Asymptotic expansion for the quadratic form of the diffusion process

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Summary In [8], asymptotic expansion of the martingale with mixed normal limit was provided. The expansion formula is expressed by the adjoint of a random symbol with coefficients described by the Malliavin calculus, differently from the standard invariance principle. As an application, an asymptotic expansion for a quadratic form of a diffusion process was derived in the same paper. This article gives some details of the derivation, after a short review of the martingale expansion in mixed normal limit.

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1 Introduction

The quadratic form of the increments of a diffusion type process under finite time horizon, the “realized volatility” in financial context for example, is in general asymptotically mixed normal. When the limit is normal, the asymptotic expansion of the quasi-likelihood type estimator was derived in [6] as an application of the martingale expansion. The expansion for the quadratic form with asymptotically mixed normal limit is then indispensable to develop the higher-order approximation and inference for the volatility. However, the classical approaches in limit theorems, where the limit is a process with independent increments, do not work.

The previous paper [8] presented asymptotic expansion of the martingale with mixed normal limit. The expansion formula is expressed by the adjoint of a random symbol with coefficients described by the Malliavin calculus, differently from the standard invariance principle. As an application, an asymptotic expansion for a quadratic form of a diffusion process was derived in [8]. The aim of this article is to give a short review of this result and some details of its derivation.

2 Asymptotic expansion of a quadratic form of a diffusion process

We consider a diffusion process satisfying the Itô stochastic integral equation

\[ X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dw_s. \]  

(1)

Here \( b \) and \( \sigma \) are assumed to be smooth functions with bounded derivatives of positive order. We only treat one-dimensional case for notational simplicity, however, multivariate analogue is straightforward. Extension to
general Itô processes is also possible but the descriptions would become involved. We will consider a quadratic form

$$ U_n = \sum_{j=1}^{n} c(X_{t_{j-1}})(\Delta_j X)^2, \quad (2) $$

of the increments of $X$, where $t_j = j/n$ and $\Delta_j X = X_{t_j} - X_{t_{j-1}}$. The function $c$ is in $C_+^\infty(\mathbb{R})$.\footnote{$C_+^\infty(\mathbb{R}; R^k)$ is the set of $\mathbb{R}^k$-valued smooth functions defined on $\mathbb{R}$ with all derivatives of at most polynomial growth, $C_+^\infty(\mathbb{R}; \mathbb{R})$ is simply denoted by $C_+^\infty(\mathbb{R})$.}

The quadratic form (2) of the increments of $X$ appears in applications in statistics and finance. In the high-frequency sampling of $n$ tending to $\infty$, $U_n$ converges in probability to

$$ U_\infty = \int_0^1 c(X_t)\sigma(X_t)^2 dt. $$

The normalized error is

$$ Z_n = \sqrt{n}(U_n - U_\infty). \quad (3) $$

It is well known that $Z_n$ has a mixed normal limit distribution in general. However, the limit theorem is not always sufficient for approximation nor for theoretical statistics. Our interest is in more precise approximation to the distribution of $Z_n$.

We write $f_t$ for $f(X_t)$, given function $f$. For differentiable $f$, the Itô decomposition of $f_t = f(X_t)$ is denoted by

$$ f_t = f_0 + \int_0^t f_t^{[1]}dw_s + \int_0^t f_t^{[0]}ds. $$

Obviously,

$$ f_t^{[1]} = \sigma(X_t)\partial_x f(X_t) \quad \text{and} \quad f_t^{[0]} = Lf(X_t) \quad \text{with} \quad L = b\partial_x + \frac{1}{2}\sigma^2\partial_x^2. $$

For a $d_1$-dimensional reference variable, we will consider

$$ F_n = \frac{1}{n} \sum_{j=1}^{n} \beta(X_{t_{j-1}}) \quad \text{or} \quad F_n = F_{\infty} := \int_0^1 \beta(X_t)dt, \quad (4) $$

where $\beta \in C_+^\infty(\mathbb{R}; \mathbb{R}^{d_1})$. The results will be the same in these cases up to the first order asymptotic expansion we will discuss in this paper. It is standard in theoretical statistics to treat the joint distribution of $Z_n$ and $F_n$ because $F_n$ can be the quadratic variation of the score martingale and the LAMN property is then established on the joint convergence. The studentization also motivates the joint convergence.

Let $a(x) = c(x)\sigma(x)^2$. Define $1 + d_1$-dimensional vector fields $V_0$ and $V_1$ by

$$ V_0(x_1, x_2) = \begin{bmatrix} b(x_1) - \frac{1}{\beta(x_1)}\sigma(x_1)\partial_x, \sigma(x_1) \\ \beta(x_1) \end{bmatrix} \quad \text{and} \quad V_1(x_1, x_2) = \begin{bmatrix} \sigma(x_1) \\ 0 \end{bmatrix} $$

for $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^{d_1}$. Denote by $\text{Lie}[V_0; V_1](x_1, x_2)$ the Lie algebra generated by

$$ V_1, [V_i, V_j] (i, j = 0, 1), [V_i, [V_j, V_k]] (i, j, k = 0, 1), .... $$

at $(x_1, x_2)$, where $[\cdot, \cdot]$ is the Lie bracket.

Assume that the support $\text{supp} P_{X_0}$ of the law of $X_0$ is compact. We will assume the following non degeneracy conditions.

$$ [\text{Hi}] \quad \inf_{x \in \mathbb{R}} |a(x)| > 0. $$
Let $[V_0; V_1](X_0, 0) = \mathbb{R}^{1+d_1}$ a.s.

The asymptotic expansion formula will be described with certain random symbols. The full random symbol, denoted by $\sigma(z, iu, iv)$ for $(z, u, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d_1}$, consists of the adaptive random symbol $\phi$ and the anticipative random symbol $\bar{\sigma}$. These random symbols are defined as follows.

Let
\[
h_t = c_t b_t^2 + c_t b_t^{[1]} \sigma_t - \frac{1}{2} c_t^{[0]} \sigma_t^2 - c_t^{[1]} \sigma_t \sigma_t^{[1]}\]
and
\[
k_t = 2c_t b_t \sigma_t + c_t \sigma_t \sigma_t^{[1]} - \frac{1}{2} c_t^{[1]} \sigma_t^2.\]

In the present situation, the adaptive random symbol is given by
\[
\sigma(z, iu, iv) = \frac{2}{3} \int_0^1 a(X_s)^3 ds \left( \int_0^1 a(X_s)^2 ds \right)^{-1} (iu)^2 + iu \int_0^1 k_t dw_t + iu \int_0^1 h_t dt. \tag{5}
\]

The processes $D_s X_t$ and $D_r D_s X_t$ satisfy a system of partially linear equations:
\[
D_s X_t = \sigma(X_s) + \int_s^t b'(X_t) D_s X_t dt_1 + \int_s^t \sigma'(X_t) D_s X_t dw_t,
\]
for $s \leq t$, and
\[
D_r D_s X_t = \sigma'(X_s) D_r X_s + \int_s^t b''(X_t) D_r X_t D_s X_t dt_1 + \int_s^t b'(X_t) D_r D_s X_t dt_1
+ \int_s^t \sigma''(X_t) D_r X_t D_s X_t dw_t_1 + \int_s^t \sigma'(X_t) D_r D_s X_t dw_t,
\]
for $r<s \leq t$. The $L^p$-estimates of the solution are at hand. Then the anticipative random symbol is given by the formula
\[
\bar{\sigma}(iu, iv) = \int_0^1 iu a(X_s) \sigma_{s,s}(iu, iv) ds \tag{6}
\]
with
\[
\sigma_{s,s}(iu, iv) = \left(-u^2 \int_s^1 \alpha'(X_t) D_s X_t dt + i \int_s^1 \beta'(X_t) [v] D_s X_t dt \right) + \int_s^1 \left\{ \alpha''(X_t) (D_s X_t)^2 + \alpha'(X_t) D_s D_s X_t \right\} dt
+ i \int_s^1 \left\{ \beta''(X_t) [v] (D_s X_t)^2 + \beta'(X_t) [v] D_s D_s X_t \right\} dt, \tag{7}
\]
where $\alpha(x) = a(x)^2$. Let $C_\infty = 2 \int_0^1 a(X_t) dt$.

The density of the multi-dimensional normal distribution with mean vector $m$ and variance matrix $C$ is denoted by $\phi(z; m, C)$. With the full random symbol
\[
\sigma(z, iu, iv) = \sigma(z, iu, iv) + \bar{\sigma}(iu, iv), \tag{8}
\]
the density function $p_n(z, x) \in C^\infty(\mathbb{R}^{1+d_1})$ is defined by
\[
p_n(z, x) = E \left[ \phi(z; 0, C_\infty) \delta_x(F_n) \right] + \frac{1}{\sqrt{n}} E \left[ \sigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; 0, C_\infty) \delta_x(F_n) \right\} \right]. \tag{9}
\]
Here $\delta_x(F_\infty)$ is Watanabe’s delta function ([5]). The adjoint operation $\sigma(z, \partial_z, \partial_x)^*$ is defined by

$$\sigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; 0, C_\infty)\delta_x(F_\infty) \right\} = \sum_j (-\partial_x)^{m_j} (-\partial_x)^{n_j} \left\{ c_j(z)\phi(z; 0, C_\infty)\delta_x(F_\infty) \right\}$$

for the random symbol $\sigma(z, \partial_z, \partial_x)$ having a representation

$$\sigma(z, \partial_z, \partial_x) = \sum_j c_j(z)(iu)^{m_j}(iv)^{n_j} \quad \text{(finite sum)}$$

where $c_j$ are random functions of $z$, $m_j \in \mathbb{Z}_+^d$ (in the present case $d = 1$) and $n_j \in \mathbb{Z}_+^d$. If $c_j$, $C_\infty$ and $F_\infty$ are smooth in Malliavin’s sense and $F_\infty$ satisfies a suitable nondegeneracy condition, $C_\infty$ being nondegenerate as well, then this adjoint operation is well defined. These conditions are satisfied in the present situation, therefore $p_n(z, x)$ is well defined. See [8] for details of random symbols and the adjoint operation.

The following theorem gives an error bound for the approximate density $p_n(z, x)$.

**Theorem 1.** Suppose that $[H1]$ and $[H2]$ are satisfied. Then for any positive numbers $M$ and $\gamma$,

$$\sup_{f \in \mathcal{E}(M, \gamma)} \left| E[f(Z_n, F_n)] - \int_{\mathbb{R}^{1+d_1}} f(z, x)p_n(z, x)dzdx \right| = o\left(\frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$, where $\mathcal{E}(M, \gamma)$ is the set of measurable functions $f : \mathbb{R}^{1+d_1} \to \mathbb{R}$ satisfying $|f(z, x)| \leq M(1+|z|+|x|)^\gamma$ for all $(z, x) \in \mathbb{R} \times \mathbb{R}^{d_1}$.

**Remark 1.** Condition $[H1]$ is usually ensured by the uniform ellipticity of the diffusion process $X_t$ and a reasonable choice of the estimator. In this sense, it is a natural assumption in statistical context.

**Remark 2.** The hybrid I method (a rough Monte-Carlo method in the first order asymptotic expansion term) is useful in the application of the expansion formula to numerical approximation. Applications to volatility derivatives are in our scope.

**Remark 3.** In the present article, we have a conditioning variable as $F_n$. On the other hand, it is also possible to consider versions of our results without $F_n$ if we were interested in a single (not joint) expansion. It will reduce the differentiability conditions of variables. However, considering the joint distribution is natural in non-ergodic statistics. Studentization is important in any case.

**Remark 4.** It is also possible to obtain asymptotic expansion of the conditional distribution.

Section 4 will give some details of derivation of Theorem 1.

## 3 Review of the asymptotic expansion of a double stochastic integral having a mixed normal limit

In this section, we will give a short review of the martingale expansion. We refer the reader to [8] for details.

On a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0,1]}, P)$ with $\mathcal{F} = \mathcal{F}_1$, we shall consider a sequence of $d$-dimensional functionals with decomposition

$$Z_n = M_n + W_n + r_nN_n.$$

(10)

Here, for every $n \in \mathbb{N}$, $M^n = (M^n_t)_{t \in [0,1]}$ denotes a $d$-dimensional continuous martingale with respect to $\mathbf{F}$ and $M_n = M^n_1$. In this decomposition, we assume $W_n, N_n \in \mathcal{F}(\Omega; \mathbb{R}^d)^3$ and $(r_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers tending to zero as $n \to \infty$.

In this section, the reference variables are $F_n \in \mathcal{F}(\Omega; \mathbb{R}^{d_1})$ ($n \in \mathbb{N}$) are general and we do not assume a specific structure like (4). It is possible to give asymptotic expansion of $\mathcal{E}\{Z_n\}$ under certain conditions; see

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\footnote{The set of $d$-dimensional measurable mappings.}
Let $(\mathbb{W}, \mathbb{P})$ be an $r$-dimensional Wiener space over time interval $[0, 1]$ and let $H$ be the Cameron-Martin subspace of $\mathbb{W}$. Suppose that the probability space $(\Omega, \mathcal{F}, P)$ is such that $\Omega = \Omega' \times \mathbb{W}$, $\mathcal{F} = \mathcal{F'} \times \mathcal{B}(\mathbb{W})$ and $P = P' \times \mathbb{P}$ for some probability space $(\Omega', \mathcal{F}', P')$. We will use the partial Malliavin calculus on $\Omega$ based on the shifts in the direction of $H$. For a Hilbert space $E$, the Sobolev space of $E$-valued functionals on $\Omega$ with indices $s \in \mathbb{R}$ for differentiability and $p \in (1, \infty)$ for integrability is denoted by $\mathbb{D}_{s,p}(E)$.

Let $d = d + d_1$ and $\ell = d + 6$. Let $K_n \in \mathbb{D}_{\ell+1,\infty}(H \otimes \mathbb{R}^d)$ and $\tilde{K}_n \in \mathbb{D}_{\ell+1,\infty}(H \otimes \mathbb{R}^d \otimes \mathbb{R}^d)$. Since $H$ can be identified with $L^2([0, 1]; \mathbb{R}^d)$, the functionals $K_n$ and $\tilde{K}_n$ are respectively regarded as $\mathbb{R}^d \times \mathbb{R}$-valued and $\mathbb{R}^d \otimes \mathbb{R} \otimes \mathbb{R}$-valued $L^2$-functions on $[0, 1]:$

$$
\tilde{K}_n = (\tilde{K}_n(s)_{\alpha})_{0 \leq s \leq 1, i = 1, \ldots, d, \alpha = 1, \ldots, r} \quad \text{and} \quad \tilde{K}_n = (\tilde{K}_n(s)_{\alpha\beta})_{0 \leq s \leq 1, i = 1, \ldots, d, \alpha, \beta = 1, \ldots, r}.
$$

The sequence $\{t_j\}_{j=0,1,\ldots,j^n}$ ($n \in \mathbb{N}$) is a triangular array of numbers such that $t_j = t_j^n$ depending on $n$ and that $0 = t_0 < t_1 < \cdots < t_{j^n} = 1$. Functional $K_n(s, r)$ is defined as the $\mathbb{R}^d \otimes \mathbb{R} \otimes \mathbb{R}$-valued function

$$
K_n(s, r) = \left[ r_n^{-1} \sum_j 1(t_{j-1}, t_j)(s) \tilde{K}_n(s)_{\alpha} 1(t_{j-1}, t_j)(r) \tilde{K}_n(r)_{\alpha\beta} \right]_{i = 1, \ldots, d, \alpha = 1, \ldots, r}.
$$

Suppose that $\tilde{K}_n$ and $\tilde{K}_n$ are progressively measurable. More strongly, we assume the strong predictability condition that $\tilde{K}_n(s)$ is $\mathcal{F}_{t_{j-1}}$-measurable for $s \in (t_{j-1}, t_j)$. Corresponding to the kernel $K_n$, we consider $M_n$ given by

$$
M_n = \left[ r_n^{-1} \sum_j \sum_{\alpha} \int_{t_{j-1}}^{t_j} \tilde{K}_n(s)_{\alpha} \left( \sum_{\beta} \int_{t_{j-1}}^{s} \tilde{K}_n(r)_{\alpha\beta} dw_r^\beta \right) dw_s^\beta \right]_{i = 1, \ldots, d}.
$$

Write $C_n = \langle M^n \rangle_t$ and $C_n = \langle M^n \rangle_1$. Suppose $\max_j |I_j| = o(r_n)$, where $I_j = (t_{j-1}, t_j]$, and that the sequence of measures

$$
\mu^n = r_n^{-2} \sum_j |I_j|^2 \delta_{t_{j-1}} \to \mu
$$

weakly for some measure $\mu$ on $[0, 1]$ with a bounded derivative. We will assume that $r_n^{-8} \sum_j |I_j|^5 = O(1)$. In this case,

$$
C_n = \left[ \sum_{\alpha} \int_0^1 r_n^{-2} \sum_j 1(t_j)(s) \tilde{K}_n(s)_{\alpha} \left( \sum_{\beta} \int_{t_{j-1}}^{s} \tilde{K}_n(r)_{\alpha\beta} dw_r^\beta \right) \cdot \tilde{K}_n(s)_{\alpha} \left( \sum_{\beta} \int_{t_{j-1}}^{s} \tilde{K}_n(r)_{\alpha\beta} dw_r^\beta \right) ds \right]_{i_1, i_2 = 1, \ldots, d}
$$

and the in-p limit of $C_n$ will be

$$
C_\infty = \left[ \frac{1}{2} \sum_{\alpha, \beta} \int_0^1 \tilde{K}_\infty(t, t)_{\alpha\beta} \tilde{K}_\infty(t, t)_{\alpha\beta} \mu(dt) \right]_{i_1, i_2 = 1, \ldots, d}
$$

$$
= \frac{1}{2} \text{Tr}^* \int_0^1 \tilde{K}_\infty(t) \otimes \tilde{K}_\infty(t) \mu(dt)
$$

under the conditions we will assume, where $\tilde{K}_\infty$ is the limit of $\tilde{K}_n$, and $\text{Tr}^*$ is the trace on $(\mathbb{R} \otimes \mathbb{R})^* \otimes (\mathbb{R} \otimes \mathbb{R})^*$.

Let $q \in (1/3, 1/2)$. Let $\Delta = \{(s, r); 0 \leq r \leq s \leq 1\}$ and $\Delta^n = \cup_j \{(s, r); t_{j-1} \leq s \leq t \leq t_j\}$. Let $\mathbb{I}_s = -\frac{1}{2}(C_s - C_t)$. [8] The same paper applied the expansion to the case where $M_n$ is given as a sum of double Itô integrals, as reviewed in what follows.
Similarly we define $\bar{\Omega}, 1(H \otimes H \otimes \mathbb{R}^d)$ and a representation density of each derivative admits

$$\text{ess.} \sup_{r_1,\ldots,r_k \in (0,1), \ (r,r) \in \Delta^n, \ n \in \mathbb{N}} \left\| D_{r_1,\ldots,r_k} \bar{K}^n(s,r) \right\|_p < \infty$$

for every $p \in (1, \infty)$ and $k \leq \ell + 1$.

(ii) For every $\eta > 0$ and $p \in (1, \infty)$,

$$\sup_{s \in (0,1), r \in S^d} \left\| \left[ \frac{I_s[e^{\eta^2}]}{(1-s)^{1+\eta^2}} \right]^{-1} \right\|_p < \infty.$$

(iii) For every $p > 1$,

$$\sup_j \sup_{s \in i_j} \| \bar{K}^n(s,r) - \bar{K}^n(t_{j-1}, t_{j-1}) \|_{\ell,p} = O(r^{2q})$$

and

$$\sup_j \sup_{t \in I_j} \| \bar{K}^n(t_{j-1}, t_{j-1}) - \bar{K}^n(t,t) \|_{\ell,p} = O(r^{2q})$$

as $n \to \infty$.

**Remark 5.** Condition [A1](iii) is [A1](iv) of [8]. In [8], [A1] was “Conditions in [A1] except (iii) hold.”

**Remark 6.** (i) In typical cases $r_n = n^{-1/2}$ so that $r_n^{2q} = n^{-q} > n^{-\frac{q}{2}}$ for $n > 1$.

(ii) Under [A1] (i), for every $p \in (1, \infty)$ and $k \leq \ell$,

$$\text{ess.} \sup_{r_1,\ldots,r_k \in (0,1), s \in (0,1)} \left\| \frac{D_{r_1,\ldots,r_k} \bar{K}^n}{1-s} \right\|_p < \infty.$$

(iii) As for [A1] (ii), we need the nondegeneracy of the derivative of $C_s^\infty$ in $s$, or a large deviation argument, in order to control $\exp(\int \frac{1}{2} (C_s^\infty - C_1^\infty)(u\otimes 2))$ for $s$ near 1.

Let $C_\infty \in \mathcal{F}(\Omega; \mathbb{R}^d \otimes \mathbb{R}^d)$, $W_\infty \in \mathcal{F}(\Omega; \mathbb{R}^d)$ and $F_\infty \in \mathcal{F}(\Omega; \mathbb{R}^{d_1})$. Set $\bar{C}_n = r_n^{-1}(C_n - C_\infty)$, $\bar{W}_n = r_n^{-1}(W_n - W_\infty)$ and $\bar{F}_n = r_n^{-1}(F_n - F_\infty)$. Consider an extention

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega \times \bar{\Omega}, \mathcal{F} \times \bar{\mathcal{F}}, P \times \bar{P})$$

of $(\Omega, \mathcal{F}, P)$ by a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$. Suppose that $M_\infty \in \mathcal{F}(\bar{\Omega}; C([0,1]; \mathbb{R}^d))$, $N_\infty \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^d)$, $\bar{C}_\infty \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^d \otimes \mathbb{R}^d)$, $\bar{W}_\infty \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^d)$ and $\bar{F}_\infty \in \mathcal{F}(\bar{\Omega}; \mathbb{R}^{d_1})$.

For $\bar{\mathcal{F}} = \mathcal{F} \vee \sigma(M_\infty)$, there exists a measurable mapping $\bar{C}_\infty : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that

$$\bar{C}_\infty(\omega, M_\infty) = E[\bar{\mathcal{F}} | \bar{\mathcal{F}}].$$

Similarly we define $\bar{W}_\infty(\omega, z)$, $\bar{F}_\infty(\omega, z)$ and $\bar{N}_\infty(\omega, z)$ by

$$\bar{W}_\infty(\omega, M_\infty) = E[\bar{W}_\infty | \bar{\mathcal{F}}]$$

$$\bar{F}_\infty(\omega, M_\infty) = E[\bar{F}_\infty | \bar{\mathcal{F}}]$$

$$\bar{N}_\infty(\omega, M_\infty) = E[\bar{N}_\infty | \bar{\mathcal{F}}].$$

Further, we introduce the notation

$$\bar{C}_\infty(z) \equiv \bar{C}_\infty(\omega, z) := \bar{C}_\infty(\omega, z - W_\infty)$$

and similarly $\bar{W}_\infty(\omega, z)$, $\bar{F}_\infty(\omega, z)$ and $\bar{N}_\infty(\omega, z)$.

Let $s_\infty$ be a positive functional defined on $\Omega$. It is said that a functional $c : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a $D_{s,\infty}$-polynomial if $c$ is a polynomial of $z \in \mathbb{R}^d$ with coefficients in $D_{s,\infty}$.

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4Condition [A1] (iv) of [8].
For the full random symbol 

\[ \sigma \in \mathcal{D}_{t+1,\infty}(\mathbb{R}^d) \] 

and \( W \in \mathcal{D}_{t+1,\infty}(\mathbb{R}^d) \). Moreover,

\[
\sup \left\{ \| W_n \|_{\ell+1,p} + \| F_n \|_{\ell+1,p} + \| N_n \|_{\ell+1,p} + \| s_n \|_{\ell,p} \right\} < \infty.
\]

for every \( p \geq 2 \).

(iii) \( \hat{C}^j_{\infty}, \hat{W}^j_{\infty}, \hat{N}^j_{\infty} (j,k = 1, \ldots, d) \) and \( \hat{F}^{d}_{\infty} (l = 1, \ldots, d_1) \) are \( \mathcal{D}_{t_0,\infty} \)-polynomials in \( z \in \mathbb{R}^d \), where \( t_0 = 2[(d_1 + 3)/2] \).

(iv) \( (M_n^\sigma, N_n^\sigma, C_n^\sigma, W_n^\sigma, F_n^\sigma) \rightarrow d_{\mathcal{F}}^\sigma(x) \) \( (M^\infty, N^\infty, C^\infty, W^\infty, F^\infty) \).

(v) For \( G = W^\infty \) and \( F^\infty \),

\[
\text{ess.} \sup_{r_1,\ldots,r_\ell \in (0,1)} \| D_{r_1,\ldots,r_\ell} G \|_p < \infty
\]

for every \( p \in [2, \infty) \) and \( k \leq \ell + 1 \). Moreover, \( r \mapsto D_r G \) and \( (r,s) \mapsto D_{r,s} G \) are continuous a.e. with respect to the Lebesgue measures.

The nondegeneracy of \((M^n + W^\infty, F^\infty)\) will be necessary.

For every \( p \geq 2 \), \( \limsup_{n \to \infty} E[s_n^{-p}] < \infty \).

Remark 7. The nondegeneracy \( C_\infty^1 \in \cap_{p \geq 2} L^p \) follows from \([A1] (ii) \). Indeed, it implies

\[
\sup_{\epsilon \in \mathbb{S}_{d-1}} P[C_{\infty}[e^{\otimes 2}] < \epsilon] \leq \epsilon^p \sup_{\epsilon \in \mathbb{S}_{d-1}} \| n[e^{\otimes 2}]^{-1} \|_p \leq C_p \epsilon^p \quad (\epsilon > 0)
\]

for some constant \( C_p \) for every \( p > 0 \). Then the desired inequality is obtained; see e.g. Lemma 2.3.1 of Nualart [4].

Now we recall the martingale expansion ([7, 8]). For \( Z_n \) given by (10) and (11), the random symbols are specified as follows. The adaptive random symbol is

\[
\sigma(z, iu, iv) = \frac{1}{2} \hat{C}_{\infty}(z)[(iu)^{\otimes 2}] + \hat{W}_{\infty}(z)[iu] + \hat{N}_{\infty}(z)[iu] + \hat{F}_{\infty}(z)[iv].
\]

(14)

For the double stochastic integral in question, the anticipative random symbol is given by

\[
\tilde{\sigma}(iu, iv) = \frac{1}{2} \text{Tr}^* \int_0^1 \bar{K}^\infty(t, t)[iu] \otimes \sigma_{t,i}(iu, iv) \mu(dt),
\]

where the random symbol \( \sigma_{t,i}(iu, iv) \) has the expression

\[
\sigma_{t,i}(iu, iv) = \left( i D_t W_{\infty}[u] - \frac{1}{2} D_t C_{\infty}[u^{\otimes 2}] + i D_t F_{\infty}[v] \right) \otimes \left( i D_t W_{\infty}[u] - \frac{1}{2} D_t C_{\infty}[u^{\otimes 2}] + i D_t F_{\infty}[v] \right)
\]

\[
\quad + \left( i D_t D_t W_{\infty}[u] - \frac{1}{2} D_t D_t C_{\infty}[u^{\otimes 2}] + i D_t D_t F_{\infty}[v] \right)
\]

with the representation densities of the Malliavin derivatives of functionals. The approximate density \( p_n(z, x) \) is defined by

\[
p_n(z, x) = E \left[ \phi(z; W^\infty, C^\infty) \delta_x(F^\infty) \right] + r_n E \left[ \sigma(z, \partial_z, \partial_x)^* \left\{ \phi(z; W^\infty, C^\infty) \delta_x(F^\infty) \right\} \right].
\]

(15)

for the full random symbol

\[
\sigma = \sigma + \tilde{\sigma}.
\]

Note that \( u \) is \( d \)-dimensional here. Class \( \mathcal{E}(M, \gamma) \) will be abused for functions on \( \mathbb{R}^d \).
Theorem 2. Suppose that Conditions [A1], [A2] and [A3] are fulfilled. Then for any positive numbers $M$ and $\gamma$,

$$\sup_{f \in \mathcal{E}(M, \gamma)} \left| E[f(Z_n, F_n)] - \int_{\mathbb{R}^d} f(z, x)p_n(z, x)dzdx \right| = o(r_n)$$

as $n \to \infty$.

4 Some details of derivation of the expansion for the quadratic form

In this section, we will give somewhat detailed proof of Theorem 1, which was originally presented in [8].

4.1 Stochastic expansion

We will work with the Itô stochastic integral equation (1). The following lemma gives a stochastic expansion of the targeted variable $Z_n$ of (3).

Lemma 1. $Z_n$ has the following stochastic expansion:

$$Z_n = M^n_1 + \frac{1}{\sqrt{n}}N_n,$$

where

$$M^n_1 = \sqrt{n} \sum_{j=1}^{n} 2c_{t_{j-1}}\sigma^2_{t_{j-1}} \int_{t_{j-1}}^{t_{j}} \int_{t_{j-1}}^{s} dw_r dw_s$$

and

$$N_n = 6n \sum_{j=1}^{n} c_{t_{j-1}}\sigma_{t_{j-1}} [1] \int_{t_{j-1}}^{t_{j}} \int_{t_{j-1}}^{s} dw_u dw_s dw_t$$

$$+ 2 \sum_{j=1}^{n} c_{t_{j-1}} b_{t_{j-1}} \sigma_{t_{j-1}} \int_{t_{j-1}}^{t_{j}} dw_t + 2n \sum_{j=1}^{n} c_{t_{j-1}}\sigma_{t_{j-1}}[1] \int_{t_{j-1}}^{t_{j}} (t - t_{j-1})dw_t$$

$$+ n^{-1} \sum_{j=1}^{n} c_{t_{j-1}} b^2_{t_{j-1}} + n^{-1} \sum_{j=1}^{n} c_{t_{j-1}}\sigma_{t_{j-1}} b_{t_{j-1}}[1]$$

$$- n \sum_{j=1}^{n} c_{t_{j-1}}^{[1]} \sigma_{t_{j-1}} \int_{t_{j-1}}^{t_{j}} \int_{t_{j-1}}^{t} dw_s dt$$

$$- \frac{1}{2n} \sum_{j=1}^{n} c_{t_{j-1}}^{[0]} \sigma^2_{t_{j-1}} - \frac{1}{n} \sum_{j=1}^{n} c_{t_{j-1}}^{[1]} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^{[1]} + o_M(1).$$

Here $o_M(1)$ denotes a term of $o(1)$ as $n \to \infty$ with respect to $\mathbb{D}_{s,p}$-norms of any order. The families $\{M^n_t\}_{t \in [0,1], n \in \mathbb{N}}$ and $\{N_n\}_{n \in \mathbb{N}}$ are bounded in every $\mathbb{D}_{s,p}$-norm.

It is possible to obtain the above lemma by somewhat long computations. See Section 5.

4.2 Random symbols

4.2.1 Adaptive random symbol

The discrete filtration $\mathcal{F}^n = (\mathcal{F}^n_t)_{t \in [0,1]}$ with $\mathcal{F}^n_t = \mathcal{F}_{[nt]/n}$ will be used. The predictable quadratic covariation for $\mathcal{F}^n_t$ is denoted by $\langle \cdot, \cdot \rangle$. Note that $\langle \cdot, \cdot \rangle$ depends on $n$. Let $H_1(x) = x$ and $H_2(x) = (x^2 - 1)/\sqrt{2}$. Denote $\Delta_j w = w_{t_j} - w_{t_{j-1}}$, which depends on $n$ as well as $j$. The discrete version of $M^n_t$ is given by

$$M^{2,n}_t = \frac{1}{\sqrt{n}} \sum_{j:t_j \leq t} \sqrt{2a(X_{t_{j-1}})}H_2(\sqrt{n}\Delta_j w).$$
For $\tilde{C}_t^n := \sqrt{n}(C_t^n - C_t^\infty)$, we have

$$\tilde{C}_t^n = \sum_{j:t_j \leq t} \int_{t_{j-1}}^{t_j} 4n \sqrt{n} a(X_{t_{j-1}})^2 \left\{ \left( \int_{t_{j-1}}^{s} dw_r \right)^2 - (s - t_{j-1}) \right\} ds - 2\sqrt{n} \sum_{j:t_j \leq t} \int_{t_{j-1}}^{t_j} \left( a(X_s)^2 - a(X_{t_{j-1}})^2 \right) ds + O_p\left( \frac{1}{\sqrt{n}} \right)$$

$$= \sum_{j:t_j \leq t} \int_{t_{j-1}}^{t_j} 4n \sqrt{n} a(X_{t_{j-1}})^2 \left\{ \left( \int_{t_{j-1}}^{s} dw_r \right)^2 - (s - t_{j-1}) \right\} ds - 2\sqrt{n} \sum_{j:t_j \leq t} \int_{t_{j-1}}^{t_j} 2a(X_{t_{j-1}}) a'(X_{t_{j-1}})(w_s - w_{t_{j-1}}) ds + O_p\left( \frac{1}{\sqrt{n}} \right)$$

$$= \sum_{j:t_j \leq t} \int_{t_{j-1}}^{t_j} 4n \sqrt{n} a(X_{t_{j-1}})^2 \left\{ \left( \int_{t_{j-1}}^{s} dw_r \right)^2 - (s - t_{j-1}) \right\} ds + O_p\left( \frac{1}{\sqrt{n}} \right).$$

Here the supremum of “$O_p(n^{-1/2})$” in $t \in [0, 1]$ is of $O_p(n^{-1/2})$ with respect to $L^p$-norms. Therefore, the principal part of $\tilde{C}_t^n$ is $F^n$-martingale

$$\tilde{M}_t^x \cdot n = \sum_{j:t_{j-1} \leq t} \int_{t_{j-1}}^{t_j} 4n \sqrt{n} a(X_{t_{j-1}})^2 \left\{ \left( \int_{t_{j-1}}^{s} dw_r \right)^2 - (s - t_{j-1}) \right\} ds$$

The discrete version of $w$ is denoted by $\tilde{w}_t^n = w_{\lfloor nt \rfloor / n}$.

By a similar argument, we have

$$\sqrt{n}(F_n - F^\infty) \rightarrow^p 0 = F^\infty.$$

In the present situation, $\tilde{W}_\infty(z) = 0$ and $\tilde{F}_\infty(z) = 0$. We need to identify the limit $(M_\infty, C^\circ, N_\infty)$ to write the adaptive random symbol. The “martingale part” of $N_n$ with respect to $F^n$ is given by

$$\tilde{N}_t^n = 6n \sum_{j:t_j \leq t} c_{t_{j-1}} \sigma_{t_{j-1}} [\sigma_{t_{j-1}}] \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} dw_u dw_v dw_t$$

$$+ 2 \sum_{j:t_j \leq t} c_{t_{j-1}} b_{t_{j-1}} \sigma_{t_{j-1}} \int_{t_{j-1}}^{t_j} dw_t + 2n \sum_{j:t_j \leq t} c_{t_{j-1}} \sigma_{t_{j-1}} [\sigma_{t_{j-1}}] \int_{t_{j-1}}^{t_j} (t - t_{j-1}) dw_t$$

$$- n \sum_{j:t_j \leq t} c_{t_{j-1}} [\sigma_{t_{j-1}}] \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_u dt.$$
Thus with the representation of $\sigma$:

$$\bar{\Omega}. \text{Since } u = t \leftarrow \infty \text{ for each } \sigma,$$

and 4.2.2 Anticipative random symbol

$$\int_{0}^{t} k_s ds$$

$$\int_{0}^{t} q_s^2 ds$$

as $n \to \infty$ for each $t \in [0, 1]$, where $\mathbb{R}_+$-valued process $q_t$ takes the form

$$q_t^2 = p(c_t, c_t^{[1]}, b_t, \sigma_t, \sigma_t^{[N]})$$

for some polynomial $p$; it is possible to give an explicit expression of $p$, however we do not need the precise form of $q_t$ later. The orthogonality of $\bar{M}^{2,n}, \bar{M}^{\xi,n}$ and $\bar{N}^n$ to any bounded martingale orthogonal to $w$ is obvious, thus with the representation of $C^*_t$, $M^*_n$ and $N_n$, we obtain

$$\begin{align*}
(M_\infty, C_\infty, N_\infty) &=\mathbb{d} \left( \int_{0}^{1} \sqrt{2a(X_s)} dB_s, \int_{0}^{1} \frac{4\sqrt{2}}{3} a(X_s)^2 dB_s + \int_{0}^{1} \frac{4}{3} a(X_s)^2 dB'_s, \right. \\
&\quad \left. \int_{0}^{1} k_s dw_s + \int_{0}^{1} \sqrt{q_s^2 - k_s^2} dB''_s + \int_{0}^{1} h_s ds, \right)
\end{align*}$$

where $(B, B', B'')$ is a three-dimensional standard Wiener process, independent of $\mathcal{F}$, defined on the extension $\bar{\Omega}$. Since

$$\tilde{N}_\infty(z) = \int_{0}^{1} k_s dw_s + \int_{0}^{1} h_s dt,$$

the random symbol $\sigma(z, iu, iv)$ is given by (5). Moreover we see Condition [A2] holds.

### 4.2.2 Anticipative random symbol

Let us find the anticipative random symbol $\sigma^{\prime}(iu, iv)$. Recall that $\alpha(x) = a(x)^2$,

$$C_s^\infty = 2 \int_{0}^{s} \alpha(X_t) dt, \quad C_s = 2 \int_{0}^{1} \alpha(X_t) dt, \quad F_s = \int_{0}^{1} \beta(X_t) dt \quad \text{and} \quad W_\infty = 0.$$  

To describe $\sigma_{s,s}(iu, iv)$ more precisely, we consider the random symbol $\sigma_{s,r}(iu, iv)$ that admits the expression $\sigma_{s,r}(iu, iv)$

$$\begin{align*}
\sigma_{s,s}(iu, iv) &= u^2 \int_{r}^{s} \alpha'(X_t) D_t X_t dt \left( -u^2 \int_{s}^{1} \alpha'(X_t) D_s X_t dt + i \int_{s}^{1} \beta'(X_t) [v] D_s X_t dt \right) \\
&\quad + \left( -u^2 \int_{r}^{s} \alpha'(X_t) D_r X_t dt + i \int_{s}^{1} \beta'(X_t) [v] D_r X_t dt \right) \left( -u^2 \int_{s}^{1} \alpha'(X_t) D_s X_t dt + i \int_{s}^{1} \beta'(X_t) [v] D_s X_t dt \right) \\
&\quad + \left( -u^2 \int_{s}^{1} \{ \alpha'(X_t) D_r X_t D_s X_t + \alpha'(X_t) D_s X_t D_t X_t \} dt + i \int_{s}^{1} \{ \beta'(X_t) [v] D_r X_t D_s X_t + \beta'(X_t) [v] D_s X_t D_t X_t \} dt \right)
\end{align*}$$

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for $r \leq s$, where the prime $'$ stands for the derivative in $x_1 \in \mathbb{R}$. The random symbol $\sigma_{s,s}$ is the limit $\lim_{r \to t} \sigma_{s,r}(iu, iv)$, that is, (7), and this gives the anticipative random symbol (6).

### 4.3 Nondegeneracy

Here we will consider the nondegeneracy of $(M^n_t, F_\infty)$ in Malliavin’s sense. Let

$$\eta_j(t) = \sqrt{n}(w(t_j \wedge t) - w(t_{j-1} \wedge t))$$

and

$$\xi_j(t) = n \left( (w(t_j \wedge t) - w(t_{j-1} \wedge t))^2 - (t_j \wedge t - t_{j-1} \wedge t) \right).$$

Then

$$M^n_t = \frac{1}{\sqrt{n}} \sum_{j=1}^n a(X_{t_{j-1}})\xi_j(t).$$

The representing density of the Malliavin derivative of $D^n_t$ is

$$D_r M^n_t = \sum_{j=1}^n 2a(X_{t_{j-1}})\eta_j(t)1_{(t_{j-1} \wedge t, t_j \wedge t)}(r)$$

$$+ \frac{1}{\sqrt{n}} \sum_{j=1}^n a'(X_{t_{j-1}})D_r X_{t_{j-1}} \xi_j(t)1_{(r \leq t_{j-1} \leq t)}$$

$$=: D_1(n, t)_r + D_2(n, t)_r.$$ 

Now

$$D_2(n, t)_r = \frac{1}{\sqrt{n}} \sum_{j=2}^n \sum_{j=1}^n a'(X_{t_{j-1}})D_r X_{t_{j-1}} \xi_j(t)1_{(r \leq t_{j-1} \leq t)}$$

$$= \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n-1} \sum_{j=1}^{\ell} a'(X_{t_{\ell-1}})\xi_{\ell+1}(t)D_r X_{t_{\ell}} 1_{(r \leq t_{\ell} \leq t)}$$

$$= \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n-1} \sum_{j=1}^{\ell} a'(X_{t_{\ell}})\xi_{\ell+1}(t)D_r X_{t_{\ell}} 1_{I_{\ell}(t)}(r)$$

$$= \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n-1} \left( \sum_{j=1}^{n-1} a'(X_{t_{k-1}})\xi_k(t)D_r X_{t_{k-1}} \right) 1_{I_{\ell}(t)}(r)$$

where $I_j(t) = (t_{j-1} \wedge t, t_j \wedge t]$. Hence the Malliavin covariance of $M^n_t$ is

$$\sigma_{11}(n, t)_r := \sigma_{M^n_t} = \sum_{j=1}^n \int_{t_{j-1} \wedge t}^{t_j \wedge t} \left[ 2a(X_{t_{j-1}})\eta_j(t) + \frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(X_{t_{k-1}})D_r X_{t_{k-1}} \xi_k(t) \right] dr$$

for $u \in \mathbb{R}$, where we read $\sum_{k=n+1}^n \ldots = 0$. Since

$$D_t F_\infty = \int_r^1 \beta'(X_s)D_s X_s ds,$$
we have
\[
\sigma_{12}(n, t)[v] := (M^n_t, F^\infty_t[v])_t
\]
\[
= \sum_{j=1}^{n} \int_{r=t_{j-1} \wedge t}^{t_j} \int_{s=r}^{1} \left[ 2a(X_{t_{j-1}})\eta_j(t) + \frac{1}{\sqrt{n}} \sum_{k=j+1}^{n} a'(X_{t_{k-1}})\xi_k(t)D_rX_{t_{k-1}} \right] \beta'(X_s)D_sX_sdsdr[v]
\]
for \( v \in \mathbb{R}^d \). Let
\[
\sigma_{22}(t)[v^\otimes 2] = \int_0^t \left[ \int_r^1 \beta'(X_s)D_sX_sds[v] \right]^2 dr.
\]
and let
\[
\sigma(n, t) = \begin{bmatrix}
\sigma_{11}(n, t) & \sigma_{12}(n, t)* \\
\sigma_{12}(n, t) & \sigma_{22}(t)
\end{bmatrix}.
\]
Let
\[
\tilde{\sigma}_{11}(n, t) = \frac{1}{n} \sum_{j=1}^{n} \left[ 2a(X_{t_{j-1}})\eta_j(t) \right]^2 + \sum_{j=1}^{n} \int_{t_{j-1} \wedge t}^{t_j} \left[ \frac{1}{\sqrt{n}} \sum_{k=j+1}^{n} a'(X_{t_{k-1}})\xi_k(t)D_rX_{t_{k-1}} \right]^2 dr,
\]
\[
\tilde{\sigma}_{12}(n, t)[v] = \sum_{j=1}^{n} \int_{r=t_{j-1} \wedge t}^{t_j} \int_{s=r}^{1} \frac{1}{\sqrt{n}} \sum_{k=j+1}^{n} a'(X_{t_{k-1}})\xi_k(t)D_rX_{t_{k-1}} \beta'(X_s)D_sX_sdsdr[v]
\]
and
\[
\tilde{\sigma}(n, t) = \begin{bmatrix}
\tilde{\sigma}_{11}(n, t) & \tilde{\sigma}_{12}(n, t)* \\
\tilde{\sigma}_{12}(n, t) & \tilde{\sigma}_{22}(t)
\end{bmatrix}.
\]
We shall show
\[
\|\sigma(n, t) - \tilde{\sigma}(n, t)\|_p = O(n^{-\frac{1}{p}})
\]
for every \( p > 1 \) and \( t \in [0, 1] \) (in particular, for \( t = 1/2 \)). For this purpose, we need a lemma. Let \( \mathcal{I} \) denote the set of sequences \( J^{(i)} = (J^{(i)}_{n,j}) \) of multiple Itô stochastic integrals taking the form
\[
J^{(i)}_{n,j} = n^\frac{1}{2} \int_{t_{j-1}}^{t_j} dw_{a_1^{(i)}}(s_1) \int_{t_{j-1}}^{s_1} dw_{a_2^{(i)}}(s_2) \int_{t_{j-1}}^{s_2} dw_{a_3^{(i)}}(s_3) \cdots \int_{t_{j-1}}^{s_m} dw_{a_m^{(i)}}(s_m),
\]
where \( \{a_i^{(i)} : i = 1, \ldots, \nu, \ j = 1, \ldots, n, \ n \in \mathbb{N} \} \) is a family of progressively measurable processes, bounded in \( \mathbb{D}^\infty = \cap_{s,p} \mathbb{D}_{s,p} \). In the following lemma, \( J^{(\nu_1)}, \ldots, J^{(\nu_m)} \), \( J^{(\mu_1)}, \ldots, J^{(\mu_m)} \) are in \( \mathcal{I} \), and each of them has \( a_i^{(i)} \) which may possibly differ from those of other indices \( \nu \)'s and \( \mu \)'s even if the values of indices coincide each other.

**Lemma 2.** Suppose that
\[
\sup_{\nu_1, \ldots, \nu_m, \mu_1, \ldots, \mu_q \in \mathbb{N}, \gamma \in [0,1], j \in \{1, \ldots, n\}} \left\| D_{r_1, \ldots, r_q} \mathcal{O} \right\|_p < \infty
\]
for all \( \mathcal{O} = a^{(e)}_{n,j}, b^{(e)}_{n,j} \) and \( a^{(s)}_{n,j,i} (s) \), and for every \( p > 1 \). Then

(a) Suppose that \( b^{(d)}_{n,k} \) are \( \mathcal{F}_{t_{k-1}} \)-measurable. Then for \( \nu_1, \ldots, \nu_m, \mu_1, \ldots, \mu_q \in \mathbb{N} \),
\[
\frac{1}{n^m} \sum_{j_1, \ldots, j_m=1}^{n} E \left[ \prod_{j=1}^{m} \left( \frac{1}{\sqrt{n}} \sum_{k_1=j_1+1}^{\nu_1} b^{(e)}_{n,j_1,j} \right) \prod_{k=1}^{\nu_m} b^{(d)}_{n,k} \right] = O \left( \frac{1}{n^{m/2}} \right).
\]

\( ^{5} \gamma = 0 \) denotes the case with no derivative.
(b) For \( \nu_1, \ldots, \nu_m \in \mathbb{N} \),
\[
\frac{1}{n^m} \sum_{j_1, \ldots, j_m = 1}^{n} E \left[ a_{n,j_1}^{(1)} J_{n,j_1}^{(\nu_1)} \cdots a_{n,j_m}^{(m)} J_{n,j_m}^{(\nu_m)} \right] = O \left( \frac{1}{n^{m/2}} \right).
\]

The constants in the above estimates depend only on the given supremums.

By using smoothness of appearing functions, apply Lemma 2 (a) to
\[
a_{n,j}^{(c)} = a(X_{t_{j-1}}), \quad J_{n,j}^{(c)} = \eta_j(t),
\]
\[
\tilde{b}_{n,k}^{(c)} = n a'(X_{t_{k-1}}) \int_{t_{k-1}}^{t_{k+1}} D_r X_t X_t dr \quad (j < k) \quad \text{and} \quad J_{n,k}^{(c)} = \xi_k(t)
\]
to obtain
\[
\sup_{t \in [0,1]} \left\| \sigma_{11}(n,t) - \tilde{\sigma}_{11}(n,t) \right\|_p = O \left( \frac{1}{\sqrt{n}} \right)
\]
Furthermore, Lemma 2 (b) applied to
\[
a_{n,j}^{(c)} = \text{element of } 2a(X_{t_{j-1}}) \times n \int_{t_{j-1}}^{t_{j+1}} \int_{s=r}^{1} \beta'(X_s) D_r X_s ds dr [v] \quad \text{and} \quad J_{n,j}^{(c)} = \eta_j(t)
\]
yields
\[
\sup_{t \in [0,1]} \left\| \sigma_{12}(n,t) - \tilde{\sigma}_{12}(n,t) \right\|_p = O \left( \frac{1}{\sqrt{n}} \right)
\]
as \( n \to \infty \) for every \( p > 1 \). Consequently, we obtain (16).

Let \( m_n = n^{-1} \sum_{j=1}^{n} \eta_j(1/2)^2 \). For a positive number \( c_1 \), define \( s_n \) by
\[
s_n = \frac{1}{2} \det \left[ \hat{\sigma} \left( n, \frac{1}{2} \right) + \psi \left( \frac{m_n}{2c_1} \right) I_{1+d_1} \right],
\]
where \( \psi : \mathbb{R} \to [0,1] \) is a smooth function such that \( \psi(x) = 1 \) if \( |x| \leq 1/2 \) and \( \psi(x) = 0 \) if \( |x| \geq 1 \). Then \( s_n \geq 2^{-1} \) if \( m_n \leq c_1 \), and \( s_n \geq 2^{-1} \det \hat{\sigma}(n,1/2) \) otherwise. Thus, it suffices to show
\[
\sup_n E \left[ 1_{\{m_n \geq c_1\}} \left( \det \hat{\sigma}(n,1/2) \right)^{-p} \right] < \infty \quad (p > 1)
\]
for the nondegeneracy
\[
\sup_n E[s_n^{-p}] < \infty \quad (18)
\]
for every \( p > 1 \). Following precisely, e.g., the proof of Lemma 2.3.1 of [4], in order to obtain (17), it is sufficient to show that for every \( p > 1 \), there exists a constant \( C_p \) such that
\[
\sup_{u \in \mathbb{R}^{1+d_1} : |u| = 1} P[m_n \geq c_1, \hat{\sigma}(n,1/2) |u|^{2}] \leq C_p \epsilon^p \quad (19)
\]
for all \( \epsilon \in (0,1) \) and all \( n \in \mathbb{N} \). [The reasoning there is valid even for the measures \( 1_{\{m_n \geq c_1\}} dP \) in place of \( P \).] Here we use \( L^{\infty} \)-boundedness of \( \{\hat{\sigma}(n,1/2)\}_{n \in \mathbb{N}} \). Besides, for a while we shall assume the nondegeneracy condition: for some constant \( C_p \),
\[
\sup_{u \in \mathbb{R}^{1} : |u| = 1} P[\sigma_{22}(1/2) |u|^{2}] \leq C_p \epsilon^p \quad (20)
\]
for all $\epsilon \in (0, 1)$. Suppose that $|u| = 1$ for $u = (u, v) \in \mathbb{R}^{1+d_1}$. For simplicity, we write $\tilde{\sigma}_{ij}$ for $\tilde{\sigma}_{ij}(n, 1/2)$ and $\sigma_{22}$ for $\sigma_{22}(1/2)$. Let $p > 1$. When $|u| < \epsilon^{1/8}$,

$$
P[m_n \geq c_1, \tilde{\sigma}(n, 1/2)[u^{\otimes 2}] \leq \epsilon]
\leq P[m_n \geq c_1, \tilde{\sigma}_{11}u^2 + 2\tilde{\sigma}_{12}[uv] + \sigma_{22}[v^{\otimes 2}] \leq \epsilon]
\leq P[m_n \geq c_1, 2|\tilde{\sigma}_{12}u| > \tilde{\sigma}_{11}u^2 / 2\epsilon^{1/8}] + P[m_n \geq c_1, \tilde{\sigma}_{11}u^2 \leq 2\epsilon]
\leq P[4|\tilde{\sigma}_{12}| > \epsilon^{-1/16}|u|] + P[m_n \geq c_1, \tilde{\sigma}_{11}u^2 < \epsilon^{1/16}] + P[m_n \geq c_1, \tilde{\sigma}_{11}u^2 \leq 2\epsilon]
\lesssim \epsilon^p
$$

(21)

uniformly in $n \in \mathbb{N}$ and $u$ satisfying $|u| = 1$ and $|v| < \epsilon^{1/8}$, where $c_2 := \inf_x |a(x)| > 0$, since $(1 - \epsilon^{1/4})^{1/2} < |u| \leq 1$ and

$$
\tilde{\sigma}_{11}u^2 \geq 2c_1|v|^2
$$

for any $\epsilon \in (0, 2^{-4})$ on the event $\{m_n \geq c_1\}$.

We will assume $|v| \geq \epsilon^{1/8}$. From

$$
\tilde{\sigma}_{11}u^2 + 2\tilde{\sigma}_{12}[uv] + \sigma_{22}[v^{\otimes 2}] \geq \tilde{\sigma}_{11}^{-1}\left\{\tilde{\sigma}_{11}\sigma_{22}[v^{\otimes 2}] - (\tilde{\sigma}_{12}[v])^2\right\},
$$

it follows that

$$
P[m_n \geq c_1, \tilde{\sigma}(n, 1/2)[u^{\otimes 2}] \leq \epsilon]
\leq P[m_n \geq c_1, \tilde{\sigma}_{11}u^2 + 2\tilde{\sigma}_{12}[uv] + \sigma_{22}[v^{\otimes 2}] \leq \epsilon]
\leq P[\tilde{\sigma}_{11} > \epsilon^{-1/4}] + P[m_n \geq c_1, \tilde{\sigma}_{11}\sigma_{22}[v^{\otimes 2}] - (\tilde{\sigma}_{12}[v])^2 \leq \epsilon^{3/4}].
$$

(22)

On the other hand, we have

$$
\tilde{\sigma}_{11}\sigma_{22}[v^{\otimes 2}] - (\tilde{\sigma}_{12}[v])^2
= \frac{1}{n}\sum_{j=1}^n \left[2a(X_{t_{j-1}})n_j(t)\right]^2 \int_0^t \left[\int_r^1 \beta'(X_s)D_sX_sds[v]\right]^2 dr
+ \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left[\frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(X_{t_{k-1}})\xi_k(t)D_{s}X_{s-}\right] \int_0^t \left[\int_r^1 \beta'(X_s)D_sX_sds[v]\right] dr
- \left\{\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left[\frac{1}{\sqrt{n}} \sum_{k=j+1}^n a'(X_{t_{k-1}})\xi_k(t)\right] \left(\int_0^1 \beta'(X_s)D_sX_sds[v]\right) dr\right\}^2
= \frac{1}{n}\sum_{j=1}^n \left[2a(X_{t_{j-1}})n_j(t)\right]^2 \int_0^t \left[\int_r^1 \beta'(X_s)D_sX_sds[v]\right]^2 dr
$$

for $t = 1/2$, where we used the Schwarz inequality. Hence,

$$
\tilde{\sigma}_{11}\sigma_{22}[v^{\otimes 2}] - (\tilde{\sigma}_{12}[v])^2 \geq \frac{4c_1c_2^2}{\epsilon} \sigma_{22}(1/2)[v^{\otimes 2}] (23)
$$

on $\{m_n \geq c_1\}$. Combining (20) with scaling $v \mapsto \epsilon^{-1/8}v$ and (23), we obtain

$$
\sup_{v \in \mathbb{R}^{1+d_1}, |v| \leq \epsilon} P[m_n \geq c_1, \tilde{\sigma}_{11}\sigma_{22}[v^{\otimes 2}] - (\tilde{\sigma}_{12}[v])^2 \leq \epsilon^{3/4}]
\leq \sup_{v \in \mathbb{R}^{1+d_1}, |v| \leq \epsilon} P[4c_1c_2^2 \sigma_{22}(1/2)[v^{\otimes 2}] \leq \epsilon^{3/4}]
\leq \sup_{v \in \mathbb{R}^{1+d_1}, |v| = 1} P[\sigma_{22}(1/2)[v^{\otimes 2}] \leq (4c_1c_2^2)^{-1} \epsilon^{1/2}]
\lesssim \epsilon^p
$$

(24)
for every \( p > 1 \). By connecting (24) to (22), we obtain

\[
\sup_{u \in \mathbb{R}^{1+d_1} : |u| = 1} P \left[ m_n \geq c_1, \, \tilde{\sigma}(n, 1/2)[u] \leq \epsilon \right] \lesssim \epsilon^p
\]  

(25)

Thus from (21) and (25), we obtain (19) and hence (18) under the assumption (20).

We consider the \((1 + d_1)\)-dimensional process \( X = (X_t)_{t \in [0,1]} \) satisfying the stochastic differential equation

\[
d\tilde{X}_t = V_0(\tilde{X}_t)dt + V_1(\tilde{X}_t) \circ dw_t, \quad \tilde{X}_0 = (X_0, 0)
\]

in the Stratonovich form. Then the Hörmander condition \([H_2]\) together with the compactness of \(\text{supp} P^X_0\) ensures that for \( t \in (0,1) \) and for every \( p > 1 \), there exists a constant \( C_p \) such that

\[
\sup_{v \in \mathbb{R}^{1+d_1} : |v| = 1} P \left[ \int_0^t D_s \tilde{X}_1 \otimes D_s \tilde{X}_1 ds[v \otimes 2] \leq \epsilon \right] \leq C_p \epsilon^p
\]  

(26)

for all \( \epsilon \in (0,1) \). See Kusuoka and Stroock [2, 3], Ikeda and Watanabe [1], Nualart [4]. In particular, (20) follows from (26) applied to \( v = (0, v) \) for \( v \in \mathbb{R}^{d_1} \).

We choose \( c_1 \) such that \( 2c_1 < 1/2 = \lim_{n \to \infty} E[m_n] \). Since

\[
\sigma_{(M^p, F_n)} = \begin{bmatrix}
\sigma_{11}(n, t) & \sigma_{12}(n, t) \\
\sigma_{12}(n, t) & \sigma_{22}(1)
\end{bmatrix} \geq \sigma(n, t)
\]

by definition, we see that for every \( K > 0 \),

\[
\sup_{t \geq 1/2} P \left[ \det \sigma_{(M^p, F_n)} < s_n \right] \leq P \left[ \det \sigma(n, 1/2) < s_n \right] + O(n^{-K})
\]

\[
\leq P \left[ \det \tilde{\sigma}(n, 1/2) < 2s_n \right] + O(n^{-K})
\]

\[
\leq P \left[ m_n < 2c_1 \right] + O(n^{-K})
\]

\[
= O(n^{-K})
\]  

(27)

as \( n \to \infty \). Here we used (16). Properties (27) and (18) verify \([A3] \).

### 4.4 Proof of Theorem 1

It is easy to verify Condition \([A1] \) under \([H1] \). Conditions \([A2] \) and \([A3] \) have been proved in the preceding sections. Now we can apply Theorem 2 to obtain Theorem 1.

### 5 Proof of Lemma 1

Let

\[
f^{(1)} = \sigma, \quad f^{(2)} = b
\]

\[
f^{(1,1)} = \sigma^{[1]} = \sigma' \sigma, \quad f^{(1,2)} = \sigma^{[0]} = \sigma' b + \frac{1}{2} \sigma'' \sigma^2.
\]

We have

\[
\sum_j c_{t_j-1} (X_{t_j} - X_{t_{j-1}})^2
\]

\[
= \sum_j c_{t_j-1} \int_{t_{j-1}}^{t_j} (f^{(1)}_t)^2 dt + 2 \sum_j c_{t_j-1} \int_{t_{j-1}}^{t_j} \left( \int_{t_{j-1}}^t f^{(1)}_sdw + \int_{t_{j-1}}^t f^{(2)}_sds \right) f^{(1)}_t dw + \int_{t_{j-1}}^{t_j} f^{(2)}_sds f^{(2)}_t dt.
\]  

(28)
In the decomposition (28),

\[
\sum_{j} c_{t_{j-1}} \int_{t_{j-1}}^{t_j} (f^{(1)}_t)^2 dt
\]

\[
= \frac{1}{2n} \sum_{j} c_{t_{j-1}} (f^{(1)}_{t_{j-1}})^2 + \sum_{j} c_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} 2f^{(1)}_s f^{(1,1)}_s dw_s dt
\]

\[
+ \sum_{j} c_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} 2f^{(1)}_s f^{(1,2)}_s ds dt + \sum_{j} c_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} (f^{(1,1)}_t)^2 ds dt
\]

\[
= \frac{1}{2n} \sum_{j} c_{t_{j-1}} (f^{(1)}_{t_{j-1}})^2 + \sum_{j} 2c_{t_{j-1}} f^{(1)}_{t_{j-1}} f^{(1,1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_s dt
\]

\[
+ \frac{1}{n^2} \sum_{j} c_{t_{j-1}} f^{(1)}_{t_{j-1}} f^{(1,2)}_{t_{j-1}} + \frac{1}{2n^2} \sum_{j} c_{t_{j-1}} (f^{(1)}_{t_{j-1}})^2 + o_p \left( \frac{1}{n} \right)
\]

as \( n \to \infty \). Next,

\[
2 \sum_{j} c_{t_{j-1}} \int_{t_{j-1}}^{t_j} \left( \int_{t_{j-1}}^{t} f^{(1)}_s dw_s + \int_{t_{j-1}}^{t} f^{(2)}_s ds \right) f^{(1)}_t dw_t
\]

\[
= 2 \sum_{j} c_{t_{j-1}} f^{(1)}_{t_{j-1}} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_s dw_t + 2 \sum_{j} c_{t_{j-1}} f^{(1,1)}_{t_{j-1}} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_s dw_t
\]

\[
+ 2 \sum_{j} c_{t_{j-1}} f^{(1)}_{t_{j-1}} f^{(2)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} ds dw_t + 2 \sum_{j} c_{t_{j-1}} f^{(1,1)}_{t_{j-1}} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \left( \int_{t_{j-1}}^{t} dw_s \right)^2 dw_t + o_p \left( \frac{1}{n} \right)
\]

\[
= 2 \sum_{j} c_{t_{j-1}} f^{(1)}_{t_{j-1}} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_s dw_t + 6 \sum_{j} c_{t_{j-1}} f^{(1,1)}_{t_{j-1}} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_s dw_t
\]

\[
+ 2 \sum_{j} c_{t_{j-1}} f^{(2)}_{t_{j-1}} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} ds dw_t + 2 \sum_{j} c_{t_{j-1}} f^{(1,1)}_{t_{j-1}} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} ds dw_t + o_p \left( \frac{1}{n} \right)
\]

(29)
Further,

\[ 2 \sum_j c_{t,j-1} \int_{t_{j-1}}^{t_j} \left( \int_{t_{j-1}}^{t} f_s^{(1)} dw_s + \int_{t_{j-1}}^{t} f_s^{(2)} ds \right) f_t^{(2)} dt \]

\[ = 2 \sum_j c_{t,j-1} f_{t,j-1}^{(1)} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_s f_t^{(2)} dt + \frac{1}{n^2} \sum_j c_{t,j-1} (f_{t,j-1}^{(2)})^2 + o_p \left( \frac{1}{n} \right) \]

\[ = 2 \sum_j c_{t,j-1} f_{t,j-1}^{(1)} f_{t,j-1}^{(2)} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_s dt + 2 \sum_j c_{t,j-1} f_{t,j-1}^{(1)} f_{t,j-1}^{(2)} \int_{t_{j-1}}^{t_j} \left( \int_{t_{j-1}}^{t} dw_s \right)^2 dt \]

\[ + \frac{1}{n^2} \sum_j c_{t,j-1} (f_{t,j-1}^{(2)})^2 + o_p \left( \frac{1}{n} \right) \]

\[ = 2 \sum_j c_{t,j-1} f_{t,j-1}^{(1)} f_{t,j-1}^{(2)} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} dw_s dt \]

\[ + \frac{1}{n^2} \sum_j c_{t,j-1} f_{t,j-1}^{(2)} + \frac{1}{n^2} \sum_j c_{t,j-1} f_{t,j-1}^{(2)} + o_p \left( \frac{1}{n} \right). \]  

Let

\[ c^{(1)} = c^{[1]} = c' \sigma = c' f^{(1)}, \]

\[ c^{(2)} = c^{[0]} = c' b + \frac{1}{2} c'' \sigma^2 = c' f^{(2)} + \frac{1}{2} c'' (f^{(1)})^2. \]

Since

\[ \sigma_t = f^{(1)}(X_t) = f_{t,j-1}^{(1)} + \int_{t_{j-1}}^{t} f_{s}^{(1,1)} dw_s + \int_{t_{j-1}}^{t} f_{s}^{(1,2)} ds, \]

we have

\[ \sigma_t^2 = (f_{t,j-1}^{(1)})^2 + \int_{t_{j-1}}^{t} 2 f_{s}^{(1)} f_{s}^{(1,1)} dw_s + \int_{t_{j-1}}^{t} 2 f_{s}^{(1)} f_{s}^{(1,2)} ds + \int_{t_{j-1}}^{t} (f_{s}^{(1,1)})^2 ds. \]
Now
\[
\int_0^1 c_t \sigma_t^2 \, dt = \sum_j \int_{t_{j-1}}^{t_j} c_t \sigma_t^2 \, dt
\]
\[
= \sum_j \int_{t_{j-1}}^{t_j} \left\{ c_{t_{j-1}} + \int_{t_{j-1}}^t c_s^{(1)} \, dw_s + \int_{t_{j-1}}^t c_s^{(2)} \, ds \right\} \left\{ (f_s^{(1)})^2 + \int_{t_{j-1}}^t 2 f_s^{(1)} f_s^{(1,1)} \, dw_s \right\} dt
\]
\[
+ \int_{t_{j-1}}^t 2 f_s^{(1)} f_s^{(1,2)} \, ds + \int_{t_{j-1}}^t (f_s^{(1,1)})^2 \, ds \right\} dt
\]
\[
= \frac{1}{n} \sum_j c_{t_{j-1}} (f_s^{(1)})^2 + \frac{1}{2n^2} \sum_j 2 c_{t_{j-1}} f_s^{(1)} f_s^{(1,1)} \int_{t_{j-1}}^t \int_{t_{j-1}}^t \, dw_s \, dt + \frac{1}{n^2} \sum_j c_{t_{j-1}} f_s^{(1)} f_s^{(1,2)} + \frac{1}{2n^2} \sum_j c_{t_{j-1}} (f_s^{(1,1)})^2
\]
\[
+ \sum_j c_{t_{j-1}} (f_s^{(1)})^2 \int_{t_{j-1}}^t \int_{t_{j-1}}^t \, dw_s \, dt
\]
\[
+ \frac{1}{2n^2} \sum_j c_{t_{j-1}} f_s^{(1,1)} \, dw_s + \frac{1}{2n^2} \sum_j c_{t_{j-1}} (f_s^{(1,1)})^2 + \frac{1}{2n^2} \sum_j c_{t_{j-1}} (f_s^{(1,1)})^2 + \frac{1}{2n^2} \sum_j c_{t_{j-1}} (f_s^{(1,1)})^2 + o_p \left( \frac{1}{n} \right)
\]
\[
= \frac{1}{n} \sum_j c_{t_{j-1}} (f_s^{(1)})^2 + \frac{1}{2n^2} \sum_j 2 c_{t_{j-1}} f_s^{(1)} f_s^{(1,1)} \int_{t_{j-1}}^t \int_{t_{j-1}}^t \, dw_s \, dt + \frac{1}{n^2} \sum_j c_{t_{j-1}} f_s^{(1)} f_s^{(1,2)} + \frac{1}{2n^2} \sum_j c_{t_{j-1}} (f_s^{(1,1)})^2
\]
\[
+ \sum_j c_{t_{j-1}} (f_s^{(1)})^2 \int_{t_{j-1}}^t \int_{t_{j-1}}^t \, dw_s \, dt
\]
\[
+ \frac{1}{n^2} \sum_j c_{t_{j-1}} f_s^{(1,1)} \, dw_s + \frac{1}{2n^2} \sum_j c_{t_{j-1}} (f_s^{(1,1)})^2 + o_p \left( \frac{1}{n} \right).
\]
From (28)-(31) we obtain

\[
\sum_j c_{t_j-1} (X_{t_j} - X_{t_{j-1}})^2 - \int_0^1 c_0 \sigma_t^2 dt = \frac{1}{2n} \sum_j c_{t_j-1} (f^{(1)}_{t_j-1})^2 + \frac{1}{2} \sum_j 2c_{t_j-1} f^{(1)}_{t_j-1} f^{(1,1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt \\
+ \frac{1}{n^2} \sum_j c_{t_j-1} (f^{(1)}_{t_j-1})^2 + \frac{1}{2n^2} \sum_j c_{t_j-1} (f^{(1)}_{t_j-1})^2 \\
+ 2 \sum_j c_{t_j-1} f^{(1)}_{t_j-1} f^{(1)}_{t_j-1} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dw_t + 6 \sum_j c_{t_j-1} f^{(1,1)}_{t_j-1} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \int_{t_{j-1}}^u dw_u dw_s dw_t \\
+ 2 \sum_j c_{t_j-1} f^{(2)}_{t_j-1} f^{(1)}_{t_j-1} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t ds dw_t + 2 \sum_j c_{t_j-1} f^{(1,1)}_{t_j-1} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t ds dw_t \\
+ 2 \sum_j c_{t_j-1} f^{(1)}_{t_j-1} f^{(2)}_{t_j-1} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt \\
+ \frac{1}{n^2} \sum_j c^{(1)}_{t_j-1} (f^{(1)}_{t_j-1})^2 + \frac{1}{2n^2} \sum_j c^{(2)}_{t_j-1} (f^{(1)}_{t_j-1})^2 \} + o_p \left( \frac{1}{n} \right) \\
= 2 \sum_j c_{t_j-1} f^{(1)}_{t_j-1} f^{(1)}_{t_j-1} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dw_t \\
+ 6 \sum_j c_{t_j-1} f^{(1,1)}_{t_j-1} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \int_{t_{j-1}}^u dw_u dw_s dw_t \\
+ 2 \sum_j c_{t_j-1} f^{(1,1)}_{t_j-1} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t ds dw_t + 2 \sum_j c_{t_j-1} f^{(1,1)}_{t_j-1} f^{(1)}_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t ds dw_t \\
+ 2 \sum_j c_{t_j-1} f^{(2)}_{t_j-1} f^{(1)}_{t_j-1} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt + \frac{1}{n^2} \sum_j c^{(1)}_{t_j-1} (f^{(1)}_{t_j-1})^2 + \frac{1}{n^2} \sum_j c^{(2)}_{t_j-1} f^{(1)}_{t_j-1} f^{(2,1)}_{t_{j-1}} \} + o_p \left( \frac{1}{n} \right)
Thus we obtained
\[
\sqrt{n} \left( \sum_j c_{t_{j-1}} (X_{t_j} - X_{t_{j-1}})^2 - \int_0^1 c_t \sigma_t^2 \, dt \right)
\]
\[= \sqrt{n} \sum_j 2 c_{t_{j-1}} \sigma_{t_{j-1}}^2 \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dw_t \]
\[+ \frac{1}{\sqrt{n}} \left\{ 6n \sum_j c_{t_{j-1}} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^1 \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t \int_{t_{j-1}}^u dw_u dw_s dw_t \right. \]
\[+ 2n \sum_j c_{t_{j-1}} b_{t_{j-1}} \sigma_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t ds dw_t \]
\[+ 2n \sum_j c_{t_{j-1}} \sigma_{t_{j-1}} b_{t_{j-1}} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt + \frac{1}{n} \sum_j c_{t_{j-1}} b_{t_{j-1}}^2 + \frac{1}{n} \sum_j c_{t_{j-1}} \sigma_{t_{j-1}} b_{t_{j-1}}^1 \]
\[-n \sum_j c_{t_{j-1}} \sigma_{t_{j-1}}^2 \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^t dw_s dt \]
\[\left. \right. - \frac{1}{2n} \sum_j c_{t_{j-1}} \sigma_{t_{j-1}}^2 - \frac{1}{n} \sum_j c_{t_{j-1}} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^1 + o_p \left( \frac{1}{n} \right) \right\}.
\]
Moreover, we can obtain $D_{x,p}$-estimates of the residual term as well as each term on the right-hand side. This completes the proof of Lemma 1.

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