RELAXATION AND 3D-2D PASSAGE WITH DETERMINANT TYPE CONSTRAINTS: AN OUTLINE

OMAR ANZA HAFSA AND JEAN-PHILIPPE MANDALLENA

Abstract. We outline our work (see [1, 2, 3, 4]) on relaxation and 3d-2d passage with determinant type constraints. Some open questions are addressed. This outline-paper comes as a companion to [5].

Contents

1. Relaxation with determinant type constraints 1
   1.1. Statement of the problem 1
   1.2. Representation of $\mathbf{Q}W$ and $\mathbf{T}$: finite case 2
   1.3. Representation of $\mathbf{Q}W$: non-finite case 3
   1.4. Representation of $\mathbf{T}$: non-finite case 3
   1.5. Application 1: “non-zero-Cross Product Constraint” 6
   1.6. Application 2: “weak-Determinant Constraint” 6
   1.7. From $p$-ample to non-$p$-ample case 7
2. 3d-2d passage with determinant type constraints 7
   2.1. Statement of the problem 7
   2.2. The $\Gamma(\pi)$-convergence 7
   2.3. $\Gamma(\pi)$-convergence of $I_\varepsilon$: finite case 8
   2.4. $\Gamma(\pi)$-convergence of $I_\varepsilon$: “weak-Determinant Constraint” 9
   2.5. $\Gamma(\pi)$-convergence of $I_\varepsilon$: “strong-Determinant Constraint” 10
   References 11

1. Relaxation with determinant type constraints

1.1. Statement of the problem. Let $m, N \in \mathbb{N}$ (with $\min\{m, N\} > 1$), let $p > 1$ and let $W : \mathbb{M}^{m \times N} \to [0, +\infty]$ be Borel measurable and $p$-coercive, i.e.,

$$\exists C > 0 \forall F \in \mathbb{M}^{m \times N} \ W(F) \geq C|F|^p,$$

where $\mathbb{M}^{m \times N}$ denotes the space of real $m \times N$ matrices. Define the functional $I : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty]$ by

$$I(\phi) := \int_\Omega W(\nabla \phi(x))dx$$
where $\Omega \subset \mathbb{R}^N$ is a bounded open set, and consider $\mathbf{T} : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty]$ (the relaxed functional of $I$) given by

$\mathbf{T}(\phi) := \inf \left\{ \liminf_{n \to +\infty} I(\phi_n) : \phi_n \overset{L^p}{\rightharpoonup} \phi \right\}.$

Denote the quasiconvex envelope of $W$ by $QW : \mathbb{M}^{m \times N} \to [0, +\infty]$. The problem of the relaxation is the following:

$(P_1)$ prove (or disprove) that

$$\forall \phi \in W^{1,p}(\Omega; \mathbb{R}^m) \quad T(\phi) = \int_{\Omega} QW(\nabla \phi(x)) dx$$

and find a representation formula for $QW$.

At the beginning of the eighties, Dacorogna answered to $(P_1)$ in the case where $W$ is “finite and without singularities” (see §1.2). Recently, we extended the Dacorogna theorem as Theorem A and Theorem B (see §1.3 and §1.4) and we showed that these theorems can be used to deal with $(P_1)$ under the “weak-Determinant Constraint”, i.e., when $m = N$ and $W : \mathbb{M}^{N \times N} \to [0, +\infty]$ is compatible with the following two conditions:

$(w-DC) \begin{cases} W(F) = +\infty \iff -\delta \leq \det F \leq 0 \text{ with } \delta \geq 0 \text{ (possibly very large)} \\ W(F) \to +\infty \text{ as } \det F \to 0^+ \end{cases}$

(see §1.6). However, the results of this section do not allow to treat $(P_1)$ under the “strong-Determinant Constraint”, i.e., when $m = N$ and $W : \mathbb{M}^{N \times N} \to [0, +\infty]$ is compatible with the two basic conditions of nonlinear elasticity:

$(s-DC) \begin{cases} W(F) = +\infty \iff \det F \leq 0 \quad \text{(non-interpenetration of matter)} \\ W(F) \to +\infty \text{ as } \det F \to 0^+ \quad \text{(necessity of an infinite amount of energy to compress a finite volume into zero volume)} \end{cases}$

(see §1.7).

### 1.2. Representation of $QW$ and $\mathbf{T}$: finite case.

Let $Z_{\infty}W, ZW : \mathbb{M}^{m \times N} \to [0, +\infty]$ be respectively defined by:

- $Z_{\infty}W(F) := \inf \left\{ \int_{Y} W(F + \nabla \varphi(y)) dy : \varphi \in W^{1,\infty}_0(Y; \mathbb{R}^m) \right\};$
- $ZW(F) := \inf \left\{ \int_{Y} W(F + \nabla \varphi(y)) dy : \varphi \in \text{Aff}_0(Y; \mathbb{R}^m) \right\},$

where $Y := (0,1)^N$, $W^{1,\infty}_0(Y; \mathbb{R}^m) := \{ \varphi \in W^{1,\infty}(Y; \mathbb{R}^m) : \varphi = 0 \text{ on } \partial Y \}$ and $\text{Aff}_0(Y; \mathbb{R}^m) := \{ \varphi \in \text{Aff}(Y; \mathbb{R}^m) : \varphi = 0 \text{ on } \partial Y \}$.

**Remark.** One always has $W \geq ZW \geq Z_{\infty}W \geq QW$.

**Theorem** (Dacorogna [12] 1982).

(a) Representation of $QW$: if $W$ is continuous and finite then

$$QW = ZW = Z_{\infty}W.$$  

(b) Integral representation of $\mathbf{T}$: if $W$ is continuous and

$$\exists c > 0 \quad \forall F \in \mathbb{M}^{m \times N} \quad W(F) \leq c(1 + |F|^p)$$
1.3. Representation of QW: non-finite case.

The part (a) of the Dacorogna theorem can be extended as follows.

**Theorem A** (see [2, 3, 5]).

- If $Z_{\infty}W$ is finite then $QW = Z_{\infty}W$.
- If $ZW$ is finite then $QW = ZW = Z_{\infty}W$.

**Proof.** We need (the two last assertions, the first one being used at the end of §1.3, of) the following result.

**Theorem** (Fonseca [16] 1988).

1. If $Z_{\infty}W$ (resp. $ZW$) is finite then $Z_{\infty}W$ (resp. $ZW$) is rank-one convex.
2. If $Z_{\infty}W$ (resp. $ZW$) is finite then $Z_{\infty}W$ (resp. $ZW$) is continuous.
3. $Z_{\infty}W \leq ZZ_{\infty}W$ and $ZZW = ZW$.

One always has $W \geq ZW \geq Z_{\infty}W \geq QW$. Hence:

(i) $QZ_{\infty}W = QW \leq Z_{\infty}W$;
(ii) $QZW = QZ_{\infty}W = QW$.

- If $Z_{\infty}W$ is finite then $Z_{\infty}W$ is continuous by the property (2) of Fonseca. From the first part of the Dacorogna theorem it follows that $QZ_{\infty}W = ZZ_{\infty}W$. But $Z_{\infty}W \leq ZZ_{\infty}W$ by the property (3) of Fonseca, and so $QW = Z_{\infty}W$ by using (i).

- If $ZW$ is finite then also is $Z_{\infty}W$. Hence $QW = Z_{\infty}W$ by the previous reasoning. On the other hand, $ZW$ is continuous by the property (2) of Fonseca. From the first part of the Dacorogna theorem it follows that $QZW = ZZW$. But $ZZW = ZW$ by the property (3) of Fonseca, and so $QW = ZW$ by using (ii). □

**Question.** Prove (or disprove) that if $Z_{\infty}W$ is finite, also is $ZW$.

1.4. Representation of $T$: non-finite case.

The part (b) of the Dacorogna theorem can be extended as follows.

**Theorem B** (see [2, 3, 5]).

- If $\exists c > 0 \ \forall F \in M^{m \times N} \ Z_{\infty}W(F) \leq c(1 + |F|^p)$ then
  \[ \forall \phi \in W^{1,p}(\Omega; \mathbb{R}^m) \ \ T(\phi) = \int_{\Omega} QW(\nabla \phi(x))dx. \]

- If $\exists c > 0 \ \forall F \in M^{m \times N} \ ZW(F) \leq c(1 + |F|^p)$ then
  \[ \forall \phi \in W^{1,p}(\Omega; \mathbb{R}^m) \ \ T(\phi) = T_{aff}(\phi) = \int_{\Omega} QW(\nabla \phi(x))dx \]
  with $T_{aff} : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty]$ defined by
  \[ T_{aff}(\phi) := \inf \left\{ \liminf_{n \to +\infty} I(\phi_n) : \text{Aff}(\Omega; \mathbb{R}^m) \ni \phi_n \overset{L^p}{\to} \phi \right\}. \]

**Outline of the proof.** Let $Z_{\infty}I, T_{\infty}I, T_{\infty}aff : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty]$ be respectively defined by:

- $Z_{\infty}I(\phi) := \int_{\Omega} Z_{\infty}W(\nabla \phi(x))dx$;
- $T_{\infty}I(\phi) := \inf \left\{ \liminf_{n \to +\infty} Z_{\infty}I(\phi_n) : \phi_n \overset{L^p}{\to} \phi \right\}$;
we deduce that the integrands (that we will call the class of
$p$-ample integrands) as follows:
\[
W \text{ is } p\text{-ample } \iff \exists c > 0 \ \forall F \in \mathbb{M}^{m \times N} \ Z_\infty W(F) \leq c(1 + |F|^p).
\]
Thus, Theorems A and B can be summarized as follows.

**Theorem A-B.** If $W$ is $p$-ample then
\[
\forall \phi \in W^{1,p}(\Omega; \mathbb{R}^m) \quad T(\phi) = \int_\Omega QW(\nabla \phi(x)) dx \text{ and } QW = Z_\infty W.
\]

\[^1\text{We use the term “p-ample” because of some analogies with the concept (developed in differential geometry by Gromov) of amplitude of a differential relation (see [13] for more details).}\]
**Question.** Prove (or disprove) that $W$ is $p$-ample if and only if $QW$ is of $p$-polynomial growth.

An analogue result of Theorem B was proved by Ben Belgacem (who is in fact the first that obtained an integral representation for $T$ in the non-finite case). Let \( \{R_i W\}_{i \in \mathbb{N}} \) be defined by \( R_0 W := W \) and for each \( i \in \mathbb{N}^* \) and each \( F \in \mathbb{M}^{m \times N} \),

\[
R_{i+1} W(F) := \inf_{a \in \mathbb{R}^N \atop b \in \mathbb{R}^m \atop t \in [0,1]} \{ (1 - t)R_i W(F - ta \otimes b) + tR_i W(F + (1 - t)a \otimes b) \}.
\]

By Kohn et Strang (see [19]) we have \( R_{i+1} W \leq R_i W \) for all \( i \in \mathbb{N} \) and \( RW = \inf_{i \geq 0} R_i W \), where \( RW \) denotes the rank-one convex envelope of \( W \). The Ben Belgacem theorem can be stated as follows.

**Theorem** (Ben Belgacem [8, 10] 1996). Assume that:

- (BB1) \( \Omega_W := \text{int} \left\{ F \in \mathbb{M}^{m \times N} : \forall i \in \mathbb{N} \ Z R_i W(F) \leq R_{i+1} W(F) \right\} \) is dense in \( \mathbb{M}^{m \times N} \);
- (BB2) \( \forall i \in \mathbb{N}^* \ \forall F \in \mathbb{M}^{m \times N} \ \forall \{F_n\}_n \subset \Omega_W \)

\[
F_n \to F \Rightarrow R_i W(F) \geq \limsup_{n \to +\infty} R_i W(F_n);
\]

\( \exists c > 0 \ \forall F \in \mathbb{M}^{m \times N} \ \mathcal{Z} W(F) \leq c(1 + |F|^p) \).

Then

\[
\forall \phi \in W^{1,p}(\Omega; \mathbb{R}^m) \quad T(\phi) = \int_{\Omega} Q W(\nabla \phi(x)) dx.
\]

Generally speaking, as rank-one convexity and quasiconvexity do not coincide, Theorem B and the Ben Belgacem theorem are not identical. However, we have

**Lemma.** If either \( \mathcal{Z}_\infty W \) or \( \mathcal{Z} W \) is finite then \( QR W = Q W \).

**Proof.** If \( \mathcal{Z}_\infty W \) (resp. \( \mathcal{Z} W \)) is finite then \( \mathcal{Z}_\infty W \) (resp. \( \mathcal{Z} W \)) is rank-one convex by the property (1) of Fonseca. Consequently \( \mathcal{Z}_\infty W \leq RW \) (resp. \( \mathcal{Z} W \leq RW \)) and (Theorem B' below follows by applying Theorem B). Thus, we have \( \mathcal{Z}_\infty W \leq RW \leq W \) (resp. \( \mathcal{Z}_\infty W \leq RW \leq W \)), hence \( Q \mathcal{Z}_\infty W \leq QR W \leq Q W \) (resp. \( Q \mathcal{Z} W \leq QR W \leq Q W \)) and so \( QR W = Q W \) since one always has \( Q \mathcal{Z}_\infty W = Q W \) (resp. \( Q \mathcal{Z} W = Q W \)).

**Theorem B'.** Assume that \( \exists c > 0 \ \forall F \in \mathbb{M}^{m \times N} \ \mathcal{Z} W(F) \leq c(1 + |F|^p) \). Then:

- if \( \mathcal{Z}_\infty W \) is finite then

\[
\forall \phi \in W^{1,p}(\Omega; \mathbb{R}^m) \quad T(\phi) = \int_{\Omega} Q W(\nabla \phi(x)) dx;
\]

- if \( \mathcal{Z} W \) is finite then

\[
\forall \phi \in W^{1,p}(\Omega; \mathbb{R}^m) \quad T(\phi) = T_{\text{aff}}(\phi) = \int_{\Omega} Q W(\nabla \phi(x)) dx.
\]

**Question.** Prove (or disprove) that if (BB1) and (BB2) hold then \( \mathcal{Z} W \) is finite.
1.5. Application 1: “non-zero-Cross Product Constraint”.  
Consider $W_0 : \mathbb{M}^{3 \times 2} \to [0, +\infty]$ Borel measurable and $p$-coercive and the following condition  

$$(\ast\CD) \quad \exists \alpha, \beta > 0 \forall \xi = (\xi_1 | \xi_2) \in \mathbb{M}^{3 \times 2} \ (|\xi_1 \wedge \xi_2| \geq \alpha \Rightarrow W_0(\xi) \leq \beta (1 + |\xi|^p))$$

with $\xi_1 \wedge \xi_2$ denoting the cross product of vectors $\xi_1, \xi_2 \in \mathbb{R}^3$. When $W_0$ satisfies $(\ast\CD)$ it is compatible with the “non-zero-Cross Product Constraint”, i.e., with the following two conditions:  

$$\begin{align*}
(W) & \quad \text{it is compatible with the “non-zero-Cross Product Constraint”, i.e., with the condition } \\
\forall \xi_1, \xi_2 \in \mathbb{R}^3, \quad \text{then } W_0(\xi_1 | \xi_2) = +\infty \iff |\xi_1 \wedge \xi_2| = 0 \\
\forall \xi_1, \xi_2 \in \mathbb{R}^3, \quad \text{as } |\xi_2 \wedge \xi_2| \to 0.
\end{align*}$$

The interest of considering $(\ast\CD)$ comes from the 3d-2d problem (see §2): if $W$ is compatible with (s-DC) then $W_0$ given by $W_0(\xi) := \inf_{\zeta \in \mathbb{R}^3} W(\xi | \zeta)$ is compatible with $(\ast\CD)$. One can prove that  

$$(\ast\CD) \Rightarrow \exists c > 0 \forall F \in \mathbb{M}^{3 \times 2} \ ZW(F) \leq c(1 + |F|^p)$$

(see [2, 3, 5]) which roughly means that the “weak-Determinant Constraint” is $p$-ample. Applying Theorem B we obtain  

**Corollary 1.** If $W_0$ satisfies $(\ast\CD)$ then  

$$\forall \psi \in W^{1,p}(\Omega; \mathbb{R}^3) \quad \overline{T}(\psi) = T_{\text{aff}}(\psi) = \int_{\Omega} QW_0(\nabla \psi(x)) dx.$$  

1.6. Application 2: “weak-Determinant Constraint”. 

The following condition on $W$ is compatible with (w-DC).  

$$(\ast\DC) \quad \exists \alpha, \beta > 0 \forall F \in \mathbb{M}^{N \times N} \ (|\det F| \geq \alpha \Rightarrow W(F) \leq \beta(1 + |F|^p)).$$

One can prove that  

$$(\ast\DC) \Rightarrow \exists c > 0 \forall F \in \mathbb{M}^{N \times N} \ ZW(F) \leq c(1 + |F|^p)$$

(see [3, 5]) which roughly means that the “weak-Determinant Constraint” is $p$-ample. Applying Theorem B we obtain  

**Corollary 2.** If $W$ satisfies $(\ast\DC)$ then  

$$\forall \phi \in W^{1,p}(\Omega; \mathbb{R}^N) \quad \overline{T}(\phi) = T_{\text{aff}}(\phi) = \int_{\Omega} QW(\nabla \phi(x)) dx.$$  

**Proof of a part of Corollary 2.** Taking the first part of Theorem B’ into account, it suffices to verify the following two points:  

- $(\ast\DC) \Rightarrow \exists c > 0 \forall F \in \mathbb{M}^{N \times N} \ R(W(F)) \leq c(1 + |F|^p);$  
- $(\ast\DC) \Rightarrow \exists \nu \in \mathbb{R}^N \ |\nu| < +\infty,$

which will give us the desired integral representation for $\overline{T}$. The first point is due to a lemma by Ben Belgacem (see [3], see also [5]). For the second point, it is obvious that $\nu \in \mathbb{R}^N < +\infty$ for all $F \in \mathbb{M}^{N \times N}$ with $|\det F| \geq \alpha$. On the other hand, we have  

**Lemma** (Dacorogna-Ribeiro [13] 2004, see also [11]),  

$$\forall F \in \mathbb{M}^{N \times N} \ (|\det F| < \alpha \Rightarrow \exists \varphi \in W^{1,\infty}(Y; \mathbb{R}^N) \ |\det(F + \nabla \varphi)| = \alpha \ p.p. \ dans \ Y).$$

Hence, if $F \in \mathbb{M}^{N \times N}$ is such that $|\det F| < \alpha$ then $\exists \omega \in W^{1,\infty}(Y; \mathbb{R}^N)$ given by the lemma above, and so $Z_{\omega}W(F) \leq 2\beta(1 + |F|^p + \|\nabla \varphi\|^p_{L^p}) < +\infty.$
1.7. From \( p \)-ample to non-\( p \)-ample case.

Because of the following theorem, none of the theorems of this section can be directly used for dealing with \((P_1)\) under (s-DC).

**Theorem** (Fonseca [16] 1988),

If \( W \) satisfies (s-DC) then:

(F1) \( \mathcal{Q}W \) is rank-one convex;

(F2) \( \mathcal{Q}W(F) = +\infty \) if and only if \( \det F \leq 0 \) and \( \mathcal{Q}W(F) \to +\infty \) as \( \det F \to 0^+ \).

The assertion (F2) roughly says that the “strong-Determinant Constraint” is not \( p \)-ample, i.e., \( Z_{\infty}W \) cannot be of \( p \)-polynomial growth, and so neither Theorem A nor Theorem B is consistent with (s-DC). From the assertion (F1) we see that \( \mathcal{Q}W \leq \mathcal{R}W \) which shows that \( \mathcal{R}W \) cannot be of \( p \)-polynomial growth when combined with (F2). Hence, the theorem of Ben Belgacem is not compatible with (s-DC).

**Question.** Develop strategies for passing from \( p \)-ample to non-\( p \)-ample case.

2. 3D-2D passage with determinant type constraints

2.1. Statement of the problem.

Let \( W : \mathbb{M}^{3 \times 3} \to [0, +\infty) \) be Borel measurable and \( p \)-coercive (with \( p > 1 \)) and, for each \( \varepsilon > 0 \), let \( I_{\varepsilon} : W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3) \to [0, +\infty] \) be defined by

\[
I_{\varepsilon}(\phi) := \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} W(\nabla\phi(x,x_3))dx_3,
\]

where \( \Sigma_\varepsilon := \Sigma \times \frac{-\varepsilon}{2, \frac{\varepsilon}{2}} \subset \mathbb{R}^2 \) with \( \Sigma \subset \mathbb{R}^2 \) Lipschitz, open and bounded, and a point of \( \Sigma_\varepsilon \) is denoted by \( (x,x_3) \) with \( x \in \Sigma \) and \( x_3 \in ]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[ \). The problem of 3d-2d passage is the following.

\((P_2)\) Prove (or disprove) that

\[
\forall \psi \in W^{1,p}(\Sigma;\mathbb{R}^3) \quad \Gamma(\pi)\text{-}\lim_{\varepsilon \to 0} I_{\varepsilon}(\phi) = \int_{\Sigma} W_{\text{mem}}(\nabla\psi(x))dx
\]

and find a representation formula for \( W_{\text{mem}} : \mathbb{M}^{3 \times 2} \to [0, +\infty] \).

At the beginning of the nineties, Le Dret and Raoult answered to \((P_2)\) in the case where \( W \) is “finite and without singularities” (see §2.3). Recently, we extended the Le Dret-Raoult theorem to the case where \( W \) is compatible with (w-DC) and (s-DC) as Theorem C and Theorem D (see §2.4 and §2.5).

2.2. The \( \Gamma(\pi)\)-convergence.

The concept of \( \Gamma(\pi)\)-convergence was introduced Anzellotti, Baldo and Percivale in order to deal with dimension reduction problems in mechanics. Let \( \pi = \{\pi_\varepsilon\} \) be the family of \( L^p \)-continuous maps \( \pi_\varepsilon : W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3) \to W^{1,p}(\Sigma;\mathbb{R}^3) \) defined by

\[
\pi_\varepsilon(\phi) := \frac{1}{\varepsilon} \int_{] \frac{\varepsilon}{2}, \frac{\varepsilon}{2}[} \phi(\cdot,x_3)dx_3.
\]

**Definition** (Anzellotti-Baldo-Percivale [8] 1994).

We say that \( \{I_{\varepsilon}\}_\varepsilon \) \( \Gamma(\pi)\)-converge to \( I_{\text{mem}} \) as \( \varepsilon \) goes to zero, and we write

\[
I_{\text{mem}} = \Gamma(\pi)\cdot\lim_{\varepsilon \to 0} I_{\varepsilon}.
\]
if and only if
\[ \forall \psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \left( \Gamma(\pi) \cdot \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\psi) \right) = \left( \Gamma(\pi) \cdot \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\psi) \right) = I_{\text{mem}}(\psi) \]

with \( \Gamma(\pi) \cdot \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon, \Gamma(\pi) \cdot \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty] \) respectively given by:

\[ \Gamma(\pi) \cdot \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\psi) := \inf \left\{ \liminf_{\varepsilon \to 0} I_{\varepsilon}(\phi_\varepsilon) : \pi_\varepsilon(\phi_\varepsilon) \xrightarrow{L^p} \psi \right\} ; \]

\[ \Gamma(\pi) \cdot \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\psi) := \inf \left\{ \limsup_{\varepsilon \to 0} I_{\varepsilon}(\phi_\varepsilon) : \pi_\varepsilon(\phi_\varepsilon) \xrightarrow{L^p} \psi \right\} . \]

Anzellotti, Baldo and Percivale proved that their concept of \( \Gamma(\pi) \)-convergence is not far from that of \( \Gamma \)-convergence introduced by De Giorgi and Franzoni. For each \( \varepsilon > 0 \), consider \( \mathcal{I}_\varepsilon : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty] \) defined by

\[ \mathcal{I}_\varepsilon(\psi) := \inf \left\{ I_{\varepsilon}(\phi) : \pi_\varepsilon(\phi) = \psi \right\} . \]

**Definition** (De Giorgi-Franzoni [15, 14] 1975).
We say that \( \{\mathcal{I}_\varepsilon\}_\varepsilon \) \( \Gamma \)-converge to \( I_{\text{mem}} \) as \( \varepsilon \) goes to zero, and we write

\[ I_{\text{mem}} = \Gamma \cdot \lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon, \]

if and only if

\[ \forall \psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \left( \Gamma \cdot \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\psi) \right) = \left( \Gamma \cdot \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\psi) \right) = I_{\text{mem}}(\psi) \]

with \( \Gamma \cdot \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon, \Gamma \cdot \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty] \) respectively given by:

\[ \Gamma \cdot \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\psi) := \inf \left\{ \liminf_{\varepsilon \to 0} I_{\varepsilon}(\psi_\varepsilon) : L^p \psi \xrightarrow{\psi} \psi \right\} ; \]

\[ \Gamma \cdot \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon(\psi) := \inf \left\{ \limsup_{\varepsilon \to 0} I_{\varepsilon}(\psi_\varepsilon) : L^p \psi \xrightarrow{\psi} \psi \right\} . \]

The link between \( \Gamma(\pi) \)-convergence and \( \Gamma \)-convergence is given by the following lemma.

**Lemma** (see [6]).
\[ I_{\text{mem}} = \Gamma(\pi) \cdot \lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon \] if and only if \( I_{\text{mem}} = \Gamma \cdot \lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon \).

2.3. \( \Gamma(\pi) \)-convergence of \( I_{\varepsilon} \): finite case.
Let \( W_0 : \mathbb{M}^{3 \times 2} \to [0, +\infty] \) be defined by

\[ W_0(\xi) := \inf_{\zeta \in \mathbb{R}^3} W(\xi \mid \zeta). \]

**Theorem** (Le Dret-Raoult [20, 21] 1993).
If \( W \) is continuous and \( \exists c > 0 \forall F \in \mathbb{M}^{3 \times 3} W(F) \leq c(1 + |F|^p) \) then

\[ \forall \psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \Gamma(\pi) \cdot \lim_{\varepsilon \to 0} I_{\varepsilon}(\psi) = \int_{\Sigma} Q W_0(\nabla \psi(x)) \, dx. \]

Although the Le Dret-Raoult theorem is compatible neither with \( (w-\text{DC}) \) nor \( (s-\text{DC}) \) it established a suitable variational framework to deal with dimensional reduction problems: it is the point of departure of many works on the subject.
2.4. $\Gamma(\pi)$-convergence of $I_\varepsilon$: “weak-Determinant Constraint”.

By using the Le Dret-Raoult theorem we can prove the following result.

**Theorem C** (see [1, 5]).

Assume that

$$(D) \exists \alpha, \beta > 0 \forall F \in M^{3 \times 3} \left( |\det F| \geq \alpha \Rightarrow W(F) \leq \beta (1 + |F|^{p}) \right).$$

Then

$$\forall \psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \Gamma(\pi) \lim_{\varepsilon \to 0} I_\varepsilon(\psi) = \int_{\Sigma} QW_0(\nabla \psi(x))dx.$$  

**Outline of the proof.** As the $\Gamma(\pi)$-limit is stable by substituting $I_\varepsilon$ by its relaxed functional $\overline{T}_\varepsilon$, i.e., $\overline{T}_\varepsilon : W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \to [0, +\infty]$ given by

$$I_\varepsilon(\phi) := \inf \left\{ \liminf_{n \to +\infty} I_\varepsilon(\phi_n) : \phi_n \rightharpoonup^p \phi \right\} = \frac{1}{\varepsilon} \inf \left( \liminf_{n \to +\infty} \int_{\Sigma} W(\nabla \phi_n)dx : \phi_n \rightharpoonup^p \phi \right).$$

it suffices to prove that

$$\forall \psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \Gamma(\pi) \lim_{\varepsilon \to 0} \overline{T}_\varepsilon(\psi) = \int_{\Sigma} QW_0(\nabla \psi(x))dx.$$  

As $W$ satisfies (D) it is $p$-ample (see §1.6), and so by Theorem A-B we have

$$\forall \varepsilon > 0 \forall \phi \in W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \quad \overline{T}_\varepsilon(\phi) = \frac{1}{\varepsilon} \int_{\Sigma} QW(\nabla \psi(x, x_3))dx.$$  

with $QW = Z_0 W$ (which is of $p$-polynomial growth and so continuous by the property (2) of Fonseca).

Applying the Le Dret-Raoult theorem we deduce that

$$\forall \psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \Gamma(\pi) \lim_{\varepsilon \to 0} \overline{T}_\varepsilon(\psi) = \int_{\Sigma} Q[QW]_0(\nabla \psi(x))dx$$

with $[QW]_0 : M^{3 \times 2} \to [0, +\infty]$ given by

$$[QW]_0(\xi) := \inf_{\xi \in \mathbb{R}^3} QW(\xi | \xi).$$

Finally, we prove that $Q[QW]_0 = QW_0$, and the proof is complete. \(\square\)

Theorem C highlights the fact that the concept of $p$-amplitude has a “nice” behavior with respect to the $\Gamma(\pi)$-convergence. More generally, let $\{\pi_\varepsilon\}_\varepsilon$ be a family of $L^p$-continuous maps $\pi_\varepsilon$ from $W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^m)$ to $W^{1,p}(\Sigma; \mathbb{R}^m)$, where $\Sigma_\varepsilon \subset \mathbb{R}^N$ (resp. $\Sigma \subset \mathbb{R}^k$ with $k \in \mathbb{N}^+$) is a bounded open set, let $\{W_\varepsilon\}_\varepsilon$ be an uniformly $p$-coercive family of measurable integrands $W_\varepsilon : M^{m \times N} \to [0, +\infty]$ and, for each $\varepsilon > 0$, let $I_\varepsilon, QI_\varepsilon : W^{1,p}(\Sigma; \mathbb{R}^m) \to [0, +\infty]$ be respectively defined by

- $I_\varepsilon(\phi) := \int_{\Sigma_\varepsilon} W_\varepsilon(\nabla \phi(x))dx$;
- $QI_\varepsilon(\phi) := \int_{\Sigma_\varepsilon} QW_\varepsilon(\nabla \phi(x))dx$.

The following theorem says that the $\Gamma(\pi)$-limit is stable by substituting $I_\varepsilon$ by $QI_\varepsilon$ whenever every $W_\varepsilon$ is $p$-ample.

**Theorem** (see [5]).

Assume that:

- $\forall \varepsilon > 0$ $W_\varepsilon$ is $p$-ample;
- $\exists I_0 : W^{1,p}(\Sigma; \mathbb{R}^m) \to [0, +\infty]$ $\Gamma(\pi) \lim_{\varepsilon \to 0} QI_\varepsilon = I_0$.

Then $\Gamma(\pi) \lim_{\varepsilon \to 0} I_\varepsilon = I_0$. 
Proof. As every $W_ε$ is $p$-ample, from Theorem A-B we deduce that $\mathcal{T}_ε = \mathcal{Q}I_ε$ for all $ε > 0$. On the other hand, as every $π_ε$ is $L^p$-continuous, it is easy to see that $Γ(π)-\lim inf_{ε→0} I_ε = Γ(π)-lim inf_{ε→0} \mathcal{T}_ε$ and $Γ(π)-\lim sup_{ε→0} I_ε = Γ(π)-\lim sup_{ε→0} \mathcal{T}_ε$, and the theorem follows.

2.5. $Γ(π)$-convergence of $I_ε$: “strong-Determinant Constraint”.

The following theorem gives an answer to $(P_2)$ in the framework of nonlinear elasticity (it is consistent with (s-DC)) in the same spirit as the theorem of Ball in 1977 (see [7]). It is the result of several works on the subject: mainly, the attempt of Per- civalle in 1991 (see [22]), the rigorous answer to $(P)$ in the same spirit as the theorem of Ball in 1977 by Le Dret and Raoult in the $p$-polynomial growth case (see [20, 21]) and especially the substantial contributions of Ben Belgacem (see [8, 9, 10]).

Theorem D (see [3, 5]).

Assume that:

$(D_0)$ $W$ is continuous;

$(D_1)$ $W(F) = +∞ ⇐⇒ detF ≤ 0$;

$(D_2)$ $∀δ > 0 ∃δ_3 > 0 ∀F ∈ M^3×3 (detF ≥ δ ⇒ W(F) ≤ C_3(1 + |F|^p))$.

Then

\[ ∀ψ ∈ W^{1,p}(Σ; \mathbb{R}^3) \quad Γ(π)-\lim_{ε→0} I_ε(ψ) = \int_{Σ} QW_0(\nabla ψ(x))dx. \]

Outline of the proof. |

- It is easy to see that if $W$ satisfies $(D_0)$, $(D_1)$ and $(D_2)$ then:

$(P_0)$ $W_0$ is continuous;

$(P_1)$ $∀α > 0 ∃β_α > 0 ∀ψ ∈ M^3×2 (\{ξ, \xi| ≥ α ⇒ W_0(ξ) ≤ β_α(1 + |ξ|^p))$. In particular, $W_0$ satisfies $(P)$ since clearly $(P_1)$ implies $(P)$.

- Let $\mathcal{T}, \mathcal{T}_p, \mathcal{T}_p$ : $W^{1,p}(Σ; \mathbb{R}^3) → [0, +∞]$ be respectively defined by:

$\mathcal{T}(ψ) := \int_{Σ} W_0(\nabla ψ(x))dx$;

$\mathcal{T}_p(ψ) := \inf \left\{ \lim inf_{n→+∞} \mathcal{T}(ψ_n) : ψ_n L^p → ψ \right\}$;

$\mathcal{T}_p(ψ) := \inf \left\{ \lim inf_{n→+∞} \mathcal{T}(ψ_n) : C^1(Σ; \mathbb{R}^3) \ni ψ_n L^p → ψ \right\}$,

where $C^1(Σ; \mathbb{R}^3)$ is the set of $C^1$-immersions from $Σ$ to $\mathbb{R}^3$, i.e.,

$C^1(Σ; \mathbb{R}^3) := \left\{ ψ ∈ C^1(Σ; \mathbb{R}^3) : ∀x ∈ Σ \partial_x ψ(x) \wedge ∂_x ψ(x) ≠ 0 \right\}$.

As $W_0$ satisfies $(P)$, by Corollary 1 we have

$∀ψ ∈ W^{1,p}(Σ; \mathbb{R}^3) \quad \mathcal{T}(ψ) = \int_{Σ} QW_0(\nabla ψ(x))dx.$

On the other hand, we can prove the following two lemmas.

Lemma. $\mathcal{T} ≤ Γ(π)-\lim inf_{ε→0} I_ε$.

Lemma. If $(D_0)$, $(D_1)$ and $(D_2)$ hold then $Γ(π)-lim sup_{ε→0} I_ε ≤ \mathcal{T}_p$.

Hence, it suffices to prove that $\mathcal{T}_p ≤ \mathcal{T}$.

- Let $\mathcal{T}_p, \mathcal{R}I, \mathcal{R}I, \mathcal{R}T : W^{1,p}(Σ; \mathbb{R}^3) → [0, +∞]$ be respectively defined by:

$\mathcal{T}_p(ψ) := \inf \left\{ \lim inf_{n→+∞} \mathcal{T}(ψ_n) : Aff_3(Σ; \mathbb{R}^3) \ni ψ_n L^p → ψ \right\}$;

$\mathcal{R}I(ψ) := \int_{Σ} RW_0(\nabla ψ(x))dx$;
\( \mathcal{RT}(\psi) := \inf \left\{ \liminf_{n \to +\infty} \mathcal{RT}(\psi_n) : \psi_n \xrightarrow{L^p} \psi \right\}; \)
\( \mathcal{RT}_{aff}(\psi) := \inf \left\{ \liminf_{n \to +\infty} \mathcal{RT}(\psi_n) : \text{Aff}_i(\Sigma; \mathbb{R}^3) \ni \psi_n \xrightarrow{L^p} \psi \right\} \)

with \( \text{Aff}_i(\Sigma; \mathbb{R}^3) := \{ \psi \in \text{Aff}(\Sigma; \mathbb{R}^3) : \psi \text{ is locally injective} \} \). As \( \mathcal{RT} \leq \mathcal{T} \), a way for proving \( \mathcal{T}_{diff} \leq \mathcal{T}_{aff} \) is to establish the following three inequalities:
\( \mathcal{T}_{diff} \leq \mathcal{T}_{aff} \);
\( \mathcal{T}_{aff} \leq \mathcal{RT}_{aff} \);
\( \mathcal{RT}_{aff} \leq \mathcal{RT} \).

The first inequality follows by using the fact that \( W_0 \) satisfies (P0) and (P1) together with the following lemma.

**Lemma** (Ben Belgacem-Bennequin [8] 1996, see also [5]).

For all \( \psi \in \text{Aff}_i(\Sigma; \mathbb{R}^3) \) there exists \( \{ \psi_n \}_{n \geq 1} \subset C_1^1(\Sigma; \mathbb{R}^3) \) such that:

\( \psi_n \xrightarrow{W^{1,p}} \psi; \)
\( \exists \delta > 0 \ \forall x \in \Sigma \ \exists n \geq 1 \ \partial_1 \psi_n(x) \wedge \partial_2 \psi_n(x) \geq \delta. \)

The second inequality is obtained by exploiting the Kohn-Strang representation of \( \mathcal{RT}_0 \) (see [8], see also [5]). Finally, we establish the next inequality by combining the following two lemmas.

**Lemma** (Ben Belgacem [8] 1996, see also [5]).

If \( W_0 \) satisfies (P0) and (P1) then:

\( \mathcal{RT}_0 \) is continuous;

\( \exists c > 0 \ \forall \xi \in \mathbb{M}^{3 \times 2} \ \mathcal{RT}_0(\xi) \leq c(1 + |\xi|^p). \)

**Lemma** (Gromov-Éliashberg [18] 1971, see also [5]).

\( \text{Aff}_i(\Sigma; \mathbb{R}^3) \) is strongly dense in \( W^{1,p}(\Sigma; \mathbb{R}^3) \).

**Question.** Try to simplify the proof of Theorem D as follows: first, approximate \( W \) satisfying (D0), (D1) and (D2) or maybe weaker conditions compatible with \((s-DC)\) by a supremum of \( p \)-ample integrands \( W_\delta \) satisfying (D) with \( \alpha, \beta > 0 \) which can depend on \( \delta \), then, apply Theorem C to each \( W_\delta \), and finally, pass to the limit as \( \delta \) goes to zero.

**References**

[1] Anza Hafsa, O., and Mandallena, J.-P. The nonlinear membrane energy: variational derivation under the constraint “\( \det \nabla u \neq 0 \)”. *J. Math. Pures Appl. (9)* 86, 2 (2006), 100–115.
[2] Anza Hafsa, O., and Mandallena, J.-P. Relaxation of variational problems in two-dimensional nonlinear elasticity. *Ann. Mat. Pura Appl. (4)* 186, 1 (2007), 187–198.
[3] Anza Hafsa, O., and Mandallena, J.-P. The nonlinear membrane energy: variational derivation under the constraint “\( \det \nabla u > 0 \)”. *Bull. Sci. Math.* 132, 4 (2008), 272–291.
[4] Anza Hafsa, O., and Mandallena, J.-P. Relaxation theorems in nonlinear elasticity. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 25, 1 (2008), 135–148.
[5] Anza Hafsa, O., and Mandallena, J.-P. Relaxation et passage 3d-2d avec contraintes de type déterminant. *Submitted* (2009). Preprint available on [http://arxiv.org/abs/0901.3688](http://arxiv.org/abs/0901.3688).
[6] Anzellotti, G., Baldo, S., and Pericivale, D. Dimension reduction in variational problems, asymptotic development in \( \Gamma \)-convergence and thin structures in elasticity. *Asymptotic Anal.* 9, 1 (1994), 61–100.
[7] Ball, J. M. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.* 63, 4 (1976/77), 337–403.
[8] Ben Belgacem, H. *Modélisation de structures minces en élasticité non linéaire*. PhD thesis, Université Pierre et Marie Curie, 1996.
[9] Ben Belgacem, H. Une méthode de Γ-convergence pour un modèle de membrane non linéaire. C. R. Acad. Sci. Paris Sér. I Math. 324, 7 (1997), 845–849.
[10] Ben Belgacem, H. Relaxation of singular functionals defined on Sobolev spaces. ESAIM Control Optim. Calc. Var. 5 (2000), 71–85 (electronic).
[11] Celada, P., and Perrotta, S. Functions with prescribed singular values of the gradient. NoDEA Nonlinear Differential Equations Appl. 5, 3 (1998), 383–396.
[12] Dacorogna, B. Quasiconvexity and relaxation of nonconvex problems in the calculus of variations. J. Funct. Anal. 46, 1 (1982), 102–118.
[13] Dacorogna, B., and Ribeiro, A. M. Existence of solutions for some implicit partial differential equations and applications to variational integrals involving quasi-affine functions. Proc. Roy. Soc. Edinburgh Sect. A 134, 5 (2004), 907–921.
[14] De Giorgi, E. Sulla convergenza di alcune successioni d’integrai del tipo dell’area. Rend. Mat. (6) 8 (1975), 277–294. Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday.
[15] De Giorgi, E., and Franzeni, T. Su un tipo di convergenza variazionale. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 58, 6 (1975), 842–850.
[16] Fonseca, I. The lower quasiconvex envelope of the stored energy function for an elastic crystal. J. Math. Pures Appl. (9) 67, 2 (1988), 175–195.
[17] Gromov, M. Partial differential relations, vol. 9 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1986.
[18] Gromov, M. L., and Éliashberg, J. M. Construction of nonsingular isoperimetric films. Trudy Mat. Inst. Steklov. 116 (1971), 18–33, 235, (Translated in Proc. Steklov Inst. Math. 116 (1971) 13–28).
[19] Kohn, R. V., and Strang, G. Optimal design and relaxation of variational problems. II. Comm. Pure Appl. Math. 39, 2 (1986), 139–182.
[20] Le Dret, H., and Raoult, A. Le modèle de membrane non linéaire comme limite variationnelle de l’élasticité non linéaire tridimensionnelle. C. R. Acad. Sci. Paris Sér. I Math. 317, 2 (1993), 221–226.
[21] Le Dret, H., and Raoult, A. The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity. J. Math. Pures Appl. (9) 74, 6 (1995), 549–578.
[22] Percivale, D. The variational method for tensile structures. Preprint 16, Dipartimento di Matematica Politecnico di Torino, 1991.

UNIVERSITE MONTPELLIER II, UMR-CNRS 5508, LMGC, PLACE EUGÈNE BATAILLON, 34095 MONTPELLIER, FRANCE.
E-mail address: Omar.Anza-Hafsa@univ-montp2.fr

UNIVERSITE DE NIMES, SITE DES CARMES, PLACE GABRIEL PÉRI, 30021 NIMES, FRANCE.
E-mail address: jean-philippe.mandallena@unimes.fr