ESTIMATES FOR L-FUNCTIONS IN THE CRITICAL STRIP UNDER GRH WITH EFFECTIVE APPLICATIONS

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Abstract. Assuming the Generalized Riemann Hypothesis, we provide explicit upper bounds for moduli of \( \log L(s) \) and \( L'(s)/L(s) \) in the neighborhood of the 1-line when \( L(s) \) are the Riemann, Dirichlet and Dedekind zeta-functions. To do this, we generalize Littlewood’s well known conditional result to functions in the Selberg class with a polynomial Euler product, for which we also establish a suitable convexity estimate. As an application we provide conditional and effective estimate for the Mertens function.

1. Introduction

Let \( s = \sigma + it \), where \( \sigma \) and \( t \) are real numbers. Determination of the true order of \( |\zeta(s)| \) in the critical strip, or any other respectable \( L \)-function, is one of the great problems in zeta-function theory with far-reaching consequences in analytic number theory. It is believed that \( \zeta(\sigma + it) \ll \varepsilon, \sigma_0 (\log |t|)^{2(1-\sigma)} + \varepsilon \) for every \( \varepsilon > 0 \) and \( \sigma \geq 1/2 \), which in the case \( \sigma = 1/2 \) is known as the Lindelöf Hypothesis, but any unconditional approach to such bounds seems to be a very hard problem, e.g., see [Bon17] for the latest result when \( \sigma = 1/2 \). For some explicit results in this direction see [For02, Tru14, Tru15, PT15, Hia16a, Hia16b, Pat20].

Assuming the Riemann Hypothesis (RH), Littlewood proved in 1912 that

\[
\log |\zeta(s)| \ll \varepsilon, \sigma_0 (\log |t|)^{2(1-\sigma)} + \varepsilon \tag{1}
\]

for \( \varepsilon > 0, 1/2 < \sigma_0 \leq \sigma \leq 1 \) and \( t \) large. Techniques in the proof ([Tit86, Theorem 14.2], [Ivi03, Theorem 1.12]) are purely complex analytic: the Hadamard–Borel–Carathéodory (HBC) inequality is used to estimate \( |\log \zeta(s)| \) with \( \log |\zeta(s)| \) on the particular circles right to the critical line by using convexity estimates to bound the latter expression, while Hadamard’s three-circles theorem then guarantees the “correct” exponent. Given the ensemble of classical ideas, inequality (1) can be generalized without much difficulty to a broader family of functions, e.g., to the Selberg class of functions with a polynomial Euler product, see Section 2 for definitions and properties. A similar approach was taken also in [CG06] where authors generalized the Lindelöf Hypothesis to some functions in the Selberg class. Our main result is the following theorem.

**Theorem 1.** Let \( L(s) \) be an element in the Selberg class of functions with a polynomial Euler product of order \( m \). Then there exist \( C > 0, C_0 > 0, c \geq 1 \) and \( T \geq e \) such that

\[
\log |L(s)| \leq \frac{1}{4} d_L \log (c|t|) + C_0 + \frac{1}{2} \log \log (c|t|) + \log^+ C \tag{2}
\]
for $|t| \geq T$, $\sigma \geq 1/2$ and $L(s) \neq 0$, where $d_L$ is the degree of $L(s)$ and $\log^+ u := \max\{0, \log u\}$ for $u > 0$. For $C_3 > 0$ and $|t| \geq c$ define

$$R(C_3, c, t) := \left\{ w \in \mathbb{C}: \Re\{w\} > \frac{1}{2}, |\Im\{w\} - t| \leq C_3 \log \log \left(c(|t| + 1)\right) + 3 \right\},$$

where $c$ is from (2). If there exists $C_3 \geq 1$ such that $L(z) \neq 0$ for $z \in R(C_3, c, t)$, then there exist positive and computable constants $a_1, b_1$ and $a_2, b_2$ which are dependent only on $d_L$, $m$, $\ell$ and $C$, such that

$$|\log L(s)| \leq a_1 \log \log (c)|t|, \quad \left|\frac{L'(s)}{L(s)}\right| \leq a_2 \left(\log \log (c)|t|\right)^2$$

for $|t| \geq t_0(c, T) > 0$ and

$$\sigma \in \mathcal{A}(A, B, c, t) := \left[ \frac{1}{2} + \frac{A}{\log \log (c)|t|}, 1 + \frac{B}{\log \log (c)|t|} \right],$$

where $A$ and $B$ are some positive constants.

Theorem 1 is contained in Theorem 3 and Corollary 4 where $a_1, b_1$ and $a_2, b_2$ are explicitly given as functions in the variables $d_L$, $m$, $\ell$ and $C$. This enables us to obtain explicit estimates (4) and (5) if we know (2) effectively. Inequality (5) is a consequence of (4) by Cauchy’s integral formula. Note that bounds for $|\log L(s)|$ and $|L'(s)/L(s)|$ can be easily deduced by elementary methods when $\sigma \geq 1 + B/\log \log (c)|t|$, see Remark 2.

Observe that the condition $L(z) \neq 0$ for $z \in R(C_3, c, t)$ is always true under the Generalized Riemann Hypothesis (GRH), i.e., $L(s) \neq 0$ for $\sigma > 1/2$. However, our result is valid under the slightly less restrictive condition which can be viewed as a “local” GRH. Small intervals in $t$-aspect come from the radii of the largest circles in the proof being $\ll \log \log |t|$.

Estimate (2) is nothing more than a precise form of a well known convexity result for the Selberg class of functions, see [Ste07] Theorem 6.8. Our inequality follows by taking the same approach, but with the crucial assumption that $L(s) \ll \log^\ell |t|$ on the 1-line for some $\ell > 0$ which may depend on $L$. Although we are not able to prove this for the full Selberg class, we show that it is true with $\ell = m$ if we have a polynomial Euler product of order $m$, see Theorem 5 (c). Similar result exists also when axioms on classical zero-free region and mild growth condition left to the critical strip are assumed instead of having the axiom on a functional equation, i.e., for the class $G$ from [DM21] Definition 1.2, see Theorem 6 (a). We must also emphasize that one could use subconvexity estimates in place of (2), but with the method presented here this would only result into numerical improvements upon $a_1, b_1$ and $a_2, b_2$.

Better conditional (RH) estimates than (4) and (5) for the region specified by (6) exist for the Riemann zeta-function, see [Tit86] pp. 383–384, [MV07] Corollaries 13.14 and 13.16 for a different approach, and [CC11] Theorem 1 for the latest improvements on constants for the leading terms. To some extension better estimates exist even for general $L$-functions (in the framework of Iwaniec and Kowalski [IK04] Chapter 5) when assuming GRH and the Ramamun–Petersson conjecture, see [IK04] Theorems 5.17 and 5.19, and also [Chi19] Theorem 1 when $L$ is entire and satisfies only GRH. The main objective of this paper is thus not to obtain some conditional bounds for fairly large family of $L$-functions, but rather simultaneously provide their explicit counterparts for three important members: the Riemann, Dirichlet, and Dedekind zeta-functions. It might be interesting to generalize the previously mentioned results to the Selberg class, and then explore possibilities to make them effective.
Corollary 1 (Riemann zeta-function). Let $\mathcal{L}(s) = \zeta(s)$, where $s = \sigma + it$ with $\sigma > 1/2$ and $|t| \geq 10^4$. Assume that $\zeta(z) \neq 0$ for $z \in \mathcal{R} \left(10^3, 1, t\right)$. Then the following is true:

(a) Inequality (1) is valid for $c = 1$, $a_1 = 5.44$, $0.95 < b_1 < 0.951$ and $\sigma \in \mathcal{S} (0.5, 0.5, 1, t)$.

(b) Inequality (5) is valid for $c = 1$, $a_2 = 33.281$, $0.97 < b_2 < 0.971$ and $\sigma \in \mathcal{S} (1.0051, 0.3349, 1, t)$.

The sets $\mathcal{R}$ and $\mathcal{S}$ are defined by (3) and (6), respectively.

Corollary 2 (Dirichlet $L$-functions). Let $\mathcal{L}(s) = L(s, \chi)$, where $\chi$ is a primitive character modulo $q \geq 2$ and $s = \sigma + it$, where $\sigma > 1/2$ and $|t| \geq 10450 + 10^3 \log \log q$.

Assume that $L(z, \chi) \neq 0$ for $z \in \mathcal{R} \left(10^3, q, t\right)$. Then the following is true:

(a) Inequality (1) is valid for $c = q$, $a_1 = 5.44$, $0.95 < b_1 < 0.951$ and $\sigma \in \mathcal{S} (0.5, 0.5, q, t)$.

(b) Inequality (5) is valid for $c = q$, $a_2 = 33.281$, $0.97 < b_2 < 0.971$ and $\sigma \in \mathcal{S} (1.0051, 0.3349, q, t)$.

The sets $\mathcal{R}$ and $\mathcal{S}$ are defined by (3) and (6), respectively.

Corollary 3 (Dedekind zeta-functions). Let $\mathcal{L}(s) = \zeta_K(s)$, where $K \neq \mathbb{Q}$ is a number field of degree $n_K$ and discriminant $\Delta_K$, and $s = \sigma + it$, where $\sigma > 1/2$ and $|t| \geq 9650 + 10^3 \log \log \left(5.552 |\Delta_K|^{1/n_K}\right)$.

Assume that $\zeta_K(z) \neq 0$ for $z \in \mathcal{R} \left(10^3, 5.552 |\Delta_K|^{1/n_K}, t\right)$.

Then the following is true:

(a) Inequality (1) is valid for $c = 5.552 |\Delta_K|^{1/n_K}$, $a_1 = 5.44 n_K$, $0.949 + \frac{1}{n_K} 0.0913 < b_1 < 0.95 + \frac{1}{n_K} 0.0914$ and $\sigma \in \mathcal{S} (0.5, 0.5, c, t)$.

(b) Inequality (5) is valid for $c = 5.552 |\Delta_K|^{1/n_K}$, $a_2 = 33.711 n_K$, $0.964 + \frac{1}{n_K} 0.0961 < b_2 < 0.965 + \frac{1}{n_K} 0.0962$ and $\sigma \in \mathcal{S} (0.961, 0.3199, c, t)$.

The sets $\mathcal{R}$ and $\mathcal{S}$ are defined by (3) and (6), respectively.

Although applications of conditional and effective estimates for $L$-functions in the critical strip to various number-theoretic problems exist, see [CHJ21, p. 20] for instance, results in this direction are quite obscure. Chandee [Cha09] obtained fully explicit bounds for $L$-functions on the critical line when analytic conductor is at least of the order exp$(\exp(10))$, while the author [Sim22, Corollary 1] derived a bound for the Riemann zeta-function which is valid for all $t \geq 2\pi$. Effective upper and lower bounds for $\zeta(s)$ right to the critical line were provided in [Sim21], thus also covering the region not enclosed by (6), i.e., near the critical line.

In [Sim21] the main purpose of having such bounds was establishing conditional (RH) and explicit estimates for the Mertens function $M(x) = \sum_{n \leq x} \mu(n)$ and for the number of $k$-free numbers, see [Sim21, Theorem 2], where $x > 1$ and $\mu(n)$ is the Möbius function. Unfortunately, bounds we have obtained are valid for a very large $x$, for example

$$|M(x)| \leq 0.505 x^{0.99} \log x, \quad x \geq 10^{10^{4.545}}.$$
Similarly, estimates for \( m(x) = \sum_{n \leq x} \mu(n)/n \) were also provided. A method [Sim21, Remark 1] was proposed to extend validity of estimates of the form \( M(x) \ll x^{\alpha} \) for fixed \( \alpha \in (1/2, 1) \) by employing bounds like (4). Here we are able to prove the following.

**Theorem 2.** Assume the Riemann Hypothesis. Then

\[
|M(x)| \leq 555.71 x^{0.99} + 1.94 \cdot 10^{14} x^{0.98},
\]

(7)

\[
|m(x)| \leq \frac{56126.71}{x^{0.01}} + \frac{9.894 \cdot 10^{15}}{x^{0.02}},
\]

(8)

for \( x \geq 1 \).

Theorem 2 follows from a more general Theorem 5. Observe that (7) improves on the trivial estimate \( |M(x)| \leq x \) when \( x \geq 10^{1714.4} \) and improves unconditional estimate [Ram13, Theorem 1.1] for \( x \geq 10^{976.8} \), while (8) improves [Ram13, Corollary 1.2] for \( x \geq 10^{1052.1} \). The constants in the second terms in (7) and (8) may be improved by solving a specific computational problem, see Remark 3. It might be interesting to generalize Theorem 2 to the Mertens function in arithmetic progressions by using Corollary 2.

The outline of this paper is as follows. In Section 2 we revise some properties of functions in the Selberg class and prove inequality (2) by establishing a result on the growth of such functions on the 1-line (Section 2.2) and on deriving an explicit convexity estimates right to the critical line for the Riemann, Dirichlet, and Dedekind zeta-functions (Section 2.3).

### 2. The Selberg class of functions

In this section we are providing a brief overview of some properties of the Selberg class of functions. The emphasis is on studying the growth of such functions on the 1-line (Section 2.2) and on deriving an explicit convexity estimates right to the critical line for the Riemann, Dirichlet, and Dedekind zeta-functions (Section 2.3).

#### 2.1. Preliminaries

The Selberg class \( \mathcal{S} \) of functions with a polynomial Euler product consists of Dirichlet series

\[
\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},
\]

(9)

satisfying the following axioms:

1. **Ramanujan hypothesis.** \( a(n) \ll n^\varepsilon \) for any \( \varepsilon > 0 \).
2. **Analytic continuation.** There exists \( k \in \mathbb{N}_0 \) such that \( (s - 1)^k \mathcal{L}(s) \) is an entire function of finite order.
3. **Functional equation.** \( \mathcal{L}(s) \) satisfies \( \Lambda_{\mathcal{L}}(s) = \omega \Lambda_{\mathcal{L}}(1 - s) \), where

\[
\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s) Q^f \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j),
\]

with \( (Q, \lambda_j) \in \mathbb{R}_+^2 \), and \( (\mu_j, \omega) \in \mathbb{C}^2 \) with \( \Re\{\mu_j\} \geq 0 \) and \( |\omega| = 1 \).
4. **Polynomial Euler product.** There exists \( m \in \mathbb{N} \), and for every prime number \( p \) there are \( \alpha_j(p) \in \mathbb{C} \), \( 1 \leq j \leq m \), such that

\[
\mathcal{L}(s) = \prod_{p} \prod_{j=1}^{m} \left( 1 - \frac{\alpha_j(p)}{p^s} \right)^{-1}.
\]
It is well known that axiom (1) implies the absolute convergence of \( \zeta(s) \) in the half-plane \( \sigma > 1 \), and that axioms (1) and (4) imply that \( |\alpha_j(p)| \leq 1 \) for \( 1 \leq j \leq m \) and all prime numbers \( p \), see [Ste07, Lemma 2.2]. Therefore, if \( L \in \mathcal{SP} \), then

\[
|\log L(s)| = \left| \sum_{p} \sum_{k=1}^{\infty} \frac{\alpha_j(p)}{kp^{k\sigma}} \right| \leq m \sum_{p} \sum_{k=1}^{\infty} \frac{1}{kp^{k\sigma_0}} = m \log \zeta(\sigma_0) \tag{10}
\]

is true for \( \sigma > \sigma_0 > 1 \). Inequality (10) implies the following two approximations

\[
|\log L(s)| \leq m \log \left( 1 + \frac{1}{\sigma_0 - 1} \right) \leq \frac{m}{\sigma_0 - 1}, \tag{11}
\]

\[
|\log L(s)| \leq m \log \frac{1}{\sigma_0 - 1} + m\gamma(\sigma_0 - 1), \tag{12}
\]

where \( \gamma \) is the Euler–Mascheroni constant. Estimate (11) follows by comparison with the integral, while (12) is a consequence of [Ram16, Lemma 5.4] and is better than (11) when \( \sigma_0 \) is close to 1. Note that \( \mathcal{SP} \subseteq \mathcal{S} \) where \( \mathcal{S} \) is the classical Selberg class of functions introduced in [Sel92], i.e., axiom (4) is replaced by

\[
L(s) = \prod_p \exp \left( \sum_{k=1}^{\infty} \frac{b_k(p)}{p^{ks}} \right)
\]

where coefficients \( b_k(p) \) satisfy \( b_k(p) \ll p^\theta \) for some \( 0 \leq \theta < 1/2 \). It is conjectured that \( \mathcal{SP} = \mathcal{S} \).

The degree of \( L \in \mathcal{S} \) is defined by \( d_L = 2 \sum_{j=1}^{L} \lambda_j \). Because \( N_L(T) \sim \frac{1}{\pi} d_L T \log T \), where \( N_L(T) \) counts the number of zeros \( \zeta(s) \) with \( \sigma \in [0, 1] \) and \( |t| \leq T \), it follows that \( d_L \) is well-defined although parameters from axiom (3) are not unique. Note that \( d_1 = 0 \) and \( d_\zeta = 1 \). It is known [Ste07] Theorem 6.1 that \( d_L \geq 1 \) for every \( L \in \mathcal{S} \setminus \{1\} \), and it is conjectured that \( d_L \) is always a positive integer.

Kaczorowski and Perelli proved that \( \zeta(s) \) and shifts \( L(s+i\theta, \chi) \), \( \theta \in \mathbb{R} \), of Dirichlet L-functions \( (\sigma > 1) \)

\[
L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1}
\]

attached to a primitive character \( \chi \) modulo \( q > 1 \), are the only functions in \( \mathcal{S} \) with degree 1, see [Sou05] for a simplified proof. Important examples are also Dedekind zeta-functions \( (\sigma > 1) \)

\[
\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \prod_p \left( 1 - \frac{1}{N(p)^s} \right)^{-1} = \prod_p \prod_{j=1}^{r} \left( 1 - \frac{1}{p^{-f_j}} \right)^{-1},
\]

where \( \mathcal{K} \) is a number field, \( N(\cdot) \) is the norm of an ideal, \( \mathfrak{a} \) runs through all non-zero ideals and \( p \) runs through all prime ideals of the ring of integers of \( \mathcal{K} \). The last equality follows because any rational prime number \( p \) has a unique factorization \( p = \prod_{j=1}^{r} p_j^{e_j} \) with \( N(p_j) = p_j^{f_j} \) and \( \sum_{j=1}^{r} e_j f_j = n_\mathcal{K} := [\mathcal{K} : \mathbb{Q}] \), where the non-negative integers \( e_j, f_j \) and \( r \) depend on \( p \). Therefore, \( r \leq n_\mathcal{K} \), which implies a polynomial Euler product representation for \( m = n_\mathcal{K} \). We have that \( \zeta_\mathcal{K} \) belongs to \( \mathcal{SP} \) and \( d_\zeta_\mathcal{K} = n_\mathcal{K} \). Observe also that \( \zeta_\mathbb{Q}(s) = \zeta(s) \).

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1 This is an asymptotic formulation of the Riemann–von Mangoldt formula for the Selberg class, see [Ste07, Theorem 7.7] for more general version and [Pal19] for an effective estimate.
The functional equation from axiom (3) can be written as \( \mathcal{L}(s) = \Delta_{\mathcal{L}}(s)\overline{\mathcal{L}(1-s)} \), where
\[
\Delta_{\mathcal{L}}(s) := \omega Q^{1-2s} \prod_{j=1}^{f} \frac{\Gamma(\lambda_j(1-s) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)}.
\]
Taking \( \mathcal{L} \in \mathcal{S} \), we can use Stirling’s formula to prove
\[
\Delta_{\mathcal{L}}(s) \ll |t|^d e^{\frac{s}{2}-s},
\]
where this estimate is uniform in \( \sigma \in [\sigma_1, \sigma_2] \) for fixed \( \sigma_1 \leq \sigma_2 \), see [Sim07, Lemma 6.7]. It is possible to make (13) uniform also in \( \mathcal{L} \) by means of the data of the functional equation, but such an approach is not needed in the present paper.

2.2. On the growth of \( \mathcal{L}(s) \) on the 1-line. It is convenient to introduce an additional axiom which concerns the growth of \( \mathcal{L}(1+it) \) when \( \mathcal{L} \in \mathcal{S} \) and \( |t| \to \infty \).

(5) Growth on the 1-line. \( \mathcal{L}(1+it) \ll \log^\ell |t| \) for some \( \ell > 0 \).

In the case of the Riemann zeta-function it is a standard result that we can take \( \ell = 1 \), while a substantial improvement to \( \ell = 2/3 \) requires techniques from the proof of the Vinogradov–Korobov’s zero-free region, see [Ivi03, Chapter 6]. Note that the former result can be proved by using the approximate functional equation for \( \zeta(s) \). Similar approach is also used in the proof of Theorem 3 (b).

Dixit and Mahatab introduced in [DM21, Definition 1.2] a new class of functions \( \mathcal{G} \). We say that \( \mathcal{L} \in \mathcal{G} \), if the series (9) is absolutely convergent for \( \sigma > 1 \), \( a(1) = 1 \), and \( \mathcal{L} \) satisfies beside axioms (2) and (4) also the following two axioms:

(6) Zero-free region. There exists \( c_{\mathcal{L}} > 0 \) such that \( \mathcal{L} \) has no zeros in the region
\[
\left\{ z \in \mathbb{C} : \Re(z) \geq 1 - \frac{c_{\mathcal{L}}}{\log (|\Im(z)| + 2)} \right\},
\]
except the possible Siegel zero, i.e., real exceptional zero of \( \mathcal{L} \) in the neighbourhood of 1.

(7) Growth condition. Define \( \mu^*_L(\sigma) := \inf \{ \lambda > 0 : \mathcal{L}(\sigma + it) \ll |t|^\lambda \} \). Then \( \mu^*_L(\sigma) \ll 1 - 2\sigma \) uniformly for \( \sigma < 0 \).

Observe that class \( \mathcal{G} \) does not require a functional equation; axiom (3) implies axiom (7), but the latter is sufficient to show that then \( \mathcal{L}(s) \) is polynomially bounded in vertical strips by using the Phragmén–Lindelöf principle. It is expected that \( \mathcal{S} \subseteq \mathcal{G} \).

The next theorem explores possible connections between classes \( \mathcal{G}, \mathcal{S}, \mathcal{SP} \) and axiom (5).

As usual, \( d_\alpha(n) \) denotes the number of ways positive integer \( n \) can be written as a product of \( \alpha \geq 2 \) factors, and we extend this to \( d_1(n) \equiv 1 \).

**Theorem 3.** The following is true:

(a) Let \( \mathcal{L} \in \mathcal{G} \) and take \( \varepsilon > 0 \). Then \( \mathcal{L} \) satisfies axiom (5) with \( \ell = m + \varepsilon \).

(b) Let \( \mathcal{L} \in \mathcal{S} \) and assume \( a(n) \ll d_\alpha(n) \) for some positive integer \( \alpha \). Then \( \mathcal{L} \) satisfies axiom (5) with \( \ell = \alpha \).

(c) Let \( \mathcal{L} \in \mathcal{SP} \). Then \( \mathcal{L} \) satisfies axiom (5) with \( \ell = m \).

**Proof.** Firstly we are going to prove the assertion (a) by following the method from [DM21]. Take \( \mathcal{L} \in \mathcal{G} \). Let \( X \geq 2, \sigma > 1 \) and
\[
\mathcal{L}(s; X) := \prod_{p \leq X} \prod_{j=1}^{m} \left( 1 - \frac{\alpha_j(p)}{p^s} \right)^{-1}, \quad \log \mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{\beta_{\mathcal{L}}(n)}{n^s}.
\]
Axiom (4) asserts that \( b_\mathcal{L}(n) = 0 \) if \( n \neq p^k \) and \( |b_\mathcal{L}(n)| \leq m \) otherwise. For \( \sigma \geq 1 \) it follows that

\[
\log \mathcal{L}(s; X) = \sum_{p \leq X} \sum_{k=1}^{\infty} \frac{b_\mathcal{L}(p^k)}{p^{ks}}
= \sum_{n \leq X} \frac{b_\mathcal{L}(n)}{n^s} + \left( \sum_{\sqrt{X} < p \leq X} \sum_{p^k > X} \frac{b_\mathcal{L}(p^k)}{p^{ks}} \right)
= \sum_{n \leq X} \frac{b_\mathcal{L}(n)}{n^s} + O\left( \sum_{\sqrt{X} < p \leq X} \sum_{k=2}^{\infty} \frac{1}{p^k} + \sum_{p \leq X} \frac{1}{X} \right)
\]

since \( |p^{ks}| = p^{k\sigma} \geq p^k \).

For \( t \geq 2, \alpha > 0 \) and \( \varepsilon > 0 \) define

\[
\sigma_1 := \frac{1}{\alpha \log t}, \quad \sigma_2 := \frac{1}{(\log t)^{1+\varepsilon/m}}.
\]

By Perron’s formula we have

\[
\sum_{n \leq X} \frac{b_\mathcal{L}(n)}{n^{1+it}} = \frac{1}{2\pi i} \int_{\sigma_2 - \frac{\varepsilon}{2}}^{\sigma_2 + \frac{\varepsilon}{2}} \log \mathcal{L}(1 + it + z) \frac{X^z}{z} \, dz + O\left( \frac{X^{1+\alpha}}{t^{\sigma_2}} + \frac{\log X}{t} + \frac{1}{X} \right).
\] (15)

Let \( \mathcal{E} := \left\{ z \in \mathbb{C} : 1 - \sigma_1 \leq \Re\{z\} \leq 1 + \sigma_2, \frac{t}{2} \leq \Im\{z\} \leq \frac{3t}{2} \right\} \).

By axiom (6) there exist \( \alpha \) and \( t_0 > 0 \) such that there are no zeros of \( \mathcal{L} \)-function in a neighbourhood of \( \mathcal{E} \) for \( t \geq t_0 \). Moreover, one can use HBC inequality together with axiom (7) and inequality [14] to prove that \( \log \mathcal{L}(z) \ll \log t \) for \( z \in \partial \mathcal{E} \). Take \( X = \exp\left( (\log t)^{1+\varepsilon/m} \right) \). By [14], [15] and Cauchy’s formula we then have

\[
\log \mathcal{L}(1 + it) = \log \mathcal{L}(1 + it; X) + O(1)
+ \frac{1}{2\pi i} \left( \int_{\sigma_2 + \frac{\varepsilon}{2}}^{1} \log \mathcal{L}(1 + it + z) \frac{X^z}{z} \, dz \right)
\]

with the same result also for the third integral while the second integral may be bounded as

\[
\int_{-\sigma_1 + \frac{\varepsilon}{2}}^{-\sigma_1 + \frac{\varepsilon}{2}} \log \mathcal{L}(1 + it + z) \frac{X^z}{z} \, dz \ll \frac{X^{1-\sigma_1} \log t}{t \log X} \ll 1.
\]

Therefore, \( \log \mathcal{L}(1 + it) = \log \mathcal{L}(1 + it; X) + O(1) \). Because

\[
|\mathcal{L}(1 + it; X)| \leq \prod_{p \leq X} \left( 1 - \frac{1}{p} \right)^{-m} \ll \log^m X
\]

by Mertens’ third theorem, it follows that

\[
\mathcal{L}(1 + it) \ll |\mathcal{L}(1 + it; X)| \ll \log^m X = (\log t)^{m+\varepsilon}.
\]

In a similar way we can obtain such estimate also when \( t \) is negative. The proof of Theorem [8] (a) is thus complete.
We are going to prove the assertion (b). Take $\mathcal{L} \in \mathcal{S}$, $x \geq 1$ and $t_0 > 0$ sufficiently large. We can assume that $\mathcal{L} \not\equiv 1$ since otherwise the result is trivial. For $|t| \geq t_0$ we have

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{1+it}} e^{-\left(\frac{x}{2}\right)\log |t|} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^z}{z} \mathcal{L}(1+it+z) \Gamma \left(1 + \frac{z}{\log |t|}\right) dz$$

$$= \frac{1}{2\pi i} \int_{-\frac{3}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{x^z}{z} \Delta \mathcal{L}(1+it+z) \Gamma \left(1 + \frac{z}{\log |t|}\right) dz$$

$$+ \mathcal{L}(1+it) + \frac{1}{\log |t|} \int \Delta \mathcal{L}(s, 1) \text{Res} \left(\mathcal{L}(s), 1\right).$$

The first equality follows from the classical Mellin integral, while the second equality follows by moving the line of integration to $\Re \{z\} = -3/2$, using the functional equation, and detecting two poles at $z = 0$ and $z = -it$ of the integrand which are inside the contour. It is clear that the second residue is $O(1)$. We are going to demonstrate that this is also true for the second integral in the latter expression if we take $x$ large enough. Denote this integral by $\mathcal{I}$ and let $z = -3/2 + it$, $u \in \mathbb{R}$. Then

$$\Delta \mathcal{L}(1+it+z) \ll (|u|+t) + 1)^{d_L} \ll \left\{ \begin{array}{ll} |t|^{d_L}, & |u| \leq \log |t|, \\
|t|^{d_L} |u|^{d_L}, & |u| > \log |t|, \end{array} \right.$$ and

$$\Gamma \left(1 + \frac{z}{\log |t|}\right) \ll \left\{ \begin{array}{ll} 1, & |u| \leq \log |t|, \\
\left(\frac{|u|}{\log |t|}\right)^{\frac{z}{2}} \exp \left(-\frac{|u|}{\log |t|}\right), & |u| > \log |t|, \end{array} \right.$$ while the implied constants are uniform in $u$ and $t$. Obviously, $\mathcal{L}(it-z) \ll 1$. Splitting the range of integration in $\mathcal{I}$ into two parts, $|u| \leq \log |t|$ and $|u| > \log |t|$, we obtain

$$\mathcal{I} \ll x^{-\frac{3}{2}} |t|^{d_L} \left(\log |t| + \Gamma \left(\frac{3}{2} + d_L, 1\right) (\log |t|)^{d_L}\right) \ll x^{-\frac{3}{2}} |t|^{d_L} \log^{d_L} |t|.$$

From the last expression we can see that $\mathcal{I} = O(1)$ if $x = |t|^{d_L}$. With such choice for $x$ we also have

$$\sum_{n>ex} \frac{a(n)}{n^{1+it}} e^{-\left(\frac{x}{2}\right)\log |t|} \ll \sum_{n>ex} e^{-\left(\frac{x}{2}\right)\log |t|} \leq e^{-|t|} + x \int_{e}^{\infty} e^{-u\log |t|} du$$

$$\ll e^{-|t|} + x |t|^{-\log |t|} \ll 1,$$ where we used $a(n) \ll n$ and $\log^2 |t| + u \leq u \log |t|$, the last inequality valid for $u \geq 2$ and $\log |t| \geq 5$. All that finally implies

$$\mathcal{L}(1+it) = \sum_{n \leq \infty} \frac{a(n)}{n^{1+it}} e^{-\left(n/|t|^{d_L}\right)^{\log |t|}} + O(1).$$

Because $a(n) \ll d_n(n)$ by the assumption, and $\sum_{n \leq \infty} d_n(n) \ll X (\log X)^{\alpha-1}$, it follows that

$$\mathcal{L}(1+it) \ll \sum_{n \leq \infty} \frac{d_n(n)}{n} \ll \log^{\alpha} |t|$$

by partial summation. The proof of statement (b) is thus complete.

The proof of statement (c) now easily follows from the assertion (b) since one can observe that the estimate for the local roots $|\alpha_j(p)| \leq 1$ implies $|a(n)| \leq d_m(n)$, see [Ste07, Lemma 2.2], and the former is true for functions in $\mathcal{SP}$. ■
Our proof of Theorem 3 (b) follows similar approach as the proof of a smooth version of the approximate functional equation for \( \zeta(s) \), see [TV03] Theorem 4.4, and also [MO93] Proposition 2.3 for a generalization to the Selberg class and [IK04] Theorem 5.3 for a generalization to \( L \)-functions. In correspondence with the latter functions, our condition \( a(n) < d_n(n) \) can be viewed as an analog to the Ramanujan–Petersson conjecture. However, for our purpose we do not require a complete result, so the proof can be simplified.

2.3. Convexity estimates for \( L(s) \). Assuming axiom (5), it is easy to prove the precise form of the convexity-type result for \( L(s) \).

**Proposition 1.** Take \( \sigma_0 < 0 \), \( L \in S \) and assume that \( L \) also satisfies axiom (5). Then

\[
L(s) \ll \begin{cases} 
|t|^d_L \left( \frac{t}{s} \right) \log^{k'} |t|, & \sigma_0 \leq \sigma < 0, \\
|t|^{\frac{1}{2}d_L (1 - \sigma)} \log^k |t|, & 0 \leq \sigma \leq 1, \\
\log^k |t|, & \sigma > 1,
\end{cases}
\]

where the implied constants are uniform in \( \sigma \).

**Proof.** Axiom (5), estimate (13) and the functional equation imply

\[
L(it) \ll |t|^{\frac{1}{2}d_L (1 - \sigma)} \log^k |t|.
\]

Also, the estimate for \( \sigma \in [\sigma_0, 0) \) follows from the estimate for \( \sigma > 1 \) and the functional equation, so it remains to prove the bounds for \( \sigma \geq 0 \).

For \( \sigma > -1 \) define

\[
f_L(s) := \frac{(s-1)^k + d_L L(s)}{(s+1)^{d_L (3-\sigma)} / 2 + k} \log^k (s + 2),
\]

\[
g_L(s) := \frac{(s-1)^k L(s)}{(s+1)^{d_L (3-\sigma)} / 2 + k} \log^k (s + 2).
\]

where \( k \) is from axiom (2). Then \( f_L(s) \) and \( g_L(s) \) are holomorphic functions of finite order in the half-plane \( \{ z \in \mathbb{C} : \Re(z) > -1 \} \).

Because \( |f_L(1 + it)| \) and \( |f_L(it)| \) are bounded for all \( t \in \mathbb{R} \), the Phragmén–Lindelöf theorem implies that also \( |f_L(s)| \) is bounded for \( \sigma \in [0, 1] \) and \( t \in \mathbb{R} \). This proves the first estimate.

Trivially, \( L(s) \ll \log^k |t| \) for \( \sigma \geq 2 \). As before, because \( |g_L(1 + it)| \) and \( |g_L(2 + it)| \) are bounded for all \( t \in \mathbb{R} \), this implies that also \( |g_L(s)| \) is bounded for all \( \sigma \in [1, 2] \) and \( t \in \mathbb{R} \). The proof is thus complete. \( \blacksquare \)

**Remark 1.** Note that inequality (2) from Theorem 1 immediately follows from Proposition 1 and Theorem 3 (c).

We are going to provide numerical values for the constants \( C, c \) and \( T \) in the case when \( L(s) = \zeta(s) \), \( L(s, \chi) \), and \( \zeta_K(s) \).

**Example 1** (Riemann zeta-function). Let \( |t| \geq 50 \). Backlund [Bac18] Equations (54) and (56) proved that \( |\zeta(s)| \leq \log |t| \) for \( \sigma > 1 \), and

\[
|\zeta(s)| \leq \frac{t^2}{t^2 - 4} \left( \frac{|t|}{2\pi} \right)^{\frac{1}{2}} \log |t|
\]

for \( \sigma \in [0, 1] \). It follows that in the case \( L(s) = \zeta(s) \), inequality (2) is valid for the values \( d_L = \ell = C = c = 1 \) and \( T = 50 \).

**Example 2** (Dirichlet \( L \)-functions). Let \( \chi \) be a primitive character modulo \( q > 1 \). Rademacher [Rad60] Theorem 3 proved that

\[
|L(s, \chi)| \leq \left( \frac{q |1 + s|}{2\pi} \right)^{\frac{1}{2}} \zeta(1 + \eta)
\]
for \( \sigma \in [\eta, 1 + \eta] \) and \( \eta \in (0, 1/2) \). Take \( \eta = \alpha/\log(q|t|) \), \( \alpha \geq 1 \), \( \sigma \in [1/2, 1 + \eta] \) and \( |t| \geq t_0 \geq e^{2\alpha} \). Because
\[
\zeta(1+\eta) \leq \frac{1}{\eta^\gamma} \leq \frac{1}{\alpha} \exp\left( \frac{\gamma \alpha}{\log t_0} \right) \log(q|t|),
\]
(16)
\[
1 \leq \frac{q(1 + s)}{2\pi} \leq \frac{1}{2\pi} \sqrt{1 + \left( \frac{2 + \frac{\alpha}{\log t_0}}{t_0} \right)^2 q|t|}, \quad 0 \leq \frac{1 + \eta - \sigma}{2} \leq \frac{1}{4} + \frac{\alpha}{2 \log (q|t|)}
\]
with the first set of inequalities true by (12), it follows that
\[
|L(s, \chi)| \leq \frac{1}{\alpha} \exp\left( \alpha \left( \frac{1}{2} + \frac{\gamma}{\log t_0} \right) \right) \left( \frac{1}{2\pi} \sqrt{1 + \left( \frac{2 + \frac{\alpha}{\log t_0}}{t_0} \right)^2 q|t|} \right)^{1/2} \zeta(1+\eta)^{nk}
\]
for \( \sigma \in [-\eta, 1 + \eta] \), \( \eta \in (0, 1/2) \) and \( s \neq 1 \). Take \( \eta = \alpha/\log\left( |\Delta_K|^{1/nk} |t| \right) \), \( \alpha \geq 1 \), \( \sigma \in [1/2, 1 + \eta] \) and \( |t| \geq t_0 \geq e^{2\alpha} \). Because \( |\Delta_K| \geq 1 \), similar procedure as in Example 2 guarantees
\[
|\zeta(s)| \leq \frac{3}{(2\pi)^2} \left( \frac{2 + \frac{\alpha}{\log t_0}}{t_0} \right)^{1/2} \times
\]
\[
\exp\left( \alpha \left( \frac{1}{2} + \frac{\gamma}{\log t_0} \right) \right)^{nk} \left( |\Delta_K|^{1/nk} |t| \right)^{1/2} \log^{nk} \left( |\Delta_K|^{1/nk} |t| \right).
\]
Take \( t_0 = 7778 \) and \( \alpha = 1.8 \), and let \( c = 5.552 |\Delta_K|^{1/nk} \). Then the latter inequality implies that \( |\zeta(s)| \leq 1.9 (c|t|)^{1/2} \log^{nk} (c|t|) \) for \( 1/2 \leq \sigma \leq 1 + 1.8/\log (c|t|) \) and \( |t| \geq 7778 \). The same bound holds also for \( \sigma \geq 1 + 1.8/\log (c|t|) \). We deduce that in the case \( L(s) = \zeta(s) \), inequality (2) is valid for the values \( d_L = \ell = n_K, C = 1.9, c = 5.552 |\Delta_K|^{1/nk} \) and \( T = 7778 \).

3. Proof of Theorem 1 and its corollaries

In this section we prove the estimates on \( \log L(s) \) and \( L'(s)/L(s) \) from Theorem 1 by explicitly expressing the corresponding constants as functions in variables from our convexity estimate (2), see Theorem 4 and Corollary 4. Next, we use these results in combination with Examples 1 and 2 to prove Corollaries 1 and 3.

Firstly, we will isolate a result which compares \( |\log L(z)| \) with the estimate (2) on some particular circles by means of HBC inequality.

**Lemma 1.** Take \( L \in \mathcal{S}P \). Let \( |t'| \geq t'_0 \geq \max \left\{ T + 1, \exp\left( e^2 \right) \right\} \) where \( T \) is from Theorem 1 \( \begin{cases} 0 < C_1 \leq 1 \quad \text{and} \quad 0 < \delta \leq 1/2 \end{cases} \) Assume that \( L(z) \neq 0 \) for \( \Re(z) > 1/2 \) and \( |\Im(z) - t'| \leq 1 \). Define
\[
\mathcal{D}(C_1, \delta, t') := \left\{ z \in \mathbb{C} : \left| 1 + \frac{C_1}{\log \log |t'|} + it' - z \right| \leq \frac{1}{2} + \frac{C_1}{\log \log |t'| - \delta} \right\}.
\]

Then
\[
|\log \mathcal{L}(z)| \leq \frac{1}{2} K \log (c(|t'| + 1))
\]  \hfill (18)
for \( z \in \mathcal{D}(C_1, \delta, t') \), where \( c \geq 1 \) is from (2),
\[
K(d_c, m, \ell, C, C_1, t'_0) := \frac{1}{4} d_c + \frac{C_1 d_c}{2 \log \log t'_0} + \left(1 + \frac{2C_1}{\log \log t'_0}\right) \times
\left(\frac{\ell \log \log t'_0}{\log t'_0} + \frac{m}{\log t'_0} \left(\log \log \log t'_0 + \log \frac{1}{C_1} + \frac{\gamma C_1}{\log \log t'_0} + \log^+ C\right)\right),
\]  \hfill (19)
m is from axiom (4), and \( \ell \) and \( C \) are from inequality (2).

Proof. Let \( \lambda \in (0, 1) \) and define
\[
\mathcal{D}_0 := \left\{ z \in \mathbb{C} : \left|1 + \frac{C_1}{\log \log |t'|} + it' - z\right| \leq \frac{1}{2} + \frac{C_1}{\log \log |t'|} - \lambda \delta\right\}.
\]
Observe that \( \mathcal{D} = \mathcal{D}(C_1, \delta, t') \) and \( \mathcal{D}_0 \) are closed discs with the same centre, and
\[
\mathcal{D} \subseteq \mathcal{D}_0 \subset \left\{ z \in \mathbb{C} : \Re\{z\} > \frac{1}{2}, |\Im\{z\} - t'| < 1\right\}.
\]
Because \( \log \mathcal{L}(z) \) is a holomorphic function on the latter domain, HBC inequality implies
\[
\max_{z \in \partial \mathcal{D}} \{|\log \mathcal{L}(z)|\} \leq \left(\frac{1}{1 - \lambda}\right) \left(1 + \frac{2C_1}{\log \log |t'|} - 2\delta\right) \max_{z \in \partial \mathcal{D}_0} \{|\Re\{\log \mathcal{L}(z)\}|\}
\]
\[
+ \frac{1}{(1 - \lambda)\delta} \left(1 + \frac{2C_1}{\log \log |t'|} - (1 + \lambda)\delta\right) \left|\log \mathcal{L} \left(1 + \frac{C_1}{\log \log |t'|} + it'\right)\right|.
\]  \hfill (20)
By (2) we have
\[
\max_{z \in \partial \mathcal{D}_0} \{|\Re\{\log \mathcal{L}(z)\}|\} \leq \left(\frac{1}{4} d_c + \frac{\ell \log \log t'_0}{\log t'_0}\right) \log (c(|t'| + 1)) + \log^+ C,
\]  \hfill (21)
while (12) guarantees that
\[
\left|\log \mathcal{L} \left(1 + \frac{C_1}{\log \log |t'|} + it'\right)\right| \leq m \left(\log \log \log |t'| + \log \frac{1}{C_1} + \frac{\gamma C_1}{\log \log |t'|}\right).
\]  \hfill (22)
Inequality (18) easily follows after using (21) and (22) in (20), and then taking \( \lambda \to 0 \) while also using the maximum-modulus principle. \( \square \)

**Theorem 4.** Take \( \mathcal{L} \in \mathcal{SP} \). Let \( 0 < C_1 \leq 1, \ 0 < C_2 \leq 2C_1 \) and \( C_3 \geq 1 \). Let \( s = \sigma + it \) and
\[
|t| \geq t_0 \geq T_1 \geq \max\left\{\exp\left(e^{2C_2}\right), e^{4m/\delta_C}, C_3, \exp\left(e^2\right)\right\}
\]
with \( m \) from axiom (4), such that
\[
t_0 - C_3 \log \log (ct_0) - \frac{1}{2} \geq T_2 \geq \max\{T + 1, \exp(e^2)\},
\]
\[
T_1 - 2C_3 \log \log T_1 \geq 0,
\]
where \( T \) and \( c \) are as in Theorem 3. Assume that \( \mathcal{L}(z) \neq 0 \) for \( \Re\{z\} > 1/2 \) and \( |\Im\{z\} - t| \leq C_3 \log \log (c|t|) + 2 \). Then (4) is true for
\[
\sigma \in \mathcal{D}(C_2, C_2, c, t),
\]  \hfill (25)
where $\mathcal{I}$ is defined by (3),

$$a_1 := \frac{m}{C_2} \exp \left( \left( 1 + \frac{\log T_1}{\log \log T_1} \right) R_1 \right),$$

$$b_1 := b_1 (d, m, \ell, C, C_1, C_3, T_1, T_2)$$

$$:= \frac{1}{m} K (d, m, \ell, C, C_1, T_2) \left( 1 + \frac{\log (1 + C_2 \log \log T_1 + 3/2)}{\log T_1} \right)$$

$$\times \left( 1 + \frac{1}{\log T_1} \log \left( 1 + \frac{2 C_3 \log \log T_1 + 1}{T_1} \right) \right).$$

(26) and

$$R_1 = R (C_2, C_3, T_1) := \left( 2 C_2 + \frac{1}{2 C_3} \right) \left( 1 - \frac{1}{4 C_3 \log \log T_1} \right)^{-1}.$$

(27)

while $K$ is defined by (19) and $\ell, C$ are from inequality (2).

**Proof.** We can assume that $\mathcal{L} \neq 1$ since otherwise the result is trivial. Let $\delta_0 := C_2 / \log \log (c|\ell|)$ and $\sigma_0 := C_3 \log \log (c|\ell|) + 1 + \delta_0$, and define also

$$\mathcal{D}_1 := \{ z \in \mathbb{C} : |\sigma_0 + it - z| \leq \sigma_0 - 1 - \delta_0 \},$$

$$\mathcal{D}_2 := \{ z \in \mathbb{C} : |\sigma_0 + it - z| \leq \sigma_0 - \sigma \},$$

$$\mathcal{D}_3 := \{ z \in \mathbb{C} : |\sigma_0 + it - z| \leq \sigma_0 - 1/2 - \delta_0 \}. $$

Observe that $\mathcal{D}_j$ are closed discs with the same centre, and

$$\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_3 \subseteq \left\{ z \in \mathbb{C} : \Re \{ z \} > \frac{1}{2}, |\Im \{ z \} - t| < C_3 \log \log (c|\ell|) + 2 \right\}. $$

Because $\log \mathcal{L}(z)$ is a holomorphic function on the latter domain, Hadamard’s three-circles theorem implies $M_2 \leq M_1^{1-n} \bar{M_3}^n$, where

$$M_j := \max_{z \in \mathcal{D}_j} \{ \log \mathcal{L}(z) \},$$

$$\mu := \left( \log \frac{\sigma_0 - \sigma}{\sigma_0 - 1 - \delta_0} \right) \left( \log \frac{\sigma_0 - 1/2 - \delta_0}{\sigma_0 - 1 - \delta_0} \right)^{-1}. $$

Note that $|\log \mathcal{L}(s)| \leq M_2$ since $s \in \mathcal{D}_2$.

We need to estimate $M_1$ and $M_3$. By (11) we have

$$M_1 \leq \sup_{\sigma \geq 1 + \delta_0} \{ \log \mathcal{L}(s) \} \leq \frac{m}{\delta_0}. $$

We are using Lemma (1) in order to estimate $M_3$. Let

$$\mathcal{A}_1 := \left\{ z \in \mathbb{C} : \frac{1}{2} + \delta_0 \leq \Re \{ z \} \leq \frac{3}{2}, |\Im \{ z \} - t| \leq \sigma_0 - \frac{1}{2} - \delta_0 \right\};$$

$$\mathcal{A}_2 := \left\{ z \in \mathbb{C} : \Re \{ z \} \geq \frac{3}{2}, |\Im \{ z \} - t| \leq \sigma_0 - \frac{1}{2} - \delta_0 \right\}. $$

Observe that $\mathcal{D}_3 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$. Inequality (10) implies

$$\sup_{z \in \mathcal{A}_2} \{ \log \mathcal{L}(z) \} \leq m \log \zeta \left( \frac{3}{2} \right) < m. $$

(28)

Remember that $c \geq 1$. Take

$$\delta(t') := \frac{C_2}{\log \log (c(|t'| + \sigma_0 - 1/2 - \delta_0))}, \quad t' \in \left[ t - \sigma_0 + \frac{1}{2} + \delta_0, t + \sigma_0 - \frac{1}{2} - \delta_0 \right].$$

Because $2C_1 - C_2 \geq 0$, we have

$$\mathcal{A}_1 \subseteq \bigcup_{t' : |t' - t| \leq \sigma_0 - \frac{1}{2} - \delta_0} \mathcal{D}_1 (C_1, \delta(t'), t').$$
where the closed disc $D(C_1, \delta(t'), t')$ is defined by (27). Because
\[ |t| - C_3 \log \log (c|t|) - \frac{1}{2} \leq |t'| \leq |t| + C_3 \log \log (c|t|) + \frac{1}{2} \]
and $|t| - C_3 \log \log (c|t|) - 1/2$ is an increasing function in $|t|$ since $|t| \log |t| \geq C_3$, we have
\[ |t'| \geq t_0 := t_0 - C_3 \log \log (c|t_0|) - \frac{1}{2} \geq T_2 \geq \max \{ T + 1, \exp (e^2) \} \]
due to (28). Also, $0 < \delta(t') \leq 1/2$ since $|t'| + \sigma_0 - 1/2 - \delta_0 \geq |t|$ and $\log \log (c|t|) \geq 2C_2 > 0$. Furthermore, $L(z) \neq 0$ for $\Re(z) > 1/2$ and $|3\{z\} - t'| \leq 1$. Conditions of Lemma 1 are thus satisfied, therefore
\[
\sup_{z \in \mathbb{S}^1} \{ \log L(z) \} \leq \frac{1}{C_2} K (d_{\mathcal{L}}, m, t, C, C_1, T_2) \times \log \log (c (|t| + 2C_3 \log \log (c|t|) + 1)) \log \left( c \left( |t| + C_3 \log \log (c|t|) + \frac{3}{2} \right) \right).
\]
Note that $K \geq d_{\mathcal{L}}/4$ and $\log (c|t|) \geq 4m/d_{\mathcal{L}}$. This implies that the right-hand side of (29) is always greater than $m$, which, together with (28), guarantees that
\[
\frac{\delta_0 M_3}{m} \leq b_1 \log (c|t|),
\]
where $b_1$ is defined by (29). Here we also used the fact that
\[
\frac{1}{\log (c|t|)} \log \left( 1 + \frac{\alpha_1 C_3 \log \log (c|t|) + \alpha_2}{|t|} \right)
\]
is a decreasing function in $c \geq 1$ for $0 < \alpha_1 \leq 2$ and $\alpha_2 > 0$. By using $\log (1 + u) \geq u \log 2$ for $u \in [0, 1]$, simple derivative analysis shows that this is true because
\[
(|t| + \alpha_2 + \alpha_1 C_3 \log \log (c|t|)) \log \left( 1 + \frac{\alpha_1 C_3 \log \log (c|t|) + \alpha_2}{|t|} \right) \geq |t| \log \left( 1 + \frac{\alpha_1 C_3 \log \log |t|}{|t|} \right) \geq \alpha_1 (\log 2) C_3 \log \log |t| \geq \alpha_1 C_3
\]
since
\[
\frac{\alpha_1 C_3 \log \log |t|}{|t|} \leq \frac{2C_3 \log \log T_1}{T_1} \leq 1
\]
by (24). Furthermore, observe that $b_1 \geq d_{\mathcal{L}}/(4m)$.

Writing $\log (1 + u) = u + R_1(u)$, where $u \geq 0$ and $|R_1(u)| \leq u^2/2$, one can easily deduce that $\mu = 2(1 - \sigma) + R$, where
\[
R = \frac{2\delta_0 + 2 (\sigma_0 - 1 - \delta_0) \left( R_1 \left( \frac{1 + \delta_0 - \sigma}{\sigma_0 - 1 - \delta_0} \right) - 2 (1 - \sigma) R_1 \left( \frac{1}{2(\sigma_0 - 1 - \delta_0)} \right) \right)}{1 + 2 (\sigma_0 - 1 - \delta_0) R_1 \left( \frac{1}{2(\sigma_0 - 1 - \delta_0)} \right)}.
\]
Because
\[
1 - 2 (\sigma_0 - 1 - \delta_0) \left| R_1 \left( \frac{1}{2(\sigma_0 - 1 - \delta_0)} \right) \right| \geq 1 - \frac{1}{4C_3 \log \log T_1} > 0,
\]
\[
-1/2 \leq -\delta_0 \leq 1 - \sigma \leq -\delta_0 < \frac{1}{2},
\]
it follows that $0 \leq 1 + \delta_0 - \sigma \leq 1/2$ and
\[
|R| \leq \frac{R_1}{\log \log (c|t|)},
\]
where $R_1$ is defined by (27).
We are now in the position to estimate $M_2$. Because $0 < \mu \leq 1$, we now have
\[ M_2 \leq \frac{m}{\delta_0} \left( \frac{\delta_0 M_3}{m} \right)^{\mu} \leq \frac{m}{C_2} \left( b_1 \log (c|t|) \right)^R \left( b_2 \log (c|t|) \right)^{2(1-\sigma)} \log \log (c|t|) \]
by inequality (30). Because $b_1 \log (c|t|) \geq 1$, this and (31) then imply
\[ (b_1 \log (c|t|))^R \leq (b_1 \log (c|t|))^\frac{m}{\delta_0} \exp \left( 1 + \frac{\log b_1}{\log \log (c|t|)} \right) R_2. \]

The proof of Theorem 4 is thus complete. ■

Corollary 4. Take $L \in \mathcal{SP}$. Let $0 < C_1 \leq 1$, $0 < C_2 \leq 2C_1$, $C_3 \geq 1$ and $0 < C_4 \leq C_2/2.0001$. Let $s = \sigma + it$ and
\[ |t| \geq t_0 \geq T_1 \geq \max \left\{ \exp \left( e^{2(1.00006C_2 + C_3)}, e^{4m/4c}, C_3, \exp (e^2) \right) \right\} + 1 \]
with $m$ from axiom (4), such that
\[ t_0 - C_3 \log \log (ct_0) - \frac{3}{2} \geq T_2 \geq \max \left\{ T + 1, \exp \left( e^2 \right) \right\}, \]
\[ T_1 - 2C_3 \log \log T_1 \geq 1, \]
where $T$ and $c$ are as in Theorem 4. Assume that $L(z) \neq 0$ for $\Re \{z\} > 1/2$ and $|\Im \{z\} - t| \leq C_3 \log \log (c|t| + 1)) + 3$. Then (15) is true for
\[ \sigma \in \mathcal{S} \left( 0.00006C_2 + C_4, C_4, c, t \right), \]
where $\mathcal{S}$ is defined by (13),
\[ a_2 := \frac{1.0002m}{C_2C_4} \exp \left( 2C_4 \left( 1 + \frac{\log^+ b_2}{\log \log T_1} \right) + \left( 1 + \frac{\log^+ b_2}{\log \log (T_1 - 1)} \right) R_2 \right), \]
\[ b_2 := b_1 (dC_3, m, t, C_1, C_3, T_1 - 1, T_2), \]
where $b_1$ and $R$ are defined by (20) and (21), respectively.

Proof. We can assume that $L \neq 1$ since otherwise the result is trivial. Let $\delta := C_4/\log \log (c|t|)$. Then $\delta \in (0, 1)$. Observe that
\[ \{z \in \mathbb{C}: |z - s| \leq \delta \} \subset \left\{ z \in \mathbb{C}: \Re \{z\} > \frac{1}{2}, \Im \{z\} - t < \frac{1}{2} \right\}. \]
Because $\log L(z)$ is a holomorphic function on the latter domain, we can write
\[ \frac{L'}{L}(s) = \frac{1}{2\pi i} \int_{|z-s|=\delta} \frac{\log L(z)}{(z-s)^2} \, dz, \]
which implies
\[ \left| \frac{L'}{L}(s) \right| \leq \frac{1}{\delta} \max_{|z-s|=\delta} \{|\log L(z)|\} \leq \frac{1}{C_4} \max_{z \in \mathcal{S}} \{|\log L(z)|\} \log \log (c|t|), \]
where $\mathcal{S} := \left\{ z \in \mathbb{C}: |\Re \{z\} - \sigma| \leq \delta, |\Im \{z\} - t| \leq \delta \right\}$. We are going to use Theorem 4 for $\sigma = \Re \{z\}$ and $t = \Im \{z\}$ while $z \in \mathcal{S}$ in order to estimate the right-hand side of (37).

Take $z \in \mathcal{S}$. Because $|t| - 1 \leq |\Im \{z\}| \leq |t| + 1$ and $|t| \geq \exp (e^2)$, we have
\[ 0.99995 \leq \frac{\log \log (|t| - 1)}{\log \log |t|} \leq \frac{\log \log (c|\Im \{z\}|)}{\log \log (c|t|)} \leq \frac{\log \log (|t| + 1)}{\log \log |t|} \leq 1.00005. \]

The latter inequality, together with (37), implies that
\[ \frac{1}{2} + \frac{C_2}{\log \log (c|\Im \{z\}|)} \leq \frac{1}{2} + \frac{1.00006C_2}{\log \log (c|t|)} \leq \Re \{z\} \]
\[ \leq 1 + \frac{2C_4}{\log \log (c|t|)} \leq 1 + \frac{C_2}{\log \log (c|\Im \{z\}|)}. \]
This confirms the validity of (25). Replace $T_1$ and $t_0$ in Theorem 4 with $T'_1$ and $t'_0$, where $T'_1 := T_1 - 1$, $t'_0 := t_0 - 1$, and $T_1$ and $t_0$ are as in Corollary 4. Because $t'_0 \leq |\Im(z)|$, inequalities (31), (32) and (33) guarantee the conditions of Theorem 4 on $T'_1$ and $t'_0$ are satisfied. Also,

$$
\bigcup_{z \in \mathcal{L}} \left\{ w \in \mathbb{C} : \Re\{w\} > \frac{1}{2}, |\Im\{w\} - \Im\{z\}| \leq C_3 \log \log (c|\Im\{z\}|) + 2 \right\}
$$

$$
\subset \left\{ w \in \mathbb{C} : \Re\{w\} > \frac{1}{2}, |\Im\{w\} - t| \leq C_3 \log \log (c(|t| + 1)) + 3 \right\}
$$

and the latter set is free of zeros of $L(s)$ by the assumption. Therefore, all conditions of Theorem 4 are satisfied, thus

$$
|\log L(z)| \leq \frac{m}{C_2} \exp \left( 1 + \frac{\log^+ b_2}{\log \log (T_1 - 1)} \right) R_2 \times (b_2 \log (c|\Im\{z\}|))^{2(1-\Re\{z\})} \log \log (c|\Im\{z\}|),
$$

where $b_2$ and $R_2$ are defined as (36). Because

$$
2(1 - \Re\{z\}) \leq 2(1 - \sigma) + \frac{2C_4}{\log \log (c|t|)} \leq 1 - \frac{2.00012C_2}{\log \log (c|t|)} < 1
$$

with the second expression being non-negative, and $b_2 \log (c|\Im\{z\}|) \geq 1$, we have

$$
(b_2 \log (c|\Im\{z\}|))^{2(1-\Re\{z\})} \leq 1.00009 \exp \left( 2C_4 \left( 1 + \frac{\log^+ b_2}{\log \log T_1} \right) \right) (b_2 \log (c|t|))^{2(1-\sigma)}.
$$

Using the latter inequality and (38) in (39), which is then used in (37), gives the final estimate from Corollary 4.

**Proof of Theorem 4** We already proved the first part of the theorem, see Remark 1. The second part is essentially the content of Theorem 4 and Corollary 4.

**Proof of Corollary 4** By Example 1 we have $d_L = \ell = C = c = 1$ and $T = 50$, while also $m = 1$. Take $t_0 = T_1 = 10^4$, $C_3 = 10^8$ and $T_2 = 7778$ in Theorem 4 and Corollary 4. With these parameters conditions (23), (24), (35) and (34) are satisfied. We are optimizing $C_1$, $C_2$ and $C_4$ in order to get the smallest possible values for $a_1$ and $a_2$, separately in each of the cases (a) and (b). We obtain the following values:

(a) $C_1 = 0.25, C_2 = 0.5.$

(b) $C_1 = 0.34, C_2 = 0.67, C_4 = 0.67/2.0001.$

It is easy to verify that all other conditions of Theorem 4 and Corollary 4 are satisfied with such choice of parameters, and the values from Corollary 4 follow immediately.

**Proof of Corollary 5** By Example 2 we have $d_L = \ell = C = c = q$ and $T = 7778$, while also $m = 1$. Take $t_0 = t_0(q) = 10450 + 10^3 \log \log q$, $T_1 = 10^4$, $C_3 = 10^3$ and $T_2 = 7788$ in Theorem 4 and Corollary 4. It is not hard to see that conditions (23), (24), (35) and (34) are then satisfied for every $q \geq 2$ since

$$
t_0(q) - 10^3 \log \log (t_0(q)|q|) - \frac{3}{2}
$$

is an increasing function in $q \geq 2$. Taking similar approach as in the proof of Corollary 4 we obtain the same values for the parameters $C_1$, $C_2$ and $C_4$. With this, all other conditions of Theorem 4 and Corollary 4 are also satisfied, and the values from Corollary 4 follow immediately.
Proof of Corollary 3 By Example 3 we have $d_L = \ell = n_K$, $C = 1.9$, $c = 5.552 \left| \Delta_{n_K} \right|^{1/n_K}$ and $T = 7778$, while also $m = n_K$. Note that $n_K \geq 2$ because $K \not= Q$, and $c \geq 5.552$. Take $t_0 = 9650 + 10^3 \log \log c$, $T_1 = 10188$, $C_3 = 10^3$ and $T_2 = 7794$ in Theorem 4 and Corollary 4. As in the proof of Corollary 3 it is not hard to see that conditions (23), (24), (33) and (34) are then satisfied. Observe that we can write

$$b_1 = K_1(C_1, T_1) + \frac{1}{n_K}K_2(C_1, T_1),$$

$$b_2 = K_1(C_1, T_1 - 1) + \frac{1}{n_K}K_2(C_1, T_1 - 1)$$

for some positive functions $K_1$ and $K_2$ which can be easily derived from (26). Taking $n_K = 2$ and performing optimization on $C_1$, $C_2$ and $C_4$ as in the proof of Corollary 1 we obtain the following values:

(a) $C_1 = 0.25, C_2 = 0.5$.

(b) $C_1 = 0.32, C_2 = 0.64, C_4 = 0.64/2.0001$.

With this all other conditions of Theorem 4 and Corollary 4 are also satisfied. Bounds for $b_1$ and $b_2$ now follow from (40) and (41). Values for $a_1$ and $a_2$ are also true for $n_K > 2$ since $b_1$ and $b_2$ are decreasing in $n_K$ according to (40) and (41). Corollary 3 is thus proved.

Remark 2. It is easy to find (unconditional) bounds for $|\log L(s)|$ and $|L'(s)/L(s)|$ if $\sigma \geq 1 + B/\log \log (c|t|)$, where $c \geq 1$, $B > 0$ and $|t| \geq t_0 > e$: if $L \in SP$, then

$$|\log L(s)| \leq m \log \log \log (c|t|) + m \log \frac{1}{B} + \frac{m B}{\log \log t_0},$$

$$\left| \frac{L'}{L}(s) \right| \leq m \sum_{p} \sum_{k=1}^{\infty} \frac{\log p}{p^{k\sigma}} = -m \sum_{p} \frac{\log p}{p^{\sigma}} (\sigma) \leq \frac{m}{\sigma - 1} \leq \frac{m}{B} \log \log (c|t|)$$

by (12) and [Del87].

4. Proof of Theorem 2

Before proceeding to the proof of Theorem 2 we will provide general bound for the Mertens function which is a consequence of Theorem 4 for the Riemann zeta-function. We are using the approach outlined in [Sim21, Remark 1].

Theorem 5. Assume the Riemann Hypothesis. Let $0 < C_1 \leq 1$, $0 < C_2 \leq 2C_1$ and $C_3 \geq 1$. Let

$$\frac{1}{2} + \frac{C_2}{\log \log T_1} \leq \sigma_0 < 1,$$

$$T_1 \geq \max \left\{ \exp \left( e^{2C_1} \right), \exp \left( e^2 \right), C_3, \exp \left( \frac{1}{2\sigma_0 - 1} \right) \right\},$$

$$T_1 - C_3 \log \log T_1 - \frac{1}{2} \geq T_2 \geq \exp \left( e^2 \right), \quad T_1 - 2C_3 \log \log T_1 \geq 0.$$

Define

$$\varepsilon_0 := \frac{1}{C_2} b_0 \exp \left( \left( 1 + \frac{\log^+ b_0}{\log \log T_1} \right) \frac{\log \log T_1}{(\log T_1)^{\sigma_0}} \right),$$

where $R = R(C_2, C_3, T_1)$ is defined by (27) and

$$b_0 = b_0(C_1, C_3, T_1, T_2) := b_1(1, 1, 1, 1, C_1, C_3, T_1, T_2)$$
with $b_1$ defined by (48). Take $\lambda \in (0, T_1]$. If $\varepsilon_0 < 1$, then

\[
|M(x)| \leq 1 + \left(\frac{1}{\pi \sigma_0} \left(1 + \frac{\lambda}{T_1}\right)^{\sigma_0} \int_0^{T_1} \frac{du}{\zeta(\sigma_0 + iu)} \right) x^{\sigma_0} \\
+ \left(1 + \frac{\lambda^2}{\pi} \left(1 + \frac{\lambda}{T_1}\right)^{\sigma_0} \left(\frac{1}{\varepsilon_0} + \frac{2}{\lambda (1 - \varepsilon_0)} \left(1 + \frac{\lambda}{T_1}\right)\right)\right) x^{\sigma_0 + \varepsilon_0 / 2} \tag{43}
\]

for

\[
x \geq \left(\frac{T_1}{\lambda}\right)^{\frac{1}{1 - \varepsilon_0}}. \tag{44}
\]

**Proof.** Let $x \geq 1$, $\widetilde{M}(x) := \sum_{n \leq x} (x - n) \mu(n)$ and $0 < x \leq x$. One can use

\[
\frac{1}{2\pi i} \int_{c - iy}^{c + iy} y^s ds = \begin{cases} 0, & 0 < y \leq 1, \\ 1 - y^{-1}, & y \geq 1, \end{cases}
\]

which is valid for every $c > 0$, to deduce that

\[
\widetilde{M}(x + h) - \widetilde{M}(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{(x + h)^{s+1} - x^{s+1}}{s(s+1)\zeta(s)} ds \tag{45}
\]

is true on RH by following the proof of Perron’s formula. Note that

\[
\left|\left(\widetilde{M}(x + h) - \widetilde{M}(x)\right) h^{-1} - M(x)\right| \leq h + 1. \tag{46}
\]

Let $\kappa \in (0, 1)$ and take $h = x^\kappa$. Assume that $\lambda x h^{-1} \geq T_1$, which is equivalent to

\[
x \geq \left(\frac{T_1}{\lambda}\right)^{\frac{1}{1 - \kappa}}. \tag{47}
\]

The integral in (45) can be written as

\[
\left(\int_{\sigma_0 - iT_1}^{\sigma_0 + iT_1} + \int_{\sigma_0 - i\lambda x / h}^{\sigma_0 + i\lambda x / h} \right) + \left(\int_{\sigma_0 - i\infty}^{\sigma_0 - i\lambda x / h} + \int_{\sigma_0 + i\lambda x / h}^{\sigma_0 + i\infty} \right) \frac{(x + h)^{s+1} - x^{s+1}}{s(s+1)\zeta(s)} ds.
\]

Denote by $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ the latter integrals, grouping as indicated and writing in the same order. Then (45) and (46) imply

\[
|M(x)| \leq 1 + x^\kappa + \frac{1}{2\pi h} (|\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3|). \tag{48}
\]

In the estimation of the first two integrals we are using

\[
\left|\frac{(x + h)^{s+1} - x^{s+1}}{s+1}\right| \leq h (x + h)^{\sigma_0} \leq \left(1 + \frac{\lambda}{t_0}\right)^{\sigma_0} h x^{\sigma_0},
\]

while the last integral is bounded with the help of

\[
|(x + h)^{s+1} - x^{s+1}| \leq 2 (x + h)^{\sigma_0 + 1} \leq 2 \left(1 + \frac{\lambda}{t_0}\right)^{\sigma_0 + 1} x^{\sigma_0 + 1}.
\]

In derivation of both inequalities we used (47). By Example 1 and Theorem 2 for $\mathcal{L}(s) = \zeta(s)$, we obtain

\[
\log \left|\frac{1}{\zeta(\sigma_0 + it)}\right| \leq \varepsilon_0 \log t
\]
for \( t \geq T_1 \), where \( \varepsilon_0 \) is defined by \((42)\). Note that all conditions of Theorem \(4\) are satisfied by the assumptions of Theorem \(5\). In derivation of the latter inequality we also used the fact that \((\log u)^{1-2\varepsilon_0} \log \log u\) is decreasing function for \( u \geq \exp(\exp(1/(2\sigma_0 - 1)))\). Then

\[
\frac{1}{2\pi h} \left( |I_1| + |I_2| + |I_3| \right) \leq \frac{1}{\pi \sigma_0} \left( 1 + \frac{\lambda}{T_1} \right)^{3\sigma_0} \int_0^{T_1} \frac{du}{|\zeta(\sigma_0 + iu)|} + \frac{\lambda^{3\sigma_0}}{\pi} \left( 1 + \frac{\lambda}{T_1} \right)^{3\sigma_0} \frac{1}{\varepsilon_0} \frac{2}{\lambda (1-\varepsilon_0)} \left( 1 + \frac{\lambda}{T_1} \right) x^{\sigma_0 + (1-\varepsilon_0)\sigma_0}. \tag{49}
\]

Comparison between \((48)\) and \((49)\) reveals that the optimal choice for \( \kappa = \sigma_0 + (1 - \kappa)\varepsilon_0 \), that is when \( \kappa = (\sigma_0 + \varepsilon_0) / (1 + \varepsilon_0) \). Inequality \((47)\) then gives \((44)\). Taking \((49)\) into \((48)\) then implies \((43)\). \( \square \)

**Proof of Theorem \(2\)** Firstly we will prove \((7)\). We are using Theorem \(5\). Take \( \sigma_0 = 0.98 \), \( T_1 = 2.6 \cdot 10^7 \) and \( T_2 = T_1 - 10^3 \log \log T_1 - 1/2 \), together with \( C_1 = C_2 = 1/2 \) and \( C_3 = 10^3 \). Then all conditions of Theorem \(5\) are satisfied. Values for \( \sigma_0 \) and \( T_1 \) were obtained by searching for the smallest possible \( T_1 \) such that \((\sigma_0 + \varepsilon_0) / (1 + \varepsilon_0) \leq 0.99\).

By computer (see Remark \(3\)) we calculated that

\[
\int_0^{11520} \frac{du}{|\zeta(\sigma_0 + iu)|} \leq 12951, \tag{50}
\]

while using Corollary \(1\) gives

\[
\int_0^{T_1} \frac{du}{|\zeta(\sigma_0 + iu)|} \leq \int_0^{T_1} \frac{5.44 \log \log u}{u^{\sigma_0 + (1-\varepsilon_0)\sigma_0}} du \leq 5.946 \cdot 10^{14}. \tag{51}
\]

Therefore, we can take \(5.95 \cdot 10^{14}\) as an upper bound for the integral in \((13)\). We choose \( \lambda = 2 \) in order to make the third term in \((13)\) as small as possible. Then \((7)\) is true for \( x \geq 10^{711} \). The proof is complete since

\[
|M(x)| \leq 555.71 x^{0.99} + 1.94 \cdot 10^{14} x^{0.98}
\]

is true for \( 1 \leq x \leq 10^{711} \).

For the proof of \((8)\) we are using

\[
\sum_{n \leq x} \frac{\mu(n)}{n^\sigma} - \frac{1}{\zeta(s)} \leq \frac{|M(x)|}{x^\sigma} + |s| \int_x^\infty \frac{|M(u)|}{u^\sigma} du,
\]

valid for \( x \geq 1 \) and \( \sigma > 1/2 \) on RH. Estimate \((8)\) now follows by taking \( s = 1 \) in the latter inequality while bounding the Mertens function with \((7)\). \( \square \)

**Remark 3.** Computation \((50)\) was done on Gadi, an HPC cluster at NCI Australia, using 192 cores of Intel Xeon Cascade Lake processors. The integral was approximated by Romberg’s method on intervals of length 10 by using SciPy function \texttt{scipy.integrate.romberg}. It is expected that \((51)\) should be of the order \(10^7\), thus improving the second term in \((7)\). However, the author has found such computations very time consuming when pushing them even only to \(10^5\).

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