The four-loop $\beta$-function in Quantum Chromodynamics

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Abstract

We present the analytical calculation of the four-loop QCD $\beta$-function within the minimal subtraction scheme.
The renormalization group $\beta$-function in Quantum Chromodynamics (QCD) has a history of more than 20 years. The calculation of the one-loop $\beta$-function in QCD has led to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions [1].

The two-loop QCD $\beta$-function was calculated in [2]. Calculations of the three-loop QCD $\beta$-function were done in [3, 4] within the minimal subtraction (MS) scheme [5]. The MS-scheme belongs to the class of massless schemes where the $\beta$-function does not depend on masses of the theory and the first two coefficients of the $\beta$-function are scheme-independent.

In this article we present the analytical four-loop result for the QCD $\beta$-function. Throughout the calculations we use dimensional regularization [6] and the MS-scheme. The dimension of space-time is defined as $D = 4 - 2\varepsilon$, where $\varepsilon$ is the regularization parameter fixing the deviation of the space-time dimension from its physical value 4.

The Lagrangian for a massless non-abelian Yang-Mills theory with fermions is

$$L = \frac{-1}{4} G^a_{\mu\nu} G^{a\mu\nu} + i \sum_q \overline{\psi}_q D^\mu \psi_q + L_{gf} + L_{gc}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$[D_\mu]_{ij} = \delta_{ij} \partial_\mu - ig A^a_\mu [T^a]_{ij}$$

where the gauge-fixing and gauge-compensating parts of the Lagrangian in the covariant gauge are

$$L_{gf} = -\frac{1}{2\xi} (\partial_\mu A^a_\mu)^2$$

$$L_{gc} = \partial_\mu \omega^a (\partial_\mu \omega^a - g f^{abc} \omega^b A^c_\mu)$$

The fermion fields (the quark fields in QCD) $\psi_q$ transform as the fundamental representation of a compact semi-simple Lie group, $q = 1, ..., n_f$ is the flavour index. The Yang-Mills fields (gluons in QCD) $A^a_\mu$ transform as the adjoint representation of this group. $\omega^a$ are the ghost fields, and $\xi$ is the gauge parameter of the covariant gauge.

$T^a$ are the generators of the fundamental representation and $f^{abc}$ are the structure constants of the Lie algebra,

$$T^a T^b - T^b T^a = i f^{abc} T^c$$

In the case of QCD we have the Lie group $SU(3)$ but we will perform the calculation for an arbitrary compact semi-simple Lie group $G$. Since the $\beta$-function does not depend on masses in the MS-scheme, we will consider the massless theory.

The definition of the 4-dimensional $\beta$-function is:

$$\frac{\partial a_s}{\partial \ln \mu^2} = \beta(a_s)$$

$$= -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6)$$

in which $a_s = \alpha_s/4\pi = g^2/16\pi^2$, $g = g(\mu^2)$ is the renormalized strong coupling constant of the standard QCD Lagrangian of eq.[1], $\mu$ is the 't Hooft unit of mass, the renormalization point in the MS-scheme.
To calculate the $\beta$-function we need to calculate the renormalization constant $Z_{a_s}$ of the coupling constant

$$a_B = Z_{a_s} a_s$$

where $a_B$ is the bare (unrenormalized) charge. We obtain this renormalization constant in the 4-loop order by calculating the following three renormalization constants of the Lagrangian: $Z_{hhg}$ for the ghost-ghost-gluon vertex, $Z_h$ for the inverted ghost propagator and $Z_g$ for the inverted gluon propagator. Then $Z_{a_s} = Z_{hhg}^2/Z_h^2/Z_g$. This is from a calculational point of view one of the simplest (and most straightforward) ways to obtain this renormalization constant at higher orders. However, several other choices are possible such as calculating the renormalization factors of the quark propagator, the gluon propagator and the quark-gluon vertex. One could also use the background field method [7], which reduces the calculation of the $\beta$-function to the calculation of the gluon propagator only. However, we should note that the Feynman rules are more complicated in that case, and this complication would lead at the 4-loop level to a complexity of the calculations that is comparable to our more standard approach.

The expression of the $\beta$-function via $Z_{a_s}$ is given by the following chain of equations

$$\frac{d(a_B \mu^{2\varepsilon})}{d\ln\mu^2} = 0 = \varepsilon Z_{a_s} a_s \mu^{2\varepsilon} + \frac{\partial Z_{a_s}}{\partial a_s} \frac{da_s}{d\ln\mu^2} a_s \mu^{2\varepsilon} + Z_{a_s} \frac{da_s}{d\ln\mu^2} \mu^{2\varepsilon}$$

$$\Rightarrow \frac{da_s}{d\ln\mu^2} = -\frac{\varepsilon Z_{a_s} a_s}{\partial Z_{a_s}/a_s + Z_{a_s}} = -\varepsilon a_s - a_s \frac{\partial}{\partial a_s} \left( a_s Z_{a_s}^{(1)} \right) = -\varepsilon a_s + \beta(a_s) \quad (5)$$

where one uses the fact that the dimensional object $a_B \mu^{2\varepsilon}$ is invariant under the renormalization group transformations. $[-\varepsilon a_s + \beta(a_s)]$ is the $D$-dimensional $\beta$-function, $Z_{a_s}^{(1)}$ is the coefficient of the first $\varepsilon$-pole in $Z_{a_s}$ defined below.

Renormalization constants within the MS-scheme do not depend on dimensional parameters (masses, momenta) [8] and have the following structure:

$$Z_{a_s}(a_s) = 1 + \sum_{n=1}^{\infty} \frac{Z_{a_s}^{(n)}(a_s)}{\varepsilon^n}, \quad (6)$$

Since $Z_{a_s}$ does not depend explicitly on $\mu$, the $\beta$-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter $\mu$. That is why the $\beta$-function is the same in the MS-scheme [5] and in the $\overline{\text{MS}}$-scheme [4].

In general, the most straightforward way to obtain renormalization constants is by multiplicative renormalization of the relevant Green functions

$$\Gamma_{\text{Renormalized}}(a_s) = Z(a_s) \Gamma_{\text{Bare}}(a_B), \quad (7)$$

This direct approach was used for the independent calculation of the 3-loop $\beta$-function in [4]. However the extension of this approach to 4-loops requires the calculation of 4-loop massless propagator type integrals which are still difficult to evaluate (at this moment).
Another approach to find the renormalization constants $Z_i$ is to obtain them as the sum of the counterterms of individual diagrams. This means that one applies the $R$-operation to each individual diagram that contributes to a propagator or vertex function. This allows a greater freedom in choosing the type of integrals that one needs to evaluate. This approach (based on massless propagator type integrals) was used for the first 3-loop calculation of the $\beta$-function [3].

For the calculation presented in this article we use massive integrals, instead of the massless integrals used in previous calculations. The calculation of renormalization constants within the MS-scheme can be reduced to the calculation of massive vacuum bubble integrals (i.e. massive integrals with no external momenta) using the general method of infrared rearrangement [10]. This method uses the property that within dimensional regularization overall ultra-violet divergences are polynomial in external momenta and masses, also for individual diagrams. For the renormalization constants of the ghost-ghost-gluon vertex, ghost propagator and gluon propagator this means, that we can safely apply Taylor expansions in the external ghost- and gluon momenta if we introduce a non-zero auxiliary mass $M$ for all internal propagators (also for the gluons). It is understood that this auxiliary mass serves only as an infrared cutoff parameter that is nullified after renormalization of the individual diagrams. For simplicity, we introduce the mass only in the denominators of the propagators, not in the numerators. The difference between the overall divergences of the diagrams with and without the mass is a term that is polynomial in this mass and vanishes when $M$ is nullified.

The procedure of renormalization with an auxiliary mass works well for individual diagrams. However the introduction of the mass $M$ in the gluon propagators spoils multiplicative renormalizability of the (massive) Green functions.

Here we use an intermediate approach to renormalization in order to get the 4-loop counterterms for the sum of the diagrams. We compute poles in $\varepsilon$ of the corresponding 4-loop massive diagrams. But we do not renormalize each diagram separately. The subtraction of subdivergences is done for the whole sum of the 4-loop diagrams. This is done by means of adding to the sum of the 4-loop diagrams the sum of the necessary bare diagrams of 1, 2 and 3-loops with all vertices replaced by effective vertices and all propagators replaced by effective propagators. The effective vertices contain the necessary vertex renormalization constants up to the appropriate order in $a_s$ and similarly, the effective propagators contain the necessary propagator counterterms. The various vertex and propagator renormalization constants that are needed in the effective vertices and propagators are already known from the lower order massless calculations (we emphasize that they are mass independent) except for the overall uv divergence proportional to $M^2$ of the gluon propagator, which is only needed up to 3-loops.

Special routines for the symbolic manipulations program FORM [11] were constructed to efficiently evaluate the 4-loop massive bubble integrals up to pole parts in $\varepsilon$ and correspondingly of the 3-loop massive bubbles up to finite parts. For the 4-loop integrals, we only needed to deal with two master bubble topologies, see fig. 1. The various vertex and propagator diagrams were generated by means of the diagram generator QGRAF [12]. For the present calculation we evaluated of the order of 50,000 4-loop diagrams.
Figure 1. The basic (master) vacuum bubble topologies up to 4 loops. It is understood that all lines have the same non-zero mass, \( M \). There are two basic 4-loop topologies (the last one shown is non-planar).

We obtained in this way the following result for the 4-loop beta function in the \( \overline{\text{MS}} \)-scheme

\[
\begin{align*}
\beta_0 &= \frac{11}{3} C_A - \frac{4}{3} T_F n_f \\
\beta_1 &= \frac{34}{3} C_A^2 - 4 C_F T_F n_f - \frac{20}{3} C_A T_F n_f \\
\beta_2 &= \frac{2857}{54} C_A^3 + 2 C_F^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f \\
&\quad - \frac{1415}{27} C_A^2 T_F n_f + \frac{44}{9} C_F T_F^2 n_f^2 + \frac{158}{27} C_A T_F^2 n_f^2 \\
\beta_3 &= C_A^4 \left( \frac{150653}{486} - \frac{44}{9} \zeta_3 \right) + C_A^3 T_F n_f \left( \frac{-39143}{81} + \frac{136}{3} \zeta_3 \right) \\
&\quad + C_A^2 C_F T_F n_f \left( \frac{7073}{243} - \frac{656}{9} \zeta_3 \right) + C_A C_F^2 T_F n_f \left( \frac{-4204}{27} + \frac{352}{9} \zeta_3 \right) \\
&\quad + 46 C_F^3 T_F n_f + C_A C_F^2 T_F n_f^2 \left( \frac{7930}{81} + \frac{224}{9} \zeta_3 \right) + C_F^2 T_F^2 n_f^2 \left( \frac{1352}{27} - \frac{704}{9} \zeta_3 \right) \\
&\quad + C_A C_F T_F^2 n_f^2 \left( \frac{17152}{243} + \frac{448}{9} \zeta_3 \right) + \frac{424}{243} C_A T_F^3 n_f^3 + \frac{1232}{243} C_F T_F^3 n_f^3 \\
&\quad + \frac{d_{abcd} F_{abcd}}{N_A} \left( \frac{-80}{9} + \frac{704}{3} \zeta_3 \right) + n_f d_{abcd} F_{abcd} \frac{1}{N_A} \left( \frac{512}{9} - \frac{1664}{3} \zeta_3 \right) \\
&\quad + n_f^2 d_{abcd} F_{abcd} \frac{1}{N_A} \left( \frac{-704}{9} + \frac{512}{3} \zeta_3 \right)
\end{align*}
\]

(8)

Here \( \zeta \) is the Riemann zeta-function (\( \zeta_3 = 1.202056903 \cdots \)). \( [T^a T^a]^2 \) are the Casimir operators of the fundamental and the adjoint representation of the Lie algebra. \( \text{tr}(T^a T^b) = T_F \delta^{ab} \) is the trace normalization of the fundamental representation. \( N_A \) is the number of generators of the group (i.e. the number of gluons) and \( n_f \) is the number of quark flavours. We expressed the higher order group invariants in terms of contractions between the following fully symmetrical tensors:

\[
d_F^{abcd} = \frac{1}{6} \text{Tr} \left[ T^a T^b T^c T^d + T^a T^b T^d T^c + T^a T^c T^b T^d + T^a T^c T^d T^b + T^a T^d T^b T^c + T^a T^d T^c T^b \right]
\]

(9)
\[ d_{A}^{abcd} = \frac{1}{6} \text{Tr} \left[ C^{a}C^{b}C^{c}C^{d} + C^{a}C^{d}C^{b}C^{c} + C^{a}C^{c}C^{b}C^{d} \right. \]
\[ \left. + C^{a}C^{b}C^{d}C^{c} + C^{a}C^{c}C^{b}C^{d} + C^{a}C^{d}C^{c}C^{b} \right] \] (10)

where the matrices \([C^{a}]_{bc} = \pm i f^{abc}\) are the generators in the adjoint representation. The result of eq. (8) is valid for an arbitrary semi-simple compact Lie group. The result for QED (i.e. the group U(1)) is included in eq. (8) by substituting \(C^{A} = 0, d_{F}^{abcd} = 0, C^{F} = 1, T^{F} = 1, d_{F}^{abcd} = 1, N_{A} = 1\). This result for QED agrees with the literature [13]. A second independent check of eq. (8) is provided by the calculation [14] where the large-\(n_{f}\) terms for the QCD beta-function were calculated in all orders of the coupling constant. Our \(n_{f}^3\) terms agree with [14].

The result of eq. (8) is obtained in an arbitrary covariant gauge for the gluon field. This means that we can take the gauge parameter \(\xi\) that appears in the gluon propagator \(i \left[ -g^{\mu\nu} + (1 - \xi) q^{\mu} q^{\nu} / (q^2 + i\epsilon) \right] / (q^2 + i\epsilon)\) as a free parameter in the calculations. The explicit cancellation of the gauge dependence in the beta-function gives an important check of the results. The results for individual diagrams that contribute to the beta function also contain (apart from the constant \(\zeta_3\)) the constants \(\zeta_4, \zeta_5\) and several other constants specific for massive vacuum integrals. The cancellation of these constants at various stages in the calculation provides additional checks of the result.

For the standard normalization of the SU(\(N\)) generators we find the following expressions for the colour factors

\[ T_{F} = \frac{1}{2}, \quad C^{A} = N, \quad C^{F} = \frac{N^2 - 1}{2N}, \quad d_{F}^{abcd} d_{A}^{abcd} / N_{A} = \frac{N^2 (N^2 + 36)}{24}, \]
\[ d_{F}^{abcd} / N_{A} = \frac{N(N^2 + 6)}{48}, \quad d_{F}^{abcd} d_{F}^{abcd} / N_{A} = \frac{N^4 - 6N^2 + 18}{96N^2} \]

Substitution of these colour factors for \(N = 3\) into eq. (8) yields the following result for QCD

\[ \beta_{0} = 11 - \frac{2}{3} n_{f} \]
\[ \beta_{1} = 102 - \frac{38}{3} n_{f} \]
\[ \beta_{2} = \frac{2857}{2} - \frac{5033}{18} n_{f} + \frac{325}{54} n_{f}^2 \]
\[ \beta_{3} = \left( \frac{149753}{6} + 3564 \zeta_3 \right) - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_{f} \]
\[ + \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_{f}^2 + \frac{1093}{729} n_{f}^3 \] (11)

Or in a numerical form

\[ \beta_{0} \approx 11 - 0.66667 n_{f} \]
\[ \beta_{1} \approx 102 - 12.6667 n_{f} \]
\[ \beta_2 \approx 1428.50 - 279.611 n_f + 6.01852 n_f^2 \]
\[ \beta_3 \approx 29243.0 - 6946.30 n_f + 405.089 n_f^2 + 1.49931 n_f^3 \] (12)

We note that \( \beta_3 \) is positive for all positive values of \( n_f \).

It is interesting to compare our result with a recent prediction [15] for the 4-loop coefficient of the QCD \( \beta \)-function in the \( \overline{\text{MS}} \) scheme using Padé Approximants. For \( n_f = 3 \) this prediction is within a factor 2 of the exact result (for \( n_f = 5 \) this factor is about 9).

Considering that one might want to use the results of equation (12) for different groups or representations we will also express its constants in a different way [16]. In general one can write for a representation \( R \) of a simple Lie group

\[
d^{abcd}_{R} = I_4(R) d^{abcd} + \frac{I_{2,2}(R)}{3} (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc})
\]

in which the tensor \( d \) is now traceless. The only exceptions to this are the spinor representations of SO(8) for which there are two fully symmetric traceless tensors with 4 indices. We will not consider this special case here. The normalization of the tensor \( d \) is fixed by the definition of \( I_4(F) \). Contraction with \( \delta^{ab} \delta^{cd} \) gives

\[
I_{2,2}(R) = \frac{3}{N_A + 2} J_{2,2}(R)
\]
\[
J_{2,2}(R) = \left( \frac{N_A}{N_R} - \frac{1}{6} \frac{I_2(A)}{I_2(R)} \right) (I_2(R))^2
\]
\[
= T_R(C_R - \frac{1}{6} C_A)
\] (13)

with \( I_2(R) = T_R, I_2(A) = T_A = C_A \) and \( N_A T_R = C_R N_R \). We use here that \( N_R \) and \( N_A \) are the dimensions of the representation \( R \) and the adjoint representation respectively. For the adjoint representation we have that \( J_{2,2}(A) = 5 C_A / 6 \).

After this the following identities hold:

\[
\frac{d^{abcd}_{R} d^{abcd}_{R}}{N_A} = (I_4(R))^2 \frac{d^{abcd}_{abcd}}{N_A} + \frac{3}{N_A + 2} (J_{2,2}(R))^2
\]
\[
\frac{d^{abcd}_{R} d^{abcd}_{A}}{N_A} = I_4(R) I_4(A) \frac{d^{abcd}_{abcd}}{N_A} + \frac{3}{N_A + 2} J_{2,2}(R) J_{2,2}(A)
\]
\[
\frac{d^{abcd}_{A} d^{abcd}_{A}}{N_A} = (I_4(A))^2 \frac{d^{abcd}_{abcd}}{N_A} + \frac{3}{N_A + 2} (J_{2,2}(A))^2
\] (14)

These \( d \)'s have some nice properties. They are zero for all exceptional groups and for SU(3). In addition they are representation independent. Hence we need to give them only for the classical groups:

\[
d^{abcd}_{R} d^{abcd}_{R}(SU(N)) = \frac{N_A (N_A - 3) (N_A - 8)}{96 (N_A + 2)}
\]
For the overall normalization we define the constant $b$ with the relation

$$Tr \left[ C^\alpha C^\beta \right] = b \, g \, \delta^{\alpha\beta}$$

in which $g$ is the dual Coxeter number. The factor $b$ is related to the normalization factor $a$ in the article by Cvitanovic [17]. The relation is $b = 2a$ for the groups $SU(N)$ and $SP(N)$ and $b = a$ for $SO(N)$. The canonical choice of $b$ is 1 for all groups. One should however be aware of the fact that sometimes different choices are used, especially for the exceptional groups.

Some values for the fundamental and adjoint representations are:

|          | $SU(N)$ | $SO(N)$ | $SP(N)$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|----------|---------|---------|---------|-------|-------|-------|-------|-------|
| $C_A$    | $bN$    | $b(N-2)$ | $b(N+2)/2$ | 4$b$ | 9$b$ | 12$b$ | 18$b$ | 30$b$ |
| $I_4(A)$ | $2b^2N$ | $b^2(N-8)$ | $b^2(N+8)$ | 0 | 0 | 0 | 0 | 0 |
| $T_F$    | $b/2$   | $b$     | $b/2$   | $b$ | 3$b$ | 3$b$ | 6$b$ | - |
| $N_F$    | $N$     | $N$     | $N$     | 7    | 26   | 27   | 56   | - |
| $N_A$    | $N^2 - 1$ | $N(N-1)/2$ | $N(N+1)/2$ | 14   | 52   | 78   | 133  | 248  |

For all groups we have $C_F = T_F \cdot N_A/N_F$. $I_4(F) = b^2$ for the classical groups and zero for the exceptional groups. For other representations one would have to obtain values for the quantities $T_R$, $N_R$ and $I_4(R)$. By comparing the above values with the equations 4 and 8 one may observe that the choice of a different value for $b$ corresponds to a redefinition of the coupling constant.

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References

[1] D.J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343;
    H.D. Politzer, Phys. Rev. Lett. 30 (1973) 1346;
    G. ’t Hooft, report at the Marseille Conference on Yang-Mills Fields, 1972.
[2] W.E. Caswell, Phys. Rev. Lett. 33 (1974) 244;
   D.R.T. Jones, Nucl. Phys. B 75 (1974) 531;
   E.S. Egorian, O.V. Tarasov, Theor. Mat. Fiz. 41 (1979) 26.

[3] O.V. Tarasov, A.A. Vladimirov, A.Yu. Zharkov, Phys. Lett. B 93 (1980) 429.

[4] S.A. Larin, J.A.M. Vermaseren, Phys. Lett. B 303 (1993) 334.

[5] G. ’t Hooft, Nucl. Phys. B 61 (1973) 455.

[6] G. ’t Hooft, M. Veltman, Nucl. Phys. B 44 (1972) 189;

[7] L.F. Abbott, Nucl. Phys. B 185 (1981) 189.

[8] J.C. Collins, Nucl. Phys. B 80 (1974) 341.

[9] W.A. Bardeen, A.J. Buras, D.W. Duke, T. Muta, Phys. Rev. D 18 (1978) 3998.

[10] A.A. Vladimirov, Theor. Mat. Fiz. 43 (1980) 210.

[11] J.A.M. Vermaseren, Symbolic Manipulation with Form, Computer Algebra Nederland,
    Amsterdam, 1991.

[12] P. Nogueira, J. Comp. Phys. 105 (1993) 279.

[13] S.G. Gorishny, A.L. Kataev, S.A. Larin, Phys. Lett. 194B (1987) 429;
    S.G. Gorishny, A.L. Kataev, S.A. Larin, L.R. Surguladze, Phys. Lett. B256 (1991) 81.

[14] J.A. Gracey, Phys. Lett. B373 (1996) 178.

[15] M.A. Samuel, J. Ellis, M. Karliner, Phys. Rev. Lett. 74 (1995) 4380;
    J. Ellis, M. Karliner, M.A. Samuel, hep-ph/9612202 (1996).

[16] S. Okubo and J. Patera, J. Math Phys. 25 (1984) 219, ibid 24 (1983) 2722.

[17] P. Cvitanović, Phys. Rev. D14 (1976) 1536.