Cameron-Liebler sets in Hamming graphs

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Abstract

In this paper, we discuss Cameron-Liebler sets in Hamming graphs, obtain several equivalent definitions and present all classification results.

AMS classification: 05E30, 51E30

Key words: Cameron-Liebler set, Hamming graph, Word

1 Introduction

Cameron-Liebler sets of lines were first introduced by Cameron and Liebler [3] in their study of collineation groups of $\text{PG}(3, q)$. There have been many results for Cameron-Liebler sets of lines in the projective space $\text{PG}(3, q)$. See [14, 15, 16] for classification results, and [2, 4, 8, 10] for the constructions of two non-trivial examples. Over the years, there have been many interesting extensions of this result. See [1, 11, 13, 17] for Cameron-Liebler sets of $k$-spaces in $\text{PG}(n, q)$, [5] for Cameron-Liebler sets of generators in polar spaces, and [6] for Cameron-Liebler classes in finite sets.

One of the main reasons for studying Cameron-Liebler sets is that there are many connections to other geometric and combinatorial objects, such as blocking sets, intersecting families, linear codes, and association schemes. Filmus and Ihringer [9] investigated recently Cameron-Liebler sets for several classical distance-regular graphs, including Johnson graphs, Grassmann graphs, dual polar graphs, and bilinear forms graphs. Their research stimulates us to consider Cameron-Liebler sets in Hamming graphs.

For positive integers $n$ and $q$ with $q \geq 2$, let $[n] = \{1, 2, \ldots, n\}$ and $Q$ be an alphabet of $q$ symbols. For $i = 1, 2, \ldots, n$, a pair $(A, f)$ is called an $i$-word if $A$ is an $i$-element subset of $[n]$ and $f$ is a function from $A$ to $Q$. In particular, $n$-words are called words. Let $\mathcal{M}(i; n, q)$ be the collection of all $i$-words. For $(A, f) \in \mathcal{M}(i; n, q)$ and $(B, g) \in \mathcal{M}(j; n, q)$, we say that $(B, g)$ contains $(A, f)$, denoted by $(A, f) \preceq (B, g)$, if $A \subseteq B$ and $g|_A = f$, where $g|_A$ is the restriction of $g$ on $A$.

For convenience, we write $\mathcal{M}(i; n, q)$ as $\mathcal{M}_i$ for $i = 1, 2, \ldots, n$. Let $1 \leq i \leq j \leq n$. For a fixed $(B, g) \in \mathcal{M}_j$, let $\mathcal{M}_i(B, g)$ be the collection of all $i$-words contained in $(B, g)$. For a fixed $(A, f) \in \mathcal{M}_i$, let $\mathcal{M}_j(A, f)$ be the collection of all $j$-words containing $(A, f)$. Two words $([n], f)$ and $([n], g)$ are called intersecting if there exists some $(\{a\}, h) \in \mathcal{M}_1$ such that $(\{a\}, h) \preceq ([n], f)$ and $(\{a\}, h) \preceq (B, g)$. For a fixed $([n], f) \in \mathcal{M}_n$, let $\overline{\mathcal{M}}_n([n], f)$ be the collection of all words disjoint to $([n], f)$.

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The Hamming graph $H(n, q)$ has the vertex set $M_n$, and two vertices $([n], f)$ and $([n], g)$ are adjacent if $|(\{a \in [n] : f(a) \neq g(a)\})| = 1$. The Hamming graph $H(n, q)$ is a distance-transitive graph with $q^n$ vertices and diameter $n$. Note that the distance between two vertices $([n], f)$ and $([n], g)$ is $n - |\{a \in [n] : f(a) \neq g(a)\}|$.

We always assume that vectors are regarded as column vectors. For any vector $\alpha$ whose positions correspond to elements in a set, we denote its value on the position corresponding to an element $a$ by $(\alpha)_a$. The characteristic vector $\chi_S$ of a subset $S$ of $M_n$ is the vector whose positions correspond to the elements of $M_n$, such that $(\chi_S)_{([a], f)} = 1$ if $([a], f) \in S$ and 0 otherwise. The all-one vector will be denoted by $1$. Let $M$ be the incidence matrix with rows indexed with $M_1$ and columns indexed with $M_n$ such that entry $M_{([a], f), ([n], g)} = 1$ if and only if $([n], g)$ contains $(A, f)$.

A subset $L$ of $\mathcal{M}_n$ is called a Cameron-Liebler set in the Hamming graph $H(n, q)$ with parameter $x = q^{-(n-1)}|L|$ if $\chi_L \in \text{Im}(M^t)$, where $M^t$ is the transpose of $M$. A family $\mathcal{F} \subseteq \mathcal{M}_n$ is called intersecting if $([n], f)$ and $([n], g)$ are intersecting for all $([n], f), ([n], g) \in \mathcal{F}$. A Cameron-Liebler set $L$ in $H(n, q)$ with parameter $x$ is called trivial if $L$ is a union of $x$ intersecting families.

In this paper, we consider Cameron-Liebler sets in the Hamming graph $H(n, q)$. The rest of this paper is structured as follows. In Section 2, we give several equivalent definitions for these Cameron-Liebler sets in $H(n, q)$. In Section 3, we obtain several properties of these Cameron-Liebler sets in $H(n, q)$. By using the properties, we give the following classification result: If $q \geq 2$, then any non-empty Cameron-Liebler set in $H(n, q)$ is trivial.

## 2 Several equivalent definitions

For $([a], f), ([b], g) \in \mathcal{M}_1$ with $a \neq b$, let $([a], f) \lor ([b], g) = ([a, b], h)$, where $h(a) = f(a)$ and $h(b) = g(b)$. Let $G$ be the graph with the vertex set $\mathcal{M}_1$, and two vertices $([a], f)$ and $([b], g)$ are adjacent if $([a], f) \lor ([b], g) \in \mathcal{M}_2$. Then $G$ is a complete $n$-partite graph.

**Lemma 2.1** (See Lemma 2 in [1]) The distinct eigenvalues of $G$ are $q(n-1), 0$ and $-q$, and the corresponding multiplicities are $1, (q-1)n$ and $n-1$, respectively.

**Lemma 2.2** Let $1 \leq i \leq j \leq n$. Then the following hold:

(i) The size of $\mathcal{M}_i$ is $q^i \binom{n}{i}$.

(ii) For a fixed $(B, g) \in \mathcal{M}_j$, the size of $\mathcal{M}_i(B, g)$ is $\binom{j}{i}$.

(iii) For a fixed $(A, f) \in \mathcal{M}_i$, the size of $\mathcal{M}_j'(A, f)$ is $q^{j-i} \binom{n-i}{j-i}$.

(iv) For a fixed $([n], f) \in \mathcal{M}_n$, the size of $\mathcal{M}_i([n], f)$ is $(q-1)^n$.

(v) For fixed $([a], f) \in \mathcal{M}_1$ and $([n], g) \in \mathcal{M}_n$, the size of $\mathcal{M}_i([a], f) \cap \mathcal{M}_i([n], g)$ is 0 if $([a], f) \not\leq ([n], g)$, and $(q-1)^{n-1}$ otherwise.

**Lemma 2.3** The rank of the incidence matrix $M$ is $(q-1)n + 1$ over the real field $\mathbb{R}$.

**Proof.** Let $N = MM^t$. Then both $N$ and $M$ have the same rank over $\mathbb{R}$. Note that $N$ is a $qn \times qn$ matrix with rows and columns indexed with the elements in $\mathcal{M}_1$. For any
$(\{a\}, f), (\{b\}, g) \in \mathcal{M}_1$, the entry $N_{\{(a), f\},\{(b), g\}}$ is the number of elements in $\mathcal{M}_n$ containing both $(\{a\}, f)$ and $(\{b\}, g)$. By Lemma 2.2 (iii), we have

$$N_{\{(a), f\},\{(b), g\}} = \begin{cases} q^{n-1} & \text{if } (\{a\}, f) = (\{b\}, g), \\ q^{n-2} & \text{if } (\{a\}, f) \lor (\{b\}, g) \in \mathcal{M}_2, \\ 0 & \text{otherwise}, \end{cases}$$

which implies that

$$N = q^{n-1}I + q^{n-2}A, \quad (1)$$

where $I$ is the identity matrix of order $qn$, and $A$ is the adjacency matrix of the graph $G$. By Lemma 2.1 and 1, we obtain that the distinct eigenvalues of $N$ are $q^{n-1}n, q^{n-1}0$, and the corresponding multiplicities are $1, (q - 1)n$ and $n - 1$, respectively. It follows that the rank of $N$ is $(q - 1)n + 1$.

The Hamming Kneser graph $HK(n, q)$ has the vertex set $\mathcal{M}_n$, and two vertices $(\{a\}, f)$ and $(\{b\}, g)$ are adjacent if $(\{a\}, f)$ and $(\{b\}, g)$ are disjoint. Let $K$ be the adjacent matrix of $HK(n, q)$. The eigenvalues and the dimensions of the eigenspaces of $HK(n, q)$ is described in [12].

**Lemma 2.4** (See Theorem 10.1.2 in [12].) The distinct eigenvalues of $HK(n, q)$ are $\lambda_j = (-1)^j(q - 1)^{n-j}$, $j = 0, 1, \ldots, n$, and the eigenspace $V_j$ corresponding to $\lambda_j$ has dimension $(q - 1)^j(n-j)$.

**Lemma 2.5** $\text{Im}(M^f) = V_0 \oplus V_1$, where $V_0 = \langle j \rangle$.

**Proof.** By Lemma 2.3 the matrix $M$ has $qn$ rows with rank($M$) = $(q - 1)n + 1$. By Lemma 2.4, $\dim V_1 = (q - 1)n$, and therefore $\dim(V_0 \oplus V_1) = \dim \text{Im}(M^f)$.

From Lemma 2.2 (iv) and (v), we deduce that $K\chi_{\mathcal{M}_n^f(\{a\}, f)} = (q - 1)^{n-1}(j - \chi_{\mathcal{M}_n^f(\{a\}, f)})$ and $K_j = (q - 1)^n j$, which imply that

$$K(\chi_{\mathcal{M}_n^f(\{a\}, f)} - q^{-1}j) = -(q - 1)^{n-1}(\chi_{\mathcal{M}_n^f(\{a\}, f)} - q^{-1}j).$$

It follows that $\chi_{\mathcal{M}_n^f(\{a\}, f)} - q^{-1}j \in V_1$. Therefore, we have $\chi_{\mathcal{M}_n^f(\{a\}, f)} \in V_0 \oplus V_1$. Since $\chi_{\mathcal{M}_n^f(\{a\}, f)}$ is the column of $M^f$ corresponding to the element $(\{a\}, f)$, we have $\text{Im}(M^f) \subseteq V_0 \oplus V_1$. From $\dim(V_0 \oplus V_1) = \dim \text{Im}(M^f)$, we deduce that $\dim \text{Im}(M^f) = V_0 \oplus V_1$.

The incidence vector $v_S$ of a subset $S$ of $\mathcal{M}_1$ is the vector whose positions correspond to the elements of $\mathcal{M}_1$, such that $(v_S)_\tau = 1$ if $\tau \in S$ and 0 otherwise.

**Lemma 2.6** Let $(\{a\}, f) \in \mathcal{M}_n$. Then

$$\chi_{\mathcal{M}_n^f(\{a\}, f)} - (q - 1)^{n-1}(q^{-(n-1)}j - \chi(\{\{a\}, f\})) \in \ker(M).$$

**Proof.** By Lemma 2.2 (v), we have

$$M\chi_{\mathcal{M}_n^f(\{a\}, f)} = (q - 1)^{n-1}(j - v_{\mathcal{M}_1(\{a\}, f)}).$$

By $Mj = q^{n-1}j$ and $M\chi(\{\{a\}, f\}) = v_{\mathcal{M}_1(\{a\}, f)}$, we obtain

$$M\chi_{\mathcal{M}_n^f(\{a\}, f)} = (q - 1)^{n-1}(q^{-(n-1)}Mj - M\chi(\{\{a\}, f\})).$$
By Lemma 2.6, we have
\[ K_{j} = (q - 1)^{n-1}(q^{-(n-1)}j - \chi([(\{1\}, f)]) \in \ker(M), \]
as desired. \hfill \Box

An \textit{n-partition} of \(M_{1}\) is a set \(P\) of words in \(M_{n}\) such that any one-word is contained exactly in one word of \(P\). Let \(P\) be an \(n\)-partition of \(M_{1}\). Every subset of \(P\) is called a \textit{partial \(n\)-partition} of \(M_{1}\). A \textit{pair of conjugate switching sets} is a pair of disjoint \(n\)-partitions of \(M_{1}\) that cover the same subset of \(M_{1}\).

Now, we give several equivalent definitions for a Cameron-Liebler set in \(M_{n}\).

\textbf{Theorem 2.7} Let \(L\) be a non-empty set in \(M_{n}\) with \(|L| = xq^{n-1}\). Then the following properties are equivalent.

(i) \(\chi_{L} \in \text{Im}(M^{t})\).
(ii) \(\chi_{L} \in \ker(M)^{\perp}\).
(iii) For every \([n], f) \in M_{n}\), the number of elements in \(L\) disjoint to \([n], f\) is \((q - 1)^{n-1}(x - (\chi_{L})([n], f))\).
(iv) The vector \(v = \chi_{L} - xq^{-1}j\) is a vector in \(V_{1}\).
(v) \(\chi_{L} \in V_{0} \oplus V_{1}\).
(vi) \(|L \cap P| = x\) for every \(n\)-partition \(P\) of \(M_{1}\).
(vii) For every pair of conjugate switching sets \(R\) and \(R'\), we have \(|L \cap R| = |L \cap R'|\).

\textbf{Proof.} (i) \(\Leftrightarrow\) (ii): Since \(\text{Im}(M^{t}) = \ker(M)^{\perp}\), the desired result follows.
(ii) \(\Rightarrow\) (iii): Let \([n], f) \in M_{n}\). By Lemma 2.6 we obtain
\[ \chi_{M_{n}([n], f)} = (q - 1)^{n-1}(q^{-(n-1)}j - \chi([(\{1\}, f)]) \in \ker(M). \]

Since \(\chi_{L} \in \ker(M)^{\perp}\), we have
\[ \chi_{M_{n}([n], f)} \cdot \chi_{L} - (q - 1)^{n-1}(q^{-(n-1)}j \cdot \chi_{L} - \chi([(\{1\}, f)]) \cdot \chi_{L}) = 0 \]
\[ \Leftrightarrow |M_{n}([n], f) \cap L| - (q - 1)^{n-1}(q^{-(n-1)}|L| - (\chi_{L})([n], f)) = 0 \]
\[ \Leftrightarrow |M_{n}([n], f) \cap L| = (q - 1)^{n-1}(x - (\chi_{L})([n], f)). \]

The last equality shows that the desired result follows.
(iii) \(\Rightarrow\) (iv): From (iii), we deduce that
\[ K\chi_{L} = (q - 1)^{n-1}(xj - \chi_{L}) = -\lambda_{1}(xj - \chi_{L}). \]
By Lemma 2.6 we have \(K_{j} = -\lambda_{1}(q - 1)j\), and therefore
\[ K \chi_{L} = -\lambda_{1}(xj - \chi_{L}) + \lambda_{1}(q - 1)xq^{-1}j = \lambda_{1}(\chi_{L} + x((q - 1)q^{-1} - 1)j) = \lambda_{1}(\chi_{L} - xq^{-1}j) = \lambda_{1}v. \]
By Lemma 2.4, we obtain \( v \in V_1 \).

(iv) \( \Rightarrow (\nu) \): From \( V_0 = (j) \), we deduce that the desired result follows.

(v) \( \Rightarrow (i) \): By Lemma 2.5, the desired result follows.

Now we show that the property (vi) is also equivalent to the other properties.

(ii) \( \Rightarrow (vi) \): Let \( P \) be an \( n \)-partition of \( M_1 \). Since \( M \chi_P = j \), by Lemma 2.2 (iii), \( \chi_P - q^{-(n-1)} j \in \ker(M) \). Since \( \chi_L \in \ker(M)^\perp \), we have

\[
0 = \chi_L \cdot (\chi_P - q^{-(n-1)} j) = |L \cap P| - q^{-(n-1)} |L|,
\]

which implies that \( |L \cap P| = q^{-(n-1)} |L| = x \).

(vi) \( \Rightarrow (iii) \): Let \( \ell_i \), for \( i = 1, 2 \), be the number of \( n \)-partitions of \( M_1 \) that contain \( i \) fixed pairwise disjoint elements in \( M_n \). Since the Hamming graph \( H(n, q) \) is distance-transitive, this number only depends on \( i \), and not on the chosen elements.

For a fixed \( ([n], f) \in M_n \), if we count the number of couples \( ([n], g, P) \), where \( ([n], g) \in M_n \) such that \( ([n], f) \) and \( ([n], g) \) are disjoint and \( P \) is an \( n \)-partition of \( M_1 \) containing \( ([n], f) \) and \( ([n], g) \), by Lemma 2.2 (iv), we have \( \ell_1(q - 1) = \ell_2(q - 1)^n \), which implies that \( \ell_1 / \ell_2 = (q - 1)^{n-1} \).

For a fixed \( ([n], f) \in M_n \), if we count the number of couples \( ([n], g, P) \), where \( ([n], g) \in L \) such that \( ([n], f) \) and \( ([n], g) \) are disjoint and \( P \) is an \( n \)-partition of \( M_1 \) containing \( ([n], f) \) and \( ([n], g) \), then the number of subspaces in \( L \) disjoint to \( ([n], f) \) is

\[
(x - (\chi_L)([n], f)) \frac{\ell_1}{\ell_2} = (x - (\chi_L)([n], f))(q - 1)^{n-1}.
\]

Next, we show that property (vii) is equivalent with the other properties.

(ii) \( \Rightarrow (vii) \): Since \( R \) and \( R' \) cover the same subset of \( M_1 \), we have \( \chi_R - \chi_{R'} \in \ker(M) \), which implies that \( \chi_L \cdot (\chi_R - \chi_{R'}) = \chi_L \cdot \chi_R - \chi_L \cdot \chi_{R'} = 0 \). It follows that \( |L \cap R| = \chi_L \cdot \chi_R = |L \cap R'| \).

(vii) \( \Rightarrow (vi) \): For any two \( n \)-partitions \( P_1 \) and \( P_2 \) of \( M_1 \), the the sets \( P_1 \setminus P_2 \) and \( P_2 \setminus P_1 \) form a pair of conjugate switching sets. So \( |L \cap (P_1 \setminus P_2)| = |L \cap (P_2 \setminus P_1)| \), which implies that \( |L \cap P_1| = |L \cap P_2| = c \).

Now we prove \( c = x = |L|q^{-(n-1)} \). Let \( \ell_i \), for \( i = 0, 1 \), be the number of \( n \)-partitions of \( M_1 \) that contain \( i \) fixed pairwise disjoint elements in \( M_n \). We count the number of couples \( ([n], f, P) \), where \( ([n], f) \in M_n \) and \( P \) is an \( n \)-partition of \( M_1 \) containing \( ([n], f) \), then \( \ell_0q = \ell_1q^n \), which implies that \( \ell_0 / \ell_1 = q^{n-1} \).

By counting the number of couples \( ([n], f, P) \), where \( ([n], f) \in L \) and \( P \) is an \( n \)-partition of \( M_1 \) containing \( ([n], f) \), then the number of elements in \( L \cap S \) equals \( |L| \frac{\ell_0}{\ell_1} = |L|q^{-(n-1)} = x \). \( \square \)

3 Classification results

In this section, we give some examples and list all classification results for Cameron-Liebler sets in the Hamming graph \( H(n, q) \). We begin with a simple lemma.

**Lemma 3.1** Let \( L \) and \( L' \) be two Cameron-Liebler sets in \( H(n, q) \) with parameters \( x \) and \( x' \), respectively. Then the following hold:

(i) \( 0 \leq x \leq q \).
(ii) The set of all elements in \( M_n \) not in \( L \) is a Cameron-Liebler set in \( H(n, q) \) with parameter \( q - x \).

(iii) If \( L \cap L' = \emptyset \), then \( L \cup L' \) is a Cameron-Liebler set in \( H(n, q) \) with parameter \( x + x' \).

(iv) If \( L' \subseteq L \), then \( L \setminus L' \) is a Cameron-Liebler set in \( H(n, q) \) with parameter \( x - x' \).

Now, we give some examples of Cameron-Liebler sets in \( H(n, q) \).

Lemma 3.2  
(i) For a fixed element \( \{a\}, f \in M_1 \), the set \( M'_n(\{a\}, f) \) is a Cameron-Liebler set in \( H(n, q) \) with parameter 1.

(ii) For each integer \( x \) with \( 0 \leq x \leq q \), there exists a Cameron-Liebler set in \( H(n, q) \) with parameter \( x \).

Proof.  
(i). Since the characteristic vector \( \chi_{M'_n(\{a\}, f)} \) is the row of \( M \) corresponding to the element \( \{a\}, f \), by Theorem 2.7 (i), the set \( M'_n(\{a\}, f) \) is a Cameron-Liebler set in \( H(n, q) \) with parameter 1.

(ii). First note that a Cameron-Liebler set in \( H(n, q) \) with parameter 0 is the empty set. Let \( \{\{a\}, f_1\}, \{\{a\}, f_2\}, \ldots, \{\{a\}, f_q\} \) be \( q \) pairwise different elements in \( M_1 \). Then \( M'_n(\{a\}, f_i) \cap M'_n(\{a\}, f_j) = \emptyset \) for all \( i \neq j \), and the set \( M'_n(\{a\}, f_i) \) is a Cameron-Liebler set in \( H(n, q) \) with parameter 1 for each \( i \). By Lemma 3.2 (iii), the set \( \bigcup_{i=1}^q M'_n(\{a\}, f_i) \) is a Cameron-Liebler set in \( H(n, q) \) with parameter \( x \).

Next, we give all classification results for Cameron-Liebler sets in \( H(n, q) \). We need the following definition.

Let \( \omega_S \) be a vector over \( \mathbb{R} \) whose positions correspond to the elements of a finite set \( S \). The spectrum of \( \omega_S \) is the set \( \{\omega_S\}_s : s \in S \} := \text{spec}(\omega_S) \), and the spectrum volume of \( \omega_S \) is the size of the set \( \text{spec}(\omega_S) \). Note that the spectrum volume of the characteristic vector of each Cameron-Liebler set in \( H(n, q) \) is at most 2.

Lemma 3.3  
Let \( \omega \in \text{Im}(M^t) \) with \( |\text{spec}(\omega)| = 2 \). Then there exist \( d, k \in \mathbb{R} \) with \( d \neq 0 \), and \( t < q \) pairwise different elements \( \{\{a\}, f_{i_1}\}, \ldots, \{\{a\}, f_{i_t}\} \) in \( M_1 \) such that

\[
\omega = kj + d \sum_{i=1}^t \chi_{M'_n(\{a\}, f_i)}.
\]

Moreover, if \( \text{spec}(\omega) = \{0, 1\} \), then either \( \omega = \sum_{i=1}^t \chi_{M'_n(\{a\}, f_i)} \) or \( \omega = \sum_{i=t+1}^q \chi_{M'_n(\{a\}, f_i)} \).

Proof.  
For each \( a \in [n] \), let \( \{\{a\}, f_{a1}\}, \{\{a\}, f_{a2}\}, \ldots, \{\{a\}, f_{aq}\} \) be \( q \) pairwise different elements in \( M_1 \). Then there exist \( k_{ai} \in \mathbb{R} \) for all \( a \in [n] \) and \( 1 \leq i \leq q \), such that

\[
\omega = \sum_{a \in [n]} \sum_{i=1}^q k_{ai} \chi_{M'_n(\{a\}, f_{ai})}.
\]

Let

\[
k = \sum_{a \in [n]} k_a, \text{ where } k_a = \begin{cases} 
\min\{k_{ai} : 1 \leq i \leq q\} & \text{if } \prod_{i=1}^q k_{ai} \neq 0; \\
0 & \text{otherwise}.
\end{cases}
\]
For each $a \in [n]$, the set $K_a = \{ i : k_{ai} - k_a \neq 0 \}$ is a proper subset of $[q] = \{1,2,\ldots,q\}$, which implies that there exists some $i_a \in [q]$ such that $i_a \not\in K_a$. Let $A = \{ a \in [n] : K_a \neq \emptyset \}$.

Since $\sum_{i=1}^{q} \chi_{\mathcal{M}_x}(\{a\},f_{ai}) = j$ for each $a \in [n]$, we have

$$\omega - kj = \sum_{a \in [n]} \sum_{i=1}^{q} (k_{ai} - k_a) \chi_{\mathcal{M}_x}(\{a\},f_{ai}) = \sum_{a \in A} \sum_{i \in K_a} (k_{ai} - k_a) \chi_{\mathcal{M}_x}(\{a\},f_{ai}).$$

Next, we show $|A| = 1$. Suppose $A \ni \{ b, c \}$. Since $K_a \neq \emptyset$ for each $a \in A$, there exist some $i_b, i_c \in [q]$ such that $i_b \in K_b$ and $i_c \in K_c$. Let

$$f(a) = \hat{i}_a \quad \text{for each } a \in [n],$$

$$g_1(b) = i_b \quad \text{and } g_1(a) = \hat{i}_a \quad \text{for each } a \in [n] \setminus \{b\},$$

$$g_2(c) = i_c \quad \text{and } g_2(a) = \hat{i}_a \quad \text{for each } a \in [n] \setminus \{c\},$$

$$g(b) = i_b, g(c) = i_c \quad \text{and } g(a) = \hat{i}_a \quad \text{for each } a \in [n] \setminus \{b, c\}.$$}

Then

$$\begin{align*}
(\omega - kj)([n],f) &= 0, \\
(\omega - kj)([n],g_1) &= k_{bi} - k_b, \\
(\omega - kj)([n],g_2) &= k_{ci} - k_c, \\
(\omega - kj)([n],g) &= k_{bi} - k_b + k_{ci} - k_c,
\end{align*}$$

which imply that $|\text{spec}(\omega - kj)| \geq 3$. Since $\text{spec}(\omega - kj) = \{b - k : b \in \text{spec}(w)\}$, we have $|\text{spec}(\omega - kj)| = |\text{spec}(w)| = 2$, a contradiction. So, we complete the proof of $|A| = 1$.

Let $A = \{a\}$. Then

$$\omega - kj = \sum_{i \in K_a} (k_{ai} - k_a) \chi_{\mathcal{M}_x}(\{a\},f_{ai}).$$

Since $\sum_{i=1}^{q} \chi_{\mathcal{M}_x}(\{a\},f_{ai}) = j$, we have $\text{spec}(\omega - kj) = \{0\} \cup \bigcup_{i \in K_a} \{k_{ai} - k_a\}$, which implies that there exists some $d$ such that $d = k_{ai} - k_a$ for each $i \in K_a$ since $|\text{spec}(\omega - kj)| = 2$. Then $w = kj + d \sum_{i \in K_a} \chi_{\mathcal{M}_x}(\{a\},f_{ai})$. Therefore, the desired result follows.

If $\text{spec}(\omega) = \{0, 1\}$, by the conclusion above, we have $\text{spec}(\omega) = \{k, k+d\} = \{0, 1\}$, then either $k = 0$ and $d = 1$, which imply that $\omega = \sum_{i \in K_a} \chi_{\mathcal{M}_x}(\{a\},f_{ai})$; or $k = 1$ and $d = -1$, which imply that $w = j - \sum_{i \in K_a} \chi_{\mathcal{M}_x}(\{a\},f_{ai}) = \sum_{i \in [q] \setminus K_a} \chi_{\mathcal{M}_x}(\{a\},f_{ai})$. \hfill\□

**Theorem 3.4** Let $1 \leq x \leq q - 1$. Then any Cameron-Liebler set in $H(n, q)$ with parameter $x$ is trivial.

**Proof.** Let $\mathcal{L}$ be a Cameron-Liebler set in $H(n, q)$ with parameter $x$. Then $\text{spec}(\chi_{\mathcal{L}}) = \{0, 1\}$. By Lemma 3.3 there exist $t (< q)$ pairwise different elements $\{\{a\}, f_1\}, \ldots, \{\{a\}, f_t\}$ in $\mathcal{M}_1$ such that $\omega = \sum_{i=1}^{t} \chi_{\mathcal{M}_x}(\{a\},f_i)$. Then $\mathcal{L} = \bigcup_{i=1}^{t} \mathcal{M}_x(\{a\}, f_i)$. Since $|\mathcal{L}| = xq^{n-1} = tq^{n-1}$, we have $x = t$, and thus $\mathcal{L}$ is trivial. \hfill\□

By Lemma 3.2 (ii), the Cameron-Liebler set in $H(n, q)$ with parameter $x = q$ is a union of $q$ intersecting families, and therefore is trivial. By Theorem 3.4 we obtain the following classification result.

**Theorem 3.5** Let $q \geq 2$. Then any non-empty Cameron-Liebler set in $H(n, q)$ is trivial.
Acknowledgment

This research is supported by National Natural Science Foundation of China (11971146), Foundation of Langfang Normal University (LSLB201707, LSPY201819, LSPY201915), Scientific Research Innovation Team of Langfang Normal University, and the Key Programs of Scientific Research Foundation of Hebei Educational Committee (ZD2019056).

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