Induced vacuum energy-momentum tensor in the background of a $d - 2$ - brane in $d + 1$ - dimensional space-time

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Abstract

Charged scalar field is quantized in the background of a static $d - 2$ - brane which is a core of the magnetic flux lines in flat $d + 1$ - dimensional space-time. We find that vector potential of the magnetic core induces the energy-momentum tensor in the vacuum. The tensor components are periodic functions of the brane flux and holomorphic functions of space dimension. The dependence on the distance from the brane and on the coupling to the space-time curvature scalar is comprehensively analysed.

1 Introduction

Since Casimir’s seminal paper [1] it has become clear that the effect of external boundary conditions in quantum field theory can be exposed as the emergence of a non-zero vacuum expectation value of the energy-momentum tensor (see, e.g. Refs. [2, 3]). This may have far reaching consequences; in particular, the vacuum energy-momentum tensor serves as a source of gravitation, and the so-called self-consistent cosmological models of the Universe are proposed, where matter is absent and its role is played by the vacuum quantum effects [4].

In this respect it seems to be of interest to look for various situations when the vacuum energy-momentum tensor is calculable and finite. Let $X$ be the base space manifold of dimension $d$ and $Y$ be a submanifold of dimension less than $d$. The matter field is quantized under certain boundary condition imposed at $Y$. In most

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implications of the Casimir effect, \( Y \) is chosen to be noncompact disconnected (e.g. two parallel infinite plates, as generically in Ref. \[ \text{[1]} \]) or closed compact (e.g. box or sphere), see Ref. \[ \text{[3]} \]. In the present paper we choose \( Y \) to be noncompact connected and possessing dimension \( d - 2 \), i.e. a \( d - 2 \)-brane in \( d \)-dimensional space. If such a brane is filled with the magnetic flux lines, then it can be regarded as a generalization of the Bohm-Aharonov \[ \text{[5]} \] singular magnetic vortex in 3-dimensional space. If the matter field vanishes at \( Y \), then the region where the matter field is nonvanishing (out of \( Y \)) does not overlap with the region where the background magnetic field is nonvanishing (inside \( Y \)). Thus there is no effect of the background field on the matter field in the framework of classical theory, and such an effect, if exists, is of purely quantum nature. Conventional Bohm-Aharonov effect pertains to the quantum-mechanical framework \[ \text{[5]} \]. As is clear from the above, our interest will be in the quantum-field-theoretical framework (i.e. vacuum polarization in the background of the brane), which, therefore, may be generally denoted as the Casimir-Bohm-Aharonov effect, see also Ref. \[ \text{[6]} \].

Throughout the present paper, we restrict ourselves to the case of scalar matter. A peculiarity of this case is that the energy-momentum tensor depends on the coupling (\( \xi \)) of the scalar field to the scalar curvature of space-time even when space-time is flat. If scalar field is massless, then conformal invariance of the theory is achieved at \( \xi = \xi_c \), where \[ \text{[7, 8, 9]} \]

\[
\xi_c = \frac{d - 1}{4d}; \tag{1.1}
\]

note that \( \xi_c \) varies from 0 to 1/4 when \( d \) varies from 1 to \( \infty \). Our analysis of vacuum polarization effects in the background of the brane will be carried out for arbitrary values of \( \xi \); however results will be most impressive in the case of conformal coupling, \( \xi = \xi_c \). We shall find out components of the induced vacuum energy-momentum tensor as functions of the brane flux, distance from the brane and space dimension. Expressions for components are especially simple in form in the case of massless scalar field, whereas in the massive case they are presented in terms of convergent integrals of the Macdonald functions.

In the next section which is also introductory but more detailed, a general definition of the energy-momentum tensor for the quantized charged scalar field is reviewed and a starting expression for its regularized vacuum expectation value in the background of the brane is given. In Section 3 which is central in the paper, the regularized vacuum tensor components are computed, and in Section 4 the renormalized vacuum tensor components are obtained. Various aspects of the latter result are examined in the following sections: asymptotic behaviour at large and small distances from the brane (Section 5), expressions at fixed values of the brane flux when tensor components have maximal absolute values (Section 6), dependence on the \( \xi \)-parameter (Section 7). Finally, results are summarized in Section 8. Some details in the derivation of results are outlined in Appendices A and B.
2 Energy-momentum tensor and its vacuum expectation value

The energy-momentum tensor for the quantized charged scalar field $\Psi(x)$ is given by expression

$$T_{\mu\nu} = T_{\text{can}}^{\mu\nu} + \xi (g^{\mu\nu} \Box - \nabla^\mu \nabla^\nu - R^{\mu\nu}) \left[ \Psi^\dagger, \Psi \right]_+ ,$$  \hspace{1cm} (2.1)

where

$$T_{\text{can}}^{\mu\nu} = \frac{1}{2} \left[ \nabla^\mu \Psi^\dagger, \nabla^\nu \Psi \right]_+ + \frac{1}{2} \left[ \nabla^\nu \Psi^\dagger, \nabla^\mu \Psi \right]_+ - g^{\mu\nu} L ,$$  \hspace{1cm} (2.2)

and

$$L = \frac{1}{2} \left[ \nabla^\mu \Psi^\dagger, \nabla_\mu \Psi \right]_+ - \frac{1}{2} \left( m^2 + \xi R \right) \left[ \Psi^\dagger, \Psi \right]_+ ,$$  \hspace{1cm} (2.3)

$\nabla_\mu$ is the covariant derivative involving both affine and bundle connections, $\Box = \nabla_\mu \nabla^\mu$ is the covariant d’Alembertian, $R^{\mu\nu}$ is the Ricci tensor and $R = g_{\mu\nu} R^{\mu\nu}$ is the scalar curvature of space-time, signature of space-time metric $g_{\mu\nu}$ is chosen as $(+ - \ldots -)$. Canonical tensor (2.2) is obtained by applying Noether’s theorem to lagrangian (2.3), whereas tensor (2.1) is obtained by variating $L$ (2.3) over metric tensor $g_{\mu\nu}$.

Eq. (2.1) can be rewritten in a form

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + (\xi - 1/4) (g^{\mu\nu} \Box - \nabla^\mu \nabla^\nu) \left[ \Psi^\dagger, \Psi \right]_+ - \xi R^{\mu\nu} \left[ \Psi^\dagger, \Psi \right]_+ ,$$  \hspace{1cm} (2.4)

where

$$\tilde{T}_{\mu\nu} = T_{\text{can}}^{\mu\nu} + \frac{1}{4} \left( g^{\mu\nu} \Box - \nabla^\mu \nabla^\nu \right) \left[ \Psi^\dagger, \Psi \right]_+ ,$$  \hspace{1cm} (2.5)

is the canonical tensor corresponding to lagrangian $\tilde{L}$ which differs from $L$ (2.3) by a total divergence:

$$\tilde{L} = L - \frac{1}{4} \Box \left[ \Psi^\dagger, \Psi \right]_+ = - \frac{1}{4} \left[ \Psi^\dagger, \Box \Psi \right]_+ - \frac{1}{4} \left[ \Box \Psi^\dagger, \Psi \right]_+ - \frac{1}{2} \left( m^2 + \xi R \right) \left[ \Psi^\dagger, \Psi \right]_+ .$$  \hspace{1cm} (2.6)

Both $L$ (2.3) and $\tilde{L}$ (2.6) yield the same equations of motion,

$$\left[ \Box + (m^2 + \xi R) \right] \Psi = 0 , \quad \left[ \Box + (m^2 + \xi R) \right] \Psi^\dagger = 0 ,$$  \hspace{1cm} (2.7)

but $\tilde{L}$ is strictly vanishing on the solutions to the equations of motion. In fact, $\tilde{L}$ is used as a lagrangian in the path integral approach to quantization, since namely $\tilde{L}$ is directly related to the inverse propagator of the quantized scalar field.

If $\Psi$ is a solution to Eq. (2.7), then one gets

$$\tilde{T}_{\mu\nu} = \frac{1}{2} \left[ \nabla^\mu \Psi^\dagger, \nabla^\nu \Psi \right]_+ + \frac{1}{2} \left[ \nabla^\nu \Psi^\dagger, \nabla^\mu \Psi \right]_+ - \frac{1}{4} \nabla^\mu \nabla^\nu \left[ \Psi^\dagger, \Psi \right]_+ ,$$  \hspace{1cm} (2.8)

and

$$g_{\mu\nu} T_{\mu\nu} = \left( \xi d - \frac{d-1}{4} \right) \Box \left[ \Psi^\dagger, \Psi \right]_+ + m^2 \left[ \Psi^\dagger, \Psi \right]_+ ,$$  \hspace{1cm} (2.9)
where $d$ is the dimension of space. Thus, there is a distinctive value of parameter $\xi$ ($\xi = \xi_c$, see Eq. (1.1)) under which the trace of the energy-momentum tensor becomes proportional to the mass squared,

$$g_{\mu\nu} T^{\mu\nu}|_{\xi = \xi_c} = m^2 [\Psi^*, \Psi]_+,$$

and the tracelessness of $T^{\mu\nu}$ and conformal invariance are achieved in the massless limit ($m = 0$).

In the case of a static background ($\nabla_0 = \partial_0, \ g_{00} = 1$), the operator of the quantized charged scalar field is represented in the form

$$\Psi(x^0, x) = \sum_{\lambda} \int \frac{1}{\sqrt{2E_\lambda}} \left[ e^{-iE_\lambda x^0} \psi_\lambda(x) a_\lambda + e^{iE_\lambda x^0} \psi_{-\lambda}(x) b^\dagger_\lambda \right].$$

(2.11)

Here, $a^\dagger_\lambda$ and $a_\lambda$ ($b^\dagger_\lambda$ and $b_\lambda$) are the scalar particle (antiparticle) creation and annihilation operators satisfying commutation relation; $\lambda$ is the set of parameters (quantum numbers) specifying the state; $E_\lambda = E_{-\lambda} > 0$ is the energy of the state; symbol $\sum_{\lambda}$ denotes summation over discrete and integration (with a certain measure) over continuous values of $\lambda$; wave functions $\psi_\lambda(x)$ are the solutions to the stationary equation of motion,

$$\{-\Delta + [m^2 + \xi R(x)]\} \psi_\lambda(x) = E_\lambda^2 \psi(x),$$

(2.12)

$\Delta = \nabla^2$ is the covariant laplacian. For components of the vacuum expectation value of the energy-momentum tensor,

$$t^{\mu\nu} = \langle \text{vac} | T^{\mu\nu} | \text{vac} \rangle,$$

(2.13)

one gets expressions

$$t^{00} = \sum_{\lambda} \int E_\lambda \psi^*_\lambda(x) \psi_\lambda(x) - (\xi - 1/4) \Delta \sum_{\lambda} E^{-1}_{\lambda} \psi^*_\lambda(x) \psi_\lambda(x),$$

(2.14)

$$t^{jj'} = \frac{1}{2} \sum_{\lambda} \int E^{-1}_{\lambda} \left\{ \left[ \nabla^j \psi_\lambda(x) \right]^* \left[ \nabla^{j'} \psi_\lambda(x) \right] + \left[ \nabla^{j'} \psi_\lambda(x) \right]^* \left[ \nabla^j \psi_\lambda(x) \right] \right\} +$$

$$+ \left\{ \frac{1}{4} g^{jj'}(x) \Delta - \xi \left[ g^{jj'}(x) \Delta + \nabla^j \nabla^{j'} + R^{jj'}(x) \right] \right\} \sum_{\lambda} E^{-1}_{\lambda} \psi^*_\lambda(x) \psi_\lambda(x),$$

(2.15)

Note that the $t^{0j}$ components are vanishing, and relations $R^{00}(x) = 0$ and $\partial_0 [\Psi^*, \Psi]_+ = 0$ have been taken into account.

However, relations (2.14) and (2.15) can be regarded as purely formal and, strictly speaking, meaningless: they are ill-defined, suffering from ultraviolet divergencies. The well-defined quantities are obtained by inserting an inverse energy in a sufficiently high power

$$t^{00}_{\text{reg}}(s) = \sum_{\lambda} \int E^{-2s}_{\lambda} \psi^*_\lambda(x) \psi_\lambda(x) - (\xi - 1/4) \Delta \sum_{\lambda} E^{-2(s+1)}_{\lambda} \psi^*_\lambda(x) \psi_\lambda(x),$$

(2.16)
\[ t_{\text{reg}}^{j j'}(s) = \frac{1}{2} \sum_{\lambda} E_\lambda^{-2(s+1)} \left\{ \left[ \nabla^j \psi_\lambda(x) \right]^* \left[ \nabla^{j'} \psi_\lambda(x) \right] + \left[ \nabla^j \psi_\lambda(x) \right]^* \left[ \nabla^{j'} \psi_\lambda(x) \right] \right\} + \right.
\[ + \left\{ \frac{1}{4} g^{jj'}(x) \Delta - \xi \left[ g^{jj'}(x) \Delta + \nabla^j \nabla^{j'} + R^{jj'}(x) \right] \right\} \sum_{\lambda} E_\lambda^{-2(s+1)} \psi^*_\lambda(x) \psi_\lambda(x), \]
\[ j, j' = 0, \ldots, d. \quad (2.17) \]

Sums (integrals) are convergent in the case of \( \text{Re} \, s > d/2 \). Thus, summation (integration) is performed in this case, and then the result is analytically continued to the case of \( s = -1/2 \). This way of dealing with ultraviolet divergencies is known as the zeta function regularization procedure [10, 11, 12].

It is amazing, as is already mentioned in Introduction, that the energy-momentum tensor and, consequently, its vacuum expectation value remain to be dependent on parameter \( \xi \) even in the case of flat space-time (\( R = 0 \)). If scalar field is quantized in the background of a static magnetic field in flat space-time, then the covariant derivative is defined as
\[ \nabla \psi = \left( \partial - i V \right) \psi, \quad \nabla \psi^\dagger = \left( \partial + i V \right) \psi^\dagger, \quad (2.18) \]
and the magnetic field strength takes form
\[ B^{j_1 \ldots j_d-2}(x) = -\varepsilon^{j_1 \ldots j_d} \partial_{j_d-1} V_{j_d}(x), \quad (2.19) \]
where \( V(x) \) is the bundle connection (vector potential of the magnetic field), and \( \varepsilon^{j_1 \ldots j_d} \) is the totally antisymmetric tensor, \( \varepsilon^{12 \ldots d} = 1 \).

In the present paper we consider the bundle curvature (magnetic field strength) to be nonvanishing in the \( d - 2 \) - brane (i.e. point in the \( d = 2 \) case, line in the \( d = 3 \) case, plane in the \( d = 4 \) case, and \( d - 2 \) - hypersurface in the \( d > 4 \) case). Denoting the location of the \( d - 2 \) - brane by \( x^1 = x^2 = 0 \), one gets
\[ B^{3 \ldots d}(x) = 2\pi \Phi \delta(x^1) \delta(x^2), \quad (2.20) \]
where \( \Phi \) is the total flux (in the units of \( 2\pi \)) of the bundle curvature; then the bundle connection can be chosen in the form:
\[ V^1(x) = -\Phi \frac{x^2}{(x^1)^2 + (x^2)^2}, \quad V^2(x) = \Phi \frac{x^1}{(x^1)^2 + (x^2)^2}, \]
\[ V^j(x) = 0, \quad j = 3, \ldots, d. \quad (2.21) \]
The complete set of regular solutions to Eq. (2.12) in background (2.20)-(2.21) is given by functions (see, e.g., Ref [3])
\[ \psi_{knp}(x) = (2\pi)^{\frac{d-1}{2}} J_{|n-\Phi|}(kr) e^{inx} e^{ipx_{d-2}}, \quad (2.22) \]
where
\[ 0 < k < \infty, \quad n \in \mathbb{Z}, \quad -\infty < p^j < \infty, \quad j = 3, \ldots, d, \quad (2.23) \]
$J_{\mu}(u)$ is the Bessel function of order $\mu$, \( r = \sqrt{(x^1)^2 + (x^2)^2} \), \( \varphi = \arctan(\frac{x^2}{x^1}) \), \( x_{d-2} = (0, 0, x^3, \ldots, x^d) \), and \( \mathbb{Z} \) is the set of integers. Since solutions (2.22) correspond to the continuous spectrum \( E_{knp} = \sqrt{p^2 + k^2 + m^2} > m \), they obey orthonormality condition

$$
\int d^d x \, \psi^*_{knp} (x) \psi^*_{k'n'p'} (x) = \frac{1}{k} \delta (k - k') \delta_{nn'} \delta (p - p').
$$

(2.24)

Taking all the above into account, we get following expressions for the nonvanishing components of the regularized vacuum expectation value of the energy-momentum tensor:

$$
t_{00}^{reg}(s) = \tilde{t}_{00}^{reg}(s) + \left( \frac{1}{4} - \xi \right) \Delta_r \tilde{t}_{00}^{reg}(s + 1),
$$

(2.25)

$$
t_{rr}^{reg}(s) = (2\pi)^{1-d} \int d^{d-2} p \int_0^\infty dk \, k \left( p^2 + k^2 + m^2 \right)^{-s-1} \sum_{n \in \mathbb{Z}} \left[ \partial_r J_{|n-\Phi|}(kr) \right]^2
$$

$$
- \left[ \frac{1}{4} \Delta_r - \frac{\xi}{r} \partial_r \right] \tilde{t}_{00}^{reg}(s + 1),
$$

(2.26)

$$
t_{\Phi\Phi}^{reg}(s) = (2\pi)^{1-d} \int d^{d-2} p \int_0^\infty dk \, k \left( p^2 + k^2 + m^2 \right)^{-s-1} \sum_{n \in \mathbb{Z}} (n - \Phi)^2 J_{|n-\Phi|}(kr) -
$$

$$
- r^{-2} \left[ \frac{1}{4} \Delta_r - \frac{\xi}{r} \partial_r \right] \tilde{t}_{00}^{reg}(s + 1),
$$

(2.27)

$$
t_{jj}^{reg}(s) = (2\pi)^{1-d} \int d^{d-2} p \int_0^\infty dk \, k \left( p^2 + k^2 + m^2 \right)^{-s-1} \sum_{n \in \mathbb{Z}} J_{|n-\Phi|}(kr) +
$$

$$
- \left( \frac{1}{4} - \xi \right) \Delta_r \tilde{t}_{00}^{reg}(s + 1), \quad j = 3, d,
$$

(2.28)

where \( \Delta_r = \partial_r^2 + r^{-1} \partial_r \) is the transverse radial part of the laplacian and

$$
\tilde{t}_{00}^{reg}(s) = (2\pi)^{1-d} \int d^{d-2} p \int_0^\infty dk \, k \left( p^2 + k^2 + m^2 \right)^{-s} \sum_{n \in \mathbb{Z}} J_{|n-\Phi|}(kr).
$$

(2.29)

Defining the fractional part of the flux,

$$
F = \Phi - [\Phi], \quad 0 \leq F < 1,
$$

(2.30)

where \([u]\) is the integer part of quantity \( u \) (i.e. the integer which is less than or equal to \( u \)), one can note that tensor components (2.25)-(2.28) are periodic functions of flux \( \Phi \), since they depend only on \( F \) (being symmetric under \( F \to 1 - F \)).
3 Regularized vacuum expectation value

Let us start by considering quantity

\[ \tilde{t}^{jj}_{reg}(s) = t^{jj}_{reg}(s) \big|_{\xi=1/4}. \]  

(3.1)

Performing the integration over \( p \), one gets

\[ \tilde{t}^{jj}_{reg}(s) = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s + 1 - \frac{d}{2})}{\Gamma(s + 1)} \int_0^\infty dk \frac{k}{(k^2 + m^2)^{\frac{d}{2} - s}} \Sigma_0(kr), \]  

(3.2)

where \( \Gamma(z) \) is the Euler gamma function and

\[ \Sigma_0(kr) = \sum_{n \in \mathbb{Z}} J_{|n - \Phi|}^2(kr). \]  

(3.3)

Using relation (see, e.g., Ref. [13])

\[ \sum_{n \geq 1, n \notin \mathbb{Z}} J_{n+\mu}^2(z) = \mu \int_0^z \frac{d\tau}{\tau} J_{\mu}^2(\tau) - \frac{1}{2} J_{\mu}^2(z), \quad \mu > 0, \]  

(3.4)

the summation over \( n \) in Eq. (3.3) is performed in the case of \( \Phi \neq n \),

\[ \Sigma_0(kr) = \int_0^{kr} d\tau [J_F(\tau)J_{1-F}(\tau) + J_{-F}(\tau)J_{1-F}(\tau)], \quad 0 < F < 1. \]  

(3.5)

Thus, Eq. (3.2) takes form (after integration by parts):

\[ \tilde{t}^{jj}_{reg}(s) = \frac{r}{2(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s + 1)} \int_0^\infty dk \frac{k}{(k^2 + m^2)^{\frac{d}{2} - s}} \times \]  

\[ \times [J_F(kr)J_{1+F}(kr) + J_{-F}(kr)J_{1-F}(kr)]. \]  

(3.6)

Using relations

\[ (k^2 + m^2)^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty dy y^{z-1} \exp \left[-y(k^2 + m^2)\right], \quad \text{Re } z > 0, \]  

(3.7)

and

\[ \int_0^\infty dk e^{-y k^2} J_\mu(kr)J_{\mu-1}(kr) = \frac{1}{2r} \int_0^{r(2y)^{-1}} du e^{-u}|I_{\mu-1}(u) - I_{\mu}(u)|, \quad \mu > 0, \]  

(3.8)
where $I_\mu(u)$ is the modified Bessel function of order $\mu$, we get

$$
\tilde{t}_{ij}^{reg}(s) = \frac{2 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \frac{m^{d-2s}}{\Gamma(s+1)} \int_0^\infty \! du \, e^{-u} [K_F(u) + K_{1-F}(u)] \gamma \left( s - \frac{d}{2}, \frac{m^2r^2}{2u} \right), \quad (3.9)
$$

where

$$
K_\mu(u) = \frac{\pi}{2 \sin(\mu \pi)} [I_{\mu}(u) - I_\mu(u)]
$$

is the Macdonald function of order $\mu$ and

$$
\gamma(z, w) = \int_0^w \! d\tau \, \tau^{z-1} e^{-\tau}
$$

is the incomplete gamma function. Although relation (3.9) has been derived at $\Re s > \frac{d}{2} - 1$, it can be continued analytically to the whole complex $s$-plane. Introducing complementary function

$$
\Gamma(z, w) = \Gamma(z) - \gamma(z, w) = \int_0^\infty \! d\tau \, \tau^{z-1} e^{-\tau},
$$

and decomposing Eq.(3.9) appropriately into a sum of two terms, we get

$$
\tilde{t}_{ij}^{reg}(s) = \frac{m^{d-2s}}{2(4\pi)^{\frac{d}{2}}} \frac{\Gamma \left( s - \frac{d}{2} \right)}{\Gamma(s+1)} - \frac{2 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \frac{m^{d-2s}}{\Gamma(s+1)} \int_0^\infty \! du \, e^{-u} [K_F(u) + K_{1-F}(u)] \Gamma \left( s - \frac{d}{2}, \frac{m^2r^2}{2u} \right) = \frac{m^{d-2s}}{2(4\pi)^{\frac{d}{2}}} \frac{\Gamma \left( s - \frac{d}{2} \right)}{\Gamma(s+1)} \frac{8 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \left( \frac{m}{r} \right)^{\frac{d}{2}-s} \times \int_1^\infty \! \frac{dv}{\sqrt{v^2-1}} \cosh[(2F-1) \arccosh v] v^{s-\frac{d}{2}-1} K_{s-\frac{d}{2}}(2mr^2), \quad (3.10)
$$

where in deriving the last line we have used relations (see, e.g., Ref.[13])

$$
K_\mu(u) = \frac{1}{2} \int_0^\infty \! d\tau \, \tau^{\mu-1} \exp \left[ -\frac{u}{2}(\tau + \tau^{-1}) \right], \quad (3.11)
$$

and

$$
\int_0^\infty \! \frac{d\tau}{\tau^2} \exp \left( -\frac{p}{2\tau} \right) \Gamma \left( q, \frac{c\tau}{2} \right) = \left( \frac{p}{4} \right)^{\frac{q-2}{2}} c^\frac{q}{2} K_q(\sqrt{pc}). \quad (3.12)
$$
In a similar way we get\(^1\)

\[
\tilde{t}^{00}_{\text{reg}}(s) = \frac{m^{d-2s}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} - \\
- \frac{16 \sin(F\pi)}{(4\pi)^{\frac{d}{2} + 1} \Gamma(s)} \left(\frac{m}{r}\right)^{\frac{d}{2}-s} \int_{1}^{\infty} \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \arccosh v] v^{s-\frac{d}{2}-1} K_{\frac{d}{2}-s}(2mr) .
\]

(3.13)

Then we derive relations

\[
- \frac{1}{r} \partial_{r} \tilde{t}^{00}_{\text{reg}}(s + 1) = - \frac{32 \sin(F\pi)}{(4\pi)^{\frac{d}{2} + 1} \Gamma(s + 1)} \left(\frac{m}{r}\right)^{\frac{d}{2}-s} \times \\
\left(\int_{1}^{\infty} \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \arccosh v] v^{1+s-\frac{d}{2}} K_{\frac{d}{2}-s}(2mr) \right) ,
\]

(3.14)

\[
\partial_{r}^{2} \tilde{t}^{00}_{\text{reg}}(s + 1) = - \frac{32 \sin(F\pi)}{(4\pi)^{\frac{d}{2} + 1} \Gamma(s + 1)} \left(\frac{m}{r}\right)^{\frac{d}{2}-s} \int_{1}^{\infty} \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \arccosh v] \times \\
v^{1+s-\frac{d}{2}} \left[ K_{\frac{d}{2}-s}(2mr) - 2mrK_{\frac{d}{2}-s+1}(2mr) \right] .
\]

(3.15)

Consequently, temporal and longitudinal components of the regularized vacuum energy-momentum tensor, Eqs. (2.25) and (2.28), are given by expressions:

\[
\tilde{t}^{00}_{\text{reg}}(s) = \frac{m^{d-2s}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} + \\
+ \frac{16 \sin(F\pi)}{(4\pi)^{\frac{d}{2} + 1} \Gamma(s + 1)} \left(\frac{m}{r}\right)^{\frac{d}{2}-s} \int_{1}^{\infty} \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \arccosh v] \times \\
v^{s-\frac{d}{2}-1} \left\{[-s + (1 - 4\xi)v^2] K_{\frac{d}{2}-s}(2mr) - (1 - 4\xi)mr^3 K_{\frac{d}{2}-s+1}(2mr) \right\} ,
\]

(3.16)

\[
\tilde{t}^{ij}_{\text{reg}}(s) = \frac{m^{d-2s}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s + 1)} - \\
- \frac{16 \sin(F\pi)}{(4\pi)^{\frac{d}{2} + 1} \Gamma(s + 1)} \left(\frac{m}{r}\right)^{\frac{d}{2}-s} \int_{1}^{\infty} \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \arccosh v] \times \\
v^{s-\frac{d}{2}-1} \left\{\left[ \frac{1}{2} + (1 - 4\xi)v^2 \right] K_{\frac{d}{2}-s}(2mr) - (1 - 4\xi)mr^3 K_{\frac{d}{2}-s+1}(2mr) \right\} .
\]

(3.17)

\(^1\)Note that this quantity was first computed in Ref. [6], where it is called as the zeta function density.
Let us turn now to transverse components of the regularized vacuum energy-momentum tensor. Integrating over $p$ in the first lines of Eq. (2.26) and (2.27), we get

$$t_{rr}^{\text{reg}}(s) = \frac{\Gamma(s + 2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(s + 1)} \left[ \frac{\Omega_1(s) - m^2 \Omega_1(s + 1)}{\Gamma(s + 1)} - \left[ \frac{1}{4} \partial_r^2 - \left( \xi - \frac{1}{4} \right) r^{-1} \partial_r \right] \tilde{t}^{00}_{\text{reg}}(s + 1) \right], \quad (3.18)$$

$$t_{\varphi\varphi}^{\text{reg}}(s) = \frac{r^{-2} \Gamma(s + 2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(s + 1)} \left[ \frac{\Omega_2(s) - m^2 \Omega_2(s + 1)}{\Gamma(s + 1)} - r^{-2} \left[ \frac{1}{4} r^{-1} \partial_r - \left( \xi - \frac{1}{4} \right) \partial_r^2 \right] \tilde{t}^{00}_{\text{reg}}(s + 1) \right], \quad (3.19)$$

where

$$\Omega_a(s) = 2 \int_0^\infty dk k \left( k^2 + m^2 \right)^{\frac{d}{2} - s - 1} \Sigma_a(kr), \quad a = 1, 2, \quad (3.20)$$

$$\Sigma_1(kr) = k^{-2} \sum_{n \in \mathbb{Z}} \left[ \partial_r J_{|n-\Phi|}(kr) \right]^2 = \frac{1}{4} \sum_{n \in \mathbb{Z}} \left[ J_{|n-\Phi|+1}(kr) - J_{|n-\Phi|-1}(kr) \right]^2, \quad (3.21)$$

and

$$\Sigma_2(kr) = (kr)^{-2} \sum_{n \in \mathbb{Z}} (n - \Phi)^2 J_{|n-\Phi|}^2(kr) = \frac{1}{4} \sum_{n \in \mathbb{Z}} \left[ J_{|n-\Phi|+1}(kr) + J_{|n-\Phi|-1}(kr) \right]^2. \quad (3.22)$$

Using Eq. (3.21) and relation [13]

$$\sum_{\substack{n \in \mathbb{Z} \cap [1, \mu] \atop n \geq 1}} J_{n+\mu+1}(z) J_{n+\mu-1}(z) = \frac{1}{2} \mu J_{\mu}^2(z) + \frac{1}{2} (1 + \mu) J_{2\nu+\mu}^2(z) - \frac{\mu(1 + \mu)}{z} J_{\mu}(z) J_{-\mu-1}(z), \quad (3.23)$$

we get

$$\Sigma_a(kr) = \frac{1}{2} \left[ \Sigma_0(kr) + \Sigma_a(kr) \right], \quad a = 1, 2, \quad (3.24)$$
where
\[ \tilde{\Sigma}_1(kr) = \frac{1}{2}(1 + F) \left[ J^2_{-F}(kr) - J^2_F(kr) \right] + \frac{1}{2}(2 - F) \left[ J^2_{-1+F}(kr) - J^2_{1-F}(kr) \right] - \frac{F(1 - F)}{kr} [J_F(kr)J_{-1+F}(kr) + J_{-F}(kr)J_{1-F}(kr)] , \] (3.25)

\[ \tilde{\Sigma}_2(kr) = \frac{1}{2}(1 - F) \left[ J^2_{-F}(kr) - J^2_F(kr) \right] + \frac{1}{2} F \left[ J^2_{-1+F}(kr) - J^2_{1-F}(kr) \right] + \frac{F(1 - F)}{kr} [J_F(kr)J_{-1+F}(kr) + J_{-F}(kr)J_{1-F}(kr)] , \] (3.26)

and \( \Sigma_0(kr) \) is given by Eq. (3.3). Thus, Eq. (3.20) takes form
\[ \Omega_0(s) = \tilde{\Omega}_0(s) + \tilde{\Omega}_a(s) , \quad a = 1, 2 , \] (3.27)

where
\[ \Omega_0(s) = \frac{r}{2s - d} \int_0^\infty dk \left( k^2 + m^2 \right)^{\frac{d}{2} - s} [J_F(kr)J_{-1+F}(kr) + J_{-F}(kr)J_{1-F}(kr)] , \] (3.28)

\[ \tilde{\Omega}_a(s) = \int_0^\infty dk k \left( k^2 + m^2 \right)^{\frac{d}{2} - s} \tilde{\Sigma}_a(kr) , \quad a = 1, 2 . \] (3.29)

The integral in Eq. (3.28) has been already encountered during the analysis of \( \bar{t}^{ij}_{reg}(s) \), see Eq. (3.6). Similarly, using relations (3.7), (3.8) and (see Ref. [13])
\[ \int_0^\infty dk k e^{-y^2} J^2_\mu(kr) = \frac{1}{2\sigma^2} \exp \left( -\frac{r^2}{2y} \right) I_\mu \left( \frac{r^2}{2y} \right) , \quad \mu > -1 , \] (3.30)

we get
\[ \Omega_1(s) = \frac{\sin(F\pi)}{\pi \Gamma \left( s + 1 - \frac{d}{2} \right)} \left\{ \frac{1}{4} \left( \frac{r^2}{2} \right)^{s - \frac{d}{2}} \int_0^\infty d\tau \tau^{\frac{d}{2} - s - 2} \exp \left( -\tau - \frac{m^2r^2}{2\tau} \right) \times \left[ (1 + F)K_F(\tau) + (2 - F)K_{1-F}(\tau) \right] - F(1 - F)r^{-2}m^{-2s-2} \times \int_0^\infty du e^{-u}[K_F(u) + K_{1-F}(u)]\gamma \left( 1 + s - \frac{d}{2}, \frac{m^2r^2}{2u} \right) \right\} , \] (3.31)

\[ \tilde{\Omega}_2(s) = \frac{\sin(F\pi)}{\pi \Gamma \left( s + 1 - \frac{d}{2} \right)} \left\{ \frac{1}{4} \left( \frac{r^2}{2} \right)^{s - \frac{d}{2}} \int_0^\infty d\tau \tau^{\frac{d}{2} - s - 2} \exp \left( -\tau - \frac{m^2r^2}{2\tau} \right) \times \left[ (1 - F)K_F(\tau) + FK_{1-F}(\tau) \right] + F(1 - F)r^{-2}m^{-2s-2} \times \int_0^\infty du e^{-u}[K_F(u) + K_{1-F}(u)]\gamma \left( 1 + s - \frac{d}{2}, \frac{m^2r^2}{2u} \right) \right\} . \] (3.32)
Although relations (3.31) and (3.32) have been derived at \( \text{Re } s > \frac{d}{2} - 1 \), they can be continued analytically to the whole complex \( s \)-plane. Using integration by parts in \( \tau \)-integrals and relations (3.11), (3.12) and

\[
e^{-\tau}K_\mu(\tau) = 2 \int_1^\infty dv \left\{ \frac{v}{\sqrt{v^2 - 1}} \cosh[(2\mu - 1) \text{arccosh } v] + \right.
\]

\[
+ \sinh[(2\mu - 1) \text{arccosh } v] \right\} e^{-2\tau v^2}, \quad (3.33)
\]

we get

\[
\tilde{\Omega}_1(s) - m^2\tilde{\Omega}_1(s + 1) =
\]

\[
= \frac{4 \sin(F\pi)}{\pi \Gamma(2 + s - \frac{d}{2})} \left( \frac{m}{r} \right)^{\frac{d}{2} - s} \int_1^\infty \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \text{arccosh } v] \times
\]

\[
\times v^{s-\frac{d}{2}} \left[ vK_{\frac{d}{2} - s}(2mr) - \frac{1}{2} mr(1 + 2v^2)K_{\frac{d}{2} - s + 1}(2mr) \right], \quad (3.34)
\]

\[
\tilde{\Omega}_2(s) - m^2\tilde{\Omega}_2(s + 1) =
\]

\[
= \frac{4 \sin(F\pi)}{\pi \Gamma(2 + s - \frac{d}{2})} \left( \frac{m}{r} \right)^{\frac{d}{2} - s} \int_1^\infty \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \text{arccosh } v] \times
\]

\[
\times v^{s-\frac{d}{2}} \left[ vK_{\frac{d}{2} - s}(2mr) + \frac{1}{2} mr(1 - 2v^2)K_{\frac{d}{2} - s + 1}(2mr) \right]. \quad (3.35)
\]

Thus, we obtain following expressions for transverse components of the regularized vacuum energy-momentum tensor, Eqs. (2.26) and (2.27):

\[
t_{rr}^{\text{reg}}(s) = \frac{m^{d-2s} \Gamma \left( s - \frac{d}{2} \right)}{2(4\pi)^{\frac{d}{2}} \Gamma(s + 1)} -
\]

\[
- \frac{8 \sin(F\pi)}{(4\pi)^{\frac{d}{2} + 1} \Gamma(s + 1)} \left( \frac{m}{r} \right)^{\frac{d}{2} - s} \int_1^\infty \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \text{arccosh } v] \times
\]

\[
\times v^{s-\frac{d}{2} - 1} (1 - 4\xi v^2)K_{\frac{d}{2} - s}(2mr), \quad (3.36)
\]

\[
t_{\phi\phi}^{\text{reg}}(s) = \frac{r^{-2} m^{d-2s} \Gamma \left( s - \frac{d}{2} \right)}{2(4\pi)^{\frac{d}{2}} \Gamma(s + 1)} -
\]

\[
- \frac{8 \sin(F\pi)}{(4\pi)^{\frac{d}{2} + 1} \Gamma(s + 1)} r^{-2} \left( \frac{m}{r} \right)^{\frac{d}{2} - s} \int_1^\infty \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \text{arccosh } v] \times
\]

\[
\times v^{s-\frac{d}{2} - 1} (1 - 4\xi v^2)[K_{\frac{d}{2} - s}(2mr) - 2mrK_{\frac{d}{2} - s + 1}(2mr)]. \quad (3.37)
\]
If, instead of Eq. (3.7), we use relation
\[(k^2 + m^2)^{-z} = \frac{2 \sin(z \pi)}{\pi} \int_0^\infty dy \frac{y^{1-2z}}{k^2 + m^2 + y^2}, \quad 0 < \text{Re } z < 1, \quad (3.38)\]
then we get components of the regularized vacuum energy-momentum tensor in the following representation (see Appendix A):

\[t_{00}^{\text{reg}}(s) = \frac{m^{d-2s} \Gamma(s - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(s)} - \frac{16 \sin(F \pi) r^{2s-d}}{(4\pi)^{\frac{d}{2}+1} \Gamma(s+1)} \left\{ \frac{s}{\Gamma(\frac{d}{2} - s + 1)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s} K_F(w) K_{1-F}(w) + \right. \]
\[\left. + \frac{1 - 4\xi}{2 \Gamma(\frac{d}{2} - s - 1)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s - 2} \left[ K_F(w) + K_{1-F}(w) \right] \right\}, \quad (3.39)\]

\[t_{ij}^{\text{reg}}(s) = \frac{m^{d-2s} \Gamma(s - \frac{d}{2})}{2(4\pi)^{\frac{d}{2}} \Gamma(s+1)} - \frac{8 \sin(F \pi) r^{2s-d}}{(4\pi)^{\frac{d}{2}+1} \Gamma(s+1)} \left\{ \frac{1}{\Gamma(\frac{d}{2} - s + 1)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s} K_F(w) K_{1-F}(w) - \right. \]
\[\left. - \frac{1 - 4\xi}{\Gamma(\frac{d}{2} - s - 1)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s - 2} \left[ K_F(w) + K_{1-F}(w) \right] \right\}, \quad (3.40)\]

\[t_{rr}^{\text{reg}}(s) = \frac{m^{d-2s} \Gamma(s - \frac{d}{2})}{2(4\pi)^{\frac{d}{2}} \Gamma(s+1)} - \frac{8 \sin(F \pi) r^{2s-d}}{(4\pi)^{\frac{d}{2}+1} \Gamma(s+1)} \left\{ \frac{1}{\Gamma(\frac{d}{2} - s + 1)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s} \times \right. \]
\[\times \left\{ K_F(w) K_{1-F}(w) - \frac{w}{2} \left[ K_F^2(w) + K_{1-F}^2(w) \right] \right\} + \right. \]
\[\left. + \frac{1}{\Gamma(\frac{d}{2} - s - 1)} \int_{mr}^\infty dw w^2 \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s - 2} \left[ 2[F(1-F) - 2\xi] K_F(w) K_{1-F}(w) + \right. \]
\[\left. + w \left[ FK_F^2(w) + (1-F) K_{1-F}^2(w) \right] \right\}, \quad (3.41)\]
\[
t_{\text{reg}}(s) = \frac{r^{d-2s}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s + 1)} - \frac{8 \sin(F\pi) r^{2s-d-2}}{(4\pi)^{\frac{d}{2}+1} \Gamma(s + 1)} \left\{ \frac{1}{\Gamma(\frac{d}{2} - s + 1)} \int_{m_r}^{\infty} dw \ (w^2 - m^2 r^2)^{\frac{d}{2} - s} \times \right. \\
\times \left\{ K_F(w) K_{1-F}(w) - \frac{w}{2} [K_F^2(w) + K_{1-F}^2(w)] \right\} - \\
- \frac{1}{\Gamma(\frac{d}{2} - s - 1)} \int_{m_r}^{\infty} dw \ w^2 (w^2 - m^2 r^2)^{\frac{d}{2} - s - 2} \{ 2[F(1 - F) - 2\xi] K_F(w) K_{1-F}(w) + \\
w [(F - 4\xi) K_F^2(w) + (1 - F - 4\xi) K_{1-F}^2(w)] \} \right\}. \quad (3.42)
\]

Whereas expressions (3.16), (3.17), (3.36), and (3.37) are defined on the whole complex s-plane, expressions (3.39) - (3.42) are defined on a half-plane \( \Re s < \frac{d}{2} - 1 \).

### 4 Renormalized vacuum expectation value

In the absence of the \( d - 2 \) - brane (i.e. at \( \Phi = 0 \)) expressions (3.16), (3.17), (3.36), and (3.37) (or (3.39) - (3.42) take form

\[
t_{00}^{\text{reg}}(s)|_{F=0} = m^{d-2s} \frac{\Gamma(s - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(s)}, \]

\[
t_{jj}^{\text{reg}}(s)|_{F=0} = t_{rr}^{\text{reg}}(s)|_{F=0} = r^2 t_{\varphi \varphi}^{\text{reg}}(s)|_{F=0} = \frac{m^{d-2s} \Gamma(s - \frac{d}{2})}{2(4\pi)^{\frac{d}{2}} \Gamma(s + 1)}. \quad (4.1)
\]

These quantities are eliminated by the requirement of the normal ordering of the operator product in the case of noninteracting quantized field. For consistency with the noninteracting case, one has to subtract quantities (4.1) from the regularized expressions corresponding to the interaction with the background. Then, taking limit \( s \to -\frac{1}{2} \), one defines renormalized vacuum energy-momentum tensor:

\[
t_{\text{ren}}^{\mu\nu} = \lim_{s \to -\frac{1}{2}} \left[ t_{\text{reg}}^{\mu\nu}(s) - t_{\text{reg}}^{\mu\nu}(s)|_{F=0} \right], \quad (4.2)
\]

which is of physical interest. Its components are given by expressions:

\[
t_{00}^{\text{ren}} = -t_{jj}^{\text{ren}} = \frac{16 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} \left( \frac{m}{r} \right)^{\frac{d+1}{2}} \int_{1}^{\infty} \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \arccosh v] \times \\
\times v^{-\frac{d+3}{2}} \left\{ [1 + 2(1 - 4\xi)v^2] K_{\frac{d+1}{2}}(2mrv) - 2(1 - 4\xi)mrv^3 K_{\frac{d+3}{2}}(2mrv) \right\}, \quad (4.3)
\]
\[
\begin{align*}
\rho_{rr}^{\text{ren}} &= -\frac{16 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} \left( \frac{mr}{r} \right)^{\frac{d+1}{2}} \int_1^\infty \frac{dv}{\sqrt{v'^2 - 1}} \cosh[(2F - 1) \arccosh v] \times \\
&\quad \times v^{-\frac{d+3}{2}} \left( 1 - 4\xi v^2 \right) K_{\frac{d+3}{2}}(2mr) \quad (4.4)
\end{align*}
\]

\[
\begin{align*}
\rho_{\phi\phi}^{\text{ren}} &= -\frac{16 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} \left( \frac{mr}{r} \right)^{\frac{d+1}{2}} \int_1^\infty \frac{dv}{\sqrt{v'^2 - 1}} \cosh[(2F - 1) \arccosh v] \times \\
&\quad \times v^{-\frac{d+3}{2}} \left( 1 - 4\xi v^2 \right) \left\{ K_{\frac{d+3}{2}}(2mr) - 2mrK_{\frac{d+1}{2}}(2mr) \right\} \quad (4.5)
\end{align*}
\]

or, in the alternative representation,

\[
\begin{align*}
\rho_{rr}^{\text{00}} &= -\rho_{rr}^{\text{jj}} = \frac{16 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} r^{-d-1} \left\{ \frac{1}{\Gamma\left(\frac{d+3}{2}\right)} \int_{mr}^\infty \frac{dw}{w^2 - m^2 r^2} \frac{d+1}{2} K_F(w)K_{1-F}(w) - \\
&\quad - \frac{1 - 4\xi}{\Gamma\left(\frac{d-1}{2}\right)} \int_{mr}^\infty dw w^3 \left( w^2 - m^2 r^2 \right)^{\frac{d+3}{2}} \left[ K_F^2(w) + K_{1-F}^2(w) \right] \right\} \quad (4.6)
\end{align*}
\]

\[
\begin{align*}
\rho_{rr}^{\text{ren}} &= -\frac{16 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} r^{-d-1} \left\{ \frac{1}{\Gamma\left(\frac{d+3}{2}\right)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d+1}{2}} \times \\
&\quad \times \left\{ K_F(w)K_{1-F}(w) - \frac{w}{2} \left[ K_F^2(w) + K_{1-F}^2 \right] \right\} + \frac{1}{\Gamma\left(\frac{d-1}{2}\right)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d-3}{2}} \times \\
&\quad \times \left\{ 2 [F(1 - F) - 2\xi] K_F(w)K_{1-F}(w) + w \left[ F K_F^2(w) + (1 - F) K_{1-F}^2(w) \right] \right\} \right\} \quad (4.7)
\end{align*}
\]

\[
\begin{align*}
\rho_{\phi\phi}^{\text{ren}} &= -\frac{16 \sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} r^{-d-3} \times \\
&\quad \times \left\{ \frac{1}{\Gamma\left(\frac{d+3}{2}\right)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d+1}{2}} \left\{ K_F(w)K_{1-F}(w) - \frac{w}{2} \left[ K_F^2(w) + K_{1-F}^2 \right] \right\} - \\
&\quad - \frac{1}{\Gamma\left(\frac{d-1}{2}\right)} \int_{mr}^\infty dw \left( w^2 - m^2 r^2 \right)^{\frac{d+1}{2}} \left\{ 2 [F(1 - F) - 2\xi] K_F(w)K_{1-F}(w) + \\
&\quad + w \left[ (F - 4\xi) K_F^2(w) + (1 - F - 4\xi) K_{1-F}^2(w) \right] \right\} \right\} \quad (4.8)
\end{align*}
\]
One can verify that relation
\[(\partial_r + r^{-1})t_{rr}^\text{ren} - r t_{\phi\phi}^\text{ren} = 0, \quad (4.9)\]
is valid; consequently, the vacuum energy-momentum tensor is conserved:
\[\nabla_\mu t^\mu_{\text{ren}} = 0. \quad (4.10)\]

Taking trace of the tensor, we get
\[g_{\mu\nu}t^\mu_{\nu\text{ren}} = 32 \sin(F\pi) \int_1^\infty \frac{dv}{\sqrt{v^2 - 1}} \cosh[(2F - 1) \arccosh v] v^{-\frac{d+1}{2}} \times \]
\[\times \left\{ (d - 1 - 4\xi d) v \left(\frac{m}{\tau}\right)^{\frac{d+1}{2}} \left[ K_{d+1}(2mrv) - mrvK_{d+3}(2mrv) \right] - \right. \]
\[-m^2 \left(\frac{m}{\tau}\right)^{\frac{d+1}{2}} K_{d+1}(2mrv) \right\}. \quad (4.11)\]

As follows from the last relation, the trace becomes proportional to the mass squared in the case of \(\xi = \xi_c\) (1.1), which is in accordance with general relation (2.10).

To conclude this Section, we present the vacuum energy-momentum tensor in the Cartesian coordinate frame:
\[
\begin{pmatrix}
\varepsilon & 0 & 0 & 0 & 0 & 0 \\
0 & P_1 & \frac{1}{2}(P_1 - P_2) & \frac{x_1^2 x_2^2}{(x_1^2 - x_2^2)^2} & 0 & P_2 \\
0 & \frac{1}{2}(P_1 - P_2) & P_2 & 0 & 0 & P_3 \\
\varepsilon & P_1 & P_2 & P_3 & 0 & P_d \\
0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots 
\end{pmatrix},
\]

where
\[
\varepsilon = -P_j = t_{\text{ren}}^{00}, \quad j = 3, d, \\
P_1 = \frac{(x_1^2)^2 + (x_2^2)^2}{(x_1^2 - x_2^2)^2} t_{rr}^\text{ren} + (x_2^2)^2 t_{\phi\phi}^\text{ren}, \\
P_2 = \frac{(x_2^2)^2}{(x_1^2 + x_2^2)^2} t_{rr}^\text{ren} + (x_1^2)^2 t_{\phi\phi}^\text{ren}.
\]

5 Asymptotics at large and small distances from the brane

Components of the vacuum energy-momentum tensor depend on the distance from the brane in the transverse direction. Using representation (4.6)-(4.8), it is straig...
forward to determine the large distance behaviour of the tensor components:

\[
\begin{align*}
t_{00}^{\text{ren}} &= -\frac{2 \sin(F\pi)}{(4\pi)^{d+1}} e^{-2mr} \left( \frac{m}{r} \right)^{\frac{d+1}{2}} \left( 1 - 4\xi - \frac{1}{2} - (1 - 4\xi) \left[ \left( \frac{d+2}{4} \right)^2 - \frac{5}{16} - F(1 - F) \right] \right) (mr)^{-1} + O \left[(mr)^{-2}\right], \quad mr \gg 1, \quad (5.1) \\
t_{rr}^{\text{ren}} &= -\frac{\sin(F\pi)}{(4\pi)^{d+1}} e^{-2mr} \left( \frac{m}{r} \right)^{\frac{d+1}{2}} r^{-2} \left( 1 - 4\xi - \frac{1}{2} - (1 - 4\xi) \left[ \left( \frac{d+2}{4} \right)^2 + \frac{3}{16} - F(1 - F) \right] \right) (mr)^{-1} + O \left[(mr)^{-2}\right], \quad mr \gg 1, \quad (5.2) \\
t_{\phi\phi}^{\text{ren}} &= \frac{2 \sin(F\pi)}{(4\pi)^{d+1}} e^{-2mr} \left( \frac{m}{r} \right)^{\frac{d+1}{2}} r^{-2} \left( 1 - 4\xi - \frac{1}{2} - (1 - 4\xi) \left[ \left( \frac{d+2}{4} \right)^2 - \frac{3}{16} - F(1 - F) \right] \right) (mr)^{-1} + O \left[(mr)^{-2}\right], \quad mr \gg 1. \quad (5.3)
\end{align*}
\]

The small-distance behaviour is given by expressions:

\[
\begin{align*}
t_{\text{ren}}^{00} &= -\frac{\sin(F\pi)}{(4\pi)^{d+1}} \frac{\Gamma \left( \frac{d+1}{2} - F \right) \Gamma \left( \frac{d-1}{2} + F \right)}{\Gamma \left( \frac{d}{2} + 1 \right)} \left( d - 1 \right)^2 - 4 \frac{F(1 - F)}{d + 1} - 4\xi d(d - 1) \right] r^{-d-1} \left\{ 1 + O \left[(mr)^2\right] \right\}, \quad mr \ll 1, \quad (5.4) \\
t_{\text{ren}}^{rr} &= -\frac{\sin(F\pi)}{(4\pi)^{d+1}} \frac{\Gamma \left( \frac{d+1}{2} - F \right) \Gamma \left( \frac{d-1}{2} + F \right)}{\Gamma \left( \frac{d}{2} + 1 \right)} \left[ d - 1 + 4 \frac{F(1 - F)}{d + 1} - 4\xi d \right] r^{-d-1} \left\{ 1 + O \left[(mr)^2\right] \right\}, \quad mr \ll 1, \quad (5.5) \\
t_{\text{ren}}^{\phi\phi} &= \frac{\sin(F\pi)}{(4\pi)^{d+1}} \frac{\Gamma \left( \frac{d+1}{2} - F \right) \Gamma \left( \frac{d-1}{2} + F \right)}{\Gamma \left( \frac{d}{2} + 1 \right)} \left[ d - 1 + 4 \frac{F(1 - F)}{d + 1} - 4\xi d \right] r^{-d-3} \left\{ 1 + O \left[(mr)^2\right] \right\}, \quad mr \ll 1. \quad (5.6)
\end{align*}
\]

Evidently, leading terms in the expressions in Eqs. (5.4)–(5.6) yield us the vacuum energy-momentum tensor in the strictly massless case \((m = 0)\). We see that in this case the tensor components are characterized by a simple power dependence on the distance from the brane.
6 Half-integer values of the brane flux

As has been already noted, the vacuum energy-momentum tensor is periodic in the value of the brane flux (i.e. depends on its fractional value only), vanishing at its integer values \((F = 0)\) and being symmetric under change \(F \rightarrow 1 - F\). Moreover, as follows from Eqs.(4.3)-(4.5) (or (4.6)-(4.8)), maximal absolute values of the tensor components are achieved at half-integer values of the brane flux \((F = \half)\).

Using representation (4.3) - (4.5) (see, e.g., first line in Eq.(3.10)), we get

\[
 t^{00}_{\text{ren}}|_{F = \half} = \frac{2m^{d+1}}{(4\pi)^{\frac{d}{2}+1}} \left\{ \int_0^\infty d\tau \tau^{-\frac{d}{2}} e^{-\tau\Gamma} \left( -\frac{d + 1}{2}, \frac{m^2r^2}{\tau} \right) + 
 + 2(1 - 4\xi)(mr)^{-\frac{d}{2}-1} \left[ K_{\frac{d}{2}}(2mr) - 2mr K_{\frac{d}{2}+1}(2mr) \right] \right\}, \quad (6.1)
\]

\[
 t^{rr}_{\text{ren}}|_{F = \half} = -\frac{2m^{d+1}}{(4\pi)^{\frac{d}{2}+1}} \left\{ \int_0^\infty d\tau \tau^{-\frac{d}{2}} e^{-\tau\Gamma} \left( -\frac{d + 1}{2}, \frac{m^2r^2}{\tau} \right) - 
 - 8\xi(mr)^{-\frac{d}{2}-1} K_{\frac{d}{2}}(2mr) \right\}, \quad (6.2)
\]

\[
 t^{\phi\phi}_{\text{ren}}|_{F = \half} = -\frac{2r^{-2m^{d+1}}}{(4\pi)^{\frac{d}{2}+1}} \left\{ \int_0^\infty d\tau \tau^{-\frac{d}{2}} e^{-\tau\Gamma} \left( -\frac{d + 1}{2}, \frac{m^2r^2}{\tau} \right) - 
 - 4(1 - 4\xi)(mr)^{-\frac{d}{2}} K_{\frac{d}{2}+1}(2mr) \right\}. \quad (6.3)
\]

In Appendix B we show that integration in Eqs.(6.1) - (6.3) can be performed in the case of physical values of space dimension \((d \in \mathbb{Z}, d \geq 2)\). As a result, quantities \((6.1) - (6.3)\) are expressed in terms of Macdonald function \(K_\mu(u)\) and modified Struve function \(L_\mu(u)\) of integer order in the case of even \(d\), and in terms of the integral exponential function \(E_1(u)\) (see, e.g., Ref.[14]) and elementary functions in the case of odd \(d\). In particular, we get

\[
 t^{00}_{\text{ren}}|_{F = \half} = \frac{m^3}{3\pi^2} \left\{ \frac{\pi}{2} - \pi mr \left[ K_0(2mr)L_{-1}(2mr) + K_1(2mr)L_0(2mr) \right] - 
 - (1 - 6\xi)(mr)^{-1} K_0(2mr) - \left[ 1 + \frac{1}{2} \left( \frac{1}{2} - 6\xi \right) (mr)^{-2} \right] K_1(2mr) \right\}, \quad d = 2, \quad (6.4)
\]
\[ t_{\nu \nu}^{\text{ren}} \big|_{F = \frac{1}{2}} = -\frac{m^2}{3\pi^2} \left\{ \frac{\pi}{2} - \pi mr \left[ K_0(2mr) L_{-1}(2mr) + K_1(2mr) L_0(2mr) \right] + \right. \\
\left. \frac{1}{2} (mr)^{-1} K_0(2mr) - \left[ 1 - \frac{1}{2} (1 - 6\xi)(mr)^{-2} \right] K_1(2mr) \right\}, \quad d = 2, \quad (6.5) \]

\[ t_{\phi \phi}^{\text{ren}} \big|_{F = \frac{1}{2}} = -\frac{r^{-2}m^2}{3\pi^2} \left\{ \frac{\pi}{2} - \pi mr \left[ K_0(2mr) L_{-1}(2mr) + K_1(2mr) L_0(2mr) \right] - \right. \\
\left. (1 - 6\xi)(mr)^{-1} K_0(2mr) - [1 + (1 - 6\xi)(mr)^{-2}] K_1(2mr) \right\}, \quad d = 2, \quad (6.6) \]

\[ t_{\nu \nu}^{\text{ren}} \big|_{F = \frac{1}{4}} = \frac{m^4}{(4\pi)^2} \left\{ E_1(2mr) - \frac{1}{2} e^{-2mr} \left[ (mr)^{-1} + \left( \frac{7}{2} - 16\xi \right)(mr)^{-2} + \right. \right. \\
\left. \left. \left( \frac{5}{2} - 16\xi \right)(mr)^{-3} + \frac{1}{2} \left( \frac{5}{2} - 16\xi \right)(mr)^{-4} \right] \right\}, \quad d = 3, \quad (6.7) \]

\[ t_{\nu \nu}^{\text{ren}} \big|_{F = \frac{3}{4}} = -\frac{m^4}{(4\pi)^2} \left\{ E_1(2mr) - \frac{1}{2} e^{-2mr} \left[ (mr)^{-1} - \frac{1}{2} (mr)^{-2} - \right. \right. \\
\left. \left. \frac{1}{2} (3 - 16\xi)(mr)^{-3} - \frac{1}{4} (3 - 16\xi)(mr)^{-4} \right] \right\}, \quad d = 3, \quad (6.8) \]

\[ t_{\phi \phi}^{\text{ren}} \big|_{F = \frac{3}{4}} = -\frac{r^{-2}m^4}{(4\pi)^2} \left\{ E_1(2mr) - \frac{1}{2} e^{-2mr} \left[ (mr)^{-1} + \left( \frac{7}{2} - 16\xi \right)(mr)^{-2} + \right. \right. \\
\left. \left. \left( \frac{3}{2} - 16\xi \right)(mr)^{-3} + \frac{3}{4} (3 - 16\xi)(mr)^{-4} \right] \right\}, \quad d = 3, \quad (6.9) \]

Expressions in cases of arbitrary even and odd values of space dimension are given in Appendix B (see Eqs. (B.6) - (B.11)).

7 The strong energy condition and its violation

Energy-momentum tensor of the physically reasonable classical matter satisfies the strong energy condition \[ \left. \begin{array}{c}
T^{\mu \nu} u_\mu u_\nu - \frac{1}{2} g_{\mu \nu} T^{\mu \nu} \geq 0,
\end{array} \right. \quad (7.1) \]

where \( u_\mu \) is a time-like vector (\( u^\mu u_\mu = 1 \)). To check, whether the vacuum energy-momentum tensor in the background of the \( d - 2 \)-brane \footnote{In this Section we consider physical values of space dimension: \( d \geq 2 \).} satisfies condition \( (7.1) \), it is sufficient to analyse three quantities: \( t_{\nu \nu}^{\text{ren}} - \frac{1}{2} g_{\mu \nu} t_{\nu \nu}^{\text{ren}}, t_{\nu \nu}^{\text{ren}} + t_{\phi \phi}^{\text{ren}}, t_{\nu \nu}^{\text{ren}} + r^2 t_{\phi \phi}^{\text{ren}} \).
In the massless case, using Eqs. (5.4)-(5.6), we get

\[
\left. \left( t_{\text{ren}}^{00} - \frac{1}{2} g_{\mu \nu} t_{\text{ren}}^{\mu \nu} \right) \right|_{m=0} = \left. 4 \sin \left( F \pi \right) \frac{\Gamma \left( \frac{d+1}{2} - F \right) \Gamma \left( \frac{d-1}{2} + F \right)}{(4\pi)^{d+1}} \times \right.
\]
\[
\times \left[ \frac{1}{2} (d-2)(d-1) d (\xi_c - \xi) + \frac{F(1-F)}{d+1} \right] r^{-d-1}, \quad (7.2)
\]

\[
\left. \left( t_{\text{ren}}^{00} + t_{\text{ren}}^{rr} \right) \right|_{m=0} = \left. -4 \sin \left( F \pi \right) \frac{\Gamma \left( \frac{d+1}{2} - F \right) \Gamma \left( \frac{d-1}{2} + F \right)}{(4\pi)^{d+1}} \times \right.
\]
\[
\times d^2 (\xi_c - \xi) r^{-d-1}, \quad (7.3)
\]

\[
\left. \left( t_{\text{ren}}^{00} + r^2 t_{\text{ren}}^{\varphi \varphi} \right) \right|_{m=0} = \left. 4 \sin \left( F \pi \right) \frac{\Gamma \left( \frac{d+1}{2} - F \right) \Gamma \left( \frac{d-1}{2} + F \right)}{(4\pi)^{d+1}} \times \right.
\]
\[
\times \left[ d (\xi_c - \xi) + F(1-F) \right] r^{-d-1}, \quad (7.4)
\]

where \( \xi_c \) is given by Eq. (1.1). All quantities (7.2)-(7.4) are simultaneously nonnegative at

\[
\xi_c \leq \xi \leq \xi_c + \frac{1}{2} F(1-F), \quad d = 2,
\]
\[
\xi_c \leq \xi \leq \xi_c + \frac{2F(1-F)}{(d-2)(d-1)d(d+1)}, \quad d \geq 3,
\]

then the strong energy condition is satisfied by the vacuum energy-momentum tensor of the quantized massless scalar matter in the background of the brane. If \( F \) is infinitesimally close to 0 or to 1, then the condition is satisfied only at \( \xi = \xi_c \), i.e. when conformal invariance is maintained. Note, that in the conformally invariant case the strong energy condition coincides, as a consequence of the vanishing trace, with the weak energy one \([15]\).

A similar analysis is carried out for the quantized massive scalar matter, and we find that both the strong and weak energy conditions are violated for all values of \( \xi \).

The temporal component of the vacuum tensor (i.e., energy density) is positive at \( \xi \geq \frac{1}{4} \) and negative at \( \xi \leq 0.3 \) Transverse components of the vacuum tensor are also of opposite signs at \( \xi \geq \frac{1}{4} \) and at \( \xi \leq 0 \): the radial one is of the same and the angular one is of the opposite to the sign of the temporal component. The region \( 0 < \xi < \frac{1}{4} \) or, more precisely, the vicinity of \( \xi = \xi_c \) is distinguished as the region where all components change their signs. Transverse components change their signs at a certain value of \( \xi \) simultaneously for all distances from the brane. In contrast to this, the temporal component is positive at small distances and negative at large distances for a certain, dependent on the value of the brane flux, vicinity of the point \( \xi = \xi_c \).

This situation is illustrated by Figures 1-3. Here variable \( mr \) is along \( x \)-axis, and dimensionless products of tensor components at half-integer values of the brane

\(^3\)The vacuum energy density at \( \xi = \frac{1}{4} \) was considered in Ref. [3].
flux, Eqs. (6.4)-(6.9), and appropriate powers of $r$ are along $y$-axis: $r^{d+1}t^0_{\text{ren}}|_{F=\frac{1}{2}}$ is presented by a solid line, $r^{d+1}t^r_{\text{ren}}|_{F=\frac{1}{2}}$ - by a dotted line, and $r^{d+3}t^{\phi\phi}_{\text{ren}}|_{F=\frac{1}{2}}$ - by a dashed line. We consider cases of $\xi = 0$, $\xi = \xi_c$, $\xi = \frac{1}{4}$, each one at $d = 2$ and $d = 3$. We see from Fig.2 that the vacuum energy density at $\xi = \xi_c$ has minimum at $mr \approx 1.0$ ($d = 2$) or $mr \approx 1.2$ ($d = 3$); the minimal value is $r^{d+1}t^0_{\text{ren}}|_{F=\frac{1}{2}} \approx -0.0018$ ($d = 2$) or $r^{d+1}t^0_{\text{ren}}|_{F=\frac{1}{2}} \approx -0.0039$ ($d = 3$).

To conclude this Section, we present general expressions for the vacuum tensor components in the case of conformal coupling:

$$
t^0_{\text{ren}}|_{\xi=\xi_c} = -\frac{16 \sin(F\pi)}{(4\pi)^{d+3}} \left(\frac{m}{r}\right)^{\frac{d+1}{2}} \int_{1}^{\infty} dv \cosh[(2F-1)\arccosh v]v^{-\frac{d+3}{2}} \times \\
\times \left\{ \sqrt{v^2 - 1}K_{d+1}(2mrv) + \frac{v^2}{d\sqrt{v^2-1}} \left[ 2mrvK_{d+1}(2mrv) - K_{d+1}(2mrv) \right] \right\},
$$

(7.6)

$$
t^r_{\text{ren}}|_{\xi=\xi_c} = \frac{16 \sin(F\pi)}{(4\pi)^{d+3}} \left(\frac{m}{r}\right)^{\frac{d+1}{2}} \int_{1}^{\infty} dv \cosh[(2F-1)\arccosh v]v^{-\frac{d+3}{2}} \times \\
\times \left( \sqrt{v^2 - 1} - \frac{v^2}{d\sqrt{v^2-1}} \right) K_{d+1}(2mrv),
$$

(7.7)

$$
t^{\phi\phi}_{\text{ren}}|_{\xi=\xi_c} = -\frac{16 \sin(F\pi)}{(4\pi)^{d+3}} \frac{1}{r^2} \left(\frac{m}{r}\right)^{\frac{d+1}{2}} \int_{1}^{\infty} dv \cosh[(2F-1)\arccosh v]v^{-\frac{d+3}{2}} \times \\
\times \left( \sqrt{v^2 - 1} - \frac{v^2}{d\sqrt{v^2-1}} \right) \left[ 2mrvK_{d+1}(2mrv) + dK_{d+1}(2mrv) \right].
$$

(7.8)

Vacuum energy density (7.6) has minimum at $r = r_{\text{min}}$ where $r_{\text{min}}$ is determined by equation

$$\left( \frac{d}{dr}t^0_{\text{ren}}|_{\xi=\xi_c} \right)_{r=r_{\text{min}}} = 0,$$

or

$$\int_{1}^{\infty} dv \cosh[(2F-1)\arccosh v]v^{-\frac{d+1}{2}} \left\{ \sqrt{v^2 - 1}K_{d+1}(2mr_{\text{min}}v) + \\
+ \frac{v^2}{d\sqrt{v^2-1}} \left[ 2mr_{\text{min}}vK_{d+1}(2mr_{\text{min}}v) - K_{d+1}(2mr_{\text{min}}v) \right] \right\} = 0.
$$

(7.9)
8 Summary

We have shown that the vacuum of the quantized charged scalar field is polarized in the background of a static magnetic $d - 2$ - brane in flat $d + 1$ - dimensional space-time. Vector potential of the brane induces a finite energy-momentum tensor in the vacuum; therefore, this effect may be denoted as the Casimir-Bohm-Aharonov effect. Tensor components depend periodically on the brane flux ($\Phi$), vanishing at its integer values ($\Phi = n$), and attaining maximal absolute values at its half-integer values ($\Phi = n + \frac{1}{2}$). A remarkable feature is a possibility of analytic continuation in space dimension: representation (4.3)-(4.5) yields holomorphic functions of $d$ on the whole complex $d$-plane, and representation (4.6)-(4.8) yields holomorphic functions of $d$ on a half-plane $\text{Re } d > 1$. The tensor components decrease exponentially at large distances from the brane, see Eqs.(5.1)-(5.3). If the mass of scalar field is zero, then expressions for tensor components are simplified considerably, see Eqs.(5.4)-(5.6).

Strong and weak energy conditions [15] are usually violated by vacuum polarization effects of quantized fields: in particular, this happens in most cases of the conventional Casimir effect [3]. However, we have found that these conditions are satisfied by the vacuum energy-momentum tensor of conformally invariant ($\xi = \xi_c$) massless scalar field quantized in the background of the $d - 2$ - brane. Thus, the latter vacuum is somewhat similar to the medium of classical matter.

Although the strong and weak energy conditions are violated for all values of $\xi$ in the case of massive scalar field quantized in the background of the brane, the conformal coupling ($\xi = \xi_c$) is distinctive in the massive case also. The vacuum tensor components at $\xi = \xi_c$ are given by Eqs.(7.6)-(7.8). Qualitatively, the temporal component (energy density) is positive power divergent at small distances from the brane, decreases with the increase of the distance, passes zero, becomes negative and reaches minimum at $r \sim m^{-1}$, then increases and reaches zero asymptotically from below with exponential behaviour. This is distinct from the conventional Casimir effect, which corresponds to the vacuum energy density being space independent constant, either negative or positive [3]. We present also expressions for the vacuum tensor components at half-integer values of the brane flux in the case of physical values of space dimension ($d = n \geq 2$), see Eqs.(6.4)-(6.9) and, generally, Eqs.(B.6)-(B.11).

It should be noted that temporal components of the vacuum tensor for the quantized scalar (with $\xi = \frac{1}{4}$) and spinor matter in magnetic backgrounds in low-dimensional ($d = 2, 3$) spaces were considered in Refs.[16, 17, 18, 19, 20]. Since the authors of these works are concerned with the case when the region of nonvanishing background field is of nonvanishing transverse size and is overlapped with the region of nonvanishing quantized matter, their results differ considerably from ours: in particular, the dependence on the flux of the background magnetic field is not periodic. Moreover, in their case in addition to the vacuum energy of quantized matter also the classical energy of magnetic background has to be consistently taken into account, as it is done in Ref.[19]. On the contrary, when the region of nonvanishing background field is impenetrable for quantized matter, then the quantum and classical energies are stored in different non-overlapping parts of space. Certainly
our neglect of transverse size of the magnetic brane is an idealization which allowed us to solve the problem analytically. But we believe that this idealization, as in the case of the conventional Bohm-Aharonov effect \cite{5}, grasps some essential features of the more realistic case of the magnetic brane with finite transverse size and the quantized scalar field satisfying the boundary condition on the edge of the brane. In particular, it is almost evident that then the vacuum energy-momentum tensor components will depend periodically on the brane flux and possess large-distance asymptotics which is exponential for \( m > 0 \), see Eqs. (5.1)-(5.3), and negative-power-behaved for \( m = 0 \), see Eqs. (5.4)-(5.6) with limit \( m \to 0 \) taken first before \( r \to \infty \).

A less evident, but still rather plausible conjecture is that tensor components will behave less divergent, as compared to \((r-r_B)^{-d-3}\) for the transverse angular and \((r-r_B)^{-d-1}\) for all other components, near brane edge \( r = r_B \). And a really challenging task is to find out whether the vacuum energy density integrated over transverse coordinates will appear to be finite and somehow capable to compensate the classical energy per transverse section of the brane.

An intriguing question is about the underlying physics of the negativity of the vacuum energy density and the violation of strong and weak energy conditions. Recent examples of low-dimensional (without magnetic background) models \cite{21} indicate that the negativeness of the vacuum energy density at large distances is related to effectively perfect reflection of quantized matter at small distances; the negativeness might persist self-consistently for the whole, classical plus quantum, energy density. In this respect Ref.\cite{22} is worth mentioning, where the vacuum energy density for the quantized spinor matter in the same background, as in the present paper, but exclusively in the \( d = 2 \) case, was considered. As is known, spinor field cannot be made zero at the location of a singular magnetic brane (which is a point in the \( d = 2 \) case). The whole set of permissible boundary conditions is parametrized by real quantity \( \Theta \), and at \( \cos \Theta < 0 \) a bound state appears in the gap between positive and negative frequency continua \cite{23}. Thus, for sure, perfect reflection at the point of singularity is excluded at \( \cos \Theta < 0 \). As is shown in Ref.\cite{22}, namely at these values of \( \Theta \) the vacuum energy density is strictly positive at all distances, whereas, otherwise, its behaviour is similar to that of the vacuum energy density for the quantized scalar matter with \( \xi = \xi_c \), see, qualitatively, solid curves on Fig.2.

At last, it should be noted that pitfalls of the zeta function regularization procedure (see, e.g., Refs.\cite{24,25,26}) do not appear in the case of singular backgrounds \cite{22,27}. Owing to this circumstance, it is possible to show that the renormalized vacuum energy-momentum tensor in the present paper is independent of the choice of a regularization procedure. In fact, this has been already shown for its temporal component at \( \xi = \frac{1}{4} \) in Ref.\cite{27}. In a similar way this can be done for other values of \( \xi \), and for other components. The key point of Ref.\cite{27} is that the heat kernel in the presence of the brane coincides actually with the heat kernel in its absence, and then, basing on this fact, it can be proved that the use of zeta function regularization leads to renormalized quantities which are regularization procedure independent.
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Appendix A

Using Eq. (3.38), we present Eqs. (3.28) and (3.29) in the following form

\[
\Omega_0(s) = \frac{r \sin \left[ \left( s - \frac{d}{2} \right) \pi \right]}{(s - \frac{d}{2}) \pi} \int_0^\infty dy \int_0^\infty \frac{dk}{k^2 + y^2 + m^2} \times
\]

\[
\times \left[ J_F(kr)J_{-1+F}(kr) + J_{-F}(kr)J_{-F}(kr) \right], \quad (A.1)
\]

\[
\tilde{\Omega}_1(s) = \frac{2 \sin \left[ \left( 1 + s - \frac{d}{2} \right) \pi \right]}{\pi} \int_0^\infty dy \int_0^\infty \frac{dk}{k^2 + y^2 + m^2} \times
\]

\[
\times \left\{ \frac{1}{2} (1 + F) [J^2_F(kr) - J^2_F(kr)] + \frac{1}{2} (2 - F) [J^2_{-1+F}(kr) - J^2_{-F}(kr)] + \frac{F(1 - F)}{kr} \left[ J_F(kr)J_{-1+F}(kr) + J_{-F}(kr)J_{-F}(kr) \right] \right\}, \quad (A.2)
\]

\[
\tilde{\Omega}_2(s) = \frac{2 \sin \left[ \left( 1 + s - \frac{d}{2} \right) \pi \right]}{\pi} \int_0^\infty dy \int_0^\infty \frac{dk}{k^2 + y^2 + m^2} \times
\]

\[
\times \left\{ \frac{1}{2} (1 - F) [J^2_F(kr) - J^2_F(kr)] + \frac{1}{2} F [J^2_{-1+F}(kr) - J^2_{-F}(kr)] + \frac{F(1 - F)}{kr} \left[ J_F(kr)J_{-1+F}(kr) + J_{-F}(kr)J_{-F}(kr) \right] \right\}. \quad (A.3)
\]

Using relation (see Ref. [13])

\[
\int_0^\infty dk \frac{k^{\nu-\mu+1}}{k^2 + \omega^2} J_\nu(kr)J_\mu(kr) = \omega^{\nu-\mu} I_\mu(\omega r) K_\nu(\omega r), \quad -1 < \nu < \mu + 1, \quad (A.4)
\]

we get

\[
\Omega_0(s) = \frac{r \sin \left[ \left( s - \frac{d}{2} \right) \pi \right]}{(s - \frac{d}{2}) \pi} \int_0^\infty dy \int_0^\infty \frac{y^{d-2s+1}}{\sqrt{m^2 + y^2}} \times
\]

\[
\times \left[ I_F(r \sqrt{m^2 + y^2})K_{-F}(r \sqrt{m^2 + y^2}) + I_{-F}(r \sqrt{m^2 + y^2})K_F(r \sqrt{m^2 + y^2}) \right]. \quad (A.5)
\]
Introducing integration variable \( w = r \sqrt{m^2 + y^2} \) and using identity

\[
I_F(w)K_{1-F}(w) + I_{1-F}(w)K_F(w) = \frac{1}{w} - \frac{2 \sin(F \pi)}{\pi} K_F(w)K_{1-F}(w),
\]

we get

\[
\Omega_0(s) = \frac{m^{d-2s}}{2s-d} - \frac{4 \sin(s - \frac{d}{2})\pi}{(2s-d)\pi^2} \sin(F \pi) r^{2s-d} \times \int_{mr}^{\infty} dw \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s} K_F(w)K_{1-F}(w) . \quad (A.6)
\]

Although our derivation is valid for a strip \( \frac{d}{2} < \text{Re } s < \frac{d}{2} + 1 \) (see Eq.(3.38)), the result, Eq.(A.6), is analytically continued to half-plane \( \text{Re } s < \frac{d}{2} + 1 \). Accordingly, \( \Omega_0(s + 1) \) is defined on half-plane \( \text{Re } s < \frac{d}{2} \). Taking relations

\[
\Omega_{00}(s) = \frac{2}{(4\pi)^{\frac{d}{2}}} \Gamma \left( 1 + s - \frac{d}{2} \right) \Gamma(s) \left[ \Omega_0(s) + \left( \frac{1}{4} - \xi \right) \frac{1 + s - \frac{d}{2}}{s} \Delta_r \Omega_0(s + 1) \right] , \quad (A.7)
\]

\[
\Omega_{00}^{ij}(s) = \frac{1}{(4\pi)^{\frac{d}{2}}} \Gamma \left( 1 + s - \frac{d}{2} \right) \Gamma(1+s) \left[ \Omega_0(s) - \left( \frac{1}{4} - \xi \right) 2 \left( 1 + s - \frac{d}{2} \right) \Delta_r \Omega_0(s + 1) \right] , \quad (A.8)
\]

into account, we obtain relations (3.39) and (3.40).

In a similar way as Eq.(A.6), we get relations

\[
\tilde{\Omega}_1(s) = -r^{-2}m^{d-2s-2} F(1 - F) - \frac{2 \sin(s - \frac{d}{2})\pi}{\pi^2} \sin(F \pi) r^{2s-d} \int_{mr}^{\infty} dw \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s - 1} \times \left[ 2F(1 - F)K_F(w)K_{1-F}(w) + (1 + F)wK_F^2(w) + (2 - F)wK_{1-F}^2(w) \right] , \quad (A.9)
\]

\[
\tilde{\Omega}_2(s) = r^{-2}m^{d-2s-2} F(1 - F) + \frac{2 \sin(s - \frac{d}{2})\pi}{\pi^2} \sin(F \pi) r^{2s-d} \int_{mr}^{\infty} dw \left( w^2 - m^2 r^2 \right)^{\frac{d}{2} - s - 1} \times \left[ 2F(1 - F)K_F(w)K_{1-F}(w) - (1 - F)wK_F^2(w) - FwK_{1-F}^2(w) \right] , \quad (A.10)
\]

which are analytically continued from strip \( \frac{d}{2} - 1 < \text{Re } s < \frac{d}{2} \) to half-plane \( \text{Re } s < \frac{d}{2} \). Accordingly, \( \tilde{\Omega}_1(s + 1) \) and \( \tilde{\Omega}_2(s + 1) \) are defined on half-plane \( \text{Re } s < \frac{d}{2} - 1 \). Thus, we can obtain expression for differences \( \tilde{\Omega}_1(s) - m^2 \tilde{\Omega}_1(s+1) \) and \( \tilde{\Omega}_2(s) - m^2 \tilde{\Omega}_2(s+1) \) which are valid on half-plane \( \text{Re } s < \frac{d}{2} - 1 \); note also that contribution of first terms in the right-hand sides of Eqs.(A.9) and (A.10) is cancelled.

Taking relations (3.18) and (3.19) into account, we obtain relations (3.41) and (3.42).
Appendix B

Using repeatedly the recurrency relations for the incomplete gamma functions, one can get

\[
\Gamma \left( -N - \frac{1}{2}, w \right) = \frac{(-1)^N}{\Gamma(N + \frac{3}{2})} \left[ -\pi \text{erfc} \left( \sqrt{w} \right) + e^{-w} \sum_{l=0}^{N} (-1)^l \Gamma \left( l + \frac{1}{2} \right) w^{-l-\frac{1}{2}} \right],
\]  

(B.1)

\[
\Gamma(-N - 1, w) = \frac{(-1)^N}{\Gamma(N + 2)} \left[ -E_1(w) + e^{-w} \sum_{l=0}^{N} (-1)^l \Gamma(l + 1) w^{-l-1} \right],
\]  

(B.2)

where

\[
\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} d\tau e^{-\tau^2}
\]

is the complementary error function, and

\[
E_1(u) = \int_{u}^{\infty} \frac{d\tau}{\tau} e^{-\tau}
\]

is the integral exponential function. Using Eqs. (B.1), (B.2) and relations (13)

\[
\int_{0}^{\infty} d\tau \tau^{-\frac{1}{2}} e^{-\tau} \text{erfc} \left( \frac{mr}{\sqrt{\tau}} \right) =
\]

\[
= \sqrt{\pi} \left\{ 1 - 2mr \left[ K_0(2mr)L_{-1}(2mr) + K_1(2mr)L_0(2mr) \right] \right\},
\]  

(B.3)

\[
\int_{0}^{\infty} d\tau \tau^{-\frac{1}{2}} e^{-\tau} E_1 \left( \frac{m^2r^2}{\tau} \right) = 2\sqrt{\pi} E_1(2mr),
\]  

(B.4)

we perform integration in Eqs. (6.1) - (6.3). Further, in the case of odd space dimension we use representation of the Macdonald function of half-integer order through a finite sum:

\[
K_{l+\frac{1}{2}}(u) = \sqrt{\pi} e^{-u} \sum_{n=0}^{l} \frac{\Gamma(l + n + 1)}{\Gamma(n + 1)\Gamma(l - n + 1)} (2u)^{-n-\frac{1}{2}}.
\]  

(B.5)
Consequently, we get

\[
\left. t_{\rho\ro}^{\rho\ro} \right|_{F=\frac{1}{2}} = \frac{2m^{2N+1}}{(4\pi)^{N+\frac{3}{2}}} \frac{(-1)^N}{\Gamma(N + \frac{3}{2})} \left\{ -\frac{\pi}{2} + \pi mr [K_0(2mr)L_{-1}(2mr) + K_1(2mr)L_0(2mr)] + \right.
\]
\[
+ \frac{1}{\sqrt{\pi}} \sum_{l=0}^{N-2} (-1)^l \Gamma \left( l + \frac{1}{2} \right) (mr)^{-l} K_{l+1}(2mr) - \left. \frac{(-1)^N}{\sqrt{\pi}} \Gamma \left( N - \frac{1}{2} \right) \left[ 1 - \frac{1}{4}(1 - 4\xi)(4N^2 - 1)(mr)^{-2} \right] (mr)^{-N+1} K_N(2mr) + \right.
\]
\[
+ \frac{(-1)^N}{\sqrt{\pi}} \Gamma \left( N + \frac{1}{2} \right) \left[ 1 - (1 - 4\xi)(2N + 1) \right] (mr)^{-N} K_{N+1}(2mr) \right\}, \quad d = 2N, \quad (B.6)
\]

\[
\left. t_{\rho\ro}^{\rho\ro} \right|_{F=\frac{1}{2}} = -\frac{2m^{2N+1}}{(4\pi)^{N+\frac{3}{2}}} \frac{(-1)^N}{\Gamma(N + \frac{3}{2})} \left\{ -\frac{\pi}{2} + \pi mr [K_0(2mr)L_{-1}(2mr) + K_1(2mr)L_0(2mr)] + \right.
\]
\[
+ \frac{1}{\sqrt{\pi}} \sum_{l=0}^{N-2} (-1)^l \Gamma \left( l + \frac{1}{2} \right) (mr)^{-l} K_{l+1}(2mr) - \left. \frac{(-1)^N}{\sqrt{\pi}} \Gamma \left( N - \frac{1}{2} \right) \left[ 1 + \xi(4N^2 - 1)(mr)^{-2} \right] (mr)^{-N+1} K_N(2mr) + \right.
\]
\[
+ \frac{(-1)^N}{\sqrt{\pi}} \Gamma \left( N + \frac{1}{2} \right) (mr)^{-N} K_{N+1}(2mr) \right\}, \quad d = 2N, \quad (B.7)
\]

\[
\left. t_{\rho\phi}^{\rho\phi} \right|_{F=\frac{1}{2}} = -\frac{2m^{2N+1}}{r^2(4\pi)^{N+\frac{3}{2}}} \frac{(-1)^N}{\Gamma(N + \frac{3}{2})} \left\{ -\frac{\pi}{2} + \pi mr [K_0(2mr)L_{-1}(2mr) + K_1(2mr)L_0(2mr)] + \right.
\]
\[
+ \frac{1}{\sqrt{\pi}} \sum_{l=0}^{N-1} (-1)^l \Gamma \left( l + \frac{1}{2} \right) (mr)^{-l} K_{l+1}(2mr) + \left. \frac{(-1)^N}{\sqrt{\pi}} \Gamma \left( N + \frac{1}{2} \right) \left[ 1 - (1 - 4\xi)(2N + 1) \right] (mr)^{-N} K_{N+1}(2mr) \right\}, \quad d = 2N, \quad (B.8)
\]

\[
\left. t_{\rho\rho}^{\rho\rho} \right|_{F=\frac{1}{2}} = \frac{2m^{2N+2}}{(4\pi)^{N+1}} \left\{ \frac{-(-1)^N}{\Gamma(N + 2)} E_1(2mr) + \right.
\]
\[
+ \frac{e^{-2mr}}{\Gamma(N + 2)} \sum_{l=0}^{N-1} (-1)^{-l-l}(l + 1) \sum_{n=0}^{l+1} \frac{\Gamma(l + n + 2)(mr)^{-l-n-1}}{2^{2n+1}\Gamma(n + 1)\Gamma(l - n + 2)} + \right.
\]
\[
+ (1 - 4\xi)e^{-2mr} \sum_{l=0}^{N} \frac{\Gamma(N + l + 1)(mr)^{-N-l-2}}{2^{2l+1}\Gamma(l + 1)\Gamma(N - l + 1)} + \left. \frac{1}{N + 1} - 2(1 - 4\xi) \right] e^{-2mr} \sum_{l=0}^{N+1} \frac{\Gamma(N + l + 2)(mr)^{-N-l-1}}{2^{2l+1}\Gamma(l + 1)\Gamma(N - l + 2)} \right\}, \quad d = 2N + 1, \quad (B.9)
\]
\[
t^{rr}_{\text{ren}} \big|_{F=\frac{1}{2}} = -\frac{2m^{2N+2}}{(4\pi)^{N+1}} \left\{ -\frac{(-1)^N}{\Gamma(N+2)} E_1(2mr) + \right. \\
+ e^{-2mr} \sum_{l=0}^{N-1} (-1)^{N-l} \Gamma(l+1) \sum_{n=0}^{l+1} \frac{\Gamma(l+n+2)(mr)^{l-n-1}}{2^{l+1} \Gamma(n+1) \Gamma(l-n+2)} - \\
- 4\xi e^{-2mr} \sum_{l=0}^{N} \frac{\Gamma(N+l+1)(mr)^{-N-l-2}}{2^{l+1} \Gamma(l+1) \Gamma(N-l+1)} + \\
+ e^{-2mr} N^{-1} \sum_{l=0}^{N+1} \frac{\Gamma(N+l+2)(mr)^{-N-l-1}}{2^{l+1} \Gamma(l+1) \Gamma(N-l+2)} \right\}, \quad d = 2N + 1, \quad (B.10)
\]

\[
t^{\varphi\varphi}_{\text{ren}} \big|_{F=\frac{1}{2}} = -\frac{2m^{2N+2}}{r^2(4\pi)^{N+1}} \left\{ -\frac{(-1)^N}{\Gamma(N+2)} E_1(2mr) + \right. \\
+ e^{-2mr} \sum_{l=0}^{N-1} (-1)^{N-l} \Gamma(l+1) \sum_{n=0}^{l+1} \frac{\Gamma(l+n+2)(mr)^{-l-n-1}}{2^{l+1} \Gamma(n+1) \Gamma(l-n+2)} + \\
+ \left[ \frac{1}{N+1} - 2(1 - 4\xi) \right] e^{-2mr} \sum_{l=0}^{N+1} \frac{\Gamma(N+l+2)(mr)^{-l-1}}{2^{l+1} \Gamma(l+1) \Gamma(N-l+2)} \right\}, \quad d = 2N + 1. \quad (B.11)
\]

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Figure 1: $\xi = 0$  a) $d = 2$,  b) $d = 3$. 
Figure 2: $\xi = \xi_c$  a) $d = 2$,  b) $d = 3$. 
Figure 3: $\xi = 1/4$  a) $d = 2$,  b) $d = 3$. 