Sharp Global Bounds for the Hessian on Pseudo-Hermitian Manifolds

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Dedicated to the memory of our friend and colleague Carlos Segovia.

1 Introduction

In PDE theory, Harmonic Analysis enters in a fundamental way through the basic estimate valid for \(f \in C^\infty_0(\mathbb{R}^n)\), which states,

\[
\sum_{i,j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^n)} \leq c(n,p) \| \Delta f \|_{L^p(\mathbb{R}^n)}, \quad \text{for } 1 < p < \infty.
\]

This estimate is really a statement of the \(L^p\) boundedness of the Riesz transforms, and thus (1) is a consequence of the multiplier theorems of Marcinkiewicz and Hörmander-Mikhlin, [15]. More sophisticated variants of (1) can be proved by relying on the square function [15] and [14]. In particular (1) leads to a-priori \(W^{2,p}\) estimates for solutions of

\[
\Delta u = f, \quad \text{for } f \in L^p.
\]

Knowledge of \(c(p,n)\) allows one to perform a perturbation of (2) and study

\[
\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f
\]

as was done by Cordes [4], where \(A = (a^{ij})\) is bounded, measurable, elliptic and close to the identity in a sense made precise by Cordes. The availability of the estimates of Alexandrov-Bakelman-Pucci and the Krylov-Safonov theory
allows one to obtain estimates for (3) in full generality without relying on a perturbation argument. See also [12].

Our focus here will be to study the CR analog of (3). Since at this moment in time there is no suitable Alexandrov-Bakelman-Pucci estimate for the CR analog of (3) we will be seeking a perturbation approach based on an analog of (1) on a CR manifold. Our main interest is the case \( p = 2 \) in (1). In this case a simple integration by parts suffices to prove (1) in \( \mathbb{R}^n \). We easily see that for \( f \in C_0^\infty (\mathbb{R}^n) \) we have

\[
\sum_{i,j=1}^n \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{L^2(\mathbb{R}^n)}^2 = \| \Delta f \|_{L^2(\mathbb{R}^n)}^2.
\]  

(4)

In the case of (1) on a CR manifold a result has been recently obtained by Domokos-Manfredi [6] in the Heisenberg group. The proof in [6] makes uses of the harmonic analysis techniques in the Heisenberg group developed by Strichartz [16] that will not apply to studying such inequalities for the Hessian on a general CR manifold, although other nilpotent groups of step 2 can be treated similarly [5].

Instead we shall proceed by integration by parts and use of the Bochner technique. A Bochner identity on a CR manifold was obtained by Greenleaf [8] and will play an important role in our computations.

We now turn to our setup. We consider a smooth orientable manifold \( M^{2n+1} \). Let \( V \) be a vector sub-bundle of the complexified tangent bundle \( \mathbb{C}TM \). We say that \( V \) is a CR bundle if

\[
V \cap \overline{V} = \{0\}, \quad [V, V] \subset V, \quad \text{and} \quad \dim \mathbb{C}V = n.
\]  

(5)

A manifold equipped with a sub-bundle satisfying (5) will be called a CR manifold. See the book by Trèves [18]. Consider the sub-bundle

\[
H = \text{Re} (V \oplus \overline{V}).
\]  

(6)

\( H \) is a real \( 2n \)-dimensional vector sub-bundle of the tangent bundle \( TM \). We assume that the real line bundle \( H^\perp \subset T^*M \), where \( T^*M \) is the cotangent bundle, has a smooth non-vanishing global section. This is a choice of a non-vanishing 1-form \( \theta \) on \( M \) and \( (M, \theta) \) is said to define a pseudo-hermitian structure. \( M \) is then called a pseudo-hermitian manifold. Associated to \( \theta \) we have the Levi form \( L_\theta \) given by

\[
L_\theta(V, W) = -i \, d\theta(V \wedge \overline{W}), \quad \text{for} \ V, W \in V.
\]  

(7)

We shall assume that \( L_\theta \) is definite and orient \( \theta \) by requiring that \( L_\theta \) is positive definite. In this case, we say that \( M \) is strongly pseudo-convex. We shall always assume that \( M \) is strongly pseudo-convex.

On a manifold \( M \) that carries a pseudo-hermitian structure, or a pseudo-hermitian manifold, there is a unique vector field \( T \), transverse to \( H \) defined in (6) with the properties
\[ \theta(T) = 1 \quad \text{and} \quad d\theta(T, \cdot) = 0. \quad (8) \]

\( T \) is also called the Reeb vector field. The volume element on \( M \) is given by

\[ dV = \theta \wedge (d\theta)^n. \quad (9) \]

A complex valued 1-form \( \eta \) is said to be of type \((1, 0)\) if \( \eta(W) = 0 \) for all \( W \in V \), and of type \((0, 1)\) if \( \eta(W) = 0 \) for all \( W \in V \).

An admissible co-frame on an open subset of \( M \) is a collection of \((1, 0)\)-forms \( \{ \theta_1, \ldots, \theta_\alpha, \ldots, \theta_n \} \) that locally form a basis for \( V^* \) and such that \( \theta_\alpha(T) = 0 \) for \( 1 \leq \alpha \leq n \). We set \( \theta_\alpha = \theta_\alpha^\alpha \).

We then have that \( \{ \theta, \theta_\alpha, \theta_\alpha^\alpha \} \) locally form a basis of the complex co-vectors, and the dual basis are the complex vector fields \( \{ T, Z_\alpha, \overline{Z}_\alpha \} \).

For \( f \in C^2(M) \) we set \( T f = f_0, \quad Z_\alpha f = f_\alpha, \quad \overline{Z}_\alpha f = \overline{f_\alpha} \). \quad (10)

We note that in the sequel all our functions \( f \) will be real valued.

It follows from (5), (7), and (8) that we can express

\[ d\theta = i h_{\alpha\beta} \theta_\alpha \wedge \overline{\theta_\beta}. \quad (11) \]

The hermitian matrix \( (h_{\alpha\beta}) \) is called the Levi matrix.

On pseudo-hermitian manifolds Webster [19] has defined a connection, with connection forms \( \omega_\alpha^\beta \) and torsion forms \( \tau_\beta = A_\beta^\alpha \theta_\alpha \), with structure relations

\[ d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau_\beta, \quad \omega_\alpha^\beta + \omega_\beta^\alpha = dh_{\alpha\beta} \quad (12) \]

and

\[ A_\alpha^\beta = A_\beta^\alpha. \quad (13) \]

Webster defines a curvature form

\[ \Pi_\alpha^\beta = d\omega_\alpha^\beta - \omega_\gamma^\beta \wedge \omega_\gamma^\alpha, \]

where we have used the Einstein summation convention. Furthermore in [19] it is shown that

\[ \Pi_\alpha^\beta = R_\alpha^\beta_\rho_\sigma \theta^\rho \wedge \overline{\theta_\sigma} + \text{other terms}. \]

Contracting two indices using the Levi matrix \( (h_{\alpha\beta}) \) we get

\[ R_{\alpha\beta}^\rho = h_{\rho\sigma} R_{\alpha\beta}^\rho_\sigma. \quad (14) \]

The Webster-Ricci tensor \( \text{Ric}(V, V) \) for \( V \in V \) is then defined as

\[ \text{Ric}(V, V) = R_{\alpha\beta}^\rho x_\alpha x_\beta \overline{x_\rho}, \quad \text{for} \quad V = \sigma_\alpha x_\alpha Z_\alpha. \quad (15) \]

The torsion tensor is defined for \( V \in V \) as follows
\[
\text{Tor}(V, V) = i \left( A_{\alpha\beta} \bar{x}^\alpha \bar{x}_\beta - A_{\alpha\beta} x^\alpha x^\beta \right). 
\] (16)

In [19], Prop. (2.2), Webster proves that the torsion vanishes if \( \mathcal{L}_T \) preserves \( H \), where \( \mathcal{L}_T \) is the Lie derivative. In particular if \( M \) is a hypersurface in \( \mathbb{C}^{n+1} \) given by the defining function \( \rho \)

\[
\text{Im} z_{n+1} = \rho(z, \bar{z}), \quad z = (z_1, z_2, \ldots, z_n) 
\] (17)

then Webster’s hypothesis is fulfilled and the torsion tensor vanishes on \( M \). Thus for the standard CR structure on the sphere \( S^{2n+1} \) and on the Heisenberg group the torsion vanishes.

Our main focus will be the sub-Laplacian \( \Delta_b \). We define the horizontal gradient \( \nabla_b \) and \( \Delta_b \) as follows:

\[
\nabla_b f = \sum_{\alpha} f_{\alpha} \bar{Z}_\alpha, 
\] (18)

\[
\Delta_b f = \sum_{\alpha} f_{\bar{\alpha} \alpha} + f_{\alpha \bar{\alpha}}. 
\] (19)

When \( n = 1 \) we will need to frame our results in terms of the CR Paneitz operator. Define the Kohn Laplacian \( \Box_b \) by

\[
\Box_b = \Delta_b + i T. 
\] (20)

Then the CR Paneitz operator \( P_0 \) is defined by

\[
P_0 f = (\Box_b \Box_b + \Box_b \Box_b) f - 2 (Q + \bar{Q}) f, 
\] (21)

where

\[
Q f = 2i (A^{11} f_1)_1. 
\]

See [10] and [9] for further details.
2 The Main Theorem

Theorem 1. Let $M^{2n+1}$ be a strictly pseudo-convex pseudo-hermitian manifold. When $M$ is non compact assume that $f \in C_0^\infty(M)$. When $M$ is compact with $\partial M = \emptyset$ we may assume $f \in C^\infty(M)$. When $f$ is real valued and $n \geq 2$ we have

$$
\sum_{\alpha,\beta} \int_M ||f_{\alpha\beta}||^2 + ||f_{\alpha\overline{\beta}}||^2 + \int_M \left( \text{Ric} + \frac{n}{2} \text{Tor} \right) (\nabla_b f, \nabla_b f) \leq \frac{(n+2)^2}{2n} \int_M |\Delta_b f|^2. \tag{22}
$$

When $n = 1$ assume that the CR Paneitz operator $P_0 \geq 0$. For $f \in C_0^\infty(M)$ we then have

$$
\int_M ||f_{11}||^2 + ||f_{1\overline{1}}||^2 + \int_M \left( \text{Ric} - \frac{3}{2} \text{Tor} \right) (\nabla_b f, \nabla_b f) \leq \frac{3}{2} \int_M |\Delta_b f|^2. \tag{23}
$$

Here by $\sum_{\alpha,\beta} ||f_{\alpha\beta}||^2$ we mean the Hilbert-Schmidt norm square of the tensor and similarly for $\sum_{\alpha,\beta} ||f_{\alpha\overline{\beta}}||^2$.

Proof. We begin by noting the Bochner identity established by Greenleaf, Lemma 3 in [8]:

$$
\frac{1}{2} \Delta_b (|\nabla_b f|^2) = \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\overline{\beta}}|^2 + \text{Re} (\nabla_b f, \nabla_b (\Delta_b f)) \tag{24}
$$

$$
+ \left( \text{Ric} + \frac{n-2}{2} \text{Tor} \right) (\nabla_b, \nabla_b) + i \sum_\alpha (f_{\alpha'\alpha} - f_{\alpha\alpha'}) .
$$

where for $V, W \in \mathcal{V}$ we use the notation $(V, W) = L_0(V, \overline{W})$ and $|V| = (V, V)^{1/2}$. We have also abused notation above and represented the Hilbert-Schmidt norm of the tensor $f_{\alpha\beta}$ in terms of its expression in the local frame which we will continue to do in the rest of the proof. Using the fact that $f \in C_0^\infty(M)$ or if $\partial M = \emptyset$, $M$ is compact, integrate (24) over $M$ using the volume (9) to get

$$
\int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\overline{\beta}}|^2 + \left( \text{Ric} + \frac{n-2}{2} \text{Tor} \right) (\nabla_b f, \nabla_b f) \tag{25}
$$

$$
+ i \int_M \sum_\alpha (f_{\alpha'\alpha} - f_{\alpha\alpha'}) = - \int_M \text{Re} (\nabla_b f, \nabla_b (\Delta_b f)) .
$$

Integration by parts in the term on the right yields (see (5.4) in [8])

$$
- \int_M \text{Re}(\nabla_b f, \nabla_b (\Delta_b f)) = \frac{1}{2} \int_M |\Delta_b f|^2. \tag{26}
$$

Combining (25) and (26) we get
\[
\int_M \sum_{\alpha, \beta} |f_{\alpha \beta}|^2 + |f_{\alpha \bar{\beta}}|^2 + \int_M \left( \text{Ric} + \frac{n-2}{2} \text{Tor} \right) (\nabla_b f, \nabla_b f) 
\] (27)
\[
+ i \int_M \sum_{\alpha} (f_{\alpha a_0} - f_{a \alpha 0}) = \frac{1}{2} \int_M |\Delta_b f|^2.
\]

To handle the third integral in the left-hand side, we use Lemmas 4 and 5 of [8] (valid for real functions) according to which we have
\[
i \int_M \sum_{\alpha} (f_{\alpha a_0} - f_{a \alpha 0}) = \frac{2}{n} \int_M \left( \sum_{\alpha, \beta} (|f_{\alpha \bar{\beta}}|^2 - |f_{\alpha \beta}|^2) - \text{Ric}(\nabla_b f, \nabla_b f) \right)
\] (28)

and
\[
i \int_M \sum_{\alpha} (f_{\alpha a_0} - f_{a \alpha 0}) = -\frac{4}{n} \int_M \sum_{\alpha} |f_{\alpha \bar{\alpha}}|^2
\] (29)
\[
+ \frac{1}{n} \int_M |\Delta_b f|^2
\]
\[
+ \int_M \text{Tor}(\nabla_b f, \nabla_b f).
\]

Applying the Cauchy-Schwarz inequality to the first term in the right-hand side of (29) we get
\[
i \int_M \sum_{\alpha} (f_{\alpha a_0} - f_{a \alpha 0}) \geq -4 \int_M \sum_{\alpha, \beta} |f_{\alpha \bar{\beta}}|^2
\] (30)
\[
+ \frac{1}{n} \int_M |\Delta_b f|^2
\]
\[
+ \int_M \text{Tor}(\nabla_b f, \nabla_b f).
\]

Multiply (28) by \(1 - c\) and (30) by \(c, 0 < c < 1\), and where \(c\) will eventually be chosen to be \(1/(n+1)\), and add to get
\[
i \int_M \sum_{\alpha} (f_{\alpha a_0} - f_{a \alpha 0}) \geq \frac{2}{n} \int_M \sum_{\alpha, \beta} (|f_{\alpha \bar{\beta}}|^2 - |f_{\alpha \beta}|^2)
\] (31)
\[
- 2 \frac{(1-c)}{n} \int_M \text{Ric}(\nabla_b f, \nabla_b f)
\]
\[
- 4c \int_M \sum_{\alpha, \beta} |f_{\alpha \beta}|^2
\]
\[
+ \frac{c}{n} \int_M |\Delta_b f|^2 + c \int_M \text{Tor}(\nabla_b f, \nabla_b f).
\]

We now insert (31) into (27) and simplify. We have
\[
\left(1 - \frac{2(1-c)}{n}\right) \int_M \text{Ric}(\nabla_b f, \nabla_b f) + \\
\left(\frac{(n-2)}{2} + c\right) \int_M \text{Tor}(\nabla_b f, \nabla_b f) + \\
\left(1 + \frac{2(1-c)}{n} - 4c\right) \int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + \\
\left(1 - \frac{2(1-c)}{n}\right) \int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 \leq \left(\frac{1}{2} - \frac{c}{n}\right) \int_M |\Delta_b f|^2.
\]

Let \(c = 1/(n+1)\). Then (32) becomes
\[
\left(\frac{n-1}{n+1}\right) \left[ \int_M \sum_{\alpha,\beta} (|f_{\alpha\beta}|^2 + |f_{\alpha\beta}|^2) + \int_M \left(\text{Ric} + \frac{n}{2} \text{Tor}\right) (\nabla_b f, \nabla_b f) \right] \leq \left(\frac{n-1}{n+1}\right) \left(\frac{n+2}{2n}\right) \int_M |\Delta_b f|^2.
\]

Since \(n \geq 2\), \(n-1 > 0\) and we can cancel the factor \(\frac{n-1}{n+1}\) from both sides to get (22).

We now establish (23) using some results by Li-Luk [11] and [9]. When \(n = 1\), identity (27) becomes
\[
\int_M |f_{1\bar{1}}|^2 + |f_{1\bar{1}}|^2 + \int_M \left(\text{Ric} - \frac{1}{2} \text{Tor}\right) (\nabla_b f, \nabla_b f) + i \int_M (f_{10\bar{1}} f_{1\bar{0}} - f_{1\bar{0}} f_{10\bar{1}}) = \frac{1}{2} \int_M |\Delta_b f|^2.
\]

By (3.8) in [11] we have
\[
i \int_M (f_{01\bar{1}} f_{1\bar{0}} - f_{01\bar{1}} f_{01\bar{1}}) = - \int_M f_{01\bar{1}}^2.
\]

Moreover, by (3.6) in [11] we also have
\[
i (f_{10\bar{1}} f_{1\bar{0}} - f_{1\bar{0}} f_{10\bar{1}}) = i (f_{01\bar{1}} f_{1\bar{0}} - f_{01\bar{1}} f_{01\bar{1}}) + \text{Tor}(\nabla_b f, \nabla_b f)
\]

and combining the last two identities we get
\[
i \int_M (f_{10\bar{1}} f_{1\bar{0}} - f_{1\bar{0}} f_{10\bar{1}}) = - \int_M f_{01\bar{1}}^2 + \int_M \text{Tor}(\nabla_b f, \nabla_b f).
\]

Substituting (35) into (34) we obtain
\[
\int_M |f_{1\bar{1}}|^2 + |f_{1\bar{1}}|^2 + \int_M \left(\text{Ric} + \frac{1}{2} \text{Tor}\right) (\nabla_b f, \nabla_b f) - \int_M f_{01\bar{1}}^2 = \frac{1}{2} \int_M |\Delta_b f|^2.
\]
Next, we use (3.4) in [9],
\[
\int_M f_0^2 = \int_M |\Delta_b f|^2 + 2 \int_M \text{Tor}(\nabla_b f, \nabla_b f) - \frac{1}{2} \int_M P_0 f \cdot f. \tag{37}
\]
Finally, substitute (37) into (36) and simplify to get
\[
\int_M |f_{11}|^2 + |f_{11}|^2 + \int_M \left( \text{Ric} - \frac{3}{2} \text{Tor} \right) (\nabla_b f, \nabla_b f) + \frac{1}{2} \int_M P_0 f \cdot f
\]
\[
= \frac{3}{2} \int_M |\Delta_b f|^2.
\]
Assuming $P_0 \geq 0$ we obtain (23). \qed

We now wish to make some remarks about our theorem:

(a) It is shown in [6] that on the Heisenberg group the constant $(n+2)/2n$ is sharp. Since the Heisenberg group is a pseudo-hermitian manifold with $\text{Ric} \equiv 0$ and $\text{Tor} \equiv 0$, we easily conclude our theorem is sharp and contains the result proved in [6].

(b) We notice that when we consider manifolds such that $\text{Ric} + (n/2) \text{Tor} > 0$, then for $n \geq 2$, in general we have the strict inequality
\[
\sum_{\alpha, \beta} \int_M |f_{\alpha \beta}|^2 + |f_{\alpha \beta}|^2 < \frac{n + 2}{2n} \int_M |\Delta_b f|^2.
\]
On the Heisenberg group $\text{Ric} \equiv 0$, $\text{Tor} \equiv 0$ and the constant $(n+2)/2n$ is achieved by a function with fast decay [6]. Thus, the Heisenberg group is, in a sense, extremal for inequality (22) in Theorem 1. A similar remark holds for inequality (23).

(c) The hypothesis on the Paneitz operator in the case $n = 1$ in our theorem is satisfied on manifolds with zero torsion. A result from [2] shows that if the torsion vanishes the Paneitz operator is non-negative.

(d) We note that Chiu [9] shows how to perturb the standard pseudo-hermitian structure in $S^3$ to get a structure with non-zero torsion, for which $P_0 > 0$ and $\text{Ric} - (3/2) \text{Tor} > 1$. To get such a structure, let $\theta$ be the contact form associated to the standard structure on $S^3$. Fix $g$ a smooth function on $S^3$. For $\epsilon > 0$ consider
\[
\tilde{\theta} = e^{2f} \theta, \text{ where } f = \epsilon^3 \sin(\frac{g}{\epsilon}). \tag{38}
\]
Since the sign of the Paneitz operator is a CR invariant and $\theta$ has zero torsion we conclude by [2] that the CR Paneitz operator $\tilde{P}_0$ associated to $\tilde{\theta}$ satisfies $\tilde{P}_0 > 0$. Furthermore following the computation in Lemma (4.7) of [9], we easily have for small $\epsilon$ that
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\[ \text{Ric} - \frac{3}{2} \text{Tor} \geq (2 + O(\epsilon)) e^{-2f} \geq 1 \geq 0. \]

Thus, the hypothesis of the case \( n = 1 \) in our theorem are met, and for such \( (M, \tilde{\theta}) \) we have, for \( f \in C^\infty(M) \) the estimate

\[
\int_M |f_{i1}|^2 + |f_{1i}|^2 \, dV \leq \frac{3}{2} \int_M |\Delta_b f|^2 \, dV.
\]

(e) Compact pseudo-hermitian 3-manifolds with negative Webster curvature may be constructed by considering the co-sphere bundle of a compact Riemann surface of genus \( g \), \( g \geq 2 \). Such a construction is given in [3].

3 Applications to PDE

For applications to subelliptic PDE it is helpful to re-state our main result Theorem 1 in its real version. We set

\[ X_i = \text{Re}(Z_i) \text{ and } X_{i+n} = \text{Im}(Z_i) \]

for \( i = 1, 2, \ldots, n \). The horizontal gradient of a function is the vector field

\[ \mathcal{X}(f) = \sum_{i=1}^{2n} X_i(f) X_i. \]

Its sublaplacian is given by

\[ \Delta_\mathcal{X} f = \sum_{i=1}^{2n} X_i X_i(f), \]

and the horizontal second derivatives are the \( 2n \times 2n \) matrix

\[ \mathcal{X}^2 f = (X_i X_j(f)). \]

For \( f \) real we have the following relationships

\[ \nabla_b f = \mathcal{X}(f) + i \left( \sum_{i=1}^{n} X_i(f) X_{i+n} - X_{i+n}(f) X_i \right), \]

\[ \Delta_b f = 2 \Delta_\mathcal{X} f, \]

and

\[ \sum_{\alpha, \beta} |f_{\alpha\beta}|^2 + |f_{\alpha\beta}|^2 = 2 \sum_{i,j} |X_i X_j(f)|^2 = 2 |\mathcal{X}^2 f|^2, \]

where the expression on the extreme right is the Hilbert-Schmidt norm square of the tensor taken by viewing the Levi form as a metric on \( H \).
Theorem 2. Let $M^{2n+1}$ be a strictly pseudo-convex pseudo-hermitian manifold. When $M$ is non compact assume that $f \in C^\infty_0(M)$. When $M$ is compact with $\partial M = \emptyset$ we may assume $f \in C^\infty(M)$. When $f$ is real valued and $n \geq 2$ we have

$$\int_M |X^2 f|^2 + \int_M \frac{1}{2} \left( \text{Ric} + \frac{n}{2} \text{Tor} \right) (\nabla_b f, \nabla_b f) \leq \frac{(n+2)}{n} \int_M |\Delta_X f|^2. \quad (39)$$

When $n = 1$ assume that the CR Paneitz operator $P_0 \geq 0$. For $f \in C^\infty_0(M)$ we then have

$$\int_M |X^2 f|^2 + \int_M \frac{1}{2} \left( \text{Ric} - \frac{3}{2} \text{Tor} \right) (\nabla_b f, \nabla_b f) \leq 3 \int_M |\Delta_X f|^2. \quad (40)$$

Let $A(x) = (a_{ij}(x))$ a $2n \times 2n$ matrix. Consider the second order linear operator in non-divergence form

$$A u(x) = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u(x), \quad (41)$$

where coefficients $a_{ij}(x)$ are bounded measurable functions in a domain $\Omega \subset M^{2n+1}$. Cordes [4] and Talenti [17] identified the optimal condition expressing how far $A$ can be from the identity and still be able to understand (41) as a perturbation of the case $A(x) = I_{2n}$, when the operator is just the sublaplacian. This is the so called Cordes condition that roughly says that all eigenvalues of $A$ must cluster around a single value.

Definition 1. ([4],[17],[6]) We say that $A$ satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0,1]$ and $\sigma > 0$ such that

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{2n} a_{ij}^2(x) \leq \frac{1}{2n-1+\varepsilon} \left( \sum_{i=1}^{2n} a_{ii}(x) \right)^2 \quad (42)$$

for a.e. $x \in \Omega$.

Let $c_n = \frac{(n+2)}{n}$ for $n \geq 2$ and $c_1 = 3$ the constants in the right-hand sides of Theorem 2. We can now adapt the proof of Theorem 2.1 in [6] to get

Theorem 3. Let $M^{2n+1}$ be a strictly pseudo-convex pseudo-hermitian manifold such that $\text{Ric} + \frac{n}{2} \text{Tor} \geq 0$ if $n \geq 2$ and $\text{Ric} - \frac{3}{2} \text{Tor} \geq 0$, $P_0 \geq 0$ if $n = 1$. Let $0 < \varepsilon \leq 1$, $\sigma > 0$ such that $\gamma = \sqrt{(1-\varepsilon)c_n} < 1$ and $A$ satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in C^\infty_0(\Omega)$ we have the a-priori estimate

$$\|X^2 u\|_{L^2} \leq \sqrt{1 + \frac{2}{n} \frac{1}{1-\gamma}} \|\alpha\|_{L^\infty} \|Au\|_{L^2}, \quad (43)$$

where

$$\alpha(x) = \frac{\langle A(x), I \rangle}{\|A(x)\|^2} = \frac{\sum_{i=1}^{2n} a_{ii}(x)}{\sum_{i,j=1}^{2n} a_{ij}^2(x)}. \quad (44)$$
**Proof.** We start from formula (2.7) in [6] which gives

$$
\int_\Omega |\Delta_X u(x) - \alpha(x)A_xu(x)|^2 \, dx \leq (1 - \varepsilon) \int_\Omega |Xu|^2 \, dx.
$$

We now apply Theorem 2 to get

$$
\int_\Omega |\Delta_X u(x) - \alpha(x)A_xu(x)|^2 \, dx \leq (1 - \varepsilon)c_n \int_\Omega |\Delta_X f|^2.
$$

The theorem then follows as in [6]. □

**Remark:** The hypothesis of Theorem 2, \( n \geq 2 \), can be weakened to assume only a bound from below

$$
\text{Ric} + \frac{n}{2} \text{Tor} \geq -K, \text{ with } K > 0
$$

to obtain estimates of the type

$$
\int_M |\mathcal{X}^2 f|^2 \leq \frac{(n + 2)}{n} \int_M |\Delta_X f|^2 + 2K \int_M |\mathcal{X} f|^2.
$$

(44)

A similar remark applies to the case \( n = 1 \).

We finish this paper by indicating how the a priori estimate of Theorem 3 can be used to prove regularity for \( p \)-harmonic functions in the Heisenberg group \( H^n \) when \( p \) is close to 2. We follow [6], where full details can be found.

Recall that, for \( 1 < p < \infty \), a \( p \)-harmonic function \( u \) in a domain \( \Omega \subset H^n \) is a function in the horizontal Sobolev space

$$
W^{1,p}_{X,\text{loc}}(\Omega) = \{ u: \Omega \mapsto \mathbb{R} \text{ such that } u, X u \in L^p_{\text{loc}}(\Omega) \}
$$

such that

$$
\sum_{i=1}^{2n} X_i (|Xu|^{p-2} X_i u) = 0, \text{ in } \Omega
$$

(45)
in the weak sense. That is, for all \( \phi \in C_0^\infty(\Omega) \) we have

$$
\int_\Omega |X u(x)|^{p-2} (X u(x), X \phi(x)) \, dx = 0.
$$

(46)

Assume for the moment that \( u \) is a smooth solution of (45). We can then differentiate to obtain

$$
\sum_{i,j=1}^{2n} a_{ij} X_i X_j u = 0, \text{ in } \Omega
$$

(47)

where
\[ a_{ij}(x) = \delta_{ij} + (p - 2) \frac{X_i u(x) X_j u(x)}{|X u(x)|^2}. \]

A calculation shows that this matrix satisfies the Cordes condition (42) precisely when

\[ p - 2 \in \left( \frac{n - n\sqrt{4n^2 + 4n - 3}}{2n^2 + 2n - 2}, \frac{n + n\sqrt{4n^2 + 4n - 3}}{2n^2 + 2n - 2} \right). \]  

(48)

In the case \( n = 1 \) this simplifies to

\[ p - 2 \in \left( \frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right). \]

We then deduce \( a \) priori estimates for \( X^2 u \) from Theorem 3. To apply the Cordes machinery to functions that are only in \( W^{1,p}_{X} \) we need to know that the second derivatives \( X^2 u \) exist. This is done in the Euclidean case by a standard difference quotient argument applied to a regularized \( p \)-Laplacian. In the Heisenberg case this would correspond to proving that solutions to

\[ \sum_{i=1}^{2n} X_i \left( \left( \frac{1}{m} + |X u|^2 \right)^{\frac{p-2}{2}} \right) X_i u = 0 \]

(49)

are smooth. Contrary to the Euclidean case (where solutions to the regularized \( p \)-Laplacian are \( C^\infty \)-smooth) in the subelliptic case this is known only for \( p \in [2, c(n)] \) where \( c(n) = 4 \) for \( n = 1, 2 \), and \( \lim_{n \to \infty} c(n) = 2 \) (see [13].) The final result will combine the limitations given by (48) and \( c(n) \).

**Theorem 4.** (Theorem 3.1 in [6]) For

\[ 2 \leq p < 2 + \frac{n + n\sqrt{4n^2 + 4n - 3}}{2n^2 + 2n - 2} \]

we have that \( p \)-harmonic functions in the Heisenberg group \( \mathcal{H}^n \) are in \( W^{2,2}_{X, loc}(\Omega) \).

At least in the one-dimensional case \( \mathcal{H}^1 \) one can also go below \( p = 2 \). See Theorem 3.2 in [6]. We also note that when \( p \) is away from 2, for example \( p > 4 \) nothing is known regarding the regularity of solutions to (45) or its regularized version (49) unless we assume a priori that the length of the gradient is bounded below and above

\[ 0 < \frac{1}{M} \leq |X u| \leq M < \infty. \]

See [1] and [13].

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References

1. Capogna, L., Regularity of quasi-linear equations in the Heisenberg group. Comm. Pure Appl. Math. 50 (1997), no. 9, 867–889.
2. Chang, S.C., Cheng, J.H., Chiu, H.L., A fourth order $Q$-curvature flow on a CR 3-manifold, to appear in Indiana Math. J., http://arxiv.org/abs/math.DG/0510494.
3. Chern, S. S., Hamilton, R. S., On Riemannian metrics adapted to three-dimensional contact manifolds. With an appendix by Alan Weinstein. Lecture Notes in Math., 1111, Workshop Bonn 1984 (Bonn, 1984), 279–308, Springer, Berlin, 1985.
4. Cordes, H.O., Zero order a-priori estimates for solutions of elliptic differential equations, Proceedings of Symposia in Pure Mathematics IV (1961).
5. Domokos, A., Fanciullo, M.S., On the best constant for the Friedrichs-Knapp-Stein inequality in free nilpotent Lie groups of step two and applications to subelliptic PDE, The Journal of Geometric Analysis, 17(2007), 245-252.
6. Domokos, A., Manfredi, J.J., Subelliptic Cordes estimates. Proc. Amer. Math. Soc. 133 (2005), no. 4, 1047–1056.
7. Gilbarg, D., Trudinger, N. S., Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
8. Greenleaf, A., The first eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold. Comm. Partial Differential Equations 10 (1985), no. 2, 191–217.
9. Chiu, H.L., The sharp lower bound for the first positive eigenvalue of the sub-Laplacian on a pseudohermitian 3-manifold, Ann. Global Anal. Geom. 30 (2006), no. 1, 81–96.
10. Lee, J.M., The Fefferman metric and pseudo-Hermitian invariants, Trans. Amer. Math. Soc. 296 (1986), no. 1, 411–429.
11. Li, S.Y., Luk, H.S., The sharp lower bound for the first positive eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold. Proc. Amer. Math. Soc. 132 (2004), no. 3, 789–798.
12. Lin, F.H., Second derivative $L^p$-estimates for elliptic equations of nondivergent type. Proc. Amer. Math. Soc. 96 (1986), no. 3, 447–451
13. Manfredi, J.J., Mingione, G., Regularity Results for Quasilinear Elliptic Equations in the Heisenberg Group, to appear in Mathematische Annalen, 2007.
14. Segovia, C., On the area function of Lusin, Studia Math. 33 1969 311–343.
15. Stein, E., Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.
16. Strichartz, R.S., Harmonic analysis and Radon transforms on the Heisenberg group, J. Funct. Analysis, 96(1991), 350-406.
17. Talenti, G., Sopra una classe di equazioni ellittiche a coefficienti misurabili. (Italian) Ann. Mat. Pura Appl. (4) 69, 1965, 285–304
18. Trèves, F., Hypo-analytic structures. Local theory, Princeton Mathematical Series, 40. Princeton University Press, Princeton, NJ, 1992.
19. Webster, S. M., Pseudo-Hermitian structures on a real hypersurface, J. Differential Geom. 13 (1978), no. 1, 25–41.