Two-loop renormalization constants and high energy $2 \rightarrow 2$
scattering amplitudes in the Higgs sector of the standard model

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Abstract

We calculate the complete matrix of two-body scattering amplitudes for the
scattering of longitudinally polarized gauge bosons $W_L^\pm, Z_L$ and Higgs bosons
to two loops in the high-energy, heavy-Higgs limit $\sqrt{s} \gg M_H \gg M_W$. Use of
the Goldstone boson equivalence theorem reduces the problem to one involv-
ing only the scalar fields $w^\pm, z$ (the Goldstone bosons of the original theory)
and the Higgs boson. Renormalization of the scattering amplitudes requires
the calculation of the self-energy functions $\Pi^i_0(M_i^2)$, the renormalization con-
stants $Z_i$, and the bare quartic Higgs coupling $\lambda_0$ to two loops. The results
will be useful in other calculations. To facilitate the calculations, we introduce
a powerful new technique for evaluating integrals over Feynman parameters
in dimensional regularization which is based on a Barnes’ type representation
of the binomial expansion. We also collect some useful integrals which extend
the tables given by Devoto and Duke.
I. INTRODUCTION

It was pointed out some time ago by Dicus and Mathur [3] and Lee, Quigg, and Thacker [2] that the presently unknown mass $M_H$ of the Higgs boson could be bounded above in a weakly-interacting standard model by using the constraint on the magnitude of partial-wave scattering amplitudes implied by unitarity. This was examined in detail in [2], where it was shown that the only real constraints from two-body scattering arise from processes which involve only longitudinally polarized gauge bosons $W_L^\pm, Z_L$, and Higgs bosons $H$. These processes are enhanced by factors of $M_H^2 / M_W^2 \propto \lambda / g^2$ relative to those which involve transversely polarized gauge bosons and the small gauge couplings $g$. Here $\lambda$ is the quartic coupling for the Higgs boson, and is related to $M_H$ and the vacuum expectation value $v = (\sqrt{2} G_F)^{-1/2} = 246$ GeV by $\lambda = M_H^2 / 2v^2$. When $\lambda$ (or $M_H$) is large enough, the interactions of the Higgs sector of the standard model become strong, and the unitarity constraints on $\lambda$ or $M_H$ are to be interpreted in terms of a transition from weak to strong coupling [2–4], rather than as upper bounds on either.

The original tree-level analysis of scattering in the neutral channels $W^+_L W^-_L, Z_L Z_L, H H, Z_L H$ given in [3] has been extended to one loop by a number of authors [5–9, 11–13], and sharpened at tree level [8,10] and at one loop [11,13] using renormalization-group arguments. The most restrictive analysis was that in [11,13] where it was shown that strong coupling sets in for $\lambda \gtrsim 2.2–2.5$, a value substantially below the tree-level bound $\lambda \approx 8\pi / 3 \approx 8.38$ (or $M_H = 1.007$ TeV) derived by Dicus and Mathur [1] and Lee, Quigg, and Thacker [2]. It is desirable to extend the analysis to two loops. This is important as a check that the conclusions of [11,13] are in fact correct, and that an increase of one order in perturbation theory does not loosen the (unexpectedly low) upper bound on $\lambda$ in a weakly interacting theory.

The present paper is devoted to the calculation of the two-body scattering amplitudes for $W^+_L, Z_L, H$ scattering to two loops in the high-energy, heavy-Higgs limit $M_W \ll M_H \ll \sqrt{s}$. Our results are used in the following paper [14] to reexamine the limits on $\lambda$ and $M_H$ to
two loops in a weakly interacting theory.

The organization of this paper is as follows. In Sec. II, we discuss the general formulation of the problem, its reduction through the use of the Goldstone boson equivalence theorem to a problem involving only the scalar sector of the unbroken standard model, and the renormalization of the scalar Lagrangian. In Sec. III, we discuss the calculation of the self-energies and wave function renormalization constants to two loops. The most difficult parts of the calculation are in the extraction of the finite parts of the two-loop terms. We sketch the methods which we found useful for various diagrams, but give only minimal details. The results are summarized by diagram in Appendix A, while some technical points of the methods used are discussed in Appendices B, C and D. In particular, we illustrate in Appendix B a powerful new method for evaluating integrals over Feynman parameters in dimensional regularization which is based on a Barnes’-type representation of the binomial expansion. We sketch the use of Kotikov’s method of differential equations to evaluate certain two-loop graphs in Appendix C, and collect a number of useful integrals in Appendix D.

The complete self-energy counterterms and the wave function and coupling renormalizations are collected in Sec. IID. These and the renormalized two-loop scattering amplitudes given in Sec. IV are the principal results of the paper, and constitute the input for [14]. The counterterms and renormalization constants are also directly relevant to calculations of $W_L^\pm, Z_L$ scattering at “low” energies for very massive Higgs bosons, that is, for $M_W \ll \sqrt{s} \lesssim M_H$.

II. GENERAL FORMULATION OF THE PROBLEM

A. Framework

We will be concerned in the following sections with the calculation to two loops ($O(\lambda^3)$) of the complete set of two-body scattering amplitudes for high-energy scattering in the coupled
neutral channels $W_L^+W_L^-, Z_LZ_L, HH, HZ_L$. The calculation can be simplified enormously in
the limit we will consider, $\sqrt{s} \gg M_H \gg M_W$, by the use of the Goldstone boson equivalence
theorem \[2,16,18\]. This theorem states that the scattering amplitudes for $n$ longitudinally-
polarized vector bosons $W_L^\pm, Z_L$ and any number of other external particles are related to the
corresponding scattering amplitudes for the scalar bosons $w^\pm, z$ to which $W_L^\pm, Z_L$ reduce
for vanishing gauge couplings $g, g'$ by

$$T(W_L^\pm, Z_L, H, \ldots) = (iC)^nT(w^\pm, z, H, \ldots) + O(M_W/\sqrt{s}).$$  (1)

The constant $C$ depends on the normalization scheme used in the calculation \[16,18,19\],
$$C = \frac{M_W^0}{M_W} \left( \frac{Z_W}{Z_w} \right)^{1/2} \left[ 1 + O(g^2) \right],$$  (2)
where the $Z$'s are the wave function renormalization constraints for the physical fields $W^\pm$
and the scalar fields $w^\pm$. $C$ is equal to unity for $g, g' \to 0$ in schemes in which the renor-
malizations are defined at mass scales $m \ll M_H$. We will renormalize the $w^\pm, z$ fields at
$m = 0$, a choice which corresponds to massless Goldstone bosons and is consistent with the
assumption that $M_W \ll M_H$. Then \[16,18\],
$$C = 1 + O(g^2).$$  (3)

Since we are interested in upper limits on $\lambda$ or $M_H$, the limit $M_W^2/M_H^2 \propto g^2/\lambda \ll 1$ required
for the validity of the theorem is natural for our purposes. We will henceforth set the gauge
couplings to zero, and work entirely with the scalar fields.

**B. Lagrangian and renormalization**

The original Lagrangian for the symmetry-breaking sector of the standard model,
$$\mathcal{L}_0 = \frac{1}{2}(D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{\lambda}{4}(\Phi^\dagger \Phi - u^2)^2$$  (4)
with $D^\mu = \partial^\mu + igW^\mu \cdot T + \frac{1}{2}g'B^\mu$ and
\[ \Phi = \begin{pmatrix} w_1 + iw_2 \\ h + iw_3 \end{pmatrix}, \quad (5) \]

reduces for \( g, g' \to 0 \) to an SO(4)-symmetric Lagrangian involving only the scalar fields \( h \)
and \( w = (w_1, w_2, w_3) \). The field \( h \) will be taken as usual as the component of \( \Phi \) which
acquires a vacuum expectation value, \( u \) at tree level and \( v \) in the full theory. We will write
\( h \) as \( h = v + H \) where \( \langle \Omega | H | \Omega \rangle = 0 \) with respect to the physical vacuum \( | \Omega \rangle \). Adding
all possible SO(4)-symmetric counterterms and rewriting the potential in \( L_0 \) in terms of \( v \)
rather than \( u \), we obtain the complete Lagrangian for \( g = g' = 0 \),
\[
L = \frac{1}{2} (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) + \frac{1}{2} A (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) 
- \left( \frac{1}{4} \lambda (\Phi^2 - v^2)^2 + \frac{1}{2} \lambda (v^2 - u^2)(\Phi^2 - v^2) + \frac{1}{2} \delta m^2 \Phi^2 + \frac{1}{4} \delta \lambda \Phi^4 \right). \quad (6)
\]

\( \Phi \) is now to be written in terms of \( w, H, \) and \( v \).

The SO(4) symmetry of the unbroken scalar theory and the condition \( \langle \Omega | H | \Omega \rangle = 0 \)
impose constraints on \( u^2, A, \delta m^2, \) and \( \delta \lambda \). For example, Higgs tadpole diagrams shift
the vacuum expectation value and contribute apparent mass terms for the Goldstone bosons
which must be absent in the final theory. Taylor \[20\] has outlined a procedure which sums the
tadpole diagrams to all orders, and enforces the vanishing of \( \langle \Omega | H | \Omega \rangle \). The same condition
guarantees the vanishing of the masses of the Goldstone bosons \( w \), and allows the tree-level
vacuum expectation value \( u \) to be eliminated in terms of \( v \), the self-energy function \( \Pi_w(0) \),
and the remaining parameters \[21\]. The parameter \( A \) appears in the kinetic terms for both
the \( w \) and \( H \) fields because of the underlying symmetry in the problem. It is related to the
wave function renormalization constant \( Z_w \) by \( Z_w = 1 + A \). Finally, \( \delta m^2 \) and \( \delta \lambda \) are related
through the constraint \( M_H^2 = 2 \lambda v^2 \), which is taken to be exact, that is, to hold to all orders in
perturbation theory, with \( M_H \) the physical mass of the Higgs boson defined by the condition
\( \text{Re}(i \Delta_H^{-1}(M_H^2)) = 0 \) \[21\]. In enforcing this constraint, we are choosing a renormalization
scheme and definitions of \( M_H \) and \( \lambda \) which are consistent with those introduced by Sirlin
\[22\] and by Sirlin and Zucchini \[23\] for the full theory, with \( g, g' \neq 0 \). The result of this
analysis \[21\] is a modified Lagrangian
\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} w_0 \cdot \partial^{\mu} w_0 + \frac{1}{2} \partial_{\mu} H_0 \partial^{\mu} H_0 - \frac{1}{2} \left( M^2_H - \text{Re} \Pi^0_H(M^2_H) \right) H_0^2 + \frac{1}{2} \Pi^0_w(0) w_0^2 \\
- \frac{1}{4} \lambda \left( 1 - \frac{\text{Re} \Pi^0_H(M^2_H) - \Pi^0_w(0)}{M^2_H} \right) \left[ \frac{1}{Z_w}(w_0^2 + H_0^2)^2 + \frac{4v}{Z^{1/2}_w}(w_0^2 + H_0^2) \right]. \quad (7) \]

We have introduced bare fields \( w_0 = Z^{1/2}_w w \) and \( H_0 = Z^{1/2}_w H \) to keep the kinetic terms and free propagators in standard form \[24\]. \( \Pi^0_w \) and \( \Pi^0_H \) are the self-energy functions for the bare fields and are to be calculated using the Lagrangian in Eq. (7). We will later drop the superscript on \( \Pi^0 \) for simplicity in denoting the one- and two-loop contributions to these functions, \( \Pi^0 = \Pi^{(1)} + \Pi^{(2)} + \cdots \). The scattering amplitudes calculated in terms of \( w_0 \) and \( H_0 \) require the usual wave function renormalization, so must be multiplied by a factor \( Z^{-1}_i \) for each external line.

The form of the new Lagrangian clearly ensures that the Goldstone bosons \( w \) are massless and that the Higgs boson has the physical mass \( M_H \). The constraint \( M^2_H = 2\lambda v^2 \) can be used to eliminate either \( M^2_H \) or \( \lambda \) given the physical value of \( v \). However, it will be most convenient to keep \( \mathcal{L} \) as written, and only use the constraint as needed. Finally, in using this form of the Lagrangian, all Higgs tadpole graphs should be omitted since the shift in the vacuum expectation value from \( u \) to \( v \) has been suppressed following Taylor \[20\].

The SO(4) symmetry of the original Lagrangian is broken by the mass terms and the trilinear couplings in Eq. (7), but remains in the quartic couplings. There is still a residual SO(3) symmetry for the \( w \) fields, so the self-energy functions and renormalization constants for the charged and neutral particles \( w^\pm = (w_1 \mp iw_2)/\sqrt{2} \) and \( z = w_3 \) are identical,

\[ Z^{-1}_z = Z^{-1}_w = 1 - \frac{d}{dp^2} \Pi^0_w(p^2)|_{p^2=0}. \quad (8) \]

However, the self-energy function and wave function renormalization constant for the Higgs boson are different, and the subtraction point is at \( p^2 = M^2_H \) rather than \( p^2 = 0 \),

\[ Z^{-1}_H = 1 - \frac{d}{dp^2} \text{Re} \Pi^0_H(p^2)|_{p^2=M^2_H}. \quad (9) \]

The SO(4) symmetry of the theory is partially restored in scattering processes in the high energy limit \( \sqrt{s} \gg M_H \). In particular, scattering graphs which involve the dimension-three
trilinear couplings in Eq. (7) are suppressed by powers of \(s, t,\) or \(u\) relative to graphs which involve only the quartic dimension-four couplings [25], and can be neglected for \(s \gg M_H^2\) subject to some subtleties discussed later. The only contributions of the trilinear couplings for \(\sqrt{s} \to \infty\) occur through the renormalization constants \(Z_w, Z_H\) which are fixed at low energies or, at two loops, in the unitarity sum for the partial-wave amplitudes, through \(2 \to 3\) processes which have a low-energy component. The \(Z\)’s and low-energy processes break the SO(4) symmetry even though it is present in the high-energy interactions.

C. The Feynman diagrams

We used the programs DIAGRAMMAR and DRAW [26] to generate the self-energy and scattering diagrams which are needed through two loops, and to calculate their weights, including the couplings. The results were checked by hand and by a Wick expansion. The thirty-eight distinct diagrams for the one- and two-loop self-energies are shown in Figs. 1–3 without the couplings or weights. All tadpole graphs have been omitted as discussed above. The coupling counterterms are not shown as these do not alter the Feynman integrals to be calculated.

For ease of reference, the distinct topologies for the self-energy diagrams are named Scoop (\(Q\)), Bubble (\(B\)), Triangle (\(T\)), Acorn (\(A\)), Eye-in-the-Sky (\(E\)), Sunset (\(S\)), and Lemon (\(L\)). Individual graphs are labeled by the number of internal Higgs lines as shown in the figures, with extra labels \(m, 0,\) and \(Q\) to distinguish otherwise-ambiguous cases. In the analytic expressions developed in the following sections, the symbols \(Q, B, \ldots\) will refer specifically to the Feynman integrals associated with the corresponding diagrams, with the coupling constants and weights from the Wick reduction omitted, e.g., from Fig. 1

\[
B_2 = \int Dk \frac{i}{(k^2 - M_H^2)} \frac{i}{((k + q)^2 - M_H^2)}
\]

for incoming momentum \(q\).

The scattering diagrams which are needed through two loops are shown in Fig. 4, with a similar labeling. The tree, one- and two-loop graphs \(V, B_S, D_S\) and \(A_S\) involve only the
SO(4)-symmetric quartic couplings. The complete set of scattering diagrams also includes, where appropriate, contributions from the $t$ and $u$ channels. These scattering amplitudes are SO(4)- and crossing-symmetric and increase in magnitude for $s \to \infty$. In contrast, the sample diagrams shown in Fig. 5, which include trilinear couplings, vanish for $s, |t|, |u| \to \infty$ as is easily checked.

The calculation of the self-energy terms will be discussed in the next section. The scattering graphs will be considered in Sec. IV.

III. CALCULATION OF SELF ENERGIES AND RENORMALIZATION CONSTANTS

A. The general problem

The self-energy function $\Pi_w(p^2)$ for the Goldstone bosons $w$ is given to two loops by

$$
- i\Pi^0_w = \frac{-i\lambda}{Z_w} \left( 1 - \frac{\text{Re} \, \Pi^{(1)}_{H} (M_H^2) - \Pi^{(1)}_{w}(0)}{M_H^2} \right) Q \\
+ \frac{(-i\lambda v)^2}{Z_w} \left( 1 - \frac{\text{Re} \, \Pi^{(1)}_{H} (M_H^2) - \Pi^{(1)}_{w}(0)}{M_H^2} \right)^2 4B_1 \\
- i\lambda B_Q[i\text{Re} \, \Pi^{(1)}_{H} (M_H^2) - 3i\lambda Q] - i\lambda(-i\lambda v)^2[18A_{4Q} + 6A_{2Q} + 20A_{1Q}] \\
+(-i\lambda v)^24[(i\text{Re} \, \Pi^{(1)}_{H} (M_H^2) - i3\lambda Q)T_2 + (i\Pi_w(0) - i\lambda Q)T_1] \\
+(-i\lambda v)^2[10S_0 + 2S_2] \\
- i\lambda(-i\lambda v)^2[40A_{10} + 24A_3 + 16A_{20} + 8(B_1)^2] \\
+(-i\lambda v)^4[72E_4 + 24E_2^* + 16E_2 + 48L_3 + 16L_{20}],
$$

(11)

where $Q, B, \text{ etc.}$ refer to the Feynman integrals which are associated with the diagrams in Figs. 1-3. We have not given the explicit expressions for the factors in the first two terms which arise from the wave function and coupling renormalizations so as to indicate their origin more clearly. These factors can be expanded systematically using the one-loop results from Eq. (11) and the following equation and their analogs for $d\Pi^0_w/dp^2$ and $d\Pi^0_H/dp^2$. 

8
The corresponding result for the Higgs boson is

$$-i\Pi_0^H = 3 \frac{-i \lambda}{Z_w} \left( 1 - \frac{\text{Re} \Pi_0^{(1)}(M_H^2) - \Pi_0^{(1)}(0)}{M_H^2} \right) Q$$

$$+ \frac{(-i\lambda v)^2}{Z_w} \left( 1 - \frac{\text{Re} \Pi_0^{(1)}(M_H^2) - \Pi_0^{(1)}(0)}{M_H^2} \right)^2 [18B_2 + 6B_0]$$

$$-i3\lambda B_Q [i\text{Re} \Pi_0^{(1)}(M_H^2) - i3\lambda Q]$$

$$-i\lambda (-i\lambda v)^2 [54A_{4Q} + 18A_{2Q} + 12A_{1Q}]$$

$$+ (-i\lambda v)^2 [36(i\text{Re} \Pi_0^{(1)}(M_H^2) - i3\lambda Q)T_3 + 12(i\Pi_w(0) - i\lambda Q)T_0]$$

$$+ (-i\lambda)^2 [6S_1 + 6S_3] - i\lambda (-i\lambda v)^2 [216A_4 + 72A_{2m} + 48A_{1m}]$$

$$-i\lambda (-i\lambda v)^2 [30(B_0)^2 + 36B_0B_2 + 54(B_2)^2]$$

$$+ (-i\lambda v)^4 [648E_5 + 216E_3 + 48E_1 + 648L_5 + 144L_{2m} + 24L_1]. \quad (12)$$

We will only need these functions evaluated for $p^2 = 0$ ($\Pi_0^0(0)$) and $p^2 = M_H^2$ ($\Pi_0^0(M_H^2)$) for use in Eq. (7).

The wave function renormalization constants $Z_w^{-1}$ and $Z_H^{-1}$, Eqs. (5) and (7), involve similar but more complicated expressions obtained by formally differentiating the expressions for $\Pi_0^0$ and $\Pi_0^H$ with respect to $p^2$, and evaluating the results at $p^2 = 0$ and $p^2 = M_H^2$, respectively. We thus need expressions for all of the self-energy graphs and their derivatives with respect to the external momentum, but only for $p^2 = 0$ or $p^2 = M_H^2$.

We have calculated the Feynman integrals using dimensional regularization with the number of dimensions $D$ given by $D = 2\omega = 4 - 2\epsilon$. To keep the coupling $\lambda$ dimensionless in $2\omega$ dimensions, we multiply the quartic coupling in Eq. (7) by $\mu^{4-2\omega}$ and the trilinear coupling by $\mu^{2-2\omega}$, where $\mu$ is an arbitrary scale mass which will drop out in the final results. For the diagrams we need, each loop then appears with a factor $\mu^{4-2\omega}$ which we will incorporate into the measure of the momentum integrals, defining $Dk$ as

$$Dk = \mu^{4-2\omega} \frac{d^{2\omega}k}{(2\pi)^{2\omega}}. \quad (13)$$

Our definitions for the momenta and particle masses in the Feynman integrals are shown in Figs. 6, 7. Ordinary Feynman parameter techniques were used to combine propagators, and
are sufficient for most of the calculations. (However, we used other methods—hyperspherical or Gegenbauer polynomial expansions and Kotikov’s differential equation method—to evaluate some of the Lemon \( (\mathcal{L}) \) graphs, as discussed later.) The basic integral needed after combining propagators is \( [27] \)

\[
\int \frac{d^{2}\omega k}{[k^2 + 2k \cdot q - m^2 + i\epsilon]^\alpha} = i\pi \omega \frac{\Gamma(\alpha - \omega)}{\Gamma(\alpha)} \frac{e^{-i\pi \omega}}{[-q^2 - m^2 + i\epsilon]^{\alpha-\omega}} \tag{14}
\]

where \(-M^2 + i\epsilon = e^{i\pi}(M^2 - i\epsilon)\).

Since a number of functions and structures appear repeatedly, we will use some abbreviated notations. Thus, we will scale all masses or momenta by \( M_H \), and denote a scaled quantity by a caret, e.g., \( \hat{s} = s/M_H^2 \). Each loop of a divergent graph contributes a factor

\[
\xi^\epsilon = (4\pi \mu^2/M_H^2)^\epsilon = (4\pi \hat{\mu}^2)^\epsilon \tag{15}
\]

to the final result; we will generally not display this factor except when needed. A derivative of an integral with respect to an external momentum, \( \partial \) (graph) / \( \partial q^2 \), will often be written as \( \partial \) (graph). Finally, the first Feynman parameter introduced, \( x \), often appears in the combination \( X = x(1-x) \).

Our results for the integrals which appear in the self-energy functions and their derivatives are given in Appendix A. In the following subsections, we will simply sketch the methods used in the calculations, with minimal detail.

**B. The one-loop diagrams**

The calculation of the one-loop graphs is straightforward. However, it is necessary to keep terms of \( O(\epsilon) \) in the expansion of the resulting expressions for \( \epsilon \to 0 \) in order to cover cases in which the graphs are multiplied by terms of order \( 1/\epsilon \), and for articulated graphs.

1. **Scoop**

The momentum integral corresponding to Fig. 6 can be evaluated directly from Eq. (14):
\[ Q = \int \mathcal{D}k \frac{i}{(k^2 - m^2)} = \frac{\xi^\epsilon \Gamma(-1 + \epsilon)}{(4\pi)^2} m^2. \]  

(16)

\( Q \) is independent of the incoming momentum, and proportional to the mass in the loop. Hence, Goldstone boson Scoops are always zero \((m_w = 0)\) and have simply been ignored here. For the case of a Higgs scoop with a self energy insertion, we also need

\[ B_Q = i \frac{\partial Q}{\partial M_H^2} = \frac{i \xi^\epsilon \Gamma(\epsilon)}{(4\pi)^2}. \]  

(17)

For the \( w \), \( Q \) vanishes identically, and \( B_Q = 0 \) as well.

2. Bubbles

The momentum integral for the bubble diagrams is given in the convention in Fig. 6 by:

\[ B = \int \mathcal{D}k \frac{i}{(k^2 - a^2)} \frac{i}{((k + q)^2 - b^2)}. \]  

(18)

We will need \( B \) to order \( \epsilon \), an order higher than retained in most calculations. Combining the denominators using a Feynman parameter \( x \) and performing the \( \mathcal{D}k \) integration gives

\[ B = \frac{-i \Gamma(\epsilon) \xi^\epsilon}{(4\pi)^2} \int_0^1 dx \left[ \hat{a}^2 (1 - x) + \hat{b}^2 x - \hat{s}X \right]^{-\epsilon}, \]  

(19)

where \( X = x(1 - x) \). Similarly,

\[ \frac{\partial B}{\partial q^2} = \frac{-i \Gamma(1 + \epsilon) \xi^\epsilon}{(4\pi)^2 M_H^2} \int_0^1 dx X \left[ \hat{a}^2 (1 - x) + \hat{b}^2 x - \hat{s}X \right]^{-1-\epsilon}. \]  

(20)

The cases we will need correspond to the choices

- \( B_0 \): \( \hat{s} = 1 \), \( \hat{a} = \hat{b} = 0 \);
- \( B_1 \): \( \hat{a} = 1 \), \( \hat{b} = \hat{s} = 0 \);
- \( B_2 \): \( \hat{a} = \hat{b} = \hat{s} = 1 \).

If \( \hat{a} = \hat{b} = 0 \) or if \( \hat{s} = 0 \), the integrals give beta functions and the answers can be written down immediately, as can the derivatives \( \partial B \) for the same cases. For \( B_2 \), we expand the denominator in powers of \( \epsilon \). The result at overall order \( \epsilon \) involves the integrals:
\[ \int dx \ln(1 - x + x^2); \quad \int dx \ln^2(1 - x + x^2); \quad \int dx \frac{\ln(1 - x + x^2)}{1 - x + x^2}. \] (21)

These can be evaluated using the factorization
\[ 1 - x + x^2 = (1 - c_+ x)(1 - c_- x), \quad c_\pm = e^{\pm i\pi/3}. \] (22)

The last two integrals then lead to dilogarithms and, in particular, to the imaginary part of dilogarithms, Clausen functions \[28\]. These integrals contribute such characteristic quantities as \( C = \text{Cl}(\pi/3) = 1.01494 \cdots, \pi \sqrt{3}, \) and \( \pi \sqrt{3} \ln 3 \). DeVoto and Duke \[29\] give a number of results which are useful in these and later calculations in an extensive compilation. Further results are given in Appendix D.

We will also need \( B_0(s) = s^{-\epsilon} B_0 \) and \( B_2(s) \) for \( s = p^2 \neq M_H^2 \). The second function can again be evaluated by expansion in powers of \( \epsilon \), or by using a method based on a Barnes’-type representation for the denominator in Eq. (20). We have found the second method—which is apparently new in this context—to be a very powerful way of handling the integrals over Feynman parameters encountered in dimensional regularization. The method is discussed in detail in Appendix B.

3. Triangle

The Triangle graph in Fig. 6 is related to the Bubble by \( T = i \partial B/\partial a^2 \),
\[ T = \int \mathcal{D}k \left( \frac{i}{(k^2 - a^2)} \right)^2 \frac{i}{((k + q)^2 - b^2)}. \] (23)

Hence, from Eq. (19),
\[ T = i \frac{\partial B}{\partial a^2} = -\frac{\Gamma(1 + \epsilon)\xi}{(4\pi)^2 M_H^2} \int_0^1 dx (1 - x) \left[ a^2 (1 - x) + \hat{b}^2 x - \hat{s} X \right]^{-1-\epsilon}, \] (24)
and
\[ \frac{\partial T}{\partial q^2} = -\frac{\Gamma(2 + \epsilon)\xi}{(4\pi)^2 M_H^2} \int_0^1 dx (1 - x) X \left[ a^2 (1 - x) + \hat{b}^2 x - \hat{s} X \right]^{-2-\epsilon}. \] (25)

There are four cases of interest here:
\[ \mathcal{T}_0: \hat{s} = 1, \quad \hat{a} = \hat{b} = 0; \]
\[ \mathcal{T}_1: \hat{b} = 1, \quad \hat{a} = \hat{s} = 0; \]
\[ \mathcal{T}_2: \hat{a} = 1, \quad \hat{b} = \hat{s} = 0; \]
\[ \mathcal{T}_3: \hat{a} = \hat{b} = \hat{s} = 1. \]

The first three lead to beta functions, as for the analogous bubble \( \mathcal{B}_i \). By power counting, or by comparison to the analogous \( \mathcal{B}_i \), it can be seen that the divergences of \( \mathcal{T}_0 \) and \( \mathcal{T}_1 \) are infrared divergences. \( \mathcal{T}_3 \) was evaluated by expanding the integrand in powers of \( \epsilon \) following the method used for \( \mathcal{B}_2 \). We will also need \( \mathcal{T}_0(s) = s^{-1-\epsilon} \mathcal{T}_0 \) and \( \mathcal{T}_3(s) \). The latter was evaluated both by the Barnes’ method in Appendix B and by direct expansion.

C. The two-loop diagrams

1. Sunset

The Sunset graph in Fig. 7 is one of two new topologies at this order, and is the only two-loop self-energy graph in a symmetric \( \phi^4 \) theory other than the double Scoops. The momentum integral is given in the conventions of Fig. 7 by

\[ S = \int DkD\ell \frac{i}{(k^2 - a^2)} \frac{i}{((k - \ell + q)^2 - b^2)} \frac{i}{(\ell^2 - c^2)}. \]  

(26)

Combining the first two denominators using a Feynman parameters \( x \) and performing the \( k \) integration, then combining the remaining denominators using a second parameter \( y \) and integrating over \( \ell \) gives the expressions

\[ S = \frac{-iM_H^2 \xi^{2\epsilon} \Gamma(-1 + 2\epsilon)}{(4\pi)^4} \int_0^1 dx \ X^{-\epsilon} \int_0^1 dy \ y^{\epsilon-1} \left[ \hat{c}^2 (1-y) + \hat{\alpha}^2 y - \hat{s} y (1-y) \right]^{1-2\epsilon}, \]  

(27)

and

\[ \frac{\partial S}{\partial q^2} = \frac{-i\xi^{2\epsilon} \Gamma(2\epsilon)}{(4\pi)^4} \int_0^1 dx \ X^{-\epsilon} \int_0^1 dy \ y^{\epsilon} (1-y) \left[ \hat{c}^2 (1-y) + \hat{\alpha}^2 y - \hat{s} y (1-y) \right]^{-2\epsilon}, \]  

(28)

where \( \hat{\alpha}^2 = \alpha^2/M_H^2 \) with
\[ \alpha^2 = \frac{a^2}{x} + \frac{b^2}{1 - x}. \]  
\[ (29) \]

The four cases of interest here are

- \( S_0 \): \( \hat{a} = \hat{b} = \hat{c} = \hat{s} = 0 \);
- \( S_1 \): \( \hat{c} = \hat{s} = 1, \quad \hat{a} = \hat{b} = 0 \);
- \( S_2 \): \( \hat{a} = \hat{b} = 1, \quad \hat{c} = \hat{s} = 0 \);
- \( S_3 \): \( \hat{a} = \hat{b} = \hat{c} = \hat{s} = 1 \).

The \( S_0 \) case is curious because there is no internal scale in the graph: all the masses are zero and the integral is to be evaluated at \( q^2 = s = 0 \). For vanishing internal masses, \( S_0 \) is proportional to \( \hat{s}^{1-2\epsilon} \) multiplied by finite integrals, hence vanishes for \( \hat{s} \to 0 \). The case of \( \partial S_0 \) is more delicate since \( \partial S_0 \propto \hat{s}^{-2\epsilon} \) and there is a final infrared divergence unless \( \epsilon < 0 \). We will adopt this limit, with \( \epsilon \to 0^- \). Alternatively, to avoid this problem, we note that \( S_0 \) can also be rewritten in a form which contains a mass scale and avoids conflicts between the infrared and ultraviolet regularizations by using the identity

\[ \frac{i}{k^2} = \frac{(k^2 - m^2)}{(k^2 - m^2)} \frac{i}{k^2} \]
\[ = \frac{i}{(k^2 - m^2)} - \frac{m^2}{(k^2 - m^2)} \frac{i}{(k^2 - m^2)} \frac{i}{k^2}. \]  
\[ (30) \]

This substitution gives

\[ S_0 = S_1 |_{q^2=0} - \frac{m^2}{i} \int \mathcal{D}k \mathcal{D}\ell \frac{i}{k^2} \frac{i}{(k^2 - m^2)} \frac{i}{(k - \ell)^2} \frac{i}{(\ell + q)^2}, \]  
\[ (31) \]

with a similar expression for \( \partial S_0 \). In either case, the two contributions, calculated independently, are identical, and both \( S_0 \) and \( \partial S_0 \) vanish.

Since all lines in a Sunset graph are formally equivalent, a judicious choice can be made of which to choose as massive in the cases \( S_1 \) and \( S_2 \). The calculations simplify enormously with the choices above. For both \( S_1 \) and \( S_2 \), the \( x \) and \( y \) integrations then factor in Eqs. (27) and (28) and lead to beta functions.

The all-massive case \( S_3 \) is troublesome, because the last factor in Eq. (27) mixes singularities in the \( x \) and \( y \) integrations. One useful approach, the “\( \partial p \)” or “partial p” method,
makes use of integration by parts in the original momentum-space integral in Eq. (26) and in the modified integral over Feynman parameters which results to make the divergences explicit. It is then possible to expand the remaining integrand in powers of $\epsilon$. The calculation proceeds along the general lines sketched for this case by Ramond [30], but a complete calculation including finite parts is what is needed here. The details are given in [21]. $S_3$ can also be calculated directly using the Barnes’ representation technique discussed in Appendix B after a factor $(y/X)^{1-2\epsilon}$ is extracted from the last factor in Eq. (27). For $\partial S_3$, the original expression in Eq. (28) is sufficient, though the Barnes method is also simple.

2. Acorn

As is evident from Fig. 7, the Acorn diagram has the character of a self energy bubble with a renormalized coupling. The $A_Q$ diagram in Fig. 6 is a Scoop with a self-energy insertion. The two types of diagrams give Feynman integrals of the same form with the restriction $q = 0$ for $A_Q$. The common momentum integral is

$$A = \int DkD\ell \frac{i}{(k^2 - a^2)} \frac{i}{((k - \ell)^2 - b^2)} \frac{i}{(\ell^2 - c^2)} \frac{i}{((\ell + q)^2 - d^2)}. \quad (32)$$

Combining the propagators of the $Dk$ loop and integrating,

$$A = \frac{i^2(4\pi\mu^2)^\epsilon \Gamma(2 - \omega)}{(4\pi)^2} \int_0^1 \frac{dx}{X^{2-\omega}} e^{-i\pi\omega} \int D\ell \frac{i}{(\ell^2 - \alpha^2)^{2-\omega}} \frac{i}{(\ell^2 - c^2)} \frac{i}{((\ell + q)^2 - d^2)}. \quad (33)$$

where $\alpha^2$ is defined in Eq. (29). Introducing two more Feynman parameters and performing the $D\ell$ integration, we obtain the expression

$$A = -\frac{\xi^2 \epsilon \Gamma(2\epsilon)}{(4\pi)^4} \int_0^1 \frac{dx}{X^\epsilon} \int_0^1 dy y^\epsilon \int_0^1 dz z^{\epsilon-1}$$

$$\times \left[ c^2 y + d^2 (1 - y) - \tilde{s} y (1 - y) - (\hat{c}^2 - \hat{\alpha}^2) y z \right]^{-2\epsilon}. \quad (34)$$

The particular Acorn graphs which we need correspond to the following sets of parameters grouped according to the diagrams in Figs. 2 and 3:
\[ \mathcal{A}_4: \hat{a} = \hat{b} = \hat{c} = \hat{d} = \hat{s} = 1; \]
\[ \mathcal{A}_{2m}: \hat{c} = \hat{d} = \hat{s} = 1, \quad \hat{a} = \hat{b} = 0; \]
\[ \mathcal{A}_{1m}: \hat{a} = \hat{s} = 1, \quad \hat{b} = \hat{c} = \hat{d} = 0; \]
\[ \mathcal{A}_3: \hat{a} = \hat{b} = \hat{c} = 1, \quad \hat{d} = \hat{s} = 0; \]
\[ \mathcal{A}_{20}: \hat{b} = \hat{d} = 1, \quad \hat{a} = \hat{c} = \hat{s} = 0; \]
\[ \mathcal{A}_{10}: \hat{c} = 1, \quad \hat{a} = \hat{b} = \hat{d} = \hat{s} = 0; \]
\[ \mathcal{A}_{4Q}: \hat{a} = \hat{b} = \hat{c} = \hat{d} = 1, \quad \hat{s} = 0; \]
\[ \mathcal{A}_{2Q}: \hat{c} = \hat{d} = 1, \quad \hat{a} = \hat{b} = \hat{s} = 0; \]
\[ \mathcal{A}_{1Q}: \hat{a} = 1; \quad \hat{b} = \hat{c} = \hat{d} = \hat{s} = 0. \]

The integral for \( \mathcal{A}_{10} \) factors and gives a product of beta functions. The remaining integrals can be evaluated by writing the last factor in Eq. (34) as \( [\cdots] = [P + Qz] \) and integrating by parts with respect to \( z \) to eliminate the \( z^{\epsilon-1} \) singularity:

\[
\mathcal{A} = \frac{-\xi^2 \Gamma(2\epsilon)}{(4\pi)^4} \int_0^1 \frac{dx}{x^\epsilon} \int_0^1 dy y^\epsilon \left[ \frac{1}{\epsilon (P + Q)^{2\epsilon}} + 2 \int_0^1 dz \frac{z^\epsilon Q}{[P + Qz]^{1+2\epsilon}} \right]. \tag{35}
\]

We can now expand the factor \( z^\epsilon \) in the last integrand to \( O(\epsilon) \) and evaluate the remaining integral on \( z \), and find that

\[
\mathcal{A} = \frac{-\xi^2 \Gamma(2\epsilon)}{(4\pi)^4} \int_0^1 \frac{dx}{x^\epsilon} \int_0^1 dy y^\epsilon \left[ \frac{1}{\epsilon P^{2\epsilon}} + 2\epsilon Li_2 \left( -\frac{Q}{P} \right) + O(\epsilon^2) \right], \tag{36}
\]

where \( Li_2(z) \) is the dilogarithm function [28], and

\[
P = \hat{c}^2 y + \hat{d}^2 (1 - y) - \hat{s} y (1 - y), \quad Q = -y (\hat{c}^2 - \hat{a}^2). \tag{37}
\]

In most cases the dilogarithm needs to be integrated by parts with respect to \( y \). This creates a set of awkward but finite integrals.

Differentiating of the expression in Eq. (36) with respect to \( s \) gives an expression for \( \partial \mathcal{A} \) with much the same character,

\[
\partial \mathcal{A} = -\frac{\xi^2 \Gamma(1 + 2\epsilon)}{(4\pi)^4 M_H^2} \int_0^1 \frac{dx}{x^\epsilon} \int_0^1 dy y^{1+\epsilon} (1 - y) \left[ \frac{1}{\epsilon P^{1+2\epsilon}} - \frac{1}{P} \ln \left( \frac{P + Q}{P} \right) + O(\epsilon) \right]. \tag{38}
\]
The $A_Q$ graphs are Scoops with a particle exchange across the loop; $\partial A_Q$ therefore vanishes identically. For $A_4$, $P = 1 - y(1 - y)$ and, as for $B_2$, the $1/P^{2\epsilon}$ term in Eq. (36) must be expanded to $O(\epsilon^2)$, and leads to integrals related to dilogarithms. The same term appears in $A_{2m}$. In all other cases, the $1/P^{2\epsilon}$ term leads to a beta function when integrated, and the hardest part of the calculation is the extraction of the finite pieces.

We will also need $A_1m, \partial A_1m/\partial a^2, \partial A_1m/\partial b^2, \partial A_1m/\partial d^2, A_2m, \partial A_2m/\partial a^2$ and $\partial A_2m/\partial d^2$, all as functions of $s$, for our calculation of the Lemon graphs $L_1, L_{2m}$, and their derivatives. These integrals were all evaluated using the Barnes’ representation method. The results of the Acorn calculations are summarized in Appendix A.

3. Eye-in-the-Sky

The Eye-in-the-Sky diagrams in Fig. 7 are simply self-energy bubbles with self-energy bubble insertions. The most singular terms from these diagrams will cancel exactly against the mass counterterm and Scoop insertions. The generic momentum integral is

$$E = \int \mathcal{D}k \mathcal{D}\ell \left( \frac{i}{(k^2 - a^2)} \right) \left( \frac{i}{((k - \ell)^2 - b^2)} \right) \left( \frac{i}{(\ell^2 - c^2)} \right) \left( \frac{i}{((\ell + q)^2 - d^2)} \right).$$

and is simply a derivative of an Acorn integral, $E = i\partial A/\partial c^2$. Thus, from Eq. (34),

$$E = \frac{i\xi^2 \Gamma(1 + 2\epsilon)}{(4\pi)^4 M_H^2} \int_0^1 dx \frac{X^{-\epsilon}}{x} \int_0^1 dy \frac{y^{1+\epsilon}}{y} \int_0^1 dz \frac{z^{\epsilon-1}(1 - z)}{(1 - y)^{1-2\epsilon}}$$

$$\times \left[ c^2 y + d^2 (1 - y) - \hat{s} y (1 - y) - (\hat{c}^2 - \hat{\alpha}^2) y z \right]^{-1-2\epsilon}.$$ (40)

Alternatively,

$$E = \frac{i\xi^2 \Gamma(1 + 2\epsilon)}{(4\pi)^4 M_H^2} \int_0^1 dx \frac{X^{-\epsilon}}{x} \int_0^1 dy \frac{y^{1+\epsilon}}{y}$$

$$\times \left[ \frac{1}{\epsilon P^{1+2\epsilon} + \frac{1}{P} + \frac{1}{Q}} \ln \left( \frac{P + Q}{P} \right) + O(\epsilon) \right],$$ (41)

$$\partial E = \frac{i\xi^2 \Gamma(2 + 2\epsilon)}{(4\pi)^4 M_H^4} \int_0^1 dx \frac{X^{-\epsilon}}{x} \int_0^1 dy \frac{y^{2+\epsilon}(1 - y)}{y}$$

$$\times \left[ \frac{1 - \epsilon}{\epsilon P^{2+2\epsilon} + \frac{1}{P^2}} \ln \left( \frac{P + Q}{P} \right) + O(\epsilon) \right],$$ (42)
when these expressions are well-defined.

We will need the same six sets of parameters as appeared for the ordinary Acorn diagrams:

\[
\begin{align*}
\mathcal{E}_5 &= \partial_{c^2} \mathcal{A}_4 : \quad \hat{a} = \hat{b} = \hat{c} = \hat{d} = \hat{s} = 1; \\
\mathcal{E}_3 &= \partial_{c^2} \mathcal{A}_{2m} : \quad \hat{c} = \hat{d} = \hat{s} = 1, \quad \hat{a} = \hat{b} = 0; \\
\mathcal{E}_1 &= \partial_{c^2} \mathcal{A}_{1m} : \quad \hat{a} = \hat{s} = 1, \quad \hat{b} = \hat{c} = \hat{d} = 0; \\
\mathcal{E}_4 &= \partial_{c^2} \mathcal{A}_3 : \quad \hat{a} = \hat{b} = \hat{c} = 1, \quad \hat{d} = \hat{s} = 0; \\
\mathcal{E}_2^* &= \partial_{c^2} \mathcal{A}_{10} : \quad \hat{c} = 1, \quad \hat{a} = \hat{b} = \hat{d} = \hat{s} = 0; \\
\mathcal{E}_2 &= \partial_{c^2} \mathcal{A}_{20} : \quad \hat{b} = \hat{d} = 1, \quad \hat{a} = \hat{c} = \hat{s} = 0.
\end{align*}
\]

These graphs display characteristics similar to the parent Acorn graphs. Thus, \( \mathcal{E}_2^* = i \partial \mathcal{A}_{10} / \partial c^2 \) was calculated using the original Feynman parametrization in Eq. (40). \( \mathcal{E}_5 \) and \( \mathcal{E}_3 \) were calculated by expanding the factor \( P^{-2e} \) terms in powers of \( \epsilon \). No expansion was necessary for \( \mathcal{E}_2^* \) or \( \mathcal{E}_4 \). \( \mathcal{E}_1 \) was calculated as a function of \( \hat{s} \) and for \( \hat{s} = 0 \) using the Barnes’ technique.

\( \mathcal{E}_2 = i \partial \mathcal{A}_{20} \) was calculated by a different route. The problem here is that the \( \ln P/P \) term is \( \ln(1 - y)/(1 - y) \), i.e., divergent. The original momentum integral was rewritten using partial fractions, as

\[
\frac{i}{((\ell + q)^2 - M_H^2)} \frac{i}{\ell^2} = \frac{i}{M_H^2} \left( \frac{i}{(\ell + q)^2 - M_H^2} - \frac{i}{\ell^2} \right) + \frac{i^2}{M_H^2} \frac{2\ell \cdot q + q^2}{((\ell + q)^2 - M_H^2) \ell^2}. \tag{43}
\]

The last term is proportional to \( q^2 \) after integration, so vanishes in the limit \( q^2 = 0 \) needed for \( \mathcal{E}_2 \). The first two terms give Acorn integrals, and \( \mathcal{E}_2 \) therefore splits into a difference of known terms,

\[
\mathcal{E}_2 = \frac{i}{M_H^2} (\mathcal{A}_{20} - \mathcal{A}_{1Q}). \tag{44}
\]

\( \partial \mathcal{E}_2 \) was calculated using the Barnes method.

4. Lemon

The Lemon topology shown in Fig. 7 was by far the most troublesome, and three very different techniques were used to handle these diagrams [31]. In addition, four separate
numbers were ultimately extracted by numerical evaluation of two-dimensional integrals using the program VEGAS \[32\]. The momentum integral is

\[
\mathcal{L} = \int \mathcal{D}k \mathcal{D}\ell \frac{i}{(k^2 - a^2)} \frac{i}{((k + q)^2 - b^2)} \frac{i}{((k - \ell)^2 - c^2)} \frac{i}{(\ell^2 - d^2)} \frac{i}{((\ell + q)^2 - e^2)}. \tag{45}
\]

This integral is ultraviolet convergent, but one case, \(\partial \mathcal{L}_{20}\) is infrared divergent. The parameter sets needed are:

\[
\mathcal{L}_5 : \quad a^2 = b^2 = c^2 = d^2 = e^2 = s = M_H^2; \\
\mathcal{L}_{2m} : \quad a^2 = b^2 = s = M_H^2, \quad c^2 = d^2 = e^2 = 0; \\
\mathcal{L}_1 : \quad c^2 = s = M_H^2, \quad a^2 = b^2 = d^2 = e^2 = 0; \\
\mathcal{L}_{20} : \quad a^2 = e^2 = M_H^2, \quad b^2 = c^2 = d^2 = s = 0. \\
\mathcal{L}_3 : \quad a^2 = c^2 = d^2 = M_H^2, \quad b^2 = e^2 = s = 0.
\]

The momentum integral can be performed formally using Feynman parameters. Combining the \(a\) and \(b\) propagators first, adding in the \(c\) propagator and integrating over \(k\), then combining the result with the remaining \(d\) and \(e\) propagators in one step and integrating over \(\ell\) gives

\[
\mathcal{L} = \frac{i\xi^2 \Gamma(1 + 2\epsilon)}{(4\pi)^2 M_H^2} \int_0^1 dx x X^\epsilon \int_0^1 dy \int_0^1 du \int_0^1 dv \int_0^1 dw \, u^\epsilon \delta(1 - u - v - w) (M - \hat{s}S)^{-1-2\epsilon} \tag{46}
\]

and

\[
\partial \mathcal{L} = \frac{i\xi^2 \Gamma(2 + 2\epsilon)}{(4\pi M_H^4)} \int_0^1 dx x X^\epsilon \int_0^1 dy \int_0^1 du \int_0^1 dv \int_0^1 dw \, u^\epsilon \delta(1 - u - v - w) \frac{S}{(M - \hat{s}S)^{2+2\epsilon}}, \tag{47}
\]

where

\[
M = u \left[ \hat{a}x(1 - y) + \hat{b}xy + \hat{c}(1 - x) \right] + \hat{d}vX + \hat{e}wX, \\
S = ux^2y(1 - y) + X(uy + w)(1 - uy - w). \tag{48}
\]

The Feynman parameter integrals are generally too complicated for direct integration except for the \(\Pi_w\) cases where \(s = 0\) and just mass terms survive. In particular, \(\mathcal{L}_{20}, \partial \mathcal{L}_{20}\) and \(\mathcal{L}_3\) can all be calculated analytically. \(\partial \mathcal{L}_3\) was calculated partially analytically and partially
by a two-dimensional numerical integration. However, the Feynman parameter expressions are not generally well-suited to numerical integration. For example, the expression for $\mathcal{L}_3$ had to be reduced analytically to a two-dimensional integral before the result obtained by numerical integration agreed with the analytic integration. This is presumably because VEGAS could not adapt to the complicated singularity structure.

A more useful technique for $\mathcal{L}_5$, the all-massive case, was based on the hyperspherical formalism [33]. In this technique, a Euclidean propagator is expanded in Gegenbauer polynomials $C_n^1$ (or equivalently, Chebyshev polynomials of the first kind $U_n$):

$$\frac{1}{(k - \ell)^2 + m^2} = \frac{Z_{KL}}{KL} \sum_{n=0}^{\infty} Z_{KL}^n C_n^1(\hat{K} \cdot \hat{L}),$$

where $K$, $L$ and $\hat{K}$, $\hat{L}$ are the magnitudes and four-dimensional unit vectors of the Euclidean momenta $k$, $\ell$ and

$$Z_{KL} = \frac{K^2 + L^2 + m^2 - \sqrt{(K^2 + L^2 + m^2)^2 - 4K^2L^2}}{2KL}, \quad m \neq 0,$$

$$= \min \left( \frac{K}{L}, \frac{L}{K} \right) \quad \text{for} \ m^2 = 0. \quad (50)$$

The angular integrals are easily carried out for planar graphs using the orthogonality relations

$$\int C_n^1(\hat{a} \cdot \hat{b}) C_m^1(\hat{b} \cdot \hat{c}) \frac{d\Omega_b}{2\pi^2} = \frac{\delta_{mn}}{n+1} C_n^1(\hat{a} \cdot \hat{c}).$$

Only propagators with angular dependence need be expanded, i.e., for the Lemon, the propagators of the $b$, $c$ and $e$ lines (see Fig. 7). The result for $\mathcal{L}_5$ after performing the angular integrations, resumming the series, and making the analytic continuation back to Minkowski space, is

$$\mathcal{L}_5 = -\frac{i}{Q^2} \int_0^{\infty} \frac{dK^2}{(4\pi)^2} \int_0^{\infty} \frac{dL^2}{(4\pi)^2} \ln(1 - Z_{KL} Z_{KQ} Z_{LQ}) \left| \frac{\ln(1 - Z_{KL} Z_{KQ} Z_{LQ})}{Q^2 = -M_H^2} \right|.$$ 

There is no known change of variables that will reduce this case to an analytically integrable form [33]. Equation (52) was integrated numerically instead. The advantage of the hyperspherical formalism is that the integral has been reduced from four to two dimensions, and the integrand is a much smoother function.
The remaining integrals $L_1, \partial L_1, L_{2m}, \partial L_{2m},$ and $\partial L_5$ are all finite, but numerical integration of either the Feynman parameter or hyperspherical expressions is complicated by singularities or, in the hyperspherical case, a complicated complex analytic structure. For these a third technique was used, Kotikov’s method of differential equations \cite{34}. Kotikov uses the method to find a first order differential equation in the particle mass with, hopefully, “easily calculable” inhomogeneous terms. In the present case, we obtain a differential equation in $\hat{s}$ (the only free variable is the ratio $\hat{s} = s/M_H^2$) by taking the derivative of the momentum integral in Eq. (46) with respect to the incoming momentum. The equation can be simplified by using the six identities \cite{27}

$$\int dk \int d\ell \frac{\partial}{\partial h_{\mu}} [p_{\mu}(\cdots)] = 0$$

(53)

where $(\cdots)$ is a product of propagators, and $p_{\mu}$ is any of the variables $k_{\mu}, \ell_{\mu},$ and $q_{\mu},$ and $h_{\mu}$ is either $k_{\mu}$ or $\ell_{\mu}$.

The result is an inhomogeneous first order differential equation with the inhomogeneous term expressible in terms of Bubbles, Triangles, Acorns, and their derivatives. $L$ can then be found by direct integration. This is an interesting technique, and the remaining cases were handled this way as discussed in Appendix C. Our exposition there is less general than Kotikov’s \cite{34} since we treat only specific diagrams.

D. The self-energy terms and renormalization constants

We will write the expansions for the self-energy terms and wave function renormalization constants in powers of the coupling $\lambda$ as

$$\Pi^0 = \Pi^{(1)} + \Pi^{(2)} + \cdots$$

$$Z = 1 + Z^{(1)} + Z^{(2)} + \cdots$$

(54)

The one-loop self energies are given by the first two terms in Eqs. (11) and (12) with the wave function and coupling renormalizations ignored,
\[ \Pi^{(1)}_w(0) = -\frac{3\lambda}{16\pi^2} M_H^2 \xi^\epsilon \left( \frac{1}{\epsilon} + (1 - \gamma) + \frac{\epsilon}{2} (2 - 2\gamma + \gamma^2 + \zeta(2)) + O(\epsilon^2) \right) \]  \hspace{1cm} (55)

and

\[ \Pi^{(1)}_H(M_H^2) = -\frac{3\lambda}{16\pi^2} M_H^2 \xi^\epsilon \left[ \frac{5}{\epsilon} + (9 - \pi\sqrt{3} - 5\gamma) \right. \\
+ \epsilon \left( \frac{5}{2} \gamma^2 - 9\gamma + 17 - \frac{\pi^2}{4} - 2\pi\sqrt{3}(1 - \frac{1}{2}\gamma) + \pi\sqrt{3}\ln 3 - 4\sqrt{3}C \right) \\
+ i\pi + (2 - \gamma)i\pi\epsilon + O(\epsilon^2) \].  \hspace{1cm} (56)

Here \( \gamma = 0.5772 \cdots \) is Euler's constant, \( \zeta(2) = \pi^2/6 \), and \( C = C\ell \left( \frac{\pi}{3} \right) = 1.01494 \cdots \). We have not expanded the factors \( \xi^\epsilon \) for simplicity. This will also make the disappearance of the arbitrary scale \( \mu \) manifest in our final results: the complete cancellation of the poles in \( \epsilon \) will allow us to take the limit \( \epsilon \to 0 \), \( \xi^\epsilon = (4\pi\mu^2/M_H^2)^\epsilon \to 1 \) with no pieces left over. The one-loop contributions to the wave function renormalization constants are similarly given by

\[ Z^{(1)}_w = -\frac{\lambda}{16\pi^2} \xi^\epsilon \left[ 1 + \epsilon \left( \frac{3}{2} - \gamma \right) + O(\epsilon^2) \right], \]  \hspace{1cm} (57)

and

\[ Z^{(1)}_H = \frac{\lambda}{16\pi^2} \xi^\epsilon \left[ 12 - 2\pi\sqrt{3} + \epsilon (24 - 12\gamma - 3\pi\sqrt{3}(1 - \frac{2}{3}\gamma)) + 2\pi\sqrt{3}\ln 3 - 8\sqrt{3}C + O(\epsilon^2) \right]. \]  \hspace{1cm} (58)

The leading terms in these expressions agree with the results of previous one-loop calculations \[\{11,13,5,4,7,9,35\}\].

The real parts of the self energies at \( O(\lambda^2) \) involve the corrections to the one-loop self energies from the one-loop wave function and coupling renormalizations as well as the new two-loop contributions. The algebraic programming system **REDUCE** was used to manipulate the expressions in Eqs. \(11\) and \(12\) and obtain the expansions needed. The results, evaluated on mass shell, are

\[ \Pi^{(2)}_w(0) = \frac{\lambda^2}{(16\pi^2)^2} \xi^{2\epsilon} M_H^4 \left( \frac{-45}{\epsilon^2} + \frac{6}{\epsilon} (-22 + 15\gamma + 3\sqrt{3}\pi) \right. \\
- 72\sqrt{5} \ln \left( \frac{\sqrt{5} + 1}{2} \right) - 24 \ln^2 \left( \frac{\sqrt{5} + 1}{2} \right) + 32 \ln 2 - 90\gamma^2 \]
+264\gamma + 40\zeta(2) - \frac{753}{2} + 66\sqrt{3}\mathcal{C} - 36\sqrt{3}\pi\gamma + 45\sqrt{3}\pi -18\pi\sqrt{3}\ln 3 + O(\epsilon) \right).

(59)

and

\text{Re} \Pi^{(2)}_H(M^2_H) = \frac{\lambda^2}{(16\pi^2)^2}\xi^{2\epsilon}M^2_H\left(\frac{-189}{\epsilon^2} + \frac{9}{\epsilon}(-71 + 42\gamma + 10\sqrt{3}\pi)
\right.
-144\ln 2 - 45\pi^2 - 378\gamma^2
-36K_2 - 162K_5 + 1278\gamma - 63\zeta(2) + 90\zeta(3)
-1524 + 126\sqrt{3}\mathcal{C} - 180\sqrt{3}\pi\gamma + 408\sqrt{3}\pi
\left. -81\xi(2)\ln 2 - 90\pi\sqrt{3}\ln 3 + O(\epsilon) \right).

(60)

The constants $K_i$ are from numerical integrations, and are given in Appendix A. These numbers were calculated to accuracies which vary from a part in $10^4$ to a part in $10^6$, more than sufficient for our purposes despite the large topological weights and the cancellations which occur.

The $O(\lambda^2)$ contributions to the wave function renormalization constants obtained from the derivatives of Eqs. (11) and (12) using REDUCE are

\begin{align*}
Z^{(2)}_w &= \frac{\lambda^2}{(16\pi^2)^2}\xi^{2\epsilon}\left(\frac{-3}{\epsilon} + 42\sqrt{5}\ln\left(\frac{\sqrt{5} + 1}{2}\right) - 96\ln^2\left(\frac{\sqrt{5} + 1}{2}\right)
+ \frac{400}{9}\ln 2 + 6\gamma - \frac{38}{3}\zeta(2) - 12K_3 - \frac{1525}{54} + 3\sqrt{3}\pi + O(\epsilon) \right) \quad (61)
\end{align*}

and

\begin{align*}
Z^{(2)}_H &= \frac{\lambda^2}{(16\pi^2)^2}\xi^{2\epsilon}\left(\frac{-3}{\epsilon} + 144\ln 2 + 42\pi^2 + 81\zeta(2)\ln 2
+ 36K_2 + 162K_5 + 6\gamma + 33\zeta(2) - 90\zeta(3)
-16\sqrt{3}\pi\ln 3 + 334\sqrt{3}\mathcal{C} + \frac{273}{2} - 292\sqrt{3}\pi + O(\epsilon) \right) .
\end{align*}

(62)

The leading $1/\epsilon$ divergences in $Z^{(2)}_w$ and $Z^{(2)}_H$ are identical because of the original SO(4) symmetry of the theory which is preserved in the interactions at large momenta.

Finally, the bare coupling $\lambda_0$ can be identified as the coefficient of $-\frac{1}{4}(w_0^2 + H_0^2)^2$ in Eq. (4).
\[ \lambda_0 = \frac{\lambda}{Z_w} \left( 1 - \frac{\text{Re} \pi^0_H (M^2_H) - \Pi^0_w (0)}{M^2_H} \right). \] (63)

Substitution of the results above using REDUCE gives

\[ \lambda_0 = \lambda + \frac{\lambda^2 \xi^e}{16 \pi^2} \left( \frac{12}{\epsilon} + 25 - 12 \gamma - 3 \sqrt{3} \pi + \epsilon \left[ 3 \pi \sqrt{3} \ln 3 - 12 \sqrt{3} \mathcal{C} - 6 \pi \sqrt{3} (1 - \frac{1}{2} \gamma) \right] - \pi^2 + 6 \gamma^2 - 25 \gamma + \frac{99}{2} \right) + O(\epsilon^2) + \frac{\lambda^3 \xi^e}{(16 \pi^2)^2} \left( \frac{144}{\epsilon^2} + \frac{18}{\epsilon} (29 - 16 \gamma - 4 \sqrt{3} \pi) \right) + \frac{32906}{27} + 12 K_3 + 162 K_5 + 36 K_2 - 90 \zeta(3) \]

\[ + \frac{185}{3} \zeta(2) - 1044 \gamma + 288 \gamma^2 + 54 \pi^2 + \frac{1184}{9} \ln 2 \]

\[ + 72 \ln^2 \left( \frac{\sqrt{5} + 1}{2} \right) - 114 \sqrt{5} \ln \left( \frac{\sqrt{5} + 1}{2} \right) \]

\[ + 81 \zeta(2) \ln 2 - 369 \pi \sqrt{3} + 144 \pi \sqrt{3} \gamma - 60 \sqrt{3} \mathcal{C} \]

\[ + 72 \pi \sqrt{3} \ln 3 + O(\epsilon) \] + O(\lambda^4). \] (64)

IV. THE SCATTERING AMPLITUDES

The two-particle scattering diagrams generated by the quartic interactions in Eq. (7) are shown in Fig. 4. There is just one generic one-loop diagram with an associated amplitude \( B_S \), and two two-loop diagrams with amplitudes \( D_S \) and \( A_S \), the last named after the similar Acorn self-energy diagram. The small number of diagrams is the result of the suppression of the dimension-three trilinear couplings in the limit \( \sqrt{s} \gg M^2_H \), as was noted earlier. The functions \( B_S, D_S, \) and \( A_S \) are given in Appendix A.

We note that \( M_H \) can be neglected on internal lines in the high energy limit, and that the scattering amplitudes consequently have the SO(4) symmetry of the bare quartic couplings. This symmetry is reflected in the structure of the \( 4 \times 4 \) matrix \( \mathcal{F} \) of unrenormalized transition amplitudes. All of these amplitudes can be expressed in terms of a single function \( A(s, t, u) \), where

\[ A(s, t, u) = -2 \lambda_0 - i (-i \lambda_0)^2 A^{(1)} - i (-i \lambda_0)^3 A^{(2)}, \] \( A^{(1)}(s, t, u) = 16 B_S(s) + 4 B_S(t) + 4 B_S(u), \)

\[ A^{(2)}(s, t, u) = \frac{1}{4} \left( 16 B_S(s) - 16 B_S(t) + 16 B_S(u) - 32 B_S(s) + 8 B_S(t) - 8 B_S(u) + 8 B_S(s) - 16 B_S(t) + 16 B_S(u) \right). \] (65)
\[
A^{(2)}(s, t, u) = 104D_S(s) + 8D_S(t) + 8D_S(u) + 176A_S(s) + 80A_S(t) + 80A_S(u),
\]
(67)

and \( \lambda_0 \) is the bare coupling defined in Eqs. (63) and (64). We will write the two-body channels in the order \( w^+w^-, \, zz, \, HH, \) and \( zH \). \( \mathbf{F} \) then has the structure

\[
\mathbf{F} = \begin{pmatrix}
A(s) + A(t) & A(s) & A(s) & 0 \\
A(s) & A(s) + A(t) + A(u) & A(s) & 0 \\
A(s) & A(s) & A(s) + A(t) + A(u) & 0 \\
0 & 0 & 0 & A(t)
\end{pmatrix},
\]
(68)

where we have indicated only the first variable in \( A \) since this function is unchanged by an interchange of the remaining two variables.

As noted after Eq. (62), the leading terms in \( 1/\epsilon \) in the renormalization constants \( Z_W \) and \( Z_H \) are identical because of the SO(4) symmetry of the initial theory. We will therefore make an intermediate renormalization which preserves the SO(4) structure by multiplying \( A \) by the field renormalization factor \((Z_W^{1/2})^4\) and replacing \( \lambda_0 \) in Eq. (65) by the expression in Eq. (63). This gives

\[
A_R(s, t, u) = -2\lambda \left( 1 - \frac{\text{Re}\Pi_0^H(M_H^2) - \Pi_0^w(0)}{M_H^2} \right) Z_w \\
+ i\lambda^2 A^{(1)} \left( 1 - \frac{\text{Re}\Pi_1^H(M_H^2) - \Pi_1^w(0)}{M_H^2} \right)^2 + \lambda^3 A^{(2)},
\]
(69)

where each term is to be expanded to \( O(\lambda^3) \). \( A_R \) is just the physical amplitude needed to describe the scattering in the \( w^+w^- \), \( zz \) sector. Channels involving external Higgs bosons are multiplied by a finite renormalization factor \((Z_H/Z_w)^{1/2}\) for each external Higgs boson.

After a number of nontrivial cancellations, \( A_R \) reduces in the limit \( \epsilon \to 0 \) to a finite expression independent of the arbitrary scale \( \mu \) which appeared at intermediate steps of the dimensional regularization,

\[
A_R(s, t, u) = -2\lambda + \frac{\lambda^2}{16\pi^2} \left( -16\ln(-\hat{s}) - 4\ln(-\hat{t}) - 4\ln(-\hat{u}) + 2 + 6\sqrt{3}\pi \right)
\]
\[
+ \frac{\lambda^3}{(16\pi^2)^2} \begin{pmatrix}
-192 \ln^2(-\hat{s}) + 176 \ln(-\hat{s}) + 96\sqrt{3}\pi \ln(-\hat{s}) \\
-48 \ln^2(-\hat{t}) + 80 \ln(-\hat{t}) + 24\sqrt{3}\pi \ln(-\hat{t}) \\
-48 \ln^2(-\hat{u}) + 80 \ln(-\hat{u}) + 24\sqrt{3}\pi \ln(-\hat{u}) \\
+60\sqrt{5} \ln \frac{\sqrt{5}+1}{2} - 456\sqrt{3}\pi C + 138\sqrt{3}\pi \\
+240 \ln^2 \frac{\sqrt{5}+1}{2} - \frac{3968}{9} \ln 2 - 180\pi^2 \\
-72K_2 - \frac{794}{3} \zeta(2) + 180\zeta(3) \\
-324K_5 + 24K_3 + \frac{3388}{27} - 162\zeta(2) \ln 2
\end{pmatrix}
\] (70)

where \(-\hat{s} = e^{-i\pi} \hat{s}\), and the quantities \(-\hat{t}\) and \(-\hat{u}\) are real and positive in the physical region.

Substitution of \(A_R\) for \(A\) in Eq. (69) gives the partially renormalized transition matrix \(\mathcal{F}_R\); the fully renormalized matrix of (Feynman) transition amplitudes is

\[\mathcal{M} = Z\mathcal{F}_R Z\] (71)

where \(Z\) is a finite diagonal matrix of ratios of renormalization constants,

\[Z = \text{diag}(1, 1, Z_H/Z_w, Z_{H}^{1/2}/Z_{w}^{1/2})\] (72)

and the products in \(\mathcal{M}\) are to be expanded to \(O(\lambda^3)\). The SO(4) symmetry of \(\mathcal{F}_R\) is lost in \(\mathcal{M}\).

V. COMMENTS

The principal results of this paper are the expressions we have obtained for the renormalization constants \(Z_w\) and \(Z_H\), the self-energy functions \(\Pi^0_w(0)\) and \(\text{Re}\Pi^0_H(M_H^2)\), the bare coupling \(\lambda_0\), and the renormalized two-body scattering amplitude \(A_R(s, t, u)\) at two loops.

We will use these results in a following paper to study limits on the range of validity of low-order perturbation theory in the Higgs sector of the standard model.

More generally, the quantities connected with the on-mass-shell renormalization of the theory can be used in other calculations of \(W_L^+ Z_L\) scattering based on the use of the equivalence theorem, for example, in possible calculations to extend the analysis of the
scattering in the “low-energy” limit $M_w \ll \sqrt{s} \ll M_H$ to two loops. This extension would be of considerable interest with respect to studies of the energy at which the effects of a very massive Higgs boson would become evident in future experiments.

We note finally that the new Barnes’-type method we have introduced for factoring and evaluating integrals over Feynman parameters has a wide range of applicability as illustrated in the calculations above and in Appendix B. The results collected in Appendix A and Appendix D should also be useful in future calculations.

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APPENDIX A: RESULTS FOR THE SELF-ENERGY GRAPHS

This appendix is a listing of the results of the Feynman graph calculations. Some intermediate integrals used in the differential equation method are also listed. Since only the real parts of the self-energy graphs were needed for the counterterms, strict attention was only paid to the real parts of those integrals when expansions were made.

A factor of $\xi^\epsilon/16\pi^2$ should be included for each loop in a graph. Factors of $M_H^2$ should be included so that, with the couplings given in Eqs. (11) and (12), the self-energy graphs have dimensions of mass squared. That is, $Q$, $S \propto M_H^2$; $B$, $A \propto 1$; and $T$, $E$, $L \propto M_H^2$. The derivatives of these functions will have one power of $M_H^2$ less. Note that the graphs with a $Q$ subscript have no dependence on the external momentum, so $\partial(\text{graph}_Q) \equiv 0$.

Expressions which involve explicit expansions in powers of $\epsilon$ have been evaluated using expansions in $\epsilon$ in intermediate steps. These expressions are not correct beyond the highest
power of \( \epsilon \) indicated. Terms involving gamma or beta functions have been kept in the forms given for compactness, and are to be expanded in powers of \( \epsilon \) when used. These terms can frequently be rearranged, and may be replaced in some cases by quite different expressions which are equivalent to the relevant order in \( \epsilon \) if an intermediate expansion is done differently; only the final expanded expression is meaningful.

In the following, \( B(a, b) \) is the beta function,

\[
B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)},
\]

\( \text{(A1)} \)

\( \text{Li}_n(x) \) is the polylogarithm \([28]\),

\[
\text{Li}_n(x) = \int_0^x \text{Li}_{n-1}(t) \frac{dt}{t}, \tag{A2}
\]

\[
\text{Li}_2(x) = -\int_0^x \ln(1 - t) \frac{dt}{t} = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \tag{A3}
\]

and \( \zeta(n) \) is the Riemann zeta function.

Constants:

\[
\gamma = 0.57721566 \ldots
\]

\[
C = \text{Cl}(\pi/3) = 1.014942 \ldots
\]

\[
K_2 = -0.86518
\]

\[
K'_2 = 0.548311
\]

\[
K_3 = -0.1066639
\]

\[
K_5 = 0.9236306
\]

Scattering graphs:

\[
\mathcal{B}_S = -i(-\hat{s})^{-\epsilon} \Gamma(\epsilon) B(1 - \epsilon, 1 - \epsilon), \tag{A4}
\]

\[
\mathcal{D}_S = (\mathcal{B}_S)^2, \tag{A5}
\]

\[
\mathcal{A}_S = -(\hat{s})^{-2\epsilon} \Gamma(2\epsilon) B(2 - 4\epsilon, \epsilon) B(1 - 2\epsilon, 1 - 2\epsilon) B(1 - \epsilon, 1 - \epsilon). \tag{A6}
\]

Scoops:
\[ Q = - \frac{\Gamma(\epsilon)}{1 - \epsilon}, \tag{A7} \]
\[ B_Q = -i \Gamma(\epsilon). \tag{A8} \]

**Bubbles:**

\[ B_0(\hat{s}) = -i(-\hat{s})^{-\epsilon} \Gamma(\epsilon) B(1 - \epsilon, 1 - \epsilon), \tag{A9} \]
\[ B_0 = -ie^{i\pi\epsilon} \Gamma(\epsilon) B(1 - \epsilon, 1 - \epsilon) \tag{A10} \]
\[ \partial B_0 = -\epsilon B_0 \tag{A11} \]
\[ B_1(\hat{s}) = -i \Gamma(\epsilon) \left[ 1 + \epsilon \left( 2 + \frac{1}{\hat{s}}(1 - \hat{s}) \ln(1 - \hat{s}) \right) + O(\epsilon^2) \right] \tag{A12} \]
\[ \partial B_1(s) = -i \Gamma(1 + \epsilon) \left( \frac{1}{\hat{s}^2} \ln(1 - \hat{s}) - \frac{1}{\hat{s}} + O(\epsilon) \right) \tag{A13} \]
\[ B_1 = -i \frac{\Gamma(\epsilon)}{1 - \epsilon}, \tag{A14} \]
\[ \partial B_1 = -i \frac{\Gamma(1 + \epsilon)}{(1 - \epsilon)(2 - \epsilon)}, \tag{A15} \]
\[ B_2(\hat{s}) = -i \Gamma(\epsilon) \left[ 1 + 2\epsilon - \frac{4 - \hat{s}}{\sqrt{4\hat{s} - \hat{s}^2}} \arctan \frac{\hat{s}}{\sqrt{4\hat{s} - \hat{s}^2}} + 2\epsilon \right. \]
\[ + \left. \frac{1}{2} \epsilon^2 \int_0^1 dx \ln^2 \left[ 1 - \hat{s}x(1 - x) \right] + O(\epsilon^3) \right], \tag{A16} \]
\[ B_2 = -i \Gamma(\epsilon) \left[ 1 + \epsilon \left( 2 - \frac{\pi}{\sqrt{3}} \right) + \epsilon^2 \left( 4 - \frac{2\pi}{\sqrt{3}} + \frac{\pi \ln 3}{\sqrt{3}} - \frac{4C}{\sqrt{3}} \right) + O(\epsilon^3) \right], \tag{A17} \]
\[ \partial B_2 = i \Gamma(1 + \epsilon) \left[ 1 - \frac{2\pi}{3\sqrt{3}} + \epsilon \left( 2 - \frac{\pi}{\sqrt{3}} + \frac{2\pi \ln 3}{3\sqrt{3}} - \frac{8C}{3\sqrt{3}} \right) + O(\epsilon^2) \right]. \tag{A18} \]

**Triangles:**

\[ T_0(\hat{s}) = -(-\hat{s})^{-1-\epsilon} \Gamma(1 + \epsilon) B(1 - \epsilon, -\epsilon), \tag{A19} \]
\[ T_0 = e^{i\pi\epsilon} \Gamma(1 + \epsilon) B(1 - \epsilon, -\epsilon) \tag{A20} \]
\[ \partial T_0 = -e^{i\pi\epsilon} \Gamma(2 + \epsilon) B(-\epsilon, 1 - \epsilon) \tag{A21} \]
\[ T_1 = \frac{\Gamma(\epsilon)}{1 - \epsilon}, \tag{A22} \]
\[ \partial T_1 = -\Gamma(2 + \epsilon) B(3, -\epsilon), \tag{A23} \]
\[ T_2 = -\frac{\Gamma(1 + \epsilon)}{1 - \epsilon}, \tag{A24} \]
\[ \partial T_2 = -\frac{\Gamma(2 + \epsilon)}{(1 - \epsilon)(2 - \epsilon)}. \tag{A25} \]
\[ \mathcal{T}_3(\hat{s}) = -\Gamma(1 + \epsilon) \left( \frac{2}{\sqrt{4\hat{s} - s^2}} \arctan \frac{\hat{s}}{\sqrt{4\hat{s} - s^2}} + \epsilon \frac{1}{1 - \hat{s}x(1 - x)} \right) + O(\epsilon^2), \] (A26)

\[ \mathcal{T}_3 = -\frac{\Gamma(1 + \epsilon)}{3\sqrt{3}} \left[ \pi - \epsilon(\pi \ln 3 - 4\mathcal{C}) + O(\epsilon^2) \right], \] (A27)

\[ \partial \mathcal{T}_3 = -\frac{\Gamma(2 + \epsilon)}{3} \left[ 1 - \frac{\pi}{3\sqrt{3}} + \epsilon \left( 1 - \frac{14\pi}{3\sqrt{3}} - \frac{\pi \ln 3}{3\sqrt{3}} + \frac{4\mathcal{C}}{3\sqrt{3}} \right) + O(\epsilon^2) \right]. \] (A28)

**Sunset:**

\[ S_0 = \partial S_0 = 0, \] (A29)

\[ S_1 = 2\epsilon \frac{\Gamma(2\epsilon)}{\epsilon} B(1 - \epsilon, 1 - \epsilon) B(1 + \epsilon, 2 - 4\epsilon), \] (A30)

\[ \partial S_1 = -\frac{\epsilon}{2} S_1, \] (A31)

\[ S_2 = i \frac{\left[ \Gamma(\epsilon) \right]^2}{(1 - 2\epsilon)(1 - \epsilon)}, \] (A32)

\[ \partial S_2 = -\frac{\Gamma(2\epsilon) B(1 + \epsilon, 1 + \epsilon)}{(2 - \epsilon)(1 - \epsilon)}, \] (A33)

\[ S_3 = i \frac{\left[ \Gamma(\epsilon) \right]^2}{1 - \epsilon} \left( \frac{1}{1 - 2\epsilon} + 1 - \frac{1}{(1 + 2\epsilon)(2 - \epsilon)} \right) + \frac{3}{4} i + O(\epsilon), \] (A34)

\[ \partial S_3 = -\frac{1}{4\epsilon} \left[ \Gamma(1 + \epsilon) \right]^2 + i \frac{3}{8} + O(\epsilon), \] (A35)

**Acorns:**

\[ A_{10} = -\Gamma(2\epsilon) B(\epsilon, 1 - 2\epsilon) B(1 - \epsilon, 1 - \epsilon), \] (A36)

\[ \partial A_{10} = \frac{\epsilon}{2 - \epsilon} A_{10}, \] (A37)

\[ A_{1m} = \frac{\Gamma(\epsilon) \Gamma(2\epsilon) \Gamma(1 - \epsilon)}{1 - \epsilon} - \frac{\left[ \Gamma(\epsilon) \Gamma(1 - \epsilon) \right]^2}{(1 - \epsilon) \Gamma(2 - 2\epsilon)} e^{i\pi\epsilon} + 2\zeta(2) - 3 + O(\epsilon), \] (A38)

\[ \partial A_{1m} = \frac{\Gamma(\epsilon) \Gamma(1 + \epsilon)}{1 - \epsilon} B(1 - \epsilon, 1 - \epsilon) + 2 - \zeta(2) + O(\epsilon), \] (A39)

\[ A_{20} = -\frac{\epsilon}{\epsilon} B(1 - \epsilon, 1 - \epsilon) B(1 + \epsilon, 1 - 2\epsilon) + 2\zeta(2) + O(\epsilon), \] (A40)

\[ \partial A_{20} = -\frac{\Gamma(1 + 2\epsilon) B(1 - \epsilon, 1 - \epsilon)}{(2 + \epsilon)\epsilon} - 2 + \zeta(2) + O(\epsilon), \] (A41)

\[ A_{2m} = A_4 - 2\zeta(2) + O(\epsilon), \] (A42)

\[ \partial A_{2m} = \partial A_4 + \frac{1}{3} \zeta(2) - \frac{2\mathcal{C}}{\sqrt{3}} + O(\epsilon), \] (A43)
\[ \mathcal{A}_3 = -\frac{\Gamma(2\epsilon)B(1-\epsilon,1-\epsilon)}{(1-\epsilon)\epsilon} + 2\sqrt{3}C + O(\epsilon), \] (A44)

\[ \partial \mathcal{A}_3 = -\frac{\Gamma(1+2\epsilon)B(1-\epsilon,1-\epsilon)}{(2-\epsilon)(1-\epsilon)\epsilon} + 1 + O(\epsilon), \] (A45)

\[ \mathcal{A}_4 = -\Gamma(2\epsilon) \left[ \frac{B(1-\epsilon,1-\epsilon)}{(1+\epsilon)\epsilon} + 2B(1-\epsilon,1-\epsilon) \left( 2 - \frac{\pi}{\sqrt{3}} \right) \right] 
- \left( 4 - \frac{2\pi}{\sqrt{3}} + \frac{\pi\ln 3}{\sqrt{3}} - 1 \right) 2\zeta(2) - \frac{7}{2} \zeta(3) + O(\epsilon), \] (A46)

\[ \partial \mathcal{A}_4 = \frac{\Gamma(1+2\epsilon)B(1-\epsilon,1-\epsilon)}{\epsilon} \left( 1 - \frac{2\pi}{3\sqrt{3}} \right) 
+ \left( 3 - \frac{\pi}{\sqrt{3}} + \frac{2\pi\ln 3}{3\sqrt{3}} - \frac{1}{3} \zeta(2) - \frac{8}{3\sqrt{3}} \right) + O(\epsilon), \] (A47)

\[ \mathcal{A}_{1Q} = \frac{\Gamma(2\epsilon)}{\epsilon(1-\epsilon)} \Gamma(1+\epsilon)\Gamma(1-\epsilon), \] (A48)

\[ \mathcal{A}_{2Q} = -\frac{\Gamma(2\epsilon)B(1-\epsilon,1-\epsilon)}{\epsilon(1-\epsilon)} + 1 - \zeta(2) + O(\epsilon), \] (A49)

\[ \mathcal{A}_{4Q} = -\frac{\Gamma(2\epsilon)B(1-\epsilon,1-\epsilon)}{\epsilon(1+\epsilon)} + 1 + \frac{2C}{\sqrt{3}} + O(\epsilon), \] (A50)

\[ \partial \mathcal{A}_{iQ} = 0, \quad i = 1, 2, 4. \] (A51)

**Auxiliary Acorns:**

\[ \mathcal{A}_{1m}(\hat{s}) = \Gamma(\epsilon) \left[ \Gamma(1-\epsilon) \right]^2 \left( \frac{\Gamma(2\epsilon)}{\Gamma(2-\epsilon)} - \frac{\Gamma(\epsilon)(\hat{s})^{-\epsilon}}{(1-\epsilon)\Gamma(2-2\epsilon)} \right) 
+ \left( 1 + \frac{1}{\hat{s}} \right) \text{Li}_2(\hat{s}) + 2 \left( 1 - \frac{1}{\hat{s}} \right) \ln(1-\hat{s}) - 3 + O(\epsilon), \] (A52)

\[ \frac{\partial \mathcal{A}_{1m}}{\partial \hat{s}^2}(\hat{s}) = -\ln(-\hat{s}) + \left( 1 - \frac{1}{\hat{s}} \right) \ln(1-\hat{s}) + \frac{1}{\hat{s}} \text{Li}_2(\hat{s}) + O(\epsilon), \] (A53)

\[ \frac{\partial \mathcal{A}_{1m}}{\partial \hat{s}^2}(\hat{s}) = -\frac{\left[ \Gamma(\epsilon)\Gamma(1-\epsilon) \right]^2}{(1-\epsilon)\Gamma(2-2\epsilon)} (-\hat{s})^{-\epsilon} 
+ \Gamma(-\epsilon) \Gamma(1-\epsilon) \sum_{n=0}^{\infty} \frac{(n+1+\epsilon)\Gamma(n+\epsilon)\Gamma(n+1+2\epsilon)}{\Gamma(n+2)\Gamma(n+2-\epsilon)} \hat{s}^n, \] (A54)

\[ \frac{\partial \mathcal{A}_{1m}}{\partial \hat{s}^2}(\hat{s}) = -\frac{\left[ \Gamma(\epsilon)\Gamma(1-\epsilon) \right]^2}{(1-\epsilon)\Gamma(1-2\epsilon)} (-\hat{s})^{-1-\epsilon} 
- \Gamma(-\epsilon) \Gamma(1-\epsilon) \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\epsilon)\Gamma(n+1+2\epsilon)}{(n+2)\Gamma(n+2-\epsilon)\Gamma(n+1)} \hat{s}^n \]
\[
\frac{\partial A_{2m}}{\partial \hat{a}^2}(\hat{s}) = \Gamma(1 + 2\epsilon)B(-\epsilon, 1 - \epsilon) \left[ - \left( 1 - \frac{1}{\hat{s}} \right) \ln(1 - \hat{s}) + \frac{2(4 - \hat{s})}{\sqrt{4\hat{s} - \hat{s}^2}} \arctan \frac{\hat{s}}{\sqrt{4\hat{s} - \hat{s}^2}} \right] \\
- \int_0^1 dy \left( \ln^2(A - y) - \ln^2 A + \ln y [ \ln A - \ln(A - y) ] - \text{Li}_2 \left( \frac{y}{A} \right) \right) + O(\epsilon), \quad \text{where } A = 1 - \hat{s}y(1 - y), \ \hat{s} \leq 4,
\]

\[
\frac{\partial A_{2m}}{\partial d^2}(\hat{s}) = \frac{1}{\epsilon} \Gamma(1 + 2\epsilon)B(1 - \epsilon, 1 - \epsilon) \frac{2}{\sqrt{4\hat{s} - \hat{s}^2}} \arctan \frac{\hat{s}}{\sqrt{4\hat{s} - \hat{s}^2}} \\
+ \int_0^1 dy \frac{1 - y}{A} \left[ \ln y - \ln A - \ln(A - y) \right] + O(\epsilon)
\]

\[
\frac{\partial A_4}{\partial d^2} = \Gamma(2\epsilon)B(1 - \epsilon, 1 - \epsilon) \frac{2\pi}{3\sqrt{3}} - \frac{\pi \ln 3}{\sqrt{3}} - \zeta(2) + \frac{7C}{3\sqrt{3}} + O(\epsilon).
\]

**Eye-in-the-Sky:**

\[
\mathcal{E}_1(\hat{s}) = -i \frac{[\Gamma(\epsilon)\Gamma(1 - \epsilon)]^2}{(1 - \epsilon)\Gamma(1 - 2\epsilon)} (-\hat{s})^{-1-\epsilon} + i \left( \frac{1}{2} \ln(-\hat{s}) - \frac{1}{2} \left( 1 - \frac{1}{\hat{s}^2} \right) \ln(1 - \hat{s}) + \frac{1}{2\hat{s}} - \frac{1}{\hat{s}} \text{Li}_2(\hat{s}) \right) + O(\epsilon),
\]

\[
\mathcal{E}_1 = i \frac{[\Gamma(\epsilon)\Gamma(1 - \epsilon)]^2}{(1 - \epsilon)\Gamma(1 - 2\epsilon)} e^{i\pi\epsilon} + i \left( \frac{1}{2} - \zeta(2) - i \frac{\pi}{2} \right) + O(\epsilon),
\]

\[
\partial \mathcal{E}_1 = -i(1 + \epsilon) \frac{[\Gamma(\epsilon)\Gamma(1 - \epsilon)]^2}{(1 - \epsilon)\Gamma(1 - 2\epsilon)} e^{i\pi\epsilon} + i [\zeta(2) - 1] + O(\epsilon),
\]

\[
\mathcal{E}_2 = i(A_{2m} - A_{1q}),
\]

\[
\partial \mathcal{E}_2 = i \left( \frac{\Gamma(2 + \epsilon)\Gamma(\epsilon)}{(1 - \epsilon)} B(3, -\epsilon) + 2\zeta(2) - \frac{7}{2} \right) + O(\epsilon),
\]

\[
\mathcal{E}_2^* = i\Gamma(1 + 2\epsilon) \frac{B(1 - \epsilon, 1 - \epsilon)}{1 - \epsilon} B(\epsilon, 1 - 2\epsilon),
\]

\[
\partial \mathcal{E}_2^* = \frac{1 + 2\epsilon}{2(2 - \epsilon)} \mathcal{E}_2^*,
\]

\[
\mathcal{E}_3 = \mathcal{E}_5 + i \frac{5}{6} \zeta(2) + O(\epsilon),
\]

\[
\partial \mathcal{E}_3 = i \Gamma(2 + 2\epsilon)B(1 - \epsilon, 1 - \epsilon) \frac{1}{\epsilon} \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) \\
- i \left( \frac{2}{3} + \frac{\pi^2}{12} - \frac{\pi \ln 3}{9\sqrt{3}} + \frac{7C}{9\sqrt{3}} - \frac{\pi}{\sqrt{3}} \right) + O(\epsilon),
\]

\[
\mathcal{E}_4 = i \frac{\Gamma(1 + 2\epsilon)B(1 - \epsilon, 1 - \epsilon)}{(1 - \epsilon)\epsilon} + i \left( 4 - \frac{4C}{\sqrt{3}} \right) + O(\epsilon),
\]

32
\[ \partial E_4 = i \frac{\Gamma(2 + 2\epsilon) B(1 - \epsilon, 1 - \epsilon)}{(2 - \epsilon)(1 - \epsilon)\epsilon} - i \frac{3}{2} + O(\epsilon), \quad (A69) \]

\[ \mathcal{E}_5 = i \Gamma(2\epsilon) B(1 - \epsilon, 1 - \epsilon) \frac{2\pi}{3\sqrt{3}} \]
\[ + i \left( - \frac{\pi \ln 3}{3\sqrt{3}} + \frac{7C}{3\sqrt{3}} - \frac{\pi}{\sqrt{3}} - \frac{1}{6} \zeta(2) \right) + O(\epsilon), \quad (A70) \]

\[ \partial E_5 = i \Gamma(2 + 2\epsilon) B(1 - \epsilon, 1 - \epsilon) \frac{1}{\epsilon} \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) \]
\[ - i \left( \frac{2}{3} - \frac{\pi^2}{36} - \frac{\pi \ln 3}{9\sqrt{3}} + \frac{31C}{9\sqrt{3}} - \frac{\pi}{\sqrt{3}} \right) + O(\epsilon). \quad (A71) \]

**Lemons:**

\[ \mathcal{L}_1(\hat{s}) = \frac{1}{\hat{s}} \left[ 2 \text{Li}_3(-\hat{s}) - 2 \text{Li}_3(\hat{s}) - 4 \text{Li}_3(1 - \hat{s}) - 4 \text{Li}_3(1 + \hat{s}) \right. \]
\[ - 2 \ln \hat{s} \text{Li}_2(-\hat{s}) - \ln^2 \hat{s} \ln(1 + \hat{s}) - \pi^2 \ln(1 + \hat{s}) + \zeta(3) + 2i\pi \ln^2(1 + \hat{s}) \]
\[ + 4 \text{Li}_3 \left( \frac{1 + \hat{s}}{1 - \hat{s}} \right) - 4 \text{Li}_3 \left( \frac{-1 + \hat{s}}{1 - \hat{s}} \right) \]
\[ + 4 \ln(1 - \hat{s}) \left( \text{Li}_2(1 + \hat{s}) + \frac{3}{2} \zeta(2) \right) \] + O(\epsilon), \quad (A72)

\[ \mathcal{L}_1 = -i6\zeta(3) + O(\epsilon), \quad (A73) \]

\[ \partial \mathcal{L}_1 = i \left[ 6\zeta(3) - 2\zeta(2) \right] + O(\epsilon), \quad (A74) \]

\[ \mathcal{L}_{20} = i \left[ \zeta(2) + 4 \ln 2 \right] + O(\epsilon), \quad (A75) \]

\[ \partial \mathcal{L}_{20} = i \frac{\Gamma(2 + 2\epsilon) B(1 + \epsilon, 1 - \epsilon) B(3, -2\epsilon)}{(2 - \epsilon)(1 - \epsilon)} \]
\[ + i \left( - \frac{17}{54} + \frac{8}{9} \ln 2 + \frac{2}{3} \zeta(2) \right) + O(\epsilon), \quad (A76) \]

\[ \mathcal{L}_{2m} = i \left[ 4 \ln 2 - \frac{3}{2} \zeta(3) - \frac{1}{2} \zeta(2) + \frac{9}{4} \zeta(2) \ln 2 + K_2 + 2\pi i K'_2 \right] + O(\epsilon) \quad (A77) \]

\[ \partial \mathcal{L}_{2m} = -\mathcal{L}_{2m} + i \left( \zeta(2) - \frac{2C}{\sqrt{3}} + \frac{i2\pi^2}{3\sqrt{3}} \right) + O(\epsilon), \quad (A78) \]

\[ \mathcal{L}_3 = i \left[ -4 \ln 2 - \zeta(2) + 6\sqrt{5} \ln \left( \frac{\sqrt{5} + 1}{2} \right) + 2 \ln^2 \left( \frac{\sqrt{5} + 1}{2} \right) \right] + O(\epsilon), \quad (A79) \]

\[ \partial \mathcal{L}_3 = i \left[ 4 - 4 \ln 2 - \frac{1}{2} \zeta(2) + 8 \ln^2 \left( \frac{\sqrt{5} + 1}{2} \right) \right. \]
\[ - \frac{7\sqrt{5}}{2} \ln \left( \frac{\sqrt{5} + 1}{2} \right) + K_3 + O(\epsilon) \], \quad (A80)

\[ \mathcal{L}_5 = iK_5, \quad (A81) \]
\[ \partial \mathcal{L}_5 = -i \left( K_5 + \frac{1}{6} \zeta(2) + \frac{C}{\sqrt{3}} \right) + O(\epsilon). \]  

(A82)

**Auxiliary Lemon integrals:**

\[ K_2 = - \int_0^1 ds \frac{4 \ln s}{\sqrt{4s - s^2}} \arctan \frac{s}{\sqrt{4s - s^2}} \]

\[ + \int_0^1 ds \int_0^1 dy \frac{s}{2 - s} \left[ \frac{1 - 2y}{a} \ln y - \frac{1 - y}{a} \ln(1 - sy) \right] \]

\[ - \frac{1}{s} \left( \ln^2(a - y) - \frac{1}{2} \ln^2 a + \ln y \ln \frac{a}{a - y} - \text{Li}_2 \left( \frac{a}{y} \right) \right), \quad a = 1 - sy(1 - y), \]  

(A83)

\[ K_2' = \int_0^1 ds \frac{2}{\sqrt{4s - s^2}} \arctan \frac{s}{\sqrt{4s - s^2}} \]  

(A84)

\[ K_3 = \int_0^1 dx \int_0^1 dy \frac{x(1 - y)}{1 - xy} \left[ \frac{1}{1 - xy} \left( 4x(1 - x) + x^2 y + \frac{x^2(1 - x)^2(1 - y)}{1 - xy} \right) \right] \]

\[ \times \ln \left( 1 + \frac{1 - xy}{x(1 - x)} \right) - 3 - \frac{x(1 - x)(1 - y)}{1 - xy} \]  

(A85)

\[ K_5 = \int_0^1 dx \int_0^1 dy \frac{1}{xy} \ln \left( 1 + \frac{b_x b_y}{2} (1 + a_x + a_y - \sqrt{(1 + a_x + a_y)^2 - 4a_x a_y}) \right), \quad a_z = \frac{1 - z}{z}, \ b_z = \frac{1}{2} \left( 1 - \sqrt{1 + \frac{4z}{1 - z}} \right). \]  

(A86)

**APPENDIX B: USE OF A BARNES'-TYPE REPRESENTATION TO EVALUATE INTEGRALS OVER FEYNMAN PARAMETERS**

We have found it very useful in evaluating a number of the integrals over Feynman parameters to use a Barnes’ type representation for the binomial expansion [37]

\[ \Gamma(a)(1 - z)^{-a} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt \Gamma(t + a) \Gamma(-t) (-z)^t, \quad |\arg(-z)| < \pi, \]  

(B1)

where the contour of integration runs to the right of the poles of \( \Gamma(t + a) \), i.e., \( t = -a, -a - 1, \ldots \), and to the left of the poles of \( \Gamma(-t) \), i.e., \( t = 0, 1, \ldots \). The binomial expansion of \((1 - z)^{-a}\) for \(|z| < 1\) (or \(|z| > 1\)) is recovered when the contour of integration is closed at infinity in the right half (left half) of the complex \( t \) plane. The advantage of the integral representation is that it holds for any magnitude of \(|z|\) and any \( a \).
We will illustrate the use of the Barnes’ representation by example. We first consider $A_{1m}(\hat{s})$ which is given up to a factor $\xi^2/(4\pi)^4$ by the expression in Eq. (34) with $a = 1$, $b = c = d = 0$:

$$A_{1m}(\hat{s}) = -\Gamma(2\epsilon)\int_0^1 dx x^{-\epsilon}(1-x)^{-\epsilon} \int_0^1 dy y^{-\epsilon} \int_0^1 dz z^{-\epsilon-1} \left[-\hat{s} y(1-y) + \frac{yz}{x}\right]^{-2\epsilon}$$

$$= -\Gamma(2\epsilon)\int_0^1 dx x^{\epsilon}(1-x)^{-\epsilon} \int_0^1 dy y^{-\epsilon} \int_0^1 dz z^{-\epsilon-1} \left(1 - s x(1-y) / z\right)^{-2\epsilon}.$$  (B2)

We can use the Barnes’ representation for the last factor to reexpress $A_{1m}(\hat{s})$ as

$$A_{1m}(\hat{s}) = -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dt}{t^2} \Gamma(t+2\epsilon)\Gamma(-t)(\hat{s})^t$$

$$\times \int_0^1 dx x^{t+\epsilon}(1-x)^{-\epsilon} \int_0^1 dy y^{-\epsilon} \int_0^1 dz z^{-\epsilon-1} \left(1 - \hat{s} x(1-y) / z\right)^{-2\epsilon}. \quad (B3)$$

The integrals on $x$ and $y$ converge and can be expressed in terms of $\Gamma$ functions for $\text{Re} \ t > -1$, $\epsilon \to 0^+$. The final integral converges for $\text{Re} \ t < -\epsilon$. We find that

$$A_{1m}(\hat{s}) = \frac{[\Gamma(1-\epsilon)]^2}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dt}{t^2} \frac{\Gamma(t+2\epsilon)\Gamma(t+\epsilon)\Gamma(-t)}{(t+1)\Gamma(t+2-\epsilon)}(-\hat{s})^t, \quad (B4)$$

where the contour of integration must cross the real axis between $-1$ and $-\epsilon$. If we take $|\hat{s}| < 1$, we can close the contour of integration to the right and find that

$$A_{1m} = \Gamma(\epsilon)[\Gamma(1-\epsilon)]^2 \left(\frac{\Gamma(2\epsilon)}{\Gamma(2-\epsilon)} - \frac{\Gamma(\epsilon)(-\hat{s})^{-\epsilon}}{(1-\epsilon)\Gamma(2-2\epsilon)}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{\Gamma(n+2\epsilon)\Gamma(n+\epsilon)}{\Gamma(n+2-\epsilon)\Gamma(n+2)} \hat{s}^n. \quad (B5)$$

The series is nonsingular for $\epsilon \to 0$ and can be summed in that limit to obtain the expression in Eq. (A52), or treated more generally in terms of hypergeometric functions for $\epsilon \neq 0$.

This calculation illustrates a fairly typical result, that we can use the Barnes’ representation to at least partially factor a multidimensional integral over Feynman parameters, and reduce the calculation to one of contour integration. It was essential that the representation held in this case for arbitrary values of $\hat{s} x(1-y) / z$ subject only to the constraint $|\text{arg}(-\hat{s})| < \pi$. This requires that $-\hat{s} = e^{-i\pi} \hat{s}$ for $\hat{s}$ identified as usual with $\hat{s} + i\epsilon$.

In some cases, partial factorization of the integrals is sufficient. For example, if we write the expression for $\partial A_{2m}/\partial \hat{d}^2$ as
\[
\frac{\partial A_{2m}}{\partial d^2}(\hat{s}) = \Gamma(1 + 2\epsilon) \int_0^1 dx \, x^{-\epsilon}(1 - x)^{-\epsilon} \int_0^1 dy \, y^\epsilon(1 - y) \int_0^1 dz \, z^{-1+\epsilon} \\
x \left[1 - 2\hat{s}y(1 - y)\right]^{-1-2\epsilon} \left(1 - \frac{yz}{1 - \hat{s}y(1 - y)}\right)^{-1-2\epsilon}
\]
(B6)

and apply the Barnes’ representation to the last factor and integrate over \(z\), we obtain a simple series in the variable \(y/[1 - \hat{s}y(1 - y)]\). The singular term in \(\epsilon\) can be extracted and the remainder summed in the limit \(\epsilon \to 0\). The result is the expression in Eq. (A57).

The method can also be used to advantage to make multiple expansions. For example, if the propagators in the expression for \(\mathcal{E}_1(\hat{s})\) are combined in an order different from that which leads to the expressions in Eq. (34), one obtains the awkward expression

\[
\mathcal{E}_1(\hat{s}) = i\Gamma(1 + 2\epsilon) \int_0^1 dx \, x^{1+\epsilon}(1 - x)^{-\epsilon} \int_0^1 dy \, (1 - y) \int_0^1 dz \, z^2(1 - z)^{-2-\epsilon} \\
x \left(1 - \frac{\hat{s}xyz(1 - yz)}{1 - z}\right)^{-1-2\epsilon}
\]
(B7)

After applying the Barnes’ representation to the last factor, one obtains an expression which involves \((1 - yz)^t\). This may be factored by a second application of the Barnes’ representation. After the \(x, y,\) and \(z\) integrations are performed, the contour integrals are easily evaluated, and lead to the result in Eq. (A59). A simpler calculation based on Eq. (40) gives the same result.

Finally, it is not always necessary or even desirable to evaluate all the integrals at an early stage even if that is possible. An example is provided by \(\partial \mathcal{E}_2\) which is given by the integral

\[
\partial \mathcal{E}_2 = i\Gamma(2 + 2\epsilon) \int_0^1 dx \, x^{-\epsilon}(1 - x)^{-\epsilon} \int_0^1 dy \, y^{2+\epsilon}(1 - y)^{-1-2\epsilon} \int_0^1 dz \, z^{-1+\epsilon}(1 - z) \\
\times \left(1 + \frac{yz}{x(1 - y)}\right)^{-2-2\epsilon}
\]
(B8)

Use of the Barnes’ representation on the last term gives a complete factorization of the integrals over the Feynman parameters. However, the singularities at \(y = 1\) in the resulting expression cause difficulties if the \(y\) integration is performed at this stage since convergence of the \(y\) and \(z\) integrations require different signs for \(Re\, t\) or \(\epsilon\). However, it is clear from
the original expression that the apparent singularities in $y$ are absent. In this case, we therefore integrate only over $x$ and $z$. After closing the contour of the $t$ integration in the left half plane, we obtain a series in positive powers of $(1 - y)$. The singular term in $\epsilon$ can be extracted, the remainder of the series summed to simple functions for $\epsilon = 0$, and the final integration on $y$ performed to obtain the result in Eq. (A63).

APPENDIX C: AN APPLICATION OF KOTIKOV’S METHOD TO THE LEMON GRAPHS

In this Appendix we describe the application of Kotikov’s differential equation method [34] to the Lemon graphs which appear in $\Pi^{(2)}_{H_{H}}$. The method is based on the identity

$$0 = i^5 \int \mathcal{D}k \mathcal{D}\ell \frac{\partial}{\partial k_\mu} (p_\mu \Delta_a \Delta_b \Delta_c \Delta_d \Delta_e),$$

(C1)

where the propagators in the Lemon graph will be denoted by $i\Delta_a, i\Delta_b, \ldots$ in the conventions of Fig. 4. The notation $\hat{\mathcal{L}}[\cdot]$ will be used to indicate the action of the Lemon operator in Eq. (46) on the quantity in the bracket, i.e., multiplication by the Lemon integrand and integration over the loop momenta $k$ and $\ell$. For example $\hat{\mathcal{L}}_i[1] = \mathcal{L}_i$ while $\hat{\mathcal{L}}_1[1/\Delta_a] = iA_{1m}$.

If we choose $p_\mu = k_\mu$ in Eq. (C1) and perform the derivatives we obtain the identity

$$0 = \hat{\mathcal{L}} [2\omega - 2k^2 \Delta_a - 2k \cdot (k + q) \Delta_b - 2k \cdot (k - \ell) \Delta_c],$$

(C2)

where $2\omega = 4 - 2\epsilon$. The scalar products can be expressed in terms of inverse propagators and masses, e.g.,

$$2k \cdot q = \frac{1}{\Delta_b} - \frac{1}{\Delta_a} - q^2 - a^2 + b^2.$$  
(C3)

The identity can then be rewritten as

$$(4 - 2\omega)\mathcal{L} = \hat{\mathcal{L}} \left[ \Delta_c \left( -\frac{1}{\Delta_a} + \frac{1}{\Delta_d} - a^2 - c^2 + d^2 \right) + \Delta_b \left( -\frac{1}{\Delta_a} + q^2 - a^2 - b^2 \right) - 2a^2 \Delta_a \right].$$

(C4)

If we choose $p_\mu = \ell_\mu$ in Eq. (C1), we obtain instead
\[
0 = \hat{\mathcal{L}} \left[ \Delta_c \left( \frac{1}{\Delta_d} - \frac{1}{\Delta_a} + c^2 + d^2 - a^2 \right) \\
+ \Delta_a \left( \frac{1}{\Delta_c} - \frac{1}{\Delta_d} + c^2 - d^2 - a^2 \right) \\
+ \Delta_b \left( \frac{1}{\Delta_c} - \frac{1}{\Delta_a} + q^2 + c^2 - e^2 - a^2 \right) \right].
\]  
(C5)

Further identities can be derived by replacing \(\partial/\partial k\) by \(\partial/\partial \ell\) in Eq. (C1). These amount to interchanges of labels in the relations above, with \(a^2 \leftrightarrow d^2\) and \(b^2 \leftrightarrow e^2\). Several of the Lemon graphs are symmetric under this interchange, or under \(a^2 \leftrightarrow b^2\) corresponding to the interchange of momenta \(k \rightarrow -k - q, \ell \rightarrow -\ell - q\) which leads to further identities.

The result of differentiating a Lemon with respect to the square of the incoming momentum is the identity

\[
\frac{\partial \mathcal{L}}{\partial q^2} = \frac{q \mu}{2q^2} \frac{\partial \mathcal{L}}{\partial q \mu} = \frac{1}{2q^2} \hat{\mathcal{L}} \left[ -2q \cdot (k + q) \Delta_b - 2q \cdot (\ell + q) \Delta_c \right].
\]  
(C6)

This can be rewritten as the differential equation

\[
\frac{\partial \mathcal{L}}{\partial q^2} + \frac{\mathcal{L}}{q^2} = \frac{1}{2q^2} \hat{\mathcal{L}} \left[ \Delta_b \Delta_a + (b^2 - a^2 - q^2) \Delta_a + \Delta_d \Delta_e + (e^2 - d^2 - q^2) \Delta_d \right].
\]  
(C7)

The key observation at this point is that the terms in the expressions above which involve ratios of \(\Delta\)'s are related to derivatives of Acorns or Double bubbles. For example, the ratio \(\Delta_a/\Delta_b\) on the right hand side of Eq. (C7) removes the \(b\) line in a Lemon, thus contracting two vertices, and introduces an extra \(a\) propagator. The result is \(\hat{\mathcal{L}}(\Delta_a/\Delta_b) = i\partial A(a, c, d, e)/\partial a^2\), i.e., \(a \rightarrow a, b \rightarrow c, c \rightarrow d,\) and \(d \rightarrow e\) in the Acorn graph in Fig. [4]. Terms of the form \((\text{mass})^2 \times \Delta\) are derivatives of Lemons. The problem is to use the identities above and the symmetries to eliminate the unwanted Lemon terms in Eq. (C7) to obtain a differential equation for \(\mathcal{L}\) with the inhomogeneous term expressed in terms of known graphs. The results for the cases of interest are:

\[
\frac{\partial \mathcal{L}_5}{\partial s} + \frac{\mathcal{L}_5}{s} = \frac{1}{s} \hat{\mathcal{L}} \left[ \frac{\Delta_a}{\Delta_e} - \frac{\Delta_a}{\Delta_d} \right] = \frac{1}{s} \left( \mathcal{T}_3 \mathcal{B}_2 - i \frac{\partial A_4}{\partial d^2} \right),
\]  
(C8)

\[
\frac{\partial \mathcal{L}_{2m}}{\partial s} + \frac{\mathcal{L}_{2m}}{s} = \frac{1}{2 - s} \hat{\mathcal{L}} \left[ \frac{1}{s} \left( \frac{\Delta_a}{\Delta_b} + \frac{\Delta_d}{\Delta_e} \right) + \frac{\Delta_a}{\Delta_d} + \frac{\Delta_d}{\Delta_a} + \frac{\Delta_a}{\Delta_e} - \frac{\Delta_d}{\Delta_e} \right] = \frac{1}{2 - s} \left( \frac{1}{s} \left[ \frac{\partial A_{1m}}{\partial a^2} + \frac{\partial A_{2m}}{\partial a^2} \right] + \frac{\partial A_{2m}}{\partial d^2} \right) \right.
\left. - \frac{1}{m^2(2 - s)} (\mathcal{T}_3 \mathcal{B}_0 + \mathcal{B}_2 \mathcal{T}_0), \right)
\]  
(C9)
\[ \frac{\partial L_1}{\partial s} + \frac{L_1}{s} = \frac{1}{1 + \hat{s}} \hat{L}_1 \left[ \frac{\Delta_3}{s \Delta_b} - \frac{\Delta_a}{\Delta_d} + \frac{\Delta_0}{\Delta_c} \right] \\
= \frac{i}{1 + \hat{s}} \left[ \frac{1}{s} \frac{\partial A_{1m}}{\partial b^2} - \frac{\partial A_{1m}}{\partial d^2} \right] + \frac{1}{1 + \hat{s}} \mathcal{T}_0 B_0. \quad (C10) \]

All the terms on the left are finite for \( \epsilon \to 0 \), but most of those on the right have single or double poles.

The differential equations have general solutions of the form

\[ L(\hat{s}) = \frac{1}{\hat{s}} \left( tL(t) + \int_t^{\hat{s}} d\hat{s} \hat{s} f(\hat{s}) \right), \quad (C11) \]

where \( f(\hat{s}) \) is the inhomogeneous term in the differential equation. A careful study of the limiting behavior of the integrands for \( \hat{s} \to 0 \) with fixed \( \epsilon > 0 \) shows that \( \hat{s}L(\hat{s}) \to 0 \) for \( \hat{s} \to 0 \) for the graphs which appear in \( \Pi_{H}^{(2)} \). (We did not use this method for the graphs which appear in \( \Pi_{w}^{(2)} \) which must be evaluated for \( \hat{s} = 0 \).) We can therefore take the lower limit of integration in Eq. (C11) as zero, and write the \( L \)'s as

\[ L(\hat{s}) = \lim_{\epsilon \to 0} \frac{1}{\hat{s}} \int_0^{\hat{s}} d\hat{s} \hat{s} f(\hat{s}), \quad (C12) \]

where an overall factor of \( 1/M_H^2 \) has been suppressed following the convention of Appendix A.

The homogeneous terms need to be calculated for arbitrary \( \hat{s} \). This is trivial for \( B_0(\hat{s}) \) and \( T_0(\hat{s}) \). \( B_2(\hat{s}), T_3(\hat{s}) \), and the derivatives of the Acorn graphs with respect to internal masses were all evaluated using the method based on Barnes’ representation for the binomial expansion discussed in Appendix B and illustrated there for \( A_{1m}(\hat{s}) \), and also by direct expansions in powers of \( \epsilon \) using the expression in Eqs. (19), (24), and (36). The singularities in the individual terms in the \( f \)'s cancel explicitly, and the problem was reduced to one of integration.

Some integrals in \( L_{2m} \) and \( \partial L_{2m} \) were evaluated analytically. The rest, while perhaps not insuperable, were evaluated numerically. The quantities actually needed are \( L_{2m} = L_{2m}(1) \) and \( \partial L_{2m} = (\partial L_{2m}/\partial \hat{s})(1) \). The second can be evaluated immediately using Eq. (C12),
\[ \partial L \equiv \frac{\partial L}{\partial \hat{s}} (1) = -L(1) + f(1). \]  

(C13)

The results are given in Eqs. (A77) and (A78) \[38\].

\( \mathcal{L}_1 \) was considerably easier to calculate, and was evaluated analytically. Curiously,

\[ \mathcal{L}_1 = \mathcal{L}_0 = -i6\zeta(3) \]  

(C14)

where \( \mathcal{L}_0 \) is the result for all internal masses zero and \( \hat{s} = 1 \) \[39\], but

\[ \partial \mathcal{L}_1 = \partial \mathcal{L}_0 - i2\zeta(2). \]  

(C15)

Finally, \( \partial \mathcal{L}_5 \) was evaluated using Eqs. (C8) and (C13) and the result for \( \mathcal{L}_5 \) obtained using the hyperspherical method, Eq. (52).

**APPENDIX D: A COLLECTION OF USEFUL INTEGRALS**

DeVoto and Duke \[29\] give a useful compilation of integrals involving logarithms and polylogarithms which occur in integrals over Feynman parameters. Lewin \[28\] is the standard reference on polylogarithms. We give here a further collection of integrals which occurred repeatedly in the calculations of the finite terms in the self-energy functions. The first few are elementary; the remainder are not.

*Integrals involving \( X = x(1-x) \):

\[ \int_0^1 dx \frac{1}{1-X} = \frac{2\pi}{3\sqrt{3}}, \]  

(D1)

\[ \int_0^1 dx \frac{1}{(1-X)^2} = \frac{2}{3} + \frac{4\pi}{9\sqrt{3}}, \]  

(D2)

\[ \int_0^1 dx \ln(1-X) = -2 + \frac{\pi}{\sqrt{3}}, \]  

(D3)

\[ \int_0^1 dx \ln^2(1-X) = 8 - \frac{4\pi}{\sqrt{3}} + \frac{2\pi \ln 3}{\sqrt{3}} - \frac{8C}{\sqrt{3}}, \]  

(D4)

\[ \int_0^1 dx x \ln(1-X) = -\frac{\pi}{2\sqrt{3}} - 1, \]  

(D5)

\[ \int_0^1 dx x^2 \ln(1-X) = -\frac{1}{18}, \]  

(D6)
\[
\int_0^1 dx \ln x \ln(1-X) = 4 - \frac{\pi}{\sqrt{3}} - \sqrt{3}C - \frac{\pi^2}{36}, \quad (D7)
\]
\[
\int_0^1 dx \frac{\ln(1-X)}{1-X} = \frac{2\pi \ln 3}{\sqrt{3}} - \frac{8C}{3\sqrt{3}}, \quad (D8)
\]
\[
\int_0^1 dx \frac{\ln(1-X)}{x} = -\frac{\pi^2}{18}, \quad (D9)
\]
\[
\int_0^1 dx \left( \frac{\ln(1-X)}{x^2} + \frac{1}{x} \right) = -1 + \frac{\pi}{\sqrt{3}}, \quad (D10)
\]
\[
\int_0^1 dx \frac{\ln x}{1-X} = -2C, \quad (D11)
\]
\[
\int_0^1 dx \frac{x \ln x}{1-X} = -\frac{C}{\sqrt{3}} + \frac{\pi^2}{36}, \quad (D12)
\]
\[
\int_0^1 dx \frac{\ln x}{(1-X)^2} = -\frac{\pi}{3\sqrt{3}} - \frac{4C}{3\sqrt{3}}, \quad (D13)
\]
\[
\int_0^1 dx \frac{x \ln x}{(1-X)^2} = -\frac{2C}{3\sqrt{3}}, \quad (D14)
\]
\[
\int_0^1 dx \frac{\ln(1+X)}{x} = 2 \ln^2 \left( \frac{\sqrt{5} - 1}{2} \right). \quad (D15)
\]

Note that many further useful identities can be obtained by substituting \( x \rightarrow (1-x) \), and also by using the identity

\[
\int_0^1 dx f(x) = \frac{1}{2} \int_0^1 dx f(X). \quad (D16)
\]

**Integrals involving the dilogarithm \( \text{Li}_2(z) \):**

\[
\int_0^1 dx \ln x \ln(1-x) = \zeta(2) - 1, \quad (D17)
\]
\[
\int_0^1 dx \ln x \text{Li}_2(x) = 3 - 2\zeta(2), \quad (D18)
\]
\[
\int_0^1 dx \ln(1-x) \text{Li}_2(x) = 3 - 2\zeta(3) - \zeta(2), \quad (D19)
\]
\[
\int_0^1 dx \text{Li}_2 \left( \frac{-1}{x} \right) = -\frac{1}{2} \zeta(2) - 2 \ln 2, \quad (D20)
\]
\[
\int_0^1 dx \text{Li}_2(X) = \frac{\pi^2}{18} + \frac{2\pi}{\sqrt{3}} - 4, \quad (D21)
\]
\[
\int_0^1 dx \text{Li}_2 \left( \frac{1-X}{-X} \right) = -2\sqrt{3}C, \quad (D22)
\]
\[
\int_0^1 dx \left( \frac{\text{Li}_2(X)}{x^2} - \frac{1}{x} \right) = -\frac{\pi}{\sqrt{3}} - \frac{\pi^2}{9}. \quad (D23)
\]
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[24] We can also write $\mathcal{L}$ entirely in terms of the physical fields $w$ and $H$ which have unit normalization,

$$\mathcal{L} = \frac{1}{2} Z_w \partial_\mu w \cdot \partial^\mu w + \frac{1}{2} Z_H \partial_\mu H \partial^\mu H - \frac{1}{2} \left( Z_H M_H^2 - \text{Re}\Pi_H(M_H^2) \right)
\quad + \frac{1}{2} \Pi_w(0) w^2 - \frac{\lambda}{4 Z_w} \left( 1 - \frac{\text{Re}\Pi_H(M_H^2)}{Z_H M_H^2} + \frac{\Pi_w(0)}{Z_w M_H^2} \right) \left[ (Z_w w^2 + Z_H H^2)^2
\quad + 4 v Z_w^{1/2} Z_H^{1/2} H (Z_w w^2 + Z_H H^2) \right].$$

The Feynman rules now involve extra derivative interactions proportional to $Z_w - 1$ and $Z_H - 1$ when the propagators for the $w$ and $H$ fields are written in the standard form. $\Pi_H$ and $\Pi_w$ are to be calculated with the Lagrangian above. The renormalization constants are the same as those used in the text, but are given in terms of the new self-energy functions by the relations

$$Z_w = 1 + \frac{d}{dp^2} \Pi_w(p^2)|_{p^2=0},
Z_H = 1 + \frac{d}{dp^2} \Pi_H(p^2)|_{p^2=M_H^2}.$$

The self-energy functions $\Pi_i$ and $\Pi_i^0$ are related by $Z_i \Pi_i^0 = \Pi_i$. We have carried through the calculations in terms of the physical fields as a check. Although the $Z$’s are defined differently and appear in very different ways, the results of the two calculations are identical.

[25] More precisely, $2 \to 2$ diagrams which involve dimension-three interactions in scattering processes are suppressed at a fixed scattering angle by powers of $M_H^2/s$. The partial-wave scattering amplitudes with which we will be concerned will similarly be suppressed by powers of $M_H^2/s$; the extra factors of $\ln(s/M_H^2)$ which appear do not cause problems.

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FIGURES

FIG. 1. The one-loop self-energy graphs needed in the calculation of $\Pi^{(1)}_H$ and $\Pi^{(1)}_w$. The heavy lines represent Higgs bosons, and the thin lines, Goldstone bosons. Tadpole graphs cancel exactly, and are omitted.

FIG. 2. The two-loop self-energy graphs needed in the calculation of $\Pi^{(2)}_w$. The weights with which individual graphs appear in $\Pi^0_w$ can be read off from Eq. (11).

FIG. 3. The two-loop self-energy graphs needed in the calculation of $\Pi^{(2)}_H$. The weights with which individual graphs appear in $\Pi^0_H$ can be read off from Eq. (12).

FIG. 4. The scattering graphs through two loops which involve only the dimension-four quartic couplings and give finite or logarithmically divergent contributions to the renormalized scattering amplitude for $s, -t, -u \to \infty$. Particle masses are irrelevant in this limit so all internal lines may be taken as massless.

FIG. 5. Examples of scattering graphs which contain dimension-three cubic couplings and give vanishing contributions to the renormalized scattering amplitude for $s, -t, -u \to \infty$. For example, the scattering Eye graph $E_1(\hat{s})$ with the topology of (d) given in Eq. (A59) vanishes as $s^{-1} \ln^2 s$ for $s \to \infty$ after renormalization.

FIG. 6. The conventions used for the particle masses and momenta in the one-loop self-energy graphs.

FIG. 7. The conventions used for the particle masses and momenta in the two-loop self-energy graphs.