CANARD CYCLES AND POINCARÉ INDEX OF
NON-SMOOTH VECTOR FIELDS ON THE PLANE

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Abstract. This paper is concerned with closed orbits of non-smooth vector fields on the plane. For a subclass of non-smooth vector fields we provide necessary and sufficient conditions for the existence of canard kind solutions. By means of a regularization we prove that the canard cycles are singular orbits of singular perturbation problems which are limit periodic sets of a sequence of limit cycles. Moreover, we generalize the Poincaré Index for non-smooth vector fields.

1. Introduction

Piecewise-smooth systems are widespread within application areas such as engineering, economics, medicine, biology and ecology. The most common piecewise-smooth systems involve either a discontinuity in the vector field, or in the orbit given by the integral solution $x(t)$. In this paper we consider the former, that is, general systems where the vector field is independently defined on either side of a smooth codimension one switching manifold. Three possible regions of the manifold are then apparent. At a crossing region the component of the vector field normal to the switching manifold has the same direction on both sides of the manifold (sometimes called sewing instead of crossing). At a stable sliding region both normal components of the vector field point toward the manifold. At an unstable sliding region both normal components point away from the manifold. Piecewise-smooth systems with sliding are also known as Filippov systems. Clearly these three different scenarios lead to vastly different dynamics. An orbit that meets the switching manifold at a crossing region passes through it, but is non-differentiable at the crossing point. An orbit that impacts at a stable sliding region sticks becomes constrained (sticks) to the manifold. An orbit in an unstable sliding region slides along the switching manifold, but will depart it under any infinitesimal perturbation. Consequently, the only means by which a stable sliding orbit can escaping the switching manifold is tangentially, at the boundary of the sliding region. This leads to the observation that, under parameter variation, orbits in Filippov systems can

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undergo a large variety of bifurcations, commonly called sliding bifurcations.

In this paper we study piecewise-smooth system on open regions on the plane. Let \( \mathcal{U} \subseteq \mathbb{R}^2 \) be an open set and \( \Sigma \subseteq \mathcal{U} \) given by \( \Sigma = f^{-1}(0) \), where \( f : \mathcal{U} \to \mathbb{R} \) is a smooth function having 0 \( \in \mathbb{R} \) as a regular value (i.e. \( \nabla f(p) \neq 0 \), for any \( p \in f^{-1}(0) \)). Clearly \( \Sigma \) is the separating boundary of the regions \( \Sigma_+ = \{ q \in \mathcal{U} | f(q) \geq 0 \} \) and \( \Sigma_- = \{ q \in \mathcal{U} | f(q) \leq 0 \} \). We can assume that \( \Sigma \) is represented, locally around a point \( q = (x, y) \), by the function \( f(x, y) = y \).

Designate by \( \chi^r \) the space of \( C^r \) vector fields on \( \mathcal{U} \) endowed with the \( C^r \)-topology with \( r \geq 1 \) or \( r = \infty \), large enough for our purposes. Call \( \Omega^r = \Omega^r(\mathcal{U}, f) \) the space of vector fields \( \hat{X}_0 : \mathcal{U} \setminus \Sigma \to \mathbb{R}^2 \) such that

\[
X_0(x, y) = \begin{cases} X_1(x, y), & \text{for } (x, y) \in \Sigma_+, \\ X_2(x, y), & \text{for } (x, y) \in \Sigma_-,
\end{cases}
\]

where \( X_i = (f_i, g_i) \in \chi^r, i = 1, 2 \). We write \( X_0 = (X_1, X_2) \), which we will accept to be multivalued in the points of \( \Sigma \). The trajectories of \( X_0 \) are solutions of \( \dot{q} = X_0(q) \), which has, in general, discontinuous righthand side. The basic results of differential equations, in this context, were stated by Filippov in [5]. Related theories can be found in [6, 9, 12]. In this paper we consider finite discontinuities, i.e., the vector fields \( X_1 \) and \( X_2 \) are defined in the set \( f^{-1}(0) \). Another kind of discontinuity of which the vector field tends to infinity when it approximates to the switching manifold can be obtained in the equations with impasse (more details in [10]).

In what follows we will use the notation \( X_i.f(p) = \langle \nabla f(p), X_i(p) \rangle \).

We distinguish the following regions on the discontinuity set \( \Sigma \):

(i) \( \Sigma_1 \subseteq \Sigma \) is the sewing region if \( (X_1.f)(X_2.f) > 0 \) on \( \Sigma_1 \).

(ii) \( \Sigma_2 \subseteq \Sigma \) is the escaping region if \( (X_1.f) > 0 \) and \( (X_2.f) < 0 \) on \( \Sigma_2 \).

(iii) \( \Sigma_3 \subseteq \Sigma \) is the sliding region if \( (X_1.f) < 0 \) and \( (X_2.f) > 0 \) on \( \Sigma_3 \).

Consider \( X_0 \in \Omega^r \). The sliding vector field associated to \( X_0 \) is the vector field \( X_0^s \) tangent to \( \Sigma_3 \) and defined at \( q \in \Sigma_3 \) by \( X_0^s(q) = m - q \) with \( m \) being the point where the segment joining \( q + X_1(q) \) and \( q + X_2(q) \) is tangent to \( \Sigma_3 \) (see Figure 1). It is clear that if \( q \in \Sigma_3 \) then \( q \in \Sigma_2 \) for \( -X_0 \) and then we can define the escaping vector field on \( \Sigma_2 \) associated to \( X_0 \) by \( X_0^e = -(X_0)^s \). In what follows we use the notation \( X_0^\Sigma \) for both cases.

Our main interest here is to study a special kind of typical minimal sets of non-smooth vector fields which will be called non-smooth “canard cycles” (see Figure 2). A canard cycle is a graphic composed by pieces of orbit of \( X_1 \), pieces of orbit of the sliding vector field \( X_0^s \) and/or pieces of orbit of \( X_2 \). See Section 2 for a more precise definition.
An approximation of the non-smooth vector field $X_0 = (X_1, X_2)$ by a 1-parameter family $X_\epsilon$ of smooth vector fields is called an $\epsilon$-regularization of $X_0$. We give the details about this process in section 4. A transition function is used to average $X_1$ and $X_2$ in order to get a family of smooth vector fields that approximates $X_0$. The main aim is to deduce certain dynamical properties of the non-smooth dynamical system from the regularized system. What is familiar may or may not be a matter of taste, at least it depends a lot on the dynamical properties of one’s interest. The regularization process developed by Sotomayor and Teixeira produces a singular problem for which the discontinuous set is a center manifold. Via a blow up we establish a bridge between non-smooth systems and the geometric singular perturbation theory.

Roughly speaking, the main results of this paper are the following:

- In our first result (Theorem 1), for a subclass of non-smooth vector fields, we provide necessary and sufficient conditions for the existence of canard kind solutions.
- In our second result (Theorem 2), following the ideas exposed in [8], we prove that hyperbolic canard cycles are limit sets, according Hausdorff distance, of families of (smooth) hyperbolic limit cycles (this fact is not proved in [8]). The regularization process plus a blow up produce a singular perturbation problem $P_\epsilon$. Our result implies that the canard cycle is the periodic limit set of closed orbits of $P_\epsilon$, with $\epsilon \to 0$. An open problem is to use the geometric singular perturbation theory proposed by Dumortier and Roussarie (center manifolds obtained via saturation by the flow plus blow up techniques, see [3] for details) to obtain the same result.
- In our third result (Theorem 3) we found an analogous for Poincaré Index in the case of non-smooth vector fields.

2. Preliminaries and statements of the main results

Consider $X_0 \in \Omega^r$. We say that $q \in \Sigma$ is a $\Sigma$-regular point if

(i) $X_1.f(q)X_2.f(q) > 0$ or
(ii) $X_1.f(q)X_2.f(q) < 0$ and $X_0^\Sigma (q) \neq 0$ (that is $q \in \Sigma_2 \cup \Sigma_3$ and it is not a singular point of $X_0^\Sigma$).

The points of $\Sigma$ which are not $\Sigma$-regular are called $\Sigma$-singular. We distinguish two subsets in the set of $\Sigma$-singular points: $\Sigma^c$ and $\Sigma^f$. We say that $q \in \Sigma^f$ is a pseudo equilibrium of $X_0$ if $X_0^\Sigma(q) = 0$ and we say that $q \in \Sigma^c$ is a $\Sigma$-contact point if $X_0^\Sigma(q) \neq 0$ and $X_1.f(q)X_2.f(q) = 0$ ($q$ is a contact point of $X_0^\Sigma$).

A $\Sigma$-contact point $q \in \Sigma^c$ is a $\Sigma$-fold point of $X_1$ if $X_1.f(q) = 0$ but $X_1^2.f(q) \neq 0$. Moreover, $q \in \Sigma$ is a visible (resp. invisible) $\Sigma$-fold point of $X_1$ if $X_1.f(q) = 0$ and $X_1^2.f(q) > 0$ (resp. $X_1^2.f(q) < 0$). We say that $q$ is a $\Sigma$-fold point of $X_0$ if it is a $\Sigma$-fold point either of $X_1$ or of $X_2$.

A pseudo equilibrium $q \in \Sigma^f$ is a $\Sigma$-saddle provided one of the following condition is satisfied: (i) $q \in \Sigma_2$ and $q$ is an attractor for $X_0^\Sigma$ or (ii) $q \in \Sigma_3$ and $q$ is a repeller for $X_0^\Sigma$. A pseudo equilibrium $q \in \Sigma$ of $X_0$ is a $\Sigma$-repeller (resp. $\Sigma$-attractor) provided $q \in \Sigma_2$ (resp. $q \in \Sigma_3$) and $q$ is a repeller (resp. attractor) for $X_0^\Sigma$. A point $q \in \Sigma$ is a hyperbolic pseudo equilibrium of $X_0$ if $q$ is a hyperbolic equilibrium point of $X_0^\Sigma$.

**Definition 1.** Consider $X_0 \in \Omega^r$.

1. A curve $\Gamma$ is a canard cycle if $\Gamma$ is closed and
   - $\Gamma$ contains arcs of at least two of the vector fields $X_1|_{\Sigma_{-}}$, $X_2|_{\Sigma_{-}}$ and $X_0^\Sigma$ or is composed by a single arc of $X_0^\Sigma$;
   - the transition between arcs of $X_1$ and arcs of $X_2$ happens in sewing points (and vice versa);
   - the transition between arcs of $X_1$ (or $X_2$) and arcs of $X_0^\Sigma$ happens through $\Sigma$-fold points or regular points in the escape or sliding arc, respecting the orientation. Moreover if $\Gamma \neq \Sigma$ then there exists at least one visible $\Sigma$-fold point on each connected component of $\Gamma \cap \Sigma$.

2. Let $\Gamma$ be a canard cycle of $X_0$. We say that
   - $\Gamma$ is a canard cycle of kind I if $\Gamma$ meets $\Sigma$ just in sewing points;
   - $\Gamma$ is a canard cycle of kind II if $\Gamma = \Sigma$;
   - $\Gamma$ is a canard cycle of kind III if $\Gamma$ contains at least one visible $\Sigma$-fold point of $X_0$.

   In Figures 3, 4 and 5 appear canard cycles of kind I, II and III respectively.

3. Let $\Gamma$ be a canard cycle. We say that $\Gamma$ is hyperbolic if
   - $\Gamma$ is of kind I and $\eta(p) \neq 1$ where $\eta$ is the first return map defined on a segment $T$ with $p \in T \cap \gamma$;
   - $\Gamma$ is of kind II;
   - $\Gamma$ is of kind III and or $\Gamma \cap \Sigma \subseteq \Sigma_1 \cup \Sigma_2$ or $\Gamma \cap \Sigma \subseteq \Sigma_1 \cup \Sigma_3$. 
In [8] is proved that the ε-regularization of non-smooth vector fields $X_0$ with hyperbolic canard cycles has hyperbolic limit cycles.

**Definition 2.** Let $\overrightarrow{AB}$ be an arc of $X_i$ joining the visible $\Sigma$-fold point $A$ to the point $B = X_i \cap \Sigma$. We say that $\overrightarrow{AB}$ has **focal kind** if there is not $\Sigma$-fold points between $A$ and $B$ (see Figure 5) and we say that $\overrightarrow{AB}$ has **graphic kind** if it has only one $\Sigma$-fold point between $A$ and $B$ (see Figure 6), $i = 1, 2$.

**Theorem 1.** Let $X_0 = (X_1, X_2) \in \Omega^r$ be a non-smooth vector field with $X_0$ presenting only one $\Sigma$-fold point $A$ which is visible. Denote $\gamma_1$ the arc of $X_i$ ($i = 1$ or $i = 2$) which passes through $A$ and call $B$ the transversal contact point of $\gamma_1$ with $\Sigma$. Then $X_0$ has a canard cycle $\Gamma$ if and only if the following conditions are satisfied: (i) the component $\gamma_1$ of $\Gamma$ which passes through $A$ is a focal kind arc; (ii) $X_1 f . X_2 f < 0$ in $(A, B)$ and (iii) $\{X_1, X_2\}$ is a linearly independent set in $[A, B]$. Moreover, $\Gamma$ is of kind III.

**Theorem 2.** Let $\Gamma_0$ be a hyperbolic canard cycle of $X_0$. Then for any $\epsilon > 0$ the regularized vector field $X_{\epsilon}$ has a hyperbolic limit cycle $\Gamma_{\epsilon}$ such that $\Gamma_{\epsilon} \rightarrow \Gamma_0$ when $\epsilon \rightarrow 0$.

We remark that for a hyperbolic canard cycle we have that each connected component of $\Gamma \cap \Sigma$ has only one $\Sigma$-fold point (See [8] for more details).

By using the previous notation, our results are:

**Theorem 1.** Let $X_0 = (X_1, X_2) \in \Omega^r$ be a non-smooth vector field with $X_0$ presenting only one $\Sigma$-fold point $A$ which is visible. Denote $\gamma_1$ the arc of $X_i$ ($i = 1$ or $i = 2$) which passes through $A$ and call $B$ the transversal contact point of $\gamma_1$ with $\Sigma$. Then $X_0$ has a canard cycle $\Gamma$ if and only if the following conditions are satisfied: (i) the component $\gamma_1$ of $\Gamma$ which passes through $A$ is a focal kind arc; (ii) $X_1 f . X_2 f < 0$ in $(A, B)$ and (iii) $\{X_1, X_2\}$ is a linearly independent set in $[A, B]$. Moreover, $\Gamma$ is of kind III.

**Theorem 2.** Let $\Gamma_0$ be a hyperbolic canard cycle of $X_0$. Then for any $\epsilon > 0$ the regularized vector field $X_{\epsilon}$ has a hyperbolic limit cycle $\Gamma_{\epsilon}$ such that $\Gamma_{\epsilon} \rightarrow \Gamma_0$ when $\epsilon \rightarrow 0$.

We remark that the Hausdorff distance between compact sets of $\mathbb{R}^2$ is:

$$D(K_1, K_2) = \max_{z_1 \in K_1, z_2 \in K_2} \{d(z_1, K_2), d(z_2, K_1)\}.$$
Theorem 3. Let $\Gamma_0$ be a hyperbolic canard cycle of the non-smooth vector field $X_0$. If $\{p_1, \ldots, p_k\}$ is the set of fixed or pseudo equilibrium points (all hyperbolic) of $X_0$ inside $\Gamma_0$ then the index of $\Gamma_0$ with respect to $X_0$ is the sum of the index of $p_i$, for $i = 1, \ldots, k$. Moreover, this sum is equal to one.

In section 5 we will define index of non-smooth vector fields.

The paper is organized as follows. In Sections 3, 4 and 5 we prove Theorems 1, 2 and 3, respectively. In section 6 we apply Theorem 1 to study a class of non-smooth vector fields $X_0 \in \Omega^r$ with just one focal kind arc and its bifurcation and we use the singular perturbation theory to study hyperbolic canard cycles.

3. Proof of the Theorem 1

In this section we prove the first result of the paper.

Proof. First we prove that (i),(ii) and (iii) imply the existence of the canard cycle. Since $X_1 f(X_2 f) < 0$ in $(A, B)$ the piece of $\Sigma$ between $A$ and $B$ is part of an escaping region or a sliding region. Moreover since $\{X_1, X_2\}$ is a linearly independent set in $[A, B]$ the system does not have pseudo equilibrium points in $[A, B]$. Without lost of generality, $[A, B]$ is part of the sliding region like in Figure 7. The curve $\Gamma = \gamma_1 \cup [B, A]$ is a hyperbolic canard cycle of kind III. We remark that this canard cycle takes place in just one side of $\Sigma$.

Now we prove that (i),(ii) and (iii) are necessaries conditions for the existence of this particular kind of canard cycle. Since $\Gamma$ is a hyperbolic canard cycle of kind III with just one $\Sigma$-fold point, $\Gamma$ takes place in just one side of $\Sigma$. In fact, if it does not occur, then $\Gamma$ returns to $\Sigma$ at least twice and so there exists at least a second $\Sigma$-fold point. Without lost of generality we suppose that $\Gamma$ is on the side corresponding to $X_1$. We denote by $\gamma_1$ the part of the cycle $\Gamma$ which is a trajectory of $X_1$. Thus we have that $\gamma_1$ is a focal kind arc because if it is a graphic kind arc then there is another $\Sigma$-fold point on $(A, B)$ (see Figure 7). Since $\gamma_1$ meets $\Sigma$ in the point $B$, the flow slides via $X_0^\Sigma$ until the point $A$ because there are not another $\Sigma$-fold point between $A$ and $B$; therefore $X_1 f(X_2 f) < 0$ in $(A, B)$. Moreover, the linear independence of $\{X_1, X_2\}$ on $[A, B]$ follows from the non-existence of pseudo equilibrium points on $[A, B]$.

Now, we will define an auxiliar function which will be useful in the sequel.

Take $(A, B) \subset \Sigma_2 \cup \Sigma_3$ contained in the escaping or in the sliding region. In $(A, B)$ consider the point $C = (C_1, C_2)$, the vectors $X_1(C) = (D_1, D_2)$ and $X_2(C) = (E_1, E_2)$ (as illustrated in Figure 8). The straight segment passing through $C + X_1(C)$ and $C + X_2(C)$ meets $\Sigma$ in a point $p(C)$. We
define the $C^r$-application
\[ p : (A, B) \rightarrow \Sigma \]
\[ z \mapsto p(z). \]
We can choose local coordinates such that $\Sigma$ is the $x$-axis; so $C = (C_1, 0)$ and $p(C) \in \mathbb{R} \times \{0\}$. The direction function on $\Sigma$ is defined by
\[ H : (A, B) \rightarrow \mathbb{R} \]
\[ z \mapsto p(z) - z. \]

\[ p(C_1) = \frac{(D_1+C_1)(E_2)-(D_2)(E_1+C_1)}{(E_2)-(D_2)}. \]

Assuming all the hypothesis of Theorem 1 we have the following corollary.

**Corollary 1.** The non-smooth vector field $X_0$ has a canard cycle $\Gamma$ if and only if the direction function $H : [A, B] \rightarrow \mathbb{R}$ is a well defined function and it has no zeros. Moreover, $\Gamma$ is of kind III.

### 4. Proof of Theorem 2

First of all we present the concept of $\epsilon$-regularization of non-smooth vector fields. It was introduced by Sotomayor and Teixeira in [9]. The regularization gives the mathematical tool to study the stability of these systems,
according to the program introduced by Peixoto. The method consists in
the analysis of the regularized vector field which is a smooth approximation
of the non-smooth vector field. Using this process we get a 1-parameter
family of vector fields $X_{\epsilon} \in \chi^r(K, \mathbb{R}^2)$ such that for each $\epsilon_0 > 0$ fixed we have

(i) $X_{\epsilon_0}$ is equal to $X_1$ in all points of $\Sigma_+$ whose distance to $\Sigma$ is bigger
than $\epsilon_0$;

(ii) $X_{\epsilon_0}$ is equal to $X_2$ in all points of $\Sigma_-$ whose distance to $\Sigma$ is bigger
than $\epsilon_0$.

**Definition 3.** A $C^\infty$ function $\varphi : \mathbb{R} \to \mathbb{R}$ is a transition function if
$\varphi(x) = -1$ for $x \leq -1$, $\varphi(x) = 1$ for $x \geq 1$ and $\varphi'(x) > 0$ if $x \in (-1, 1)$. The $\epsilon$-
regularization of $X_0 = (X_1, X_2)$ is the 1-parameter family $X_{\epsilon} \in \chi^r$ given by

$$(2) \quad X_{\epsilon}(q) = \left(\frac{1}{2} + \frac{\varphi_\epsilon(f(q))}{2}\right)X_1(q) + \left(\frac{1}{2} - \frac{\varphi_\epsilon(f(q))}{2}\right)X_2(q).$$

with $\varphi_\epsilon(x) = \varphi(x/\epsilon)$, for $\epsilon > 0$.

In order to prove Theorem 2 we need to construct a special neighborhood of arbitrary diameter for hyperbolic canard cycles.

**Construction of a neighborhood of diameter $\mu$ around a hyperbolic canard cycle.** Here we describe a method to construct a tubular neighborhood of diameter $\mu$ around a hyperbolic canard cycle. This presentation is done for canard cycles of kind III, but the ideas can also be extended for kinds I or II. We will be particularly interested in two of them: the ones that take place on just one side of $\Sigma$ and with just one visible $\Sigma$-fold point and the ones that take place on the two sides of $\Sigma$ with two visible $\Sigma$-fold points (one for $X_1$ and another one for $X_2$).

**Case 1- One $\Sigma$-fold point.** Denote by $\Gamma$ the hyperbolic canard cycle of
kind III with just one $\Sigma$-fold point and with orientation showed in Figure 7
(the reverse orientation is treated in a similar way). Consider the strip of
diameter $\mu$ around $\Sigma = \{y = 0\}$. Let $p_1$ and $q_1$ be points in $\{y = \mu\} \cap \Gamma$. Take an arc $\gamma_2$ of the vector field $X_1$ passing to the point $p_2 \in \{y = \mu\}$ in such a way that $p_2$ stays on the left of $p_1$ and such that $\gamma_2$ returns to the line $y = \mu$ in a point $q_2$ which is in a neighborhood of $q_1$. Take this trajectory satisfying $d(\Gamma, \gamma_2) < \frac{\mu}{2}$ (this is possible by the continuity of $X_1$). Let $r_1$ be the point where the arc of $X_1$ through by $p_1$ first meets the straight line $y = \mu$ for negative time. Analogously take an arc $\gamma_1$ of the field $X_1$ passing by the point $r_2$ in such a way that $r_2$ stays on the left of $r_1$ and such that $\gamma_1$ has second return to $y = \mu$ in a point $q_3$ which is in a neighborhood of $q_1$. Take this trajectory satisfying $d(\Gamma, \gamma_1) < \frac{\mu}{2}$. On $\Sigma$, on the left of the $\Sigma$-fold point $A$, the flow of $X_1$ is oriented to up, so it is possible to construct a transversal section $\sigma_2$ joining $p_2$ to the straight line $y = 0$ in such a way
that the same trajectories of \( X_1 \) cross transversally \( \sigma_2 \) and the segment \( p_2 \overline{p_1} \). Take \( t_2 \) the point where \( \sigma_2 \) meets the straight line \( y = 0 \) satisfying \( d(\Gamma, \sigma_2) < \mu \), as before. Moreover, on \( \Sigma \), on the right of the point \( B \), the flow of \( X_1 \) is oriented for down, so it is possible to construct a transversal section \( \theta_2 \) joining \( q_2 \) to the straight line \( y = 0 \) in such a way that the trajectories of \( X_1 \) that cross transversally \( \theta_2 \) do not cross the segment \( \overline{q_1 q_2} \). Let \( s_2 \) be the point where \( \theta_2 \) meets the straight line \( y = 0 \) (here we also need to take care for \( d(\Gamma, \theta_2) < \mu \)). Since \([A, B]\) is a sliding region, the flow of \( X_2 \) is transversal to \( \Sigma \). In the straight line \( y = -\mu \), consider the points \( u_2 \) and \( v_2 \) (\( u_2 \) is on the left of \( v_2 \)) satisfying that the trajectories of \( X_2 \) crossing the transversal sections \( \lambda_2 = \overline{t_2 u_2} \) and \( \delta_2 = \overline{s_2 v_2} \) meet transversally \( \Sigma \) at the segment \( \overline{t_2 s_2} \) (again, we need to take care for \( d(\Gamma, \lambda_2) < \mu \) and \( d(\Gamma, \delta_2) < \mu \)).

![Figure 9. Tubular neighborhood of a canard cycle with one \( \Sigma \)-fold.](image)

In this way, the strip defined by the closed curve \( \gamma_1 \cup \overline{q_3 r_2} \) and by the closed curve \( \gamma_2 \cup \theta_2 \cup \delta_2 \cup \overline{t_2 u_2} \cup \lambda_2 \cup \sigma_2 \) is a tubular neighborhood of \( \Gamma \) of diameter \( \mu \). Note that the flow of \( X_0 = (X_1, X_2) \) is arriving in this neighborhood and never it departs from it.

**Case 2. Two \( \Sigma \)-fold points.** Now we study the hyperbolic canard cycles of kind III with two visible \( \Sigma \)-fold points, being one for \( X_1 \) and the other one for \( X_2 \), like showed in Figure [10]. We work with canard cycles \( \Gamma \) that have only escaping regions on \( \Sigma = \{ y = 0 \} \) (the case with sliding regions is treated similarly). Consider the strip of diameter \( \mu \) around \( \Sigma \). Let \( p_1 \) and \( q_1 \) be points in \( \{ y = \mu \} \cap \Gamma \). Take an arc \( \gamma_1 \) of the vector field \( X_1 \) through \( t_1 \in \{ y = \mu \} \) satisfying that \( t_1 \) stays on the left of \( p_1 \) and such that \( \gamma_1 \) returns to the line \( y = \mu \) in a point \( u_1 \) which is in a neighborhood of \( q_1 \). Take this trajectory satisfying that \( d(\Gamma, \gamma_1) < \frac{\mu}{2} \). Take an arc \( \sigma_1 \) of the vector field \( X_1 \) through \( v_1 \in \{ y = \mu \} \) satisfying that \( v_1 \) stays on the right of \( p_1 \) and such that \( \sigma_1 \) has second return on the straight line \( y = \mu \) in a point \( x_1 \), even take this trajectory with the particularity that \( d(\Gamma, \sigma_1) < \frac{\mu}{2} \). We repeat the same argument for the vector field \( X_2 \) and we found the points
Let \( p_2, q_2, t_2, u_2, v_2 \) and \( x_2 \) respectively, and the curves \( \gamma_2 \) and \( \sigma_2 \). Let \( c \) be the point on \( \Sigma \cap \sigma_2 \), \( d \) be the point on \( \Sigma \cap \gamma_1 \), \( e \) be the point on \( \Sigma \cap \gamma_2 \) and \( f \) be the point on \( \Sigma \cap \sigma_1 \) as indicated in Figure 10. On \( \Sigma \), take the points \( g \) on the left of \( c \), \( h \) between \( A \) and \( d \), \( i \) between \( e \) and \( A' \) and \( j \) on the right of \( f \); satisfying that the arcs \( \theta_1 \) (joining \( g \) to \( x_1 \)), \( \rho_1 \) (joining \( u_1 \) to \( h \)), \( \eta_1 \) (joining \( i \) to \( t_1 \)) and \( \pi_1 \) (joining \( v_1 \) to \( j \)) are transversal sections for \( X_1 \) and the arcs \( \theta_2 \) (joining \( g \) to \( v_2 \)), \( \rho_2 \) (joining \( h \) to \( t_2 \)), \( \eta_2 \) (joining \( u_2 \) to \( i \)) and \( \pi_2 \) (joining \( j \) and \( x_2 \)) are transversal sections for \( X_0 \) with the distance from \( \Gamma \) to any one of this arcs less than \( \mu \).

\[ X_1 \]

\[ X_2 \]

**Figure 10.** Tubular neighborhood for canard cycles of kind III with two \( \Sigma \)-fold points.

In this way, the strip defined by the closed curve \( \gamma_1 \cup \rho_1 \cup \rho_2 \cup \gamma_2 \cup \eta_2 \cup \eta_1 \) and by the closed curve \( \sigma_1 \cup \theta_1 \cup \theta_2 \cup \sigma_2 \cup \pi_2 \cup \pi_1 \) is a tubular neighborhood for \( \Gamma \) of diameter \( \mu \). Note that the flow of \( X_0 = (X_1, X_2) \) is departing from the tubular neighborhood and never it arrives in it.

- Since this neighborhood bounds a region where the non-smooth vector field \( X_0 = (X_1, X_2) \) is arriving in or it is departing from them, it makes sense to say **attractor canard cycle** or **repeller canard cycle**.
- In the neighborhoods constructed before we allow that trajectories can make part of them, however it is possible to do it with the flow of \( X_0 \) being transversal to the boundaries of the tubular neighborhoods. In fact, it is enough to replace the trajectories by transversal curves. It is important for the construction of the tubular neighborhood of the canard cycles of kind I. Thus we make a construction like we made before but now we can use for this, the first return application \( \eta \) and thus if \( \eta' < 1 \) we have an attractor canard cycle and if \( \eta' > 1 \) we have a repeller canard cycle.
- For canard cycles of kind II is enough to take the strip of diameter \( \mu \) in the beginning of the construction as the tubular neighborhood.
Any other hyperbolic canard cycle is an arrangement of pieces of the canard cycles described above and so we can construct a tubular neighborhood for it arranging the previous tubular neighborhoods.

Proof of Theorem 2 Let $\Gamma_0$ be a canard cycle of $X_0$ and let $V_\epsilon$ be a tubular neighborhood of diameter $\epsilon$ around $\Gamma_0$. Since $X_0$ is transversal to the boundary of $V_\epsilon$, by continuity, the regularized vector field $X_\epsilon$ also is transversal to the boundary of $V_\epsilon$. Assume that $\Gamma_0$ is an attractor canard cycle, so the flow of $X_0$ is arriving in the neighborhood $V_\epsilon$ and consequently the flow of $X_\epsilon$ also is arriving in the neighborhood $V_\epsilon$. As there are not fixed points in $V_\epsilon$, applying the Poincaré-Bendixson Theorem we conclude that there exists an attractor limit cycle $\Gamma_\epsilon$ inside $V_\epsilon$. Moreover with a more detailed analysis we can prove that it is hyperbolic (see [8] for instance). Since every paths that compose $V_\epsilon$ depends continuously of $\epsilon$ we have that the diameter of the tubular neighborhood is a continuous function of the variable $\epsilon$. Therefore making $\epsilon \to 0$ we conclude that $\Gamma_\epsilon \to \Gamma_0$ (see Figure 11).

![Figure 11. Cycles convergence.](image)

We remark that if $\Gamma_0$ is an attractor (resp. repeller) hyperbolic canard cycle of $X_0$, then the same occurs for $\Gamma_\epsilon$ and $X_\epsilon$.

5. Proof of Theorem 3

Now we start our discussion about the third result. Let $I = [0, 1]$ be an interval and $\sigma : I \to U$ be an oriented closed continuous path. Suppose that there are no critical points of $X_0$ on $\sigma$. Let us move a point $P$ along the curve in the counterclockwise direction. The vector $X_0(P)$ will rotate during the motion. When $P$ returns to its starting place after one revolution along the curve $\sigma$, $X_0(P)$ also returns to its original position. During the journey $X_0(P)$ will make some whole number of revolutions. Counting these revolutions positively if they are counterclockwise, negatively if they are clockwise, the resulting algebraic sum of the number of revolutions is called the index of $\sigma$ with respect to $X_0$, and is denoted by $I(X_0, \sigma)$. 
To calculate $I(X_0, \sigma)$ it is convenient normalize $X(\sigma(t))$ as an unit vector at the origin. In this way, we can define a function $\theta : I \to \mathbb{R}$ such that
\[
\lim_{t \to t^-} \frac{X(\sigma(t))}{\|X(\sigma(t))\|} = \lim_{t \to t^-} (\cos \theta(t), \sin \theta(t))
\]
for every $t \in I$. The function $\theta$ is called angle function.

We observe that in the case of smooth vector fields, the angle function is always continuous, but in the case of non-smooth vector fields it admits a “jump” when the path pass to a point $s_i \in \Sigma$, $i \in \mathbb{N}$. Therefore, we establish a rule for this jump: at $s_i = \sigma(t_i)$ the angle function oscillates from $\lim_{t \to t^-} \theta(t) \in I_{k-1} = (2(k-1)\pi, 2k\pi)$ and $\lim_{t \to t^+} \theta(t) \in I_k = (2k\pi, 2(k+1)\pi)$, where $k \in \mathbb{Z}$, we add $1$ to the number $I(X_0, \sigma)$; if $\lim_{t \to t^-} \theta(t) \in I_k$ and $\lim_{t \to t^+} \theta(t) \in I_{k-1}$ we add $-1$ to the number $I(X_0, \sigma)$. We always consider that the jump of the vector $\lim_{t \to t^-} X(\sigma(t))$ to $\lim_{t \to t^+} X(\sigma(t))$ occurs by the smallest angle between this vectors.

![Figure 12. Angle function: at $s_1$ it has a jump of size $\alpha_1$ and at $s_2$ it has a jump of size $\alpha_2$.](image)

The difference $\theta(1) - \theta(0)$ is a multiple of $2\pi$, and
\[
I(X_0, \sigma) = \frac{\theta(1) - \theta(0)}{2\pi}
\]
is an integer independent of the chosen $\sigma$-parametrization. This number also is called Poincaré Index of the curve $\sigma$ with relation to the non-smooth vector field $X_0$.

Our interest here is to calculate the index of canard cycles surrounding fixed or pseudo equilibrium points that are the critical points of $X_0$. We will see that, different from the smooth case, given two canard cycles $\Gamma_1$ and $\Gamma_2$ surrounding the same critical points of $X_0$ we have $I(X_0, \Gamma_1) \neq I(X_0, \Gamma_2)$ in general.

**Example.** Consider the configuration described in Figure 13. Let $\gamma_1$ be an arc of $X_1$ joining the $\Sigma$-fold points $a$ and $b$; $e$ the $\Sigma$-fold point of $X_2$; $c$ and
The previous situation just occurs because the canard cycle given, for example, by $\gamma_1 \cup \delta_2 \cup \delta_1$, is non-hyperbolic because in its composition we can found pieces of escaping region and pieces of sliding region. If we eliminate this possibility we have the next theorem, which has an analogous in the case of smooth vector fields.

**Remark 1.** We recall that if $X_0$ has a hyperbolic $\Sigma$-saddle (or a hyperbolic $\Sigma$-focus) $s_0$ then the regularized vector field $X_\epsilon$ has a hyperbolic saddle (hyperbolic focus) $s_\epsilon$ where $s_\epsilon \to s_0$ when $\epsilon \to 0$ (for details see [8]). Moreover, we can verify that if $s_0$ is a saddle or a $\Sigma$-saddle then we can take a sufficiently small closed path $\sigma$ around $s_0$ and prove that $I(X_0, \sigma) = -1$ (and if $s_0$ is a focus or a $\Sigma$-attractor or a $\Sigma$-repeller then we can take a sufficiently small closed path $\tilde{\sigma}$ around $s_0$ and prove that $I(X_0, \tilde{\sigma}) = 1$). When the path $\sigma$ is sufficiently small to have just one critical point of $X_0$, named $s_0$, in its interior we use the notation $I_{s_0}(X_0, \sigma)$ to denote its Poincaré Index.

### 5.1. Proof of Theorem 8

We want to prove that if $\Gamma_0$ is a hyperbolic canard cycle of the non-smooth vector field $X_0$ and if $p_1, \ldots, p_k$ are the only ones critical points (all hyperbolic) of $X_0$ inside $\Gamma_0$ then $I(X_0, \Gamma_0)$ is well defined and

$$I(X_0, \Gamma_0) = \sum_{i=1}^{k} I_{p_i}(X_0, \Gamma_0) = 1.$$
First of all we assume that the index is well defined. Let $X_0$ be a non-smooth vector field with a hyperbolic canard cycle $\Gamma_0$ and $k$ hyperbolic critical points of $X_0$ inside $\Gamma_0$. Thus, the regularized vector field $X_\epsilon$ has a hyperbolic limit cycle $\Gamma_\epsilon$ and $k$ hyperbolic fixed points inside $\Gamma_\epsilon$. So, by the Poincaré Index Theorem for smooth vector fields the index calculated in $\Gamma_\epsilon$ in relation to $X_\epsilon$ is the sum of the index of the fixed points of $X_\epsilon$ inside $\Gamma_\epsilon$ and this sum is equal to 1. Since $\Gamma_\epsilon \to \Gamma_0$ we conclude that the index calculated in $\Gamma_0$ in relation to $X_0$ is the sum of the index of the critical points of $X_0$ inside $\Gamma_0$ and this sum is equal to 1 (see remark 1). In order to finish the proof we verify that the index is well defined in the case that the closed curve $\Gamma_0$ is a hyperbolic canard cycle. In fact, let $\Gamma_0$ be a hyperbolic canard cycle of $X_0$. Let us assume that $\Gamma_0$ is of one kind described in Figures 14 or 15 below.

**Figure 14.** $p_1$ is repeller.

**Figure 15.** $p_1$ is attractor.

We will prove that there is not danger of ambiguity in the choose of the closed paths, differently that what happens in the previous example. In Figure 14 we consider the hyperbolic canard cycle given by $\Gamma_0 = \gamma_1 \cup B \overrightarrow{A}$. Any canard cycle of $X_0$ having pieces of sliding region must to pass by the $\Sigma$-fold point $A$, now walk in $\gamma_1$ (which is the only one possibility that we have!) and meet the point $B$ on $\Sigma$. The unique choice we have is return to $A$, closing the path, without ambiguity. For Figure 15 the analysis is more interesting. Obviously we can use the trick of take the vector field $-X_0$ and obtain an analogous result that the previous, however we prefer give here a complete idea to the case described in Figure 15. Since in the semi straight line $r = \overrightarrow{A B}$ we have an escaping region, the canard cycle must have only escaping region in its composition. Note that, if we choose to depart from $r$ by a point in the segment $(A, B)$ then this path does not return to $(A, B) \subset \Sigma$ (it will move spirally around the focus $p_1$), if we choose going out from $r$ by a point after $B$ in $r$ then this path also does not return to $(A, B) \subset \Sigma$ (to this path return to $(A, B)$ it must return in a sliding region, what is not allowed because only escaping regions compose this hyperbolic canard cycle). So we must leave $r$ by the point $B$ and to close the curve. Therefore in any case there is not danger of ambiguity in the choose of the closed curve and so the Poincaré index for non-smooth vector fields is well defined. To hyperbolic canard cycles of kind III with another particularities is enough repeat the ideas exposed here. To hyperbolic canard cycles of
kinds I and II clearly there is not danger of ambiguity in the choose of the closed curves once there are not escaping or sliding region in its composition.

**Corollary 2.** Under the hypothesis of the previous theorem and assuming that all canard cycles of $X_0$ are hyperbolic, we have that:

1. If $Γ_0$ is a canard cycle then inside $Γ_0$ there exist $(2n + 1)$ critical points of $X_0$, being $n$ saddles or $Σ$-saddles and $(n + 1)$ focus, $Σ$-repeller or $Σ$-attractor.

2. If all critical points of $X_0$ are saddle or $Σ$-saddle then $X_0$ does not have canard cycles.

**Proof.** Since the index of each saddle and each $Σ$-saddle point is equal to $−1$ and the index of each other critical point of $X_0$ is equal to $1$ the result is an immediate consequence of Theorem 3.

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### 6. Applications and Examples

#### 6.1. Heteroclinic Orbits.

Consider the notation of the Theorem 1. We give now an example of a curve that satisfies all the hypothesis in this theorem except that there exists a point $C ∈ (A, B)$ such that the vectors $X_1(C)$ and $X_2(C)$ are not linearly independent; instead of $Γ$ obtained in the theorem we have here a “$Σ$-loop”, that is a $Σ$-saddle-attractor with connection between $Σ$-separatrices.

**Example.** Consider the non-smooth vector field $X_0 = (X_1, X_2)$ with $X_1(x, z) = (x + z - 1, -x + z - 1)$, $X_2(x, z) = (-x^2 + \frac{3}{2}x - \frac{1}{2}, 1)$ and discontinuity set given by the $x$-axis, i.e., $f(x, z) = z$. On $z = 0$, we have $X_1(x, 0) = (x - 1, -x - 1)$ and $X_2(x, 0) = (-x^2 + \frac{3}{2}x - \frac{1}{2}, 1)$ and so,

$$(X_1, f)(x, 0) = -x - 1, \quad (X_2, f)(x, 0) = 1.$$

In this way, we can conclude that $x = -1$ is a $Σ$-fold point of $X_1$ which determines a focal kind arc. For $x > -1$ we have that $Σ$ is a sliding region and for $x < -1$ it is a sewing region (see Figure 16). We show now that there exists a point $C ∈ (A, B)$ for which $X_1(C) = λX_2(C)$.

In fact, if $X_1(x, 0) = λX_2(x, 0)$ then $h(x) = (x - 1)^2(x + \frac{3}{2}) = 0$. The graphic of $h(x)$ is given in Figure 17. We observe that $h$ is equal to $−H$, where $H$ is the direction function defined previously. So, we have the situation described in the Theorem 1 except that $X_1$ and $X_2$ are not linearly independent in $x = 1$ where $X_0^{Σ}$ has an equilibrium point. The orientation of $X_0^{Σ}$ is in direction to the $Σ$-fold point because for $x = \frac{1}{2}$ we have $X_1(\frac{1}{2}, 0) = (\frac{1}{2}, \frac{5}{2})$, $X_2(\frac{1}{2}, 0) = (0, 1)$ and so, the direction function $H$ is negative ($H(\frac{1}{2}) = -\frac{1}{5}$), analogously for $x = \frac{3}{2}$ we have $X_1(\frac{3}{2}, 0) = (\frac{1}{2}, \frac{5}{2})$, $X_2(\frac{3}{2}, 0) = (\frac{1}{2}, 1)$ and so, the direction function $H$ also is negative ($H(\frac{3}{2}) = -\frac{1}{7}$). The pseudo equilibrium $p = (1, 0)$ is a $Σ$-saddle-attractor where the $Σ$-separatrices are connected.
Example (Bifurcation of the previous example) In the previous example, the pseudo equilibrium with a “loop” was found because the function $h$ has a double-zero at $x = 1$. So, we can conclude that putting small variations on the fields $X_1$ and $X_2$ a new function $h$ appears, with two simple real zeros (see Figure 21) or without real zeros (see Figure 19) in a neighborhood of $x = 1$. The phase portrait of $X_0$ for small variations on $X_1$ and $X_2$ are showed in Figures 18 and 20. Note that for the case showed in Figure 18, the direction function $H$ is always negative in a neighborhood of $x = 1$ and we can apply the corollary 1 to conclude that there exists a hyperbolic canard cycle of kind III and, for the case showed in Figure 20, the direction function $H$ has two simple real zeros in a neighborhood of $x = 1$ and assumes positive values between this two points and negative values in the rest of the sliding region. In this way we have a bifurcation model where imposing small variations we can have a hyperbolic canard cycle or we can have two pseudo equilibrium points with a stable connection between its separatrices. We call this a Σ-Loop Bifurcation.

Following the notation of Theorem 1 and the ideas just exposed we can state the next Proposition:

Proposition 1. Let $X_0$ a non-smooth vector field.
(1) $X_0$ has an unstable configuration topologically equivalent to that one in Figure 16 if and only if (i) the direction function $H$ is well defined in $[A, B]$, (ii) $H$ has a single zero in $(A, B)$ and (iii) $H(B) < 0$.

(2) $X_0$ has a stable configuration topologically equivalent to that one in Figure 22 if and only if (i) the direction function $H$ is well defined in $[A, B]$, (ii) $H$ has a single zero in $(A, B)$ and (iii) $H(B) > 0$.

(3) $X_0$ has an unstable configuration topologically equivalent to that one in Figure 23 if and only if (i) the direction function $H$ is well defined in $[A, B]$, (ii) $H$ do not have zeros in $(A, B)$ and (iii) $H(B) = 0$.

**Proof.** It is straightforward following what is done in the previous example.

Moreover, concerning with the item (1) of the Proposition 1 small perturbations in $X_0$ produces the effects showed in the previous example and concerning with the item (3) of the Proposition 1 small perturbations in $X_0$ produces effects described in Theorem 1 in the previous example and in the item (2) of the Proposition 1.

### 6.2. Canard Cycles and Singular Perturbations Problems

In this section we show how the regularization process gives a singular perturbation problem. In this context, the canard cycles defined in this paper can be considered as limit periodic sets of singular problems. First of all we present
some basic definitions.

**Definition 4.** Let \( U \subseteq \mathbb{R}^2 \) be an open subset and take \( \epsilon \geq 0 \). A singular perturbation problem in \( U \) (SP–Problem) is a differential system which can be written like

\[
x' = dx/d\tau = l(x, y, \epsilon), \quad y' = dy/d\tau = \epsilon m(x, y, \epsilon)
\]

or equivalently, after the time re-scaling \( t = \epsilon \tau \)

\[
\epsilon \dot{x} = \epsilon dx/dt = l(x, y, \epsilon), \quad \dot{y} = dy/dt = m(x, y, \epsilon),
\]

with \((x, y) \in U \) and \( l, m \) smooth in all variables.

The understanding of the phase portrait of the vector field associated to a SP-problem is the main goal of the geometric singular perturbation theory (GSP-theory). The techniques of GSP-theory can be used to obtain information on the dynamics of \( X_\epsilon \) for small values of \( \epsilon > 0 \), mainly in searching limit cycles. System \( (4) \) is called the fast system, and \( (5) \) the slow system of SP-problem. Observe that for \( \epsilon > 0 \) the phase portraits of the fast and the slow systems coincide. For \( \epsilon = 0 \), let \( S \) be the set

\[
S = \{ (x, y) : f(x, y, 0) = 0 \}
\]

of all singular points of \( \mathbb{H} \). We call \( S \) the slow manifold of the singular perturbation problem and it is important to notice that equation \( (5) \) defines a dynamical system, on \( S \), called the reduced problem:

\[
f(x, y, 0) = 0, \quad \dot{y} = g(x, y, 0).
\]

Combining results on the dynamics of these two limiting problems, with \( \epsilon = 0 \), one obtains information on the dynamics of \( X_\epsilon \) for small values of \( \epsilon \). We refer to [1] for an introduction to the general theory of singular perturbations. Related problems can be seen in [1], [3] and [11]. Let us apply the techniques of GSP-Theory to study hyperbolic canard cycles.

**Example.** Consider the non-smooth vector field \( X_0 = (X_1, X_2) \) with \( X_1(x, y) = (x + y - 1, -x + y + 1) \), \( X_2(x, y) = (1, 2) \) and \( f(x, y) = x \).

The regularized vector field becomes

\[
\dot{x} = \frac{x+y}{2} + \varphi \left( \frac{x}{r} \right) \frac{x+y-2}{2}, \quad \dot{y} = \frac{-2+y+3}{2} + \varphi \left( \frac{y}{r} \right) \frac{-2+y-1}{2},
\]

where \( \varphi \left( \frac{x}{r} \right) \) is the transition function. Making the change of variables \( x = r \cos \theta \) and \( \epsilon = r \sin \theta \) we obtain

\[
r \dot{\theta} = -\sin \theta \left( \frac{r \cos \theta + y}{2} + \varphi \left( \frac{y}{r} \right) \frac{r \cos \theta + y-2}{2} \right),
\]

\[
\dot{y} = -\frac{r \cos \theta + y + 3}{2} + \varphi \left( \frac{y}{r} \right) \frac{-r \cos \theta + y - 1}{2}.
\]

In the blowing up locus \( r = 0 \) the fast dynamics is determined by the system

\[
\theta' = -\sin \theta \left( \frac{y}{2} + \varphi \left( \frac{y}{r} \right) \frac{y-2}{2} \right), \quad y' = 0;
\]
and the slow dynamics on the slow manifold is determined by the reduced system

\[
\frac{y}{2} + \varphi(\cot \theta) \frac{y - 2}{2} = 0, \quad \dot{y} = \frac{y + 3}{2} + \varphi(\cot \theta) \frac{y - 1}{2}.
\]

We remark that the slow manifold is implicitly defined by

\[
\frac{y}{2} + \varphi(\cot \theta) \frac{y - 2}{2} = 0
\]

and \(y(\theta)\) defined in this way is such that \(\lim_{\theta \to \pi \over 4} y(\theta) = 1, \lim_{\theta \to 3\pi \over 4} y(\theta) = -\infty\).

In the phase portrait on the blowing up locus double arrow over one the trajectory means that the trajectory is of the fast dynamical system, and simple arrow means that the trajectory is of the slow dynamical system. So, we can draw the slow variety and its orientation and give the orientation of the fast flow (see Figure 24).

![Figure 24](image1.png)  ![Figure 25](image2.png)

**Figure 24.** Singular perturbation of a Canard with one \(\Sigma\)-fold point.  
**Figure 25.** Singular perturbation of a Canard with two \(\Sigma\)-fold points.

In the final example of [2] the authors apply the GSP-Theory to the non-smooth vector field \(X_0(x, y) = (X_1(x, y), X_2(x, y)) = ((3y^2 - y - 2, 1), (-3y^2 - y + 2, -1))\) and obtain the SP-problem which behavior is described in Figure 25.

In [7] the authors prove that \(\Sigma^2 \cap \Sigma^3\) is homeomorphic to the slow variety and that the sliding vector field \(X^\Sigma_0\) is topologically equivalent to the reduced problem. So, we can apply step-by-step the method described in section 4 of this paper and found tubular neighborhoods for the canard cycles in Figures 24 and 25. Moreover, we can apply the Theorem 2 and conclude that it is a limit set (making \(r \to 0\)) of hyperbolic limit cycles. This also is true to any one hyperbolic canard cycle.

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