CONFORMAL TRANSFORMATIONS AND QUANTUM GRAVITY

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October 20, 2018

Abstract

Recently[1], it was shown that quantum effects of matter could be identified with the conformal degree of freedom of the space–time metric. Accordingly, one can

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introduce quantum effects either by making a scale transformation (i.e. changing the metric), or by making a conformal transformation (i.e. changing all physical quantities). These two ways are investigated and compared. Also, it is argued that, the ultimate formulation of such a quantum gravity theory should be in the framework of the scalar–tensor theories.

1 Quantum Effects, Scale, and Conformal Transformations

In a previous paper it was shown that the application of an idea of de-Broglie leads to the fact that the quantum effects of matter are equivalent to a specific conformal factor of the space–time metric. In the de-Broglie–Bohm relativistic quantum theory, the Klein–Gordon equation

$$\square \Psi + \frac{m^2}{\hbar^2} \Psi = 0 \quad (1)$$

is replaced with the quantum Hamilton–Jacobi equation:

$$\overline{g}^{\mu\nu} \nabla_{\mu} S \nabla_{\nu} S = \mathcal{M}^2 \quad (2)$$

and the continuity equation:

$$\nabla_{\mu} (\rho \nabla^{\mu} S) = 0 \quad (3)$$

1 A "bar" sign over any quantity indicates that it is expressed in terms of \( \overline{g}_{\mu\nu} \) metric. This is introduced for later convenience.
where all covariant derivatives are calculated with respect to the space–time metric $\mathfrak{g}_{\mu\nu}$ and

$$\Psi = \sqrt{\rho}e^{iS/h}$$

and

$$\mathcal{M} = \mathfrak{m}(1 + \mathcal{Q})^{1/2}$$

$$\mathcal{Q} = \alpha \frac{\Box \sqrt{\rho}}{\sqrt{\rho}}; \quad \alpha = \frac{h^2}{m^2}$$

It must be noted that $\mathcal{M}$ is a variable mass, originating from the matter quantum potential $\mathcal{Q}$.

If the above metric is written as conformally transformed of another metric ($g_{\mu\nu}$) which

$$\mathfrak{g}_{\mu\nu} = \phi^{-1} g_{\mu\nu}$$

where

$$\phi^{-1} = 1 + \mathcal{Q}$$

Then the Hamilton–Jacobi equation in terms of this metric is:

$$g^{\mu\nu} \nabla_\mu S \nabla_\nu S = \mathfrak{m}^2$$

This means that quantum effects has been removed in this new metric. In other words, in terms of $\mathfrak{g}_{\mu\nu}$ (the background metric) the quantum effects are included in the variable mass $\mathcal{M}$, while in terms of $g_{\mu\nu}$ metric some parts of the curvature of the space–time represents the quantum effects. This new metric is called the physical metric because it has physical interpretation. For example the singularities of FRW universe would be removed in $g_{\mu\nu}$.
In our previous work [1], the conformal transformation was applied only to the space–time metric. Other quantities like mass, density and so on were assumed to possess no transformation. This is because the above conformal transformation which incorporates the quantum effects of matter into a specific conformal factor, is in fact a scale transformation guessed from the Hamilton–Jacobi equation (2). As the conformal transformation is more general than scale transformation which is used in [1], it seems preferable to make a conformal transformation, in which all physical quantities are transformed, instead of making only a scale transformation.

Now, applying the conformal transformation given by the equation (7), we have:

\[ m = \phi^{1/2}m \]  

(10)

and

\[ \rho = \phi^{3/2}\rho \]  

(11)

Therefore the equation (2) reads:

\[ \nabla \mu S \nabla^\mu S = \phi^2m^2(1 + \overline{Q}) \]  

(12)

instead of (9).

In order to have no quantum effects, when physics is expressed in terms of the physical metric, it is necessary to set:

\[ \phi^{-2} = 1 + \overline{Q} \]  

(13)

This equation determines the conformal degree of freedom of the space–time metric.
With the aid of the guiding equation in the physical metric \( P^\mu = mu^\mu = \nabla^\mu S \) and the equation (12), the equation of motion takes the form:

\[
\frac{du_\mu}{d\tau} - \frac{1}{2} \left( \partial_\mu g_{\alpha\beta} \right) u^\alpha u^\beta = 0 \tag{14}
\]

which is the geodesic equation for a free particle. But with respect to the background metric \( (g_{\mu\nu}) \), the particle is affected by quantum force and therefore it doesn’t move on the geodesic. In this case from the guiding formula \((P^\mu = M u^\mu = \nabla^\mu S)\) and the equation (2), we have

\[
\frac{d\bar{u}_\mu}{d\tau} - \frac{1}{2} \left( \partial_\mu g_{\alpha\beta} \right) \bar{u}^\alpha \bar{u}^\beta = \frac{1}{\mathcal{M}} \left( g_{\mu\nu} - \bar{u}_\mu \bar{u}_\nu \right) \nabla^\nu \mathcal{M} \tag{15}
\]

where the expression on the right hand side is the quantum force.

It can be shown that these two geodesic equations (14,15), as expected, are conformal transformations of each other. Using the conformal transformation, equation (14) reads:

\[
\frac{d\bar{u}_\mu}{d\tau} - \frac{1}{2} \left( \partial_\mu g_{\alpha\beta} \right) \bar{u}^\alpha \bar{u}^\beta = - \frac{1}{2} \phi^{-1/2} \bar{u}_\mu \frac{d\phi}{d\tau} - \frac{1}{2} \bar{u}^\alpha \bar{u}^\beta g_{\alpha\beta} \frac{\partial_\mu \phi}{\phi} \tag{16}
\]

Since \( d/d\tau = \bar{u}_\alpha \nabla^\alpha \) and \( \phi = m/\mathcal{M} \), a simple calculation shows that the right hand side of the above equation is the quantum force.

Here, it must be noted that the conformal factor is obtained only from one of the equations of motion, i.e. the Hamilton–Jacobi equation. The continuity equation in the physical metric is:

\[
\nabla_\mu \left( \rho \nabla^\mu S \right) = 0 \tag{17}
\]

So

\[
\nabla_\mu \left( \rho \nabla^\mu S \right) = - \frac{\nabla_\mu \phi}{2\phi} \rho \nabla^\mu S \tag{18}
\]
The right hand side of this expression is of the order of \( h^2 \) and higher\(^1\).

The conformal transformation used to show the equivalence between quantum effects and curvature of space–time has many applications. It is possible to discuss the physical properties of the conformal factor (13) as it was done in our earlier paper\(^1\) about the scale factor (8). Correspondingly, one can start from gravity–matter action and then proceed to eliminate the matter quantum potential and transform metric and physical quantities conformally. Therefore, the quantum gravity action is:

\[
A = \int d^4x \sqrt{-\bar{g}} \left\{ \phi \bar{\mathcal{R}} - \frac{3}{2} \frac{\nabla_{\mu} \phi \nabla_{\nu} \phi}{\phi^2} + 2\pi \left( \phi^3 \frac{\bar{\rho}}{m^2} \nabla_{\mu} \nabla_{\nu} \phi - \phi \bar{\rho} \phi \right) + \Lambda \left( \phi - (1 + \bar{Q})^{-1/2} \right) \right\}
\]

(19)

where \( \Lambda \) is a lagrangian multiplier introduced in order to fix the conformal factor. Variation of this action with respect to \( \bar{g}_{\mu\nu}, \phi, \bar{\rho}, S \) and \( \Lambda \) leads to:

1. The equation of motion of \( \phi \):

\[
\bar{\mathcal{R}} - \frac{3}{2} \frac{\nabla_{\mu} \phi \nabla_{\nu} \phi}{\phi^2} + 3 \frac{\Box \phi}{\phi} + \Lambda + 6\pi \phi^2 \frac{\bar{\rho}}{m^2} \nabla_{\mu} \nabla_{\nu} \phi - 2\pi \bar{\rho} \bar{m} = 0
\]

(20)

2. The continuity equation:

\[
\nabla_{\mu} \left( \bar{\rho} \phi^3 \nabla^\mu S \right) = 0
\]

(21)

3. The equation of motion of particles:

\[
\frac{4\pi}{m^2} \phi^3 \sqrt{\bar{p}} \nabla_{\mu} S \nabla^\mu S - 4\pi \bar{m} \sqrt{\bar{p}} \phi - \frac{\Lambda}{2} (1 + \bar{Q})^{-3/2} \frac{\bar{Q}}{\sqrt{\bar{p}}} + \frac{\Lambda \alpha}{2} \Box \left( \frac{1 + \bar{Q})^{-3/2} \right) = 0
\]

(22)

4. The modified Einsten equations for \( \bar{g}_{\mu\nu} \):

\[
\bar{g}^{\mu\nu} - \frac{1}{2} \Lambda \bar{g}^{\mu\nu} = \frac{1}{\phi} \left[ \bar{g}^{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu} \right] \phi + \frac{3}{2} \left( \frac{\nabla_{\mu} \phi \nabla_{\nu} \phi}{\phi^2} - \frac{1}{2} \frac{\nabla_{\alpha} \phi \nabla_{\alpha} \phi}{\phi^2} \bar{g}^{\mu\nu} \right)
\]
\[ + \frac{\kappa}{m} \phi^2 \nabla^\alpha S \nabla_\alpha S g^{\mu\nu} - \frac{2\kappa}{m} \rho \phi^2 \nabla^\mu S \nabla^\nu S - \kappa \frac{\rho}{m} \phi \nabla^\mu \nabla^\nu g^{\mu\nu} - \frac{1}{2} \frac{\Lambda}{\phi} (1 + \overline{Q})^{-1/2} g^{\mu\nu} \]

\[-\frac{\Lambda}{4} (1 + \overline{Q})^{-3/2} \overline{Q} g^{\mu\nu} - \frac{\alpha}{4\phi} \nabla^\mu \nabla^\nu \left( \Lambda \phi \frac{(1 + \overline{Q})^{-3/2}}{\sqrt{\rho}} \right) + \frac{\alpha}{4\phi} \nabla^\nu \nabla^\mu \left( \Lambda \phi \frac{(1 + \overline{Q})^{-3/2}}{\sqrt{\rho}} \right) \]

(23)

5. The constraint equation:

\[ \phi^{-2} = 1 + \overline{Q} \]

(24)

Then, the equations of motion can be solved, leading to the background metric and other physical quantities.

2 Concluding Remarks

Recently [1], it was shown that quantum effects could be contained in the conformal factor of the space–time metric. In this paper, the difference between introducing the quantum effects by conformal transformation or scale transformation are discussed. No essential difference is found. But, some points must be noted here. The first important point is about the geodesic equation (14,15). In the background metric, this equation resembles the geodesic equation in Brans–Dicke theory. Consideration of the matter quantum effects, leads to the physical metric in which a particle moves on the geodesic of Brans–Dicke theory written in Einstein guage. This point supports the suggestion that the discussion of quantum gravity requires a scalar–tensor theory. Previously this was suggested when discusing Bohmian quantum gravity [4].
Secondly, since as the dimensional coupling between matter and gravity is resulted from the breaking of the conformal invariance, this formulation shows that the conformal frame is fixed by the distribution of matter at quantum level. Thus, the Bohmian quantum theory singles out the preferred frame.

In the present paper and the earlier one \cite{1} we have used lagrangian multiplier, in order to fix the conformal factor by quantum potential. We suggested now that it must be possible to write a scalar–tensor theory which automatically leads to the correct equations of motion. This has the advantage that the conformal factor would be fixed by the equations of motion, and not by introducing a lagrangian multiplier by hand. We shall elaborate on this point in a forthcoming paper \cite{2}.

References

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