Lattice fermions with Majorana couplings

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We analyse stability of almost massless Dirac mode in gauge models with boundary (domain wall) fermions, and consider the possibility of decoupling one of its chiral component by giving it a Majorana mass of the order of the inverse lattice spacing. We argue that the chiral spectrum in such models is always uncharged, so they can be implemented for defining the Weyl fermions only in the real representation of the gauge group, for instance, in SUSY models.

1. Both the Wilson and the domain wall formulations of lattice fermions involve coupled pair(s) of left-handed (ψ) and right-handed (χ) Weyl fermions. One of the conceivable ways to define within such formulations a chiral theory, is to decouple, say, χ by giving it a Majorana mass of the order of the inverse lattice spacing. It can be done directly, if the fermions belong to a real representation of the gauge group, or through the Higgs mechanism, if the representation is complex. In the latter case the model must have strong coupling paramagnetic (PMS) phase, where fermions acquire masses $O(1)$ without spontaneous symmetry breaking.

In both formulations the ψ and χ are coupled through the momentum dependent Dirac mass terms. Therefore one of the problem on this way is that the introduction of the Majorana mass for χ induces a Majorana mass also for ψ, and due to radiative corrections such a mass can get $O(1)$. So, fine tuning of the Majorana mass of ψ may be necessary. In the Wilson formulation such a Dirac mass term is the Wilson term $m_D(p) = \frac{1}{2}p^2$ ($p_\mu = 2 \sin \frac{1}{2}p_\mu$), and the problem of fine tuning of both the Dirac and the Majorana masses does arise.

The boundary fermions \[\psi\] can be viewed as a coupled system of $N_s$ pairs of ψ and χ with the action

\[A_0 = \sum_{s,t=1}^{N_s} (\bar{\psi}_s \delta_{st} D \psi_t + \bar{\chi}_s \delta_{st} \overline{D} \chi_t + \bar{\psi}_s W_{st} \chi_t + \bar{\chi}_s W^+_t \psi_t),\]

where

\[D = \nabla_0 + i \sum_i \sigma_i \nabla_i, \quad \overline{D} = \nabla_0 - i \sum_i \sigma_i \nabla_i,\]

\[W^\pm = \delta_{s \pm 1, t} - \delta_{st} (1 - M - \frac{1}{2} \Delta),\]

and $M \in (0, 2)$ is intrinsic mass parameter of the formulation (for more detail see [4]). The propagators of such a system have the form

\[\langle \bar{\psi}_s \psi_t \rangle_0 = -\overline{D} G_L \equiv \overline{D} \frac{1}{p^2 + W^- W^+},\]

\[\langle \chi_s \bar{\chi}_t \rangle_0 = -D G_R \equiv -D \frac{1}{p^2 + W^+ W^-},\]

\[\langle \bar{\psi}_s \chi_t \rangle_0 = W^- G_R, \quad \langle \bar{\chi}_s \psi_t \rangle_0 = W^+ G_L,\]

with $\overline{p}_\mu = \sin p_\mu$. The point is that mass matrices $W^- W^+$ and $W^+ W^-$ have $N_s - 1$ eigenvalues $O(1)$ and exactly one eigenvalue that at $p \sim 0$ have the form $m_D^2 \sim (1 - M)^2 N_s$, with the corresponding eigenvectors $\psi \sim \sum_s (1 - M)^s - 1 \psi_s$ and $\chi \sim \sum_s (1 - M)^{N_s - s} \chi_s$.

Like in the Wilson case, the radiative corrections leads to both a multiplicative and an additive renormalization of the mass $M$. However, unlike the Wilson fermions, in this case $\delta m_D \neq \delta M$. 

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This is illustrated in Fig. 1–2, where numerical results of a mean field estimate\(^2\) of the mass renormalization are shown. They demonstrate that for a given value of the gauge coupling there exists an interval for \(M\) with the midpoint \(M = 5 - 4\langle U \rangle (g^2)\) within which the renormalized \(m_D \sim (M - M)^N_s \ll 1\), and the larger \(N_s\), the wider this interval. For QCD with \(g^2 \leq O(1)\) we have
\[
M \simeq 1 + 0.4g^2. 
\] (4)

The situation is not changed with taking into account the nondiagonal parts of the fermion self energy in perturbation theory (for more detail see \(\text{[3]}\)). So no fine tuning for the Dirac mass is needed in such a formulation. This gives rise to a hope that introducing of the Majorana mass for \(\chi\) will not lead to the need for fine tuning of the Majorana mass of \(\psi\).

2. This, in fact, is an underlying motivation of two recent proposals for the lattice formulations of the Standard Model \(\text{[2]}\) and of \(N = 1\) SUSY model \(\text{[3]}\). In both proposals it has been suggested to introduce the Majorana mass only for field \(\chi_{N_s}\). Thus the first question arising in this approach is (i) whether such an introduction of the Majorana mass does provide a chiral low-lying spectrum and whether the fine tuning is not needed. Besides, in the case of complex representation of the gauge group \(\text{[3]}\) the questions of (ii) the existence of the PMS phase and of (iii) the properties of the system within this phase arise. Here we present the answers to these questions obtained in \(\text{[2]}\).

We consider the system defined by the action
\[
A_m = A_0 + \sum (\chi_{N_s}^T H \chi_{N_s} + \chi_{N_s}^T H^\dagger \chi_{N_s}^T), 
\] (5)
where \(H\) either is a constant \(H = m\sigma_2\), \(m = O(1)\), if the representation of the gauge group is real, or is a Higgs field \(H = y\Phi\sigma_2\) if the representation is complex. So the action (5) in both cases is gauge invariant.

(i) To answer to the first question consider how the propagators (3) are modified by the constant Majorana mass term in (5). In the approximation neglecting terms \(O((1 - M)^N_s)\) the functions \(G_{L,R}\) have the form
\[
G_{L(R)} = A_{L(R)} e^{-\alpha(s+t-2)} + A_{R(L)} e^{\alpha(s+t-2N_s)} + B e^{-\alpha|s-t|}, 
\] (6)
where \(\Re \alpha > 0\), \(A_L, A_R\) and \(B\) are functions of \(p\), and only \(A_L\) has a pole at \(p^2 = 0\) that corresponds
to the massless mode. The introduction of the Majorana mass (5) leads to such a modification of the function $G_{L,R}$, that in $G_L$ only function $A_R$ is modified, so that no new poles appear in $G_L$, while in $G_R$ only function $A_L$ is modified, and this modification is such that exactly cancels the pole in $A_L$. Thus only one (nearly) massless mode of $\psi$ survives in the system (5). Like in the massless case (3) the main contribution to this mode at low momenta comes from $\psi_{s=1}$, with exponential dumping of the contributions of higher $s$’s.

It turns out also that at $p \sim 0$ the induced propagator $\langle \psi \psi \rangle$ has the form

$$\langle \psi_s \psi_t \rangle \propto p^2(1 - M)^{2N_s + 2 - s - t}, \quad (7)$$

that shows that the effect of the Majorana mass (5) is suppressed exponentially in the physical sector. This justifies the hope that no fine tuning of the Majorana mass is needed, as well.

(ii) In order that similar situation is realised for complex representation of the gauge group, the system (5) must be within PMS phase. That such a phase exists in this system has been demonstrated in 

For instance, in the case of group $U(1)$ the system is in the PMS phase at $y > 9.7$, and $\kappa < \kappa_{cr}(y)$, where $\kappa$ is the standard hopping parameter of the Higgs field and $\kappa_{cr}(y) \simeq 1/8 - 11.7/y^2$.

(iii) To get some idea of the properties of the system (5) within the PMS phase, we represent the Higgs field $\hat{\Phi}$ as $\hat{\Phi} = \hat{\Phi}_L \hat{\Phi}_R$, and use a mean field technique in terms of the link expectation value $z^2 = \langle \hat{\Phi}_L \hat{\Phi}_R \rangle$, which is known to be nonzero in the PMS phase, though $\langle \hat{\Phi} \rangle = 0$ (for more detail and references see [2]). We find three possible scenarios. Namely, at tree level the system may consist:

- either of one neutral (i.e. singlet under the gauge group) field $\bar{\chi} = z\hat{\Phi}_N$, with the Majorana mass $m = y/z^2$ and charged $\psi$’s and $\chi$’s with naive spectrum;
- or of pair of massive neutral fields $\bar{\chi} = z\hat{\Phi}_N$, and $\bar{\psi} = z\hat{\Phi}_N$, and the decoupled system (1) with $N_s - 1$ pairs of $\psi$ and $\chi$;
- or of $N_s$ pair of the neutral fields $\bar{\chi}_s = z\hat{\Phi}_s$, and $\bar{\psi}_s = z\hat{\Phi}_s$, described by the action (5) with modified term $\bar{W}^{\pm}_{st} = \delta_{s\pm 1}/z^2 - \delta_{st}(1 - M)/z^2 + \sum_{\mu}(1/z^2 - \cos p_{\mu})$ and the Majorana mass $m = y/z^2$.

Which of these possibilities is actually realised depends on the parameters of the system and the answer to this question requires special investigation. Only the third scenario leads to the chiral spectrum, provided the mass $M$ is chosen properly, i.e. within the interval $4 - 4z^2 < M < 6 - 4z^2$. However all the fermions in this case are neutral, and the crucial question to such a model is what are the gauge interactions of such neutral fermions.

3. Thus, we have demonstrated that for the real representations of the gauge group the boundary fermions with the Majorana mass have chiral spectrum at tree level, and argued that it is stable at least in perturbation theory. So, such models can be implemented for a non-perturbative formulation of the SUSY models without problem of fine tuning. As concerns the chiral gauge theories, our conclusion is less optimistic, since we have found that only neutral states may have chiral spectrum within the PMS phase, and there is a strong evidence that in the continuum limit such neutral states become non-interacting (see [2] for more detail and the references).

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