The limitations of the Poincaré inequality

Derek W. Robinson¹ and Adam Sikora²

May 2013

Abstract

We examine the validity of the Poincaré inequality for degenerate, second-order, elliptic operators \( H \) in divergence form on \( L^2(\mathbb{R}^n \times \mathbb{R}^m) \). We assume the coefficients are real symmetric and \( a_1 H_\delta \geq H \geq a_2 H_\delta \) for some \( a_1, a_2 > 0 \) where \( H_\delta \) is a generalized Grušin operator,

\[
H_\delta = -\nabla_{x_1} |x_1|^{(2\delta_1, 2\delta'_1)} \nabla_{x_1} - |x_1|^{(2\delta_2, 2\delta'_2)} \nabla_{x_2}^2.
\]

Here \( x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^m, \delta_1, \delta'_1 \in [0, 1), \delta_2, \delta'_2 \geq 0 \) and \( |x_1|^{(2\delta, 2\delta')} = |x_1|^{2\delta} \) if \( |x_1| \leq 1 \) and \( |x_1|^{(2\delta, 2\delta')} = |x_1|^{2\delta'} \) if \( |x_1| \geq 1 \).

We prove that the Poincaré inequality, formulated in terms of the Riemannian geometry corresponding to \( H \), is valid if \( n \geq 2 \), or if \( n = 1 \) and \( \delta_1 \lor \delta'_1 \in [0, 1/2) \) but it fails if \( n = 1 \) and \( \delta_1 \lor \delta'_1 \in [1/2, 1) \). The failure is caused by the leading term. If \( \delta_1 \in [1/2, 1) \) it is an effect of the local degeneracy \( |x_1|^{2\delta} \) but if \( \delta_1 \in [0, 1/2) \) and \( \delta'_1 \in [1/2, 1) \) it is an effect of the growth at infinity of \( |x_1|^{2\delta'_1} \).

If \( n = 1 \) and \( \delta_1 \in [1/2, 1) \) then the semigroup \( S \) generated by the Friedrichs’ extension of \( H \) is not ergodic. The subspaces \( x_1 \geq 0 \) and \( x_1 \leq 0 \) are \( S \)-invariant and the Poincaré inequality is valid on each of these subspaces. If, however, \( n = 1, \delta_1 \in [0, 1/2) \) and \( \delta'_1 \in [1/2, 1) \) then the semigroup \( S \) is ergodic but the Poincaré inequality is only valid locally.

Finally we discuss the implication of these results for the kernel of the semigroup \( S \).

AMS Subject Classification: 35J70, 35H20, 35L05, 58J35.

1. Centre for Mathematics and its Applications
   Mathematical Sciences Institute
   Australian National University
   Canberra, ACT 0200
   Australia
   derek.robinson@anu.edu.au

2. Department of Mathematics
   Macquarie University
   Sydney, NSW 2109
   Australia
   sikora@ics.mq.edu.au
1 Introduction

In this paper we continue the analysis of the class of degenerate elliptic operators in divergence form introduced in [RS08]. The evolution determined by these operators describes diffusion around and across the surface in $\mathbb{R}^d$ on which the corresponding flows are degenerate. If the degeneracy surface has codimension one then several interesting phenomena can occur depending on the nature of the degeneracy. In [RS08] it was established that for sufficiently strong degeneracy ergodicity can fail; the diffusion can have non-trivial invariant subspaces. In this situation discontinuous and non-Gaussian behaviour occurs. In this paper we will demonstrate that non-Gaussian behaviour can also occur even in the ergodic situation; the heat kernel satisfies Gaussian upper bounds but the matching lower bounds are not necessarily valid. We will, however, establish continuity properties and Gaussian bounds for most situations by combining the results of [RS08] with the criteria of Grigory’ an [Gri92] and Saloff-Coste [SC92a, SC92b, SC95]. These authors show that Gaussian upper and lower bounds follow from two geometric properties, the Poincaré inequality and volume doubling. The crucial feature is that the latter properties are equivalent to the parabolic Harnack inequality of Moser [Mos64]. Since the volume doubling property was established for the class of operators we consider by [RS08], Corollary 5.2, the validity of lower Gaussian bounds hinges on the Poincaré inequality. The latter property is the main focus of the subsequent analysis. We demonstrate that the validity of the Poincaré inequality is dependent on the order of degeneracy.

The operators we examine are formally expressed on $\mathbb{R}^d$ by

$$H = -\sum_{i,j=1}^{d} \partial_i c_{ij} \partial_j,$$

where $\partial_i = \partial/\partial x_i$, the $c_{ij}$ are real-valued measurable functions and the coefficient matrix $C = (c_{ij})$ is symmetric and positive-definite almost-everywhere. We assume that $d = n + m$ and adopt the notation $x = (x_1, x_2)$ with $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. Further we assume that $C \sim C_\delta$ where $C_\delta$ is a block diagonal matrix, $C_\delta(x) = |x_1|^{(2\delta_1, 2\delta'_1)} I_n + |x_1|^{(2\delta_2, 2\delta'_2)} I_m$, with $I_n$ and $I_m$ the identity matrices on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The indices $\delta_1, \delta_2, \delta'_1, \delta'_2$ are all non-negative, $\delta_1, \delta'_1 < 1$ and we use the notation, introduced in [RS08], that $a^{(\alpha, \alpha')} = a^{\alpha}$ if $a \leq 1$ and $a^{(\alpha, \alpha')} = a^{\alpha'}$ if $a \geq 1$. Moreover, the equivalence relation $f \sim g$ for two functions $f, g$ with values in a real ordered space indicates that there are $a, a' > 0$ such that $af \leq g \leq a'f$.

The operators are defined more precisely through the quadratic forms $h$ and $h_\delta$ given by $D(h) = C^{-1}_c(\mathbb{R}^d) = D(h_\delta),$

$$h(\varphi) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^{n+m}} dx \, c_{ij}(x) (\partial_i \varphi)(x) (\partial_j \varphi)(x)$$

and

$$h_\delta(\varphi) = \int_{\mathbb{R}^d} dx \, |x_1|^{(2\delta_1, 2\delta'_1)} ((\nabla x_1 \varphi)(x))^2 + \int_{\mathbb{R}^d} dx \, |x_1|^{(2\delta_2, 2\delta'_2)} ((\nabla x_2 \varphi)(x))^2.$$

It follows that $h_\delta$ is closable (see, for example, [MR92] Section II.2) and since $h \sim h_\delta$, in the sense of ordering of quadratic forms, $h$ is also closable. But by standard arguments
the degenerate elliptic operator $H$ and $H_\delta$ are Dirichlet forms. Then $H$ and $H_\delta$ are defined as the positive self-adjoint operators associated with these Dirichlet forms. Formally $H_\delta$ is given by

$$H_\delta = -\nabla x_1 |x_1|^{(2\delta_1, 2\delta_2)} \nabla x_1 - |x_1|^{(2\delta_2, 2\delta_2)} \nabla^2$$

(2)

and $H_\delta \sim H$ in the sense of ordering of positive self-adjoint operators. In the analysis of the degenerate elliptic operator $H$ the comparison operator $H_\delta$ plays a role analogous to that of the Laplacian in the framework of strongly elliptic operators. The first problem is to establish properties of the operators $H_\delta$ and the second problem is to show that the equivalence relation $H_\delta \sim H$ implies that these properties are shared by $H$.

The Poincaré inequality is formulated in terms of the Riemannian geometry defined by the metric $C^{-1}$. The Riemannian distance $d(\cdot ; \cdot)$ can be defined in several equivalent ways. In particular

$$d(x; y) = \sup\{\psi(x) - \psi(y) : \psi \in C^1(\mathbb{R}^d), \Gamma(\psi) \leq 1\}$$

(3)

for all $x, y \in \mathbb{R}^d$ where $\Gamma(\psi) = \sum_{i,j=1}^d c_{ij}(\partial_i \psi)(\partial_j \psi)$ denotes the carré du champ associated with $H$. Then the Riemannian ball $B(x; r)$ centred at $x \in \mathbb{R}^d$ with radius $r > 0$ is defined by $B(x; r) = \{y \in \mathbb{R}^d : d(x; y) < r\}$. The volume (Lebesgue measure) of the ball is denoted by $|B(x; r)|$. In addition if $n = 1$ we set $\Omega_+ = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^m : x_1 \geq 0\}$, $\Omega_- = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^m : x_1 \leq 0\}$ and then define ‘balls’ $B_\pm(x; r)$ by $B_\pm(x; r) = B(x; r) \cap \Omega_\pm$.

Our principal result is the following.

**Theorem 1.1** I. If $n \geq 2$, or if $n = 1$ and $\delta_1 \vee \delta_1' \in [0, 1/2]$, then there exist $\lambda > 0$ and $\kappa \in (0, 1]$ such that

$$\int_{B(x; r)} dy \Gamma(\varphi)(y) \geq \lambda r^{-2} \int_{B(x; \kappa r)} dy (\varphi(y) - \langle \varphi \rangle_B)^2$$

(4)

for all $x \in \mathbb{R}^{n+m}$, $r > 0$ and $\varphi \in C^1(\mathbb{R}^{n+m})$ where $\langle \varphi \rangle_B = |B(x; \kappa r)|^{-1} \int_{B(x; \kappa r)} dy \varphi(y)$.  

II. If $n = 1$ and $\delta_1 \vee \delta_1' \in [1/2, 1]$ then the uniform Poincaré inequality (I) fails.  

III. If $n = 1$ and $\delta_1 \in [1/2, 1]$ then there then then there exist $\lambda > 0$ and $\kappa \in (0, 1]$ such that

$$\int_{B_\pm(x; r)} dy \Gamma(\varphi)(y) \geq \lambda r^{-2} \int_{B_\pm(x; \kappa r)} dy (\varphi(y) - \langle \varphi \rangle_{B_\pm})^2$$

(5)

for all $x \in \Omega_\pm$, $r > 0$ and $\varphi \in C^1(\mathbb{R}^{1+m})$.

Theorem 1.1 will be established in Section 3 The proof is based on the straightforward observation in Section 2 that the Poincaré inequality (I) for $H$ is equivalent to the analogous inequality for $H_\delta$. This allows exploitation of the characterization derived in [RS08], Section 5, of the Riemannian geometry defined by the metric $C^{-1}_\delta$. Note that (I) and (II) are strong forms of the usual Poincaré inequality insofar they are uniform for balls of all position and all radii but they are weak forms insofar they involve a large ball $B(x; r)$ on the left hand side but a small ball $B(x; \kappa r)$ on the right hand side. It follows, however, from the work of Jerison [Jer86] (see also [Lu94]) that the weak form together with the volume doubling property implies the stronger version with $\kappa = 1$. 

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The failure of the Poincaré inequality in Case II has different origins in each of the cases $\delta_1 \in [1/2, 1)$ and $\delta'_1 \in [1/2, 1)$. In the first situation it is caused by the local degeneracy $|x_1|^{2\delta}$ of the leading term of $H_\delta$. Although the inequality fails on the whole space $\mathbb{R}^{n+m}$ it nevertheless holds on the subspaces $\Omega_\pm$ by the third statement of the theorem. If, however, $\delta_1 \in [0, 1/2)$ and $\delta'_1 \in [1/2, 1)$ then the inequality fails because of the growth at infinity of the coefficient $|x_1|^{2\delta}$ in the leading term. Nevertheless the inequality holds for all $R > 0$ and all $B(x; r)$ with $r \leq R$ but it does not hold uniformly for all $r$. It should be emphasized that all these conclusions are independent of the degeneracy parameters $\delta_2$, $\delta'_2$.

The Poincaré inequality is of relevance for the properties of the heat kernel corresponding to $H$ because in combination with the volume doubling property it implies both Hölder continuity and upper and lower Gaussian bounds. This observation follows from the work of Grigor’yan [Gri92] and Saloff-Coste [SC92a, SC92b, SC95] which extends and simplifies earlier arguments of Moser [Mos64, Mos71]. Since $H$ is defined by the Dirichlet form $\tilde{\mathcal{H}}$ it generates a self-adjoint submarkovian semigroup $S$ on $L_2(\mathbb{R}^n \times \mathbb{R}^m)$ which, by Proposition 3.1 of [RS08], is bounded as an operator from $L_1(\mathbb{R}^n \times \mathbb{R}^m)$ to $L_\infty(\mathbb{R}^n \times \mathbb{R}^m)$. Therefore $S$ is determined by a positive bounded integral kernel $K$. The first statement of Theorem 1.1 then leads to the following extension of the results obtained in [RS08].

**Theorem 1.2** Assume $n \geq 2$, or $n = 1$ and $\delta_1 \lor \delta'_1 \in [0, 1/2)$. Then the semigroup kernel $x, y \in \mathbb{R}^{n+m} \mapsto K_t(x; y)$ is jointly Hölder continuous and there exist $a, \omega, a', \omega' > 0$ such that

$$a |B(x; t^{1/2})|^{-1} e^{-\omega d(x; y)^2/t} \leq K_t(x; y) \leq a' |B(x; t^{1/2})|^{-1} e^{-\omega' d(x; y)^2/t}$$

for all $x, y \in \mathbb{R}^{n+m}$ and $t > 0$.

If $n = 1$ and $\delta_1 \in [1/2, 1)$ then it follows from Remark 6.9 of [RS08] that $S_t$ leaves the subspaces $L_2(\Omega_\pm)$ invariant, i.e. the semigroup is not ergodic. Consequently the kernel $K$ is discontinuous and $K_t(x; y) = 0$ if $x_1 > 0$ and $y_1 < 0$. Nevertheless the submarkovian semigroups $S_t^{(\pm)}$ obtained by restricting $S$ to $L_2(\Omega_\pm)$ have bounded, continuous, integral kernels $K^{(\pm)}$ which satisfy similar Gaussian bounds.

**Theorem 1.3** Assume $n = 1$ and $\delta_1 \in [1/2, 1)$. Then the kernels $x, y \in \Omega_\pm \mapsto K_t^{(\pm)}(x; y)$ are jointly Hölder continuous and there exist $a, \omega, a', \omega' > 0$ such that

$$a |B_{\pm}(x; t^{1/2})|^{-1} e^{-\omega d(x; y)^2/t} \leq K_t^{(\pm)}(x; y) \leq a' |B_{\pm}(x; t^{1/2})|^{-1} e^{-\omega' d(x; y)^2/t}$$

for all $x, y \in \Omega_\pm$ and $t > 0$.

The foregoing operators are related to two classes of degenerate operators which have previously received wide attention. First if $\delta_1 = \delta'_1 = 0$ then $H$ is equivalent to an operator

$$H_G = -\nabla_{x_1}^2 - c(x_1)\nabla_{x_2}^2.$$

Operators of this form, with $c(x_1) = x_1^{2k}$ and $k \in \mathbb{N}$, were introduced by Grušin [Gru70]. They are subelliptic operators of Hörmander type [Hör67] and clearly fall within the class we consider. The Poincaré inequality [41] was established for operators of the form $H_G$ by Franchi, Gutiérrez and Wheeden [FGW94]. These authors considered a wide class of coefficients $c$, including $c(x_1) \sim |x_1|^{(2\delta_2; 2\delta'_2)}$ with $\delta_2, \delta'_2 \geq 0$, but their methods are
completely different to the arguments we use for the operators $H$ and $H_\delta$. The distinctive feature of the latter operators is the presence of the coefficient $|x_1|^{(2\delta_2,2\delta_1)}$ in the leading term. If $n = 1$ these coefficients have a dramatic influence as evinced by Theorems 1.1 and 1.3. The resulting effects can be partially understood through the associated diffusion process. The operator $H_\delta$ has a degeneracy $|x_1|^{(\delta_2,\delta_1)}$ in the components of the underlying flow tangential to the hyperplane $x_1 = 0$ and an additional degeneracy $|x_1|^{(\delta_1,\delta_1')}$ in the normal component of the corresponding flow. If $\delta_1 \in [1/2, 1)$ the normal component of the flow, which is not present in the Grushin class, leads to an evolution which is non-ergodic. The corresponding diffusion separates into two distinct components $\Omega_{\pm}$ and the Poincaré inequality on $\mathbb{R}^{n+m}$ fails. The somewhat surprising conclusion is that (1) also fails for $n = 1$, $\delta_1 \in [0, 1/2)$ and $\delta_1' \in [1/2, 1)$ although the diffusion is ergodic. There is, however, an approximate failure of ergodicity. For example, if the one-dimensional diffusion process determined by $-d_x (1 \lor |x|) d_x$ begins at the right (left) of the origin then with large probability it diffuses to infinity on the right (left). Therefore the two half-lines, $x \geq 0$ and $x \leq 0$ are approximately invariant. This behaviour is analogous to diffusion on manifolds with ends [CF91] [BCF96] [Dav97] [GSC09] and leads to more complicated lower bounds. This will be discussed in Section 5.

A second class of degenerate operators considered by Trudinger [Tru73] and later by Fabes, Kenig and Serapioni [FKS82] expresses the degeneracy in terms of the largest and smallest eigenvalues $\mu_M$, $\mu_m$ of the coefficient matrix. Typically Poincaré and Harnack inequalities, Hölder continuity etc. follow from local integrability of $\mu_M$ and $\mu_m^{-1}$. These conditions place direct restraints on the order of the local degeneracy, e.g. for the operators $H$ under consideration the local integrability of $\mu_m^{-1}$ would require that $\delta_1 \lor \delta_2 < n/2$, and this limits the analysis to weakly degenerate operators. This type of condition not only rules out operators such as $H_\delta$ with $n = 1$ and $\delta_1 \in [1/2, 1)$ but also rules out simple examples such as the Heisenberg sublaplacian $H_{\text{Heis}} = -\partial_1^2 - (\partial_2 + x_1 \partial_3)^2$ on $L_2(\mathbb{R}^3)$ for which $\mu_m$ is identically zero.

## 2 Preliminaries

In this section we make some preliminary observations which simplify the subsequent discussion of the Poincaré inequality (1). We begin by recalling some standard consequences of equivalence properties and then we derive some approximate scaling estimates.

First we note that the Poincaré inequality for $H$ is equivalent to the Poincaré inequality for the comparison operator $H_\delta$. This equivalence follows by first remarking that both integrals in the Poincaré inequality (1) are monotonic functions of the radius of the ball $B = B(x; r)$. This is evident for the left hand integral $\int_B \Gamma(\varphi)$ but it is also true for the right hand integral since

$$\int_{B(x;r)} dy (\varphi(y) - \langle \varphi \rangle_B)^2 = \inf_{M \in \mathbb{R}} \int_{B(x;r)} dy (\varphi(y) - M)^2.$$  \hspace{1cm} (6)

This identification is a direct consequence of the identity

$$\int_B dy (\varphi(y) - M)^2 = \int_B dy (\varphi(y) - \langle \varphi \rangle_B)^2 + |B| (M - \langle \varphi \rangle_B)^2.$$  

Secondly, since $C \sim C_\delta$ one has $\Gamma(\varphi) \sim \Gamma_\delta(\varphi)$ where $\Gamma_\delta$ is the carré du champ associated with $H_\delta$. In particular if $a C \leq C_\delta \leq b C$ then $a \Gamma(\varphi) \leq \Gamma_\delta(\varphi) \leq b \Gamma(\varphi)$. Consequently, the
Riemannian distance $d(\cdot,\cdot)$ defined by $\Gamma$ is equivalent to the distance $d_\delta(\cdot,\cdot)$ defined by $\Gamma_\delta$. Specifically, $b^{-1/2}d(x;y) \leq d_\delta(x;y) \leq a^{-1/2}d(x;y)$ for all $x,y \in \mathbb{R}^{n+m}$. Therefore the corresponding balls $B,B_\delta$ satisfy

$$B_\delta(x;b^{-1/2}r) \subseteq B(x;r) \subseteq B_\delta(x;a^{-1/2}r)$$

for all $x \in \mathbb{R}^{n+m}$ and $r > 0$. The equivalence of the Poincaré inequalities for $H$ and $H_\delta$ now follows from combination of these remarks. For example, if (4) is valid then

$$\int_{B_\delta(x;r)} \Gamma_\delta(\varphi) \geq a \int_{B(x;r)} \Gamma(\varphi) \geq a \lambda r^{-2} \inf_{M \in \mathbb{R}} \int_{B(x;\kappa a^{-1/2}r)} (\varphi(y) - M)^2 \geq a \lambda r^{-2} \inf_{M \in \mathbb{R}} \int_{B_\delta(x;\kappa a^{-1/2}r)} (\varphi(y) - M)^2$$

i.e. the analogous inequality is valid for the operator $H_\delta$ but with $\lambda$ replaced by $a \lambda$ and $\kappa$ replaced by $(a/b)^{1/2} \kappa$. The converse implication follows by an identical argument.

Thirdly, one can replace the Riemannian distance $d_\delta(\cdot,\cdot)$ by any other equivalent distance without destroying the equivalence property of the Poincaré inequality. In Section 3 we use the distance function $D_\delta(\cdot,\cdot)$ introduced in Section 5 of [RS08]. This function is not strictly a distance as it does not satisfy the triangle inequality. Nevertheless $D_\delta(\cdot,\cdot)$ is equivalent to $d_\delta(\cdot,\cdot)$, by Proposition 5.1 of [RS08], and the triangle inequality is not used in the foregoing discussion of equivalence of the balls and the Poincaré inequalities.

A key method for analyzing differential operators is scaling. Clearly if $\delta_i = \delta'_i$ then $|t,x|^{(2\delta_i,2\delta'_i)} = |t,x|^{2\delta_i} = t^{2\delta_i}|x|^{(2\delta_i,2\delta'_i)}$ for all $t \geq 0$. Consequently simple scaling arguments can then be used to analyze $H_\delta$. If, however, $\delta_i \neq \delta'_i$ one no longer has a scaling identity. But one does have scaling estimates.

**Proposition 2.1** If $s,t > 0$ then

$$2^{-2(\delta \vee \delta')} \cdot s^{(\delta,\delta')} t^{(\delta \vee \delta',\delta \wedge \delta')} \leq (s t)^{(\delta,\delta')} \leq 2^{2(\delta \wedge \delta')} s^{(\delta,\delta')} t^{(\delta \vee \delta',\delta \wedge \delta')}$$

for all $\delta,\delta' \geq 0$.

**Proof** The proof can be established in two steps. First one argues that if $s > 0$ and $t \in (0,1]$ then

$$2^{-s' - \delta} s^{(\delta,\delta')} t^{s'} \quad 2^{-\delta} \quad \leq \quad (s t)^{(\delta,\delta')} \leq \quad \begin{cases} 2^{s'} s^{(\delta,\delta')} t^{\delta} & \text{if } \delta' \geq \delta \\ 2^{s' + \delta} s^{(\delta,\delta')} t^{\delta} & \text{if } \delta \geq \delta' \end{cases} \quad (7)$$

for all $\delta,\delta' \geq 0$. Secondly, if $s > 0$ and $t \geq 1$ then

$$2^{-s' - \delta} s^{(\delta,\delta')} t^{s'} \quad \leq \quad (s t)^{(\delta,\delta')} \leq \quad \begin{cases} 2^{s'} s^{(\delta,\delta')} t^{\delta} & \text{if } \delta' \geq \delta \\ 2^{s' + \delta} s^{(\delta,\delta')} t^{\delta} & \text{if } \delta \geq \delta' \end{cases} \quad (8)$$

for all $\delta,\delta' \geq 0$. The statement of the proposition is an immediate consequence of these bounds.
The proofs of (7) and (8) are elementary. First assume $s > 0$ and $t \in (0, 1]$ and consider (7). It is evident that $s^{(\delta, \delta')} = (s \wedge 1)^{\delta} (s \vee 1)^{\delta'}$. Consequently one has estimates 
\[ (s/(1+s))^{\delta} \leq (s \wedge 1)^{\delta} \leq 2^{\delta} (s/(1+s))^{\delta} \] 
and 
\[ 2^{-\delta'} (1+s)^{\delta-\delta} \leq s^{\delta}(1+s)^{\delta-\delta} \] 
for all $s > 0$ and all $\delta, \delta' > 0$. Then replacing $s$ with $st$ one obtains 
\[ (s t)^{\delta}(s t)^{\delta'}(1+s t)^{\delta-\delta}. \]
But if $\delta' \geq \delta$ then 
\[ (1+s t)^{\delta-\delta} \geq (t(1+s))^{\delta-\delta} \]
since $t \leq 1$. Therefore 
\[ (s t)^{\delta}(s t)^{\delta'}(t(1+s))^{\delta-\delta} \]
\[ = 2^{-\delta'} s^{\delta}(1+s)^{\delta-\delta} t^\delta \geq 2^{-\delta-\delta} s^{(\delta, \delta')} t^\delta \]
where the last estimate uses (9). Alternatively if $\delta' \leq \delta$ then 
\[ (1+s t)^{\delta-\delta} \geq (1+s)^{\delta-\delta}(1+t)^{\delta-\delta}. \]
Therefore 
\[ (s t)^{\delta}(s t)^{\delta'}(1+s)^{\delta-\delta}(1+t)^{\delta-\delta} \]
\[ = 2^{-\delta'} s^{\delta}(1+s)^{\delta-\delta} t^\delta(1+t)^{\delta-\delta} \geq 2^{-2\delta} s^{(\delta, \delta')} t^\delta \]
where the last estimate uses (9) and $t \leq 1$. Combining these conclusions gives the lower bound of (7). The upper bound follows analogously using the upper bound in (9). The proof of (8) is similar. We omit the details. \(\square\)

Finally we examine the action of the scaling semigroup $t > 0 \mapsto \sigma_t$ defined on $\mathbb{R}^{n+m}$ by 
\[ \sigma_t(x_1, x_2) = (t^{(\alpha, \alpha')} x_1, t^{(\beta, \beta')} x_2) \] 
\[ \alpha = (1 - \delta_1)^{-1}, \quad \alpha' = (1 - \delta'_1)^{-1}, \]
\[ \beta = (1 + \delta_2 - \delta_1) \alpha \quad \text{and} \quad \beta' = (1 + \delta'_2 - \delta'_1) \alpha'. \]
The semigroup acts by transposition on $L_2(\mathbb{R}^{n+m})$ and we denote the transpose action by $\tilde{\sigma}_t$. Thus 
\[ (\tilde{\sigma}_t \varphi)(x_1, x_2) = \varphi(t^{(\alpha, \alpha')} x_1, t^{(\beta, \beta')} x_2) \]
for all $\varphi \in L_2(\mathbb{R}^{n+m})$.

For orientation note that if $\delta_1 = \delta_1'$ and $\delta_2 = \delta_2'$ then $\alpha = \alpha'$, $\beta = \beta'$ and 
\[ \Gamma_{\delta}(\tilde{\sigma}_t \varphi)(x) = t^{2\alpha-2\delta_1} |\sigma_t(x_1)|^{\delta_1} |(\tilde{\sigma}_t \partial_{x_1} \varphi)(x)|^2 + t^{2\beta-2\delta_2} |\sigma_t(x_1)|^{\delta_2} |(\tilde{\sigma}_t \nabla_{x_2} \varphi)(x)|^2 \]
\[ = t^{2} (\tilde{\sigma}_t \Gamma_{\delta}(\varphi))(x). \]
This explains the choice of the scaling parameters. They are chosen to ensure that the carré du champ scales quadratically. The situation is more complicated if $\delta_i \neq \delta'_i$ because there is no exact scaling. Nevertheless the scaling semigroup has an approximate intertwining property.
Proposition 2.2 Let \( \hat{\Gamma} \) denote the carré du champ of the operator \( \hat{H}_\delta \) with coefficients \( |x_1|^{2(\delta_i,\delta_i',\delta_i,\delta_i')} \) and \( \tilde{\Gamma} \) the carré du champ of the operator \( \tilde{H}_\delta \) with coefficients \( |x_1|^{2(\delta_i,\delta_i',\delta_i,\delta_i')} \). Then
\[
2^{4\delta_M} t^2 (\tilde{\sigma}_t \tilde{\Gamma}_\delta(\varphi)) \geq \Gamma_\delta(\tilde{\sigma}_t \varphi) \geq 2^{-4\delta_M} t^2 (\tilde{\sigma}_t \hat{\Gamma}_\delta(\varphi))
\]
where \( \delta_M = \max\{\delta_1, \delta_1', \delta_2, \delta_2'\} \).

Proof It follows by the definition of the scaling semigroup that
\[
\Gamma_\delta(\tilde{\sigma}_t \varphi)(x) = t^{(2\alpha,2\alpha')} |x_1|^{(2\delta_i,2\delta_i')} |(\tilde{\sigma}_t \partial_{x_1} \varphi)(x)|^2
+ \ell(2\beta,2\beta') |x_1|^{(2\delta_i,2\delta_i')} |(\tilde{\sigma}_t \nabla_{x_2} \varphi)(x)|^2.
\]
(11)
But the lower bound in Proposition 2.1 gives
\[
|x_1|^{(2\delta_i,2\delta_i')} = \left( t^{(-\alpha,-\alpha')} |t^{(\alpha,\alpha')} x_1| \right)^{(2\delta_i,2\delta_i')}
\geq 2^{-4(\delta_i,\delta_i')} t^{(-2\alpha_i,-2\alpha_i')} |t^{(\alpha,\alpha')} x_1|^{2(\delta_i,\delta_i',\delta_i,\delta_i')}
\]
for both \( i = 1 \) and \( i = 2 \). Combination of these estimates then gives
\[
\Gamma_\delta(\tilde{\sigma}_t \varphi)(x) \geq 2^{-4(\delta_i,\delta_i')} t^{(2\alpha-2\delta_i,2\alpha-2\delta_i')} |\sigma_t(x_1)|^{2(\delta_i,\delta_i',\delta_i,\delta_i')} |(\tilde{\sigma}_t \partial_{x_1} \varphi)(x)|^2
+ 2^{-4(\delta_i,\delta_i')} t^{(2\beta-2\delta_i,2\beta-2\delta_i') |\sigma_t(x_1)|^{2(\delta_i,\delta_i',\delta_i,\delta_i')} |(\tilde{\sigma}_t \nabla_{x_2} \varphi)(x)|^2
\]
\[
\geq 2^{-4(\delta_i,\delta_i')} t^2 |\sigma_t(x_1)|^{2(\delta_i,\delta_i',\delta_i,\delta_i')} |(\tilde{\sigma}_t \partial_{x_1} \varphi)(x)|^2
+ 2^{-4(\delta_i,\delta_i')} t^2 |\sigma_t(x_1)|^{2(\delta_i,\delta_i',\delta_i,\delta_i')} |(\tilde{\sigma}_t \nabla_{x_2} \varphi)(x)|^2
\]
because \( 2\alpha - 2\delta_i \alpha = 2, 2\beta - 2\alpha_i \delta_i = 2 \) etc. Therefore
\[
\Gamma_\delta(\tilde{\sigma}_t \varphi)(x) \geq 2^{-4\delta_M} t^2 (\tilde{\sigma}_t \hat{\Gamma}_\delta(\varphi))(x).
\]
The upper bound is derived similarly but using the upper bound of Proposition 2.1 \( \square \)

3 Poincaré inequality

In this section we prove Theorem 1.1. The main onus of the proof consists of establishing the Poincaré inequality \( 1 \). The proof of the analogous inequality \( 5 \) for \( n = 1 \) and \( \delta_1 \in \left[ 1/2, 1 \right) \) is an almost direct consequence of the argument.

It follows from the discussion of equivalences in Section 2 that it suffices to prove the Poincaré inequality for the operator \( \hat{H}_\delta \). Then since \( \hat{H}_\delta \) is invariant under translations in the \( x_2 \)-directions it is sufficient to consider balls with centres \((x_1, 0)\). The proof will be broken down into three distinct cases. First we examine balls centred at the origin \((0, 0)\) and secondly balls that do not contain the origin. Finally we deduce the result for general balls from the two special cases.

Case I–Balls centred at the origin. This case is handled in three steps. First we derive the Poincaré inequality for a unit cube centred at the origin. Secondly we extend the result to parallelepipeds obtained by scaling the cube with the semigroup of scale transformations introduced in Section 2. Finally we establish embedding properties involving the balls and parallelepipeds which allow the deduction of the desired inequality for balls.
Proposition 3.1 Assume \( n \geq 2 \) or \( n = 1 \) and \( \delta_1 \in [0,1/2) \). Then there is a \( \lambda > 0 \) such that
\[
\int_{[-1,1]^{n+m}} dx \Gamma_\delta(\varphi)(x) \geq \lambda \int_{[-1,1]^{n+m}} dx \left( \varphi(x) - \langle \varphi \rangle \right)^2
\]  
for all \( \varphi \in C^1_c(\mathbb{R}^{n+m}) \) where \( \langle \varphi \rangle = 2^{-(n+m)} \int_{[-1,1]^{n+m}} dx \varphi(x) \).

Proof First since the proposition only involves \( x \) with \( |x_1| \leq 1 \) one can assume the coefficients of \( H_\delta \) are given by \( |x_1|^{2\delta} \). Thus we may assume that
\[
\Gamma_\delta(\varphi)(x) = |x_1|^{2\delta_1} ((\nabla_{x_1} \varphi(x))^2 + |x_1|^{2\delta_2}((\nabla_{x_2} \varphi)(x))^2).
\]
Secondly, the quadratic form
\[
\varphi \in C^\infty(\mathbb{R}^{n+m}) \mapsto h_\delta(\varphi) = \int_{[-1,1]^{n+m}} dx \Gamma_\delta(\varphi)(x)
\]
is closable (see, for example, [MR92] Section II.2a). Then by standard arguments the subspace \( \mathcal{D} \) of \( C^1_c(\mathbb{R}^{n+m}) \) consisting of functions whose normal derivative is zero on the boundary of the parallelepiped \([-1,1]^{n+m}\) is a core of \( \mathcal{T}_\delta \). (Formally the closure \( \mathcal{T}_\delta \) determines the self-adjoint version of the operator \( (2) \) corresponding to Neumann boundary conditions.) Hence for the first statement of the proposition it suffices to verify \( (12) \) on \( \mathcal{D} \).

Thirdly let \( \tilde{\varphi} \) denote the Fourier cosine-transform with respect to the \( x_2 \)-variables of \( \varphi \in \mathcal{D} \) and \( \tilde{\varphi}_{x_1} \) the cosine-transform of the gradient \( \nabla_{x_1} \varphi \). Then
\[
h_\delta(\varphi) = \sum_{k \in \mathbb{Z}^m} \int_{[-1,1]^n} dx_1 \left( |x_1|^{2\delta_1} (\tilde{\varphi}_{x_1}(x_1,k))^2 + (\pi k/2)^2 |x_1|^{2\delta_2} (\tilde{\varphi}(x_1,k))^2 \right)
\]
and
\[
\int_{[-1,1]^{n+m}} dx \left( \varphi(x) - \langle \varphi \rangle \right)^2 = \int_{[-1,1]^n} dx_1 \left( \tilde{\varphi}_{x_1}(x_1,0) - \langle \tilde{\varphi}_{x_1} \rangle_0 \right)^2 + \sum_{k \in \mathbb{Z}^m \setminus \{0\}} \int_{[-1,1]^n} dx_1 (\tilde{\varphi}(x_1,k))^2
\]
where
\[
\langle \varphi \rangle = 2^{-n-m} \int_{[-1,1]^{n+m}} dx \varphi(x) = 2^{-n} \int_{[-1,1]^n} dx_1 \tilde{\varphi}(x_1,0) = \langle \tilde{\varphi} \rangle_0.
\]
Therefore to establish \( (12) \) it suffices to prove that one can choose \( \lambda > 0 \) such that
\[
\int_{[-1,1]^n} dx_1 |x_1|^{2\delta_1} (\tilde{\varphi}_{x_1}(x_1,0))^2 \geq \lambda \int_{[-1,1]^n} dx_1 (\tilde{\varphi}(x_1,0) - \langle \tilde{\varphi} \rangle_0)^2
\]
and in addition
\[
\int_{[-1,1]^n} dx_1 \left( |x_1|^{2\delta_1} (\tilde{\varphi}_{x_1}(x_1,k))^2 + (\pi k/2)^2 |x_1|^{2\delta_2} (\tilde{\varphi}(x_1,k))^2 \right) \geq \lambda \int_{[-1,1]^n} dx_1 (\tilde{\varphi}(x_1,k))^2
\]
for all \( k \in \mathbb{Z}^m \setminus \{0\} \).

Fourthly, if for \( x \in \mathbb{R}^n \) and \( k \in \mathbb{Z}^m \) one defines \( \psi_k \) by setting \( \psi_k(x) = \tilde{\varphi}(x,k) \) then \( \psi \in C^1_c(\mathbb{R}^n) \). Therefore it now suffices to prove that there is a \( \lambda > 0 \) such that
\[
\int_{[-1,1]^n} dx |x|^{2\delta_1}((\nabla_x \psi)(x))^2 \geq \lambda \int_{[-1,1]^n} dx (\psi(x) - \langle \psi \rangle)^2
\]  
(13)
and, in addition,
\[ \tilde{h}_\delta(\psi) \geq \lambda \int_{[-1,1]^n} dx (\psi(x))^2 \]  
for all \( \psi \in C^1_c(\mathbb{R}^n) \) where \( \langle \psi \rangle = 2^{-n} \int_{[-1,1]^n} \psi \) and \( \tilde{h}_\delta \) denotes the form
\[
\tilde{h}_\delta(\psi) = \int_{[-1,1]^n} dx \left( |x|^{2\delta_1} (\nabla_x \psi(x))^2 + (\pi/2)^2 |x|^{2\delta_2}(\psi(x))^2 \right)
\](15)
on \( L_2([-1,1]^n) \) with domain given by the restrictions of \( C^1_c(\mathbb{R}^n) \) to the parallelepiped \([-1,1]^n\). These two properties are established by the following lemmas.

**Lemma 3.2** If \( n \geq 2 \) or \( n = 1 \) and \( \delta_1 \in [0,1/2) \) then one may choose \( \lambda > 0 \) such that (13) is valid.

**Proof** Let \( \tilde{H} \) denote the positive self-adjoint operator associated with the closure of the form \( \psi \in C^1_c(\mathbb{R}^n) \rightarrow \int_{[-1,1]^n} dx |x|^{2\delta_1}((\nabla_x \phi)(x))^2 \). It follows by standard arguments that \( \tilde{H} \) has compact resolvent. Now zero is an eigenvalue and if \( \varphi \) is a corresponding eigenfunction then \( \int_{[-1,1]^n} dx |x|^{2\delta_1}((\nabla_x \varphi)(x))^2 = 0 \). Therefore \( \varphi = 0 \) on the complement of the origin. Thus if \( n \geq 2 \) one must have \( \varphi = 0 \) and zero is a simple eigenvalue. But if \( n = 1 \) then \( \varphi \) is constant on \([-1,0)\) and on \((0,1]\) and the zero eigenvalue has multiplicity two.

If \( n \geq 2 \) it follows that there is a \( \lambda > 0 \) such that \( \tilde{H} \geq \lambda I \) on the orthogonal complement of the constants. But this is just an alternative formulation of (13).

If \( n = 1 \) the foregoing argument does not work and indeed (13) fails if \( \delta_1 \in [1/2,1) \). But if \( \delta_1 \in (0,1/2) \) and \( \psi \in C^1_c(\mathbb{R}) \) it follows from the Cauchy–Schwarz inequality that
\[
|\psi(x) - \psi(0)|^2 = \left| \int_0^x ds \, |s|^{-\delta_1} \left( |s|^{\delta_1} \psi'(s) \right)^2 \right|^2 
\leq \int_0^1 ds \, s^{-2\delta_1} \int_{s}^1 ds' \, |s'|^{2\delta_1}(\psi'(s'))^2 = (1 - 2\delta_1)^{-1} \int_{1}^1 ds \, |s|^{2\delta_1}(\psi'(s))^2
\]for all \( x \in [-1,1] \). Therefore, using (6), one has
\[
\int_{-1}^1 dx \, |x|^{2\delta_1}(\psi'(x))^2 \geq (1/2) \int_{-1}^1 dx \, (\psi(x) - \psi(0))^2 
\geq (1/2) \int_{-1}^1 dx \, (\psi(x) - \langle \psi \rangle)^2
\]and (13) is valid with \( \lambda = (1 - 2\delta_1)/2 \). \( \square \)

**Lemma 3.3** If \( n \geq 2 \) or \( n = 1 \) and \( \delta_1 \in [0,1/2) \) then one may choose \( \lambda > 0 \) such that (14) is valid.

**Proof** It follows from Lemma 3.2 that one may choose \( \lambda > 0 \) such that (13) is valid for all \( \psi \in C^1_c(\mathbb{R}^n) \). Set \( R_{l} = [-l, l]^n \) with \( l \in (0,1) \) and let \( \| \cdot \|_2 \) denote the \( L_2(R_{l}) \)-norm.

First assume \( \|\psi - \langle \psi \rangle\|_2 \geq \|\psi\|_2^2 / 4 \). Then it follows from (13) that \( \tilde{h}_\delta(\psi) \geq \lambda/4 \|\psi\|_2^2 \).

Secondly, assume \( \|\psi - \langle \psi \rangle\|_2 \leq \|\psi\|_2^2 / 4 \). Then
\[
\int_{R_{l}} dx \, |x|^{2\delta_2} |\psi(x)|^2 \geq \int_{R_{l} \setminus R_{l}} dx \, |\psi(x)|^2 = \int_{R_{l}} dx \, |\psi(x)|^2
\]
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for all \( l \in \langle 0, 1 \rangle \). But
\[
\int_{R_t} dx \, |\psi(x)|^2 \leq 2 \int_{R_t} |\psi(x) - \langle \psi \rangle|^2 + 2 \langle \psi \rangle^2 |R_t|
\]
\[
\leq 2 \|\psi - \langle \psi \rangle\|_2^2 + 2 \|\psi\|_2^2 \left( |R_t|/|R_1| \right) \leq 2 (1/4 + l^n) \|\psi\|_2^2 .
\]
Now combining the last two estimates and setting \( l = 1/8 \) one has
\[
\int_{R_1} dx \, |x|^{2\delta_2} |\psi(x)|^2 \geq (2^{-6\delta_2}/4) \|\psi\|_2^2 .
\]
Then it follows from (13) that \( \tilde{h}_\delta(\psi) \geq (\pi/2)^2 (2^{-6\delta_2}/4) \|\psi\|_2^2 . \)

One concludes that \( \tilde{h}_\delta(\psi) \geq \lambda_0 \|\psi\|_2^2 \) for all \( \psi \in C^1_c(\mathbb{R}^n) \) with \( \lambda_0 = (\lambda \wedge 2^{-6\delta_2})/4 . \) \( \Box \)

The statement of Proposition 3.1 now follows from Lemmas 3.2 and 3.3 by the discussion preceding the lemmas. \( \Box \)

Proposition 3.1 establishes the Poincaré inequality on the Euclidean cube \( C_1 = [-1, 1]^{n+m} \) and we now extend the result to more general cubes by the scaling transformation (10) introduced in Section 2. Let \( C_t = \sigma_t(C_1) \) for all \( t > 0 \). Explicitly
\[
C_t = \{ x \in \mathbb{R}^{n+m} : |x_1|_\infty < t^{(\alpha, \alpha')}, |x_2|_\infty < t^{(\beta, \beta')} \}
\]
\[
= \{ x \in \mathbb{R}^{n+m} : (|x_1|_\infty)^{(1-\delta_1, 1-\delta'_1)} < t, (|x_2|_\infty)^{(1-\gamma, 1-\gamma')} < t \}
\]
where \( \gamma = \delta_2(1+\delta_2 - \delta_1)^{-1}, \gamma' = \delta'_2(1+\delta'_2 - \delta'_1)^{-1} \) and \( |x_1|_\infty, |x_2|_\infty \) denote the \( l_\infty \)-norms of \( x_1 \in \mathbb{R}^n \) and \( x_2 \in \mathbb{R}^m \).

Next we apply Proposition 3.1 to the operator with the coefficients \( 2(\delta_1 \wedge \delta'_1, \delta_1 \wedge \delta'_1) \) to reach the following conclusion.

**Proposition 3.4** Assume \( n \geq 2 \) or \( n = 1 \) and \( \delta_1 \vee \delta'_1 \in [0, 1/2] \). Then there is a \( \lambda > 0 \) such that
\[
\int_{C_t} dx \, \Gamma_\delta(\varphi)(x) \geq \lambda t^{-2} \int_{C_t} dx \, (\varphi(x) - \langle \varphi \rangle)^2
\]
for all \( \varphi \in C^1_c(\mathbb{R}^{n+m}) \) and \( t > 0 \) where \( \langle \varphi \rangle = |C_1|^{-1} \int_{C_t} dx \, \varphi(x) . \)

**Proof** First by a change of coordinates \( x \in C_t \rightarrow y = \sigma_t(x) \in C_1 \) one has
\[
\int_{C_t} dx \, \Gamma_\delta(\varphi)(x) = J(t) \int_{C_1} dy \, (\tilde{\sigma}_t \Gamma_\delta(\varphi))(y)
\]
with \( J(t) = t^{(n a + m b, n a' + m b')} \) the Jacobian of the coordinate change. Secondly, it follows from the lower bound of Proposition 2.2 that \( (\tilde{\sigma}_t \Gamma_\delta(\varphi)) \geq 2^{-4\delta M} t^{-2} \tilde{\Gamma}_\delta(\tilde{\sigma}_t \varphi) . \) Therefore
\[
J(t) \int_{C_1} dy \, (\tilde{\sigma}_t \Gamma_\delta(\varphi))(y) \geq 2^{-4\delta M} J(t) t^{-2} \int_{C_1} dy \, \tilde{\Gamma}_\delta(\tilde{\sigma}_t \varphi)(y) .
\]
Thirdly, \( \tilde{\Gamma}_\delta \) is the carré du champ of the operator with coefficients \( |x_1|^{2(\delta_1 \vee \delta'_1, \delta \wedge \delta')} \). Then since \( n \geq 2 \) or \( \delta_1 \vee \delta'_1 \in [0, 1/2] \) if \( n = 1 \) one can apply the Poincaré inequality of Proposition 3.1 together with the identification (9) to deduce that
\[
\int_{C_1} dy \, \tilde{\Gamma}_\delta(\tilde{\sigma}_t \varphi)(y) \geq \lambda \inf_{M \in \mathbb{R}} \int_{C_1} dy \, ((\tilde{\sigma}_t \varphi)(y) - M)^2 .
\]
Therefore by combination of these observations and another coordinate change one finds
\[
\int_{C_t} dx \Gamma_\delta(\varphi)(x) \geq \lambda_\delta t^{-2} J(t) \inf_{M \in \mathbb{R}} \int_{C_t} dx \left((\tilde{\varphi}_t(x) - M)^2 \right)
\]
\[
= \lambda_\delta t^{-2} \inf_{M \in \mathbb{R}} \int_{C_t} dx \left(\varphi(x) - M\right)^2 = \lambda_\delta t^{-2} \int_{C_t} dx \left(\varphi(x) - \langle \varphi \rangle\right)^2
\]
for all \( t > 0 \) where \( \lambda_\delta = \lambda 2^{-4\delta M} \).
\[]

At this point we appeal to the discussion given in Section 5 of \[RS08\] of the Riemannian geometry defined by the metric \( C_{\delta}^{-1} \). The corresponding Riemannian distance \( d_\delta(\cdot;\cdot) \) is equivalent to the distance given by the function \( D_\delta(\cdot;\cdot) \) where
\[
D_\delta(x;y) = \|x_1 - y_1\|/(|x_1| + |y_1|)^{\delta_1,\delta_1'} + |x_2 - y_2| \left((|x_1| + |y_1|)^{\delta_2,\delta_2'} + (|x_2| + |y_2|)^{\gamma,\gamma'}\right)^{-1}
\]
with \( |x_i| \) the \( l_2 \)-norm of \( x_i \). In fact \( D_\delta(\cdot;\cdot) \) is not strictly a distance since it does not satisfy the triangle inequality but, as mentioned in Section 2, this does not affect the discussion of the Poincaré inequality. It suffices that \( D_\delta(\cdot;\cdot) \) is equivalent to the Riemannian distance.

Now we extend the Poincaré inequality of Proposition 3.4 from the parallelepipeds \( C_t \) to the centred balls \( B_\Delta(0;r) \) defined by the distance function \( D_\delta(\cdot;\cdot) \). The extension is based on the following two embedding lemmas.

**Lemma 3.5** \( C_t \subseteq B_\Delta(0;4(n+m)t) \) for all \( t > 0 \).

**Proof** First if \( x_1 \in \mathbb{R}^n \) then \( |x_1| \leq n^{1/2} |x_1|_1 \leq n |x_1|_\infty \). Therefore using Proposition 2.1 one has \( |x_1|^{(1-\delta_1,1-\delta_1')} \leq 4n (|x_1|_\infty)^{(1-\delta_1,1-\delta_1')} \). Similarly \( |x_2|^{(1-\gamma,1-\gamma')} \leq 4n (|x_2|_\infty)^{(1-\gamma,1-\gamma')} \) for \( x_2 \in \mathbb{R}^m \).

Secondly, it follows from the characterization \[10\] of \( C_t \) that
\[
C_t \subseteq \{ x \in \mathbb{R}^{n+m} : 4n (|x_1|_\infty)^{(1-\delta_1,1-\delta_1')} + 4m (|x_2|_\infty)^{(1-\gamma,1-\gamma')} < 4(n+m)t \}
\]
\[
\subseteq \{ x \in \mathbb{R}^{n+m} : |x_1|^{(1-\delta_1,1-\delta_1')} + |x_2|^{(1-\gamma,1-\gamma')} < 4(n+m)t \}
\]
\[
\subseteq \{ x \in \mathbb{R}^{n+m} : D_\delta(x;0) < 4(n+m)t \} = B_\Delta(0;4(n+m)t)
\]
for all \( t > 0 \) where we have used \( |x_2| \left(|x_1|^{(\delta_2,\delta_2')} + (|x_2|)^{\gamma,\gamma'}\right)^{-1} \leq (|x_2|)^{(1-\gamma,1-\gamma')} \).
\[]

**Lemma 3.6** There is a \( \kappa \in \langle 0,1 \rangle \) such that \( B_\Delta(0;\kappa t) \subseteq C_t \) for all \( t > 0 \).

**Proof** If \( x \in B_\Delta(0;t) \) then \( |x_1|^{(1-\delta_1,1-\delta_1')} < t \) and
\[
|x_2| < t \left(|x_1|^{(\delta_2,\delta_2')} + |x_2|^{(\gamma,\gamma')}\right).
\]
Therefore \( |x_1| < t^{(\alpha,\alpha')} \) and
\[
|x_2| < t^{(1+\alpha\delta_2,1+\alpha'\delta_2')} + t |x_2|^{(\gamma,\gamma')} = t^{(\beta,\beta')} + t |x_2|^{(\gamma,\gamma')}
\]
where \( \alpha, \alpha', \beta \) and \( \beta' \) are the parameters introduced in the definition \((10)\) of the scaling semigroup.

Now we consider the cases \( t \leq 1 \) and \( t \geq 1 \) separately.

First, if \( t \leq 1 \) then \( |x_1| \leq t^\alpha \). Moreover, \( |x_2| \leq t^\beta + t |x_2|^{(\gamma \vee \gamma')} \leq 1 + |x_2|^{(\gamma \vee \gamma')} \). Then since \( \gamma, \gamma' < 1 \) it follows that there is an \( a > 0 \) such that \( |x_2| \leq a \). There are two possibilities, \( a \leq 1 \) or \( a > 1 \). If \( a \leq 1 \) then \( |x_2|^{(\gamma \vee \gamma')} = |x_2|^{\gamma} \) and \( |x_2| \leq t^\beta + t |x_2|^{\gamma} \) or, equivalently,

\[
(|x_2|/t^\beta) \leq 1 + (|x_2|/t^\beta)^\gamma
\]

for all \( t \in [0, 1] \). Then since \( \gamma < 1 \) one can choose \( b > 0 \) such that \( |x_2| \leq bt^\beta \) for all \( t \leq 1 \). Alternatively if \( a > 1 \) then

\[
|x_2|^{(\gamma \vee \gamma')} = (a (|x_2|/a))^{(\gamma \vee \gamma')} \leq 2^{(\gamma \vee \gamma')} a^{\gamma \vee \gamma'} (|x_2|/a)^\gamma ,
\]

by the upper bound of Proposition \((2.1)\) and one now has

\[
(|x_2|/t^\beta) \leq 1 + a' (|x_2|/t^\beta)^\gamma
\]

with \( a' = 2^{(\gamma \vee \gamma')} a^{\gamma \vee \gamma' - \gamma} \). Therefore one again deduces a bound \( |x_2| \leq bt^\beta \) for all \( t \leq 1 \). Thus if \( \kappa \leq (1 + b)^{-\beta} \) then \( B_\Delta(0; \kappa t) \subseteq C_t \) for all \( t \leq 1 \).

Secondly suppose \( t \geq 1 \). Then it follows from \((19)\) that

\[
(|x_2|/t^\beta) \leq 1 + t^{1-\beta} |x_2|^{(\gamma \vee \gamma')}
\]

\[
= 1 + t^{\beta \gamma'} (t^\beta (|x_2|/t^\beta))^{(\gamma \vee \gamma')} \leq 1 + 2^{\gamma \vee \gamma'} (|x_2|/t^\beta)^{\gamma \vee \gamma'}
\]

by another application of the upper bounds of Proposition \((2.1)\). Since \( \gamma \vee \gamma' < 1 \) it follows that there is a \( b' > 0 \) such that \( |x_2| \leq b' t^\beta \) uniformly for all \( t \geq 1 \). But one also has the bound \( |x_1| < t^{\alpha'} \) for all \( t \geq 1 \). (This is evident if \( |x_1| \leq 1 \) but if \( |x_1| \geq 1 \) then \( |x_1|^{-\delta'_1} \leq t \) and the bound again follows.) Therefore one now concludes that if \( \kappa \leq (1 + b')^{\beta'} \) then \( B_\Delta(0; \kappa' t) \subseteq C_t \) for all \( t \geq 1 \). The statement of the lemma follows immediately.

The Poincaré inequality now extends to the balls \( B_\Delta \).

**Proposition 3.7** Assume \( n \geq 2 \) or \( n = 1 \) and \( \delta_1 \vee \delta'_1 \in [0, 1/2] \). Then there are \( \lambda_1 > 0 \) and \( \kappa_1 \in (0, 1] \) such that

\[
\int_{B_\Delta(0;r)} dx \, \Gamma_\delta(\varphi)(x) \geq \lambda_1 r^{-2} \int_{B_\Delta(0;\kappa_1 r)} dx \, (\varphi(x) - \langle \varphi \rangle)^2
\]

for all \( \varphi \in C_c^1(\mathbb{R}^{n+m}) \) and \( r > 0 \) where \( \langle \varphi \rangle \) is the average of \( \varphi \) over \( B_\Delta(0; \kappa_1 r) \).

**Proof** Set \( \hat{r} = r/(4(n + m)) \). It follows from Proposition \((3.4)\) together with Lemmas \((3.5)\) and \((3.6)\) that

\[
\int_{B_\Delta(0;r)} dx \, \Gamma_\delta(\varphi)(x) \geq \int_{C_r} dx \, \Gamma_\delta(\varphi)(x)
\]

\[
\geq \lambda \hat{r}^{-2} \inf_{M \in \mathbb{R}} \int_{C_r} dx \, (\varphi(x) - M)^2
\]

\[
\geq 16 (n + m)^2 \lambda r^{-2} \inf_{M \in \mathbb{R}} \int_{B_\Delta(0;\kappa_1 \hat{r})} dx \, (\varphi(x) - M)^2
\]

\[
= 16 (n + m)^2 \lambda r^{-2} \int_{B_\Delta(0;\kappa_1 r)} dx \, (\varphi(x) - \langle \varphi \rangle)^2
\]
which gives the desired conclusion with $\lambda_1 = 16(n + m)^2 \lambda$ and $\kappa_1 = \kappa/(4(n + m))$. \qed

The last proposition establishes the Poincaré inequality for the Riemannian balls $B_{\Delta}(0; r)$ for all $r > 0$.

**Case II—Balls not containing the origin.**

Next we consider balls $B_{\Delta}((\xi_1, 0); r)$ determined by the metric $D_\delta$ which do not contain the origin, i.e. balls with radius $r \leq D_\delta((\xi_1, 0); (0, 0))$. Our aim is to prove the following.

**Proposition 3.8** There are $\lambda_2 > 0$ and $\kappa_2 \in (0, 1]$ such that

$$\int_{B_{\Delta}((\xi_1, 0); r)} dx \Gamma_\delta(\varphi)(x) \geq \lambda_2 r^{-2} \int_{B_{\Delta}((\xi_1, 0); \kappa_2 r)} dx (\varphi(x) - \langle \varphi \rangle)^2 \quad (21)$$

for all $\varphi \in C^1_c(\mathbb{R}^{n+m})$ and $r \in (0, r_\xi]$ where $r_\xi = D_\delta((0, 0); (\xi_1, 0))$ and $\langle \varphi \rangle$ is the average of $\varphi$ over $B_{\Delta}((\xi_1, 0); \kappa_2 r)$.

The proof has several features in common with Case I. It relies in part on estimating on special sets which are are embedded in an appropriate manner in the Riemannian balls. These sets are defined for each $\xi_1 \in \mathbb{R}^n$ and $\kappa \in (0, 1]$ by

$$C(\xi; \kappa) = \{(x_1, x_2) : |x_1 - \xi_1| < (\kappa/2) r_\xi^{(\alpha, \alpha')}, |x_2| < (\kappa/2) r_\xi^{(\beta, \beta')} \}.$$

Thus $C(\xi; \kappa)$ is the product of an $n$-dimensional Euclidean ball centred at $\xi_1$ and an $m$-dimensional Euclidean ball centred at 0, both with diameter $\kappa$, rescaled by the Riemannian shape factors $r_\xi^{(\alpha, \alpha')}$ and $r_\xi^{(\beta, \beta')}$. The choice of balls instead of cubes is for convenience in the following estimates and is of no great significance.

The Riemannian rescaling ensures the following embedding in analogy with Lemma 3.5.

**Lemma 3.9** $C(\xi; \kappa) \subseteq B_{\Delta}((\xi_1, 0); \kappa r_\xi)$ for all $\kappa \in (0, 1]$.

**Proof** First note that $r_\xi = |\xi_1|^{1-\delta_1, 1-\delta'}$. Secondly, if $x \in C(\xi; \kappa)$ then

$$D_\delta((\xi_1, 0); (x_1, x_2)) < |x_1 - \xi_1| |\xi_1|^{(-\delta_1, -\delta')} + |x_2| |\xi_1|^{(-\delta_2, -\delta')} < (\kappa/2) r_\xi^{(\alpha, \alpha')} r_\xi^{(-\alpha\delta_1, -\alpha'\delta')} + (\kappa/2) r_\xi^{(\beta, \beta')} r_\xi^{(-\delta_2\alpha, -\delta'\alpha')} = (\kappa/2) r_\xi + (\kappa/2) r_\xi = \kappa r_\xi.$$

Thus $x \in B_{\Delta}((\xi_1, 0); \kappa r_\xi)$ and the embedding is established. \qed

The starting point for the derivation of the Poincaré inequality for the balls $B_{\Delta}((\xi, 0); r)$ is the following analogue of Proposition 3.1.

**Proposition 3.10** There is a $\lambda > 0$ such that

$$\int_{C(\xi, \kappa)} dx \Gamma_\delta(\varphi)(x) \geq \lambda (\kappa r_\xi)^{-2} \int_{C(\xi, \kappa)} dx (\varphi(x) - \langle \varphi \rangle)^2 \quad (22)$$

for all $\kappa \in (0, 1]$ and all $\varphi \in C^1_c(\mathbb{R}^{n+m})$ where $\langle \varphi \rangle$ is the average of $\varphi$ over $C(\xi; \kappa)$. 13
Proof Let \( x \in C(\xi; \kappa) \). Since \( |x_1| \geq |\xi_1| - |x_1 - \xi_1| \) and \( |\xi_1| = r^{(\alpha, \beta)}_\xi \) it follows that
\[
|x_1| \geq r^{(\alpha, \beta)}_\xi - (\kappa/2) r^{(\alpha, \beta)}_\xi \geq 2^{-1} r^{(\alpha, \beta)}_\xi.
\]
Therefore
\[
|x_1|^{(2\delta_i, 2\delta_i')} \geq (2^{-1} r^{(\alpha, \beta)}_\xi)^{(2\delta_i, 2\delta_i')} \geq (1/8) r^{(2\alpha\delta_i, 2\alpha\delta_i')}_\xi
\]
where the second estimate uses Proposition 2.1. Consequently,
\[
\Gamma_\delta(\varphi)(x) \geq (1/8) \left( r^{(2\alpha\delta_i, 2\alpha\delta_i')}_\xi((\nabla x_{1}\varphi)(x))^2 + r^{(2\alpha\delta_i, 2\alpha\delta_i')}_\xi((\nabla x_{2}\varphi)(x))^2 \right)
\]
for all \( x \in C(\xi; \kappa) \).

Next changing integration variables to \( y_1 = (x_1 - \xi_1)/r^{(\alpha, \beta)}_\xi \) and \( y_2 = x_2/r^{(\beta, \beta')}_\xi \), one calculates that
\[
\int_{C(\xi; \kappa)} dx \Gamma_\delta(\varphi)(x) \geq J \int_{B_\kappa} dy_1 \int_{C_\kappa} dy_2 \left( r^{(-2\alpha, -2\alpha')}_\xi r^{(2\alpha\delta_i, \alpha\delta_i')}((\nabla y_1 \varphi)(y))^2 \right.
\]
\[
+ r^{(-2\beta, -2\beta')}_\xi r^{(2\alpha\delta_i, \alpha\delta_i')}((\nabla y_2 \varphi)(y))^2 \right)
\]
\[
= J r^{-2}_\xi \int_{B_\kappa} dy_1 \int_{C_\kappa} dy_2 \left( (\nabla y_1 \varphi)(y))^2 + (\nabla y_2 \varphi)(y))^2 \right)
\]
where \( B_\kappa = \{ y_1 \in \mathbb{R}^n : |y_1| < (\kappa/2) \}, C_\kappa = \{ y_2 \in \mathbb{R}^m : |y_2| < (\kappa/2) \} \) and \( J \) is the Jacobian of the coordinate transformation.

Now one can use the usual Poincaré inequality for the Laplacian on the set \( B_\kappa \times C_\kappa \) to deduce that there is a \( \lambda > 0 \), independent of \( \kappa \), such that
\[
\int_{B_\kappa} dy_1 \int_{C_\kappa} dy_2 (\nabla y \varphi)(y))^2 \geq \lambda \kappa^{-2} \int_{B_\kappa} dy_1 \int_{C_\kappa} dy_2 (\varphi(y) - \langle \varphi \rangle)^2
\]
\[
= \lambda \kappa^{-2} \inf_{M \in \mathbb{R}} \int_{B_\kappa} dy_1 \int_{C_\kappa} dy_2 (\varphi(y) - M)^2.
\]
In particular \( \lambda \) is the lowest eigenvalue of the Laplacian on the set of \( y \in \mathbb{R}^{n+m} \) with \( |y_1| \leq 1/2 \) and \( |y_2| \leq 1/2 \). Consequently it is independent of all the parameters \( \xi, \kappa, \delta_i \), etc. Combining these estimates and reverting to the original coordinates one deduces that
\[
\int_{C(\xi; \kappa)} dx \Gamma_\delta(\varphi)(x) \geq J \lambda (\kappa r^{-2}_\xi) \inf_{M \in \mathbb{R}} \int_{B_\kappa} dy_1 \int_{C_\kappa} dy_2 (\varphi(y) - M)^2
\]
\[
= \lambda (\kappa r^{-2}_\xi) \inf_{M \in \mathbb{R}} \int_{C(\xi; \kappa)} dx (\varphi(x) - M)^2
\]
\[
= \lambda (\kappa r^{-2}_\xi) \int_{C(\xi; \kappa)} dx (\varphi(x) - \langle \varphi \rangle)^2
\]
for all \( \kappa \in (0, 1] \) and all \( \varphi \in C^1_c(\mathbb{R}^{n+m}) \) \( \square \).

Next one can transfer the Poincaré inequality on the \( C(\xi; \kappa) \) to the balls \( B_\Delta((\xi_1, 0); r) \) by the following embedding analogous to Lemma 3.6.
Lemma 3.11  There is a $\kappa_0 \in \langle 0, 1 \rangle$ such that $B_\Delta((\xi_1, 0); \kappa_0 r_\xi) \subseteq C(\xi; \kappa)$ for all $\kappa \in \langle 0, 1 \rangle$.

Proof  Consider the family of balls $B_\Delta((\xi_1, 0); \kappa r_\xi)$ for $\kappa \in \langle 0, 1 \rangle$ and introduce the set $B_n = \{x_1 \in \mathbb{R}^n : (x_1, 0) \in B_\Delta((\xi_1, 0); \kappa r_\xi)\}$. Then $x_1 \in B_n$ if and only if

$$|x_1 - \xi_1| < \kappa r_\xi (|x_1| + |\xi_1|)^{\delta_1, \delta'_1}$$

i.e. $D_\delta((x_1, 0); (\xi_1, 0)) < \kappa r_\xi$. Therefore $B_n \subseteq C_n$ where

$$C_n = \{x_1 \in \mathbb{R}^n : |x_1 - \xi_1| < \kappa r_\xi (|x_1 - \xi_1| + 2|\xi_1|)^{\delta_1, \delta'_1}\} = \{x_1 \in \mathbb{R}^n : |x_1 - \xi_1| < \kappa |\xi_1|^{(1-\delta_1, 1-\delta'_1)} (|x_1 - \xi_1| + 2|\xi_1|)^{\delta_1, \delta'_1}\}.$$

Since $|\xi_1| \neq 0$ one deduces that $x_1 \in C_n$ if and only if

$$|x_1 - \xi_1|/|\xi_1| < \kappa |\xi_1|^{(1-\delta_1, -\delta'_1)} \left(|\xi_1|(2 + |x_1 - \xi_1|/|\xi_1|)^{\delta_1, \delta'_1}\right).$$

Therefore by Proposition [2.1] it is necessary that

$$|x_1 - \xi_1|/|\xi_1| < 4 \kappa (2 + |x_1 - \xi_1|/|\xi_1|)^{\delta_1 \vee \delta'_1}. $$

Then setting $\sigma = |x_1 - \xi_1|/(\kappa |\xi_1|)$ one must have $\sigma \leq 4 (2 + \sigma)^{\delta_1 \vee \delta'_1}$. But $\delta_1 \vee \delta'_1 < 1$ and one concludes that $\sigma \leq a_1$ where $a_1 > 0$ is the unique solution of $a_1 = 4 (2 + a_1)^{\delta_1 \vee \delta'_1}$. Consequently

$$|x_1 - \xi_1| \leq a_1 \kappa |\xi_1| = a_1 \kappa r_\xi^{(\alpha, \alpha')}$$

for all $\kappa \in \langle 0, 1 \rangle$.

Next observe that if $x \in B_\Delta((\xi_1, 0); \kappa r_\xi)$ then $x_1 \in B_n$. This follows by contradiction. Assume $x \in B_\Delta((\xi_1, 0); \kappa r_\xi)$ but $x_1 \notin B_n$. Then

$$\kappa r_\xi > D_\delta((x_1, x_2); (\xi_1, 0)) \geq |x_1 - \xi_1|/|x_1| + |\xi_1|^{\delta_1, \delta'_1} = D_\delta((x_1, 0); (\xi_1, 0)) \geq \kappa r_\xi.$$

The first inequality follows since $x \in B_\Delta((\xi_1, 0); \kappa r_\xi)$ and the last follows since $x_1 \notin B_n$. But this gives a contradiction.

It now follows that there is a $\rho > 0$, dependent on $\xi$, such that

$$\inf_{x_1 \in B_n, |x_2| \geq \rho} D_\delta((x_1, x_2); (\xi_1, 0)) = \kappa r_\xi.$$

Then $B_\Delta((\xi_1, 0); \kappa r_\xi) \subseteq B_n \times \{x_2 \in \mathbb{R}^m : |x_2| < \rho\}$. Next we estimate $\rho$.

First it follows from the observation

$$\kappa r_\xi = \inf_{x_1 \in B_n, |x_2| \geq \rho} D_\delta((x_1, x_2); (\xi_1, 0)) \leq \inf_{|x_2| \geq \rho} D_\delta((\xi_1, x_2); (\xi_1, 0))$$

that

$$\kappa r_\xi \leq \rho \left(|\xi_1|^{(\delta_2, \delta'_2)} + \rho^{(\gamma, \gamma')}ight)^{-1}.$$

Therefore one obtains a lower bound on $\rho$,

$$\rho \geq \kappa r_\xi |\xi_1|^{(\delta_2, \delta'_2)} \geq \kappa r_\xi^{(\beta, \beta')}$$
where the second step uses $r_\xi = |\xi_1|^{(1-\delta_1,1-\delta'_1)}$.

Secondly, one obtains an upper bound on $\rho$ by observing that

$$
\kappa r_\xi \geq \inf_{x_1 \in B_n, |x_2| \geq \rho} |x_2| \left( (|x_1| + |\xi_1|)^{(\delta_2,\delta'_2)} + (|x_2|)^{(\gamma,\gamma')} \right)^{-1} 
\geq \inf_{x_1 \in B_n} \rho \left( (|x_1| + |\xi_1|)^{(\delta_2,\delta'_2)} + \rho^{(\gamma,\gamma')} \right)^{-1}.
$$

Since by the previous estimate $x_1 \in B_n$ satisfies the bound $|x_1| \leq (1 + a \kappa) |\xi_1| \leq (1 + a) |\xi_1|$ it follows that

$$
\rho \leq \kappa r_\xi \left( ((1 + a) |\xi_1|)^{(\delta_2,\delta'_2)} + \rho^{(\gamma,\gamma')} \right)
$$

for all $\kappa \in (0,1]$. Then by Proposition 2.1 one deduces that there is a $b > 0$ such that

$$
\rho \leq \kappa r_\xi \left( b |\xi_1|^{(\delta_2,\delta'_2)} + \rho^{(\gamma,\gamma')} \right)
$$

for all $\kappa \in (0,1]$. But $r_\xi = |\xi_1|^{(1-\delta_1,1-\delta'_1)}$ so

$$
\rho \leq b \kappa |\xi_1|^{(\tau,\tau')} + \kappa |\xi_1|^{(1-\delta_1,1-\delta'_1)} \rho^{(\gamma,\gamma')}.
$$

with $\tau = 1 + \delta_2 - \delta_1$ and $\tau' = 1 + \delta'_2 - \delta'_1$. Now we estimate $\rho$ in two separate cases, $|\xi_1| \leq 1$ and $|\xi_1| \geq 1$.

If $|\xi_1| \leq 1$ then

$$
\rho \leq b \kappa + \kappa \rho^{(\gamma,\gamma')}.
$$

Since $\gamma,\gamma' < 1$ it follows that there is a $b_0 > 0$ such that $\rho \leq b_0$. Now if $b_0 \leq 1$ then $\rho \leq (1 + b \kappa)$. Alternatively if $b_0 \geq 1$ then $b_0/\kappa \leq b + b'_0 \leq b + (b_0/\kappa)^{\gamma'}$ and $b_0/\kappa$ must be uniformly bounded. Therefore there is a $b_0 > 0$ such that $\rho \leq b_0 \kappa$ for all $\kappa \in (0,1]$. Hence by another application of Proposition 2.1 one deduces that there is a $c > 0$ such that $\rho^{(\gamma,\gamma')} \leq c (\rho/(b \kappa))^{\gamma}$ for all $\kappa \in (0,1]$. The value of $c$ is independent of $\kappa$ and $\xi_1$. Then it follows from (23) that

$$
\rho/(b_0 \kappa |\xi_1|^{\gamma'}) \leq (b/b_0) + (c/b_0) |\xi_1|^{1-\delta_1-\tau} (\rho/(b_0 \kappa))^{\gamma} 
= (b/b_0) + (c/b_0) (\rho/(b_0 \kappa |\xi_1|^{\gamma'}))^{\gamma}.
$$

Therefore, since $\gamma < 1$ one has

$$
\rho \leq a_2 \kappa |\xi_1|^{\gamma'} = a_2 \kappa r_\xi^{\beta}
$$

where $a_2$ satisfies $a_2 = b + c (a_2/b_0)^{\gamma}$.

Next suppose $|\xi_1| \geq 1$. It then follows from the above discussion of the lower bound on $\rho$ that

$$
\rho \geq \kappa r_\xi |\xi_1|^{\delta_2} \geq \kappa |\xi_1|^{\gamma'} = \kappa r_\xi^{\beta'}
$$

because $r_\xi = |\xi_1|^{1-\delta'_1}$. In particular $\rho/\kappa \geq 1$. But then

$$
\rho^{(\gamma,\gamma')} = (\kappa (\rho/\kappa))^{(\gamma,\gamma')} \leq 2^{(\gamma \vee \gamma')} \kappa^{(\gamma \wedge \gamma')} (\rho/\kappa)^{\gamma'}
$$
by Proposition 2.1. Hence it follows from (23) that
\[
\frac{\rho}{(\kappa |\xi_1|^\gamma')} \leq b + 4 |\xi_1|^{-\gamma'} |\xi_1|^{1-\delta} \kappa^{(\gamma'\gamma')} (\rho/\kappa)^{\gamma'}
\]
\[
\leq b + 4 \left( \frac{\rho}{(\kappa |\xi_1|^\gamma')} \right)^{\gamma'}
\]
where the second step uses \(\kappa \leq 1\). Consequently one deduces as before that
\[
\rho \leq a_2 \kappa |\xi_1|^\gamma' = a_2 \kappa r^{\gamma'}_\xi
\]
for all \(\kappa \in (0, 1]\) and \(|\xi_1| \geq 1\) where \(a_2 = b + 4 a'_2\).

Finally combination of these results leads to the conclusion that there are \(a_2, a'_2 > 0\) such that
\[
a_2 \kappa |\xi_1|^{(\tau,\tau')} \leq \rho \leq a_2 \kappa |\xi_1|^{(\tau,\tau')}
\]
or, equivalently,
\[
a_2 \kappa r^{(\beta,\beta')}_\xi \leq \rho \leq a_2 \kappa r^{(\beta,\beta')}_\xi
\]
for all \(\kappa \in (0, 1]\). The values of \(a_2\) and \(a'_2\) are independent of \(\xi_1\) and \(\kappa\).

Now we can complete the proof of Lemma 3.11.

If \(x = (x_1, x_2) \in B_\Delta((\xi_1, 0); \kappa_0 \kappa r_\xi)\) the foregoing estimates are valid with \(\kappa\) replaced by \(\kappa_0 \kappa\) with \(\kappa_0 \in (0, 1]\) and \(\kappa \in (0, 1]\). Therefore
\[
|x_1 - \xi_1| \leq a_1 \kappa_0 \kappa r^{(a,a')}_\xi \quad \text{and} \quad |x_2| \leq a_2 \kappa_0 \kappa r^{(\beta,\beta')}_\xi.
\]
Hence if \((a_1 \lor a_2) \kappa_0 < 1/2\) it follows that \(x \in C(\xi; \kappa)\). \(\square\)

At this point the proof of Proposition 3.8 is immediate. First there is a \(\lambda > 0\) such that
\[
\int_{B_\Delta((\xi_1, 0); \kappa_0 \kappa r_\xi)} dx \Gamma_\delta(\varphi)(x) \geq \int_{C(\xi; \kappa)} dx \Gamma_\delta(\varphi)(x)
\]
\[
\geq \lambda (\kappa r_\xi)^{-2} \int_{C(\xi; \kappa)} dx (\varphi(x) - \langle \varphi \rangle)^2
\]
for all \(\kappa \in (0, 1]\) by Lemma 3.9 and Proposition 3.8.

Secondly, there is a \(\kappa_0 \in (0, 1]\) such that
\[
\int_{C(\xi; \kappa)} dx (\varphi(x) - \langle \varphi \rangle)^2 = \inf_{M \in \mathbb{R}} \int_{C(\xi; \kappa)} dx (\varphi(x) - M)^2
\]
\[
\geq \inf_{M \in \mathbb{R}} \int_{B_\Delta((\xi_1, 0); \kappa_0 \kappa r_\xi)} dx (\varphi(x) - M)^2
\]
\[
= \int_{B_\Delta((\xi_1, 0); \kappa_0 \kappa r_\xi)} dx (\varphi(x) - \langle \varphi \rangle)^2
\]
for all \(\kappa \in (0, 1]\) by two more applications of (3) and by Lemma 3.11.

Therefore one concludes that
\[
\int_{B_\Delta((\xi_1, 0); \kappa r_\xi)} dx \Gamma_\delta(\varphi)(x) \geq \lambda (\kappa r_\xi)^{-2} \int_{B_\Delta((\xi_1, 0); \kappa_0 \kappa r_\xi)} dx (\varphi(x) - \langle \varphi \rangle)^2
\]
for all $\kappa \in [0, 1]$ and the statement of Proposition 3.8 with $\lambda_2 = \lambda$ and $\kappa_2 = \kappa_0$, follows by setting $r = \kappa r_\xi$. \qed

Thus Proposition 3.8 establishes the Poincaré inequality for the balls $B_\Delta((\xi_1, 0); r)$ for $\xi \neq 0$ and for all $r \leq D_\delta((\xi_1, 0); (0, 0))$.

**Remark 3.12** Note that in the foregoing proof of Case II we do not need to assume that $\delta_1 \vee \delta'_1 \in [0, 1/2)$ if $n = 1$.

**Case III—General balls.** To complete the proof of the Poincaré inequality (4) it suffices to verify it for the balls $B_\Delta((\xi_1, 0); r)$ with $\xi_1 \neq 0$ and $r \geq r_\xi$ where again $r_\xi = D_\delta((0, 0); (\xi_1, 0))$. (If $\xi_1 = 0$ the inequality follows for all $r > 0$ by Case I and if $r \leq r_\xi$ then it follows from Case II.) The general case is a corollary of the two special cases.

First assume $r \geq K r_\xi$ with $K = 2 (1 + \kappa_1)/\kappa_1$ where $\kappa_1$ is the parameter of Proposition 3.7. Then $r \geq 2 r_\xi$ and $B_\Delta((0, 0); r - r_\xi) \subseteq B_\Delta((\xi_1, 0); r)$. Therefore

$$\int_{B_\Delta((\xi_1, 0); r)} \Gamma(\varphi) \geq \int_{B_\Delta((0, 0); r - r_\xi)} \Gamma(\varphi) \geq \lambda (r - r_\xi)^{-2} \inf_{M \in \mathbb{R}} \int_{B_\Delta((0, 0); \kappa_1 r - r_\xi)} (\varphi - M)^2$$

by Proposition 3.7. But $0 < r - r_\xi < r$. Hence $(r - r_\xi)^{-2} > r^{-2}$. Moreover, one has the inclusion $B((\xi_1, 0); \kappa_1 (r - r_\xi) - r_\xi)) \subseteq B_\delta((0, 0); \kappa_1 (r - r_\xi))$. Therefore

$$\int_{B_\Delta((\xi_1, 0); r)} \Gamma(\varphi) \geq \lambda r^{-2} \inf_{M \in \mathbb{R}} \int_{B_\Delta((\xi_1, 0); \kappa_1 (r - r_\xi) - r_\xi)} (\varphi - M)^2 .$$

Since $\kappa_1 (r - r_\xi) - r_\xi \geq \kappa_1 r/2$ it then follows that

$$\int_{B_\Delta((\xi_1, 0); r)} \Gamma(\varphi) \geq \lambda r^{-2} \inf_{M \in \mathbb{R}} \int_{B_\Delta((\xi_1, 0); \kappa_1 r/2)} (\varphi - M)^2$$

for all $r \geq K r_\xi$.

Secondly suppose $K r_\xi \geq r \geq r_\xi$. Then

$$\int_{B_\Delta((\xi_1, 0); r)} \Gamma(\varphi) \geq \int_{B_\Delta((\xi_1, 0); r)} \Gamma(\varphi) \geq \lambda (r_\xi)^{-2} \inf_{M \in \mathbb{R}} \int_{B_\Delta((\xi_1, 0); \kappa_2 r_\xi)} (\varphi - M)^2$$

by Proposition 3.8. But $\kappa_2 r_\xi \geq (\kappa_2/K) r$ and $(r_\xi)^{-2} \geq r^{-2}$. Therefore

$$\int_{B_\Delta((\xi_1, 0); r)} \Gamma(\varphi) \geq \lambda r^{-2} \inf_{M \in \mathbb{R}} \int_{B_\Delta((\xi_1, 0); (\kappa_2/K)r)} (\varphi - M)^2$$

for all $r \in [r_\xi, 2(1 + \kappa_1) r_\xi/\kappa_1]$.

Combination of these two results establishes the Poincaré inequality for the balls $B_\Delta((\xi_1, 0); r)$ and all $r \geq r_\xi$ with the value of $\kappa$ given by $(\kappa_2/K) \wedge (\kappa_1/2)$. But the bounds were established for $\xi_1 = 0$ and all $r$ in Case I and for $\xi_1 \neq 0$ and $r \leq r_\xi$ in Case II. Thus it follows that the Poincaré inequality is valid for the $B_\Delta((\xi_1, 0); r)$ for all $\xi_1 \in \mathbb{R}^n$ and all $r > 0$. Finally, since the Riemannian metric $d(\cdot; \cdot)$ is equivalent to the metric $D_\delta(\cdot; \cdot; \cdot)$ which defines the balls $B_\Delta(0; r)$ the Poincaré inequality is valid for the
Riemannian balls by the discussion in Section 2. The change to an equivalent metric only requires a change in the value of the parameter $\kappa$ in the inequality.

At this stage we have established the first statement of Theorem 1. Next we prove the second statement, the failure of the Poincaré inequality for $n = 1$ and $\delta_1 \vee \delta'_1 \in [1/2, 1)$. We will establish this in two steps.

First assume $\delta_1 \in [1/2, 1)$. Let $\varphi = \chi \psi$ where $\chi \in C^1_c(\mathbb{R})$, $\psi \in C^1_c(\mathbb{R}^m)$ and $\psi$ is equal to one on $[-1, 1]^m$. Then

\[
\int_{[-1,1]^{1+m}} dx \, \Gamma_\delta(\varphi)(x) = 2^m \int_{-1}^1 dx_1 \, \Gamma_\delta^{(1)}(\chi)(x_1)
\]

where $\Gamma_\delta^{(1)}$ is the carré du champ of $H_\delta^{(1)} = -d_{x_1} |x_1|^{2\delta_1} \, dx_1$ acting on $L_2(-1, 1)$. Explicitly $\Gamma_\delta^{(1)}(\chi(x_1)) = |x_1|^{2\delta_1} (\chi'(x_1))^2$. Moreover,

\[
\int_{[-1,1]^{1+m}} dx \, (\varphi(x) - \langle \varphi \rangle)^2 = 2^m \int_{-1}^1 dx_1 \, (\chi(x_1) - \langle \chi \rangle)^2
\]

where $\langle \varphi \rangle$ is the average of $\varphi$ over the cube $[-1, 1]^{1+m}$ and $\langle \chi \rangle$ is the average of $\chi$ over the interval $[-1, 1]$. Therefore the Poincaré inequality fails for $H_\delta$ on $[-1, 1]^{1+m}$ if it fails for $H_\delta^{(1)}$ on $[-1, 1]$.

Define $\chi_n : \mathbb{R} \to [-1, 1]$ by

\[
\chi_n(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq n^{-1} \\ 
1 - \eta_n^{-1} \eta(x) & \text{if } n^{-1} \leq x \leq 1 \\ 
1 & \text{if } x \geq 1
\end{cases}
\]

and

\[
\chi_n(x) = \begin{cases} 
0 & \text{if } -n^{-1} \leq x \leq 0 \\ 
-1 + \sigma_n^{-1} \sigma(x) & \text{if } -1 \leq x \leq -n^{-1} \\ 
-1 & \text{if } x \leq -1
\end{cases}
\]

where

\[
\eta(x) = \int_x^1 ds \, |s|^{-2\delta_1}, \quad \sigma(x) = \int_{-1}^x ds \, |s|^{-2\delta_1}, \quad \eta_n = \eta(n^{-1}) \quad \text{and} \quad \sigma_n = \sigma(-n^{-1}).
\]

Note that $\chi_n$ is an absolutely continuous increasing function. Moreover, since $\delta_1 \in [1/2, 1)$, it follows that $\lim_{n \to \infty} \chi_n(x) = 1$ if $x > 0$ and $\lim_{n \to \infty} \chi_n(x) = -1$ if $x < 0$. For example if $\delta_1 = 1/2$ then $\eta(x) \sim \log |x| \sim \sigma(x)$ and $\eta_n \sim \log n \sim \sigma_n$. Therefore $\langle \chi_n \rangle \to 0$ as $n \to \infty$ and

\[
\lim_{n \to \infty} \int_{-1}^1 dx \, (\chi_n(x) - \langle \chi_n \rangle)^2 = \lim_{n \to \infty} \int_{-1}^1 dx \, \chi_n(x)^2 = 2.
\]

But

\[
\lim_{n \to \infty} \int_{-1}^1 dx \, |x|^{2\delta_1} (\chi_n'(x))^2 = \lim_{n \to \infty} (\sigma_n + \eta_n) = 0.
\]

Therefore the Poincaré inequality for $H_\delta^{(1)}$ on $[-1, 1]$ must fail for $\chi_n$ if $n$ is sufficiently large. Consequently the Poincaré inequality (12) for $H_\delta$ must fail for $\varphi = \chi_n \psi$ for large $n$. 

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Secondly, assume $\delta_1 \in [0, 1/2)$ and $\delta_1' \in [1/2, 1)$. We aim to show that the Poincaré inequality fails for Riemannian balls of large radius centred at the origin. But it follows from the discussion of Case I above that it suffices to prove that \( \text{(I')} \) fails for centred cubes $C_t$ with $t$ large. This can again be reduced to a one-dimensional problem.

For each $t > 0$ set $\varphi = \chi \psi$ with $\chi \in C^1_c(\mathbb{R})$ and $\psi \in C^1_c(\mathbb{R}^m)$ with $\psi(x_2) = 1$ if $|x_2|_1 < t^{1/(\beta, \beta')}. Then one has

$$\int_{C_t} dx \Gamma_\delta(\varphi)(x) = 2m \int_{|x_1| < t^{(\alpha, \alpha')}} dx_1 \Gamma_\delta(\chi)(x_1)$$

and

$$\int_{C_t} dx (\varphi(x) - \langle \varphi \rangle)^2 = 2m \int_{|x_1| < t^{(\alpha, \alpha')}} dx_1 (\chi(x_1) - \langle \chi \rangle)^2$$

where $\langle \varphi \rangle$ is the average of $\varphi$ over the cube $C_t$ and $\langle \chi \rangle$ is the average of $\chi$ over the interval $I_t = \{ x_1 \in \mathbb{R} : |x_1| < t^{(\alpha, \alpha')} \}$. Thus to establish that \( \text{(I')} \) fails for $C_t$ it suffices to establish that the one-dimensional analogue fails on $I_t$.

First we consider the particular case $\delta_1 \in [0, 1/2)$ but $\delta_1' = 1/2$. Then $\alpha \in [1, 2)$ but $\alpha' = 2$. Let $\chi$ be an odd function with $\chi(x_1) = \int_0^{x_1} ds s^{-2\delta_1, -1}$ for $x_1 \geq 0$. Then $\chi$ is locally bounded and $\chi(x_1) \sim \pm \log |x_1|$ as $x_1 \to \pm \infty$. Now $\Gamma_\delta(\chi)(x_1) = |x_1|^{-2\delta_1, -1}$ and it follows that

$$\int_{|x_1| < t^{(\alpha, \alpha')}} dx_1 \Gamma_\delta(\chi)(x_1) \sim \int_1^{t^2} ds s^{-1} \sim \int_1^{t^2} ds \sim (\log t)$$

as $t \to \infty$. But $\langle \chi \rangle = 0$ because the function is odd and

$$t^{-2} \int_{|x_1| < t^{(\alpha, \alpha')}} dx_1 (\chi(x_1) - \langle \chi \rangle)^2 \sim t^{-2} \int_{|x_1| < t^{(\alpha, \alpha')}} dx_1 (\chi(x_1))^2 \sim \int_1^{t^2} ds (\log s)^2 \sim (\log t)^2$$

as $t \to \infty$. Thus the Poincaré inequality must fail for large $t$.

Secondly consider the case $\delta_1 \in [0, 1/2)$ but $\delta_1' \in (1/2, 1)$. Again $\alpha \in [1, 2)$ but now $\alpha' > 2$. Let $\chi \in C^1(\mathbb{R})$ be an odd increasing function with $\chi(x_1) = 1$ if $x_1 \geq 1$. Then $\Gamma_\delta(\chi)$ is a positive bounded function with support in the interval $[-1, 1]$. Hence

$$\int_{|x_1| < t^{(\alpha, \alpha')}} dx_1 \Gamma_\delta(\chi)(x_1) \text{ is bounded uniformly for } t \geq 1.$$ 

On the other hand

$$t^{-2} \int_{|x_1| < t^{(\alpha, \alpha')}} dx_1 (\chi(x_1) - \langle \chi \rangle)^2 \sim t^{-2} \int_{|x_1| < t^{(\alpha, \alpha')}} dx_1 (\chi(x_1))^2 \sim t^{-2}$$

as $t \to \infty$. Since $\alpha' > 2$ the Poincaré inequality must again fail.

These examples establish the second statement of Theorem 1.1 and it remains to prove the third statement.

The proof is by modification of the above argument for the Poincaré inequality on $\mathbb{R}^{n+m}$. The only significant modification occurs in the discussion of the (half) balls centred at the origin. Consider the case of $\mathbb{R}_+ \times \mathbb{R}^m$. Then $B_+(0 ; r) = B(0 ; r) \cap \{ x_1 : x_1 > 0 \}$. The proof of \( \text{(5)} \) for $B_+(0 ; r)$ begins with the analogue of Proposition 3.11.

**Proposition 3.13** Assume $\delta_1 \in [1/2, 1)$. Then there is a $\lambda > 0$ such that

$$\int_0^1 dx_1 \int_{[-1,1]^m} dx_2 \Gamma_\delta(\varphi)(x_1, x_2) \geq \lambda \int_0^1 dx_1 \int_{[-1,1]^m} dx_2 (\varphi(x_1, x_2) - \langle \varphi \rangle)^2$$

for all $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}^m)$.  

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The argument used to prove Proposition 3.1 is easily adapted to the half-space and again reduces the problem to a pair of one-dimensional problems. It is reduced to proving (13) and (14) with the interval $[-1, 1]$ replaced by $[0, 1]$ where $\hat{h}_\delta$ is given by (15) modified similarly. But (13) follows because $\psi \in C^1_c(\mathbb{R}) \mapsto \int_0^1 dx x^{2\delta_1}(\psi'(x))^2$ is a closable form and its closure corresponds to the self-adjoint extension of the operator $-d_x x^{2\delta_1} d_x$ on $L_2(0, 1)$ with Neumann boundary conditions at each endpoint. This operator has, however, a compact resolvent and the lowest eigenvalue is zero with the constant function one as corresponding eigenvalue. Since the condition $\int_0^1 dx x^{2\delta_1}(\psi'(x))^2 = 0$ implies that $\psi' = 0$ and $\psi$ is constant it follows that the zero eigenvalue is simple. Thus (13) is satisfied with $\lambda$ the second eigenvalue. Note that this argument, in contrast to that used to prove Lemma 3.2, does not require $\delta < 1/2$ but works equally well for all $\delta \in [0, 1)$. The point is that it is for the operator on $[0, 1]$ Next the proof of (14) as given in Lemma 3.3 remains unchanged. It is based on Lemma 3.2 and hence it also does not require $\delta < 1/2$ but is valid for all $\delta \in [0, 1)$. 

The second step in the proof is an analogue of Proposition 3.4.

**Proposition 3.14** Assume $\delta_1 \in [1/2, 1)$. Then there is a $\lambda > 0$ such that

$$\int_{C_t^-} dx \Gamma_\delta(\varphi)(x) \geq \lambda t^{-2} \int_{C_t^+} dx (\varphi(x) - \langle \varphi \rangle_\pm)^2$$

for all $\varphi \in C^1_c(\mathbb{R}^{1+m})$ and $t > 0$ where $C_t^+ = C_t \cap \{x_1 > 0\}$, $C_t^- = C_t \cap \{x_1 < 0\}$, $\langle \varphi \rangle_\pm = |C_t^\pm|^{-1} \int_{C_t^\pm} dx \varphi(x)$.

**Proof** The proof is identical to the proof of Proposition 3.4 but is now based on Proposition 3.13 and scaling in the half-space. The key point is that Proposition 3.4 has to be applicable to the operator on the half-space with coefficients $|x_1|^{2(\delta \vee \delta_1', \delta \wedge \delta')}$ This, however, only requires $\delta \vee \delta_1' \in [1/2, 1)$ and this is ensured if $\delta_1 \in [1/2, 1)$. There is no restraint on $\delta_1'$ it can take all values in $[0, 1)$. 

The rest of the proof of the Poincaré inequality (5) now follows by the argument used earlier to establish (4). The inequality for half balls centred at the origin follows from Proposition 3.13 by slight modification of the earlier embedding arguments for cubes and balls. Then the proof for balls completely contained in the appropriate half-space follows by the discussion of Case II in of the proof of (4). This did not require the condition $\delta \vee \delta_1' \in [0, 1/2)$ (see Remark 3.12) and applies equally well to the current situation with $\delta_1 \in [1/2, 1)$ and $\delta_1' \in [0, 1)$. Finally the inequality for general ‘balls’ $B_\pm(\xi; r)$ follows as in the argument of Case III above. 

**4 Heat kernel bounds**

In this section we establish Theorems 1.2 and 1.3. The upper Gaussian bounds follow from Corollary 6.6 of [RS08] once one establishes continuity of the kernel. Therefore it suffices to prove the continuity and the lower Gaussian bounds. These results are indirect corollaries of Statements I and III of Theorem 1.1. The key observation of Grigor’yan [Gri92] and Saloff-Coste [SC92a] is that the Poincaré inequality (4) combined with the
volume doubling property of the Riemannian metric, [RS08] Corollary 5.2, implies the parabolic Harnack inequality of Moser [Mos64] on $\mathbb{R}^{n+m}$. Similarly (5) and volume doubling imply the Harnack inequality on $\Omega_\pm$.

The operator $H$ is defined to satisfy the (global) parabolic Harnack inequality on $\mathbb{R}^{n+m}$ if there exists an $a > 0$ such that for any $x \in \mathbb{R}^{n+m}$ and $t > 0$ any non-negative (weak) solution $\varphi$ of the parabolic equation $(\partial_t + H)\varphi = 0$ in the cylinder $Q = \langle t, t + r^2 \rangle \times B(x; 2r)$ satisfies

$$\sup_{Q^-} \varphi \leq a \inf_{Q^+} \varphi \tag{25}$$

where $Q^- = [t + r^2/4, t + r^2/2] \times B(x; r)$ and $Q^+ = [t + 3r^2/4, t + r^2] \times B(x; r)$. This definition is the key to establishing the continuity of the heat kernel and the lower Gaussian bounds.

**Proof of Theorem 1.2** First it follows from Theorem 1.1 that the Poincaré inequality (4) is valid.

Secondly, it follows from [RS08] Corollary 5.2 that the Riemannian balls $B(x; r)$ satisfy the volume doubling property, i.e. there is a $b > 0$ such that

$$|B(x; 2r)| \leq b |B(x; r)| \tag{26}$$

for all $x \in \mathbb{R}^{n+m}$ and all $r > 0$.

Thirdly, Theorem 3.1 of [SC92a] establishes that (4) together with (26) imply that $H$ satisfies the parabolic Harnack inequality (25). Then, however, a straightforward argument of Moser, [Mos61] Section 5 or [Mos64] pages 108–109, establishes that each non-negative solution $\varphi$ of $(\partial_t + H)\varphi = 0$ is Hölder continuous. Hence one deduces that the heat kernel $K_t(x; y)$ is jointly Hölder continuous. Then the Gaussian upper bounds follow from Corollary 6.6 of [RS08]. Moreover, the continuity ensures that the kernel $K_t$ is well-defined on the diagonal $x = y$ and it follows from Corollary 6.7 and Remark 6.8 of [RS08] that there is a $c > 0$ such that

$$K_t(x; x) \geq c |B(x; t^{1/2})|^{-1} \tag{27}$$

for all $x \in \mathbb{R}^{n+m}$ and $t > 0$.

Fourthly, fix $x$ and define define $\varphi$ by $\varphi(t, y) = K_t(x; y)$ for all $t > 0$. Then $\varphi$ is a non-negative weak solution of $(\partial_t + H)\varphi = 0$ in the cylinder $Q = \langle 0, r^2 \rangle \times B(x; 2r)$. Now if $Q^- = [r^2/4, r^2/2] \times B(x; r)$ and $Q^+ = [3r^2/4, r^2] \times B(x; r)$ the parabolic Harnack inequality gives

$$K_{r^2/2}(x; x) \leq \sup_{(t, y) \in Q^-} \varphi(t, y) \leq a \inf_{(t, y) \in Q^+} \varphi(t, y) \leq a K_{r^2}(x; y)$$

for all $y \in B(x; r)$. Therefore, setting $r^2 = t$, this estimate combined with (27) gives

$$K_t(x; y) \geq (c/a) |B(x; t^{1/2})|^{-1} \tag{28}$$

for all $x \in \mathbb{R}^{n+m}$ all $t > 0$ and all $y \in B(x; t^{1/2})$. Thus (28) is valid for all $x, y \in \mathbb{R}^{n+m}$ and all $t > 0$ with $d(x; y)^2/t \leq 1$. Under the latter restraint one can of course introduce a Gaussian factor to obtain the desired lower bound. Therefore it remains to derive the bound for $d(x; y)^2/t \geq 1$. This can be achieved by combination of the semigroup property, the volume doubling property and the bound (28) by adaptation of an argument of Jerison and Sanchez-Callé, [JSC86] Section 5.
Let $\rho = d(x; y)$ and assume $\rho^2/t \geq 1$. Choose a continuous path of length $l \leq 2\rho$ connecting $x$ and $y$. Next let $k \geq 4$ be the integer satisfying $k \geq 4\rho^2/t > k - 1$. Then fix points $x_1, \ldots, x_{k-1}$ in the path with $d(x_j; x_{j+1}) \leq 2\rho/k$ for $j \in \{0, \ldots, k-1\}$ where $x_0 = x$ and $x_k = y$. Now set $B_j = B(x_j; 2\rho/k)$ and $I_j = B_j \cap B_{j+1}$. Then

$$K_t(x; y) \geq \int_{I_1 \times \cdots \times I_{k-1}} d\xi_1 \cdots d\xi_{k-1} K_{t/k}(x; \xi_1) K_{t/k}(\xi_1; \xi_2) \cdots K_{t/k}(\xi_{k-1}; y). \tag{29}$$

If $\xi_j \in I_j$ and $\xi_{j+1} \in I_{j+1}$ then $d(\xi_j; \xi_{j+1}) \leq 2\rho/k$. Thus since $4\rho^2/t \leq k$ one has $d(\xi_j; \xi_{j+1})^2/(t/k) \leq (4\rho^2/t)/k \leq 1$. Therefore it follows from (28) that

$$K_{t/k}(\xi_j; \xi_{j+1}) \geq (c/a)|B(\xi_j; (t/k)^{1/2})|^{-1}.$$

Moreover, for each $\xi_j \in B(x_j; (t/k)^{1/2})$ one has $B(\xi_j; (t/k)^{1/2}) \subseteq B(x_j; 2(t/k)^{1/2})$. Therefore

$$|B(\xi_j; (t/k)^{1/2})| \leq |B(x_j; 2(t/k)^{1/2})| \leq b|B(x_j; (t/k)^{1/2})|$$

where the second step uses the volume doubling property (26). Hence

$$K_{t/k}(\xi_j; \xi_{j+1}) \geq (c/ab)|B(x_j; (t/k)^{1/2})|^{-1}$$

for each $j \in \{0, \ldots, k-1\}$. Further $I_j$ contains a ball $B(\xi; \rho/k)$ with $B(x_j; 2\rho/k) \subseteq B(\xi; 4\rho/k)$. Since $|B(\xi; 4\rho/k)| \leq b^2|B(\xi; \rho/k)|$ by (26) one then has

$$|I_j| \geq b^{-2}|B(x_j; 2\rho/k)| \geq b^{-2}|B(x_j; (t/4k)^{1/2})| \geq b^{-3}|B(x_j; (t/k)^{1/2})|.$$

The second inequality follows because $4\rho^2/t > k - 1$. Hence $(2\rho/k)^2 \geq t/4k$. The third uses volume doubling. Combination of these estimates then gives

$$K_t(x; y) \geq (c/ab)^k \left( \prod_{j=0}^{k-1} |B(x_j; (t/k)^{1/2})|^{-1} \right) b^{-3(k-1)} \left( \prod_{j=1}^{k-1} |B(x_j; (t/k)^{1/2})| \right)$$

$$= (c/ab)(c/ab)^{k-1}|B(x; (t/k)^{1/2})|^{-1} \geq (c/ab)(c/ab)^{k-1}|B(x; t^{1/2})|^{-1}.$$

Since $k - 1 < 4\rho^2/t$ one then obtains the lower bounds

$$K_t(x; y) \geq (c/ab)|B(x; t^{1/2})|^{-1} e^{-\omega d(x; y)^2/t}$$

with $\omega = \log(ab^4/c)$ for all $x, y \in \mathbb{R}^{n+m}$ and $t > 0$ with $d(x; y)^2/t \geq 1$. This completes the proof of Theorem 1.2. $\square$

**Proof of Theorem 1.3** The proof is very similar but it relies on Sturm’s extension [Stu95, Stu96] of Grigor’yan and Saloff-Coste’s work characterizing the parabolic Harnack inequality. Sturm establishes that the parabolic Harnack inequality holds for a large class of strictly local regular Dirichlet spaces $X$ with an intrinsic distance $\rho$ if the volume doubling property and the Poincaré inequality are satisfied. The key point is that the intrinsic distance is finite, continuous, defines the original topology of the space and $(X, \rho)$ is complete. (For an extensive discussion in a setting similar to ours see [GSC11] and especially Theorem 2.31.) Therefore the proof of Theorem 1.3 reduces to verifying the assumptions of Sturm’s theorem for the Dirichlet forms $h_{\pm}$ forms associated with the generators $H_{\pm}$ of
the submarkovian semigroups $S^{(+)}$ on the spaces $L_2(\Omega_\pm)$ and for the distance functions $\rho_\pm$ obtained by restricting the Riemannian distance $d(\cdot, \cdot)$ to $\Omega_\pm$.

The Dirichlet forms $h_\pm$ are, however, clearly strictly local and regular. Thus it remains to consider properties of the Riemannian distance $d(\cdot, \cdot)$. Let $d_\epsilon(\cdot, \cdot)$ denote the standard Euclidean distance on $\mathbf{R}^{n+m}$. Since the Riemannian distance is equivalent to the distance $D_\delta(\cdot, \cdot)$ given by (18) it follows that for each $x \in \mathbf{R}^{n+m}$ and $r > 0$ there exists a positive real number $r' > 0$ such that $d_\epsilon(x ; y) < r'$ implies that $d(x ; y) < r$ and conversely $d(x ; y) < r$ implies $d_\epsilon(x ; y) < r'$. Thus the distances $d_\epsilon(\cdot, \cdot)$ and $d(\cdot, \cdot)$ determine the same topology. Then a standard argument establishes that $\mathbf{R}^{n+m}$ and any of its closed subsets are complete with respect to both distances. It follows that the spaces $\mathbf{R}^{n+m}$ and $\mathbf{R}_+ \times \mathbf{R}^m \subset \mathbf{R}^{m+1}$ satisfy assumptions (A1) and (A2) of [GSC11], page 24. Note that this implies that these spaces are geodesic length spaces in the terminology of Theorem 2.11 of [GSC11].

Finally these observations establish that the theorem of Sturm applies to the operators $H_\pm$. Hence they satisfy the parabolic Harnack inequality on $\Omega_\pm$. Then the proof of Theorem [L3] is a repetition of the arguments used to establish Theorem [L2].

Theorems [L2] and [L3] demonstrate that the heat semigroup corresponding to the degenerate operator $H$ has a Gaussian character similar to that of a non-degenerate strongly elliptic operator. Even in the non-ergodic situation $n = 1$, $\delta_1 \in [1/2, 1)$, the Gaussian characteristics persist in the ergodic components. The Gaussian upper bounds on the kernel are, however, not optimal. These estimates can be improved as in the strongly elliptic case, e.g. for each $\epsilon > 0$ one can choose $a'$ such that $\omega' = (4 + \epsilon)^{-1}$.

5 The exceptional case

The discussion of the heat kernel in Section 4 does not cover the case $n = 1$, $\delta_1 \in [0, 1/2)$ and $\delta'_1 \in [1/2, 1)$. Moreover, it follows from Theorem [L1] II that in this case the uniform Poincaré inequality is not valid. The proof in Section 3 that the inequality is invalid demonstrates that the problem is a global one. In this section we establish that the inequality is nevertheless valid locally and subsequently discuss the implications for the heat semigroup.

The principal result is the following.

**Theorem 5.1** Assume $n = 1$, $\delta_1 \in [0, 1/2)$ and $\delta'_1 \in [1/2, 1)$. Then there is a $\kappa \in (0, 1]$ and for each $R > 0$ there is a $\lambda_R > 0$ such that

$$\int_{B(x;r)} dy \Gamma(\varphi)(y) \geq \lambda_R r^{-2} \int_{B(x;kr)} dy \langle \varphi(y) - \langle \varphi \rangle_B \rangle^2 \quad (30)$$

for all $x \in \mathbf{R}_1^{1+m}$, $r \in (0, R]$ and $\varphi \in C^1(\mathbf{R}^{n+m})$ where $\langle \varphi \rangle_B = |B(x ; kr)|^{-1} \int_{B(x;kr)} dy \varphi(y)$.

The conclusion of the theorem is considerably weaker than Statement I of Theorem [L1] since $\lambda_R$ tends to zero as $R \to \infty$. In fact the proof of the theorem establishes that the rate of convergence is given by a power of $R^{-\alpha'(\delta'_1 - \delta_1)}$.

The proof is similar to the proof [L1]. It consists of three steps. First one proves the inequality (30) for balls centred at the origin, secondly for balls which do not contain the origin and finally one deduces the result for general balls from the two special cases. The
only major change occurs in the first step, the discussion of balls centred at the origin. The essential feature is the following analogue of Proposition 3.4.

**Proposition 5.2** Assume $n = 1$, $\delta_1 \in [0, 1/2)$ and $\delta'_1 \in [1/2, 1)$. Then for each $T > 0$ there is a $\lambda_T > 0$ such that

\[
\int_{C_t} dx \Gamma_{\delta}(\varphi)(x) \geq \lambda_T t^{-2} \int_{C_t} dx (\varphi(x) - \langle \varphi \rangle)^2 \tag{31}
\]

for all $\varphi \in C^1_c(\mathbb{R}^{1+m})$ and $t \in (0, T]$ where $\langle \varphi \rangle = |C_t|^{-1} \int_{C_t} dx \varphi(x)$.

**Proof** First if $t \leq 1$ then the integral on the left hand side of (31) is independent of $\delta'_1$ and the inequality is a corollary of Proposition 3.4.

Secondly if $t \geq 1$ then $C_t$ is the product of an interval $-t^{\alpha'} \leq x_1 \leq t^{\alpha'}$ and a cube $|x_2| \leq t^{\beta'}$ in $\mathbb{R}^m$. Then changing variables to $y_1 = t^{-\alpha'}x_1$ and $y_2 = t^{-\beta'}x_2$ and setting $\psi(y) = \varphi(x)$ one has

\[
\int_{C_t} dx \Gamma_{\delta}(\varphi)(x) = t^{4\alpha' + m}\beta'} \int_{C_1} dy \left( t^{-2\alpha'} |t^{\alpha'} y_1|^{2m} |(\partial_{y_1} \psi)(y)|^2 \right)
\]

\[
+ t^{-2\beta'} |t^{\alpha'} y_1|^{2m} |(\nabla_{y_2} \psi)(y)|^2 .
\]

Since $\delta'_1 > \delta_1$, $t \geq 1$ and $|y_1| \leq 1$ it follows from Proposition 2.1 that

\[
t^{-2\alpha'} |t^{\alpha'} y_1|^{2m} \delta_1 \geq 2^{-4\delta_1} t^{-2\alpha'} t^{2\alpha' \delta_1} |y_1|^{2\delta_1} \geq 2^{4} t^{-2} T^{-2\alpha'(\delta'_1 - \delta_1)}
\]

for all $t \in (0, T]$ where we have used $1 - \alpha' + \alpha' \delta_1 = -\alpha'(\delta'_1 - \delta_1) < 0$. Similarly

\[
t^{-2\beta'} |t^{\alpha'} y_1|^{2m} \delta_2 \geq 2^{-2(\delta_2 + \delta'_2)} t^{-2\beta'} t^{2\alpha'(\delta_2 \vee \delta'_2)} |y_1|^{2\delta_2} \geq 2^{2}(\delta_2 + \delta'_2) t^{-2} |y_1|^{2\delta_2}
\]

where the last step uses $1 - \beta' + \alpha'(\delta_2 \vee \delta'_2) = \alpha'(\delta_2 \vee \delta'_2 - \delta_2) \geq 0$. Combining these estimates one has

\[
\int_{C_t} dx \Gamma_{\delta}(\varphi)(x) \geq a_{\delta}(T) t^{-2} t^{4\alpha' + m}\beta'} \int_{C_1} dy \left( |y_1|^{2\delta_1} |(\partial_{y_1} \psi)(y)|^2 + |y_1|^{2\delta_2} |(\nabla_{y_2} \psi)(y)|^2 \right)
\]

\[
= a_{\delta}(T) t^{-2} t^{4\alpha' + m}\beta'} \int_{C_1} dy \Gamma_{\delta}(\psi)(y)
\]

for all $t \in (0, T]$ where $a_{\delta}(T) = 2^{4} T^{-2\alpha'(\delta'_1 - \delta_1)} (1 \wedge 2^{-2(\delta_2 + \delta'_2 - 2)})$. But $C_1 = [-1, 1]^{1+m}$ so it follows from Proposition 3.4 that there is a $\lambda > 0$ such that

\[
t^{4\alpha' + m}\beta'} \int_{C_1} dy \Gamma_{\delta}(\psi)(y) \geq \lambda t^{4\alpha' + m}\beta'} \int_{C_1} dy (\psi(y) - \langle \psi \rangle)^2
\]

\[
= \lambda \int_{C_t} dx (\varphi(x) - \langle \varphi \rangle)^2
\]

where the last step follows by reverting to the original $x$-coordinates. Finally one concludes by combination of these estimates that (31) is valid with $\lambda_T = \lambda a_{\delta}(T)$. □

Now the proof of Theorem 5.1 is essentially a corollary of Proposition 5.2 and the arguments used to prove Theorem 1.1.
Proof of Theorem 5.1.} First, it follows from Proposition 5.2 and and the embedding statements, Lemmas 3.5 and 3.6 that there is a $\kappa \in [0, 1]$ and for each $R > 0$ there is a $\lambda_R > 0$ such that the Poincaré inequality (30) is valid for all balls $B_{\Delta}(0; r)$ with $r \in (0, R]$. Note that the embedding lemmas are general geometric results which are valid for all $n$ all $\delta, \delta' \in [0, 1)$ and balls of arbitrary radius. Moreover, the value of $\kappa$ which occurs in Lemma 3.6 is independent of the radius of the balls. It also follows from these lemmas together with the estimates in the foregoing proof that $\lambda_R$ converges to zero as $R \to \infty$ and the rate of convergence is given by a power of $R^{-\alpha(\delta - \delta')}$.

Secondly, it follows from the discussion of Case II of the proof of Theorem 1.1.I that one may also choose $\kappa$ and $\lambda_R$ such that (30) is valid for all balls $B_{\Delta}((\xi_1, 0); r)$ with $r < r_\xi$. Note that the arguments in Case II are independent of the assumption $\delta \vee \delta' \in [0, 1/2)$ if $n = 1$ (see Remark 3.12). Hence the arguments apply with the current assumptions $\delta, \delta' \in [0, 1/2]$ and $\delta' \in [1/2, 1]$.

It now remains to establish (30) for the balls $B_{\Delta}((\xi_1, 0); r)$ with $r_\xi < R$ and $r \in [r_\xi, R]$. But this is again a corollary of the foregoing special cases.

Set $K = 2(1 + \kappa)/\kappa$. First suppose $K r_\xi \leq R$. Then if $r \in [K r_\xi, R]$ the Poincaré inequality follows from the previous two special cases by the first argument in the discussion of Case III of the proof of Theorem 1.1.I. If, however, $r \in [r_\xi, K r_\xi]$ it follows from the special cases by the second argument in the discussion of Case III. Secondly, if $R \leq K r_\xi$ then the condition $r \in [r_\xi, R]$ implies that $r \in [r_\xi, K r_\xi]$ and the Poincaré inequality follows again.

Finally combination of these conclusions gives the Poincaré inequality (30) as stated in Theorem 5.1

The Poincaré inequality (30) of Theorem 5.1 is a local version of the inequality (4) established in Theorem 1.1 for $n \geq 2$ or for $n = 1$ and $\delta \vee \delta' \in [0, 1/2)$ insofar the value of $\lambda$ depends on the scale $R$ of the balls. But the results of Grigor’yan and Saloff-Coste establish that much of the discussion of Section 4 still applies to the heat kernel although the conclusions are of a local nature. In particular the volume doubling property in combination with the local Poincaré inequality gives a local version of the parabolic Harnack inequality. Explicitly, if $n = 1$, $\delta \in [0, 1/2]$ and $\delta' \in [1/2, 1]$ then for each $R > 0$ there exists an $a > 0$ such that for any $x \in R_{1+m}, t > 0$ and all $r \in (0, R]$ any non-negative (weak) solution $\varphi$ of the parabolic equation $(\partial_t + H)\varphi = 0$ in the cylinder $Q = (t, t + r^2) \times B(x; 2r)$ satisfies

$$
\sup_{Q^-} \varphi \leq a \inf_{Q^+} \varphi \tag{32}
$$

where $Q^- = [t + r^2/4, t + r^2/2] \times B(x; r)$ and $Q^+ = [t + 3 r^2/4, t + r^2) \times B(x; r)$.

The local version of the Harnack inequality again implies Hölder continuity of the heat kernel by Moser’s argument. Therefore Gaussian upper bounds on the kernel follow from the almost everywhere bounds of Corollary 6.6 in [RS08].

**Corollary 5.3** Assume $n = 1$, $\delta \in [0, 1/2]$ and $\delta' \in [1/2, 1]$. Then the semigroup kernel $x, y \in R_{1+m} \mapsto K_t(x; y)$ is jointly Hölder continuous and there exist $a', \omega' > 0$ such that

$$
K_t(x; y) \leq a' |B(x; t^{1/2})|^{-1} e^{-\omega' d(x,y)^2/t}
$$

for all $x, y \in R_{1+m}$ and $t > 0$. 

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Although the heat kernel satisfies Gaussian upper bounds this type of bound is not expected to be optimal for reasons we discuss below. In addition one cannot expect matching Gaussian lower bounds as these would imply the global parabolic Harnack inequality which in turn would imply the global Poincaré inequality in contradiction with Theorem 1.1. Nevertheless one has the on-diagonal lower bounds (27) established in [RS08] and then arguing with the local Harnack inequality as in the proof of Theorem 1.2 one obtains the small time off-diagonal lower bound
\[ K_t(x; y) \geq (c/a) |B(x; t^{1/2})|^{-1} \] (33)
with the restrictions \( d(x; y) \leq t^{1/2} \leq R \) where \( a \) and \( R \) are the parameters in (32). These small-time lower bounds then imply that the kernel \( K_t \) is strictly positive for all \( t > 0 \). This is a consequence of the semigroup property and an estimate of the type (29). Consequently the semigroup \( S_t \) is ergodic, i.e. there are no non-trivial \( S \)-invariant subspaces of the form \( L^2(\Omega) \).

The complications with this exceptional case arise because the subspaces \( \Omega_{\pm} \) are ‘approximately’ invariant. We will not discuss the precise meaning of approximately invariant but instead argue that there is a similarity of the evolution in the exceptional case and the evolution on manifolds with ends as described by Grigor’yan and Saloff-Coste [GSC09]. In the special case of manifolds with two ends with the same dimension the situation can be described as two copies of \( \mathbb{R}^n \) connected by a compact cylinder. Assuming the manifold is rotationally invariant it can be identified as \( \mathbb{R} \times S^n \) with polar coordinates \( (r, \sigma) \in \mathbb{R} \times S^n \) the Riemannian metric is given by \( d\sigma^2 + f(r)d\sigma^2 \), where \( f(r) > 0 \) is continuous and \( f(r) = r^{-2} \) for \( |r| \geq 1 \). Then the quadratic form corresponding to the Laplace Beltrami operator can be defined as
\[ Q(\psi) = \int_{\mathbb{R}} \int_{S^n} (|\partial_r \psi|^2 + f(r)^{-1} |\partial_\sigma \psi|^2) g(r) dr d\sigma \]
where \( g(r) > 0 \) is continuous and \( g(r) = r^{n-1} \) for \( |r| \geq 1 \). If one is just interested in the evolution corresponding to Brownian motion in the radial direction then we can restrict attention to functions which are invariant in \( \sigma \). This leads to the one-dimensional operator \( L \) acting on \( L^2(\mathbb{R}; g(r) dr) \) corresponding to the quadratic form
\[ Q_1(\phi) = \int_{\mathbb{R}} |\partial_r \phi|^2 g(r) dr . \]
After a simple change of variable, which we describe in Example 5.4 below, the operator \( L \) is equivalent to a one-dimensional degenerate elliptic operator. It is worth noting that if we consider an operator \( H \) acting on \( L^2(\mathbb{R} \times \mathbb{R}^m) \) and are only interested in the evolution of \( x_1 \) then we again obtain the same operator or rather its equivalent version discussed in Example 5.4. This means that at least in the radial case the approximate invariance corresponds is characterized by the heat kernel bounds described in [GSC09].

We conclude with an example, adapted from [HS09], which illustrates the structure of the \( H_{\delta,0} \) and \( S^{\delta,0} \) in the simplest case \( \delta_1 = 0 \) and the connection with the end problem.

**Example 5.4** Let \( H_0 = -d_x (1 \lor |x|)^{2\delta} d_x \) with \( \delta \in [1/2, 1) \) be the operator on \( L^2(\mathbb{R}) \) with domain \( C^\infty_c(\mathbb{R}) \). Since the coefficient of \( H_0 \) is strictly positive the operator is essentially self-adjoint. Let \( H \) denote the self-adjoint closure and \( h \) the corresponding Dirichlet form. Now for \( \varphi \in C^\infty_c(\mathbb{R}) \) define \( \Phi = \varphi \circ f \) where \( f(x) = (-|x|^\alpha) \land x \) if \( x \leq 0 \) and \( f(x) = x \lor |x|^\alpha \).
if $x > 0$ with $\alpha = (1 - \delta)^{-1} \in [2, \infty)$. The mapping $\varphi \mapsto \Phi$ extends to an isometric isomorphism $U$ from $L_2(\mathbb{R})$ to $L_2(\mathbb{R} : \mu)$ where $d\mu = f'$. Thus the $L_2(\mathbb{R} : \mu)$-norm $\| \cdot \|_{2, \mu}$ is given by

$$\|\Phi\|_{2, \mu}^2 = \int_{\mathbb{R}} dy f'(y) |\Phi(y)|^2$$

$$= \alpha \int_{-\infty}^{-1} dy |y|^\alpha |\Phi(y)|^2 + \int_{-1}^{1} dy |\Phi(y)|^2 + \alpha \int_{1}^{\infty} dy |y|^{\alpha-1} |\Phi(y)|^2.$$ 

But if $\alpha = k$ is a positive integer, i.e. if $\delta = 1 - k^{-1}$, then $dy |y|^{k-1}$ is the radial measure on $\mathbb{R}^k$. Therefore the first and last integrals can be identified as the square of the radial part of the $L_2(\mathbb{R}^k)$-norm restricted to $\mathbb{R}^k \setminus B(0; 1)$ where $B(0; 1)$ denotes the unit Euclidean ball centred at the origin. Next one calculates that $h(\varphi) = h_{\mu}(\Phi)$ for each $\varphi \in C_c^\infty(\mathbb{R})$ where

$$h_{\mu}(\Phi) = \alpha^{-1} \int_{-\infty}^{-1} dy |y|^{\alpha-1} |\Phi'(y)|^2 + \int_{-1}^{1} dy |\Phi'(y)|^2 + \alpha^{-1} \int_{1}^{\infty} dy |y|^{\alpha-1} |\Phi'(y)|^2.$$ 

This identification then extends by closure to all $\varphi \in D(h)$. The form $h_{\mu}$ is a Dirichlet form on $L_2(\mathbb{R} : \mu)$ with $D(h_{\mu}) = UD(h)$. Therefore the corresponding self-adjoint operator $H_{\mu} = UHU^{-1}$ generates the submarkovian semigroup $S_t^{\mu} = US_tU^{-1}$. If $\alpha = k$ is a positive integer the operator $H_{\mu}$ models the radial part of the Laplace-Beltrami operator acting on a manifold which consists of two copies $\mathbb{R}^k \setminus B(0; 1)$ with a cylindrical channel joining the two balls. The semigroup $S^{\mu}$ describes a diffusion process for which the two ends $\mathbb{R}^k \setminus B(0; 1)$ are largely invariant since the probability of passing from one end of the manifold to the other is small. Such processes have been studied by Grigor’yan and Saloff-Coste [GSC09]. In particular they have derived matching upper and lower bounds which describe non-Gaussian behaviour.

**Acknowledgement**

This collaboration was carried out during numerous visits of the second author to the Mathematical Sciences Institute at ANU with the support of ARC Discovery grant DP130101302.

**References**

[BCF96] Benjamini, I., Chavel, I., and Feldman, E. A., Heat kernel lower bounds on Riemannian manifolds using the old ideas of Nash. *Proc. London Math. Soc.* 72 (1996), 215–240.

[CF91] Chavel, I., and Feldman, E. A., Isoperimetric constants, the geometry of ends, and large time heat diffusion in Riemannian manifolds. *Proc. London Math. Soc.* 62 (1991), 427–448.

[Dav97] Davies, E. B., Non-Gaussian aspects of heat kernel behaviour. *J. London Math. Soc.* 55 (1997), 105–125.
[SC92b] , Uniformly elliptic operators on Riemannian manifolds. *J. Diff. Geom.* **36** (1992), 417–450.

[SC95] , Parabolic Harnack inequality for divergence-form second-order differential operators. *Potential Anal.* **4** (1995), 429–467.

[Stu95] STURM, K.-T., Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. *Osaka J. Math.* **32** (1995), 275–312.

[Stu96] , Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. *J. Math. Pures Appl.* **75** (1996), 273–297.

[Tru73] TRUDINGER, N. S., Linear elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa* **27** (1973), 265–308.