phi \otimes w^i_{\eta, \varepsilon} + \nabla (\phi \tilde{U}_\varepsilon) : \nabla w^i_{\eta, \varepsilon} - \nabla \phi \otimes \tilde{U}_\varepsilon : \nabla w^i_{\eta, \varepsilon} \, dx \, dt \\
&= \int_0^T \int_\Omega \nabla \tilde{U}_\varepsilon : \nabla \phi \otimes w^i_{\eta, \varepsilon} + \varepsilon^{-1} \text{div} (\phi \tilde{U}_\varepsilon) q^i_{\eta, \varepsilon} + \varepsilon^{-2} (\phi \tilde{U}_\varepsilon) \cdot \varepsilon^i - \nabla \phi \otimes \tilde{U}_\varepsilon : \nabla w^i_{\eta, \varepsilon} \, dx \, dt.
\end{align*}
\]

For the first term and the last term on the right-hand side of (3.25), we have

\[
\begin{align*}
\int_0^T \int_\Omega \nabla \tilde{U}_\varepsilon : \nabla \phi \otimes w^i_{\eta, \varepsilon} \, dx \, dt &\leq \| \nabla \tilde{U}_\varepsilon \|_{L^2 L^2} \| w^i_{\eta, \varepsilon} \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2 L^\infty(\Omega)} \leq C \sigma_\varepsilon \to 0, \\
\int_0^T \int_\Omega \nabla \phi \otimes \tilde{U}_\varepsilon : \nabla w^i_{\eta, \varepsilon} \, dx \, dt &\leq \| \tilde{U}_\varepsilon \|_{L^2 L^2} \| \nabla w^i_{\eta, \varepsilon} \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2 L^\infty(\Omega)} \leq C \sigma_\varepsilon \to 0.
\end{align*}
\]

Using \(\text{div} \tilde{U}_\varepsilon = 0\) and \(\varepsilon^{-1} c_\eta = \sigma_\varepsilon^2\) implies

\[
\int_0^T \int_\Omega \varepsilon^{-1} \text{div} (\phi \tilde{U}_\varepsilon) q^i_{\eta, \varepsilon} \, dx \, dt \leq \varepsilon^{-1} \| \nabla \phi \|_{L^2 L^\infty(\Omega)} \| \tilde{U}_\varepsilon \|_{L^2 L^2} \| q^i_{\eta, \varepsilon} \|_{L^2(\Omega)} \leq C \varepsilon^{-1} \sigma_\varepsilon^2 c_\eta = C \sigma_\varepsilon \to 0.
\]

By the weak convergence of \(\sigma_\varepsilon^{-2} \tilde{U}_\varepsilon\) in (3.7), we have

\[
\int_0^T \int_\Omega \sigma_\varepsilon^{-2} \phi \tilde{U}_\varepsilon \cdot \varepsilon^i \, dx \, dt \to \int_0^T \int_\Omega \phi \tilde{U}_\varepsilon \cdot \varepsilon^i \, dx \, dt.
\]

Now we deal with the term related to the pressure. By the decomposition in (3.14)–(3.16) we have

\[
\begin{align*}
\int_0^T \int_\Omega \tilde{P}_\varepsilon \text{div} (w^i_{\eta, \varepsilon} \phi) \, dx \, dt &= \int_0^T \int_\Omega \tilde{P}^{(1)}_{\varepsilon} w^i_{\eta, \varepsilon} \cdot \nabla \phi \, dx \, dt \\
&\quad + \sigma_\varepsilon^{1/2} \int_0^T \int_\Omega \tilde{P}^{(2)}_{\varepsilon} w^i_{\eta, \varepsilon} \cdot \nabla \phi \, dx \, dt + \sigma_\varepsilon^{1/2} \int_0^T \int_\Omega \tilde{P}^{(3)}_{\varepsilon} w^i_{\eta, \varepsilon} \cdot \nabla \phi \, dx \, dt.
\end{align*}
\]
By the uniform estimates in (3.16), there holds
\[
\sigma_{\varepsilon}^2 \int_0^T \int_\Omega \tilde{P}_\varepsilon^{(2)} w_{n,\varepsilon} \cdot \nabla \phi \, dx \, dt \lesssim \sigma_{\varepsilon}^2 \| \tilde{P}_\varepsilon^{(2)} \|_{L^\frac{4}{3} L^2} \| w_{n,\varepsilon} \|_{L^2} \| \nabla \phi \|_{L^4 L^\infty} \lesssim C \sigma_{\varepsilon}^{\frac{1}{2}} \to 0,
\]
\[
\sigma_{\varepsilon}^2 \int_0^T \int_\Omega \tilde{P}_\varepsilon^{(3)} w_{n,\varepsilon} \cdot \nabla \phi \, dx \, dt \lesssim \sigma_{\varepsilon}^2 \| \tilde{P}_\varepsilon^{(3)} \|_{L^2 L^2} \| w_{n,\varepsilon} \|_{L^2} \| \nabla \phi \|_{L^2 L^\infty} \lesssim C \sigma_{\varepsilon}^2 \to 0.
\]
\[
(3.29)
\]
Since \( \tilde{P}_\varepsilon^{(1)} \to P \) weakly in \( W^{1,2}(0,T; W^{1,2}(\Omega)) \), by Sobolev compact embedding, one has \( \tilde{P}_\varepsilon^{(1)} \to P \) strongly in \( L^2(0,T; L_0^2(\Omega)) \). Thus,
\[
\int_0^T \int_\Omega \tilde{P}_\varepsilon^{(1)} w_{n,\varepsilon} \cdot \nabla \phi \, dx \, dt \to \int_0^T \int_\Omega P (w^t \cdot \nabla \phi) \, dx \, dt = \int_0^T \int_\Omega P \text{div} (w^t \phi) \, dx \, dt.
\]
\[
(3.30)
\]
From (3.28)–(3.30) we deduce that
\[
\int_0^T \int_\Omega \tilde{P}_\varepsilon \text{div} (w_{n,\varepsilon} \phi) \, dx \, dt \to \int_0^T \int_\Omega P \text{div} (w^t \phi) \, dx \, dt.
\]
\[
(3.31)
\]
By (3.23)–(3.31), passing \( \varepsilon \to 0 \) in (3.22) implies our desired equation (3.20). The proof of Proposition 3.2 is completed.

\[
\square
\]

3.4. End of the Proof

By Proposition 3.2, differentiating in \( t \) to Eq. (3.20) implies
\[
\mu u = A(f - \nabla p), \text{ in } D'(\Omega \times (0,T)),
\]
with \( p = \partial_t P \). By (3.16), (3.18) and (3.19), we know that
\[
\tilde{p}_\varepsilon = \tilde{p}_\varepsilon^{(1)} + \sigma_{\varepsilon}^2 \tilde{p}_\varepsilon^{(2)} + \sigma_{\varepsilon}^2 \tilde{p}_\varepsilon^{(3)},
\]
with
\[
\tilde{p}_\varepsilon^{(1)} \to p \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)),
\]
\[
\tilde{p}_\varepsilon^{(2)} \text{ bounded in } L^\frac{4}{3}(0,T; L_0^2(\Omega)),
\]
\[
\tilde{p}_\varepsilon^{(3)} \text{ bounded in } W^{-1,2}(0,T; L_0^2(\Omega)),
\]
which are exactly (1.19) and (1.20).

Since \( \tilde{u}_\varepsilon \in L^2(0,T; W_0^{1,2}(\Omega)), \text{ div } \tilde{u}_\varepsilon = 0 \) and \( \sigma_{\varepsilon}^{-2} \tilde{u}_\varepsilon \to u \) weakly in \( L^2(0,T; L^2(\Omega)) \), then \( \text{div } u = 0 \) in \( \Omega \times (0,T) \) and \( u \cdot n = 0 \) on \( \partial \Omega \times (0,T) \) (see [21]). We thus complete the Proof of Theorem 1.2.

Acknowledgements. Yong Lu has been supported by the Recruitment Program of Global Experts of China. Both authors are partially supported by the NSF of China under Grant 12171235. On behalf of all authors, the corresponding author states that there is no conflict of interest.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.
References

[1] Allaire, G.: Homogenization of the Navier–Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. Arch. Ration. Mech. Anal. 113(3), 209–259 (1990)

[2] Allaire, G.: Homogenization of the Navier–Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes. Arch. Ration. Mech. Anal. 113(3), 261–298 (1990)

[3] Allaire, G.: Continuity of the Darcy’s law in the low-volume fraction limit. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4, 475–499 (1991)

[4] Bella, P., Oschmann, F.: Homogenization and low Mach number limit of compressible Navier-Stokes equations in critically perforated domains. J. Math. Fluid Mech., 24 (2022), Paper No. 79

[5] Bella, J.; Oschmann, F.: Inverse of divergence and homogenization of compressible Navier-Stokes equations in randomly perforated domains, arXiv: 2103.04323

[6] Cioranescu, D., Murat, F.: Un terme étrange venu d’ailleurs, Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Brezis, H., Lions, J. L. (Eds.) Research Notes in Mathematics 60, Vols. 2 & 3, pp. 98-138, and 70, pp. 154-178, Pitman, London (1982)

[7] Diening, L., Feireisl, E., Lu, Y.: The inverse of the divergence operator on perforated domains with applications to homogenization problems for the compressible Navier-Stokes system. ESAIM: Control Optim. Calc. Var. 23, 851–868 (2017)

[8] Feireisl, E., Novotný, A., Takahashi, T.: Homogenization and singular limits for the complete Navier-Stokes Fourier system. J. Math. Pures Appl. 94(1), 33–57 (2010)

[9] Feireisl, E., Lu, Y.: Homogenization of stationary Navier-Stokes equations in domains with tiny holes. J. Math. Fluid Mech. 17, 381–392 (2015)

[10] Feireisl, E., Namlyeyeva, Y., Nečasová, S.: Homogenization of the evolutionary Navier-Stokes system. Manuscr. Math. 149, 251–274 (2016)

[11] Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems. Springer, Berlin (2011)

[12] Hőfer, R.M., Kowalczyk, K., Schwarzacher, S.: Darcy’s law as low Mach and homogenization limit of a compressible fluid in perforated domains. Math. Model. Method. Appl. Sci. 31(09), 1787–1819 (2021)

[13] Jing, W.: A unified homogenization approach for the Dirichlet problem in perforated domains. SIAM: J. Math. Anal. 52(2), 1192–1220 (2020)

[14] Leray, J.: Essai sur le mouvement d’un liquide visqueux emplissant l’espace. Acta Math. 63, 193–248 (1934)

[15] Lions, P.L.: Mathematical Topics in Fluid Dynamics. Incompressible Models, vol. I. Oxford Science Publication, Oxford (1996)

[16] Lions, P.L.: Mathematical Topics in Fluid Dynamics. Compressible Models, vol. II. Oxford Science Publication, Oxford (1998)

[17] Lu, Y.: Homogenization of Stokes equations in perforated domains: A unified approach. J. Math. Fluid Mech., 22 (2020), Paper No. 44

[18] Lu, Y., Schwarzacher, S.: Homogenization of the compressible Navier-Stokes equations in domains with very tiny holes. J. Differ. Equ. 265(4), 1371–1406 (2018)

[19] Masmoudi, N.: Homogenization of the compressible Navier–Stokes equations in a porous medium. ESAIM: Control Optim. Calc. Var. 8, 885–906 (2002)

[20] Mikelić, A.: Homogenization of nonstationary Navier Stokes equations in a domain with a grained boundary. Ann. Mat. Pura Appl. 158, 167–179 (1991)

[21] Sanchez-Palencia, E.: Non homogeneous media and vibration theory. Lecture Notes in Physics, vol 127, Springer-Verlag (1980)

[22] Tartar, L.: Incompressible fluid flow in a porous medium: convergence of the homogenization process, In: Sánchez-Palencia, E. (Ed.) Nonhomogeneous Media and Vibration Theory, 368-377 (1980)

[23] Temam, R.: Navier-Stokes Equations. North-Holland, Amsterdam (1979)

Yong Lu and Peikang Yang
Department of Mathematics
Nanjing University
Nanjing 210093
China
e-mail: luyong@nju.edu.cn

Peikang Yang
e-mail: ypk@smail.nju.edu.cn

(accepted: October 31, 2022; published online: November 23, 2022)