Validity of amplitude equations for non-local non-linearities

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Abstract

Amplitude equations are used to describe the onset of instability in wide classes of partial differential equations (PDEs). One goal of the field is to determine simple universal/generic PDEs, to which many other classes of equations can be reduced, at least on a sufficiently long approximating time scale. In this work, we study the case, when the reaction terms are non-local. In particular, we consider quadratic and cubic convolution-type non-linearities. As a benchmark problem, we use the Swift-Hohenberg equation. The resulting amplitude equation is a Ginzburg-Landau PDE, where the coefficients can be calculated from the kernels. Our proof relies on separating critical and non-critical modes in Fourier space in combination with suitable kernel bounds.

Keywords: Swift-Hohenberg, Ginzburg-Landau, amplitude equation, modulation equation, non-local non-linearity, convolution operators.

1 Introduction

In this work we study the non-local Swift-Hohenberg (SH) equation

\[ \partial_t u = -(1 + \partial_x^2)^2 u + pu + uQ * u + uK * u^2, \]

where \((x, t) \in \mathbb{R} \times [0, \infty), p \in \mathbb{R}\) is a (small) parameter, \(u = u(x, t) \in \mathbb{R}\), and \(K, Q\) are given symmetric, finite measures; here * denotes convolution in the spatial coordinate. The terms \(uQ * u\) and \(uK * u^2\) are the quadratic and cubic non-local non-linearities in (1.1). Before we discuss our main result, we provide a brief overview of amplitude (or modulation) equations as well as recent results on non-local PDEs, which provide considerable motivation to consider (1.1). The rigorous analysis of (1.1) and our mathematical contribution starts in the next section.

In dynamical systems, one common approach to deal with local instabilities is to derive a standard system, which represents the dynamics of an entire class [24]. Consider an ordinary differential equation (ODE)

\[ \frac{dz}{dt} = f(z; p), \quad z = z(t) \in \mathbb{R}^d, \quad p \in \mathbb{R}, \]

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with an equilibrium point \( z_* \) undergoing a local bifurcation upon variation of a parameter \( p \), say at \( p = 0 \). The standard method consists in first deriving a low-dimensional center manifold \( \mathcal{M}^c \) [14]. The manifold \( \mathcal{M}^c \) is tangent to the center eigenspace of \( Df(z_*;0) \). On \( \mathcal{M}^c \), the dynamics is low-dimensional and can be brought into a normal form by first Taylor-expanding and then using coordinate transformations to eliminate as many polynomial terms up to a given order [24, 30]. This procedure yields several generic classes of low-dimensional ODEs, which can then be analysed.

A similar strategy is available for many PDEs [17, 18, 26]. A typical class is

\[
\partial_t u = \mathcal{L}u + F(u; p), \quad u = u(x, t), \quad (x, t) \in \Omega \times [0, T), \quad \Omega \subseteq \mathbb{R}^d, \tag{1.2}
\]

where \( \mathcal{L} \) is a linear differential operator and \( F(u; p) = F \) is the non-linearity. Suppose \( u_* \) is a steady state of (1.2) for all \( p \). If the spectrum \( \sigma(\mathcal{L} + DF(u_*; p)) \) is contained in the left-half of the complex plane, then \( u_* \) is locally linearly stable [25]. Upon parameter variation of \( p \), say without loss of generality at \( p = 0 \), a bifurcation occurs, when \( \sigma(\mathcal{L} + DF(u_*; 0)) \cap i\mathbb{R} \neq \emptyset \) and suitable genericity conditions hold, i.e., transversal crossing of the spectrum and non-degeneracy of the non-linearity [18, 26]. As for the ODE, we may ask, whether there is a simple generic normal form, now called amplitude or modulation equation, to describe the formation of non-trivial patterns near \( p = 0 \). For a bounded domain \( \Omega \) and suitable \( \mathcal{L} \), one may often use standard centre manifold reduction for point spectrum crossing \( i\mathbb{R} \) at \( p = 0 \) [7, 48]. However, for cases involving unbounded domains, one usually faces essential spectrum crossing \( i\mathbb{R} \), which presents substantial challenges as one expects the amplitude equation to be a PDE, not an ODE, in this context [18].

The development of the field of amplitude equations has a long history and a benchmark problem is to consider the local Swift-Hohenberg equation [13, 31, 33, 41]

\[
\partial_t u = -(1 + \Delta)^2 u + pu + \mathcal{N}(u), \quad x \in \mathbb{R}^d, \tag{1.3}
\]

where \( \mathcal{N}(u) \) is a given non-linearity, frequently taken as a quadratic-cubic polynomial. The spectrum of the linearised operator has two quadratic tangencies with \( i\mathbb{R} \) for \( p = 0 \). To derive an amplitude equation formally, one possibility is to use the method of multiple scales [8, 26, 27, 29] in combination with the ansatz

\[
u(x, t) \approx \psi_A(x, t) = \varepsilon(A(X, T)e^{ik \cdot x} + \overline{A}(X, T)e^{-ik \cdot x}), \tag{1.4}\]

where \( X \) are the new scaled variables with \( X_i = x_i \varepsilon^{a_i} \) for some exponents \( a_i > 0 \), \( T = t \varepsilon^b \) is a scaled time for some \( b > 0 \), \( k \in \mathbb{Z}^d \) is a suitably chosen wave vector, and \( A \) is a slowly modulated amplitude governing the envelope of the fast Fourier modes. One re-writes (1.3) using the doubled number of variables \((x, t; X, T)\) via the chain rule, inserts an asymptotic series

\[
u = \nu_0(x, t; X, T) + \varepsilon \nu_1(x, t; X, T) + \varepsilon^2 \nu_2(x, t; X, T) + \cdots
\]

into the resulting PDE, and then uses (1.4) to derive a PDE for \( A(X, T) \). For example, \( d = 1, \mathcal{N}(u) = -u^3, \) and \( k = 1 \) yield the (real) Ginzburg-Landau equation [5, 32, 35]

\[
\partial_T A = 4\partial^2_A + \hat{p}A - 3A|A|^2, \quad p/\varepsilon^2 =: \hat{p} \in \mathbb{R}. \tag{1.5}
\]

The next step is to prove rigorous validity of the approximation, which has been discussed in many publications for local PDEs; see e.g., [28, 40, 42, 43, 47]. The typical structure of the
approximation results is the following: Assume the amplitude equation has a solution $A$ of a certain regularity over a time scale $T \in [0, T_\ast]$ and $\|\psi_A(\cdot, 0) - u(\cdot, 0)\| \lesssim \varepsilon^{\alpha_0}$ for all $x$ in the spatial domain. Then one proves

$$\|\psi_A(\cdot, t) - u(\cdot, t)\| \lesssim \varepsilon^{\alpha}, \quad \text{for all } t \in \left[0, \frac{T_\ast}{\varepsilon^2}\right],$$

(1.6)

where various choices of the (space-variable) norm $\| \cdot \|$ can be considered. For example, in the case (1.3)–(1.5) with $\alpha_0 = 2$, $\alpha = 2$, and $\beta = 2$ one may prove a uniform pointwise $O(\varepsilon^2)$-approximation over a long time scale of order $O(T_\ast/\varepsilon^2)$ [28].

There have been several recent works, using a multiple scales approach to formally derive amplitude equations also in the case when the non-linearity is non-local. Morgan and Dawes [37] studied a Swift-Hohenberg equation (1.3) for $d = 1$ with non-local non-linearity

$$\mathcal{N}(u) = c_2 u^2 - u^3 - c_3 u \int_{\mathbb{R}} K(\cdot - y) u(y, t)^2 \, dy,$$

(1.7)

where $c_2, c_3 \in \mathbb{R}$ are parameters. They provided the formal derivation of the amplitude equation in the case (1.7), calculated the coefficients in the Ginzburg-Landau equation for two classes of the kernel $K$ explicitly, and provided numerical bifurcation studies of the amplitude equation. Hence, their work provides immediate motivation to investigate the rigorous validity of amplitude equations for our non-local Swift-Hohenberg equation (1.1). Indeed, (1.7) is just a special case of the non-linearity in (1.1) as we allow $\delta$-measures to appear in the kernels $Q, K$. Faye and Holzer [20] have studied the non-local Fisher-Kolmogorov-Petrovskii-Piscounov (FKPP) equation [23]

$$\partial_t u = \partial_x^2 u + c_2 u \left(1 - \int_{\mathbb{R}} Q(\cdot - y) u(y, t) \, dy\right),$$

(1.8)

which has raised considerably interest recently in the literature; see e.g. [2, 4, 9, 10, 22]. Faye and Holzer are interested in modulated travelling fronts bifurcating from the monotone FKPP invasion wave upon variation of $c_2$. Part of their work [20, Sec.3], contains a multiple scales ansatz to derive amplitude equations for the modulated fronts, which again yields a Ginzburg-Landau equation with coefficients that can be calculated from the kernel $Q$. As in the case of (1.7), also (1.8) provides strong motivation to investigate non-local non-linearities and related amplitude equations in more detail.

The model problem (1.1) can also be motivated more abstractly. It contributes to the general interest to obtain a better understanding of non-local non-linearities. Examples include neural field equations [12, 16], phase-field models [6, 15], non-local singular perturbation problems [11, 21], various types of reaction-diffusion PDEs [38, 44, 46], non-local Schrödinger equations [1, 3], non-local models in vegetation pattern formation [34, 45] and vast classes of PDEs with constraints, e.g., elliptic-parabolic systems with elliptic part solvable in integral form. For all these scenarios, rigorous results on amplitude equations are going to be relevant.

Our main result for (1.1) can informally be stated as follows: Recall $Q, K$ are finite measures, which are symmetric, so that they obey the same symmetry of the spectrum of the linearised Swift–Hohenberg equation. Consider the local Ginzburg-Landau equation for an amplitude $A$, where the coefficients in this equation can be calculated from the Fourier transforms of $Q, K$. Suppose $A(X, T)$ is a sufficiently regular solution for $T \in [0, T_\ast]$ and $\|\psi_A(\cdot, 0) - u(\cdot, 0)\|_{C^4} \lesssim \varepsilon^2$, then

$$\|\psi_A(\cdot, t) - u(\cdot, t)\|_{C^4} \lesssim \varepsilon^2, \quad \text{for all } t \in \left[0, \frac{T_\ast}{\varepsilon^2}\right],$$

(1.9)
with $\psi_A(x, t) = \varepsilon(A(\varepsilon x, \varepsilon^2 t) e^{i x} + \overline{A}(\varepsilon x, \varepsilon^2 t) e^{-i x})$; see also Theorem 2.5.

In some sense, our result is the natural analogue to the classical local result, and we include
the local result as a special case in our approach. The key proof strategy is to generalise
techniques from [36, 39] to the non-local case using suitable a priori bounds. Although these
bounds do not yield the exact cancellation property initially developed in [28] via an improved
higher-order approximation, the kernel bounds do still yield the correct error order, i.e., only
produce terms of order $O(\varepsilon^3)$ in the final result. Our method is designed to be general enou gh
to handle larger classes of PDEs, not just (1.1), as we only use the spectral information from
the linear part, and the non-linearity contains the first two important forms of quadratic and
cubic terms. However, the Swift-Hohenberg model problem already shows very clearly the key
steps required in the analysis. In summary, our results provide a step towards a more general
theory of amplitude equations for non-local PDEs.

2 Assumptions and main result

We now specify the assumptions used throughout this work and we precisely state the main
result we are going to show. Therefore, we recall (1.1) and note that the considerations in
Section 1 suggest the scaling $p = \varepsilon^2$ with a small parameter $\varepsilon > 0$. Thus, we are led to study
the equation

$$
\partial_t u = - (1 + \partial_x^2)^2 u + \varepsilon^2 u - uQ\ast u - uK\ast u^2 \quad \text{on } \mathbb{R}.
$$

(2.1)

The precise assumptions on the convolution kernels $Q$ and $K$ will be given below (see Sec-
tion 2.1).

2.1 Assumptions on $Q$ and $K$

In the remainder of this work, the convolution kernels $Q$ and $K$ are assumed to be finite,
symmetric measures on $\mathbb{R}$, i.e. $Q, K \in M^{\text{fin}}(\mathbb{R})$ and symmetric, such that it holds

$$
\int_{\mathbb{R}} |x| |Q|(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}} |x| |K|(dx) < \infty.
$$

(2.2)

Remark 2.1. Note that throughout this work, we will use the notation $Q(dx) = Q(x) dx$ and
$K(dx) = K(x) dx$ for simplicity although $Q$ and $K$ might not have a Lebesgue density. Thus
all integrals occurring have to be interpreted accordingly. In particular, we write by abuse of
notation $\|fQ\|_{L^1}$ for the expression $\int_{\mathbb{R}} |f| |Q|(dx)$ and equivalently also for $K$.

Since our analysis relies crucially on the use of the Fourier transform, we have to restrict
moreover at certain places to measures $Q$ and $K$ which can be decomposed as

$$
Q = Q_r + Q_s \quad \text{and} \quad K = K_r + K_s \quad \text{with } Q_r, K_r \in \mathcal{S}
$$

and $Q_s, K_s \in M^{\text{fin}}(\mathbb{R})$ compactly supported.

(2.3)

Here, $\mathcal{S}$ denotes the Schwartz space of smooth and rapidly decaying functions, while the dual
space of tempered distributions will be denoted by $\mathcal{S}'$ throughout this work. We emphasise
that the usual embedding $M^{\text{fin}}(\mathbb{R}) \subset \mathcal{S}'$ yields that we may consider $Q_s$ and $K_s$ also as elements in $\mathcal{S}'$.

Remark 2.2. We note that (2.3) in particular implies (2.2) while (2.3) also allows for purely
‘local’ non-linearities if we choose $Q = q\delta(\cdot)$ and $K = k\delta(\cdot)$. 

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To simplify the notation at several places, we define
\[ L(v) := -(1 + \partial^2_x) v + \varepsilon^2 v, \quad N_Q(v) := -uQ * u, \quad N_K(v) = -uK * u^2 \]
and \( \mathcal{N}(v) = \mathcal{N}_Q(v) + \mathcal{N}_K(v) \)
such that (2.1) can be written as
\[ \frac{\partial}{\partial t} u = L(u) + \mathcal{N}(u). \]

Moreover, we have the following continuity property for the convolution operators induced by \( Q \) and \( K \).

**Lemma 2.3.** For each \( n \in \mathbb{Z} \) and \( k \in \mathbb{N}_0 \) there exists a constant \( C > 0 \) such that it holds
\[ \| B_1(Qe^{in\cdot}) * B_2 \|_{C^k} \leq C \| B_1 \|_{C^k} \| B_2 \|_{C^k} \] (2.4)
and
\[ \| B_1(Ke^{in\cdot}) * (B_2B_3) \|_{C^k} \leq C \| B_1 \|_{C^k} \| B_2 \|_{C^k} \| B_3 \|_{C^k} \] (2.5)
for all \( B_\ell \in C^k_b(\mathbb{R}) \) with \( \ell \in \{1, 2, 3\} \).

**Proof.** Due to the assumptions of \( Q \) it holds
\[ \| B_1(Qe^{in\cdot}) * B_2 \|_{C^0} \leq \| B_1 \|_{C^0} \| B_2 \|_{C^0} \| Qe^{in\cdot} \|_{L^1} \leq C \| B_1 \|_{C^0} \| B_2 \|_{C^0} \] (2.6)
This proves (2.4) for \( k = 0 \), while the case of general \( k \in \mathbb{N} \) then follows immediately from (2.6) together with Leibniz’ rule. The proof of (2.5) is analogously.

Finally, we introduce some notation, i.e. the assumption (2.2) allows to define the constants
\[ q_n := \int_{\mathbb{R}} e^{inx} Q(x) \, dx \quad \text{and} \quad k_n := \int_{\mathbb{R}} e^{inx} K(x) \, dx \quad \text{for all} \quad n \in \mathbb{Z}. \] (2.7)
Moreover, we note that due to the symmetry of \( Q \) and \( K \) it also holds
\[ k_{-n} = k_n \quad \text{and} \quad q_{-n} = q_n \quad \text{for all} \quad n \in \mathbb{Z} \]
and thus in particular \( k_n, q_n \in \mathbb{R} \) for all \( n \in \mathbb{Z} \).

### 2.2 Main result

As explained in Section 1, one expects that a solutions \( u \) of (2.1) can be approximated by a function of the form
\[ \psi(x, t) = \varepsilon (A(\varepsilon x, \varepsilon^2 t) e^{ix} + \bar{A}(\varepsilon x, \varepsilon^2 t) e^{-ix}) \] (2.8)
provided that the initial condition \( u_0 = u(\cdot, 0) \) is sufficiently close to \( \psi(\cdot, 0) \) and \( A \) is a solution to the Ginzburg-Landau equation. Precisely, under our assumptions on \( Q \) and \( K \) formal calculations suggest to take \( A \) as solution of
\[ \partial_T A(X, T) = (1 + 4\partial^2_X) A(X, T) - \left( 2k_0 + k_2 - \frac{q_1q_2}{9} - \frac{q_1^2}{9} - 2q_0q_1 - 2q_1^2 \right) |A(X, T)|^2 A(X, T). \] (2.9)

The following proposition guarantees the existence of a solution to (2.9) at least locally in time.
Proposition 2.4. Assume that $A_0(\cdot) \in C^1_t(\mathbb{R})$. Then there exists $T_* > 0$ such that there exists a unique solution $A = A(X, T) \in C([0, T_*], C^4)$ of (2.9) with $A(\cdot, 0) = A_0$.

Proof. This statement follows easily by an application of the contraction mapping theorem. □

We can now state the main result that we will show in this work.

Theorem 2.5. Let $A \in C([0, T_*], C^4)$ be a solution of (2.9). Then for each $d > 0$ there exist constants $\varepsilon_*, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$ the following statement holds. If $\|u_0 - \psi(\cdot, 0)\|_{C^4} \leq d\varepsilon^2$ then there exists a unique solution $u$ of (2.1) on the time interval $[0, T_*/\varepsilon^2]$ with $u(\cdot, 0) = u_0$ and moreover we have the estimate

$$\|u(\cdot, t) - \psi(\cdot, t)\|_{C^4} \leq C\varepsilon^2 \quad \text{for all} \ (x, t) \in \mathbb{R} \times [0, T_*/\varepsilon^2].$$

2.3 Notation and outline

In order to prove Theorem 2.5, we will follow the same approach as in [39] and thus, instead of showing that $\psi$ is a good approximation for solutions of (2.1), we will consider the intermediate approximation

$$\phi(x, t) = \varepsilon \left( (E_0A)(X, T)e^{ix} + (E_0\bar{A})(X, T)e^{-ix} \right) + \varepsilon^2 \left( (E_0A_2)(X, T)e^{2ix} + (E_0\bar{A}_2)(X, T)e^{-2ix} + (E_0A_0)(X, T) \right)$$

(2.10)

with $X = \varepsilon x$ and $T = \varepsilon^2 t$. The operator $E_0$ acts as a cut-off function in Fourier variables to select modes which are sufficiently close to zero. The precise definition of $E_0$ is given in (3.1). The coefficients $A$, $A_0$ and $A_2$ are chosen such that $A$ is a solution of (2.9) while $A_2$ and $A_0$ are given by

$$A_0 = -2q_1|A|^2 \quad \text{and} \quad A_2 = -\frac{q_1}{9}A^2.$$ 

(2.11)

One key ingredient in the proof of Theorem 2.5 is to consider the critical Fourier modes $e^{\pm ix}$ separately from the uncritical ones. Therefore, one defines

$$\phi_c = (E_0A)e^{ix} + (E_0\bar{A})e^{-ix} \quad \text{and} \quad \phi_s = (E_0A_2)e^{2ix} + (E_0\bar{A}_2)e^{-2ix} + (E_0A_0)$$

(2.12)

such that $\phi = \varepsilon \phi_c + \varepsilon^2 \phi_s$. We then have the following lemma which states that $\phi_s$ is uniformly bounded and $\phi$ is uniformly close to $\psi$ up to an error of $O(\varepsilon^3)$.

Lemma 2.6. Let $A \in C([0, T_*], C^4(\mathbb{R}))$ be a solution of (2.9) and $\phi_c$ and $\phi_s$ be given by (2.12) together with (2.11). Then there exists a constant $C > 0$ such that it holds

$$\sup_{t \in [0, T_*/\varepsilon^2]} \|\phi_s\|_{C^4} \leq C \quad \text{and} \quad \sup_{t \in [0, T_*/\varepsilon^2]} \|\phi - \psi\|_{C^4} \leq C\varepsilon^2,$$

where $\phi = \varepsilon \phi_c + \varepsilon^2 \phi_s$ and $\psi$ is defined in (2.8).

Proof. The bound on $\phi_s$ is an immediate consequence of the assumptions on $A$, the definition of $A_0$ and $A_2$ in (2.11) and Leibniz’ rule, while we also note that the operator $E_0$ commutes with $\partial_x$.

To verify the second estimate of the lemma, we note that

$$(\phi - \psi)(x) = \varepsilon \left( (E_0A(\varepsilon \cdot))(x)e^{ix} + (E_0\bar{A}(\varepsilon \cdot))(x)e^{-ix} - A(\varepsilon x)e^{ix} - \bar{A}e^{-ix} \right) + \varepsilon^2 \phi_s$$

$$= \varepsilon^2 \phi_s + \varepsilon \left( (E_0^cA(\varepsilon \cdot))(x)e^{ix} + (E_0^c\bar{A}(\varepsilon \cdot))e^{-ix} \right),$$

(2.8)
where $E_0^*$ is defined in (3.1). Thus, combining Lemma 3.3 below with Leibniz’ rule as well as the first estimate of the lemma, the claim easily follows. 

The main strategy to prove Theorem 2.5 is now as follows. First, we note that Lemma 2.6 yields that $\psi$ can be approximated by $\phi$ on the time interval $[0,T_*/\varepsilon^2]$ up to an error of $O(\varepsilon^2)$. As a consequence, it is enough to prove Theorem 2.5 with $\psi$ replaced by $\phi$. The general approach for this will be to consider the approximation error $R = u - \phi$ and to show that this quantity remains of $O(\varepsilon^2)$ on $[0,T_*/\varepsilon^2]$. To this end, we will derive an evolution equation for $R$ and show that there exists a unique solution which is small on $[0,T_*/\varepsilon^2]$. Since $u$ is on the other hand uniquely determined on a small time interval, this then yields that $u$ also exists on $[0,T_*/\varepsilon^2]$ by a standard continuation argument. One crucial part within this approach consists in obtaining suitable estimates for the residuum of $\phi$ which is defined as

$$\text{Res}(\phi) := -\partial_t \phi + \mathcal{L}(\phi) + \mathcal{N}(\phi). \quad (2.13)$$

The study of this expression will be contained in Section 4. Moreover, in Section 5, we will derive the equation which has to be satisfied by $\phi$ to obtain that $R$ stays of $O(\varepsilon^2)$ on $[0,T_*/\varepsilon^2]$, it will be necessary to consider the critical and uncritical modes separately. Based on these preparations, we will then provide the proof of Theorem 2.5 in Section 6. Moreover, in Section 3, we recall several technical definitions and properties from [39] which will be used frequently.

2.4 Main difference to local non-linearity

To conclude this section, we will finally point out one main difference between the proof of Theorem 2.5 and the corresponding result for local non-linearities, i.e. the equation

$$\partial_t u = -(1 + \partial_x^2)u + \varepsilon^2 u - qu^2 - ku^3$$

as for example considered in [36]. However, as mentioned in Remark 2.2, this equation is still contained as special case in our Theorem 2.5.

As explained above, we follow the same main approach as in [36,39] by computing and estimating $\text{Res}(\phi)$ in order to show that the approximation error $R = u - \phi$ remains small. However, in the case where the non-linearity is given as $N(u)$ with a polynomial $N$, the choice of $A$ together with (2.11) yields that in $\text{Res}(\phi)$ several expressions of lower order in $\varepsilon$ exactly cancel. In contrast to this, when we consider the more general non-local non-linearities as in (2.1) this is no longer the case. To circumvent this problem, we have to use the following result which states that although the lower order expressions do not cancel, we can still gain at least one order in $\varepsilon$.

Lemma 2.7. For each $n \in \mathbb{Z}$ there exists a constant $C > 0$ such that it holds

$$\|B_1(\varepsilon)\langle Qe^{nt} \rangle * B_2(\varepsilon) - q_n(B_1 B_2)(\varepsilon)\|_{C^1} \leq C\varepsilon\|B_1\|_{C^1}\|B_2\|_{C^1}$$

$$\|B_1(\varepsilon)\langle K e^{nt} \rangle * (B_2 B_3)(\varepsilon) - k_n(B_1 B_2 B_3)(\varepsilon)\|_{C^1} \leq C\varepsilon\|B_1\|_{C^1}\|B_2\|_{C^1}\|B_3\|_{C^1}$$

for all $B_\ell \in C^1_b(\mathbb{R})$ with $\ell \in \{1,2,3\}$.

Proof. Since $\partial_x B_\ell(\varepsilon x) = \varepsilon \partial_x B_\ell(\varepsilon x)$ for $\ell = 1,2,3$ and $B_\ell \in C^1_b(\mathbb{R})$ it suffices to prove that

$$\|B_1(\varepsilon)\langle Qe^{nt} \rangle * B_2(\varepsilon) - q_n(B_1 B_2)(\varepsilon)\|_{C^0} \leq C\varepsilon\|B_1\|_{C^1}\|B_2\|_{C^1} \quad (2.14)$$

$$\|B_1(\varepsilon)\langle K e^{nt} \rangle * (B_2 B_3)(\varepsilon) - k_n(B_1 B_2 B_3)(\varepsilon)\|_{C^0} \leq C\varepsilon\|B_1\|_{C^1}\|B_2\|_{C^1}\|B_3\|_{C^1}. \quad (2.15)$$
We first consider (2.14) and notice that
\[ \left| B_1(\varepsilon x) \left( (Qe^{ni}) \ast B_2(\varepsilon \cdot) \right)(x) - q_n(B_1B_2)(\varepsilon x) \right| \]
\[ = \left| B_1(\varepsilon x) \int_{\mathbb{R}} Q(y)e^{iny}(B_2(\varepsilon(x-y)) - B_2(\varepsilon x)) \, dy \right| \]
\[ \leq \varepsilon \| B_1 \|_{C^0} \| B_2 \|_{C^1} \int_{\mathbb{R}} |yQ(y)| \, dy \leq C \varepsilon \| B_1 \|_{C^0} \| B_2 \|_{C^1}. \]

Here we also used that \( |B_2(\varepsilon(x-y)) - B_2(\varepsilon x)| \leq \varepsilon \| B_2 \|_{C^1} |y| \) for all \( x, y \in \mathbb{R} \). Thus, (2.14) immediately follows.

To prove (2.15) we can argue analogously since we have the relation
\[ \left| B_1(\varepsilon x) \left( (Ke^{ni}) \ast (B_2B_3)(\varepsilon \cdot) \right)(x) - k_n(B_1B_2B_3)(\varepsilon x) \right| \]
\[ = \left| B_1(\varepsilon x) \int_{\mathbb{R}} K(y)e^{iny}(B_2(\varepsilon(x-y)) - B_2(\varepsilon x))B_3(\varepsilon(x-y)) \, dy \right| \]
\[ + \left| B_1(\varepsilon x) \int_{\mathbb{R}} K(y)e^{iny}B_2(\varepsilon x)(B_3(\varepsilon(x-y)) - B_3(\varepsilon x)) \, dy \right|. \]

From this, the estimate (2.15) follows in the same way as for the quadratic term. \( \Box \)

### 3 Technical preparation

Our strategy to prove Theorem 2.5 follows closely that one in [39], where the equation
\[ \partial_t u = -(1 + \partial_x^2)u + \varepsilon^2 u + u \partial_x u \]
has been considered and we thus recall in this section several technical fundamentals. Moreover, we provide the necessary adaptations and extensions that we need for the situation that we consider in this work. More precisely, we will work in the space \( C^4_b(\mathbb{R}) \) of four times differentiable functions with globally bounded derivatives. As already indicated before, one key ingredient is to consider the critical Fourier modes \( e^{\pm ix} \) separately from the uncritical ones which will be achieved by suitable multiplication operators in Fourier space the so-called mode filters. This approach makes it necessary, to work with the Fourier transform which is not directly defined on the space \( C^4_b(\mathbb{R}) \). However, as also pointed out in [39] we can embed \( C^4_b(\mathbb{R}) \) into \( \mathcal{S}' \), where the Fourier transform is defined in the usual way by duality.

We recall now the definition of the mode filters as given in [39] and for this, we will denote by \( I_r(x) \) the open interval of radius \( r \) centred around \( x \), i.e. \( I_r(x) = (x-r, x+r) \). One then fixes non-negative and even functions \( \chi_c, \chi_0 \in C^\infty_c(\mathbb{R}) \) which satisfy
\[ \chi_c(k) = \begin{cases} 
1 & \text{if } k \in I_{1/8}(-1) \cup I_{1/8}(1) \\
0 & \text{if } k \in \mathbb{R} \setminus (I_{1/4}(-1) \cup I_{1/4}(1)) 
\end{cases} \]
and
\[ \chi_0(k) = \begin{cases} 
1 & \text{if } k \in I_{1/8}(0) \\
0 & \text{if } k \in \mathbb{R} \setminus I_{1/4}(0). 
\end{cases} \]

For these functions we additionally define \( G_c \) and \( G_0 \) as the inverse Fourier transforms, i.e.
\[ G_c(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \chi_c(k) \, dk \quad \text{and} \quad G_0(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \chi_0(k) \, dk. \]

The mode filters \( E_c, E_0, E^c_0 \) and \( E_s \) are then defined as
\[ E_cv := G_c \ast v, \quad E_0v := G_0 \ast v, \quad E^c_0 := E_0 - \text{Id} \quad \text{and} \quad E_s := \text{Id} - E_c. \quad (3.1) \]
Remark 3.1. If we denote by $\mathcal{F}$ the Fourier transform, it is well-known that for $v \in \mathcal{S}$ it holds $\mathcal{F}(E_c v) = \chi_c \mathcal{F}(v)$ as well as $\mathcal{F}(E_0 v) = \chi_0 \mathcal{F}(v)$. Moreover, since $\chi_c$ and $\chi_0$ have compact support it holds in particular $\chi_c, \chi_0 \in \mathcal{S}$ such that (3.1) makes even sense for $v \in \mathcal{S}'$ since the convolution between tempered distributions and Schwartz functions is well-defined.

Remark 3.2. We additionally remark that $E_s$ and $E_0^s$ can also be represented as convolution operators, with kernels $G_s = \delta - G_c$ as well as $G_0^s = G_0 - \delta$. The corresponding Fourier transforms are given by $\chi_s = 1 - \chi_c$ and $\chi_0^s = \chi_0 - 1$. Note that in this case $G_0^s$ and $G_s$ are only measures due the fact that the Fourier transforms have unbounded support.

For technical reasons it is also necessary to introduce further operators $E_c^h$ and $E_s^h$ which satisfy $E_c^h E_c = E_c$ and $E_s^h E_s = E_s$ and which are defined via $C^\infty$-functions $\chi^h_c$ and $\chi^h_s$. More precisely, $\chi^h_c \in C^\infty_c(\mathbb{R})$ is chosen such that it vanishes outside $I_{3/8}(-1) \cup I_{3/8}(1)$ while $\chi^h_s$ vanishes in $I_{1/{16}}(-1) \cup I_{1/{16}}(1)$.

With these definitions, we can cite three results on the mode filters which are contained in [39] as Lemmas 3–5.

Lemma 3.3. The operators $E_c$ and $E_0$ are linear and continuous mappings from $C^0$ to $C^m$. For every $m \geq 0$ there exists $C_m > 0$ with $\|E_0 u\|_{C^m} + \|E_c u\|_{C^m} \leq C_m \|u\|_{C^0}$.

Lemma 3.4. For $n \in \mathbb{N}$ there is a $C_n > 0$ such that $\|(E_0^s A(\varepsilon \cdot))\|_{C^m} = \|(E_0 A(\varepsilon \cdot)) - A(\varepsilon)\|_{C^n} \leq C_n \varepsilon^n \|A\|_{C^m}$.

Lemma 3.5. For $u_1, u_2 \in C^n$ and $r_1, r_2 \in \mathbb{N}$ it is true that

$$E_c (\partial^{r_1}_{x^1} E_c u_1 \cdot \partial^{r_2}_{x^2} E_c u_2) = 0.$$  

The last statement essentially says that the product of two functions with critical Fourier modes only contains uncritical modes. However, since we have to deal with non-linearities which are in general convolutions, we will need an extension of Lemma 3.5. In order to proof this, we also require the following well-known result about the convolution of distributions (see for example [19]).

Lemma 3.6. Let $u, v \in \mathcal{S}'$ and assume that either $u$ or $v$ has compact support. Then the convolution $u * v$ exists in $\mathcal{S}'$ and moreover, it holds $\mathcal{F}(u * v) = \mathcal{F}(u) \mathcal{F}(v)$. This means in particular that the product on the right-hand side exists in $\mathcal{S}'$.

Remark 3.7. As a consequence of Lemma 3.6 it also holds that $\mathcal{F}(uv) = (2\pi)^{-1} \mathcal{F}(u) * \mathcal{F}(v)$ provided that $u, v \in \mathcal{S}'$ such that either $\mathcal{F}(u)$ or $\mathcal{F}(v)$ has compact support.

We can then show the following generalisation of Lemma 3.5.

Lemma 3.8. For all $n \in \mathbb{Z}$ and $B_1, B_2 \in \mathcal{S}'$ with Fourier transform supported in $\overline{T}_{1/4}(-1) \cup \overline{T}_{1/4}(1)$ it holds

$$E_c (B_1(Qe^{ni}) * B_2) = 0.$$

In particular $B_1(Qe^{ni}) * B_2 \in \mathcal{S}'$ is well-defined.

Proof. Due to the assumptions on $Q$ it is well-known that $\mathcal{F}(Qe^{in\cdot}) \in C^\infty$. Thus, since $\mathcal{F}(B_1)$ and $\mathcal{F}(B_2)$ are assumed to have compact support, Lemma 3.6 yields that $B_1(Qe^{ni}) * B_2 \in \mathcal{S}'$ exists and

$$\mathcal{F}(B_1(Qe^{ni}) * B_2) = \mathcal{F}(B_1) * (\mathcal{F}(Qe^{ni}) \mathcal{F}(B_2)).$$  

(3.2)
Since \( \mathcal{F}(B_2) \) is supported in \( \mathcal{T}_{1/4}(-1) \cup \mathcal{T}_{1/4}(1) \), the same is true for \( \mathcal{F}(Qe^{ni})\mathcal{F}(B_2) \) as well as for \( \mathcal{F}(B_1) \) by assumption. Thus, we immediately obtain from (3.2) that the support of \( \mathcal{F}(B_1(Qe^{ni}) * B_2) \) is contained in \( \Omega := \mathcal{T}_{1/2}(-1) \cup \mathcal{T}_{1/2}(0) \cup \mathcal{T}_{1/2}(1) \) while \( \chi_c \equiv 0 \) on \( \Omega \). Thus the claim immediately follows from the definition of \( E_c \). \( \square \)

In a similar fashion, we have the following result which provides information on the support in Fourier space for the operators induced by \( Q \) and \( K \).

**Lemma 3.9.** For all \( n \in \mathbb{Z} \) and \( B_1, B_2, B_3 \in \mathcal{S}' \) with Fourier transform supported in \( \mathcal{T}_{1/4}(0) \) the expressions \( B_1(Qe^{ni}) * B_2 \) and \( B_1(Ke^{ni}) * (B_2B_3) \) are well-defined in \( \mathcal{S}' \) and it holds

\[
\text{supp} \mathcal{F}(B_1(Qe^{ni}) * B_2) \subset \mathcal{T}_{1/2}(0) \quad \text{and} \quad \text{supp} \mathcal{F}(B_1(Ke^{ni}) * (B_2B_3)) \subset \mathcal{T}_{3/4}(0).
\]

**Proof.** Similarly as in the proof of Lemma 3.8 one finds together with Lemma 3.6 and Remark 3.7 that \( B_1(Qe^{ni}) * B_2 \) and \( B_1(Ke^{ni}) * (B_2B_3) \) are well-defined and it holds

\[
\mathcal{F}(B_1(Qe^{ni}) * B_2) = \mathcal{F}(B_1) * (\mathcal{F}(Qe^{ni})\mathcal{F}(B_2))
\]

and

\[
\mathcal{F}(B_1(Ke^{ni}) * (B_2B_3)) = \mathcal{F}(B_1) * (\mathcal{F}(Ke^{ni})\mathcal{F}(B_2) * \mathcal{F}(B_3)).
\]

From these relations, the claim immediately follows due to the assumptions on the support of \( \mathcal{F}(B_1), \mathcal{F}(B_2) \) and \( \mathcal{F}(B_3) \). \( \square \)

For later use, we also recall the following semi-group estimates which are stated in [39].

**Lemma 3.10.** Let \( e^{\mathcal{L}t} \) denote the semi-group associated to the operator \( \mathcal{L} \). Then there exist constants \( C, \sigma > 0 \) which are independent of \( \varepsilon \) such that it holds

\[
\| e^{\mathcal{L}t} E_c^h \|_{\mathcal{L}(C^4)} \leq Ce^{\varepsilon^2 t} \quad \text{and} \quad \| e^{\mathcal{L}t} E_s^h \|_{\mathcal{L}(C^4)} \leq Ce^{-\sigma t} \max \{ 1, t^{-3/4} \}.
\]

4 The residuum

In this section, we will compute the residuum as defined in (2.13) and moreover, we will derive several estimates which we will need for the proof of the main statement.

4.1 Computing the residuum

Since we only need estimates on the \( C^4 \)-norm of \( \text{Res}(\phi) \) one can easily verify, that the assumptions of Theorem 2.5 together with Lemma 3.3 yield that all derivatives which occur during the computation of \( \text{Res}(\phi) \) are uniformly bounded on the relevant time interval. More precisely, this is immediately clear for the purely spatial derivatives. However, the following lemma states that also the \( C^1 \)-norm of the time derivative is uniformly bounded.

**Lemma 4.1.** Let \( A \in C^0([0,T_*],C^4_b(\mathbb{R})) \) be a solution of (2.9) and \( A_0 \) and \( A_2 \) be given as in (2.11). Then it holds

\[
\| \partial_T A_0 \|_{C^1} + \| \partial_T A_2 \|_{C^1} \leq C(\| A \|_{C^3} + \| A \|_{C^1}^3) \| A \|_{C^1}
\]

for some constant \( C > 0 \).
Proof. Due to (2.11) it holds $A_0 = -2q_1 A \bar{A}$ and $A_2 = -(q_1 A^2)/9$. Thus, we have
\[\partial_T A_0 = -2q_1 (\partial_T A \bar{A} + A \partial_T \bar{A}) \quad \text{and} \quad \partial_T A_2 = -\frac{2}{9} q_1 A \partial_T A.\]

Since both $A$ and $\bar{A}$ solve (2.9) the claim easily follows. \qed

As a consequence, it suffices to consider only terms up to $O(\varepsilon^3)$ and we will therefore only compute explicitly these terms while all expressions of $O(\varepsilon^4)$ are just estimated by a constant.

To simplify the presentation, we first compute the different expressions separately and then finally collect all the terms. Moreover, we skip the argument of the functions in order to improve the readability and we use the common notation c.c. to indicate complex conjugate.

First of all, we obtain
\[\partial_t \phi = -\varepsilon^2 \partial_T (E_0 A) e^{ix} + \text{c.c.} + O(\varepsilon^4).\]

Moreover, it holds $(1 + \partial_x^2)^2 = 1 + 2\partial_x^2 + \partial_x^4$ and we have
\[-2\partial_x^2 \phi = \left[ -2\varepsilon^3 \partial_X (E_0 A) e^{ix} - 4\varepsilon^2 \partial_X (E_0 A) e^{ix} + 2\varepsilon (E_0 A) e^{ix} \right.\]
\[\left. + 8\varepsilon^2 (E_0 A_2) e^{2ix} - 8\varepsilon^3 \partial_X (E_0 A_2) e^{2ix} \right] + \text{c.c.} + O(\varepsilon^4).\]

Similarly, we obtain
\[-\partial_x^4 \phi = \left[ 6\varepsilon^3 \partial_X (E_0 A) e^{ix} + 4\varepsilon^2 \partial_X (E_0 A) e^{ix} - \varepsilon (E_0 A) e^{ix} \right.\]
\[\left. - 16\varepsilon^2 (E_0 A_2) e^{2ix} + 32\varepsilon^3 \partial_X (E_0 A_2) e^{2ix} \right] + \text{c.c.} + O(\varepsilon^4).\]

If we also note that $\varepsilon^2 \phi = \varepsilon^3 (E_0 A) e^{ix} + \text{c.c.} + O(\varepsilon^4)$ we already get
\[
\mathcal{L}(\phi) = \left[ -9\varepsilon^2 (E_0 A_2) e^{2ix} + 4\varepsilon^3 \partial_X (E_0 A) e^{ix} + (E_0 A) e^{ix} + 24\varepsilon^3 \partial_X (E_0 A_2) e^{-2ix} \right] + \text{c.c.}
\]
\[= -\varepsilon^2 (E_0 A_0) + O(\varepsilon^4). \quad (4.1)\]

In order to compute the non-linear terms, we will use the general relation
\[
V(\varepsilon x) (N(\cdot) * W(\varepsilon \cdot e^{m \cdot i}) (x)) = V(\varepsilon x) e^{n \cdot i x} \int_{\mathbb{R}} N(y) W(\varepsilon(x - y)) e^{m i(x - y)} \, dy
\]
\[= V(\varepsilon x) e^{(n + m) i x} \int_{\mathbb{R}} N(y) e^{-m i y} W(\varepsilon(x - y)) \, dy = V(\varepsilon x) e^{(n + m) i x} (N(\cdot) e^{-m i \cdot} * W(\varepsilon \cdot))(x).\]

We note that these manipulations are rigorously justified in the expressions where we will use this below. In particular, we find
\[
\mathcal{N}_Q(\phi) = -\varepsilon^2 \left[ (E_0 A) e^{2ix} ((Q e^{-i}) * (E_0 A)(\varepsilon \cdot)) + (E_0 A) ((Q e^{i}) * (E_0 A)(\varepsilon \cdot)) \right] + \text{c.c.}
\]
\[= -\varepsilon^3 \left[ (E_0 A) e^{3ix} ((Q e^{-2i}) * (E_0 A_2)(\varepsilon \cdot)) + (E_0 A_2) e^{3ix} ((Q e^{2i}) * (E_0 A)(\varepsilon \cdot)) \right.\]
\[\left. + (E_0 A) e^{ix} ((Q e^{-2i}) * (E_0 A_2)(\varepsilon \cdot)) + (E_0 A_2) e^{ix} ((Q e^{2i}) * (E_0 A)(\varepsilon \cdot)) \right.\]
\[\left. + (E_0 A) e^{ix} (Q * (E_0 A_0)(\varepsilon \cdot)) + (E_0 A_0) e^{ix} ((Q e^{2i}) * (E_0 A)(\varepsilon \cdot)) \right] + \text{c.c.} + O(\varepsilon^4). \quad (4.2)\]
For the cubic terms we obtain in the same way

\[ \mathcal{N}_K(\phi) = -\varepsilon^3 \left[ (E_0 A) e^{3i\varepsilon} ((K e^{-2i}) \ast (E_0 A)^2(\varepsilon)) + (E_0 \bar{A}) e^{i\varepsilon} ((K e^{-2i}) \ast (E_0 A)^2(\varepsilon)) \right. \]
\[ + \left. 2(E_0 A) e^{i\varepsilon} (K \ast ((E_0 A)(E_0 \bar{A}))(\varepsilon)) \right] + c.c. + \mathcal{O}(\varepsilon^4). \] (4.3)

Summarising (4.1)–(4.3) we find that

\[ \text{Res}(\phi) = \sum_{\ell=-3}^{3} a_{\ell} e^{i\varepsilon \ell} + \mathcal{O}(\varepsilon^4) \]

with \( a_{-\ell} = \bar{a}_{\ell} \) and

\[ a_0 = -\varepsilon^2 \left[ (E_0 A_0) + (E_0 A)((Q e^i) \ast (E_0 \bar{A})(\varepsilon)) + (E_0 \bar{A})((Q e^{-i}) \ast (E_0 A)(\varepsilon)) \right] \]
\[ a_1 = \varepsilon^3 \left[ \partial_T (E_0 A) + 4i\varepsilon^2 (E_0 A) + (E_0 A) - (E_0 \bar{A})(Q e^{-2i}) \ast (E_0 A_2)(\varepsilon) \right. \]
\[ - \left. (E_0 A) Q \ast (E_0 A_0)(\varepsilon) - (E_0 A_0)(Q e^{-i}) \ast (E_0 A)(\varepsilon) \right) - 2(E_0 A) K \ast ((E_0 A)(E_0 \bar{A}))(\varepsilon) - (E_0 \bar{A})(K e^{-2i}) \ast (E_0 A)^2(\varepsilon) \] (4.4)
\[ a_2 = -9e^2(E_0 A_2) + 24i\varepsilon^3 \partial_X (E_0 A_2) - \varepsilon^2 (E_0 A)(Q e^{-i}) \ast (E_0 A)(\varepsilon) \]
\[ a_3 = -\varepsilon^3 \left[ (E_0 A)(Q e^{-2i}) \ast (E_0 A_2)(\varepsilon) \right. \]
\[ + \left. (E_0 A_2)(Q e^{-i}) \ast (E_0 A)(\varepsilon) + (E_0 A)(K e^{-2i}) \ast (E_0 A)^2(\varepsilon) \right]. \]

4.2 Estimating the residuum

In this section, we provide several estimates on \( \text{Res}(\phi) \) that we will need later on. More precisely, the next lemma states that the pre-factor for the uncritical modes is of order \( \varepsilon^3 \) while that one for the critical modes can even be bounded by \( \varepsilon^4 \).

**Lemma 4.2.** There exists a constant \( C > 0 \) such that it holds

\[ \sup_{t \in [0,T, \varepsilon^2]} \left( \| a_0 \|_{C^1} + \| a_2 \|_{C^1} + \| a_3 \|_{C^1} \right) \leq C \varepsilon^3 \] (4.5)
\[ \sup_{t \in [0,T, \varepsilon^2]} \| a_1 \|_{C^1} \leq C \varepsilon^4 \] (4.6)

where \( a_{\ell} \) is given by (4.4) for \( \ell \in \{0,1,2,3\} \)

The following relations will be used in the proof of the lemma.

**Remark 4.3.** For each \( n \in \mathbb{Z} \) and functions \( B, B_1, B_2, B_3 \) we have the relations \( (E_0 B) = (E_0^2 B) + B \) as well as

\( (E_0 B_1)(Q e^{ni}) \ast (E_0 B_2) = (E_0^2 B_1)(Q e^{ni}) \ast (E_0 B_2) + B_1(Q e^{ni}) \ast (E_0 B_2) + B_1(Q e^{ni}) \ast B_2 \)

and

\( (E_0 B_1)(K e^{ni}) \ast ((E_0 B_2)(E_0 B_3)) = (E_0^2 B_1)(K e^{ni}) \ast ((E_0 B_2)(E_0 B_3)) \)
\[ + B_1(K e^{ni}) \ast ((E_0 B_2)(E_0 B_3)) + B_1(K e^{ni}) \ast (B_2(E_0 B_3)) + B_1(K e^{ni}) \ast (B_2 B_3). \)
Proof of Lemma 4.2. We consider first \( a_0 \). Since \( A_0 = -2q_1 A \bar{A} \) we obtain by means of Remark 4.3 that

\[
- \varepsilon^2 \left[ (E_0 A_0) + (E_0 A)((Qe^i) \ast (E\bar{A})(\varepsilon)) + (E_0 \bar{A})((Qe^{-i}) \ast (E_0 A)(\varepsilon)) \right] \\
= - \varepsilon^2 \left[ (E_0 A_0) - 2q_1 A \bar{A} + (E_0 \bar{A})((Qe^i) \ast (E_0 \bar{A})(\varepsilon)) + A((Qe^i) \ast (E_0 A)(\varepsilon)) + A(Qe^{-i}) \ast \bar{A}(\varepsilon) \\
+ (E_0 \bar{A})((Qe^{-i}) \ast (E_0 A)(\varepsilon)) + \bar{A}((Qe^{-i}) \ast (E_0 \bar{A})(\varepsilon)) + \bar{A}((Qe^{-i}) \ast A(\varepsilon)) \right].
\]

Due to Lemmas 2.3 and 2.7 and \( A_0 = -2q_1 A \bar{A} \) we thus obtain

\[
\|(E_0 A_0) + (E_0 A)((Qe^i) \ast (E\bar{A})(\varepsilon)) + (E_0 \bar{A})((Qe^{-i}) \ast (E_0 A)(\varepsilon))\|_{C^1}
\leq C \varepsilon \|A\|_{C^1} \|\bar{A}\|_{C^1}
+ C \left[ \|E_0 A\|_{C^1} \|E_0 \bar{A}\|_{C^1} + \|A\|_{C^1} \|E_0 \bar{A}\|_{C^1} + \|\bar{A}\|_{C^1} \|E_0 A\|_{C^1} \right].
\]

Lemmas 3.3 and 3.4 together with the uniform boundedness of \( A \) thus yield

\[
\sup_{t \in [0,T_*/\varepsilon^2]} \|a_0\|_{C^1} \leq C \varepsilon^3.
\]

To estimate \( a_2 \) we can proceed in the same way, i.e. Lemma 3.3 together with the boundedness of \( A \) yields \( \|24 \varepsilon^3 \partial_X (E_0 A_2)\|_{C^1} \leq C \varepsilon^3 \). Thus, it remains to estimate \(-9(E_0 A_2) - (E_0 A)(Qe^{-i}) \ast (E_0 A)(\varepsilon) \) which can be rewritten by means of Remark 4.3 and (2.11) as

\[
-9(E_0 A_2) - (E_0 A)(Qe^{-i}) \ast (E_0 A)(\varepsilon)
\leq -9(E_0 A_2) + q_1 A_2 - (E_0 A)(Qe^{-i}) \ast (E_0 A)(\varepsilon) - A(Qe^{-i}) \ast (E_0 A) - A(Qe^{-i}) \ast A(\varepsilon).
\]

Lemmas 2.3, 2.7, 3.3 and 3.4 as well as \( A_2 = -(q_1 A^2)/9 \) and the uniform boundedness of \( A \) then imply that

\[
\| -9(E_0 A_2) - (E_0 A)(Qe^{-i}) \ast (E_0 A)(\varepsilon) \|_{C^1} \leq C \varepsilon
\]

uniformly with respect to \( t \in [0,T_*/\varepsilon^2] \).

Moreover, due to the choice of \( A_2 \) together with Lemmas 2.3 and 3.3 one immediately gets \( \|a_3\|_{C^1} \leq C \varepsilon^3 \) for all \( t \in [0,T_*/\varepsilon^2] \). Summarising, this shows (4.5).

Thus, it only remains to prove (4.6) and for this we proceed similarly as before. More precisely, we first note that Remark 4.3 allows to rewrite

\[
- \partial_T (E_0 A) + 4 \partial_X^2 (E_0 A) + (E_0 A) = - \partial_T (E_0^c A) + 4 \partial_X^2 (E_0^c A) + (E_0^c A) - \partial_T A + 4 \partial_X^2 A + A.
\]

Since \( A \) solves (2.9) we further get

\[
- \partial_T (E_0 A) + 4 \partial_X^2 (E_0 A) + (E_0 A) = - \partial_T (E_0^c A) + 4 \partial_X^2 (E_0^c A) + (E_0^c A)
+ \left( 2k_0 + k_2 - \frac{q_1 q_2}{9} - \frac{q_1^2}{9} - 2q_0 q_1 - 2q_1^2 \right) |A|^2 A.
\]

Therefore, it remains to estimate the \( C^1 \)-norm of

\[
- \partial_T (E_0^c A) + 4 \partial_X^2 (E_0^c A) + (E_0 A) + \left( 2k_0 + k_2 - \frac{q_1 q_2}{9} - \frac{q_1^2}{9} - 2q_0 q_1 - 2q_1^2 \right) |A|^2 A
+ (E_0 A) - (E_0 A)(Qe^{-2i}) \ast (E_0 A_2)(\varepsilon) - (E_0 A)(Qe^{-i}) \ast (E_0 A)(\varepsilon) - (E_0 A)(Qe^{-i}) \ast (E_0 A)(\varepsilon)
- 2(E_0 A)K \ast ((E_0 A)(E_0 A))(\varepsilon) - (E_0 A)(K e^{-2i}) \ast (E_0 A)^2(\varepsilon).
\]

However, since \( |A|^2 A = A^2 \bar{A} \) this can be done in the same way as for \( a_0 \) and \( a_2 \).
As a consequence of Lemma 4.2, we can now prove the following result which provides bounds on the restrictions of $\text{Res}(\phi)$ to critical and uncritical Fourier modes.

**Proposition 4.4.** For each solution $A \in C([0,T_*], C^1_b)$ of (2.9) and $\phi$ as in (2.10) there exists a constant such that it holds

$$\sup_{t \in [0,T_*/\varepsilon^2]} \|E_{\varepsilon}(\text{Res}(\phi))\|_{C^1} \leq C\varepsilon^3$$

and

$$\sup_{t \in [0,T_*/\varepsilon^2]} \|E_{\varepsilon}(\text{Res}(\phi))\|_{C^1} \leq C\varepsilon^4.$$

**Proof.** The proof follows easily from Lemma 4.2. Precisely, we note that $\text{Res}(\phi) = \sum_{\ell = -3}^{3} a_{\ell} e^{i\ell x}$ with $a_{\ell}$ as in (4.4) and $a_{-\ell} = \bar{a}_{\ell}$. Moreover, $E_{\varepsilon} = 1 - E_{\varepsilon}$ and thus, due to Lemma 3.3 we deduce that $E_{\varepsilon} : C^1 \rightarrow C^1$ is linear and bounded. Therefore, in order to verify the first claimed estimate, it suffices to show that

$$\sup_{t \in [0,T_*/\varepsilon^2]} \|\text{Res}(\phi)\|_{C^1} \leq C\varepsilon^3$$

which is however an immediate consequence of Lemma 4.2.

To prove the second claim of the lemma, we note that the definition of $E_{\varepsilon}$ together with Lemma 3.9 yields that $a_{\ell}$ is supported in $[-3/4, 3/4]$. Thus, we find that the Fourier transform of $a_{\ell} e^{i\ell x}$ is supported in $\mathcal{B}_{3/4}(\ell)$. Since $\chi_{c} \equiv 0$ on $\mathcal{B}_{3/4}(\ell)$ for $\ell \in \{0, \pm 2, \pm 3\}$ we get that $E_{\varepsilon}(\text{Res}(\phi)) = E_{\varepsilon}(a_{\ell} e^{i\ell x} + \bar{a}_{\ell} e^{-i\ell x} + O(\varepsilon))$. However, Lemma 3.3 implies that $E_{\varepsilon} : C^1 \rightarrow C^1$ is bounded and thus the second claim of the lemma follows immediately from Lemma 4.2. \qed

### 5 An equation for the approximation error

In this section, we will derive the equation which the approximation error $R$ has to satisfy and we will mainly use the same notation as in [39]. As already mentioned before, it will be necessary to treat the critical Fourier modes $e^{\pm i\ell x}$ separately from the uncritical ones and we therefore write

$$u = \varepsilon \phi_c + \varepsilon^2 \phi_s + \varepsilon^3 R_c + \varepsilon^3 R_s$$

where $\phi_c$ and $\phi_s$ have been defined in (2.12). Moreover, to shorten the notation we also use $R := \varepsilon^2 R_c + \varepsilon^3 R_s$ such that it holds $u = \phi + R$. If we plug this into (2.1) it follows

$$0 = \partial_t u - \mathcal{L}(u) - N(u)$$

$$= \partial_t R + \partial_t \phi - \mathcal{L}(R) - \mathcal{L}(\phi) - N_Q(R + \phi) - N_K(R + \phi)$$

$$= \partial_t R + \partial_t \phi - \mathcal{L}(R) - \mathcal{L}(\phi) - N_Q(R) - N_Q(\phi) + R(Q + \phi) + \phi(Q * R)$$

$$- N_K(R) - N_K(\phi) + 2R K * (R \phi) + R K * \phi^2 + 2\phi K * (R \phi) + \phi K * R^2.$$

If we now insert $R = \varepsilon^2 R_c + \varepsilon^3 R_s$ and recall that $\text{Res}(\phi) = -\partial_t \phi + \mathcal{L}(\phi) + N(\phi)$ this can be further rearranged as

$$\varepsilon^2 \partial_t R_c + \varepsilon^3 \partial_t R_s = \varepsilon^2 \mathcal{L}(R_c) + \varepsilon^3 \mathcal{L}(R_s) + \text{Res}(\phi) - \varepsilon^4 R_c Q * R_c - \varepsilon^5 R_s Q * (R_c + \varepsilon R_s)$$

$$- \varepsilon^6 (R_c + \varepsilon R_s) (K * (R_c + \varepsilon R_s)^2) - \varepsilon^3 R_s Q * \phi_c - \varepsilon^4 R_c Q * \phi_s$$

$$- \varepsilon^4 R_s Q * \phi_c - \varepsilon^5 R_s Q * \phi_s - \varepsilon^3 \phi_c Q * R_c - \varepsilon^4 \phi_s Q * R_c$$

$$- \varepsilon^4 \phi_c Q * R_s - \varepsilon^5 \phi_s Q * R_s - 2\varepsilon^5 (R_c + \varepsilon R_s) (K * ((R_c + \varepsilon R_s)(\phi_c + \varepsilon \phi_s)))$$

$$- \varepsilon^4 R_c (K * \phi_c^2) - \varepsilon^5 R_c (K * (2\phi_c \phi_s + \varepsilon \phi_s^2)) - \varepsilon^5 R_s (K * (\phi_c + \varepsilon \phi_s)^2) - 2\varepsilon^4 \phi_c (K * (R_c \phi_c))$$

$$- 2\varepsilon^5 \phi_c (K * (R_c \phi_s + R_s \phi_c + \varepsilon R_s \phi_s)) - 2\varepsilon^5 \phi_s (K * ((R_c + \varepsilon R_s)(\phi_c + \varepsilon \phi_s)))$$

$$- 2\varepsilon^5 (\phi_c + \varepsilon \phi_s)(K * (R_c + \varepsilon R_s)^2).$$

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If we divide by \( \varepsilon^2 \) and reorganise, we finally end up with

\[
\partial_t R_c + \varepsilon \partial_t R_s = \mathcal{L}(R_c) + \varepsilon \mathcal{L}(R_s) - \varepsilon L_2(R_c) - \varepsilon^2 N_2(R_c) - \varepsilon^2 L_1(R_c, R_s) + \varepsilon^3 N_1(R_c, R_s, \varepsilon) + \frac{1}{\varepsilon^2} \text{Res}(\phi),
\]

(5.1)

where we write

\[
L_2(R_c) = R_c Q * \phi_c + \phi_c Q * R_c
\]

\[
N_2(R_c) = R_c Q * R_c
\]

\[
N_1(R_c, R_s, \varepsilon) = -R_s Q * (R_c + \varepsilon R_s) - \varepsilon(R_c + \varepsilon R_s)(K * (R_c + \varepsilon R_s)^2) - R_s Q * \phi_s - \phi_s Q * R_s - 2(R_c + \varepsilon R_s)(K * (R_c + \varepsilon R_s)(\phi_c + \varepsilon \phi_s)) - R_c(K * (2 \varepsilon \phi_s + \varepsilon \phi_s^2)) - R_s(K * (\phi_c + \varepsilon \phi_s)^2) - 2 \phi_c(K * (R_c \phi_s + R_s \phi_c + \varepsilon R_s \phi_s)) - 2 \phi_s(K * ((R_c + \varepsilon R_s)(\phi_c + \varepsilon \phi_s))) - 2(\phi_c + \varepsilon \phi_s)(K * (R_c + \varepsilon R_s)^2).
\]

As in [39] we now exploit that Lemma 3.8 implies \( E_c L_2(R_c) = 0 \) and \( E_s N_2(R_c) = 0 \) to separate the equation for \( R \). Precisely, we apply the identity operator \( \text{Id} = E_c + E_s \) to (5.1) such that we obtain

\[
\begin{align*}
\partial_t R_c &= \mathcal{L}(R_c) - \varepsilon^2 L_c(R_c, R_s) + \varepsilon^3 N_c(R_c, R_s) + \varepsilon^2 \delta_c, \\
\partial_t R_s &= \mathcal{L}(R_s) - L_s(R_c) + \varepsilon N_s(R_c, R_s) + \delta_s,
\end{align*}
\]

(5.2)

with the abbreviations

\[
\begin{align*}
L_c(R) &= E_c(L_1(R_c, R_s)) & L_s(R_c) &= E_s(L_2(R_c)) \\
N_c(R) &= E_c(N_1(R_c, R_s)) & N_s(R_c, R_s) &= E_s(L_1(R_c, R_s) + N_2(R_c) + \varepsilon N_1(R_c, R_s)) \\
\delta_c &= \varepsilon^{-4} E_c(\text{Res}(\phi)) & \delta_s &= \varepsilon^{-3} E_s(\text{Res}(\phi)).
\end{align*}
\]

(5.3)

**Remark 5.1.** Note that if \( R_c \) and \( R_s \) solve (5.2) the sum \( R_c + \varepsilon R_s \) gives a solution to (5.1).

**Remark 5.2.** The existence of a unique solution to (5.2) locally in time can be shown by a standard fixed-point argument similarly as in [39]. Note that for this it is important that the non-linear terms are locally Lipschitz continuous which might be easily deduced from Lemma 2.3.

Moreover, we have the following estimates on the linear operators \( L_c \) and \( L_s \).

**Lemma 5.3.** There exists a constant \( C > 0 \) such that it holds

\[
\|L_c(R_c, R_s)\|_{C^1} \leq C(\|R_c\|_{C^1} + \|R_s\|_{C^1}) \quad \text{and} \quad \|L_s(R_c)\|_{C^1} \leq C\|R_c\|_{C^1}
\]

for the operators \( L_c \) and \( L_s \) as given in (5.3).

**Proof.** These estimates follow immediately from Lemma 2.3 together with the boundedness of the operators \( E_c \) and \( E_s \).
6 Proof of Theorem 2.5

Based on the preparations in Sections 4 and 5 we will now give the proof of our main result.

Proof of Theorem 2.5. We first introduce some notation, namely for fixed $T \geq 0$ and $n \in \mathbb{N}$ we define the Banach space

$$
B^n_T := C([0, T], C^n(\mathbb{R})) \quad \text{with norm } \|f\|_{B^n_T} = \sup_{t \in [0, T]} \|f(t)\|_{C^n}.
$$

Moreover, we note that one may easily deduce from Lemma 2.3 together with the boundedness of $E_c$ and $E_s$ that for each $D > 0$ there exists $M_D > 0$ such that it holds for all $t > 0$ that

$$
\|N_c(R_c, R_s)\|_{B^1_T} + \varepsilon \|N_s(R_c, R_s)\|_{B^1_T} \leq M_D \quad \text{if } \|R_c\|_{B^1_T} + \varepsilon \|R_s\|_{B^1_T} \leq D. \quad (6.1)
$$

Furthermore, we recall from Proposition 4.4 that

$$
\|\delta_s\|_{B^n_{T^2/4}} \leq C \quad \text{and } \|\delta_s\|_{B^n_{T^2/4}} \leq C. \quad (6.2)
$$

Finally, due to the assumptions on the initial data we have

$$
\|R_c(0)\|_{C^4} = \|R_c\|_{B^4_0} \leq C \quad \text{and } \|R_s(0)\|_{C^4} = \|R_s\|_{B^4_0} \leq C/\varepsilon. \quad (6.3)
$$

By means of the semi-group $e^{L_c t}$ and the relations $E^h_cE_c = E_c$ as well as $E^h_sE_s = E_s$ we can rewrite (5.2) as

$$
R_c(t) = R_c(0) + \varepsilon^2 \int_0^t e^{L_c(t-\tau)}E^h_c[-L_c(R_c, R_s) + \varepsilon N_c(R_c, R_s) + \delta_c] \, d\tau
$$
$$
R_s(t) = R_s(0) + \int_0^t e^{L_s(t-\tau)}E^h_s[-L_s(R_c) + \varepsilon N_s(R_c, R_s) + \delta_s] \, d\tau.
$$

From Lemmas 3.10 and 5.3 and (6.2) we thus obtain that

$$
\|R_s\|_{B^1_T} \leq \|R_s\|_{B^1_0} + C \int_0^t \max\{1, \tau^{-3/4}\}e^{-\sigma \tau} \, d\tau \left[\|R_c\|_{B^1_T} + M_D + C\right] \quad (6.4)
$$

as long as the condition in (6.1) holds. For $R_c$ we proceed similarly, while we additionally exploit (6.3) and (6.4) and the assumption $t \leq T^2/\varepsilon^2$ to find

$$
\|R_c\|_{B^1_T} \leq \|R_c\|_{B^1_0} + C \varepsilon^2 \int_0^t e^{C^2(t-\tau)}(\|R_c\|_{B^1_T} + \|R_s\|_{B^1_T} + \varepsilon M_D + C) \, d\tau
$$
$$
\leq C + C_T \varepsilon^2 \int_0^t \|R_c\|_{B^1_T} \, d\tau + C_T (\varepsilon M_D + C). \quad (6.5)
$$

Due to Gronwall’s inequality and the assumption $t \leq T^2/\varepsilon^2$ we obtain

$$
\|R_c\|_{B^1_T} \leq C_T (\varepsilon M_D + 1) e^{C_T \varepsilon^2 t} \leq C_T (\varepsilon M_D + 1). \quad (6.6)
$$

If we use this estimate together with (6.3) it follows from (6.4) that

$$
\|R_s\|_{B^1_T} \leq C/\varepsilon + C M_D + C_T (\varepsilon M_D + 1). \quad (6.7)
$$
If we now fix first \( D_\ast > 0 \) sufficiently large and then \( \varepsilon_\ast = \varepsilon_\ast(D_\ast) > 0 \) sufficiently small one immediately deduces from (6.6) and (6.7) that it holds \( \| R_c \|_{B^4_t} + \varepsilon \| R_s \|_{B^4_t} \leq D_\ast \) for all \( t \in [0, T_\ast/\varepsilon^2] \) provided \( \varepsilon \leq \varepsilon_\ast \). Thus, the error \( R = \varepsilon^2 R_c + \varepsilon^3 R_s \) remains in the ball of radius \( D_\ast \) (with respect to \( \| \cdot \|_{C^4_t} \)) for all \( t \in [0, T_\ast/\varepsilon^2] \).

Since \( R = u - \phi \) we thus find together with Lemma 2.6 that

\[
\|u - \psi\|_{C^4_t} \leq \|R\|_{C^4_t} + \|\phi - \psi\|_{C^4_t} \leq C\varepsilon^2.
\]

The existence and uniqueness of \( u \) now follows straightforward. Precisely, by a standard fixed-point argument one gets that there exists a unique solution \( u \) to (2.1) on a small time interval. Due to the approximation result that we have just shown, this solution cannot blow up—and can thus be extended uniquely—on the interval \([0, T_\ast/\varepsilon^2]\).

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