THE BAOUENDI-TREVES APPROXIMATION THEOREM FOR GEVREY CLASSES AND APPLICATIONS

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ABSTRACT. In this work we show how to extend the seminal Baouendi-Treves approximation theorem for Gevrey functions and ultradistributions. As applications we present a Gevrey version of the approximate Poincaré Lemma and study ultradistributions vanishing on maximally real submanifolds.

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1. Introduction

The goal of this paper is to extend the celebrated Baouendi-Treves approximation theorem to Gevrey functions and ultradistributions. The classical Baouendi-Treves theorem has deep implications in the theory of CR geometry and in the theory of local solvability of locally integrable structures.

Let us denote by Ω an open subset of \( \mathbb{R}^N \). A locally integrable structure is a subbundle \( \mathcal{L} \) of the complexified tangent bundle \( \mathbb{C}T\Omega \) if given an arbitrary point \( p_0 \in \Omega \) there are an open neighborhood \( U_0 \) of \( p_0 \) and functions \( Z_1, \ldots, Z_m \in C^\infty(U_0) \) such that the orthogonal of \( \mathcal{L} \) is generated over \( U_0 \) by their differentials \( dZ_1, \ldots, dZ_m \). We say that \( u \) is a solution of \( \mathcal{L} \) if, for every (smooth) local section \( L \in \mathcal{L} \), we have \( Lu = 0 \). The Baouendi-Treves approximation theorem states that any \( u \) in \( C^k(\Omega) \), \( k \in \{0, 1, 2, \ldots, \infty\} \), that is solution of \( \mathcal{L} \) can be approximated in a small neighborhood of any given point of \( \Omega \) in the \( C^k \)-topology by polynomials in \( Z_1, \ldots, Z_m \) and if \( u \in \mathcal{D}'(\Omega) \) is a solution a similar result holds in the topology of \( \mathcal{D}' \). Further generalizations for Lebesgue spaces \( L^p \), \( 1 \leq p < \infty \); Sobolev spaces; Hölder spaces; and (localizable) Hardy spaces \( h^p \), \( 0 < p < \infty \) were given in \cite{HM98, BCR08}.

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Our main theorem extends to the class of Gevrey functions and their dual spaces.

**Theorem 1.1** (Baouendi-Treves approximation formula). Let \( \mathcal{L} \) be a \( G^s \)-locally integrable structure on \( \Omega \). Let us assume that there is \( Z = (Z_1, \ldots, Z_m) : \Omega \to \mathbb{C}^m \) of class \( G^s \) such that \( \partial Z_1, \ldots, \partial Z_m \) spans \( \mathcal{L}^\perp \) over \( \Omega \). Then, for any \( p \in \Omega \), there exist two open sets \( U \) and \( W \) with \( \overline{U} \subset W \subset \Omega \) such that

1. any \( u \in G^s(W) \) that is a solution of \( \mathcal{L} \) in \( W \) is the limit in \( G^s(U) \) of a sequence of polynomial solutions \( P_j(Z) \).
2. any \( u \in \mathcal{D}'_s(W) \) that is a solution of \( \mathcal{L} \) in \( W \) is the limit in \( \mathcal{D}'_s(U) \) of a sequence of polynomial solutions \( P_j(Z) \).

The Baouendi-Treves approximation formula was already proved in the special case when the Gevrey locally integrable structure has corank zero, i.e., every point has a neighborhood \( U \) where we have defined \( Z_1, \ldots, Z_N \) Gevrey functions whose differential generate \( \mathbb{C}T^*\Omega \) (see [Cae01] and [Rag19]).

Our first application is a Gevrey version of a result called approximate Poincaré Lemma (see [Tre92]). It is a useful lemma in the theory of local solvability of locally integrable structures that essentially says that a form that is \( \mathcal{L} \)-closed is the limit of \( \mathcal{L} \)-exact forms, here \( \mathcal{L} \) is a differential operator induced by the de Rham operator.

Another application says that an ultradistribution solution of \( \mathcal{L} \) that vanishes in a submanifold maximally real with respect to \( \mathcal{L} \) must be zero in a neighborhood of the submanifold.

The proof of Theorem 1.1 will be divided in two steps: first for ultradifferentiable functions (classical solutions) and second for ultradistributions (weak solutions).

The novelty here, in one hand, to provide a finer way to write the commutator formula first given by [BT81, BCH08], see (3.12), which allow us to obtain optimal control on the constants appearing in the process of differentiating indefinitely the approximation operators when the solution is classic. On the other hand, when the solution is only an ultradistribution, we need to justify the approximation operator by proving that the traces of solutions of \( \mathcal{L} \) are well defined (by the same formula given in [Hor90, Section 8] in the case of a distribution) and then take the full advantage of this formula (B.2) to obtain the approximation scheme in these case.

We point out that the original arguments for weak solutions cannot be applied in our situation since ultradistributions cannot be represented by a finite order differential operator. However our argument can be used to recover the original result without making use of either: the representation of distributions by means of a finite order partial differential operators, or Sobolev embedding theorems. Thus we strongly believe that our arguments can be used to simplify the original arguments.

The paper is organized as follows: in Section 2 we recall some definitions and basic results of the Gevrey functions and introduce the locally integrable structures. The proof of Theorem 1.1 is presented in Section 3, first for Gevrey functions, Subsection 3.1, then for Gevrey ultradistributions, Subsection 3.2. We present two main applications: the first is given in Section 4 where we use Theorem 1.1 to prove the approximate Poincaré Lemma in the Gevrey topology; and the second is treated in Section 5 where we study ultradistributions vanishing on maximally real submanifolds. The approximation theorem for more general classes of ultradifferentiable functions and ultradistributions is discussed in Section 6. Finally, we conclude with two sections in the Appendix regarding some technicalities needed throughout the paper.
2. Definitions and Preliminary Results

Let $\Omega$ be an open subset of $\mathbb{R}^N$ and fix $s \geq 1$. A Gevrey function of order $s$ in $\Omega$ is a smooth function $f \in C^\infty(\Omega)$ such that for every $K$ compact subset of $\Omega$ there is a $h > 0$ such that
\[
\|f\|_{h,K} \doteq \sup_{x \in K} \left(\frac{1}{h^{s|\alpha|}} \sup_{x \in K} |\partial^\alpha f(x)|\right) < \infty.
\] (2.1)

We will denote the space of Gevrey functions of order $s$ in $\Omega$ by $G^s(\Omega)$. We recall that $G^1(\Omega)$ is the space of real-analytic functions in $\Omega$. In this work we will always assume that $s > 1$ and we shall denote by $G^s_c(\Omega)$ the space of Gevrey functions of order $s$ with compact support.

If $V \subset \subset \Omega$ and $h > 0$, we shall denote by $G^{s,h}(\mathbf{V})$ the space of all smooth functions $f$ in $\mathbf{V}$ for which $\|f\|_{h,\mathbf{V}} < \infty$. Moreover, we denote by $G^s(\mathbf{V})$ the space of all smooth functions $f$ in $\mathbf{V}$ for which there is a $h > 0$ such that $f \in G^{s,h}(\mathbf{V})$ and we denote by $G^{s,h}_c(\mathbf{V})$ the space of all $f \in C^\infty(\mathbb{R}^N)$ with support in $\mathbf{V}$ such that $f|_{\mathbf{V}} \in G^{s,h}(\mathbf{V})$.

The topological dual of $G^s_c(\Omega)$ will be called the space of ultradistributions of order $s$ and will be denote by $\mathcal{D}'(\Omega)$. The continuity of $u \in \mathcal{D}'(\Omega)$ can be expressed in the following way: for every $V \subset \subset \Omega$ and every $h > 0$ there is $C_h > 0$ such that
\[
|u(\varphi)| \leq C_h \|\varphi\|_{h,\mathbf{V}},
\]
for every $\varphi \in G^{s,h}_c(\mathbf{V})$. We will denote by $\mathcal{E}'(\Omega)$ the space of ultradistributions with compact support in $\Omega$.

Let us assume that 0 belongs to $\Omega$, $N = m + n$ and consider a $G^s$-locally integrable structure of rank $n$ in $\Omega$, i.e., a subbundle $\mathcal{L}$ of $\mathbb{T}\Omega$ of rank $n$ over $\Omega$ which the orthogonal, $\mathcal{L}^\perp$, is locally generated by the differentials of $m$ Gevrey functions of order $s$ in $\Omega$.

According to \cite{BCH08, Tre92} we can assume, shrinking $\Omega$ around 0 if necessary, the existence of a local system of $G^s$ coordinates $(x, t) = (x_1, \ldots, x_m, t_1, \ldots, t_n)$ in $\Omega$ as well as a map $\phi : \Omega \rightarrow \mathbb{R}^m$, $\phi = (\phi_1, \ldots, \phi_m)$ of class $G^s$ satisfying
\[
\phi_k(0, 0) = 0, \quad d_x \phi_k(0, 0) = 0, \quad k = 1, \ldots, m
\] (2.2)
such that
\[
Z_k(x, t) = x_k + i\phi_k(x, t), \quad k = 1, \ldots, m.
\] (2.3)

Denote by $B_R^{\mathbb{R}^p}(0) = \{x \in \mathbb{R}^p : |x| < R\}$ and define $V := B_R^{\mathbb{R}^m}(0) \times B_R^{\mathbb{R}^n}(0)$. Since $d_x \phi_1(0) = \cdots = d_x \phi_m(0) = 0$, we can choose a positive number $R$ such that $V \subset \subset \Omega$ and
\[
|\phi_k(x, t) - \phi_k(y, t)| \leq \frac{1}{2}|x - y|, \quad \forall (x, t), (y, t) \in \mathbf{V}.
\] (2.4)

Fix an open neighborhood $W \subset \subset \Omega$ of $\mathbf{V}$. Modifying the imaginary part of $Z$ outside of $W$ using cutoff functions of class $G^s$ we can obtain a locally integrable structure defined globally in $\mathbb{R}^N$ that agrees with $\mathcal{L}$ in $W$. Abusing of notation we will still denote this new structure by $\mathcal{L}$ and assume that (2.4) holds globally in $\mathbb{R}^N$. Note that the conclusions that we will obtain for this new structure $\mathcal{L}$ will also be true for the old structure in $V$.

\footnote{It is easy to check that their proofs also work in the $G^s$-category.}
Since \( dZ_1, \ldots, dZ_m, dt_1, \ldots, dt_n \) is a global frame for \( \mathbb{C}T^*\mathbb{R}^N \) we can consider its dual frame in \( \mathbb{C}T\mathbb{R}^N \), i.e., consider \( N \) vector fields:

\[
M_1, \ldots, M_m, L_1, \ldots, L_n
\]

with the property that

\[
dZ_k(M_{k'}) = \delta_{kk'}, \quad dZ_k(L_j) = 0, \quad k, k' \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, n\},
\]

\[
dt_j(M_k) = 0, \quad dt_j(L_{j'}) = \delta_{jj'}, \quad k \in \{1, \ldots, m\}, \quad j, j' \in \{1, \ldots, n\}.
\]

Finally, note that the differential of any \( C^1 \) function \( w(x,t) \) can be expressed in the basis \( \{dZ_1, \ldots, dZ_m, dt_1, \ldots, dt_n\} \) of \( \mathbb{C}T^*\mathbb{R}^N \) as

\[
dw = \sum_{j=1}^n L_jw\, dt_j + \sum_{k=1}^m M_kw\, dZ_k.
\]

Let \( X_1, \ldots, X_N \) be a family of \( N \) pairwise commuting smooth vector fields that form a global frame to \( \mathbb{C}T\mathbb{R}\Omega \). We can define the space of Gevrey functions regarding \( X_1, \ldots, X_N \) as the space of all \( f \in C^\infty(\Omega) \) such that for every \( K \subset \Omega \) compact there is a \( h > 0 \) such that

\[
\sup_{\alpha \in \mathbb{Z}_N^+} \left( \frac{1}{h^{\|\alpha\|}} \sup_{(x,t) \in K} |X^\alpha f(x,t)| \right) < \infty.
\]

We shall denote this space by \( G^s(\Omega; X) \). A sequence of functions \( f_\nu \in G^s(\Omega; X) \) converges to \( f \in G^s(\Omega; X) \) if for every \( K \subset \Omega \) compact there is \( h > 0 \) such that for every \( \epsilon > 0 \) there is \( \nu_0 \) such that

\[
\sup_{\alpha \in \mathbb{Z}_N^+} \left( \frac{1}{h^{\|\alpha\|}} \sup_{(x,t) \in K} |X^\alpha f_\nu(x,t) - X^\alpha f(x,t)| \right) < \epsilon,
\]

for every \( \nu > \nu_0 \).

Analogously, we denote by \( G^s(\Omega; L, M) \) the space of Gevrey functions with respect to the vector fields considered in (2.5), associated with a locally integrable structure. Since \( L \) is a \( G^s \)-locally integrable structure, it was proved in [Rag19] that

\[
G^s(\Omega; L, M) \text{ is isomorphic to } G^s(\Omega) \text{ as topological spaces}
\]

and the same holds for compact sets. These spaces will play an important role in this work since part of the proof will be to show that a sequence of functions converges in \( G^s(\bar{V}; M, L) \) (and consequently in \( G^s(\bar{V}) \)) for a relatively compact open subset \( V \) of \( \Omega \). We will also use the following notation: for every \( k \) positive integer we denote

\[
\|f\|_{C^k(\bar{V})} = \sum_{|\alpha| \leq k} \sup_{x \in \bar{V}} |\partial^\alpha f(x)|
\]

where \( f \in C^k(\bar{V}) \).

3. The ultradifferentiable Baouendi-Treves approximation formula

In this section we will present the Baouendi-Treves approximation formula for ultradifferentiable functions and ultradistributions that are solutions of a locally integrable structure of arbitrary rank. It is easy to see that the theorem follows if we prove that the solutions are limit in the appropriate topology of entire functions in \( Z \).
3.1. Proof of Baouendi-Treves approximation theorem in $G^s$. Let $u \in G^s(\Omega)$ be a solution of $L$ in $W$. For each $\chi \in G^s_c(B^m_R(0))$ and for each $\tau > 0$, define the function $E^\chi_\tau[u]$ by

$$E^\chi_\tau[u](x, t) := \left(\frac{\tau}{\pi}\right)^{\frac{m}{2}} \int_{\mathbb{R}^m} e^{-\tau(\langle x, t \rangle - \langle y, 0 \rangle)^2} \chi(y) u(y, 0) \det Z_x(y, 0) \, dy, \quad (x, t) \in \mathbb{R}^N.$$  \hspace{1cm} (3.1)

For each $\tau > 0$, $E^\chi_\tau[u]$ is an entire function of $Z(x, t)$. Thus $E^\chi_\tau[u] \in G^s(\mathbb{R}^N)$ and is a solution of $L$. Consider also the functions defined by

$$G^\chi_\tau[u](x, t) := \left(\frac{\tau}{\pi}\right)^{\frac{m}{2}} \int_{\mathbb{R}^m} e^{-\tau(\langle x, t \rangle - \langle y, 0 \rangle)^2} \chi(y) u(y, t) \det Z_x(y, t) \, dy,$$

and,

$$R^\chi_\tau[u](x, t) := G^\chi_\tau[u](x, t) - E^\chi_\tau[u](x, t).$$  \hspace{1cm} (3.3)

We note that $G^\chi_\tau[u]$ converges to $\chi u$ even when $u$ is not a solution of $L$.

**Proposition 3.1.** Let $\chi \in G^s_c(B^m_R(0))$ and $u \in G^s(\mathbb{V})$. Then $G^\chi_\tau[u]$ converges to $\chi u$ in $G^s(\mathbb{V})$ when $\tau \to \infty$.

**Proof.** It is enough to prove (see (2.8)) that

$$G^\chi_\tau[u](x, t) \to \chi u \quad \text{in} \quad G^s(\mathbb{V}; M, L).$$  \hspace{1cm} (3.4)

Note that we may write

$$G^\chi_\tau[u](x, t) - \chi(x)u(x, t) = I^\chi_\tau[u](x, t) - J^\chi_\tau[u](x, t)$$

where $I^\chi_\tau[u]$ and $J^\chi_\tau[u]$ can be written, after the change of variables $y \mapsto x + \tau^{-1/2}y$ in (3.2), as

$$I^\chi_\tau[u](x, t) = \pi^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-\tau(\langle z, x \rangle \cdot y)^2} (v(x + \tau^{-1/2}y, t) - v(y, t)) \, dy,$$

$$J^\chi_\tau[u](x, t) = \pi^{-\frac{m}{2}} \int_{\mathbb{R}^m} \left( e^{-\tau(\langle z, x \rangle - \langle z + \tau^{-1/2}y, t \rangle)^2} - e^{-\langle z, x \rangle y^2} \right) v(x + \tau^{-1/2}y, t) \, dy,$$

and the function $v$ is defined by

$$v(y, t) = \begin{cases} \chi(y) u(y, t) \det Z_x(y, t), & (y, t) \in B^m_R(0) \times B^n_R(0), \\ 0, & (y, t) \in (\mathbb{R}^m \setminus B^m_R(0)) \times B^n_R(0). \end{cases}$$

We have

$$|v(x + \tau^{-1/2}y, t) - v(x, t)| \leq \tau^{-\frac{1}{2}} \|\nabla v\|_{C(\mathbb{V})},$$

$$\leq \tau^{-\frac{1}{2}} \|\chi\|_{C^1(B_R(0))} \|u\|_{C^1(\mathbb{V})} \|\det Z_x\|_{C^1(\mathbb{V})},$$

therefore,

$$|I^\chi_\tau[u](x, t)| \leq \pi^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-\|y\|^2 + \phi(x, t)\|y\|^2} \left|v(x + \tau^{-1/2}y, t) - v(x, t)\right| \, dy$$

$$\leq \tau^{-\frac{1}{2}} \frac{B}{\pi^\frac{m}{2}} \|\chi\|_{C^1(B_R(0))} \|u\|_{C^1(\mathbb{V})} \int_{\mathbb{R}^m} e^{-\frac{3}{4}y^2} \, dy,$$  \hspace{1cm} (3.6)

where $B := \|\det Z_x\|_{C^1(\mathbb{V})}$. 

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To estimate \( J^\chi_{\tau}[u](x,t) \), we use the fact that \( |e^{-\tau(Z(x,t)-Z(x+\tau^{-1/2}y,t))} - e^{-3|y|^2/4} \) and \( |e^{-(Z(x,t)y)^2}| \leq e^{-3|y|^2/4} \), to obtain
\[
|J^\chi_{\tau}[u](x,t)| \leq \pi^{-\frac{m}{2}}\|v\|_{C(\mathbb{V})} \int_{\mathbb{R}^m} \left| e^{-\tau(Z(x,t)-Z(x+\tau^{-1/2}y,t))^2} - e^{-(Z(x,t)y)^2} \right| dy 
\]
\[
\leq \pi^{-\frac{m}{2}}\|v\|_{C(\mathbb{V})} \int_{|y|<A} \left| e^{-\tau(Z(x,t)-Z(x+\tau^{-1/2}y,t))^2} - e^{-(Z(x,t)y)^2} \right| dy 
\]
\[
+ \pi^{-\frac{m}{2}}\|v\|_{C(\mathbb{V})}e^{-A^2/2} \int_{|y|\geq A} 2e^{-|y|^2/4} dy, \quad (3.7)
\]
for every \( A > 0 \). To estimate the first integral on the rightmost hand-side of \((3.7)\), we fix \( y \) and \( t \) and note that \( \zeta_1 = [Z(x,t) - Z(x+\tau^{-1/2}y,t)]/\tau^{-1/2} \) converges to \( \zeta_2 = -Z_x(x,t)y \) uniformly in \( x \in \mathbb{R}^m \) as \( \tau \) goes to \( 0 \) and so there is \( C > 0 \) such that \( ||\zeta_1||^2 - ||\zeta_2||^2 \leq CT^{-1/2} \). This implies that Re \( [\zeta_1]^2 \geq 0 \) and Re \( [\zeta_2]^2 \geq 0 \) and using that \( e^{-\zeta} \) is a Lipschitz function on Re \( \zeta \geq 0 \) we conclude that
\[
|J^\chi_{\tau}[u](x,t)| \leq \frac{B}{\pi^{\frac{m}{2}}}||\chi||_{C(B_R(0))}\|v\|_{C(\mathbb{V})} \left( CA^{m-1/2} + e^{-A^2/2} \int_{|y|\geq A} 2e^{-|y|^2/4} dy \right). \quad (3.8)
\]
Using \((3.8)\), we may rewrite Lemma II.1.4 and Lemma II.1.6 in \([BCH08]\) as
\[
M_k G^\chi_{\tau}[u] = G^\chi_{\tau}[M_k u] + G^M_{\chi}\xi[u], \quad \forall k = 1, \ldots, m \quad (3.9)
\]
and
\[
L_j G^\chi_{\tau}[u] = G^\chi_{\tau}[L_j u] + G^L_{\chi}\xi[u], \quad \forall j = 1, \ldots, n. \quad (3.10)
\]
In order to simplify the notation define \( X_j = L_j \) for \( j = 1, \ldots, n \) and \( X_{n+k} = M_k \), for \( k = 1, \ldots, m \). Since \( X_1, \ldots, X_{n+m} \) are pairwise commuting, we have that
\[
X^\alpha G^\chi_{\tau}[u] = \sum_{\alpha' + \alpha'' = \alpha} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) G^\alpha\xi' [X^\alpha'' u], \quad \forall \alpha \in \mathbb{Z}_+^N. \quad (3.11)
\]
Thus it follows that
\[
X^\alpha G^\chi_{\tau}[u] - X^\alpha(\chi u) = \sum_{\alpha' + \alpha'' = \alpha} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) (G^{\alpha''\chi} [X^\alpha'' u] - (X^\alpha) [X^\alpha'' u]). \quad (3.12)
\]
To estimate \( X^\alpha G^\chi_{\tau}[u] - X^\alpha(\chi u) \) on \( \mathbb{V} \) we will make use of \((3.5)\), \((3.6)\) and \((3.8)\), together with \((3.12)\) to obtain
\[
|X^\alpha G^\chi_{\tau}[u](x,t) - X^\alpha(\chi u)(x,t)| \leq \sum_{\alpha' + \alpha'' = \alpha} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) \left| J^{\alpha''\chi}_{\tau} [X^\alpha'' u](x,t) \right| + |J^{\alpha\chi}_{\tau} [X^\alpha u](x,t)| 
\]
\[
\leq \tau^{-1/2} \frac{BC_A}{\pi^{\frac{m}{2}}} \sum_{\alpha' + \alpha'' = \alpha} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) \|X^\alpha(\chi)\|_{C^1(B_R(0))} \|X^\alpha'' u\|_{C^1(\mathbb{V})} 
\]
\[
+ e^{-A^2/2} \frac{B\hat{C}}{\pi^{\frac{m}{2}}} \sum_{\alpha' + \alpha'' = \alpha} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) \|X^\alpha(\chi)\|_{C(B_R(0))} \|X^\alpha'' u\|_{C(\mathbb{V})} \quad (3.13)
\]
where

$$\tilde{C}_A := \int e^{-\frac{4}{7}|y|^2} \, dy + CA^m \quad \text{and} \quad \tilde{C} := \int 2e^{-|y|^2/4} \, dy.$$ 

Now we assume that $\chi \in G^s_0(B^m_R(0))$ and $u \in G^s(V)$ thus, it follows from (2.8) that we can find $h > 0$ such that for every $\alpha', \alpha'' \in \mathbb{Z}^N_+$, we have

$$\|X^{\alpha''}u\|_{C^1(V)} \leq h^{|\alpha''|+1}\|u\|_{h,V}(\alpha'') + 1)^s \quad \text{and} \quad \|X^{\alpha'}\chi\|_{C^1(B^m_R)} \leq h^{|\alpha'|+1}\|\chi\|_{h,B^m_R(0)}(|\alpha'| + 1)^s.$$ 

We may use (3.13) and (3.14) to obtain

$$\sup_{\mathcal{V}} \frac{|X^\alpha G^s_\tau[u] - X^\alpha(\chi u)|}{(2h)^{|\alpha|}} \leq \left(\tau^{-1/2}\tilde{C} + e^{-A^2/2}\tilde{C}_A\right) \frac{BA}{\pi^\frac{m}{2}} h^2 2^3 \|\chi\|_{h,B^m_R(0)} \|u\|_{h,V}, \quad (3.15)$$

where we used that $(|\alpha| + 2)^s \leq 2^{s(|\alpha|+3)}|\alpha|!^s$. Now, for a given $\epsilon > 0$ choose $A > 0$ so that $e^{-A^2/2}\tilde{C} \leq \epsilon/2$ and then choose $\tau > 1$ so that $\tau^{-1/2}\tilde{C}_A \leq \epsilon/2$ to conclude that $G^s_\tau[u] \converges \chi u$ in $G^s(V;M,L)$.

We would like to point out that the proof yields a slightly stronger version of Proposition 3.1. Denote by $\mathcal{B}(G^s_0(B^m_R(0)) \times G^s(V), G^s(V))$ the space of the bilinear continuous operator and denote by $P$ the bilinear operator defined by the usual product, i.e., $P(\chi,u) = \chi u$.

**Proposition 3.2.** The operator $G_\tau : G^s_0(B^m_R(0)) \times G^s(V) \rightarrow G^s(V)$ define by $G_\tau(\chi,u) = G^s_\tau[u]$ is a bilinear and continuous. Moreover, the sequence of operators $G_\tau \converges$ to $P$ in $\mathcal{B}(G^s_0(B^m_R(0)) \times G^s(V); G^s(V))$ as $\tau \rightarrow \infty$.

Observe that if we take $\chi = 1$ in $B^m_{R/2}(0)$ and define $U := B^m_S(0) \times B^m_T(0)$, where $0 < S \leq R/2$ and $0 < T \leq R$, then $G^s_\tau[u] \converges u$ in $G^s(U)$.

Next, we recall (see, for instance, [BCH08, pag. 59-60]) that there exists a positive constant $T < R$ such that

$$|e^{-\tau(Z(x,t)-Z(y,t'))^2}| \leq e^{-\tau R^2/33}, \quad (3.16)$$

for all $(x,t) \in B^m_{R/4}(0) \times B^m_T(0)$ and $(y,t') \in \{y \in \mathbb{R}^m : |y| \geq R/2\} \times B^m_T(0)$. From now on, we fix the open set $U$ in the statement of Theorem 3.1 to be $B^m_{R/4}(0) \times B^m_T(0)$. The proof of Theorem 1.1 in $G^s$, will be complete once we prove the following result.

**Proposition 3.3.** Let $\chi \in G^s_0(B^m_R(0))$, with $\chi = 1$ in $B^m_{R/2}(0)$ and $u \in G^s(\Omega)$ that is a solution of $\mathcal{L}$ in $W$. Then $R^s_\tau[u] \converges$ to $0$ in $G^s(U)$ when $\tau \rightarrow \infty$.

**Proof.** It is a consequence of Stokes’ theorem that we may write $R^s_\tau[u]$ given in (3.3) as

$$R^s_\tau[u](x,t) = \left(\frac{T}{\pi}\right)^{m^2} \sum_{j=1}^{m^2} \int_{\mathbb{R}^m \times [0,t]} e^{-\tau(Z(x,t)-Z(y,t'))^2} (L_j\chi)(y,t') u(y,t') \det Z_x(y,t') \, dt' \wedge dy.$$ (3.17)

Since $\chi(y) = 1$ for $|y| < R/2$ and supp $\chi \subset B^m_R(0)$, $L_j\chi$ vanishes for $\{|y| \leq R/2\} \cup \{|y| \geq R\}$, we can write

$$R^s_\tau[u](x,t) = \left(\frac{T}{\pi}\right)^{m^2} \sum_{j=1}^{m^2} \int_{A_{R^2/4,T}} \int_0^1 e^{-\tau(Z(x,t)-Z(y,rt))^2} (L_j\chi)(y,rt) u(y,rt) \det Z_x(y,rt) t \, dr \, dy,$$ (3.18)
where $A(R, R) := \{ y \in \mathbb{R}^m : R < \| y \| < R \}$. For each $(\alpha, \beta) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^n$ we may differentiate under the integration sign the expression in the right hand side of (3.18) to obtain

\[
\partial_x^\alpha \partial_t^\beta R_\gamma^3[u](x, t) = \left( \frac{\tau}{\pi} \right)^m \sum_{j=1}^n \int_{A(\frac{R}{2}, R)} \int_0^1 \partial_x^\alpha \partial_t^\beta \left\{ e^{-\tau(Z(x,t) - Z(y,rt))^2} (L_j \chi)(y, rt) u(y, rt) \det Z_x(y, rt) t_j \right\} dr dy
\]

\[
= \left( \frac{\tau}{\pi} \right)^m \sum_{j=1}^n \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \int_{A(\frac{R}{2}, R)} \int_0^1 \partial_x^\alpha \partial_t^\gamma \left\{ e^{-\tau(Z(x,t) - Z(y,rt))^2} \right\} \times
\]

\[
\times \partial_t^{\beta-\gamma} \left\{ (L_j \chi)(y, rt) u(y, rt) \det Z_x(y, rt) t_j \right\} dr dy.
\]

We can use Lemma A.3 with $f(y, r, x, t) = [Z(x, t) - Z(y, rt)]^2$ yielding that there are constants $C, h > 0$ for which we have

\[
\left| \partial_x^\alpha \partial_t^\beta \left\{ e^{-\tau(Z(x,t) - Z(y,rt))^2} \right\} \right| \leq C h^{\alpha+|\gamma|} (|\alpha| + |\gamma|)! s e^{-\tau \Re (Z(x,t) - Z(y,rt))^2 + s \tau R^2/33}
\]

for every $(x, y, t, r) \in D_{R^2/33}^m(0) \times A(\frac{R}{2}, R) \times D_{R^2/33}^n(0) \times [0, 1]$.

Since there exist $\bar{h} > 0$ so that $(L_j \chi) u \det Z_x \in G^{s, \bar{h}}(V)$ for each $j \in \{1, \ldots, n\}$, we can use (3.16), (3.19) and (3.20) to find a constant $\bar{C} > 0$ independent of $\alpha, \beta, \gamma$ and $\tau$ such that

\[
\left| \partial_x^\alpha \partial_t^\beta R_\gamma^3[u](x, t) \right| \leq \bar{C} \left( \sum_{\gamma \leq \beta} h^{\alpha+|\gamma|} \bar{h}^{\beta-\gamma} \left( \frac{\beta}{\gamma} \right) (|\alpha| + |\gamma|)! s (|\beta| - |\gamma|)! s \right) \tau^{m/2} e^{s R^2/33}
\]

\[
\leq \bar{C} \tau^{m/2} e^{s R^2/33} \left( h + \bar{h} \right)^{|\alpha|+|\beta|} 2^{|\beta|} (|\alpha| + |\beta|)! s,
\]

for every $(x, t) \in U$. Thus, $R_\gamma^3[u]$ converges to 0 in $G^s(U)$ when $\tau$ converges to $\infty$. □

3.2. Proof of Baouendi-Treves approximation theorem in $\mathcal{D}_s'$. Given $\chi \in G^s_c(B_{R}^{2m}(0))$ we will first need to extend the definitions of $E^3_\gamma[u], G^3_\gamma[u]$ and $R^3_\gamma[u]$ when $u \in \mathcal{D}_s'(W)$ is a solution of $\mathcal{L}$.

The definitions of $E^3_\gamma[u]$ and consequently $R^3_\gamma[u]$ will strongly use the fact $u$ is a solution of $\mathcal{L}$ which guarantee that the pullback of $u$ to the submanifolds $\{(x, t) : t = constant\}$ are well defined in the sense of ultradistributions, see Appendix [3].

We can, however, provide a definition for $G^3_\gamma[u]$ for every $u \in \mathcal{D}_s'(V)$ and as a consequence we will proof the convergence of $G^3_\gamma[u]$ to $\chi u$ in $\mathcal{D}_s'(V)$ when $\tau$ goes to $+\infty$ regardless whether $u$ is a solution of $\mathcal{L}$ or not.

Definition 3.4. Let $u \in \mathcal{D}_s'(V)$ and fix $\chi \in G^s_c(B_{R}^{2m}(0))$. We define $G^3_\gamma[u] \in \mathcal{D}_s'(V)$, acting on $\varphi \in G^s_c(V)$ as

\[
G^3_\gamma[u](\varphi) := u(x', t) \chi(x') G^3_\gamma[\psi](x', t) \det Z_x(x', t),
\]

where $\tilde{\chi}$ is any element of $G^s_c(B_{R}^{2m}(0))$ equal to 1 in the projection of the support of $\varphi$ in $\mathbb{R}^m$ and $\psi(x, t) := \varphi(x, t)/ \det Z_x(x, t)$.

Remark 3.5. Note that, it follows immediately from (3.22) and Proposition 3.1 that for every $\varphi \in G^s(V), G^3_\gamma[u](\varphi)$ converges to $(\chi u)(\varphi)$, consequently, $G^3_\gamma[u]$ converges to $\chi u$ in $\mathcal{D}_s'(V)$.  

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Now we will define \( E^\chi[u] \) when \( u \in \mathcal{D}'(V) \) is a solution of \( \mathcal{L} \) in \( W \). To do so, we will follow the results and notations from Appendix \([3]\) in particular, the definition of the trace of an ultradistribution \([3,2]\). Given \( t \in B_{R}^{m}(0) \) we can consider \( \iota_{t} : B_{R}^{m}(0) \rightarrow V \) defined by \( \iota_{t}(x) = (x, t) \). Since \( u \) is a solution of \( \mathcal{L} \) it holds that \( W_{F_{s}}(u) \cap \{(x, t, 0, \theta), (x, t) \in W, \tau \neq 0\} \) is empty. This means that, for each fixed \( \tau \), \( t \) is empty. This means that, for each \( u \), one can check that

\[
(\iota_{t}^{*}u)(\phi) = \frac{1}{(2\pi)^{N}} \int_{\mathbb{R}^{N}} \left( \mathcal{F}(\lambda u)(\eta)e^{it\theta} \int_{\mathbb{R}^{m}} \varphi(x)e^{ix\eta}dx \right)d\sigma d\theta, \quad \forall \phi \in C_{c}(B_{R}^{m}(0)),
\]

where \( \lambda \in C_{c}(W) \) is identically 1 on \( V \) and \( \mathcal{F}(\lambda u) \) stands for the Fourier transform of \( \lambda u \). In the Appendix \([3]\) it is shown that \( \iota_{t}^{*}u \) is an ultradistribution with the property that, for each fixed \( \phi \in C_{c}(W) \), the application \( B_{R}^{m}(0) \ni t \mapsto \iota_{t}^{*}u(\phi) \) is of class \( G_{s}^{\chi} \).

Moving on, notice that

\[
(\iota_{t}^{*}u)(\varphi) = \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} \int_{\mathbb{R}^{N}} (\iota_{0}^{*}u)(\varphi(x)e^{-\tau(x-Z(x',0)^{2})2}\chi(x') \det Z_{x}(x', 0))
\]

is an entire function in \( z \in \mathbb{C}^{m} \). So we can define \( E^\chi[u] \in G^{s}(\mathbb{R}^{N}) \) as

\[
E^\chi[u](x, t) := \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} \int_{\mathbb{R}^{N}} (\iota_{0}^{*}u)(\varphi(x)e^{-\tau(Z(x,Z(x',0)^{2})2}\chi(x') \det Z_{x}(x', 0))
\]

Also, when \( u \) is a solution of \( \mathcal{L} \), one can verify that \( G_{\chi}^{\chi} [u] \) given in \([3,2]\) can be rewritten as

\[
G_{\chi}^{\chi}[u](x, t) = \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} (\iota_{t}^{*}u)(e^{-\tau(Z(x,t)-Z(x',t)^{2})2}\chi(x') \det Z_{x}(x', t))
\]

in this case one can check that \( G_{\chi}^{\chi}[u] \in G^{s}(V) \), see Proposition \([3,2]\).

Still assuming that \( u \) is a solution of \( \mathcal{L} \), we note that, for each \( \varphi \in C_{c}(B_{R}^{m}(0)) \), we can use \([3,26] \) and then \([3,2] \) to write

\[
\int_{B_{R}^{m}(0)} G_{\chi}^{\chi}[u](x, t) \varphi(x)dx = \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} \int_{B_{R}^{m}(0)} (\iota_{0}^{*}u)(e^{-\tau(Z(x,t)-Z(x',t)^{2})2}\chi(x') \det Z_{x}(x', 0)) \varphi(x)dx
\]

\[
= (\iota_{t}^{*}u)(\psi(x', t)\chi(x') \det Z_{x}(x', t))
\]

where \( \psi(x, t) = \varphi(x)/\det Z_{x}(x, t) \) and \( \chi \in C_{c}(B_{R}^{m}(0)) \) is equal to 1 on the support of \( \varphi \). Thus, one can use Proposition \([3,1] \) to obtain that for each fixed \( t \), it holds

\[
\lim_{\tau \rightarrow \infty} \int_{B_{R}^{m}(0)} G_{\chi}^{\chi}[u](x, t) \varphi(x)dx = (\iota_{t}^{*}u)(\varphi(x') \chi(x'))
\]

for every \( \varphi \in C_{c}(B_{R}^{m}(0)) \). Moving on, we will now work on the error term \( R_{\chi}^{\chi}[u] = E_{\chi}^{\chi}[u] = G_{\chi}^{\chi}[u] \) when \( u \in \mathcal{D}'(W) \) is a solution of \( \mathcal{L} \). The goal is to obtain an expression analogous to \([3,18] \).

**Proposition 3.6.** Let \( u \in \mathcal{D}'(W) \) be a solution of \( \mathcal{L} \) and \( \chi \in C_{c}(B_{R}^{m}(0)) \) then

\[
R_{\chi}^{\chi}[u](x, t) = \frac{1}{(2\pi)^{N}} \int_{[0, t]} \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} \sum_{j=1}^{n} (\iota_{j}^{*}u)(e^{-\tau(Z(x,j)-Z(x',j)^{2})2}L_{j}\chi(x') \det Z_{x}(x', t'))dt_{j}
\]

for every \((x, t) \in V\).
Proof. To begin with, consider $\omega_{\tau, \tilde{\tau}}$ to be the sequence of $m$-forms with $G^s$ coefficients defined by

$$\omega_{\tau, \tilde{\tau}}^{(x,t)}(x', t') := \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} e^{-\tau(Z(x,t) - Z(x', t'))^2} G^s_\tilde{\tau}[u](x', t') \tilde{\chi}(x') dZ(x', t')$$

where $dZ = dZ_1 \wedge \cdots \wedge dZ_m$ and $\tilde{\chi} \in G^s_\tilde{\tau}(B_{R}^{m}(0))$ is equal to 1 in supp $\chi$. Also, let us define

$$I_{1}^{\tau, \tilde{\tau}}(x, t) = \int_{\mathbb{R}^m \times [0, t]} d\omega_{\tau, \tilde{\tau}}^{(x,t)}(x', t'),$$

$$I_{2}^{\tau, \tilde{\tau}}(x, t) = \int_{\mathbb{R}^m} \omega_{\tau, \tilde{\tau}}^{(x,t)}(x', t),$$

$$I_{3}^{\tau, \tilde{\tau}}(x, t) = \int_{\mathbb{R}^m} \omega_{\tau, \tilde{\tau}}^{(x,t)}(x', 0).$$

Thanks to Stokes’ theorem it holds that $I_{1}^{\tau, \tilde{\tau}}(x, t) = I_{2}^{\tau, \tilde{\tau}}(x, t) - I_{3}^{\tau, \tilde{\tau}}(x, t)$. Now for any given $\varphi_1 \in G^s_\tau(B_{R}^{m}(0))$, $\varphi_2 \in G^s_\tau(B_{R}^{m}(0))$. Applying $I_{2}^{\tau, \tilde{\tau}}$ to $\varphi_1 \otimes \varphi_2$ in the sense of ultradistributions we obtain

$$I_{2}^{\tau, \tilde{\tau}}(\varphi_1 \otimes \varphi_2) = \int_{B_{R}^{m}(0)} T_{\varphi_2}^{\tau, \tilde{\tau}}(x) \varphi_1(x) dx,$$

where

$$T_{\varphi_2}^{\tau, \tilde{\tau}}(x) := \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} \int_{B_{R}^{m}(0)} \int_{\mathbb{R}^m} e^{-\tau(Z(x,t) - Z(x', t'))^2} G^s_{\tilde{\tau}}[u](x', t) \tilde{\chi}(x') \varphi_2(t) \det Z_x(x', t) dx' dt.$$

Thus, one can use (3.28) to conclude that

$$\lim_{\tilde{\tau} \to \infty} T_{\varphi_2}^{\tau, \tilde{\tau}}(x) = \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} \int_{B_{R}^{m}(0)} \int_{\mathbb{R}^m} e^{-\tau(Z(x,t) - Z(x', t'))^2} \chi(x') \varphi_2(t) \det Z_x(x', t) dt$$

$$= \int_{B_{R}^{m}(0)} G^s_{\tau}[u](x, t) \varphi_2(t) dt. \quad (3.30)$$

If follows from identity (3.30) that $I_{2}^{\tau, \tilde{\tau}}$ converges to $G^s_{\tau}[u]$ in $\mathcal{D}'(V)$ as $\tilde{\tau} \to \infty$. Analogously, $I_{3}^{\tau, \tilde{\tau}}$ converges to $E^s_{\tau}[u]$ in $\mathcal{D}'(V)$ as $\tilde{\tau} \to \infty$. Therefore, all we have to do now is to focus on the next identity

$$R^s_{\tau}[u](\varphi) = \lim_{\tilde{\tau} \to \infty} I_{1}^{\tau, \tilde{\tau}}(\varphi),$$

for every $\varphi \in G^s_{\tau}(V)$. Note that $d\omega_{\tau, \tilde{\tau}}^{(x,t)}$ can be written as

$$d\omega_{\tau, \tilde{\tau}}^{(x,t)}(x', t') = \left( \frac{\tau}{\pi} \right)^{\frac{m}{2}} \sum_{j=1}^{n} e^{-\tau(Z(x,t) - Z(x', t'))^2} L_j(G^s_{\tilde{\tau}}[u](x', t') \tilde{\chi}(x')) dt' \wedge dZ(x', t')$$

and using the following equality

$$L_j(G^s_{\tilde{\tau}}[u](x', t') \tilde{\chi}(x')) = G^s_{\tilde{\tau}}[L_j u](x', t') \tilde{\chi}(x') + G^s_{\tilde{\tau}}[L_j u](x', t') \tilde{\chi}(x') + G^s_{\tilde{\tau}}[u](x', t') L_j \tilde{\chi}(x')$$
together with the convergence stated in (3.28) we obtain

\[ R^x_\tau[u](\varphi) = \int_V \int_{[0,t]} \left( \frac{\tau}{\pi} \right)^\frac{m}{2} \sum_{j=1}^{n} (\tau_{*j}^x u)_{x^j} \left( e^{-\tau(Z(x,t)-Z(x',t'))} \right)^2 L_j(x') \det Z_x(x', t') \varphi(x, t) \text{d}t' \text{d}x \wedge \text{d}t \]

since \( u \) is a solution of \( \mathcal{L} \) and \( L_j \chi = 0 \) over \( \text{supp} \chi \), as we wished to prove. \( \square \)

Now we can use (3.29) to conclude the proof of Theorem 1.1 in \( \mathcal{D}'_s \).

**Proposition 3.7.** Let \( \chi \in G^s_c(B^{Rm}_{R^m}(0)) \), with \( \chi = 1 \) in \( B^{Rm}_{R^m}(0) \) and \( u \in \mathcal{D}'(W) \) a solution of \( \mathcal{L} \). Then \( R^x_\tau[u] \) converges to 0 in \( G^s(U) \) when \( \tau \to \infty \).

**Proof.** For every \( j = 1, \ldots, n \), we define

\[ \Phi_j(x, t, x', r) = e^{-\tau(Z(x,t)-Z(x',t'))} \left( L_j \chi(x') \det Z_x(x', rt) \right) \]

Note that there exist \( \rho > 0 \) such that \( \Phi_j \in G^{s,\rho}(U \times B^{Rm}_{R^m}(0) \times (0,1)) \) for every \( j = 1, \ldots, n \). Now we differentiate \( R^x_\tau[u](x,t) \) using (B.2) to obtain

\[ \partial^\alpha_x \partial^\beta_t \rho \partial^\gamma_r R^x_\tau[u](x,t) = \left( \frac{\tau}{\pi} \right)^\frac{m}{2} \sum_{j=1}^{n} \int_0^1 \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \partial^\beta - \partial^\gamma \left( \rho \partial^\alpha_x \partial^\gamma_{x'} \right) \Phi_j(x, t, x', r) \right) \text{d}r. \quad (3.31) \]

Thus we can consider \( \partial^\alpha_x \partial^\beta_t \Phi_j \) as an element of \( G^{s,\rho}(B^{Rm}_{R^m}(0)) \) in \( x' \in B^{Rm}_{R^m}(0) \) (where \( \rho \) could be any number greater than \( \rho \), let us take \( \rho = 2^s \rho \)) and we can apply estimate (B.9) from Appendix A to obtain

\[ \partial^\beta - \partial^\gamma \left( \rho \partial^\alpha_x \partial^\gamma_{x'} \right) \Phi_j(x, t, x', r) \] \( \leq \| \partial^\alpha_x \partial^\beta_t \Phi_j \|_{\rho, B^{Rm}_{R^m}(0)} C \tilde{H} \| \beta - \gamma \| \}

Now for every \( (x, t) \in U \) and each \( r \in (0,1) \) we have

\[
\| \partial^\alpha_x \partial^\beta_t \Phi_j \|_{\rho, B^{Rm}_{R^m}(0)} = \sup_{\theta \in \mathbb{Z}^m_x} \sup_{x' \in B^{Rm}_{R^m}(0)} \left| \partial^\alpha_x \partial^\beta_t \partial^\gamma_{x'} \Phi_j(x, t, x', r) \right| \bigg| \frac{\rho^{|\theta| + |\alpha + |\gamma|}}{2^s \| \theta \| \| \beta - \gamma \| \}
\leq \sup_{\theta \in \mathbb{Z}^m_x} \sup_{x' \in B^{Rm}_{R^m}(0)} \left| \partial^\alpha_x \partial^\beta_t \partial^\gamma_{x'} \Phi_j(x, t, x', r) \right| \bigg| \frac{\rho^{|\theta| + |\alpha + |\gamma|}}{2^s \| \theta \| \| \beta - \gamma \| \}
\leq \| \Phi_j \|_{\rho, U \times B^{Rm}_{R^m}(0) \times (0,1)} (|\alpha| + |\gamma|)^s (2^s \rho)^{|\alpha + |\gamma|}. \quad (3.33) \]

Let us denote by \( \mathcal{U} = U \times B^{Rm}_{R^m}(0) \times (0,1) \), then using (3.31), (3.32) and (3.33) we obtain

\[
| \partial^\alpha_x \partial^\beta_t R^x_\tau[u](x,t) | \leq \left( \frac{\tau}{\pi} \right)^\frac{m}{2} \sum_{j=1}^{n} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \| \Phi_j \|_{\rho, \mathcal{U}} (|\alpha| + |\gamma|)^s (2^s \rho)^{|\alpha + |\gamma|} C \tilde{H} \| \beta - \gamma \| \}
\leq C \left( \frac{\tau}{\pi} \right)^\frac{m}{2} \sum_{j=1}^{n} \| \Phi_j \|_{\rho, \mathcal{U}} (2^s \rho + \tilde{H})^{|\alpha + |\beta|} (|\alpha| + |\beta|)^s. \quad (3.34) \]
Now let us estimate \( \| \Phi_j \|_{p,U \times B_R^m(0) \times (0,1)} \), we have

\[
\begin{align*}
\partial^a_t \partial^{\beta}_{\mu} \partial^{\gamma}_{\nu} \partial^\sigma \Phi_j(x, t, x', r) & = \\
= \sum_{S_{\beta, \gamma, \sigma}} \left( \frac{\beta}{\beta'} \right) \left( \frac{\gamma}{\gamma'} \right) \left( \frac{\sigma}{\sigma'} \right) \partial^a_t \partial^{\beta'}_{x} \partial^{\gamma'}_{x'} \partial^\rho \partial^{\sigma'}_{x'} e^{-\tau(Z(x,t)-Z(x',rt))} \partial^\sigma \partial^\nu \partial^\mu \Lambda(t, x', r),
\end{align*}
\]

where \( S_{\beta, \gamma, \sigma} = \{ (\beta', \beta'', \gamma', \gamma'', \sigma', \sigma'') : \beta' + \beta'' = \beta; \gamma' + \gamma'' = \gamma; \sigma' + \sigma'' = \sigma \} \) and

\[
\Lambda(t, x', r) = (L_j \chi(x') \det Z(x', rt) t_j).
\]

Using Lemma \( \Lambda.3 \) with \( f(x, t, x', r) = (Z(x, t) - Z(x', rt))^2 \) we see that there are constants \( C' > 0 \) and \( h > 0 \) such that

\[
\left| \partial^a_t \partial^{\beta'}_{x} \partial^{\gamma'}_{x'} \partial^\rho \partial^\mu \Lambda(t, x', r) \right| \leq \tilde{C} h^{3|\beta'| + \gamma' + |\sigma'|} (|\beta| + \gamma' + |\sigma'|) t_j.
\]

Consequently, using (3.35), (3.36), (3.37) and (3.16), we obtain

\[
\left| \partial^a_t \partial^{\beta'}_{x} \partial^{\gamma'}_{x'} \partial^\rho \partial^\mu \Phi_j(x, t, x', r) \right| \leq \tilde{C} h^{3|\beta'| + \gamma' + |\sigma'|} (|\beta| + \gamma' + |\sigma'|) t_j.
\]

Therefore we can take \( \rho = 2(h + \tilde{h}) \), it follows from (3.34) and (3.38) that

\[
\left| \partial^a_t \partial^{\beta'}_{x} \partial^{\gamma'}_{x'} \partial^\rho \partial^\mu \Phi_j(x, t, x', r) \right| \leq \frac{C C' \tilde{C}}{2(2 \rho + H)^{s + |\beta| + |\gamma|} |\alpha| + |\beta| + |\gamma| + |\rho|}.
\]

Proving that \( R_j^m[u] \) converges to 0 in \( G^s(U) \) as desired. \( \square \)

### 4. Approximate Poincaré Lemma

Let \( \Omega \) be an open neighborhood of the origin in \( \mathbb{R}^N \) and assume that we have a locally integrable structure in \( \Omega \) where the orthogonal \( L^\perp \) is defined globally in \( \Omega \) by the differential of \( Z_1, \ldots, Z_m \) and denote \( \lambda(x, t) = (Z_1(x, t), \ldots, Z_m(x, t), t_1, \ldots, t_n) \). We define \( G^s(\Omega, \Lambda^{p,q}) \) the space of all \( (p, q) \)-forms

\[
f(x, t) = \sum_{[t] = p} \sum_{[J] = q} f_{IJ}(x, t) dZ_I \wedge dt_J,
\]

where the coefficients \( f_{IJ} \in G^s(\Omega) \), where \( dZ_I = dZ_{i_1} \wedge \ldots \wedge dZ_{i_p} \) and \( dt_J = dt_{j_1} \wedge \ldots \wedge dt_{j_q} \) for \( I = \{ 1 \leq i_1 < \cdots < i_p \leq m \} \) and \( J = \{ 1 \leq j_1 < \cdots < j_q \leq n \} \). The notation \( K = \{ 1 \leq k_1 < \cdots < k_s \leq r \} \) means that \( K = \{ k_1, \ldots, k_s \} \subset \{ 1, \ldots, r \} \) and that \( k_1 < \cdots < k_s \).
Let us define a linear differential operator $\mathbb{L} : G^s(\Omega, \Lambda^{p,q}) \rightarrow G^s(\Omega, \Lambda^{p,q+1})$ by

$$\mathbb{L}f = \sum_{|I| = p} \sum_{|J| = q} \sum_{j = 1}^n L_j f_{I,J} dt_j \wedge dZ_I \wedge dt_I,$$

for every $f \in G^s(\Omega, \Lambda^{p,q})$. The Gevrey local solvability of $\mathbb{L}$ in degree $(p, q)$ at a point $p_0 \in \Omega$ here means that there is $\Omega_f$ for every $f \in G^s(\Omega, \Lambda^{p,q})$ such that for any other neighborhood $\Omega_1$ of $p_0$ with $\Omega_1 \subset \Omega_0$, we can find a neighborhood $\Omega_2$ of $p_0$ such that $\Omega_2 \subset \Omega_1$ and for any $f_0 \in G^s(\Omega_1, \Lambda^{p,q})$ such that $\mathbb{L}f_0 = 0$ there is $g \in G^s(\Omega_2, \Lambda^{p,q-1})$ such that $\mathbb{L}g = f_0$ in $\Omega_2$.

We recall that the approximate Poincaré lemma is a result concerning approximate solvability of $\mathbb{L}$. Before we actually enunciate and prove this result let us fix more notation and recall an important trick. Let $J = \{1 \leq j_1 < \cdots < j_q \leq n\}$ and $j \in \{1, \ldots, n\} \setminus J$ and we define $\epsilon(j, J)$ to be the sign of permutations to orderate the $q + 1$-form $dt_j \wedge dZ_I$, i.e., $\epsilon(j, J) = 1$ if this number is odd.

Assume that $q \geq 2$ and define, for any $J$ with $|J| = q$, the $q - 1$-form

$$\omega_J = \sum_{j \in J} \epsilon(j, J \setminus \{j\}) t_j dt_{J \setminus \{j\}},$$

and, when $q = 1$, $\omega_J = t_J$.

Now we follow [Tre92], and, for any $q$-form

$$F = \sum_{|J| = q} F_J dt_J,$$

we define an operator for $q$-forms to $q - 1$-forms

$$K^{(q)}F = \sum_{|J| = q} \left\{ \int_0^1 F_J(\sigma t) \sigma^{q-1} d\sigma \right\} \omega_J.$$

This operator satisfies the following formula:

$$F = d_I K^q F + K^{(q+1)} d_I F. \quad (4.2)$$

Assume that $W$ and $V$ are as in the Baouendi-Treves approximation formula, i.e., $V = B_R^{\mathbb{R}^m}(0) \times B_R^{\mathbb{R}^n}(0)$, $V \subset \subset W \subset \subset \Omega$ and such that (2.4) holds and let $U = B_{R/2}^{\mathbb{R}^m}(0) \times B_R^{\mathbb{R}^n}(0)$. For every $\chi \in G^s_c(B_R^{\mathbb{R}^m}(0))$ and $g \in G^s(\Omega)$ we define

$$G^\chi_T[g](z, t) := \left( \frac{m}{\pi} \right)^{\frac{m}{2}} \int e^{-\tau(z-Z(x', t))^2} \chi(x') g(x', t) \det Z(x', t) dx'.$$

Note that $G^\chi_T[g](Z(x, t), t) = G^\chi_T[g](x, t)$. If $f$ is a $(p, q)$-form as in (4.1) we define

$$G^\chi_T[f](z, t) = \sum_{|I| = p} \sum_{|J| = q} G^\chi_T[f_{I,J}](z, t) dz_I \wedge dt_I, \quad (4.4)$$

then $\lambda^*(G^\chi_T[f])(x, t) = G^\chi_T[f](x, t)$ where $G^\chi_T[f]$ is defined by allowing $G^\chi_T$ acts coefficientwise.

We now can define

$$K^{(p,q)}_\chi[f, \chi](z, t) = (-1)^p K^{(q)} G^\chi_T[f](z, t)$$

$$= (-1)^p \sum_{|I| = p} \sum_{|J| = q} \left\{ \int_0^1 G^\chi_T[f_{I,J}](z, \sigma t) \sigma^{q-1} d\sigma \right\} dz_I \wedge \omega_J.$$
It follows from \([4.2]\) that we have
\[ G^*_t[f] = (-1)^p d_t[K^{(q)} G^*_t[f]] + (-1)^p K^{(q+1)} d_t G^*_t[f]. \]  
\[ (4.5) \]

**Theorem 4.1.** Assume \(0 \leq p \leq m, 1 \leq q \leq n\). There are open neighborhoods of the origin, \(W\) and \(U\) as above, such that given any \(f \in G^s(W; \Lambda^{p,q})\) that is \(\mathbb{L}\)-closed and any \(\chi \in G^s_c(B_R^m(0))\) that is equal to 1 in \(B_R^m(0)\) we have
\[ f = \lim_{\tau \to \infty} \mathbb{L}(\lambda^* K^{(p,q)}_t[f, \chi]) \]
in \(G^s(U, \Lambda^{p,q})\).

**Proof.** From \([4.5]\), it follows that it is enough to prove that \(K^{(q+1)} d_t G^*_t[f]\) converges to 0 in \(G^s(U, \Lambda^{p,q})\), note that \(\mathbb{L}\lambda^* = \lambda^* d_t\). Now we use that
\[ d_t G^*_t[f](z, t) = \sum_{|I|=p} \sum_{|J|=q} n \left( G^L_{t}^{I,J}[f_{I,J}](z, t) + G^L_{t}^{I,J}[L_{J,f_{I,J}}](z, t) \right) dt_J \wedge dz_I \wedge dt_J. \]

Since \(f\) is \(\mathbb{L}\)-closed it follows, for every \(|I| = p\) and every \(|K| = q + 1\), that
\[ \left( -\frac{\tau}{n} \right)^m \int e^{-\tau(z-Z(x', t))^2} \chi(x') \sum_{K=J,J\cup\{j\}} \epsilon(j, J) L_{J,I}(x', t) \det Z_x(x', t) dx' = 0. \]

Consequently,
\[ \sum_{|I|=p} \sum_{|J|=q} n G^L_{t}^{I,J}[L_{J,f_{I,J}}](z, t) dt_J \wedge dz_I \wedge dt_J = 0. \]

Therefore all we need to show is that
\[ \sum_{|I|=p} \sum_{|J|=q} n \lambda^* K^{(q+1)} \left( G^L_{t}^{I,J}[f_{I,J}] dt_J \wedge dz_I \wedge dt_J \right)(x, t) \to 0 \text{ in } G^s(U, \Lambda^{p,q}). \]

Since
\[ \lambda^* K^{(q+1)} \left( G^L_{t}^{I,J}[f_{I,J}] dt_J \wedge dz_I \wedge dt_J \right)(x, t) = (-1)^p \left( \int_{0}^{1} G^L_{t}^{I,J}[Z(x, t), \sigma t] \sigma^q d\sigma \right) dZ_I \wedge \omega_J \]  
\[ (4.6) \]
one can use the same argument to prove that \(R^n_x[u](x, t)\) given by \([3.18]\) converges to 0 in \(G^s(U)\) to conclude that coefficient of the form in the left hand side of \([4.6]\) converges to 0 in \(G^s(U)\).

5. Ultradoxistributions vanishing on maximally real submanifolds

Let \(\Omega\) an open subset of \(\mathbb{R}^N\) and \(\mathcal{L}\) be a locally integrable structure of corank \(m\). Let \(\Sigma \subset \Omega\) be an embedded Gevrey submanifold of dimension \(m\), i.e., the defining functions of \(\Sigma\) are Gevrey functions. We recall that \(\Sigma\) is maximally real with respect to \(\mathcal{L}\) if for every \(p \in \Sigma\), any nonvanishing section of \(\mathcal{L}\) defined in a neighborhood of \(p\) is transversal to \(\Sigma\) at \(p\).

**Theorem 5.1.** Let \(\Sigma\) be an embedded submanifold in \(\Omega\) maximally real with respect to \(\mathcal{L}\). If \(u \in D'_s(\Omega)\) is a solution of \(\mathcal{L}\) and \(u|_{\Sigma} = 0\), then \(u\) vanishes in a neighborhood of \(\Sigma\).
Proof. It is enough to prove that every \( p \in \Sigma \) has a neighborhood where \( u \) vanishes. Fix \( p \in \Sigma \) so that we can find local coordinates \((x, t)\) centered at \( p \) and \( Z_1, \ldots, Z_m \) such that properties (2.3) and (2.2) hold and \( \Sigma = \{(x, 0)\} \) in a neighborhood of \( p \) as proved in [EG03].

Now thanks to the Baouendi-Treves approximation formula there is \( U \) a neighborhood of \( p \) where \( u \) is the limit of \( E \chi \tau \int_0^1 \tau \pi \left( \tau Z(x, t) - Z(x', 0) \right)^2 \chi(x') \text{det} Z_x(x', 0) \right). \)

Since \( \Sigma = \{(x, 0)\} \), \( u|_{\Sigma} = 0 \) means that \( \iota^* u = u|_{\Sigma} = 0 \). Therefore, \( E \chi \tau \int_0^1 \) vanishes in a neighborhood of \( p \) and so does \( u \). □

6. BAOUENDI-TREVES THEOREM IN DENJOY-CARLEMAN CLASSES

One can use the ideas of Section 3.1 to prove of the Baouendi-Treves approximation theorem for more general classes of ultradifferentiable functions. More precisely, consider the strongly non-quasianalytic Denjoy-Carleman classes of Roumieu type associated with a non-decreasing sequence of positive numbers \( (M_p)_{p \in \mathbb{Z}_+} \) satisfying:

- **Initial condition:** \( M_0 = M_1 = 1 \). (6.1)
- **Strong logarithmic convexity:** \( \frac{M_j}{M_{j-1}} \leq \frac{M_{j+1}}{M_j} \), \( j = 1, 2, 3, \ldots \). (6.2)
- **Stability under ultradifferential operators:** There exist \( A, H > 0 \) such that \( M_{j+k} \leq AH^{j+k}M_jM_k \), \( \forall j, k \in \mathbb{Z}_+ \). (6.3)
- **Strong Non-quasianalyticity condition:** there exist a constant \( A > 0 \) such that \( \sum_{j=p+1}^{\infty} \frac{M_{j-1}}{M_j} < Ap \frac{M_p}{M_{p+1}} \), \( p = 1, 2, 3, \ldots \). (6.4)

We refer to [Kom73] for more details about these classes. The techniques used strongly the fact that \( G^s(V) \) and \( G^s(V; M, L) \) are isomorphic as topological spaces. This result can be adapted to strongly non-quasianalytic Denjoy-Carleman classes. With this equality of topological spaces proved it is not difficult to see that with minor changes in our proof the Baouendi-Treves approximation theorem also holds for these spaces of strongly non-quasianalytic Denjoy-Carleman functions and ultradistributions.

APPENDIX A. FAÀ DI BRUNO FORMULA

Next we recall the Faà di Bruno generalized formula.

**Theorem A.1** ([BM04]). Let \( \Omega \subset \mathbb{R}^p \) and \( U \subset \mathbb{R}^n \) open subsets. Let \( f \in C^\infty(\Omega) \) and \( g \in C^\infty(U; \mathbb{R}^p) \) such that \( g(U) \subset \Omega \) and denote by \( h \) the composition \( f \circ g \). For all \( \alpha \in \mathbb{Z}_+^n \setminus \{0\} \), we have that

\[
\partial^\alpha h(x) = \sum_{\beta} \partial^\alpha (g(x)) \alpha! \frac{(\partial^\beta g(x))^{\beta_1}}{\beta_1! \delta_1^{\beta_1}} \cdots \frac{(\partial^\beta g(x))^{\beta_\ell}}{\beta_\ell ! \delta_\ell^{\beta_\ell}},
\]

where \( \delta = ||\beta|| \) and \( \beta = (\beta_1, \ldots, \beta_\ell) \).
where $\kappa = \beta_1 + \cdots + \beta_\ell$ and $S_\alpha$ is the set of all $\{\delta_1, \ldots, \delta_\ell\}$ distinct elements of $(\mathbb{Z}_+^n \setminus \{0\})^\ell$ and all $(\beta_1, \ldots, \beta_\ell) \in (\mathbb{Z}_+^p \setminus \{0\})^\ell$, $\ell = 1, 2, 3, \ldots$, such that

$$\alpha = \sum_{j=1}^\ell |\beta_j| \delta_j.$$

We will also need the following result from [BM04]:

**Lemma A.2.** Given $\alpha \in \mathbb{Z}_+^n \setminus \{0\}$ and $p \in \mathbb{Z}_+^+ \setminus \{0\}$ let $S_\alpha$ the set defined in the Theorem A.1. For every $(\beta_1, \ldots, \beta_\ell; \delta_1, \ldots, \delta_\ell) \in S_\alpha$, we have

$$|\kappa|!^t |\delta_1|!^t \cdots |\delta_\ell|!^t \leq |\alpha|!^t$$

for every $t > 0$ and, for every positive constant $A$, there are constants $L, D > 0$, depending only on $A$, $n$ and $p$, such that

$$\sum_{S_\alpha} \frac{\kappa!}{\beta_1! \cdots \beta_\ell!} A|\kappa| \leq LD|\alpha|.$$  

(A.2)

**Lemma A.3.** Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set, $f \in G^s(\Omega)$ and $\tau > 1$ is a parameter. Then, for each compact subset $K \subset \Omega$ there exist constants $C, h > 0$ such that and $\alpha \in \mathbb{Z}_+^n$, it holds

$$\sup_{(x,y) \in K} \left| \partial^\alpha_x \{ e^{\tau f(x,y)} \} \right| \leq C h|\alpha|!^s e^{\tau Re f(x,y)+s\tau^{1/s}}.$$  

(A.3)

**Proof.** It is enough to prove for $\theta \in (0, 1]$. Using Theorem A.1, we have

$$\partial^\alpha \{ e^{\tau f(x,y)} \} = \sum_{S_\alpha} e^{\tau f(x,y)} \alpha! \left( \frac{\tau (\partial^\delta_1 f(x,y))^{\beta_1}}{\beta_1! \delta_1!^{\beta_1}} \right) \cdots \left( \frac{\tau (\partial^\delta_\ell f(x,y))^{\beta_\ell}}{\beta_\ell! \delta_\ell!^{\beta_\ell}} \right).$$

Since $f$ is in $G^s(\Omega)$ there exist constants $\tilde{C}, \tilde{h} > 0$ such that,

$$\left| \partial^\alpha_x \{ e^{\tau f(x,y)} \} \right| \leq \alpha! e^{\tau Re f(x,y)} \sum_{S_\alpha} \frac{\tau^\kappa}{\beta_1! \cdots \beta_\ell!} \prod_{j=1}^\ell \left| \partial^\delta_j \{ f(x,y) \} \right|^{\beta_j}$$

$$\leq \alpha! e^{\tau Re f(x,y)} \sum_{S_\alpha} \frac{\tau^\kappa}{\beta_1! \cdots \beta_\ell!} \prod_{j=1}^\ell \left( \tilde{C} \tilde{h} |\delta_j|!^s \right)^{\beta_j}$$

$$= \tilde{h}^{\alpha!} \alpha! e^{\tau Re f(x,y)} \sum_{S_\alpha} \left( \tilde{C} \tau \right)^\kappa \prod_{j=1}^\ell \delta_j!(s-1)|\beta_j|$$

$$\leq \tilde{h}^{\alpha!} \alpha! e^{\tau Re f(x,y)} \sum_{S_\alpha} \left( \tilde{C} \tau \right)^\kappa \prod_{j=1}^\ell \delta_j!(s-1)|\beta_j|$$

$$\leq \tilde{h}^{\alpha!} \alpha! s^s e^{\tau Re f(x,y)} \sum_{S_\alpha} \tilde{C}^{\alpha!} \kappa! \left( \frac{\tau^{\kappa/s}}{\kappa!} \right)^s$$

$$\leq \tilde{h}^{\alpha!} \alpha! s^s e^{\tau Re f(x,y)+s\tau^{1/s}} \sum_{S_\alpha} \frac{\kappa!}{\beta_1! \cdots \beta_\ell!} \tilde{C}^{\alpha!}.$$
Now we use Lemma A.2 to find constants \( C \) and \( h \) such that
\[
\left| \partial_x \{ e^{\tau f(x,y)} \} \right| \leq C h^{\alpha_1} |\alpha|^s e^{\tau \Re f(x,y) + s^{1/s}},
\]
as we wished to prove. \( \square \)

**APPENDIX B. TRACE OF ULTRADISTRIBUTIONS**

Let \( u \in \mathcal{D}'(W) \) be an ultradistribution such that
\[
WF_s(u) \cap \{(x,t,0,\theta) : (x,t) \in W, \theta \neq 0\} = \emptyset. \tag{B.1}
\]

We will see that condition \((B.1)\) is enough to define the trace of \( u \) at \( t \), \( \iota_t^* u \in \mathcal{D}'(B^m_R(0)) \), as
\[
(\iota_t^* u)(\varphi) = \frac{1}{(2\pi)^N} \int \mathcal{F}(\lambda u)(\sigma,\theta)e^{it\theta} \left( \int \varphi(x)e^{ix\sigma} \, dx \right) \, d\sigma d\theta, \quad \forall \varphi \in \mathcal{C}_c^o(B^m_R(0)), \tag{B.2}
\]
where \( \lambda \in \mathcal{C}_c^o(W) \) is equal to 1 in \( V \) and \( \mathcal{F}(\lambda u) \) stands for the Fourier transform of \( \lambda u \). Note that, when \( u \in \mathcal{D}'(W) \), this is the classical definition of trace of a distribution, see \cite{Hor90} Section 8.

**Proposition B.1.** Let \( u \in \mathcal{D}'(W) \) be an ultradistribution such that \((B.1)\) is valid then \( \iota_t^* u \) given by \((B.2)\) is in \( \mathcal{D}'(B^m_R(0)) \). Moreover, for each fixed \( \varphi \in \mathcal{C}_c^o(B^m_R(0)) \) the function \( B^m_R(0) \ni t \mapsto \iota_t^* u(\varphi) \) is in \( \mathcal{C}_o^o(B^m_R(0)) \).

**Proof.** Since \( WF_s(u) \) and \( \{(x,t,0,\theta), (x,t) \in \text{supp} \lambda, \theta \neq 0\} \) are disjoint closed cones, there is \( \rho > 0 \) such that
\[
WF_s(u) \cap \{(x,t,\sigma,\theta), (x,t) \in \text{supp} \lambda, \theta \neq 0, \text{ and } |\sigma| \leq \rho |\theta|\} = \emptyset.
\]
Let \( A_1 = \{(\sigma,\theta) : |\sigma| \leq \rho |\theta|\} \) and \( A_2 = \mathbb{R}^N \setminus A_1 \). Thus,
\[
\partial^\beta \iota_t^* u(\varphi) = I_1 + I_2 \tag{B.3}
\]
where
\[
I_k = \frac{1}{(2\pi)^N} \int_{A_k} \mathcal{F}(\lambda u)(\sigma,\theta)(i\theta)^\beta e^{it\theta} \left( \int \varphi(x)e^{ix\sigma} \, dx \right) \, d\sigma d\theta, \quad k = 1, 2. \tag{B.4}
\]
Then, there are \( h > 0 \) and \( C > 0 \) such that
\[
|\mathcal{F}(\lambda u)(\sigma,\theta)| \leq C e^{-h |(\sigma,\theta)|^{1/s}}, \tag{B.5}
\]
for every \((\sigma,\theta) \in A_1\) and for every \( \epsilon > 0 \) there exist a positive constant \( C_\epsilon \) such that
\[
|\mathcal{F}(\lambda u)(\sigma,\theta)| \leq C_\epsilon e^{\epsilon |(\sigma,\theta)|^{1/s}}, \tag{B.6}
\]
for every \((\sigma,\theta) \in A_2\). Moving on, assume that \( \varphi \in \mathcal{C}_c^o(B^m_R(0)) \), therefore there is a constant \( \tilde{C} \) depending only on \( m \) and \( R \) and \( a \) depending only on \( \rho, r \) and \( s \) (see inequality \((B.11)\) below) such that
\[
|\mathcal{F} \varphi(\sigma)| \leq \tilde{C} \| \varphi \|_{B^m_R(0)} e^{-r |\sigma|^{1/s}} e^{-a |(\sigma,\theta)|^{1/s}} \tag{B.7}
\]
\[
\leq \tilde{C} \| \varphi \|_{B^m_R(0)} e^{-a |(\sigma,\theta)|^{1/s}}.
\]
for every \((\sigma, \theta) \in A_2\). In one hand, if we choose \(\tilde{h} = h^{-s}(2s)^s\) it follows that

\[
|I_1| \leq \|\varphi\|_{r,B_R^{m}(0)} \frac{C\tilde{C}}{(2\pi)^N} \int_{A_1} |t\|^{\beta}|e^{-h[(\sigma, \theta)]^{1/s}}| d\sigma d\theta
\]
\[
\leq \|\varphi\|_{r,B_R^{m}(0)} \tilde{h}^{|\beta|} t^s \frac{C\tilde{C}}{(2\pi)^N} \int_{A_1} |(\sigma, \theta)|^{\beta}|e^{-h[(\sigma, \theta)]^{1/s}}| d\sigma d\theta
\]
\[
\leq \|\varphi\|_{r,B_R^{m}(0)} \tilde{h}^{|\beta|} t^s \frac{C\tilde{C}}{(2\pi)^N} \int_{A_1} e^{-\frac{\tilde{h}}{2}(\sigma, \theta)^{1/s}} d\sigma d\theta. \tag{B.8}
\]

On the other hand, we can choose \(\epsilon < a/2\) and \(\tilde{a} = a^{-s}(4s)^s\) to obtain that

\[
|I_2| \leq \|\varphi\|_{r,B_R^{m}(0)} \frac{C\epsilon \tilde{C}}{(2\pi)^N} \int_{A_2} |t\|^{\beta}|e^{-\frac{\epsilon}{2}|t\|^{1/s}}| d\sigma d\theta
\]
\[
\leq \|\varphi\|_{r,B_R^{m}(0)} \tilde{a}^{|\beta|} t^s \frac{C\epsilon \tilde{C}}{(2\pi)^N} \int_{A_2} e^{-\frac{\epsilon}{2}d|t\|^{1/s}} d\sigma d\theta.
\]

Taking \(b = \max\{\tilde{a}, \tilde{h}\}\) we conclude that there is \(C' > 0\) such that then it holds

\[
|\partial^{\beta}(\iota^*_t u)(\varphi)| \leq C'\|\varphi\|_{r,B_R^{m}(0)} \tilde{h}^{|\beta|} t^s. \tag{B.9}
\]

This means that, if \(t \in B_R^{m}(0)\) is fixed, then \(\partial^{\beta}(\iota^*_t u)\) is a continuous linear functional in \(G^s(B_R^{m}(0))\), i.e., \(\partial^{\beta}(\iota^*_t u) \in \mathcal{D}'(B_R^{m}(0))\). Moreover if \(\varphi\) is fixed, then \(B.9\) shows that \((\iota^*_t u)(\varphi) \in G^s(B_R^{m}(0))\). \(\square\)

**Proposition B.2.** Let \(u \in \mathcal{D}'(W)\) be an ultradistribution such that \(B.1\) is valid and \(\psi \in G^s(W)\) then the function \(B_R^{m}(0) \ni t \mapsto \iota^*_t u(\varphi(\cdot, t))\) is in \(G^s(B_R^{m}(0))\) and the following Leibniz formula holds

\[
\partial^{\beta}(\iota^*_t u(\varphi(\cdot, t)) = \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} (\partial^{\alpha}(\iota^*_t u))_x (\partial^{\beta-\alpha} \varphi(\cdot, t)). \tag{B.10}
\]

**Proof.** It follows from the same arguments as in the proof of Proposition \(B.1\). \(\square\)

Note that if \(\varphi \in G^s_r(B_R^{m}(0))\), then

\[
|\xi^\alpha \mathcal{F} \varphi(\xi)| \leq \int \left| (D_x^\alpha \varphi(\xi)) e^{ix\xi} dx \right|
\]
\[
\leq \mu(B_R^{m}(0)) \|\varphi\|_{r,B_R^{m}(0)} r^{\|\alpha\|} \xi^{1/s},
\]
where \(\mu\) stands for the Lebesgue measure in \(\mathbb{R}^m\). Moreover, since there exist a positive constant \(C_m\) depending only of the dimension \(m\) such that

\[
|\mathcal{F} \varphi(\xi)| \leq C_m \mu(B_R^{m}(0)) \|\varphi\|_{r,B_R^{m}(0)} \frac{r^{\|\alpha\|} \xi^{1/s}}{|\xi|^{1/s}}, \quad \xi \neq 0
\]
holds for every \(\alpha \in \mathbb{Z}^m_+\), we obtain

\[
|\mathcal{F} \varphi(\xi)| \leq C_m \mu(B_R^{m}(0)) \|\varphi\|_{r,B_R^{m}(0)} e^{-\frac{\alpha}{m+1} \xi^{1/s}}. \tag{B.11}
\]
B.1. **Final Remark: definition of the restriction.** Observe that $\iota^*_t u$ in $V$ is independent of the choice of $\lambda$ in the following way: for any $\varphi \in G^s_c(B_{R}^m(0))$ and any $t \in B_{R}^n(0)$, function $\iota^*_t u(\varphi)$ does not depend on the choice of $\lambda$ in $G^s_c(W)$ as long as $\lambda = 1$ in $V$. To see use that if $\psi \in G^s_c(B_{R}^n(0))$, then

\[
\begin{align*}
\psi \otimes \varphi &= (\lambda u)(\psi \otimes \varphi) \\
&= \frac{1}{(2\pi)^N} \int \mathcal{F}(\lambda u)(\sigma, \theta) \left( \int e^{it\theta} \Psi(t) dt \right) \left( \int \varphi(x) e^{ix\sigma} dx \right) d\sigma d\theta \\
&= \int (\iota^*_t u)(\varphi) \psi(t) dt.
\end{align*}
\]

The equality above implies that the function $t \mapsto \iota^*_t u(\varphi)$ is uniquely determined in $B_{R}^n(0)$. 

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