Co-domain Symmetry for Complex-Valued Deep Learning

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Abstract

We study complex-valued scaling as a type of symmetry natural and unique to complex-valued measurements and representations. Deep Complex Networks (DCN) extends real-valued algebra to the complex domain without addressing complex-valued scaling. SurReal takes a restrictive manifold view of complex numbers, adopting a distance metric to achieve complex-scaling invariance while losing rich complex-valued information.

We analyze complex-valued scaling as a co-domain transformation and design novel equivariant and invariant neural network layer functions for this special transformation. We also propose novel complex-valued representations of RGB images, where complex-valued scaling indicates hue shift or correlated changes across color channels.

Benchmarked on MSTAR, CIFAR10, CIFAR100, and SVHN, our co-domain symmetric (CDS) classifiers deliver higher accuracy, better generalization, robustness to co-domain transformations, and lower model bias and variance than DCN and SurReal with far fewer parameters.

1. Introduction

Symmetry is one of the most powerful tools in the deep learning repertoire. Naturally occurring symmetries lead to structured variation in natural data, and so modeling these symmetries greatly simplifies learning [1]. A key factor behind the success of Convolutional Neural Networks [2] is their ability to capture the translational symmetry of image data. Similarly, PointNet [3] captures permutation symmetry of point-clouds. These symmetries are formalized as invariance or equivariance to a group of transformations [4]. This view is taken to model a general class of spatial symmetries, including 2D/3D rotations [5–7]. However, this line of research has primarily focused on real-valued data.

We explore complex-valued data which arise naturally in 1) remote sensing such as synthetic aperture radar (SAR), medical imaging such as magnetic resonance imaging (MRI), and radio frequency communications; 2) spectral representations of real-valued data such as Fourier Transform [8,9]; and 3) physics and engineering applications [10]. In deep learning, complex-valued models have shown several benefits over their real-valued counterparts: larger representational capacity [11], more robust embedding [12] and associative memory [13], more efficient multi-task learning [14], and higher quality MRI image reconstruction [15]. We approach complex-valued deep learning from a symmetry perspective: Which symmetries are inherent in complex-valued data, and how do we exploit them in modeling?

One type of symmetry inherent to complex-valued data is complex-valued scaling ambiguity [18]. For example, consider a complex-valued MRI or SAR signal $z$. Due to the nature of signal acquisition, $z$ could be subject to global magnitude scaling and phase offset represented by a complex-valued scalar $s$, thus becoming $s \cdot z$.

A complex-valued classifier takes input $z$ and ideally should focus on discriminating among instances from differ-
Figure 2. Our method learns invariant features with respect to complex-scaling of the input. All examples are from CIFAR 10 with our LAB encoding, undergoing multiplication by a unit complex number. (b, e) tSNE embedding trajectories from DCN [16] and our model. Each color represents a different example. Embeddings form tight clusters for our model, and irregular overlapping curves for DCN. (c) Visualization of our complex-valued embedding of LAB information. The $L^*$ channel is visualized as a grayscale image, and the complex-valued $a^* + ib^*$ visualized as a color image. (d) Model confidence of the correct class for a single example. Higher confidence means larger radius. DCN predictions are highly variable, while our model is robust to complex-scaling and thus constant. (f) Accuracy under complex-scaling and color jitter. Red bars represent complex-rotations sampled from different rotation ranges. Blue bars represent color jitter (as used in [17]). Our method maintains high accuracy across complex-rotations and color jitter, whereas DCN and Real-valued CNN fail. SurReal [18] is robust, but has low overall accuracy. Our method combines high accuracy with robustness. (g) Average accuracy under different rotation ranges, comparing DCN with phase normalization (dotted blue line) and without phase normalization (solid blue line) against our method. The color encoding has a complicated phase distribution, and phase normalization fails to estimate the amount of rotation, resulting in poor accuracy. Our method is thus more suitable for complicated phase distributions.
with the content of the image (Figure 2g). SurReal [18] applies manifold-valued deep learning to complex-valued data, but this framework only captures the manifold aspect and not the complex algebra of complex-valued data. Thus, a more general, principled method is needed. We propose novel layer functions for complex-valued deep learning by studying how they preserve co-domain symmetry. Specifically, we study whether each layer-wise transformation achieves equivariance or invariance to complex-valued scaling.

**Our contributions:** 1) We derive complex-scaling equivariant and invariant versions of common layers used in computer vision pipelines. Our model circumvents the limitations of SurReal [18] and scales to larger models and datasets. 2) Our experiments on MSTAR, CIFAR 10, CIFAR 100, and SVHN datasets demonstrate a significant gain in generalization and robustness. 3) We introduce novel complex-valued encodings of color, demonstrating the utility of using complex-valued representations for real-valued data. Complex-scaling invariance under our LAB encoding automatically leads to color distortion robustness without the need for color jitter augmentation.

2. Related Work

**Complex-Valued Processing:** Complex numbers are ubiquitous in mathematics, physics, and engineering [10, 19, 20]. Traditional complex-valued data analysis involves higher-order statistics [21, 22]. [11] demonstrates higher representational capacity of complex-valued processing on the XOR problem. [23] proposes a sparse coding layer utilizing complex basis functions. [24] proposes a biologically meaningful complex-valued model. [25, 26] encodes data features in a complex vector and learns a metric for this embedding. [15] applies complex-valued neural networks to MRI image reconstruction. [27] investigates the role of critical points in complex neural networks. [28] demonstrates that complex networks have smaller generalization error than real-valued, and offers an overview of their convergence and stability. We refer the reader to [16] for a more detailed account of complex-valued deep learning.

**Transformation Equivariance/Invariance:** An important line of work [5, 6, 29, 30] aims to develop convolutional layers equivariant to domain transformations like rotation/scaling. [6] introduces a principled method for producing group-equivariant layers for finite groups. [5] extends this work to Lie groups on continuous data. [31] uses circular harmonics to produce deep neural networks equivariant to rotation and translation. [32] attempts to produce a general theory of group-equivariant CNNs on the Euclidean space and the sphere. [33] further extends the framework to local gauge transformations on the manifold. This class of methods is well-suited for domain transformations. However, complex-valued scaling is a co-domain transformation, and these methods are not applicable. [7] generalizes neurons to \( \mathbb{R}^3 \) vectors with 3D rotations as a co-domain transformation, introducing rotation-equivariant layers for point-clouds. In contrast, our method handles both the complex algebra and the geometry of complex-valued scaling.

**Complex-Valued Scaling:** Despite the increasing research interest in complex-valued neural networks, the problem of effectively dealing with complex-scaling ambiguity remains open. [16, 34–36] propose an extension of real neural architectures to the complex field by redefining basic building blocks such as complex convolution, batch normalization, and non-linear activation functions. However, these methods are not robust against complex-valued scaling. SurReal [18] tackles the problem of complex-scale invariance by adopting a manifold-based view of complex numbers. It models each complex number as an element of a manifold where complex-scaling corresponds to translation and uses tools from manifold-valued learning to create complex-scale invariant models. This results in better generalization to unseen complex-valued data with leaner models. However, the SurReal framework is restrictive (Sec. 3.1), and the complex-valued processing is forced to be linear (Sec. 3.2), preventing SurReal from scaling to large datasets (Table 1).

3. Equivariance and Invariance for Complex-Valued Deep Learning

3.1. Equivariant Convolution

In contrast to domain transformations which group-specific convolution layers for equivariance [5, 6], any linear layer is equivariant to complex-valued scaling: For a linear function \( L : \mathbb{C}^n \to \mathbb{C}^n \) with an input vector \( x \in \mathbb{C}^n \) and complex scalar \( s \in \mathbb{C} \), \( L(s \cdot x) = s \cdot L(x) \). While a bias term is useful in real-valued neural networks, it turns the convolution layer into an affine function, destroying complex-scale equivariance. Thus, we remove the bias term from the complex-valued convolution used in DCN [16], restoring its equivariance. Additionally, we use Gauss' multiplication trick to speed up the convolution by 25%.

Given a complex-valued input feature map \( z = a + i b \in \mathbb{C}^{C \times H \times W} \) with \( C \) channels and \( H \times W \) pixels, and given a convolutional filter of size \( K \times K \) with weight \( W = X + i Y \in \mathbb{C}^{C \times C \times K \times K} \), we define the Complex-Scale Equivariant Convolution as:

\[
E_{\text{conv}}(z; W) = W \ast z = (X + iY) \ast (a + ib) = \mathbf{t}_1 - \mathbf{t}_2 + i(\mathbf{t}_3 - \mathbf{t}_1 - \mathbf{t}_2)
\]

where \( \mathbf{t}_1 = X \ast a, \mathbf{t}_2 = Y \ast b, \mathbf{t}_3 = (X + Y) \ast (a + b), \) and \( X \ast a \) represents the convolution operation on \( a \) with weight \( X \). In contrast, SurReal uses weighted Frechet Mean (wFM), a restricted convolution where the weights are constrained to be real-valued, positive, and to sum to 1. This restrictive formulation drastically reduces accuracy.
which act on the relative phase information between features (Fig. 3). Given a complex-valued input feature map $\hat{f}$, the normalized mean activations from a set of neighboring activations. However, for complex numbers, this scheme selects for phase $0 \leq \phi \leq \frac{\pi}{2}$ over other phases even though they may encode similar salience. Additionally, this process destroys complex-scale equivariance. To remove this dependence on phase, we select the pixels with the highest magnitude. The result is equivariant to both magnitude and phase.

3.4. Phase-Equivariant Batch Normalization

We follow Deng et. al [7], computing Batch Normalization [38] only on the magnitude of each complex-valued feature, thus preserving the phase information. Given an input feature map $f \in \mathbb{C}^{H \times W}$, we compute:

$$f_{BN} = BN(|f|) \odot \frac{f}{|f|} \quad (3)$$

where $BN$ refers to real-valued BatchNorm and $\odot$ is elementwise multiplication.

3.5. Complex-Valued Invariant Layers

In order to produce invariant complex-valued features, we introduce the Division Layer and the Conjugate Multiplication Layer.

**Division Layer**

Given two complex-valued features $z_1, z_2 \in \mathbb{C}^{H \times W}$, we define:

$$\text{Div}(z_1, z_2) = \frac{|z_1|}{|z_2|} \exp\{i(\angle z_1 - \angle z_2)\} \quad (4)$$

$$\text{Conj}(z_1, z_2) = z_1 \overline{z_2} \quad (5)$$

In practice, the denominator for division can be small, so we offset the magnitude of the denominator by $\epsilon = 10^{-7}$.

While the division layer induces invariance to all complex-valued scaling, the conjugate layer only induces invariance to phase. This layer also captures some second-degree interactions similar to a bilinear layer [39]. In contrast to our layers which capture relative phase and magnitude offsets of input features, SurReal’s Distance Layer achieves invariance by extracting real-valued distances between features, discarding rich complex-valued information in the process.

3.6. Generalized Tangent ReLU

In practice, TReLU slows down convergence compared to CReLU. We remedy this through three modifications: a) a learned complex-valued scaling factor for each input channel, enabling the layer to adapt to input magnitude and phase, b) hyperparameter $r$ to control the magnitude threshold. Notably, $r = 0$ produces a phase-only version of TangentReLU, which is equivariant to input magnitude, and c) learned scaling constant for the output phase of each channel, allowing the non-linearity to adapt the output phase distribution. Our proposed method generalizes TReLU both as a transformation and as a thresholding function. It is defined as:

$$\text{GTRelu}(x; r, \omega) = \max(r, |c \cdot \omega|) \exp\{i \omega \angle (c \cdot x)\}$$
In order to produce real-valued outputs from complex-valued features, we propose using feature distances, which capture hue shift and channel correlations respectively, demonstrating the utility of complex-valued representations for real-valued data.

Our first so-called "Sliding" encoding takes an \([R,G,B]\) image and encodes it with two complex-valued channels:

\[
[R,G,B] \rightarrow [R + iG, G + iB]
\]

The complex phase in this encoding corresponds to the ratio of R, G, B values, so the phase in this encoding captures the correlation between the adjacent color channels.

Our second proposed encoding uses \(L \cdot a^* b^*\), a perceptually uniform color representation with luminance represented by the \(L\) channel and chromaticity by the \(a\) and \(b\) channels. [40] uses this color space for image colorization.
Figure 6. Our CIFARnet models demonstrate two methods of constructing complex-scale invariant models. Green arrows represent equivariant features, and blue arrows represent invariant features. top: Type I architecture uses a Division Layer in early stages, producing complex-scale invariant features which can be used with any following layers. bottom: Type E uses equivariant layers throughout the network, retaining phase information until the final Invariant Prototype Distance Layer. This class of models is more restrictive but can achieve higher accuracy (See Table 1) as a consequence of retaining more information.

We use it to represent color as a two-channel, complex-valued representation, with the first channel containing the luminance ($L^*$ channel), and the second channel containing chromaticity ($a^*$ and $b^*$ channels) as $a^* + ib^*$ (Figure 2c):

$$ [R, G, B] \rightarrow [L^*, a^* + ib^*] \quad (10) $$

Color distortions as co-domain transformations: Color distortion can be approximated with complex-valued scaling of our LAB color representation (Figure 2a). A complex-scale invariant network is thus automatically robust to color distortions without the need for data augmentation.

5. Experiments

We conduct three kinds of experiments: Accuracy: 1) Classification of naturally complex-valued images, 2) real-valued images with real and complex representations; Robustness against complex scaling and color distortion; Generalization: 1) Bias-variance analysis, 2) generalization on smaller training sets, 3) Feature redundancy analysis.

5.1. Complex-Valued Dataset: MSTAR

MSTAR contains 15,716 complex-valued synthetic aperture radar (SAR) images divided into 11 classes [41]. Each image has one channel and size $128 \times 128$. We discard the last "clutter" class and follow [42], training on the depression angle $17^\circ$ and testing on $15^\circ$. We train each model on varying proportions of the dataset to evaluate the accuracy and generalization capabilities of each model.

SurReal: We replicate the architecture described in Table 1 of [18]. Since the paper does not mention the learning rate, we use the same learning rate and batch size as our model. DCN: We use author-provided code$^1$, creating a complex ResNet with $\mathbb{C}$ReLU and 10 blocks per stage. By default, this model accepts $32 \times 32$ images, so we append $2 \times \{\text{ComplexConv, ComplexBatchNorm}\}$ with stride 2 to downsample the input. The model is trained for 200 epochs using SGD with batch size 64 and the learning rate schedule in [16]. We select the epoch with the best validation accuracy. Real-valued baseline: We use a 3-stage ResNet with 3 layers per residual block and convert the complex input into two real-valued channels.

CDS: We use a Type I model based on SurReal [18]. We extract equivariant features using an initial equivariant block containing $\mathbb{C}$Conv, Eq. GTReluU, Eq. MaxPool layers, and then obtain complex-scale invariant features by using a Division Layer. These features are then fed to a real-valued ResNet. For more details about every model, please refer to supplementary materials.

Training: We optimize both SurReal and CDS models using the AdamW optimizer [43,44] with learning rate $10^{-3}$, momentum $(0.9, 0.99)$, weight decay 0.1, and batch size 256 for $2.5 \times 10^5$ iterations. We validate every 1000 steps, picking the model with the best validation accuracy.

5.2. Real-valued Datasets: CIFAR10/100, SVHN

Datasets: CIFAR10 [48] (and CIFAR100) consists of 10 (100) classes containing 6000 (600) images each. Both CIFAR10 and CIFAR100 are partitioned into 50000 training images and 10000 test images. SVHN [49] consists of house number images from Google Street View, divided into 10 classes with 73,257 training digits and 26,032 testing digits.

$^1$https://github.com/ChihebTrabelsi/deep_complex_networks
Figure 7. Our model generalizes across various dataset sizes, has lower bias and variance, and avoids learning redundant filters. (a): We produce trend curves (similar to [45]) for the MSTAR accuracy table (Table 2). Our method has the lowest test error for measured dataset sizes, a trend that is predicted to scale to even smaller sizes. (b): We followed [46] for CIFAR10 models with LAB encoding. Classes are ordered in ascending order of bias for our model. Our model consistently shows the lowest bias for each class, and the lowest variance for 9 out of 10 classes, indicating overall superior generalization ability. (c): Filter similarity histogram from conv2 layer of each CIFARNet model, following [47]. Our distribution mean is closest to 0, indicating our method achieves the least redundant filters.

| Method       | # Param | CIFAR10 | CIFAR100 | SVHN   |
|--------------|---------|---------|----------|--------|
|              |         | RGB     | LAB      | Sliding| RGB     | LAB      | Sliding |
| DCN [16]     | 66,858  | 65.17   | 58.64    | 63.83  | 32.52   | 27.36    | 28.87   | 85.26  |
| SurReal [18] | 35,274  | 50.68   | 53.02    | 54.61  | 23.57   | 25.97    | 26.66   | 80.51  |
| Real-valued CNN | 34,282 | 64.43   | 63   | 63.43  | 31.93   | 31.72    | 31.93   | 87.47  |
| Ours (Type-I) | 24,241  | **69.23** | 67.17 | 68.7   | 36.92   | 37.81    | 38.51   | **89.39** |
| Ours (Type-E) | 25,745  | 68.48   | **67.58** | **69.19** | 41.83   | **39.55** | **42.08** | 77.19  |

Table 1. Our models outperform the baselines’ CIFARNet versions on real-valued datasets. Type-I model performs better on easier datasets like SVHN, and Type-E performs better on difficult datasets like CIFAR100. In contrast, SurReal does not scale to large datasets.

Models: To ensure equal footing for each model, all networks in this experiment are based off CIFARNet, i.e., 3 Convolution Layers (stride 2) and 2 fully connected layers. We also replace average pooling with a depthwise-separable convolution as a learnable pooling layer. All models are optimized with AdamW [43,44] using momentum (0.99, 0.999), for $5 \times 10^5$ steps with batch size 256, learning rate $10^{-3}$, weight decay 0.1, and validated every 1000 iterations. DCN: We use ComplexConv for convolutions and CReLU as the non-linearity. We do not use Residual Blocks or Complex BatchNorm from [16] to ensure fairness. SurReal: We use wFM for convolutions and Use Distance Transform after Layer 3 to extract invariant real-valued features. Real-Valued CNN: We use the CIFARNet architecture, converting each complex input channel into two real-valued channels. CDS: We evaluate two models: Type I: We use EConv for convolutions and GTRelu ($r = 0$) for non-linearity. We use a Division layer after the first Econv to achieve invariance. The final fully-connected layer is replaced with Prototype Distance layer to predict class logits (Figure 6). Type E: We use Econv for convolutions and Equivariant GTRelu for non-linearity. The final FC layer is replaced with Invariant Prototype Distance layer to predict logits (Figure 6), and the prototype distance inputs are normalized with Equivariant BatchNorm to preserve equivariance.

CDS-Large: We train a 1.7M parameter Type I model on CIFAR 10 with LAB encoding, and compare it against equivalently sized DCN (WS with CReLU from [16]). CDS-Large is based on the simplified 4-stage ResNet provided by Page et al. [50] for DAWNBench [51]. We use the conjugate layer after the first Econv to get complex-scale invariant features and feed them to the Complex ResNet. Like DCN, we optimize the model using SGD with horizontal flipping and random cropping augmentations with a varying learning rate schedule (see supplementary material for more details).

5.3. Model Performance Analysis

Accuracy and scalability: Our approach achieves complex-scale invariance of manifold-based methods while retaining high accuracy and scalability. On MSTAR, our model beats the baselines across a diverse range of splits with less than half the parameters used by SurReal (Table 2). On the smallest training split (5% training data), our model shows a gain of 19.7% against DCN and real-valued CNN and 8.4% against SurReal. On the largest split (100% training data), our model beats real-valued CNN by 29.2%, DCN
### Table 2. Our method achieves the best accuracy and generalization with the fewest parameters. We report accuracy on varying proportions of MSTAR training data. The performance gap is wider for smaller train-sets, with Real-CNN and DCN failing to generalize.

| Model    | # Params | 5%  | 10%  | 50%  | 90%  | 100% |
|----------|----------|-----|------|------|------|------|
| Real     | 33,050   | 47.4| 46.6 | 60.6 | 73   | 66.9 |
| SurReal [18] | 63,690   | 61.1| 68   | 90.3 | 95.6 | 94.9 |
| DCN [16] | 863,587  | 49.8| 47   | 81.9 | 89.1 | 89.1 |
| Ours     | 29,536   | 69.5| 78.3 | 91.3 | 95.2 | 96.1 |

by 7%, and SurReal by 1.2%, demonstrating our advantage across an extensive range of dataset sizes.

On CIFAR10, CIFAR100, and SVHN under different encodings, our models obtain the highest accuracy across every setting (Table 1). Unlike SurReal, our model scales to these large classification datasets while retaining complex-scale invariance. For the complex-valued color encodings, which require precise processing of phase information, our model consistently beats baselines by 4%-8%. These results highlight the advantage of our approach for precise complex-valued processing across a variety of real-valued datasets.

**Phase normalization and color jitter:** A natural pre-processing trick to address complex-scaling invariance is to compute the average input phase $\bar{\phi}$ and to scale the input by $e^{-i\bar{\phi}}$ to cancel it. We test this approach by applying random complex-valued scaling with different rotation ranges and comparing DCN’s accuracy with and without phase normalization against our method (Figure 2g). When the input phase distribution is simple (e.g., phase set to 0), phase normalization successfully protects DCN against complex-valued scaling. However, for complicated phase distributions such as LAB encoding, this method fails. Our method succeeds in both situations, and this robustness transfers to the color jitter (as used by [17], see Figure 2f). Our model is more robust against color jitter without data augmentation.

**Bias and variance analysis:** While model accuracy across different datasets is useful, a better measure for the generalization of supervised models is the bias-variance decomposition. We follow [46]: given model $f$, dataset $D$, ground truth $Y$ and instance $x$, [46] defines the bias-variance decomposition of the prediction error (per instance) as:

$$\text{Error}(x; f) = E [(f(x; D) - Y)^2]$$  \hspace{1cm} (11)

$$= \text{Bias}(x; f) + \text{Var}(x; f) + R(x)$$  \hspace{1cm} (12)

where bias measures the accuracy of the predictions with respect to the ground-truth, variance measures the stability of the predictions, and $R$ denotes the irreducible error. Using the 0-1 loss $\mathcal{L}_{0-1}$, [46] calculates the bias and variance terms (per instance per model) for the classification task as such:

$$\text{Bias}(x; h) = \mathcal{L}_{0-1}(y_m; t)$$  \hspace{1cm} (13)

$$\text{Var}(x; h) = \frac{1}{n} \sum_{k=1}^{n} \mathcal{L}_{0-1}(y^{(k)}; y_m)$$  \hspace{1cm} (14)

where $y_m$ is the mode or the main prediction. We compute this metric for each instance, averaging bias and variance over classes. Compared on CIFAR10 with LAB encoding, our model achieves the lowest bias among all classes and the lowest variance among 9 out of 10 classes (Figure 7).

**Generalization from less training data:** [45] derives empirical trends for scaling of language models under different conditions, including the overfitting regime where the training set is small compared to parameters. We produce similar trend curves for the MSTAR test results by fitting linear regression curves to log accuracy and dataset size reported in Table 2. We plot the results in Figure 7. The extrapolated least-squares linear fit suggests our model might continue to generalize better on yet smaller datasets.

**Feature redundancy comparison:** [47] shows that common CNN architectures learn highly correlated filters. This increases model size and reduces the ability to capture diversity. We follow [47], measuring correlations between guided backpropagation maps of different filters in layer 2 for each model on CIFAR10 with the LAB encoding. We find that our model displays the highest filter diversity. This observa-
tion is consistent with higher test accuracy, lower bias and variance, and leaner models from previous experiments.

Scaling to large models: Small models are essential for applications such as edge computing, thus motivating leaner models (Table 1 and 2). However, the best models on large real-valued datasets, like ViT-H/14 [52] with 99.5% test accuracy on CIFAR10, have millions of parameters. We test the scalability of our approach by comparing CDS-Large with a DCN model of equivalent size on CIFAR10 with LAB encoding (Table 8a). While we focus on leaner complex-scale invariant models, our method beats DCN while additionally achieving complex-scale invariance even for large models. This observation is consistent with our results for small models (Table 1), showing the effectiveness of our method for diverse model sizes.

Summary: We analyze complex-scaling as a co-domain transformation and derive equivariant/invariant versions of commonly used layers. We also present novel complex encodings. Our approach combines complex-valued algebra with complex-scaling geometry, resulting in leaner and more robust models with better accuracy and generalization.

Acknowledgements: The authors gratefully acknowledge DARPA, AFRL, and NGA for funding various applications of this research. We also thank Matthew Tancik, Ren Ng, Connelly Barnes, and Claudia Tischler for their thoughtful comments.

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Table 3. wFM results in significant reductions in accuracy. We train models on CIFAR10 with LAB encoding and tabulate the resulting test accuracy. Compared to a real-valued CNN, wFM results in significantly lower accuracy due to its restricted formulation.

| Layer Type | Params | Acc (%) |
|------------|--------|---------|
| Complex    | 66,858 | 58.6    |
| Real       | 34,282 | 63.4    |
| wFM        | 42,154 | 52.2    |

6. Supplementary Material

Table of contents:

1. Extensive bias-variance evaluation, including plots for both LAB and RGB encodings.
2. Limitations of wFM as a convolutional layer in CNNs.
3. Ablation tests
4. Architecture details for models used in our experiments.

6.1. Bias-Variance Evaluation

In this section, we run extensive evaluations for bias and variance on each encoding for the CIFAR10 dataset with 6. Supplementary Material (Table 3).

6.2. Limitations of wFM

We demonstrate the theoretical and empirical limitations of wFM. From a theoretical perspective, we show that wFM using the Manifold Distance Metric of SurReal processes magnitude and phase separately and is thus unable to process the joint distribution. Our experiments show that in practice, wFM results in a significant loss in accuracy compared to real-valued and complex-valued convolutional filters.

**Decomposability:** We discuss the weighted-Fréchet Mean for complex-valued neural networks and show that the magnitude and phase computations are decoupled. wFM is defined as the minimum of weighted distances to a given set of points. In specific cases (like the Euclidean distance metric), closed-form solutions (like the euclidean weighted mean) exist, but there is no general closed-form solution.
In order to pass the complex-valued features into the real-valued ResNet, we convert complex features to real-valued using the \((\log \text{mag}, \sin \theta, \cos \theta)\) encoding, treating each as a separate real-valued channel (resulting in 15 real-valued channels from 5 complex-valued channels).

**CDS-Large:** For the model architecture, please see Table 10. We train this model with SGD, using momentum 0.9, weight decay constant \(5 \times 10^{-4}\), using a piece-wise linear learning rate schedule starting at 0.01, increasing to 0.2 by epoch 10, then decreasing to 0.01 by epoch 100, 0.001 by 120, 0.0001 by 150, and staying constant until 200. To ensure fair comparison, use horizontal flips and random cropping augmentation as used in [16]. All models are implemented in PyTorch [53].

6.4. Architecture Details

In this section, we discuss details of the architectures used in our experiments.

**CIFARnet architectures:** For CIFARnet architectures, please refer to tables 5-9. Please note that our replication of wFM [18] uses the \((\log \text{mag}, \sin \theta, \cos \theta)\) encoding for the manifold values, and uses the weighted average formulation.

**MSTAR architectures:** For DCN, please refer to [16], and for the downsampling block, see Table 11. Our SurReal replication is based on Table 1 in [18], and our model is based on the SurReal architecture (see Table 13). We use the same real-valued ResNet as the real-valued baseline (see Table 12). In order to pass the complex-valued features into the real-valued ResNet, we convert complex features to real-valued using the \((\log \text{mag}, \sin \theta, \cos \theta)\) encoding, treating each as a separate real-valued channel (resulting in 15 real-valued channels from 5 complex-valued channels).

| Method          | Accuracy |
|-----------------|----------|
| Division Layer  | 67.17    |
| Conjugate Layer | 66.73    |
| Euclidean Distance | 67.17 |
| Manifold Distance | 68.54 |
| GTRelu (r=0)    | 67.17    |
| GTRelu (r=0.1)  | 68.14    |
| GTRelu (r=1)    | 49.15    |

Table 4. Ablation test results for our Type-I model on CIFAR 10 with LAB encoding. We find that Division Layer, Manifold Distance, and a GTRelu threshold of \(r = 0.1\) perform the best.

![Figure 9](image.png)

Figure 9. Our method demonstrates lowest overall bias and variance on CIFAR10. We followed [46] for CIFAR10 models with RGB encoding. Classes are ordered in ascending order of bias for our model. Our model consistently shows the lowest bias for each class in every encoding, and lowest variance in 8, 9, and 10 out of 10 classes with RGB, LAB, and Sliding encodings respectively.

**CDS-Large:** For the model architecture, please see Table 10. We train this model with SGD, using momentum 0.9, weight decay constant \(5 \times 10^{-4}\), using a piece-wise linear learning rate schedule starting at 0.01, increasing to 0.2 by epoch 10, then decreasing to 0.01 by epoch 100, 0.001 by 120, 0.0001 by 150, and staying constant until 200. To ensure fair comparison, use horizontal flips and random cropping augmentation as used in [16]. All models are implemented in PyTorch [53].
### Table 5. SurReal CIFAR Model Architecture

| Layer Type   | Input Shape | Kernel | Stride | Padding | Output Shape |
|--------------|-------------|--------|--------|---------|--------------|
| Complex CONV | [3, 32, 32] | 3 × 3  | 2      | 1       | [16, 16, 16] |
| G-transport  | [16, 16, 16]| -      | -      | -       | [16, 16, 16] |
| Complex CONV | [16, 16, 16]| 3 × 3  | 2      | 1       | [32, 8, 8]   |
| G-transport  | [32, 8, 8]  | -      | -      | -       | [32, 8, 8]   |
| Complex CONV | [32, 8, 8]  | 3 × 3  | 2      | 1       | [64, 4, 4]   |
| G-transport  | [64, 4, 4]  | -      | -      | -       | [64, 4, 4]   |
| Distance Layer | [64, 4, 4] | -      | -      | -       | [64, 4, 4]   |
| Average Pooling | [64, 4, 4] | 4 × 4  | -      | -       | [64, 1, 1]   |
| FC           | [64]        | -      | -      | -       | [128]        |
| ReLU         | [128]       | -      | -      | -       | [128]        |
| FC           | [128]       | -      | -      | -       | [10]         |

### Table 6. DCN CIFAR Model Architecture

| Layer Type   | Input Shape | Kernel | Stride | Padding | Output Shape |
|--------------|-------------|--------|--------|---------|--------------|
| Complex CONV | [3, 32, 32] | 3 × 3  | 2      | 1       | [16, 16, 16] |
| ReLU         | [16, 16, 16]| -      | -      | -       | [16, 16, 16] |
| Complex CONV | [16, 16, 16]| 3 × 3  | 2      | 1       | [32, 8, 8]   |
| ReLU         | [32, 8, 8]  | -      | -      | -       | [32, 8, 8]   |
| Complex CONV | [32, 8, 8]  | 3 × 3  | 2      | 1       | [64, 4, 4]   |
| ReLU         | [64, 4, 4]  | -      | -      | -       | [64, 4, 4]   |
| Average Pooling | [64, 4, 4] | 4 × 4  | -      | -       | [64, 1, 1]   |
| Complex-to-Real | [64, 1, 1] | 4 × 4  | -      | -       | [128]        |
| FC           | [128]       | -      | -      | -       | [128]        |
| ReLU         | [128]       | -      | -      | -       | [128]        |
| FC           | [128]       | -      | -      | -       | [10]         |
Table 7. Our (Type-E) CIFAR Model Architecture

| Layer Type    | Input Shape | Kernel | Stride | Padding | Output Shape |
|---------------|-------------|--------|--------|---------|--------------|
| Econv         | [3, 32, 32] | 3 x 3  | 2      | 1       | [16, 16, 16] |
| Eq. GReLU     | [16, 16, 16]| -      | -      | -       | [16, 16, 16] |
| Econv         | [16, 16, 16]| 3 x 3  | 2      | 1       | [32, 8, 8]   |
| Eq. GReLU     | [32, 8, 8]  | -      | -      | -       | [32, 8, 8]   |
| Econv         | [32, 8, 8]  | 3 x 3  | 2      | 1       | [64, 4, 4]   |
| Average Pooling| [64, 4, 4]| 4 x 4  | -      | -       | [64, 1, 1]   |
| Equivariant FC| [64]        | -      | -      | -       | [128]        |
| Invariant Prototype Distance| [128]| -      | -      | -       | [10]         |

Table 8. Our (Type-I) CIFAR Model Architecture

| Layer Type    | Input Shape | Kernel | Stride | Padding | Output Shape |
|---------------|-------------|--------|--------|---------|--------------|
| Econv         | [3, 32, 32] | 3 x 3  | 2      | 1       | [16, 16, 16] |
| Division Layer| [16, 16, 16]| 3 x 3  | -      | -       | [16, 16, 16] |
| GReLU         | [16, 16, 16]| -      | -      | -       | [16, 16, 16] |
| Econv         | [16, 16, 16]| 3 x 3  | 2      | 1       | [32, 8, 8]   |
| GReLU         | [32, 8, 8]  | -      | -      | -       | [32, 8, 8]   |
| Econv         | [32, 8, 8]  | 3 x 3  | 2      | 1       | [64, 4, 4]   |
| GReLU         | [64, 4, 4]  | -      | -      | -       | [64, 4, 4]   |
| Average Pooling| [64, 4, 4]| 4 x 4  | -      | -       | [64, 1, 1]   |
| Equivariant FC| [64]        | -      | -      | -       | [128]        |
| Prototype Distance| [128]| -      | -      | -       | [10]         |

Table 9. 2-Channel Real-Valued CIFAR Model Architecture

| Layer Type    | Input Shape | Kernel | Stride | Padding | Output Shape |
|---------------|-------------|--------|--------|---------|--------------|
| CONV          | [3, 32, 32] | 3 x 3  | 2      | 1       | [16, 16, 16] |
| ReLU          | [16, 16, 16]| -      | -      | -       | [16, 16, 16] |
| CONV          | [16, 16, 16]| 3 x 3  | 2      | 1       | [32, 8, 8]   |
| ReLU          | [32, 8, 8]  | -      | -      | -       | [32, 8, 8]   |
| CONV          | [32, 8, 8]  | 3 x 3  | 2      | 1       | [64, 4, 4]   |
| ReLU          | [64, 4, 4]  | -      | -      | -       | [64, 4, 4]   |
| Average Pooling| [64, 4, 4]| 4 x 4  | -      | -       | [64, 1, 1]   |
| FC            | [64]        | -      | -      | -       | [128]        |
| ReLU          | [128]       | -      | -      | -       | [128]        |
| FC            | [128]       | -      | -      | -       | [10]         |
### Table 10. Our CDS-Large Model Architecture

| Layer Type            | Input Shape | Kernel | Stride | Padding | Output Shape |
|-----------------------|-------------|--------|--------|---------|--------------|
| Econv                 | [3, 32, 32] | 3 × 3  | 1      | 1       | [64, 32, 32] |
| Conjugate Layer       | [64, 32, 32]| 1 × 1  | -      | -       | [64, 32, 32] |
| Econv (Groups=2)      | [64, 32, 32]| 3 × 3  | 1      | 1       | [64, 32, 32] |
| ComplexBatchNorm      | [64, 32, 32]| -      | -      | -       | [64, 32, 32] |
| ReLU                  | [64, 32, 32]| -      | -      | -       | [64, 32, 32] |
| Econv (Groups=2)      | [64, 32, 32]| 3 × 3  | 1      | 1       | [128, 32, 32]|
| ComplexBatchNorm      | [128, 32, 32]| -      | -      | -       | [128, 32, 32]|
| ReLU                  | [128, 32, 32]| -      | -      | -       | [128, 32, 32]|
| Eq. MaxPool           | [128, 32, 32]| 2 × 2  | -      | -       | [128, 16, 16]|
| ResBlock(groups=2)    | [128, 16, 16]| -      | -      | -       | [128, 16, 16]|
| Econv (Groups=4)      | [128, 16, 16]| 3 × 3  | 1      | 1       | [256, 16, 16]|
| ComplexBatchNorm      | [256, 16, 16]| -      | -      | -       | [256, 16, 16]|
| ReLU                  | [256, 16, 16]| -      | -      | -       | [256, 16, 16]|
| Eq. MaxPool           | [256, 16, 16]| 2 × 2  | -      | -       | [256, 8, 8]  |
| Econv (Groups=2)      | [256, 8, 8]  | 3 × 3  | 1      | 1       | [512, 8, 8]  |
| ComplexBatchNorm      | [512, 8, 8]  | -      | -      | -       | [512, 8, 8]  |
| ReLU                  | [512, 8, 8]  | -      | -      | -       | [512, 8, 8]  |
| Eq. MaxPool           | [512, 8, 8]  | 2 × 2  | -      | -       | [512, 4, 4]  |
| ResBlock(groups=4)    | [512, 4, 4]  | -      | -      | -       | [512, 4, 4]  |
| Eq. MaxPool           | [512, 4, 4]  | 2 × 2  | -      | -       | [512, 1, 1]  |
| Fully Connected       | [1024]       | -      | -      | -       | [10]         |

### Table 11. DCN Down-sampling Block for MSTAR

| Layer Type            | Input Shape | Kernel | Stride | Padding | Output Shape |
|-----------------------|-------------|--------|--------|---------|--------------|
| Complex CONV          | [1, 128, 128]| 3 × 3  | 2      | 1       | [12, 64, 64] |
| ComplexBatchNorm      | [12, 64, 64]| -      | -      | -       | [12, 64, 64] |
| Complex CONV          | [1, 64, 64] | 3 × 3  | 2      | 1       | [12, 32, 32] |
| ComplexBatchNorm      | [12, 32, 32]| -      | -      | -       | [12, 32, 32] |
Table 12. MSTAR Real-valued Model Architecture

| Layer Type     | Input Shape | Kernel | Stride | Output Shape |
|----------------|-------------|--------|--------|--------------|
| CONV           | [2, 100, 100]| 5 x 5  | 1      | [30, 96, 96] |
| GroupNorm + ReLU | [30, 96, 96]| -      | -      | [30, 96, 96] |
| ResBlock       | [30, 96, 96]| -      | -      | [40, 96, 96] |
| MaxPool        | [40, 96, 96]| 2 x 2  | 2      | [40, 48, 48] |
| CONV           | [40, 48, 48]| 5 x 5  | 3      | [50, 15, 15] |
| GroupNorm + ReLU | [50, 15, 15]| -      | -      | [50, 15, 15] |
| ResBlock       | [50, 15, 15]| -      | -      | [60, 15, 15] |
| MaxPool        | [60, 15, 15]| 2 x 2  | 1      | [70, 14, 14] |
| GroupNorm + ReLU | [70, 14, 14]| -      | -      | [70, 14, 14] |
| AveragePool    | [70, 14, 14]| -      | -      | [70]         |
| FC             | [70]        | -      | -      | [30]         |
| ReLU           | [30]        | -      | -      | [30]         |
| FC             | [30]        | -      | -      | [10]         |

Table 13. Our MSTAR Model Architecture

| Layer Type     | Input Shape | Kernel | Stride | Padding | Output Shape |
|----------------|-------------|--------|--------|---------|--------------|
| Econv (Groups=5) | [1, 100, 100]| 5 x 5  | 1      | 0       | [5, 96, 96]  |
| Eq. GReLU      | [5, 96, 96] | -      | -      | -       | [5, 96, 96]  |
| Eq. MaxPool    | [5, 96, 96] | 2 x 2  | 2      | -       | [5, 48, 48]  |
| Econv          | [5, 48, 48] | 3 x 3  | 2      | 0       | [5, 23, 23]  |
| Eq. GReLU      | [5, 23, 23] | -      | -      | -       | [5, 21, 21]  |
| Division Layer | [5, 23, 23] | 3 x 3  | -      | -       | [5, 21, 21]  |
| Complex-to-Real | [5, 21, 21]| -      | -      | -       | [15, 21, 21] |
| CONV (Groups=5) | [15, 21, 21]| 5 x 5  | 1      | -       | [30, 17, 17] |
| GroupNorm + ReLU | [30, 17, 17]| -      | -      | -       | [30, 17, 17] |
| ResBlock       | [30, 17, 17]| -      | -      | -       | [40, 17, 17] |
| MaxPool        | [40, 17, 17]| 2 x 2  | 2      | -       | [40, 8, 8]   |
| CONV (Groups=5) | [40, 8, 8] | 5 x 5  | 3      | -       | [50, 2, 2]   |
| GroupNorm + ReLU | [50, 2, 2] | -      | -      | -       | [50, 2, 2]   |
| ResBlock       | [50, 2, 2] | -      | -      | -       | [60, 2, 2]   |
| CONV (Groups=5) | [60, 2, 2] | 2 x 2  | 1      | -       | [70, 1, 1]   |
| GroupNorm + ReLU | [70, 1, 1] | -      | -      | -       | [70, 1, 1]   |
| FC             | [70]        | -      | -      | -       | [30]         |
| ReLU           | [30]        | -      | -      | -       | [30]         |
| FC             | [30]        | -      | -      | -       | [10]         |