Abstract. The purpose of this paper is to give an easy to understand with step-by-step explanation to allow interested people to fully appreciate the power of natural deduction for first-order logic. Natural deduction as a proof system can be used to prove various statements in propositional logic, but we will see its extension to cover quantifiers which gives it more power over propositional logic in solving more complex, real-world problems. We started by going over logical connectives and quantifiers to agree on the symbols that will be used throughout the paper, as some authors use different symbols to refer to the same thing. Besides, we showed the inference rules that are used the most. Furthermore, we presented the soundness and completeness of natural deduction for first-order logic. Finally, we solved examples ranging from easy to complex to give you different circumstances in which you can apply the proof system to solve problems you may encounter. Hopefully, this paper will be helpful makes the subject easy to understand.

Keywords: First-order logic, Predicate logic, Natural deduction
1 Introduction

First-order logic (a.k.a. predicate logic) is extending the propositional logic power to give it more power to solve an advanced level of problems that can not be solved with propositional logic. To stay consistent throughout this paper, *predict* as a name will not be used, instead, *first-order* is will be used instead [?]. By demonstrating with an example the power of first-order logic, let give an example. stating with a proposition P "Every lion drinks coffee", and proposition Q "Cat is a lion", therefore, proposition R "Cat drinks coffee". Using propositional logic, you can not derive the conclusion R from premises P and Q, but with first-order logic, you can. From your brain’s logical point of view, you can conclude R from P and Q easily as following.

\[
\begin{align*}
1 & P \quad \text{Every lion drinks milk} \\
2 & Q \quad \text{Cat is a lion} \\
3 & \therefore R \quad \text{Therefore, Cat drinks milk}
\end{align*}
\]

As far as propositional logic is concerned it is hard to arrive at such a conclusion. Propositional logic did not give you the tools to solve such a problem, yet, your brain can logically solve it, also first-order logic can too [?]. Quantifiers from first-order logic such as ∀ or ∃, i.e. universal and existential quantifiers, respectively can help solve such a problem. Let us write the predicates in short notation:

- Lion(x) = x is a lion
- Milk(x) = x drinks milk

**Note.** x is the subject, \{is a lion, drinks milk\} are the predicates. Also, the domain of discourse will be all the animals.

\[
\begin{align*}
1 & \forall x (\text{Lion}(x) \rightarrow \text{Milk}(x)) \quad \text{premise} \\
2 & \text{Lion}(\text{Cat}) \quad \text{premise} \\
3 & \text{Lion}(\text{Cat}) \rightarrow \text{Milk}(\text{Cat}) \quad \forall E, 1 \\
4 & \therefore \text{Milk}(\text{Cat}) \quad \rightarrow E, 3, 2
\end{align*}
\]

We discussed (∀E) and (→E) in the next section **Basic Concepts** beside connectives, quantifiers, and other inference rules that will be used to solve such a problem using natural deduction for first-order logic.
2 Basic Concepts

Different authors used different symbols, in order to eliminate confusion and stay consistent, we will be using the symbols that is widely used in mathematical logic books. Connectives and quantifiers are shown in table 1 and table 2 respectively.

2.1 Logical Connectives

The following table is sorted based on the precedence of each connective, starting from the higher priority.

| Symbol | Connective | Description |
|--------|------------|-------------|
| ⊤      | Truth      | True.       |
| ⊥      | Falsehood  | False.      |
| ¬      | Negation   | ¬P is true if and only if P is false. |
| ∧      | Conjunction| P∧Q is true if and only if both P and Q are true. |
| ∨      | Disjunction| P∨Q is true if and only if either P or Q is true. |
| →      | Conditional| P→Q is true if and only if either P is false or Q is true (or both). |
| ↔      | Biconditional| P↔Q is true if and only if P and Q have the same truth value. |

Table 1: Logical Connectives

2.2 Quantifiers

| Symbol | Quantifier | Description |
|--------|------------|-------------|
| ∀      | Universal  | ∀xP, P is true for every object x. |
| ∃      | Existential| ∃xP, P is true for at least one object x. |

Table 2: Quantifiers
2.3 Inference Rules

There are many inference rules such as basic, derived, and others. This paper covers the basic rules of inference that can be used to derive a proof which is a chain of conclusions that leads to the desired outcome [?].

2.3.1 Negation

*Negation Introduction* (¬I) also known as *reductio ad absurdum* is to derive a negation of a sentence if the sentence leads to contradiction, As shown on the left, ⊥ could be Q ∧ ¬Q from assumption P. *Negation Elimination* (¬E) is to remove the double negation [?].

| Negation Introduction | Negation Elimination |
|-----------------------|----------------------|
| φ                     | ¬¬φ                  |
| ⊥                     | φ                    |

2.3.2 Conjunction

*Conjunction Introduction* (∧I) is to derive a conjunction from its conjuncts, i.e. if A is true, and B is true, then A ∧ B must be true. *Conjunction Elimination* (∧E) is to remove the conjunction and pick one of its conjuncts, i.e. if A ∧ B is true, then A must be true, and B must be true [?].

| Conjunction Introduction | Conjunction Elimination |
|--------------------------|-------------------------|
| φ₁                       | φ₁ ∧ ... ∧ φₙ            |
|                          | φᵢ                      |
| ...                      |                         |
| φₙ                       |                         |
| φ₁ ∧ ... ∧ φₙ            |                         |
2.3.3 Disjunction

*Disjunction Introduction* ($\lor I$) is to add as many disjunct as you prefer if at least one of the disjuncts is in the proof. *Disjunction Elimination* ($\lor E$) is to remove conjunction and pick one of the true sentences [?].

| Disjunction Introduction | Disjunction Elimination |
|--------------------------|-------------------------|
| $\phi_i$                 | $\phi_1 \lor \ldots \lor \phi_n$ |
| $\phi_1 \lor \ldots \lor \phi_n$ | $\phi_1 \rightarrow \psi$ |
|                          | $\vdots$ |
|                          | $\phi_n \rightarrow \psi$ |
|                          | $\psi$ |

2.3.4 Conditional

*Conditional Introduction* ($\rightarrow I$) is to use subproof assuming $\phi$ and prove $\psi$, then it follows that $\phi \rightarrow \psi$. *Conditional Elimination* ($\rightarrow E$) also known as *Modus Ponens* is to conclude $\psi$ if $\phi \rightarrow \psi$ and $\phi$ are proven [?].

| Conditional Introduction | Conditional Elimination |
|--------------------------|-------------------------|
| $\phi \vdash \psi$       | $\phi \rightarrow \psi$ |
| $\phi \rightarrow \psi$  | $\phi$                  |
|                          | $\psi$                  |

2.3.5 Biconditional

*Biconditional Introduction* ($\leftrightarrow I$) is to use subproof by assuming $\phi$ and prove $\psi$, also assuming $\psi$ and prove $\phi$, then it follows that $\phi \leftrightarrow \psi$ [?]. *Biconditional Elimination* ($\leftrightarrow E$) is to replace biconditional by $\phi \rightarrow \psi$ or $\psi \rightarrow \phi$ as both of them are true by the definition of biconditional [?].

| Biconditional Introduction | Biconditional Elimination |
|----------------------------|---------------------------|
| $\phi \rightarrow \psi$    | $\phi \leftrightarrow \psi$ |
| $\psi \rightarrow \phi$    | $\phi \rightarrow \psi$   |
| $\phi \leftrightarrow \psi$| $\psi \rightarrow \phi$   |
2.3.6 Universal

Universal Introduction ($\forall I$) also known as Universal Generalization if arbitrary $x$ has a property $\phi$ i.e $\phi(x)$, then we can conclude that for all $x$, such that $x$ has property $\phi \forall x \phi(x)$. Variable $x$ should not be free in any hypothesis on which $\phi(x)$ depends. Universal Elimination ($\forall E$) also known as Universal Instantiation if all $x$ in the universe has a property $\phi(x)$, $\forall x \phi(x)$, then there must be a $t$ in that universe that has the property $\phi$, $\phi(t)$, $t$ must be free for $x$ [?].

\[
\begin{array}{c}
\text{Universal Introduction} \\
\phi(x) \\
\hline
\forall x \phi(x)
\end{array}
\quad
\begin{array}{c}
\text{Universal Elimination} \\
\forall x \phi(x) \\
\hline
\phi(t)
\end{array}
\]

2.3.7 Existential

Existential Introduction ($\exists I$) also known as Existential Generalization if an object $c$ has property $P$, then there must exist $x$ in a universe that has property $P$. Existential Elimination ($\exists E$) also known as Existential Instantiation if exist in a universe that object $x$ has property $P$, then there must be an object $c$ that has property $P$ [?].

\[
\begin{array}{c}
\text{Existential Introduction} \\
\phi[t/x] \\
\hline
\exists x \phi
\end{array}
\quad
\begin{array}{c}
\text{Existential Elimination} \\
\exists x \phi \\
\hline
P(c)
\end{array}
\]

2.3.8 Reiteration

As the proof gets complicated and long, iteration rule can be used to bring an earlier step within the proof or to bring it to the subproof. It works as a reminder that "we have already shown that $P" [?].

\[
\begin{array}{c|c}
\text{Reiteration (Re)} \\
1 & P & \text{Premise} \\
2 & Q & \text{Assumption} \\
3 & \vdots & \\
4 & P & \text{Re, 1}
\end{array}
\]

7
3 Natural Deduction Calculus

Natural Deduction was first introduced as a term by the German logician Gentzen, Gerhard. It was introduced as a formalism that mimics how humans naturally reason, hence the name. By applying inference rules, one can infer conclusions from the premises. In other words, it is a method for showing that the logical reasoning (premises logically entails conclusion) is valid [?].

Calculi, which means the way of calculating or reasoning [?], in our case, natural deduction calculus is calculating the truth values of an argument by means of natural deduction proof system. Proofs in natural deduction will be as follows [?]:

1. Start with zero or more premises.
2. Prove formula (e.g. \( P \land Q \)) is provable from 1.
3. Justify each formula by using rules of inference or other proper justification.

In order to use any proof system, we have to make sure it is reliable and correct. Reliability and correctness of a proof system are shown through its soundness and completeness.

3.1 Soundness

The Soundness Theorem for a proof system will assure that we can only construct proofs of valid arguments. That is, we want to prove that every sentence in a proof is entailed by the previous sentences [?].

**Theorem 3.1** (Soundness)

Let \( \varphi \) be any formula and \( \Gamma \) a set of formulas in first-order language \( L \). If \( \Gamma \vdash \varphi \), then \( \Gamma \models \varphi \) [?].

\[ \Gamma \vdash \varphi \rightarrow \Gamma \models \varphi \]

**Proof.** Using mathematical induction, we can prove the soundness of natural deduction for first-order logic. We start with the base case, i.e. the first step \( n = 1 \), if it holds, we do the induction step, where we assume it holds for step \( n = k \) and prove that it also holds for the next step \( n = k + 1 \) [?].
Base Case

\[ \Gamma_1 \models \varphi_1 \]
\[ \varphi_1 \models \varphi_1 \]

Indeed, any formula is model of itself.

\[ \therefore \varphi_1 \vdash \varphi_1 \rightarrow \varphi_1 \models \varphi_1 \]

Inductive Step

Assume \( \Gamma_k \models \varphi_k \)
Show \( \Gamma_{k+1} \models \varphi_{k+1} \)

Because you are going to use the inference rules in any step of a proof, you need to prove that it will still be valid at any step, in our case, step \( k+1 \). Therefore, we will prove the soundness of the proof system case by case. Note that, \( \Gamma_{k+1} \models \varphi_{k+1} \) means the valuation of \( \alpha \) at step \( k+1 \) is a model for all formulas in \( \Gamma \) at step \( k+1 \) and the \( k \) previous formulas in \( \Gamma \). Furthermore, you can use the truth table instead of the small boxes on the right as proof of what is presented on the left.

Case 1: Negation Introduction

\[
\begin{array}{c|c|c|c}
   & \alpha & \Gamma_i \models \alpha \\
   i & \bot & \Gamma_j \models \bot \\
   j & \neg \alpha & \Gamma_{k+1} \models \neg \alpha (= \varphi_{k+1}) \\
   k+1 & \neg \alpha \rightarrow \bot, i-j & \\
\end{array}
\]

Note that from \( i \) to \( j \) is a subproof, starting with \( \alpha \) as an assumption and reaching a contradiction by the end of the subproof.

Case 2: Negation Elimination

\[
\begin{array}{c|c|c|c}
   & \neg \neg \alpha & \Gamma_i \models \neg \neg \alpha \\
   i & \alpha & \Gamma_{k+1} \models \alpha (= \varphi_{k+1}) \\
   k+1 & \alpha \rightarrow \bot, i-j & \\
\end{array}
\]

Case 3: Conjunction Introduction

\[
\begin{array}{c|c|c|c}
   & \alpha & \Gamma_i \models \alpha \\
   i & \phi & \Gamma_j \models \phi \\
   j & \alpha \land \phi \rightarrow \bot, i-j & \\
   k+1 & \alpha \land \phi (= \varphi_{k+1}) & \\
\end{array}
\]
Case 4: Conjunction Elimination

\[
\begin{array}{c|c}
  & \alpha \land \phi \\
  i & \alpha \land \phi \\
 k+1 & \phi \land E, i \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_i \models \alpha \land \phi \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_{k+1} \models \alpha (= \varphi_{k+1}) \\
\end{array}
\]

Case 5: Disjunction Introduction

\[
\begin{array}{c|c}
  & \alpha \\
  i & \alpha \\
 k+1 & \alpha \lor \phi \lor I, i \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_i \models \alpha \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_{k+1} \models \alpha \lor \phi (= \varphi_{k+1}) \\
\end{array}
\]

Case 6: Disjunction Elimination

\[
\begin{array}{c|c}
  & \alpha \lor \phi \\
  i & \alpha \lor \phi \\
 j & \alpha \rightarrow \psi \\
 m & \phi \rightarrow \psi \\
 k+1 & \psi \lor E, i, j, m \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_i \models \alpha \lor \phi \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_j \models \alpha \rightarrow \psi \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_m \models \phi \rightarrow \psi \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_{k+1} \models \psi (= \varphi_{k+1}) \\
\end{array}
\]

Case 7: Conditional Introduction

\[
\begin{array}{c|c}
  & \alpha \\
  i & \alpha \\
 j & \phi \\
 k+1 & \alpha \rightarrow \phi \rightarrow I, i, j \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_i \models \alpha \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_j \models \phi \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_{k+1} \models \alpha \rightarrow \phi (= \varphi_{k+1}) \\
\end{array}
\]

Case 8: Conditional Elimination

\[
\begin{array}{c|c}
  & \alpha \rightarrow \phi \\
  i & \alpha \rightarrow \phi \\
 j & \alpha \\
 k+1 & \phi \rightarrow E, i, j \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_i \models \alpha \rightarrow \phi \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_j \models \alpha \\
\end{array}
\]

\[
\begin{array}{c|c}
  & \Gamma_{k+1} \models \phi (= \varphi_{k+1}) \\
\end{array}
\]
Case 9: Biconditional Introduction

\[
\begin{array}{l|l}
i & \alpha \rightarrow \phi \\
k+1 & \alpha \leftrightarrow \phi \quad \leftrightarrow I, i, j \\
\end{array}
\]

\[
\begin{array}{l}
\Gamma_i \models \alpha \rightarrow \phi \\
\Gamma_j \models \phi \rightarrow \alpha \\
\Gamma_{k+1} \models \alpha \leftrightarrow \phi \quad (= \varphi_{k+1})
\end{array}
\]

Case 10: Biconditional Elimination

\[
\begin{array}{l|l}
i & \alpha \leftrightarrow \phi \\
j & \alpha \\
k+1 & \phi \quad \leftrightarrow E, i, j \\
\end{array}
\]

\[
\begin{array}{l}
\Gamma_i \models \alpha \leftrightarrow \phi \\
\Gamma_j \models \alpha \\
\Gamma_{k+1} \models \phi \quad (= \varphi_{k+1})
\end{array}
\]

\[
\begin{array}{l|l}
i & \alpha \leftrightarrow \phi \\
j & \phi \\
k+1 & \alpha \quad \leftrightarrow E, i, j \\
\end{array}
\]

\[
\begin{array}{l}
\Gamma_i \models \alpha \leftrightarrow \phi \\
\Gamma_j \models \phi \\
\Gamma_{k+1} \models \alpha \quad (= \varphi_{k+1})
\end{array}
\]

Case 11: Universal Introduction

\[
\begin{array}{l|l}
i & c \\
j & \phi(c) \\
k+1 & \forall x \phi(x) \quad \forall I, i, j \\
\end{array}
\]

\[
\begin{array}{l}
\Gamma_i \models c \\
\Gamma_j \models \phi(c) \\
\Gamma_{k+1} \models \forall x \phi(x) \quad (= \varphi_{k+1})
\end{array}
\]

Note that \( c \) is an arbitrary object from the domain of discourse that must be introduced as a new constant in a subproof, then prove that \( c \) has a property \( \phi \), i.e. \( \phi(c) \). \( \phi(c) \) must not contain any constant introduced by existential elimination after we introduced the constant \( c \) [?].

Case 12: Universal Elimination

\[
\begin{array}{l|l}
i & \forall x \phi(x) \\
k+1 & \phi(c) \quad \forall E, i \\
\end{array}
\]

\[
\begin{array}{l}
\Gamma_i \models \forall x \phi(x) \\
\Gamma_{k+1} \models \phi(c) \quad (= \varphi_{k+1})
\end{array}
\]
Case 13: Existential Introduction

\[
\begin{array}{c|c}
\text{i} & \phi(c) \\
\text{k+1} & \exists x \phi(x) & \exists E, i
\end{array}
\]

\[\Gamma_i \models \phi(c)\]

\[\Gamma_{k+1} \models \exists x \phi(x) \quad (= \varphi_{k+1})\]

Case 14: Existential Elimination

\[
\begin{array}{c|c}
\text{i} & \exists x \phi(x) \\
\text{k+1} & \phi(c) & \exists E, i
\end{array}
\]

\[\Gamma_i \models \exists x \phi(x)\]

\[\Gamma_{k+1} \models \phi(c) \quad (= \varphi_{k+1})\]

Note that \(c\) is an object that satisfies property \(\phi\). Therefore, you may assume \(\phi(c)\) [?].

Case 15: Reiteration

\[
\begin{array}{c|c}
\text{i} & \alpha \\
\text{k+1} & \alpha & \text{Reit, i}
\end{array}
\]

\[\Gamma_i \models \alpha_h\]

\[\Gamma_{k+1} \models \alpha_{k+1}\]

From the 15 cases that have been shown, we can conclude that the natural deduction for first-order logic is sound.

\[\square\]

3.2 Completeness

Gödel’s completeness theorem states that a deduction system is said to be complete if every universally valid formula in the language \(L\) has a proof under the proof system, (natural deduction) in our case [?].

Proving the completeness of a formal proof system is a huge and complex task, it was Gödel’s doctoral dissertation that was finished in 1929 and published in 1930 [?]. Therefore, to stay consistent in the way we present the proof to fit with the overall presentation of the paper in which we aim to make various concepts as easy to understand as possible, we will give a sketch of the proof, but before doing so, we need to stop at a couple of definitions and lemmas on our way to reach the proof of completeness for the natural deduction for first-order logic [?].

---

1The procedure we followed to tackle the proof is taken from the book *Logic and Structure* by Dirk van Dalen [?].
Definition 3.1
(i) A theory $T$ is a collection of sentences with property $T \vdash \varphi \rightarrow \varphi \in T$ ($T$ is closed under derivability).
(ii) A set $\Gamma$ such that $T = \{ \varphi \mid \Gamma \vdash \varphi \}$ is called an axiom set of theory $T$.
(iii) Theory $T$ is called Henkin theory, if for each sentence $\exists x \varphi(x)$ there is a constant $c$ such that $\exists x \varphi(x) \rightarrow \varphi(c) \in T$ ($c$ is called a witness for $\exists x \varphi(x)$).

Definition 3.2
Let $T, T'$ be theories in language $L, L'$.
(i) $T'$ is an extension of $T$ if $T \subseteq T'$.
(ii) $T'$ is a conservative extension of $T$ if $T' \cap L = T$, i.e. all theorems of $T'$ in the language $L$ are already theorems of $T$.

Definition 3.3
Let a theory $T$ be with language $L$. By adding a constant $c_\varphi$ for each sentence of the form $\exists x \varphi(x)$ in language $L$, we obtain $L^*$. $T^*$ is the theory with axiom set $T \cup \{ \exists x \varphi(x) \rightarrow \varphi(c) \mid \exists x \varphi(x) \text{ closed, with witness } c_\varphi \}$

Lemma 3.2
Let language $L$ have cardinality $\kappa$. If $\Gamma$ is a consistent set of sentences, then $\Gamma$ has a model of cardinality $\leq \kappa$.

Lemma 3.3
$T^*$ is conservative over $T$.

Proof. (a) Let $\exists x \alpha(x) \rightarrow \alpha(c)$ be one of the new axioms.
Suppose set of sentences $\Gamma, \exists x \alpha(x) \rightarrow \alpha(c) \vdash \psi$, where the constant $c$ is neither in $\Gamma$ nor in $\psi$. We will show that $\Gamma \vdash \psi$:

1. $\Gamma \vdash (\exists x \alpha(x) \rightarrow \alpha(c)) \rightarrow \psi$.
2. $\Gamma \vdash (\exists x \alpha(x) \rightarrow \alpha(y)) \rightarrow \psi$. Note that $y$ is a variable that does not occur in the associated derivation. 2 follows from 1, it harmless to replace $c$ by $y$, the derivation remains intact).
3. $\Gamma \vdash \forall y[(\exists x \alpha(x) \rightarrow \alpha(y)) \rightarrow \psi]$. Since $c$ does not occur in $\Gamma$, the application of $\forall$ is valid.
4. $\Gamma \vdash \exists y(\exists x \alpha(x) \rightarrow \alpha(y)) \rightarrow \psi$.
5. $\Gamma \vdash (\exists x \alpha(x) \rightarrow \exists y \alpha(y)) \rightarrow \psi$.
6. $\exists x \alpha(x) \rightarrow \exists y \alpha(y)$.
7. From [5] [6] $\Gamma \vdash \psi$. 


(b) Let $T^* \vdash \psi$, we know that $T \cup \{\delta_1, ..., \delta_n\} \vdash \psi$ from derivability’s definition, where $\delta_i$ is the new axiom of the form $\exists x \alpha(x) \to \alpha(c)$. We will prove $T \vdash \psi$ by induction. For the base case, where $n = 0$, is done by (a). For inductive step, let $T \cup \{\delta_1, ..., \delta_n\} \vdash \psi$. Set $T' = T \cup \{\delta_1, ..., \delta_n\} \vdash \psi$, then $T', \delta_{n+1} \vdash \psi$. By induction hypothesis, $T' \vdash \psi$.

Lemma 3.4
Define $T_0 := T$, $T_{n+1} := (T_n)^*$ $T_\omega := \cup\{T_n \mid n \geq 0\}$. Then $T_\omega$ is a Henkin theory and it is conservative over $T$.

Proof. Call $L_n$ the language of $T_n$ and $L_\omega$ the language of $T_\omega$.

(i) $T_n$ is conservative over $T$.

(ii) $T_\omega$ is a theory. Suppose $T_\omega \vdash \delta$, then $\alpha_0, ..., \alpha_n \in T_\omega$. For each $i \leq n$, $\alpha_i \in T_m$ for some $m_i$. Let $m = \max \{m_i \mid i \leq n\}$. $T_m \subseteq T_m(i \leq n)$ since for all $k$, $T_k \subseteq T_{k+1}$. Therefore, $T_m \vdash \delta$. $T_m$ is a theory by definition, so $\delta \in T_m \subseteq L_\omega$.

(iii) $T_\omega$ is a Henkin theory. Let $\exists x \alpha(x) \in L_\omega$, then $\exists x \alpha(x) \in L_n$, $\exists x \alpha(x) \to \alpha(c) \in L_{n+1}$ (by definition) for a certain $c$. So, $\exists x \alpha(x) \to \alpha(c) \in L_\omega$.

(iv) $T_\omega$ is conservative over $T$. Note that $T_\omega \vdash \delta$ if $T_n \vdash \delta$ for some $n$.

Lemma 3.5 (Lindenbaum)
Each consistent theory is contained in a maximally consistent theory.

Proof. Let $T$ be consistent. Consider the partially ordered by inclusion set $A$ of all consistent extensions $T'$ of $T$. We claim that $A$ has a maximal element.

1. All chains in $A$ have an upper bound. Let $\{T_i \mid i \in I\}$ be a chain, then $T' = \bigcup T_i$ is a consistent extension of $T$ containing each $T_i$. So $T'$ is an upper bound.

2. From $A$ has a maximal element $T_m$ (Zorn’s lemma).

3. Trivially, we can see that $T_m$ is a maximally consistent extension of $T$, in the sense of $\subseteq$, therefore, $T$ is contained in the maximally consistent theory $T_m$. 

14
Lemma 3.6
An extension of a Henkin theory with the same language is again a Henkin theory.

Lemma 3.7 (Model Existence Lemma)
If $\Gamma$ is consistent, then $\Gamma$ has a model.

Proof. Let the theory given by $T$ to be $T = \{ \delta | \Gamma \vdash \delta \}$. Trivially, any model of $T$ is also a model of $\Gamma$. Let the maximally consistent Henkin extension of $T$ to be $T_m$ using $T_m$ itself. Recall, that the language is nothing but a set of strings of symbols.

1. $A = \{ t \in L_m | t \text{ is closed} \}$.
2. We define a function $\hat{f}(t_1, ..., t_k) := f(t_1, ..., t_k)$, for each function symbol $\bar{f}$.
3. We define a relation $\hat{P} \subseteq A^p$ by $< t_1, ..., t_p > \in \hat{P} \iff T_m \vdash P(t_1, ..., t_p)$, for each predicate symbol $\bar{P}$.
4. We define a constant $\hat{c} := c$, for each constant $c$.

We can assert that:

(a) The relation $t \sim s$ defined by $T_m \vdash t = s$ for $t, s \in A$ is an equivalence relation.

(b) $t_i \sim s_i$ ($i \leq p$) and $< t_1, ..., t_p > \in \hat{P} \iff < s_1, ..., s_p > \in \hat{P}$.

As we have the equivalence relation, it is natural to introduce the quotient structure.

Denote the equivalence class of $t$ under $\sim$ by $[t]$.

Define $\mathfrak{A} := \langle A/ \sim, \tilde{P}_1, ..., \tilde{P}_n, \tilde{f}_1, ..., \tilde{f}_m, \{ \tilde{c}_i | i \in I \} \rangle$, where:

- $\tilde{P}_i := \{ < [t_1], ..., [t_r] > | < [t_1], ..., [t_r] > \in \hat{P}_i \}$.
- $\tilde{f}_j([t_1], ..., [t_a]) = [\hat{f}_j(t_1, ..., t_a)]$.
- $\tilde{c}_i := [\hat{c}_i]$. 

15
By induction we can prove $\mathfrak{A} \models \alpha(t) \iff T_m \vdash \alpha(t)$ for all sentences in the language $L_m$ of $T_m$ (a.k.a $L(\mathfrak{A})$)

(i) $\alpha$ is atomic. $\mathfrak{A} \models P(t_1, ..., t_p) \iff \langle t_1^\mathfrak{A}, ..., t_p^\mathfrak{A} \rangle \in \hat{P} \iff T_m \vdash P(t_1, ..., t_p)$.

(ii) Trivially, $\alpha = \delta \land \tau$.

(iii) $\alpha = \delta \rightarrow \tau$. We can see that $T_m \vdash \delta \rightarrow \tau \iff (T_m \vdash \delta \rightarrow T_m \vdash \tau)$.

(iv) $\alpha = \forall x \psi(x)$. $\mathfrak{A} \models \forall x \psi(x) \iff \mathfrak{A} \not\models \exists x \neg \psi(x) \iff \mathfrak{A} \not\models \neg \psi(\bar{a})$, for all $a \in |\mathfrak{A}| \iff$ for all $a \in |\mathfrak{A}|(\mathfrak{A} \models \psi(\bar{a}))$. We assume $\mathfrak{A} \models \forall x \psi(x)$, we get $\mathfrak{A} \models \psi(\bar{c})$ for witness $c$ belong to $\exists x \neg \psi(x)$. By induction hypothesis $T_m \vdash \psi(\bar{c})$. $T_m \vdash \exists x \neg \psi(x) \rightarrow \neg \psi(\bar{c})$, so $T_m \vdash \psi(\bar{c}) \rightarrow \neg \exists x \neg \psi(x)$. Thus, $T_m \vdash \forall x \alpha(\bar{x})$.

Contrarily, $T_m \vdash \forall x \psi(x) \rightarrow T_m \vdash \psi(t)$, so $T_m \vdash \psi(t)$ for all closed $t$. By induction hypothesis, $\mathfrak{A} \models \psi(t)$ for all closed $t$. Thus, $\mathfrak{A} \models \forall x \psi(x)$.

We can see that $\mathfrak{A}$ is a model of $\Gamma$, as $\Gamma \subseteq T_m$. $\square$

The model constructed above is known canonical model or the closed term model.

From 3.7 we can immediately deduce Gödel’s completeness theorem

**Theorem 3.8** (Completeness)

Let $\varphi$ be any formula and $\Gamma$ a set of formulas in first-order language $L$. If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

$$\Gamma \models \varphi \rightarrow \Gamma \vdash \varphi$$
4 Examples

There are different styles for representing the proof of an argument, e.g. Gentzen-style, Fitch-style, and others. This paper will follow Fitch-style to solve the examples in this section [?].

4.1 Mortality and Socrates

All humans are mortal, Socrates is human. Therefore, someone is mortal [?].

- H(x): x is human
- M(x): x is mortal
- s: Socrates

Indirect Proof (Proof by Contradiction)

1. \(\forall x(H(x) \rightarrow M(x))\) Premise
2. \(H(s)\) Premise
3. \(\neg \exists x M(x)\) Assumption
4. \(\forall x \neg M(x)\) Def, 3
5. \(H(s) \rightarrow M(s)\) \(\forall E, 1\)
6. \(\neg M(s)\) \(\forall E, 4\)
7. \(M(s)\) \(\rightarrow E, 5, 2\)
8. \(\bot\) \(\bot, 6, 7\)
9. \(\exists x M(x)\) \(\neg I, 3-8\)

Start with assuming \(\neg \exists x M(x)\) and try to find a counter-example, hence the name (proof by contradiction). Reductio ad absurdum is a rule to show that if an assumption leads to a contradiction, then the negation of that assumption must be true [?].

Direct Proof

1. \(\forall x(H(x) \rightarrow M(x))\) Premise
2. \(H(s)\) Premise
3. \(H(s) \rightarrow M(s)\) \(\forall E, 1\)
4. \(M(s)\) \(\rightarrow E, 3, 2\)
5. \(\exists x M(x)\) \(\exists I, 4\)
4.2 Living Trees

All trees are plants. All plants are living things. Therefore, all trees are living things.

- $T(x)$: $x$ is tree
- $P(x)$: $x$ is plant
- $L(x)$: $x$ is a living thing

Indirect Proof (Proof by Contradiction)

| No. | Formula                                      | Rule   |
|-----|----------------------------------------------|--------|
| 1   | $\forall x (T(x) \rightarrow P(x))$        | Premise |
| 2   | $\forall x (P(x) \rightarrow L(x))$        | Premise |
| 3   | $\neg \forall x (T(x) \rightarrow L(x))$   | Assumption |
| 4   | $\exists x \neg (T(x) \rightarrow L(x))$   | Def, 3 |
| 5   | $\neg (T(a) \rightarrow L(a))$             | $\exists E$, 4 |
| 6   | $T(a) \land \neg L(a)$                     | EQUIV, 5 |
| 7   | $T(a) \rightarrow P(a)$                    | $\forall E$, 1 |
| 8   | $P(a) \rightarrow L(a)$                    | $\forall E$, 2 |
| 9   | $T(a)$                                       | $\land E$, 6 |
| 10  | $P(a)$                                       | $\rightarrow E$, 7, 9 |
| 11  | $\neg L(a)$                                 | $\land E$, 6 |
| 12  | $L(a)$                                       | $\rightarrow E$, 8, 10 |
| 13  | $\bot$                                      | $\bot$, 11, 12 |
| 14  | $\forall x (T(x) \rightarrow L(x))$        | $\neg I$, 3-13 |
### Direct Proof

1. \(\forall x (T(x) \rightarrow P(x))\)  
   Premise
2. \(\forall x (P(x) \rightarrow L(x))\)  
   Premise
3. \(T(a) \rightarrow P(a)\)  
   \(\forall E, 1\)
4. \(P(a) \rightarrow L(a)\)  
   \(\forall E, 2\)
5. \(T(a)\)  
   Assumption
6. \(P(a)\)  
   \(\rightarrow E, 3, 5\)
7. \(L(a)\)  
   \(\rightarrow E, 4, 6\)
8. \(T(a) \rightarrow L(a)\)  
   \(\rightarrow I, 5-7\)
9. \(\forall x (T(x) \rightarrow L(x))\)  
   \(\forall I, 8\)

### 4.3 Cats and Rabbits

Some cats have fur or some cat are rabbits. Therefore, some cats are rabbits or have fur.

- \(F(x)\): \(x\) has fur
- \(R(x)\): \(x\) is a rabbit

### Indirect Proof (Proof by Contradiction)

1. \(\exists x F(x) \lor \exists x R(x)\)  
   Premise
2. \(\neg \exists x (F(x) \lor R(x))\)  
   Assumption
3. \(F(c)\)  
   Assumption
4. \(\forall x \neg (F(x) \lor R(x))\)  
   Def, 2
5. \(\neg F(c)\)  
   \(\forall E, 5\)
6. \(\bot\)  
   \(\bot, 3, 6\)
7. \(\exists x (F(x) \lor R(x))\)  
   \(\neg I, 2-6\)
Direct Proof

1. \( \exists x F(x) \lor \exists x R(x) \)  Premise
2. \( \exists x F(x) \)  Assumption
3. \( F(c) \)  \( \exists \text{E, 2} \)
4. \( F(c) \lor R(c) \)  \( \lor \text{I, 3} \)
5. \( \exists x (F(x) \lor R(x)) \)  \( \exists \text{I, 4} \)