Integrable Hierarchy of Multi-Component Kaup-Boussinesq Equations

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Abstract

By using the Lax approach we find the integrable hierarchy of the two and three field Kaup-Boussinesq equations. We then give a multi-component Kaup-Boussinesq equations and their recursion operators. Finally we show that all multi-component Kaup-Boussinesq equations are the degenerate Svinolupov KdV systems.

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1 Introduction

Systems of nonlinear evolution equations which are integrable are not so many. Well-known ones are the coupled nonlinear Schrödinger equations on simple Lie algebras [1], [2], coupled KdV equations on Jordan algebras [3], [4], and coupled KdV, mKdV and Nonlinear Schrödinger equations and their super extensions on symmetric and homogeneous spaces [5], [6]. The way one obtains such systems is either by starting with soliton connections on the corresponding algebras [5]-[8] and imposing the zero-curvature condition or by assuming the form of the recursion operator of the corresponding system [9], [10]. Most cases these two approaches are in agreement [11]. This means that one imply the other.

In this work we start with a Lax representation of the KB equations and then derive the whole hierarchy and the recursion operator. This approach is similar to the one to obtain the hierarchy of the KdV equation [12], [13]. The Lax pair for the KdV equation \( u_t = -u_{xxx} + 6uu_x \) is given by

\[
\begin{align*}
\psi_{xx} &= (-\lambda + u(t,x)) \psi, \\
\psi_t &= A \psi - \frac{1}{2} A_x \psi
\end{align*}
\]

(1.1)  
(1.2)

where \( A = 2u + 4\lambda \) for the KdV equation and it is a polynomial of the spectral parameter \( \lambda \) for the integrable hierarchy of the KdV equation. One can first find the recursion operator \( R = -D^2 + 4u + 2u_x D^{-1} \) and then the whole hierarchy of the KdV equation. For the multi-component KB equations the Lax operator in (1.1) is also a second order differential operator but it is a polynomial of degree \( \ell \) in the spectral parameter \( \lambda \).

By using the above Lax approach for the KdV equation we will give a new integrable system of evolution equations named as multi-component Kaup-Boussinesq (KB) equations. We shall do this starting from two and three field KB equations. Two component Kaup-Boussinesq equation has been studied earlier by several authors [14]-[17]. We introduce two component KB equations in sections 2 and 3. In these sections we give the full hierarchy of two field KB equations with their recursion operator. In section 4 we give the three component extension of KB equations with the corresponding recursion operator. Then, in section 5 we give the multi-component KB equations with the recursion operator. It is shown in section 6 that the system of multi-component KB equations is a sub-class of degenerate Svinolupov KdV system.

2 Two Field KB Equations

Recently Ivanov and Lyons [17] introduced a Lax pair
\[ \psi_{xx} = [-\lambda^2 + \lambda u(x,t) + \frac{k}{2}u^2(x,t) + v(x,t)] \psi, \quad (2.1) \]

\[ \psi_t = -(\lambda + \frac{1}{2}u(x,t)) \psi_x + \frac{1}{4}u_x \psi, \quad (2.2) \]

leading to the coupled evolution equations

\[ u_t + v_x + \frac{3}{2}(1 + k)uu_x = 0, \quad (2.3) \]

\[ v_t - \frac{1}{4}u_{xxx} + (uv)_x - (\frac{1}{2} + k)uv_x - k(\frac{1}{2} + k)u^2u_x = 0 \quad (2.4) \]

where \( \lambda \) is the spectral parameter and \( k \) is an arbitrary constant.

We shall now find the hierarchy of the above equations just like determining the hierarchy of the KdV equation. Let the Lax operator \( \mathbf{L} \) be given by

\[ \mathbf{L} = D^2 - \lambda u + \frac{k}{2}u^2 + v \quad (2.5) \]

Then the first part of the Lax equation is the eigen-value equation for \( \mathbf{L} \)

\[ \mathbf{L} \psi = -\lambda^2 \psi \quad (2.6) \]

Let the time evolution of \( \psi \) be given by

\[ \psi_t = A \psi_x + B \psi, \quad (2.7) \]

where \( A \) and \( B \) depend on the dependent variables \( u, v \) and their derivatives with respect to \( x \) and also on the spectral parameter \( \lambda \). Compatibility of the Lax equations (2.6) and (2.7) gives \( B = -\frac{1}{2}A_x \) and

\[ A_{xxx} + [-2ku^2 + 4\lambda^2 - 4\lambda u - 4v]A_x - [2kuu_x + 2\lambda u_x + 2v_x]A + 2k uu_t + 2\lambda u_t + 2v_t = 0 \quad (2.8) \]

One can solve this equation by assuming \( A \) as a polynomial of \( \lambda \) (analytic in \( \lambda \)). For instance, by assuming \( A \) as a third order polynomial in \( \lambda \), a solution of this equation is obtained as

\[ A = a_0 [16\lambda^3 + 8\lambda^2 u + 2\lambda (2ku^2 + 3u^2 + 4v) - 2u_{xx} + 6ku^3 + 5u^3 + 12uv] \]

\[ +a_1 [16\lambda^2 + 8\lambda u + 2(2ku^2 + 3u^2 + 4v)] + a_2 [4\lambda + 2u] + a_3 \quad (2.9) \]

where \( a_0, a_1, a_2 \) and \( a_3 \) are arbitrary constant. The corresponding evolution equations are given as follows:
The evolution equations presented in (2.3) and (2.4) correspond to the choices: All constants $a_i = 0$ and $a_3 = 0$ and $a_2 = -4$. Then we can find the first four members of the hierarchy as follows

$N=0$: All constants $a_i = 0$ ($i = 0, 1, 2$) in the above equations except $a_3$.

\[
\begin{align*}
  u_t &= \frac{a_0}{16} [-(4k + 10)u_{xxx} - 12(k + 1)u_xu_{xx} + (12k^2 + 60k + 35)u^3u_x \\
  &\quad + (24k + 60)uvu_x + (12k + 30)u^2v_x + 24vv_x] + \frac{a_1}{16} [-4u_{xxx} + (36k + 30)u^2u_x + 24(uv)_x] + \frac{a_2}{16} [2(2k + 3)uu_x + 4v_x] - \frac{a_3}{16} u_x \quad (2.11)
\end{align*}
\]

\[
\begin{align*}
  v_t &= \frac{a_0}{32} [2u_{xxxx} + (8k^2 - 2k - 15)u^2u_{xxx} - 20vu_{xxx} + (2k^2 - 72k - 90)u_x \\
  &\quad + (60 - 48k^2)vu^2u_x + 48v^2u_x + (8k - 12)uw_{xxx} - (2k^2 + 24k - 10)u^3v_x \\
  &\quad + (48k + 72)uvv_x] + \frac{a_1}{32} [12uu_{xxx} - (36 + 24k)u_xu_{xx} \\
  &\quad - 24k(2k + 1)u^3u_x + 48uvu_x - 8v_{xxx} + 12(1 - 2k)u^2v_x + 48vv_x] \\
  &\quad + \frac{a_2}{32} [-2u_{xxx} - 4(k(2k + 1)u^2u_x + 8vu_x + 4(-2k + 1)uv_x] - \frac{a_3}{16} v_x \quad (2.12)
\end{align*}
\]

The evolution equations presented in (2.3) and (2.4) correspond to the choices $a_0 = a_1 = a_3 = 0$ and $a_2 = -4$. Then we can find the first four members of the hierarchy as follows

$N=0$: All constants $a_i = 0$ ($i = 0, 1, 2$) in the above equations except $a_3$.

\[
\begin{align*}
  u_{t0} + u_x &= 0, \quad (2.13) \\
  v_{t0} + v_x &= 0, \quad (2.14)
\end{align*}
\]

$N=1$: All constants $a_i = 0$ ($i = 0, 1, 3$) in the above equations except $a_2$.

\[
\begin{align*}
  u_{t1} + v_x + \left(\frac{3}{2} + k\right)uu_x &= 0, \quad (2.15) \\
  v_{t1} - \frac{1}{4}u_{xxx} + (uv)_x - \left(\frac{1}{2} + k\right)uv_x - k\left(\frac{1}{2} + k\right)u^2u_x &= 0 \quad (2.16)
\end{align*}
\]

$N=2$: All constants $a_i = 0$ ($i = 0, 2, 3$) in the above equations except $a_1$.

\[
\begin{align*}
  u_{t2} &= \frac{a_1}{32} [-12uu_{xxx} - (36 + 24k)u_xu_{xx} \\
  &\quad - 24k(2k + 1)u^3u_x + 48uvu_x - 8u_{xxx} + 12(1 - 2k)u^2v_x + 48vv_x], \quad (2.17) \\
  v_{t2} &= \frac{a_1}{32} [-12uu_{xxx} - (36 + 24k)u_xu_{xx} \\
  &\quad - 24k(2k + 1)u^3u_x + 48uvu_x - 8u_{xxx} + 12(1 - 2k)u^2v_x + 48vv_x] \quad (2.18)
\end{align*}
\]
N=3: All constants $a_i = 0 \ (i = 1, 2, 3)$ in the above equations except $a_0$.

$$u_t = \frac{a_0}{16} \left[-(4k + 10u)u_{xxx} - 12(k + 1)u_xu_{xx} + (12k^2 + 60k + 35)u^3u_xight. \\
+ (24k + 60)uvu_x + (12k + 30)u^2v_x + 24vv_x], \\
$$

$$v_t = \frac{a_0}{32} \left[2u_{xxxx} + (8k^2 - 2k - 15)u^2u_{xxx} - 20vu_{xxx} + (24k^2 - 72k - 90)uv_xight. \!
\!
\! - u_{xx} - 40v_xu_{xx} - (36k + 30)(u_x)^3 - 36u_xv_{xx} - (24k^3 + 72k^2 + 30k)u^4u_x \!
\!
\! + (60 - 48k^2)vu^2u_x + 48v^2u_x + (8k - 12)uw_{xxx} - (24k^2 + 24k - 10)u^3v_x \!
\!
\! + (-48k + 72)uvv_x]$$

By defining a new variable $\bar{v} = v + \frac{k}{2} u^2$ it is possible to eliminate $k$ dependence of the equations (2.3) and (2.4). The new equations are

$$u_t + \bar{v}_x + \frac{3}{2} uu_x = 0, \quad (2.21)$$
$$v_t - \frac{1}{4} uu_{xx} + \bar{v}u_x + \frac{1}{2} u\bar{v}_x = 0. \quad (2.22)$$

Although Eqs. (2.3) and (2.4) are equivalent, for any value of $k$, to the above system of evolution equations corresponding to $k = 0$, we will keep them in the sequel because the case with $k = -\frac{1}{2}$ corresponds to the Kaup-Boussinesq (KB) equations [14]

$$u_t + v_x + uu_x = 0, \quad (2.23)$$
$$v_t - \frac{1}{4} uu_{xx} + (uv)_x = 0. \quad (2.24)$$

KB equation later studied by several authors [15], [16]. We shall call the system of evolution equations obtained here as generalized KB equations. In section 3 we shall find $\ell = 3$ (three dependent variables) system of KB equations. In section 4 we shall show that, for any $\ell$, the KB systems turn out to be the Fokas-Liu extension of the Svinolopov KdV systems [9], [10].

3 Hierarchy and the Recursion Operator of Two Field KB Equations

The first three members of the hierarchy are given in the previous section. Here in this section we shall determine the full hierarchy by computing the recursion operator of the three field KB equations. Let the function $A$ in (4.4) be an analytic function of $\lambda$, then

$$A = \sum_{n=0}^{N} A_n \lambda^{N-n} \quad (3.1)$$
where \(A_n\)'s are functions of \(u, v\) and their \(x\) partial derivatives. We obtain

\[
2u_{tN} - 2(2uD + u_x)A_N + M_2A_{N-1} = 0, \quad (3.2)
\]

\[
2v_{tN} + [M_2 + 2ku(2uD + u_x)]A_N - kuMA_{N-1} = 0, \quad (3.3)
\]

and the recursion relations among \(A_m, (0 \leq m \leq N - 2)\).

\[
M_2A_m + 4A_{m+2} - 2(2uD + u_x)A_m + 1 = 0, \quad 0 \leq m \leq N - 2 \quad (3.4)
\]

with \(A_0 = a_0 = \text{constant and } A_1 = \frac{1}{2}a_0u\) and so on. In the above expressions \(D\) is the differential operator with respect \(x\) and

\[
M_2 = D^3 - 2(2v + ku^2)D - 2(kuu_x + v_x) \quad (3.5)
\]

Using (3.2), (3.3) and the recursion relations (3.4) we obtain that

\[
\left(\begin{array}{c}
u_{tN} \\
v_{tN}
\end{array}\right) = R\left(\begin{array}{c}
u_{tN-1} \\
v_{tN-1}
\end{array}\right), \quad (3.6)
\]

where \(R\) is the recursion operator of the hierarchy

\[
R = \left(\begin{array}{cc}
-(k + 1)u - \frac{1}{2}u_xD^{-1} & -1 \\
\frac{1}{4}D^2 - v + \left(\frac{1}{2} + k\right)ku^2 - \frac{1}{2}v_xD^{-1} & ku
\end{array}\right) \quad (3.7)
\]

Then the hierarchy of the shallow water wave equations are given by

\[
\left(\begin{array}{c}
u_{tN} \\
v_{tN}
\end{array}\right) = R^N\left(\begin{array}{c}
u_x \\
v_x
\end{array}\right), \quad N = 1, 2, \cdots \quad (3.8)
\]

To prove that the operator in (3.7) is the recursion operator we write the coupled evolution equations (2.3) and (2.4) as

\[
\left(\begin{array}{c}
u_t \\
v_t
\end{array}\right) = K = \left(\begin{array}{c}
v_x - \left(\frac{3}{2} + k\right)uu_x \\
\frac{1}{4}u_{xxx} - (uv)_x + \left(\frac{1}{2} + k\right)uv_x + k\left(\frac{1}{2} + k\right)u^2u_x
\end{array}\right), \quad (3.9)
\]

Then

\[
\left(\begin{array}{c}
(\delta u)_t \\
(\delta v)_t
\end{array}\right) = K^*\left(\begin{array}{c}
(\delta u) \\
(\delta v)
\end{array}\right), \quad (3.10)
\]

where

\[
K^* = \left(\begin{array}{cc}
-(\frac{3}{2} + k)(u_x + uD) & -D \\
\frac{1}{4}D^3 + k(u_x + uD) & -(\frac{1}{2} - k)(v_x + uD - vD + 2k(u_x + uD) - (\frac{3}{2} + k)uu_x - (\frac{1}{2} - k)v_x
\end{array}\right) \quad (3.11)
\]

Then it is easy to show that the recursion operator for symmetries satisfies (see [19], [20])

\[
R_t = [K^*, R] \quad (3.12)
\]
We have the Hamilton operators

\[ \theta_0 = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \]  \quad (3.13)

\[ \theta_1 = \begin{pmatrix} -2D & \frac{1}{2}D^3 + (-2v + (1 + 2k)ku^2)D - v_x \\ 2ukD & -2(k + 1)uD - u_x \end{pmatrix} \]  \quad (3.14)

so that \( R = \theta_1 \theta_0^{-1} \).

4 Three field KB Equations

Let the Lax operator \( L \) be given by

\[ L = D^2 - \lambda^2 u + \lambda v + w \]  \quad (4.1)

where \( u, v, \) and \( w \) are functions of \( x \) and \( t \). Then the first part of the Lax equation is the eigen-value equation for \( L \)

\[ L \psi = -\lambda^3 \psi \]  \quad (4.2)

Similarly let the time evolution of \( \psi \) be given by

\[ \psi_t = A\psi_x + B\psi, \]  \quad (4.3)

where \( A \) and \( B \) depend on the dependent variables \( u, v \) and their derivatives with respect to \( x \) and also on the spectral parameter \( \lambda \). Compatibility of the Lax equations (4.2) and (4.3) gives \( B = -\frac{1}{2}A_x \) and

\[ A_{xxx} + [4\lambda^3 - 4\lambda^2 u - 4\lambda v - 4w]A_x - [2\lambda^2 u_x + 2\lambda v_x + 2w_x]A + 2\lambda^2 u_t + 2\lambda v_t + 2w_t = 0 \]  \quad (4.4)

One can solve this equation by assuming \( A \) as a polynomial of \( \lambda \) (analytic in \( \lambda \)). For instance, by assuming \( A \) as a third order polynomial in \( \lambda \), a solution of this equation is obtained as

\[ A = (a_0 [128\lambda^4 + 64\lambda^3 u + 16\lambda^2 (3u^2 + 4v) + 8\lambda (5u^3 + 12uv + 8w) - 16u_{xx} + 35u^4 + 120u^2 v + 96uv + 8w] + a_1 [128\lambda^3 + 64\lambda^2 u + 16\lambda (3u^2 + 4v) + 8(5u^3 + 12uv + 8w)] + a_2 [128\lambda^2 + 64\lambda u + 16(3u^2 + 4v)] + a_3 [128\lambda + 64u] + 128a4)/128 \]  \quad (4.5)

where \( a_0, a_1, a_2, a_3 \) and \( a_4 \) are arbitrary constant. The corresponding evolution equations are given as follows:

\( N=1. \) \( a_0 = a_1 = a_2 = a_4 = 0 \) and \( a_3 = 1. \)
\[ u_t = \frac{3}{2} uu_x + v_x, \quad (4.6) \]
\[ v_t = vu_x + \frac{1}{2} uv_x + w_x, \quad (4.7) \]
\[ w_t = -\frac{1}{4} u_{xxx} + wu_x + \frac{1}{2} uw_x \quad (4.8) \]

\( N=2 \). \( a_0 = a_1 = a_3 = a_4 = 0 \) and \( a_2 = 1 \).

\[ u_t = \frac{15}{8} u^2 u_x + \frac{3}{2} (uv)_x + w_x, \quad (4.9) \]
\[ v_t = -\frac{1}{4} u_{xxx} + \frac{3}{2} uv u_x + \frac{3}{8} 3u^2 v_x + wu_x + \frac{3}{2} vv_x + \frac{1}{2} uw_x, \quad (4.10) \]
\[ w_t = -\frac{3}{8} uu_{xxx} - \frac{1}{4} v_{xxx} - \frac{9}{8} u_x uu_x + wv x + \frac{3}{2} uv u_x + \frac{3}{8} u^2 w_x + \frac{1}{2} vw_x, \quad (4.11) \]

etc. To find all members the of hierarchy of three field evolution equations we let

\[ A = \sum_{n=0}^{N} A_n \lambda^{N-n} \quad (4.12) \]

where \( A_n \)'s are functions of \( u, v \) and \( w \) and their \( x \) partial derivatives. Then we obtain

\[ A_0 = a_0, \quad A_1 = \frac{a_0}{2} u + a_1, \quad A_2 = \frac{3a_0}{8} u^2 + \frac{1}{2} a_1 u + \frac{1}{2} a_0 v + a_2 \quad (4.13) \]

where \( a_0, a_1 \) and \( a_2 \) are arbitrary constants. The evolution equations are given as follows

\[ 2u_{tN} + M_3 A_{N-2} - 2(2uD + u_x) A_N - 2(2vD + v_x) A_{N-1} = 0, \quad (4.14) \]
\[ 2v_{tN} + M_3 A_{N-1} - 2(2vD + v_x) A_N = 0, \quad (4.15) \]
\[ 2w_{tN} + M_3 A_N = 0 \quad (4.16) \]

with the recursion relations

\[ M_3 A_n + 4A_{n+3, x} - 2(2uD + u_x) A_{n+2} - 2(2vD + v_x) a_{n+1} = 0 \quad (4.17) \]

where \( 0 \leq n \leq N - 3 \) and

\[ M_3 = D^3 - 4wD - 2w_x \quad (4.18) \]

It is straight forward to show

\[ \begin{pmatrix} u_{tN} \\ v_{tN} \\ w_{tN} \end{pmatrix} = R \begin{pmatrix} u_{t(N-1)} \\ v_{t(N-1)} \\ w_{t(N-1)} \end{pmatrix}, \quad N = 1, 2, \cdots \quad (4.19) \]
where \( R \) is the recursion operator

\[
R = \begin{pmatrix}
(uD + \frac{1}{2}u_x)D^{-1} & 1 & 0 \\
vD + \frac{1}{2}v_x)D^{-1} & 0 & 1 \\
-\frac{1}{4}M_3D^{-1} & 0 & 0
\end{pmatrix}
\] (4.20)

One can verify easily that \( R \) satisfies the equation (3.12) where \( K^* \) is given by

\[
K^* = \begin{pmatrix}
\frac{3}{2}uD + \frac{3}{2}u_x & D & 0 \\
vD + \frac{1}{2}v_x & \frac{1}{2}uD + u_x & 0 \\
-\frac{1}{4}D^3 + wD + \frac{1}{2}w_x & 0 & \frac{1}{2}uD + u_x
\end{pmatrix}
\] (4.21)

Then the hierarchy of the three field KB is given by

\[
\begin{pmatrix}
u_t.N \\
v_t.N^N \\
w_t.N^N
\end{pmatrix} = R^N \begin{pmatrix}
u_x \\
v_x \\
w_x
\end{pmatrix}, \quad N = 1, 2, \cdots
\] (4.22)

\( N = 1 \) gives the three field KB Equations and all \( N > 1 \) cases are generalized symmetries of the three field KB equations.

5 Multi-Component KB Equations

Multi-component KB equations can be obtained from the Lax operator

\[
L = D^2 - \sum_{k=1}^{\ell} \lambda^{k-1} q^k(x,t)
\] (5.1)

where \( q^k(x,t), \ (k = 1, 2, \cdots, \ell) \) are the multi-KB fields, as we did in the earlier sections. Here \( \ell \) is a positive integer greater or equal to two. The Lax equations in this case take the form

\[
\begin{align*}
L \psi &= -\lambda^\ell \psi, \\
\psi_t &= A \psi_x - \frac{1}{2} A_x \psi
\end{align*}
\] (5.2, 5.3)

where \( A \) is a polynomial of the spectral parameter \( \ell \). To obtain the multi-component KB equations and their recursion operators by the Lax operator given above for arbitrary positive integer \( \ell \) is very lengthy. Instead we shall make use our experience from the two-field and three field KB equations and their recursion operators in sections 2-4. The multi system of KB equations are given as follows.
\[ u_t = \frac{3}{2} u u_x + q_x^2, \]
\[ q_x^2 = q_x^2 u_x + \frac{1}{2} w q_x^2 + q_x^3, \]
\[ q_x^3 = q_x^3 u_x + \frac{1}{2} w q_x^3 + q_x^4, \]
\[ \vdots \]
\[ q_{x}^{\ell-1} = q_{x}^{\ell-1} u_x + \frac{1}{2} w q_{x}^{\ell-1} + w_x, \]
\[ w_t = -\frac{1}{4} u_{xxx} + w u_x + \frac{1}{2} u w_x, \]

where we took \( q^1 = u \) and \( q^\ell = w \). The recursion operator of the system can be given by

\[
R = \begin{pmatrix}
\begin{pmatrix} u + \frac{1}{2} u_x D^{-1} & 1 & 0 & 0 & \cdots & 0 \\
q_x^2 + \frac{1}{2} q_x^2 D^{-1} & 0 & 1 & 0 & \cdots & 0 \\
q_x^3 + \frac{1}{2} q_x^3 D^{-1} & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
q_{x}^{\ell-1} + \frac{1}{2} q_{x}^{\ell-1} D^{-1} & 0 & 0 & 0 & \cdots & 1 \\
-\frac{1}{4} M_{\ell} D^{-1} & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\end{pmatrix}
\]

(5.10)

where

\[
M_{\ell} = D^3 - 4 w D - 2 w_x
\]

(5.11)

It is not difficult to show that the KB equations (5.4)-(5.9) are integrable and the operator in (5.10) is the recursion operator of the system.

One can show that the recursion operator \( R \) satisfies the equation (3.12) where

\[
K^* = \begin{pmatrix}
\begin{pmatrix} \frac{3}{2} u_x + \frac{3}{2} u D & D & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{2} q_x^2 + q_x^2 D & u_x + \frac{1}{2} u D & D & 0 & \cdots & 0 & 0 \\
\frac{1}{2} q_x^3 + q_x^3 D & 0 & u_x + \frac{1}{2} u D & D & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{2} q_{x}^{\ell-1} + q_{x}^{\ell-1} D & 0 & 0 & 0 & \cdots & u_x + \frac{1}{2} u D & D \\
-\frac{1}{4} D^3 + \frac{1}{2} w_x + w D & 0 & 0 & 0 & \cdots & 0 & u_x + \frac{1}{2} u D
\end{pmatrix}
\end{pmatrix}
\]

(5.12)

Hence the KB equations (5.4)-(5.9) are integrable and the operator in (5.10) is the recursion operator of the system.

### 6 Multi-Component KB Equations as Svinolupov KdV System

The system of KB equations (5.4)-(5.9) and the corresponding recursion operator (5.10) look like the coupled integrable KdV systems studied earlier [9], [10]. Let \( q^\ell = (q^1, q^2, \cdots, q^\ell) \) be...
\( q^i_t = b^i_k q^k + S^i_{jk} q^j q^k_x + \chi^i_k q^k_x, \quad i = 1, 2, \ldots, \ell \) \tag{6.1}

where all repeated indices are summed up from 1 to \( \ell \). The coupled evolution equations (6.1) are integrable if the constants \( b^i_k \), \( S^i_{jk} \), and \( \chi^i_k \) satisfy certain conditions [9], [10]. If \( b^i_j = \delta^i_j \) and \( \chi^i_j = 0 \) and \( S^i_{jk} \) are the structure constants of a Jordan algebra then the system of equations (6.1) are integrable [3]-[4]. All other cases were studied in [9], [10]. In particular if \( \det(b) = 0 \) and \( \chi^i_j \neq 0 \) is known as the Fokas-Liu extension of the degenerate Svinolupov KdV systems [18], [10]. For the integrable cases the corresponding recursion operators are given by

\[
R^i_{jk} = b^i_j D^2 + A^i_{jk} q^k + C^i_{jk} q^k_x D^{-1} + w^i_j, \quad i, j = 1, 2, \ldots, \ell \tag{6.2}
\]

where \( A^i_{jk}, C^i_{jk}, w^i_j \) are given in terms of \( S^i_{jk} \) and \( \chi^i_j \), [9], [10]. The form of the recursion operator is valid for all cases discussed above.

The multi-components KB systems fall into the Fokas-Liu extension of the degenerate Svinolupov KdV system where the corresponding constants are given by

\[
\begin{align*}
    b^i_j &= -\frac{1}{4} \delta^i_k \delta^j_1, \quad \chi^i_j = w^i_j = \delta^i_{j-1}, \\
    A^i_{jk} &= \delta^i_j \delta^j_k, \quad B^i_{jk} = \frac{1}{2} A^i_{jk}, \\
    S^i_{jk} &= \delta^i_i \delta^i_j + \frac{1}{2} \delta^j_k \delta^j_1
\end{align*} \tag{6.3-6.5}
\]

where \( \delta^i_j \) is the Kronecker \( \delta \)-symbol and \( \det(b) = 0 \). Then the multi-component KB equations (5.4)-(5.9) take the form

\[
q^i_t = -\frac{1}{4} \delta^i_k u_{xxx} + q^i u_x + \frac{1}{2} u q^i_x + q^{i+1}_x, \quad i = 1, 2, \ldots, \ell \tag{6.6}
\]

where we took \( q^{\ell+1} = 0 \).

The hierarchy of multi-component KB equations is given by

\[
q^i_{1,N} = (R^N)^i_k q^k_x, \quad i = 1, 2, \ldots, \ell \tag{6.7}
\]

where \( N = 1, 2, \cdots \). The case \( N = 1 \) is the multi-component KB equations (6.6). All others \( N > 1 \) correspond the integrable family or higher generalized symmetries of multi-component KB equations. Components of the \( \ell \times \ell \) matrix recursion operator (5.10) which is compatible with (6.2), in index notation, is given by
\[
\mathbf{R}_{ij}^\ell = -\frac{1}{4} \delta^i_\ell \delta^1_j D^2 + \delta^1_j q^i + \frac{1}{2} \delta^1_j q_x^i D^{-1} + \delta^i_{j-1}, \quad i, j = 1, 2, \ldots, \ell \tag{6.8}
\]

with

\[
(K^*)_{ij}^\ell = -\frac{1}{4} \delta^i_\ell \delta^1_j D^3 + \delta^1_j (q^i D + \frac{1}{2} q_x^i) + \delta^i_j (u_x + \frac{1}{2} u D) + \delta^i_{j-1} D \quad i, j = 1, 2, \ldots, \ell \tag{6.9}
\]

The evolution equation (3.12) for the recursion operator for arbitrary \( \ell \) in index notation is given by

\[
\frac{d}{dt} \mathbf{R}_{ij}^\ell = (K^*)_{k}^i \mathbf{R}_{j}^k - \mathbf{R}_{k}^i (K^*)_{j}^k \quad i, j = 1, 2, \ldots, \ell \tag{6.10}
\]

It is easy to verify this equation with \( \mathbf{R} \) and \( K^* \) given in (6.8) and (6.9) respectively for arbitrary \( \ell \) provided that \( q^i \)’s satisfy the multi-component KB equations (6.6).

7 Concluding Remarks

We found the hierarchy (generalized symmetries) of the integrable two field and three field Kaup-Boussinesq equations. Integrability of these equations are supported by giving the corresponding recursion and compatible Hamilton operators.

We presented a new multi-component (arbitrary number of fields) generalization of the Kaup-Boussinesq equations. This system of equations is integrable. It has both a Lax representation and a symmetry recursion operator. It is also noticed that this system is a special case of the Fokas-Liu extension of the degenerate Svinolupov KdV system. Conservation laws and the bi-Hamiltonian structure of the multi-component KB equations will be communicated in a subsequent work.

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