Solvability of interior transmission problem for the diffusion equation

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Abstract

Consider the interior transmission problem arising in inverse boundary value problems for the diffusion equation with discontinuous diffusion coefficients. We prove its unique solvability using the method of Green function. The key point is to construct the Green function for the interior transmission problem. We complete this procedure in the following way. First, we construct a local parametrix for the interior transmission problem near the boundary in the Laplace domain, by using the theory of pseudo-differential operators with a large parameter. Second, by carefully analyzing the analyticity of the Green function in the Laplace domain and estimating it there, a local parametrix near the boundary for the original parabolic interior transmission problem is obtained via the inverse Laplace transform. Finally, using a partition of unity, we patch all the local parametrices and the fundamental solution of the diffusion equation to have a parametrix for the parabolic interior transmission problem, and then compensate it to get the Green function. The uniqueness of the Green function is justified by using the duality argument. We would like to emphasize that the Green function for the interior transmission problem is constructed for the first time in this paper. It has many applications. For example, it can be used for the active tomography and diffuse optical tomography modeled by diffusion equations to identify an unknown inclusion and its physical property.

Keywords. Inverse problems; Interior transmission problem; diffusion equation; Solvability.

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1 Introduction

Interior transmission problem plays an important role in inverse scattering theory for inhomogeneous media. It is a non-classical boundary value problem for a pair of partial differential equations in a bounded domain coupled on the boundary. There are many works on interior transmission problems for elliptic equations; see, for example, [2–11, 12–16, 20–22, 26–31], focusing on the solvability of interior transmission problems for different kinds of inhomogeneous media, the existence and efficient computations of the transmission eigenvalues, and their applications to
inverse scattering problems. Recently, it was found in [14, 21] that interior transmission problems are closely related to the invisibility cloak in acoustic and electromagnetic wave scattering.

As we explained in [23], the interior transmission problem for the diffusion equation naturally arises in inverse boundary value problems for the diffusion equation with discontinuous coefficients when we consider a reconstruction method called the linear sampling method. More explicitly, the problem is formulated to study the solvability of the so-called Neumann-to-Dirichlet map equation, which motivates our linear sampling method for reconstructing unknown inclusions inside a diffusive medium from boundary measurements. Let $D$ be an inclusion embedded in the diffusive medium. Assume that the diffusion coefficients of $D$ and the background are $k$ and 1, respectively. We assume for simplicity that $k$ is a constant with $k > 1$. Suppose that $D \subset \mathbb{R}^n (n = 2, 3)$ is a bounded domain with $C^\infty$ smooth boundary $\partial D$. The interior transmission problem for the diffusion equation is described by the following initial boundary value problem:

\[
\begin{aligned}
(\partial_t - \Delta)v &= 0 \quad \text{in } D_T := D \times (0, T), \\
(\partial_t - k\Delta)u &= 0 \quad \text{in } D_T := D \times (0, T), \\
u - v &= f \quad \text{on } (\partial D)_T := \partial D \times (0, T), \\
k\partial_{\nu}u - \partial_{\nu}v &= g \quad \text{on } (\partial D)_T := \partial D \times (0, T), \\
u = v &= 0 \quad \text{at } t = 0,
\end{aligned}
\]

where $\nu$ is the unit outer normal vector to $\partial D$.

One may think that the interior transmission problem is a special problem attached to the linear sampling method for the aforementioned inverse boundary value problem. We would like to show by giving a general example that this is not true. For example, consider $D$ as a heat conductor with thermal conductivity $c$ located in a room with thermal conductivity $c_0$. Put some heat source $p(x, t)$ over the time interval $(0, T)$ and let it radiate. Then the heat temperature $u = u(x, t)$ generated by this heat source satisfies the following initial value problem:

\[
\begin{aligned}
\partial_t u - \nabla \cdot (\gamma \nabla u) &= p \quad \text{in } \mathbb{R}^3 \times (0, T), \\
u = 0 \quad \text{at } t = 0,
\end{aligned}
\]

where $\gamma = c_0 + (c - c_0)\chi_D$ with $\chi_D$ being the characteristic function of $D$. Suppose we can measure

\[
u_t |_{(\partial D)_T}, \quad \partial_t u := \nu \cdot (c\nabla u) |_{(\partial D)_T} = c\partial_{\nu}u |_{(\partial D)_T},
\]

and consider the uniqueness of identifying an unknown $c$ from this measurement. That is, for given two unknowns $c_1$ and $c_2$, show that they are equal if we have $u_1 = u_2, c_1\partial_{\nu}u_1 = c_2\partial_{\nu}u_2$ on $(\partial D)_T$, where $u_j (j = 1, 2)$ satisfies

\[
\begin{aligned}
(\partial_t - c_j\Delta)u_j &= p \quad \text{in } D_T, \\
u_j &= 0 \quad \text{at } t = 0.
\end{aligned}
\]

Combining (1.4) with the measurement (1.3) for $u = u_j (j = 1, 2)$, we have an interior transmission problem for $(u_1, u_2)$ in $D_T$.

In (1.1), we assume that $f = g = 0$ at $t = 0$ and $f, g$ satisfy certain regularity assumption. Then we can remove $f$ and $g$ so that we have homogeneous boundary conditions and inhomogeneous terms.
in the equations. So we are led to the following problem:

\[
\begin{aligned}
(\partial_t - \Delta)v &= N_1 \quad \text{in } D_T, \\
(\partial_t - k\Delta)u &= N_2 \quad \text{in } D_T, \\
u - v &= 0 \quad \text{on } (\partial D)_T, \\
k\partial_\nu u - \partial_\nu v &= 0 \quad \text{on } (\partial D)_T, \\
u &= v = 0 \quad \text{at } t = 0.
\end{aligned}
\] (1.5)

We refer it as ITP.

In this paper, we will show the unique solvability of (1.5) by the method of Green function. It suffices to construct the Green function for (1.5) and show its uniqueness. We start by constructing the local parametrix for (1.5) near the boundary. For this, let us consider the following interior transmission problem in the Laplace domain:

\[
\begin{aligned}
(\tau - \Delta)G &= e^{-\tau s}\delta(x - y) \quad \text{in } D, \\
(\tau - k\Delta)H &= e^{-\tau s}\delta(x - y) \quad \text{in } D, \\
H - G &= 0 \quad \text{on } \partial D, \\
k\partial_\nu H - \partial_\nu G &= 0 \quad \text{on } \partial D,
\end{aligned}
\] (1.6)

where \(y \in D\), and \(\tau \in \mathbb{C}\) denotes the Laplace variable with \(\text{Re } \tau \gg 1\). By using the theory of pseudo-differential operators with a large parameter, we localize the problem to a neighborhood of \(x_0 \in \partial D\). Then we construct a local parametrix for (1.6) near the boundary, which is holomorphic with respect to \(\tau\) in some restricted domain and has a good estimate. This allows us to construct a local parametrix for the original parabolic interior transmission problem (1.5) near the boundary via the inverse Laplace transform. Using a partition of unity, we patch all local parametrices and the fundamental solution of the diffusion equation to have a global parametrix, and then compensate it to obtain the desired Green function by Levi’s method. Finally, we show the uniqueness of the Green function by the duality argument.

The novelty and new contributions of the present work are as follows. First, by showing the solvability of the interior transmission problem (1.1) as a by-product of the construction of its Green function, we could clearly clarify the solvability of the Neumann-to-Dirichlet map equation which supplies our theoretical analysis on the sampling method proposed in [23]. Second, our argument of constructing the Green function for the interior transmission problem is new and efficient. It is an adaptation of Seeley’s argument [28] for elliptic boundary value problems to the interior transmission problems for diffusion equations. In [15], one of the author of this paper showed how to adapt Seeley’s argument to construct the Green function of the interior transmission problem for elliptic equations assuming its unique solvability. Concerning the construction of local parametrix for the Green function, the argument in [15] was much more simple than the direct application of Seeley’s argument. Since in our case we are studying the unique solvability of the interior transmission problem for the diffusion equations via constructing its Green function, we have to compensate a parametrix to have the Green function of the interior transmission problem without using its unique solvability. We could successfully use the Levi method to compensate a parametrix to have the Green function instead of using the solvability of the interior transmission problem. Here we note that
the solvability of the interior transmission problem is the consequence of the existence of its Green function and it is not available beforehand. The efficiency is that the argument may also be utilized to analyze the asymptotic behavior of indicator functions in non-iterative reconstruction methods for inverse scattering problems and inverse boundary value problems for parabolic equations, since the indicator functions are closely related to the corresponding Green functions \[13, 15, 24, 25\].

The paper is organized as follows. In Section 2, we show how to construct a local parametrix for the interior transmission problem in the Laplace domain. Some lengthy details are given in the appendix. Then, in Section 3, the local parametrix for the original interior transmission problem is obtained by taking the inverse Laplace transform, and an estimate of this local parametrix is derived. In Section 4, using a partition of unity, we patch the local parametrices and the fundamental solutions to the operators \( \partial_t - \Delta \), \( \partial_t - k \Delta \) so that we have a global parametrix for our interior transmission problem \( \text{(1.5)} \). This parametrix can be compensated to construct the Green function by Levi’s method. The uniqueness of the Green function is justified, and the unique solvability of \( \text{(1.5)} \) is concluded in Section 5. Finally, we give some concluding remarks in Section 6.

## 2 Local parametrix in the Laplace domain

In this section, we construct the local parametrix for the interior transmission problem in the Laplace domain by studying \( \text{(1.6)} \). Our argument is based on the theory of pseudo-differential operators with a large parameter. We only consider the case that \( n = 3 \).

Let us first locally flatten the boundary \( \partial D \) near a point \( x_0 \in \partial D \) by a coordinate transformation \( \Phi_{x_0} : U(x_0) \to \mathbb{R}^3 \), where \( U(x_0) \) is an open neighbourhood of \( x_0 \) in \( \mathbb{R}^3 \). Under this coordinate transformation, we can locally express \( \partial D \) and \( D \) by \( \partial D = \{ \xi_3 = 0 \} \) and \( D = \{ \xi_3 < 0 \} \) in terms of the local coordinates \( \xi = (\xi_1, \xi_2, \xi_3) \). Denote by \( J := (\nabla \xi) \) the Jacobian of the coordinate transformation. Define \( M = (m_{ji})_{3 \times 3} := JJ^T \) and \( J := \det(\nabla \xi) \). Without loss of generality, we assume \( J > 0 \) by fixing the orientation of \( \partial D \) as a manifold. Let \( \xi = \Phi_{x_0}(x), \eta = \Phi_{x_0}(y), \bar{G}(\xi) = G(\Phi_{x_0}^{-1}(\xi)), \bar{H}(\xi) = H(\Phi_{x_0}^{-1}(\xi)) \). Then from \( \text{(1.6)} \) we locally have

\[
\begin{cases}
\tau \bar{G} - J^{-1}(\xi) \nabla \xi \cdot (J(\xi)M \nabla \xi \bar{G}) = e^{-\tau s} J^{-1}(\eta) \delta(\xi - \eta) & \text{in } \mathbb{R}^3_-, \\
\tau \bar{H} - kJ^{-1}(\xi) \nabla \xi \cdot (J(\xi)M \nabla \xi \bar{H}) = e^{-\tau s} J^{-1}(\eta) \delta(\xi - \eta) & \text{in } \mathbb{R}^3_-, \\
\bar{G} - \bar{H} = 0 & \text{on } \partial \mathbb{R}^3_-, \\
e_3 \cdot M \nabla \xi \bar{G} - ke_3 \cdot M \nabla \xi \bar{H} = 0 & \text{on } \partial \mathbb{R}^3_-.
\end{cases}
\tag{2.1}
\]

In the sequel, for convenience, we will still use the notations \( G, H, x, y \) in the local coordinates system, instead of \( \bar{G}, \bar{H}, \xi, \eta \). Let \( y = (y_1, y_2, y_3) \in \mathbb{R}^3 \) with \( y_3 \) near to 0. To clarify the dependency of \( G \) and \( H \) on \( y \), we denote them by \( G(x, y) \) and \( H(x, y) \), respectively. Set

\[
G(x, y) = \begin{cases}
G^+(x, y), & \text{if } x_3 - y_3 > 0, \\
G^-(x, y), & \text{if } x_3 - y_3 < 0,
\end{cases}
\]

and

\[
H(x, y) = \begin{cases}
H^+(x, y), & \text{if } x_3 - y_3 > 0, \\
H^-(x, y), & \text{if } x_3 - y_3 < 0.
\end{cases}
\]
Define
\[ -P := \tau - \nabla \cdot (M(x)\nabla) - (J^{-1}M(x)\nabla J) \cdot \nabla, \]
\[ -Q := \tau - k\nabla \cdot (M(x)\nabla) - k(J^{-1}M(x)\nabla J) \cdot \nabla. \]

Then \( G^\pm \) and \( H^\pm \) satisfy the following equations:
\[ PG^\pm = QH^\pm = 0 \quad \text{for } \pm (x_3 - y_3) > 0, \quad x_3 < 0, \]
meeting the transmission conditions on \( x_3 = y_3 \)
\[ \begin{cases} 
G^+ = G^-, \\
H^+ = H^-, \\
e_3 \cdot M(x)\nabla (G^+ - G^-) = -e^{-\tau s}J^{-1}(y) \delta(x' - y'), \\
ke_3 \cdot M(x)\nabla (H^+ - H^-) = -e^{-\tau s}J^{-1}(y) \delta(x' - y')
\end{cases} \tag{2.2} \]
with \( x' = (x_1, x_2) \), \( y' = (y_1, y_2) \) and the boundary conditions on \( x_3 = 0 \)
\[ \begin{cases} 
G^+ = H^+, \\
e_3 \cdot M(x)\nabla G^+ = ke_3 \cdot M(x)\nabla H^+.
\end{cases} \tag{2.3} \]

To construct the local parametrix by solving (2.1), we first recall the definition of the set of symbols.

**Definition 2.1** a\((x, \xi', \tau)\) is in \( S(m) \) if the followings hold:

1. \( a(x, \xi', \tau) \in C^\infty(\mathbb{R}_x^2 \times \mathbb{R}_\xi^2 \times \Sigma); \)
2. For arbitrary multi-indices \( \alpha, \beta \in \mathbb{Z}_+^2 \) with \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \), there exists a constant \( C_{\alpha, \beta} > 0 \) such that
\[ |D_x^\alpha D_\xi^\beta a(x, \xi', \tau)| \leq C_{\alpha, \beta}(1 + |\xi'| + |\tau|^{1/2})^{m-|\beta|}, \quad (x', \xi', \tau) \in \mathbb{R}_x^2 \times \mathbb{R}_\xi^2 \times \{\Sigma \cap \{|\tau| > 1\}\}, \]
where \( \Sigma := \{re^{i\theta} : r > 0, \theta_1 < \theta < \theta_2\} \) with the constants \( \theta_1 \) and \( \theta_2 \).

We call \( a \) a symbol of order \( m \).

If \( a(x, \xi', \tau) = a(x, \xi', \tau, y, s) \) depends on \( (y, s) \), the above estimate has to hold uniformly with respect to \( (y, s) \in D_\tau \).

Consider the operator
\[ P = \nabla \cdot (M(x)\nabla) + W \cdot \nabla - \tau = -D \cdot (M(x)D) + iW \cdot D - \tau, \]
where \( W = (w_1, w_2, w_3)^T := J^{-1}M(x)\nabla J \) and \( D = -i\nabla \). Denote by \( p(x, \xi', \tau) \) the total symbol of \( P \). We expand \( p(x, \xi', \tau) \) into its Taylor series around \( x_3 = y_3 \). That is,
\[ p(x, \xi', \tau) = \sum_{j=0}^{\infty} (j!)^{-1}(x_3 - y_3)^j (\partial_{x_3}^j p)(x', y_3, \xi', \tau). \]

For our further arguments, we introduce the concept of order.
Definition 2.2 The multiplications by $\xi_1, \xi_2, \tau^{1/2}$ and $D_{x_3} = -i\partial_{x_3}$ are considered as operators of order 1, and the multiplication by $x_3 - y_3$ is considered as an operator of order $-1$. Further, $D_{x'}$ is considered as an operator of order 0.

Then the total symbol $p$ of $P$ can be decomposed as 

$$p = p_2 + p_1$$

with respect to the order, where

$$p_2 = -m_{33}D_3^2 - 2\sum_{j=1}^2 m_{3j}(x)\xi_jD_3 - \sum_{j,l=1}^2 m_{jl}(x)\xi_j\xi_l - \tau,$$

$$p_1 = i\sum_{j=1}^3 \partial_jm_{3j}(x)D_3 + i\sum_{j=1}^3 \sum_{l=1}^2 \partial_jm_{jl}(x)\xi_l + iw_3D_3 + i\sum_{j=1}^2 w_j\xi_j$$

with $\text{ord} p_2 = 2$ and $\text{ord} p_1 = 1$. Here $\text{ord} p_j$ denotes the order of $p_j$.

We now look for the amplitude $a = \sum_{L=0}^{\infty} a_{-1-L}$ with $a_{-1-L} \in S(-1-L)$ of the pseudo-differential operator $G^+$ given by

$$G^+ \varphi(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix'\cdot \xi'} a(x, \xi', \tau)(\mathcal{F}\varphi)(\xi') d\xi'$$

which satisfies

$$0 = (PG^+ \varphi)(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix'\cdot \xi'} A(x, \xi', \tau)(\mathcal{F}\varphi)(\xi') d\xi',$$

where

$$A(x, \xi', \tau) = \sum_{\alpha} (\alpha!)^{-1} \partial_\xi^\alpha p(x, \xi', \tau) D_\xi^\alpha a(x, \xi', \tau),$$

and

$$(\mathcal{F}\varphi)(\xi') = \int_{\mathbb{R}^2} e^{-ix'\cdot \xi'} \varphi(x') dx'$$

is the Fourier transform of $\varphi(x') \in C_0^\infty(\mathbb{R}^2)$. Here we have suppressed $(y, s) \in DT$ for simplifying notations. This simplification will be used in the sequel if we do not need to clarify the dependence on $(y, s)$. We arrange $A(x, \xi', \tau)$ in terms of the order as follows:

$$A = \sum_{l=0}^{\infty} A_{1-l},$$

where $\text{ord} A_{1-l} = 1-l$, and $A_{1-l}$ can be explicitly expressed; see, for example,

$$A_1 = p_{2,0}^{(0)} a_{-1},$$

$$A_0 = \sum_{j+k+|\alpha|=1} (x_3 - y_3)^j p_{2,j}^{(\alpha)} D_\xi^\alpha a_{-1-k} + p_{1,0}^{(0)} a_{-1}$$

$$= p_{2,0}^{(0)} a_{-2} + \sum_{j+|\alpha|=1} (x_3 - y_3)^j p_{2,j}^{(\alpha)} D_\xi^\alpha a_{-1} + p_{1,0}^{(0)} a_{-1}$$
with $p^{(\alpha)}_{l,j} = \partial_{x'_j} \partial_{\xi'^{\alpha}_l} p_l(x', y_3, \xi', \tau)$, $l = 1, 2$, $j \in \mathbb{Z}_+$, $a \in \mathbb{Z}_+^{n-1}$.

We deal with the amplitudes $b$, $d$ and $e$ of $G^-$, $H^+$ and $H^-$ in the same manner. Corresponding to $A$, we use the notations $B$, $D$ and $E$ for $b$, $d$ and $e$. That is, $B$, $D$ and $E$ are associated with $\mathcal{P}G^-$, $QH^+$ and $QH^-$, respectively. Then we have the following representations of the amplitudes.

**Theorem 2.3** Let $M^1 = M|x_3=y_3$ and $M^0 = M|x_3=0$. Apply these notations even for their components, for example, $m^1_{33} = m_{33}|x_3=y_3$, $m^0_{33} = m_{33}|x_3=0$. Define

$$
\begin{align*}
\lambda_\pm := (m^1_{33})^{-1} & \left[ -i \sum_{j,l=1}^{2} m^1_{j,l} \xi_j \pm \sqrt{m^1_{33} \left( \sum_{j,l=1}^{2} m^1_{j,l} \xi_j + \tau \right) - \left( \sum_{j,l=1}^{2} m^1_{j,l} \xi_j \right)^2} \right], \\
\mu_\pm := (m^1_{33})^{-1} & \left[ -i \sum_{j,l=1}^{2} m^1_{j,l} \xi_j \pm \sqrt{m^1_{33} \left( \sum_{j,l=1}^{2} m^1_{j,l} \xi_j + \tau \right) - \left( \sum_{j,l=1}^{2} m^1_{j,l} \xi_j \right)^2} \right],
\end{align*}
$$

(2.5)

where the real parts of the square roots in $\lambda_\pm$ and $\mu_\pm$ are positive. Then we have

$$
a = \sum_{L=0}^{\infty} a_{-L}, \quad b = \sum_{L=0}^{\infty} b_{-L}, \quad d = \sum_{L=0}^{\infty} d_{-L}, \quad e = \sum_{L=0}^{\infty} e_{-L},$$

where

$$
a_{-L} = \sum_{l=0}^{2L-2} f^L_{11}(x_3 - y_3)^l \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy^l \cdot \xi' \right) + \sum_{l=0}^{2L-2} f^L_{12}(x_3 - y_3)^l \exp \left( \lambda_+ x_3 - \mu_- y_3 - \tau s - iy^l \cdot \xi' \right) + \sum_{l=0}^{2L-2} f^L_{13}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \lambda_- y_3 - \tau s - iy^l \cdot \xi' \right),
$$

(2.6)

$$
b_{-L} = \sum_{l=0}^{2L-2} f^L_{11}(x_3 - y_3)^l \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy^l \cdot \xi' \right) + \sum_{l=0}^{2L-2} f^L_{12}(x_3 - y_3)^l \exp \left( \lambda_+ x_3 - \mu_- y_3 - \tau s - iy^l \cdot \xi' \right) + \sum_{l=0}^{2L-2} f^L_{13}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \lambda_- y_3 - \tau s - iy^l \cdot \xi' \right),
$$

(2.7)

$$
d_{-L} = \sum_{l=0}^{2L-2} f^L_{15}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \lambda_- y_3 - \tau s - iy^l \cdot \xi' \right) + \sum_{l=0}^{2L-2} f^L_{16}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \mu_- y_3 - \tau s - iy^l \cdot \xi' \right) + \sum_{l=0}^{2L-2} f^L_{17}(x_3 - y_3)^l \exp \left( \mu_- x_3 - \mu_- y_3 - \tau s - iy^l \cdot \xi' \right),
$$

(2.8)
and

\[ e_L = \sum_{l=0}^{2L-2} f_{l,j}^L(x_3 - y_3)^l \exp (\mu_+ x_3 - \lambda_+ y_3 - \tau s - iy' \cdot \xi') \]
\[ + \sum_{l=0}^{2L-2} f_{l,j}^L(x_3 - y_3)^l \exp (\mu_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi') \]
\[ + \sum_{l=0}^{2L-2} f_{l,j}^L(x_3 - y_3)^l \exp (\mu_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi') \]

(2.9)

with \( \text{ord } f_{l,j}^L = l - L \) for \( j = 1, \ldots, 8 \). In the above formulae, all \( f_{l,j}^L \)'s can be explicitly given in the following proof. Actually, \( f_{l,j}^L \)'s for \( L = 1 \) are given in (2.18) with \( A_1, A_2, B_1, B_2 \) defined in (2.17). For general \( L \geq 2 \), \( f_{l,j}^L \)'s are determined by \( F_{l,j} \) in (2.28)-(2.29) and \( A_5, A_7, B_5, B_7 \) defined in (A.8), (A.10).

**Proof.** We prove the result by induction. At first, let us find \( a_{-1} \), \( b_{-1} \), \( d_{-1} \) and \( e_{-1} \). It implies from \( A_1 = B_1 = D_1 = \mathcal{E}_1 = 0 \) that

\[ p_{2,0}^{(0)} a_{-1} = p_{2,0}^{(0)} b_{-1} = q_{2,0}^{(0)} d_{-1} = q_{2,0}^{(0)} e_{-1} = 0, \]

(2.10)

where

\[ p_{2,0}^{(0)} = m^1_{33} \partial_{x_3}^2 + 2i \sum_{j=1}^2 m^1_{3j} \xi_j \partial_{x_3} - \left( \sum_{j,l=1}^2 m^1_{jl} \xi_j \xi_l + \tau \right), \]
\[ q_{2,0}^{(0)} = km^1_{32} \partial_{x_3}^2 + 2ik \sum_{j=1}^2 m^1_{3j} \xi_j \partial_{x_3} - \left( k \sum_{j,l=1}^2 m^1_{jl} \xi_j \xi_l + \tau \right). \]

The solutions to the above ordinary differential equations (2.10) can be given in the form of

\[ a_{-1} = C_1 \exp (\lambda_+ x_3) + C_2 \exp (\lambda_- x_3), \]
\[ b_{-1} = C_3 \exp (\lambda_+ x_3), \]
\[ d_{-1} = C_4 \exp (\mu_+ x_3) + C_5 \exp (\mu_- x_3), \]
\[ e_{-1} = C_6 \exp (\mu_+ x_3). \]

Notice here that we took \( b_{-1} \) and \( e_{-1} \) satisfying \( \lim_{x_3 \to -\infty} b_{-1} = \lim_{x_3 \to -\infty} e_{-1} = 0 \). From the transmission conditions and boundary conditions

\[ a_{-1} - b_{-1} = 0, \quad ie_3 \cdot M^1 \left( \frac{\xi'}{D_{x_3}} \right) (a_{-1} - b_{-1}) = -\mathcal{J}^{-1}(y) \exp (-\tau s - iy' \cdot \xi') \quad \text{on } x_3 = y_3, \]
\[ d_{-1} - e_{-1} = 0, \quad ike_3 \cdot M^1 \left( \frac{\xi'}{D_{x_3}} \right) (d_{-1} - e_{-1}) = -\mathcal{J}^{-1}(y) \exp (-\tau s - iy' \cdot \xi') \quad \text{on } x_3 = y_3, \]
\[ a_{-1} - d_{-1} = 0, \quad ie_3 \cdot M^0 \left( \frac{\xi'}{D_{x_3}} \right) a_{-1} = ike_3 \cdot M^0 \left( \frac{\xi'}{D_{x_3}} \right) d_{-1} \quad \text{on } x_3 = 0, \]
we can easily derive the following system of equations for constants $C_j$ $(1 \leq j \leq 6)$:

\[
C_1 \exp (\lambda_+ y_3) + C_2 \exp (\lambda_- y_3) - C_3 \exp (\lambda_+ y_3) = 0, \tag{2.11}
\]

\[
\lambda_+ C_1 \exp (\lambda_+ y_3) + \lambda_- C_2 \exp (\lambda_- y_3) - \lambda_+ C_3 \exp (\lambda_+ y_3) = -(m_{33}^1)^{-1} J^{-1}(y) \exp (-\tau s - iy' \cdot \xi'), \tag{2.12}
\]

\[
C_4 \exp (\mu_+ y_3) + C_5 \exp (\mu_- y_3) - C_6 \exp (\mu_+ y_3) = 0, \tag{2.13}
\]

\[
\mu_+ C_4 \exp (\mu_+ y_3) + \mu_- C_5 \exp (\mu_- y_3) - \mu_+ C_6 \exp (\mu_+ y_3) = -(km_{33}^1)^{-1} J^{-1}(y) \exp (-\tau s - iy' \cdot \xi'), \tag{2.14}
\]

\[
C_1 + C_2 - C_4 - C_5 = 0, \tag{2.15}
\]

\[
(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0)C_1 + (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_- m_{33}^0)C_2 = k(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_+ m_{33}^0)C_4 + k(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_- m_{33}^0)C_5. \tag{2.16}
\]

From \textbf{(2.11)-(2.14)}, we get

\[
C_2 = \frac{J^{-1}(y) \exp (-\tau s - iy' \cdot \xi')}{m_{33}^1 (\lambda_+ - \lambda_-)} \exp (-\lambda_- y_3),
\]

\[
C_5 = \frac{J^{-1}(y) \exp (-\tau s - iy' \cdot \xi')}{km_{33}^1 (\mu_+ - \mu_-)} \exp (-\mu_- y_3).
\]

Define

\[
\begin{cases}
A_1 := \frac{J^{-1}(y)}{m_{33}^1 (\lambda_+ - \lambda_-)}, \\
B_1 := \frac{J^{-1}(y)}{km_{33}^1 (\mu_+ - \mu_-)}, \\
A_2 := \left\{ k(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_+ m_{33}^0) - (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0) \right\}^{-1} \\
\times (\lambda_- - \lambda_+) m_{33}^0 A_1, \\
B_2 := \left\{ k(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_+ m_{33}^0) - (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0) \right\}^{-1} \\
\times \{(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0) - k(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_- m_{33}^0)\} B_1.
\end{cases} \tag{2.17}
\]

Note that \text{ord} \, A_j = \text{ord} \, B_j = -1 \text{ for } j = 1, 2. \text{ Then we derive from (2.15) and (2.16) that}

\[
C_4 = A_2 \exp (-\lambda_- y_3 - \tau s - iy' \cdot \xi') + B_2 \exp (-\mu_- y_3 - \tau s - iy' \cdot \xi'),
\]

and therefore

\[
C_1 = C_4 - A_1 \exp (-\lambda_- y_3 - \tau s - iy' \cdot \xi') + B_1 \exp (-\mu_- y_3 - \tau s - iy' \cdot \xi')
\]

\[
= (-A_1 + A_2) \exp (-\lambda_- y_3 - \tau s - iy' \cdot \xi') + (B_1 + B_2) \exp (-\mu_- y_3 - \tau s - iy' \cdot \xi').
\]
Thus, we have
\[
C_3 = (-A_1 + A_2) \exp \left(-\lambda_- y_3 - \tau s - iy' \cdot \xi'\right) + (B_1 + B_2) \exp \left(-\mu_- y_3 - \tau s - iy' \cdot \xi'\right) + A_1 \exp \left(-\lambda_- y_3 - \tau s - iy' \cdot \xi'\right),
\]
\[
C_6 = A_2 \exp \left(-\lambda_- y_3 - \tau s - iy' \cdot \xi'\right) + B_2 \exp \left(-\mu_- y_3 - \tau s - iy' \cdot \xi'\right) + B_1 \exp \left(-\mu_+ y_3 - \tau s - iy' \cdot \xi'\right).
\]
So we finally get
\[
\begin{align*}
a_{-1} &= (-A_1 + A_2) \exp \left(\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'\right) + (B_1 + B_2) \exp \left(\lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'\right) + A_1 \exp \left(\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'\right), \\
b_{-1} &= (-A_1 + A_2) \exp \left(\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'\right) + (B_1 + B_2) \exp \left(\lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'\right) + A_1 \exp \left(\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'\right), \\
d_{-1} &= A_2 \exp \left(\mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'\right) + B_2 \exp \left(\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'\right) + B_1 \exp \left(\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'\right), \\
e_{-1} &= A_2 \exp \left(\mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'\right) + B_2 \exp \left(\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'\right) + B_1 \exp \left(\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'\right).
\end{align*}
\]
(2.18)

These show that (2.16) - (2.19) are true for \( L = 1 \).

Next, let us prove for the case \( L = 2 \). From \( A_0 = B_0 = D_0 = \xi_0 = 0 \), we know that
\[
q_{2,0}^{(0)} a_{-2} = - \sum_{j+|\alpha|=1} (x_3 - y_3)^j p_{2,j}^{(0)} D_x^\alpha a_{-1} - p_{1,0}^{(0)} a_{-1} =: \Theta_a,
\]
\[
p_{2,0}^{(0)} b_{-2} = - \sum_{j+|\alpha|=1} (x_3 - y_3)^j p_{2,j}^{(0)} D_x^\alpha b_{-1} - p_{1,0}^{(0)} b_{-1} =: \Theta_b,
\]
\[
q_{2,0}^{(0)} d_{-2} = - \sum_{j+|\alpha|=1} (x_3 - y_3)^j q_{2,j}^{(0)} D_x^\alpha d_{-1} - q_{1,0}^{(0)} d_{-1} =: \Theta_d,
\]
\[
q_{2,0}^{(0)} e_{-2} = - \sum_{j+|\alpha|=1} (x_3 - y_3)^j q_{2,j}^{(0)} D_x^\alpha e_{-1} - q_{1,0}^{(0)} e_{-1} =: \Theta_e.
\]

Note that
\[
\Theta_a = - \left[ (x_3 - y_3) \partial_{x_3} p_2 |_{x_3=y_3} - i \sum_{j=1}^2 \partial_{x_j} p_2 |_{x_3=y_3} \partial_{x_j} + p_1 |_{x_3=y_3} \right] a_{-1},
\]
and \( \Theta_b, \Theta_d, \Theta_e \) can be expressed analogously. From the forms of \( a_{-1}, b_{-1}, d_{-1} \) and \( e_{-1} \), we have
\[
\Theta_a = \sum_{l=0}^1 E_{l,1} (x_3 - y_3)^l \exp \left(\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'\right)
+ \sum_{l=0}^1 E_{l,2} (x_3 - y_3)^l \exp \left(\lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'\right)
+ \sum_{l=0}^1 E_{l,3} (x_3 - y_3)^l \exp \left(\lambda_- x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'\right),
\]
\[ \Theta_b = \sum_{l=0}^{1} E_{l,1}(x_3 - y_3)^l \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right) + \sum_{l=0}^{1} E_{l,2}(x_3 - y_3)^l \exp \left( \lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right) + \sum_{l=0}^{1} E_{l,4}(x_3 - y_3)^l \exp \left( \lambda_+ x_3 - \lambda_+ y_3 - \tau s - iy' \cdot \xi' \right), \]

\[ \Theta_d = \sum_{l=0}^{1} E_{l,5}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right) + \sum_{l=0}^{1} E_{l,6}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right) + \sum_{l=0}^{1} E_{l,7}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi' \right), \]

\[ \Theta_e = \sum_{l=0}^{1} E_{l,5}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right) + \sum_{l=0}^{1} E_{l,6}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right) + \sum_{l=0}^{1} E_{l,8}(x_3 - y_3)^l \exp \left( \mu_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi' \right), \]

where \( E_{l,j} \) can be computed explicitly and \( \text{ord} E_{l,j} = l \) for \( l = 0, 1 \) and \( j = 1, \cdots, 8 \).

Let us first consider \( a_{-2} \). From the form of \( \Theta_a \), we know that

\[ a_{-2} = \sum_{j=1}^{3} a_{-2,j}^3, \quad a_{-2,j} = \sum_{l=0}^{2} F_{l,j}(x_3 - y_3)^l \exp \left( \beta_j x_3 - \delta_j y_3 - \tau s - iy' \cdot \xi' \right) \quad \text{for } j = 1, 2, 3 \]

satisfying

\[ p_{2,0}^{(0)} a_{-2} = \sum_{l=0}^{1} E_{l,j}(x_3 - y_3)^l \exp \left( \beta_j x_3 - \delta_j y_3 - \tau s - iy' \cdot \xi' \right), \]

where \( \beta_1 = \beta_2 = \lambda_+ \), \( \beta_3 = \delta_1 = \delta_3 = \lambda_- \) and \( \delta_2 = \mu_- \). From the above equation, we can express \( F_{l,j} \) in terms of \( E_{l,j} \):

\[ F_{2,j} = \frac{E_{1,j}}{4 \gamma_j (\beta_j m_{33}^1 + i \sum_{j=1}^{2} m_{3j}^1 \xi_j)}, \quad \text{(2.19)} \]

\[ F_{1,j} = \frac{E_{0,j} - 2 \gamma_j m_{33}^1 F_{2,j}}{2 \gamma_j (\beta_j m_{33}^1 + i \sum_{j=1}^{2} m_{3j}^1 \xi_j)}, \quad \text{(2.20)} \]
where
\[
\gamma_j := \begin{cases} 1, & 1 \leq j \leq 4, \\ k, & 5 \leq j \leq 8, \end{cases}
\]
and \(\text{ord} F_{l,j} = l - 2\) \((l = 1, 2)\). We do the same calculations for \(b_{-2}\), \(d_{-2}\) and \(e_{-2}\). Then, we have

\[
a_{-2} = \left\{ \sum_{l=1}^{2} F_{l,1}(x_3 - y_3)^l + C_1 \right\} \exp (\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2} F_{l,2}(x_3 - y_3)^l + C_2 \right\} \exp (\lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2} F_{l,3}(x_3 - y_3)^l + C_3 \right\} \exp (\lambda_- x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'), \tag{2.21}
\]

\[
b_{-2} = \left\{ \sum_{l=1}^{2} F_{l,1}(x_3 - y_3)^l + C_4 \right\} \exp (\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2} F_{l,2}(x_3 - y_3)^l + C_5 \right\} \exp (\lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2} F_{l,4}(x_3 - y_3)^l + C_6 \right\} \exp (\lambda_- x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi'), \tag{2.22}
\]

\[
d_{-2} = \left\{ \sum_{l=1}^{2} F_{l,5}(x_3 - y_3)^l + C_7 \right\} \exp (\mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2} F_{l,6}(x_3 - y_3)^l + C_8 \right\} \exp (\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2} F_{l,7}(x_3 - y_3)^l + C_9 \right\} \exp (\mu_- x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'), \tag{2.23}
\]

\[
e_{-2} = \left\{ \sum_{l=1}^{2} F_{l,5}(x_3 - y_3)^l + C_{10} \right\} \exp (\mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2} F_{l,6}(x_3 - y_3)^l + C_{11} \right\} \exp (\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2} F_{l,8}(x_3 - y_3)^l + C_{12} \right\} \exp (\mu_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi'), \tag{2.24}
\]

where \(C_j\) for \(j = 1, \ldots, 12\) are constants with respect to \(x_3\) to be determined. From the following
transmission and boundary conditions:

\[ \mathbf{a}_{-2} - \mathbf{b}_{-2} = 0, \quad i \mathbf{e}_3 \cdot \mathbf{M}^1 \left( \frac{\xi}{D_{x_3}} \right) (\mathbf{a}_{-2} - \mathbf{b}_{-2}) = 0 \quad \text{on } x_3 = y_3, \]  

(2.25)

\[ \mathbf{d}_{-2} - \mathbf{e}_{-2} = 0, \quad i k \mathbf{e}_3 \cdot \mathbf{M}^1 \left( \frac{\xi}{D_{x_3}} \right) (\mathbf{d}_{-2} - \mathbf{e}_{-2}) = 0 \quad \text{on } x_3 = y_3, \]  

(2.26)

\[ \mathbf{a}_{-2} - \mathbf{d}_{-2} = 0, \quad i \mathbf{e}_3 \cdot \mathbf{M}^0 \left( \frac{\xi}{D_{x_3}} \right) \mathbf{a}_{-2} = i k \mathbf{e}_3 \cdot \mathbf{M}^0 \left( \frac{\xi}{D_{x_3}} \right) \mathbf{d}_{-2} \quad \text{on } x_3 = 0, \]  

(2.27)

we can derive a system of linear equations for \( C_j (j = 1, \ldots, 12) \), which will be explicitly given and solved in the appendix. Then, the proof for \( L = 2 \) is completed by inserting these constants \( C_j (j = 1, \ldots, 12) \) into the above expressions of \( \mathbf{a}_{-2}, \mathbf{b}_{-2}, \mathbf{d}_{-2} \) and \( \mathbf{e}_{-2} \).

Suppose that (2.6)-(2.9) are true for \( L \geq 2 \). We will show that they also hold for \( L + 1 \). Note that

\[ A_{1-L} = p_{2,0}^{(0)} \mathbf{a}_{-1-L} + \sum_{j+k+|\alpha|=L} (\alpha!)^{-1} (j!)^{-1} (x_3 - y_3)^j p_{2,j}^{(\alpha)} D_{x_3}^{\alpha} \mathbf{a}_{-1-k} \]

\[ + \sum_{j+k+|\alpha|=L-1} (\alpha!)^{-1} (j!)^{-1} (x_3 - y_3)^j p_{1,j}^{(\alpha)} D_{x_3}^{\alpha} \mathbf{a}_{-1-k} \]

\[ + \sum_{j+k=L-2} (j!)^{-1} (x_3 - y_3)^j p_{0,j}^{(0)} \mathbf{a}_{-1-k}. \]

Then \( A_{1-L} = 0 \) implies that

\[ p_{2,0}^{(0)} \mathbf{a}_{-1-L} = -\sum_{k=0}^{L-1} \left[ \sum_{j+k+|\alpha|=L} (\alpha!)^{-1} (j!)^{-1} (x_3 - y_3)^j p_{2,j}^{(\alpha)} D_{x_3}^{\alpha} \mathbf{a}_{-1-k} \right. \]

\[ + \sum_{j+k+|\alpha|=L-1} (\alpha!)^{-1} (j!)^{-1} (x_3 - y_3)^j p_{1,j}^{(\alpha)} D_{x_3}^{\alpha} \mathbf{a}_{-1-k} \]

\[ - \sum_{j+k=L-2} (j!)^{-1} (x_3 - y_3)^j p_{0,j}^{(0)} \mathbf{a}_{-1-k}. \]

From the form of \( \mathbf{a}_{-1-k} \), we know that \( \Theta_a^{1-L} \) is the sum of \( \exp(\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi') \), \( \exp(\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi') \) and \( \exp(\lambda_- x_3 - \lambda_+ y_3 - \tau s - iy' \cdot \xi') \) with \((2L-1)\)-th degree’s polynomials in \( x_3 - y_3 \) as the coefficients of exponentials. That is,

\[ \Theta_a^{1-L} = \sum_{l=0}^{2L-1} E_{l,1}(x_3 - y_3)^l \exp(\lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi') \]

\[ + \sum_{l=0}^{2L-1} E_{l,2}(x_3 - y_3)^l \exp(\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi') \]

\[ + \sum_{l=0}^{2L-1} E_{l,3}(x_3 - y_3)^l \exp(\lambda_- x_3 - \lambda_+ y_3 - \tau s - iy' \cdot \xi'), \]
where \( \text{ord} E_{l,j} = l + 1 - L \) for \( j = 1, 2, 3 \). So we have

\[
\mathbf{a}_{-L-1}^j = \sum_{j=1}^{3} a_{-1-L}^j, \quad \mathbf{a}_{-L-1}^j = \sum_{l=0}^{2L} F_{l,j}(x_3 - y_3)^l \exp \left( \beta_j x_3 - \delta_j y_3 - \tau s - iy' \cdot \xi' \right), \quad j = 1, 2, 3
\]
satisfying

\[
p_{2,0}^{(0)} \mathbf{a}_{-L-1}^j = \sum_{l=0}^{2L-1} E_{l,j}(x_3 - y_3)^l \exp \left( \beta_j x_3 - \delta_j y_3 - \tau s - iy' \cdot \xi' \right),
\]

where \( \beta_1 = \beta_2 = \lambda_+ \), \( \beta_3 = \delta_1 = \delta_3 = \lambda_- \) and \( \delta_2 = \mu_- \). Then we can express \( F_{l,j} \) in terms of \( E_{l,j} \):

\[
F_{2l,j} = \frac{E_{2L-1,j}}{4L(\beta_j m_{33}^1 + i \sum_{j=1}^{2} m_{3j}^1 \xi_j)},
\]

\[
F_{l+1,j} = \frac{E_{l,j} - (l + 2)(l + 1)m_{33}^1 F_{l+2,j}}{2(l + 1)(\beta_j m_{33}^1 + i \sum_{j=1}^{2} m_{3j}^1 \xi_j)},
\]

where \( \text{ord} F_{l+1,j} = l - L \) for \( l = 0, 1, \cdots, 2L - 1 \). Therefore, we have

\[
\mathbf{a}_{-L-1} = \left\{ \sum_{l=1}^{2L} f_{l,1}^{L+1}(x_3 - y_3)^l + C_1 \right\} \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right)
\]

\[
+ \left\{ \sum_{l=1}^{2L} f_{l,2}^{L+1}(x_3 - y_3)^l + C_2 \right\} \exp \left( \lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right)
\]

\[
+ \left\{ \sum_{l=1}^{2L} f_{l,3}^{L+1}(x_3 - y_3)^l + C_3 \right\} \exp \left( \lambda_- x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right)
\]

where \( f_{l,j}^{L+1} = F_{l,j} \) for \( l = 1, \cdots, 2L \) and \( j = 1, 2, 3 \). In the same way, we can get the similar expressions for \( \mathbf{b}_{-L-1}, \mathbf{d}_{-L-1} \) and \( \mathbf{e}_{-L-1} \) as follows:

\[
\mathbf{b}_{-L-1} = \left\{ \sum_{l=1}^{2L} f_{l,1}^{L+1}(x_3 - y_3)^l + C_4 \right\} \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right)
\]

\[
+ \left\{ \sum_{l=1}^{2L} f_{l,2}^{L+1}(x_3 - y_3)^l + C_5 \right\} \exp \left( \lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right)
\]

\[
+ \left\{ \sum_{l=1}^{2L} f_{l,3}^{L+1}(x_3 - y_3)^l + C_6 \right\} \exp \left( \lambda_+ x_3 - \lambda_+ y_3 - \tau s - iy' \cdot \xi' \right),
\]

\[
\mathbf{d}_{-L-1} = \left\{ \sum_{l=1}^{2L} f_{l,5}^{L+1}(x_3 - y_3)^l + C_7 \right\} \exp \left( \mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right)
\]

\[
+ \left\{ \sum_{l=1}^{2L} f_{l,6}^{L+1}(x_3 - y_3)^l + C_8 \right\} \exp \left( \mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right)
\]

\[
+ \left\{ \sum_{l=1}^{2L} f_{l,7}^{L+1}(x_3 - y_3)^l + C_9 \right\} \exp \left( \mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right),
\]
\[
e_{-1-L} = \left\{ \sum_{l=1}^{2L} f_{l,5}^{L+1} (x_3 - y_3)^l + C_{10} \right\} \exp \left( \mu_{+} x_3 - \lambda_{-} y_3 - \tau s - i y' \cdot \xi' \right) \\
+ \left\{ \sum_{l=1}^{2L} f_{l,6}^{L+1} (x_3 - y_3)^l + C_{11} \right\} \exp \left( \mu_{+} x_3 - \mu_{-} y_3 - \tau s - i y' \cdot \xi' \right) \\
+ \left\{ \sum_{l=1}^{2L} f_{l,8}^{L+1} (x_3 - y_3)^l + C_{12} \right\} \exp \left( \mu_{+} x_3 - \mu_{+} y_3 - \tau s - i y' \cdot \xi' \right),
\]

where \( f_{l,j}^{L+1} \) are determined from \( E_{i,j} \) (\( l = 1, \cdots, 2L, \ i = 0, 1, \cdots, 2L - 1, \ j = 1, \cdots, 8 \)) in \( \Theta_m (m = b, d, e) \) like (2.28) and (2.29). To determine the constants \( C_j \) (\( j = 1, \cdots, 12 \)), we use the transmission conditions on \( x_3 = y_3 \) and the boundary conditions \( x_3 = 0 \), namely,

\[
a_{-1-L} - b_{-1-L} = 0, \quad i e_3 \cdot M^1 \left( \frac{\xi'}{D_{x_3}} \right) (a_{-1-L} - b_{-1-L}) = 0 \quad \text{on} \ x_3 = y_3,
\]

\[
d_{-1-L} - e_{-1-L} = 0, \quad i k e_3 \cdot M^1 \left( \frac{\xi'}{D_{x_3}} \right) (d_{-1-L} - e_{-1-L}) = 0 \quad \text{on} \ x_3 = y_3,
\]

\[
a_{-1-L} - d_{-1-L} = 0, \quad i e_3 \cdot M^0 \left( \frac{\xi'}{D_{x_3}} \right) a_{-1-L} = i k e_3 \cdot M^0 \left( \frac{\xi'}{D_{x_3}} \right) d_{-1-L} \quad \text{on} \ x_3 = 0.
\]

These equations lead to the system of equations for \( C_j \) (\( j = 1, \cdots, 12 \)), and it is solved through the same process as we did for \( L = 2 \). Then we have the following expressions:

\[
a_{-1-L} = \left\{ \sum_{l=1}^{2L} f_{l,1}^{L+1} (x_3 - y_3)^l + (A_5 + A_7) \right\} \exp \left( \lambda_{+} x_3 - \lambda_{-} y_3 - \tau s - i y' \cdot \xi' \right) \\
+ \left\{ \sum_{l=1}^{2L} f_{l,2}^{L+1} (x_3 - y_3)^l + (B_5 + B_7) \right\} \exp \left( \lambda_{+} x_3 - \mu_{-} y_3 - \tau s - i y' \cdot \xi' \right) \\
+ \left\{ \sum_{l=1}^{2L} f_{l,3}^{L+1} (x_3 - y_3)^l + \frac{f_{l,1}^{L+1} - f_{l,4}^{L+1}}{\lambda_{+} - \lambda_{-}} \right\} \exp \left( \lambda_{-} x_3 - \lambda_{-} y_3 - \tau s - i y' \cdot \xi' \right),
\]

\[
b_{-1-L} = \left\{ \sum_{l=1}^{2L} f_{l,1}^{L+1} (x_3 - y_3)^l + (A_5 + A_7) \right\} \exp \left( \lambda_{+} x_3 - \lambda_{-} y_3 - \tau s - i y' \cdot \xi' \right) \\
+ \left\{ \sum_{l=1}^{2L} f_{l,2}^{L+1} (x_3 - y_3)^l + (B_5 + B_7) \right\} \exp \left( \lambda_{+} x_3 - \mu_{-} y_3 - \tau s - i y' \cdot \xi' \right) \\
+ \left\{ \sum_{l=1}^{2L} f_{l,4}^{L+1} (x_3 - y_3)^l + \frac{f_{l,1}^{L+1} - f_{l,4}^{L+1}}{\lambda_{+} - \lambda_{-}} \right\} \exp \left( \lambda_{+} x_3 - \lambda_{+} y_3 - \tau s - i y' \cdot \xi' \right),
\]
\[ d_{-1-L} = \left\{ \sum_{l=1}^{2L} f^{L+1}_{i,5}(x_3 - y_3)^l + A_7 \right\} \exp (\mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2L} f^{L+1}_{i,6}(x_3 - y_3)^l + B_7 \right\} \exp (\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2L} f^{L+1}_{i,7}(x_3 - y_3)^l + \frac{f^{L+1}_{i,7} - f^{L+1}_{i,8}}{\mu_+ - \mu_-} \right\} \exp (\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'), \]

\[ e_{-1-L} = \left\{ \sum_{l=1}^{2L} f^{L+1}_{i,5}(x_3 - y_3)^l + A_7 \right\} \exp (\mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2L} f^{L+1}_{i,6}(x_3 - y_3)^l + B_7 \right\} \exp (\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi') \\
+ \left\{ \sum_{l=1}^{2L} f^{L+1}_{i,7}(x_3 - y_3)^l + \frac{f^{L+1}_{i,7} - f^{L+1}_{i,8}}{\mu_+ - \mu_-} \right\} \exp (\mu_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi'), \]

where \( \text{ord} A_i = \text{ord} B_i = -1 - L \) for \( i = 5, 7 \) and \( \text{ord} f^{L+1}_{i,j} = l - 1 - L \) for \( j = 1, \cdots, 8 \). Thus we have shown (2.26), (2.29) for \( L + 1 \). The proof of the theorem is complete. \( \square \)

Once we have determined the amplitudes \( a, b, d, e \), the pseudo-differential operators \( G^\pm, H^\pm \) can be expressed in terms of (2.4).

### 3 Construction of the local parametrix with error estimate

In the section, we construct the local parametrix for the parabolic interior transmission problem (1.5) by taking the inverse Laplace transform of \((G^\pm, H^\pm)\) given in the last section. The error estimates coming from the truncation of the amplitudes are derived. We only show how to treat with \( G^+ \), since the arguments for the others are the same. To proceed, we need the following result:

**Lemma 3.1** [1, Lemma 2 and Lemma 3] For each \( \rho \geq 0 \), let \( g(\xi', \eta, \rho) \) be a holomorphic function of \((\xi', \eta)\) in \( L^2_\mu \subset \mathbb{C}^2 \times \mathbb{C} \) for some \( \mu > 0 \) with

\[ L^2_\mu = \{(\xi', \eta) \in \mathbb{C}^2 \times \mathbb{C}: \text{Im} \eta < \mu(|\text{Re} \eta| + |\text{Re} \xi'|^2) - \mu^{-1} |\text{Im} \xi'|^2 \}. \]

Assume that

\[ |g(\xi', \eta, \rho)| \leq C(|\xi'| + |\eta|^{1/2}) \exp[-c\rho(|\xi'| + |\eta|^{1/2})] \quad (3.1) \]

for \((\xi', \eta) \in L^2_\mu, l < 0 \) and \( \rho \geq 0 \). Set

\[
G(x', t; \rho) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix' \cdot \xi'} \int_{-\infty - i\rho}^{\infty - i\rho} e^{itn} g(\xi', \eta, \rho) \, d\eta \, d\xi' \quad \text{for } \rho > 0, \\
G(x', t; 0) \equiv \lim_{\rho \downarrow 0} G(x', t; \rho),
\]

where \( q \) is an arbitrary positive number. Then we have

\[ |G(x', t; \rho)| \leq Ct^{-\frac{1}{2}} \exp \left[ -c \frac{|x'|^2 + \rho^2}{t} \right]. \]
Recall that
\[
(G^+ \varphi)(x, \tau; y, s) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix \cdot \xi'} a(x, \xi', \tau, y, s)(F \varphi)(\xi') \, d\xi'
\]
\[
= (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix \cdot \xi'} \sum_{j=0}^{\infty} a_{-1-j}(x, \xi', \tau, y, s)(F \varphi)(\xi') \, d\xi'.
\] (3.2)

Since we will truncate \( \sum_{j=0}^{\infty} a_{-1-j} \) at \( j = N \) with large \( N \), we don’t care about the convergence of the formal sum. Denote this truncated one by \( G^+_N \). We also define \( G^-_N \) and \( H^+_N \) in the same way.

Take the leading term of (3.2) and consider its inverse Laplace transform
\[
\hat{G}^+(x, t; y, s) = (2\pi)^{-1}(2\pi)^{-2} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\mathbb{R}^2} \exp \left( t \tau + ix' \cdot \xi' \right) a_{-1}(x, \xi', \tau, y, s) \, d\xi' \, d\tau,
\]
where
\[
a_{-1} = (-A_1 + A_2) \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right)
+ (B_1 + B_2) \exp \left( \lambda_+ x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right)
+ A_1 \exp \left( \lambda_- x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right)
=: a_{-1}^{(1)} + a_{-1}^{(2)} + a_{-1}^{(3)}.
\]

Define
\[
\hat{G}^+_{(j)}(x, t; y, s) := -(2\pi)^{-3} i \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\mathbb{R}^2} \exp \left( t \tau + ix' \cdot \xi' \right) a_{-1}^{(j)}(x, \xi', \tau, y, s) \, d\xi' \, d\tau
\]
for \( j = 1, 2, 3 \). We only estimate \( \dot{G}^+_{(1)} \), since \( \dot{G}^+_{(2)} \) and \( \dot{G}^+_{(3)} \) can be done analogously. Note that
\[
\dot{G}^+_{(1)}(x, t; y, s) = -(2\pi)^{-3} i \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\mathbb{R}^2} \exp \left( t \tau + ix' \cdot \xi' \right) (-A_1 + A_2)
\times \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right) \, d\xi' \, d\tau
\]
\[
= (2\pi)^{-3} \int_{-\infty-i\sigma}^{\infty-i\sigma} \int_{\mathbb{R}^2} \exp \left( i\eta(t-s) + i(x' - y') \cdot \xi' \right) g(x, \xi', \eta, y_3, s) \, d\xi' \, d\eta,
\]
where
\[
g(x, \xi', \eta, y_3, s) = \left[ (-A_1 + A_2) \exp \left( \lambda_+ x_3 - \lambda_- y_3 \right) \right]_{\tau=i\eta}.
\] (3.3)

Notice that Lemma 3.1 still holds when the amplitude \( g \) depends on \( (x', y_3, s) \). So, if \( g \) satisfies the assumptions of Lemma 3.1 with \( l = -1 \), then we have the desired estimate
\[
|\dot{G}^+_{(1)}(x, t; y, s)| \leq c_1 (t-s)^{-3/2} \exp \left( -c_2 \frac{|x' - y'|^2 + (x_3 - y_3)^2}{t-s} \right)
\] (3.4)

with some positive constants \( c_1 \) and \( c_2 \), where we evaluated the inverse Laplace transform at \( t-s \).

Let us first verify that \( \dot{G}^+(x, t; y, s) \) satisfies the initial condition at \( t = s \). We change the contour \( \{ \tau = \sigma + i\mu : \mu \in \mathbb{R} \} \) of the integration with respect to \( \tau \) for \( \dot{G}^+(x, t; y, s) \) with \( t = s \) to an infinitely large half circle \( C_\infty := \lim_{\rho \to \infty} C_\rho \) with \( C_\rho = \{ \sigma + re^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \} \). Then we can easily see that
\[
\dot{G}^+(x, t; y, s) = 0 \quad \text{at} \quad t = s.
\] (3.5)
We next show that the assumptions of Lemma 3.1 are satisfied for \( g(\xi', \eta, \rho) \) with \( \rho = |x_3 - y_3| \). Let us first verify the holomorphicity assumption. Then the other assumption is easily justified. By (2.5), (2.17) and (2.18), it is enough to prove that \( \lambda_{\pm}, \mu_{\pm} \) and their terms with square root are holomorphic, and for \( y_3 \) close to 0

\[
k(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_{+} m_{33}^0) - (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_{+} m_{33}^0) = (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_{-} m_{33}^0)
\]

(3.6)
does not vanish for \((\xi', \eta) \in L_{\mu}^2\). Here we note that \( x_3 < 0 \) is confined to \((-\delta, 0)\) with small \( \delta > 0 \), so if \( y_3 \leq -2\delta \) it is easy to see that \( \hat{G}^+ \) becomes a smoothing operator. This is why we can assume that \( y_3 \) is close to 0.

Consider the characteristic equation for the operator \( p_{2,0}^{(0)} \), that is,

\[
p_0(x') \xi_3^2 + p_1(x', \xi') \xi_3 + (p_2(x', \xi') - \tau) = 0,
\]

where

\[
p_0(x') = -m_{33}, \quad p_1(x', \xi') = -2 \sum_{j=1}^{2} m_{3j} \xi_j, \quad p_2(x', \xi') = - \sum_{i,j=1}^{2} m_{ij} \xi_i \xi_j, \quad \tau = i\eta.
\]

Then the roots are given by \( \xi_3 = -p_1 \pm z^\pm \), where \( z^\pm = \sqrt{p_1^2 - 4p_0(p_2 - \tau)} \) and \( \pm \text{Im} z^\pm > 0 \). By the ellipticity, there exists a constant \( c > 0 \) such that

\[
p_1^2 - 4p_0p_2 < -c|\xi'|^2, \quad \xi' \in \mathbb{R}^2 \setminus \{0\}, \quad x' \in U,
\]

where \( U \) is a bounded open set in which \( p_0, p_1, p_2 \) are smooth. Then we have the following claim:

**Claim 3.2** There exists \( \mu > 0 \) such that \( p_1^2 - 4p_0p_2 + 4p_0i\eta \not\in [0, \infty) \) for \((\xi', \eta) \in L_{\mu}^2\).

**Proof.** We prove the claim by a contradiction argument. Note that

\[
L_{\mu}^2 \ni (\xi', \eta) \iff \text{Im} \eta < \mu(|\text{Re} \eta| + |\text{Re} \xi'|^2) - \mu^{-1}|\text{Im} \xi'|^2.
\]

(3.7)

Suppose that for \((\xi', \eta) \in L_{\mu}^2\), there is a positive constant \( m \) such that \( p_1^2 - 4p_0p_2 + 4p_0i\eta = m \). Set \( \alpha := -m_{33}, \beta := (\beta_1, \beta_2) := -2(m_{31}, m_{32}) \) and \( \gamma = (\gamma_{ij})_{i,j=1,2} := (-m_{ij})_{i,j=1,2} \). Then we obtain that

\[
p_0(x') = \alpha < 0, \quad p_1(x', \xi') = \beta(x') \cdot \xi',
\]

\[
p_2(x', \xi') = (\gamma(x') \xi') \cdot \xi' = (\gamma \xi') \cdot \xi' < 0 \quad \text{for} \; \xi' \in \mathbb{R}^2 \setminus \{0\},
\]

\[
p_1^2 = \sum_{j,k=1}^{2} \beta_j \beta_k \xi_j \xi_k = (\beta \otimes \beta) : (\xi' \otimes \xi'),
\]

\[
m = (\beta \otimes \beta) : (\xi' \otimes \xi') - 4\alpha (\gamma \xi') \cdot \xi' + 4i\alpha \eta.
\]

For simplicity of notations, we denote \( \xi_R = \text{Re} \xi', \xi_I = \text{Im} \xi', \eta_R = \text{Re} \eta \) and \( \eta_I = \text{Im} \eta \). Note that

\[
m = (\beta \otimes \beta) : (\xi_R \otimes \xi_R) - (\beta \otimes \beta) : (\xi_I \otimes \xi_I) + 2i(\beta \otimes \beta) : (\xi_R \otimes \xi_I)
\]

\[
-4\alpha((\xi_R \cdot \xi_R) - (\gamma \xi') \cdot \xi') - 8i\alpha(\xi_R \cdot \xi_I) - 4i\alpha \eta,
\]

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which yields

\[(\beta \otimes \beta) : (\xi_R' \otimes \xi_R') - (\beta \otimes \beta) : (\xi_i' \otimes \xi_i') - 4\alpha \{(\gamma \xi_R') \cdot \xi_i' - (\gamma \xi_i') \cdot \xi_i'\} - 4\alpha \eta_I = m,\]  
\[(\beta \otimes \beta) : (\xi_R' \otimes \xi_i') - 4\alpha (\gamma \xi_R') \cdot \xi_i' + 2\alpha \eta_R = 0.\]  

(3.8) (3.9) 

We obtain from (3.8) that

\[-(\beta \otimes \beta) : (\xi_R' \otimes \xi_R') + 4\alpha (\gamma \xi_R') \cdot \xi_R' + \{(\beta \otimes \beta) : (\xi_i' \otimes \xi_i') - 4\alpha (\gamma \xi_i') \cdot \xi_i'\} = -m - 4\alpha \eta_I.\]  

(3.10) 

The left hand side (LHS) of (3.10) has the estimate that LHS \(>c^'|\xi_R'|^2 - c''|\xi_i'|^2\) for some positive constants \(c'\) and \(c''\). For the right hand side (RHS) of (3.10), by the definition of \(L^2_\mu\), we deduce from (3.9) that

\[\text{RHS} \leq -4\alpha \eta_I < (-4\alpha)\{\mu(|\eta_R| + |\xi_R'|^2) - \mu^{-1}|\xi_i'|^2\}\]
\[= (-4\alpha)\{\mu((-2\alpha)^{-1}|(\beta \otimes \beta) : (\xi_i' \otimes \xi_i') - 4\alpha (\gamma \xi_R') \cdot \xi_i'| + |\xi_R'|^2) - \mu^{-1}|\xi_i'|^2\}\]
\[\leq \mu K(|\xi_R'|^2 + |\xi_i'|^2) - (-4\alpha)\mu^{-1}|\xi_i'|^2\]

for some positive constant \(K\). Thus, we have

\[c(|\xi_R'|^2 + |\xi_i'|^2) \leq \mu K(|\xi_R'|^2 + |\xi_i'|^2), \quad \mu > 0.\]

By taking \(\mu > 0\) small, we have \(\xi_R' = \xi_i' = 0\) and hence \(\eta_R = 0\) by (3.9). Then, \(-4\alpha \eta_I = m\) gives \(\eta_I \geq 0\). This contradicts to (3.7). \(\square\)

Using the above claim, we can easily see that \(\lambda_\pm, \mu_\pm\) and their terms with square root are holomorphic for \((\xi', \eta) \in L^2_\mu\). In addition, for \(y_3\) near to 0, there exists a constant \(\mu > 0\) such that (3.6) will not vanish. We give the proof as follows. For simplicity of notations, we define

\[R_0 := \sum_{j=1}^{2} m_{3j}^0 \xi_j, \quad R_1 := \sum_{j=1}^{2} m_{3j}^1 \xi_j, \quad Q := \sum_{j,l=1}^{2} m_{jl}^1 \xi_j \xi_l.\]

Recalling the definitions of \(\lambda_\pm\) and \(\mu_\pm\), we have

\[k(i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_+ m_{33}^0) - (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0)\]
\[= k(iR_0 + \mu_+ m_{33}^0) - (iR_0 + \lambda_+ m_{33}^0)\]
\[= i(k - 1) (R_0 - m_{33}^0 (m_{33}^1)^{-1} R_1) + m_{33}^0 (m_{33}^1)^{-1} (\sqrt{m_{33}^1 (k^2 Q + k \tau)} - k^2 R_1^2 - \sqrt{m_{33}^1 (Q + \tau) - R_1^2})\].

When \(y_3\) is near to 0, \(R_0 - m_{33}^0 (m_{33}^1)^{-1} R_1\) is also near to 0. Using the same argument as that for Claim 3.2, we can show that there exists a constant \(\mu > 0\) such that

\[(k - 1)[m_{33}^1 (k + 1)Q + m_{33}^1 \tau - (k + 1)R_1^2] \notin [0, \infty) \quad \text{for } (\xi', \eta) \in L^2_\mu,\]

which implies that

\[\sqrt{m_{33}^1 (k^2 Q + k \tau)} - k^2 R_1^2 - \sqrt{m_{33}^1 (Q + \tau) - R_1^2} \neq 0.\]
This completes the proof of the holomorphic property by the scaling argument.

In conclusion, we have justified that $g$ defined in (3.3) satisfies the assumptions of Lemma 3.1 and hence we obtain the desired estimate (3.4) for $\hat{G}^+$ where we only take the leading term $a_{-1}$ of the amplitude $a$. In the following, we estimate the error term if we truncate the amplitude at $j = N$. Let

$$\hat{G}_N(x, t; y, s) = \hat{G}_N^\pm(x, t; y, s), \quad \pm(x_3 - y_3) > 0,$$

where $\hat{G}_N^\pm(x, t; y, s)$ is the inverse Laplace transform with respect to $\tau$ of $C_N^\pm(x, \tau; y, s)$. Then, by the construction of local parametrix and Lemma 3.1, the error defined by

$$R_N := R_N(x, t; y, s) = \mathcal{L}G_N - I$$

is smooth enough for $x_3 \leq 0$ with respect to all the variables and satisfies the estimate

$$|\partial_t^j \partial_x^\alpha R_N(x, t; y, s)| \leq C_{N,M}^{j}(t - s)^{-2 + N\cdot|\alpha|-j} \exp \left( -C_{N,M} \frac{|x - y|^2}{t - s} \right)$$

(3.11)

for any $j \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^3$ such that $2j + |\alpha| \leq M$ for any given $N \geq M \leq N - 4$ and some positive constants $C_{N,M}, C'_{N,M}$.

We define $\hat{H}_N(x, t; y, s)$ and the associated error $S_N(x, t; y, s)$ similarly as $\hat{G}_N(x, t; y, s)$ and $R_N(x, t; y, s)$, respectively. Then they satisfy similar properties as those of $\hat{G}_N$ and $R_N$. Since the pair $(\hat{G}_N(x, t; y, s), \hat{H}_N(x, t; y, s))$ satisfies the boundary condition of (1.6) locally in terms of the coordinates $\xi = (\xi_1, \xi_2, \xi_3) = \Phi_{x_0}(x)$, this pair can be considered as a local parametrix in the open neighborhood $U(x_0)$ of $x_0 \in \partial D$.

4 Construction of the Green function

In this section, using a partition of unity, we first patch the local parametrices constructed above and the fundamental solution to the diffusion equation so that we have the parametrix for our parabolic interior transmission problem (1.5). Then, using Levi’s method, we construct the Green function from this parametrix.

Take $x_0^{(j)} \in \partial D (j = 1, 2, \cdots, J)$ so that $U_j := U(x_0^{(j)}) (j = 1, 2, \cdots, J)$ is an open covering of $\partial D$. Let $U_0$ be an open set such that $\overline{U_0} \subset D$ and $U_j (j = 0, 1, \cdots, J)$ give an open covering of $\overline{D}$. Let $\varphi_j \in C_0^\infty(U_j) (j = 0, 1, \cdots, J)$ be a partition of unity subordinated to this cover and $\psi_j \in C_0^\infty(U_j) (j = 0, 1, \cdots, J)$ satisfy $\psi_j = 1$ on supp $\varphi_j (j = 0, 1, \cdots, J)$. We will abuse the notations $(G_j, H_j)$ to denote the local parametrix $(\hat{G}_N(x, t; y, s), \hat{H}_N(x, t; y, s))$ constructed for each $U_j := U(x_0^{(j)}) (j = 1, 2, \cdots, J)$ in the previous section.

Set

$$G_0(x, t; y, s) := H(t - s) (4\pi(t - s))^{-3/2} \exp \left( -(4(t - s))^{-1}|x - y|^2 \right),$$

$$H_0(x, t; y, s) := H(t - s) (4k\pi(t - s))^{-3/2} \exp \left( -(4k(t - s))^{-1}|x - y|^2 \right),$$

where $H$ is the Heaviside function defined by

$$H(t - s) = \begin{cases} 
1, & t - s > 0, \\
0, & t - s \leq 0.
\end{cases}$$
Define
\[
\tilde{G}'(x, t; y, s) := \sum_{j=0}^{J} \psi_j(x)G_j(x, t; y, s)\varphi_j(y),
\]
\[
\tilde{H}'(x, t; y, s) := \sum_{j=0}^{J} \psi_j(x)H_j(x, t; y, s)\varphi_j(y).
\]

Let \(\tilde{G}'(s), \tilde{H}'(s), G_j(s), H_j(s)\) for \(j = 0, 1, \cdots, J\) be the integral transforms with the corresponding kernels \(\tilde{G}'(x, t; y, s), \tilde{H}'(x, t; y, s), G_j(x, t; y, s), H_j(x, t; y, s)\). For example, \((\tilde{G}'\varphi)(x, t)\) is defined as
\[
\left(\tilde{G}'(s)\varphi\right)(x, t) = \int_{s}^{T} \int_{D} \tilde{G}'(x, t; y, s')\varphi(y, s')\,dy\,ds'
\]
for some function \(\varphi(x, t)\). Note that
\[
(\partial_t - \Delta)\tilde{G}'(s) = \sum_{j=0}^{J} \left\{ \psi_j(\partial_t - \Delta)G_j(s)\varphi_j - [\Delta, \psi_j]G_j(s)\varphi_j \right\}
\]
\[
= I + S_N^{\tilde{G}'}(s) - \sum_{j=0}^{J} [\Delta, \psi_j]G_j(s)\varphi_j
\]
\[
= I + \hat{R}_N^{\tilde{G}'}(s),
\]
where \([\Delta, \psi_j] := \nabla\psi_j \cdot \nabla - \Delta \psi_j\) is the commutator of \(\Delta\) and the multiplication by \(\psi_j\). Using the estimate \((3.11)\) and its derivation, we know that
\[
S_N^{\tilde{G}'}(s) : C^m([s, T]; H^r(D)) \to C^m([s, T]; H^{r+m}(D))
\]
with \(m \in \mathbb{Z}_+, 3m \leq N - 4\) is a bounded operator for any \(r \in \mathbb{Z}_+\) vanishing at \(t = s\) by order \(m\). In the sequel we fix \(m\) as above.

Since \(\text{supp } \varphi_j \cap \text{supp } [\Delta, \psi_j] = \emptyset\), it can be seen that \(\sum_{j=0}^{J} [\Delta, \psi_j]G_j(s)\varphi_j\) is a smoothing operator flat at \(t = s\). Hence, \(\hat{R}_N^{\tilde{G}'}(s) : C^m([s, T]; H^r(D)) \to C^m([s, T]; H^{r+m}(D))\) is a bounded operator for any \(r \in \mathbb{Z}_+\) vanishing at \(t = s\) by order \(m\). Similarly, we have
\[
(\partial_t - k\Delta)\tilde{H}'(s) = I + \hat{R}_N^H(s)
\]
with a bounded operator \(\hat{R}_N^H(s) : C^m([s, T]; H^r(D)) \to C^m([s, T]; H^{r+m}(D))\) for any \(r \in \mathbb{Z}_+\) vanishing at \(t = s\) by order \(m\). From the construction of the local parametrix \((G_j, H_j)\), it is clear that
\[
\tilde{H}'(s) - \tilde{G}'(s) = 0 \quad \text{on } \partial D.
\]

Further, since \(\text{supp } \varphi_j \cap \text{supp } [\partial_\nu, \psi_j] = \emptyset\), we have
\[
k\partial_\nu \tilde{H}'(s) - \partial_\nu \tilde{G}'(s) = \sum_{j=1}^{J} \left\{ \psi_j(k\partial_\nu H_j(s) - \partial_\nu G_j(s))\varphi_j - (k[\partial_\nu, \psi_j]H_j(s) - [\partial_\nu, \psi_j]G_j(s))\varphi_j \right\}
\]
\[
=: \Lambda(s)
\]
with a bounded smoothing operator $\Lambda(s)$ flat at $t = s$. By noticing $G_j(s)|_{t=s} = H_j(s)|_{t=s} = 0$ for $j = 0, 1, \cdots, J$ form their definitions, we have

$$\tilde{G}'(s)|_{t=s} = \tilde{H}'(s)|_{t=s} = 0.$$  

Finally, by moving $\Lambda(s)$ to inhomogeneous terms via the inverse trace operator, there exists a parametrix $(\tilde{G}(s), \tilde{H}(s))$ which has the same estimates as those of $(\tilde{G}'(s), \tilde{H}'(s))$ and satisfies

$$\begin{cases}
(\partial_t - \Delta)\tilde{G}(s) = I + R^N_G(s) & \text{in } D_T, \\
(\partial_t - k\Delta)\tilde{H}(s) = I + R^N_H(s) & \text{in } D_T, \\
\tilde{H}(s) - \tilde{G}(s) = 0 & \text{on } (\partial D)_T, \\
k\partial_x \tilde{H}(s) - \partial_x \tilde{G}(s) = 0 & \text{on } (\partial D)_T, \\
\tilde{G}(s) = \tilde{H}(s) = 0 & \text{at } t = s,
\end{cases} \tag{4.2}$$

where $R^N_G(s), R^N_H(s) : C^m([s, T]; H^r(D)) \to C^m([s, T]; H^{r+m}(D))$ are bounded operators for any $r \in \mathbb{Z}_+$ vanishing at $t = s$ by order $m$. Thus we have $\tilde{G}(t, s)$ and $\tilde{H}(t, s)$ such that

$$\mathcal{L}_1 \tilde{G} = I + \mathcal{S}_N \quad \text{with } \tilde{G}(s, s) = 0,$$

$$\mathcal{L}_k \tilde{H} = I + \mathcal{S}_N \quad \text{with } \tilde{H}(s, s) = 0$$

with $\mathcal{L}_1 = \partial_t - \Delta$ and $\mathcal{L}_k = \partial_t - k\Delta$. Consequently, we can construct the Green function for our interior transmission problem (1.3) using Levi’s method. In the following, we only show the argument for $G$. Fixing $r \in \mathbb{Z}_+$, let

$$W_1(t, s) = -R_N(t, s),$$

$$W_j(t, s) = \int_s^t W_1(t, s')W_{j-1}(s', s) \, ds', \quad j \geq 2.$$  

Here we note that for instance the operator $R_N(t, s)$ should be understood as an integral operator on $C^m([s, T]; H^r(D))$ with kernel $R_N(x, t; y, s)$, that is,

$$(R_N(t, s)\phi)(x) := \int_s^t \int_D R_N(x, t; y, s')\phi(y, s') \, dy \, ds'.$$

Then, we have

$$\sum_{j=1}^l W_j(t, s) = -R_N(t, s) - \int_s^t R_N(t, s')\sum_{j=1}^{l-1} W_j(s', s) \, ds'. \tag{4.3}$$

Note that $R_N(t, s) : C^m([s, T]; H^r(D)) \to C^m([s, T]; H^r(D))$ is uniformly bounded for each $r \in \mathbb{Z}_+$ vanishing at $t = s$ by order $m \in \mathbb{Z}_+$. Let $\| \cdot \|$ denote the operator norm for operators on $C^m([s, T]; H^r(D))$. Then, we have the following estimates:

$$\|W_1(t, s)\| \leq C_0,$$

$$\|W_j(t, s)\| \leq \frac{C_0^{j-1}}{(j-1)!}(t-s)^{j-1}, \quad j \geq 2.$$
Then it can be easily concluded that

\[ W(t, s) := \sum_{j=1}^{\infty} W_j(t, s) \]

converges as a bounded operator on \( C^m([s, T]; H^r(D)) \) and vanishes at \( t = s \) by order \( m \). From (4.3) we observe that

\[ W(t, s) = -R_N(t, s) - \int_s^t R_N(t, s') W(s', s) \, ds'. \]  

(4.4)

Set

\[ G = \tilde{G} + \int_s^t \tilde{G}(t, s') W(s', s) \, ds'. \]  

(4.5)

Then direct calculations give that

\[ \mathcal{L}_1 G = \mathcal{L}_1 \tilde{G} + \int_s^t \mathcal{L}_1 \tilde{G}(t, s') W(s', s) \, ds' \]

\[ = I + R_N(t, s) + W(t, s) + \int_s^t R_N(t, s') W(s', s) \, ds' \]

\[ = I. \]

Since we can define \( H \) by compensating \( \tilde{H} \) via the Levi method and it has a similar formula as (4.5), the pair \((G, H)\) satisfies the boundary condition of the interior transmission problem. Thus we have completed our argument for constructing the Green function \((G, H)\) of the interior transmission problem.

### 5 Uniqueness of the Green function and the solvability of ITP

In this section, we show the uniqueness of the Green function for (1.5). Equivalently, we only need to show that for the following problem:

\[
\begin{cases}
(\partial_t - \Delta)v = 0 & \text{in } D_T, \\
(\partial_t - k\Delta)u = 0 & \text{in } D_T, \\
u - v = 0 & \text{on } (\partial D)_T, \\
k\partial_n u - \partial_n v = 0 & \text{on } (\partial D)_T, \\
u = u_0, \ v = v_0 & \text{at } t = s.
\end{cases}
\]  

(5.1)

To this end, we use the duality argument \[10\] Chapter II] and introduce the adjoint problem of (5.1):

\[
\begin{cases}
(-\partial_t - \Delta)w = 0 & \text{in } D_T, \\
(-\partial_t - k\Delta)z = 0 & \text{in } D_T, \\
z + w = 0 & \text{on } (\partial D)_T, \\
k\partial_n z + \partial_n w = 0 & \text{on } (\partial D)_T, \\
z = z_0, \ w = w_0 & \text{at } t = s.
\end{cases}
\]  

(5.2)
Let \( U = (v, u)^T, Z = (w, z)^T \). Then we have
\[
\int_D \partial_t (U \cdot Z) \, dx = \int_D \left[ (w \partial_t v + v \partial_t w) + (z \partial_t u + u \partial_t z) \right] \, dx
= \int_D \left[ (w \Delta v - v \Delta w) + k(z \Delta u - u \Delta z) \right] \, dx
= \int_{\partial D} \left[ (w \partial_\nu v - v \partial_\nu w) + k(z \partial_\nu u - u \partial_\nu z) \right] \, dx
= \int_{\partial D} [-k(z \partial_\nu u - u \partial_\nu z) + k(z \partial_\nu u - u \partial_\nu z)] \, dx
= 0. \tag{5.3}
\]

In order to prove the uniqueness of the Green function for \( (5.1) \), it is enough to show that, for any Green function \( G_1(x, t; y, s) \) of \( (5.1) \) and any Green function \( G_2(x, t; y, s) \) of \( (5.2) \), we have
\[
G_1(x, t; y, s) = G_2^*(x, t; y, s). \tag{5.4}
\]
Indeed, it can be justified in the following way. Let \( U_0, Z_0 \in C_0^\infty(D) \). For any fixed \( t \) and \( s \) such that \( 0 < s < \tau < t < T \), set
\[
U(z, \tau) = \int_D G_1(z, \tau; y, s)U_0(y) \, dy, \quad Z(z, \tau) = \int_D G_2(x, \tau; z, \tau)Z_0(x) \, dx.
\]
We see from (5.3) that
\[
F(\tau) := \int_D U(z, \tau) \cdot Z(z, \tau) \, dz
\]
is independent of \( \tau \). This implies that
\[
\int_D U(z, t) \cdot Z(z, t) \, dz = \int_D U(z, s) \cdot Z(z, s) \, dz, \tag{5.5}
\]
On the other hand, we notice that
\[
\int_D U(z, t) \cdot Z(z, t) \, dz = \int_D \left( \int_D G_1(x, t; y, s)U_0(y) \, dy \right) \cdot Z_0(x) \, dx,
\]
\[
\int_D U(z, s) \cdot Z(z, s) \, dz = \int_D U_0(y) \cdot \left( \int_D G_2(x, t; y, s)Z_0(x) \, dx \right) \, dy
= \int_D \left( \int_D G_2^*(x, t; y, s)U_0(y) \, dy \right) \cdot Z_0(x) \, dx.
\]
Therefore, (5.5) gives (5.4).

We are now in a position to conclude that the interior transmission problem \((1.3)\) has a unique Green function. Therefore, for any given inhomogeneous term \((N_1, N_2) \in \tilde{H}^{-1, \frac{1}{2}}(D_T) \times \tilde{H}^{-1, -\frac{1}{2}}(D_T)\), our problem \((1.3)\) has a unique solution in \( \tilde{H}^{1, \frac{1}{2}}(D_T) \times \tilde{H}^{1, -\frac{1}{2}}(D_T)\), which can be expressed as usual by the Green function and \((N_1, N_2)\). Further, we have the following estimate:
\[
\|v\|_{\tilde{H}^{1, \frac{1}{2}}(D_T)}^2 + \|u\|_{\tilde{H}^{1, -\frac{1}{2}}(D_T)}^2 \leq C \left( \|N_1\|_{\tilde{H}^{-1, \frac{1}{2}}(D_T)}^2 + \|N_2\|_{\tilde{H}^{-1, -\frac{1}{2}}(D_T)}^2 \right).
\]

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6 Concluding remarks

In this paper, we investigated the interior transmission problem for the diffusion equation which is a non-classical initial boundary value problem for a pair of the diffusion equations with coupled boundary conditions. This work was motivated by our previous studies on the sampling method for reconstructing unknown inclusions in a diffusive conductor from boundary measurements. The unique solvability of the interior transmission problem was shown by the method of Green function. Our argument constructing the Green function is based on the theory of pseudo-differential operators. It can be used to construct the fundamental solutions for the diffusion operators with discontinuous coefficients which arise in the mathematical modeling of active thermography and diffusion optical tomography. It may also be applied to analyze the asymptotic behavior of indicator functions in non-iterative reconstruction methods for inverse boundary problems. These are our future works.

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Appendix: Solving the linear system for $L = 2$ in Theorem 2.3

In this appendix, we show some details of the proof for $L = 2$ in Theorem 2.3. From (2.25)-(2.27), we obtain the following system of equations for constants $C_j (j = 1, \cdots, 12)$ in (2.21)-(2.24):

\[
(C_1 - C_4) \exp ((\lambda_+ - \lambda_-) y_3) + (C_2 - C_5) \exp ((\lambda_+ - \mu_-) y_3) + C_3 - C_6 = 0, \quad (A.1)
\]

\[
\lambda_+(C_1 - C_4) \exp ((\lambda_+ - \lambda_-) y_3) + \lambda_+(C_2 - C_5) \exp ((\lambda_+ - \mu_-) y_3) + \lambda_-C_3 - \lambda_-C_6 = -F_{1,3} + F_{1,4}, \quad (A.2)
\]

\[
(C_7 - C_{10}) \exp ((\mu_+ - \lambda_-) y_3) + (C_8 - C_{11}) \exp ((\mu_+ - \mu_-) y_3) + C_9 - C_{12} = 0, \quad (A.3)
\]

\[
\mu_+(C_7 - C_{10}) \exp ((\mu_+ - \lambda_-) y_3) + \mu_+(C_8 - C_{11}) \exp ((\mu_+ - \mu_-) y_3) + \mu_-C_9 - \mu_+C_{12} = -F_{1,7} + F_{1,8}, \quad (A.4)
\]

\[
(C_1 + C_3 - C_7) \exp (-\lambda_- y_3) + (C_2 - C_8 - C_9) \exp (-\mu_- y_3)
\]

\[
= -\sum_{l=1}^{2}(F_{l,1} + F_{l,3} - F_{l,5})(-y_3)^l \exp (-\lambda_- y_3) - \sum_{l=1}^{2}(F_{l,2} - F_{l,6} - F_{l,7})(-y_3)^l \exp (-\mu_- y_3)
\]

\[
=: A_3 \exp (-\lambda_- y_3) + B_3 \exp (-\mu_- y_3), \quad (A.5)
\]
\( (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0) (C_1 \exp (-\lambda_- y_3) + C_2 \exp (-\mu_- y_3)) \)
\( + (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_- m_{33}^0) C_3 \exp (-\lambda_- y_3) \)
\( - k (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_+ m_{33}^0) (C_7 \exp (-\lambda_- y_3) + C_8 \exp (-\mu_- y_3)) \)
\( - k (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_- m_{33}^0) C_9 \exp (-\mu_- y_3) \)
\( = - (i \sum_{j=1}^{2} m_{3j}^0 \xi_j) [\sum_{l=1}^{2} (F_{l,1} + F_{l,3})(-y_3)^l \exp (-\lambda_- y_3) + \sum_{l=1}^{2} F_{l,2}(-y_3)^l \exp (-\mu_- y_3)] \)
\( - m_{33}^0 [\sum_{l=1}^{2} l (F_{l,1} + F_{l,3})(-y_3)^{l-1} + \sum_{l=1}^{2} (\lambda_+ F_{l,1} + \lambda_- F_{l,3})(-y_3)^l] \exp (-\lambda_- y_3) \)
\( + \{ \sum_{l=1}^{2} l F_{l,2}(-y_3)^{l-1} + \lambda_+ \sum_{l=1}^{2} F_{l,2}(-y_3)^l \} \exp (-\mu_- y_3) \]
\( + (ik \sum_{j=1}^{2} m_{3j}^0 \xi_j) [\sum_{l=1}^{2} F_{l,5}(-y_3)^l \exp (-\lambda_- y_3) + \sum_{l=1}^{2} (F_{l,6} + F_{l,7})(-y_3)^l \exp (-\mu_- y_3)] \)
\( + km_{33} [\sum_{l=1}^{2} l F_{l,5}(-y_3)^{l-1} + \mu_+ \sum_{l=1}^{2} F_{l,5}(-y_3)^l] \exp (-\lambda_- y_3) \)
\( + \{ \sum_{l=1}^{2} l (F_{l,6} + F_{l,7})(-y_3)^{l-1} + \sum_{l=1}^{2} (\mu_+ F_{l,6} + \mu_- F_{l,7})(-y_3)^l \} \exp (-\mu_- y_3) \]
\( =: A_4 \exp (-\lambda_- y_3) + B_4 \exp (-\mu_- y_3), \) \hspace{1cm} (A.6) 

where \( \text{ord} A_3 = \text{ord} B_3 = -2 \) and \( \text{ord} A_4 = \text{ord} B_4 = -1. \)

We remark that, looking at the structures of the amplitudes, \( C_3, C_9 \) and the four linear combinations given by \( C_1 \exp (-\lambda_- y_3) + C_2 \exp (-\mu_- y_3), C_4 \exp (-\lambda_- y_3) + C_5 \exp (-\mu_- y_3) + C_6 \exp (-\lambda_+ y_3), C_7 \exp (-\lambda_- y_3) + C_8 \exp (-\mu_- y_3), C_{10} \exp (-\lambda_- y_3) + C_{11} \exp (-\mu_- y_3) + C_{12} \exp (-\mu_+ y_3) \) are the substantial unknowns for the system \( (A.1)-(A.6) \). Hence, \( a_{-2}, b_{-2}, d_{-2}, e_{-2} \) are uniquely determined from \( (A.1)-(A.6) \).

From \( (A.1)-(A.4) \), we can easily get
\[
C_3 = \frac{F_{1,3} - F_{1,4}}{\lambda_+ - \lambda_-}, \quad C_9 = \frac{F_{1,7} - F_{1,8}}{\mu_+ - \mu_-}. \) \hspace{1cm} (A.7)
\]

where \( \text{ord} C_3 = \text{ord} C_9 = -2. \) Then \( (A.5) \) and \( (A.6) \) become
\[
C_1 \exp (-\lambda_- y_3) + C_2 \exp (-\mu_- y_3) - C_7 \exp (-\lambda_- y_3) - C_8 \exp (-\mu_- y_3) \\
= (A_3 - C_3) \exp (-\lambda_- y_3) + (B_3 + C_9) \exp (-\mu_- y_3) \\
=: A_5 \exp (-\lambda_- y_3) + B_5 \exp (-\mu_- y_3) \) \hspace{1cm} (A.8)
respectively, where \( \text{ord} \ A_5 = \text{ord} \ B_5 = -2 \) and \( \text{ord} \ A_6 = \text{ord} \ B_6 = -1 \). By direct calculations, we derive from (A.8) and (A.9) that

\[
C_7 \exp \left( -\lambda_y \right) + C_8 \exp \left( -\mu_y \right) = \left\{ k \left( i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \mu_+ m_{33}^0 \right) - (i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0) \right\}^{-1} \times \left[ \left\{ i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0 \right\} A_5 - A_6 \right] \exp \left( -\lambda_y \right) + \left[ \left\{ i \sum_{j=1}^{2} m_{3j}^0 \xi_j + \lambda_+ m_{33}^0 \right\} B_5 - B_6 \right] \exp \left( -\mu_y \right) =: A_7 \exp \left( -\lambda_y \right) + B_7 \exp \left( -\mu_y \right), \tag{A.10}
\]

\[
C_1 \exp \left( -\lambda_y \right) + C_2 \exp \left( -\mu_y \right) = (C_7 + A_5) \exp \left( -\lambda_y \right) + (C_8 + B_5) \exp \left( -\mu_y \right) = (A_5 + A_7) \exp \left( -\lambda_y \right) + (B_5 + B_7) \exp \left( -\mu_y \right) \tag{A.11}
\]

with \( \text{ord} \ A_7 = \text{ord} \ B_7 = -2 \). Consequently, we have

\[
C_4 \exp \left( -\lambda_y \right) + C_5 \exp \left( -\mu_y \right) + C_6 \exp \left( -\lambda_+ \right) = \exp \left( -\lambda_+ \right) [C_4 \exp \left( \lambda_+ - \lambda_+ \right) y_3] + C_5 \exp \left( \lambda_+ - \mu_+ \right) y_3 + C_6 = \exp \left( -\lambda_+ \right) [C_1 \exp \left( \lambda_+ - \lambda_+ \right) y_3] + C_2 \exp \left( \lambda_+ - \mu_+ \right) y_3 + C_3 = (A_5 + A_7) \exp \left( -\lambda_y \right) + (B_5 + B_7) \exp \left( -\mu_y \right) \tag{A.12}
\]

\[
C_10 \exp \left( -\lambda_y \right) + C_{11} \exp \left( -\mu_y \right) + C_{12} \exp \left( -\mu_+ \right) = \exp \left( -\mu_+ \right) [C_{10} \exp \left( \mu_+ - \lambda_+ \right) y_3] + C_{11} \exp \left( \mu_+ - \mu_+ \right) y_3 + C_{12} = \exp \left( -\mu_+ \right) [C_7 \exp \left( \mu_+ - \lambda_+ \right) y_3] + C_8 \exp \left( \mu_+ - \mu_+ \right) y_3 + C_9 = A_7 \exp \left( -\lambda_y \right) + B_7 \exp \left( -\mu_y \right) + \frac{F_{1,7} - F_{1,8}}{\mu_+ - \mu_-} \exp \left( -\mu_+ y_3 \right). \tag{A.13}
\]
By substituting \( \text{A.7}, \text{A.10} - \text{A.13} \) into \( 2.21 - 2.24 \), we finally obtain the expressions of the amplitudes \( a_{-2}, b_{-2}, d_{-2} \) and \( e_{-2} \) as follows:

\[
\begin{align*}
\mathbf{a}_{-2} &= \left\{ \sum_{l=1}^{2} F_{l,1}(x_3 - y_3)^l + (A_5 + A_7) \right\} \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right) \\
&\quad + \left\{ \sum_{l=1}^{2} F_{l,2}(x_3 - y_3)^l + (B_5 + B_7) \right\} \exp \left( \lambda_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi' \right) \\
&\quad + \left\{ \sum_{l=1}^{2} F_{l,3}(x_3 - y_3)^l + \frac{F_{1,3} - F_{1,4}}{\lambda_+ - \lambda_-} \right\} \exp \left( \lambda_- x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right), \\
\mathbf{b}_{-2} &= \left\{ \sum_{l=1}^{2} F_{l,1}(x_3 - y_3)^l + (A_5 + A_7) \right\} \exp \left( \lambda_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right) \\
&\quad + \left\{ \sum_{l=1}^{2} F_{l,2}(x_3 - y_3)^l + (B_5 + B_7) \right\} \exp \left( \lambda_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi' \right) \\
&\quad + \left\{ \sum_{l=1}^{2} F_{l,4}(x_3 - y_3)^l + \frac{F_{1,3} - F_{1,4}}{\lambda_+ - \lambda_-} \right\} \exp \left( \lambda_+ x_3 - \lambda_+ y_3 - \tau s - iy' \cdot \xi' \right), \\
\mathbf{d}_{-2} &= \left\{ \sum_{l=1}^{2} F_{l,5}(x_3 - y_3)^l + A_7 \right\} \exp \left( \mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right) \\
&\quad + \left\{ \sum_{l=1}^{2} F_{l,6}(x_3 - y_3)^l + B_7 \right\} \exp \left( \mu_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi' \right) \\
&\quad + \left\{ \sum_{l=1}^{2} F_{l,7}(x_3 - y_3)^l + \frac{F_{1,7} - F_{1,8}}{\mu_+ - \mu_-} \right\} \exp \left( \mu_- x_3 - \mu_- y_3 - \tau s - iy' \cdot \xi' \right), \\
\mathbf{e}_{-2} &= \left\{ \sum_{l=1}^{2} F_{l,5}(x_3 - y_3)^l + A_7 \right\} \exp \left( \mu_+ x_3 - \lambda_- y_3 - \tau s - iy' \cdot \xi' \right) \\
&\quad + \left\{ \sum_{l=1}^{2} F_{l,6}(x_3 - y_3)^l + B_7 \right\} \exp \left( \mu_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi' \right) \\
&\quad + \left\{ \sum_{l=1}^{2} F_{l,8}(x_3 - y_3)^l + \frac{F_{1,7} - F_{1,8}}{\mu_+ - \mu_-} \right\} \exp \left( \mu_+ x_3 - \mu_+ y_3 - \tau s - iy' \cdot \xi' \right).
\end{align*}
\]

These are the forms for \( L = 2 \) in \( 2.6 - 2.9 \).

References

[1] R. Arima. On general boundary value problem for parabolic equations. J. Math. Kyoto Univ., 4-1 (1964) 207-243.
[2] F. Cakoni, D. Colton, D. Gintides. The interior transmission eigenvalue problem. SIAM J. Math. Anal., 42 (2010), 2912-2921.

[3] F. Cakoni, D. Colton, H. Haddar. The interior transmission problem for regions with cavities. SIAM J. Math. Anal., 42 (2010), 145-162.

[4] F. Cakoni, D. Colton, H. Haddar. Inverse Scattering Theory and Transmission Eigenvalues. SIAM Publications, 2016.

[5] F. Cakoni, D. Gintides, H. Haddar. The existence of an infinite discrete set of transmission eigenvalues. SIAM J. Math. Anal., 42 (2010), 237-255.

[6] D. Colton, L. Päivärinta, J. Sylvester. The interior transmission problem. Inverse Problems and Imaging, 1 (2007), 13-28.

[7] A. Cossonnière, H. Haddar. The electromagnetic interior transmission problem for regions with cavities. SIAM J. Math. Anal., 43 (2011), 1698-1715.

[8] M. Faierman. The interior transmission problem: spectral theory. SIAM J. Math. Anal., 46 (2014), 803-819.

[9] A. García, E.V. Vesalainen, M. Zubeldia. Discreteness of transmission eigenvalues for higher-order main terms and perturbations. SIAM J. Math. Anal., 48 (2016), 2382-2398.

[10] I.M. Gel’fand, G.E. Shilov. Generalized Functions, Vol. 3: Theory of Differential Equations. Academic Press Inc., 1967.

[11] M. Hitrik, K. Krupchyk, P. Ola, L. Päivärinta. Transmission eigenvalues for operators with constant coefficients. SIAM J. Math. Anal., 42 (2010), 2965-2986.

[12] M. Hitrik, K. Krupchyk, P. Ola, L. Päivärinta. Transmission eigenvalues for elliptic operators. SIAM J. Math. Anal., 43 (2011), 2630-2639.

[13] V. Isakov, K. Kim, G. Nakamura. Reconstruction of an unknown inclusion by thermography. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 9 (2010), 725-758.

[14] X. Ji, H. Liu. On isotropic cloaking and interior transmission eigenvalue problems. arXiv:1604.05498.

[15] K. Kim, G. Nakamura, M. Sini. The Green function of the interior transmission problem and its applications. Inverse Problems and Imaging, 6 (2012), 487-521.

[16] A. Kirsch. The denseness of the far field patterns for the transmission problem. IMA J. Appl. Math., 37 (1986), 213-225.

[17] A. Kirsch. On the existence of transmission eigenvalues. Inverse Problems and Imaging, 3 (2009), 155-172.

[18] E. Lakshtanov, B. Vainberg. Ellipticity in the interior transmission problem in anisotropic media. SIAM J. Math. Anal., 44 (2012), 1165-1174.
[19] E. Lakshtanov, B. Vainberg. Sharp Weyl law for signed counting function of positive interior transmission eigenvalues. SIAM J. Math. Anal., 47 (2015), 3212-3234.

[20] A. Lechleiter, M. Rennoch. Inside-outside duality and the determination of electromagnetic interior transmission eigenvalues. SIAM J. Math. Anal., 47 (2015), 684-705.

[21] J. Li, X. Li, H. Liu, Y. Wang. Electromagnetic interior transmission eigenvalue problem for inhomogeneous media containing obstacles and its applications to near cloaking. arXiv:1701.05301.

[22] P. Monk, J. Sun. Finite element methods for Maxwell’s transmission eigenvalues. SIAM J. Sci. Comput., 34 (2012), B247-B264.

[23] G. Nakamura, H. Wang. Linear sampling method for the heat equation with inclusions. Inverse Problems, 29 (2013), 104015.

[24] G. Nakamura, H. Wang. Reconstruction of an unknown cavity with Robin boundary condition inside a heat conductor. Inverse Problems, 31 (2015), 125001.

[25] G. Nakamura, H. Wang. Numerical reconstruction of unknown Robin inclusions inside a heat conductor by a non-iterative method. Inverse Problems, 33 (2017), 055002.

[26] L. Päivärinta, J. Sylvester. Transmission eigenvalues. SIAM J. Math. Anal., 40 (2008), 738-753.

[27] B.P. Rynne, B.D. Sleeman. The interior transmission problem and inverse scattering from inhomogeneous media. SIAM J. Math. Anal., 22 (1991), 1755-1762.

[28] R. Seeley. The resolvent of an elliptic boundary problem. Amer. J. Math., 91 (1969), 889-920.

[29] J. Sun. Iterative methods for transmission eigenvalues. SIAM J. Numer. Anal., 49 (2011), 1860-1874.

[30] J. Sylvester. Discreteness of transmission eigenvalues via upper triangular compact operators. SIAM J. Math. Anal., 44 (2012), 341-354.

[31] J. Sylvester. Transmission eigenvalues in one dimension. Inverse Problems, 29 (2013), 104009.