Dilute Bose gas in a quasi two-dimensional trap

Brandon P. van Zyl, R.K. Bhaduri, and Justin Sigetich
Department of Physics and Astronomy,
McMaster University, Hamilton, Ontario,
Canada L8S 4M1

Abstract

We investigate the behavior of a dilute quasi two-dimensional, harmonically confined, weakly interacting Bose gas within the finite-temperature Thomas-Fermi approximation. We find that the thermodynamic properties of the system are markedly different for repulsive and attractive interactions. Specifically, in contrast to the repulsive case, there appears to be a phase transition when the atoms interact with an attractive pseudo-potential, in the sense that there is no self-consistent solution for the normal ground state below a certain temperature $T^*$. These numerical findings are supported by analytical investigations of the thermodynamics of the system in the complex fugacity plane, and within the random-phase approximation. We also show that the temperature $T^*$ can be interpreted as the limiting temperature below which the system cannot be described as a collection of noninteracting haldons.

PACS: 03.75.Fi, 05.30.Jp, 31.14.Bs
I. INTRODUCTION

Over the last decade or so, there have been a number of theoretical studies of Bose-Einstein condensation (BEC) in inhomogeneous low-dimensional Bose systems [1–9]. In principle, such low-dimensional systems can be created in the laboratory through a suitable manipulation of the potential used to trap the ultra-cold atoms. In this paper, we will be primarily interested in the situation for which the confinement in the \(z\)-direction is much steeper than the in-plane confinement, so that the system can be viewed as being (quasi) two-dimensional (2D) in the sense that the motion of the atoms in the \(z\)-direction is “frozen out”.

It is well-known that a homogeneous 2D Bose gas has no long range order (at \(T \neq 0\)), and so a BEC transition at any (non-zero) finite temperature cannot occur. However, in a 2D harmonically confined system with oscillator frequency \(\omega_0\), it is easily shown that an ideal Bose gas has a critical BEC temperature

\[
T_c^{(0)} = (6/\pi^2)^{1/2}N^{1/2}\hbar\omega_0, \tag{1}
\]

which, for \(N \gg 1\), is much larger than the oscillator gap \(\hbar\omega_0\). It is interesting to consider the limit \(N \to \infty, \omega_0 \to 0\), such that \(N^{1/2}\omega_0 = \text{constant}\). In this situation, the critical BEC transition temperature \(T_c^{(0)}\) remains the same. Note that this is not the usual thermodynamic limit, which in 2D would demand that \(N\omega_0 = \text{constant}\), resulting in no BEC. We have previously shown that a self-consistent Thomas-Fermi (SCTF) model [7] (i.e., with no condensate density), has a solution at all temperatures provided the mean-field contribution of a repulsive zero range interaction is included, no matter how weak. This result contradicts a previous claim in the literature [4] that no self-consistent solution of the TF equation in a trap is possible below a certain critical temperature \(T_c\). (It is worth pointing out that in Ref. [4], the absence of a solution below \(T_c\) was interpreted as a phase transition in the system, but not a phase transition to the BEC state.) The absence of a phase transition for the repulsive case was also confirmed analytically by examining the branch-points of the density in the complex fugacity plane. Note, however, that the SCTF method cannot address the important issue of phase fluctuations in the condensate [8]. It was also found that the such a system could be described by noninteracting haldons obeying the generalized fractional exclusion statistics (FES) [10]. This has been recently confirmed by Hansson et al. [11].

For a dilute Bose gas in a three dimensional trap, where the average distance between the atoms is much larger than the scattering length, the Gross-Pitaevskii (GP) approach [12] of using a zero-range pseudo-potential with a strength given by \(4\pi\hbar^2a/m\) (where \(a\) is the s-wave scattering length, and \(m\) is the mass of the atom) is very successful. For a strictly 2D system however, the strength has a logarithmic energy dependence [2]. Recently, Petrov et al. [8] have deduced the strength of the quasi-2D system by considering free two-particle scattering in the plane of the gas, and strong harmonic confinement in the \(z\)-direction (i.e., \(\omega_0 \ll \omega_z\)). The effective two dimensional pseudo-potential may be written as a zero-range momentum-dependent interaction with a strength given by [13]

\[
\frac{2\pi\hbar^2}{m}g = \frac{2\sqrt{2\pi}\hbar^2}{m}[^{1}_{z}/a - \frac{1}{\sqrt{2\pi}}\ln(\pi p_z^2/f_z^2)]^{-1}, \tag{2}
\]
where $\mathbf{p}_{12} = (\mathbf{p}_1 - \mathbf{p}_2)/2$ is the relative momentum between two bosons. In the above equation, $m$ is the mass of the atom, $a$ the s-wave scattering length, and $\ell_z = \sqrt{\hbar/m\omega_z}$. Note that the kinetic energy of relative motion, $E = p^2_{12}/m$, reduces to twice the single-particle energy in the frame where the CM momentum $\mathbf{P} = (\mathbf{p}_1 + \mathbf{p}_2)/2 = 0$. To simplify actual calculations in the many-body problem with this coupling strength, the momentum-dependence in Eq.(2) is replaced [8] by a density dependence by putting $p^2_{12} = mE = 2m|\mu|$, where $\mu$ is the chemical potential. For the bosons in a harmonic oscillator trap, the spatial density is inhomogeneous, and the local chemical potential $\mu(\mathbf{r}) = \mu - \frac{1}{2}m\omega^2_0 r^2$ will be used. For the situation when $\ell_z \gg a$, the first term on the right-hand side (RHS) of Eq. (2) dominates over the second, and the logarithmic term may be neglected. This yields a “constant” interaction strength,

$$\frac{2\pi\hbar^2}{m} \tilde{g} = \frac{2\sqrt{2\pi\hbar^2}}{m} a/\ell_z,$$

which agrees with our definition for the coupling constant in an earlier work [7].

In this paper, we perform finite-temperature SCTF calculations with the above density dependent strength factor in the pseudo-potential. Both the repulsive, and the attractive cases are considered for realistic values of the physical parameters. For the repulsive case ($g > 0$), the earlier conclusion that there is no BEC at any temperature remains unaltered. For the attractive case ($g < 0$) on the other hand, we find compelling numerical and analytical evidence suggesting the existence of a phase transition at a $T^*$ which is greater than the critical temperature for BEC given by Eq. (1). The outline for the rest of the paper is as follows. In Sec. II, we present numerical results of the SCTF calculations for both $\tilde{g}$ and $g$ as defined above, as well as some analytical arguments explaining the origin of the critical temperature $T^*$. In Sec. III we provide additional support for our numerical results by analytically investigating the thermodynamic properties of the system at zero and finite-temperatures. Finally, in Sec. IV, we present our concluding remarks.

II. THOMAS-FERMI CALCULATION

A. Repulsive Interaction

We consider bosonic atoms in a two-dimensional harmonic oscillator trap above the critical temperature, if any. In this subsection, a repulsive interaction is used, while in the next subsection the interaction is taken to be attractive. In the SCTF method [7], the inhomogeneous number density of the interacting bosons is given by

$$n(\mathbf{r}) = \int \frac{d^2p/(2\pi\hbar^2)}{[\exp[(\frac{p^2}{2m} + \frac{1}{2}m\omega^2_0 r^2 + U(n(\mathbf{r})) - \mu)\beta] - 1]},$$

where $U(n(\mathbf{r})) = \frac{2\pi\hbar^2}{m} \tilde{g} n(\mathbf{r})$ is the mean-field potential [14] generated by the zero-range two-body pseudo-potential with the constant strength $\tilde{g}$, and $\mu$ is the chemical potential that serves to fix the total number of atoms in the trap,

$$N = \int n(\mathbf{r}) \ d^2\mathbf{r}.$$
To obtain \( n(r) \) for a fixed \( N \), Eqs. (4) and (5) have to be solved self-consistently. (In our numerical work, we have scaled all lengths by the characteristic in-plane oscillator length \( \ell_0 = \sqrt{\hbar/m\omega_0} \), and all energies by \( \hbar\omega_0 \).) The case for \( \tilde{g} > 0 \), has already been addressed in an earlier paper \[7\], where it was shown that the self-consistent solution for \( n(r) \) could be obtained for any \( T \geq 0 \), indicating that there is no BEC. We first want to check if the same conclusion holds when \( \tilde{g} \) is replaced by the more realistic momentum dependent \( g \), as given by Eq. (2). As noted before, we replace the \( p_{12} \)-dependence in \( g \) by an effective density dependence. Under this assumption, the \( p \)-integration in (4) can be done analytically. At zero-temperature, the result is given by

\[
\frac{2\pi\hbar^2}{m}gn(r) = (\mu - \frac{1}{2}m\omega_0^2r^2).
\]

The RHS of Eq. (5) is precisely the local chemical potential \( \mu(r) \). Hence we replace the \( p_{12}^2 \) on the RHS of Eq. (2) by \( (2\pi\hbar)^2gn(r) \) and \( g \) itself is determined self-consistently by Eqs. (5) and (6). At finite-temperatures, we continue to follow the same procedure, but in this case \( g(r) \) and \( n(r) \) must be found self-consistently from Eqs. (4) and (5). Note that Petrov et al. \[8\] in one of their footnotes recommend an identical prescription. Following this method, we still find that for a fixed boson number \( N \), Eqs. (4) and (5) may be solved self-consistently for all \( T \geq 0 \), indicating that there is no phase transition for a repulsive interaction. The behavior of the chemical potential \( \mu \) as a function of \( T/T_c^{(0)} \) is shown in Fig. 1. The lowest solid curve, for noninteracting bosons, shows the usual discontinuity in \( \mu \) at \( T/T_c^{(0)} = 1 \). The next higher pair of closely spaced curves show the results for \( N = 10^4 \) Rb\(^{87} \) atoms in a trap with with \( \hbar\omega_0 = 0.6 \) nK, \( \omega_z = 10^3\omega_0 \), and \( a = 5.8 \) nm. The solid curve shows the result for \( \tilde{g} = 0.05 \), obtained from the above physical parameters, while the slightly lower dashed curve is for the corresponding \( g \) given by Eq. (2). We note that the difference between the two is negligible for the parameters chosen above, which suggests that FES is still obeyed to a good approximation for relatively small values of \( g \). In the same figure, the top three curves are for an artificially boosted interaction, with the continuous curve given by \( \tilde{g} = 0.25 \). Note that this could in principle be achieved in a variety of ways by altering both \( \ell_z \) and \( a \). Two limiting cases are given by keeping \( \ell_z \) constant and decreasing \( a \) by a factor of five, or keeping \( a \) fixed and increasing \( \ell_z \) by a factor of five. Although both procedures yield the same \( \tilde{g} \), we see from Eq. (2) that \( g \) gets affected more when \( \ell_z \) alone is altered. In Fig. 1, the lowest (dashed) curve is for this case, while the next higher one is for the case where only \( a \) is decreased by a factor of five. We see that in all cases with a repulsive interaction, the main conclusion that there is no strict BEC continues to hold.

Even though there is no BEC in the sense that the SCTF solution is obtained without paying any special attention to the lowest single-particle quantum state, it is found that there is a significant enhancement in the peak of the density distribution as the temperature is reduced below \( T_c^{(0)} \), the critical temperature for the noninteracting case. This is shown in Fig. 2 for \( \tilde{g} = 0.25 \) (continuous curves) at \( T/T_c^{(0)} \) = 0.8 and 1.1. When the density-dependent version of the coupling \( g \) is used, the enhancement is even more pronounced. The reason for this behavior is clear from the inset, where the effective \( g(r) \), as obtained from the self-consistent solution, is shown as a function of \( r \). We note that for the density-dependent case, the interaction \( g(r) \) is considerably weaker than the constant \( \tilde{g} \), thus causing less suppression of the central density.
B. Attractive Interaction

We first take the example of Li\textsuperscript{7} atoms for which the s-wave scattering length is $a = -1.45$ nm. For $\hbar \omega_0 = 7$ nK (the corresponding oscillator angular frequency $\omega_0$ in the plane is $2\pi \times 145 \text{ rads/s}$) and $\omega_z = 10^3 \omega_0$, the coupling constant $\tilde{g} = -0.012$. On the other hand, the interaction strength could be considerably larger for Cs\textsuperscript{135} atoms, which have $a = -60$ nm. In Fig. 3, we display for $N = 10^4$ atoms of Li\textsuperscript{7} the behavior of the chemical potential $\mu$ as a function of $T$ obtained self-consistently. No solution is found for $T \leq T_\star$, irrespective of whether one uses $\tilde{g}$ or $g$. The highest lying solid curve is for $\tilde{g} = -0.012$, and difference between this and the density-dependent form of $g$ for this weak interaction is too small to be shown in the same scale. In the same figure, we also display the effect of boosting the attractive interaction strength nearly ten-fold to $\tilde{g} = -0.1$. Not only does the temperature $T_\star$ increases significantly, but it is also possible now to differentiate the curves for the cases where the density-dependent form of $g$ is used rather than $\tilde{g}$. Furthermore, when the boosting of the coupling $g$ is done by altering $\ell_z$ rather than $a$, $T_\star$ is larger. Nevertheless, the behavior of the chemical potential with temperature, for $T > T_\star$ does not alter significantly with the strength of the attractive interaction. Comparing Fig. 3 with the corresponding repulsive cases (see Fig. 1), we note that $\mu$ showed a much larger variation, but only in the range $T \leq T_{c(0)}$, which is in any case inaccessible for the attractive interactions.

Some insight as to why no self-consistent solution for attractive interaction strength $\tilde{g}$ may be found by performing the $p-$integration in Eq. (4) analytically (valid for either sign of $\tilde{g}$). The result is given by

$$\left(\mu - \frac{1}{2}m\omega_0^2 r^2\right) = \frac{2\pi \hbar^2}{m}\tilde{g} n(r) + \frac{1}{\beta} \ln[1 - \exp(-2\pi \hbar^2 n(r) \beta/m)]. \tag{7}$$

For a negative $\tilde{g}$, both terms on the RHS are negative, and therefore $\mu$, which is independent of $r$, must be negative at all temperatures. For this case, if we take the above expression literally at $T = 0$, the density is given by

$$n(r) = \frac{m}{2\pi |\tilde{g}| \hbar^2} (|\mu| + \frac{1}{2}m\omega_0^2 r^2). \tag{8}$$

This solution for $n(r)$ is rejected since it keeps increasing with $r$! We demand a normalized density $n(r)$ that is a monotonically decreasing function of $r$, which is only found for $T > T_\star$. By differentiating Eq. (7) with respect to $r$, we find that for $dn/dr \leq 0$ for all values of $r$, we must have

$$[\exp(\frac{2\pi \hbar^2}{m} n(r) \beta) - 1]^{-1} - |\tilde{g}| \geq 0. \tag{9}$$

This may be satisfied only for $T \geq T_\star$, which is related to the central density at this temperature $n(0)$ by the relation

$$T_\star = \frac{2\pi \hbar^2 n(0)}{m \ln(1 + 1/|\tilde{g}|)}. \tag{10}$$

It is interesting that the above equation may also be derived using the the generalized exclusion statistics of Haldane [10]. It has been shown [7,11] that when the interaction
strength of the zero-range pseudo-potential is taken to be a constant (which we denote by $\tilde{g}$ in this paper), the interacting boson system may be mapped on to a noninteracting system of haldons, whose statistical occupancy factor of a single particle state $i$ with energy $\epsilon_i$ is given by

$$\eta_i = (w_i + \tilde{g})^{-1}, \quad (11)$$

which must be $\geq 0$. Note that $w_i$ obeys the relation

$$w_i \tilde{g}(1 + w_i)^{1-\tilde{g}} = \exp[(\epsilon_i - \mu)\beta].$$

In what follows, we go to the limit $\omega_0 \to 0, N \to \infty$, such that $N^{1/2}\omega_0 = \text{constant}$. We then identify the central density $n(0)$ of the harmonic oscillator potential as the uniform density in this limit. The number density $n(0)$ of these haldons at the center of the harmonic oscillator potential is given by

$$n(0) = \frac{m}{2\pi\hbar^2\beta} \ln \frac{1 + w_0}{w_0}, \quad (12)$$

where $w_0$ defines the occupancy of the lowest quantum state through Eq. $(11)$. We now use these relations for the attractive interaction strength $\tilde{g} = -|\tilde{g}|$. Note from Eq. $(11)$ that for the occupancy $\eta_0$ to be positive definite, $w_0 \geq |\tilde{g}|$, and that the occupancy tends to infinity when $w_0 = |\tilde{g}|$. Substituting this limiting value of $w_0$ in Eq. $(12)$ at $T^*$, we immediately obtain Eq. $(14)$. We should regard $T^*$ as the limiting temperature below which the system cannot be described as a collection of ideal haldons.

In Fig. 4, the self-consistent spatial density $n(r)$ is shown for $N = 10^4 \text{ Li}^7$ atoms in the quasi 2D trap, with $\tilde{g} = -0.012$. For this very weak interaction, the corresponding curve for the density-dependent coupling strength $g$ cannot be differentiated on this scale. We see that not only is there there a marked enhancement in the central density when $T$ is decreased from $1.1 T^*$ to $T^*$, but the shape of the density distribution is also different. We cannot, of course, get any solution for $T < T^*$. It may be instructive to find a relation between $T^*$ and $N$ analogous to Eq. $(1)$, which will of course also depend on the interaction strength $\tilde{g}$. Empirically, we find that this is given by

$$T^* = \gamma(|\tilde{g}|) \left( \frac{N}{\zeta(2)} \right)^{1/2} \hbar\omega_0, \quad (13)$$

where $\gamma(|\tilde{g}|) = |1 + \frac{2}{5}(|\tilde{g}|)^{2/3}|$. This relation is quite accurate even for small $N$ and large $\tilde{g}$, as depicted in Fig. 5. In this figure, we show the numerically calculated points by circles, and the continuous curves are derived from Eq. $(13)$. Although the solid curves fitting the data have been extended down to $N = 0$ at $T = 0$, we were not able to numerically access stable values of $N$ for $T^* \lesssim 8$, and so Eq. $(13)$ is only used to fit the data down to a finite temperature $T^* \gtrsim 8$. Also note from Eqs. $(1)$ and $(13)$ that $T^*/T_c(0)$ is independent of $N$, and therefore we can draw a “universal” curve of $\tilde{g}$ versus $T^*/T_c(0)$, which is shown by the inset in Fig. 5. The normal bosonic phase is above the continuous curve shown in this inset, but the SCTF method used cannot access the region below.
III. ANALYTICAL CONSIDERATIONS

In this section, we wish to address the question that naturally arises from the numerical work presented in Sec. IIB; namely, are our numerical calculations suggesting that there is phase transition in the quasi-2D trap for attractive interactions? Since we are unable to answer this question within the framework of the SCTF model, we now turn our attention to various analytical approaches which may aid in our assessment of the numerical results presented above.

A. Zero-Temperature Variational Calculation

Let us consider the 2D analogue of the GP energy functional, namely

\[ E = \int d^2 r \left[ \frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{m}{2} r^2 \omega_0^2 |\psi|^2 + \frac{12\pi \hbar^2}{2m} \tilde{g} |\psi|^4 \right] , \]

where \( \tilde{g} \) is given by Eq. (3), and may be of either sign. In Eq. (14), the function \( \psi(r) \) represents the wave function of a system for which there is macroscopic occupation of the lowest quantum state (i.e., all \( N \) bosons are in the ground state):

\[ \psi(r) = \frac{1}{b} \sqrt{\frac{N}{\pi}} \exp\left(-\frac{r^2}{b^2}\right) , \]

where \( b \) is taken to be a variational parameter. We find that the energy is a minimum for

\[ b = \sqrt{\frac{\hbar}{m \omega_0}} (1 + N \tilde{g})^{1/4} . \]

For an attractive interaction, \( \tilde{g} = -|\tilde{g}| \), Eq. (16) implies that the condensate collapses for \( N |\tilde{g}| \geq 1 \).

This rough estimate is in good agreement with the numerical estimate of \( N |\tilde{g}| \geq 0.94 \) by Adhikari [9] which was carefully obtained through a self-consistent solution of the 2D GP equations. For a weakly interacting gas like Li\(^7\) with \( \tilde{g} = -0.012 \), we find that the condensate is unstable for \( N > 83 \).

B. Finite-Temperature Thermodynamics

This subsection deals with the study of the behavior of the branch points of the quantity \( u = 2\pi \hbar^2 \beta n(r)/m \) in the complex fugacity plane \( Z = \exp(\beta \mu) \) as a function of the coupling constant \( \tilde{g} \). This is analogous to the analysis that was made by Sutherland while studying the thermodynamics of a one-dimensional gas interacting with an inverse square potential [15]. The important point to be taken from Sutherland’s work is that a branch point of \( u \) on the positive real axis of \( Z \) indicates a phase transition in the system.

We begin by noting that Eq. (7) can be rewritten as
\[
Z \ e^{-\beta m \omega_0^2 r^2/2} = 2 \ (\tilde{g}^{1/2}) \ \sinh \frac{u}{2}.
\]

As before, going to the limit \(N \to \infty, \omega_0 \to 0\) with \(N^{1/2} \omega_0 = \text{constant}\), Eq. (17) reduces to the \(r\)-independent relation

\[
Z = e^{\tilde{g}u} - e^{(\tilde{g}-1)u}.
\]

We note that for \(\tilde{g} = 0\), the above equation gives \(u = -\ln(1-Z)\), implying that \(u\) has a branch point at \(Z = 1\), i.e. \(\mu = 0\). This is precisely where BEC takes place for a noninteracting gas in a harmonic trap. For \(\tilde{g} = 1/2\), the branch points of \(u\) are at \(Z = \pm 2i\), and for \(\tilde{g} = 1\), the branch point is at \(Z = -1\). For arbitrary values of \(\tilde{g}\), the branch points of \(u\) can be obtained by examining the analytical structure of the solutions to the equation \((x \equiv \exp(u))\)

\[
x^{\tilde{g}} - x^{\tilde{g}-1} - Z = 0.
\]

Following Sutherland’s prescription [15], we obtain the branch points of \(u\), denoted by \(Z_0\), for all \(\tilde{g} \geq 0\):

(i) \(\tilde{g} \geq 1\)

\[
Z_0 = -\frac{1}{\sqrt[\tilde{g}](\tilde{g} - 1)} \left( \frac{\tilde{g} - 1}{\tilde{g}} \right)^{(\tilde{g}-1)/2},
\]

(ii) \(1 > \tilde{g} \geq 0\)

\[
Z_0 = e^{\pm \pi i \tilde{g}} \frac{1}{\sqrt[\tilde{g}](1-\tilde{g})} \left( \frac{1 - \tilde{g}}{\tilde{g}} \right)^{(\tilde{g}-1)/2}.
\]

In Fig. 6, the loop and the line along the negative real axis \(-1 \leq Z \leq 0\) in the complex fugacity plane shows the branch points of \(u\) for positive values of \(\tilde{g}\) given by Eqs. (20) and (21). Since there is no branch point on the real fugacity axis for \(\tilde{g} > 0\), there is no phase transition, in agreement with our numerical calculations.

The situation for \(\tilde{g} < 0\) can be addressed by either solving Eq. (19) directly, or by noticing that the transformation \(\tilde{g} \to -\tilde{g} + 1, \quad u \to -u, \quad Z \to -Z\)

leaves Eq. (18) unchanged. More specifically, under Eq. (22), the branch points of \(u\) on the negative \(Z\)-axis between \(Z_0 \in [-1, 0]\) for \(1 \leq \tilde{g} < \infty\) are now shifted to the positive side of the axis, namely, \(Z_0 \in [0, 1]\) for all \(-\infty < \tilde{g} \leq 0\). The branch points for \(0 < \tilde{g} < 1\) are all mapped back onto Eq. (21), thereby leaving the loop structure in the complex \(Z\)-plane invariant under (22) (see Fig. 6). In other words, for every \(\tilde{g} < 0\) we have an associated branch point on the positive real \(Z\)-axis, suggesting that there is a phase transition in the system for attractive interactions. It is important to note, however, that the above analysis can offer us no insight into the precise nature of the phase transition; is it BEC? Is it the Kosterlitz-Thouless transition [17]? Is it the collapse of the gas?
C. Random-Phase Approximation

In this subsection, we will try to address the nature of the phase transition discussed above by applying the well-known random-phase approximation (RPA) \[^{18}\] to investigate the stability of the trapped 2D Bose gas with attractive interactions. Our approach parallels the recent investigation by Meuller and Baym \[^{19}\], with the main difference being that we confine ourselves to strictly two-dimensions. The basic idea behind their method is to identify the collapse of the gas with a divergence in the density-response function \(\chi(k, \omega; \beta)\), which is a measure of the response of the system to some external probe with wave vector \(k\) and frequency \(\omega\). As in Ref. \[^{19}\], \(\chi\) is calculated within the local-density approximation (LDA), which amounts to replacing the response function of the inhomogeneous system by that of a system of uniform density \(n\). In order to make contact with the trapped gas, the uniform density \(n\) is then taken to represent the central density of the trap \(n(r = 0)\).

For a confined Bose gas, the collapse is associated with an instability of the lowest energy “breathing” mode of the system. In 2D, the breathing mode is given by

\[
\delta \rho(r) \propto \left( \frac{r^2}{\ell_0^2} - 1 \right) \exp(-r^2/\ell_0^2),
\]

where \(r\) is the 2D radial coordinate, and \(\ell_0 = \sqrt{\hbar/m\omega_0}\) is the characteristic size of the cloud. The Fourier transform of \(\delta \rho(r)\) has a maximum at \(k = 2/\ell_0\), and so we look for an instability at this wave vector. Owing to the fact the breathing mode has a vanishing frequency, the line of collapse is characterized by

\[
\chi(k = 2/\ell_0, \omega = 0; T^0)^{-1} = 0,
\]

where \(T^0\) denotes the temperature for which the gas undergoes collapse. Note that \(T^0\) need not coincide with the BEC transition temperature \(T_c\) of the interacting gas. Since our SCTF calculations deal only with the normal phase of the Bose gas, we will restrict our attention to the noncondensate contribution to \(\chi(k, \omega; T)\) above, and present a more detailed analysis of the RPA in 2D attractive Bose gases elsewhere \[^{20}\]. Within the RPA, the response function for the interacting Bose gas has the structure \[^{21}\]

\[
\chi^{\text{RPA}}(k, \omega; T) = \frac{1}{\left(1 - \tilde{g}\chi_0^n\right)},
\]

where \(\chi_0^n\) is the “bare” noncondensate response of the noninteracting system. In 2D, we have

\[
\chi_0^n(k, \omega; T) = \frac{1}{(2\pi)^2} \int d^2q \frac{f(q - k/2) - f(q + k/2)}{\hbar \omega - (\varepsilon(q + k/2) - \varepsilon(q - k/2))}.
\]

In Eq. (25), \(f(q)\) is the Bose distribution function, and \(\varepsilon(k) = \hbar^2k^2/2m\) is the free particle kinetic energy. After integrating out the angular dependence, we have

\[
\chi_0^n(k, 0; T) = -\frac{8\pi m}{(2\pi \hbar)^2 k} \int dq f(q) \frac{q}{\sqrt{k^2 - 4q^2}}.
\]
Using the above expression for $\chi_0^n$ we can obtain the line of collapse by setting $k = 2/\ell_0$ and solving for the roots of the denominator in Eq. (24). This procedure yields the line of collapse in terms of $\mu$ and $T_{\text{RPA}}^0$. Following the arguments made in Ref. [19], we can then relate $\mu$ to the total number of normal phase atoms, $N$, in the harmonically confined gas through the relation

$$N = \left( \frac{k_B T}{\hbar \omega_0} \right)^2 g_2(e^{\beta \mu}),$$  

where $g_\nu(z) = \sum_p z^p/p^\nu$ is the polylogarithm function. Note that evaluating Eq. (27) at $\mu = 0$ defines the line of condensation for a noninteracting 2D Bose gas. (This result can easily be derived by integrating Eq. (4) with $\tilde{g} = 0$ over all space.)

The results of our calculations for $\tilde{g} = -0.012, -0.1, -0.2$ are shown in Fig. 7. For comparison, we have also included as dashed lines in the figure the $\tilde{g} = -0.1, -0.2$ SCTF results from Sec. IIB (see also Fig. 5). This figure suggests that the line of collapse derived from the RPA has the same functional relationship as the empirical formula given by Eq. (13) in Sec. IIB. In order to confirm this observation, we recall that the relationship used to link the uniform gas results to the harmonically confined gas, viz., Eq. (27), is strictly valid only for a noninteracting system. As we saw in Sec. IIB, the attractive interactions have the effect of “renormalizing” the ideal $N$ vs. $T$ dependence (see Eq. (13)), which will result in different curves for the RPA and SCTF data. With this in mind, we rescale Eq. (27) by $1/\gamma(|\tilde{g}|)$, and compare the results with the SCTF data. After this procedure, we find that the SCTF and RPA curves are almost indistinguishable on the scale of Fig. 7. In this sense, we can identify $T_{\text{RPA}}^0$ with the $T^*$ of Sec. IIB. This analysis leads us to believe that the phase-boundary generated from the SCTF in Sec. IIB is in fact the line of collapse of the Bose gas as one approaches the instability from above the BEC transition temperature $T_c$.

To close this section, we mention that we have also performed RPA calculations for the 3D interacting Li$^7$ Bose gas with attractive interactions (exactly as in Ref. [19] but without exchange) and found that the 3D gas is more stable than its 2D counterpart; that is, for the same temperature and particle number, the 3D system is always in the stable uncollapsed phase relative to the 2D gas. To illustrate this, we have included as an inset to Fig. 7, the lines of collapse (solid curve) and condensation (dashed curve) for the 3D system, focusing only on the $N$ and $T$ regimes that are relevant for a comparison with the corresponding 2D system.

**IV. SUMMARY**

One of the first results of this paper was a confirmation of the fact that, at least within the SCTF method, there is no strict BEC in a quasi-2D trap for atoms with repulsive interactions, even if the more “realistic” momentum-dependent pseudo-potential recently derived by Petrov et al. [8], and given by Eq. (2) is used. Indeed, our results indicate that the much simpler density-independent coupling constant $\tilde{g}$ (Eq. (3)) is quite adequate for numerical work on 2D Bose systems. This observation provides further support to the conclusion obtained earlier [4] using a momentum-independent coupling constant $\tilde{g}$.

In the case of attractive interactions, we have found strong numerical and analytical evidence for the existence of a phase transition in the system. Since the qualitative features
of this phase transition are not strongly influenced by using a more complicated momentum-dependent pseudo-potential, it is again sufficient to use the simpler $\tilde{g}$ in numerical work. Our numerical investigations reveal that the phase transition is signaled by a “critical” temperature $T^* > T_{c}^{(0)}$ below which there is no self-consistent solution for the normal ground state of the system. From the point of view of FES, the $T^*$ was interpreted as the limiting temperature below which the system cannot be described as a collection of noninteracting haldons.

In order to further our understanding of the numerical results, we considered several analytical approaches. This began in Sec. II IA with a zero-temperature 2D variational calculation of the GP energy functional. Surprisingly, this simple analysis revealed that if there is a BEC for attractive interactions in 2D, then the region of stability for the condensate (i.e., prior to collapse) is severely restricted by the condition $N|\tilde{g}| \geq 1$. For Li$^7$, this implies an instability for $N > 83$. The $T = 0$ calculation was followed in Sec. IIIB by an investigation of the branch points of $u$ in the complex fugacity plane. We found that for every $\tilde{g} < 0$, there is an associated branch point of $u$ on the positive real $Z$-axis. This result strongly supports our numerical calculations which indicate that there is a phase transition taking place in the system for attractive interactions.

Finally, in Sec. IIIC, an RPA-type calculation was used to investigate the line of collapse for a 2D Bose gas with attractive interactions. The purpose of this study was to try and determine the nature of the phase transition suggested by the analysis in Sec. IIIB. We found that the line of instability in the RPA calculation had the same $N$ vs. $T$ dependence as the SCTF curves in Sec. IIIB. The correspondence between the two formulations is highly suggestive that the phase transition is in fact associated with the collapse of the 2D gas as one approaches the instability from above $T_c$. We also performed analogous RPA calculations for the 3D Bose gas with attractive interactions, and found that for the same $N$ and $T$, the 3D system is always in a stable, uncollapsed phase relative to the 2D gas. We will present a more detailed comparison of the 2D and 3D RPA calculations elsewhere.

V. ACKNOWLEDGMENTS

One of us (BVZ) would like to thank Dr. J. P. Carbotte for financial support as well as stimulating discussions. RKB would like to thank Dr. Niall Whelan for relation (22). This research was supported by grants from the Natural Sciences and Engineering Research Council of Canada.
REFERENCES

[1] V. Bagnato and D. Kleppner, Phys. Rev. A 44, 7439 (1991).
[2] S. I. Shevchenko, Sov. Phys.-JETP 73, 1010 (1991).
[3] W. Ketterle and N. J. van Druten, Phys. Rev. A 54, 656 (1996).
[4] W. J. Mullin, J. Low Temp. Phys. 110, 167 (1998).
[5] L. P. Pitaevskii and A. Rosch, Phys. Rev. A 55, R853 (1997).
[6] T. Haugset and H. Haugerud, Phys. Rev. A 57, 3809 (1998).
[7] R. K. Bhaduri, S. M. Reimann, S. Viefers, S. Ghose Choudhury, and M. K. Srivastava, J. Phy. B 33, 3895 (2000).
[8] D. S. Petrov, M. Holzmann, and G. V. Shlyapnikov, Phys. Rev. Lett. 84, 2551 (2000).
[9] S. K. Adhikari, cond-mat/0005455; Phys. Lett (2000).
[10] F. D. M. Haldane, Phys. Rev. Lett. 67, 937 (1991); Yong Shi Wu, Phys. Rev. Lett. 73, 922 (1994).
[11] T. H. Hansson, J. M. Leinaas, and S. Viefers, Phys. Rev. Lett. 86, 2930 (2001).
[12] L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. 40, 646 (1961) [Sov. Phys. JETP 13, 451(1961)]; E. P. Gross, Nuovo Cimento, 20, 454 (1961).
[13] Our $g$ defined by Eq.(2) is dimensionless. Petrov et. al’s $g$ is given by all of the LHS of Eq. (3).
[14] This expression for $n(r)$ does not include the exchange energy. Nevertheless, exchange effects can easily be included in our SCTF by taking $U = 4\pi\hbar^2\tilde{g}n(r)/m$ in Eq. (4).
[15] B. Sutherland, J. Math. Phys. 12, 251 (1971).
[16] This transformation was pointed out to us by Niall D. Whelan.
[17] M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. 6, 1181 (1973).
[18] A. Griffin, *Excitations in a Bose-Condensed Liquid* (Cambridge University Press, Cambridge, 1993).
[19] E. J. Mueller and G. Baym, Phys. Rev. A62, 053605 (2000).
[20] We have performed similar RPA calculations for $T < T_c$ which at $T = 0$ are in reasonable agreement with the simple variational calculation in Sec. IIIA. Furthermore, (within the approximations of the formalism) the RPA calculations indicate that the 2D Bose gas with attractive interactions has a stable condensed phase at finite temperatures provided $N$ is below some critical particle number $N_c(T)$. This is not necessarily in contradiction to the results of Ref. [8] (which suggest the existence of a stable quasi-condensate with fluctuating phase for $T < T_c$), since our simple RPA formulation cannot examine fluctuations in the phase.
[21] This is the usual RPA without exchange. To include exchange effects, one should replace the denominator in Eq. (24) by $(1 - 2\tilde{g}\chi_0^a)$. See Ref. [18] for details.
FIGURES

FIG. 1. Chemical potential $\mu$ as a function of $T/T_c^{(0)}$. The parameters used to generate this figure are consistent with those of the Rb$^{87}$ experiments. Solid lines are for $\tilde{g}$, dashed lines are for $g(r)$ with fixed $a$ and variable $\ell_z$, while long-dashed-short-dashed lines are for $g(r)$ with fixed $\ell_z$ and variable $a$. Unless stated otherwise, this convention is used in all subsequent figures.

FIG. 2. Calculated self-consistent density profile for constant $\tilde{g} = 0.25$ and an artificially boosted density-dependent interaction $g(r)$. The lowest set of curves correspond to $T/T_c^{(0)} = 1.1$ and the highest set to $T/T_c^{(0)} = 0.8$. The inset illustrates the self-consistent $g(r)$ evaluated at $T/T_c^{(0)} = 0.8$. Notice that $g(r)$ always lies below the corresponding constant $\tilde{g}$ value.

FIG. 3. Chemical potential $\mu$ as a function of $T$ for attractive interactions. Open circles denote the last point for which a self-consistent solution to Eqs. (4) and (5) could be obtained while the arrows serve to clarify the location of $T^\star$. The parameters used to generate this figure are consistent with those of the Li$^{7}$ experiments. The same line labeling convention as in the repulsive case has been used. See text for details.

FIG. 4. Calculated self-consistent density profile for an attractive interaction $\tilde{g} = -0.012$. The lowest curve is for a temperature $T = 1.1T^\star$ and the highest lying curve is for a temperature $T = T^\star$. Li$^{7}$ parameters have been used.

FIG. 5. Total number of normal phase atoms $N$ as a function of the critical temperature $T^\star$. From left to right, each curve is evaluated at $\tilde{g} = 0.0, -0.1, -0.2, -1.0$. Open circles denote numerical data, and the continuous curves are obtained from Eq. (13). Inset: Universal “phase-boundary” separating the normal bosonic phase from a region which is inaccessible within the SCTF approximation.

FIG. 6. The thick continuous curves represent the location of branch points of $u$ in the complex fugacity plane as a function of $\tilde{g}$. Branch points of $u$ lying to the positive real Z-axis indicate a phase transition in the system.

FIG. 7. Total number of normal phase atoms as a function of temperature $T$. The solid curves (RPA), from left to right, are for $\tilde{g} = -0.012, -0.1, -0.2$ and correspond to $N$ vs. $T_{RPA}^0$. The dashed curves (SCTF), from left to right, are for $\tilde{g} = -0.1, -0.2$ and correspond to $N$ vs. $T^\star$. Inset: Lines of collapse (solid) and condensation (dashed) for the 3D Li$^{7}$ Bose gas with attractive interactions. In this inset, we have zoomed in to show only the $N$ and $T$ ranges relevant for a comparison with the corresponding 2D system. See Ref. [19] for details.
$\tilde{g} = -0.012$

$N = 10^4$ Li$^7$ atoms
