Source integrals of multipole moments for static space–times

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Abstract

The definition of Komar for the mass of a relativistic source is used as a starting point to introduce volume integrals for relativistic multipole moments. A certain generalisation of the classical Gauss theorem is used to rewrite these multipole moments as integrals over a surface at infinity. It is shown that this generalisation leads to asymptotic relativistic multipole moments, recovering the multipoles of Geroch or Thorne, when the integrals are evaluated in asymptotically cartesian harmonic coordinates. Relationships between this result and the Thorne definition and the classical theory of moments are shown.

Keywords: multipole moments, source integrals, Komar mass

1. Introduction

The early definition of relativistic multipole moments (RMMs) for vacuum static asymptotically flat space–times appeared in the pioneering work of Geroch [1]. This definition was extended later by Geroch and Hansen [2] to the stationary case. They were also defined by Thorne [3], and others [4, 5] and various approaches for their calculation have been presented [6, 7].

Multipole moments have been very useful especially in the case of axial symmetry to describe some physical quantities of gravitating sources. They can be used to compute observable physical features of compact objects by means of test particles orbiting within the gravitational field they produce (precession, geodesic deviation, ISCOs, ...) [8, 9]. These estimates and calculations have been possible because vacuum solutions constructed in terms of the RMM have been expressed as deviations from the spherical symmetry case given by the Schwarzschild solution (see for instance [10]).
In classical gravity multipole moments have a double meaning that allow us to identify these quantities not only with the coefficients of the asymptotic expansion of the gravitational potential but also with the source of the field through integral expressions extended over the volume of the gravitating object. There is recent interest in works (see [11] and references therein) trying to link the RMM to gravitational sources (in general relativity), so that quantities defined on the outside can be related to physical magnitudes of the stellar object and the source itself. In this sense Gürlebeck [11] managed to express Newtonian multipole moments of the Weyl metrics as source integrals. In the relativistic case there are not even precise definitions that allow us to write RMM as integrals over the sources, and that is the main objective of this work.

The paper is organised as follows. Section 2 is devoted to showing the generalised Gauss theorem (GGT) in classical gravity. First, we recall the approach of the multipole series of the gravitational field generated by a compact source, insisting the fact that the asymptotic moments are also integrals extended over the volume of the stellar object that creates the field. In particular, we restrict ourselves to the case of sources with axial symmetry, which is a common symmetry in astronomical objects. In this work we restrict ourselves to static spacetimes and in section 3 we recall the Einstein equations in this setting. These equations are the basis of our developments and are written in two different ways; by using the quotient metric related to the time-like Killing congruence and by using a conformal metric.

In section 4 the definition of Komar mass [12] and its connection with Tolman mass [13] is addressed. It is also shown that, as is well known, the result of Komar for the static case is simply the relativistic version of the Gauss theorem (GT), which serves as the basis for subsequent sections. Section 5 contains the main contribution of this work. By taking as a starting point the Komar mass, source integrals of static multipole moments are proposed. In addition, the structure of Weyl metrics in harmonic coordinates, previously obtained in [14, 15], can be used to prove that these integrals also provide the asymptotic Thorne moments in the axially symmetric case. The computation is performed by using both the quotient and the conformal metric and we conclude that the second one is more satisfactory for our purposes. This result should be considered as the relativistic version of the generalised Gauss theorem (RGGT).

In section 6 we revisit the integrals used by Thorne to define static multipole moments and show that the RGGT can be trivially applied to obtain coincidence with the asymptotic moments. Finally, the appendix is devoted to detail the approximate expressions obtained in previous works for the Weyl metric when it is developed in spherical harmonic coordinates, up to order nine in the inverse of the radial coordinate.

Latin indices $i, j, k, \ldots$ take values $1, 2, 3$. Greek ones $\alpha, \beta, \lambda, \mu, \ldots$ carry from 0 to 3. Einstein summation rule is used for equal indexes in different positions. The signature of the space–time is $(−, +, +, +)$ and we use relativistic units $G = c = 1$.

2. GGT in classical gravity

2.1. The Poisson equation and the GT

In newtonian theory the gravitational field $\tilde{g}$ associated to an isolated compact stellar object is determined, up to a sign, by the gradient of a potential $\Phi(\vec{x})$ which satisfies the Poisson equation:

$$\Delta \Phi = 4\pi \, \mu(\vec{x}), \quad (1)$$
where $\Delta$ represents the ordinary Laplacian operator in three dimensional space and $\mu(\vec{x})$ denotes the density of the object, which is a function of the position vector $\vec{x}$ and obviously vanishes outside the source.

As it is well known, an immediate consequence of equation (1) is the classical GT according to which the total field flux across the surface boundary $\partial V$ of any volume $V$ containing the source is proportional to its mass. Indeed, according to (1) and the definition of the Laplacian operator, the mass $M$ can be written as follows:

$$ M = \int_V \mu(\vec{x}) d^3x = \frac{1}{4\pi} \int_V \text{div} \Phi d^3x, $$

and taking into account the divergence theorem of Gauss

$$ \oint_{\partial V} \vec{g} \cdot d\vec{\sigma} = -4\pi M, $$

where $d\vec{\sigma}$ is the surface element of the boundary $\partial V$.

It should be specified here that stellar objects are self-gravitating, and so their density not only depends, in general, on the position but also on the potential $\Phi$ itself. Indeed, let us consider for example a static and spherically symmetric barotropic perfect fluid object, that is,

$$ \mu = \mu(p), \quad p(r), \quad \mu(r), \quad \Phi(r), \quad r \equiv |\vec{x}|, $$

with $p$ denoting the pressure. From the Euler equation

$$ dp = -\mu(p) d\Phi $$

the following conclusion is obtained by integration:

$$ p = f_1(\Phi_S - \Phi) \Rightarrow \mu = f_2(\Phi_S - \Phi), $$

for appropriate function $f_1$ and $f_2$ where $\Phi_S$ is the potential (constant) on the surface of the star ($p = 0$). Therefore we conclude that, in general, the Poisson equation is nonlinear (see for instance Lane–Emden equation in [16]).

2.2. Lagrange–Poisson integral. Multipole series

The Poisson equation can be solved using the Green function of the Laplacian operator, which allows one to rewrite the potential $\Phi(\vec{x})$ by means of the Lagrange–Poisson integral:

$$ \Phi(\vec{x}) = -\int \frac{\mu(\vec{y})}{|\vec{x} - \vec{y}|} d^3y, $$

where the integral extends to all space. In this expression we have imposed the condition that the potential vanishes at infinity, and it turns out that $\Phi$ is at least of class $C^1$.

The integral (7) is the basis for the asymptotic multipole expansion of the potential. Indeed, the Green function admits the following series expansion

$$ \frac{1}{|\vec{x} - \vec{y}|} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial (1/|\vec{x} - \vec{y}|)}{\partial y^{i_1} \cdots \partial y^{i_n}} \bigg|_{y=0} y^{i_1} y^{i_2} \cdots y^{i_n}. $$

Now, it is trivial to see that

$$ \frac{\partial (1/|\vec{x} - \vec{y}|)}{\partial y^{i_1} \cdots \partial y^{i_n}} \bigg|_{y=0} = (-1)^n \frac{\partial 1/r}{\partial x^{i_1} \cdots \partial x^{i_n}} = \frac{(2n - 1)!!}{r^{n+1}} n_{i_1i_2\ldots i_n}, $$

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where
\[ n_{i_1\ldots i_n} \equiv (n_i \, n_{i_2} \ldots n_{i_n})^{TF}, \quad n^i \equiv x^i/r. \]  
(10)

Indices here are risen and lowered with the Euclidean metric \( \delta_{ij} \) and the notation TF (trace free) means traceless part (obviously we are using Cartesian coordinates everywhere). Substituting now (8) and (9) in the integral (7), the following expression is obtained for the potential:

\[
\Phi(x) = -\frac{M}{r} - \sum_{n=1}^{\infty} \frac{(2n-1)!}{n!} \frac{1}{r^{n+1}} M^{n\ldots i_n} n_{i_1\ldots i_n},
\]
(11)

where
\[
M^{n\ldots i_n} \equiv \int_V y^{n\ldots i_n} \mu(\vec{y}) d^3y, \quad y^{n\ldots i_n} \equiv (y^1 y^2 \ldots y^n)^{TF},
\]
(12)

the quantity \( M^{n\ldots i_n} \) is the multipole moment of order \( n \), and it is obviously a completely symmetric object without trace. Therefore, it is clear that it only has \( 2n + 1 \) independent components.

In view of the above it is important to remember that the multipole moments have a double meaning. On the one hand they are integrals over the source that measure the deviation from spherical symmetry and on the other hand they are the coefficients of an asymptotic expansion.

A classic question now arises. Do multipole moments completely determine the source? The theory of function moments ensures that the set of all mathematical moments of the density \( \mu(x) \) fixes it completely. Nevertheless the multipole moments \( M^{n\ldots i_n} \) are completely symmetric traceless quantities, so they are only a subset of all mathematical moments. Accordingly, they do not fully determine the source, i.e., many different sources may exist which lead to the same potential (11). It follows easily that multipole moments describe only the ‘skeleton’ of the source, i.e., the following distribution density [17]

\[
\mu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} M^{n\ldots i_n} \partial_{i_1\ldots i_n} \delta(x),
\]
(13)

\( \delta(x) \) being the Dirac delta function and \( \partial_k \equiv \partial / \partial x^k \).

2.3. Generalised Gauss theorem

Expression (13) shows that the first element of the ‘skeleton’ of the stellar object creating the gravitational field is simply its mass (monopole moment). On the other hand GT states that mass is proportional to the flux of the field (which is, up to a sign, the derivative of the potential) across any closed surface containing the star. So naturally the question is whether the remaining elements, i.e., the other multipole moments, can be expressed by means of fluxes of some expression containing the potential and its derivative. The answer lies in what we call GGT, which is probably well known, but it does not appear in standard texts. The proof of this theorem is almost as simple as the proof of the ordinary GT itself. In fact, it starts from the following identity:

\[
\chi^{i_1\ldots i_n} \Delta \Phi = \partial^k [\chi^{i_1\ldots i_k} \partial_k \Phi - \Phi \partial_k \chi^{i_1\ldots i_k}] + \Phi \Delta \chi^{i_1\ldots i_n},
\]
(14)

3 We had knowledge of this theorem thanks to a preprint of Augusto Espinoza (Universidad Zapoteco). Private communication.
In addition, it is easy to show the following equation:

$$\Delta x^{i_1 \cdots i_n} = 0,$$  \hspace{1cm} (15)

so that, taking into account the definition of moments (12) as well as the Poisson equation (1) and the divergence theorem, the desired result is obtained:

$$M^{i_1 \cdots i_n} = \frac{1}{4\pi} \oint [x^{i_1 \cdots i_n} \partial_i \Phi - \Phi \partial_i x^{i_1 \cdots i_n}] d\sigma^k.$$  \hspace{1cm} (16)

That is, similarly as with the mass, higher-order multipole moments can be expressed as the flux across any surface containing the star, of certain combination of the potential and its gradient.

It is important to note that the Poisson equation plays an essential role in the proof and, in addition, the fact that the moments (12) are expressed as integrals over the source. It is checked without difficulty that the fluxes (16) coincide with the coefficients (moments) of the asymptotic multipole expansion (11), whereby the consistency of the result is ensured.

* Axial symmetry

In this paper we are interested only in the case where the source of the field has axial symmetry, which is the usual symmetry of celestial objects. In this case, the multipole moments have the following structure

$$M^{i_1 \cdots i_n} \equiv M_n e^{i_1 \cdots i_n}, \quad e^{i_1 \cdots i_n} \equiv (e^{i_1} e^{i_2} \ldots e^{i_n})^T,$$  \hspace{1cm} (17)

where $e^k$ is a unit vector along the positive direction of the symmetry axis, so that the corresponding moment has a single component $M_n$. Now, considering the formula

$$e^{i_1 \cdots i_n} n_{i_1 \cdots i_n} = \frac{n!}{(2n - 1)!!} P_n(\cos \theta),$$  \hspace{1cm} (18)

where $P_n$ is a Legendre polynomial and $\theta$ the polar angle with respect to the symmetry axis, the multipole expansion (11) leads to the following expression

$$\Phi(\vec{x}) = -\sum_{n=0}^{\infty} \frac{M_n}{n+1} P_n(\cos \theta).$$  \hspace{1cm} (19)

Furthermore, using the formula

$$e^{i_1 \cdots i_n} e^{i_1 \cdots i_n} = \frac{n!}{(2n - 1)!!},$$  \hspace{1cm} (20)

which is a consequence of (17) and (12), the following expression for the moment $M_n$ as a source integral is obtained:

$$M_n = \frac{(2n - 1)!!}{n!} M^{i_1 \cdots i_n} e^{i_1 \cdots i_n} = \oint_D \mu(r, \theta) r^n P_n(\cos \theta) d^3\vec{x}, \quad M_0 \equiv M.$$  \hspace{1cm} (21)

With regard to the GGT (16), it is now written (in axial symmetry) as follows

$$M_n = \frac{1}{4\pi} \oint \left( r^n P_n(\cos \theta) \partial_k \Phi - \Phi \partial_k [r^n P_n(\cos \theta)] \right) d\sigma^k.$$  \hspace{1cm} (22)

Given the next standard formulas ($e^k$ is the unit vector in the direction of the meridians):

$$\partial_k f(r, \theta) = \partial_r f \partial_r r + \partial_\theta f \partial_\theta \theta = \partial_r f n_k + \partial_\theta f \frac{1}{r} e_{0k},$$

$$d\sigma^k = n^k d\sigma = n^k r^2 \sin \theta d\theta d\phi,$$  \hspace{1cm} (23)
and using a sphere of radius \( r \) as surface of integration, the following expression is obtained:

\[
M_n = \frac{1}{2} r^{n+1} \int_{-1}^{+1} P_n(\cos \theta)(r \partial_\theta \Phi - n \Phi) d(\cos \theta).
\]

(24)

It is trivial to verify that (24) reproduces the coefficients of the asymptotic expansion. Indeed, taking into account (19) one finds:

\[
r \partial_\theta \Phi - n \Phi = \sum_{n=0}^{\infty} \frac{(q + 1 + n)M_q}{r^{q+1}} P_q(\cos \theta),
\]

(25)

which substituted in (24) and recalling orthogonality relations

\[
\int_{-1}^{+1} P_n(\cos \theta) P_m(\cos \theta) d(\cos \theta) = \frac{2}{2n + 1} \delta_{nm},
\]

(26)

concludes the verification.

3. Static fields. Einstein equations

As already mentioned in the Introduction, one of the aims of this work is to look for a possible relativistic extension of the GGT in the static case. This means that we will consider a gravitational static compact source, which obviously will generate a static and asymptotically flat space–time.

A space–time of this type is characterised by the existence of an integrable time-like Killing vector field, i.e., it admits orthogonal space-like hypersurfaces (we assume that all of this has a global character). In an asymptotically cartesian coordinate system adapted to the Killing, the metric is written as follows:

\[
dx^2 = \hat{g}_{00}(x^k) dx^2 + \hat{g}_{ij}(x^k) dx^i dx^j, \quad G = c = 1,
\]

(27)

with the conditions

\[
g_{00} \equiv - \xi^2 < 0, \quad \hat{g}_{ij}(x^k) dx^i dx^j > 0, \quad \hat{g}_{ij} \equiv g_{ij},
\]

(28)

\[
g_{00} \to -1 + \frac{2m}{r}, \quad g_{ij} \to \delta_{ij} + \frac{2m}{r} n_i n_j,
\]

(29)

where \( m \) is the total mass–energy of the system, \( r \) is the associated radial coordinate and \( n_i \) is the unit radial vector at infinity.

In this case, the Einstein equations for the quotient metric \( \hat{g}_{ij} \) are written as follows (see [18] for instance):

\[
(V_3, \hat{g}) : \begin{cases}
\Delta \xi = 4\pi \xi (-T^0_0 + \hat{T}) \equiv 4\pi \hat{\rho}_0 \\
\hat{R}_{ij} - \xi^{-1} \hat{\nabla}_i \hat{\nabla}_j \xi = 8\pi \left(T_{ij} - \frac{1}{2} T \hat{g}_{ij}\right),
\end{cases}
\]

(30)

where \( \hat{\nabla} \) denotes covariant derivative with respect to \( \hat{g}_{ij}, \Delta \equiv \hat{\nabla}^a \hat{\nabla}_a \) is the Laplacian operator and \( \hat{R}_{ij} \) is the Ricci tensor. The sub-index ‘to’ by the density \( \hat{\rho}_0 \) refers to Tolman [13], and will be justified in the next section. Finally, \( T^a_j \) is the energy-momentum tensor, \( T \equiv T^a_a \) and \( \hat{T} \equiv T^j_j \) being the traces. In the particular case of a perfect fluid we have that \( \mu \) being the quadri-vector of the congruence.
\( T^\rho_\beta = \rho u^\rho u_\beta + p(\delta^\rho_\beta + u^\rho u_\beta) \)
\( u^\alpha = \xi^{-1} \delta^0_\alpha, \quad u_\beta = -\xi \delta^0_\beta \)  
(31)

\[ \Rightarrow \hat{\rho}_{\text{to}} = \xi [\rho + p(-1 + 1) + 3p] = \xi (\rho + 3p). \]  
(32)

If one takes into account the Euler equations, the nonlinearity of the Poisson equation for the self-gravitating case becomes obvious, since the density turns out to be a non trivial function of the potential \( \xi \).

Some authors [1, 3, 19] consider it more convenient to use the conformal metric \( \tilde{g}_{ij} \) instead of the metric \( g_{ij} \). In [19] it is argued that \( \tilde{g}_{ij} \) must be regarded as the true space metric since the non-relativistic limit of general relativity should lead not only to the classical Poisson equation but also to a flat three-dimensional space. References [1, 3] are restricted, however, to use \( \tilde{g}_{ij} \) for mathematical convenience. A standard computation shows that the Einstein equations can be written as follows:

\[
\begin{align*}
(V_3, g) : & \\
\Delta \log \xi &= 4\pi \xi^{-2} (\hat{T}^0_0 + \hat{T}) \equiv 4\pi \hat{\rho}_{\text{to}}.
\end{align*}
\]  
(33)

Let us note that densities \( \hat{\rho}_{\text{to}} \) and \( \tilde{\rho}_{\text{to}} \) are related to each other as follows:

\[
\hat{\rho}_{\text{to}} \sqrt{\tilde{g}} = \tilde{\rho}_{\text{to}} \sqrt{\tilde{g}}.
\]  
(34)

We will see later the influence of the use of each metric in what we will call ‘relativistic generalised Gauss theorem’ (RGGT).

4. Mass and flux of Komar

The quantity usually called Komar mass is defined in the original article [12] by means of the following expression:

\[
M_K = \frac{1}{\chi} \int_{\Sigma_1} \nabla_\lambda (\nabla^\lambda \xi^\mu - \nabla^\mu \xi^\lambda) d\sigma_{\lambda},
\]
(35)

\( \xi^\lambda \) being a vector field, \( \chi = 8\pi \) the Einstein constant of gravitation and \( d\sigma_{\lambda} \) the normal 1-form to a three-dimensional surface \( \Sigma_3 \) of the space–time considered, i.e.

\[
d\sigma_{\lambda} = \frac{1}{3!} \eta_{\lambda\mu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho,
\]
(36)

where \( \eta_{\lambda\mu\rho} \) denotes the element of volume and \( \wedge \) the exterior product. We assume that \( \Sigma_3 \) represents the whole ordinary three-dimensional space in a certain admissible coordinate system or at least a part of it.

Using the covariant GT for the divergence, the three-dimensional volume integral (35) can be written as a flux integral across the two-dimensional surface \( \Sigma_2 = \partial \Sigma_3 \). The following result is obtained:

\[
M_K = -\frac{1}{\chi} \oint_{\Sigma_2 = \partial \Sigma_3} \nabla^\lambda \xi^\mu \ d\sigma_{\lambda},
\]
(37)

d\sigma_{\lambda} being the 2-form normal to \( \Sigma_2 \), i.e.,

\[
d\sigma_{\lambda} = \frac{1}{2} \eta_{\lambda\mu\rho} dx^\mu \wedge dx^\nu,
\]
(38)
or equivalently, in three-dimensional common language, the normal vector to the surface.
We are interested in the case where $\xi^\alpha$ is a Killing vector field, so we have
\begin{equation}
\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \Rightarrow \nabla_\alpha \xi^\beta = -\xi^\beta R^\alpha_{\rho \mu}.
\end{equation}
and thus, given the Einstein equations, the integral of Komar (35) is written as follows:
\begin{equation}
M_K = -2 \int_{\Sigma} \left( \xi^\alpha T^\mu_\alpha - \frac{1}{2} T^\alpha_{\mu \nu} \right) d\sigma\nu.
\end{equation}
If $\xi^\alpha$ is the time-like killing vector of a stationary space–time and we use coordinates adapted to it ($\xi^\alpha = \delta^\alpha_0$), and the three-surface is $\Sigma_3 : x^0 = Cte$, the following expression is obtained:
\begin{equation}
M_K = \int_{\Sigma} (-T^0_0 + \hat{T}) \xi d^3x,
\end{equation}
which coincides with the Tolman mass [13], which justifies the use of sub-index ‘$\iota$’ in the density $\hat{\rho}_{\iota\iota}$ (30). The following notation has been used:
\begin{equation}
\hat{T} \equiv T_i^i, \quad \xi \equiv \sqrt{-\xi^{\alpha} \xi_{\alpha}} = \sqrt{-g_{00}}, \quad d^3x \equiv dx^1 \wedge dx^2 \wedge dx^3.
\end{equation}

Given the above, when the space–time is generated by a compact object we end up with the following conclusions:
(a) If the energy–momentum tensor $T_{\iota\iota}$ is zero outside the stellar object (no electromagnetic field) then the Komar flux (35) is independent of $\Sigma_3$ and in addition it can be expressed as a volume integral there on (this is the classic result of Tolman [13])

(b) If $T_{\iota\iota}$ is not zero in the exterior (existence of electromagnetic field) then the Komar flux defines the mass by taking the sphere of infinity (assuming that space–time is asymptotically flat and $T_{\iota\iota}$ goes to zero quickly enough).

In order to illustrate these ideas we present below the expression of Komar mass for the Kerr–Newman metric [20] written in Boyer–Lindquist coordinates (the computation is involved but it can be done in a straightforward way by using a computer)
\begin{equation}
M_K = m - \frac{e^2}{2r} - \frac{e^2}{2a} \left[ 1 + \frac{a^2}{r^2} \arctg \frac{a}{r} \right] \rightarrow m,
\end{equation}
where $\Sigma_2$ has been considered to be a regular sphere of radius $r$ (radial coordinate) and the parameters $\{m, a, e\}$ denote, as usual, mass, angular momentum per unit of mass, and electric charge respectively. As long as $r$ remains finite both the charge and angular momentum contribute to the Komar mass. Nevertheless, as we go to infinity ($r \rightarrow \infty$) the following result is obtained:
\begin{equation}
M_K = m - \frac{e^2}{r} - \frac{e^2 a^2}{3r^3} \left[ 1 + O\left( \frac{a}{r} \right) \right] \rightarrow m,
\end{equation}
i.e. Komar mass reduces to the parameter $m$.

It is worthwhile noticing that, at least in the case of a static space–time ($\{g_{00} = 0\}$ in coordinates adapted to the time-like Killing vector), the result of Komar is simply the gravitational GT in its relativistic version. Indeed, recall the first Einstein equation (30):
\begin{equation}
\hat{\Delta} \xi = 4 \pi \rho_{\iota\iota}, \quad \hat{\rho}_{\iota\iota} \equiv \xi (T^0_0 - \hat{T}),
\end{equation}
which is a Poisson equation with corresponding density $\hat{\rho}_{\iota\iota}$. Then it seems natural to use this density as bulk mass density, and henceforth the total mass will be written as
\begin{equation}
M = \int_V \hat{\rho}_{\iota\iota} \hat{\nu},
\end{equation}
$V$ being any compact volume containing the source, and $\tilde{\eta}$ being the three-dimensional volume element, i.e.

$$\tilde{\eta} = \sqrt{\tilde{g}} \, dx^1 \wedge dx^2 \wedge dx^3.$$  \hfill (47)

As a result, by applying the divergence theorem in its covariant form, we have that

$$M = \frac{1}{4\pi} \int_V \tilde{\Delta} \xi \tilde{\eta} = \frac{1}{4\pi} \int_V \tilde{\nabla}_i \tilde{\nabla}^i \xi \tilde{\eta} = \frac{1}{4\pi} \oint_{\partial V} \tilde{\nabla}^i \xi \, d\sigma_i,$$  \hfill (48)

and it is trivial to see that (48) is precisely the result of Komar (37) for the static case, $\partial V$ being the boundary and $d\sigma_i$ its corresponding surface element:

$$d\sigma_i = \frac{1}{2} \eta_{ij} \, dx^i \wedge dx^j = \sqrt{\tilde{g}} \frac{1}{2} \epsilon_{ij} \, dx^i \wedge dx^j.$$  \hfill (49)

We can also apply the divergence theorem in its non-covariant form as follows:

$$M = \frac{1}{4\pi} \int_V \tilde{\Delta} \xi \tilde{\eta} = \frac{1}{4\pi} \int_V \frac{1}{\sqrt{\tilde{g}}} \partial_k (\sqrt{\tilde{g}} \tilde{g}^{ij} \partial_j \xi) \sqrt{\tilde{g}} \, d^3\tilde{\nu}$$  \hfill (50)

$$\Rightarrow M = \frac{1}{4\pi} \oint_{\partial V} \tilde{g}^{ij} \partial_j \xi \sqrt{\tilde{g}} \, n_k \, d\sigma_k.$$  \hfill (51)

5. Static multipoles ‘à la Komar’. RGRT

As we restrict ourselves to the static case, it is clear from (34), (46) and (48) that Komar mass can be treated in two different ways: either by using the quotient metric $\tilde{g}_{ij}$ or by using the conformal quotient metric $\bar{g}_{ij} = \xi^2 \delta_{ij}$. This is what we will do in this section.

5.1. Quotient metric

Expression (46) provides the Komar mass of the system, which obviously coincides with the first coefficient (monopole) of the asymptotic expansion of the metric. In addition, taking into account (48) it is clear that it can be written as an integral over the whole space of a function obtained from the gravitational field. In this way we will generalise (46) and define the other multipole moments as follows:

$$M_k^{i_1 \ldots i_k} = \int x^{i_1} \ldots x^{i_k} \tilde{\rho}_\mu \tilde{\eta} = \frac{1}{4\pi} \int \xi \tilde{\Delta} x^{i_1} \ldots x^{i_k} \tilde{\eta}$$

$$= \frac{1}{4\pi} \int [x^{i_1} \ldots x^{i_k} \tilde{\Delta} \xi - \xi \tilde{\Delta} x^{i_1} \ldots x^{i_k}] \tilde{\eta},$$  \hfill (52)

where $x^{i_1} \ldots x^{i_k}$ are, as before, the traceless part of the products $x^{i_1} \ldots x^{i_k}$ with respect to the euclidean metric $\delta_{ij}$, i.e., as an example

$$x^{ij} = x^i x^j - \frac{1}{3} \delta^{ij} \eta_x x^k x^l,$$  \hfill (53)

in such a way that $M_k^{i_1 \ldots i_k}$ turns out to be a completely symmetric object without trace with respect to $\delta_{ij}$. We also require $\{x^i\}$ to be asymptotically cartesian harmonic coordinates. The ultimate justification of this definition is that, as we will see, these moments coincide, at least in the case of axial symmetry, with the standard multipole moments of Thorne [3] or Geroch [1].
The same process used to prove the GGT in classical gravitation can be used now to express the above definition as a flux integral. Indeed, given that

\[ x^{(\omega)} \Delta \xi - \xi \Delta x^{(\omega)} \sqrt{g} \equiv x^{(\omega)} \partial_{k} \left[ \sqrt{g} \ g^{kl} \partial_{l} \xi \right] - \xi \partial_{k} \left[ \sqrt{g} \ g^{kl} \partial_{l} x^{(\omega)} \right], \tag{54} \]

\((I_{\alpha}) \equiv i_{i_{2}} \cdots i_{n})\) as well as,

\[ x^{(\omega)} \partial_{k} \left[ \sqrt{g} \ g^{kl} \partial_{l} \xi \right] = \partial_{k} \left[ x^{(\omega)} \sqrt{g} \ g^{kl} \partial_{l} \xi \right] - \sqrt{g} \ g^{kl} \partial_{k} x^{(\omega)}, \tag{55} \]

\[ \xi \partial_{k} \left[ \sqrt{g} \ g^{kl} \partial_{l} x^{(\omega)} \right] = \partial_{k} \left[ \xi \sqrt{g} \ g^{kl} \partial_{l} x^{(\omega)} \right] - \sqrt{g} \ g^{kl} \partial_{k} x^{(\omega)}, \tag{55} \]

it turns out that we can rewrite (52) as follows:

\[ 4\pi M_{K}^{(\omega)} = \int \partial_{k} \left[ x^{(\omega)} \sqrt{g} \ g^{kl} \partial_{l} \xi - \xi \sqrt{g} \ g^{kl} \partial_{k} x^{(\omega)} \right] d^{3}x, \tag{56} \]

and once again making use of the divergence theorem, it is finally obtained

\[ M_{K}^{(\omega)} = \frac{1}{4\pi} \int_{s} \left[ x^{(\omega)} \sqrt{g} \ g^{kl} \partial_{l} \xi - \xi \sqrt{g} \ g^{kl} \partial_{k} x^{(\omega)} \right] \sqrt{g} \ d\sigma, \tag{57} \]

result that can be considered as the relativistic RGTT for the static case. Let us note that the application of the divergence theorem requires the gravitational potentials to be of class \( C^{1} \), which in principle can be ensured by the use of certain harmonic coordinates [21].

We shall restrict ourselves again to the axial-symmetry case. As usual we will use a coordinate system adapted to both Killing vector fields and we assume that the metric has Papapetrou structure in associated spherical coordinates [21], i.e.

\[ ds^{2} = g_{tt} dt^{2} + g_{rr} dr^{2} + 2g_{rt} dtdr + g_{\theta\theta} d\theta^{2} + g_{\phi\phi} d\phi^{2}, \tag{58} \]

where the azimuthal coordinate \( \phi \) defines the axial symmetry and hence all metric components are only functions of the radial coordinate \( r \) and the polar coordinate \( \theta \). In particular, we write the metric in the following way:

\[ ds^{2} = \gamma_{tt} dt^{2} + \gamma_{rr} dr^{2} + 2\gamma_{rt} dtdr + \gamma_{\theta\theta} (r d\theta)^{2} + \gamma_{\phi\phi} (r \sin \theta d\phi)^{2}, \tag{59} \]

that is, we are using the 'orthonormal euclidean cobasis'

\[ \{ dt, r d\theta, r \sin \theta d\phi \}, \tag{60} \]

so that,

\[ g_{tt} = \gamma_{tt}, \quad g_{rr} = \gamma_{rr}, \quad g_{\theta\theta} = r^{2} \gamma_{\theta\theta}, \quad g_{\phi\phi} = r^{2} \sin^{2} \theta \gamma_{\phi\phi}. \tag{61} \]

Now considering the above and the results of section 2 concerning the axial symmetry, a short calculation shows that the flux integral (57) for axisymmetric multipole moments ‘à la Komar’ are written as follows:

\[ M_{K}^{(\omega)} = \frac{1}{4} \int_{-1}^{+1} r^{n+1} p_{n}(w)(-g)^{-1/2} \gamma_{\phi\phi} \gamma_{\theta\theta} [2n \gamma_{tt} - r \partial_{r} \gamma_{tt}] dw \]

\[ + \frac{1}{4} \int_{-1}^{+1} r^{n+1} p_{n}(w)(-g)^{-1/2} \gamma_{\phi\phi} \gamma_{\theta\theta} \partial_{\theta} \gamma_{tt} dw \]

\[ - \frac{1}{2} \int_{-1}^{+1} r^{n+1} p_{n}(w)(-g)^{-1/2} \gamma_{tt} \gamma_{\phi\phi} \gamma_{\theta\theta} dw, \tag{62} \]

where \( g \) is the determinant of the metric of the space–time in the euclidean basis

\[ g = \gamma_{tt}(\gamma_{\phi\phi} \gamma_{\theta\theta} - \gamma_{\phi\phi}^{2}) \gamma_{\phi\phi}, \tag{63} \]
and \( w = \cos \theta \) is our notation for the variable of integration. The circle used in integration sign means that the sphere of infinity must be considered, i.e., the limit \( r \to \infty \) needs to be taken. Therefore all terms of the form \( 1/r^k \) with \( k \geq 1 \) should be excluded.

The calculation of the flux integrals (62) requires the use of the structure (104) of the metric in harmonic coordinates set out in the appendix. From that, it is easy to obtain the following structures for different terms of (62):

\[
(-g)^{-1/2}_{\gamma_{\phi}\gamma_{\phi}} r \gamma_{\phi}\gamma_{\phi} = -1 + \sum_{q=0}^{\infty} \frac{1}{r^{q+2}} X_{1}^{(q)}(w),
\]

\[
(-g)^{-1/2}_{\gamma_{\phi}\gamma_{\phi}} r \partial_r \gamma_{\phi}\gamma_{\phi} = -2 \sum_{q=0}^{\infty} \frac{q+1}{r^{q+1}} M_q(w) + \sum_{q=2}^{\infty} \frac{1}{r^{q+3}} X_{2}^{(q)}(w),
\]

\[
(-g)^{-1/2}_{\gamma_{\phi}\gamma_{\phi}} \partial_{\gamma_{\phi}}\gamma_{\phi}\gamma_{\phi} = \sum_{q=0}^{\infty} \frac{1}{r^{q+5}} X_{3}^{(q+2)}(w),
\]

\[
(-g)^{-1/2}_{\gamma_{\phi}\gamma_{\phi}} \partial_{\gamma_{\phi}}\gamma_{\phi}\gamma_{\phi} = \sqrt{1 - w^2} \sum_{q=1}^{\infty} \frac{1}{r^{q+3}} X_{4}^{(q)}(w),
\]

\( X_{a}^{(q)}(w) (a = 1, \ldots, 4) \) being polynomials of degree \( q \) in the variable \( w \).

As an example to fix ideas on how to do the flux integrals we choose the terms (65) and (67). Regarding (65) the following integrals are evaluated:

\[
+ \frac{1}{2} \int_{-1}^{+1} r^{n+1} P_n(w) \sum_{q=0}^{\infty} \frac{q+1}{r^{q+1}} M_q(w) dw = \frac{n+1}{2n+1},
\]

\[
- \frac{1}{4} \int_{-1}^{+1} r^{n+1} P_n(w) \sum_{q=2}^{\infty} \frac{1}{r^{q+3}} X_{2}^{(q)}(w) dw = 0.
\]

The result in (68) is a consequence of the orthogonality relation (26) whereas for the integral (69) the inequality \( q + 3 \leq n + 1 \) must be taken into account, in order to avoid terms like \( 1/r \) or higher, which vanish at infinity. But this implies that \( q \leq n - 2 \), which means that the maximum degree of a Legendre polynomial from \( X_{2}^{(q)}(w) \) is \( n - 2 \). Applying again the relation (26) it follows that this integral vanishes.

Let us now analyse the term from (67), which leads to the following integral:

\[
- \frac{1}{2} \int_{-1}^{+1} r^{n+1} P_n(w) \sum_{q=1}^{\infty} \frac{1}{r^{q+3}} \sqrt{1 - w^2} X_{4}^{(q)}(w) dw.
\]

Inequality \( q + 3 \leq n + 1 \) must be considered again, which implies that \( q + 1 \leq n - 1 \). However, the product of the factor \( \sqrt{1 - w^2} \) by the polynomial \( X_{4}^{(q)} \) generates associated Legendre functions \( P_n^m \) whose maximum degree is \( m = q + 1 \). Consequently, the integral (70) is zero under the orthogonality relation

\[
\int_{-1}^{+1} P_n^m(w) P_n^m(w) dw = \frac{2n(n+1)}{2n+1} \delta_{mn}.
\]

Arguing similarly with the other terms (64) and (66), we can conclude that both of them lead to vanishing integrals. Consequently the unique non-zero integral is (68), whereby the expression (62) is finally reduced to the following:
It is concluded that the axisymmetric moments ’à la Komar’ are proportional to the known moments of Thorne [3] (the proportionality constant depends on the multipole order).

5.2. Conformal quotient metric

The result (72) would be more satisfactory if the Komar moments would coincide exactly with the Thorne moments. In this subsection we will modify the definition (52) to get that matching. To do this we exploit the fact that the Komar mass can also be written respect to the conformal metric, accordingly with (34), as follows

\[ M_K = \int \rho_0 \eta. \]  

(73)

Hence, given Einstein equation (33), this equality suggests replacing the definition (52) of the multipole moments for the following one:

\[ \tilde{M}_K^{l_1 \ldots l_n} = \frac{1}{4\pi} \int [x^{e^{l_1 \ldots l_n}} \Delta \log \xi - \log \xi \Delta x^{e^{l_1 \ldots l_n}}] \eta, \]  

(74)

whereby it follows that, similar to (57),

\[ \tilde{M}_K^{l_1} = \frac{1}{4\pi} \int [x^{e^{l_1}} \psi^l \log \xi - \log \xi \psi^l] \sqrt{g} \, \mathrm{d}S, \]  

(75)

and restricting ourselves to the axial symmetry case, a standard calculation leads to the following:

\[ \tilde{M}_K^{n} = \frac{1}{4} \int_{-1}^{+1} r^{n+1} P_n(w)(-g)^{-1/2} \gamma_{\varphi \varphi} \gamma_{\theta \theta} [n \gamma_\theta \log(-\gamma_\theta) - r \partial_r \gamma_\theta] \, \mathrm{d}w \]

\[ + \frac{1}{4} \int_{-1}^{+1} r^{n+1} P_n(w)(-g)^{-1/2} \gamma_{\varphi \theta} \partial_\varphi \gamma_\theta \, \mathrm{d}w \]

\[ - \frac{1}{4} \int_{-1}^{+1} r^{n+1} P_n(w)(-g)^{-1/2} \gamma_{\varphi \varphi} \gamma_{\theta \theta} \gamma_\theta \log(-\gamma_\theta) \, \mathrm{d}w. \]  

(76)

To calculate these integrals we proceed in a similar way than the previous case where we used the metric \( \tilde{g}_{ij} \). We need structures (65), (66) and the following expressions:

\[ (-g)^{-1/2} \gamma_{\varphi \varphi} \gamma_{\theta \theta} \gamma_\theta \log(-\gamma_\theta) = 2 \sum_{q=0}^{\infty} \frac{M_q}{r^{q+1}} P_q(w) + \sum_{q=0}^{\infty} \frac{1}{r^{q+3}} X_3^{(q)}(w), \]  

(77)

\[ (-g)^{-1/2} \gamma_{\varphi \theta} \partial_\varphi \gamma_\theta = \sqrt{1 - \nu^2} \sum_{q=1}^{\infty} \frac{1}{r^{q+1}} X_6^{(q)}(w), \]  

(78)

where \( X_6^{(q)}(w) \) (\( a = 5, 6 \)) are again polynomials of degree \( q \) in the variable \( w \). This time the unique non-zero integrals are the following ones:

\[ \frac{1}{2} \left[ \int_{-1}^{+1} r^{n+1} P_n(w) \sum_{q=0}^{\infty} \frac{q + 1}{r^{q+1}} M_q P_q(w) \, \mathrm{d}w \right] = \frac{n + 1}{2n + 1}, \]  

(79)

\[ \frac{1}{2} \left[ \int_{-1}^{+1} r^{n+1} P_n(w) \sum_{q=0}^{\infty} \frac{1}{r^{q+1}} M_q P_q(w) \, \mathrm{d}w \right] = \frac{n}{2n + 1}. \]  

(80)
whereby the following result is now obtained:

$$M_n^K = \frac{n}{2n+1} M_n + \frac{n+1}{2n+1} M_n = M_n. \quad (81)$$

As we see, the use of the conformal metric $\tilde{g}_{ij}$ leads to a more satisfactory result than the one obtained with the quotient metric. This fact represents an additional argument for the use of the conformal metric against the quotient metric as a metric of space.

5.3. Comments

(A) Firstly, it should be noted here that although it may seem somewhat artificial the use of the ‘euclidean’ factor $x^{i_1\ldots i_k}$ in (52) and (74), however we should consider some important items: (1) The moments are required to be numerical objects completely symmetric and without trace, and hence this condition should be verified with respect to a metric independently of any coordinate system. (2) The definitions (52) and (74) lead to flux integrals across the surface at infinity, where we assumed that the metric is flat. (3) Henceforth the moments $\tilde{M}_{k}^{i_1\ldots i_k}$ (as well as $M_{k}^{i_1\ldots i_k}$) should be considered as tensors in a neighbourhood of infinity so that everything becomes consistent.

(B) Secondly, another relevant feature of the definitions (52) and (74) has to do with the requirement imposed on the calculation of the integrals in asymptotically cartesian harmonic coordinates. This fact means that the above definitions are not covariant and so they depend on the system of coordinates. Let us see what happens, for example, if we write the definition (52) in canonical spherical coordinates of Weyl for the static-axisymmetric case (metric (100) at the appendix).

In this case the integral (52) is reduced to the following:

$$\int [r^n P_n \Delta e^\Psi - e^\Psi \Delta (r^n P_n)] \sqrt{g} = \partial_k [\beta [r^n P_n \partial^k \Psi - \partial^k (r^n P_n)]], \quad (82)$$

and the following result is easily obtained:

$$M_n^K = \frac{1}{4\pi} \oint_{\text{ext}} \left[ r^n P_n \partial^k \Psi - \partial^k (r^n P_n) \right] n_k d\sigma = -\frac{n+1}{2n+1} a_n, \quad (83)$$

which compared to (72) highlights the importance of using harmonic coordinates. What happens if we do this calculation with the conformal quotient metric? A similar calculation leads to the following:

$$\tilde{M}_n^K = -a_n, \quad (84)$$

as expected if we remember (72) and (81).

(C) We should mention recent work of Gürlebeck [11] who obtains an integral expression, restricted to the interior of the source, for the Weyl coefficients $a_n$ (newtonian multipole moments) which apparently is independent of the coordinate system. It is stated in [11] that this result is equivalent to getting the $M_n$ multipole moments, as their relationship with $a_n$ is known (see for example [14, 22]). In a subsequent work we will establish a comparison between the result of Gürlebeck with ours by explicitly taking an inner solution which describes an anisotropic fluid.

(D) We think that it is worthwhile providing here some arguments and comments devoted to clarify the physical significance of the achieved results. And in that sense, let us comment on the advantages of these results (and other source integrals definitions) and their potentiality to provide a useful tool for studying the relation between RMMs and the properties of the sources.
Firstly, we remind the reader that there already exist different definitions of RMM in the literature but almost all of them are defined with respect to the exterior gravitational field. By means of these definitions, RMM are constructed starting from the vacuum solution to compute them at infinity. Here we have presented a way to obtain a definition of RMM as a volume integral that recovers Thorne’s definition and, in our opinion, it is simpler and does not need the rather cumbersome approach based in the post-newtonian deviation of the metric density (see section 6). Instead, definition (52) is compact and easy to handle, and only involves the $g_{00}$ component of the global metric. As it is well known, FHP method [6], or the expansion of the exterior metric in harmonic coordinates are successful procedures to calculate explicitly the RMM of the exterior gravitational field of a self gravitating isolated source. However there is no known method to relate, in general, those RMM and the interior of the source. It is widely believed that these RMM are related to some properties of the source and they surely describe some aspects of the physical distribution that generates the gravitational field. A clear example is the intrinsic relationship between the sphericity of the source and the vanishing RMM of order higher than the monopole. Also, the equatorial symmetry of the source leads to the vanishing odd order RMM.

We should clarify that the volume integrals (52) are evaluated at infinity for two main reasons: one is for simplicity since that definition of the integral is just a flux and can be expressed as a surface integral (57). The second reason is that we know explicit expressions of the axial symmetric metric for the vacuum space–time in harmonic coordinates outside the source. In contrast, there exist few examples of global solutions for this case. But, nevertheless the evaluation of the volume integral (52) can be explicitly done if one has a global solution for an axially symmetric space–time. In fact, in [25] interior solutions for any exterior metric of the Weyl family are constructed. Hence we have a global static solution available in the axisymmetric case to perform the calculation of RMM by means of the volume integrals (52). In particular, if we consider the global space–time with exterior metric of Schwarzschild constructed in [25], it is relatively easy to calculate the volume integral in two parts, one integral for the volume extended from the boundary to infinity and another one for the interior volume of the source. For any order of the calculated multipole moment, the part of the integral restricted to the exterior always vanishes and then the definition of the RMM becomes reduced to a source integral.

As we already said in the introduction multipole moments in classical gravity can be connected with the source by means of volume integrals (12), and hence the density of the source and multipole moments are directly related. The aim of the work is focused towards looking for a generalisation of this feature into general relativity. The way as the definition presented here may lead to establishing explicit connections between RMM and properties of the sources is work still in progress, but some results can be already pointed out:

(i) As it is well-known some solutions of the vacuum field equations can be smoothly matched with many different interior solutions, all of them providing the same RMM. Source integrals can be applied to restrict the equation of state of a rotating perfect fluid from the observed multipole moments. This approach is of astrophysical interest since often only asymptotics of the gravitational field as well as the RMM are accessible to experiments. In contrast, once we have fixed our source, its exterior gravitational field is unique and it is completely determined by the RMM. Hence these source integrals must be a tool of highly physical interest for establishing conditions on the matching problem. Source integrals will

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4 We will argue below that this volume integral can be explicitly calculated with respect to the volume bounded by the surface of the source.
prove useful in the search of global solutions of Einstein equations describing figures of equilibrium or relativistic stars [28].

(ii) It is clear from (52) that the expression appearing in the integral depends on the $g_{00}$ metric component (or equivalently $\xi$), and hence by means of the Einstein field equations the volume integral can be related with physical magnitudes of the source, in particular its density. For example, a calculation is done in [25] that may illustrate the above comment; the order zero of the definition (52) corresponds to the Monopole and it is shown in [25] that this integral is exactly equal to the Komar mass and it must equal the Tolman mass (as we already showed in section 3). The Tolman mass is explicitly calculated for static axisymmetric space–times in terms of the density distribution of the source. Another interesting and open problem consists in proving that any static matter distribution of perfect fluid must be spherical, as we all assume [33–36]. The calculation of the volume integrals (52) should lead to a single non-vanishing moment corresponding to the monopole.

(iii) In [11] the author uses source integrals to recover easily an old result concerning non-existence results for dust configurations [26]. It is shown that static, axially symmetric and isolated dust configurations do not exist in general relativity. This indicates possible application of source integrals (defined in this case for newtonian moments), providing a tool to solve the task of obtaining (or not) a global asymptotically flat solution starting from an assumed matter distribution. This example shows in a concisive way how the source integrals can be applied in more difficult physical situations like rotating stars. Another possible application is the already known source integrals for isolated horizons [11, 27] (and references therein) since source integrals might prove useful for identifying the contributions to the so defined multipole moments, to explain the observed discrepancies between the isolated horizon multipole moments and those of Geroh–Hansen.

(iv) Finally, source integrals can be used to compare numerical solutions, analytical solutions, and analytical approximations by calculating their RMM [29]. Let us note the difficulties of extracting the RMM of a given numerically determined metric [30]. Additionally, source integrals provide the means to test the accuracy of numerical methods, which are used to determine relativistic stars (see for example [31, 32]).

6. RGGM ‘à la Thorne’

In this section we first recall Thorne’s approach [3] to define the multipole moments using volume integrals over all space. From there we show that the GGT can be applied almost identically to that of classical gravitation. This allows us to conclude in a simple way that our moments coincide with the multipole moments of the asymptotic expansion used by Thorne himself. As it is well known Einstein equations can be written in terms of the metric density $g^{\alpha \beta} \equiv \sqrt{-g} g^{\alpha \beta}$ as follows (see for instance [23, 24]):

$$\partial_\mu H^{\alpha \mu \lambda \nu} = 2 \chi (-g)(\chi^{\alpha \beta} + T^{\alpha \beta}) \quad \Rightarrow \partial_\mu T^{\alpha \beta} = 0,$$

where

$$H^{\alpha \mu \lambda \nu} \equiv (-g)g^{\alpha \lambda}g^{\nu \mu} - g^{\alpha \mu}g^{\lambda \nu} = g^{\alpha \beta}g^{\lambda \nu} - g^{\alpha \mu}g^{\lambda \beta}$$

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and \( \tau^{\alpha\beta}_I \) denotes the pseudo-tensor of Einstein–Landau, defined by the following expression:

\[
2\chi (-g) \tau^{\alpha\beta}_I \equiv \partial_\mu g^{\alpha\beta} \partial_\nu g^{\rho\sigma} - \partial_\mu g^{\alpha\rho} \partial_\nu g^{\beta\sigma} + \frac{1}{2} g^{\alpha\beta} \partial_\rho g^{\mu\nu} \partial_\sigma g^{\rho\sigma} + g_{\rho\sigma} \partial_\nu g^{\alpha\rho} \partial_\sigma g^{\beta\tau} + \frac{1}{8} \left( g^{\alpha\beta} g^{\rho\sigma} - 2 g^{\alpha\rho} g^{\beta\sigma} \right) (g_{\rho\sigma} \partial_\nu g^{\alpha\rho} - 2 g_{\rho\sigma} \partial_\sigma g^{\nu\rho} \partial_\nu g^{\beta\sigma}) \partial_\sigma g^{\mu\nu} \partial_\tau g^{\rho\sigma} \subset 87.
\]

Thorne’s approach consists firstly on introducing the deviation of the metric density respect to the Minkowski metric

\[
h^{\alpha\beta} \equiv g^{\alpha\beta} - \eta^{\alpha\beta},
\]

and then writing the Einstein equation (81) as follows:

\[
\square h^{\alpha\beta} + E^{\alpha\beta}_G = 2\chi T^{\alpha\beta} - E^{\alpha\beta}_\theta \equiv 2\chi T^{\alpha\beta}_\theta,
\]

\( \square \) being the flat D’Alembert operator, and the following notation is used:

\[
E^{\alpha\beta}_G \equiv g^{\alpha\beta} \partial_\mu h^{\mu\nu} - g^{\rho\sigma} \partial_\mu h^{\rho\sigma} - h^{3\beta} \partial_\mu h^{\mu\Lambda} + 2 \partial_\Lambda h^{\alpha\beta} \partial_\mu h^{\mu\nu} - \partial_\mu h^{\alpha\lambda} \partial_\nu h^{\beta\lambda}.
\]

\[
E^{\alpha\beta}_\theta \equiv h^{\alpha\beta} \partial_\mu h^{\mu\nu} - \partial_\mu h^{\alpha\beta} \partial_\nu h^{\beta\lambda}.
\]

By demanding the coordinates to be harmonic, i.e., satisfying \( \partial_i h^{i\Lambda} = 0 \), it turns out to be the following conclusion obtained:

\[
E^{\alpha\beta}_G = 0, \quad \square E^{\alpha\beta}_\theta = 0,
\]

and so, Einstein equation (85) are reduced to

\[
\square h^{\alpha\beta} = 2\chi T^{\alpha\beta}_\theta \quad \Rightarrow \quad \partial_\nu T^{\alpha\beta}_\theta = 0,
\]

where

\[
T^{\alpha\beta}_\theta \equiv T^{\alpha\beta} - \frac{1}{2\chi} \left( h^{\mu\nu} \partial_\mu h^{\alpha\beta} - \partial_\mu h^{\alpha\nu} \partial_\nu h^{\beta\lambda} \right),
\]

which represents an effective energy–momentum pseudo-tensor (containing ‘matter’ and ‘field’) that is conserved in the ordinary sense (let us note that according to (85) and the condition of harmonic coordinates, both terms in (93) are divergence free). Subsequently Thorne defines the static multipoles as follows:

\[
M^{h^{i_1\ldots i_n}}_\theta = \int x^{i_1\ldots i_n} T^{\alpha\beta}_\theta \, d^3x = \frac{1}{2\chi} \int x^{i_1\ldots i_n} \Delta h^{00} \, d^3x,
\]

where integrals are extended to all space and \( \Delta \) is the flat Laplacian operator.

With this expression of Thorne we can proceed identically to the classic case of GGT and conclude with the following flux integral:

\[
M^{h^{i_1\ldots i_n}}_\theta = \frac{1}{2\chi} \oint_{\infty} \left( h^{i_1\ldots i_n} \partial_k h^{00} - h^{00} \partial_k x^{i_1\ldots i_n} \right) \, d\delta^k,
\]

having taken into account that this time \( \Delta x^{i_1\ldots i_n} = 0 \). If we restrict ourselves now to the axial symmetry case, then
which is virtually identical to (22) if we substitute the Newtonian potential $\Phi$ by the component $h^{00}$ of the deviation of the metric density. Now considering (23) an expression similar to (24) is obtained:

$$M_n^{th} = \frac{(2n-1)!!}{n!} \varepsilon_{i_1i_2...i_n} M_{i_1i_2...i_n}^{th},$$

$$= 1 \frac{1}{2} \int d\sigma \{ r^p P_n^{(p)}(\cos\theta) \partial_p h^{00} - h^{00} \partial_p [r^p P_n^{(p)}(\cos\theta)] \},$$

(96)

where once again the circle at the sign of integration means that the limit $r \to \infty$ must be taken.

To calculate the integral (97) we must use the following structure of $h^{00}$ (obtained in the appendix):

$$h^{00} = -4 \sum_{q=0}^{\infty} \frac{M_q}{r^{q+1}} P_r^{(q)}(\cos\theta) + \sum_{q=0}^{\infty} \frac{1}{r^{q+2}} X^{(q)}(\cos\theta),$$

(98)

where $X^{(q)}(\cos\theta)$ represents a polynomial of degree $q$. From here we obtain the following result:

$$r \partial_r h^{00} - n h^{00} = 4 \sum_{q=0}^{\infty} \frac{(q+1+n)M_q}{r^{q+1}} P_r^{(q)}(\cos\theta) - \sum_{q=0}^{\infty} \frac{q+2+n}{r^{q+2}} X^{(q)}(\cos\theta),$$

(99)

which inserted in (97) leads to the desired equality $M_n^{th} = M_n$, since the second summation yields a zero integral (by a similar reasoning as in the classic case described in section 5).

### 7. Conclusions

In this work we have described what we call GGT in classical gravitation, a virtually unknown result in standard bibliography. As we have seen, it allows to express the multipole moments of the source of the gravitational field as fluxes across any surface containing it. This generalises the GT, which allows one to write the mass of the object as the field flux across any enclosing surface.

We have shown that the definition of Komar mass (35) can be understood, at least in the static case, as a relativistic generalisation of the GT. Indeed, the natural introduction of the mass (46), via the Einstein equation (45), allows one to identify it with the mass of Komar by means of the flux integral (48).

In our opinion the most important aspect of this work is the generalisation of the previous result. That is, we have defined the static RMMs as integrals over all space which can then be identified with the moments of Thorne (or Geroch) by means of flux integrals across the surface of infinity. Flux integrals are calculated in associated asymptotically cartesian harmonic polar coordinates, and the structure of the Weyl metrics in these coordinates, obtained by the authors in previous articles, is used.

Finally we have reviewed, in the static case, the approach of the multipole moments of Thorne. As it is known, they are defined as integrals over all the space involving the time component of a pseudo-tensor $T_{\mu \nu}^{\text{th}}$ which is divergence free, provided that harmonic coordinates are used. As the starting equation possess a Minkowskian character, it is almost immediate that the application of the GGT leads to flux integrals that reproduce the asymptotic multipoles.
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Appendix. Approximate Weyl metrics in ‘spherical harmonic coordinates’

Weyl metrics are the most general solution of vacuum axially symmetric Einstein equations for the static case. In ‘Canonical spherical coordinates’ of Weyl \{t, \tilde{r}, \theta, \varphi\} these metrics are written as follows:

\[ ds^2 = -e^{2\beta} dr^2 + e^{-2\beta} [e^{2\Psi}(dr^2 + r^2 d\theta^2) + \beta^2 r^2 \sin^2 \theta \, d\varphi^2], \]  

(100)

\( \beta, \Psi \) y \( \Gamma \), are functions of \( (\tilde{r}, \tilde{\theta}) \), given by the following expressions:

\[
\begin{align*}
\beta(\tilde{r}, \tilde{\theta}) &= 1, \quad \Psi(\tilde{r}, \tilde{\theta}) = \sum_{n=0}^{\infty} \frac{\alpha_n}{\tilde{r}^{n+1}} P_n(\cos \tilde{\theta}) \\
\Gamma(\tilde{r}, \tilde{\theta}) &= \sum_{n,k=0}^{\infty} \frac{(n+1)(k+1)}{n+k+2} \frac{\alpha_n \alpha_k}{\tilde{r}^{n+k+2}} (P_{n+1}P_{k+1} - P_nP_k),
\end{align*}
\]

(101)

where the argument of the Legendre polynomials in the second formula is \( \cos \tilde{\theta} \). We have included the function \( \beta = 1 \) to indicate that, in general, inside of the source this function is not trivial (see for instance [10, 11]).

In [14] the relativistic moments \( M_n \) of these metrics were obtained in terms of the Weyl constants \( \alpha_n \) up to a fairly high order. The procedure used to do that relies primarily on the definition of Thorne, carrying out the following steps. First of all, spherical coordinates \{t, r, \theta, \varphi\} associated to asymptotically cartesian harmonic ones are determined. Then, the moments \( M_n \) are the coefficients corresponding to the terms like

\[
\frac{1}{r^{n+1}} P_n(\cos \theta)
\]

(102)

arising in the series expansion of the component \( g_{tr} \) of the metric.

The calculation of the harmonic coordinates \{t, r, \theta, \varphi\} is approximate, so they are provided by series expansion in the inverse of the radial coordinate \( r \) up to a high order. These expressions allow us to write the components of the metric up to order \( 1/r^3 \) (those expressions are shown at the end of this appendix), and the line element is taken in the following form:

\[
\begin{align*}
ds^2 &= \gamma_t dt^2 + \gamma_r dr^2 + 2 \gamma_{t\theta} dtd\theta + \gamma_{\theta\theta} (d\theta)^2 + \gamma_{\varphi\varphi} (r \sin \theta d\varphi)^2,
\end{align*}
\]

(103)

i.e., the ‘euclidean orthonormal co-basis’ (60) has been used.

As can be seen, the components of the metric have two distinguished types of terms; the first ones reproduce the Newtonian multipole expansion and the second terms (summations involving the notation \( \mathcal{R}^{\ell\theta} \)) contain the so-called rests of Thorne, with the following structure:
\[\gamma_n = -1 + 2 \sum_{q=0}^{\infty} \frac{M_q}{r^{q+1}} P_q(w) + 2 \sum_{q=0}^{\infty} \frac{1}{r^{q+2}} R_{nq}^{(q)}(w),\]

\[\gamma_i = 1 + 2 \sum_{q=0}^{\infty} \frac{M_q}{r^{q+1}} P_q(w) + 2 \sum_{q=0}^{\infty} \frac{1}{r^{q+2}} R_{iq}^{(q)}(w),\]

\[\gamma_\theta = \sum_{q=0}^{\infty} \frac{1}{r^{q+3}} R_{\theta q}^{(q)}(w),\]

where \(i = \{r, \theta, \varphi\}\) and \(w \equiv \cos \theta\). The rests of Thorne \(R_{nq}^{(q)}(w)\) and \(R_{iq}^{(q)}(w)\) are polynomials of degree \(q\) in the variable \(w\) and they appear in this expansion as certain combinations of the Legendre polynomials (from Weyl series (101)):

\[R_{nq}^{(q)}(w) \equiv \sum_{s=0}^{q} A_{nq}^{qs} P_s(w),\]

\[R_{iq}^{(q)}(w) \equiv \sum_{s=0}^{q} A_{iq}^{qs} P_s(w).\]

Regarding to the rests \(R_{\theta q}^{(q)}(w)\) they appear as certain combinations of the associated Legendre functions of first kind \(P_{nq}(w)\)

\[R_{\theta q}^{(q)}(w) \equiv \sum_{s=0}^{q} A_{\theta q}^{qs} P_{s+1}(w),\]

and so they are polynomials of degree \(q\) in the variable \(w\) multiplied by a factor \(\sin \theta\), i.e.

\[R_{\theta q}^{(q)}(w) = \sqrt{1 - w^2} Q_{nq}^{(q)}(w).\]

Let us note that the absence of odd degree polynomials (as well as associated functions) is due to the equatorial symmetry considered in our case.

\[
\gamma_n = -1 + \frac{2M_0}{r} + \frac{2M_7}{r^3} P_2 + \frac{2M_4}{r^5} P_4 + \frac{2M_6}{r^7} P_6 + \frac{2M_8}{r^9} P_8
\]

\[- \frac{2}{r^2} M_0^2 + \frac{2}{r^3} M_0^3 - \frac{2}{r^4} M_0 (M_0^3 + 2M_2 P_2) + \frac{2}{r^5} M_0^2 \left( M_0^3 + \frac{22}{7} M_2 P_2 \right)\]

\[- \frac{2}{r^6} \left[ M_0^6 + \frac{1}{5} M_2^2 + \frac{2}{7} M_2 (15M_0^3 + M_2 P_2) + 2 \left( \frac{9}{35} M_2^2 + M_0 M_4 \right) P_2 \right]\]

\[+ \frac{2}{r^7} \left[ \frac{71}{105} M_0 M_2^2 + \frac{1}{21} M_0 M_2 (115M_0^3 + 17M_2) P_2 \right]\]

\[+ \frac{3}{11} M_0 \left( \frac{166}{35} M_2^2 + 13M_0 M_4 \right) P_2\]

\[+ \frac{2}{r^2} \left[ M_0^8 + \frac{148}{105} M_0^6 M_2^2 + 2M_2 \left( \frac{10}{3} M_0^5 + \frac{125}{147} M_0^3 M_2^2 + \frac{2}{7} M_4 \right) P_2 \right].\]
\[\gamma_\theta = -\frac{2}{3r^4} M_0 M_2 P_{21} - \frac{4}{3r^5} M_0^2 M_2 P_{21} + \frac{1}{r^6} \left[ \frac{4}{21} M_2 (-8 M_0^2 + M_2) P_{21} - \frac{1}{5} \left( \frac{9}{7} M_0^2 + 2 M_0 M_4 \right) P_{21} \right] \]
\begin{align}
+ \frac{2}{r^7} \left[ \frac{4}{7} M_0 M_2 \left( -3 M_0^3 + \frac{1}{3} M_2 \right) P_{21} - M_0 \left( \frac{3}{7} M_2^2 + \frac{2}{5} M_0 M_4 \right) P_{01} \right] \\
+ \frac{1}{r^8} \left[ \frac{1}{21} M_0 \left( -38 M_0^6 + \frac{943}{700} M_0^2 M_2 + \frac{4}{3} M_4 \right) P_{21} \\
+ \left( -\frac{15782}{13475} M_0^3 M_2^2 - \frac{62}{55} M_0^3 M_4 + \frac{20}{77} M_2 M_4 \right) P_{41} \\
- \frac{2}{7} \left( \frac{145}{99} M_2 M_4 + M_0 M_6 \right) P_{01} \right] \\
+ \frac{1}{r^9} \left[ -\frac{1}{21} M_2 \left( 40 M_0^6 + \frac{151}{50} M_0^3 M_2 + 4 M_2^3 \right) P_{21} \\
+ \frac{2}{11} \left( -\frac{1571}{175} M_0^3 M_2^2 - \frac{3}{35} M_2^3 - 8 M_0^3 M_4 + \frac{8}{5} M_0 M_3 M_4 \right) P_{41} \\
- \frac{4}{7} \left( \frac{3}{11} M_3^2 + \frac{74}{33} M_0 M_2 M_4 + M_2^2 M_6 \right) P_{01} \right].
\end{align}

\texttt{\textbullet \; \gamma_{\phi \phi} = 1 + \frac{2 M_0}{r} + \frac{2 M_2}{r^3} P_2 + \frac{2 M_4}{r^5} P_4 + \frac{2 M_6}{r^7} P_6 + \frac{2 M_8}{r^9} P_8 \\
+ \frac{1}{r^7} M_0^2 + \frac{1}{3 r^8} M_0 M_2 (1 + 5 P_2) + \frac{4}{5 r^8} M_0^2 M_2 \left( 1 - \frac{4}{7} P_2 \right) \\
+ \frac{1}{r^9} \left[ \frac{13}{21} M_0^3 M_2 + \frac{4}{15} M_2^2 + \frac{2}{15} M_0 M_4 \right. \\
- \frac{1}{3} M_0 \left( \frac{5}{7} M_0^2 M_2 - \frac{19}{7} M_2^2 - 2 M_0 M_4 \right) P_2 - \frac{6}{5} \left( \frac{1}{7} M_2^2 - M_0 M_4 \right) P_4 \\
+ \frac{2}{7} \left[ \frac{2}{105} M_0^3 M_2 + \frac{8}{15} M_0 M_2^2 + \frac{2}{15} M_0^2 M_4 \right. \\
- \frac{1}{3} M_0 \left( \frac{5}{7} M_0^2 M_2 - \frac{17}{7} M_2^2 - 2 M_0 M_4 \right) P_2 \\
\left. - \frac{1}{11} M_0 \left( \frac{426}{35} M_2^2 + \frac{14}{5} M_0 M_4 \right) P_4 \right] \\
+ \frac{1}{r^7} \left[ \frac{11}{21} M_0^3 M_2 + \frac{1651}{4620} M_0^2 M_2^2 + \frac{52}{165} M_0^2 M_4 \right. \\
+ \frac{1}{42} M_2 M_4 + \frac{1}{14} M_0 M_6 \\
+ \left( -\frac{3}{7} M_0^3 M_2 + \frac{76843}{40425} M_0^2 M_2^2 + \frac{52}{33} M_0^3 M_4 + \frac{37}{42} M_2 M_4 + \frac{5}{14} M_0 M_6 \right) P_2 \\
+ \left( -\frac{2952}{1225} M_0^2 M_2^2 - \frac{4}{5} M_0^3 M_4 + \frac{317}{154} M_2 M_4 + \frac{9}{14} M_0 M_6 \right) P_4 \\
+ \left( -\frac{445}{462} M_2 M_4 + \frac{13}{14} M_0 M_6 \right) P_6 \right].
\end{align}
\[ + \frac{1}{r^9} \left[ \frac{10}{21} M_0^6 M_2 + \frac{879}{1925} M_0^3 M_2^2 + \frac{2}{21} M_2^3 + \frac{4}{11} M_2^4 M_4 \right. \\
+ \frac{11}{35} M_0 M_2 M_4 + \frac{1}{7} M_0^2 M_6 \\
+ \left( - \frac{100}{231} M_0^6 M_2 + \frac{161383}{80850} M_0^3 M_2^2 + \frac{4}{231} M_2^3 + \frac{20}{11} M_2^4 M_4 \right. \\
+ \frac{97}{77} M_0 M_2 M_4 + \frac{5}{7} M_0^2 M_6 \right] P_2 \\
+ \left( - \frac{481934}{175175} M_0^3 M_2^2 + \frac{300}{1001} M_2^3 - \frac{222}{143} M_2^4 M_4 \right. \\
+ \frac{17887}{5005} M_0 M_2 M_4 + \frac{9}{7} M_2^2 M_6 \right] P_4 \\
- \frac{1}{7} \left( \frac{40}{11} M_2^3 + \frac{461}{11} M_0 M_2 M_4 + M_0^2 M_6 \right) P_6. \]  

(111)

\[ \phi^{\nu\nu} = 1 + \frac{2M_0}{r} + \frac{2M_2}{r^3} P_2 + \frac{2M_4}{r^5} P_4 + \frac{2M_6}{r^7} P_6 + \frac{2M_8}{r^9} P_8 \]

\[ + \frac{1}{r^9} \left[ - \frac{13}{21} M_0 M_2 (1 - P_2) - \frac{2}{3} M_2^3 M_2 \left( 1 - \frac{10}{7} P_2 \right) \right. \\
+ \frac{1}{r^6} \left[ \frac{19}{21} M_0^3 M_2 + \frac{3}{7} M_2^2 - \frac{2}{15} M_0 M_4 \right] P_2 + \frac{2}{5} \left( \frac{3}{7} M_2^2 + 7 M_0 M_4 \right) P_4 \right] \\
+ \frac{2}{r^7} \left[ - \frac{2}{7} M_0^3 M_2 + \frac{12}{35} M_0 M_2^2 - \frac{2}{15} M_2^3 M_4 \right] P_2 \\
+ M_0 \left[ \frac{1}{3} M_0^3 M_2 - \frac{1}{7} M_2^2 - \frac{2}{3} M_0 M_4 \right] P_2 - \frac{2}{11} M_0 \left( \frac{81}{35} M_2^2 + \frac{37}{5} M_0 M_4 \right) P_0 \right] \\
+ \frac{2}{r^8} \left[ - \frac{11}{21} M_0^5 M_2 + \frac{3227}{3300} M_0^3 M_2^2 - \frac{52}{165} M_0^3 M_4 - \frac{1}{42} M_2 M_4 - \frac{1}{14} M_0 M_6 \right] P_2 \\
+ \left[ \frac{13}{21} M_0^5 M_2 + \frac{5039}{16170} M_0^3 M_2^2 - \frac{52}{33} M_0^3 M_4 + \frac{43}{42} M_2 M_4 - \frac{5}{14} M_0 M_6 \right] P_2 \\
+ \left. \left( - \frac{2204}{2625} M_0^3 M_2^2 + \frac{164}{35} M_0^3 M_4 + \frac{9}{14} M_2 M_4 - \frac{9}{14} M_0 M_6 \right) P_0 \right] \\
+ \frac{1}{14} (5 M_2 M_4 + 43 M_0 M_6) P_0 \right] \\
+ \frac{1}{r^9} \left[ - \frac{10}{21} M_0^6 M_2 + \frac{6724}{5775} M_0^3 M_2^2 - \frac{2}{21} M_2^3 - \frac{4}{11} M_2^4 M_4 - \frac{11}{35} M_0 M_2 - \frac{1}{7} M_0^2 M_6 \right] \\
+ \frac{1}{14} (5 M_2 M_4 + 43 M_0 M_6) P_0 \right].
\[
\begin{aligned}
&+ \left( \frac{40}{77} M_0^6 M_2 - \frac{42379}{80850} M_0^3 M_2^3 + \frac{16}{77} M_0^3 M_4^2 - \frac{20}{11} M_0^4 M_2^4 - \frac{5}{7} M_0^2 M_0^2 M_2 \right) P_2 \\
&+ \left[ \frac{164422}{175175} M_0^2 M_2^2 - \frac{12}{1001} M_0^2 M_4^2 + \frac{402}{143} M_0^4 M_2^4 - \frac{51}{65} M_0^2 M_2 M_4 - \frac{9}{7} M_0^3 M_0 M_2 \right] P_4 \\
&+ \frac{1}{7} \left[ -\frac{16}{11} M_2^3 - 9 M_0^2 M_2 M_4 + 29 M_0^2 M_2 \right] [P_2].
\end{aligned}
\] (112)

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