Parameter estimation for an affine two factor model

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Abstract

For an affine two factor model, we study the asymptotic properties of the maximum likelihood and least squares estimators of some appearing parameters in the so-called subcritical (ergodic) case based on continuous time observations. We prove strong consistency and asymptotic normality of the estimators in question.

1 Introduction

We consider the following 2-dimensional affine process (affine two factor model)

\begin{align*}
\begin{aligned}
dY_t &= (a - bY_t)\, dt + \sqrt{Y_t} \, dL_t, & \quad t \geq 0, \\
x_t &= (m - \theta X_t)\, dt + \sqrt{Y_t} \, dB_t, & \quad t \geq 0,
\end{aligned}
\end{align*}

where \(a > 0, b, m, \theta \in \mathbb{R}, (L_t)_{t \geq 0} \) and \((B_t)_{t \geq 0}\) are independent standard Wiener processes. Note that the process \((Y_t)_{t \geq 0}\) given by the first SDE of (1.1) is the so-called Cox-Ingersol-Ross (CIR) process which is a continuous state branching process with branching mechanism \(bz + z^2/2, z \geq 0\), and with immigration mechanism \(az, z \geq 0\). Chen and Joslin [8] have found several applications of the model (1.1) in financial mathematics, see their equations (25) and (26).

The process \((Y, X)\) given by (1.1) is a special affine diffusion process. The set of affine processes contains a large class of important Markov processes such as continuous state branching processes and Ornstein–Uhlenbeck processes. Further, a lot of models in financial mathematics are affine such as the Heston model [11], the model of Barndorff-Nielsen and Shephard [3] or the model due to Carr and Wu [7]. A precise mathematical formulation and a complete characterization of regular affine processes are due to Duffie et al. [10].

This article is devoted to estimate the parameters \(a, b, m\) and \(\theta\) from some continuously observed real data set \((Y_t, X_t)_{t \in [0, T]}\), where \(T > 0\). To the best knowledge of the authors the parameter estimation problem for multi-dimensional affine processes has not been tackled so far. Since affine processes are frequently used in financial mathematics, the question of parameter estimation for them is of high importance. In Barczy et al. [1] we started the discussion with a simple non-trivial two-dimensional affine diffusion process given by (1.1) in the so called critical case: \(b \geq 0, \theta = 0\) or \(b = 0, \theta \geq 0\) (for the definition of criticality, see Section 2). In the special critical case \(b = 0, \theta = 0\) we described the asymptotic behavior of least squares estimator (LSE) of \((m, \theta)\) from some discretely observed low frequency real data set \(X_0, X_1, \ldots, X_n\) as \(n \to \infty\). The description of

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the asymptotic behavior of the LSE of $(m, \theta)$ in the other critical cases $b = 0$, $\theta > 0$ or $b > 0$, $\theta = 0$ remained opened. In this paper we deal with the same model but in the so-called subcritical (ergodic) case: $b > 0$, $\theta > 0$, and we consider the maximum likelihood estimator (MLE) of $(a, b, m, \theta)$ using some continuously observed real data set $(Y_t, X_t)_{t \in [0, T]}$, where $T > 0$, and the LSE of $(m, \theta)$ using some continuously observed real data set $(X_t)_{t \in [0, T]}$, where $T > 0$. For studying the asymptotic behaviour of the MLE and LSE in the subcritical (ergodic) case, one first needs to examine the question of existence of a unique stationary distribution and ergodicity for the model given by (1.1). In a companion paper Barczy et al. [2] we solved this problem, see also Theorem 2.5. Further, in a more general setup by replacing the CIR process $(Y_t)_{t \geq 0}$ in the first SDE of (1.1) by a so-called $\alpha$-root process (stable CIR process) with $\alpha \in (1, 2)$, the existence of a unique stationary distribution for the corresponding model was proved in Barczy et al. [2].

In general, parameter estimation for subcritical (also called ergodic) models has a long history, see, e.g., the monographs of Kutoyants [18], Bishwal [6] and the research paper of Klimko and Nelson [17]. In case of the one-dimensional CIR process $Y$, the parameter estimation of $a$ and $b$ goes back to Overbeck and Rydén [23] (conditional LSE), Overbeck [24] (MLE), and see also Bishwal [6, Example 7.6] and the very recent papers of Ben Alaya and Kebaier [4], [5] (MLE). In Ben Alaya and Kebaier [4], [5] one can find a systematic study of the asymptotic behavior of the quadruplet $(\log(Y_t), Y_t, \int_0^t Y_s \, ds, \int_0^t 1/Y_s \, ds)$ as $t \to \infty$. Finally, we note that Li and Ma [20] started to investigate the asymptotic behaviour of the (weighted) conditional LSE of the drift parameters for a CIR model driven by a stable noise (they call it a stable CIR model) from some discretely observed low frequency real data set. To give another example besides the one-dimensional CIR process, we mention a model that is somewhat related to (1.1) and parameter estimation of the appearing parameters based on continuous time observations has been considered. It is the so-called Ornstein–Uhlenbeck process driven by $\alpha$-stable Lévy motions, i.e.,

$$dU_t = (m - \theta U_t) \, dt + dZ_t, \quad t \geq 0,$$

where $\theta > 0$, $m \neq 0$, and $(Z_t)_{t \geq 0}$ is an $\alpha$-stable Lévy motion with $\alpha \in (1, 2)$. For this model Hu and Long investigated the question of parameter estimation, see [13], [14] and [15].

We give a brief overview of the structure of the paper. Section 2 is devoted to a preliminary discussion of the existence and uniqueness of a strong solution of the SDE (1.1), we make a classification of the model (see Definition 2.4), we also recall our results in Barczy et al. [2] on the existence of a unique stationary distribution and ergodicity for the affine process given by SDE (1.1), see Theorem 2.5. Further, we recall some limit theorems for continuous local martingales that will be used later on for studying the asymptotic behaviour of the MLE of $(a, b, m, \theta)$ and the LSE of $(m, \theta)$, respectively. In Sections 3–8 we study the asymptotic behavior of the MLE of $(a, b, m, \theta)$ and LSE of $(m, \theta)$ proving that the estimators are strongly consistent and asymptotically normal under appropriate conditions on the parameters. We call the attention that for the MLE of $(a, b, m, \theta)$ we require a continuous time observation $(Y_t, X_t)_{t \in [0, T]}$ of the process $(Y, X)$, but for the LSE of $(m, \theta)$ we need a continuous time observation $(X_t)_{t \in [0, T]}$ only for the process $X$. We note that in the critical case we obtained a different limit behaviour for the LSE of $(m, \theta)$, see Barczy et al. [1, Theorem 3.2].
2 Preliminaires

Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of positive integers, non-negative integers, real numbers and non-negative real numbers, respectively. By $\|x\|$ and $\|A\|$ we denote the Euclidean norm of a vector $x \in \mathbb{R}^m$ and the induced matrix norm $\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^m, \|x\| = 1\}$ of a matrix $A \in \mathbb{R}^{n \times m}$, respectively.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e., $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $(L_t)_{t \geq 0}$ and $(B_t)_{t \geq 0}$ be independent standard $(\mathcal{F}_t)_{t \geq 0}$-Wiener processes.

The next proposition is about the existence and uniqueness of a strong solution of the SDE (1.1). Further, it clarifies that the process given by the SDE (1.1) belongs to the family of regular affine processes, see Barczy et al. [2, Theorem 2.2] which follows from the theorem due to Yamada and Watanabe.

2.1 Proposition. Let $(\eta_0, \zeta_0)$ be a random vector independent of $(L_t, B_t)_{t \geq 0}$ satisfying $\mathbb{P}(\eta_0 \geq 0) = 1$. Then, for all $a > 0$, and $b, m, \theta \in \mathbb{R}$, there is a (pathwise) unique strong solution $(Y_t, X_t)_{t \geq 0}$ of the SDE (1.1) such that $\mathbb{P}((Y_0, X_0) = (\eta_0, \zeta_0)) = 1$ and $\mathbb{P}(Y_t \geq 0, \forall t \geq 0) = 1$. Further, we have

\[
Y_t = e^{-(t-s)} \left( Y_s + a \int_s^t e^{-(s-u)} \, du + \int_s^t e^{-(s-u)} \sqrt{Y_u} \, dL_u \right), \quad 0 \leq s \leq t,
\]

and

\[
X_t = e^{-(t-s)} \left( X_s + m \int_s^t e^{-(s-u)} \, du + \int_s^t e^{-(s-u)} \sqrt{X_u} \, dB_u \right), \quad 0 \leq s \leq t.
\]

Moreover, $(Y_t, X_t)_{t \geq 0}$ is a regular affine process with infinitesimal generator

\[
(Af)(y, x) = (a - by)f_1'(y, x) + (m - \theta x)f_2'(y, x) + \frac{1}{2}y(f_{1,1}''(y, x) + f_{2,2}''(y, x)),
\]

where $(y, x) \in \mathbb{R}_+ \times \mathbb{R}$, $f \in C^2_c(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $f'_i$, $i = 1, 2$, and $f''_{i,j}$, $i, j \in \{1, 2\}$, denote the first and second order partial derivatives of $f$ with respect to its $i$-th and $i$-th and $j$-th variables, respectively, and $C^2_c(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ is the set of twice continuously differentiable real-valued functions defined on $\mathbb{R}_+ \times \mathbb{R}$ having compact support.

2.2 Remark. In Proposition 2.1 it is the assumption $a > 0$ which ensures $\mathbb{P}(Y_t \geq 0, \forall t \geq 0) = 1$.

In what follows we will make a classification of the affine processes given by the SDE (1.1). First we recall a result about the first moment of $(Y_t, X_t)_{t \in \mathbb{R}_+}$, see Proposition 3.2 in Barczy et al. [1].

2.3 Proposition. Let $(Y_t, X_t)_{t \in \mathbb{R}_+}$ be an affine diffusion process given by the SDE (1.1) with a random initial value $(\eta_0, \zeta_0)$ independent of $(L_t, B_t)_{t \geq 0}$ such that $\mathbb{P}(\eta_0 \geq 0) = 1$, $\mathbb{E}(\eta_0) < \infty$ and $\mathbb{E}(|\zeta_0|) < \infty$. Then

\[
\begin{bmatrix}
\mathbb{E}(Y_t) \\
\mathbb{E}(X_t)
\end{bmatrix}
= \begin{bmatrix}
e^{-bt} & 0 \\
0 & e^{-\theta t}
\end{bmatrix}
\begin{bmatrix}
\mathbb{E}(\eta_0) \\
\mathbb{E}(\zeta_0)
\end{bmatrix}
+ \begin{bmatrix}
\int_0^t e^{-bs} \, ds & 0 \\
0 & \int_0^t e^{-\theta s} \, ds
\end{bmatrix}
\begin{bmatrix}
(a) \\
m
\end{bmatrix}, \quad t \in \mathbb{R}_+.
\]
Proposition 2.3 shows that the asymptotic behavior of the first moment of \((Y_t, X_t)_{t \in \mathbb{R}^+}\) as \(t \to \infty\) is determined by the spectral radius of the diagonal matrix \(\text{diag}(e^{-bt}, e^{-\theta t})\), which motivates our classification of the affine processes given by the SDE (1.1).

2.4 Definition. Let \((Y_t, X_t)_{t \in \mathbb{R}^+}\) be an affine diffusion process given by the SDE (1.1) with a random initial value \((\eta_0, \zeta_0)\) independent of \((L_t, B_t)_{t \geq 0}\) satisfying \(P(\eta_0 \geq 0) = 1\). We call \((Y_t, X_t)_{t \in \mathbb{R}^+}\) subcritical, critical or supercritical if the spectral radius of the matrix

\[
\begin{bmatrix}
e^{-bt} & 0 \\
0 & e^{-\theta t}
\end{bmatrix}
\]

is less than 1, equal to 1 or greater than 1, respectively.

Note that, since the spectral radius of the matrix given in Definition 2.4 is \(\max(e^{-bt}, e^{-\theta t})\), the affine process given in Definition 2.4 is

- subcritical if \(b > 0\) and \(\theta > 0\),
- critical if \(b = 0\), \(\theta \geq 0\) or \(b \geq 0\), \(\theta = 0\),
- supercritical if \(b < 0\) or \(\theta < 0\).

Further, under the conditions of Proposition 2.3, by an easy calculation, if \(b > 0\) and \(\theta > 0\), then

\[
\lim_{t \to \infty} \begin{bmatrix}
\frac{\mathbb{E}(Y_t)}{\mathbb{E}(X_t)}
\end{bmatrix} = \begin{bmatrix}
\frac{m}{b}
\end{bmatrix},
\]

if \(b = 0\) and \(\theta = 0\), then

\[
\lim_{t \to \infty} \frac{1}{t} \begin{bmatrix}
\mathbb{E}(Y_t) \\
\mathbb{E}(X_t)
\end{bmatrix} = \begin{bmatrix}
a \\
m
\end{bmatrix},
\]

if \(b = 0\) and \(\theta > 0\), then

\[
\lim_{t \to \infty} \frac{1}{t} \begin{bmatrix}
\mathbb{E}(Y_t) \\
\mathbb{E}(X_t)
\end{bmatrix} = \begin{bmatrix}
a \\
m
\end{bmatrix},
\]

if \(b > 0\) and \(\theta = 0\), then

\[
\lim_{t \to \infty} \frac{1}{t} \begin{bmatrix}
\mathbb{E}(Y_t) \\
\mathbb{E}(X_t)
\end{bmatrix} = \begin{bmatrix}
\frac{a}{b} \\
m
\end{bmatrix},
\]

and if \(b < 0\) and \(\theta < 0\), then

\[
\lim_{t \to \infty} \begin{bmatrix}
e^{bt} \mathbb{E}(Y_t) \\
e^{\theta t} \mathbb{E}(X_t)
\end{bmatrix} = \begin{bmatrix}
\mathbb{E}(\eta_0) - \frac{a}{b} \\
\mathbb{E}(\zeta_0) - \frac{m}{b}
\end{bmatrix}.
\]

Remark also that Definition 2.4 of criticality is in accordance with the corresponding definition for one-dimensional continuous state branching processes, see, e.g., Li [19, page 58].

In the sequel \(\overset{p}{\longrightarrow}\) and \(\overset{c}{\longrightarrow}\) will denote convergence in probability and in distribution, respectively.

The following result states the existence of a unique stationary distribution and the ergodicity for the affine process given by the SDE (1.1), see Theorems 3.1 with \(\alpha = 2\) and Theorem 4.2 in Barczy et al. [2].
2.5 Theorem. Let us consider the 2-dimensional affine model \((Y_t, X_t)\) with \(a > 0, b > 0, m \in \mathbb{R}, \theta > 0\), and with a random initial value \((\eta_0, \zeta_0)\) independent of \((L_t, B_t)_{t \geq 0}\) satisfying \(\mathbb{P}(\eta_0 \geq 0) = 1\). Then

(i) \((Y_t, X_t) \xrightarrow{\mathcal{L}} (Y_\infty, X_\infty)\) as \(t \to \infty\), and the distribution of \((Y_\infty, X_\infty)\) is given by

\[
\mathbb{E}(e^{-\lambda_1 Y_\infty + i \lambda_2 X_\infty}) = \exp \left\{ -a \int_0^\infty v_s(\lambda_1, \lambda_2) \, ds + i \frac{m}{\theta} \lambda_2 \right\}, \quad (\lambda_1, \lambda_2) \in \mathbb{R}_+ \times \mathbb{R},
\]

where \(v_t(\lambda_1, \lambda_2), t \geq 0\), is the unique non-negative solution of the (deterministic) differential equation

\[
\begin{cases}
\frac{\partial v_t}{\partial t}(\lambda_1, \lambda_2) = -bv_t(\lambda_1, \lambda_2) - \frac{1}{2}(v_t(\lambda_1, \lambda_2))^2 + \frac{1}{2}e^{-\theta t} \lambda_2^2, & t \geq 0, \\
v_0(\lambda_1, \lambda_2) = \lambda_1.
\end{cases}
\]

(ii) supposing that the random initial value \((\eta_0, \zeta_0)\) has the same distribution as \((Y_\infty, X_\infty)\) given in part (i), we have \((Y_t, X_t)_{t \geq 0}\) is strictly stationary.

(iii) for all Borel measurable functions \(f : \mathbb{R}^2 \to \mathbb{R}\) such that \(\mathbb{E}(\|f(Y_\infty, X_\infty)\|) < \infty\), we have

\[
\mathbb{P}\left( \lim_{T \to \infty} \frac{1}{T} \int_0^T f(Y_s, X_s) \, ds = \mathbb{E}(f(Y_\infty, X_\infty)) \right) = 1,
\]

where the distribution of \((Y_\infty, X_\infty)\) is given by (2.3) and (2.4).

Moreover, the random variable \((Y_\infty, X_\infty)\) is absolutely continuous, the Laplace transform of \(Y_\infty\) takes the form

\[
\mathbb{E}(e^{-\lambda_1 Y_\infty}) = \left(1 + \frac{\lambda_1}{2b}\right)^{-2a}, \quad \lambda_1 \in \mathbb{R}_+,
\]

i.e., \(Y_\infty\) has Gamma distribution with parameters \(2a\) and \(2b\), all the (mixed) moments of \((Y_\infty, X_\infty)\) of any order are finite, i.e., \(\mathbb{E}(Y_\infty^p | X_\infty) < \infty\) for all \(p \in \mathbb{Z}_+\), and especially,

\[
\begin{align*}
\mathbb{E}(Y_\infty) &= \frac{a}{b}, \quad \mathbb{E}(X_\infty) = \frac{m}{\theta}, \\
\mathbb{E}(Y_\infty^2) &= \frac{a(2a + 1)}{2b^2}, \quad \mathbb{E}(Y_\infty X_\infty) = \frac{ma}{\theta b}, \quad \mathbb{E}(X_\infty^2) = \frac{a\theta + 2bm^2}{2b^2\theta^2}, \\
\mathbb{E}(Y_\infty X_\infty^2) &= \frac{a}{(b + 2\theta)^2 b^2 \theta^2} \left( \theta(ab + 2a\theta + \theta) + 2m^2 b(2\theta + b) \right).
\end{align*}
\]

In all what follows we will suppose that we have continuous time observations for the process \((Y, X)\), i.e., \((Y_t, X_t)_{t \in [0, T]}\) can be observed for some \(T > 0\), and our aim is to deal with parameter estimation of \((a, b, m, \theta)\). We also deal with parameter estimation of \(\theta\) provided that the parameter \(m \in \mathbb{R}\) is supposed to be known.

Next we recall some limit theorems for continuous local martingales. We will use these limit theorems in the sequel for studying the asymptotic behaviour of different kinds of estimators for \((a, b, m, \theta)\). First we recall a strong law of large numbers for continuous local martingales, see, e.g., Liptser and Shiryayev [22] Lemma 17.4].
2.6 **Theorem.** Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space satisfying the usual conditions. Let \((M_t)_{t \geq 0}\) be a continuous local martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\) started from 0. Let \((\xi_t)_{t \geq 0}\) be a progressively measurable process such that

\[
\mathbb{P} \left( \int_0^t (\xi_u)^2 \, d\langle M \rangle_u < \infty \right) = 1, \quad t \geq 0,
\]

and

\[
(2.7) \quad \mathbb{P} \left( \lim_{t \to \infty} \int_0^t (\xi_u)^2 \, d\langle M \rangle_u = \infty \right) = 1,
\]

where \((\langle M \rangle_t)_{t \geq 0}\) denotes the quadratic variation process of \(M\). Then

\[
(2.8) \quad \mathbb{P} \left( \lim_{t \to \infty} \frac{\int_0^t \xi_u \, dM_u}{\int_0^t (\xi_u)^2 \, d\langle M \rangle_u} = 0 \right) = 1.
\]

In case of \(M_t = B_t\), \(t \geq 0\), where \((B_t)_{t \geq 0}\) is a standard Wiener process, the progressive measurability of \((\xi_t)_{t \geq 0}\) can be relaxed to measurability and adaptedness to the filtration \((\mathcal{F}_t)_{t \geq 0}\).

The next theorem is about the asymptotic behaviour of continuous multivariate local martingales.

2.7 **Theorem.** (van Zanten [27, Theorem 4.1]) Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space satisfying the usual conditions. Let \((M_t)_{t \geq 0}\) be a \(d\)-dimensional continuous local martingale with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\) started from 0. Suppose that there exists a function \(Q : [0, \infty) \to \mathbb{R}^{d \times d}\) such that \(Q(t)\) is a non-random, invertible matrix for all \(t \geq 0\), \(\lim_{t \to \infty} \|Q(t)\| = 0\) and

\[
Q(t)^\top M(t) Q(t) \xrightarrow{\mathbb{P}} \eta \eta^\top \quad \text{as} \quad t \to \infty,
\]

where \(\eta\) is a \(d \times d\) random matrix defined on \((\Omega, \mathcal{F}, \mathbb{P})\). Then, for each \(\mathbb{R}^k\)-valued random variable \(V\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\), it holds that

\[
(Q(t)M_t, V) \xrightarrow{L} (\eta Z, V) \quad \text{as} \quad t \to \infty,
\]

where \(Z\) is a \(d\)-dimensional standard normally distributed random variable independent of \((\eta, V)\).

We note that Theorem 2.7 remains true if the function \(Q\), instead of the interval \([0, \infty)\), is defined only on an interval \([t_0, \infty)\) with some \(t_0 > 0\).

### 3 Existence and uniqueness of maximum likelihood estimator

Let \(\mathbb{P}_{(a,b,m,\theta)}\) denote the probability measure on the measurable space \((\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}), \mathcal{B}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})))\) induced by the process \((Y_t, X_t)_{t \geq 0}\) corresponding to the parameters \((a, b, m, \theta)\) and initial value \((Y_0, X_0)\). Here \(\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})\) denotes the set of continuous \(\mathbb{R}_+ \times \mathbb{R}\)-valued functions defined on \(\mathbb{R}_+\), \(\mathcal{B}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}))\) is the Borel \(\sigma\)-algebra on it, and we suppose that the space \((\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}), \mathcal{B}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})))\) is endowed with the natural filtration \((\mathcal{A}_t)_{t \geq 0}\), given by \(\mathcal{A}_t := \varphi_t^{-1}(\mathcal{B}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})))\), where \(\varphi_t : \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R}) \to \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+ \times \mathbb{R})\) is the mapping \(\varphi_t(f)(s) := f(t \wedge s), s \geq 0\). For all \(T > 0\), let \(\mathbb{P}_{(a,b,m,\theta),T} := \mathbb{P}_{(a,b,m,\theta)}|_{\mathcal{A}_T}\) be the restriction of \(\mathbb{P}_{(a,b,m,\theta)}\) to \(\mathcal{A}_T\).
3.1 Lemma. Let \( a > 1/2, \ b, m, \theta \in \mathbb{R}, \ T > 0, \) and suppose that \( \mathbb{P}(Y_0 > 0) = 1. \) Let \( \mathbb{P}_{(a,b,m,\theta)} \) and \( \mathbb{P}_{(1,0,0,0)} \) denote the probability measures induced by the unique strong solutions of the SDE (1.1) corresponding to the parameters \((a,b,m,\theta)\) and \((1,0,0,0)\) with the same initial value \((Y_0, X_0)\), respectively. Then \( \mathbb{P}_{(a,b,m,\theta),T} \) and \( \mathbb{P}_{(1,0,0,0),T} \) are absolutely continuous with respect to each other, and the Radon-Nykodim derivative of \( \mathbb{P}_{(a,b,m,\theta),T} \) with respect to \( \mathbb{P}_{(1,0,0,0),T} \) (so called likelihood ratio) takes the form

\[
L^{(a,b,m,\theta),(1,0,0,0)}_T((Y_s, X_s)_{s \in [0,T]}) = \exp \left\{ \int_0^T \left( \frac{a - b Y_s - 1}{Y_s} \, dY_s + \frac{m - \theta X_s}{Y_s} \, dX_s \right) - \frac{1}{2} \int_0^T \left( a - b Y_s - 1 \right) \left( a - b Y_s + 1 \right) \left( m - \theta X_s \right)^2 \, ds \right\},
\]

where \((Y_t, X_t)_{t \geq 0}\) denotes the unique strong solution of the SDE (1.1) corresponding to the parameters \((a,b,m,\theta)\) and the initial value \((Y_0, X_0)\). The process

\[
(3.1) \quad \left( L^{(a,b,m,\theta),(1,0,0,0)}_T((Y_s, X_s)_{s \in [0,T]}) \right)_{T \geq 0}
\]

is an \( \mathcal{F}_T \)-martingale (where the filtration \( \mathcal{F}_T \) is introduced before Proposition 2.1).

Proof. First note that the SDE (1.1) can be written in the form:

\[
\begin{bmatrix}
Y_t \\
X_t
\end{bmatrix} = \begin{bmatrix}
-b & 0 \\
0 & -\theta
\end{bmatrix} \begin{bmatrix}
Y_t \\
X_t
\end{bmatrix} + \begin{bmatrix}
a \\
m
\end{bmatrix} \, dt + \sqrt{\begin{bmatrix}
y_t \\
\theta X_t
\end{bmatrix}} \, dB_t, \quad t \geq 0.
\]

Note also that under the condition \( a \geq \frac{1}{2} \), for all \( y_0 > 0 \), we have \( \mathbb{P}(Y_t > 0, \forall t \in \mathbb{R}_+ \mid Y_0 = y_0) = 1 \), see, e.g., page 442 in Revuz and Yor [25]. Since \( \mathbb{P}(Y_0 > 0) = 1 \), by the law of total probability, \( \mathbb{P}(Y_t > 0, \forall t \in \mathbb{R}_+) = 1 \).

We intend to use formula (7.139) in Section 7.6.4 of Liptser and Shiryaev [21]. Their condition (7.137) has to be checked which takes the form

\[
\int_0^T \frac{(a - b Y_s)^2 + (m - \theta X_s)^2 + 1}{Y_s} \, ds < \infty \quad \text{a.s. for all } T \in \mathbb{R}_+.
\]

Since \((Y,X)\) has continuous sample paths almost surely, this holds if

\[
(3.2) \quad \int_0^T \frac{1}{Y_s} \, ds < \infty \quad \text{a.s. for all } T \in \mathbb{R}_+.
\]

Under the conditions \( a \geq 1/2 \) and \( \mathbb{P}(Y_0 > 0) = 1 \), Theorems 1 and 3 in Ben-Alaya and Kebaier [5] yield (3.2). More precisely, if \( a \geq \frac{1}{2} \) and \( y_0 > 0 \), then Theorems 1 and 3 in Ben-Alaya and Kebaier [5] yield

\[
\mathbb{P} \left( \int_0^T \frac{1}{Y_s} \, ds < \infty \mid Y_0 = y_0 \right) = 1, \quad T \in \mathbb{R}_+.
\]

Since \( \mathbb{P}(Y_0 > 0) = 1 \), by the law of total probability, we get (3.2). We give another, direct proof for (3.2). Namely, since \( Y \) has continuous sample paths almost surely and \( \mathbb{P}(Y_t > 0, \forall t \in \mathbb{R}_+) = 1 \), we have \( \mathbb{P}(\inf_{t \in [0,T]} Y_t > 0) = 1 \) for all \( T \in \mathbb{R}_+ \), which yields (3.2).
We call the attention that conditions (4.110) and (4.111) are also required for Section 7.6.4 in Liptser and Shiryaev [21], but the Lipschitz condition (4.110) in Liptser and Shiryaev [21] does not hold for the SDE (1.1). However, we can use formula (7.139) in Liptser and Shiryaev [21], since they use their conditions (4.110) and (4.111) only in order to ensure the SDE they consider in Section 7.6.4 has a unique strong solution (see, the proof of Theorem 7.19 in Liptser and Shiryaev [21]). By Proposition 2.1, under the conditions of the present lemma, there is a (pathwise) unique strong solution of the SDE (1.1).

The martingale property of the process (3.1) is a consequence of Jacod and Shiryaev [16, Chapter III, Theorem 3.4].

By Lemma 3.1, under its conditions the log-likelihood function takes the form

\[
\log L_{T(a,b,m,\theta),(1,0,0,0)}((Y_s, X_s)_{s \in [0,T]}) = (a - 1) \int_0^T \frac{1}{Y_s} dY_s - b(Y_T - Y_0) + m \int_0^T \frac{1}{Y_s} dX_s - \theta \int_0^T \frac{X_s}{Y_s} dX_s - \frac{a^2 - 1}{2} \int_0^T \frac{1}{Y_s} ds + abT - \frac{b^2}{2} \int_0^T Y_s ds - \frac{m^2}{2} \int_0^T \frac{1}{Y_s} ds + m\theta \int_0^T \frac{X_s}{Y_s} ds - \theta^2 \int_0^T \frac{X_s^2}{Y_s} ds
\]

\[
=: f_T(a, b, m, \theta), \quad T > 0.
\]

We remark that for all \( T > 0 \) and all initial values \((Y_0, X_0)\), the probability measures \( P_{(a,b,m,\theta),T} \), \( a \geq 1/2, b, m, \theta \in \mathbb{R} \), are absolutely continuous with respect to each other, and hence it does not matter which measure is taken as a reference measure for defining the MLE (we have chosen \( P_{(1,0,0,0),T} \)). For more details, see, e.g., Liptser and Shiryaev [21, page 35]. Then the equation \( \frac{\partial f_T}{\partial \theta}(a, b, m, \theta) = 0 \) and the system of equations

\[
\frac{\partial f_T}{\partial a}(a, b, m, \theta) = 0, \quad \frac{\partial f_T}{\partial b}(a, b, m, \theta) = 0, \quad \frac{\partial f_T}{\partial m}(a, b, m, \theta) = 0, \quad \frac{\partial f_T}{\partial \theta}(a, b, m, \theta) = 0,
\]

respectively, take the forms

\[
- \int_0^T \frac{X_s}{Y_s} dX_s + m \int_0^T \frac{X_s}{Y_s} ds - \theta \int_0^T \frac{X_s^2}{Y_s} ds = 0,
\]

and

\[
\begin{bmatrix}
T & 0 & 0 & \int_0^T \frac{1}{Y_s} ds \\
- \int_0^T Y_s ds & 0 & 0 & \int_0^T \frac{X_s}{Y_s} ds \\
0 & \int_0^T \frac{1}{Y_s} ds & - \int_0^T \frac{X_s}{Y_s} ds & m \int_0^T \frac{1}{Y_s} ds \\
0 & 0 & - \int_0^T \frac{X_s}{Y_s} ds & \int_0^T \frac{X_s^2}{Y_s} ds
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
m \\
\theta
\end{bmatrix}
=
\begin{bmatrix}
\int_0^T \frac{1}{Y_s} dY_s \\
(Y_T - Y_0) \\
- \int_0^T \frac{X_s}{Y_s} dX_s \\
\int_0^T \frac{X_s^2}{Y_s} dX_s
\end{bmatrix}.
\]

First, we suppose that \( a \geq 1/2, \) and \( b, m \in \mathbb{R} \) are known. By maximizing \( \log L_{T(a,b,m,\theta),(1,0,0,0)} \) in \( \theta \in \mathbb{R} \), we get the MLE of \( \theta \) based on the observations \((Y_t, X_t)_{t \in [0,T]}\),

\[
\hat{\theta}_T^{\text{MLE}} := - \int_0^T \frac{X_s}{Y_s} dX_s + m \int_0^T \frac{X_s}{Y_s} ds \int_0^T \frac{X_s^2}{Y_s} ds, \quad T > 0,
\]

\(3.3\)
provided that \( f_0^T X_s^2 ds > 0 \). Indeed,

\[
\frac{\partial^2 f_T}{\partial \theta^2}(\theta, m) = - \int_0^T X_s^2 \frac{1}{Y_s} ds < 0.
\]

Using the SDE (1), one can also get

\[
(3.4) \quad \hat{\theta}_T^{\text{MLE}} - \theta = - \frac{\int_0^T X_s \frac{1}{Y_s} dB_s}{\int_0^T X_s^2 \frac{1}{Y_s} ds}, \quad T > 0,
\]

provided that \( \int_0^T X_s^2 \frac{1}{Y_s} ds > 0 \). Note that the estimator \( \hat{\theta}_T^{\text{MLE}} \) does not depend on the parameters \( a \geq 1/2 \) and \( b \in \mathbb{R} \). In fact, if we maximize \( \log L^{(a,b,m,\theta)}_{T,(1,0,0,0)} \) in \( (a, b, \theta) \in \mathbb{R}^3 \), then we obtain the MLE of \( (a, b, \theta) \) supposing that \( m \in \mathbb{R} \) is known, and one can observe that the MLE of \( \theta \) by this procedure coincides with \( \hat{\theta}_T^{\text{MLE}} \).

By maximizing \( \log L^{(a,b,m,\theta)}_{T,(1,0,0,0)} \) in \( (a, b, m, \theta) \in \mathbb{R}^4 \), the MLE of \( (a, b, m, \theta) \) based on the observations \( (Y_t, X_t)_{t \in [0,T]} \) takes the form

\[
(3.5) \quad \hat{a}_T^{\text{MLE}} := \frac{\int_0^T Y_s ds \int_0^T X_s \frac{1}{Y_s} ds - T(Y_T - Y_0)}{\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds - T^2}, \quad T > 0,
\]

\[
(3.6) \quad \hat{b}_T^{\text{MLE}} := \frac{\int_0^T Y_s ds \int_0^T X_s \frac{1}{Y_s} ds - (Y_T - Y_0) \int_0^T \frac{1}{Y_s} ds}{\int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds - T^2}, \quad T > 0,
\]

\[
(3.7) \quad \hat{m}_T^{\text{MLE}} := \frac{\int_0^T X_s^2 \frac{1}{Y_s} ds \int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds - \int_0^T X_s \frac{1}{Y_s} ds \int_0^T X_s \frac{1}{Y_s} ds \int_0^T X_s \frac{1}{Y_s} ds}{\int_0^T X_s \frac{1}{Y_s} ds \int_0^T \frac{1}{Y_s} ds - \left( \int_0^T X_s \frac{1}{Y_s} ds \right)^2}, \quad T > 0,
\]

\[
(3.8) \quad \hat{\theta}_T^{\text{MLE}} := \frac{\int_0^T X_s \frac{1}{Y_s} ds \int_0^T \frac{1}{Y_s} ds \int_0^T \frac{1}{Y_s} ds - \int_0^T X_s \frac{1}{Y_s} ds \int_0^T X_s \frac{1}{Y_s} ds \int_0^T X_s \frac{1}{Y_s} ds}{\int_0^T X_s \frac{1}{Y_s} ds \int_0^T \frac{1}{Y_s} ds - \left( \int_0^T X_s \frac{1}{Y_s} ds \right)^2}, \quad T > 0,
\]

provided that \( \int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds - T^2 > 0 \) and \( \int_0^T X_s^2 \frac{1}{Y_s} ds \int_0^T \frac{1}{Y_s} ds - \left( \int_0^T X_s \frac{1}{Y_s} ds \right)^2 > 0 \). Indeed,

\[
\begin{bmatrix}
\frac{\partial^2 f_T}{\partial a^2}(a, b, m, \theta) \\
\frac{\partial^2 f_T}{\partial b^2}(a, b, m, \theta)
\end{bmatrix}
= \begin{bmatrix}
- \int_0^T \frac{1}{Y_s} ds & T \\
T & - \int_0^T Y_s ds
\end{bmatrix},
\]

\[
\begin{bmatrix}
\frac{\partial^2 f_T}{\partial m^2}(a, b, m, \theta) \\
\frac{\partial^2 f_T}{\partial \theta^2}(a, b, m, \theta)
\end{bmatrix}
= \begin{bmatrix}
- \int_0^T \frac{1}{Y_s} ds & \int_0^T X_s \frac{1}{Y_s} ds \\
\int_0^T X_s \frac{1}{Y_s} ds & - \int_0^T X_s^2 \frac{1}{Y_s} ds
\end{bmatrix},
\]

and the positivity of \( \int_0^T Y_s ds \int_0^T X_s \frac{1}{Y_s} ds ds - T^2 \) and \( \int_0^T X_s^2 \frac{1}{Y_s} ds \int_0^T \frac{1}{Y_s} ds - \left( \int_0^T X_s \frac{1}{Y_s} ds \right)^2 \) yield \( \int_0^T \frac{1}{Y_s} ds > 0 \), respectively. Using the SDE (1), one can check that

\[
\begin{bmatrix}
\hat{a}_T^{\text{MLE}} - a \\
\hat{b}_T^{\text{MLE}} - b
\end{bmatrix}
= \begin{bmatrix}
\int_0^T \frac{1}{Y_s} ds & - T \\
-T & \int_0^T Y_s ds
\end{bmatrix}^{-1}
\begin{bmatrix}
\int_0^T \frac{1}{Y_s} ds & - T \\
-T & \int_0^T Y_s ds
\end{bmatrix}
\begin{bmatrix}
\int_0^T X_s \frac{1}{Y_s} dL_s \\
- \int_0^T \sqrt{Y_s} dL_s
\end{bmatrix},
\]

\[
\begin{bmatrix}
\hat{m}_T^{\text{MLE}} - m \\
\hat{\theta}_T^{\text{MLE}} - \theta
\end{bmatrix}
= \begin{bmatrix}
\int_0^T \frac{1}{Y_s} ds & - \int_0^T X_s \frac{1}{Y_s} ds \\
- \int_0^T \frac{1}{Y_s} ds & \int_0^T \frac{1}{Y_s} ds
\end{bmatrix}^{-1}
\begin{bmatrix}
\int_0^T \frac{1}{Y_s} ds & - \int_0^T X_s \frac{1}{Y_s} ds \\
- \int_0^T \frac{1}{Y_s} ds & \int_0^T \frac{1}{Y_s} ds
\end{bmatrix}
\begin{bmatrix}
\int_0^T \frac{1}{Y_s} dB_s \\
- \int_0^T \sqrt{Y_s} dB_s
\end{bmatrix}.
\]
and hence

\[ \hat{b}_{T}^{MLE} - b = \frac{\int_0^T \frac{1}{s} dL_s - \int_0^T \frac{1}{\sqrt{s}} dL_s}{\int_0^T \frac{1}{s} ds - T^2}, \quad T > 0, \]

\[ \hat{m}_{T}^{MLE} - m = \frac{\int_0^T \frac{X^2}{Y_s} ds - \int_0^T \frac{X}{\sqrt{Y_s}} dB_s}{\int_0^T \frac{X^2}{Y_s} ds - \left( \int_0^T \frac{X}{\sqrt{Y_s}} ds \right)^2}, \quad T > 0, \]

and

\[ \hat{\theta}_{T}^{MLE} - \theta = \frac{\int_0^T \frac{X}{\sqrt{Y_s}} dB_s}{\int_0^T \frac{X^2}{Y_s} ds - \left( \int_0^T \frac{X}{\sqrt{Y_s}} ds \right)^2}, \quad T > 0, \]

provided that \( \int_0^T \frac{1}{s} ds - T^2 > 0 \) and \( \int_0^T \frac{X^2}{Y_s} ds - \left( \int_0^T \frac{X}{\sqrt{Y_s}} ds \right)^2 > 0 \).

3.2 Remark. In order that the stochastic integrals \( \int_0^T \frac{1}{s} dY_s \), \( \int_0^T \frac{X}{\sqrt{Y_s}} dX_s \) and \( \int_0^T \frac{1}{s} dX_s \) in (3.5), (3.6), (3.7) and (3.8) can be observable, we define them pathwise in the following way. With the notations of Jacod and Shiryaev [16], \( \tau_n := \left( \frac{1}{n} \wedge T \right)_{t \in \mathbb{N}} \) is a Riemann sequence of (adapted) subdivisions and hence, by Jacod and Shiryaev [16 Proposition I.4.44], the sequence of processes

\[
\left( \sum_{i=1}^{[nT]} \frac{X_{\frac{i-1}{n} \wedge t} - X_{\frac{i}{n} \wedge t}}{Y_{\frac{i}{n} \wedge t}} \right)_{t \in \mathbb{R}^+}, \quad n \in \mathbb{N},
\]

converges to \( \int_0^T \frac{X}{\sqrt{Y_s}} dX_s \) in probability as \( n \to \infty \), uniformly on every compact interval. Similarly,

\[
\left( \sum_{i=1}^{[nT]} \frac{1}{Y_{\frac{i}{n} \wedge t}} \frac{Y_{\frac{i}{n} \wedge t} - Y_{\frac{i-1}{n} \wedge t}}{Y_{\frac{i-1}{n} \wedge t}} \right)_{t \in \mathbb{R}^+}, \quad n \in \mathbb{N},
\]

and

\[
\left( \sum_{i=1}^{[nT]} \frac{1}{Y_{\frac{i}{n} \wedge t}} \frac{X_{\frac{i}{n} \wedge t} - X_{\frac{i-1}{n} \wedge t}}{Y_{\frac{i-1}{n} \wedge t}} \right)_{t \in \mathbb{R}^+}, \quad n \in \mathbb{N},
\]

converge to \( \int_0^T \frac{1}{s} dY_s \) and \( \int_0^T \frac{1}{s} dX_s \), respectively, in probability as \( n \to \infty \), uniformly on every compact interval. This means that in practice the continuous time observations \( (Y_t, X_t)_{t \in [0,T]} \) is used for giving a discrete time approximation of the appearing stochastic integrals in (3.5), (3.6), (3.7) and (3.8).

The next lemma is about the existence of \( \hat{\theta}_{T}^{MLE} \) (supposing that \( a \geq \frac{1}{2}, b \in \mathbb{R} \) and \( m \in \mathbb{R} \) are known).
3.3 Lemma. If \( a \geq \frac{1}{2}, \ b, m, \theta \in \mathbb{R}, \) and \( \mathbb{P}(Y_0 > 0) = 1, \) then

\[
\mathbb{P}\left( \int_0^T \frac{X_s^2}{Y_s} \, ds \in (0, \infty) \right) = 1 \quad \text{for all} \ T > 0, \tag{3.13}
\]

and hence there exists a unique MLE \( \widehat{\theta}_{\text{MLE}} \) which has the form given in (3.3).

Proof. First note that under the condition \( a \geq \frac{1}{2}, \) for all \( y_0 > 0, \) we have \( \mathbb{P}(Y_t > 0, \forall \ t \in \mathbb{R}_+ \ | \ Y_0 = y_0) = 1, \) see, e.g., page 442 in Revuz and Yor [25]. Since \( \mathbb{P}(Y_0 > 0) = 1, \) by the law of total probability, we get \( \mathbb{P}(Y_t > 0, \forall \ t \in \mathbb{R}_+) = 1. \) Note also that, since \( X \) has continuous trajectories almost surely, by the proof of Lemma 3.1 we have \( \mathbb{P}\left( \int_0^T \frac{X_s^2}{Y_s} \, ds \in (0, \infty) \right) = 1 \) for all \( T > 0. \) Further, for any \( \omega \in \Omega, \) \( \int_0^T \frac{X_s^2(\omega)}{Y_s(\omega)} \, ds = 0 \) holds if and only if \( X_s(\omega) = 0 \) for almost every \( s \in [0, T]. \) Using that \( X \) has continuous trajectories almost surely, we have

\[
\mathbb{P}\left( \int_0^T \frac{X_s^2}{Y_s} \, ds = 0 \right) > 0
\]

holds if and only if \( \mathbb{P}(X_s = 0, \forall \ s \in [0, T]) > 0. \) Due to \( a \geq \frac{1}{2}, \) there does not exist a constant \( c \in \mathbb{R} \) such that \( \mathbb{P}(X_s = c, \forall \ s \in [0, T]) > 0. \) Indeed, if \( c \in \mathbb{R} \) is such that \( \mathbb{P}(X_s = c, \forall \ s \in [0, T]) > 0, \) then using (2.22), we have

\[
c = e^{-\theta s} \left( c + m \int_0^s e^{\theta u} \, du + \int_0^s \sqrt{Y_u} \, dB_u \right), \quad s \in [0, T],
\]

on the event \( \{ \omega \in \Omega : X_s(\omega) = c, \forall \ s \in [0, T] \}. \) In case \( \theta \neq 0, \) the process

\[
\left( \int_0^s e^{\theta u} \sqrt{Y_u} \, dB_u \right)_{s \in [0, T]}
\]

would be equal to the deterministic process \( \left( (c - m/\theta)(e^{\theta s} - 1) \right)_{s \in [0, T]} \) on the event \( \{ \omega \in \Omega : X_s(\omega) = c, \forall \ s \in [0, T] \} \) having positive probability. Since the quadratic variation of the deterministic process \( \left( (c - m/\theta)(e^{\theta s} - 1) \right)_{s \in [0, T]} \) is the identically zero process (due to the fact that the quadratic variation process is a process starting from 0 almost surely), the quadratic variation of \( \left( \int_0^s e^{\theta u} \sqrt{Y_u} \, dB_u \right)_{s \in [0, T]} \) should be identically zero on the event \( \{ \omega \in \Omega : X_s(\omega) = c, \forall \ s \in [0, T] \}. \) This would imply that \( \int_0^s e^{2\theta u} Y_u \, du = 0 \) for all \( s \in [0, T] \) on the event \( \{ \omega \in \Omega : X_s(\omega) = c, \forall \ s \in [0, T] \}. \) Using the almost sure continuity and non-negativeness of the sample paths of \( Y, \) we have \( Y_s = 0 \) for all \( s \in [0, T] \) on the event

\[
\{ \omega \in \Omega : X_s(\omega) = c, \forall \ s \in [0, T] \} \cap \{ \omega \in \Omega : (Y_s(\omega))_{s \in [0, T]} \text{ is continuous} \}
\]

having positive probability. If \( \theta = 0, \) then replacing \( \left( (c - m/\theta)(e^{\theta s} - 1) \right)_{s \in [0, T]} \) by \( -(ms)_{s \in [0, T]}, \) one can arrive at the same conclusion. Hence \( \mathbb{P}(\inf \{ t \in \mathbb{R}_+ : Y_t = 0 \} = 0) > 0 \) which leads us to a contradiction. This implies (3.13).

The above argument also shows that we can make a shortcut by arguing in a little bit different way. Indeed, if \( C \) is a random variable such that \( \mathbb{P}(X_s = C, \forall \ s \in [0, T]) > 0, \) then on the event \( \{ \omega \in \Omega : X_s(\omega) = C(\omega), \forall \ s \in [0, T] \}, \) the quadratic variation of \( X \) would be identically zero. Since, by the SDE (1.1), the quadratic variation of \( X \) is the process \( \left( \int_0^t Y_u \, du \right)_{t \geq 0}, \) it would imply that \( \int_0^s Y_u \, du = 0 \) for all \( s \in [0, T] \) on the event \( \{ \omega \in \Omega : X_s(\omega) = C(\omega), \forall \ s \in [0, T] \}. \) Using the almost
sure continuity and non-negativeness of the sample paths of $Y$, we have $Y_s = 0$ for all $s \in [0, T]$ on the event

$$\{\omega \in \Omega : X_s(\omega) = C(\omega), \forall s \in [0, T]\} \cap \{\omega \in \Omega : (Y_s(\omega))_{s \in [0, T]} \text{ is continuous}\}$$

having positive probability. As before, this leads us to a contradiction. \hfill \Box

The next lemma is about the existence of $\left(\hat{\alpha}_T^{\text{MLE}}, \hat{b}_T^{\text{MLE}}, \hat{m}_T^{\text{MLE}}, \hat{\theta}_T^{\text{MLE}}\right)$.

**3.4 Lemma.** If $a \geq \frac{1}{2}$, $b, m, \theta \in \mathbb{R}$, and $\mathbb{P}(Y_0 > 0) = 1$, then

$$\mathbb{P} \left( \int_0^T Y_s \, ds \int_0^T \frac{1}{Y_s} \, ds - T^2 \in (0, \infty) \right) = 1 \quad \text{for all } T > 0, \tag{3.14}$$

$$\mathbb{P} \left( \int_0^T X_s^2 \, ds \int_0^T \frac{1}{Y_s} \, ds - \left( \int_0^T X_s \, ds \right)^2 \in (0, \infty) \right) = 1 \quad \text{for all } T > 0, \tag{3.15}$$

and hence there exists a unique MLE $\left(\hat{\alpha}_T^{\text{MLE}}, \hat{b}_T^{\text{MLE}}, \hat{m}_T^{\text{MLE}}, \hat{\theta}_T^{\text{MLE}}\right)$ which has the form given in (3.5), (3.6) (3.7) and (3.8).

**Proof.** First note that under the condition $a \geq \frac{1}{2}$, as it was detailed in the proof of Lemma 3.3, we have $\mathbb{P}(Y_t > 0, \forall t \in \mathbb{R}_+) = 1$. By Cauchy-Schwarz’s inequality, we have

$$\int_0^T Y_s \, ds \int_0^T \frac{1}{Y_s} \, ds - T^2 \geq 0, \quad T > 0, \tag{3.16}$$

and hence, using also that $Y$ has continuous trajectories almost surely and $\int_0^T \frac{1}{Y_s} \, ds < \infty$ (see the proof of Lemma 3.1), we have

$$\mathbb{P} \left( \int_0^T Y_s \, ds \int_0^T \frac{1}{Y_s} \, ds - T^2 \in (0, \infty) \right) = 1, \quad T > 0. \tag{3.17}$$

Further, equality holds in (3.16) if and only if $K Y_s = L / Y_s$ for almost every $s \in [0, T]$ with some $K, L \geq 0, K^2 + L^2 > 0$ ($K$ and $L$ may depend on $\omega \in \Omega$ and $T > 0$) or equivalently $KY_s^2 = L$ for almost every $s \in [0, T]$ with some $K, L \geq 0, K^2 + L^2 > 0$. Note that if $K = 0$, then $L$ would be 0, too, hence $K$ can not be 0 implying that equality holds in (3.16) if and only if $Y_s^2 = L / K$ for almost every $s \in [0, T]$ with some $K > 0$ and $L \geq 0$ ($K$ and $L$ may depend on $\omega \in \Omega$ and $T$). Using that $Y$ has continuous trajectories almost surely, we have

$$\mathbb{P} \left( \int_0^T Y_s \, ds \int_0^T \frac{1}{Y_s} \, ds - T^2 = 0 \right) > 0 \tag{3.17}$$

holds if and only if $\mathbb{P}(Y_s^2 = L / K, \forall s \in [0, T]) > 0$ with some random variables $K$ and $L$ such that $\mathbb{P}(K > 0, L \geq 0) = 1$ ($K$ and $L$ may depend on $T$). Similarly, as it was explained at the end of the proof of Lemma 3.3, this implies that the quadratic variation of the process $\{Y_s^2\}_{s \in [0, T]}$ would be identically zero on the event $\{\omega \in \Omega : Y_s^2(\omega) = L(\omega) / K(\omega), \forall s \in [0, T]\}$ having positive probability. Since, by Itô’s formula,

$$dY_t^2 = 2Y_t \, dY_t + Y_t \, dt = (2Y_t(a - bY_t) + Y_t) \, dt + 2Y_t \sqrt{Y_t} \, dL_t, \quad t \geq 0,$$
the quadratic variation of \((Y_t^2)_{t \geq 0}\) is the process \(\left( \int_0^t 4Y_u^3 \, du \right)_{t \geq 0}\). If \((3.17)\) holds, then \(\int_0^s 4Y_u^3 \, du = 0\) for all \(s \in [0, T]\) on the event \(\{\omega \in \Omega : Y_s^2(\omega) = L(\omega)/K(\omega), \, \forall \, s \in [0, T]\}\) having positive probability. Using the almost sure continuity and non-negativeness of the sample paths of \(Y\), we have \(Y_s = 0\) for all \(s \in [0, T]\) on the event \(\{\omega \in \Omega : (Y_t(\omega))_{t \geq 0} \text{ is continuous}\}\) having positive probability. This yields us to a contradiction since \(P(Y_t > 0, \forall \, t \geq 0) = 1\), implying \((3.14)\).

Now we turn to prove \((3.15)\). By Cauchy-Schwarz’s inequality, we have

\[
(3.18) \quad \int_0^T \frac{X_s^2}{Y_s} \, ds - \left( \int_0^T \frac{X_s}{Y_s} \, ds \right)^2 \geq 0, \quad T > 0,
\]

and hence, since \(X\) has continuous trajectories almost surely, by the proof of Lemma 3.1, we have

\[
P\left( \int_0^T \frac{X_s^2}{Y_s} \, ds - \left( \int_0^T \frac{X_s}{Y_s} \, ds \right)^2 \in [0, \infty) \right) = 1, \quad T > 0.
\]

Further, equality holds in \((3.18)\) if and only if \(KX_s^2/Y_s = L/Y_s\) for almost every \(s \in [0, T]\) with some \(K, L \geq 0, K^2 + L^2 > 0\) \((K \text{ and } L \text{ may depend on } \omega \in \Omega \text{ and } T > 0)\) or equivalently \(KX_s^2 = L\) for almost every \(s \in [0, T]\) with some \(K, L \geq 0, K^2 + L^2 > 0\). Note that if \(K = 0\), then \(L = 0\), too, hence \(K\) cannot be 0 implying that equality holds in \((3.18)\) if and only if \(X_s^2 = L/K\) for almost every \(s \in [0, T]\) with some \(K > 0\) and \(L \geq 0\) \((K \text{ and } L \text{ may depend on } \omega \in \Omega \text{ and } T)\). Using that \(X\) has continuous trajectories almost surely, we have

\[
(3.19) \quad \mathbb{P}\left( \int_0^T \frac{X_s^2}{Y_s} \, ds - \left( \int_0^T \frac{X_s}{Y_s} \, ds \right)^2 = 0 \right) > 0
\]

holds if and only if \(\mathbb{P}(X_s^2 = L/K, \, \forall \, s \in [0, T]) > 0\) with some random variables \(K\) and \(L\) such that \(\mathbb{P}(K > 0, L \geq 0) = 1\) \((K \text{ and } L \text{ may depend on } T)\). Similarly, as it was explained at the end of the proof of Lemma 3.3 this implies that the quadratic variation of the process \((X_s^2)_{s \in [0, T]}\) would be identically zero on the event \(\{\omega \in \Omega : X_s^2(\omega) = L(\omega)/K(\omega), \, \forall \, s \in [0, T]\}\) having positive probability. Since, by Itô’s formula,

\[
dX_t^2 = 2X_t \, dX_t + Y_t \, dt = (2X_t(m - \theta X_t) + Y_t) \, dt + 2X_t \sqrt{Y_t} \, dB_t, \quad t \geq 0,
\]

the quadratic variation of \((X_t^2)_{t \geq 0}\) is the process \(\left( \int_0^t 4X_u^2 \, du \right)_{t \geq 0}\). If \((3.19)\) holds, then \(\int_0^s 4X_u^2 \, du = 0\) for all \(s \in [0, T]\) on the event \(\{\omega \in \Omega : X_s^2(\omega) = L(\omega)/K(\omega), \, \forall \, s \in [0, T]\}\) having positive probability. Using the almost sure continuity and non-negativeness of the sample paths of \(X^2Y\), we have \(X_s^2Y_s = 0\) for all \(s \in [0, T]\) on the event

\[
\{\omega \in \Omega : X_s^2(\omega) = L(\omega)/K(\omega), \, \forall \, s \in [0, T]\} \cap \{\omega \in \Omega : (X_t(\omega)Y_t(\omega))_{t \geq 0} \text{ is continuous}\} =: A
\]

having positive probability. Since \(\mathbb{P}(Y_t > 0, \forall \, t \geq 0) = 1\), we have \(X_s = 0\) for all \(s \in [0, T]\) on the event \(A\) having positive probability. Repeating the argument given in the proof of Lemma 3.3 we arrive at a contradiction. This implies \((3.15)\).
4 Existence and uniqueness of least squares estimator

Studying LSE for the model \((\ref{eq:1.1})\), the parameters \(a > 0\) and \(b \in \mathbb{R}\) will be not supposed to be known. However, we will not consider the LSEs of \(a\) and \(b\), we will focus only on the LSEs of \(m\) and \(\theta\), since we would like use a continuous time observation only for the process \(X\), and not for the process \((Y,X)\), studying LSEs.

First we give a motivation for the LSE based on continuous time observations using the form of the LSE based on discrete time low frequency observations.

Let us suppose that \(m \in \mathbb{R}\) is known \((a > 0\) and \(b \in \mathbb{R}\) are not supposed to be known). The LSE of \(\theta\) based on the discrete time observations \(X_i, i = 0,1, \ldots, n\), can be obtained by solving the following extremum problem

\[
\hat{\theta}^{LSE,D}_n := \arg \min_{\theta \in \mathbb{R}} \sum_{i=1}^{n} (X_i - X_{i-1} - (m - \theta X_{i-1}))^2.
\]

Here in the notation \(\hat{\theta}^{LSE,D}_n\) the letter \(D\) refers to discrete time observations, and we note that \(X_0\) denotes an observation for the second coordinate of the initial value of the process \((Y,X)\). This definition of LSE of \(\theta\) can be considered as the corresponding one given in Hu and Long \([13\text{, formula (1.2)}]\) for generalized Ornstein-Uhlenbeck processes driven by \(\alpha\)-stable motions, see also Hu and Long \([15\text{, formula (3.1)}]\). For a heuristic motivation of the LSE of \(\theta\) based on the discrete observations \(X_i, i = 0,1, \ldots, n\), see, e.g., Hu and Long \([13\text{, page 178}]\) (formulated for Langevin equations), and for a mathematical one, Remark 3.4 in Barczy et al. \([1]\). Further, by Barczy et al. \([1\text{, formula (3.5)}]\),

\[
\hat{\theta}^{LSE,D}_n = - \sum_{i=1}^{n} (X_i - X_{i-1})X_{i-1} - m \left(\sum_{i=1}^{n} X_{i-1}\right)
\]

provided that \(\sum_{i=1}^{n} X_{i-1}^2 > 0\). Motivated by \((\ref{eq:1.1})\), the LSE of \(\theta\) based on the continuous time observations \((X_t)_{t \in [0,T]}\) is defined by

\[
\hat{\theta}^{LSE}_T := - \int_0^T X_s dX_s - m \int_0^T X_s ds
\]

provided that \(\int_0^T X_s^2 ds > 0\), and using the SDE \((\ref{eq:1.1})\) we have

\[
\hat{\theta}^{LSE}_T - \theta = - \int_0^T X_s \sqrt{Y_s} dB_s - \int_0^T X_s^2 ds,
\]

provided that \(\int_0^T X_s^2 ds > 0\).

Let us suppose that the parameters \(a > 0\) and \(b, m \in \mathbb{R}\) are not known. The LSE of \((m,\theta)\) based on the discrete time observations \(X_i, i = 0,1, \ldots, n\), can be obtained by solving the following extremum problem

\[
(m^{LSE,D}_n, \hat{\theta}^{LSE,D}_n) := \arg \min_{(\theta,m) \in \mathbb{R}^2} \sum_{i=1}^{n} (X_i - X_{i-1} - (m - \theta X_{i-1}))^2.
\]

By Barczy et al. \([1\text{, formulas (3.16) and (3.17)}]\),

\[
m^{LSE,D}_n = \frac{\sum_{i=1}^{n} X_{i-1}^2 \sum_{i=1}^{n} (X_i - X_{i-1}) - \sum_{i=1}^{n} X_{i-1} \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} (X_i - X_{i-1})X_{i-1}}{n \sum_{i=1}^{n} X_{i-1}^2 - \left(\sum_{i=1}^{n} X_{i-1}\right)^2}
\]
and

$$\hat{\theta}^\text{LSE,D}_n = \frac{\sum_{i=1}^n X_{i-1} \sum_{i=1}^n (X_i - X_{i-1}) - n \sum_{i=1}^n (X_i - X_{i-1})^2}{n \sum_{i=1}^n X_{i-1}^2 - (\sum_{i=1}^n X_{i-1})^2},$$

provided that $n \sum_{i=1}^n X_{i-1}^2 - (\sum_{i=1}^n X_{i-1})^2 > 0$. Motivated by (4.4) and (4.5), the LSE of $(m, \theta)$ based on the continuous time observations $(X_t)_{t\in[0,T]}$ is defined by

$$\hat{m}^\text{LSE}_T := \frac{(X_T - X_0) \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2 \left(\int_0^T X_s^2 \, ds\right)}{T \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2},$$

and

$$\hat{\theta}^\text{LSE}_T := \frac{(X_T - X_0) \int_0^T X_s \, ds - T \int_0^T X_s \, ds X_s}{T \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2},$$

provided that $T \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2 > 0$. Note that, by Cauchy-Schwarz’s inequality, $T \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2 \geq 0$, and $T \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2 > 0$ yields that $\int_0^T X_s^2 \, ds > 0$. Then

$$\begin{bmatrix} \hat{m}^\text{LSE}_T - m \\ \hat{\theta}^\text{LSE}_T - \theta \end{bmatrix} = \begin{bmatrix} T & - \int_0^T X_s \, ds \\ - \int_0^T X_s \, ds & \int_0^T X_s^2 \, ds \end{bmatrix}^{-1} \begin{bmatrix} X_T - X_0 \\ - \int_0^T X_s \, ds \end{bmatrix},$$

and using the SDE (1.1) one can check that

$$\begin{bmatrix} \hat{m}^\text{LSE}_T - m \\ \hat{\theta}^\text{LSE}_T - \theta \end{bmatrix} = \begin{bmatrix} T & - \int_0^T X_s \, ds \\ - \int_0^T X_s \, ds & \int_0^T X_s^2 \, ds \end{bmatrix}^{-1} \begin{bmatrix} \int_0^T \sqrt{Y_s} \, dB_s \\ - \int_0^T X_s \, ds \end{bmatrix},$$

and hence

$$\hat{m}^\text{LSE}_T - m = - \int_0^T X_s \, ds \int_0^T X_s \sqrt{Y_s} \, dB_s + \int_0^T X_s^2 \, ds \int_0^T \sqrt{Y_s} \, dB_s$$

$$T \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2,$$

and

$$\hat{\theta}^\text{LSE}_T - \theta = - \frac{T \int_0^T X_s \sqrt{Y_s} \, dB_s + \int_0^T X_s \, ds \int_0^T \sqrt{Y_s} \, dB_s}{T \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2},$$

provided that $T \int_0^T X_s^2 \, ds - \left(\int_0^T X_s \, ds\right)^2 > 0$.

4.1 Remark. In order that the stochastic integral $\int_0^T X_s \, ds$ in (4.2), (4.6) and (4.7) can be observable, we define it pathwise in the following way. By Ito’s formula, we have $dX_t^2 = 2X_t \, dX_t + Y_t \, dt$, $t \in \mathbb{R}_+$, and then

$$\int_0^T X_s \, ds = \frac{1}{2} \left( X_T^2 - X_0^2 - \int_0^T Y_s \, ds \right).$$

The next lemma is about the existence of $\hat{\theta}^\text{LSE}_T$ (supposing that $m \in \mathbb{R}$ is known, but $a > 0$ and $b$ are unknown).
4.2 Lemma. If \( a > 0, \ b, m, \theta \in \mathbb{R}, \) and \( P(Y_0 > 0) = 1, \) then

\[
P\left( \int_0^T X_s^2 \, ds \in (0, \infty) \right) = 1 \quad \text{for all } T > 0,
\]

and hence there exists a unique LSE \( \tilde{\theta}_T \) which has the form given in \((4.12)\).

**Proof.** First note that under the condition on the parameters, for all \( y_0 > 0, \) we have

\[
P(\inf\{t \in \mathbb{R}_+ : Y_t = 0\} > 0 \mid Y_0 = y_0) = 1,
\]

and hence, by the law of total probability, \( P(\inf\{t \in \mathbb{R}_+ : Y_t = 0\} > 0) = 1. \) Since \( X \) has continuous trajectories almost surely, we readily have

\[
P\left( \int_0^T X_s^2 \, ds \in [0, \infty) \right) = 1, \quad T > 0.
\]

Observe also that for any \( \omega \in \Omega, \int_0^T X_s^2(\omega) \, ds = 0 \) holds if and only if \( X_s(\omega) = 0 \) for almost every \( s \in [0, T]. \) Using that \( X \) has continuous trajectories almost surely, we have

\[
P\left( \int_0^T X_s^2 \, ds = 0 \right) > 0
\]

holds if and only if \( P(X_s = 0, \forall s \in [0, T]) > 0. \) By the end of the proof of Lemma \( 3.3 \) if \( P(X_s = 0, \forall s \in [0, T]) > 0, \) then \( Y_s = 0 \) for all \( s \in [0, T] \) on the event

\[
\{\omega \in \Omega : X_s(\omega) = 0, \forall s \in [0, T]\} \cap \{\omega \in \Omega : (Y_s(\omega))_{s \in [0, T]} \text{ is continuous}\}
\]

having positive probability. This would yield that \( P(\inf\{t \in \mathbb{R}_+ : Y_t = 0\} = 0) > 0 \) leading us to a contradiction, which implies \((4.10).\) \( \square \)

The next lemma is about the existence of \( (\tilde{m}_T, \tilde{\theta}_T). \)

4.3 Lemma. If \( a > 0, \ b, m, \theta \in \mathbb{R}, \) and \( P(Y_0 > 0) = 1, \) then

\[
P\left( T \int_0^T X_s^2 \, ds - \left( \int_0^T X_s \, ds \right)^2 \in (0, \infty) \right) = 1 \quad \text{for all } T > 0,
\]

and hence there exists a unique LSE \( (\tilde{m}_T, \tilde{\theta}_T) \) which has the form given in \((4.6)\) and \((4.7).\)

**Proof.** Just as in the proof of Lemma \( 4.2 \) we have \( P(\inf\{t \in \mathbb{R}_+ : Y_t = 0\} > 0) = 1. \) By Cauchy-Schwarz’s inequality, we have

\[
T \int_0^T X_s^2 \, ds - \left( \int_0^T X_s \, ds \right)^2 \geq 0, \quad T > 0,
\]

and hence, since \( X \) has continuous trajectories almost surely, we readily have

\[
P\left( T \int_0^T X_s^2 \, ds - \left( \int_0^T X_s \, ds \right)^2 \in [0, \infty) \right) = 1, \quad T > 0.
\]
Further, equality holds in (4.12) if and only if $KX_s^2 = L$ for almost every $s \in [0, T]$ with some $K, L \geq 0, K^2 + L^2 > 0$ ($K$ and $L$ may depend on $\omega \in \Omega$ and $T$). Note that if $K$ were 0, then $L$ would be 0, too, hence $K$ can not be 0 implying that equality holds in (4.12) if and only if $X_s^2 = L/K$ for almost every $s \in [0, T]$ with some $K > 0$ and $L \geq 0$ ($K$ and $L$ may depend on $\omega \in \Omega$ and $T$). Using that $X$ has continuous trajectories almost surely, we have

\begin{equation}
\mathbb{P} \left( \int_0^T X_s^2 \, ds - \left( \int_0^T X_s \, ds \right)^2 = 0 \right) > 0 \tag{4.13}
\end{equation}

holds if and only if $\mathbb{P}(X_s^2 = L/K, \forall s \in [0, T]) > 0$ with some random variables $K$ and $L$ such that $\mathbb{P}(K > 0, L \geq 0) = 1$ ($K$ and $L$ may depend on $T$). Using again that $X$ has continuous trajectories almost surely, we have (4.13) holds if and only if $\mathbb{P}(X_s = C, \forall s \in [0, T]) > 0$ with some random variable $C$ (note that $C = \sqrt{L/K}$ if $X$ is non-negative and $C = -\sqrt{L/K}$ if $X$ is negative). This leads us to a contradiction by the end of the proof of Lemma 3.3. Indeed, if $\mathbb{P}(X_s = C, \forall s \in [0, T]) > 0$ with some random variable $C$, then, by the end of the proof of Lemma 3.3, we have $Y_s = 0$ for all $s \in [0, T]$ on the event \( \{ \omega \in \Omega : X_s(\omega) = C(\omega), \forall s \in [0, T] \} \cap \{ \omega \in \Omega : (Y_s(\omega))_{s \in [0, T]} \text{ is continuous} \} \) having positive probability. This would yield that $\mathbb{P}(\inf\{t \in \mathbb{R}_+ : Y_t = 0\} = 0) > 0$ leading us to a contradiction, which implies (4.11).

5 Consistency of maximum likelihood estimator

5.1 Theorem. If $a \geq \frac{1}{2}, b > 0, m \in \mathbb{R}, \theta > 0,$ and $\mathbb{P}(Y_0 > 0) = 1$, then the MLE of $\theta$ is strongly consistent: $\mathbb{P}(\lim_{T \to \infty} \hat{\theta}^{\text{MLE}}_T = \theta) = 1$.

Proof. By Lemma 3.3 there exists a unique MLE $\hat{\theta}^{\text{MLE}}_T$ of $\theta$ which has the form given in (3.3). Further, by (3.4),

\[ \hat{\theta}^{\text{MLE}}_T - \theta = -\frac{\int_0^T X_s \, dB_s}{\int_0^T X_s^2 \, ds}, \quad T > 0. \]

The strong consistency of $\hat{\theta}^{\text{MLE}}_T$ will follow from a strong law of large numbers for continuous local martingales (see, e.g., Theorem 2.6) provided that we check that

\begin{equation}
\mathbb{P} \left( \int_0^T \frac{X_s^2}{Y_s} \, ds < +\infty \right) = 1, \quad T > 0, \tag{5.1}
\end{equation}

\begin{equation}
\mathbb{P} \left( \lim_{T \to \infty} \int_0^T \frac{X_s^2}{Y_s} \, ds = +\infty \right) = 1. \tag{5.2}
\end{equation}

Since $(Y, X)$ has continuous trajectories almost surely, we have (5.1). Next we check (5.2). By Theorem 2.5 we have

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{X_s^2}{1 + Y_s} \, ds = \mathbb{E} \left( \frac{X_\infty^2}{1 + Y_\infty} \right) \quad \text{a.s.,} \]

17
where
\[ \mathbb{E} \left( \frac{X_{\infty}^2}{1 + Y_{\infty}} \right) \leq \mathbb{E} \left( X_{\infty}^2 \right) = \frac{a\theta + 2bm^2}{2b\theta^2} < \infty. \]

Next we check that \( \mathbb{E}(X_{\infty}^2/(1 + Y_{\infty})) > 0 \). Since \( X_{\infty}^2/(1 + Y_{\infty}) \geq 0 \), we have \( \mathbb{E}(X_{\infty}^2/(1 + Y_{\infty})) = 0 \) holds if and only if \( \mathbb{P}(X_{\infty}^2/(1 + Y_{\infty}) = 0) = 1 \) or equivalently \( \mathbb{P}(X_{\infty} = 0) = 1 \) which leads us to a contradiction since \( X_{\infty} \) is absolutely continuous by Theorem 2.5. Hence we have
\[
\mathbb{P} \left( \lim_{T \to \infty} \int_0^T \frac{X_s^2}{1 + Y_s} \, ds = \infty \right) = 1,
\]
which yields \[ \text{5.2}. \]

**5.2 Theorem.** If \( a > \frac{1}{2}, b > 0, m \in \mathbb{R}, \theta > 0, \) and \( \mathbb{P}(Y_0 > 0) = 1 \), then the MLE of \((a, b, m, \theta)\) is strongly consistent: \( \mathbb{P} \left( \lim_{T \to \infty} \hat{a}_T, \hat{b}_T, \hat{m}_T, \hat{\theta}_T = (a, b, m, \theta) \right) = 1. \)

**Proof.** By Lemma 3.4, there exists a unique MLE \((\hat{a}_T, \hat{b}_T, \hat{m}_T, \hat{\theta}_T)\) of \((a, b, m, \theta)\) which has the form given in (3.5), (3.6), (3.7) and (3.8). First we check that \( \mathbb{P}(\lim_{T \to \infty} \hat{\theta}_T = \theta) = 1. \) By (3.12), using also that \( \int_0^T \frac{X_s^2}{Y_s} \, ds \int_0^T \frac{T \cdot \frac{X_s}{Y_s}}{\frac{T}{Y_s}} \, ds - \left( \int_0^T \frac{X_s}{Y_s} \, ds \right)^2 > 0 \) yields \( \int_0^T \frac{X_s^2}{Y_s} \, ds \int_0^T \frac{T}{Y_s} \, ds > 0, \) we have
\[
\hat{\theta}_T - \theta = \frac{1}{\int_0^T \frac{T \cdot \frac{X_s}{Y_s}}{\frac{T}{Y_s}} \, ds} \int_0^T \frac{X_s^2}{Y_s} \, ds - \left( \int_0^T \frac{X_s}{Y_s} \, ds \right)^2, \tag{5.3}
\]
provided that \( \int_0^T \frac{X_s^2}{Y_s} \, ds \int_0^T \frac{T}{Y_s} \, ds - \left( \int_0^T \frac{X_s}{Y_s} \, ds \right)^2 > 0. \) Next, we show that \( \mathbb{E}(1/Y_{\infty}) < \infty, \mathbb{E}(X_{\infty}/Y_{\infty}) < \infty \) and \( \mathbb{E}(X_{\infty}^2/Y_{\infty}) < \infty, \) which, by part (iii) of Theorem 2.5, will imply that
\[
\mathbb{P} \left( \lim_{T \to \infty} \frac{1}{\mathbb{E} \left( \frac{T}{Y_{\infty}} \right)} \int_0^T \frac{1}{Y_s} \, ds = \mathbb{E} \left( \frac{1}{Y_{\infty}} \right) \right) = 1, \tag{5.4}
\]
\[
\mathbb{P} \left( \lim_{T \to \infty} \frac{1}{\mathbb{E} \left( \frac{T}{Y_{\infty}} \right)} \int_0^T X_s \, ds = \mathbb{E} \left( \frac{X_{\infty}}{Y_{\infty}} \right) \right) = 1, \tag{5.5}
\]
\[
\mathbb{P} \left( \lim_{T \to \infty} \frac{1}{\mathbb{E} \left( \frac{T}{Y_{\infty}} \right)} \int_0^T X_s^2 \, ds = \mathbb{E} \left( \frac{X_{\infty}^2}{Y_{\infty}} \right) \right) = 1. \tag{5.6}
\]
We only prove \( \mathbb{E}(X_{\infty}^2/Y_{\infty}) < \infty \) noting that \( \mathbb{E}(1/Y_{\infty}) < \infty \) and \( \mathbb{E}(X_{\infty}/Y_{\infty}) < \infty \) can be checked in the very same way. First note that, by Theorem 2.5, \( \mathbb{P}(Y_0 > 0) = 1, \) and hence the random variable \( X_{\infty}^2/Y_{\infty} \) is well-defined with probability one. By Hölder’s inequality, for all \( p, q > 0 \) with \( \frac{1}{p} + \frac{1}{q} = 1, \) we have
\[
\mathbb{E} \left( \frac{X_{\infty}^2}{Y_{\infty}} \right) \leq \left( \mathbb{E}(X_{\infty}^p) \right)^{1/p} \left( \mathbb{E}(Y_{\infty}^q) \right)^{1/q}. \]

Since, by Theorem 2.5, \( \mathbb{E}(|X_{\infty}|^n) < \infty \) for all \( n \in \mathbb{N} \), to prove that \( \mathbb{E}(X_{\infty}^2/Y_{\infty}) < \infty \) it is enough to check that there exists some \( \varepsilon > 0 \) such that \( \mathbb{E} \left( \frac{X_{\infty}^2}{Y_{\infty}} \right) < \infty. \) Using that, by Theorem 2.5, \( Y_{\infty} \) has Gamma distribution with density function \( \frac{(2b)^{2a}}{\Gamma(2a)} x^{2a-1} e^{-2bx} 1_{x > 0}, \) we have, for any \( \varepsilon > 0, \)
\[
\mathbb{E} \left( \frac{1}{Y_{\infty}^{1+\varepsilon}} \right) = \int_0^\infty \frac{1}{x^{1+\varepsilon}} \cdot \frac{(2b)^{2a}}{\Gamma(2a)} x^{2a-1} e^{-2bx} \, dx = \frac{(2b)^{2a}}{\Gamma(2a)} \int_0^\infty x^{2a-2-\varepsilon} e^{-2bx} \, dx.
\]
Due to our assumption $a > 1/2$, one can choose an $\varepsilon$ such that $\max(0, 2a - 2) < \varepsilon < 2a - 1$, and hence for all $M > 0$,

\[
\int_0^\infty x^{2a-2-\varepsilon} e^{-2bx} \, dx \leq \int_0^M x^{2a-2-\varepsilon} \, dx + M^{2a-2-\varepsilon} \int_\infty^M e^{-2bx} \, dx
\]

\[
= \frac{M^{2a-1-\varepsilon}}{2a-1-\varepsilon} + M^{2a-2-\varepsilon} \lim_{L \to \infty} \frac{e^{-2bL} - e^{-2bM}}{-2b}
\]

\[
= \frac{M^{2a-1-\varepsilon}}{2a-1-\varepsilon} + M^{2a-2-\varepsilon} \frac{e^{-2bM}}{2b} < \infty,
\]

which yields that $\mathbb{E}(1/Y_\infty^{1+\varepsilon}) < \infty$ for $\varepsilon$ satisfying $\max(0, 2a - 2) < \varepsilon < 2a - 1$.

Further, since $\mathbb{E}(1/Y_\infty) > 0$ and $\mathbb{E}(X_\infty^2/Y_\infty) > 0$ (due to the absolutely continuity of $X_\infty$, as it was explained in the proof of Theorem 5.1), (5.4) and (5.6) yield that

\[
\mathbb{P}\left( \lim_{T \to \infty} \int_0^T \frac{1}{Y_s} \, ds = \infty \right) = \mathbb{P}\left( \lim_{T \to \infty} \int_0^T \frac{X_s^2}{Y_s} \, ds = \infty \right) = 1.
\]

Using (5.3), (5.4), (5.5), (5.6), Slutsky’s lemma and a strong law of large numbers for continuous local martingales (see, e.g., Theorem 2.6), we get

\[
\lim_{T \to \infty} \left( \hat{\theta}^\text{MLE}_T - \theta \right) = \frac{\mathbb{E}\left( \frac{X_\infty}{Y_\infty} \right) \cdot 0 - 0}{1 - \frac{\mathbb{E}\left( \frac{X_\infty^{2}}{Y_\infty^{2}} \right) E\left( \frac{1}{Y_\infty} \right)}{\mathbb{E}\left( \frac{X_\infty}{Y_\infty} \right)}} = 0 \quad \text{a.s.,}
\]

where we also used that the denominator is strictly positive. Indeed, by Cauchy-Schwarz’s inequality,

\[
\left( \mathbb{E}\left( \frac{X_\infty}{Y_\infty} \right) \right)^2 \leq \mathbb{E}\left( \frac{X_\infty^{2}}{Y_\infty^{2}} \right) \mathbb{E}\left( \frac{1}{Y_\infty} \right),
\]

and equality would hold if and only if $\mathbb{P}(KX_\infty^2/Y_\infty = L/Y_\infty) = 1$ with some $K, L \geq 0, K^2 + L^2 > 0$. By Theorem 2.3, $(Y_\infty, X_\infty)$ is absolutely continuous and then $\mathbb{P}(X_\infty = c) = 0$ for all $c \in \mathbb{R}$. This implies that (5.7) holds with strict inequality.

Similarly, by (3.9), (3.10) and (3.11), we have

\[
\hat{a}^\text{MLE}_T - a = \int_0^T \frac{1}{Y_s} \frac{dL_s}{Y_s} ds - \frac{1}{T} \int_0^T \frac{1}{Y_s} \, ds \cdot \int_0^T \frac{\sqrt{L_s}}{Y_s} \, ds,
\]

\[
\hat{b}^\text{MLE}_T - b = \frac{1}{T} \int_0^T \frac{1}{Y_s} \, ds - \frac{1}{T} \int_0^T \frac{\sqrt{L_s}}{Y_s} \, ds,
\]

\[
\hat{m}^\text{MLE}_T - m = \frac{1}{T} \int_0^T \frac{\sqrt{L_s}}{Y_s} \, ds - \frac{1}{T} \int_0^T \frac{\sqrt{X_s}}{Y_s} \, ds,
\]

\[
1 - \frac{\left( \frac{1}{T} \int_0^T \frac{\sqrt{X_s}}{Y_s} \, ds \right)^2}{1 - \frac{\left( \frac{1}{T} \int_0^T \frac{\sqrt{L_s}}{Y_s} \, ds \right)^2}{\frac{\left( \frac{1}{T} \int_0^T \frac{\sqrt{X_s}}{Y_s} \, ds \right)^2}}.
\]
provided that \( \int_0^T Y_s \, ds \int_0^T \frac{1}{Y_s} \, ds - T^2 > 0 \) and \( \int_0^T \frac{X_s^2}{Y_s} \, ds \int_0^T \frac{1}{Y_s} \, ds - \left( \int_0^T \frac{X_s}{Y_s} \, ds \right)^2 > 0 \), respectively. Since \( \mathbb{E}(Y_\infty) < \infty \), by part (iii) of Theorem 2.5.

(5.8) \[
P\left( \lim_{T \to \infty} \frac{1}{T} \int_0^T Y_s \, ds = \mathbb{E}(Y_\infty) \right) = P\left( \lim_{T \to \infty} \int_0^T Y_s \, ds = \infty \right) = 1,
\]

and then, similarly as before, one can argue that

- \[
\lim_{T \to \infty} (\hat{a}_{MLE}^T - a) = 0 - \mathbb{E}\left( \frac{1}{Y_\infty} \right) \cdot 0 = 0 \quad \text{a.s.,}
\]
- \[
\lim_{T \to \infty} (\hat{b}_{MLE}^T - b) = 0 - \mathbb{E}(Y_\infty) \cdot 0 = 0 \quad \text{a.s.,}
\]
- \[
\lim_{T \to \infty} (\hat{m}_{MLE}^T - m) = 0 - \mathbb{E}(X_\infty) \cdot 0 = 0 \quad \text{a.s.,}
\]

where we also used that \( \mathbb{E}(Y_\infty) \mathbb{E}\left( \frac{1}{Y_\infty} \right) > 1 \) (which can be checked using Cauchy-Schwarz’s inequality and the absolute continuity of \( Y_\infty \)). Using that the intersection of four events with probability one is an event with probability one, we have the assertion. \( \Box \)

5.3 Remark. If \( a = \frac{1}{\theta}, \ b > 0, \ \theta > 0, \ m \in \mathbb{R} \), and \( \mathbb{P}(Y_0 > 0) = 1 \), then one should find another approach for studying the consistency behaviour of the MLE of \( (a, b, m, \theta) \), since in this case

\[
\mathbb{E}\left( \frac{1}{Y_\infty} \right) = \int_0^\infty \frac{2b e^{-2b x}}{x} \, dx = \infty,
\]

and hence one cannot use part (iii) of Theorem 2.5. In this paper we renounce to consider it. \( \Box \)

6 Consistency of least squares estimator

6.1 Theorem. If \( a > 0, \ b > 0, \ m \in \mathbb{R}, \ \theta > 0, \) and \( \mathbb{P}(Y_0 > 0) = 1 \), then the LSE of \( \theta \) is strongly consistent: \( \mathbb{P} \left( \lim_{T \to \infty} \hat{\theta}_{LSE}^T = \theta \right) = 1. \)

Proof. By Lemma 4.2 there exists a unique \( \hat{\theta}_{LSE}^T \) of \( \theta \) which has the form given in (4.2). By (4.3), we have

(6.1) \[
\hat{\theta}_{LSE}^T - \theta = -\frac{\int_0^T X_s \sqrt{Y_s} \, dB_s}{\int_0^T X_s^2 \, ds} - \frac{\int_0^T X_s \sqrt{Y_s} \, dB_s}{\int_0^T X_s^2 Y_s \, ds} \cdot \frac{1}{T} \int_0^T X_s^2 Y_s \, ds = \frac{1}{T} \int_0^T X_s^2 \, ds.
\]

By Theorem 2.5, we have

(6.2) \[
P\left( \lim_{T \to \infty} \frac{1}{T} \int_0^T X_s^2 \, ds = \mathbb{E}(X_\infty^2) \right) = 1 \quad \text{and} \quad P\left( \lim_{T \to \infty} \frac{1}{T} \int_0^T X_s^2 \, ds = \mathbb{E}(X_\infty^2) \right) = 1.
\]

20
We note that \( \mathbb{E}(X_{\infty}^2 Y_{\infty}) \) and \( \mathbb{E}(X_{\infty}^2) \) are calculated explicitly in Theorem 2.5. Note also that \( \mathbb{E}(X_{\infty}^2 Y_{\infty}) \) is positive (due to that \( X_{\infty}^2 Y_{\infty} \) is non-negative and absolutely continuous), and hence we also have

\[
P \left( \lim_{T \to \infty} \int_0^T X_s^2 Y_s ds = \infty \right) = 1.
\]

Then, by a strong law of large numbers for continuous local martingales (see, e.g., Theorem 2.6), we get

\[
P \left( \lim_{T \to \infty} \frac{\int_0^T X_s \sqrt{Y_s} dB_s}{\int_0^T X_s^2 Y_s ds} = 0 \right) = 1,
\]

and hence (6.1) yields the assertion. \( \square \)

### 6.2 Theorem

If \( a > 0, \ b > 0, \ m \in \mathbb{R}, \ \theta > 0, \) and \( \mathbb{P}(Y_0 > 0) = 1, \) then the LSE of \( (m, \theta) \) is strongly consistent: \( \mathbb{P} \left( \lim_{T \to \infty} (\hat{m}_T, \hat{\theta}_T) = (m, \theta) \right) = 1. \)

**Proof.** By Lemma 4.3 there exists a unique LSE \( (\hat{m}_T, \hat{\theta}_T) \) of \( (m, \theta) \) which has the form given in (4.6) and (4.7). By (4.9), we have

\[
|LSE - \theta| = \frac{-\frac{1}{T} \int_0^T X_s \sqrt{Y_s} dB_s \cdot \frac{\int_0^T X_s Y_s ds}{\int_0^T X_s^2 Y_s ds} + \frac{1}{T} \int_0^T X_s ds - \frac{1}{T} \int_0^T Y_s ds}{1 - \left( \frac{\int_0^T X_s ds}{\int_0^T X_s^2 ds} \right)^2}
\]

provided that \( T \int_0^T X_s^2 ds - \left( \int_0^T X_s ds \right)^2 > 0. \) Using (5.8), (6.2) and that (6.3)

\[
P \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T X_s ds = \mathbb{E}(X_{\infty}) \right) = 1,
\]

similarly to the proof of Theorem 5.2, we get

\[
\lim_{T \to \infty} (\hat{LSE} - \theta) = \frac{-\mathbb{E}(X_{\infty}^2 Y_{\infty}) \cdot 0 + \mathbb{E}(X_{\infty}) \cdot \mathbb{E}(Y_{\infty}) \cdot 0}{1 - \left( \frac{\mathbb{E}(X_{\infty})}{\mathbb{E}(X_{\infty})} \right)^2} = 0 \quad a.s.,
\]

where for the last step we also used that \( \left( \mathbb{E}(X_{\infty}) \right)^2 < \mathbb{E}(X_{\infty}^2) \) (which holds since there does not exist a constant \( c \in \mathbb{R} \) such that \( \mathbb{P}(X_{\infty} = c) = 1 \) due to the fact that \( X_{\infty} \) is absolutely continuous).

Similarly, by (4.8),

\[
\lim_{T \to \infty} (\hat{m}_T - m) = \lim_{T \to \infty} \frac{-\frac{1}{T} \int_0^T X_s ds - \frac{1}{T} \int_0^T X_s \sqrt{Y_s} dB_s \cdot \frac{\int_0^T X_s Y_s ds}{\int_0^T X_s^2 Y_s ds} + \frac{1}{T} \int_0^T Y_s ds \cdot \frac{\int_0^T \sqrt{Y_s} dB_s}{\int_0^T Y_s ds}}{1 - \left( \frac{\int_0^T X_s ds}{\int_0^T X_s^2 ds} \right)^2}
\]

\[
= \frac{-\mathbb{E}(X_{\infty}) \cdot \frac{\mathbb{E}(X_{\infty}^2 Y_{\infty})}{\mathbb{E}(X_{\infty})} \cdot 0 + \mathbb{E}(Y_{\infty}) \cdot 0}{1 - \left( \frac{\mathbb{E}(X_{\infty})}{\mathbb{E}(X_{\infty})} \right)^2} = 0 \quad a.s.
\]

Using that the intersection of two events with probability one is an event with probability one, we have the assertion. \( \square \)
7 Asymptotic behaviour of maximum likelihood estimator

7.1 Theorem. If \( a > 1/2, \ b > 0, \ m \in \mathbb{R}, \ \theta > 0, \) and \( \mathbb{P}(Y_0 > 0) = 1, \) then the MLE of \( \theta \) is asymptotically normal, i.e.,

\[
\sqrt{T}(\hat{\theta}_T^{MLE} - \theta) \xrightarrow{L} \mathcal{N}(0, \frac{1}{\mathbb{E}\left(\frac{X_\infty^2}{Y_\infty}\right)}) \quad \text{as} \ T \to \infty,
\]

where \( \mathbb{E}\left(\frac{X_\infty^2}{Y_\infty}\right) \) is positive and finite.

Proof. First note that, by (3.4),

\[
(7.1)
\]

provided that \( \int_0^T \frac{X_s^2}{Y_s} \, ds > 0. \) Recall that, by (5.2), \( \mathbb{P}\left(\lim_{T \to \infty} \int_0^T \frac{X_s^2}{Y_s} \, ds = +\infty\right) = 1. \) Using Theorem 2.7 with the following choices

\[
d := 1, \quad M_t := \int_0^t \frac{X_s}{\sqrt{Y_s}} \, dB_s, \quad t \geq 0, \quad (\mathcal{F}_t)_{t \geq 0} \quad \text{defined before Proposition 2.1}
\]

\[
Q(t) := \frac{1}{\sqrt{t}}, \quad t > 0, \quad \eta := \sqrt{\mathbb{E}\left(\frac{X_\infty^2}{Y_\infty}\right)},
\]

we have

\[
\frac{1}{\sqrt{T}} \int_0^T \frac{X_s}{\sqrt{Y_s}} \, dB_s \xrightarrow{L} \sqrt{\mathbb{E}\left(\frac{X_\infty^2}{Y_\infty}\right)} \xi \quad \text{as} \ T \to \infty,
\]

where \( \xi \) is a standard normally distributed random variable. Then Slutsky’s lemma, 5.6 and (7.1) yield the assertion. \( \square \)

7.2 Theorem. If \( a > 1/2, \ b > 0, \ m \in \mathbb{R}, \ \theta > 0, \) and \( \mathbb{P}(Y_0 > 0) = 1, \) then the MLE of \( (a, b, m, \theta) \) is asymptotically normal, i.e.,

\[
\sqrt{T} \begin{bmatrix}
\hat{a}_T^{MLE} - a \\
\hat{b}_T^{MLE} - b \\
\hat{m}_T^{MLE} - m \\
\hat{\theta}_T^{MLE} - \theta
\end{bmatrix} \xrightarrow{L} \mathcal{N}_4(0, \Sigma^{MLE}) \quad \text{as} \ T \to \infty,
\]

where \( \mathcal{N}_4(0, \Sigma^{MLE}) \) denotes a 4-dimensional normally distribution with mean vector \( 0 \in \mathbb{R}^4 \) and with covariance matrix \( \Sigma^{MLE} := \text{diag}(\Sigma_1^{MLE}, \Sigma_2^{MLE}) \) with blockdiagonals given by

\[
\Sigma_1^{MLE} := \frac{1}{\mathbb{E}\left(\frac{1}{Y_\infty}\right)\mathbb{E}(Y_\infty) - 1} D_1, \quad D_1 := \begin{bmatrix}
\mathbb{E}(Y_\infty) & 1 \\
1 & \mathbb{E}\left(\frac{1}{Y_\infty}\right)
\end{bmatrix},
\]

\[
\Sigma_2^{MLE} := \frac{1}{\mathbb{E}\left(\frac{1}{Y_\infty}\right)\mathbb{E}\left(\frac{X_\infty^2}{Y_\infty}\right) - \left(\mathbb{E}\left(\frac{X_\infty}{Y_\infty}\right)\right)^2} D_2, \quad D_2 := \begin{bmatrix}
\mathbb{E}\left(\frac{X_\infty^2}{Y_\infty}\right) & \mathbb{E}\left(\frac{X_\infty}{Y_\infty}\right) \\
\mathbb{E}\left(\frac{X_\infty}{Y_\infty}\right) & \mathbb{E}\left(\frac{1}{Y_\infty}\right)
\end{bmatrix}.
\]
Proof. By (3.9), (3.10), (3.11) and (3.12), we have

\[
\sqrt{T}(\hat{a}_T^{MLE} - a) = \frac{\frac{1}{T} \int_0^T Y_s^T dL_s - \frac{1}{\sqrt{T}} \int_0^T \sqrt{Y_s} dL_s}{\frac{1}{T} \int_0^T Y_s^T dL_s - \frac{1}{T} \int_0^T \frac{1}{Y_s} ds - 1}, \quad T > 0,
\]

\[
\sqrt{T}(\hat{b}_T^{MLE} - b) = \frac{\frac{1}{T} \int_0^T \sqrt{Y_s} dL_s - \frac{1}{T} \int_0^T \frac{1}{Y_s} ds \cdot \frac{1}{\sqrt{T}} \int_0^T \sqrt{Y_s} dL_s}{\frac{1}{T} \int_0^T Y_s dL_s - \frac{1}{T} \int_0^T \frac{1}{Y_s} ds - 1}, \quad T > 0,
\]

\[
\sqrt{T}(\hat{m}_T^{MLE} - m) = \frac{\frac{1}{T} \int_0^T X_s^T \frac{1}{Y_s} ds \cdot \frac{1}{\sqrt{T}} \int_0^T \frac{1}{Y_s} dL_s - \frac{1}{T} \int_0^T \frac{1}{Y_s} ds \cdot \frac{1}{\sqrt{T}} \int_0^T \frac{X_s}{\sqrt{Y_s}} dL_s}{\frac{1}{T} \int_0^T X_s^2 \frac{1}{Y_s} ds \cdot \frac{1}{T} \int_0^T \frac{1}{Y_s} ds - \left( \frac{1}{T} \int_0^T \frac{X_s}{\sqrt{Y_s}} ds \right)^2}, \quad T > 0,
\]

\[
\sqrt{T}(\hat{\theta}_T^{MLE} - \theta) = \frac{\frac{1}{T} \int_0^T X_s^2 \frac{1}{Y_s} ds \cdot \frac{1}{\sqrt{T}} \int_0^T \frac{1}{Y_s} dL_s - \frac{1}{T} \int_0^T \frac{1}{Y_s} ds \cdot \frac{1}{\sqrt{T}} \int_0^T \frac{X_s}{\sqrt{Y_s}} dL_s}{\frac{1}{T} \int_0^T X_s^2 \frac{1}{Y_s} ds \cdot \frac{1}{T} \int_0^T \frac{1}{Y_s} ds - \left( \frac{1}{T} \int_0^T \frac{X_s}{\sqrt{Y_s}} ds \right)^2}, \quad T > 0,
\]

provided that \( \int_0^T Y_s ds \int_0^T \frac{1}{Y_s} ds - T^2 > 0 \) and \( \int_0^T X_s^2 \frac{1}{Y_s} ds \int_0^T \frac{1}{Y_s} ds - \left( \int_0^T \frac{X_s}{\sqrt{Y_s}} ds \right)^2 > 0 \). Next, we show that

\[
(7.2) \quad \frac{1}{\sqrt{T}}M_T := \frac{1}{\sqrt{T}} \begin{pmatrix} \int_0^T \sqrt{Y_s} dL_s \\ \int_0^T \frac{1}{Y_s} dL_s \\ \int_0^T X_s \frac{1}{Y_s} dL_s \\ \int_0^T \frac{1}{Y_s} dL_s \end{pmatrix} \xrightarrow{a.s.} \eta Z, \quad \text{as } T \to \infty,
\]

where \( Z \) is a 4-dimensional standard normally distributed random variable and \( \eta \) is a non-random 4 \( \times \) 4 matrix such that

\[
\eta^T = \text{diag}(D_1, D_2).
\]

Here the matrices \( D_1 \) and \( D_2 \) are positive definite, since their principal minors are positive, and \( \eta \) denotes the unique symmetric positive definite square root of \( \text{diag}(D_1, D_2) \) (see, e.g., Horn and Johnson [12, Theorem 7.2.6]). Indeed, by the absolutely continuity of \( (Y_\infty, X_\infty) \) (see Theorem 2.5), there does not exist constants \( c, d \in \mathbb{R}_+ \) such that \( \mathbb{P}(Y_\infty^2 = c) = 1 \) and \( \mathbb{P}(X_\infty^2 = d) = 1 \), and, by Cauchy-Schwarz’s inequality,

\[
\mathbb{E} \left( \frac{1}{Y_\infty} \right) \mathbb{E}(Y_\infty) - 1 \geq 0 \quad \text{and} \quad \mathbb{E} \left( \frac{X_\infty^2}{Y_\infty} \right) \mathbb{E} \left( \frac{1}{Y_\infty} \right) - \mathbb{E} \left( \frac{X_\infty}{Y_\infty} \right) \geq 0,
\]

where equalities would hold if and only if \( \mathbb{P}(Y_\infty^2 = c) = 1 \) and \( \mathbb{P}(X_\infty^2 = d) = 1 \) with some constants \( c, d \in \mathbb{R}_+ \), respectively. Note also that the quantity \( \mathbb{E}(1/Y_\infty) \mathbb{E}(Y_\infty) - 1 \) could have been calculated explicitly, since \( Y_\infty \) has Gamma distribution with parameters \( 2a \) and \( 2b \).

Let us use Theorem 2.7 with the choices \( d = 4, \ M_t, \ t \geq 0 \), defined in (7.2), \( (\mathcal{F}_t)_{t \geq 0} \) defined before Proposition 2.1 and \( Q(t) := \text{diag}(t^{-1/2}, t^{-1/2}, t^{-1/2}, t^{-1/2}) \), \( t > 0 \). We have

\[
\langle M \rangle_t = \begin{bmatrix}
\int_0^t Y_s ds & t & 0 & 0 \\
t & \int_0^t \frac{1}{Y_s} ds & 0 & 0 \\
0 & 0 & \int_0^t X_s^2 \frac{1}{Y_s} ds & \int_0^t \frac{X_s}{Y_s} ds \\
0 & 0 & \int_0^t \frac{X_s}{Y_s} ds & \int_0^t \frac{1}{Y_s} ds
\end{bmatrix}, \quad t \geq 0,
\]

23
where we used the independence of \((L_t)_{t \geq 0}\) and \((B_t)_{t \geq 0}\). Recall that (under the conditions of the theorem) in the proof of Theorem 5.2 it was shown that \(\mathbb{E}(Y_\infty) < \infty\), \(\mathbb{E}(1/Y_\infty) < \infty\), \(\mathbb{E}(X_\infty/Y_\infty) < \infty\), and \(\mathbb{E}(X_\infty^2/Y_\infty) < \infty\), and, hence by Theorem 2.5, we have

\[
Q(t)(M_t)Q(t)^\top \rightarrow \text{diag}(D_1, D_2) \quad \text{as} \quad t \to \infty \text{ a.s.}
\]

Hence, Theorem 2.7 yields (7.2). Then Slutsky’s lemma and the continuous mapping theorem yield that

\[
\sqrt{T} \begin{bmatrix}
\hat{\alpha}_T^{\text{MLE}} - a \\
\hat{b}_T^{\text{MLE}} - b \\
\hat{m}_T^{\text{MLE}} - m \\
\hat{\theta}_T^{\text{MLE}} - \theta
\end{bmatrix} \xrightarrow{L} \text{diag}(A_1, A_2)\eta Z \quad \text{as} \quad T \to \infty,
\]

where

\[
A_1 := \frac{1}{\mathbb{E}\left( \frac{1}{Y_\infty} \right) \mathbb{E}(Y_\infty) - 1} B_1, \quad B_1 := \begin{bmatrix}
-1 \\
-\mathbb{E}\left( \frac{1}{Y_\infty} \right)
\end{bmatrix},
\]

\[
A_2 := \frac{1}{\mathbb{E}\left( \frac{1}{Y_\infty} \right) \mathbb{E}\left( \frac{X_\infty^2}{Y_\infty} \right) - \left( \mathbb{E}\left( \frac{X_\infty}{Y_\infty} \right) \right)^2} B_2, \quad B_2 := \begin{bmatrix}
-\mathbb{E}\left( \frac{X_\infty}{Y_\infty} \right) \\
-\mathbb{E}\left( \frac{1}{Y_\infty} \right)
\end{bmatrix} \mathbb{E}\left( \frac{X_\infty^2}{Y_\infty} \right).
\]

Using that \(\eta Z\) is a 4-dimensional normally distributed random variable with mean vector zero and with covariance matrix \(\eta \eta^\top = \text{diag}(D_1, D_2)\), the covariance matrix of \(\text{diag}(A_1, A_2)\eta Z\) takes the form

\[
\text{diag}(A_1, A_2) \text{diag}(D_1, D_2) \text{diag}(A_1^\top, A_2^\top) = \text{diag}(A_1 D_1 A_1^\top, A_2 D_2 A_2^\top),
\]

which yields the assertion. Indeed,

\[
B_1 D_1 B_1^\top = \left( \mathbb{E}\left( \frac{1}{Y_\infty} \right) \mathbb{E}(Y_\infty) - 1 \right) D_1,
\]

and

\[
B_2 D_2 B_2^\top = \left( \mathbb{E}\left( \frac{X_\infty^2}{Y_\infty} \right) \mathbb{E}\left( \frac{1}{Y_\infty} \right) - \left( \mathbb{E}\left( \frac{X_\infty}{Y_\infty} \right) \right)^2 \right) D_2.
\]

\[\square\]

7.3 Remark. The asymptotic variance \(1/\mathbb{E}\left( \frac{X_\infty^2}{Y_\infty} \right)\) of \(\hat{\theta}_T^{\text{MLE}}\) in Theorem 7.1 is less than the asymptotic variance

\[
\mathbb{E}\left( \frac{1}{Y_\infty} \right) \mathbb{E}\left( \frac{X_\infty^2}{Y_\infty} \right) - \left( \mathbb{E}\left( \frac{X_\infty}{Y_\infty} \right) \right)^2
\]

of \(\hat{\theta}_T^{\text{MLE}}\) in Theorem 7.2. This is in accordance with the fact that \(\hat{\theta}_T^{\text{MLE}}\) is the MLE of \(\theta\) provided that the value of the parameter \(m\) is known, which gives extra information, so the MLE estimator of \(\theta\) becomes better. \[\square\]
8 Asymptotic behaviour of least squares estimator

8.1 Theorem. If \( a > 0, \ b > 0, \ m \in \mathbb{R}, \ \theta > 0, \) and \( \mathbb{P}(Y_0 > 0) = 1, \) then the LSE of \( \theta \) is asymptotically normal, i.e.,
\[
\sqrt{T} (\hat{\theta}_T^{LSE} - \theta) \overset{\mathcal{L}}{\to} \mathcal{N} \left( 0, \frac{\mathbb{E}(X^2_s Y_s)}{\left( \mathbb{E}(X^2_s) \right)^2} \right) \quad \text{as } T \to \infty,
\]
where \( \mathbb{E}(X^2_s Y_s) \) and \( \mathbb{E}(X^2_s) \) are given explicitly in Theorem 2.7.

Proof. First note that, by (4.3),
\[
\sqrt{T} (\hat{\theta}_T^{LSE} - \theta) = -\frac{1}{\sqrt{T}} \int_0^T X_s \sqrt{Y_s} \, dB_s + \frac{1}{T} \int_0^T X^2_s \, ds, \quad T > 0,
\]
provided that \( \int_0^T X^2_s \, ds > 0. \) Using that \( \mathbb{E}(X^2_s Y_s) \) is positive (since \( X^2_s Y_s \) is non-negative and absolutely continuous), by (6.2), we have
\[
\mathbb{P} \left( \lim_{T \to \infty} \int_0^T X^2_s Y_s \, ds = +\infty \right) = 1.
\]
Further, an application of Theorem 2.7 with the following choices
\[
d := 1, \quad M_t := \int_0^t X_s \sqrt{Y_s} \, dB_s, \quad t \geq 0, \quad (F_t)_{t \geq 0} \text{ defined before Proposition 2.1},
\]
\[
Q(t) := \frac{1}{\sqrt{t}}, \quad t > 0, \quad \eta := \sqrt{\mathbb{E}(X^2_s Y_s)},
\]
yields that
\[
\frac{1}{\sqrt{T}} \int_0^T X_s \sqrt{Y_s} \, dB_s \overset{\mathcal{L}}{\to} \sqrt{\mathbb{E}(X^2_s Y_s)} \eta \quad \text{as } T \to \infty,
\]
where \( \eta \) is a standard normally distributed random variable. Using again (6.2), Slutsky’s lemma and (8.1), we get the assertion. \( \square \)

8.2 Remark. The asymptotic variance \( \frac{\mathbb{E}(X^2_s Y_s)}{\left( \mathbb{E}(X^2_s) \right)^2} \) of the LSE \( \hat{\theta}_T^{LSE} \) in Theorem 8.1 is greater than the asymptotic variance \( 1/\mathbb{E} \left( X^2_s / Y_s \right) \) of the MLE \( \hat{\theta}_T^{MLE} \) in Theorem 7.1 since, by Cauchy and Schwarz’s inequality,
\[
\left( \mathbb{E}(X^2_s) \right)^2 = \left( \mathbb{E} \left( \frac{X^2_s}{Y_s} \sqrt{Y_s} \right) \right)^2 < \mathbb{E} \left( \frac{X^2_s}{Y_s} \right) \mathbb{E} \left( X^2_s Y_s \right).
\]
\( \square \)

8.3 Theorem. If \( a > 0, \ b > 0, \ m \in \mathbb{R}, \ \theta > 0, \) and \( \mathbb{P}(Y_0 > 0) = 1, \) then the LSE of \( (m, \theta) \) is asymptotically normal, i.e.,
\[
\sqrt{T} \left[ \hat{m}_T^{LSE} - m \right] \overset{\mathcal{L}}{\to} \mathcal{N}_2 \left( 0, \Sigma^{LSE} \right) \quad \text{as } T \to \infty,
\]
where \( \mathcal{N}_2 \left( 0, \Sigma^{\text{LSE}} \right) \) denotes a 2-dimensional normally distribution with mean vector \( 0 \in \mathbb{R}^2 \) and with covariance matrix \( \Sigma^{\text{LSE}} = (\Sigma^{\text{LSE}})_{i,j=1} \), where

\[
\Sigma^{\text{LSE}}_{1,1} := \frac{(\mathbb{E}(X_\infty))^2 \mathbb{E}(X_\infty^2 Y_\infty) - 2 \mathbb{E}(X_\infty) \mathbb{E}(X_\infty^2) \mathbb{E}(X_\infty Y_\infty) + (\mathbb{E}(X_\infty^2))^2 \mathbb{E}(Y_\infty)}{(\mathbb{E}(X_\infty^2) - (\mathbb{E}(X_\infty))^2)^2},
\]

\[
\Sigma^{\text{LSE}}_{1,2} = \Sigma^{\text{LSE}}_{2,1} := \frac{\mathbb{E}(X_\infty) \left( \mathbb{E}(X_\infty^2 Y_\infty) + \mathbb{E}(X_\infty^2) \mathbb{E}(Y_\infty) \right) - \mathbb{E}(X_\infty Y_\infty) \left( \mathbb{E}(X_\infty^2) + (\mathbb{E}(X_\infty))^2 \right)}{(\mathbb{E}(X_\infty^2) - (\mathbb{E}(X_\infty))^2)^2},
\]

\[
\Sigma^{\text{LSE}}_{2,2} := \frac{\mathbb{E}(X_\infty^2 Y_\infty) - 2 \mathbb{E}(X_\infty) \mathbb{E}(X_\infty Y_\infty) + (\mathbb{E}(X_\infty))^2 \mathbb{E}(Y_\infty)}{(\mathbb{E}(X_\infty^2) - (\mathbb{E}(X_\infty))^2)^2}.
\]

**Proof.** By (1.3) and (4.5), we have

\[
\sqrt{T}(m_{\text{LSE}} - m) = -\frac{1}{T} \int_0^T X_s ds \frac{1}{\sqrt{T}} \int_0^T X_s \sqrt{Y_s} dB_s + \frac{1}{T} \int_0^T X_s^2 ds \frac{1}{\sqrt{T}} \int_0^T \sqrt{Y_s} dB_s,
\]

and

\[
\sqrt{T}(\theta_{\text{LSE}} - \theta) = -\frac{1}{\sqrt{T}} \int_0^T X_s \sqrt{Y_s} dB_s + \frac{1}{T} \int_0^T X_s ds \frac{1}{\sqrt{T}} \int_0^T \sqrt{Y_s} dB_s,
\]

provided that \( T \int_0^T X_s^2 ds - \left( \int_0^T X_s ds \right)^2 > 0 \). Next we show that

\[
(8.2) \quad \left( \frac{1}{\sqrt{T}} \int_0^T X_s \sqrt{Y_s} dB_s, \frac{1}{\sqrt{T}} \int_0^T \sqrt{Y_s} dB_s \right) \xrightarrow{\mathcal{L}} (\eta Z_1, (\eta Z)_2) \quad \text{as} \ T \to \infty,
\]

where \( Z \) is a 2-dimensional standard normally distributed random variable and \( \eta \) is a non-random \( 2 \times 2 \) matrix such that

\[
\eta \eta^\top = \begin{bmatrix} \mathbb{E}(X_\infty^2 Y_\infty) & \mathbb{E}(X_\infty Y_\infty) \\ \mathbb{E}(X_\infty Y_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix}.
\]

Here the matrix

\[
\begin{bmatrix} \mathbb{E}(X_\infty^2 Y_\infty) & \mathbb{E}(X_\infty Y_\infty) \\ \mathbb{E}(X_\infty Y_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix}
\]

is positive definite, since its principal minors are positive, and \( \eta \) denotes its unique symmetric positive definite square root. Indeed, by the absolutely continuity of \( (Y_\infty, X_\infty) \) (see Theorem 2.5), we have \( \mathbb{P}(X_\infty^2 Y_\infty = 0) = 0 \) and, by Cauchy-Schwarz’s inequality,

\[
\mathbb{E}(X_\infty^2 Y_\infty) \mathbb{E}(Y_\infty) - (\mathbb{E}(X_\infty Y_\infty))^2 \geq 0,
\]

where equality would hold if and only if \( \mathbb{P}(K X_\infty^2 Y_\infty = L Y_\infty) = 1 \) with some constant \( K, L \in \mathbb{R}_+ \) such that \( K^2 + L^2 > 0 \) or equivalently (using that \( \mathbb{P}(Y_\infty > 0) = 1 \) since \( Y_\infty \) has Gamma distribution) if and only if \( \mathbb{P}(X_\infty^2 = L/K) = 1 \), which leads us to a contradiction refereeing to the absolutely continuity of \( X_\infty \).
Let us use Theorem 2.7 with the following choices

d := 2, \quad (F_t)_{t \geq 0} \text{ defined before Proposition 2.1}

\begin{align*}
M_t := \begin{bmatrix} f_0^t X_s \sqrt{Y_s} \, ds \\ f_0^t \sqrt{Y_s} \, ds \end{bmatrix}, \quad t \geq 0, \\
Q(t) := \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{bmatrix}, \quad t > 0.
\end{align*}

Then

\begin{align*}
\langle M \rangle_t = \begin{bmatrix} \frac{1}{2} f_0^t X_s^2 \, ds + \frac{2}{t} f_0^t X_s \, ds \\ f_0^t X_s \, ds + \frac{t}{2} f_0^t \, ds \end{bmatrix}, \quad t \geq 0,
\end{align*}

and hence, by Theorem 2.5 (similarly as detailed in the proof of Theorem 7.2),

\begin{align*}
Q(t) \langle M \rangle_t Q(t)^\top \rightarrow \begin{bmatrix} \mathbb{E}(X_\infty^2 Y_\infty) & \mathbb{E}(X_\infty Y_\infty) \\ \mathbb{E}(X_\infty Y_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix} \text{ as } t \to \infty \text{ a.s.}
\end{align*}

By [5.8], [6.2], Slutsky’s lemma and the continuous mapping theorem, we get

\begin{align*}
\sqrt{T} \left[ \frac{\hat{\theta}^{LSE} - \theta}{\partial \hat{\theta}^{LSE} - \theta} \right] \leq T \rightarrow \mathbb{E}(X_\infty^2) - (\mathbb{E}(X_\infty))^2 \begin{bmatrix} -\mathbb{E}(X_\infty) & \mathbb{E}(X_\infty^2) \\ \mathbb{E}(X_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix} \eta Z \text{ as } T \to \infty.
\end{align*}

Using that $\eta Z$ is a 2-dimensional normally distributed random variable with mean vector zero and with covariance matrix

\begin{align*}
\eta \eta^\top = \begin{bmatrix} \mathbb{E}(X_\infty^2) & \mathbb{E}(X_\infty Y_\infty) \\ \mathbb{E}(X_\infty Y_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix},
\end{align*}

the covariance matrix of

\begin{align*}
\begin{bmatrix} -\mathbb{E}(X_\infty) & \mathbb{E}(X_\infty^2) \\ \mathbb{E}(X_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix} \eta Z
\end{align*}

takes the form

\begin{align*}
\begin{bmatrix} -\mathbb{E}(X_\infty) & \mathbb{E}(X_\infty^2) \\ \mathbb{E}(X_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix} \eta \mathbb{E}(Z Z^\top) \eta^\top = \begin{bmatrix} \mathbb{E}(X_\infty^2) & \mathbb{E}(X_\infty Y_\infty) \\ \mathbb{E}(X_\infty Y_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix}
\end{align*}

\begin{align*}
\begin{bmatrix} -\mathbb{E}(X_\infty) & \mathbb{E}(X_\infty^2) \\ \mathbb{E}(X_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix}
\end{align*}

\begin{align*}
= \begin{bmatrix} \mathbb{E}(X_\infty^2) & \mathbb{E}(X_\infty Y_\infty) \\ \mathbb{E}(X_\infty Y_\infty) & \mathbb{E}(Y_\infty) \end{bmatrix}
\end{align*}

\begin{align*}
\mathbb{E}(X_\infty^2) & \mathbb{E}(X_\infty Y_\infty) - \mathbb{E}(X_\infty) \mathbb{E}(X_\infty^2 Y_\infty) - \mathbb{E}(X_\infty Y_\infty) \mathbb{E}(X_\infty^2) + \mathbb{E}(X_\infty) \mathbb{E}(X_\infty Y_\infty)
\end{align*}

\begin{align*}
\times \begin{bmatrix} -\mathbb{E}(X_\infty) & 1 \\ \mathbb{E}(X_\infty^2) & \mathbb{E}(X_\infty) \end{bmatrix},
\end{align*}

which yields the assertion. \hfill \Box

**8.4 Remark.** Using the explicit forms of the mixed moments given in (iii) of Theorem 2.5, one can check that the asymptotic variance $\mathbb{E}(X_\infty^2 Y_\infty)/(\mathbb{E}(X_\infty^2))^2$ of $\hat{\theta}^{LSE}$ in Theorem 8.1 is less than the asymptotic variance $\Sigma_{LSE}^{1,1}$ of $\hat{\theta}^{LSE}$ in Theorem 8.3. This can be interpreted similarly as in Remark 7.3. \hfill \Box
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