Do manifolds have little symmetry?

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Key words: Symmetry, group actions, manifolds, cohomology
Subject classification: 57S17, 55N91

Introduction.

This note is surveying certain aspects of the following problem stated by F. Raymond and R. Schultz (s. [BH], p. 260).

"It is generally felt that a manifold 'chosen at random' will have very little symmetry. Can this intuitive notion be made more precise? In connection with this intuitive feeling, we have the following specific question. Does there exist a closed simply connected manifold, on which no finite group acts effectively? (A weaker question, no involution?)"

The problem is also listed as one of five conjectures in the survey [AD]. But since in that reference very little information is given about this problem, I hope that the present note can serve as a useful complement.

As a general assumption we consider here closed, connected (topological) manifolds with continuous group actions.

1. Some remarks about non-simply connected asymmetric manifolds.

In the beginning of the 1970's several people have shown the existence of asymmetric (i.e. not admitting any effective action of a finite group) manifolds (see, e.g. [CRW], [B]). In fact, e.g. R. Schultz has shown that, for dimension \( \geq 4 \), in any cobordism class there are infinitely many asymmetric manifolds (s. [Sc1], [Sc2]) ; and several
authors have given examples of asymmetric 3-dimensional manifolds (cf. [E1]). (It is, of course, easy to see that there are no asymmetric manifolds of dimension 1 or 2.)

All of these examples have non-trivial fundamental group. An essential tool for many of the examples mentioned above is the following result due to A. Borel, which uses the fundamental group to detect asymmetry.

**Theorem (Borel)** If $M$ is an aspherical (i.e., $\pi_i(M) = 0$ for $i \geq 2$) manifold, such that

(i) $\pi_1(M)$ is centerless

(ii) $Out(\pi_1(M))$, the outer automorphism group of $\pi_1(M)$, is torsion free,

then $M$ is asymmetric (cf. [Bo],Cor.2 and [CR], Thm.3.2).

R. Waldmüller found the first example of a centerless Bieberbach group $B$ (i.e., a torsion free subgroup of the group of isometries $\mathbb{R}^n$, such that $M := \mathbb{R}^n/B$ is a compact aspherical manifold) with $Out(B) = \{1\}$ (s. [Wd]). Hence $M$ is asymmetric by the above theorem.

### 2. How to use cohomology to detect asymmetry

At first glance it might seem unlikely that cohomological information could suffice to detect asymmetry. Of course, an action of a finite group $G$ on a manifold $M$ induces an action of $G$ on the cohomology algebra $H^*(M)$, which clearly could be trivial without the original action being so. Hence the question is, how a non-trivial, but cohomologically trivial action, is reflected in cohomology.

We first consider involutions, i.e., $G \cong \mathbb{Z}/2\mathbb{Z}$ and we use cohomology with coefficients in $k = \mathbb{F}_2$. In case of a cohomologically trivial $G$–action on $M$ the $E_2$–term of the Serre spectral sequence of the Borel construction $M \to EG \times_G M \to BG$ is isomorphic to $k[t] \otimes H^*(M;k)$, $\deg(t) = 1$.

The first non-trivial higher differential is given by a derivation
\[ \partial : H^*(M; k) \to H^*(M; k) \] of negative degree with \( \partial^2 = 0 \). If all higher differentials vanish, the equivariant cohomology \( H^*_G(M; k) \) is isomorphic to \( k[t] \otimes H^*(M) \) as \( k[t] \)-module, but not necessarily as \( k[t] \)-algebra. The famous Localization Theorem for equivariant cohomology (see, e.g. [AP]) then implies that there is a filtration on the cohomology \( H^*(M^G; k) \) of the fixed point set \( M^G \) such that the associated graded algebra is isomorphic to \( H^*(M; k) \) (s. [Pu3], p. 131/132).

This means in particular that \( H^*(M^G; k) \) (as a filtered algebra with filtration \( F_i(H^*(M^G; k)) := \oplus_{j=0}^i H^j(M^G; k) \)) is a deformation of negative weight of the graded algebra \( H^*(M; k) \) (s. [Pu2]).

If this deformation is trivial then \( H^*(M; k) \) and \( H^*(M^G; k) \) are isomorphic as algebras yet not necessarily as graded algebras. But if \( H^*(M; k) \) has ”minimal formal dimension”, i.e. any graded algebra (occurring as the cohomology algebra of a manifold), which is isomorphic to \( H^*(M; k) \), as algebra (ignoring the grading), has formal dimension bigger or equal to that of \( H^*(M) \), then Smith theory implies that the inclusion \( M^G \to M \) induces an isomorphism of graded algebras (cf. [AHsP]). Hence in this case \( M^G = M \), i.e. the action is trivial.

Putting all this together, we get the following result, which in a sense is analogous to Borel’s result above, but uses the cohomology algebra instead of the fundamental group to detect asymmetry.

**Theorem 1.** Let \( M \) be a compact manifold such that

(i) \( H^*(M; k) \) has no automorphism of order 2

(ii) \( H^*(M; k) \) has no non-trivial derivation of negative degree

(iii) \( H^*(M; k) \) has no non-trivial deformation of negative weight

(iv) \( H^*(M; k) \) has minimal formal dimension, then \( M \) does not admit any non-trivial involution.

**Remark 1.** The condition (iii) in Theorem 1 can be replaced by

(iii’) \( H^*(M; k) \) cannot be given as the associated graded algebra of a filtration of a product of Poincaré algebras of formal dimension \( < \dim M \) (see above).
An analogous result holds for \( \mathbb{Z}/p\mathbb{Z} \)-actions, \( p \) prime, choosing \( k = \mathbb{F}_p \).

Now the question is whether there exist examples fulfilling (i) - (iv) in Theorem 1, and one might expect that they are even 'generic'. In fact, this is true in a certain sense. If we consider 3-dimensional manifold there is no classification available, but we can use the 'parametrization' by their cohomology algebras (with \( k = \mathbb{F}_2 \) coefficients), which correspond to trilinear, symmetric forms on \( H^1(M; k) \), to give a meaning to terms like 'generic' or 'chosen at random'. Or, at least to say, what is meant by 'most \( \mathbb{F}_2 \)-cohomology types of 3-manifolds' in Theorem 2 below.

Namely, if \( \dim_k H^1(M; k) = m \) then the space of trilinear, symmetric forms on \( H^1(M; k) \) is isomorphic to the space \( \mathcal{S}^3(k^m) \cong k^{\alpha(m)} \), \( \alpha = \left( \frac{m+2}{3} \right) \). Let \( \mathcal{R}(m) \subset \mathcal{S}^3(k^m) \) be the subset of forms that can be realized by the cohomology algebras of 3-manifolds with \( \dim_k H^1(M; k) = m \). According to M. Postnikov \( \mathcal{R}(m) = \mathcal{R}^o(m) \cup \mathcal{R}^n(m) \), where \( \mathcal{R}^o(m) := \{ \mu \in \mathcal{S}^3(k^m) ; \mu(x, x, y) + \mu(x, y, y) = 0 \} \) for all \( x, y \in k^n \), and \( \mathcal{R}^n(m) := \{ \mu \in \mathcal{S}^3(k^m) ; \exists x^o \in k^m \text{ s. } x^o \neq 0 \text{ such that } \mu(x, x, y) + \mu(x, y, y) = \mu(x^o, x, y) \} \) (s. [Po]). And let \( \mathcal{I}(m) \subset \mathcal{R}(m) \) be the subset of forms, which can be realized by the cohomology algebras of 3-manifolds admitting non-trivial involutions. By \( |A| \) we denote the number of elements of a subset \( A \subset \mathcal{S}^3(m) \).

Using Theorem 1 one gets the following result.

**Theorem 2.** Most 3-manifolds do not admit a non-trivial involution; more precisely:

\[
\lim_{m \to \infty} \frac{|\mathcal{I}(m)|}{|\mathcal{R}(m)|} = 0 .
\]

See [Pu6] for details, where in particular the connection with binary, self-dual codes is studied.

Actions of \( \mathbb{Z}/p\mathbb{Z} \), \( p \) odd prime, on 3-manifold can be treated in a similar way. As an illustration of the method we give a very simple proof of the following result, which was proved independently by Su (where it is somewhat hidden in [Su], Theo-
rem \( (3.9) \), J.H. Przytycki and M.V. Sokolov (s. [PS], Theorem 2.1) and A. Sikora (s. [Si], Prop.(1.7)); cf. also Example (2.9) in [AHkP].

**Proposition.** If a closed orientable 3-manifold \( M \) admits an action of a cyclic group \( G \cong \mathbb{Z}/p\mathbb{Z} \) where \( p \) is an odd prime and the fixed point set of the action is \( S^1 \) then \( H_1(M;\mathbb{F}_p) \neq \mathbb{F}_p \).

**Proof.** Let us assume that \( H^1(M;\mathbb{F}_p) \cong H_1(M;\mathbb{F}_p) \cong \mathbb{F}_p \) and that \( M^G \neq \phi \). We will then show that the Serre spectral sequence of the Borel construction collapses and hence \( \dim H^*(M^G;\mathbb{F}_p) = \dim H^*(M;\mathbb{F}_p) = 4 \) by the Localization Theorem, which implies the Proposition. Since \( \dim H^i(M;\mathbb{F}_p) = 1 \) for \( i = 0, 1, 2, 3 \) the action must be cohomologically trivial. It remains to show that the higher differentials in the Serre spectral sequence vanish. Since \( E_2 \cong H^*(BG;\mathbb{F}_p) \otimes H^*(M;\mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda(s) \otimes H^*(M;\mathbb{F}_p) \). It suffices to show that \( H^*(M;\mathbb{F}_p) \) does not admit any non-trivial derivations of negative degree.

Let \( 1, a_1, a_2, a_3 \) be generators of \( H^i(M;\mathbb{F}_p) \) for \( i = 0, 1, 2, 3 \), such that \( a_1 \cup a_2 = a_3 \). Since \( M^G \neq \phi \), 1 can not be a boundary. So any derivation \( \partial \) of negative degree vanishes on \( a_1 \). It follows that \( \partial(a_3) = \partial(a_1 \cup a_2) = (\partial a_1) \cup a_2 - a_1 \cup \partial a_2 = 0 \), if \( \partial \) has degree \((-1)\), since \( a_1^2 = 0 \). Also \( \partial a_2 \) must vanish, since \( 0 = \partial(a_2^2) = 2(\partial_2) \cup a_2 \); so \( \partial a_2 = \lambda a_1 \) must be zero (for \( p \) odd). A derivation of degree \((-m)\), \( m \geq 2 \) must vanish on \( a_1 \) and \( a_2 \) (since 1 is not a boundary), and hence also on \( a_3 = a_1 \cup a_2 \).

\( \square \)

Of course, one can not get simply connected 3-manifolds without symmetry by the above approach, but the method of proof does not refer to the fundamental group and hence could be applied to simply connected manifolds of higher dimension. It does not work for dimensions 4 and 5, though.

A. Edmonds’ discussion of cyclic group actions on simply connected 4-manifolds (s. [E2], [E3]) in particular implies that there are no asymmetric ones, and in dimension 5 the information given by the cohomology algebra of a simply connected manifold is certainly to weak to detect asymmetry. Hence we discuss simply connected 6-manifolds in the next section.
3. Simply connected 6-manifolds

Classification theorems for certain types of simply connected 6-manifolds have been given by C.T.G. Wall ([Wa]), P.E. Jupp ([J]) and A.V. Žubr ([Z]). For the class $\mathcal{M}$ of simply connected, 6-dimensional spin-manifolds $M$ with $H^3(M;\mathbb{Z}) = 0$ the following result is contained in [Wa].

**Theorem (Wall).** The diffeomorphism classes of elements of $\mathcal{M}$ correspond bijectively to isomorphism classes of invariants.

1. $H$ free $\mathbb{Z}$-module of finite rank (corresponding to $H^2(M;\mathbb{Z})$ for $M \in \mathcal{M}$)
2. $\mu : H \times H \times H \rightarrow \mathbb{Z}$ trilinear, symmetric form (corresponding to the cup product in $H^*(M;\mathbb{Z})$)
3. $P : H \rightarrow \mathbb{Z}$ linear map (corresponding to the dual of the first Pontrjagin class)

Subject to the following conditions:

(a) $\mu(x, x, y) \equiv \mu(x, y, y) \pmod{2}$ for $x, y \in H$

(b) $P(x) \equiv 4\mu(x, x, x) \pmod{24}$ for $x \in H$.

Similar to Section 2 we can parametrize the elements in $\mathcal{M}$ by the corresponding trilinear, symmetric form in $S^3(\mathbb{Z}^m) \cong \mathbb{Z}^{\alpha(m)}$, $\alpha(m) = \binom{m+2}{3}$.

By Wall’s result this is much closer to an actual classification up to diffeomorphism or homeomorphism than the parametrization in Section 2.

Let $\mathcal{R}(m)$ again denote the set of forms, which can be realized by the cohomology of elements in $\mathcal{M}$.

We define the density of a subset $\mathcal{A} \subset \mathcal{R}(m)$ by

$$d_m(\mathcal{A}) := \lim_{N \to \infty} \sup \frac{|\mathcal{A} \cap [-N, N]^{\alpha(m)}|}{|\mathcal{R}(m) \cap [-N, N]^{\alpha(m)}|}$$
Using Theorem 1 and its analogue for $G = \mathbb{Z}/p\mathbb{Z}$, $p$ prime, one obtains the following result (s. [Pu5] for details).

Theorem 3.

(a) For $m \geq 6$ the subset of $\mathcal{R}(m)$ corresponding to those manifolds, which admit a cohomologically non-trivial, orientation preserving action of a finite group has density zero.

(b) For $m \geq 6$ the subset of $\mathcal{R}(m)$ corresponding to those manifolds which admit non-trivial $\mathbb{Z}/p\mathbb{Z}$–actions for infinitely many primes $p$ has density zero.

(c) For a given prime $p$, let $\mathcal{C}_p(m) \subset \mathcal{R}(m)$ denote the subset corresponding to those manifolds, which admit a non-trivial, orientation preserving $\mathbb{Z}/p\mathbb{Z}$–action. Then

$$\lim_{m \to \infty} d_m(\mathcal{C}_p(m)) = 0.$$ 

Theorem 3 gives a precise meaning to the somewhat vague statement that most manifolds in $\mathcal{M}$ have little symmetry.

Example (Iarrobino). The following polynomial of degree 3 in 6 variables gives a trilinear, symmetric form, and hence a Poincare duality algebra over $\mathbb{Z}$.

$$f(x_1, \ldots, x_6) = 6(x_1x_2^2 - x_1^2x_4 + x_2x_1^2 + x_2x_4^2 - x_2^2x_5 + x_2x_5^2 + x_3^2x_4 - x_3x_4^2 + x_3^2x_6 + x_3x_6^2 + x_5^2x_6 + x_5x_6^2 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_6 + x_2x_4x_6 + x_3x_5x_6 + x_4x_5x_6 + x_4x_5x_6 + x_4x_5x_6 + x_5x_6^2 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_6 + x_2x_4x_6 + x_3x_5x_6 + x_4x_5x_6 + x_4x_5x_6 + x_5x_6^2 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_6 + x_2x_4x_6 + x_3x_5x_6 + x_4x_5x_6 + x_4x_5x_6 + x_5x_6^2)$$

G. Nebe has verified that this algebra has no orientation preserving automorphisms of finite order (cf. condition (i) in Theorem 1) and T. Iarrobino and A. Suciu have verified that it has no deformations of negative weight modulo any prime (cf. condition (iii) with the help of computer calculations. Conditions (ii) (for derivations of odd degree) and (iv) (for a non-trivial trilinear form and $p$ odd) are easily seen to hold for all elements in $\mathcal{R}(m)$. But clearly (iv) is not fulfilled for $p = 2$. This does not matter in the case at hand if one assumes the $\mathbb{Z}/2\mathbb{Z}$–action to be orientation preserving (because then the fixed point set must have even codimension), but it
shows that one can not exclude the possibility of orientation reversing involutions by applying Theorem 1. So one gets the following result.

**Theorem 4.** There exist simply connected manifolds on which no finite group can act effectively and orientation preserving.

(See [Pu5]).

For orientation reversing involutions one can imitate the arguments leading to Theorem 3 up to a certain point. This is sketched in [Pu5], Remark 1.3. But since some of the consequences are prerequisites for Kreck’s result (s.[K]), we give some more details here.

Any algebra \( H^*(M, \mathbb{Z}) \), \( M \in \mathcal{M} \), admits an orientation reversing involution, given by the identity on \( H^0 \) and \( H^4 \), and by multiplication with \((-1)\) on \( H^2 \) and \( H^6 \), in contrast to the fact that most \( H^*(M; \mathbb{Z}) \) do not admit any non-trivial orientation preserving involution. Since the composition of an arbitrary orientation reversing involution with the particular one given above is an orientation preserving involution, it follows that most \( H^*(M; \mathbb{Z}) \) only admit the one orientation reversing involution given above.

The following results imply for most \( M \in \mathcal{M} \), if \( \tau: M \to M \) is a non-trivial involution, then \((M, \tau)\) is a conjugation space in the sense of [HHHP], in particular \( H^*(M; \mathbb{F}_2) \cong H^*(M^\tau; \mathbb{F}_2) \) by a degree halving isomorphism of algebras, \( M^\tau = \text{fixed point set} \) (cf. [Pu5], Remark 1.3)

**Theorem 5.** Let \( M \in \mathcal{M} \), and let \( H^*(M; \mathbb{F}_2) \) be generated by \( H^2(M; \mathbb{F}_2) \) as an algebra. Assume that \( \tau: M \to M \) is an involution with non-empty fixed point set \( M^\tau \), which acts on \( H^2(M; \mathbb{Z}) \) by multiplication with \((-1)\). Then \((M, \tau)\) is a conjugation space.

**Proof.** We first consider the Serre spectral sequence of the Borel construction of the \( C \)-space \( M, C := \{id, \tau\} \cong \mathbb{Z}/2\mathbb{Z} \) with coefficients in \( \mathbb{F}_2 \). Since \( \tau \) acts trivially
By the Universal-Coefficients-Theorem we get an inclusion of algebras

\[ E_2^{s,*} \cong H^*(BC; \mathbb{F}_2) \otimes H^*(M; \mathbb{F}_2). \]

The differentials in the spectral sequence correspond to derivations of negative degree on \( H^*(M; \mathbb{F}_2) \). Since \( M^T \) is assumed to be non-empty, all these derivations vanish on \( H^2(M; \mathbb{F}_2) \). (Otherwise \( H^*_C(M; \mathbb{F}_2) \) would be annihilated by a power of \( u \in \mathbb{F}_2[u] \cong H^*(BC; \mathbb{F}_2), deg(u) = 1 \), and hence \( M^T \) would have to be empty by the Localization Theorem.) Since \( H^*(M; \mathbb{F}_2) \) is generated by \( H^2(M; \mathbb{F}_2) \), all differentials in the spectral sequence vanish, and the inclusion \( M^T \to M \) induces an injection \( H^*_C(M; \mathbb{F}_2) \to H^*_C(M^T; \mathbb{F}_2) = H^*(BC; \mathbb{F}_2) \otimes H^*(M^T; \mathbb{F}_2) \) (see, e.g. [AP], Prop. (1.3.14)).

We next try to calculate the analogous map for \( \mathbb{Z} \) coefficients. The \( E_2 \)-term of the Serre spectral sequence for the Borel construction of \( M \) is given by \( E_2^{s,*} \cong H^*(BC; H^*(M; \mathbb{Z})) \). The map induced by coefficient change, \( \mathbb{Z} \to \mathbb{F}_2 \), on the \( E_2 \)-terms is injective for all \( E_2^{i,j} \) with \( i > 0 \). So the spectral sequence with \( \mathbb{Z} \) coefficients also collapses at the \( E_2 \)-level. Since, up to periodicity (i.e. multiplication by \( t \in \mathbb{Z}[t]/(2t) \cong H^*(BC; \mathbb{Z}), deg(t) = 2 \), the only non-zero terms in the \( E_2 \)-term are \( E_2^{4,0}, E_2^{1,2} \) and \( E_2^{1,6} \), it is easy to see that there are no extension problems for \( E_2 = E_2 \).

This is shown in a more general context by M. Olbermann (s.[O1]). Hence, as \( H^*(BC; \mathbb{Z}) \)-module, \( H^*_C(M; \mathbb{Z}) \) is isomorphic to \( H^*(BC; \mathbb{Z}) \otimes (H^0 \oplus H^4) \oplus m \otimes (H^2[-1] \oplus H^6[-1]), \) where \( m := ker(H^*(BC; \mathbb{Z}) \to \mathbb{Z}), t \to 0, H^i := H^i(M; \mathbb{Z}), \) and "\([-1]" indicate a degree shift by \(-1\), i.e. we identify \( t \otimes H^2(M; \mathbb{Z})[-1] \) with \( E_2^{1,2} = H^1(BC; H^2(M; \mathbb{Z})) \), etc. Here, the assumption is used that \( \tau \) acts on \( H^2 \) (and \( H^6 \)) by multiplication with \(-1\).

Next we calculate the equivariant cohomology of the fixed point set \( M^T \) with \( \mathbb{Z} \) coefficients. This is not completely obvious since \( H^*(M^T; \mathbb{Z}) \) may have \( \mathbb{Z} \)-torsion. By the Universal-Coefficients-Theorem we get an inclusion of algebras

\[ H^*_C(M^T; \mathbb{Z}) \otimes \mathbb{F}_2 \longrightarrow H^*_C(M^T, \mathbb{F}_2) = H^*(BC; \mathbb{F}_2) \otimes H^*(M^T; \mathbb{F}_2). \]

The image of this map is contained in the kernel of the Bockstein operator \( \beta : H^*_C(M^T; \mathbb{F}_2) \to M^*_C(M^T; \mathbb{F}_2), \) more precisely: The intersection of this image with the kernel of the restriction to the fibre \( H^*_C(M^T; \mathbb{F}_2) \to H^*(M^T; \mathbb{F}_2) \) is the subalgebra...
of $H^*(BC; \mathbb{F}_2) \otimes H^*(M^\tau; \mathbb{F}_2)$ given by all elements of the form $u^{2k} \otimes x + u^{2k-1} \otimes \beta(x)$, where $x \in H^*(M^\tau; \mathbb{F}_2)$ and $u^{2k}, u^{2k-1} \in H^*(BC; \mathbb{F}_2) \cong \mathbb{F}_2[u]; k > 0$. (Recall that $\beta(u^{2k}) = 0$ and $\beta(u^{2k-1}) = u^{2k}$.)

We consider the following commutative diagram

$$
\begin{array}{ccc}
H^*_C(M; \mathbb{Z}) & \xrightarrow{r} & H^*_C(M^\tau; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^*_C(M; \mathbb{F}_2) & \xrightarrow{\rho} & H^*_C(M^\tau; \mathbb{F}_2)
\end{array}
$$

where the horizontal maps are induced by the inclusion $M^\tau \to M$, and the vertical maps by coefficient change $\mathbb{Z} \to \mathbb{F}_2$.

An element of the form $t \otimes a \in t \otimes H^2(M; \mathbb{Z})[-1] \subset H^*_C(M; \mathbb{Z})$ is mapped to $u \otimes \tilde{a} \in H^*_C(M; \mathbb{F}_2) \cong H^*(BC; \mathbb{F}_2) \otimes H^*(M; \mathbb{F}_2)$ under coefficient change, where $\tilde{a}$ is the image of $a$ in $H^2(M; \mathbb{F}_2)$. Since the restriction of $t \otimes a$ to the fibre, $H^*(M; \mathbb{Z})$, vanishes, the image of the element $r(t \otimes a)$ in $H^*_C(M^\tau; \mathbb{F}_2)$ is of the form $u^2 \otimes x + u \otimes \beta(x)$, with $x \in H^1(M^\tau; \mathbb{F}_2)$. It follows that $\tilde{r}(1 \otimes \tilde{a}) = u \otimes x + 1 \otimes \beta(x)$. Since $H^2(M; \mathbb{F}_2)$ generates $H^*(M; \mathbb{F}_2); 1 \otimes H^2(M; \mathbb{F}_2)$ generates $H^*_C(M; \mathbb{F}_2) \cong H^*(BC; \mathbb{F}_2) \otimes H^*(M; \mathbb{F}_2)$ as $H^*(BC; \mathbb{F}_2)$-algebra, even though the multiplication might - a priori - be twisted (which is indicated by the tilde sign). One therefore gets that $H^*(M^\tau; \mathbb{F}_2)$ is generated, as $\mathbb{F}_2$- algebra, by those $x \in H^1(M^\tau; \mathbb{F}_2)$ which occur in $\tilde{r}(1 \otimes \tilde{a})$ for $\tilde{a} \in H^2(M^\tau, \mathbb{F}_2)$ (cf. [Pu1]). This implies that the composition

$$
H^*_C(M; \mathbb{F}_2) \to H^*_C(M^\tau; \mathbb{F}_2) = H^*(BC; \mathbb{F}_2) \otimes M^*(M^\tau; \mathbb{F}_2)
$$

$$
\to H^*(BC; \mathbb{F}_2) \otimes H^*(M^\tau; \mathbb{F}_2)/ \oplus_{i<j} H^i(BC; \mathbb{F}_2) \otimes H^j(M^\tau; \mathbb{F}_2)
$$

is an isomorphism of $\mathbb{F}_2$-vector spaces. This property is the dual equivalent of a characterization of conjugation spaces due to M. Olbermann (s. [O2]). Hence $(M, \tau)$ is a conjugation space. \qed

**Remark 2.** The proof of the above theorem carries over from $M \in \mathcal{M}$ to manifolds $M$ of arbitrary (even) dimension with $H^{\text{odd}}(M; \mathbb{Z}) = 0$.

The following Lemma implies that for most $M \in \mathcal{M}$ the algebras $M^*(M; \mathbb{F}_2)$ do not admit any non-trivial derivation of negative degree (cf. [Pu6], Remark 5).
Lemma 1. Let \( A^* = A^0 \oplus A^2 \oplus A^4 \oplus A^6 \) be a graded connected Poincarè duality algebra over \( \mathbb{F}_2 \) which is generated by \( A^2 \). The algebra \( A^* \) admits a non-trivial derivation of negative degree if and only if there exists a subspace \( K \subset A^2 \) of codimension 1 with the following properties:

(i) \( K \times K \xrightarrow{\mu} K^\perp \subset A^4 \), where \( K^\perp \) denotes the orthogonal complement with respect to the Poincarè pairing

(ii) \( \exists a \in A^2 \setminus K \) such that

(1) the map \( K \xrightarrow{\mu a} A^4 \rightarrow A^4 \setminus K^\perp \) is an isomorphism where \( \mu_a : K \rightarrow A^4 \) denotes the multiplication by \( a \).

(2) \( aa \in K^\perp \).

Proof. Assume that \( \partial : A^* \rightarrow A^* \) is a non-trivial derivation of negative degree. Put \( K := \ker \partial|_{A^2} \). Condition (i) means that \( k_1k_2k_3 = 0 \) for all \( k_i \in K, i = 1, 2, 3 \). Clearly \( \partial(k_1k_2k_3) = 0 \), since \( \partial(k_i) = 0 \) for \( i = 1, 2, 3 \). But because of Poincarè duality (and \( 1 \in \partial(A^2) \)), \( \partial|_{A^6} \) is injective. This shows (i).

For any \( k \in K \) one has \( \partial(aa)k = \partial(aa)k + aa\partial k = 0 \). Hence \( aak = 0 \), and \( aa \in K^\perp \). If \( k \neq 0 \) then, by Poincarè duality there exists a \( k_1 \in k \) with \( aak \neq 0 \); and therefore for any \( x \in A^2, x \neq 0 \) there exists a \( k \in K \) such that \( xk \neq 0 \). If \( b \in K^\perp \) and \( k \in K \), then \( 0 = \partial(bk) = \partial(b)k \). Hence \( \partial(b) = 0 \). So, \( \partial : A^4 \rightarrow A^2 \) factors through \( A^4/K^\perp \) and for the composition \( K \xrightarrow{\mu a} A^4 \rightarrow A^4/K^\perp \rightarrow K \) is the identity since \( \partial(ak) = \partial(a)k = k \). Hence also (ii) holds.

Assume now that (i) and (ii) are fulfilled. Define \( \partial : A^* \rightarrow A^* \) by \( \partial^2 : A^2 \rightarrow A^2/K \cong \mathbb{F}_2 = A^0, \partial^4 : A^4 \rightarrow A^4/K^\perp \xrightarrow{\mu^{-1}} K \subset A^2, \partial^6 : A^6 \xrightarrow{\Delta} K^\perp \subset A^4 \). Clearly \( \partial^* \partial^* = 0 \). We still have to check the derivation property. For \( k_1, k_2 \in K \) we have \( 0 = \partial(k_1k_2) = (\partial k_1)k_2 + k_1(\partial k_2) \); for \( a \) and \( k \in K \) one gets \( \partial(ak) = (\partial a)k + a(\partial k) = k \) by definition of \( \partial^4 \), \( \partial(aa) = (\partial a)a + a(\partial a) = 0 \), since \( aa \in K^\perp \). Using the decomposition \( A^2 = K \oplus < a > \) and \( A^4 = K^\perp \oplus < a >^\perp \), where \( < a > \) is the \( \mathbb{F}_2 \)-vector space generated by \( a \in A^2 \setminus K \), it is straight forward to check for all products that the derivation property holds. \( \square \)
Remark 3. In the presence of condition (i) of the above lemma conditions (ii), (1) and (2), are equivalent to conditions (1) and (2) for all \( a \in A^2 \setminus K \).

For the example above a straightforward calculation gives the following structure constants for the trilinear form mod 2, \( \mu : \mathbb{F}_2^6 \times \mathbb{F}_2^6 \times \mathbb{F}_2^6 \rightarrow \mathbb{F}_2^6 \).

\[ \mu_{ijk} := \mu(e_i, e_j, e_k), i, j, k = 1, ..., 6 \], corresponding to the cubic polynomial with respect to the coordinate system \( x_1, ..., x_6 \): \( \mu_{124} = \mu_{125} = \mu_{136} = \mu_{246} = \mu_{356} = \mu_{456} = 1 \); and all other \( \mu_{ijk} \) (not obtained by permutation of indices from the former) vanish.

It is easy to check in this example that, mod 2, \( A^2 \) generates \( A^* \), and also that the conditions in Lemma 1 for the existence of a non-trivial derivation of negative degree are not fulfilled.

Corollary. If \( (M, \tau) \) is a manifold with an orientation reversing involution, \( M \in \mathcal{M} \), and \( H^*(M; \mathbb{Z}) \) given by the polynomial in the above example, then \( (M, \tau) \) is a conjugation space.

Recently M. Kreck (s. [K]), using completely different methods, has shown that in the situation of the above corollary the first Pontrjagin class of \( M \) has to fulfill an additional condition, if the action is differentiable, or - at least - locally linear. He obtained the following result, which completely answers the question, stated in the introduction in the smooth and locally linear category.

Theorem (Kreck). There are infinitely many closed asymmetric simply connected smooth 6-manifolds.

The above Theorem (Wall) is only a part of Wall’s classification result in that he considers the bigger class \( \mathcal{N} \) of simply connected, 6-dimensional spin-manifolds with free integral cohomology, so \( H^3(M; \mathbb{Z}) \) is a free module of even rank (because of Poincaré duality) for \( M \in \mathcal{N} \). Wall reduces the classification of \( \mathcal{N} \) to that of \( M \) by showing that a manifold \( M \in \mathcal{N} \) is diffeomorphic to \( M' \sharp S^3 \times S^3 \cdots \sharp S^3 \times S^3 \) with \( M' \in \mathcal{M} \). So the only additional invariant needed is the rank of \( H^3(M; \mathbb{Z}) \).
It is clear that no manifold $M \in \mathcal{N}$ with $H^3(M; \mathbb{Z}) \neq 0$ fulfills assumption (i) of Theorem 1, since then $H^*(M; \mathbb{F}_2)$ admits non-trivial involutions (as graded algebra). Hence for the following we restrict to cohomologically trivial actions, i.e. the induced action on $H^*(M; \mathbb{Z})$ is assumed to be trivial.

Considering the Serre spectral sequence of the Borel construction with coefficients in $\mathbb{Z}$ (cf. [Pu5], Prop. 1) the assumption (ii) in Theorem 1 is fulfilled for $M \in \mathcal{M}$ already for degree reasons. But for elements in $\mathcal{N}$ one needs an extra argument, namely the following simple lemma.

**Lemma 2.** Let $A^*$ be a Poincaré duality algebra over $\mathbb{F}_p$, $p$ prime, of formal dimension 6, and let $A^1 = A^5 = 0$. Assume that the even dimensional part $A^e$ is generated by $A^2$ (as an algebra with unit), then $A^*$ does not admit a non-trivial derivation of negative, odd degree.

**Proof.** Let $A^0 = \langle 1 \rangle$, $A^2 = \langle a_i \mid i \in I \rangle$, $A^3 = \langle b_j, \tilde{b}_j \mid j \in J \rangle$, $A^4 = \langle c_i \mid i \in I \rangle$, $A^6 = \langle d \rangle$, where the $\tilde{b}_j$, $c_i$, $d$ form the dual basis of $b_j$, $a_i$, 1 with respect to the Poincaré duality pairing, and $\langle \rangle$ denotes the vector space generated by the indicated basis. Assume that $\partial : A^* \rightarrow A^*$ is a derivation of degree (-1). Then $\partial$ vanishes on $A^0$, $A^2$ and $A^6$ for degree reasons, and on $A^4$ since $A^e$ is generated by $A^2$.

Let $b$ be a non-zero element in $A^3$ and let $a := \partial b$. Assume that $a \neq 0$. If $c$ is the dual of $a$, so $ac = d$, then $\partial(bc) = (\partial b)c - b \partial c = ac = d$; but $bc = 0$. Hence we get a contradiction. So $a = \partial b = 0$ for any $b \in A^3$. Therefore $\partial$ must be trivial.

The argument for derivations of lower (negative, odd) degree is completely analogous. \qed

**Remark 4.** It is obvious that one can generalize Lemma 1 to Poincaré duality algebras $A^*$ of formal dimension $2m$ with $A^1 = 0$ and $A^e$ generated by $A^2$.

Parametrizing $\mathcal{N}$ by integral cohomology type one gets the following generalization of Theorem 3.
**Theorem 6.** Most integral cohomology types in $\mathcal{N}$ do not admit non-trivial but cohomologically trivial $\mathbb{Z}/p\mathbb{Z}$–actions.

Here 'most' can be given a precise meaning similar to Theorem 3.

For the proof one uses the integral version of Theorem 1 for $\mathbb{Z}/p\mathbb{Z}$–actions. By assumption the considered actions are cohomologically trivial and $H^*(M;\mathbb{Z})$ is free. So the $E_2$–term of the Serre spectral sequence of the Borel construction with integral coefficients is given by

$$E_2 \cong H^*(BG; H^*(M;\mathbb{Z}) \otimes H^*(M;\mathbb{Z}) \cong \mathbb{Z}[t]/(pt) \otimes H^*(M;\mathbb{Z}),$$

where $\deg(t) = 2$.

The first non-trivial boundary in the spectral sequence would give a non-trivial derivation of odd, negative degree on $H^*(M;\mathbb{Z}) \otimes \mathbb{F}_p \cong H^*(M;\mathbb{F}_p)$. Hence, by Lemma 1, the spectral sequence collapses if $H^{ev}(M;\mathbb{F}_p)$ is generated by $H^2(M;\mathbb{F}_p)$, which is the case if the trilinear form, given by the cup product, is non-degenerate. This holds for most cohomology types. Since $t$ above has degree 2, $H^{ev}(M^G;\mathbb{F}_p)$ is a deformation of negative weight of $H^{ev}(M;\mathbb{F}_p)$ (cf. [Pu5], Prop. 1 and 4). In most cases there are no such non-trivial deformations. So $H^{ev}(M^G;\mathbb{F}_p)$ is isomorphic to $H^{ev}(M;\mathbb{F}_p)$ as filtered algebras. In case of a non-degenerate trilinear form the cup length of $H^{ev}(M;\mathbb{F}_p)$ (and hence of $H^{ev}(M^G;\mathbb{F}_p)$) is 3. So the dimension of $M^G$ must be 6, and hence $M^G = M$; i.e. the action is trivial. □

**Remark 5.** If $p$ is large compared to the size of $H^*(M;\mathbb{Z})$ (more precisely: $p > rkH^i(M;\mathbb{Z}) + 1$, for all $i$), then an action of $\mathbb{Z}/p\mathbb{Z}$ on $M$ must be cohomologically trivial by elementary representation theory. So Theorem 5 holds for 'large $p$' without the restriction 'cohomologically trivial'. But recently M.Olbermann has improved Theorem 6 considering not only cohomologically trivial actions, but assuming that $rkH^3(M;\mathbb{Z}) = o((rkH^2(M;\mathbb{Z}))^{3/2})$ (s. [O1]).

The classification of simply connected 6-manifolds, without the assumptions 'spin' and '$H^*(-;\mathbb{Z})$ free over $\mathbb{Z}$' involves further invariants (s. [Z]). But for a given manifold $M$ we can kill the torsion in $H^*(M;\mathbb{Z})$ localizing $\mathbb{Z}$ by inverting those primes which occur in the torsion. The above arguments can then be applied...
to $\mathbb{Z}/p\mathbb{Z}$–action, where $p$ does not belong to the (finitely many) inverted primes. Parametrizing the class of simply connected manifolds by their rational cohomology algebras (which is, of course, far from a classification up to homeomorphism or diffeomorphism) one gets the following result.

**Theorem 7.** Most rational cohomology types of simply connected 6-manifolds do admit non-trivial $\mathbb{Z}/p\mathbb{Z}$–action for at most finitely many primes.

This generalizes Theorem 2 in [Pu4].

**Remark 6.** It is to be expected that similar results hold for higher (even) dimensions. But in particular the discussion of condition (iii) or (iv) in Theorem 1 gets more and more involved with increasing dimension. Certain results for $S^1$–actions on 8-manifolds in this direction are contained in [I].

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