M5-branes and D4-branes wrapped on a direct product of spindle and Riemann surface

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Abstract

We construct multi-charged $\text{AdS}_3 \times \Sigma \times \Sigma_g$ and $\text{AdS}_2 \times \Sigma \times \Sigma_g$ solutions from M5-branes and D4-branes wrapped on a direct product of spindle, $\Sigma$, and Riemann surface, $\Sigma_g$. Employing uplift formula, we obtain these solutions by uplifting the multi-charged $\text{AdS}_3 \times \Sigma$ and $\text{AdS}_2 \times \Sigma$ solutions to seven and six dimensions, respectively. We further uplift the solutions to eleven-dimensional and massive type IIA supergravity and calculate the holographic central charge and the Bekenstein-Hawking entropy, respectively. We perform the gravitational block calculations and, for the $\text{AdS}_3 \times \Sigma \times \Sigma_g$ solutions, the result precisely matches the holographic central charge from the supergravity solutions.

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1 Introduction

Recently, there was a discovery of novel class of anti-de Sitter solutions obtained from branes wrapped on an orbifold, namely, a spindle, [2]. The spindle, $\Sigma$, is an orbifold, $\mathbb{WP}^{1}_{[n_-,n_+]}$, with conical deficit angles at two poles. The spindle numbers, $n_-, n_+$, are arbitrary coprime positive integers. Interestingly, these solutions realize the supersymmetry in different ways from very well studied topological twist in field theory, [3], and in gravity, [4]. It was first constructed from D3-branes, [2] [5] [6], and then generalized to other branes: M2-branes, [7] [8] [9] [10], M5-branes, [11], and D4-branes, [12] [13]. Furthermore, two possible ways of realizing supersymmetry, topologically topological twist and anti-twist, were studied, [14, 15].
The spindle solutions were then generalized to an orbifold with a single conical deficit angle, namely, a topological disk. These solutions were first constructed from M5-branes, [16, 17], and proposed to be a gravity dual to a class of 4d \( \mathcal{N} = 2 \) Argyres-Douglas theories, [18]. See also [19] for further generalizations. Brane solutions wrapped on a topological disk were then constructed from D3-branes, [20, 21], M2-branes, [22, 10], D4-branes, [23], and more from M5-branes, [24]. See also [25] for defect solutions from different completion of global solutions.

An interesting generalization would be to find AdS solutions from branes wrapped on an orbifold of dimensions more than two. Four-dimensional orbifolds are natural place to look for such constructions and some solutions were found. First, by uplifting AdS$_3 \times \Sigma$ solutions, where $\Sigma$ is a spindle, [6], or a disk, [21], AdS$_3 \times \Sigma \times \Sigma_g \times S^4$ solutions from M5-branes were obtained where $\Sigma_g$ is a Riemann surface of genus $g$. Also AdS$_2 \times \Sigma \times \Sigma_g$ solutions with spindle, $\Sigma$, from D4-branes were obtained, [13, 12]. More recently, performing and using a consistent truncation on a spindle, AdS$_3 \times \Sigma_1 \times \Sigma_2$ solutions from M5-branes wrapped on a spindle fibered over another spindle were found, [26]. Also AdS$_3 \times \Sigma_1 \times \Sigma_g$ solutions on a spindle fibered over Riemann surface were found, [26].

In this work, we fill in the gaps in the literature. First, we construct multi-charged AdS$_3 \times \Sigma \times \Sigma_g$ solutions from M5-branes. Employing the consistent truncation of [1], we obtain the solutions by uplifting the multi-charged AdS$_3 \times \Sigma$ solutions, [6], to seven-dimensional gauged supergravity. When the solutions are uplifted to eleven-dimensional supergravity, they precisely match the previously known AdS$_3 \times \Sigma \times \Sigma_g \times S^4$ solutions in [6] and [21], which were obtained by uplifting the AdS$_3 \times \Sigma$ solutions of five-dimensional gauged supergravity. However, it is the first time to construct the AdS$_3 \times \Sigma \times \Sigma_g$ solutions in seven-dimensional gauged supergravity.

Second, we construct multi-charged AdS$_2 \times \Sigma \times \Sigma_g$ solutions from D4-branes. Inspired by the consistent truncation in [27], we construct them by uplifting the multi-charged AdS$_2 \times \Sigma$ solutions, [14], to matter coupled $F(4)$ gauged supergravity. Our solutions generalize the minimal AdS$_2 \times \Sigma \times \Sigma_g$ solutions in [12] and also the solutions obtained in [13]. We then uplift the solutions to massive type IIA supergravity to obtain AdS$_2 \times \Sigma \times \Sigma_g \times \tilde{S}^4$.

Finally, we perform the gravitational block calculations and, for the AdS$_3 \times \Sigma \times \Sigma_g$ solutions, the result precisely matches the holographic central charge obtained from the supergravity solutions.

In section 2 we construct AdS$_3 \times \Sigma \times \Sigma_g$ solutions from M5-branes. We uplift the solutions to eleven-dimensional supergravity and calculate the holographic central charge. In section 3 we construct AdS$_2 \times \Sigma \times \Sigma_g$ solutions from D4-branes. We uplift the solutions to massive type IIA supergravity and calculate the Bekenstein-Hawking entropy. In section 4 we present the gravitational block calculations. In section 5 we conclude. We present the equations of motion in appendix A and briefly review the consistent truncations of [1] in appendix B.
2 M5-branes wrapped on $\Sigma \times \Sigma_g$

2.1 $U(1)^2$-gauged supergravity in seven dimensions

We review $U(1)^2$-gauged supergravity in seven dimensions, [28], in the conventions of [26]. The bosonic field content consist of the metric, two $U(1)$ gauge fields, $A^{12}$, $A^{34}$, a three-form field, $S^5$, and two scalar fields, $\lambda_1$, $\lambda_2$. The Lagrangian is given by

$$\mathcal{L} = (R - V) \operatorname{vol}_7 - 6 * d\lambda_1 \wedge d\lambda_1 - 6 * d\lambda_2 \wedge d\lambda_2 - 8 * d\lambda_1 \wedge d\lambda_2$$

$$- \frac{1}{2} e^{-4\lambda_1} * F^{12} \wedge F^{12} - \frac{1}{2} e^{-4\lambda_2} * F^{34} \wedge F^{34} - \frac{1}{2} e^{-4\lambda_1 - 4\lambda_2} * S^5 \wedge S^5$$

$$+ \frac{1}{2g} S^5 \wedge dS^5 - \frac{1}{g} S^5 \wedge F^{12} \wedge F^{34} + \frac{1}{2g} A^{12} \wedge F^{12} \wedge F^{34} \wedge F^{34},$$

(2.1)

where $F^{12} = dA^{12}$, $F^{34} = dA^{34}$ and the scalar potential is

$$V = g^2 \left[ \frac{1}{2} e^{-8(\lambda_1 + \lambda_2)} - 4 e^{2(\lambda_1 + \lambda_2)} - 2 e^{-2(2\lambda_1 + \lambda_2)} - 2 e^{-2(\lambda_1 + 2\lambda_2)} \right].$$

(2.2)

The equations of motion are presented in appendix A.

2.2 Multi-charged $AdS_3 \times \Sigma$ solutions

We review the $AdS_3 \times \Sigma$ solutions of $U(1)^3$-gauged $\mathcal{N} = 2$ supergravity in five dimensions, [6]. These solution are obtained from D3-branes wrapped on a spindle, $\Sigma$. The metric, gauge fields and scalar fields read

$$ds^2_5 = H^{1/3} \left[ ds^2_{AdS_3} + \frac{1}{4P} dy^2 + \frac{P}{H} dz^2 \right],$$

$$A^{(I)} = \frac{y - \alpha}{y + 3K_I} dz, \quad X^{(I)} = \frac{H^{1/3}}{y + 3K_I},$$

(2.3)

where $I = 1, \ldots, 3$ and the functions are defined to be

$$H = (y + 3K_1) (y + 3K_2) (y + 3K_3), \quad P = H - (y - \alpha)^2,$$

(2.4)

where $K_I$ and $\alpha$ are constant and satisfy the constraint, $K_1 + K_2 + K_3 = 0$.

In the case of three distinct roots, $0 < y_1 < y_2 < y_3$, of cubic polynomial, $P(y)$, the solution is positive and regular in $y \in [y_1, y_2]$. The spindle, $\Sigma$, is an orbifold, $\mathbb{CP}^1_{[n-,n+]}$, with conical deficit angles at $y = y_1, y_2$, [6]. The spindle numbers, $n_-$, $n_+$, are arbitrary coprime positive integers. The Euler number of the spindle is given by

$$\chi(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} R_{\Sigma} \operatorname{vol}_\Sigma = \frac{n_- + n_+}{n_- n_+},$$

(2.5)
where $R$ and $\text{vol}_\Sigma$ are the Ricci scalar and the volume form on the spindle. The magnetic flux through the spindle is given by

$$Q_I = \frac{1}{2\pi} \int_\Sigma F^{(I)} = \frac{(y_2 - y_1) (\alpha + 3K_I)}{(y_1 + 3K_I) (y_2 + 3K_I)} \frac{\Delta z}{2\pi} = \frac{p_I}{n_- n_+},$$

(2.6)

and we demand $p_I \in \mathbb{Z}$. One can show that the R-symmetry flux is given by

$$Q_1 + Q_2 + Q_3 = \frac{p_1 + p_2 + p_3}{n_- n_+} = \frac{\eta_1 n_+ - \eta_2 n_-}{n_- n_+},$$

(2.7)

where the supersymmetry is realized by, [14],

$$\text{Anti-twist} : (\eta_1, \eta_2) = (+1, +1), \quad \text{Twist} : (\eta_1, \eta_2) = (-1, +1).$$

(2.8)

In minimal gauged supergravity, $K_1 = K_2 = K_3$, only the anti-twist solutions are allowed. Otherwise, both anti-twist and twist are allowed.

One can express $\Delta z$, $y_1$, $y_2$, and the parameters, $K_I$, $\alpha$, in terms of the spindle numbers, $n_-$, $n_+$, $p_1$, and $p_2$, [6]. The period of the coordinate, $z$, is given by

$$\frac{\Delta z}{2\pi} = \frac{(n_- - n_+) (p_1 + p_2) + n_- n_+ - p_1^2 - p_1 p_2 - p_2^2}{n_- n_+ (n_+ + n_-)},$$

(2.9)

In the special case of

$$K_1 = K_2, \quad X^{(1)} = X^{(2)}, \quad A^{(1)} = A^{(2)},$$

(2.10)

expressions of $y_1$, $y_2$, and $K_1 = K_2$ are simpler,

$$y_1 = \frac{q (n_+ + q) [2n_+^2 - 2n_- (n_+ + 4q) + q (5n_+ + 9q)]}{3 [n_- (n_+ + 2q) - q (2n_+ + 3q)]^2},$$

$$y_2 = -\frac{q (n_- - q) [2n_-^2 - 2n_+ (n_- - 4q) - q (5n_- - 9q)]}{3 [n_- (n_+ + 2q) - q (2n_+ + 3q)]^2},$$

$$K_1 = K_2 = \frac{q (n_- - n_+ - 3q) (n_+ + q) (n_- - q)}{9 [n_- (n_+ + 2p) - q (2n_+ + 3q)]^2},$$

(2.11)

where we define $q \equiv p_1 = p_2$. For the expression of $\alpha$, we leave the readers to [6]. For this special case, the $AdS_3 \times \Sigma$ solutions are also solutions of $SU(2) \times U(1)$-gauged $\mathcal{N} = 4$ supergravity in five dimensions, [29]. The solutions can be uplifted to eleven-dimensional supergravity, [30], as it was done for a spindle, [6], and for a disk, [21].
2.3 Multi-charged $AdS_3 \times \Sigma \times \Sigma_g$ solutions

A consistent reduction of seven-dimensional maximal gauged supergravity, $[31]$, on a Riemann surface was performed in $[1]$. Employing the consistent truncation, we uplift the $AdS_3 \times \Sigma$ solutions in section 2.2 with

$$K_1 = K_2 \neq K_3,$$  (2.12)

to $U(1)^2$-gauged supergravity in seven dimension. We briefly summarize the uplift by consistent truncation in appendix B. As a result, we find the $AdS_3 \times \Sigma \times \Sigma_g$ solutions,

$$ds_7^2 = e^{-4\varphi} H^{1/3} \left( ds_{AdS_3}^2 + \frac{1}{4P} dy^2 + \frac{P}{H} dz^2 \right) + \frac{1}{g^2} e^{6\varphi} ds_{\Sigma_g},$$

$$e^{-\frac{10}{3}\lambda_1} = 2^{1/3} X, \quad e^{\frac{2}{3}\lambda_2} = 2^{1/3} Y, \quad e^{10\varphi} = 2^{1/3} X,$$

$$S^5 = 2^{2/3} (3K + \alpha) \text{vol}_{AdS_3},$$

$$F^{12} = \frac{1}{g} \frac{d}{dy} \left( \frac{y - \alpha}{y + 3K_3} \right) dy \wedge dz + \frac{1}{g} \text{vol}_{\Sigma_g},$$

$$F^{34} = \frac{2}{g} \frac{d}{dy} \left( \frac{y - \alpha}{y + 3K_1} \right) dy \wedge dz,$$  (2.13)

where $\Sigma_g$ is a Riemann surface and we define

$$H = (y + 3K_1)^2 (y + 3K_3), \quad P = H - (y - \alpha)^2, \quad X = X^{(1)} = X^{(2)} = \frac{H^{1/3}}{y + 3K_1},$$  (2.14)

and $g^2 L_{AdS_5}^2 = 2^{4/3}$. The gauge coupling and the radius of asymptotic $AdS_5$ are fixed to be $g = 2^{2/3}$ and $L_{AdS_5} = 1$, respectively.

The flux quantization through the Riemann surface is given by

$$s_1 = \frac{g}{2\pi} \int_{\Sigma_g} F^{12} = 2 \left( 1 - \frac{g}{2} \right) \in \mathbb{Z},$$

$$s_2 = \frac{g}{2\pi} \int_{\Sigma_g} F^{34} = 0,$$  (2.15)

where we find $s_1 + s_2 = 2 \left( 1 - \frac{g}{2} \right)$. Fluxes through the spindle are quantized to be

$$n_1 \equiv -\frac{g}{2\pi} \int_{\Sigma} F^{12} = -\frac{(y_2 - y_1) (\alpha + 3K_3) \Delta z}{(y_1 + 3K_3) (y_2 + 3K_3) 2\pi} = -\frac{p_3}{n_- n_+},$$

$$2n_2 \equiv -\frac{g}{2\pi} \int_{\Sigma} F^{34} = -2 \frac{(y_2 - y_1) (\alpha + 3K_1) \Delta z}{(y_1 + 3K_1) (y_2 + 3K_1) 2\pi} = -2 \frac{p_1}{n_- n_+},$$  (2.16)

where $p_1$ and $p_3$ are introduced in (2.6) and $p_i \in \mathbb{Z}$. The minus signs in the definition of $n_i$ are introduced for later convenience in the gravitational block calculations. By (2.7) the total flux is obtained to be

$$n_1 + 2n_2 = -\frac{2p_1 + p_3}{n_- n_+} = \frac{\eta_1 n_+ - \eta_2 n_-}{n_- n_+},$$  (2.17)

where $\eta_1$ and $\eta_2$ are given in (2.8) and, thus, both twist and anti-twist solutions are allowed.
2.4 Uplift to eleven-dimensional supergravity

We review the uplift formula, \([32]\), of \(U(1)^2\)-gauged supergravity in seven dimensions to eleven-dimensional supergravity, \([33]\), as presented in \([26]\). The metric is given by

\[
L^{-2} ds_{11}^2 = \Delta^{1/3} ds_7^2 + \frac{1}{g^2} \Delta^{-2/3} \left\{ e^{4\lambda_1+4\lambda_2} dw_0^2 + e^{-2\lambda_1} \left[ dw_1^2 + w_1^2 (d\chi_1 - gA^{12})^2 \right] \\
+ e^{-2\lambda_2} \left[ dw_2^2 + w_2^2 (d\chi_2 - gA^{34})^2 \right] \right\},
\]

(2.18)

where

\[
\Delta = e^{-4\lambda_1-4\lambda_2} dw_0^2 + e^{2\lambda_1} w_1^2 + e^{2\lambda_2} w_2^2,
\]

(2.19)

and \(L\) is a length scale. We employ the parametrizations of coordinates of internal four-sphere by

\[
\mu^1 + i\mu^2 = \cos\xi \cos\theta e^{i\lambda_1}, \quad \mu^3 + i\mu^4 = \cos\xi \sin\theta e^{i\lambda_2}, \quad \mu^5 = \sin\xi,
\]

(2.20)

with

\[
w_0 = \sin\xi, \quad w_1 = \cos\xi \cos\theta, \quad w_2 = \cos\xi \sin\theta,
\]

(2.21)

where \(w_0^2 + w_1^2 + w_2^2 = 1\) and \(\xi \in [-\pi/2, \pi/2], \theta \in [0, \pi/2], \chi_1, \chi_2 \in [0, 2\pi]\). The four-form flux is given by

\[
L^{-3} F_{(4)} = \frac{w_1 w_2}{g^3 w_0} U \Delta^{-2} dw_1 \wedge dw_2 \wedge (d\chi_1 - gA^{12}) \wedge (d\chi_2 - gA^{34}) \\
+ \frac{2 w_1^2 w_2^2}{g^3} \Delta^{-2} e^{2\lambda_1+2\lambda_2} (d\lambda_1 - d\lambda_2) \wedge (d\chi_1 - gA^{12}) \wedge (d\chi_2 - gA^{34}) \wedge dw_0 \\
+ \frac{2 w_0 w_1 w_2}{g^3} \Delta^{-2} \left[ e^{-4\lambda_1-4\lambda_2} \wedge (3d\lambda_1 + 2d\lambda_2) - e^{-4\lambda_1-2\lambda_2} w_2 dw_1 \wedge (2d\lambda_1 + 3\lambda_2) \right] \\
\wedge (d\chi_1 - gA^{12}) \wedge (d\chi_2 - gA^{34}) \\
+ \frac{1}{g^2} \Delta^{-1} F_{12} \wedge \left[ w_0 w_2 e^{-4\lambda_1-4\lambda_2} dw_2 - w_2^2 e^{2\lambda_2} dw_0 \right] \wedge (d\chi_2 - gA^{34}) \\
+ \frac{1}{g^2} \Delta^{-1} F_{34} \wedge \left[ w_0 w_1 e^{-4\lambda_1-4\lambda_2} dw_1 - w_1^2 e^{2\lambda_1} dw_0 \right] \wedge (d\chi_1 - gA^{12}) \\
- w_0 e^{-4\lambda_1-4\lambda_2} \ast_7 S^5 + \frac{1}{g} S^5 \wedge dw_0,
\]

(2.22)

where

\[
U = \left( e^{-8\lambda_1-8\lambda_2} - 2e^{-2\lambda_1-4\lambda_2} - 2e^{-4\lambda_1-2\lambda_2} \right) w_0^2 \\
- \left( e^{-2\lambda_1-4\lambda_2} + 2e^{2\lambda_1+2\lambda_2} \right) w_1^2 - \left( e^{-4\lambda_1-2\lambda_2} + 2e^{2\lambda_1+2\lambda_2} \right) w_2^2,
\]

(2.23)

and \(\ast_7\) is a Hodge dual in seven dimensions.
We find a quantization condition of four-form flux through the internal four-sphere,

\[ \frac{1}{(2\pi l_p)^3} \int_{S^4} F(4) = \frac{L^3}{(2\pi l_p)^3} \int_{S^4} \frac{w_1 w_2}{g^3 w_0} U \Delta^{-2} dw_1 \wedge dw_2 \wedge d\chi_1 \wedge d\chi_2 \]

\[ = \frac{L^3}{\pi g^3 l_p^3} \equiv N \in \mathbb{Z}, \quad (2.24) \]

where \( l_p \) is the Planck length and \( N \) is the number of M5-branes wrapping \( \Sigma \times \Sigma_g \).

For the metric of the form,

\[ ds_{11}^2 = e^{2A} (ds_{AdS_3}^2 + ds_{M_8}^2), \quad (2.25) \]

the central charge of dual two-dimensional conformal field theory is given by [34, 35], and we follow [6],

\[ c = \frac{3}{2G_N^{(3)}} = \frac{3}{2G_N^{(11)}} \int_{M_8} e^{9A} \text{vol}_{M_8}, \quad (2.26) \]

where the eleven-dimensional Newton’s gravitational constant is \( G_N^{(11)} = \frac{(2\pi)^{\frac{8}{16}} g_0^2}{16\pi} \). For the solutions, with (2.11), we find the holographic central charge to be

\[ c = \frac{L^3 \Delta z}{8\pi^3 g_0^6 l_p^9} (y_2 - y_1) vol_{\Sigma_g} = \frac{\Delta z}{2\pi^2} N^3 (y_1 - y_2) vol_{\Sigma_g} \]

\[ = \frac{4q^2 (n_- - n_+ - 2q)}{n_- n_+ [n_- (n_+ + 2q) - q (2n_+ + 3q)]} (g - 1) N^3, \quad (2.27) \]

where \( vol_{\Sigma_g} = 4\pi (g - 1) \). This precisely matches the result obtained from the solutions by uplifting \( AdS_3 \times \Sigma \) to eleven-dimensional supergravity, [6].

### 3 D4-branes wrapped on \( \Sigma \times \Sigma_g \)

#### 3.1 Matter coupled \( F(4) \) gauged supergravity

We review \( F(4) \) gauged supergravity, [36], coupled to a vector multiplet in six dimensions, [37, 38], in the conventions of [12]. The bosonic field content is consist of the metric, two \( U(1) \) gauge fields, \( A_i \), a two-form field, \( B \), and two scalar fields, \( \varphi_i \), where \( i = 1, 2 \). We introduce a parametrization of the scalar fields,

\[ X_i = e^{-\frac{1}{2} \tilde{a}_i \cdot \varphi}, \quad \tilde{a}_1 = \left( 2^{1/2}, 2^{-1/2} \right), \quad \tilde{a}_2 = \left( -2^{1/2}, 2^{-1/2} \right), \quad (3.1) \]

with

\[ X_0 = (X_1 X_2)^{-3/2}. \quad (3.2) \]
The field strengths of the gauge fields and two-form field are, respectively,

\[ F_i = dA_i, \quad H = dB. \]  

(3.3)

The action is given by

\[
S = \frac{1}{16\pi G_N^{(6)}} \int d^6x \sqrt{-g} \left[ R - V - \frac{1}{2} |d\varphi|^2 - \frac{1}{2} \sum_{i=1}^{2} X_i^{-2} |F_i|^2 - \frac{1}{8} (X_1 X_2)^2 |H|^2 
\right.

\[ \left. - \frac{m^2}{4} (X_1 X_2)^{-1} |B|^2 - \frac{1}{16} \varepsilon^{\mu\nu\rho\sigma\tau\lambda} B_{\mu\nu} \left( F_{1\rho\sigma} F_{2\tau\lambda} + \frac{m^2}{12} B_{\rho\sigma} B_{\tau\lambda} \right) \right], \]

(3.4)

where the scalar potential is

\[ V = m^2 X_0^2 - 4g^2 X_1 X_2 - 4gmX_0 (X_1 + X_2), \]

(3.5)

and \( \varepsilon_{012345} = +1. \) The norm of form fields are defined by

\[ |\omega|^2 = \frac{1}{p!} \omega_{\mu_1...\mu_p} \omega^{\mu_1...\mu_p}. \]

(3.6)

The equations of motion are presented in appendix A.

### 3.2 Multi-charged AdS\(_2 \times \Sigma\) solutions

We review the AdS\(_2 \times \Sigma\) solutions of \( U(1)^4\)-gauged \( \mathcal{N} = 2 \) supergravity in four dimensions, [10, 14]. These solution are obtained from M2-branes wrapped on a spindle, \( \Sigma \). The metric, gauge fields and scalar fields read

\[
d s_4^2 = H^{1/2} \left[ \frac{1}{4} d s_{\text{AdS}_2}^2 + \frac{1}{P} d y^2 + \frac{P}{4H} d z^2 \right],
\]

\[
A^{(I)} = \frac{y}{y + q_I} d z, \quad X^{(I)} = \frac{H^{1/4}}{y + q_I},
\]

(3.7)

where \( I = 1, \ldots, 4 \) and the functions are defined to be

\[ H = (y + q_1)(y + q_2)(y + q_3)(y + q_4), \quad P = H - 4y^2. \]

(3.8)

In the case of four distinct roots, \( y_0 < y_1 < y_2 < y_3 \), of quartic polynomial, \( P(y) \), the solution is positive and regular in \( y \in [y_1, y_2] \). The spindle, \( \Sigma \), is an orbifold, \( \mathbb{C} P^1_{[n_1, n_2]} \), with conical deficit angles at \( y = y_1, y_2 \), [10, 14]. The spindle numbers, \( n_1, n_2 \), are arbitrary coprime positive integers. The Euler number of the spindle is given by

\[ \chi(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} R_{\Sigma} \text{vol}_{\Sigma} = \frac{n_1 + n_2}{n_1 n_2}, \]

(3.9)
where $R_\Sigma$ and $\text{vol}_\Sigma$ are the Ricci scalar and the volume form on the spindle. The magnetic flux through the spindle is given by

$$Q_I = \frac{1}{2\pi} \int_\Sigma F^{(I)} = \left( \frac{y_2}{y_2 + q_I} - \frac{y_1}{y_1 + q_I} \right) \frac{\Delta z}{2\pi} = \frac{2p_I}{n_1 n_2}, \quad (3.10)$$

and we demand $p_I \in \mathbb{Z}$. One can show that the R-symmetry flux is given by

$$Q^R = \frac{1}{2} (Q_1 + Q_2 + Q_3 + Q_4) = \frac{p_1 + p_2 + p_2 + p_4}{n_1 n_2} = \frac{\eta_1 n_2 - \eta_2 n_1}{n_1 n_2}, \quad (3.11)$$

where the supersymmetry is realized by, $[14, 15]$

$$\text{Anti-twist} : (\eta_1, \eta_2) = (+1, +1),
\text{Twist} : (\eta_1, \eta_2) = (\pm 1, \mp 1). \quad (3.12)$$

When parameters, $q_I$, $I = 1, \ldots, 4$, are all identical or identical in pairwise, only the anti-twist solutions are allowed. Otherwise, for all distinct or three identical with one distinct parameters, both the twist and anti-twist solutions are allowed.

Unlike five-dimensional $U(1)^3$-gauged supergravity which has a unique $U(1)^2$ subtruncation, there are two distinct $U(1)^2$ subtruncations from four-dimensional $U(1)^4$-gauged supergravity,

$$\text{ST}^2 \text{ model} : \quad A^{(1)} = A^{(2)} \neq A^{(3)} = A^{(4)}, \quad X^{(1)} = X^{(2)} \neq X^{(3)} = X^{(4)},$$
$$\text{T}^3 \text{ model} : \quad A^{(1)} = A^{(2)} = A^{(3)} \neq A^{(4)}, \quad X^{(1)} = X^{(2)} = X^{(3)} \neq X^{(4)}, \quad (3.13)$$

and their permutations.

### 3.3 Multi-charged $AdS_2 \times \Sigma \times \Sigma_\mathfrak{g}$ solutions

A consistent reduction of matter coupled $F(4)$ gauged supergravity on a Riemann surface was performed in [27]. Inspired by the consistent truncation in [27], the $AdS_3 \times \Sigma \times \Sigma_\mathfrak{g}$ solutions in [2.13], and the minimal $AdS_2 \times \Sigma \times \Sigma_\mathfrak{g}$ solutions in [12], we construct the $AdS_2 \times \Sigma \times \Sigma_\mathfrak{g}$ solutions. However, only the $T^3$ model is obtained from the truncation of $F(4)$ gauged supergravity and not the $ST^2$ model. Thus, we only find solutions by uplifting multi-charged $AdS_2 \times \Sigma$ solutions in section 3.2 with

$$q_1 = q_2 = q_3 \neq q_4, \quad (3.14)$$
to six dimensions. After some trial and error we find the solutions to be

\[
\begin{align*}
    ds_6^2 &= e^{-2C} L_{AdS_4}^2 H^{1/2} \left[ \frac{1}{4} ds_{AdS_2}^2 + \frac{1}{P} dy^2 + \frac{P}{4H} dz^2 \right] + e^{2C} ds_{\Sigma_g}^2, \\
    X_1 &= k_{8}^{1/8} k_2^{1/2} \frac{H^{1/4}}{y + q_1}, \quad X_2 = k_{8}^{1/8} k_2^{-1/2} \frac{H^{1/4}}{y + q_1}, \quad e^{-2C} = m^2 k_{8}^{1/4} k_4 \frac{H^{1/4}}{y + q_1} \\
    B &= q_1 \frac{9 k_{8}^{1/2}}{4 g^2} \text{vol}_{AdS_2}, \\
    F_1 &= \frac{3 k_{8}^{1/2} k_2^{1/2} q_1}{2g} \frac{(y + q_1)^2}{(y + q_1)^2} dy \wedge dz + \frac{\kappa + z}{2g} \text{vol}_{\Sigma_g}, \\
    F_2 &= \frac{3 k_{8}^{1/2} k_2^{-1/2} q_4}{2g} \frac{(y + q_1)^2}{(y + q_4)^2} dy \wedge dz + \frac{\kappa - z}{2g} \text{vol}_{\Sigma_g},
\end{align*}
\]  

(3.15)

where we define

\[
H = (y + q_1)^3 (y + q_4), \quad P = H - 4y^2,
\]  

(3.16)

and

\[
g = \frac{3m}{2}, \quad L_{AdS_4} = \frac{k_{8}^{1/4} k_4^{-1/2}}{m^2}.
\]  

(3.17)

There are parameters, \(\kappa = 0, \pm 1\), for the curvature of Riemann surface, and, \(z\), which define

\[
k_2 = \frac{3z + \sqrt{\kappa^2 + 8z^2}}{z - \kappa}, \quad k_8 = \frac{16k_2}{9(1 + k_2)^2}, \quad k_4 = \frac{18}{-3\kappa + \sqrt{\kappa^2 + 8z^2}}.
\]  

(3.18)

If we set \(q_1 = q_2 = q_3 = q_4\), it reduces to the minimal \(AdS_2 \times \Sigma \times \Sigma_g\) solutions in [12]. For our solutions, in order to satisfy the equations of motion, we find that we should choose

\[
\kappa = -1, \quad z = 1,
\]  

(3.19)

and we find \(k_2 = k_4 = k_8^{-1} = 3\). Then the solutions are given by

\[
\begin{align*}
    ds_6^2 &= e^{-2C} L_{AdS_4} H^{1/2} \left[ \frac{1}{4} ds_{AdS_2}^2 + \frac{1}{P} dy^2 + \frac{P}{4H} dz^2 \right] + e^{2C} ds_{\Sigma_g}^2, \\
    X_1 &= 3^{3/8} \frac{H^{1/4}}{y + q_1}, \quad X_2 = 3^{-5/8} \frac{H^{1/4}}{y + q_1}, \quad e^{-2C} = \frac{4g^2}{3^{5/4}} \frac{H^{1/4}}{y + q_1} \\
    B &= q_1 \frac{3\sqrt{3}}{4 g^2} \text{vol}_{AdS_2}, \\
    F_1 &= \frac{3}{2g} \frac{q_1}{(y + q_1)^2} dy \wedge dz, \\
    F_2 &= \frac{1}{2g} \frac{q_4}{(y + q_4)^2} dy \wedge dz - \frac{1}{g} \text{vol}_{\Sigma_g}.
\end{align*}
\]  

(3.20)
where we have
\[ g = \frac{3m}{2}, \quad L_{\text{AdS}} = \frac{3^{5/4}}{4g^2}. \]  
(3.21)

Notice that the components of \( F_1 \) on the Riemann surface is turned off by the choice of \( (3.19) \).

The flux quantization through the Riemann surface is given by
\[ s_1 = \frac{g}{2\pi} \int_{\Sigma_g} F_1 = 0, \]
\[ s_2 = \frac{g}{2\pi} \int_{\Sigma_g} F_2 = 2(1 - g) \in \mathbb{Z}, \]  
(3.22)

where we find \( s_1 + s_2 = 2(1 - g) \). Fluxes through the spindle are quantized to be
\[ 3n_1 \equiv \frac{g}{2\pi} \int_{\Sigma} F_1 = \frac{3}{2} \left( \frac{y_2}{y_2 + q_1} - \frac{y_1}{y_1 + q_1} \right) \Delta \frac{z}{2\pi} = \frac{3p_1}{n_1 n_2}, \]
\[ n_2 \equiv \frac{g}{2\pi} \int_{\Sigma} F_2 = \frac{1}{2} \left( \frac{y_2}{y_2 + q_4} - \frac{y_1}{y_1 + q_4} \right) \Delta \frac{z}{2\pi} = \frac{p_4}{n_1 n_2}, \]  
(3.23)

where \( p_1 \) and \( p_4 \) are introduced in \( (3.10) \) and \( p_i \in \mathbb{Z} \). By \( (3.11) \) the total flux is obtained to be
\[ 3n_1 + n_2 = \frac{3p_1 + p_4}{n_1 n_2} = \frac{\eta_1 n_2 - \eta_2 n_1}{n_1 n_2}, \]  
(3.24)

and both the twist and anti-twist solutions are allowed.

### 3.4 Uplift to massive type IIA supergravity

We review the uplift formula, \([39]\), of matter coupled \( F(4) \) gauged supergravity to massive type IIA supergravity, \([40]\), presented in \([12]\). Although the uplift formula is only given for vanishing of two-form field, \( B \), in \( F(4) \) gauged supergravity, it correctly reproduces the metric, the dilaton and the internal four-sphere part of four-form flux. The metric in the string frame and the dilaton field are
\[ ds_{\text{s.f.}}^2 = \lambda^2 \mu_0^{-1/3} (X_1 X_2)^{-1/4} \left\{ \Delta^{1/2} d\ell_{\text{s.f.}}^2 + g^{-2} \Delta^{-1/2} \left[ X_0^{-1} d\mu_0^2 + X_1^{-1} (d\mu_1^2 + \mu_1^2 \sigma_1^2) + X_2^{-1} (d\mu_2^2 + \mu_2^2 \sigma_2^2) \right] \right\}, \]  
(3.25)
\[ e^\Phi = \lambda^2 \mu_0^{-5/6} \Delta^{1/4} (X_1 X_2)^{-5/8}, \]  
(3.26)

where the function, \( \Delta \), is defined by
\[ \Delta = \sum_{a=0}^2 X_a \mu_a^2, \]  
(3.27)

\[ ^{\text{\footnotesize 1}}\text{We would like to thank Chris Couzens for discussion on this.} \]
and the one-forms are \( \sigma_i = d\phi_i - gA_i \). The angular coordinates, \( \phi_1, \phi_2 \), have canonical periodicities of \( 2\pi \). We employ the parametrization of coordinates,

\[
\mu_0 = \sin \xi, \quad \mu_1 = \cos \xi \sin \eta, \quad \mu_2 = \cos \xi \cos \eta, \quad (3.28)
\]

where \( \sum_{a=0}^{2} \mu_a^2 = 1 \) and \( \eta \in [0, \pi/2] \), \( \xi \in (0, \pi/2] \). The internal space is a squashed four-hemisphere which has a singularity on the boundary, \( \xi \to 0 \). The four-form flux is given by

\[
\lambda^{-1} F(4) = gU \text{vol}_6 - \frac{1}{g^2} \sum_{i=1}^{2} X_i^{-2} \mu_i (\ast_6 F_1) \wedge d\mu_i \wedge \sigma_i + \frac{1}{g} \sum_{a=0}^{2} X_a^{-1} \mu_a (\ast_6 dX_a) \wedge d\mu_a, \quad (3.29)
\]

where the function, \( U \), is defined by

\[
U = 2 \sum_{a=0}^{2} X_a^{2} \mu_a^2 - \left[ \frac{4}{3} X_0 + 2(X_1 + X_2) \right] \Delta, \quad (3.30)
\]

and \( \ast_6 \) is a Hodge dual in six dimensions. The Romans mass is given by

\[
F(0) = \frac{2g}{3\lambda^3}. \quad (3.31)
\]

The positive constant, \( \lambda \), is introduced from the scaling symmetry of the theory. It plays an important role to have regular solutions with proper flux quantizations, [12]. The uplift formula implies \( m = 2g/3 \).

The relevant part of the four-form flux for flux quantization is the component on the internal four-sphere,

\[
F(4) = \frac{\lambda \mu_0^{1/3}}{g^3 \Delta} \frac{U \mu_1 \mu_2}{\mu_0} d\mu_1 \wedge d\mu_2 \wedge \sigma_1 \wedge \sigma_2 + \ldots. \quad (3.32)
\]

We impose quantization conditions on the fluxes,

\[
(2\pi l_s) F(0) = n_0 \in \mathbb{Z}, \quad \frac{1}{(2\pi l_s)^3} \int_{\hat{S}_4} F(4) = N \in \mathbb{Z}, \quad (3.33)
\]

where \( l_s \) is the string length. For the solutions, these imply that

\[
g^8 = \frac{1}{(2\pi l_s)^8} \frac{18\pi^6}{N^3 n_0} , \quad \lambda^8 = \frac{8\pi^2}{9N n_0^3}, \quad (3.34)
\]

where we have \( n_0 = 8 - N_f \) and \( N_f \) is the number of D8-branes. These results are identical to the case of minimal \( AdS_2 \times \Sigma \times \Sigma_6 \) solutions in [12].

For the metric of the form in the string frame,

\[
ds_{s.f.}^2 = e^{2A} \left( ds^2_{AdS_2} + ds^2_{M_6} \right), \quad (3.35)
\]
the Bekenstein-Hawking entropy is by, \[34, 35\], and in \[12\],
\[
S_{BH} = \frac{1}{4G_N^{(2)}} = \frac{8\pi^2}{(2\pi l_s)^8} \int e^{8A_{\phi}} \text{vol}_{M_8}. \tag{3.36}
\]

For the solutions, we obtain the Bekenstein-Hawking entropy to be
\[
S_{BH} = \frac{1}{(2\pi l_s)^8} \frac{9(3\pi \lambda)^4 k_8^{1/2}}{20g^8 \kappa_4} 4\pi \kappa (1 - g) A_h = \frac{1}{(2\pi l_s)^8} \frac{\sqrt{3}(3\pi \lambda)^4}{20g^8} 4\pi \kappa (1 - g) A_h, \tag{3.37}
\]
where the area of the horizon of black hole, multi-charged AdS$_2 \times \Sigma$, in (3.7) is
\[
A_h = \frac{1}{2} (y_2 - y_1) \Delta z, \tag{3.38}
\]
and $y_1$ and $y_2$ are two relevant roots of $P(y)$. The free energy of 5d $USp(2N)$ gauge theory on $S^3 \times \Sigma_g$ is given by, \[41, 42, 12\],
\[
\mathcal{F}_{S^3 \times \Sigma_g} = \frac{16\pi^3}{(2\pi l_s)^8} \int e^{8A_{\phi}} \text{vol}_{M_6} = \frac{16\pi \kappa (1 - g) N^{5/2} (z^2 - \kappa^2)^{3/2} (\sqrt{\kappa^2 + 8z^2} - \kappa)}{5\sqrt{8 - N_f} (\kappa\sqrt{\kappa^2 + 8z^2} - \kappa^2 + 4z^2)^{3/2}}. \tag{3.39}
\]

By comparing (3.39) with (3.36), we find the Bekenstein-Hawking entropy to be
\[
S_{BH} = \frac{1}{2} \mathcal{F}_{S^3 \times \Sigma_g} A_h = \frac{8\kappa (1 - g) N^{5/2} (z^2 - \kappa^2)^{3/2} (\sqrt{\kappa^2 + 8z^2} - \kappa)}{5\sqrt{8 - N_f} (\kappa\sqrt{\kappa^2 + 8z^2} - \kappa^2 + 4z^2)^{3/2}} A_h, \tag{3.40}
\]
and, for $\kappa = -1$ and $z = 1$, (3.19), we obtain\footnote{We would like to thank Hyojoong Kim for comments on this limit.}
\[
S_{BH} = \left(\frac{3}{8}\right)^{3/2} \frac{32 (g - 1) N^{5/2}}{5\sqrt{8 - N_f}} A_h. \tag{3.41}
\]

Although formally the Bekenstein-Hawking entropy is in the identical expression of the one for minimal $AdS_2 \times \Sigma \times \Sigma_g$ solutions in \[12\], note that the black holes that give the area, $A_h$, are different: it was minimal $AdS_2 \times \Sigma$ in \[12\], but now it is multi-charged $AdS_2 \times \Sigma$. \[14\]. We refer \[15\] for the explicit expression of $A_h$ for the multi-charged solutions.
4 Gravitational blocks

In this section, we briefly review the off-shell quantities from gluing gravitational blocks, \cite{43}, and show that extremization of off-shell quantity correctly reproduces the Bekenstein-Hawking entropy, central charge, and free energy, depending on the dimensionality, \cite{12}. Then apply the gravitational block calculations to the solutions we constructed in the previous sections.

Depending on the dimensionality, the Bekenstein-Hawking entropy, central charge, and free energy are obtained by extremizing the off-shell quantity, \cite{12},

\[
F^\pm_d (\Delta, \epsilon; n_i, n_+, n_-, \sigma) = \frac{1}{\epsilon} \left( F_d (\Delta_i^+) \pm F_d (\Delta_i^-) \right),
\]

(4.1)

where \(F_d\) are the gravitational blocks, \cite{43}. We also define quantities,

\[
\Delta_i^\pm \equiv \varphi_i \pm n_i \epsilon,
\]

(4.2)

and

\[
\varphi_i \equiv \Delta_i + \frac{r_i n_+ - \sigma n_-}{2 n_+ n_-} \epsilon,
\]

(4.3)

where \(\sigma = +1\) and \(\sigma = -1\) for twist and anti-twist solutions, respectively. The expressions of gravitational blocks are

\[
F_3 = b_3 (\Delta_1 \Delta_2 \Delta_3 \Delta_4)^{1/2}, \quad F_4 = b_4 (\Delta_1 \Delta_2 \Delta_3), \quad F_5 = b_5 (\Delta_1 \Delta_2)^{3/2}, \quad F_6 = b_6 (\Delta_1 \Delta_2)^2,
\]

(4.4)

and the constants, \(b_d\), will be given later. The relative sign for gluing gravitational blocks in (4.1) is \(-\sigma\) for \(d = 3, 5\) and \(-\) for \(d = 4, 6\). The twist conditions on the magnetic flux through the spindle, \(n_i\), is given by

\[
\sum_{i=1}^b n_i = \frac{n_+ + \sigma n_-}{n_+ n_-},
\]

(4.5)

where \(n_+\) and \(n_-\) are the orbifold numbers of spindle and \(\varnothing\) is the rank of global symmetry group of dual field theory, \(i.e., \varnothing = 4\) for \(d = 3\), \(\varnothing = 3\) for \(d = 4\), and \(\varnothing = 2\) for \(d = 5, 6\). The constants are constrained by

\[
\sum_{i=1}^b r_i = 2,
\]

(4.6)

and they parametrize the ambiguities of defining the flavor symmetries. The \(U(1)\) R-symmetry flux gives

\[
\frac{1}{2\pi} \int_X dA_R = \frac{n_+ + \sigma n_-}{n_+ n_-},
\]

(4.7)

and the fugacities of dual field theories are normalized by

\[
\sum_{i=1}^b \Delta_i = 2.
\]

(4.8)
The off-shell quantity can be written by
\[ F_{\pm}^d (\varphi, \epsilon; n) = \frac{1}{\epsilon} \left( F_{d} (\varphi_i + n_i \epsilon) \pm F_{d} (\varphi_i - n_i \epsilon) \right), \tag{4.9} \]
where the variables satisfy the constraint,
\[ \sum_{i=1}^{g} \varphi_i - \frac{n_+ - \sigma n_-}{n_+ n_-} \epsilon = 2, \tag{4.10} \]
which originates from (4.6) and (4.8).

### 4.1 M5-branes wrapped on \( \Sigma \times \Sigma_g \)

For the \( AdS_3 \times \Sigma \times \Sigma_g \) solutions, there is standard topological twist on \( \Sigma_g \) for the magnetic charges, \( s_i \), and anti-twist on \( \Sigma \) for \( n_i \). Then the off-shell central charge is given by
\[ S(\varphi_i, \epsilon_1, \epsilon_2; n_i, s_i) = -\frac{1}{4\epsilon_1 \epsilon_2} \left[ F_6 (\varphi_i + n_i \epsilon_1 + s_i \epsilon_2) - F_6 (\varphi_i - n_i \epsilon_1 + s_i \epsilon_2) \right. \]
\[ -\left. F_6 (\varphi_i + n_i \epsilon_1 - s_i \epsilon_2) + F_6 (\varphi_i - n_i \epsilon_1 - s_i \epsilon_2) \right], \tag{4.11} \]
with the constraints,
\[ n_1 + 2n_2 = \frac{n_+ - n_-}{n_+ n_-}, \quad s_1 + s_2 = 2(1 - g), \quad \varphi_1 + 2\varphi_2 - \frac{n_+ + n_-}{n_+ n_-} \epsilon_1 = 2. \tag{4.12} \]

For the calculations, we employ
\[ b_4 = -\frac{3}{2} N^2, \quad b_6 = -N^3. \tag{4.13} \]

Extremizing it with respect to \( \epsilon_2 \) gives \( \epsilon_2 = 0 \) and renaming \( \epsilon_1 \rightarrow \epsilon \), we find the off-shell central charge expressed by
\[ S(\varphi_i, \epsilon; n_i, s_i) = 2N^3 s_1 \left( n_1 \varphi_2 \varphi_3 + \varphi_1 n_2 \varphi_3 + \varphi_1 \varphi_2 n_3 + n_1 n_2 n_3 \epsilon^2 \right) \bigg|_{3 \rightarrow 2} \]
\[ + 2N^3 s_2 \left( n_1 \varphi_2 \varphi_3 + \varphi_1 n_2 \varphi_3 + \varphi_1 \varphi_2 n_3 + n_1 n_2 n_3 \epsilon^2 \right) \bigg|_{3 \rightarrow 1} \]
\[ = 2N^3 s_1 \left( -\frac{1}{3N^2} F_4^{-} \right) \bigg|_{3 \rightarrow 2} + 2N^3 s_2 \left( -\frac{1}{3N^2} F_4^{-} \right) \bigg|_{3 \rightarrow 1}. \tag{4.14} \]

We have started with the \( d = 6 \) gravitational blocks, \( F_6 \), and we observe the \( d = 4 \) structure, \( F_4^- \), naturally emerges. See section 5.2 of [12] for the calculations of \( d = 4 \) gravitational blocks. From the \( d = 4 \) point of view, the \( s_1 \) term of \( S(\varphi_i, \epsilon; n_i, s_i) \) in (4.14) is the off-shell central charge for \( n_1 \neq n_2 = n_3 \) and the \( s_2 \) terms is for \( n_1 = n_3 \neq n_2 \). Thus, extremization gives disparate results for each term. However, for the solution, as we have
\[ s_1 = 2(1 - g), \quad s_2 = 0, \tag{4.15} \]
the solution chooses the $s_1$ term in the off-shell central charge. Extremizing this we find the values,

$$\epsilon^* = \frac{n_+ - n_-}{2 \left( \frac{\sigma}{n_+ n_-} - (n_1 n_2 + n_2 n_3 + n_3 n_1) \right)} \bigg|_{3 \to 2}, \quad \varphi^*_2 = \frac{n_2 (n_2 - n_3 - n_1)}{2 \left( \frac{\sigma}{n_+ n_-} - (n_1 n_2 + n_2 n_3 + n_3 n_1) \right)} \bigg|_{3 \to 2}.$$  

(4.16)

Then the off-shell central charge gives

$$S(\varphi^*_i, \epsilon^*; n_i) = 4N^3 (g - 1) \left( \frac{\sigma}{n_+ n_-} - (n_1 n_2 + n_2 n_3 + n_3 n_1) \right) \bigg|_{3 \to 2},$$  

(4.17)

which precisely matches the holographic central charge from the supergravity solutions, (2.27), with $\sigma = -1$.

4.2 D4-branes wrapped on $\Sigma \times \Sigma_g$

For the $AdS_2 \times \Sigma \times \Sigma_g$ solutions, there is standard topological twist on $\Sigma_g$ for the magnetic charges, $s_i$, and anti-twist on $\Sigma$ for $n_i$. Then the entropy function is given by

$$S(\varphi_i, \epsilon_1, \epsilon_2; n_i, s_i) = \frac{1}{4 \epsilon_1 \epsilon_2} \left[ F_5(\varphi_i + n_i \epsilon_1 + s_i \epsilon_2) + F_5(\varphi_i - n_i \epsilon_1 + s_i \epsilon_2) \right] \right] \bigg|_{3 \to 2},$$  

(4.18)

with the constraints,

$$n_1 + 3n_2 = \frac{n_+ - n_-}{n_+ n_-}, \quad s_1 + s_2 = 2(1 - g), \quad \varphi_1 + 3\varphi_2 - \frac{n_+ + n_-}{n_+ n_-} \epsilon_1 = 2.$$  

(4.19)

For the calculations, we employ

$$b_3 = -\sqrt{2\pi} \frac{N^{3/2}}{N}, \quad b_5 = -\frac{2^{5/2}}{15} \frac{N^{5/2}}{\sqrt{8 - N_f}}.$$  

(4.20)

Extremizing it with respect to $\epsilon_2$ gives $\epsilon_2 = 0$ and renaming $\epsilon_1 \mapsto \epsilon$, we find the entropy function expressed by

$$S(\varphi_i, \epsilon; n_i, s_i) = \frac{c}{\epsilon} \left[ s_1 \left( \sqrt{(\varphi_1 + n_1 \epsilon)(\varphi_2 + n_2 \epsilon)} + \sqrt{(\varphi_1 - n_1 \epsilon)(\varphi_2 - n_2 \epsilon)} \right) \right] + s_2 \left( \sqrt{(\varphi_1 + n_1 \epsilon)^3 (\varphi_2 + n_2 \epsilon)} + \sqrt{(\varphi_1 - n_1 \epsilon)^3 (\varphi_2 - n_2 \epsilon)} \right),$$  

(4.21)

where we have

$$c \equiv \sqrt{2\pi} \frac{N^{5/2}}{5 \sqrt{8 - N_f}}.$$  

(4.22)
We have started with the $d = 5$ gravitational blocks, $\mathcal{F}_5$, and we observe the $d = 3$ structure naturally emerges. See section 5.1 of [12] for the calculations of $d = 3$ gravitational blocks. From the $d = 3$ point of view, the $s_1$ term of $S(\varphi_i, \epsilon; n_i, s_i)$ in (4.21) is the entropy function for $n_1 \neq n_2 = n_3 = n_4$ and the $s_2$ terms is for $n_1 = n_2 = n_3 \neq n_4$. Thus, extremization gives disparate results for each term. However, for the solution, as we have
\[
s_1 = 2(1 - g), \quad s_2 = 0,
\]
the solution chooses the $s_1$ term in the entropy function. However, in this case, the algebraic equations appearing in the extremization procedure are quite complicated and we do not pursue it further here.

5 Conclusions

In this work, we have constructed multi-charged $AdS_3 \times \Sigma \times \Sigma_g$ and $AdS_2 \times \Sigma \times \Sigma_g$ solutions from M5-branes and D4-branes. We have uplifted the solutions to eleven-dimensional and massive type IIA supergravity, respectively. We have also studied their spindle properties and calculated the holographic central charge and the Bekenstein-Hawking entropy, respectively.

Although we have only considered the $AdS_{2,3} \times \Sigma \times \Sigma_g$ solutions for spindle, $\Sigma$, the local form of our solutions naturally allows solutions for disk, $\Sigma$, by different global completion. However, the $AdS_3 \times \Sigma \times \Sigma_g$ solution for disk, $\Sigma$, was already constructed and studied in [21]. Thus, it would be interesting to analyze the $AdS_2 \times \Sigma \times \Sigma_g$ solutions for disk, $\Sigma$, from the solutions we have constructed.

Unlike the minimal $AdS_2 \times \Sigma \times \Sigma_g$ solutions in [12] where $z$ is a free parameter, only $z = 1$ is allowed for our multi-charged $AdS_2 \times \Sigma \times \Sigma_g$ solutions, (3.19). We would like to understand why it is required to fix the parameter for the solutions and if there are more general multi-charged solutions with additional parameters.

The solutions we have obtained could be seen as generalizations of $AdS_3 \times \Sigma_{g_1} \times \Sigma_{g_2}$ solutions in [44] and $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ solutions in [45, 46, 47]. In particular, via the AdS/CFT correspondence, [48], the Bekenstein-Hawking entropy of $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ solutions was microscopically counted by the topologically twisted index of 5d $USp(2N)$ gauge theories, [49, 42]. It would be most interesting to derive the Bekenstein-Hawking entropy of the $AdS_2 \times \Sigma \times \Sigma_g$ solutions from the field theory calculations.

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A The equations of motion

A.1 $U(1)^2$-gauged supergravity in seven dimensions

We present the equations of motion derived from the Lagrangian in \([2.1]\),

\[
R_{\mu\nu} = 6\partial_\mu \lambda_1 \partial_\nu \lambda_1 + 6\partial_\mu \lambda_2 \partial_\nu \lambda_2 + 8\partial_\mu (\lambda_1 \partial_\nu \lambda_2) + 1 \frac{1}{5} g_{\mu\nu} V \\
+ \frac{1}{2} e^{-4\lambda_1} \left( F_{\mu\rho} F_{\nu}^{\mu\rho} - \frac{1}{10} g_{\mu\nu} F_{\rho\sigma} F_{\rho\sigma} \right) + \frac{1}{2} e^{-4\lambda_2} \left( F_{\mu\rho} F_{\nu}^{\mu\rho} - \frac{1}{10} g_{\mu\nu} F_{\rho\sigma} F_{\rho\sigma} \right) \\
+ \frac{1}{4} e^{-4\lambda_1-4\lambda_2} \left( S_{\mu\rho\sigma}^5 S_{\nu}^{5\rho\sigma} - \frac{2}{15} g_{\mu\nu} S_{\rho\sigma\delta}^5 S_{5\rho\sigma\delta} \right), \tag{A.1}
\]

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu (3\lambda_1 + 2\lambda_2)) + \frac{1}{4} e^{-4\lambda_1} F_{\mu\nu}^1 F_{\mu\nu}^{12} + \frac{1}{12} e^{-4\lambda_1-4\lambda_2} S_{\mu\rho\sigma}^5 S_{\nu\rho\sigma}^5 - g^2 \frac{\partial V}{\partial \lambda_1} = 0, \\
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu (2\lambda_1 + 3\lambda_2)) + \frac{1}{4} e^{-4\lambda_2} F_{\mu\nu}^3 F_{\mu\nu}^{34} + \frac{1}{12} e^{-4\lambda_1-4\lambda_2} S_{\mu\rho\sigma}^5 S_{\nu\rho\sigma}^5 - g^2 \frac{\partial V}{\partial \lambda_2} = 0, \tag{A.2}
\]

\[
d \big( e^{-4\lambda_1} * F^{12} \big) + e^{-4\lambda_1-4\lambda_2} * S^5 \wedge F^{34} = 0, \\
d \big( e^{-4\lambda_2} * F^{34} \big) + e^{-4\lambda_1-4\lambda_2} * S^5 \wedge F^{12} = 0, \\
d S^5 - g e^{-4\lambda_1-4\lambda_2} * S^5 - F^{12} \wedge F^{34} = 0. \tag{A.3}
\]

A.2 Matter coupled $F(4)$ gauged supergravity

We present the equations of motion derived from the action in \([3.4]\),

\[
R_{\mu\nu} - \frac{1}{2} \sum_{i=1}^{2} \partial_\mu \varphi_i \partial_\nu \varphi_i - \frac{1}{4} V g_{\mu\nu} - \frac{1}{2} \sum_{i=1}^{2} X_i^{-2} \left( F_{i\mu\rho} F_{i\nu}^{\mu\rho} \right) - \frac{m^2}{4} (X_1 X_2)^{-1} \left( B_{\mu\nu} B_{\nu}^{\mu} \right) - \frac{1}{16} (X_1 X_2)^2 \left( H_{\mu\nu\rho} H_{\nu}^{\mu\rho} \right) = 0, \tag{A.4}
\]

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi_1) - \frac{\partial V}{\partial \varphi_1} = - \frac{1}{2\sqrt{2}} X_1^{-2} F_{1\mu\nu} F_1^{\mu\nu} + \frac{1}{2\sqrt{2}} X_2^{-2} F_{2\mu\nu} F_2^{\mu\nu} = 0, \\
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi_2) - \frac{\partial V}{\partial \varphi_2} = - \frac{1}{4\sqrt{2}} X_1^{-2} F_{1\mu\nu} F_1^{\mu\nu} - \frac{1}{4\sqrt{2}} X_2^{-2} F_{2\mu\nu} F_2^{\mu\nu} \\
- \frac{m^2}{8\sqrt{2}} (X_1 X_2)^{-1} B_{\mu\nu} B_\mu^{\mu\nu} + \frac{1}{24\sqrt{2}} (X_1 X_2)^2 H_{\mu\nu\rho} H_\mu^{\mu\rho} = 0, \tag{A.5}
\]
\[ D_\nu (X_1^{-2} F_{1}^{\nu \mu}) = \frac{1}{24} \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma \tau \lambda} F_{2 \nu \rho} H_{\sigma \tau \lambda}, \]
\[ D_\nu (X_2^{-2} F_{2}^{\nu \mu}) = \frac{1}{24} \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma \tau \lambda} F_{1 \nu \rho} H_{\sigma \tau \lambda}, \]
\[ D_\nu ((X_1 X_2)^{-1} B_{\nu \mu}) = \frac{1}{24} \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma \tau \lambda} B_{\nu \rho} H_{\sigma \tau \lambda}, \]
\[ D_\rho ((X_1 X_2)^{2} H^{\rho \nu \mu}) = -\frac{1}{4} \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma \tau \lambda} \left( \frac{m^2}{2} B_{\rho \sigma} B_{\tau \lambda} + F_{\rho \sigma} F_{\tau \lambda} \right) - 2m^2 (X_1 X_2)^{-1} B^{\mu \nu}. \quad (A.6) \]

## B Consistent truncations of [1]

In this appendix, we briefly review the consistent truncation of seven-dimensional maximal gauged supergravity, [31], on a Riemann surface in [1] and explain the setup to uplift our solutions by employing the truncation ansatz.

The consistent truncation ansatz for the seven-dimensional metric on a Riemann surface, \( \Sigma_g \), is given by
\[ ds^2_7 = e^{-4\varphi} ds^2_5 + \frac{1}{g^2} e^{6\varphi} ds^2_{\Sigma_g}, \quad (A.1) \]
which introduces a scalar field, \( \varphi \), in five dimensions. Also \( g^2 L_{AdS_5}^2 = 2^{4/3} \) for the gauge coupling, \( g \), and the radius of asymptotic AdS_5, \( L_{AdS_5} \). The \( SO(5) \) gauge fields are decomposed by \( SO(5) \rightarrow SO(2) \times SO(3) \),
\[ A^{ab} = \varepsilon^{ab} A + \frac{1}{g} \omega^{ab}, \]
\[ A_{a\alpha} = -A^{a\alpha} = \psi^{1\alpha} e^a - \psi^{2\alpha} e^b, \]
\[ A^{\alpha\beta} = A^{\alpha\beta}, \quad (A.2) \]
where \( a, b = 1, 2 \), \( \alpha, \beta = 3, 4, 5 \), \( ds^2_{\Sigma_g} = e^{\varphi} e^a \), and \( \omega^{ab} \) is the spin connection on \( \Sigma_g \). The ansatz introduces an \( SO(2) \) one-form, \( A \), \( SO(3) \) one-forms, \( A^{\alpha\beta} \), transforming in the \((1, 3)\) of \( SO(2) \times SO(3) \), and six scalar fields, \( \psi^{a\alpha} = (\psi^{1\alpha}, \psi^{2\alpha}) \), transforming in the \((2, 3)\). The scalar fields are given by
\[ T^{ab} = e^{-6\lambda} \delta^{ab}, \quad T^{a\alpha} = 0, \quad T^{\alpha\beta} = e^{4\lambda} T^{\alpha\beta}, \quad (A.3) \]
which introduces a scalar field, \( \lambda \), and five scalar fields in \( T^{\alpha\beta} \) which live on the coset manifold, \( SL(3)/SO(3) \). The three-form field is given by
\[ S^a = K^{1(2)}_a \wedge e^a - \varepsilon^{ab} K^{2(2)}_a \wedge e^b, \]
\[ S^\alpha = h^\alpha_{(3)} + \chi^\alpha_{(1)} \wedge \text{vol} \Sigma_g, \quad (A.4) \]
which introduces an \( SO(2) \) doublet of two-forms, \( K^{(2)}_a \), three-forms, \( h^{\alpha}_{(3)} \), and one-forms, \( \chi^\alpha_{(1)} \).
To be particular, we consider a subtruncation of the general consistent truncations which reduces to $SU(2) \times U(1)$-gauged $\mathcal{N} = 4$ supergravity in five dimensions, \cite{29}, which is presented in section 5.1 of \cite{1}. In this case, we have the scalar fields to be

\[ \lambda = 3 \varphi, \quad T_{\alpha\beta} = \delta_{\alpha\beta}, \quad \psi^{\alpha\alpha} = 0. \quad (A.5) \]

From the three-form field, we have a complex two-form field,

\[ C_{(2)} = K_{(2)}^1 + iK_{(2)}^2, \quad (A.6) \]

and a three-form field,

\[ *h_{(3)}^\alpha = \frac{1}{2} e^{-20\varphi} \epsilon_{\alpha\beta\gamma} F^{\beta\gamma}, \quad (A.7) \]

with $\chi_{(1)}^\alpha = 0$.

In order to match with the special case of $U(1)^2$-gauged supergravity in seven dimensions, \cite{2.10}, we further impose $A_{\alpha\alpha}^{(1)} = 0$ and $C_{(2)} = 0$. In $U(1)^2$-gauged supergravity in seven dimensions, the scalar fields of are given by

\[ T_{ij} = \text{diag} \left( e^{2\lambda_1}, e^{2\lambda_1}, e^{2\lambda_2}, e^{2\lambda_2}, e^{-4\lambda_1 - 4\lambda_2} \right). \quad (A.8) \]

By matching it with the consistent truncation ansatz,

\[ T_{ij} = \text{diag} \left( e^{-6\lambda}, e^{-6\lambda}, e^{4\lambda}, e^{4\lambda}, e^{4\lambda} \right), \quad (A.9) \]

we identify the scalar fields to be

\[ \lambda_1 = -3\lambda, \quad \lambda_2 = 2\lambda. \quad (A.10) \]

The non-trivial three-form field, $S^5$, is given by $h_{(3)}^\alpha$ in \cite{A.7}.

Finally, we compare the actions of $SU(2) \times U(1)$-gauged $\mathcal{N} = 4$ supergravity in five dimensions, \cite{29}, presented in (5.4) of \cite{1} and in (2.1) with (3.1) of \cite{6} to fix

\[ X^{(1)} = 2^{-1/3} e^{10\varphi}, \quad X^{(2)} = 2^{-1/3} e^{10\varphi}, \quad X^{(3)} = 2^{2/3} e^{-20\varphi}. \quad (A.11) \]

With $X = X^{(1)} = X^{(2)}$, this determines the scalar fields to be

\[ e^{-\frac{40}{9}\lambda_1} = 2^{1/3} X, \quad e^{\frac{2}{3}\lambda_2} = 2^{1/3} X, \quad e^{10\varphi} = 2^{1/3} X. \quad (A.12) \]
References

[1] K. C. M. Cheung, J. P. Gauntlett and C. Rosen, Consistent KK truncations for M5-branes wrapped on Riemann surfaces, *Class. Quant. Grav.* **36** (2019) 225003, [1906.08900](https://arxiv.org/abs/1906.08900).

[2] P. Ferrero, J. P. Gauntlett, J. M. Pérez Ipiña, D. Martelli and J. Sparks, *D3-Branes Wrapped on a Spindle*, *Phys. Rev. Lett.* **126** (2021) 111601, [2011.10579](https://arxiv.org/abs/2011.10579).

[3] E. Witten, *Topological Quantum Field Theory*, *Commun. Math. Phys.* **117** (1988) 353.

[4] J. M. Maldacena and C. Nunez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys. A* **16** (2001) 822–855, [hep-th/0007018](https://arxiv.org/abs/hep-th/0007018).

[5] S. M. Hosseini, K. Hristov and A. Zaffaroni, *Rotating multi-charge spindles and their microstates*, *JHEP* **07** (2021) 182, [2104.11249](https://arxiv.org/abs/2104.11249).

[6] A. Boido, J. M. P. Ipiña and J. Sparks, *Twisted D3-brane and M5-brane compactifications from multi-charge spindles*, *JHEP* **07** (2021) 222, [2104.13287](https://arxiv.org/abs/2104.13287).

[7] P. Ferrero, J. P. Gauntlett, J. M. P. Ipiña, D. Martelli and J. Sparks, *Accelerating black holes and spinning spindles*, *Phys. Rev. D* **104** (2021) 046007, [2012.08530](https://arxiv.org/abs/2012.08530).

[8] D. Cassani, J. P. Gauntlett, D. Martelli and J. Sparks, *Thermodynamics of accelerating and supersymmetric AdS4 black holes*, *Phys. Rev. D* **104** (2021) 086005, [2106.05571](https://arxiv.org/abs/2106.05571).

[9] P. Ferrero, M. Inglese, D. Martelli and J. Sparks, *Multicharge accelerating black holes and spinning spindles*, *Phys. Rev. D* **105** (2022) 126001, [2109.14625](https://arxiv.org/abs/2109.14625).

[10] C. Couzens, K. Stemerdink and D. van de Heisteeg, *M2-branes on discs and multi-charged spindles*, *JHEP* **04** (2022) 107, [2110.00571](https://arxiv.org/abs/2110.00571).

[11] P. Ferrero, J. P. Gauntlett, D. Martelli and J. Sparks, *M5-branes wrapped on a spindle*, *JHEP* **11** (2021) 002, [2105.13344](https://arxiv.org/abs/2105.13344).

[12] F. Faedo and D. Martelli, *D4-branes wrapped on a spindle*, *JHEP* **02** (2022) 101, [2111.13660](https://arxiv.org/abs/2111.13660).

[13] S. Giri, *Black holes with spindles at the horizon*, *JHEP* **06** (2022) 145, [2112.04431](https://arxiv.org/abs/2112.04431).

[14] P. Ferrero, J. P. Gauntlett and J. Sparks, *Supersymmetric spindles*, *JHEP* **01** (2022) 102, [2112.01543](https://arxiv.org/abs/2112.01543).
[15] C. Couzens, A tale of (M)2 twists, JHEP 03 (2022) 078, 2112.04462.

[16] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, Holographic Duals of Argyres-Douglas Theories, Phys. Rev. Lett. 127 (2021) 211601, 2105.11567.

[17] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, M5-brane sources, holography, and Argyres-Douglas theories, JHEP 11 (2021) 140, 2106.01322.

[18] P. C. Argyres and M. R. Douglas, New phenomena in SU(3) supersymmetric gauge theory, Nucl. Phys. B 448 (1995) 93–126, hep-th/9505062.

[19] C. Couzens, H. Kim, N. Kim and Y. Lee, Holographic duals of M5-branes on an irregularly punctured sphere, JHEP 07 (2022) 102, 2204.13537.

[20] C. Couzens, N. T. Macpherson and A. Passias, $\mathcal{N} = (2, 2)$ AdS$_3$ from D3-branes wrapped on Riemann surfaces, JHEP 02 (2022) 189, 2107.13562.

[21] M. Suh, D3-branes and M5-branes wrapped on a topological disc, JHEP 03 (2022) 043, 2108.01105.

[22] M. Suh, M2-branes wrapped on a topological disk, JHEP 09 (2022) 048, 2109.13278.

[23] M. Suh, D4-branes wrapped on a topological disk, JHEP 06 (2023) 008, 2108.08326.

[24] P. Karndumri and P. Nuchino, Five-branes wrapped on topological disks from 7D N=2 gauged supergravity, Phys. Rev. D 105 (2022) 066010, 2201.05037.

[25] M. Gutperle and N. Klein, A note on co-dimension 2 defects in N = 4, d = 7 gauged supergravity, Nucl. Phys. B 984 (2022) 115969, 2203.13839.

[26] K. C. M. Cheung, J. H. T. Fry, J. P. Gauntlett and J. Sparks, M5-branes wrapped on four-dimensional orbifolds, JHEP 08 (2022) 082, 2204.02990.

[27] S. M. Hosseini and K. Hristov, 4d F(4) gauged supergravity and black holes of class $F$, JHEP 02 (2021) 177, 2011.01943.

[28] J. T. Liu and R. Minasian, Black holes and membranes in AdS(7), Phys. Lett. B 457 (1999) 39–46, hep-th/9903269.

[29] L. J. Romans, Gauged N = 4 Supergravities in Five-dimensions and Their Magnetovac Backgrounds, Nucl. Phys. B 267 (1986) 433–447.

[30] J. P. Gauntlett and O. Varela, D=5 SU(2) x U(1) Gauged Supergravity from D=11 Supergravity, JHEP 02 (2008) 083, 0712.3560.
[31] M. Pernici, K. Pilch and P. van Nieuwenhuizen, *Gauged Maximally Extended Supergravity in Seven-dimensions*, Phys. Lett. B 143 (1984) 103–107.

[32] M. Cvetic, H. Lu, C. N. Pope, A. Sadrzadeh and T. A. Tran, *S**3 and S**4 reductions of type IIA supergravity*, Nucl. Phys. B 590 (2000) 233–251, hep-th/0005137.

[33] E. Cremmer, B. Julia and J. Scherk, *Supergravity Theory in Eleven-Dimensions*, Phys. Lett. B 76 (1978) 209–226.

[34] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, Commun. Math. Phys. 104 (1986) 207–226.

[35] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, JHEP 07 (1998) 023, hep-th/9806087.

[36] L. J. Romans, *The F(4) Gauged Supergravity in Six-dimensions*, Nucl. Phys. B 269 (1986) 691.

[37] L. Andrianopoli, R. D’Auria and S. Vaula, *Matter coupled F(4) gauged supergravity Lagrangian*, JHEP 05 (2001) 065, hep-th/0104155.

[38] P. Karndumri, *Twisted compactification of N = 2 5D SCFTs to three and two dimensions from F(4) gauged supergravity*, JHEP 09 (2015) 034, 1507.01515.

[39] M. Cvetic, S. S. Gubser, H. Lu and C. N. Pope, *Symmetric potentials of gauged supergravities in diverse dimensions and Coulomb branch of gauge theories*, Phys. Rev. D 62 (2000) 086003, hep-th/9909121.

[40] L. J. Romans, *Massive N=2a Supergravity in Ten-Dimensions*, Phys. Lett. B 169 (1986) 374.

[41] I. Bah, A. Passias and P. Weck, *Holographic duals of five-dimensional SCFTs on a Riemann surface*, JHEP 01 (2019) 058, 1807.06031.

[42] P. M. Crichigno, D. Jain and B. Willett, *5d Partition Functions with A Twist*, JHEP 11 (2018) 058, 1808.06744.

[43] S. M. Hosseini, K. Hristov and A. Zaffaroni, *Gluing gravitational blocks for AdS black holes*, JHEP 12 (2019) 168, 1909.10550.

[44] J. P. Gauntlett and N. Kim, *M five-branes wrapped on supersymmetric cycles. 2.*, Phys. Rev. D 65 (2002) 086003, hep-th/0109039.
[45] M. Suh, *Supersymmetric AdS\textsubscript{6} black holes from F(4) gauged supergravity*, JHEP \textbf{01} (2019) 035, [1809.03517].

[46] S. M. Hosseini, K. Hristov, A. Passias and A. Zaffaroni, *6D attractors and black hole microstates*, JHEP \textbf{12} (2018) 001, [1809.10685].

[47] M. Suh, *Supersymmetric AdS\textsubscript{6} black holes from matter coupled F(4) gauged supergravity*, JHEP \textbf{02} (2019) 108, [1810.00675].

[48] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. \textbf{2} (1998) 231–252, [hep-th/9711200].

[49] S. M. Hosseini, I. Yaakov and A. Zaffaroni, *Topologically twisted indices in five dimensions and holography*, JHEP \textbf{11} (2018) 119, [1808.06626].