Compton and Raman free electron laser stability properties for a cold electron beam propagating through a helical magnetic wiggler

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This paper gives an extensive characterization of the range of validity of the Compton and Raman approximations to the exact free electron laser dispersion relation for a cold, relativistic electron beam propagating through a constant-amplitude helical wiggler magnetic field. The electron beam is treated as infinite in transverse extent. Specific properties of the exact and approximate dispersion relations are investigated analytically and numerically. In particular, a detailed numerical analysis is carried out to determine the range of validity of the Compton approximation.

1. Introduction

Recently Davidson & Uhm (1980, hereafter referred to as I) have developed a fully self-consistent treatment of the free electron laser instability based on the Vlasov-Maxwell equations. Their analysis treats an intense relativistic electron beam, with uniform cross-section, propagating through a constant-amplitude magnetic field approximating a helical wiggler field (equation (1)). The description includes beam kinetic effects and coupling to higher harmonics of the wiggler wavenumber \( k_0 \), and makes no \textit{a priori} assumptions that any off-diagonal elements of the dispersion matrix \( \mathbf{D} \) are negligibly small. The class of distribution functions considered in I is a product of delta functions of the transverse canonical momenta, including electromagnetic components, times a function \( G(z, p_z, t) \) of axial position, momentum, and time (equation (2)). The complete dispersion relation (20) obtained in I is referred to as the full dispersion relation (FDR).

We are engaged in analytic and numerical studies of the properties of the FDR.

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(20) and the corresponding matrix dispersion equation (9). The work has proceeded at three levels: a study of the cold-fluid (cold-beam) FDR, of the warm-fluid FDR, and of the complete Vlasov–Maxwell FDR. There are two general goals of this work. The first is to identify important qualitative properties of the solutions of the FDR. Because of the few approximations made in the derivation of this dispersion relation, these properties can be expected to be reflected in the physics of the free electron laser instability and are not simply characteristic of the model. The second is to compare solutions of the FDR with those of approximate dispersion relations. Such comparisons enable one to determine the range of validity of such approximations and, more importantly, to identify the key physical processes contributing to the instability.

In this paper, we restrict the detailed stability analysis to the cold-beam version of the FDR (28). The cold-beam stability properties discussed below, many of which are new, provide a point of reference for the inclusion of thermal effects which will be considered in subsequent papers. Most of the techniques applied here for a cold electron beam can be extended directly to the case where the electrons are treated as a warm fluid. However, in the latter case, the introduction and application of these techniques are not so simple and straightforward as in the cold-beam case.

§ 2 begins with a concise outline of the principal analytic results of the Vlasov–Maxwell treatment in [I], including the matrix dispersion equation (9) and the full dispersion relation (FDR) in (20). Analytic expressions are obtained for the ratios of the electromagnetic energies (averaged over one cycle) contained in the left- and right-hand circularly polarized radiation fields and in the longitudinal fields ((24)–(26)). An important result in this section is the full Compton dispersion relation (CDR) given in (27). Its derivation is the same as that of the FDR in (20), with the additional assumption that the longitudinal field is negligibly small ($\delta \phi \simeq 0$). The cold-beam limits of the FDR and CDR are then obtained ((28) and (33)), as are the corresponding limits of the field energy ratios ((35)–(37)). These cold-beam results are then used in the remainder of the paper.

Detailed properties of the cold-beam FDR and CDR growth curves ($\text{Im} (\phi)$ versus $k$) are rigorously derived in §§ 3 and 5. As one result of this analysis, we conclude that the growth rate curves obtained from the cold-beam CDR (33) are not generally valid, except possibly in the vicinity of the maximum growth regions (as functions of $\tilde{k}$) obtained from the FDR in (28) (§ 6). A criterion is established for classifying cold-beam systems as Compton or Raman based upon the accuracy of the growth rate obtained from the CDR at the upshifted maximum of the growth rate curve obtained from the FDR. (In this paper, the Raman classification is applied to any system for which the CDR is not a valid approximation at the maximum growth of the upshifted peak.) Using a combination of analytic and numerical studies, we obtain a condition for the validity of the Compton approximation for a cold electron beam. The validity condition for the Compton approximation is given by (64):

$$\frac{\omega_p ck_b}{\omega_e^2} < \frac{\gamma_b^2 (1 + \beta_b)}{25 \beta_b},$$
where $\omega_p$, $\omega_c$, $\gamma_b$, and $\beta_b$ are defined in (18), (19), (30), and (31), respectively.

An immediate result of the analysis in §§3 and 4 is the derivation of a sufficient condition for stability of a cold-beam system, i.e. $\text{Im}(\hat{\omega}) = 0$ for all $\hat{k}$. This condition is given in (53). A numerical example showing the approach to stability with increasing beam density is presented in figure 6.

In §7, we obtain an approximate dispersion relation (68) known as the Raman approximation (RA) to the FDR. This dispersion relation is applicable when the primary coupling is between the left-hand polarized radiation field and the negative-energy longitudinal field.

Detailed numerical examples comparing stability results from the FDR, CDR and RA dispersion relations are presented in §8. These confirm several analytic results obtained in previous sections. Plots of frequency mismatches as functions of $\hat{k}$ over the (upshifted) growth regions are also presented, since assumptions regarding the relative values of these mismatches form the basis for most approximate dispersion relations applied to the free electron laser instability. The numerical analysis in §8 shows that the relative values of the frequency mismatches may vary significantly over the growth interval. Thus, approximations to the FDR which are valid at maximum growth do not necessarily give a valid description of the detailed shape of the growth curve, $\text{Im}(\hat{\omega})$ versus $\hat{k}$.

The numerical analysis shows that an important feature of the FDR growth curves for cold-beam Compton systems is a tail extending from the maximum growth in the direction of increasing $\hat{k}$. In §9, it is shown that the instability in the tail region is produced by a coupling of the positive- and negative-energy longitudinal oscillations with the wiggler and radiation fields. We derive a condition for the existence of this tail, applicable to high-gamma systems with $2\gamma_b^2 > 1 + \omega_p^2/c^2 k_0^2$.

2. Dispersion relation

2.1. Introduction and background

In I, Davidson & Uhm develop a self-consistent description of the free electron laser instability for a relativistic electron beam propagating in the z direction through a left-hand circularly polarized helical wiggler field of the form

$$B_w = -B_0 \cos k_0 z \hat{e}_x - B_0 \sin k_0 z \hat{e}_y,$$

where $B_0 = \text{const.}$ is the wiggler amplitude, and $\lambda_0 = 2\pi/k_0$ is the wiggler wavelength. We briefly outline the model and analysis below.

The analysis is based on the Vlasov–Maxwell equations. Since only one-dimensional spatial variations are considered ($\partial/\partial x = 0 = \partial/\partial y$), and equilibrium self-fields are neglected, the Vlasov equation can be expressed as

$$\left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial z} - e \left( \frac{v \times (B_w + \delta B)}{c} \right) \cdot \frac{\partial}{\partial p} \right] f_b(z, p, t) = 0.$$
Maxwell's equations, expressed in the Coulomb gauge, are given by

\[
\begin{align*}
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right) & \delta A_x = -\frac{4\pi e}{c} \int d^3p v_x (f_b - f_0^b), \\
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right) & \delta A_y = -\frac{4\pi e}{c} \int d^3p v_y (f_b - f_0^b), \\
\frac{\partial^2 \delta \phi}{\partial z^2} & = 4\pi e \int d^3p (f_b - f_0^b).
\end{align*}
\]

In the above equations, \( f_0^b \) and \( f_b - f_0^b \) are the equilibrium and perturbed distribution functions, respectively, and \( \delta E, \delta B, \delta \phi, \) and \( \delta A \) are the electromagnetic field and potential perturbations. It is further assumed that the beam distribution is of the form

\[
f_b(z, p, t) = n_0 \delta(P_x) \delta(P_y) G(z, p_z, t),
\]

where \( P_x = p_x - eA_x/c \) and \( P_y = p_y - eA_y/c \) are the canonical momenta transverse to the beam propagation direction, and \( p_z \) is the mechanical momentum in the \( z \) direction. (Thus, it is assumed that the beam is cold in the transverse direction.) Substituting (3) into (2), it is readily verified that the distribution function \( G(z, p_z, t) \) satisfies the one-dimensional nonlinear Vlasov equation

\[
\left( \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial v_z} - \frac{\partial}{\partial z} \hat{H}(z, p_z, t) \right) G(z, p_z, t) = 0.
\]

Here, \( \hat{H}(z, p_z, t) = \gamma T mc^2 - e\delta \phi(z, t) \), where

\[
\gamma T mc^2 = [m^2c^4 + c^2p_z^2 + e^2(A_y + \delta A_y)^2 + e^2(A_x + \delta A_x)^2]^\frac{1}{2}
\]

is the electron energy when \( P_x = P_y = 0 \). The quantity \( \hat{H}(z, p_z, t) \) is the effective potential for the ponderomotive and longitudinal electrostatic forces on an electron.

The one-dimensional equilibrium distribution function is denoted by \( G_0(p_z) \). In the equilibrium configuration, the energy

\[
\gamma mc^2 = [m^2c^4 + p_z^2c^2 + B_0^2e^2/k_0^2]^\frac{1}{2}
\]

is conserved, as is the axial momentum \( p_z = mv_z \). The stability properties of the system are analysed by linearizing (4) and Maxwell's equations. The analysis is facilitated by introducing \( \delta A_+ = \delta A_x + i\delta A_y \) and \( \delta A_- = \delta A_x - i\delta A_y \), which are the vector potentials for the right- and left-hand circularly polarized radiation fields, respectively. We introduce the following three perturbed field quantities: (a) the vector potential for the left-hand circularly polarized radiation field with wavenumber \( k - k_0 \) and frequency \( \omega \),

\[
\delta A_{k+k_0}^+ = \frac{A_{k+k_0}^-}{2i} \exp[i(k-k_0)z - i\omega t],
\]

(6)

(b) the vector potential for the right-hand circularly polarized radiation field with wavenumber \( k + k_0 \) and frequency \( \omega \),

\[
\delta A_{k+k_0}^- = \frac{A_{k+k_0}^+}{2i} \exp[i(k+k_0)z - i\omega t],
\]

(7)
and (c) the scalar potential for the longitudinal electric field with wavenumber \( k \) and frequency \( \omega \),
\[
\delta \phi_k = \phi_k \exp (ikz - i\omega t).
\] (8)
Here, \( \omega \) is the complex oscillation frequency with \( \text{Im}(\omega) > 0 \) corresponding to instability. After some algebraic manipulation that makes use of the linearized Vlasov–Maxwell equations, we obtain the matrix dispersion equation
\[
\begin{pmatrix}
\hat{D}_k^{+} + \frac{1}{2} \hat{\omega}_c^2 (\alpha_3 \hat{\omega}_p^2 + \hat{\chi}_k^{(2)}), & \frac{1}{2} \hat{\omega}_c^2 (\alpha_3 \hat{\omega}_p^2 + \hat{\chi}_k^{(2)}), & -\hat{\omega}_c \hat{\chi}_k^{(1)} \\
\frac{1}{2} \hat{\omega}_c^2 (\alpha_3 \hat{\omega}_p^2 + \hat{\chi}_k^{(2)}), & \hat{D}_{-k}^- + \frac{1}{2} \hat{\omega}_c^2 (\alpha_3 \hat{\omega}_p^2 + \hat{\chi}_k^{(2)}), & -\hat{\omega}_c \hat{\chi}_k^{(1)} \\
-\hat{\omega}_c \hat{\chi}_k^{(1)}, & -\hat{\omega}_c \hat{\chi}_k^{(1)}, & 2\hat{D}_k^- 
\end{pmatrix}
\begin{pmatrix}
A_{k+K_0}^+
A_{-k-k_0}^-
\phi_k
\end{pmatrix} = 0.
\] (9)
Higher-order couplings are included in the derivation of (9). Definitions of the quantities appearing in the above matrix dispersion equation are the following. The (dimensionless) longitudinal dielectric function and the transverse dielectric functions are given by
\[
\hat{D}_k^- = \hat{k}^2 + \hat{\chi}_k^{(0)}(\omega),
\] (10)
\[
\hat{D}_{-k}^- = \hat{\omega}_c^2 - (\hat{k} - 1)^2 - \alpha_3 \hat{\omega}_p^2,
\] (11)
\[
\hat{D}_{k+K_0}^+ = \omega_c^2 - (\hat{k} + 1)^2 - \alpha_3 \hat{\omega}_p^2,
\] (12)
The dimensionless susceptibilities are defined by
\[
\hat{\chi}_k^{(0)} = \gamma mc \alpha_3^2 \int dp_z \frac{\hat{k} \hat{G}_0 \hat{\partial}_p z}{\hat{k} - \gamma \hat{v}_e / c},
\] (13)
\[
\hat{\chi}_k^{(1)} = \gamma mc \alpha_3 \int dp_z \frac{\hat{k} \hat{G}_0 \hat{\partial}_p z}{\gamma \hat{v}_e / c},
\] (14)
\[
\hat{\chi}_k^{(2)} = \gamma mc \alpha_3 \int dp_z \frac{\hat{k} \hat{G}_0 \hat{\partial}_p z}{\gamma^2 \hat{v}_e / c},
\] (15)
where \( \gamma mc^2 = \text{const.} \) is the characteristic electron energy and \( \alpha_1 \) and \( \alpha_3 \) are defined by
\[
\alpha_n = \gamma_n \int dp_z \hat{G}_0(p_z).
\] (16)
The remaining dimensionless quantities are defined by
\[
\hat{k} = k / k_0,
\] (17)
\[
\hat{\omega}_c = \frac{\omega_c}{ck_0}, \quad \text{where} \quad \omega_c = \frac{eB_0}{\gamma mc},
\] (18)
\[
\hat{\omega}_p^2 = \frac{\omega_p^2}{c^2 k_0^2}, \quad \text{where} \quad \omega_p^2 = \frac{4\pi n_0 e^2}{\gamma m}.
\] (19)
We refer to the secular equation corresponding to the matrix dispersion equation (9) as the full dispersion relation (FDR). The FDR is given by
\[
\hat{D}_k^- (\hat{\omega}) \hat{D}_{k+k_0}^- (\hat{\omega}) \hat{D}_{-k-k_0}^- (\hat{\omega}) = \frac{1}{2} \hat{\omega}_c^2 \left[ \hat{D}_{k+k_0}^- (\hat{\omega}) + \hat{D}_{-k-k_0}^- (\hat{\omega}) \right] \left[ \left[ \hat{\chi}_k^{(1)}(\hat{\omega}) \right]^2 - \hat{D}_k^- (\hat{\omega}) \left[ \alpha_3 \hat{\omega}_p^2 + \hat{\chi}_k^{(2)}(\hat{\omega}) \right] \right].
\] (20)
No assumption that any of the dielectric functions, the wiggler field, or the beam density is small has been made in deriving (20).

We return briefly to the effective potential \( \tilde{H}(z, p_z, t) \) appearing in (4). In I, it is shown (after linearization) that the portion of \( \tilde{H}(z, p_z, t) \) contributing to the ponderomotive and longitudinal electrostatic forces on an electron is proportional to

\[
\frac{1}{2} \gamma \omega_c \gamma \left[ \exp(ik_0z) \delta A_- + \exp(-ik_0z) \delta A_+ \right] - \delta \phi,
\]

where \( \delta A_\pm = \delta A_x \pm i \delta A_y \) is the vector potential for the left- and right-hand circularly polarized radiation fields and \( \delta \phi \) is the longitudinal potential. It follows that the harmonics of the pondermotive and longitudinal potentials with normalized wavenumber \( \hat{k} \) and frequency \( \hat{\omega} \) have phase velocity \( \omega/k \). Also note from (6) and (7) that, for a given frequency \( \omega \), the phase velocities of the left- and right-hand circularly polarized radiation fields are \( \omega/(k - k_0) \) and \( \omega/(k + k_0) \), respectively. In particular, for \( \omega > 0 \), the left-hand circularly polarized field propagates in the negative \( z \)-direction for \( k < k_0 \) (i.e. \( \hat{k} < 1 \)), and in the positive \( z \) direction for \( k > k_0 \) (i.e. \( \hat{k} > 1 \)).

2.2. Polarization and electromagnetic energies

Solving the matrix equation (9) with the aid of the FDR (20), we obtain the following amplitude ratios:

\[
\frac{\phi_k}{A_{k-k_0}^-} = \frac{\omega_c}{2} \left[ \frac{\tilde{D}_{k-k_0}^T + \tilde{D}_{k-k_0}^L}{\tilde{D}_k \tilde{D}_k^T} \right]^{(1)},
\]

\[
\frac{\phi_k}{A_{k+k_0}^+} = \frac{\omega_c}{2} \left[ \frac{\tilde{D}_{k+k_0}^T + \tilde{D}_{k-k_0}^L}{\tilde{D}_k \tilde{D}_k^T} \right]^{(1)},
\]

\[
A_{k-k_0}^- A_{k+k_0}^+ = \frac{\tilde{D}_{k} \tilde{D}_{k}^T}{\tilde{D}_{k-k_0}^L \tilde{D}_{k-k_0}^T}.
\]

For the special case of monochromatic perturbation with \( |\text{Im} (\hat{\omega})| \ll |\text{Re} (\hat{\omega})| \), the electromagnetic energy density, averaged over one cycle, is given by

\[
E = \frac{1}{2} \left[ \frac{\delta \mathbf{B} \cdot \delta \mathbf{B}^*}{8\pi} + \frac{\delta \mathbf{E} \cdot \delta \mathbf{E}^*}{8\pi} \right].
\]

The perturbed fields are

\[ \delta \mathbf{B} = \nabla \times \delta \mathbf{A}, \]

and

\[ \delta \mathbf{E} = -\nabla \delta \phi - \frac{1}{c} \frac{\partial}{\partial t} \delta \mathbf{A}, \]

where

\[ \delta \mathbf{A} = 2^{-\frac{1}{2}} \left( \hat{\mathbf{e}}_x + i \hat{\mathbf{e}}_y \right) \delta A_{k-k_0}^- + 2^{-\frac{1}{2}} \left( \hat{\mathbf{e}}_x - i \hat{\mathbf{e}}_y \right) \delta A_{k+k_0}^+ + \delta \mathbf{A}_L, \]

and \( \delta \phi = \delta \phi_k \) (see (6)-(8)).

From (23), it follows that the contributions of the longitudinal oscillations and of the right- and left-hand circularly polarized radiation fields to the total electromagnetic energy are separable. Denoting these contributions by \( E_L(\hat{\omega}, \hat{k}) \),
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E_+(\omega, \hat{k} + 1) and E_-(\omega, \hat{k} - 1), respectively, and making use of (22), we obtain the electromagnetic energy ratios,

\begin{align}
\frac{E_L(\omega, \hat{k})}{E_+(\omega, \hat{k} + 1)} &= \frac{\sqrt{2}}{2} \alpha_0 \frac{\hat{k}^2}{(\hat{k} + 1)^2 + |\hat{\omega}(\hat{k})|^2} \left| \frac{\hat{D}_{\hat{k} + k_0} T \hat{D}_{\hat{k} - k_0} T}{\hat{D}_k T \hat{D}_k T} \right|^2, \\
\frac{E_L(\omega, \hat{k})}{E_-(\omega, \hat{k} - 1)} &= \frac{\sqrt{2}}{2} \alpha_0 \frac{\hat{k}^2}{(\hat{k} - 1)^2 + |\hat{\omega}(\hat{k})|^2} \left| \frac{\hat{D}_{\hat{k} + k_0} T \hat{D}_{\hat{k} - k_0} T}{\hat{D}_k T \hat{D}_k T} \right|^2, \\
\frac{E_+(\omega, \hat{k} + 1)}{E_-(\omega, \hat{k} - 1)} &= \frac{(\hat{k} + 1)^2 + |\hat{\omega}(\hat{k})|^2}{(\hat{k} - 1)^2 + |\hat{\omega}(\hat{k})|^2} \left| \frac{\hat{D}_{\hat{k} + k_0} T \hat{D}_{\hat{k} - k_0} T}{\hat{D}_k T \hat{D}_k T} \right|^2.
\end{align}

The cold-fluid limits of the above energy ratios are used later in this paper.

2.3. The Compton dispersion relation

If the longitudinal potential \( \delta \phi \) is neglected in the derivation of (9), then we obtain a two-dimensional matrix dispersion equation involving only the vector potentials \( A_{\hat{k} \pm k_0} \). We refer to the corresponding secular equation as the full Compton dispersion relation (CDR). The CDR is given by

\[ \hat{D}_{\hat{k} + k_0} T \hat{D}_{\hat{k} - k_0} T = -\frac{1}{2} \omega_0^2 (\hat{D}_{\hat{k} + k_0} T + \hat{D}_{\hat{k} - k_0} T) (\alpha_0 \omega_0^2 + \hat{\chi}^{(2)}). \]

2.4. Cold-beam dispersion relations

In this paper, we deal with the FDR and CDR for the case of a cold electron beam. To obtain dispersion relations for a cold beam, the equilibrium distribution function is specified by \( G_0(p) = \delta(p - p_0) \) in (13)-(16). The resulting cold-beam FDR (20) is given by

\[ [(\hat{\omega} - \hat{\omega}_0)^2 - \frac{\omega_0^2}{\gamma_0^2}] [\hat{\omega}^2 - (\hat{k} + 1)^2 - \omega_0^2] [\hat{\omega}^2 - (\hat{k} - 1)^2 - \omega_0^2] = -\hat{\omega}_0^2 \frac{\omega_0^2}{\gamma_0^2} [\hat{\omega}^2 - (\hat{k} + 1)^2 - \omega_0^2] [\hat{\omega}^2 - (\hat{k} - 1)^2 - \omega_0^2]. \]

To obtain the above dispersion relation, we have set the constant \( \gamma_0 \) (appearing in (13)-(15) and (18)-(19)) equal to \( \gamma_0 \), where \( \gamma_0 \) is defined by

\[ \gamma_0 mc^2 = [m^2 c^4 + \omega_0^2 c^2 + e^2 B_0^2/k_0^2]^\frac{1}{2}. \]

Moreover, the quantity \( \gamma_b \) is defined by

\[ \gamma_b = (1 - \beta_b^2)^{-\frac{1}{2}}, \]

where \( \beta_b \), the ratio of the unperturbed axial beam velocity to the speed of light, is given by

\[ \beta_b = p_0/\gamma_0 mc. \]

The quantities \( \gamma_0 \) and \( \gamma_b \) are related by the expression

\[ \frac{1}{\gamma_b} = \frac{1}{\gamma_0^2} + \omega_0^2, \]

where \( \omega_c = (eB_0/\gamma_0 mc)/ck_0 \), and \( \gamma_0 > \gamma_b \) necessarily follows from (29)-(32). We note that it is (incorrectly) assumed that \( \gamma_b > \gamma_0 \) in several of the cold-beam numerical calculations in I. These non-physical examples are those whose
temporal growth rate curves fail to vanish above some finite value of the wave-number. (See figures 2 and 5–11 of I.)

The cold-beam CDR (obtained from (27)) is

\[
[\omega - \bar{k}\beta_b]^2 [\omega^2 - (\bar{k} + 1)^2 - \omega_p^2] [\omega^2 - (\bar{k} - 1)^2 - \omega_p^2] = -\frac{\omega_b^2 \omega_p^2}{\omega_b^2 - \omega_p^2} \left[ \frac{\omega^2 - (\bar{k}^2 + 1)}{\omega^2 - \omega_p^2} \right] \left[ \frac{\omega^2 - (\bar{k}^2 - 1)}{\omega^2 - \omega_p^2} \right].
\] (33)

Comparing (33) with (28), we find that a condition for validity of the cold-beam CDR is that

\[
|\omega - \bar{k}\beta_b| \gg \omega_p/\gamma_b.
\] (34)

Clearly, a second requirement for validity of the CDR is that \(|\omega^2 - \bar{k}^2| \gg \omega_p^2\). However, it can be shown that (34) implies this second inequality if \(\omega_p/\gamma_b \ll 1\) and \(\bar{k}^2 \gg \gamma_b^2\). Thus, (34) assures the validity of the Compton approximation at the unshifted peak for all systems of moderate density.

For the case of a cold beam, (24)–(26) reduce to

\[
\frac{E_L(\omega, \bar{k})}{E_+(\omega, \bar{k} + 1)} = 2 \frac{\omega_b^2 \omega_p^4}{(\bar{k} + 1)^2 + |\omega|^2} \left[ \frac{\omega^2 - (\bar{k} - 1)^2 - \omega_p^2}{(\omega - (\bar{k}^2 + 1) - \omega_p^2/\gamma_b)} \right]^2,
\] (35)

\[
\frac{E_L(\omega, \bar{k})}{E_-(\omega, \bar{k} - 1)} = 2 \frac{\omega_b^2 \omega_p^4}{(\bar{k} + 1)^2 + |\omega|^2} \left[ \frac{\omega^2 - (\bar{k}^2 - 1) - \omega_p^2}{(\omega - \bar{k}\beta_b - \bar{k})^2} \right]^2,
\] (36)

\[
\frac{E_+(\omega, \bar{k} + 1)}{E_-(\omega, \bar{k} - 1)} = \frac{[(\bar{k} + 1)^2 + |\omega|^2] \omega^2 - (\bar{k} - 1)^2 - \omega_p^2}{[(\bar{k} - 1)^2 + |\omega|^2] \omega^2 - (\bar{k} + 1)^2 - \omega_p^2}.
\] (37)

In (35)–(37), the quantities \(E_L(\omega, \bar{k})\), \(E_-(\omega, \bar{k} - 1)\), and \(E_+(\omega, \bar{k} + 1)\) are the electromagnetic energy densities associated with the longitudinal oscillations and the left- and right-hand circularly polarized electromagnetic fields, respectively, for the case of a monochromatic wave perturbation with frequency \(\omega\).

It should be noted that both \(E_L(\omega, \bar{k})/E_+(\omega, \bar{k} + 1)\) and \(E_L(\omega, \bar{k})/E_-(\omega, \bar{k} - 1)\) approach infinity as \((\omega - \bar{k}\beta_b)^2 \to \omega_p^2/\gamma_b^2\). On the other hand, non-zero electrostatic energy remains in the longitudinal plasma oscillations as \(\omega^2 \to (\bar{k} \pm 1)^2 + \omega_p^2\).

3. Properties of the cold-beam dispersion relation

The cold-beam full dispersion relation FDR (28) is a sixth-degree polynomial in \(\omega\). Thus, complex roots of the FDR will occur in complex conjugate pairs, one of which will represent growth, when \(\text{Im} (\omega') > 0\). The occurrence of complex roots can be analysed by writing the cold-beam FDR in the form

\[
\text{LHS} = \text{RHS},
\] (38)

where LHS is the parabola defined by \(\text{LHS} = (\omega - \bar{k}\beta_b)^2 - \omega_p^2/\gamma_b^2\), and RHS is the discontinuous curve

\[
\text{RHS} = -\omega_b^2 \omega_p^2 \left[ \frac{\omega^2 - \omega_b^2}{\omega^2 - \omega_p^2} \right] \left[ \frac{\omega^2 - \omega_b^2}{\omega^2 - \omega_p^2} \right] \left[ \frac{\omega^2 - \omega_b^2}{\omega^2 - \omega_p^2} \right].
\]
Here we have defined the frequencies

\[ \dot{\omega}_1 = (\hat{k}^2 + \dot{\omega}_2^2)^{\frac{1}{2}}, \]
\[ \dot{\omega}_2 = (\hat{k}^2 + 1 + \dot{\omega}_3^2)^{\frac{1}{2}}, \]
\[ \dot{\omega}_+ = [(\hat{k} + 1)^2 + \dot{\omega}_4^2]^{\frac{1}{2}}, \]
\[ \dot{\omega}_- = [(\hat{k} - 1)^2 + \dot{\omega}_5^2]^{\frac{1}{2}}. \]

(The approach used in this section and in §5 is suggested by an analysis by Bernstein & Hirshfield (1980).)

For \( \hat{k} \geq 0 \), these frequencies satisfy the orderings

\[ \dot{\omega}_+ > \dot{\omega}_2 > \dot{\omega}_1 > \dot{\omega}_- \quad \text{for} \quad \hat{k} > \frac{1}{2}, \]
\[ \dot{\omega}_+ > \dot{\omega}_2 > \dot{\omega}_- > \dot{\omega}_1 \quad \text{for} \quad 0 < \hat{k} < \frac{1}{2}. \]

Graphs of LHS and RHS versus real \( \dot{\omega} \) for fixed \( \hat{k} \) are shown schematically in figures 1(a) and 1(b) for \( 0 < \hat{k} < \frac{1}{2} \) and \( \hat{k} > \frac{1}{2} \), respectively. For the values of \( \hat{k} \) shown, there are six real roots and no complex roots of the FDR because the LHS
and RHS curves are shown to have six intersections. Additional frequencies shown on the graphs are

\[ \omega_u = \hat{k}\beta_b + \omega_p/\gamma_b, \]
\[ \omega_l = \hat{k}\beta_b - \omega_p/\gamma_b, \]

which are the positive- and negative-energy longitudinal space-charge wave frequencies, respectively. The occurrence of complex roots \( \hat{\omega} \) of the FDR for positive \( \hat{k} \) can be determined by considering the manner in which the graphs in figure 1 (a) and (b) change as \( \hat{k} \) varies. Consider the behaviour of the quantity \( \omega_u - \omega_l \). Using (39) and (45), we find that \( \omega_u - \omega_l = -\omega_p(1 - 1/\gamma_b) < 0 \), at \( \hat{k} = 0 \).

The quantity \( \omega_u - \omega_l \) attains a maximum value of zero at \( \hat{k} = \omega_p\gamma_b\beta_b \) (where \( \omega = \omega_u \) is a real root of the FDR), and then approaches \(-\infty \) as \( \hat{k} \) approaches \(+\infty \).

With the aid of figure 1 (a), we find that the FDR has six real roots (and therefore exhibits no instability) for all \( \hat{k} \) in the interval \( 0 < \hat{k} < \frac{1}{2} \) by using the following argument. Making use of (38), it is easily shown that RHS \( (\hat{\omega}, \hat{k}) \) is a monotonically increasing function of \( \hat{\omega} \) for \( 0 \leq \hat{\omega} < \omega_1 \) provided \( 0 \leq \hat{k} < \frac{1}{2} \). The minimum of RHS at \( \hat{\omega} = 0 \) is given by

\[ \text{RHS} (0, \hat{k}) = -\omega_p^2 \omega_0^2 \frac{\partial^2 \omega_0}{\partial \omega_0^2} \frac{\partial^2 \omega_0}{\partial \omega_0^2}, \]

whereas the minimum of the parabola is \(-\omega_p^2/\gamma_b^2 \). According to the inequality in (44), \( |\text{RHS} (0, \hat{k})| < \omega_p^2 \omega_0^2 \), whereas it follows from (32) that

\[ \omega_p^2/\gamma_b^2 = \omega_p^2 \left( \frac{1}{\gamma_b^2} + \omega_0^2 \right) > \omega_p^2 \omega_0^2. \]

It was noted in the preceding paragraph that \( \omega_u \leq \omega_1 \). Therefore, the FDR has two real roots in the interval \( 0 \leq |\hat{\omega}| < \omega_- \) and four elsewhere, for a total of six.

For the FDR, it therefore follows that \( \text{Im} (\hat{\omega}) = 0 \) for \( \hat{k} \) in the interval \( 0 < \hat{k} < \frac{1}{2} \).

Next we consider the case where \( \hat{k} > \frac{1}{2} \). We give here a qualitative description of how the complex roots of the FDR appear and disappear as \( \hat{k} \) ranges from \( \frac{1}{2} \) to infinity. Mathematical details are given later in this section. Referring to figure 1 (b), we note that all the quantities \( \omega_0, \hat{k}\beta_b, \omega_u, \omega_l, \omega_1, \omega_2 \) and \( \omega_\pm \) increase with increasing \( \hat{k} \). However, the parabola (LHS) at first shifts to the right relative to the RHS curve, for increasing \( \hat{k} \). Then, with further increase in \( \hat{k} \), the parabola shifts back to the left. If the parabola shifts sufficiently far to the right, it will no longer intersect with that portion of the RHS curve between \( 0 \) and \( \omega_- \) in figure 1 (b), and two complex conjugate roots of the FDR will appear. The parabola may then shift back to the left. As a result, a plot of \( \text{Im} (\hat{\omega}) \) versus \( \hat{k} \) will exhibit a single growth interval where \( \text{Im} (\hat{\omega}) > 0 \) (see figure 3 (a)). A second possibility is that the parabola shifts sufficiently far to the right that it intersects the RHS curve in the interval \( \omega_- < \hat{\omega} < \omega_1 \), as shown in figure 2. Then a plot of \( \text{Im} (\hat{\omega}) \) versus \( \hat{k} \) will exhibit two growth intervals, the first being produced as the parabola shifts to the right and the second as it shifts back to the left. We refer to these as the downshifted and upshifted peaks, respectively (see figure 3 (b)). Since \( \omega_u \leq \omega_1 \), the parabola will always intersect that portion of the RHS curve in the interval...
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\[ \omega_0 < \omega < \omega_+ \]. Thus, the FDR has at most two complex roots, one of which corresponds to growth (\( \text{Im} (\omega) > 0 \)).

The mathematical details justifying the above description are the following. Making use of (42), one can easily show that the quantity \((k\beta_b - \omega_0)\) increases monotonically with increasing \(k\) from the value \(-\omega_p\) at \(k = 0\), to a maximum value of \(\omega_p/\gamma_b\) at \(k = 1 + \beta_b\gamma_p\). With a further increase in \(k\), the quantity \((k\beta_b - \omega_0)\) decreases monotonically, approaching \(-\infty\) as \(k\) approaches \(+\infty\). Further, if we evaluate RHS at \(\omega = k\beta_b\) (the position of the minimum of the parabola) and let \(k \to \infty\), we find

\[
\text{Im} \text{ RHS} (k\beta_b, k) = -\omega_p^2/\gamma_b^2.
\]

Since \(\omega_0^2/\omega_p^2 < \omega_p^2/\gamma_b^2 = \) the distance of the minimum of the parabola below the real \(\omega\) axis, we find that the parabola must intersect the RHS curve between \(0\) and \(\omega_\) for all \(\beta\) larger than some finite value. Therefore, there is some finite value of \(\beta\) beyond which \(\text{Im} (\omega) = 0\).

The boundaries of the growth peak (or peaks) in a plot of \(\text{Im} (\omega)\) versus \(\beta\) are those values of \(\beta\) for which LHS and RHS versus \(\beta\) have points of tangency; that is, those real values of \(\beta\) which obey the two equations LHS = RHS and \(\partial \text{LHS} / \partial \omega = \partial \text{RHS} / \partial \omega\). However, we have had little success in obtaining analytic solutions to these equations. Upper and lower bounds \((\beta_{ub} \text{ and } \beta_{ub})\) of the entire growth region, however, can be obtained as follows. Referring to figure 1 (b) and assuming that the parabola shifts to the right (with increasing \(\beta\)) relative to the RHS curve, we note that \(\omega = k\beta_b\) will become a root of the FDR before the leftmost boundary of a growth peak in a plot of \(\text{Im} (\omega)\) versus \(\beta\) is reached. Then, assuming that the parabola shifts to the left with increasing \(\beta\), we find that \(\omega = k\beta_b\) again becomes a root, after the right-most boundary of a growth peak is attained. Substituting \(\omega = k\beta_b\) into the FDR and solving for \(\beta\), we obtain the solutions

\[
\begin{bmatrix} \beta_{ub} \\ \beta_{ub} \end{bmatrix} = \frac{1}{2} \left[ -b \pm (b^2 - 4c)^{1/2} \right],
\]

where

\[
b = -[4\gamma_b^2\gamma_0^2 - 2\gamma_b^2\omega_p^2 - \gamma_0^2 - \gamma_b^2],
\]

and

\[
c = (1 + \omega_p^2)\gamma_b^2(\gamma_0^2 + \omega_p^2\gamma_b^2).
Thus, $\hat{k}_{ub} > 0$ and $\hat{k}_{lb} > 0$ provide upper and lower bounds respectively on the entire growth region for non-negative $k$ as shown in figures 3(a) and 3(b). (For a high-energy beam satisfying $\gamma_0 \gg 1, \gamma_b \gg 1$ and $\bar{\omega}_p \gtrsim 1$, it follows that $\hat{k}_{ub} \simeq 2\gamma_b \gamma_0$.)

In the numerical analysis, we often find that $\hat{k}_{ub}$ and $\hat{k}_{lb}$ provide excellent approximations to the upper and lower boundaries of the growth region. This situation occurs when the RHS curve between 0 and $\bar{\omega}_-$ does not deviate appreciably from the horizontal at its point of tangency with the parabola. On the other hand, we find numerically that $\hat{k}_{ub}$ and $\hat{k}_{lb}$ are less accurate approximations to the boundaries of the instability region if the point of tangency of the LHS parabola with the RHS curve for $0 < \bar{\omega} < \bar{\omega}_-$ occurs close to $\bar{\omega}_-$, where the magnitude of the slope of the RHS curve is relatively large.

Upper and lower bounds ($\hat{k}_{ub}'$ and $\hat{k}_{lb}'$) on the interval between the downshifted
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Single, connected growth interval

Separate downshifted and upshifted growth intervals possible

FIGURE 4. The region above the curve $\omega_\gamma = \frac{1}{2} \gamma_\beta \beta_\gamma$ satisfies the sufficiency condition for overlap of the downshifted and upshifted growth regions in $\hat{k}$ space.

and upshifted peaks can be obtained by noting that with increasing $\hat{k}$, $\hat{\omega}_i$ coincides with $\hat{\omega}_-$ before the situation depicted in figure 2 occurs and coincides again with $\hat{\omega}_-$ after the situation ceases to occur. Setting $\hat{\omega}_i = \hat{\omega}_-$ and solving for $\hat{k}$, with the aid of (42) and (46), we obtain

$$\frac{\hat{k}'_{ub}}{\hat{k}'_{ib}} = \gamma_\beta^2 \left( 1 - \frac{\beta_\beta \hat{\omega}_E}{\gamma_\beta} \right) \pm \left( \beta_\beta^2 - \frac{2 \beta_\beta \hat{\omega}_E}{\gamma_\beta} \right)^{1/4}. \quad (49)$$

The quantities $\hat{k}'_{ub}$ and $\hat{k}'_{ib}$ are upper and lower bounds, respectively, on the interval between the unstable $\hat{k}$ ranges.

Equation (49) is normally used in the literature to estimate positions of the maxima of the upshifted and downshifted peaks and not to bound the interval between the unstable $\hat{k}$-ranges. From our numerical studies, we find that if two well-defined peaks are present, (49) gives better estimates of the maxima of Im ($\hat{\omega}$) than of the actual marginal stability boundaries. Nevertheless, (49) does provide us with a sufficient condition that no interval with Im ($\hat{\omega}$) = 0 exists (as in figure 3 (a)). Clearly, real solutions for $\hat{k}'_{ub}$ and $\hat{k}'_{ib}$ do not exist if the discriminant
in (49) is negative. Thus, the upshifted and downshifted unstable regions will
overlap provided
\[ \omega_p \geq \frac{1}{2} \gamma_b \beta_b. \]  

(50)

In figure 4, we plot the minimum \( \omega_p (= \frac{1}{2} \gamma_b \beta_b) \) for which the inequality in (50)
holds versus \( \gamma_b \). Note that if \( \omega_p < 1 \), then the sufficiency condition is satisfied only
if \( \gamma_b < 2 \). Of course, unstable regions can overlap before the sufficiency condition
is satisfied.

The discussion in this section is concerned mainly with the behaviour of two of
the six branches \( \omega(\hat{k}) \) of the cold-beam FDR, namely, those two branches which
become complex conjugates in the unstable \( \hat{k} \) intervals. The large-\( \hat{k} \) behaviour of
the two branches is given by
\[ \omega \rightarrow k \beta_b \pm \left( \frac{\omega_p^2 - \omega_p^2 \omega_p^2}{\gamma_b^2} \right)^{\frac{1}{2}} = k \beta_b \pm \frac{\omega_p}{\gamma_b}. \]  

(51)

The above result is obtained by co-locating the intersections of the parabola in
figure 1(b), with the horizontal line \( \text{RHS} \rightarrow - \omega_p^2 \omega^2 \) as \( \hat{k} \rightarrow \infty, \ 0 < \omega < \omega \). 

Equation (51) shows that these branches become longitudinal plasma oscillations
for large \( \hat{k} \), provided that \( \gamma_0 \sim \gamma_b \). However, for a sufficiently large wiggler field,
\( \gamma_0 \) may differ significantly from \( \gamma_b \) (as is evident from (32)) and an appreciable
amount of the energy in these branches may be contained in the radiation fields
at large \( \hat{k} \). In the limit of large \( \hat{k} \) for these two branches, the energy ratios in (35)
and (36) become
\[ \lim_{\hat{k} \rightarrow \infty} \frac{E_L(\omega, \hat{k})}{E_L(\omega, \hat{k} + 1)} = \lim_{\hat{k} \rightarrow \infty} \frac{E_L(\omega, \hat{k})}{E_L(\omega, \hat{k} - 1)} = \frac{2}{\omega_p^2 \gamma_b^2(1 + \beta_b^2)}. \]  

(52)

4. A cold-beam stability criterion

Equation (48) gives upper and lower bounds (\( \hat{k}_{ub} \) and \( \hat{k}_{ib} \)) on the entire unstable
\( \hat{k} \) region for the case of a cold beam. Thus, if distinct, real \( \hat{k}_{ub} \) and \( \hat{k}_{ib} \) do not exist,
then there will be no growth region. It follows from (48) that a sufficient condition
for the full dispersion relation (FDR) in (28) to give stable solutions (Im (\( \omega \)) = 0)
for all \( \hat{k} \) is
\[ \beta^2 \leq 4c. \]

This stability condition can be written in the form
\[ \omega_p^2 \geq \gamma_0^2 + \frac{1}{16 \gamma_0^2} + \frac{1}{16 \gamma_0^2} - \frac{1}{2} \left( 1 + \frac{\gamma_0^2}{\gamma_b^2} + \frac{1}{4 \gamma_b^2} \right) \equiv [\omega_p^2]_{\text{min}}. \]  

(53)

A free electron laser for which the cold-beam FDR (28) is applicable will become
stable at sufficiently high beam densities. However, for moderate densities
(\( \omega_p^2 \leq 1 \)), the above inequality is not satisfied except for beams with low values
of \( \gamma_0 \).

Figure 5 shows contours of constant \( [\omega_p^2]_{\text{min}} \) in \( (\gamma_0, \gamma_b) \) space. Referring to (32),
we note that \( \omega_p^2 = 0 \) on the diagonal \( (\gamma_b = \gamma_0) \) and that \( \omega_p^2 \) attains a maximum
value of \( \omega_p^2 = (\gamma_0^2 - 1)/\gamma_0^2 \) on the \( \gamma_0 \) axis. For \( \omega_p^2 \geq [\omega_p^2]_{\text{min}} \), the system is stable
for all values of \( \gamma_0 \) and \( \gamma_b \) below the corresponding contour in figure 5.
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The condition \( \tilde{\omega}_p \geq \frac{1}{2} \gamma_b \beta_b \) (equation (50)) is sometimes given as an approximate cold-beam stability criterion (Cary & Kwan 1981). However, this condition bears little similarity to the inequality in (53). For example, contours of constant \( \frac{1}{2} \gamma_b \beta_b \) appear as vertical lines in figure 5. In § 7, we investigate an example of a system satisfying the inequality in (50) which exhibits a large growth rate. In the limit \( \gamma_b = \gamma_0 \), the inequality in (53) reduces to \( \tilde{\omega}_p \geq \gamma_b \beta_b \).

Figure 6 gives numerical results obtained from the FDR (28) illustrating the onset of stability, with increasing \( \tilde{\omega}_p \), for fixed \( \gamma_0 = 1.3 \) and \( \gamma_b = 1.1 \). The sufficient condition for stability (53) is satisfied by all \( \tilde{\omega}_p^2 \geq 0.498 \). On the other hand, our numerical analysis shows that instability ceases when \( \tilde{\omega}_p^2 \geq 0.37 \).

5. Properties of the cold-beam Compton dispersion relation

The analysis of the cold-beam Compton dispersion relation (CDR) is analogous to that of the FDR in (28). We express the CDR in the form

\[
\text{LHS} = \text{RHS},
\]

where

\[
\text{LHS} = [\tilde{\omega} - \tilde{\kappa} \beta_b]^2
\]

and

\[
\text{RHS} = -\tilde{\omega}_p^2 \tilde{\omega}_p^2 \frac{\left[\tilde{\omega}_p^2 - \tilde{\omega}_+^2\right] \left[\tilde{\omega}_p^2 - \tilde{\omega}_-^2\right]}{\left[\tilde{\omega}_+^2 - \tilde{\omega}_-^2\right] \left[\tilde{\omega}^2 - \tilde{\omega}_p^2\right]}
\]

The frequencies appearing in the above definitions are

\[
\tilde{\omega}_+ = \left[(\tilde{\kappa} + 1)^2 + \tilde{\omega}_p^2\right]^{\frac{1}{2}}
\]

(55)
FIGURE 6. Plots of growth rate $\text{Im} (\omega)$ versus $\hat{k}$ obtained numerically from the FDR for several values of $\omega_p^2$, shown against the curves, and for fixed $\gamma_b = 1.3$, $\gamma_b = 1.1$ and $\omega_c = 0.484$.

\[
\omega_- = [(\hat{k} - 1)^2 + \omega_p^2 \hat{k}]^{1/2}
\]
where the following orderings are satisfied:

\[
\omega_+ > \omega_0 > \hat{k} > \omega_+ \quad \text{if} \quad \hat{k} > \frac{1}{2}(1 + \omega_p),
\]

\[
\omega_+ > \omega_0 > \omega_- > \hat{k} \quad \text{if} \quad 0 < \hat{k} < \frac{1}{2}(1 + \omega_p).
\]

Schematic plots of LHS and RHS versus $\omega$, for fixed $\hat{k}$, are shown in figures 7(a) and (b) for the cases where $0 < \hat{k} < \frac{1}{2}(1 + \omega_p)$, and $\hat{k} > \frac{1}{2}(1 + \omega_p)$, respectively.

It is evident from figure 7(a) that the CDR has exactly two complex conjugate roots (one of which represents growth) for all $\hat{k}$ in the interval $0 < \hat{k} < \frac{1}{2}(1 + \omega_p)$. Referring to (33) for the CDR, we see that these complex conjugate roots reduce to a double root at $\omega = 0$ for $\hat{k} = 0$. The behaviour of the complex conjugate branches of the CDR for small $\hat{k} > 0$ is determined by neglecting all powers of $\omega$ higher than quadratic in (33) and solving for $\omega$ to linear order in $\hat{k}$. The result is

\[
\omega = \frac{\hat{k} \beta_p (1 + \omega_p^2)}{[(1 + \omega_p^2) - \omega_c^2 \omega_p^2]^{1/2}} \pm \frac{i \hat{k} \omega_c \omega_p (1/\gamma_b^2 + \omega_p^2/\gamma_b^2) \hat{k}}{[(1 + \omega_p^2) - \omega_c^2 \omega_p^2]^{1/2}}.
\]
It follows that a plot of $\text{Im} (\tilde{\omega})$ versus $\tilde{k} \geq 0$ for the unstable branch gives zero growth rate at $\tilde{k} = 0$, increases linearly in the neighbourhood of $\tilde{k} = 0$, and remains positive over the interval $0 < \tilde{k} < \frac{1}{2}(1 + \omega_p)$. Also note that the phase velocity $\text{Re} (\tilde{\omega})/\tilde{k}$ is greater than $\beta_b$ in this small-$\tilde{k}$ growth region.

The treatment of the CDR for $\tilde{k} > \frac{1}{2}(1 + \omega_p)$ is somewhat similar to that of the FDR for $\tilde{k} > \frac{1}{2}$. As shown in §3, the quantity $(\tilde{k} \beta_b - \omega_\omega)$ increases monotonically with increasing $\tilde{k}$, from the value $-\omega_\omega$ at $\tilde{k} = 0$, to a maximum value of $(\beta_b - \omega_p/\gamma_b)$ at $\tilde{k} = 1 + \beta_b \omega_p \gamma_b$. Then, as $\tilde{k} \rightarrow \infty$, the quantity $(\tilde{k} \beta_b - \omega_\omega)$ decreases monotonically to $-\infty$. The process can be pictured as a shift to the right of the LHS parabola relative to the RHS curve in figure 7(b), followed by a relative shift back to the left. If, as a result of this process, the parabola fails to form additional intersections with the RHS curve (in addition to the original four depicted in figure 7(b)), then the cold-beam CDR exhibits instability for all $\tilde{k} > 0$. However, an interval of no growth will exist over a finite interval of $\tilde{k}$ if the situation shown in figure 8 exists over that interval. The CDR will then be unstable in two regions of $\tilde{k}$ space. The first region will extend from $\tilde{k} = 0$ to the
lowest value of $\hat{k}$ for which the parabola is tangential to the RHS curve in the interval $\hat{\omega}_- < \hat{\omega} < \hat{\omega}_+$. The second region will extend to infinity from the large value of $\hat{k}$ for which there is a tangency.

The LHS parabola always has at least two intersections with the RHS curve in the interval $\hat{\omega}_- < \hat{\omega} < \hat{\omega}_+$ (see figure 7(b)). Thus, the cold-beam CDR in (33) has at most two complex roots, one of which corresponds to growth ($\text{Im}(\hat{\omega}) > 0$).

The discussion in this section is concerned with those two of the six branches of the CDR which are complex conjugates in the unstable $\hat{k}$ regions. The behaviour of these branches for large $\hat{k}$ can be determined by letting $\hat{k} \rightarrow \infty$ in the RHS of (54), and solving for $\hat{\omega}$. We obtain

$$\hat{\omega} \rightarrow \hat{k}\beta_b \pm i\omega_c \omega_p.$$  

Therefore, in a plot of $\text{Im}(\hat{\omega})$ versus $k$ for the CDR, $\text{Im}(\hat{\omega})$ approaches the asymptote $\text{Im}(\hat{\omega}) = \omega_c \omega_p$ as $\hat{k}$ approaches infinity.

The behaviour of the cold-beam CDR growth rate curves discussed in this section is shown schematically in figure 3(a) and (b).

6. Condition for validity of the Compton approximation

From the discussion in §§3–5, it is clear that the cold-beam FDR (28) and CDR (33) are qualitatively different for both small and large values of $\hat{k}$ and that the Compton approximation is not valid in those limits. On the other hand, the two treatments are qualitatively similar for those values of $\hat{k}$ in the region of maximum growth. We adopt the following criterion for the validity of the Compton approximation. We compute the growth rate at the maxima of the (upshifted) peaks for the FDR and CDR, using (28) and (33), respectively. If these values agree to within 5 % of the FDR value, and the corresponding real parts of $\hat{\omega}$ also agree to within 5 %, then we consider the Compton approximation to be valid. If not, but both of the above quantities agree to within 10 %, then we consider the Compton approximation to be marginally valid. In either case, we refer to the system as a Compton system. All other systems are classified as Raman.
Cold-beam systems governed by the FDR or CDR are characterized by the three parameters $\gamma_0$, $\omega_c$, and $\omega_p$. Strictly speaking, an exhaustive study of the validity of the Compton approximation for cold beams would require a determination of the region in a three-dimensional parameter space in which the approximation is valid. We reduce the parameter space from three to two dimensions in the following way. First, we use a procedure similar to that of Kroll & McMullin (1978) and McMullin (1980) to obtain a condition for validity of the Compton approximation for cold beams in the neighbourhood of the upshifted peak. If it is assumed that $D_{\omega-k,0}(\omega) \approx 0$ and $D_{\omega-k,0}(\omega) \neq 0$ in (27), then the cold-beam CDR (33) is given approximately by

$$ (\phi - \ddot{k} \beta_b)^2 (\phi^2 - \phi_0^2) = -\frac{1}{4} \omega_0^2 \beta_b^2 (\phi^2 - \ddot{k}^2), $$

(60)

where $\phi^2 = (\ddot{k} - 1)^2 - \omega_0^2$. Assuming that $\phi \approx \ddot{k} \beta_b \approx \phi_0$ at the upshifted peak, we find that the solution to (60) corresponding to growth ($\operatorname{Im} \phi > 0$) is given approximately by

$$ \phi = \ddot{k} \beta_b = \left( \frac{\omega_0^2 \ddot{k}}{4 \gamma_0^2 \beta_b} \right)^{1/2} \left(-\frac{1}{2} + \frac{i3^{1/2}}{2}\right). $$

(61)

A condition for validity of the CDR is $|\phi - \ddot{k} \beta_b| \gg \omega_p/\gamma_b$ (equation (34)). Substituting the approximate solution (61), one obtains the validity condition

$$ \frac{\gamma_0 \ddot{k}}{4 \beta_b} \gg \frac{\omega_p}{\omega_0^2}. $$

(62)

If we approximate $\ddot{k}$ at the upshifted peak maximum by $\ddot{k} \approx 1/(1 - \beta_b)$ (see I), then the condition for validity of the cold-beam Compton approximation can be expressed as

$$ y \gg 4x^4 (1 - x)^4 [1 - (1 - x)^4], $$

(63)

where

$$ x = \frac{1}{\gamma_0^2} + \phi_0^2 = \frac{1}{\gamma_b}, \quad y = \frac{\omega_0^2}{\omega_p}. $$

By numerically determining the roots of the FDR and CDR ((28) and (33)), we have classified a large number of systems as Compton, marginally Compton, and Raman according to the criteria established earlier in this section.

We find that the curve $y = 25x^4 (1 - x)^4 [1 - (1 - x)^4]$, shown in figures 9(a) and 9(b), separates the Raman region of the two-dimensional parameter space from the Compton and marginally Compton regions. Moreover, the curve $y = 70x^4 (1 - x)^4 [1 - (1 - x)^4]$ shown in figure 9(a) and (b), separates the Compton and marginally Compton regions. Thus, we adopt the condition

$$ y > 25x^4 (1 - x)^4 [1 - (1 - x)^4], $$

where $y = \phi_0^2/\omega_p$ and $x = 1/\gamma_0^2$, as the numerically deduced validity condition for the cold-beam Compton approximation. If a system satisfies this condition, the Compton approximation, at maximum growth of the upshifted peak, will be valid (for both $\operatorname{Re} \phi$ and $\operatorname{Im} \phi$) to within an accuracy of approximately 10%.

This validity criterion can be expressed in the equivalent form

$$ \left( \frac{\omega_p \ddot{k}_0}{\omega_0^2} \right) < \frac{\gamma_b (1 + \beta_b)}{25 \beta_b}. $$

(64)

Details of the above analysis are given in Appendix A.
Figure 9: Plot showing the Raman and Compton regions of the parameter space $y = \frac{x}{y'}$, versus $x = \frac{1}{y''}$ for: (a) large $y''$, (b) the full range of $y''$, $y = 25z(1-z)[1-(1-z)^{1}]$, $y = 70zt(1-a)^{1} (1-(1-a)^{1})$.
It is interesting to note that for fixed \( \frac{\gamma^2}{\beta p} \), the Compton approximation becomes valid both in the limit of large \( \gamma_b \), and in the limit that \( \gamma_b \) approaches unity. The latter results follow directly from (62), since \( \hat{k} \) must be greater than \( \frac{1}{4} \) for growth to occur.

7. The Raman approximation

In this paper, the term Raman is used to designate any system for which the Compton approximation is not valid. In a more restricted sense, a system is considered to satisfy the Raman approximation if its upshifted growth peak is due to a coupling of the negative-energy longitudinal wave and the left-hand polarized radiation field through the presence of the wiggler field. In such a case \( \hat{\omega}_1 \approx \hat{\omega}_- \), \( |\hat{\omega} - \hat{\omega}_u| > |\hat{\omega} - \hat{\omega}_l| \) and \( |\hat{\omega} - \hat{\omega}_-| \) at the maximum growth peak. In the following, we apply this approximation to the FDR (28).

The cold-beam Raman approximation, corresponding to that derived by Kwan, Dawson & Lin (1977) and Kwan (1978) for the fluid case, is the following. We express the FDR in the form

\[
[\hat{\omega} - \hat{\omega}_l][\hat{\omega} - \hat{\omega}_u][\hat{\omega} - \hat{\omega}_-] = R(\hat{\omega}, \hat{k}),
\]

and make the approximation \( R(\hat{\omega}, \hat{k}) \approx R(\hat{\omega}_-, \hat{k}) \equiv R \), where

\[
R = \frac{\omega^2 \omega^2_p (2\hat{k} - 1)/4\hat{\omega}_-}.
\]

It is also assumed that

\[
[\hat{\omega} - \hat{\omega}_u] \approx [\hat{\omega}_l - \hat{\omega}_u] = -2L,
\]

where \( L = \frac{\omega_p}{\gamma_b} \). Solving the resulting quadratic equation, we obtain the solution

\[
\hat{\omega} - \hat{\omega}_- \approx \frac{1}{2}[\mu + i(2R/L - \mu^2)\hat{i}],
\]

for the unstable branch. Here \( \mu = (\hat{\omega}_- - \hat{\omega}_l) \) is the frequency mismatch. We refer to (68) as the Raman approximation (RA).

A validity condition for the Raman approximation (68) is obtained by following the procedure of Kroll & McMullin (1978) and McMullin (1980). For the case of maximum growth (\( \mu = 0 \)), (68) reduces to

\[
\hat{\omega} \approx \hat{\omega}_- + \frac{1}{2}i(2R/L)\hat{i}.
\]

At the growth maximum for a Raman system, it follows that

\[
\hat{\omega}_- \approx \hat{\omega}_l = \hat{k} \beta_b - L.
\]

Consistency of (67), (69), and (70) requires that

\[
(2R/L)\hat{i} \ll 4L.
\]

8. Numerical analysis of Compton and Raman systems

In this section we present detailed numerical solutions to the FDR (28) and the CDR (33) for three choices of cold-beam parameters, in order to illustrate several of the analytic results and stability properties described in the preceding
sections. One of the systems is classified as Compton and two of the systems as Raman.

We also present numerical plots of the frequency mismatches $|\hat{\omega} - \omega_l|$, $|\hat{\omega} - \omega_u|$, $|\hat{\omega} - \omega_a|$, $|\hat{\omega} - \hat{\epsilon} \beta_b|$ and $|\hat{\omega} - \hat{\omega}_s|$ as functions of $\hat{k}$ over the range of the (upshifted) peaks. Assumptions concerning the magnitudes of these mismatches are the basis of such approximations as (33), (61) and (68). An interesting property of the results is that, in some cases, the relative values of the mismatches may vary.
Figure 11. Plots of Re (ω) versus k for (a) the downshifted and (b) the upshifted growth regions in example 1. System parameters are the same as in figure 10.
significantly over the range of the peak. Thus, an approximation which predicts
the maximum growth rate accurately may not necessarily give the correct
detailed shape for $\text{Im}(\phi)$ versus $\hat{k}$.

8.1. Example 1: Compton system ($\gamma_0 = 2.0, \omega_p = 0.01, \omega_c = 0.5$)
In figure 10(a) and (b), we present numerical solutions to the CDR and FDR for
the downshifted and upshifted growth rate curves, respectively, for a typical
system classified as Compton according to the terminology in §6. The system
parameters are $\gamma_0 = 2.0, \omega_p = 0.01$ and $\omega_c = 0.5$. It is located outside the range
of figure 9(b) with $1/\gamma_0^2 = 0.5$ and $\omega_c^2/\omega_p = 25$. Referring to figure 10(a) and (b), it is
evident that the respective upper and lower bounds on the FDR growth region
($\hat{k}_{ub}$ and $\hat{k}_{ib}$ given by (48)) are very good approximations to the boundaries of the
FDR growth region. Moreover, the respective upper and lower bounds on the gap
between the FDR peaks ($\hat{k}_{ub}$ and $\hat{k}_{ib}$ as given in (49)) are found to provide good
approximations to the respective maxima of the peaks. (Recall that $\hat{k}_{ub}$ and $\hat{k}_{ib}$
are obtained by setting the frequency of the negative-energy longitudinal
oscillations equal to the frequency of the forward-scattered radiation field.) A pro-
minent feature of the FDR upshifted growth curve is the tail extending from the
growth rate maximum in the direction of increasing $\hat{k}$. This feature will be
discussed later in §9.

From figure 10(a) and (b), in agreement with the discussion in §6, the CDR
growth rate curve provides an adequate approximation to that of the FDR only
over the interval of $\hat{k}$ extending from the downshifted peak to somewhat beyond
the maximum of the upshifted peak. In accordance with the discussion in §5,
the CDR exhibits growth over the entire region extending from $\hat{k} = 0$ to the
FDR downshifted peak, and $\text{Im}(\phi)$ for the CDR approaches the asymptote
$\omega_c \omega_p = 0.005$ as $\hat{k}$ approaches infinity.

Plots of Re($\phi$) versus $\hat{k}$ for the CDR and FDR in the downshifted growth
intervals of $\hat{k}$ are shown in figure 11(a). Note that both the FDR and CDR exhibit
positive and negative group velocities over the interval of the downshifted peak.
We also note that below $\hat{k} \simeq 0.5$, the real frequency $\text{Re}(\phi)$ for the CDR exceeds
$\hat{k}_p$, in agreement with (58), which gives the solution for $\phi$ that solves the CDR
for small values of $\hat{k}$.

Figure 11(b) shows plots of Re($\phi$) versus $\hat{k}$ for the CDR and FDR in the up-
shifted growth regions. It follows that Re($\phi$) < $\hat{k}_p$ for both the CDR and FDR in
these regions. However, the difference between $\hat{k}_p$ and Re($\phi$) (for both the FDR
and CDR) becomes very small as $\hat{k}$ increases from its value at the FDR peak
towards the upper boundary of the FDR growth curve. The above behaviour for the
FDR conforms with the discussion in §3. Consider figure 1(b) in circum-
stances where the LHS parabola shifts to the left (with increasing $\hat{k}$) in order to
relink with the RHS curve in the interval $0 < \phi < \phi_-$. The minimum of the LHS
parabola lies below $\phi = \hat{k}_p$. If the slope of the RHS curve below this minimum is
approximately zero, then the parabola minimum can approach close to the RHS
curve at values of $\hat{k}$ which are much smaller than $\hat{k} \simeq \hat{k}_{ub}$, where the relinking
finally takes place. At such values of $\hat{k}$ we have Re($\phi$) $\simeq \hat{k}_p$. These values of $\hat{k}$
FIGURE 12. Plots of the frequency mismatches (a) $|\omega - \tilde{k}\beta_b|$, (b) $|\omega - \omega_1|$, (c) $|\omega - \omega_2|$ and (d) $|\omega - \omega_3|$ versus $\tilde{k}$ over the interval of the upshifted growth curve in example 1. Parameters are the same as in figure 10, and the complex $\omega$ solves the FDR in (28).

constitute the tail region which follows the FDR upshifted maximum in figure 10(b).

In figure 12, we plot the frequency mismatches for this system as functions of $\tilde{k}$, over the interval of the FDR upshifted peak. Consider the plot of the mismatch $|\omega - \tilde{k}\beta_b|$ versus $\tilde{k}$. A validity condition (34) for the CDR is $|\omega - \omega \beta_b| > \tilde{k}/\gamma_b$. For values of $\tilde{k}$ at the onset and maximum of the FDR growth curve, $|\omega - \tilde{k}\beta_b|$ is about four times $\omega_p/\gamma_b$. However, $|\omega - \tilde{k}\beta_b|$ decreases very rapidly with increasing $\tilde{k}$, and the CDR is not valid over most of the FDR growth region. The behaviour in figure 12 conforms closely to the relative behaviour of the FDR and CDR growth curves shown in figure 10(b). Further properties of figure 12 will be considered in §9.

8.2. Example 2: Raman system ($\gamma_0 = 2.0$, $\omega_p = (0.4)^\dagger$, $\omega_c = 0.5$)

We increase the value $\omega_p$ of example 1 from 0.01 to (0.4)$\dagger$ to obtain a system with parameters $\gamma_0 = 2.0$, $\omega_p = 0.6324 \ldots$ and $\omega_c = 0.5$. This system is located at the point $1/\gamma_0^2 = 0.5$ and $\omega_c^2/\omega_p = 0.395$ in figure 9(b). Figure 13 shows both the
CDR and FDR growth rate curves \( \text{Im}(\omega) \) versus \( \hat{k} \), for this system. The classification of the system is clearly Raman according to the terminology in §§ 6 and 7. The downshifted and upshifted peaks are combined into a single peak, and the upper and lower bounds on the gap (\( \hat{k}_{ub} \) and \( \hat{k}_{lb} \) of (49)) do not appear since the sufficiency condition \( \omega_p \geq \frac{1}{2} \gamma_b \beta_b \) (50) for no gap is also satisfied. (It follows that the equation \( \omega_l(\hat{k}) = \omega_u(\hat{k}) \) cannot be satisfied for this system.) We see that \( \hat{k}_{lb} \) and \( \hat{k}_{ub} \) (equation (48)) provide a useful estimate of the \( \hat{k} \) interval for which there is growth, but not a good approximation for its boundaries.

Figure 14 shows a plot of \( \text{Re}(\omega) \) versus \( \hat{k} \) in the FDR and CDR growth intervals. Near \( \hat{k} = 0 \), \( \text{Re}(\omega) > \hat{k}\beta_b \) for the CDR. Both the CDR and FDR solutions approach the curve \( \omega = \hat{k}\beta_b \) as \( \hat{k} \) increases, but do not approach it rapidly, as they do in the Compton example 1. Neither the CDR nor FDR curves show a region of negative group velocity below \( \hat{k} = 1 \), as they do in example 1 (figure 11(b)).

Frequency mismatches for the FDR are shown in figure 15. Clearly the CDR validity condition, \( |\omega - \hat{k}\beta_b| \geq \omega_p/\gamma_b \), is satisfied nowhere within the interval of the FDR growth curve. The strong inequality in (71) for the validity of the
Raman dispersion relation (68), is satisfied marginally by this system (with \(0.534 \ll 1.79\)). For \(\hat{k} = 1.8\) (approximately the FDR peak maximum), the mismatches are \(|\hat{\omega} - \hat{\omega}_j| \simeq |\hat{\omega} - \hat{\omega}_e| \simeq 0.36|\hat{\omega} - \hat{\omega}_u|\). Thus, the assumptions in the Raman approximation (68) to the dispersion relation are marginally satisfied. However, at the right-hand boundary of the FDR peak, \(|\hat{\omega} - \hat{\omega}_u| \simeq |\hat{\omega} - \hat{\omega}_e|\), and the assumptions of the derivation are clearly invalid. This behaviour is reflected in the accuracy of the Raman approximation (RA) growth rate curve (68) shown in figure 13. The RA gives a marginally accurate estimate of the FDR maximum growth rate (an error of 13\%), but the detailed RA growth curve is not a good approximation to the FDR growth curve at larger values of \(\hat{k}\).

8.3. Example 3: Raman system \((\gamma_0 = 10, \omega_p = 1.0, \omega_c = 0.03)\)

An example of a Raman system at relatively high \(\gamma_b\) is given by the choice of parameters \(\gamma_0 = 10, \omega_p = 1.0\) and \(\omega_c = 0.03\). Apart from the cold-beam approximation, these parameters are similar to those given for an Astron beam by Kwan et al. (1977). This system is located at the point \(1/\gamma_0^2 = 0.0109\) and \(\omega_c^2/\omega_p = 0.0009\) in figures 9(a) and (b). Figure 16 shows both the FDR and CDR growth rate curves in the neighbourhood of the upshifted peak. (The downshifted peak is not shown.) As is typical for Raman systems, the upper bound \(\hat{k}_{ub}\) on the growth region.
provides a useful (but approximate) estimate of the upper boundary of the unstable $k$ range. Moreover, $k_{ub}'$ provides a good approximation to the location of the maximum growth rate. Plots of $\text{Re}(\hat{\omega})$ versus $k$ in the upshifted growth region are not included. For the scales of $k$ and $\text{Re}(\hat{\omega})$ involved, such plots are indistinguishable from $\text{Re}(\hat{\omega}) = \hat{k}\beta_b$.

Frequency mismatches are shown in figure 17, where $\hat{\omega}$ is computed from the FDR in (28). As expected, the validity condition $|\hat{\omega} - \hat{k}\beta_b| > \hat{\omega}_p/\gamma_b$ for the CDR is not satisfied in the interval of the upshifted peak. This system obeys the strong inequality in (71) (with $0.093 \leq 0.42$), and the Raman approximation in (68) is a good approximation to the dispersion relation. In contrast with example 2, the assumptions in the Raman approximation are satisfied (as least marginally) over the entire interval of the FDR growth curve (with $|\hat{\omega} - \hat{\omega}_i| \simeq |\hat{\omega} - \hat{\omega}_u| \simeq 0.21|\hat{\omega} - \hat{\omega}_u|$ at the peak maximum and $|\hat{\omega} - \hat{\omega}_i| \simeq 0.35|\hat{\omega} - \hat{\omega}_u|$ at the upper boundary). The RA growth curve in figure 16 is seen to provide a good approximation to that of the FDR, except for a small shift to the left.
9. Properties of the tail region

9.1. Coupled longitudinal oscillations

The FDR growth rate curve in example 1 of the previous section shows a tail extending from the upshifted growth rate maximum (figure 10(b)) in the direction of increasing $\hat{k}$. Our numerical analysis indicates that such a tail occurs for all systems classified as Compton or marginally Compton according to the criteria developed in § 6. The underlying reason for such a tail can be seen by analysing the plots of frequency mismatches versus $\hat{k}$ (e.g. in figure 12).

Consider the numerical results in figure 12, which show the frequency mismatches for example 1 ($\gamma_0 = 2.0$, $\omega_p = 0.01$, $\omega_0 = 0.5$) in § 7 (a Compton system). Three of the mismatches illustrated in figure 12 are $|\hat{\varphi} - \hat{\varphi}_+|$, $|\hat{\varphi} - \hat{\varphi}_-|$ and $|\hat{\varphi} - \hat{\varphi}_u|$, where $\hat{\varphi}_-(\hat{k}) = [(\hat{k} - 1)^2 + \hat{\omega}_b^2]^{1/2}$ is the frequency of the left-hand circularly polarized radiation field, $\hat{\varphi}_+(\hat{k}) = \hat{k}\hat{\omega}_b - \hat{\omega}_p/\gamma_b$ is the frequency of the negative-energy longitudinal oscillation, and $\hat{\varphi}_u(\hat{k}) = \hat{k}\hat{\omega}_b + \hat{\omega}_p/\gamma_b$ is the frequency of the positive-energy longitudinal oscillation. At the maximum of the FDR growth curve ($\hat{k} \simeq 3.4$), the differences between the values of these three mismatches are small. However, with increasing $\hat{k}$, the mismatch $|\hat{\varphi} - \hat{\varphi}_-|$ grows in magnitude while $|\hat{\varphi} - \hat{\varphi}_+|$ and $|\hat{\varphi} - \hat{\varphi}_u|$ decrease toward the value $\hat{\omega}_p/\gamma_b$ (as a result of the mismatch $|\hat{\varphi} - \hat{k}\hat{\omega}_b|$ becoming very small). Thus, in the tail region, $|\hat{\varphi} - \hat{\varphi}_-|$ becomes almost two orders of magnitude larger than $|\hat{\varphi} - \hat{\varphi}_+|$ and $|\hat{\varphi} - \hat{\varphi}_u|$. The mismatch $|\hat{\varphi} - \hat{\varphi}_+|$, not shown in figure 12, is approximately 2.0, which is much greater than the mismatches shown in figure 12.

The above behaviour of the frequency mismatches indicates that the tail in the
growth rate curve is produced by a coupling of the two longitudinal modes by the wiggler field and the radiation fields. Although the uncoupled dispersion relations for these modes do not have a common $\omega(k)$, the frequencies in the tail region are very close to the natural frequencies of both longitudinal oscillations.

If the tails are due to such a coupling, an approximate dispersion relation, valid in the tail region, can be derived as follows. We express the FDR (28) in the approximate form

$$\omega - \omega_0 = \text{RHS}(\omega = \hat{k} \beta_b, \hat{k})$$

where RHS $(\omega, \hat{k})$ is defined below (38) and $\omega$ in the expression for RHS is replaced by $\hat{k} \beta_b$. We refer to (72) as the longitudinal–longitudinal (LL) approximation to the FDR. From it we obtain the following analytic expression for the growth rate
or $\mathrm{Im}(\tilde{\omega}) = 0$, if the right-hand side of (73) is pure imaginary. The quantity $\mathrm{Im}(\tilde{\omega})$ approaches infinity as $\tilde{k}$ approaches $\tilde{k}_{LL1}$ from below or $\tilde{k}_{LL2}$ from above, where

$$\frac{\tilde{k}_{LL1}}{\tilde{k}_{LL2}} = \gamma_0^2\{1 + [1 - (1 + \tilde{\omega}_p^2)\gamma_0^2]^{1/2}\}. \quad (74)$$

A plot of $\mathrm{Im}(\tilde{\omega})$ versus $\tilde{k}$ for the LL approximation (73) is included in figure 10(b) for example, 1. In figure 18, we show the FDR upshifted peak and the LL approximation for a system with $\gamma_0 = 50$, $\tilde{\omega}_p = 0.006$, $\tilde{\omega}_c = 0.015$, $\gamma_b = 40$. The LL approximation (73) in the tail region is also shown.

In § 2, we derived (35)–(37) for the field energy ratios $E_L/E_-$, $E_L/E_+$ and $E_+/E_-$. Here, $E_L$, $E_-$ and $E_+$ are, respectively, the energy densities of the longitudinal waves and the left- and right-hand circularly polarized radiation fields. We consider the behaviour of these ratios as $\tilde{k}$ varies from the region of maximum growth to the tail region. Plots of these ratios as functions of $\tilde{k}$, over the interval of the upshifted growth region, are shown in figure 19 for the parameters used in example 1 of § 8.

We note that most of the energy resides in the left-hand circularly polarized radiation field. However, as $\tilde{k}$ increases from the left boundary of the upshifted growth curve to the edge of the tail region in figure 10(b), the ratio $E_L/E_-$ increases over two orders of magnitude from the (very small) value $E_L/E_- \approx 3.6 \times 10^{-4}$ to the (small) value $E_L/E_- \approx 1.4 \times 10^{-1}$.
Figure 19. Plot of the energy ratios (a) $E_+/E_-$, (b) $E_L/E_-$ and (c) $E_L/E_+$ versus $\hat{k}$ over the interval of the upshifted peak for example 1 analysed in figures 10–12.

An estimate of the ratio $E_L/E_-$ at the edge of the tail region is obtained by setting $\hat{\omega} = \hat{k}\beta_b$ in (36). We obtain

$$\left( \frac{E_L}{E_-} \right)_{\hat{\omega} = \hat{k}\beta_b} = \frac{\hat{\omega}_c^2}{(1 + \beta_b^2 + 2/\hat{k} + 1/\hat{k}^2)(\hat{k}^2\beta_b^2 - \hat{\omega}_c^2)} \times \frac{2}{(\hat{k}^2\beta_b^2 - \hat{\omega}_c^2)} (75)$$

The factor $2/(\ldots)$ in (75) is of order unity since $\hat{k} > 1$ at the edge of the tail region. Referring to figure 1 (b), and noting that $\hat{k}\beta_b$ is to the left of $\hat{\omega}_-$ at the edge of the tail region, we find that the final ratio in (75) is also of order unity. We therefore conclude that $E_L/E_-$ is of order $\hat{\omega}_c^2$ at the edge of the tail region and consequently can be quite small. For a high-gamma system such as that in figure 18, (75) simplifies to give

$$\left( \frac{E_L}{E_-} \right)_{\hat{\omega} = \hat{k}\beta_b} \simeq \frac{2\hat{\omega}_c^2}{(1 + \beta_b^2)(\gamma_0 + \gamma_b)^2} \simeq \hat{\omega}_c^2 \frac{\gamma_0^2}{(\gamma_0 + \gamma_b)^2}. \quad (76)$$

We conclude that although the detailed structure of the tail region is associated with coupled longitudinal oscillations, the major portion of the field energy
associated with this region resides in the radiation field. However, the ratio $E_L/E_-$ can be expected to increase significantly as $\hat{k}$ varies from the left side of the upshifted peak to the edge of the tail region.

9.2. Conditions for existence of a tail region for large-gamma systems

Two conditions are required for a tail region to exist as in figure 10(b). First, for an interval $\hat{k}$ within the growth region, it is necessary that

$$|\hat{\omega} - \hat{k}\beta_b| \ll \hat{\omega}_p/\gamma_b,$$

(77)

in order that $|\hat{\omega} - \hat{\omega}_u|$ and $|\hat{\omega} - \hat{\omega}_u|$ have approximately the same value. Secondly, it is required that

$$|\hat{\omega} - \hat{\omega}_u| \ll |\hat{\omega} - \hat{\omega}_-|,$$

(78)

in order that the coupling is primarily between the longitudinal modes. Note that the inequalities in (34) and (77) are not contradictory, since they apply to different $\hat{k}$ regions of the upshifted FDR growth curve.

Recalling the discussion in § 3, and considering figure 1(b) for the case where $\hat{k}$ corresponds to the upper boundary of the upshifted peak, it follows that the LHS parabola, shifting to the left, is tangent to the RHS curve for $0 < \hat{\omega} < \hat{\omega}_-$. The double real root $\hat{\omega}$ of the FDR corresponding to this tangency is the frequency at the upper boundary of the growth curve. To satisfy the condition in (77), we require that the magnitude of the common slope of the LHS and RHS curves at the point of tangency be much less than $2\hat{\omega}_p/\gamma_b$, since the magnitude of the slope of the parabola is $2|\hat{\omega} - \hat{k}\beta_b|$. If we approximate the slope of the RHS curve at this tangency by the value at $\hat{k}_{ub}$ (the upper bound on the growth region defined in (48)), then the condition in (77) is replaced by

$$\left| \frac{\partial}{\partial \hat{\omega}} \text{RHS} (\hat{\omega}, \hat{k}_{ub}) \right|_{\hat{\omega} = \hat{k}_{ub}\beta_b} \ll \frac{2\hat{\omega}_p}{\gamma_b}.$$

(79)

If (79) holds (assuring a small slope), then (78) can be approximated by substituting $\hat{k} = \hat{k}_{ub}$ and $\hat{\omega} = \hat{k}_{ub}\beta_b$, which gives

$$\frac{\hat{\omega}_p/\gamma_b}{|\hat{\omega} - (\hat{k}_{ub}\beta_b)|} \ll 1.$$

(80)

It is not difficult to apply the inequalities in (79) and (80) to most high-gamma systems. We differentiate RHS, using (38). For systems satisfying

$$2\gamma_b^2 \gg 1 + \hat{\omega}_p^2,$$

(81)

the inequality in (79) then reduces to

$$\frac{16\gamma_0\gamma_b^2(\gamma_0 - \gamma_b)\hat{\omega}_p}{(\gamma_0 + \gamma_b)[4\gamma_0(\gamma_0 - \gamma_b) + 1 + \hat{\omega}_p^2]^2} \ll 1.$$

(82)

Making use of (81), (42) and (48), the inequality in (80) becomes

$$\hat{\omega}_p \ll \gamma_0.$$

(83)

Since large-gamma systems are assumed, (83) is similar to (81). In addition, it is assumed that

$$4\gamma_0(\gamma_0 - \gamma_b) \gg 1 + \hat{\omega}_p^2.$$

(84)
This condition would be violated by a high-gamma system only for extremely low wiggler field (equation (32)). If the inequality in (84) holds, then with the aid of (32) we simplify the inequality in (82) to give

$$\frac{\omega_p}{\gamma_b^2 \omega_c^2} \ll 1,$$

as the condition for the existence of a tail region in a high-gamma system.

Consider the condition in (62) for the validity of the Compton approximation at the maximum of the upshifted growth curve. For high-gamma systems, \(\beta_b \simeq 1\) and \(\hat{k} \simeq 2\gamma_b^2\) at this maximum. Then the inequality in (62) reduces to

$$2\frac{\omega_p}{\gamma_b^2 \omega_c^2} \ll 1.$$

Comparing (85) and (86), we conclude that all cold-beam high-gamma Compton systems have a tail region produced by coupled longitudinal oscillations.

10. Conclusions

In this paper, we have given rigorous derivations of properties of the cold-fluid full dispersion relation (FDR) in (28). Such properties include the upper and lower bounds on the unstable growth region in \(\hat{k}\) space as given by (48), and on the gap between the downshifted and upshifted peaks as given by (50) From these bounds, we have derived sufficient conditions that a cold beam be stable for all values of \(\hat{k}\) and that the downshifted and upshifted peaks merge together. We have also verified that the cold-fluid FDR (28) as well as the cold-fluid CDR (33) have at most one unstable branch with \(\text{Im}(\phi) > 0\) for any value of \(\hat{k}\).

The full Compton dispersion relation (CDR) in (33) has been derived by making the single additional assumption that the electrostatic perturbations may be neglected (\(\delta \phi \simeq 0\)). We have shown that the detailed growth curves obtained from the cold-fluid CDR differ substantially from those obtained from the cold-fluid FDR except possibly at maximum growth and in the region between the growth peaks. Equation (64) is the (numerically-deduced) condition for the Compton approximation to be valid to within 10% at the maximum growth rate of the upshifted peak for cold-beam systems.

Results of detailed numerical analysis of the dispersion relations are presented in §8. This analysis shows that the relative values of frequency mismatches may vary significantly over the interval of the upshifted FDR peak. Thus, approximations which adequately predict the maximum growth rate do not necessarily give an adequate description of the detailed shape of the peak. As \(\hat{k}\) increases over the region of the upshifted peak, \(|\hat{\omega} - \hat{\omega}_\omega|\) increases and \(|\hat{\omega} - \hat{k}\beta_b|\) decreases. The decrease of the latter causes both \(|\hat{\omega} - \hat{\omega}_\omega|\) and \(|\hat{\omega} - \hat{\omega}_u|\) to approach the value \(\omega_p/\gamma_b\). For Compton systems, the result is a change in the appropriate approximation to the FDR (28) from the CDR (33) in the maximum growth region, to the LL approximation (73) for \(\hat{k}\) in the tail region. The Raman approximation (68) also becomes invalid if \(|\hat{\omega} - \hat{\omega}_\omega|\) and \(|\hat{\omega} - \hat{\omega}_u|\) are approximately equal before the growth rate has decreased to zero with increasing \(\hat{k}\).

An interesting property of the upshifted FDR growth curves for Compton systems is the tail which extends from maximum growth in the direction of...
increasing $\hat{k}$. In § 9, we showed that the mechanism producing this tail is associated with a coupling of the positive- and negative-energy longitudinal oscillations by the wiggler and radiation fields. In most cases, the dominant field energy associated with this instability is concentrated in the radiation field.

We conclude with some important remarks concerning thermal effects. The above result, that the CDR (33) can never adequately approximate the detailed shape of the FDR growth curves (28), holds in the cold-fluid approximation. It is not a general result applicable to systems with finite temperature described by either Vlasov or warm-fluid versions of the FDR or CDR. We also note that the tail regions for Compton systems decrease in size when finite-temperature effects are included. The important influence of thermal effects will be presented in a subsequent paper.

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Appendix A. Validity of the Compton approximation

In deriving the criterion presented in (64) and figure 9 for validity of the Compton approximation, we have solved numerically the full dispersion relation (28) and the Compton dispersion relation (33) in the region of maximum growth for the upshifted peak for a wide range of system parameters. Detailed comparisons of growth rate and real frequency have been made for system parameters covering more than seventy-five points in the parameter space $(x, y)$, where

$$x = \frac{1}{\gamma_b^2} = \frac{1}{\gamma_b^2} + \tilde{\omega}_c^2, \quad y = \frac{\tilde{\omega}_c^2}{\tilde{\omega}_p^2}. \quad (A1)$$

Rather than tabulating here these extensive numerical results, we summarize stability properties for several points $(x^*, y^*)$ located exactly on the curve

$$y^* = 25x^* (1 - x^*) \left( 1 - (1 - x^*)^\frac{1}{3} \right). \quad (A2)$$

Table 1 shows a comparison of numerical results for $x^*$ covering the range from 0.005 to 0.992 and $y^*$ from 0.0044 to 5.530. The values of $\text{Im} \tilde{\omega}$, $\text{Re} \tilde{\omega}$ and $\hat{k}$ listed in table 1 are calculated from the full dispersion relation (28). The corresponding percentage errors $\text{Im} (\tilde{\omega})$ and $\text{Re} (\tilde{\omega})$ incurred by using the Compton dispersion relation (33) are also shown. Table 1 illustrates that the inequality in (64) is indeed a good criterion for the Compton dispersion relation (33) to be marginally valid (within 10% error).

Finally, for completeness, table 2 shows a similar comparison of stability results from (28) and (33) for values of $(x, y)$ just above the broken curve in figure 9, i.e. for $(x, y)$ satisfying

$$y > 70x\left( 1 - x \right)^\frac{1}{3} \left( 1 - (1 - x)^\frac{1}{3} \right). \quad (A3)$$

It is evident from table 2 that the error incurred by using the Compton dispersion relation (33) is indeed less than 5% when the inequality in (A3) is satisfied.
| \( \gamma_0 \) | \( \hat{\omega}_p \) | \( \hat{\omega}_e \) | \( x = \frac{1}{\gamma_0^2 + \hat{\omega}_e^2} \) | \( y = \frac{\hat{\omega}_e^2}{\hat{\omega}_p} \) | \( \text{Equation (28)} \) | \( \Delta \text{Im} \hat{\omega} \) | \( \Delta \text{Re} \hat{\omega} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 1.006 | 0.00200 | 0.0637 | 0.9920 | 2.0279 | 0.00209 | 0.0106 | 1.098 | 9.3% | 0.7% |
| 1.026 | 0.0317 | 0.2146 | 0.9660 | 1.4780 | 0.00570 | 0.03197 | 1.005 | 5.0% | 9.8% |
| 2.159 | 0.0100 | 0.2352 | 0.8000 | 5.3300 | 0.01313 | 0.794 | 1.800 | 9.6% | 0.9% |
| 2.732 | 0.1000 | 0.6050 | 0.5000 | 3.6003 | 0.1088 | 2.287 | 3.34 | 10.0% | 4.5% |
| 3.993 | 0.1300 | 0.3705 | 0.2000 | 1.0560 | 0.0887 | 8.26 | 9.3 | 10.2% | 3.4% |
| 5.774 | 0.1011 | 0.1000 | 0.060 | 0.00399 | 0.02988 | 48.085 | 49.1 | 9.9% | 1.0% |
| 20.000 | 0.5664 | 0.0500 | 0.0050 | 0.00044 | 0.0602 | 393 | 394 | 10.1% | 2.0% |

**Table 1.** Comparison of stability properties obtained from FDR (28) and CDR (33) for a range of system parameters \((x, y)\) located on the solid curve \(y^* = 25\gamma_x^*(1 - x^*)^\frac{1}{3} (1 - (1 - x^*)^\frac{1}{3})\) in figure 9.

| \( \gamma_0 \) | \( \hat{\omega}_p \) | \( \hat{\omega}_e \) | \( x = \frac{1}{\gamma_0^2 + \hat{\omega}_e^2} \) | \( y = \frac{\hat{\omega}_e^2}{\hat{\omega}_p} \) | \( \text{Equation (28)} \) | \( \Delta \text{Im} \hat{\omega} \) | \( \Delta \text{Re} \hat{\omega} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 1.009 | 0.0010 | 0.0894 | 0.9900 | 8.0000 | 0.002342 | 0.1094 | 1.1104 | 4.1% | 0.0% |
| 1.200 | 0.00060 | 0.1000 | 0.7044 | 16.6667 | 0.00134 | 1.002 | 2.191 | 4.5% | 0.0% |
| 1.504 | 0.0224 | 0.5981 | 0.8000 | 16.0000 | 0.0445 | 0.7739 | 1.796 | 4.7% | 1.2% |
| 2.000 | 0.208 | 0.5000 | 0.5000 | 12.0000 | 0.0340 | 2.3829 | 3.401 | 4.5% | 0.7% |
| 7.071 | 0.0400 | 0.5292 | 0.3000 | 7.0000 | 0.0563 | 5.0797 | 6.11 | 4.2% | 0.7% |
| 10.000 | 0.0200 | 0.0300 | 0.0109 | 0.0450 | 0.004695 | 181.85 | 182.85 | 4.5% | 0.1% |
| 10.000 | 0.0400 | 0.2000 | 0.0500 | 1.0000 | 0.001378 | 3198.4 | 3199.4 | 0.3% | 0.0% |
| 20.000 | 0.1250 | 0.0500 | 0.0050 | 0.0200 | 0.0282 | 399.0 | 399.0 | 3.6% | 0.0% |
| 50.000 | 0.0060 | 0.0150 | 0.00063 | 0.0375 | 0.001378 | 3199.4 | 3199.4 | 0.3% | 0.0% |
| 50.000 | 0.1000 | 0.0100 | 0.0104 | 1.0000 | 0.00086 | 191.0 | 192.0 | 0.6% | 0.0% |
| 50.000 | 0.1000 | 0.0100 | 0.0104 | 1.0000 | 0.0317 | 191.0 | 192.0 | 2.6% | 0.0% |

**Table 2.** Comparison of stability properties obtained from FDR (28) and CDR (33) for a range of system parameters \((x, y)\) located above the dashed curve \(y = 70\gamma_x^*(1 - x^*)^\frac{1}{3} (1 - (1 - x^*)^\frac{1}{3})\) in figure 9.
Stability of electron beam

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