IMPROVED BOUND ON SETS INCLUDING NO SUNFLOWER WITH THREE PETALS

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Abstract. A sunflower with k petals, or k-sunflower, is a family of k sets every two of which have a common intersection. Known since 1960, the sunflower conjecture states that a family \( F \) of sets each of cardinality \( m \) includes a k-sunflower if \( |F| \geq c_m^k \) for some \( c_m \in \mathbb{R}_{>0} \) depending only on \( k \). The case \( k = 3 \) of the conjecture was especially emphasized by Erdős, for which Kostochka’s bound \( cm \left( \frac{\log \log m}{\log \log \log m} \right)^m \) on \( |F| \) without a 3-sunflower had been the best-known since 1997 until the recent development to update it to \( c \log m \).

This paper proves with an entirely different combinatorial approach that \( F \) includes three mutually disjoint sets if it satisfies the \( \Gamma \left( cm^{\frac{1}{2} + \delta} \right) \)-condition for any given \( \delta \in (0, 1/2) \). Here \( c \) is a constant depending only on \( \delta \), and the \( \Gamma \)-condition refers to

\[ |\{ U : U \in F \text{ and } S \subset U \}| < \left( cm^{\frac{1}{2} + \delta} \right)^{|S|} |F|, \]

for every nonempty set \( S \). This poses an alternative proof of the 3-sunflower bound \( \left( cm^{\frac{1}{2} + \delta} \right)^m \).

1. Motivation and Approach

In this paper we verify the following statement.

**Theorem 1.1.** For each \( \delta \in (0, 1/2) \), there exists \( c \in \mathbb{R}_{>0} \) such that a family \( F \) of sets each of cardinality \( m \in \mathbb{Z}_{>0} \) includes three mutually disjoint sets if it satisfies the \( \Gamma \left( cm^{\frac{1}{2} + \delta} \right) \)-condition. \( \square \)

This means that \( F \) includes a 3-sunflower if \( |F| > \left( cm^{\frac{1}{2} + \delta} \right)^m \), since for such an \( F \), there exists a set \( S \) with \( |S| < m \) such that the family \( \{ U - S : U \in F \text{ and } S \subset U \} \) satisfies the \( \Gamma \left( cm^{\frac{1}{2} + \delta} \right) \)-condition in the universal set minus \( S \). The claim asymptotically updates Kostochka’s bound \( \Gamma \left( cm^{\frac{1}{2} + \delta} \right) \) that had been the best-known related to the three-petal sunflower problem noted in \( [3] \) since 1997, until the recent development \( [4] \) to reduce the upper bound to \( c \log m \).

We prove the statement with a new combinatorial theory describing its approach in the rest of the section.

1.1. l-Extension of a Family of m-Sets. Let the universal set \( X \) have cardinality \( n \). Denote a subset of \( X \) by a capital alphabetical letter. It is an \( m \)-set if its cardinality is \( m \). Use the standard notation \( [p] := [1, p] \cap \mathbb{Z} \) for \( p \in \mathbb{Z} \), and \( \left( X \right)_m := \)

2010 Mathematics Subject Classification. 05D05: Extremal Set Theory (Primary).
Key words and phrases. sunflower lemma, sunflower conjecture, \( \Delta \)-system.
\{U : U \subset X', |U| = m\} for X' \subset X. For \( F \subset \binom{X}{m} \), we denote \( F[S] := \{U : U \in F, S \subset U\} \).

The family \( F \) satisfies the \( \Gamma(b) \)-condition \( (b \in \mathbb{R}_{>0}) \) if \( |F[S]| < b^{-|S|}|F| \) for every nonempty set \( S \).

The \( l \)-extension of \( F \) for \( l \in [n] - [m] \) is defined as
\[
Ext(F, l) := \left\{ T : T \in \binom{X}{l}, \text{ and } \exists U \in F, U \subset T \right\}.
\]

It is shown in \([5]\) that
\[
|Ext(F, l)| \geq \binom{n}{l} \left\{ 1 - m \exp \left[ -\frac{(l - m + 1)|F|}{8m!\binom{n}{m}} \right] \right\},
\]
for any \( F \subset \binom{X}{m} \) and \( l \in [n] - [m] \). The result means that an \( n \)-vertex graph \( G \) with \( \binom{n}{2} - k \) edges contains at most \( 2^{|F|} \binom{n}{l} \exp \left[ -\frac{(l - 1)k}{8m(n - 1)} \right] \) cliques of size \( l \): let the vertex set of \( G \) be \( X \), and \( F \) be the set of non-edges in \( G \) regarded as a family of \( 2 \)-sets. Then \( Ext(F, l) \) equals the family of \( l \)-sets each not a clique of size \( l \) in \( G \), which means the claim. Similar facts can be seen for \( m \)-uniform hypergraphs for small \( m \) such as 3.

1.2. Existence of a Bounded Set \( T \) with Dense \( Ext(F[T], l) \). An \((l, \lambda)\)-extension generator of \( F \) is a set \( T \subset X \) such that
\[
|Ext(F[T], l)| \geq \binom{n - |T|}{l - |T|}(1 - e^{-\lambda}),
\]
where \( \lambda \in \mathbb{R}_{>0} \), and \( e = 2.71 \ldots \) is the natural logarithm base. If \( \lambda \) is much larger than a constant, the \( l \)-sets in \( Ext(F[T], l) \) form a vast majority of \( \binom{X}{l}[T] \), the family of \( l \)-sets each containing \( T \).

We have a fact shown in \([6]\).

**Theorem 1.2.** (Extension Generator Theorem) There exists \( \epsilon \in (0, 1) \) satisfying the following statement: let \( X \) be the universal set of cardinality \( n \), \( m \in [n - 1] \), \( l \in [n] - [m] \), and \( \lambda \in (1, \frac{\epsilon m}{2m}) \). For every nonempty family \( F \subset \binom{X}{m} \), there exists an \((l, \lambda)\)-extension generator \( T \) of \( F \) with \( |T| \leq \left[ \ln \binom{n}{m} - \ln |F| \right]/\ln \frac{4}{m^2\lambda} \). \( \square \)

We will also confirm it in Section 2. The theorem could help us understand the structure of \( Ext(F, l) \): for some large family \( F \), we can find bounded sets \( T_1, T_2, \ldots, T_k \) such that \( Ext(F, l) \) is close to \( \bigcup_{i \in [k]} \binom{X}{l}[T_i] \).

In addition, an alternative proof has been given \([6]\) with the theorem that the monotone complexity of detecting cliques in an \( n \)-vertex graph is exponential. For any given polynomial-sized monotone circuit \( C \) for the \( k \)-clique problem \( (k = n^\epsilon \) for some constant \( \epsilon \in (0, 1) \)), the proof explicitly constructs a graph containing no \( k \)-clique for which \( C \) returns true. The standard method to show the exponential complexity uses the sunflower lemma or its variant with random vertex coloring \([7, 8]\).
1.3. To Show Theorem [1.1]. Given \( \mathcal{F} \subset \binom{X}{m} \), we first partition \( X \) into equal sized disjoint sets \( X_1, X_2, \ldots, X_r \) (\( r \approx m^{2/r} \)) such that \( \frac{m}{2^r} < |U \cap X_j| < \frac{2m}{r} \) for every \( j \in [r] \) and most \( U \in \mathcal{F} \). Find such \( X_j \) by the claims we show in Section 3. Then we will inductively construct three families \( \mathcal{F}_i \) (\( i \in [3] \)) of \( U \) for each \( j \) such that \( U_i \cap \bigcup_{j' \in [j]} X_{j'} \) are mutually disjoint for any three \( U_i \in \mathcal{F}_i \). The recursive invariant is verified by claims closely related to Theorem 1.2 which we will prove in the following section.

2. Proof of Theorem 1.2 and Other Facts

2.1. A Structural Lemma. Denote \( U \times U \times \cdots \times U \) by \( U^g \) for \( U \subset 2^X \) and \( g \in \mathbb{Z}_{>0} \), also writing

\[
\text{union}(U) = \bigcup_{p=1}^{g} U_p, \quad \text{for } U = (U_1, U_2, \ldots, U_g) \in U^g.
\]

Let \( w : (2^X)^g \to \mathbb{R}_{\geq 0} \) and \( m \in [n] \) be given in addition to \( g \). These define the norm \( \|U\| \) of \( U \) and sparsity \( \kappa(F) \) of \( F \subset \binom{X}{m} \) by

\[
\|U\| = \left[ \sum_{U \in U^g} w(U) \right]^\frac{1}{g}, \quad \text{and} \quad \kappa(F) = \ln \left( \frac{n}{m} \right) - \ln \|F\|,
\]

respectively.

Given such an \( F \), and numbers \( l \in [n] - |m| \) and \( j \in \mathbb{Z}_{>0} \), denote

\[
\mathcal{P}_{j,g} = \{ U : U \in F^g, \ |\text{union}(U)| = gm - j \},
\]

\[
\mathcal{D}_g = \left\{ (U, Y) : U \in F^g, \ Y \in \binom{X}{l}, \ \text{union}(U) \subset Y \right\},
\]

\[
\|\mathcal{P}_{j,g}\| = \sum_{U \in \mathcal{P}_{j,g}} w(U), \quad \text{and} \quad \|\mathcal{D}_g\| = \sum_{(U, Y) \in \mathcal{D}_g} w(U),
\]

extending the norm \( \cdot \) for \( \mathcal{P}_{j,g} \) and \( \mathcal{D}_g \), for which we say \( w \) induces \( \cdot \) and also the sparsity \( \kappa \). The family \( F \) satisfies the \( \Gamma_g(b, h) \)-condition on \( \cdot \) \((b, h \in \mathbb{R}_{>0})\) if

\[
\|U\| = \left[ \sum_{U \in (U \cap F)^g} w(U) \right]^\frac{1}{g}, \quad \text{for all } U \subset 2^X,
\]

and

\[
\|\mathcal{P}_{j,g}\| < hb^{-j} \|F\|^g, \quad \text{for every } j \in [(g-1)m].
\]

We may drop the subscript \( g \) if it is obvious from the context, so \( \mathcal{P}_{j,g} \) can be written as \( \mathcal{P}_j \), \( \Gamma_g(b, h) \) as \( \Gamma(b, h) \) etc.

In this subsection, we prove the following lemma that is a structural claim we will use to show Theorem 1.1 and to derive Theorem 1.2.

**Lemma 2.1.** Let

i) \( X \) be the universal set weighted by \( w : (2^X)^g \to \mathbb{R}_{\geq 0} \) for some \( g \in \mathbb{Z}_{\geq 2} \),

ii) \( l \in [n] \), \( m \in [l-1] \), and \( h, \gamma \in \mathbb{R}_{>0} \), such that \( \gamma \) and \( \frac{1}{gm} \) are both sufficiently large,

iii) and \( F \subset \binom{X}{m} \) satisfy the \( \Gamma_g \left( \frac{4m}{\gamma}, h \right) \)-condition on the norm \( \cdot \) induced by \( w \).
Then
\[ \|\mathcal{D}_g\| < \left(1 + \frac{h}{g}\right) \binom{n}{l} \binom{l}{m}^g \|\mathcal{F}\|^g. \quad \square \]

Since
\[ \|\mathcal{D}_g\| = \sum_{(U,Y) \in \mathcal{D}_g} w(U) = \sum_{Y \in \binom{X}{l}} \sum_{U \in [\mathcal{F} \cap \binom{Y}{m}]} w(U), \]
it means:

**Corollary 2.2.** For such objects and \( \epsilon \in (0, 1) \), there are \( \lceil (1 - \epsilon) \binom{g}{l} \rceil \) sets \( Y \in \binom{X}{l} \) such that
\[ \sum_{U \in [\mathcal{F} \cap \binom{Y}{m}]} w(U) < \frac{1 + h}{\epsilon} \binom{n}{m}^g \sum_{U \in \mathcal{F}_g} w(U). \quad \square \]

Write
\[ b = \frac{4\gamma n}{l}, \quad \text{and} \quad v(n') = \binom{n'}{m}^{-g+1} \prod_{p=1}^{g-1} \binom{n' - pm}{m}, \]
for \( n' \in [n] - [gm] \) for the proof. Observe the five remarks.

A) \( \mathcal{P}_j = \emptyset \) if \( j > (g - 1)m \), so
\[ \|\mathcal{D}\| = \sum_{Y \in \binom{X}{l}} \sum_{U \in \binom{Y}{m}} w(U) = \sum_{Y \in \binom{X}{l}} \left\| \binom{Y}{m} \right\|^g = \sum_{j=0}^{(g-1)m} \|\mathcal{P}_j\| \binom{n - gm + j}{l - gm + j}, \]
\[ < h \sum_{j=1}^{(g-1)m} b^{-j} \binom{n}{m}^g e^{-g\kappa(\mathcal{F})} \binom{n - gm + j}{l - gm + j}, \]
by the \( \Gamma_g(b, h) \)-condition of \( \mathcal{F} \), and \( \|\mathcal{F}\|^g = \binom{n}{m}^g e^{-g\kappa(\mathcal{F})}. \)

B) \( v(l) \leq v(n) \), since
\[ \prod_{i=0}^{m-1} \frac{1 - pm}{1 - \frac{pm}{n-i}} \leq 1, \quad \text{for} \ p \in [g - 1], \]
\[ \Rightarrow \frac{(l-pm)}{(l-1)} \frac{\binom{l}{m}}{\binom{n}{m}} \leq 1, \quad \Rightarrow \frac{v(l)}{v(n)} \leq 1. \]
C) By the identity \( \binom{x}{m} \binom{x-z}{y} = \binom{x}{y} \binom{x-y}{m} \),
\[
\binom{n}{m}^g \binom{n - gm}{l - gm} = \frac{1}{v(n)} \binom{n - gm}{l - gm} \prod_{p=0}^{g-1} \binom{n - pm}{m} 
\]
\[= \frac{(l-(g-1)m)^m}{v(n)} \binom{n - (g-1)m}{l - (g-1)m} \prod_{p=0}^{g-2} \binom{n - pm}{m} \]
\[= \frac{(l-(g-1)m)(l-(g-2)m)^m}{v(n)} \binom{n - (g-2)m}{l - (g-2)m} \prod_{p=0}^{g-3} \binom{n - pm}{m} \]
\[= \ldots = \prod_{p=0}^{g-1} \frac{\binom{n-(pm)}{m}}{v(n)} \binom{n}{l} \]
\[= \frac{v(l)}{v(n)} \binom{n}{l} \binom{l}{m}^g \leq \binom{n}{l} \binom{l}{m}^g. \]

D) So,
\[
\|P_0\| \binom{n - gm}{l - gm} \leq \|P\| \binom{n - gm}{l - gm} = \binom{n}{m}^g \binom{n - gm}{l - gm} e^{-g\kappa(F)} \]
\[\leq \binom{n}{l} \binom{l}{m}^g e^{-g\kappa(F)}. \]

E) For each \( j \in [(g-1)m] \),
\[
\binom{n - gm + j}{l - gm + j} = \binom{n - gm}{l - gm} \prod_{i=0}^{j-1} \binom{n - gm + i}{l - gm + i} < \binom{2n}{l} \binom{l}{m}^g \]
since \( l \) and \( n \) are both sufficiently larger than \( gm \).
\( \Box \)

It suffices to show by the remarks that
\[
(2.1) \quad \|D\| < 1 + h \sum_{j=1}^{(g-1)m} (2\gamma)^{-j} \binom{n}{l} \binom{l}{m}^g e^{-g\kappa(F)},
\]
as its RHS is less than \( \frac{(1 + \frac{1}{2})^g (\frac{1}{m})^g}{(\frac{1}{m})^g} \sum_{U \in \mathcal{F}, w(U)} \). We see from A), C) and E) that
\[
\sum_{j=1}^{(g-1)m} \|P_j\| \binom{n - gm + j}{l - gm + j}
\]
\[< h \sum_{j=1}^{(g-1)m} b^{-j} \binom{n}{m}^g e^{-g\kappa(F)} \binom{n - gm + j}{l - gm + j}
\]
\[< he^{-g\kappa(F)} \binom{n}{m}^g \binom{n - gm}{l - gm} \sum_{j=1}^{(g-1)m} (2\gamma)^{-j}
\]
\[\leq he^{-g\kappa(F)} \binom{n}{l} \binom{l}{m}^g \sum_{j=1}^{(g-1)m} (2\gamma)^{-j}. \]

Also by D), (2.1) is confirmed completing the proof of Lemma (2.1).
Lemma 2.4. Let any $F \tilde{\rightarrow} \text{primitive with } \epsilon$ sufficiently small.

Theorem 2.3. Let $X$ be primitively weighted inducing the norm $\| \cdot \|$ for every sufficiently small $\epsilon \in (0, 1)$, and $F \subset \binom{X}{m}$ satisfying the $\Gamma_2 \left( \frac{m}{2^{m}}, 1 \right)$-condition on $\| \cdot \|$ for some $l \in [n]$, $m \in [l]$, and $\gamma \in [e^{-2}, lm^{-1}]$, there are $\binom{n}{m} (1 - \epsilon)$ sets $Y \in \binom{X}{m}$ such that

$$\left( 1 - \sqrt{\frac{2}{\epsilon \gamma}} \frac{l}{m} \right) \| F \| < \left( \frac{Y}{m} \right) < \left( 1 + \sqrt{\frac{2}{\epsilon \gamma}} \frac{l}{m} \right) \| F \| . \quad \square$$

The rest of this subsection proves the theorem. Given such $\epsilon, m, l, \gamma$ and $F$, use the same $D$ and $P_l$ as Section 2.1. Find the following remarks.

F) Let the weight $w$ of $X$ be primitive with $\tilde{w}$, then

$$\| G \| = \sum_{V \in G} \tilde{w}(U), \quad \text{for any } G \subset 2^X,$$

since $\sum_{V \in G} \tilde{w}(U)^2 = \sum_{U \in G^2} w(U) = \| G \|^2$.

G) By this linearity of the norm $\| \cdot \|$ for primitive weight,

$$\sum_{Y \in \binom{X}{m}} \left( \frac{Y}{m} \right) = \| F \| \left( n - m \right) = \left( \frac{n}{m} \right) \left( \frac{n - m}{l - m} \right) e^{-\kappa(F)} = \left( \frac{n}{l} \right) \left( \frac{l}{m} \right) e^{-\kappa(F)}.$$

H) $\| D \| = \sum_{Y \in \binom{X}{m}} \left( \frac{Y}{m} \right)^2 > 0$, since

$$\| D \| = \sum_{j=0}^{m} \| P_j \| \left( \frac{n - 2m + j}{l - 2m + j} \right) < \sum_{j=0}^{m} h b^{-j} \| F \|^2 \left( \frac{n - 2m + j}{l - 2m + j} \right)$$

as in A). So $\| F \| > 0$ meaning $\| D \| > 0$ by definition.

I) By (2.1) for $g = 2$,

$$\| D \| < \frac{e^{-2\kappa(F)}}{1 - (2\gamma)^{-1}} \left( \frac{n}{m} \right) \left( \frac{l}{m} \right)^2 . \quad \square$$

We find another property on $\| D \|$ below. The statement is general holding for any $F$ that meets the conditions.

Lemma 2.4. Let

i) $X$ be primitively weighted by $\left( 2^X \right)^2 \rightarrow \mathbb{R}_{\geq 0}$ inducing the norm $\| \cdot \|$,

ii) $l \in [n]$, $m \in [l]$, $t \in \mathbb{R}_{\geq 0}$,

iii) $F \subset \binom{X}{m}$ such that

$$0 < \| D \| \leq t \left( \frac{n}{l} \right) \left( \frac{l}{m} \right)^2 e^{-2\kappa(F)},$$

iv) and $u, v \in \mathbb{R}_{\geq 0}$ with

$$u < 1, \quad u \left( \frac{n}{l} \right) \in \mathbb{Z}, \quad \text{and} \quad t < 1 + \frac{u(v - 1)^2}{1 - u}.$$
The two statements hold.

a) If \( v \geq 1 \), more than \( (1-u)(\binom{n}{l}) \) sets \( Y \in \binom{X}{l} \) satisfy \( \|Y_m\| < v(l_m) e^{-\kappa(F)} \).
b) If \( v \leq 1 \), more than \( (1-u)(\binom{n}{l}) \) sets \( Y \in \binom{X}{l} \) satisfy \( \|Y_m\| > v(l_m) e^{-\kappa(F)} \).

Proof. a): Put
\[
z = e^{-\kappa(F)}, \quad \text{and} \quad x_j = \|Y_j\|,
\]
where \( Y_j \) is the \( j \)-th \( l \)-set in \( \binom{X}{l} \). Suppose to the contrary that \( x_j \geq vz(l_m) \) if \( 1 \leq j \leq u(n) \).

Noting \( \sum_{1 \leq j \leq u(n)} x_j = z(l_m) > 0 \) from \( G \) and \( \|D\| > 0 \), let \( y \in (0,1) \) satisfy
\[
\sum_{1 \leq j \leq u(n)} x_j = yz(l_m)(l_m),
\]
so \( y \geq uv \). Find that
\[
\sum_{1 \leq j \leq u(n)} x_j^2 \geq \left[ \frac{yz(l_m)}{u(n)} \right]^2 \frac{u(n)}{l_m} = \frac{y^2z^2}{u} \binom{n}{l_m}^2,
\]
and
\[
\sum_{u(n) < j \leq \binom{n}{l}} x_j^2 \geq \left( 1 - y \right) \binom{n}{l_m}^2 \left( 1 - u \right) \binom{n}{l_m}^2 = \frac{1 - y^2}{1 - u} \binom{n}{l_m}^2,
\]
meaning
\[
(2.2) \quad \|D\| = \sum_{Y \in \binom{n}{l}} \|Y_m\|^2 \geq fz^2(l_m)^2, \quad \text{where} \quad f = \frac{y^2}{u} + \frac{(1 - y^2)}{1 - u},
\]
From \( y \geq uv \geq u \),
\[
(2.3) \quad f \geq uv^2 + \frac{(1 - uv^2)}{1 - u} = 1 + \frac{u(v - 1)^2}{1 - u} > t.
\]
This contradicts the given condition proving a).

b): Suppose \( x_j \leq vz(l_m) \) if \( 1 \leq j \leq u(n) \). Use the same \( y \) and \( f \) so \( y \leq uv \) and (2.2). These also imply (2.3) producing the same contradiction. Thus b). \( \square \)

Set
\[
t = \frac{1}{1 - (2\gamma)^{-1}}, \quad u = \frac{\binom{n}{l}}{\binom{n}{l}}, \quad v = 1 + \frac{u}{(\frac{n}{2})^{\gamma} \sqrt{\gamma}}, \quad \text{and} \quad v' = 1 - \frac{u}{(\frac{n}{2})^{\gamma} \sqrt{\gamma}}.
\]
Then
\[
1 + \frac{u(v - 1)^2}{1 - u} = 1 + \frac{u(v' - 1)^2}{1 - u} = 1 + \gamma^{-1} \frac{u^3}{(\frac{n}{2})^{3} (1 - u)} > t,
\]
since \( u > \frac{\gamma}{2} - e^2 \) from \( \epsilon^2 \leq \gamma \leq \binom{n}{l} \). Here \( l < n \) is assumed as the theorem is trivially true if \( l = n \). By 1) and Lemma 2.4,
\[
v'(l_m) e^{-\kappa(F)} < \|Y_m\| < v(l_m) e^{-\kappa(F)},
\]
for some \((1 - 2n) \binom{n}{l} \) sets \(Y \in \binom{X}{l} \). As \(e^{-\kappa(F)} = \|F\| \binom{n}{m}^{-1} \), this means there are \([ \binom{n}{l} (1 - \epsilon) \) sets \(Y \in \binom{X}{l} \) such that

\[
\left(1 - \sqrt{\frac{2}{e\gamma}} \right) \binom{m}{n} \|F\| < \left\| \binom{Y}{m} \right\| < \left(1 + \sqrt{\frac{2}{e\gamma}} \right) \binom{m}{n} \|F\|,
\]

completing the proof of Theorem \ref{thm:2.3}.

2.3. Deriving Theorem \ref{thm:1.2}. Given an \(F\), let us assume for a while that \(X\) is primitively weighted with \(\tilde{w} : U \mapsto |F[U]| \). The norm of \(G \subset 2^X\) and sparsity of \(F\) in this default case are

\[
\|G\| = \sum_{U \in G} |F[U]|, \quad \text{and} \quad \kappa(F) = \ln \binom{n}{m} - \ln |F|,
\]

respectively, by the linearity of \(\| \cdot \|\). The latter depends on \(X\) as well as \(|F|\). Generalize the default sparsity to any uniform family \(G \in \binom{X'}{m'}\) of \(m'\)-sets in the universal set \(X' \subset X\) (\(m' \in \{X'\}\)) to write \(\kappa(G) = \ln \binom{|X'|}{m'} - \ln |G|\).

Remarks.

J) The notation could be useful to express \(\ln |G|\): for example, \(|T| \leq \kappa(F) / \ln \frac{d}{\sqrt{n-m}}\)

for Theorem \ref{thm:1.2} and \(\kappa(F_X) < \kappa(F) + m\) in Lemma \ref{lem:3.3} we will see in the next section.

K) As we use on the bottom of the subsection, the same definition can apply to the projection \(G_m'\) of \(G\) onto \(X'\), i.e., \(G_m' = \{U \cap X' : U \in G, |U \cap X'| = m'\}\).

L) \(\kappa(Ext(F, l)) \leq \kappa(F)\) for \(l \leq [n] - [m]\). For there are \(|F| \binom{n-m}{l} = \binom{n}{m} e^{-\kappa(F)} \binom{n-m}{l} = \binom{n}{l} e^{-\kappa(F)}\) set pairs \((S, T)\) such that \(S \in F, T \in \binom{X}{l}\) and \(S \subset T\). This means

\[
|Ext(F, l)| \geq \binom{n}{l} e^{-\kappa(F)}
\]

leading to the claim.

M) Join a \(p\)-set \(P\) to \(X\) such that \(P \cap X = \emptyset\). The sparsity of \(Ext(F, m + p)\) in the universal set \(X \cup P\) at most \(\kappa(F)\) in \(X\) since

\[
|Ext(F, m + p)| \geq \sum_{j=0}^{p} \binom{n}{m + j} e^{-\kappa(F)} \binom{p}{p - j} = \binom{n + p}{m + p} e^{-\kappa(F)}.
\]

N) The following lemma is proven in \cite{3} and Appendix 1.

Lemma 2.5. For \(F \subset \binom{X}{m}\) such that \(m \leq \frac{n}{2}\),

\[
\kappa \left[ \binom{X}{2m} - Ext(F, 2m) \right] \geq 2\kappa \left[ \binom{X}{m} - F \right]. \quad \square
\]

Assume \(|F| > b^m\) for some \(b \in \mathbb{R}_{\geq 1}\). There exists \(T \subset X\) such that \(|T| < m\), \(|F[T]| \geq |F|b^{-|T|}\), and \(|F[T \cup S]| < b^{-|T \cup S|}\) for any nonempty \(S \subset X - T\). We use such a family \(F[T]\) projected onto the universal set \(X - T\) in place of \(F\) in our proof of Theorem \ref{thm:1.1}. Observe that the \(F\) satisfies not only the \(\Gamma(b)\)-condition, but also the \(\Gamma_2 (bm^{-1}, 1)\)-condition on \(\| \cdot \|\) since

\[
\|P_j\| = \sum_{\{U_{1}, U_{2} \in F \atop U_{1} \cap U_{2} = \emptyset\}} \tilde{w}(U_{1}) \tilde{w}(U_{2}) \leq \sum_{S \in \binom{X}{m}} |F[S]|^2 < |F|^2 \binom{m}{j} b^{-j},
\]

for each \(j \in [m]\).

By Theorem \ref{thm:2.3}.
Corollary 2.6. Let $X$ be the universal set of cardinality $n$, $m \in [n-1]$, $l \in [n]-[m]$ and $\gamma \in \mathbb{R}_{>0}$ be sufficiently large not exceeding $\frac{1}{m}$. For any $\mathcal{F} \subset \binom{X}{m}$ satisfying the $\Gamma\left(\frac{4\gamma mn}{l}\right)$-condition, there are $\left(\binom{n}{j} \left(1 - \frac{2}{\sqrt{\lambda}}\right)\right)$ sets $Y \in \binom{X}{\gamma}$ such that

$$\frac{(\binom{n}{m}|\mathcal{F}|}{(\binom{n}{m})} \left(1 - \frac{1}{\sqrt{\lambda}}\right) < |\mathcal{F} \cap \binom{Y}{m}| < \frac{(\binom{n}{m}|\mathcal{F}|}{(\binom{n}{m})} \left(1 + \frac{1}{\sqrt{\lambda}}\right).$$

□

We show Theorem 1.2 from the corollary. Given $m,l,\lambda$, sufficiently small $\epsilon$, and $\mathcal{F}$ as the statement, set

$$l_0 = \left[\frac{l\sqrt{\epsilon}}{\lambda}\right], \quad \gamma = \frac{1}{\sqrt{\epsilon}}, \quad \text{and} \quad b = \frac{4\gamma mn}{l_0}.$$  

Then $\gamma$ is sufficiently large and less than $\frac{1}{m}$ since $1 < \lambda < \frac{1}{m}$.

There exists a set $T$ such that $|T| \leq \kappa(\mathcal{F})/\ln \frac{d}{mX}$ and $\mathcal{F}[T]$ satisfies the $\Gamma(b)$-condition in $X - T$: because the cardinality $j$ of such $T$ satisfies

$$\left(\binom{n}{m} e^{-\kappa(\mathcal{F})} b^{-j} = |\mathcal{F}| b^{-j} \leq |\mathcal{F}[T]| \leq \left(\binom{n-j}{m-j}\right),
\Rightarrow b^{-j} \left(\binom{n}{m}\right)^{-j} \leq b^{-j} \prod_{j'=0}^{j-1} \frac{n-j}{m-j} = \left(\frac{\binom{n}{m}}{\binom{n-j}{m-j}}\right)b^{-j} \leq e^{\kappa(\mathcal{F})},
\Rightarrow j \leq \frac{\kappa(\mathcal{F})}{\ln \frac{d}{mX}}.$$  

Assume $j < m$, otherwise the desired claim is trivially true.

Apply Corollary 2.6 to $\mathcal{F}[T]$ in the universal set $X - T$ noting $\frac{l_0}{m^2} \leq \frac{l_0 - j}{(m-j)^2}$ and $b \geq \frac{4\gamma(m-j)(n-j)}{l_{0-j}}$. We see

$$|\text{Ext}(\mathcal{F}[T], l_0)| > \left(\binom{n-j}{l_0-j} \left(1 - \frac{2}{\sqrt{\lambda}}\right)\right),$$

from which

$$|\text{Ext}(\mathcal{F}[T], l)| > \left(\binom{n-j}{l-j} \left(1 - e^{-\lambda}\right)\right),$$

proving Theorem 1.2. The truth of the last inequality is due to Lemma 2.5 as $\frac{l_0 - j}{l_{0-j}} \geq \lambda^{-1/2}$, it means

$$\kappa \left[\left(\binom{X}{l}\right)[T] - \text{Ext}(\mathcal{F}[T], l)\right] \geq 2^{\left|\log_2 \frac{l_{0-j}}{l_{0-j}}\right|} \kappa \left[\left(\binom{X}{l_0}\right)[T] - \text{Ext}(\mathcal{F}[T], l_0)\right] > \lambda,$$

in the universal set $X - T$ leading to the inequality.

3. Splitting the Universal Set

Given $m \in [n]$ with $m|n$ and $q \in [m]$, let

$$d = \frac{nq}{m}, \quad \text{and} \quad r = \left\lfloor \frac{m}{q} \right\rfloor.$$  

Assume for a while

$$r \in \mathbb{Z}_{\geq 2}, \quad \Rightarrow d = \frac{n}{r},$$
Denote
\[ \mathcal{X}_j := \{(X_1, X_2, \ldots, X_j) : X_i \text{ are mutually disjoint } d\text{-sets}\}, \]
for \( j \in [r] \). Call an element of \( \mathcal{X}_r \) \( r \)-split of \( X \) noting the given \( q \) decides \( r \).

When a \( j \) is also given, define
\[ \mathcal{F}_X = \{ U : U \in \mathcal{F}, \text{ and } |U \cap X_i| = q \text{ for every } i \in [j]\}. \]
for \( \mathcal{F} \subset \binom{\mathbf{X}}{m} \) and \( X \in \mathcal{X}_j \), and
\[ T_{\mathcal{F}, j} := \{(U, X) : X \in \mathcal{X}_j \text{ and } U \in \mathcal{F}_X\}. \]
Let \( X \) be primitively weighted with \( \hat{w} : 2^X \to \mathbb{R}_{\geq 0} \), inducing the norm \( \| \cdot \| \) and sparsity \( \kappa \). Extend \( \| \cdot \| \) to write
\[ \| T_{\mathcal{F}, j} \| = \sum_{(U, X) \in T_{\mathcal{F}, j}} \hat{w}(U). \]
For \( j = 0 \), let \( \mathcal{X}_0 = \{\emptyset\} \) and \( \mathcal{F}_\emptyset = \mathcal{F} \) so \( \| T_{\mathcal{F}, 0} \| = \| \mathcal{F} \| \). Assume \( \| \mathcal{F} \| > 0 \).

Prove the following lemma.

**Lemma 3.1.**
\[ \| T_{\mathcal{F}, j} \| = \binom{d}{q}^j \binom{n - dj}{m - qj} e^{-\kappa(\mathcal{F})} \prod_{i=0}^{j-1} \binom{n - di}{d}, \]
for every \( j \in [0, r) \cap \mathbb{Z} \) and \( \mathcal{F} \subset \binom{\mathbf{X}}{m} \).

**Proof.** We show the claim by induction on \( j \) with the trivial basis \( j = 0 \). Assume true for \( j \) and prove for \( j + 1 \). Fix any \( X = (X_1, X_2, \ldots, X_j) \in \mathcal{X}_j \) putting
\[ X' = X - \bigcup_{i=1}^{j} X_i, \quad n' = |X'|, \quad m' = m - jq, \quad \text{and} \quad \gamma_X = \frac{\| \mathcal{F}_X \|}{\binom{d}{q}^j \binom{n'}{m'}}. \]
Also write \( X' = (X_1, X_2, \ldots, X_j, X_{j+1}) \), for a \( d \)-set \( X_{j+1} \in \binom{\mathbf{X}}{d} \).

For each \( U \in \mathcal{F}_X \), there are \( \binom{m'}{q} \binom{n' - m'}{d - q} \) sets \( X_{j+1} \in \binom{\mathbf{X}}{d} \) such that \( |U \cap X_{j+1}| = q \). So the sum of \( \hat{w}(U) \) for \( (U, X') \in T_{\mathcal{F}, j+1} \) constrained by the fixed \( X \) is
\[ \binom{m'}{q} \binom{n' - m'}{d - q} \| \mathcal{F}_X \| = \binom{m'}{q} \binom{n' - m'}{d - q} \gamma_X \binom{d}{q}^j \binom{n'}{m'}. \]
\[ = \binom{n'}{q} \binom{n' - q}{m' - q} \binom{n' - m'}{d - q} \gamma_X \binom{d}{q}^j. \]
Here
\[ \binom{n'}{q} \binom{n' - q}{m' - q} \binom{n' - m'}{d - q} = \frac{(n' - q)!}{(m' - q)!((d - q)!(n - m' - d + q)!} = \binom{n' - q}{d} \binom{n' - d}{m' - q}. \]
So the above equals
\[ \binom{n'}{q} \binom{n' - q}{m' - q} \binom{n' - d}{m' - q} \gamma_X \binom{d}{q}^j \binom{n'}{m'}. \]
Note \( n' \geq m' + d - q \) from \( \frac{m'}{q} = r \geq j + 1 \) and \( n' = n - dj \).
By induction hypothesis,
\[ \sum_{X \in \mathcal{X}_j} \|\mathcal{F}_X\| = \|\mathcal{T}_{\mathcal{F},j}\| = \left(\frac{d}{q}\right)^{j} \frac{n^r}{m^r} e^{-\kappa(\mathcal{F})} \prod_{i=0}^{j-1} \left(\frac{n-di}{d}\right), \]

\[ \Rightarrow \sum_{X \in \mathcal{X}_j} \gamma_X = e^{-\kappa(\mathcal{F})} \prod_{i=0}^{j-1} \left(\frac{n-di}{d}\right). \]

Hence,
\[ \|\mathcal{T}_{\mathcal{F},j+1}\| = \sum_{X \in \mathcal{X}_j} \gamma_X \left(\frac{d}{q}\right)^{j+1} \frac{n'-d}{m'-q} \frac{m'}{q} \left(\frac{n'}{d}\right) \]
\[ = \left(\frac{d}{q}\right)^{j+1} \frac{n-d(j+1)}{m-q(j+1)} e^{-\kappa(\mathcal{F})} \prod_{i=0}^{j} \left(\frac{n-di}{d}\right), \]

proving the induction step. The lemma follows. \(\square\)

It means \( \|\mathcal{T}_{\mathcal{F},r-1}\| = \left(\frac{d}{q}\right)^r e^{-\kappa(\mathcal{F})} |\mathcal{X}_{r-1}|. \) By the natural bijection between \( \mathcal{X}_{r-1} \) and \( \mathcal{X}_r, \)
\[ \sum_{X \in \mathcal{X}_r} \|\mathcal{F}_X\| = \|\mathcal{T}_{\mathcal{F},r}\| = \left(\frac{d}{q}\right)^r e^{-\kappa(\mathcal{F})} |\mathcal{X}_r| = \frac{\left(\frac{d}{q}\right)^r \|\mathcal{F}\|}{\binom{n}{m}} |\mathcal{X}_r| \]

Considering the case \( r = 1 \) as well, we have:

**Corollary 3.2.** Let \( X \) be primitively weighted inducing the norm \( \| \cdot \| \). Given \( m \in [n] \) and \( q \in [m] \), let \( r = m/q \) and \( d = n/r \) be both positive integers. For a family \( \mathcal{F} \subseteq \binom{X}{m} \) with \( \|\mathcal{F}\| > 0 \), there exists an \( r \)-split \( X \) of \( X \) such that \( \|\mathcal{F}_X\| \geq \left(\frac{d}{q}\right)^r \|\mathcal{F}\| \binom{n}{m}. \) \(\square\)

Note that if \( q \) does not divide \( m \) where \( |U \cap \mathcal{X}_r| = q' \in [q, 2q] \cap \mathbb{Z} \) for \( U \in \mathcal{F} \), some \( X \) meets \( \|\mathcal{F}_X\| \geq \left(\frac{d}{q}\right)^{r-1} \frac{n-q-r-1}{q'} \binom{n-r}{m}. \) by the same argument. For \( q = 1 \):

**Corollary 3.3.** For a universal set \( X \) primitively weighted inducing the sparsity \( \kappa \), and any family \( \mathcal{F} \subseteq \binom{X}{m} \) with \( m/n \) and finite \( \kappa(\mathcal{F}) \), there exists an \( m \)-split \( X \) of \( X \) such that \( \kappa(\mathcal{F}_X) < \kappa(\mathcal{F}) + m \).

**Proof.** Since \( \kappa(\mathcal{F}_X) \leq \ln \binom{n}{m} - \ln \left(\frac{m}{n}\right)^m e^{-\kappa(\mathcal{F})} \big/ \frac{n}{m} \leq \kappa(\mathcal{F}) + m \) by the standard estimate of a binomial coefficient that is also derived in Appendix 2. \(\square\)

Let us now focus on the first case \( j = 1 \) of the lemma. Relax the constraints on \( q \) and \( d \) to see the following statement.

**Corollary 3.4.** Let \( X \) be primitively weighted inducing \( \| \cdot \| \), \( m, d \in [n] \) and \( q \in [0, m] \cap \mathbb{Z} \) such that \( n-d-m+q \geq 0 \). For each \( \mathcal{F} \subseteq \binom{X}{m} \) and \( \epsilon \in (0, 1) \), there exist no more than \( \left\lfloor \frac{\epsilon n^m}{m!} \right\rfloor \) sets \( X_1 \in \binom{X}{m} \) each with
\[ \|\mathcal{F}_{X_1}\| > \frac{(m-d-q)/d}{\\epsilon(n/m)} \|\mathcal{F}\|, \] where \( X = (X_1) \).
4. Proof of Theorem \[ \text{111} \]

We prove the theorem in this section. Given \( F \subset \binom{X}{m} \) and a sufficiently small \( \delta \in (0, 1/2) \) by the statement, let

\[
\epsilon = e^{-1/\delta}, \quad g = \left[ e^{1/\epsilon} \right], \quad c = e^{\epsilon}, \quad \text{and} \quad b_x = e^{\epsilon} m^{\frac{1}{2} + \delta},
\]

assuming \( F \) satisfies the \( \Gamma(b_x) \)-condition. WLOG \( m > e^c \), otherwise \( F \) includes three mutually disjoint sets similarly to the proof of the sunflower lemma \[ \text{9} \]: select any \( U_1 \in F \) eliminating all sets in \( F \) that intersect with \( U_1 \). By the \( \Gamma(b_x) \)-condition with \( b_x > 3m \), this removes less than a third of the original \( F \). Find \( U_2 \) and \( U_3 \) in the remaining \( F \) similarly, and the obtained three are mutually disjoint.

Further let

\[
z = \left\lfloor \log_2 m^{(1-\delta)/2} \right\rfloor, \quad r = 2^z, \quad \text{and} \quad q = \frac{m}{r}.
\]

Assume \( n = |X| \) is larger than \( m^4 \) and divisible by \( mr \). Otherwise add some extra elements to \( X \).

4.1. Preprocess. On such objects, we first perform our initial construction. Prove a recursive statement.

**Lemma 4.1.** Let

i) \( j \in [0, z] \cap \mathbb{Z} \), \( \delta_j = \sum_{j=0}^{j} \left( 2^{-j} m \right)^{\frac{1}{2} + \epsilon} \),

ii) \( X' \in \binom{X}{2^{-j} n} \), weighted primitively inducing the norm \( \| \cdot \| \),

iii) and \( G \subset 2^{X'} \) such that \( \|G\| > 0 \), and \( |U| - 2^{-j} m < \delta_j \) for every \( U \in G \).

There exists an \( 2^{z-j} \)-split \( X = (X_1, X_2, \ldots, X_{2^{z-j}}) \) of \( X' \), and \( G' \subset G \) such that

\[
\|G'\| > \left( 1 - 4^{1-j} e^{-m^c} \right) \|G\|, \quad \text{and} \quad |U \cap X_{j'}| - 2^{-j} m < \delta_z,
\]

for every \( j' \in [2^{z-j}] \) and \( U \in G' \).

**Proof.** Proof by induction on \( j \) with the trivial basis \( j = z \). Assume true for \( j + 1 \) and prove true for \( j \).

Let

\[
G_{m_1, m_2, Y} = \{ U : U \in G, \ |U \cap Y| = m_1, \ \text{and} \ |U \cap X' - Y| = m_2 \},
\]

for each \( Y \in \binom{X'}{m_1/2} \), and \( m_1, m_2 \in [m] \) with \( |m_1 + m_2 - 2^{-j} m| < \delta_j \). By Corollary \[ \text{3.4} \] there are no more than \( m^{-3} \binom{|X'|}{m_1/2} \binom{|X'|}{m_2} \) sets \( Y \) such that

\[
\|G_{m_1, m_2, Y}\| > \frac{m^3 \binom{|X'|}{m_1/2} \binom{|X'|}{m_2}}{\binom{|X'|}{m_1 + m_2}} \|G\|.
\]
We also have
\[
\ln \left( \left( \frac{|X'_1|}{m_1} \right)^2 \left( \frac{|X'_2|}{m_2} \right)^2 \right) < -\left( \frac{m_1}{2(m_1 + m_2)} \right)^2 < -m^{\epsilon + \epsilon'},
\]
if \( m_1 - m_2 > m^{-\epsilon}(m_1 + m_2)^{2/1 + \epsilon}, \)
by Lemma A.3, since \( m_1 + m_2 > 2^{-z+1}m - \delta_j > m^{1/2}. \)

For every possible combination of \( m_i \), exclude all \( Y \) with (4.1) from consideration. Fix any one remaining \( Y \), and all \( G_{m_1,m_2,Y} \) meet \( (1.1) \). Delete from \( G \) the union of \( G_{m_1,m_2,Y} \) each with \( |m_1 - m_2| > m^{-\epsilon}(m_1 + m_2)^{2/1 + \epsilon}. \) Then
- this reduces \( \|G\| \) only by its \( e^{-m^a} \) or less,
- and \( |m_i - 2^{-j-1}m| < \delta_j + 1 \) for each remaining \( G_{m_1,m_2,Y} \) and \( i \in [2] \), since \( m_1 + m_2 - 2^{-j}m < \delta_j \) and \( |m_1 - m_2| \leq m^{-\epsilon}(m_1 + m_2)^{2/1 + \epsilon}. \)

Now we obtain recursive solutions in both \( Y \) and \( X' - Y \). Weight \( Y \) primitively with \( U \to |F[U]| \). Apply \( G_1 = \{U \cap Y : U \in G\} \) to the induction hypothesis to obtain a \( 2^{z-j-1}\)-split \( X_1 = (X_1, X_2, \ldots, X_{2^{z-j-1}}) \) of \( Y \) and subfamily \( G' \subset G \) such that \( \|G'\| > (1 - 4^{z-j}e^{-m^a}) \|G\| \), and \( |U \cap X_j - 2^{-z}m| < \delta_z \) for every \( j' \in [2^{z-j-1}] \) and \( U \in G' \).

Replace \( G \) by \( G' \). Similarly construct \( G_2 \subset 2^{X' - Y} \) to obtain a \( 2^{z-j-1}\)-split \( X_2 \) of \( X' - Y \) and new \( G' \) that satisfy the two conditions.

Concatenate the two splits \( X_i \) to construct the \( 2^{z-j}\)-split \( X \) of \( X' \). As
\[
\left( 1 - 4^{z-j}e^{-m^a} \right)^2 \left( 1 - e^{-m^a} \right) > \left( 1 - 4^{r-j}e^{-m^a} \right),
\]
the obtained \( X \) and \( G' \) meet the two desired conditions. We have proven the induction step completing the proof. \( \square \)

For \( G = F \) in \( X \) weighted primitively with \( U \to |F[U]| \), obtain such an \( r \)-split \( X = (X_1, X_2, \ldots, X_r) \) of \( X \) and \( G' \) by the lemma. Replace \( F \) by \( G' \) and \( b_* \) by \( b_*/2 \), then the new \( F \) satisfies \( |U \cap X_j - \tilde{q}| < \delta_z \) for each \( j \in [r] \) and \( U \in F \), in addition to all the conditions seen above.

We now construct three sets \( C_i \) and subfamilies \( F_i \subset F \) \( (i \in [3]) \) by a recursive process with the index \( j \in [r] \): initially set \( C_i = \emptyset \) and \( F_i = F \) for all \( i \). At the beginning of the \( j \)th trial, we are given \( C_i \) and \( F_i \) with \( |C_i| < jqm^{-\epsilon} \), and the \( \Gamma(b_j, 2) \)-condition of \( F_i \), i.e., \( |F_i[S]| < 2b_j^{-1} |F_i| \) for every nonempty \( S \subset X \), where
\[
b_j = \epsilon b_* \left( 1 - \frac{1}{r} \right)^{j-1}.
\]

Putting
\[
Q = [(1 - \epsilon)q, (1 + \epsilon)q] \cap \mathbb{Z},
\]
find and fix \( q_{i,j} \in Q \) such that \( |U : U \in F_i, |U \cap X_j| = q_{i,j}| \) is maximum. Also let \( S_i \) be a maximal set in \( X - C_i \) such that \( |F_i[S_i]| \geq b_j^{-1} |F_i| \). Update \( F_i \) and \( C_i \) by
\[
F_i \leftarrow F_i[S_i], \quad \text{and} \quad C_i \leftarrow C_i \cup S_i,
\]
where \( \leftarrow \) represents substitution for update. Let the other two \( F_i' \) \( (i' \in [3] - \{i\}) \) exclude \( S_i \), i.e., update them by \( F_i' \leftarrow F_i' \cap (X \setminus S_i) \). Also performing \( j \leftarrow j + 1 \), continue to the next trial if \( j \leq r \). This completes the description of our recursive process.
Right after the update $F_i \leftarrow F_i[S_i]$, we have $|S_i| < qm^{-\epsilon}$ and the $\Gamma(b_j)$-condition of $F_i$; by the $\Gamma(b_j-1,2)$-condition given at the beginning of the $j$th trial,

\[
|F_i[S]| < \frac{2(1+\epsilon)q}{b_{j-1}}|F_i| < m \left(1 - \frac{1}{r}\right)^{|S|} b_j^{-|S|}|F_i|, 
\]

so $|S_i|$ must be less than $qm^{-\epsilon}$ while the new $F_i$ satisfies the $\Gamma(b_j)$-condition. After excluding $S_i$ of the other two $F_i$, the family $F_i$ correctly satisfies the $\Gamma(b_j,2)$-condition.

By these, the obtained objects satisfy that:

A) $C_i$ are three mutually disjoint sets each with $|C_i| < m^{1-\epsilon}$,

B) $F_i \subset F[C_i] \cap (X-U_{\ell \in [m]} C_{\ell^i})$ with $|F_i| > m^{-\epsilon} b_j^{-|C_i|}|F|$ and the $\Gamma(b_r,2)$-condition,

C) and $|U \cup X_j| = q_i, j \in \{1,2\}$ and $U \in F_i$. □

4.2. **Recursive Updates on $X$.** Put $F_{i,0} = F_i$ and $C_{i,0} = C_i$ freeing the variables $F_i$ and $C_i$. Also update $b_i \leftarrow eb_i$ with which we use the same $b_j$ as above. The families satisfy $|F_{i,0}| > b_i^{m-|C_{i,0}|}$ and the $\Gamma(b_r)$-condition in $X - C_{i,0}$, embedded in $X$ the way C) describes.

We show the following property for every $j \in [r+1]$.

**Property** $\Pi_j$: there exist three mutually disjoint sets $C_i \supset C_{i,0}$ and subfamilies $F_i \subset F_{i,0}$ satisfying the following conditions.

i) $F_i \subset F_{i,0}[C_i] \cap (X-U_{\ell \in [m]} C_{\ell^i})$ such that $|F_i| > \epsilon^{j}(eb_i)^{-|C_{i,0}|} |F_{i,0}|$, 

ii) If $j \leq r$,

a) \[
\sum_{u \in [r], m \geq 2} \frac{|F_i[S]|^g}{\frac{b_j^{-g-1} u}{m_u}} < |F_i|^g, 
\]

where

\[m_u = (g-1)m, \quad \text{and} \quad Z_j = \bigcup_{p=j}^r X_p - C_i, \]

b) and the $\Gamma(b_j m^{-\epsilon})$-condition of $F_i$ in $Z_j$,

iii) $U \cap U' \cup \bigcup_{j \in [j-1]} X_{j'} = \emptyset$ for each $U \in F_i$ and $U' \in F_{i'}$ with $i' \in [3] - \{i\}$.

As $\Pi_{r+1}$-iii) means three mutually disjoint sets in $F_i$, our task here is to prove $\Pi_j$ by induction on $j$. For the basis $j = 1$, choose $C_i = C_{i,0}$ and $F_i = F_{i,0}$ satisfying $\Pi_1$-i) to $\Pi_1$-ii). Here $\Pi_1$-ii)-a) holds since $\sum_{S \in (X)} |F_i[S]|^g < b_j^{g-1} u \left(\frac{m_u}{u}\right)|F_i|^g$ for every $u \in [m]$, by the $\Gamma(b_*)$-condition of $F_i$. This confirms the basis.

Assume $\Pi_j$ and prove $\Pi_{j+1}$. When we are given $C_i$ and $F_i$ of $\Pi_j$, write for simplicity

\[b = b_j, \quad Z = Z_{j+1}, \quad X_0 = X_j - C_i, \]

\[q_* = q_j - |X_j \cap C_i|, \quad n_* = |X_j|, \quad \text{and} \quad H = \left(\frac{X_*}{q_*}\right). \]

\footnote{We say $G \subset (X)$ satisfies the $\Gamma(b_*)$-condition in $X' \subset X$ if $|G[S]| < b_*^{-|S|} |G|$ for every nonempty $S \subset X'$.}
We may use \( s, t, u \in \mathbb{Z}_{\geq 0} \) as summation/product indices. Obvious floor functions are omitted in the rest of the proof.

The induction step will update \( C_i \) and \( \mathcal{F}_i \) given by \( \Pi_j \), so they satisfy \( \Pi_{j+1} \). We complete it in seven steps.

**Step 1.** Construct a family \( \mathcal{Y}_i \) of \( Y \in \binom{X^*_i}{n_{v, i}^*} \) such that \( \mathcal{F}_i \cap (X^*_m \cup Y) \) is sufficiently large. Fix each \( i \in [3] \) assuming \( q_\ast > 0 \). Weight \( X_\ast \) by \( w : (2^{X_\ast})^2 \to \mathbb{Z}_{\geq 0} \) primitively with \( V \mapsto |\mathcal{F}_i[V]| \), inducing the norm \( \| \cdot \| \). Then the family \( \mathcal{H} \) satisfies the \( \Gamma_2 \left( \frac{b}{q_\ast m^*}, 1 \right) \)-condition on \( \| \cdot \| \), since

\[
\sum_{V_1, V_2 \in \mathcal{H}} w(V_1, V_2) = \sum_{V_1, V_2 \in \mathcal{H}} |\mathcal{F}_i[V_1]| \cdot |\mathcal{F}_i[V_2]| \leq \sum_{T \in (X^*_i)} |\mathcal{F}_i[T]|^2
\]

\[
< \left| \mathcal{F}_i \right|^2 (bm^-\epsilon)^u \left( \frac{q_\ast}{u} \right) \leq \left( \frac{b}{q_\ast m^*} \right)^{-u} \| \mathcal{H} \|^2,
\]

for each \( u \in [q_\ast] \), by \( \Pi_{j-1}) \)-b) and \( \| \mathcal{H} \| = |\mathcal{F}_i| \).

Apply Theorem 2.3 to \( \mathcal{H} \). There exists a family \( \mathcal{Y}_i \subset \binom{X^*_i}{n_{v, i}^*} \) such that \( |\mathcal{Y}_i| > \binom{n_{v, i}^*}{n_{v, i}^*(1 - \epsilon)} \), and

\[
|\mathcal{F}_{Y, i}| > \frac{\binom{n_{v, i}^*}{q_\ast} \cdot |\mathcal{F}_i| (1 - \epsilon)}{\binom{n_{v, i}^*}{q_\ast}} \text{ for every } Y \in \mathcal{Y}_i,
\]

where \( \mathcal{F}_{Y, i} := \mathcal{F}_i \cap \left( X - X_\ast \cup Y \right) \).

**Step 2.** With another weight \( w \) on \( X_\ast \), confirm some \( \Gamma_g \)-condition of \( \mathcal{H} \). For each \( i \), skip this step, Steps 3, 4 and 6 if \( j = r + 1 \) or \( q_\ast = 0 \). Denote by \( S \) a subset of \( X_\ast \), and by \( T \) a nonempty subset of \( Z \). Define

\[
w_T : (2^{X^*_i})^g \to \mathbb{R}_{\geq 0}, \quad (V_1, V_2, \ldots, V_g) \mapsto \prod_{t=1}^g |\mathcal{F}_i[V_t \cup T]|^{\frac{1}{b - (g - 1)T}} (|T|)^{\frac{m^*}{|T|}}.
\]

for each \( T \) inducing the norm \( \| \cdot \|_T \). Reset \( w \) and \( \| \cdot \| \) by

\[
w : (2^{X^*_i})^g \to \mathbb{R}_{\geq 0}, \quad V \mapsto \sum_{s \leq u \leq m} w_T(V).
\]

Also denote

\[
w_S, T := \frac{|\mathcal{F}_i[S \cup T]|}{b^{-(1 - \frac{1}{u})T} (m^* |T|)^{\frac{1}{g}}}, \quad \text{for each } S \text{ and } T,
\]

\[
\gamma_T := |\mathcal{F}_i|^{-g} \sum_{s \leq u \leq m} \frac{|\mathcal{F}_i[S \cup T]|^g}{b^{-(g - 1)T} (m^* |T|)} (\frac{m^*}{s^*}), \quad \text{for each } T,
\]

\[
b_g := \left( \frac{b^{1 - \frac{1}{u}}}{2^g m^*} \right)^{\frac{b_g}{m^*}}, \quad b_1 := \frac{b_g}{(g - 1)q_\ast}, \quad \text{and } h := \left( \frac{|\mathcal{F}_i|}{\| \mathcal{H} \|} \right)^g.
\]

This step shows the \( \Gamma_g \left( b_1, h \right) \)-condition of \( \mathcal{H} \) on \( \| \cdot \| \).

See the following remarks.
D) For each $S$ and $T$,
\[
\sum_{(V_1, V_2, \ldots, V_g) \in \mathcal{H}[S]^g} \prod_{i=1}^g |\mathcal{F}_i[V_i \cup T]| = |\mathcal{F}_i[S \cup T]|^g,
\]
so
\[
|\mathcal{H}[S]|^g = \sum_{V \in \mathcal{H}[S]^g} w(V) = \sum_{r \leq s \leq m} \sum_{T \in \binom{\mathcal{S}}{r}} w_T(V) = \sum_{r \leq s \leq m} \sum_{T \in \binom{\mathcal{S}}{r}} |\mathcal{F}_i[S \cup T]|^g.\]

E) $\sum_{r \leq s \leq m} \gamma_T < 1$ by $\Pi_j$-ii)-a).

F) For each $T$,
\[
\sum_{o \leq r \leq s, \quad s \in \binom{\mathcal{S}}{r}} \frac{|\mathcal{H}[S]|^g}{b^{-(g-1)s}(m_*)^s} = \sum_{o \leq r \leq s, \quad s \in \binom{\mathcal{S}}{r}} \sum_{T \in \binom{\mathcal{S}}{s} \cap T} \frac{w_T(V)}{b^{-(g-1)s}(m_*)^s}
\leq \sum_{o \leq r \leq s, \quad s \in \binom{\mathcal{S}}{r}} \frac{|\mathcal{F}_i[S \cup T]|^g}{b^{-(g-1)(s+|T|)}(m_*)^{s+|T|}} = \gamma_T |\mathcal{F}_i|^g.
\]

The inequality holds by D) and
\[
\left(\frac{m_0}{|T|}\right)^s \left(\frac{m_1}{|T|}\right)^{m_0-s} \left(\frac{m_0}{s+|T|}\right)^s \left(\frac{m_0}{|T|}\right)^s \geq \left(\frac{m_0}{s+|T|}\right)^s.
\]

G) $h$ is greater than 1, otherwise
\[
|\mathcal{F}_i|^g \leq |\mathcal{H}|^g = \sum_{o \leq r \leq s, \quad s \in \binom{\mathcal{S}}{r}} \frac{|\mathcal{F}_i[T]|^g}{b^{-(g-1)s}(m_*)^s} < |\mathcal{F}_i|^g,
\]

by D) and $\Pi_j$-ii)-a).

H) For each $S$ and $T$,
\[
w_{S,T} = |\mathcal{H}[S]|_T \leq \frac{\gamma_T}{\mathcal{S}[|S|]} b^{-(1-\frac{1}{\gamma_T})|S|}(m_*)^{\frac{|S|}{m_0}}.
\]
due to F) and
\[
w_{S,T}^g = \frac{|\mathcal{F}_i[S \cup T]|^g}{b^{-(g-1)|T|}(m_*)^{|T|}} = \sum_{V \in \mathcal{H}[S]^g} w_T(V) = |\mathcal{H}[S]|_T^g. \quad \Box
\]

Let us show the $\Gamma_g$-condition with the remarks. It suffices to confirm
\[
(4.3) \quad \|P_{s,g}\|_T < \gamma_T b^{-s}(g-1)^{q_*} |\mathcal{F}_i|^g,
\]
for every $T$ and $s \in [(g-1)q_*]$; it is due to E) and
\[
\|P_{s,g}\| = \sum_{V \in P_{s,g}} w(V) = \sum_{V \in P_{s,g}} \sum_{r \leq s \leq m} \sum_{T \in \binom{\mathcal{S}}{r}} w_T(V) = \sum_{r \leq s \leq m} \sum_{T \in \binom{\mathcal{S}}{r}} \|P_{s,g}\|_T.
\]

Here $P_{s,g}$ is defined for $\mathcal{H}$ as in Section 2, i.e.,
\[
(4.4) \quad P_{s,g} = \{V : V \in \mathcal{H}^g, |\text{union}(V)| = gq_* - s\},
\]
for every \( s \geq 0 \). So if we show (4.3) for all \( T \) and \( s \), we have the \( \Gamma_g (b_1, h) \)-condition of \( \mathcal{H} \) on \( \| \cdot \| \).

Fix each \( T \) for the proof. Define

\[
w_{g'} : (2^X)^{g'} \to \mathbb{R}_{\geq 0}, \quad (V_1, V_2, \ldots, V_{g'}) \mapsto \frac{\prod_{i=1}^{g'} |F_i[V_i \cup T]|}{b^{-(1-\frac{1}{s})g'|T|} (m_s)^{\frac{2}{s}}},
\]

for \( g' \in [2, g] \cap \mathbb{Z} \) inducing the norm \( \| \cdot \|_{g'} \). We verify

\[
(4.5) \quad \|P_{s,g'}\|_{g'} < \gamma_{T}^{g'} b_{g'}^{-s} \left( \frac{(g'-1)}{s} \right)^{q_s} |F_i|^{g'},
\]

for every \( g' \) and \( s \), where \( b_{g'} := 2^{g'-g} b_g \), and \( P_{s,g'} \) is given by replacing \( g \) by \( g' \) in (4.4). The case \( g' = g \) means (4.3).

Proof of (4.5) by induction on \( g' \). Fix each \( s \in [(g'-1)q_s] \) for the basis \( g' = 2 \). From H) above,

\[
\|P_{s,2}\|_2 = \sum_{V \in \mathcal{P}_{s,2}} w_2(V) = \sum_{(V_1, V_2) \in \mathcal{P}_{s,2}} \frac{|F_i[V_1 \cup T]| |F_i[V_2 \cup T]|}{b^{-(1-\frac{1}{s})2|T|} (m_s)^{\frac{2}{s}}} \leq \sum_{V_1 \in \mathcal{H}} |F_i[V_1 \cup T]| b^{-(1-\frac{1}{s})|T|} (m_s)^{\frac{2}{s}} \sum_{V_2 \in \mathcal{H} \text{ with } |V_1 \cup V_2| = 2q_s - s} \frac{|F_i[V_2 \cup T]|}{b^{-(1-\frac{1}{s})2|T|} (m_s)^{\frac{2}{s}}} \leq w_{\emptyset, T} \left( \frac{q_s}{s} \right) \max_{S \in \binom{X_s}{2}} w_{S, T} \leq \gamma_T^2 |F_i|^2 b^{-(1-\frac{1}{s})s} m_s^{\frac{2}{s}} \left( \frac{q_s}{s} \right) \leq \gamma_T^2 b^{-s} \left( \frac{q_s}{s} \right) \cdot |F_i|^2,
\]

proving the basis.

Assume true for \( g'-1 \) and prove true for \( g' \). By induction hypothesis,

\[
\sum_{V \in \mathcal{P}_{s,g'-1}} w_{g'-1}(V) = \|P_{s,g'-1}\|_{g'-1} < \gamma_T^{g'-1} b_{g'-1}^{-(1-\frac{1}{s})} \left( \frac{(g'-2)q_s}{s} \right)^{q_s} |F_i|^{g'-1},
\]
for $v \in [(g'-2)q_*]$. Fix any $s \in [(g'-1)q_*]$. Since $P_{v,g'-1} = \emptyset$ if $v > (g'-2)q_*$,
\[
\sum_{v \in \pi} w_{g'-1}(V) \sum_{S \in \text{union}(V)} w_{S,T} \leq \sum_{v \in \min\{s, (g'-2)q_*\}} w_{g'-1}(V) \left( \frac{(g'-1)q_* - v}{s - v} \right) \sum_{S \in \text{union}(V)} w_{S,T}
\]
\[
< \sum_{v=1}^{s} \gamma_T \frac{q_*}{t} b_0^{-v} \left( \frac{(g'-1)q_*}{s} \right) \left| F_i \right|^{g' - 1} \left( \frac{(g'-1)q_* - v}{s - v} \right) \gamma_T \left| F_i \right|^{g' - 1}
\]
\[
< \gamma_T \left( 2 b_0 \right)^{-s} \left( \frac{(g'-1)q_*}{s} \right) \left| F_i \right|^{g'} \sum_{v=1}^{s} \left( \frac{s}{v} \right).
\]
The last line is due to $\left( \frac{(g'-2)q_*}{v} \right) \left( \frac{(g'-1)q_* - v}{s - v} \right) < \left( \frac{(g'-1)q_*}{v} \right) \left( \frac{(g'-1)q_* - v}{s - v} \right)$
$= \left( \frac{(g'-1)q_*}{s} \right) \left( \frac{s}{v} \right)$ for every $v$.

For $v = 0$, we have
\[
\sum_{v \in \pi} w_{g'-1}(V) \leq w_{g',T} \leq \gamma_T \left( 2 b_0 \right)^{-s} \left( \frac{(g'-1)q_*}{s} \right) \left| F_i \right|^{g'},
\]
by H), so
\[
\sum_{V \in P_{v,g'}} w_{g'-1}(V) \sum_{S \in \text{union}(V)} w_{S,T} < \gamma_T \left( 2 b_0 \right)^{-s} \left( \frac{(g'-1)q_*}{s} \right) \left| F_i \right|^{g'}.
\]
As $w_{S,T} = \sum_{V \in \mathcal{M}[S]} \frac{|F_i[V \cup U]|}{b_0^{-\left(1 - \frac{1}{2}\right)T(i)^{m_*}}} \left( \frac{T(i)}{T} \right)^{\frac{1}{2}}$ for each $S$, we conclude that
\[
\|P_{s,g'}\|_{g'} = \sum_{V \in P_{s,g'}} w_{g'}(V) = \sum_{V_1,V_2,...,V_{g'} \in P_{s,g'}} \frac{\prod_{v=1}^{g'} |F_i[V_i \cup U]|}{b_0^{-\left(1 - \frac{1}{2}\right)g'|U|^m}} \left( \frac{T(i)}{T} \right)^{\frac{g'}{2}}
\]
\[
\leq \sum_{v \in \pi} w_{g'-1}(V) \sum_{S \in \text{union}(V)} w_{S,T}
\]
\[
< \gamma_T \left( 2 b_0 \right)^{-s} \left( \frac{(g'-1)q_*}{s} \right) \left| F_i \right|^{g'} \sum_{v=1}^{s} \left( \frac{s}{v} \right)
\]
\[
= \gamma_T \left( 2 b_0 \right)^{-s} \left( \frac{(g'-1)q_*}{s} \right) \left| F_i \right|^{g'},
\]
completing the induction step.

This confirms (5), hence the $\Gamma_g(b_t, h)$-condition of $\mathcal{H}$ on $\| \cdot \|$ as well.

**Step 3.** Remove $Y$ from $\mathcal{Y}_i$ such that $\sum_{T \in \mathcal{G}(z)} |F_{Y,T}|^g$ is too large for any $u \geq r$.

With the $\Gamma_g$-condition meeting
\[
b_t > \frac{4q^{2n_*}}{n_* / 4}, \quad \text{and} \quad h > 1 \text{ from G),}
\]
apply Corollary 2.2 to \( \mathcal{H} \). There are \([1 - \epsilon) \binom{n_*/4}{(n_*/4)}] \) sets \( Y \in \binom{X_*}{n_*/4} \) such that

\[
\sum_{r \leq u \leq m \atop T \in \binom{Z}{1}} b^{-3(r-1)u} \frac{|F_{Y,T}|^g}{b^{-(g-1)u}\binom{n_*/4}{u}} = \sum_{V \in \binom{\mathcal{H} \cap (Y,b)}{1}} w(V) < \frac{(1 + \frac{h}{g})^{\binom{n_*/4}{q_*/4}}}{\epsilon^{\binom{n_*/4}{q_*/4}}} \|\mathcal{H}\|^g < \frac{2^{\binom{n_*/4}{g}}}{\epsilon^{\binom{n_*/4}{g}}} |F_i|^g < \frac{3}{\epsilon} |F_{Y,i}|^g,
\]

since (4.2). As \( b = b_{j+1} (1 - r^{-1})^{-1} \), the inequality means

\[
(4.6) \quad \sum_{r \leq u \leq m \atop T \in \binom{Z}{1}} b^{-3(r-1)u} \frac{|F_{Y,T}|^g}{b^{-(g-1)u}\binom{n_*/4}{u}} < \epsilon |F_{Y,i}|^g.
\]

Delete \( Y \) such that \(-4.6\) from \( \mathcal{Y}_i \). Now the family satisfies \( |\mathcal{Y}_i| > (1 - 2\epsilon) \binom{n_*/4}{(n_*/4)} \), and (4.2) \& (4.6) for every \( Y \in \mathcal{Y}_i \).

**Step 4.** Further delete some undesired \( Y \) from \( \mathcal{Y}_i \). Reset \( b \) by \( b \leftarrow eb_{j-m^{-\epsilon}} \). Let \( w_T, \| \cdot \|_T, w_{S,T}, \gamma_T, \) and \( b_T \) be the same as Step 2 with the updated \( b \). Also let

\[
w : \mathcal{H}^g \rightarrow \mathbb{R}_{\geq 0}, \quad V \mapsto \sum_{T \in \binom{Z}{1}} w_T(V),
\]

re-defining \( \| \cdot \| \) and \( h = |\mathcal{F}|^g \|\mathcal{H}\|^{-g} \) accordingly.

This just considers \( u = 1 \) instead of all \( u \in [r, m] \cap \mathbb{Z} \). The following statements can be verified similarly to Steps 2 and 3.

- From \( \Pi_j \text{-ii)-b)},

\[
\sum_{s \in \binom{Z}{1}} b^{-3(s-1)u} \frac{|F_{i,T}|^g}{b^{-(g-1)u}\binom{n_*/4}{u}} < |F_i|^g, \quad \Rightarrow \quad \sum_{T \in \binom{Z}{1}} \gamma_T < 1.
\]

- \( h > 1, \) and \( w_{S,T} = \|\mathcal{H}[S]\|_T \) for all \( S \subseteq X_* \) and \( T \in \binom{Z}{1} \).

- \( \mathcal{H} \) satisfies the \( \Gamma_g (b_1, h) \)-condition on \( \| \cdot \| \).

- \( \sum_{T \in \binom{Z}{1}} \frac{|F_{Y,T}|^g}{b^{-(g+1)u}\binom{n_*/4}{u}} < \frac{3}{\epsilon} |F_{Y,i}|^g, \) for more than \( 1 - \epsilon \) of all \( Y \in \binom{X_*}{n_*/4} \), meaning

\[
(4.7) \quad |F_{Y,i}|^g < \frac{3}{\epsilon b_{j+1}} |F_{Y,i}|^g, \quad \text{for every } T \in \binom{Z}{1}.
\]
Eliminate all $Y$ with $\neg (4.7)$ from $\mathcal{V}$.

**Step 5. Update $\mathcal{F}_i$ so they satisfy $\Pi_{j+1-i)}$ and $iii)$**. The obtained $\mathcal{V}$ is a subfamily of $\binom{X_j}{n_s/4}$ such that $|\mathcal{V}| > \binom{n_s/4}{2} (1 - 3\epsilon), (4.8), \text{ and } (4.6) \land (4.7)$ for every $Y \in \mathcal{V}$.

Extend the $n_s/4$-sets in $\mathcal{V}$ to $n'$-sets in the universal set $X_j$ where $n' := 3q + |X_j|/4$. Noting Remarks L) and M) of Section 2 with $q_s + n_s/4 < n' < |X_j|/3$, we see the obtained family $\mathcal{V}' := Ext(\mathcal{V}, n')$ has a cardinality at least $\left(\binom{|X_j|}{n_s/4}(1 - 3\epsilon)\right)$. If $q_s = 0$, the same goes for $\mathcal{V}' := \binom{X_j}{n_s/4}$.

Performing the above for the three $i$, we have

$$\left[\binom{|X_j|}{n'} \left(\frac{|X_j| - n'}{n'} \left(\frac{|X_j| - 2n'}{n'} \left(1 - 9\epsilon\right)\right)\right)\right]$$

triples $(Y_1, Y_2, Y_3)$ such that $Y_i \in \mathcal{V}'_i$, and $Y_i$ are mutually disjoint. Fix such a triple $(Y_1, Y_2, Y_3)$.

Choose any $Y \in \mathcal{V}_i \cap \binom{n_s/4}{2}$ for each $i$, and update $\mathcal{F}_i$ by $\mathcal{F}_i \leftarrow \mathcal{F}_Y$. By construction so far, the new $\mathcal{F}_i$ satisfy:

I) $\Pi_{j+1-i)$ and $iii)$.
J) If $j \leq r$,

$$\sum_{s \in \binom{\mathcal{V}}{2}} \frac{|\mathcal{F}_i[S]|^g}{b_{(g-1)u}(m_s - v)} < \epsilon|\mathcal{F}_i|^g$$

K) and $|\mathcal{F}_i[T]| < \frac{m_s^{v}}{b_{(g-1)u}|\mathcal{F}_i|}$ for each $T \in \binom{\mathcal{V}}{2}$.

**Step 6. Find a set $S \subset Z$ meeting some desired conditions**. Reset $b$ by $b \leftarrow b_{j+1}$. Let $S$ denote a subset of $Z$. Find the maximum $v \in [0, m] \cap Z$ such that

$$(4.8) \sum_{s \in \binom{\mathcal{V}}{2}} \frac{|\mathcal{F}_i[S \cup T]|^g}{b_{(g-1)u}(m_s - v)} \geq |\mathcal{F}_i|^g.$$

There does exist such a $v$ less than $r$, as (4.8) is true for $v = 0$ and false for $v \geq r$ by J).

Below we show the existence of $S \in \binom{\mathcal{V}}{2}$ satisfying the three conditions.

1) $\sum_{r \leq u \leq m} \frac{|\mathcal{F}_i[S \cup T]|^g}{b_{(g-1)u}(m_s - v)} < \frac{1}{2}|\mathcal{F}_i|^g$.

2) $\sum_{1 \leq u \leq r} \frac{|\mathcal{F}_i[S \cup T]|^g}{b_{(g-1)u}(m_s - v)} < |\mathcal{F}_i|^g$.

3) $|\mathcal{F}_i[S]| > \frac{1}{2b^v}|\mathcal{F}_i|$.

Assume $v > 0$, otherwise these are clearly true by J).

Observe here that

$$\sum_{s \in \binom{\mathcal{V}}{2}} \frac{|\mathcal{F}_i[S]|^g}{b_{(g-1)u}(m_s - v)} \leq 2^{-g+1}|\mathcal{F}_i|^g.$$
similarly to having found $\Pi_{1-ii})$-a) before Step 1. Therefore, if there were no $S \in \binom{Z}{v}$ such that 1) $\land$ 2) $\land$ 3), one of the following would be true:

$$- \sum_{s \in \binom{Z}{v}} \frac{\left| \mathcal{F}_i[S] \right|^g}{b^{-(g-1)v} \binom{m_*}{v}} \geq \frac{1}{3} \left| \mathcal{F}_i \right|^g;$$

$$- \sum_{s \in \binom{Z}{v}} \frac{\left| \mathcal{F}_i[S] \right|^g}{b^{-(g-1)v} \binom{m_*}{v}} \geq \frac{1}{3u} \left| \mathcal{F}_i \right|^g, \text{ for some } u \in [r - 1],$$

where 2)-u means

$$\sum_{T \in \binom{Z - S}{u}} \frac{\left| \mathcal{F}_i[S \cup T] \right|^g}{b^{-(g-1)u} \binom{m_*}{u}} < \frac{1}{2u} \left| \mathcal{F}_i[S] \right|^g.$$

Call the two cases Cases 1 and 2, respectively.

We show a contradiction in Case 1 from

$$\sum_{s \in \binom{Z}{v}} \sum_{r \leq s \leq m} \frac{\left| \mathcal{F}_i[S \cup T] \right|^g}{b^{-(g-1)(v+u)} \binom{m_*}{v+u}}$$

$$= \sum_{s \in \binom{Z}{v}} \frac{1}{b^{-(g-1)v} \binom{m_*}{v}} \sum_{r \leq s \leq m} \frac{\left| \mathcal{F}_i[S \cup T] \right|^g}{b^{-(g-1)u} \binom{m_*-v}{u}}$$

$$\geq \sum_{s \in \binom{Z}{v}} \frac{\left| \mathcal{F}_i[S] \right|^g}{2b^{-(g-1)v} \binom{m_*}{v}}$$

The inequality means

$$\sum_{r \leq s \leq m} \frac{\left| \mathcal{F}_i[S'] \right|^g}{b^{-(g-1)v'} \binom{m_*}{v'}} \geq \frac{1}{6} \left| \mathcal{F}_i \right|^g.$$ 

See it as follows. Let

$$r_{v'} = \left| \mathcal{F}_i \right|^{-g} \sum_{s \in \binom{Z}{v'}} \frac{\left| \mathcal{F}_i[S'] \right|^g}{b^{-(g-1)v'} \binom{m_*}{v'}},$$

for each $v' \in [r, m] \cap Z$. It satisfies

$$\sum_{s \in \binom{Z}{v}, T \in \binom{Z - S}{u}} \left| \mathcal{F}_i[S \cup T] \right|^g \leq \sum_{s' \in \binom{Z}{v'}} \left| \mathcal{F}_i[S'] \right|^g \binom{v'}{v} \left( \frac{v'}{v} \right)^g = r_{v'} \left| \mathcal{F}_i \right|^g b^{-(g-1)v'} \binom{m_*}{v'} \binom{v'}{v},$$

since there are at most $\binom{v'}{v}$ pairs $(S, T)$ such that $S \cup T$ equals each given $S' \in \binom{Z}{v'}$. Summing up

$$\sum_{s \in \binom{Z}{v}, T \in \binom{Z - S}{u}} \frac{\left| \mathcal{F}_i[S \cup T] \right|^g}{b^{-(g-1)v'} \binom{m_*}{v'}} \binom{v'}{v} \leq r_{v'} \left| \mathcal{F}_i \right|^g.$$
for all $v'$, we see $\neg (1.10) \Rightarrow \neg (1.9)$. Hence (1.10). This contradicts J), so Case 1 is impossible to occur.

Given $u \in [r - 1]$ in Case 2, similarly find
\[
\sum_{s \in \binom{Z}{u}} \sum_{t \in \binom{T^c}{u}} |F_i[T \cup T]|^g b^{-(g-1)(v+u)}(m_v)(v+u)\]
\[
\geq 2^{(g-2)u} \sum_{s' \in \binom{Z}{u}} |F_i[s']|^g b^{-(g-1)v}(m_v) > \left(\frac{2^{g-2}}{3}\right)^u |F_i|^g,
\]
meaning
\[
\sum_{s' \in \binom{Z}{u}} |F_i[s']|^g b^{-(g-1)v}(m_v) > |F_i|^g.
\]
It is against the maximality of $v$ such that (4.8).

For the $F_i$ updated by Step 5, we have proven the existence of $S \subset Z$ such that $|S| < r$ and 1) $\land$ 2) $\land$ 3). Denote it by $S_i$.

**Step 7. Perform the final updates on $C_i$ and $F_i$ for $\Pi_j+1$.** Let $i = 1$. With the obtained $S_i$, update $C_i$ by $C_i \leftarrow C_i \cup S_i$ and $F_i$ by $F_i \leftarrow F_i[C_i]$. Then let the other two $F_i'$ ($i' \in [3] \setminus \{i\}$) exclude $S_i$ as we did before in the preprocess with $F_i' \leftarrow F_i \cap (X - S_i)$. By K) and $|S_i| < r$, this could reduce $|F_i'|$ only by a factor greater than $1 - m^{-\delta/2}$, thus $|F_i'|^g$ by one greater than $1 - \epsilon$, affecting the subsequent updates trivially. We note here that the same result at the end of Step 6 holds even if $\epsilon$ in J) is replaced by 2ε.

Set $i = 2$ to perform the same construction, where $F_i$ can exclude $S_2$ by 2) of Step 6 since it means the $\Gamma (2bm^{-r})$-condition of $F_i$ in Z.

Finally set $i = 3$ to perform the same construction on $F_i$ with the other $F_i'$ excluding $S_3$. Then the three new $C_i$ and $F_i$ all satisfy $\Pi_j+1$: if $j \leq r$, the property $\Pi_j+1$-i) is true by I), $|S_i| < r$ and 3) of Step 6. Also ii)-a) holds by 1), and ii)-b) by 2).

This completes our updates proving the induction step $\Pi_j \Rightarrow \Pi_j+1$. We now have Theorem 1.1.

**APPENDIX 1: PROOF OF LEMMA 2.5**

**Lemma 2.5.** For $F \subset (X_m)$ such that $m \leq \frac{n}{2}$,
\[
\kappa \left(\binom{X}{2m} - \text{Ext}(F, 2m)\right) \geq 2\kappa \left(\binom{X}{m} - F\right).
\]

**Proof.** For each $S \in \binom{X_m}{m} - F$ and $j \in [0, m] \cap \mathbb{Z}$, let
\[
F_j = \{T - S : T \in F, |T - S| = j\}.
\]
There exists $j$ such that $\kappa(F_j)$ in the universal set $X - S$ is at most $\kappa(F)$ in X, otherwise
\[
|F| < \sum_{j \geq 0} \binom{m-j}{n-m} \binom{n-m}{j} e^{-\kappa(F)} = \binom{n}{m} e^{-\kappa(F)} = |F|.
\]
Taking $Ext(F_j, m)$ in $X - S$ with Remark A of Sec. 2.3, we see there are $\lceil \binom{n-m}{m} e^{-\kappa(F)} \rceil$ pairs $(S, U)$ such that $U \in \binom{X-S}{m}$ and $S \cup U \in Ext(F, 2m)$ for each $S \in \binom{X}{m} - F$.

Now consider all pairs $(S, U)$ such that $S$ and $U$ are disjoint $m$-sets, and $S \cup U \in Ext(F, 2m)$. Their total number is at least $\binom{n}{m} \binom{2m}{m} (1-z)^n$ where $z = e^{-\kappa(F)}$.

As a $2m$-set produces at most $\binom{2m}{m}$ pairs $(S, U)$, there are at least $\lceil \binom{n}{m} (1-z^2) \rceil$ sets in $Ext(F, 2m)$. The lemma follows. 

\textbf{Appendix 2: Asymptotics of Binomial Coefficients}

Let
\begin{equation}
(A.11) \quad s : (0, 1) \to (0, 1), \quad t \mapsto 1 - \left(1 - \frac{1}{t}\right) \ln(1-t).
\end{equation}

By the Taylor series of $\ln(1-t)$, the function is also expressed as
\begin{equation}
(A.12) \quad s(t) = 1 + \left(1 - \frac{1}{t}\right) \sum_{j \geq 1} \frac{t^j}{j} = 1 + \sum_{j \geq 1} \frac{t^j}{j} - \sum_{j \geq 0} \frac{t^j}{j+1} = \sum_{j \geq 1} \frac{t^j}{j(j+1)}.
\end{equation}

We have the following double inequality.

\textbf{Lemma A.2.} For $x, y \in \mathbb{Z}_{>0}$ such that $x < y$,
\[
\frac{1}{12x+1} - \frac{1}{12y} - \frac{1}{12(x-y)} < z < \frac{1}{12x} - \frac{1}{12y} - \frac{1}{12(x-y) + 1},
\]
where $z = \ln\left(\frac{x}{y}\right) - y \left[\ln \frac{x}{y} + 1 - s\left(\frac{y}{x}\right)\right] - \frac{1}{2} \ln\left\frac{x}{2\pi y(x-y)}\right..$

\textbf{Proof.} Stirling’s approximation in form of double inequality is known as
\[
\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n+1}\right) < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n}\right),
\]
for any $n \in \mathbb{Z}_{>0}$ \cite{10}. By this we find
\[
\frac{1}{12x+1} - \frac{1}{12y} - \frac{1}{12(x-y)} < \ln\left(\frac{x}{y}\right) < \frac{1}{12x} - \frac{1}{12y} - \frac{1}{12(x-y) + 1},
\]
where $u = \sqrt{\frac{x}{2\pi y(y-x)} y^y (x-y)^{x-y}}$.

Since $\ln\left(\frac{x}{y}\right)^y y^y (x-y)^{x-y} = y \left[\ln \frac{x}{y} + 1 - s\left(\frac{y}{x}\right)\right]$ by (A.11), it proves the lemma. \hfill \Box

The lemma is useful to approximate $\ln\left(\frac{x}{y}\right)$ by $z$ to an error less than $1/6$. It derives the standard estimate of the binomial coefficient, i.e., \((\frac{x}{y})^y < \left(\frac{ex}{y}\right)^y\), due to $-y s\left(\frac{y}{x}\right) + \frac{1}{10x} < 0$ from (A.12) and $\frac{x}{2\pi y(x-y)} < 1$. So
\[
\left(\frac{x}{y}\right)^y \leq \left(\frac{x}{y}\right)^y < \left(\frac{ex}{y}\right)^y,
\]
for every $x, y \in \mathbb{Z}_{>0}$ with $x \geq y$, as \((\frac{x}{y})^y = \prod_{j=0}^{y-1} \frac{x}{y-j} \geq \left(\frac{x}{y}\right)^y\).

We also have:
Lemma A.3. For \( x \in \mathbb{Z}_{>1}, \ y \in (1, x) \cap \mathbb{Z}, \ x' \in [1, x/2] \cap \mathbb{Z}, \) and \( y' \in [y-1] \) with \( y' > x' + y - x, \)

\[
\ln \left( \frac{x}{y} \right) - \ln \left( \frac{x - x'}{y - y'} \right) \left( \frac{x'}{y'} \right) > \frac{7(y' - x')^2}{8y} + \frac{-1 + \ln 2\pi z}{2},
\]

where \( z = \left( 1 - \frac{1}{y} \right) \left( 1 - y' \right) \left( 1 - \frac{y'}{x} \right) \left( 1 - \frac{y}{x} \right)^{-1}. \)

Proof. Apply Lemma A.2 to the three binomial coefficients to see

\[
\ln \left( \frac{x - x'}{y - y'} \right) \left( \frac{x'}{y'} \right) - \ln \left( \frac{x}{y} \right) < u \sum_{j \geq 1} \frac{v_j}{j(j+1)} + \frac{1 + \ln \frac{y}{x}}{2},
\]

where

\[
u_j = \frac{(y - y')^j}{(x - x')^j} + \frac{y^{j+1}}{x^{j+1}},
\]

and

\[
w = \frac{x - x'}{y(y - y')(x - x' - y + y')}, \quad \frac{x'}{y'(x' - y')} \frac{y(x - y)}{x}.
\]

As \( w \leq \frac{1}{z} \) from \( \frac{y}{y'(y - y')} \leq \left( 1 - \frac{1}{y} \right)^{-1}, \) it suffices to show

(A.13) \( v_j \geq 0, \) for all \( j \in \mathbb{Z}_{>0}, \)

and

(A.14) \( u < -\frac{7\Delta^2}{8y}, \) where \( \Delta = y' - \frac{x' y}{x}. \)

To see (A.13), put

\[
a = \frac{x'}{x}, \quad t = \frac{y}{y'}, \quad \text{and} \quad f = \frac{(1 - t)^{j+1}}{(1 - a)^j} + \frac{a^{j+1}}{a^j},
\]

for a given \( j. \) Then the desired condition holds if \( f \geq 1 \) for each fixed \( a \in (0,1) \)

and all \( t \in \mathbb{R}_{>0}. \) It is straightforward to check its truth.

We show (A.14) finding that

\[
u = y' \ln \frac{x'y}{xy'} + (y - y') \ln \frac{1 - x'}{1 - x'},
\]

\[
= -\Delta \left( \frac{1}{p} + 1 \right) \ln (1 + p) - \Delta \left( \frac{1}{q} - 1 \right) \ln (1 - q),
\]

where \( p = \frac{x\Delta}{x'y}, \) and \( q = \frac{\Delta}{y(1 - \frac{y}{x})}. \)

Observe facts. 

- \( |q| < 1 \Leftrightarrow \frac{2x'}{x} - 1 < \frac{y'}{y} < 1 \) is true by \( x' \leq x/2. \)
- By the Taylor series of the natural logarithm,
  \[
  \left( \frac{1}{p} + 1 \right) \ln (1 + p) = 1 - \sum_{j \geq 1} \frac{(-p)^j}{j(j+1)}, \quad \text{if } |p| < 1,
  \]
  and 
  \[
  \left( \frac{1}{q} - 1 \right) \ln (1 - q) = -1 + \sum_{j \geq 1} \frac{q^j}{j(j+1)}, \quad \text{as } |q| < 1,
  \]
- So
  \[
  u = \Delta \sum_{j \geq 1} \frac{(-p)^j}{j(j+1)} - \Delta \sum_{j \geq 1} \frac{q^j}{j(j+1)} < \Delta \left( \frac{-p^2 + p^2}{6} - \frac{q^2}{2} \right) < \Delta^2 \frac{y}{y},
  \]
  if |p| < 1.
- \[
  \left( 1 + \frac{1}{p} \right) \ln (1 + p) \geq 2 \ln 2 \text{ if } p \geq 1, \text{ so }
  \]
  \[
  u < (-2 \ln 2 + 1) \Delta - \Delta^2 \frac{y}{2y} < -\frac{7\Delta^2}{8y},
  \]
  since \( \frac{2}{x} = \frac{\Delta}{y} = \frac{\Delta}{y} - \frac{2}{2} \in (0, 1) \).
  Hence (A.14), completing the proof.

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