NATURALLY OF THE HYPERHOLOMORPHIC SHEAF OVER THE
CARTESIAN SQUARE OF A MANIFOLD OF $K3^{[n]}$-TYPE

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Abstract. Let $M$ be a $2n$-dimensional smooth and compact moduli space of stable sheaves
on a $K3$ surface $S$ and $U$ a universal sheaf over $S \times M$. Over $M \times M$ there exists a natural
reflexive sheaf $E$ of rank $2n - 2$, namely the first relative extension sheaf of the two pullbacks
of $U$ to $M \times S \times M$. We prove that $E$ is $\omega \boxplus \omega$-slope-stable with respect to every Kähler class
$\omega$ on $M$. The sheaf $E$ is known to deform to a sheaf $E'$ over $X \times X$, for every manifold $X$
deformation equivalent to $M$, and we prove that $E'$ is $\omega \boxplus \omega$-slope-stable with respect to every
Kähler class $\omega$ on $X$. This triviality of the stability chamber structure combines with a result of
S. Mehrotra and the author to show that the deformed sheaf $E'$ is canonical; each component
of the the moduli space of marked triples $(X, \eta, E')$, where $\eta$ is a marking of $H^2(X, \mathbb{Z})$, maps
isomorphically onto the component of the moduli space of marked pairs $(X, \eta)$ by forgetting $E'$.
Consequently, the pretriangulated $K3$ category associated to the pair $(X \times X, E')$ in $\text{MM}_2$
depends only on the isomorphism class of $X$.

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1. Introduction

1.1. Stability of modular sheaves. An irreducible holomorphic symplectic manifold is a
simply connected compact Kähler manifold, such that $H^0(X, \Omega^2_X)$ is one dimensional spanned
by an everywhere non-degenerate holomorphic 2-form. Every Kähler manifold $X$ deformation
equivalent to the Hilbert scheme $S^{[n]}$ of length $n$ subschemes of a $K3$ surface $S$ is an irreducible
holomorphic symplectic manifold [Be]. Such $X$ is said to be of $K3^{[n]}$-type. Every smooth and
projective moduli space $\mathcal{M}$ of stable sheaves on a $K3$ surface is an irreducible holomorphic
symplectic manifold of $K3^{[n]}$-type, by results of Huybrechts, Mukai, O’Grady, and Yoshioka
[O’G, Y].

Let $S$ be a $K3$ surface, $H$ an ample line bundle on $S$, $v$ a primitive Mukai vector, and
$\mathcal{M} := \mathcal{M}_H(v)$ a smooth and compact moduli space of $H$-stable sheaves on $S$ of Mukai vector
$v$. Assume that $2n := \dim_\mathbb{C} \mathcal{M} \geq 4$. Denote by $\pi_S$ and $\pi_\mathcal{M}$ the two projections from $S \times \mathcal{M}$.

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Let \( \mathcal{U} \) be a universal sheaf over \( S \times \mathcal{M} \). There is a Brauer class \( \theta \) in the cohomology group \( H^2_{an}(\mathcal{M}, \mathcal{O}^*_\mathcal{M}) \), with respect to the analytic topology of \( \mathcal{M} \), such that \( \mathcal{U} \) is \( \pi^*_\mathcal{M} \theta \)-twisted. Let \( \pi_{ij} \) be the projection from \( \mathcal{M} \times S \times \mathcal{M} \) onto the product of the \( i \)-th and \( j \)-th factors. Let

\[
E := \text{Ext}^1_{\pi_{13}^* \mathcal{U}} (\pi_{12}^* \mathcal{U}, \pi_{23}^* \mathcal{U})
\]

be the relative extension sheaf over \( \mathcal{M} \times \mathcal{M} \). Let \( f_i \) be the projection from \( \mathcal{M} \times \mathcal{M} \) onto the \( i \)-th factor. \( E \) is a reflexive \( f_1^* \theta^{-1} f_2^* \theta \)-twisted sheaf of rank \( 2n - 2 \), which is locally free away from the diagonal, by [M2 Prop. 4.1]. Given a Kähler class \( \omega \) on \( \mathcal{M} \), denote by \( \tilde{\omega} := f_1^* \omega + f_2^* \omega \) the corresponding Kähler class over \( \mathcal{M} \times \mathcal{M} \).

**Definition 1.1.** Let \( X \) be a \( d \)-dimensional compact Kähler manifold and \( \omega \) a Kähler class on \( X \). The \( \omega \)-degree of a coherent sheaf \( G \) on \( X \) is \( \deg_\omega(G) := \int_X \omega^{d-1} c_1(G) \). If \( G \) is torsion free of rank \( r \), its \( \omega \)-slope is \( \mu_\omega(G) := \deg_\omega(G) / r \). Let \( E \) be a torsion free coherent sheaf over \( X \), which is \( \theta \)-twisted with respect to some Brauer class \( \theta \). The sheaf \( E \) is \( \omega \)-slope-semistable, if for every subsheaf \( F \) of \( E \), satisfying \( 0 < \text{rank}(F) < \text{rank}(E) \), we have

\[
\deg_\omega(\text{Hom}(E, F)) \leq 0.
\]

\( E \) is said to be \( \omega \)-slope-stable if strict inequality holds above. \( E \) is said to be \( \omega \)-slope-polystable, if it is \( \omega \)-slope-semistable as well as the direct sum of \( \omega \)-slope-stable sheaves.

**Theorem 1.2.** The sheaf \( E \), given in Equation (1.1), is \( \tilde{\omega} \)-slope-stable with respect to every Kähler class \( \omega \) on \( \mathcal{M} \).

The Theorem follows from the more general Theorem [1,4] stated below. The Chern class \( c_2(\text{End}(E)) \), of the sheaf \( E \) in Theorem [1,2] flatly deforms to a class of Hodge type \( (2,2) \) on the cartesian square \( X \times X \) of every manifold of \( K3^{[n]} \)-type [M2 Prop. 1.2]. This fact combines with Theorem [1,2] to imply that the sheaf \( E \) is \( \tilde{\omega} \)-hyperholomorphic in the sense of Verbitsky [V] (see also [M2 Cor. 6.11]). Verbitsky proved a very powerful deformation theoretic result for such sheaves [V Theorem 3.19]. Associated to a Kähler class \( \omega \) on \( \mathcal{M} \) is a twistor family \( \mathcal{X} \rightarrow \mathbb{P}^1_\omega \) deforming \( \mathcal{M} \) [Hu1]. Verbitsky’s theorem implies that \( E \) extends to a reflexive sheaf over the fiber square \( \mathcal{X} \times_{\mathbb{P}^1_\omega} \mathcal{X} \) of the twistor family associated to every Kähler class \( \omega \) on \( \mathcal{M} \). Verbitsky’s theorem, applied to the sheaf \( E \) in Theorem [1,2] is central to our proof with F. Charles of the Standard Conjectures for projective manifolds of \( K3^{[n]} \)-type and to our work with S. Mehrotra on pretriangulated K3-categories associated to manifolds of \( K3^{[n]} \)-type [CM1, MM1, MM2].

The first example of a moduli space \( \mathcal{M} \), for which the above Theorem holds, was given in [M2 Theorem 7.4]. In that example the order of the Brauer class was shown to be equal to the rank of \( E \), and so \( E \) does not have any non-zero subsheaf of lower rank. Such a maximally twisted reflexive sheaf is thus \( \tilde{\omega} \)-slope-stable with respect to every Kähler class \( \omega \) on \( X \).

The special case of Theorem [1,2] where \( \mathcal{M} = S^{[n]} \) is the Hilbert scheme of length \( n \) subschemes of a K3 surface \( S \) with a trivial Picard group, was proven earlier in [M3 Theorem 1.1(1)]. In that case it was proven that, though untwisted, \( E \) again does not have any non-zero subsheaf of lower rank.

1.2. Stability in terms of the singularities along the diagonal. Let \( X \) be an irreducible holomorphic symplectic manifold of complex dimension \( 2n > 2 \). Let \( \beta : Y \rightarrow X \times X \) be the
are isomorphic as sheaves of algebras. The sheaf $E$ isomorphism $f$ and the forgetful morphism sending a triple $(X, \eta, E)$ to the corresponding connected component of $\tilde{M}$ of $\Lambda$, which implies Assumption 1.3 above. Fix a connected component $K\tilde{\Lambda}$ of $\Lambda$ with a lattice $\Lambda$. The moduli space $\tilde{M}$ of isomorphism classes of $\Lambda$-marked irreducible holomorphic symplectic manifolds $(X, \eta)$ is a non-Hausdorff manifold of dimension $\text{rank}(\Lambda) - 2$ $\text{[H]}$.

The pair $(M \times M, E)$ in Theorem 1.2 is known to deform to all cartesian squares $X \times X$ of manifolds of $K3^{[\text{type}]}$ $\text{[M2 Theorem 1.3]}$. The sheaf $E$ is infinitesimally rigid, $\text{Ext}^1(E, E) = 0$, by $\text{[MM2 Lemma 5.2]}$. We describe next a global analogue of these two facts.

In $\text{[MM]}$ we constructed a moduli space $\tilde{\mathcal{M}}_{\Lambda}$ of equivalence classes of triples $(X, \eta, E)$, where $X$ is of $K3^{[\text{type}]}$, $n \geq 2$, $\eta$ is a $\Lambda$-marking for $X$, and $E$ is a rank $2n - 2$ reflexive infinitesimally rigid twisted sheaf over $X \times X$, which is $\omega$-slope-stable with respect to some Kähler class $\omega$ on $X$. Two pairs $(X, \eta, E)$ and $(X', \eta', E')$ are equivalent, if there exists an isomorphism $f : X \to X'$, such that $\eta' = \eta \circ f^*$, and the sheaves $\mathcal{E}nd(E)$ and $(f \times f)^*\mathcal{E}nd(E')$ are isomorphic as sheaves of algebras. The sheaf $E$ of every triple in $\tilde{\mathcal{M}}_{\Lambda}$ is assumed to satisfy $\text{[MM]}$ Condition 1.6], which implies Assumption 1.3 above. Fix a connected component $\tilde{\mathcal{M}}^0_{\Lambda}$ of $\tilde{\mathcal{M}}_{\Lambda}$ and let

$$\phi : \tilde{\mathcal{M}}^0_{\Lambda} \to \mathcal{M}^0_{\Lambda}$$

be the forgetful morphism sending a triple $(X, \eta, E)$ to the marked pair $(X, \eta)$, where $\mathcal{M}^0_{\Lambda}$ is the corresponding connected component of $\mathcal{M}_{\Lambda}$. The forgetful morphism $\phi$ is a surjective local

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1In $\text{[MM]}$ it was also assumed that the sheaf $\mathcal{E}nd(E)$ is $\omega$-slope-polystable, but the latter property follows from the $\omega$-slope-stability of $E$, as was proven later in $\text{[M2 Prop. 6.5]}$ for twisted sheaves (and is well known for untwisted sheaves).
Corollary 1.6. The above morphism \( \phi \) is an isomorphism.

The Corollary is proven in Section 3 (conditional on [MM1, Conj. 1.12]). Taking the quotient by the monodromy action, Corollary 1.6 may be reformulated as follows.

Corollary 1.7. On the cartesian square \( X \times X \) of every manifold \( X \) of \( K^3[n] \)-type there exists a canonical pair of isomorphism classes of sheaves of algebras \( \mathcal{E}_{nd}(E) \) and \( \mathcal{E}_{nd}(E^*) \), where \( E \) is a reflexive rank \( 2n - 2 \) twisted sheaf, locally free away from the diagonal, which satisfies Assumption 1.3 and is \( \omega \)-slope-stable with respect to every Kähler class \( \omega \) on \( X \).

The Corollary is proven in Section 3 (conditional on [MM1, Conj. 1.12]). Following is the main application of Theorem 1.4. Let \( E \) be the sheaf in Theorem 1.4 and let \( F \) be a saturated subsheaf of \( E \) of rank in the range \( 0 < \text{rank}(F) < 2n - 2 \). Let \( \hat{G} \) be the image of the composition \( (\beta^* F)(D) \to (\beta^* E)(D) \to V \). Denote by \( G \) the saturation of \( \hat{G} \) as a subsheaf of \( V \). The sheaf \( G \) is reflexive, since \( V \) is locally free. The restriction of \( G \) to \( D \) maps to the restriction \( \ell^t / \ell \) of \( V \) and we denote by \( G' \) the saturation of its image as a subsheaf of \( \ell^t / \ell \). We provide in this section an upper bound for \( \deg_{\ell^t} (\text{Hom}(E, F)) \) in terms of the first Chern class of \( G' \) (Lemma 2.5). We then provide information on \( c_1(G') \) (Lemma 2.6).

Lemma 2.1. The equality \( c_1(\text{Hom}(E, F)) = \beta_*(c_1(\text{Hom}(V, G))) \) holds.

Proof. The higher direct image \( R^i \beta_*(\text{Hom}(V, G)) \), \( i > 0 \), is supported on the diagonal and so its first Chern class vanishes. The equality \( c_1(\beta_* \text{Hom}(V, G)) = c_1(R\beta_* \text{Hom}(V, G)) \) follows. Similarly, \( c_1(\beta_* \text{Hom}(V, G)) = c_1(\text{Hom}(E, F)) \), as the two sheaves coincide away from the diagonal. Hence, \( c_1(\text{Hom}(E, F)) \) is the graded summand in degree 2 of \( \beta_*(\text{ch}(\text{Hom}(V, G))^t d_{\beta}) \), by Grothendieck-Riemann-Roch [OTT]. We have \( c_1(TY) = (1 - 2n)[D] \) and so
\[
\begin{align*}
\text{td}_{\beta} &= \text{td}(TY)/\text{td}(T[X \times X]) = 1 + \frac{1}{2n} [D] + \ldots
\end{align*}
\]
The statement follows, by the vanishing of \( \beta_*([D]) \). \( \Box \)

Set \( h := c_1(\ell^{-1}) \in H^2(D, \mathbb{Z}) \).

Lemma 2.2. \( p_*(h^i) = \begin{cases} 0 & \text{if } i < 2n - 1 \text{ or } i \text{ is even}, \\ 1 & \text{if } i = 2n - 1, \end{cases} \) and

\[
(2.1) \quad p_*(h^{2n+k}) = -c_{k+1}(TX) - \sum_{j=2}^{k-1} c_j(TX)p_*(h^{2n+k-j}),
\]

where \( c_j \) is the \( j \)th Chern class.
for any positive odd integer $k$.

**Proof.** The statement is well known. The vanishing of $p_i(h^i)$, for $i < 2n - 1$, follows for dimension reasons. The equality $p_*(h^{2n-1}) = 1$ is proven in [F, Appendix B.4 Lemma 9]. Consider the short exact sequence $0 \to \ell \to p^*TX \to q \to 0$ defining $q$. Then $c_{2n}(q) = 0$, since rank($q$) = $2n - 1$. On the other hand, the Chern polynomial of $q$ satisfies

$$c_t(q) = p^*c_t(TX)/c_t(\ell) = p^*c_t(TX)(1 + h + h^2 + \cdots + h^{4n-1}).$$

Consequently, $0 = c_{2n}(q) = \sum_{j=0}^{2n} p^*c_j(TX)h^{2n-j}$ and

$$h^{2n} = -\left[ \sum_{j=1}^{2n} p^*c_j(TX)h^{2n-j} \right].$$

The projection formula yields

$$p_*(h^{2n+k}) = -p_*\left( \sum_{j=1}^{2n} p^*c_j(TX)h^{2n+k-j} \right) = -c_{k+1}(TX) - \sum_{j=1}^{k} c_j(TX)p_*(h^{2n+k-j}),$$

for every positive integer $k$. The vanishing of $p_*(h^i)$ for even $i$ follows, by induction, from the vanishing of $c_j(TX)$ for odd $j$. The recursive formula (2.1) follows. $\square$

**Lemma 2.3.** The $t$-th symmetric power $\text{Sym}^t(TX)$ of the tangent bundle is $\omega$-slope-stable with respect to every Kähler class $\omega$ on $X$, for all $t \geq 0$. The space $H^0(X, (\wedge^3 TX) \otimes \text{Sym}^t(T^*X))$ vanishes, for all $j \geq 0$ and all $t > 1$.

**Proof.** The vector bundle $TX$ admits a Hermite-Einstein metric whose $(1,1)$-form represents $\omega$, for every Kähler class $\omega$ on $X$, by Yau’s proof of the Calabi Conjecture [H2, Cor. 4.8.22]. In particular, $TX$ is $\omega$-slope-polystable with respect to every Kähler class $\omega$. The holonomy group of the tangent bundle of an irreducible holomorphic symplectic manifold of complex dimension $2n$ is the symplectic group $Sp(n)$ [B1, Prop. 4]. Consequently, the vector bundle $TX$ is slope-stable with respect to every Kähler class, its tensor powers are poly-stable, and the indecomposable direct summands of the tensor powers correspond to irreducible representations of $Sp(n)$. Let $U$ be the standard representation of $Sp(n)$. The symmetric powers $\text{Sym}^t(U)$ are irreducible representations, for all $t \geq 0$. If $t > 1$ then $\text{Sym}^t(U)$ does not appear as a subrepresentation of the exterior product $\wedge^3 U$, for any $j \geq 0$. Hence, $H^0(X, (\wedge^3 TX) \otimes \text{Sym}^t(T^*X))$ vanishes, for $t > 1$. $\square$

**Lemma 2.4.** Let $Z$ be a non-zero effective divisor on $D$ and $[Z] \in H^2(D, \mathbb{Z})$ its class. Then

$$\int_D p^*(\omega)^{2n-1}h^{2n-1}[Z] > 0,$$

for every Kähler class $\omega$ on $X$.

**Proof.** We have a direct sum decomposition $\text{Pic}(D) = p^*\text{Pic}(X) \oplus \mathbb{Z}\ell$. The restriction of $\mathcal{O}_D(Z)$ to each fiber of $p$ is effective. Hence, $\mathcal{O}_D(Z)$ is isomorphic to $\ell^{-a} \otimes p^*L$, for some line bundle $L \in \text{Pic}(X)$ and for some nonnegative integer $a$. The space $H^0(D, \mathcal{O}_D(Z))$ does not vanish and is isomorphic to $H^0(X, L \otimes \text{Sym}^a T^*X)$. Hence, $L^{-1}$ is a subsheaf of $\text{Sym}^a T^*X$. The vector bundle $\text{Sym}^a T^*X$ is an $\omega$-slope-stable bundle with a trivial determinant, by Lemma 2.3. If $a > 0$, then $L^{-1}$ is a subsheaf of lower rank. If $a = 0$, then $L^{-1}$ is the ideal sheaf of a non-zero
effective divisor on $X$, since $Z$ was assumed to be such. In both cases we get the inequality $\text{deg}_\omega(L) > 0$ and so
\[
\int_D p^*(\omega)^{2n-1}h^{2n-1}[Z] = \int_X \omega^{2n-1}p_*(h^{2n-1}(ah + p^*c_1(L))) = \int_X \omega^{2n-1}c_1(L) = \text{deg}_\omega(L) > 0,
\]
where the second equality is due to the vanishing of $p_*(h^{2n})$ and the equality $p_*(h^{2n-1}) = 1$ of Lemma 2.2.

Recall that the restriction of $V$ to $D$ is isomorphic to $\ell^\perp/\ell$, by Assumption 1.3. Let $G'$ be the saturation of the image in $\ell^\perp/\ell$ of the restriction to $D$ of the subsheaf $G$ of $V$. Note that the sheaves $\ell^\perp/\ell$ and $G'$ are untwisted.

**Lemma 2.5.** The following inequality holds for every Kähler class $\omega$ on $X$.

\[
\text{deg}_\omega(\text{Hom}(E, F)) \leq (2n - 2) \left(\frac{4n - 1}{2n}\right) \left(\int_X \omega^{2n}\right) \int_D (p^*\omega)^{2n-1}h^{2n-1}c_1(G').
\]

**Proof.** Let $a, b \in H^2(X, \mathbb{Z})$ be the classes satisfying $c_1(\text{Hom}(E, F)) = f_1^*a + f_2^*b$. We have
\[
\text{deg}_\omega(\text{Hom}(E, F)) = \int_{X \times X} \tilde{\omega}^{4n-1}c_1(\text{Hom}(E, F))
\]
\[
= \int_{X \times X} (f_1^*\omega + f_2^*\omega)^{4n-1}(f_1^*a + f_2^*b)
\]
\[
= \left(\frac{4n - 1}{2n}\right) \left(\int_X \omega^{2n}\right) \int_X \omega^{2n-1}(a + b)
\]
\[
= \left(\frac{4n - 1}{2n}\right) \left(\int_X \omega^{2n}\right) \int_X \omega^{2n-1}\delta^*c_1(\text{Hom}(E, F))
\]
\[
(2.2)
\]
where the last equality is by Lemma 2.1. Lemma 2.2 yields
\[
\delta^*\beta_*c_1(\text{Hom}(V, G)) = p_* \left[h^{2n-1}p^*\delta^*(\beta_*c_1(\text{Hom}(V, G)))\right] = p_* \left[h^{2n-1}\iota^*\beta^*(\beta_*c_1(\text{Hom}(V, G)))\right].
\]
The equality $\beta_*\beta^*\beta_* = \beta_*$ implies that the difference between the classes $\beta^*(\beta_*c_1(\text{Hom}(V, G)))$ and $c_1(\text{Hom}(V, G))$ belongs to the kernel of $\beta_*$ and is hence a multiple of $[D]$. The vanishing $p_*(h^{2n-1}\iota^*[D]) = -p_*(h^{2n}) = 0$ yields the equality
\[
\delta^*\beta_*c_1(\text{Hom}(V, G)) = p_* \left[h^{2n-1}\iota^*(c_1(\text{Hom}(V, G)))\right].
\]

We have $\iota^*c_1(\text{Hom}(V, G)) = c_1(L\iota^*\text{Hom}(V, G)) = c_1([\ell^\perp/\ell]^* \otimes L^*(G))$. The sheaf $G$ is reflexive, by construction, and hence its singular locus has codimension $\geq 3$ in $Y$. It follows that $c_1(L\iota^*G) = c_1(\iota^*G) = c_1(\iota^*G)|_Y$. The subscheme of $Y$, where the rank of the homomorphism $G \to V$ is lower than the rank of $G$, has codimension at least 2 in $Y$, since $G$ is a saturated subsheaf of $V$. Hence, the natural homomorphism $(\iota^*G)|_Y \to G'$ is injective. We conclude that $c_1(L\iota^*G) + [Z] = c_1(G')$, for some effective divisor $Z$ on $D$. The rank $2n - 2$ vector bundle $\ell^\perp/\ell$ is symplectic, and hence $c_1(\ell^\perp/\ell) = 0$. Hence
\[
\iota^*c_1(\text{Hom}(V, G)) = (2n - 2) (c_1(G') - [Z]).
\]
The Projection Formula and the two displayed formulas above yield
\[ \int_X \omega^{2n-1} \delta^s \beta_s c_1(\operatorname{Hom}(V, G)) = (2n - 2) \int_D (p^* \omega)^{2n-1} h^{2n-1} (c_1(G') - [Z]). \]
Lemma 2.4 yields the inequality
\[ \int_X \omega^{2n-1} \delta^s \beta_s c_1(\operatorname{Hom}(V, G)) \leq (2n - 2) \int_D (p^* \omega)^{2n-1} h^{2n-1} c_1(G'). \]
Lemma 2.5 follows from the above inequality and Equation (2.2). \(\square\)

Lemma 2.6. \(c_1(G') = p^* \alpha - kh,\) for a non-zero class \(\alpha \in H^2(X, \mathbb{Z})\) and a positive integer \(k.\)

Proof. There exist an integer \(k\) and a class \(\alpha \in H^2(X, \mathbb{Z})\) satisfying \(c_1(G') = p^* \alpha - kh,\) since \(H^2(D, \mathbb{Z}) = p^* H^2(X, \mathbb{Z}) \oplus \mathbb{Z} h.\) The integer \(k\) is positive, since the restriction of \(\ell^{-1}/\ell\) to each fiber of \(p : D \to X\) is slope-stable, by [MM1, Lemma 7.4].

The rest of the proof is by contradiction. Assume that \(\alpha = 0.\) Let \(g\) be the rank of \(G.\) Then \(0 < g < 2n - 2\) and the top exterior power of \(G'\) yields a line subbundle of \(\Lambda^g[\ell^{-1}/\ell]\) isomorphic to \(\ell^k.\) In particular, the vector space
\[(2.3) \quad H^0(D, \Lambda^g[\ell^{-1}/\ell] \otimes \ell^{-k})\]
contains a non-zero section.

We have the short exact sequence
\[0 \to \ell^{-1} \to p^* TX \to \ell^{-1} \to 0.\]
Dualizing, we get
\[0 \to \ell \to p^* T^* X \to (\ell^{-1})^* \to 0.\]
Hence also the short exact
\[0 \to \ell \otimes \Lambda^{j-1}(\ell^{-1})^* \to p^* \Lambda^j (T^* X) \to \Lambda^j (\ell^{-1})^* \to 0.\]
Dualizing the latter and tensoring by \(\ell^{-t}\) we get the short exact sequence
\[0 \to \Lambda^j(\ell^t) \otimes \ell^{-t} \to p^* (\Lambda^j T^* X) \otimes \ell^{-t} \to \Lambda^{j-1}(\ell^t) \otimes \ell^{-t-1} \to 0.\]
The inclusion of \(\Lambda^{j-1}(\ell^t) \otimes \ell^{-t-1}\) in \(p^* (\Lambda^j T^* X) \otimes \ell^{-t-1}\) yields the left exact sequence
\[0 \to p_* \left[ \Lambda^j(\ell^t) \otimes \ell^{-t} \right] \to \left( \Lambda^j T^* X \right) \otimes \operatorname{Sym}^t(T^* X) \to \left( \Lambda^{j-1} T^* X \right) \otimes \operatorname{Sym}^{t+1}(T^* X).\]
The vector space \(H^0 \left( X, (\Lambda^j T^* X) \otimes \operatorname{Sym}^t(T^* X) \right)\) vanishes, for \(t > 1,\) by Lemma 2.3. We conclude that \(H^0 \left( D, \Lambda^j(\ell^t) \otimes \ell^{-t} \right)\) vanishes, for all pairs \((j, t)\) of integers satisfying \(j \geq 0\) and \(t > 1.\) The short exact sequence \(0 \to \ell \to \ell^t \to [\ell^{-1}/\ell] \to 0\) yields
\[0 \to \ell \otimes \Lambda^{j-1}[\ell^{-1}/\ell] \to \Lambda^j \ell^t \to \Lambda^j[\ell^{-1}/\ell] \to 0.\]
Tensoring by \(\ell^{-t}\) we get
\[(2.4) \quad 0 \to \Lambda^{j-1}[\ell^{-1}/\ell] \otimes \ell^{-t} \to \left( \Lambda^j \ell^t \right) \otimes \ell^{-t} \to \Lambda^j[\ell^{-1}/\ell] \otimes \ell^{-t} \to 0.\]
We conclude that \(H^0 \left( D, \Lambda^{j-1}[\ell^{-1}/\ell] \otimes \ell^{-t} \right)\) vanishes, for all pairs \((j, t)\) of integers satisfying \(j \geq 1\) and \(t > 1.\) The vanishing of the space \((2.3)\) follows, by taking \(j = g + 1\) and \(t = k + 1.\) This provides the desired contradiction. \(\square\)

3Under the identification of \(TX\) with \(T^* X\) via the symplectic form, the rightmost homomorphism is the homomorphism \((\Lambda^j T^* X) \otimes \operatorname{Sym}^t(T^* X) \to (\Lambda^{j-1} T^* X) \otimes \operatorname{Sym}^{t+1}(T^* X)\) appearing in the Koszul complex. We will not use this fact.
3. Proof of Theorem 1.4

The proof of Theorem 1.4 reduces to that of the inequality
\[ \int_D (p^* \omega)^{2n-1} h^{2n-1} c_1(G') < 0, \]
for every Kähler class \( \omega \) and for every non-zero saturated proper subsheaf \( G' \) of \( \ell \perp / \ell \), by Lemma 2.5. Set \( c_1(G') = p^* \alpha - kh \) as in Lemma 2.6. The left hand side of the above inequality is equal to \( \int_X \omega^{2n-1} \alpha \), by the projection formula and the vanishing of \( p_*(h^{2n}) \).

Given a positive integer \( N \) we have the sequence of inclusions
\[ p_*(G' \otimes \ell^{-N}) \subset p_*([\ell^1/\ell] \otimes \ell^{-N}) \subset p_* (\wedge^2(\ell^1) \otimes \ell^{-N-1}) \subset p_* (\wedge^2(p^*TX) \otimes \ell^{-N-1}), \]
where the second inclusion follows from the sequence (2.4). The projection formula yields the inclusion \( p_*(G' \otimes \ell^{-N}) \subset (\wedge^2TX) \otimes \text{Sym}^{N+1}T^*X \). The latter is an \( \omega \)-polystable vector bundle with zero first Chern class. We conclude the inequality
\[ \int_X \omega^{2n-1} c_1(p_*(G' \otimes \ell^{-N})) \leq 0, \]
for every Kähler class \( \omega \) on \( X \).

The higher direct images \( R^i p_*(G' \otimes \ell^{-N}) \) vanish, for all \( N \) sufficiently large. Hence, there exists an integer \( N_0 \), such that
\[ c_1(p_*(G' \otimes \ell^{-N})) = c_1(Rp_*(G' \otimes \ell^{-N})), \]
for all \( N > N_0 \). The right hand side is the graded summand \( P(N) \) in \( H^2(X, \mathbb{Z})[N] \) of the polynomial
\[ p_*(\text{ch}(G') \exp(Nh)td_p) \in H^*(X, \mathbb{Q})[N] \]
in the variable \( N \) with coefficients in the cohomology ring. Let \( g \) be the rank of \( G' \). We have \( c_1(T_p) = 2nh \),
\[ \begin{align*}
  td_p &= 1 + nh + \ldots, \\
  \text{ch}(G') &= g + (p^* \alpha - kh) + \ldots \\
  \exp(Nh) &= \sum_{j=0}^{4n-1} h^j N^j ./ j!.
\end{align*} \]
Hence, the graded summand of \( \text{ch}(G') \exp(Nh)td_p \) in \( H^{4n}(D, \mathbb{Q}) \) is a polynomial of degree 2n in \( N \) whose first two leading terms are
\[ gh^{2n} N^{2n}/2n! + [(p^* \alpha - kh) + gn]h^{2n-1} N^{2n-1}/(2n-1)! + \ldots \]
The polynomial \( P(N) \) is the image of the above polynomial under \( p_* \). The vanishing of \( p_*(h^{2n}) \) yields that \( P(N) \) has degree 2n - 1 in \( N \) and its leading term is
\[ P(N) := c_1(Rp_*(G' \otimes \ell^{-N})) = N^{2n-1}/(2n-1)! \alpha + \ldots \]
The values of the left hand side of the inequality (3.1) for \( N > N_0 \) are thus the values of a polynomial in \( \mathbb{R}[N] \) of degree \( 2n - 1 \), whose leading coefficient is
\[ \frac{1}{(2n-1)!} \int_X \omega^{2n-1} \alpha. \]
The inequality (3.1) thus implies the inequality

\[ \int_X \omega^{2n-1} \alpha \leq 0, \]

for every Kähler class \( \omega \).

Let \( K_X \) be the Kähler cone of \( X \). The map \( K_X \to H^{2n-1,2n-1}(X, \mathbb{R}) \) sending \( \omega \) to \( \omega^{2n-1} \) is an open map. Indeed, its differential at \( \omega \) is the cup product

\[ (2n-1)\omega^{2n-2} : H^{1,1}(X, \mathbb{R}) \to H^{2n-1,2n-1}(X, \mathbb{R}), \]

which is an isomorphism by the Hard Lefschetz Theorem. The class \( \alpha \) does not vanish, by Lemma 2.6. Hence, the linear functional

\[ H^{2n-1,2n-1}(X, \mathbb{R}) \to \mathbb{R}, \]

sending \( \lambda \) to \( \int_X \lambda \alpha \), is an open map as well. We conclude that the image of \( K_X \) under the composition of these two open maps is an open subset and the inequality (3.2) is strict. This completes the proof of Theorem 1.4. \( \square \)

**Proof of Corollary 1.6.** Any two points \( (X, \eta, E_1) \) and \( (X, \eta, E_2) \) in the fiber of \( \phi \) are inseparable, by [MM1, Theorem 6.1]. If, furthermore, \( E_1 \) and \( E_2 \) are \( \tilde{\omega} \)-slope-stable, with respect to the same Kähler class \( \omega \) on \( X \), then \( \mathcal{E}nd(E_1) \) and \( \mathcal{E}nd(E_2) \) are isomorphic as sheaves of algebras, by [MM1, Lemma 5.3]. The sheaves \( E_i \), \( i = 1, 2 \), are assumed to satisfy [MM1, Condition 1.6], which implies Assumption 1.3 by [MM1, Lemma 7.6]. Hence, the sheaves \( E_i \), \( i = 1, 2 \), are \( \tilde{\omega} \)-slope-stable with respect to every Kähler class \( \omega \) on \( X \), by Theorem 1.4. It follows that \( \mathcal{E}nd(E_1) \) and \( \mathcal{E}nd(E_2) \) are isomorphic as sheaves of algebras, and the two triples \( (X, \eta, E_i) \), \( i = 1, 2 \), represent the same equivalence class in \( \mathfrak{M}_A \). \( \square \)

**Proof of Corollary 1.7.** The isometry group \( O(\Lambda) \) of \( \Lambda \) acts on \( \mathfrak{M}_A \) by \( g(X, \eta) \mapsto (X, g\eta) \). Let \( \text{Mon}(\Lambda) \subset O(\Lambda) \) be the subgroup leaving the connected component \( \mathfrak{M}^0_\Lambda \) invariant. \( \text{Mon}(\Lambda) \) is determined in [M1, Theorem 1.2], which implies, in particular, that elements of \( \text{Mon}(\Lambda) \) act on the discriminant group \( \Lambda^*/\Lambda \cong \mathbb{Z}/(2n-2)\mathbb{Z} \) as \( \pm 1 \). Denote by

\[ \text{cov} : \text{Mon}(\Lambda) \to \{\pm 1\} \]

the corresponding character. The set of \( \text{Mon}(\Lambda) \)-orbits in \( \mathfrak{M}^0_\Lambda \) is in bijection with the set of isomorphism classes of manifolds of \( K3^{[n]} \)-type. Conjugating the \( \text{Mon}(\Lambda) \)-action via the isomorphism \( \phi \) of Corollary 1.6 lifts it to an action on \( \mathfrak{M}^0_\Lambda \). The latter action is given by \( g(X, \eta, E) \mapsto (X, g\eta, E^{\text{cov}(g)}) \), where \( E^{\text{cov}(g)} = E \), if \( \text{cov}(g) = 1 \), and \( E^{\text{cov}(g)} = E^* \), if \( \text{cov}(g) = -1 \), by the proof of [MM1, Theorem 1.11]. The inverse image under \( \phi \) of the \( \text{Mon}(\Lambda) \) orbit of \( (X, \eta_0) \) in \( \mathfrak{M}^0_\Lambda \) consists of equivalence classes of triples \( (X, \eta, E) \) in a unique \( \text{Mon}(\Lambda) \) orbit of, say, \( (X, \eta_0, E_0) \), by Corollary 1.6. The above description of the monodromy action on \( \mathfrak{M}^0_\Lambda \) implies that \( \mathcal{E}nd(E) \) is isomorphic as a sheaf of algebras to \( \mathcal{E}nd(E_0) \) or \( \mathcal{E}nd(E_0^*) \). \( \square \)

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