EXACT SOLUTIONS FOR A QUANTUM-MECHANICAL PARTICLE WITH SPIN 1 IN THE EXTERNAL HOMOGENEOUS MAGNETIC FIELD

With the use of the general covariant matrix 10-dimensional Petiau – Duffin – Kemmer formalism in cylindrical coordinates and tetrad there are constructed exact solutions of the quantum-mechanical equation for a particle with spin 1 in presence of an external homogeneous magnetic field. There are separated three linearly independent types of solutions; in each case the formula for energy levels has been found.

1 Introduction, setting the problem

The problem of a quantum-mechanical particle in the external homogeneous magnetic field is well-known in theoretical physics. In fact, only two cases are considered: a scalar (Schrödinger’s) non-relativistic particle with spin 0, and fermions (non-relativistic Pauli’s and relativistic Dirac’s) with spin 1/2 (the first investigation were [1, 2, 3, 4]). In the present paper, exact solutions for a vector particle with spin 1 will be constructed explicitly. The most popular quantum-mechanical problem for such a particle is that in presence of external Coulomb potential [4].

To treat the problem we take the matrix Petiau – Duffin – Kemmer approach in the theory of the vector particle extended to a general covariant form on the base of tetrad formalism (recent consideration and list of references see in [5, 6]).

The main equation in tetrad form is [6]

$$\left[ i \beta^\alpha (x) (\partial_\alpha + B_\alpha - i \frac{e}{\hbar} A_\alpha) - \frac{Mc}{\hbar} \right] \Psi (x) = 0,$$

$$\beta^\alpha (x) = \beta^\alpha e^{\alpha (a)} (x), \quad B_\alpha (x) = \frac{1}{2} J^{ab} e^{\beta (a)} \nabla_\alpha e^{(b)\beta};$$

(1)

$e^{\alpha (a)} (x)$ is a tetrad, $J^{ab}$ stands for generators for 10-dimensional representation of the Lorentz group referred to 4-vector and anti-symmetric tensor (for brevity we note $Mc/\hbar$ as $M$). To the homogeneous magnetic field $B = (0, 0, B)$ corresponds 4-potential

$$A^a = (0, \vec{A}) = (0, \frac{1}{2} \vec{B} \times \vec{r}) = \frac{B}{2} (0, -x^2, +x^1, 0);$$

in the cylindric coordinates it is given by a simple expression

$$(ct, r, \phi, z), \quad dS^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \quad A_0 = 0, \quad A_r = 0, \quad A_\phi = -\frac{Br^2}{2}, \quad A_z = 0.$$  

(2)

Choosing a diagonal cylindric tetrad

$$e^{\alpha (0)} = (1, 0, 0, 0), \quad e^{\alpha (1)} = (0, 1, 0, 0), \quad e^{\alpha (2)} = (0, 0, 1, 0), \quad e^{\alpha (3)} = (0, 0, 0, 1).$$  

(3)
after simple calculation, the main equation (1) is reduced to the form
\[
\begin{pmatrix}
i\beta^0 \partial_0 + i\beta^1 \partial_r + \frac{i\beta^2}{r} (\partial_\phi + \frac{ieB}{2\hbar} r^2 + J^{12}) + i\beta^3 \partial_z - M
\end{pmatrix}\Psi(t,r,\phi,z) = 0.
\] (4)

For brevity we will note \((eB/2\hbar)\) as \(B\). It is best to chose the matrices \(\beta^a\) in the so-called cyclic form, where the generator \(J^{12}\) has a diagonal structure. In block-form \((1 - 3 - 3 - 3)\) these matrices are
\[
\beta^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i \tau_i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & \tau_i \\ -e_i^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{pmatrix},
\]
where \(e_i, \tau_i\) denote
\[
e_1 = \frac{1}{\sqrt{2}}(-i, 0, i), \quad e_2 = \frac{1}{\sqrt{2}}(1, 0, 1), \quad e_3 = (0, i, 0),
\]
\[
\tau_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau_3 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix} = s_3.
\]

Entering eq. (4) generator \(J^{12}\) is given by
\[
J^{12} = \beta^1\beta^2 - \beta^2\beta^1 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{pmatrix} = -i s_3.
\]

2 Separation of the variables

With the use of special substitution (it corresponds to diagonalization of the third projections of momentum \(P_3\) and angular momentum \(J_3\) for a particle with spin 1)
\[
\Psi = e^{-i\epsilon t} e^{i m \phi} e^{ik z} \begin{pmatrix} \Phi_0 \\ \bar{\Phi} \\ \vec{E} \\ \bar{\vec{E}} \end{pmatrix}, \quad \left[ ie^0 + i\beta^1 \partial_r - \frac{\beta^2}{r} (m + Br^2 - S_3) - k\beta^3 - M \right] \begin{pmatrix} \Phi_0 \\ \bar{\Phi} \\ \vec{E} \\ \bar{\vec{E}} \end{pmatrix} = 0. \] (5)

after calculations we arrive at the radial system of ten equations
\[
\begin{align*}
-b_{m-1}E_1 - a_{m+1}E_3 - i k E_2 &= M \Phi_0, \\
-ib_{m-1}H_1 + ia_{m+1}H_3 + ie E_2 &= M \Phi_2, \\
i a_m H_2 + ie E_1 - k H_1 &= M \Phi_1, \\
-ib_m H_2 + ie E_3 + k H_3 &= M \Phi_3,
\end{align*}
\]
\[
\begin{align*}
a_m \Phi_0 - ie \Phi_1 &= ME_1, \\
i a_m \Phi_2 + k \Phi_1 &= MH_1, \\
b_m \Phi_0 - ie \Phi_3 &= ME_3, \\
ib_m \Phi_2 - k \Phi_3 &= MH_3, \\
-ie \Phi_2 - ik \Phi_0 &= ME_2, \\
ib_{m-1} \Phi_1 - ia_{m+1} \Phi_3 &= MH_2.
\end{align*}
\] (6)
Because, we can readily get

\[ \frac{1}{\sqrt{2}} \left( \frac{d}{dr} + \frac{m + Br^2}{r} \right) = a_m, \quad \frac{1}{\sqrt{2}} \left( -\frac{d}{dr} + \frac{m + Br^2}{r} \right) = b_m. \]

From (6) – (7) it follow 4 equations for the components \( \Phi_a \)

\[
\begin{align*}
- b_{m-1} a_m - a_{m+1} b_m - k^2 - M^2 \right) \Phi_0 - \epsilon k \Phi_2 + i \epsilon (b_{m-1} \Phi_1 + a_{m+1} \Phi_3) &= 0, \\
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 \right) \Phi_2 + \epsilon k \Phi_0 - i \epsilon k (b_{m-1} \Phi_1 + a_{m+1} \Phi_3) &= 0, \\
- a_m b_{m-1} + \epsilon^2 - k^2 - M^2 \right) \Phi_1 + a_m a_{m+1} \Phi_3 + i \epsilon a_m \Phi_0 + i \epsilon k a_m \Phi_2 &= 0, \\
- b_{m} a_{m+1} + \epsilon^2 - M^2 - k^2 \right) \Phi_3 + b_m b_{m-1} \Phi_1 + i \epsilon b_m \Phi_0 + i \epsilon b_m \Phi_2 &= 0; \quad (8)
\end{align*}
\]

3 Special simple class of solutions

There exists a simple linear condition on 4-vector \( \Phi_a \), leading to a second order differential equation. Let it be \( \Phi_1 = 0, \Phi_3 = 0 \), the system (8) gives

\[
\begin{align*}
- b_{m-1} a_m - a_{m+1} b_m - k^2 - M^2 \right) \Phi_0 - \epsilon k \Phi_2 &= 0, \\
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 \right) \Phi_2 + \epsilon k \Phi_0 &= 0, \\
i a_m (\epsilon \Phi_0 + i \epsilon k \Phi_2) &= 0, \\
i b_m (\epsilon \Phi_0 + i \epsilon k \Phi_2) &= 0. \quad (9)
\end{align*}
\]

From two last equations in (9) we conclude that

\[ \epsilon \Phi_0 + k \Phi_2 = 0 \]

(10)
correspondingly, the first two in (9) take the form

\[
\begin{align*}
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2 \right) \Phi_0 &= 0, \\
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2 \right) \Phi_2 &= 0. \quad (11)
\end{align*}
\]

Because, we can readily get

\[
- b_{m-1} a_m - a_{m+1} b_m = \frac{d^2}{dr^2} + \frac{1}{r} \frac{dr}{dr} - \frac{(m + Br^2)^2}{r^2} = \Delta,
\]

eqs. (11) are differential equations of one the same type that is operative in the theory of a scalar particle in magnetic field

\[
(\Delta + \epsilon^2 - k^2 - M^2) \Phi_0 = 0, \quad (\Delta + \epsilon^2 - k^2 - M^2) \Phi_2 = 0. \quad (12)
\]

All the remaining component of the 10-dimensional function can be found straightforwardly as in accordance with the relations

\[
\begin{align*}
\Phi_1 &= 0, \Phi_3 = 0, \quad \epsilon \Phi_0 + k \Phi_2 = 0, \\
a_m \Phi_0 &= ME_1, \quad a_m \Phi_2 = i MH_1, \quad b_m \Phi_0 = ME_3, \\
b_m \Phi_2 &= -i MH_3, \quad (\epsilon \Phi_2 + k \Phi_0) = i ME_2, \quad 0 = H_2. \quad (13)
\end{align*}
\]

In general, there must exist three types of solutions for the particle with spin 1, we have found only one that.
4 General analysis of the radial equations

Eqs. (18) can be transformed to the form

\[
\begin{align*}
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 - k^2 & ( k \Phi_0 + \epsilon \Phi_2 ) = 0 , \\
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2 & ( \epsilon \Phi_0 + k \Phi_2 ) = \\
= (\epsilon^2 - k^2) & \left[ ( \epsilon \Phi_0 + k \Phi_2 ) - ( i b_{m-1} \Phi_1 + i a_{m+1} \Phi_3 ) \right] ; \\
( -a_m b_{m-1} + \epsilon^2 - k^2 - M^2 ) & \Phi_1 + a_m a_{m+1} \Phi_3 + i \epsilon a_m \Phi_0 + i k a_m \Phi_2 = 0 , \\
( -b_m a_{m+1} + \epsilon^2 - M^2 - k^2 ) & \Phi_3 + b_m b_{m-1} \Phi_1 + i \epsilon b_m \Phi_0 + i k b_m \Phi_2 = 0 .
\end{align*}
\]

(14)

Let us introduce new variables

\[
F(r) = k \Phi_0(r) + \epsilon \Phi_2(r) , \quad G(r) = \epsilon \Phi_0(r) + k \Phi_2(r) ,
\]

then eqs. (14) - (15) read

\[
\begin{align*}
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 - k^2 & F = 0 , \\
- b_{m-1} a_m - a_{m+1} b_m - M^2 & G = - (\epsilon^2 - k^2) ( i b_{m-1} \Phi_1 + i a_{m+1} \Phi_3 ) , \\
( -a_m b_{m-1} + \epsilon^2 - k^2 - M^2 ) & \Phi_1 + a_m a_{m+1} \Phi_3 + i \epsilon a_m G = 0 , \\
( -b_m a_{m+1} + \epsilon^2 - M^2 - k^2 ) & \Phi_3 + b_m b_{m-1} \Phi_1 + i b_m G = 0 .
\end{align*}
\]

(17)

For two equations in (18), let us multiply the first (from the left) by \( b_{m-1} \) and the second by the \( a_{m+1} \), which result in

\[
\begin{align*}
- b_{m-1} a_m (b_{m-1} \Phi_1) + (\epsilon^2 - k^2 - M^2) (b_{m-1} \Phi_1) + b_{m-1} a_m (a_{m+1} \Phi_3) + i b_{m-1} a_m G &= 0 , \\
- a_{m+1} b_m (a_{m+1} \Phi_3) + (\epsilon^2 - M^2 - k^2) (a_{m+1} \Phi_3) + a_{m+1} b_m (b_{m-1} \Phi_1) + i a_{m+1} b_m G &= 0 .
\end{align*}
\]

(19)

Again, let us introduce two new variables

\[
b_{m-1} \Phi_1 = Z_1 , \quad a_{m+1} \Phi_3 = Z_3 ;
\]

(20)
eqs. (19) read as follows

\[
\begin{align*}
- b_{m-1} a_m Z_1 + (\epsilon^2 - k^2 - M^2) Z_1 + b_{m-1} a_m Z_3 + i b_{m-1} a_m G &= 0 , \\
- a_{m+1} b_m Z_3 + (\epsilon^2 - M^2 - k^2) Z_3 + a_{m+1} b_m Z_1 + i a_{m+1} b_m G &= 0 .
\end{align*}
\]

(21)

With the aid of new functions \( f(r), g(r) \)

\[
Z_1 = \frac{f + g}{2} , \quad Z_3 = \frac{f - g}{2} , \quad Z_1 + Z_3 = f , \quad Z_1 - Z_3 = g ,
\]

(22)
the system (21) is transformed to the following ones

\[
\begin{align*}
- b_{m-1} a_m g + (\epsilon^2 - k^2 - M^2) \frac{f + g}{2} + i b_{m-1} a_m G &= 0 , \\
a_{m+1} b_m g + (\epsilon^2 - M^2 - k^2) \frac{f - g}{2} + i a_{m+1} b_m G &= 0 .
\end{align*}
\]

(23)
Combining these equations we get

\[
- b_{m-1}a_m - a_{m+1}b_m + \epsilon^2 - k^2 - M^2 \right] g + i(b_{m-1}a_m - a_{m+1}b_m) G = 0 , \\
( -b_{m-1}a_m + a_{m+1}b_m ) g + (\epsilon^2 - k^2 - M^2) f + i(b_{m-1}a_m + a_{m+1}b_m)G = 0 .
\] (24)

In these variables, eqs. (17) can be written as

\[
( -b_{m-1}a_m - a_{m+1}b_m + \epsilon^2 - M^2 - k^2 ) F = 0 , \\
( -b_{m-1}a_m - a_{m+1}b_m - M^2 ) G = -i(\epsilon^2 - k^2 ) f .
\] (25)

Further, with the use of identities

\[
- b_{m-1}a_m - a_{m+1}b_m = \Delta , \\
- b_{m-1}a_m + a_{m+1}b_m = 2B .
\] (26)

eqs. (25) and (24) can be written down as follows

\[
(\Delta + \epsilon^2 - M^2 - k^2) F = 0 , \\
\Delta G = M^2 G - i(\epsilon^2 - k^2) f , \\
(\Delta + \epsilon^2 - k^2 - M^2) g = 2iB G , \\
(\epsilon^2 - k^2 - M^2) f - i\Delta G + 2B g = 0 .
\] (27)

With the help of the second equation, from the forth one it follows the linear relationship

\[
f = -i G + \frac{2B}{M^2} g .
\] (28)

Now, excluding the function \( f \) in the second one in (27)

\[
(\Delta + \epsilon^2 - k^2 - M^2) G = -i(\epsilon^2 - k^2)\frac{2B}{M^2} g .
\] (29)

Thus, the general problem is reduced to the system of four equations

\[
(\Delta + \epsilon^2 - M^2 - k^2) F = 0 , \\
f = -i G + \frac{2B}{M^2} g , \\
(\Delta + \epsilon^2 - k^2 - M^2) g = 2iB G , \\
(\Delta + \epsilon^2 - k^2 - M^2) G = -2iB \frac{\epsilon^2 - k^2}{M^2} g ,
\] (30)

The structure of this system allows to separate an evident linearly independent solution as follows

\[
f(r) = 0, \quad g(r) = 0 , \quad H(r) = 9 , \\
F(r) \neq 0 , \quad (\Delta - k^2 - M^2 + \epsilon^2) F = 0 .
\] (31)

corresponding functions and energy spectrum are known (also see below). We are to solve the system of two last equations in (30), in matrix form it reads (let \( \gamma = (\epsilon^2 - k^2)/M^2 \))

\[
\begin{pmatrix}
\Delta + \epsilon^2 - M^2 - k^2 \\
\epsilon^2 - k^2 - M^2
\end{pmatrix}
\begin{pmatrix}
g(r) \\
G(r)
\end{pmatrix}
= 
\begin{pmatrix}
0 & 2iB \\
-2iB\gamma & 0
\end{pmatrix}
\begin{pmatrix}
g(r) \\
G(r)
\end{pmatrix} .
\] (32)
Let us construct transformation changing the matrix on the right to a diagonal form

\[
\begin{pmatrix}
\Delta + e^2 - M^2 - k^2
\end{pmatrix}
\begin{pmatrix}
g' \\
G'
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\begin{pmatrix}
g' \\
G'
\end{pmatrix},
\]

\[
\begin{pmatrix}
g' \\
G'
\end{pmatrix} = S \begin{pmatrix}
g \\
G
\end{pmatrix},
\]

\[S = \begin{pmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{pmatrix}.
\]

(33)

The problem to solve is

\[
S \begin{pmatrix}
0 & 2iB \\
-2iB\gamma & 0
\end{pmatrix} S^{-1} = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix},
\]

which results in two linear systems

\[
\begin{cases}
-\lambda_1 s_{11} - 2iB\gamma s_{12} = 0 , \\
2iB s_{11} - \lambda_1 s_{12} = 0 ,
\end{cases}
\]

and

\[
\begin{cases}
-\lambda_2 s_{21} - 2iB\gamma s_{22} = 0 , \\
2iB s_{21} - \lambda_2 s_{22} = 0 .
\end{cases}
\]

The values of \(\lambda_1\) and \(\lambda_2\) are given by

\[
\lambda_1 = \pm 2B\sqrt{\gamma} , \quad \lambda_2 = \pm 2B\sqrt{\gamma}.
\]

The matrix \(S\) must be degenerate, so we must use different \(\lambda_1, \lambda_2\):

Variant (A)

\[
\lambda_1' = +2B\sqrt{\gamma} , \quad \lambda_2' = -2B\sqrt{\gamma} ,
\]

\[
i s_{11} - \sqrt{\gamma} s_{12} = 0 , \quad i s_{21} + \sqrt{\gamma} s_{22} = 0 ;
\]

let it be

\[
s_{12} = 1 , \quad s_{22} = 1 , \quad s_{11} = -i\sqrt{\gamma} , \quad s_{21} = +i\sqrt{\gamma} , \quad S = \begin{pmatrix}
-i\sqrt{\gamma} & 1 \\
+i\sqrt{\gamma} & 1
\end{pmatrix}.
\]

(34)

Variant (B)

\[
\lambda_1'' = -2B\sqrt{\gamma} = \lambda_2'' , \quad \lambda_2'' = +2B\sqrt{\gamma} = \lambda_1'' ,
\]

\[
i s_{11} + \sqrt{\gamma} s_{12} = 0 , \quad i s_{21} - \sqrt{\gamma} s_{22} = 0 ;
\]

let it be

\[
s_{12} = 1 , \quad s_{22} = 1 , \quad s_{11} = +i\sqrt{\gamma} , \quad s_{21} = -i\sqrt{\gamma} , \quad S = \begin{pmatrix}
+ i \sqrt{\gamma} & 1 \\
- i \sqrt{\gamma} & 1
\end{pmatrix}.
\]

(35)

In the new (primed) basis, eqs. (32) take the form of two separated differential equations

(A) \[
\begin{pmatrix}
\Delta + e^2 - k^2 - M^2 - 2B\sqrt{\gamma}
\end{pmatrix} g' = 0 ,
\]

\[
\begin{pmatrix}
\Delta + e^2 - k^2 - M^2 + 2B\sqrt{\gamma}
\end{pmatrix} G' = 0 ;
\]

(36)

(B) \[
\begin{pmatrix}
\Delta + e^2 - k^2 - M^2 + 2B\sqrt{\gamma}
\end{pmatrix} g'' = 0 ,
\]

\[
\begin{pmatrix}
\Delta + e^2 - k^2 - M^2 - 2B\sqrt{\gamma}
\end{pmatrix} G'' = 0 .
\]

(37)
Recalling the meaning of \( \Delta \), let us detail the second order equation

\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} + \lambda^2 \right) \varphi(r) = 0 ,
\]

\[
\lambda^2 = \epsilon^2 - k^2 - M^2 \pm 2B \sqrt{\gamma} , \quad \sqrt{\gamma} = \frac{\sqrt{\epsilon^2 - k^2}}{M} .
\]  (38)

It is convenient to introduce a new variable \( x = Br^2 \), then eq. (38) reads

\[
\frac{d^2 \varphi}{dx^2} + \frac{d}{dx} - \left( \frac{m^2}{4x} + \frac{x}{4} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) \varphi = 0 .
\]  (39)

With the substitution \( \varphi(x) = x^A e^{-Cx} f(x) \), for \( f(x) \) we get

\[
x \frac{d^2 f}{dx^2} + (2A + 1 - 2Cx) \frac{df}{dx} + \left[ \frac{A^2 - m^2/4}{x} + \left( C^2 - \frac{1}{4} \right)x - 2AC \right] f = 0 .
\]

When \( A, C \) are taken as \( A = + |m|/2 \), \( C = +1/2 \) the previous equation becomes simpler

\[
x \frac{d^2 R}{dx^2} + (2A + 1 - x) \frac{dR}{dx} - \left( A + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B} \right) R = 0 ,
\]

which is of (degenerate) hypergeometric type

\[
x Y'' + (\gamma - x)Y' - \alpha Y = 0 , \quad \alpha = \frac{|m|}{2} + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B} ; \quad \gamma = |m| + 1 .
\]

To obtain polynomials we must impose additional condition \( \alpha = -n \); which leads to the following quantization for \( \lambda^2 \)

\[
\lambda^2 = 4B \left( n + \frac{1}{2} + \frac{|m| + m}{2} \right) .
\]  (40)

Taking into account (36) – (37), we have relations

(A) \( \left( \Delta + (\epsilon^2 - k^2) - M^2 - 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) g' = 0 \), \( \sqrt{\epsilon^2 - k^2} = +B + \sqrt{B^2 + M^2(M^2 + \lambda^2)} \),

\( \left( \Delta + (\epsilon^2 - k^2) - M^2 + 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) G' = 0 \), \( \sqrt{\epsilon^2 - k^2} = -B + \sqrt{B^2 + M^2(M^2 + \lambda^2)} \);  

(B) \( \left( \Delta + (\epsilon^2 - k^2) - M^2 + 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) g'' = 0 \), \( \sqrt{\epsilon^2 - k^2} = -B + \sqrt{B^2 + M^2(M^2 + \lambda^2)} \),

\( \left( \Delta + (\epsilon^2 - k^2) - M^2 - 2B \frac{\sqrt{\epsilon^2 - k^2}}{M} \right) G'' = 0 \), \( \sqrt{\epsilon^2 - k^2} = +B + \sqrt{B^2 + M^2(M^2 + \lambda^2)} \).

\footnote{For definiteness let us consider \( B \) to be positive, which does not affect generality of the analysis. So, to infinite values of \( r \) corresponds infinite and positive values of \( x \).}

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In fact, here there exist only two different possibilities (and correspondingly two formulae for energy spectrum):

\[
\sqrt{\epsilon^2 - k^2} = \frac{B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}, \quad q'(r) \neq 0, \ G' = 0; \\
\sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}, \quad q'(r) = 0, \ G' \neq 0.
\]

In turn, energy spectrum for the case (31) is given by

\[
\epsilon^2 = M^2 + k^2 + \lambda^2
\]

Thus, on the base of the use of general covariant formalism in the Petiau – Duffin – Kemmer theory of the vector particle, exact solutions for such a particle are constructed in presence of external homogeneous magnetic field. There are separated three types of linearly independent solutions, and energy spectra are found.

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EXACT SOLUTIONS FOR A QUANTUM-MECHANICAL PARTICLE WITH SPIN 1 IN THE EXTERNAL HOMOGENEOUS MAGNETIC FIELD

With the use of the general covariant matrix 10-dimensional Petiau – Duffin – Kemmer formalism in cylindrical coordinates and tetrad there are constructed exact solutions of the quantum-mechanical equation for a particle with spin 1 in presence of an external homogeneous magnetic field. There are separated three linearly independent types of solutions; in each case the formula for energy levels has been found.

1 Introduction, setting the problem

The problem of a quantum-mechanical particle in the external homogeneous magnetic field is well-known in theoretical physics. In fact, only two cases are considered: a scalar (Schrödinger’s) non-relativistic particle with spin 0, and fermions (non-relativistic Pauli’s and relativistic Dirac’s) with spin 1/2 (the first investigation were [1, 2, 3, 4]). In the present paper, exact solutions for a vector particle with spin 1 will be constructed explicitly. The most popular quantum-mechanical problem for such a particle is that in presence of external Coulomb potential [4].

To treat the problem we take the matrix Petiau – Duffin – Kemmer approach in the theory of the vector particle extended to a general covariant form on the base of tetrad formalism (recent consideration and list of references see in [5, 6]).

The main equation in tetrad form is [6]

\[
\left[ i \beta^\alpha(x) \left( \partial_\alpha + B_\alpha - \frac{e^\alpha}{\hbar} A_\alpha \right) - \frac{M c}{\hbar} \right] \Psi(x) = 0 ,
\]

\[
\beta^\alpha(x) = \beta^a e^\alpha_a(x), \quad B_\alpha(x) = \frac{1}{2} J^{ab} e^\beta_{(a)} \nabla_\alpha e^{(b)\beta} ;
\]

(1)

e^\alpha_a(x) is a tetrad, \(J^{ab}\) stands for generators for 10-dimensional representation of the Lorentz group referred to 4-vector and anti-symmetric tensor (for brevity we note \(M c/\hbar\) as \(M\)). To the homogeneous magnetic field \(B = (0, 0, B)\) corresponds 4-potential

\[
A^a = \left( 0, \vec{A} \right) = \left( 0, \frac{1}{2} \vec{B} \times \vec{r} \right) = \frac{B}{2} \left( 0, -x^2, +x^1, 0 \right) ;
\]

in the cylindric coordinates it is given by a simple expression

\[
(ct, r, \phi, z) , \quad ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2 ,
\]

\[
A_0 = 0, \quad A_r = 0, \quad A_\phi = -\frac{Br^2}{2}, \quad A_z = 0 .
\]

(2)

Choosing a diagonal cylindric tetrad

\[
e^\alpha_{(0)} = (1, 0, 0, 0) , \quad e^\alpha_{(1)} = (0, 1, 0, 0) , \quad e^\alpha_{(2)} = (0, 0, \frac{1}{r}, 0) , \quad e^\alpha_{(3)} = (0, 0, 0, 1) .
\]

(3)
after simple calculation, the main equation (1) is reduced to the form
\[ [i\beta^0 \partial_0 + i\beta^1 \partial_r + \frac{i\beta^2}{r} (\partial_\phi + \frac{ieB}{2\hbar} r^2 + J^{12}) + i\beta^3 \partial_z - M] \Psi(t, r, \phi, z) = 0. \] (4)
For brevity we will note \((eB/2\hbar)\) as \(B\). It is best to chose the matrices \(\beta^a\) in the so-called cyclic form, where the generator \(J^{12}\) has a diagonal structure. In block-form \((1 - 3 - 3 - 3)\) these matrices are
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & e_i & 0 \\
0 & 0 & 0 & \tau_i \\
e^{-i\beta_i^+} & 0 & 0 & 0 \\
0 & -\tau_i & 0 & 0
\end{pmatrix},
\]
where \(e_i, \tau_i\) denote
\[
e_1 = \frac{1}{\sqrt{2}}(-i, 0, i), \quad e_2 = \frac{1}{\sqrt{2}}(1, 0, 1), \quad e_3 = (0, i, 0),
\]
\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -i & 0 \\
i & 0 & -i \\
i & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & \tau_3 & 0 \\
0 & 0 & \tau_3
\end{pmatrix} = \tau_2, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & \tau_3 & 0 \\
0 & 0 & 0
\end{pmatrix} = \tau_1.
\]
Entering eq. (4) generator \(J^{12}\) is given by
\[
J^{12} = \beta^1\beta^2 - \beta^2\beta^1 = -i \begin{pmatrix}
0 & 0 & 0 \\
0 & \tau_3 & 0 \\
0 & 0 & \tau_3
\end{pmatrix} = -i\tau_3.
\]

2 Separation of the variables

With the use of special substitution (it corresponds to diagonalization of the third projections of momentum \(P_3\) and angular momentum \(J_3\) for a particle with spin 1)
\[
\Psi = e^{-i\epsilon_0 E^0 \epsilon^kz} \begin{pmatrix}
\Phi_0 \\
\Phi_0 \\
\Phi_0
\end{pmatrix} \begin{pmatrix}
\epsilon \beta^0 + i\beta^1 \partial_r - \frac{\beta^2}{r} (m + Br^2 - S_3) - k\beta^3 - M \\
\epsilon \beta^0 + i\beta^1 \partial_r - \frac{\beta^2}{r} (m + Br^2 - S_3) - k\beta^3 - M \\
\epsilon \beta^0 + i\beta^1 \partial_r - \frac{\beta^2}{r} (m + Br^2 - S_3) - k\beta^3 - M
\end{pmatrix} \begin{pmatrix}
\Phi_0 \\
\Phi_0 \\
\Phi_0
\end{pmatrix} = 0. \quad (5)
\]
after calculations we arrive at the radial system of ten equations
\[
- b_{m-1} E_1 - a_{m+1} E_3 - ik E_2 = M\Phi_0, \\
-ib_{m-1} H_1 + ia_{m+1} H_3 + ie E_2 = M\Phi_2, \\
ia_m H_2 + ie E_1 - k H_1 = M\Phi_1, \\
-ib_m H_2 + ie E_3 + k H_3 = M\Phi_3,
\]
\[
a_m \Phi_0 - ie \Phi_1 = ME_1, \quad -ia_m \Phi_2 + k \Phi_1 = MH_1, \\
b_m \Phi_0 - ie \Phi_3 = ME_3, \quad ib_m \Phi_2 - k \Phi_3 = MH_3, \\
-ie \Phi_2 - ik \Phi_0 = ME_2, \quad ib_{m-1} \Phi_1 - ia_{m+1} \Phi_3 = MH_2.
\] (7)
Because, we can readily get

\[
\frac{1}{\sqrt{2}} \left( \frac{d}{dr} + \frac{m + Br^2}{r} \right) = a_m, \quad \frac{1}{\sqrt{2}} \left( \frac{d}{dr} + \frac{m + Br^2}{r} \right) = b_m. 
\]

From (6) – (7) it follow 4 equations for the components \( \Phi_a \)

\[
(-b_{m-1} a_m - a_{m+1} b_m - k^2 - M^2) \Phi_0 - \epsilon k \Phi_2 + i\epsilon (b_{m-1} \Phi_1 + a_{m+1} \Phi_3) = 0, \\
(-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2) \Phi_2 + \epsilon k \Phi_0 - ik (b_{m-1} \Phi_1 + a_{m+1} \Phi_3) = 0, \\
(-a_m b_{m-1} + \epsilon^2 - k^2 - M^2) \Phi_1 + a_m a_{m+1} \Phi_3 + i\epsilon a_m \Phi_0 + ik a_m \Phi_2 = 0, \\
(-b_m a_{m+1} + \epsilon^2 - M^2 - k^2) \Phi_3 + b_m b_{m-1} \Phi_1 + i\epsilon b_m \Phi_0 + ik b_m \Phi_2 = 0; \quad (8)
\]

3 Special simple class of solutions

There exists a simple linear condition on 4-vector \( \Phi_a \), leading to a second order differential equation. Let it be \( \Phi_1 = 0, \ \Phi_3 = 0 \), the system (8) gives

\[
(-b_{m-1} a_m - a_{m+1} b_m - k^2 - M^2) \Phi_0 - \epsilon k \Phi_2 = 0, \\
(-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2) \Phi_2 + \epsilon k \Phi_0 = 0, \\
i \epsilon a_m(\epsilon \Phi_0 + ik \Phi_2) = 0, \quad i \epsilon b_m(\epsilon \Phi_0 + ik \Phi_2) = 0. \quad (9)
\]

From two last equations in (9) we conclude that

\[
\epsilon \Phi_0 + k\Phi_2 = 0 \quad (10)
\]

correspondingly, the first two in (9) (9) take the form

\[
(-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2) \Phi_0 = 0, \\
(-b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2) \Phi_2 = 0. \quad (11)
\]

Because, we can readily get

\[-b_{m-1} a_m - a_{m+1} b_m = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} = \Delta, \]

eqs. (11) are differential equations of one the same type that is operative in the theory of a scalar particle in magnetic field

\[
(\Delta + \epsilon^2 - k^2 - M^2) \Phi_0 = 0, \quad (\Delta + \epsilon^2 - k^2 - M^2) \Phi_2 = 0. \quad (12)
\]

All the remaining component of the 10-dimensional function can be found straightforwardly as in accordance with the relations

\[
\Phi_1 = 0, \ \Phi_3 = 0, \quad \epsilon \Phi_0 + k\Phi_2 = 0, \\
a_m \Phi_0 = ME_1, \quad a_m \Phi_2 = iMH_1, \quad b_m \Phi_0 = ME_3, \\
b_m \Phi_2 = -iMH_3, \quad (\epsilon \Phi_2 + k \Phi_0) = iME_2, \quad 0 = H_2. \quad (13)
\]

In general, there must exist three types of solutions for the particle with spin 1, we have found only one that.
4 General analysis of the radial equations

Eqs. [8] can be transformed to the form

\[ \begin{align*}
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 - k^2 \right] ( k \Phi_0 + \epsilon \Phi_2 ) &= 0 , \\
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - k^2 - M^2 \right] ( \epsilon \Phi_0 + k \Phi_2 ) &= 0 \\
&= (\epsilon^2 - k^2) \left[ ( \epsilon \Phi_0 + k \Phi_2 ) - ( i b_{m-1} \Phi_1 + i a_{m+1} \Phi_3 \right] ;
\end{align*} \]

\[ ( -a_m b_{m-1} + \epsilon^2 - k^2 - M^2 ) \Phi_1 + a_m a_{m+1} \Phi_3 + i \epsilon a_m \Phi_1 + i k a_m \Phi_2 = 0 , \\
( -b_m a_{m+1} + \epsilon^2 - M^2 - k^2 ) \Phi_3 + b_m b_{m-1} \Phi_1 + i \epsilon b_m \Phi_0 + i k b_m \Phi_2 = 0 . \]

Let us introduce new variables

\[ F(r) = k \Phi_0(r) + \epsilon \Phi_2(r) , \quad G(r) = \epsilon \Phi_0(r) + k \Phi_2(r) , \]

then eqs. [14] - [15] read

\[ \begin{align*}
- b_{m-1} a_m - a_{m+1} b_m + \epsilon^2 - M^2 - k^2 \right] F &= 0 , \\
- b_{m-1} a_m - a_{m+1} b_m - M^2 \right] G &= - (\epsilon^2 - k^2) ( i b_{m-1} \Phi_1 + i a_{m+1} \Phi_3 \right] ;
\end{align*} \]

\[ ( -a_m b_{m-1} + \epsilon^2 - k^2 - M^2 ) \Phi_1 + a_m a_{m+1} \Phi_3 + i a_m G = 0 , \\
( -b_m a_{m+1} + \epsilon^2 - M^2 - k^2 ) \Phi_3 + b_m b_{m-1} \Phi_1 + i b_m G = 0 . \]

For two equations in [18], let us multiply the first (from the left) by \( b_{m-1} \) and the second by the \( a_{m+1} \), which result in

\[ \begin{align*}
- b_{m-1} a_m (b_{m-1} \Phi_1) + (\epsilon^2 - k^2 - M^2))(b_{m-1} \Phi_1) + b_{m-1} a_m (a_{m+1} \Phi_3) + i b_{m-1} a_m G &= 0 , \\
- a_{m+1} b_m (a_{m+1} \Phi_3) + (\epsilon^2 - M^2 - k^2)(a_{m+1} \Phi_3) + a_{m+1} b_m (b_{m-1} \Phi_1) + i a_{m+1} b_m G &= 0 .
\end{align*} \]

Again, let us introduce two new variables

\[ b_{m-1} \Phi_1 = Z_1 , \quad a_{m+1} \Phi_3 = Z_3 ; \]

eqs. [19] read as follows

\[ \begin{align*}
- b_{m-1} a_m Z_1 + (\epsilon^2 - k^2 - M^2) Z_1 + b_{m-1} a_m Z_3 + i b_{m-1} a_m G &= 0 , \\
- a_{m+1} b_m Z_3 + (\epsilon^2 - M^2 - k^2) Z_3 + a_{m+1} b_m Z_1 + i a_{m+1} b_m G &= 0 .
\end{align*} \]

With the aid of new functions \( f(r), g(r) \)

\[ Z_1 = \frac{f + g}{2} , \quad Z_3 = \frac{f - g}{2} , \quad Z_1 + Z_3 = f , \quad Z_1 - Z_3 = g ; \]

the system [21] is transformed to the following ones

\[ \begin{align*}
- b_{m-1} a_m G + (\epsilon^2 - k^2 - M^2) \frac{f + g}{2} + i b_{m-1} a_m G &= 0 , \\
a_{m+1} b_m G + (\epsilon^2 - M^2 - k^2) \frac{f - g}{2} + i a_{m+1} b_m G &= 0 .
\end{align*} \]
Combining these equations we get
\[
\begin{align*}
- b_{m-1} a_m - a_{m+1} b_m + e^2 - k^2 - M^2 \mid g + i(b_{m-1} a_m - a_{m+1} b_m) G &= 0 , \\
( - b_{m-1} a_m + a_{m+1} b_m ) g + (\epsilon^2 - k^2 - M^2) f + i(b_{m-1} a_m + a_{m+1} b_m) G &= 0 .
\end{align*}
\]
(24)

In these variables, eqs. (17) can be written as
\[
\begin{align*}
( - b_{m-1} a_m - a_{m+1} b_m + e^2 - M^2 - k^2 ) F &= 0 , \\
( - b_{m-1} a_m - a_{m+1} b_m - M^2 ) G &= -i(\epsilon^2 - k^2) f .
\end{align*}
\]
(25)

Further, with the use of identities
\[
\begin{align*}
- b_{m-1} a_m - a_{m+1} b_m = \Delta , \\
- b_{m-1} a_m + a_{m+1} b_m = 2B .
\end{align*}
\]
(26)

eqs. (25) and (24) can be written down as follows
\[
\begin{align*}
(\Delta + \epsilon^2 - M^2 - k^2 ) F &= 0 , \\
\Delta G &= M^2 G - i(\epsilon^2 - k^2) f , \\
(\Delta + \epsilon^2 - k^2 - M^2 ) g &= 2iB G , \\
(\epsilon^2 - k^2 - M^2 ) f - i\Delta G + 2B g &= 0 .
\end{align*}
\]
(27)

With the help of the second equation, from the forth one it follows the linear relationship
\[
f = -i G + \frac{2B}{M^2} g .
\]
(28)

Now, excluding the function \( f \) in the second one in (27)
\[
(\Delta + \epsilon^2 - k^2 - M^2 ) G = -i(\epsilon^2 - k^2)\frac{2B}{M^2} g .
\]
(29)

Thus, the general problem is reduced to the system of four equations
\[
\begin{align*}
(\Delta + \epsilon^2 - M^2 - k^2 ) F &= 0 , \\
f &= -i G + \frac{2B}{M^2} g , \\
(\Delta + \epsilon^2 - k^2 - M^2 ) g &= 2iB G , \\
(\Delta + \epsilon^2 - k^2 - M^2 ) G &= -2iB \frac{\epsilon^2 - k^2}{M^2} g .
\end{align*}
\]
(30)

The structure of this system allows to separate an evident linearly independent solution as follows
\[
\begin{align*}
f(r) &= 0 , \\
g(r) &= 0 , \\
H(r) &= 9 , \\
F(r) &\neq 0 , \\
(\Delta - k^2 - M^2 + \epsilon^2 ) F &= 0 .
\end{align*}
\]
(31)

corresponding functions and energy spectrum are known (also see below). We are to solve the system of two last equations in (30), in matrix form it reads (let \( \gamma = (\epsilon^2 - k^2)/M^2 \))
\[
\begin{pmatrix}
\Delta + \epsilon^2 - M^2 - k^2 \\
\epsilon^2 - k^2 - M^2 \\
\epsilon^2 - M^2 - k^2 \\
\Delta + \epsilon^2 - k^2 - M^2
\end{pmatrix}
\begin{pmatrix}
g(r) \\
G(r) \\
G(r) \\
g(r)
\end{pmatrix}
=
\begin{pmatrix}
0 \\
2iB \\
-2iB \gamma \\
0
\end{pmatrix}
\begin{pmatrix}
g(r) \\
G(r)
\end{pmatrix} .
\]
(32)
Let us construct transformation changing the matrix on the right to a diagonal form

\[
(\Delta + \epsilon^2 - M^2 - k^2) \begin{vmatrix} g' \\ G' \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \begin{vmatrix} g' \\ G' \end{vmatrix},
\]

\[
\begin{vmatrix} g' \\ G' \end{vmatrix} = S \begin{vmatrix} g \\ G \end{vmatrix}, \quad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}.
\]

(33)

The problem to solve is

\[
S \begin{vmatrix} 0 & 2iB \\ -2iB\gamma & 0 \end{vmatrix} S^{-1} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix},
\]

which results in two linear systems

\[
\begin{aligned}
-\lambda_1 s_{11} - 2iB\gamma s_{12} &= 0, \\
2iB s_{11} - \lambda_1 s_{12} &= 0,
\end{aligned}
\]

\[
\begin{aligned}
-\lambda_2 s_{21} - 2iB\gamma s_{22} &= 0, \\
2iB s_{21} - \lambda_2 s_{22} &= 0.
\end{aligned}
\]

The values of \(\lambda_1\) and \(\lambda_2\) are given by

\[
\lambda_1 = \pm 2B\sqrt{\gamma}, \quad \lambda_2 = \pm 2B\sqrt{\gamma}.
\]

The matrix \(S\) must be degenerate, so we must use different \(\lambda_1, \lambda_2\):

**Variant (A)** \(\lambda_1' = +2B\sqrt{\gamma}, \quad \lambda_2' = -2B\sqrt{\gamma}, \quad i s_{11} - \sqrt{\gamma} s_{12} = 0, \quad i s_{21} + \sqrt{\gamma} s_{22} = 0;\)

let it be

\[
s_{12} = 1, \quad s_{22} = 1, \quad s_{11} = -i\sqrt{\gamma}, \quad s_{21} = +i\sqrt{\gamma}, \quad S = \begin{vmatrix} -i\sqrt{\gamma} & 1 \\ +i\sqrt{\gamma} & 1 \end{vmatrix}.
\]

(34)

**Variant (B)** \(\lambda_1'' = -2B\sqrt{\gamma} = \lambda_2'', \quad \lambda_2'' = +2B\sqrt{\gamma} = \lambda_1'', \quad i s_{11} + \sqrt{\gamma} s_{12} = 0, \quad i s_{21} - \sqrt{\gamma} s_{22} = 0;\)

let it be

\[
s_{12} = 1, \quad s_{22} = 1, \quad s_{11} = +i\sqrt{\gamma}, \quad s_{21} = -i\sqrt{\gamma}, \quad S = \begin{vmatrix} +i\sqrt{\gamma} & 1 \\ -i\sqrt{\gamma} & 1 \end{vmatrix}.
\]

(35)

In the new (primed) basis, eqs. (32) take the form of two separated differential equations

**A)** \(\left(\Delta + \epsilon^2 - k^2 - M^2 - 2B\sqrt{\gamma}\right) g' = 0, \quad \left(\Delta + \epsilon^2 - k^2 - M^2 + 2B\sqrt{\gamma}\right) G' = 0;\)

(36)

**B)** \(\left(\Delta + \epsilon^2 - k^2 - M^2 + 2B\sqrt{\gamma}\right) g'' = 0, \quad \left(\Delta + \epsilon^2 - k^2 - M^2 - 2B\sqrt{\gamma}\right) G'' = 0.
\)

(37)
Recalling the meaning of $\Delta$, let us detail the second order equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m + Br^2)^2}{r^2} + \lambda^2\right) \varphi(r) = 0,$$

$$\lambda^2 = \epsilon^2 - k^2 - M^2 \pm 2B \sqrt{\gamma}, \quad \sqrt{\gamma} = \frac{\sqrt{\epsilon^2 - k^2}}{M}.$$  (38)

It is convenient to introduce a new variable $x = Br^2$, then eq. (38) reads

$$\frac{d^2 \varphi}{dx^2} + \left(\frac{m^2}{4x} + \frac{x}{4} + \frac{m - \lambda^2}{4B}\right) \varphi = 0.$$  (39)

With the substitution $\varphi(x) = x^A e^{-Cx} f(x)$, for $f(x)$ we get

$$x \frac{d^2 f}{dx^2} + (2A + 1 - 2Cx) \frac{df}{dx} + \left[\frac{A^2 - m^2/4}{x} + (C^2 - \frac{1}{4})x - 2AC - C - \frac{m - \lambda^2}{4B}\right] f = 0.$$  

When $A, C$ are taken as $A = + \frac{|m|}{2}, C = +1/2$ the previous equation becomes simpler

$$x \frac{d^2 R}{dx^2} + (2A + 1 - x) \frac{dR}{dx} - \left(A + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B}\right) R = 0,$$

which is of (degenerate) hypergeometric type

$$x Y'' + (\gamma - x)Y' - \alpha Y = 0, \quad \alpha = \frac{|m|}{2} + \frac{1}{2} + \frac{m}{2} - \frac{\lambda^2}{4B}; \quad \gamma = |m| + 1.$$  

To obtain polynomials we must impose additional condition $\alpha = -n$; which leads to the following quantization for $\lambda^2$

$$\lambda^2 = 4B \left( n + \frac{1}{2} + \frac{|m| + m}{2} \right).$$  (40)

Taking into account (36) – (37), we have relations

(A) $$(\Delta + (\epsilon^2 - k^2) - M^2 - 2B \sqrt{\epsilon^2 - k^2} \frac{M}{M}) g' = 0, \quad \sqrt{\epsilon^2 - k^2} = \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M},$$

(B) $$(\Delta + (\epsilon^2 - k^2) - M^2 + 2B \sqrt{\epsilon^2 - k^2} \frac{M}{M}) G' = 0, \quad \sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}. $$

For definiteness let us consider $B$ to be positive, which does not affect generality of the analysis. So, to infinite values of $r$ corresponds infinite and positive values of $x$. 

1For definiteness let us consider $B$ to be positive, which does not affect generality of the analysis. So, to infinite values of $r$ corresponds infinite and positive values of $x$. 

7
In fact, here there exist only two different possibilities (and correspondingly two formulae for energy spectrum):

\[
\sqrt{\epsilon^2 - k^2} = \frac{+B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}, \quad q'(r) \neq 0, \ G' = 0 ;
\]

\[
\sqrt{\epsilon^2 - k^2} = \frac{-B + \sqrt{B^2 + M^2(M^2 + \lambda^2)}}{M}, \quad q'(r) = 0, \ G' \neq 0 . \tag{41}
\]

In turn, energy spectrum for the case (31) is given by

\[
\epsilon^2 = M^2 + k^2 + \lambda^2 \tag{42}
\]

Thus, on the base of the use of general covariant formalism in the Petiau – Duffin – Kemmer theory of the vector particle, exact solutions for such a particle are constructed in presence of external homogeneous magnetic field. There are separated three types of linearly independent solutions, and energy spectra are found.

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