Solution of the Continuous Time Bilinear Quadratic Regulator Problem by Krotov’s Method

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Abstract—This article contributes to the field of optimal control of bilinear systems. It concerns a continuous time finite-dimensional bilinear state equation with a quadratic performance index to be minimized. The state equation is non-autonomous and comprises a deterministic a priori known excitation. The control trajectory is constrained to an admissible set without a specific structure. The performance index is a functional quadratic in the state variables and control signals. The Krotov’s method is used for solving this problem by means of an improving sequence. To this end, the required sequence of improving functions is formulated. Finally, the solution is encapsulated in an algorithm form, and a numerical example of structural control problem is provided.

Index Terms—Bilinear systems, Krotov’s method, optimal control, structural control.

I. INTRODUCTION

Bilinear state-space models are simple nonlinear models, useful for capturing dynamic attributes of systems in various fields, such as quantum mechanics [1], chaotic dynamics [2], biology [3]–[5], mechanical damping [6]–[9], and structural control [10]–[14]. Even when more complex nonlinear plants are addressed, they can sometime be well-approximated by bilinear models [1], [15], [16].

The ongoing research, conducted on such systems, has been yielding diverse results, including optimal control design tools. Among the published works, discussing finite-dimensional bilinear systems, one can find results concerning homogeneous [10]–[12], [17]–[21] or inhomogeneous [13], [22], [23] plants, continuous [10]–[12], [17], [18], [20], [22], [23] or discrete [16], [19], [21] time, problems with control constraints [10]–[14], [16], [21], [23], and quadratic [10], [11], [17], [19]–[21], [23] or biquadratic [12], [14] performance index. These solutions are provided in a form of an algorithm. Some furnishes solutions that meet necessary optimality conditions, such as the Pontryagin’s minimum principle [10]–[14], [22], while others rely on sufficient conditions, such as the Hamilton–Jacobi–Bellman equation [24], [25] and two-sided algorithms [21].

In the 60s of the previous century, series of intriguing results on sufficient conditions for global optimum of optimal control problems have begun to be published by Krotov [26]. One of the outcomes of the author’s work is a method for solving optimal control problems—“a global method of successive improvements of control,” or in its contemporary name—the Krotov’s method. For illustrating the gist of it, one can take as an example the Lyapunov’s direct method for stability. Even though the method outlines properties for the required Lyapunov function, additional effort should be exerted for finding one. In this sense, Krotov’s and Lyapunov’s methods take similar approaches. Basically, the Krotov’s method successively improves the control feedback until convergence is reached. However, for this to happen, a special sequence of functions should be formulated. This subproblem stands at the method’s heart and is still, in general, an open problem that should be solved in the context of a given case. The method has the following benefits.

1) First, the iteration is proven to furnish a process whose performance is either equal or better than its former.

2) Second, it is suitable for problems where small variations in the control are not allowed, e.g., when the set of admissible control vectors is closed.

3) Third, the resulting control is formulated in a feedback form [27]. This is unlike methods inspired by the Pontryagin’s minimum principle [22], which are more convenient for obtaining open-loop solutions [27].

The Krotov’s method was used for optimal control of bilinear systems in several works. For example, an optimal control of quantum systems by laser radiation can be defined as a single input homogeneous in the state bilinear system. Solving it through gradient methods can turn problematic, because there are trajectory portions where there is no information about the recommended variation of the control trajectory. The Krotov’s method, however, does not suffer of this deficiency [27]. A slightly more generalized version of this class of problems was addressed for two forms of performance index: 1) first is quadratic in the control [1]; and 2) other is in the state [28]. Additional types of optimal bilinear control problems, solved by the Krotov’s method, can be found in the literature [13], [14].

This article provides another contribution to optimal control of finite-dimensional bilinear systems by generalizing previous works. Here, a continuous time bilinear state equation is concerned with a quadratic performance index to be minimized. The state equation is time varying, given in a general bilinear form and comprises a deterministic a priori known excitation. The control trajectory is constrained to an admissible set without a specific structure. The performance index is a functional quadratic in the state variables and control signals, and includes a terminal cost. The Krotov’s method is used here for solving this problem. The novelties include the derived sequence of improving functions as well as some additional theoretical remarks.

The rest of this article is organized as follows. Section II defines the addressed problem. It is addressed by the Krotov’s method, which is described in Section III. The main results are described in Section IV. A numerical example is given in Section V. Finally, Section VI concludes this article.

Throughout this article, bold lowercase notation is used for vectors, e.g., \( \mathbf{x} = (x_i)_{i=1}^n \), normally column ones. Bold uppercase notation is used for matrices. A trajectory is some vector function, e.g., \( \mathbf{x} \) is a state trajectory \( \mathbb{R} \to \mathbb{R}^n \). \( \mathbf{x}(t) \) refers to the trajectory \( \mathbf{x} \) evaluated at \( t \).
II. PROBLEM STATEMENT

A set of admissible control trajectories is denoted as $U(x)$ in the following. Its state dependency implies that it might be involved in the constraints laid upon the trajectories’ elements, i.e., the control signals.

**Definition II.1:** Let $x : \mathbb{R} \rightarrow \mathbb{R}^n$ be a state trajectory and $u : \mathbb{R} \rightarrow \mathbb{R}^{nu}$ be a control trajectory. Let $U_t(x)$ be a set of control signals, admissible at a state trajectory $x$. If the pair of trajectories $(x, u)$ satisfies the following bilinear state equation:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + (uN(t))x(t) + g(t)$$  
$$x(0) \forall t \in (0, t_f) \quad (1)$$

and $u_i \in U(x)$ for all $i = 1, \ldots, n_u$, then $(x, u)$ is called an “admissible process”. Here, $uN(t) \triangleq \sum_{i=1}^{n_u} u_i(t)N_i(t)$. $A, N_i : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}, B : \mathbb{R} \rightarrow \mathbb{R}^{n \times n},$ and $g : \mathbb{R} \rightarrow \mathbb{R}^n$, where $g$ is a trajectory of external excitations.

The continuous time bilinear regulator (CBQR) problem is defined as follows.

**Definition II.2 (CBQR):** Find an optimal and admissible process $(x^*, u^*)$ that minimizes the quadratic performance index

$$J(x, u) = \left(\frac{1}{2}\right) \int_0^{t_f} x(t)^T Q(t)x(t) + u(t)^T R(t)u(t) dt$$  
$$+ \frac{1}{2} \int x(t)^T H x(t) dt \quad (2)$$

where $Q : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ such that $Q(t) \geq 0$ and $R : \mathbb{R} \rightarrow \mathbb{R}^{n_u \times n_u}$ such that $R(t) > 0$ and $H \geq 0$.

III. BACKGROUND – THE KROTOV’S METHOD

This section succinctly overviews portions of the Krotov’s theory, tailored and relevant to the treated problem. The definitions and theorem, stated here, are needed for understanding the results provided in Section IV.

Let

$$\dot{x}(t) = f(t, x(t), u(t)); \quad x(0) \forall t \in (0, t_f) \quad (3)$$

be a state equation, $U \subseteq \{R \rightarrow \mathbb{R}^{nu}\}$ be a set of admissible control trajectories, and $X \subseteq \{R \rightarrow \mathbb{R}^n\}$ be a set of state trajectories, which are reachable from $U$ and $(x(0))$. In what follows, the term process refers to some pair $(x \in X, u \in U)$. A pair $(x, u)$ that satisfies (3) is called an admissible process. Let the performance index be a non-negative function $J : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, and $f$ and $J$ define an optimal control problem as follows.

**Definition III.1 (Constrained Optimal Control Problem):** Find an admissible process $(x^*, u^*)$ that minimizes

$$J(x, u) = l_f(x(t_f)) + \int_0^{t_f} l(t, x(t), u(t))dt \quad (4)$$

In some problems, it is easier to embrace sequential approach, instead of directly trying to synthesize $(x^*, u^*)$. In this context, a useful type of sequence is given as follows.

**Definition III.2 (Improving Sequence (Section 6.2 of [27])):** Let $\{(x_k, u_k)\}$ be a sequence of admissible processes and assume that inf$_{u \in U}$ $J(x, u)$ exists. If

$$J(x_k, u_k) \geq J(x_{k+1}, u_{k+1}) \quad (5)$$

for all $k = 0, 1, 2, \ldots$, and

$$\lim_{k \to \infty} J(x_k, u_k) \quad (6)$$

exists, then $\{(x_k, u_k)\}$ is an improving sequence.

The Krotov’s method is aimed at computing such a sequence by successively improving admissible processes. This is done by repeatedly solving a subproblem of process improving as follows. In the following derivations, $\xi$ and $\nu$ are some vectors in $\mathbb{R}^n$ and $\mathbb{R}^{nu}$, respectively. $X(t) \subseteq \mathbb{R}^n$ refers to an intersection of $X$ at $t$, i.e., the set of vectors obtained by evaluating the trajectories in $X$ at $t$. Let $g : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function of time and the state vector. $q_k$ stands for the $q_k$’s partial derivative with respect to (w.r.t.) the time argument. $q_k$ is its gradient w.r.t. the state vector. In order to improve a given process, it is suffice to find $q$ and a feedback $\hat{u}$, meeting the requirements described in the following theorem.

**Theorem III.1 ([27], Th. 6.1):** Let a given admissible process be $(x_1, u_1)$, and let $g : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. The functions $s : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $\nu : \mathbb{R} \rightarrow \mathbb{R}^n$ are constructed from $q$ according to

$$s(t, \xi, \nu) \triangleq l(t, \xi, \nu) + q_1(t, \xi) + q_n(t, \xi) f(t, \xi, \nu) \quad (7)$$

$$s(t, \xi, \nu) \triangleq l_f(t) - q(t, \xi). \quad (8)$$

If the following three statements hold, then $(x_2, u_2)$ is an improved process.

1) $q$ grants $s$ and $\nu$ the property

$$s(t, x_1(t), u_1(t)) = \max_{\xi \in X(t)} s(t, \xi, u_1(t)) \quad \forall t \in (0, t_f) \quad (9)$$

2) $\hat{u}$ is a feedback that satisfies

$$\hat{u}(t, \xi) = \arg \min_{\nu \in (t)} s(t, \xi, \nu) \quad (10)$$

for all $t \in [0, t_f], \xi \in X(t)$.

3) $x_2$ is a state trajectory that solves

$$\dot{x}_2(t) = f(t, x_2(t), \hat{u}(t, x_2(t))) \quad (11)$$

at any $\forall t \in (0, t_f)$, and $x_2$ is a control trajectory such that $u_2(t) = \hat{u}(t, x_2(t))$.

The method repeats process improvements over and over, thereby requiring to derive a sequence of improving functions $\{q_k\}$. If such a sequence can be found, it allows to compute a process, which is a candidate global optimum.

Generally, the procedure is summarized in the following algorithm. Hereinafter, $s_k$ and $s_{f,k}$ are used for signifying $s$ and $s_{f,k}$, which are constructed by $q_k$. First, some initial admissible process $(x_0, u_0)$ should be computed. Afterwards, the following steps are iterated for $k = 0, 1, 2, \ldots$.

1) Find $q_k$ that grants $s_k$ and $s_{f,k}$ the property

$$s_k(t, x_k(t), u_k(t)) = \max_{\xi \in X(t)} s_k(t, \xi, u_k(t)) \quad (12)$$

$$s_{f,k}(x_k(t)) = \max_{\xi \in X(t)} s_{f,k}(\xi) \quad (13)$$

at the current $(x_k, u_k)$ and for all $t \in [0, t_f]$. 

2) Find an improving feedback $\hat{u}_{k+1}$ such that

$$\hat{u}_{k+1}(t, x(t)) = \arg \min_{\nu \in (t)} s_k(t, x(t), \nu) \quad (14)$$

for all $t \in [0, t_f]$. 

3) Compute an improved process by solving

$$\dot{x}_{k+1}(t) = f(t, x_{k+1}(t), \hat{u}_{k+1}(t, x_{k+1}(t))) \quad (15)$$

and setting $u_{k+1}(t) = \hat{u}_{k+1}(t, x_{k+1}(t))$. 


Remarks:
1) This algorithm generates an improving sequence of processes \( \{x_k, u_k\} \) such that \( J(x_k, u_k) \geq J(x_{k+1}, u_{k+1}) \) [27].
2) If at some point, the processes in the improving sequence stop changing, then the process satisfies the Pontryagin’s minimum principle [27], thereby inferring that it is a candidate optimum of the given problem.
3) The method has a significant advantage over algorithms, which are based on small variations. The latter are constrained to small process variations, which is troublesome as it leads to slow convergence rate, and for some optimal control problems, small variations are impossible [26].

IV. MAIN RESULTS

In order to solve the CBQR problem by the Krotov’s method, a class of suitable improving functions should be formulated. Such a class is defined in the following lemmas. The notation \( M(t, \xi) \) signifies an \( \mathbb{R}^{n \times n} \) matrix \( [N_1(t) \xi \quad N_2(t) \xi \quad \cdots \quad N_n(t) \xi] \).

Lemma IV.1: Let

\[
q(t, \xi) = \frac{1}{2} \xi^T P(t) \xi + p(t)^T \xi; \quad P(t) = \mathbf{H}; \quad p(t) = 0
\]

where \( \xi \in \mathbb{R}^n \), \( P : \mathbb{R} \to \mathbb{R}^{n \times n} \) is a smooth and symmetric matrix function, and \( p : \mathbb{R} \to \mathbb{R}^n \) is a smooth vector function.

The vector of control laws \( (\hat{u}_i)_{i=1}^n \), which minimizes \( s(t, x(t), u(t)) \), is given by

\[
\hat{u}(t, x(t)) = \arg \min_{\nu \in L(t, x)} \left( \frac{1}{2} \nu^T R(t) \nu \right)
+ q_x(t, x(t))(B(t) + M(t, x(t))\nu).
\]

Proof: Let \( \nu \in \mathbb{R}^n \). By substituting the state equation into \( s \), we get

\[
s(t, x(t), \nu) = q_t(t, x(t)) + q_x(t, x(t))f(t, x(t), \nu)
+ \frac{1}{2} (x(t)^T Q(t) x(t) + \nu^T R(t) \nu)
\]

\[
= q_t(t, x(t)) + q_x(t, x(t))\left(A(t) x(t) + B(t) \nu
+ [\nu N(t)] x(t) + g(t)\right)
+ \frac{1}{2} x(t)^T Q(t) x(t) + \nu^T R(t) \nu
\]

\[
= f_2(t, x(t)) + q_x(t, x(t))(B(t)
+ M(t, x(t))\nu) + \frac{1}{2} \nu^T R(t) \nu.
\]

Lemma IV.2: Let \( (x, u) \) be a given process, and let \( P \) and \( p \) be solutions to

\[
\dot{P}(t) = -P(t)(A(t) + [uN(t)])
- (A(t) + [uN(t)])^T P(t) - Q(t)
\]

with \( P(t_f) = \mathbf{H} \), and

\[
\dot{p}(t) = -(A(t) + [uN(t)])^T p(t)
- P(t)(B(t)u(t) + g(t))
\]

with \( p(t_f) = 0. \) Then,

\[
q(t, x(t)) = \frac{1}{2} x(t)^T P(t) x(t) + p(t)^T x(t)
\]

grants \( s \) and \( s_f \) the property

\[
s(t, x(t), u(t)) = \max_{\xi \in \mathcal{X}(t_f)} s(t, \xi, u(t))
\]

\[
s_f(x(t_f)) = \max_{\xi \in \mathcal{X}(t_f)} s_f(\xi).
\]

Proof: For any \( x(t_f) \in \mathcal{X}(t_f) \), we get

\[
s_f(x(t_f)) = \frac{1}{2} x(t_f)^T H x(t_f) - q(t, x(t_f)) = 0.
\]

Therefore, it is obvious that \( s_f(\xi) \leq s_f(x(t_f)) \) for all \( \xi \in \mathcal{X}(t_f) \). The partial derivatives of \( q \) and \( q_x \) into (13), and then arranging it in a canonical form for \( x(t) \), yields

\[
s(t, x(t), u(t))
= \frac{1}{2} x(t)^T \left( \dot{P}(t) + P(t) A(t) + [uN(t)] \right)
+ (A(t) + [uN(t)])^T P(t) + Q(t) x(t)
+ x(t)^T \left( p(t) + (A(t) + [uN(t)])^T p(t) + P(t)(B(t)u(t) + g(t)) \right)
+ p(t)^T (B(t)u(t) + g(t)) + \frac{1}{2} u(t)^T R(t) u(t).
\]

Because the dependency on \( x(t) \) has dropped, it is obvious that \( s(t, x(t), u(t)) = s(t, \xi, u(t)) \), and that

\[
\left. s(t, \xi, u(t)) \right| \leq s(t, x(t), u(t))
\]

for all \( \xi \in \mathcal{X}(t). \)

Let \( (x_1, u_1) \) be a given admissible process. Solve (14) and (15) to \( u_1 \) and obtain \( P_1 \) and \( p_1 \). An improving feedback \( u_2 \), which is constructed by (12) in conjunction with \( P_1 \) and \( p_1 \), allows to obtain an improved process \( (x_2, u_2) \).
In other words, lemmas IV.1 and IV.2 allow to compute two sequences: 1) \( \{ q_k \} \); and 2) \( \{ (x_k, u_k) \} \), such that the second one is an improving sequence. As \( J \) is non-negative, it has an infimum and \( \{ J(x_k, u_k) \} \) converges. In case that the processes stop changing, then a candidate optimum is obtained (see the second remark in theorem III.1).

The corresponding algorithm is summarized in Fig. 1. Its output is an arbitrarily close approximation of \( P^* \) and \( p^* \), which define the candidate optimal control law [see (12)] through \( q^* \).

Remarks:
1) Equation (18) provides an alternative approach for computing \( J(x, u) \). As \( s_j(x(t_f)) = 0 \), it can be shown that

\[
J(x, u) = J_{eq}(x, u) \triangleq q(0, x(0)) + \int_0^{t_f} s(t, x(t), u(t)) dt \]

\[ = \frac{1}{2} x(0)^T P(0)x(0) + p(0)^T x(0) \]

\[ + \left( \int_0^{t_f} p(t)^T (B(t)u(t) + g(t)) \right. \]

\[ + \left. \frac{1}{2} u(t)^T R(t)u(t) dt \right). \]

2) The minimization described in (12) defines the improving feedback. It should be solved for each \( t \in [0, t_f] \) independently and can have different attributes, depending on \( U \) and \( R \); e.g., let \( R(t) = R_0 > 0 \) and \( U(t, x) = \mathbb{R}^{n_u} \forall t \in [0, t_f] \). Then,

\[
\hat{u}(t, x(t)) = R_c^{-1}(B(t) + M(t, x(t)))^T (P(t)x(t) + p(t)).
\]

Another example is the singular case, e.g., when \( R(t) = 0 \) at some \( t \). Then, in this time instance, the improving feedback is

\[
\hat{u}(t, x(t)) = \arg \min_{\nu \in U(t, x)} \left( (x(t)^T P(t) + p(t)^T) \right.
\]

\[ \times \left. (B(t) + M(t, x(t)) \nu) \right). \]

This is in general a nonlinear programming problem [29]. An existence of solution, in this case, depends on the nature of \( U(t, x) \), and therefore, should be discussed in the context of a given problem.
Fig. 2. Dynamic scheme of a two-floor structure, configured with two SAVS devices.

Fig. 3. Control signals of the CBQR case. (a) \( u_1 \) — unlocking pattern 1: only device 2 is unlocked (b) \( u_2 \) — unlocking pattern 2: both devices are unlocked

3) Treat \( u \) as a given trajectory. Consequently, (14) is a linear matrix ordinary differential equation (ODE) w.r.t. \( x \). Using the Kronecker product, it can be reorganized into a standard linear ODE form [30]. Next, assume that the elements of \( A, B, (N_i)_{i=1}^n, g, \) and \( Q \) are all integrable on \((0, t_f)\). If \( u \) is integrable on \((0, t_f)\), then this linear ODE meets Caratheodory’s conditions. Thereby, it has a unique absolutely continuous solution \( P \) [31]. As the domain is finite and \( P \) is absolutely continuous, then \( P \) is bounded. It follows that \( P(Bu + g) \) is integrable on \((0, t_f)\) and (15) is a linear ODE that meets Caratheodory’s conditions too. Hence, there exists a unique bounded solution \( p \).

V. NUMERICAL EXAMPLE

The use of this method is exemplified here by a structural control problem. The method is applied to an optimal control problem of a structure with two degrees of freedom (DOF) and two semi-active variable stiffness (SAVS) devices [32]. A shear model is used for plant modeling. Its dynamic scheme and the devices are shown in Fig. 2. The masses are lumped and located at the floors’ ceilings. They are 100 [ton] and 200 [ton] at the first and second floors, respectively. The floors’ horizontal stiffnesses are identical and equal to \( 100 \times 10^6 \) [N/m]. Let \( z_1 \) and \( z_2 \) be the horizontal DOFs in the ceilings. The related mass and stiffness matrices are

\[
M = \begin{bmatrix} 100 & 0 \\ 0 & 200 \end{bmatrix} \times 10^3, \quad K = \begin{bmatrix} 200 & -100 \\ -100 & 100 \end{bmatrix} \times 10^6.
\]
The structure is assumed to have inherent damping, described by a proportional damping matrix $C_d = 0.001M + 0.0001K$. A horizontal seismic external excitation was applied to the structure by ground acceleration signal of $\ddot{z}_g(t) = 3\sin(16.55t)$. The equation of motion is

$$M\ddot{z}(t) + C_d\dot{z}(t) + Kz(t) = -\gamma \ddot{z}_g(t) + \Psi w(t) \quad (19)$$

with

$$\Psi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}; \quad \gamma = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $z \triangleq (z_1, z_2)$ is the trajectory of the DOF displacements and $w \triangleq (w_1, w_2)$ is the trajectory of the control forces. Spring and variable dashpot were used for modeling each SAVS device. The device’s spring stiffness is $k_{savs} = 25 \times 10^6 \text{ [N/m]}$, accounting for the bracing stiffness and the fluid’s bulk modulus. The variable dashpot can provide either a finite damping coefficient of $5 \times 10^3 \text{ [N/(m/s)]}$ or infinity, depending whether it is unlocked or locked. Here, the control policy is restricted to one out of three locking patterns [32]: 1) both devices are unlocked; 2) only the second device is unlocked; or 3) both devices are locked. Each device provides another state variable to the system. Here, these states variables are represented by the control forces’ signals, providing another two state equations as follows:

$$\dot{w}(t) = -5 \times 10^4 \begin{bmatrix} u_1(t) & 0 \\ 0 & 1 \end{bmatrix} + u_2(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t)$$

$$- 25 \times 10^6 \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \dot{z}(t)$$

where $u \triangleq (u_1, u_2)$ is the control input trajectory. These settings lead to an admissible set of control inputs $u(t) \in U = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Every object in this set reflects the SAVS devices’ unlocking patterns 1, 2, and 3, respectively. Note that $U$’s finiteness prohibits the use of variational methods with respect to $u$.

Putting these equations altogether boils down to the following bilinear state equation:

$$\dot{x}(t) = Ax(t) + u_1(t)N_1x(t) + u_2(t)N_2x(t) + g(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -M^{-1}K & -M^{-1}C & M^{-1}\Psi \\ 0 & -\Psi^T / k_{savs} & 0 \end{bmatrix}$$

$$N_1 = \text{diag}(0, 0, 0, 0, 0, -5000)$$

$$N_2 = \text{diag}(0, 0, 0, 0, -5000, -5000)$$

$$g(t) = \begin{bmatrix} 0 & 0 & -3 & -3 & 0 & 0 \end{bmatrix}^T \sin(16.55t)$$

where the state trajectory is $x = (z_1, z_2, \dot{z}_1, \dot{z}_2, w_1, w_2)$. The time span for the problem was set to $t = 0, 5$ and the initial conditions were set to zero.

The performance evaluation accounts for interstory drifts and control forces, it is

$$J(x, u) = \frac{1}{2} \int_0^5 x_1(t)^2 \cdot 10^5 + (x_2(t) - x_1(t))^2 \cdot 10^5$$

TABLE I

| $\Delta J$ | $\Delta J$ |
|---|---|
| $i$ | $J$ | $i$ | $J$ |
| 1 | $4.59 \times 10^4$ | 8 | $4.51 \times 10^4$ |
| 2 | $4.54 \times 10^4$ | 9 | $4.49 \times 10^4$ |
| 3 | $4.47 \times 10^4$ | 10 | $4.49 \times 10^4$ |
| 4 | $4.45 \times 10^4$ | 11 | $4.48 \times 10^4$ |
| 5 | $4.45 \times 10^4$ | 12 | $4.47 \times 10^4$ |
| 6 | $4.44 \times 10^4$ | 13 | $4.49 \times 10^4$ |
| 7 | $4.43 \times 10^4$ | 14 | $4.47 \times 10^4$ |
| 8 | $4.42 \times 10^4$ | 15 | $4.47 \times 10^4$ |

The matrices, $Q$ and $H$ were constructed accordingly. It follows that $R \equiv 0$ as the control inputs have no weight in $J$.

According to step 1 of the iterations stage, an improving feedback should be found at each time instance. In the addressed problem, it reduces to

$$u_{k+1}(t, x(t)) = \arg\min_{w \in U} y_k(t, x(t))$$

where $y_k$ is the row vector defined in the algorithm. This minimization should be carried over a finite set of points, and therefore, can be calculated by a simple table-search technique.

The method was realized by MATLAB through a standard desktop PC. Solving the ODEs was based on a fourth-order Runge-Kutta method. The following two cases were computed:

1) The first case is a straightforward application of the suggested method. It is referred to as CBQR.

2) The second one, referred to as MPC, describes a model predictive control with a CBQR optimizer. Its horizon was set to 1 s and three improvements were carried out in each control update.

In the CBQR case, the improvement process was iterated 15 times, each lasting approx 5.9 s in average. Table I provides $J$’s values of the processes obtained in this case. In this table, $i$ stands for the process number. The table also provides the relative change in $J$, signified by $\Delta J \triangleq J_i - J_{i-1}$. $J$’s value at the MPC process is $1.78 \times 10^13$.

The form of the control signals, synthesized by the feedback obtained in the CBQR case, are showed in Fig. 3. As the signals are rapidly varying, they are presented over a representing time interval $t \in [0.5, 1.2]$. Careful inspection of $u$ will tell that $u_1(t)u_2(t) = 0$ for all $t$, meaning that only one unlocking pattern is active in each moment, as it was required by the problem’s definition. Generally, SAVS controllers fall into one out of two operating modes: 1) resetting mode; or 2) switching mode. The former refers to devices, which are normally closed and are opened momentarily. The latter is more general, referring to devices that are switching from open to close and vice-versa, without favoring any of these states [33]. Fig. 3 shows finite unlocking segments, thereby manifesting that switching mode is the optimal in this case. Fig. 4 shows the control force signals of each case. When a device is unlocked, the elastic energy that it gradually accrued during its locked state rapidly drops and zeros the control force in the device. This is the reason for $w$’s saw-tooth shape. Each of these drops is related with one of the unlocking patterns related to $u$. An effective control is expected to reduce the amplitude of the vibrating structure during the ground motion. This goal was indeed attained by each of the synthesized controllers, as shown in Fig. 5.
VI. CONCLUSION

The CBQR is an optimal control problem, concerning a finite-dimensional continuous time bilinear state equation and a quadratic performance index to be minimized. In this article, the state equation was time varying, given in a general bilinear form and comprises a deterministic a priori known excitation. The control trajectory was constrained to an admissible set, which was left here in a general form. This form should be set in accordance to the nature of a specific/class of addressed CBQR problem. The performance index is a functional quadratic in the state variables and control signals, and includes a terminal cost. This problem, which was written here in general structure, was addressed by the Krotov’s method. The main results comprise a definition of a suitable sequence of improving functions, allowing to generate an improving sequence of processes. This sequence is convergent by means of the performance index. If the processes converge too, then the obtained process provides an arbitrarily close approximation to a candidate optimal feedback. The trajectory, synthesized by this feedback, satisfies Pontryagin’s minimum principle. Even though it is merely a necessary condition, it is normally enough in many practical applications. As a matter of fact, assuming that the improving sequence, obtained by the Krotov’s method, consists of at least two different elements and the processes converge, then it is not only that the outcome satisfies Pontryagin’s minimum principle, but it is also guaranteed not to be maximal [27]. Solving by this method, which is not relying on small variations, is a significant advantage over such that do. It makes the suggested solution valid in problems with control inputs that do not allow small variations, e.g., control vectors that are constrained to closed subsets of $\mathbb{R}^n$. Finally, this article summarizes the results in an algorithm and furnishes a numerical example of structural control. The example consists of a structure with two floors, which is controlled by two SAVS devices. This configuration constrains the control signals to a discrete set of values, hence prohibiting the use of methods that rely on small variations of the control signals. Two control solutions are presented to this problem: 1) a formal CBQR solution; and 2) an MPC with a CBQR optimizer.

These results provide a useful tool for the control design of bilinear systems and/or as a benchmark for other control strategies for such systems.

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