Wilson loop and Wilczek-Zee phase from a non-Abelian gauge field

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Quantum states can acquire a geometric phase called the Berry phase after adiabatically traversing a closed loop, which depends on the path not the rate of motion. The Berry phase is analogous to the Aharonov-Bohm phase derived from the electromagnetic vector potential, and can be expressed in terms of an Abelian gauge potential called the Berry connection. Wilczek and Zee extended this concept to include non-Abelian phases — characterized by the gauge independent Wilson loop — resulting from non-Abelian gauge potentials. Using an atomic Bose-Einstein condensate, we quantum-engineered a non-Abelian SU(2) gauge field, generated by a Yang monopole located at the origin of a 5-dimensional parameter space. By slowly encircling the monopole, we characterized the Wilczek-Zee phase in terms of the Wilson loop, that depended on the solid-angle subtended by the encircling path: a generalization of Stokes’ theorem. This observation marks the observation of the Wilson loop resulting from a non-Abelian point source.

INTRODUCTION

The seemingly abstract geometry of a quantum system’s eigenstates now finds application in fields ranging from condensed-matter and quantum information science to high-energy physics. The Berry curvature — a geometric gauge field present for a single (non-degenerate) quantum state moving in any parameter space [1] — is a prime observable associated with this geometry. The Berry phase is the direct analogue to the Aharonov-Bohm phase for motion along a closed loop with the enclosed Berry curvature playing the role of magnetic field. Berry’s curvature and phase have been measured in a variety of physical systems throughout physics and chemistry [2,3], and even have an analog for planetary-scale atmospheric waves [7]. Monopoles, or conical intersections, singular points in the energy landscapes of a range of physical systems [4,5,11] where the curvature diverges, play a crucial role in geometric effects, since particles encircling the singular point can acquire non-zero Berry phase.

The Wilczek-Zee (W.-Z.) phase [12] extends these ideas to include non-Abelian ‘operator-valued’ geometric phases possible for adiabatically evolving systems with a degenerate subspace (DS). Initial nuclear magnetic resonance experiments [13,14] inspired holonomic quantum computation utilizing the W.-Z. phase to affect noise-resistant geometric quantum gates [15,19]. A non-Abelian phase has been also studied and characterized in ‘non-degenerate’ multi-band optical lattice on non-cyclic paths at the strong-force limit [10]. Despite the universality of Wilczek and Zee’s concept and the tremendous theoretical and experimental interest in synthetic non-Abelian gauge fields [19,30], there has been no realistic cold-atom scheme for robust control of non-Abelian geometric phase in an adiabatic matter, nor a measurement of non-trivial gauge-independent Wilson loop on a closed path that characterizes the non-trivial non-Abelian geometric phase in cold-atom systems.

Here, we observed and characterized the W.-Z. phase in an atomic Bose-Einstein condensate (BEC) as it underwent near-adiabatic motion in a five-dimensional (5D) parameter space with a non-Abelian Yang monopole at its origin [31,32]. Each point in this synthetic dimensional parameter space defined the Hamiltonian for four atomic hyperfine states, and the W.-Z. phase, governed by a non-Abelian SU(2) gauge field, described the adiabatic response within a spin-1/2 DS. We obtain the analog to Stokes’ theorem, connecting W.-Z. phase to the solid-angle subtended by a closed path by characterizing the phase with gauge-independent Wilson loop (WL) [33].

This geometric process, shown in Fig. 1c, can be viewed as moving a test particle in 5D around the Yang monopole, the source of the SU(2) gauge field [31,32]. The Yang monopole is a non-Abelian generalization of Dirac monopole [34], and characterized by a non-zero second Chern number. Our manuscript is organized as follows: (1) we introduce the essential physics of the WL, (2) describe our experimental setup. We then (3) show the non-Abelian operator character of the W.-Z. phase factor using quantum process tomography, and (4) characterize it in terms of the gauge-independent WL. Lastly, (5) we comment on extensions of these techniques to larger gauge groups, such as the SU(3) gauge group of the strong nuclear force.

Although the usual Berry connection, the gauge potential associated with the Berry curvature, is gauge dependent [1], both the Berry phase and Berry curvatures are gauge independent. Specifically, they are invariant under the local gauge transformation $U = e^{i\Phi(q)}$ for any choice of position-dependent phase $\Phi(q)$. In contrast, non-Abelian extensions of Berry’s phase and cur-
vatures need not be gauge invariant. The WL, defined as trace of the W.-Z. phase factor (non-Abelian holonomy) is a gauge-independent geometric quantity that reduces to the Berry phase for a single non-degenerate state [35]. It was originally considered for the problem of quark-confinement [36, 39] and is often used in formulating gauge theories. In topological quantum computation, the WL describes braiding evolution of non-Abelian anyons [37, 38]. Moreover, in crystalline systems – including both conventional materials and synthetic quantum matter – the eigenspectrum of the WL can characterize the topology of multiple Bloch bands [39, 42].

In the framework of differential geometry [43], adiabatic motion is described in terms of fiber bundles, where the fibers represent the gauge degree of freedom. As the state adiabatically evolves, a parallel transport condition sets the choice of basis state leading to a vertical lift along the fiber (Fig. 1d). After tracing out a closed loop C in space, the state will have evolved according to the unitary transformation

$$\hat{U}_C = \mathcal{P} \exp \left( i \int_C \hat{A}_q \cdot dq \right),$$

the W.-Z. geometric phase factor, i.e., the non-Abelian holonomy. Here \( \mathcal{P} \) indicates that the exponential should be evaluated in a path-ordered manner and \( \hat{A}_q \) is a non-Abelian gauge field (non-Abelian Berry connection). The cyclic property of the trace makes the WL \( W = \text{tr}(\hat{U}) \) manifestly gauge independent. Previous experimental work on a multi-band system in an optical lattice characterized the matrix elements of the non-Abelian holonomy along a non-cyclic path in crystal-momentum space and reconstructed the gauge-dependent Wilson line [40]. Similar to this work, we characterize the non-Abelian holonomy. However, we measured its process matrices instead to include potential imperfection in the analysis to prove the near-unity fidelity of our robust control, in addition to the reconstruction of the WL. We demonstrate that a gauge-independent WL on a closed path can be fully tuned, in contrast to the previous study in multi-band system [40], where gauge-independent WL on a closed path was trivial.

**RESULTS**

**Experimental setup** We prepared \(^{87}\text{Rb} \) BEC with \( \approx 1 \times 10^5 \) atoms in \( |F, m_F\rangle = |1, -1\rangle \) in a crossed optical dipole trap formed by two horizontal 1064 nm optical trapping beams. We engineered a non-Abelian SU(2) gauge field with the BECs, using four \( |F, m_F\rangle \) hyperfine ground states [22]: \{1, 0\}, \{1, -1\}, \{2, 0\}, \{2, 1\} respectively labeled \{1\}, \{2\}, \{3\}, \{4\}. The 19.8 G bias magnetic field (with 2.5 ppm long-term stability), resolved the rf and microwave transition frequencies within the hyperfine states (Fig. 1a). As shown in Fig. 1b, we coupled these states with rf and microwave fields parameterized by two Rabi frequencies \( \Omega_A \) and \( \Omega_B \) with phases \( \phi_A \) and \( \phi_B \). We parameterize the coupling ratio \( \Omega_B / \Omega_A = \tan \theta_2 \) in terms of an angle \( \theta_2 \). The system then evolved according to the Hamiltonian

$$\hat{H} = -\frac{\hbar}{2} \sum_{i=1}^{5} q_i \hat{\Gamma}_i,$$

expressed in terms of the reduced Planck constant \( \hbar \), and the five Dirac gamma matrices \( \hat{\Gamma}_i \). In addition, the vector \( \mathbf{q} = (q_1, q_2, q_3, q_4, q_5) \) defines coordinates in a 5D parameter space, and is determined by laboratory parameters \( q_1 = -\Omega_B \cos \phi_B \), \( q_2 = -\Omega_A \cos \phi_A \), \( q_3 = -\Omega_A \sin \phi_A \), \( q_4 = \delta z \), and \( q_5 = -\Omega_B \sin \phi_B \). The detuning \( \delta z \), from the linear Zeeman shift, is set to zero throughout our measurement. The Hamiltonian can be represented by a 4-by-4 matrix by taking the four hyperfine states as the basis, where the diagonal part shows the detuning, and the non-zero off-diagonal elements shows the coupling between the hyperfine states. The resulting spectrum, insensitive to environmental noise such as magnetic field fluctuations, always consisted of a pair of two-fold degenerate energy manifolds with eigenstates \{\{\uparrow, \downarrow\}(\mathbf{q})\}, \{\{\downarrow, \uparrow\}(\mathbf{q})\} for the ground state manifold (See Eq. 5) and \{\{\uparrow, \uparrow\}(\mathbf{q})\}, \{\{\downarrow, \downarrow\}(\mathbf{q})\} for the excited state manifold. Throughout this manuscript, the gap \( \langle \Delta E(q) \rangle = h |q| = h \sqrt{\Omega_A^2 + \Omega_B^2} \) is \( h \times 2.0 \) kHz, and was measured by inducing coherent Rabi-like oscillations between the eigenstates [see Methods]. Due to the two-fold DS, the underlying gauge field \( \hat{A} \) is non-Abelian and has SU(2) symmetry. When \( q \) is adiabatically changed along a closed path, the quantum state evolves within the subspace and acquires W.-Z. phases.

**Wilczek-Zee phase** The consequence of the acquired W.-Z. phase can be experimentally captured by explicitly following an initial state as it evolves within the DS as a result of adiabatically moving \( q \) in parameter space. We demonstrated this by preparing an eigenstate \( \{\{\uparrow, \downarrow\}(\mathbf{q}_0)\} = 1/\sqrt{2} \{1\} - 1/\sqrt{2} \{2\} \) at \( \mathbf{q}_0 = (-\Omega_B, -\Omega_A, 0, 0, 0) \) with \( \theta_2 = \pi/4 \) in the ground state manifold. After the state preparation, we linearly ramped the rf phase \( \phi_A(t) = 2\pi t / T \), where \( T = 2 \) ms, tracing out a closed loop \( C_+ \). We performed state tomography within the DS to compare the initial and final states.

The state within the DS is described by the Bloch vector on a Bloch sphere. The states before (blue) and after (red) the control sequence are shown in the left panel of Fig. 2a. The Bloch vector is rotated even though the final control parameters are the same as the initial ones, manifesting the operator content of the W.-Z. phase factor. This is striking in contrast with the Abelian Berry phase, which would leave the orientation on the Bloch sphere unchanged.

Geometric phases depend on ramp-direction, however,
for the Abelian case reversing the ramp direction along the same closed path simply inverts the sign of the phase. For the non-Abelian case, this relation does not hold, indeed, the right panel of Fig. 2 shows the dependance of the final state on $\theta_2$. The trajectories can be varied by changing $\theta_2$. Figure 2 shows the dependence of the final state on $\theta_2$ for both ramp directions. The control vector $\mathbf{q}$ traces out a closed loop $\mathcal{C}$ in parameter space with $\mathbf{q}(t) = (-\Omega_B, -\Omega_A \cos(2\pi t/T), -\Omega_A \sin(2\pi t/T), 0, 0)$, a circle in 5-space subtending a solid angle $\Xi = 2\pi(1 - \sin \theta_2)$ with respect to the origin. In the following sections, we will see that the states resulting from the $\pm$ ramps can be related by process tomography and WL measurement.

**Quantum process tomography** We fully characterize the process of W.-Z. phase acquisition using quantum process tomography which allows us to completely and uniquely represent arbitrary transformations. In our experiment, the path-dependent process matrix $\chi$ describes the transformation from the initial quantum state at $\mathbf{q}(t=0) = \mathbf{q}_0$ to the final state at $\mathbf{q}(t=T) = \mathbf{q}_0$, characterizing the W.-Z. phase acquisition process including any potential experimental imperfection. Under ideal unitary evolution, each element $\chi_{ij} = \text{tr}(U E_i U^\dagger)\text{tr}(U E_j^\dagger)$ is derived from the non-Abelian W.-Z. phase.

We experimentally obtain the process matrix $\chi$ by repeating the measurement illustrated in Fig. 2 for four-independent initial states ($|A\rangle = |\uparrow\rangle$, $|B\rangle = |\downarrow\rangle$, $|C\rangle = (|A\rangle + |B\rangle)/\sqrt{2}$, $|D\rangle = (|A\rangle + i|B\rangle)/\sqrt{2}$) and applying maximum likelihood estimation to obtain a
positive semi-definite and Hermitian matrix $\chi$. We took $\{E_i\} = \{I_0, \sigma_x, \sigma_y, \sigma_z\}$ as the basis.

Figure 3a, b illustrates the reconstructed process matrices of the non-Abelian W.-Z. phase obtained for the forward and the reverse ramps at $\theta_2 = \pi/4$. The two results for opposite ramps along the same path show that the real part of $\chi$ takes almost the same values, whereas the imaginary parts of $\chi$ have the opposite sign. This trend can be explained from the definition of the W.-Z. phase (Eq. 4) satisfying the relation $\dot{U}_{C+} = \dot{U}_{C-}$ and thus $\chi_{ij}(C+) = \chi_{ij}^*(C-)$ for the process matrices of the non-Abelian W.-Z. phase. The above behavior of the process matrix elements holds for different coupling ratios (i.e. $\theta_2$) as shown in Fig. 3c, where different non-Abelian W.-Z. phases are realized (See Methods). The result, which is in stark contrast to the Abelian Berry phase, is in excellent agreement with the generalized relation for holonomy.

The process matrix allows us to evaluate the fidelity of our holonomic control within the DS. Using the analytical expression for the non-Abelian holonomy $\dot{U}_c$, the fidelity of the process shown in Fig. 3a, b reached as high as $F_{\dot{U}_{C+}} = 0.98$ for forward ramp and $F_{\dot{U}_{C-}} = 0.96$ for reverse ramp even for finite ramp time. Here the fidelity is defined as $F = \text{tr}(\chi_{th}\chi)$, where $\chi_{th}$ is theoretical expected process matrix. This high fidelity then enabled us to characterize the W.-Z. phase with high accuracy. It has been argued that the non-adiabatic effect does not contribute to the state evolution in the DS up to first order, even though the state deflects from the adiabatic trend [32, 46].

Wilson loop The above measurements depend on a choice of basis, i.e., of gauge, whereas the WL does not. The absolute value of the WL is $|W_C| = 2\sqrt{\chi_{00}}$, derived from a single component of the process matrix shown in Fig. 3. Figure 4 shows $|W_C|$ for forward ($C_+$) and reverse ramps ($C_-$) as the path $C$ is varied by changing the solid angle $\Xi(\theta_2)$. The expected relation $|W_{C_+}| = |W_{C_-}|$ is evidenced in experimental data, which also shows good agreement with the analytical curve for adiabatic control. At $\Xi = 2\pi$ ($\theta_2 = 0$) our system decomposes into two uncoupled two-level systems; the geometric phase is $\dot{U}_C = -I_0$ and $W_{C_+} = -2$ results from the $\pm \pi$ Berry
WL measurement in eigenvalues of W.-Z phase factor with circular paths reduce to a point, which we trivially measure \(W\) and green points), and sum to complex plane, the eigenvalues appear on the unit circle (pink elements) along with theory (solid curve). At \(\Xi = 0\) \((\theta_2 = \pi/2)\), the circular paths reduce to a point, which we trivially measure with \(T=0\) ms (green diamond).

The phase difference of the eigenvalues of W.-Z phase factor \(\delta\lambda_{\pm}\) obtained from the WL measurement in a, with the same symbols. Inset: In the complex plane, the eigenvalues appear on the unit circle (pink and green points), and sum to \(W\) (red arrow).

**DISCUSSION**

Our experiment realized Wu and Yang’s gedanken experiment \([48]\) to apply the generalized Aharonov-Bohm effect to the SU(2) isospin doublet of neutron and a proton, as an isospin gauge field detector. Such experiments remain impractical for probing the standard model’s combined U(1) \(\times\) SU(2) \(\times\) SU(3) gauge symmetry, but further progress in quantum analogues such as ours can shed light on the operation of such experiments. An exciting next step in this direction would be creating a monopole source of a SU(3) gauge field (requiring a three-fold degenerate manifold), in analog to the Dirac monopole’s U(1) gauge field (for a single non-degenerate state) and the Yang monopole’s SU(2) gauge field (the two-fold degenerate manifold discussed here).

Our experiments demonstrated essentially the full set of high-fidelity SU(2) holonomic control in a subspace which was protected against environmental noise and imperfections. In the Bloch sphere picture, the process can be regarded as holonomic single qubit gate operation \([15]\), where the Bloch vector is rotated by an angle of \(\pm 2\pi \sin \theta_2\) around an axis \((-\cos \theta_2, 0, \sin \theta_2)\) depending on the path \(C\). Universal operation are possible with more general path for Eq. \(\Xi\) since there is no experimental limitation in our implementation. We note that our four level system can be used to code two qubits simultaneously – one per degenerate manifold – this may have application for redundant encoding, or possibly even independent holonomic control.

This scheme forms a building-block broadly applicable to a wide range of systems including trapped ions, superconducting qubits \([27]\), NV centers and other solid-state spins. Applications in this broader setting include precision measurement (e.g. magnetometry \([49]\), quantum gate operations, and quantum simulation using adiabatic W.-Z phases.

**METHODS**

**Atom preparation and atom number counting**

Bose-Einstein Condensates (BECs) of rubidium-87 of \(\approx 1 \times 10^5\) were prepared in a crossed optical dipole trap formed by two horizontal 1064 nm optical trapping beams with the trapping frequencies \((f_x, f_y, f_z) \approx (50, 110, 70)\) Hz, where the y-axis is along the direction of gravity.
Initially, the BECs were prepared in the $|1, -1 \rangle$ state. Atoms were then transferred to prepare a superposition state of $|1, 0 \rangle$ and $|2, 0 \rangle$ by rf and microwave pulses, which is the ground state of our Hamiltonian in Eq. (2) at $\mathbf{q}_N = |\mathbf{q}_N\rangle(0, 0, 0, 1, 0)$. The bias magnetic field of 19.8 G pointing along the z-axis was stabilized for long term drift at 2.5 ppm.

We performed an absorption imaging and Stern-Gerlach measurements to resolve the atoms in the hyperfine ground states. After the rf and microwave control was finished, we abruptly turned off the optical dipole trap beams for time-of-flight (TOF). During the TOF, a magnetic field gradient pulse was applied to perform Stern-Gerlach measurement, which separated atoms in $|1, 0 \rangle$ and $|2, 0 \rangle$ from those in $|1, 1 \rangle$ and $|2, -1 \rangle$ in space. We imaged the atoms in $F = 2$ manifold by illuminating a probe pulse resonant to $5S_{1/2}, F = 2 \rightarrow 5P_{3/2}, F = 3$ transition after TOF of 23.2 ms. A short repump laser pulse resonant with the $5S_{1/2}, F = 1 \rightarrow 5P_{3/2}, F = 2$ transition was applied before the probe pulse in order to image atoms in $F = 1$ and $F = 2$ manifolds. When we focused on the state in ground DS, we apply a $\pi$-pulse resonant with the microwave transition $|1, 0 \rangle \leftrightarrow |2, 1 \rangle$ to swap the population between the two states right before resonant with the microwave transition.

We take the following eigenstates for the basis of the ground DS at $\mathbf{q}$ for the region in parameter space we have experimentally explored ($\delta = 0, \phi_B = 0, \phi_A \in [0, 2\pi], \theta_2 \in [0, \pi/2]$).

\[
|\uparrow_{-}(\mathbf{q})\rangle = (|1\rangle - e^{i\varphi_A} \cos \theta_2 |2\rangle + \sin \theta_2 |4\rangle)/\sqrt{2}
\]

\[
|\downarrow_{-}(\mathbf{q})\rangle = (-\sin \theta_2 |2\rangle + |3\rangle - e^{i\varphi_A} \cos \theta_2 |4\rangle)/\sqrt{2}
\]

Using the basis states, the pure state within the DS at $\mathbf{q}$ is described by

\[
|\Psi(\mathbf{q})\rangle = c_1|\uparrow_{-}(\mathbf{q})\rangle + c_2|\downarrow_{-}(\mathbf{q})\rangle,
\]

which can be represented by a two-component spinor $\Psi = (c_1, c_2)^T$, where $|c_1|^2 + |c_2|^2 = 1$ is met.

Each eigenstate can be prepared by applying the cyclic coupling pulse as described above to one of the bare spin states.

\[
|\uparrow_{-}(\mathbf{q})\rangle = \hat{U}_{\text{trans}}(t_{\text{prep}}, \mathbf{q})|\uparrow\rangle,
\]

\[
|\downarrow_{-}(\mathbf{q})\rangle = \hat{U}_{\text{trans}}(t_{\text{prep}}, \mathbf{q})|\downarrow\rangle,
\]

where $|\uparrow\rangle = |1\rangle$ and $|\downarrow\rangle = |3\rangle$ is the basis state of the ground DS at $\mathbf{q}_N$. Therefore, the four initial eigenstates ($|\uparrow\rangle, |\downarrow\rangle, |\uparrow_{-}\rangle, |\downarrow_{-}\rangle$) at $\mathbf{q}_0$ can be prepared by applying the pulse for the duration $t_{\text{prep}}$ with the parameter vector $\mathbf{q}_0$ to the states $|\uparrow\rangle, |\downarrow\rangle, |\uparrow_{-}\rangle, |\downarrow_{-}\rangle$/$\sqrt{2}$, and $|\uparrow_{-}\rangle + i|\downarrow_{-}\rangle$/$\sqrt{2}$, respectively. For the state mapping, the basis states of the DS at $\mathbf{q}$ can be mapped to the bare spin states.

\[
|\uparrow_{-}(\mathbf{q})\rangle = -\hat{U}_{\text{trans}}(t_{\text{map}}, \mathbf{q})|\uparrow_{-}(\mathbf{q})\rangle,
\]

\[
|\downarrow_{-}(\mathbf{q})\rangle = -\hat{U}_{\text{trans}}(t_{\text{map}}, \mathbf{q})|\downarrow_{-}(\mathbf{q})\rangle,
\]

The unitary evolution during the pulsing is then

\[
\hat{U}_{\text{trans}}(t, \mathbf{q}) = \exp(-i\hat{H}_{\text{map}}(\mathbf{q})t),
\]
Quantum state tomography

After the state acquired the W.-Z. phase, we measured the final state within the DS by evaluating the Bloch vector \((\langle \sigma_x(q) \rangle, \langle \sigma_y(q) \rangle, \langle \sigma_z(q) \rangle)\). Here the Pauli operators are defined from the basis states of the DS at \(q\).

Using the state mapping procedure described above, the target Bloch vector is obtained by performing state tomography for the superposition states in the microwave clock transition \((1, 0) \leftrightarrow (2, 0)\). The z-component was obtained from the population imbalance \((N_t - N_s)/(N_t + N_s)\). The x or y-component was obtained by rotating the Bloch vector with a \(\pi/2\)-pulse with an appropriate microwave phase before measuring the population imbalance.

Quantum process tomography using maximum likelihood estimation

Quantum process tomography is a scheme to characterize unknown quantum process by the knowledge of final output states for different input states. This information is used to reconstruct the process matrix that characterizes an arbitrary transformation. To determine \(\chi\), \(d^2\) linearly independent input states are required for a \(d\)-dimensional Hilbert space. For our DS with \(d = 2\), four inputs states, \(|A\rangle, |B\rangle, |C\rangle, |D\rangle\) are taken as a set of inputs. In order to the find physical process matrix \(\chi\) that represents the W.-Z. phase from our measurement, we adopted maximum likelihood estimation in the quantum process tomography. For the process matrix to be physical, we define \(\chi\) as

\[
\chi = T^\dagger T / \text{tr}(T^\dagger T),
\]

where \(T\) is the lower triangular matrix of the form

\[
T = \begin{bmatrix}
t_1 & 0 & 0 & 0 \\
t_5 + it_6 & t_2 & 0 & 0 \\
t_{11} + it_{12} & t_7 + it_8 & t_3 & 0 \\
t_{15} + it_{16} & t_{13} + it_{14} & t_9 + it_{10} & t_4
\end{bmatrix}
\]

(9)

(10)

where \(t_i, (i = 1, \ldots, 16)\) is real. We define a minimizing function \(f(t)\) as

\[
f(t) = \sum_j \left[ \hat{\rho}_{\text{fin},j} - \left\{ \sum_{m,n} \hat{E}_m \hat{\rho}_{\text{ini}} \hat{E}_m^\dagger \left( \frac{T^\dagger T}{\text{tr}(T^\dagger T)} \right)_{mn} \right\} \right]^2,
\]

(11)

where \(t = (t_1, t_2, \ldots, t_{16})\), \(j \in \{A, B, C, D\}\) distinguish the four initial states in the W.-Z. phase measurements, \(\hat{\rho}_{\text{fin}} = |j\rangle \langle j|\), and \(\hat{\rho}_{\text{ini}}\) is the density operator for the state after it traced out the (open or closed) loop \(C\). An average of measurements was used for each \(\hat{\rho}_{\text{fin},j}\). We numerically minimize \(f(t)\) for the parameter vector \(t\) to obtain optimum \(T\) and the process matrix \(\chi\).

Synthetic non-Abelian SU(2) gauge field and Wilson loop

Consider a quantum system with a Hamiltonian \(\hat{H}(q)\) that depends continuously on the position vector \(q = (q_1, q_2, \ldots)\). The system is described by eigenstates and eigenenergies

\[
\hat{H}(q)|\Psi_{n\alpha}(q)\rangle = E_n(q)|\Psi_{n\alpha}(q)\rangle,
\]

(12)

where \(|\Psi_{n\alpha}(q)\rangle\) (\(\alpha = 1, 2, \ldots, N_o\)) is \(N_o\)-fold degenerate eigenstate with energy \(E_n\) forming an \(N_o\)-fold DS. For quantum states in a single energy level \(E_n\), a gauge potential called the Berry connection

\[
A^{\alpha\beta}_{q_m}(q) = i [\Psi_{\alpha}(q)|\partial/\partial q_m|\Psi_{\beta}(q)],
\]

(13)

is encoded in the systems’ eigenstates, where \(A^{\alpha\beta}_{q_m}(q)\) is the \(m\)-th component of the vector gauge field \(A\) represented as \(N_o\)-by-\(N_o\) matrix. Here, we omitted \(n\) in the l.h.s. for simplicity, and the matrix indices take \(\alpha, \beta \in \{1, 2, \ldots, N_o\}\). The gauge field (Berry connection) is non-Abelian when two components of the gauge field do not commute with each other.

Now, we consider the gauge field for the Hamiltonian in Eq. (2). We focus on the parameters relevant to the experiment \((\Omega_A = \Omega \cos \theta_2, \Omega_B = \sin \theta_2, \delta = 0, \text{ and } \phi_B = 0)\). The non-Abelian SU(2) Berry connection for the two-fold degenerate ground states is

\[
A_{\phi_A}(q) = \frac{1}{2} \begin{pmatrix}
\cos^2 \theta_2 & e^{i\phi_A} \sin \theta_2 \cos \theta_2 \\
e^{-i\phi_A} \sin \theta_2 \cos \theta_2 & -\cos^2 \theta_2
\end{pmatrix}
\]

(14)

\[
= (\cos \phi_A \sin \theta_2 \cos \theta_2 \sigma_x - \sin \phi_A \sin \theta_2 \sin \theta_2 \sigma_y + \cos^2 \theta_2 \sigma_z)/2.
\]

From the definition in Eq. (1), we obtain W.-Z. phases

\[
U_{\mathcal{C}_\pm} = \begin{pmatrix}
-\cos(\pi \sin \theta_2) & \pm i \sin \theta_2 \sin(\pi \sin \theta_2) \\
\mp i \cos \theta_2 \sin(\pi \sin \theta_2) & -\cos(\pi \sin \theta_2) \mp i \sin \theta_2 \sin(\pi \sin \theta_2)
\end{pmatrix}
\]

(15)

factors and Wilson loops (WLs) for the paths \(\mathcal{C}_\pm\).
\[ W_{C_\pm} = -2 \cos(\pi \sin \theta_2). \] (16)

The physical process can be regarded as holonomic single qubit gate operation, where the two eigenstates of the degenerate level are taken as the qubit basis states and the Bloch vector representing the qubit is rotated by an angle of \( \pm 2\pi \sin \theta_2 \) around an axis \((-\cos \theta_2, 0, \sin \theta_2)\). The dependence on the ramp direction for the W.-Z. phase factors \((\hat{U}_{C_+} = \hat{U}_{C_-}^\dagger)\), and the WLs \((W_{C_+} = W_{C_-})\) can be clearly seen. Both the W.-Z. phases factor and the WL do not depend on \(\Omega\), thus they are robust against fluctuation in the coupling strength. By varying the rf phase \(\phi_A\), the SU(2) WL covers the full range \((-2 \leq W_C \leq 2)\), realizing various non-Abelian SU(2) holonomic controls.

### Wilson line for an open path (theory)

In the following we give an argument on non-cyclic W.-Z. phase and Wilson line for an open path. The definition for non-cyclic W.-Z. phase and Wilson line are essentially the same as the cyclic case, except the integral is taken over for an open path \(C\).

\[
W_C = \text{tr}(\hat{U}_C) = \text{tr} \left[ \mathcal{P} \exp \left( i \int_{\mathbf{C}} \hat{A}_q \cdot d\mathbf{q} \right) \right], \tag{17}
\]

\[
\hat{U}_C = \begin{pmatrix}
\exp \left( \frac{i\phi}{2} \left[ \cos \left( \frac{\phi}{2\sqrt{2}} \right) - \frac{i}{\sqrt{2}} \sin \left( \frac{\phi}{2\sqrt{2}} \right) \right] \right)
& \exp \left( \frac{ie^{i\phi/2}}{2} \sin \left( \frac{\phi}{2\sqrt{2}} \right) \right) / \sqrt{2} \\
\exp \left( \frac{ie^{-i\phi/2}}{2} \sin \left( \frac{\phi}{2\sqrt{2}} \right) \right) / \sqrt{2}
& \exp \left( \frac{ie^{-i\phi/2}}{2} \sin \left( \frac{\phi}{2\sqrt{2}} \right) \right) / \sqrt{2}
\end{pmatrix}.
\tag{20}
\]

\[
W_C = \sqrt{2} \sin \left( \frac{\phi}{2} \right) \sin \left( \frac{\phi}{2\sqrt{2}} \right) + 2 \cos \left( \frac{\phi}{2} \right) \cos \left( \frac{\phi}{2\sqrt{2}} \right). \tag{21}
\]

Here we took the basis in Eq. (5) for the matrix representation.

### Wilson line for an open path (experiment)

We show measurement on the Wilson line on open paths by observing non-cyclic W.-Z. phases. The non-cyclic W.-Z. phases is defined by simply replacing the closed path for the integral in Eq. (11) with an open path \(C\). The Wilson line, defined as the trace of the non-cyclic W.-Z. phase factor, is not gauge-independent, and thus it depends on the choice of the basis at both ends of the path. The experimental procedure is the same as the WL measurement, except the rf phase ramp is halted at variable phase \(\phi_A = \phi\) ranging from 0 to \(2\pi\). After preparing the eigenstates at \(q_0\), we ramp the control vector as \(q(t) = (-\Omega_B, -\Omega_A \cos(2\pi t/T), -\Omega_A \sin(2\pi t/T), 0, 0)\) for \(t = [0, \phi T/2\pi]\) until the control vector reaches \(q_f = (-\Omega_B, -\Omega_A \cos \phi, -\Omega_A \sin \phi, 0, 0)\). The final state within the DS at \(q_f\) is always mapped to the DS at \(q = q_0\) for the state tomography. By performing the process tomography for the four-independent initial eigenstates, the process matrix of the non-cyclic non-Abelian W.-Z phase and Wilson lines along a segment are

\[
\hat{U}_C \rightarrow \hat{V}(q_f)\hat{U}_C\hat{V}^\dagger(q_0),
\]

where \(\hat{V}(q)\) is a position-dependent unitary operator. This can be regard as a change in the basis states for the WL and is gauge-independent.

For our experimental parameters for Wilson line measurement in Fig. 5 \((\delta = 0, \phi_B = 0, \phi_T = \pi/4)\), the non-cyclic non-Abelian W.-Z phase and Wilson lines along a segment are

\[
\hat{U}_C \rightarrow \hat{V}(q_f)\hat{U}_C\hat{V}^\dagger(q_0),
\]

where \(q_0\) and \(q_f\) are the start point and end point of the open path \(C\), respectively. Manifestly, the r.h.s depends on the unitary operators, \(\hat{V}(q_0)\) and \(\hat{V}(q_f)\). When the trace is closed \((q_f = q_0)\), the Wilson line is equivalent to the WL and is gauge-independent.

The whole unitary process including the state preparation and the state mapping processes can be viewed as a local gauge transformation of the W.-Z phase: \(\hat{U}_C \rightarrow \hat{V}(q_f)\hat{U}_C\hat{V}^\dagger(q_0) = \hat{U}_{\text{map}}(\phi)\hat{U}_C\hat{U}_{\text{prep}}\), where \(\hat{V}(q)\) is a position-dependent unitary operator, and \(\hat{U}_{\text{prep}}\) \((\hat{U}_{\text{map}})\) is a unitary operator that represents the pulse that maps the state within the DS at \(q_N\) \((q_f)\) to the state within the
FIG. 5. Wilson line. Absolute values of measured Wilson line $|W(\phi)|$ for open paths with variable path length characterized by rf phase range $\phi$. Theory curve (solid line) also shown. The inset illustrates the control sequence for Wilson line measurement of a segment from $q_0$ to $q_f$ on a circular loop. After preparing one of the eigenstates at $q_0$, the rf phase $\phi_A$ is ramped from 0 to $\phi$. The red and green curves represent the pulse controls for the state preparation and the state mapping for the read-out. The laboratory parameters are $\Omega_A = \Omega_B = h \times 1.4$ kHz.

DS at $q_0$. This clearly illustrates that the Wilson line is gauge-independent only when $q_f = q_0$, where it becomes equivalent to the WL.

Non-adiabatic effect due to finite ramp time

Although we have focused on evolution within the ground-state manifold, a small fraction of atoms can be populated to the excited state manifold due to the finite ramp time. We experimentally confirmed this by measuring the fraction of atoms in the excited state manifold.

After the state mapping, we evaluated the fraction $N_e/(N_e + N_g)$, where $N_e = N_{[2]} + N_{[4]}$ is the atom number in the excited state manifold, $N_g = N_{[1]} + N_{[3]}$ is the atom number in the ground state manifold, and $N_{[i]}$ is the atom number in the bare spin state $|i\rangle$, $(i = 1, 2, 3, 4)$. The observed fraction of atoms, which depends on the initial state, is negligibly small, and consistent with the numerical simulation (See Fig. 6a). The dependence of the excited atomic fraction on the initial state can be understood by the state-dependent nature of the state deflection due to local non-Abelian gauge field [32].

Longer ramp time led to a smaller fraction in the excited state manifold as confirmed by the numerical simulation [Fig. 6b]. Experimentally, the fidelity of the WL-Z. phase by varying the ramp time. For our experimental parameters with $\theta_2 = \pi/4$, the fidelity reaches 0.998% at $T = 2$ ms.

Measurement of the energy gap

The energy gap can be clearly measured by inducing coherent Rabi-like oscillations between the eigenstates. Figure 8 shows that the time evolution of the population imbalance $(N_e - N_g)/(N_e + N_g)$ after abruptly turning on the cyclic coupling described by the Hamiltonian in Eq. (2). Since the system has only two eigenenergies, the state oscillates between the ground and excited eigenstates at the frequency determined by the energy gap.
FIG. 7. Infidelity of the non-Abelian W.-Z. phase \((1 - \mathcal{F})\) due to finite ramp time \(T\). The non-Abelian W.-Z. phase factor is numerically evaluated by solving time-dependent Schrödinger equation for our experimental condition for the path \(C\) with \(\theta_2 = \pi/4\) and analyzed by following the same procedure used in the experiment to numerically obtain the process matrix. The solid line is a power-law fit to the numerical results.

FIG. 8. Rabi-like oscillation between the two eigenstates. The population imbalance \(\frac{N_g - N_e}{N_g + N_e}\) was measured after the cyclic coupling for \(\theta_2 = \pi/4\) was abruptly turned on with the BEC in state \(\left|2\right\rangle\). The observed oscillation frequency of 2.0 kHz determines the energy gap of our system.

**DATA AVAILABILITY**

The data are available from the corresponding author upon reasonable request.

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**COMPETING INTERESTS**

The authors declare no competing interests.

**AUTHOR CONTRIBUTIONS**

S.S., F.S.-C., Y.Y., and A.P. measured the data. S.S. conceived the experiment and analyzed the data. I.B.S. supervised the project. All the authors contributed to discussion and preparation of the manuscript.

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