Weak Lax pairs for lattice equations

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Abstract
We consider various 2D lattice equations and their integrability, from the point of view of 3D consistency, Lax pairs and Bäcklund transformations. We show that these concepts, which are associated with integrability, are not strictly equivalent. In the course of our analysis, we introduce a number of black and white lattice models, as well as variants of the functional Yang–Baxter equation.

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1. Introduction

Recent progress in the description of integrable partial difference equations is to a great part due to the consistency approach [1–4], in particular in the form of three-dimensional consistency-around-a-cube (CAC). One of the highlights of this approach is the immediate existence of a Lax pair and Bäcklund transforms (BTs), which can be directly constructed from the equations on the vertical sides of the consistency cube (the 'side equations') [3, 5]. One can say, in effect, that the set of side equations yields the BT and the Lax pair. One situation where side equations have been used effectively is in constructing soliton solutions to the lattice equations [6–8].

Originally it was assumed that on all faces of the cube the equations were the same in form, depending on the relevant corner variables (one component at each corner) and spectral parameters. Recently, 3D consistent sets have appeared, with different equations on the faces [9–13]. In this more general context it is of interest to take a closer look at the Lax pairs, BTs and consistency, and investigate what they are good for. It should be noted that in the context of partial differential equations the existence of trivial Lax pairs is well known [14] and similar examples have also been noted for some discrete equations (see [15, chapter 6]).

We will first recall how consistency around the cube, existence of Lax pairs and Bäcklund transformations are intimately related for lattice maps on the square lattice given by multiaffine relations (section 2). These considerations apply to the elementary cells, and are local.
section 3, we describe specific examples having the CAC property, which we use in this paper. In particular, we find that in some cases the zero curvature condition (ZCC) yields two different equations that can be used to define rational evolution in the lattice. We then address the global problem of defining equations over the whole lattice, with the guideline given by the three-dimensional structure coming from the CAC construction (section 4), and check integrability by computing the algebraic entropy. In section 5, we push the use of $2 \times 2$-matrix Lax pairs to its limits, by constructing discrete systems over a larger sublattice of the original lattice. This brings to light interesting (integrable) structures related to a generalized form of the functional Yang–Baxter equations [16].

2. 3D consistency, a reminder

The starting point is a regular 2D square lattice, with vertices labelled by integers $n, m$. Functions $x_{n,m}$ are associated with the vertices, and they are subject to a constraint at all elementary cells, expressed by an equation of the form $Q(x_{n,m}, x_{n+1,m}, x_{n,m+1}, x_{n+1,m+1}) = 0$, assumed to be multiaffine in the four vertex variables. It should depend on all four vertex variables, and it should not factorize. It may also depend on some parameters. Sometimes the parameters can be associated with specific directions of the lattice, in which case they appear as 'spectral parameters'. To ease the notation, one usually denotes the running value $x_{n,m}$ by $x_{n,m} = x$, and for neighbouring values one only indicates the shifts: $x_{n,m+1} = x_2$, or in 3D setting, $x_1 = x_{n+1,m,k}$, $x_{113} = x_{n+2,m,k+1}$, etc.

For multidimensional consistency one needs to build a cube on top of a square, as given in figure 1, and give equations on all six faces of the cube, the bottom equation being the original one.

Supposing $x, x_1, x_2, x_3$ are given, one can compute $x_{12}$ using the bottom equation, $x_{13}$ using the left side equation and $x_{23}$ using the back side equation. We may then calculate the final value $x_{123}$ in three different ways, using the front-, right- and top-equations, respectively. The equations are said to be CAC if the three ways yield the same value for $x_{123}$. Note that it is also possible to check CAC with other initial values that allow full evolution, for example, $x_3, x, x_1, x_{12}$.

The construction of a Lax pair from the (multiaffine) equations associated with the various faces of the cube of figure 1 is straightforward, along the lines of [3, 5]

Introducing the homogeneous coordinates $f, g$ for $x_3$ and its shifts by

\[
    x_3 = f/g, \quad x_{23} = f_2/g_2, \quad x_{13} = f_1/g_1, \quad x_{123} = f_{12}/g_{12},
\]

Figure 1. The consistency cube.
we denote by \( \psi \) the pair
\[
\psi = \left( \begin{array}{c} f \\ g \end{array} \right)
\]
which is defined up to a global factor. The equations on the vertical sides of the cube can then be written in the form
\[
\psi_1 \simeq L \psi, \quad (\psi_2)_1 \simeq \overline{L} \psi_2, \quad \psi_2 \simeq M \psi, \quad (\psi_1)_2 \simeq \overline{M}(\psi_1),
\]
which provides the Lax pair.

The Lax matrices describe parallel transport of \( \psi \) along the bonds of the lattice. The ZCC means that the parallel transport along any closed path on the lattice is trivial. It is necessary and sufficient to ensure that taking \( \psi \) from position \((0, 0)\) to \((1, 1)\) via the two routes \((0, 0) \to (0, 1) \to (1, 1)\) and \((0, 0) \to (1, 0) \to (1, 1)\) gives the same result:
\[
(\psi_1)_2 \simeq (\psi_2)_1, \quad \text{i.e.} \quad \overline{L}(x_{12}, x_2)M(x_2, x) \simeq \overline{M}(x_{12}, x_1)L(x_1, x).
\]

(1)

In the case we consider (multiaffine relations on square cells), one obtains Lax matrices of size \( 2 \times 2 \). As a consequence, relation (1) implies three scalar equations\(^3\), written in terms of the variables \( x, x_1, x_2, x_{12} \). In standard cases, they are equivalent to the bottom equation. Indeed, in integrable cases the three equations in (1) have the bottom equation as their greatest common divisor (GCD).

In order to derive the bottom equation from the side equations one can also proceed more directly by treating the side equations as providing a BT. One can use three of the side equations to solve for \( x_{13}, x_{23} \) and \( x_{123} \), respectively, and then the fourth equation is a polynomial in \( x_3 \), and the GCD of its coefficients yields the bottom equation. Similarly, by eliminating \( x_1, x_2, x_{12} \) from three side equations, the fourth one will be a polynomial in \( x \), and the GCD of its coefficients yields the top equation.

It may also happen that the ZCC is satisfied automatically, if the side equations are simple enough, or sometimes the common factor may factorize, yielding multiple choices. We will examine specific examples of this phenomenon below.

\section*{3. Examples}

We describe here the models which we will use in the rest of the paper.

\subsection*{3.1. H1}

The lattice potential KdV (lpKdV), which describes the permutability property of continuous KdV, is a paradigm of integrable lattice equations (it is called H1 in the ABS list [4]). For this system everything works well. (We will return to this model in a new context in section 5.2.)

The model is given by
\[
H1 := (x_1 - x_2)(x - x_{12}) - p + q = 0.
\]

(2)

The Lax matrices in this case are
\[
L(x_1, x) = \begin{pmatrix} x & p - r - xx_1 \\ 1 & -x_1 \end{pmatrix}, \quad M(x_2, x) = \begin{pmatrix} x & q - r - xx_2 \\ 1 & -x_2 \end{pmatrix},
\]

(3)

and one easily finds that
\[
M(x_{12}, x_1) L(x_1, x) - L(x_{12}, x_2) M(x_2, x) = H1 \times \begin{pmatrix} 1 & -(x_1 + x_2) \\ 0 & -1 \end{pmatrix}.
\]

We will return later to the question of gluing the cubes to fill the space.

\(^3\) One could do the computations in some particular representative of the equivalence class, e.g., by requiring the matrices to be uni-modular. However, this is not necessary, and may in fact be cumbersome, if it introduces square roots.
3.2. Flipped \( H_1 \).

This model was proposed in [13] (see section 3.1, case \((\epsilon, 0, 0, \epsilon)\)). In that paper the model was given in a cube with flipped coordinates \( x_2 \leftrightarrow x_{23}, x_1 \leftrightarrow x_{13} \). In the coordinates of figure 1 the side equations of this model are given by

\[
\begin{align*}
\text{left} : & \quad (x - x_1)(x_{13} - x_3) - (p - r)(1 + \epsilon xx_1) = 0, \quad (4a) \\
\text{right} : & \quad (x_2 - x_{12})(x_{123} - x_3) - (p - r)(1 + \epsilon xx_{12}) = 0, \quad (4b) \\
\text{back} : & \quad (x - x_2)(x_{23} - x_3) - (q - r)(1 + \epsilon xx_2) = 0, \quad (4c) \\
\text{front} : & \quad (x_1 - x_{12})(x_{123} - x_{13}) - (q - r)(1 + \epsilon xx_{12}) = 0. \quad (4d)
\end{align*}
\]

**BT derivation of bottom and top equations.** Now computing the values of \( x_{13}, x_{23}, x_{123} \) from left-, back- and right-equations, respectively, gives for the front equation an expression that does not contain \( x_3 \) at all. This expression factorizes into two factors and thus we could have two different bottom equations:

\[
\begin{align*}
\text{bottom1} & \quad \epsilon xx_{123}(1/x - 1/x_1 - 1/x_2 + 1/x_{12}) - x + x_1 + x_2 - x_{12} = 0, \quad (5a) \\
\text{bottom2} & \quad p(x - x_2)(x_1 - x_{12}) + q(x - x_1)(x_{12} - x_2) + r(x - x_{12})(x_2 - x_1) = 0. \quad (5b)
\end{align*}
\]

Similarly, performing the usual CAC computations with the given sides and different top and bottom equations reveals that the set of equations is consistent in two cases: with the pair \((\text{bottom1, top1})\), or with \((\text{bottom2, top2})\), which was given in [9]. In other words, given the side equations (4) there are two consistent ways of completing the cube. Continuing further, we can also interpret the cube to provide a BT between the left and right equations. Indeed, if we do the above BT construction on the corners of the left side equation the result is identically zero with \((\text{bottom1, top1})\), while \((\text{bottom2, top2})\) produces the right equation.

**The Lax matrices.** The standard procedure gives

\[
\begin{align*}
L(x_1, x) = \begin{pmatrix} 1 & \lambda(p, x, x_1) \\ 0 & 1 \end{pmatrix}, \quad M(x_2, x) = \begin{pmatrix} 1 & \lambda(q, x, x_2) \\ 0 & 1 \end{pmatrix},
\lambda(a, x, y) := \frac{(a - r)(1 + \epsilon xy)}{x - y}.
\end{align*}
\]

Since the matrices are upper triangular the ZCC implies

\[
\Sigma := \lambda(p, x, x_1) + \lambda(q, x, x_2) - \lambda(q, x_1, x_1) - \lambda(p, x, x_1) = 0. \quad (8)
\]

Remarkably enough, the above sum factorizes as

\[
\Sigma = \text{bottom1} \cdot \text{bottom2} / [ (x - x_1)(x - x_2)(x_1 - x_{12})(x_2 - x_{12}) ].
\]

Note that we can write (8) also in the form

\[
\Sigma = (T - 1)\lambda(p, x, x_1) - (S - 1)\lambda(q, x, x_2) = 0,
\]

where \( T \) is a shift in \( m \) and \( S \) a shift in \( n \). This of course is in the form of a conservation law.
4. Filling the space with consistent cubes

So far we have only considered a single cube and its CAC/Lax/BT. But as the name indicates, lattice equations should be defined over the whole lattice. This brings further complications, for example with one cube we could freely do different Möbius transformations in each corner of a cube, but when the cube is part of a lattice such seemingly innocuous actions will affect neighbouring cubes as well and can destroy the lattice structure.

The rule is simple: cover the two-dimensional lattice with consistent cubes, with the condition that adjacent vertical faces coincide exactly, that is to say their four corner values satisfy the same equation. This is expected to produce integrable lattice equations.

We will follow this guideline for various models mentioned before, and systematically check integrability of the lattice equations so obtained, by calculating their algebraic entropy [17, 18]: the vanishing of the entropy is a yes/no test which gives a clear cut separation between integrable and non-integrable cases.

We may briefly recall how to calculate the entropy. The local equation determines an evolution, starting from initial conditions given for example on a diagonal staircase (lattice points of coordinates \((m, n)\) with \(m + n = 0\) or \(1\)). The solution is then calculated on diagonals moving away from the diagonal of initial conditions, explicitly in terms of these initial conditions. The algebraic entropy is defined as the rate of growth of the degrees on these diagonals. Exponential growth is generic, and polynomial growth is characteristic of integrability, while linear growth is associated with linearizable equations [17–19].

The exact shape of the diagonal line on which the initial values are given is not important. If one modifies this shape locally, the sequence of degrees will change, but not its asymptotic rate of growth. This should be kept in mind for some of the models studied below. For example one could very well change the initial diagonal (steps of height 1 and width 1) to a diagonal with bigger steps, for example with height 2 and width 2, but this is irrelevant for the calculation of the entropy.

4.1. Flipped H1\(\epsilon\): black and white lattices

In the flipped H1\(\epsilon\) case (section 3.2), the side equations are the same, allowing simple gluing together of the cubes, but the side equations are somewhat weak and allow two different compatible pairs of bottom/top equations. Thus we can construct a lattice of consistent cubes and to each elementary cell of the lattice we can assign either equation bottom1 (top1), which we call white, or equation bottom2 (top2) which we call black. This can be done in an arbitrary way if one just insists on having a compatible 3D structure of cubes over the 2D lattice. It is then natural to ask which of the configurations obtained in this way are integrable.

Let us consider periodic distributions. The lattice is divided into rectangular groups of cells of width \(h\) and height \(v\). Within such a rectangle, a fixed assignment is made, and the pattern is repeated periodically in both directions. (A pattern with \(v = 1\) and \(h = 1\) gives a uni-coloured assignment.)

Consider for example \((h, v) = (2, 2)\). There are \(2^4\) possible patterns of that size, but only three inequivalent ones which cannot be reduced to configurations having smaller periods (see figure 2). The naming convention is to list the colours starting from the lower left corner onwards, denoting bottom1/top1 with 0, alias white, bottom2/top2 with 1, alias black. The equivalence of patterns comes from the fact that we have to look at the lattice globally. It is easy to see, for example, that in the case \((h, v) = (2, 2)\) we have the equivalences \([0100] \simeq [0010] \simeq [1000] \simeq [0001], [1011] \simeq [1101] \simeq [0111] \simeq [1110] and
1960 J Hietarinta and C Viallet

Figure 2. The three inequivalent (2, 2) patterns.

\[ [1010] \simeq [0101] \] (checkerboard lattice). Moreover, \([0000]\) and \([1111]\) have periods \((1, 1)\), \([0101]\) and \([1010]\), \([0011]\), \([1100]\) have periods \((2, 1)\) and \((1, 2)\).

**Claim.** Some of the distributions are integrable, and some are not. Although the pattern is 3D consistent, the Lax pair is weak and cannot precisely fix the bottom and top equations.

\[ 1 \times 1 \text{ patterns (unicolour distributions).} \] Both unicolour distributions \((h, v) = (1, 1)\) have vanishing entropy. The purely white one is linear. The purely black one is non-trivially integrable, showing quadratic growth of the sequence of degrees

\[ \{d_n\} = 1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, 172 \ldots \] (9)

\[ 1 \times 2 \text{ patterns.} \] Both \(1 \times 2\) patterns \((h, v) = (1, 2)\) or \((h, v) = (2, 1)\), that is to say alternating black and white stripes, are integrable, with quadratic growth of the degrees.

\[ 2 \times 2 \text{ patterns.} \] For \((h, v) = (2, 2)\) we have different results for the different patterns in figure 2.

- Both \([1010]\) and \([0100]\) are integrable, with quadratic growth of the degrees.
- The calculation of the degrees for \([1011]\) yields the sequence

\[ \{d_n\} = 1, 2, 4, 8, 18, 41, 93, 215, 493, 1132, 2600, 5970, 13710, 31487, 72313, 166077, 381417, 875974, 2011788, 4620332 \ldots \] (10)

This sequence if fitted by the rational generating function

\[ g(s) = \sum_n d_n s^n = \frac{1 - s^2 - 2 s^3 + s^5 - s^6 - s^8 + s^9}{(1 - s)(s + 1)(s^4 - 2 s^3 - 2 s + 1)(s^4 + 1)}, \] (11)

and gives a non vanishing entropy \(\epsilon = \log(\sigma)\) with \(\sigma\) the largest root of \(s^4 - 2 s^3 - 2 s + 1\), approximately \(\epsilon = \log(2.296 63)\).

**Caveat.** When computing sequences of degrees, one should in principle consider iterations of the whole pattern, but that tends to make the calculations heavier. For the sequence (10), this would mean considering only the subsequence formed by odd terms, leading to a growth given by the maximal root \(\tau\) of \(t^8 - 4 t^7 - 6 t^3 - 4 t + 1\). Of course \(\tau = \sigma^2\).

\[ 2 \times 3 \text{ patterns.} \] We have examined all the period \((2, 3)\) patterns. The various nonequivalent patterns are depicted in figure 3.
The computations show that what matters is not just the proportion of black and white cells, but the actual conformation of the pattern. For example the period (2, 3) patterns [010110] and [001011] have equal numbers of black and white cells. The first one is integrable (quadratic growth of the degrees) while the latter is not, as may be seen from the sequence:

\[ \{d_n\} = 1, 1, 2, 3, 5, 9, 19, 41, 84, 169, 329, 631, 1199, 2287, 4412, 8627, 17059, 33941, 67573, 134071, 264576, 519343, 1015531, 1982461, 3871597, 7574863, 14855790 \ldots \]  

(12)

This sequence has exponential growth, but is not long enough to determine an exact value of the entropy. The approximate value is \( \log(1.96) \).

Out of the 14 period (2, 3) nonequivalent patterns, we have one linear case (all white [000000]), eight integrable cases ([010000], [010100], [110000], [011000], [010101], [010110], [111000], [111111]) and four non-integrable ones ([110100], [001011], [110101], [111001]). The following pictures show the aspect of two integrable cases in figure 4 and two non-integrable ones in figure 5.

**Remark.** The entropy calculations for these patterns can be made equally well with the relations [white = bottom1, black = bottom2] or with [white = top1, black = top2]. Both would give the same results.

From the above results, one may already conclude that random distributions are expected to be non-integrable.
5. Lax pair for a $2 \times 2$ sublattice

We will next push the Lax concept and ZCC to a $2 \times 2$ sublattice described in figure 6. (Such sublattices have been discussed previously, e.g., in [11].) To determine the evolution we need five initial values, marked with black discs and expect to get values for the vertices at the open circles. Since the Lax matrices belong to $PGL(2, C)$ the ZCCs can provide at most three equations, and thus if everything works well the evolution is determined. It is also clear that this will not work for bigger sublattices as we would then need to provide more than three values.

The ZCC for the $2 \times 2$ sublattice is given by

$$M(x_{1122}, x_{112}) M(x_{112}, x_{11}) L(x_{11}, x_1) L(x_1, x)$$

$$\simeq L(x_{1122}, x_{112}) L(x_{122}, x_{22}) M(x_{22}, x_2) M(x_2, x). \quad (13)$$
5.1. Flipped H1ε

In the flipped H1ε case the Lax matrices were given in equation (7). Now using this on a 2 × 2 sublattice we obtain just one condition, namely
\[ λ(p, x_{122}, x_{112}) + λ(p, x_{22}, x_{122}) + λ(q, x_2, x_{22}) + λ(q, x, x_2) = λ(q, x_{112}, x_{122}) + λ(q, x_{11}, x_{112}) + λ(p, x_1, x_{11}) + λ(p, x, x_1). \]

From this one can in principle solve \( x_{112} \) in terms of the other variables. However, this equation does not determine the values for \( x_{122} \) or \( x_{112} \) and therefore these Lax matrices fail to give the evolution.

5.2. H1

For this basic model the Lax matrices were given in equation (3). Condition (13) leads to equations that have two rational solutions: The regular one

\begin{align}
\text{regular solution:} & \quad x_{112} = x_1 + \frac{(p - q)(x_1 - x_2)}{(x_1 - x_2)(x - x_{11}) - (p - q)}, \tag{14a} \\
x_{122} = x_2 + \frac{(q - p)(x_2 - x_1)}{(x_2 - x_1)(x - x_{22}) - (q - p)}, \tag{14b} \\
x_{112} = x + \frac{(p - q)(x_{11} + x_{22} - 2x) + 2(x - x_{11})(x - x_{22})(x_1 - x_2)}{(p - q)^2 - (x - x_{11})(x - x_{22})(x_1 - x_2)^2}, \tag{14c}
\end{align}

and an exotic solution

\begin{align}
\text{exotic solution:} & \quad x_{112} = x_{11} + x_{22} - x, \quad (15a) \\
x_{122} = x_{11}x_{22} - x_1 + x_2, \quad (15b) \\
x_{112} = x_1 - \frac{(x_1 - x_2)\left((p - r)(x - x_{22}) + (q - r)(x - x_{11})\right)}{(p - r)(x - x_{22}) - (q - r)(x - x_{11}) - (x - x_{11})(x - x_{22})(x_1 - x_2)}, \quad (15c) 
\end{align}

The regular solution could also be obtained using the evolution on the original lattice, first solving for \( x_{12} \). As a consequence \( x_{122} \) depends only on \( x, x_1, x_2, x_{22} \) and \( x_{112} \) only on \( x, x_1, x_2, x_{11} \). The exotic solution is different, as \( x_{122} \) and \( x_{112} \) both depend on \( all \) initial values. Furthermore, it depends on \( p - r \) and \( q - r \), and not solely on \( p - q \), as is the case for the regular solution.

We may view the variables \( x, x_{11}, x_{22}, x_{112} \) as associated with the vertices and \( x_1, x_2, x_{112}, x_{122} \) as associated with the bonds of the 2 × 2 sublattice.

In the algebraic entropy analysis the vertex variables are linear and for the bond variables we find the sequence of degrees
\[ \{d_n\} = 1, 4, 13, 28, 49, 76, 109, 148, 193, 244, 301, 364, 433, \ldots \tag{16} \]

This sequence can be fitted with the generating function
\[ \zeta(s) = \sum_n d_n s^n = \frac{1 + 4s^2 + s}{(1 - s)^3}. \tag{17} \]

The sequence has quadratic growth, signalling integrability.

What is the nature of the exotic solution? Since the vertex variables have independent linear evolution we can solve the equation with \( x_{2n,2m} = F(n) + G(m) \). When this is substituted into the bond equations they give a non-autonomous generalization of a Yang–Baxter map: using the coarse grained indexing \( u_{n,m} = x_{2n,2m}, X_{n,m} = x_{2n+1,2m}, Y_{n,m} = x_{2n,2m+1}, \) i.e. \( x_1 = X, x_2 = Y, x_{112} = Y_1, x_{122} = X_2, \) we have
\[ Y_1 - X = P(X, Y), \quad X_2 - Y = P(X, Y). \tag{18a} \]
The solution \( w = \text{constant} \) is not allowed and if either \( F \) or \( G \) is constant, \( P \) collapses to \( P = \pm (x - y) \). In the generic case, denoting
\[
f(n) := \frac{p - r}{F(n) - F(n + 1)}, \quad g(m) := \frac{q - r}{G(m) - G(m + 1)},
\]
we obtain
\[
P = \frac{(X - Y)[f(n) + g(m)]}{X - Y - f(n) + g(m)}.
\]
After the further translation \( X \mapsto X + f(n) + T(n, m), \ Y \mapsto Y + g(n) + T(n, m), \) where \( T \) is a solution of \( T(n, m + 1) - T(n, m) = 2g(m), \ T(n + 1, m) - T(n, m) = 2f(n) \) we finally obtain
\[
P = \frac{f(n)^2 - g(m)^2}{X - Y}, \quad (18b)
\]
which is a non-autonomous version of the Adler map [20] (aka \( F_V \) in the classification [21].

5.3. \( H^3 \)

The phenomenon described in the previous section is not generic. Indeed, applying the same coarse-graining to an arbitrary integrable quad-equation will lead to a system having only one rational solution (the regular one coming from the original lattice).

We have, however, found more examples where an exotic rational solution exists. Here is one, provided by the lattice modified KdV (lmKdV) (aka \( H^3_{\delta = 0} \)). In that case the defining relations of the exotic solution are
\[
x_{1122} x = x_{11} x_{22}, \quad (19a)
x_{122} x_1 = x_{112} x_2, \quad (19b)
x_{122} x_2 = x_{112} x_1 = \frac{(q^2 x_{22} - r^2 x)(x - x_{11})p x_1 - (p^2 x_{11} - r^2 x)(x - x_{22})q x_2}{(r^2 x_{22} - q^2 x)(x_{11} - x) p x_2 - (r^2 x_{11} - p^2 x)(x_{22} - x) q x_1}. \quad (19c)
\]

In the algebraic entropy analysis the sequence of degrees for the vertex variables has linear growth as expected, while the sequence for the bonds is the same as for \( H^1 \) (see above).

Now the equation on the vertex variables (19a) can be solved with
\[
x_{2n,2m} = F(n)G(m),
\]
and if we introduce
\[
f(n) := \frac{r^2 F(n + 1) - p^2 F(n)}{F(n + 1) - F(n)}, \quad g(m) := \frac{r^2 G(m + 1) - q^2 G(m)}{G(m + 1) - G(m)},
\]
we obtain the bond equations in the form
\[
\frac{X_2}{Y} = \frac{Y_1}{X} = \frac{(q^2 + r^2 - g(m))p X - (p^2 + r^2 - f(n))q Y}{f(n)q X - g(m)p Y}, \quad (20)
\]
using the previously introduced notation. With the further scaling
\[
X(n, m) \mapsto p T(n, m) X(n, m)/f(n), \quad Y(n, m) \mapsto q T(n, m) Y(n, m)/g(m),
\]
where $T$ solves

\[ T(n, m + 1)/T(n, m) = (q^2 + r^2 - g(m))/g(m), \]
\[ T(n + 1, m)/T(n, m) = (p^2 + r^2 - f(n))/f(n), \]

equation (20) reduces to

\[ X_2 = \frac{Y}{\alpha(n)} P, \quad Y_1 = \frac{X}{\beta(m)} P, \quad P = \frac{\alpha(n) X - \beta(m) Y}{X - Y}, \] (21)

where

\[ \alpha(n) = \lambda p^2/[f(n)(f(n) - p^2 - r^2)], \quad \beta(n) = \lambda q^2/[g(n)(g(n) - q^2 - r^2)]. \]

This is nothing but a non-autonomous version of $F_{111}$ in the classification of [21]. The diagram presented for H1 works also for H3($\delta = 0$).

We have found that this phenomenon occurs also for the lattice modified KdV and for the lattice Schwarzian KdV. We have examined in some detail the properties of the models defined in this way [16].

6. Discussion

We have discussed the strength of the Lax pairs (or the zero curvature condition). We have found several cases where the ZCC does not uniquely determine the evolution but allows two possibilities. This happens, e.g., in the flipped H1 $\epsilon$ model. If one then builds an infinite lattice by arbitrarily choosing for each cell one of the two allowed relations, the result is sometimes integrable and sometimes shows nonzero entropy.

If the ZCC is pushed to a $2 \times 2$ sublattice we obtain more examples where it is ambiguous and yields both the regular solution as well as an exotic one. The latter cannot be generated by some equation in the sublattice, because in the exotic solution the variables $x_{122}$ and $x_{112}$ depend on both $x_{11}$ and $x_{22}$, which is not possible using the equations on the elementary squares.

The equations we have obtained in this way can be interpreted as having vertex and edge variables: the variables with even number of indices live at the vertices while the ones with an odd number of indices live on the edges. The edge variables evolve as in a non-autonomous functional Yang–Baxter equation.

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