Experimental demonstration of four-party quantum secret sharing

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Secret sharing is a multiparty cryptographic task in which some secret information is split into several pieces which are distributed among the participants such that only an authorized set of participants can reconstruct the original secret. Similar to quantum key distribution, in quantum secret sharing, the secrecy of the shared information relies not on computational assumptions, but on laws of quantum physics. Here, we present an experimental demonstration of four-party quantum secret sharing via the resource of four-photon entanglement.

Entanglement is a fundamental resource for many quantum communication schemes. Bipartite entanglement has been used for the experimental demonstration of two-party quantum communication schemes like quantum dense coding [1], quantum teleportation [2] or entanglement based quantum cryptography [3]. Similar to such two-party quantum communication schemes, multiparty entanglement allows the experimental implementation of multiparty quantum communication schemes, like multiparty quantum teleportation [4], quantum teleportation of multiparty quantum communication schemes, multipartite entanglement based quantum cryptography [3]. Similar to two-party quantum communication schemes, multiparty quantum communication schemes. Bipartite entanglement has been used for the experimental demonstration of three-party QSS [12, 13]. Here, we consider QSS of classical information. Different information [6] and QSS of quantum information [10].

In this scheme, any one of the four participants can act as the dealer, while the remaining three participants form the access set. For the experimental implementation, we use the following four-photon polarization-entangled state [14, 15]:

$$|\Psi^+_4\rangle = \frac{1}{\sqrt{2\sqrt{3}}} \left( 2|HHVV\rangle - |HVHV\rangle - |VHHV\rangle - |VVHH\rangle \right)_{abcd}$$

where $H$ and $V$ denotes horizontal and vertical polarization of photons in the four spatial modes $a, b, c$ and $d$. This state shows perfect four-photon polarization correlations as indicated by the four-photon polarization correlation function defined as the expectation value of the product of the four operators $\hat{s}_x = |+\rangle\langle+| - |\rangle\langle\rangle$ with eigenstates $|\pm, \Phi\rangle = 1/\sqrt{2}(|R\rangle \pm e^{i\phi_s}|L\rangle)$ and eigenvalues $\pm 1$, where $R$ and $L$ denote right- and lefthanded circular polarization. The explicit expression of this correlation function for the four-photon state given by Eq. (1), is:

$$E(\phi_a, \phi_b, \phi_c, \phi_d) = \frac{2}{3} \cos(\phi_a + \phi_b - \phi_c - \phi_d) \quad \frac{1}{3} \cos(\phi_a - \phi_b) \cos(\phi_c - \phi_d).$$

Another property of this state, which is useful for the realization of multiparty secure quantum communication, is its ability to violate a four-party Bell inequality [14, 15]:

$$S = \frac{1}{2} \sum_{s_x = \pm 1} \sum_{y = 1,2} \sum_{z = 1,2} \sum_{w = 1,2} s^a_x s^b_y s^c_z s^d_w E(\phi^a_{s_x}, \phi^b_{s_y}, \phi^c_{s_z}, \phi^d_{s_w}) \leq 1$$

where each index $y = k, l, m, n$ denotes a pair of angles defining the local polarization measurement settings required for the evaluation. A strong violation of this Bell inequality (with $S = 1.886$) is obtained e.g. for $\phi^1_k = 0, \pi/2$ and $\phi^1_{a,c,d} = \pi/4, -\pi/4$, or any other setting resulting by permutation of the indices.

The four-party QSS protocol works as follows: Alice, Bob, Claire and David share a photon from...
the $|\Psi^+\rangle$ state. In the following, we assume that Alice is the dealer. Each party chooses randomly between two complementary measurement bases. This can be for example either the $\{H, V\}$ and $\{P, M\}$ basis ($\phi_x = 0, \pi/2$) analog to the BB84 protocol [17], or a basis set like $\{(22.5^\circ, 112.5^\circ), (-22.5^\circ, 67.5^\circ)\}$ ($\phi_x = \pi/4, -\pi/4$), which can also be used to violate the Bell inequality given by Eq. (3), in analogy to the Ekert scheme [18].

To transfer the measurement results into a key sequence, each participant identifies his result either with a bit value of 0 or 1. The measurements will be repeated until they have established a raw key of desired length. For key sifting, each participant announces publicly whenever he has registered a photon and which measurement basis he has used, but not the results. For the announcement, Bob, Claire and David send their information about their measurement to Alice. After that, Alice can decide which quadruples will be used for secure communication: According to Eq. (2), she will keep those results where all participants used the same basis setting and will drop all others.

The information splitting works as follows: Consider e.g. the case where all four parties have measured in the $\{H, V\}$ basis. Suppose Bob obtained the result $|H\rangle_b$ ($|V\rangle_b$), then he can not predict with certainty the measurement result of Alice, because there are two different possible outcomes for her. Similarly, each other participant is not able to obtain the secret of Alice without cooperation. Let us now consider pairwise cooperation. If Bob and Claire obtained the result $|HV\rangle_{bc}$ ($|VH\rangle_{bc}$), both together are not able to get the secret. They need the help of David to determine the key bit of Alice. Only in cases where Bob and Claire have obtained the result $|HH\rangle_{bc}$ ($|VV\rangle_{bc}$) they can infer the key bit without the help of David. The same is true for Bob and David, or Claire and David. This implies that always two of them have some partial information on Alice’s key. However, Bob, Claire and David must cooperate to retrieve the complete key and to ensure perfect information-theoretic security, the partial information of any semi-access set can be removed by the application of a hash function (privacy amplification) [19, 20].

After key sifting, all participants have to check for external eavesdropping. Depending on their chosen basis settings they can proceed as follows: If they used a basis set similar to the BB84 protocol, they can use a fraction of their measurement results (which should be perfectly correlated) for the evaluation of the quantum bit error rate (QBER) defined as the ratio of wrong bits to all bits. If the QBER is low enough, they can use their results to distill a secure key, otherwise they have to discard their bits [21]. If they used a basis set which can be used to violate the Bell inequality given by Eq. (3), they can use a fraction of their measurement results for the evaluation. If the violation is high enough, they can use their key bits, otherwise they have to drop them [22].

To finally obtain a common secure key, they have to perform key reconciliation and privacy amplification. For this, different strategies developed for quantum key distribution (QKD) can be adapted [19, 23]. After that, Alice can use the final key for an unbreakable encryption of her secret information via the Vernam cipher [24] and broadcasts the encrypted message to Bob, Claire and David. Cooperation of Bob, Claire and David allows the reconstruction of the key and therefore to obtain the secret information of Alice.

The four-party QSS protocol was experimentally implemented as sketched in Fig. 1. We used a mode-locked Ti:Sapphire laser emitting pulses with a pulse length of about 120 fs at a repetition rate of 82 MHz. This radiation is frequency-doubled with a lithium-triborate crystal to $\lambda = 390$ nm which is used to pump a 2 mm thick beta-barium-borate crystal to generate the four-photon polarization-entangled state given by Eq. (1) via pulsed type-II parametric down-conversion. This state results directly from the four-photon emission of the pulsed down-conversion source and the usage of two beam splitters to distribute these photons into the four spatial modes $a, b, c$ and $d$. Each party uses a random number generator (RNG) which sets a half-wave plate (HWP) angle and a polarizing beam splitter (PBS) followed by two avalanche photodiodes for the analysis and registration of the distributed photons.
observed a four-photon state rate of about 0.4 per second. To build up a shared key, the acquisition time for each randomly chosen analysis setting was set to 1 s. If more than one four-photon coincidence event was registered during that time, only the first one was chosen.

An analysis of the four-photon source used for the experimental implementation, where Alice, Bob, Claire and David have measured in the \{H, V\} basis is shown in Fig. 2. The data acquisition time was 24 hours. Due to differences in the efficiencies of the detectors, the presented four-photon coincidences have been corrected without changing the overall raw detection rate. The corresponding four-photon correlation is \( E = 0.945 \pm 0.002 \), which demonstrates the high quality and stability of this four-photon source.

To demonstrate the perfect four-party correlations, necessary for four-party QSS, we analyzed the four-photon correlations under different polarization analyzer orientations. In Fig. 3 (a) the results of two measurements are shown, where Alice, Claire and David have analyzed their photons in the \{H, V\} \{P, M\} basis corresponding to \( \phi_{a,c,d} = 0 (\pi/2) \), while Bob varied his analysis direction \( \phi_b \) from 0 to 4\( \pi \). The maximal absolute value of \( E(\phi_a, \phi_b, \phi_c, \phi_d) \) can be expressed by a visibility \( V \) according to \( E = V \mathcal{F} \), where \( \mathcal{F} \) denotes the theoretical value. Fitting the experimental data leads to a visibility of \( V_{H/V} = 92.3\% \pm 0.8\% \) and \( V_{P/M} = 88.2\% \pm 1.2\% \). Figure 3 (b) shows the experimental results obtained from two measurements, where Alice, Claire and David analyzed their photons in the \{22.5\( ^\circ \), 112.5\( ^\circ \}\} \{−22.5\( ^\circ \), 67.5\( ^\circ \}\} basis corresponding to \( \phi_{a,c,d} = \pi/4 (-\pi/4) \), while Bob varied his analyzer setting \( \phi_b \). The resulting visibilities are \( V_{22.5\%/112.5\%} = 90.2 \pm 1.1\% \) and \( V_{-22.5\%/67.5\%} = 89.0 \pm 0.7\% \), respectively. The overall data acquisition time for each four-photon correlation function was 8.5 hours. The average visibility can be translated into the QBER via \( QBER = (1 - V)/2 \) leading to a QBER of 4.88 \pm 0.50\% and 5.22 \pm 0.45\% for each basis set. 

For a full experimental demonstration of four-party QSS, we performed a key exchange according to the four-party QSS protocol described above using two different complementary basis sets. Using a key sifting analog to the BB84 protocol, we have exchanged 2000 key bits in a complete transfer time of about 16 hours with a QBER of 5.20\%. A fraction of 100 bits of the sifted key is shown in Fig. 4. Neglecting the dead time required to set the polarization analysis direction and considering in addition those events which have been registered in the same time interval, leads to a bit rate of 196 bits per hour being in well agreement with the theoretical expected rate. To check for eavesdropping, Alice chooses a random subset of 200 bits (10\%) of the sifted key and asks Bob, Claire and David for their results at those positions. From this subset, she evaluates the QBER and obtains a value of 4\%. This value lies well below several known security threshold values required for two-party QKD and should therefore be low enough to distill finally a perfect secure key. 

Thus, using the remaining 1800 bits...
to perform key reconciliation and privacy amplification, Alice ends up with a totally secure key with about 200 bits [20]. For the reconstruction of the final key, Bob, Claire and David have to combine their individual bits of the reconciled key and have to perform the same privacy amplification procedure as Alice.

Using the Bell angles ($\phi_{\bar{x}}^{1,2} = \pi/4, -\pi/4$) for key sifting, while Bob switches every fifth measurement to $\phi_{b}^{1,2} = 0, \pi/2$, we obtained 1342 key bits in about 21 hours with a QBER of 5.96%. Since 1/3 of the measured data can be used to evaluate the Bell inequality given by Eq. 3, we obtained a value of $S = 1.78 \pm 0.07$, which is well above the classical limit of 1 and very close to the theoretical value of 1.886. Since the value for $S$ scales linear with the visibility, the achieved value of $S$ can be translated into a QBER of about 3%, which should be low enough to ensure secure quantum communication [21, 27].

We demonstrated four-party QSS via the resource of four-photon entanglement. We obtained bit rates of about 100 bits per hour with QBERs of about 5%, which should be low enough to ensure perfect security. Comparing this with QBERs usually obtained in QKD between two parties, we obtain similar results demonstrating the feasibility of secure multiparty quantum communication via multi-photon entanglement. For future implementations, it would be useful to use fast optical switches for an efficient registration. Using in addition known techniques to increase the efficiency of the key exchange, this scheme can be extended to obtain the maximal reachable efficiency [30].

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