Elliptic Rydberg States as Direction Indicators

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The orientation in space of a Cartesian coordinate system can be indicated by the two vectorial constants of motion of a classical Keplerian orbit: the angular momentum and the Laplace-Runge-Lenz vector. In quantum mechanics, the states of a hydrogen atom that mimic classical elliptic orbits are the coherent states of the SO(4) rotation group. It is known how to produce these states experimentally. They have minimal dispersions of the two conserved vectors and can be used as direction indicators. We compare the fidelity of this transmission method with that of the idealized optimal method.

I. UNSPEAKABLE QUANTUM INFORMATION

Information theory usually deals with the transmission of a sequence of discrete symbols, such as 0 and 1. Even if the information to be transmitted is of continuous nature, such as the position of a particle, it can be represented with arbitrary accuracy by a string of bits. However, there are situations where information cannot be encoded in such a way. For example, the emitter (conventionally called Alice) wants to indicate to the receiver (Bob) a direction in space. If they have a common coordinate system to which they can refer, or if they can create one by observing distant fixed stars, Alice simply communicates to Bob the components of a unit vector \( \mathbf{n} \) along that direction, or its spherical coordinates \( \theta \) and \( \phi \). But if no common coordinate system has been established, all she can do is to send a real physical object, such as a gyroscope, whose orientation is deemed stable.

In the quantum world, the role of the gyroscope is played by a system with large spin. For example, Alice can send angular momentum eigenstates satisfying \( \mathbf{n} \cdot \mathbf{J} |\psi\rangle = j |\psi\rangle \). This is essentially the solution proposed by Massar and Popescu [1], who took \( N \) parallel spins, polarized along \( \mathbf{n} \). This, however, is not the most efficient procedure: for two spins, a higher accuracy is achieved by preparing them with opposite polarizations [2]. For more than two spins, optimal results are obtained with entangled states [3, 4].

The above discussion can be generalized to the transmission of a Cartesian frame. If \( N \) spins are available, one can encode a Cartesian frame in an entangled state of these spins, as in [7]. However, a more accurate transmission is then obtained if Alice uses half of the spins to indicate the \( x \)-axis, and the other half for her \( y \)-axis [5]. In this case the \( x \) and \( y \) directions found by Bob may not be exactly perpendicular and some adjustment will be needed to obtain Bob’s best estimate of the \( x \) and \( y \) axes. Finally, the \( z \)-axis can be inferred from the estimates of the \( x \) and \( y \) axes.

However, it is not possible to proceed in this way if a single quantum messenger is available. The optimal transmission of a Cartesian frame by a hydrogen atom (formally, a spinless particle in a Coulomb potential) was derived by Peres and Scudo [7]. The results of [3] can also be used for a hydrogen atom if one considers the angular momentum eigenstates to be those of the atom, rather than those of \( N \) spins.

In this paper we show how to transmit a Cartesian frame by using the elliptic Rydberg states of a hydrogen atom. These are the quantum mechanical analog of a classical Keplerian orbit, and it is known how to produce these states experimentally. Elliptic Rydberg states, just as their classical counterparts, define three orthogonal directions in space, and thus are natural candidates for encoding a Cartesian frame.

In the following section we discuss the properties of quantum elliptic states. Section III deals with the transmission of one direction by means of them, and in Sec. IV we use them to transmit two orthogonal axes (and thus a Cartesian frame). In Sec. III and IV the detection procedure is based on SO(3) coherent states as in [7]. SO(4) coherent states are employed in Sec. V to produce a positive operator valued measure (POVM) which enables the use of elliptic states for the transmission of two directions that are not orthogonal. Even when these states are used by Alice to transmit two orthogonal axes, the two directions found by Bob are not necessarily perpendicular and further adjustment is needed, as explained above. As shown in the Appendix, these adjustments increase the fidelity of transmission of two orthogonal axes. However, a higher fidelity is achieved with a POVM based on the SO(3) rotation group, especially when the energy quantum number \( n \) is large.

II. CONSTRUCTION OF AN ELLIPTIC STATE

A classical bounded Keplerian orbit in a potential \(-k/r\) can be defined by its constants of motion: the energy \( E < 0 \), the angular momentum \( \mathbf{L} \) which is an axial vector perpendicular to the plane of the orbit, and the Laplace-Runge-Lenz (LRL) vector \( \mathbf{K} \)

\[
\mathbf{K} = (-2H)^{-1/2} (\mathbf{p} \times \mathbf{L} - \mu \mathbf{k} r / r),
\]

where \( \mu \) is the particle’s reduced mass, and we introduced a prefactor \((-2H)^{-1/2}\) for later convenience. This pref-
actor, which is a constant of motion, does not appear in the usual definition of the LRL vector \[8\]. The Hamiltonian is \[H = \mathbf{p}^2/2m - k/r\]. We consider only bounded motion for which the energy \(E\), which is the numerical value of \(H\), is negative. The classical orbit is then an ellipse, and the LRL vector is a polar vector directed along its major axis. It satisfies:

\[
\mathbf{L} \cdot \mathbf{K} = 0, \tag{2}
\]

and

\[
\mathbf{K}^2 + \mu \mathbf{L}^2 = -\mu^2 k^2 / 2E. \tag{3}
\]

Because of these relations, only five out of the seven constants are independent. They uniquely determine the shape and orientation of the ellipse. Its eccentricity is \[5\]

\[
e = \frac{|\mathbf{K}| \sqrt{-2E}}{\mu k}. \tag{4}
\]

We now turn to the quantum version. We use natural units, \(\mu = k = \hbar = 1\), so that the energy levels for bound states are \(E = -1/2n^2\), where \(n\) is an integer. The operator \(\mathbf{K}\) is defined by

\[
\mathbf{K} = (-2H)^{-1/2} \left( \frac{i}{2} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - r/r \right). \tag{5}
\]

Note that \(H\) commutes with \(\left( \frac{i}{2} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - r/r \right)\). The commutation relations for the operators \(\mathbf{L}\) and \(\mathbf{K}\) are \[9\]

\[
[L_i, K_j] = i \epsilon_{ijk} K_k, \tag{6}
\]

\[
[K_i, K_j] = i \epsilon_{ijk} L_k. \tag{7}
\]

Together with \([L_i, L_j] = i \epsilon_{ijk} L_k\), these are the commutation rules of infinitesimal rotations in four dimensional Euclidean space, which leave the \(n\)-th energy level subspace invariant \[10\]. The coherent states of SO(4), \(\text{i.e.}\) the states for which the dispersion of \(\mathbf{L}^2 + \mathbf{K}^2\) is minimal, can be built from the coherent states of SO(3), since SO(4) = SO(3)×SO(3). Define

\[
\mathbf{J}_1 = \frac{1}{2}(\mathbf{L} - \mathbf{K}) \quad \text{and} \quad \mathbf{J}_2 = \frac{1}{2}(\mathbf{L} + \mathbf{K}). \tag{8}
\]

These two operators have the commutation relations of two independent three-dimensional angular momenta:

\[
[J_{1i}, J_{1j}] = i \epsilon_{ijk} J_{1k}, \tag{9}
\]

\[
[J_{2i}, J_{2j}] = i \epsilon_{ijk} J_{2k}, \tag{10}
\]

\[
[J_{1i}, J_{2j}] = 0. \tag{11}
\]

Instead of the classical equations \[2\] and \[3\] we now have \[11\]

\[
\mathbf{L} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{L} = 0, \tag{12}
\]

and

\[
\mathbf{L}^2 + \mathbf{K}^2 = -1 - 1/2H = n^2 - 1, \tag{13}
\]

where the last form of the equality holds for energy eigenstates. In the classical limit, \(n \gg 1\), Eq. \[13\] reduces to the classical one \[8\].

In the rest of this article we consider only energy eigenstates. Owing to Eq. \[8\], we have

\[
j_1(j_1 + 1) = j_2(j_2 + 1) = \frac{1}{4}(n^2 - 1). \tag{14}
\]

where \(j_1\) and \(j_2\) are the quantum numbers referring to the operators \(J_1^2\) and \(J_2^2\) respectively. It follows that \(j_1\) and \(j_2\) are equal: \(j_1 = j_2 \equiv j\), and \(j(j + 1) = \frac{1}{4}(n^2 - 1)\), so that

\[
j = \frac{1}{2}(n - 1). \tag{15}
\]

The coherent states of a three-dimensional angular momentum will be denoted by \(|J, \mathbf{u}\rangle\). They obey \(\mathbf{u} \cdot J |J, \mathbf{u}\rangle = j |J, \mathbf{u}\rangle\) for an arbitrary classical unit vector \(\mathbf{u}\). For the coherent states the dispersion \(\Delta J \equiv (\langle J^2 \rangle - \langle J \rangle^2)^{1/2}\) is minimal: \((\Delta J)^2 = j\). In particular, \(\Delta J_{\mathbf{u}} = 0\), and \(\Delta J_{\mathbf{v}} = \sqrt{j/2}\), where \(J_0 = J \cdot \mathbf{u}\) and \(J_{\mathbf{v}} = J \cdot \mathbf{v}\) with \(\mathbf{v} \perp \mathbf{u}\). The coherent states of SO(3) are obtained by a rotation of a fiducial coherent state \(|J, \mathbf{z}\rangle\),

\[
|J, \mathbf{u}_0\rangle = e^{-iL_+ \theta} e^{-iL_- \theta} |J, \mathbf{z}\rangle. \tag{16}
\]

The coherent states of SO(4) are now obtained as direct products of coherent states for each of the SO(3) subgroups,

\[
|n, \mathbf{u}_1 \mathbf{u}_2\rangle = |J_1, \mathbf{u}_1\rangle \otimes |J_2, \mathbf{u}_2\rangle. \tag{17}
\]

where the unit vectors \(\mathbf{u}_1\) and \(\mathbf{u}_2\) are again classical. The coherent state \(|J_1, \mathbf{u}_1\rangle\) obeys

\[
\mathbf{u}_1 \cdot J_1 |J_1, \mathbf{u}_1\rangle = j |J_1, \mathbf{u}_1\rangle = \frac{1}{2}(n - 1) |J_1, \mathbf{u}_1\rangle, \tag{18}
\]

and likewise we have

\[
\mathbf{u}_2 \cdot J_2 |J_2, \mathbf{u}_2\rangle = j |J_2, \mathbf{u}_2\rangle = \frac{1}{2}(n - 1) |J_2, \mathbf{u}_2\rangle. \tag{19}
\]

As from now, we shall omit the symbols \(n, J_1, J_2\) in state vectors, since the quantum numbers \(n, j_1, j_2\) have fixed values, related by Eq. \[14\]. For example the state \(|nlm\rangle\) which obeys \(H|nlm\rangle = n|nlm\rangle\), \(L^2|nlm\rangle = l(l + 1)|nlm\rangle\), and \(L_z|nlm\rangle = m|nlm\rangle\) will be written simply as \(|lm\rangle\), \(|J, \mathbf{u}\rangle\) becomes \(|\mathbf{u}\rangle\), and \(|J, \mathbf{z}\rangle\) becomes \(|jj\rangle\), etc. The symbol \(j\) will always denote the fixed value \(j = \frac{1}{2}(n - 1)\).

Owing to Eq. \[8\], the dispersion of \(\mathbf{L}^2 + \mathbf{K}^2\) is minimal for coherent states \[11\]:

\[
(\Delta \mathbf{L})^2 + (\Delta \mathbf{K})^2 = 2[(\Delta J_1)^2 + (\Delta J_2)^2] = 2(n - 1). \tag{20}
\]
To obtain the expansion of the coherent state $|n, u_1 u_2\rangle$ in the familiar $nlm$ basis, we first expand each of the $|u_i\rangle$ in Eq. (17):

$$|u_i\rangle = \sum_{m=-j}^{j} D_{m}^{j}(\theta_i, \phi_i) |jm\rangle, \quad i = 1, 2,$$

(21)

where the $D_{m}^{j}(\theta_i, \phi_i)$ are related to the usual rotation matrices:

$$D_{m}^{j}(\theta, \phi) \equiv D^{(j)}(\theta \theta \theta)_{m},$$

(22)

$$= \left(\frac{2j}{j + m}\right)^{1/2} (\cos \frac{\phi}{2})^{j+m} (\sin \frac{\phi}{2})^{-m} e^{-im\phi}.$$ 

(23)

Substitution into Eq. (17) gives

$$|u_1 u_2\rangle = \sum_{m_1=-j}^{j} \sum_{m_2=-j}^{j} D_{m_1}^{j}(\theta_1 \phi_1) D_{m_2}^{j}(\theta_2 \phi_2) |jm_1\rangle \otimes |jm_2\rangle.$$ 

(24)

We then use the angular momentum addition formula,

$$|jm_1\rangle \otimes |jm_2\rangle = \sum_{l=0}^{2j} \sum_{m=-l}^{l} C_{m_1 m_2 m}^{j j l} |lm\rangle,$$

(25)

where $C_{m_1 m_2 m}^{j j l}$ is the Clebsch-Gordan coefficient which vanishes for $m \neq m_1 + m_2$. Combining Eqs. (24) and (25), we finally get

$$|u_1 u_2\rangle = \sum_{l=0}^{2j} \sum_{m=-l}^{l} \left( \sum_{m_1=-j}^{j} \sum_{m_2=-j}^{j} D_{m_1}^{j}(\theta_1 \phi_1) D_{m_2}^{j}(\theta_2 \phi_2) C_{m_1 m_2 m}^{j j l} \right) |lm\rangle.$$ 

(26)

The classical orbit that corresponds to the coherent state $|u_1 u_2\rangle$, in the limit of large $n$, can be obtained as follows. From (23), we have

$$L = (J_1 + J_2) \quad \text{and} \quad K = (J_2 - J_1).$$ 

(27)

Let $\zeta$ be half the angle between $u_1$ and $u_2$, i.e. $\sin \zeta \equiv |u_1 \times u_2|/|u_1 + u_2|$, and define three orthogonal classical unit vectors

$$\ell = \frac{u_1 + u_2}{|u_1 + u_2|}, \quad k = \frac{u_2 - u_1}{|u_2 - u_1|},$$ 

(28)

and $w \equiv \ell \times k$. Denoting by $u_{1\perp}$ an arbitrary vector orthogonal to $u_1$, we have

$$J_1 = (J_1 \cdot u_1) u_1 + (J_1 \cdot u_{1\perp}) u_{1\perp},$$ 

(29)

$$J_1 \cdot u_2 = (J_1 \cdot u_1)(u_1 \cdot u_2) + (J_1 \cdot u_{1\perp})(u_{1\perp} \cdot u_2).$$ 

(30)

Then from

$$\langle u_1 | (u_1 \cdot J_1) | u_1 \rangle = j,$$

(31)

and

$$\langle u_1 | u_{1\perp} \cdot J_1 | u_1 \rangle = 0,$$

(32)

we get

$$\langle u_1 | (u_2 \cdot J_1) | u_1 \rangle = \langle u_2 | (u_1 \cdot J_2) | u_2 \rangle,$$

(33)

$$= j u_1 \cdot u_2 = j \cos \zeta.$$ 

(34)

Noting that

$$|u_1 + u_2| = 2|\cos \zeta|,$$

(35)

and

$$|u_1 - u_2| = 2|\sin \zeta|,$$

(36)

we obtain from (24) the expectation values of the components of $K$ and $L$ along the directions of $k$, $\ell$ and $w$, for the coherent state $|u_1 u_2\rangle$

$$\langle K_\ell \rangle = (n - 1) \sin \zeta,$$

(37)

$$\langle L_\ell \rangle = (n - 1) \cos \zeta,$$

(38)

where $K_\ell \equiv k \cdot K$, etc. In the perpendicular directions the expectation values vanish:

$$\langle K_\perp \rangle = \langle K_w \rangle = \langle L_k \rangle = \langle L_w \rangle = 0.$$ 

(39)

From Eqs. (37)-(39) we see that in the limit of large $n$, the coherent state $|u_1 u_2\rangle$ corresponds to a classical
elliptic trajectory in the $kw$ plane with the LRL vector in the $k$ direction, the angular momentum in the $l$ direction, and eccentricity $e = \langle K_k \rangle / (n - 1) = \sin \zeta$, which is the quantum mechanical analog to Eq. (3). The unit vector $w$ is parallel to the minor axis of the ellipse.

III. TRANSMISSION OF ONE DIRECTION

We now turn to the use of elliptic wave functions as direction indicators. Consider two observers (Alice and Bob) who do not have a common reference frame. Alice wants to indicate to Bob her $z$-axis by using a hydrogen atom in a Rydberg state. In the next section, we shall likewise discuss the transmission of two orthogonal axes by a single hydrogen atom. We use as much as possible the same notations in both sections. Alice’s signal is

$$|A⟩ = \sum_{l=0}^{2j} \sum_{m=-l}^{l} a_{lm} |lm⟩, \tag{40}$$

where

$$\sum_{l=0}^{2j} \sum_{m=-l}^{l} |a_{lm}|^2 = 1. \tag{41}$$

Bob’s detectors have labels $ψθφ$ which indicate the unknown Euler angles relating his Cartesian axes to those used by Alice. The mathematical representation of his apparatus is a POVM

$$\int dE(ψθφ) = 1, \tag{42}$$

where

$$dE(ψθφ) = dψdθdφ/d8π2 is the SO(3) Haar measure for Euler angles \[12\]. As usual, $U(ψθφ)$ is the unitary operator for a rotation by Euler angles $ψθφ$, and $|B⟩$ is Bob’s fiducial vector defined as in $\[5\].$  

$$|B⟩ = \sum_{l=0}^{2j} \sqrt{2l+1} \sum_{m=-l}^{l} b_{lm} |lm⟩, \tag{44}$$

where for each $l$,

$$\sum_{m=-l}^{l} |b_{lm}|^2 = 1. \tag{45}$$

Note that Eq. (40) was written with Alice’s notation, while Eq. (43) is in Bob’s notation (recall that they use different coordinate systems).

Optimizing the transmission fidelity, defined by Eq. (45) below, leads to $\[5\]$

$$b_{lm} = a_{lm} \left( \sum_{n=-l}^{l} |a_{mn}|^2 \right)^{-1/2}, \tag{46}$$

for each $l$. Since $|B⟩$ is a direct sum of vectors, one for each value of $l$, then likewise $U(ψθφ)$ is a direct sum with one term for each irreducible representation:

$$U(ψθφ) = \sum_{l} φ_d^{(l)}(ψθφ), \tag{47}$$

where the $D^{(l)}(ψθφ)$ are the usual irreducible unitary rotation matrices $\[12\]. A generalization of Schur’s lemma $\[14\] confirms that Eq. (43) is indeed satisfied, owing to the coefficients $√2l+1$ in Eq. (43).

The fidelity of the transmission of a single direction is defined as usual:

$$F = \langle \cos^2(ω/2) \rangle = \frac{1}{2} (1 + \langle \cos ω \rangle), \tag{48}$$

where $ω$ is the angle between the direction indicated by Alice and the one that is estimated by Bob. If Alice indicates her $z$-axis, we thus want to maximize $\langle \cos ω_z \rangle$. Following the method of Peres and Scudo $\[7\] we define Euler angles $αβγ$ whose effect is rotating Bob’s Cartesian frame into his estimate of Alice’s frame, and then rotating back the result by the true angles from Alice’s to Bob’s frame. The angles $αβγ$ thus indicate Bob’s measurement error. Since in this case Bob’s estimate refers to Alice’s $z$-axis only, the angle $ω_z$ is identical to the second Euler angle $β$:

$$\langle \cos ω_z \rangle = \int dαdβdγ |⟨A|U(αβγ)|B⟩|^2 \cos β. \tag{49}$$

Let us examine two extreme cases. First, we take Alice’s vector to be a circular state, with null eccentricity ($\sin ζ = 0$), i.e.,

$$|A⟩ = |II⟩, \tag{50}$$

with

$$l = 2j = n - 1. \tag{51}$$

Bob’s vector is obtained from (40), which in this case gives

$$|B⟩ = \sqrt{2n-1} |II⟩. \tag{52}$$

We then have $\[12\]

$$⟨A|U(αβγ)|B⟩ = \sqrt{2n-1} e^{i(n-1)(α+γ)} \cos (2n-1) β/2. \tag{53}$$

Inserting the last equation into Eq. (49) gives

$$\langle \cos ω_z \rangle = (n - \frac{3}{2}) \int_0^π \sin βdβ \cos 4(n-1)(β/2) \cos β, \tag{54}$$

$$= (n - 1)/n. \tag{55}$$

The “infidelity” $1 - F$, whose typical meaning is Bob’s mean square error $\[12\]$, is

$$\frac{1}{2} (1 - \langle \cos ω_z \rangle) = 1/2n. \tag{56}$$
owing to Eqs. (6) and (7). Thus the rotated extreme coherent states is (15)

\[ |K, z\rangle \equiv | -z \rangle \otimes |z\rangle = |j, -j\rangle \otimes |j-j\rangle. \] 

(57)

This state satisfies \( L_z |K, z\rangle = 0 \) and it is also an eigenstate of \( K_z \):

\[ K_z |K, z\rangle = (J_{z2} - J_{1z}) |j, -j\rangle \otimes |j-j\rangle, \]
\[ = (n-1) |K, z\rangle, \]

(58)

(59)

owing to

\[ (J_{z2} - J_{1z}) |j, -j\rangle \otimes |j-j\rangle = -J_{1z} |j, -j\rangle \otimes J_{z2} |j-j\rangle. \]

(60)

In the \( nlm \) basis we have

\[ |K, z\rangle = \sum_{l=0}^{2j} \sum_{m=-l}^{l} C^{j,l}_{m,2m} |lm\rangle = \sum_{l=0}^{2j} C^{j,l}_{-j0} |0\rangle, \]

(61)

since \( m_1 = -j \) and \( m_2 = j \). The fidelity of transmission by this state will be evaluated at the end of this Section.

Both the circular state and the extreme Stark state are coherent states of \( SO(4) \), but only the circular state is also an angular momentum coherent state. Moreover, the circular state is symmetric, \( \langle ll | r | ll \rangle = 0 \), while the extreme Stark state is not. This can be seen from (16):

\[ \langle nlm | r | nlm \rangle = \frac{2}{3} \langle nlm | K | nlm \rangle. \]

(62)

Let us examine which one of these states gives better results when used by Alice to transmit the directions of her \( z \)-axis. The overlap between two angular momentum coherent states is (15)

\[ \langle u_1 | u_2 \rangle^2 = \cos^{4j} (\chi/2), \]

(63)

where \( \chi \) is the angle between the directions of \( u_1 \) and \( u_2 \). It is noteworthy that the overlap between two extreme Stark states is the same, as we will see shortly. First, a rotation of the \( |K, z\rangle \) state by angles \( (\theta \phi) \) gives

\[ |K, u_{\theta \phi}\rangle = e^{-iL_z \phi} e^{-iL_y \theta} |K, z\rangle, \]

(64)

where again the operator \( e^{-iL_z \phi} e^{-iL_y \theta} \) performs an active rotation of the vector \( |K, z\rangle \). Using (27) we have

\[ |K, u_{\theta \phi}\rangle = e^{-i(J_{1z} + J_{z1}) \phi} e^{-i(J_{1y} + J_{y1}) \theta} \}
\[ = e^{-iJ_{1z} \phi} e^{-iJ_{1y} \theta} | -z \rangle \otimes e^{-iJ_{z1} \phi} e^{-iJ_{y1} \theta} |z\rangle. \]

(65)

(66)

owing to Eqs. (10) and (17). Thus the rotated extreme Stark state is just

\[ |K, u_{\theta \phi}\rangle = | -u_{\theta \phi}\rangle \otimes |u_{\theta \phi}\rangle, \]

(67)

where the \( SO(3) \) coherent states \( |u_{\theta \phi}\rangle \) are defined as in Eq. (21). This Stark state is an eigenstate of \( u \cdot K \) with the maximal eigenvalue \( n - 1 \), and it satisfies \( k = \mathbf{u} \), as can be seen from Eq. (28). The overlap between two such states, \( \langle K, \mathbf{u} | K, \mathbf{u}' \rangle^2 \), is

\[ \langle -u| u' \rangle^2 \langle u | u' \rangle^2, \]

(68)

which by using (63) is just:

\[ \cos^{4j} (\chi/2) \cos^{4j} (\chi/2) = \cos^{4(n-1)} (\chi/2), \]

(69)

where \( \chi \) is the angle between the vectors \( u \) and \( u' \).

Such a simple expression cannot hold for the overlap of two generic elliptic states whose eccentricities are not 0 or 1. Let a generic elliptic state

\[ |u_1 u_2\rangle = |u_1\rangle \otimes |u_2\rangle, \]

(70)

be an elliptic state with eccentricity \( 0 < e < 1 \). Unlike the \( e = 1 \) and \( e = 0 \) cases, this state does not define one direction, but two independent ones \( u_1 \) and \( u_2 \). If it is rotated by Euler angles \( \alpha \beta \gamma \) the result is

\[ e^{-iL_z \alpha} e^{-iL_y \beta} e^{-iL_z \gamma} |u_1 u_2\rangle = U_1 |u_1\rangle \otimes U_2 |u_2\rangle. \]

(71)

where

\[ U_1 = e^{-iJ_{1z} \alpha} e^{-iJ_{1y} \beta} e^{-iJ_{1z} \gamma}, \]

(72)

and likewise for \( U_2 \). To obtain this result we have used (27) and the commutation relations (6) and (7). The rotation \( e^{-iL_z \alpha} e^{-iL_y \beta} e^{-iL_z \gamma} \) opens an angle \( \chi_1 \) between the classical vectors \( u_1 \) and \( R(\alpha \beta \gamma) u_1 \), and an angle \( \chi_2 \) (which is generally different from \( \chi_1 \)) between \( u_2 \) and \( R(\alpha \beta \gamma) u_2 \). Here \( R(\alpha \beta \gamma) \) denotes the classical rotation matrix \( \mathbf{R} \). It follows that

\[ \langle u_1 u_2| e^{-iL_z \alpha} e^{-iL_y \beta} e^{-iL_z \gamma} |u_1 u_2\rangle = \left( \cos \frac{\chi_1}{2} \cos \frac{\chi_2}{2} \right)^{2(n-1)}. \]

(73)

Generally, both \( \chi_1 \) and \( \chi_2 \) are different from the angle between the directions \( k \) and \( k' = R(\alpha \beta \gamma) k \), or between the directions \( \ell \) and \( \ell' = R(\alpha \beta \gamma) \ell \).

We now calculate the transmission fidelity for the case where Alice sends an extreme Stark state \( |K, z\rangle \). Since \( |A\rangle \) contains only \( m = 0 \) terms, so does Bob’s fiducial vector

\[ b_{lm} = a_{00} |a_{00}|^{-1/2} \delta_{m0}. \]

(74)

We thus have

\[ b_{lm} = \delta_{m0} |a_{00}| |a_{00}|, \]

(75)

\[ |B\rangle = \sum_{l=0}^{n-1} \sqrt{2l+1} \left( |a_{00}| / |a_{00}| \right) |l0\rangle. \]

(76)
In order to determine \( \langle \cos \omega_z \rangle \) in Eq. 19, we note that
\[
\langle A|U(\alpha \beta \gamma)|B \rangle = \sum_{i=0}^{n-1} \sqrt{2l+1} a_i^* b_i |l \rangle |D^{(l)}(\alpha \beta \gamma)|0 \rangle,
\]
\[
= \sum_{i=0}^{n-1} \sqrt{2l+1} |a_i \rangle |d_i(\beta) \rangle. \tag{77}
\]
We insert this expression into 19. The result, obtained by using Eqs. (19)–(21) of ref. 3, is
\[
\langle \cos \omega_z \rangle = \sum_{k,l} A_{ik} |a_{i0} a_{k0}|, \tag{78}
\]
where \( A_{ik} \) is a real symmetric matrix whose non-vanishing elements are
\[
A_{l,l-1} = A_{l-1,l} = l/\sqrt{4l^2 - 1}, \tag{79}
\]
and \( a_{i0} = C_{l,j}^l \). The results are summarized in Fig. 1, in which the mean square error is plotted versus \( n \). The |\( K, z \rangle \) state gives better fidelity than the circular state |\( l \rangle \), but for \( n > 3 \) its fidelity is substantially less than optimal 3,4 and goes asymptotically to 1/(4n−2). This raises the question whether it is possible to build a “natural” POVM by setting Bob’s vector to |\( B \rangle = \sqrt{N}|K, z \rangle\), so that POVM elements are
\[
N|K, u_{\theta \phi}\rangle \langle K, u_{\theta \phi}|, \tag{80}
\]
where |\( K, u_{\theta \phi}\rangle \) was defined in Eq. 66, and \( N \) is a normalization factor. Unfortunately, |\( K, z \rangle \) contains a superposition of all values of \( l \), as can be seen from 61. Thus |\( K, z \rangle \) does not belong to one irreducible subspace of the representation of the SO(3) rotation group. As a result, the operator
\[
B = \int d_{\theta \phi}|K, u_{\theta \phi}\rangle \langle K, u_{\theta \phi}| \tag{81}
\]
is not proportional to the identity, but is a block-diagonal matrix with different blocks for each irreducible representation of the rotation group. Moreover, the resulting POVM includes an element which corresponds to the absence of any answer, thus reducing fidelity. A natural POVM which uses the SO(4) group will be discussed in Sec. IV.

The direction of the minor axis of a classical nondegenerate ellipse is that of \( L \times K \). A quantum ellipse also has this property. Taking Alice’s state as a quantum ellipse with eccentricity \( 0 < e < 1 \), with both \( k \) and \( L \) lying in the \( xy \) plane so that \( w = z \), the resulting fidelity can be compared with the cases where \( k \) or \( L \) points along the \( z \)-axis and the eccentricity of the ellipse is 0 or 1, respectively. The fidelity for transmission using the semi-minor axis reaches a maximum at eccentricity of about \( e = 0.7 \) (a different eccentricity for each value of \( n \)) . A comparison of the mean square error for using the three options is given in Table 1.

## IV. TRANSMISSION OF TWO AXES

Alice now wants to transmit a Cartesian frame by indicating the directions of two axes, the third one being inferred from them. Which elliptic state is optimal? Obviously, states with \( e = 0 \) and \( e = 1 \) will not do in this case, since they define only one direction. We have to find the optimal eccentricity. Let
\[
\Delta K_\perp = \sqrt{(\Delta K_\ell)^2 + (\Delta K_w)^2}, \tag{86}
\]
\[
\Delta L_\perp = \sqrt{(\Delta L_\ell)^2 + (\Delta L_w)^2}, \tag{87}
\]
where
\[
\Delta K_\ell = \sqrt{\langle K_\ell^2 \rangle - \langle K_\ell \rangle^2} = \sqrt{\langle K_\ell^2 \rangle}, \tag{88}
\]
owing to Eq. 39. We define similar expressions for the other components. When we want to transmit \( k \), namely the direction of the classical LRL vector, then a smaller \( \Delta K_\perp /\langle K_\ell \rangle \) improves the fidelity. A similar argument holds for the transmission of \( L \). Thus when transmitting two axes, a heuristic guideline is to look for states that satisfy
\[
\frac{\Delta K_\perp}{\langle K_\ell \rangle} \approx \frac{\Delta L_\perp}{\langle L_\ell \rangle}. \tag{89}
\]
A straightforward calculation gives
\[
\Delta K_w = \Delta L_w = \sqrt{\frac{1}{2}(n-1)},
\]
\[
\Delta K_\ell = \sqrt{\frac{1}{2}(n-1)\sin \zeta},
\]
\[
\Delta L_k = \sqrt{\frac{1}{2}(n-1)\cos \zeta}.
\]
Together with Eqs. (37) and (35), this gives an equation for the eccentricity,
\[
\frac{1 + \sin^2 \zeta}{2(n-1)\sin \zeta} \approx \frac{1 + \cos^2 \zeta}{2(n-1)\cos \zeta}.
\]
Therefore we expect that the optimal eccentricity is approximately
\[
e = \sin \zeta = \cos \zeta = 1/\sqrt{2}.
\]

More accurate numerical results are given below.

We now evaluate the fidelity for the transmission of two axes. Alice uses an elliptic state with \(k = x\) and \(\ell = y\) (the unit vectors in the \(x\) and \(y\) directions respectively). The eccentricity \(e = \sin \zeta\) has to be optimized. Recall the \(\zeta\) is defined to be half the angle between \(u_1\) and \(u_2\). The definitions of \(k\) and \(\ell\) are given in Eq. (28).

Thus in order to meet the above requirements we set in \(|A\rangle = |u_{\theta_1,\phi_1}, u_{\theta_2,\phi_2}\rangle\) the parameters \(\theta_1 = \theta_2 = \pi/2\), and
\[
\phi_1 = \frac{1}{2}\pi - \zeta, \quad \phi_2 = \frac{3}{2}\pi - \zeta.
\]

Fidelities now must be defined for each one of the axes. Note that \(\cos \omega_k\) (for the \(k\)th axis) is given by the corresponding diagonal element of the orthogonal (classical) rotation matrix. For the transmission of the \(x\) and \(y\) axes, we thus need \[8\]
\[
\langle \cos \omega_x + \cos \omega_y \rangle = ((1 + \cos \beta)(\cos(\alpha + \gamma))).
\]

We expand \(|A\rangle\) and \(|B\rangle\) as in \[40\] and \[14\]. Bob’s optimal fiducial vector is still given by \[40\], and \(\langle \cos \omega_x + \cos \omega_y \rangle\) is calculated using equations (23)–(26) of ref. \[7\]. The mean square error per axis is plotted in Fig. 3 for \(n = 5, 10,\) and 20. The error is minimal at \(e \approx 0.708\) for \(n = 5,\) at \(e \approx 0.704\) for \(n = 10,\) and \(e \approx 0.674\) for \(n = 20.\)

Note that the shape of the curve flattens with increasing \(n\), so that the minimum is hard to find numerically. The intuitive explanation is that in the limit of large \(n\), as if Alice were to transmit a “classical atom,” i.e., a classical two body Kepler system, then the direction of the classical angular momentum and LRL vectors could be found irrespective of the eccentricity. Therefore, the transmission accuracy would be the same for any eccentricity that is not close to zero or one.

The deviation of the optimum from \(e = 1/\sqrt{2}\) was expected, since transmission of the \(k\) direction (\(e = 1\)) achieved higher fidelity than the transmission of the \(\ell\) direction (\(e = 0\)). Thus the ellipse with optimal eccentricity for transmission of two axes is biased to give \(\Delta L_\perp / L < \Delta K_\perp / K\) in order to compensate the difference and make the contribution to the error from the \(k\) direction about equal to that from the \(\ell\) direction.

Elliptic states give results very close to the optimal ones. The mean square error for transmission of two axes by elliptic states with optimal eccentricity is compared to the optimal results \[7\] in Table III.

V. POVM FOR SO(4)

We now construct a POVM based on the SO(4) group and use it in order to transmit two axes. This POVM is naturally built with the SO(4) coherent states which are, as we have seen, direct products of two SO(3) coherent states. We shall use for each one of the SO(3) subspaces the notation
\[
|\psi\theta\phi\rangle_u = \sqrt{2j+1} U(\psi\theta\phi)|u\rangle,
\]
and
\[
dE(\psi\theta\phi) = d\psi d\theta d\phi d\psi /8\pi^2\text{ as in Eq. (143).}
\]
By applying Schur’s lemma to each of the SO(3) subspaces we have
\[
\int dE(\psi_1\theta_1\phi_1) \otimes dE(\psi_2\theta_2\phi_2) = 1_1 \otimes 1_2 = 1.
\]

We are now ready to discuss the transmission of Alice’s \(x\) and \(y\) axes by means of an elliptic state. We take
\[
|A\rangle = |xy\rangle = |x\rangle \otimes |y\rangle.
\]
This equation was written in Alice’s notation. We also define a fiducial vector for Bob
\[
|B\rangle = (2j + 1)|x\rangle \otimes |y\rangle,
\]
written in Bob’s notations. Thus the POVM element is constructed from the vector
\[
|\psi_1\theta_1\phi_1\rangle_x \otimes |\psi_2\theta_2\phi_2\rangle_y = (2j + 1)(U_1 \otimes U_2)|B\rangle.
\]

The result of Bob’s measurement consists of two sets of Euler angles, \(\psi_1\theta_1\phi_1\) and \(\psi_2\theta_2\phi_2\). The first one gives Bob’s estimate of the active rotation needed to bring his \(x\)-axis to Alice’s \(x\)-axis. Likewise, the second set gives Bob’s estimate of the active rotation needed to bring his \(y\)-axis to Alice’s \(y\)-axis. The detection probability of these sets of angles is
\[
dP(\psi_1\ldots\phi_2) = d\psi_1 d\theta_1 d\psi_2 d\theta_2 \langle A|U_1 \otimes U_2|B\rangle^2.
\]
Recall that Eq. (100) was written in Alice’s notations, while (101) is in Bob’s notations. To compute the result
explicitly, we need a uniform system of notations. For this we introduce, as in 7, the Euler angles $\xi \eta \zeta$ that rotate Bob’s $x'y'z'$ axes into Alice’s axes. (The Euler angle $\zeta$ should not be confused with the eccentricity parameter introduced in Sec. II.) The unitary operator $U(\xi \eta \zeta)$ represents an active transformation of Bob’s state vectors to the corresponding state vectors of Alice’s system. Therefore, $U(\xi \eta \zeta)$ is also the passive transformation from Alice’s notations to Bob’s notations. Written in Bob’s notations, Alice’s vector

$$\langle A \rangle = U(\xi \eta \zeta) |A\rangle$$

and $|A\rangle$ becomes $\langle A |U(\xi \eta \zeta)^{\dagger}\rangle$. Owing to the commutation relations 6 and 7

$$U(\xi \eta \zeta) = e^{-iL_2 \xi} e^{-iL_3 \eta} e^{-iL_2 \zeta},$$

$$= U_1(\xi \eta \zeta) \otimes U_2(\xi \eta \zeta),$$

where again $U_1$ and $U_2$ are defined as in Eq. 62. We thus have

$$U(\xi \eta \zeta)|u_1 u_2\rangle = U_1(\xi \eta \zeta)|u_1\rangle \otimes U_2(\xi \eta \zeta)|u_2\rangle.$$  \hspace{2cm} \text{(106)}$$

Let us therefore define

$$U_1(\alpha_1 \beta_1 \gamma_1) = U_1^\dagger(\xi \eta \zeta) U_1(\psi_1 \theta_1 \phi_1),$$

$$\text{and}$$

$$U_2(\alpha_2 \beta_2 \gamma_2) = U_2^\dagger(\xi \eta \zeta) U_2(\psi_2 \theta_2 \phi_2).$$

We shall henceforth use the left hand sides of Eqs. 107 and 108 as the new definitions of the symbols $U_1$ and $U_2$. As before, the Euler angles $\alpha_1 \beta_1 \gamma_1$ have the effect of rotating Bob’s $x$-axis into his estimate of Alice’s $x$-axis and then rotating back the result by the true rotation from Alice’s to Bob’s frame. The action of the Euler angles $\alpha_2 \beta_2 \gamma_2$ is similar for the $y$-axis. Thus the Euler angles $\alpha_i \beta_i \gamma_i$ indicate Bob’s measurement error, and the probability of that error is

$$dP(\alpha_1...\gamma_2) = d_{\alpha_1 \beta_1 \gamma_1} d_{\alpha_2 \beta_2 \gamma_2} |\langle A |U_1 \otimes U_2 |B\rangle|^2.$$  \hspace{2cm} \text{(109)}$$

Note the similarity with Eq. 103. The difference is that 103 referred to the probability of detection of a particular set of Euler angles, while 109 gives the probability of error in that detection.

The transmission mean square error per axis is, as in Eq. 69,

$$R = \frac{1}{4} \langle 1 - \cos \omega_x \rangle + \frac{1}{4} \langle 1 - \cos \omega_y \rangle,$$  \hspace{2cm} \text{(110)}$$

where we have used 103 and Schur’s lemma for the second set of angles, namely

$$(2j + 1) \int d_{\alpha_2 \beta_2 \gamma_2} |\langle y |U_2\rangle|^2 = \mathbb{1}_2.$$  \hspace{2cm} \text{(112)}$$

The evaluation of Eq. 111 is identical to the one performed in Eq. 54, with $n$ replaced by $\frac{1}{2}(n + 1)$ everywhere, and we get

$$\langle \cos \omega_x \rangle = (n - 1)/(n + 1).$$  \hspace{2cm} \text{(113)}$$

Likewise

$$\langle \cos \omega_y \rangle = (n - 1)/(n + 1).$$  \hspace{2cm} \text{(114)}$$

Thus the infidelity (mean square error) per axis is

$$\frac{1}{4}(1 - \langle \cos \omega_x \rangle) + \frac{1}{4}(1 - \langle \cos \omega_y \rangle) = 1/(n + 1).$$  \hspace{2cm} \text{(115)}$$

In ref. 7, it was found that the optimal POVM (not restricted to elliptic states) for transmission of two axes using a hydrogen atom is of the form given by Eq. 103. It was shown that using this POVM, the infidelity per axis for Alice’s optimal signal approaches $1/(3n)$ asymptotically. Using SO(4) instead of SO(3) as in 7, we obtain an infidelity per axis that is exactly $1/(n + 1)$ for all values of $n$. As shown in the Appendix, an adjustment procedure to obtain to orthogonal axes will further decrease the mean square error by a factor which for large values of $n$ tends to $3/4$.

The SO(4) POVM also enables the transmission of two directions which are not orthogonal, by means of a single hydrogen atom in an elliptic state. To transmit the directions of two general unit vectors $v_1$ and $v_2$, Alice’s prepares the elliptic state

$$|A\rangle = |v_1 v_2\rangle = |v_1\rangle \otimes |v_2\rangle,$$  \hspace{2cm} \text{(116)}$$

(in her notations) while Bob’s vector is (in his notations)

$$|B\rangle = (2j + 1)|v_1\rangle \otimes |v_2\rangle.$$  \hspace{2cm} \text{(117)}$$

As before, the infidelity for each direction is $1/(n + 1)$. It should be noted that transmission of two non-orthogonal directions with one hydrogen atom is not possible with the SO(3) POVM.

**VI. SUMMARY AND CONCLUDING REMARKS**

We have shown how elliptic Rydberg states can transfer information on the orientation of one direction, or more generally that of a Cartesian frame. For increasing values of $n$, the fidelity obtained for a single direction falls rapidly below the optimal ones. However, for a Cartesian frame the results are very close to the optimal ones. Furthermore, elliptic states have the advantage of being
experimentally accessible, while preparation of the optimal states seems much more difficult. Note that we have assumed Alice and Bob have the same chirality. If their chiralities are opposite, then when angular momenta are used for the transmission, the direction inferred by Bob should be reversed (because directions are polar vectors while angular momentum is an axial vector). However, the LRL vector is also a polar vector, thus even if Bob and Alice have opposite chiralities, the direction inferred by Bob is correct. We have also shown how elliptic Rydberg states can be prepared to encode two arbitrary states, when the measurement is based on the SO(4) rotation group.

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**APPENDIX A: REDUCTION OF ERRORS BY ORTHOGONALIZATION**

As we have seen in Sec. V, Bob’s estimates of Alice’s $x$ and $y$ axes may not be exactly orthogonal. The probability for the estimate of the $x$ axis to have an angular error $\omega_x$, as can be seen from Eq. (43), is

$$\rho(\omega_x) \propto \cos^{2n-2}(\omega_x/2),$$

and likewise for the $y$ axis. Thus for large values of $n$, the error probability distribution will be highly peaked. We now calculate the gain in fidelity achieved if Bob performs a simple orthogonalization of his two estimates $\hat{r}_x$ and $\hat{r}_y$, by rotating the two vectors in their plane by the same angle, so that they become orthogonal.

Let us define two pairs of spherical angles that give the position of the estimated directions with respect to the (unknown) true axes. These positions are given by

$$\hat{r}_x = (\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1),$$

and

$$\hat{r}_y = (\sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \cos \theta_2).$$

The probability distributions will be denoted $\rho_i(\theta_i, \phi_i)$. In the limit of large $n$, the deviation angles $\omega_x$ and $\omega_y$ are small. Hence the distribution are centered as

$$\rho_x = \rho(\theta_1 - \frac{1}{2}\pi, \phi_1),$$

$$\rho_y = \rho(\theta_2 - \frac{1}{2}\pi, \phi_2 - \frac{1}{2}\pi),$$

where $\rho(\xi, \mu)$ is peaked around $(0, 0)$. Here we used the fact that the SO(4) POVM gives probabilities of error for each axis which are identical and independent. Define new variables

$$\tilde{\theta}_i = \theta_i - \frac{1}{2}\pi,$$

and

$$\tilde{\phi}_i = \phi_i - \frac{1}{2}\pi.$$

The deviation angles are given by $\cos \omega_x = \hat{r}_x \cdot \hat{x}$ and $\cos \omega_y = \hat{r}_y \cdot \hat{y}$, namely

$$\cos \omega_x = \sin \theta_1 \cos \phi_1 \approx 1 - \frac{1}{2}\tilde{\theta}_1^2 - \frac{1}{2}\tilde{\phi}_1^2,$$

and

$$\cos \omega_y = \sin \theta_2 \cos \phi_2 \approx 1 - \frac{1}{2}\tilde{\theta}_2^2 - \frac{1}{2}\tilde{\phi}_2^2.$$  

Let $g$ denote the infidelity per axis before the adjustment. The infidelities for both axes are equal, thus

$$g \equiv \frac{1}{2}(1 - \langle \cos \omega_x \rangle),$$

$$\approx \frac{1}{4} \int (\tilde{\theta}_1^2 + \tilde{\phi}_1^2) \, d\rho_1 \equiv \frac{1}{4}(\tilde{\theta}_1^2 + \tilde{\phi}_1^2),$$

where

$$d\rho_i = \rho(\tilde{\theta}_i, \tilde{\phi}_i) \sin \tilde{\theta}_i \, d\tilde{\theta}_i \, d\tilde{\phi}_i.$$  

fullfills

$$\int d\rho_i = 1.$$  

Equivalently, we can write the infidelity in terms of $\tilde{\theta}_2$ and $\tilde{\phi}_2$ as

$$g \approx \frac{1}{4} \int (\tilde{\theta}_2^2 + \tilde{\phi}_2^2) \, d\rho_2 \equiv \frac{1}{4}(\tilde{\theta}_2^2 + \tilde{\phi}_2^2).$$

In first order we have, by combining (A2) and (A3) with the definitions (A6) and (A7),

$$\hat{r}_x \approx (1, \phi_1, -\tilde{\theta}_1), \quad \hat{r}_y \approx (-\tilde{\phi}_2, 1, -\tilde{\theta}_2),$$

and the angle $\Omega$ between them is given by

$$\cos \Omega = \hat{r}_x \cdot \hat{r}_y \approx \phi_1 - \tilde{\phi}_2.$$  

The bisector of $\hat{r}_x$ and $\hat{r}_y$ is given by the unit vector $\hat{b} = (\hat{r}_1 + \hat{r}_2)/|\hat{r}_1 + \hat{r}_2|$. Using (A16) and keeping only first order terms we have

$$\hat{b} \approx [1 - \frac{1}{2}(\phi_1 + \tilde{\phi}_2), 1 + \frac{1}{2}(\phi_1 + \tilde{\phi}_2), -\tilde{\theta}_1 - \tilde{\theta}_2] / \sqrt{2},$$  

where we used

$$|\hat{r}_1 + \hat{r}_2| \approx \sqrt{2}(1 + \frac{1}{2}\phi_1 - \frac{1}{2}\tilde{\phi}_2).$$  

We can also express the bisector $\hat{b}$ in terms of its spherical angles which we shall denote by $(\tau, \varphi)$. Since the errors are small, we have $\varphi \approx \frac{1}{2}\pi$, and it is convenient to define

$$\varphi = \varphi - \frac{1}{2}\pi.$$
Comparison of the two expressions for $\hat{b}$ gives

$$\xi = \frac{1}{2} \pi + \sqrt{2} (\tilde{\theta}_1 + \tilde{\theta}_2), \quad \tilde{\varphi} = \frac{1}{2} (\varphi_1 + \tilde{\varphi}_2).$$  \hfill (A20)

In first order, as Eq. (A16) shows, the orthogonalization consists in changing the angles $\varphi_i$ irrespective of $\tilde{\theta}_i$, without changing the $\tilde{\theta}_i$ themselves. Hence, in first order, the procedure defines

$$\varphi'_1 = \varphi - \frac{1}{4} \pi, \quad \varphi'_2 = \varphi + \frac{1}{4} \pi,$$  \hfill (A21)

i.e.,

$$\varphi'_1 = \tilde{\varphi}'_2 = \tilde{\varphi} = \frac{1}{2} (\varphi_1 + \tilde{\varphi}_2),$$  \hfill (A22)

where again $\tilde{\varphi}'_2 = \tilde{\varphi}_2 - \frac{1}{2} \pi$. The change in $\hat{b}$ is of higher order, $\tilde{\theta}'_1 = \tilde{\theta}_i + O(\theta^2, \varphi^2)$. The new infidelity per axis $g^{\text{new}}$ is

$$g^{\text{new}} = \frac{1}{4} (\varphi'_1^2 + \tilde{\theta}'_1) = \frac{1}{4} (\tilde{\varphi}'_2^2 + \tilde{\varphi}'_2^2).$$  \hfill (A23)

Returning to (A11) and (A14), consider the integrals over $\varphi_i$. Define

$$\rho_{\varphi}(\varphi) = \int \rho(\theta, \varphi) \sin \theta d\theta$$  \hfill (A24)

Keeping in mind that the distributions for $\varphi_1$ and $\tilde{\varphi}_2$ are identical, the $\varphi$-part of the infidelity per axis before the adjustment is

$$g_{\varphi} = \frac{1}{4} (\varphi_1^2) = \frac{1}{4} (\tilde{\varphi}_2^2),$$  \hfill (A25)

$$= \int \varphi_1^2 \rho_{\varphi}(\varphi_1) \rho_{\varphi}(\tilde{\varphi}_2) d\varphi_1 d\tilde{\varphi}_2 / 4,$$  \hfill (A26)

$$= \int \tilde{\varphi}_2^2 \rho_{\varphi}(\tilde{\varphi}_1) \rho_{\varphi}(\tilde{\varphi}_2) d\tilde{\varphi}_1 d\tilde{\varphi}_2 / 4.$$  \hfill (A27)

The $\varphi$-parts of the infidelities after the adjustment, denoted by $g_{\varphi}^{\text{new}}$, are

$$g_{\varphi}^{\text{new}} = \frac{1}{4} (\varphi'_1^2) = \frac{1}{4} (\tilde{\varphi}'_2^2) = \frac{1}{16} (\varphi_1^2 + 2 \varphi_1 \tilde{\varphi}_2 + \tilde{\varphi}_2^2).$$  \hfill (A28)

The functions $\rho_{\varphi}(\varphi_1)$ and $\rho_{\varphi}(\tilde{\varphi}_2)$ are even, because the probability distribution $\rho$ depends only on the angles $\omega_x$ or $\omega_y$, which are independent of the sign of $\varphi_1$ and $\tilde{\varphi}_2$. Thus

$$\langle \varphi_1 \tilde{\varphi}_2 \rangle = \int \varphi_1 \tilde{\varphi}_2 \rho_{\varphi}(\varphi_1) \rho_{\varphi}(\tilde{\varphi}_2) d\varphi_1 d\tilde{\varphi}_2 = 0.$$  \hfill (A29)

With the help of Eq. (A25) we obtain

$$g_{\varphi}^{\text{new}} = \frac{1}{16} (\varphi_1^2 + \tilde{\varphi}_2^2) = \frac{1}{8} (\tilde{\varphi}_1^2).$$  \hfill (A30)

Thus the $\varphi$-parts of the infidelity are halved,

$$g_{\varphi}^{\text{new}} = \frac{1}{4} g_{\varphi}.$$  \hfill (A31)

As already stated, the angles $\tilde{\theta}_i$ are unchanged in first order. Since the probability function $\rho_x(\varphi_1, \tilde{\theta}_1)$ depends only on the angle $\omega_x = r_x \cdot \hat{x}$, it is symmetric with respect to rotations around the $x$ axis. A similar argument holds for the $y$-axis. Thus

$$\langle \tilde{\varphi}'_1^2 \rangle = \langle \varphi_1^2 \rangle = \langle \tilde{\varphi}'_2^2 \rangle = \langle \varphi_2^2 \rangle,$$  \hfill (A32)

and we have finally

$$g^{\text{new}} = \frac{1}{4} g.$$  \hfill (A33)

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TABLE I: Eccentricities $e$ and mean square errors $\eta$ for transmission of a single direction using $z = w$, $z = \ell$, or $z = k$, for $n = 5$ or $10$.

| $n$ | $z = w$ | $z = \ell$ | $z = k$ |
|-----|---------|-------------|---------|
| 5   | $e = 0.6963$ | $e = 0$ | $e = 1$ | $\eta = 0.193967$ | $\eta = 0.1$ | $\eta = 0.0573645$ |
| 10  | $e = 0.701261$ | $e = 0$ | $e = 1$ | $\eta = 0.0861934$ | $\eta = 0.05$ | $\eta = 0.0264067$ |

TABLE II: Coefficients $|a_l|$ for Alice’s optimal state and for the extreme Stark state when $n = 10$.

| $l$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|---|
| $|K, z|\rangle$ | 0.3162 | 0.4954 | 0.5222 | 0.4534 | 0.3365 | 0.2148 | 0.1167 | 0.0526 | 0.0186 | 0.0045 |
| Optimal | 0.1825 | 0.3079 | 0.3767 | 0.4098 | 0.4130 | 0.3894 | 0.3422 | 0.2751 | 0.1923 | 0.0989 |

TABLE III: Mean square error $\eta$ for transmission of two axes by an elliptic state with optimal eccentricity, and by the optimal method [7] for $n = 5, 10,$ and $20$.

| $n$ | elliptic optimal |
|-----|-------------------|
| 5   | $\eta = 0.14765$ | $\eta = 0.14465$ |
| 10  | $\eta = 0.06822$ | $\eta = 0.06793$ |
| 20  | $\eta = 0.03190$ | $\eta = 0.03088$ |
Figure 1

FIG. 1: Mean square error as a function of $n$ for the transmission of a single axis using the circular state (open circles), the extreme Stark state (squares), and the optimal state (closed circles).
Figure 2

FIG. 2: Mean square error (per axis) as a function of eccentricity for $n=5$, 10 and 20.