ON THE NUMBER OF MATROIDS

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Abstract. We consider the problem of determining \( m_n \), the number of matroids on \( n \) elements. The best known lower bound on \( m_n \) is due to Knuth (1974) who showed that \( \log \log m_n \) is at least \( n - 4 \log n - O(1) \). On the other hand, Piff (1973) showed that \( \log \log m_n \leq n - \log n + \log \log n + O(1) \), and it has been conjectured since that the right answer is perhaps closer to Knuth’s bound.

We show that this is indeed the case, and prove an upper bound on \( \log \log m_n \) that is within an additive \( 1 + o(1) \) term of Knuth’s lower bound. Our proof is based on using some structural properties of non-bases in a matroid together with some properties of stable sets in the Johnson graph to give a compressed representation of matroids.

1. Introduction

Matroids, introduced by Whitney in his seminal paper [28], are fundamental combinatorial objects and have been extensively studied due to their very close connection to combinatorial optimization, see e.g. [26], and their ability to abstract core notions from areas such as graph theory and linear algebra [16, 22].

There are several ways to define a matroid. Perhaps the most natural one is using the notion of independence. A matroid \( M \) is a pair \((E, \mathcal{I})\), where \( E \) is the ground set of elements, and \( \mathcal{I} \) is a nonempty collection of subsets of \( E \) called the independent sets with the following properties:

(1) Subset property: \( A \in \mathcal{I} \) implies \( A' \in \mathcal{I} \) for all \( A' \subset A \), and

(2) Exchange property: If \( A, B \in \mathcal{I} \) with \( |A| > |B| \), then there exists an element \( x \) in \( A \setminus B \), such that \( B \cup \{x\} \in \mathcal{I} \).

A basic question is: how many distinct matroids are there on a ground set of \( n \) elements? We denote this number by \( m_n \). Clearly, there are \( 2^n \) subsets of \( E \) and hence at most \( 2^{2^n} \) ways to choose \( \mathcal{I} \), which gives the trivial upper bound \( \log \log m_n \leq n \). Here, and throughout the paper, \( \log \) denotes the logarithm to the base 2.

In 1973, Piff [24] improved this bound further to \( \log \log m_n \leq n - \log n + \log \log n + O(1) \), by focussing on matroids of a fixed rank. In a matroid, the maximal independent sets are called bases, and by the exchange property all bases of a matroid have the same cardinality. This common cardinality is the rank of the matroid. Let \( m_{n,r} \) be the number of matroids of rank \( r \). As \( m_n = m_{n,0} + m_{n,1} + \ldots + m_{n,n} \), it must hold that \( m_{n,r} \geq m_n/(n+1) \) for some \( r \). By the subset property, any matroid of rank \( r \) is completely determined by specifying its bases. As there are at most \( \binom{n}{\text{\ell}} \leq \left(\frac{n}{\lceil n/2 \rceil}\right) = O(2^n/\sqrt{n}) \) (call this \( \ell \)) such bases, this gives \( m_{n,r} \leq 2^\ell \) and thus

\[
\log \log m_n \leq \log \log((n+1)m_{n,r}) \leq \log \log((n+1)2^\ell) = n - \frac{1}{2} \log n + O(1).
\]

In 1973, Piff [24] improved this bound further to \( \log \log m_n \leq n - \log n + \log \log n + O(1) \), by observing that a matroid is also completely determined by the closures of its circuits, and using a counting argument to show that there “only” \( O(2^n/n) \) such closures (we describe Piff’s proof in Section 2.5). This is the best upper bound known to date.
In the other direction, the best known lower bound is due to Knuth \[15\] from 1974, who showed that \( \log \log m_n \geq n - \frac{3}{2} \log n - O(1) \). Knuth’s bound is based on an elegant construction of matroids whose non-bases\(^1\) satisfy a particular property. Specifically, he constructs a large family of so-called \textit{sparse paving matroids}. These are matroids of rank \( r \), where any two non-bases intersect in at most \( r - 2 \) elements (i.e. their incidence vectors have Hamming distance 4 or more). Such sets of non-bases are precisely the stable sets\(^2\) in the so-called \textit{Johnson graph} \( J(n, r) \). This is the graph with vertex set \( \{ X \subseteq \{ n \} \mid |X| = r \} \), in which two vertices are adjacent if and only if their intersection contains \( r - 1 \) elements.

Knuth’s bound follows by taking a collection of \( k = \frac{1}{n} \binom{n}{n/2} \) such non-bases — equivalently, a stable set in the graph \( J(n, n/2) \) of this size, of which an explicit description is provided in Section \( 2.4 \) — and considering the family of size \( 2^k \) of sparse paving matroids obtained by taking each possible subset of this family. Thus \( m_n \geq s_n \geq 2^k \), where \( s_n \) is the number of sparse paving matroids on \( n \) elements. This gives the lower bound

\[
\log \log m_n \geq \log \log s_n \geq \log k = n - \frac{3}{2} \log n + \frac{1}{2} \log \frac{2}{\pi} - o(1).
\]

We explain Knuth’s bound in more detail in Section \( 2.4 \).

Historically, the interest in paving matroids seems to be a response to the publication of the catalog of matroids on at most 8 elements by Blackburn, Crapo, and Higgs \[4\] in the early 1970’s. With reference to such numerical evidence, Crapo and Rota consider it probable that paving matroids “would actually predominate in any asymptotic enumeration of geometries” \[9, p. 3.17\]. In his book “Matroid Theory”, Welsh also notes that paving matroids predominate among the small matroids, and puts the questions whether this pattern extends to matroids in general as an exercise \[27, p. 41\]. An earlier lower bound on the number of matroids due to Piff and Welsh \[25\] was also based on a bound on the number of (sparse) paving matroids. Mayhew and Royle recently confirmed that the predominance of sparse paving matroids extends to the matroids on 9 elements \[19\].

In recent years, (sparse) paving matroids have received attention in relation to a wide variety of matroid topics \[13, 10, 21, 5\]. These authors all suggest that the class of sparse paving matroids is probably a very substantial subset of all matroids, pointing out Knuth’s argument for the lower bound.

Mayhew, Newman, Welsh and Whittle \[17\] present a very nice collection of conjectures on the asymptotic behavior of matroids. In particular, they conjecture that asymptotically almost every matroid is sparse paving:

\textbf{Conjecture 1} (Mayhew, Newman, Welsh and Whittle \[17\]). \( \lim_{n \to \infty} s_n/m_n = 1 \).

If true, this would imply:

\textbf{Conjecture 2}. \( \log \log m_n = \log \log s_n + o(1) \).

Note that this is in fact a much weaker statement as \( \log \log (\cdot) \) is a very “forgiving” function, e.g. if \( m_n = \Omega(n s_n) \) or even if \( m_n = \Omega(2^{2\sqrt{n}} s_n) \), then \( \frac{s_n}{m_n} \to 0 \), while still \( \log \log m_n = \log \log s_n + o(1) \).

1.1. \textbf{Our results}. Our main result is a substantial strengthening of the upper bound on \( m_n \).

\begin{itemize}
\item[1.] For a matroid of rank \( r \), a non-basis is an \( r \)-subset of the ground set that is dependent.
\item[2.] We use \textit{stable set} as a synonym for what is often called \textit{independent set} in graph theory (i.e. a set of vertices, no two of which are adjacent). As independent set has a different meaning in matroid theory, this serves to avoid confusion.
\end{itemize}
Theorem 3. The number of matroids $m_n$ on $n$ elements satisfies

$$\log \log m_n \leq n - \frac{3}{2} \log n + \frac{1}{2} \log \frac{2}{\pi} + 1 + o(1).$$

Combining Theorem 3 with Knuth’s lower bound (1) on the number of sparse paving matroids $s_n$, this gives:

Corollary 4. $\log \log m_n \leq \log \log s_n + 1 + o(1)$.

Thus, this result comes quite close to Conjecture 2, except for the additive $+1$ term. In particular, it implies that the number of matroids is indeed much closer to Knuth’s lower bound, and perhaps also lends support to the conjecture that most matroids are indeed sparse paving.

1.2. Our Techniques. The proof of Theorem 3 is based on a combination of the following:

1. A technique for proving refined upper bounds on the total number of stable sets in a graph.
2. Defining a notion of local cover of a matroid, which serves as a short certificate to identify the bases in the neighborhood of an $r$-set. Combining the local covers for a carefully chosen set of $r$-sets then serves as a compressed representation of any matroid.

To see the connection to the total number of stable sets, note that any upper bound on $m_n$ is also an upper bound on $s_n$. As $s_n = s_{n,0} + s_{n,1} + \ldots + s_{n,n}$, where $s_{n,r}$ denotes the number of sparse paving matroids of rank $r$, and $s_{n,r}$ is precisely the total number of stable sets in the Johnson graph $J(n, r)$, any method to upper bound $m_n$ must also bound the number of such sets.

We first give an overview of each of these two ideas, and then describe how they are combined to prove Theorem 3. These ideas are already useful by themselves to improve the currently known bounds on $s_n$ and $m_n$. In Section 3 we show how local covers can be used in a very simple way to obtain the following bound.

Theorem 5. $\log \log m_n \leq n - \frac{3}{2} \log n + 2 \log \log n + O(1)$.

While this bound is weaker than the one in Theorem 3 it already improves Piff’s upper bound substantially, and matches Knuth’s lower bound up to the additive $O(\log \log n)$ term.

Similarly, in Section 5 we show how the refined counting technique for stable sets implies the following bound on the number of sparse paving matroids.

Theorem 6. $\log \log s_n \leq n - \frac{3}{2} \log n + \frac{1}{2} \log \frac{2}{\pi} + 1 + o(1)$.

Previously, the best known upper bound on $s_n$ seems to be $\log \log s_n \leq n - \frac{3}{2} \log n + O(\log \log n)$ [20] (we sketch an argument below).

Finally, we prove Theorem 3 in Section 6.

1.2.1. Upper-bounding $m_n$ via local covers: Recall that $m_{n,r}$ denotes the number of matroids of rank $r$ on $n$ elements. As $m_n = m_{n,0} + m_{n,1} + \ldots + m_{n,n}$, it suffices to bound each $m_{n,r}$ separately. For a matroid of rank $r$, let us call a collection of flats a flat cover if it completely describes the matroid by certifying for each $r$-set whether it is a basis or not.

A related notion is that of a local cover: a collection of flats that allows us to identify the bases in the neighborhood of some fixed $r$-set. Our main observation is that given any matroid, for every $r$-set, one can associate to it a local cover consisting of at most $r$ flats. This implies that if we pick any dominating set $D$ in the Johnson graph and list all the local covers for the vertices in $D$, then this gives a valid flat cover consisting of at most $|D|r$ flats. Together with standard arguments about the existence of small dominating sets in any regular graph, this implies that each matroid $M \in M_{n,r}$ can be described by a “small” flat cover, which gives the bound in Theorem 5.
1.2.2. Upper-bounding \( s_n \) via stable sets: As \( s_n = s_{n,0} + s_{n,1} + \ldots + s_{n,n} \) it suffices to bound each of these terms separately. For a graph \( G \), let \( i(G) \) denote the number of stable sets in \( G \), and recall that \( s_{n,r} = i(J(n,r)) \). While it is hard to obtain any reasonable estimate of \( i(G) \) for general graphs, it was shown in [20] that

\[
\log \log s_n \leq - \frac{3}{2} \log n + \log \log n + O(1)
\]

One may argue this as follows. Let \( G = (V,E) \) be a \( d \)-regular graph and let \(-\lambda\) denote the smallest eigenvalue of its adjacency matrix. Then the size of a maximum stable set of \( G \) is at most \(|V|\lambda/(d+\lambda)\) by Hoffmann’s bound (see e.g. Theorem 3.5.2 of [8], or our Corollary [18]). Let us denote \( \alpha = \lambda/(d+\lambda) \). This implies that

\[
i(G) \leq \sum_{j=0}^{\lfloor \alpha |V| \rfloor} \binom{|V|}{j}.
\]

For the graph \( J(n,r) \) it is known that \( \alpha \) is at most \( 2/n \), which implies that the maximum stable set has size at most \( \alpha N \) where \( N = \binom{n}{r} \). Note that this bound is quite good and is within a factor 2 of the size of the explicit stable set used in Knuth’s lower bound. Applying (3) to \( J(n,r) \) then gives \( s_{n,r} \leq 2^{(1+o(1))2N \log n/n} \), which implies the bound (2). We note that the proof of (2) in [20] is similar, except that there the same bound on the maximal size of a stable set of \( J(n,r) \) was shown by a combinatorial argument.

It turns out however that counting all the subsets in (3) is rather wasteful and that this bound can be improved. In particular, we show:

**Theorem 7.** Let \( G \) be a \( d \)-regular graph on \( N \) vertices with smallest eigenvalue \(-\lambda\), with \( d > 0 \). Then \( i(G) \leq \sum_{i=0}^{\lfloor \alpha N \rfloor} \binom{N}{i} 2^\sigma N \), where \( \alpha = \frac{\lambda}{d+\lambda} \) and \( \sigma = \frac{\ln(d+1)}{d+\lambda} \).

For the graph \( J(n,r) \), we find that \( \sigma N \leq \frac{2 \ln n}{n^2} N \) and hence this gives the stronger bound \( i(G) \leq 2^{(2+o(1))N/n} \). As \( \alpha N = 2N/n \) was our bound on the size of the maximum stable set, this bound on \( i(G) \) roughly implies that most stable sets occur as subsets of a few large stable sets of size \( \alpha N \). Using standard bounds on the binomial coefficients, this directly implies Theorem [6].

Our proof of Theorem 7 is based on a procedure for encoding stable sets that is originally due to Kleitman and Winston [14]. We remain very close to the description of the procedure as given in Alon, Balogh, Morris and Samotij [2], to which we also refer for detailed references on the earlier uses of the procedure. Compared to [2], we give a somewhat improved analysis (specifically, Lemma [16]) to obtain a sufficient bound in the parameter range that is of interest to us.

1.2.3. The improved upper bound on \( m_n \): To obtain the bound in Theorem 8 we combine the two ideas above. The main observation is that given a matroid \( M \), if \( X \) is a dependent \( r \)-set (i.e. a non-basis) in \( M \), then \( X \) has a local cover consisting of at most 2 flats (as opposed to up to \( r \) flats if \( X \) was an arbitrary \( r \)-set). Thus if we could construct a flat cover using few such local covers, then we would obtain a much smaller description of a matroid. To this end, we generalize the procedure of Alon et al. [2] for encoding stable sets to more generally encode flat covers of the kind described above using a few number of bits. This gives the improved bound on \( m_{n,r} \) and hence on \( m_n \).

Finally, we remark that the +1 additive gap in our upper bound on \( m_n \) arises only because of the factor \( 2 + o(1) \) gap between the known upper and lower bounds on the size of the maximum stable set in the graphs \( J(n,r) \) for \( r \approx n/2 \). It is likely that reducing this gap could lead to improved bounds for \( m_n \). In Section 7 we elaborate on this issue a bit further.
2. Preliminaries

2.1. Matroids. As mentioned previously, a matroid $M$ is specified by $M = (E, \mathcal{I})$, where the
sets in the collection $\mathcal{I}$ satisfy the independence axioms. The elements of $\mathcal{I}$ are independent, the
remaining elements of $2^E \setminus \mathcal{I}$ are dependent. The set $E$ is the ground set, and we say that $M$ is a
matroid on $E$. There are various set systems and functions defined on $M$ that each allow one to
distinguish between dependent and independent sets, such as the set of bases, the rank function, the
circuits, the closure operator, etc. We define these notions and state some of their basic properties
here, but for a detailed account of their interrelations and for proofs we refer to Oxley [22].

A basis of $M$ is an inclusionwise maximal independent set of $M$. It follows from the independence
axioms, that each basis has the same cardinality. In this paper, we will present matroids as
here, but for a detailed account of their interrelations and for proofs we refer to Oxley [22].

We define these notions and state some of their basic properties
by

$$
(4) \quad \text{for each } B, B' \in B \text{ and each } e \in B \setminus B' \text{ there exists an } f \in B' \setminus B \text{ such that } B - e + f \in B.
$$

Here, we write $X + y := X \cup \{y\}$ and $X - y := X \setminus \{y\}$.

The rank of a set $X \subseteq E$ is $r_M(X) := \max\{|I| \mid I \subseteq X, I \in \mathcal{I}\}$, i.e. the cardinality of any
maximal independent set in $X$. The rank function is submodular:

$$
r_M(X \cap Y) + r_M(X \cup Y) \leq r_M(X) + r_M(Y).
$$

We write $r(M) := r_M(E)$. Then $r(M)$ is the common cardinality of all bases, the rank of $M$. We
say that an $r$-set $X$ is a non-basis if $r_M(X) < r$. Clearly, a matroid of rank $r$ with set of bases $B$
is also uniquely defined by its set of non-bases, $(E) \setminus B$.

A circuit of $M$ is an inclusionwise minimal dependent set of $M$. We denote the set of circuits of $M$ by $\mathcal{C}(M)$. By definition, each dependent set contains some circuit. We will use that if $X$ is an
$r$-set with $r_M(X) = r(M) - 1$, then it contains a unique circuit $C \subseteq X$.

In $M$, the closure of a set $X \subseteq E$ is the set $cl_M(X) := \{e \in E \mid r_M(X + e) = r_M(X)\}$. We will
often use that $r_M(cl_M(X)) = r_M(X)$ for any set $X$, which follows easily from induction and the
submodularity of the rank function. A set $F \subseteq E$ is called a flat of $M$ if $cl_M(F) = F$, and $F(M)$
denotes the set of all flats of $M$. As $cl_M(cl_M(X)) = cl_M(X)$ for any set $X$, every closure $cl_M(X)$
is a flat.

The following simple property of flats will be crucially used in our construction of flat covers: A
set $X \subseteq E$ is dependent if and only if there exists a flat $F$ such that $|X \cap F| > r_F(F)$. In other
words, $F$ acts as witness that $X$ contains a dependency when restricted to $F$.

The dual of $M$ is the matroid $M^*$ whose bases are $B^* = \{E \setminus B \mid B \in B\}$. The bases, circuits,
rank, and closure of sets in $M^*$ are called the cobases, cocircuits, corank, and coclosure of sets in
$M$, and we write $r^*_M(X) := r_{M^*}(X)$, $C^*(M) := C(M^*)$, $cl^* := cl_{M^*}$, etc.

The rank and corank functions of $M$ are related by

$$
r^*_M(X) = r_M(E \setminus X) - r(M) + |X|.
$$

We write

$$
M_n := \{M \text{ a matroid } \mid E(M) = \{1, \ldots, n\}\}, \quad M_{n,r} := \{M \in M_n \mid r(M) = r\}.
$$

Also, we put $m_n := |M_n|, \quad m_{n,r} := |M_{n,r}|$.

A matroid $M$ is paving if $|C| \geq r(M)$ for each circuit $C$ of $M$ (or equivalently if there is no
dependent set of size $< r(M)$), and sparse if $M^*$ is paving. $M$ is said to be sparse paving if it is both sparse and paving. We write

$$
s_n := |\{M \in M_n \mid M \text{ is sparse paving}\}|, \quad s_{n,r} := |\{M \in M_{n,r} \mid M \text{ is sparse paving}\}|.
$$
2.2. Bounds on binomial coefficients. We will frequently use the following standard bounds.

\[
\binom{n}{r} \leq \left(\frac{en}{r}\right)^r
\]

We will also use the following bound on the sum of binomial coefficients for \( k < n/2 \):

\[
\sum_{i=0}^{k} \binom{n}{i} \leq \frac{n-(k-1)}{n-(2k-1)} \binom{n}{k}.
\]

This follows by upper bounding \( \binom{n}{k-j} \leq \binom{n}{k} \left(\frac{k}{n-k+1}\right)^j \), and summing up the resulting geometric series. We are particularly interested in the case \( k = o(n) \), which yields

\[
\sum_{i=0}^{k} \binom{n}{i} \leq (1+o(1)) \binom{n}{k}.
\]

2.3. The Johnson graph. If \( E \) is a finite set and \( r \leq |E| \), then we write

\[
\binom{E}{r} := \{ X \subseteq E \mid |X| = r \}
\]

for the collection of \( r \)-subsets of \( E \). We say that \( X, Y \in \binom{E}{r} \) are adjacent (notation: \( X \sim Y \)) if they have Hamming distance \(|X \triangle Y| = 2\) (or equivalently: \(|X \cap Y| = r-1\)). The Johnson graph \( J(E,r) \) is defined as the graph with vertex set \( \binom{E}{r} \), in which two vertices \( X \) and \( Y \) are adjacent if and only if \( X \sim Y \). We abbreviate \( J(n,r) := J([n],r) \). For any \( r \)-set \( X \in \binom{E}{r} \), we write \( N(X) := \{ Y \in \binom{E}{r} \mid X \sim Y \} \) for the neighborhood of \( X \) in \( J(E,r) \). Obviously, \( J(E,r) \cong J(n,r) \) for any \( n \)-set \( E \).

The following lemma points out the connection between the Johnson graph and sparse paving matroids. It was essentially shown by Piff and Welsh [25] in proving an earlier lower bound on \( s_n \).

**Lemma 8.** For \( 0 < r < n \), sparse paving matroids \( M \in \mathcal{M}_{n,r} \) correspond one-to-one to stable sets in \( J(n,r) \).

**Proof.** Let \( E = [n] \). We first show that the non-bases of a rank-\( r \) sparse paving matroid \( M \) on \( E \) form a stable set in \( J(E,r) \). Suppose that there are non-bases \( X,Y \in \binom{E}{r} \setminus \mathcal{B}(M) \) such that \( X \sim Y \), then we would have

\[
r_M(X \cap Y) + r_M(X \cup Y) \leq r_M(X) + r_M(Y) < 2r-1,
\]

so that either \( r_M(X \cap Y) < r-1 = |X \cap Y| \) or \( r_M(X \cup Y) < r \). In the former case, \( X \cap Y \) is a dependent set of size \(< r(M) \), which contradicts that \( M \) is paving. In the latter case, it follows from \( [5] \) that

\[
r_M^*(E \setminus (X \cup Y)) = r_M(X \cup Y) - r(M) + |E \setminus (X \cup Y)| < r - r + |E \setminus (X \cup Y)| = n - r - 1 = r^*(M) - 1,
\]

so that \( E \setminus (X \cup Y) \) is a dependent set of \( M^* \) of size \(< r(M^*) \), which contradicts that \( M^* \) is paving.

Next, suppose that \( I \) is a stable set in \( J(E,r) \). We will show that \( \mathcal{B} := \binom{E}{r} \setminus I \) forms a valid collection of bases for some matroid on \( E \).
First, it cannot be that $B = \emptyset$ as this would imply that $I = \binom{E}{r}$ and hence that $J(E, r)$ has no edges. So the only way $B$ may fail to be a family of bases is if it fails the basis exchange axiom (4). That is, there are distinct $B, B' \in B$ and an $e \in B \setminus B'$ such that $B - e + f \not\in B$ for all $f \in B' \setminus B$. Now, it must be the case that $|B' \setminus B| > 1$, for otherwise it would hold that $B - e + f = B' \in B$ for the only $f \in B' \setminus B$. So, let $f, f'$ be distinct elements of $B' \setminus B$, and consider $N = B - e + f$ and $N' = B - e + f'$. Since the base exchange axiom fails, it follows that both $N, N' \in I$. On the other hand, $|N \triangle N'| = |\{f, f'\}| = 2$, i.e. $N \sim N'$, contradicting independence of $I$. □

2.4. Knuth’s lower bound. In [13], Knuth argues that if $J(n, r)$ has a stable set $I$ of size $k$, then $J(n, r)$ has at least $2^k$ stable sets, as each subset of $I$ is itself stable. Knuth constructed a stable set of size $k = \frac{1}{2n} \binom{n}{r}$, but Theorem 1 in [11] shows the existence of a stable set of size at least $k = \frac{1}{n} \binom{n}{r}$.

We sketch the construction in [11]. Identifying the vertices of $J(n, r)$ with their incidence vectors, we view them as $\{0, 1\}$ vectors $(x_1, \ldots, x_n)$ with exactly $r$ 1’s. It is easily verified that the functional $\{0, 1\}^n \rightarrow \mathbb{Z}/n\mathbb{Z}$, defined by

$$ (x_1, x_2, \ldots, x_n) \mapsto \sum_{i=1}^n ix_i \mod n $$

gives a valid $n$-vertex-coloring of $J(n, r)$. As there are $n$ color classes, at least one of them should contain at least $\frac{n}{n+1} \binom{n}{r}$ vertices.

Picking such a stable set gives $\log(s_{n, r}) \geq \frac{1}{n} \binom{n}{r}$, and in particular $\log(s_n) \geq \log(s_{n, \lfloor n/2 \rfloor}) \geq \frac{2n}{n+1} \left( \sqrt{\frac{2}{\pi}} - o(1) \right)$ by (7). Therefore,

$$ \log \log s_n \geq n - 3 \log n + \frac{1}{2} \log \frac{2}{\pi} - o(1). $$

2.5. Piff’s upper bound. To prove his upper bound on $m_n$, Piff [24] uses that any matroid $M$ is characterized by the set of all closures of circuits and their ranks, i.e. by the collection

$$ K(M) := \{ (\text{cl}_M(C), r_M(C)) \mid C \text{ a circuit of } M \}. $$

This completely defines $M$ as a set $X \subseteq E(M)$ is dependent in $M$ if and only if $|X \cap \text{cl}_M(C)| > r_M(C)$ for some circuit $C$ of $M$. He then uses the following counting argument to bound the size of $K(M)$.

Lemma 9. If $M \in \mathcal{M}_n$, then $|K(M)| \leq \frac{1}{n+1} 2^{n+1}$.

Proof. Fix an $i < n$. Let $C \in \mathcal{C}(M)$ be a circuit such that $|C| = i + 1$. Then for each $e \in C$, we have $\text{cl}_M(C) = \text{cl}_M(C - e)$, i.e. there are $i + 1$ sets $C - e \in \binom{E}{i}$ that map to $\text{cl}_M(C)$. It follows that

$$ |\{ (\text{cl}_M(C), r) \in K(M) \mid r = i \}| \leq \frac{1}{i+1} \binom{n}{i} = \frac{1}{n+1} \binom{n+1}{i}. $$

Summing these upper bounds over all $i$, we get

$$ |K(M)| = \sum_{i=0}^{n-1} |\{ (\text{cl}_M(C), r) \in K(M) \mid r = i \}| \leq \sum_{i=0}^{n-1} \frac{1}{n+1} \binom{n+1}{i} \leq \frac{1}{n+1} 2^{n+1}. \quad \square $$

It follows that the number of matroids on a set $E$ of $n$ elements is at most the number of subsets $\mathcal{K} \subseteq 2^E \times \{0, \ldots, n\}$ with $|\mathcal{K}| \leq \frac{1}{n+1} 2^{n+1}$. Thus, together by (9)

$$ m_n \leq \sum_{i=0}^{2^{n+1}/(n+1)} \left( \frac{2^n(n+1)}{i} \right) = (1 + o(1)) \left( \frac{2^n(n+1)}{1/n+1} 2^{n+1} \right). $$
By (1), this gives \( \log m_n \leq \frac{2^{n+1}}{n+1} \log \frac{e(n+1)^2}{2} + o(1) \) and hence \( \log \log m_n \leq n - \log n + \log \log n + O(1) \).

\[ \square \]

3. A WEAKER UPPER BOUND ON THE NUMBER OF MATROIDS

In this section, we introduce the notion of flat covers and local covers and use them to show that each matroid in \( M_{n,r} \) has a concise description. Using this, we then bound \( m_{n,r} \).

**Definition 10** (Flat cover). Let \( M = (E, B) \) be a matroid with \( n \) elements, of rank \( r \). For a set \( X \subseteq E \), we say that a flat \( F \in \mathcal{F}(M) \) covers \( X \) if \( |F \cap X| > r_M(F) \). We say that a set of flats \( Z \) is a flat cover of \( M \) if each non-basis \( X \in \binom{E}{r} \setminus B \) is covered by some \( F \in Z \).

Note that if \( Z \) covers \( M \), then \( M \) is characterized by \( E, r \) and the collection \( \{(F,r_M(F)) \mid F \in Z\} \), since by definition of a cover, we have \( B = \{X \in \binom{E}{r} \mid |X \cap F| \leq r_M(F) \text{ for all } F \in Z\} \).

**Definition 11** (Local cover). For an \( r \)-set \( X \in \binom{E}{r} \), we say that a collection of flats \( Z_X \subseteq \mathcal{F}(M) \) is a local cover at \( X \) if \( Z_X \) covers all the non-bases \( Y \in (N(X) \cup \{X\}) \setminus B \).

**Lemma 12.** Let \( M \in M_{n,r} \). For each \( r \)-set \( X \in \binom{E}{r} \), there is a local cover \( Z_X \) such that \( |Z_X| \leq r \).

**Proof.** Let \( X \) be a fixed \( r \)-set. Take \( Z_X := \{\text{cl}_M(X - x) \mid x \in X\} \). Then clearly \( |Z_X| \leq r \). We consider a \( Y \in N(X) \cup \{X\} \).

If \( Y = X \) and \( X \) is dependent, then \( X \subseteq \text{cl}_M(X - x_0) \) for some \( x_0 \in X \). Then \( \text{cl}_M(X - x_0) \) covers \( X \), as

\[
|X \cap \text{cl}_M(X - x_0)| = |X| = r > r_M(X) \geq r_M(X - x_0) = r_M(\text{cl}_M(X - x_0)).
\]

If \( Y \in N(X) \), then \( Y = X - x + y \) for some \( x \in X \) and \( y \in E \setminus X \). If \( \text{cl}_M(X - x) \) covers \( Y \), we are done. Otherwise,

\[
r - 1 = |X - x| \leq |\text{cl}_M(X - x) \cap Y| \leq r_M(\text{cl}_M(X - x)) \leq r - 1
\]
so that equality holds throughout, and in particular \( r_M(X - x) = r - 1 \) and \( y \notin \text{cl}_M(X - x) \). It follows that \( r_M(Y) = r_M(X - x + y) = r \), so that \( Y \) is a basis and it is not required to cover \( Y \).

If \( G = (V,E) \) is a graph, then a set \( D \subseteq V \) is dominating if \( D \cup N(D) = V \). The point of introducing local covers is that one can construct a small flat cover from a collection of local covers at the vertices in some small dominating set, as every non-basis in the matroid will be covered by this collection. By standard probabilistic arguments (see Theorem 1.2.2 of [3]), one has:

**Lemma 13.** \( J(n,r) \) has a dominating set of cardinality \( \frac{\ln(r(n-r)+1)+1}{r(n-r)+1} \binom{n}{r} \).

**Corollary 14.** Let \( M \in M_{n,r} \). Then \( M \) has a flat cover \( Z \) with \( |Z| \leq r \frac{\ln(r(n-r)+1)+1}{r(n-r)+1} \binom{n}{r} \).

**Proof.** By Lemma 13, \( J(n,r) \) has a dominating set \( D \) with \( |D| \leq \frac{\ln(r(n-r)+1)+1}{r(n-r)+1} \binom{n}{r} \). For each \( X \in D \), let \( Z_X \) be a local cover of \( M \) at \( X \) as in Lemma 12. Then \( |Z_X| \leq r \) for each \( X \in D \). Take \( Z := \bigcup_{X \in D} Z_X \). Then \( Z \) is a cover of \( M \), and \( |Z| \leq r |D| \).

**Theorem 5.** \( \log \log m_n \leq n - \frac{3}{2} \log n + 2 \log \log n + O(1) \).
we refer to the following procedure to order the set $A$ of vertices of $G$. We proceed by describing a better technique for bounding the number of sparse paving matroids. (which is clearly the best possible) could improve this gap to $\log \log n$. We note that this approach cannot give a gap of $O(1)$. In particular, even an upper bound on the cardinality of a dominating set of the Johnson graph $J(n, r)$ that is closer to $\binom{n}{r}/(2r(n-r)+1)$ (which is clearly the best possible) could improve this gap to $\log \log n + O(1)$ at best. We cannot expect to do better: indeed, if we consider the above proof applied to bound the number of sparse paving matroids, it is inherently as wasteful as using the trivial counting bound (3) (as a minimal cover of a sparse paving matroid just lists the non-bases). We proceed by describing a better technique for bounding the number of sparse paving matroids.

4. A PROCEDURE FOR ENCODING VERTEX SETS

4.1. The procedure. We now describe a procedure given by Alon, Balogh, Morris and Samotij [2], for which they refer to Kleitman and Winston [14] as the original source. They use the procedure to encode a stable set $I$ as a pair $(S, I \setminus S)$, such that $S \subseteq I$ and the number of possibilities for both $S$ and $I \setminus S$ can be controlled. We will use it for that purpose in this section as well, but to prepare for other uses in this paper we generalize the procedure so that it takes a general vertex set $K$ and produces a pair $(S, A)$, satisfying

\[(S) \subseteq K \subseteq S \cup N(S) \cup A.\]

We stress that the encoding is not one-to-one, and several sets $K$ may produce the same pair $(S, A)$. We will later describe why such a pair $(S, A)$ is useful.

Throughout this section, $G$ is a $d$-regular graph on $N$ vertices, with $d > 0$, and the smallest eigenvalue of its adjacency matrix is $-\lambda$. We denote $\alpha := \frac{\lambda}{d+\lambda}$. For a subset $A \subseteq V$, let $G[A]$ denote the subgraph of $G$ induced by $A$. Let $e(A)$ denote the number of edges with both end points in $A$, i.e. the number of edges in $G[A]$. Let us assume there is some fixed linear ordering $\leq_V$ of the vertices of $G$ (say according to their indices $1, \ldots, N$). By the canonical ordering of $A \subseteq V$, we refer to the following procedure to order the set $A$ linearly. Let $v$ be the vertex with maximum degree in $G[A]$; if there are multiple such $v$, take the one that is smallest with respect to $\leq_V$. Call $v$ the first vertex in the canonical ordering, and apply the procedure iteratively to $A \setminus \{v\}$.

The procedure to produce $(S, A)$ (see Figure 1) maintains two disjoint sets of vertices: $S$ for selected and $A$ for available. Initially, no vertices are selected ($S = \emptyset$) and all vertices are available.

**Proof.** Denote the upper bound in Corollary [14] by $k_{n,r} := \frac{\ln(r(n-r)+1)+1}{r(n-r)+1} \binom{n}{r}$. As each matroid in $\mathcal{M}_{n,r}$ is uniquely determined by the set $\{(F, r_M(F)) \mid F \in \mathcal{Z}\} \subseteq 2^E \times \{0, \ldots, n\}$, where $\mathcal{Z}$ is a cover of size bounded by $k_{n,r}$, the number of matroids in $\mathcal{M}_{n,r}$ is bounded by the number of subsets of a set of size $2^n(n+1)$ of cardinality at most $k_{n,r}$.

Now, for sufficiently large $n$,

$$\max_{r \leq n/2} k_{n,r} \leq \max_{n \leq r \leq n/2} \frac{\ln(n^2) + 1}{n-r} \binom{n}{r} \leq \frac{2 + 4 \ln n}{n} \binom{n}{n/2};$$

By (7) this is at most $O(2^n(\ln n)/(n^{3/2})$ and hence by (9) and (6),

$$\max_{n \leq r \leq n/2} m_{n,r} \leq (1 + o(1)) \frac{2^n(n+1)}{O(2^n(\ln n)/n^{3/2})} \leq (O(n^{5/2}))O(2^n(\ln n)/n^{3/2})$$

The same bound applies to $\max_{r \geq n/2} m_{n,r}$, since $m_{n,r} = m_{n-r}$ by matroid duality. Thus,

$$\max_{r \leq n} \log m_{n,r} \leq O\left(\frac{2^n \ln n}{n^{3/2}}\right) \cdot O(\log n),$$

and hence $\log \log m_n \leq n - \frac{3}{2} \log n + 2 \log \log n + O(1)$ as required. \[\square\]

Remark: The difference between this upper bound and the lower bound of Knuth is $2\log \log n + O(1)$. We note that this approach cannot give a gap of $O(1)$. In particular, even an upper bound on the cardinality of a dominating set of the Johnson graph $J(n, r)$ that is closer to $\binom{n}{r}/(2r(n-r)+1)$ (which is clearly the best possible) could improve this gap to $\log \log n + O(1)$ at best. We cannot expect to do better: indeed, if we consider the above proof applied to bound the number of sparse paving matroids, it is inherently as wasteful as using the trivial counting bound (3) (as a minimal cover of a sparse paving matroid just lists the non-bases). We proceed by describing a better technique for bounding the number of sparse paving matroids.
\( A = V \). During the procedure, the set \( S \) will expand and the set \( A \) will shrink, until \(|A| \leq \alpha N\). Throughout we will maintain (12) as an invariant.

**Input**: graph \( G = (V, E) \), vertex set \( K \subseteq V \)

**Output**: \((S, A)\)

Set \( A \leftarrow V \) and \( S \leftarrow \emptyset \);

while \(|A| > \alpha N\) do

Pick the first vertex \( v \) in the canonical ordering of \( A \);

if \( v \in K \) then

set \( S \leftarrow S \cup \{v\} \) and set \( A \leftarrow A \setminus (N(v) \cup \{v\}) \);

else

set \( A \leftarrow A \setminus \{v\} \);

end

end

Output \((S, A)\).

**Figure 1**: The encoding procedure.

The following is a simple but subtle and crucial observation from [2].

**Lemma 15.** Upon the termination of the algorithm, the set \( A \) is completely determined by \( S \) (irrespective of the set \( K \)).

This follows as at any step in the algorithm, the vertices chosen thus far in \( S \) completely determine the remaining vertices and their ordering. In particular, given \( S \) one can recover \( A \) as follows: Initialize \( X = V \) and \( T = S \). Repeating the following steps until \(|X| \leq \alpha N\) (the resulting set \( X \) when the algorithm terminates will be \( A \)). (i) Consider the canonical ordering of \( X \), and let \( v \) be the first vertex in this ordering. (ii) If \( v \in T \), discard \( v \) from \( T \) and \( \{v\} \cup N(v) \) from \( X \) and go to step 1. Otherwise, discard \( v \) from \( X \) and go to step 1.

**4.2. Application to counting stable sets.** Later we will show:

**Lemma 16.** The number of vertices selected into \( S \) is at most \( \left\lceil \frac{\ln(d+1)}{d+\lambda} \right\rceil N \).

Let us first see how this implies the following upper bound on \( i(G) \), the number of stable sets in \( G \).

**Theorem 7.** Let \( G \) be a \( d \)-regular graph on \( N \) vertices with smallest eigenvalue \(-\lambda \), with \( d > 0 \). Then \( i(G) \leq \sum_{i=0}^{\left\lceil \sigma N \right\rceil} \binom{N}{i} 2^{\alpha N} \), where \( \alpha = \frac{\lambda}{d+\lambda} \) and \( \sigma = \frac{\ln(d+1)}{d+\lambda} \).

**Proof.** Let \( K \) be any stable set. Running the procedure yields a pair \( S, A \) with \(|A| \leq \alpha N\), such that (i) \( A \) is completely determined by \( S \) (by Lemma 15) and (ii) \( S \subseteq K \subseteq S \cup N(S) \cup A \). Now, since \( K \) is a stable set and \( S \subseteq K \), we have \( N(S) \cap K = \emptyset \). Together with (ii) above, this implies that \( K \subseteq S \cup A \). Thus, \( K = S \cup (K \cap A) \) and hence \( K \) is completely determined by \( S \) and \( K \cap A \).

As \( A \) is completely determined by \( S \), for a fixed \( S \), there are at most \( 2^{\alpha N} \) possibilities for \( K \cap A \). Moreover, as \(|S| \leq \lceil \sigma N \rceil \), the number of ways of choosing \( S \) is at most \( \sum_{i=0}^{\left\lceil \sigma N \right\rceil} \binom{N}{i} \).

**4.3. Analysis.** We now prove Lemma 16. We first need the following lemma that was proved by Alon and Chung in [1], and earlier by Haemers (Theorem 2.1.4 (i) of [2]). We use the version of the lemma stated in [2].

**Lemma 17.** For all \( A \subseteq V(G) \), we have \( 2e(A) \geq |A| \left( \frac{d}{N} |A| - \lambda \frac{N-|A|}{N} \right) \).
Proof. Let \( x \) denote the incidence vector of set \( A \), and let \( B \) denote the adjacency matrix of \( G \). Then the number of edges is given by \((1/2)x^TBx\). Let \( v \) be the all 1’s vector scaled by \(|A|/N\), then \( v \cdot (x-v) = 0 \), and \( v \) is an eigenvector of \( B \) with eigenvalue \( d \). As \( v \) is an eigenvector of \( B \), \( v \perp (x-v) \) implies that \( Bv \perp (x-v) \). Thus
\[
2e(A) = x^TBx = (x-v+v)^TB(x-v+v) = (x-v)^TB(x-v) + v^TBv \\
\geq -\lambda \|x-v\|^2 + d \|v\|^2 \\
= -\lambda \left( \frac{2(N-|A|)|A|^2}{N^2} + \frac{|A|^2}{N^2} \right) + d \frac{|A|^2}{N} \\
= \frac{|A|}{N} \left( d \frac{|A|}{N} - \lambda \frac{2(N-|A|)}{N} \right). \quad \square
\]

**Corollary 18.** For any \( \varepsilon > 0 \), if \( |A| = (\alpha + \varepsilon)N \), then \( G[A] \) contains a vertex of degree at least \( \varepsilon(d + \lambda) \).

**Proof.** Let \( A \) be a vertex set of size \((\alpha + \varepsilon)N\). By Lemma 17, \( 2e(A) \geq |A|(d + \lambda)\varepsilon \). Hence the average degree in \( G[A] \) is at least \( \varepsilon(d + \lambda) \), and so there must be some vertex in \( G[A] \) of degree \( \geq \varepsilon(d + \lambda) \).

In particular, it follows that a stable set \( A \subseteq V(G) \) has size at most \( \alpha N \) (this is Hoffmann’s bound).

**Proof of Lemma 16** Say that the procedure is in phase \( j \) if the current \( A \) satisfies
\[
\frac{|A|}{N} \in \left( \alpha + \frac{j-1}{d + \lambda}, \alpha + \frac{j}{d + \lambda} \right], \quad (j = d, d-1, \ldots, 1).
\]
Then each phase sees the removal of at most \( N/(d+\lambda) \) vertices from \( A \). By Corollary 18, any vertex that gets selected into \( S \) during phase \( j \) has degree \( > j - 1 \), hence removes at least \( j + 1 \) vertices from \( A \) (the vertex selected into \( S \), and its neighbors). Let \( S(j) \) be the set of vertices that get selected into \( S \) during phase \( j \), then the above argument shows that \( |S(j)| \leq \left\lfloor \frac{N}{(d+\lambda)(j+1)} \right\rfloor \).

Summing up we obtain
\[
|S| = \sum_{j=1}^{d} |S(j)| \leq \sum_{j=1}^{d} \left\lfloor \frac{N}{(d+\lambda)(j+1)} \right\rfloor \leq \frac{\ln(d+1)}{d + \lambda}N + d,
\]
as \( \sum_{j=1}^{d} \frac{1}{j+1} \leq \ln(d+1) \).

To obtain the slightly sharper bound stated in the lemma, we make a more refined analysis. Let \( u_j \) be the fractional number of vertices that get removed in phase \( j \) due to the insertion of a vertex in \( S(j+1) \) for \( j = d, \ldots, 0 \). That is, if \( v \) is the last vertex which is inserted in \( S(j+1) \) during phase \( j+1 \), then \( u_j := \max\{0, (\alpha + \frac{j}{d+\lambda})N - |A|\} \) just after removing \( N(v) \cup \{v\} \) from \( A \) in the procedure. Then \( |S(j)| \leq \frac{N}{(d+\lambda)(j+1)} + \frac{u_{j-1} - u_j}{j+1} \), since inserting vertices in \( S(j) \) removes vertices from \( A \) only while
\[
|A| \in \left( \alpha N + \frac{j-1}{d + \lambda} N - u_{j-1}, \alpha N + \frac{j}{d + \lambda} N - u_j \right].
\]
It follows that
\[
|S| = \sum_{j=1}^{d} |S(j)| \leq \frac{N}{d + \lambda} \sum_{j=1}^{d} \frac{1}{j+1} + \sum_{j=1}^{d} \frac{u_{j-1} - u_j}{j+1} \leq \frac{\ln(d+1)}{d + \lambda}N + 1,
\]
as \( 0 \leq u_j < j + 2, u_d = 0 \), and hence \( \sum_{j=1}^{d} \frac{u_{j-1} - u_j}{j+1} \leq \frac{\ln d}{2} < 1 \). This proves the lemma. \quad \square
5. An Upper Bound on the Number of Sparse Paving Matroids

As shown in Lemma 8, sparse paving matroids of rank \(r\) on groundset \([n]\) correspond one-to-one to stable sets in the Johnson graph \(J(n, r)\). Thus \(s_{n, r} = i(J(n, r))\), and we may apply Theorem 7 to bound the number of sparse paving matroids. We first investigate the parameters that occur in this application of the theorem.

Recall that \(J(n, r)\) is regular of degree \(r(n - r)\) and has \(\binom{n}{r}\) vertices. The eigenvalues of the adjacency matrix of \(J(n, r)\) are \(r(n - r) - i(n + 1 - i)\) for \(i = 0, 1, \ldots, r\) if \(r \leq n/2\) (see [7]). The minimal eigenvalue is attained for \(i = r\), which identifies the smallest eigenvalue as \(-\lambda_{n, r}\), where \(\lambda_{n, r} = r\). For \(r, n\) such that \(0 < r \leq n/2\), we define

\[
\alpha_{n, r} := \frac{\lambda_{n, r}}{r(n - r) + \lambda_{n, r}} = \frac{1}{n - r + 1} \quad \text{and} \quad \sigma_{n, r} := \frac{\ln(r(n - r) + 1)}{r(n - r) + \lambda_{n, r}} = \frac{\ln(r(n - r) + 1)}{r(n - r + 1)}
\]

Lemma 19. If \(n \geq 4\) and \(0 < r \leq n/2\), then \(\alpha_{n, r} \lesssim \frac{2}{n} \binom{n}{\lfloor n/2 \rfloor}\) and \([\sigma_{n, r}] \lesssim \frac{8 \ln(n^2)}{n^2} \binom{n}{\lfloor n/2 \rfloor}\).

Proof. Clearly \(\alpha_{n, r} = 1/(n - r + 1) \leq 2/n\) whenever \(0 < r \leq n/2\). If \(n \geq 4\), we have

\[
\sigma_{n, r} \left\lfloor \frac{n}{r} \right\rfloor \leq f(n, r) := \frac{2 \ln(n^2)}{(r + 1)(n - r + 1)} \binom{n}{r}.
\]

Since \(f(n, r + 1)/f(n, r) = (n - r + 1)/(r + 2) \geq 1\) while \(r \leq n/2 - 1\), it follows that

\[
\left\lceil \sigma_{n, r} \left\lfloor \frac{n}{r} \right\rfloor \right\rceil \leq \lceil f(n, \lfloor n/2 \rfloor) \rceil \leq \frac{1}{1 + 2/n} \frac{8 \ln(n^2)}{n^2} \left\lfloor \frac{n}{\lfloor n/2 \rfloor} \right\rfloor \leq \frac{8 \ln(n^2)}{n^2} \left\lfloor \frac{n}{\lfloor n/2 \rfloor} \right\rfloor
\]

whenever \(0 < r \leq n/2\) and \(n \geq 4\), where we use that \(1 + 2/n \leq 1 - 2/n\). \(\Box\)

This gives sufficient control over the parameters to prove Theorem 6 from Theorem 7.

Theorem 6. \(\log \log s_n \leq -\frac{1}{2} \log n + \frac{1}{2} \log \frac{2}{\pi} + 1 + o(1)\).

Proof. Let \(n > 2\) and \(0 < r \leq n/2\). Then \(\lceil \sigma_{n, r} N \rceil < N/2\), and using Theorem 7 we have

\[
s_{n, r} = i(J(n, r)) \leq \left\lceil \sigma_{n, r} N \right\rceil \left(\frac{N}{\lceil \sigma_{n, r} N \rceil}\right)^{2^{\alpha_{n, r} N}},
\]

where \(N = \binom{n}{r}\). Using (6) to bound the factor \(\left(\frac{N}{\lceil \sigma_{n, r} N \rceil}\right)^{2^{\alpha_{n, r} N}}\) and applying logarithms, we obtain

\[
\log s_{n, r} \leq \log N + \lceil \sigma_{n, r} N \rceil \log \left(\frac{e N}{\lceil \sigma_{n, r} N \rceil}\right) + \alpha_{n, r} N.
\]

Applying Lemma 19 and using that \(N \leq \left(\frac{n}{\lfloor n/2 \rfloor}\right) \leq 2^n\), we obtain

\[
\log s_{n, r} \leq \log n + \frac{8 \ln(n^2)}{n^2} \left(\frac{n}{\lfloor n/2 \rfloor}\right) \log(n^2) + \frac{2}{n} \left(\frac{n}{\lfloor n/2 \rfloor}\right)
\]

for sufficiently large \(n\) and \(0 < r \leq n/2\). As \(\log(n^2)/n^2 \leq o(1/n)\), the bound on \(\alpha_{n, r} N\) dominates as \(n \to \infty\). Using that \(s_{n, 0} = 1\) and \(s_{n, r} = s_{n, n - r}\), we have

\[
\max_{0 \leq r \leq n} \log s_{n, r} \leq \frac{2}{n} \left(\frac{n}{\lfloor n/2 \rfloor}\right) (1 + o(1)) \text{ as } n \to \infty.
\]

As \(s_n = \sum_{r=0}^n s_{n, r} \leq (n + 1) \max_r s_{n, r}\), we also have \(\log s_n \leq \frac{2}{n} \left(\frac{n}{\lfloor n/2 \rfloor}\right) (1 + o(1))\). Taking logarithms and applying (7) to bound the binomial coefficient, the result follows. \(\Box\)
6. An upper bound on the number of matroids

We will now show the upper bound on \( m_n \) claimed in Theorem 3. To do this, we first show that substantially better local covers at \( X \) exist if \( X \) is a non-basis. Later, we combine this fact with the encoding procedure in Section 4 to find a very concise encoding of a matroid \( M \in \mathcal{M}_{n,r} \).

6.1. Improved Local Covers.

**Lemma 20.** Let \( M \in \mathcal{M}_{n,r} \). For each \( r \)-set \( X \in \binom{E}{r} \) that is dependent in \( M \), there exists a set \( Z_X \subseteq F(M) \) such that \(|Z_X| \leq 2\), and each non-basis \( Y \in N(X) \cup \{X\} \) is covered by some \( F \in Z_X \).

**Proof.** Let \( X \) be a fixed \( r \)-set. If \( r_M(X) < r - 1 \), take \( Z_X := \{cl_M(X)\} \). Then if \( Y \in N(X) \) or \( Y = X \), we have

\[
|cl_M(X) \cap Y| \geq |X \cap Y| \geq r - 1 > r_M(X) = r_M(cl_M(X)).
\]

If \( r_M(X) = r - 1 \), then \( X \) contains a unique circuit \( C \) of \( M \). Take \( Z_X := \{cl_M(C), cl_M(X)\} \). If \( Y \in N(X) \) is not a basis, then by submodularity

\[
r_M(X \cup Y) + r_M(X \cap Y) \leq r_M(X) + r_M(Y) < 2r - 1
\]

so that \( r_M(X \cup Y) < r \) or \( r_M(X \cap Y) < r - 1 \). In the former case, we have \( Y \subseteq cl_M(X) \), hence \( |cl_M(X) \cap Y| = r > r_M(X \cup Y) \geq r_M(X) = r_M(cl_M(X)) \). In the latter case, \( X \cap Y \) is dependent and hence must contain a circuit \( C' \), and as \( C \) is the unique circuit contained in \( X \), we must have \( C' = C \). Then \( |cl_M(C) \cap Y| \geq |C| > r_M(C) = r_M(cl_M(C)) \). \( \Box \)

6.2. Matroid Encoding. The crucial difference from Lemma 12 is the assumption that \( X \) is a dependent set in \( M \). This allows us to obtain a much smaller bound on the size of a cover of \( M \), if we can identify a small collection of non-bases of \( M \) such that their neighborhood contains all non-bases in a large fraction of the \( r \)-sets. This is exactly what the encoding algorithm will accomplish. We now give the details.

**Theorem 3.** The number of matroids \( m_n \) on \( n \) elements satisfies

\[
\log \log m_n \leq n - \frac{3}{2} \log n + \frac{1}{2} \log \frac{2}{\pi} + 1 + o(1).
\]

**Proof.** Let \( n, r \) be such that \( 0 < r < n/2 \). Consider a matroid \( M \in \mathcal{M}_{n,r} \), and let \( K := \binom{E}{r} \setminus \mathcal{B}(M) \) be the set of its non-bases. Then \( K \) is a set of vertices of the graph \( G = J(n, r) \). As before, let \( N = \binom{n}{r} \) denote the number of vertices of \( G \), let \( d = r(n-r) \) be its degree and \( -\lambda = -r \) be the smallest eigenvalue of \( G \). Let \( \alpha = \frac{\lambda}{d+\lambda} = 1/(n-r+1) \) and \( \sigma = \frac{\ln(d+1)}{d+\lambda} \). We describe how to obtain a concise description of \( M \).

Apply the encoding procedure to \( K \) and obtain sets \( S, A \) such that \( |A| \leq \alpha N \), \( |S| \leq [\sigma N] \), \( A \) is determined by \( S \), and

\[
S \subseteq K \subseteq S \cup N(S) \cup A.
\]

By Lemma 20 there exists a local cover \( Z_X \) with \(|Z_X| \leq 2\) for each \( X \in S \), where we use that each such \( X \) is a dependent set of \( M \). Then

\[
Z := \bigcup_{X \in S} Z_X
\]

covers all \( Y \in (S \cup N(S)) \setminus \mathcal{B} \), and \(|Z| \leq 2|S| \). As all members of \( K \setminus A \) lie in \( S \cup N(S) \), the set \( K \setminus A \) is fully determined by \( \{(F, r_M(F)) \mid F \in Z\} \). For the remaining non-bases in \( K \cap A \), we can simply list them. Thus, \( \{(F, r_M(F)) \mid F \in Z\}, K \cap A \) gives a complete and concise description of the non-bases of the matroid \( M \).
This bounds the number of matroids in $\mathbb{M}_{n,r}$ by the number of ways of choosing $S$ from an $N$-set, times the number of ways of choosing the collection $\{(F, r_M(F)) \mid F \in \mathcal{Z}\}$ from a set of size $2^n(n+1)$, times the number of possible subsets from $A$. As $|A| \leq \alpha N$, $|S| \leq \lceil \sigma N \rceil$ and $|\mathcal{Z}| \leq 2|S|$, this yields

$$m_{n,r} \leq \lceil \sigma N \rceil \left( \frac{N}{\lceil \sigma N \rceil} \right)^{2\lceil \sigma N \rceil} \left( \frac{2^n(n+1)}{2\lceil \sigma N \rceil} \right)^2$$

for $n$ sufficiently large so that $2\lceil \sigma N \rceil < N/2$ for all $r$. Using (6) to bound the factors $\left( \frac{N}{\lceil \sigma N \rceil} \right)$ and $\left( \frac{2^n(n+1)}{2\lceil \sigma N \rceil} \right)$, and applying logarithms, we obtain

$$\log m_{n,r} \leq \log([\sigma N]) + [\sigma N] \log \frac{eN}{\lceil \sigma N \rceil} + \log(2[\sigma N]) + 2[\sigma N] \log \frac{e2^n(n+1)}{2\lceil \sigma N \rceil} + \alpha N$$

By Lemma 19 we have $[\sigma N] \leq \frac{8\ln n}{n^2} \binom{n}{\lceil n/2 \rceil}$ and $\alpha N \leq \frac{2}{n} \binom{n}{\lceil n/2 \rceil}$ for sufficiently large $n$, and we bound $N \leq \binom{n}{\lceil n/2 \rceil} \leq 2^n$ to obtain

$$\log m_{n,r} \leq n + \frac{8\ln(n^2)}{n^2} \binom{n}{\lceil n/2 \rceil} \log(n^2) + (n+1) + 2 \frac{8\ln(n^2)}{n^2} \binom{n}{\lceil n/2 \rceil} \log(n^4) + \frac{2}{n} \binom{n}{\lceil n/2 \rceil}$$

for $0 < r < n/2$ and sufficiently large $n$. As $\log(n^2)/n^2 = o(1/n)$, the bound on $\alpha N$ dominates as $n \to \infty$. Using that $m_{n,0} = 1$ and $m_{n,r} = m_{n,n-r}$ we conclude that

$$\max_{0 \leq r \leq n} \log m_{n,r} \leq 2 \frac{n}{\lceil n/2 \rceil} (1 + o(1)) \text{ as } n \to \infty.$$

Since $m_n = \sum_{r=0}^n m_{n,r} \leq (n+1) \max_r m_{n,r}$, we also have

$$\log m_n \leq 2 \frac{n}{\lceil n/2 \rceil} (1 + o(1)).$$

(13)

Taking logarithms and applying (7) to bound the binomial coefficient, the theorem follows. \qed

Combining the above theorem with Knuth’s lower bound on $s_n$ (10), we obtain:

**Corollary 4.** $\log \log m_n \leq \log \log s_n + 1 + o(1)$.

### 7. Further Directions

#### 7.1. The maximal stable sets of the Johnson graph.

By Knuth’s lower bound and our result,

$$\frac{1}{n} \binom{n}{\lceil n/2 \rceil} \leq \log s_n \leq \log m_n \leq 2 \frac{n}{\lceil n/2 \rceil} (1 + o(1)).$$

So asymptotically, there is a factor 2 gap between the lower and upper bounds on $\log m_n$ (which gives the +1 gap in Corollary 4). Any improvement of this gap, would immediately reduce the gap between the known lower and upper bounds on the size of the maximum stable set in $J(n,n/2)$ (this question has been studied in coding, see e.g. [6] and references therein).

Conversely, it is possible that a better understanding of the maximum size of a stable set in $J(n,r)$ could lead to better bounds for $m_n$. If it could be shown that $J(n, \lfloor n/2 \rfloor)$ actually has a stable set of size $\approx 2 \binom{n}{\lfloor n/2 \rfloor}/n$, then the gap of +1 in Corollary 4 would reduce to $o(1)$. On the other hand, if the maximum size of a stable set is at most $\approx \binom{n}{\lfloor n/2 \rfloor}/n$, then the technique for showing such an upper bound could potentially be useful for bounding $m_n$.  

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7.2. The number of matroids without circuit-hyperplanes. If \( M = (E, B) \) is a matroid of rank \( r \), then the following are equivalent for a subset \( X \subseteq E \):

- \( X \) is both a circuit and a hyperplane of \( M \); and
- \( X \) is an isolated vertex of \( J(E, r) \setminus B \).

It is well-known that if \( X \) is a circuit-hyperplane of \( M \), then relaxing \( X \) gives another matroid \( (E, B \cup \{X\}) \). More strongly, we have:

**Lemma 21.** Let \( 0 < r < |E| \), let \( B \subseteq \binom{E}{r} \), and let \( U \) be a set of isolated vertices of \( J(E, r) \setminus B \). Then

\[
(E, B) \text{ is a matroid} \iff (E, B \cup U) \text{ is a matroid.}
\]

By the lemma, each matroid \( M = (E, B) \) can be decomposed uniquely as \( (\tilde{M}, U) \), where \( \tilde{M} := (E, B \cup U) \) and \( U \) is the set of all isolated vertices of \( J(E, r) \setminus B \). Then \( \tilde{M} \) has no circuit-hyperplanes, and \( U \) is a stable set of the Johnson graph. Hence,

\[
m_{n,r}/i(J(n, r)) \leq \{|M \in \mathcal{M}_{n,r} | M \text{ has no circuit-hyperplanes }\}.
\]

It follows that any upper bound on the number of circuit-hyperplane-free matroids will yield an upper bound on all matroids, using Theorem 6 to bound \( i(J(n, r)) = s_{n,r} \). We think that it may be possible to directly prove Theorem 3 along these lines, and conjecture that the number of circuit-hyperplane-free matroids is relatively small.

**Conjecture 22.**

\[
\lim_{n \to \infty} \frac{|\{M \in \mathcal{M}_n | M \text{ has no circuit-hyperplanes}\}|}{m_n} = 0.
\]

This would also follow from Conjecture 1 as the only sparse paving matroids without circuit-hyperplanes are the uniform matroids.

Mayhew, Newman and Whittle have recently shown that real-representability of matroids cannot be captured by a ‘natural’ axiom, that is, a sentence of monadic second-order logic for matroids, which is defined in their paper [18]. We believe that the difficulty may be in the seemingly unmanageable set of stable sets of the Johnson graph, and conjecture that there is not even a natural axiom that captures whether a sparse paving matroid is real-representable.

7.3. The cover complexity of a matroid. Inspired by the flat covers from Section 3, we define the cover complexity of a matroid \( M \) as

\[
\kappa(M) := \min \{|Z| | Z \subseteq \mathcal{F}(M), Z \text{ is a flat cover of } M\},
\]

i.e. the minimal size of such a flat cover. The cover complexity has a number of properties that one would expect from a complexity measure on matroids; e.g. it is invariant under taking duals, and monotone under taking minors.

In [17 Conj. 1.7] it is conjectured that if \( N \) is any sparse paving matroid, then

\[
\lim_{n \to \infty} \frac{|\{M \in \mathcal{M}_n | M \text{ does not have an } N\text{-minor}\}|}{m_n} = 0.
\]

In [23] two of the current authors show that this conjecture holds for each of the matroids \( N = U_{2,k}, U_{3,6}, P_6, Q_6, \) and \( R_6 \), by deriving bounds on the cover complexity of matroids that do not have such a minor.

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References

[1] N. Alon and F. R. K. Chung. Explicit construction of linear sized tolerant networks. *Discrete Math.*, 72(1-3):15–19, 1988.

[2] Noga Alon, József Balogh, Robert Morris, and Wojciech Samotij. Counting sum-free sets in Abelian groups. arXiv:1201.6654 (to appear in Israel J. Math.), 2012.

[3] Noga Alon and Joel H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[4] John E. Blackburn, Henry H. Crapo, and Denis A. Higgs. A catalogue of combinatorial geometries. *Math. Comp.*, 27:155–166; addendum, ibid. 27 (1973), no. 121, loose microfiche suppl. A12-G12, 1973.

[5] Joseph E. Bonin. Sparse paving matroids, basis-exchange properties, and cyclic flats. arXiv:1011.1010v1, 2011.

[6] A. E. Brouwer and T. Etzion. Some new constant weight codes. *Advances in Mathematics of Communications*, 5:417–424, 2011.

[7] Andrés E. Brouwer, Arjeh M. Cohen, and Arnold Neumaier. *Distance-regular graphs*. Springer, 1989.

[8] Andrés E. Brouwer and Willem H. Haemers. *Spectra of graphs*. Universitext. Springer, New York, 2012.

[9] Henry H. Crapo and Gian-Carlo Rota. *On the foundations of combinatorial theory: Combinatorial geometries*. The M.I.T. Press, Cambridge, Mass.-London, preliminary edition, 1970.

[10] Jim Geelen and Peter J. Humphries. Rota’s basis conjecture for paving matroids. *SIAM J. Discrete Math.*, 20(4):1042–1045 (electronic), 2006.

[11] R. L. Graham and N. J. A. Sloane. Lower bounds for constant weight codes. *IEEE Trans. Inform. Theory*, 26(1):37–43, 1980.

[12] Wilhelmus Hubertus Haemers. *Eigenvalue techniques in design and graph theory*, volume 121 of *Mathematical Centre Tracts*. Mathematisch Centrum, Amsterdam, 1980. Dissertation, Technische Hogeschool Eindhoven, Eindhoven, 1979.

[13] Mark Jerrum. Two remarks concerning balanced matroids. *Combinatorica*, 26(6):733–742, 2006.

[14] Daniel J. Kleitman and Kenneth J. Winston. On the number of graphs without 4-cycles. *Discrete Math.*, 41(2):167–172, 1982.

[15] Donald E. Knuth. The asymptotic number of geometries. *J. Combinatorial Theory Ser. A*, 16:398–400, 1974.

[16] Joseph P. S. Kung. Matroids. In *Handbook of algebra, Vol. 1*, pages 157–184. North-Holland, Amsterdam, 1996.

[17] Dillon Mayhew, Mike Newman, Dominic Welsh, and Geoff Whittle. On the asymptotic proportion of connected matroids. *European J. Combin.*, 32(6):882–890, 2011.

[18] Dillon Mayhew, Mike Newman, and Geoff Whittle. Is the missing axiom of matroid theory lost forever? arXiv:1204.3365, 2012.

[19] Dillon Mayhew and Gordon F. Royle. Matroids with nine elements. *J. Combin. Theory Ser. B*, 98(2):415–431, 2008.

[20] Dillon Mayhew and Dominic Welsh. On the number of sparse paving matroids. http://homepages.ecs.vuw.ac.nz/~mayhew/Publications/MW.pdf, 2010.

[21] Criel Merino, Steven D. Noble, Marcelino Ramírez-Ibáñez, and Rafael Villarroel-Flores. On the structure of the h-vector of a paving matroid. *Eur. J. Comb.*, 33(8):1787–1799, 2012.

[22] James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[23] Robert A. Pendavingh and J.G. van der Pol. Counting matroids in minor-closed classes. arXiv:1302.1315v3, 2013.

[24] M. J. Piff. An upper bound for the number of matroids. *J. Combinatorial Theory Ser. B*, 14:241–245, 1973.

[25] M. J. Piff and D. J. A. Welsh. The number of combinatorial geometries. *Bull. London Math. Soc.*, 3:55–56, 1971.

[26] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency*. Vol. B, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69.

[27] D. J. A. Welsh. *Matroid theory*. Academic Press [Harcourt Brace Jovanovich Publishers], London, 1976. L. M. S. Monographs, No. 8.

[28] Hassler Whitney. On the Abstract Properties of Linear Dependence. *Amer. J. Math.*, 57(3):509–533, 1935.
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