ENumerating permutation polynomials over finite fields by degree

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Abstract. we prove an asymptotic formula for the number of permutation for which the associated permutation polynomial has degree smaller than \( q - 2 \).

Let \( \mathbb{F}_q \) be a finite field with \( q = p^f > 2 \) elements and let \( \sigma \in S(\mathbb{F}_q) \) be a permutation of the elements of \( \mathbb{F}_q \). The permutation polynomial \( f_\sigma \) of \( \sigma \) is

\[
f_\sigma(x) = \sum_{c \in \mathbb{F}_q} \sigma(c) \left(1 - (x - c)^{q-1}\right) \in \mathbb{F}_q[x].
\]

\( f_\sigma \) has the property that \( f_\sigma(a) = \sigma(a) \) for every \( a \in \mathbb{F}_q \) and this explains its name.

For an account of the basic properties of permutation polynomials we refer to the book of Lidl and Niederreiter [5].

From the definition, it follows that for every \( \sigma \)

\[
\partial(f_\sigma) \leq q - 2.
\]

A variety of problems and questions regarding permutation polynomials have been posed by Lidl and mullen in [3, 4]. Among these there is problem of determining the number \( N_d \) of permutation polynomials of fixed degree \( d \). In [6] and [7], Malvenuto and the second author address the problem of counting the permutations that move a fixed number of elements of \( \mathbb{F}_q \) and whose permutation polynomials have “low” degree.

Here we consider all permutations and we want to prove the following

Theorem 1. Let

\[
N = \# \{ \sigma \in S(\mathbb{F}_q) \mid \partial(f_\sigma) < q - 2 \}.
\]

Then

\[
|N - (q - 1)!| \leq \sqrt{2e/\pi q^{q/2}}.
\]

This confirms the common believe that \textit{almost all permutation polynomials have degree} \( q - 2 \).

The first few values of \( N \) are listed below:

| \( q \) | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 11 |
|---|---|---|---|---|---|---|---|---|
| \( N \) | 0 | 0 | 12 | 20 | 630 | 5368 | 42120 | 3634950 |
| \( (q - 1)! \) | 1 | 2 | 6 | 24 | 720 | 5040 | 40320 | 3628800 |

Proof. The proof uses exponential sums and a similar argument as the one in [6].

By extracting the coefficient of \( x^{q-2} \) in \( f_\sigma(x) \), we obtain that the degree of \( f_\sigma(x) \) is strictly smaller than \( q - 2 \) if and only if

\[
\sum_{c \in \mathbb{F}_q} c \sigma(c) = 0.
\]
For a fixed subset $S$ of $\mathbb{F}_q$, we introduce the auxiliary set of functions

$$N_S = \left\{ f : \mathbb{F}_q \to S, \text{ and } \sum_{c \in S} cf(c) = 0 \right\}$$

and set $n_S = \#N_S$. By inclusion exclusion, it is easy to check that

$$N = \sum_{S \subseteq \mathbb{F}_q} (-1)^{q-|S|} n_S. \quad (1)$$

Now if $e_p(u) = e^{2\pi i u/p}$, consider the identity

$$n_S = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \left( \sum_{f : \mathbb{F}_q \to S} e_p(\sum_{c \in \mathbb{F}_q} \text{Tr}(acf(c))) \right)$$

which follows from the standard property

$$\frac{1}{q} \sum_{a \in \mathbb{F}_q} e_p(\text{Tr}(ax)) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

By exchanging the sum with the product, we obtain

$$n_S = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \left( \prod_{c \in \mathbb{F}_q} \sum_{t \in S} e_p(\text{Tr}(act)) \right).$$

By isolating the term with $a = 0$ in the external sum and noticing that the internal product does not depend on $a$ (for $a \neq 0$), we get

$$n_S = \frac{|S|^q}{q} + \frac{1}{q} \sum_{a \in \mathbb{F}_q} \left( \prod_{b \in \mathbb{F}_q^*} \sum_{t \in S} e_p(\text{Tr}(bt)) \right).$$

Finally

$$n_S = \frac{|S|^q}{q} + \frac{q-1}{q} \prod_{b \in \mathbb{F}_q^*} \sum_{t \in S} e_p(\text{Tr}(bt)). \quad (2)$$

Now let us plug equation (2) in equation (1) and obtain

$$N - \sum_{S \subseteq \mathbb{F}_q} \frac{(-1)^{q-|S|}}{q} |S|^q = \frac{q-1}{q} \sum_{S \subseteq \mathbb{F}_q} (-1)^{q-|S|} \prod_{b \in \mathbb{F}_q^*} \sum_{t \in S} e_p(\text{Tr}(bt)).$$

Note that inclusion exclusion gives

$$\sum_{S \subseteq \mathbb{F}_q} \frac{(-1)^{q-|S|}}{q} |S|^q = (q - 1)!. $$

Therefore

$$N - (q - 1)! = \frac{q-1}{q} \sum_{S \subseteq \mathbb{F}_q} (-1)^{q-|S|} |S| \prod_{b \in \mathbb{F}_q^*} \sum_{t \in S} e_p(\text{Tr}(bt)).$$

Using the fact that for $b \in \mathbb{F}_q^*$

$$\sum_{t \in S} e_p(\text{Tr}(bt)) = - \sum_{t \notin S} e_p(\text{Tr}(bt))$$
and grouping together the term relative to $S$ and the term relative to $F_q \setminus S$, we get

\[(3) \quad |N - (q - 1)!| \leq \frac{q - 1}{2q} \sum_{S \subseteq F_q} (q - 2|S|)! \prod_{b \in F_q^*} \left| \sum_{i \in S} e_p(\text{Tr}(bt)) \right|.
\]

Now let us also observe that

\[\sum_{b \in F_q^*} \left| \sum_{i \in S} e_p(\text{Tr}(bt)) \right|^2 = q|S|,
\]

so that

\[\sum_{b \in F_q^*} \left| \sum_{i \in S} e_p(\text{Tr}(bt)) \right|^2 = (q - |S||S|).
\]

From the fact that the geometric mean is always bounded by the arithmetic mean (i.e. $(\prod_{i=1}^{k} |a_i|^2)^{1/k} \leq \frac{1}{k} \sum_{i=1}^{k} |a_i|^2$), we have that

\[(4) \quad \prod_{b \in F_q^*} \left| \sum_{i \in S} e_p(\text{Tr}(bt)) \right| \leq \left( \frac{1}{q - 1} \sum_{b \in F_q^*} \left| \sum_{i \in S} e_p(\text{Tr}(bt)) \right|^2 \right)^{(q-1)/2}
\]

\[= \left( \frac{q - |S||S|}{q - 1} \right)^{(q-1)/2}.
\]

Furthermore, using (3) and (4) we obtain

\[(5) \quad |N - (q - 1)!| \leq \frac{q - 1}{2q(q - 1)^{(q-1)/2}} \sum_{S \subseteq F_q} (q - 2|S|)! ((q - |S||S|)^{(q-1)/2}.
\]

We want to estimate the above sum. Consider the inequality

\[(6) \quad ((q - |S||S|)^{(q-1)/2} \leq \left( \frac{q}{2} \right)^{q-1},
\]

and the identity

\[(7) \quad \sum_{S \subseteq F_q} |q - 2|S|| = 2q \left( \frac{q - 1}{[q/2]} \right),
\]

which holds since

\[2 \sum_{S \subseteq F_q, |S| \leq q/2} (q - 2|S|) = 2 \left[ \sum_{j=0}^{[q/2]} \binom{q}{j} (q - j) - \sum_{j=1}^{[q/2]} \binom{q}{j} (j) \right]
\]

\[= 2q \left[ \sum_{j=0}^{[q/2]} \binom{q-1}{j} - \sum_{j=1}^{[q/2]} \binom{q-1}{j-1} \right] = 2q \left( \frac{q-1}{[q/2]} \right).
\]

From the standard inequality

\[\binom{2n}{n} \leq \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{2n + 1/2}}
\]

which can be found for example in [3], we deduce

\[(8) \quad \binom{q-1}{[q/2]} \leq \sqrt{\frac{2}{\pi}} \frac{2^{q-1}}{\sqrt{q - 1/2}}.
\]
Therefore, (5), (6), (7) and (8) imply
\[|N - (q - 1)!| \leq \left(\frac{q - 1}{\sqrt{q - 1/2}}\right)^{\frac{1}{2}} \sqrt{\frac{2}{\pi}} \left(\frac{q}{q - 1}\right)^{\frac{q}{q/2}}\]
and in view of the inequalities
\[
\frac{q - 1}{\sqrt{q - 1/2}} < 1, \quad \left(\frac{q}{q - 1}\right)^{\frac{q}{q/2}} < e\]
we finally obtain
\[|N - (q - 1)!| \leq \sqrt{\frac{2e}{\pi}} q^{q/2}\]
and this completes the proof. 

**Conclusion.** Computations suggest that a more careful estimate of the sum in (9) would yield to a constant \(\sqrt{\frac{2e}{\pi}}\) instead of \(\sqrt{\frac{2e}{\pi}}\) as coefficient in \(q^{q/2}\) in the statement of Theorem 1. However we feel that such a minor improvement does not justify the extra work.

The ideas in the proof of Theorem 1 can be used to deal with the analogous problem of enumerating the permutation polynomials that have the \(i\)-th coefficient equal to 0 and also to the problem of enumerating the permutation polynomials with degree less than \(q - k\) (for fixed \(k\)). However, the exponential sums that need to be considered are significantly more complicated.

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