SOERGEL BIMODULES AND MATRIX FACTORIZATION

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Abstract. We establish an isomorphism between the Khovanov-Rozansky triply graded link homology and the geometric triply graded homology due to the authors. Hence we provide an interpretation of the Khovanov-Rozansky homology of the closure of a braid $\beta$ as the space of derived sections of a $C^* \times C^*$-equivariant sheaf $T r(\beta)$ on the Hilbert scheme $\text{Hilb}_n(C^2)$, thus proving a version of Gorsky-Negut-Rasmussen conjecture [GRN16]. As a consequence we prove that Khovanov-Rozansky homology of knots satisfies the $q \rightarrow t/q$ symmetry conjectured by Dunfield-Gukov-Rasmussen [DGR06]. We also apply our main result to compute the Khovanov-Rozansky homology of torus links.

1. Introduction

In this paper we explain why the knot homology theory developed in our previous work [OR18f], [OR18d], [OR18c], [OR17], [OR18a] produces the same link homology $\text{HHH}_{\text{alg}}$ as the homology $\text{HHH}_{\text{geo}}$ coming from the Soergel bimodule construction [KR08], [Kho07].

Theorem 1.0.1. For any braid $\beta \in \mathfrak{B}_n$ the homologies $\text{HHH}_{\text{alg}}(\beta)$ and $\text{HHH}_{\text{geo}}(\beta)$ are canonically isomorphic:

\begin{equation}
\text{HHH}_{\text{alg}}(\beta) \cong \text{HHH}_{\text{geo}}(\beta).
\end{equation}

The algebraic triply-graded homology $\text{HHH}_{\text{alg}}(\beta)$ of the closure $L(\beta) \subset \mathbb{R}^3$ defined in [KR08], [Kho07] is based on Rouquier’s construction of the homomorphism $\Phi_R$ from the braid group $\mathfrak{B}_n$ to the homotopy category $\text{Ho}(\text{SBim}_n)$ of complexes of Soergel bimodules. We remind the details of the construction in the second half of the introduction.

The geometric version of the homology is more recent and is constructed in the series of papers by the authors: [OR18f], [OR18c], [OR19]. We constructed a trace on the braid group $\mathfrak{B}_n$ with values in the category of two-periodic complexes of $T_{q,t} = (C^* \times C^*)$-equivariant coherent sheaves $D_{T_{q,t}}^\text{per}(\text{Hilb}_n(C^2))$ on the Hilbert scheme of points on the plane:

\begin{equation}
\tau r : \mathfrak{B}_n \rightarrow D_{T_{q,t}}^\text{per}(\text{Hilb}_n(C^2)), \quad \text{HHH}_{\text{geo}}(\beta) = \text{RHom}(\mathcal{O} \otimes \Lambda^* \mathcal{B}, \tau r(\beta)),
\end{equation}

where $\mathcal{B}$ is the vector bundle on $\text{Hilb}_n(C^2)$ which is dual to the tautological bundle. We proved that $\text{HHH}_{\text{geo}}(\beta)$ is an isotopy invariant of the closure $L(\beta)$.

In [OR19] we have shown that if the closure $L(\beta)$ of a braid $\beta$ is a knot then the complex of sheaves $\tau r(\beta)$ does not change if we switch the factors $\mathbb{C}$ in the product $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ underlying the Hilbert scheme.
Corollary 1.0.2. If the closure of $\beta$ is a knot then the homology $\text{HHH}_{\text{alg}}(\beta)$ is symmetric with respect to switching of homological and polynomial gradings:

$$\text{HHH}_{\text{alg}}(\beta) = \text{HHH}_{\text{alg}}(\beta)|_{q=t/q}.$$ 

Thus our main result implies the conjecture of [DGR06] which resisted an algebraic proof for almost fifteen years.

The existence of the geometric model (1.2) for link the Soergel bimodule-based link homology was [GRN16] and some special classes of braids in [ORS18, GORS14, GN15]. In particular, our result immediately imply the formula homology of torus links $T_{n,k}$ conjectured in the above mentioned papers (for purely algebraic proofs of these conjectures see [Hog17, Mel17]). The simplest version of the conjecture is in terms of punctual Hilbert scheme $\text{Hilb}_n(\mathbb{C}^2,0) \subset \text{Hilb}_n(\mathbb{C}^2)$:

Corollary 1.0.3. For any $n,k$ we have

$$(1-q^2) \cdot \text{HHH}_{\text{alg}}(T_{n,nk+1}) = H^0(\text{Hilb}_n(\mathbb{C}^2,0), \det(\mathcal{B})^k \otimes \Lambda(\mathcal{B})).$$

For a general braid $\beta$ the geometric answer is more complicated. However, we have the following property of the geometric trace functor [OR18a]:

$$\mathcal{T}r(\beta \cdot FT) = \mathcal{T}r(\beta) \otimes \det(\mathcal{B}),$$

where $FT$ is the full twist braid. In general, we hope that our result would allow to transfer geometric results of moduli spaces to the algebraic side. For example, it is natural to expect that the categorical version of the localization by Halpern-Leistner [HL14] transfers to the categorical diagonalization of Elias-Hogancamp [EH17].

In the next subsection we describe the monoidal category of stable matrix factorizations $\text{MF}^\text{st}_n$ that was used in [OR18a] to provide the geometric realization of the braid group $\mathfrak{B}_n$. The key step of our proof is the construction of an additive subcategory and a functor

$$\text{MF}^\text{b}_n \subset \text{MF}^\text{st}_n, \quad \mathcal{B} : \text{MF}^\text{b}_n \rightarrow \text{SBim}_n.$$

Theorem 1.0.4. For any $n$ the functor $\mathcal{B}$ is monoidal and fully-faithful.

Establishing the isomorphism between the algebraic and geometric homologies requires a refinement of this theorem. Namely, we have to relate derived homomorphisms on the Soergel side with the $\Lambda^*\mathcal{B}$ refinement of the homomorphism space on the matrix factorization side. In the next section we describe the refinement and outline our main argument.

1.1. Link homology from Soergel bimodules. Denote $x = x_1, \ldots, x_n$ and the same for $y$. The category of Soergel bimodules $\text{SBim}_n$ is a monoidal additive subcategory of $\mathbb{C}[x,y]$-modules that is a Karoubi envelope of tensor category generated by the standard Soergel bimodules $B_i$ [Soe01]. The standard bimodule $B_i$ is the quotient of $\mathbb{C}[x,y]$ modulo relations:

$$x_k = y_k, \quad k \neq i, i+1, \quad x_i^m + x_{i+1}^m = y_i^m + y_{i+1}^m, \quad m = 1, 2.$$ 

The category $\text{SBim}_n$ has the (derived) duality endo-functor $\vee$. The convolution of bimodules endows $D(\mathbb{C}[x,y] - \text{mod})$ with the monoidal structure, the unit object being
$B_1 = \mathbb{C}[x,y] / (y-x)$. The Hochschild homology of a bimodule $B$ is defined as $\text{HH}_*(B) = \text{Ext}^*_{\mathcal{E}}(B^c, B)$.

Soergel [Soe01] constructed a homomorphism

$$\Phi_S : \mathfrak{Br}_n^\varphi \rightarrow \mathcal{S}Bim_n$$

from the semi-group $\mathfrak{Br}_n^\varphi$ of flat braid-graphs to Soergel bimodules. In [Rou04] Rouquier extended this homomorphism from graphs to braids:

$$\Phi_R : \mathfrak{Br}_n \rightarrow \text{Ho}(\mathcal{S}Bim_n),$$

where $\mathfrak{Br}_n$ is the braid group and $\text{Ho}(\mathcal{S}Bim_n)$ is the homotopy category over $\mathcal{S}Bim_n$. The spaces of morphisms in $\text{Ho}(\mathcal{S}Bim_n)$ have three gradings: $q$-grading is the polynomial degree, $t$-grading is the homological degree of the outer (homotopy) category and $a$-grading is the homological degree of the inner (derived) category. The category $\text{Ho}(\mathcal{S}Bim_n)$ is equipped with the (derived) duality endo-functor $\vee$.

Modifying the construction of [KR08], Khovanov defined the triply graded homology of a braid $\beta \in \mathfrak{Br}_n$ as a vector space

$$(1.4) \quad \text{HHH}_{\text{alg}}(\beta) := \text{Hom}_{\text{Ho}(\mathcal{S}Bim_n)}(\Phi_R(1), \Phi_R(\beta)).$$

In other words, the space $\text{HHH}_{\text{alg}}(\beta)$ results from, first, replacing the Soergel bimodules in the Rouquier complex $\Phi_R(\beta)$ with their Hochschild homology and, second, taking the homology of the resulting complex:

$$\text{HHH}_{\text{alg}}(\beta) := \text{H}(\text{HH}(\Phi_R(\beta))).$$

Denote $L(\beta)$ the link constructed by closing a braid $\beta$. Khovanov proved that $\text{HHH}_{\text{alg}}(\beta)$ is invariant under Markov moves, hence the triply graded homology is the isotopy invariant of $L(\beta)$: $\text{HHH}_{\text{alg}}(L(\beta)) := \text{HHH}_{\text{alg}}(\beta)$.

1.2. Link homology from Hilbert schemes.

1.2.1. Matrix factorizations. In our recent work [OR18] we construct another triply graded link homology by using a different homomorphism. Denote $\mu : \mathfrak{gl}_n \rightarrow \mathfrak{gl}_n$ the Grothendieck resolution. Explicitly, $\mathfrak{gl}_n = (\text{GL}_n \times \mathfrak{b}) / \mathfrak{b}$, where $\mathfrak{b} \subset \mathfrak{gl}_n$ is the Borel subalgebra of upper-triangular matrices, $B \subset \text{GL}_n$ is the Borel subgroup of upper-triangular matrices, the action of $B$ on $\text{GL}_n \times \mathfrak{b}$ is $h \cdot (g,Y) = (gh^{-1}, \text{Ad}_h Y)$ and $\mu(g,Y) = \text{Ad}_{\beta} Y$.

Denote $V_n = \mathbb{C}^n$ the defining representation of $\text{GL}_n$. A triple $(X,Y,v) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \times V_n$ is called stable if $C\langle X,Y \rangle v = V_n$. Define

$$(1.5) \quad (\mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathfrak{gl}_n \times V_n)^{st}$$

$$= \{(X,z_1,z_2,v) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathfrak{gl}_n \times V_n \mid (X,\mu(z_1),v), (X,\mu(z_2),v) \text{ are stable}\}.$$ 

Define the polynomial $W \in \mathbb{C}[\mathfrak{gl}_n \times \mathfrak{gl}_n]$ by the formula $W_{FL}(X,z) = \text{Tr}(X \mu(z))$. In [OR18] we consider the category

$$\text{MF}^{st}_n := \text{MF}_{\text{GL}_n}((\mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathfrak{gl}_n \times V_n)^{st}; W_{FL}(X,z_2) - W_{FL}(X,z_1)),$$
which is equipped with the duality endo-functor $\vee$. We define two homomorphisms

\begin{equation}
\Phi^5 : \mathcal{B}r_n^5 \to \text{MF}^\text{st}_n, \quad \Phi : \mathcal{B}r_n \to \text{MF}^\text{st}_n
\end{equation}

and a triply graded homology $\text{HHH}_{\text{geo}}$ for graphs $\gamma \in \mathcal{B}r_n^5$ and for braids $\beta \in \mathcal{B}r_n$ by similar formulas:

\begin{align}
\text{HHH}_{\text{geo}}(\gamma) & = \text{Hom}_{\text{MF}^\text{st}_n}(\Phi^5(\mathbb{1})^\vee, \Phi^5(\beta) \otimes \Lambda^* V_n), \\
\text{HHH}_{\text{geo}}(\beta) & = \text{Hom}_{\text{MF}^\text{st}_n}(\Phi(\mathbb{1})^\vee, \Phi(\beta) \otimes \Lambda^* V_n),
\end{align}

(note that by our definition, $\Phi^5(\mathbb{1}) = \Phi(\mathbb{1})$). We proved that $\text{HHH}_{\text{geo}}$ is invariant under the Markov moves, thus defining a triply graded link homology $\text{HHH}_{\text{geo}}(L(\beta)) := \text{HHH}_{\text{geo}}(\beta)$, for links presented as braid closures.

1.2.2. Sheaves on a Hilbert scheme. In order to see the indirect relation between the category $\text{MF}^\text{st}_n$ and the Hilbert scheme $\text{Hilb}_n(\mathbb{C}^2)$, define $\text{Hilb}_n(\mathbb{C}^2)$ as the quotient

\begin{equation}
\text{Hilb}_n(\mathbb{C}^2) = \{(X, Y, v) \in (\mathfrak{gl}_n \times \mathfrak{gl}_n \times V_n)^\text{st} \mid [X, Y] = 0\}/\text{GL}_n.
\end{equation}

Then the matrix $X$ and the vector $v$ from (1.5) resemble $X$ and $v$ of (1.9), whereas $\mu(z_1)$ and $\mu(z_2)$ resemble $Y$ of (1.5).

In [OR18c] we presented an alternative construction of the link homology $\text{HHH}_{\text{geo}}(\beta)$ in which $\text{Hilb}_n(\mathbb{C}^2)$ appears explicitly. We defined the Chern character functor $\text{CH} : \text{MF}^\text{st}_n \to \text{D}^\text{per}(\text{Hilb}_n(\mathbb{C}^2))$

and proved the relation

$$\text{HHH}_{\text{geo}}(\beta) = \text{RHom}_{\text{D}^\text{per}(\text{Hilb}_n(\mathbb{C}^2))} \left( \mathcal{O} \otimes \Lambda^* \mathcal{B}, \text{CH}(\Phi(\beta)) \right),$$

where $\mathcal{B}$ is the dual of the tautological vector bundle on $\text{Hilb}_n(\mathbb{C}^2)$. In other words, the functor $\mathcal{T} \tau$ of (1.2) appears as the composition of the second homomorphism of (1.6) and the Chern character functor:

$$\mathcal{T} \tau = \text{CH} \circ \Phi.$$

1.3. Equivalences. The main result of this paper is the (partly canonical) isomorphism (1.11) between both link homologies. This isomorphism originates from a special functor

\begin{equation}
\mathcal{B} : \text{MF}^\text{st}_n \to \text{SBim}_n
\end{equation}

defined as a composition of a pull-back and a push-forward. Namely, consider a projection $\lambda$ from the Borel subalgebra $\mathfrak{b}$ to the Cartan subalgebra $\mathfrak{h}$:

$$\lambda : \mathfrak{b} \to \mathfrak{h} = \mathbb{C}^n, \quad Y \mapsto (Y_{11}, \ldots, Y_{nn}).$$

There is an associate map $\tilde{\lambda} : \tilde{\mathfrak{gl}}_n \to \mathbb{C}^n$, $\tilde{\lambda}(g, Y) = \lambda(Y)$. Consider two maps

$$\begin{array}{ccc}
(g \mathfrak{l}_n \times \tilde{\mathfrak{gl}}_n \times V_n)^\text{st} & \xrightarrow{\text{res}} & (\tilde{\mathfrak{gl}}_n \times \tilde{\mathfrak{gl}}_n \times \tilde{\mathfrak{gl}}_n \times V_n)^\text{st} \\
& & \xrightarrow{\pi_B} & \mathbb{C}^n \times \mathbb{C}^n
\end{array}$$
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where

$$\text{res}(z_1, z_2, v) = (0, z_1, z_2, v), \quad \pi_b(z_1, z_2, v) = (\tilde{\lambda}(z_1), \tilde{\lambda}(z_2)).$$

Thus we define the functor $B$ of (1.10) as a composition of a pull-back and a push-forward:

$$B = \pi_b^* \circ \text{res}^*.$$

In section 3 we construct a monoidal additive subcategory $MF^\beta_n$ and monoidal functor as image of the homomorphism:

$$\Phi^\beta : \mathcal{B}r^\beta_n \rightarrow MF^\beta_n.$$

**Theorem 1.3.1.**

For any $n$ we have

1. The functor $B$ is monoidal.
2. The functor $B$ intertwines the homomorphisms $\Phi_S$ and $\Phi^\beta$:

$$\Phi^\beta \downarrow \mathcal{B}r^\beta_n \quad \Phi_S \downarrow \mathcal{B}r^\beta_n \quad \mathcal{B} \quad MF^\text{st}_n \supset MF^\beta_n \quad \mathcal{B} \quad MF^\beta_n \rightarrow \text{SBim}_n$$

3. The functor $B$ converts the exterior power of $V_n$ into the homological degree of the derived category $D(\mathbb{C}[x, y] - \text{mod})$: for any two graphs $\gamma_1, \gamma_2 \in \mathcal{B}r^\beta_n$

$$\text{Hom}_{MF^\text{st}_n}(\Phi^\beta(\gamma_1), \Phi^\beta(\gamma_2) \otimes \Lambda^\gamma V_n) \cong \text{Ext}_{D(\mathbb{C}[x, y] - \text{mod})}^\ast(\Phi_S(\gamma_1), \Phi_S(\gamma_2))$$

4. The functor $B$ intertwines duality endo-functors up to a ‘twisting’:

$$B \left( \Phi^\beta(\gamma) \otimes \text{det} V_n \right) = \Phi_S(\gamma)^\gamma [n]_a.$$

Since $\text{det} V_n \otimes \Lambda^\gamma V_n = \Lambda^{n-\gamma} V_n$, this theorem implies the isomorphism between homologies of graph closures:

**Corollary 1.3.2.** For any graph $\gamma \in \mathcal{B}r^\beta_n$ the functor $B$ induces an isomorphism

$$\text{HHH}^\text{alg}(\gamma) \cong \text{HHH}_\text{geo}(\gamma).$$

It remains to extend the isomorphism (1.11) from graphs to braids. Recall the Murakami-Ohtsuki-Yamada construction [MOY98] presenting a braid $\beta$ (considered as an element of the Hecke algebra) as a (weighted) alternating sum of associated braid graphs:

$$\beta = \sum_{\gamma \in \mathcal{B}r^\beta(\beta)} (-1)^{s(\beta, \gamma)} \gamma,$$

where $s(\beta, \gamma)$ is a $\mathbb{Z}$-valued function and we ignore the powers of $q$ for simplicity. Categorifying this formula, Rouquier [Rou04] represented a braid $\beta$ by a complex $\Phi_R(\beta)$ of Soergel bimodules $\Phi_S(\gamma)$:

$$\Phi(\beta) = \bigoplus_{\gamma \in \mathcal{B}r^\beta(\beta)} \Phi_S(\gamma), d_\text{alg}.$$
We show that the image $\Phi(\beta)$ of the braid $\beta$ admits a similar presentation within the geometric construction as a complex of sheaves associated with graphs:

$$\Phi(\beta) = \left( \bigoplus_{\gamma \in \mathfrak{Br}_5^\ell(\beta)} \Phi^\beta(\gamma), d_{\text{geo}} \right)$$

while the functor $\mathcal{B}$ intertwines the differentials:

$$\bigoplus_{\gamma \in \mathfrak{Br}_5^\ell(\beta)} \Phi^\beta(\gamma) \xrightarrow{d_{\text{geo}}} \bigoplus_{\gamma \in \mathfrak{Br}_5^\ell(\beta)} \Phi^\beta(\gamma)$$

In order to relate homologies $\text{HHH}_{\text{alg}}^\beta(\beta)$ and $\text{HHH}_{\text{geo}}^\beta(\beta)$ we substitute the presentations (1.12) and (1.13) into the formulas (1.4) and (1.8). The difference between the computation of resulting homologies is that the cone of $\Phi^\beta_R(\beta)$ is in the outer homotopy category, hence the homology of $d_{\text{alg}}$ is computed after the Hochschild homology is applied to the bimodules $\Phi_S(\gamma)$, whereas the cone of $\Phi^\beta(\beta)$ is in the same matrix factorization category $\text{MF}^\ast_n$, so the differential $d_{\text{geo}}$ is added to the matrix factorization differentials $D_\gamma$ of the matrix factorizations $\Phi^\beta(\gamma)$. As a result, the homology in (1.8) is computed with respect to the total differential $d_{\text{geo}} + \sum D_\gamma$. However, this homology can be computed by spectral sequence, first taking homology with respect to $\sum D_\gamma$ (which matches the Hochschild homology), then taking the homology with respect to $d_{\text{geo}}$ (which matches $\text{HHH}_{\text{alg}}^\beta(\beta)$) and, finally, taking homology with respect to secondary differentials. The counting of $t$-degree indicates that the spectral sequence converges in the second term, and this implies the isomorphism $\text{HHH}_{\text{geo}}(\beta) \cong \text{HHH}_{\text{alg}}(\beta)$.

1.4. Previous work. The relation between the knot homology $\text{HHH}_{\text{alg}}^\beta(\beta)$, $\beta \in \mathfrak{Br}_n$ and the sheaves on the Hilbert scheme of points on the plane $\text{Hilb}_n(\mathbb{C}^2)$ was proposed in various forms by several groups. Here we briefly outline their contributions to the problem.

The physics prospective on the question originates in the work of Aganagic and Shakirov [AS12], the authors proposed formulas for the homology of torus knots in terms of Macdonald polynomials. Their motivation comes from the fact that the compliment to the torus knot has a $U(1)$-symmetry thus one can compute the index of the usual Chern-Simons theory with respect to this action and upgrade the HOMFLYPT polynomial to the super polynomial. Using the TQFT perspective on Chern-Simons theory they compute the index in terms of $\text{SL}_2(\mathbb{Z})$ action from [Kir96]. The last work explains the connection with the Macdonald polynomials.

The work of Haiman [Hai02a] provides a bridge between the theory of Macdonald polynomials and $K$-theory of the Hilbert schemes on the plane. Gorsky and Negut [GN15] exploited this connection together some more recent developments in the theory of elliptic Hall algebras in order to provide a geometric set of conjectures that imply the formulas of [AS12]. More precisely, they constructed the classes $S_{n,k}$ in $\mathbb{C}^* \times \mathbb{C}^*$-equivariant $K$-theory $K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}_n(\mathbb{C}^2))$ such that the $\mathbb{C}^* \times \mathbb{C}^*$ Euler characteristic of $S_{n,k} \otimes \Lambda^* \mathcal{B}$ reproduces the
stable limit of the Aganagic-Shakirov formulas. The more recent papers by Elias, Mellit and Hogancamp [EH16], [Mel17], [Hog17] provide an algebraic proof of the conjectures.

From another perspective, a torus knot $T_{n,k}$ is a link of the singularity of the toric curve $x^n = y^k$. The relation between a Hilbert scheme of points on a singular curve and knot invariants was studied first in [GZDC99] and developed further in [OS12], [ORS18]. In the latter work the isomorphism between the homology of the Hilbert scheme of points on the curve $x^n = y^{1+kn}$ and the space of sections of $\text{det}(B)^k$ on $\text{Hilb}_n(\mathbb{C}^2, 0)$ was observed.

On the other hand, the earlier conjecture of Gorsky [Gor12] on the Catalan numbers and super-polynomials of $T_{n,n}$ combined the geometric constructions of the modules over the rational Cherednik algebra [GS06], [OY16] led to work [GORST14] in which the statement of theorem 1.0.3 was presented and motivated.

The first precise conjecture relating $\text{HHH}_{\text{alg}}$ to $K$-theory of coherent sheaves on $\text{Hilb}_n(\mathbb{C}^2)$ appears in the work of Gorsky, Negut and Rasmussen [GRN16]. The authors proposed a conjecture that relates the knot homology to the $K$-theory of the flag Hilbert scheme $F\text{Hilb}_n(\mathbb{C}^2)$. The conjecture from [GRN16] identifies the Picard group of the flag Hilbert scheme with the Jucys-Murphy commutative subgroup of $\text{Br}_n$.

In [OR18f] and [OR18c] we constructed a link homology $\text{HHH}_{\text{geo}}$ which is directly related to the Hilbert scheme $\text{Hilb}_n(\mathbb{C}^2)$ as well as to the flag Hilbert scheme. The invariant $\text{HHH}_{\text{geo}}$ has most of the properties predicted in [GRN16] but initial it was not clear why in would be related to $\text{HHH}_{\text{alg}}$. This paper provides a proof the equality between the two link homologies.

Finally, for a braid $\beta$, Gorsky and Hogancamp [GH17] construct a complex of sheaves on the isospectral Hilbert scheme with the property that there is a spectral sequence connecting its homology and $\text{HHH}_{\text{alg}}(\beta)$.

1.5. The structure of the paper. In section 2 we recall the definition of the category of equivariant matrix factorizations from [OR18f]. We also explain our construction of the equivariant push-forward for matrix factorizations and explain basic properties of the push-forward. In section 3 we provide the details of the construction of our main functor $B$. We prove that $B$ is monoidal and intertwines the induction functors. Also in this section the value of $B$ of the elementary braid graph is computed.

In section 4 we prove the MOY relations in the category of matrix factorizations that correspond to the braid relations. In section 5 we prove the MOY relations that correspond to the strand removing Markov moves. The MOY relations allow us to use Hao Wu induction strategy [Wu08] in section 6 where we prove our main categorical theorem 1.3. In the same section we show how the categorical theorem implies the comparison theorem 1.0.1. Finally, in section 7 we prove corollaries that we discuss in the introduction. We also state conjectures suggested by our results.

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2. Preliminaries on matrix factorizations

The goal of this section is to collect the definitions and results on matrix factorizations that are used in the main body of the paper.

Let $Z$ be an affine manifold and $W \in \mathbb{C}[Z] = R$ then a matrix factorization [Eis80] is a pair

$$M = M_1 \oplus M_2, \quad D \in \text{Hom}(M_1, M_0) \oplus \text{Hom}(M_0, M_1), \quad D^2 = W,$$

where we assume that $M_i$ are free $R$-modules.

Given $F = (M, D)$ and $G = (M', D')$ the space morphisms $\text{Hom}(F, G)$ consists of $\mathbb{Z}_2$ graded $R$-linear maps $\phi \in \text{Hom}_R(M, M')$ that respects the differentials:

$$D' \circ \phi = \phi \circ D.$$

A homotopy $h$ between the morphisms $\phi, \psi \in \text{Hom}_R(M, M')$ shifts the $\mathbb{Z}_2$ grading by 1 and $\phi - \psi = D' \circ h - h \circ D$. Thus the matrix factorizations are elements of the homotopy category

$$\text{MF}(Z, W)$$

and it was shown by Orlov [Orl04] that this category is triangulated.

There is a natural generalization of the category of matrix factorizations to the case when $X$ is a quasi-projective manifold or even Artin stack [PV11]. However, in our theory the spaces that host matrix factorizations are usually quotients of affine manifolds by group actions. Hence we prefer to work with equivariant matrix factorizations which were introduced in [OR18f].

Before we define the First let us remind the construction of the Chevalley-Eilenberg complex.

2.1. Chevalley-Eilenberg complex. Suppose that $\mathfrak{h}$ is a Lie algebra. Chevalley-Eilenberg complex $\text{CE}_\mathfrak{h}$ is the complex $(V_\bullet(\mathfrak{h}), d)$ with $V_p(\mathfrak{h}) = U(\mathfrak{h}) \otimes_\mathbb{C} \Lambda^p \mathfrak{h}$ and differential $d_{ce} = d_1 + d_2$ where:

$$d_1(u \otimes x_1 \wedge \cdots \wedge x_p) = \sum_{i=1}^{p} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p,$$

$$d_2(u \otimes x_1 \wedge \cdots \wedge x_p) = \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p,$$

Let us denote by $\Delta$ the standard map $\mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$ defined by $x \mapsto x \otimes 1 + 1 \otimes x$. Suppose $V$ and $W$ are modules over the Lie algebra $\mathfrak{h}$ then we use notation $V \overset{\Delta}{\otimes} W$ for the $\mathfrak{h}$-module which is isomorphic to $V \otimes W$ as a vector space, the $\mathfrak{h}$-module structure being defined by $\Delta$. Respectively, for a given $\mathfrak{h}$-equivariant matrix factorization $F = (M, D)$ we denote by $\text{CE}_\mathfrak{h} \overset{\Delta}{\otimes} F$ the $\mathfrak{h}$-equivariant matrix factorization $(\text{CE}_\mathfrak{h} \overset{\Delta}{\otimes} F, D + d_{ce})$. The $\mathfrak{h}$-equivariant
structure on $\text{CE}_h \otimes \mathcal{F}$ originates from the left action of $U(\mathfrak{h})$ that commutes with right action on $U(\mathfrak{h})$ used in the construction of $\text{CE}_h$.

A slight modification of the standard fact that $\text{CE}_h$ is the resolution of the trivial module implies that $\text{CE}_h \otimes M$ is a free resolution of the $\mathfrak{h}$-module $M$.

### 2.2. Equivariant matrix factorizations

Let us assume that there is an action of the Lie algebra $\mathfrak{h}$ on $\mathcal{Z}$ and $F$ is a $\mathfrak{h}$-invariant function. Then we can construct the following triangulated category $\text{MF}_h(\mathcal{Z}, W)$.

The objects of the category are triples:

$$\mathcal{F} = (M, D, \partial), \quad (M, D) \in \text{MF}(\mathcal{Z}, W)$$

where $M = M^0 \oplus M^1$ and $M^i = \mathbb{C}[\mathcal{Z}] \otimes V^i, V^i \in \text{Mod}_{\mathfrak{h}}, \partial \in \bigoplus_{i > j} \text{Hom}_{\mathbb{C}[\mathcal{Z}]}(\Lambda^i \mathfrak{h} \otimes M, \Lambda^j \mathfrak{h} \otimes M)$ and $D$ is an odd endomorphism $D \in \text{Hom}_{\mathbb{C}[\mathcal{Z}]}(M, M)$ such that

$$D^2 = F, \quad D_{\text{tot}}^2 = F, \quad D_{\text{tot}} = D + d_{\text{ce}} + \partial,$$

where the total differential $D_{\text{tot}}$ is an endomorphism of $\text{CE}_h \otimes M$, that commutes with the $U(\mathfrak{h})$-action.

Note that we do not impose the equivariance condition on the differential $D$ in our definition of matrix factorizations. On the other hand, if $\mathcal{F} = (M, D) \in \text{MF}(\mathcal{Z}, F)$ is a matrix factorization with $D$ that commutes with $\mathfrak{h}$-action on $M$ then $(M, D, 0) \in \text{MF}_h(\mathcal{Z}, F)$.

Given two $\mathfrak{h}$-equivariant matrix factorizations $\mathcal{F} = (M, D, \partial)$ and $\tilde{\mathcal{F}} = (\tilde{M}, \tilde{D}, \tilde{\partial})$ the space of morphisms $\text{Hom}(\mathcal{F}, \tilde{\mathcal{F}})$ consists of homotopy equivalence classes of elements $\Psi \in \text{Hom}_{\mathbb{C}[\mathcal{Z}]}(\text{CE}_h \otimes M, \text{CE}_h \otimes \tilde{M})$ such that $\Psi \circ D_{\text{tot}} = \tilde{D}_{\text{tot}} \circ \Psi$ and $\Psi$ commutes with $U(\mathfrak{h})$-action on $\text{CE}_h \otimes M$. Two maps $\Psi, \Psi' \in \text{Hom}(\mathcal{F}, \tilde{\mathcal{F}})$ are homotopy equivalent if there is

$$\eta \in \text{Hom}_{\mathbb{C}[\mathcal{Z}]}(\text{CE}_h \otimes M, \text{CE}_h \otimes \tilde{M})$$

such that $\Psi - \Psi' = \tilde{D}_{\text{tot}} \circ \eta - \eta \circ D_{\text{tot}}$ and $\eta$ commutes with $U(\mathfrak{h})$-action on $\text{CE}_h \otimes M$.

Given two $\mathfrak{h}$-equivariant matrix factorizations $\mathcal{F} = (M, D, \partial) \in \text{MF}_h(\mathcal{Z}, F)$ and $\tilde{\mathcal{F}} = (\tilde{M}, \tilde{D}, \tilde{\partial}) \in \text{MF}_h(\mathcal{Z}, \tilde{F})$ we define $\mathcal{F} \otimes \tilde{\mathcal{F}} \in \text{MF}_h(\mathcal{Z}, F + \tilde{F})$ as the equivariant matrix factorization $(M \otimes \tilde{M}, D + \tilde{D}, \partial + \tilde{\partial})$.

### 2.3. Push forwards, quotient by the group action

The technical part of [OR18f] is the construction of push-forwards of equivariant matrix factorizations. Here we state the main results, the details may be found in section 3 of [OR18f]. We need push forwards along projections and embeddings. We also use the functor of taking quotient by group action for our definition of the convolution algebra.

The projection case is more elementary. Suppose we have a smooth projection $\pi : \mathcal{Z} \to \mathcal{X}$ both $\mathcal{Z}$ and $\mathcal{X}$ have $\mathfrak{h}$-action and the map $\pi : \mathcal{Z} \to \mathcal{X}$ is $\mathfrak{h}$-equivariant. Then for any $\mathfrak{h}$ invariant element $w \in \mathbb{C}[\mathcal{X}]^\mathfrak{h}$ there is a functor

$$\pi_* : \text{MF}_h(\mathcal{Z}, \pi^*(w)) \to \text{MF}_h(\mathcal{X}, w)$$
which is induced by the push-forwards of the modules underlying the matrix factorizations.

We define an embedding-related push-forward in the case when the subvariety $Z_0 \hookrightarrow Z$ is the common zero of an ideal $I = (f_1, \ldots, f_n)$ such that the functions $f_i \in \mathbb{C}[Z]$ form a regular sequence. We assume that the Lie algebra $\mathfrak{h}$ acts on $Z$ and $I$ is $\mathfrak{h}$-invariant. Then there exists an $\mathfrak{h}$-equivariant Koszul complex $K(I) = (\Lambda^* \mathbb{C}^n \otimes \mathbb{C}[Z], d_K)$ over $\mathbb{C}[Z]$ which has non-trivial homology only in degree zero. Then in section 3 of [OR18f] we define the push-forward functor

$$j_* : \text{MF}_b(Z_0, W|_{Z_0}) \rightarrow \text{MF}_b(Z, W),$$

for any $\mathfrak{h}$-invariant element $W \in \mathbb{C}[Z]^\mathfrak{h}$.

Finally, let us discuss the quotient map. The complex $\text{CE}_b$ is a resolution of the trivial $\mathfrak{h}$-module by free modules. Thus the correct derived version of taking $\mathfrak{h}$-invariant part of the matrix factorization $F = (M, D, \partial) \in \text{MF}_b(Z, W)$, $W \in \mathbb{C}[Z]^\mathfrak{h}$ is

$$\text{CE}_b(F) := (\text{CE}_b(M), D + d_{ce} + \partial) \in \text{MF}(Z/H, W),$$

where $Z/H := \text{Spec}(\mathbb{C}[Z]^\mathfrak{h})$ and use the general definition of $\mathfrak{h}$-module $V$:

$$\text{CE}_b(V) := \text{Hom}_b(\text{CE}_b, \text{CE}_b \Delta V).$$

2.4. **Base change.** Unlike push-forward functor, the pull-back of matrix factorizations is defined for any regular map $f : \mathcal{X} \rightarrow Z$. If we assume that both $\mathcal{X}, Z$ have $\mathfrak{h}$ action and $f$ is $\mathfrak{h}$-equivariant then pull-back of the $\mathbb{C}[Z]$ modules induces the functor

$$f^* : \text{MF}_b(Z, W) \rightarrow \text{MF}_b(\mathcal{X}, W).$$

Just as in the case of coherent sheaves we have the smooth base change isomorphism. In more details, suppose we have affine manifolds $\mathcal{X}, \mathcal{X}', \mathcal{S}, \mathcal{S}$ and the corresponding potentials $W_{\mathcal{X}}, W_{\mathcal{X}'}, W_{\mathcal{S}}, W_{\mathcal{S}'}$ that fit into the commuting diagrams:

$$\begin{align*}
\mathcal{X}' & \xrightarrow{g'} \mathcal{X} \\
\downarrow f' & \quad \downarrow f \\
\mathcal{S}' & \xrightarrow{g} \mathcal{S}
\end{align*}$$

$$\begin{align*}
\text{MF}(\mathcal{X}', W_{\mathcal{X}'}) & \xleftarrow{g^*} \text{MF}(\mathcal{X}, W_{\mathcal{X}}) \\
\downarrow f'^* & \quad \downarrow f^* \\
\text{MF}(\mathcal{S}', W_{\mathcal{S}'}) & \xleftarrow{g^*} \text{MF}(\mathcal{S}, W_{\mathcal{S}})
\end{align*}$$

If $g$ is a flat map then we have an natural transformation that identifies the functors:

$$g^* \circ f_* = f'_* \circ g^*.$$

The identity also holds in the equivariant setting.

3. **Construction of the functor**

In this section we recall a construction of the functor:

$$B : \text{MF}_n \rightarrow \text{D}^\text{per}(\mathbb{C}^n \times \mathbb{C}^n).$$

and prove that the functor is monoidal and compute the functor on the some important collection of matrix factorizations. The matrix factorization that we study are the key ingredients for our braid group realization from [OR18f].
From now on we fix the conventions for the most used groups and Lie algebras:

\[ G_n = GL_n, \quad \mathfrak{g}_n = \mathfrak{gl}_n, \quad \mathfrak{h}_n \subset \mathfrak{n}_n \subset \mathfrak{b}_n \subset \mathfrak{g}_n. \]

When the rank of the group is clear from the context we suppress the lower index.

3.1. **Motivation and the main definition.** In our previous work [OR18b] we introduced the three-category \( \text{Cat}_{\mathfrak{gl}_n} \), the objects in the category are labeled by the positive integers. The two-category of morphisms has objects:

\[ \text{Obj} \hom(p \mathfrak{Z}, W) \]

The composition of the morphisms is described in the cited paper. The one-category of morphisms between \( p \mathfrak{Z}_1, W_1 q \) and \( p \mathfrak{Z}_2, W_2 q \) has objects:

\[ \text{Obj} \hom(p \mathfrak{Z}_1, W_1 q, p \mathfrak{Z}_2, W_2 q) = \text{MF}_{G_n \times G_m}(\mathfrak{g}_1 \times Z_1 \times Z_2 \times \mathfrak{g}_m, W_2 - W_1). \]

There are two particular objects that play central role in our constructions:

\[ \text{A}_n = (\mathfrak{g}_n, W_{F1}), \quad \text{O}_n = ((\mathfrak{g}_n \times \mathbb{C}^n)^{st}, W_{pt}) \in \hom(n, 0), \]

\[ \text{W}_{F1}(X, g, Y) = \text{Tr}(X \text{Ad}_g Y), \quad X \in \mathfrak{g}_n, \quad Y \in \mathfrak{b}_n, \quad g \in G G_n, \]

\[ \text{W}_{pt}(X, Y) = \text{Tr}(XY), \quad X, Y \in \mathfrak{g}_n, \]

here we use the identification \( \mathfrak{g}_n = G_n \times \mathfrak{b} / B \) and \( (\mathfrak{g}_n \times \mathbb{C}^n)^{st} \) stands for the stable sublocus that consists of the pairs \( (Y, v) \) such that \( \mathbb{C}[Y]v = \mathbb{C}^n \).

The one-category of the endomorphisms of the object \( \text{A}_n \) is a dg category of the equivariant matrix factorizations:

\[ \text{Hom}(\text{A}_n, \text{A}_n) = \text{MF}_{G_n \times B^n}(\mathfrak{g}_n \times (G_n \times \mathfrak{b})^2, W), \]

\[ \text{W}(X, g_1, Y_1, g_2, Y_2) = \text{Tr}(X (\text{Ad}_{g_1} Y_1 - \text{Ad}_{g_2} Y_2)), \]

we abbreviate the notation for this category by \( \text{MF}_n \).

The composition of elements of \( \text{Hom}(\text{A}_n, \text{A}_n) \) endows the category \( \text{MF}_n \) with the monoidal structure which we denote by \( * \) to keep our notations consistent with [OR18f]. In the last mentioned paper we constructed a group homomorphism from the braid group

\[ \Phi : \mathcal{B} \rightarrow (\text{MF}_n, *). \]

Thus we have an action of the braid group on category \( \text{Hom}(\text{A}_n, \text{O}_n) \), that is for every element \( \beta \in \mathcal{B} \) we obtain a functor:

\[ \Phi_\beta : \text{Hom}(\text{A}_n, \text{O}_n) \rightarrow \text{Hom}(\text{A}_n, \text{O}_n). \]

The category \( \text{Hom}(\text{A}_n, \text{O}_n) = \text{MF}_{G_n \times B_n}(\mathfrak{g}_n \times G_n \times \mathfrak{b}_n \times \mathfrak{b}_n, W_A - W_O) \) has a simpler model:

**Proposition 3.1.1.** For any \( n \) we have an isomorphism of the linear categories:

\[ \text{Hom}(\text{A}_n, \text{O}_n) = \text{D}_{\text{per}}(\mathbb{C}^n). \]
Proof. We use Knörrer periodicity, the potential in our matrix factorizations is quadratic:

\[ W_{Fl} - W_{pt} = \text{Tr}(X(Y_1 - \text{Ad}_g Y_2)). \]

Thus we conclude that

\[ \text{Hom}(A_n, O_n) = D^\text{per}_{G_n \times B_n}((G_n \times b \times \mathbb{C}^n)^{st}) = D^\text{per}_{B_n}((b \times \mathbb{C}^n)^{st}). \]

To complete our argument we observe that we have a natural slice to the \( B_n \)-action:

\[
\kappa: \mathbb{C}^n \to (b \times \mathbb{C}^n)^{st}, \quad \kappa(y) = (Y(y), e_n),
\]

\[
Y(y)_{ij} = \delta_{i-j} y_i + \delta_{j-i-1}.
\]

Since all points of the slice have a trivial stabilizer the statement follows. \( \square \)

Let us emphasize that we choose double grading of the variable as

\[
\deg(Y_1)_{ij} = \deg(Y_2)_{ij} = q^2, \quad \deg(X_{ij}) = t^2 q^{-2}.
\]

Thus a general theorem implies that for a braid \( \beta \in \mathfrak{B}_n \) we have a two-periodic complex of bimodules \( C_\beta \) that is a Fourier-Mukai kernel for \( \Phi_\beta \). Let us denote by \( \mathcal{B}\text{im} \) the abelian category of the \( R_n \)-bimodules, \( R_n = \mathbb{C}[y_1, \ldots, y_n] \). Respectively, we have constructed the functor of the dg categories:

\[ \mathcal{B}: \text{MF}_n \to D^\text{per}(\mathbb{C}^n \times \mathbb{C}^n). \]

The argument of the previous proposition implies an elementary description of the above functor. The Knorrrer periodicity used in the proof amounts to the pull-back along the embedding:

\[ \text{res}: G_n \times b \times G_n \times b \rightarrow \mathfrak{g} \times G_n \times b \times G_n \times b, \]

that is induced by the embedding of the zero into the Lie algebra \( \mathfrak{g}_n \).

The stability condition in the definition of \( O_n \) translates in the stability condition in the definition below:

\[
\left( G_n \times b \times G_n \times b \times \mathbb{C}^n \right)^{st} = \{(g_1, Y_1, g_2, Y_2, v) | \mathbb{C}[\text{Ad}_g Y]v = \mathbb{C}^n\}.
\]

Respectively, the inclusion of 0 inside \( \mathfrak{g} \) combined with the projection along \( \mathbb{C}^n \) gives us map:

\[ \text{res}: \left( G_n \times b \times G_n \times b \times \mathbb{C}^n \right)^{st} \rightarrow \mathfrak{g} \times G_n \times b \times G_n \times b. \]

There is a natural projection from \( b_n \) to \( h_n \) that extracts the diagonal part of the matrix. In the introduction we fixed notation \( \lambda \) for this map.

With these notations we define the \( G_n \times B_n \times B_n \) equivariant map:

\[ \pi_h: \left( G_n \times b \times G_n \times b \times \mathbb{C}^n \right)^{st} \rightarrow h_n \times h_n, \quad (g_1, Y_1, g_2, Y_2, v) \mapsto (h(Y_1), h(Y_2)). \]

As we mentioned in the introduction, the functor from above can be constructed in terms of \( \text{res} \) and \( \pi_h \) as follows. For any \( C \in \text{MF}_n \) we have

\[ \mathcal{B}(C) = \text{CE}_{\mathbb{Z}^2}(\pi_h^* \circ \text{res}^*(C))^{T^2 \times G_n}. \]
From now on we use the last formula as definition of the functor and in the rest of the paper we do not use any TQFT results from [OR18a].

3.2. Monoidal properties. Our motivational construction of functor B is strongly suggests that the functor is monoidal. The monoidal structure on $D^{perv}(\mathbb{C}^n \times \mathbb{C}^n)$ is an extension of the monoidal structure on the $R_n = \mathbb{C}[x_1, \ldots, x_n]$-bimodules:

$$B_1 \ast B_2 = B_1 \otimes_{R_n} B_2.$$ 

We show that this monoidal structure is compatible with our functor $B$.

**Proposition 3.2.1.** For any $\mathcal{C}_1, \mathcal{C}_2 \in MF_n$ we have:

$$B(\mathcal{C}_1 \ast \mathcal{C}_2) = B(\mathcal{C}_1) \ast B(\mathcal{C}_2).$$

**Proof.** Let us recall that the convolution for the matrix factorizations is defined with push-pull construction that evolves the space:

$$\mathcal{X}_{\text{con}} = \mathfrak{g} \times (G_n \times \mathfrak{b})^3$$

and the $G_n \times B^3$-equivariant projections

$$\pi_{ij} : \mathcal{X}_{\text{con}} \to \mathcal{X}_n.$$ 

The convolution product of two matrix factorizations $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined as

$$\mathcal{C}_1 \ast \mathcal{C}_2 = \pi_{13*}(\text{CE}_{n^{(2)}}(\pi_{12}^*(\mathcal{C}_1) \otimes \pi_{23}^*(\mathcal{C}_2))^{T^{(2)}},$$

where $n^{(i)}$ and $T^{(i)}$ stands for the $i$-th copy of the corresponding group in inside $B^3$.

Thus the object $B(\mathcal{C}_1 \ast \mathcal{C}_2)$ can be described in terms of the open piece inside $\mathcal{X}_{\text{con}} \times \mathbb{C}^n$ with the following stability conditions:

$$\left(\mathcal{X}_{\text{con}} \times \mathbb{C}^n\right)_{\text{sst}}^{(1),(3)} = \{(X, g_1, Y_1, g_2, Y_2, g_3, Y_3, v) | \mathbb{C}[\text{Ad}_{g_i}Y_i]v = \mathbb{C}^n, i = 1, 3\}.$$ 

We can naturally extend the maps $\pi_{ij}$ to the maps between $\mathcal{X}_{\text{con}} \times \mathbb{C}^n$ and $\mathcal{X}_n \times \mathbb{C}^n$. After the extension we have:

$$\pi_{13}^{-1}\left(\text{res}^{-1}\left(\left(\left(\mathcal{X}_{\text{con}} \times \mathbb{C}^n\right)^{\text{sst}}\right)_{\text{sst}}^{(1),(3)}\right)\right) = \left(\mathcal{X}_{\text{con}} \times \mathbb{C}^n\right)_{\text{sst}}^{(1),(3)}.$$ 

The element $B(\mathcal{C}_1 \ast \mathcal{C}_2)$ can be rewritten as result of application of the functor $\text{CE}_{n^{(2)}}(\cdot)^{T^{(2)}}$ to the pull-back:

$$\text{res}^*\left(\pi_{13*}\text{CE}_{n^{(2)}}\left(\pi_{12}^*(\mathcal{C}_1) \otimes \pi_{23}^*(\mathcal{C}_2)\right)_{(1),(3)}^{T^{(2)}}\right),$$

where the subscript $(1), (3)$ indicates that we work with the matrix factorization on $(\mathcal{X}_{\text{con}} \times \mathbb{C}^n)^{\text{sst}}_{(1),(3)}$. Next we observe that the $\pi_{12}$ and $\pi_{23}$ images of the strongly stable spaces also has a stability-condition description:

$$\pi_{12}\left(\left(\mathcal{X}_{\text{con}} \times \mathbb{C}^n\right)^{\text{sst}}_{(1),(3)}\right) = \left(\mathcal{X}_n \times \mathbb{C}^n\right)^{\text{sst}}_{(1)}, \quad \pi_{23}\left(\left(\mathcal{X}_{\text{con}} \times \mathbb{C}^n\right)^{\text{sst}}_{(1),(3)}\right) = \left(\mathcal{X}_n \times \mathbb{C}^n\right)^{\text{sst}}_{(2)},$$
\((\mathcal{X}_n \times \mathbb{C}^n)^{\text{sst}}_{(i)} = \{(X, g_1, Y_1, g_2, Y_2, v) | \mathbb{C}[\text{Ad}_{g_i}Y_i]v = \mathbb{C}^n\}\)

On the critical locus of \(W\) we have \(\text{Ad}_{g_1} Y_1 = \text{Ad}_{g_2} Y_2\). Thus by the shrinking lemma in [OR18] the restriction from \((\mathcal{X}_n \times \mathbb{C}^n)^{\text{sst}}_{(i)}\) to

\[
\left(\mathcal{X}_n \times \mathbb{C}^n\right)^{\text{sst}}_{(1),(2)} = \{(X, g_1, Y_1, g_2, Y_2, v) | \mathbb{C}[\text{Ad}_{g_i}Y_i]v = \mathbb{C}^n, i = 1, 2\},
\]

induces the equivalence of the categories.

Thus the element \(B(\mathcal{C}_1 \star \mathcal{C}_2)\) can be rewritten as result of application of the functor \(\text{CE}_{n^2}(\cdot)^T\) to the pull-back:

\[
\text{res}^*\left(\pi_{13*} \text{CE}_{n^2}(\cdot) \left(\left(\pi_{12}^*(\mathcal{C}_1)_{(1),(2)} \otimes \pi_{23}^*(\mathcal{C}_2)_{(1),(2)}\right)(1),(2),(3)\right)\right),
\]

where the subscript \((1), (2)\) indicates that we apply the pull-back \(\pi_{ij}\) to the the matrix factorization on \((\mathcal{X}_n \times \mathbb{C}^n)^{\text{sst}}_{(1),(2)}\). Respectively, the subscript \((1), (2), (3)\) indicates that we work with the matrix factorizations on

\[
\left(\mathcal{X}_{\text{con}} \times \mathbb{C}^n\right)^{\text{sst}}_{(1),(2),(3)} = \{(X, g_1, Y_1, g_2, Y_2, Y_3, v) | \mathbb{C}[\text{Ad}_{g_i}Y_i]v = \mathbb{C}^n, i = 1, 2, 3\}.
\]

Finally, let us observe that \(B^{(2)}\)-action on \((\mathcal{X}_{\text{con}} \times \mathbb{C}^n)^{\text{sst}}_{(1),(2),(3)}\) is free. Hence on this space the functor \(\text{CE}_{n^2}(\cdot)^{T}\) is equivalent to the functor of restriction to the slice to \(B^{(2)}\)-action:

\[
\left(\mathcal{X}_{\text{con}} \times \mathbb{C}^n\right)^{\text{sst}}_{(1),(\kappa),(3)} \subset \left(\mathcal{X}_{\text{con}} \times \mathbb{C}^n\right)^{\text{sst}}_{(1),(2),(3)}
\]

that is defined by the condition:

\[(Y_2, g_2v) \in \kappa(\mathbb{C}^n),\]

where \(\kappa\) is defined by \([3.1]\).

The last observation implies that the element \(B(\mathcal{C}_1 \star \mathcal{C}_2)\) can be rewritten as result of application of the functor \(\text{CE}_{n^2}(\cdot)^T\) to the pull-back:

\[
\text{res}^*\left(\pi_{13*} \left(\pi_{12}^*(\mathcal{C}_1)_{(1),\kappa} \otimes \pi_{23}^*(\mathcal{C}_2)_{(1),(2)}\right)_{(1),\kappa,(3)}\right),
\]

where the subscripts \((1), \kappa\) and \((1), (2)\) indicate that we apply the pull-back to the matrix factorizations on the spaces

\[
\left(\mathcal{X}_2 \times \mathbb{C}^n\right)^{\text{sst}}_{(1),\kappa}, \quad \left(\mathcal{X}_2 \times \mathbb{C}^n\right)^{\text{sst}}_{\kappa,(2)}
\]

which are the images of \((\mathcal{X}_{\text{con}} \times \mathbb{C}^n)^{\text{sst}}_{(1),\kappa,(3)}\) under projections \(\pi_{12}\) and \(\pi_{23}\), respectively.

Since the spaces \((\mathcal{X}_n \times \mathbb{C}^n)^{\text{sst}}_{(1),\kappa}\) and \((\mathcal{X} \times \mathbb{C}^n)^{\text{sst}}_{\kappa,(2)}\) are slices to \(B\)-actions, we have:

\[
\mathcal{B}(\mathcal{C}_1) = \text{CE}_n (\text{res}^*(\mathcal{C}_1)_{(1),\kappa})^T, \quad \mathcal{B}(\mathcal{C}_2) = \text{CE}_n (\text{res}^*(\mathcal{C}_2)_{\kappa,(2)})^T,
\]
where the subindices \((1), \kappa\) and \((2)\) indicate the restriction to the \(B\)-slices \((\mathcal{X}_n \times \mathbb{C}^n)_{\text{st}(1), \kappa}\) and \((\mathcal{X} \times \mathbb{C}^n)_{\kappa,(2)}\).

Thus to complete our proof we observe that

\[
\text{CE}_n^{\mathbb{C}} \left( \text{res}^* \left( \pi_{13}\left( \pi_{12}^*(\mathcal{C}_1)_{(1), \kappa} \otimes \pi_{23}^*(\mathcal{C}_2)_{\kappa,(2)} \right) \right) \right)_{(1), \kappa,(3)}^{T^2} = \pi_{13}\left( \pi_{12}^* \left( \text{CE}_n(\text{res}^*(\mathcal{C}_1)_{(1), \kappa})^T \right) \otimes \pi_{23}^* \left( \text{CE}_n(\text{res}^*(\mathcal{C}_2)_{\kappa,(2)})^T \right) \right).
\]

\[\square\]

In the rest of the text we work with the stable of the category that is defined to be

\[\text{MF}_{\text{st}}^n = \text{MF}_{G_n \times B^2}(\mathcal{X}_{\text{st}}, W), \quad \mathcal{X}_{\text{st}} \xrightarrow{j_{\text{st}}} \mathcal{X}_n = \mathfrak{g}_n \times (G_n \times \mathfrak{b})^2 \times V_n,\]

where \((X, g_1, Y_1, g_2, Y_2, v)\) is stable if

\[\mathbb{C}^n = \mathbb{C}\langle X, \text{Ad}_{g_1} Y_1 \rangle v.\]

It is shown in [OR18f] that the pull-back map \(j_{\text{st}}^*\) is monoidal.

The functor \(B\) naturally extends to \(\text{MF}_{\text{st}}^n\) and the previous proposition holds for the functor:

\[B : \text{MF}_{\text{st}}^n \to \text{D}_{\text{per}}(\mathbb{C}^n \times \mathbb{C}^n).\]

Moreover, since the stability condition restrict naturally:

\[\text{res}^{-1}(\mathcal{X}_{\text{st}}^n) = \left( G_n \times \mathfrak{b} \times G_n \times \mathfrak{b} \times \mathbb{C}^n \right)^{\text{st}}\]

we have

**Corollary 3.2.2.** For any \(n\) there is a commuting diagram of monoidal functors:

\[\begin{array}{ccc}
\text{MF}_n & \xrightarrow{B} & \text{D}_{\text{per}}(\mathbb{C}^n \times \mathbb{C}^n) \\
\downarrow j_{\text{st}}^* & & \downarrow B \\
\text{MF}_{\text{st}}^n & & \\
\end{array}\]

3.3. **Knorrer reduction for** \(B\). In this section we explain how the Knorrer periodicity allows us to simplify the functor \(B\). Computationally, it is easier to work with the simplified form of the functor.

The \(G_n\) action on the space \(\mathcal{X}\) is free. We choose a slice to this action as

\[\mathcal{X}^\circ = \mathfrak{g}_n \times \mathfrak{b}_n \times G_n \times \mathfrak{b}_n\]

that is the image of \(B^2\)-equivariant projection

\[\mathcal{X} \to \mathcal{X}^\circ, \quad (X, g_1, Y_1, g_2, Y_2) \mapsto (\text{Ad}_{g_1}^{-1} X, Y_1, g_1^{-1} g_2, Y_2).\]
Respectively, we have the induced $B^2$-invariant potential:

$$W^\circ(X_1, Y_1, g_{12}, Y_2) = \text{Tr}(X_1(Y_1 - \text{Ad}_{g_{12}}Y_2)).$$

and we have an equivalence of categories:

$$\text{MF}_n \cong \text{MF}_n^\circ = \text{MF}_{B^2}(\mathcal{X}^\circ, W^\circ).$$

Next we observe that the potential $W^\circ$ has a quadratic summand:

$$W^\circ = \overline{W}^\circ + W_{kn}, \quad W_{kn}(X_1, Y_1, g_{12}, Y_2) = \text{Tr}((X_1)_{--}((Y_1)_{++} - (\text{Ad}_{g_{12}}Y_2)_{++})).$$

Above and everywhere in the text we use notations for the projections on the upper triangular subspaces $b, n$ and the lower-triangular $b, n$:

$$X = X_+ + X_{--}, \quad X_+ \in b, \quad X_{--} \in n, \quad X = X_{++} + X_-, \quad X_{++} \in n, \quad X_- \in b.$$ The potential is $W_{kn}$ is quadratic of $(X_1)_{--}$ and $(Y_1)_{++} - (\text{Ad}_{g_{12}}Y_2)_{++}$ and the embedding:

$$j_{kn}: \quad b \times b \times G_n \times b \rightarrow g_n \times b \times G_n \times b,$$ is $B^2$-equivariant and regular. Also the $B^2$-equivariant projection

$$\pi_{kn}: \quad b \times b \times G_n \times b \rightarrow b \times b \times G_n \times b = \overline{\mathcal{X}}^\circ,$$

is smooth with fibers affine spaces that have the coordinates $(Y_1)_{++} - (\text{Ad}_{g_{12}}Y_2)_{++}$. Hence Knorrer functor

$$KN = j_{kn*} \circ \pi_{kn}^*: \quad \text{MF}_n^\circ = \text{MF}_{B^2}(\overline{\mathcal{X}}^\circ, \overline{W}^\circ) \rightarrow \text{MF}_n,$$

is an equivalence of categories. The inverse functor $\pi_{kn*} \circ j_{kn*}$ is conjugate to $KN$.

Thus we want to construct a version of the functor $\mathcal{B}$ for the reduced category

$$\overline{\mathcal{B}} = \mathcal{B} \circ KN: \quad \text{MF}_n \rightarrow \text{D}^\text{per}(\mathbb{C}_n \times \mathbb{C}^n).$$

First let us simplify the construction of the functor $\mathcal{B}$ by eliminating the Chevalley-Eilenberg step. That could be achieved with use of the slices $\mathbb{C}_n^\kappa$ for $B$-action on the space $g_n \times \mathbb{C}^n$:

$$\mathbb{C}_n^\kappa = \kappa(\mathbb{C}^n) \subset g_n \times \mathbb{C}^n.$$ Thus we can define $B^2$-slice in the strongly stable locus:

$$\left(\mathcal{X}^\circ \times \mathbb{C}^n\right)_{\kappa, \kappa}^{\text{st}} = \{(X_1, Y_1, g_{12}, Y_2, v)| (Y_1, v) \in \mathbb{C}_n^\kappa, (Y_2, g_{12}^{-1}v) \in \mathbb{C}_n^\kappa\}.$$ Let us denote by $i_{\kappa, \kappa}$ the embedding of last space into $\mathcal{X}^\circ \times \mathbb{C}^n$. Thus we have a simplified formula for the functor:

$$\mathcal{B} = i_{\kappa, \kappa}^* \circ \text{res}^* \circ \pi_{b, *}^*.$$
The maps underlying the Knörrer functor and the functor $B$ fits into the commutative diagram:

$$
\begin{array}{c}
\mathcal{X}_n^\circ \times \mathbb{C}^n \\
\downarrow j_{kn} \\
\mathcal{X}_n^\circ
\end{array}
\overset{i_{\kappa,\kappa}}{\longleftrightarrow}
\begin{array}{c}
(\mathcal{X}_n^\circ \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st} \\
\downarrow \text{res} \\
(\mathfrak{h} \times G_n \times \mathfrak{b} \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st}
\end{array}
\overset{\pi_\mathfrak{b}}{\longrightarrow}
\begin{array}{c}
\mathfrak{b}^2 \\
\downarrow i_{\kappa,\kappa} \\
\mathfrak{b}^2 \times G_n \times \mathfrak{b} \times \mathbb{C}^n
\end{array}
\overset{j}{\longrightarrow}
\begin{array}{c}
\mathfrak{h} \times G_n \times \mathfrak{b} \times \mathbb{C}^n \\
\downarrow \text{res} \\
(\mathfrak{h} \times G_n \times \mathfrak{b} \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st}
\end{array}
\overset{\pi_\mathfrak{b}}{\longrightarrow}
\begin{array}{c}
\mathfrak{b} \\
\downarrow j
\end{array}
\begin{array}{c}
\mathfrak{h}^2
\end{array}

In the last diagram we set

$$
(b \times G_n \times b \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st} = \{(Y_1, g, Y_2, v) | (Y_1, v) \in \mathbb{C}_\mathcal{X}^n, (Y_2, g^{-1}v) \in \mathbb{C}_\mathcal{X}^n\}
$$

the the closed embedding

$$
\tilde{j} : (\mathfrak{h} \times G_n \times \mathfrak{b} \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st} \rightarrow (b \times G_n \times \mathfrak{b} \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st}
$$

is defined by the equation:

$$
\tilde{j}(h, g, Y_2, v) = (h + (\text{Ad}_g Y_2)_{++}, g, Y_2, v).
$$

The inclusion $\tilde{i}_{\kappa,\kappa}$ is uniquely defined by the commutativity of the diagram.

Now we observe that the composition of the maps:

$$
i_{\kappa,\kappa} = \pi_{kn} \times \text{Id}_{\mathbb{C}^n} \circ \tilde{i}_{\kappa,\kappa} : (\mathfrak{h} \times G_n \times \mathfrak{b} \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st} \rightarrow \mathcal{X}_n^\circ \times \mathbb{C}^n,
$$

defines a natural embedding of $(\mathfrak{h} \times G_n \times \mathfrak{b} \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st}$ into $\mathcal{X}_n^\circ$. That is the map $\tilde{i}_{\kappa,\kappa}$ is the section of the projection $\pi_{kn} \times \text{Id}_{\mathbb{C}^n}$.

The image of $j_{kn} \circ \tilde{i}_{\kappa,\kappa}$ consists of $(X_1, Y_1, g, Y_2, v) \in (\mathcal{X}_n^\circ \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st}$ in the vanishing locus of the ideal with generators

$$(X_1)_{--}, (Y_1)_{++} - (\text{Ad}_g Y_2)_{++}.
$$

Since the push-forward $j_{kn*}$ is defined in terms of Koszul matrix factorization of the last two group of elements, we conclude that

$$
i_{\kappa,\kappa}^* \circ j_{kn*} \circ \pi_{kn}^* = \tilde{j}_* \circ \tilde{i}_{\kappa,\kappa}^* \circ \pi_{kn}^*.
$$

Hence we obtain a simplified formula for the functor:

$$
\overline{B} = \pi_\mathfrak{b}^* \circ \tilde{i}_{\kappa,\kappa}^*.
$$

The further simplification comes from the following observation:

**Proposition 3.3.1.** For any $n$ we have:

$$(\mathfrak{h} \times G_n \times \mathfrak{b} \times \mathbb{C}^n)_{\mathcal{X},\mathcal{X}}^{st} = \mathbb{C}^n \times U_+ \times \mathbb{C}^n
$$

where $U_+$ is the group of strictly-lower triangular matrices.

**Proof.** We need to show that if

$$(Y_1, v), (Y_2, g^{-1}v) \in \mathbb{C}_\mathcal{X}^n \quad \text{and} \quad Y_1 - \text{Ad}_g Y_2 \in \mathfrak{b}_-
$$

then $g \in U_-$. 
By the first assumption we have:
\[ v = e_n, \quad Y_i = h(Y_i) + J, \quad g_m = \delta_n. \]
where \( J \) is the Jordan block of size \( n \).

The second assumption defines subgroup \( G' \subset G_n \). The Lie algebra \( g' \) of this group is defined by
\[ \text{ad}_\alpha \in b_. \]

An element \( \alpha \in g' \) uniquely decomposes into the sum
\[ \alpha = \alpha_- + \alpha_+, \quad \alpha_+ \in b, \quad \alpha_- \in n_. \]
The defining equation of \( g' \) only constraints \( \alpha_+ \) since
\[ \text{ad}_{\alpha_-}(h(Y_2) + J) \in b_. \]
Moreover, the condition on \( \alpha_+ \) is equivalent to
\[ \text{ad}_{\alpha_+}(J) = 0 \]
Hence \( \alpha_+ \) is a polynomial of \( J \):
\[ \alpha_+ = \sum_{i=0}^{n-1} c_i J^i. \]
However, by the first assumption \( (\alpha)_{in} = 0, \ i < n \) and thus \( c_i = 0, \ i > 0 \) and we have shown that \( g' = n_-. \)

Let us summarize the discussion of the section by reminding that the embedding
\[ j' : b \times h \times G_n \times b = \mathcal{X}_n^o \to X_n, \quad j'(X_1, Y_1, g, Y_2) \mapsto (X_1, Y_1 + (\text{Ad}_g Y_2)_{++}, g, Y_2), \]
and the potential \( W^o \) on \( \mathcal{X}_n^o \) is obtained from \( W \) by pull-back along \( j' \). The inverse to the Knörrer equivalence is given by:
\[ KN^{-1} = \pi_{kn*} \circ j_{kn}^* : \text{MF}_n \to \text{MF}_n^o = \text{MF}_{B^2}(\mathcal{X}_n^o, W^o). \]

As we have shown the reduced functor \( \overline{\text{B}} \), which is obtain by pre-composing \( \text{B} \) with \( KN \) has the following simple decryption:

**Theorem 3.3.2.** The reduced functor
\[ \overline{\text{B}} : \text{MF}_n^o = \text{MF}_{B^2}(\mathcal{X}_n^o, W^o) \to D(\mathbb{C}^n \times \mathbb{C}^n), \]
is defined by
\[ \overline{\text{B}} = \pi_{h*} \circ \tilde{i}_{\mathfrak{g}, \mathfrak{g}}^* \]
where
\[ \tilde{i}_{\mathfrak{g}, \mathfrak{g}} : \mathbb{C}^n \times U_- \times \mathbb{C}^n \to \mathcal{X}_n^o, \quad \tilde{i}_{\mathfrak{g}, \mathfrak{g}}(\mathfrak{g}_1, g, \mathfrak{g}_2) = (0, h(\mathfrak{g}_1), g, h(\mathfrak{g}_2) + J), \]
and \( \pi_{h} \) is the projection to \( \mathbb{C}^n \times \mathbb{C}^n \).
3.4. **Unit element.** In this section we apply the results of the previous section and check that $B$ sends the convolution unit to the unit. Let us first recall that $R_n$ has a natural structure of $R_n$ bimodule and is naturally a unit of the bimodule monoidal structure.

To describe the unit on the geometric side we introduce $B^2$-equivariant space 

$$\bar{X}_n^\circ(B_n) = b \times h \times B_n \times b, \quad \bar{j}_B : \bar{X}_n^\circ(B_n) \rightarrow \bar{X}_n^\circ,$$

where the last map is induced by the natural inclusion of $B_n$ inside $G_n$. The pull-back of the potential $\bar{W}$ is quadratic:

$$\bar{j}_B^*(\bar{W}) = \sum_{i=1}^n X_{1,ii}(Y_{1,ii} - Y_{2,ii}).$$

In particular, the category $MF_{B^2}(\bar{X}_n^\circ(B_n), \bar{j}_B^*(\bar{W}))$ contains a canonical Koszul matrix factorization:

$$K_\Delta = \begin{bmatrix} X_{1,11} & Y_{1,11} - Y_{2,11} \\ \vdots & \vdots \\ X_{1,nn} & Y_{1,nn} - Y_{2,nn} \end{bmatrix}$$

Respectively, the unit in the monoidal category $MF_n$ is defined by :

$$C_\parallel = K \cdot N(C_\parallel) \in MF_n, \quad C_\parallel = j_{B^*}(K_\Delta)$$

**Proposition 3.4.1.** For any $n$ we have 

$$B(C_\parallel) = R_n.$$ 

**Proof.** By above discussion it is enough to show that 

$$\bar{B}(\bar{C}_\parallel) = R_n.$$ 

The diagram of maps that participate the construction of $\bar{B}$ could be completed to the commuting diagram:

$$\begin{array}{c}
\bar{X}_n^\circ(B_n) \xrightarrow{\bar{j}_B} \bar{X}_n^\circ \\
\uparrow \quad \uparrow \\
\mathbb{C}^n \times \mathbb{C}^n \xrightarrow{\bar{j}_B} \mathbb{C}^n \times U_- \times \mathbb{C}^n \xrightarrow{\pi_b} \mathbb{C}^n \times \mathbb{C}^n.
\end{array}$$

where

$$\bar{i}_{\kappa,\kappa}(Y_1, Y_2) = (0, Y_1, 1, Y_2 + J_n).$$

The map $\bar{j}_B$ is uniquely determined by the commutativity of the diagram and 

$$\pi_b \circ \bar{j}_B = id.$$ 

Thus we can apply the base change formula to obtain

$$\bar{B}(\bar{C}_\parallel) = \pi_{b^*} \circ \bar{i}_{\kappa,\kappa}^* \circ \bar{j}_{B^*}(K_\Delta) = \pi_{b^*} \circ \bar{j}_{B^*} \circ \bar{i}_{\kappa,\kappa}^*(K_\Delta) = \bar{i}_{\kappa,\kappa}^*(K_\Delta).$$

Since $K_\Delta|_{X=0}$ is the Koszul complex of the diagonal in $\mathbb{C}^n \times \mathbb{C}^n$ the statement follows. \qed
3.5. Elementary braid diagrams. For our argument we would need to calculate the functor $B$ on the matrix factorizations that represent the braid diagrams.

Let us recall that the basic building blocks for the Soergel bimodules are the singular Soergel bimodules:

$$B_i = R_n \otimes_{R_i} R_n,$$

where $R_n = \mathbb{C}[y_1, \ldots, y_n]$ and $R_i = R_{n+1}^{(i, i+1)}$ is the subring of polynomials invariant with respect to switching $y_i$ and $y_{i+1}$. The bimodule $R_n$ is the unit of the convolution.

In this section we would like to discuss the case $n = 2$ for both Soergel bimodules and for matrix factorizations.

The analog of the bimodule $B_1 \in S\text{Bim}_2$ is the element $C_*$ of the category $\text{MF}_n$ which is easier to describe in the different model $\overline{\text{MF}}_n$ of the category $\text{MF}_n$. The category $\overline{\text{MF}}_n$ is defined as category of matrix factorizations on the space

$$\overline{\mathcal{X}}_n = n \times G_n \times b,$$

with the potential

$$\overline{W}(X, g, Y) = \text{Tr}(X \text{Ad}_g Y).$$

The left and right $B$-actions on $G$ could be extended in a unique to preserve the potential. Respectively, we use the following notations for the $B^2$-equivariant category of matrix factorizations:

$$\overline{\text{MF}}_n = \text{MF}_{B^2}(\overline{\mathcal{X}}_n, \overline{W}).$$

As it is shown in [OR19] there is a natural Knörrer equivalence functor:

$$KN : \overline{\text{MF}}_n \to \text{MF}_n.$$

If $n = 2$ the potential $\overline{W}$ factors:

$$\overline{W}(X, g, Y) = x_{12} g_{21} \tilde{y} \det(g)^{-1}, \quad \tilde{y} = g_{22}(y_{11} - y_{22}) - g_{21} y_{12}.$$

Thus we can define three Koszul matrix factorizations:

$$\mathcal{C}_\parallel = [g_{21}, x_{12} \tilde{y} / \det(g)], \quad \mathcal{C}_* = [x_{12} g_{21}, \tilde{y} / \det(g)], \quad \mathcal{C}_+ = [x_{12}, g_{12} \tilde{y} / \det(g)].$$

Using the Knörrer functor we get the elements:

$$K \mathcal{C}_\parallel = KN(\mathcal{C}_\parallel), \quad K \mathcal{C}_* = KN(\mathcal{C}_*), \quad K \mathcal{C}_+ = KN(\mathcal{C}_+).$$

Below we show the basic case of our main result:

**Proposition 3.5.1.** If $n = 2$ then

$$B(\mathcal{C}_*) = B_1.$$

**Proof.** Since the $G_n$-action on the space $\mathcal{X}^{st}$ is free it is more convenient to work with the $G_n$-quotient space with the potential

$$\mathcal{X}^{\circ, st} \cong \mathfrak{g}_n \times b \times G \times b \times V, \quad W^\circ(X, Y^1, g_{12}, Y^2, v) = \text{Tr}(X(\text{Ad}_{g_{12}} Y_1 - Y_2)),$$

and the stability condition $\langle X, Y_2 \rangle v = \mathbb{C}^n, \langle X, \text{Ad}_{g_{12}} Y_1 \rangle v = \mathbb{C}^n$ and $B^2$ action:

$$(b_1, b_2) \cdot v = b_2 v, \quad (b_1, b_2) \cdot Y_i = \text{Ad}_b Y_i, \quad (b_1, b_2) \cdot X = \text{Ad}_b X, \quad (b_1, b_2) \cdot g_{12} = b_2 g_{12} b_1^{-1}.$$
To distinguish two $B$-actions we write $B^2 = B^{(1)} \times B^{(2)}$. We can use the $B^{(2)}$-action on $V$ to put the vector $v$ in the standard position

$$v_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ 

Let $G' \subset G$ be a open $B^{(1)}$-equivariant locus inside the group:

$$G' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a \neq 0 \right\}.$$

The pull-back along the inclusion map

$$j' : g_n \times b \times G' \times b \times v_0 \to g_n \times b \times G \times b \times v_0$$

is induces the equivalence of categories of $B^{(1)}$-equivariant matrix factorizations.

Indeed, if $g \in G \setminus G'$ then for any $Y \in b$ there is vanishing of one of the matrix coefficients of the conjugate:

$$(\text{Ad}_g(Y))_{12} = 0.$$

On the other hand on the critical locus of $W^\circ$ we have

$$\text{Ad}_{g_{12}} Y_1 = Y_2.$$

Thus if $g_{12} \in G \setminus G'$ the critical relations imply that $Y_2$ is diagonal. On the other hand $v = v_0$ thus $\mathbb{C}[Y_2]v \neq \mathbb{C}^2$ and we get a contradiction to the strong stability.

The $B^{(1)}$-action and $T^{(2)}$-actions allows us to move the point of $X^{\text{st}}$ to the standard position:

$$g_{12} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 1 \end{bmatrix}, \quad Y_i = \begin{bmatrix} y_{11}^{(i)} & 1 \\ 0 & y_{22}^{(i)} \end{bmatrix}, \quad v = v_0$$

This closed locus has a trivial stabilizer inside $B^2$.

By direct computation we obtain:

$$\text{Ad}_{g_{12}} Y_1 = \begin{bmatrix} a_{11} \\ (y_{11}^{(1)} - y_{22}^{(1)} - a_{21})a_{21}/a_{11} & a_{21} + y_{22}^{(1)} \end{bmatrix},$$

respectively the potential is equal:

$$W^\circ = x_{11}(y_{11}^{(1)} - y_{11}^{(2)} - a_{21}) + x_{21}(a_{11} - 1) + x_{22}(y_{22}^{(1)} - y_{22}^{(2)} + a_{21}) + x_{12}a_{21}(y_{11}^{(1)} - y_{22}^{(1)} - a_{21}).$$

The first line of the formula for the potential is exactly the quadratic term that is responsible for the the Knörrer functor $KN$. Thus obtain a presentation of the matrix factorizations as curved Koszul complexes:

$$CE_n(C_\bullet)^{T^2} = Q \otimes [x_{12}, a_{21}(y_{11}^{(1)} - y_{22}^{(1)} - a_{21})],$$

$$Q = [x_{11}, y_{11}^{(1)} - y_{11}^{(2)} - a_{21}] \otimes [x_{21}, a_{11} - 1] \otimes [x_{22}, y_{22}^{(1)} - y_{22}^{(2)} + a_{21}].$$
The functor $B$ sets variables $x_{ij}$ to zero. Thus $B (C_\bullet)$ is a Koszul complex for the regular sequence
\[ a_{21}(y_{11}^{(1)} - y_{22}^{(1)} - a_{21}), \quad y_{11}^{(1)} - y_{11}^{(2)} - a_{21}, \quad a_{11} - 1, \quad y_{22}^{(1)} - y_{22}^{(2)} + a_{21}. \]
This sequence is equivalent to the sequence
\[ (y_{22}^{(2)} - y_{22}^{(1)})(y_{11}^{(2)} - y_{11}^{(1)}), \quad y_{11}^{(1)} + y_{11}^{(1)} - y_{22}^{(1)} - y_{22}^{(2)}, \quad a_{11}, \quad y_{22}^{(1)} - y_{22}^{(2)} + a_{21}. \]
The first two terms of the sequence define the bimodule $B_1$ and the statement follows. □

3.6. Induction functors. The presentation of $C^n$ as product $C^k \times C^{n-k}$ induces the induction functor:

\[ \text{ind}_k : SBim_k \times SBim_{n-k} \to SBim_n. \]

Similar functor was constructed for the category of matrix factorization, we provide the details below. In the next section we use the induction functor to define the generators $C^\bullet (i)$ of $MF_n^b$. In this section we show the compatibility of these induction functors:

**Proposition 3.6.1.** We have:

\[ B \circ \text{ind}_k = \text{ind}_k \circ B \times B. \]

Before proceed with the proof let us remind the construction for the induction functors for categories $MF_n$. The induction functor is easier to describe in the reduced case:

\[ \overline{\text{ind}}_k : MF^\circ_k \times MF^\circ_{n-k} \to MF^\circ_n. \]

To define the functor we introduce the intermediate space:

\[ \mathcal{X}^\circ_n (P_k) = b_n \times b_n \times P_k \times b_n, \quad P_k \subset G_n. \]

where $P_k$ the standard parabolic subgroups.

The last space has a natural $B^2_n$-action. Since the standard group homomorphisms:

\[ P_k \to G_k \times G_{n-k}, \quad P_k \to G_n \]

are $B^2$-equivariant we get the corresponding $B^2$-maps and the induction functor:

\[ \mathcal{X}^\circ_n (P_k) \xrightarrow{i_k} \mathcal{X}^\circ_n, \quad \mathcal{X}^\circ_n (P_k) \xrightarrow{\overline{\text{ind}}_k} \mathcal{X}^\circ_{n-k}, \quad \text{s.t.} \quad \overline{\text{ind}}_k = \overline{i}_k \circ \overline{p}_k. \]

To define the functor in the non-reduced version of the category we introduce the intermediate space:

\[ \mathcal{X}^\circ_n (P_k) \xrightarrow{i_k} \mathcal{X}^\circ_n, \quad \mathcal{X}^\circ_n (P_k) = p_k \times b_n \times P_k \times b_n, \]

where $p_k = \text{Lie}(P_k)$. 
Analogously to the reduced case we have $B_n^2$-equivariant maps $p_k, i_k$ which allow us to define the induction functor:

$$
\mathcal{X}_n^\circ(P_k) \xrightarrow{i_k} \mathcal{X}_n^\circ, \quad \text{ind}_k = i_{k*} \circ p_k^*.
$$

$\mathcal{X}_k^\circ \times \mathcal{X}_{n-k}^\circ$

Let us list the properties of these functors.

**Proposition 3.6.2.** For any $n$ and $1 \leq k < n$ we have

1. $\text{ind}_k \circ KN_k \times KN_{n-k} = KN_n \circ \text{ind}_k$.
2. $\text{ind}_k(\mathcal{F} \star \mathcal{F}' \times \mathcal{G} \star \mathcal{G}') = \text{ind}_k(\mathcal{F} \times \mathcal{G}) \star \text{ind}_k(\mathcal{F}' \times \mathcal{F}')$.

The second property is shown in [OR18f, Proposition 6.2] the first property is an easy modification of [OR18f, Proposition 6.1]. It is also shown in [OR18f] that the induction functor is transitive:

**Proposition 3.6.3.** [OR18f, Proposition 6.4] The following functors

$$
\text{MF}_k^\circ \times \text{MF}_{m-k}^\circ \times \text{MF}_{n-m}^\circ \to \text{MF}_n^\circ
$$

are naturally isomorphic:

$$
\text{ind}_k \circ \text{Id} \times \text{ind}_{m-k} \simeq \text{ind}_m \circ \text{ind}_k \times \text{Id}.
$$

**Proof of.** By proposition 3.6.3 it is enough to show

$$
\mathcal{B} \circ \text{ind}_k = \text{ind}_k \circ \mathcal{B} \times \mathcal{B}.
$$

To prove the last statement we observe there is a commutative diagram that includes all the map that participate in both sides of the equation:

$$
\begin{array}{cccccc}
\mathbb{C}^k \times U_k^- \times \mathbb{C}^k \times \mathbb{C}^{n-k} \times U_{n-k}^- \times \mathbb{C}^{n-k} & \xrightarrow{\pi_h \times \pi_h} & \mathbb{C}^k \times \mathbb{C}^{n-k} \\
\mathbb{C}^n \times U_k^- \times U_{n-k}^- \times \mathbb{C}^n & \xrightarrow{\tilde{i}_k} & \mathbb{C}^n \times U_{n-k}^- \times \mathbb{C}^n & \xrightarrow{\pi_h} & \mathbb{C}^n \times \mathbb{C}^n \\
\tilde{i}_{\nu, \kappa} \times \tilde{i}_{\nu, \kappa} & \xrightarrow{\tilde{g}_k} & \tilde{X}_n^\circ(P_k) & \xrightarrow{\tilde{i}_k} & \tilde{X}_n^\circ \\
\end{array}
$$

where the new maps are

$$
\tilde{i}_{\nu, \kappa}(Y_1, g', g'', Y_2) = (0, Y_1, g' \times g'', Y_2 + J_n),
$$

and the map $\tilde{i}_\kappa$ is induced by the inclusion of $U_k^- \times U_{n-k}^- \to U_n^-$. 


Thus we can use base-change relation to show:

\[
\mathcal{B} \circ \text{ind}_k = \pi_{h*} \circ \tilde{i}_{\kappa,\xi}^* \circ \tilde{i}_k \circ \tilde{p}_k = \pi_{h*} \circ \tilde{i}_{\kappa,\xi}^* \circ \tilde{i}_{\kappa,\xi}^* \circ \tilde{p}_k = \pi_{h*} \circ \tilde{i}_{\kappa,\xi}^* \circ \tilde{i}_{\kappa,\xi}^* \times \tilde{i}_{\kappa,\xi}^*
\]

\[
= \pi_{h*} \times \pi_{h*} \circ \tilde{i}_{\kappa,\xi}^* \times \tilde{i}_{\kappa,\xi}^* = \text{ind}_k \circ \mathcal{B} \times \mathcal{B}.
\]

\[\square\]

### 3.7. Generators of MF\(_n^b\). Let us set as in [OR18f]:

\[
C_{p_i}^{(i)} = \text{ind}_i(C'' \times \text{ind}_2(C \times C')) \in \text{MF}_n,
\]

where \(C' = C_{||} \in \text{MF}_{n-i-1}\) and \(C'' = C_{||} \in \text{MF}_{i-1}\).

Thus combination of propositions 3.6.3, 3.4.1 and 3.5.1 implies

**Proposition 3.7.1.** For any \(i \in 1, \ldots, n-1\) we have:

\[
\mathcal{B}(C_{p_i}^{(i)}) = B_i.
\]

### 4. MOY relations for matrix factorizations

In this section we discuss the properties of the functor \(\mathcal{B}\). In context of our previous work we assign a matrix factorization to a braid graph \(D \in \mathcal{B}r_n^b\) on \(n\) strands.

The building blocks of our category of graphs are elementary graphs \(D_{p_i}^{(i,i+1)}\). We label the strand and show only non-trivial part of the graph, the strands \(j\)-th strand for \(j \neq i, i+1\) go up.

We compose the graphs vertically and use notation \(\circ\) for the composition. The set of all possible compositions of \(D_{p_i}^{(i,i+1)}, i = 1, \ldots, n-1\) we denote by \(\mathcal{B}r_n^b\).

There is a natural homomorphism

\[
\Phi^b : \mathcal{B}r_n^b \to \text{MF}_n,
\]

\[
\Phi^b(D_{p_i}^{(i,i+1)}) = C_{p_i}^{(i)}.
\]

Respectively, \(\Phi^b\) sends composition \(\circ\) to \(\ast\).

The graphs \(\mathcal{B}r_n^b\) are particular cases of so-called MOY graphs [MOY98]. To be more precise the graphs from \(\mathcal{B}r_n^b\) are braid graphs. In [MOY98] it is explained how one can the braid diagram in term of the braid graphs. After resolution the braid relations become so called MOY relations between the braid diagrams. The main result of this section is a proof of the braid MOY relations inside \(\text{MF}_n\).

**Lemma 4.0.1.** For any \(1 \leq i \leq n-2\) we have:

\[
C_{p_i}^{(i+1)} \ast C_{p_i}^{(i)} \ast C_{p_i}^{(i+1)} \oplus q^2 C_{p_i}^{(i)} = C_{p_i}^{(i)} \ast C_{p_i}^{(i+1)} \ast C_{p_i}^{(i)} \oplus q^2 C_{p_i}^{(i+1)}
\]

The skein-relation (or quadratic Hecke relation) becomes the MOY relation depicted below and prove the relation also holds in category \(\text{MF}_n\);

**Lemma 4.0.2.** For any \(1 \leq i \leq n-1\) we have

\[
C_{p_i}^{(i)} \ast C_{p_i}^{(i)} = C_{p_i}^{(i)} \oplus q^2 C_{p_i}^{(i)}.
\]
4.1. **Coherent sheaf version.** The categories $\text{MF}_n$ that we study are closely related to the categories from the work of Riche [Ric08] and further developed in the work of Bezrukavnikov and Riche [BR12].

The relation between the category of Bezrukavnikov and Riche and matrix factorizations were studied by Arkhipov and Kanstrup [AK15a], [AK15b] in the context of derived algebraic geometry. Here we would like to present a link between the categories studied by Bezrukavnikov-Riche and the equivariant matrix factorization in the sense of [OR18f].

We use this link to give a more geometric proofs of some arguments. It is also possible to do all proofs entirely in the matrix factorization setting but the reference to the work of Bezrukavnikov and Riche might be helpful for a reader familiar with the coherent sheaf version of the construction.

The results of this section hold in the generality of [BR12], that is the group $\text{GL}_n$ can be replaced by any complex reductive group $G$ with Lie algebra $\mathfrak{g}$. The category $\text{MF}_n$ also can be defined in this generality.

The Grothendieck alteration $\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ has a quotient description $\tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ such that the homology $H^\ast(\mathfrak{C}^\ast)$ are sheaves with the support proper over each copy of $\mathfrak{g}$.

Using the projections $\pi_{ij}: \tilde{\mathfrak{g}}^3 \to \tilde{\mathfrak{g}}^2$ Riche defines the convolution product

$$\ast: \mathbb{D}^b_{\text{prop}}(\tilde{\mathfrak{g}}^2) \times \mathbb{D}^b_{\text{prop}}(\tilde{\mathfrak{g}}^2) \to \mathbb{D}^b_{\text{prop}}(\tilde{\mathfrak{g}}^2).$$

In [Ric10] the homomorphism $\mathfrak{B}^\ast_{\text{aff}} \to (\mathbb{D}^b_{\text{prop}}(\tilde{\mathfrak{g}}^2), \ast)$ is constructed.

The category that is most relevant for our model is the category $T_q$-equivariant complexes $\mathbb{D}^b_{G,T_q}(\tilde{\mathfrak{g}}^2)$ that $G$-equivariant. The torus $T_q = \mathbb{C}^\ast_q$ acts on the space $\tilde{\mathfrak{g}}$ by scaling of the $\mathfrak{g}$ with weight $q$.

We can enhance the category by enlarging the torus $T_q$ to $T_{qt} = \mathbb{C}_q \times \mathbb{C}_t$. The other factor $\mathbb{C}^\ast_t$ acts trivially on $\tilde{\mathfrak{g}}^2$ but the differentials in the complexes have degree $t$.

There is a natural folding functor $\text{Per}: \mathbb{D}^b_{G,T_q}(\tilde{\mathfrak{g}}^2) \to \mathbb{D}^b_{G,T_{qt}}(\tilde{\mathfrak{g}}^2)$ that folds the homological grading to the 2-periodic grading. The homological grading could be restored from the 2-periodic grading since the differential in the two-periodic complexes is of degree $t$.

Since $\tilde{\mathfrak{g}}^2$ is smooth and every coherent sheaf has a finite projective resolution, this category is equivalent to the category of the strictly $G \times B^2$-equivariant matrix factorizations with zero potential

$$\text{MF}^{\text{str}}_{G \times B^2}(b \times G \times b \times G, 0) \cong \mathbb{D}^b_{G,T_{qt}}(\tilde{\mathfrak{g}}^2).$$

An equivariant matrix factorization $(M, D, \partial)$ is strict if $D$ is equivariant and $\partial = 0$. Thus a pull-back of a strictly equivariant matrix factorization is a strictly equivariant. Similarly, the push-forward along the smooth map and group quotient from section 2.3...
preserve strict equivariance. Thus the subcategory
\[ \text{MF}_{G \times B^2}(b \times G \times b \times G, 0) \subset \text{MF}_{G \times B^2}(b \times G \times b \times G, 0) \]
is a monoidal subcategory (the monoidal structure is \( \cdot \)).

The pull-back along the inclusion
\[ i : b \times G \times b \times G \to g \times b \times G \times b \times G, \quad i(z) = (0, z) \]
annihilates the potential \( W \). Since the inclusion is clearly a regular lci inclusion by the results from \([OR18f]\) there is a well-defined functor:
\[ i_* : \text{D}^b_{G, \text{prop}}(\tilde{g}^2) \to \text{MF}_n. \]

Let us also use notation \( i_* \) for the composition of the folding functor and the push-forward. If we specify the domain of our functor there is no chance for a confusion.

The subcategory \( \text{D}^b_{G, \text{prop}}(\tilde{g}^2) \) that consists of the complexes with the homological supports proper with respect to the projections \( \pi_i : \tilde{g}^2 \to \tilde{g} \) is the analog of the category \( \text{D}^b_{\text{prop}}(\tilde{g}^2) \) and thus has a convolution product \( \cdot \).

**Proposition 4.1.1.** The push-forward functor \( i_* \) is monoidal:
\[ i_*(B) \star i_*(B') = i_*(B \star B'). \]

**4.2. Outline of proof of MOY relations.** The geometry of irreducible components of the Steinberg varieties their behavior under convolution product are well-studied (see for example \([CG97]\)). Thus we can use the previous proposition to transfer some of these result into matrix factorization setting and prove the MOY relation from the beginning of the section. We outline our approach in this section.

There are natural Koszul complexes that are sent to the generators \( C^{(i)} \) by \( i_* \)
\[ B^{(i)}_\bullet \in \text{D}^b_{T_q}(\tilde{g}^2), \quad i_*(B^{(i)}_\bullet) = C^{(i)}_\bullet. \]

The \( B^{(i)}_\bullet \) is a Koszul complex of the subvariety \( St^{(i)} \subset \tilde{g}_n \times \tilde{g}_n \) that is a \( B^2 \)-quotient of the subvariety \( \tilde{St}^{(i)} \). The subvariety \( \tilde{St}^{(i)} \) consists of quadruples \( (Y_1, g_1, Y_2, g_2) \) that satisfy equations:
\[ g_{12} = g_1^{-1}g_2 \in P_i, \quad \text{Ad}_{g_1}Y_1 = \text{Ad}_{g_2}Y_2. \]

Here and everywhere below we define a parabolic subgroup \( P_i \subset G_n, I \subset \{1, \ldots, n-1\} \) as a group with the Lie algebra \( \text{Lie}(P_i) \subset \mathfrak{g} \) generated by \( b \) and \( E_{i+1,i}, i \in I \).

The variety \( \tilde{St}^{(i)} \) is a complete intersection and the \( G \times B^2 \)-equivariant Koszul complex provides a resolution of its structure sheaf:
\[ B^{(i)}_\bullet = K(\tilde{St}^{(i)}) \in \text{D}^b_{T_q}(\tilde{g}^2). \]

The equation (4.1) is almost immediate from our construction of the induction functor. Thus we need to show two propositions which imply the MOY relations.

**Proposition 4.2.1.** For \( B^{(1)}_\bullet \in \text{D}^b_{G_2, \text{prop}}(\tilde{g}^2) \) we have:
\[ B^{(1)}_\bullet \star B^{(1)}_\bullet = B^{(1)}_\bullet \oplus q^2 \cdot B^{(1)}_\bullet. \]
For the next proposition we need a slight generalization of the main generators. We define $S_{\hat{(i,i+1)}} \subset \hat{g}^2$ as variety of quadruples $(g_1, Y_1, g_2, Y_2)$ that satisfy the system of equations:

$$g_{12} = g_1^{-1} g_2 \in P_{i,i+1}, \quad \text{Ad}_{g_1} Y_1 = \text{Ad}_{g_2} Y_2.$$  

This variety is a complete intersection and the corresponding $G \times B^2$-equivariant Koszul complex defines the element:

$$B^{(i,i+1)} = K(S_{\hat{(i,i+1)}}) \in D^\text{per,T}_{G,\text{prop}}(\hat{g}^2).$$

Respectively, we have the corresponding element of the category of matrix factorization:

$$C^{(i,i+1)} = i_* (B^{(i,i+1)}).$$

**Proposition 4.2.2.** For $C^{(1)}_\bullet, C^{(2)}_\bullet, C^{(1,2)}_\bullet \in \text{MF}_3$ we have:

$$C^{(1)}_\bullet \ast C^{(2)}_\bullet \ast C^{(1)}_\bullet = C^{(1)}_\bullet \oplus C^{(1,2)}_\bullet.$$  

The first proposition is an easy computation. The second of the proposition could be proven by a laborious computation but we provide a proof of the lemma that relies on a computations of the extension groups and two lemmas.

**Lemma 4.2.3.** There is a short exact sequence of sheaves on $\hat{g}^2$:

$$0 \rightarrow B^{(1)}_\bullet \rightarrow B^{(1)}_\bullet \ast B^{(2)}_\bullet \ast B^{(1)}_\bullet \rightarrow B^{(1,2)}_\bullet \rightarrow 0.$$  

The push forward $i_*$ sends the short exact sequence from the proposition to the short exact sequence in $\text{MF}_3$. Thus to complete our argument we need to compute the extension groups:

**Lemma 4.2.4.** For $C^{(1,2)}_\bullet \in \text{MF}_3$ we have the following vanishing of the extension groups:

$$\text{Ext}^i(C^{(1,2)}_\bullet, C_\bullet) = 0, \quad i > 0.$$  

To derive the proposition from the lemmas we need to explain the relation between the extension groups and convolution product. It leads us to study of duality on our category. Given two characters $\lambda, \mu \in T^\vee$ and a module $M$ with $B^2$ action

$$(b', b'', m) \mapsto (b', b'') \cdot m, \quad b', b'' \in B, m \in M,$$  

we denote $M(\lambda, \mu)$ the module $M$ with the twisted $B^2$-action:

$$(b', b'', m) \mapsto \lambda(b') \mu(b'')(b', b'') \cdot m, \quad b', b'' \in B, m \in M.$$  

Respectively, we use the same convention for twisting objects from $\text{MF}_n$ and $D^{k,T}_{G,n}(\hat{g}^2)$. Let us fix $\rho$ for the half sum of positive roots of $G_n$. Using these notations let us list the properties of the $B^2$-twisting.
4.3. Duality. In this section we describe the action of the duality functor on the categories \( \text{MF}_n \) and \( \text{D}_{G_n}(\bar{\mathfrak{g}}_n^2) \). Just as in the case of matrix factorizations we can introduce the an additive subcategory of \( \text{D}^b \subset \text{D}^b_{G_n}(\bar{\mathfrak{g}}_n^2) \) that is spanned by all products of elements \( \mathcal{B}^{(i)} \) and by the products of the direct summands of these products.

The both categories \( \text{MF}_n \) and \( \text{D}_{G_n}(\bar{\mathfrak{g}}_n^2) \) have duality functor:

\[
\mathcal{F} \mapsto \mathcal{F}^\vee = \mathcal{H}\text{om}(\mathcal{F}, \mathcal{O}).
\]

The duality preserves the objects of the both graph categories:

**Proposition 4.3.1.** For any \( \mathcal{A} \) from \( \text{MF}_n \) or \( \text{D}^b \) we have:

\[
\mathcal{A}^\vee = \mathcal{A}.
\]

The statement is almost immediate from the observation below and the self-duality of the generators \( \mathcal{B}^{(i)} \) and \( \mathcal{C}^{(i)} \) which is discussed next.

**Proposition 4.3.2.** For any \( \mathcal{A}_1, \mathcal{A}_2 \) from \( \text{MF}_n \) or \( \text{D}_{G_n}(\bar{\mathfrak{g}}_n^2) \) we have

\[
(\mathcal{A}_1 \star \mathcal{A}_2)^\vee = \mathcal{A}_1^\vee \star \mathcal{A}_2^\vee.
\]

**Proof.** Since \( \bar{\mathfrak{g}}_n \) is holomorphic symplectic it has a trivial canonical class. Hence the statement follows from the Serre duality:

\[
\mathcal{A}_1^\vee \star \mathcal{A}_2^\vee = \pi_{13*}(\pi_{12}^*(\mathcal{A}_1)^\vee \otimes \pi_{23}^*(\mathcal{A}_2)^\vee) = \pi_{13*}(\pi_{12}^*(\mathcal{A}_1) \otimes \pi_{23}^*(\mathcal{A}_2))^\vee,
\]

for the second equality we used the Serre duality. \( \square \)

Let us denote the unit object in both categories as \( 1 \):

\[
1 = \mathcal{C}_\parallel \in \text{MF}_n, \quad 1 = \mathcal{B}_\parallel \in \text{D}_{G_n}(\bar{\mathfrak{g}}_n^2).
\]

**Corollary 4.3.3.** For any \( \mathcal{A}_1, \mathcal{A}_2 \) from \( \text{MF}_n \) or \( \text{D}_{G_n}(\bar{\mathfrak{g}}_n^2) \) we have

\[
\text{Ext}^1(\mathcal{A}_1, \mathcal{A}_2) = \text{Tor}^1(\mathcal{A}_1 \star \mathcal{A}_2, 1).
\]

The only thing that we check to complete a proof of proposition 4.3.1 is the action of the duality on the generators. For that let us fix notations for the twisting of \( B^2 \)-action.

Given two characters \( \lambda, \mu \in T^\vee \) and a module \( M \) with \( B^2 \) action

\[
(b', b'', m) \mapsto (b', b'') \cdot m, \quad b', b'' \in B, m \in M,
\]

we denote \( M\langle \lambda, \mu \rangle \) the module \( M \) with the twisted \( B^2 \)-action:

\[
(b', b'', m) \mapsto \lambda(b') \mu(b'') (b', b'') \cdot m, \quad b', b'' \in B, m \in M.
\]

Respectively, we use the same convention for twisting objects from \( \text{MF}_n \) and \( \text{D}_{G_n}(\bar{\mathfrak{g}}_n^2) \).

Let us fix \( \rho \) for the half sum of positive roots of \( G_n \).

**Proposition 4.3.4.** For any \( \mathcal{A} \) from \( \text{D}_{G_n}(\bar{\mathfrak{g}}_n^2) \) (or \( \text{MF}_n \)) and \( \lambda, \mu \in T^\vee \) we have:

1. \( \mathcal{A} \star 1\langle \lambda, \mu \rangle = \mathcal{A}\langle 0, \lambda + \mu \rangle \)
2. \( 1\langle \lambda, \mu \rangle \star \mathcal{A} = \mathcal{A}\langle \lambda + \mu, 0 \rangle \).
3. \( 1\langle \lambda, \mu \rangle = 1\langle \lambda + \mu, 0 \rangle = 1\langle 0, \lambda + \mu \rangle.
4. \( 1^\vee = 1 \).
For the generators of \( \pi \) in these coordinates the pull-back defines the variety \( \sim \). The convolution space \( \sim \) is described by two equivalent systems of equations. The first system is

\[
g_2^{-1}g_1 \in B, \quad (\text{Ad}_{g_2^{-1}g_1}Y_1)_{ij} = (Y_2)_{ij}, \quad i \geq j
\]

The sum of the \( B^2 \) weights of the first group of equations is \( \langle -\rho, -\rho \rangle \) and of the second is \( \langle 0, 2\rho \rangle \). To compute the dual of \( \sim \) we need to invert the Borel weights in the Koszul complex, hence by the third property

\[
\sim^\vee = \sim \langle \rho, -\rho \rangle = \sim.
\]

Finally, let us compute the dual of \( \mathcal{B}_i^{(i)} \in \mathcal{D}_n^\vee \). The corresponding variety \( \mathcal{S}^{(i)} \) can be described by two equivalent systems of equations. The first system is

\[
g_2^{-1}g_1 \in P, \quad (\text{Ad}_{g_2^{-1}g_1}Y_1)_{kj} = (Y_2)_{kj}, \quad k \geq j, \quad (\text{Ad}_{g_2^{-1}g_1}Y_1)_{i+1,i} = 0.
\]

The other system is

\[
g_1^{-1}g_2 \in P, \quad (\text{Ad}_{g_1^{-1}g_2}Y_2)_{kj} = (Y_1)_{kj}, \quad k \geq j, \quad (\text{Ad}_{g_1^{-1}g_2}Y_2)_{i+1,i} = 0.
\]

The sum of the weights of the first system of equations is \( \langle -\rho + \epsilon_{i+1}, -\rho - \epsilon_{i+1} \rangle \), respectively the sum of the weights of the equations of the second system is \( \langle -\rho - \epsilon_{i+1}, -\rho + \epsilon_{i+1} \rangle \). Thus the Koszul complexes of these two systems are dual to each other. Since these two Koszul complexes are two presentations of the element \( \mathcal{B}_i^{(i)} \) we obtain:

\[
(\mathcal{B}_i^{(i)})^\vee = \mathcal{B}_i^{(i)}.
\]

The same argument implies the self-duality of the corresponding matrix factorization:

\[
(\mathcal{C}_i^{(i)})^\vee = \mathcal{C}_i^{(i)}.
\]

4.4. **Two strands.** Before we proceed with the proof of lemma \[ \text{[4.2.1]} \] let us write explicitly the generators of the Koszul complex \( \mathcal{B}_i^{(1)} \in \mathcal{D}_{G,\text{prop}}^{bT} \). If the space \( \mathfrak{b} \times G \times \mathfrak{b} \times G \) has coordinates \( (Y_1, g_1, Y_2, g_2) \) then collection of equations:

\[
(\text{Ad}_{g_2^{-1}g_1}Y_1)_{21} = 0, \quad (\text{Ad}_{g_2^{-1}g_1}Y_1)_{ij} = (Y_2)_{ij}, \quad ij = 11, 22, 12,
\]

defines the variety \( \mathcal{S}_i^{(1)} \) as complete intersection. Thus these elements provide a choice of generators for the Koszul complex \( \mathcal{B}_i^{(1)} \).

*Proof of Lemma \[ \text{[4.2.1]} \] The convolution space \( (\mathfrak{b} \times G)^3 \) has coordinates \( (Y_1, g_1, Y_2, g_2, Y_3, g_3) \). In these coordinates the pull-back \( \pi_{12}^*(\mathcal{B}_i^{(1)}) \) is a Koszul complex with generators as in \[ \text{[4.2]} \]. For the generators of \( \pi_{23}^*(\mathcal{B}_i^{(1)}) \) we choose a system of generators of the Koszul complex as follows:

\[
(\text{Ad}_{g_3^{-1}g_2}Y_3)_{21} = 0, \quad (\text{Ad}_{g_3^{-1}g_2}Y_3)_{ij} = (Y_2)_{ij}, \quad ij = 11, 22, 12.
\]
Thus the combination of the systems (4.2) and (4.3) is equivalent to the system:

\[
\begin{align*}
    \text{Ad}_{g_2^{-1} g_1} Y_1 &= \text{Ad}_{g_2^{-1} g_3} Y_3, \\
    (\text{Ad}_{g_2^{-1} g_1} Y_1)_{ij} &= (Y_2)_{ij}, \quad i j = 11, 22, 12, \\
    (\text{Ad}_{g_2^{-1} g_1} Y_1)_{21} &= 0.
\end{align*}
\]

The equations in the first line of the last system allow us to eliminate variable \( Y_2 \). Also the first group of equations in the first line is equivalent to the system

\[
\text{Ad}_{g_1} Y_1 = \text{Ad}_{g_3} Y_3.
\]

Thus the convolution of interest is given by

\[
\text{CE}_n(K[(\text{Ad}_{g_2^{-1} g_1} Y_1)_2]) \otimes B^{(1)}.
\]

Finally let us observe that the element

\[
(\text{Ad}_{g_2^{-1} g_1} Y_1)_{21} = a_{21}(a_{22}(y_{22} - y_{11}) + a_{21}y_{12}), \quad g = (a_{ij})
\]

has weight \((2, 0)\) with respect to the action of the \( T = \mathbb{C}^* \times \mathbb{C}^* \) since the elements \( a_{ij} \) have weight \( e_i \).

Since \( G/B = \mathbb{P}^1 \) we need to compute the push-forward along the projection of the two step complex

\[
\pi_{p_1}([O \rightarrow q^2 O(-2)]), \quad \pi_{p_1} : \mathbb{P}^1 \times b \times G \rightarrow b \times G.
\]

Since \( \pi_{p_1*}(O) = O \) and \( \pi_{p_1*}(O(-2)) = O[1] \) the statement follows. \( \square \)

4.5. Three stands. Before we prove the lemma 4.2.2 let us provide the details on the complexes that appear in the statement of the lemma.

Let us fix coordinates on the space \( b_3 \times G_3 \times b_3 \times G_3 \) as \( (Y_1, g_1, Y_2, g_2) \). A minimal set of equations defining \( \text{St}^{(1)} \subset b_3 \times G_3 \times b_3 \times G_3 \) is

\[
(g_1^{-1} g_2)_{ij}, \quad i j = 31, 32, \quad (\text{Ad}_{g_1} Y_1)_{21} = (\text{Ad}_{g_2} Y_2)_{21}.
\]

These equations define \( \text{St}^{(1)} \) as a complete intersection.

Similarly, a minimal set of equations for \( \text{St}^{(2)} \) is

\[
(g_1^{-1} g_2)_{ij} = 0, \quad i j = 21, 23, \quad (\text{Ad}_{g_1} Y_1)_{32} = (\text{Ad}_{g_2} Y_2)_{32}.
\]

Finally, the variety \( \text{St}^{(1,2)} \) has a description as a complete intersection:

\[
\text{Ad}_{g_1} Y_1 = \text{Ad}_{g_2} Y_2.
\]

4.5.1. In the next arguments we need a convolution statements that is parallel to the lemma 4.2.1

**Lemma 4.5.2.** For \( B^{(1)} \) and \( B^{(1,2)} \) we have

\[
B^{(1)} \ast B^{(1,2)} = B^{(1,2)} \ast B^{(1)} = B^{(1,2)} \oplus q^2 B^{(1,2)}.
\]
Proof. We show the last equality since the others are analogous. The convolution space \( \mathcal{X}_{conv} \subset (b \times G)^3 \) has coordinates \( (Y_1, g_1, Y_2, g_2, Y_3, g_3) \) with constraint \( g_2^{-1} g_3 \in P_1 \). In these coordinates the pull-back \( \pi_{12}^* (\mathcal{B}_{(1)}^*) \) is a Koszul complex with generators as in (4.4). For the generators of \( \pi_{23}^* (\mathcal{B}_{(1)}^*) \) we choose a system of generators of the Koszul complex as follows:

\[
\text{(4.5)} \quad (\text{Ad}_{g_2^{-1} g_3} Y_3)_{21} = 0, \quad (\text{Ad}_{g_2^{-1} g_3} Y_3)_{ij} = (Y_2)_{ij}, \quad i \leq j.
\]

Thus the combination of the systems (4.4) and (4.5) is equivalent to the system:

\[
\begin{align*}
\text{Ad}_{g_2^{-1} g_1} Y_1 &= \text{Ad}_{g_2^{-1} g_3} Y_3, \\
(\text{Ad}_{g_2^{-1} g_3} Y_3)_{ij} &= (Y_2)_{ij}, \quad i = 11, 22, 12, \\
(\text{Ad}_{g_2^{-1} g_3} Y_3)_{21} &= 0.
\end{align*}
\]

The equations in the first line of the last system allow us to eliminate variable \( Y_2 \). Also the first group of equations in the first line is equivalent to the system

\[
\text{Ad}_{g_1} Y_1 = \text{Ad}_{g_3} Y_3.
\]

Thus the convolution of interest is given by

\[
\text{CE}_{n}(K[(\text{Ad}_{g_2^{-1} g_3} Y_3)_{21}])^T \otimes \mathcal{B}_{(1,2)}^*.
\]

Finally let us observe that the element

\[
(\text{Ad}_{g_2^{-1} g_3} Y_3)_{21} = a_{21} (a_{22}(y_{22} - y_{11}) + a_{21} y_{12}), \quad g_2^{-1} g_3 = (a_{ij})
\]

has weight \((2, 0)\) with respect to the action of the \( T = \mathbb{C}^* \times \mathbb{C}^* \) since the elements \( a_{ij} \) have weight \( \epsilon_i \).

Since \( P_1 / B = \mathbb{P}^1 \) we need to compute the push-forward along the projection of the two step complex

\[
\pi_{P_1} ([\mathcal{O} \rightarrow a^3 \mathcal{O}(-2)]), \quad \pi_{P_1} : \mathbb{P}^1 \times b \times G \rightarrow b \times G.
\]

Since \( \pi_{P_1*} (\mathcal{O}) = \mathcal{O} \) and \( \pi_{P_1*} (\mathcal{O}(-2)) = \mathcal{O}[1] \) the statement follows.

\[\Box\]

**Lemma 4.5.3.** For \( \mathcal{B}_{(1)}^* \in D^b_{\mathcal{T}_q G_{3,\text{prop}}}(\mathfrak{g}_3^2) \) we have

\[
\mathcal{B}_{(1)}^* \ast \mathcal{B}_{(2)}^* = \mathcal{O}_{Z_{12}},
\]

where \( Z_{12} \subset \mathfrak{g}_3^2 \) is a complete intersection defined by

\[
(g_1^{-1} g_2)_{31} = 0, \quad \text{Ad}_{g_1}^{-1} \text{Ad}_{g_1} (Y_1)_{ij} = 0, \quad i = 32, 21, 31.
\]

\[
(Y_2)_{ij} = (\text{Ad}_{g_2}^{-1} \text{Ad}_{g_1} (Y_1))_{ij}, \quad i = 11, 22, 33, 12, 23, 13.
\]

Proof. The statement follows immediately from the observation that the projection: \( \pi_{13} : \pi_{12}^{-1} (\mathcal{S}^{(1)}) \cap \pi_{23}^{-1} (\mathcal{S}^{(2)}) \rightarrow Z_{12} \) is an isomorphism of reduced schemes.

\[\Box\]

Let us discuss a geometric description of \( Z_{12} \). It is piece of the Steinberg variety \( \widetilde{\text{St}}^{(1,2)} \). The Steinberg variety \( \text{St}^{(1,2)} \subset \mathfrak{g}_3^2 \) has six irreducible components \( \text{[CG97]} \):

\[
\text{St}^{(1,2)} = \bigcup_{w \in \Sigma_3} \Gamma_w.
\]
The generic point of $\Gamma_w$ consists of triples $(F', F'', X), X \in g_3$ and $F', F''$ are in relative position $w$.

The small Steinberg varieties are unions of subsets of irreducible components:

\[ S_t^{(1)} = \Gamma_1 \cup \Gamma_{s_1}, \quad S_t^{(2)} = \Gamma_1 \cup \Gamma_{s_2}. \]

Let us use notation $\tilde{\Gamma}_w$ for the corresponding subvariety of $(G_3 \times h_3)^2$. The computation above shows that

\[ Z_{12} = \Gamma_1 \cup \Gamma_{s_1} \cup \Gamma_{s_2} \cup \Gamma_{s_1s_2}. \]

**Proof of Lemma 4.2.3.** We need to compute the product

\[ \mathcal{O}_{Z_{12}} \ast \mathcal{B}^{(1)} = \left( \mathcal{B}^{(1)} \ast \mathcal{B}^{(2)} \right) \ast \mathcal{B}^{(1)}. \]

In geometric terms, the convolution is the push-forward:

\[ \pi_{13*}(\mathcal{O}_{Z_{123}}), \quad Z_{123} = \pi_{12}^{-1}(Z_{12}) \cap \pi_{23}^{-1}(S_t^{(1)}) \subseteq \tilde{g}_3^3. \]

Let us present the variety $Z_{123}$ as union of two collections of irreducible components:

\[ Z_{123} = Z_{123}' \cup Z_{123}'' , \quad Z_{123}' = \pi_{12}^{-1}(\Gamma_1 \cup \Gamma_{s_1}) \cap \pi_{23}^{-1}(\Gamma_{s_1}), \]
\[ Z_{123}'' = \pi_{12}^{-1}(Z_{12}) \cap \pi_{23}^{-1}(\Gamma_1) \bigcup \pi_{12}^{-1}(\Gamma_{s_2} \cup \Gamma_{s_1s_2}) \cap \pi_{23}^{-1}(\Gamma_{s_1}). \]

The intersection of the graph closures $\Gamma_w', \Gamma_w''$ is sent by the projection

\[ \mu_1 : (G \times h)^2 \to h \]
to the locus defined by the equations:

\[ w' \cdot \lambda(Y) = w'' \cdot \lambda(Y). \]

Thus the intersection $Z_{123}' \cap Z_{123}''$ is a divisor inside $Z_{123}'$, the defining equation of the divisor is $(Y_{12}^1 - Y_{11}^3)(Y_{12}^3 - Y_{22}^3)$. The vanishing locus of the first factor of the equation is the intersection $\pi_{12}^{-1}(\Gamma_1 \cup \Gamma_{s_1}) \cap \pi_{23}^{-1}(\Gamma_{s_1})$ and the second defines the intersection $\pi_{23}^{-1}(\Gamma_1) \cap \pi_{23}^{-1}(\Gamma_{s_1})$. Hence, we have the short exact sequence:

\[ 0 \to q^4\mathcal{O}_{Z_{123}'} \to \mathcal{O}_{Z_{123}} \to \mathcal{O}_{Z_{123}''} \to 0. \]

As a last step of the proof we need to compute the push forward $\pi_{13*}$ of the last short exact sequence. For that we first notice that

\[ \pi_{13*}(\mathcal{O}_{Z_{123}'}) = \mathcal{B}^{(1)} \ast \mathcal{O}_{\Gamma_w}. \]

Thus the computation of the last convolution product can done in the category $D_{G_2}(\tilde{g}_2^2)$ since both $\mathcal{B}^{(1)}$ and $\mathcal{O}_{\Gamma_{s_1}}$ are induced from $G_2 \subset G_3$. The argument analogous to the previously discussed lemma 4.5.2 yields:

\[ \mathcal{B}^{(1)} \ast \mathcal{O}_{\Gamma_{s_1}} = q^{-2}\mathcal{B}^{(1)}. \]

Next let use the following general fact about the graph closures $\Gamma_w$. Suppose that $w, w'' \in S_n$ such that $\ell(w'w'') = \ell(w') + \ell(w'')$ then the projection $\pi_{13} : \tilde{g}_3^3 \to \tilde{g}_2^2$ provides an isomorphism

\[ \pi_{12}^{-1}(\Gamma_w) \cap \pi_{23}^{-1}(\Gamma_{w''}) \to \Gamma_{w'w''}. \]
Thus the fact implies that in our situation \( \pi_{13*} \) restricts to the isomorphism: \( Z''_{123} \rightarrow St. \) Hence we have shown that \( \pi_{13*}(\mathcal{O}_{Z''_{123}}) = \mathcal{B}^{(1)} \) and \( \pi_{13*}(\mathcal{O}_{Z''_{123}}) = \mathcal{O}_{St} \) and the statement of the lemma follows.

\[ \square \]

4.5.4. **Computation of the extension group.** The statements from the previous section imply the statements for the matrix factorizations, since

\[ i_*(\mathcal{B}^{(i)}_*) = C^{(i)}_*, \quad i_*(\mathcal{B}^{(1,2)}_*) = C^{(1,2)}_. \]

To show that the short exact sequence (or may be triangle since we are in the triangulated category) of the matrix factorizations

\[ 0 \rightarrow C^{(1)}_* \rightarrow C^{(1)}_* \ast C^{(2)}_* \ast C^{(1)}_* \rightarrow C^{(1,2)}_* \rightarrow 0. \]

split we need to compute the corresponding extension group.

**Proposition 4.5.5.** For \( C^{(1)}_, C^{(1,2)}_\in \text{MF}_3 \) we have:

\[ \text{Ext}^0(C^{(1,2)}_, C^{(1)}_) = 0, \quad \text{Ext}^0(C^{(1,2)}_, C^{(1)}_) = C[y_1, y_2, y_3]_{\mathbb{O}^2}. \]

**Proof.** First we observe that \( \tilde{S}t \subset (G \times b)^2 \) is defined by

\[ (4.6) \quad \text{Ad}_{g_1}Y_1 = \text{Ad}_{g_2}Y_2, \]

where \((g_1, Y_1, g_2, Y_2)\) are the coordinates on \((G \times b)^2\). The last system of equations is \( B^2\)-invariant. Thus corresponding Koszul matrix factorization:

\[ C^{(1,2)} = K^W[\text{Ad}_{g_1}Y_1 - \text{Ad}_{g_2}Y_2] \]

is self-dual and

\[ \text{Ext}^*(C^{(1,2)}_, C^{(1)}_) = \text{Tor}^*(C^{(1,2)}_, C^{(1)}_) = \text{Tor}^*(C^{(1,2)}_, C^{(1)}_\parallel) = \text{Tor}^*(C^{(1,2)}_, C^{(1)}_\parallel)_{\mathbb{O}^2}. \]

The last equality follows from \[4.5.2\]

The last group can be computed by the standard method:

\[ \text{Tor}^*(C^{(1,2)}_, C^{(1)}_\parallel) = \text{CE}_{n^2}(K^W[\text{Ad}_{g_1}Y_1 - \text{Ad}_{g_2}Y_2] \otimes K^{-W}[g_1g_2^{-1} \in B, Y_1 - Y_2])_{T^2 \times G}, \]

where we used used the standard notation for the Koszul matrix factorizations on the space \( g \times (G \times b)^2 \) with coordinates \((X, g_1, Y_1, g_2, Y_2)\). As it is explained in section 13 of [OR18], the last group is equal to the pull-back

\[ \text{CE}_n(j_e^*(K^W[\text{Ad}_{g_1}Y_1 - \text{Ad}_{g_2}Y_2]))^T, \]

where \( j_e : g \times b \rightarrow g \times (G \times b)^2 \) is the \( B\)-equivariant embedding:

\[ j_e(X, Y) = (X, e, Y, e, Y). \]

Since the pull-back \( j_e^* \) turns the system \[4.6\] to the trivial system we obtain

\[ \text{CE}_n(j_e^*(K^W[\text{Ad}_{g_1}Y_1 - \text{Ad}_{g_2}Y_2]))^T = \text{CE}_n(K[X])^T = H_{Lie}^*(n, \mathbb{C}[b])^T. \]
The last group is the homology of the dg algebra $A = \mathbb{C}[b] \otimes \Lambda^\bullet(\theta_{ij})_{i < j}$ with the differential

$$D = \sum_{i < j} \theta_{ij} E_{ij},$$

where $E_{ij}$ is the action of $E_{ij}$ on $A$. The element $E_{ij}$ acts on $\Lambda^\bullet$ by the Lie bracket. Thus elements $\theta_{ij}$ have the $T$-weight $\epsilon_i - \epsilon_j$. Hence all elements in the algebra $A$ have the positive $T$-grading and thus $T$-invariant part of the complex is $\mathbb{C}[y_1, y_2, y_3]$. □

5. Markov-type MOY relations

5.1. Markov relations. In this section we prove the last two statements that are needed for the comparison of Hom spaces. To state the main result of this section it is convenient to assemble the traces $\text{Tr}^i$ from the section [6] into one triply graded vector space:

$$\text{Tr}^*(\mathcal{F}) = \bigoplus_i \text{Tr}^i(\mathcal{F}), \quad \mathcal{F} \in \mathcal{M}F^*_n.$$

Let us recall that each of the spaces $\text{Tr}^i(\mathcal{F})$ is a doubly graded vector space. We call these two grading $q, t$-gradings and use notation $q^\cdot, t^\cdot$ for the functors that shift corresponding grading. The additional grading $*$ in $\text{Tr}^*$ name $a$-grading and use $a^\cdot$ for the corresponding grading shifting functor.

In section [6] we introduce notation $D_{i,i+1}^\bullet$ for the generators of free monoid of planar diagram. We denote by $\mathcal{B}r_n^\bullet$ the free monoid of the diagram on $n$ strings. Respectively, to each $D \in \mathcal{B}r_n^\bullet$ we attach an element $\mathcal{F}_D \in \mathcal{M}F^*_n$. We use same notation for the pull-back of $\mathcal{F}_D$ on the stable locus.

The main statement of this section is analog of the second Markov moves for the plane diagrams:

**Proposition 5.1.1.** Suppose $D \in \mathcal{B}r_n^\bullet$ is element of the sub-monoid

$$\langle D_{(1,2)}, \ldots, D_{(n-2,n-1)} \rangle.$$

Respectively, $D' \in \mathcal{B}r_{n-1}^\bullet$ is the diagram $D$ with the last strand removed. Then we have:

1. $\text{Tr}^*(\mathcal{F}_D) = \frac{1-a^2}{1-q^2} \cdot \text{Tr}^*(\mathcal{F}_{D'}).$
2. $\text{Tr}^*(\mathcal{F}_{D \circ D'_{(n-1,n)}}) = \frac{1-a^2}{1-q^2} \cdot \text{Tr}^*(\mathcal{F}_{D'}).$

5.2. Geometric Markov moves. Before we prove the proposition let us recall key points of the construction of $\text{Tr}^i$ from [OR18]. Our proof is very much analogous to the proof of the Markov move [OR18].

In [OR18] we work with the free Hilbert scheme $\text{Hilb}_{1,n}^{\text{free}}$ instead of the usual Hilbert scheme of points on plane $\text{Hilb}_n(\mathbb{C}^2)$. We define the trace in terms of two-periodic sheaves on the this free Hilbert scheme and in subsequent paper [OR18e] we explain that the trace from [OR18] could be interpreted in terms of $\text{Hilb}_n(\mathbb{C}^2)$. We remind the details of this relation in the next section but for now let us recall the basic geometric constructions from [OR18].
The free nested Hilbert scheme $\widehat{\text{FHilb}}_{n}^{\text{free}}$ is a $B \times \mathbb{C}^*$-quotient of the sublocus
\[ \widehat{\text{FHilb}}_{n}^{\text{free}} \subset b \times n \times V_n \]
of the cyclic triples
\[ \widehat{\text{FHilb}}_{n}^{\text{free}} = \{(X, Y, v)|\mathbb{C}(X, Y)v = V_n\}, \]
here $V_n = \mathbb{C}^n$. The usual nested Hilbert scheme $\text{FHilb}_{n}$ is the subvariety of $\widehat{\text{FHilb}}_{n}^{\text{free}}$, it is defined by the condition that $X, Y$ commute.

**Remark 5.2.1.** There is a bit of confusion in the notations, what we denote here by $\text{FHilb}_{n}$ is denoted in [OR18f] by $\text{Hilb}_{1,n}$ and by $\widehat{\text{FHilb}}_{n}$ in [GRN16].

The torus $T_{qt} = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\widehat{\text{FHilb}}_{n}^{\text{free}}$ by scaling the matrices. We denote by $D_{T_{qt}}^{\text{per}}(\text{FHilb}_{n}^{\text{free}})$ a derived category of the two-periodic complexes of the $T_{qt}$-equivariant quasi-coherent sheaves on $\widehat{\text{FHilb}}_{n}^{\text{free}}$. Let us also denote by $\mathcal{B}^\vee$ the descent of the trivial vector bundle $V_n$ on $\widehat{\text{FHilb}}_{n}^{\text{free}}$ to the quotient $\text{FHilb}_{n}^{\text{free}}$. Respectively, $\mathcal{B}$ stands for the dual of $\mathcal{B}^\vee$. Below we construct for every $\beta \in \mathfrak{Br}_n$ an element
\[ S_\beta \in D_{T_{qt}}^{\text{per}}(\text{FHilb}_{n}^{\text{free}}) \]
such that space of hyper-cohomology of the complex:
\[ H^k(S_\beta) := H(S_\beta \otimes \Lambda^k \mathcal{B}) \]
defines an isotopy invariant.

**Theorem 5.2.2.** [OR18f] For any $\beta \in \mathfrak{Br}_n$ the doubly graded space
\[ H^k(\beta) := H((k+\text{writhe}(\beta))-n)/2(S_\beta) \]
is an isotopy invariant of the braid closure $L(\beta)$.

The variety $\widehat{\text{FHilb}}_{n}^{\text{free}}$ embeds inside $\overline{\mathcal{X}}_n^0$ via a map $j_e : (X, Y, v) \rightarrow (X, \text{diag}(Y), e, Y, v)$. The diagonal copy $B = B_\Delta \hookrightarrow B^2$ respects the embedding $j_e$ and since $j_e^*(\overline{W}) = 0$, we obtain a functor:
\[ j_e^* : \text{MF}_{B^2}(\overline{\mathcal{X}}_n^0, \overline{W}) = \text{MF}_{B^2}(\overline{\text{FHilb}}_{n}^{\text{free}}, 0). \]

Respectively, we get a geometric version of ”closure of the braid” map:
\[ L_{\text{st}} : \text{MF}_{B^2}(\overline{\mathcal{X}}_n^0, \overline{W}) = \text{MF}_{B^2}(\overline{\text{FHilb}}_{n}^{\text{free}}, 0). \]

As we explained earlier, there is an isomorphism of categories:
\[ K\mathcal{N}^\circ : \overline{\text{MF}}_n \rightarrow \text{MF}^\circ_n. \]
Moreover in [OR18f] we show that the category $\overline{\text{MF}}_n$ has a natural monoidal structure $\ast$ and $K\mathcal{N}^\circ$ is monoidal. We also constructed the corresponding homomorphism $\Phi_n$ from...
the braid group that fits into the commuting diagram:

\[
\begin{array}{ccc}
\mathcal{B}_n & \xrightarrow{\pi_n} & (MF_n, \ast) \\
\Phi_n & \downarrow & \downarrow KN^\circ \\
& & (MF_n^\circ, \ast)
\end{array}
\]

The main result of [OR18f] could be restated in more geometric term via geometric trace map:

\[\mathcal{T} : \mathcal{B}_n \to \text{D}^\text{per}_{\text{Tr}}(\text{FHilb}_{n}^{\text{free}}), \quad \mathcal{T}(\beta) := \oplus_k \mathbb{L}^\text{st}(\Phi_n(\beta) \otimes \Lambda^k \mathcal{B}).\]

The above mentioned complex \(S_\beta\) is the complex \(\mathbb{L}^\text{st}(\Phi_n(\beta))\). The differentials in the complex \(S_\beta\) are of degree \(t\) thus the differentials are invariant with respect to the anti-diagonal torus \(T_a\).

For any braid graph \(D \in \mathcal{B}_n^0\) we define

\[S_D = \mathbb{L}^\text{st}(KN^\circ(F_D)), \quad H^k(D) = \mathbb{H}(S_D \otimes \Lambda^k \mathcal{B}).\]

In our proof of the Markov moves we use nested nature of the scheme \(\text{F Hilb}_n\) to define the intermediate map:

\[\pi : \text{F Hilb}_n^{\text{free}} \to \mathbb{C} \times \text{F Hilb}_n^{\text{free}},\]

where the first component of the map \(\pi\) is \(x_{11}\) and the second component is just forgetting of the first rows and rows of the matrices \(X, Y\) and the first component of the vector \(v\). Let us also fix notation for the line bundles on \(\text{F Hilb}_n^{\text{free}}\): we denote by \(\mathcal{O}_k(-1)\) the line bundle induced from the twisted trivial bundle \(\mathcal{O} \otimes \chi_k\). It is quite elementary to show

**Proposition 5.2.3.** The fibers of the map \(\pi\) are projective spaces \(\mathbb{P}^{n-1}\) and

1. \(B_n/\pi^*(B_{n-1}) = \mathcal{O}_n(-1)\).
2. \(\mathcal{O}_n(-1)|_{\pi^{-1}(z)} = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)\).

We can combine the last proposition with the observation that the total homology \(H^*(\mathbb{P}^{n-1}, \mathcal{O}(-l))\) vanish if \(l \in (1, n-1)\) and is one-dimensional for \(l = 0, n:\)

**Corollary 5.2.4.** For any \(n\) we have:

- \(\pi_*(\Lambda^k B_n) = \Lambda^k B_{n-1}\)
- \(\pi_*(\mathcal{O}_n(m) \otimes \Lambda^k B_n) = 0\) if \(m \in [-n+2, -1]\).
- \(\pi_*(\mathcal{O}_n(-n+1) \otimes \Lambda^k B_n) = \Lambda^{k-1} B_{n-1}[n]\)

The geometric version of the Markov moves is the following

**Theorem 5.2.5.** For any \(D \in \mathcal{B}_n^0, D' \in \mathcal{B}_n^{n-1}\) as in the statement of proposition 5.1.1 we have

\[
\begin{align*}
\mathbb{H}^k(D \circ D') &= (\mathbb{H}^k(D') \oplus \mathbb{H}^{k-1}(D')) \otimes \mathbb{C}[x_{11}] \\
\mathbb{H}^k(D) &= (\mathbb{H}^k(D') \oplus \mathbb{Q}^2 \cdot \mathbb{H}^{k-1}(D')) \otimes \mathbb{C}[x_{11}].
\end{align*}
\]

Our proof of the theorem follows the method of [OR18f] where similar statement is shown in section 13. To make this paper self-contain we present a slightly more geometric proof of the main technical step of the proof. We dedicate the next section to these results.
5.3. Induction and closure. In this section we explore interaction between the induction, convolution and the closure functor $\mathbb{L}$. Before we state our main result we need to extend some of the definitions from the reduced category $\text{MF}_n$ to $\text{MF}_n^\circ$. In particular, we define:

$$L^\circ : \text{MF}_n^\circ \rightarrow D^\text{per}_{\mathfrak{a},\mathfrak{t} \times B^2}(n_n \times b_n), \quad L^\circ = j_e^* \circ (KN^\circ)^{-1},$$

$$L = i_{e^*} : \text{MF}_n \rightarrow D^\text{per}_{\mathfrak{a},\mathfrak{t} \times B^2}(n_n \times b_n)$$

To state our first result let us recall notation for the parabolic subgroups: $P_I \subset G_n$, $I \subset \{1, \ldots, n-1\}$, is the subgroup such that $\text{Lie}(P_I) = p_I$ is generated by $\mathfrak{b}$ and $E_{i,i+1}$, $i \notin I$. We also define $P_{\geq k} \subset P_{k,n-1}$ as a kernel of homomorphism $P_{k,n-1} \rightarrow (\mathbb{C}^*)^{n-k}$, that is the projection on the last $n-k$ diagonal elements. Using this group we can define the inclusion functor:

$$\mathcal{X}_n^\circ(P_{\geq k}) \xrightarrow{i_{\geq k}} \mathcal{X}_n^\circ$$

$$\downarrow p_{\geq k} \quad , \quad \text{inc}_{\geq k} = i_{\geq k,*} \circ p_{\geq k}^*,$$

$$\mathcal{X}_k^\circ$$

where here and everywhere below

$$\mathcal{X}_n^\circ(P) = \text{Lie}(P) \times b_n \times P \times b_n.$$

There is a natural relation between the inclusion and induction functors:

**Proposition 5.3.1.** For any $n$ and $k \leq n-1$ we have:

$$\text{ind}_k(F \times C_{\|}) = \text{inc}_{\geq k}(F).$$

**Proof.** The formula follows from the construction of the unit matrix factorization as push-forward $C_{\|} = i_{\geq 1,*}(O)$, since $P_{\geq 1} = U$. The map $i_{\geq 1}$ naturally fits into the commutative diagram:

$$\mathcal{X}_n^\circ(P_{\geq k}) \xrightarrow{i'} \mathcal{X}^\circ(P_{k}) \xrightarrow{i_k} \mathcal{X}_n^\circ$$

$$\downarrow p' \quad \downarrow p_k \quad \quad , \quad \quad \text{inc}_{\geq k} = i_{\geq k,*} \circ p_{\geq k}^*,$$

$$\mathcal{X}_k^\circ \times \mathcal{X}_{n-k}(U_{n-k}) \xrightarrow{1 \times i_{\geq 1}} \mathcal{X}_k^\circ \times \mathcal{X}_{n-k}^\circ.$$

Now we can use the base-change relations:

$$\text{ind}_k(F \times C_{\|}) = i_{k,*} \circ p_{k}^*(F \times C_{\|}) = i_{k,*} \circ (1 \times i_{\geq 1})_* \circ (p')^*(F) = i_{k,*} \circ i'_{\geq 1} \circ (p')^*(F) = i_{\geq k,*} \circ p_{\geq k}^*(F) = \text{inc}_{\geq k}(F),$$

where we used the base-change and $p'' \circ p' = p_{\geq k}$, $i_k \circ i' = i_{\geq k}$. □
To connect with the convolution operation with the closure operation $\mathbb{L}^\circ$ we need to discuss the Knorrer functor $\mathcal{K} \mathcal{N}^\circ$. Observe that the potential $\bar{\mathcal{W}}^\circ$ has a quadratic summand: 

$$\bar{\mathcal{W}}^\circ = \text{Tr}(XY_1) - \text{Tr}(X + \text{Ad}_g Y_2).$$

Thus we have an isomorphism of categories $\overline{\mathsf{MF}}^\circ_n \cong \overline{\mathsf{MF}}_n$. To obtain the isomorphism $\mathcal{K} \mathcal{N}^\circ$ we need to compose the last isomorphism with $\mathcal{K} \mathcal{N}$. The composition fits into the commuting diagram:

$$
\begin{array}{ccccccccc}
g_n \times G_n \times b_n & \xrightarrow{\rho^\circ} & \mathcal{X}^\circ_n & \xrightarrow{j_{kn}} & X_n \\
\uparrow & & \uparrow & & \uparrow \\
b_n \times G_n \times b_n & \xrightarrow{j^\circ} & \mathcal{X}^\circ_n & \xrightarrow{\pi_{kn}} & b_n^2 \times G_n b \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_n & \xrightarrow{\rho^\circ} & b_n \times G_n \times b_n
\end{array}
$$

where $p^\circ(X, Y_1, g, Y_2) = (X, g, Y_2)$ and $j^\circ$ is the natural inclusion of $X_n = n_n \times G_n \times b_n$. Thus the base-change relation implies that

$$\mathcal{K} \mathcal{N}^\circ = p^\circ_* \circ j^\circ_*.$$

Using base-change formula we can simplify the formula for the functor $\mathbb{L}^\circ$. We can describe the functor in terms of maps:

$$j^\circ_e : n_n \times b_n \times b_n \to \mathcal{X}^\circ, \quad \rho_e : n_n \times b_n \times b_n \to n_n \times b_n,$$

$$j^\circ_e(X, Y_1, Y_2) = (X, Y_1, 1, Y_2), \quad \rho_e(X, Y_1, Y_2) = (X, Y_2).$$

**Proposition 5.3.2.** For any $\mathcal{F} \in \mathsf{MF}^\circ_n$ we have:

$$\mathbb{L}^\circ(\mathcal{F}) = \rho_{e*} \circ j^\circ_e(\mathcal{F}).$$

**Proof.** The above maps and the functor $\mathcal{K} \mathcal{N}^\circ$ fit into commuting diagram:

$$
\begin{array}{ccccccccc}
g_n \times G_n \times b_n & \xrightarrow{\rho^\circ} & \mathcal{X}^\circ_n & \xrightarrow{j_{kn}} & X_n \\
\uparrow & & \uparrow & & \uparrow \\
n_n \times b_n & \xrightarrow{j_e} & \mathcal{X}_n & \xrightarrow{\rho_e} & n \times b_n \times b_n
\end{array}
$$

Thus we can apply the base-change to complete the proof:

$$\mathbb{L}^\circ = j^\circ_e \circ j^\circ_e^* \circ \rho^\circ_\ast = \rho_{e*} \circ j^\circ_e^*.$$

\[ \square \]

We need a slight generalization of the previous formula that explain how to compute a closure of the convolution of two elements.

First, let us recall how we define the monoidal structure on $\mathsf{MF}^\circ_n$:

$$\mathcal{X}_{\text{con}}^\circ = g \times b \times G_n \times b \times G_n \times b, \quad \mathcal{M}_{ij}^\circ : \mathcal{X}_{\text{con}}^\circ \to \mathcal{X}^\circ_n,$$
$$\pi_{12}(X, Y, g_{12}, Y, g_{23}, Y) = (X, Y, g_{12}, Y), \quad \pi_{13}(X, Y, g_{12}, Y, g_{23}, Y) = (X, Y, g_{12}g_{23}, Y),$$
$$\pi_{23}(X, Y, g_{12}, Y, g_{23}, Y) = (\text{Ad}_{g_{12}}^{-1}(X), Y, g_{23}, Y),$$
$$\mathcal{F} \star \mathcal{G} = \pi_{13*}(\text{CE}_n(\pi_{12}^*(\mathcal{F}) \otimes \pi_{23}^*(\mathcal{G}))^T).$$

To state the next statement we set
$$j_{s,G}^\circ : n \times b_n^3 \times G \to X_n^\circ, \quad \rho : n \times b_n^3 \times G \to n \times b_n.$$
$$j_{l,G}^\circ(X, Y, Y, g) = (X, Y, g, Y), \quad j_{r,G}^\circ(X, Y, Y, g) = (\text{Ad}_g^{-1}X, Y, g^{-1}, Y),$$
$$\rho_G(X, Y, Y, g) = (X, Y).$$

**Proposition 5.3.3.** For any $\mathcal{F}, \mathcal{G} \in \text{MF}_n^\circ$ we have:
$$\mathbb{L}^\circ(\mathcal{F} \star \mathcal{G}) = \rho_{G*}(\text{CE}_n(j_{l,G}^\circ)^*(\mathcal{F}) \otimes j_{r,G}^\circ)^*(\mathcal{G}))^T),$$
where the $B$-action in the quotient is
$$b \cdot (X, Y, Y, g) = (X, Y, \text{Ad}_b(Y), Y, g^{-1}).$$

**Proof.** The statement follows from the base-change formula applied to the commuting diagram of maps:

$$\begin{array}{ccc}
n \times b_n^3 \times G & \xrightarrow{j_{e,G}} & X_{\text{con}}^\circ \\
p_{e,G} \downarrow & & \downarrow \pi_{13}^\circ \\
n \times b_n & \xrightarrow{j_e} & X_n^\circ
\end{array}$$

where the new maps are
$$j_{e,G}(X, Y, Y, g) = (X, Y, g, Y, g^{-1}, Y), \quad p_{e,G}(X, Y, Y, g) = (X, Y, Y).$$
To complete the proof one can use the base-change and
$$\rho \circ p_{e,G} = \rho_G, \quad \pi_{12}^\circ \times \pi_{13}^\circ \circ j_{e,G} = j_{l,G}^\circ \times j_{r,G}^\circ.$$

Now we need combine the previous computation with the inclusion functor. The pair of maps needed for this combination is:
$$j_{l,G}^\circ : n \times b_n^3 \times P_{\geq k} \to X_n^\circ, \quad p_{l,G}^\circ : n \times b_n^3 \times P_{\geq k} \to X_k^\circ,$$
where $j_{l,G}^\circ$ is a restriction of $j_{l,G}$ to the subspace and $p_{r,\geq k}$ is a restriction of $j_{r,G}$ post-composed with the map $X_n(P_{\geq k}) \to X_k^\circ$.

**Proposition 5.3.4.** For any $\mathcal{F} \in \text{MF}_n^\circ, \mathcal{G} \in \text{MF}_k^\circ$ we have:
$$\mathbb{L}^\circ(\mathcal{F} \star \text{inc}(\mathcal{G})) = \rho_{G*}(\text{CE}_n(j_{l,G}^\circ)^*(\mathcal{F}) \otimes p_{r,\geq k}^\circ)^*(\mathcal{G}))^T),$$
Proof. There is a unique map $\phi$ such that the diagram below commutes:

$$
\begin{array}{c}
\mathcal{X}_n^0 \times \mathcal{X}_n^0 \\
\downarrow j_{l,G} \times j_{l,G} \\
\mathcal{X}_n^0(P_{\geq k}) \times \mathcal{X}_n^0 \\
\downarrow \phi \\
\mathcal{X}_k^0 \times \mathcal{X}_k^0 \\
\end{array}
$$

Thus the statement follow from the base-change formula.

The closure functor in reduced category $\overline{\text{MF}}_n$ is simpler than in the full category $\text{MF}^0$. Thus we use the the matrix factorizations from the reduced category in our main lemma and we give a version of the previous result in the reduced category.

To state the reduced version of the result let us recall that we the automorphism of the space $\mathcal{X}_n$:

$$
n_n \times G_n \times b_n \ni (X, g, Y) \to (X, g, -Y)
$$

induces an equivalence of categories $\overline{\text{MF}}_n$ and $\overline{\text{MF}}^*_n = \text{MF}^{B_2}(\mathcal{X}_n, -W)$. For $\mathcal{F} \in \overline{\text{MF}}_n$ we denote by $\mathcal{F}^*$ the corresponding element of $\overline{\text{MF}}^*_n$.

Next we fix a pair of natural maps from the space $\mathcal{X}_n(P_{\geq k}) = n_n \times P_{\geq k} \times b_n$:

$$
\bar{j}_{l,\geq k} : \mathcal{X}_n(P_{\geq k}) \to \mathcal{X}_n, \quad \bar{p}_{r,\geq k} : \mathcal{X}_n(P_{\geq k}) \to \mathcal{X}_k
$$

Proposition 5.3.5. For any $\mathcal{F} \in \overline{\text{MF}}_n$, $\mathcal{G} \in \overline{\text{MF}}_k$ we have:

$$
\mathcal{L}^0(KN^0(\mathcal{F}) \star \text{inc}_{\geq k}(KN^0(\mathcal{G}))) = \rho_{G^*}(\text{CE}_n(\bar{j}_{l,\geq k}^*, \mathcal{F}) \otimes \bar{p}_{r,\geq k}^*(\mathcal{G}^*)^T)
$$

Proof. In the diagram below the dashed arrows correspond to the unique maps that makes diagram commute:

$$
\begin{array}{c}
\mathcal{X}_n^0 \times \mathcal{X}_n^0 \\
\downarrow j_{l,\geq k} \times \rho_{G^*} \times j_{l,\geq k} \\
\mathcal{X}_k^0 \times \mathcal{X}_k^0 \\
\end{array}
$$

The $B^2_n$-equivariant projection

$$
\pi : n_n \times b_n \to b_{n-1} \times b_{n-1},
$$

that projects matrices to the subspace spanned by the matrix units $E_{ij}, i, j > 1$ descends to the projection $\pi$ between the free flag Hilbert scheme $[5.1]$. On the other hand the projection

$$
\pi_{\geq k} : n_n \times b_n \to n_k \times b_k.
$$

that projects matrices to the subspace spanned by the matrix units $E_{ij}, i, j \leq k$ does not descend to to the map between the free flag Hilbert schemes. The failure is due to the fact
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that the stability condition is incompatible with the projection. However, we can use this projection to understand the homology of the graphs that appear in the Markov move theorem:

**Lemma 5.3.6.** Let \( \overline{F} \in \text{MF}_{n-1} \), \( \overline{G} \in \text{MF}_2 \) and \( V = \mathbb{C}^{n-2}(\epsilon_1) \) is a vector space with the \( B_n \)-action given by the character \( \chi_1 = \exp(\epsilon_1) \). Suppose we have

\[
L(F) = (M, d) \in D^\text{per}_{B^2}(n_{n-1} \times b_{n-1})
\]

then there is a deformed \( \mathbb{Z}_2 \)-graded complex

\[
\overline{F}' = (\pi^* M \otimes \Lambda^{\text{even}} V \oplus \pi^* M \otimes \Lambda^{\text{odd}} V, D_V + \pi^* d) \in D^\text{per}(n_n \times b_n)
\]

\[
D_V : \pi^* M \to \pi^* M \otimes V \to \pi^* M \otimes \Lambda^2 V \to \pi^* M \otimes \Lambda^3 V \to \pi^* M \otimes \Lambda^4 V \to \cdots
\]

such that

\[
L \left( \text{ind} \circ KN^\circ(\overline{F}) \ast \text{inc} \circ KN^\circ(\overline{G}) \right) = \overline{F}' \otimes \pi^*_{\geq 2}(L(\overline{G}^*))
\]

In the proof of the lemma we use the reduced version of the induction and inclusion functors:

\[
\overline{X}_n(P_{\geq k}) \xrightarrow{i_{\geq k}} \overline{X}_n \quad \overline{X}_n(P_{k}) \xrightarrow{i_{k}} \overline{X}_n \quad \text{inc}_{\geq k} = i_{\geq k} \ast \overline{P}_{\geq k}^*,
\]

\[
\overline{X}_n \to \overline{X}_n \times \overline{X}_{n-k} \quad \text{ind}_{k} = i_{k} \ast \overline{P}_{k}^*
\]

where here and everywhere below

\[
\overline{X}_n(P) = n_n \times P \times b_n.
\]

The Knorrer functor \( KN^\circ \) intertwines these functors with the functors \( \text{inc}_{\geq k} \) and \( \text{ind}_k \) thus the lemma above is equivalent to the formula:

\[
L \left( KN^\circ \circ \text{ind}_1(\overline{F}) \ast KN^\circ \circ \text{inc}_2(\overline{G}) \right) = \overline{F}' \otimes \pi^*_{\geq 2}(L(\overline{G}^*))
\]

**Proof of Lemma 5.3.6.** Let us denote by \( P' \subset G_n \) the \( B^2 \) invariant subspace defined by

\[
P' = \{ g \in G_n | g_i = 0, i = 3, \ldots, n \}.
\]

The natural inclusions

\[
j'_1 : P_1 \to P', \quad j_{\geq 2} : P_{\geq 2} \to P', \quad i' : P' \to G_n,
\]
induces the dashed arrow maps in the diagram:

\[
\begin{array}{c}
\xymatrix{
\mathcal{X}_n(P') & \mathcal{X}_n(P_1) \ar[r]^\rho_1 & \mathcal{X}_{n-1} \times \mathcal{X}_1 \\
\mathcal{X}_2 \ar[u]^{\tilde{j}_2} \ar[r]_{\tilde{\rho}_2} & \mathcal{X}_n(P_{\geq 2}) \ar[r]^\gamma & \mathcal{X}_n \ar[r]_{\tilde{\gamma}_1} & \mathcal{X}_n(P') \\
}
\end{array}
\]

There are natural maps \(\rho_G\) from the spaces in the second and third diagram to \(n_n \times b_n\) which intertwine the maps in the diagram.

Now we show how one can use base-change and projection formula to prove (5.2). According to proposition 5.3.5 the LHS of the formula is obtained by application of \(CE_n(\cdot)^T\) to

\[
\rho_{G^*}(\tilde{j}_{1*} \circ \tilde{p}_1^*(\mathcal{F}) \otimes \tilde{\iota}_{\geq 2*} \circ \tilde{p}_{i > 2}^*(\mathcal{G}^*)) = \rho_{G^*}(\tilde{i}^* \circ \tilde{j}_{1*} \circ \tilde{p}_1^*(\mathcal{F}) \otimes \tilde{j}_{\geq 2*} \circ \tilde{p}_{\geq 2}^*(\mathcal{G}^*))
\]

here we used that \(\tilde{i}_{\geq 2*} = \tilde{\iota}_* \circ \tilde{j}_*\) and the projection formula for the map \(\tilde{\iota}_*\). Similarly we have \(\tilde{j}_{1*} = \tilde{\iota}_* \circ \tilde{j}_*\), hence

\[
\tilde{i}^* \circ \tilde{j}_{1*} \circ \tilde{p}_1^*(\mathcal{F}) = \tilde{i}^* \circ \tilde{\iota}_* \circ \tilde{j}_* \circ \tilde{p}_1^*(\mathcal{F}) = \tilde{i}^* \circ \tilde{j}_*^\prime(\mathcal{F}'), \quad \mathcal{F}' = \tilde{j}_* \circ \tilde{p}_1^*(\mathcal{F}).
\]

The key observation follows from the construction of the push-forward for matrix factorizations from [OR18]. Indeed, if \(\mathcal{F}' = (M', d')\) then the composition \(\tilde{i}^* \circ \tilde{j}_*^\prime(\mathcal{F}')\) is equal to the matrix factorization

\[
(M' \otimes \Lambda^* V, d' + D'_v), \quad D'_v \in \oplus_{k, j \geq 0} \text{Hom}(M' \otimes \Lambda^j V, M' \otimes \Lambda^{j+2k} V),
\]

where \(V\) is a vector space spanned by the matrix elements \(g_{i1}, i \geq 3\). These are exactly the matrix elements vanish on \(P'\). We note the last matrix factorization by \(\mathcal{F}' \otimes \Lambda^* V\) for brevity.

To complete our proof we need to apply the base-change relation one more time. We work with the diagram:

\[
\begin{array}{c}
\xymatrix{
\mathcal{X}_n(B_n) \ar[r]^\delta & \mathcal{X}_n(P_1) \ar[r]^\rho_1 & \mathcal{X}_{n-1} \times \mathcal{X}_1 \\
\mathcal{X}_2 \ar[u]^{\tilde{j}_2} \ar[r]_{\tilde{\rho}_2} & \mathcal{X}_n(P_{\geq 2}) \ar[r]^\gamma & \mathcal{X}_n \ar[r]_{\tilde{\gamma}_1} & \mathcal{X}_n(P') \\
}
\end{array}
\]

Thus by the previous argument:

\[
\rho_{G^*}(\tilde{j}_*^\prime \circ \tilde{p}_1^*(\mathcal{F}) \otimes \Lambda^* V \otimes \tilde{j}_{\geq 2*} \circ \tilde{p}_{\geq 2}^*(\mathcal{G}^*)) = \rho_{G^*}(\tilde{p}_1^*(\mathcal{F}) \otimes \tilde{j}_*^\prime(\Lambda V^* \otimes \delta \circ \gamma^* \circ \tilde{p}_{\geq 2}^*(\mathcal{G}^*)) = \rho_{G^*}(\delta^* \circ \tilde{p}_1^*(\mathcal{F}) \otimes \delta^* \circ \tilde{j}_*^\prime(\Lambda V^* \otimes \gamma^* \circ \tilde{p}_{\geq 2}^*(\mathcal{G}^*))
\]

The \(B_n\)-action on \(\mathcal{X}_n(B_n)\) is free. Thus the functor \(CE_n(\cdot)^T\) on this space is equivalent to the functor of restriction on the \(B_n\)-action slice:

\[
\tilde{j}_c : \mathcal{X}_n \rightarrow n_n \times b_n.
\]

Thus by the previous argument the LHS of (5.2) is equal

\[
\tilde{j}_c^*(\delta^* \circ \tilde{p}_1^*(\mathcal{F}) \otimes \delta^* \circ \tilde{j}_*^\prime(\Lambda V^*) \otimes \gamma^* \circ \tilde{p}_{\geq 2}^*(\mathcal{G}^*)).
\]
The statement follows the last formula because
\[ \tilde{j}_e \circ \gamma^* \circ \tilde{p}_{i \geq 2}^*(G^*) = \pi_{\geq 2}^*(L(V^*)), \quad \tilde{j}_e^*(\delta^* \circ \tilde{p}_i^*(\overline{F}) \otimes \delta^* \circ \tilde{j}_e^*(\Lambda^i V^*)) = \overline{F}', \]
where \( D_V = \tilde{j}_e \circ \delta^* \circ \tilde{j}_e^*(D_V) \).

5.4. Proof of Markov move relations. Let us set
\[ \mathcal{F}_\bullet = \mathcal{F}_{D \circ D_{i \circ (n-1,n)}} \in MF(\mathcal{X}_n, \mathcal{W}) \]
and \( \bullet \) can be either \( \bullet \) or \( \parallel \).

The lemma 5.3.6 is directly applicable in the setting of the proof, that is:
\[ \mathbb{L}(\mathcal{F}_\bullet) = \mathcal{F}_D' \otimes \pi_{\geq 2}^*(\mathbb{L}(\mathcal{F}_\parallel^*)). \]

To describe the second factor in the last formula let us fix coordinates on \( \mathcal{X}_n = n_n \times G_n \times \mathfrak{b}_n \) as \( X \in n_n, Y \in \mathfrak{b}_n \). Then the pull-backs are the following Koszul complexes:
\[ \pi_{\geq 2}^*(\mathbb{L}(\mathcal{F}_\bullet)) = K_\bullet = K[x_{12}], \quad \pi_{\geq 2}^*(\mathbb{L}(\mathcal{F}_\parallel^*)) = K_\parallel = K[x_{12}y_{11}]. \]

After restricting to the stable locus these Koszul complexes define the divisors \( \text{FHilb}_{n,\bullet} \subset \text{FHilb}^{\text{free}}_n \). Let us describe these divisors.

The projection \( \pi \) is surjective on the divisors \( \text{FHilb}_{n,\bullet} \). Moreover, every fiber of the projection \( \pi|_{\text{FHilb}_{n,\bullet}} \) is a projective space \( \mathbb{P}^{n-2} \). The divisor \( \text{FHilb}_{n,\parallel} \) is bigger,
\[ \text{FHilb}_{n,\parallel} = \text{FHilb}_{n,\bullet} \cup \pi^{-1}(0 \times \text{FHilb}^{\text{free}}_{n-1}). \]

That is the fibers of \( \pi|_{\text{FHilb}_{n,\parallel}} \) are \( \mathbb{P}^{n-2} \) outside the divisor \( x_{11} = 0 \) and the fiber is \( \mathbb{P}^{n-1} \) over the divisor.

The vector space \( V \) in the definition of complex \( \overline{F}_D' \) becomes the vector bundle \( \mathcal{O}_n(-1)^{\otimes n-2} \). Hence to compute \( \pi_*(\overline{F}_D' \otimes \Lambda^k \mathcal{B}_n) \), we need to modify the previous corollary.

We denote \( \pi_\bullet \) and \( \pi_\parallel \) the restriction of the projection \( \pi \) on the corresponding divisor:
\[ \pi_\bullet (\mathcal{C}) = \pi_\parallel (K_\bullet \otimes \mathcal{C}). \]

Let us discuss the case \( \bullet = \bullet \). Then for any \( z \in \mathbb{C} \times \text{FHilb}_{n-1} \) the fiber \( F_z = \pi^{-1}(z) \) is the projective space \( \mathbb{P}^{n-2} \). Hence
\[
\begin{align*}
\bullet \pi_\bullet (\Lambda^k \mathcal{B}_n) &= \Lambda^k \mathcal{B}_{n-1} \\
\bullet \pi_\bullet (\mathcal{O}_n(m) \otimes \Lambda^k \mathcal{B}_n) &= 0 \text{ if } m \in [-n+3, -1]. \\
\bullet \pi_\bullet (\mathcal{O}_n(-n+2) \otimes \Lambda^k \mathcal{B}_n) &= \Lambda^{k-1} \mathcal{B}_{n-1}[n-2]
\end{align*}
\]

The statement is similar for \( \pi_\parallel \) except we have take into account that over the divisor \( \mathcal{D} = \{ x_{11} = 0 \} \) the dimension of the fiber jumps:
\[
\begin{align*}
\parallel \pi_\parallel (\Lambda^k \mathcal{B}_n) &= \Lambda^k \mathcal{B}_{n-1} \\
\parallel \pi_\parallel (\mathcal{O}_n(m) \otimes \Lambda^k \mathcal{B}_n) &= 0 \text{ if } m \in [-n+3, -1]. \\
\parallel \pi_\parallel (\mathcal{O}_n(-n+2) \otimes \Lambda^k \mathcal{B}_n) &= \mathcal{O}(-\mathcal{D}) \otimes \Lambda^{k-1} \mathcal{B}_{n-1}[n-2].
\end{align*}
\]
Here we used the derived version of the push-forward $\pi_*(\mathcal{F}) := \pi_*(\mathcal{F} \otimes \mathcal{O}_{\text{Hilb}_n})$.

Thus in both case then only the left and the right extreme terms of $\Lambda^*V, D_V$ survive the push-forward $\pi_*$. The contraction of the $\pi_*$-acyclic terms could potential lead to appearance of new correction arrows. However, this does not happen since only two extreme ends of the complex $\Lambda^*V, D_V$ survive the push-forward. Hence the standard spectral sequence (or Gauss elimination lemma ) argument implies that

$$\pi_*(\mathcal{F} \otimes \Lambda^k \mathcal{B}_n) = \mathbb{C}[x_{11}] \otimes \left( \mathbb{L}(\mathcal{F}_{D'}) \otimes \Lambda^k \mathcal{B}_{n-1} \oplus \mathbb{L}(\mathcal{F}_{D'}) \otimes \Lambda^{k-1} \mathcal{B}_{n-1}[n-2] \right).$$

$$\pi_*(\mathcal{F} \otimes \Lambda^k \mathcal{B}_n) = \mathbb{C}[x_{11}] \otimes \mathbb{L}(\mathcal{F}_{D'}) \otimes \Lambda^k \mathcal{B}_{n-1} \oplus x_{11} \mathbb{C}[x_{11}] \otimes \mathbb{L}(\mathcal{F}_{D'}) \otimes \Lambda^{k-1} \mathcal{B}_{n-1}[n-2].$$

6. Properties of the functor

6.1. Geometric Soergel category. The general braid graph $D$ is a composition of the elementary graphs $D_{i+1}^{(i)}$, $D = D_{i+1}^{(i+1)} \circ \cdots \circ D_{i+1}^{(i+1)}$. Thus $\Phi_n$ assigns to the graph $D$ the matrix factorization:

$$F_D := \mathcal{C}_{(i)} \ast \cdots \ast \mathcal{C}_{(m)}. $$

Categories of matrix factorizations are triangulated [Orl04] hence additive. We defined an additive a $\text{MF}_n^\phi$ as Karoubi envelope of the smallest full subcategory of additive category $\text{MF}_n$ that contains the elementary matrix factorizations $\mathcal{C}_{(i)}$, $i = 1, \ldots, n - 1$ and closed under convolution $\ast$. The matrix factorizations $F_D$ are objects of this category but it contains more objects that correspond to more singular diagrams.

The main technical statement of the paper now follows from the MOY relations by application of the technique developed by Hao Wu [Wu08]. The same technique used for example in the work of Jake Rasmussen [Ras15] we refer for the details to his work and below we the key steps of the proof

Proof of Theorem 1.3. The first part of the theorem is proven in 3.2.1 The second part follows from the MOY relations from the section 4, see lemma 4.0.1 and lemma 4.0.2.

For third part we use Hao Wu technique [Wu08]. He shows (see also [Ras15]) that if one has homology theory for braid graphs such that the MOY relation from lemma 4.0.1, lemma 4.0.2 and proposition 5.1.1 hold then there is a unique inductive procedure for the reduction to the unknot. It was shown in [Ras15] that

$$\text{Ext}_D(\mathcal{C}_n \otimes \mathcal{C}_n)(\Phi_S(1), \Phi_S(\gamma))$$

satisfies the MOY relations. Thus we have the statement for the case $\gamma_1 = 1$. The general case follows from the special since the corollary 4.3.3

The part follows from the previous ones after combining with $B^\gamma \otimes \det(B) = B$. □

Thus we immediately obtain

Corollary 6.1.1. The functor $B$ is a faithful functor when restricted to $\text{MF}_n^\phi$:

$$B : \text{MF}_n^\phi \to \text{SBim}_n.$$
The Hochschild homology functor defines the natural traces on the category $\text{SBim}_n$:
\[
B \to \text{Tor}^i_{R_n \otimes R_n}(B, R_n).
\]

As we have shown in [OR18f] the category $\text{MF}_n$ also has natural trace functors
\[
\text{Tr}^i(F) := \text{CE}_{n^2} \left( \mathcal{E}xt(F, F) \otimes \Lambda^i B \right)^{T^2 \times G_n}.
\]

Thus the theorem [1.3] and the properties of the duality functor from the section 4.3 imply that these functors are intertwined by the functor $B$:

**Corollary 6.1.2.** For any $F \in \text{MF}^b_n$ we have an isomorphism of graded vector spaces
\[
\text{Tr}^i(F) = \text{Tor}^i_{R_n \otimes R_n}(B(F), R_n).
\]

Let us notice that element $\text{Tr}^i(F_D)$ naturally has gradings: $q$ and $t$-gradings. On the other hand the Soergel bimodule side of the equality has only one grading: $q$-grading. Thus we see that the elements $F_D$ are actually pure:

**Corollary 6.1.3.** For any diagram $D$ all elements of $\text{Tr}^i(F_D)$ are of the same $t$-degree.

The trace functor also computes the space of morphisms between the objects $\text{MF}^b_n$ since:
\[
\text{Tr}^0(F_D \bullet F_D') = \text{Hom}(F_D, F_D').
\]

Similarly, any two Soergel bimodules $B, B'$ we have
\[
\text{Tor}^0_{R_n}(B \otimes R_n B', R_n) = \text{Hom}(B, B').
\]

Thus the matching of the traces implies
\[
\text{Hom}(B(F), B(F')) = \text{Hom}(F, F'),
\]
for any $F, F' \in \text{MF}^b_n$.

### 6.2. Extension to Rouquier complexes.

Let us denote by $\text{Ho}(\text{SBim}_n)$ and $\text{Ho}(\text{MF}^b_n)$ the homotopy categories of the bounded complexes of the objects of these two categories. Rouquier constructed the homomorphism from the braid group $\mathfrak{B}t_n$ to the monoidal category $\text{Ho}(\text{SBim}_n)$, $\Phi_R : \mathfrak{B}t_n \to \text{Ho}(\text{SBim}_n)$:
\[
\mathfrak{B}t_n \ni \beta \mapsto C_\beta = (C_{\beta, \bullet}, d^R) \in \text{Ho}(\text{SBim}_n).
\]

Imitating Rouquier’s construction we define the geometric version of the homomorphism
\[
\Phi_{ho} : \mathfrak{B}t_n \to \text{Ho}(\text{MF}^b_n), \quad \beta \mapsto C_\beta,
\]
and this homomorphism is compatible with the Rouquier’s construction
(6.1) $\Phi_R = B \circ \Phi_{ho}$.

Probably, the most important result is the homotopy of the complexes

**Theorem 6.2.1.** For any $\beta \in \mathfrak{B}t_n$ and $i$ we have a homotopy of the complexes of graded vector spaces
\[
\text{Tr}^i(\Phi_{ho}(\beta)) \sim \text{Tor}^i_{R_n}(R_n, \Phi_R(\beta)).
\]
There is a natural projection $\Pi$ from the homotopy category $\text{Ho}(\text{MF}_n)$ to the triangulated category $\text{MF}_n$. This projection turns the $\mathbb{Z}$-grading of complexes to two-periodic grading of matrix factorizations. In our previous work we constructed the homomorphism 

$$\Phi : \mathfrak{Br} \to \text{MF}_n$$

and we show that our new construction is consistent with the previous result:

$$(6.2) \Phi = \Pi \circ \Phi_{ho}.$$ 

It was shown in [OR18f] that the total homology

$$H^*(\beta) = \mathbb{H}(\mathcal{E}xt(\Phi_{ho}(\beta), \Phi(1) \otimes \Lambda^i \mathcal{B})), $$

is an isotopy invariant of the closure of $\beta$. In particular, the triply-graded invariant from the introduction is

$$\text{HHH}_{geo}(\beta) = \oplus_i H^i(\beta).$$

Finally, using purity result from above we show

**Lemma 6.2.2.** For any $\beta$ we have

$$\mathbb{H}(\text{CE}_n^2(\mathcal{E}xt(\Phi_{ho}(\beta), \Phi_{ho}(1) \otimes \Lambda^i \mathcal{B})))^T = H^*(\text{Tr}^i(\Phi_{ho}(\beta)), d^R),$$

and homological grading $*$ matches with $t$-grading on the other side as well as $q$-grading matches with the polynomial grading.

**Proof.** Let us fix the notation the two-periodic complex:

$$S^i_{\beta,*} = (\Phi_{ho}(\beta)_*, \Phi_{ho}(1_+) \otimes \Lambda^i \mathcal{B}, d^R),$$

where $d^R : S^i_{\beta,*} \to S^i_{\beta,*+1}$ is the Rouquier differential.

In general, given an element $F \in \text{MF}_n$ the two-periodic complex

$$C = \text{CE}_n^2(\mathcal{E}xt(F, \mathcal{F}_\parallel \otimes \Lambda^i \mathcal{B}))^T$$

is has $t$-degree one. That is if we decompose the $C$ into the $t$-homogeneous pieces $C = \oplus_i C[i]$ then the differential of the complex $d^{mf}$ is graded $d^{mf} : C[i] \to C[i + 1]$. The differential $d^{mf}$ is the sum of the matrix factorization differential and Chevalley-Eilenberg differential with the corrections.

On the other hand the complex $\text{Tr}^i(\Phi_{ho}(\beta))$ is naturally a bi-complex. Indeed, $\Phi_{ho}(\beta)$ is a complex whose terms are elements of $\text{MF}^3_n$, we denote these differential by $d^R$ since we imitate Rouquier construction here:

$$\Phi_{ho}(\beta) = (\oplus j C_{\beta,j}, d^R), \quad C_{\beta,j} \in \text{MF}^3_n$$

where the differential $d^R$ is of $t$-degree 1.

On the other hand $\text{Tr}^i(C_{\beta,j})$ has its own differential $d^{mf,*i}$ and the total differential $d^{tot,i}$ is the sum $d^{tot,i} = d^R + d^{mf,*i}$.

Tautologically we have

$$\mathbb{H}(S^i_\beta) = H^*(S^i_\beta, d^{tot,i}).$$
Since $H^*(S^i_j, d^{mf,*}) = \text{Tr}^i(C_{\beta,j})$ we need to show that $H^*(S^i_j, d^{tot,*}) = H^*(H^*(S^i_j, d^{mf,*}), d^R)$.

Now we let us recall that both differentials are of $t$-degree 1. Thus we need to study the spectral sequence of the bi-complex

\[
\begin{array}{cccc}
   & S^i_{\beta,j}[1] & S^i_{\beta,j}[2] & S^i_{\beta,j}[3] & S^i_{\beta,j}[4] \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
S^i_{\beta,j+1}[2] & S^i_{\beta,j+1}[3] & S^i_{\beta,j+1}[4] & S^i_{\beta,j+1}[5] \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
S^i_{\beta,j+2}[3] & S^i_{\beta,j+2}[4] & S^i_{\beta,j+2}[5] & S^i_{\beta,j+2}[6] \\
\end{array}
\]

The horizontal differential in the bi-complex is $d^{mf,*}$ and the vertical differential is $d^R$.

Thus to compute $E_1$ page of the spectral sequence we compute the homology of the $d^{mf}$ differential and we obtain:

\[
\begin{array}{cccc}
   & \text{Tr}^i(C_{\beta,1})[1] & \text{Tr}^i(C_{\beta,1})[2] & \text{Tr}^i(C_{\beta,1})[3] & \text{Tr}^i(C_{\beta,1})[4] \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\text{Tr}^i(C_{\beta,2})[2] & \text{Tr}^i(C_{\beta,2})[3] & \text{Tr}^i(C_{\beta,2})[4] & \text{Tr}^i(C_{\beta,2})[5] \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\text{Tr}^i(C_{\beta,3})[3] & \text{Tr}^i(C_{\beta,3})[4] & \text{Tr}^i(C_{\beta,3})[5] & \text{Tr}^i(C_{\beta,3})[6] \\
\end{array}
\]

where the solid arrows are the differentials $d^R$ and the dashed arrows are the induced differentials that govern page $E_3$ of the spectral sequence.

Since the complexes $C_{\beta,j}$ are elements of $\text{MF}^n$ shifted by $t^j$, we conclude that the implies that $\text{Tr}^i(C_{\beta,j})[k]$ is zero if $k \neq j$. Thus only the first column of the last diagram is non-zero and the spectral sequence converges at $E_2$ page. \hfill \Box

6.3. **Proof of the comparison theorem.** Thus we can conclude our main result since

\[
\text{HHH}_{\text{alg}}(\beta) = \oplus_{i,j} H^j(\text{Tor}^i_{R_n}(\Phi_R(\beta), R_n)) = \oplus_{i,j} H^j(\text{Tr}(\Phi_{ho}(\beta))) \\
= \oplus_i \mathbb{H}(\text{CE}_{n^2}(\mathcal{E}xt(\Phi_{ho}(\beta), \Phi_{ho}(1) \otimes \Lambda^i \mathcal{B})))^T = \text{HHH}_{\text{geo}}(\beta).
\]
7. Applications and further directions

7.1. Torus knots. In this subsection we explain how the results of the current paper provide a geometric proof of the conjectures for the torus knots. More algebraic approach is used in work [Hog17, Mel17].

Here we only treat the case of the torus knots $T_{n,nk+i}$. The case of the general torus link follows from the localization computation in our earlier work [OR17] if know the parity statement for the homology. The parity statement for the torus link was shown algebraically in the work of Hogancamp and Mellit. There is a geometric approach to the parity that will be discussed in our forthcoming work.

The Hilbert-Chow map sends an ideal $I$ to the support of the quotient $\mathbb{C}[x,y]/I$:

$$HC : \text{Hilb}_n(\mathbb{C}^2) \rightarrow \text{Sym}^n(\mathbb{C}^2).$$

Thus we define the punctual Hilbert scheme as $\text{Hilb}_n(\mathbb{C}^2,0) := HC^{-1}(0^n)$. Respectively, we define $\text{Hilb}_n(\mathbb{C}^2,0) \times \mathbb{C}_{q^2} \subset \text{Hilb}_n(\mathbb{C}^2)$ to be the $HC$ preimage of the diagonal embedding of $\mathbb{C}_{q^2}$. Let us remind that $\mathbb{C}^2 = \mathbb{C}_{q^2} \times \mathbb{C}_{q^{-2}}$ is the $T_{q^2}$-weight decomposition.

The results of [OR17] allowed us to compute the geometric trace for the simplest braid that closes to unknot:

$$\mathcal{T}_r(\text{cox}_n) = \mathcal{O}_{\text{Hilb}_n(\mathbb{C}^2,0) \times \mathbb{C}_{q^2}}, \quad \text{cox}_n = \sigma_1 \cdot \sigma_2 \cdots \sigma_{n-1} \in \mathfrak{Br}_n$$

The torus link $T_{n,m}$ is defined as $L(\text{cox}_n^m)$. The full twist $FT = \text{cox}_n^m$ generates the center of the braid group $\mathfrak{Br}_n$. Hence $T_{n,nk+1} = L(\text{cox}_n \cdot FT^k)$. Thus the relation (1.3) implies

$$(1 - q^2) \cdot \text{HHHA}_0(T_{n,nk+1}) = H^*(\text{Hilb}_n(\mathbb{C}^2,0), \det(B)^k \otimes \Lambda(B)).$$

The vanishing of the higher cohomology of this sheaf is proven by Haiman [Hai02b]. Thus the statement of [1.0.3] follows. In the same paper of Haiman one can find a localization for formula for $T_{q^2}$-character of the space of global sections.

7.2. Poincare duality for knot homology. In our earlier work [OR19] we constructed a categorical invariant of the link:

$$\mathcal{E}(L) \in \text{D}_{\text{T}_{q^2}}^{\text{per}}(R(L))$$

where $R(L) = \mathbb{C}[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell], \ell = |\pi_0(L)|$ and $\deg(x_i) = q^2$, $\deg(y_i) = t^2/q^2$. The object $\mathcal{E}(L)$ also has additional a-grading and it is a direct sum of a-isotypical components.

The category $\text{D}_{\text{T}_{q^2}}^{\text{per}}(R(T))$ has a natural automorphism $\mathfrak{F}$ that is induced by switching $x_i$ and $y_i$ for all $i$. We show in [OR19] that $\mathcal{E}(L)$ has the following properties:

1. $\mathfrak{F}(\mathcal{E}(L)) = \mathcal{E}(L)$.
2. $\text{HHHA}_0(\beta) = \mathcal{E} \otimes_{\mathcal{O}(L(\beta))} \mathbb{C}[x_1, \ldots, x_\ell], \ell = |\pi_0(L(\beta))|.$
3. $\mathcal{E}(L)$ is the sum of free $\mathbb{C}[x,y]$-modules if $|\pi_0(L(\beta))| = 1$.

The combination of these three properties together with the comparison property (1.1) implies the duality theorem [1.0.2].
7.3. **Conjectures.** As we show the functor $B$ is fully faithful. Thus we have a realization of the category of category Soergel bimodules $\text{SBim}_n$ inside of the category of matrix factorizations $\text{MF}_n^{st}$. It is natural to ask whether the functor $B$ is surjective:

**Conjecture 7.3.1.** The functor $B$ extends to the equivalence of categories:

$$\text{MF}_n^{st} \cong \text{Ho} (\text{SBim}_n).$$

Finally, let us mention that the object $E_{pLq}$ allows one to define dualizable homology $\text{OR19}$ of the link as

$$\text{HXY}(L) = R\Gamma(E(L)).$$

On the other hand Gorsky and Hogancamp $\text{GH17}$ constructed a deformation of the homology theory $\text{HY}$ which they call $y$-fied homology such that $\text{HY}(L)$ is a module over $R(L)$. They conjecture that their homology are preserved by the involution $\tilde{\delta}$. Thus we expect the following

**Conjecture 7.3.2.** For any link $L$ we have

$$\text{HY}(L) = \text{HXY}(L).$$

To prove this conjecture we need to study the analog of the functor $B$ for the deformations of the categories $\text{MF}_n^{st}$ from $\text{OR19}$.

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