Optimal oracle inequalities for solving projected fixed-point equations

Wenlong Mou⋄ Ashwin Pananjady⋆ Martin J. Wainwright⋄†

Department of Electrical Engineering and Computer Sciences⋄
Simons Institute for the Theory of Computing⋆
Department of Statistics†
University of California Berkeley

Abstract

Linear fixed point equations in Hilbert spaces arise in a variety of settings, including reinforcement learning, and computational methods for solving differential and integral equations. We study methods that use a collection of random observations to compute approximate solutions by searching over a known low-dimensional subspace of the Hilbert space. First, we prove an instance-dependent upper bound on the mean-squared error for a linear stochastic approximation scheme that exploits Polyak–Ruppert averaging. This bound consists of two terms: an approximation error term with an instance-dependent approximation factor, and a statistical error term that captures the instance-specific complexity of the noise when projected onto the low-dimensional subspace. Using information-theoretic methods, we also establish lower bounds showing that both of these terms cannot be improved, again in an instance-dependent sense. A concrete consequence of our characterization is that the optimal approximation factor in this problem can be much larger than a universal constant. We show how our results precisely characterize the error of a class of temporal difference learning methods for the policy evaluation problem with linear function approximation, establishing their optimality.

1 Introduction

Linear fixed point equations over a Hilbert space, with the Euclidean space being an important special case, arise in various contexts. Depending on the application, such fixed point equations take different names, including estimating equations, Bellman equations, Poisson equations and inverse systems [Ber11, KVZ+72, Woo16]. More specifically, given a Hilbert space $X$, we consider a fixed point equation of the form

$$v = Lv + b,$$

where $b$ is some member of the Hilbert space, and $L$ is a linear operator mapping $X$ to itself.

When the Hilbert space is infinite-dimensional—or has a finite but very large dimension $D$—it is common to seek approximate solutions to equation (1). A standard approach is to choose a subspace $S$ of the Hilbert space, of dimension $d \ll D$, and to search for solutions within this subspace. In particular, letting $\Pi_S$ denote the orthogonal projection onto this subspace, various methods seek (approximate) solutions to the projected fixed point equation

$$v = \Pi_S(Lv + b).$$

In order to set the stage, let us consider some generic examples that illustrate the projected fixed point equation (2). We eschew a fully rigorous exposition at this stage, deferring technical details and specific examples to Section 2.2.
Applying the operator $A$ Galerkin method constructs an approximate solution to the differential equation $L$, which is a linear fixed point relation of the form (1) with $A(v) = b$. This represents a particular case of our fixed point equation with $L = I - A$.

Let $S$ be a finite-dimensional subspace of $X$, say spanned by a set of basis functions $\{\phi_j\}_{j=1}^d$. A Galerkin method constructs an approximate solution to the differential equation $A(v) = b$ by solving the projected fixed point equation (2) over a subspace of this type. Concretely, any function $v \in S$ has a representation of the form $v = \sum_{j=1}^d \vartheta_j \phi_j$ for some weight vector $\vartheta \in \mathbb{R}^d$.

Applying the operator $A$ to any such function yields the residual $A(v) = \sum_{j=1}^d \vartheta_j A(\phi_j)$, and the Galerkin method chooses the weight vector $\vartheta \in \mathbb{R}^d$ such that $v$ satisfies the equation $v = \Pi_S((I - A)v + f)$. A specific version of the Galerkin method for a second-order differential equation called the **elliptic boundary value problem** is presented in detail in Section 2.2.2.

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**Example 1** (Galerkin methods for differential equations). Let $X$ be a Hilbert space of suitably differentiable functions, and let $A$ be a linear differential operator of order $k$, say of the form $A(v) = \omega_0 v + \sum_{j=1}^k \omega_j v^{(j)}$, where $v^{(j)}$ denotes the $j$th-order derivative of the function $v \in X$. Given a function $b \in X$, suppose that we are interested in solving the differential equation $A(v) = b$. This is a particular case of our fixed point equation with $L = I - A$.

Let $S$ be a finite-dimensional subspace of $X$, say spanned by a set of basis functions $\{\phi_j\}_{j=1}^d$. A Galerkin method constructs an approximate solution to the differential equation $A(v) = b$ by solving the projected fixed point equation (2) over a subspace of this type. Concretely, any function $v \in S$ has a representation of the form $v = \sum_{j=1}^d \vartheta_j \phi_j$ for some weight vector $\vartheta \in \mathbb{R}^d$.

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**Example 2** (Instrumental variable methods for nonparametric regression). Let $X$ denote a suitably constrained space of square-integrable functions mapping $\mathbb{R}^p \to \mathbb{R}$, and suppose that we have a regression model of the form $Y = f^*(X) + \epsilon$. Here $X$ is a random vector of covariates taking values in $\mathbb{R}^p$, the pair $(Y, \epsilon)$ denote scalar random variables, and $f^* \in X$ denotes an unknown function of interest. In the classical setup of nonparametric regression, it is assumed that $\mathbb{E}[\epsilon | X] = 0$, an assumption that can be violated. Instead, suppose that we have a vector of *instrumental variables* $Z \in \mathbb{R}^p$ such that $\mathbb{E}[\epsilon | Z] = 0$. Now let $T : X \to X$ denote a linear operator given by $T(f) = \mathbb{E}[f(X) | Z]$, and denote by $r = \mathbb{E}[Y | Z]$ a point in $X$. Instrumental variable approaches to estimating $f^*$ are based on the equality

$$\mathbb{E}[Y - f^*(X) | Z] = r - T(f^*) = 0,$$

which is a linear fixed point relation of the form (1) with $L = I - T$ and $b = r$.

Now let $\{\phi_j\}_{j \geq 1}$ be an orthonormal basis of $X$, and let $S$ denote the subspace spanned by the first $d$ such eigenfunctions. Then each function $f \in S$ can be represented as $f = \sum_{j=1}^d \vartheta_j \phi_j$, and approximate solutions to the fixed point equation (3) may be obtained via solving a projected variant (2), i.e., the equation $f = \Pi_S((I - T)f + r)$.

A specific example of an instrumental variables method is the class of temporal difference methods for policy evaluation, introduced and discussed in detail in Section 2.2.3.

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In particular instantiations of both of the examples above, it is typical for the ambient dimension $D$ to be very large (if not infinite) and for us to only have sample access to the pair $(L, b)$. This paper treats the setting in which $n$ observations $\{(L_i, b_i)\}_{i=1}^n$ are drawn i.i.d. from some distribution with mean $(L, b)$. Letting $v^*$ denote the solution to the fixed point equation (1), our goal is to use these observations in order to produce an estimate $\widehat{v}_n$ of $v^*$ that satisfies an oracle inequality of the form

$$\mathbb{E}[\|\widehat{v}_n - v^*\|^2] \leq \alpha \cdot \inf_{v \in S} \|v - v^*\|^2 + \varepsilon_n,$$

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1 For a more detailed discussion of existence and uniqueness of the various objects in this model, see Darolles et al. [DFFR11].

2 Especially for applications in reinforcement learning, another natural setting is that of Markov noise, which we handle in a companion paper.
Here we use $\|\cdot\|$ to denote the Hilbert norm associated with $X$. The three terms appearing on the RHS of inequality (4) all have concrete interpretations. The term

$$A(S, v^*) := \inf_{v \in S} \| v - v^* \|^2$$

(5)

defines the approximation error; this is the error incurred by an oracle procedure that knows the fixed point $v^*$ in advance and aims to output the best approximation to $v^*$ within the subspace $S$. The term $\alpha$ is the approximation factor, which indicates how poorly the estimator $\hat{v}_n$ performs at carrying out the aforementioned approximation; note that $\alpha \geq 1$ by definition, and it is most desirable for $\alpha$ to be as small as possible. The final term $\varepsilon_n$ is a proxy for the statistical error incurred due to our stochastic observation model; indeed, one expects that as the sample size $n$ goes to infinity, this error should tend to zero for any reasonable estimator, indicating consistent estimation when $v^* \in S$. More generally, we would like our estimator to also have as small a statistical error as possible in terms of the other parameters that define the problem instance.

In an ideal world, we would like both desiderata to hold simultaneously: the approximation factor should be as close to one as possible while the statistical error stays as small as possible. As we discuss shortly, such a “best-of-both-worlds” guarantee can indeed be obtained in many canonical problems, and “sharp” oracle inequalities—meaning ones in which the approximation factor is equal to 1—are known [RST17, DS12]. On the other hand, such oracle inequalities with unit factors are not known for the fixed point equation (1). Tsitsiklis and Van Roy [TVR97] show that if the operator $L$ is $\gamma_{\max}$-contractive in the norm $\|\cdot\|$, then the (deterministic) solution $\tilde{v}$ to the projected fixed point equation (2) satisfies the bound

$$\| \tilde{v} - v^* \|^2 \leq \frac{1}{1 - \gamma_{\max}^2} \inf_{v \in S} \| v - v^* \|^2.$$  

(6)

The bound (6) has a potentially large approximation factor that can be quite far from one (as would be the case for a “sharp” oracle inequality). One motivating question for our work is whether or not this bound can be improved, and if so, to what extent.3

Our work is also driven by the complementary question of whether a sharp bound can be obtained on the statistical error of an estimator that, unlike $\tilde{v}$, has access only to the samples $\{(L_i, b_i)\}_{i=1}^n$. In particular, we would like the statistical error $\varepsilon_n$ to depend on some notion of complexity within the subspace $S$, and not on the ambient space. Recent work by Bhandari et al. [BRS18] provides worst-case bounds on the statistical error of a stochastic approximation scheme, showing that the parametric rate $\varepsilon_n \lesssim d/n$ is attainable. In this paper, we study how to derive a more fine-grained bound on the statistical error that reflects the practical performance of the algorithm and depends optimally on the geometry of our problem instance.

### 1.1 Contributions and organization

The main contribution of this paper is to resolve both of the aforementioned questions, in particular by deriving upper bounds and information-theoretic lower bounds on both the approximation factor and statistical error that are instance-dependent.

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3Note that one can achieve an approximation factor arbitrarily close to one provided that $n \gg D$. One way to do so is as follows: form the plug-in estimate that solves the original fixed point relation (1) on the sample averages $\frac{1}{n} \sum_{i=1}^n L_i$ and $\frac{1}{n} \sum_{i=1}^n b_i$, and then project this solution onto the subspace $S$. In this paper, our principal interest—driven by the practical examples of Galerkin approximation and temporal difference learning—is in the regime $d \ll n \ll D$. 
On one hand, these bounds demonstrate that in most cases, the optimal oracle inequality no longer has approximation factor 1 in our general setting, but on the other hand, that the optimal approximation factor can be much better in many cases than what is suggested by the worst-case bound (6). We also derive a significantly sharper bound on the statistical error of a stochastic approximation scheme that is instance-optimal in a precise sense. In more detail, we present the following results:

- Theorem 1 establishes an instance-dependent risk upper bound of the form (4) for the Polyak–Ruppert averaged stochastic approximation estimator, whose approximation factor $\alpha$ depends in a precise way on the projection of the operator $L$ onto the subspace $S$, and the statistical error $\epsilon_n$ matches the Cramér–Rao lower bound for the instance within the subspace.
- In Theorem 2, we prove an information-theoretic lower bound on the approximation factor. It is a local analysis, in that the bound depends critically on the projection of the population-level operator. This lower bound certifies that the approximation factor attained by our estimator is optimal. To the best of our knowledge, this is also the first instance of an optimal oracle inequality with a non-constant and problem-dependent approximation factor.
- In Theorem 3, we establish via a Bayesian Cramér-Rao lower bound that the leading statistical error term for our estimator is also optimal in an instance-dependent sense.
- In Section 4, we derive specific consequences of our results for several examples, including for the problem of Galerkin approximation in second-order elliptic equations and temporal difference methods for policy evaluation with linear function approximation. A particular consequence of our results shows that in a minimax sense, the approximation factor (6) is optimal for policy evaluation with linear function approximation (cf. Proposition 1).

The remainder of this paper is organized as follows. Section 1.2 contains a detailed discussion of related work. We introduce formal background and specific examples in Section 2. Our main results under the general model of projected fixed point equations are introduced and discussed in Section 3. We then specialize these results to our examples in Section 4, deriving several concrete corollaries for Galerkin methods and temporal difference methods. Our proofs are postponed to Section 5, and technical results are deferred to the appendix.

1.2 Related work

Our paper touches on various lines of related work, including oracle inequalities for statistical estimation, stochastic approximation and its application to reinforcement learning, and projected linear equation methods. We provide a brief discussion of these connections here.

Oracle inequalities: There is a large literature on misspecified statistical models and oracle inequalities (e.g., see the monographs [Mas07, Kol11] for overviews). Oracle inequalities in the context of penalized empirical risk minimization (ERM) are quite well-understood (e.g., [BBM05, Kol06, MN06]). Typically, the resulting approximation factor is exactly 1 or arbitrarily close to 1, and the statistical error term depends on the localized Rademacher complexity or metric entropy of this function class. Aggregation methods have been developed in order to obtain sharp oracle inequalities with approximation factor exactly 1 (e.g. [Tsy04, BTW07b, DS12, RST17]). Sharp oracle inequalities are now available in a variety of settings including for sparse linear models [BTW07a], density estimation [DS18], graphon estimation [KTV17], and shape-constrained estimation [Bel18]. As previously noted, our setting differs qualitatively from the ERM setting, in that as shown in this paper, sharp
oracle inequalities are no longer possible. There is another related line of work on oracle inequalities of density estimation. Yatracos [Yat85] showed an oracle inequality with the non-standard approximation factor 3, and with a statistical error term depending on the metric entropy. This non-unit approximation factor was later shown to be optimal for the class of one-dimensional piecewise constant densities [CDSS14, BKM19, ZJT20]. The approximation factor lower bound in these papers and our work both make use of the birthday paradox to establish information-theoretic lower bounds.

Stochastic approximation: Stochastic approximation algorithms for linear and non-linear fixed-point equations have played a central role in large-scale machine learning and statistics [RM51, Lai03, NJLS09]. See the books [BMP12, Bor09] for a comprehensive survey of the classical methods of analysis. The seminal works by Polyak, Ruppert, and Juditsky [Pol90, PJ92, Rup88] propose taking the average of the stochastic approximation iterates, which stabilizes the algorithm and achieves a Gaussian limiting distribution. This asymptotic result is also known to achieve the local asymptotic minimax lower bound [DR16]. Non-asymptotic guarantees matching this asymptotic behavior have also been established for stochastic approximation algorithms and their variance-reduced variants [MB11, KPR+20, MLW+20, LMWJ20].

Stochastic approximation is also a fundamental building block for reinforcement learning algorithms, wherein the method is used to produce an iterative, online solution to the Bellman equation from data; see the books [Sze10, Ber19] for a survey. Such approaches include temporal difference (TD) methods [Sut88] for the policy evaluation problem and the Q-learning algorithm [WD92] for policy optimization. Variants of these algorithms also abound, including LSTD [Boy02], SARSA [RN94], actor-critic algorithms [KT00], and gradient TD methods [SMP+09]. The analysis of these methods has received significant attention in the literature, ranging from asymptotic guarantees (e.g., [BB96, TVR97, TVR99]) to more fine-grained finite-sample bounds (e.g., [BRS18, SY19, LS18, PW20, Wai19b, Wai19c]). Our work contributes to this literature by establishing finite-sample upper bounds for temporal difference methods with Polyak–Ruppert averaging, as applied to the policy evaluation problem with linear function approximation.

Projected methods for linear equations: Galerkin [Gal15] first proposed the method of approximating the solution to a linear PDE by solving the projected equation in a finite-dimensional subspace. This method later became a cornerstone of finite-element methods in numerical methods for PDEs; see the books [Fle84, BS07] for a comprehensive survey. A fundamental tool used in the analysis of Galerkin methods is Céa’s lemma [Cea64], which corresponds to a special case of the approximation factor upper bounds that we establish. As mentioned before, projected linear equations were also considered independently by Tsitsiklis and Van Roy [TVR97] in the context of reinforcement learning; they established the worst-case upper bound (6) on the approximation factor under contractivity assumptions. These contraction-based bounds were further extended to the analysis of Q-learning in optimal stopping problems [TVR99]. The connection between the Galerkin method and TD methods was discovered by Yu and Bertsekas [YB10, Ber11], and the former paper shows an instance-dependent upper bound on the approximation factor. This analysis was later applied to Monte–Carlo methods for solving linear inverse problems [PWB09, PWB12].

We note that the Bellman equation can be written in infinitely many equivalent ways—by using powers of the transition kernel and via the formalism of resolvents—leading to a
continuous family of projected equations indexed by a scalar parameter $\lambda$ (see, e.g., Section 5.5 of Bertsekas [Ber19]). Some of these forms can be specifically leveraged in other observation models; for instance, by observing the trajectory of the Markov chain instead of i.i.d. samples, it becomes possible to obtain unbiased observations for integer powers of the transition kernel. This makes it possible to efficiently estimate the solution to the projected linear equation for various values of $\lambda$, and underlies the family of TD(\lambda) methods [Sut88, Boy02]. Indeed, Tsitsiklis and Van Roy [TVR97] also showed that the worst-case approximation factor in equation (6) can be improved by using larger values of $\lambda$. Based on this observation, a line of work has studied the trade-off between approximation error and estimation measure in model selection for reinforcement learning problems [Ber16, Sch10, MS08, VR06]. However, unlike this body of work, our focus in the current paper is on studying the i.i.d. observation model; we postpone a detailed investigation of the Markov setting to a companion paper.

1.3 Notation

Here we summarize some notation used throughout the paper. For a positive integer $m$, we define the set $[m] := \{1, 2, \ldots, m\}$. For any pair $(X, Y)$ of real Hilbert spaces and a linear operator $A : X \to Y$, we denote by $A^* : Y \to X$ the adjoint operator of $A$, which by definition, satisfies $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $(x, y) \in X \times Y$. For a bounded linear operator $A$ from $X$ to $Y$, we define its operator norm as: $\|A\|_{X \to Y} := \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X}$. We use the shorthand notation $\|A\|_X$ to denote its operator norm when $A$ is a bounded linear operator mapping $X$ to itself. When $X = \mathbb{R}^{d_1}$ and $Y = \mathbb{R}^{d_2}$ are finite-dimensional Euclidean spaces equipped with the standard inner product, we denote by $\|A\|_{op}$ the operator norm in this case. We also use $\|\cdot\|_2$ to denote the standard Euclidean norm, in order to distinguish it from the Hilbert norm $\|\cdot\|$.

For a random object $X$, we use $\mathcal{L}(X)$ to denote its probability law. Given a vector $\mu \in \mathbb{R}^d$ and a positive semi-definite matrix $\Sigma \in \mathbb{R}^{d \times d}$, we use $\mathcal{N}(\mu, \Sigma)$ to denote the Gaussian distribution with mean $\mu$ and covariance $\Sigma$. We use $\mathcal{U}(\Omega)$ to denote the uniform distribution over a set $\Omega$. Given a Polish space $\mathcal{S}$ and a positive measure $\mu$ associated to its Borel $\sigma$-algebra, for $p \in [1, +\infty)$, we define $L^p(\mathcal{S}, \mu) := \{ f : \mathcal{S} \to \mathbb{R}, \|f\|_{L^p} := (\int_\mathcal{S} |f|^p d\mu)^{1/p} < +\infty \}$. When $\mathcal{S}$ is a subset of $\mathbb{R}^d$ and $\mu$ is the Lebesgue measure, we use the shorthand notation $L^p(\mathcal{S})$. For a point $x \in \mathbb{R}^d$, we use $\delta_x$ to denote the Dirac $\delta$-function at point $x$.

We use $(e_j)_{j=1}^{d}$ to denote the standard basis vectors in the Euclidean space $\mathbb{R}^d$, i.e., $e_i$ is a vector with a 1 in the $i$-th coordinate and zeros elsewhere. For two matrices $A \in \mathbb{R}^{d_1 \times d_2}$ and $B \in \mathbb{R}^{d_3 \times d_4}$, we denote by $A \otimes B$ their Kronecker product, a $d_1 d_3 \times d_2 d_4$ real matrix. For symmetric matrices $A, B \in \mathbb{R}^{d \times d}$, we use $A \preceq B$ to denote the fact $B - A$ is a positive semi-definite matrix, and denote by $A \prec B$ when $B - A$ is positive definite. For a positive integer $d$ and indices $i, j \in [d]$, we denote by $E_{ij}$ a $d \times d$ matrix with a 1 in the $(i, j)$ position and zeros elsewhere. More generally, given a set $\mathcal{S}$ and $s_1, s_2, \in \mathcal{S}$, we define $E_{s_1, s_2}$ to be the linear operator such that $E_{s_1, s_2} f(x) := f(s_2) 1_{x=s_1}$ for all $f : \mathcal{S} \to \mathbb{R}$.

2 Background

We begin by formulating the projected fixed point problem more precisely in Section 2.1. Section 2.2 provides illustrations of this general set-up with some concrete examples.
2.1 Problem formulation

Consider a separable Hilbert space $\mathbb{X}$ with (possibly infinite) dimension $D$, equipped with the inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{L}$ denote the set of all bounded linear operators mapping $\mathbb{X}$ to itself. Given one such operator $L \in \mathcal{L}$ and some $b \in \mathbb{X}$, we consider the fixed point relation $v = Lv + b$, as previously defined in equation (1). We assume that the operator $I - L$ has a bounded inverse, which guarantees the existence and uniqueness of the fixed point satisfying equation (1). We let $v^*$ denote this unique solution.

As previously noted, in general, solving a fixed point equation in the Hilbert space can be computationally challenging. Consequently, a natural approach is to seek approximations to the fixed point $v^*$ based on searching over a finite-dimensional subspace of the full Hilbert space. More precisely, given some $d$-dimensional subspace $S$ of $\mathbb{X}$, we seek to solve the projected fixed point equation (2).

Existence and uniqueness of projected fixed point: For concreteness in analysis, we are interested in problems for which the projected fixed equation has a unique solution. Here we provide a sufficient condition for such existence and uniqueness. In doing so and for future reference, it is helpful to define some mappings between $\mathbb{X}$ and the subspace $S$. Let us fix some orthogonal basis $\{\phi_j\}_{j \geq 1}$ of the full space $\mathbb{X}$ such that $S = \text{span}\{\phi_1, \ldots, \phi_d\}$. In terms of this basis, we can define the projection operator $\Phi_d : \mathbb{X} \to \mathbb{R}^d$ via $\Phi_d(x) := \left(\langle x, \phi_j \rangle\right)_{j=1}^d$. The adjoint operator of $\Phi_d$ is a mapping from $\mathbb{R}^d$ to $\mathbb{X}$, given by

$$\Phi_d^*(v) := \sum_{j=1}^d v_j \phi_j.$$ (7)

Using these operators, we can define the projected operator associated with $L$—namely

$$M := \Phi_d L \Phi_d^*.$$ (8)

Note that $M$ is simply a $d$-dimensional matrix, one which describes the action of $L$ on $S$ according to the basis that we have chosen. As we will see in the main theorems, our results do not depend on the specific choice of the orthonormal basis, but it is convenient to use a given one, as we have done here.

Consider the quantity

$$\kappa(M) := \frac{1}{2} \lambda_{\text{max}} \left( M + M^T \right),$$ (9)

corresponding to the maximal eigenvalue of the symmetrized version of $M$. One sufficient condition for there be a unique solution to the fixed point equation (2) is the bound $\kappa(M) < 1$. When this bound holds, the matrix $(I_d - M)$ is invertible, and hence for any $b \in \mathbb{X}$, there is a unique solution $\bar{v}$ to the equation $v = \Pi_S(Lv + b)$.

Stochastic observation model: As noted in the introduction, this paper focuses on an observation model in which we observe i.i.d. random pairs $(L_i, b_i)$ for $i = 1, \ldots, n$ that are unbiased estimates of the pair $(L, b)$ so that

$$\mathbb{E}[L_i] = L, \quad \text{and} \quad \mathbb{E}[b_i] = b.$$ (10)

In addition to this unbiasedness, we also assume that our observations satisfy a certain second-moment bound. A weaker and a stronger version of this assumption are both considered.
Assumption 1(W) (Second-moment bound in projected space). There exist scalars $\sigma_L, \sigma_b > 0$ such that for any unit-norm vector $u \in \mathbb{S}$ and any basis vector in $\{\phi_j\}_{j=1}^d$ we have the bounds
\[
\mathbb{E} \langle \phi_j, (L_i - L)u \rangle^2 \leq \sigma_L^2 \|u\|^2, \quad \text{and} \quad \mathbb{E} \langle \phi_j, b_i - b \rangle^2 \leq \sigma_b^2.
\] (11a) (11b)

Assumption 1(S) (Second-moment bound in ambient space). There exist scalars $\sigma_L, \sigma_b > 0$ such that for any unit-norm vector $u \in \mathbb{X}$ and any basis vector in $\{\phi_j\}_{j=1}^D$ we have the bounds
\[
\mathbb{E} \langle \phi_j, (L_i - L)u \rangle^2 \leq \sigma_L^2 \|u\|^2, \quad \text{and} \quad \mathbb{E} \langle \phi_j, b_i - b \rangle^2 \leq \sigma_b^2.
\] (12a) (12b)

In words, Assumption 1(W) guarantees that the random variable obtained by projecting the “noise” onto any of the basis vectors $\phi_1, \ldots, \phi_d$ in the subspace $\mathbb{S}$ has bounded second moment. Assumption 1(S) further requires the projected noise onto any basis vector of the entire space $\mathbb{X}$ to have bounded second moment. In Section 4, we show that there are various settings—including Galerkin methods and temporal difference methods—for which at least one of these assumptions is satisfied.

2.2 Examples

We now present some concrete examples to illustrate our general formulation. In particular, we discuss the problems of linear regression, temporal difference learning methods from reinforcement learning\(^4\), and Galerkin methods for solving partial differential equations.

2.2.1 Linear regression on a low-dimensional subspace

Our first example is the linear regression model when true parameter is known to lie approximately in a low-dimensional subspace. This example, while rather simple, provides a useful pedagogical starting point for the others to follow.

For this example, the underlying Hilbert space $\mathbb{X}$ from our general formulation is simply the Euclidean space $\mathbb{R}^D$, equipped with the standard inner product $\langle \cdot, \cdot \rangle$. We consider zero-mean covariates $X \in \mathbb{R}^D$ and a response $Y \in \mathbb{R}$, and our goal is to estimate the best-fitting linear model $x \mapsto \langle v, x \rangle$. In particular, the mean-square optimal fit is given by $v^* := \arg \min_{v \in \mathbb{R}^D} (Y - \langle v, X \rangle)^2$. From standard results on linear regression, this vector must satisfy the normal equations $\mathbb{E}[XX^\top]v^* = \mathbb{E}[YX]$. We assume that the second-moment matrix $\mathbb{E}[XX^\top]$ is non-singular, so that $v^*$ is unique.

Let us rewrite the normal equations in a form consistent with our problem formulation. An equivalent definition of $v^*$ is in terms of the fixed point relation
\[
v^* = \left( I - \frac{1}{\beta} \mathbb{E}[XX^\top] \right) v^* + \frac{1}{\beta} \mathbb{E}[YX],
\] (13)

where $\beta := \lambda_{\max}(\mathbb{E}[XX^\top])$ is the maximum eigenvalue. This fixed point condition is a special case of our general equation (1) with the operator $L = I - \frac{1}{\beta} \mathbb{E}[XX^\top]$ and vector $b = \frac{1}{\beta} \mathbb{E}[YX]$. Note that we have
\[
\|L\|_{op} = \|I - \frac{1}{\beta} \mathbb{E}[XX^\top]\|_{op} \leq 1 - \frac{\mu}{\beta} < 1,
\]

\(^4\) As noted by Bradtke and Barto [BB96], this method can be understood as an instrumental variable method [Woo16], and our results also apply to this more general setting.
where \( \mu = \lambda_{\text{min}}(\mathbb{E}[XX^\top]) > 0 \) is the minimum eigenvalue of the covariance matrix.

In the well-specified setting of linear regression, we observe i.i.d. pairs \((X_i, Y_i) \in \mathbb{R}^D \times \mathbb{R}\) that are linked by the standard linear model

\[
Y_i = \langle v^*, X_i \rangle + \varepsilon_i, \quad \text{for } i = 1, 2, \ldots, n,
\]

where \( \varepsilon_i \) denotes zero-mean noise with finite second moment. Each such observation can be used to form the matrix-vector pair

\[
L_i = I - \beta^{-1}X_iX_i^\top, \quad \text{and} \quad b_i = \beta^{-1}X_iY_i,
\]

which is in the form of our assumed observation model.

Thus far, we have simply reformulated linear regression as a fixed point problem. In order to bring in the projected aspect of the problem, let us suppose that the ambient dimension \( D \) is much larger than the sample size \( n \), but that we have the prior knowledge that \( v^* \) lies (approximately) within a known subspace \( \mathbb{S} \) of \( \mathbb{R}^D \), say of dimension \( d \ll D \). Our goal is then to approximate the solution to the associated projected fixed-point equation.

Using \( \{\phi_j\}_{j=1}^d \) to denote an orthonormal basis of \( \mathbb{S} \), the population-level projected linear equation (2) in this case takes the form

\[
\mathbb{E} \left[ (\Pi_\mathbb{S}X)(\Pi_\mathbb{S}X)^\top \right] \sigma = \mathbb{E} [Y \cdot \Pi_\mathbb{S}X],
\]

Thus, the population-level projected problem (15) corresponds to performing linear regression using the projected version of the covariates, thereby obtaining a vector of weights \( \hat{\sigma} \in \mathbb{S} \) in this low-dimensional space.

### 2.2.2 Galerkin methods for second-order elliptic equations

We now turn to the Galerkin method for solving differential equations, a technique briefly introduced in Section 1. The general problem is to compute an approximate solution to a partial differential equation based on a limited number of noisy observations for the coefficients. Stochastic inverse problems of this type arise in various scientific and engineering applications [Nic17, AMOS19].

For concreteness, we consider a second-order elliptic equation with Dirichlet boundary conditions.\(^5\) Given a bounded, connected and open set \( \Omega \subseteq \mathbb{R}^m \) with unit Lebesgue measure, let \( \partial \Omega \) denote its boundary. Consider the Hilbert space of functions

\[
\mathbb{X} := \left\{ v : \Omega \to \mathbb{R}, \int_\Omega \|
abla v(x)\|^2 dx < \infty, \; v|_{\partial \Omega} = 0 \right\}
\]

equipped with the inner product \( \langle u, v \rangle_{\mathbb{H}^1} := \int_\Omega \nabla u(x)^\top \nabla v(x) dx \).

Given a symmetric matrix-valued function \( a \) and a square-integrable function \( f \in \mathbb{L}^2 \), the boundary-value problem is to find a function \( v : \Omega \to \mathbb{R} \) such that

\[
\begin{cases}
\nabla \cdot (a(x)\nabla v(x)) + f = 0 & \text{in } \Omega, \\
v(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

\(^5\)It should be noted that Galerkin methods apply to a broader class of problems, including linear PDEs of parabolic and hyperbolic type [LT08], as well as kernel integral equations [PWB09, PWB12].
We impose a form of uniform ellipticity by requiring that \( \mu I_m \preceq a(x) \preceq \beta I_m \), for some positive scalars \( \mu \leq \beta \), valid uniformly over \( x \).

The problem can be equivalently stated in terms of the elliptic operator \( A := -\nabla \cdot (a \nabla) \); as shown in Appendix C.3.1, the pair \((A, f)\) induces a bounded, self-adjoint linear operator \( \tilde{A} \) on \( X \) and a function \( g \in X \) such that the solution to the boundary value problem can be written as

\[
v^* = \left( I - \frac{1}{\beta} \tilde{A} \right) v^* + \frac{1}{\beta} g.
\]

By construction, this is now an instance of our general fixed point equation (1) with \( L := I - \frac{1}{\beta} \tilde{A} \) and \( b := \beta^{-1} g \). Furthermore, our assumptions imply that \( \|L\|_X \leq 1 - \frac{\mu}{\beta} \).

We consider a stochastic observation model that is standard in the literature (see, e.g., the paper [GN20]). Independently for each \( i \in [n] \), let \( W_i \) denote an \( m \times m \) symmetric random matrix with entries on the diagonal and upper-diagonal given by i.i.d. standard Gaussian random variables. Let \( w'_i \sim \mathcal{N}(0, 1) \) denote a standard Gaussian random variable. Suppose now that we observe the pair \( x_i, y_i \sim \mathcal{U}(\Omega) \); the observed values for the \( i \)-th sample are then given by

\[
(a_i, f_i) := (a(x_i) + W_i, f(y_i) + w'_i) \quad \text{with} \quad x_i, y_i \sim \mathcal{U}(\Omega).
\]

The unbiased observations \((L_i, b_i)\) can then be constructed by replacing \((a, f)\) with \((a_i\delta_{x_i}, f_i\delta_{y_i})\) in the constructions above.

For such problems, the finite-dimensional projection not only serves as a fast and cheap way to compute solutions from simulation [LWK20], but also makes the solution stable and robust to noise [KKV11]. Given a finite-dimensional linear subspace \( S \subseteq X \) spanned by orthogonal basis functions \((\phi_i)_{i=1}^d\), we consider the projected version of equation (17), with solution denoted by \( \bar{v} \):

\[
\bar{v} = \Pi_S(L\bar{v} + b).
\]

Straightforward calculation in conjunction with Lemma 17 shows that equation (19) is equivalent to the conditions \( \bar{v} \in S \), and

\[
\langle \tilde{A}\bar{v}, \phi_j \rangle_{H^1} = \langle g, \phi_j \rangle_{H^1} \quad \text{for all} \quad j \in [d],
\]

with the latter equality better known as the Galerkin orthogonality condition in the literature [BS07].

\subsection*{2.2.3 Temporal difference methods for policy evaluation}

Our final example involves the policy evaluation problem in reinforcement learning. This is a special case of an instrumental variable method, as briefly introduced in Section 1. We require some additional terminology to describe the problem of policy evaluation. Consider a Markov chain on a state space \( S \) and a transition kernel \( P : S \times S \to \mathbb{R} \). It becomes a discounted Markov reward process when we introduce a reward function \( r : S \to \mathbb{R} \), and discount factor \( \gamma \in (0, 1) \). The goal of the policy evaluation problem to estimate the value function, which is the expected, long-term, discounted reward accrued by running the process. The value function exists under mild assumptions such as boundedness of the reward, and is given by the solution to the Bellman equation \( v^* = \gamma P v^* + r \), which is a fixed point equation of the form (1) with \( L = \gamma P \) and \( b = r \).
Throughout our discussion, we assume that the transition kernel $P$ is ergodic and aperiodic, so that its stationary distribution $\xi$ is unique. We define $X$ to be the Hilbert space $L^2(S, \xi)$, and for any pair of vectors $v, v' \in X$, we define the inner product as follows

$$\langle v, v' \rangle := \int_S v(s)v'(s)d\xi(s).$$

In the special case of a finite state space, the Hilbert space $X$ is a finite-dimensional Euclidean space with dimension $D = |S|$ and equipped with a weighted $\ell_2$-norm.

We consider the i.i.d. observation model in this paper. For each $i = 1, 2, \cdots, n$, suppose that we observe an independent tuple $(s_i, s_i^+, R_i(s_i))$, such that

$$s_i \sim \xi, \quad s_i^+ \sim P(s_i, \cdot), \quad \text{and} \quad \mathbb{E}[R_i(s_i)|s_i] = r(s_i). \quad (21)$$

The $i$-th observation $(L_i, b_i)$ is then obtained by plugging in these observations to compute unbiased estimates of $P$ and $r$, respectively.

A common practice in reinforcement learning is to employ function approximation, which in its simplest form involves solving a projected linear equation on a subspace. In particular, consider a set $\{\psi_1, \psi_2, \cdots, \psi_d\}$ of basis functions in $X$, and suppose that they are linearly independent on the support of $\xi$. We are interested in projections onto the subspace $S = \text{span}(\psi_1, \ldots, \psi_d)$, and in solving the population-level projected fixed point equation (2), which takes the form

$$\bar{v} = \Pi_S(\gamma P\bar{v} + r). \quad (22)$$

Equation (22) is the population relation underlying the canonical least squares temporal difference (LSTD) learning method [BB96, Boy02].

### 3 Main results for general projected linear equations

Having set-up the problem and illustrated it with some examples, we now turn to the statements of our main results. We begin in Section 3.1 by stating an upper bound on the mean-squared error of a stochastic approximation scheme that uses Polyak–Ruppert averaging. We then discuss the form of this upper bound for various classes of operator $L$, with a specific focus on producing transparent bounds on the approximation factor. Section 3.2 is devoted to information-theoretic lower bounds that establish the sharpness of our upper bound.

---

As mentioned before, we undertake a more in-depth study of the case with Markov observations, which is particularly relevant to the MRP example, in a companion paper.
3.1 Upper bounds

In this section, we describe a standard stochastic approximation scheme for the problem based on combining ordinary stochastic updates with Polyak–Ruppert averaging [Pol90, PJ92, Rup88]. In particular, given an oracle that provides observations \((L_t, b_t)\), consider the stochastic recursion parameterized by a positive stepsize \(\eta\):

\[
v_{t+1} = (1 - \eta)v_t + \eta \Pi_S(L_{t+1}v_t + b_{t+1}), \quad \text{for } t = 1, 2, \ldots.
\]

This is a standard stochastic approximation scheme for attempting to solve the projected fixed point relation. In order to improve it, we use the standard device of applying Polyak–Ruppert averaging so as to obtain our final estimate. For a given sample size \(n \geq 2\), our final estimate \(\hat{v}_n\) is given by taking the average of these iterates from time \(n_0\) to \(n\)—that is

\[
\hat{v}_n := \frac{1}{n - n_0} \sum_{t=n_0+1}^{n} v_t.
\]

Here the “burn-in” time \(n_0\) is an integer parameter to be specified.

The stochastic approximation procedure (24) is defined in the entire space \(X\); note that it can be equivalently written as iterates in the projected space \(\mathbb{R}^d\), via the recursion

\[
\vartheta_{t+1} = (1 - \eta)\vartheta_t + \eta (\Phi_d L_{t+1} \vartheta_t + \Phi_d b_{t+1}).
\]

The original iterates can be recovered by applying the adjoint operator—that is, \(v_t = \Phi_d^* \vartheta_t\) for \(t = 1, 2, \ldots\).

3.1.1 A finite-sample upper bound

Having introduced the algorithm itself, we are now ready to provide a guarantee on its error. Two matrices play a key role in the statement of our upper bound. The first is the \(d\)-dimensional matrix \(M := \Phi_d L \Phi_d^*\) that we introduced in Section 2.1. We show that the mean-squared error is upper bounded by the approximation error \(\inf_{v \in S} \|v - v^*\|^2\) along with a pre-factor of the form

\[
\alpha(M, s) = 1 + \lambda_{\max}\left((I - M)^{-1}(s^2 I_d - MM^T)(I - M)^{-T}\right),
\]

for \(s = \|L\|_{op}\). Our bounds also involve the quantity \(\kappa(M) = \frac{1}{2} \lambda_{\max}(M + M^T)\), which we abbreviate by \(\kappa\) when the underlying matrix \(M\) is clear from the context.

The second matrix is a covariance matrix, capturing the noise structure of our observations, given by

\[
\Sigma^* := \text{cov} (\Phi_d (b_1 - b)) + \text{cov} (\Phi_d (L_1 - L) \bar{v}).
\]

This matrix, along with the constants \((\sigma_L, \sigma_b)\) from Assumption 1(W), arise in the definition of two additional error terms, namely

\[
\mathcal{E}_n(M, \Sigma^*) := \frac{\text{trace}\left(\left((I - M)^{-1}\Sigma^*(I - M)^{-T}\right)\right)}{n}, \quad \text{and}
\]

\[
\mathcal{H}_n(\sigma_L, \sigma_b, \bar{v}) := \frac{\sigma_L}{(1 - \kappa)^3} \left(\frac{d}{n}\right)^{3/2} \left(\|\bar{v}\|^2 \sigma_L^2 + \sigma_b^2\right).
\]

As suggested by our notation, the error \(\mathcal{H}_n(\sigma_L, \sigma_b, \bar{v})\) is a higher-order term, decaying as \(n^{-3/2}\) in the sample size, whereas the quantity \(\mathcal{E}_n(M, \Sigma^*)\) is the dominant source of statistical error. With this notation, we have the following:
Theorem 1. Suppose that we are given $n$ i.i.d. observations $\{(L_i, b_i)\}_{i=1}^n$ that satisfy the noise conditions in Assumption 1(W). Then there are universal constants $(c_0, c)$ such that for any sample size $n \geq \frac{c_0 \sigma^2 L^2 d}{(1-\kappa)^2} \log^2 \left( \frac{\|\mu - \bar{v}\|^2 d}{1-\kappa} \right)$, then running the algorithm (24) with 

$$\text{stepsize } \eta = \frac{1}{c\sigma L \sqrt{d n}}, \quad \text{and burn-in period } n_0 = n/2$$

yields an estimate $\hat{v}_n$ such that 

$$E\|\hat{v}_n - v^*\|^2 \leq (1 + \omega) \alpha(M, \|L\|_\infty) \inf_{v \in \mathbb{S}} \|v - v^*\|^2 + c \left( 1 + \frac{1}{\omega} \right) \{ \mathcal{E}_n(M, \Sigma^\ast) + \mathcal{H}_n(\sigma_L, \sigma_b, \bar{v}) \}; \quad (28)$$

valid for any $\omega > 0$.

We prove this theorem in Section 5.1.

A few comments are in order. First, the quantity $\alpha(M, \|L\|_\infty) \inf_{v \in \mathbb{S}} \|v - v^*\|^2$ is an upper bound on the approximation error $\|\delta - v^*\|^2$ incurred by the (deterministic) projected fixed point $\delta$. The pre-factor $\alpha(M, \|L\|_\infty) \geq 1$ measures the instance-specific deficiency of $\delta$ relative to an optimal approximating vector from the subspace, and we provide a more in-depth discussion of this factor in Section 3.1.2 to follow. Note that Theorem 1 actually provides a family of bounds, indexed by the free parameter $\omega > 0$. By choosing $\omega$ arbitrarily close to zero, we can make the pre-factor in front of $\inf_{v \in \mathbb{S}} \|v - v^*\|^2$ arbitrarily close to $\alpha(M, \|L\|_\infty)$—albeit at the expense of inflating the remaining error terms. In Theorem 2 to follow, we prove that the quantity $\alpha(M, \|L\|_\infty)$ is, in fact, the smallest approximation factor that can be obtained in any such bound.

The latter two terms in the bound (28) correspond to estimation error that arises from estimating $\delta$ based on a set of $n$ stochastic observations. While there are two terms here in principle, we show in Corollary 1 to follow that the estimation error is dominated by the term $\mathcal{E}_n(M, \Sigma^\ast)$ under some natural assumptions. Note that the leading term $\mathcal{E}_n(M, \Sigma^\ast)$ scales with the local complexity for estimating $\delta$, and we show in Theorem 3 that this term is also information-theoretically optimal.

In the next subsection, we undertake a more in-depth exploration of the approximation factor in this problem, discussing prior work in the context of the term $\alpha(M, \|L\|_\infty)$ appearing in Theorem 1.

3.1.2 Detailed discussion of the approximation error

As mentioned in the introduction, upper bounds on the approximation factor have received significant attention in the literature, and it is interesting to compare our bounds.

Past results: In the case where $\gamma_{\max} := \|L\|_\infty < 1$, the approximation-factor bound (6) was established by Tsitsiklis and Van Roy [TVR97], via the following argument. Letting $\bar{v} := \Pi_S(\lam v^* + b)$, we have 

\begin{align*}
\|\bar{v} - v^*\|^2 &\leq \|ar{v} - \bar{v}\|^2 + \|ar{v} - v^*\|^2 = \|\Pi_S(\lam \bar{v} + b) - \Pi_S(\lam v^* + b)\|^2 + \|\bar{v} - v^*\|^2 \\
&\leq (i) \|\lam \bar{v} - \lam v^*\|^2 + \|\bar{v} - v^*\|^2 \\
&\leq (ii) \|\lam \bar{v} - \lam v^*\|^2 + \|\bar{v} - v^*\|^2 \\
&\leq (iii) \gamma_{\max}^2 \|\bar{v} - v^*\|^2 + \|\bar{v} - v^*\|^2. \quad (29)
\end{align*}
Step (i) uses Pythagorean theorem; step (ii) follows from the non-expansiveness of the projection operator; and step (iii) makes use of the contraction property of the operator $L$. Note that by definition, we have $\alpha(M, \|L\|_X) \leq (1 - \|L\|_X)^{-2}$, and so the approximation factor in Theorem 1 recovers the bound (6) in the worst case. In general, however, the factor $\alpha(M, \|L\|_X)$ can be significantly smaller.

Yu and Bertsekas [YB10] derived two fine-grained approximation factor upper; in terms of our notation, their bounds take the form

$$\alpha^{(1)}_{YB} := 1 + \|L\|^2_X \cdot \lambda_{\max} \left( (I - M)^{-1}(I - M)^{-\top} \right),$$

$$\alpha^{(2)}_{YB} := 1 + \|(I - \Pi S)L^{-1}\Pi S\Pi S^\perp\|^2_L.$$

It is clear from the definition that $\alpha(M, \|L\|_X) \leq \alpha^{(1)}_{YB}$, but $\alpha(M, \|L\|_X)$ can often provide an improved bound. This improvement is indeed significant, as will be shown shortly in Lemma 1. On the other hand, the term $\alpha^{(2)}_{YB}$ is never larger than $\alpha(M, \|L\|_X)$, and is indeed the smallest possible bound that depends only on $L$ and not $b$. However, as pointed out by Yu and Bertsekas, the value of $\alpha^{(2)}_{YB}$ is not easily accessible in practice, since it depends on the precise behavior of the operator $L$ over the orthogonal complement $S^\perp$. Thus, estimating the quantity $\alpha^{(2)}_{YB}$ requires $O(D)$ samples. In contrast, the term $\alpha(M, \|L\|_X)$ depends only on the projected operator $M$ and the operator norm $\|L\|_X$. The former can be easily estimated using $d$ samples and at smaller computational cost, while the latter is usually known a priori. The discussion in Section 4 to follow fleshes out these distinctions.

**A simulation study:** In order to compare different upper bounds on the approximation factor, we conducted a simple simulation study on the problem of value function estimation, as previously introduced in Section 2.2.3. For this problem, the approximation factor $\alpha(M, \gamma)$ is computed more explicitly in Corollary 5. The Markov transition kernel is given by the simple random walk on a graph. We consider Gaussian random feature vectors and associate them with two different random graph models, Erdös-Rényi graphs and random geometric graphs, respectively. The details for these models are described and discussed in Appendix D.

In Figure 1, we show the simulation results for the values of the approximation factor. Given a sample from above graphs and feature vectors, we plot the value of $\alpha(M, \gamma)$, $\alpha^{(1)}_{YB}$ and $\alpha^{(2)}_{YB}$ against the discount rate $1 - \gamma$, which ranges from $10^{-5}$ to $10^{-0.5}$. Note that the two plots use different scales: Panel (a) is a linear-log plot, whereas panel (b) is a log-log plot. Figure 1 shows that the approximation factor $\alpha(M, \gamma)$ derived in Theorem 1 is always between $\alpha^{(1)}_{YB}$ and $\alpha^{(2)}_{YB}$. As mentioned before, the latter quantity depends on the particular behavior of the linear operator $L$ in the subspace $S^\perp$, which can be difficult to estimate. The improvement over $\alpha^{(1)}_{YB}$, on the other hand, can be significant.

In the Erdös-Rényi model, all the three quantities are bounded by relatively small constant, regardless of the value of $\gamma$. The bound $\alpha(M, \gamma)$ is roughly at the midpoint between the bounds $\alpha^{(1)}_{YB}$ and $\alpha^{(2)}_{YB}$. The differences are much starker in the random geometric graph case: The bound improves over $\alpha^{(1)}_{YB}$ by several orders of magnitude, while being off from $\alpha^{(2)}_{YB}$ by a factor of 10 for large $\gamma$. As we discuss shortly in Lemma 1, this is because the approximation factor $\alpha(M, \gamma)$ scales as $O\left(\frac{1}{1 - \kappa(M)}\right)$ while $\alpha^{(1)}_{YB}$ scales as $O\left(\frac{1}{(1 - \kappa(M))^2}\right)$, making a big difference for the case with large correlation.
Figure 1. Plots of various approximation factor as a function of the discount factor $\gamma$ in the policy evaluation problem. (See the text for a discussion.) (a) Results for an Erdős-Rényi random graph model with $N = 3000$, projected dimension $d = 1000$, and $a = 3$. The resulting number of vertices in the graph $\tilde{G}$ is 2813. The value of $1 - \gamma$ is plotted in log-scale, and the value of approximation factor is plotted on the standard scale. (b) Results for a random geometric graph model with $N = 3000$, projected dimension $d = 2$, and $r = 0.1$. The resulting number of vertices in the graph $\tilde{G}$ is 2338. Both the discount rate $1 - \gamma$ and the approximation factor are plotted on the log-scale.

Some useful bounds on $\alpha(M, \|L\|_X)$: We conclude our discussion of the approximation factor with some bounds that can be derived under different assumptions on the operator $L$ and its projected version $M$. The following lemma is useful in understanding the behavior of the approximation factor as a function of the contractivity properties of the operator $L$; this is particularly useful in analyzing convergence rates in numerical PDEs.

Lemma 1. Consider a projected matrix $M \in \mathbb{R}^{d \times d}$ such that $(I - M)$ is invertible and $\kappa(M) < 1$.

(a) For any $s > 0$, we have the bound

$$\alpha(M, s) \leq 1 + \|(I - M)^{-1}\|_{\text{op}}^2 \cdot s^2 \leq 1 + \frac{s^2}{(1 - \kappa(M))^2}. \quad (30a)$$

(b) For $s \in [0, 1]$, we have

$$\alpha(M, s) \leq 1 + 2\|(I - M)^{-1}\|_{\text{op}} \leq 1 + \frac{2}{1 - \kappa(M)}. \quad (30b)$$

See Appendix B.1 for the proof of this lemma.

A second special case, also useful, is when the matrix $M$ is symmetric, a setting that appears in least-squares regression, value function estimation in reversible Markov chains, and self-adjoint elliptic operators. The optimal approximation factor $\alpha(M, \gamma_{\text{max}})$ can be explicitly computed in such cases.

Lemma 2. Suppose that $M$ is symmetric with eigenvalues $\{\lambda_j(M)\}_{j=1}^d$ such that $\lambda_{\text{max}}(M) < 1$. Then for any $s > 0$, we have

$$\alpha(M, s) = 1 + \max_{j=1,\ldots,d} \frac{s^2 - \lambda_j^2}{(1 - \lambda_j)^2}. \quad (31)$$
Lemma 1 reveals that there is a qualitative shift between the non-expansive case $\|L\|_X \leq 1$ and the complementary expansive case. In the latter case, the optimal approximation factor always scales as $O\left(\frac{1}{(1-\kappa(M))^2}\right)$, but below the threshold $\|L\|_X = 1$, the approximation factor drastically improves to become $O\left(\frac{1}{1-\kappa(M)}\right)$. It is worth noting that both bounds can be achieved up to universal constant factors. In the context of differential equations, the bound of the form (a) in Lemma 1 is known as Céa’s lemma [Céa64], which plays a central role in the convergence rate analysis of the Galerkin methods for numerical differential equations. However, the instance-dependent approximation factor $\alpha(M, \|L\|_X)$ can often be much smaller: the global coercive parameter needed in Céa’s estimate is replaced by the bounds on the behavior of the operator $L$ in the finite-dimensional subspace. The part (b) in Lemma 1 generalizes Céa’s energy estimate from the symmetric positive-definite case to the general non-expansive setting. See Corollary 4 for a more detailed discussion on the consequences of our results to elliptic PDEs.

Lemmas 1 and 2 yield the following corollary of the general bound (28) under different conditions on the operator $L$.

**Corollary 1.** Under the conditions of Theorem 1 and given a sample size $n \geq \frac{c_0\sigma^2 d}{(1-\kappa)^2} \log^2 \left(\frac{\|v_0 - v^*\|^2 d}{1-\kappa}\right)$:

(a) There is a universal positive constant $c$ such that

$$
\mathbb{E}\|\hat{v}_n - v^*\|^2 \leq c \left\{ \frac{\|L\|_X^2}{(1-\kappa(M))^2} \cdot \inf_{v \in S} \|v - v^*\|^2 + \left(\frac{\sigma^2 + \|\sigma\|}{(1-\kappa(M))^2}\right) \frac{d}{n} \right\}
$$

for any operator $L$, and its associated projected operator $M = \Phi_d L \Phi_d$.

(b) Moreover, when $L$ is non-expansive ($\|L\|_X \leq 1$), we have

$$
\mathbb{E}\|\hat{v}_n - v^*\|^2 \leq c \left\{ \frac{1}{1-\kappa(M)} \cdot \inf_{v \in S} \|v - v^*\|^2 + \left(\frac{\sigma^2 + \|\sigma\|}{(1-\kappa(M))^2}\right) \frac{d}{n} \right\}.
$$

See Section 5.2 for the proof of this claim.

As alluded to before, the simplified form of Corollary 1 no longer has an explicit higher order term, and the statistical error now scales at the parametric rate $d/n$. It is worth noting that the lower bound on $n$ required in the assumption of the corollary is a mild requirement: in the absence of such a condition, the statistical error term $\left(\frac{\sigma^2 + \|\sigma\|}{(1-\kappa)^2}\right) \frac{d}{n}$ in both bounds would blow up, rendering the guarantee vacuous.

### 3.2 Lower bounds

In this section, we establish information-theoretic lower bounds on the approximation factor, as well as the statistical error. Our eventual result (in Corollary 2) shows that the first two terms appearing in Theorem 1 are both optimal in a certain instance-dependent sense. However, a precise definition of the local neighborhood of instances over which the lower bound holds requires some definitions. In order to motivate these definitions more transparently and naturally arrive at both terms of the bound, the following section presents individual bounds on the approximation and estimation errors, and then combines them to obtain Corollary 2.
3.2.1 Lower bounds on the approximation error

As alluded to above, the first step involved in a lower bound is a precise definition of the collection of problem instances over which it holds; let us specify a natural such collection for lower bounds on the approximation error. Each problem instance is specified by the joint distribution of the observations \((L_i, b_i)\), which implicitly specifies a pair of means \((L, b) = (\mathbb{E}[L_i], \mathbb{E}[b_i])\). For notational convenience, we define this class by first defining a collection comprising instances specified solely by the mean pair \((L, b)\), and then providing restrictions on the distribution of \((L_i, b_i)\). Let us define the first such component. For a given matrix \(M_0 \in \mathbb{R}^{d \times d}\) and vector \(h_0 \in \mathbb{R}^d\), write

\[
C_{\text{approx}}(M_0, h_0, D, \delta, \gamma_{\text{max}}) := \left\{ (L, b) \mid \|L\|_X \leq \gamma_{\text{max}}, \ A(S, v^*) \leq \delta^2, \ \dim(X) = D, \ \Phi_d L \Phi_d^* = M_0, \ \text{and} \ \Phi_d h = h_0. \right\}.
\]

In words, this is a collection of all instances of the pair \((L, b) \in \mathcal{L} \times \mathbb{R}^D\) whose projections onto the subspace of interest are fixed to be the pair \((M_0, h_0)\), and whose approximation error is less than \(\delta^2\). In addition, the operator \(L\) satisfies a certain bound on its operator norm.

Having specified a class of \((L, b)\) pairs, we now turn to the joint distribution over the pair of observations \((L_i, b_i)\), which we denote for convenience by \(\mathbb{P}_{L, b}\). Now define the collection of instances

\[
G_{\text{var}}(\sigma_L, \sigma_b) := \left\{ \mathbb{P}_{L, b} \mid (L_i, b_i) \text{ satisfies Assumption 1(S) with constants } (\sigma_L, \sigma_b) \right\}.
\]

This is simply the class of all distributions such that our observations satisfy Assumption 1(S) with pre-specified constants. As a point of clarification, it is useful to recall that our upper bound in Theorem 1 only needed Assumption 1(W) to hold, and we could have chosen to match this by defining the \(G_{\text{var}}\) under Assumption 1(W). We comment further on this issue following the theorem statement.

We are now ready to state Theorem 2, which is a lower bound on the worst-case approximation factor over all problem instances such that \((L, b) \in C_{\text{approx}}(M_0, h_0, D, \delta, \gamma_{\text{max}})\) and \(\mathbb{P}_{L, b} \in G_{\text{var}}(\sigma_L, \sigma_b)\). Note that such a collection of problem instances is indeed local around the pair \((M_0, h_0)\). Two settings are considered in the statement of the theorem: proper estimators when \(\hat{v}_n\) is restricted to take values in the subspace \(S\); and improper estimators, where \(\hat{v}_n\) can take values in the entire space \(X\). We use \(\hat{V}_S\) and \(\hat{V}_X\) to denote the class of proper and improper estimators, respectively. Finally, we use the shorthand \(C_{\text{approx}} \equiv C_{\text{approx}}(M_0, h_0, D, \delta, \gamma_{\text{max}})\) for convenience.

**Theorem 2.** Suppose \(M_0 \in \mathbb{R}^{d \times d}\) is a matrix such that \(I - M_0\) is invertible, and that the scalars \((\sigma_L, \sigma_b)\) are such that \(\sigma_L \geq \gamma_{\text{max}}\) and \(\sigma_b \geq \delta\). If the ambient dimension satisfies \(D \geq d + \frac{12}{\omega} n^2\) for some scalar \(\omega \in (0, 1)\), then we have the lower bounds

\[
\inf_{\hat{v}_n \in \hat{V}_S} \sup_{\mathbb{P}_{L, b} \in C_{\text{approx}}} \mathbb{E} \|\hat{v}_n - v^*\|^2 \geq (1 - \omega) \cdot \alpha(M_0, \gamma_{\text{max}}) \cdot \delta^2 \quad \text{and} \quad (33a)
\]

\[
\inf_{\hat{v}_n \in \hat{V}_X} \sup_{\mathbb{P}_{L, b} \in C_{\text{approx}}} \mathbb{E} \|\hat{v}_n - v^*\|^2 \geq (1 - \omega) \cdot (\alpha(M_0, \gamma_{\text{max}}) - 1) \cdot \delta^2. \quad (33b)
\]

See Section 5.3 for the proof of this claim.

A few remarks are in order. First, Theorem 2 shows that the approximation factor upper bound in Theorem 1 is information-theoretically optimal in the instance-dependent sense.
in the case of proper estimators, the upper and lower bound can be made arbitrarily close by choosing the constant $\omega$ arbitrarily small in both theorems. Both bounds depend on the projected matrix $M_0$, characterizing the fundamental impact of the geometry in the projected space on the complexity of the estimation problem. The lower bound for improper estimators is slightly smaller, but for most practical applications we have $\alpha(M_0, \gamma_{\text{max}}) \gg 1$ and so this result should be viewed as almost equivalent.

Second, note that we may also extract a worst-case lower bound on the approximation factor from Theorem 2. Indeed, for a scalar $\gamma_{\text{max}} \in (0, 1)$, consider the family of instances in the aforementioned problem classes satisfying $\|L\|_\infty \leq \gamma_{\text{max}}$. Setting $M_0 = \frac{\gamma_{\text{max}}^2}{2} I_d$ and applying Theorem 2, we see that (in a worst-case sense over this class), the risk of any estimator is lower bounded by $\frac{1}{1 - \gamma_{\text{max}}} A(S, v^*)$. This establishes the optimality of the classical worst-case upper bound (6).

Third, notice that theorem requires the noise variances $(\sigma_L, \sigma_b)$ to be large enough, and this is a natural requirement in spite of the fact that we seek lower bounds on the approximation error. Indeed, in the extreme case of noiseless observations, we have access to the population pair $(L, b)$ with a single sample, and can compute both $v^*$ and its projection onto the subspace $S$ without error. From a more quantitative standpoint, it is worth noting that our requirements $\sigma_L \geq \gamma_{\text{max}}$ and $\sigma_b \geq \delta$ are both mild, since the scalars $\gamma_{\text{max}}$ and $\delta$ are typically order 1 quantities. Indeed, if both of these bounds held with equality, then Corollary 1 yields that the statistical error would be of the order $O(d/n)$, and so strictly smaller than the approximation error we hope to capture.

Observe that Theorem 2 requires the ambient dimension $D$ to be larger than $n^2$. As mentioned in the introduction, we should not expect any non-trivial approximation factor when $n \geq D$, but this leaves open the regime $n \ll D \ll n^2$. Is a smaller approximation factor achievable when $D$ is not extremely large? We revisit this question in Section 3.2.4, showing that while there are some quantitative differences in the lower bound, the qualitative nature of the message remains unchanged.

Regarding our noise assumptions, it should be noted that the class of instances satisfying Assumption 1(W) is strictly larger than the corresponding class satisfying Assumption 1(S), and so our lower bound to follow extends immediately to the former case. Second, it is important to note that imposing only Assumption 1(W) would in principle allow the noise in the orthogonal complement $S^\perp$ to grow in an unbounded fashion, and one should expect that it is indeed optimal to return an estimate of the projected fixed point $\bar{v}$. To establish a more meaningful (but also more challenging) lower bound, we operate instead under the stronger Assumption 1(S), enforcing second moment bounds on the noise not only for basis vectors in $S$, but also its orthogonal complement. Assumption 1(S) allows for other natural estimators: For instance, the plug-in estimator of $v^*$ via the original fixed point equation (1) would now incur finite error. Nevertheless, as shown by our lower bound, the stochastic approximation estimator analyzed in Theorem 1 is optimal even if the noise in $S^\perp$ behaves as well as that in $S$.

\footnote{As a side remark, we note that our noise conditions can be further weakened, if desired, via a mini-batching trick. To be precise, given any problem instance $P_{L, b} \in G_{\text{var}}(\sigma_L, \sigma_b)$ and any integer $m > 0$, one could treat the sample mean of $m$ independent samples as a single sample, resulting in a problem instance in the class $G_{\text{var}}(\frac{\sigma_L}{\sqrt{m}}, \frac{\sigma_b}{\sqrt{m}})$. The same lower bound still applies to the class $G_{\text{var}}(\frac{\sigma_L}{\sqrt{m}}, \frac{\sigma_b}{\sqrt{m}})$, at a cost of stronger dimension requirement $D \geq d + \frac{12}{\omega^2} m^2$.}
3.2.2 Lower bounds on the estimation error

We now turn to establishing a minimax lower bound on the estimation error that matches the statistical error term in Theorem 1. This lower bound takes a slightly different form from Theorem 2: rather than studying the total error $\|\hat{v}_n - v^*\|$ directly, we establish a lower bound on the error $\|\hat{v}_n - \bar{v}\|$ instead.

Indeed, the latter term is more meaningful to study in order to characterize the estimation error—which depends on the sample size $n$—since for large sample sizes, the total error $\|\hat{v}_n - v^*\|$ will be dominated by a constant approximation error. As we demonstrate shortly, the term $\|\hat{v}_n - \bar{v}\|$ depends on noise covariance and the geometry of the matrix $M_0$ in the projected space, while having the desired dependence on the sample size $n$. It is worth noting also that this automatically yields a lower bound on the error $\|\hat{v}_n - v^*\|$ when we have $\bar{v} = v^*$.

We are now ready to prove a local minimax lower bound for estimating $\bar{v} \in \mathbb{S}$, which is given by the solution to the projected linear equation $\bar{v} = \Pi_{\mathbb{S}}(L\bar{v} + b)$. While our objective is to prove a local lower bound around each pair $(L_0, b_0) \in \mathcal{L} \times \mathbb{X}$, the fact that we are estimating $\bar{v}$ implies that it suffices to define our set of local instances in the $d$-dimensional space of projections. In particular, our means $(L, b)$ are specified by those pairs for which $\Phi_d L \Phi_d^*$ is close to $M_0 := \Phi_d L_0 \Phi_d^*$, and $\Phi_d b$ is close to $h_0 := \Phi_d b_0$. Specifically, let $\tilde{v}_0$ denote the solution to the projected linear equation $\tilde{v}_0 = \Pi_{\mathbb{S}}(L_0 \tilde{v}_0 + b_0)$, and define the neighborhood

$$\mathcal{N}(M_0, h_0) := \left\{ (M', h') : \|M' - M_0\|_F \leq \sigma_L \sqrt{\frac{d}{n}}, \text{ and } \|h' - h_0\|_2 \leq \sigma_b \sqrt{\frac{d}{n}} \right\},$$

which, in turn, defines a local class of problem instances $(L, b)$ given by

$$\mathcal{C}_{est} := \left\{ (L, b) \mid (\Phi_d L \Phi_d^*, \Phi_d b) \in \mathcal{N}(M_0, h_0) \right\}.$$

We have thus specified our local neighborhood in terms of the mean pair $(L, b)$, and as before, it remains to define a local class of distributions on these instances. Toward this end, define the class

$$G_{cov}(\Sigma_L, \Sigma_b, \sigma_L, \sigma_b)$$

$$:= G_{var}(\sigma_L, \sigma_b) \cap \left\{ P_{L,b} : \text{cov}(\Phi_d(b_1 - b)) \preceq \Sigma_b \quad \text{and} \quad \text{cov}(\Phi_d(L_1 - L) \tilde{v}_0) \preceq \Sigma_L \right\},$$

corresponding to distributions on the observation pair $(L_i, b_i)$ that satisfy Assumption 1(S) and whose “effective noise” covariances are dominated by the PSD matrices $\Sigma_L$ and $\Sigma_b$.

Note that Assumption 1(S) implies the diagonal elements of above two covariance matrices are bounded by $\sigma_b^2$ and $\sigma_L^2 \|\tilde{v}_0\|^2$, respectively. In order to avoid conflicts between assumptions, we assume throughout that for all indices $j \in [d]$, the diagonal entries of the covariance matrices satisfy the conditions

$$\begin{align*}
\langle \Sigma_b \rangle_{j,j} &\leq \sigma_b^2 \quad \text{and} \quad \langle \Sigma_L \rangle_{j,j} \leq \sigma_L^2 \|\tilde{v}_0\|^2.
\end{align*}$$

We then have the following theorem for the estimation error $\|\hat{v}_n - \bar{v}\|$, where we use the shorthand $G_{cov} \equiv G_{cov}(\Sigma_L, \Sigma_b, \sigma_L, \sigma_b)$ for brevity.

**Theorem 3.** Under the setup above, suppose the matrix $I - M_0$ is invertible, and suppose that $n \geq 16 \sigma_L^2 \|I - M_0\|^{-1}_{op}^2 d$. Then there is a universal constant $c > 0$ such that

$$\inf_{\hat{v}_n} \sup_{(L,b) \in \mathcal{C}_{est}} \mathbb{E}\|\hat{v}_n - \bar{v}\|^2 \geq c \cdot \mathcal{E}_n(M_0, \Sigma_L + \Sigma_b).$$
See Section 5.4 for the proof of this claim.

The estimation error lower bound in Theorem 3 matches the statistical error term \( \mathcal{E}_\alpha(M, \Sigma^*) \) in Theorem 1, up to a universal constant. Indeed, in the asymptotic limit \( n \to \infty \), the regularity of the problem can be leveraged in conjunction with classical Le Cam theory (see, e.g., [vdV00]) to show that the asymptotic optimal limiting distribution is a Gaussian law with covariance \((I - M)^{-1}\Sigma^*(I - M)^{-T}\). (See the paper [KPR+20] for a detailed analysis of this type in the special case of policy evaluation in tabular MDPs.) This optimality result holds in a “local” sense: it is minimax optimal in a small neighborhood of radius \( O(1/\sqrt{n}) \) around a given problem instance \((M_0, h_0)\). Theorem 3, on the other hand, is non-asymptotic, showing that a similar result holds provided \( n \) is lower bounded by an explicit, problem-dependent quantity of the order \( \sigma_L^2 d \|(I - M_0)^{-1}\|_{op}^2 \). This accommodates a broader range of sample sizes than the upper bound in Theorem 1.

### 3.2.3 Combining the bounds

Having presented separate lower bounds on the approximation and estimation errors in conjunction with definitions of local classes of instances over which they hold, we are now ready to present a corollary which combines the two lower bounds in Theorems 2 and 3.

We begin by defining the local classes of instances over which our combined bound holds. Given a matrix-vector pair \((M_0, h_0)\), covariance matrices \((\Sigma_L, \Sigma_b)\), ambient dimension \( D > 0 \), and scalars \( \delta, \gamma_{\text{max}}, \sigma_L, \sigma_b > 0 \), we begin by specifying a collection of mean pairs \((L, b)\) via

\[
\mathcal{C}_{\text{final}}(M_0, h_0, D, \delta, \gamma_{\text{max}}) := \bigcup_{(M', h') \in \mathcal{G}_{\text{approx}}(M_0, h_0)} \mathcal{C}_{\text{approx}}(M', h', D, \delta, \gamma_{\text{max}}).
\]

Clearly, this represents a natural combination of the classes \( \mathcal{C}_{\text{approx}} \) and \( \mathcal{C}_{\text{est}} \) introduced above. We use the shorthand \( \mathcal{C}_{\text{final}} \) for this class for brevity. Our collection of distributions \( \mathbb{P}_{L,b} \) is still given by the class \( \mathcal{G}_{\text{cov}} \) from equation (35).

With these definitions in hand, we are now ready to state our combined lower bound.

**Corollary 2.** Under the setup above, suppose that the pair \((\sigma_L, \sigma_b)\) satisfies the conditions in Theorem 2 and equation (36), and that the matrix \( M_0 \) satisfies \( \|M_0\|_{op} \leq \gamma_{\text{max}} - \sigma_L \sqrt{d/n} \). Moreover, suppose that the sample size and ambient dimension satisfy \( n \geq 16\sigma_L^2 \|(I - M_0)^{-1}\|_{op}^2 d \) and \( D \geq d + 36n^2 \), respectively. Then the following minimax lower bound holds for a universal positive constant \( c \):

\[
\inf_{\tilde{v}_n \in \tilde{V}_N} \sup_{(L,b) \in \mathcal{C}_{\text{final}}} \mathbb{E}\|\tilde{v}_n - v^*\|^2 \geq c \cdot \left\{ \alpha(M_0, \gamma_{\text{max}}) - 1 \right\} \cdot \delta^2 + \mathcal{E}_\alpha(M_0, \Sigma_L + \Sigma_b).
\]

We prove this corollary in Section 5.5. It is a relatively straightforward consequence of combining Theorems 2 and 3.

The combined lower bound matches the expression \( \alpha(M_0, \gamma_{\text{max}})A(S, v^*) + \mathcal{E}_\alpha(M_0, \Sigma_L + \Sigma_b) \), given by the first two terms of Theorem 1, up to universal constant factors. Recall from our discussion of Theorem 1 that the high-order term \( \mathcal{H}_{\text{est}}(\sigma_L, \sigma_b, \tilde{v}) \) represents the “optimization error” of the stochastic approximation algorithm, which depends on the coercive condition \( \kappa(M_0) \) instead of the natural geometry \( I - M_0 \) of the problem. While we do not expect this term to appear in an information-theoretic lower bound, the leading estimation error term \( \mathcal{E}_\alpha(M_0, \Sigma_L + \Sigma_b) \) will dominate the high-order term when the sample size \( n \) is large.
We now discuss the consequences of our main theorems for the three examples introduced in Theorem 4. Suppose Assumption 1(S), with $\kappa(M) \leq \kappa$ and $\|L\|_X \leq 1$. Then the bound in Theorem 1 is optimal, in a worst-case sense, over this class as long as the sample size exceeds the threshold $\frac{\sigma^2}{(1-\kappa)^2} d$.

### 3.2.4 The intermediate regime

It remains to tie up some loose ends. Note that the lower bound in Theorem 2 requires a condition $D \gg n^2$. On the other hand, it is easy to see that the approximation factor can be made arbitrarily close to 1 when $n \gg D$. (For example, one could run the estimator based on stochastic approximation and averaging—which was analyzed in Theorem 1—with the entire Euclidean space $\mathbb{R}^d$, and project the resulting estimate onto the subspace $\mathbb{S}$.) In the middle regime $n \ll D \ll n^2$, however, it is not clear which estimator is optimal.

In the following theorem, we present a lower bound for the approximation factor in this intermediate regime, which establishes the optimality of Theorem 1 up to a constant factor.

**Theorem 4.** Suppose $M_0 \in \mathbb{R}^{d \times d}$ is a matrix such that $I - M_0$ is invertible, and that the scalars $(\sigma_L, \sigma_b)$ satisfy $\sigma_L \geq 1 + \gamma_{\text{max}}$ and $\sigma_b \geq \delta$. If the ambient dimension satisfies $D \geq d + 3q n^{1+1/q}$ for some integer $q \in \left[2, \log n \wedge \frac{1}{\sqrt{2(1-\gamma_{\text{max}})^2}} \right]$, then we have the lower bound

$$\inf_{\tilde{v}_n \in V_X} \sup_{(L,b) \in C_{\text{approx}}} \mathbb{E}\|\tilde{v}_n - v^*\|^2 \geq \frac{\alpha(M_0, \gamma_{\text{max}}) - 1}{4q^2} \delta^2.$$ 

See Appendix A for the proof of this theorem.

Theorem 4 resolves the gap in the intermediate regime, up to a constant factor that depends on $q$. In particular, the stochastic approximation estimator (24) for projected equations still yields a near-optimal approximation factor. Compared to Theorem 2, Theorem 4 weakens the requirement on the ambient dimension $D$ and covers the entire regime $D \gg n$. Furthermore, using the same arguments as in Corollary 2, this theorem can also be combined with Theorem 3 to obtain the following lower bound in the regime $D \geq d + 3q n^{1+1/q}$, for any integer $q > 0$:

$$\inf_{\tilde{v}_n \in V_X} \sup_{(L,b) \in C_{\text{final}}} \mathbb{E}\|\tilde{v}_n - v^*\|^2 \geq c \cdot \left\{ \frac{\alpha(M_0, \gamma_{\text{max}}) - 1}{q^2} \cdot \delta^2 + \mathcal{E}_n(M_0, \Sigma_L + \Sigma_b) \right\}.$$ 

Let us summarize our approximation factor lower bounds in the various regimes. Consider a sequence of problem instances $(\mathbb{P}_{L,b}^{(n)})_{n=1}^\infty$ with increasing ambient dimension $D_n$. Let the projected dimension $d$, noise variances $(\sigma_L, \sigma_b)$, oracle error $\delta$, projected matrix $\Phi_d L^{(n)} \Phi_d^* = M$, and the operator norm bound $\|L\|_X \leq \gamma_{\text{max}}$ be all fixed. The following table then presents a combination of our results from Theorems 1, 2, and 4; our results suggest that the optimal approximation factor exhibits a “slow” phase transition phenomenon. It is an interesting open question whether the phase transition is sharp, and to identify the asymptotically optimal approximation factor in the regime $\lim_{n \to \infty} \log \frac{D_n}{\log n} = 1$ since our lower bounds do not apply in this linear regime.

### 4 Consequences for specific models

We now discuss the consequences of our main theorems for the three examples introduced in Section 2.2. For brevity, we state only upper bounds for the first two examples; our third
\[
q = \lim_{n \to \infty} \frac{\log L_n}{\log n}
\]

| Lower bound | Upper bound |
|-------------|-------------|
| \(\alpha(M_0, \gamma_{\max})\) | \(\alpha(M_0, \gamma_{\max})\) |
| \(c_q \cdot \alpha(M_0, \gamma_{\max})\) | 1 |
| 1 | 1 |

Table 1. Bounds on the approximation factor \(\frac{E[\|v_n - v^\ast\|^2]}{\lambda(S, \Sigma)^{\dagger}}\) for proper estimators in different ranges of ambient dimension. Here, \(c_q \in (0, 1)\) represents a constant depending only on the aspect ratio \(q\).

example for temporal difference learning methods includes both upper and lower bounds.

### 4.1 Linear regression

Recall the setting of linear regression from Section 2.2.1, including our i.i.d. observation model (14). We assume bounds on the second moment of \(\varepsilon\) and fourth moment of \(X\)—namely, the existence of some \(\varsigma > 0\) such that

\[
E(u, X)^4 \leq \varsigma^4, \quad \text{and} \quad E(\varepsilon)^2 \leq \varsigma^2 \quad \text{for all } u \in S^{d-1}.
\]  

(38)

These conditions ensure that Assumption 1(W) is satisfied with \((\sigma_L, \sigma_b) = (\beta^{-1}\varsigma^2, \beta^{-1}\varsigma^2)\).\(^8\)

Recall that the (unprojected) covariance matrix satisfies the PSD relations \(\mu I \preceq E[XX^\dagger] \preceq \beta I\), and define the \(d\)-dimensional covariance matrix \(\Sigma := E[(\Phi_\beta X)(\Phi_\beta X)^\dagger]\) for convenience.

In this case, our stochastic approximation iterates (24a) take the form

\[
v_{t+1} = v_t - \eta \left( \Pi_S X_{t+1} X_{t+1}^\dagger \Pi_S v_t + Y_{t+1} \Pi_S X_{t+1} \right), \quad \text{for all } t = 0, 1, 2, \ldots,
\]

(39)

and we take the averaged iterates \(\hat{v}_n := \frac{2}{n} \sum_{t=n/2}^{n-1} v_t\). For this procedure, we have the following guarantee:

**Corollary 3.** Suppose that we have \(n\) i.i.d. observations \(\{(X_i, Y_i)\}_{i=1}^n\) from the model (14) satisfying the moment conditions (38). Then there are universal positive constants \((c, c_0)\) such that given a sample size \(n \geq \frac{c_0^4 d}{\lambda_{\min}(\Sigma)} \log^2 \left( \frac{2}{\mu} \|v_0 - v^\ast\|_2^2 d \right)\), if the stochastic approximation scheme (39) is run with step size \(\eta = \frac{1}{c_0 \varsigma^2 \sqrt{d} n}\), then the averaged iterate satisfies the bound

\[
E[\|\hat{v}_n - v^\ast\|^2] \leq (1 + \omega) \cdot \alpha \left( I_d - \frac{\Sigma}{\beta}, 1 - \frac{\mu}{\beta} \right) \lambda(S, \Sigma)^{\dagger} + c \cdot \frac{\text{trace}(\Sigma^{-1}) \cdot E(\varepsilon)^2}{\omega n} + c \oint \frac{\varsigma^2^2}{\lambda_{\min}(\Sigma) \cdot \sqrt{d} n}^2 \quad \text{for each } \omega > 0.
\]

This result is a direct consequence of Theorem 1 in application to this model.

Note that the statistical error term \(\frac{\text{trace}(\Sigma^{-1}) \cdot E(\varepsilon)^2}{n}\) in this case corresponds to the classical statistical rates for linear regression in this low-dimensional subspace. The approximation factor, by Lemma 2 admits the closed-form expression

\[
\alpha \left( I_d - \frac{\Sigma}{\beta}, 1 - \frac{\mu}{\beta} \right) = \max_{i \in [d]} \frac{\mu^2 + 2 \beta (\lambda_i - \mu)}{\lambda_i^2},
\]

\(^8\)Note that the stochastic approximation iterates are invariant under translation, and consequently we can assume without loss of generality that \(\nu = 0\).
where \( \{\lambda_j\}_{j=1}^d \) denote the eigenvalues of the matrix \( \Sigma \). Since \( \lambda_j \in [\mu, \beta] \) for each \( j \in [d] \), the approximation factor is at most of the order \( O\left( \frac{\beta}{\lambda_{\min}(\Sigma)} \right) \).

Compared to known sharp oracle inequalities for linear regression (e.g., [RH15]), the approximation factor in our bound is not 1 but rather a problem-dependent quantity. This is because we study the estimation error under the standard Euclidean metric \( \| \cdot \|_2 \), as opposed to the prediction error under the data-dependent metric \( \| \cdot \|_{L^2(P_X)} \). When the covariance matrix \( \mathbb{E}[XX^\top] \) is identity, the approximation factor \( \alpha(I_d - \frac{\Sigma}{\beta}, 1 - \frac{d}{\beta}) \) is equal to 1, recovering classical results. Another error metric of interest, motivated by applications such as transfer learning [LCL20], is the prediction error when the covariates \( X \) follow a different distribution \( Q \). For such a problem, the result above can be modified straightforwardly by choosing the Hilbert space \( X \) to be \( \mathbb{R}^D \), equipped with the inner product \( \langle u, v \rangle := u^\top (\mathbb{E}_Q[X X^\top])^{-1} v \).

### 4.2 Galerkin methods

We now return to the example of Galerkin methods, as previously introduced in Section 2.2.2, with the i.i.d. observation model (18). We assume the basis functions \( \phi_1, \ldots, \phi_d \) to have uniformly bounded function value and gradient, and define the scalars

\[
\sigma_L := \left( 1 + \frac{2}{\beta} \right) \max_{j \in [d]} \max_{x \in \Omega} \| \nabla \phi_j(x) \|_2, \quad \text{and} \quad \sigma_b := \frac{\| f \|_{L^2} + 1}{\beta} \max_{j \in [d]} \max_{x \in \Omega} | \phi_j(x) |. \tag{40}
\]

These boundedness conditions are naturally satisfied by many interesting basis functions such as the Fourier basis\(^9\), and ensure—we verify this concretely in the proof of Corollary 4 to follow—that our observation model satisfies Assumption 1(W) with parameters \((\sigma_L, \sigma_b)\).

Taking the finite-dimensional representation \( v = \vartheta^\top \phi \), the stochastic approximation estimator for solving equation (19) is given by

\[
\vartheta_{t+1} = \vartheta_t - \beta^{-1} \eta \left( \nabla \phi(x_{t+1})^\top a_{t+1} \nabla \phi(x_{t+1}) \vartheta_t - f_{t+1} \phi(x_{t+1}) \right), \quad \text{for } t = 0, 1, \ldots
\]

\[
\hat{\vartheta}_{n} := \frac{2}{n} \sum_{t=n/2}^{n-1} \vartheta_t, \quad \text{and} \quad \hat{v}_n := \hat{\vartheta}_{n}^\top \phi.
\]

In order to state our statistical guarantees for \( \hat{v}_n \), we define the following matrices:

\[
M := I_d - \beta^{-1} \int_{\Omega} \nabla \phi(x)^\top a(x) \nabla \phi(x) dx,
\]

\[
\Sigma_L := \frac{1}{\beta^2} \int_{\Omega} (\nabla \phi)^\top a \nabla \phi (\nabla \phi)^\top a \nabla \phi dx - \frac{1}{\beta^2} \left( \int_{\Omega} (\nabla \phi)^\top a \nabla \phi dx \right) \left( \int_{\Omega} (\nabla \phi)^\top a \nabla \phi dx \right)^\top + \frac{1}{\beta^2} \int_{\Omega} (\nabla \phi)^\top (\nabla \phi)^\top + \text{diag}(\| \nabla \phi \|_2, (\partial_1 \nabla \phi)_{j=1}^m)^m (\nabla \phi) dx,
\]

\[
\Sigma_b := \frac{1}{\beta^2} \int_{\Omega} (f(x)^2 + 1) \phi(x) \phi(x)^\top dx - \frac{1}{\beta^2} \left( \int_{\Omega} f(x) \phi(x) dx \right) \left( \int_{\Omega} f(x) \phi(x) dx \right)^\top.
\]

With these definitions in hand, we are ready to state the consequence of our main theorems to the estimation problem of elliptic equations.

---

\(^9\)In the typical application of finite-element methods, basis functions based on local interpolation are widely used [BS07]. These basis functions can have large sup-norm, but via application of the Walsh–Hadamard transform, a new basis can be obtained satisfying condition (40) with dimension-independent constants. Since the stochastic approximation algorithm is invariant under orthogonal transformation, this modification is only for the convenience of analysis and does not change the algorithm itself.
Corollary 4. Under the setup above, there are universal positive constants \((c, c_0)\) such that if \(n \geq \frac{c_0 \sigma_l^2 d}{(1 - \kappa(M))^2} \log^2 \left( \frac{\|v_0 - \bar{v}\|^2 \beta d}{\mu} \right)\) and the stochastic approximation scheme is run with step size \(\eta = \frac{1}{c_0 \sigma_L \sqrt{d} n}\), then the averaged iterates satisfy
\[
E\|\hat{v}_n - v^*\|^2 \leq (1 + \omega) \alpha \left( M, 1 - \frac{\mu}{\beta} \right) \inf_{v \in S} \|v - v^*\|^2_X + c \left( 1 + \frac{1}{\omega} \right) \cdot \left( E_n(M, \Sigma_L + \Sigma_b) + \mathcal{H}_n(\sigma_L, \sigma_b, v) \right)
\]
for any \(\omega > 0\).

See Appendix C.3.2 for the proof of this corollary.

Note that the approximation factor \(\alpha(M, 1 - \frac{\mu}{\beta})\) is uniformly bounded on the order of \(O(\beta/\mu)\), which recovers Céa’s energy estimates in the symmetric and uniform elliptic case [Céa64]. On the other hand, for a suitable choice of basis vectors, the bound in Corollary 4 can often be much smaller: the parameter \(\mu\) corresponding to a global coercive condition can be replaced by the smallest eigenvalue of the projected operator \(M\). Furthermore, our analysis can also directly extend to the more general asymmetric and semi-elliptic case, for which case the global coercive condition may not hold true.

It is also worth noting that the bound in Corollary 4 is given in terms of Sobolev norm \(\|\cdot\|_X = \|\cdot\|_{\mathcal{H}^1}\), as opposed to standard \(L^2\)-norm used in the nonparametric estimation literature. By the Poincaré inequality, and \(\mathcal{H}^1\)-norm bound implies an \(L^2\)-norm bound, and ensures stronger error guarantees on the gradient of the estimated function.

4.3 Temporal difference learning

We now turn to the final example previously introduced in Section 2.2.3, namely that of the TD algorithm in reinforcement learning. Recall the i.i.d. observation model (21). Also recall the equivalent form of the projected fixed point equation (23), and note that the population-level operator \(L\) satisfies the norm bound
\[
\|L\|_X = \gamma \cdot \sup_{\|v\| \leq 1} \|Pv\| \leq \gamma := \gamma_{\text{max}},
\]
since \(\xi\) is the stationary distribution of the transition kernel \(P\).

4.3.1 Upper bounds on stochastic approximation with averaging

As mentioned before, this example is somewhat non-standard in that the basis functions \(\psi_i\) are not necessarily orthonormal; indeed the classical temporal difference (TD) learning update in \(\mathbb{R}^d\) involves the stochastic approximation algorithm
\[
\vartheta_{t+1} = \vartheta_t - \eta \left( \psi(s_{t+1}) \psi(s_{t+1})^\top \vartheta_t - \gamma \psi(s_{t+1}) \psi(s_{t+1})^\top \vartheta_t - R_{t+1} s_{t+1}^\top \psi(s_{t+1}) \right).
\]
(41a)

The Polyak–Ruppert averaged estimator is then given by the relations
\[
\hat{\vartheta}_n = \frac{2}{n} \sum_{t=n/2}^{n-1} \vartheta_t, \quad \text{and} \quad \hat{v}_n := \hat{\vartheta}_n^\top \psi.
\]
(41b)

Note that the updates (22) are, strictly speaking, different from the canonical iterates (25), but this should not be viewed as a fundamental difference since we are ultimately interested in
the value function iterates $\widehat{v}_n$; these are obtained from the iterates $\widehat{\vartheta}_n$ by passing back to the original Hilbert space.

Nevertheless, this cosmetic difference necessitates some natural basis transformations before stating our results. Define the matrix $^\top B \in \mathbb{R}^{d \times d}$ by $B_{ij} := \langle \psi_i, \psi_j \rangle$ for $i, j \in [d]$; this defines an orthonormal basis given by
\[
[\phi_1, \phi_2, \ldots, \phi_d] := [\psi_1, \psi_2, \ldots, \psi_d] B^{-1/2}.
\]
Let
\[
\beta := \lambda_{\text{max}}(B) \quad \text{and} \quad \mu := \lambda_{\text{min}}(B),
\]
so that $\beta/\mu$ is the condition number of the covariance matrix of the features.

Having set up this transformation, we are now ready to state the implication of our main theorem to the case of LSTD problems. We assume the following fourth-moment condition:
\[
\forall u \in \mathbb{S}^{d-1}, \quad \mathbb{E}_\xi \left( u^\top B^{-1/2} \psi(s) \right)^4 \leq \varsigma^4, \quad \text{and} \quad \mathbb{E}_\xi \left[ R^4(s) \right] \leq \varsigma^4. \tag{42}
\]
As verified in the proof of Corollary 5 to follow, equation (42) suffices to guarantee that Assumption 1(W) is satisfied with parameters $(\sigma_L, \sigma_b) = (2\varsigma^2, \varsigma^2/\sqrt{\beta})$. We also require the following matrices to be defined:
\[
M := \gamma B^{-1/2} \mathbb{E}_\xi [\psi(s) \psi(s^\top)] B^{-1/2}, \quad \Sigma_L := \text{cov}_\xi \left[ B^{-1/2} \psi(s) \left( \psi(s) - \gamma \psi(s^+) \right) \right],
\]
\[
\Sigma_b := \text{cov}_\xi \left[ R(s) B^{-1/2} \psi(s) \right].
\]

The following corollary then provides a guarantee on the Polyak–Ruppert averaged TD(0) iterates (41).

**Corollary 5.** *Under the set-up above, there are universal positive constants $(c, c_0)$ such that given a sample size $n \geq \frac{c_0 \varsigma^4 \beta d^2}{\mu^3 (1 - \kappa(M))} \log^2 \left( \frac{\|v_0 - \bar{v}\|_2^2 \beta d}{\mu (1 - \kappa(M))} \right)$, then when the stochastic approximation scheme (41a) is run with step size $\eta = \frac{1}{c_0 \varsigma^2 \beta \sqrt{d} n}$, then the averaged iterates satisfy the bound
\[
\mathbb{E} \|\widehat{v}_n - v^*\|^2 \leq (1 + \omega)\alpha(M, \gamma) A(S, v^*)
\]
\[
+ c \left( 1 + \frac{1}{\omega} \right) \left[ \mathcal{E}_n(M, \Sigma_L + \Sigma_b) + (1 + \|\bar{v}\|^2) \left( \frac{\varsigma^2 \beta}{(1 - \kappa(M)) \mu} \sqrt{\frac{d}{n}} \right)^3 \right] \tag{43}
\]
for any $\omega > 0$.*

See Appendix C.1 for the proof of this corollary.

In the worst case, the approximation factor $\alpha(M, \gamma)$ scales as $1/\gamma^2$, recovering the classical result (6), but more generally gives a more fine-grained characterization of the approximation factor depending on the one-step auto-covariance matrix for the feature vectors. By Lemma 1, we have $\alpha(M, \gamma) \leq O\left( \frac{1}{1 - \kappa(M)} \right)$, so intuitively, the approximation factor is large when the Markov chain transitions slowly in the feature space along a certain directions. On the other hand, if the one-step-transition is typically a big jump, then approximation factor is smaller.

The statistical error term $\mathcal{E}_n(M, \Sigma_L + \Sigma_b)$ matches the Cramér–Rao lower bound, and gives a finer characterization than both worst-case upper bounds [BRS18] and existing instance-dependent upper bounds [LS18]. Note that the final, higher-order term depends on the condition number $\beta/\mu$ of the covariance matrix $B$. This ratio is 1 when the basis vectors are orthonormal, but in general, the speed of algorithmic convergence depends on this parameter.

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\(^{10}\)Since the functions $\psi_i$ are linearly independent, we have $B \succ 0$. 

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4.3.2 Approximation factor lower bounds for MRPs

We conclude our discussion of discounted MRPs with an information-theoretic lower bound for policy evaluation. This bound involves technical effort over and above Theorem 2 since our construction for MRPs must make use only of operators $L$ that are constructed using a valid transition kernel. To set the stage, we say that a Markov reward process $(P, \gamma, r)$ and associated basis functions $\{\psi_j\}_{j=1}^d$ are in the canonical set-up if the following conditions hold:

- The stationary distribution $\xi$ of $P$ exists and is unique.
- The reward function and its observations are uniformly bounded. In particular, we have $\|r\|_{\infty} \leq 1$, and $\|R\|_{\infty} \leq 1$ almost surely.
- The basis functions are orthonormal, i.e., $\mathbb{E}_{\xi}[\psi(s)\psi(s)^{\top}] = I_d$.

The three conditions are standard assumptions in Markov reward processes.

Now given scalars $\nu \in (0,1]$ and $\gamma \in (0,1)$, integer $D > 0$ and scalar $\delta \in (0,1/2)$, we consider the following class of MRPs and associated feature vectors:

$$C_{\text{MRP}}(\nu, \gamma, D, \delta) := \left\{ (P, \gamma, r, \psi) \mid (P, \gamma, r, \psi) \text{ is in the canonical setup, } |S| = D, \right\}.$$

Note that under the canonical set-up, we have $M = \gamma \mathbb{E}_{\xi}[\psi(s)\psi(s)^{\top}]$, and consequently, a problem instance in the class $C_{\text{MRP}}(\nu, \gamma, D, \delta)$ satisfies $\kappa(M) \leq \nu \gamma$ in the set-up of Corollary 5. The condition $\kappa(\mathbb{E}_{\xi}[\psi(s)\psi(s)^{\top}]) \leq \nu$ can be seen as a “mixing” condition in the projected space: when $\nu$ is bounded away from 1, the feature vector cannot have too large a correlation with its next-step transition in any direction.

We have the following minimax lower bound for this class, where we use the shorthand $C_{\text{MRP}} \equiv C_{\text{MRP}}(\nu, \gamma, D, \delta)$ for convenience.

**Proposition 1.** There are universal positive constants $(c, c_1)$ such that if $D \geq c_1(n^2 + d)$, then for all scalars $\nu \in (0,1]$ and $\gamma \in (0,1)$, we have

$$\inf_{\tilde{v}_n \in \mathbb{V}_n} \sup_{(P, \gamma, r, \psi) \in C_{\text{MRP}}} \|\tilde{v}_n - v^*\| \geq \frac{c}{1 - \nu \gamma} \delta^2 \wedge 1. \quad (44)$$

See Appendix C.2 for the proof of this proposition.

A few remarks are in order. First, in conjunction with Corollary 5 and the second upper bound in Lemma 1, we can conclude that the TD algorithm for policy evaluation with linear function approximation attains the minimax-optimal approximation factor over the class $C_{\text{MRP}}$ up to universal constants. It is also worth noting that Proposition 1 also shows that the worst-case upper bound (6) due to Tsitsiklis and Van Roy [TVR97] is indeed sharp up to a universal constant; indeed, note that for all $\gamma \in (0,1)$, we have $\frac{1}{1-\gamma^2} \asymp \frac{1}{1-\gamma}$, and that the latter factor can be obtained from the lower bound (44) by taking $\nu = 1$.

Second, note that the class $C_{\text{MRP}}$ is defined in a more “global” sense, as opposed to the “local” class $C_{\text{approx}}$ used in Theorem 2. This class contains all the MRP instances satisfying the approximation error bound and the constraint on $\kappa(M)$, and a minimax lower bound over this larger class is weaker than the lower bound over the local class that imposes restrictions on the projected matrix. That being said, Proposition 1 still captures more structure in the Markov transition kernel than the fact that it is contractive in the $\xi$-norm. For example, when the Markov chain makes “local moves” in the feature space, the correlation between feature vectors
can be large, leading to large value of $\nu$ and larger values of optimal approximation factor. On the other hand, if the one-step transition of the feature vector jumps a large distance in all directions, the optimal approximation factor will be small.

Finally, it is worth noticing that Proposition 1 holds true only for the i.i.d. observation models. If we are given the entire trajectory of the Markov reward process, the approximation factor can be made arbitrarily close to 1, using TD($\lambda$) methods [TVR97]. The trade-off inherent to the Markov observation model is left for our companion paper.

5 Proofs

We now turn to the proofs of our main results.

5.1 Proof of Theorem 1

We divide the proof into two parts, corresponding to the two components in the mean-squared error of the estimator $\hat{v}_n$. The first term is the approximation error $\|v - v^*\|_2$ that arises from the difference between the exact solution $v^*$ to the original fixed point equation, and the exact solution $\bar{v}$ to the projected set of equations. The second term is the estimation error $E\|\hat{v}_n - v\|_2$, measuring the difficulty of estimating $v$ on the basis of $n$ noisy samples.

In particular, under the conditions of the theorem, we prove that the approximation error is upper bounded as

$$\|v - v^*\|_2 \leq \alpha(M, |||L|||_X) \inf_{v \in S} \|v - v^*\|_2,$$ (45a)

whereas the estimation error is bounded as

$$E\|\hat{v}_n - \bar{v}\|_2 \leq \frac{c \text{trace} ((I - M)^{-1}\Sigma^*(I - M)^{-\top})}{n} + c \frac{\sigma_L}{(1 - \kappa)^3} \left( \frac{d}{n} \right)^{3/2} (\|\bar{v}\|^2 \sigma_L^2 + \sigma_b^2).$$ (45b)

Given these two inequalities, it is straightforward to prove the bound (28) stated in the theorem. By expanding the square, we have

$$E\|\hat{v}_n - v\|_2 = E\|\hat{v}_n - \bar{v}\|_2 + \|\bar{v} - v^*\|^2 + 2E\langle \hat{v}_n - \bar{v}, \bar{v} - v^* \rangle$$

$$(i) \leq E\|\hat{v}_n - \bar{v}\|^2 + \|\bar{v} - v^*\|^2 + 2\sqrt{E\|\hat{v}_n - \bar{v}\|^2 \cdot \|\bar{v} - v^*\|^2}$$

$$(ii) \leq E\|\hat{v}_n - \bar{v}\|^2 + \|\bar{v} - v^*\|^2 + \frac{1}{2}E\|\hat{v}_n - \bar{v}\|^2 + \omega\|\bar{v} - v^*\|^2$$

$$= (1 + \omega)\|\bar{v} - v^*\|^2 + (1 + \frac{1}{\omega})E\|\hat{v}_n - \bar{v}\|^2$$

where step (i) follows from the Cauchy–Schwarz inequality; and step (ii) follows from the arithmetic-geometric mean inequality, and is valid for any $\omega > 0$. Substituting the bounds from equations (45a) and (45b) yields the claim of the theorem.

The remainder of our argument is devoted to the proofs of the bounds (45a) and (45b).

5.1.1 Proof of approximation error bound (45a)

We begin with some decomposition relations for vectors and operators. Note that $S$ is a finite-dimensional subspace, and therefore is closed. We use

$$S^\perp := \{ u \in X \mid \langle u, v \rangle = 0 \text{ for all } v \in S. \}$$
The fixed point equation follows by combining these two lemmas, as claimed.

Subtracting equation (47) from equation (48) yields

For the projected solution each element of \( S \) condition implies that \( \alpha \) and recalling our definition of \( \alpha(M, L) \). We now prove these two lemmas in turn.

5.1.2 Proof of Lemma 3

For any vector \( v \in \mathbb{X} \), we perform the orthogonal decomposition \( v = v_S + v_L \), where \( v_S := \Pi_S(v) \) is a member of the set \( S \), and \( v_L := \Pi_S^\perp \) is a member of the set \( S^\perp \). With this notation, the operator \( L \) can be decomposed as

\[
L = (\Pi_S + \Pi_S^\perp) L (\Pi_S + \Pi_S^\perp) = \Pi_S L \Pi_S + \Pi_S L \Pi_S^\perp + \Pi_S^\perp L \Pi_S + \Pi_S^\perp L \Pi_S^\perp.
\]

The four operators \( L_S, L_S^\perp, L_L, L_L^\perp \) defined in the equation above are also bounded linear operators. By the properties of projection operators, we note that \( L_S \) and \( L_L^\perp \) both map each element of \( S^\perp \) to \( 0 \), and \( L_S^\perp \) and \( L_L \) both map each element of \( S \) to \( 0 \).

Decomposing the target vector \( v^* \) in an analogous manner yields the two components

\[
\bar{v} := \Pi_S(v^*), \quad \text{and} \quad v^\perp := v^* - \bar{v}.
\]

The fixed point equation \( v^* = L v^* + b \) can then be written using \( S \) and its orthogonal complement as

\[
\bar{v} = (a) L_S \bar{v} + L_S^\perp v^\perp + b_S, \quad \text{and} \quad v^\perp = (b) L_L \bar{v} + L_L^\perp v^\perp + b_L.
\]

For the projected solution \( \bar{v} \), we have the defining equation

\[
\bar{v} = L_S \bar{v} + b_S.
\]

Subtracting equation (47)(a) from equation (48) yields

\[
(I - L_S) (\bar{v} - \bar{v}) = L_S^\perp v^\perp.
\]

Recall the quantity \( M = \Phi_d L \Phi_d^* \), and our assumption that \( \kappa(M) \lambda_{\text{max}}(M + M^\top) < 1 \). This condition implies that \( I - L_S \) is invertible on the subspace \( S \). Since this operator also maps each element of \( S^\perp \) to itself, it is invertible on all of \( \mathbb{X} \), and we have \( \bar{v} - \bar{v} = (I - L_S) (I - L_S)^{-1} L_S^\perp v^\perp \).

Applying the Pythagorean theorem then yields

\[
\| \bar{v} - v^* \|^2 = \| \bar{v} - \bar{v} \|^2 + \| \bar{v} - v^* \|^2 = \| (I - L_S) (I - L_S)^{-1} L_S^\perp v^\perp \|^2 + \| v^\perp \|^2
\]

\[
\leq (1 + \| (I - L_S) (I - L_S)^{-1} L_S^\perp \|^2) \| v^\perp \|^2, \tag{49}
\]

as claimed.
5.1.3 Proof of Lemma 4

By the definition of operator norm for any vector \( v \in X \) such that \( \|v\| = 1 \), we have
\[
\|L\|^2 \geq \|Lv\|^2 = \|L_{S,S}v_S + L_{S,\perp}v_{\perp}\|^2 + \|L_{\perp,S}v_S + L_{\perp,\perp}v_{\perp}\|^2 \geq \|L_{S,S}v_S + L_{S,\perp}v_{\perp}\|^2.
\]

Noting the fact that \( L_{S,S}v_{\perp} = 0 = L_{S,\perp}v_S \), we have the following norm bound on the linear operator \( L_{S,S} + L_{S,\perp} \):
\[
\|L_{S,S} + L_{S,\perp}\|_X = \sup_{\|v\|=1} \|(L_{S,S} + L_{S,\perp})v\|
= \sup_{\|v\|=1} \|L_{S,S}v_S + L_{S,\perp}v_{\perp}\| \leq \|L\|_X.
\]

By definition, the operator \( L_{S,\perp}^* = \Pi_{S,\perp}L^*\Pi_S \) maps any vector to \( \mathbb{S}^\perp \), and the operator \( L_{S,S} \) maps any element of \( \mathbb{S}^\perp \) to 0. Therefore, we have the identity \( L_{S,S}L_{S,\perp}^* = 0 \). A similar argument yields that \( L_{S,\perp}L_{S,S}^* = 0 \). Consequently, we have
\[
\|L\|^2 \geq \|L_{S,S} + L_{S,\perp}\|^2 = \|(L_{S,S} + L_{S,\perp})(L_{S,S} + L_{S,\perp})^*\|_X
= \|L_{S,S}L_{S,\perp}^* + L_{S,\perp}L_{S,S}^*\|_X
= G. \tag{50}
\]

Note that the operator \( G \) can be expressed as \( G = \Pi_S(L\Pi_S L^* + L\Pi_S L^*)\Pi_S \). From this representation, we see that:

- For any vector \( x \in X \), we have \( Gx \in \mathbb{S} \).
- For any vector \( y \in \mathbb{S}^\perp \), we have \( Gy = 0 \).

Consequently, there exists a matrix \( \tilde{G} \in \mathbb{R}^{d \times d} \), such that \( G = \Phi_d^* \tilde{G} \Phi_d \). Since \( G \) is a positive semi-definite operator, the matrix \( \tilde{G} \) is positive semi-definite. Equation (50) implies that
\[
\lambda_{\max}(\tilde{G}) = \|\tilde{G}\|_{op} = \|G\|_X \leq \|L\|^2_X. \tag{51}
\]

Now defining \( \tau := \|(I - L_{S,S})^{-1}L_{S,\perp}\|_X \), note that
\[
\tau^2 = \|(I - L_{S,S})^{-1}L_{S,\perp}L_{S,\perp}^*(I - L_{S,S})^{-1}\|_X. \tag{52}
\]

Moreover, the operator \( H \) is self-adjoint, and we have the following properties:

- The operator \( L_{S,\perp} \) maps any vector to \( \mathbb{S} \), and \( (I - L_{S,S})^{-1} \) maps \( \mathbb{S} \) to itself. Consequently, for any \( x \in X \), the vector \( Hx = (I - L_{S,S})^{-1}L_{S,\perp} \left( L_{S,\perp}^*(I - L_{S,S})^{-1} \right) x \) is a member of the set \( \mathbb{S} \).
- The operator \( L_{S,\perp}^* = \Pi_{S,\perp}L^*\Pi_S \) maps any vector from \( \mathbb{S}^\perp \) to 0. Consequently, for any \( y \in \mathbb{S}^\perp \), we have \( Hy = (I - L_{S,S})^{-1}L_{S,\perp} \left( L_{S,\perp}^*(I - L_{S,S})^{-1} \right) y = 0 \).

Owing to the facts above, there exists a matrix \( \tilde{H} \in \mathbb{R}^{d \times d} \), such that \( H = \Phi_d^* \tilde{H} \Phi_d \). Since the operator \( H \) is positive semi-definite, so is the matrix \( \tilde{H} \). Consequently, by equation (52), we obtain the identity \( \tau^2 = \|H\|_X = \|\tilde{H}\|_{op} = \lambda_{\max}(H) \). In particular, letting \( u \in \mathbb{S}^{d-1} \) be a maximal eigenvector of \( \tilde{H} \), we have
\[
\tilde{H} \geq \tau^2 uu^\top. \tag{53}
\]
We now turn to the proof of our claimed bound on the estimation error. Our analysis relies on second moment bounds from Assumption 1(W) combined with the assumption that $\kappa$ matches a term in the bound (45b). As for the remaining two terms in equation (55), the second moment bounds from Assumption 1(W) combined with the assumption that $\kappa(M) < 1$

\[
\|L\|_2^2 I_d \succeq \tilde{G}
\]

\[
= \Phi_d \left(L_{SS, SS} L_{SS, SS}^* + L_{SL, SL}^* I_d\right) \Phi_d^*
\]

\[
= \Phi_d L_{SS, SS} L_{SS, SS}^* + \left(\Phi_d (I - L_{SS, SS}) \Phi_d^*\right) \cdot \left(\Phi_d (I - L_{SS, SS})^{-1} L_{SL, SL} (I - L_{SS, SS})^{-1} \Phi_d\right)
\]

\[
= MM^\top + (I - M) \bar{H} (I - M^\top)
\]

\[
\succeq MM^\top + \tau^2(I - M) uu^\top (I - M^\top).
\]

Re-arranging and noting that $u \in \mathbb{S}^{d-1}$, we arrive at the inequality

\[
\tau^2 \leq u^\top \left[(I - M)^{-1} \left(\|L\|_2^2 I_d - MM^\top\right) (I - M)^{-\top}\right] u \leq \lambda_{\max} \left[(I - M)^{-1} \left(\|L\|_2^2 I_d - MM^\top\right) (I - M)^{-\top}\right],
\]

which completes the proof of Lemma 4.

### 5.1.4 Proof of estimation error bound (45b)

We now turn to the proof of our claimed bound on the estimation error. Our analysis relies on two auxiliary lemmas. The first lemma provides bounds on the mean-squared error of the standard iterates $\{v_t\}_{t \geq 0}$—that is, without the averaging step:

**Lemma 5.** Suppose that the noise conditions in Assumption 1(W) hold. Then for any stepsize $\eta \in \left(0, \frac{1 - \kappa}{4\sigma_L^2 d + 1 + \|L\|_2^2}\right)$, we have the bound

\[
E\|v_t - \bar{\sigma}\|^2 \leq e^{-(1 - \kappa)\eta t/2} E\|v_0 - \bar{\sigma}\|^2 + \frac{8\eta}{1 - \kappa} (\|\bar{\sigma}\|^2 \sigma_L^2 d + \sigma_0^2 d)
\]

valid for $t = 1, 2, \ldots$ (54)

See Section 5.1.5 for the proof of this claim.

Our second lemma provides a bound on the PR-averaged estimate $\hat{v}_n$ based on $n$ observations in terms of a covariance term, along with the error of the non-averaged sequences $\{v_t\}_{t \geq 1}$:

**Lemma 6.** Under the setup above, we have the bound

\[
E\|\hat{v}_n - \bar{\sigma}\|^2 \leq \frac{6}{n - n_0} \text{trace} \left((I - M)^{-1} \Sigma^* (I - M)^{-\top}\right)
\]

\[
+ \frac{6}{(n - n_0)^2} \sum_{t = n_0}^n E\| (I - M)^{-1} \Phi_d (L_{t+1} - L)(v_t - \bar{\sigma})\|^2 + \frac{3E\|v_n - v_{n_0}\|^2}{\eta^2 (n - n_0)^2 (1 - \kappa)^2}.
\]

See Section 5.1.6 for the proof of this claim.

Equipped with these two lemmas, we can now complete the proof of the claimed bound (45b) on the estimation error. Recalling that $n_0 = n/2$, we see that the first term in the bound (55) matches a term in the bound (45b). As for the remaining two terms in equation (55), the second moment bounds from Assumption 1(W) combined with the assumption that $\kappa(M) < 1$
imply that
\[
\mathbb{E}\|(I - M)^{-1}\Phi_d(L_{t+1} - L)(v_t - \bar{v})\|^2 \leq \frac{1}{(1 - \kappa)^2} \mathbb{E}\|\Phi_d(L_{t+1} - L)(v_t - \bar{v})\|^2
\]
\[
\leq \frac{1}{(1 - \kappa)^2} \sum_{j=1}^{d} \mathbb{E}\langle \phi_j, (L_{t+1} - L)(v_t - \bar{v}) \rangle^2
\]
\[
\leq \frac{\sigma_d^2 d\|v_t - \bar{v}\|^2}{(1 - \kappa)^2}.
\]

On the other hand, we can use Lemma 5 to control the third term in the bound (55). We begin by observing that
\[
\|v_n - v_{n_0}\|^2 \leq 2\|v_n - \bar{v}\|^2 + 2\|v_{n_0} - \bar{v}\|^2 \leq 4 \sup_{n_0 \leq t \leq n} \mathbb{E}\|v_t - \bar{v}\|^2.
\]
If we choose a burn-in time \( n_0 > \frac{c_\eta}{(1 - \kappa)\eta} \log \left( \frac{\|v_0 - \bar{v}\|^2 d}{\eta} \right) \), then Lemma 5 ensures that
\[
\sup_{n_0 \leq t \leq n} \mathbb{E}\|v_t - \bar{v}\|^2 \leq \frac{16\eta}{1 - \kappa} \left( \|\bar{v}\|^2 \sigma_d^2 d + \sigma_b^2 d \right).
\]
Finally, taking the step size \( \eta = \frac{1}{24\sigma_L \sqrt{dn}} \), recalling that \( n_0 = n/2 \), and putting together the pieces yields
\[
\mathbb{E}\|\hat{v}_n - \bar{v}\|^2 \leq \frac{12}{n} \text{trace} \left( (I - M)^{-1} \Sigma^*(I - M)^{-T} \right) + \frac{1}{(1 - \kappa)^2} \left( \frac{12\sigma_d^2 d}{n} + \frac{48}{\eta^2 n^2} \right) \sup_{n_0 \leq t \leq n} \mathbb{E}\|v_t - \bar{v}\|^2
\]
\[
\leq \frac{12}{n} \text{trace} \left( (I - M)^{-1} \Sigma^*(I - M)^{-T} \right) + \frac{48\sigma_L}{(1 - \kappa)^3} \left( \frac{d}{n} \right)^{3/2} \left( \|\bar{v}\|^2 \sigma_d^2 + \sigma_b^2 \right),
\]
as claimed.

It remains to prove our two auxiliary lemmas, which we do in the following subsections.

### 5.1.5 Proof of Lemma 5

We now prove Lemma 5, which provides a bound on the error of the non-averaged iterates \( \{v_t\}_{t \geq 1} \), as defined in equation (24a). Using the form of the update, we expand the mean-squared error to find that
\[
\mathbb{E}\|v_{t+1} - \bar{v}\|^2 = \mathbb{E}\|(I - \eta I + \eta \Pi_S L)(v_t - \bar{v}) + \eta \Pi_S (L_{t+1} - L)v_t + \eta \Pi_S (b_{t+1} - b)\|^2
\]
\[
\overset{(i)}{=} \mathbb{E}\|(I - \eta I + \eta \Pi_S L)(v_t - \bar{v})\|^2 + \eta^2 \mathbb{E}\|\Pi_S (L_{t+1} - A)v_t + \Pi_S (b_{t+1} - b)\|^2
\]
\[
\overset{(ii)}{\leq} (1 - \eta(1 - \kappa))\mathbb{E}\|v_t - \bar{v}\|^2 + 2\eta^2 \mathbb{E}\|\Pi_S (L_{t+1} - L)(v_t - \bar{v})\|^2
\]
\[
+ 2\eta^2 \mathbb{E}\|\Pi_S (L_{t+1} - L)\bar{v} + \Pi_S (b_{t+1} - b)\|^2.
\]
In step (i), we have made use of the fact that the noise is unbiased, and in step (ii), we have used that for any \( \Delta \) in the subspace \( S \) and any stepsize \( \eta \in (0, \frac{1 - \kappa}{1 + L_\Sigma L}) \), we have
\[
\|(I - \eta I + \eta \Pi_S L)\Delta\|^2 = (1 - \eta)^2 \|\Delta\|^2 + \eta^2 \|\Pi_S L \Delta\|^2 + 2(1 - \eta)\eta \langle \Delta, \Pi_S L \Delta \rangle
\]
\[
\leq \left\{ 1 - 2\eta + \eta^2 \|L\|_2^2 + 2(1 - \eta)\eta \kappa \right\} \|\Delta\|^2
\]
\[
\leq (1 - \eta(1 - \kappa)) \|\Delta\|^2.
\]
Turning to the second term of equation (56), the moment bounds in Assumption 1(W) imply that

\[ \mathbb{E}\|\Pi S (L_{t+1} - L)(v_t - \bar{v})\|^2 = \sum_{j=1}^{d} \mathbb{E}\langle \phi_j, (L_{t+1} - L)(v_t - \bar{v}) \rangle^2 \leq \mathbb{E}\|v_t - \bar{v}\|^2 \sigma_d^2. \]

Finally, the last term of equation (56) is also handled by Assumption 1(W), whence we obtain

\[ \mathbb{E}\|\Pi S (L_{t+1} - L)\bar{v} + \Pi S (b_{t+1} - b)\|^2 \leq 2\|\bar{v}\|^2 \sigma_d^2 + 2\|b\|^2 \sigma_b^2. \]

Putting together the pieces, we see that provided \( \eta < \frac{1 - \kappa}{4\sigma_d^2 + \|L\|}, \) we have

\[ \mathbb{E}\|v_{t+1} - \bar{v}\|^2 \leq (1 - \eta(1 - \kappa)) + 2\eta^2 \sigma_d^2 \mathbb{E}\|v_t - \bar{v}\|^2 + 4\eta^2 (\|\bar{v}\|^2 \sigma_d^2 + \sigma_b^2) \]

\[ \leq \left(1 - \frac{\eta(1 - \kappa)}{2}\right) \mathbb{E}\|v_t - \bar{v}\|^2 + 4\eta^2 (\|\bar{v}\|^2 \sigma_d^2 + \sigma_b^2). \]

Finally, rolling out the recursion yields the bound

\[ \mathbb{E}\|v_n - \bar{v}\|^2 \leq e^{-\eta(1 - \kappa)\eta n/2} \mathbb{E}\|v_0 - \bar{v}\|^2 + \frac{8\eta}{1 - \kappa} (\|\bar{v}\|^2 \sigma_d^2 + \sigma_b^2), \]

which completes the proof.

5.1.6 Proof of Lemma 6

Recall that \( \bar{v} \) satisfies the fixed point equation \( \bar{v} = \Pi S L\bar{v} + \Pi S b. \) Using this fact, we can derive the following elementary identity:

\[ \frac{v_{n_0} - v_n}{\eta(n - n_0)} = \sum_{t=n_0}^{n-1} (v_t - \Pi S L_{t+1} v_t - \Pi S b_{t+1}) \]

\[ = (I - \Pi S L)(\bar{v}_n - \bar{v}) + \sum_{t=n_0}^{n-1} \Pi S (L_{t+1} - L)v_t + \sum_{t=n_0}^{n-1} \Pi S (b_{t+1} - b) \cdot (57) \]

Re-arranging terms and applying the Cauchy–Schwarz inequality, we have

\[ \|\bar{v}_n - \bar{v}\|^2 \leq \frac{3}{(n - n_0)^2} \left( \frac{1}{\eta^2} \| (I - \Pi S L)^{-1} (v_n - v_{n_0}) \|^2 + \| (I - \Pi S L)^{-1} \Psi_n^{(1)} \|^2 + \| (I - \Pi S L)^{-1} \Psi_n^{(2)} \|^2 \right). \]
Note that the quantities $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$ are martingales adapted to the filtration $\mathcal{F}_n := \sigma(\{L_i, b_i\}_{i=1}^n)$, so that

$$
\mathbb{E}\|\tilde{v}_n - \bar{v}\|^2 \leq \frac{3}{(n - n_0)^2} \sum_{t=n_0}^{n-1} \mathbb{E}\|(I - \Pi_S L)^{-1}\Pi_S(L_{t+1} - L)v_t\|^2 \\
+ \frac{3}{(n - n_0)^2} \sum_{t=n_0}^{n-1} \mathbb{E}\|(I - \Pi_S A)^{-1}\Pi_S(b_{t+1} - b)\|^2 \\
+ \frac{3}{(n - n_0)^2 \eta^2} \mathbb{E}\|(I - \Pi_S L)^{-1}(v_n - v_{n_0})\|^2.
$$

We claim that for any vector $v \in \mathbb{X}$, we have

$$(I - \Pi_S L)^{-1}\Pi_S v = \Phi_d^*( (I - M)^{-1}\Phi_d v). \quad (58)$$

Taking this claim as given for the moment, by applying equation (58) with $v = (L_{t+1} - L)v_t$ and $v = b_{t+1} - b$, we find that

$$
\mathbb{E}\|(I - \Pi_S L)^{-1}\Pi_S(L_{t+1} - L)v_t\|^2 = \mathbb{E}\|(I - M)^{-1}\Phi_d(L_{t+1} - L)v_t\|^2 \\
\leq 2\mathbb{E}\|(I - M)^{-1}\Phi_d(L_{t+1} - L)\bar{v}\|^2 + 2\mathbb{E}\|(I - M)^{-1}\Phi_d(L_{t+1} - L)(v_t - \bar{v})\|^2,
$$

and

$$
\mathbb{E}\|(I - L)^{-1}\Pi_S(b_{t+1} - b)\|^2 = \mathbb{E}\|(I - M)^{-1}\Phi_d(b_{t+1} - b)\|^2.
$$

Putting together the pieces, we obtain

$$
\mathbb{E}\|\tilde{v}_n - \bar{v}\|^2 \leq \frac{3}{n - n_0} \text{trace}\left( (I - M)^{-1} \cdot \text{cov}(\Phi_d(b_1 - b)) \cdot (I - M)^{-\top}\right) \\
+ \frac{6}{n - n_0} \text{trace}\left( (I - M)^{-1} \cdot \text{cov}(\Phi_d(L_1 - L)\bar{v}) \cdot (I - M)^{-\top}\right) \\
+ \frac{6}{(n - n_0)^2} \sum_{t=n_0}^{n} \mathbb{E}\|(I - M)^{-1}\Phi_d(L_{t+1} - L)(v_t - \bar{v})\|^2 + \frac{3\mathbb{E}\|v_n - v_{n_0}\|^2}{\eta^2(n - n_0)^2(1 - \kappa)^2},
$$

as claimed.

It remains to prove the identity (58).

**Proof of claim (58):** Note that for any vector $v \in \mathbb{X}$, the vector $z := (I - \Pi_S L)^{-1}\Pi_S v$ is a member of $\mathcal{S}$, since $z = \Pi_S Lz + \Pi_S v$. Furthermore, since $\{\phi_j\}_{j=1}^d$ is a standard basis for $\mathcal{S}$, we have $z = \Pi_S z = \Phi_d^*\Phi_d z$, and consequently,

$$
\Phi_d z = \Phi_d Lz + \Phi_d v = (\Phi_d L\Phi_d^*)\Phi_d z = \Phi_d v = M\Phi_d z + \Phi_d v.
$$

Since the matrix $M$ is invertible, we have $\Phi_d z = (I_d - M)^{-1}\Phi_d v$. Consequently, we have the identity $z = \Phi_d^*\Phi_d z = \Phi_d^*(I_d - M)^{-1}\Phi_d v$, which proves the claim.
5.2 Proof of Corollary 1

We begin by applying Theorem 1 with \( \omega = 1 \). Applying Lemmas 1 and 2 yield the desired bounds on the approximation error in parts (a) and (b). We also claim that

\[
\mathcal{E}_n(M, \Sigma^*) \leq \frac{(\sigma_L^2 \|\bar{\mathbf{v}}\|^2 + \sigma_b^2) d}{(1 - \kappa)^2 n}, \tag{59a}
\]

and that given a sample size such that \( n \geq \frac{c \sigma_L^2 d}{(1 - \kappa)^2} \log^2 \left( \frac{\|v_0 - \mathbf{v}\|^2 d}{1 - \kappa} \right) \), we have

\[
\mathcal{H}_n(\sigma_L, \sigma_b, \mathbf{v}) \leq \frac{\sigma_L}{1 - \kappa} \sqrt{\frac{d}{n}} \cdot \frac{(\sigma_L^2 \|\bar{\mathbf{v}}\|^2 + \sigma_b^2) d}{(1 - \kappa)^2 n} \leq \frac{(\sigma_L^2 \|\bar{\mathbf{v}}\|^2 + \sigma_b^2) d}{(1 - \kappa)^2 n}, \tag{59b}
\]

Combining these two auxiliary claims establishes the corollary. It remains to establish the bounds (59).

**Proof of claim (59):** Let us first handle the contribution to this error from the noise variables \( b_i \). We begin with the following sequence of bounds:

\[
\text{trace} \left( (I - M)^{-1} \text{cov}(\Phi_d(b_1 - b))(I - M)^{-\top} \right) = \text{trace} \left( (I - M)^{-\top} (I - M)^{-1} \cdot \text{cov}(\Phi_d(b_1 - b)) \right) \leq \|(I - M)^{-\top} (I - M)^{-1}\|_{\text{op}} \cdot \| \text{cov}(\Phi_d(b_1 - b)) \|_{\text{nuc}} \leq \|(I - M)^{-1}\|_{\text{op}}^2 \text{trace} \left( \text{cov}(\Phi_d(b_1 - b)) \right).
\]

By the assumption \( \kappa(M) < 1 \), for any vector \( u \in \mathbb{R}^d \), we have that

\[
(1 - \kappa)\|u\|^2 \leq \langle (I - M)u, u \rangle \leq \|(I - M)u\|_2 \cdot \|u\|_2.
\]

Consequently, we have the bound \( \|(I - M)^{-1}\|_{\text{op}} \leq \frac{1}{1 - \kappa(M)} \). For the trace of the covariance, we note by Assumption 1(W) that

\[
\text{trace} \left( \text{cov}(\Phi_d(b_1 - b)) \right) = \sum_{j=1}^{d} \langle \phi_j, b_1 - b \rangle^2 \leq \sigma_b^2 d.
\]

Putting together the pieces yields\( \text{trace} \left( (I - M)^{-1} \text{cov}(\Phi_d(b_1 - b))(I - M)^{-\top} \right) \leq \frac{\sigma_b^2 d}{(1 - \kappa)^2} \).

Turning now to the contribution to the error from the random observation \( L_i \), we have

\[
\text{trace} \left( (I - M)^{-1} \text{cov}(\Phi_d(L_1 - L)\bar{v})(I - M)^{-\top} \right) \leq \|(I - M)^{-1}\|_{\text{op}}^2 \text{trace} \left( \text{cov}(\Phi_d(L_1 - L)\bar{v}) \right).
\]

Once again, Assumption 1(W) yields the bound

\[
\text{trace} \left( \text{cov}(\Phi_d(L_1 - L)\bar{v}) \right) = \sum_{j=1}^{d} \langle \phi_j, (L_1 - L)\bar{v} \rangle^2 \leq \sigma_L^2 \|\bar{\mathbf{v}}\|^2 d,
\]

and combining the pieces proves the claim.
At a high level, our proof of the lower bound proceeds by constructing two ensembles of problem instances indexed by a binary string \( \varepsilon \) and each instance is, in turn, obtained as a mixture over \( 2^z \) instances that are hard to distinguish from each other, and such that the approximation error is\( \leq \gamma_{\max}^2 \).

Specifically, let \( u \in \mathbb{S}^{d-1} \) be an eigenvector associated to the largest eigenvalue of the matrix \((I-M_0)^{-1}(\gamma_{\max}^2 I - M_0 M_0^\top)(I-M_0)^{-\top}\). By the definition of the approximation factor \( \alpha(M_0, \gamma_{\max}) \), we have:

\[
(\alpha(M_0, \gamma_{\max}) - 1) \cdot (I - M_0) uu^\top (I - M_0)^\top \preceq \gamma_{\max}^2 I - M_0 M_0^\top.
\]

Based on the eigenvector \( u \), we further define the \( d \)-dimensional vectors:

\[
w := \sqrt{\alpha(M_0, \gamma_{\max}) - 1} \cdot (I - M_0) u, \quad \text{and} \quad y := \sqrt{\alpha(M_0, \gamma_{\max}) - 1} \cdot \delta u. \tag{60}
\]

Substituting into the above PSD domination relation yields that

\[
w w^\top + M_0 M_0^\top \preceq \gamma_{\max}^2 I. \tag{61}
\]

Now consider the following class of (population-level) problem instances \((L^{(\varepsilon), (\varepsilon, z)}, b^{(\varepsilon, z)}, v_{\varepsilon, z}^*)\) indexed by a binary string \( \varepsilon \in \{-1, 1\}^{D-d} \) and a bit \( z \in \{-1, 1\} \):

\[
L^{(\varepsilon, z)} := \begin{bmatrix}
M_0 & \sqrt{d} \delta_{\varepsilon_{D+1}} w \\
0 & \ddots & \sqrt{d} \delta_{\varepsilon_D} w \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & 0 & \ddots & \sqrt{2d} \delta_{\varepsilon_{D+1}} \delta_{\varepsilon_D}
\end{bmatrix}, \quad v_{\varepsilon, z}^* := \begin{bmatrix}
\sqrt{2d} (zy + (I-M_0)^{-1} h_0) \\
\sqrt{2d} \delta_{\varepsilon_{D+1}} \delta_{\varepsilon_D} \\
\vdots \\
\sqrt{2d} \delta_{\varepsilon_{D+1}} \\
\sqrt{2d} \delta_{\varepsilon_D}
\end{bmatrix}, \tag{62}
\]

\[
b^{(\varepsilon, z)} := (I - L^{(\varepsilon, z)}) v_{\varepsilon, z}^* = \begin{bmatrix}
\sqrt{2d} h_0 \\
\sqrt{2z \delta_{\varepsilon_{D+1}} \delta_{\varepsilon_D}} \\
\vdots \\
\sqrt{2z \delta_{\varepsilon_{D+1}} \delta_{\varepsilon_D}}
\end{bmatrix}.
\]

We take the weight vector \( \xi \) to be

\[
\xi = \begin{bmatrix}
\frac{1}{\sqrt{d}} & \cdots & \frac{1}{\sqrt{d}} \\
\frac{1}{\sqrt{(D-d)^3}} & \cdots & \frac{1}{\sqrt{(D-d)^3}}
\end{bmatrix},
\]

and the weighted inner product \( \langle \cdot, \cdot \rangle \) on the space \( \mathbb{X} = \mathbb{R}^D \) is defined via

\[
\langle p, q \rangle := \sum_{j=1}^D p_j \xi_j q_j \quad \text{for each pair } p, q \in \mathbb{R}^D.
\]
This choice of inner product then induces the vector norm $\| \cdot \|$ and operator norm $\| \cdot \|_\infty$.

Next, we define the basis vectors via

$$
\phi_i = \begin{cases} 
\sqrt{2d} e_i & \text{for } i = 1, 2, \ldots, d, \text{ and} \\
\sqrt{2(D-d)} e_i & \text{for } i = d+1, \ldots, D.
\end{cases}
$$

By construction, we have ensured that $\| \phi_i \| = 1$ for each $i \in [D]$. We let the subspace $S$ be the span of the first $d$ standard basis vectors, i.e., $S := \text{span}(e_1, e_2, \ldots, e_d)$.

For each binary string $\varepsilon \in \{-1, 1\}^{D-d}$ and signed bit $z \in \{-1, 1\}$, a straightforward calculation reveals that the projected problem instance satisfies the identities

$$
\Phi_d L^{(\varepsilon, z)} \Phi_d^* = M_0, \quad \Phi_d b^{(\varepsilon, z)} = h_0.
$$

(63a)

Also note that for any pair $(\varepsilon, z)$, we have by construction that

$$
\inf_{v \in S} \| v^{*\varepsilon, z} - v \|^2 = \frac{1}{2(D-d)} \sum_{j=d+1}^D (\sqrt{2} z \delta \varepsilon_j)^2 = \delta^2.
$$

(63b)

In words, this shows that the $\| \cdot \|$-error of approximating $v^{*\varepsilon, z}$ with the linear subspace $S$ is always $\delta$, irrespective of which $\varepsilon \in \{-1, 1\}^{D-d}$ and $z \in \{-1, 1\}$ are chosen.

Next, we construct the random observation models for the $i$-th $d$ observations, which are also indexed by the pair $(\varepsilon, z)$. In particular, we construct the random matrix $L_i^{(\varepsilon, z)}$ and random vector $b_i^{(\varepsilon, z)}$ via

$$
L_i^{(\varepsilon, z)} := \begin{bmatrix}
M_0 & 0 & \cdots & \sqrt{\delta \varepsilon_{\tau_L(i)}} w & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad
b_i^{(\varepsilon, z)} := \begin{bmatrix}
\sqrt{2} d h_0 \\
0 \\
\vdots \\
0 \\
\sqrt{2(D-d)z \delta \varepsilon_{\tau_b(i)}} \\
0 \\
\vdots \\
0
\end{bmatrix}.
$$

(64)

where the random indices $\tau_L(i)$ and $\tau_b(i)$ are chosen independently and uniformly at random from the set $\{d+1, d+2, \ldots, D\}$. By construction, we have ensured that for each $\varepsilon \in \{-1, 1\}^{D-d}$ and $z \in \{-1, 1\}$, the observations have mean

$$
\mathbb{E} \left[ L_i^{(\varepsilon, z)} \right] = L^{(\varepsilon, z)}, \quad \text{and} \quad \mathbb{E} \left[ b_i^{(\varepsilon, z)} \right] = b^{(\varepsilon, z)}.
$$

This concludes our description of the problem instances themselves. Since our proof proceeds via Le Cam’s lemma, we require some more notation for product distributions and mixtures under this observation model. Let $\mathbb{P}_{\varepsilon, z}^{(n)}$ denote the $n$-fold product of the probability laws of the pair $(L_i^{(\varepsilon, z)}, b_i^{(\varepsilon, z)})$. We also define the following mixture of product measures for each $z \in \{-1, 1\}$:

$$
\mathbb{P}_z^{(n)} := \frac{1}{2^{D-d}} \sum_{\varepsilon \in \{\pm 1\}^{D-d}} \mathbb{P}_{\varepsilon, z}^{(n)}.
$$
We seek bounds on the total variation distance $d_{TV}\left(\mathbb{P}_{1}^{(n)} , \mathbb{P}_{-1}^{(n)} \right)$.

With this setup, the following lemmas assert that (a) Our construction satisfies the conditions in Assumption 1(S), and (b) The total variation distance is small provided $n \ll \sqrt{D-d}$.

**Lemma 7.** For each binary string $\varepsilon \in \{-1,1\}^{D-d}$ and bit $z \in \{-1,1\}$:

(a) The population-level matrix $L^{(\varepsilon,z)}$ defined in equation (62) satisfies $\|L^{(\varepsilon,z)}\|_{X} \leq \gamma_{\text{max}}$.

(b) The random observations $\left(\mathcal{L}_{i}^{(\varepsilon,z)}, b_{i}^{(\varepsilon,z)}\right)$ defined in equation (64) satisfies Assumption 1(S), for any scalar pair $(\sigma_{L}, \sigma_{b})$ such that $\sigma_{L} \geq \gamma_{\text{max}}$ and $\sigma_{b} \geq \delta$.

**Lemma 8.** Under the set-up above, we have $d_{TV}\left(\mathbb{P}_{1}^{(n)} , \mathbb{P}_{-1}^{(n)} \right) \leq \frac{12n^{2}}{D-d}$.

Part (a) of Lemma 7 and equations (63a)–(63b) together ensure that population-level problem instance $(L, b)$ we constructed belongs to the class $\mathcal{C}_{\text{approx}}(M_{0}, h_{0}, D, \delta, \gamma_{\text{max}})$. Part (b) of Lemma 7 further ensures the probability distribution $\mathbb{P}_{L,b}$ belongs to the class $\mathcal{G}_{\text{var}}(\sigma_{L}, \sigma_{b})$. Lemma 8 ensures that the two mixture distributions corresponding to different choices of the bit $z$ are close provided $n$ is not too large. The final step in applying Le Cam’s mixture-vs-mixture result is to show that the approximation error is large for at least one of the choices of the bit $z$.

We carry out this step by splitting the rest of the proof into two cases, depending on whether or not we enforce that our estimator $\hat{v}$ is constrained to lie in the subspace $S$. Throughout, we use the decomposition $\hat{v} = [\hat{v}_{1} \hat{v}_{2}]$, where $\hat{v}_{1} \in \mathbb{R}^{d}$ and $\hat{v}_{2} \in \mathbb{R}^{D-d}$. Also recall the definition of the vector $y$ from equation (60).

**Case I:** $\hat{v} \in S$. This corresponds to the “proper learning” case where the estimator is restricted to take values in the subspace $S$ and $\hat{v}_{2} = 0$. Note that for any $\varepsilon \in \{-1,1\}^{D-d}$, we have

$$
\|v_{\varepsilon,z}^{*} - \hat{v}\|^2 = \|v_{\varepsilon,z}^{*} - \Pi_{S}(v_{\varepsilon,z}^{*})\|^2 + \|v_{\varepsilon,z}^{*} - \hat{v}\|^2 = \delta^2 + \frac{1}{2d} \|\hat{v}_{1} - \sqrt{2d}z y\|_{2}^2.
$$

Therefore, for any $\varepsilon, \varepsilon' \in \{-1,1\}^{D-d}$, the following chain of inequalities holds:

$$
\frac{1}{2} \left(\|v_{\varepsilon,1}^{*} - \hat{v}\|^2 + \|v_{\varepsilon',1}^{*} - \hat{v}\|^2\right) = \delta^2 + \frac{1}{4d} \left(\|\hat{v}_{1} - \sqrt{2d}y\|_{2}^2 + \|\hat{v}_{1} + \sqrt{2d}y\|_{2}^2\right)
$$

$$
= \delta^2 + \frac{1}{2d} \left(\|\hat{v}_{1}\|_{2}^2 + 2d \|y\|_{2}^2\right)
$$

$$
\geq \delta^2 + \|y\|_{2}^2
$$

$$
= \alpha(M_0, \gamma_{\text{max}}) \cdot \delta^2.
$$

By Le Cam’s lemma, we thus have

$$
\inf_{\hat{v}_{n} \in \mathcal{V}_{\mathcal{S}}} \sup_{(L,b) \in \mathcal{C}_{\text{approx}}} \mathbb{E}\|\hat{v}_{n} - v^{*}\|^2 \geq \alpha(M_0, \gamma_{\text{max}}) \delta^2 \cdot \left(1 - d_{TV}(\mathbb{P}_{-1}^{(n)} , \mathbb{P}_{1}^{(n)})\right)
$$

$$
\geq \left(1 - \omega\right) \cdot \alpha(M_0, \gamma_{\text{max}}) \cdot \delta^2,
$$

where in step (i), we have applied Lemma 8 in conjunction with the inequality $D \geq d + \frac{12n^2}{\omega}$. 37
Case II: \( \hat{v} \notin \mathbb{S} \). This corresponds to the case of “improper learning” where the estimator can take values in the entire space \( \mathbb{X} \). In this case, for any pair \( \varepsilon, \varepsilon' \in \{-1, 1\}^{D-d} \), we obtain
\[
\|v_{\varepsilon,1} - v_{\varepsilon',1}\| \geq \|2\sqrt{2d}y^\top \ 0 \cdots 0\| = 2\|y\|_2 = 2\delta \sqrt{\alpha(M_0, \gamma_{\text{max}})} - 1.
\]
Applying triangle inequality and Young’s inequality yields the bound
\[
\frac{1}{2}(\|\hat{v} - v_{\varepsilon,1}\|^2 + \|\hat{v} - v_{\varepsilon',1}\|^2) \geq \frac{1}{4}(\|\hat{v} - v_{\varepsilon,1}\| + \|\hat{v} - v_{\varepsilon',1}\|)^2 \geq \frac{1}{4}\|v_{\varepsilon,1} - v_{\varepsilon',1}\|^2 \geq (\alpha(M_0, \gamma_{\text{max}}) - 1) \cdot \delta^2.
\]
By Le Cam’s lemma, we once again have
\[
\inf_{\hat{v}_n \in \mathbb{V}_X} \sup_{(L, b) \in \mathcal{L}_{\text{approx}}, F_{L, b} \in \mathcal{F}_{\text{var}}(\sigma_L, \sigma_b)} \mathbb{E}\|\hat{v}_n - v^*\|^2 \geq (\alpha(M_0, \gamma_{\text{max}}) - 1) \cdot \delta^2 \cdot \left(1 - d_{\text{TV}}(\mathbb{P}_1^{(n)}, \mathbb{P}_1^{(n)})\right)
\]
\[
\geq (1 - \omega) \cdot (\alpha(M_0, \gamma_{\text{max}}) - 1) \cdot \delta^2.
\]
Putting together the two cases completes the proof.

5.3.1 Proof of Lemma 7

We prove the two parts of the lemma separately. Once again, recall our definition of the pair \((w, y)\) from equation (60).

Proof of part (a): In order to study the operator norm of the matrix \(L^{(\varepsilon, z)}\) in the Hilbert space \(\mathbb{X}\), we consider a vector \(p = [p^{(1)}, p^{(2)}] \in \mathbb{R}^D\), with \(p^{(1)} \in \mathbb{R}^d\) and \(p^{(2)} \in \mathbb{R}^{D-d}\). Assuming \(\|p\| = 1\), we have
\[
\|L^{(\varepsilon, z)}p\|^2 = \frac{1}{2d}\|M_0p^{(1)} + w \cdot \sqrt{\frac{d}{d-d^2}} \sum_{j=d+1}^D \varepsilon_j p^{(2)}_j\|^2.
\]

By the Cauchy–Schwarz inequality, we have
\[
\left|\sqrt{\frac{d}{d-d^2}} \sum_{j=d+1}^D \varepsilon_j p^{(2)}_j\right|^2 \leq \frac{d}{d-d^2} \left(\sum_{j=d+1}^D \varepsilon_j^2\right) \left(\sum_{j=d+1}^D (p^{(2)}_j)^2\right) = \frac{d}{d-d^2}\|p^{(2)}\|^2.
\]

Define the vector \(a_1 := \frac{1}{\sqrt{2d}}p^{(1)} \in \mathbb{R}^d\) and \(a_2 := \frac{1}{\sqrt{2d(d-d^2)}}\|p^{(2)}\|_2\). Clearly, we have \(1 = \|p\|^2 = \|a_1\|^2 + \|a_2\|^2\), and so
\[
\|L^{(\varepsilon, z)}p\|^2 \leq \frac{1}{2d} \cdot \sup_{r \in [-1, 1]} \|M_0p^{(1)} + \sqrt{2d}a_2w\|_2^2
\]
\[
= \frac{1}{2d} \cdot \max\left(\|M_0p^{(1)} + \sqrt{2d}a_2w\|^2, \|M_0p^{(1)} - \sqrt{2d}a_2w\|^2\right)
\]
\[
= \max\left(\|M_0a_1 + a_2w\|^2, \|M_0a_1 - a_2w\|^2\right)
\]
\[
\leq \|M_0 w\|_2^2 = \lambda_{\text{max}} (M_0M_0^\top + w^\top) \leq \gamma_{\text{max}}^2,
\]
Equation (61) implies that \(\|M_0 w\|_2^2 = \lambda_{\text{max}} (M_0M_0^\top + w^\top) \leq \gamma_{\text{max}}^2\), and therefore, for all \(\varepsilon \in \{-1, 1\}^{D-d}\) and \(z \in \{-1, 1\}\), we have
\[
\|L^{(\varepsilon, z)}\|_\infty = \sup_{\|p\| = 1} \|L^{(\varepsilon, z)}p\| \leq \gamma_{\text{max}},
\]
as desired.
Proof of part (b): Consider any pair of vectors \( p, q \in \mathbb{R}^D \) such that \( ||p|| = ||q|| = 1 \). Using the decompositions \( p = \begin{bmatrix} p^{(1)} \\ p^{(2)} \end{bmatrix} \) and \( q = \begin{bmatrix} q^{(1)} \\ q^{(2)} \end{bmatrix} \), with \( p^{(1)}, q^{(1)} \in \mathbb{R}^d \), \( p^{(2)}, q^{(2)} \in \mathbb{R}^{D-d} \), we have

\[
\mathbb{E}(p, (L^{(\epsilon,z)}_i - L^{(\epsilon,z)})q)^2 \leq \frac{1}{(2d)^2} \mathbb{E} \left( \sqrt{d} \varepsilon_{\tau_L}^{(2)}(w^\top p^{(1)})^2 \right) = \frac{1}{4d} \cdot \left( w^\top p^{(1)} \right)^2 \cdot \mathbb{E} \left( q^{(2)}_{\tau_L} \right)^2 \leq \frac{1}{4d} \|p^{(1)}\|^2_2 \cdot \|w\|^2_2 \cdot \frac{1}{D-d} \|q^{(2)}\|^2_2 \leq \|w\|^2_2 \cdot ||p||^2 \cdot ||q||^2 \leq \|w\|^2_2.
\]

Recall that \( M_0 M_0^\top + w w^\top \leq \gamma^2_{\max} I_d \) by equation (61). Consequently, we have \( \|w\|^2_2 \leq \|M_0^\top w\|^2 + \|w\|^2_2 \leq \gamma^2_{\max} \|w\|^2_2 \), which implies that \( \|w\|_2 \leq \gamma_{\max} \). Therefore, the noise assumption in equation (12a) is satisfied with parameter \( \sigma_L = \gamma_{\max} \).

For the noise on the vector \( b \), we note that

\[
\mathbb{E}(b^{(\epsilon,z)}_i - b, p)^2 \leq \frac{1}{4(D-d)^2} \mathbb{E} \left( \sqrt{2}(D-d)\varepsilon_{\tau_b}^{(2)}(p^{(2)}_{\tau_b})^2 \right) = \frac{\delta^2}{2} \mathbb{E} \left( p^{(2)}_{\tau_b} \right)^2 \leq \frac{\delta^2}{2} \cdot \frac{1}{D-d} \|p^{(2)}\|^2_2 \leq \delta^2 \|p\|^2 = \delta^2,
\]

showing the the noise assumption (12b) is satisfied with \( \sigma_b = \delta \).

5.3.2 Proof of Lemma 8

Recall that \( \tau_L^{(i)}, \tau_b^{(i)} \) are the random indices in the \( i \)-th sample. We define \( \mathcal{E} \) to be the event that the indices \( (\tau_L^{(i)})_{i=1}^n, (\tau_b^{(i)})_{i=1}^n \) are not all distinct, i.e.,

\[
\mathcal{E} := \left\{ \exists i_1, i_2 \in [n], \text{ s.t. } \tau_L^{(i_1)} = \tau_b^{(i_2)} \right\} \cup \left\{ \exists i_1 \neq i_2, \text{ s.t. } \tau_L^{(i_1)} = \tau_L^{(i_2)} \right\} \cup \left\{ \exists i_1 \neq i_2, \text{ s.t. } \tau_b^{(i_1)} = \tau_b^{(i_2)} \right\}.
\]

We claim that

\[
\mathbb{P}_1^{(n)}|\mathcal{E}^C = \mathbb{P}_2^{(n)}|\mathcal{E}^C.
\]

Assuming equation (65), we now give a proof of the upper bound on the total variation distance \( \delta \). We use the following lemma:

**Lemma 9.** Given two probability measures \( \mathbb{P}_1, \mathbb{P}_2 \) and an event \( \mathcal{E} \) with \( \mathbb{P}_1(\mathcal{E}), \mathbb{P}_2(\mathcal{E}) < 1/2 \), we have

\[
d_{TV}(\mathbb{P}_1, \mathbb{P}_2) \leq d_{TV}(\mathbb{P}_1|\mathcal{E}^C, \mathbb{P}_2|\mathcal{E}^C) + 3 \mathbb{P}_1(\mathcal{E}) + 3 \mathbb{P}_2(\mathcal{E}).
\]

In order to bound the probability of \( \mathcal{E} \), we apply a union bound. Under either of the probability measures \( \mathbb{P}_1^{(n)} \) and \( \mathbb{P}_2^{(n)} \), we have the following bound:

\[
\mathbb{P}(\mathcal{E}) \leq \sum_{i,j \in [n]} \mathbb{P}(\tau_L^{(i)} = \tau_b^{(j)}) + \sum_{i_1 < i_2} \mathbb{P}(\tau_L^{(i_1)} = \tau_L^{(i_2)}) + \sum_{i_1 < i_2} \mathbb{P}(\tau_b^{(i_1)} = \tau_b^{(i_2)}) \leq \frac{2n^2}{D-d}.
\]

Applying Lemma 9 in conjunction with equation (65) yields

\[
d_{TV}(\mathbb{P}_1^{(n)}, \mathbb{P}_2^{(n)}) \leq \frac{12n^2}{D-d}.
\]

It remains to prove claim (65) and Lemma 9.
Proof of equation (65): For $D \geq 2n + d$, we define a probability measure $Q$ through the following sampling procedure:

- Sample a subset $S \subseteq \{d + 1, \cdots, D\}$ of size $2n$ uniformly at random over all possible $\binom{D-d}{2n}$ possible subsets.
- Partition the set $S$ into two disjoint subsets $S = S_L \cup S_b$, each of size $n$. The partition is chosen uniformly at random over all $\binom{2n}{n}$ possible partitions. Let
  \[
  S_L := \left\{ \tau_L^{(1)}, \tau_L^{(2)}, \cdots, \tau_L^{(n)} \right\} \quad \text{and} \quad S_b := \left\{ \tau_b^{(1)}, \tau_b^{(2)}, \cdots, \tau_b^{(n)} \right\}.
  \]
- For each $i \in [n]$, sample two random bits $\zeta_L^{(i)}, \zeta_b^{(i)} \sim \mathcal{U}(-1, 1)$.
- Let $Q$ be the probability distribution of the observations $(L_i, b_i)_{i=1}^n$, that are constructed from the tuple $(\tau_L^{(i)}, \tau_b^{(i)}, \zeta_L^{(i)}, \zeta_b^{(i)})$ defined above. Specifically, we let
  \[
  L_i := \begin{bmatrix}
  M_0 & 0 & \cdots & 0 & \sqrt{d \zeta_L^{(i)}} w & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
  0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{bmatrix}, \quad
  b_i := \begin{bmatrix}
  \sqrt{2d} h_0 \\
  0 \\
  \vdots \\
  0 \\
  \sqrt{2(D-d) \zeta_b^{(i)}} \delta \\
  0 \\
  \vdots \\
  0 \\
  \end{bmatrix}
  \]
where the vector $\sqrt{d} \zeta_L^{(i)} w$ appears at the $\tau_L^{(i)}$-th column of the matrix $L_i$, and the scalar $\sqrt{2(D-d)} \zeta_b^{(i)} \delta$ appears at the $\tau_b^{(i)}$-th row of the vector $b_i$.

For either choice of the bit $z \in \{\pm 1\}$, we claim that the probability measure $\mathbb{P}_z^{(n)}|C$ is identical to the distribution $Q$. To prove this claim, we first note that conditioned on the event $\mathcal{E}^C$, the indices $(\tau_L^{(i)}, \tau_b^{(i)})_{i=1}^n$ actually form a uniform random subset of $\{d+1, \cdots, D\}$ with cardinality $2n$, and the partition into $(\tau_L^{(i)})_{i=1}^n$ and $(\tau_b^{(i)})_{i=1}^n$ is a uniform random partition, i.e.,
  \[
  \left( \tau_L^{(i)}, \tau_b^{(i)} \right)_{i=1}^n | \mathcal{E}^C \overset{d}{\sim} \left( \tau_L^{(i)}, \tau_b^{(i)} \right)_{i=1}^n, \quad \text{under both } \mathbb{P}_1^{(n)} \text{ and } \mathbb{P}_{-1}^{(n)}. \quad (67)
  \]

Given an index subset $(t_L^{(i)}, t_b^{(i)})_{i=1}^n \subseteq \{d+1, \cdots, D\}$ that are mutually distinct, conditioned on the value of $(\tau_L^{(i)}, \tau_b^{(i)})_{i=1}^n = (t_L^{(i)}, t_b^{(i)})_{i=1}^n$, the observed random bits under the probability distribution $\mathbb{P}_z^{(n)}$ are given by
  \[
  \varepsilon_{\tau_L^{(1)}}, \varepsilon_{\tau_L^{(2)}}, \cdots, \varepsilon_{\tau_L^{(n)}}, \varepsilon_{\tau_b^{(1)}}, \varepsilon_{\tau_b^{(2)}}, \cdots, \varepsilon_{\tau_b^{(n)}},
  \]
which are $2n$ independent Rademacher random variables.

On the other hand, the random bits $\zeta_L^{(1)}, \zeta_L^{(2)}, \cdots, \zeta_L^{(n)}, \zeta_b^{(1)}, \zeta_b^{(2)}, \cdots, \zeta_b^{(n)}$ are also $2n$ independent Rademacher random variables. Consequently, for any index subset $(t_L^{(i)}, t_b^{(i)})_{i=1}^n \subseteq \{d+1, \cdots, D\}$ that are mutually distinct, we have the following equality-in-distribution:
  \[
  \left( \varepsilon_{\tau_L^{(i)}}, \varepsilon_{\tau_b^{(i)}} \right)_{i=1}^n | (\tau_L^{(i)} = t_L^{(i)}, \tau_b^{(i)} = t_b^{(i)})_{i=1}^n \overset{d}{=} \left( \zeta_L^{(i)}, \zeta_b^{(i)} \right)_{i=1}^n | (\tau_L^{(i)} = t_L^{(i)}, \tau_b^{(i)} = t_b^{(i)})_{i=1}^n. \quad (68)
  \]
Putting equations (67) and (68) together completes the proof. □
Proof of Lemma 9: Given a function $f$ with range contained in $[0,1]$, we have
\[
\left| \int f(x)\mathbb{P}_1(dx) - \int f(x)\mathbb{P}_2(dx) \right| \\
\leq \left| \int \varepsilon f(x)\mathbb{P}_1(dx) \right| + \left| \int (1-\varepsilon) f(x)\mathbb{P}_2(dx) \right| + \left| \int \varepsilon f(x)\mathbb{P}_2(dx) - \int \varepsilon f(x)\mathbb{P}_1(dx) \right| \\
\leq \mathbb{P}_1(\varepsilon) + \mathbb{P}_2(1-\varepsilon) + \mathbb{P}_1(\varepsilon^C) \cdot \left| \frac{\int \varepsilon f(x)\mathbb{P}_1(dx)}{\mathbb{P}_1(\varepsilon^C)} - \frac{\int (1-\varepsilon) f(x)\mathbb{P}_2(dx)}{\mathbb{P}_2(\varepsilon^C)} \right| + \mathbb{P}_1(\varepsilon^C) - \mathbb{P}_2(\varepsilon^C) \\
\leq 3(\mathbb{P}_1(\varepsilon) + \mathbb{P}_2(1-\varepsilon)) + d_{TV}(\mathbb{P}_1|\varepsilon^C,\mathbb{P}_2|\varepsilon^C) + 2(\mathbb{P}_1(\varepsilon) - \mathbb{P}_2(\varepsilon)) \\
\leq 3(\mathbb{P}_1(\varepsilon) + \mathbb{P}_2(1-\varepsilon)) + d_{TV}(\mathbb{P}_1|\varepsilon^C,\mathbb{P}_2|\varepsilon^C),
\]
which completes the proof. \(\square\)

5.4 Proof of Theorem 3

In order to prove our local minimax lower bound, we make use of the Bayesian Cramér–Rao bound, also known as the van Trees inequality. In particular, we use a functional version of this inequality. It applies to a parametric family of densities $\{p_\eta, \eta \in \Theta\}$ w.r.t Lebesgue measure, with sufficient regularity so that the Fisher information matrix $I(\eta) := \mathbb{E}_\eta[\frac{\partial}{\partial n} \log p_\eta(X_1)))(\frac{\partial}{\partial n} \log p_\eta(X_1))^{\top}]$ is well-defined.

Proposition 2 (Theorem 1 of [GL95], special case). Given a prior distribution $\rho$ with bounded support contained within $\Theta$, let $T : \text{supp}(\rho) \mapsto \mathbb{R}^p$ denote a locally smooth functional. Then for any estimator $\hat{T}$ based on i.i.d. samples $X^n_i = \{X_i\}_{i=1}^n$ and for any smooth matrix-valued function $C : \mathbb{R}^d \mapsto \mathbb{R}^{p \times d}$, we have
\[
\mathbb{E}_{\eta \sim X^n_i \sim \rho_\eta} \mathbb{E} \|\hat{T}(X^n_i) - T(\eta)\|^2 \geq \frac{\left(\int \text{tr}(C(\eta) \frac{\partial}{\partial n}(\eta) ) \rho(\eta) d\eta \right)^2}{n \int \text{tr}(C(\eta)I(\eta)(C(\eta))^{\top}) \rho(\eta) d\eta + \int \|\nabla C(\eta) + C(\eta) \nabla \log \rho(\eta)\|^2 \rho(\eta) d\eta}.
\]

Recall that our lower bound is local, and holds for problem instances $(L, b)$ such that the pair $(\Phi_d L \Phi_d^*, \Phi_d b)$ is within a small $\ell_2$ neighborhood of a fixed pair $(M_0, h_0)$. We proceed by constructing a careful prior on such instances that will allow us to apply Proposition 2; note that it suffices to construct our prior over the $d \times d$ matrix $\Phi_d L \Phi_d^*$ and $d$-dimensional vector $\Phi_d b$. For the rest of the proof, we work under the orthonormal basis $\{\phi_j\}_{j=1}^d$. We also use the convenient shorthand $\tilde{x}_0 := \Phi_{\theta_0} \hat{v}_0$.

As a building block for our construction, we consider the one-dimensional density function $\mu(t) := \cos^2\left(\frac{\pi t}{2}\right) \cdot 1_{t \in [-1,1]}$, borrowed from Section 2.7 of Tsybakov [Tsy08]. It can be verified that $\mu$ defines a probability measure supported on the interval $[-1,1]$. We denote by $\mu^{\otimes d}$ the $d$-fold product measure of $\mu$. Let $Z$ and $Z'$ denote two random vectors drawn i.i.d. from the distribution $\mu^{\otimes d}$.

We use an auxiliary pair of $\mathbb{R}^d$-valued random variables $(\psi, \lambda)$ given by
\[
\psi := \frac{1}{\sqrt{n}} \Sigma_b^\frac{1}{2} Z \quad \text{and} \quad \lambda := \frac{1}{\sqrt{n}} \Sigma_L^\frac{1}{2} Z'.
\]
(69)

Our choice of this pair is motivated by the fact that the Fisher information matrix of this distribution takes a desirable form. In particular, we have the following lemma.

Lemma 10. Let $\rho : \mathbb{R}^{2d} \mapsto \mathbb{R}_+$ denote the density of $(\psi, \lambda)$ defined in equation (69). Then
\[
I(\rho) = n\pi \begin{bmatrix} \Sigma_b^{-1} & 0 \\ 0 & \Sigma_L^{-1} \end{bmatrix}.
\]
We now use the pair \((\psi, \lambda)\) in order to define the ensemble of population-level problem instances

\[
M^{(\psi, \lambda)} := M_0 + \|\bar{x}_0\|_2^2\lambda(\bar{x}_0)^\top \quad \text{and} \quad h^{(\psi, \lambda)} := h_0 + \psi.
\]  

(70)

In order to define the problem instance in the Hilbert space \(X\), we simply let \(L^{(\psi, \lambda)} := \Phi_d^* M^{(\psi, \lambda)} \Phi_d\) and \(b^{(\psi, \lambda)} := \Phi_d^* h^{(\psi, \lambda)}\), for a given basis \((\phi_i)_{i=1}^d\) in the space \(\mathcal{S}\).

The matrix-vector pair \((L^{(\psi, \lambda)}, b^{(\psi, \lambda)})\) induces the fixed point equation \(\bar{x}_{\psi, \lambda} = M^{(\psi, \lambda)} \bar{x}_{\psi, \lambda} + h^{(\psi, \lambda)}\), and its solution is given by

\[
\bar{x}_{\psi, \lambda} = (I - M^{(\psi, \lambda)})^{-1} h^{(\psi, \lambda)} = (I - M_0 - \|\bar{x}_0\|_2^2\lambda(\bar{x}_0)^\top)^{-1}(h_0 + \psi).
\]

Note that by construction, the Jacobian matrix formed by taking the partial derivative of \(\bar{x}_{\psi, \lambda}\) with respect to \(\psi\) and \(\lambda\) is given by

\[
\nabla_{\psi, \lambda} \bar{x}_{\psi, \lambda} = \left[ (I - M^{(\psi, \lambda)})^{-1} \|\bar{x}_0\|_2^2(\bar{x}_0)^\top (I - M^{(\psi, \lambda)})^{-1}(h_0 + \psi) \cdot (I - M^{(\psi, \lambda)})^{-1} \right].
\]

Now define the observation model via \(L_i^{(\psi, \lambda)} := \Phi_d^* M_i^{(\psi, \lambda)} \Phi_d\) and \(b_i^{(\psi, \lambda)} := \Phi_d^* h_i^{(\psi, \lambda)}\), where

\[
M_i^{(\psi, \lambda)} := M^{(\psi, \lambda)} + \|\bar{x}_0\|_2^2 w_i(\bar{x}_0)^\top \quad \text{and} \quad h_i^{(\psi, \lambda)} := h^{(\psi, \lambda)} + w_i',
\]

where \(w_i \sim N(0, \Sigma_L)\) and \(w_i' \sim N(0, \Sigma_b)\) are independent.

The following lemma certifies some basic properties of observation model constructed above.

**Lemma 11.** Consider the ensemble of problem instances defined in equations (70) and (71). For each pair \((\psi, \lambda)\) in the support of \(\rho\), each index \(j \in [d]\) and each unit vector \(u \in \mathbb{S}^{d-1}\), we have

\[
\|M^{(\psi, \lambda)} - M_0\|_F \leq \sigma_L \sqrt{\frac{d}{n}}, \quad \text{and} \quad \|h^{(\psi, \lambda)} - h_0\|_2 \leq \sigma_b \sqrt{\frac{d}{n}},
\]

(72a)

\[
\text{cov} \left( (M_i^{(\psi, \lambda)} - M^{(\psi, \lambda)}) \bar{x}_0 \right) = \Sigma_L, \quad \text{and} \quad \text{cov} \left( h_i^{(\psi, \lambda)} - h^{(\psi, \lambda)} \right) = \Sigma_b,
\]

(72b)

\[
\mathbb{E} \left( e_j^\top (M_i^{(\psi, \lambda)} - M^{(\psi, \lambda)}) u \right)^2 \leq \sigma_L^2, \quad \text{and} \quad \mathbb{E} \left( e_j^\top h_i^{(\psi, \lambda)} - h^{(\psi, \lambda)} \right)^2 \leq \sigma_b^2.
\]

(72c)

Lemma 11 ensures that our problem instance lies in the desired class. In particular, equation (72a) guarantees that the population-level problem instance \((L^{(\psi, \lambda)}, b^{(\psi, \lambda)})\) lies in the class \(\mathcal{C}_{\text{est}}\); on the other hand, equation (72b) and (72c) guarantees that the probability distribution \(\mathbb{P}_{L, b}\) we constructed lies in the class \(\mathcal{G}_{\text{cov}}\).

Some calculations yield that the Fisher information matrix for this observation model is given by

\[
I(\psi, \lambda) = \begin{bmatrix}
\Sigma_b^{-1} & 0 \\
0 & \Sigma_L^{-1}
\end{bmatrix},
\]

for any \(\psi, \lambda \in \mathbb{R}^d\).

We will apply Proposition 2 shortly, for which we use the following matrix \(C\):

\[
C(\psi, \lambda) := \nabla_{\psi, \lambda} \bar{x}_{\psi, \lambda}|_{(0, 0)} \cdot I(\psi, \lambda)^{-1} = \left[ (I - M_0)^{-1} \Sigma_b \ (I - M_0)^{-1} \Sigma_L \right].
\]

Note that by construction, the matrix \(C\) does not depend on the pair \((\psi, \lambda)\).
We also claim that if \( n \geq 16 \sigma^2 I \| (I - M_0)^{-1} \|_{op}^2 d \), then the following inequalities hold for our construction:

\[
T_b := \mathbb{E}_\rho \left[ \text{trace} \left( (I - M_0)^{-1} \Sigma_b (I - M^{(\psi, \lambda)})^{-\top} \right) \right] \geq \frac{1}{2} \text{trace} \left( (I - M_0)^{-1} \Sigma_b (I - M_0)^{-\top} \right) \quad \text{(73a)}
\]

\[
T_L := \mathbb{E}_\rho \left[ \frac{(\bar{x}_0)^\top \bar{x}_\psi,\lambda}{\|\bar{x}_0\|_2^2} \text{trace} \left( (I - M_0)^{-1} \Sigma_L (I - M^{(\psi, \lambda)})^{-\top} \right) \right] \geq \frac{1}{3} \text{trace} \left( (I - M_0)^{-1} \Sigma_L (I - M_0)^{-\top} \right). \quad \text{(73b)}
\]

Taking these two claims as given for the moment, let us complete the proof of the theorem. First, note that

\[
\mathbb{E}_\rho \left[ \text{trace} \left( C(\psi, \lambda) \cdot \nabla_{\psi, \lambda} \bar{x}^\top_{\psi, \lambda} \right) \right] = \mathbb{E}_\rho \left[ \text{trace} \left( (I - M_0)^{-1} \Sigma_b (I - M^{(\psi, \lambda)})^{-\top} \right) \right] + \mathbb{E}_\rho \left[ \frac{(\bar{x}_0)^\top \bar{x}_\psi,\lambda}{\|\bar{x}_0\|_2^2} \text{trace} \left( (I - M_0)^{-1} \Sigma_L (I - M^{(\psi, \lambda)})^{-\top} \right) \right] \geq \frac{5}{6} \text{trace} \left( (I - M_0)^{-1} \Sigma_b (I - M_0)^{-\top} \right). \quad \text{(74)}
\]

Second, since \( I(\cdot, \cdot) \) and \( C(\cdot, \cdot) \) are both constant functionals, we have that

\[
\mathbb{E} [\text{trace} (C(\psi, \lambda)I(\psi, \lambda)C(\psi, \lambda))] = \text{trace} \left( (I - M_0)^{-1} (\Sigma_L + \Sigma_b) (I - M_0)^{-\top} \right). \quad \text{(75)}
\]

Additionally, Lemma 10 yields

\[
\mathbb{E} \| \nabla \cdot C(\psi, \lambda) + C(\psi, \lambda) \cdot \nabla \log \rho(\psi, \lambda) \|^2_2 = \text{trace} \left( C(0, 0) \cdot \mathbb{E} \left[ \nabla \log \rho(\psi, \lambda) \nabla \log \rho(\psi, \lambda)^\top \right] C(0, 0)^\top \right) = n \pi \cdot \text{trace} \left( (I - M_0)^{-1} (\Sigma_L + \Sigma_b) (I - M_0)^{-\top} \right). \quad \text{(76)}
\]

We are finally in a position to put together the pieces. Applying Proposition 2 and combining equations (74), (75), and (76), we obtain the lower bound

\[
\int \mathbb{E} \| \bar{x}_n(L^n_1, b^n_1) - \bar{x}_{\psi, \lambda} \|^2_2 \rho(d\psi, d\lambda) \geq \frac{\text{trace} \left( (I - M_0)^{-1} (\Sigma_L + \Sigma_b) (I - M_0)^{-\top} \right)}{9(1 + \pi)n}, \quad \text{(77)}
\]

for any estimator \( \hat{x}_n \) that takes values in \( \mathbb{R}^d \).

For the problem instances we construct, note that

\[
\bar{x}^{(\psi, \lambda)} = (I - \Pi_S L^{(\psi, \lambda)})^{-1} \Pi_S b^{(\psi, \lambda)} = \Phi_d^* (I - M^{(\psi, \lambda)})^{-1} h^{(\psi, \lambda)} = \Phi_d^* \bar{x}_{\psi, \lambda}.
\]

For any estimator \( \hat{v}_n \in \hat{V}_X \), we note that

\[
\| \bar{x}^{(\psi, \lambda)} - \hat{v}_n \|^2 \geq \| \Pi_S (\bar{x}^{(\psi, \lambda)} - \hat{v}_n) \|^2 = \| \Phi_d \hat{v}_n - \bar{x}_{\psi, \lambda} \|^2.
\]

Recall by Lemma 11 that on the support of the prior distribution \( \rho \), the population-level problem instance \( (L^{(\psi, \lambda)}, b^{(\psi, \lambda)}) \) lies in the class \( \mathbb{C}_{est} \), and that the probability distribution \( \mathbb{P}_{L,b} \) we constructed lies in the class \( \mathbb{G}_{cov} \). We thus have the minimax lower bound

\[
\inf_{\hat{v}_n} \sup_{(L,b) \in \mathbb{C}_{est}} \mathbb{E} \| \hat{v}_n(L^n_1, b^n_1) - \bar{x}^{(\psi, \lambda)} \|^2 \geq \inf_{\hat{x}_n} \sup_{(\psi, \lambda) \in \text{supp}(\rho)} \mathbb{E} \| \hat{x}_n(L^n_1, b^n_1) - \bar{x}_{\psi, \lambda} \|^2_2 \geq \int \mathbb{E} \| \hat{x}_n(L^n_1, b^n_1) - \bar{x}_{\psi, \lambda} \|^2_2 \rho(d\psi, d\lambda) \geq c \cdot \frac{\text{trace} \left( (I - M_0)^{-1} \Sigma^* (I - M_0)^{-\top} \right)}{n}
\]

for \( c = \frac{1}{9(1 + \pi)} > 0 \), which completes the proof of the theorem.
5.4.1 Proof of Lemma 10

We first note that $\lambda$ is independent of $\psi$, and consequently $\rho = \rho_b \otimes \rho_a$, where $\rho_b, \rho_L$ are the marginal densities for $\psi$ and $\lambda$ respectively. Since the Fisher information tensorizes over product measures, it suffices to compute the Fisher information of $\rho_L$ and $\rho_b$ separately.

By a change of variables, we have

$$\rho_L(\lambda) = n^d \det(\Sigma_L)^{-\frac{1}{2}} \cdot \mu^{\otimes d} \left( \sqrt{n} \Sigma_L^{-1/2} \lambda \right).$$

Substituting this into the expression for Fisher information, we obtain

$$I(\rho_a) = \int (\nabla \log \rho_L(\lambda))(\nabla \log \rho_a(\lambda))^\top \rho_L(\lambda) d\lambda$$
$$= \int \left( \sqrt{n} \Sigma_L^{-1/2} \nabla \log \mu^{\otimes d}(y) \right) \cdot \left( \sqrt{n} \Sigma_L^{-1/2} \nabla \log \mu^{\otimes d}(z) \right)^\top \mu^{\otimes d}(z) dz$$
$$= n \Sigma_L^{-1/2} \cdot E_{Z \sim \mu^{\otimes d}} \left[ (\nabla \log \mu^{\otimes d}(Z))(\nabla \log \mu^{\otimes d}(Z))^\top \right] \cdot \Sigma_L^{-1/2}. $$

Finally, since $\mu^{\otimes d}$ is a product measure, we have $I(\mu^{\otimes d}) = I(\mu) \cdot I_d = \pi I_d$, and hence $I(\rho_L) = \pi n \Sigma_L^{-1}$. Reasoning similarly for $\rho_b$, we have that $I(\rho_b) = \pi n \Sigma_b^{-1}$. This completes the proof.

5.4.2 Proof of Lemma 11

We prove the three facts in sequence.

**Proof of equation (72a):** Note that the scalars $\sigma_L$ and $\sigma_b$ satisfies the compatibility condition (36), we therefore have the bounds

$$\| M^{(\psi,\lambda)} - M_0 \|_F = \| \bar{x}_0 \|_2^{-1} \cdot \| \lambda \|_2 \leq n^{-1/2} \| \bar{x}_0 \|_2^{-1} \cdot \sqrt{\text{trace}(\Sigma_L)} \leq \sigma_L \sqrt{\frac{d}{n}},$$

$$\| h_1^{(\psi,\lambda)} - h_0 \|_2 = \| \psi \|_2 \leq \sqrt{n \cdot \text{trace}(\Sigma_b)} \leq \sigma_b \sqrt{\frac{d}{n}},$$

which completes the proof of the first bound.

**Proof of equation (72b):** Straightforward calculation leads to the following identities

$$\text{cov} \left( (M_1^{\psi,\lambda} - M^{\psi,\lambda}) \bar{x}_0 \right) = \text{cov} \left( \| \bar{x}_0 \|_2^{-2} w_1 \bar{x}_0^\top \bar{x}_0 \right) = \text{cov}(w_1) = \Sigma_L;$$

$$\text{cov} \left( h_1^{\psi,\lambda} - h^{\psi,\lambda} \right) = \text{cov}(w'_1) = \Sigma_b.$$

**Proof of equation (72c):** Given any index $j \in [d]$ and vector $u \in S^{d-1}$, we note that:

$$\mathbb{E} \left( e_j^\top (M_1^{\psi,\lambda} - M^{\psi,\lambda}) u \right)^2 = \frac{1}{\| \bar{x}_0 \|^2} \mathbb{E} \left( e_j^\top w_1 \cdot \bar{x}_0^\top u \right)^2 \leq \frac{1}{\| \bar{x}_0 \|^2} \mathbb{E} \left( e_j^\top w_1 \right)^2 = \frac{1}{\| \bar{x}_0 \|^2} \| e_j^\top \Sigma_L e_j \|_2 \leq \sigma_L^2,$$

where the last inequality is due to the compatibility condition (36).

Similarly, for the noise $h_1^{\psi,\lambda} - h^{\psi,\lambda}$, we have:

$$\mathbb{E} \left( e_j^\top (h_1^{\psi,\lambda} - h^{\psi,\lambda}) \right)^2 = \mathbb{E} \left( e_j^\top w'_1 \right)^2 = e_j^\top \Sigma_b e_j \leq \sigma_b^2,$$

which completes the proof of the last condition.
5.4.3 Proof of claims (73)

We prove the two bounds separately, using the convenient shorthand $I_b$ for the LHS of claim (73a) and $I_a$ for the LHS of claim (73b).

**Proof of claim (73a):** We begin by applying the matrix inversion formula, which yields

$$(I - M^{(\psi, \lambda)})^{-1} = (I - M_0)^{-1} - \frac{1}{\|\bar{x}_0\|^2 - (\bar{x}_0)^\top (I - M_0)\lambda} (I - M_0)^{-1} \lambda (\bar{x}_0)^\top (I - M_0)^{-1}. \quad =: H$$

For $n \geq 16\sigma_L^2 d$, we have

$$\left| (\bar{x}_0)^\top (I - M_0)\lambda \right| \leq 2\|\bar{x}_0\|_2 \cdot \|\lambda\|_2 \leq 2\|\bar{x}_0\|_2 \sqrt{n^{-1} \text{trace}(\Sigma_L)} \leq 2\sigma_L \|\bar{x}_0\|_2^2 \sqrt{\frac{d}{n}} \leq \frac{1}{2} \|\bar{x}_0\|_2^2.$$ 

To bound $T_b$ from below, we note that

$$T_b = \text{trace} \left( (I - M_0)^{-1} \Sigma_b (I - M_0)^{-\top} \right) - \text{trace} \left( (I - M_0)^{-1} \Sigma_b \mathbb{E}[H^\top] \right)$$

$$\geq \text{trace} \left( (I - M_0)^{-1} \Sigma_b (I - M_0)^{-\top} \right) - \| (I - M_0)^{-1} \Sigma_b (I - M_0)^{-\top} \|_{\text{nuc}} \cdot \| (I - M_0)^\top \mathbb{E}[H]^\top \|_{\text{op}}$$

$$= \text{trace} \left( (I - M_0)^{-1} \Sigma_b (I - M_0)^{-\top} \right) \cdot (1 - \| \mathbb{E}[H](I - M_0) \|_{\text{op}}).$$

When $n \geq 4\sigma_L^2 d$, we have

$$\| \mathbb{E}[H](I - M_0) \|_{\text{op}} \leq \sum_{k=0}^{\infty} \frac{1}{\|\bar{x}_0\|^2_2} \mathbb{E} \left[ \left( \frac{(\bar{x}_0)^\top (I - M_0)\lambda}{\|\bar{x}_0\|^2_2} \right)^k (I - M_0)^{-1} \lambda (\bar{x}_0)^\top \right] \|_{\text{op}}$$

$$\leq \frac{1}{\|\bar{x}_0\|^2_2} \sum_{k=1}^{\infty} \mathbb{E} \left( \left| \frac{(\bar{x}_0)^\top (I - M_0)\lambda}{\|\bar{x}_0\|^2_2} \right|^k \cdot \| (I - M_0)^{-1} \lambda (\bar{x}_0)^\top \|_F \right)$$

$$\leq 2\sigma_L^2 \| (I - M_0)^{-1} \|_{\text{op}} \frac{d}{n}.$$ 

Therefore, for $n \geq 16\sigma_L^2 \| (I - M_0)^{-1} \|_{\text{op}} d$, we obtain the lower bound

$$T_b \geq \frac{1}{2} \text{trace} \left( (I - M_0)^{-1} \Sigma_b (I - M_0)^{-\top} \right),$$

as desired.

**Proof of claim (73b):** We note that

$$\bar{x}_{\psi, \lambda} = \bar{x}_0 - Hh_0 + (I - M_0)\psi - H\psi.$$

Consequently, we have

$$T_L = \mathbb{E} \left[ \frac{(\bar{x}_0)^\top (\bar{x}_0 - Hh_0 + (I - M_0)\psi - H\psi)}{\|\bar{x}_0\|^2} \cdot \text{trace} \left( (I - M_0)^{-1} \Sigma_L ((I - M_0)^{-\top} - H^\top) \right) \right].$$
Since $\lambda$ is independent of $\psi$ and $H$ is dependent only upon $\lambda$, by taking expectation with respect to $\psi$, we have that

$$T_L = \mathbb{E} \left[ \frac{(\bar{x}_0)\top (\bar{x}_0 - Hh_0)}{\|\bar{x}_0\|^2_2} \cdot \text{trace} \left( (I - M_0)^{-1} \Sigma_L ((I - M_0)^{-\top} - H\top) \right) \right].$$

We note that

$$\frac{|(\bar{x}_0)\top Hh_0|}{\|\bar{x}_0\|^2_2} = \frac{|(\bar{x}_0)\top (I - M_0)^{-1} \lambda|}{\|\bar{x}_0\|^2_2 - (\bar{x}_0)\top (I - M_0)} \leq 2 \|(I - M_0)^{-1} \|_{\text{op}} \sigma_L \sqrt{\frac{d}{n}}.$$

Therefore, for $n \geq 16\sigma_L^2 \|(I - M_0)^{-1}\|_{\text{op}}^2 d$, we have the bound $\frac{1}{2} \leq \frac{(\bar{x}_0)\top (\bar{x}_0 - Hh_0)}{\|\bar{x}_0\|^2_2} \leq \frac{3}{2}$, and consequently, we have

$$T_L \geq \frac{1}{2} \text{trace} \left( (I - M_0)^{-1} \Sigma_L (I - M_0)^{-\top} \right) - \frac{3}{2} \mathbb{E} \left| \text{trace} \left( (I - M_0)^{-1} \Sigma_L H\top \right) \right|$$

$$\geq \frac{1}{2} \text{trace} \left( (I - M_0)^{-1} \Sigma_L (I - M_0)^{-\top} \right) - \frac{3}{2} \|(I - M_0)^{-1} \Sigma_L (I - M_0)^{-\top}\|_{\text{nuc}} \cdot \mathbb{E} \|H(I - M_0)\|_{\text{op}}$$

$$\geq \frac{1}{2} \text{trace} \left( (I - M_0)^{-1} \Sigma_L (I - M_0)^{-\top} \right) \cdot (1 - 3\mathbb{E} \|H(I - M_0)\|_{\text{op}}).$$

For the matrix $H(I - M_0)$, we have the almost-sure upper bound

$$\|H(I - M_0)\|_{\text{op}} \leq \frac{2}{\|\bar{x}_0\|^2_2} \|(I - M_0)^{-1} \lambda(\bar{x}_0)^\top\|_{\text{op}} \leq \frac{2}{\|\bar{x}_0\|^2_2} \cdot \|(I - M_0)^{-1} \lambda\|_2 \leq 2 \|(I - M_0)^{-1} \|_{\text{op}} \sigma_L \sqrt{\frac{d}{n}}.$$

Thus, provided $n \geq 18\sigma_L^2 \|(I - M_0)^{-1}\|_{\text{op}}^2 d$, putting together the pieces yields the lower bound

$$T_L \geq \frac{1}{3} \text{trace} \left( (I - M_0)^{-1} \Sigma_L (I - M_0)^{-\top} \right).$$

### 5.5 Proof of Corollary 2

Define the terms $\Delta_1 := \frac{2}{3} \alpha(M_0, \gamma_{\text{max}}) \delta^2$ and $\Delta_2 := c \cdot \mathcal{E}_n(M_0, \Sigma_L + \Sigma_b)$. We split our proof into two cases.

**Case I: if $\Delta_1 \leq \Delta_2$.** Consider the function class:

$$\bar{C} := \bigcup_{(M', h') \in \mathcal{G}(M_0, h_0)} C_{\text{approx}}(M', h', D, 0, \gamma_{\text{max}})$$

Clearly, we have the inclusion $\bar{C} \subseteq C_{\text{final}}$. Moreover, for a problem instance in $\bar{C}$, we have $A(S, v^*) = 0$, and consequently $v^* = \bar{v}$. Note that the construction of problem instances in Theorem 3 can be embedded in $\mathbb{X}$ of any dimension $D$, and the linear operator $L$ constructed in the proof of Theorem 3 satisfies the bound

$$\|L(\psi, \lambda)\|_\mathbb{X} \leq \|M(\psi, \lambda)\|_{\text{op}} \leq \|M_0\|_{\text{op}} + \sigma_L \sqrt{\frac{d}{n}} \leq \gamma_{\text{max}}.$$
Consequently, the class \( \tilde{C} \) contains the population-level problem instances \( (M(\psi, \lambda), h(\psi, \lambda)) \) constructed in the proof of Theorem 3, for any choice of \( \psi, \lambda \in \mathbb{R}^d \). Invoking Theorem 3, we thus obtain the sequence of bounds

\[
\inf_{\hat{v}_n \in \hat{V}_X} \sup_{(L, b) \in \mathcal{C}_{\text{final}}} \mathbb{E}\|\hat{v}_n - v^*\|^2 \geq \inf_{\hat{v}_n \in \hat{V}_X} \sup_{(L, b) \in \tilde{C}} \mathbb{E}\|\hat{v}_n - v^*\|^2 = \inf_{\hat{v}_n \in \hat{V}_X} \sup_{(L, b) \in \mathcal{C}_{\text{cov}}} \mathbb{E}\|\hat{v}_n - \hat{v}\|^2 = \inf_{\hat{v}_n \in \hat{V}_X} \sup_{(L, b) \in \hat{C}_{\text{cov}}} \mathbb{E}\|\hat{v}_n - \hat{v}\|^2 \geq \Delta_2.
\]

**Case II: if \( \Delta_1 > \Delta_2 \).** In this case, we consider the class of noise distributions

\( \hat{G} := \mathcal{G}_{\text{cov}}(0, 0, \sigma_L, \sigma_b) \).

Clearly, \( \hat{G} \) is a sub-class of \( \mathcal{G}_{\text{cov}} \). Note that the observation model \( \left( L_i^{(\varepsilon, \gamma)} b_i^{(\varepsilon, \gamma)} \right)_{i=1}^n \) constructed in the proof of Theorem 2 satisfies the following identities almost surely:

\[
\Phi_d L_i^{(\varepsilon, \gamma)} \Phi_d^* = \Phi_d L_i^{(\varepsilon, \gamma)} \Phi_d^*, \quad \Phi_d b_i^{(\varepsilon, \gamma)} = \Phi_d b_i^{(\varepsilon, \gamma)}.
\]

So the problem instances constructed in the proof of Theorem 2 belongs to class \( \tilde{G} \). Invoking Theorem 2, we obtain the bound

\[
\inf_{\hat{v}_n \in \hat{V}_X} \sup_{(L, b) \in \mathcal{C}_{\text{final}}} \mathbb{E}\|\hat{v}_n - v^*\|^2 \geq \inf_{\hat{v}_n \in \hat{V}_X} \sup_{(L, b) \in \mathcal{C}_{\text{final}}} \mathbb{E}\|\hat{v}_n - v^*\|^2 \geq \frac{2}{3} (\alpha(M_0, \gamma_{\text{max}}) - 1) \delta^2 = \Delta_1.
\]

Combining the results in two cases, we arrive at the lower bound:

\[
\inf_{\hat{v}_n \in \hat{V}_X} \sup_{(L, b) \in \mathcal{C}_{\text{final}}} \mathbb{E}\|\hat{v}_n - v^*\|^2 \geq \max(\Delta_1, \Delta_2)
\]

\[
\geq \frac{1}{2} \Delta_1 + \frac{1}{2} \Delta_2 \geq \frac{1}{3} (\alpha(M_0, \gamma_{\text{max}}) - 1) \delta^2 + \frac{c}{2} \mathcal{E}_n(M, \Sigma_L + \Sigma_b),
\]

which completes the proof of the corollary.

### 6 Discussion

In this paper, we studied methods for computing approximate solutions to fixed point equations in Hilbert spaces, using methods that search over low-dimensional subspaces of the Hilbert space, and operate on stochastic observations of the problem data. We analyzed a standard stochastic approximation scheme involving Polyak–Ruppert averaging, and proved non-asymptotic instance-dependent upper bounds on its mean-squared error. This upper bound involved a pure approximation error term, reflecting the discrepancy induced by searching over a finite-dimensional subspace as opposed to the Hilbert space, and an estimation error term, induced by the noisiness in the observations. We complemented this upper bound with an information-theoretic analysis, that established instance-dependent lower bounds for both the approximation error and the estimation error. A noteworthy consequence of our analysis is that the optimal approximation factor in the oracle inequality is neither unity nor constant,
but a quantity depending on the projected population-level operator. As direct consequences of our general theorems, we showed oracle inequalities for three specific examples in statistical estimation: linear regression on a linear subspace, Galerkin methods for elliptic PDEs, and value function estimation via temporal difference methods in Markov reward processes.

The results of this paper leave open a number of directions for future work:

• This paper focused on the case of independently drawn observations. Another observation model, one which arises naturally in the context of reinforcement learning, is the Markov observation model. As discussed in Section 2.2.3, we consider the problem with $L = \gamma P$ and $b = r$, where $P$ is a Markov transition kernel, $\gamma$ is the discount factor and $r$ is the reward function. The observed states and rewards in this setup are given by a single trajectory of the Markov chain $P$, instead of i.i.d. from the stationary distribution. It is known [TVR97] that the resolvent formalism (a.k.a. TD($\lambda$)) leads to an improved approximation factor with larger $\lambda \in [0, 1)$. On the other hand, larger choice of $\lambda$ may lead to larger variance and slower convergence for the stochastic approximation estimator, and a model selection problem exists (See Section 2.2 in the monograph [Sze10] for a detailed discussion). It is an important future work to extend our fine-grained risk bounds to the case of TD($\lambda$) methods with Markov data. Leveraging the instance-dependent upper and lower bounds, one can also design and analyze estimators that achieve the optimal trade-off.

• This paper focused purely on oracle inequalities defined with respect to a subspace. However, the framework of oracle inequalities is far more general; in the context of statistical estimation, one can prove oracle inequalities for any star-shaped set with bounds on its metric entropy. (See Section 13.3 in the monograph [Wai19a] for the general mechanism and examples.) For all the three examples considered in Section 2.2, one might imagine approximating solutions using sets with nonlinear structure, such as those defined by $\ell_1$-constraints, Sobolev ellipses, or the function class representable by a given family neural networks. An interesting direction for future work is to understand the complexity of projected fixed point equations defined by such approximating classes.

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A Proof of Theorem 4

We begin by defining some additional notation needed in the proof. For a non-negative integer \( k \), we use \( H_k \in \mathbb{R}^{2^k \times 2^k} \) to denote the Hadamard matrix of order \( k \), recursively defined as:

\[
H_0 := 1, \quad H_k := \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{bmatrix}, \quad \text{for } k = 1, 2, \ldots
\]
For any integer $q \geq 2$, we define

$$J_q := \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix},$$

which is a $q \times q$ Jordan block with zeros in the diagonal.

Now we turn to the proof of the theorem. We assume that $D$ is an integer multiple of $q$, and that $m := \frac{D-d}{q}$ is an integer power of 2; the complementary case can be handled by adjusting the constant factors in our bounds. Similarly to the proof of Theorem 2, we let $u \in \mathbb{S}^{d-1}$ be an eigenvector associated to the largest eigenvalue of the matrix $(I - M_0)^{-1}(\gamma_{\max}^2 I - M_0 M_0^T) (I - M_0)^{-T}$, and define the $d$-dimensional vectors:

$$w := \sqrt{\alpha(M_0, \gamma_{\max}) - 1} \cdot (I - M_0) u,$$

and

$$y := \sqrt{\alpha(M_0, \gamma_{\max}) - 1} \cdot \delta u.$$

We first construct the following $(D-d) \times (D-d)$ block matrix, indexed by the bits $(\varepsilon_{ij})_{1 \leq i \leq q, 1 \leq j \leq m}$:

$$J(\varepsilon) := \begin{bmatrix}
I_m & \text{diag}(\varepsilon_{1j} \cdot \varepsilon_{2j})_{j=1}^m & 0 & \cdots & 0 & 0 \\
0 & I_m & \text{diag}(\varepsilon_{2j} \cdot \varepsilon_{3j})_{j=1}^m & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & I_m & \text{diag}(\varepsilon_{(q-1)j} \cdot \varepsilon_{qj})_{j=1}^m & 0 \\
0 & 0 & \cdots & 0 & I_m
\end{bmatrix}. \quad (78a)$$

Each submatrix depicted above is an $m \times m$ matrix, and the diagonal blocks are given by identity matrices. We use this construction to define the population-level instance $(L(\varepsilon,z), b(\varepsilon,z))$ as follows:

$$L(\varepsilon,z) = \begin{bmatrix}
M_0 & \sqrt{\frac{d}{d-q}} \varepsilon_{11} w & \sqrt{\frac{d}{d-q}} \varepsilon_{12} w & \cdots & \sqrt{\frac{d}{d-q}} \varepsilon_{1m} w & 0 & \cdots & 0 \\
0 & 0 & \cdots & \varepsilon_{11}^2 w & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \varepsilon_{1m}^2 w & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad b(\varepsilon,z) = \begin{bmatrix}
\sqrt{2d} h_0 \\
0 \\
\vdots \\
0 \\
\delta \sqrt{2} \varepsilon_{q1} \\
\vdots \\
\delta \sqrt{2} \varepsilon_{qm}
\end{bmatrix}. \quad (78b)$$

It can then be verified that the solution to the fixed point equation $v_{\varepsilon,z}^* = (I - L(\varepsilon,z))^{-1} b(\varepsilon,z)$ is given by

$$v_{\varepsilon,z}^* = \begin{bmatrix}
\sqrt{\frac{d}{q}} z y + \sqrt{2d}(I - M_0)^{-1} h_0 \\
\sqrt{2} \delta \varepsilon_{11} \\
\sqrt{2} \delta \varepsilon_{12} \\
\vdots \\
\sqrt{2} \delta \varepsilon_{q1} \\
\vdots \\
\sqrt{2} \delta \varepsilon_{qm}
\end{bmatrix}. \quad (78c)$$
Similarly to the proof of Theorem 2, we take the subspace $S$ to be the one spanned by first $d$ coordinates, and take the weight vector be

$$\xi = \begin{bmatrix} \frac{1}{2d} & \cdots & \frac{1}{2d} & \frac{1}{2(D-d)} & \cdots & \frac{1}{2(D-d)} \end{bmatrix}.$$  

The inner product is then given by \( \langle p, p' \rangle := \sum_{j=1}^{D} p_j \xi_j p'_j \) for vectors $p, p' \in \mathbb{R}^D$.

It remains to define our basis vectors. For $j \in [d]$, we let $\phi_j := \sqrt{2d} e_j$. For the orthogonal complement $S^\perp$, we use the basis vectors

$$\begin{bmatrix} \phi_{d+1} & \phi_{d+2} & \cdots & \phi_{d+q} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2q} H_{\log_2 m} \otimes I_q \end{bmatrix}.$$  

Recall that $H_k \in \mathbb{R}^{2k \times 2^k}$ denotes the Hadamard matrix of order $k$, and $\otimes$ denotes Kronecker product. The first $d$ rows of the matrix are zeros, while the following $D - d = mq$ columns are given by the Kronecker product. By the definition of Hadamard matrix, we have $\|\phi_i\| = 1$ and $\langle \phi_i, \phi_j \rangle = 0$ for $i \neq j$.

As before, the construction of equations (78a)-(78c) ensures that for any choice of the binary string $\epsilon \in \{-1, 1\}^{mq}$ and bit $y \in \{-1, 1\}$, the oracle approximation error is equal to

$$A(S, v^*_\epsilon, y) = \inf_{v \in S} \|v^*_\epsilon - v\|^2 = \frac{1}{2(D-d)} \sum_{i=1}^{q} \sum_{j=1}^{m} (\sqrt{2} \delta_{ij})^2 = \delta^2. \quad (79a)$$

Furthermore, straightforward calculation yields that the projected matrix-vector pair takes the form

$$\Phi_d^* L^{(\epsilon, z)} \Phi_d^* = M_0, \quad \text{and} \quad \Phi_d^* L^{(\epsilon, z)} = h_0. \quad (79b)$$

Now, we construct our observation model from which samples $(L_i^{(\epsilon, z)}, b_i^{(\epsilon, z)})_{i=1}^n$ are generated. For each $i \in [n]$ and $j \in [m]$, we sample independently Bernoulli random variables $\xi_{0j}^{(i)}, \xi_{1j}^{(i)}, \ldots, \xi_{qj}^{(i)} \sim \text{Ber}(1/m)$. The random observations are then generated by the random matrix

$$J_i^{(\epsilon)} := \begin{bmatrix} I_m & \text{diag} \left( m \chi_{1j}^{(i)} \varepsilon_{1j} \varepsilon_{2j} \right)_{j=1}^m & 0 & \cdots & 0 & 0 \\
0 & I_m & \text{diag} \left( m \chi_{2j}^{(i)} \varepsilon_{2j} \varepsilon_{3j} \right)_{j=1}^m & \cdots & 0 & 0 \\
0 & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I_m & \text{diag} \left( m \chi_{q(j-1)}^{(i)} \varepsilon_{(q-1)j} \varepsilon_{qj} \right)_{j=1}^m & 0 \\
0 & 0 & \cdots & 0 & I_m \\
0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}, \quad (80a)$$

where once again, the diagonal blocks correspond to $m \times m$ identity matrices. We use this random matrix to generate the observations

$$L_i^{(\epsilon, z)} = \begin{bmatrix} M_0 & \chi_{01} \sqrt{\frac{d}{q}} z \varepsilon_{11} w & \chi_{02} \sqrt{\frac{d}{q}} z \varepsilon_{12} w & \cdots & \chi_{0m} \sqrt{\frac{d}{q}} z \varepsilon_{1m} w & 0 & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \frac{1}{2} J_i^{(\epsilon)} \\
\end{bmatrix} \quad (80b)$$

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\[ b_i^{(ε,z)} = \left[ \sqrt{2dh_0^\top} \begin{array}{cccc} 0 & \cdots & 0 & m\chi_{q1} \frac{\delta}{\sqrt{2}} \varepsilon_{q1} & \cdots & m\chi_{qm} \frac{\delta}{\sqrt{2}} \varepsilon_{qm} \end{array} \right]^\top. \] (80c)

This concludes our description of the problem instances themselves. As before, our proof proceeds via Le Cam’s lemma, and we use similar notation for product distributions and mixtures under this observation model. Let \( P_i^{(n)} \) denote the \( n \)-fold product of the probability laws of \( (L_i^{(ε,z)}, b_i^{(ε,z)}) \). We also define the following mixture of product measures for each \( z \in \{−1, 1\} \):

\[ P_z^{(n)} := \frac{1}{2^{2-d}} \sum_{ε \in \{±1\}^{m \times q}} P_i^{(n)} \varepsilon,z. \]

We seek bounds on the total variation distance

\[ ∆ := d_{TV} \left( P_1^{(n)}, P_{−1}^{(n)} \right). \]

With this setup, the following lemmas assert that (a) Our construction satisfies the operator norm condition and the noise conditions in Assumption 1(S) with the associated parameters bounded by dimension-independent constants, and (b) The total variation distance \( ∆ \) is small provided \( n \ll m^{1+1/q} \).

**Lemma 12.** For \( q \in \left[ 2, \frac{1}{2(1−1/γ_{\max})} \right] \), and any \( ε \in \{−1, 1\}^{m \times q} \) and \( z \in \{−1, 1\} \),

(a) The construction in equation (78b) satisfies the bound \( \|L^{(ε,z)}\|_{\infty} ≤ γ_{\max} \).

(b) The observation model defined in equation (80a)-(80c) satisfies Assumption 1(S) with \( σ_{L} = γ_{\max} + 1 \) and \( σ_{b} = \frac{δ}{q} \).

**Lemma 13.** Under the setup above, we have

\[ ∆ ≤ \frac{12nq^2 + 1}{m^q}. \]

Part (a) of Lemma 12 and equation (79a)-(79b) together ensure that population-level problem instance \( (L, b) \) we constructed belongs to the class \( C_{\text{approx}}(M_0, h_0, D, δ, γ_{\max}) \). Part (b) of Lemma 12 further ensures the probability distribution \( P_{L,b} \) belongs to the class \( G_{\text{var}}(σ_{L}, σ_{b}) \). Lemma 13 ensures that the two mixture distributions corresponding to different choices of the bit \( z \) are close provided \( n \) is not too large. The final step in applying Le Cam’s mixture-vs-mixture result is to show that the approximation error is large for at least one of the choices of the bit \( z \).

Given any pair \( ε, ε' \in \{−1, 1\}^{q \times m} \), we note that

\[ \|v_{ε,1}^* − v_{ε',1}^*\| ≥ \left\| \begin{array}{cccc} 2\sqrt{\frac{d}{q}} y^\top & 0 & \cdots & 0 \end{array} \right\|_2 = \frac{\sqrt{d}}{q} \sqrt{\alpha(M_0, γ_{\max}) − 1} · δ. \]

Applying triangle inequality and Young’s inequality, we have the bound

\[
\frac{1}{2} \left( \|\hat{v} − v_{ε,1}^*\|^2 + \|\hat{v} − v_{ε',1}^*\|^2 \right) ≥ \frac{1}{4} \left( \|\hat{v} − v_{ε,1}^*\| + \|\hat{v} − v_{ε',1}^*\| \right)^2 ≥ \frac{1}{4} \|v_{ε,1}^* − v_{ε',1}^*\|^2 ≥ \frac{\alpha(M_0, γ_{\max}) − 1}{2q^2} δ^2.
\]
Finally, applying Le Cam’s lemma yields
\[
\inf_{\tilde{v}_n \in \mathcal{V}_X} \sup_{(L,b) \in \mathcal{C}_{\text{approx}}} \mathbb{E}\|\tilde{v}_n - v^*\|^2 \geq \frac{\alpha(M_0, \gamma_{\max}) - 1}{2q^2} \cdot \left(1 - d_{TV}(P_{1,n}, P_{2,n})\right)
\]
and using Lemma 13 in conjunction with the condition \(D \geq d + 3q\nu^{1+1/q}\), we arrive at the final bound
\[
\inf_{\tilde{v}_n \in \mathcal{V}_X} \sup_{(L,b) \in \mathcal{C}_{\text{approx}}} \mathbb{E}\|\tilde{v}_n - v^*\|^2 \geq \frac{\alpha(M_0, \gamma_{\max}) - 1}{2q^2} \delta^2
\]
as desired.

A.1 Proof of Lemma 12

We prove the two parts of the lemma separately.

**Proof of part (a):** We first show the upper bound on the operator norm. For any vector \(p \in \mathbb{R}^D\), we employ the decomposition \(p = [p_1 \, p_2]\) with \(p_1 \in \mathbb{R}^d\) and \(p_2 \in \mathbb{R}^{qm}\). Assuming that \(\|p\|^2 = \frac{1}{2d}\|p_1\|^2 + \frac{1}{2(D-d)}\|p_2\|^2 = 1\), we have that
\[
\|L^{(\varepsilon,z)}p\|^2 = \frac{1}{2d}\|M_0p_1 + w \cdot \frac{z\sqrt{d}}{D-d} \sum_{j=1}^{m} \varepsilon_j p_2(j)\|^2 + \frac{1}{2(D-d)}\|J^{(\varepsilon)}p_2\|^2.
\]

Define the vector \(a_1 := \frac{1}{\sqrt{2d}}p_1^{(1)}\) and scalar \(a_2 := \frac{1}{\sqrt{2(D-d)}}\|p_2^{(2)}\|_2\) for convenience; we have the identity \(\|a_1\|^2 \approx a_2^2 = 1\). Following the same arguments as in the proof of Lemma 7, we then have
\[
\frac{1}{2d}\|M_0p_1 + w \cdot \frac{z\sqrt{d}}{D-d} \sum_{j=1}^{m} \varepsilon_j p_2(j)\|^2 \leq \frac{1}{2d} \sup_{t \in [-1,1]} \|M_0p_1 + \frac{\sqrt{d}}{q}a_2 tw\|^2 \\
\leq \max \left(\|M_0a_1 + \frac{1}{q\sqrt{2}}a_2 w\|_2^2, \|M_0a_1 - \frac{1}{q\sqrt{2}}a_2 w\|_2^2\right) \leq \|M_0 w\|_{\text{op}}^2 \left(\|a_1\|^2 + \frac{1}{2q^2a_2^2}\right).
\]

By the definition of the vector \(w\), we have the bound
\[
\|M_0 w\|_{\text{op}}^2 = \lambda_{\max} \left(M_0M_0^T + ww^T\right) \leq \gamma_{\max}^2.
\]
On the other hand, note that
\[
\frac{1}{2(D-d)}\|J^{(\varepsilon)}p_2\|^2 \leq \frac{1}{8(D-d)}\|J^{(\varepsilon)}\|_{\text{op}}^2 \cdot \|p_2\|^2 = \frac{1}{4}\|J^{(\varepsilon)}\|_{\text{op}}^2 a_2^2,
\]
and consequently, that
\[
\|L^{(\varepsilon,z)}p\|^2 \leq \gamma_{\max}^2 \left(\|a_1\|^2 + \frac{1}{2q^2a_2^2}\right) + \frac{1}{4}\|J^{(\varepsilon)}\|_{\text{op}}^2 a_2^2.
\]
In order to bound the operator norm of the matrix $J^{(ε)}$, we use the following fact about operator norm, proved at the end of this section for convenience. For any block matrix $T = [T_{ij}]_{1 \leq i,j \leq q}$, with each block $T_{ij} \in \mathbb{R}^{m \times m}$, we have

$$\| [T_{ij}]_{1 \leq i,j \leq q} \|_{op} \leq \| [T_{ij}]_{1 \leq i,j \leq q} \|_{op}. \quad (82)$$

Applying equation (A.1) to matrix $J^{(ε)}$ yields the bound

$$\| J^{(ε)} \|_{op} \leq \| I_q + J_q \|_{op}.$$ 

Recall that $J_q$ is the Jordan block of size $q$ with zeros in the diagonal. Straightforward calculation yields the bound

$$(I_q + J_q)(I_q + J_q)^T \preceq \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix} =: C_q.$$ 

Note that $C_q$ is a tridiagonal Toeplitz matrix, whose norm admits the closed-form expression

$$\| C_q \|_{op} = 2 + 2 \cos \left( \frac{π}{q + 1} \right) \leq 4 - \frac{4}{q^2}.$$ 

Therefore, we have $\| J^{(ε)} \|_{op} \leq \sqrt{\| C_q \|_{op}} \leq \sqrt{4 - 4/q^2}$. Substituting into equation (81), we obtain

$$\| L^{(ε,z)}p \|^2 \leq \gamma_{\text{max}}^2 \left( \| a_1 \|^2 + \frac{1}{2q^2} a_2^2 \right) + \left( 1 - \frac{1}{q^2} \right) a_2^2.$$ 

Invoking the condition $q \leq \frac{1}{\sqrt{2(1 - \gamma_{\text{max}})}},$ we have the bound

$$\| L^{(ε,z)}p \|^2 \leq \gamma_{\text{max}}^2 \left( \| a_1 \|^2 + \frac{a_2^2}{2q} \right) + \left( 1 - \frac{1}{q^2} \right) a_2^2 \leq \gamma_{\text{max}}^2 (\| a_1 \|^2 + a_2^2) = \gamma_{\text{max}}^2.$$ 

Since the choice of the vector $p$ is arbitrary, this yields

$$\| L^{(ε,z)} \|_{X} \leq \gamma_{\text{max}},$$

which completes the proof.

**Proof of part** (b): Next, we verify the noise conditions in Assumption 1(S). For a vector $p \in X$ such that $\| p \| = 1$, denote it with $p = \begin{bmatrix} p^{(1)} \\ p^{(2)} \end{bmatrix}$, where $p^{(1)} \in \mathbb{R}^d$ and $p^{(2)} \in \mathbb{R}^{D-d}$. We have the identity $\frac{1}{2d} \| p^{(1)} \|_2^2 + \frac{1}{2(D-d)} \| p^{(2)} \|_2^2 = 1$.

For the noise $b_i^{(ε,z)} - b_i^{(ε,z)}$, we note that

$$\mathbb{E} \langle p, b_i^{(ε,z)} - b_i^{(ε,z)} \rangle^2 \leq \frac{1}{4(D-d)^2} \sum_{j=1}^{m} \mathbb{E} \left( \left( mX_{qj} - 1 \right) \frac{\delta}{\sqrt{2}} \varepsilon_{qj} \right)^2 \left( p^{(2)}_{(q-1)m+j} \right)^2 \leq \frac{1}{4(D-d)^2} \cdot \frac{m\delta^2}{2} \| p^{(2)} \|_2^2 \leq \frac{\delta^2}{q^2}.$$ 

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Consequently, equation (12b) is satisfied for $\sigma_b = \delta/q$.

In order to bound the noise in the $L$ component, we consider the first $d$ basis vectors and the last $(D - d)$ basis vectors separately. First, note that for each $k \in [d]$, we have

$$\mathbb{E}\langle \phi_k, (L_i^{(\varepsilon, z)} - L^{(\varepsilon, z)}) p \rangle^2 = \frac{1}{4d^2} \mathbb{E}\left(\frac{\sqrt{d/2}}{D - d} \sum_{j=1}^{m} z(m\chi_{k_0j}^{(i)} - 1)\varepsilon_{k_0j}p_j^{(2)} \cdot \sqrt{2dw^\top e_k}\right)^2$$

$$\leq \frac{1}{4} \|w\|_2^2 \cdot \frac{m}{(D - d)^2} \sum_{j=1}^{m} (p_j^{(2)})^2$$

$$\leq \|w\|_2^2 \cdot \frac{1}{4q}.$$ 

Following the derivation in Lemma 7, we have $\|w\|_2 \leq \gamma_{\text{max}}$, and consequently, we have the bound

$$\mathbb{E}\langle \phi_k, (L_i^{(\varepsilon, z)} - L^{(\varepsilon, z)}) p \rangle^2 \leq \gamma_{\text{max}}^2.$$ 

On the other hand, for $k \geq d + 1$, the basis vector $\phi_k$ is constructed through the Hadamard matrix. Let $\psi^{(k)} := (2q)^{-1/2} \cdot \phi_k$, and note that the entries of $\psi^{(k)}$ are uniformly bounded by 1. Letting $k - d = (k_0 - 1)m + k_1$ for some choice of integers $k_0 \in [q]$ and $k_1 \in [m]$, we have

$$\mathbb{E}\langle \phi_k, (L_i^{(\varepsilon, z)} - L^{(\varepsilon, z)}) p \rangle^2$$

$$\leq \frac{1}{4(D - d)^2} \mathbb{E}\left(\sum_{j=1}^{m} (m\chi_{k_0j}^{(i)} - 1)\varepsilon_{k_0j}p_j^{(2)} \cdot \sqrt{2mq\psi_{(d+(k_0-1)m+j)}^{(k)} - \psi_{d+(k_0-1)m+j}} \cdot p_{k_0m+j}^{(2)} \right)^2$$

$$\leq \frac{2mq}{4(D - d)^2} \sum_{j=1}^{m} \left(\psi_{d+(k_0-1)m+j}^{(k)}\right)^2 \cdot (p_{k_0m+j}^{(2)})^2$$

$$\leq \frac{1}{2(D - d)} \|p^{(2)}\|_2^2 = \|p^{(2)}\|_2^2 \leq 1.$$ 

This verifies that equation (12a) is satisfied with parameter $\sigma_L = \gamma_{\text{max}} + 1$.

**Proof of equation (A.1):** For any vector $x \in \mathbb{R}^{mq}$, consider the decomposition $x^\top = [x_1^\top \cdots x_q^\top]$ with $x_j \in \mathbb{R}^m$ for each $j \in [q]$. We have the bound

$$\|Tx\|_2^2 = \sum_{i=1}^{q} \sum_{j=1}^{q} T_{ij}x_j \leq \sum_{i=1}^{q} \left(\sum_{j=1}^{q} \|T_{ij}\|_\text{op} \|x_j\|_2\right)^2$$

$$= \|T_{ij}\|_\text{op}_{1 \leq i, j \leq q} \cdot \|x_j\|_2 \leq \|T_{ij}\|_\text{op}_{1 \leq i, j \leq q}^2 \cdot \|x\|_2^2,$$

which proves this inequality.

**A.2 Proof of Lemma 13**

For each $j \in [m]$, define the event

$$E_j := \left\{ \text{for all } i \in \{0, 1, \cdots, q\}, \text{ there exists } \ell_i \in [n] \text{ such that } \chi_{ij}^{(\ell_i)} = 1 \right\}. \quad (83)$$
Let $\mathcal{E} := \bigcup_{j=1}^{m} \mathcal{E}_j$. We note that under both $\mathbb{P}_{1}^{(n)}$ and $\mathbb{P}_{-1}^{(n)}$, we have the inequality

$$\mathbb{P}(\mathcal{E}) \leq \sum_{j=1}^{m} \mathbb{P}(\mathcal{E}_j) = \sum_{j=1}^{m} \prod_{i=0}^{q} \left\{ 1 - \mathbb{P} \left( \chi_{ij}^{(\ell)} = 1 \text{ for all } \ell \in [n] \right) \right\} = m \cdot \left( 1 - \left( \frac{m-1}{m} \right)^{n+1} \right) \leq \frac{n^{q+1}}{m^{q}}.$$  

As before, our choice of the event $\mathcal{E}$ was guided by the fact that the two mixture distributions are identical on its complement. In particular, we claim that

$$\mathbb{P}_{1}^{(n)} |_{\mathcal{E}^C} = \mathbb{P}_{-1}^{(n)} |_{\mathcal{E}^C}. \quad (84)$$

Taking this claim as given for the moment and applying Lemma 9, we arrive at the bound

$$d_{TV}(\mathbb{P}_{1}^{(n)}, \mathbb{P}_{-1}^{(n)}) \leq \frac{12n^{q+1}}{m^{q}}.$$  

This completes the proof of this lemma. It remains to prove equation (84).

**Proof of equation (84):** We first note that both $\mathbb{P}_{1}^{(n)}$ are $\mathbb{P}_{-1}^{(n)}$ are $m$-fold i.i.d. product distributions: given $z = 1$ or $z = -1$, the random objects $\left( (\varepsilon_{ij})_{1 \leq i \leq q}, (\chi_{ij}^{(\ell)})_{0 \leq \ell \leq q, 1 \leq i \leq n} \right)_{j=1}^{m}$ are independent and identically distributed. Now for each $j \in [m]$, let $\mathbb{Q}_{z,j}^{(n)}$ be the joint law of the random object $(\rho^{(\ell)})_{\ell=1}^{n}$, where $\rho^{(\ell)} = (z\chi_{0j}^{(\ell)} \varepsilon_{1j}, \chi_{1j}^{(\ell)} \varepsilon_{1j} \varepsilon_{2j}, \ldots, \chi_{(q-1)j}^{(\ell)} \varepsilon_{(q-1)j} \varepsilon_{qj}, \chi_{aj}^{(\ell)} \varepsilon_{qj})$. It suffices to show that

$$\forall j \in [m], \quad \mathbb{Q}_{1,j}^{(n)} |_{\mathcal{E}_j^C} = \mathbb{Q}_{-1,j}^{(n)} |_{\mathcal{E}_j^C}. \quad (85)$$

To prove equation (85), we construct a distribution $\mathbb{Q}_{z,j}^{(n)}$, and show that it is equal to both of the conditional laws above. In analogy to the proof of Theorem 2, we construct $\mathbb{Q}_{z,j}^{(n)}$ according to the following sampling procedure:

- Sample the indicators $(\chi_{ij}^{(\ell)})_{0 \leq i \leq q, 1 \leq \ell \leq n}$, each from the Bernoulli distribution $\text{Ber}(1/m)$, conditioned on the event $\mathcal{E}_j^C$.
- For each $i \in [0, 1, \ldots, q]$, sample a random bit $\zeta^{(i)} \sim \mathcal{U}\{\{-1, 1\}\}$ independently.
- For each $\ell \in [n]$, generate the random object

$$\rho^{(\ell)} = (\chi_{0j}^{(\ell)} \zeta^{(0)}, \chi_{1j}^{(\ell)} \zeta^{(1)}, \ldots, \chi_{(q-1)j}^{(\ell)} \zeta^{(q-1)}, \chi_{aj}^{(\ell)} \zeta^{(q)}).$$

In the following, we construct a coupling between $\mathbb{Q}_{z,j}^{(n)} |_{\mathcal{E}_j^C}$ and $\mathbb{Q}_{z,j}^{(n)}$, for any $z \in \{-1, 1\}$, and show that they are actually the same.

First, we couple the random indicators $(\chi_{ij}^{(\ell)})_{0 \leq i \leq q, 1 \leq \ell \leq n}$ under $\mathbb{Q}_{z,j}^{(n)} |_{\mathcal{E}_j^C}$ and $\mathbb{Q}_{z,j}^{(n)}$ directly so that they are equal almost surely. By the first step in the sampling procedure, we know that the conditional law of these indicators are the same under both probability distributions.

By definition (83), we note that

$$\mathcal{E}_j^C = \left\{ \text{there exists } i \in \{0, 1, \ldots, q\}, \text{ such that } \chi_{ij}^{(\ell)} = 0 \text{ for all } \ell \in [n] \right\}.$$  

Let the random variable $i \in \{0, 1, \ldots, q\}$ be the smallest such index$^{12}$ $i$. We construct the joint distribution by conditioning on different values of $i$. First, note that the value of the random variable $\zeta^{(i)}$ is never observed, so we may set it to be an independent Rademacher random variable without loss of generality, and this does not affect the law of the random object $(\rho^{(\ell)})_{\ell=1}^{n}$ under consideration. Now consider the following three cases:

$^{11}$Note that $\mathbb{P}(\mathcal{E}_j^C) > 0$ in this sampling procedure. So the conditional distribution is well-defined.

$^{12}$Note that under the joint distribution we construct, $i$ is well-defined almost surely.
Case I: \( \ell = 0 \): In this case, we have \( \chi_{ij}^{(0)} = 0 \) for all \( \ell \in [n] \). So the first coordinate of each \( \rho^{(\ell)} \) is always zero. We define the following random variables in the probability space of \( (\varepsilon_{ij})_{i=1}^{q} \):

\[
\zeta^{(q)'} := \varepsilon_{qj}, \quad \text{and} \quad \zeta^{(i)'} := \varepsilon_{ij}\varepsilon_{(i+1)j} \quad \text{for } i \in \{1, 2, \cdots, q-1\}.
\]

Since \( (\varepsilon_{ij})_{i=1}^{q} \) are i.i.d. Rademacher random variables, it is easy to show by induction that the random sequence \( (\zeta^{(i)'})_{i=1}^{q} \) is also i.i.d. Rademacher, which has the same law as \( (\zeta^{(0)'}_{i})_{i=1}^{q} \). Consequently, we can construct the coupling such that \( (\zeta^{(i)})_{i=1}^{q} = (\zeta^{(i)'})_{i=1}^{q} \) almost surely. Under this coupling, when \( \ell = 0 \), the random objects \( (\rho^{(\ell)})_{\ell=1}^{n} \) generated under both probability distributions are almost-surely the same.

Case II: \( \ell \in \{1, 2, \cdots, q-1\} \): In this case, we have \( \chi_{ij}^{(\ell)} = 0 \) for any \( \ell \in [n] \). We define the following random variables in the probability space of \( (\varepsilon_{ij})_{i=1}^{q} \):

\[
\zeta^{(0)'} := z\varepsilon_{1j}, \quad \text{and} \quad \zeta^{(i)'} := \varepsilon_{ij}\varepsilon_{(i+1)j} \quad \text{for } i \in \{1, 2, \cdots, q-1\} \setminus \{\ell\}, \quad \text{and} \quad \zeta^{(q)'} := \varepsilon_{qj}.
\]

Note that the tuple of random variables \( (\zeta^{(i)'}_{i=1}^{q-1}) \) lives in the sigma-field \( \sigma(\varepsilon_{1j}, \cdots, \varepsilon_{ij}) \), and \( (\zeta^{(i)'}_{i=q+1}^{q}) \), on the other hand, lives in the sigma-field \( \sigma(\varepsilon_{(i+1)j}, \cdots, \varepsilon_{qj}) \), so these two tuples are independent. For any fixed bit \( z \in \{-1, 1\} \), it is easy to show by induction that \( (\zeta^{(i)'}_{i=0}^{q-1}) \) is an i.i.d. Rademacher random sequence. Similarly, applying the induction backwards from \( q \) to \( \ell+1 \), we can also show that \( (\zeta^{(i)'}_{i=q+1}^{q}) \) is also an i.i.d. Rademacher random sequence. Putting together the pieces, we see that the random variables \( (\zeta^{(i)'})_{0 \leq i \leq q, i \neq \ell} \) are i.i.d. Rademacher, and have the same law as the tuple \( (\zeta^{(i)})_{0 \leq i \leq q, i \neq \ell} \). We can therefore construct the coupling such that they are equal almost surely. Under this coupling, the random objects \( (\rho^{(\ell)})_{\ell=1}^{n} \) generated under both probability distributions are almost-surely the same.

Case III: \( \ell = q \): In this case, we have \( \chi_{ij}^{(q)} = 0 \) for any \( \ell \in [n] \). We define the following random variables in the probability space of \( (\varepsilon_{ij})_{i=1}^{q} \):

\[
\zeta^{(0)'} := z\varepsilon_{1j}, \quad \text{and} \quad \zeta^{(i)'} := \varepsilon_{ij}\varepsilon_{(i+1)j} \quad \text{for } i \in \{1, 2, \cdots, q-1\}.
\]

Note that \( (\varepsilon_{ij})_{i=1}^{q} \) are i.i.d. Rademacher random variables. For each choice of the bit \( z \in \{-1, 1\} \), we can show by induction that the random sequence \( (\varepsilon^{(i)'})_{i=1}^{q} \) is also i.i.d. Rademacher, which has the same law as the tuple \( (\varepsilon^{(i)})_{i=1}^{q} \). Making them equal almost surely in the coupling leads to the corresponding random object \( (\rho^{(q)})_{\ell=1}^{n} \) being almost-surely equal.

Therefore, we have constructed a coupling between \( Q_{z,ij}^{(n)} | \mathcal{E}_{C,j}^{n} \) and \( Q_{ij}^{(n)} \) so that the generated random objects are always the equal, for any \( z \in \{-1, +1\} \). This shows that

\[
Q_{1,ij}^{(n)} | \mathcal{E}_{C,j}^{n} = Q_{z,ij}^{(n)} = Q_{-1,ij}^{(n)} | \mathcal{E}_{C,j}^{n},
\]

which completes the proof of equation (85), and hence, the lemma.

B Proof of the bounds on the approximation factor

In this section, we prove the claims on the quantity \( \alpha(M, \gamma_{\text{max}}) \) that defines the optimal approximation factor.
B.1 Proof of Lemma 1

Recall that

\[ \alpha(M, s) = 1 + \lambda_{\max} \left( (I - M)^{-1} (s^2 I_d - MM^\top)(I - M)^{-\top} \right). \]  

(86)

In the following, we prove upper bounds for the two different cases separately.

**Bounds in the general case:** By assumption, we have \( \|M\|_{\text{op}} \leq s \), and consequently,

\[ 0 \preceq s^2 I_d - MM^\top \preceq s^2. \]

Thus, we have the sequence of implications

\[
\alpha(M, s) - 1 = \lambda_{\max} \left( (I - M)^{-1} (s^2 I - MM^\top)(I - M)^{-\top} \right)
\]

\[
= \| (I - M)^{-1} (s^2 I - MM^\top)(I - M)^{-\top} \|_{\text{op}}
\]

\[
\leq \| (I - M)^{-1} \|_{\text{op}} \cdot \| s^2 I_d - MM^\top \|_{\text{op}} \cdot \| (I - M)^{-1} \|_{\text{op}}
\]

\[
\leq \| (I - M)^{-1} \|_{\text{op}}^2 \cdot s^2,
\]

which proves the bound.

**Bounds under non-expansive condition:** When \( s \leq 1 \), we have

\[ s^2 I - MM^\top \preceq I - MM^\top = \frac{1}{2} (I - M)(I + M^\top) + \frac{1}{2} (I + M)(I - M^\top). \]

Consequently, we have the chain of bounds

\[
\alpha(M, s) - 1 \leq \lambda_{\max} \left( (I - M)^{-1} (I - MM^\top)(I - M)^{-\top} \right)
\]

\[
= \frac{1}{2} \lambda_{\max} \left( (I + M)^\top (I - M^\top)^{-1} + (I - M)^{-1} (I + M) \right)
\]

\[
\leq \frac{1}{2} \| (I + M)^\top (I - M^\top)^{-1} \|_{\text{op}} + \frac{1}{2} \| (I - M)^{-1} (I + M) \|_{\text{op}}
\]

\[
\leq \| (I - M)^{-1} \|_{\text{op}} + \| (I - M^\top)^{-1} \|_{\text{op}}
\]

\[
= 2 \| (I - M)^{-1} \|_{\text{op}}.
\]

Finally, we note that if \( \kappa(M) < 1 \), then for any \( u \in \mathbb{R}^d \), we have

\[
(1 - \kappa(M)) \| u \|_2^2 \leq \langle (I - M)u, u \rangle \leq \| (I - M)u \|_2 \cdot \| u \|_2.
\]

Consequently, we have \( \| (I - M)^{-1} \|_{\text{op}} \leq \frac{1}{1 - \kappa(M)} \), which completes the proof of this lemma.

B.2 Proof of Lemma 2

Once again, recall the definition

\[ \alpha(M, s) = 1 + \lambda_{\max} \left( (I - M)^{-1} (s^2 I_d - MM^\top)(I - M)^{-\top} \right). \]
Since $M$ is symmetric, let $M = P\Lambda P^\top$ be its eigen-decomposition, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_d)$, and note that
\[
\alpha(M, s) = 1 + \lambda_{\text{max}} \left(P(I - \Lambda)^{-1}(s^2 - 2\lambda)(I - \Lambda)^{-1}P^\top\right)
\]
\[
= 1 + \lambda_{\text{max}} \left((I - \Lambda)^{-2}(s^2 - 2\lambda_1^2)\right)
\]
\[
= 1 + \max_{1 \leq i \leq d} \left(\frac{\gamma_{\text{max}}^2 - \lambda_i^2}{(1 - \lambda_i)^2}\right),
\]
which completes the proof.

C Proofs for the examples

In this section, we provide proofs for the results related to three examples discussed in Section 4. Note that Corollary 3 follows directly from Theorem 1. Moreover, the proof of Corollary 4 builds on some technical results, and so we postpone the proof of all results related to elliptic equations to Appendix C.3. We begin this section with proofs of results related to temporal difference methods, i.e., Corollary 5 and Proposition 1.

C.1 Proof of Corollary 5

Recall our definition of the positive definite matrix $B$, with $B_{ij} = \langle \psi_i, \psi_j \rangle$. Letting $\theta_t := B^{1/2} \bar{\theta}_t$, the iterates (41a) can be equivalently written as
\[
\theta_{t+1} = \theta_t - \eta B \cdot \left(\phi(s_t+1)\phi(s_t+1)^\top \theta_t - \gamma \phi(s_t+1)\phi(s_t+1)^\top \theta_t + R_{t+1}(s_t+1)\phi(s_t+1)\right),
\]
and the Polyak–Ruppert averaged iterate is given by $\bar{\theta}_n := \frac{2}{n} \sum_{t=n/2}^{n-1} \theta_t$. We also define $\bar{\theta} := \Phi_d \bar{v}$, which is the solution to projected linear equations under the orthogonal basis. Clearly, we have $\bar{\theta} = B^{1/2} \bar{\theta}$.

We now claim that if $n \geq \frac{\epsilon_0^4 \beta^2}{\mu^4(1 - \kappa(M))^2} d \log^2 \left(\frac{\|\theta_0 - \bar{\theta}\|_2^2 d \beta}{\beta \mu(1 - \kappa(M))}\right)$, then
\[
\|\Phi_d \bar{\theta} - v^*\|^2 \leq \alpha(M, \gamma) A(S, v^*), \quad \text{and}
\]
\[
\mathbb{E}\|\bar{\theta}_n - \bar{\theta}\|_2^2 \leq c \mathcal{E}(M, \Sigma_L + \Sigma_b) + c(1 + \|\bar{v}\|^2) \left(\frac{\varsigma^2 \beta}{(1 - \kappa(M))\mu} \sqrt{\frac{d}{n}}\right)^3.
\]
Taking both inequalities as given for now, we proceed with the proof of this corollary. Combining equation (88a) and equation (88b) via Young’s inequality, we arrive at the bound
\[
\mathbb{E}\|\hat{\theta}_n - \hat{v}\|^2 \leq (1 + \omega)\|\Phi_d \bar{\theta} - v^*\|^2 + \left(1 + \frac{1}{\omega}\right) \mathbb{E}\|\bar{\theta}_n - \bar{\theta}\|_2^2
\]
\[
\leq (1 + \omega) A(S, v^*) + c \left(1 + \frac{1}{\omega}\right) \mathcal{E}(M, \Sigma_L + \Sigma_b) + (1 + \|\bar{v}\|^2) \left(\frac{\varsigma^2 \beta}{(1 - \kappa(M))\mu} \sqrt{\frac{d}{n}}\right)^3,
\]
which completes the proof of this corollary.
Proof of equation (88a): By equation (23) and the definition of $\bar{\theta}$, we have

$$\bar{\theta} = \gamma M \bar{\theta} + \mathbb{E}_\xi [R(s)\phi(s)].$$

It is easy to see that $\Phi_d^* \bar{\theta}$ solves the projected Bellman equation (22). Note furthermore that the projected linear operator is given by

$$\Phi_d L \Phi_d^* = \gamma \Phi_d P \Phi_d^* = M.$$

Invoking the bound in equation (45a), we complete the proof of this inequality.

Proof of equation (88b): Following the proof strategy for the bound (45b), we first show an upper bound on the iterates $\mathbb{E}\|\theta_t - \bar{\theta}\|^2$ under the non-orthogonal basis $(\psi_j)_{j \in [d]}$, and then use this bound to establish the final estimation error guarantee under $\| \cdot \|$-norm.

Recall the stochastic approximation procedure under the non-orthogonal basis:

$$\theta_{t+1} = \theta_t - \eta \left( \psi(s_{t+1})\psi(s_{t+1})^\top \theta_t - \gamma \psi(s_{t+1})\psi(s_{t+1})^\top \theta_t - R_{t+1}(s_{t+1}) \psi(s_{t+1}) \right).$$

Let $\tilde{M} := I_d - \frac{1}{\beta} B^{1/2} (I_d - M) B^{1/2}$ and $\tilde{h} := \frac{1}{\beta} \mathbb{E}[R(s)\psi(s)]$. We can view equation (41a) as a stochastic approximation procedure for solving the linear fixed-point equation $\bar{\theta} = \tilde{M} \bar{\theta} + \tilde{h}$, with stochastic observations

$$\tilde{M}_t := I_d - \beta^{-1} \left( \psi(s_t)\psi(s_t)^\top - \gamma \psi(s_t)\psi(s_t)^\top \right), \quad \text{and} \quad \tilde{h}_t := \beta^{-1} R(s_t)\psi(s_t).$$

To verify Assumption 1(W), we note that for $p, q \in \mathbb{S}^{d-1}$, the following bounds directly follows from the condition (42):

$$\mathbb{E} \left( p^\top (\tilde{M}_t - \tilde{M}) q \right)^2 \leq 2\beta^{-2} \mathbb{E} \left( (p^\top \psi(s_t))^4 \cdot (\psi(s_t)^\top q)^4 \right)^2 = \frac{4^4}{\beta^2} \| B^{1/2} p \|^2_2 \cdot \| B^{1/2} q \|^2_2 \leq 4\varsigma^4,$$

and

$$\mathbb{E} \left( p^\top (\tilde{h}_t - \tilde{h}) \right)^2 \leq \beta^{-2} \mathbb{E} \left( R(s_t) \cdot p^\top \psi(s_t) \right)^2 \leq \beta^{-2} \mathbb{E} [R(s_t)^4] \cdot \mathbb{E} (p^\top B^{1/2} \phi(s_t))^4 \leq \varsigma^4 / \beta.$$

Consequently, for the stochastic approximation procedure in equation (41a), Assumption 1(W) is satisfied with $\sigma_L = 2\varsigma^2$ and $\sigma_b = \varsigma^2 / \sqrt{\beta}$.

To establish an upper bound on $\kappa(M)$, we note that

$$1 - \kappa(M) = \frac{1}{\beta} \lambda_{\min} \left( B - B^{1/2} \frac{M + M^\top}{2} B^{1/2} \right)$$

$$= \frac{1}{\beta} \inf_{u \in S^{d-1}} (B^{1/2} u)^\top \left( I_d - \frac{M + M^\top}{2} \right) (B^{1/2} u) \geq \frac{\mu}{\beta} \inf_{u \in S^{d-1}} u^\top \left( I_d - \frac{M + M^\top}{2} \right) u \geq \frac{\mu}{\beta} (1 - \kappa(M)).$$
Invoking Lemma 5, for $\eta < \frac{c_0(1-\kappa(M))\mu}{(\kappa\sigma+1)\beta^2}$, we have

$$
\mathbb{E}\|\vartheta_t - \tilde{\vartheta}\|^2 \leq e^{-\frac{\eta(1-\kappa(M))\mu}{\kappa\sigma+1}}\mathbb{E}\|\vartheta_0 - \tilde{\vartheta}\|^2 + \frac{8\eta\beta}{(1-\kappa(M))\mu} (\|\vartheta\|^2_{S^2}d + \varsigma^4d/\beta). \tag{89}
$$

On the other hand, applying Lemma 6 to the stochastic approximation procedure (87) under the orthogonal coordinates, we have the bound

$$
\mathbb{E}\|\tilde{\vartheta}_n - \bar{\vartheta}\|^2 \leq \frac{6}{n-n_0} \text{trace} \left( (I-M)^{-1}\Sigma^*(I-M)^{-\top} \right) + \frac{6}{(n-n_0)^2} \sum_{t=n_0}^{n} \mathbb{E}\| (I-B^{1/2}\tilde{M}B^{-1/2})^{-1} B^{1/2}(\tilde{M}_{t+1} - \tilde{M})B^{-1/2}(\vartheta_t - \tilde{\vartheta}) \|^2
$$

$$
+ \frac{3\mathbb{E}\| (I_d - B^{1/2}\tilde{M}B^{-1/2})^{-1}(\vartheta_n - \vartheta_{n_0}) \|^2}{n^2\beta^2(n-n_0)^2}. \tag{90}
$$

Straightforward calculation yields

$$
\mathbb{E}\| (I - B^{1/2}\tilde{M}B^{-1/2})^{-1} B^{1/2}(\tilde{M}_{t+1} - \tilde{M})B^{-1/2}(\vartheta_t - \tilde{\vartheta}) \|^2 = \beta^2 \mathbb{E}\| (I - M)^{-1} B^{-1/2}(\tilde{M}_{t+1} - \tilde{M})(\vartheta_t - \tilde{\vartheta}) \|^2.
$$

For any vector $p \in \mathbb{R}^d$, using condition (42), we note that

$$
\mathbb{E}\| B^{-1/2}(\tilde{M}_{t+1} - \tilde{M})p \|^2 \leq 2\beta^{-2}\mathbb{E}\| \phi(s_t)^{\top}B^{1/2}p \|^2 + 2\beta^{-2}\mathbb{E}\| \phi(s_t)^{\top}B^{1/2}p \|^2
$$

$$
\leq 2\beta^{-2}\sqrt{\mathbb{E}\| \phi(s_t)^{\top}B^{1/2}p \|^4} + 2\beta^{-2}\sqrt{\mathbb{E}\| \phi(s_t)p \|^4}. \tag{91}
$$

Substituting into the identity above, we obtain

$$
\mathbb{E}\| (I - B^{1/2}\tilde{M}B^{-1/2})^{-1} B^{1/2}(\tilde{M}_{t+1} - \tilde{M})B^{-1/2}(\vartheta_t - \tilde{\vartheta}) \|^2 \leq \frac{4\beta\varsigma^4d}{(1-\kappa(M))\mu} \mathbb{E}\| \vartheta_t - \tilde{\vartheta} \|^2.
$$

For the third term in equation (90), we note that

$$
\mathbb{E}\| (I_d - B^{1/2}\tilde{M}B^{-1/2})^{-1}(\vartheta_n - \vartheta_{n_0}) \|^2 = \beta^2 \mathbb{E}\| (I - M)^{-1} B^{-1/2}(\vartheta_n - \vartheta_{n_0}) \|^2
$$

$$
\leq \frac{2\beta^2}{\mu(1-\kappa(M))^2} (\mathbb{E}\| \vartheta_n - \vartheta \|^2 + \mathbb{E}\| \vartheta_{n_0} - \vartheta \|^2). \tag{92}
$$

Putting together the pieces and invoking the bound (89), we see that if $n_0 \geq c_0\frac{1}{\mu(1-\kappa)} \log \left( \frac{d^3}{\mu(1-\kappa)} \right)$, then

$$
\mathbb{E}\|\tilde{\vartheta}_n - \bar{\vartheta}\|^2 \leq 6\mathbb{E}_n(M, \Sigma^*) + \left[ \frac{24\beta\varsigma^4d}{(1-\kappa(M))^2n^2} + \frac{48}{\mu(1-\kappa(M))^2n^2} \right] \sup_{n_0 \leq t \leq n} \mathbb{E}\| \vartheta_t - \tilde{\vartheta} \|^2
$$

$$
\leq 6\mathbb{E}_n(M, \Sigma^*) + c\frac{\beta^3}{\mu^2(1-\kappa(M))^3} \left[ \frac{\varsigma^4d}{n} + \frac{1}{\eta\beta n^2} \right] (\|\tilde{\vartheta}\|^2_{S^2}d + \varsigma^4d/\beta). \tag{93}
$$

Now note that $\|\tilde{\vartheta}\|^2 = \|B^{-1/2}\tilde{\vartheta}\|^2 \leq \mu^{-1}\|\bar{\vartheta}\|^2$, and so choosing the step size $\eta := \frac{1}{c\varsigma^2\beta\sqrt{dn}}$ yields

$$
\mathbb{E}\|\tilde{\vartheta}_n - \bar{\vartheta}\|^2 \leq 6\mathbb{E}_n(M, \Sigma^*) + c\frac{\beta^3\varsigma^6}{\mu^3(1-\kappa(M))^3} \left( \frac{d^3}{n} \right)^{3/2}.
$$

This completes the proof of equation (88b), and thus the corollary.
C.2 Proof of Proposition 1

Our construction and proof is inspired by the proof of Theorem 2, with some crucial differences in the analysis that result from the specific noise model in the MRP setting. Letting $D$ and $d$ be integer multiples of four without loss of generality, we denote the state space by $S = \{1, 2, \cdots, D\}$. We decompose the state space into $S = S_0 \cup S_1 \cup S_2$, with $S_0 := \{1, 2, \cdots, 2d\}$, $S_1 := \{2d + 1, \cdots, d + D\}$, and $S_2 := \{d + \frac{D}{2} + 1, \cdots, D\}$. Define the scalars $\rho = \min(\gamma, \nu) \in (0, 1)$ and $\tau := \delta \sqrt{2(1-\rho)} \wedge 1$.

Given a sign $z \in \{-1, 1\}$ and subsets $\Gamma_1 \subseteq S_1$ and $\Gamma_2 \subseteq S_2$ such that $|\Gamma_i| = \frac{1}{2} |S_i|$ for each $i \in \{1, 2\}$, we let $\bar{\Gamma}_i := S_i \setminus \Gamma_i$ for $i \in \{1, 2\}$. We then construct Markov reward processes $(P(\Gamma_1, \Gamma_2, z), r(\Gamma_1, \Gamma_2, z))$ and feature vectors $(\psi(\Gamma_1, \Gamma_2, z)(s_i))_{i=1}^D$, indexed by the tuple $(\Gamma_1, \Gamma_2, z)$.

Entry $(i, j)$ of the transition matrix is given by

$$P(\Gamma_1, \Gamma_2, z)(i, j) := \begin{cases} \rho & i = j \in S_0, \\ \frac{1-\rho}{2} & i, j \in S_0, \ |i-j| = d, \\ \frac{1-\rho}{|S_1|} & (i, j) \in \{1, \cdots, d\} \times \bar{\Gamma}_1 \cup \{(d+1, \cdots, 2d) \times \bar{\Gamma}_1\}, \\ \frac{1-\rho}{|S_2|} & (i, j) \in (\Gamma_1 \times \Gamma_2) \cup (\bar{\Gamma}_1 \times \bar{\Gamma}_2), \\ \frac{1}{d} & (i, j) \in (\Gamma_2 \times \{1, 2, \cdots, d\}) \cup (\bar{\Gamma}_2 \times \{d+1, \cdots, 2d\}) \end{cases} \quad (91a)$$

Figure 2. A graphical illustration of the MRP instance constructed above. For this instance, we let $d = 1$, $|S_1| = 4$ and $|S_2| = 4$, so that the total number of states is $D = 10$. In the graph, solid rounds stand for states, and arrows stand for the possible transitions. The numbers associated to the arrows stand for the probability of the transitions, and the equations $r = \cdots$ standard for the reward at a state. The sets $S_0$, $S_1$ and $S_2$ are separated by red dotted lines, and the sets $\Gamma_1$, $\bar{\Gamma}_1$, $\Gamma_2$, and $\bar{\Gamma}_2$ are marked by transparent rectangles. A blue round stands for a state with positive value function, and an orange round stands for a state with negative value function.
The reward function at state $i$ is given by

$$r_{(\Gamma_1, \Gamma_2, z)}(i) := \begin{cases} z\tau & i \in \Gamma_1, \\ -z\tau & i \in \bar{\Gamma}_1, \\ 0 & \text{otherwise.} \end{cases} \quad (91b)$$

This MRP is illustrated in Figure 2 for convenience. It remains to specify the feature vectors, and we use the same set of features for each tuple $(\Gamma_1, \Gamma_2, z)$. The $i$-th such feature vector is given by

$$\psi(i) := \begin{cases} \sqrt{3\rho \cdot d}e_i & i \in \{1, \ldots, d\}, \\ -\sqrt{3\rho \cdot d}e_{i-d} & i \in \{d+1, \ldots, 2d\}, \\ 0 & \text{otherwise.} \end{cases} \quad (91c)$$

It is easy to see that for any tuple $(z, \Gamma_1, \Gamma_2)$, the Markov chain is irreducible and aperiodic, and furthermore, that the stationary distribution of the transition kernel $P^{(\Gamma_1, \Gamma_2, z)}$ is independent of the tuple $(\Gamma_1, \Gamma_2, z)$, and given by

$$\xi = \left[ \frac{1}{(3-\rho)d} \cdots \frac{1}{(3-\rho)d} \frac{1-\rho}{(3-\rho)(D-2d)} \cdots \frac{1-\rho}{(3-\rho)(D-2d)} \right].$$

Clearly, we have $\mathbb{E}_\xi[\psi(s)^T\psi(s)] = I_d$ under the stationary distribution. For the projected transition kernel, we have

$$\mathbb{E}[\psi(s)^T\psi(s^+)] = \frac{3-\rho}{2} \left( \rho - \frac{1-\rho}{2} \right) I_d \preceq \rho I_d \preceq \nu I_d. \quad (92)$$

Given the discount factor $\gamma \in (0, 1)$, let $c_0 := \frac{(1-\rho)/2}{1-\gamma(1-\rho)(1-\gamma^2)/2}$ for convenience. Straightforward calculation then yields that the value function for the problem instance $(P^{(\Gamma_1, \Gamma_2, z)}, r_{(\Gamma_1, \Gamma_2, z)})$ at state $i$ is given by

$$v_{\Gamma_1, \Gamma_2, z}^*(i) = \begin{cases} c_0 z\tau & i \in \{1, \ldots, d\}, \\ -c_0 z\tau & i \in \{d+1, \ldots, 2d\}, \\ (1+\gamma^2 c_0) z\tau & i \in \Gamma_1, \\ -(1+\gamma^2 c_0) z\tau & i \in \bar{\Gamma}_1, \\ \gamma c_0 z\tau & i \in \Gamma_2, \\ -\gamma c_0 z\tau & i \in \bar{\Gamma}_2. \end{cases}$$

For $\rho > 1/2$, we have the bounds

$$c_0 \geq \frac{1}{4} \cdot \frac{1-\rho}{1-\gamma\rho} \geq \frac{1-\rho}{4(1-\rho^2)} \geq \frac{1}{8}, \quad \text{and} \quad c_0 \leq \frac{1-\rho}{1-\gamma\rho} \leq 1.$$

Consequently, we have $|v_{\Gamma_1, \Gamma_2, z}^*(i)| \preceq |v_{\Gamma_1, \Gamma_2, z}^*(j)|$ for each pair $(i, j)$.

Note that by our construction, the subspace $S$ spanned by the basis functions $\psi(1), \psi(2), \cdots, \psi(2d)$ is given by

$$S = \{ v \in L^2(S, \xi) : v(s) = 0 \text{ for } s \notin S_0, \text{ and } v(i+d) = -v(i) \text{ for all } i \in [d] \}.$$
Consequently, we have

\[
\inf_{v \in \mathbb{S}} \|v - v_{\Gamma_1, \Gamma_2, z}^*\|^2 = \frac{1 - \rho}{3 - \rho} \cdot \left( \frac{1}{2} (1 + \gamma^2 c_0)^2 \sigma^2 + \frac{1}{2} \gamma^2 c_0^2 \sigma^2 \right) \leq 2(1 - \rho) \sigma^2 = \delta^2. \tag{93}
\]

Putting the equations (92) and (93) together, for any tuple \((\Gamma_1, \Gamma_2, z)\), we conclude that the problem instance \((P(\Gamma_1, \Gamma_2, z), r(\Gamma_1, \Gamma_2, z), \gamma, \psi((\Gamma_1, \Gamma_2, z)))\) belongs to the class \(C_{\text{MRP}}(\nu, \gamma, D, \delta)\).

In order to apply Le Cam’s lemma, we define the following mixture distributions for each \(z \in \{-1, 1\}:
\[
\mathbb{P}_{\Gamma_1, \Gamma_2, z}^{(n)} := \left( \frac{|S_1|}{|S_1|/2} \right)^2 \sum_{\Gamma_1 \subseteq S_1, \Gamma_2 \subseteq S_2 \atop |\Gamma_1| = |\Gamma_2| = \frac{1}{2} |S_1|} \mathbb{P}_{\Gamma_1, \Gamma_2, z}^\otimes n,
\]

where \(\mathbb{P}_{\Gamma_1, \Gamma_2, z}\) is the law of an observed tuple \((s_i, s_i^+, r(s_i))\) under the MRP \((P(\Gamma_1, \Gamma_2, z), r(\Gamma_1, \Gamma_2, z))\), and \(\mathbb{P}_{\Gamma_1, \Gamma_2, z}^\otimes n\) denotes its \(n\)-fold product. Our next result gives a bound on the total variation distance.

**Lemma 14.** Under the set-up above, we have \(d_{TV}(\mathbb{P}_{1, \Gamma_2, -1}^{(n)}, \mathbb{P}_{-1, \Gamma_2, -1}^{(n)}) \leq C_{\nu}n^2 D^{-2d}\).

Taking this lemma as given, we now turn to the proof of the proposition. Consider any estimator \(\hat{v}\) for the value function. For any pair \(\Gamma_1, \Gamma_2\) and \(\Gamma'_1, \Gamma'_2\), we have

\[
\|\hat{v} - v_{\Gamma_1, \Gamma_2, 1}^*\|^2 + \|\hat{v} - v_{\Gamma'_1, \Gamma'_2, -1}^*\|^2 \geq \frac{1}{2} \|v_{\Gamma_1, \Gamma_2, 1}^* - v_{\Gamma'_1, \Gamma'_2, -1}^*\|^2 \geq \frac{1}{2} c_0^2 \sigma^2 \geq \frac{\delta^2}{64(1 - \rho)}.
\]

Invoking Le Cam’s lemma, for \(D > 2C(n^2 + d)\), we have

\[
\inf_{\hat{v}} \sup_{(P, \gamma, r, \psi) \in C_{\text{MRP}}} \frac{c}{1 - \rho} \delta^2 \left( 1 - d_{TV}(\mathbb{P}_{1, \Gamma_2, -1}^{(n)}, \mathbb{P}_{-1, \Gamma_2, -1}^{(n)}) \right) \geq \frac{c'}{1 - \nu \gamma} \delta^2,
\]

which completes the proof. \(\square\)

### C.2.1 Proof of Lemma 14

In contrast to the proof of Theorem 2, the underlying mixing components here are indexed by random subsets of a given size, instead of random bits. This sampling procedure introduces additional dependency, so that the arguments in the proof of Theorem 2 do not directly apply. Instead, we use an induction-type argument by constructing the coupling directly.

Similarly to before, we construct a probability distribution \(Q^{(n)}\) and bound the total variation distance between \(Q^{(n)}\) and \(\mathbb{P}_z^{(n)}\) for each \(z \in \{-1, 1\}\). In particular, for \(k \in [n]\), we let \(Q^{(k)}\) be the law of \(k\) independent samples drawn from the following observation model:

1. (Initial state:) Generate the state \(s_i \sim \xi\).
2. (Next state:) If \(s_i \in S_1\), then generate \(s_i^+ \sim \mathcal{U}(S_2)\). If \(s_i \in S_2\), then generate \(s_i^+ \sim \mathcal{U}(S_0)\). On the other hand, if \(s_i \in S_0\), then generate \(S \sim \mathcal{U}(S_1)\) and let

\[
s_i^+ = \begin{cases} s_i & \text{w.p. } \rho, \\ (s_i + d) \mod 2d & \text{w.p. } \frac{1 - \rho}{2}, \\ S & \text{w.p. } \frac{1 - \rho}{2}. \end{cases} \tag{94}
\]

\(^{13}\)The expression \(a \mod b\) denotes the remainder of \(a\) divided by \(b\), when \(a\) and \(b\) are integers.
• (Reward:) If $s_i \in S_1$, randomly draw $R_i = \zeta^{(i)} \sim \mathcal{U}(\{-1,1\})$, and output $\zeta^{(i)} \tau$ as the reward. Otherwise, output the reward $R_i = 0$.

To bound the total variation distance $d_{TV}(\mathbb{Q}^{(n)}, \mathbb{P}^{(n)})$, we use the following recursive relation, which holds for each $k = 0, 1, \cdots, n - 1$:

$$d_{TV}(\mathbb{Q}^{(k+1)}, \mathbb{P}^{(k+1)}) \leq d_{TV}(\mathbb{Q}^{(k)}, \mathbb{P}^{(k)}) + \sup_{(s_i, s_i^+, R_i)_i} d_{TV}(\mathbb{Q}^{(k+1)}|(s_i, s_i^+, R_i)_i, \mathbb{P}^{(k+1)}|(s_i, s_i^+, R_i)_i) \quad \text{(95)}$$

Owing to the i.i.d. nature of the sampling model for $\mathbb{Q}^{(k+1)}$, note that we have the equivalence $(s_{k+1}, s_{k+1}^+, R_{k+1})|(s_i, s_i^+, R_i)_i = (s_{k+1}, s_{k+1}^+, R_{k+1})$.

At this juncture, it is helpful to view the probability distributions $\mathbb{P}_1^{(k)}$ and $\mathbb{P}_2^{(k)}$ via the following two-step sampling procedure: First, for $j \in \{1, 2\}$, sample the subsets $\Gamma_j \subseteq S_j$ uniformly at random from the collection of all subsets of size $|S_j|/2$. Then, generate $k$ i.i.d. samples $(s_i, s_i^+, R_i)_i$ according to the observation model (91a)-(91b). Consequently, for the rest of this proof, we view $\Gamma_1$ and $\Gamma_2$ as random sets. With this equivalence at hand, the following technical lemma shows that the posterior distribution of the subsets $(\Gamma_1, \Gamma_2)$ conditioned on sampling the tuple $(s_i, s_i^+, R_i)_i$ is very close to the distribution of subsets chosen uniformly at random.

**Lemma 15.** There is a universal positive constant $c$ such that for each bit $z \in \{\pm 1\}$ and indices $j \in \{1, 2\}$ and $k \in [n]$, the following statement is true almost surely. For each tuple $(s_i, s_i^+, R_i)_i$ in the support of $\mathbb{P}_z^{(k)}$, the posterior distribution of $\Gamma_j$ conditioned on $(s_i, s_i^+, R_i)_i \sim \mathbb{P}_z^{(k)}$ satisfies

$$\max_{s \in S_j \setminus \bigcup_{i=1}^k \{s_i, s_i^+\}} \left| \mathbb{P}(\Gamma_j \ni s \mid (s_i, s_i^+, R_i)_i) - \frac{1}{2} \right| \leq \frac{ck}{D - d}.$$ 

In words, for any “observable” tuple $(s_i, s_i^+, R_i)_i$ and each state $s \in S_j \setminus \bigcup_{i=1}^k \{s_i, s_i^+\}$, the posterior probability of the event $\{\Gamma_j \ni s\}$ conditioned on observing the tuple $(s_i, s_i^+, R_i)_i$ is close to $1/2$ provided $D - d$ is large relative to $k$. In addition to the sets $\Gamma_j, j = 1, 2$ being close to uniformly random, we also require the following analog of a “birthday-paradox” argument in this setting. For convenience, we let $T_k := \bigcup_{i=1}^k \{s_i, s_i^+\}$ denote the subset of states seen up until sample $k$.

**Lemma 16.** There is a universal positive constant $c$ such that for each $k \in [n]$ and each distribution $\mathbb{M}^{(k+1)} \in \left\{ \mathbb{P}_1^{(k+1)}, \mathbb{P}_2^{(k+1)}, \mathbb{Q}^{(k+1)} \right\}$, the following statement holds almost surely. For each tuple $(s_i, s_i^+, R_i)_i$ in the support of $\mathbb{M}^{(k+1)}$, the probability the tuple of states $(s_{k+1}, s_{k+1}^+, R_{k+1})$ conditioned on $(s_i, s_i^+, R_i)_i \sim \mathbb{M}^{(k)}$ satisfies

$$\mathbb{P}\left( (s_{k+1}, s_{k+1}^+) \cap T_k \cap \text{(S1 \cup S2)} \neq \emptyset \mid (s_i, s_i^+, R_i)_i \right) \leq \frac{ck}{D - d}.$$ 

In words, Lemma 16 ensures that if $D - d$ is large relative to $k$, then the states seen in sample $k + 1$ are different from those seen up until that point (provided we only count states in the set $S_1 \cup S_2$). Lemmas 15 and 16 are both proved at the end of this section; we take them as given for the rest of this proof.
Now consider tuples \((s_{k+1}, s^+_{k+1}, R_{k+1}) \sim \mathbb{P}^{(k+1)}(s_i, s^+_i, R_i^{\text{th}})\) and \((\bar{s}_{k+1}, \bar{s}^+_{k+1}, \bar{R}_{k+1}) \sim Q^{(k+1)}(s_i, s^+_i, R_i^{\text{th}})\); we will now construct a coupling between these two tuples in order to show that the total variation between between the respective laws is small. First, note that under both \(\mathbb{P}^{(k+1)}\) and \(Q^{(k+1)}\), the initial state is drawn from the stationary distribution, i.e., \(s_{k+1}, \bar{s}_{k+1} \sim \xi\), regardless of the sequence \((s_i, s^+_i, R_i^{\text{th}})\). We can therefore couple the two conditional laws together so that \(s_{k+1} = \bar{s}_{k+1}\) almost surely. To construct the coupling for the rest, we consider the following three cases:

**Coupling on the event \(s_{k+1} \in S_0\):** We begin by coupling the reward random variables; we have \(R_{k+1} = \bar{R}_{k+1} = 0\) under both conditional distributions, so this component of the distribution can be coupled trivially. Next, we couple the next state: By construction of the observation models \((91a)\) and \((94)\), we have

\[
\mathbb{P}(s^+_{k+1} = s_{k+1} | s_{k+1}) = \mathbb{P}(\bar{s}^+_{k+1} = \bar{s}_{k+1} | \bar{s}_{k+1}) = \rho, \quad \text{and} \quad \mathbb{P}(s^+_{k+1} = s_{k+1} + d \mod 2d | s_{k+1}) = \mathbb{P}(\bar{s}^+_{k+1} = \bar{s}_{k+1} + d \mod 2d | \bar{s}_{k+1}) = \frac{1 - \rho}{2},
\]

and so these two components of the distribution can be coupled trivially. It remains to handle the case where \(s_{k+1} \in S_0\) and \(s^+_{k+1} \in S_1\). By the symmetry of elements within set \(S_1\), we note that on the event \((\epsilon_{k+1}^{(1)})^C\), both random variables \(\bar{s}^+_{k+1}\) and \(s^+_{k+1}\) are uniformly distributed on the set \(S_1 \setminus \mathcal{T}_k\). Consequently, on the event \((\epsilon_{k+1}^{(1)})^C\), we can couple the conditional laws so that \(s^+_{k+1} = \bar{s}^+_{k+1}\) almost surely.

**Coupling on the event \(s_{k+1} \in S_1\):** As before, we begin by coupling the rewards, but first, note that on the event \((\epsilon_{k+1}^{(2)})^C\), we have \(s_{k+1} \in S_1 \setminus \mathcal{T}_k\). Invoking Lemma 15, under \(\mathbb{P}^{(k)}\) and conditionally on the value of \(s_{k+1}\), we have the bound

\[
\left| \mathbb{P}\left( s_{k+1} \in \Gamma_1 \mid (s_i, s^+_i, R_i^{\text{th}})_{i=1}^{\text{th}} \right) - \frac{1}{2} \right| \leq \frac{ck}{D - d}.
\]

Now the reward function \((91b)\) satisfies \(r(s) = z\tau\) for \(s \in \Gamma_1\) and \(r(s) = -z\tau\) for \(s \in \bar{\Gamma}_1\). On the other hand, under \(Q^{(k+1)}\), the reward \(\bar{R}_{k+1}\) takes value of \(\tau\) and \(-\tau\), each with probability half. Consequently, there exists a coupling between \(R_{k+1}\) and \(\bar{R}_{k+1}\), such that

\[
\mathbb{P} \left( \bar{R}_{k+1} \neq R_{k+1}, s_{k+1} \in S_1 \mid (s_i, s^+_i, R_i^{\text{th}})_{i=1}^{\text{th}} \right) \leq \frac{ck}{D - d}.
\]

Next, we construct the coupling for next-step transition conditionally on the current step. By the symmetry of elements within set \(S_2\), we note that under \((\epsilon_{k+1}^{(1)})^C\), both random variables \(\bar{s}^+_{k+1}\) and \(s^+_{k+1}\) are uniformly distributed on the set \(S_2 \setminus \mathcal{T}_k\). Consequently, on the event \((\epsilon_{k+1}^{(1)})^C\), we can couple the conditional laws so that \(s^+_{k+1} = \bar{s}^+_{k+1}\) almost surely.

**Coupling on the event \(s_{k+1} \in S_2\):** In this case, we have \(R_{k+1} = \bar{R}_{k+1} = 0\) under both conditional distributions, so this coupling is once again trivial. It remains to construct a coupling between next-step transitions \(s^+_{k+1}\) and \(\bar{s}^+_{k+1}\). On the event \((\epsilon_{k+1}^{(1)})^C\), we have
$s_{k+1} \in S_2 \setminus T_k$. Under $P_2^{(k)}$ and conditionally on the value of $s_{k+1}$, Lemma 15 leads to the bound

$$\left| \mathbb{P} \left( s_{k+1} \in \Gamma_2 \mid (s_i, s_i^+, R_i)_{i=1}^k \right) - \frac{1}{2} \right| \leq \frac{ck}{D - d}.$$  

By definition, under $P_2^{(n)}$, we have that $s_{k+1}^+ \sim \mathcal{U}\{1, 2, \ldots, d\}$ when $s_{k+1} \in \Gamma_2$, and $s_{k+1}^+ \sim \mathcal{U}\{d + 1, \ldots, 2d\}$ when $s_{k+1} \in \bar{\Gamma}_2$. Under $Q^{(n)}$, we have $\tilde{s}_{k+1} \sim \mathcal{U}\{1, 2, \ldots, 2d\}$. Consequently, there exists a coupling such that

$$\mathbb{P} \left( s_{k+1}^+ \neq \tilde{s}_{k+1}^+, s_{k+1} \in S_2 \mid (s_i, s_i^+, R_i)_{i=1}^k \right) \leq \frac{ck}{D - d}.$$  

Putting together our bounds from the three cases, note that for any sequence $(s_i, s_i^+, R_i)_{i=1}^k$ on the support of $Q^{(k)}$ and $P_2^{(k)}$, we almost surely have

$$d_{TV} \left( \mathcal{L} \left[ (s_{k+1}, s_{k+1}^+, R_{k+1}) \mid (s_i, s_i^+, R_i)_{i=1}^k \right] \right) \leq \sum_{j=1}^{3} \mathbb{P} \left( e_{k+1}^{(j)} \mid (s_i, s_i^+, R_i)_{i=1}^k \right) \leq \frac{c'k}{D - d},$$

where the final inequality follows from applying Lemma 16. Substituting into the recursion (95), we conclude that for any $z \in \{-1, 1\}$, we have

$$d_{TV} \left( Q^{(n)}, P_2^{(n)} \right) \leq \sum_{k=0}^{n-1} \sum_{j=1}^{3} \sup_{(s_i, s_i^+, R_i)_{i=1}^k} \mathbb{P} \left( e_{k+1}^{(j)} \mid (s_i, s_i^+, R_i)_{i=1}^k \right) \leq \frac{c'n^2}{D - d},$$

which completes the proof of this lemma.  

It remains to prove the two helper lemmas.

**Proof of Lemma 15:** Given $z \in \{\pm 1\}$, we define the sets

$$Z_1 := \{ s_i : i \in [k], s_i \in S_1, R_i = z \tau \}, \quad \bar{Z}_1 := \{ s_i : i \in [k] \cap S_1 \} \setminus Z_1, \quad \text{and}$$

$$Z_2 := \{ s_i : i \in [k], s_i \in S_2, s_i \in [d] \}, \quad \bar{Z}_2 := \{ s_i : i \in [k] \cap S_2 \} \setminus Z_2.$$  

By the reward model (91b) in our construction, for any valid pair of subsets $(\Gamma_1, \Gamma_2)$, under the law $P_{1,2}^{\otimes k}$, the observations $(s_i, s_i^+, R_i)_{i=1}^k$ have positive probability if and only if $Z_1 \subseteq \Gamma_1$ and $\Gamma_1 \cap \bar{Z}_1 = \emptyset$. Furthermore, by the symmetry between the elements in $\Gamma_1$, for any $\Gamma_1$ such that $Z_1 \subseteq \Gamma_1$ and $\Gamma_1 \cap \bar{Z}_1 = \emptyset$, the probability of observing $(s_i, s_i^+, R_i)_{i=1}^k$ under $P_{1,2}^{\otimes k}$ is independent of the choice of $\Gamma_1$. Consequently, the probability under the mixture distribution

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Applying a union bound, we arrive at the inequality

\[ \mathbb{P} \left( \Gamma_1 \ni s \mid (s_i, s_i^+, R_i)_{i=1}^k \right) = \sum_{s \in \Gamma'} \frac{\mathbb{P} \left( (s_i, s_i^+, R_i)_{i=1}^k \mid \Gamma_1 = \Gamma' \right) \cdot \mathbb{P} (\Gamma_1 = \Gamma')}{\mathbb{P} \left( (s_i, s_i^+, R_i)_{i=1}^k \right)} \]

which completes the proof.

C.3 Proof for elliptic equations

In this section, we prove the results for the elliptic equation example in Section 2.2.2.
C.3.1 Technical results from Section 2.2.2

The main technical result that was assumed in Section 2.2.2 is collected as a lemma below.

**Lemma 17.** There exists a bounded, self-adjoint, linear operator \( \tilde{A} \in L \) and a function \( g \in \dot{H}^1 \), such that, for all \( u, v \in \dot{H}^1 \),

\[
\langle u, \tilde{A}v \rangle_{\dot{H}^1} = \langle u, Av \rangle_{L^2}, \tag{97a}
\]

\[
\langle u, g \rangle_{\dot{H}^1} = \langle u, f \rangle_{L^2}, \tag{97b}
\]

and such that

\[
\mu \| u \|_{\dot{H}^1}^2 \leq \langle u, \tilde{A}u \rangle_{\dot{H}^1} \leq \beta \| u \|_{\dot{H}^1}^2. \tag{97c}
\]

We prove the three claims in turn.

**Proof of equation (97a):** For any pair of test functions \( u, v \in \dot{H}^1 \), integration by parts and the uniform ellipticity condition yield

\[
\langle u, Av \rangle_{L^2} = -\int_\Omega u(x) \nabla \cdot (a(x) \nabla v(x)) \, dx = \int_\Omega \nabla u^\top a \nabla v \, dx \leq \beta \| u \|_{\dot{H}^1} \cdot \| v \|_{\dot{H}^1}.
\]

Now given a fixed function \( v \in \dot{H}^1 \), the above equation ensures that \( \langle \cdot, Av \rangle_{L^2} \) is a bounded linear functional. By the Riesz representation theorem, there exists a unique function \( v' \in \dot{H}^1 \) with \( \| v' \|_{\dot{H}^1} \leq \beta \| v \|_{\dot{H}^1} \) for any \( v \in \dot{H}^1 \), such that

\[
\forall u \in \dot{H}^1, \quad \langle u, Av \rangle_{L^2} = \langle u, v' \rangle_{\dot{H}^1}.
\]

Clearly, the mapping from \( v \) to \( v' \) is linear, and we have \( \| v' \|_{\dot{H}^1} \leq \beta \| v \|_{\dot{H}^1} \) for any \( v \in \dot{H}^1 \). Thus, the mapping \( v \mapsto v' \) is a bounded linear operator. Using \( \tilde{A} \) to denote this operator, equation (97a) then directly follows. It remains to verify that \( \tilde{A} \) is self-adjoint. Indeed, for \( u, v \in \dot{H}^1 \), we have the identity

\[
\langle u, \tilde{A}v \rangle_{\dot{H}^1} = \langle u, Av \rangle_{L^2} = \int_\Omega \nabla u^\top a \nabla v \, dx = -\int_\Omega v \nabla \cdot (a \nabla u) \, dx = \langle v, Au \rangle_{L^2} = \langle v, \tilde{A}u \rangle_{\dot{H}^1},
\]

which proves the self-adjoint property.

**Proof of equation (97b):** Since the domain \( \Omega \) is bounded and connected, there exists a constant \( \rho_P \) depending only on \( \Omega \), such that the following Poincaré equation holds:

\[
\forall v \in \dot{H}^1, \quad \| v \|_{L^2}^2 \leq \frac{1}{\rho_P} \| v \|_{\dot{H}^1}^2. \tag{98}
\]

For any test function \( u \in \dot{H}^1 \), equation (98) leads to the bound

\[
\langle u, f \rangle_{L^2} \leq \| f \|_{L^2} \cdot \| u \|_{L^2} \leq \frac{1}{\rho_P} \| f \|_{L^2} \cdot \| u \|_{\dot{H}^1}.
\]

So \( \langle \cdot, f \rangle_{L^2} \) is a bounded linear functional on \( \dot{H}^1 \). Again, by the Riesz representation theorem, there exists a unique \( g \in \dot{H}^1 \), such that \( \langle u, f \rangle_{L^2} = \langle u, g \rangle_{\dot{H}^1} \) for all \( u \in \dot{H}^1 \), which completes the proof.
Proof of equation (97c): For any test function \( u \in \mathbb{H}^1 \), we note that
\[
\langle u, \tilde{A}u \rangle_{\mathbb{H}^1} = \langle u, Au \rangle_{L^2} = \int_{\Omega} (\nabla u(x))^\top a(x)(\nabla u(x)) \, dx.
\]
From our uniform ellipticity condition, we know that \( \mu I_m \leq a(x) \leq \beta I_m \) for any \( x \in \Omega \). Substituting this relation yields \( \mu \|u\|_{\mathbb{H}^1}^2 \leq \langle u, \tilde{A}u \rangle_{\mathbb{H}^1} \leq \beta \|u\|_{\mathbb{H}^1}^2 \), as claimed.

C.3.2 Proof of Corollary 4

The matrices \( M, \Sigma_L, \Sigma_h \) for the projected problem instances can be obtained by straightforward calculation. In order to apply Theorem 1, it remains to verify the assumptions.

By Lemma 17, the operator \( L \) is self-adjoint in \( \mathbb{X} \), and is sandwiched as \( 0 \leq \langle u, Lu \rangle_{\mathbb{H}^1} \leq (1 - \frac{\mu}{\beta}) \|u\|_{\mathbb{H}^1}^2 \) for all \( u \in \mathbb{X} \). This yields the operator norm bound \( \|L\|_{\mathbb{X}} \leq 1 - \frac{\mu}{\beta} \).

Now we verify the conditions in Assumption 1(W). For any basis function \( \phi_j \) with \( j \in [d] \) and vector \( v \in \mathbb{H}^1 \), we have
\[
\mathbb{E}\langle \phi_j, (L_i - L)u \rangle_{\mathbb{H}^1}^2 \leq \frac{1}{\beta^2} \mathbb{E}\left( \int_{\Omega} \delta_{x_i} \nabla \phi_j(x)^\top a(x) \nabla u(x) \, dx \right)^2 + \frac{1}{\beta^2} \mathbb{E}\left( \int_{\Omega} \delta_{x_i} \nabla \phi_j(x)^\top W_i \nabla u(x) \, dx \right)^2
\]
\[
= \frac{1}{\beta^2} \int_{\Omega} (\nabla \phi_j(x)^\top a(x) \nabla u(x))^2 \, dx + \frac{1}{\beta^2} \int_{\Omega} (\nabla \phi_j(x)^\top W_i \nabla u(x))^2 \, dx
\]
\[
\leq \frac{1}{\beta^2} \int_{\Omega} \|\nabla \phi_j(x)\|^2_2 \|a(x)\|_W^2 \|\nabla u(x)\|^2_2 \, dx + \frac{2}{\beta^2} \int_{\Omega} \|\nabla \phi_j(x)\|^2_2 \cdot \|\nabla u(x)\|^2_2 \, dx
\]
\[
\leq \left( 1 + \frac{2}{\beta^2} \right) \max_{j \in [d]} \sup_{x \in \Omega} \|\nabla \phi_j\|^2_2 \cdot \int_{\Omega} \|\nabla u(x)\|^2_2 \, dx
\]
\[
\leq \sigma_L^2 \|u\|^2_{\mathbb{H}^1},
\]
and
\[
\mathbb{E}\langle \phi_j, b_i - b_i \rangle_{\mathbb{H}^1}^2 \leq \frac{1}{\beta^2} \mathbb{E}\left( \int_{\Omega} \delta_{y_i} \phi_j(y) f(y_i) \, dy \right)^2 + \frac{1}{\beta^2} \mathbb{E}\left( \int_{\Omega} \delta_{y_i} \phi_j(y) g_i \, dy \right)^2
\]
\[
= \frac{1}{\beta^2} \int_{\Omega} \phi_j(y)^2 f(y)^2 \, dy + \frac{1}{\beta^2} \int_{\Omega} \phi_j(y)^2 \mathbb{E}[g_i^2] \, dy
\]
\[
\leq \frac{1}{\beta^2} \sup_{x \in \Omega} |\phi_j|^2 \int_{\Omega} (f(y)^2 + 1) \, dy
\]
\[
= \frac{\|f\|^2_2 + 1}{\beta^2} \max_{j \in [d]} \sup_{x \in \Omega} |\phi_j|^2.
\]

Therefore, Assumption 1(W) is satisfied with constants \( (\sigma_L, \sigma_h) \). Invoking Theorem 1 completes the proof.

D Models underlying simulations in Figure 1

The simulation results shown in Figure 1 are generated by constructing random transition matrices based on the following random graph models:

**Erdős-Rényi random graph**: Given \( d, N \in \mathbb{N}_+ \) and \( a > 1 \), we consider the following sampling procedure. Let \( G \) be an Erdős-Rényi random graph with \( N \) vertices and edge probability \( p = \frac{a}{N} \), and take \( \tilde{G} \) to be its largest connected component. (When \( c > 1 \), the number of vertices in \( \tilde{G} \) is...
of order $\Theta(N)$. See the monograph [Dur07] for details.) For each vertex $v \in V(\tilde{G})$, we associate it with an independent standard Gaussian random vector $\phi_v \sim \mathcal{N}(0, I_d)$. Let $V(\tilde{G})$ be the state space and let the Markov transition kernel $P$ be the simple random walk on $\tilde{G}$.

In Figure 1 (a), we take the number of vertices to be $N = 3000$ and the feature dimension to be $d = 1000$. The edge density parameter is chosen as $a = 3$. The resulting giant connected component contains 2813 vertices.

**Random geometric graph:** Given a pair of positive integeres $(d, N)$ and scalar $r > 0$, we consider the following sampling procedure. For each vertex $i \in [N]$, we associate it with an independent standard Gaussian random vector $\phi_i \sim \mathcal{N}(0, I_d)$. The graph $G$ is then constructed such that $(i, j) \in E(G)$ if and only if $\|\phi_i - \phi_j\|_2 \leq r$. (See the monograph [Pen03] for more details of this random graph model.) Take $\tilde{G}$ to be the largest connected component of $G$. We take $V(\tilde{G})$ as the state space, and let $P$ be the simple random walk on $\tilde{G}$.

In Figure 1 (b), we take the number of vertices to be $N = 3000$ and the feature dimension to be $d = 2$. The distance threshold is chosen as $r = 0.1$. The resulting giant connected component contains 2338 vertices.

Despite their simplicity, the two random graph models capture distinct types of the behavior of the resulting random walk in feature space: in the former model, the transition kernel makes “big jumps” in the feature space, and the correlation between two consecutive states is small; in the latter model, the transition kernel makes “local moves” in the feature space, leading to large correlation.