On the cohomology of Calogero-Moser spaces
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Abstract. — We compute the equivariant cohomology of smooth Calogero-Moser spaces and some associated symplectic resolutions of symplectic quotient singularities.

1. Notation and main results

Throughout this note, we will abbreviate $\otimes_C$ as $\otimes$. By an algebraic variety, we mean a reduced scheme of finite type over $\mathbb{C}$.

1.A. Reflection group. — Let $V$ be a $\mathbb{C}$-vector space of finite dimension $n$ and let $W$ be a finite subgroup of $\text{GL}_C(V)$. We set

$$\text{Ref}(W) = \{ s \in W \mid \dim_C V^s = n - 1 \}$$

and we assume that $W = \langle \text{Ref}(W) \rangle$.

We set $\varepsilon : W \to \mathbb{C}^*$, $w \mapsto \det(w)$.

If $s \in \text{Ref}(W)$, we denote by $\alpha_s^V$ and $\alpha_s^*$ two elements of $V$ and $V^*$, respectively, such that $V^s = \text{Ker}(\alpha_s)$ and $V^{*s} = \text{Ker}(\alpha_s^*)$, where $\alpha_s^*$ is viewed as a linear form on $V^*$.

If $w \in W$, we set

$$\text{cod}(w) = \text{codim}_C(V^w)$$

and we define a filtration $\mathcal{F}_i(CW)$ of the group algebra of $W$ as follows: let

$$\mathcal{F}_i(CW) = \bigoplus \mathbb{C}w, \quad \text{cod}(w) \leq i$$

Then

$$\mathbb{C}\text{Id}_V = \mathcal{F}_0(CW) \subset \mathcal{F}_1(CW) \subset \cdots \subset \mathcal{F}_n(CW) = CW = \mathcal{F}_{n+1}(CW) = \cdots$$

is a filtration of $CW$. For any subalgebra $A$ of $CW$, we set $\mathcal{F}(A) = A \cap \mathcal{F}(CW)$, so that

$$\mathbb{C}\text{Id}_V = \mathbb{C}\mathcal{F}_0(A) \subset \mathcal{F}_1(A) \subset \cdots \subset \mathcal{F}_n(A) = A = \mathcal{F}_{n+1}(A) = \cdots$$

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The second author is partly supported by the ANR (Project No ANR-13-8501-001-01 VARGEN).
The “only if” part is essentially due to Gordon (tensor algebra $T_{\mathcal{F}}(A) \subset \mathbb{C}[t] \otimes A$) and Thiel (9.6.6 and (16.1.2)) while the “if” part follows from the work of Bellamy, Schedler (this descends to a $/\mathbb{Z}$-filtration). Since the center of a graded algebra is always graded, the subalgebra $\mathcal{Z}$ is also $\mathcal{Z}$-linear.

1.B. Rational Cherednik algebra at $t = 0$. — Throughout this note, we fix a function $c : \text{Ref}(W) \to \mathbb{C}$ which is invariant under conjugacy. We define the rational Cherednik algebra $\mathcal{H}_c$ to be the quotient of the algebra $T(V \oplus V*) \rtimes W$ (the semi-direct product of the tensor algebra $T(V \oplus V*)$ with the group $W$) by the relations

\[
\begin{cases}
[x, x'] = [y, y'] = 0, \\
[y, x] = \sum_{s \in \text{Ref}(W)} (s - 1)c_s \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} \cdot s,
\end{cases}
\]

for all $x, x' \in V, y, y' \in V$. Here $(\cdot, \cdot) : V \times V^* \to \mathbb{C}$ is the standard pairing. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \to \mathcal{H}_c$ and $\mathbb{C}[V^*] \to \mathcal{H}_c$. Recall [5, Theorem 1.3] that we have an isomorphism of $\mathcal{C}$-vector spaces

\[
\mathbb{C}[V] \otimes \mathcal{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathcal{H}_c
\]

induced by multiplication (this is the so-called PBW-decomposition).

We denote by $\mathcal{Z}_c$ the center of $\mathcal{H}_c$: it is well-known [5] that $\mathcal{Z}_c$ is an integral domain, which is integrally closed and contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras (so it contains $\mathcal{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$), and that it is a free $\mathcal{P}$-module of rank $|W|$. We denote by $\mathcal{L}_c$ the affine algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{L}_c]$ is $\mathcal{Z}_c$: this is the Calogero-Moser space associated with the datum $(V, W, c)$.

Using the PBW-decomposition, we define a $\mathbb{C}$-linear map $\Omega: \mathcal{H}_c \to \mathcal{C}W$ by

\[
\Omega(fwg) = f(0)g(0)w
\]

for all $f \in \mathbb{C}[V], g \in \mathbb{C}[V^*]$ and $w \in \mathcal{C}W$. This map is $W$-equivariant for the action on both sides by conjugation, so it induces a well-defined $\mathbb{C}$-linear map

\[
\Omega : \mathcal{L}_c \to \mathbb{C}[\mathcal{L}].
\]

Recall from [4, Corollary 4.2.11] that $\Omega$ is a morphism of algebras, and that

\[
\mathcal{L}_c \text{ is smooth if and only if } \Omega \text{ is surjective.}
\]

The “only if” part is essentially due to Gordon [7, Corollary 5.8] (see also [4, Proposition 9.6.6 and (16.1.2)]) while the “if” part follows from the work of Bellamy, Schedler and Thiel [3, Corollary 1.4].

1.C. Grading. — The algebra $T(V \oplus V^*) \rtimes W$ can be $\mathbb{Z}$-graded in such a way that the generators have the following degrees

\[
\begin{cases}
\deg(y) = -1 & \text{if } y \in V, \\
\deg(x) = 1 & \text{if } x \in V^*, \\
\deg(w) = 0 & \text{if } w \in W.
\end{cases}
\]

This descends to a $\mathbb{Z}$-grading on $\mathcal{H}_c$, because the defining relations $(\mathcal{H}_c)$ are homogeneous. Since the center of a graded algebra is always graded, the subalgebra $\mathcal{Z}_c$ is also

\[
\mathcal{Z}_c = \mathcal{Z}_c^1 \oplus \mathcal{Z}_c^{\mathbb{Z}}
\]
Z-graded. So the Calogero-Moser space \( \mathcal{X}_c \) inherits a regular \( \mathbb{C}^* \)-action. Note also that by definition \( \mathcal{P} = \mathbb{C}[V]^D \otimes \mathbb{C}[V]^W \) is clearly a graded subalgebra of \( \mathbb{Z}_c \).

1.D. Main results. — For a complex algebraic variety \( \mathcal{X} \) (equipped with its classical topology), we denote by \( H^i(\mathcal{X}) \) its \( i \)-th singular cohomology group with coefficients in \( \mathbb{C} \). If \( \mathcal{X} \) carries a regular action of a torus \( T \), we denote by \( H^i_T(\mathcal{X}) \) its \( i \)-th \( T \)-equivariant cohomology group (still with coefficients in \( \mathbb{C} \)). Note that \( H^{2*}(\mathcal{X}) = \bigoplus_{i \geq 0} H^{2i}(\mathcal{X}) \) is a graded \( \mathbb{C} \)-algebra and \( H^*_T(\mathcal{X}) = \bigoplus_{i \geq 0} H^{2i}_T(\mathcal{X}) \) is a graded \( H^*_T(\mathfrak{pt}) \)-algebra. The following result [5, Theorem 1.8] describes the algebraic structure on the cohomology of \( \mathcal{X}_c \) (with coefficients in \( \mathbb{C} \)):

**Theorem 1.3 (Etingof-Ginzburg).** — Assume that \( \mathcal{X}_c \) is smooth. Then:

(a) \( H^{2i+1}_c(\mathcal{X}_c) = 0 \) for all \( i \).

(b) There is an isomorphism of graded \( \mathbb{C} \)-algebras \( H^{2*}_c(\mathcal{X}_c) \cong \text{gr}_c^*(Z(CW)) \).

In this note, we prove an equivariant version of this statement (we identify \( H^*_c(\mathfrak{pt}) \) with \( \mathbb{C}[\hbar] \) in the usual way):

**Theorem A.** — Assume that \( \mathcal{X}_c \) is smooth. Then:

(a) \( H^{2i+1}_c(\mathcal{X}_c) = 0 \) for all \( i \).

(b) There is an isomorphism of graded \( \mathbb{C}[\hbar] \)-algebras \( H^{2*}_c(\mathcal{X}_c) \cong \text{Rees}^*_c(Z(CW)) \).

Note that Theorem A(a) just follows from the statement (a) of Etingof-Ginzburg Theorem by Proposition 2.4(a) below. As a partial consequence of Theorem A, we also obtain the following application to the equivariant cohomology of some symplectic resolutions.

**Theorem B.** — Assume that the symplectic quotient singularity \( (V \times V^*)/W \) admits a symplectic resolution \( \pi : \mathcal{X} \rightarrow (V \times V^*)/W \). Recall that the \( \mathbb{C}^* \)-action on \( (V \times V^*)/W \) lifts (uniquely) to \( \mathcal{X} \) (see [12, Theorem 1.3(ii)]) Then:

(a) \( H^{2i+1}_c(\mathcal{X}) = 0 \) for all \( i \).

(b) There is an isomorphism of graded \( \mathbb{C}[\hbar] \)-algebras \( H^{2*}_c(\mathcal{X}) \cong \text{Rees}^*_c(Z(CW)) \).

Note that for \( W = S_n \) acting on \( \mathbb{C}^n \), Theorem B describes the equivariant cohomology of the Hilbert scheme of \( n \) points in \( \mathbb{C}^2 \); this was already proved by Vasserot [14]. In [6, Conjecture 1.3], Ginzburg-Kaledin proposed a conjecture for the equivariant cohomology of a symplectic resolution of a symplectic quotient singularity \( E/G \), where \( E \) is a finite dimensional symplectic vector space and \( G \) is a finite subgroup of \( \text{Sp}(E) \). However, their conjecture cannot hold as stated, because they considered the \( \mathbb{C}^* \)-action by dilatation, which is contractible. Theorem B shows that the correct equivariant cohomological realization of the Rees algebra is provided by the symplectic \( \mathbb{C}^* \)-action, which exists only when \( G \) stabilizes a Lagrangian subspace of \( E \).

**Remark 1.4.** — Recall from the works of Etingof-Ginzburg [5], Ginzburg-Kaledin [6], Gordon [7] and Bellamy [1] that the existence of a symplectic resolution of \( (V \times V^*)/W \) is equivalent to the existence of a parameter \( c \) such that \( \mathcal{X}_c \) is smooth, and that it can only occur if all the irreducible components of \( W \) are of type \( G(d,1,n) \) (for some \( d, n \geq 1 \)) or \( G_4 \) in Shephard-Todd classification. ■
1.E. Conjectures. — In [4, Chapter 16, Conjectures COH and ECOH], Rouquier and the first author proposed the following conjecture which aims to generalize the above Etingof-Ginzburg Theorem 1.3 into two directions: it includes singular Calogero-Moser spaces and it extends to equivariant cohomology.

**Conjecture 1.5.** — With the above notation, we have:

1. \( H_{2i+1}(Z_c) = 0 \) for all \( i \).
2. We have an isomorphism of graded \( \mathbb{C} \)-algebras \( H^\bullet(Z_c) \cong \text{gr}(\text{Im}(\Omega^c)) \).
3. We have an isomorphism of graded \( \mathbb{C}[\hbar] \)-algebras \( H^\bullet_c(Z_c) \cong \text{Rees}_\mathcal{F}(\text{Im}(\Omega^c)) \).

By (1.2), when \( Z_c \) is smooth, the image of \( \Omega^c \) coincide with the center of \( CW \). So Theorem A proves this conjecture for smooth \( Z_c \).

We will also prove in Example 3.6 the following result:

**Proposition 1.6.** — If \( \dim_{\mathbb{C}}(V) = 1 \), then Conjecture 1.5 holds.

1.F. Structure of the paper. — The paper is organized as follows. The proof of Theorem A relies on classical theorems on restriction to fixed points in cohomology and K-theory. In Section 2, we first recall basic properties on equivariant cohomology and equivariant K-theory and restriction to fixed points. Section 3 explains how these general principles can be applied to Calogero-Moser spaces. Theorem A will be proved in Section 4. The cyclic group case (Proposition 1.6) will be handled in Example 3.6. The proof of Theorem B will be given in Section 5.

2. Equivariant cohomology, K-theory and fixed points

2.A. Equivariant cohomology. — Let \( \mathcal{X} \) be a complex algebraic variety equipped with a regular action of a torus \( T \). Recall that the *equivariant cohomology* of \( \mathcal{X} \) is defined by

\[
H^*_T(\mathcal{X}) = H^*(E_T \times_T \mathcal{X}),
\]

where \( E_T \to B_T \) is a universal \( T \)-bundle. The pullback of the structural morphism \( x : \mathcal{X} \to \text{pt} \) yields a ring homomorphism \( H^*_T(\mathcal{X}) \to H^*_T(\text{pt}) \), which makes \( H^*_T(\mathcal{X}) \) a graded \( H^*_T(\text{pt}) \)-algebra.

Denote by \( X(T) \) the character lattice of \( T \). Then for each \( \chi \in X(T) \), denote by \( C_\chi \) the one dimensional \( T \)-module of character \( \chi \), the first Chern class \( c_\chi \) of the line bundle \( E_T \times_T C_\chi \) on \( B_T \) is an element in \( H^2(B_T) \). Identify the vector space \( \mathbb{C} \otimes_{\mathbb{Z}} X(T) \) with the dual \( \mathfrak{t}^* \) of the Lie algebra \( \mathfrak{t} \) of \( T \) via \( \chi \mapsto d\chi \). Then the assignment \( \chi \mapsto c_\chi \) yields an isomorphism of graded \( \mathbb{C} \)-algebras \( S(\mathfrak{t}^*) = H^*_T(\text{pt}) \).
2.B. Equivariant K-theory. — We denote by $K_T(\mathcal{X})$ the Grothendieck ring of the category of $T$-equivariant vector bundles on $\mathcal{X}$. Note that a $T$-equivariant vector bundle on a point is the same as a finite dimensional $T$-module. We have a canonical isomorphism $K_T(pt) = \mathbb{Z}[X(T)]$ which sends the class of a $T$-module $M$ to
\[
\dim^T(M) = \sum_{\chi \in X(T)} \dim_C(M_\chi) \chi,
\]
where $M_\chi$ is the $\chi$-weight space in $M$.

Let $\hat{H}^*_T(\mathcal{X})$ be the completion of $H^*_T(\mathcal{X})$ with respect to the ideal $\bigoplus_{i > 0} H^i_T(\mathcal{X})$. The *equivariant Chern character* provides a ring homomorphism
\[
\text{ch}_\mathcal{X} : K_T(\mathcal{X}) \to \hat{H}^*_T(\mathcal{X})
\]
with the following properties. First, when $\mathcal{X}$ is a point $pt$, we have
\[
\text{ch}_{pt} : K_T(pt) = \mathbb{Z}[X(T)] \to \hat{H}^*_T(pt) = \hat{S}(t^*)
\]
\[
\chi \to \exp(d\chi).
\]
Next the Chern character commutes with pullback. More precisely, if $\mathcal{V}$ is another variety with a regular action of the same torus $T$ and if $\varphi : \mathcal{X} \to \mathcal{V}$ is a $T$-equivariant morphism, then the following diagram commutes
\[
\begin{align*}
\xymatrix{ K_T(\mathcal{X}) \ar[r]^(0.5){\varphi^*} & K_T(\mathcal{V}) \ar[d] \ar[d] \\
\hat{H}^*_T(\mathcal{X}) \ar[r]^(0.5){\varphi^*} & \hat{H}^*_T(\mathcal{V}).}
\end{align*}
\]
Here $\varphi^*$ denotes both the pullback map in $K$-theory or in equivariant cohomology, and $\hat{\varphi}^*$ is the map induced after completion by the pullback map in equivariant cohomology. In particular, by applying the above diagram to $\mathcal{V} = pt$ and the structural morphism $\mathcal{X} \to pt$, we may view $\text{ch}_\mathcal{X}$ as a morphism of algebras over $K_T(pt)$, with the $K_T(pt)$-algebra structure on $\hat{H}^*_T(\mathcal{X})$ provided by the embedding (2.2).

2.C. Fixed points. — We denote by $\mathcal{X}^T$ the (reduced) variety consisting of fixed points of $T$ in $\mathcal{X}$. Let $i_\mathcal{X} : \mathcal{X}^T \hookrightarrow \mathcal{X}$ be the natural closed immersion. Since $T$ acts trivially on $\mathcal{X}^T$, we have $H^*_T(\mathcal{X}^T) = H^*_T(pt) \otimes H^*(\mathcal{X}^T)$ as $H^*_T(pt)$-algebras. Recall that the $T$-action on $\mathcal{X}$ is called *equivariantly formal* if the Leray-Serre spectral sequence
\[
E_2^{pq} = H^p(B_T; \mathcal{H}^q(\mathcal{X}^T)) \Rightarrow H^{p+q}_T(\mathcal{X}^T)
\]
for the fibration $E_T \times_T \mathcal{X} \to B_T$ degenerates at $E_2$. We have the following standard result on equivariant cohomology (see for instance [10, Proposition 2.1]):

**Proposition 2.4.** — Assume that $H^{2i+1}(\mathcal{X}) = 0$ for all $i$. Then $\mathcal{X}$ is equivariantly formal, and:
(a) There is an isomorphism of graded $H^*_T(pt)$-modules $H^*_T(\mathcal{X}) \simeq H^*_T(pt) \otimes H^*(\mathcal{X})$. In particular $H^i_T(\mathcal{X}) = 0$ for all $i$.
(b) The pullback map $i^*_\mathcal{X} : H^*_T(\mathcal{X}) \to H^*_T(\mathcal{X}^T)$ is injective.
(c) Let $m$ be the unique graded maximal ideal of $H^*_T(pt)$. Then we have an isomorphism of algebras $H^*(\mathcal{X}) \simeq H^*_T(\mathcal{X})/mH^*_T(\mathcal{X})$. 
In particular, this shows that Conjectures 1.5(1) and (3) imply Conjecture 1.5(2).

**Example 2.5 (Blowing-up).** — Let \( \mathcal{Y} \) be an affine variety with a \( T \)-action and let \( \mathcal{C} \) be a \( T \)-stable closed subvariety (not necessarily reduced) of \( \mathcal{Y} \). Let \( \mathcal{X} \) be the blowing-up of \( \mathcal{Y} \) along \( \mathcal{C} \). Write \( \pi : \mathcal{X} \to \mathcal{Y} \) for the natural morphism, and we equip \( \mathcal{X} \) with the unique \( T \)-action such that \( \pi \) is equivariant. We assume that \( \mathcal{X}^T \) is finite. We write

\[
H^2_T(\mathcal{X}^T) = \bigoplus_{x \in \mathcal{X}^T} S(t^e)x,
\]

where \( e_x \in H^0_T(\mathcal{X}^T) \) is the primitive idempotent associated with \( x \) (i.e., the fundamental class of \( x \)).

Then \( \mathcal{D} = \pi^*(\mathcal{C}) \) is a \( T \)-stable effective Cartier divisor, and we denote by \([\mathcal{D}]\) the class in \( K_T(\mathcal{X}) \) of its associated line bundle (which is \( T \)-equivariant). We denote by \( ch^1_T([\mathcal{D}]) \in H^2_T(\mathcal{X}) \) its first \( T \)-equivariant Chern class. We want to compute \( i_*^p(ch^1_T([\mathcal{D}])) \).

First, let \( I \) be the ideal of \( C[\mathcal{Y}] \) associated with \( \mathcal{C} \). As it is \( T \)-stable, we can find a family of \( T \)-homogeneous generators \( (a_i)_{i \in I} \) of \( I \). We denote by \( \lambda_i \in X(T) \) the \( T \)-weight of \( a_i \). The choice of this family of generators induces a \( T \)-equivariant closed immersion \( \mathcal{X} \to \mathcal{Y} \times P^{k-1}(C) \). We denote by \( \mathcal{X}_i \) the affine chart corresponding to \( a_i \neq 0 \). If \( x \in \mathcal{X}^T \), we denote by \( i(x) \in \{ 1, 2, ..., k \} \) an element such that \( x \in \mathcal{X}_{i(x)} \). Then

\[
(2.6) \quad i_*^p(ch^1_T([\mathcal{D}])) = -h \sum_{x \in \mathcal{D}} (d\lambda_{i(x)})e_x.
\]

Indeed, we just need to compute the local equation of \( \mathcal{D} \) around \( x \in \mathcal{X}^T \), and this can be done in \( \mathcal{X}_{i(x)} \). But \( \mathcal{D} \cap \mathcal{X}_{i} \) is principal for all \( i \), defined by

\[
\mathcal{D} \cap \mathcal{X}_i = \{ (y, \xi) \in \mathcal{Y} \times P^{k-1}(C) \mid (y, \xi) \in \mathcal{X}_i \text{ and } a_i(y) = 0 \}.
\]

So \( \mathcal{D} \cap \mathcal{X}_i \) is defined by a \( T \)-homogeneous equation of degree \( \lambda_i \in X(T) \), and so (2.6) follows. ■

### 3. Localization and Calogero-Moser spaces

In this section, we apply the previous discussions to \( \mathcal{X} = \mathcal{Z}_c \) and \( T = C^{\times} \). Denote by \( q : C^{\times} \to C^{\times} \) the identity map. Then \( X(C^{\times}) = qZ \), and we have

\[
K_{C^{\times}}(pt) = \mathbb{Z}[q, q^{-1}], \quad H^2_{C^{\times}}(pt) = C[h],
\]

with \( h = c_q \), following the notation of Section 2.A. So \( \hat{H}^2_{C^{\times}}(pt) = C[[h]] \) and the Chern map in this case is given by

\[
ch_{pt} : \mathbb{Z}[q, q^{-1}] \to C[[h]], \quad q \to \exp(h).
\]

Note also that a finite \( C^{\times} \)-module is nothing but a finite dimensional \( \mathbb{Z} \)-graded vector space \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) such that \( C^{\times} \) acts on \( M_i \) by the character \( q^i \). The identification \( K_{C^{\times}}(pt) = \mathbb{Z}[q, q^{-1}] \) sends the class of \( M \) to its graded dimension (or Hilbert series):

\[
\dim^\hat{\sigma}(M) = \sum_{i \in \mathbb{Z}} \dim_{C}(M_i)q^i \in \mathbb{N}[q, q^{-1}].
\]
3.A. Fixed points. — For \( \chi \in \text{Irr}(W) \), we denote by \( \omega_{\chi} : Z(CW) \to \mathbb{C} \) the associated morphism of algebras, that is, \( \omega_{\chi}(z) = \chi(z)/\chi(1) \) is the scalar through which \( z \) acts on the irreducible \( CW \)-module with the character \( \chi \). We denote by \( e_{\chi}^{W} \) the unique primitive idempotent of \( Z(CW) \) such that \( \omega_{\chi}(e_{\chi}^{W}) = 1 \). If \( \mathcal{E} \) is a subset of \( \text{Irr}(W) \), then we set \( e_{\mathcal{E}}^{W} = \sum_{\chi \in \mathcal{E}} e_{\chi}^{W} \).

Now, consider the algebra homomorphism

\[
\Omega_{\chi}^{e} = \omega_{\chi} \circ \Omega^{e} : \mathbb{Z}_{c} \to \mathbb{C}.
\]

Its kernel is a maximal ideal of \( \mathbb{Z}_{c} \); we denote by \( z_{\chi} \) the corresponding closed point in \( \mathcal{Z}_{c} \). It follows from \([4, \text{Lemma } 10.2.3 \text{ and } (14.2.2)]\) that \( z_{\chi} \in \mathcal{Z}_{c}^{C^{e}} \) and that the map

\[
z : \text{Irr}(W) \to \mathcal{Z}_{c}^{C^{e}}
\]

\[
\chi \mapsto z_{\chi}
\]

is surjective. The fibers of this map are called the Calogero-Moser \( c \)-families. They were first consider by Gordon \([7]\) and Gordon-Martino \([9]\); see also for instance \([4, \text{§9.2}]\). Let \( \text{CM}_{c}(W) \) be the set of Calogero-Moser \( c \)-families. For \( \mathcal{E} \in \text{CM}_{c}(W) \), we denote by \( z_{\mathcal{E}} \in \mathcal{Z}_{c}^{C^{e}} \) its image under the map \( z \). On the other hand, by \([4, (16.1.2)]\) we have

\[
\text{Im}(\Omega^{e}) = \bigoplus_{\mathcal{E} \in \text{CM}_{c}(W)} \mathbb{C} e_{\mathcal{E}}^{W}.
\]

Hence we get an isomorphism of \( \mathbb{C} \)-algebras

\[
H^{*}(\mathcal{Z}_{c}^{C^{e}}) \cong \text{Im}(\Omega^{e}), \quad e_{\mathcal{E}} \mapsto e_{\mathcal{E}}^{W},
\]

which extends to an isomorphism of \( \mathbb{C}[h] \)-algebras \( H^{*}(\mathcal{Z}_{c}^{C^{e}}) \cong \mathbb{C}[h] \otimes \text{Im}(\Omega^{e}) \). For simplification, we set \( i_{c} = i_{\mathcal{Z}_{c}} : \mathcal{Z}_{c}^{C^{e}} \to \mathcal{Z} \) and, under the above identification, we view the pullback map \( i_{c}^{*} \) as a morphism of algebras

\[
i_{c}^{*} : H^{*}(\mathcal{Z}_{c}) \to \mathbb{C}[h] \otimes \text{Im}(\Omega^{e}).
\]

So, by Proposition 2.4, Conjecture 1.5 is implied by the following one:

**Conjecture 3.3.** — With the above notation, we have:

1. \( H^{2i+1}(\mathcal{Z}_{c}) = 0 \) for all \( i \).
2. \( \text{Im}(i_{c}^{*}) = \text{Rees}_{H}(\text{Im}(\Omega^{e})) \).

**Remark 3.4.** — Set

\[
\mathcal{F}^{H}_{t}(\text{Im}(\Omega^{e})) = \{ z \in \text{Im}(\Omega^{e}) \mid h^{t} z \in \text{Im}(i_{c}^{*}) \}.
\]

Then, by construction and Proposition 2.4, \( \mathcal{F}^{H}_{t}(\text{Im}(\Omega^{e})) \) is the filtration of \( \text{Im}(\Omega^{e}) \) satisfying

\[
H^{2i}(\mathcal{Z}_{c}) \cong \text{Im}(i_{c}^{*}) = \text{Rees}_{H}(\text{Im}(\Omega^{e}))
\]

and

\[
H^{2i}(\mathcal{Z}_{c}) \cong \text{gr}_{H}(\text{Im}(\Omega^{e})).
\]

So showing Conjecture 3.3 is then equivalent to showing that the filtrations \( \mathcal{F}_{t}(\text{Im}(\Omega^{e})) \) and \( \mathcal{F}^{H}_{t}(\text{Im}(\Omega^{e})) \) coincide.

Note also that, since \( \mathcal{Z}_{c} \) is an affine variety of dimension \( 2n \), we have \( H^{i}(\mathcal{Z}_{c}) = 0 \) for \( i > 2n \), so this shows that

\[
\mathcal{F}_{t}(\text{Im}(\Omega^{e})) = \text{Im}(\Omega^{e}) = \mathcal{F}^{H}_{t}(\text{Im}(\Omega^{e})).
\]
On the other hand, since $\mathcal{X}_c$ is connected, we have
\[(3.7) \quad \mathcal{F}_0(\text{Im}(\Omega^i)) = \mathcal{F}_0^H(\text{Im}(\Omega^i)) = \mathbb{C}.\]
These two particular cases will be used below. ■

The following result, based on the previous remark, can be viewed as a reduction step for the proof of Conjecture 3.3:

**Proposition 3.5.** — Assume that $H^{2i+1}(\mathcal{X}_c) = 0$ and
\[
\dim_C(\mathcal{H}^{2i}(\mathcal{X}_c)) = \dim_C(\mathcal{F}_i(\text{Im}(\Omega^i)) / \mathcal{F}_{i-1}(\text{Im}(\Omega^i)))
\]
for all $i$. Then Conjecture 3.3 holds if and only if $\text{Rees}_\mathcal{F}(\text{Im}(\Omega^i)) \subset \text{Im}(\imath^*_i)$.

**Proof.** — Assume that $H^{2i+1}(\mathcal{X}_c) = 0$ and
\[
\dim_C(\mathcal{H}^{2i}(\mathcal{X}_c)) = \dim_C(\mathcal{F}_i(\text{Im}(\Omega^i)) / \mathcal{F}_{i-1}(\text{Im}(\Omega^i)))
\]
for all $i$. We keep the notation of Remark 3.4. It then follows from this remark and the hypothesis that
\[
\dim_C(\mathcal{F}_i(\text{Im}(\Omega^i)) / \mathcal{F}_{i-1}(\text{Im}(\Omega^i))) = \dim_C(\mathcal{F}_i^H(\text{Im}(\Omega^i)) / \mathcal{F}_{i-1}^H(\text{Im}(\Omega^i)))
\]
for all $i$. Then, by induction, we get that $\dim_C(\mathcal{F}_i(\text{Im}(\Omega^i)) / \mathcal{F}_{i-1}(\text{Im}(\Omega^i))) = \dim(\mathcal{F}_i^H(\text{Im}(\Omega^i)) / \mathcal{F}_{i-1}^H(\text{Im}(\Omega^i)))$ for all $i$ (by the equality (7) of Remark 3.4). This shows that $\text{Rees}_\mathcal{F}(\text{Im}(\Omega^i)) = \text{Im}(\imath^*_i)$ if and only if $\text{Rees}_\mathcal{F}(\text{Im}(\Omega^i)) \subset \text{Im}(\imath^*_i)$, as desired. ■

**Example 3.6.** — Assume in this example, and only in this example, that $\dim_C(V) = 1$. It is proved in [4, Theorem 18.5.8] that in this case $H^{2i+1}(\mathcal{X}_c) = 0$ and
\[
\dim_C(\mathcal{H}^{2i}(\mathcal{X}_c)) = \dim_C(\mathcal{F}_i(\text{Im}(\Omega^i)) / \mathcal{F}_{i-1}(\text{Im}(\Omega^i)))
\]
for all $i$. Since $\mathcal{X}_c$ is affine of dimension 2, we have $H^i(\mathcal{X}_c) = 0$ if $i \notin \{0, 2\}$. So it follows from the equalities (6) and (7) of Remark 3.4 that Conjecture 3.3 holds in this case (this proves Proposition 1.6). ■

### 3.B. Chern map

If $\mathcal{E}$ is a Calogero-Moser family, we denote by $m_\mathcal{E} \subset \mathbb{Z}_c$ the ideal of functions vanishing at $z_\mathcal{E} \in \mathcal{Z}_c^{\mathcal{E}}$. We also set
\[
\text{Im}(\Omega^c)_\mathbb{Z} = \bigoplus_{\mathcal{E} \in \text{CM}_c(W)} \mathbb{Z} e_\mathcal{E}^c.
\]
We make the natural identification $K_{\mathcal{E}}(\mathcal{Z}_c^{\mathcal{E}}) = \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \text{Im}(\Omega^c)_\mathbb{Z}$. Through these identifications, the Chern map $\text{ch}_{\mathcal{Z}_c^{\mathcal{E}}}$ just becomes the natural inclusion $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \text{Im}(\Omega^c)_\mathbb{Z} \hookrightarrow \mathbb{C}[[\hbar]] \otimes \text{Im}(\Omega^c)_\mathbb{Z}$. Moreover, if $\mathcal{P}$ is a $\mathbb{Z}$-graded finitely generated projective $\mathbb{Z}_c$-module, then the commutativity of the diagram (2.3) just says that
\[
(3.7) \quad \imath^*_i(\text{ch}_{\mathcal{Z}_c^{\mathcal{E}}}(\text{Im}(\mathcal{P}))) = \sum_{\mathcal{E} \in \text{CM}_c(W)} \dim^\mathcal{P}(\mathcal{P} / m_\mathcal{E} \mathcal{P}) e_\mathcal{E}^c \subset \mathbb{C}[[\hbar]] \otimes \text{Im}(\Omega^c)_\mathbb{Z}.
\]
4. Proof of Theorem A

**Hypothesis and notation.** We assume in this section, and only in this section, that \( \mathcal{Z} \) is smooth.

If \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) and \( N = \bigoplus_{i \in \mathbb{Z}} N_i \) are two finite-dimensional graded CW-modules, we set

\[
\langle M, N \rangle^g_W = \sum_{i, j \in \mathbb{Z}} \langle M_i, N_j \rangle_W q^{i+j},
\]

where \( \langle E, F \rangle_W = \dim \text{Hom}_{CW}(E, F) \) for any finite dimensional CW-modules \( E \) and \( F \).

We extend this notation to the case where \( M \) or \( N \) is a graded virtual character (i.e. an element of \( \mathbb{Z}[q, q^{-1}] \otimes \mathbb{Z} \text{Irr}(W) \)). Finally, we denote by \( \mathbb{C}[V]^{\text{co}(W)} \) the coinvariant algebra, that is, the quotient of the algebra \( \mathbb{C}[V] \) by the ideal generated by the elements \( f \in \mathbb{C}[V]^W \) such that \( f(0) = 0 \). Then \( \mathbb{C}[V]^{\text{co}(W)} \) is a graded CW-module, which is isomorphic to the regular representation \( CW \) when one forgets the grading.

**4.A. Localization in K-theory.** — Recall from (1.2) that the smoothness of \( \mathcal{Z} \) implies that \( \text{Im}(\Omega^\ell) = \mathbb{Z}(CW) \), so that \( z : \text{Irr}(W) \to \mathcal{Z}_{\text{co}}^c \) is bijective. Let \( e = e^W_1 = (1/|W|) \sum_{w \in W} w \). The smoothness of \( \mathcal{Z} \) also implies that the functor

\[
\mathcal{H}_c \text{-mod} \longrightarrow \mathcal{Z}_c \text{-mod}
\]

\[
M \longmapsto eM = e\mathcal{H}_c \otimes_{\mathcal{H}_c} M
\]

is an equivalence of categories [5, Theorem 1.7]. If \( E \) is a finite dimensional \( \mathbb{Z} \)-graded CW-module, the \( \mathcal{H}_c \)-module \( \mathcal{H}_c \otimes_{CW} E \) is finitely generated, \( \mathbb{Z} \)-graded and projective. Therefore, \( e\mathcal{H}_c \otimes_{CW} E \) is a graded doubly generated graded projective \( \mathcal{Z}_c \)-module, which can be viewed as a \( \mathcal{Z}_c \)-equivariant vector bundle on \( \mathcal{Z}_{\text{co}}^c \). For simplification, we set

\[
\text{ch}_c(E) = i^c(\text{ch}_\mathcal{Z}(e\mathcal{H}_c \otimes_{CW} E)) = c_{([H])} \otimes \text{Im}(\Omega^\ell).
\]

**Proposition 4.1.** — Assume that \( \mathcal{Z} \) is smooth. Let \( E \) be a finite dimensional graded CW-module. Then

\[
\text{ch}_c(E) = \sum_{\chi \in \text{Irr}(W)} \frac{\langle \chi, \mathcal{C}[V]^{\text{co}(W)} \otimes E \rangle^g_W}{\langle \chi, \mathcal{C}[V]^{\text{co}(W)} \rangle^g_W} e^W_{\chi}.
\]

**Proof.** — As the formula is additive, we may, and we will, assume that \( E \) is an irreducible CW-module, concentrated in degree 0.

Now, let \( \chi \in \text{Irr}(W) \): we denote by \( m_\chi \) the maximal ideal of \( \mathcal{Z}_c \) corresponding to the fixed point \( z_\chi \). We set \( p = m_\chi \cap \mathcal{P} \): it does not depend on \( \chi \) (it is the maximal ideal of \( \mathcal{P} = \mathbb{C}[V/W \times V^*/W] \) of functions which vanishes at \((0,0)\); see for instance [4, (14.2.2)]). By (3.7),

\[
\text{ch}_c(E) = \sum_{\chi \in \text{Irr}(W)} \dim_{\mathbb{C}}((e\mathcal{H}_c \otimes_{CW} E)/m_\chi(e\mathcal{H}_c \otimes_{CW} E)) e^W_{\chi}.
\]

Now, let \( \mathcal{Z}_c = \mathcal{Z}_c/p\mathcal{Z}_c \) and \( \mathcal{H}_c = \mathcal{H}_c/p\mathcal{H}_c \). We set \( \tilde{m}_\chi = m_\chi/p\mathcal{Z}_c \). Then \( \tilde{\mathcal{H}}_c \) is a finite dimensional \( \mathbb{C} \)-algebra (called the \textit{restricted rational Cherednik algebra}) and again the bimodule \( e\mathcal{H}_c \) induces a Morita equivalence between \( \tilde{\mathcal{H}}_c \) and \( \mathcal{Z}_c \). This implies that \( e\mathcal{H}_c/\tilde{m}_\chi \mathcal{H}_c = \mathcal{Z}_c/\tilde{m}_\chi \mathcal{Z}_c \) for any fixed point \( z_\chi \).
(\mathcal{Z}_c/\mathfrak{m}_\chi) \otimes_{\mathcal{Z}} e \mathfrak{H}_c is a simple right \mathfrak{H}_c\text{-module (which will be denoted by } \mathcal{L}_c(\chi)): it is isomorphic to the shift by some \( r_\chi \in \mathbb{Z} \) of the quotient of the baby Verma module denoted by \( L(\chi) \) in [7].

Since \( CW \) is semisimple, the \( CW \)-module \( E \) is flat, and so
\[
(\mathcal{E}_c \otimes_{CW} E)/\mathfrak{m}_\chi (\mathcal{E}_c \otimes_{CW} E) \cong (\mathcal{E}_c/\mathfrak{m}_\chi (\mathcal{E}_c)) \otimes_{CW} E \\
\cong (\mathcal{E}_c/\mathfrak{m}_\chi (\mathcal{E}_c)) \otimes_{CW} E.
\]
But the graded dimension of \( \mathcal{L}_c(\chi) \otimes_{CW} E \) is known whenever \( \mathcal{Z}_c \) is smooth and is given by the expected formula (see [1, Lemma 3.3 and its proof]), up to a shift in grading:
\[
\text{ch}_c(E) = \sum_{\chi \in \text{Irr}(W)} q^{r_\chi} \langle \chi, \mathcal{C}[V]^{\otimes V}\otimes E \rangle_{E}^{W} (\chi, \mathcal{C}[V]^{\otimes V})_{W}^{gr} \epsilon_{\chi}^{W},
\]
where \( r_\chi \in \mathbb{Z} \) does not depend on \( E \). Now, if \( C \) denotes the trivial \( CW \)-module concentrated in degree 0, then \( \text{ch}_c(C) = 1 \), which shows that \( r_\chi = 0 \) for all \( \chi \), as desired. \( \square \)

Let \( K_{C_0}(CW) \) denote the Grothendieck group of the category of finite dimensional graded \( CW \)-modules. If \( F \) is a finite dimensional graded \( CW \)-module, we denote by \([ F ]\) its class in \( K_{C_0}(CW) \). We still denote by \( \text{ch}_c : K_{C_0}(CW) \to \mathbb{C}[h] \otimes Z(CW) \) the map defined by
\[
\text{ch}_c([ F ]) = \text{ch}_c(F).
\]
Now, let \( W' \) be a parabolic subgroup of \( W \) and set \( V' = V^{W'} \) and \( r = \text{codim}_C(V') \). We identify the dual \( V'^* \) of \( V' \) with \( V'^{W'} \) and note that
\[
(4.2) V'^* = V'^\otimes (V')^\perp.
\]
We denote by \( \wedge(V')^\perp \) the element of \( K_{C_0}(CW) \) defined by
\[
\wedge(V')^\perp = \sum_{i \geq 0} (-1)^i [ \wedge^i(V')^\perp ].
\]
Recall also that there exist \( n \) algebraically independent homogeneous polynomials \( f_1, \ldots, f_n \) in \( \mathbb{C}[V]^W \) such that \( \mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n] \), and we denote by \( d_i \) the degree of \( f_i \).

**Corollary 4.3.** — Assume that \( \mathcal{E}_c \) is smooth. Let \( E' \) be a finite dimensional graded \( CW' \)-module. Then
\[
\text{ch}_c(\text{Ind}^W_{W'} (\wedge(V')^\perp \otimes [E'])) = (1-q^{d_1}) \cdots (1-q^{d_n}) \sum_{\chi \in \text{Irr}(W)} \langle \chi, \text{Ind}^W_{W'} E'^{gr} \rangle_{E'}^{W} \epsilon_{\chi}^{W}.
\]

**Proof.** — The group \( W' \) acts trivially on \( V' \) so it acts trivially on \( \wedge^i V'^* \) for all \( i \). Therefore,
\[
(1-q)^{n-r} \text{ch}_c(\text{Ind}^W_{W'} (\wedge(V')^\perp \otimes [E'])) = \text{ch}_c(\text{Ind}^W_{W'} (\wedge V'^* \otimes (V')^\perp \otimes [E'])).
\]
But \( \wedge V'^* \otimes (V')^\perp = \text{Res}^W_{W'} (\wedge V'^* ) \) by (4.2), so, by Frobenius formula,
\[
(1-q)^{n-r} \text{ch}_c(\text{Ind}^W_{W'} (\wedge(V')^\perp \otimes [E'])) = \text{ch}_c(\wedge V'^* \otimes \text{Ind}^W_{W'} E').
\]
So it follows from Proposition 4.1 that

\[
(1 - q)^{n - r} \text{ch}_c(\text{Ind}^W_W((\Lambda V')^\perp \otimes [E']))) = \sum_{\chi \in \text{Irr}(W)} \frac{1}{(\chi, C[V]^\text{co}(W))_W} (\chi, C[V]^\text{co}(W))_W \otimes (\Lambda V^* \otimes \text{Ind}^W_W(E'))_W e^W.
\]

But, if \( w \in W \), the Molien’s formula implies that the graded trace of \( w \) on \( C[V]^\text{co}(W) \) is equal to

\[
\frac{(1 - q^{d_1}) \cdots (1 - q^{d_n})}{\det(1 - w^{-1}q)},
\]

while its graded trace on \( \Lambda V^* \) is equal to \( \det(1 - w^{-1}q) \). So the class of \( C[V]^\text{co}(W) \otimes \Lambda V^* \) is equal to \( (1 - q^{d_1}) \cdots (1 - q^{d_n}) \) times the class of the trivial module, and the corollary follows. \( \Box \)

**Proof of Theorem A.** — Assume that \( \mathcal{K}_c \) is smooth. This implies in particular that Conjecture 1.5(1) and (2) hold (Etingof-Ginzburg Theorem 1.3), and so the hypotheses of Proposition 3.5 are satisfied. Note also that \( \text{Im}(\Omega^c) = Z(CW) \) by (1.2). It is then sufficient to prove that \( \mathcal{H}^i \mathcal{P}(Z(CW)) \subseteq \text{Im}(i^* \mathcal{P}) \) for all \( i \).

Let us introduce some notation. If \( G \) is a finite group and \( H \) is a subgroup, we define a \( C \)-linear map \( \text{Tr}^G_H : Z(CW) \longrightarrow Z(CG) \) by

\[
\text{Tr}^G_H(z) = \sum_{g \in (G/H)} g z = \frac{1}{|H|} \sum_{g \in G} g z
\]

(here, \([G/H]\) denotes a set of representatives of elements of \( G/H \) and \( g z = g z g^{-1} \)). It is easy to check that

\[
\text{Tr}^G_H(e_H) = \frac{\eta(1)}{|H|} \sum_{\gamma \in \text{Irr}(G)} \frac{|G|}{|\gamma(1)|} \langle \gamma, \text{Ind}^G_H(\eta) \rangle_G e^G
\]

for all \( \eta \in \text{Irr}(H) \). Also, if \( h \in H \) and \( \Sigma_H(h) \in Z(CH) \) denotes the sum of the conjugates of \( h \) in \( H \), then

\[
\text{Tr}^G_H(\Sigma_H(h)) = \frac{|C_G(h)|}{|C_H(h)|} \Sigma_G(h),
\]

where \( C_G(h) \) and \( C_H(h) \) denote the centralizers of \( h \) in \( G \) and \( H \) respectively. Let \( \mathcal{P}(W) \) denote the set of parabolic subgroups \( W' \) of \( W \) such that \( \text{codim}_C(V^{W'}) = r \). It follows from (4.5) that

\[
\mathcal{P}(Z(CW)) = \sum_{W' \in \mathcal{P}(W)} \text{Tr}^W_W(Z(CW')).
\]

Therefore, by Proposition 3.5, it is sufficient to prove that

\[
(\star) \quad \mathcal{H}^i \text{Tr}^W_W(e^W_{\chi'}) \subseteq \text{Im}(i^*_\mathcal{P}) \text{ for all } W' \in \mathcal{P}(W) \text{ and all } \chi' \in \text{Irr}(W').
\]

But it turns out that the coefficient of \( h^i \) in

\[
\frac{\chi'(1)}{|W'|} \text{ch}_c(\text{Ind}^W_W((\Lambda V')^\perp \otimes \chi'))
\]
is equal, according to Corollary 4.3, to
\[
(-1)^r \frac{\chi'(1)}{|W|} \sum_{\chi \in \text{Irr}(W)} \frac{|W|}{\chi(1)} \langle \chi, \text{Ind}_W^G(V) \rangle_W e^W.
\]
Indeed, since \( q = \exp(h) \), this follows from the fact that the polynomial \( \langle \chi, C[V]^{\text{co}(W)} \rangle_W \) takes the value \( \chi(1) \) whenever \( q = 1 \) (i.e. \( h = 0 \)) and from the fact that
\[
\frac{(1-1-q^{d_1}) \cdots (1-q^{d_n})}{(1-q)^n} \equiv (-1)^r d_1 \cdots d_n h^r \mod h^{r+1}
\]
and that \( |W| = d_1 \cdots d_n \). Note that by definition \( \frac{\chi'(1)}{|W|} \text{ch}_e(\text{Ind}_W^G(V) \otimes \chi') \) belongs to \( \mathbb{C}[[h]] \otimes \mathbb{C}[h] \text{Im}(i^*_\chi) \), so each of its homogeneous components belongs to \( \text{Im}(i^*_\chi) \). By (4.4), this proves that \( \ast \) holds, and the proof of Theorem A is complete. \( \square \)

5. Proof of Theorem B

**Hypothesis and notation.** We assume in this section, and only in this section, that the symplectic quotient singularity \( \mathcal{Z}_0 = (V \times V^*)/W \) admits a symplectic resolution \( \mathcal{X} \to \mathcal{Z}_0 \).

Recall [12, Theorem 1.3(ii)] that the \( \mathbb{C}^* \)-action on \( \mathcal{Z}_0 \) lifts uniquely to \( \mathcal{X} \). As it is explained in Remark 1.4, the existence of a symplectic resolution of \( \mathcal{Z}_0 \) implies that all the irreducible components of \( W \) are of type \( G(d,1,n) \) or \( G_4 \). Since the proof of Theorem B can be easily reduced to the irreducible case, we will separate the proof in two cases.

**Proof of Theorem B for \( W = G(d,1,n) \).** — Assume here that \( W = G(d,1,n) \). Let \( S^1 \) be the group of complex numbers of modulus 1. In this case, it follows from [8] that \( \mathcal{X} \) is diffeomorphic to some smooth \( \mathcal{Z}_0 \), and that the diffeomorphism might be chosen to be \( S^1 \)-equivariant. As the \( S^1 \)-equivariant cohomology is canonically isomorphic to the \( \mathbb{C}^* \)-equivariant cohomology, this proves Theorem B in this case. \( \square \)

**Proof of Theorem B for \( W = G_4 \).** — Assume here that \( W = G_4 \). It is possible (probable?) that again \( \mathcal{X} \) is \( S^1 \)-diffeomorphic to some smooth \( \mathcal{Z}_0 \), but we are unable to prove it (it is only known that they are diffeomorphic). So we will prove Theorem B in this case by brute force computations.

We fix a primitive third root of unity \( \zeta \) and we assume that \( V = \mathbb{C}^2 \) and that \( W = (s,t) \), where
\[
s = \begin{pmatrix} \zeta & 0 \\ \zeta^2 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & -\zeta^2 \\ 0 & \zeta \end{pmatrix}.
\]
By the work of Bellamy [2], there are only two symplectic resolutions of \( \mathcal{Z}_0 = (V \times V^*)/W \). They have both been constructed by Lehn and Sörger [13]: one can be obtained from the other by exchanging the role of \( V \) and \( V^* \), so we will only prove Theorem B for one of them. Let us describe it.

Let \( H = V^* \) and let \( \mathcal{H} \) denote the image of \( H \times V^* \) in \( \mathcal{X}_0 \), with its reduced structure of closed subvariety. We denote by \( \beta : \mathcal{Y} \to \mathcal{X}_0 \) the blowing-up of \( \mathcal{X}_0 \) along \( \mathcal{H} \) and we denote by \( a : \mathcal{X} \to \mathcal{Y} \) the blowing-up of \( \mathcal{Y} \) along its reduced singular locus \( \mathcal{Y} \). Then [13]
\[
\pi = \beta \circ a : \mathcal{X} \to \mathcal{X}_0
\]
is a symplectic resolution.

We now give more details, which all can be found in [13, §1]. First, \( C[Z_0] \) is generated by 8 homogeneous elements \((z_i)_{1 \leq i \leq 8}\) whose degrees are given by the following table:

\[
\begin{array}{cccccccc}
  z & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\
  \text{deg}(z) & 0 & 4 & -4 & 2 & -2 & -6 & 6 & 0 \\
\end{array}
\]

The defining ideal of \( \mathcal{H} \) in \( C[Z_0] \) is generated by 6 homogeneous elements \((b_j)_{1 \leq j \leq 6}\) whose degrees are given by the following table:

\[
\begin{array}{ccccccc}
  b & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\
  \text{deg}(b) & 2 & 6 & 0 & 12 & 8 & 4 \\
\end{array}
\]

This defines a \( \mathbb{C}^\times \)-equivariant closed immersion \( \mathcal{U} \hookrightarrow Z_0 \times \mathbb{P}^3(\mathbb{C}) \). We denote by \( \mathcal{U}_i \) the affine chart defined by \( \gamma_i \neq 0 \). The equations of the zero fiber \( \beta^{-1}(0) \) given in [13, §1] show that \( \mathcal{U}_i^{\mathbb{C}} = \{ p_2, p_3, p_4, p_5, p_6 \} \), where \( p_i \) is the unique element of \( \mathcal{U}_i^{\mathbb{C}} \). We use the notation of Example 2.5. By (2.6) and (5.2), we have

\[
\left( \gamma \right) \quad i^*_\gamma (\text{ch}^1_{\mathbb{P}}([\beta^* \mathcal{H}])) = -h(6e_{p_2} + 12e_{p_4} + 4e_{p_6}).
\]

Now, \( \mathcal{U} \) is contained in \( \mathcal{U}_2 \cup \mathcal{U}_3 \), so \( \alpha \) is an isomorphism in a neighborhood of \( p_4 \) and \( p_6 \). So let \( q_4 = \alpha^{-1}(p_4) \) and \( q_6 = \alpha^{-1}(p_6) \). These are elements of \( \mathcal{X}_{\mathbb{C}}^{\mathbb{C}} \).

On the other hand, \( \mathcal{U}_2 \) is a transversal \( A_1 \)-singularity, so the defining ideal of \( \mathcal{X} \cap \mathcal{U}_2 \) in \( C[\mathcal{U}_2] \) is generated by three homogeneous elements \( a_+, a_- \) and \( a_0 \) (of degree 6, 0 and \(-6\), by [13, §1]), and it is easily checked that \( \alpha^{-1}(p_2)^{\mathbb{C}} = \{ q_2^+, q_2^- \} \) where \( q_2^- \) is the unique \( \mathbb{C}^\times \)-fixed element in the affine chart defined by \( a_+ \neq 0 \).

Also, \( \mathcal{U}_3 \) is isomorphic to \((h \times h^*)/\mathbb{S}_3\), where \( h \) is the diagonal Cartan subalgebra of \( \mathfrak{sl}_2(\mathbb{C}) \) and \( \mathbb{S}_3 \) is the symmetric group on 3 letters, viewed as the Weyl group of \( \mathfrak{sl}_2(\mathbb{C}) \). Let \( \text{Hilb}_3(\mathbb{C}^2) \) denote the Hilbert scheme of 3 points in \( \mathbb{C}^2 \), and let \( \text{Hilb}_3(\mathbb{C}^2) \) denote the (reduced) closed subscheme defined as the Hilbert scheme of three points in \( \mathbb{C}^2 \) whose sum is equal to \((0, 0)\). By [11, Proposition 2.6],

\[
\mathcal{X}_3 \simeq \text{Hilb}_3(\mathbb{C}^2).
\]

It just might be noticed that the isomorphism \( \mathcal{X}_3 \simeq (h \times h^*)/\mathbb{S}_3 \) becomes \( \mathbb{C}^\times \)-equivariant if one “doubles the degrees” in \((h \times h^*)/\mathbb{S}_3\), that is, if \( \mathbb{C}^\times \) acts on \( h \) (respectively \( h^* \)) with weight 2 (respectively \(-2\)). Recall that \( \mathbb{C}^\times \)-fixed points in \( \text{Hilb}_3(\mathbb{C}^2) \) are parametrized by partitions of 3: we denote by \( q_3^+, q_3^- \) and \( q_3 \) the fixed points in \( \mathcal{X}_3 \) corresponding respectively to the partitions \((3), (2, 1)\) and \((1, 1, 1)\) of 3, so that

\[
\mathcal{X}_3^{\mathbb{C}} = \{ q_3^+, q_3^-, q_3 \}.
\]

Finally, we have

\[
\mathcal{X}_{\mathbb{C}}^{\mathbb{C}} = \{ q_2^+, q_2^-, q_3^+, q_3^-, q_4, q_6 \}.
\]

Also, \( \pi^*(\mathcal{H}) = a^*(\beta^*(\mathcal{H})) \) is an effective Cartier divisor of \( \mathcal{X} \), and it follows from \((\gamma)\) and the commutativity of the diagram (2.3) that

\[
\left( \delta \right) \quad i^*(\text{ch}^1_{\mathbb{P}}([\pi^* \mathcal{H}])) = -h(6e_{q_2} + 6e_{q_4} + 12e_{q_6} + 4e_{q_6}).
\]
We now wish to compute \( i_x^*(\text{ch}_I^1([\alpha^* \mathcal{X}])) \). As the singular locus \( \mathcal{Y} \) is contained in \( \mathcal{Y}_2 \cup \mathcal{Y}_3 \), and contains \( p_1 \) and \( p_2 \), there exists \( n_1^+, n_2^-, n_3^+, n_4^+ \) in \( \mathbb{Z} \) such that
\[
i_x^*(\text{ch}_I^1([\alpha^* \mathcal{Y}])) = -h(n_1^+e_{d_1} + n_2^-e_{d_2} + n_3^+e_{d_3} + n_4^+e_{d_4}).
\]
Since \( a_1 \) and \( a_2 \) have degree 6 and \( -6 \), it follows from (2.6) that \( n_2^- = 6 \) and \( n_4^+ = -6 \).

As we can exchange the roles of \( h \) and \( h^* \) in the description of \( \mathcal{Y}_3 \) and its singular locus (because \( h \simeq h^* \) as an \( \mathcal{O}_\mathcal{X} \)-module), this shows that \( n_3^- = -n_3^+ \) and \( n_2^+ = -n_2^- \). So \( n_2^+ = 0 \) and it remains to compute \( n_3^+ \). So let \( U_i \) be the open subset of \( \text{Hilb}_{\mathbb{H}}^3(\mathbb{C}^2) \) consisting of ideals \( I \) of codimension 3 of \( \mathbb{C}[x, y] \) such that the classes of 1, \( x \) and \( x^2 \) form a basis of \( \mathbb{C}[x, y]/I \).

Then we have an isomorphism \( J_\alpha : \mathbb{C}^4 \cong U_i \) given by
\[
(a, b, c, d) \mapsto J_\alpha(a, b, c, d) = (x^3 + a x + b, y - c x^2 - d x - \frac{2}{3} a c).
\]
The form of the generators of the ideal \( J_\alpha(a, b, c, d) \) is here to ensure that \( J_\alpha(a, b, c, d) \in \text{Hilb}_{\mathbb{H}}^3(\mathbb{C}^2) \). The fixed point \( q_3^+ \) is the unique one in \( U_i \). Through this identification, the action of \( \mathbb{C}^* \) on \( \mathbb{C}^4 \) is given by
\[
\xi \cdot (a, b, c, d) = (\xi^4 a, \xi^6 b, \xi^6 c, \xi^4 d)
\]
(remember that we must “double the degrees” of the usual action). The equation of \( \alpha^*(\mathcal{Y}) \) on this affine chart \( \simeq \mathbb{C}^4 \) can then be computed explicitly and is given by \( 4a^3 + 27b^2 = 0 \), so is of degree 12. This shows that \( n_3^+ = 12 \). Finally,
\[
(\forall) \quad i_x^*(\text{ch}_I^1([\alpha^* \mathcal{Y}])) = -h(6e_{d_1} - 6e_{d_2} + 12e_{d_3} - 12e_{d_4}).
\]
Let us now conclude. First, recall from [6, Theorem 1.2] that
\[
\begin{align*}
H^{2i+1}(\mathcal{X}) &= 0 \quad \text{for all } i, \\
H^{2i}(\mathcal{X}) &= \text{gr}_x(Z(\mathcal{C}W)).
\end{align*}
\]
By Proposition 2.4, we get
\[
(\spadesuit) \quad \dim_{\mathbb{C}} H^{2i}_{\mathbb{C}^2}(\mathcal{X}) = \begin{cases} 
1 & \text{if } i = 0, \\
3 & \text{if } i = 1, \\
7 & \text{if } i \geq 2.
\end{cases}
\]
Now, \( W \) has seven irreducible characters \( 1, \varepsilon, \varepsilon^2, \chi, \chi \varepsilon, \chi \varepsilon^2 \) and \( \theta \), where \( \chi \) is the unique irreducible character of degree 2 with rational values and \( \theta \) is the unique one of degree 3. We denote by
\[
\Psi : H_{\mathbb{C}^2}^{2i}(\mathcal{X}^{ac}) \to \mathbb{C}[h] \otimes Z(\mathcal{C}W)
\]
the isomorphism of \( \mathbb{C}[h] \)-algebras such that
\[
\Psi(e_{d_1}) = e_1^W, \quad \Psi(e_{d_2}) = e_0^W, \quad \Psi(e_{d_3}) = e_1^W, \quad \Psi(e_{d_4}) = e_2^W
\]
By Proposition 2.4, \( H_{\mathbb{C}^2}^{2i}(\mathcal{X}) \) is isomorphic to its image by \( \Psi \circ i^*_x \) in \( \mathbb{C}[h] \otimes Z(\mathcal{C}W) \). But, by (\spadesuit) and (\forall), and after investigation of the character table of \( W \), this image contains
\[
h(6e_1^W - 6e_2^W + 12e_1^W + 4e_0^W) = h(4 + \Sigma_W(s) + \Sigma_W(s^2))
\]
and
\[
h(12e_2^W - 12e_1^W + 6e_2^W - 6e_2^W) = h((1 + 2\zeta)\Sigma_W(s) + (1 + 2\zeta^2)\Sigma_W(s^2)).
\]
So it contains $\hbar \Sigma_W(s)$ and $\hbar \Sigma_W(s^2)$. Also, by (♠) and Proposition 2.4, it also contains $\hbar^2 Z(CW)$, so

$$\text{Rees}^*_\chi(Z(CW)) \subset \text{Im}(\Psi \circ i^*_\chi).$$

Using again (♠) and Proposition 2.4, a comparison of dimensions yields that

$$\text{Rees}^*_\chi(Z(CW)) = \text{Im}(\Psi \circ i^*_\chi),$$

and the proof is complete. \qed

References

[1] G. Bellamy, On singular Calogero-Moser spaces, Bull. LMS 41 (2009), 315-326.
[2] G. Bellamy, Counting resolutions of symplectic quotient singularities, Compos. Math. 152 (2016), 99-114.
[3] G. Bellamy, T. Schedler & U. Thiel, Hyperplane arrangements associated to symplectic quotient singularities, preprint (2017), arXiv:1702.04881.
[4] C. Bonnafé & R. Rouquier, Cherednik algebras and Calogero-Moser cells, preprint (2017), arXiv:1708.09764.
[5] P. Etingof & V. Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, Inventiones Mathematicae 147, no. 2 (2002), 243-348.
[6] V. Ginzburg & D. Kaledin, Poisson deformations of symplectic quotient singularities, Adv. in Math. 186 (2004), 1-57.
[7] I. Gordon, Baby Verma modules for rational Cherednik algebras, Bull. London Math. Soc. 35 (2003), 321-336.
[8] I. Gordon, Quiver varieties, category $\mathcal{O}$ for rational Cherednik algebras, and Hecke algebras, Int. Math. Res. Papers 2008 (2008), Art. ID rpn006, 69 pages.
[9] I. Gordon & M. Martin, Calogero-Moser space, restricted rational Cherednik algebras, and two-sided cells, Math. Res. Lett. 16 (2009), 255-262.
[10] M. Goresky & R. MacPherson, On the spectrum of the equivariant cohomology ring, Canadian Journal of Mathematics 62, no. 2 (2010), 262-283.
[11] M. Haiman, $t$, $q$-Catalan numbers and the Hilbert scheme, Discrete Math. 193 (1998), 201-224. Selected papers in honor of Adriano Garsia (Taormina, 1994).
[12] D. Kaledin, On crepant resolutions of symplectic quotient singularities, Selecta Math. 9 (2003), 529-555.
[13] M. Lehn & C. Sorger, A symplectic resolution for the binary tetrahedral group, Geometric methods in representation theory II, 429-435, Sémin. Congr., 24-II, Soc. Math. France, Paris, 2012.
[14] É. Vasserot, Sur l’anneau de cohomologie du schéma de Hilbert de $\mathbb{C}^2$, C. R. Acad. Sci. 332 (2001), 7-12.