FROM CRACKED POLYTOPES TO FANO THREEFOLDS

THOMAS PRINCE

ABSTRACT. We construct Fano threefolds with very ample anti-canonical bundle and Picard rank greater than one from cracked polytopes — polytopes whose intersection with a complete fan forms a set of unimodular polytopes — using Laurent inversion; a method developed jointly with Coates–Kasprzyk. We also give constructions of rank one Fano threefolds from cracked polytopes, following work of Christophersen–Ilten and Galkin. We explore the problem of classifying polytopes cracked along a given fan in three dimensions, and classify the unimodular polytopes which can occur as ‘pieces’ of a cracked polytope.

1. INTRODUCTION

We explain how to construct Fano threefolds with very ample anti-canonical bundle from cracked polytopes; special classes of polytopes introduced in [29]. Fixing a complete fan — or shape — we say a polytope is cracked along Σ if its intersection with each maximal cone of Σ is unimodular, see Definition 2.1.

Fixing a polytope P such that the polar polytope $P^\circ$ is cracked along Σ, we use Laurent inversion – developed in joint work with Coates and Kaspzryk [11] – to embed $X_P$ into a non-singular toric variety $Y$. Such embeddings correspond to special combinatorial decorations of $P^\circ$ called full scaffoldings; see Definition 2.6 and Theorem 2.7. The ideal of $X_P$ in the homogeneous co-ordinate ring of $Y$ is determined by the choice of shape: for example, if $TV(\Sigma)$ is a product of projective spaces, a full scaffolding with this shape realises $X_P$ as a toric complete intersection. We show that every Fano threefold with $-K_X$ very ample and $b_2 \geq 2$ – famously classified by Mori–Mukai [19–23] – can be obtained from a full scaffolding $S$ of a cracked polytope via an explicit deformation of the corresponding toric embedding.

We extend these constructions to the rank one case in §3 by interpreting the toric degenerations constructed by Christophersen–Ilten [5,6] as smoothings of toric varieties associated to cracked polytopes. Moreover, we provide Laurent inversion constructions and toric degenerations of Fano varieties with $-K_X$ not very ample in §4.2.

Many of our constructions follow those given in work of Coates–Corti–Galkin–Kasprzyk [9], in which the authors obtain mirror partners for each family of Fano threefolds. These mirror symmetry results rely on explicit constructions which are usually compatible with Laurent inversion. We note that the connection between toric degenerations and mirror symmetry is further explored by Ilten–Lewis–Przyjalkowski [18].

Theorem 1.1. Every smooth Fano threefold with a very ample anti-canonical bundle and $b_2 \geq 2$ can be obtained by smoothing a Gorenstein toric Fano variety. In particular these can be constructed as deformations of toric embeddings provided by Laurent inversion, applied to a cracked polytope together with a full scaffolding $S$. We assume that the shape of the scaffolding $S$ appears in Table 1.

Extending the list of shapes given in Table 1 to include the varieties $Z_{2g-2}$ for $g \in \{2, 8, 9, 10, 12\}$ defined in §3, we obtain members of every family of Fano threefolds with

2000 Mathematics Subject Classification. 14J45 (Primary), 14M25 (Secondary).

Key words and phrases. Fano manifolds, toric degenerations.
very ample anti-canonical bundle from a cracked polytope and full scaffolding. We consider
the Fano threefolds for which $-K_X$ is not very ample in §4.2.

We suggest that four-dimensional cracked polytopes form classes of polytopes from which it
is natural to algorithmically construct Fano fourfolds. We note, by way of example, that each of
the 738 families of Fano fourfolds which appear in [7] can be constructed from a polytope
cracked along the fan of a product of projective spaces.

Given a cracked polytope $P$ there is a natural degeneration of $X_P$ to a union of smooth
toric varieties. Moreover, when $X_P$ smooths to a Fano threefold $X$, we expect $X$ to de-
egenerate to this union of smooth toric varieties. This is close to the notion of semi-simple
degeneration, although the total space of the degeneration is generally singular. Such de-
egenerations also play a key role in the work of Christophersen–Ilten [5, 6], via the work of
Altmann–Christophersen [1, 2]; in particular they consider the deformation spaces of certain
Stanley–Reisner rings. Indeed, cracked polytopes can be regarded as a direct non-simplicial
generalisation of the Stanley–Reisner rings considered in [1, 2].

The extension of the notion of semi-simple degeneration to singular families plays an impor-
tant role in mirror symmetry. Indeed, such a notion of toric degeneration plays a key role in
the approach pioneered by Gross–Siebert via logarithmic and integral affine geometry [16,17]
to the Strominger–Yau–Zaslow conjecture. In [30] we show how to (partially) smooth toric
log del Pezzo surfaces using the Gross–Siebert algorithm. In a similar way we can associate
a polarised tropical manifold – the input to the Gross–Siebert algorithm – to any cracked
polytope. Alternatively one could attempt to smooth the toric variety associated to a cracked
polytope by compactifying families of log Calabi–Yau varieties constructed in [14, 15] using
theta functions. In two dimensions we expect that the toric embeddings defined by scaffolding
extend, under appropriate conditions, to compactifications of these families of log Calabi–Yau
varieties.

Acknowledgements. We thank Tom Coates and Alexander Kasprzyk for our many conver-
sations about Laurent inversion. The author is supported by a Fellowship by Examination at
Magdalen College, Oxford.

Conventions. Throughout this article $N$ will refer to a 3-dimensional lattice, and $M :=
\text{hom}(N, \mathbb{Z})$ will refer to the dual lattice. Given a ring $R$ we write $N_R := N \otimes \mathbb{Z} R$ and
$M_R := M \otimes \mathbb{Z} R$. For brevity we let $\{k\}$ denote the set $\{1, \ldots, k\}$ for each $k \in \mathbb{Z}_{\geq 1}$. We
work over the field $\mathbb{C}$ of complex numbers throughout this article. Given a reflexive polytope
$P \subset N_\mathbb{R}$, we assume throughout that $X_P$ is the toric variety associated to the fan of cones over
faces of $P$. Cracked polytopes will always be contained in $M_\mathbb{R}$; in particular if $Q$ is a polytope
cracked along a fan $\Sigma$, $\Sigma$ is a fan in $M_\mathbb{R}$. Given a variety $Y$, and an identification $\text{Pic}(Y) \cong \mathbb{Z}^r$,
we write $\mathcal{O}(a_1, \ldots, a_r)$ for the line bundle of (multi) degree $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$.

Table 1. The various shape varieties used to construct Fano threefolds.

| $Z$   | $\rho(Z)$ | $Z$   | $\rho(Z)$ |
|-------|-----------|-------|-----------|
| $pt$  | 0         | $dP_7$| 3         |
| $\mathbb{P}^1$ | 1     | $dP_6$| 4         |
| $\mathbb{P}^2$ | 1     | $Z_{10} = dP_7 \times \mathbb{P}^1$| 4 |
| $\mathbb{P}^3$ | 1     | $dP_6 \times \mathbb{P}^1$| 5         |
| $\mathbb{P}^1 \times \mathbb{P}^1$ | 2     | $Z_{12}$| 5         |
| $\mathbb{P}^2 \times \mathbb{P}^1$ | 2     | $dP_5' \times \mathbb{P}^1$| 6         |
| $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 3     |         |           |
2. Cracked polytopes and Laurent Inversion

The method *Laurent inversion* – introduced in [11] – can be used to construct models of Fano manifolds as follows. First, fix a Fano polytope $P$, together with a combinatorial decoration of $P$ called a *scaffolding* $S$ – see Definition 2.3. Loosely, a scaffolding is a collection of polytopes associated to nef divisors on a fixed toric variety $Z$ – the *shape* – whose convex hull is equal to $P$. From this information we construct a polytope $Q_S$ which projects to $P^\circ$. Letting $Y_S$ denote the toric variety associated to the normal fan of $Q_S$, the toric variety $X_P$ embeds into $Y_S$, and the corresponding ideal in the homogeneous co-ordinate ring of $Y_S$ is determined by $Z$. We can then test explicit deformations of the equations cutting out $X_P$ in $Y_S$ to attempt to construct an embedded smoothing.

For general choices of $S$, the variety $Y_S$ may be highly singular: for example $Y_S$ need not be $\mathbb{Q}$-Gorenstein. In [29] we explore the (restrictive) conditions on $S$ which ensure that $Y_S$ is non-singular.

**Definition 2.1** ([29, Definition 2.1]). Fix a convex polyhedron $P \subset M_\mathbb{R}$ containing the origin in its interior, and a unimodular fan $\Sigma$. We say $P$ is cracked along $\Sigma$ if every tangent cone of $P \cap C$ is unimodular for every maximal cone $C$ of $\Sigma$.

**Remark 2.2.** Let $P$ be a polytope cracked along a fan $\Sigma$. We do not assume that the minimal cone of the fan $\Sigma$ is not necessarily zero dimensional; these are sometimes called *generalised fans*. The shape $Z$ is the toric variety associated to the quotient $\bar{\Sigma}$ of $\Sigma$ by its minimal cone. Slightly abusing terminology, we also say that $P$ is cracked along the fan $\bar{\Sigma}$.

We know – [29, Proposition 2.5] – that any cracked polytope is reflexive. In three dimensions the converse holds, in the sense that any reflexive polytope is cracked along *some* complete unimodular fan. Indeed, consider the fan $\Sigma$ defined by taking the cone over every face of a maximal triangulation of the boundary of $P$; the polytopes obtained by intersecting maximal cones of $\Sigma$ with $P$ are all standard simplices. Examples of cracked polytopes are shown in Figure 1.

Our principal application for cracked polytopes – constructing toric degenerations of Fano 3-folds – makes heavy use of the notion of *scaffolding*. We first fix a splitting $N = \bar{N} \oplus N_U$, adopting the notation used in [11].

![Figure 1. Examples of cracked polytopes.](image-url)
Definition 2.3 ([11, Definition 3.1]). Fix a smooth projective toric variety $Z$ with character lattice $N$. A **scaffolding** of a polytope $P$ is a set of pairs $(D, \chi)$ – where $D$ is a nef divisor on $Z$ and $\chi$ is an element of $N_U$ – such that

$$P = \text{conv}\left(P_D + \chi \mid (D, \chi) \in S\right).$$

We refer to $Z$ as the *shape* of the scaffolding, and elements $(D, \chi) \in S$ as *struts*. We also assume that there is a unique $s = (D, \chi)$ such that $v \in P_D + \chi$ for every vertex $v \in \text{verts}(P)$.

Scaffolding a polytope $P$ determines an embedding of $X_P$ into an ambient space $Y_S$. This is the main result of [11]; see also the treatment given in [29, §3].

Definition 2.4 ([11, Definition A.1]). Given a scaffolding $S$ of $P$ we define a toric variety $Y_S$, associated to the normal fan $\Sigma_S$ of the polytope $Q_S \subset M_{\mathbb{R}} := (\text{Div}_{T_M} Z \oplus M_U) \otimes_{\mathbb{Z}} \mathbb{R}$, itself defined by the inequalities

$$\left\{ \langle (D, \chi), - \rangle \geq -1 \right\} \text{ for all } (D, \chi) \in S;$$

$$\left\{ \langle 0, e_i \rangle, - \rangle \geq 0 \right\} \text{ for } i \in [\ell];$$

where $e_i$ denotes the standard basis of $\text{Div}_{T_M} Z \cong \mathbb{Z}^\ell$.

We let $\rho$ denote the ray map of the fan $\tilde{\Sigma}$ determined by $Z$, and let $\rho_s := (-D, \chi)$ for each $s = (D, \chi) \in S$. We also define a map of lattices $\theta$, setting

$$\theta := \rho^* \oplus \text{Id}: \tilde{N} \oplus N_U \longrightarrow \text{Div}_{T_M}(Z) \oplus N_U,$$

$$\begin{array}{ccc}
\tilde{N} & & \text{Div}_{T_M}(Z) \oplus N_U, \\
\| & & \\
\tilde{N} & & \tilde{N}.
\end{array}$$

Theorem 2.5 ([11, Theorem 5.5]). A scaffolding $S$ of a polytope $P$ determines a toric variety $Y_S$ and an embedding $X_P \to Y_S$. This map is induced by the map $\theta$ on the corresponding lattices of one-parameter subgroups.

Note that the ideal, in homogeneous co-ordinates on $Y_S$, of $X_P$ is determined by the map $\theta$: a hyperplane containing the image of $\theta$ defines a function $h$ on the set of ray generators of $\Sigma_S$. $X_P$ then satisfies the equation

$$\prod_{\{v; h(v) \geq 0\}} z_v^{h(v)} = \prod_{\{v; h(v) < 0\}} z_v^{h(v)} = 0,$$

where products are taken over the ray generators of $\Sigma_S$, and $z_v$ is the homogeneous co-ordinate on $Y_S$ corresponding to the ray generated by $v$.

In [29] we describe the facets of the polar polytope $P^\circ$ to a polytope $P$ cracked along a complete unimodular fan $\Sigma$. In particular, each facet $F$ of $P^\circ$ is dual to a vertex $F^*$ of $P$, which is contained in a cone of $\Sigma$. Assume that $\sigma$ is minimal among such cones, then $\sigma$ corresponds to a non-singular toric stratum $Z(\sigma)$ of the toric variety $TV(\Sigma)$. The facet $F$ of $P^\circ$ is the Cayley sum $P_{D_1} \star \cdots \star P_{D_k}$, where $\{D_i : 1 \leq i \leq k\}$ is a set of nef divisors on $Z(\sigma)$ and $k = \dim(\sigma) + 1$; see [29, Proposition 2.8]. We call a face $A$ of $P^\circ$ **vertical** if it is contained in $P_{D_i}$ for some facet containing $A$ and some $D_i$ as above.

Definition 2.6 ([29, Definition 4.1]). Given a Fano polytope $P \subset N_{\mathbb{R}}$ cracked along a fan $\Sigma$ in $M_{\mathbb{R}}$ we say a scaffolding $S$ of $P$ with shape $Z := TV(\Sigma)$ is **full** if every vertical face of $P$ is contained in a polytope $P_D + \chi$ for a (unique) element $(D, \chi) \in S$.

We show in [29] that full scaffoldings on cracked polytopes give rise to embeddings $X_P \to Y_S$ where $Y_S$ is smooth in a neighbourhood of $X_P$. 

Theorem 2.7 ([29, Theorem 1.1]). Fix a polytope $P \subset M_\mathbb{R}$, and a rational fan $\Sigma$ in $M_\mathbb{R}$ such that the toric variety $Z := TV(\Sigma)$ is smooth and projective. Given a scaffolding $S$ of $P$ with shape $Z$, we have that the target of the corresponding embedding is smooth in a neighbourhood of the image of $X_P$ if and only if $P$ is cracked along $\Sigma$ and $S$ is full.

2.1. Torus quotients. Every $n$-dimensional toric variety $X$ (over $\mathbb{C}$) may be described as the quotient of a Zariski open set of affine space $\mathbb{C}^{n+r}$ by a complex torus $T := (\mathbb{C}^*)^r$. Recalling that, if $X$ is determined by a fan in $N$, with $n + r$ rays with primitive generators $\nu_1, \ldots, \nu_{n+r}$ we have an exact sequence

$$0 \rightarrow L \rightarrow \mathbb{Z}^{n+r} \nu \rightarrow N \rightarrow 0$$

where $\nu : e_i \rightarrow \nu_i$ for each $i \in \{1, \ldots, n + r\}$, the character lattice of $T$ is $\mathbb{L}^*$, and right exactness of the above sequence is assumed. This lattice fits into the dual sequence,

$$0 \rightarrow M \rightarrow (\mathbb{Z}^{n+r})^* \nu \rightarrow R \rightarrow L^* \rightarrow 0.$$ 

Moreover we recall that – if $X$ is smooth – there is a canonical identification $\mathbb{L}^* \cong \text{Pic}(X)$, while if $X$ is $\mathbb{Q}$-factorial there is a canonical identification of $\mathbb{L}_{\mathbb{R}}^* := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{R} \cong \text{Pic}(X)_{\mathbb{R}}$. The map $R : (\mathbb{Z}^{n+r})^* \rightarrow \mathbb{L}^*$ is called the weight data for the toric variety. Assuming that $X$ is a projective variety, the possible fans in $N$, with rays generated by a subset of $\{\nu_1, \ldots, \nu_{n+r}\}$, such that the associated toric variety is projective are indexed by cones in a fan contained in the effective cone $\text{Eff}(X) \subset \text{Pic}(X)_{\mathbb{R}}$. This fan is called the secondary fan or GKZ decomposition.

Fixing a maximal cone (or chamber) $\sigma$ in the secondary fan, the corresponding toric variety can be described as the torus quotient

$$X_\sigma = (\mathbb{C}^{n+r} \setminus Z(\sigma)) / T,$$

where $T := (\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{C}^*)$, the weights of the torus action are specified by $R$, and the $Z(\sigma)$ is the irrelevant locus. Choosing a point (or stability condition) $\omega$ in the interior of $\sigma$, the irrelevant locus is defined by setting

$$Z(\sigma) := V(x_{i_1} \cdots x_{i_r} : \omega \in \langle R_{i_1}, \ldots, R_{i_r} \rangle),$$

where $R_i = R(e_i)$ for each standard basis vector $e_i$, $i \in [n + r]$. Some of the constructions described in §4 make use of stability conditions contained in a codimension one cone (or wall) in the secondary fan. The toric variety corresponding to the wall formed by the intersection of chambers $\sigma_1$ and $\sigma_2$ is determined by the coarsest common refinement of the fans associated to $X_{\sigma_1}$ and $X_{\sigma_2}$.

We can use the GIT presentation of a toric variety to streamline the construction of the variety $Y_S$ from a scaffolding $S$. To do this we first assume that, writing $N = \tilde{N} \oplus N_U$, that there is a basis $B = \{b_i \in N_U : i \in [\dim N_U]\}$, such that $\{(0, b) : b \in B\} \subset S$. If this condition holds the cone generated jointly by the vectors in $B$ and the standard basis elements $\{e_i : i \in [\dim \text{Div}_{al}(Z)]\}$ define a smooth torus invariant point in $Y_S$. We assume throughout this section that every scaffolding satisfies this condition. The second Betti number of the toric variety $Y_S$ is $s = \dim N_U$, where $s := |S|$. We explain how to form a weight matrix and stability condition which determine the variety $Y_S$ directly from the scaffolding $S$. This construction follows [11, Algorithm 5.1].

Construction 2.8. Given a scaffolding $S$ with shape $Z$ of a polytope $P$, index the elements of $S$ by $[s]$, and let $(D_i, \chi_i)$ denote the $i^{th}$ element of $S$. It follows from our assumptions on $S$ that the ray matrix of $\Sigma_S$ is in echelon form

$$\begin{pmatrix} I_n & -D_1 & \cdots & -D_r \\ -\chi_1 & \cdots & -\chi_r \end{pmatrix},$$
where \([s] \backslash [r] \) indexes the elements \((D_i, \chi_i) \in S \) of the form \((0, b_i)\), for a basis \( \{b_i : i \in [\dim N_U]\} \). Hence \( R \), the transpose of the kernel matrix, is given by

\[
R = \begin{pmatrix}
I_r & \chi_1 & D_1 \\
\vdots & \vdots & \vdots \\
\chi_r & D_r
\end{pmatrix}.
\]

The variety \( Y_S \) is defined using the a polarising torus invariant divisor given by the sum of all rays corresponding to elements of \( S \). The degree of this divisor is given by the sum of the first \( s \) columns of \( R \). That is, the stability condition used to define \( Y_S \) is given by the sum of \((1, \ldots, 1)^T\) with the columns of the matrix \((\chi_1, \ldots, \chi_r)^T\).

In the case that \( Z \) is a product of \( c \) projective spaces there is a partition of the columns of \( R \) containing the vectors \( D_i \in \text{Div}_{T_{\bar{M}}}(Z) \). In particular, the standard basis in \( \text{Div}_{T_{\bar{M}}}(Z) \) partitions into \( c \) sets \( C_1, \ldots, C_c \), such that \( C_i \) consists of divisors pulled back from the projection to the \( i \)th projective space factor. For each \( i \in [c] \) the degree of the line bundle \( L_i \) cutting out \( X_P \) in \( Y_S \) is given by the sum of the columns in \( C_i \). In particular, there is a distinguished binomial \( z^{m_1} - z^{m_2} \) in \( L_i \), where \( m_1 \) is the sum of standard basis vectors in \((\mathbb{Z}^{n+r})^*\) corresponding to the columns of \( C_i \), and \( m_2 \) is the unique lift of \( L_i \in \mathbb{L}^* \) to \((\mathbb{Z}')^*\): the subspace of \((\mathbb{Z}^{n+r})^*\) corresponding to the first \( r \) columns of \( R \). It is shown in [11] – see also [29, §3] – that \( X_P \) is the vanishing locus of these binomials.

**Example 2.9.** Fix a 3-dimensional reflexive polytope \( P \), and let \( Z \) be a crepant resolution of the toric variety determined by the normal fan of \( P \). In particular, \( \tilde{N} = N \) and \( N_U = \{0\} \). Let \( S := \{(D, 0)\} \), where \( D \in [-K_Z] \) is the toric boundary of \( Z \). Hence \( P = P_D \), and the corresponding weight matrix \( R \) is equal to \((1 \ 1 \ \cdots \ 1)\), and contains \( n := 1 + \dim \text{Div}_{T_{\bar{M}}}(Z) \) columns. The stability condition is equal to \( 1 \in \mathbb{L}^* \cong \mathbb{Z} \), and hence \( Y_S \cong \mathbb{P}^{n-1} \). This is nothing but the anti-canonical embedding of \( X_P \) into projective space.

**Example 2.10.** In [11, Example 3.5] we consider two distinct scaffoldings for the polygon \( P \) associated with the toric del Pezzo surface of degree six. One of these is illustrated in Figure 2.

The scaffolding illustrated in Figure 2 has shape \( Z = \mathbb{P}^1 \times \mathbb{P}^1 \) and – letting \( D_{i,a} \) denote the pullback of \( \{a\} \subset \mathbb{P}^1 \) along the \( i \)th projection for each \( i \in \{1, 2\} \) and \( a \in \{0, \infty\} \) – we define \( S := \{\{D_{1,0} + D_{2,0}\}, \{D_{1,\infty} + D_{2,\infty}\}\} \).
invariant divisors of \( dP \) is rather trivial; see Example 2.9. In this case occur in the anti-canonical embedding, the use of cracked polytopes in this context degenerations of these 15 Fano threefolds. We remark that, since the toric degenerations in one Fano threefolds with very ample anti-canonical bundle. In particular, we describe toric small toric degenerations – we obtain cracked polytopes corresponding to each of the 15 rank by Altmann–Christophersen [1, 2]. Using these results – and the work of Galkin [13] on and Christophersen [5, 6], using the deformation theory of Stanley–Reisner rings developed

\[
\begin{align*}
\mathbb{P}^3 & \quad 0 & - & \text{pt} \\
Q^3 & \quad 1 & x_1 x_2 - x_0^2 & \mathbb{P}^1 \\
B_5 & \quad 245 & \{x_3 x_4 - x_0^2\} & \mathbb{P}^1 \times \mathbb{P}^1 \\
B_6 & \quad 433 & \{x_1 x_2 - x_0^2, x_3 x_4 - x_0^2\} & \mathbb{P}^1 \times \mathbb{P}^1 \\
B_3 & \quad 741 & x_1 x_2 x_3 - x_0^3 & \mathbb{P}^2 \\
B_2 & \quad 427 & \{x_3 x_4 - x_0^2\} & \mathbb{P}^2 \\
V_4 & \quad 4311 & x_1 x_2 x_3 x_4 - x_0^4 & \mathbb{P}^3 \\
V_6 & \quad 4286 & \{x_1 x_2 - x_0^2, x_3 x_4 x_5 - x_0^2\} & \mathbb{P}^1 \times \mathbb{P}^2
\end{align*}
\]

| Rank one Fano threefolds |
|--------------------------|

Applying Construction 2.8 to \( S \) we obtain the weight matrix

\[
\begin{pmatrix}
I_2 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1
\end{pmatrix}
\]

and stability condition \( \omega = (1, 1) \). This is a GIT presentation of the toric variety \( \mathbb{P}^2 \times \mathbb{P}^2 \). The variety \( X_P \cong dP_0 \) is the vanishing locus of the binomials \( x_1 y_1 = x_0 y_0 \) and \( x_2 y_2 = x_0 y_0 \), where \( x_i \) and \( y_i \) denote homogeneous co-ordinates on the \( \mathbb{P}^2 \) factors.

3. Rank one Fano threefolds

Toric degenerations of rank one, index one, Fano toric varieties have been obtained by Ilten and Christophersen [5, 6], using the deformation theory of Stanley–Reisner rings developed by Altmann–Christophersen [1, 2]. Using these results – and the work of Galkin [13] on small toric degenerations – we obtain cracked polytopes corresponding to each of the 15 rank one Fano threefolds with very ample anti-canonical bundle. In particular, we describe toric degenerations of these 15 Fano threefolds. We remark that, since the toric degenerations in this case occur in the anti-canonical embedding, the use of cracked polytopes in this context is rather trivial; see Example 2.9.

To specify the toric varieties \( Z_{2n} \) for \( n \in \{6, 7, 8, 9, 11\} \), we let \( \ell_1^a, \ldots, \ell_5^a \) denote the torus invariant divisors of \( dP_7 \times \{a\} \subset Z_{10} := dP_7 \times \mathbb{P}^1 \) where \( a \in \{0, \infty\} \):

- \( Z_{12} \) is the blow up of \( Z_{10} := dP_7 \times \mathbb{P}^1 \) in a toric invariant line \( \ell_1^0 \subset Z_{10} \).
- \( Z_{14} \) is the blow up of \( Z_{12} \) in the strict transform (and pre-image) of \( \ell_1^\infty \subset Z_{10} \).
- \( Z_{16} \) is the blow up of \( Z_{14} \) in the strict transform of the line \( \ell_5^0 \subset Z_{10} \).
- \( Z_{18} \) is the blow up of \( Z_{16} \) in the strict transform of the line \( \ell_3^\infty \subset Z_{10} \).

The fans determined the varieties are define triangulations of the sphere via radial projection. The sequence of blow up maps described induces the starring operations on these triangulations described in [6]. We define the variety \( Z_{22} \) to be a crepant resolution of the toric variety determined by the normal fan of the reflexive polytope with ID 1941. Similarly, we define the
variety $Z_2$ to be a crepant resolution of the toric variety determined by the normal fan of the (self-dual) reflexive polytope with ID 427.

The Fano variety $\mathbb{P}^3$ is toric, while $Q^3, B_2, B_3, B_4, V_4, V_6, V_8$ are well known to be toric complete intersections. These admit toric degenerations to the varieties defined by the equations in Table 2. To describe the scaffolding associated to each of these Fano threefolds, let $d$ be the dimension of the shape variety $Z$, set $\tilde{N} := \mathbb{Z}^d$ and $N_U := \mathbb{Z}^{3-d}$. Letting $\{e_1, \ldots, e_{3-d}\}$ denote the standard basis of $N_U$, we define

$$S := \{(0, e_1), \ldots, (0, e_{3-d}), (D, \chi)\},$$

where $D \in |-K_Z|$ is the toric boundary of $Z$, and $\chi = (-1, \ldots, -1) \in N_U$. This scaffolding is illustrated in the case $B_3$ in Figure 24 (setting $a = 1$ and $b = 3$).

3.1. Pfaffian equations and $B_5$. The Fano threefold $B_5$ is a linear section of the Grassmannian $\text{Gr}(2, 5)$. We make heavy use of the fact the equations of $\text{Gr}(2, n)$ can be written as the $4 \times 4$ Pfaffians of a skew-symmetric $n \times n$ matrix; entries of which are the $\binom{n}{3}$ Plücker co-ordinates of $\text{Gr}(2, n)$. Hyperplane sections can then be obtained by replacing entries with linear combinations of a subset of the Plücker co-ordinates. For example, $B_5$ can be described as the Pfaffians of the matrix

$$
\begin{pmatrix}
  x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\
  tx_0 & x_3 & x_4 & x_5 & x_0 & x_2 \\
  x_0 & x_5 & & & & \\
  tx_0 & & & & &
\end{pmatrix}
$$

for a fixed value of $t \neq 0$. Varying $t$ defines a flat family, the central fibre of which is the projective cone over a toric variety with two ordinary double points, obtained from $dP_5$ by moving the four points at which $\mathbb{P}^2$ is blown up to two pairs of infinitely close points, and contracting the pair of resulting $-2$ curves. Setting $t = 0$ recovers five equations generating the ideal of a toric variety in $\mathbb{P}^5$. This toric variety is isomorphic to $X_P$, where $P$ denote the toric variety with ID 741. The embedding $X_P \rightarrow \mathbb{P}^5$ is the embedding of $X_P$ determined by the scaffolding $S = \{(0, 1), (D, 0)\}$, where $1 \in N_U \cong \mathbb{Z}$, and $D \in -K_Z$ (recalling that $Z = dP_5$) is the toric boundary of $Z$.

3.2. Higher genus Fano threefolds. The varieties $V_{2n-2}$ for $n \in \{6, 7, 8, 9, 10, 12\}$ are linear sections of the Mukai varieties $M_n$ [24]. Toric degenerations of these are related – by work of Ilten–Christophersen [6] – to the convex deltahedra in the cases $n < 12$, while varieties in the family $V_{22}$ admit a toric degeneration to a variety with ordinary double point singularities, see [13].

Given a Fano toric variety $Z$, let its dual $Z^*$ be toric variety associated to the normal fan of the convex hull of the ray generators of the fan determined by $Z$.

**Proposition 3.1.** The toric varieties $V_{2n-2}$ admit toric degenerations to the Fano toric varieties $Z_{2n-2}$ dual to $Z_{2n-2}$ for each $n \in \{6, 7, 8, 9, 10, 12\}$.

**Proof.** If $n < 12$ we recover the triangulations $T_n$ of $S^2$ used in [6] to construct degenerations of Fano threefolds by removing the origin from $N_R \cong \mathbb{R}^3$ and radially projecting the fan $\Sigma_n$ determined by $Z_{2n-2}$. The result then follows immediately from [6, Proposition 2.3]. In the case $n = 12$ we observe that $Z_{22}^*$ contains only ordinary double point singularities, and hence admits a smoothing. It is shown in [13] that the general fibre of this smoothing is a member of the family $V_{22}$.

In the cases $n \in \{6, 7, 8\}$ we can provide an explicit description of the toric degeneration.
(i) $V_{10}$: varieties in this family can be described by the Pfaffians of a $5 \times 5$ skew-symmetric matrix, and one quadric equation. We can form a toric degeneration following §3.1.

(ii) $V_{12}$: varieties in this family can be described via a system of 9 Pfaffian equations, see 2–21 for a description of a toric degeneration using the same shape variety.

(iii) $V_{14}$: varieties in this family can be described as the vanishing of the $4 \times 4$ Pfaffians of a $6 \times 6$ skew matrix. An explicit toric degeneration is given by the $4 \times 4$ Pfaffians of the matrix (1) below.

The vanishing $4 \times 4$ Pfaffians of the matrix

\[
\begin{pmatrix}
-x_1 & x_2 & tf_1 & x_3 & x_4 \\
-tg_1 & x_5 & x_1 & x_6 \\
x_7 & x_2 & x_0 & \\
x_0 & x_8 & \\
 & & & & \end{pmatrix}
\]

define a toric degeneration of $V_{14}$, a general linear section of $\text{Gr}(2,6)$, where $x_i$ are homogeneous co-ordinates on $\mathbb{P}^9$ and $f_1, g_1,$ and $h_1$ are general linear forms on $\mathbb{P}^9$. The scaffolding $S$ in each case is equal to the singleton set $\{(D, 0)\}$, where $D$ is the toric boundary of $Z$.

3.3. The quartic hypersurface in $\mathbb{P}(1,1,1,1,2)$. The toric variety $Y_S$ defined by a full scaffolding of a cracked polytope $P$ is non-singular in a neighbourhood of the image of $P$. This excludes certain constructions of Fano manifolds as hypersurfaces as weighted projective spaces. In particular, consider the scaffolding of the polytope $P$ with ID 3312 with shape $\mathbb{P}^2$ illustrated in Figure 24, where $(a, b) = (1, 4)$. We have that $N \cong \mathbb{Z}^2$, $N_U \cong \mathbb{Z}$, and $S = \{(0, 1), (D_0 + D_1 + 2D_2, -1)\}$; where $D_i := \{x_i = 0\} \subset \mathbb{P}^2$. Computing the corresponding weight matrix we find

\[ R = \begin{pmatrix} I_r & \chi & D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \end{pmatrix}. \]

Thus $X_P$ is the vanishing locus of a section of $\mathcal{O}(4)$ in $\mathbb{P}(1^4, 2) := \mathbb{P}(1,1,1,1,2)$. Notice that $P^0$ is not cracked along the fan of $\mathbb{P}^2$. To obtain a construction from a cracked polytope we first embed $\mathbb{P}(1^4, 2)$ into $\mathbb{P}^{10}$ via the linear system defined by sections of $\mathcal{O}(2)$. Sections of $\mathcal{O}(2)$ define the integral points of a polytope in $\mathbb{Z}^2$ given by the convex hull of the points given by the columns of the matrix

\[
\begin{pmatrix}
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 \end{pmatrix}
\]

The quartic equation $x_0x_1x_2x_3 = y^2$ defines a projection of this polytope to the reflexive polytope $P$ with ID 427. This polytope is self-dual, and we take the scaffolding of $P$ with shape $Z$ given by a crepant resolution of $X_P$, covering $P$ with a single strut. This scaffolding corresponds to the anti-canonical embedding of $X_P$ into $\mathbb{P}^{10}$, which is the intersection of the image of the Veronese embedding of $\mathbb{P}(1^4, 2)$ with a (binomial) quadric. Deforming this quadric deforms $X_P$ to a general quartic hypersurface in $\mathbb{P}(1^4, 2)$.

4. Constructions of Fano manifolds

There are 98 Fano threefolds with very ample anti-canonical bundle. In the previous section we described constructions from cracked polytopes of the 15 of these which have Picard rank one. We now explain constructions in the remaining 83 cases. In particular, for each of these 83 Fano threefolds $X$, we exhibit a fan $\Sigma$ and polytope $P$ cracked along $\Sigma$ such that – for
Examples from ‘Quantum periods for 3-dimensional Fano manifolds’. Explicit constructions of Fano threefolds are provided in [9]. The authors use these constructions to compute (part of) the J-function of each Fano threefold using either the Quantum Lefschetz principle, or the Abelian-non Abelian correspondence. In particular, each Fano threefold $X$ is exhibited either as a complete intersection in a weak Fano toric variety, or as the degeneracy locus of a map of homogeneous vector bundles.

**Proposition 4.1.** Fix a Fano threefold $X$, and assume that the model of $X$ in [9] describes $X$ as the zero locus of a section of a split vector bundle $\Lambda = L_1 \oplus \cdots \oplus L_c$ on a toric variety $Y$ for which $-K_Y - \Lambda$ is ample. There is a reflexive polytope $P$, shape variety $Z$, and full scaffolding $S$ of $P$ such that $Y_S \cong Y$ and the image of the induced inclusion of $X_P \subset Y_S$ admits an embedded smoothing to $X$.

**Proof.** Tables 3, 4, and 5 list binomial equations cutting out toric varieties to which Fano varieties in the various families satisfying our hypotheses degenerate. The leading monomial in each case is square-free and defines a subset of the columns $C_i$ of the weight matrix listed in [9] for each $i \in \{1, \ldots, c\}$. In every case the sets $C_i$ are pairwise disjoint, and disjoint from a subset $C$ of columns which define a basis of $\text{Pic}(Y)$. Reversing Construction 2.8 we obtain a scaffolding associated to each family represented in Tables 3, 4, and 5. The rank one complete intersection cases are listed in Table 2.

It follows from by [29, Theorem 1.1], and smoothness of $Y_S$, that the polytope $P^0$ is cracked along the fan determined by $Z$, and $S$ is full. \hfill \Box

**Example 4.2.** Consider a Fano threefold $X$ in the family 2–18. $X$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$, branched in a divisor with bidegree $(2, 2)$. The construction in [9] describes this Fano threefold as a hypersurface in the projectivisation of a rank 3 split vector bundle on $\mathbb{P}^2$.

Consider the scaffolding $S$ with shape $Z = \mathbb{P}^2$ illustrated in the left hand image in Figure 3. That is, $N \cong \mathbb{Z}^2$, $N_Y \cong \mathbb{Z}$, and $S = \{(D_1 + D_2, 0), (D_0 + D_2, -1)\}$, where $D_i = \{x_i = 0\}$ and $(x_0 : x_1 : x_2)$ are homogeneous co-ordinates on $\mathbb{P}^2$. The corresponding hypersurface is given by the vanishing locus of the binomial $zy_2x_3 - y_1^2x_1^2$, in the toric variety with weight matrix

\[
\begin{array}{cccccc}
   y_1 & x_1 & x_2 & x_3 & y_2 & z \\
   1 & 0 & 0 & 0 & 1 & 1 \\
   0 & 1 & 1 & 1 & 0 & 1 \\
\end{array}
\]

such that the class $(2, 1)$ is ample. Note that the weight matrix – up to a permutation of the columns – and stability condition $\omega = (1, 2)$ are identical to those appearing in [9, p.40]. Thus the general member of the linear system $\mathcal{O}(2, 2)$ is a Fano threefold in the family 2–18.

Of the 83 Fano threefolds with very ample anti-canonical divisor and Picard rank $> 1$, 67 of the constructions given in [9] coincide with constructions from full scaffoldings on cracked polytopes. We summarise these constructions in Tables 3, 4, and 5. The column *Equations* in each table describes a generating set for the ideal in the homogeneous co-ordinate ring of the ambient variety $Y$ described in [9]. The first monomial of each binomial is always square-free, and may be used to identify columns of the weight matrix defined by $Y$. If $Y$ is a product of projective spaces the co-ordinates are not named in [9], and we name these $x_0, \ldots, x_m$ for the first projective space factor $\mathbb{P}^m$, $y_0, \ldots, y_n$ for the second, etc.

We now provide constructions from cracked polytopes of the 16 Fano threefolds whose construction in [9] is not directly related to a full scaffolding of a cracked polytope. In five
Figure 3. Constructing 2–18 via Laurent inversion.

| Fano | Equations | Shape | Fano | Equations | Shape |
|------|-----------|-------|------|-----------|-------|
| 2–4  | $x_1y_1y_2y_3 - x_0y_0^2$ | $\mathbb{P}^3$ | 2–23 | $\begin{cases} x_3s_4 - s_0x_5 \\ s_1s_2 - s_0^2 \end{cases}$ | $\mathbb{P}^2 \times \mathbb{P}^1$ |
| 2–5  | $s_1x_2x_3x_4 - x_2^3$ | $\mathbb{P}^3$ | 2–24 | $x_1y_1y_2 - x_0y_0^2$ | $\mathbb{P}^2$ |
| 2–6  | $x_1x_2y_1y_2 - x_0^2y_0^2$ | $\mathbb{P}^3$ | 2–25 | $x_1y_1y_2 - x_0y_0^2$ | $\mathbb{P}^2$ |
| 2–7  | $\begin{cases} x_1y_1y_2 - x_0y_0^2 \\ y_3y_4 - y_0^2 \end{cases}$ | $\mathbb{P}^1 \times \mathbb{P}^2$ | 2–26 | $x_3y_4 - y_0^2$ | $\mathbb{P}^2$ |
| 2–9  | $\begin{cases} x_1y_1 - x_0y_0 \\ x_2x_3y_2 - x_0^2y_0 \end{cases}$ | $\mathbb{P}^1 \times \mathbb{P}^2$ | 2–30 | $s_1s_2x - s_0y$ | $\mathbb{P}^3$ |
| 2–10 | $\begin{cases} x_4x_5 - x_2^2 \\ x_3x_1 - x_2^2 \end{cases}$ | $\mathbb{P}^1 \times \mathbb{P}^2$ | 2–31 | $s_1x_2 - s_0x_3$ | $\mathbb{P}^2$ |
| 2–11 | $s_0s_1x_4 - s_2x_3^2$ | $\mathbb{P}^3$ | 2–32 | $x_1y_1 - x_0y_0$ | $\mathbb{P}^1$ |
| 2–12 | $x_1y_i - x_0y_0, i \in [3]$ | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 2–33 | – | $pt$ |
| 2–13 | $\begin{cases} x_1y_1 - x_0y_0 \\ x_2y_2 - x_0y_0 \\ y_3y_4 - y_0^2 \end{cases}$ | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 2–34 | – | $pt$ |
| 2–15 | $s_0s_1s_2x - s_3^2x_4$ | $\mathbb{P}^3$ | 2–35 | – | $pt$ |
| 2–16 | $\begin{cases} s_0s_1x - s_2x_3 \\ x_4x_5 - x_2^2 \end{cases}$ | $\mathbb{P}^2 \times \mathbb{P}^1$ | 2–36 | – | $pt$ |
| 2–18 | $x_1y_1w - x_0^2y_0^2$ | $\mathbb{P}^2$ | 2–19 | $\begin{cases} s_1x_5 - s_0x_4 \\ x_3s_3 - s_0x_1 \end{cases}$ | $\mathbb{P}^2 \times \mathbb{P}^1$ |

Table 3. Scaffolding constructions for Picard rank 2 Fano threefolds.
| Fano | Equations | Shape | Fano | Equations | Shape |
|------|-----------|-------|------|-----------|-------|
| 3–3  | $x_1y_1z_2 - x_0y_0w_0^2$ | $\mathbb{P}^3$ | 3–19 | $s_1x x_3 - x_3^2$ | $\mathbb{P}^2$ |
| 3–6  | $x_2^2y_0 - s_1x x_3y_1$ | $\mathbb{P}^3$ | 3–20 | $s_1t_3 - s_0t_0$ | $\mathbb{P}^1$ |
| 3–7  | $\begin{cases} x_1y_1z_2 - x_0y_0w_0 \\ y_2z_2 - y_0w_0 \end{cases}$ | $\mathbb{P}^2 \times \mathbb{P}^1$ | 3–21 | $y_1s - tx_0y_0^2$ | $\mathbb{P}^1$ |
| 3–8  | $s_1x y_1y_2 - x_2y_0^2$ | $\mathbb{P}^3$ | 3–22 | $x_1s - ty_0^2$ | $\mathbb{P}^1$ |
| 3–9  | $y_1x y_1s_2 - y_0^2$ | $\mathbb{P}^3$ | 3–23 | $s_2v - s_1x_0u$ | $\mathbb{P}^1$ |
| 3–10 | $s_1t_3x y - x_1^2$ | $\mathbb{P}^3$ | 3–24 | $x_2y_1 - s_0x_0y_0$ | $\mathbb{P}^1$ |
| 3–11 | $s_1s_2x y_1 - s_0x_3y_0$ | $\mathbb{P}^3$ | 3–25 | – | $pt$ |
| 3–12 | $\begin{cases} s_3x y_1 - x_1y_0 \\ x_2y_2 - x_1y_0 \\ x_1y_1 - x_0y_0 \end{cases}$ | $\mathbb{P}^2 \times \mathbb{P}^1$ | 3–26 | – | $pt$ |
| 3–13 | $\begin{cases} x_2z_1 - x_0z_0 \\ y_2z_2 - y_0z_0 \end{cases}$ | $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 3–27 | – | $pt$ |
| 3–15 | $s_1s_2y - s_0t_3z$ | $\mathbb{P}^2$ | 3–28 | – | $pt$ |
| 3–17 | $x_1y_1z_1 - x_0y_0z_0$ | $\mathbb{P}^2$ | 3–29 | – | $pt$ |
| 3–18 | $s_1x x_3y_0 - s_0x_2y_1$ | $\mathbb{P}^3$ | 3–30 | – | $pt$ |
| 3–31 | – | $\mathbb{P}^2$ | 3–31 | – | $pt$ |

Table 4. Scaffolding constructions for Picard rank 3 Fano threefolds.

| Fano | Equations | Shape | Fano | Equations | Shape |
|------|-----------|-------|------|-----------|-------|
| 4–1  | $x_1y_1z_2w_1 - x_0y_0z_0w_0$ | $\mathbb{P}^3$ | 4–7  | $\begin{cases} y_1u_1 - x_0u_0 \\ z_1u_2 - x_0u_0 \end{cases}$ | $\mathbb{P}^1 \times \mathbb{P}^1$ |
| 4–3  | $y_0y_1 - s_0^2t_0^2x_2$ | $\mathbb{P}^1$ | 4–8  | $x_2y_2 - s_0x t_0y$ | $\mathbb{P}^1$ |
| 4–4  | $x_1y_1v - x_0y_0z_0u$ | $\mathbb{P}^2$ | 4–9  | $z_1u - x_0y_0v$ | $\mathbb{P}^1$ |
| 4–5  | $x_3y_4 - x_2^2y_2$ | $\mathbb{P}^1$ | 4–10 | – | $pt$ |

Table 5. Scaffolding constructions for Picard rank 4 Fano threefolds.
cases the corresponding construction in [9] does not describe the Fano threefold as a toric complete intersection. In the remaining eleven cases the construction given in [9] expresses the Fano threefold $X$ as the vanishing locus of a section of split vector bundle $\Lambda$ on a toric variety $Y$, such that $L := -K_Y - \Lambda$ is nef but not ample. In the latter case the embedding cannot come from a scaffolding $S$, since the construction of the ambient space uses $L$ to polarise the ambient space.

Remark 4.3. Note that the numbering for the rank 4 Fano threefolds replicates that in [9], which differs from the original list of Mori–Mukai by the insertion of the family 4–2 which was omitted from the original classification (some lists instead append this family as 4–13).

Rank 2, number 8. Varieties in the family 2–8 are either,

(i) the double cover of $B_7$ (the blow-up of $\mathbb{P}^3$ at a point) with branch locus a member $B$ of $|-K_{B_7}|$ such that $B \cap D$ is non-singular, where $D$ is the exceptional divisor of the blow-up $B_7 \to \mathbb{P}^3$, or;

(ii) the specialisation of (i) where $B \cap D$ is reduced but singular.

We make use of the construction given in [9], embeds Fano threefolds in the family 2–8 as hypersurfaces of bi-degree $(2, 4)$ in the toric variety $Y$, defined by the weight matrix

\[
\begin{array}{cccccc}
y & x_0 & z & x_1 & x_2 & x_3 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 1 & 1 \\
\end{array}
\]

and a stability condition in the chamber $\langle (0, 1), (1, 2) \rangle$. The coincidence of these two constructions is proved in [9, p.31].

We consider the scaffolding $S = \{D\}$ of the reflexive polytope $P$ with PALP ID 3262, with shape $Z = \mathbb{P}^1 \times dP'_5$; where $dP'_5$ is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three of its torus invariant points, and $D \in |-K_Z|$ is the toric boundary of $Z$.

This scaffolding corresponds to the anti-canonical embedding $X_P \to \mathbb{P}^9$, see Example 2.9. To prove that $X_P$ admits a smoothing in this embedding we consider another scaffolding of $P$ – with shape $Z' = \mathbb{P}^2 \times \mathbb{P}^1$ – shown in Figure 4. Note that $P^0$ is not cracked along the fan determined by $Z'$. The scaffolding $S'$ defines an embedding $X_P \to Y_{S'}$ where $Y_{S'}$ is the toric variety defined by weight matrix

\[
\begin{array}{cccccc}
y & x_0 & z_0 & z_1 & x_1 & x_2 & x_3 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 1 & 1 \\
\end{array}
\]

and stability condition $\omega = (1, 2)$ – note that $\omega$ is contained in a wall. The toric variety $X_P$ is the vanishing locus of a section of $E := \mathcal{O}(1, 2) \oplus \mathcal{O}(2, 4)$.

Lemma 4.4. The vanishing locus of a general section of $E$ is a Fano threefold 2–8.

Proof. General sections of $E$ do not vanish at the torus invariant point defined by the vanishing of all co-ordinates except $z_1$. There is a projection from this point to the toric variety $Y'$, the toric variety defined by the same weight matrix as $Y'$, but stability condition $\omega = (1, 2)$. The wall spanned by $(1, 2)$ is a flipping wall, and the birational transformation induced by crossing this wall is given by (the cone on) a Pachner move in the fan determined by $Y$. The intermediate variety has the non-$\mathbb{Q}$ factorial point given by the vanishing of all homogeneous co-ordinates (labelled as for $Y$) except $z$. The image of the vanishing locus $X$ of a general section of $E$ in $Y'$ misses this singularity. Hence the resolution of $Y'$ induced by moving the stability condition from $(1, 2)$ into the chamber $\langle (1, 2), (0, 1) \rangle$ restricts to an isomorphism of $X$, and the result follows from [9, p.31].
Consider the embedding $\phi_{O(1,2)}: Y_{S'} \to \mathbb{P}(H^0(Y_{S'}, O(1,2)))^* = \mathbb{P}^{10}$. Composing $\phi_{O(1,2)}$ with the embedding $\iota: X_P \to Y_{S'}$, the pull-back of the line bundle $O_{P^{10}}(1)$ is the anti-canonical class on $X_P$ by adjunction. Moreover $X_P$ is the intersection of $\phi_{O(1,2)}(Y_{S'})$ with a quadric and a hyperplane in $\mathbb{P}^{10}$. In particular, restricting to this hyperplane, we obtain the anti-canonical embedding of $X_P$ in $\mathbb{P}^{9}$. Restricting to members of a general pencil of hyperplanes – and intersecting with a general pencil of quadrics – we see that $X_P$ deforms in $\mathbb{P}^{9}$ to a variety in the family 2–8.

**Rank 2, number 14.** This example is the first of a sequence of examples – along with 2–20, 2–22, and 2–26 – to make use of polytopes cracked along the fan of $Z := dP_7$. The corresponding embeddings are defined using the five $4 \times 4$ Pfaffians of a $5 \times 5$ matrix of polynomials in the homogeneous co-ordinate ring of a toric variety. Varieties in the family 2–14 are the blow up of $B_5$ (a three dimensional linear section of $\text{Gr}(2, 5)$) in an elliptic curve which is the intersection of two hyperplane sections.

Consider the polytope $P$ with PALP ID 3027 together with the scaffolding with shape $Z$ displayed in Figure 5. We have that $N \cong \mathbb{Z}^2$, $N_U \cong \mathbb{Z}$, and $S = \{(0, 1), (D, 0), (D, -1)\}$, where $D$ is the toric boundary of $Z = dP_7$.  

---

**Figure 4.** Scaffolding used to construct 2–8

**Figure 5.** Scaffolding used to construct 2–14
The variety $Y_S$ is determined by the weight matrix
\[
\begin{array}{cccccccc}
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & y \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
and stability condition $\omega = (2,1)$. The variety $Y_S$ is consequently the blow up of $\mathbb{P}^6$ in a codimension 2 linear subspace. The ideal of $X_P$ in $Y_S$ is obtained by homogenizing the $4 \times 4$ Pfaffians of the skew-symmetric matrix
\[
\begin{pmatrix}
x_0y & x_1 & x_2 & x_0y \\
0 & x_3 & x_4 & 0 \\
x_0y & x_5 & 0 & 0 \\
\end{pmatrix}.
\]

Consider the contraction $Y_S \to \mathbb{P}^6$, and observe that the intersection of the image of $X_P$ with the centre $V := \{x_0 = x_1 = 0\}$ is a cycle of five $(-1)$-curves. Replacing the two 0 entries with general linear forms this cycle of $(-1)$-curves becomes a (codimension 3) non-singular curve of genus one, and – blowing up $V$ – we obtain a flat family deforming $X_P$ to a Fano threefold in the family 2–14.

**Rank 2 number** 17. Varieties in the family 2–17 are the blow up of a quadric threefold in an elliptic curve of degree 5. We consider the polytope $P$ with PALP ID 1527, together with the scaffolding $S$ shown in Figure 7 using the shape variety $Z = \mathbb{P}^1 \times dP_7$. 

---

**Figure 6.** Secondary fan for the variety $Y_S$ used in the construction of 2–14.

**Figure 7.** Scaffolding used to construct 2–17
The scaffolding $S$ determines the toric variety $Y_S \cong \mathbb{P}^4 \times \mathbb{P}^3$. Letting $x_0, \ldots, x_4$ and $y_0, \ldots, y_3$ denote homogeneous co-ordinates on the respective projective space factors, $X_P$ is the vanishing locus of the binomial $x_0 y_0 = x_1 y_1$, and the five $4 \times 4$ Pfaffians of the skew-symmetric matrix

$$
\begin{pmatrix}
y_0 & y_2 & y_3 & y_0 \\
t f_1 & x_2 & x_4 \\
x_0 & x_3 & \\
t f_2
\end{pmatrix},
$$

where $t = 0$ and $f_i$ are general linear equations in $x_0, \ldots, x_4$. One of these five Pfaffians describes the threefold $x_0 x_4 - x_0 x_4 = 0$ in $\mathbb{P}^4$, while the other four equations have bidegree $(1, 1)$. It is shown in [9, p.38] that varieties in the family $2–17$ may be obtained as the vanishing loci of general sections of the bundle

$$
E := (S^* \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)) \oplus (\text{det } S^* \boxtimes \mathcal{O}_{\mathbb{P}^3}(1)) \oplus (\text{det } S^* \boxtimes \mathcal{O}_{\mathbb{P}^3})
$$

in the variety $\text{Gr}(2, 4) \times \mathbb{P}^3$. The Grassmannian $\text{Gr}(2, 4) \subset \mathbb{P}^5$ is a quadric fourfold, while sections of the line bundle $\text{det } S^*$ define hyperplane sections in $\mathbb{P}^5$. Moreover, the binomial $x_0 y_0 = x_1 y_1$ defines a section of the bundle obtained by pulling back $(\text{det } S^* \boxtimes \mathcal{O}_{\mathbb{P}^3}(1))$ to the product of a hyperplane section in $\mathbb{P}^5$ with $\mathbb{P}^3$. We claim that the remaining four Pfaffian equations define a section of the pull-back of $(S^* \boxtimes \mathcal{O}_{\mathbb{P}^3}(1))$ to this hyperplane section. Representing a point in $\text{Gr}(2, 4)$ as the row-space of a $2 \times 4$ matrix

$$
M = \begin{pmatrix}
y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\
y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4}
\end{pmatrix},
$$

a section of the bundle $S^*$ is given by a vector $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, which vanishes when $z$ lies in the row space of $M$. This happens when the maximal minors of the matrix

$$
\bar{M} = \begin{pmatrix}
z_1 & z_2 & z_3 & z_4 \\
y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\
y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4}
\end{pmatrix}
$$

vanish. Writing the $2 \times 2$ minors of $M$ (the Plücker co-ordinates) as $x_0, \ldots, x_5$ we have that sections of $S^*$ are defined by four equations of degree 1 in the variables $\{x_i : i \in \{0, \ldots, 5\}\}$ and constants $\{z_i : i \in \{1, \ldots, 4\}\}$. Replacing each $z_i$ with the homogeneous co-ordinate $y_{i-1}$ we recover the 4 remaining Pfaffian equations found above, up to a linear relation eliminating $x_5$. That is, $X_P$ admits an embedded flat deformation to a variety in the family $2–17$.

**Rank 2 number 20.** Varieties in the family $2–20$ are the blow up of $B_5$ (a three dimensional linear section of $\text{Gr}(2, 5)$) in a twisted cubic. Consider the polytope $P$ with PALP ID 1909 together with the scaffolding with shape $Z = dP_7$ displayed in Figure 8.

The corresponding toric variety $Y_S$ is isomorphic to $BL_{P_7} \mathbb{P}^6$. Moreover, the variety $X_P$ is the blow up of the vanishing locus of the five $4 \times 4$ Pfaffians of

$$
\begin{pmatrix}
x_0 & x_1 & x_2 & x_0 \\
0 & x_3 & x_4 \\
x_0 & x_5 \\
0
\end{pmatrix},
$$

where $x_0, \ldots, x_6$ are homogeneous co-ordinates on $\mathbb{P}^6$, in the locus $\{x_0 = x_1 = x_6\}$. Note that the ideal $x_2 x_4 = x_3 x_5 = x_2 x_5 = 0$ defines a (degenerate) twisted cubic. Replacing the two zero entries in the above matrix with general homogeneous elements of degree one we obtain a flat deformation of $X_P \hookrightarrow Y_S$ to a blow up of $B_5$ in a twisted cubic.
Rank 2 number 21. Varieties in the family 2–21 are the blow up of a quadric threefold in a rational curve of degree 4. These are shown in [9, p.43] to be zero loci of sections of the vector bundle
\[ E = (S^* \boxtimes \mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 2} \oplus (\det S^* \boxtimes \mathcal{O}_{\mathbb{P}^4}) \]
on Gr(2,4) × P^4. Consider the polytope with PALP ID 702, with the scaffolding shown in Figure 10. This scaffolding has shape \( Z = Z_{12} \), the shape used in the construction of Fano threefolds in the family \( V_{12} \). The ambient space \( Y_S \) defined by this scaffolding is isomorphic to \( \mathbb{P}^4 \times \mathbb{P}^4 \) with co-ordinates \( x_0, \ldots, x_4 \) and \( y_0, \ldots, y_4 \) respectively. The equations cutting out \( X_P \) in \( Y_S \) can be read off as relations between labelled lattice points in Figure 9. In particular if \( u_1 + v_1 = u_2 + v_2 \), where \( u_i \) and \( v_i \) are lattice points labelled with variables \( z_i \) and \( w_i \) for each \( i \in \{1, 2\} \), points in \( X_P \) satisfy the equation \( z_1 w_1 = z_2 w_2 \). There are nine such binomial equations, which can be written as the 4 × 4 Pfaffians of the following pair of matrices (setting \( t = 0 \)).
Figure 10. Scaffolding used to construct 2–21

\[
\begin{pmatrix}
 y_0 & ty_3 & y_2 & y_1 \\
 x_2 & x_0 & x_3 & x_1 \\
 x_1 & x_0 & & \\
 & & & \end{pmatrix},
\begin{pmatrix}
 y_0 & ty_1 & y_3 & y_4 \\
 x_4 & x_1 & x_0 & x_0 \\
 x_0 & x_3 & & \end{pmatrix}.
\]

Note that these matrices share the Pfaffian \(x_1x_3 - x_0^2 + tx_2x_4\), which defines a toric degeneration of a quadric threefold.

Following the treatment of the variety 2–17, we observe that each set of five Pfaffian equations defines a section of (the pullback to a hyperplane section of) \(S^* \otimes O_{\mathbb{P}^4}(1)\). Thus the general member of the family given by the set of 9 Pfaffian equations is isomorphic to a Fano threefold in the family 2–21.

**Rank 2 number 22.** Varieties in the family 2–22 are the blow up of \(B_5\) in a conic. Consider the polytope \(P\) with PALP ID 1856, and scaffolding – with shape \(Z = dP_7\) – displayed in Figure 11. The variety \(Y_S\) is the blow up of \(\mathbb{P}^6\) in a plane; that is, the toric variety determined by the weight matrix

\[
\begin{array}{cccccccc}
 y & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

and stability condition \(\omega = (1, 2)\). \(X_P\) is cut out by the five \(4 \times 4\) Pfaffians of

\[
\begin{pmatrix}
 x_0 & tf_{0,1} & x_2 & x_3 \\
 x_4 & x_5 & x_0y & x_6 \\
 & & & \end{pmatrix}
\]

where \(f_{i,j}\) is a generic polynomial of bi-degree \((i, j)\), and \(t = 0\). The ambient variety \(Y_S\) is obtained from \(\mathbb{P}^6\) with co-ordinates \(x_0, \ldots, x_6\) by blowing up the plane \(\Pi := \{ x_0 = x_1 = x_2 = x_3 = 0 \}\). The Pfaffian equations defining \(X_P\) pull back to the single equation \(x_5x_6 = tx_4f_{1,1}\)
on this locus. Hence, for general values of $t$, the equations define the blow up of $B_5$ (cut out by 5 Pfaffian equations in $\mathbb{P}^6$) in a non-degenerate conic.

**Rank 2 number 26.** Varieties in the family 2–26 are the blow up of $B_5$ in a line. Consider the polytope $P$ with PALP ID 1433 and scaffolding with shape $Z = dP_7$ displayed in Figure 12. The variety $Y_S$ is the blow up of $\mathbb{P}^6$ in the line with homogeneous co-ordinates $\{x_4, x_5\}$. Consider the one parameter family

$$
\begin{pmatrix}
 x_0 & x_1 & x_2 & x_3 \\
 t f_1 & x_3 & x_4 & x_0 \\
 x_0 & x_5 & t g_1 
\end{pmatrix},
$$

where $f_1$ and $g_1$ are general linear forms with no terms in $x_4$ or $x_5$. Varying $t$, this family contains the line with co-ordinates $x_4$ and $x_5$ for all values of $t$. Blowing up this line we obtain a flat family embedded in $Y_S \times A^1_t$ with central fibre $X_P$, and general fibre a Fano threefold in the family 2–26.

**Rank 3 number 1.** Varieties in the family 3–1 are double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a divisor of tri-degree $(2, 2, 2)$. Our treatment of this family is similar to that of 2–8. Consider the Fano polygon $P$ with PALP ID 3874, illustrated in Figure 13. We give $P$ the ‘anti-canonical scaffolding’: covering $P$ with the polyhedron of sections of the toric boundary on the shape variety $Z = \mathbb{P}^1 \times dP_6$. This scaffolding produces the standard anti-canonical map $X_P \to \mathbb{P}^8$, see Example 2.9. Similarly to our treatment of family 2–8, we prove this smooths by factoring the embedding through a map to a toric variety obtained from a non-full scaffolding of $P$. Figure 13 shows a scaffolding $S'$ of $P$ with shape $Z' := \mathbb{P}^1 \times \mathbb{P}^2$. The scaffolding $S'$ consists of three elements, and defines the toric variety $Y_{S'}$ with weight matrix

$$
\begin{array}{cccccccc}
 x_0 & y_0 & z_0 & x_1 & y_1 & z_2 & w_0 & w_1 \\
 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 
\end{array}
$$

**Figure 11.** Scaffolding used to construct 2–22
Figure 12. Scaffolding used to construct 2–26

Figure 13. Scaffolding used to construct 3–1

and stability condition $\omega = (1,1,1)$. The hypersurface $X_P$ is the vanishing locus of the binomial $w_0w_1 = x_0^2y_0^2z_0^2$ — a section of the line bundle $L_1$ with tri-degree $(2,2,2)$ — and $x_1y_1z_1 = x_0y_0z_0$ — a section of the line bundle $L_2$ tri-degree $(1,1,1)$. Note that the variety $Y_{S'}$ is not $\mathbb{Q}$-factorial along the line on which $x_0 = y_0 = z_0 = x_1 = y_1 = z_2 = 0$. General linear sections though this non-isolated singularity are isomorphic to the affine cone $V$ over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, polarised by the line bundle of tri-degree $(1,1,1)$.

Consider a general section $s$ of $E = L_1 \oplus L_2$, and its vanishing locus $X$. Projecting away from the point at which all co-ordinates except $w_0$ vanish, $X$ is an isomorphism onto its image in a toric variety $Y'$. The variety $F$ which appears in the construction in [9, p.57] is obtained from the variety $Y'$ by a making one of the three possible small resolutions of the singularity $V$. Since the variety $X$ does not intersect the singular locus of $Y'$ this resolution restricts to an isomorphism of $X$. The rest of the example follows our treatment of the family 2–8: the complete linear system determined by $L_2$ defines an embedding $Y_{S'} \to \mathbb{P}^9$ and varying a quadric section in the anti-canonical embedding of $Y_{S'}$ smooths $X_P$.

**Rank 3 number 4.** Fano threefolds in this family are obtained by blowing up the fibre of the projection map $X_{2–18} \to \mathbb{P}^2$, where $X_{2–18}$ is a double cover of $\mathbb{P}^2 \times \mathbb{P}^1$ branched in a
Figure 14. Scaffolding used to construct 3–4 divisor of bidegree (2, 2). In [9, p.60] it is shown that varieties in this family may be obtained as hypersurfaces of tri-degree (2, 2, 2) contained in the toric variety $Y$ defined by the weight matrix

$$
\begin{array}{ccccccc}
   x_0 & x_1 & y & z & t_0 & t_1 & w \\
   1 & 1 & 1 & 0 & 0 & 0 & 1 \\
   0 & 0 & 1 & 1 & 0 & 0 & 1 \\
   0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
$$

together with a stability condition $\omega$ in the chamber $\langle (1, 0, 0), (1, 1, 0), (1, 1, 1) \rangle$. We compare these toric hypersurfaces to the threefolds obtained by scaffolding the polytope $P$ with PALP ID 2602 shown in Figure 14. This scaffolding has shape $Z = \mathbb{P}^1 \times \mathbb{P}^1$, and hence defines a codimension 2 toric complete intersection in the toric variety $Y_S$ with weights:

$$
\begin{array}{ccccccc}
   a_0 & a_1 & b_0 & b_1 & b_2 & c_0 & c_1 \\
   1 & 1 & 1 & 1 & 1 & 0 & 0 \\
   0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
$$

and stability condition $\omega = (2, 1)$. Let $X$ be the vanishing locus of a general section of the vector bundle $E := \mathcal{O}(1, 2) \oplus \mathcal{O}(2, 2)$. Note that the line bundle $\mathcal{O}(1, 2)$ is not nef on $Y_S$. We define a Segre type map $\phi: Y \to Y_S$, setting

$$
\phi: (x_0, x_1, y, z, t_0, t_1, w) \mapsto (x_0, x_1, w, yt_0, yt_1, zt_0, zt_1).
$$

It is easy to check that this map is homogeneous, and that $\phi^*: \text{Pic}(Y_S) \to \text{Pic}(Y)$ is given by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^T$. In particular the stability condition $\omega = (2, 1)$, is mapped to the wall spanned by $(1, 0, 0)$ and $(1, 1, 1)$. Let $Y'$ be the toric variety defined by weight matrix $M$ and stability condition $(2, 1, 1)$.

**Lemma 4.5.** $Y$ is obtained by a small resolution a non-isolated singularity of $Y'$ which is disjoint from the divisor $\{w = 0\}$. 
Figure 15. Scaffolding used to construct 3–5

**Proof.** There is a morphism $\pi : Y_S \to \mathbb{P}_1$ expressing $Y_S$ as a $\mathbb{P}^4$ bundle over $\mathbb{P}^1$, with co-ordinates $(a_0 : a_1)$. Similarly, $Y$ and $Y'$ admit projections to the $\mathbb{P}^1$ with co-ordinates $(x_0 : x_1)$. Each of these projections commute with the inclusion $\iota : Y' \hookrightarrow Y_S$. Given a point $a \in \mathbb{P}^1$, the intersection $\pi^{-1}(a) \cap \iota(Y')$ is the projective closure of a conifold singularity in $\mathbb{P}^4$ with co-ordinates $(b_0 : b_1 : b_2 : c_0 : c_1)$. The (smooth) variety $Y$ is obtained by making either of the two possible small resolutions of this line of conifold singularities. Note however that, for any fibre of $\pi$, the divisor $b_0 = 0$ is disjoint from the singular locus of $Y'$. Since $\iota^*b_0 = w$, the locus $w = 0$ is disjoint from the singular locus of $Y'$. □

Note that $Y'$ is a hypersurface in the class $O(1,2)$, cut out by $\det \begin{pmatrix} b_1 & c_0 \\ b_2 & c_1 \end{pmatrix}$. Moreover, we have that $\phi^*(2,2) = (2,2,2)$; hence, by Lemma 4.5 any hypersurface cut out by a member of the linear system $(2,2,2)$ on $Y$, is the vanishing locus of a section of $E$ on $Y_S$.

**Rank 3 number 5.** It was shown in [9, p.62] that varieties in the family 3–5 are codimension 2 complete intersections in the toric variety $Y$, defined by the weight matrix

| $x_0$ | $x_1$ | $y_0$ | $y_1$ | $y_2$ | $z_0$ | $z_1$ | $t$ |
|------|------|------|------|------|------|------|----|
| 1    | 1    | 0    | 0    | 0    | 1    | 1    | 0  |
| 0    | 0    | 1    | 1    | 1    | 1    | 1    | 0  |
| 0    | 0    | 0    | 0    | 0    | 1    | 1    | 1  |

and a stability condition in the chamber $\langle (1,0,0),(0,1,0),(1,1,1) \rangle$. Varieties $X$ in the family 3–5 are obtained as zero loci of sections of the bundle $O(1,2,1)^{\oplus 2}$. The secondary fan for $Y$ is illustrated in Figure 16. Consider the scaffolding of the polytope $P$ with PALP ID 1836 shown in Figure 15. The corresponding variety $Y_S$ is determined by the weight matrix

| $a_0$ | $a_1$ | $b_0$ | $b_1$ | $b_2$ | $c_0$ | $c_1$ |
|------|------|------|------|------|------|------|
| 1    | 1    | 0    | 0    | 0    | 1    | 1    |
| 0    | 0    | 1    | 1    | 1    | 1    | 1    |

and stability condition $(2,1)$. The toric variety $X_P$ is cut out of $Y_S$ by a pair of binomial sections of $O(1,2)$. Observe that the linear system $(1,2)$ is not nef on $Y_S$, and has base locus $B = \{b_0 = b_1 = b_2 = 0\}$. We claim that $Y$ is obtained from $Y_S$ by blowing up $B$. It is
Figure 16. Secondary fan of the toric variety $Y_S$, used to construct 3–5.

clear that the weight matrix defining $Y$ is the same as the defining the toric variety $\text{Bl}_B Y_S$. Moreover, the map defined by setting

$$\phi: (x_0, x_1, y_0, y_1, z_0, z_1, t) \mapsto (x_0 t, x_1 t, y_0, y_1, y_2, z_1, z_2)$$

has pull-back defined by the matrix

$$[\phi^*] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}. $$

Hence, considering the ample class $\omega = (2, 1)$, $\phi^* \omega = (2, 1, 2)$. It remains to analyse the effect of crossing the wall in the secondary fan of $Y$ generated by $(1, 0, 0)$ and $(1, 1, 1)$. We observe that moving the stability condition into this wall contracts the divisor $t = 0$ (defining the ray generated by $(0, 0, 1)$) to the locus $\{y_0 = y_1 = y_2 = 0\}$.

We claim that general sections of $E := \mathcal{O}(2, 1)$ are smooth. If so, the blow-up of the base locus is an isomorphism on general sections, as the restriction of the base locus to a general fibre is a Cartier divisor. Smoothness follows directly from the Jacobian condition. Indeed, sections of $E$ are of the form

$$c_0 f_1 + c_1 g_1 + a_0 f_2 + a_1 g_2$$

where $f_j$ and $g_j$ are homogeneous polynomials of degree $j \in \{1, 2\}$ in $b_0, b_1, b_2$. Taking two such sections the corresponding Jacobian matrix, evaluated at $b_0 = b_1 = b_2$ and – without loss of generality – $a_0 = c_0 = 1$, has the form $(0 \ L)$; a block matrix consisting of a $2 \times 2$ zero block and a $2 \times 3$ matrix $L$ of linear forms in $c_1$. Since the locus $F$ in $\mathbb{P}^5$ where a $2 \times 3$ matrix drops rank has codimension 2, any projective line in this space which misses $F$ determines a matrix $L$ which does not drop rank.

**Rank 3 number 14.** We consider the reflexive polytope $P$ with PALP ID 142, together with the scaffolding $S$ with shape $Z = \mathbb{P}^1$ shown in Figure 17. This scaffolding expresses $X_P$ as a hypersurface of tri-degree $(3, 1, 1)$ in the toric variety $Y_S$ with weight matrix

| $x_0$ | $x_1$ | $x_2$ | $y_0$ | $y_1$ | $z_0$ | $z_1$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | 1     | 1     | 0     | 1     | 0     | 2     |
| 0     | 0     | 0     | 1     | 1     | 0     | 0     |
| 0     | 0     | 0     | 0     | 0     | 1     | 1     |

and stability condition $\omega = (3, 1, 1)$. Note that $Y_S$ is not $\mathbb{Q}$-factorial around the point $w := \{x_0 = x_1 = x_2 = y_0 = z_0 = 0\}$. However, since the monomial $y_1 z_1$ defines a section of $\mathcal{O}(3, 1, 1)$ – and this does not vanish along $w$ – a general hypersurface $X$ with tri-degree $(3, 1, 1)$
Figure 17. Scaffolding used to construct 3–14

misses this locus. Moving \( \omega \in \text{Pic}(Y_S)_\mathbb{R} \) to \((4,1,1)\) induces a resolution of this singularity which restricts to an isomorphism of \( X \), and recovers the ambient space considered in \([9, p.70]\). Hence – by the argument given in \([9, p.70]\) – the hypersurface \( X \) is isomorphic to a Fano variety in the family 3–14.

**Remark 4.6.** We could also construct varieties in this family using the scaffolding \( S' \) obtained by combining the two struts containing the origin in \( N_{\mathbb{R}} \) into a single line segment of length two. This produces an embedding \( X_P \to Y_S \), where \( Y_S \) is given by the weight matrix

\[
\begin{array}{cccccc}
 s & x_0 & x_1 & x_2 & y & z \\
 1 & 0 & 0 & 0 & 1 & 1 \\
 0 & 1 & 1 & 1 & 1 & 2 \\
\end{array}
\]

and stability condition \( \omega = (1,3) \). \( X_P \) is the vanishing locus of the binomial \( yz = s^2x_0^3 \).

**Rank 3 number** 16. Varieties in this family are obtained by blowing up \( B_7 = \text{Bl}_{pt} \mathbb{P}^3 \) with centre the strict transform of a twisted cubic passing through the centre of the blow-up \( B_7 \to \mathbb{P}^3 \).

We can recover the construction used in \([9, p.71]\) using a scaffolding of a reflexive polytope. Indeed, consider the polytope \( P \) with PALP ID 1091, together with the scaffolding \( S \) displayed in Figure 18 with shape \( Z = \mathbb{P}^1 \times \mathbb{P}^1 \). Note that this scaffolding is not full, and \( P_0 \) is not cracked along the fan defined by \( Z \). The toric variety \( Y_S \) is determined by the weight matrix

\[
\begin{array}{cccccc}
 x_0 & x_1 & x_2 & y_0 & z & s & z_0 & z_1 \\
 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{array}
\]

together with the stability condition \( \omega = (2,2,1) \). The toric variety \( X_P \) is defined by the vanishing of a pair of binomial sections of \( \mathcal{O}(1,1,1) \). A stability condition which lies in the cone spanned by \( \langle (1,0,0),(1,1,0),(1,1,1) \rangle \) determines the toric variety \( \hat{Y}_S \) used in \([9]\) to construct Fano varieties in 3–16. However \( \omega \) lies in the wall spanned by a pair of these vectors. Moving \( \omega \) into the chamber used in \([9]\) resolves the singular locus \( \{x_0 = x_1 = x_2 = y_0 = z_0 = z_1 = 0\} \). However general sections of \( \mathcal{O}(1,1,1) \) do not vanish along this point, and hence the intersection of two general divisors of tri-degree \((1,1,1)\) are isomorphic to varieties in the family 3–16.
In order to provide a construction using a cracked polytope, we consider the scaffolding $S'$ of $P$ with shape $Z = dP_6$, also shown in Figure 18.

The scaffolding $S'$ defines the weight matrix

\[
\begin{array}{cccccccc}
  x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

and stability condition $(2, 1)$. Let $Y$ denote the toric variety determined by the weight matrix

\[
\begin{array}{cccccccc}
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

and stability condition $(2, 1)$. Note that general sections of $\mathcal{O}(1, 1)^{\oplus 2}$ define subvarieties of $Y$ isomorphic to $Y_{S'}$. There is a map $\theta: Y_S \hookrightarrow Y$ – analogous to the Segre embedding map $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ – sending

\[
(x_0, x_1, x_2, y_0, z, s, z_0, z_1) \mapsto (x_0y_0, x_1y_0, x_2y_0, s, x_0z_0, x_0z_1, x_1z_0, x_1z_1, x_2z_0, x_2z_1, z).
\]

We have that $\theta^*\mathcal{O}(1, 0) = \mathcal{O}(1, 1, 0)$, while $\theta^*\mathcal{O}(0, 1) = \mathcal{O}(0, 0, 1)$. Hence the ample line bundle $\mathcal{O}(2, 1)$ pulls-back to $\mathcal{O}(2, 2, 1)$. This class is not ample on $Y_S$ and the image of the induced morphism $Y_S \to Y$ factors through the contraction $\hat{Y}_S \to Y$. Indeed, we have the commutative diagram of embeddings

\[
\begin{array}{ccc}
X_P & \hookrightarrow & Y_S \\
\downarrow & & \downarrow \\
X_P & \hookrightarrow & Y_{S'} \\
\end{array}
\]

We can deform $X_P$ in $Y_S$ by moving the section of $\mathcal{O}(1, 1, 1)^{\oplus 2}$ cutting out $X_P$. In other words, we obtain varieties in the family 3–16 in $Y_{S'}$ in codimension 4 by embedding $Y_S \to Y$ and moving the sections used to cut out $Y_{S'}$.

**Rank 4, number 2.** Varieties in this family are obtained from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by blowing up a curve of tri-degree $(1, 1, 3)$.

We consider the polytope with PALP ID 1080, together with the scaffolding shown in Figure 19, with shape $Z = \mathbb{P}^2$. This scaffolding describes $X_P$ as a hypersurface of tri-degree.
Figure 19. Scaffolding used to construct 4–2

$(1, 1, 2)$ in the toric variety $Y_S$ determined by the weight matrix

| $x_0$ | $x_1$ | $y_0$ | $y_1$ | $z_0$ | $z_1$ | $z_2$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | 1     | 0     | 0     | 0     | 0     | 0     |
| 0     | 0     | 1     | 1     | 0     | 0     | 1     |
| 0     | 0     | 0     | 0     | 1     | 1     | 1     |

and stability condition $\omega = (1, 1, 2)$. The variety $Y_S$ is the projectivisation of the bundle $\mathcal{O}^\oplus 2 \oplus \mathcal{O}(0, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Note that the line bundle $\mathcal{O}(1, 1, 2)$ is not nef, and that its base locus is section of the projection $Y_S \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by $z_0 = z_1 = 0$. Blowing up this base locus we obtain the variety $F$ considered in [9, p.82]. To check smoothness of general hypersurfaces in this linear system, note that general sections of $L$ have the form

$$f = z_0^2 f_{1,1} + z_0 z_1 g_{1,1} + z_1^2 h_{1,1} + z_0 z_2 f_1 + z_1 z_2 g_1,$$

where $f_{1,1}$ and $g_{1,1}$ are polynomials of bidegree $(1, 1)$ in $x_0, x_1, y_0, y_1$, while $f_1, g_1$ are linear polynomials in $x_0, x_1$. Restricting the Jacobian to the locus $z_0 = z_1 = 0$, we see that the locus $\{f = 0\}$ is singular precisely when $f_1 = g_1 = 0$. However this locus is empty for general choices of $f_1$ and $g_1$.

Since the restriction of the base locus of this linear system to a smooth member $X$ is a Cartier divisor in $X$, its blow up is an isomorphism. Hence such hypersurfaces $X$ are members of the family 4–2, and $X_P$ is the central fibre of a toric degeneration in this family.

Rank 4 number 6. Varieties $X$ in the family 4–6 are obtained by blowing up $\mathbb{P}^2 \times \mathbb{P}^1$ in curves of bidegree $(1, 2)$ and $(0, 1)$ respectively. Consider the polytope $P$ with PALP ID 425, together with the scaffolding $S$ with shape $\mathbb{P}^1 \times \mathbb{P}^1$ illustrated in Figure 20.

The toric variety $Y_S$ is defined by the weight matrix

| $s_0$ | $s_1$ | $s_2$ | $y_0$ | $y_1$ | $x_0$ | $x_1$ | $x_2$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| 1     | 1     | 1     | 0     | 0     | 0     | 0     | 0     |
| 0     | 0     | 0     | 0     | 0     | 1     | 1     | 1     |
| 0     | 0     | 1     | 1     | 0     | 0     | 0     | 0     |

and stability condition $\omega = (1, 1, 2)$. The secondary fan of $Y_S$ is illustrated in Figure 21.
The variety $Y_S$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(1,0))$; and the two chambers in the secondary fan correspond to isomorphic varieties – despite the presence of a non-trivial flopping locus. The projection $\pi: Y_S \to \mathbb{P}^2 \times \mathbb{P}^1$ corresponds to projecting out the variables $s_i$ for all $i \in \{0, 1, 2\}$. The toric variety $X_P$ is cut out of $Y_S$ by the binomial equations

$$s_2 x_2 = s_0 x_0 y_0 \quad s_1 x_1 = s_0 x_0.$$

These are sections of the line bundles $L_1$ and $L_2$, with weights $(1, 1, 1)$ and $(1, 1, 0)$ respectively. Note that $L_1$ is nef while $L_2$ is not.

Let $X$ be the vanishing locus of a general section $s = l_1 + l_2$ of $E := L_1 \oplus L_2$. The section $l_1 \in \Gamma(Y_S, L_1)$ has the general form $s_0 f_{1,1} + s_1 g_{1,1} + s_2 h_1$, where $f_{1,1}$ and $g_{1,1}$ have bi-degree $(1, 1)$ in $x_0, x_1, x_2$ and $y_0, y_1$ respectively; while $h_1$ has bi-degree $(1, 0)$. Similarly $l_2$ has the general form $s_0 f_1 + s_1 g_1$, where $f_1$ and $g_1$ have bi-degree $(1, 1)$.

Fibres of the restriction of $\pi$ to $X$ are given by the kernel of the matrix

$$\begin{pmatrix} f_{1,1} & g_{1,1} & h_1 \\ f_1 & g_1 & 0 \end{pmatrix}.$$
T. PRINCE

Figure 22. Scaffolding used to construct 5–1

That is, \( \pi \) is a graph away from the locus at which this matrix has rank \( \leq 1 \). This locus in \( \mathbb{P}^2 \times \mathbb{P}^1 \) has two connected components, one given by \( h_1 = f_{1,1}g_1 - g_{1,1}f_1 = 0 \), a curve of bidegree \((1,2)\), and the other by \( f_1 = g_1 = 0 \), a curve of degree \((0,1)\). Thus the morphism \( \pi \) exhibits \( X \) as a Fano threefold in the family 4–6.

**Rank 5 number 1.** Varieties in this family are obtained by first blowing up a quadric in a conic – obtaining a variety \( V \) in the family 2–29 – and blowing up \( V \) in three exceptional lines. Consider the scaffolding \( S \) of the polytope with PALP ID 1082 with shape \( \mathbb{P}^2 \), illustrated in Figure 22.

That is, we consider general hypersurfaces \( X \) of tri-degree \((1,2,1)\) in the toric variety \( Y_S \) defined by the weight matrix

\[
\begin{array}{cccccc}
  s_0 & s_1 & x_0 & x_1 & x_2 & y_0 & y_1 \\
  1 & 1 & 0 & 1 & 1 & 0 & -1 \\
  0 & 0 & 1 & 1 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

and stability condition \((2,1,1)\). The variety \( Y_S \) admits a map to \( \mathbb{P}^1 \) (with co-ordinates \((s_0 : s_1)\)), giving \( Y_S \) the structure of a \( \mathbb{P}^2 \times \mathbb{P}^1 \) fibre bundle. The variety \( X \) also admits a morphism to \( \mathbb{P}^1 \), whose fibres are surfaces of bi-degree \((2,1)\) in \( \mathbb{P}^2 \times \mathbb{P}^1 \). Projecting \( \mathbb{P}^2 \times \mathbb{P}^1 \) to \( \mathbb{P}^2 \) we see that any such smooth fibre is the blow up of \( \mathbb{P}^2 \) in four (general) points; that is, isomorphic to the del Pezzo surface \( dP_5 \).

Hypersurfaces of tri-degree \((1,2,1)\) have general form

\[
y_0x_0(x_0f_1 + p_1) + y_1(x_0^2f_2 + x_0f_1q_1 + p_2),
\]

where \( p_i, q_i \in \mathbb{C}[x_1, x_2] \) and \( f_i \in \mathbb{C}[s_0, s_1] \) are homogeneous polynomials of degree \( i \) for each \( i \in \{1, 2\} \). Let \( X \) denote the vanishing locus of this polynomial. Note that \( X \) contains the surface \( \{x_0 = y_1 = 0\} \). Fixing a point \((s_0, s_1) \in \mathbb{P}^1\), the \( dP_5 \) fibre of the projection \( X \to \mathbb{P}^1 \) is obtained by blowing up the intersection points of the conics \( C_1 := \{x_0(x_0f_1 + p_1) = 0\} \) and \( C_2 := \{(x_0^2f_2 + x_0f_1q_1 + p_2) = 0\} \) in \( \mathbb{P}^2 \) (with homogeneous co-ordinates \((x_0 : x_1 : x_2)\)). First consider the case \( x_0 = p_2 = 0 \). Choosing a general \( p_2 \), we find two distinct reduced points \( \alpha_1, \alpha_2 \) in \( C_1 \cap C_2 \); these are independent of the choice of \( s = (s_0, s_1) \in \mathbb{P}^1 \). The other two solutions
depend on $s$, and lie in the line $(x_0 f_1 + p_1) = 0$. Note that we may choose co-ordinates such that $C_1$ is defined by $\{x_0 x_1 = 0\}$.

Hence we can construct four surfaces, each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, contained in $X$: two surfaces $-S_1$ and $S_2$ – swept out by $\{x_0 \times \mathbb{P}^1_{(s_0, s_1)}\}$, the surface $S_3$ swept out by $C_1$ over $\mathbb{P}^1_{(s_0, s_1)}$, and the base locus $S_4 = \{x_0 = y_1 = 0\}$. Each of these surfaces restrict to exceptional curves in the $dP_3$ fibres. Note that fibres of $X \to \mathbb{P}^1$ are not all smooth – there are two singular fibres – but they are smooth in a neighbourhood of $\bigcup_{i \in [4]} S_i$. Hence – applying a relative version of Castelnuovo’s criterion – we can have a morphism $X \to X'$ which contracts the disjoint surfaces $S_1$, $S_2$, and $S_3$ to sections of the induced morphism $\pi: X' \to \mathbb{P}^1_{(s_0, s_1)}$. The smooth fibres of $\pi$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, while singular fibres have a single nodal singularity; these are isomorphic to $\mathbb{P}(1, 1, 2)$. The surface $S_4$ is the strict transform of a surface $S'_4$, which intersects every fibre $F$ in a smooth section of $-\frac{1}{2}K_F$.

Letting $\rho(X)$ denote the Picard rank of $X$, we have that $\rho(X) = \rho(X') + 3$. Since $X_P$ – and hence $X$ – has degree 28, we can conclude from the classification of Fano 3-folds that if $\rho(X') \geq 2$, $X$ is in the family 5–1. This is easily seen from the Leray spectral sequence

$$H^i(\mathbb{P}^1, R^j\pi_*\mathcal{Q}) \Rightarrow H^{i+j}(X', \mathcal{Q});$$

indeed – since $H^1(F, \mathcal{Q}) = 0$ for all fibres $F$ of $\pi$ – we have $h^0(\mathbb{P}^1, R^2\pi_*\mathcal{Q})$. However $h^0(\mathbb{P}^1, R^2\pi_*\mathcal{Q}) \geq 1$ since the surface $S'_4$ defines a non-trivial class in $H^2(F, \mathcal{Q})$ for every fibre $F$.

**Remark 4.7.** Comparing our construction with that made by Mori–Mukai [19], they first consider the blow up of a quadric threefold in a conic. Restricting the projection $\mathbb{P}^4 \to \mathbb{P}^1$ this blow-up defines $X'$, a quadric surface bundle over $\mathbb{P}^1$ with two singular fibres (with singularities are disjoint from the exceptional locus). Note that the exceptional locus distinguishes a conic $C$ in each fibre of $\pi$. To obtain varieties in 5–1 we then blow-up $X'$ in three exceptional lines. These lines are sections of the map $X' \to \mathbb{P}^1$ defined by a triple of points on the distinguished conic $C$ in each fibre. That is, the surface $S_4$ is the strict transform of the exceptional locus obtained by the blow-up of the quadric threefold; while $S_i, i \in \{1, 2, 3\}$ are obtained by blowing up exceptional lines.

### 4.1. Products

The remaining non-toric Fano threefolds $X$ with $-K_X$ very ample are products of non-toric del Pezzo surfaces with $\mathbb{P}^1$. That is, $dP_k \times \mathbb{P}^1$ for $k \in \{3, 4, 5\}$. We can easily construct toric degenerations of these from degenerations of $dP_k$ for each $k$. Fix a reflexive polygon $Q$ such that $Q^\circ$ is cracked along the fan of a shape variety $Z'$, together with a scaffolding $S'$ of $Q$ with shape $Z'$. We can produce a scaffolding $S$ of $\text{conv}(Q, (0, 0, 1), (0, 0, -1))$ with shape $Z := Z' \times \mathbb{P}^1$ by setting $S = \{(\pi'_i(D, \chi)) : (D, \chi) \in S'\} \cup \{\pi'_a D\}$ where $D$ is the toric boundary of $\mathbb{P}^1$, and $\pi_i$ is the $i$th projection from $Z' \times \mathbb{P}^1$. The example of $dP_3 \times \mathbb{P}^1$, together with a scaffolding with shape $Z = \mathbb{P}^2 \times \mathbb{P}^1$ is illustrated in Figure 23, setting $a = 1$ and $b = 3$. We thus produce toric degenerations embedded in the following spaces.

1. $dP_3 \times \mathbb{P}^1 \to \mathbb{P}^3 \times \mathbb{P}^2$,
2. $dP_4 \times \mathbb{P}^1 \to \mathbb{P}^4 \times \mathbb{P}^2$,
3. $dP_5 \times \mathbb{P}^1 \to \mathbb{P}^8 \times \mathbb{P}^2$.

### 4.2. $-K_X$ not very ample.

There are 7 families of Fano threefolds $X$ for which $-K_X$ is not very ample. These fall into three distinct groups. We first consider the varieties

- $B_1$, a sextic in $\mathbb{P}(1, 1, 1, 2, 3)$; and,
- $V_2$, a sextic in $\mathbb{P}(1, 1, 1, 1, 3)$. 
Figure 23. Scaffolding used to construct $dP_n \times \mathbb{P}^1$ for $n \leq 3$.

Figure 24. The scaffolding used to construct $B_i$ for each $i \in [3]$.

Writing $x_i$ for homogeneous co-ordinates of degree 1, and $y, z$ for those of degree 2 and 3 respectively, $B_1$ degenerates to the toric hypersurface $x_2yz = x_0^6$; while $V_2$ degenerates to the toric variety $x_1x_2x_3z = x_0^6$. These toric varieties correspond to scaffoldings of non-reflexive toric varieties with shape $\mathbb{P}^2$ and $\mathbb{P}^3$ respectively. The scaffolding used to construct $B_1$ is illustrated in Figure 24 in the case $(a, b) = (2, 6)$. The details of these constructions follow those described in §3.3.

The next three families have Picard rank 2.

- 2–1, the blow up of $B_1$ is an elliptic curve formed by intersecting two members of $-\frac{1}{2}K_{B_1}$.
- 2–2, a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched along a divisor of bidegree $(2, 4)$.
- 2–3, the blow up of $V_2$ is an elliptic curve formed by intersecting two members of $-\frac{1}{2}K_{V_2}$. 
In each case a toric complete intersection construction is given in [9], and each construction admits a toric degeneration to an embedding described by Laurent inversion. The corresponding scaffoldings have shapes \( \mathbb{P}^2 \times \mathbb{P}^1 \), \( \mathbb{P}^3 \), and \( \mathbb{P}^2 \times \mathbb{P}^1 \) respectively. Letting \( (x_0 : x_1 : x_2 : y : z) \) be homogeneous co-ordinates on \( \mathbb{P}(1, 1, 1, 2, 3) \), and \( (s_0 : s_1) \) be co-ordinates on \( \mathbb{P}^1 \), varieties in the family 2–1 degenerate to the toric variety given by the binomial equations

\[
\begin{align*}
    x_2y^2 &= x_0^6 \\
    x_1s_1 &= x_0s_0
\end{align*}
\]

in \( \mathbb{P}(1, 1, 1, 2, 3) \times \mathbb{P}^1 \). Varieties in the family 2–3 degenerate to the toric variety given by the binomial equations

\[
\begin{align*}
    x_2x_3y &= x_0^4 \\
    x_1s_1 &= x_0s_0
\end{align*}
\]

where \( (x_0 : x_1 : x_2 : x_3 : y) \) are homogeneous co-ordinates on \( \mathbb{P}(1, 1, 1, 1, 2) \). Finally, varieties in the family 2–2 degenerate to the hypersurface \( x_1y_1y_2w = x_0^2y_0^4 \) in the variety \( F \) described in [9, p.25].

The remaining two varieties are the products

- \( dP_2 \times \mathbb{P}^1 \), recalling that \( dP_2 \) is a quartic in \( \mathbb{P}(1, 1, 1, 2) \) ; and,
- \( dP_4 \times \mathbb{P}^1 \), recalling that \( dP_4 \) is a sextic in \( \mathbb{P}(1, 1, 2, 3) \).

Let \( Q_1 \) and \( Q_2 \) denote the polygons associated to the toric varieties given by the binomials \( \{x_1x_2y = x_0^4\} \) and \( \{x_0^6 = x_1yz\} \) respectively. \( Q_1 \) and \( Q_2 \) are triangles and the corresponding scaffolding (with shape \( \mathbb{P}^2 \)) covers each of these with a single strut. Hence we can scaffold conv\((Q_i, (0, 0, 1), (0, 0, -1))\) with a pair of struts – following the constructions made in §4.1 – embedding \( dP_2 \times \mathbb{P}^1 \rightarrow \mathbb{P}(1, 1, 1, 2) \times \mathbb{P}^2 \) and \( dP_4 \times \mathbb{P}^1 \rightarrow \mathbb{P}(1, 1, 2, 3) \times \mathbb{P}^2 \). These scaffoldings are illustrated in Figure 23, setting \((a, b) = (1, 4)\) and \((a, b) = (2, 6)\) respectively.

5. Classifying cracked 3-topes

5.1. One dimensional shape variety. We refer to polytopes cracked along the fan of \( \mathbb{P}^1 \) as cracked in half, since their intersection with a pair of half spaces form unimodular polytopes. This class of polytopes is explored in greater detail – and in the four dimensional setting – in [10].

Since – by [29, Proposition 2.5] – polytopes cracked in half are reflexive, we can proceed from the classification of reflexive 3-topes. Given a reflexive polytope \( P \subset M_\mathbb{R} \), we define \( V_\mathbb{P} \) to be the vector space spanned by the vertices \( v \in P \) such that the tangent cone \( C_v \) to \( P \) at \( v \) is not unimodular. If \( P \) is cracked along \( \mathbb{P}^1 \) these must lie in a proper linear subspace of \( M_\mathbb{R} \). Moreover, by [29, Proposition 2.8], no facet of \( P^o \) contains an interior point. We use Magma to search for reflexive polytopes meeting both these conditions, and obtain a list of 91 reflexive 3-topes. In 73 cases \( V_\mathbb{P} \) is two-dimensional, and hence unique determines the direction of the line segments used to scaffold \( P^o \). The remaining polytopes contain a square facet, which admits two possible full scaffoldings.

Testing which of these 91 polytopes are cracked in half, we find there are 82 three dimensional polytopes cracked along the fan of \( \mathbb{P}^1 \); we list these reflexive polytopes in Table 6. These polytopes are specified by the Kreuzer–Skarke list of reflexive 3-topes. Note that, as elsewhere, we index this list from zero. The column \textit{Fano} indicates the families Fano threefolds \( X \) for which there is a mirror Minkowski polynomial – see [8, 9] – \( f \) such that \( \text{Newt}(f) \) is isomorphic to the reflexive polytope with the indicated ID. Note that in each case there is at most one such family of Fano threefolds. Applying Laurent inversion to a full scaffolding
on $P$ with shape $Z = \mathbb{P}^1$, we obtain $X_P$ as a Fano hypersurface. We expect to recover $X$ by passing to a general hypersurface, although we have only partial results in this direction.

**Proposition 5.1** ([27]). For each $P$ in Table 6 with no associated Fano threefold, $X_P$ is not smoothable.

**Proof.** The list of reflexive 3-topes with no associated Fano in Table 6 is a subset of the list of non-smoothable Fano threefolds which appears in work of Petracci [27, p.10].

**Proposition 5.2** ([13]). For each $P$ in Table 6 such that each torus invariant point of $X_P$ is either a smooth point, or an ordinary double point, $X_P$ smooths to the associated Fano indicated.

**Proof.** By Namikawa’s results [25] all such toric varieties admit a smoothing. The invariants of the smoothed varieties were computed by Galkin in [13].

Assuming the toric Fano varieties associated to the reflexive polyhedra listed in Table 6 all smooth as indicated, there are 22 non-toric Fano threefolds obtained from polytopes cracked along the fan of $Z = \mathbb{P}^1$; these are:

| PALP ID | Fano | PALP ID | Fano | PALP ID | Fano |
|---------|------|---------|------|---------|------|
| 1       | $Q^3$| 69      | 2–31 | 202     | 3–14 |
| 3       | $Q^3$| 71      | 2–29 | 204     | 3–23 |
| 13      | 2–30 | 72      | 3–26 | 206     | 3–24 |
| 14      | -    | 73      | 3–25 | 207     | 3–20 |
| 15      | -    | 74      | 3–19 | 211     | 3–18 |
| 17      | 3–27 | 75      | 3–22 | 213     | 3–21 |
| 18      | 2–29 | 76      | 3–23 | 214     | 4–10 |
| 19      | 3–31 | 77      | 3–24 | 215     | 4–8  |
| 20      | 2–31 | 78      | 3–24 | 216     | 4–9  |
| 21      | 2–32 | 79      | 3–20 | 217     | 4–8  |
| 22      | 2–32 | 80      | 3–28 | 288     | 3–9  |
| 23      | 2–34 | 130     | 2–29 | 340     | 3–18 |
| 33      | 2–28 | 142     | 3–14 | 343     | 3–9  |
| 45      | 2–31 | 170     | 2–29 | 345     | 4–9  |
| 51      | 3–28 | 177     | 3–23 | 353     | 3–9  |
| 54      | 2–28 | 179     | 4–10 | 373     | 3–9  |
| 56      | 3–19 | 180     | 4–12 | 392     | 3–18 |
| 57      | -    | 183     | 3–21 | 403     | 3–20 |
| 58      | -    | 185     | 3–14 | 407     | 5–2  |
| 59      | 4–13 | 189     | 4–12 | 408     | 4–6  |
| 60      | -    | 190     | 4–11 | 425     | 4–6  |
| 61      | 4–11 | 191     | -    | 426     | 4–5  |
| 62      | 3–18 | 192     | -    | 682     | 4–5  |
| 63      | 3–22 | 193     | 5–2  | 683     | 4–3  |
| 64      | -    | 194     | 5–3  | 726     | 4–5  |
| 65      | -    | 195     | 4–5  | 727     | 4–3  |
| 66      | 4–9  | 196     | -    | 734     | 4–3  |
| 68      | 2–28 |         |      |         |      |

Table 6. Reflexive polytopes cracked in two.
Given a polytope \( Q^3 \), 2–29, 2–30, 2–31, 2–32, 3–14, 3–18, 3–19, 3–20, 3–21, 3–22, 3–23, 3–24, 4–3, 4–5, 4–6, 4–8, 4–9, 4–10, 4–11, 4–12, 4–13.

5.2. Classification algorithm. We present the general form of an algorithm which we can use to classify three dimensional polytopes cracked along a given two dimensional fan \( \Sigma \).

Fixing a choice of \( Z \), use to classify three dimensional polytopes cracked along a given two dimensional fan \( \Sigma \).

5.2. Classification algorithm. We present the general form of an algorithm which we can use to classify three dimensional polytopes cracked along a given two dimensional fan \( \Sigma \). Fixing a choice of \( Z \), and letting \( \Sigma \) denote the corresponding fan, we first divide cases among possible wrapping polyhedra.

Definition 5.3. Given a polytope \( P \) cracked along a fan \( \Sigma \), let \( C_v \) denote the tangent cone to \( P \) at a point \( v \in P \). The wrapping polyhedron of \( P \) is the intersection of cones \( C_v \) as \( v \) varies over the primitive ray generators of \( \Sigma \).

Note that the set of primitive ray generators is empty in the case \( Z = \mathbb{P}^1 \), and need not be a subset of the vertex set of \( P \) for any choice of shape \( Z \).

Lemma 5.4. Given a shape variety \( Z \) determined by a fan \( \Sigma \) in \( M_{\mathbb{R}} \), and a ray \( \rho \in \Sigma[1] \), let \( Z_\rho \) denote the codimension one torus invariant subvariety of \( Z \) determined by \( \rho \). There is a canonical inclusion, with bounded image, from the set wrapping polyhedra of reflexive polytopes \( P \) cracked along \( \Sigma \) to the set of possible wrapping polyhedra is contained in the cone

\[
\prod_{\rho \in \Sigma[1]} \{ \text{Amp}(Z_\rho) \times (M_{\mathbb{R}}/\mathbb{R}\rho) \}.
\]

Proof. Fix a splitting \( M \cong \langle v \rangle \oplus M_\rho \), and let \( \Sigma_\rho \) denote the fan in \( M_\rho \) determined by \( Z_\rho \). The tangent cone at \( v \) to a wrapping polyhedron for \( \Sigma \) determines – and is determined by – a piecewise linear function \( \theta : (M_\rho) \otimes_{\mathbb{Z}} \mathbb{R} \to M_\rho \) which is linear on each cone of \( \Sigma_\rho \), sends \( 0 \mapsto v \), and sends the cones of \( \Sigma_\rho \) into their corresponding cones in \( \Sigma \). The connected component of the complement of the image of \( \theta \) which contains 0 must be a convex set. Such maps \( \theta \) are in bijection with points in \( \text{Amp}(Z_\rho) \times (M_{\mathbb{R}}/\mathbb{R}v) \subset \text{Div}_{T_{M_\rho}}(Z_\rho) \cong \mathbb{Z}^r \), for some \( r \in \mathbb{Z}_{\geq 0} \). Hence the set of possible wrapping polyhedra is contained in the cone required.

To show this region is bounded, first note that each ray \( \tau \) of \( \Sigma_\rho \) corresponds to a cone in \( \Sigma \) of dimension 2; generated by \( v \) and some \( v' \in M \). Since \( v' \) must be in the same connected component as 0 of \( M_{\mathbb{R}} \setminus \theta((M_\rho) \otimes_{\mathbb{Z}} \mathbb{R}) \), the co-ordinate of \( \theta \) – regarded as an element of \( \mathbb{Z}^r \) – corresponding to \( \tau \) is bounded. Each pair \( (\rho, \tau) \), where \( \rho \in \Sigma[1] \) and \( \tau \in \Sigma_\rho[1] \) defines a linear inequality satisfied by any tuple of piecewise linear maps \( \theta \) which define a wrapping polyhedron. The intersection of these half spaces with \( \text{Amp}(Z_\rho) \times (M_{\mathbb{R}}/\mathbb{R}\rho) \) defines a polytope, \( \mathcal{R}_\mathcal{S} \), which contains the image of each wrapping polyhedron.

Given a fan \( \Sigma \), we call a polygon contained in a two dimensional cone of \( \Sigma \) a panel. Given an element \( \varphi \in \mathcal{R}_\mathcal{S} \), we let \( \mathcal{S}(\varphi) \) denote the set of tuples of panels whose tangent cone at the generator of ray \( \rho \) of \( \Sigma \) is given by the projection of \( \varphi \) to the cone \( \text{Amp}(Z_\rho) \times (M_{\mathbb{R}}/\mathbb{R}\rho) \).

Definition 5.5. Let \( Q \) be a unimodular hollow polytope in \( M_{\mathbb{R}} \). We call \( Q \) a (reflexive) piece if \( 0 \in Q \) and – for any facet \( F \) of \( Q \) with inner normal vector \( w - w(F) = 0 \) if \( 0 \in F \), and \( w(F) = -1 \) otherwise.

The set of reflexive pieces has an obvious iterative structure: faces of reflexive pieces which contain the origin are themselves reflexive pieces. Thus the classification of reflexive pieces of dimension \( n \) makes use of the classification in dimensions \( < n \). If \( Q \) is a 3-tope there are
four cases, depending on the minimal dimension $d$ of the face of $Q$ containing 0. In particular either

(i) $Q$ is a reflexive polytope;
(ii) the origin is the unique interior point of a facet of $Q$;
(iii) the origin is the unique relative interior lattice point of an edge of $Q$, or;
(iv) the origin is a vertex of $Q$, and every edge of $Q$ containing $v$ has lattice length 1.

Note that this generalises both the notion of reflexive polytope (the first case) and the notion of top \cite{4} (the second case).

Assuming that the minimal face of $Q$ containing 0 has dimension $d$, we say that a piece $Q$ has type $3-d$. Given a smooth cone – with minimal face of dimension $d$ – and choices panels $\{p_1,\ldots,p_d\}$ in each of its facets we can attempt to classify all possible pieces of type $3-d$ whose facets are given by the specified panels. We let $P(p_1,\ldots,p_d)$ denote the set of possible pieces with facets given by the polygons $p_1,\ldots,p_d$.

**Algorithm 5.6.** Fix a complete fan $\Sigma$ in $N$ such that the dimension of the minimal cone of $\Sigma$ is at most one.

(i) Compute the integral points in the polytope $R_\Sigma$.
(ii) Exploit symmetries of $\Sigma$ to obtain a minimal subset $R$ of $R_\Sigma$, containing a representative of every isomorphism class of cracked polytope in $N_R$.
(iii) Compute the set $S(\varphi)$ for each point $\varphi \in R$, and iterate over this set of tuples of panels.
(iv) For each pair $\varphi \in R$, $p \in S(\varphi)$, and maximal cone $\sigma \in \Sigma$ let $\{p_1,\ldots,p_d\}$ be the multiset of panels contained in facets of $\sigma$ (note that $d \in \{2,3\}$). There is a finite subset $A(\varphi,p,\sigma)$ of $P(p_1,\ldots,p_d)$ such that for each polytope $Q$ in this subset, $w(v) \geq -1$ for all inner normal vectors $w$ to facets of $Q$ and vertices $v$ of polygons in $p$.
(v) Iterate over all functions from the set of maximal cones $\sigma$ in $\Sigma$ to $A(\varphi,p,\sigma)$. Test whether the union of these polytopes is itself a convex, reflexive, and cracked polytope.

5.3. **Classifying Pieces.** In order to implement Algorithm 5.6 in dimension $n$ we require a database of pieces in dimension $\leq n$. We now treat the classification of pieces in dimension $\leq 3$. Note that the classification in dimension $n$ divides into cases depending on the dimension $k$ of the minimal face containing $Q$. The cases $k = n$ and $k = n-1$ form known classes: indeed, if $k = n$, the corresponding pieces are polar dual to smooth polytopes, which have a well-known classification up to dimension 8 by Øbro \cite{26}. If $k = n-1$ the definition of reflexive piece coincides precisely with the notion of a top \cite{4,12} which is also a unimodular polytope; we call such polytopes unimodular tops.

In dimension one there are two possible cases, depending on the dimension $k$ of the minimal face of $P$ containing 0:

- If $k = 1$, $P = \text{conv}(-1,1)$ is a line segment of length two.
- If $k = 0$, $P = \text{conv}(0,1)$.

It is well-known that hollow polytopes in dimension two are either Cayley polytopes or equal to $T := \text{conv}((0,0),(2,0),(0,2))$ up to integral affine linear transformations. We have three cases for pieces $P$ in $\mathbb{R}^2$, depending on the dimension $k$ of the minimal face of $P$ containing 0:

- If $k = 2$, $P$ is a reflexive polytope, of which five are unimodular.
- If $k = 1$, $P = T$ or a quadrilateral isomorphic to $\text{conv}((0,-1),(0,1),(1,-1),(1,m))$, for some $m \in \mathbb{Z}_{\geq 0}$.
If $k = 0$, $P$ is isomorphic to
\[
\text{conv}((0,0), (0,1), (0,1), (1,m)),
\]
for some $m \in \mathbb{Z}_{\geq 0}$.

In dimension three we have four possible cases depending on $k$. In the case $k = 3$, $P$ is a unimodular reflexive polytope, of which there are 18. If $k = 2$, $P$ is a \textit{unimodular top}. We do not describe the classification of unimodular tops in dimension 3, as the algorithm used in case $Z = \mathbb{P}^1$ – see §5.1 – does not rely on this classification. Moreover, this classification is contained in that of all three dimensional tops made by Bouchard–Skarke [3].

Assume next that $k = 1$; that is, assume that 0 lies in an edge $E$ of the piece $P \subset \mathbb{R}^3$. Fixing a vertex $v \in E$, and making a change of co-ordinates, we can assume that the edges incident to $v$ are parallel to the co-ordinate lines, $E$ has direction $e_3$, and $v = (0,0,-1)$. Since $E$ is itself a reflexive piece of dimension one, $(0,0,1)$ is a vertex of $P$. Let $F_1$ and $F_2$ denote the facets of $P$ containing 0. For each $i \in \{1,2\}$, $F_i$ contains an edge $E_i$ incident to $(0,0,1)$ with direction vectors $(1,0,\alpha_i)$ and $(0,1,\alpha_2)$ respectively, such that – by the unimodularity of $F_i - \alpha_i \geq -1$. Assume without loss of generality that $\alpha_1 \geq \alpha_2$. Since $F_i$ is a reflexive piece for each $i \in \{1,2\}$ we have that, if $\alpha_i > -1$ that
\[
F_i = \text{conv}(e_3, -e_3, e_i - e_3, e_i + (\alpha_i + 1)e_3);
\]
while if $\alpha_i = -1$ we have that additional possibility that $F_i \cong T$. We let $\alpha := (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, and, fixing a value of $l \in \mathbb{Z}_{\geq 0}$, we define the Cayley polytopes $P(\alpha, l, 1)$ and $P(\alpha, l, 2)$ to be the convex hulls of the points given by the columns of the matrices
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & l & l \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & -1 & -1 & (\alpha_1 + 1) & (\alpha_2 + 1) & -1 & (\alpha_2 + l\alpha_1 + 1)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & l & l \\
1 & -1 & -1 & (\alpha_1 + 1) & (\alpha_2 + 1) & -1 & (\alpha_1 + l\alpha_2 + 1)
\end{pmatrix}
\]
respectively.

**Lemma 5.7.** Let $P_i$, $i \in [k]$ be a collection of $d$-dimensional lattice polytopes in $\mathbb{R}^d$. If $P := P_1 \star \cdots \star P_k \subset \mathbb{R}^{d+k}$ is a unimodular polytope, there is a non-singular projective toric variety $Z$ such that $P_i$ is the polyhedron of sections of an ample divisor $D_i$ on $Z$ for all $i \in [k]$.

**Proof.** Since, for any $i_1, i_2 \in [k]$, $P_{i_1} \star P_{i_2}$ is a face of $P$, we assume without loss of generality that $k = 2$. Since $P_1$ is unimodular, its normal fan defines a non-singular projective toric variety $Z$. We claim that $P_2 = P_D$ for some ample divisor on $Z$.

Note that $\text{verts}(P) = \text{verts}(P_1) \bigsqcup \text{verts}(P_2)$. Moreover, each vertex of $P_1$ is contained in $d$ edges of $P_1$ and $(d + 1)$ edges of $P$. Hence, fixing a facet $F$ of $P$ different from $P_1$ and $P_2$, $F \cap P_1$ is equal to a facet $G$ of $P_1$. $G$ contains $(d - 1)$ edges of $P_1$ incident to $v$.

The normal fan of $P$ consequently contains a ray for each facet of $P_1$ (or $P_2$), as well as rays $\rho_1, \rho_2$ dual to $P_1$ and $P_2$ respectively. Moreover, each vertex of $P_1$ is dual to a maximal cone, generated by $\rho_1$ and rays corresponding to facets of $P_1$ containing $v$. Since the same applies to vertices of $P_2$, the toric variety associated to the normal fan of $P$ has the structure of a fibre bundle over $\mathbb{P}^1$, in particular the fibres over 0 and $\infty$ are isomorphic.

**Lemma 5.8.** If $\alpha_1, \alpha_2 > -1$, then $P$ is isomorphic to $P(\alpha, l, j)$ for some $l \in \mathbb{Z}_{\geq 0}$ and $j \in \{1,2\}$. 

The point \((1, 1, 0)\) cannot lie in the interior of \(P\), and hence there is a \(u \in \mathbb{N}\) such that \(\langle u, (1, 1, 0) \rangle \leq -1\), but \(\langle u, p \rangle \geq -1\) for any point \(p \in P\). In particular, writing \(u = (u_1, u_2, u_3)\), and recalling that that \((0, 0, \pm 1) \in P\), we have that \(u_3 \in \{-1, 0, 1\}\). Similarly, \(u_1 \geq -1 + u_3\), \(u_2 \geq -1 + u_3\), \(u_1 \geq -1 - \alpha_1 u_3\) and \(u_2 \geq -1 - \alpha_2 u_3\). Hence, if \(u_3 = 1\), \(u_1 \geq 0\) and \(u_2 \geq 0\), but no such points satisfy \(u_1 + u_2 \leq -1\). If \(u_3 = 0\), we have the solutions \((u_1, u_2) = (-1, -1)\), \((-1, 0)\), or \((0, 1)\). These all define the Cayley sum of a pair of quadrilaterals, as \(T\) is not a panel of \(P\) by the assumption that \(\alpha_i > -1\) for each \(i \in \{1, 2\}\). Since the panels of \(P\) are Cayley polytopes (the sum of two line segments) – and \(P\) is unimodular – \(P\) is the Cayley sum of a pair of polyhedra of sections of ample divisors on a (fixed) Hirzebruch surface by Lemma 5.7. Such a polytope must be of the form \(P(\alpha, l, j)\).

In the case \(u_3 = 1\) the bounds \(\alpha_i > -1\) for each \(i \in [2]\), together with the inequalities \(u_1 \geq -1 - \alpha_1 u_3\) and \(u_2 \geq -1 - \alpha_2 u_3\), ensure that there are no further cases.

Note that \(P(\alpha, 0, 1) = P(\alpha, 0, 2)\) and \(P(\alpha, -1, 1) = P(\alpha, -1, 2)\). Note also that whenever \(\alpha_1 = \alpha_2\), \(P(\alpha, l, 1) \cong P(\alpha, l, 2)\), although these polytopes are not equal. The remaining cases are \((\alpha_1, \alpha_2) = (0, -1)\) and \((\alpha_1, \alpha_2) = (-1, -1)\). In the latter case \(P\) is a sub-polytope of \(\text{conv}(-e_3, e_3, 2e_1 - e_3, 2e_2 - e_3)\), and hence there are three possible polytopes, illustrated in Figure 25. In the case \((\alpha_1, \alpha_2) = (0, -1)\), we introduce another infinite class of polytopes. Fixing a value of \(l \in \mathbb{Z}_{\geq 1}\) define the ‘wedge’ polytope \(W(l)\) to be the convex hull of the points given by the columns of the following matrix,

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & l & l & 2(l-1) \\
0 & 0 & 2 & 0 & 0 & 1 & 1 & 2 \\
1 & -1 & -1 & -1 & 1 & 0 & -1 & -1
\end{pmatrix}
\]

See Figure 5.3 for an illustration of such a polytope. We also define

\[W'(l) := W(l) \cap \{x : \langle(-1, 1, 0), x \rangle \leq 1\}\]

for each \(l\). There are additional cases which appear for small values of \(l\); in particular we define the polytopes \(W_0(l)\) to be the convex hull of the points given by the columns of the following matrix,

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 2l-1 \\
0 & 0 & 2 & 0 & 0 & 2 \\
1 & -1 & -1 & -1 & 1 & -1
\end{pmatrix}
\]

and \(W_0'(l) := W_0(l) \cap \{x : \langle(-1, 1, 0), x \rangle \leq 1\}\) for each \(l \in \{1, 2\}\).

**Lemma 5.9.** If \(\alpha = (0, -1)\), then \(P\) is isomorphic to one of

(i) \(W(l)\), for some \(l \in \mathbb{Z}_{\geq 2}\).
Figure 26. Pieces $W(2)$ and $W'(2)$.

(ii) $W'(l)$, for some $l \in \mathbb{Z}_{\geq 2}$.
(iii) $W_0(l)$ for $l \in \{1, 2\}$.
(iv) $W'_0(1)$; or,
(v) $P(\alpha, l, 1)$ for some $l \in \mathbb{Z}_{\geq 0}$.

Proof. Since $\alpha = (0, -1)$ the polytope $P$ is contained in the half-space $\{x : ((0, 1, 1), x) \leq 1\}$. Moreover $P$ is assumed to be contained in the positive orthant; that is,

$$P \subset A := \mathbb{R}_{\geq -1} \times \text{conv}(\{(0, \pm 1), (2, -1)\}).$$

We claim such pieces $P$ are determined by the facet $F = P \cap \{u : \langle u, e_3^* \rangle = -1\}$. Indeed, fixing this polygon $F$ it is easy to verify that $P = A \cap (F \times \mathbb{R})$.

The possible polygons $F$ are also easily classified. Choose co-ordinates on $\mathbb{R}^2$ such that $0$ and $(1, 0)$ are vertices of $F$. If $F \cap \{y = 2\} = \emptyset$ both $F$ and $P$ are Cayley polytopes, and $P = P(\alpha, l, 1)$ for some $l \geq 0$. Otherwise $F$ is a (possibly degenerate) hexagon with vertices given by the columns of

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & a \\ 0 & 0 & k_1 & k_2 & k_3 & 0 \end{bmatrix},$$

where $a \in \{1, 2\}$. Fix a value of $k_1 \geq 0$. By convexity and unimodularity of $F$ at $(1, k_1)$, we have that $k_2 = 2(k_1 - 1)$; unless $k_1 \in \{1, 2\}$; which gives the additional cases $(k_1, k_2) = (1, 1)$ and $(k_1, k_2) = (2, 3)$. If $a = 2$, $k_3 = 0$ and $P = W(l)$ for some $l \in \mathbb{Z}_{\geq 0}$ or $W_0(l)$ for some $l \in \{0, 1\}$. Otherwise, $a = 1$, and – by unimodularity of $F$ at $(2, k_3)$ – we have that $k_3 = 1$ (note $k_3 \neq 0$ as $(1, 0)$ is vertex of $F$). In these cases $P = W'(l)$ for some $l \in \mathbb{Z}_{\geq 1}$ or $W'_0(2)$. Note that $W(1)$ and $W'_0(1)$ are not unimodular. Moreover, $P(\alpha, l, 1) = P(\alpha, l, 1)$ for $l \in \{0, 1\}$, while $P(\alpha, l, 2)$ is not unimodular if $l > 1$.

We summarise the above calculations in the following Proposition.

Proposition 5.10. If $P$ is a 3-dimensional piece such that the origin is contained in an edge of $P$, then $P$ belongs to one of the infinite families $P(\alpha, l, j)$, one of the three exceptional cases shown in Figure 25, or one of the polytopes listed in Lemma 5.9.
Finally, assume that $k = 0$. For each $l \in \mathbb{Z}_{\geq 0}$ and $j \in \{1, 2\}$, we define the Cayley polytopes $Q(\alpha, l, j)$ to be the intersection of $P(\alpha, l, j)$ with the half-space $\{u \in \mathbb{R}^3 : \langle e_3^*, u \rangle \geq 0\}$.

**Proposition 5.11.** If $P$ is a 3-dimensional piece such that the origin is contained in an edge of $P$, then $P$ belongs to the infinite family $Q(\alpha, l, j)$. The polytope $Q(\alpha, l, j)$ is a reflexive piece if and only if one of the following hold.

(i) $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $j \in \{1, 2\}$, and $l \in \mathbb{Z}_{\geq 0}$.

(ii) $\alpha_1 = 0$, $\alpha_2 = -1$, $j = 1$ and $l \in \mathbb{Z}_{\geq 0}$.

(iii) $\alpha_1 \geq 0$, $\alpha_2 = -1$, $j = 2$ and $l = \alpha_1 + 1$.

(iv) $\alpha_1 = -1$, $\alpha_2 = -1$.

Note that the only case which occurs in the fourth case is the standard simplex.

**Proof.** In a suitable co-ordinate system, the vertex set of a piece $P$ contains 0 and each of the three standard basis vectors. The polygon $F_i := \{e_i^* = 0\} \cap P$ is a two dimensional reflexive piece, which are classified above.

In particular we may assume that each polygon $F_i$ is either a standard triangle or a Cayley sum of line segments. These polygons may be oriented relative to each other in two distinct ways, illustrated in Figure 27. We show that the first case does not include any piece which is not a special case of the second. Polytopes in the first case contain vertices $(1, 0, k_1)$, $(k_2, 1, 0)$, and $(0, k_3, 1)$. Note that we can assume that $k_i \geq 2$. If $k_i > 2$ for any $i \in [3]$, the lattice point

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure27.png}
\caption{Relative arrangement of panels.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure28.png}
\caption{An impossible arrangement of panels.}
\end{figure}
Figure 29. Piece $Q((1,0),1,1)$

$(1,1,1)$ is in the interior of the convex hull of the vertices of $P$, and hence $k_1 = k_2 = k_3 = 2$. However, as $P$ is contained in the half space $\{u \in \mathbb{R}^3 : (1,1,1) \cdot u \leq 3\}$, $P$ is a sub-polytope of the convex hull $P'$ of the vertices shown in Figure 28. Note that every vertex of this polytope is contained in a panel, and hence $P = P'$. Since $P'$ is not unimodular it does not contribute to the list of pieces.

In the second case illustrated in Figure 29, we observe that $P$ is a Cayley polytope. Indeed, assuming that $P$ contains the vertices $(1,0,k_1)$, $(k_2,1,0)$, and $(0,1,k_3)$, $P$ is the Cayley sum of the facets contained in $H_0$ and $H_1$, where $H_k := \{u : \langle e_2^*, u \rangle = k\}$. These are both 2-dimensional if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$; and in this case it follows from Lemma 5.7 that $P$ is of the form $Q(\alpha,l,j)$ for some $l \in \mathbb{Z}_{\geq 0}$ and $j \in [2]$. The classification of the remaining possible pieces follows from a case-by-case analysis. The case $\alpha = (-1,-1)$ is trivial. If $\alpha = (0,-1)$, $P$ is contained in the product of a standard simplex and a ray, and equal to some $Q(\alpha,1,l)$. If $\alpha_1 > 0$ and $\alpha_2 = -1$ we note that the polytopes $Q(\alpha,1,l)$ are not unimodular, while $Q(\alpha,2,l)$ is a Cayley polytope $P_1 \star P_2$, such that $P_1$ is a standard simplex. By Lemma 5.7 $P_2$ is a dilate of a standard simplex, and hence $l = \alpha_1 + 1$. □

References

[1] Klaus Altmann and Jan Arthur Christophersen. Cotangent cohomology of Stanley-Reisner rings. *Manuscripta Math.*, 115(3):361–378, 2004.
[2] Klaus Altmann and Jan Arthur Christophersen. Deforming Stanley-Reisner schemes. *Math. Ann.*, 348(3):513–537, 2010.
[3] Vincent Bouchard and Harald Skarke. Affine Kac-Moody algebras, CHL strings and the classification of tops. *Adv. Theor. Math. Phys.*, 7(2):205–232, 2003.
[4] Philip Candelas and Anamaria Font. Duality between the webs of heterotic and type II vacua. *Nuclear Phys. B*, 511(1-2):295–325, 1998.
[5] Jan Arthur Christophersen and Nathan Ilten. Hilbert schemes and toric degenerations for low degree Fano threefolds. *J. Reine Angew. Math.*, 717:77–100, 2016.
[6] Jan Arthur Christophersen and Nathan Owen Ilten. Degenerations to unobstructed Fano Stanley-Reisner schemes. *Math. Z.*, 278(1-2):131–148, 2014.
[7] T. Coates, A. Kasprzyk, and T. Prince. Four-dimensional Fano toric complete intersections. *Proc. Royal Society A.*, 471(2175):20140704, 14, 2015.
[8] Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander M. Kasprzyk. Mirror symmetry and Fano manifolds. In European Congress of Mathematics Kraków, 2–7 July, 2012, pages 285–300, 2014.

[9] Tom Coates, Alessio Corti, Sergey Galkin, and Alexander Kasprzyk. Quantum periods for 3-dimensional Fano manifolds. Geom. Topol., 20(1):203–256, 2016.

[10] Tom Coates, Alexander Kasprzyk, and Thomas Prince. Fano compactifications of conic bundles. In preparation.

[11] Tom Coates, Alexander Kasprzyk, and Thomas Prince. Laurent Inversion. arXiv:1707.05842 [math.AG], 2017.

[12] Ryan Davis, Charles Doran, Adam Geiss, Andrey Novoseltsev, Dmitri Skjorshammer, Alexa Syryczuk, and Ursula Whitcher. Short tops and semistable degenerations. Exp. Math., 23(4):351–362, 2014.

[13] S. Galkin. Small toric degenerations of Fano 3-folds. Preprint, available at http://member.ipmu.jp/sergey.galkin/, 2007.

[14] Mark Gross, Paul Hacking, and Sean Keel. Mirror symmetry for log Calabi-Yau surfaces I. Publ. Math. Inst. Hautes Études Sci., 122:65–168, 2015.

[15] Mark Gross, Paul Hacking, and Bernd Siebert. Theta functions on varieties with effective anti-canonical class. arXiv:1601.07081 [math.AG], 2016.

[16] Mark Gross and Bernd Siebert. Mirror symmetry via logarithmic degeneration data. I. J. Differential Geom., 72(2):169–338, 2006.

[17] Mark Gross and Bernd Siebert. From real affine geometry to complex geometry. Ann. of Math. (2), 174(3):1301–1428, 2011.

[18] Nathan Owen Ilten, Jacob Lewis, and Victor Przyjalkowski. Toric degenerations of Fano threefolds giving weak Landau-Ginzburg models. J. Algebra, 374:104–121, 2013.

[19] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. Manuscripta Math., 36(2):147–162, 1981/82.

[20] Shigefumi Mori and Shigeru Mukai. On Fano 3-folds with $B_2 \geq 2$. In Algebraic varieties and analytic varieties (Tokyo, 1981), volume 1 of Adv. Stud. Pure Math., pages 101–129. North-Holland, Amsterdam, 1983.

[21] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. I. In Algebraic and topological theories (Kinokawa, 1984), pages 496–545. Kinokuniya, Tokyo, 1986.

[22] Shigefumi Mori and Shigeru Mukai. Erratum: “Classification of Fano 3-folds with $B_2 \geq 2$” [Manuscripta Math. 36 (1981/82), no. 2, 147–162]. Manuscripta Math., 110(3):407, 2003.

[23] Shigefumi Mori and Shigeru Mukai. Extremal rays and Fano 3-folds. In The Fano Conference, pages 37–50. Univ. Torino, Turin, 2004.

[24] Shigeru Mukai. Curves, $K3$ surfaces and Fano 3-folds of genus $\leq 10$. In Algebraic geometry and commutative algebra, Vol. I, pages 357–377. Kinokuniya, Tokyo, 1988.

[25] Yoshinori Namikawa. Smoothing Fano 3-folds. J. Algebraic Geom., 6(2):307–324, 1997.

[26] Mikkel Øbro. An algorithm for the classification of smooth fano polytopes. arXiv:0704.0049 [math.CO], 2007.

[27] Andrea Petracci. Some examples of non-smoothable gorenstein fano toric threefolds. arXiv:1804.07960 [math.AG], 2018.

[28] T. Prince and L. J. Barrott. Deforming toric log del Pezzo surfaces in toric varieties. In progress.

[29] Thomas Prince. Cracked polytopes and fano toric complete intersections. arXiv:1808.04590 [math.AG], 2018.

[30] Thomas Prince. Smoothing toric Fano surfaces using the Gross-Siebert algorithm. Proc. Lond. Math. Soc. (3), 117(3):617–660, 2018.