Dirichlet problem associated with Dunkl Laplacian on $W$-invariant open sets

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Abstract

Combining probabilistic and analytic tools from potential theory, we investigate Dirichlet problems associated with the Dunkl Laplacian $\Delta_k$. We establish, under some conditions on the open set $D \subset \mathbb{R}^d$, the existence of a unique continuous function $h$ in the closure of $D$, twice differentiable in $D$, such that

$$\Delta_k h = 0 \text{ in } D \text{ and } h = f \text{ on } \partial D.$$

We also give a probabilistic formula characterizing the solution $h$. The function $f$ is assumed to be continuous on the Euclidean boundary $\partial D$ of $D$.

1 Introduction

In their monograph [2], J. Bliedtner and W. Hansen developed four descriptions of potential theory using balayage spaces, families of harmonic kernels, sub-Markov semigroups and Markov processes. They proved that all these descriptions are equivalent and gave a straight presentation of balayage theory which is, in particular, applied to the generalized Dirichlet problem associated with a large class of differential and pseudo-differential operators.

Let $W$ be a finite reflection group on $\mathbb{R}^d$, $d \geq 1$, with root system $R$ and we fix a positive subsystem $R_+$ of $R$ and a nonnegative multiplicity function $k : R \to \mathbb{R}_+$. For every $\alpha \in R$, let $H_\alpha$ be the hyperplane orthogonal to $\alpha$ and $\sigma_\alpha$ be the reflection with respect to $H_\alpha$, that is, for every $x \in \mathbb{R}^d$,

$$\sigma_\alpha x = x - 2\frac{\langle x, \alpha \rangle}{|\alpha|^2} \alpha$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of $\mathbb{R}^d$. C. F. Dunkl introduced in [4] the operator

$$\Delta_k = \sum_{i=1}^{d} T_i^2,$$
which will be called later Dunkl Laplacian, where, for $1 \leq i \leq d$, $T_i$ is the differential-difference operator defined for $f \in C^1(\mathbb{R}^d)$ by

$$T_i f(x) = \frac{\partial f}{\partial x_i}(x) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$ 

Our main goal in this paper is to investigate the Dirichlet problem associated with the Dunkl Laplacian. More precisely, given a bounded open set $D \subset \mathbb{R}^d$ and a continuous real-valued function $f$ on $D^c := \mathbb{R}^d \setminus D$, we are concerned with the following problem:

$$\begin{cases}
\Delta_k h = 0 & \text{in } D, \\
h = f & \text{on } D^c.
\end{cases}$$

We mean by a solution of (1) every function $h : \mathbb{R}^d \to \mathbb{R}$ which is continuous in $\mathbb{R}^d$, twice differentiable in $D$ and such that both equations in (1) are pointwise fulfilled. In the particular case where $D$ is the unit ball of $\mathbb{R}^d$, M. Maslouhi and E. H. Youssfi [11] solved problem (1) by methods from harmonic analysis using the Poisson kernel for $\Delta_k$ which is introduced by C. F. Dunkl and Y. Xu [5]. It should be noted that, for balls with center $a \neq 0$, the Poisson kernel for $\Delta_k$ is not known up to now.

Let us briefly introduce our approach. It is well known (see [6] and references therein) that there exists a càdlàg $\mathbb{R}^d$-valued Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^x)$, which is called Dunkl process, with infinitesimal generator $\frac{1}{2} \Delta_k$. For a given bounded Borel function $h : \mathbb{R}^d \to \mathbb{R}$, we define

$$H_U h(x) = E^x[h(X_{\tau_U})]$$

for every $x \in \mathbb{R}^d$ and every bounded open subset $U$ of $\mathbb{R}^d$, where

$$\tau_U = \inf \{ t > 0; X_t \notin U \}$$

denotes the first exit time from $U$ by $X$. We first show that if $h$ is continuous in $\mathbb{R}^d$ and twice differentiable in $D$ then $\Delta_k h = 0$ in $D$ if and only if $h$ is $X$-harmonic in $D$, i.e., $H_U h(x) = h(x)$ for every open set $U$ such that $\overline{U} \subset D$ (we shall write $U \Subset D$) and for every $x \in U$. We then conclude, using the general framework of balayage spaces [2], that problem (1) admits at most one solution. Moreover, if the open set $D$ is regular for the Dunkl process, then $H_D f$ will be the solution of (1) provided it is of class $C^2$ in $D$.

For some examples of Markov processes, namely Brownian motion or $\alpha$-stable process, some additional geometric assumptions on the Euclidean boundary $\partial D$ of $D$ permit a decision on the regularity of $D$. In fact, it is well known that $D$ is regular, with respect to Brownian motion or $\alpha$-stable process, whenever each boundary point of $D$ satisfies the ”cone condition”. For a particular choice of the root system $R$, we shall prove in Section 3 that the cone condition is still sufficient for the regularity of $D$ with respect to the Dunkl process. However, we could not know whether this result holds true for arbitrary root systems. In this setting, we only show that balls of center 0 are regular.

Finally, assuming that $D$ is regular, the study of problem (1) is equivalent to the study of smoothness of $H_D f$. Indeed, as was mentioned above, (1) has a solution if and only if $H_D f \in C^2(D)$. 

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To that end, we need to assume that $D$ is $W$-invariant which means that $\sigma_\alpha(D) \subset D$ for every $\alpha \in R$. Hence, using the fact that the operator $\Delta_k$ is hypoelliptic in $D$ (see [7, 10]) we prove that $H_D f$ is infinitely differentiable in $D$. Thus, we not only deduce the existence and uniqueness of the solution to

$$
\begin{cases}
\Delta_k h = 0 & \text{in } D, \\
h = f & \text{on } \partial D,
\end{cases}
$$

but we also prove that $h$ is given by the formula $h(x) = E^x[f(X_{\tau_D})]$.

Throughout this paper, let $\lambda = \gamma + \frac{d}{2} - 1$ and assume that $\lambda > 0$.

## 2 Harmonic Kernels

For the sake of simplicity, we assume in all the following that $|\alpha|^2 = 2$ for every $\alpha \in R$. It follows from [11] that, for $f \in C^2(\mathbb{R}^d)$,

$$
\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left( \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right),
$$

where $\Delta$ denotes the usual Laplacian on $\mathbb{R}^d$. M. Rösler has shown in [13] that $\frac{1}{2}\Delta_k$ generates a Feller semigroup $P^k_t(x,dy) = p^k_t(x,y)w_k(y)dy$ which has the expression

$$
p^k_t(x,y) = \frac{1}{c_k t^{\gamma + \frac{d}{2}}} \exp \left( -\frac{|x|^2 + |y|^2}{2t} \right) E_k \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right),
$$

where $E_k(\cdot, \cdot)$ is the Dunkl kernel associated with $W$ and $k$ (see [14]), the constant $c_k$ is taken such that $P^k_1 \equiv 1$, $\gamma = \sum_{\alpha \in R_+} k(\alpha)$ and $w_k$ is the $W$-invariant weight function defined on $\mathbb{R}^d$ by

$$
w_k(y) = \prod_{\alpha \in R_+} |\langle y, \alpha \rangle|^{2k(\alpha)}.
$$

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^x)$ be the Dunkl process in $\mathbb{R}^d$ with transition kernel $P^k_t(x,dy)$. For every bounded open subset $D$ of $\mathbb{R}^d$, let $\tau_D$ be the first exit time from $D$ by $X$. A point $z \in \partial D$ is said to be regular (for $D$) if $P^z[\tau_D = 0] = 1$ and irregular if $P^z[\tau_D = 0] = 0$. Notice that by Blumenthal’s zero-one law, each boundary point of $D$ is either regular or irregular. It is also easy verified that the fact that Dunkl process has right continuous paths yields that $P^z[\tau_D = 0] = 0$ if $x \in D$ and $P^x[\tau_D = 0] = 1$ if $x \in \mathbb{R}^d \setminus \overline{D}$.

**Proposition 1.** $E^x[\tau_D] < \infty$ for every $x \in \mathbb{R}^d$ and every bounded open subset $D$ of $\mathbb{R}^d$.

**Proof.** Let $D$ be a bounded open subset of $\mathbb{R}^d$, $x \in \mathbb{R}^d$ and choose $r > 0$ such that the ball $B = B(0,r)$ contains $x$ and $D$. Then, applying Fubini’s theorem and using spherical coordinates,

$$
E^x[\tau_B] \leq \int_0^\infty E^x[1_B(X_s)]ds = \int_0^r t^{2\lambda + 1} \int_0^\infty \int_{S^{d-1}} p^k_s(x,tz)w_k(z)\sigma(dz)ds dt.
$$
Here and in all the following, \( \sigma \) denotes the surface area measure on the unit sphere \( S^{d-1} \) of \( \mathbb{R}^d \). It is well known (see [13, 14]) that for every \( x, y \in \mathbb{R}^d \) and \( s > 0 \),

\[
p^x_s(x, y) = \frac{1}{c_k} \int_{\mathbb{R}^d} e^{-\frac{d}{2}|\xi|^2} E_k(-ix, \xi) E_k(iy, \xi) w_k(\xi) d\xi
\]

and

\[
\int_{S^{d-1}} E_k(ix, \xi) w_k(\xi) \sigma(d\xi) = \frac{c_k}{2^\lambda \Gamma(\lambda + 1)} j_\lambda(|x|),
\]

where

\[
j_\lambda(z) := \Gamma(\lambda + 1) \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{4^n n! \Gamma(n + \lambda + 1)}
\]

is the Bessel normalized function. Hence

\[
E^x[\tau_D] \leq \int_0^\infty E^x[1_B(X_s)] ds
= \frac{1}{2^{2\lambda-1}(\Gamma(\lambda + 1))^2} \int_0^r t^{2\lambda + 1} \int_0^{\infty} j_\lambda(u t) j_\lambda(u |x|) u^{2\lambda - 1} du dt
= \frac{2^{2\lambda-1}\Gamma(\lambda + 1)\Gamma(\lambda)}{2^{2\lambda-1}(\Gamma(\lambda + 1))^2} \int_0^r t^{2\lambda + 1}(\max(t, |x|))^{-2\lambda} dt
= \frac{r^2}{2\lambda} - \frac{|x|^2}{2\lambda + 2} < \infty.
\]

In order to get (5) one should think about formula (11.4.33) in [1].

Let \( D \) be a bounded open subset of \( \mathbb{R}^d \). For every \( x \in \mathbb{R}^d \), the exit distribution \( H_D(x, \cdot) \) from \( D \) by the Dunkl process starting at \( x \) will be called harmonic measure relative to \( x \) and \( D \). That is, for every Borel subset \( A \) of \( \mathbb{R}^d \),

\[
H_D(x, A) = P^x(\tau_D \in A).
\]

It is clear that \( H_D(x, \cdot) = \delta_x \) the Dirac measure at \( x \) whenever \( x \in \partial D \) is regular or \( x \not\in \overline{D} \).

We define

\[
\overline{D} := \cup_{w \in W} w(D) \quad \text{and} \quad \Gamma_D := \overline{W} \setminus D.
\]

In other words, \( \overline{D} \) is the smallest open set containing \( D \) which is invariant under the reflection group \( W \). The following theorem ensures that \( H_D(x, \cdot) \) is supported by \( \Gamma_D \) for every \( x \in \overline{D} \).

**Theorem 2.** Let \( D \) be a bounded open subset of \( \mathbb{R}^d \). Then for every \( x \in \overline{D} \),

\[
P^x(\tau_D \in \Gamma_D) = 1. \tag{7}
\]

**Proof.** It is easily seen that for every regular boundary point \( x \), \( P^x(\tau_D \in \Gamma_D) = \delta_x(\Gamma_D) = 1 \). Now, assume that \( x \in D \) or \( x \in \partial D \) is irregular and consider the function \( F \) defined for every \( y, z \in \mathbb{R}^d \) by \( F(y, z) = 0 \) if \( z \in \{ \sigma_\alpha y; \alpha \in R_+ \} \) and \( F(y, z) = 1 \) otherwise. Let

\[
Y_t := \sum_{s < t} 1_{\{X_s - \neq X_s\}} F(X_s, X_s), \quad t > 0.
\]

It follows from [6, Proposition 3.2] that for every \( t > 0 \), \( P^x(Y_t = 0) = 1 \) and consequently

\[
P^x \left(1_{\{X_s - \neq X_s\}} F(X_s, X_s) = 0; \forall s > 0 \right) = 1.
\]
Then, since \( P^x(0 < \tau_D < \infty) = 1 \) we deduce that
\[
P^x \left( 1_{\{x_{\tau_D} \neq x_{\tau_D}\}} F(X_{\tau_D}, X_{\tau_D}) = 0 \right) = 1.
\]
On the other hand, seeing that \( X_{\tau_D} \in \overline{D} \) on \( \{0 < \tau_D < \infty\} \) we have
\[
\{X_{\tau_D} \notin \Gamma_D, 0 < \tau_D < \infty\} \subset \left\{ 1_{\{x_{\tau_D} \neq x_{\tau_D}\}} F(X_{\tau_D}, X_{\tau_D}) = 1 \right\}.
\]
This finishes the proof.

Let \( \mathcal{O} \) be the set of all bounded open subsets of \( \mathbb{R}^d \). In the following, we denote by \( \mathcal{B}_b(\mathbb{R}^d) \) the set of all bounded Borel measurable functions on \( \mathbb{R}^d \). For every \( D \in \mathcal{O} \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \), let \( H_D f \) be the function defined on \( \mathbb{R}^d \) by
\[
H_D f(x) = E^x[f(X_{\tau_D})] = \int f(y) H_D(x, dy).
\]
Since \( X \) is a Hunt process, it follows from the general framework of balayage spaces studied by J. Bliedtner and W. Hansen in [2] that, for every \( D \in \mathcal{O} \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \) with compact support,
\[
H_D f \text{ is continuous in } D \text{ and for every } V \Subset D,
\]
\[
H_V H_D = H_D \text{ in } V. \tag{8}
\]
Since \( \text{supp } H_D(x, \cdot) \subset \Gamma_D \) for every \( x \in W_D \), it is trivial that
\[
H_D f(x) = H_D (1_{\Gamma_D} f)(x), \quad x \in W_D.
\]
Hence, we immediately conclude that \( H_D f \) is continuous in \( D \). For every \( D \in \mathcal{O} \) and every \( f \in \mathcal{B}_b(\Gamma_D) \), it will be convenient to denote again
\[
H_D f(x) = \int f(y) H_D(x, dy), \quad x \in W_D. \tag{9}
\]
Let \( U \) be an open subset of \( \mathbb{R}^d \). A locally bounded function \( h : W_U \to \mathbb{R} \) is said to be \( X \)-harmonic in \( U \) if \( H_D h(x) = h(x) \) for every open set \( D \Subset U \) and every \( x \in D \). If \( U \) is bounded and \( h \) is continuous in \( W_U \) then \( h \) is \( X \)-harmonic in \( U \) if and only if for every \( x \in U \),
\[
h(x) = H_U h(x). \tag{10}
\]
In fact, let \( x \in U \) and let \( (U_n)_{n \geq 1} \) be a sequence of nonempty bounded open subsets of \( \mathbb{R}^d \) such that \( x \in U_n \in U_{n+1} \) and \( U = \bigcup_n U_n \). Then \( (\tau_{U_n})_n \) converges to \( \tau_U \) almost surely. Hence, the continuity of \( h \) on \( W_U \) together with the quasi-left-continuity of the Dunkl process yield that \( H_U h(x) = \lim_n H_{U_n} h(x) \). The following proposition follows immediately from (10).

**Proposition 3.** Let \( U \in \mathcal{O} \) and let \( h \) be a continuous function on \( W_U \). If \( h \) is \( X \)-harmonic in \( U \), then
\[
\max_{x \in W_U} h(x) = \max_{x \in \Gamma_U} h(x) \quad \text{and} \quad \min_{x \in W_U} h(x) = \min_{x \in \Gamma_U} h(x).
\]
We shall denote by $G^k$ the Green function of $\Delta_k$ which is defined for every $x, y \in \mathbb{R}^d$ by

$$G^k(x, y) = \int_0^\infty p_t^k(x, y)dt.$$  

Since $p_t^k$ is symmetric in $\mathbb{R}^d \times \mathbb{R}^d$, we obviously see that the Green function $G^k$ is also symmetric in $\mathbb{R}^d \times \mathbb{R}^d$. Therefore, it follows from [3, Theorem VI.1.16] that for every $D \in \mathcal{O}$ and for every $x, y \in \mathbb{R}^d$,

$$\int G^k(x, z)H_D(y, dz) = \int G^k(y, z)H_D(x, dz). \quad (11)$$

Furthermore, for every $y \in \mathbb{R}^d$, the function $G^k(\cdot, y)$ is excessive, that is, $G^k(\cdot, y)$ is lower semi-continuous in $\mathbb{R}^d$ and $\int p_t^k(x, z)G^k(z, y)w_k(z)dz \leq G^k(x, y)$ for every $t > 0$ and $x \in \mathbb{R}^d$. Consequently, it follows from [2, Theorem IV.8.1] that $G^k(\cdot, y)$ is hyperharmonic on $\mathbb{R}^d$, i.e., for every $D \in \mathcal{O}$ and for every $x \in \mathbb{R}^d$,

$$\int G^k(z, y)H_D(x, dz) \leq G^k(x, y). \quad (12)$$

**Lemma 4.** Let $f \in C^2_c(\mathbb{R}^d)$ and $D \in \mathcal{O}$. For every $x \in \mathbb{R}^d$,

$$\int G^k(x, y)\Delta_k f(y)w_k(y)dy = -2f(x). \quad (13)$$

In particular,

$$H_D f(x) - f(x) = \frac{1}{2}E^x \left[ \int_0^{\tau_D} \Delta_k f(X_s)ds \right]. \quad (14)$$

**Proof.** To get (13) it suffices to recall that

$$\frac{\partial}{\partial t}p_t^k = \frac{1}{2}p_t^k\Delta_k, \quad t > 0.$$  

Then, we integrate over $t$ and use the fact that $\lim_{t \to 0} p_t^k f(x) = f(x)$ and $\lim_{t \to \infty} p_t^k f(x) = 0$ for every $x \in \mathbb{R}^d$. Formula (14) follows from (13) and the strong Markov property. \[\Box]\n
Let $U$ be an open subset of $\mathbb{R}^d$. A function $h : \mathbb{R}^d \to \mathbb{R}$ is said to be $\Delta_k$-harmonic in $U$ if $h \in C^2(U)$ and $\Delta_k h(x) = 0$ for every $x \in U$.

**Theorem 5.** Let $U$ be an open subset of $\mathbb{R}^d$ and let $h \in C(\overline{U})$. If $h \in C^2(U)$ then $h$ is $\Delta_k$-harmonic in $U$ if and only if $h$ is $X$-harmonic in $U$.

**Proof.** Let $D \subseteq U$ and let $x \in D$. Then

$$H Dh(x) - h(x) = \frac{1}{2}E^x \left[ \int_0^{\tau_D} \Delta_k h(X_s)ds \right]. \quad (15)$$

In fact, choose an open set $V$ such that $D \subseteq V \subseteq U$, $f \in C^2_c(\mathbb{R}^d)$ which coincides with $h$ in $V$ and let $\psi = h - f$. Then using (14) we obtain

$$H Dh(x) - h(x) = \frac{1}{2}E^x \left[ \int_0^{\tau_D} \Delta_k f(X_s)ds \right] + H D\psi(x). \quad (16)$$
For every \( y \in \mathbb{R}^d \), let \( N(y, dz) \) be the Lévy kernel of the Dunkl process \( X \) which is given by the following formula \([6]\)

\[
N(y, dz) = \sum_{\alpha \in \mathbb{R}^+ \setminus \{0\}, \langle \alpha, y \rangle \neq 0} k(\alpha) \sigma_{\alpha,y}(dz).
\]  

(17)

Since \( \psi = 0 \) on \( V \), it follows from \([8, \text{Theorem 1}]\) that

\[
H_D \psi(x) = E^x \left[ \int_0^{\tau_D} \int \psi(z) N(X_s, dz) ds \right].
\]  

(18)

On the other hand, by (3) and (17) we easily see that for every \( y \in D \),

\[
\Delta_k f(y) = \Delta_k h(y) - 2 \int \psi(z) N(y, dz).
\]  

(19)

Thus formula (15) is obtained by combing (16), (18) and (19) above. Now, \( h \) is obviously \( X \)-harmonic in \( U \) whenever it is \( \Delta_k \)-harmonic in \( U \). Conversely, assume that \( h \) is \( X \)-harmonic in \( U \) and let \( x \in U \). Since \( h \in C^1(U) \cap C^2(U) \) then \( \Delta_k h \) is continuous in \( U \) and consequently for every \( \varepsilon > 0 \) there exists an open neighborhood \( D \subseteq U \) of \( x \) such that \( |\Delta_k h(y) - \Delta_k h(x)| \leq \varepsilon \) for every \( y \in D \). Using formula (15), we obtain

\[
|\Delta_k h(x)| = \frac{1}{E^x(\tau_D)} \left| E^x \left[ \int_0^{\tau_D} (\Delta_k h(X_s) - \Delta_k h(x)) ds \right] \right| \leq \varepsilon.
\]

Hence \( \Delta_k h(x) = 0 \) as desired.

\[\square\]

3 Regular Sets

A bounded open subset \( D \) of \( \mathbb{R}^d \) is said to be regular if each \( z \in \partial D \) is regular for \( D \). A complete study of regularity is developed by J. Bliedtner and W. Hansen in \([2]\). It follows that a point \( z \in \partial D \) is regular for \( D \) if and only if for every \( f \in C(\Gamma_D) \),

\[
\lim_{x \in D, x \to z} H_D f(x) = f(z).
\]

Consequently, \( H_D f \) is continuous on \( \overline{\mathbb{W}D} \) whenever \( D \) is regular and \( f \in C(\Gamma_D) \).

**Example 6.** For all \( R > r > 0 \), the ball \( B(0, R) \) and the annulus \( C(r, R) = \{ x \in \mathbb{R}^d ; r < \|x\| < R \} \) are regular.

In fact, by \([2, \text{Proposition VII.3.3}]\), it is sufficient to find a neighborhood \( V \) of \( z \in \partial D \) and a real function \( u \) such that

i) \( u \) is positive in \( V \cap D \),

ii) \( u \) is \( X \)-harmonic in \( V \cap D \),

iii) \( \lim_{x \in V \cap D, x \to z} u(x) = 0 \).
Consider $V = \mathbb{R}^d \setminus \{0\}$ and $g$ the function defined on $V$ by

$$g(x) = \frac{1}{|x|^{2\lambda}}.$$

Using formula (3), simple computation shows that $g$ is $\Delta_k$-harmonic in $V$ which yields, by theorem 5 that $g$ is $X$-harmonic in $V$. Let $z \in \mathbb{R}^d$ such that $|z| = R$ and consider

$$u(x) = g(x) - \frac{1}{R^{2\lambda}}, \quad x \in V.$$

It is clear that $u$ satisfy (i), (ii) and (iii) above with $D = B(0, R)$ or $D = C(r, R)$. Hence $z$ is regular for $D$. Similarly, taking

$$u(x) = \frac{1}{y^{2\lambda}} - g(x), \quad x \in V,$$

we conclude that all points $z \in \mathbb{R}^d$ such that $|z| = r$ are regular for $C(r, R)$.

A sufficient condition for regularity, known as the cone condition, is given in the following theorem for a particular root system $R$.

**Theorem 7.** Let $(e_1, ..., e_d)$ be the canonical basis of $\mathbb{R}^d$ and consider the root system $R = \{\pm e_i, \ 1 \leq i \leq d\}$. Let $D$ be a bounded open subset of $\mathbb{R}^d$ and let $z \in \partial D$. Assume that there exists a cone $C$ of vertex $z$ such that $C \cap B(z, r) \subset D^c$ for some $r > 0$. Then $z$ is regular for $D$.

**Proof.** It is trivial that $P^z[\tau_D \leq t] \geq P^z[X_t \in C \cap B(z, r)]$ for all $t > 0$. Therefore, in virtue of Blumenthal’s zero-one law, it is sufficient to show that $\lim \inf_{t \to 0} P^z[X_t \in C \cap B(z, r)]$ is positive. Denote $C_0 = C - z$, then

$$P^z[X_t \in C \cap B(z, r)] = \int_{C \cap B(z, r)} p^k_t(z, y)w_k(y)dy$$

$$= \frac{1}{t^{\gamma}} \int_{C_0 \cap B(0, \sqrt{t})} p^k_t(\frac{z}{\sqrt{t}}, \frac{y}{\sqrt{t}} - y)w_k(z - \sqrt{t}y)dy. \quad (20)$$

It is trivial to see, from (4), that

$$p^k_t\left(\frac{z}{\sqrt{t}}, \frac{y}{\sqrt{t}} - y\right) = e^{-\frac{|w|^2}{2}} e^{-\left(\frac{z}{\sqrt{t}} - \sqrt{t}y\right)} E_k\left(\frac{z}{\sqrt{t}}, z - \sqrt{t}y\right).$$

Let $k_i = k(e_i)$ and $y_i = \langle y, e_i \rangle$ for every $y \in \mathbb{R}^d$ and $i \in \{1, ..., d\}$. It is known [16] that for all $x, y \in \mathbb{R}^d$,

$$e^{-(x, y)} E_k(x, y) = \prod_{i=1}^{d} M(k_i, 2k_i + 1, -2x_iy_i).$$

$M(k_i, 2k_i + 1, \cdot)$ denotes the Kummer’s function defined on $\mathbb{R}$ by

$$M(k_i, 2k_i + 1, s) = \sum_{n \geq 0} \frac{(k_i)_n s^n}{(2k_i + 1)_n n!} = 1 + \frac{k_i}{2k_i + 1} s + \frac{k_i(k_i + 1)}{(2k_i + 1)(2k_i + 2)} \frac{s^2}{2!} + \cdots.$$
Therefore, for any $y \in \mathbb{R}^d$ and $t > 0$, we have
\[
\frac{1}{t^d} e^{-\left(\frac{t}{2} \|z - \sqrt{t}y\| \right)} E_k \left(\frac{z}{t}, z - \sqrt{t}y \right) w_k \left( z - \sqrt{t}y \right) = \prod_{i=1}^{d} M \left( k_i, 2k_i + 1, -2\frac{z_i}{t} (z_i - \sqrt{t}y_i) \right) (z_i - \sqrt{t}y_i)^{2k_i}. \]

First, it is clear that
\[
\frac{1}{t^{k_i}} M \left( k_i, 2k_i + 1, -2\frac{z_i}{t} (z_i - \sqrt{t}y_i) \right) (z_i - \sqrt{t}y_i)^{2k_i} = \begin{cases} 
1 & \text{if } k_i = 0 \\
y_i^{2k_i} & \text{if } z_i = 0.
\end{cases}
\]

Next, assume that $k_i > 0$ and $z_i \neq 0$ for some $i \in \{1, \cdots, d\}$. Then, it follows from the integral representation of $M(k_i, 2k_i + 1, \cdot)$ that
\[
\frac{1}{t^{k_i}} M \left( k_i, 2k_i + 1, -2\frac{z_i}{t} (z_i - \sqrt{t}y_i) \right) = \frac{\Gamma(2k_i + 1)}{\Gamma(k_i) \Gamma(k_i + 1)} \int_0^1 \frac{1}{t^{k_i}} e^{-\frac{2z_i}{t} (z_i - \sqrt{t}y_i) u} u^{k_i-1} (1 - u)^{k_i} du \\
= \frac{\Gamma(2k_i + 1)}{\Gamma(k_i) \Gamma(k_i + 1)} \int_0^1 e^{-\frac{2z_i}{t} (z_i - \sqrt{t}y_i) v} v^{k_i-1} (1 - tv)^{k_i} dv.
\]

Now, applying the Lebesgue dominated convergence theorem, we obtain
\[
\lim_{t \to 0} \frac{1}{t^{k_i}} M \left( k_i, 2k_i + 1, -2\frac{z_i}{t} (z_i - \sqrt{t}y_i) \right) = \frac{\Gamma(k_i + 1)}{\sqrt{\pi z_i^{2k_i}}}. 
\]

Thus
\[
\lim_{t \to 0} \frac{1}{t^d} e^{-\left(\frac{t}{2} \|z - \sqrt{t}y\| \right)} E_k \left(\frac{z}{t}, z - \sqrt{t}y \right) w_k \left( z - \sqrt{t}y \right) \geq \prod_{i=1}^{d} \min \left( 1, y_i^{2k_i}, \frac{\Gamma(k_i + 1)}{\sqrt{\pi z_i^{2k_i}}} \right) =: \theta(y).
\]

Hence, Fatou's lemma applied to (20) yields that
\[
\liminf_{t \to 0} P^z[X_t \in C \cap B(z, r)] \geq \int_{C_0} e^{-\frac{\|y\|^2}{2} \theta(y)} dy > 0.
\]

\[ \square \]

4 Dirichlet Problem

This section is devoted to study the following Dirichlet problem: Given a regular open subset $D$ of $\mathbb{R}^d$ and a function $f \in C(\Gamma_D)$, we shall investigate existence and uniqueness of function $h \in C^{2}(\overline{D}) \cap C^{2}(D)$ satisfying the boundary value problem
\[
\begin{cases} 
\Delta_k h = 0 & \text{in } D, \\
h = f & \text{in } \Gamma_D.
\end{cases}
\]

(21)

For every square integrable functions $\varphi$ and $\psi$ on $\mathbb{R}^d$ with respect to the measure $w_k(x)dx$, we define
\[
\langle \varphi, \psi \rangle_k = \int \varphi(x) \psi(x) w_k(x) dx.
\]
Lemma 8. For every bounded open set \( D \) and for every \( \phi, \psi \in C^2_c(\mathbb{R}^d) \),
\[
\langle H_D \psi, \Delta_k \varphi \rangle_k = \langle \Delta_k \psi, H_D \varphi \rangle_k.
\] (22)

Proof. Applying formula (13) to \( \psi \), we have
\[
\langle H_D \psi, \Delta_k \phi \rangle_k = -\frac{1}{2} \int G(z,y) \Delta_k \psi(y) w_k(y) dy H_D(x,dz) \Delta_k \varphi(x) w_k(x) dx.
\] (23)
Then (22) is obtained by Fubini’s theorem and formulas (11) and (13). Here, since \( \phi \) and \( \psi \) are with compact supports, formulas (12) and (6) justify the transformation of the integrals in (23) by Fubini’s theorem.

A set \( D \) is called \( W \)-invariant if \( W D = D \) which, in turn, is equivalent to \( \Gamma_D = \partial D \). We finally have the necessary tools at our disposal for solving the following Dirichlet problem.

Theorem 9. Let \( D \) be a \( W \)-invariant regular open subset of \( \mathbb{R}^d \). For every function \( f \in C(\partial D) \), there exists one and only one function \( h \in C(\overline{D}) \cap C^2(D) \) such that
\[
\begin{cases}
\Delta_k h = 0 & \text{in } D, \\
h = f & \text{in } \partial D.
\end{cases}
\] (24)

Moreover, \( h \) is given by
\[
h(x) = \int_{\partial D} f(y) H_D(x,dy), \quad x \in \overline{D}.
\]

Proof. In virtue of Theorem 5, we observe that for \( f \in C(\partial D) \), every solution \( h \) of (21) satisfies necessarily :
\[
\begin{cases}
h \text{ is } X\text{-harmonic in } D, \\
h = f \text{ in } \partial D.
\end{cases}
\] (25)
Then, by Proposition 8, (24) admits at most one solution. The function \( H_D f \) is \( X \)-harmonic in \( D \) by (5). Moreover, the regularity of \( D \) yields that \( H_D f \) is a continuous extension of \( f \) to \( \overline{D} \). Therefore, according to Theorem 5, \( H_D f \) will be the unique solution of (24) provided it is twice differentiable in \( D \). On the other hand, it has been shown in [7] that \( \Delta_k \) is hypoelliptic in \( D \) (see also [10]), i.e., a continuous function \( g \) in \( D \) which satisfies
\[
\langle g, \Delta_k \varphi \rangle_k = 0 \quad \text{for all } \varphi \in C^\infty_c(D)
\] (26)
is necessary infinitely differentiable in \( D \). Thus to complete the proof we only need to show that (26) holds true for \( g = H_D f \). To this end let \( \varphi \in C^\infty_c(D) \) and let \( (f_n)_{n \geq 1} \subset C^2_0(\mathbb{R}^d) \) be a sequence which converges uniformly to \( f \) in \( \partial D \). Since \( H_D \varphi(y) = 0 \) for all \( y \in \mathbb{R}^d \), applying (22) we obtain
\[
\langle H_D f_n, \Delta_k \varphi \rangle_k = 0, \quad n \geq 1.
\] (27)
On the other hand,
\[
\sup_{x \in D} |H_D f_n(x) - H_D f(x)| \leq \sup_{y \in \partial D} |f_n(y) - f(y)| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\]
Hence \( H_D f \) satisfies (26) by letting \( n \) tend to \( \infty \) in (27).
It should be noted that the hypothesis ”$D$ is $W$-invariant” is only needed to get the hypoellipticity of $\Delta_k$. For open set $D$ which is not $W$-invariant, the question whether $\Delta_k$ is hypoelliptic in $D$ or not remained open. In the case of positive answer, analogous arguments as in the proof of Theorem 9 will immediately imply that $H_D f$ is the unique solution of problem (21).

Let us notice that, using methods from harmonic analysis, M. Maslouhi and E. H. Youssfi [11] studied problem (24) in the special case where $D = B$ is the unit ball of $\mathbb{R}^d$. They proved that, for any $f \in C(\partial B)$, the function $h$ given by

$$h(x) = \int_{\partial B} P_\kappa(x,y)f(y)w_k(y)\sigma(dy), \ x \in B$$

is the unique solution of (24), where $P_\kappa$ denotes the Poisson kernel introduced by C. F. Dunkl and Y. Xu [5]. Hence, our above theorem immediately yields that for every $x \in B$,

$$H_B(x,dy) = P_\kappa(x,y)w_k(y)\sigma(dy).$$

References

[1] Abramowitz, M. and Stegun, I. A. (1984). *Handbook mathematical functions*. Verlag Harri Deutsch. Frankfurt-Main.

[2] Bliedtner, J. and Hansen, W. (1986). *Potentiel theory. An analytic and probabilistic approach to balayage*. Springer-Verlag.

[3] Blumenthal, R. M. and Getoor, R. K. (1968). *Markov processes and potential theory*. Academic Press.

[4] Dunkl, C. F. (1989). Differential-difference operators associated to reflection groups. *Trans. Am. Math. Soc.* 311 167–183.

[5] Dunkl, C. F. and Xu, Y. (2001). *Othogonal polynomials of sevaral variables*. Cambridge University Press.

[6] Gallardo, L. and Yor, M. (2006). A chaotic representation property of the multidimensional Dunkl processes. *Ann. Proba.* 34 1530–1549.

[7] Hassine, K. (2014). Mean value propoerty associated with the Dunkl Laplace opertor. *Preprint. arXiv:1401.1940v1*.

[8] Ikeda, N. and Watanabe, S. (1962). On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes. *J. Math. Kyoto Univ.* 2 79–95.

[9] Mejjaoli, H. and Trimèche, K. (2001). On a mean value property associated with the dunkl Laplacian operator and applications. *Integral Transforms Spec. Funct.* 12 279–302.

[10] Mejjaoli, H. and Trimèche, K. (2004). Hypoellipticity and hypoaanalyticity of the Dunkl Laplacian operator. *Integral Transforms Spec. Funct.* 15 523–548.

[11] Maslouhi, M. and Youssfi, E. H. (2007). Harmonic functions associated to Dunkl operators. *Monatsh. Math.* 152 337–345.
[12] Rösler, M. (1999). Positivity of Dunkl’s intertwining operator. *Duke Math. J.* **98** 445–463.

[13] Rösler, M. (1998). Generalized Hermite polynomials and heat equation for Dunkl operators. *Commun. Math. Phys.* **192** 519–542.

[14] Rösler, M. (2003). A positive radial product formula for Dunkl kernel. *Trans. Am. Math. Soc.* **355** 2413–2438.

[15] Trimèche, K. (2001). The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual. *Integral Transforms Spec. Funct.* **12** 349–374.

[16] Xu, Y. (1997). Orthogonal polynomials for a family of product weight functions on the spheres. *Can. J. Math.* **49** 175–192.