Almost analytic solutions to equilibrium sequences of irrotational binary polytropic stars for \( n=1 \)

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A solution to an equilibrium of irrotational binary polytropic stars in Newtonian gravity is expanded in a power of \( \epsilon = a_0/R \), where \( R \) and \( a_0 \) are the separation of the binary system and the radius of each star for \( R = \infty \). For the polytropic index \( n = 1 \), the solutions are given almost analytically up to order \( \epsilon^6 \). We have found that in general an equilibrium solution should have the velocity component along the orbital axis and that the central density should decrease when \( R \) decreases. Our almost analytic solutions can be used to check the validity of numerical solutions.

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Coalescing binary neutron stars (BNSs) are considered to be one of the most promising sources of gravitational waves for laser interferometers such as TAMA300, GEO600, VIRGO and LIGO [1]. We can determine the mass and the spin of neutron stars from the gravitational wave signals in the inspiraling phase. We may also extract the informations on the equation of state of a neutron star from the signals in the pre-merging phase [2].

For this purpose it is important to complete theoretical templates of gravitational waves in the pre-merging phase as well as in the inspiraling phase.

Recently Bonazzolla, Gourgoulhon and Marck have numerically calculated quasi-equilibrium configurations of irrotational BNSs in general relativity [3]. Although their results seem to be reasonable, we do not have any analytic solutions in general relativity in order to check the validity of their results.

On the other hand, Uryū and Eriguchi have numerically constructed stationary structures of irrotational BNSs in Newtonian gravity [4]. We can use semi-analytic solutions produced by Lai, Rasio and Shapiro (hereafter LRS) [5] for checking the validity of the results.

However, in numerical solutions of Uryū and Eriguchi, the velocity component along the orbital axis exists while in those of LRS such a component is assumed to be zero from the beginning. When we extend the analytic solutions to the general relativistic ones [6], we should include this velocity component. This is because in the numerical calculation, there is a possibility to obtain another solution although the binding energy of a BNS is almost the same value, and to lead a different conclusion [7].

In order to include the velocity component along the orbital axis, we solve the equation of continuity with the other basic equations. The method we use in this Letter is that we seek a solution to an equilibrium of irrotational binary polytropic stars in Newtonian gravity by expanding all physical quantities in a power of \( \epsilon \equiv a_0/R \), where \( R \) and \( a_0 \) are the separation of the binary system and the radius of each star for \( R = \infty \). We extend the method developed by Chandrasekhar more than 65 years ago for corotating fluids [8] to the one for irrotational fluids.

Although a binary system consists of two stars, we pay particular attention to one of two stars. We call it star 1 whose mass is \( M_1 \) and the companion one star 2 whose mass is \( M_2 \). In this Letter, we adopt two corotating coordinate systems. First one is \( X \) whose origin is located at the center of mass of the binary system. For calculational convenience, we choose the orbital axis as \( X_3 \), and we take the direction of \( X_1 \) from the center of mass of star 2 to that of star 1. The second coordinate system is the spherical one \( r = (r, \theta, \varphi) \) whose origin is located at the center of mass of star 1. We use units of \( G = 1 \).

Since we treat irrotational fluids in Newtonian gravity, the basic equations are the equation of state, the Euler equation, the equation of continuity and the Poisson equation:

\[
P = K \rho^{1+\frac{1}{n}},
\]

\[
\nabla \left[ \int \frac{dP}{\rho} - U + \frac{1}{2} v^2 - v \cdot (\Omega \times r) \right] = 0,
\]

\[
\nabla \cdot v = - (v - \Omega \times r) \cdot \nabla \rho, \tag{3}
\]

\[
\Delta U = -4\pi \rho, \tag{4}
\]

where \( P, \rho, n, U \) and \( \Omega \) are the pressure, the density, the polytropic index, the gravitational potential and the orbital angular velocity, respectively. The gravitational potential \( U \) is separated into two parts, i.e., the contribution from star 1 to itself \( U^{1 \rightarrow 1} \) and that from star 2 to star 1 \( U^{2 \rightarrow 1} \). \( U^{2 \rightarrow 1} \) is written as

*The irrotational state is considered to be a realistic state for binary neutron stars before merger [9].

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\[ U^{2+1} = \frac{M_2}{R} \sum_{i=0}^{\infty} (-1)^i \left( \frac{\alpha}{R} \right)^i \xi^i P_i(\sin \theta \cos \varphi) \]
\[ + 3 \left( \frac{\alpha}{R} \right)^2 \left[ 1 - 3 \left( \frac{\alpha}{R} \right) \xi \sin \theta \cos \varphi + O(R^{-2}) \right] + \cdots. \]  
(5)

Here, \( P_i \) and \( f_{11} \) denote the Legendre function and the reduced quadrupole moment, and the superscript \( ' \) means the term concerned with star 2. \( K \) is a constant related to entropy and \( v \) represents the velocity field in the inertial frame. In the irrotational fluid case, we can express \( v \) as a gradient of a scalar function \( \Phi \), i.e., \( v = -\nabla \Phi \).

Following the Lane-Emden equation we first express the density as \( \rho = \rho_c \Theta^n \) with \( \rho_c \) being the central density. By using \( \alpha \equiv (K(1+n)\rho_c)^{1/n-1} / (4\pi)^{1/2} \), we also introduce \( \xi = r/\alpha \). We expand \( \Theta \) in a power series of a parameter \( \epsilon \) as \( \Theta = \sum_{i=0}^{\infty} \epsilon^i \Theta_i \). Since the shape of star 1 is spherical when \( R \) is large, the lowest order term of \( \Theta \) is the solution of the Lane-Emden equation. Then we expand \( \Theta_i \) by spherical harmonics as \( \Theta_i = \sum_{l,m} \psi_l^m(\xi) Y_l^m(\varphi, \theta) \). The radius of a spherical star \( a_0 \) is given by \( a_0 = \alpha \xi_1 \) as usual. Note here that \( \xi_1 = \pi \) for \( n = 1 \) case.

Now we consider the orbital motion of star 1. In the spherical coordinate system, it becomes

\[ \Omega \times r = \Omega R (\tilde{\Omega} \times \xi)_{orb} + \Omega \alpha (\tilde{\Omega} \times \xi)_{fig}, \]

(6)

where

\[ (\tilde{\Omega} \times \xi)_{orb} = \frac{1}{1+p} (\sin \theta \sin \varphi, \cos \theta \sin \varphi, \cos \varphi), \]

(7)

\[ (\tilde{\Omega} \times \xi)_{fig} = (0, 0, \xi \sin \theta), \]

and \( p \equiv M_1/M_2 \). The first term on the right-hand side of Eq. (6) comes from the orbital motion of the center of mass of star 1 and the second term comes from the fluid motion around the center of mass of star 1.

Next, we rewrite the equation of continuity (4) as

\[ \Delta \Phi = -n (\nabla \Phi - \Omega \times r) \cdot \nabla \Theta. \]

(9)

The condition for \( \Phi \) at the stellar surface is \( (\nabla \Phi - \Omega \times r) \cdot (\nabla \Theta)\big|_{surf} = 0 \), since \( \Theta = 0 \) at the surface. We expand \( \Phi \) also as \( \Phi = \sum_{i=0}^{\infty} \epsilon^i \Phi_i \). The gradient of the lowest order term of \( \Phi \) should agree with the orbital motion of the center of mass of star 1, i.e., \( \Omega R (\tilde{\Omega} \times \xi)_{orb} \) because when \( R \) is large, the shape of star 1 is spherical and star 1 has only the orbital motion of the center of mass in the inertial frame with no intrinsic spin. This leads us to normalize \( \Phi \) as \( \tilde{\Phi} = \Phi (\Omega \alpha a_0) \). We again expand \( \tilde{\Phi} \), by spherical harmonics as \( \tilde{\Phi}_i = \sum_{l,m} \psi_l^{m}(\xi) Y_l^m(\varphi, \theta) \).

The orbital angular velocity is derived from the first tensor virial relation defined by [13]

\[ \int d^3x \frac{\partial P}{\partial x_1} = 0, \]

(10)

where \( x_1 = r \sin \theta \cos \varphi \). If we substitute Eq. (2) and \( \Phi_0 \) into Eq. (4), we obtain the orbital angular velocity in the lowest order as \( \Omega_0^2 = M_{tot} / R^3 \) where \( M_{tot} = M_1 + M_2 \). Note here that we also expand \( \Omega^2 \) as \( \Omega^2 = \sum_{i=0}^{\infty} \epsilon^i \Omega_i^2 \).

Finally, we express the gravitational potential by rewriting Eq. (2) as

\[ U = K(1+n)\rho_c^{4/3} + \frac{1}{2} v^2 - v \cdot (\Omega \times r) + U_0. \]

where \( U_0 \) is constant. Substituting Eq. (11) into the Poisson equation (4), we obtain the equation to determine the equilibrium figure as

\[ \alpha^2 \Delta \Theta = \hat{\Theta} - 1 \frac{\Phi}{8\pi\rho_c} \left[ (\nabla \Phi)^2 - 2(\nabla \Phi) \cdot (\Omega \times r) \right]. \]

(12)

Now a solution can be obtained iteratively. Firstly, \( \hat{\Theta}_1 \) is determined by demanding that the gravitational potential and its normal derivative are continuous at the stellar surface [14], that is, \( U_{int}\xi=z = U_{ext}\xi=z \) and \( \partial U_{int}/\partial \xi=z = \partial U_{ext}/\partial \xi=z \), where \( \Xi(\Theta, \varphi) \) expresses the surface (\( \Theta(\xi, \varphi) = 0 \)). Substituting \( \hat{\Theta}_1 \) and Eq. (2) into Eqs. (4) and (10), we obtain \( \Phi_1 \) and \( \Omega^2_1 \). After that we substitute these equations into Eq. (12) and derive \( \hat{\Theta}_{i+1} \). We continued this procedure up to order \( \epsilon^6 \) in this Letter.

Since we would like to present almost analytic results, we only calculate \( n = 1 \) case in this Letter. For other polytropic indices, we must solve differential equations numerically. The results of these cases will be given in the subsequent paper [12]. The density profile up to \( O(\epsilon^6) \) becomes \( \Theta = \Theta_0 + \epsilon^3 \Theta_3 + \epsilon^4 \Theta_4 + \epsilon^5 \Theta_5 + \epsilon^6 \Theta_6 \), where

\[ \Theta_0 = \frac{\sin \xi}{\xi}, \]

(13)

\[ \Theta_3 = \frac{5}{p} j_2(\xi) P_2(\sin \theta \cos \varphi), \]

(14)

\[ \Theta_4 = \frac{-7\xi_1}{3p} j_3(\xi) P_3(\sin \theta \cos \varphi), \]

(15)

\[ \Theta_5 = \frac{9\xi_1^2}{(15 - \xi_1^2)p} j_4(\xi) P_4(\sin \theta \cos \varphi) + \frac{405}{7(15 - \xi_1^2)p} j_4(\xi) P_4(\sin \theta \cos \varphi), \]

(16)

\[ \Theta_6 = \frac{225}{19p^2 \xi_1} j_5(\xi) P_5(\sin \theta \cos \varphi) + \frac{11\xi_1}{105 - 10\xi_1^2} j_5(\xi) P_5(\sin \theta \cos \varphi). \]

(17)

Here \( j_l \) and \( P^m_l \) denote the spherical Bessel function and the associated Legendre function. The function \( \psi_{22}(\xi) \) is obtained by solving a differential equation:

\[ \frac{d\psi}{d\xi} - \frac{\xi^2}{2} \psi - \left[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \frac{\xi^2}{2} \frac{d}{d\xi} \right) - \frac{6}{\xi^2} \right] \psi = 0 \]

(11)

(18)
where \((\psi_0^2)(\xi)\) will be defined later. We point out that the terms \(\Theta_1\) and \(\Theta_2\) disappear in the equation for \(\Theta\) since there are no tidal terms to produce \(\Theta_1\) and \(\Theta_2\) in the gravitational potential.

We can write the velocity potential as \(\Phi = \Phi_0 + e^4 \Phi_4 + e^5 \Phi_5 + e^6 \Phi_6\), where
\[
\begin{align*}
\Phi_0 &= \frac{1}{1 + p} \xi \sin \theta \sin \varphi, \\
\Phi_4 &= (\psi_0^2) \frac{P_3^3}{(\cos \theta) \sin 2\varphi}, \\
\Phi_5 &= (\Theta_3(\xi)) \left[ P_3^1 (\cos \theta) \sin \varphi - \frac{1}{2} P_3^3 (\cos \theta) \sin 3\varphi \right], \\
\Phi_6 &= (\psi_0^2) \left[ \frac{P_3^2}{(\cos \theta) \sin 2\varphi - \frac{1}{4} P_3^4 (\cos \theta) \sin 4\varphi \right].
\end{align*}
\]

The functions \((\psi_0^2), (\psi_3)\) and \((\psi_4)\) are determined by solving following differential equations:
\[
\begin{align*}
\left[\frac{d^2}{d\xi^2} + \left(2 \frac{\Theta_0^2}{\Theta_0} \right) \frac{d}{d\xi} - \frac{6}{\xi^2}\right]^{(\psi_0^2)} &= -\frac{5j_2(\xi)}{2p_2^2 \Theta_0(\xi)}, \\
\left[\frac{d^2}{d\xi^2} + \left(2 \frac{\Theta_0^2}{\Theta_0} \right) \frac{d}{d\xi} - \frac{12}{\xi^2}\right]^{(\psi_3)} &= -\frac{7j_3(\xi)}{12p_3^2 \Theta_0(\xi)}, \\
\left[\frac{d^2}{d\xi^2} + \left(2 \frac{\Theta_0^2}{\Theta_0} \right) \frac{d}{d\xi} - \frac{20}{\xi^2}\right]^{(\psi_4)} &= \frac{3\xi j_4(\xi)}{4(15 - \xi^2)p_4 \Theta_0(\xi)}.
\end{align*}
\]

where \(\Theta_0^2\) denotes \(d\Theta_0(\xi)/d\xi\). Note that the terms \(\Phi_1, \Phi_2\) and \(\Phi_3\) disappear in the equation for \(\Phi\). In Table I, we show the results of the velocity potentials and their derivatives for an identical star binary \((p = 1)\). From the expressions of \(\Phi_4, \Phi_5\) and \(\Phi_6\), it is clear that the velocity has non-zero components along the orbital axis. Because since \((\psi_0^2)\) is not proportional to \(\xi^2\), there remains the velocity component \(\partial \Phi_4/\partial x_3\) where \(\dot{x}_3 = \xi \cos \theta\). We can also show the existence of the velocity components along the orbital axis for \(\Phi_5\) and \(\Phi_6\). This analytic result does not agree with the semi-analytic solution given by LRS \cite{LRS} but qualitatively agrees with the numerical solutions given by Uryū and Eriguchi \cite{UE98}.

The orbital angular velocity is calculated as
\[
\Omega^2 = M_{tot} a_0^3 c^3 \left[ 1 + \frac{9e^5}{2a_0^5} (\tilde{T}_{11} + \frac{\tilde{T}_{11}}{M_1} + \frac{\tilde{T}_{11}}{M_2}) + O(e^7) \right],
\]
where \(\tilde{T}_{11} = T_{11}/c^3\). The effect of the quadrupole moments in \(\Omega^2\) is \(O(e^8)\). One can see from Eq. (20) that the quadrupole term is \(O(e^8)\) higher than the monopole term since \(\tilde{T}_{11} = O(e^6)\). In the case of a different mass binary, i.e., a neutron star-black hole binary, we can express the orbital angular velocity as
\[
\Omega^2 = M_{tot} a_0^3 c^3 \left( 1 + \frac{9e^5}{2a_0^5} \tilde{T}_{11} \right),
\]
where we assume that a black hole is a point source.

Now it is ready to calculate relevant physical quantities. The mass of star 1 is calculated as \(M_1 = \int d^3 x \rho = 4\pi \rho \sigma^3 \xi_1 [1 + 45e^6/(2p^2 \xi_1^2)]\). Since we consider the sequence for given baryon mass, the mass of star 1 should be the same one, which yields
\[
\rho_c = \rho_{c0} \left(1 - \frac{45}{2p^2 \xi_1^2} e^6 \right),
\]
where \(\rho_{c0}\) is the central density for \(R = \infty\). We note that the central density of star 2 should be written by changing \(p\) into \(1/p\) in Eq. (28). The equation (28) means that the central density of star 1 up to order \(e^6\) decreases from that of the spherical star as the separation decreases. Although we did not assume ellipsoidal figures like in Ref. \cite{LRS}, the result is essentially the same as that derived by Lai \cite{Lai94}, since the calculation of the central density up to \(O(e^6)\) includes only quadratic terms. In both calculations the central density of irrotational binary system decreases as \(R^{-6}\). This dependence is different from that of corotating ones, i.e., \(R^{-3}\) \cite{UE98}.

Next, we calculate the total energy of the binary system, which is written as \(E = \Pi_{tot} + (W_{self})_{tot} + (W_{int})_{tot} + T_{tot}\), where \(\Pi_{tot}, (W_{self})_{tot}, (W_{int})_{tot}\) and \(T_{tot}\) denote the total internal energy, the total self-gravity energy, the total interaction energy and the total kinetic energy, respectively. They are given as
\[
\begin{align*}
\Pi_{tot} &= \frac{M_1 M_2}{4a_0} \left( p + \frac{1}{p} \right) \left[ 1 - 5 \left( \frac{15}{\xi_1^2} - 1 \right) e^6 \right], \\
(W_{self})_{tot} &= \frac{M_1 M_2}{4a_0} \left( p + \frac{1}{p} \right) \left[ -3 + 7 \left( \frac{15}{\xi_1^2} - 1 \right) e^6 \right], \\
(W_{int})_{tot} &= -\frac{M_1 M_2}{a_0} e^6 3M_1 M_2 \left( \frac{\tilde{T}_{11}}{M_1} + \frac{\tilde{T}_{11}}{M_2} \right), \\
T_{tot} &= \frac{M_1 M_2}{2a_0} e^6 \left[ 9M_1 M_2 \left( \frac{\tilde{T}_{11}}{M_1} + \frac{\tilde{T}_{11}}{M_2} \right) \right],
\end{align*}
\]

\footnote{In the case of \(n = 0\), \((\psi_0^2)\) is proportional to \(\xi^2\).}
where the reduced quadrupole moment of star 1 is calculated as $\tau_{11} = (2M_1a_1^3/3p)(15/\xi_1^2 - 1)$. The reduced quadrupole moment of star 2 is expressed by changing $M_1$ into $M_2$ and $p$ into $1/p$. We note that the above energies satisfy the virial equation for $n = 1$; i.e. $3\Pi_\text{tot} + (W_{\text{self}})_\text{tot} + (W_{\text{int}})_\text{tot} + 2T_\text{tot} = 0$. Accordingly, the total energy is given as

$$E = \frac{M_1M_2}{a_0}\left[-\frac{1}{2}(p + \frac{1}{p}) - \frac{\epsilon}{2} + \frac{15}{\xi_1^2}(1)(p + \frac{1}{p})\epsilon^6\right].$$

(33)

In the case of a different mass binary, we obtain the total energy as

$$E = \frac{M_1^2}{a_0}\left[-\frac{1}{2} - \frac{\epsilon^6}{p^2} + \frac{15}{\xi_1^2}(1)\right].$$

(34)

FIG. 2. The total energy as functions of the orbital separation for different mass binary ($p = 0.1$). Solid line and open triangles are the same notations in Fig. 1. Filled circles are the results of LRS (1993).

Our solution is correct if $\epsilon \ll 1$ so that it can be used to check the validity of numerical solutions. For any numerical codes, one can ask to solve an equilibrium for large $R$. One can compare numerically derived density and velocity distribution with our almost analytic solutions. However, for small $R$, our expansion up to $\epsilon^6$ may not be enough. We can include the effects of the quadrupole terms of stars in the physical values such as the total energy at order $\epsilon^6$. Since the next higher order terms are the octupole ones at order $\epsilon^8$ and their coefficients are order unity, we think that this expansion converges for the effect of the deformation. However, at order $\epsilon^3$, there appears the spin kinetic energy term in the total energy. There is a possibility to change the behavior of the total energy for small $R$. Therefore, in order to apply our solution in this case, further higher order calculations in our scheme as well as the check of numerical codes using our almost analytic solutions for large $R$ are urgently necessary.

In Figs. 1 and 2, we show the total energy for binary systems with $p = 1$ and $p = 0.1$ as functions of the orbital separation for an identical star binary. We find from these figures that the results of numerical and semi-analytic calculations coincide with our analytic solutions up to $\epsilon^6$ rather well in the region $R/a_0 > 3$ for $p = 1$ and $R/a_0 > 5$ for $p = 0.1$ although they are quite different in details.

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[1] K. S. Thorne, in Relativistic Cosmology, Proc. of the 8th Nishinomiya-Yukawa Memorial Symposium, ed. M. Sasaki (Tokyo Universal Academy), p.67 (1994).
[2] L. Lindblom, Astrophys. J. 398, 569 (1992).
[3] S. Bonazzola, E. Gourgoulhon, and J.-A. Marck, Phys. Rev. Lett. 82, 892 (1999).
[4] C. S. Kochanek, Astrophys. J. 398, 234 (1992).
[5] L. Bildsten and C. Cutler, Astrophys. J. 400, 175 (1992).
[6] K. Uryū and Y. Eriguchi, Astrophys. J. Suppl. Ser. 118, 563 (1998).
[7] D. Lai, F. A. Rasio, and S. L. Shapiro, Astrophys. J. 420, 811 (1994).
[8] J. C. Lombardi, F. A. Rasio, and S. L. Shapiro, Phys. Rev. D 56, 3416 (1997).
[9] P. Marronetti, G. J. Mathews, and J. R. Wilson, Phys. Rev. D 60, 087301 (1999).
[10] S. Chandrasekhar, Mon. Not. R. Astron. Soc. 93, 390 (1933); A. Kovetz, Astrphys. J. 154, 999 (1968).
[11] D. Lai, F. A. Rasio, and S. L. Shapiro, Astrophys. J. Suppl. Ser. 88, 295 (1993).
[12] K. Taniguchi and T. Nakamura, in preparation.
[13] D. Lai, Phys. Rev. Lett. 76, 4878 (1996).

| $\xi$ | $\phi_2$ | $\partial_1 \phi_2$ | $\phi_3$ | $\phi_4$ | $\partial_4 \phi_3$ | $\partial_4 \phi_4$ | $\partial_4 \phi_4$ |
|-------|----------|-----------------|-------|-------|----------------|----------------|----------------|
| 0.00  | 0.00     | 0.00            | 0.00  | 0.00  | 0.00           | 0.00            | 0.00           |
| 0.50  | 2.46(-2) | 9.86(-2)        | 5.80(-3) | -3.40(-5) | -2.73(-4)            | 0.00            | 0.00           |
| 1.00  | 9.91(-2) | 0.200           | 7.80(-3) | 2.36(-2) | -5.52(-4)            | -2.23(-3)       | 0.00           |
| 1.50  | 0.226    | 0.307           | 2.68(-2) | 5.50(-2) | -2.87(-3)            | -7.83(-3)       | 0.00           |
| 2.00  | 0.408    | 0.424           | 6.55(-2) | 0.103  | -9.40(-3)            | -1.96(-2)       | 0.00           |
| 2.50  | 0.652    | 0.556           | 0.133  | 0.171  | -2.41(-2)            | -4.13(-2)       | 0.00           |
| 3.00  | 0.967    | 0.710           | 0.241  | 0.269  | -5.33(-2)            | -7.87(-2)       | 0.00           |

TABLE I. Velocity potentials and their derivatives for an identical star binary ($p = 1$).