Einstein static universe, GUP, and natural IR and UV cut-offs

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We study the Einstein static universe in the framework of Generalized Uncertainty Principle constructed by the Snyder non-commutative space. It is shown that the deformation parameter can induce an effective energy density subject to GUP which obeys the holographic principle (HP) and plays the role of a cosmological constant. Using the holographic feature of this effective energy density, we introduce natural IR and UV cut-offs which depend on the GUP based effective equation of state. Moreover, we propose a solution to the cosmological constant problem. This solution is based on the result that the Einstein equations just couple to the tiny holographic based surface energy density (cosmological constant) induced by the deformation parameter, rather than the large quantum gravitational based volume energy density (vacuum energy) having contributions of order $M_p^4$.

Keywords: Einstein static universe; GUP; IR and UV cut-offs.

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1. Introduction

In the last decade, a new model of dark energy (DE) so called Holographic Dark Energy (HDE) was proposed based on the holographic principle (HP) \[1,2\]. The energy density for HDE is given by \[3-5\]

\[ \rho_{\Lambda} = 3c^2 M_p^2 L^{-2} , \tag{1} \]

where $c$ is a numerical constant, $M_p$ is the Planck mass and $L$ is the infrared IR cut-off length. Different choices for IR cut-off length have been proposed such as Hubble length, particle horizon, event horizon, apparent horizon, Ricci radius,
On the other hand, the existence of a minimal length is a prediction of quantum theory of gravity \[7\]-\[12\]. Therefore, at high energy physics such as early universe we must consider the effects of such a minimal length. Such consideration is achieved by the deformation of standard Heisenberg commutation relation known as the Generalized Uncertainty Principle (GUP) \[13\]-\[18\]. The simplest form of such relation in the framework of Snyder non-commutative space is given by \[19\]
\[\Delta q \Delta p \geq \frac{1}{2} | \sqrt{1 - \alpha p^2} |,\]
where we have used the units $\hbar = 1$. In this units, $\alpha$ is a deformation parameter, with the dimension of squared length (\(\text{Length}^2\)), which is assumed to have a sufficiently small value. In fact, it is treated as a small parameter with $\sqrt{\alpha}$ being the measure of a minimum length scale \[20\]. The important point which is worth to mention here is that GUP is applicable to all energy regimes, however the question that at which energy scale it becomes more important and considerable depends on the deformation parameter $\alpha$. The smallness of $\alpha$ is considered in relation to the order of magnitude of the momentum $p$, so that at low energy regime with classical values of momentum $p$, much smaller than the Plank energy, the correction term almost vanishes and we get the Heisenberg Uncertainty Principle, whereas at high energy regime with ultra-relativistic values of momentum $p$, of the order of Plank energy, the correction term becomes relevant and we get the Generalized Uncertainty Principle. The corresponding commutation relation can be written as
\[ [q, p] = i \sqrt{1 - \alpha p^2}, \]
where the only freedom is on the sign of the deformation parameter $\alpha$. Therefore, a maximum momentum or a minimal length are predicted by the Snyder-deformed relation \[20\] if $\alpha > 0$ or $\alpha < 0$, respectively.

On the other hand, the emergent “Einstein Static Universe” (ESU) scenario was proposed by Ellis et al to solve the initial singularity problem in the standard cosmological model \[21\], \[22\]. Initial singularity is the major problem of most cosmological models constructed based on General Relativity (GR), and there are variety of classical or quantum approaches to resolve this major problem. Recently, based on the loop quantum cosmology, the search for singularity free cosmological models within the framework of GR has led to development of the so-called ESU scenario \[21\], \[22\]. In this scenario, the early universe is initially in a past-eternal ESU state and eventually evolves to a subsequent inflationary phase as follows.

The stable ESU against cosmological perturbations addresses the existence of a fixed point around which our universe in the early times was eternally fluctuating. In the case of the universe filled by a massless scalar field with a suitable inflaton potential, these fluctuations could have been disappeared and the stable state of the universe might have been turned into an inflationary phase. This transmission from a stable ESU to an inflationary phase could have been taken place around the
different values of e-folds. Of course, the quantum field theory on curved spacetime and its modifications on the Hilbert-Einstein action may affect deeply the inflationary scenarios [23]. Therefore, transmission from a stable ESU to an inflationary phase could have been also affected by these modifications.

The achievements of this cosmological model are remarkable: there is no initial singularity; the universe is ever existing and it tends to a static universe in the past infinity rather than originating from a big bang singularity. Note that the emergent ESU scenario is completely different from that of introduced first by Einstein. The model of Einstein was introduced to describe a stable large scale universe, whereas the emergent ESU scenario which was introduced first by Ellis et al is related to the very early universe and the resolution of its problems such as initial singularity problem. In fact, emergent ESU belongs to the very early era of the universe, even before the inflationary era. For this reason, the Einstein Static Universe ESU is considered to be related to the High Energy Regime of the early universe. In this letter, we study the impact of Generalized Uncertainty Principle (GUP) on the Einstein Static Universe (ESU), Holographic Dark Energy (HDE) and also the Cosmological Constant Problem (CCP), in the context of Emergent Universe.

2. GUP and ESU

In this section, we investigate the consistency of the emergent Einstein Static Universe scenario with GUP. Let us consider the isotropic and homogeneous FRW universe described by the line element

$$ds^2 = -N^2 dt^2 + a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),$$

where $N = N(t)$ and $a = a(t)$ are the lapse function and scale factor, respectively, and the spatial curvature $k$ can be zero or $\pm 1$ depending on the symmetry group. We assume that this universe is filled by a perfect fluid given by $T_{\mu\nu} = \text{diag}(\rho, p, p, p)$. Hamiltonian constraint of such models is given by

$$H = -\frac{2\pi G}{3} \frac{p_a^2}{a} - \frac{3}{8\pi G} ak + a^3 \rho = 0,$$

where $G$ is the gravitational constant, $\rho = \rho(a)$ denotes for generic energy density of the system and $p_a$ is the momentum conjugate to the scale factor $a$. The dynamics of this model universe is described by the analysis of Hamiltonian constraint system which results in the following Poisson bracket equations

$$\dot{N} = \{N, H_E\}, \quad \dot{a} = \{a, H_E\}, \quad \dot{p}_a = \{p_a, H_E\},$$

where

$$H_E = \frac{2\pi G}{3N} \frac{p_a^2}{a} + \frac{3}{8\pi G} N ak - Na^3 \rho + \lambda \pi.$$
By using the above equations, based on the Poisson bracket \( \{a, p_a\} = 1 \), we can obtain the standard Friedmann equation. Now, we study the analysis of deformed dynamics of the FRW model to obtain the modifications of the Friedmann equation resulting from the algebra (3). The modified symplectic geometry at the classical limit of the algebra (3), results in the Snyder-deformed classical dynamics. According to Dirac, we can replace the quantum-mechanical commutator (3) by the classical Poisson bracket

\[-i[\tilde{q}, p] \Rightarrow \{\tilde{q}, p\} = \sqrt{1 - \alpha p^2}. \tag{8}\]

The deformed Poisson bracket must satisfy the same properties as those of quantum mechanical commutator, i.e. it has to be bilinear and anti-symmetric, and also it must satisfy the Leibniz rules as well as the Jacobi identity. Thus, the deformed Poisson bracket in two-dimensional phase space is given by

\[\{F, G\} = \frac{\partial F}{\partial \tilde{q}} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial \tilde{q}} \sqrt{1 - \alpha p^2}. \tag{9}\]

Specially, the canonical equations for coordinate and momentum from the deformed Hamiltonian \( \mathcal{H}(\tilde{q}, p) \) are obtained

\[\dot{\tilde{q}} = \{\tilde{q}, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p} \sqrt{1 - \alpha p^2},\]

\[\dot{p} = \{p, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial \tilde{q}} \sqrt{1 - \alpha p^2}. \tag{10}\]

Now, we apply this deformation scheme to the FRW model in the presence of matter energy density, namely to the Hamiltonian (7). We assume the minisuperspace to be Snyder-deformed and consequently the commutator between the scale factor \( a \) and its conjugate momentum \( p_a \) is uniquely obtained as

\[\{a, p_a\} = \sqrt{1 - \alpha p_a^2}. \tag{11}\]

Considering the term \( \alpha p_a^2 \) in RHS, and the units \( \hbar = 1 \), it is obvious that the dimension of \( \sqrt{\alpha} \) is the same dimension of the conjugate coordinate of the momentum \( p_a \), namely the scale factor \( a \). Therefore, since the dimension of \( \sqrt{\alpha} \) is \( \text{Length} \) then one may attribute the dimension of a \( \text{Length} \) to the scale factor too, as the radius of the universe.

The Snyder-deformed commutation relation (11) does not change the equations of motion \( \dot{N} = \{N, \mathcal{H}_E\} = \lambda \) and \( \dot{\pi} = \{\pi, \mathcal{H}_E\} = \mathcal{H} = 0 \). The Poisson bracket \( \{N, \pi\} = 1 \) is not affected by the deformations induced by the \( \alpha \) parameter. Nevertheless, the equations of motion (6) are modified in this approach via the relation (11), so we have

\[\dot{a} = \{a, \mathcal{H}_E\} = \frac{4\pi G p_a}{3N} a \sqrt{1 - \alpha p_a^2}, \tag{12}\]

\[\dot{p}_a = \{p_a, \mathcal{H}_E\} = \sqrt{1 - \alpha p_a^2} N \left( \frac{2\pi G p_a^2}{3 a^2} - \frac{3}{8\pi G} k + 3a^2 \rho + a^3 \frac{d\rho}{da} \right).\]

\(^a\)With this choice, the comoving radial coordinate \( r \) becomes dimensionless [28].
The deformed Friedmann equation can be obtained by solving the constraint \( \rho = \frac{8 \pi G}{3} \rho - \frac{k}{a^4} \) with respect to \( p_a \) and considering the first equation of (12) as follows \((N = 1)\) \[26\]

\[
\left( \frac{\dot{a}}{a} \right)^2 = \left( \frac{8 \pi G}{3} \rho - \frac{k}{a^4} \right) \left[ 1 - \frac{3 \alpha}{2 \pi G} a^2 \left( \rho - \frac{3}{8 \pi G} k \right) \right]. \tag{13}
\]

The conservation equations for the matter part is also given by

\[
\dot{\rho} + 3H(\rho + p) = 0. \tag{14}
\]

By using the equations (13) and (14) for the closed FRW universe \((k = 1)\) and the equation of state \( p = \omega \rho \), the acceleration equation is obtained

\[
\ddot{a} = \left[ -9(\omega + 1) - \frac{9 \alpha a^4}{2 \pi G} + \frac{6 \alpha a^3}{3 \pi G} - 4 \pi G a(w + 1) \rho + 12 \omega a^2 \rho^2 - \frac{9 \alpha a^8}{16 \pi G^2} \right]. \tag{15}
\]

where the matter energy density \( \rho \) is given by solving the equation (13) as follows \[27\]

\[
\rho = \frac{9 \pi G \alpha a^4 + 8 G^3 \pi^3 a^2 \rho^2 \pm \sqrt{4 a^4 G^6 \pi^6 - 9 \alpha^2 G^4 H^2 \pi^2 \alpha}}{24 a^6 G^2 \pi^2 \alpha}. \tag{16}
\]

Imposing the requirements of Einstein static solution given by \( \ddot{a} = \dot{a} = 0 \) in the equations (15) and using (16), we obtain the solutions \( a_\alpha(\omega) \) and the corresponding energy density \( \rho_\alpha(\omega) \) as follows

\[
a_\alpha^2(\omega) = \frac{16 \pi^2 G^2}{9 \alpha} \left( 1 - 3w \right), \quad \rho_\alpha(\omega) = \frac{27 \alpha}{64 \pi^3 G^3} \left( 1 + 3w \right). \tag{17}
\]

where we call \( \rho_\alpha(\omega) \) as the effective GUP based energy density. For a positive \( \alpha \), the reality condition for \( a_\alpha(\omega) \) and the positivity condition of \( \rho_\alpha(\omega) \) result in the following domain for the equation of state parameter

\[-1/3 < w < 1/3. \tag{18}\]

By removing the parameter \( \alpha \), the equation (17) can be rewritten as

\[
\rho(\omega) = \frac{3}{4 \pi G (1 - 3w) a^2}. \tag{19}
\]

Although \( \alpha \) is assumed to be a very small parameter, however the size of Einstein static universe \( a_\alpha(\omega) \) in (17) can be sufficiently small, by appropriate choice of \( \omega \leq \frac{1}{3} \), such that it can describe an emergent small size Einstein static universe with large energy density \( \rho_\alpha(\omega) \) at very early time. This means that such an emergent small universe is just consistent with the upper limit of the domain \[13\], namely the ultra-relativistic state of the effective matter \( p_\alpha(\omega) = \rho_\alpha(\omega) \) whose origin is GUP. This is in complete consistency with the origin of GUP described by \[3\] (or \[11\]) at high energy regime with ultra-relativistic values of momentum \( p \), of the order of Plank energy. In fact, the GUP correction term \( \alpha p_a^2 \) in \[14\] and the term \( (1 - 3w)/\alpha \) (or \( \alpha/(1 - 3w)^2 \)) in \[17\] represent a similar and common behavior that: in both cases the small parameter \( \alpha \) is accompanied by the high energy state of matter, namely very large momentums \( p_a \) or an ultra-relativistic state of matter with equation of
Therefore, it turns out that not only the emergent Einstein Static Universe is consistent with GUP, but also the implementation of GUP at high energy regime for the description of emergent Einstein Static Universe at very early universe is inevitable, and this certifies our motivation for the study of Einstein Static Universe in the framework of GUP.

3. Natural $w$ dependent IR-UV cut-offs

Comparing (19) with the generic form of the HDE, described by the equation (1) \((8\pi G = M_P^{-2})\), we find that the energy density \(\rho_\alpha(\omega)\) corresponding to the Einstein static universe has a holographic feature. Therefore, one may consider \(L_\alpha(\omega) = a_{Es}(\omega) \sqrt{(1 - 3w)/2c^2}\) as a \(w\) dependent cut-off for the HDE. At upper limit \(\omega \lesssim \frac{1}{3}\), which describes the ultra-relativistic state, this cut-off is essentially of UV type. At lower limit \(\omega \gtrsim -\frac{1}{3}\), which describes the threshold of exotic matter, this cut-off is essentially of IR type due to the very large size of \(a_{Es}(\omega)\). The dependence of IR-UV cut-offs on the equation of state parameter makes it a natural cut-off which is uniquely set by the natural state of material content.

The extreme condition \(\omega \lesssim \frac{1}{3}\) makes \(\rho_\alpha(\omega) \gg 0, a_{Es}(\omega) \ll 1\) and causes the state of effective matter to be ultra-relativistic \(p_\alpha(\omega) \simeq \rho_\alpha(\omega)/3\). These are physical characteristic features of emergent Einstein Static Universe whose size is extremely “small” and its matter content is extremely “dense” and “hot”. Such features can describe a universe which has capability of experiencing a Hot Big Bang, starting with an arbitrarily large energy density and an infinitesimally small size. However, since we have a finite and non-vanishing radius \(a_{Es}\), this Big Bang will be free of singularity problem. The singularity is avoided by resorting to the natural UV cut-off imposed by the natural state of the GUP based matter content given by \(\omega \lesssim \frac{1}{3}\).

4. Discussion

The equations of motion (13) and (15) in the absence of \(\alpha\) and without a cosmological constant \(\Lambda\) lead to the usual Friedmann equations which describe an expanding universe. However, in the presence of \(\alpha\) and without a cosmological constant \(\Lambda\), we obtain Einstein static universe. This clearly indicates that in the absence of cosmological constant \(\Lambda\), the static state of Einstein universe can be provided by the quantum gravitational effect induced by the deformation parameter \(\alpha\). In other words, the deformation parameter \(\alpha\) induces energy densities \(\rho_\alpha\) and \(\rho_\alpha(\omega)\) which can play the role of cosmological constants capable of constructing the Einstein static universe. These energy densities obey the holographic principle and ensures us about the fact that the origin of these energy densities lie on the two dimensional surface of a sphere with the radius of Einstein static universe. The bounds on \(\alpha\)

\[\begin{align*}
\text{ Relevant works to natural cut-offs are reported in Ref. [29]}.
\end{align*}\]
should be a subject of precise cosmological observations, for instance the measurement of the cosmological constant.

For the case of $\rho_\alpha$, by identification of $\rho_\alpha$ with the cosmological constant we can estimate the value of deformation parameter $\alpha$ by considering the observational upper bound on the present value of cosmological constant as $\alpha \sim (\Lambda \sim 10^{-47} GeV^4)$. This also gives estimation on the size of the radius of Einstein static universe as $a_\alpha = a_E \sim 10^{28} m$ which coincides with the size of the observable universe.

We may also study the cosmological constant problem in the present model for the energy density $\rho_\alpha$. The cosmological constant problem arises because of the huge disagreement between the observed value of cosmological constant, of order $10^{-47}GeV^4$, and the zero-point energy suggested by quantum field theory (QFT). We know that at high energies and small scales comparable to the Compton wavelength of the particles, matter is quantized and one can employ a semiclassical description of gravitation in which the Einstein equations take the form

$$G_{\mu\nu} = \langle T_{\mu\nu} \rangle,$$

where $G_{\mu\nu}$ is the Einstein tensor and $\langle T_{\mu\nu} \rangle$ denotes the expectation value of a quantum stress-energy tensor. In curved spacetime, even in the absence of classical matter and radiation, quantum fluctuations of matter fields give non-vanishing contributions to $\langle T_{\mu\nu} \rangle$ through a non-minimal couplings between matter and the curvature. Considering such non-vanishing contributions, of order $M_p^4 \sim 10^{74} GeV^4$, reveals a large discrepancy of order $10^{121}$ between the observed and calculated values of zero point energy.

In principle, the zero-point energy in quantum field theory is provided by the vacuum quantum field fluctuations originated by the Heisenberg uncertainty principle $\Delta q \Delta p \geq \frac{1}{2}$. By applying (2), we expect that the leading “unit” term in the square root is responsible for the zero-point energy of quantum gravity which can be originated by the vacuum fluctuations corresponding to Generalized Uncertainty Principle $\Delta q \Delta p \geq \frac{1}{2\sqrt{1-\alpha p^2}}$, and the small correction term $\alpha p^2 \sim \alpha M_p^2 \sim 10^{-10} < 1$ has negligible contribution to the zero-point energy of quantum gravity, hence it is not expected that the correction term $\alpha p^2$ can cancel out the huge zero-point energy of quantum gravity and solve the cosmological constant problem. However, the application of GUP on the Einstein static universe has a capability of resolving the cosmological constant problem in this model, as is described in the following.

We know that the zero-point energy density is the zero-point energy per unit volume, namely it is a energy density distributed over the entire volume inside the Einstein static universe, whereas the energy density $\rho_\alpha$ corresponding to the deformation parameter $\alpha$ is a surface energy density distributed over the entire surface enclosing the Einstein static universe. The fact that in spite of the presence of quantum gravitational vacuum fluctuations of order $M_p^4$ distributed over the volume of Einstein static universe, the cosmological constant in the Einstein equation (which arises effectively from GUP in the Einstein static universe) is merely set and es-
established by the holographic surface energy density $\rho_\alpha$ distributed over the surface enclosing the Einstein static universe, may remark the important point that the holographic based cosmological constant $\Lambda \sim \rho_\alpha$ with the origin over the enclosing surface of Einstein static universe is the real cosmological constant which contributes to the Einstein equation, and the zero-point volume energy density raised by the quantum gravitational corrections of order $M_P^4$ distributed over the volume of Einstein static universe is an irrelevant and nonphysical background energy which has vanishing contribution to the Einstein equation. In simple words, in the spirit of holographic principle, it seems that the observed cosmological constant $\Lambda \sim \rho_\alpha$ has just a holographic nature which is merely provided by the information encoded on the surface of the sphere enclosing the Einstein static universe, rather than the volume information inside the Einstein static universe. Therefore, one may conclude that the volume information including the quantum gravitational perturbative corrections, in principle cannot contribute to the surface information including the classical energy density $\rho_\alpha$ appearing in the Einstein equations. It turns out that, according to holographic principle, the Einstein tensor just feels (and react against) the small holographic surface energy density of order $10^{-47} GeV^4$ rather than the huge volume energy density of order $10^{74} GeV^4$. Therefore, similar to the quantum field theory in flat spacetime where the zero of energy is arbitrary and the vacuum energy with infinite energy is considered as a nonphysical background energy, the volume energy density here which is not felt by the Einstein tensor, can also be regarded as a nonphysical background energy, as explained above.

It is well-known that the zero-point energy is fundamentally related to the Heisenberg Uncertainty Principle stated as $\Delta q \Delta p \geq \frac{1}{2}$ or the commutation relation $[q, p] = i$. Since the zero-point energy is distributed over the entire space, it is obviously a volume energy density which can be attributed to non-vanishing “$\frac{1}{2}$” (or “$i$”) in $\Delta q \Delta p \geq \frac{1}{2}$ (or $[q, p] = i$). Considering the Generalized Uncertainty Principle $\Delta q \Delta p \geq \frac{1}{2}\langle \sqrt{1 - \alpha p^2} \rangle$ or generalized commutation relation $[q, p] = i\sqrt{1 - \alpha p^2}$, one can still consider the zero-point energy as a volume energy attributed to the first term appearing within the square root term in $\frac{1}{2}\langle \sqrt{1 - \alpha p^2} \rangle$ (or in $i\sqrt{1 - \alpha p^2}$). However, we have already shown that the second term within the square root terms, having contributions of deformation parameter $\alpha$, corresponds to a holographic surface energy density. Therefore, one may interpret the first and second terms “$1$” and “$\alpha p^2$” within the square root term, corresponding to the bare volume energy density and physical surface energy density, respectively. Now, the solution of cosmological constant problem lies in the fact that the bare cosmological constant of order $M_P^4$, coming from the volume energy density associated to “$1$” in the Generalized Uncertainty Principle does not contribute to the holographic based physical cosmological constant $\Lambda \sim \rho_\alpha$ coming from the deformation parameter $\alpha$.

In conclusion, the physical cosmological constant which is coupled to the Einstein equations comes just from the holographic surface energy density originating from “quantum gravitational modifications included in GUP”, and the bare co-
mological constant as a volume energy density receiving huge contributions from “quantum gravitational corrections to zero point energy” is naturally ruled out from being coupled to the Einstein equations.

For the case of $\rho_\alpha(\omega)$, the holographic energy density depends on both the quantum deformation parameter $\alpha$ and the classical equation of state parameter $\omega$. For any value of $\omega$ given by (18), the radius of Einstein static universe and the holographic energy density are determined by the equation (17). Moreover, according to the equation (19), the variable holographic energy density $\rho_\alpha(\omega)$ is proportional to the constant holographic energy density $\rho_\alpha$. Therefore, one may write $\rho_\alpha(\omega) = \rho_\alpha / (1 - 3w) = \rho^{eff}_\alpha$ where $\rho^{eff}_\alpha$ is considered as effective cosmological constant. Thus, the same discussion on the holographic origin of the observed cosmological constant and the solution of cosmological constant problem can be applied to this case, as well.

Summary and Conclusion

We have studied the Einstein Static Universe in the framework of Generalized Uncertainty Principle. We have shown that the deformation parameter can induce an energy density subject to GUP which obeys the holographic feature and plays the role of a cosmological constant. Using the holographic feature of GUP energy density, we have introduced natural IR-UV cut-offs which are dependent on the effective equation of state parameter subject to GUP. Moreover, we have realized that the Einstein equations are naturally coupled to the tiny holographic surface energy density (interpreted as physical cosmological constant) instead of large volume energy density (interpreted as bare cosmological constant). Then, we have proposed a solution to the cosmological constant problem.

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