A MINKOWSKI TYPE INEQUALITY FOR HYPERSURFACES IN THE SCHWARZSCHILD MANIFOLD

HAIZHONG LI AND YONG WEI

Abstract. In this paper, we use the inverse mean curvature flow to prove a sharp Minkowski type inequality for hypersurfaces in the Schwarzschild manifold.

1. INTRODUCTION

The Schwarzschild manifold is an $n$-dimensional ($n \geq 3$) manifold $M = [s_0, \infty) \times S^{n-1}$ equipped with the metric

$$\bar{g} = \frac{1}{1 - 2ms^{2-n}} ds^2 + s^2 g_{S^{n-1}},$$

where $m > 0$ is a constant, $s_0$ is the unique positive solution of $1 - 2ms_0^{2-n} = 0$ and $g_{S^{n-1}}$ is the canonical round metric on the unit sphere $S^{n-1}$. We define the function $f = \sqrt{1 - 2ms^{2-n}}$. The Schwarzschild metric is asymptotically flat, that is the sectional curvature of $(M, \bar{g})$ approach zero near infinity. In fact, by a change of variable, the Schwarzschild manifold $(M^n, \bar{g})$ can be expressed as $\mathbb{R}^n \setminus \{0\}$ equipped with the conformal metric

$$\bar{g} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{n}{n-2}} g_{\mathbb{R}^n},$$

where $g_{\mathbb{R}^n}$ is the flat Euclidean metric. Moreover, the scalar curvature of $(M, \bar{g})$ equals to zero (see §2).

A hypersurface $\Sigma$ in $(M, \bar{g})$ is said to be mean convex if its mean curvature $H$ is positive everywhere on $\Sigma$. In this paper, we prove the following sharp inequality for mean convex and star-shaped hypersurface in $(M, \bar{g})$ (see §2 for the definition of star-shaped).

Theorem 1. Let $\Sigma$ be a closed mean convex and star-shaped hypersurface in the Schwarzschild manifold $(M, \bar{g})$. Then

$$\int_\Sigma fHd\mu \geq (n - 1)(\frac{1}{\omega_{n-1}}) \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} - 2m,$$

(2)
where $\omega_{n-1}$ is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and $|\Sigma|$ is the area of $\Sigma$. Moreover, equality holds if and only if $\Sigma$ is a coordinate sphere $\{s\} \times S^{n-1}$.

Recall that the boundary $\partial M = \{s_0\} \times S^{n-1}$ is called the horizon of the Schwarzschild manifold, its area is equal to $|\partial M| = s_0^{n-1}\omega_{n-1}$. Since $s_0$ is the unique positive solution of $1 - 2ms_0^2 - n = 0$, we have

$$2m = \left(\frac{|\partial M|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$ 

Therefore (2) is equivalent to the following inequality (compare with Theorem 1 in [3])

$$\int_{\Sigma} fHd\mu \geq (n-1)\omega_{n-1}^{1 \cdot \frac{1}{n-1}} \left(|\Sigma|^{\frac{n-2}{n-1}} - |\partial M|^{\frac{n-2}{n-1}}\right). \quad (3)$$

The classical Minkowski inequality for convex hypersurface $\Sigma$ in $\mathbb{R}^n$ states that

$$\int_{\Sigma} Hd\mu \geq (n-1)\omega_{n-1}^{1 \cdot \frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}}. \quad (4)$$

This was generalized by Guan and Li [8] to a mean convex and star-shaped hypersurface using the inverse mean curvature flow. By letting $m \to 0$, the Schwarzschild metric reduces to the Euclidean metric $\bar{g} = ds^2 + s^2g_{S^{n-1}}$ and the potential $f$ becomes $f = 1$. Thus Theorem 1 recover the Minkowski inequality (4) for mean convex and star-shaped hypersurface $\Sigma$ in $\mathbb{R}^n$. Note that Huisken’s recent work [11] showed that the assumption star-shaped in [8] can be replaced by outward-minimizing.

Our result is motivated by the recent work of Brendle, Hung and Wang [3], where they proved a sharp Minkowski-type inequality for mean convex and star-shaped hypersurfaces in Anti-deSitter-Schwarzschild manifold (which is asymptotically hyperbolic near infinity), by using the inverse mean curvature flow. The inverse mean curvature flow has many applications in geometry and general relativity, see, e.g. [1, 4, 8, 12, 15–17]. We first establish a convergence result for mean convex and star-shaped hypersurface in the Schwarzschild manifold. Note that Huisken and Ilmanen [12] considered the weak solution of the inverse mean curvature flow in asymptotically flat manifold (in a level-set formulation), which includes the Schwarzschild manifold as a special case. In this paper we consider the smooth solution in the notion of Gerhardt’s work (see [6]). We show that under the inverse mean curvature flow, if the initial hypersurface $\Sigma_0$ is mean convex and star-shaped, then the flow hypersurface $\Sigma_t$ converges to a large coordinate sphere as $t \to \infty$. Then we define a quantity

$$Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \left(\int_{\Sigma_t} fHd\mu_t + 2(n-1)m\omega_{n-1}\right),$$

and show that $Q(t)$ is monotone decreasing along the inverse mean curvature flow. Thus we could compare the initial value $Q(0)$ with the limit
\[ \lim_{t \to \infty} Q(t) \geq (n - 1) \omega_{n-1}. \]

From this we can complete the proof of Theorem 1 easily.

**Acknowledgment.** We would like to thank Professor Ben Andrews for his interest and helpful comments.

2. **Preliminaries**

In this section, we collect some facts about the Schwarzschild manifolds and star-shaped hypersurfaces.

2.1. **Schwarzschild metric.** By a change of variable, the Schwarzschild metric (1) can be written as

\[ \bar{g} = dr^2 + \lambda^2(r)g_{S^{n-1}}, \]

where \( \lambda(r) \) satisfies

\[ \lambda'(r)^2 = 2m\lambda^2 - n. \]

Let \( \theta = \{ \theta^j \}, j = 1, \cdots, n-1 \) be a coordinate system on \( S^{n-1} \) and \( \partial_{\theta^j} \) be the corresponding coordinate vector field in \( M \). Let \( \partial_r \) be the radial vector. By a direct calculation (see for example [18]), the curvature tensor of \( (M, \bar{g}) \) has the following components

\[ \bar{R}_{ijkl} = \frac{1 - \lambda'^2}{\lambda^2} (\bar{g}_{ik} \bar{g}_{jl} - \bar{g}_{il} \bar{g}_{jk}) \]

\[ \bar{R}_{irjr} = -\frac{\lambda''}{\lambda} \bar{g}_{ij}, \]

where \( 1 \leq i, j, k, l \leq n - 1 \) and \( \bar{g}_{ij} = \bar{g}(\partial_{\theta^i}, \partial_{\theta^j}) = \lambda^2 g_{S^{n-1}}(\partial_{\theta^i}, \partial_{\theta^j}) = \lambda^2 \sigma_{ij} \).

Other components of the curvature tensor are equal to zero. Noting that

\[ \frac{1 - \lambda'^2}{\lambda^2} = 2m\lambda^{-n}, \quad \frac{\lambda''}{\lambda} = -m(n - 2)\lambda^{-n}. \]

By a further calculation, we have the Ricci curvature of \( (M, \bar{g}) \)

\[ \bar{Ric} = \left( (n - 2) \frac{1 - \lambda'^2}{\lambda^2} - \frac{\lambda''}{\lambda} \right) \bar{g} - (n - 2) \left( \frac{1 - \lambda'^2}{\lambda^2} + \frac{\lambda''}{\lambda} \right) dr^2 \]

\[ = m(n - 2)\lambda^{-n} \bar{g} - mn(n - 2)\lambda^{-n} dr^2 \]

and the scalar curvature

\[ \bar{R} = (n - 1) \left( (n - 2) \frac{1 - \lambda'^2}{\lambda^2} - \frac{2\lambda''}{\lambda} \right) = 0. \]

On the other hand, by the definition of \( f \), we have \( f' = \lambda' = \sqrt{1 - 2m\lambda^{2-n}}. \)

In the sequel, we denote by \( \nabla, \nabla^2 \) and \( \Delta \) the gradient, Hessian and Laplacian.
operator on \((M, \bar{g})\). We can compute the Hessian of \(f\):
\[
\bar{\nabla}^2 f = \frac{\lambda' \lambda''}{\lambda} \bar{g} + (\lambda'' - \frac{\lambda' \lambda''}{\lambda}) dr^2
\]
\[
=m(n-2)\lambda^{-n-1} \bar{g} - mn(n-2)\lambda^{-n} \lambda' dr^2
\]  
(9)
Thus we have
\[
\bar{\Delta} f = (n-1)\frac{\lambda' \lambda''}{\lambda} + \lambda''' = 0.
\]  
(10)
Combining (7), (9) and (10), we conclude that \(f\) satisfies the following static equation:
\[
(\bar{\Delta} f) \bar{g} - \bar{\nabla}^2 f + f \bar{Ric} = 0.
\]  
(11)

2.2. Star-shaped hypersurface. We say a hypersurface \(\Sigma\) in \((M, \bar{g})\) is star-shaped if
\[
\langle \partial_r, \nu \rangle > 0 \text{ on } \Sigma.
\]
A star-shaped hypersurface could be parameterized by a graph
\[
\Sigma = \{(r(\theta), \theta) : \theta \in \mathbb{S}^{n-1}\}
\]
for a smooth function \(r\) on \(\mathbb{S}^{n-1}\). As in [3, 5, 7], we define a function \(\varphi\) on \(\mathbb{S}^{n-1}\) by
\[
\varphi(\theta) = \Phi(r(\theta)),
\]
where \(\Phi(r)\) is a positive function satisfying \(\Phi'(r) = 1/\lambda(r)\). Define
\[
v = \sqrt{1 + |D\varphi|^2_{\mathbb{S}^{n-1}}},
\]
where \(D\) denotes the Levi-Civita connection on \(\mathbb{S}^{n-1}\). Let \(\theta = \{\theta^j\}, j = 1, \cdots, n-1\) be a coordinate system on \(\mathbb{S}^{n-1}\). The unit normal vector of this hypersurface could be written as
\[
\nu = \frac{1}{v}(\partial_r - \frac{r^j}{\lambda^2} \partial_{\theta^j}).
\]
We can express the metric and second fundamental form of \(\Sigma\) as following (see [3][5][7])
\[
g_{ij} = \lambda^2 (\sigma_{ij} + \varphi_i \varphi_j),
\]  
(12)
\[
h_{ij} = \lambda \frac{\lambda'}{v^2} g_{ij} - \frac{\lambda}{v^2} \varphi_{ij},
\]  
(13)
where \(\varphi_i, \varphi_{ij}\) are covariant derivatives of \(\varphi\) with respect to the metric \(g_{\mathbb{S}^{n-1}}\). After lifting the indice \(j\), we have
\[
h^j_i = \frac{1}{v^2} \left( \lambda \delta^j_i - \tilde{\sigma}^j_k \varphi_{ki} \right),
\]  
(14)
where \(\tilde{\sigma}^{jk} = \sigma^{jk} - \frac{\varphi^j \varphi^k}{v^2}\) with \(\varphi^j = \sigma^{jk} \varphi_k\). The mean curvature \(H\) then has the form
\[
H = \frac{(n-1)\lambda' - \tilde{\sigma}^{ij} \varphi_{ij}}{v^2 \lambda}.
\]  
(15)
3. Inverse mean curvature flow

In this section, we consider the inverse mean curvature flow in the Schwarzschild manifold, which is a family \( X : \Sigma \times [0, T) \rightarrow (M, \bar{g}) \) satisfying

\[
\partial_t X = \frac{1}{H} \nu,
\]

where \( \nu \) is the unit outward normal and \( H \) is the mean curvature of \( \Sigma_t = X(\Sigma, t) \). If the initial hypersurface is star-shaped and mean convex, the short time existence result implies the flow exists on a maximum time interval \([0, T)\).

Let \( \partial_i, i = 1, 2, \ldots, n - 1 \) be coordinate vector fields on \( \Sigma_t \). Denote by \( g_{ij} \) and \( h_{ij} \) the components of the first and second fundamental form, by \( H = g_{ij}h^{ij} \) the mean curvature and \( |A|^2 = h_{ik}h_{lj}g^{il}g^{jk} \) the squared norm of the second fundamental form, by \( \chi = \langle \lambda \partial_r, \nu \rangle \) the support function and by \( d\mu_t \) the area element on \( \Sigma_t \). We first collect the evolution equations for various geometric quantities under the inverse mean curvature flow.

**Lemma 2** (Evolution equations). Under the flow \( (16) \), we have

\[
\partial_t g_{ij} = 2H^{-1}h_{ij},
\]

\[
\partial_t d\mu_t = d\mu_t,
\]

\[
\partial_t \nu = \frac{1}{H^2} \nabla H,
\]

\[
\partial_t h^j_i = \frac{\Delta h^j_i}{H^2} + \frac{|A|^2}{H^2} \delta^j_i - \frac{2}{H} h^k_i h^j_k - \frac{2}{H^3} \nabla_i H \nabla^j H - \frac{2}{H} R_{\nu\nu\nu k} g^{kj}
\]

\[
+ \frac{2}{H^2} g^{ij} g^{km} R_{\mu\nu\mu\nu} + \frac{1}{H^2} g^{ij} g^{km} R_{\mu\nu\nu} + \frac{1}{H^2} g^{ij} g^{km} R_{\mu\nu\nu} + \frac{1}{H^2} \text{Ric}(\nu, \nu) g^{ij},
\]

\[
\partial_t H = \frac{\Delta H}{H^2} - \frac{2}{H^2} \nabla \frac{|H|^2}{H^3} - \frac{|A|^2}{H^2} - \frac{\text{Ric}(\nu, \nu)}{H}
\]

\[
\partial_t \chi = \frac{1}{H^2} \Delta \chi + \frac{|A|^2}{H^2} \chi - \frac{1}{H^2} \text{Ric}(\nu, \partial_k)(\lambda \partial_r, \partial_j) g^{kj}.
\]

where \( \nabla \) and \( \Delta \) are gradient and Laplacian operator with respect to the induced metric on the flow hypersurface \( \Sigma_t \).

**Proof.** The evolution equations for the metric, area element, unit normal and the second fundamental form can be calculated in a standard way as in [10], we omit the argument here. For the support function, we have

\[ \partial_t \chi = \partial_t \langle \lambda \partial_r, \nu \rangle = \frac{\chi}{H} + \frac{1}{H^2} \langle \lambda \partial_r, \nabla H \rangle, \]

where we used the conformal property of the vector field \( \lambda \partial_r \) (see [2]) and \( (19) \). On the other hand, by using the conformal property of \( \lambda \partial_r \) again and the Codazzi equations, we have

\[ \nabla_i \chi = \langle \lambda \partial_r, h^k_i \partial_k \rangle \]
and
\[ \nabla_j \nabla_i \chi = \lambda' h_{ij} - h^k_i h_{kj} \chi + \langle \lambda \partial_r, \nabla_j h^k_i \partial_k \rangle \]

Thus we obtain
\[ \Delta \chi = \lambda' H - |A|^2 \chi + \langle \lambda \partial_r, \nabla H \rangle + \langle R_{ijkl} g^{kj} \partial_k \rangle. \]

Then (22) follows from combining the above equations. □

We could use the evolution equation (22) of the support function to show that under the inverse mean curvature flow, the evolved hypersurface \( \Sigma_t \) remains star-shaped.

**Lemma 3.** Under the inverse mean curvature flow (16), the evolved hypersurface \( \Sigma_t \) remains star-shaped if \( \Sigma_0 \) is star-shaped.

**Proof.** From the expression (7) of Ricci curvature of the Schwarzschild manifold and the evolution equation (22) of the support function, we have
\[ \partial_t \chi = \frac{1}{H^2} \Delta \chi + \frac{|A|^2}{H^2} \chi + \frac{1}{H^2}mn(n-2)\lambda^{-n} |\partial^T_r|^2 \chi, \]
where
\[ |\partial^T_r|^2 = \sum_{k,j=1}^{n-1} \langle \partial_r, \partial_k \rangle \langle \partial_r, \partial_j \rangle g^{kj} \]
is squared norm of the tangential part \( \partial^T_r \) of the radial vector \( \partial_r \). Since \( \chi > 0 \) on the initial hypersurface \( \Sigma_0 \), in view of the inequality \( |A|^2 \geq H^2/(n-1) \) and using the parabolic maximum principle, we conclude that
\[ \chi \geq e^{n-1} \min_{\Sigma_0} \chi > 0 \]
which implies the star-shapedness of \( \Sigma_t \). □

The flow equation (16) is often called the parametric form of the flow. Since each \( \Sigma_t \) is star-shaped, it can also be represented as a graph
\[ \Sigma_t = \{(r(\theta, t), \theta) : \theta \in S^{n-1}\} \]
Then the flow equation is equivalent to the following non-parametric form of the flow (cf. [3, 5, 7])
\[ \frac{\partial r}{\partial t} = \frac{v}{H}. \]
(24)
The speed function \( \frac{v}{H} \) depends on \( r, Dr, D^2 r \). It is easy to see that the flow equation (24) is parabolic. In deed, as in section 2 we define \( \varphi(\theta, t) = \Phi(r(\theta, t)) \) with \( \Phi(r) \) is a positive function satisfying \( \Phi'(r) = 1/\lambda(r) \). Then
\[ \varphi_i = \frac{r_i}{\lambda}, \quad \varphi_{ij} = \frac{r_{ij}}{\lambda} - \frac{\lambda' r_i r_j}{\lambda^2}, \]
and we have
\[ H = \frac{(n-1)\lambda'}{v \lambda} - \frac{\tilde{\sigma}_{ij}}{v \lambda^2} (r_{ij} - \frac{\lambda' r_i r_j}{\lambda}). \]
(25)
Thus
\[
\frac{\partial}{\partial r_{ij}} \left( \frac{v}{H} \right) = \tilde{\sigma}_{ij} H^2 \lambda^2
\]
which is nonnegative definite and therefore \((24)\) is parabolic.

**Lemma 4.** Let \(r_1, r_2\) be constants such that
\[
r_1 < r(\theta) < r_2
\]
holds on the initial hypersurface \(\Sigma_0\). Then on \(\Sigma_t\) we have the estimate
\[
\lambda(r_1) e^{\frac{t}{n-1}} < \lambda(r(\theta, t)) < \lambda(r_2) e^{\frac{t}{n-1}}, \quad \forall t \in [0, T).
\]

**Proof.** Let \(S_{r_i}, i = 1, 2\) be coordinate spheres \(\{r_i\} \times S^{n-1}\). We solve the inverse mean curvature flows with initial hypersurface \(S_{r_i}\) respectively. If the initial hypersurface is a coordinate sphere, the inverse mean curvature flow becomes a scalar flow:
\[
\frac{dr}{dt} = \frac{1}{H} = \frac{\lambda}{(n-1)\lambda'},
\]
where we used that the principal curvatures of a coordinate sphere are \(\lambda'/\lambda\). Then
\[
\frac{d\lambda}{dt} = \frac{\lambda}{n-1}.
\]
From this we deduce that \(\lambda(r_i(t)) = \lambda(r_i(0)) e^{\frac{t}{n-1}}\). By the parabolic maximum principle, we have
\[
r_i(t) < r(\theta, t) < r_2(t), \quad t \in [0, T).
\]
Since \(\lambda' > 0\), we also have
\[
\lambda(r_1(t)) < \lambda(r(\theta, t)) < \lambda(r_2(t)).
\]
The assertion follows from the above inequality. \(\Box\)

**Lemma 5.** There is a constant \(C_1 > 0\) such that \(He^{\frac{t}{n-1}} \leq C_1\).

**Proof.** From the evolution equation \((21)\) of \(H\), we have
\[
\partial_t H^2 = -2H \Delta \frac{1}{H} - 2|A|^2 - 2\overline{Ric}(\nu, \nu).
\]
Using \(\overline{Ric}(\nu, \nu) = O(\lambda^{-n}) = O(e^{-\frac{nt}{n-1}})\) and the inequality
\[
|A|^2 \geq \frac{1}{n-1} H^2,
\]
we obtain that
\[
\frac{d}{dt} H_{\max}^2 \leq -\frac{2}{n-1} H_{\max}^2 + O(e^{-\frac{nt}{n-1}}).
\]
From this, the assertion follows easily. \(\Box\)
To estimate the lower bound of $H$, by the definition of $\varphi$ we have
\[
\frac{\partial \varphi}{\partial t} = \frac{v}{\lambda H} = \frac{1}{F},
\]
where the function
\[
F = \frac{\lambda H}{v} = \frac{(n-1)\lambda' - \tilde{\sigma}^{ij} \varphi_{ij}}{v^2}.
\]
The flow equation (26) is also parabolic.

**Lemma 6.** There is a constant $C_2 > 0$ such that $H e^{-\frac{t}{C_2}} \geq C_2$.

*Proof.* If we differentiate (26) with respect to $t$, we obtain
\[
\partial_t (\partial_t \varphi) = \tilde{\sigma}^{ij} \varphi_{ij} - \frac{1}{F^2} \partial F (\partial_t \varphi)_i - \frac{(n-1)\lambda' \varphi_{ij}}{\lambda^2 H^2} \partial_t \varphi.
\]
So from (6) we have
\[
\frac{d}{dt} (\partial_t \varphi)_{\max} \leq 0.
\]
Noting that $\partial_t \varphi = \frac{\lambda}{\lambda H}$ and $v \geq 1$, we obtain that
\[
\lambda H \geq C > 0,
\]
The assertion follows from the above inequality and by using Lemma 4.

For the first space derivatives of $\varphi$, we have the following estimate.

**Lemma 7.** There is a constant $\beta > 0$ such that $\|D\varphi\|_{G^{n-1}} \leq O(e^{-\beta t})$.

*Proof.* Let $\omega = \frac{1}{2} |D\varphi|_{G^{n-1}}^2$. We can compute as lemma 8 in [3] to obtain the evolution of $\omega$:
\[
\partial_t \omega = \tilde{\sigma}^{ij} \omega_{ij} - \frac{1}{F^2} \partial F \omega_i - \frac{\tilde{\sigma}^{ij}}{v^2 F^2} \sigma^{kl} \varphi_{ik} \varphi_{jl}
- \frac{2(n-2)}{\lambda^2 H^2} \omega - \frac{2(n-1)\lambda''}{\lambda^2 H^2} \omega.
\]
By Lemma 4 and Lemma 5 there is a constant $\beta > 0$ such that
\[
\frac{(n-2)}{\lambda^2 H^2} \geq \beta.
\]
So from (6) we have
\[
\frac{d}{dt} \omega_{\max} \leq -2 \beta \omega_{\max},
\]
which implies the Lemma.

**Next,** we will estimate the second fundamental form of $\Sigma_t$. We define the tensor $M$ by (see [3][13])
\[
M^i_j = H h^i_j.
\]
Combining the evolution equations (20), (21), and using the fact that the Schwarzschild metric is asymptotically flat (see §2) and lemma 4, we have

\[
\partial_t M^i_j = \frac{1}{H^2} \Delta M^i_j - \frac{2}{H^3} \nabla^k H \nabla_k M^i_j - \frac{2}{H^2} \nabla^j H \nabla_i H
- \frac{2}{H^2} M^k_i M^j_k + \frac{|M|}{H^2} O(e^{-\frac{nt}{n-1}}) + o(e^{-\frac{n+1}{n-1}}).
\] (27)

If on the time interval considered we have an uniform bound \(H_0 \leq H \leq H_1\) for the mean curvature, by Hamilton’s maximum principle for parabolic system [9], we can bound the largest eigenvalue \(\mu_{n-1}\) of \(M\) above by the solution of the following ODE

\[
\frac{d}{dt} \varphi = -\frac{2}{H_1^2} \varphi^2 + \frac{\varphi}{H_0^2} O(e^{-\frac{nt}{n-1}}) + o(e^{-\frac{n+1}{n-1}}).
\]

So that \(\mu_{n-1}\) and then the largest principal curvature \(\kappa_{n-1}\) of \(\Sigma_t\) have a uniform upper bound depending on \(H_1\) and \(H_0\). Since the mean curvature is bounded from below, it follows that the full second fundamental form is bounded by

\[
|A| \leq C(n, H_0, H_1).
\] (28)

In view of the mean curvature estimate in Lemma 5 and Lemma 6, we have the long time existence of the inverse mean curvature flow.

**Proposition 8.** The solution of the inverse mean curvature flow is defined on \([0, \infty)\).

**Proof.** Let \([0, T)\) be the maximum time interval of existence for the smooth solution of the inverse mean curvature flow. If \(T < \infty\), then Lemma 5 and Lemma 6 imply that

\[
C_2 e^{-\frac{t}{n-1}} \leq H \leq C_1.
\]

From (28), we know that the second fundamental form of \(\Sigma_t\) is bounded for \(t \to T\). Then the regularity results of Krylov [13] and the short time existence theorem imply that we can extend the solution smoothly beyond \(T\), contradicting with the maximum of \(T\). So we conclude that \(T\) must be \(\infty\).

Finally we show that the solution \(\Sigma_t\) converges to a large coordinate sphere as \(t \to \infty\). Let us define

\[
\tilde{\lambda} = \lambda e^{-\frac{t}{n-1}}.
\] (29)

Then \(\tilde{\lambda}\) satisfies

\[
\partial_t \tilde{\lambda} = \frac{\lambda' v}{H} e^{-\frac{t}{n-1}} - \frac{1}{n-1} \tilde{\lambda} \quad (:= \tilde{F}),
\] (30)

where we denote the right hand side of (30) by \(\tilde{F}\). By lemma 4, the family \(\tilde{\lambda}(\cdot, t)\) is uniformly bounded. By lemma 5 and lemma 6 \(|\partial_t \tilde{\lambda}|\) is also uniformly bounded. Noting that

\[
\tilde{\lambda}_{ij} = (\lambda'' \lambda^2 + \lambda \lambda'^2) e^{-\frac{t}{n-1}} \varphi_i \varphi_j + \lambda \lambda' e^{-\frac{t}{n-1}} \varphi_{ij}
\] (31)
and the expression (15) of $H$, we deduce that
\[ \frac{\partial \tilde{F}}{\partial \tilde{\lambda}_{ij}} = \tilde{\sigma}_{ij} H^2 \lambda^2 \]
which is positive definite and therefore (30) is parabolic. Moreover, from lemma 5, lemma 6 and lemma 7, we conclude that (30) is uniformly parabolic.

By lemma 7, $D\tilde{\lambda}$ decays exponentially fast:
\[ D\tilde{\lambda} = D\lambda e^{-\frac{t}{n-1}} = \lambda' e^{-\frac{t}{n-1}} D\varphi = O(e^{-\beta t}). \]  
(32)
Thus $\tilde{\lambda}$ converges to a positive constant $\bar{\lambda}$ uniformly. From the regularity estimate of Krylov [14, §5.5], the second derivatives of $\tilde{\lambda}$ are uniformly bounded in $C^{0,\alpha}$. Using the interpolation theorem we deduce that $D^2\tilde{\lambda}$ also decays exponentially fast. In view of (31) and lemma 4, lemma 7, we have $|D^2\varphi| = O(e^{-\tilde{\beta} t})$ for some constant $\tilde{\beta} > 0$.

By (12) and lemma 7, the metric of $\Sigma_t$ satisfies
\[ e^{-\frac{2\varphi}{n-1}} g_{ij} \to \tilde{\lambda}^2 \sigma_{ij} \]  
(33)
exponentially fast. From the expression (14) of $h^i_j$, we have
\[ |\frac{\lambda}{\lambda'} h^i_j - \delta^i_j| = (\frac{1}{v} - 1)\delta^i_j - \frac{1}{v\lambda'} \tilde{\sigma}^j_k \varphi_k = O(e^{-\beta' t}) \]  
(34)
for a positive constant $\beta' = \min\{2\beta, \tilde{\beta}\}$. This implies that $\Sigma_t$ converges to a large coordinate sphere as $t \to \infty$.

4. PROOF OF THEOREM 1

In this section, we follow a similar argument in [3] to prove Theorem 1. We define the quantity
\[ Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \left( \int_{\Sigma_t} f H d\mu_t + 2(n-1)m\omega_{n-1} \right), \]
where $|\Sigma_t|$ is the area of $\Sigma_t$. We first show that $Q(t)$ is monotone decreasing under the inverse mean curvature flow.

**Proposition 9.** Under the inverse mean curvature flow (16), the quantity $Q(t)$ is monotone decreasing in $t$.

**Proof.** As the proof of Proposition 19 in [3], we first have
\[ \frac{d}{dt} \int_{\Sigma_t} f H d\mu_t \leq \int_{\Sigma_t} \left( \frac{n-2}{n-1} f H + 2(\nabla f, \nu) \right) d\mu_t, \]  
(35)
and equality holds if and only if $\Sigma_t$ is totally umbilical. For convenience of reader, we include the proof of (35) here.

$$\frac{d}{dt} \int_{\Sigma_t} fHd\mu_t = \int_{\Sigma_t} (\partial_t fH + f\partial_t H + fH) d\mu_t$$

$$= \int_{\Sigma_t} \left(\bar{\nabla} f, \nu\right) \frac{-1}{H} (\Delta f + f\bar{\nabla}(\nu, \nu) + \frac{n-2}{n-1} fH) d\mu_t$$

$$\leq \int_{\Sigma_t} \left(\bar{\nabla} f, \nu\right) \left(\Delta f + f\bar{\nabla}(\nu, \nu) + \frac{n-2}{n-1} fH\right) d\mu_t,$$

(36)

where we used $|A|^2 \geq H^2/(n-1)$ in the last inequality. Using the identity $\Delta f = \bar{\Delta} f - \bar{\nabla}^2 f(\nu, \nu) - H(\bar{\nabla} f, \nu)$ and (10), (11), we have

$$\Delta f + f\bar{\nabla}(\nu, \nu) = -H(\bar{\nabla} f, \nu).$$

Substituting this into (36) gives (35). If equality holds in (35), then $|A|^2 = H^2/n - 1$ and $\Sigma_t$ is totally umbilical.

Let $\Omega_t$ denote the region bounded by $\Sigma_t$ and the horizon $\partial M$. Using the divergence theorem and noting that $\bar{\Delta} f = 0$, we get

$$\int_{\Sigma_t} \left(\bar{\nabla} f, \nu\right)d\mu_t = \int_{\Omega_t} \bar{\Delta} f dvol + m(n-2)\omega_{n-1}$$

$$= m(n-2)\omega_{n-1}$$

which is a constant. Thus we obtain

$$\frac{d}{dt} \left(\int_{\Sigma_t} fHd\mu_t + 2(n-1)m\omega_{n-1}\right) \leq \frac{n-2}{n-1} \left(\int_{\Sigma_t} fHd\mu_t + 2(n-1)m\omega_{n-1}\right).$$

On the other hand, from the evolution (18) of the area element $d\mu_t$, the area of $|\Sigma_t|$ satisfies $\frac{d}{dt}|\Sigma_t| = |\Sigma_t|$. So we conclude that

$$\frac{d}{dt}Q(t) \leq 0,$$

equality holds if and only if (35) assumes equality and then $\Sigma_t$ is totally umbilical. □

Once we obtain the monotonicity of $Q(t)$, we need to investigate the limit of $Q(t)$ as $t \to \infty$.

**Proposition 10.** We have

$$\liminf_{t \to \infty} Q(t) \geq (n-1)\omega_{n-1}^{-1}.$$

**Proof.** From the expression (12) of the metric $g$ and lemma 7, the area element $d\mu_t$ satisfies

$$d\mu_t = \lambda^{n-1}(1 + O(\varepsilon^{-2\beta_t})) dvol_{S^{n-1}},$$
Then we have
\[ |\Sigma_t|^{\frac{n-2}{n-1}} = \left( \int_{S^{n-1}} \lambda^{n-1} d\text{vol}_{S^{n-1}} \right)^{\frac{n-2}{n-1}} \left( 1 + O(e^{-2\beta t}) \right). \] (37)

On the other hand,
\[ f = \sqrt{1 - 2m\lambda^2} = 1 - m\lambda^2 + O(\lambda^4 - 2n). \]

By the expression (15) of the mean curvature $H$ and the exponentially decay of $\varphi_i, \varphi_{ij},$
\[ \lambda H = \frac{(n-1)\lambda'}{v} - \frac{\sigma^{ij}\varphi_{ij}}{v} + \frac{\varphi^i\varphi^j\varphi_{ij}}{v^3} = n - 1 + O(e^{-\gamma t}) \]
for some positive constant $\gamma = \min\{2\beta, \tilde{\beta}, \frac{n-2}{n-1}\}$. So we have
\[ \int_{\Sigma_t} fHd\mu_t = (n-1) \int_{S^{n-1}} \lambda^{n-2} d\text{vol}_{S^{n-1}} \left( 1 + O(e^{-\gamma t}) \right). \] (38)

Then we obtain
\[ \liminf_{t \to \infty} Q(t) \geq (n-1) \liminf_{t \to \infty} \int_{S^{n-1}} \lambda^{n-2} d\text{vol}_{S^{n-1}} \left( 1 + O(e^{-\gamma t}) \right). \] (39)

Since $\tilde{\lambda} = \lambda e^{-\frac{t}{n-1}}$ converges to a constant $\bar{\lambda}$, there exists a positive function $\epsilon(t)$ such that $\lim_{t \to \infty} \epsilon(t) = 0$ and
\[ \bar{\lambda} - \epsilon < \tilde{\lambda} < \bar{\lambda} + \epsilon \]
or equivalently
\[ (\bar{\lambda} - \epsilon)e^{\frac{t}{n-1}} < \lambda < (\bar{\lambda} + \epsilon)e^{\frac{t}{n-1}} \]
when $t$ is sufficiently large. So we conclude that
\[ \liminf_{t \to \infty} Q(t) \geq (n-1) \omega^{\frac{1}{n-1}} \liminf_{t \to \infty} \left( \frac{1}{\lambda} - \epsilon \right)^{\frac{n-2}{n-1}} = (n-1) \omega^{\frac{1}{n-1}}, \] (40)
which completes the proof.

Proof of Theorem 4. Since $Q(t)$ is monotone decreasing in $t$, we have
\[ Q(0) \geq \liminf_{t \to \infty} Q(t) \geq (n-1) \omega^{\frac{1}{n-1}}. \]

Thus we obtain
\[ \int_{\Sigma_t} fHd\mu_t + 2(n-1)m\omega_{n-1} \geq (n-1) \omega^{\frac{1}{n-1}} |\Sigma_t|^{\frac{n-2}{n-1}} \]
which is equivalent to (2). If the equality holds in (2), then $Q(t)$ is a constant in $t$. From the proof of Proposition 3 the hypersurface $\Sigma_0$ is totally umbilical. It follows from the Codazzi equations that $\text{Ric}(v, e_i) = 0$ for any tangent vector fields $e_i$. Since $m > 0$, the expression (7) of Ricci curvature implies that the radial vector $\partial_r$ is either parallel or orthogonal to the unit
normal vector $\nu$ of $\Sigma_0$. By the star-shapedness of $\Sigma_0$, $\partial_r$ is parallel to $\nu$ at each point of $\Sigma_0$ and then $\Sigma_0$ is a coordinate sphere $\{s\} \times S^{n-1}$.

\[\square\]

References

[1] H. Bray and A. Neves, Classification of prime 3-manifolds with Yamabe invariant greater than $\mathbb{RP}^3$, Ann. of Math. 159 (2004), 407-424.
[2] S. Brendle, Constant mean curvature surfaces in warped product manifolds, arXiv:1105.4273, to appear in Publ Math IHES.
[3] S. Brendle, P.-K. Hung, and M.-T. Wang, A Minkowski-type inequality for hypersurfaces in the Anti-deSitter-Schwarzschild manifold, arXiv: 1209.0669.
[4] L.L. De Lima and F. Girao, An Alexandrov-Fenchel-type inequality in hyperbolic space with an application to a penrose inequality, arXiv: 1209.0438.
[5] Q. Ding, The inverse mean curvature flow in rotationally symmetric spaces, Chinese Annals of Mathematics, Series B, 1-18 (2010)
[6] C. Gerhardt, Flow of nonconvex hypersurfaces into spheres, J. Differential Geom. 32 (1990) 299-314.
[7] C. Gerhardt, Inverse curvature flows in hyperbolic space, J. Differential Geom. 89 (2011), no. 3, 487-527.
[8] P. Guan and J. Li, The quermassintegral inequalities for k-convex starshaped domains, Adv. Math. 221 (2009), 1725-1732.
[9] R.S. Hamilton, Four-manifolds with positive curvature operator, J. Differential Geom. 24 (1986) 153-179.
[10] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Differential Geom. 20 (1984) 237-266.
[11] G. Huisken, in preparation
[12] G. Huisken and T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, J. Differential Geom. 59 (2001) 353-438.
[13] G. Huisken and T. Ilmanen, Higher regularity of the inverse mean curvature flow, J. Differential Geom., 80 (2008), 433-451.
[14] N.V. Krylov, Nonlinear elliptic and parabolic equations of the second order, Dordrecht: Reidel, 1987
[15] K.-K. Kwong and P. Miao, A new monotone quantity along the inverse mean curvature flow in $\mathbb{R}^n$, arXiv:1212.1906
[16] H. Li, Y. Wei and C. Xiong, A geometric inequality on hypersurface in hyperbolic space, arXiv:1211.4109
[17] A. Neves, Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds, J. Diff. Geom. 84(2010), 191-229
[18] P. Petersen, Riemannian Geometry, GTM 171, 2nd edition, Springer-Verlag, New York, 2006.

Department of mathematical sciences, and Mathematical Sciences Center, Tsinghua University, 100084, Beijing, P. R. China
E-mail address: hli@math.tsinghua.edu.cn

Department of mathematical sciences, Tsinghua University, 100084, Beijing, P. R. China
E-mail address: wei-y09@mails.tsinghua.edu.cn