MIN-MAX FOR SWEEPOUTS BY CURVES

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0. Introduction

Given a Riemannian metric on the 2-sphere, sweep the 2-sphere out by a continuous one-parameter family of closed curves starting and ending at point curves. Pull the sweepout tight by, in a continuous way, pulling each curve as tight as possible yet preserving the sweepout. We show the following useful property (see Theorem 1.9 below); cf. [CM1], [CM2], proposition 3.1 of [CD], proposition 3.1 of [Pi], and 12.5 of [Al]:

Each curve in the tightened sweepout whose length is close to the length of the longest curve in the sweepout must itself be close to a closed geodesic. In particular, there are curves in the sweepout that are close to closed geodesics.

Finding closed geodesics on the 2-sphere by using sweepouts goes back to Birkhoff in the 1920s; see [B] and section 2 in [Cr] about Birkhoff’s ideas. The argument works equally well on any closed manifold, but only produces non-trivial closed geodesics when the width, which is defined in (1.1) below, is positive. For instance, when $M$ is topologically a 2-sphere, the width is loosely speaking the length of the shortest closed curve needed to “pull over” $M$. Thus Birkhoff’s argument gives that the width is realized as the length of a closed geodesic.

The above useful property is virtually always implicit in any sweepout construction of critical points for variational problems yet it is not always recorded since most authors are only interested in the existence of one critical point.

Similar results holds for sweepouts by 2-spheres instead of circles; cf. [CM2]. The ideas are essentially the same in the two cases, though the techniques in the curve case are purely ad hoc whereas in the 2-sphere case additional techniques, developed in the 1980s, have to be used to deal with energy concentration (i.e., “bubbling”); cf. [Jo].

1. Existence of good sweepouts by curves

Let $M$ be a closed Riemannian manifold. Fix a large positive integer $L$ and let $\Lambda$ denote the space of piecewise linear maps from $S^1$ to $M$ with exactly $L$ breaks (possibly with unnecessary breaks) such that the length of each geodesic segment is at most $2\pi$, parametrized by a (constant) multiple of arclength, and with Lipschitz bound $L$. By a linear map, we mean a (constant speed) geodesic. Let $G \subset \Lambda$ denote the set of immersed closed geodesics in $M$ of length at most $2\pi L$. (The energy of a curve in $\Lambda$ is equal to its length squared divided by $2\pi$. In other words, energy and length are essentially equivalent.)

We will use the distance and topology on $\Lambda$ given by the $W^{1,2}$ norm (Sobolev norm) on the space of maps from $S^1$ to $M$. The simplest way to define the $W^{1,2}$ norm is to isometrically

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embed the compact manifold $M$ into some Euclidean space $\mathbb{R}^N$. It will be convenient to scale $\mathbb{R}^N$, and thus $M$, by a constant so that it satisfies the following: (M1) $\sup_M |A| \leq 1/16$, where $|A|^2$ is the norm squared of the second fundamental form of $M$, i.e., the sum of the squares of the principal curvatures (see, e.g., (1.24) on page 4 of [CM3]); (M2) the injectivity radius of $M$ is at least $8\pi$ and the curvature is at most $1/64$, so that every geodesic ball of radius at most $4\pi$ in $M$ is strictly geodesically convex; (M3) if $x, y \in M$ with $|x - y| \leq 1$, then $\text{dist}_M(x, y) \leq 2|x - y|$.

1.1. The width. Let $\Omega$ be the set of continuous maps $\sigma : S^1 \times [-1, 1] \to M$ so that for each $t$ the map $\sigma(\cdot, t)$ is in $W^{1,2}$, the map $t \to \sigma(\cdot, t)$ is continuous from $[-1, 1]$ to $W^{1,2}$, and finally $\sigma$ maps $S^1 \times \{-1\}$ and $S^1 \times \{1\}$ to points. Given a map $\hat{\sigma} \in \Omega$, the homotopy class $\Omega_{\hat{\sigma}}$ is defined to be the set of maps $\sigma \in \Omega$ that are homotopic to $\hat{\sigma}$ through maps in $\Omega$. The width $W = W(\hat{\sigma})$ associated to the homotopy class $\Omega_{\hat{\sigma}}$ is defined by taking inf of max of the energy of each slice. That is, set

$$
W = \inf_{\sigma \in \Omega_{\hat{\sigma}}} \max_{t \in [-1, 1]} \text{Energy}(\sigma(\cdot, t)),
$$

where the energy is given by $\text{Energy}(\sigma(\cdot, t)) = \int_{S^1} |\partial_2 \sigma(x, t)|^2 \, dx$.

The main theorem, Theorem 1.9 that almost maximal slices in the tightened sweepout are almost geodesics, is proven in subsection 1.4. The proof of this theorem as well as the construction of the sequence of tighter and tighter sweepouts uses a curve shortening map that is defined in the next subsection. We also state the key properties of the shortening map in the next subsection, but postpone their proofs to Section 2 and the appendices.

1.2. Curve shortening $\Psi$. The curve shortening is a map $\Psi : \Lambda \to \Lambda$ so that

1. $\Psi(\gamma)$ is homotopic to $\gamma$ and $\text{Length}(\Psi(\gamma)) \leq \text{Length}(\gamma)$.

2. $\Psi(\gamma)$ depends continuously on $\gamma$.

3. There is a continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ so that

$$
\text{dist}^2(\gamma, \Psi(\gamma)) \leq \phi \left( \frac{\text{Length}^2(\gamma) - \text{Length}^2(\Psi(\gamma))}{\text{Length}^2(\Psi(\gamma))} \right).
$$

4. Given $\epsilon > 0$, there exists $\delta > 0$ so that if $\gamma \in \Lambda$ with $\text{dist}(\gamma, G) \geq \epsilon$, then $\text{Length}(\Psi(\gamma)) \leq \text{Length}(\gamma) - \delta$.

To define $\Psi$, we will fix a partition of $S^1$ by choosing $2L$ consecutive evenly spaced points

$$
x_0, x_1, x_2, \ldots, x_{2L} = x_0 \in S^1,
$$

1Recall that the square of the $W^{1,2}$ norm of a map $f : S^1 \to \mathbb{R}^N$ is

$$
\int_{S^1} (|f|^2 + |f'|^2).
$$

Thus two curves that are $W^{1,2}$ close are also $C^0$ close; cf. [LS].

2A particularly interesting example is when $M$ is a topological 2-sphere and the induced map from $S^2$ to $M$ has degree one. In this case, the width, defined below, is positive and realized by one or more non-trivial closed geodesics. In general, the width is always non-negative but may not always be positive.

3This map is essentially what is usually called Birkhoff’s curve shortening process, see section 2 of [C].

4Note that this is not necessarily where the piecewise linear maps have breaks.
so that $|x_j - x_{j+1}| = \frac{\pi}{L}$. $\Psi(\gamma)$ is given in three steps. First, we apply step 1 to $\gamma$ to get a curve $\gamma_e$, then we apply step 2 to $\gamma_e$ to get a curve $\gamma_o$. In the third and final step, we reparametrize $\gamma_o$ to get $\Psi(\gamma)$.

**Step 1:** Replace $\gamma$ on each *even* interval, i.e., $[x_{2j}, x_{2j+2}]$, by the linear map with the same endpoints to get a piecewise linear curve $\gamma_e : S^1 \to M$. Namely, for each $j$, we let $\gamma_e|_{[x_{2j}, x_{2j+2}]}$ be the unique shortest (constant speed) geodesic from $\gamma(x_{2j})$ to $\gamma(x_{2j+2})$.

**Step 2:** Replace $\gamma_e$ on each *odd* interval by the linear map with the same endpoints to get the piecewise linear curve $\gamma_o : S^1 \to M$.

**Step 3:** Reparametrize $\gamma_o$ (fixing $\gamma_o(x_0)$) to get the desired constant speed curve $\Psi(\gamma) : S^1 \to M$.

It is easy to see that $\Psi$ maps $\Lambda$ to $\Lambda$ and has property (1); cf. section 2 of [Cr]. Properties (2), (3) and (4) for $\Psi$ are established in Section 2 and Appendix B. Throughout the rest of this section, we will assume these properties and use them to prove the main theorem.

The next lemma, which combines (3) and (4), is the key to producing the desired sequence of sweepouts.

**Lemma 1.4.** Given $W \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ so that if $\gamma \in \Lambda$ and

$$2\pi (W - \delta) < \text{Length}^2 (\Psi(\gamma)) \leq \text{Length}^2 (\gamma) < 2\pi (W + \delta),$$

then $\text{dist}(\Psi(\gamma), G) < \epsilon$.

**Proof.** If $W \leq \epsilon^2/6$, then the Wirtinger inequality (see footnote 6) yields the lemma with $\delta = \epsilon^2/6$.

Assume next that $W > \epsilon^2/6$. The triangle inequality gives

$$\text{dist}(\Psi(\gamma), G) \leq \text{dist}(\Psi(\gamma), \gamma) + \text{dist}(\gamma, G).$$

Since $\Psi$ does not decrease the length of $\gamma$ by much, property (4) of $\Psi$ allows us to bound $\text{dist}(\gamma, G)$ by $\epsilon/2$ as long as $\delta$ is sufficiently small. Similarly, property (3) of $\Psi$ allows us to bound $\text{dist}(\Psi(\gamma), \gamma)$ by $\epsilon/2$ as long as $\delta$ is sufficiently small. \hfill $\Box$

1.3. **Defining the sweepouts.** Choose a sequence of maps $\hat{\sigma}^j \in \Omega_\delta$ with

$$\max_{t \in [-1, 1]} \text{Energy} (\hat{\sigma}^j (\cdot, t)) < W + \frac{1}{j}.$$ 

Observe that (1.7) and the Cauchy-Schwarz inequality imply a uniform bound for the length and uniform $C^{1/2}$ continuity for the slices, that are both independent of $t$ and $j$. The first follows immediately and the latter follows from

$$|\hat{\sigma}^j(x, t) - \hat{\sigma}^j(y, t)|^2 \leq \left( \int_x^y |\partial_s \hat{\sigma}^j(s, t)| \, ds \right)^2 \leq |y - x| \int_x^y |\partial_s \hat{\sigma}^j(s, t)|^2 \, ds \leq |y - x| (W + 1).$$

We will replace the $\hat{\sigma}^j$’s by sweepouts $\sigma^j$ that, in addition to satisfying (1.7), also satisfy that the slices $\sigma^j(\cdot, t)$ are in $\Lambda$. We will do this by using local linear replacement similar to Step 1 of the construction of $\Psi$. Namely, the uniform $C^{1/2}$ bound for the slices allows us to fix a partition of points $y_0, \ldots, y_N = y_0$ in $S^1$ so that each interval $[y_i, y_{i+1}]$ is always mapped
to a ball in $M$ of radius at most $4\pi$. Next, for each $t$ and each $j$, we replace $\hat{\sigma}^j(\cdot, t)[y_i, y_{i+1}]$ by the linear map (geodesic) with the same endpoints and call the resulting map $\tilde{\sigma}^j(\cdot, t)$. Reparametrize $\tilde{\sigma}^j(\cdot, t)$ to have constant speed to get $\sigma^j(\cdot, t)$. It is easy to see that each $\sigma^j(\cdot, t)$ satisfies (1.7). Furthermore, the length bound for $\sigma^j(\cdot, t)$ also gives a uniform Lipshitz bound for the linear maps; let $L$ be the maximum of $N$ and this Lipshitz bound.

It remains to show that $\sigma^j$ is continuous in the transversal direction, i.e., with respect to $t$, and homotopic to $\sigma$ in $\Omega$. These facts were established already by Birkhoff (see [B] and section 2 of [Cr]), but also follow immediately from Appendix B.

Finally, applying the replacement map $\Psi$ to each $\sigma^j(\cdot, t)$ gives a new sequence of sweepouts $\gamma^j = \Psi(\sigma^j)$. (By Appendix B $\Psi$ depends continuously on $t$ and preserves the homotopy class $\Omega_4$; it is clear that $\Psi$ fixes the constant maps at $t = \pm 1$.)

1.4. Almost maximal implies almost critical. Our main result is that this sequence $\gamma^j$ of sweepouts is tight in the sense of the Introduction. Namely, we have the following theorem.

**Theorem 1.9.** Given $W \geq 0$ and $\epsilon > 0$, there exist $\delta > 0$ so that if $j > 1/\delta$ and for some $t_0$

$$2\pi \text{Energy}(\gamma^j(\cdot, t_0)) = \text{Length}^2(\gamma^j(\cdot, t_0)) > 2\pi(W - \delta),$$

then for this $j$ we have $\text{dist}(\gamma^j(\cdot, t_0), G) < \epsilon$.

**Proof.** Let $\delta$ be given by Lemma 1.4. By (1.10), (1.7), and using that $j > 1/\delta$, we get

$$2\pi(W - \delta) < \text{Length}^2(\gamma^j(\cdot, t_0)) \leq \text{Length}^2(\sigma^j(\cdot, t_0)) < 2\pi(W + \delta).$$

Thus, since $\gamma^j(\cdot, t_0) = \Psi(\sigma^j(\cdot, t_0))$, Lemma 1.4 gives $\text{dist}(\gamma^j(\cdot, t_0), G) < \epsilon$, as claimed. $\square$

2. Establishing Properties (2), (3) and (4) for $\Psi$

To prove (2) and (3), it is useful to observe that there is an equivalent, but more symmetric, way to construct $\Psi(\gamma)$ using four steps:

(A1) Follow Step 1 to get $\gamma_e$.

(B1) Reparametrize $\gamma_e$ (fixing the image of $x_0$) to get the constant speed curve $\tilde{\gamma}_e$. This reparametrization moves the points $x_j$ to new points $\tilde{x}_j$ (i.e., $\gamma_e(x_j) = \tilde{\gamma}_e(\tilde{x}_j)$).

(A2) Do linear replacement on the odd $\tilde{x}_j$ intervals to get $\gamma_o$.

(B2) Reparametrize $\tilde{\gamma}_o$ (fixing the image of $x_0$) to get the constant speed curve $\Psi(\gamma)$.

The reason that this gives the same curve is that $\tilde{\gamma}_o$ is just a reparametrization of $\gamma_o$. We will also use that each of the four steps is energy non-increasing. This is obvious for the linear replacements, since linear maps minimize energy. It follows from the Cauchy-Schwarz inequality for the reparametrizations, since for a curve $\sigma : S^1 \to M$ we have

$$\text{Length}^2(\sigma) \leq 2\pi \text{Energy}(\sigma),$$

with equality if and only if $|\sigma'| = \text{Length}(\sigma)/(2\pi)$ almost everywhere.

Using the alternative way of defining $\Psi(\gamma)$ in four steps, we see that (3) follows from the triangle inequality once we bound $\text{dist}(\gamma, \gamma_e)$ and $\text{dist}(\gamma_e, \tilde{\gamma}_e)$ in terms of the decrease in length (as well as the analogs for steps (A2) and (B2)).

The bound on $\text{dist}(\gamma, \gamma_e)$ follows directly from the following, see Appendix A for the proof:
Lemma 2.2. There exists \( C \) so that if \( I \) is an interval of length at most \( 2\pi/L \), \( \sigma_1 : I \to M \) is a Lipschitz curve with \( |\sigma'_1| \leq L \), and \( \sigma_2 : I \to M \) is the minimizing geodesic with the same endpoints, then

\[
(2.3) \quad \text{dist}^2(\sigma_1, \sigma_2) \leq C \left( \text{Energy}(\sigma_1) - \text{Energy}(\sigma_2) \right).
\]

Applying Lemma 2.2 on each of the \( L \) intervals in step (A1), we get that

\[
(2.4) \quad \text{dist}^2(\gamma, \gamma_e) \leq C \left( \text{Energy}(\gamma) - \text{Energy}(\gamma_e) \right) \leq \frac{C}{2\pi} \left( \text{Length}^2(\gamma) - \text{Length}^2(\Psi(\gamma)) \right).
\]

This gives the desired bound on \( \text{dist}(\gamma, \gamma_e) \) since \( \text{Length}(\Psi(\gamma)) \leq 2\pi L \).

In bounding \( \text{dist}(\gamma_e, \tilde{\gamma}_e) \), we will use that \( \gamma_e \) is just the composition \( \tilde{\gamma}_e \circ P \), where \( P : S^1 \to S^1 \) is a monotone piecewise linear map. Using that \( |\tilde{\gamma}'_e| = \text{Length}(\tilde{\gamma}_e)/(2\pi) \) (away from the breaks) and that the integral of \( P' \) is \( 2\pi \), an easy calculation gives

\[
(2.5) \quad \int (P' - 1)^2 = \int (P')^2 - 2\pi = \int \left( \frac{|\gamma'_e|}{|\tilde{\gamma}'_e \circ P|} \right)^2 - 2\pi = \frac{4\pi^2}{\text{Length}^2(\tilde{\gamma}_e)} \int |\gamma'_e|^2 - 2\pi.
\]

Since \( \gamma_e \) and \( \tilde{\gamma}_e \) agree at \( x_0 = x_{2L} \), the Wirtinger inequality bounds \( \text{dist}^2(\gamma_e, \tilde{\gamma}_e) \) in terms of

\[
(2.6) \quad \int |(\tilde{\gamma}_e \circ P)' - \tilde{\gamma}'_e|^2 \leq 2 \int |(\tilde{\gamma}'_e \circ P) P' - \tilde{\gamma}'_e \circ P|^2 + 2 \int |\tilde{\gamma}'_e \circ P - \tilde{\gamma}'_e|^2.
\]

We will bound both terms on the right hand side of (2.6) in terms of \( \int |P' - 1|^2 \) and then appeal to (2.4). To bound the first term, use that \( |\tilde{\gamma}'_e| \) is (a constant) \( \leq L \) to get

\[
(2.7) \quad \int |(\tilde{\gamma}'_e \circ P) P' - \tilde{\gamma}'_e \circ P|^2 \leq L^2 \int |P' - 1|^2.
\]

To bound the second integral, we will use that when \( x \) and \( y \) are points in \( S^1 \) that are not separated by a break point, then \( \tilde{\gamma}_e \) is a geodesic from \( x \) to \( y \) and, thus, \( \tilde{\gamma}''_e \) is normal to \( M \) and \( |\tilde{\gamma}'_e| \leq |\tilde{\gamma}'_e|^2 \sup_M |A| \leq \frac{L^2}{16} \). Therefore, integrating \( \tilde{\gamma}''_e \) from \( x \) to \( y \) gives

\[
(2.8) \quad |\tilde{\gamma}'_e(x) - \tilde{\gamma}'_e(y)| \leq |x - y| \sup |\tilde{\gamma}''_e| \leq \frac{L^2}{16} |x - y|.
\]

Divide \( S^1 \) into two sets, \( S_1 \) and \( S_2 \), where \( S_1 \) is the set of points within distance \( (\pi \int |P' - 1|^2)^{1/2} \) of a break point for \( \tilde{\gamma}_e \). Since \( P(x_0) = x_0 \), arguing as in (1.8) gives \( |P(x) - x| \leq (\pi \int |P' - 1|^2)^{1/2} \). Thus, if \( x \in S_2 \), then \( \tilde{\gamma}_e \) is smooth between \( x \) and \( P(x) \). Consequently, (2.8) gives

\[
(2.9) \quad \int_{S_2} |\tilde{\gamma}'_e \circ P - \tilde{\gamma}'_e|^2 \leq \frac{L^4}{256} \int_{S_2} |P(s) - s|^2 \leq \frac{L^4}{64} \int |P' - 1|^2,
\]

The map \( P \) is Lipschitz, but the inverse map \( P^{-1} \) may not be if \( \gamma_e \) is constant on an interval.

The Wirtinger inequality is just the usual Poincare inequality which bounds the \( L^2 \) norm in terms of the \( L^2 \) norm of the derivative; i.e., \( \int_0^{2\pi} f^2 dt \leq 4 \int_0^{2\pi} (f')^2 dt \) provided \( f(0) = f(2\pi) = 0 \).
Lemma 2.2 and (2.5). However, this would contradict that the \( \gamma \) fact, which was used already by Birkhoff (see section 2 in [Cr]), follows immediately from any such closed immersed geodesic, completing the proof of (4).

Proof. (Lemma A.1.)

We will need a simple consequence of (M1) and (M3) in Section 1.

Lemma A.1. If \( x, y \in M \), then \(|(x - y)^\perp| \leq |x - y|^2\), where \((x - y)^\perp\) is the normal component to \( M \) at \( y \).

Proof. If \(|x - y| \geq 1\), then the claim is clear. Assume therefore that \(|x - y| < 1\) and \( \alpha : [0, \ell] \to M \) is a minimizing unit speed geodesic from \( y \) to \( x \) with \( \ell \leq 2|x - y|\). Let \( V \) be the unit normal vector \( V = (x - y)^\perp/|(x - y)^\perp| \), so \( \langle \alpha'(0), V \rangle = 0 \), and observe that

\[
|(x - y)^\perp| = \int_0^\ell \langle \alpha'(s), V \rangle \, ds = \int_0^\ell \langle \alpha'(0) + \int_0^s \alpha''(t) \, dt, V \rangle \, ds \leq \int_0^\ell \int_0^s |\alpha''(t)| \, dt \, ds
\]

\[
(A.2) \leq \int_0^\ell \int_0^s |A(\alpha(t))| \, dt \, ds \leq \frac{1}{2} \ell^2 \sup_M |A| \leq |x - y|^2.
\]

Proof. (of Lemma A.2). Integrating by parts and using that \( \sigma_1 \) and \( \sigma_2 \) are equal on \( \partial I \) gives

\[
(A.3) \quad \int_I |\sigma'_1|^2 - \int_I |\sigma'_2|^2 - \int_I |(\sigma_1 - \sigma_2)'|^2 = -2 \int_I \langle (\sigma_1 - \sigma_2), \sigma''_2 \rangle \equiv \kappa.
\]

The lemma will follow by bounding \(|\kappa|\) by \( \frac{1}{2} \int_I |(\sigma_1 - \sigma_2)'|^2 \) and appealing to Wirtinger’s inequality.

Since \( \sigma_2 \) is a geodesic on \( M \), \( \sigma''_2 \) is normal to \( M \) and \(|\sigma''_2| \leq |\sigma'_2|^2 \sup_M |A| \leq \frac{|\sigma'_2|^2}{16}\). Thus, Lemma A.1 gives

\[
(A.4) \quad |\langle (\sigma_1 - \sigma_2), \sigma''_2 \rangle| \leq |(\sigma_1 - \sigma_2)^\perp| \frac{|\sigma'_2|^2}{16} \leq |\sigma_1 - \sigma_2|^2 \frac{|\sigma'_2|^2}{16}.
\]

\(^7\)Compactness of \( \Lambda \) follows since \( \sigma \in \Lambda \) depends continuously on the images of the \( L \) break points in the compact manifold \( M \).
Integrating (A.4), using that $|\sigma'_2|$ is constant with $|\sigma'_2| \text{ Length}(I) \leq 2\pi$, and applying Wirtinger’s inequality gives

$$(A.5) \quad |\kappa| \leq \frac{|\sigma'_2|^2}{8} \int_I |\sigma_1 - \sigma_2|^2 \leq \frac{|\sigma'_2|^2}{8} \left( \frac{\text{Length}(I)}{\pi} \right)^2 \int_I |(\sigma_1 - \sigma_2)'|^2 \leq \frac{1}{2} \int_I |(\sigma_1 - \sigma_2)'|^2.$$ 

□

**Appendix B. The continuity of $\Psi$**

**Lemma B.1.** Let $\gamma : S^1 \to M$ be a $W^{1,2}$ map with Energy($\gamma$) $\leq L$. If $\gamma_e$ and $\tilde{\gamma}_e$ are given by applying steps (A.1) and (B.1) to $\gamma$, then the map $\gamma \to \tilde{\gamma}_e$ is continuous from $W^{1,2}$ to $\Lambda$ equipped with the $W^{1,2}$ norm.

**Proof.** It follows from (1.8) and the energy bound that $\text{dist}_M(\gamma(x_{2j}), \gamma(x_{2j+2})) \leq 2\pi$ for each $j$ and thus we can apply step (A.1). The lemma will follow easily from two observations:

(C1) Since $W^{1,2}$ close curves are also $C^0$ close (cf. footnote 1), it follows that the points $\gamma_e(x_{2j}) = \gamma(x_{2j})$ are continuous with respect to the $W^{1,2}$ norm.

(C2) Define $\Gamma \subset M \times M$ by $\Gamma = \{(x, y) \in M \times M | \text{dist}_M(x, y) \leq 4\pi\}$, and define a map $H : \Gamma \to C^1([0,1], M)$ by letting $H(x, y) : [0, 1] \to M$ be the linear map from $x$ to $y$. Then the map $H$ is continuous on $\Gamma$. Furthermore, the map $t \to H(x, y)(t)$ has uniformly bounded first and second derivatives $|\partial_t H(x, y)| \leq 4\pi$ and $|\partial_{tt} H(x, y)| \leq \pi^2$; the second derivative bound comes from (M1).

To prove the lemma, suppose that $\gamma^1$ and $\gamma^2$ are non-constant curves in $\Lambda$ (continuity at the constant maps is obvious). For $i = 1, 2$ and $j = 1, \ldots, L$, let $a_j^i$ be the distance in $M$ from $\gamma^i(x_{2j})$ to $\gamma^i(x_{2j+2})$. Let $S^i = \frac{1}{2\pi} \sum_{j=1}^L a_j^i$ be the speed of $\tilde{\gamma}_e^i$, so that $|\tilde{\gamma}_e^i| = S^i$ except at the $L$ break points. By (C1), the $a_j^i$’s are continuous functions of $\gamma^i$ and, thus, so are $S^1$ and $S^2$. Moreover, (C1) and (C2) imply that $\gamma_e^1$ and $\gamma_e^2$ are $C^1$-close on each interval $[x_{2j}, x_{2j+2}]$. Thus, we have shown that $\gamma \to \gamma_e$ is continuous.

To show that $\gamma_e \to \tilde{\gamma}_e$ is also continuous, we will show that the $\tilde{\gamma}_e^i$’s are close when the $\gamma_e^i$’s are. Since the point $x_0 = x_{2L}$ is fixed under the reparametrization, this will follow from applying Wirtinger’s inequality to $(\tilde{\gamma}_e^1 - \gamma_e^2) - (\tilde{\gamma}_e^1 - \gamma_e^2)(x_0)$ once we show that $\int_{S^1} |(\tilde{\gamma}_e^1 - \gamma_e^2)'|^2$ can be made small.

The piecewise linear curve $\tilde{\gamma}_e^i$ is linear on the intervals

$$(B.2) \quad I_j^i = \left[ \frac{1}{S^i} \sum_{t \leq j} a_t^i, \frac{1}{S^i} \sum_{t \geq j} a_t^i \right].$$

Set $I_j = I_j^1 \cap I_j^2$. Observe first that since the intervals $I_j^i$ in (B.2) depend continuously on $\gamma_e^i$, the measure of the complement $S^1 \setminus \bigcup_{j=1}^L I_j^i$ can be made small, so that

$$(B.3) \quad \int_{S^1 \setminus \bigcup_{j=1}^L I_j} |(\tilde{\gamma}_e^1 - \gamma_e^2)'|^2 \leq 4L^2 \text{ Length } (S^1 \setminus \bigcup_{j=1}^L I_j)$$

can also be made small. We will divide the $I_j$’s into two groups, depending on the size of $a_j$. Fix some $\epsilon > 0$ and suppose first that $a_j^2 < \epsilon$; by continuity, we can assume that $a_j^2 < 2\epsilon$.
For such a $j$, we get
\begin{equation}
(B.4) \quad \int_{I_j} |(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)'|^2 \leq 2 \int_{I_j} |(\tilde{\gamma}_e^1)'|^2 + 2 \int_{I_j} |(\tilde{\gamma}_e^2)'|^2 \leq 2 L (a_1^2 + a_2^2) \leq 6 \epsilon L.
\end{equation}

Since there are at most $L$ breaks, summing over these intervals contributes at most $6 \epsilon L^2$ to the energy of $(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)$.

The last case to consider is an $I_j$ with $a_j^1 \geq \epsilon$; by continuity, we can assume that $a_j^2 \geq \epsilon/2$. In this case, $\tilde{\gamma}_e^i$ can be written on $I_j$ as the composition $\gamma^i \circ P_j$ where $|P_j'| = 2\pi S^j/(La_j^i)$. Furthermore, $P_j^1$ and $P_j^2$ both map $I_j$ into $[x_{2j}, x_{2j+2}]$ and
\begin{equation}
(B.5) \quad \int_{I_j} |(\tilde{\gamma}_e^1 - \tilde{\gamma}_e^2)'|^2 = \int_{I_j} |(\gamma^1 \circ P_j^1 - \gamma^2 \circ P_j^2)'|^2.
\end{equation}

Finally, this can be made small since the speed $|P_j'|$ is continuous\footnote{The speed is continuous because of the lower bound for the $a_j^j$'s.} in $\gamma^i$ and the $\gamma^i_e$'s are $C^2$ bounded and $C^1$ close on $[x_{2j}, x_{2j+2}]$. Therefore, the integral over these intervals can also be made small since there are at most $L$ of them. \hfill \Box

The next result shows that $\Psi$ preserves the homotopy class of a sweepout.

**Lemma B.6.** Let $\gamma \in \Omega$ satisfy
\begin{equation}
(B.7) \quad \max_t \text{Energy}(\gamma(\cdot, t)) \leq L.
\end{equation}

If $\gamma_e$ and $\tilde{\gamma}_e$ are given by applying steps $(A_1)$ and $(A_2)$ to each $\gamma(\cdot, t)$, then $\gamma$, $\gamma_e$ and $\tilde{\gamma}_e$ are all homotopic in $\Omega$.

**Proof.** Given $x, y \in M$ with $\text{dist}_M(x, y) \leq 4\pi$, let $H(x, y) : [0, 1] \to M$ be the linear map from $x$ to $y$ as in (C2). It follows that
\begin{equation}
(B.8) \quad F(x, t, s) = H(\gamma(x, t), \gamma(x, t))(s)
\end{equation}
is an explicit homotopy with $F(\cdot, \cdot, 0) = \gamma$ and $F(\cdot, \cdot, 1) = \gamma_e$.

For each $t$ with $\text{Length}(\gamma_e(\cdot, t)) > 0$, $\gamma_e$ is given by $\gamma_e(\cdot, t) = \tilde{\gamma}_e(\cdot, t) \circ P_t$ where $P_t$ is a monotone reparametrization of $S^1$ that fixes $x_0 = x_{2L}$. Moreover, $P_t$ is continuous by (2.5) and $P_t$ depends continuously on $t$ by Lemma B.1. Since $x \to (1-s)P_t(x) + sx$ gives a homotopy from $P_t$ to the identity map on $S^1$, we conclude that
\begin{equation}
(B.9) \quad G(x, t, s) = \tilde{\gamma}_e((1-s)P_t(x) + sx, t)
\end{equation}
is an explicit homotopy with $G(\cdot, \cdot, 0) = \gamma_e$ and $G(\cdot, \cdot, 1) = \tilde{\gamma}_e$. Note that $P_t$ is not defined when $\text{Length}(\gamma_e(\cdot, t)) = 0$, but the homotopy $G$ is. \hfill \Box

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