Mutual information, bit error rate and security in Wójcik’s scheme

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In this paper the correct calculations of the mutual information of the whole transmission, the quantum bit error rate (QBER) are presented. Mistakes of the general conclusions relative to the mutual information, the quantum bit error rate (QBER) and the security in Wójcik’s paper [Phys. Rev. Lett. 90, 157901(2003)] have been pointed out.

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After the pioneering work of Bennett and Brassard published in 1984[1], a variety of quantum secret communication protocols have been proposed[ for a review see [2]]. Recently, a quite different quantum cryptographic protocol (namely, the ‘ping-pong’ protocol) has been proposed by Boström and Felbinger[3], which allows the generation of a deterministic key or even direct secret communication. The protocol has been claimed to be secure and experimentally feasible. However, since the security of the ‘ping-pong’ protocol can be impaired as far as considerable quantum losses are taken into account, very recently Wójcik has presented an undetectable eavesdropping scheme on the ‘ping-pong’ protocol[4]. The aims of this paper are as follows: (1) to present the correct calculations of various mutual information (i.e., the mutual information between the legitimate sender (Alice) and the legitimate receiver (Bob) and the mutual information between Alice and the eavesdropper (Eve)) and the quantum bit error rates (QBERs, i.e., Bob’s BQER and Eve’s QBER); (2) to point out the mistakes relative to the mutual information, the QBERs and the security in Wójcik’s paper and correct them.

Since Wójcik’s scheme is a realistic scheme, all the total numbers of Alice’s bits, Bob’s bits and Eve’s bits should be finite in his paper. This is important in pointing out the mistakes in Wójcik’s paper. This can be seen later. The mutual information in Wójcik’s paper is only and essentially a single-bit mutual information. This physical quantity is used inappropriately in Wójcik’s paper to stand for the mutual information of the whole transmissions (multi bits), alternatively, the mutual information of the whole transmissions (multi bits) are not worked out appropriately in Wójcik’s paper. This can also be seen later. So, in this paper, first, let us give the formulae of the mutual information of the whole transmissions and the QBERs. Incidentally, one should keep in mind that, provided that Alice’s bits are given, the mutual information between Alice and Bob (Eve) and the QBER in Bob’s (Eve’s) bits should be completely determined by the bits Bob (Eve) obtains.

Let $J$ be the total number of Alice’s bits and $J_0$ be the number of all ‘0’ bits and $J_1$ be the number of all ‘1’ bits in Alice’s bits. Obviously, $J = J_0 + J_1$. Let $a_0(a_1)$ be the rate of the ‘0’ (‘1’) bit in Alice’s bits, then $a_0 = J_0/J$ and $a_1 = J_1/J = 1 - a_0$. If an assumption that $a_0 = a_1 = 1/2$ is employed, then $J_0 = J_1 = J/2$. In this paper such an assumption is employed hereafter.

Let $M$ be the total number of Bob’s bits and $M_0$ be the number of all ‘0’ bits and $M_1$ be the number of all ‘1’ bits in Bob’s bits. Obviously, $M = M_0 + M_1$. Let $b_0(b_1)$ be the rate of the ‘0’
('1') bit in Bob’s bits, then \( b_0 = M_0/M \) and \( b_1 = M_1/M = 1 - b_0 \).

Obviously, only if an ideal quantum channel is assumed, then \( M = J \); otherwise, \( M \) should be less than \( J \) due to the channel losses. In this paper, later, \( J \) is assumed to be the effective number of Alice’s bits which can be transmitted to Bob. In this case \( M = J \).

Let \( L_{00} \) be the total number extracted from Alice’s \( J_0 \) '0' bits and Bob’s \( M_0 \) '0' bits in the case that when Alice sends a '0' bit Bob accordingly gets a '0' bit by his measurement. \( L_{00} \) is named as the number of the pair \((0,0)\). Similarly, \( L_{10}, \ L_{01} \) and \( L_{11} \) can be defined. Their rates can be worked out as follows:

\[
e_{00} = L_{00}/J, \quad e_{01} = L_{01}/J, \quad e_{10} = L_{10}/J, \quad e_{11} = L_{11}/J.
\]  

(1)

According to above definitions, the following relations can be built up easily:

\[
L_{00} + L_{01} = J_0 = J/2,
\]  

(2)

\[
L_{10} + L_{11} = J_1 = J/2,
\]  

(3)

\[
L_{00} + L_{10} = M_0,
\]  

(4)

\[
L_{01} + L_{11} = M_1 = J - M_0.
\]  

(5)

Let \( Q_b \) be the number of the wrong bits in Bob’s bits comparing with Alice’s bits. Easily, one can obtain \( Q_b = L_{01} + L_{10} \). Accordingly, the QBER in Bob’s bits can be obtained:

\[
q_b = Q_b/J = (L_{01} + L_{10})/J.
\]  

(6)

Taking advantage of equations 1-6, the following rates can be arrived at

\[
\begin{cases} 
    e_{00} = (2b_0 + 1 - 2q_b)/4, & e_{01} = (1 - 2b_0 + 2q_b)/4, \\
    e_{10} = (2b_0 - 1 + 2q_b)/4, & e_{11} = (3 - 2b_0 - 2q_b)/4.
\end{cases}
\]  

(7)

It seems that the values of \( b_0 \) and \( q_b \) can be chosen freely in the domain \([0,1]\), however, since \( 0 \leq e_{00}, e_{01}, e_{10}, e_{11}, b_0, q_b \leq 1 \), the following limitations exist:

\[
\begin{cases} 
\frac{1}{4} - q_b \leq b_0 \leq \frac{1}{4} + q_b, \quad 0 \leq q_b \leq \frac{1}{2}; \\
q_b - \frac{1}{4} \leq b_0 \leq \frac{3}{4} - q_b, \quad \frac{1}{2} \leq q_b \leq 1.
\end{cases}
\]  

(8)

According to the definition of the mutual information, one can obtain the mutual information between Alice and Bob as follows:

\[
I_{AB} = H(A : B) = H(A) + H(B) - H(A, B)
= -\frac{1}{2} \log_2 \frac{1}{2} - \frac{1}{2} \log_2 \frac{1}{2} - b_0 \log_2 b_0 - (1 - b_0) \log_2 (1 - b_0)
+ \frac{2b_0 + 1 - 2q_b}{4} \log_2 \frac{2b_0 + 1 - 2q_b}{4} + \frac{1 - 2b_0 + 2q_b}{4} \log_2 \frac{1 - 2b_0 + 2q_b}{4}
+ \frac{2b_0 - 1 + 2q_b}{4} \log_2 \frac{2b_0 - 1 + 2q_b}{4} + \frac{3 - 2b_0 - 2q_b}{4} \log_2 \frac{3 - 2b_0 - 2q_b}{4},
\]  

(9)

which is a function of \( b_0 \) and \( q_b \). Once Bob finishes his measurements, then in his bits the \( b_0 \) is determined and accordingly the \( q_b \) is determined provided that Alice’s bits are given. Up to now, the above equation has established the relation between the mutual information and the QBER.
Let us see the properties of the equation (9): (a) If \( q_b = 0 \), according to the equation (8) one can obtain \( b_0 = 1/2 \). Substituting \( q_b = 0 \) and \( b_0 = 1/2 \) into the equation (9), one can obtain \( I_{AB} = 1 \). This is easily understood. Since Bob gets the whole bits correctly, it is sure for him to get the whole information. (b) If \( q_b = 1/2 \), then \( I_{AB} = 0 \) disregarding the value of \( b_0 \) completely. This also can be seen from figure 1. In fact, when Bob gets his bits simply by guess instead of his measurements, it is possible for him to get \( J/2 \) wrong bits (i.e., \( q_b = 1/2 \)). In this case, he should get no information. Incidentally, although it is very possible for Bob to get \( J/2 \) wrong bits (i.e., \( q_b = 1/2 \)) by guess, there are other possibilities. (c) Substituting \( 1 - q_b \) for \( q_b \) in the equation (9), the equation (9) does not change. This means that if Bob can get the mutual complementary bits of his bits he may have the same mutual information. This can be easily seen from figure 1. The figure is symmetric about the \( q_b = 1/2 \). Specifically, in both cases of \( q_b = 0 \) and \( q_b = 1 \), \( I_{AB} = 1 \). Therefore, when one uses the mutual information as the criteria to judge how much information Bob has obtained, an assumption that the mutual complementary bits should correspond to the same information (e.g., if ‘00100’ defines a character, then ‘11011’ as the former’s mutual complementary bits should define the same character) is employed; otherwise, such a criteria is incorrect. Since the assumption is disadvantageous for the message transmission in reality, the mutual information is not the quantity which is good enough to characterize the successfully transmitted message. By the way, the strategy combining the mutual information with the QBER is feasible as an improvement. In addition, as mentioned before, once Bob gets more or less wrong bits relative to \( J/K \), combining the mutual information with the QBER is feasible as an improvement. In addition, as mentioned before, once Bob gets more or less wrong bits relative to \( J/K \), it is possible that two different \( b_0 \)’s do not correspond to a same \( I_{AB} \) but a same \( q_b \) (e.g. see the lines 7 and 8 in table 1), the . This means that the QBER is not a suitable quantity in characterizing the amount of the information Bob gains from Alice.

Similarly, one can work out the mutual information between Alice and Eve. Let \( K \) be the total number of Eve’s bits and \( K_0 \) be the number of all ‘0’ bits and \( K_1 \) be the number of all ‘1’ bits in Eve’s bits. Obviously, \( K = K_0 + K_1 \). Let \( e_0(e_1) \) be the rate of the ‘0’ (‘1’) bit in Eve’s bits, then \( e_0 = K_0/K \) and \( e_1 = K_1/K = 1 - e_0 \).

Let \( N_{00} \) be the total number extracted from Alice’s \( J_0 \) ‘0’ bits and Eve’s \( K_0 \) ‘0’ bits in the case that when Alice sends a ‘0’ bit Eve accordingly gets a ‘0’ bit by her eavesdropping. \( N_{00} \) is named as the number of the eavesdropping pair (0,0). Similarly, \( N_{10}, N_{01} \) and \( N_{11} \) can be defined. Their rates can be worked out as follows:

\[
d_{00} = N_{00}/K, \quad d_{01} = N_{01}/K, \quad d_{10} = N_{10}/K, \quad d_{11} = N_{11}/K. \tag{10}
\]

Let \( Q_e \) be the number of the wrong bits in Eve’s bits comparing with Alice’s bits. Easily, one can obtain \( Q_e = N_{01} + N_{10} \). Accordingly, the QBER in Eve’s bits can be obtained:

\[
q_e \equiv Q_e/J = (N_{01} + N_{10})/J. \tag{11}
\]

Assume Eve can attack all the bits, then \( K = J \). According to the definition of the mutual information, one can obtain the mutual information between Alice and Eve as follows:

\[
I_{AE} \equiv H(A : E) = H(A) + H(E) - H(A, E) = -\frac{1}{2} \log_2 \left( \frac{1}{2} \right) - \frac{1}{2} \log_2 \left( \frac{1}{2} \right) - e_0 \log_2 e_0 - (1 - e_0) \log_2 (1 - e_0) + \frac{2e_0 + 1 - 2q_e}{4} \log_2 \frac{2e_0 + 1 - 2q_e}{4} + \frac{1 - 2e_0 + 2q_e}{4} \log_2 \frac{1 - 2e_0 + 2q_e}{4}.
\]
Let us turn to point out the mistakes in Wójcik’s paper. First, see two obvious facts:

(i) in Wójcik’s paper, Bob’s QBER and Eve’s QBER are precisely 1/4. Then one would like to ask how many wrong bits in Bob’s (Eve’s) bits provided that the total bit number $M$ is odd or $2I + 2$. Assuming $M = 201, 202, 203$, whether the corresponding answers are $50.25, 50.50, 50.75$ respectively? If so, how ridiculous they are for the numbers of the wrong bits are noninteger. It is the wrong QBER=1/4 that leads to the ridiculous results.

(ii) According to Wójcik’s figure 4, one can find when the channel transmission efficiency is zero, both Eve and Bob still can get information from Alice and Eve’s is larger than Bob’s. Another ridiculous result!

All these have obviously shown that there are really mistakes in Wójcik’s paper. In fact, there are more mistakes in his paper. These can be seen as follows. In the ping-pong protocol with ideal quantum channel, Bob’s bits are obtained uniquely and deterministically, which are same as Alice’s. For example, if Alice’s bits are ‘100110’, then Bob gets ’100110’ after his measurements. In Wójcik eavesdropping scheme, due to Eve’s attacks with help of channel losses, Bob’s (Eve’s) bits are not unique in theory, and only after his (her) measurements his (her) bits are determined. For an example: Let ‘u’ (‘s’) be Eve’s attack without (with) the symmetry operation. Assume Alice’s bits are ’100110’ and Eve’s attacks are ‘susuus’. Also Assume that the transmission efficiency η of the quantum channel is not greater than 50%, for in this case Eve can attack also the bits. According to Wójcik’s scheme, taking advantage of the following conditional probability distributions:

\[
\begin{align*}
J_{00}^u &= 1, \quad J_{01}^u = J_{10}^u = J_{11}^u = 0, \quad J_{10}^s = J_{11}^s = 1/4; \\
J_{00}^s &= J_{01}^s = J_{10}^s = J_{11}^s = 1/4, \quad J_{10}^s = J_{11}^s = 0;
\end{align*}
\]

in theory it is possible for Eve to get any one of the following batches: ’100110’, ’100111’, ’100100’, ’100101’, ’100010’, ’100011’, ’100000’, ’100001’, ’101100’, ’101101’, ’101110’, ’101111’, ’101000’, ’101001’, ’101010’ and ’101011’. According to equation (12), it is easy to work out the mutual information for each batch of bits. Obviously these batches do not lead to a same value of $I_{AE}$ and a same QBER (See table 1). This example denies Wójcik’s conclusion that the $I_{AE}$ is always 0.311 and the QBER is always 1/4. It is very easy to be understood that when Eve gets different batches of bits it is quite possible for her to have different mutual information with Alice (See table 1). However, which batch of bits Eve will get on earth by his measurements can not be determined beforehand, for each one can occur with its own probability (See table 1). Only after Eve’s measurements, one can know which one it is exactly. From table 1, one can also deny Wójcik’s another general conclusion that $I_{AE}$ is always larger than $I_{AB}$, for in some cases $I_{AE} = 0$ (See Table 1). So there must be mistakes in Wójcik’s calculations of the mutual information. I think it is the confusion between the single-bit mutual information and the multi-bits mutual information which leads to the mistakes. Hence, intuitively, the security estimation based on the wrong mutual information in Wójcik’s paper is also not reliable anymore.

Let $J_u$($J_s$) be the number of Alice’s bits suffering the ‘u’ (‘s’) attacks, then $J = J_u + J_s \equiv J_{u0} + J_{u1} + J_{s0} + J_{s1}$. Further, still assume that after Eve’s ‘susuus’ attacks, Eve gets her ’100110’ from Alice’s ’100110’. Then $J_{u0} = 1$, $J_{u1} = 2$, $J_{s0} = 2$, $J_{s1} = 1$, $N_{00}^u = 1, N_{01}^u = 0, N_{10}^u = 0, N_{11}^u = 2$, $N_{00}^s = 2, N_{01}^s = 0, N_{10}^s = 0, N_{11}^s = 1$. The extracted probability distributions of the eavesdropping...
pairs \((0,0),(0,1),(1,0)\) and \((1,1)\) can be calculated as follows:

\[
\begin{align*}
& t_{00}^u = \frac{N_{00}^u}{J_{00}} = 1, \quad t_{01}^u = \frac{N_{01}^u}{J_{01}} = 0, \quad t_{10}^u = \frac{N_{10}^u}{J_{10}} = 0, \quad t_{11}^u = \frac{N_{11}^u}{J_{11}} = 1; \\
& t_{00}^s = \frac{N_{00}^s}{J_{00}} = 1, \quad t_{01}^s = \frac{N_{01}^s}{J_{01}} = 0, \quad t_{10}^s = \frac{N_{10}^s}{J_{10}} = 0, \quad t_{11}^s = \frac{N_{11}^s}{J_{11}} = 1. \quad (14)
\end{align*}
\]

They are apparently different with the conditional probability distributions:

\[
\begin{align*}
& t_{00}^u = p_{00}^u + p_{001}^u, \quad p_{01}^u = p_{010}^u + p_{011}^u; \quad t_{01}^u = \frac{p_{010}^u}{J_{01}} = 0, \quad t_{10}^u = \frac{p_{10}^u}{J_{10}} = 0, \quad t_{11}^u = \frac{p_{11}^u}{J_{11}} = 1; \\
& t_{00}^s = p_{00}^s + p_{001}^s, \quad p_{01}^s = p_{010}^s + p_{011}^s; \quad t_{01}^s = \frac{p_{010}^s}{J_{01}} = 0, \quad t_{10}^s = \frac{p_{10}^s}{J_{10}} = 0, \quad t_{11}^s = \frac{p_{11}^s}{J_{11}} = 1. \quad (15)
\end{align*}
\]

The essential reason is that the number \(J = 6\) of the whole bits is finite. By the way, since Wojwick’s scheme is a realistic scheme, \(J\) should be finite. Generally speaking, when \(J\) is large enough, then the extracted probability distributions may be close to the conditional probability distributions. Only when all the \(J, J_{u1}\) and \(J_{s0}\) are infinite, the extracted probability distributions are equivalent to the conditional probability distributions, that is,

\[
\begin{align*}
& t_{00}^u = \frac{N_{00}^u}{J_{00}} = p_{00}^u + p_{001}^u = 1, \quad t_{01}^u = \frac{N_{01}^u}{J_{01}} = p_{010}^u + p_{011}^u = 0, \\
& t_{10}^u = \frac{N_{10}^u}{J_{10}} = p_{10}^u + p_{101}^u = 1, \quad t_{11}^u = \frac{N_{11}^u}{J_{11}} = p_{110}^u + p_{111}^u = 1; \\
& t_{00}^s = \frac{N_{00}^s}{J_{00}} = p_{00}^s + p_{001}^s = 1, \quad t_{01}^s = \frac{N_{01}^s}{J_{01}} = p_{010}^s + p_{011}^s = 1, \\
& t_{10}^s = \frac{N_{10}^s}{J_{10}} = p_{10}^s + p_{101}^s = 1, \quad t_{11}^s = \frac{N_{11}^s}{J_{11}} = p_{110}^s + p_{111}^s = 1. \quad (16)
\end{align*}
\]

Assume that \(J_{u0} = J_{u1} = J_{s0} = J_{s1} = J/4\) (same hereafter), then one can obtain

\[
\begin{align*}
& e_0 = K_0/J = (N_{00} + N_{10})/J = (N_{00}^u + N_{10}^u + N_{00}^s + N_{10}^s)/J = 1/2, \\
& q_0 = Q_c/J = (Q_{cu} + Q_{cs})/J = (N_{00}^u + N_{10}^u + N_{00}^s + N_{10}^s)/J = 1/4. \quad (17)
\end{align*}
\]

Substituting \(e_0 = 1/2\) and \(q_0 = 1/4\) into the equation (12), then one can obtain \(I_{AE} = \frac{1}{4} \log_2 3 - 1 \approx 0.189\), which is different from the value Wojwick claims. This is easily understood. As I have mentioned before, in Wojwick’s paper, the mutual information are essentially single-bit mutual information. Since Eve knows exactly when the ‘s’ operations are performed, for each bit the mutual information can be easily worked out and really not change. However, such single-bit mutual information is not suitable for representing the multi-bit mutual information. From my deduction, one can see it is the inappropriate use of the mutual information which leads to the wrong result in Wojwick’s paper. Accordingly, there are mistakes in Wojwick’s figure 4. Intuitively, the security estimation based on the wrong mutual information in Wojwick’s paper is also not reliable anymore.

To summarize, I have presented the calculations of the multi-bit mutual information and the QBERs. A number of mistakes in Wojwick’s have been found and pointed out.

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Table 1 The possibility, the QBER, the rate of '0' bit in Eve’s bits and the mutual information between Alice and Eve for possible batches of bits.

| Eve’s bits possibility | q | $e_0$ | $I_{AB}$ |
|------------------------|---|------|---------|
| '100110'               | 1/16 | 0 | 1/2 | 1 |
| '100111'               | 1/16 | 1/6 | 1/3 | 0.459 |
| '100100'               | 1/16 | 1/6 | 2/3 | 0.459 |
| '100101'               | 1/16 | 1/3 | 1/2 | 0.082 |
| '100010'               | 1/16 | 1/6 | 2/3 | 0.459 |
| '100011'               | 1/16 | 1/3 | 1/2 | 0.082 |
| '100000'               | 1/16 | 1/3 | 5/6 | 0.134 |
| '100001'               | 1/16 | 1/3 | 2/3 | 0.093 |
| '101100'               | 1/16 | 1/3 | 1/2 | 0.082 |
| '101101'               | 1/16 | 1/2 | 1/3 | 0 |
| '101110'               | 1/16 | 1/6 | 1/3 | 0.459 |
| '101111'               | 1/16 | 1/3 | 1/6 | 0.093 |
| '101000'               | 1/16 | 1/2 | 2/3 | 0 |
| '101001'               | 1/16 | 2/3 | 1/2 | 0.082 |
| '101010'               | 1/16 | 1/3 | 1/2 | 0.082 |
| '101011'               | 1/16 | 1/2 | 1/3 | 0 |

Alice’s bits are '100110'. Eve’s attacks are 'susus'.

The transmission efficiency $\eta$ of the quantum channel is assumed to be not greater than 50%.

Caption

Figure 1  The mutual information between Alice and Bob as a function of $q$ and $b_0$. See text for $q$ and $b_0$. (a) surface diagram; (b) contour diagram.
FIG. 1:

fig. 1(a)
fig. 1(b)

FIG. 2: