Scalar propagator in the pp-wave geometry obtained from $AdS_5 \times S^5$

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Abstract

We compute the propagator for massless and massive scalar fields in the metric of the pp-wave. The retarded propagator for the massless field is found to stay confined to the surface formed by null geodesics. The algebraic form of the massive propagator is found to be related in a simple way to the form of the propagator in flat spacetime.

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1 Introduction.

Recently there has been a surge of interest in the study of string theory on pp-wave backgrounds. A 10-d pp-wave metric (with a certain background 5-form field strength) can be obtained as a limit of $AdS_5 \times S^5$: it is the metric seen by particles that rotate fast around the diameter of the $S^5$\[1\].

String states in such backgrounds were studied in \[2\]. To proceed further and study interactions between such states one needs to know the propagators of the fields that are exchanged between string states. Since we are in Type IIB string theory, the massless bosonic fields that can be exchanged are the dilaton $\Phi$ and axion $C$, the NS-NS and RR 2-form gauge fields $B_{\mu\nu}^{NSNS}$ and $B_{\mu\nu}^{RR}$, the 4-form gauge field $A_{\mu\nu\lambda\rho}$ and the graviton $h_{\mu\nu}$. In this letter we compute the propagator for massless scalar fields like $\Phi$, as well as for massive scalar fields.

We note that the propagator in $AdS$ space is known \[3\], and in principle one can try to extract the propagator in the pp-wave by starting with the propagators in $AdS$ space for different harmonics on the sphere, putting together these propagators in an appropriate way and taking an appropriate limit. This process appears to be cumbersome though, and we find it much easier to work directly with the pp-wave background instead.

Our principal results are the following. We compute the retarded and Feynman propagators for the scalar, both for the massless field and the massive field. The retarded propagator is found to stay confined to the surface formed by the null geodesics. For both the massless and the massive propagators we find algebraic forms that are closely related to the forms found in flat space. The results thus suggest that even for higher spin fields there might be a simple way to guess the pp-wave propagators from flat space computations.

One of our goals in finding the exact form of the propagator was to study the behavior of highly excited string states which can collapse into black holes under their self-gravity. We comment briefly on this problem in the discussion. While this paper was in preparation there appeared \[4\] which studied a similar question using an approximate form of the propagator.

2 Constructing the propagator from solutions of the wave equation

We consider the 10-D metric obtained from the pp-wave limit of $AdS_5 \times S^5$. The line element in pp-wave background is

$$ds^2 = -4dx^+dx^- - r^2(dx^+)^2 + dx_idx_i$$

(1)
where \( x_i, i = 1 \ldots 8 \) are the transverse coordinates and \( r^2 = x_i x_i \) is the distance squared in the transverse coordinates. The scalar laplacian is

\[
\nabla^2 = \frac{1}{\sqrt{-g}} \nabla_a (g^{ab} \sqrt{-g} \nabla_b ) \\
= \frac{r^2}{4} \partial_-^2 - \partial_+ \partial_- + \nabla_8^2
\]

(2)

where \( \nabla_8^2 = \partial_i \partial_i \). The wave-equation for mass \( m \) is

\[
(\nabla^2 - m^2) F = 0
\]

(3)

To find the propagator we will solve the eigenvalue problem for the above operator. Let \( F_\lambda \) be an eigenfunction with eigenvalue \(-\lambda\):

\[
(\nabla^2 - m^2) F_\lambda = -\lambda F_\lambda
\]

(4)

We write

\[
F_\lambda [x^+, x^-, x_i] = \mathcal{N} e^{ik^+ x^+ + ik^- x^-} f[x_i]
\]

(5)

(\( \mathcal{N} \) is a normalization) obtaining

\[
\nabla_8^2 f - \frac{k^2 r^2}{4} f + (k_+ k_- - m^2 + \lambda) f = 0
\]

(6)

This is just the equation for a non-relativistic, spherically symmetric quantum harmonic oscillator in 8 dimensions with frequency

\[
\omega = \frac{|k_-|}{2}
\]

(7)

and total energy

\[
E = \frac{(k_+ k_- - m^2 + \lambda)}{2}
\]

(8)

Writing

\[
f = \prod_i f_i
\]

(9)

we get

\[
\frac{1}{f_i} \frac{d^2 f_i}{dx_i^2} - \omega^2 x_i^2 = -g_i
\]

(10)

with

\[
\sum_{i=1}^{8} g_i = 2E.
\]

(11)

The normalised solutions are

\[
f_i = \left( \frac{\sqrt{\omega}}{2^{n_i} n_i! \sqrt{\pi}} \right)^{\frac{1}{4}} H_{n_i}(\tilde{x}_i) e^{-\frac{1}{2} \tilde{x}_i^2}
\]

(12)
where $\tilde{x}_i = \sqrt{\omega}x_i$, $H_n$ are the Hermite polynomials of order $n$ and $g_i = 2(n_i + \frac{1}{2})\omega$. We can now write the normalized solutions to (4) as

$$F_{\lambda}[x^+, x^-, x_i] = \frac{1}{\sqrt{2\pi}}e^{ik_+x^+}e^{ik_-x^-}\Pi_{i=1}^g\left(\frac{\sqrt{\omega}}{2^{n_i}n_i!}\right)^{\frac{1}{2}}H_{n_i}(\tilde{x}_i)e^{-\frac{1}{2}\tilde{x}_i^2}$$

(13)

with

$$\lambda = -[k_+k_- - |k_-|\Sigma_{i=1}^g(n_i + \frac{1}{2}) - m^2)]$$

(14)

The propagator satisfying

$$(\Delta - m^2)G(x_2, x_1) = \frac{1}{\sqrt{-g}}\delta(x_2 - x_1)$$

(15)

is then given by ($\sqrt{-g} = 2$)

$$G(x_2, x_1) = \sum_{\lambda} F_{\lambda}(x_2)F^{*}_{\lambda}(x_1)$$

$$= \int_{-\infty}^{\infty} \frac{dk_+}{2\pi}e^{ik_+(x_2^+ - \tilde{x}_i^+)}e^{ik_-(x_2^- - \tilde{x}_i^-)}\Sigma_{i=1}^g(k_+k_- - |k_-|\Sigma_{i=1}^g(n_i + \frac{1}{2}) - m^2)$$

(16)

### 3 Simplifying the expression for the propagator

Since we encounter the absolute value of $k_-$ in $\omega$ eq. (17), it is convenient to break up the $k_-$ integral into two parts: a part $(-\infty, 0)$ and a part $(0, \infty)$. We write $n \equiv \Sigma_{i=1}^g n_i$ and $\Delta x^\pm = x_2^\pm - x_1^\pm$. Then

$$G(x_2, x_1) = I_+ + I_-$$

(17)

where

$$I_+ = \Sigma_{i=1}^g \frac{1}{2}\int_0^{\infty} dk_- e^{ik_-\Delta x^-}\int_{-\infty}^{\infty} dk_+ e^{ik_+\Delta x^+}$$

$$= \Pi_{i=1}^g H_{n_i}(\sqrt{\omega}x_{1i})H_{n_i}(\sqrt{\omega}x_{2i})e^{-\frac{1}{2}(x_{1i}^2 + x_{2i}^2)}$$

(18)

and after a change of variables $k_- \to -k_-$ we can write $I_-$ as

$$I_- = -\Sigma_{i=1}^g \frac{1}{2}\int_0^{\infty} dk_- e^{ik_-\Delta x^-}\int_{-\infty}^{\infty} dk_+ e^{ik_+\Delta x^+}$$

$$= \Pi_{i=1}^g H_{n_i}(\sqrt{\omega}x_{1i})H_{n_i}(\sqrt{\omega}x_{2i})e^{-\frac{1}{2}(x_{1i}^2 + x_{2i}^2)}$$

(19)
Consider any point in the pp-wave spacetime and look at the light-cone in the infinitesimal vicinity of this point. First ignore the $x_i$, and look at the light cone in the 2-D space $x^+, x^-$. The two lines forming this cone are $x^+ = 0$, and $x^- = -\frac{x^+}{4x_1 x_2}$. Further, the vector $V^\mu$ which has $V^+ > 0$ (and all other components zero) is timelike in the metric $\Box$; this tells us which of the four sectors marked out by the two null lines is the forward light cone. We find that at least for the infinitesimal vicinity of the starting point a retarded propagator can be defined by requiring that the propagator be nonzero only for $\Delta x^+ > 0$. We adopt this as our definition of retarded propagator (we will return to a more complete discussion of null geodesics later).

Let us begin with $I_+$ and perform the integral over $k_+$. For $\Delta x^+ > 0$ we can close the $k_+$ contour in the upper half plane, so to get a nonvanishing result from each pole we shift the poles to slightly above the real axis. The $k_+$ integral just sets $k_+ = (n + 4) + m^2/k_-$. Let us define

$$z = e^{i\Delta x^+}$$

(20)

For any one choice of the index $i$ the sum in $\Sigma_{\{n_i\}}$ over the Hermite polynomials can be evaluated by using the identity [7]

$$\sum_{n_i=0}^{\infty} \frac{H_{n_i}(\sqrt{\omega x_1})H_{n_i}(\sqrt{\omega x_2})(\frac{2}{\sqrt{\omega}})^{n_i}}{n_i!} = \frac{1}{\sqrt{1 - z^2}} e^{\frac{2x_1 x_2 x_1 x_2^2 - x_1^2 - x_2^2}{1 - z^2}}$$

(21)

We need to take a product over 8 values of the index $i$. We then find

$$I_+ = \frac{2\pi i}{(2\pi)^6} \frac{z^4}{(1 - z^2)^4} \frac{\Theta(\Delta x^+)}{2} \int_0^\infty dk_- k_3^3 e^{-ik_- Y^2} e^{i m^2 \Delta x^+}$$

(22)

where

$$Y^2 = -\Delta x^- + \frac{x_1^2 + x_2^2}{4i} + \frac{x_1^2 z^2 + x_2^2 z^2 - 2x_1 x_2 z}{2i(1 - z^2)}$$

Before proceeding further with the evaluation of $I_+$ we take note of the regularizations required to define $I_+$. It is a general feature of Green’s functions that for sufficiently high spacetime dimensionality the Green’s function is not square integrable, and so the Fourier transform need not be given by a convergent integral – a damping factor must be introduced to cut off high frequencies. Lifting this cutoff at the end brings the Green’s function back to its required form as a singular ‘distribution’.

The damping at high frequencies can be provided by introducing into $I_+, I_-$ the factors:

$$e^{-\epsilon|k_-|} e^{-\epsilon n_i}$$

(23)

The first of these factors leads to the change in (22)

$$Y^2 \rightarrow Y^2 - i\epsilon$$

(24)

The second factor in (23) leads to the change that in the identity (21) we have the replacement

$$z \rightarrow ze^{-\epsilon'}$$

(25)
With these regulations we find

$$I_+ = \frac{\Theta(\Delta x^+) 2\pi i}{2} \frac{1}{(2\pi)^6 (2\sin(\Delta x^+ + i\epsilon'))^4} \int_0^\infty dk_- k_-^3 e^{-i(k_- (Y^2 - i\epsilon') - \frac{m^2}{2} \Delta x^+)}$$

(26)

and

$$Y^2 = \frac{\Phi}{4 \sin(\Delta x^+ + i\epsilon)}$$

(27)

where

$$\Phi = -4\Delta x \sin(\Delta x^+ + i\epsilon') - 2\sin^2 \frac{(\Delta x^+ + i\epsilon')}{2}(x_1^2 + x_2^2) + (x_1 - x_2)^2$$

(28)

is an expression that goes over to the invariant distance squared for small separations.

Proceeding similarly for $I_-$, we require again that the contribution be nonzero only for $\Delta x^+ > 0$, and we get

$$I_- = -\frac{\Theta(\Delta x^+) 2\pi i}{2} \frac{1}{(2\pi)^6 (2\sin(\Delta x^+ - i\epsilon'))^4} \int_0^\infty dk_- k_-^3 e^{i(k_- (Y^2 + i\epsilon) - \frac{m^2}{2} \Delta x^+)}$$

(29)

Note that $I_-$ is precisely the complex conjugate of $I_+$.

4 Closed form expressions for the propagators

4.1 The massless propagator

Let us first work out the case $m^2 = 0$. From (26) we find that

$$I_+ = \frac{\Theta(\Delta x^+) 2\pi i}{2} \frac{1}{(2\pi)^6 (2\sin(\Delta x^+ + i\epsilon'))^4} \frac{6}{(Y^2 - i\epsilon')^4} = 3\Theta(\Delta x^+) \frac{2\pi i}{(2\pi)^6 (\Phi - i\epsilon)^4}$$

(30)

where

$$\tilde{\epsilon} = (4\sin \Delta x)\epsilon$$

(31)

Similarly,

$$I_+ + I_- = 3\Theta(\Delta x^+) \frac{2\pi i}{(2\pi)^6} \left[ \frac{2^4}{(\Phi - i\epsilon)^4} - \frac{2^4}{(\Phi + i\epsilon)^4} \right]$$

(32)

Noting that

$$\frac{1}{(\Phi - i\epsilon)} - \frac{1}{(\Phi + i\epsilon)} = 2\pi i \text{ (sign } \tilde{\epsilon} \text{)} \delta(\Phi)$$

(33)

we get

$$G_{ret}(x_2, x_1) = \Theta(\Delta x^+) \frac{1}{2\pi^4} \frac{\sin \Delta x^+}{\sin \Delta x^+} \delta''(\Phi)$$

(34)

where the derivatives of the delta function are with respect to its argument $\Phi$.

It is readily verified that this propagator has the correct normalization to agree at short distances with the retarded propagator satisfying (15) (with $m^2 = 0$).
4.2 The massive propagator

Let us now consider the expression (22) for $I_+$ for the massive scalar field. Performing the integral we find

\[ I_+ = -\frac{\pi^2}{16} \Theta(\Delta x^+) \frac{1}{(2\pi)^6} \frac{m^4(\Delta x^+)^2}{(Y^2 - i\epsilon)^2 \sin^4(\Delta x^+ + i\epsilon')} H^{(1)}_{\Delta x^+} \sqrt{-2m^2(Y^2 - i\epsilon)\sqrt{\Delta x^+}} \] (35)

where $H^{(1)}$ is the Hankel function of the first kind. For $I_-$ we get the complex conjugate of $I_+$

\[ I_- = \frac{\pi^2}{16} \Theta(\Delta x^-) \frac{1}{(2\pi)^6} \frac{m^4(\Delta x^-)^2}{(Y^2 + i\epsilon)^2 \sin^4(\Delta x^- - i\epsilon')} H^{(1)}_{\Delta x^-} \sqrt{-2m^2(Y^2 + i\epsilon)\sqrt{\Delta x^-}} \] (36)

Note that the Hankel function is $H^{(1)}_{\Delta x^+} = J^{(1)}_{\Delta x^+} + iN^{(1)}_{\Delta x^+}$. The function $N^{(1)}(z)$ has a logarithmic branch cut at $z = 0$, so in principle we must decide which branch of the Hankel function we are on. With the given regulations in (22) the integral is well defined for all values of $\Delta x^+ > 0, m^2 > 0$, which suggests that there should be no ambiguity in the choice of branch, and that we should be on the principal branch which is defined by the integral.

To observe that this is indeed what happens, we look at the behavior of the argument of the Hankel function when $\Delta x^+$ passes through $n\pi$ (which causes $\sin \Delta x^+$ to vanish) or when we enter or leave the light cone. We find that with the given regulations the argument of the Hankel function in $I_+$ stays (for all values of the coordinates) in the first quadrant of the complex plane. Since the argument does not thus circle the origin, we stay on the same branch of the Hankel function. This branch is given by continuing the value of $H^{(1)}$ for real positive arguments to the first quadrant of the complex plane. A similar analysis holds for $I_-$, and so no ambiguity exists in the value of the retarded Green’s function which is

\[ G_{\text{ret}} = I_+ + I_- \] (37)

with $I_+, I_-$ given by (35), (36).

We observe that as expected the choice of branch indicated by the regularizations does not depend on the ratio of the two regularizations $\epsilon, \epsilon'$, and from now on we just write $\epsilon$ for both.

4.3 The Feynman propagator

To define the analogue of the flat space Feynman propagator we require that positive frequencies in the variable $x^+$ travel forward in $x^+$ and negative frequencies travel backwards in $x^+$. To make $I_+$ nonzero at negative $\Delta x^+$ we must shift the poles in $k^+$ integral in (18) into the lower half plane. We then get instead of (35) the result

\[ I_+ = \frac{\pi^2}{16} \Theta(-\Delta x^+) \frac{1}{(2\pi)^6} \frac{m^4(\Delta x^+)^2}{(Y^2 - i\epsilon)^2 \sin^4(\Delta x^+ + i\epsilon')} H^{(1)}_{-\Delta x^-} \sqrt{-2m^2(Y^2 - i\epsilon)\sqrt{\Delta x^-}} \] (38)
The Feynman propagator is then

\[ G_F(x_2, x_1) = \tilde{I}_+ + I_- \]  \hspace{1cm} (39)

For completeness we record the result of shifting poles in \( I_- \) such that we get a contribution only for \( \Delta x^+ < 0 \):

\[ \tilde{I}_- = -\frac{\pi^2}{16} \Theta(-\Delta x^+) \frac{1}{(2\pi)^6} \frac{m^4(\Delta x^+)^2}{(Y^2 + i\epsilon)^2 \sin^4(\Delta x^+ - i\epsilon)} H^{(1)}_{-1/4}(-2m\sqrt{-(Y^2 + i\epsilon)\sqrt{\Delta x^+}}) \] \hspace{1cm} (40)

5 Similarities with flat space propagators

5.1 The massless propagator

If spacetime is flat and its total dimension is even then the retarded propagator \( G(x_2, x_1) \) for massless fields stays confined to the null cone, which is the cone generated by the tracks of null geodesics starting at \( x_1 \). We will now observe that a similar property holds for the retarded massless propagator in the pp-wave.

Let us first compute the geodesics in the pp-wave metric. If \( \tau \) is an affine parameter along the geodesic, then the geodesic equations are

\[ \frac{d^2x^+}{d\tau^2} = 0 \] \hspace{1cm} (41)

\[ \frac{d^2x^-}{d\tau^2} + \dot{x}^+ \sum_{i=1}^8 x_i \dot{x}_i = 0 \] \hspace{1cm} (42)

\[ \frac{d^2x_i}{d\tau^2} + (\dot{x}^+)^2 x_i = 0 \] \hspace{1cm} (43)

where a dot denotes the derivative with respect to \( \tau \). The first equation gives

\[ x^+ = \alpha \tau + \beta \] \hspace{1cm} (44)

If \( \alpha \neq 0 \), then we can scale \( \tau \) to set \( x^+ = \tau \). (For massive geodesics \( \tau \) will not be the proper distance along the geodesic but rather a multiple of the proper distance.) For the initial conditions

\[ x^-(\tau = 0) = x_1^- \]
\[ \frac{dx^-}{d\tau}(\tau = 0) = \dot{x}_1^- \]
\[ x_i(\tau = 0) = x_{1i} \]
\[ \frac{dx_i}{d\tau} = \dot{x}_{1i} \] \hspace{1cm} (45)
we get the solution
\[ x^-(\tau) - x_1^- = \frac{x_1^2 + \bar{x}_1^2}{8} \sin 2\tau + \frac{\dot{x}_1 x_1}{4} \cos 2\tau + C\tau - \frac{\dot{x}_1 x_1}{4} \] (46)
\[ \bar{x}(\tau) = \dot{x}_1 \sin \tau + \bar{x}_1 \cos \tau \] (47)
where \( \bar{x} \) denotes the coordinates in the transverse 8 dimensional space, and

\[ C = \dot{x}_1 - \frac{x_1^2 - \bar{x}_1^2}{4} \] (48)

For null geodesics the initial conditions imply that \( C = 0 \). A little algebra then shows that along such geodesics
\[ \Phi(x, x_1) = 0 \] (49)
so that the retarded propagator (31) is confined to the surface formed by the null geodesics.\(^4\)

In flat space the propagator is a function of the invariant distance squared
\[ s_{\text{flat}}^2 = -4\Delta x^+ \Delta x^+ + (\bar{x}_1 - \bar{x}_2)^2. \] (50)
We have
\[ G_{\text{ret}}^{\text{flat}}(x_2, x_1) = \Theta(t) \frac{1}{2(\pi)^4} \delta'''(-s^2) \] (51)
We thus see that the pp-wave result (34) is related to the flat space result by the replacement \((-s_{\text{flat}}^2) \to \Phi\).

5.2 The massive propagator

For \( \Delta x^+ \ll 1 \) we can replace \( \sin \Delta x^+ \to \Delta x^+ \) and the propagator reduces to the propagator in flat space, as expected. What is interesting is that the algebraic form of the massive propagator in the pp-wave is closely related to the form of the propagator in flat space.

For flat space
\[ I_{\text{flat}}^{\text{flat}} = -\frac{\pi^2}{16} \Theta(\Delta x^+) \frac{1}{(2\pi)^6} \frac{16m^4}{(-s_{\text{flat}}^2)^2} H_{-4}^{(1)}(m\sqrt{-s_{\text{flat}}^2}) \] (52)
with similar expressions for the other functions involved in the propagators. Now note that for a timelike geodesic
\[ \frac{ds^2}{(dx^+)^2} = -4C \] (53)
\(^4\)If \( \alpha = 0 \) in (44) then we have the solution \( x^+ = x_1^+, x^- = \gamma \tau + \delta, \bar{x} = \bar{x}_1 \). These geodesics cannot be timelike, and the only null geodesic is (after scaling \( \tau \)) \( x^+ = \text{const.}, x^- = \gamma \tau, \bar{x} = \bar{x}_1 \). This geodesic is a limiting case of those obtained for \( \alpha \neq 0 \), and so does not affect the conclusion above regarding the surface formed by the null geodesics.
where the constant $C$ can be written in terms of the initial and final points as

$$C = -\frac{\Phi}{4\Delta x^+ \sin \Delta x^+}$$

(54)

Thus the distance measured along a timelike geodesic will be

$$\int_{x_1}^{x_2} \sqrt{-ds^2} = \sqrt{-\Phi \frac{\Delta x^+}{\sin \Delta x^+}} = 2\sqrt{-Y^2} \sqrt{\Delta x^+} \equiv \sqrt{-\sigma^2}$$

(55)

We observe that the argument of the Hankel function in $I_+$ is $m \sqrt{-\sigma^2}$. Overall we can write

$$I_+ = -\frac{\pi^2}{16} \Theta(\Delta x^+) \frac{1}{(2\pi)^6} \left(\frac{\sin \Delta x^+}{\Delta x^+}\right)^{-4} \frac{16m^4}{(-\sigma^2)^2} H_{-1}(m \sqrt{-\sigma^2})$$

(56)

which differs from (52) only by the factor $(\sin \Delta x^+)^4$ and the replacement $s_{flat}^2 \rightarrow \sigma^2$.

Let us now observe the relation to the flat space propagator in another way. In the massless case we had found that the relation to the flat space propagator was directly seen when using the variable $\Phi$. Define

$$\tilde{m}^2 = m^2 \frac{\Delta x^+}{\sin \Delta x^+}$$

(57)

Then we observe that

$$I_+ = -\frac{\pi^2}{16} \Theta(\Delta x^+) \frac{1}{(2\pi)^6} \frac{16\tilde{m}^4}{(-\Phi)^2} H_{-1}(\tilde{m} \sqrt{-\Phi})$$

(58)

which differs from (52) only by the replacements $m \rightarrow \tilde{m}$ and $s_{flat}^2 \rightarrow \Phi$.

6 Discussion

We have obtained propagators for the massless and massive scalar fields in the pp-wave geometry obtained from $AdS_5 \times S^5$. This is the geometry seen by string states moving fast around the diameter of the $S^5$, while staying in the vicinity of the origin in global $AdS_5$. We have noted several relations between these propagators and the corresponding flat space expressions. These relations should help us to guess the form of propagators for higher spin fields. The propagators for other spaces $AdS_p \times S^q$ can be obtained by direct inspection of our results here – the different dimension leads to a change in the number of harmonic oscillators obtained from the transverse coordinates $x_i$, but there is no other essential change in the computation.

The space $AdS_5 \times S^5$ is locally conformally flat, so for massless fields one should be able to recover the propagator in the pp-wave from the flat space result.\(^5\) (We have not

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\(^5\)We thank D. Berenstein for this comment.
taken a conformally coupled scalar, but the curvature scalar $R$ of the geometry vanishes so we do not have to consider a term $R\phi^2$ in the Lagrangian for a scalar field $\phi$.) Working with such conformal maps may make it more difficult however to follow the behavior of the propagator through the singularities at $\Delta x^+ = n\pi$, since the conformal map is local rather than global. The massive propagator has a length scale set by the mass, and so cannot be obtained by a conformal mapping to flat space.

The pp-wave has the property that light rays can reach the boundary and return in a finite time, a property that is also shared by $AdS$ spaces. We thus need to have a boundary condition at infinity which reflects the energy of the waves back into the geometry. The fact that we have used normalizable wavefunctions in constructing the propagator implies that we have Dirichlet boundary conditions at infinity.

One of the interesting features of the propagators is the appearance of the function $\sin \Delta x^+$, which vanishes whenever $\Delta x^+ = n\pi$. Let us write

$$x^+ = \frac{1}{2}(t + \psi), \quad x^- = \frac{1}{2}(t - \psi)$$

(59)

The pp-wave geometry is infinitely extended in the coordinate $\psi$ along the wave. However when we consider this geometry as a part of $AdS_5 \times S^5$ then we find that $\psi$ is an angular variable on a circle. The point $x^+ = \pi, x^- = x_i = 0$ is at $\psi = \pi$, halfway around the $S^5$. The point $x^+ = 2\pi, x^- = x_i = 0$ is the same as the point $x^+ = 0, x^- = x_i = 0$. Null geodesics moving on the $S^5$ cross the line $x_i = 0$ twice in each revolution, at the points $x^+ = 0, x^+ = \pi$.

Finally we comment on the physical problem that we would like to address with the help of this propagator (and similar propagators for other fields). In [2] string states were considered that move fast around the $S^5$. If we consider higher and higher excitations of this string, then there can come a point where the string collapses to a black hole under its self-gravitation. Such an effect was studied for strings in flat space in [5]. Let us note here some aspects of this problem in the pp-wave geometry.

In flat space the size of the free string is given by a ‘random walk’ made out of string bits. In the pp-wave geometry there is a potential that drives the string to the center $x_i = 0$. There is no potential confining the direction $\psi$ along the wave, so the string can extend into a cylindrical shaped object lying along the central axis of the pp-wave.

Let us assume such a geometry for the string state, and consider the effects of self-interaction created by a field with a propagator that behaves like that of the massless scalar field considered here. (We expect the dilaton, 2-form gauge field and the graviton to have propagators that are similar except for their index structure.) Note that the massless propagator is a function of the variable $\Phi$, and consider the propagator between points that are near the axis $x_i = 0$. The fact that $\Phi \sim -4\Delta x^- \sin \Delta x^+$ implies that the retarded Green’s function behaves like an inverse power of $\sin x^+$ instead of an inverse power of $x^+$. Now consider applying such a propagator to find the self-interaction of a time independent distribution. We find a ‘resonant interaction’ between points that are separated
by $\Delta \psi = n\pi$. Such an interaction permits solutions that are independent of the coordinate $\psi$, but it also suggests that for suitable interactions one may get lower energy configurations that are periodic under $\psi \rightarrow \psi + n\pi$ rather than independent of $\psi$. We hope to return to a more detailed discussion of this problem elsewhere.

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