Pullback-invariant matter couplings to degenerate tetrads

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Abstract. An algorithm is described for the construction of actions for scalar, spinor, and vector gauge fields that remains well defined when the metric is degenerate and that involves no contravariant tensor fields. These actions produce the standard matter dynamics and coupling to gravity when the tetrad is non-degenerate, but have the property that all fields on a spacetime manifold \( M \) that appear in them can be pulled back to any smooth manifold \( M' \) through an arbitrary map \( \varphi : M' \to M \), and that this pullback leaves the action invariant when \( \varphi \) has degree one.

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1. Introduction

One of the intriguing features of tetrad formulations of gravity is that they remain well defined when the tetrad becomes degenerate. This property has proved useful in the quantization of \( 2+1 \) gravity \([1]\) and may be important for the quantization of \( 3+1 \) gravity as well (see comments in \([2]\)). The quantization of \( 2+1 \) gravity in fact used the more startling property that non-degenerate tetrads are gauge equivalent to certain degenerate tetrads.

A similar property has been discussed for the \( 3+1 \) case \([3]\), where the action is:

\[
S = \int_M e^a \wedge e^b \wedge F^{cd} e_{abcd}
\]

where \( F^{cd} \) is the field strength of either a spin connection \([4]\) or a self-dual connection \([2]\), \( e^a = e^a_\mu dx^\mu \) is the tetrad one-form, and \( M \) is a smooth four-manifold. Horowitz points out in \([3]\) that these fields may all be pulled back to another four-manifold \( M' \) through an arbitrary smooth map \( \varphi : M' \to M \) and that the action (1.1) is invariant under this transformation if \( \varphi \) has degree one. He then uses this observation to construct topology-changing solutions of the equations of motion.

However, this is valid only for the purely gravitational case, since the standard matter actions are not invariant under such pullbacks, and are not well defined when the tetrad is degenerate. The problem is that both the standard variational principles and the standard equations of motion for the matter fields involve the inverse \( e^{\mu}_m \) of the tetrad \( e^m_\mu \) §. For example, the usual scalar field action contains two inverse tetrads:

\[
S = \int d^4x \sqrt{g} \, g^{\mu\nu} \partial_\mu \partial_\nu \phi = \int d^4x \, e^{\mu}_a \eta^{ab} e^b_\alpha \partial_\alpha \partial_\nu \phi.
\]

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§ We use greek letters for spacetime indices and latin letters for internal Lorentz indices.
A suggestion was made in [3] of how to avoid this problem, at least if \( e_\mu^\nu \) is degenerate only on a set of measure zero. The idea is to require the fields to solve the equations of motion (Einstein's equations and the matter equations of motion) only where \( \det(g) \neq 0 \), but also to be smooth everywhere, even when \( \det(g) = 0 \). This approach has been criticized on the grounds that it does not follow from, and may be in contradiction to, a variational principle for the system dynamics. An example for which similar problems arise is described in [5] for the pure gravity case. There, a metric is defined on \( \mathbb{R}^2 \times D^2 \) which, when their parameter \( \alpha \) is a negative integer, is smooth everywhere and is non-degenerate except at the centre of the disk. Such metrics can be constructed by pulling back a non-degenerate tetrad on \( \mathbb{R}^2 \times D^2 \) to \( \mathbb{R}^2 \times D^2 \) through a map that is a \((-\alpha)\)-fold covering map except at the centre of the disk, and which leaves this centre fixed. Thus, \( \alpha \) characterizes the strength of a conical singularity in the spacetime at the centre of the disk. The above prescription would allow solutions for any negative integer \( \alpha \), but the Einstein–Hilbert variational principle requires that the strength of this singularity vanish.

It is important to point out that it is not our goal here to describe any dynamics of matter fields in the presence of a degenerate metric that might follow from the standard variational principles for matter fields by some regularization procedure. In fact, we will not even ask that our description be consistent with any such dynamics. Instead, we take the position that we should find some variational principle that

(i) remains well defined when the tetrad is degenerate;

(ii) reproduces the usual dynamics for standard matter fields whenever the tetrad is non-degenerate;

(iii) provides the proper source terms for the gravitational field.

These need not, in general, require consistency with any dynamics prescribed by the standard variational principles when the tetrad is degenerate. Again, an example of such a discrepancy is the case of conical singularities in vacuum general relativity, for whilst the metric given in [5] is not a stationary point of the Einstein–Hilbert action, the corresponding tetrad together with a connection that vanishes everywhere, even at the singularity, is a stationary point of the 'first order action' (1.1) with, say, the tetrad specified on the boundary and the appropriate boundary terms included in the variational principle. This happens despite the fact that this variational principle reproduces the Einstein–Hilbert dynamics when the tetrad is not degenerate.

Actions for matter fields satisfying (i)–(iii) have been discussed before [6,7], but their description has always involved contravariant tensor fields. For example, the massless free scalar field can be described by the action:

\[
S = \int_M \sqrt{-g} \left( \pi^\mu \partial_\mu \varphi - \frac{1}{2} \pi^\mu \pi^\nu g_{\mu\nu} \right) d^4 x. \tag{1.3}
\]

Since such a formalism is not well suited to the pullback construction of [3], we would like to find action principles that satisfy (i)–(iii) and contain only fields that can be pulled back to another manifold \( M' \) through an arbitrary smooth map \( \varphi : M' \rightarrow M \).

By making use of differential forms as Lagrange multipliers, we will present an algorithm in section 2 which transforms actions for standard matter fields into action principles that involve no contravariant tensor fields or inverted tetrads, but that are equivalent to the original actions when the tetrad is non-degenerate. This algorithm can be applied to arbitrarily complicated systems, though the fields involved will have to be of a certain type. However, because all scalar fields, all spinor fields, and all vector gauge fields are of this type, this is not a severe restriction. We show in sections 3 and 4 that these actions have properties (ii) and (iii) above. A similar scheme exists for arbitrary fields,
but now contravariant fields must be introduced. However, this method introduces fewer contravariant fields than previous prescriptions \[6,7\], and still allows a sort of pullback in certain cases. We will concentrate on the cases in which the contravariant fields can be completely eliminated and describe the more general case in section 5. We close with a discussion of the gauge invariance of the resulting actions.

2. New actions for old

We will now present an algorithm that will construct from an action \(S^0\) an action \(S\) such that

(i) \(S\) leads to the same dynamics as \(S^0\) when the tetrad is non-degenerate;
(ii) \(S\) is well defined for arbitrary smooth tetrads, regardless of degeneracies;
(iii) \(S\) provides the same coupling to the gravitational fields as \(S^0\) when the equations of motion are satisfied;
(iv) \(S\) contains only covariant tensor fields and scalar fields, so that all fields in the action can be pulled back from any smooth manifold \(M\) to any other smooth manifold \(M'\) through an arbitrary smooth map \(\varphi : M' \to M\). In fact, the action \(S\) is invariant under such pullbacks of degree one.

Our algorithm will apply only to the limited class of actions in which both the matter fields and their derivatives appear only in an appropriately antisymmetric form, though we allow arbitrary combinations of tetrads and inverse tetrads. The restriction on matter fields should come as no surprise after consideration of the action (1.1), in which the use of differential forms eliminates any need for the inverse tetrads or contravariant tensor fields. The antisymmetry condition guarantees that the matter fields can be replaced by differential forms and that, by introducing additional differential forms as Lagrange multipliers, the entire action can be written in terms of differential forms.

There is one more intuitive idea that we should present before describing the algorithm. This idea was also inspired by \[3\], in which the suggestion was made that solutions to the field equations be considered for which scalars formed from the fields remain finite, but the individual fields may diverge or go to zero. For example, \(e^\mu_a\) might be allowed to diverge, but only if the derivatives of scalar fields \(\partial_\mu \varphi\) vanish fast enough for \(e^\mu_a \partial_\mu \varphi\) to remain well defined. We note that the contractions \(e^\mu_a \partial_\mu \varphi = \varphi_{.,a}\) are just the internal components of \(d\varphi\), and, in order to guarantee that they take on finite values, we will take these internal components to be the fundamental description of \(d\varphi\).

This brings us to the algorithm itself. The description below refers to a four-dimensional spacetime and uses the word ‘tetrad’ to refer to the fields \(e^\mu_a\). However, for our purposes the dimensionality of the manifold is completely unimportant, and so is the explicit form of the gravitational action as long as gravity is described by a ‘tetrad’ and a connection. Thus, the procedure works equally well for \(2 + 1\), \(3 + 1\), or higher dimensional systems, and also for higher derivative theories. We use four-dimensional language only for convenience.

Suppose then that we are given some action \(S^0\) that is an integral over a four manifold \(\mathcal{M}\) with boundary \(\partial \mathcal{M}\) of a four-form \(L^0\) which is a function of completely antisymmetric covariant matter fields \(f_{[\mu_1,\mu_2,\ldots]}^{(i)}\) of density weight zero and their antisymmetrized covariant derivatives \(D_{[\alpha} f_{\beta \gamma \mu_2,\ldots]}^{(i)}\), as well as the tetrad \(e^\mu_a\) and its inverse \(e_a^\mu\). This covariant derivative is to be given by the Lorentz \(\dagger\) connection \(\omega_{\alpha \beta}^{\gamma \delta}\) that describes gravity, and acts

\[\dagger\] By ‘Lorentz’ connection we mean \(SU(1, 1)\), \(SO(3, 1)\), \(SL(2, C)\), or whatever gauge group is appropriate for the dimension of spacetime and the description of gravity under consideration.
only on internal indices. Note that only spacetime indices on the fields \( f^{(i)} \) have been indicated, and that there is no antisymmetry requirement on any internal indices that may be present in the collective label. We assume that appropriate boundary terms are also included in the action so that functional derivatives are well defined (with some set of boundary conditions) but we do not keep track of such terms here.

In this case, the following procedure produces an action \( S \) that satisfies (i)-(iv) above.

Step (1): insert sufficient inverse tetrads to write all tensor fields in terms of their covariant components:

\[
f^{(i)}_{\mu \nu \ldots} \to e^a \eta e^b f^{(i)}_{\beta \mu \nu \ldots}.
\]  

(2.1)

Step (2): insert enough tetrads to write all undifferentiated tensor fields in terms of their tetrad components. More specifically, for each tensor field \( f^{(i)}_{[\mu \nu \ldots]} \) introduce a set of scalar fields \( f^{(i)}_{\mu \nu \ldots} \) with the same number of internal indices as the rank of the original tensor field. Then perform the substitution:

\[
f^{(i)}_{[\mu \nu \ldots]} \to e^\mu e^\nu \ldots f^{(i)}_{[\mu \nu \ldots]}.
\]  

(2.2)

for each undifferentiated field \( f^{(i)}_{[\mu \nu \ldots]} \) in \( \mathcal{L}^0 \).

Step (3): for each tensor field \( f^{(i)}_{[\mu \nu \ldots]} \) present in the original action, introduce another collection of spacetime scalars \( f^{(i)}_{[\mu \nu \ldots]} \) labelled by one more internal index than the rank of the tensor field. This comma is no more than a grouping symbol, and does not denote any kind of differentiation. Now introduce these new fields into the action, by using them to replace the covariant derivatives of the fields \( f^{(i)}_{[\mu \nu \ldots]} \) according to the rule:

\[
D_{[\mu} f^{(i)}_{\nu \ldots]} \to e^\alpha e^\beta e^\gamma f^{(i)}_{[\mu \nu \ldots]}.
\]  

(2.3)

Note that no derivatives remain in the matter Lagrange density after this substitution has been performed.

Step (4): replace any spacetime Levi–Civita densities with the corresponding internal symbols:

\[
\varepsilon_{ab\gamma\delta} \to \varepsilon_{abcd} e^a \epsilon^b \epsilon^c \epsilon^d \quad \text{and} \quad \varepsilon_{ab\gamma\delta} \to \varepsilon_{abcd} e^a \epsilon^b \epsilon^c \epsilon^d.
\]  

(2.4)

Step (5): formally cancel all contracted tetrads and inverse tetrads:

\[
e_m^\mu e_\mu^\nu \to \delta_m^\nu \quad \text{and} \quad e_m^\mu e_\mu^\nu \to \delta_m^\nu.
\]  

(2.5)

Note that since all matter fields and Levi–Civita tensors have been replaced by fields with no spacetime indices, and since the connection and covariant derivative no longer appear in the Lagrange density after step 5, any spacetime indices on tetrads still present in the Lagrange density must be contracted with spacetime indices from inverse tetrads, and vice versa. Because the Lagrange density is a four-form and the action contains no matter

† Nevertheless, we have just stated that the covariant derivative acts on objects \( f^{(i)}_{[\mu \nu \ldots]} \) that have spacetime indices as well. Because of the antisymmetrized form \( D_{[\mu} f^{(i)}_{\nu \ldots]} \) in which the covariant derivatives are assumed to appear, the action will in fact be independent of the extension of this covariant derivative to act on such fields, as long as it is torsion-free.

‡ Note that this case includes arbitrary couplings of scalar and gauge fields!
densities, matter differential forms, or covariant derivatives of tetrads, after this step the tetrad appears in the matter action only though the four-form \( (1/4!) e^a \wedge e^b \wedge e^c \wedge e^d \).

We now have a matter 'action functional' that depends only upon a set of scalar fields and the one-forms \( e^a \). It is therefore perfectly well defined when these one-forms are degenerate and is also invariant under pullbacks. However, since the tetrad only appears in the action through the volume element \( (1/4!) e^a \wedge e^b \wedge e^c \wedge e^d \), this action produces the wrong coupling to the gravitational field. Even worse, it contains no derivatives at all, and so cannot lead to any dynamics for the matter fields. To correct these problems we add one more step to our algorithm.

Step (6): for each field \( f^{(i)}[\mu_\nu...] \) with \( n \) spacetime indices in the original action, introduce a set of \((4 - n)\)-form fields \( k^{(i)} \) with a number of internal indices equal to the rank of the tensor field, and a set of \((3 - n)\)-form fields \( \lambda^{(i)} \) also with a number of internal indices given by the rank of the tensor field. Now, use these new fields as Lagrange multipliers and add to the above Lagrange density the constraint terms:

\[
k^{(i)} \wedge [f^{(i)}[\mu_\nu...] \, dx^\mu \wedge dx^\nu \wedge \ldots - f_{S[mn...]}^{(i)} e^m \wedge e^n \wedge \ldots] \quad (2.6)
\]

and

\[
\lambda^{(i)} \wedge [D \wedge (f^{(i)}[\mu_\nu...] \, dx^\mu \wedge dx^\nu \wedge \ldots - f_{S[mn...a]}^{(i)} e^m \wedge e^n \wedge \ldots)] \quad (2.7)
\]

where the \( D \wedge \) with no subscript is the covariant exterior derivative operator defined by \( D \).

Intuitively, these constraints link the tetrad components of the matter fields to the new fields that we have introduced. Practically, these constraints reintroduce dynamics through the derivatives in equation (2.7) and reintroduce the proper coupling of the matter to the gravitational fields through the tetrads they contain. We will demonstrate these practical properties in the next two sections, but first we note that when this algorithm is applied to the massive scalar and Yang-Mills actions, the results are:

\[
S_{\phi} = \int \left( \frac{1}{2} \dot{\varphi} \dot{\varphi} + m^2 \varphi^2 \right) \frac{1}{4!} \epsilon_{ijkl} \, e^i \wedge e^j \wedge e^k \wedge e^l + \int \lambda \wedge (D \varphi - \varphi_o e^a) \quad (2.8)
\]

\[
S_{YM} = -\frac{1}{4} \int \left( A_{[\mu,\nu]}^i - C_{jk}^i A_{\alpha}^k A_{\beta}^j \right) (A_{[\alpha,\beta]}^i - C_{m\alpha}^k A_{\nu}^m A_{\mu}^k)

\times \eta^{ac} \eta^{bd} \frac{1}{4!} \epsilon_{a_1 a_2 a_3 a_4} e^{a_1} \wedge e^{a_2} \wedge e^{a_3} \wedge e^{a_4}

+ \int k_i \wedge (A_{\mu}^i dx^\mu - A_{\alpha}^i e^a) + \int \lambda_i \wedge (D \wedge A_{(b,\alpha]}^i dx^\mu - A_{(b,\alpha]}^i e^a \wedge e^b) \quad (2.9)
\]

where in this last example \( i, j, k, l, m, n \) are Yang–Mills indices, \( \mu \) is a spacetime index, and \( a, b, c, d \) are internal Lorentz indices.

Note that our algorithm can easily be generalized to matter Lagrangians that involve spinors. To do so, we need only replace each tetrad \( e^a_{\mu} \) with a soldering form \( \varphi^A_{\mu} A^i \) in the appropriate steps above. In fact, the algorithm treats Dirac spinors much like scalar fields, since the spinors have no spacetime indices but only internal indices.

\[\vdash\] Of course, if \( n > 4 \), the field does not appear in the action at all, since its indices must be completely antisymmetrized. Similarly, if \( n = 4 \), its derivatives do not appear in the action, in which case the \( \lambda \) fields are not needed, and if \( n = 0 \) the fields \( k \) are not needed. The appropriate modifications can then be made to the following discussion, but we will not treat this case explicitly and will implicitly assume that \( 1 \leq n \leq 3 \).
3. Equivalence of new and old actions

There are several senses in which we might wish our new actions to be equivalent to the original actions. Note, for example, that the new and old actions are numerically equal when the constraints hold, as the constraints simply keep track of the various substitutions made in the early steps of the algorithm. Here, however, we show that the matter dynamics produced by the two actions are equivalent, or, more specifically, that the matter equations of motion that follow from our new action are equivalent to the old matter equations of motion when the tetrad is non-degenerate. The presence of spinor fields does not alter the discussion below.

Our new action $S$ is the integral of some four-form $L$ obtained from the original action $S^0 = \int L^0 = \int L^0 \frac{1}{4!} \varepsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l$ by following steps 1-6. Note that our fields $f^{(i)}_{mn...}$ appear in $L$ only as a result of step 2 so that the variation of $S$ with respect to these fields can be expressed as

$$0 = \frac{\delta S}{\delta f^{(i)}_{mn...}} = \frac{\partial L^0}{\partial f^{(i)}_{[\mu \nu ...]}} \left( \frac{\partial f^{(i)}_{[\mu \nu ...]} f^{(i)}_{mn...}}{\partial f^{(i)}_{[mu...]} e^m \wedge e^n \wedge e^l} \right) \frac{\delta S}{\delta f^{(i)}_{mn...}}$$

(3.1)

The substitutions of steps 1–5 are to be performed after the derivative $\partial L^0 / \partial f$ has been computed. Note that this derivative has a number of free spacetime indices which can only appear through inverse tetrads after the substitutions have taken place. Thus,

$$\frac{\partial L^0}{\partial f^{(i)}_{[\mu \nu ...]}} = D^{(i)mn...} e_{[m} e_{n]} ... \frac{1}{4!} \varepsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l$$

(3.2)

for some internal tensor field $D^{(i)mn...}$ built entirely from the matter fields. The partial derivative of $f^{(i)}_{[\mu \nu ...]}$ with respect to $f^{(i)}_{mn...}$ is to be computed from step 3, that is:

$$\frac{\partial f^{(i)}_{[\mu \nu ...]} f^{(i)}_{mn...}}{\partial f^{(i)}_{[mu...]} e^m \wedge e^n \wedge e^l} \equiv e^m e^n ...$$

(3.3)

The tetrads in (3.2) then cancel with the inverse tetrads in (3.3). Strictly speaking, since expression (3.2) involves inverse tetrads, it is not defined when the tetrad is degenerate. In that case, it is only our notation $\partial L^0 / \partial f$ and $\partial f / \partial f$ that is ill-defined, and the functional derivative (3.1) is still perfectly well defined and equal to $D^{(i)mn...} \det(e)$. However, we are interested here in the case where the tetrad is not degenerate, so that there is no ambiguity in either expression (3.2) or (3.3). Despite this complication, we will find this notation to be convenient.

Similarly, we will be interested in the variation of the new action with respect to the 'derivative fields' $f^{(i)}_{mn...}$:

$$0 = \frac{\delta S}{\delta f^{(i)}_{mn...}} = \frac{\partial L^0}{\partial f^{(i)}_{[\mu \nu ...]}} \left( \frac{\partial D^{(i)\mu \nu ...}}{\partial f^{(i)}_{[\mu \nu ...]} e^m \wedge e^n \wedge e^l} \right) - \lambda^{(i)} \wedge e^a \wedge e^n \wedge e^m$$

(3.4)

Here, the notation is similar to that used in equation (3.1) and the partial derivative $\partial D f / \partial f$ is to be computed using step 3.
In order to show that the dynamics generated by these equations is the same as what follows from the original equations of motion \( \delta S^0 / \delta f_{[\mu \nu \ldots]}^{(i)} = 0 \), we will, of course, need to relate the new fields \( f_{S[mn \ldots]}^{(i)} \) to the old fields \( f_{[\mu \nu \ldots]}^{(i)} \). The proper relationship is guaranteed by the constraints:

\[
0 = V \frac{\delta S}{\delta \lambda^{(i)}_{[\alpha \beta \ldots]}} = d x^\sigma \wedge d x^\rho \wedge \ldots \left\{ D \wedge \left( f_{[\mu \nu \ldots]}^{(i)} d x^\mu \wedge d x^\nu \wedge \ldots \right) - f_{S[mn \ldots a]}^{(i)} e^a \wedge e^m \wedge e^n \wedge \ldots \right\} \tag{3.5}
\]

\[
0 = V \frac{\delta S}{\delta \kappa^{(i)}_{[\alpha \beta \ldots]}} = d x^\sigma \wedge d x^\rho \wedge \ldots \left\{ f_{[\mu \nu \ldots]}^{(i)} d x^\mu \wedge d x^\nu \wedge \ldots - f_{S[mn \ldots]}^{(i)} e^m \wedge e^n \wedge \ldots \right\} \tag{3.6}
\]

and

\[
0 = V \frac{\delta S}{\delta f_{[\mu \nu \ldots]}^{(i)}} = \kappa^{(i)} \wedge d x^\mu \wedge d x^\nu \wedge \ldots - D \wedge \lambda^{(i)} \wedge d x^\mu \wedge d x^\nu \wedge \ldots \tag{3.7}
\]

where \( \kappa^{(i)} = (1/n!) \kappa^{(i)}_{[\alpha \beta \ldots]} d x^\alpha \wedge d x^\beta \wedge \ldots \) and \( \lambda^{(i)} = (1/m!) \lambda^{(i)}_{\alpha \beta \ldots} d x^\alpha \wedge d x^\beta \wedge \ldots \) are the ranks of these forms, and \( V = (1/4!) e_{\alpha \beta \gamma \delta} d x^\alpha \wedge d x^\beta \wedge d x^\gamma \wedge d x^\delta \) to simplify the notation. Note that equations (2.7) and (3.7) contain the only derivatives in any of the equations of motion. We will see that the second of these effectively contains all the dynamics of the theory.

Our goal now is to show that when the tetrad is non-degenerate the above equations of motion are equivalent to the original equations of motion together with the following definitions:

\[
f_{S[mn \ldots]}^{(i)} = e^\mu_m e^\nu_n f_{[\mu \nu \ldots]}^{(i)} \tag{3.8a}
\]

\[
f_{S[mn \ldots a]}^{(i)} = D_a f_{[\mu \nu \ldots]}^{(i)} e^\sigma_a e^\rho_m e^n \tag{3.8b}
\]

\[
(*\kappa^{(i)}_{[\mu \nu \ldots]} = \frac{\partial L^0}{\partial f_{[\mu \nu \ldots]}^{(i)}} \ldots \tag{3.8c}
\]

\[
\lambda_{mn \ldots}^{(i)} \equiv * \left( \frac{\partial L^0}{\partial D^2 f_{[\mu \nu \ldots]}^{(i)}} d x^\mu \wedge d x^\nu \wedge \ldots \right) \tag{3.8d}
\]

where * is the Hodge duality operator. We note that * is well defined and that such definitions are always possible when the tetrad is non-degenerate.

We also note that these are exactly the solutions of equations (3.1), (3.4), (3.5), and (3.6) when the tetrad is non-degenerate. Therefore, if the two theories are equivalent, all of the dynamics must be contained in the single unsolved equation (3.7). This follows since direct substitution of the results/definitions (3.8) into equation (3.7), gives, after taking a dual:

\[
0 = \frac{\partial L^0}{\partial f_{[\mu \nu \ldots]}^{(i)}} d x^\mu \wedge d x^\nu \wedge \ldots - * \left( D \wedge * \left( \frac{\partial L^0}{\partial (D^2 f_{[\mu \nu \ldots]}^{(i)})} d x^\alpha \wedge d x^\rho \wedge \ldots \right) \right) \tag{3.9}
\]

which is just the set of matter equations of motion of our original action \( S^0 \).
4. Source terms

If the new actions are to be satisfactory they must produce not only the correct matter field
dynamics when the tetrad is non-degenerate, but also the correct coupling to the gravitational
fields — both to the connection and to the tetrad. Specifically, we show in this section that
the gravitational source terms are identical for the old and new actions when the tetrad is
non-degenerate and the matter equations of motion hold. Again, no changes are needed
to include spinors other than the substitution of soldering forms for tetrads. Because it is
somewhat simpler, we consider the coupling to the connection first.

We assume that our description of gravity is based on a connection $\omega_\mu {}^a_b$, where $\mu$
is a one-form index and $a, b$ are matrix indices in the vector representation of the gauge
group. The covariant derivative then acts on an internal vector by $D_\mu v^a = \partial_\mu v^a + \omega_\mu {}^a_b v^b$.
Note that $D_\mu [af(i)_{\mu \nu \ldots}] = \partial_\mu [af(i)_{\mu \nu \ldots}]$ unless $f(i)$ has internal indices, so that it will be
important to display such indices explicitly. We will write $f(i)_{[\mu \nu \ldots]} = f(i)_{[\mu \nu \ldots]}abc \ldots$ and
$f(i)_{[\mu \nu \ldots]} = f(i)_{[\mu \nu \ldots]}abc \ldots$, where the extra internal indices are displayed after the bar in order
to separate them from internal indices, created by our algorithm, that replace spacetime
indices on the original fields. Note that since these extra indices were previously absorbed
into the collective label $(i)$, they should also be displayed on the Lagrange multipliers. In
particular, $\lambda(i) = \lambda(i)_{abc \ldots}$.

The source term for the connection given by our new action is:

$$ V \frac{\delta S}{\delta \omega_\mu e_b} = \sum_{n, i} \lambda(i)_{[a_1 a_2 \ldots a_n]} f(i)_{[\mu \nu \ldots]} [a_1 a_2 \ldots a_n e_{a_{n+1}}} \ldots \right) $$

$$ \wedge dx^\nu \wedge dx^\eta \wedge \ldots \quad (4.1) $$

since the connection now appears in the action only through the covariant derivative in the
constraints. However, we know that when the tetrad is non-degenerate, $\lambda(i)_{abc \ldots}$ is given by
equation (3.8d) so that we have

$$ V \frac{\delta S}{\delta \omega_\mu e_b} = \sum_{n, i} \left( \frac{\partial \mathcal{L}^0}{\partial D_\mu [a f(i)_{\mu_1 \mu_2 \ldots} a_{i_1} a_{i_2} \ldots a_{i_n}]} \right) $$

$$ \times \frac{1}{4!} \epsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l \quad (4.2) $$

and in fact the source terms for the connection are equivalent in the two actions.

This leaves only the variation with respect to the tetrad. We note that tetrad
source terms from the new action can only arise from variation of the volume element
$(1/4!) \epsilon_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l$ that still appears in the Lagrangian, or from variation of the
constraints. From our new action, then:

$$ V \frac{\delta S}{\delta e_\alpha^a} = \frac{1}{3!} \epsilon_{abcd} dx^a \wedge e^b \wedge e^c \wedge e^d \mathcal{L}^0 $$
where again the substitutions of steps 1–5 are to be made in $\mathcal{L}^0$, and we have once again absorbed any internal indices on the original fields into the collective label $(i)$.

Since none of our matter fields are densities, the original Lagrange density $\mathcal{L}^0$ must be of the form $\mathcal{L}^0 = \frac{1}{2} e_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l$, where all of the tetrads in $\mathcal{L}^0$ appear contracted with some $f^{(i)}_{[\mu_1\ldots\mu_n]}$, in the form $e_{\mu}^a f^{(i)}_{[\mu_1\ldots\mu_n]}$, $e^a D_\mu f^{(i)}_{[\mu_1\ldots\mu_n]}$, or $e^a D_\mu f^{(i)}_{[\mu_1\ldots\mu_n]}$, and every spacetime index associated with either $f^{(i)}_{[\mu_1\ldots\mu_n]}$ or $D_\mu f^{(i)}_{[\mu_1\ldots\mu_n]}$ is, in fact, contracted with an $e^a_\mu$. This means that we can compute the variation of the original Lagrangian with respect to the tetrad in terms of its variations with respect the matter fields and their covariant derivatives:

$$
\frac{\partial \mathcal{L}^0}{\partial e^a_\alpha} = \frac{1}{3!} \varepsilon_{abcd} \, dx^a \wedge e^b \wedge e^c \wedge e^d \mathcal{L}^0
$$

If we now inspect equation (4.3) term by term, we see that these terms become exactly the terms in equation (4.4) when the equations (3.8) (or, more directly, equations (3.1) and (3.4)) are used to substitute for the various Lagrange multipliers. It follows that when the matter equations of motion hold, the variation of our new action with respect to the tetrad is equal to the corresponding variation of the old action.
5. Generalization

A generalization of the algorithm presented in section 2 allows the original action to contain arbitrary matter fields, but has the disadvantage that it introduces contravariant tensor fields. However, this algorithm introduces fewer contravariant fields than previous methods [6, 7], and, despite the presence of the contravariant fields, the resulting actions are still invariant under a sort of pullback in certain cases. The algorithm is given below and is followed by a short discussion of the invariance. We do not present a separate proof that the resulting actions are equivalent to the original ones, since such a proof is very much the same as the one given in section 3.

Step (1): for each tensor density \( T^{(w)(i)}_{\mu\nu...} \) of weight \( w \neq 0 \), introduce a new tensor field \( T^{(i)}_{\mu\nu...} \) and use it to replace that density by performing the substitution:

\[
T^{(w)(i)}_{\mu\nu...} \rightarrow [\det(e)]^w T^{(i)}_{\mu\nu...}
\]  \hspace{1cm} (5.1)

where \( \det(e) \) is the determinant of the tetrad \( e \).

Step (2): insert sufficient inverse tetrads to write all tensor fields in terms of their covariant components:

\[
T^{(i)}_{\mu\nu...} \rightarrow e^\alpha_a \eta^{ab} e^\beta_b T^{(i)}_{\beta\mu\nu...}
\]  \hspace{1cm} (5.2)

Step (3): insert enough tetrads to write all tensor fields in terms of their tetrad components. More specifically, for each tensor field \( T^{(i)}_{\mu\nu...} \) (including those introduced in step (1)) introduce a set of scalar fields \( T_{mn...}^{(i)} \) with the same number of internal indices as the rank of the original tensor field, and with the same symmetries. Then perform the substitution:

\[
T^{(i)}_{\mu\nu...} \rightarrow e^m_e e^n_n ... T_{mn...}^{(i)}
\]  \hspace{1cm} (5.3)

Step (4): arrange the terms now present in the ‘action’ so that the covariant derivatives act only on spacetime scalars, though these may have an arbitrary structure of internal indices.

Step (5): for each tensor field \( T^{(i)}_{\mu\nu...} \) either present in the original action or introduced in step 1, introduce another collection of spacetime scalars \( T_{mn...}^{(i)} \) labelled by one more internal index than the rank of the tensor field. Again, this comma is no more than a grouping symbol, and does not denote any kind of differentiation. Now, introduce these new fields into the action by using them to replace the covariant derivatives of the fields \( T_{mn...}^{(i)} \) according to the rule:

\[
D_\alpha T_{mn...}^{(i)} = e^a_a e^m_e e^n_n T_{mn...a}^{(i)}
\]  \hspace{1cm} (5.4)

Again, no covariant derivatives remain in the Lagrangian after this substitution has been performed.

Step (6): replace any spacetime Levi–Civita densities with the corresponding internal densities:

\[
\varepsilon_{\alpha\beta\gamma\delta} \rightarrow \varepsilon_{abcd} e^a_a e^b_b e^c_c e^d_d \quad \text{and} \quad \varepsilon^{\alpha\beta\gamma\delta} \rightarrow \varepsilon^{abcd} e^a_a e^b_b e^c_c e^d_d
\]  \hspace{1cm} (5.5)

and replace \( \det(e) d^4x \) with \((1/4!)\ e^a \wedge e^b \wedge e^c \wedge e^d\).
Step (7): formally cancel all remaining positive and negative powers of $\det(e)$ and all contracted tetrads and inverse tetrads:

$$e^\mu_m e^n_n \to \delta^m_n \quad \text{and} \quad e^\mu_\mu e^\nu_\nu \to \delta^\nu_\nu \quad (5.6)$$

Again, the tetrad no longer appears in the action except though the four-form: $(1/4!) e^a \wedge e^b \wedge e^c \wedge e^d$.

Step (8): for each tensor field $T^{(i)}_{\mu\nu\ldots}$ in the original action or introduced in step 1, introduce a set of four-form fields $\kappa^{(i)}_{mn\ldots}$ with a number of internal indices equal to the rank of the tensor field and a set of three-form fields $\lambda^{(i)}_{mn\ldots}$ also with a number of internal indices given by the rank of the tensor field. Now, use these new fields as Lagrange multipliers and add to the above Lagrange density the constraint terms:

$$\kappa^{(i)}_{mn\ldots} [T^{(i)}_{\mu\nu\ldots} e^m_\mu e^n_\nu \ldots - T_S^{(i)mn\ldots}] \quad (5.7)$$

and

$$\lambda^{(i)}_{mn\ldots} \wedge [D(T^{(i)}_{\mu\nu\ldots} e^m_\mu e^n_\nu \ldots) - T_S^{(i)mn\ldots} a e^a] \quad (5.8)$$

where the D with no subscript is an external derivative operator that creates a one-form from the spacetime scalar field on which it acts. The internal indices on the matter tensor field are raised using the appropriate internal metric.

The resulting action contains the contravariant tensor fields $T^{(i)}_{\mu_1\mu_2\ldots}$, but only when contracted with tetrads. As a result, any transformation that leaves these contractions invariant is a gauge transformation. This result can be used to show that even these actions are invariant under any ‘pullback’ values $T^{(i)}_{\mu_1\mu_2\ldots}$ of this contravariant tensor field are chosen to be any values such that their contractions with the pulled back tetrad $e^m_\mu$ are the same as the old contractions:

$$T^{(i)}_{\mu_1\mu_2\ldots} e^{m_1}_\mu e^{m_2}_\mu \ldots = T^{(i)}_{\mu_1\mu_2\ldots} e^{m_1}_\mu e^{m_2}_\mu \ldots \quad (5.9)$$

The action is then invariant under any ‘pullback’ of degree one for which such new components $T^{(i)}_{\mu_1\mu_2\ldots}$ of $T^{(i)}_{\mu_1\mu_2\ldots}$ exist.

Since these contravariant fields appear only in constraints and only in contractions with tetrads, it might seem that they could be eliminated entirely, even from this more general formalism. This is not easy to do, and simple attempts to remove them do not work. For example, if the contravariant fields are replaced by covariant tensor fields, then the resulting covariant indices must be contracted with some other fields. This requires either the introduction of inverse tetrads to supply the needed contravariant indices, the introduction of contravariant indices on the Lagrange multipliers $\lambda$ and $\kappa$, or the contraction of these covariant indices with the forms $dx^\mu \wedge dx^\nu \wedge \ldots$, which is essentially what was done in section 2 for the class of matter fields with the appropriate symmetries. Another option is to remove the contravariant fields entirely and to replace them in the differential constraint by their tetrad components. While the resulting matter dynamics is correct, such an action contains tetrads only through the volume element and through the single tetrad in each differential constraint (5.8). But this is completely the wrong coupling to the gravitational field, and we are forced to keep the contravariant fields.
6. Discussion

We have seen that it is possible to write actions for certain kinds of matter fields that remain well defined when the tetrad is degenerate and contain only fields that can be pulled back. The relationship of the actions presented above to standard formulations involving matter is much the same as the relationship between the Palatini and Einstein–Hilbert (metric) formulations of gravity. The dynamics agree when $e^a_\mu$ is non-degenerate, but may differ when $\det(e)$ vanishes. A canonical analysis of the above actions is also similar to that of the Palatini action, and leads in the same way to second class constraints.

Again like the Palatini action, our new actions are invariant under pullback of the gravitational and matter fields from one spacetime to another through any map of degree one. It would thus seem that we should identify a set of fields and its pullback as gauge related, at least if the pullback map is continuously connected to the identity map.

This point was raised in [3], in the purely gravitational context, and is not significantly different here. There, this comment was followed by a discussion of how the gauge group might be enlarged so that this is so. We would simply like to make the comment that while it is nice to identify a gauge group, such an identification is not strictly necessary for the identification and investigation of gauge orbits. Since gauge transformations are defined in an infinitesimal form (see, for example [8]) with no stipulation that they can be integrated in such a way that the finite transformations form a group, we can use the infinitesimal transformations to define the gauge orbits. That is, two field histories are gauge related if and only if they can be continuously connected by a set of infinitesimal transformations. This relation is necessarily an equivalence relation, and so can be used to define the physical equivalence classes without reference to any gauge group.

Having said this, we would like to make one further suggestion with regard to these physical equivalence classes. Whenever the 'gauge group' is not connected, there is always the question of whether to identify field histories that are related by large gauge transformations; that is, by gauge transformations not continuously connected to the identity. A similar issue arises here: only pullbacks via maps of degree one can be continuously connected to the identity, so that only such pullbacks necessarily impose physical equivalence between field histories. However, we would like to suggest that histories related by pullbacks through maps of any degree other than zero might also be considered physically equivalent.

We suggest this, despite the fact that such transformations do not in fact leave the action invariant (as large gauge transformations would do), but multiply it by the degree of the map. Nevertheless, a number of arguments could be made that this is a physically reasonable thing to do, based on the observational indistinguishability of such histories. These arguments discuss measurements performed in the two field histories. We stress the word in because for such arguments it is important that the fields that define the laboratory and measuring apparatus also be a part of our description, and that they, too, be pulled back from one history to the other. In such a setting, any field configuration that describes an experiment and result in one spacetime is pulled back to describe the same experiment and the same result in the other spacetime. While there are, at least in principle, global measurements that could distinguish between the two spacetimes, the experimental apparatus required are not related by pullbacks, and such measurements are never performed by experimentalists living in such spacetimes.

Potential problems with this suggestion are the possibilities of excessively large equivalence classes resulting from such identifications, and of complicated topologies on the space of such classes. For example, the space of solutions of the equations of motion
on a cylinder would not be disconnected from the space of solutions on two cylinders, but would be connected through the two cylinder pullbacks of one cylinder solutions. This might also make any formulation as a sum over histories that allows topology change even more complicated, since the action would no longer be a continuous (or even well defined) functional on the space of gauge equivalence classes. I will not comment further on the possible implications for ‘topology change’, since a number of interpretations are possible without a definite structure in which to work. Note, however, that with this definition of equivalence the classical topology changing solutions described in [3] would be considered equivalent to solutions that do not change topology. It is not clear whether these solutions can be generalized in a straightforward way so that they are no longer the pullback of some solution that does not change topology.

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