POSITIVE ENERGY THEOREM AND SUPERSYMMETRY IN EXACTLY SOLVABLE QUANTUM-CORRECTED 2D DILATON-GRAVITY

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ABSTRACT

Extending the work of Park and Strominger, we prove a positive energy theorem for the exactly solvable quantum-corrected 2D dilaton-gravity theories. The positive energy functional we construct is shown to be unique (within a reasonably broad class of such functionals). For field configurations asymptotic to the LDV we show that this energy functional (if defined on a space-like surface) yields the usual (classical) definition of the ADM mass plus a certain “quantum”-correction. If defined on a null surface the energy functional yields the Bondi-mass. The latter is evaluated carefully and applied to the RST shock-wave scenario where it is shown to behave as physically expected. Motivated by the existence of a positivity theorem we construct manifestly supersymmetric (semiclassical) extensions of these quantum-corrected dilaton-gravity theories.

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1. Introduction

Dilaton-gravity in two dimensions provides a simplified model to study quantum gravity and in particular the analogues of (four-dimensional) black hole formation and evaporation. The classical action for dilaton-gravity coupled to $N$ conformal massless matter fields [1]

$$S_{cl} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ e^{-2\phi} (R + 4(\nabla \phi)^2 + 4\lambda^2) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right]$$

(1.1)

admits classical (non-radiating) black hole solutions

$$ds^2 = -\frac{dx^+dx^-}{m - \lambda^2 x^+x^-} , \quad e^{2\phi} = \frac{m}{\lambda} - \lambda^2 x^+x^- .$$

(1.2)

Here $m$ is the black hole mass, and the $m = 0$ solution is called the linear dilaton vacuum (LDV). Note that $g = e^\phi$ is the coupling constant.

The goal then is to quantize the theory described by this action $S_{cl}$. The $N$ matter fields give rise to the well-known conformal anomaly†

$$S_{anom} = -\kappa \frac{1}{8\pi} \int d^2x \sqrt{-g} R \frac{1}{\sqrt{\nabla^2}} R$$

(1.3)

with $\kappa = \frac{N}{12}$. Note that this is $O(e^{2\phi}) \equiv O(g^2)$ with respect to the gravitational part of $S_{cl}$ and may be thought of as the one-loop contribution of the matter fields. We will refer to $S_{cl} + S_{anom}$ as $S_{CGHS}$. Evidently, we also have to quantize the dilaton-gravity sector. In order to do so it is convenient to go to conformal gauge. It was argued in refs 2, 3 that quantum consistency requires the resulting theory to be a conformal field theory. It turned out that this (almost uniquely) determined the quantum-corrected action. Moreover, when written in terms of appropriately redefined field variables, the quantum-corrected action is very simple and the corresponding equations of motion can be solved exactly. The non-trivial physics comes from the (transcendental) transformation back to the “physical” dilaton and metric fields. We will review these theories briefly in section 2.

† A possible term $\mu^2 \int \sqrt{-g}$ is supposed to be fine-tuned to vanish.
These exactly-solvable quantum-corrected theories seemed to have no lower bound to the total energy (mass).\footnote{The same objection of course applies to the classical theory (1.1), but since the solutions (1.2) are static and non-radiating there will be no transitions from positive to negative mass. In the CGHS model including $S_{\text{anom}}$ there will be, a priori, such transitions, but since the theory is not exactly solvable we cannot study them to the same extent.} The static solutions of negative mass have naked singularities, just as the four-dimensional Schwarzschild geometry. This, by itself, is not really worrisome, as long as we can avoid that an initially singularity-free solution dynamically develops a naked singularity. The cosmic censorship conjecture (in 4D general relativity) states that this can indeed be avoided. In the present context of the 2D quantum-improved dilaton-gravity theories, Russo, Susskind and Thorlacius (RST) \cite{4,5} have shown that one can impose boundary conditions so as to avoid naked singularities. (This also applies to ref. 2.) This results in matching an evaporating black hole onto a vacuum configuration precisely when the singularity is about to develop.

What is the mass of the (dynamical) black hole just before it is matched to the vacuum? In their initial work \cite{4} RST claimed that it was “slightly” (i.e. of order $\kappa \lambda$) negative, and that this negative amount of energy is sent off by a “thunderpop”. In fact, there seem to be (at least) two different definitions (differing by $O(\kappa \lambda)$-terms) for the mass of the black hole \cite{6} one of them leading to a slightly positive and one to a slightly negative mass just before the black hole disappears.

More generally we would like to have a theorem that (at least one reasonable definition of) the mass or total energy is positive as long as no singularity is encountered.\footnote{By singularity we do not simply mean a curvature singularity, but a region of “space-time” where the dilaton gets complex. Of course, in general the curvature diverges at the boundary of such a region. Strictly speaking such regions should not be considered as part of the physical space-time. In the four-dimensional analogue these regions correspond to negative radius.} Such a theorem was proven by Park and Stominger \cite{7} for the classical theory (1.1) and for the CGHS-theory. They showed how to extend $S_{\text{cl}}$ to a supersymmetric theory and derived a spinorial expression $M$ (that coincides under certain assumptions with the conventionally defined total energy) from the supercharge. This $M$ then is shown to be positive using the equations of motion of the bosonic theory given by $S_{\text{cl}}$. Thus although supersymmetry is probably the underlying reason that makes things work out, the positive energy proof itself does not require supersymmetry, only the bosonic equations of motion. Park and Strominger then extend this proof to the CGHS theory by representing the anomaly action (1.3) in a local form using the “$Z$”-field.
Here we will extend their work to the exactly solvable quantum-corrected theories and prove the same result (for $\kappa > 0$): *A suitably defined mass functional $M$ given by an integral over a space-like or null surface $\Sigma$ is always non-negative as long as the dilaton-field is real on $\Sigma$. Now, the scalar curvature diverges at the boundary of a region of complex dilaton field and is complex inside. In all physically interesting situations $\Sigma$ includes at least a portion where the dilaton is real (e.g. an asymptotically flat region). Thus we can conclude that $M$ is non-negative as long as there is no curvature singularity on $\Sigma$. We also prove a uniqueness theorem: under the assumptions specified below our positive mass/energy functional is unique.*

We will show how our energy functional when evaluated with a space-like surface $\Sigma$ leads to the usual ADM-mass plus a certain “quantum”-correction. When defined over a null-surface $\Sigma$ of constant $\sigma^-$ we obtain an expression for the Bondi-mass $M_B(\sigma^-)$. The latter is shown to behave as physically expected. We evaluate $M_B$ in detail for the case of an evaporating black hole formed from an infalling shock-wave. In particular, we show that at $\sigma^- = \sigma^-_s$ when the singularity is about to develop and the configuration is matched to the LDV, its Bondi-mass is “slightly” positive and non-vanishing.

The total energy functional $M$ is again given by some spinorial expression but our proof will only rely on the (bosonic) equations of motion. The spinors are just some book-keeping device determined in terms of the metric and the dilaton. One might however ask whether the quantum-corrected theories under consideration have a supersymmetric extension or not. This is particularly interesting in view of the negative statement of Nojiri and Oda [8]. Following Park and Strominger [7] it is easy to explicitly write down these extensions although the supersymmetric extensions of the cosmological-constant term are rather non-trivial. For reasons first discussed in ref. 9 and repeated below, it is not the same to construct a (classically) supersymmetric extension of a given exact conformal theory (what we did) and to construct an exact superconformal theory (what ref. 8 attempted to do). This explains the apparent discrepancy with ref. 8.

The outline of this paper is as follows: in the next section, we briefly review the exactly-solvable quantum-corrected dilaton-gravity theories and show how they can be rewritten using the “$Z$”-field. The expert reader might choose to skip part or all of this section. In section 3, we give the mass functional $M$ and show that it is non-negative provided the dilaton is real on $\Sigma$. This is done by relating it to an expression involving (a part of) the “matter” stress-tensor...
which is manifestly non-negative. We further show that, under certain reasonable assumptions, this mass functional is uniquely determined. In particular there is no freedom to change terms that are subleading in the coupling constant \( e^\phi \). In section 4, we show in some detail how this mass functional \( M \) is related to the usual Bondi and ADM masses, discuss some of its properties and evaluate it in particular for the shock-wave scenario. In section 5, we write down the supersymmetric extensions of the quantum-corrected theories of refs. 2 and 4.

2. The exactly solvable quantum-corrected dilaton-gravity theories

Adding the matter anomaly piece (1.3) to the classical action (1.1) was a first step towards quantizing two-dimensional dilaton-gravity [1]. The resulting theory has two drawbacks, however. On the one hand, the action \( S_{\text{CGHS}} \) is conformally invariant (after shifting \( \kappa \)) only classically, but not at the quantum level. On the other hand, the equations of motion are not solvable in closed form, which makes it difficult to study the dynamical evolution. It turned out that solving the first problem also cured the second: making the action conformally invariant, at the same time leads to exactly solvable equations of motion. Let us outline how this works.

First of all, not only the matter fields contribute to the conformal anomaly, and as a result \( \kappa \) is shifted to [2]

\[
\kappa = \frac{N - 24}{12}. \tag{2.1}
\]

More generally, we expect other \( \mathcal{O}(e^{2\phi} \rho^0) \) corrections to \( S_{\text{cl}} \) (which itself is \( \mathcal{O}(e^{2\phi} \rho^{-1}) \)). It was shown in ref. 2 that in conformal gauge and after a local field redefinition the kinetic part of \( S_{\text{cl}} + S_{\text{anom}} \) takes on a free-field form. Then it was easy to identify the correction to the cosmological constant term (\( \sim \lambda^2 \)) necessary to turn it into a marginal operator. The resulting theory was shown [2] to be a conformal field theory (at the quantum level). It is given by

\[
S^{\rho,\phi} = \frac{1}{\pi} \int d^2\sigma \left[ \kappa \partial_+ \Omega \partial_- \Omega - \kappa \partial_+ \chi \partial_- \chi + \lambda^2 e^{2(\chi - \Omega)} \right] \tag{2.2}
\]

where \( \chi \) and \( \Omega \) are the new fields related to the dilaton \( \phi \) and the conformal factor of the

\footnote{Our notation is as usual: \( \sigma^\pm = \sigma^0 \pm \sigma^1 = \tau \pm \sigma, \partial_\pm = \frac{1}{2} (\partial_0 \pm \partial_1) \) and conformal gauge is defined by \( g_{++} = g_{--} = 0, g_{+-} = -\frac{1}{2} e^{2\rho} \), hence \( \nabla^2 = -4 e^{-2\rho} \partial_+ \partial_- \) on any scalar.}
metric (Liouville field) $\rho$ by†

$$\Omega = \omega \sqrt{\omega^2 - 1} - \log \left( \omega + \sqrt{\omega^2 - 1} \right) + \frac{1}{2} \left( 1 - \log \frac{2}{\Omega} \right), \quad \omega = e^{-\phi}/\sqrt{\kappa},$$

$$\chi = \rho + \omega^2.$$ (2.3)

This is valid for $\kappa > 0$. See ref. 2 for the appropriate analytic continuation to $\kappa < 0$.

Alternatively, as in ref. 4, one may modify the kinetic part of $S_{\text{cl}} + S_{\text{anom}}$ by adding an $O((e^{2\phi})^0)$ correction

$$\delta S_{\text{RST}} = -\frac{\kappa}{4\pi} \int d^2 x \sqrt{-g} \phi R.$$ (2.4)

This modifies the stress-tensor in such a way that the (original) cosmological constant term $\frac{1}{2\pi} \int d^2 x \sqrt{-g} e^{-2\phi} 4 \lambda^2$ becomes marginal. The resulting action (in conformal gauge) can again be written in terms of new fields $\chi$ and $\Omega$ so that the resulting action is identical to (2.2), but the relation with the original $\phi$ and $\rho$-fields is different (we have rescaled $\Omega$ and $\chi$ by $\sqrt{\kappa}$ with respect to ref. 4)

$$\Omega = \frac{e^{-2\phi}}{\kappa} + \frac{\phi}{2},$$

$$\chi = \frac{e^{-2\phi}}{\kappa} - \frac{\phi}{2} + \rho.$$ (2.5)

Also, in terms of $\Omega$ and $\chi$ the stress-tensor looks identical in both cases [2,4]:

$$T_{\pm \pm}^{\rho, \phi} = \kappa \left[ (\partial_{\pm} \Omega)^2 - (\partial_{\pm} \chi)^2 + \partial_{\pm}^2 \chi \right].$$ (2.6)

All this should be no surprise: there are not that many conformal field theories with a canonical kinetic term one can write down.

Although the action (2.2) is almost trivial - as shown first in ref. 2 the equations of motion can be solved exactly - the field transformations (2.3) and (2.5) certainly are not. For $\kappa > 0$ they become singular for some critical value of the dilaton field $\phi = \phi_{\text{cr}}$ where $\partial \Omega / \partial \phi = 0$ and $\Omega$ is minimum. (We have $e^{-2\phi_{\text{cr}}} = \kappa$ for (2.3) and $e^{-2\phi_{\text{cr}}} = \kappa/4$ for (2.5).) In general, this corresponds to a real singularity of the geometry, i.e. the scalar curvature diverges.‡ The

† We have rescaled $\Omega, \chi$ by a factor of 2 and shifted $\Omega$ by a constant with respect to ref. 2.
‡ The LDV is the only exception: the curvature vanishes everywhere and there is no singularity although $\phi = \phi_{\text{cr}}$ somewhere.
dilaton $\phi$ is complex beyond the line of singularity which might be interpreted as the boundary of physical space-time.

In the next section we will give a positive energy proof that does not rely on conformal gauge but is generally covariant. Now, although the anomaly action (1.3) is local in conformal gauge, it is non-local when written in the covariant form (1.3). We will need a reformulation of these exactly solvable quantum-corrected theories that is local and covariant at the same time. We now proceed to give such a reformulation, using the example of the RST-action $S_{\text{cl}} + S_{\text{anom}} + \delta S_{\text{RST}}$. Define [10]

$$S_{Z} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ -\frac{1}{2} (\nabla Z)^2 + QRZ \right] . \quad (2.7)$$

The $Z$-stress-tensor obtained from this action is [7]

$$T_{\mu\nu}^Z = \hat{T}_{\mu\nu}^Z + Q \left( \nabla_{\mu} \nabla_{\nu} Z - g_{\mu\nu} \nabla^2 Z \right) ,$$

$$\hat{T}_{\mu\nu}^Z = \frac{1}{2} \nabla_{\mu} Z \nabla_{\nu} Z - \frac{1}{4} g_{\mu\nu} (\nabla Z)^2 . \quad (2.8)$$

If we write

$$Z = \tilde{Z} - Q \frac{1}{\nabla^2} R \quad (2.9)$$

we have

$$S_{Z} = -\frac{1}{4\pi} \int d^2 x \sqrt{-g} (\nabla \tilde{Z})^2 - \frac{Q^2}{4\pi} \int d^2 x \sqrt{-g} R \frac{1}{\nabla^2} R , \quad (2.10)$$

which is a free-field action for $\tilde{Z}$ plus $S_{\text{anom}}$ of eq. (1.3) provided we take

$$2Q^2 = \kappa . \quad (2.11)$$

Note that $Z$ will be real only if $\kappa > 0$. Thus we want to consider

$$S = S_{\text{cl}} + \delta S_{\text{RST}} + S_{Z} . \quad (2.12)$$

Naively, one might think that integration over $Z$, i.e. over $\tilde{Z}$ will just reproduce the anomaly action $S_{\text{anom}}$ and dispose of $\tilde{Z}$ leaving us with the complete dilaton-gravity part of the RST-action (i.e. excluding the matter part). This is of course not the case since the $\tilde{Z}$-field is
coupled to the other fields via the $g_{\mu\nu}$-equations of motion, i.e. the $T_{\pm\pm} = 0$ constraints after adopting conformal gauge. This is much the same as for the matter fields $f_i$. Indeed, Park and Strominger [7] suggest to identify $T^Z$ with $T^M + T^\rho_{\text{anom}}$ which amounts to identifying $T^Z_{\pm\pm}$ with $T^M_{\pm\pm} + t_{\pm}$. ($t_{\pm}$ is a projective connection contained in $T^\rho\text{anom}$ describing the boundary values of the “anomalous” stress-energy flux.) With this identification it is easy to see that the dilaton and $g_{\mu\nu}$-equations of motion of the action $S$, eq. (2.12), are rigorously identical to those of the original RST-action (including the matter part).\footnote{Note that from $T^Z_{\mu\nu}$ we see that the $\tilde{Z}$-field contributes a (classical) conformal anomaly of $c_{\tilde{Z}} = 24Q^2 = 12\kappa = N - 24$. Hence the $\tilde{Z}$-field mimics not only the matter fields but also the ghosts plus certain quantum parts of $\phi$ and $\rho$. However, this interpretation of the fields themselves should not be taken too literally.} Thus the RST action is equivalent to

$$S = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ \left( e^{-2\phi} - \frac{\kappa}{2}\phi \right) R + e^{-2\phi} \left( 4(\nabla\phi)^2 + 4\lambda^2 \right) - \frac{1}{2}(\nabla Z)^2 + Q R Z \right]. \quad (2.13)$$

Going to conformal gauge and splitting

$$Z = \tilde{Z} + 2Q \rho, \quad (2.14)$$

we have

$$S = \frac{1}{\pi} \int d^2 \sigma \left[ \kappa \partial_+ \Omega \partial_- \Omega - \kappa \partial_+ \chi \partial_- \chi + \lambda^2 e^{2(\chi - \Omega)} + \frac{1}{2} \partial_+ \tilde{Z} \partial_- \tilde{Z} \right], \quad (2.15)$$

where $\Omega$, $\chi$ are given by (2.5), while the constraints are

$$T_{\pm\pm} = 0 \quad , \quad T_{\pm\pm} = T^{\rho,\phi}_{\pm\pm} + T_{\pm\pm}^Z, \quad (2.16)$$

where $T^{\rho,\phi}_{\pm\pm}$ is still given by (2.6) and

$$T_{\pm\pm}^Z = \frac{1}{2}(\partial_{\pm} \tilde{Z})^2 + Q \partial_{\pm}^2 \tilde{Z}. \quad (2.17)$$

Let us repeat again that, upon identifying $T_{\pm\pm}^Z$ with $T^M_{\pm\pm} + t_{\pm}$, eqs. (2.15)-(2.17) are exactly equivalent with the RST model (or that of ref. 2).
For later reference, let us review one solution of the equations of motion in more detail. Since they are exactly solvable it was possible \([2,4]\) to study the formation of a black hole and its subsequent evaporation exactly, automatically including the correct backreaction of the Hawking radiation on the geometry. It turned out (as first discussed in ref. 4 but also applying to ref. 2) that after the black hole formed the singularity is space-like and hidden behind an apparent horizon. Then, as the black hole evaporates, the apparent horizon recedes, until it hits the singularity which then turns time-like and naked. At this point one has to impose boundary conditions and it was shown \([4,5, \text{Note Added to 2}]\) that one can match the solution to a linear dilaton vacuum (static in some new coordinates) or its analogue. If one does so, beyond that point there is no more Hawking radiation and the black hole has disappeared.

Since we will need it below, we will write out the field configuration for this scenario. To be specific, we will use the RST variant \([4]\) since the transformation from \(\phi\) and \(\rho\) to \(\Omega\) and \(\chi\) is simplest here. The asymptotically flat static solutions to the equations of motion are given by

\[
\Omega(\phi) = e^{\frac{2\lambda\sigma}{\kappa}} + 2P\lambda\sigma + \frac{m}{\lambda\kappa}.
\] (2.18)

The LDV, \(\phi = -\lambda\sigma\), corresponds to \(P = -\frac{1}{4}\), \(m = 0\). We use coordinates \(\sigma^{\pm} = \tau \pm \sigma\) such that \(\chi = \Omega + \lambda\sigma\) or

\[
\rho = \phi + \lambda\sigma.
\] (2.19)

In these coordinates one has for the LDV solution \(\rho = 0\), i.e. the coordinates are Minkowskian. In general these coordinates are asymptotically Minkowskian coordinates as \(\sigma \to \infty\). In addition to the \(\phi, \rho\)-equations of motion we must also satisfy the \(T_{\pm\pm} = 0\) constraints (which are the \(g_{\pm\pm}\)-equations of motion before going to conformal gauge). They are

\[
T_{\pm\pm} = 0, \quad T_{\pm\pm} = T_{\pm\pm}^{\rho,\phi} + T_{\pm\pm}^{M} + t_{\pm}\tag{2.20}
\]

where \(T_{\pm\pm}^{M}\) is the matter stress-tensor and \(t_{\pm}(\sigma^{\pm})\) is required to vanish in asymptotically Minkowskian coordinates: \(t_{\pm}(\sigma^{\pm}) = 0\). This and the constraints imply \(P = -\frac{1}{4}\) for \(T_{\pm\pm}^{M} = 0\). \(P \neq -\frac{1}{4}\) is appropriate only if the solution is in equilibrium with a bath of radiation.

Now let’s suppose that we have a matter shock-wave\(^*\) travelling along the line \(\sigma^{+} = \sigma_{0}^{+}\)

\(^*\) This is the analogue of a collapsing shell of matter in four dimensions.
with stress-tensor
\[ T^M_{++}(\sigma^+) = m\delta(\sigma^+ - \sigma_0^+) \quad , \quad T^M_{--}(\sigma^-) = 0 . \] (2.21)

We use \( \lambda x^\pm = \pm e^{\pm \lambda \sigma^\pm} \) and \( m = a e^{\lambda \sigma_0^+} \). Equation (2.21) corresponds to \( T^M_{++}(x^+) = a\delta(x^+ - x_0^+) \). Note that by our preceding discussion of the \( \tilde{Z} \)-field this \( T^M_{++}(\sigma^+) \) corresponds to
\[ \partial_+ \tilde{Z} = -2Q \frac{\theta(\sigma^+ - \sigma_0^+)}{\sigma^+ - \sigma_0^+ + \frac{\kappa}{m}} . \] (2.22)

We further take the LDV solution for \( \sigma^+ < \sigma_0^+ \), i.e. in the causal past of the shock-wave trajectory. Matching across the \( \sigma^+ = \sigma_0^+ \) line (and requiring LDV asymptotics at right past null infinity) leads to
\[ \frac{e^{-2\phi}}{\kappa} + \frac{\phi}{2} \equiv \Omega(\phi) = \frac{1}{\kappa} e^{\lambda \sigma^+ - \lambda \sigma^-} - \frac{\lambda}{4} \sigma^+ + \frac{\lambda}{4} \sigma^- - \frac{a}{\lambda \kappa} \left( e^{\lambda \sigma^+ - \lambda \sigma_0^+} \right) \theta(\sigma^+ - \sigma_0^+) \] (2.23)

The curvature is singular on the line where \( \Omega = \Omega_{\text{cr}} \) (except when \( \phi \) is the LDV), where \( \Omega_{\text{cr}} = \frac{1}{4} \left( 1 - \log \frac{k}{4} \right) \) is the value of \( \Omega(\phi) \) at its minimum. As discussed in ref. 4, just above the infall-line \( \sigma^+ = \sigma_0^+ \) the line of singularity is space-like and hidden to the asymptotically flat region \( \sigma \to \infty \) by an apparent horizon. As time goes on (\( \tau \) increases) the apparent horizon and the line of singularity approach each other until they intersect at \( \sigma^\pm = \sigma_s^\pm \) where
\[ e^{\lambda \sigma_s^+} = \frac{\kappa \lambda}{4a} \left( e^{\frac{4a}{\kappa \lambda}} - 1 \right) , \quad e^{\lambda \sigma_s^-} = \frac{\lambda}{a} \left( 1 - e^{-\frac{4a}{\kappa \lambda}} \right) . \] (2.24)

The singularity turns time-like and naked. As shown by RST, this can be avoided by matching the solution (2.23) to a shifted LDV for \( \sigma^\pm > \sigma_s^\pm \). Indeed, on the half-line \( \sigma^- = \sigma_s^- \), \( \sigma^+ > \sigma_s^+ \) the solution \( \phi \) given by (2.23) takes on LDV values:
\[ \phi = -\frac{\lambda}{2} \sigma^+ + \frac{\lambda}{2} \tilde{\sigma}_s^- , \] (2.25)

where
\[ \tilde{\sigma}^- = \sigma^- - \frac{1}{\lambda} \log \left( 1 - \frac{a}{\lambda} e^{\lambda \sigma^-} \right) . \] (2.26)
One chooses to take $\phi = \frac{1}{2}\sigma^+ - \frac{1}{2}\bar{\sigma}^-$ in all of the causal future of $(\sigma^+_s, \sigma^-_s)$.

3. The positive energy theorem

In this section, we will define a functional $M$ given by an integral over a space-like or null surface $\Sigma$ of a suitable expression involving the fields. We will then prove, for $\kappa > 0$ (which we assume throughout this paper), using the equations of motion, that this functional $M$ is non-negative if $\Sigma$ is contained in the physical space-time, i.e. if $\phi$ and $\rho$ are real everywhere on $\Sigma$. The expression for $M$ and the proof are suggested by Park and Strominger’s analysis \cite{7} of the simpler CGHS-case. In the next section, we will actually evaluate this functional $M$ and show that it gives a satisfactory Bondi-mass (for $\Sigma$ a null line) and that (for space-like $\Sigma$) it reproduces the usual expression for the ADM-mass plus a certain “quantum”-correction.

We will write down the mass functional $M$ in covariant form and use the covariant equations of motion to prove positivity. This is why we had to bother about rewriting the non-local covariant anomaly term in a local form by introducing the $Z$-field. All we will need are the $g_{\mu\nu}$-equations of motion of the action (2.13). (We discuss the RST variant here, since the algebra is simpler.) They are

$$T^{g,\phi}_{\mu\nu} + T^Z_{\mu\nu} = 0, \quad \text{(3.1)}$$

where $T^Z_{\mu\nu}$ is given by (2.8) and $T^{g,\phi}_{\mu\nu}$ is the covariant form of $T^{\rho,\phi}_{\pm\pm} - T^{\rho,\text{anom}}_{\pm\pm}$, namely

$$T^{g,\phi}_{\mu\nu} = -2 \left( e^{-2\phi} + \frac{\kappa}{4} \right) (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi) - 2e^{-2\phi} g_{\mu\nu} ((\nabla \phi)^2 - \lambda^2). \quad \text{(3.2)}$$

Given the structure of the equations of motion one can easily guess how the mass functional of ref. 7 should be modified in the present case. Let

$$M = \int_{\Sigma} d\sigma^\mu \nabla_\mu \left[ 2 \left( e^{-2\phi} + \frac{\kappa}{4} \right) \tilde{\epsilon} \gamma_5 (\nabla \phi - \lambda) \epsilon - Q \tilde{\epsilon} \gamma_5 \nabla Z \epsilon \right], \quad \text{(3.3)}$$

where $2Q^2 = \kappa$, $\epsilon$ is a commuting real two-dimensional spinor, and $\nabla = \gamma^\mu \nabla_\mu = e^{\mu a} \Gamma_a \nabla_\mu$, $e^{\mu a}$ being the zwei-bein and $\Gamma_a$ Minkowski-space Dirac-matrices obeying $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$. A

\textsuperscript{1} The line where $\Omega = \Omega_{cr}$ continues through this shifted LDV (just as it was also present in the LDV region $\sigma^+_0 < \sigma^+_0$), but as in any LDV, this line does not correspond to singular curvature, but rather to $R = 0$. One might however choose to consider as physical space-time only the region to the right of this line.
convenient choice that we adopt here is $\Gamma_0 = i\sigma_y, \Gamma_1 = \sigma_x$. Let $\Gamma_5 = \Gamma_0\Gamma_1 = \sigma_z$ while (following ref. 7) $\gamma_5 = \gamma^0\gamma^1 = -\Gamma_5$. As usual, $\bar{\epsilon} = \epsilon^+\Gamma_0$. We will also use the antisymmetric tensor normalized as $\epsilon_0^1 = \epsilon_1^0 = -1$.

The functional $M$ is given by a line integral of a derivative along this line and thus reduces to the difference of the values of the expression in the square brackets at “both ends of the world”. Thus $M$ is given by the asymptotic values of the fields and of the spinor $\epsilon$. In the next section, we will discuss under which conditions this reproduces the more standard definition of mass in terms of asymptotic field variations with respect to some reference (vacuum) configuration.

We will prove that the above defined functional $M$ equals

$$M = \int_\Sigma d\sigma^\mu \left( 1 - \frac{\kappa}{4}e^{2\phi} \right) \epsilon^\rho \hat{T}_{\rho\nu} \bar{\epsilon} \gamma^\nu \epsilon$$

provided the spinor $\epsilon$ satisfies the (ordinary) differential equation

$$d\sigma^\mu \left[ \left( 1 + \frac{\kappa}{4}e^{2\phi} \right) \nabla_\mu \epsilon - \frac{1}{2} \gamma_{\mu}(\nabla \phi - \lambda) \epsilon - \frac{Q}{4}e^{2\phi} \left( 1 + \frac{\kappa}{4}e^{2\phi} \right)^{-1} \gamma_{\mu}\nabla Z \epsilon \right] = 0 \ .$$

This determines $\epsilon$ only up to two functions of integration. They are not relevant to the positivity proof, but have to be specified to obtain a physical interpretation of $M$ as the mass. This will be done in the next section. The form (3.4) of $M$ is manifestly non-negative for $\kappa > 0$ if $\phi$ is real everywhere on $\Sigma$. Indeed, it is easy to see that for any real non-zero $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ (not necessarily a solution of (3.5)), $v^a = \bar{\epsilon} \Gamma^a \epsilon$ is time-like or null: $v^a v_a = -4(\epsilon_1 \epsilon_2)^2 \leq 0$, and future-directed: $v^0 = \epsilon_1^2 + \epsilon_2^2 > 0$. The same then is obviously true for $v^\mu$. Now $\hat{T}^Z_{\mu\nu}$ (cf. (2.8)) obeys the dominant energy condition, i.e. for time-like or null, future-directed $v^\mu$ the vector $-\hat{T}^Z_{\mu\nu} v^\nu$ is again time-like or null, future-directed. Note that this is true only if $Z$ is real, i.e. for $\kappa > 0$! Indeed, for real $\nabla_a Z = (f, g)$ we have $\hat{T}^Z_{00} = \frac{1}{4}(f^2 + g^2) = \hat{T}^Z_{11} = 0, \hat{T}^Z_{01} = \frac{1}{2}fg$ and obviously then $(\hat{T}^Z)^a_{\phantom{a}b} v^b$ is a time-like or null, future-directed vector. Since $\epsilon_1^0 = -1$ it follows that $M$ as given by (3.4) is non-negative provided $\Sigma$ is space-like or null and $\phi$ real on $\Sigma$.

* As discussed below, $\epsilon$ is a solution of a differential equation and, in general, its asymptotic value depends on the values of the fields on all of $\Sigma$. This differs from 4D general relativity.
Let us now proceed to prove the equality of (3.3) and (3.4) using (3.1) and (3.5). We start with expression (3.3) and evaluate \( \nabla_\mu [ \ldots ] \). Note that since we work with real spinors, 
\[ (\nabla_\mu \bar{\epsilon}) \gamma_5 \nabla_\phi \epsilon = \bar{\epsilon} \gamma_5 \nabla_\phi \nabla_\mu \epsilon, \] etc. One uses eq. (3.5) to get rid of all derivatives of \( \epsilon \). Employing further the identities 
\[ \bar{\epsilon} \gamma_5 \gamma^\mu \epsilon = \epsilon^\mu \epsilon \] and 
\[ \epsilon^\rho \alpha_{\mu \rho} = \epsilon^\rho (g_{\nu \rho} a^2 - a_{\nu} a_{\rho}) \] one arrives at

\[
M = \int_\Sigma d\sigma^\mu \bar{\epsilon} \gamma^\nu \epsilon \epsilon^\rho \left[ 2 \left( e^{-2\phi} + \frac{\kappa}{4} \right) (\nabla_\nu \nabla_\rho \phi - g_{\nu \rho} \nabla^2 \phi) 
+ 2e^{-2\phi} g_{\nu \rho} ((\nabla_\phi)^2 - \lambda^2) 
- Q (\nabla_\nu \nabla_\rho Z - g_{\nu \rho} \nabla^2 Z) 
- \frac{\kappa e^{2\phi}}{(1 + \frac{\kappa}{4} e^{2\phi})^2} \left( \frac{1}{2} \nabla_\nu Z \nabla_\rho Z - \frac{1}{4} g_{\nu \rho}(\nabla Z)^2 \right) \right].
\]

(3.6)

By (3.2) and (2.8) this equals

\[
M = \int_\Sigma d\sigma^\mu \bar{\epsilon} \gamma^\nu \epsilon \epsilon^\rho \left[ - T_{\nu \rho}^\phi - T_{\nu \rho}^Z + \hat{T}_{\nu \rho}^Z - \frac{\kappa e^{2\phi}}{(1 + \frac{\kappa}{4} e^{2\phi})^2} \hat{T}_{\nu \rho}^Z \right].
\]

(3.7)

Finally, by the equation of motion (3.1) this reduces to (3.4).

One might wonder whether the functional (3.3) is the only one for which one can prove positivity or whether there are many others. We will show that (3.3) is indeed the only one of this form. More precisely, suppose we start with a more general functional

\[
\tilde{M} = \int_\Sigma d\sigma^\mu \nabla_\mu \left[ \bar{\epsilon} \gamma_5 (f_1(\phi) \nabla_\phi + f_2(\phi) \lambda + f_3(\phi) \nabla Z) \epsilon \right]
\]

(3.8)

with \( \epsilon \) subject to some more general first-order differential equation

\[
d\sigma^\mu \left[ \nabla_\mu \epsilon - \gamma_\mu (g_1(\phi) \nabla_\phi + g_2(\phi) \lambda + g_3(\phi) \nabla Z) \epsilon \right] = 0.
\]

(3.9)

Note that the square brackets in (3.8) and (3.9) are the most general (spinorial) expressions one can write down that are covariant, local and have the correct dimension. To prove positivity we must be able to reexpress \( \tilde{M} \) as an integral containing only \( \hat{T}_{\nu \rho}^Z \) which is the only (“matter”) piece obeying the dominant-energy condition. Using the equations of motion (3.1) we can also
write $\hat{T}^Z_{\nu\rho} = T^g_{\nu\rho} - (T^Z_{\nu\rho} - \hat{T}^Z_{\nu\rho})$. Thus we require that, using now only (3.9) and spinor identities, one can express $\tilde{M}$ as

$$\tilde{M} = \int d\sigma^\mu \bar{\epsilon} \gamma^\nu \epsilon \epsilon_\mu^\rho \left[ F_1(\phi)\hat{T}^Z_{\nu\rho} + F_2(\phi) \left( -T^g_{\nu\rho} - (T^Z_{\nu\rho} - \hat{T}^Z_{\nu\rho}) \right) \right]. \quad (3.10)$$

Under these assumptions it is straightforward algebra to show that the functions $f_i(\phi), g_i(\phi)$ and $F_i(\phi)$ are uniquely determined to be as in (3.3), (3.5) and (3.7), (3.4).

Let us sketch the proof. What we will precisely show is that equating (3.8) and (3.10) using only (3.9) fixes the functions $f_i, g_i$ and $F_i$ to be

$$f_1 = \pm f_2 = 2c(e^{-2\phi} + \frac{\kappa}{4}), \quad f_3 = -cQ$$

$$g_1 = \pm g_2 = \frac{1}{2}(1 + \frac{\kappa}{4}e^{2\phi})^{-1}, \quad g_3 = \frac{Q}{4}e^{2\phi}(1 + \frac{\kappa}{4}e^{2\phi})^{-2}$$

$$F_1 = cke^{2\phi}(1 + \frac{\kappa}{4}e^{2\phi})^{-2}, \quad F_2 = c \quad (3.11)$$

where $c$ is a constant. On the one hand, we start with (3.8) and evaluate $\nabla_\mu$ of the square bracket, and eliminate all $\nabla_\mu \epsilon$ and $\nabla_\mu \bar{\epsilon}$ using (3.9). On the other hand, we substitute the explicit expressions (3.2) and (2.8) for the energy momentum tensors into (3.10). All we have to do then is to compare the coefficients of independent terms, $\bar{\epsilon}\gamma_5\nabla_\phi \gamma_\mu \epsilon$, $\bar{\epsilon}\gamma_5\nabla_\mu \phi \nabla Z \epsilon$, etc. Some care has to be exercised since independent looking terms may be related by spinor identities. We end up with the following system of equations:

$$2f_1g_2 + 2f_2g_1 + f'_2 = 0, \quad f_3g_2 + f_2g_3 = 0$$

$$f_1g_3 + f_3g_1 = 0, \quad f'_3 = 0$$

$$QF_2 + f_3 = 0, \quad F_1 + 8f_3g_3 = 0$$

$$F_2 - e^{2\phi}f_2g_2 = 0, \quad 2(e^{-2\phi} + \frac{\kappa}{4})F_2 - f_1 = 0$$

$$2e^{-2\phi}F_2 + 2f_1g_1 + f'_1 = 0, \quad 4f_1g_1 + f'_1 = 0. \quad (3.12)$$

These are ten equations for only eight functions. Thus it appears to be non-trivial that one can solve this system, i.e. that one can prove a positive energy theorem at all. However, we actually can solve (3.12) and the only solutions are those given in (3.11). The constant $c$ only determines the overall normalization of the energy and is irrelevant to the positivity proof. We
must choose \( c > 0 \) (otherwise replace \( M \) by \(-M\)) and hence it can always be absorbed into the normalization of the spinor \( \epsilon \) (which has to be fixed anyhow). We can thus choose

\[
c = 1 .
\]  

(3.13)

We are left with the sign ambiguity of \( f_2 \) and \( g_2 \) which reflects the symmetry \( \lambda \rightarrow -\lambda \). It is thus irrelevant, too. We conclude that the functions \( f_i, g_i \) and \( F_i \) are necessarily as in (3.3), (3.5) and (3.7), (3.4).

Although the mass functional (3.8) is uniquely determined, its actual value depends on the boundary or initial conditions imposed on the spinor \( \epsilon \) upon solving its differential equation (3.9). They will be fixed in the next section by imposing physical requirements.

4. Physical interpretation and applications

Now that we have a (to a certain extent) unique functional \( M \) that is non-negative we would like to see whether it defines a reasonable mass and compute it for various physically interesting scenarios. In particular, we will evaluate \( M \) as defined in (3.3) for the case where the field configuration is asymptotic to the LDV at both ends of \( \Sigma \). If \( \Sigma \) is a space-like line one should obtain the ADM-mass while a null-line \( \Sigma \) should lead to the Bondi-mass. We will discuss the latter in considerable detail and apply it to the shock-wave scenario where we show how our functional \( M \) produces the physically expected behaviour of the Bondi-mass.

4.1. ADM-mass

We will first compute \( M \) for a space-like line \( \Sigma \) of constant \( \tau \). Denote the expression in square brackets in (3.3) by \( \mathcal{M} \), i.e.

\[
\mathcal{M}(\tau, \sigma) = 2 \left( e^{-2\phi} + \frac{\kappa}{4} \right) \bar{\epsilon} \gamma_5 (\nabla \phi - \lambda) \epsilon - Q \bar{\epsilon} \gamma_5 \nabla \epsilon
\]

(4.1)

so that

\[
M(\tau) = \mathcal{M}(\tau, \sigma = \infty) - \mathcal{M}(\tau, \sigma = -\infty) .
\]

(4.2)

Choose conformal coordinates and assume LDV asymptotics, i.e. let asymptotically as \( \sigma \rightarrow \cdots \)
\[ \pm \infty \]
\[ \phi \sim -\lambda \sigma + \delta \phi \]
\[ \rho \sim \delta \rho \]

where \( \delta \phi \) and \( \delta \rho \) vanish as \( \sigma \to \pm \infty \). Let furthermore \( \tilde{Z} \to 0 \) as \( \sigma \to \pm \infty \) so that \( Z \sim 2Q\rho \).

The key point in computing the total energy \( M \) is to specify the asymptotics of the spinor \( \epsilon \) (and hence also its normalization). Since \( \epsilon \) is a solution of the differential equation (3.5) it is determined by two functions of integration which we may take to be determined by the asymptotics of \( \epsilon \) at one end of \( \Sigma \) (e.g. \( \sigma \to +\infty \)). Obviously then, to fix the two functions of integration we need to specify the leading and the subleading term in the asymptotic expansion of \( \epsilon \) as \( \sigma \to \infty \).

The LDV asymptotics as \( \sigma \to +\infty \) implies
\[ \rho \sim a_1(\tau)e^{-\lambda \sigma} + a_2(\tau)e^{-2\lambda \sigma} + \ldots \,, \quad \phi = -\lambda \sigma + \rho \]

where we use a suitable set of coordinates so that \( \phi = -\lambda \sigma + \rho \). It is related to the “Kruskal” coordinates \( x^\pm \) where \( \phi = \rho \) by the usual transformation \( \lambda x^\pm = e^{\pm \lambda \sigma} \). The equations of motion and the constraints together with the asymptotics (4.4) imply the following relations:
\[ \dot{a}_2 = 2a_1 \dot{a}_1 \,, \quad \ddot{a}_1 = \lambda^2 a_1 \,.
\]

Asymptotically the differential equation for \( \epsilon \) reads for a spacelike surface \( \Sigma \) of constant \( \tau \):
\[ \partial_\sigma \epsilon = -\frac{\lambda}{2}(1 + a_1 e^{-\lambda \sigma})(1 + \Gamma_1)\epsilon + O(e^{-2\lambda \sigma}) \,.
\]

If we write the solution as
\[ \epsilon = \epsilon_{(0)} + \epsilon_{(1)} e^{-\lambda \sigma} + O(e^{-2\lambda \sigma}) \]

then the differential equation implies
\[ (1 + \Gamma_1)\epsilon_{(0)} = (1 - \Gamma_1)\epsilon_{(1)} = 0 \]

i.e.
\[ \epsilon_{(0)} = \frac{c_0}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \epsilon_{(1)} = \frac{c_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

where \( c_0 \) and \( c_1 \) may depend on \( \tau \) and are the two functions of integration. Note that (4.9)
ensures that \( M(\tau, \sigma = +\infty) \) and hence \( M \) does not diverge for configurations asymptotic to the LDV. Given \( c_0 \) and \( c_1 \), the differential equation completely determines \( \epsilon \), and in particular its asymptotics as \( \sigma \to -\infty \). The latter however not only depends on the asymptotics of the fields but on their values on all of \( \Sigma \).

For the LDV, the exact solution is (\( \sigma_0 \) is some fixed reference point)

\[
\epsilon_{\text{LDV}}(\sigma) = \frac{1 - \Gamma_1}{2} \epsilon(\sigma_0) + \left( \frac{e^{2\lambda \sigma_0} + \frac{\kappa}{4}}{e^{2\lambda \sigma} + \frac{\kappa}{4}} \right)^{1/2} \frac{1 + \Gamma_1}{2} \epsilon(\sigma_0) \tag{4.10}
\]

from which one may read off \( \epsilon(0) = \frac{1 - \Gamma_1}{2} \epsilon(\sigma_0) \) and \( \epsilon(1) = \left( e^{2\lambda \sigma_0} + \frac{\kappa}{4} \right)^{1/2} \frac{1 + \Gamma_1}{2} \epsilon(\sigma_0) \). Using the exact result (4.10) it is straightforward to obtain

\[
M_{\text{LDV}}(\tau, \sigma) = -2\lambda \left( e^{2\lambda \sigma_0} + \frac{\kappa}{4} \right) \bar{\epsilon}(\sigma_0) \gamma_5(1 + \Gamma_1) \epsilon(\sigma_0) \tag{4.11}
\]

which is independent of \( \sigma \), and hence by (4.2)

\[
M_{\text{LDV}} = 0 \tag{4.12}
\]

independent of the choice of \( \epsilon(\sigma_0) \), i.e. of the functions of integration \( c_0 \) and \( c_1 \). It is worthwhile noting that unless \( \epsilon(1) = 0 \) the total energy for the LDV receives contributions from both ends of \( \Sigma \) (which cancel each other). One sees that one has to take carefully into account \( M(\tau, \sigma = -\infty) \) as well as the subleading term (\( \sim \mathcal{O}(e^{-\lambda \sigma}) \)) in \( \epsilon \) when evaluating \( M(\tau, \sigma = +\infty) \).

For the general LDV-asymptotic configuration (4.4) it is straightforward to obtain

\[
M(\tau, \sigma = +\infty) = 2\lambda \left( \frac{a_1^2}{2} - a_2 \right) c_0^2 + 4\dot{a}_1 c_0 c_1 - 4\lambda c_1^2 \tag{4.13}
\]

For \( \sigma \to -\infty \), only the leading asymptotics contribute, and since these are LDV asymptotics we can read off the result from (4.11) if we take \( \sigma_0 \) in the region where the LDV asymptotics is valid, i.e. \( \sigma_0 = -\infty \):

\[
M(\tau, \sigma = -\infty) = -\frac{\kappa}{2} \lambda \epsilon \gamma_5(1 + \Gamma_1) \epsilon|_{\sigma = -\infty} \tag{4.14}
\]

Note that, by equation (4.8) the terms \( \mathcal{O}(e^{-2\lambda \sigma}) \) in the asymptotic expansion of \( \epsilon \) do not contribute to \( M(\tau, \sigma = +\infty) \). Due to the explicit factor of \( \kappa \) in \( -M(\tau, \sigma = -\infty) \) one might
want to interpret the latter as a quantum correction to $M(\tau, \sigma = +\infty)$. Looking at the LDV example, eq. (4.11), however, shows that this is not possible in general and might only be true for some particular choice of $c_0$ and $c_1$.

We will now consider such a choice and set $c_0 = 1$. The other function of integration, $c_1$ is fixed by requiring

$$\lim_{\sigma \to \infty} e^{-\phi}(\nabla \phi - \lambda)\epsilon = 0 .$$

(4.15)

Indeed, this fixes the subleading term $\epsilon(1)$ in the expansion (4.7) of $\epsilon$, since the leading term $\sim (1 + \Gamma_1)\epsilon(0)$ vanishes by the differential equation, see (4.8).* Equation (4.15) determines $c_1$ as $c_1 = \frac{\dot{a}_1}{2\lambda}c_0$. Then, choosing $c_0 = 1$ fixes the normalisation. The latter can be obtained by requiring

$$\bar{\epsilon}\gamma_5\epsilon|_{\sigma=\infty} = 1 .$$

(4.16)

Remark, that an alternative choice would be to replace (4.15) by the following condition at $\sigma = +\infty$: $(\nabla \phi - \lambda)\epsilon|_{\sigma=-\infty} = 0$. Then $M(\tau, \sigma = -\infty) = 0$ and $M = M(\tau, \sigma = +\infty)$. $c_1$ then has to be obtained by solving the $\epsilon$-differential equation for all $\sigma$. This type of approach is advocated in the next subsection for computing the Bondi-mass, and it could also be carried out for the present discussion of the ADM-mass.

At present, however, we will simply remark that if we choose to impose (4.15) and (4.16), i.e. $c_0 = 1$ and $c_1 = \frac{\dot{a}_1}{2\lambda}c_0$, we obtain

$$M(\tau, \sigma = +\infty) = \lambda a_1^2 - 2\lambda a_2 + \frac{\dot{a}_1^2}{\lambda} .$$

(4.17)

Note that using (4.5) we get

$$\frac{d}{d\tau}M(\tau, \sigma = +\infty) = 0 .$$

(4.18)

Thus we find that at least the contribution from $\sigma = +\infty$ is time-independent.† Let us compare (4.17) with other expressions for the ADM-mass given in the literature. First, for $a_1 = 0$ (and

* Note that ref. 7 requires $\lim_{\sigma \to \infty}(\nabla \phi - \lambda)\epsilon = 0$ to fix $\epsilon$. However, as we have just seen, this is an empty statement, since it cannot fix $\epsilon(1)$, while $(1 + \Gamma_1)\epsilon(0) = 0$ as a consequence of the differential equation, anyhow.

† Although we were not able to prove it, the contribution from $\sigma = -\infty$ should also be time-independent in order to produce a satisfactory definition of ADM-mass.
only in this case), expression (4.17) equals\(^\dagger\)
\[
\mathcal{M}(\tau, \sigma = +\infty) = \lim_{\sigma \to \infty} 2e^{2\lambda \sigma}(\partial_\sigma \delta \phi + \lambda \delta \rho)
\]  (4.19)
which is the standard expression for the ADM-mass usually used in the literature [11,1] (it is correct only if \(a_1 = 0\), since otherwise it is \textit{not} \(\tau\)-independent). For general \(a_1\), expression (4.17) equals \(2\lambda(a_1^2 - a_2) + \Delta\), where \(\lambda \Delta = a_1^2 - \lambda^2 a_1^2\) is a constant by equation (4.5). Up to the constant \(\Delta\)-term our expression (4.17) coincides with the conserved expression for the mass, \(\lim_{\sigma \to \infty} 2e^{2\lambda \sigma}(\partial_\sigma + \lambda)(\delta \phi - \delta \phi^2)\), derived in ref. 12.

With these remarks in mind, we have shown that for a space-like surface of constant \(\tau\) and for the more restricted asymptotics \((a_1 = 0)\), the energy functional \(M\), with \(\epsilon\) subject to (4.15), (4.16), equals the usual “classical” ADM-mass plus a “quantum”-correction \(-\mathcal{M}(\tau, \sigma = -\infty)\). The latter by itself is non-negative (since \(\bar{\epsilon} \gamma_5(1 + \Gamma_1)\epsilon = |\epsilon_1 + \epsilon_2|^2 \geq 0\)). Obviously our positivity theorem allows the classical expression for the ADM-mass to get slightly negative by just the amount that is compensated for by the “quantum”-correction.

4.2. Bondi-mass

We will now consider a light-like surface \(\Sigma\) of constant \(\sigma^-\) and obtain an expression for the Bondi-mass. Again, we work in coordinates where \(\phi = -\lambda \sigma + \rho\). First we solve the \(\epsilon\)-differential equation \textit{exactly}. It reads (with \(\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}\))
\[
\partial_+ \epsilon_1 = \left[ \frac{\partial_+ \phi}{1 + x} + \left( -\frac{1}{2} + \frac{2x}{(1 + x)^2} \right) \partial_+ \rho + \frac{2Q}{\kappa} \frac{x}{(1 + x)^2} \partial_+ \tilde{Z} \right] \epsilon_1 - \frac{\lambda}{2} \frac{\epsilon^\rho}{1 + x} \epsilon_2
\]
\[
\partial_+ \epsilon_2 = \frac{1}{2} \epsilon^\rho \epsilon_2
\]  (4.20)
where \(x = \frac{4}{\kappa} e^{2\phi}\). Making use of \(\rho = \phi + \lambda \sigma\) this is readily integrated:
\[
\epsilon = \left( e^H \left[ d_1 - d_2 \frac{\lambda}{\kappa} \int_0^{\sigma^+} \frac{d\tilde{\sigma}^+}{\sqrt{1 + \frac{\epsilon}{4} e^{2\phi}}} (1 + \frac{\epsilon}{4} e^{2\phi})^{-1} e^{\frac{3}{2} \tilde{\sigma}^+ - H} \right] \frac{e^H d_2}{d_1} \right).
\]  (4.21)
\(^\dagger\) More precisely, from (4.5) we see that \(a_1(\tau) e^{-\lambda \sigma} \sim a_+ e^{\lambda \sigma^+} + a_- e^{-\lambda \sigma^-}\), and hence there is a conformal coordinate transformation that sets \(a_1\) to zero in the new coordinates. Of course, in the new coordinates, where \(\delta \rho\) and \(\delta \phi\) are \(\mathcal{O}(e^{-2\lambda \sigma})\), we no longer have \(\phi = -\lambda \sigma + \rho\). Repeating the above computation with \(\delta \rho \neq \delta \phi\) one obtains the desired equation.
Here $\sigma_0^+$ is some arbitrary reference coordinate, and the function $H(\sigma^+, \sigma^-)$ is given by

$$H = -\frac{\lambda}{2} (\sigma^+ - \sigma_0^+) + \frac{1}{2} \left[ \rho - \log(1 + x) + \frac{2x}{1 + x} \right] \sigma_0^+ + \int_{\sigma_0^+}^{\sigma^+} \frac{d\sigma^+}{(1 + x)^2} \left( \lambda + \frac{2Q}{\kappa} \partial_+ \tilde{Z} \right).$$

(4.22)

$H$ vanishes at $\sigma^+ = \sigma_0^+$ so that

$$\epsilon(\sigma_0^+, \sigma^-) = \left( \frac{d_1(\sigma^-)}{e^{\frac{1}{2} \sigma(\sigma_0^+, \sigma^-)} d_2(\sigma^-)} \right)$$

(4.23)

i.e. $d_1(\sigma^-)$ and $d_2(\sigma^-)$ are “constants” of integration.

Next, we need to study the asymptotics as $\sigma^+ \to \pm \infty$. Consider $\sigma^+ \to +\infty$ first:

$$\rho \sim a_0(\sigma^-) + a_1(\sigma^-) e^{-\lambda \sigma^+} + O(e^{-2\lambda \sigma^+})$$

$$\phi = -\frac{\lambda}{2} \sigma^+ + \frac{\lambda}{2} \sigma^- + \rho$$

$$\partial_+ \tilde{Z} \to 0$$

as $\sigma^+ \to +\infty$.

(4.24)

Note that we do not have $\rho \to 0$ but rather $\rho \to a_0(\sigma^-)$. In order to be consistent with the LDV asymptotics as $\sigma \to \infty$ one must have $a_0(\sigma^-) \to 0$ as $\sigma^- \to -\infty$. This is satisfied for the shock-wave scenario where $a_0 = -\frac{1}{2} \log \left( 1 - \frac{4}{\lambda} e^{\lambda \sigma^-} \right)$ and also for more general solutions. Indeed, the general solution of the equations of motion and constraints with a matter stress-energy $T^M_{++}$ vanishing for large enough $\sigma^+$ (or decreasing sufficiently fast) and $T^M_{--} = 0$ yields

$$a_0 = -\frac{1}{2} \log \left( 1 - \frac{p}{\lambda} e^{\lambda \sigma^-} \right)$$

$$a_1 = -\frac{1}{2} \frac{e^{\lambda \sigma^-}}{1 - \frac{p}{\lambda} e^{\lambda \sigma^-}} \left[ \frac{m}{\lambda} + \frac{\kappa}{4} \log \left( 1 - \frac{p}{\lambda} e^{\lambda \sigma^-} \right) \right].$$

(4.25)

Here $m$ and $p$ are the total energy and momentum carried by the infalling matter. (The shock-wave corresponds to $p = a$ and $m = ae^{\lambda \sigma_0^+} = a\lambda e^{\lambda \sigma_0^+}$ in the usual notation.) Note that $a_0$
satisfies the relation

$$\frac{2}{\lambda}a_0'' + 1 - e^{2a_0} = 0$$

(4.26)

which we shall use below. The asymptotics (4.24) imply for \( \sigma^+ \to \infty \)

$$H = -\frac{\lambda}{2}(\sigma^+ - \sigma_0^+) + H_0(\sigma^-) + O(e^{-\lambda \sigma^+}),$$

$$H_0(\sigma^-) = \frac{a_0}{2} - \frac{1}{2} \left[ \rho - \log(1 + x) \right]_{\sigma_0^+}^{\infty} + \int_{\sigma_0^+}^{\infty} d\tilde{\sigma}^+ \frac{x}{(1 + x)^2} \left( \lambda + \frac{2Q}{\kappa} \partial_+ \tilde{Z} \right)$$

(4.27)

(recall that \( x = \frac{\kappa}{4} e^{2\phi} \)). Then the integrand in eq. (4.21) for \( \epsilon \) behaves as \((1 + x)^{-1} e^{\frac{3}{2} \rho - H} \sim \exp \left( \frac{3}{2} a_0 - H_0 + \frac{1}{2}(\sigma^+ - \sigma_0^+) \right) + O(e^{-\frac{3}{2} \sigma^+})\). As a result, as \( \sigma^+ \to \infty \):

$$\epsilon = \epsilon(0) + \epsilon(1) e^{-\frac{3}{2} \sigma^+} + \epsilon(2) e^{-\lambda \sigma^+} + \ldots,$$

$$\epsilon(0) = -e^{a_0} d_2 \left( \begin{array}{c} e^{\frac{3}{2} a_0} \\ -e^{-\frac{1}{2} a_0} \end{array} \right), \quad \epsilon(1) = \frac{1}{\sqrt{2}} e^{\frac{3}{2} a_0} L \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$$

(4.28)

where

$$L(\sigma^-) = \sqrt{2} e^{-\frac{3}{2} a_0 + H_0 + \frac{1}{2} \sigma_0^+} \left[ d_1 + d_2 (e^{\frac{3}{2} a_0 - H_0} - J) \right],$$

$$J(\sigma^-) = \frac{\lambda}{2} \int_{\sigma_0^+}^{\infty} d\tilde{\sigma}^+ \left[ (1 + \frac{\kappa}{4} e^{2\phi})^{-1} e^{\frac{3}{2} \rho - H} - e^{\frac{3}{2} a_0 - H_0 + \frac{1}{2}(\sigma^+ - \sigma_0^+)} \right].$$

(4.29)

At this point the reader might wonder why it looks so complicated to extract the asymptotics of \( \epsilon \). Of course, the form (4.28) of the asymptotics follows immediately from the \( \epsilon \)-differential equation. However, this does not determine the value of \( L \). We may take \( L \) as a free parameter, but then we need to use the exact solution of the differential equation to obtain \( \epsilon \) at the other end of \( \Sigma \) (as \( \sigma^+ \to -\infty \)). Our approach here is to fix \( \epsilon \) at some finite \( \sigma_0^+ \) (cf (4.23)) and to determine the value of \( L(\sigma^-) \). This will be particularly convenient for any configuration that equals the LDV for all \( \sigma^+ < \sigma_0^+ \), as is the case e.g. in the shock-wave scenario.

It is now straightforward to compute

$$\mathcal{M}(\sigma^-, \sigma^+ = +\infty) = \lim_{\sigma^+ \to \infty} \left[ 2 \left( e^{-2\phi} + \frac{\kappa}{4} \right) \bar{\epsilon} \gamma_5 (\nabla \phi - \lambda) \epsilon - \kappa \bar{\epsilon} \gamma_5 \nabla \rho \epsilon \right]$$

(4.30)

where we already used \( \partial_+ \tilde{Z} \to 0 \) as \( \sigma^+ \to \infty \). Inserting the asymptotics (4.24) and (4.28),
and using the relation (4.26), one arrives at

$$\mathcal{M}(\sigma^-, \sigma^+ = +\infty) = -4d_1^2 e^{-\lambda\sigma^-} e^{-2a_0 d_1} - \kappa \lambda d_2^2 (1 - e^{2a_0}) - \lambda e^{-\lambda\sigma^-} L^2 . \quad (4.31)$$

Using now the solution (4.25) of the equations of motion and constraints one obtains

$$\mathcal{M}(\sigma^-, \sigma^+ = +\infty) = 2d_1^2 \left[ m + \frac{\kappa}{4} \lambda \log \left( 1 - \frac{p}{\lambda} e^{\lambda\sigma^-} \right) + \frac{\kappa}{4} p e^{\lambda\sigma^-} \right] - \lambda e^{-\lambda\sigma^-} L^2(\sigma^-) . \quad (4.32)$$

Next, we evaluate $\mathcal{M}(\sigma^-, \sigma^+ = -\infty)$. We will do so under the assumption that there is some value $\sigma^*_+ \in \sigma^+$ so that we have the LDV ($\rho = 0, \phi = -\frac{1}{2} \sigma^+ + \frac{1}{2} \sigma^-, \partial_+ \tilde{Z} = 0$) for all $\sigma^+ < \sigma^*_+$. We then identify the (so far arbitrary) value of $\sigma^*_0 \in \sigma^-$ with this $\sigma^*_+$. In the LDV region ($\sigma^+ < \sigma^*_+$) our formulas simplify:

$$H = -\frac{1}{2} \log \frac{e^{\lambda(\sigma^+_0 - \sigma^-)} + \frac{\kappa}{4}}{e^{\lambda(\sigma^+_0 - \sigma^-)} + \frac{\kappa}{4}} \quad (4.33)$$

and then from equation (4.21)

$$\epsilon = \left( \begin{array}{c} -d_2 \\ d_2 \end{array} \right) + \left( \frac{e^{\lambda(\sigma^+_0 - \sigma^-)} + \frac{\kappa}{4}}{e^{\lambda(\sigma^+_0 - \sigma^-)} + \frac{\kappa}{4}} \right)^{1/2} \left( \begin{array}{c} d_1 + d_2 \\ 0 \end{array} \right) . \quad (4.34)$$

Note that this has a finite limit as $\sigma^+ \to -\infty$ for all finite $d_1(\sigma^-), d_2(\sigma^-)$. We obtain

$$\mathcal{M}(\sigma^-, \sigma^+ = -\infty) = \lim_{\sigma^+ \to -\infty} \left( -\frac{\kappa \lambda}{2} \right) \epsilon_{56}(1 + \Gamma_1) \epsilon = -2 \lambda (d_1 + d_2)^2 \left( e^{\lambda(\sigma^+_0 - \sigma^-)} + \frac{\kappa}{4} \right) \quad (4.35)$$

and the Bondi-mass equals

$$M_B(\sigma^-) = \mathcal{M}(\sigma^-, \sigma^+ = +\infty) - \mathcal{M}(\sigma^-, \sigma^+ = -\infty)$$

$$= \frac{2d_1^2(\sigma^-)}{1 - \frac{p}{\lambda} e^{\lambda\sigma^-}} \left[ m + \frac{\kappa}{4} \lambda \log \left( 1 - \frac{p}{\lambda} e^{\lambda\sigma^-} \right) + \frac{\kappa}{4} p e^{\lambda\sigma^-} \right] - \lambda e^{-\lambda\sigma^-} L^2(\sigma^-) + 2 (d_1 + d_2)^2 \lambda \left( e^{\lambda(\sigma^+_0 - \sigma^-)} + \frac{\kappa}{4} \right) . \quad (4.36)$$

So far, $M_B$ still depends on two arbitrary functions $d_1(\sigma^-)$ and $d_2(\sigma^-)$. They have to be fixed by physical requirements. First of all, the LDV should have vanishing energy. It is easy
to check that this is true independent of the choice of \( d_1, d_2 \):

\[
M_B^{LDV} = 0 .
\]  
(4.37)

Second, for \( \kappa = 0 \), we do not expect any Hawking radiation in the RST model and \( M_B \) should equal \( m \) for all \( \sigma^- \). Since, for \( \kappa = 0 \), we can solve for \( \rho \) and \( \phi \) explicitly,

\[
\rho = -\frac{1}{2} \log \left( 1 - \frac{p}{\lambda} e^{\lambda \sigma^-} + \frac{m}{\lambda} e^{-\lambda \sigma^+ + \lambda \sigma^-} \right), \quad \phi = -\lambda \sigma + \rho,
\]
we can evaluate \( J \) and hence \( L \) exactly. The final result for \( M_B \) is very simple:

\[
M_B|_{\kappa=0} = 2d_1^2|_{\kappa=0} m .
\]  
(4.38)

Thus we find \( d_1^2 = \frac{1}{2} + \mathcal{O}(\kappa) \).

Third, we consider the general case corresponding to (4.36) in the limit \( \sigma^- \to -\infty \). Again one should find \( M_B(\sigma^- = -\infty) = m \) since no Hawking radiation yet had a chance to be emitted. Evaluating carefully the asymptotics (as \( \sigma^- \to -\infty \)) of \( \rho, \phi \), and using (2.22) to obtain those of \( J \) and \( L \), we find

\[
M_B(\sigma^- = -\infty) = 2d_2^2(-\infty) + (d_1(-\infty) + d_2(-\infty))^2(2pe^{\lambda \sigma_0^+} + 2\kappa \lambda I_1)
- 2d_2(-\infty)(d_1(-\infty) + d_2(-\infty))(2pe^{\lambda \sigma_0^+} + \kappa \lambda I_{1/2})
\]  
(4.39)

where \( I_1 \) and \( I_{1/2} \) are logarithm integrals defined below. We will now argue that the only physically acceptable choice is \( d_1 + d_2 = 0 \), \( d_2^2 = \frac{1}{2} \). Indeed, if \( d_1 + d_2 \neq 0 \) the equation \( M_B(\sigma^- = -\infty) = m \) defines \( d_2(-\infty) \) as a function of \( (d_1 + d_2)(-\infty) \) in a way that depends on \( p \) and \( m \). In particular, \( \epsilon \) would depend on these quantities even in the LDV region which is in the causal past of the infalling matter distribution. Hence, we argue that by causality we should have \( d_1(-\infty) + d_2(-\infty) = 0 \), and hence \( d_2^2(-\infty) = d_1^2(-\infty) = \frac{1}{2} \). This would still leave the possibility that for finite \( \sigma^- \) we have \( d_1 + d_2 = \mathcal{O}(e^{\lambda \sigma^-}) \). However, by the same argument, in the LDV region, \( \epsilon \) should not depend on \( \sigma_0^+ \) which determines the trajectory of the shock-wave. Looking at the exact solution (4.34) for \( \epsilon \) in the LDV region we see that this implies \( d_1 + d_2 = 0 \) for all \( \sigma^- \). Thus we arrive at

\[
d_1 + d_2 = 0 , \quad d_1^2 = d_2^2 = \frac{1}{2} , \quad \forall \sigma^-,
\]  
(4.40)

so that in the LDV region \( \epsilon = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \). The conditions (4.40) can be written more
elegantly as
\[ e^{-2p} \delta \gamma 5 \epsilon |_{\sigma^+=\infty} = -1 \] (4.41)

which fixes \( d_2^2 = \frac{1}{2} \), and (note that the l.h.s. is taken at \( \sigma^+ = -\infty \), not \( +\infty \))
\[ (\nabla \phi - \lambda) \epsilon |_{\sigma^+ = -\infty} = 0 \] (4.42)

which fixes \( d_1 + d_2 = 0 \). With this choice, \( M(\sigma^-, \sigma^+) = -\infty \) = 0, and
\[ M_B(\sigma^-) = \frac{1}{1 - \frac{2}{\lambda} e^{\lambda\sigma^-}} \left[ m + \frac{\kappa}{4} \lambda \log \left( 1 - \frac{p}{\lambda} e^{\lambda\sigma^-} \right) + \frac{\kappa}{4} pe^{\lambda\sigma^-} \right] - \lambda e^{-\lambda\sigma^-} L^2(\sigma^-). \] (4.43)

We now evaluate \( M_B \) for large negative but finite \( \sigma^- \). Keeping the first subleading terms as \( \sigma^- \to -\infty \), we have
\[ J \sim -\frac{1}{2} \left[ \frac{p}{\lambda} e^{\lambda\sigma_0^+} + \frac{\kappa}{4} + \kappa I_{1/2} - \kappa I_1 \right] e^{-\lambda\sigma_0^+} e^{\lambda\sigma^-}, \]
\[ L \sim \pm \left[ \frac{p}{\lambda} e^{\lambda\sigma_0^+} + \frac{\kappa}{2} I_{1/2} \right] e^{-\frac{1}{2}\sigma_0^+} e^{\lambda\sigma^-} \] (4.44)

where \( I_\alpha \) is a logarithm integral given by
\[ I_\alpha = e^{\alpha \lambda \sigma_0^+} \int_{\sigma_0^-}^{\infty} d\sigma^+ \frac{e^{-\alpha \lambda \sigma^+}}{\sigma^+ - \sigma_0^- + \frac{\alpha m}{\lambda}} = \int_0^{\infty} dx \frac{e^{-x}}{x + \frac{\alpha m}{\lambda}} = - e^{\frac{\alpha \lambda m}{\lambda}} \log \left( e^{\frac{\alpha \lambda m}{\lambda}} \right). \] (4.45)

Thus, for the shock-wave scenario \( (pe^{\lambda\sigma_0^+} = m) \), we arrive at
\[ M_B(\sigma^-) \sim m - \kappa \left( mI_{1/2} + \frac{\lambda \kappa}{4} I_{1/2} \right) e^{-\lambda\sigma_0^+} e^{\lambda\sigma^-} + \mathcal{O}(e^{2\lambda\sigma^-}). \] (4.46)

Let us comment on this equation. First, as already observed, \( M_B \) is constant for \( \kappa = 0 \): classically there is no Hawking radiation. Second, \( M_B \) is decreasing as \( \sigma^- \) (i.e. time) increases (at least to the first order in \( e^{\lambda\sigma^-} \) we computed): Hawking radiation carries energy away from the black hole.* Note that for \( \frac{\lambda \kappa}{m} << 1 \) the leading term in (4.46) reads \( M_B \sim m - \kappa m \log \left( \frac{2m}{\lambda \kappa} \right) e^{-\lambda\sigma_0^+} e^{\lambda\sigma^-} \). This differs from the CGHS prediction for the very early Hawking radiation by the extra factor of \( \log \left( \frac{2m}{\lambda \kappa} \right) \). However, there is nothing wrong with this difference, since the RST and CGHS models represent different \( \mathcal{O}(\kappa) \) corrections to the same classical dilaton-gravity.

* Recall that the we assume \( \kappa \geq 0 \) throughout this paper.
Finally we would like to compute \( M_B(\sigma^-) \) for the shock-wave scenario at \( \sigma^- = \sigma_s^- \), the point where the singularity and apparent horizon intersect, and the solution is matched to the LDV. As for any finite \( \sigma^- \), we have no explicit functions for \( \phi \) and \( \rho \) (they are given by solving the transcendental equation (2.23) at each point) which makes it difficult to obtain an exact expression for \( L \) since it involves integrals of functions of \( \phi \) over all \( \sigma^+ > \sigma_0^+ \). One could, of course, proceed numerically. For \( \sigma^- = \sigma_s^- \), however, the situation is slightly better since we know that \( \phi \) and \( \rho \) equal the “shifted” LDV for \( \sigma^+ > \sigma_s^+ \) (cf. (2.25)):

\[
\phi(\sigma^+, \sigma_s^-) = -\frac{\lambda}{2}(\sigma^+ - \sigma_s^+) - \frac{1}{2} \log \frac{\kappa}{4}, \quad \rho(\sigma^+, \sigma_s^-) = \frac{2m}{\lambda \kappa}, \quad \sigma^+ > \sigma_s^+. \tag{4.47}
\]

On the other hand, if \( \frac{m}{\lambda \kappa} \ll 1 \), \( \sigma_s^+ \) is close to \( \sigma_0^+ : \lambda \sigma_s^+ - \lambda \sigma_0^+ = \frac{2m}{\lambda \kappa} + \mathcal{O}(\frac{2m}{\lambda \kappa})^2 \), and we have to solve for \( \phi \) and \( \rho \) on the small interval \( [\sigma_0^+, \sigma_s^+] \) only which can be done perturbatively in \( \frac{m}{\lambda \kappa} \). It is easy to see that all quantities can be developed in powers of \( \frac{m}{\lambda \kappa} \) (e.g. no \( \log \frac{\lambda \kappa}{m} \) occurs contrary to the opposite limit \( \frac{m}{\lambda \kappa} \gg 1 \)). Here we only compute \( M_B(\sigma_s^-) \) to first order in \( \frac{m}{\lambda \kappa} \) which will turn out to be very easy. Indeed, write \( L(\sigma_s^-) = \alpha + \beta \frac{m}{\lambda \kappa} + \ldots \). But we know that in the limit where \( m \to 0 \) (i.e. in the LDV) \( L \) vanishes. Hence \( \alpha = 0 \). Using the explicit expression for \( \sigma_s^- \), equation (2.24), we find

\[
M_B(\sigma_s^-) = \frac{\kappa \lambda}{4} \left[ \left( e^{\frac{4m}{\lambda \kappa}} - 1 \right) - \frac{4m}{\lambda \kappa} e^{-\lambda \sigma_0^-} L^2(\sigma_s^-) \right]. \tag{4.48}
\]

Expanding to first order in \( \frac{m}{\lambda \kappa} \), \( L^2 \) does not contribute and

\[
M_B \sim m + \mathcal{O}(\frac{m}{\lambda \kappa})^2. \tag{4.49}
\]

Thus if we start with a very small black hole (small \( m \)) or a very large number of matter fields (large \( \kappa \)), the black hole is matched to the shifted LDV before any substantial Hawking radiation has occurred: its mass is still the initial mass \( m \) up to \( \mathcal{O}(\frac{m}{\lambda \kappa})^2 \) corrections. This positive amount of energy must then be sent off by the thunderpop. In ref. 4, RST find (up to their sign ambiguity) that the thunderpop carries energy \( \frac{\lambda \kappa}{4} \left( 1 - e^{-\frac{4m}{\lambda \kappa}} \right) = m + \mathcal{O}(\frac{m}{\lambda \kappa})^2 \) in agreement with (4.49).

In conclusion, we have found that our functional \( M \) as given by (3.3) with \( \epsilon \) subject to the differential equation (3.5) and the boundary conditions (4.41) and (4.42) defines a satisfactory
Bondi-mass: it is non-negative, equals the ADM-mass \( m \) at \( \sigma^- = -\infty \), decreases for \( \kappa > 0 \) and is constant for \( \kappa = 0 \). It also gives correctly the energy of the thunderpop (at least to the order we computed).

5. The supersymmetric extension

Positive energy theorems naturally occur in supersymmetric theories. Thus the preceding results prompt the question: does there exist a supersymmetric extension of the action (2.13) or (2.15)? For the CGHS model such an extension was constructed in ref. 7. There it was also shown that starting from a general supersymmetric dilaton-gravity action in 2D

\[
S^{(1)} = \frac{i}{2\pi} \int d^2 x d^2 \theta E [J(\Phi)S + iK(\Phi)D_\alpha \Phi D^\alpha \Phi + L(\Phi)]
\]  

(5.1)

(\( \Phi \) is the dilaton superfield, \( S \) the curvature multiplet and \( E \) the super-zweibein, see ref. 7 for all notation and conventions) the purely bosonic part (all fermi fields set to zero) reads after integrating out the auxiliary fields

\[
S^{(1)}_{\text{bos}} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ J R + 2K(\nabla \phi)^2 + \left( \frac{L L'}{2J'} - \frac{K L^2}{2J'^2} \right) \right],
\]  

(5.2)

where now \( J = J(\phi), K = K(\phi), L = L(\phi) \).

We now repeat this exercise, including a supersymmetric \( Z \)-field:

\[
S^{(2)} = \frac{i}{2\pi} \int d^2 x d^2 \theta E \left[ -\frac{i}{4} D_\alpha Z D^\alpha Z + Q Z S \right].
\]  

(5.3)

The bosonic part of this action alone is just \( S_Z \) of eq. (2.7) after integrating out the auxiliary fields. When combining \( S^{(1)} \) and \( S^{(2)} \), the auxiliary field equations get modified and the resulting bosonic part is not just (5.2) plus \( S_Z \), but rather

\[
\left[ S^{(1)} + S^{(2)} \right]_{\text{bos}} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ J R + 2K(\nabla \phi)^2 - \frac{1}{2}(\nabla Z)^2 + Q R Z + F(\phi) \right]
\]  

(5.4)

where

\[
F(\phi) = \left( 1 - \frac{2\kappa K}{J'^2} \right)^{-1} \left( \frac{L L'}{2J'} - \frac{K L^2}{2J'^2} - \frac{\kappa L'^2}{4J'^2} \right).
\]  

(5.5)

All we have to do now is to identify the functions \( J, K \) and \( L \) of \( \phi \) that reproduce e.g. the RST-action (2.13).
Equation (5.4) will be identical to the RST-action (2.13) if
\[ J(\phi) = e^{-2\phi} - \frac{\kappa}{2} \phi, \quad K(\phi) = 2e^{-2\phi}, \quad F(\phi) = 4\lambda^2 e^{-2\phi}. \] (5.6)

Substituting these into eq. (5.5) we obtain a non-linear differential equation for \( L(\phi) \):
\[ (L + xL')(L + L') = -\kappa^2 \lambda^2 \frac{(1 - x)^2}{x^2} \] (5.7)
where \( L' = dL/d\phi \) and \( x = \frac{\kappa e^{2\phi}}{4} \). The solution is very simple:
\[ L(\phi) = \pm 4\lambda \left( e^{-2\phi} + \frac{\kappa}{4} \right) = \mp 2\lambda J'(\phi). \] (5.8)

Obviously there are two choices of sign since only \( \lambda^2 \) is relevant. Thus, if \( J, K \) and \( L \) are given by (5.6), (5.8), the action \( S^{(1)} + S^{(2)} \) is a supersymmetric extension of the RST-action. Note that the \( Z \)-independent part of the energy-functional \( M \) is \( \sim \bar{\epsilon} \gamma_5 (J' \nabla \phi \pm \frac{1}{2} L)\epsilon \) as expected (cf. eq. (63) of ref. 7). Thus it can be derived from the square of the supercharge.

Similarly, we can construct a supersymmetric extension of the action of ref. 2. In this case \( J(\phi) \) and \( K(\phi) \) are given by the CGHS-functions
\[ J(\phi) = e^{-2\phi}, \quad K(\phi) = 2e^{-2\phi} \] (5.9)
while the function \( F \) is more complicated [2]:
\[ F(\phi) = 4\lambda^2 e^{-2\phi} D(\phi) = \kappa \lambda^2 \frac{1}{y} \left( 1 + \sqrt{1 - y} \right)^2 \exp \left[ \frac{1 - \sqrt{1 - y}}{1 + \sqrt{1 - y}} \right] \] (5.10)
where now \( y = \kappa e^{2\phi} \). The differential equation for \( L(\phi) \) then is
\[ LL' + L^2 + \frac{y}{4} L'' = -4\kappa \frac{1 - y}{y} F. \] (5.11)

If we substitute
\[ L(\phi) = 2\lambda \kappa \left( \frac{1 + \sqrt{1 - y}}{1 - \sqrt{1 - y}} \right)^{1/2} \exp \left[ \frac{1 - \sqrt{1 - y}}{2 \frac{1}{1 + \sqrt{1 - y}}} \right] g(y) \] (5.12)
the differential equation for \( L \) simplifies to
\[ 2y \sqrt{1 - y} gg' + y^3 g'' = \frac{1 - y}{y} \] (5.13)
where \( g' = dg/dy \). If we now change variables from \( y = \kappa e^{2\phi} \) to \( \Omega \) with \( \Omega \) given by (2.3), the
differential equation becomes simply

\[ g \frac{dg}{d\Omega} - \frac{1}{4} \left( \frac{dg}{d\Omega} \right)^2 = 1 . \]  

(5.14)

This is easily integrated and the solution \( g(\Omega) \) is given implicitly by the following transcendental equation (\( c \) is a constant of integration)

\[ 4(\Omega + c) = g(g \pm \sqrt{g^2 - 1}) \pm \log(g + \sqrt{g^2 - 1}) . \]  

(5.15)

This defines the solution \( g \), and by (5.12) also \( L(\phi) \), as a transcendental function of \( \Omega(\phi) \). The first terms in an expansion for small \( \kappa e^{2\phi} \) are

\[ L(\phi) \sim \pm 4\lambda \left( e^{-2\phi} + \frac{\kappa}{4} \phi + \tilde{c} \right) . \]  

(5.16)

Although we have not derived it above, the energy-functional \( M \) should be obtained from the square of the supercharge. Following ref. 7 and our observation above we expect that the \( Z \)-independent part of the energy-functional \( M \) for the variant of ref. 2 is again \( \bar{\epsilon} \gamma_5 (J_{\nabla} \phi \pm \frac{1}{2} L) \epsilon \) (although now \( L(\phi) \) as given by (5.12) and (5.15) is a rather complicated function!), while the \( Z \)-dependent part should be the same as given in section 3 for the RST theory.

We have explicitly shown that the exactly solvable conformally invariant actions of refs. 4 and 2 have supersymmetric extensions. At first sight this seems to be in contrast with the statement of ref. 8 that such supersymmetric extensions do not exist. A closer look however shows that one is dealing with two different requirements. This was recently clarified by Danielsson [9] after a first circulation of the present paper. Indeed what we claim here is to have constructed \( (semi)classical \) theories that are the supersymmetric extensions of the exact conformal theories of refs 2 and 4. The point is [9] that integrating out the auxiliary fields is a procedure that can only be trusted semiclassically if the auxiliary fields are not set equal to zero by their field equations. The reason is very simple to see in the present case: integrating out the auxiliary fields will typically replace a vertex operator of conformal dimension \((\alpha, \alpha)\) by its square which \textit{classically} has dimension \((2\alpha, 2\alpha)\), but of course not quantum mechanically. Thus if we start with dimension \((\frac{1}{2}, \frac{1}{2})\) operators as required by exact \textit{superconformal} invariance we will not get \((1, 1)\) operators after integrating out the auxiliary
fields, and vice versa. Since we insisted here on having a $(1,1)$ operator after integrating out the auxiliary fields, we certainly did not have an exact superconformal theory to start with. The claim of ref. 8 was precisely that such an exact superconformal theory with the required bosonic part does not exist. However, it turned out [9] that by complicating the original supersymmetric action slightly one can construct an exact superconformal theory $(c = 0)$ whose bosonic part, although not identical to any of the exact bosonic conformal theories, still gives the usual dilaton gravity in the (weak-coupling) semiclassical limit.

6. Conclusions

We have proven a positive energy (mass) theorem for the exactly solvable quantum-corrected 2D dilaton-gravity theory à la RST. Although there are probably many more or less reasonable mass functionals, the one given here is (within a relatively broad class) the only one that obeys a positivity theorem. For field configurations asymptotic to the LDV we have shown that this mass functional if defined on a space-like surface coincides with the usual definition of the ADM-mass given by the field asymptotics at $\sigma = +\infty$, plus a certain (“quantum”)-correction $-\mathcal{M}(\sigma = -\infty)$ depending, via the $\epsilon$-differential equation, on the fields on all of $\Sigma$. For light-like (null) $\Sigma$, we have given a rather detailed analysis of the resulting Bondi-mass and shown that it exhibits all expected physical properties: besides being non-negative, it equals the ADM-mass in the far past, is decreasing for $\kappa > 0$ and constant for $\kappa = 0$, and gives the correct (positive) energy of the thunderpop in the shock-wave scenario (up to the order we computed).

We also explicitly constructed supersymmetric extensions of the exactly solvable theories of refs. 2 and 4. The squares of the supercharges give us the positive energy functionals, as we could check for the RST variant.

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