Inertial waves generated by circular oscillations of inner core in a rotating spherical cavity

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Abstract. The fluid flow excited by a core oscillating in a rotating spherical cavity is experimentally investigated. The core performs circular oscillations around the rotation axis under the action of an external inertial field and generates inertial waves. The main attention is paid to the effect of the oscillation frequency on the instantaneous flow structure and intensity. It is found that at a certain frequency, inertial waves experience spatial resonance, resulting in the intensification of oscillatory flow.

1. Introduction
Forced perturbations in a rotating fluid lead to the appearance of inertial waves [1]. This class of waves is typical for rotating systems and is traditionally considered in spherical shell geometry because of astro- and geophysical applications. Due to the gravitational interaction, planetary bodies can experience tidal deformations [2, 3], non-uniformly rotation (longitudinal librations) [4, 5], and precession [6]. In all cases, inertial oscillations of the fluid will be generated in liquid interiors. Oscillations in the liquid cores of rotating planets can be excited by the buoyancy translational oscillation of the inner solid core, known as the "Slichter triplet" [7]. One of these modes characterizes the natural oscillations along the axis of rotation, and the other two – circular oscillations in the equatorial plane. The case of circular oscillations of a free core in a rotating spherical cavity with a relative frequency \( n = -1 \) was considered in detail in the works [8, 9]. In experiments, the free light body was located in the liquid-filled rapidly rotating around a horizontal axis spherical cavity. The gravity field caused a stationary displacement of the core from the axis of rotation, which corresponds to circular core oscillations in the frame of rotating cavity. The main attention was paid to the non-linear response of the fluid. Averaged stresses in dynamic boundary layers lead to the differential rotation of both the fluid and the free inner core. The effect of multi-frequency core oscillations on the structure of the steady flow was studied in [10]. In addition to the static field of gravity, the rotating system was affected by an alternating inertial field generated by transverse vibrations. It was shown that the resulting differential rotation of the core and fluid is determined by the superposition of averaged effects caused by oscillations with different frequencies. The present work develops the study [10] for the case of an arbitrary frequency \( n \). The influence of the relative frequency of oscillations on the structure and the intensity of oscillatory flow is under consideration. To exclude the action of gravity, the cavity rotates around the vertical axis, and to exclude the differential rotation of the core, the latter is connected to the lower pole of the shell by the fishing line.
2. Experimental setup and technique

A spherical core of radius $r = 17.7 \text{ mm}$ and average density $\rho = 0.91 \text{ g/cm}^3$ is placed in a spherical cavity of radius $R = 36.0 \text{ mm}$ filled with a low-viscosity liquid (figure 1). The light core is located in the equatorial plane due to a thin nylon fishing line of thickness $0.37 \text{ mm}$ connected to the lower pole of the cavity. The cavity rotates around the vertical axis at a constant speed $\Omega$, while the centrifugal force keeps the core on the axis of rotation. To excite oscillations of the core relative to the cavity, the cuvette is installed on the platform of an electrodynamic shaker of type Veb GRW Teltow 3.19.39. The translational vibrations change according to the harmonic law $\sim \sin \Omega_{vib} t n$ and are directed across the axis of rotation. The rotation of the cuvette is set by a stepper motor of type FL86STH80-4208A. The vibrations of the electrodynamic shaker table are set using the ZET-210 SigmaUSB generator, the signal from which is amplified by Digisynthetic DP3200. A detailed description of the experimental setup can be found in [11].

The cavity rotation rate in all experiments is constant and amounts to $\Omega = 62.8 \text{ s}^{-1}$. At the same time, the vibration frequency in the laboratory reference frame varies in the range of $f_{vib} = \Omega_{vib}/2\pi = 15 \div 25 \text{ Hz}$. The parameter $n \equiv (\Omega_{vib} - \Omega)/\Omega = 0.5 \div 1.5$ is used as a dimensionless characteristic of the core oscillation frequency relative to the cavity. The water-glycerin solution with kinematic viscosity of $\nu = 3.1 \text{ cSt}$ and density of $\rho_L = 1.13 \text{ g/cm}^3$ is used as the working fluid. The Ekman number is $E \equiv \nu/\Omega R^2 = 3.8 \times 10^{-5}$.

![Figure 1. Problem statement.](image)

The amplitude of the core oscillations $b$ relative to the cavity in the equatorial plane is measured by video recording on stationary in the laboratory reference frame high-speed camera CamRecord CL600x2 with a resolution of $800 \times 800$ pixels. The velocity field in the meridional plane is studied using the PIV-method. For this purpose, the Resin Amberlite tracer particles of a size $\sim 60 \mu\text{m}$ are added to the working liquid. The illuminator is a light sheet generated by a continuous laser Z-Laser Z500Q. The position of the light scattering particles is recorded on the high-speed video camera. To obtain an instantaneous velocity field in the cavity reference frame the video-recording frequency is selected equal to the cavity rotation frequency. The ratio between the cavity rotation frequency and the core oscillation frequency changes arbitrarily and after a time $T_0 = 2\pi/\Omega$ the structure of the resulting flow will represent a superposition of the averaged (over the period of rotation of the cavity) and the oscillatory flow. Taking into account that the oscillatory component is obviously greater than the average one, the latter can be ignored. In this regard, a time $\Delta \tau_c$ between two successive phases of core oscillations is used to calculate the instantaneous velocity field (figure 2). It can be seen that the phase of the core oscillation changes after each period $T_0$. Thus, one can determine the value of fluid pulsations in
3. Core oscillations

In the absence of vibrations under the action of centrifugal force, the core is located on the cavity rotation axis. The inertial response of the core to the vibration action is observed near two resonant frequencies [11]. Both modes of oscillation are circular. This is because the linear vibrations in the laboratory reference frame can be decomposed into two circularly polarized components:

\[ r \sin \Omega_{\text{vib}} t \mathbf{i} = \frac{r}{2} (\sin \Omega_{\text{vib}} t \mathbf{i} + \cos \Omega_{\text{vib}} t \mathbf{j}) + \frac{r}{2} (\sin \Omega_{\text{vib}} t \mathbf{i} - \cos \Omega_{\text{vib}} t \mathbf{j}). \]

These oscillations are opposite in direction and correspond to two rotating inertial fields. In the present experiments, we consider the leading core oscillations \( n > 0 \) excited by an inertial field rotating in the direction of rotation of the cavity. In this case, the circular oscillations of the core
Figure 4. Snapshots of the velocity field in the fixed meridional plane at \( n = 0.90 \) and different phases of the core oscillations with step by phase \( \pi/2 \). The color shows the vertical component of the flow velocity; the white arrows show the theoretical trajectory of the wave beams.

in the rotating reference frame in the direction opposite to the cavity rotation are negligible. Figure 3 shows the results of measurements of the position of the core in the reference frame of the cavity during several periods of rotation. It can be seen that the oscillation trajectory is close to circular. Observations of the behavior of the core show that due to the fishing line, the core does not perform differential rotation and oscillates relative to the cavity exclusively in the equatorial plane.

4. Inertial waves

Oscillations of the core are the source of inertial waves that are generated at critical latitudes, where the Ekman boundary layers break down [13]. Inertial waves propagate along characteristic surfaces in the form of cones. According to [1] the direction of wave beams propagation is determined by the ratio of the oscillations frequency to the doubled rotation rate of the system, \( \theta = \arcsin(n/2) \). Figure 4 shows the sequence of instantaneous velocity fields in a fixed meridional plane at various time instants at a frequency \( n = 0.90 \). It can be seen that the flow structure is non-axisymmetric, which is explained by the specifics of core oscillations with an azimuthal wave number \( m = 1 \). During the oscillation process, the value and sign of the vertical velocity component \( u_y \) change periodically, and the velocity fields in the opposite phases of the oscillation (figure 4, a and c, b and d) are mirror-symmetrical. It should be noted that the direction of the group speed of wave beams (white arrows) is preserved, and the value of the angle \( \theta \) is in good agreement with the theoretical predictions [1]. The most intense fluid movement is observed near the axis of rotation.

Axisymmetric inertial fluid oscillations in a rotating spherical shell previously were observed in [4, 5, 14, 15]. For example, in a recent paper [5] the wave beams generated by librations of either the inner core or the outer shell at frequencies \( n = 1.0 \) and \( n = \sqrt{2} \) were numerically studied. At the selected frequencies the conical shear layers spawned from the critical latitudes propagated into the fluid bulk and formed a closed trajectory. In our case the type of harmonic forcing is non-axisymmetric but the main properties of the oscillating flow are consistent with previous experiments and calculations for different problem statements.
Figure 5. The time dependence of the space-averaged pulsational component of the velocity in the meridional plane. The average value of the velocity oscillations is represented by the horizontal dashed line; vertical lines indicate the period of the core oscillations.

The full instantaneous velocity is used to characterize the intensity of the oscillatory flow in the meridional plane. Since a part of the plane in experiments is located in the geometric shadow and is not illuminated by a laser sheet, a half-ring area bounded by the equator is used for calculations. The speed value is calculated using the following expression

\[ \bar{v} = \frac{1}{S} \int_S \sqrt{u_x^2 + u_y^2} \, ds / S, \]

where \( u_x \) and \( u_y \) are the horizontal and vertical components of the velocity, and \( S \) is the sectional area presented in figure 4. An example of a typical dependence of speed \( \bar{v} \) on time for the oscillation frequency \( n = 0.90 \) is presented in figure 5. It can be seen that the dependence is periodic, while during one oscillation the integral velocity reaches its maximum and minimum values twice, which corresponds to the four velocity fields presented in figure 4. It should be noted that the value of the pulsation rate for the same phases practically does not change with time. The time-averaged velocity value \( \bar{v} \) is used as a characteristic of the complete dynamic response of the fluid to perturbations introduced by the core. For the calculations using data obtained by at least ten-period oscillations.

To compare the results of experiments obtained at different values of amplitude and frequency, the time-averaged velocity of the fluid is normalized to the pulsation velocity of the core \( b(\Omega_{vib} - \Omega) \). Thus, the value of the dynamic response of the fluid is expressed in the amount of energy that is pumped into the system. Figure 6 shows the dependence of the parameter \( \bar{v}/b(\Omega_{vib} - \Omega) \) on the dimensionless frequency of vibrations \( n \). It can be seen that the dependence is a non-monotonic with a strongly pronounced peak. In the frequency range \( n = 0.80 \div 1.00 \) the average rate of pulsations practically does not change and takes the value \( \approx 1.5 \). At the same time, the direction of the group speed of wave beams changes in the range of \( \theta = 23 \div 30^\circ \), and the instantaneous flow structure is generally similar to that shown in figure 4. In the frequency range \( n = 1.00 \div 1.10 \) occurs a local reduction in the pulsation rate, which is replaced by a sharp rise when the dimensionless frequency \( n \) increases. The maximum value of the parameter \( \bar{v}/b(\Omega_{vib} - \Omega) = 3 \) is reached at the frequency \( n = 1.20 \). PIV visualization of the instantaneous velocity field shows that for \( n = 1.20 \) the flow structure is qualitatively different from that shown in figure 4. In some phases of the oscillations, the individual wave beams are not observed, while the fluid performs an intensive movement in the entire spherical shell (figure 7). The appearance of large-scale oscillating vortices is associated with the spatial resonance of inertial waves upon
multiple reflections from the outer and inner shells. In this case, we can talk about the resonant excitation of one of the inertial eigenmodes with an azimuthal number \( m = 1 \). As a rule, similar flow modes lead to a strong nonlinear response. For example, in the case of tidal forcing with azimuthal wave number \( m = 2 \) \([2, 3]\) at certain frequencies an intensive axisymmetric zonal flow was generated. A similar steady flow has been generated in the case under consideration, but a description of these results is beyond the scope of this work.

5. Conclusions
The response of the fluid in a rotating sphere under the inner core oscillations has been experimentally studied by PIV-method. Circular oscillations of the core relative to the cavity (azimuthal wave number \( m = 1 \)) are set by an external inertial force field. The experiments were carried out in a range of dimensionless frequencies \( n = 0.5 \div 1.5 \) that correspond to the leading oscillations. The main flow is a set of inertial waves generated at critical latitudes on the inner core boundary. These waves are non-axisymmetric and propagate in the fluid bulk along the characteristic conical surfaces. The time dependence of the pulsation velocity field in the meridional plane has been measured. The inertial wave propagates in the azimuthal direction so that the rate of flow oscillations in a fixed plane periodically changes between two values. At frequency \( n = 1.20 \) inertial waves experience spatial resonance due to which inertial mode with \( m = 1 \) is excited.

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