WHAT IS THE BEST APPROACH TO COUNTING PRIMES?

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As long as people have studied mathematics, they have wanted to know how many primes there are. Getting precise answers is a notoriously difficult problem, and the first suitable technique, due to Riemann, inspired an enormous amount of great mathematics, the techniques and insights penetrating many different fields. In this article we will review some of the best techniques for counting primes, centering our discussion around Riemann’s seminal paper. We will go on to discuss its limitations, and then recent efforts to replace Riemann’s theory with one that is significantly simpler.

1. How many primes are there? Predictions

You have probably seen a proof that there are infinitely many prime numbers. This was known to the ancient Greeks, and today there are many different proofs, using tools from all sorts of fields.

Once one knows that there are infinitely many prime numbers then it is natural to ask whether we can give an exact formula, or at least a good approximation, for the number of primes up to a given point. Such questions were no doubt asked, since antiquity, by those people who thought about prime numbers. With the advent of tables of factorizations, it was possible to make predictions supported by lots of data. Figure 1 is a photograph of Chernac’s Table of Factorizations of all integers up to one million, published in 1811.¹ There are exactly one thousand pages of factorization tables in the book, each giving the factorizations of one thousand numbers. For example, Page 677, seen here, enables us to factor all numbers between 676000 and 676999. The page is split into 5 columns, each in two parts, the ten half columns on the page each representing those integers that are not divisible by 2, 3 or 5, in an interval of length 100.² On the left side of a column is a number like 567, which represents, on this page, the number 676567 to be factored. On the right side of the column we see 619 · 1093 which gives the complete factorization of 676567. On the other hand for 589, which represents the prime number 676589, the right column simply contains “——”, and hence that number is prime. And so it goes for all of the numbers in this range. It only takes a minute to get used to these protocols, and then the table becomes very useful if you do not have appropriate factoring software at your disposal.

On December 24th, 1849, Gauss responded to his “most honored friend”, Encke, describing his own attempt to guess at an approximation as to the number of primes up to \( x \) (which we will denote throughout by \( \pi(x) \)). Gauss describes his work:

¹My colleague, Anatole Joffe, kindly presented his copy of these tables to me when he retired. Nowadays I happily distract myself from boring office-work, by flicking through to discover obscure factorizations!

²Chernac trusted that the reader could easily extract those factors for him-or-herself.
The kind communication of your remarks on the frequency of prime numbers was interesting to me in more than one respect. You have reminded me of my own pursuit of the same subject, whose first beginnings occurred a very long time ago, in 1792 or 1793 [when Gauss was 15 or 16] ... Before I had occupied myself with the finer investigations of higher arithmetic, one of my first projects was to direct my attention to the decreasing frequency of prime numbers, to which end I counted them up in several chiliads [within blocks of 1000 consecutive integers] and recorded the results ... I soon recognized, that under all variations of this frequency, on average, it is nearly inversely proportional to the logarithm, so that the number of all prime numbers under a given boundary \( x \) were nearly expressed through the integral

\[
\int_2^x \frac{dt}{\log t}
\]

where the logarithm is understood to be the natural logarithm.

The 1811 appearance of Chernac's Cribrum Arithmeticum gave me great joy, and (since I did not have the patience for a continuous count of the series) I have very often employed a spare unoccupied quarter of an hour in order to count up a chiliad here and there; however, I eventually dropped it completely, without having quite completed the first million ...

The first three million has now been counted (for some years), and compared with the integral value. I lay out here only a small extract:

| Under \( x \) | \( \pi(x) = \# \{ \text{primes } \leq x \} \) | \( \text{Li}(x) \pm \text{Error} \) |
|--------------|-----------------|------------------|
| 500000       | 41556           | 41606.4 - 50.4   |
| 1000000      | 78501           | 79627.5 - 126.5  |
| 1500000      | 111112          | 114263.1 - 151.1 |
| 2000000      | 148883          | 149054.8 - 171.8 |
| 2500000      | 183016          | 183245.0 - 229.0 |
| 3000000      | 216745          | 216970.6 - 225.6 |

Table 1. Primes up to various points, and a comparison with Gauss's prediction.

(Here, and throughout, we define \( \text{Li}(x) := \int_2^x \frac{dt}{\log t} \).)

In fact Legendre in his Théorie des Nombres had made the prediction

\[
\frac{x}{\log x - A}
\]

with \( A = 1.08366 \) for \( \pi(x) \), in which case the comparative errors are

\[-23.3, +42.2, +68.1, +92.8, +159.1, \text{ and } +167.6,\]

respectively, which are certainly smaller than the errors from Gauss’s prediction, though both seem to be excellent approximations. Nevertheless Gauss retained faith in his prediction:

These differences are smaller than those with the integral, though they do appear to grow more quickly than [the differences given by the integral] with increasing \( x \), so that it is possible that they could easily surpass the latter, if carried out much farther.

Today we have data that goes “much farther”. Comparing these two predictions, using this data:
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It is now obvious that Gauss’s prediction is indeed better, that Legendre’s error terms quickly surpass those of Gauss, and keep on growing bigger. Here is some of the most recent data and a comparison to Gauss’ prediction:

| $x$  | $\pi(x) = \#\{\text{primes } \leq x\}$ | Gauss’s error term | Legendre’s error term |
|------|--------------------------------------|--------------------|----------------------|
| $10^{20}$ | 2220819602560918840                | 222744644           | 2981921009910364     |
| $10^{21}$ | 21127269486018731928                | 597394254           | 27516571651291205    |
| $10^{22}$ | 201467286689315906290               | 1932355208          | 254562416350667927   |
| $10^{23}$ | 1925320391606803968923               | 7250186216          | 236082999934659157   |

Table 2. Comparing the errors in Gauss’s and Legendre’s predictions.

When looking at this data, compare the widths of the right two columns. It is pretty clear that the rightmost column is about half the width of the middle column . . . How do we interpret that? Well, the width of a column is given by the number of digits of the integer there, which corresponds to the number’s logarithm in base 10. If the log of one number is half that of a second number, then the first number is about the square-root of the first. What that means here is that the data seems to suggest that when we approximate $\pi(x)$, the number of primes up to $x$, by Gauss’s guesstimate, $\text{Li}(x)$, the error is around $\sqrt{x}$, which is really tiny in comparison to the actual number of primes. In other words, Gauss’s prediction is terrific.

As we will discuss below, we really do believe that Gauss’s $\text{Li}(x)$ is always that close to $\pi(x)$, specifically, we have good reason to conjecture that

$$|\pi(x) - \text{Li}(x)| \leq x^{1/2} \log x$$
for all \( x \geq 3 \). This would be an extraordinary thing to prove. For now we will just focus on the much simpler statement that the ratio of \( \frac{\pi(x)}{\log x} \) tends to 1 as \( x \to \infty \). Since \( \text{Li}(x) \) is well-approximated by \( x/\log x \), this quest can be more simply stated as

\[
\lim_{x \to \infty} \frac{\pi(x)}{\log x} \text{ exists and equals 1.}
\]

This is known as the Prime Number Theorem, and it took more than a hundred years, and some earth-shaking ideas, to prove it.

2. Elementary techniques to count the primes. It is not easy to find a way to count primes at all accurately. Even proving good upper and lower bounds is challenging.

One effective technique to get an upper bound is to try to use the principle of the sieve of Eratosthenes. This is where we “construct” the primes up to \( x \), by removing the multiples of all of the primes \( \leq \sqrt{x} \). One starts by removing the multiples of 2, from a list of all of the integers up to \( x \), then the remaining multiples of 3, then the remaining multiples of 5, etc. Hence once we have removed the multiples of the primes \( \leq y \) we have an upper bound:

\[
\# \{ \text{prime : } y < p \leq x \} \leq \# \{ n \leq x : p \nmid n \text{ for all primes } p \leq y \}.
\]

At the start this works quite well. If \( y = 2 \) the quantity on the right is \( \frac{1}{2}x \pm 1 \), and so bounded above by \( \frac{1}{2}x + 1 \). If \( y = 3 \) then we remove roughly a third of the remaining integers (leaving two-thirds of them) and so the bound improves to \( \frac{2}{3} \cdot \frac{1}{2}x + 2 \). For \( y = 5 \) we have four-fifths of the remaining integers to get the upper bound \( \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{1}{2}x + 4 \). And, in general, we obtain an upper bound of no more than

\[
\prod_{p \leq y} \left(1 - \frac{1}{p}\right) \cdot x + 2^{\pi(y)-1}.
\]

The problem with this bound is the second term . . . as one sieves by each consecutive prime, the second term doubles each time, and so quickly becomes larger than \( x \) (and thus this is a useless upper bound). This formula does allow us (by letting \( y \to \infty \) slowly with \( x \)) to prove that

\[
\lim_{x \to \infty} \frac{\pi(x)}{x} = 0;
\]

that is the primes are a vanishing proportion of the integers up to \( x \), as \( x \) gets larger.\(^4\)

There has been a lot of deep and difficult work on improving our understanding of the sieve of Eratosthenes, but we are still unable to get a very good upper bound for the number of primes in this way. Moreover we are unable to use the sieve of Eratosthenes (or any other sieve method) to get good lower bounds on the number of primes up to \( x \).

The first big leap in our ability to give good upper and lower bounds on \( \pi(x) \) came from an extraordinary observation of Cebyshev in 1851. The observation (as reformulated

\(^3\)To prove this, try integrating Li\((x)\) by parts.

\(^4\)To deduce this we need to know that \( \lim_{y \to \infty} \prod_{p \leq y} \left(1 - \frac{1}{p}\right) = 0 \), a fact which was shown by Euler.
by Erdős in 1933) is that the binomial coefficient \( \binom{2n}{n} \) is an integer, by definition, and is divisible, once, by each prime \( p \) in \( (n, 2n] \), which is obvious since \( p \) is a term in the expansion of the numerator \( (2n)! \), but not the denominator, \( n! \). Therefore

\[
\prod_{\substack{p \text{ prime} \\ n < p \leq 2n}} p \leq \binom{2n}{n}.
\]

Now, by the binomial theorem, \( \binom{2n}{n} \leq \sum_{j=0}^{2n} \binom{2n}{j} = (1 + 1)^{2n} = 2^{2n} \). Moreover for each \( p \in (n, 2n] \) we have \( p > n \) and so

\[
n \pi(2n) - \pi(n) = \prod_{\substack{p \text{ prime} \\ n < p \leq 2n}} p \leq \binom{2n}{n} \leq 2^{2n}.
\]

Taking logarithms this gives us the upper bound \( \pi(2n) - \pi(n) \leq \frac{2n \log 2}{\log n} \), and summing this bound yields

\[
\pi(x) \leq \sum_{j \geq 0} \left( \pi\left( \frac{x}{2^j} \right) - \pi\left( \frac{x}{2^{j+1}} \right) \right) \leq (\log 4 + \epsilon) \frac{x}{\log x},
\]

for \( x \) sufficiently large.

One can interpret this proof as a mixture of algebra and analysis: The algebra comes when we consider the primes that divide the central binomial coefficient to get a lower bound on its size; the analysis when we bound the size of the central binomial coefficient by comparing it to the size of other binomial coefficients. This sets the pattern for what is to come.

One can also obtain good lower bounds by studying the prime power divisibility of \( \binom{2n}{n} \): First we note that if \( p \) divides \( \binom{2n}{n} \) then \( p \leq 2n \), since the numerator is the product of all integers \( \leq 2n \), and hence can have no prime factor larger than 2n. The key observation, essentially due to Kummer, is that if a prime power \( p^e \) divides \( \binom{2n}{n} \) then we even have that \( p^e \leq 2n \). We couple that with the observation that \( \binom{2n}{n} \) is the largest of the binomial coefficients \( \binom{2n}{j} \), and therefore

\[
2^{2n} = (1 + 1)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} \leq (2n + 1) \binom{2n}{n}.
\]

Combining this information, we obtain

\[
\frac{2^{2n}}{2n + 1} \leq \binom{2n}{n} = \prod_{\substack{p \text{ prime} \\ p \leq 2n}} p^e \leq \prod_{\substack{p \text{ prime} \\ p \leq 2n}} 2n = (2n)^{\pi(2n)}.
\]

Taking logarithms gives us the lower bound \( \pi(2n) \geq \frac{2n \log 2 - \log(2n + 1)}{\log(2n)} \), and therefore

\[
\pi(x) \geq (\log 2 - \epsilon) \frac{x}{\log x}.
\]
if \( x \) is sufficiently large. Hence we have shown that there exist constants \( c_2 > 1 > c_1 > 0 \), such that if \( x \) is sufficiently large then

\[
\frac{c_1}{\log x} \leq \pi(x) \leq \frac{c_2}{\log x}.
\]

The **Prime Number Theorem**, that is the conjecture of Gauss and Legendre guesstimating the number of primes up to \( x \), is the claim that these inequalities hold for any constants \( c_1 \) and \( c_2 \) satisfying \( c_2 > 1 > c_1 > 0 \); and in particular we can take both \( c_1 \) and \( c_2 \) arbitrarily close to 1.

Perhaps the method of Cebyshev and Erdős can be suitably modified to prove the result. In other words, perhaps we can find some other product of factorials, which yields an integer, and in which we can track the divisibility of the large prime factors, so that we obtain constants \( c_1 \) and \( c_2 \) that are closer to 1. We might expect that the closer the constants get to 1, the more complicated the product of factorials, and that has been the case in the efforts that researchers have made to date.\(^6\) There is one remarkable identity that gives us hope. First note that in our argument above we might replace \( \left(\begin{array}{c} 2n \\ n \end{array}\right) \) by \( [x]!/[x/2]!^2 \), where \( x = 2n \). Hence the correct factorials to consider take the shape \( \lfloor x/n \rfloor! \) for various integers \( n \). Our identity is:

\[
\prod_{\substack{p \text{ prime} \\ \epsilon \geq 1 \\ p^\epsilon \leq x}} p = \prod_{n \leq x} [x/n]! \mu(n).
\]

This needs some explanation. The left-hand side is the product over the primes \( p \leq x \), each \( p \) repeated \( k_p \) times, where \( p^{k_p} \) is the largest power of \( p \) that is \( \leq x \). On the right hand side we have the promised factorials, each to the power \(-1, 0 \) or \( 1 \). Indeed the Mőbius function \( \mu(n) \) is defined as

\[
\mu(n) = \begin{cases} 
0 & \text{if there exists a prime } p \text{ for which } p^2 \text{ divides } n; \\
(-1)^k & \text{if } n \text{ is squarefree, and has exactly } k \text{ prime factors}
\end{cases}
\]

The difficulty with using the identity (3) to prove the prime number theorem is that the length of the product on the right side grows with \( x \), so there are too many terms to keep track of. One idea is to simply take a finite truncation of the right-hand side; that is

\[
\prod_{n \leq N} [x/n]! \mu(n)
\]

for some fixed \( N \). The advantage of this is that, once \( x > N^2 \), then this product is divisible by every prime \( p \in [x/(N+1), x] \) to the power 1. The disadvantage is that the product is often not an integer, though we can correct that by multiplying through by a few smaller

\(^5\)In fact with any \( c_1 < \log 2 \) and \( c_2 > \log 4 \).

\(^6\)Which is not to say that someone new, not overly influenced by previous, failed attempts, might not come up with a cleverer way to modify the previous approaches.
factorials. We can handle this and other difficulties that arise, to obtain (2) with other values of $c_1$ and $c_2$, which each appear to be getting closer and closer to 1. However when we analyze what it will take to prove that the constants (which we now denote by $c_1(N)$ and $c_2(N)$ since they depend on $N$) tend to 1 as $N \to \infty$, we find that the issue lies in the average of the exponents $\mu(n)$. In fact one can prove that the constants $c_1(N)$ and $c_2(N)$ do tend to 1 if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) \text{ exists and equals 0.}$$

One can go further and prove that this is equivalent to the prime number theorem.

The problem in (5) certainly looks more approachable than the prime number theorem itself (even though the problems are equivalent). It can be rephrased as saying that, amongst the squarefree integers, there are roughly as many integers with an even number of prime factors, as there are integers with an odd number of prime factors. This seems very plausible, and leads to many elementary approaches. More on that later.

One of the most famous old problems about primes was Bertrand’s postulate, to prove that for every $n > 1$ there is always a prime $p$ for which $n < p < 2n$. This follows easily from suitable modifications of the above discussion with the binomial coefficient $\binom{2n}{n}$. In fact it was Erdős’s beautiful proof of this (as above), at age 20, that announced his arrival onto the world stage (of mathematics) and inspired the lines:

*Chebyshev said it, and I say it again:*

*There is always a prime between $n$ and $2n$.*

Up to now we have proved that $\pi(x)$ lies between two multiples of $x/\log x$, and we have looked to see whether the ratio of $\pi(x)$ to $x/\log x$ tends to 1, as predicted by Gauss and Legendre. We should ask whether any other behaviour is possible, given what we know already? It seems to me that there are two possibilities

(i) The ratio of $\pi(x)$ to $x/\log x$ oscillates around as $x$ grows larger, never tending to a limit.

(ii) The ratio of $\pi(x)$ to $x/\log x$ tends to a limit as $x$ grows larger, but that limit is not 1.

Our goal in the rest of this section is to show that option (ii) is not possible. Indeed we will show that if there is a limit then that limit would have to be 1. Yet again the trick is to study factorials, this time $N!$, both algebraically (by determining its prime factors), and analytically (by analyzing its size).

Now, by definition,

$$\log N! = \sum_{n=1}^{N} \log n.$$ 

The right-hand side is very close to the integral of $\log t$ over the same range. Why is this obvious? The logarithm function is monotone increasing, which implies that

$$\int_{n-1}^{n} \log t \ dt < \log n < \int_{n}^{n+1} \log t \ dt$$
for every \( n \geq 1 \). Summing these inequalities over all integers \( n \) in the range \( 2 \leq n \leq N \) (since \( \log 1 = 0 \)), we obtain that for \( N \geq 2 \), the value of \( \log N! \) equals
\[
\int_1^N \log t \, dt = [t(\log t - 1)]_1^N = N(\log N - 1) + 1,
\]
plus an error that is no larger, in absolute value, than \( \log N \).

On the other hand \( N! \) is the product of the integers up to \( N \), and we want to know how often each prime divides this product. The integers \( \leq N \) that are multiples of a given integer \( m \) (which could be a prime or prime power) are \( m, 2m, \ldots, \lfloor N/m \rfloor m \), since \( \lfloor N/m \rfloor m \) is the largest multiple of \( m \) that is \( \leq N \), and therefore there are \( \lfloor N/m \rfloor \) such multiples. Now the power of prime \( p \) dividing \( N! \) is given by the number of integers \( \leq N \) that are multiples of \( p \), plus the number of integers \( \leq N \) that are multiples of \( p^2 \), etc., which yields a total of
\[
\left\lfloor \frac{N}{p} \right\rfloor + \left\lfloor \frac{N}{p^2} \right\rfloor + \left\lfloor \frac{N}{p^3} \right\rfloor + \ldots +
\]
Hence, by studying the prime power divisors of \( N! \) we deduce that
\[
\log N! = \sum_{\substack{p \text{ prime} \\ p \leq N}} \log p \left( \left\lfloor \frac{N}{p} \right\rfloor + \left\lfloor \frac{N}{p^2} \right\rfloor + \ldots + \right).
\]
The total error created by discarding those \( \lfloor N/p^k \rfloor \) terms with \( k \geq 2 \), and by replacing each \( \lfloor N/p \rfloor \) by \( N/p \), adds up to no more than a constant times \( N \). Therefore, comparing our two estimates for \( \log N! \), and dividing through by \( N \) we obtain that
\[
\sum_{p \leq N} \frac{\log p}{p} - \log N
\]
is bounded; that is, there exists a constant \( C \) such that
\[
(7) \quad \left| \sum_{p \leq N} \frac{\log p}{p} - \log N \right| \leq C
\]
for all \( N \geq 1 \).

Now let’s suppose that there exists a constant \( \eta \), such that for any \( \epsilon > 0 \)
\[
(\eta - \epsilon) \frac{x}{\log x} \leq \pi(x) \leq (\eta + \epsilon) \frac{x}{\log x},
\]
for all sufficiently large \( x \) (say \( > x_\epsilon \)). Our goal is to show that \( \eta = 1 \).

We will work with the following identity:
\[
\sum_{p \leq N} \frac{\log p}{p} = \sum_{p \leq N} \left\{ \frac{\log N}{N} + \int_p^N \frac{\log x - 1}{x^2} \, dx \right\}
\]
\[
= \pi(N) \frac{\log N}{N} + \int_2^N \pi(x) \frac{\log x - 1}{x^2} \, dx,
\]
inserting our (assumed) bounds on \( \pi(x) \) to obtain bounds on \( \sum_{p \leq N} \frac{\log p}{p} \). The part of the integral with \( x \leq x_\epsilon \) is bounded by some constant that only depends on \( \epsilon \), call that \( C_1(\epsilon) \). Therefore we obtain an upper bound

\[
\sum_{p \leq N} \frac{\log p}{p} \leq C_1(\epsilon) + \pi(N) \frac{\log N}{N} + \int_{x_\epsilon}^{N} \pi(x) \frac{\log x - 1}{x^2} \, dx \\
\leq C_1(\epsilon) + (\eta + \epsilon) \frac{N}{\log N} \frac{\log N}{N} + \int_{x_\epsilon}^{N} (\eta + \epsilon) \frac{x}{\log x} \frac{\log x - 1}{x^2} \, dx \\
\leq C_2(\epsilon) + (\eta + \epsilon) \int_{x_\epsilon}^{N} \frac{dx}{x} \leq (\eta + \epsilon) \log N + C_2(\epsilon),
\]

for some constant \( C_2(\epsilon) \), that only depends on \( \epsilon \). This implies that \( \eta \geq 1 \) else we let \( \epsilon = (1 - \eta)/2 \) and this inequality contradicts (7) for \( N \) sufficiently large. An analogous proof with the lower bound implies that \( \eta \leq 1 \). We deduce that if \( \pi(x)/\frac{x}{\log x} \) tends to a limit as \( x \to \infty \) then that limit must be 1.

3. **A first reformulation: Introducing appropriate weights.** In the techniques we have introduced so far to count primes, we actually estimated the size of the product of the primes in some interval. Taking logs, this means that we counted prime \( p \) with the weight \( \log p \), that is we bounded

\[
\sum_{\substack{p \text{ prime} \\ p \leq x}} \log p.
\]

We now denote this by \( \theta(x) \); and we also define its close cousin,\(^7\)

\[
\psi(x) := \sum_{\substack{p \text{ prime} \\ m \geq 1 \\ p^m \leq x}} \log p.
\]

When we do calculations, it seems that these functions are more natural than \( \pi(x) \) itself. Surprisingly this fits rather well with (a slight re-interpretation of) Gauss’s original musings in his letter to Encke. The key phrase is:

> I soon recognized, that under all variations of this frequency [of prime numbers], on average, it is nearly inversely proportional to the logarithm.

We re-word this as “The density of primes at around \( x \) is \( 1/\log x \).” Then we would expect that the number of primes, each weighted with \( \log p \) (that is, the sum \( \theta(x) \)) should be well-approximated by

\[
\int_{2}^{x} \log t \cdot \frac{dt}{\log t} = \int_{2}^{x} dt = x - 2.
\]

\(^7\)The reader might verify that \( \theta(x) \) and \( \psi(x) \) do not differ by more than a bounded multiple of \( \sqrt{x} \).
Occam’s razor tells us that, given two choices, one should opt for the more elegant one. There can be little question that \( x \) is a more pleasant function to work with than the complicated integral Li\((x)\), and so we will develop the theory with logarithmic weights, and therefore use the function \( \theta(x) \) rather than \( \pi(x) \).\(^8\)

We have good reason to conjecture that

\[
(RH2) \quad \left| \sum_{p \leq x} \log p - x \right| \leq x^{1/2} (\log x)^2.
\]

since this is equivalent to our conjecture (RH1) on \( \pi(x) - \text{Li}(x) \). The prime number theorem is equivalent to the much weaker assertion that

\[
\lim_{x \to \infty} \frac{1}{x} \sum_{p \leq x} \log p \text{ exists and equals 1.}
\]

4. **Riemann’s memoir.** In a nine page memoir written in 1859, Riemann outlined an extraordinary plan to attack the elementary question of counting prime numbers using deep ideas from the theory of complex functions. His approach begins with what we now call the Riemann zeta-function:

\[
\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}.
\]

To make sense of an infinite sum it needs to converge, and preferably be absolutely convergent. The sum for \( \zeta(s) \) is absolutely convergent only when \( \text{Re}(s) > 1 \). The advantage of working with an absolutely convergent sum is that we can re-arrange the order of the terms and the value does not change.\(^9\) This is especially interesting when we apply the **Fundamental Theorem of Arithmetic** to each term in the sum: Every integer \( n \geq 1 \) can be factored in a unique way and, vice-versa, every product of primes yields a unique positive integer. Then we can write

\[
n = 2^{n_2} 3^{n_3} \ldots,
\]

where each \( n_j \) is a non-negative integer, and only finitely many of them are non-zero. Hence

\[
\zeta(s) = \sum_{n_2, n_3, n_5, \ldots \geq 0} \frac{1}{(2^{n_2} 3^{n_3} 5^{n_5} \ldots)^s} = \prod_{p \text{ prime}} \left( \sum_{n_p \geq 0} \frac{1}{(p^{n_p})^s} \right) = \prod_{p \text{ prime}} \left( 1 - \frac{1}{p^s} \right)^{-1}.
\]

\(^8\)It is not difficult to show that a good estimate for one is equivalent to an analogous estimate for the other, so there is no harm done in focusing on \( \theta(x) \).

\(^9\)Riemann proved that if one has a convergent but not absolutely convergent sum then one might get different limits if one re-arranges the order of the terms in the sum.
This product over primes is an Euler product — indeed, it was Euler who first seriously explored the connection between $\zeta(s)$ and the distribution of prime numbers, though he did not penetrate the subject as deeply as Riemann.

Although the Euler product provides a connection between $\zeta(s)$ and prime numbers, this was exploited by Riemann in an interesting way. Since $\zeta(s)$ is absolutely convergent, it is safe to perform calculus operations on it. Thus

$$\frac{-\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = \sum_{p \text{ prime}} \frac{d}{ds} \log \left(1 - \frac{1}{p^s}\right)$$

using the Euler product. Using the chain rule, we then obtain

$$\frac{-\zeta'(s)}{\zeta(s)} = \sum_{p \text{ prime}} \frac{\log p}{p^s - 1} = \sum_{p \text{ prime}} \log p \left[p^{m_p}; \quad m_p \geq 1\right]$$

and notice that the sum of the coefficients of $1/n^s$ on the right-side, for $n$ up to $x$, equals $\sum_{p^m \leq x} \log p = \psi(x)$. As we remarked above, this is a close cousin of $\theta(x)$, and we now see that it arises naturally in this context.

5. Contour integration. One of the great discoveries of 19th century mathematics is that it is possible to convert problems of a discrete flavour, in number theory and combinatorics, into questions of complex analysis. The key lies in finding suitable analytic identities to describe combinatorial issues. For example, if we ask whether two integers, $a$ and $b$, are equal, that is “does $a = b$?”, then this is equivalent to asking “is $a - b = 0$?” and therefore we need some analytic device that will distinguish 0 from all other integers. This is given by the integral of the exponential function around the circle:

$$\int_0^1 e^{2\pi i nt} dt = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{otherwise}.
\end{cases}$$

So, for example, if we want to determine the number of pairs $p, q$ of primes $\leq n$, which add to give the even integer $n$, we create an integral that gives 1 if $p + q = n$, and 0 otherwise, and then sum over all such $p$ and $q$. Therefore we have

$$\#\{p, q \leq n : \ p, q \text{ primes, } p + q = n\} = \sum_{p, q \leq n \atop p, q \text{ primes}} \int_0^1 e^{2i\pi(p+q-n)t} dt$$

and this can be re-arranged as

$$\int_0^1 e^{-2\pi nt} \left( \sum_{p \leq n \atop p \text{ prime}} e^{2i\pi pt} \right)^2 dt.$$
This is of course an approach to Golbach’s conjecture (that every even integer \( \geq 4 \) is the sum of two primes), and it is rather surprising that understanding this integral is equivalent to the original combinatorial number theory question.

So we have seen how to analytically identify when two integers are equal, and why that is useful. Next we will show how to analytically verify a proposed inequality between two real numbers. As you might have guessed, we start by noting that asking whether \( u < v \) is the same as asking whether \( v - u > 0 \), and so we restrict our attention to determining whether a given real number is \( < 0 \) or \( = 0 \) or \( > 0 \) (though we are less interested in the middle case). Here the trick is that for any \( \sigma > 0 \) we have

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{y(s+it)}}{\sigma + it} \, dt = \begin{cases} 
0 & \text{if } y < 0; \\
1/2 & \text{if } y = 0; \\
1 & \text{if } y > 0,
\end{cases}
\]

which is Perron’s formula. It is convenient to write \( s \) for \( \sigma + it \) and, instead of the limits of the integral, we write “Re(\( s \)) = \sigma”, understanding that we take \( s \) along the line Re(\( s \)) = \sigma, that is \( s = \sigma + it \) as \( t \) runs from \(-\infty \) to \( \infty \). Hence we integrate \( e^{ys}/s \). Moreover if we let \( z = e^y \), the formula can be rephrased as

\[
\frac{1}{2i\pi} \int_{\text{Re}(s)=\sigma} \frac{z^s}{s} \, ds = \begin{cases} 
0 & \text{if } 0 < z < 1; \\
1/2 & \text{if } z = 1; \\
1 & \text{if } z > 1,
\end{cases}
\]

for any \( \sigma > 0 \). In number theory, the most common use of Perron’s formula is to identify when an integer \( n \) is \( < x \), that is when \( x/n > 1 \).

We are interested in estimating \( \psi(x) = \sum_{p^m \leq x} \log p \). We extend the sum to all prime powers, multiplying by 1 if \( p^m \leq x \), and by 0 otherwise, which we achieve by using Perron’s formula with \( z = x/p^m \). The outcome is \( \psi^*(x) \) which has the same value as \( \psi(x) \) except we subtract \( 1/2 \log x \) if \( x \) is a prime power. Therefore we have

\[
\psi^*(x) = \sum_{\substack{p \text{ prime} \\ m \geq 1}} \log p \cdot \frac{1}{2i\pi} \int_{\text{Re}(s)=\sigma} \frac{(x/p^m)^s}{s} \, ds,
\]

for any \( \sigma > 0 \). We would like to now swap the order of the summation and the integral, but there are convergence issues. Fortunately these are easily dealt with when the sum is absolutely convergent, as happens when \( \sigma > 1 \). Then we have, after a little re-arrangement,

\[
\psi^*(x) = \frac{1}{2i\pi} \int_{\text{Re}(s)=\sigma} \sum_{\substack{p \text{ prime} \\ m \geq 1}} \log p \cdot \frac{x^s}{p^ms} \, ds
\]

\[
= \frac{1}{2i\pi} \int_{\text{Re}(s)=\sigma} -\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s} \, ds.
\]

(11)

This seems, at first sight, like a rather strange thing to do. We have gone from a perfectly understandable question like estimating \( \pi(x) \), involving a sum that is easily interpreted,
to a rather complicated integral, over an infinitely long line in the complex plane, of a function that is delicate to work with in that it is only well-defined when $\text{Re}(s) > 1$. It is by no means obvious how to proceed from here, as we discuss in more detail in the next section.

The proof of Perron’s formula did not use many properties of $\zeta'(s)/\zeta(s)$. In fact if $a_n$ is any sequence of real numbers with each $|a_n| \leq 1$ then define the Dirichlet series $A(s) = \sum_{n \geq 1} a_n/n^s$, to obtain

$$\sum_{n \leq x} a_n = \frac{1}{2i\pi} \int_{\text{Re}(s)=\sigma} A(s) \cdot \frac{x^s}{s} \, ds.$$

6. Riemann’s genius. How do we evaluate the integral in (11)? In complex integration the idea is to shift the path of the integral to one on which the integrand is “very small”. Then the value of the integral is given by the sum of the “residues” of the integrand at its “poles.” There is a lot to explain here; indeed the main points of a first course in complex analysis. Rather than get into all of these details, let me just say that the poles are the points where the function goes to $\infty$, like the point $s = 1$ for the function $1/(s - 1)$. And if the function $f(s)$ has a pole at, say, $s = 1$, whereas $(s - 1)f(s)$ equals $r \in \mathbb{C}$ at $s = 1$, then we say that $f(s)$ has a simple pole at $s = 1$ with residue $r$.

So what new path should we take from $\sigma - i\infty$ to $\sigma + i\infty$ to be able to apply this strategy to the integral in (11)? If we are going to choose the same path for each value of $x$, then we want to be sure that the integrand (on the path) does not grow large with $x$. Now $|x^s| = x^{\text{Re}(s)} = x^{\sigma}$, so the smaller $\sigma$ is the better. In fact if we make $\sigma$ negative then the $x^s$ in the integrand will ensure that the integral over this line gets smaller as $x \to \infty$. Or, rather, that would be true if $\zeta(s)$ and $(\zeta'/\zeta)(s)$ are defined in this region, which they are not, for now.

The theory of analytic continuation is deep and subtle. Under the right conditions it allows us to take a function that is well-defined in some part of the complex plane, and define it in the rest of the complex plane, except perhaps at its poles. Moreover at each point one can express the function in terms of a Taylor series. Analytic continuations are a little bit mysterious – for example the theory allows for more than one apparently different way that one can define the function on the rest of the complex plane, but it will turn out that any two such definitions will have equal values everywhere they are both defined. Anyway, we can analytically continue $\zeta(s)$ to all of the complex plane, except for its pole at $s = 1$, and to do this, Riemann discovered some remarkable properties of $\zeta(s)$ (which we do not pursue here). There are several subtleties involved in bounding the contribution of the integrand on the new contours, and Riemann succeeded in doing that. Finally one needs to find the poles of

$$\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s}.$$

---

10 That is, for $f(s)$ at $s_0 \in \mathbb{C}$, there exist constants $a_0, a_1, \ldots$ and some constant $r$, such that if $|s - s_0| < r$ then $a_0 + a_1(s - s_0) + a_2(s - s_0)^2 + \ldots$ is absolutely convergent, and converges to $f(s)$.

11 This allows us to give the analytic continuation the same name as the original function, since we know that it can only be analytically continued in one way, if at all.
and to compute their residues: Evidently \( x^s \) has no poles in the complex plane, and \( 1/s \) has a simple pole at \( s = 0 \), which contributes a residue of

\[
- \lim_{s \to 0} s \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} = \frac{\zeta'(0)}{\zeta(0)} x^0 = -\zeta'(0) \zeta(0)
\]

to the value of the integral. The poles of \( \zeta'(s)/\zeta(s) \) are the poles and zeros of \( \zeta(s) \). The only pole of \( \zeta(s) \) is, we said, at \( s = 1 \), and so this has residue

\[
- \lim_{s \to 1} (s - 1) \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} = - \lim_{s \to 1} (s - 1) \left( \frac{-1}{(s - 1)} \right) \frac{x^1}{1} = x,
\]

the expected main term, to the value of the integral. Last, but by no means least, are the zeros of \( \zeta(s) \). The Euler product representation of \( \zeta(s) \) converges in \( \text{Re}(s) > 1 \), and so there can be no zeros of \( \zeta(s) \) in this half-plane. Otherwise the zeros of \( \zeta(s) \) are rather mysterious. All we can really say is that if \( \zeta(s) \) looks like \( c(s - \rho)^m \), near to the zero \( \rho \), for some integer \( m \geq 1 \) (and non-zero constant \( c \)) then \( -\zeta'(s)/\zeta(s) \) looks like \( -m/(s - \rho) \), and therefore the residue at \( s = \rho \) is

\[
- \lim_{s \to \rho} (s - \rho) \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} = - \lim_{s \to \rho} (s - \rho) \left( \frac{m}{(s - \rho)} \right) \frac{x^\rho}{\rho} = -m \frac{x^\rho}{\rho}.
\]

If we count a zero of multiplicity \( m \) (like this), \( m \) times in the sum, then we have evaluated the integral so as to yield Riemann’s remarkable explicit formula:

\[
\psi^*(x) = x - \sum_{\rho: \zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}.
\]

Remarkable it may be, but is it useful? We have gone from a simple question like counting the number of primes up to \( x \), to a sum over all of the zeros of the analytic continuation of \( \zeta(s) \). Ever since Riemann’s memoir, mathematical researchers have struggled to find a way to fully understand the zeros of \( \zeta(s) \), so as to make this “explicit formula” useful. We have had some, rather limited, success.

Riemann showed that there are infinitely many zeros of \( \zeta(s) \) so we have a problem in that the sum over \( \rho \), in Riemann’s explicit formula, is an infinite sum and one can easily show that it is not absolutely convergent. So to evaluate it directly, we would need to detect cancelation amongst the summands, something that we are not very skilled at. Instead, one can modify Riemann’s argument to show that one can truncate the sum, taking only those \( \rho \) in the box up to height \( T \),

\[
B(T) := \{ \rho \in \mathbb{C} : 0 \leq \text{Re}(\rho) \leq 1, \quad -T \leq \text{Im}(\rho) \leq T \},
\]

in our sum. This turns out to be a finite set, and we get the explicit formula,

\[
\psi(x) = x - \sum_{\rho: \zeta(\rho)=0} \frac{x^\rho}{\rho} + \text{a small error},
\]
where the “small error” is smallish if $T$ is appropriately chosen (as a function of $x$; typically $T = \sqrt{x}$). Then we can bound $|\psi(x) - x|$, by taking absolute values in the sum above:

$$\left| \sum_{\rho: \zeta(\rho)=0 \atop \rho \in B(T)} \frac{x^\rho}{\rho} \right| \leq \sum_{\rho: \zeta(\rho)=0 \atop \rho \in B(T)} \frac{|x^\rho|}{|\rho|} = \sum_{\rho: \zeta(\rho)=0 \atop \rho \in B(T)} \frac{x^{\text{Re}(\rho)}}{|\rho|} \leq x^{m(T)} \sum_{\rho: \zeta(\rho)=0 \atop \rho \in B(T)} \frac{1}{|\rho|}$$

where $m(T) := \max_{\rho \in B(T)} \text{Re}(\rho)$. The sum over zeros can be shown to be bounded by a multiple of $(\log T)^2$, so if we can get a good bound on $m(T)$ then we will be able to deduce the prime number theorem. By “good bound” here we mean that $m(T)$ must be somewhat less than 1, in fact

$$m(T) \leq 1 - \frac{3 \log \log T}{\log T}$$

will do.

Riemann made a few calculations of the zeros of $\zeta(s)$ and all the real parts seemed to be $1/2$. This led to him to:\footnote{Riemann actually wrote: “It is very probable that all roots are [on the $1/2$-line]. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.”}

**The Riemann Hypothesis.** If $\zeta(\rho) = 0$ with $0 \leq \text{Re}(\rho) \leq 1$ then $\text{Re}(\rho) = \frac{1}{2}$.

If, the Riemann Hypothesis is true then each $x^\rho$ has absolute value $x^{1/2}$ or $< 1$, and one can deduce, via the argument that we have just sketched, the estimates (RH1) and (RH2) for $\pi(x)$ and $\theta(x)$. Actually there is a very intimate link between upper bounds for the real parts of the zeros of $\zeta(s)$ and bounds on the error term in the prime number theorem, and one can show that if either (RH1) or (RH2) is true then the Riemann Hypothesis follows. This connection goes much further. For example, fix $1 > \beta > 1/2$. Then all zeros of $\zeta(s)$ satisfy

$$\text{Re}(s) < \beta \text{ if and only if } \left| \sum_{p \leq x} \log p - x \right| \leq C_\beta x^{\beta}.$$ 

for some constant $C_\beta > 0$. How strange! Here we are in two different worlds, counting primes, and zeros of the analytic continuation of a function, and yet a key part of understanding each is equivalent. This is the bedrock on which mathematics is formed. Surprising connections between fields that have no obvious right to be related, and yet they are, at some fundamental level. Riemann’s work gave one of the first results of this type, and now every field of research in pure mathematics is full of such links.

Riemann’s connection is not restricted to this one question. Indeed, using the explicit formula, one can reformulate many different problems about primes, as problems about zeros of zeta-functions, upon which we can use the tools of analysis. Mathematicians love bringing fields together that seem so distant, hopefully allowing a more rounded perspective of both.

These observations are so seductive that they have been the thrust of almost all research into the distribution of prime numbers ever since. Moreover there are many other
good questions about prime numbers, number fields, finite fields, curves and varieties that can be re-cast in terms of appropriate zeta-functions, so there is no end to what can be investigated by such methods.

7. The coup de grâce in the proof of the prime number theorem. What we have sketched above is not quite the end of the story of the proof of the prime number theorem. It was not proved by Riemann: Although he came up with the whole idea, and made many spectacular advances in his short memoir, he could not give an unconditional proof. He left several steps to be completed. These turned out to be very difficult indeed, and it was only 37 years later that Hadamard and de le Vallé Poussin did so, independently, in 1896. The final step that was left to them, was to show that \( \zeta(s) \) has no zeros on the line \( \text{Re}(s) = 1 \) (which, from now on, we call the 1-line). Both of their proofs (and most that followed) show that \( \zeta(s) \) cannot have a zero at \( 1 + it \) by showing:

\[
\text{If } \zeta(1 + it) = 0 \text{ then } \zeta(1 + 2it) = \infty;
\]

that is, \( \zeta(s) \) has a pole at \( 1 + 2it \). However we have already noted that that \( \zeta(s) \) only has a pole at \( s = 1 \), and therefore \( t = 0 \). But this yields a contradiction to the assumption that \( \zeta(1 + it) = 0 \). The proofs of Hadamard and de le Vallé Poussin are complicated, and the proof of Mertens that can be found in every textbook is relatively easy without being enlightening. Nonetheless it is not difficult to get an intuitive feel for why (13) should be true: Since \( \zeta(s) \) is an analytic function and equals 0 at \( 1 + it \), it must be well-approximated by the leading term in its Taylor series, \( c(s-(1+it))^r \), when \( s \) is sufficiently close to \( 1 + it \), for some integer \( r \geq 1 \) and some non-zero constant \( c \). For example, if \( s = 1 + it + \frac{1}{\log x} \)

then \( \zeta(s) \approx \frac{c}{(\log x)^r} \), which is pretty small.

Since \( \text{Re}(s) > 1 \) we can also determine \( \zeta(s) \) in terms of its Euler product, and one can then show that one can approximate \( \zeta(s) \) well, at \( s = 1 + it + \frac{1}{\log x} \), by truncating the Euler product at \( x \); in other words

\[
\zeta(s) \approx \prod_{p \text{ prime}} \left( 1 - \frac{1}{p^{1+it}} \right)^{-1}.
\]

We know that \( \zeta(s) \) is small, yet the \( p \)-th term in this Euler product has absolute value

\[
|1 - \frac{1}{p^{1+it}}|^{-1} \geq (1 + \frac{1}{p})^{-1},
\]

and one can deduce from (7) that

\[
\prod_{p \text{ prime}} \left( 1 + \frac{1}{p} \right)^{-1} \approx \frac{c'}{\log x},
\]

for some constant \( c' \). Comparing these two estimates for \( |\zeta(s)| \) allows us to deduce that \( c/(\log x)^r \geq c'/\log x \) for all sufficiently large \( x \). Hence \( r \leq 1 \), but we know that \( r \) is an
integer $\geq 1$, and so $r = 1$. Therefore (17) must be close to equality, most of the time; that is

$$-1/p^{1+it} \approx 1/p$$

for “most” primes $p$. In other words $p^{it} \approx -1$ for “most primes” $p$.

Squaring one sees that $p^{2it} \approx (-1)^2 = 1$ for “most primes” $p$, which tells us that if $s = 1 + 2it + \frac{1}{\log x}$ then

$$\zeta(s) \approx \prod_{p \text{ prime } p \leq x} \left(1 - \frac{1}{p^{1+2it}}\right)^{-1} \approx \prod_{p \text{ prime } p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \approx \epsilon'' \log x,$$

and hence $\zeta(s)$ has a pole at $s = 1 + 2it$.

This is not the only way to show that we cannot have $p^{it} \approx -1$ for “most primes” $p$. My favourite technique is to take logarithms and to show that if this were true then it implies that “most” primes are clustered in intervals of the form

$$[(1 - \epsilon)e^{i\pi(2k+1)/|t|}, (1 + \epsilon)e^{i\pi(2k+1)/|t|}]$$

for some integer $k$, with $\epsilon$ very small. One can then use sieve techniques (the direct descendants of the sieve of Eratosthenes, specifically the Brun-Titchmarsh Theorem) to show that primes cannot be clustered into very short intervals at more than double the expected density, and thus we obtain a contradiction.

We will return later to precisely formulate the hypothesis “$p^{it} \approx -1$ for “most primes” $p$” in more elementary terms. Moreover we will deduce it directly from (5) by more elementary methods.

8. Selberg’s elementary approach. Given the tautology between primes and zeros, no lesser authorities than Hardy, Ingham and Bohr had asserted that it would be impossible to find an elementary proof of the prime number theorem. After all, how could it be possible? The prime number theorem implies restrictions on the the zeros of the analytic continuation of $\zeta(s)$ – how could one have a proof of that which did not use analysis? As Ingham wrote in the introduction of his 1922 book [I1]:

Every known proof of the prime number theorem is based on a certain property of the complex zeros of $\zeta(s)$, and this conversely is a simple consequence of the prime number theorem itself. It seems therefore clear that this property must be used (explicitly or implicitly) in any proof based on $\zeta(s)$, and it is not easy to see how this is to be done if we take account only of real values of $s$.

And Hardy in Copenhagen in 1921:
No elementary proof of the prime number theorem is known, and one may ask whether it is reasonable to expect one. Now we know that the theorem is roughly equivalent to a theorem about an analytic function, the theorem that Riemann’s zeta function has no roots on a certain line. A proof of such a theorem, not fundamentally dependent on the theory of functions, seems to me extraordinarily unlikely. It is rash to assert that a mathematical theorem cannot be proved in a particular way; but one thing seems quite clear. We have certain views about the logic of the theory; we think that some theorems, as we say, ‘lie deep’ and others nearer to the surface. If anyone produces an elementary proof of the prime number theorem, he will show that these views are wrong, that the subject does not hang together in the way we have supposed, and that it is time for the books to be cast aside and for the theory to be rewritten.

The key to Selberg’s elementary approach is Selberg’s formula:

$$\log x \sum_{p \leq x, p \text{ prime}} \log p + \sum_{pq \leq x, p, q \text{ both prime}} \log p \log q \approx 2x \log x.$$  

Here by “≈” we mean that the difference of the two sides is bounded by a multiple of $x$. So instead of getting an accurate estimate for the (weighted) number of primes up to $x$, Selberg gets an accurate estimate for the (weighted) number of primes and P2’s up to $x$ (where a “P2” is an integer that is the product of two primes). Moreover Selberg [S2] gave an elementary proof that (19) is true using combinatorial methods; and it is tempting to believe that it should not then be difficult to remove the P2’s from the equation. But first we ask, how can a formula like (19) hold without any hint of the zeros of $\zeta(s)$?

Selberg does not indicate how he came up with such a formula, and why he would have guessed that it would be true, so we can only speculate. Selberg was a master analyst, so it is plausible that he reasoned as follows: The main problem in using Riemann’s formula is that if a zero, $\rho$, of $\zeta(s)$ has real part equal to 1 (which is not easy to disprove), then the corresponding error term, $x^\rho/\rho$, has size $cx$ for some non-zero constant $c$, a positive fraction of the main term. So, can we come up with a formula, for a quantity similar to the primes, where one such “bad zero” cannot have such a damaging effect?

The obvious way to approach this is to try to produce an integrand which is similar to the one that Riemann worked with, but where there is a double pole at $s = 1$, and no new higher order poles elsewhere. The easy way to get a double pole is to work with the function $\zeta''(s)/\zeta(s)$. This also has the feature that if $\rho$ is a simple zero of $\zeta(s)$ then it is a simple pole of $\zeta''(s)/\zeta(s)$. This is not the case with double zeros, but we expect them to be rare. Hence if we consider the integral

$$\frac{1}{2\pi i} \int_{\text{Re}(s) = \sigma} \frac{\zeta''(s)}{\zeta(s)} \cdot \frac{x^s}{s} \, ds$$

then we have the double pole at $s = 1$. One can compute the residue, using the Taylor expansion at $s = 1$, to be $2x(\log x - 1 - \gamma)$. If $\rho$ is a simple zero of $\zeta(s)$ then its residue

I asked Selberg, in around 1989, how he would define “elementary”. He responded that there is no good definition, but that it is perhaps best expressed as “what a good high school student could follow.”
WHAT IS THE BEST APPROACH TO COUNTING PRIMES?

is \( c_p x^p \), for some constant \( c_p \), and with a bit of luck, these all sum up to no more than a constant times \( x \). In other words, one might guess that the above integral should equal \( 2x \log x \) plus an error which is bounded by at most some multiple of \( x \). Evaluating the integral is tricky in its current form, but once we note that

\[
\frac{\zeta''(s)}{\zeta(s)} = \left( \frac{\zeta'(s)}{\zeta(s)} \right)' + \left( \frac{\zeta'(s)}{\zeta(s)} \right)^2,
\]

we can rewrite the integral as

\[
\frac{1}{2i\pi} \int_{\Re(s) = \sigma} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \, ds + \frac{1}{2i\pi} \int_{\Re(s) = \sigma} \left( \frac{\zeta'(s)}{\zeta(s)} \right)^2 \frac{x^s}{s} \, ds
\]

which, by Perron’s formula, equals

\[
\sum_{\substack{p^m \leq x \\atop p \text{ prime}}} m (\log p)^2 + \sum_{\substack{pq \leq x \\atop p,q \text{ primes}}} \log p \log q.
\]

It is easy to show that the prime powers do not contribute much to the first sum, and that \( \log p \) is close to \( \log x \) for most of the primes \( p \) counted in the sum. Hence we get the left-hand side of (19).

This is perhaps why Selberg believed that something like (19) holds, and why it should be accessible to an elementary proof. Whatever the reasons, he proceeded to obtain a highly ingenious elementary proof (see his paper [S2]).

How can we deduce the prime number theorem from (19)? The first thing to do is to recast this in terms of the function \( \theta(x) \) or, even better, the error term \( E(x) := \theta(x) - x \).

Using (7) (and interpreting \( \approx 2x \log x \) in (19) as \( 2x \log x \) plus an error bounded by a multiple of \( x \)) we obtain

\[
E(x) \log x + \sum_{p \leq x} E(x/p) \log p \approx x,
\]

where \( \approx x \) means “bounded by a multiple of \( x \).” Dividing through by \( x \log x \) we obtain

\[
\frac{E(x)}{x} \approx -\frac{1}{\log x} \sum_{p \leq x} \frac{E(x/p)}{x/p} \frac{\log p}{p}.
\]

It is a little difficult to appreciate what this tells us. The right hand side can be viewed as \(-1\) times the (suitably weighted) average of \( E(t)/t \) for \( t \leq x/2 \) (use (7) to see that this really is a weighted average). But this says that \( E(x)/x \) is minus the average of \( E(t)/t \), which is only consistent if that average is 0, and therefore, we would hope to deduce that \( E(x)/x \to 0 \) as \( x \to \infty \), as desired. One can make this deduction if one can prove that \( E(x)/x \) does not change value quickly as \( x \) varies, which is not so straightforward to do. However, one can easily use this argument to deduce that

\[
\liminf_{x \to \infty} \frac{E(x)}{x} = -\limsup_{x \to \infty} \frac{E(x)}{x}.
\]
This is as far as Selberg got with using (19) when Erdős heard about Selberg’s formula and started to work from it. Indeed both Erdős and Selberg went on to deduce the prime number theorem using entirely elementary methods.\textsuperscript{14} We will not describe their proofs here since we wish to take these ideas in a different direction.

9. Mean values of multiplicative functions. We explained above how the prime number theorem is “equivalent” to the statement that the mean value of $\mu(n)$ for $n$ up to $N$, tends to 0 as $N \to \infty$ (which is formulated in (5)). The beauty of reformulating the prime number theorem like this is that $\mu(n)$ is a multiplicative function, and this opens up many possibilities. A multiplicative function $f$ is one for which $f(mn) = f(m)f(n)$ whenever $m$ and $n$ are coprime integers. Other important examples include

- $n^it$, for fixed $t \in \mathbb{R}$;
- $\chi(n)$, where $\chi$ is a Dirichlet character, which comes up when one studies arithmetic progressions;
- $\tau(n)$, the sum-of-divisors function, which arises when studying perfect numbers;
- etc. In all these cases we might ask for the function’s mean value as we take the average up to infinity; that is

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} f(n).$$

One should ask first whether this exists and, if so, whether we can determine it. And, more importantly for the prime number theorem, can we come up with a “simple” classification of those multiplicative functions that have mean value 0?

In fact the mean value, up to $N$, of $\chi(n)$ tends to 0 as $N \to \infty$, of $\tau(n)$ tends to $\log N$, and of $\sigma(n)$ tends to $cN$ for some non-zero constant $c$. The most interesting of our examples is $n^it$ with $t \neq 0$, since its mean value is

$$\frac{1}{N} \sum_{n=1}^{N} n^it \approx \frac{1}{N} \int_{0}^{N} u^it \, dt = \frac{N^it}{1+it}.$$

That is, the mean value does not tend to a limit as $N \to \infty$, but rather rotates steadily around a circle of radius $1/\sqrt{1+t^2}$. We see here that the period works on a logarithmic scale, that is we get roughly the same mean value for $N$ and $Ne^{2\pi/|t|}$.

In 1971 Halász answered our key question: Restricting attention to multiplicative functions $f$ for which $|f(n)| \leq 1$ for all $n$, when does the mean value of $f(n)$ not tend to 0? There is the obvious example 1, or any example much like 1 (e.g. in which we perturb the value at each prime by just a small amount). There is the generalization $n^it$, for any real

\textsuperscript{14}An unfortunate and unpleasant controversy arose as to who deserved credit for this first elementary proof of the prime number theorem. The establishment (in the form of Weyl) passed down the judgement that Erdős had “muscled in” on Selberg’s breakthrough, that Selberg would have found the route to the elementary proof in time by himself. However Goldfeld \cite{G4} provides an account of the controversy in which one cannot help but be sympathetic to Erdős. To my mind, the controversy reflects two different perspectives on what is appropriate when one hears about the latest research of others, and what is not. Moreover what is appropriate changes over time and I do not think anyone would have questioned Erdős’s behaviour today, nor would have been so unkind as Weyl. (See also \cite{B1} and \cite{Gr}.)
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number $t$ (as we have just seen), and any small perturbations of that. Halász proved that these are essentially all the examples: The only multiplicative functions whose mean value do not tend to 0 are ones that look a lot like $n^t$ for some real number $t$, that is pretend to be $n^t$. His proof involves Dirichlet series to the right of 1 and Parseval’s identity, but it never uses analytic continuation.

We now apply this to (5). If $\mu(n)$ does not have mean value 0, then Halász’s theorem tells us that $\mu(n)$ must pretend to be $n^t$ for some real number $t$. Hence $\mu(n)^2$ must “pretend” to be $n^{2it}$, which implies that $t = 0$, and hence $\mu(n)$ “pretends” to be 1, a contradiction. Formulating what “pretends” means takes a little bit of doing: If $f$ and $g$ are multiplicative functions with absolute value 1, then $f$ “pretends” to be $g$ (meaning that they are not too different, at least in an appropriate average sense) if and only if $f g$ “pretends” to be 1. Since the values of a multiplicative function only depend on its values at primes and prime powers, we can restrict our attention to these. Now if $h$ “pretends” to be 1 then we might measure that by seeing how small $|1 - h(p)|$ is, averaged in some way over the primes, or even $1 - \Re h(p)$, which turns out to be more natural (because of (31), below). Thus we define the distance between $f$ and $g$, for $n \leq x$, by:

$$D(f, g; x)^2 = \sum_{p \leq x} \frac{1 - \Re (f(p)g(p))}{p}.$$  

We say that $f$ is $g$-pretentious if $D(f, g; x)$ is less than some small constant. This allows us to formalize what we meant earlier (e.g. after (17)) when we wrote “$p^t \approx -1$ for most primes $p$” — now we simply write that $D(\mu, n^t; x)$ is bounded.

$D(f, g; x)$ is not truly a distance (for example, it is only 0 if not only $f = g$ but also $|f(n)| = 1$ for all $n$). But for us it is more important that our notion of distance satisfies the triangle inequality

$$(29) \quad D(f, h; x) \leq D(f, g; x) + D(g, h; x).$$

In particular we deduce that $D(f^2, 1; x) \leq 2 D(f, \mu; x)$, and from this we easily deduce the prime number theorem. We are still looking for an elegant proof of the triangle inequality — more on that at the end of this article.

There is a direct connection between $D(f, n^t; x)$ and the Dirichlet series $F(s)$ (which is $\sum_{n \geq 1} f(n)/n^s$): If $\sigma = 1 + \frac{1}{\log x}$ then

$$(31) \quad |F(\sigma + it)| \asymp \log x \exp (-D^2(f, n^t; x)).$$

where the symbol “$\asymp$” means that the ratio of the two sides is bounded, above and below, by positive constants. Halász’s Theorem gives an upper bound for the mean value of $f$ in terms of the minimum of $D^2(f, n^t; x)$ as we range over $t$ in some box, $|t| \leq T$, where $T$ is a power of $\log x$ (that is, the minimum occurs at that $t$, with $|t| \leq T$, for which $n^t$ is “closest” to $f(n)$). From (31) this $t$ can also be thought of as the value at which $|F(\sigma + it)|$ is largest (up to a constant), out of those $t$ for which $|t| \leq T$. 
Besides the distance function, another key tool in working with mean values of multiplicative functions is a generalization of the argument we used in obtaining (7). Now we are interested in $S(x) := \sum_{n \leq x} f(n)$. The trick is evaluate

$$(37) \quad \sum_{n \leq x} f(n) \log n$$

in two ways, one analytic the other algebraic. First analytically, note that this equals $S(x) \log x - \int_1^x \frac{S(t)}{t} dt$.

Since $|S(t)| \leq \sum_{n \leq t} |f(n)| \leq t$, the integral here is $\leq \int_1^x 1 dt = x$, so the value is $S(x) \log x$ plus an error bound by $x$, in absolute value. The second way to evaluate (37) involves again writing $\log n$ as the sum of the logarithms if its prime and prime power divisors. As before we can bound the contribution of the prime power divisors, and so we are left with the “identity”

$$(41) \quad S(x) \log x \approx \sum_{p \leq x} f(p) \log p S(x/p)$$

where “$\approx$” means that the difference is bounded by a multiple of $x$. If we look at the special case $f = \mu$ and write $M(x) = \sum_{n \leq x} \mu(n)$ then we obtain

$$M(x) \log x + \sum_{p \leq x} M(x/p) \log p \asymp x$$

where “$\asymp x$” means “bounded by a multiple of $x$”. Notice that this is exactly the same functional equation as (23), with $M(.)$ replaced by $E(.)$ (which was the error term in the prime number theorem). This was a lot easier to derive than (23), and from here we can also prove that $M(x)/x \to 0$ as $x \to \infty$, much like Erdős and Selberg deduced that $E(x)/x \to 0$ as $x \to \infty$ from (23).

10. What else do we count about primes?. If one writes down the primes, it soon appears as if there are roughly equal numbers that end in a 1, 3, 7 or 9; in other words, in each residue class $a \pmod{10}$ where $\gcd(a, 10) = 1$. And it appears that, in general, there should be roughly equal numbers of primes in each arithmetic progression $a \pmod{q}$ for any positive integers $a$ and $q$ with $\gcd(a, q) = 1$. However, even proving that there are infinitely many primes in any such arithmetic progression is a rather tough challenge. It was only in 1837 that Dirichlet did so, showing that the primes are equidistributed\(^\text{15}\) in the arithmetic progressions if one weights them with a $1/p$. In doing this, Dirichlet invented a generalization of the Riemann-zeta function, called a Dirichlet $L$-function.\(^\text{16}\) To begin

\(^\text{15}\)By “equidistributed” we mean that there are roughly the same number of primes, up to $x$, in each arithmetic progression $a \pmod{q}$ with $\gcd(a, q) = 1$.

\(^\text{16}\)The astute reader might ask how Dirichlet could “generalize” the Riemann-zeta function, 22 years before Riemann’s paper! The fact is that $\zeta(s)$ was considered at length by Euler about one hundred years before Dirichlet; it was later named after Riemann, in honour of his trailblazing work.
with we have *Dirichlet characters* \( \chi \mod q \), which are multiplicative functions \( \chi : \mathbb{Z} \to \mathbb{C} \) that are periodic with minimal period \( q \), and \( \chi(n) = 0 \) if \( (n, q) > 1 \). The most interesting are the real characters (i.e., those characters that can only take the values \(-1, 1\) and \(0\), and do, in fact, take each of those values) like the Legendre symbol \( \left( \frac{a}{q} \right) \). For each Dirichlet character we create the *Dirichlet L-function*

\[
L(s, \chi) := \sum_{n \geq 1} \frac{\chi(n)}{n^s},
\]

which is absolutely convergent when \( \Re(s) > 1 \). This can be analytically continued to the whole complex plane with no poles. Dirichlet’s proof is an elegant piece of combinatorics which easily leads to his theorem that there are infinitely many primes \( \equiv a \mod q \) whenever \( (a, q) = 1 \) provided one can prove that

\[
L(1, \chi) \neq 0 \quad \text{for all real characters } \chi \mod q.
\]

This is a whole lot harder to prove than one might guess, even today (though we have many proofs of it). The most interesting proof, due to Dirichlet himself, shows that \( L(1, \chi) \) can be determined as a simple multiple of the size of a certain group that comes up in algebra. This *Dirichlet’s class number formula* was the first deep connection found between algebra and analysis and is the pre-cursor of so many of the great theorems and conjectures of the last thirty years in number theory.\(^{17}\)

Once the prime number theorem was proved, it was not difficult to modify its proof to show that, whenever \( (a, q) = 1 \) and \( (b, q) = 1 \),

\[
\lim_{x \to \infty} \frac{\# \{ p \leq x : p \text{ prime, and } p \equiv a \mod q \}}{\# \{ p \leq x : p \text{ prime, and } p \equiv b \mod q \}} \text{ exists and equals } 1;
\]

that is the primes are equidistributed amongst the plausible arithmetic progressions mod \( q \). This is called the *Prime Number Theorem for arithmetic progressions mod q*.

If we approach this with the Möbius function, it is not difficult to show that the prime number theorem for arithmetic progressions holds mod \( q \) if and only if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) \chi(n)
\]

for every Dirichlet character \( \chi \mod q \). As in the classical proof, this is not difficult to prove using Halász’s Theorem when \( \chi \) takes complex values (since then \( \chi(p)\mu(p) \) is often quite far from the real line). If \( \chi \) is a real character it also follows immediately from Halász’s Theorem unless \( L(1, \chi) = 0 \). So the “pretentious proof” hinges on the same issue as the classical proof.

\(^{17}\)Like Wiles’ Theorem, the Birch-Swinnerton Dyer conjecture, etc.
11. Is a “pretentious” proof, an elementary proof? The prime number theorem can be phrased as: For all $\epsilon > 0$, we have $|\theta(x) - x| < \epsilon x$ once $x$ is sufficiently large. However we believe that much more is true, namely $|\theta(x) - x| < x^{1/2}(\log x)^2$, so the question becomes how close we can get to our belief. We have seen that improving the error term in the prime number theorem is equivalent to exhibiting wider regions to the left of the “1-line” that contain no zeros of $\zeta(s)$, so-called zero-free regions. Proving such results is difficult (details can be found, for instance, in [Da] and [Bo]) needing complicated analysis of various explicit formulas for $\zeta(s)$, involving its zeros. The key idea is that we understand the values of $\zeta(s)$ well to the right of the 1-line, and we can use that understanding, via such explicit formulas, to get some control over $\zeta(s)$ just to the left of the 1-line (we write “just”, since the zero-free regions that have been obtained are so very narrow). These are beautiful and subtle proofs but they give relatively weak results. Moreover, the main application is to discuss issues about prime numbers that are, essentially, questions that arise in the elementary world to the right of the 1-line. It seems strange to work so hard to extrapolate our knowledge of $\zeta(s)$ to the right of the 1-line, in order to get a meagre understanding just to the left of the 1-line, so as to answer questions to the right of the 1-line. Why one earth should we cross the 1-line at all? The goal of these methods is to avoid that journey.

Pretentious methods use non-trivial techniques of complex analysis (in particular Perron’s formula, as we will see), but not analytic continuation, nor Cauchy’s Theorem and residue computations, nor subtle calculations of zeros of analytic continuations. On the other hand, the calculations used in pretentious techniques can be challenging, though usually they can be reduced to completely elementary techniques, at the cost of further complications. So, technically, one could invoke Selberg’s definition to say that these are elementary techniques, though that misses the point. The main issue is that the methods avoid needing the Dirichlet series $F(s)$ to be analytically continuable (which is rare and unreasonably convenient), and so are inevitably much more widely applicable.

12. Primes in arithmetic progressions (questions involving uniformity). If we begin computing primes in arithmetic progressions mod, say, 101, we notice that, quite soon, the primes are roughly equidistributed in all of the 100 possible progressions (for example, by the time there are 100 primes in each arithmetic progression, on average, the least is 87 and the most is 109). So there is an important, new question for primes in arithmetic progressions:

By when can we expect roughly equal numbers of primes in each arithmetic progression mod $q$?

With enough computing we might guess that there should be roughly equal numbers of primes up to $x$, in each arithmetic progression $a \pmod q$, for each $a$ with $(a, q) = 1$, once $x > c_\epsilon q^{1+\epsilon}$, for some constant $c_\epsilon$, for any fixed $\epsilon > 0$. This is far out of reach of what we can prove. Indeed, the best result we have a plan of how to prove is that the primes are roughly equidistributed mod $q$ once $x > c_\epsilon q^{2+\epsilon}$. This plan, though, involves proving the Generalized Riemann Hypothesis,\(^{18}\) which seems very far out of reach.

So what can we prove unconditionally? By now, you may not be surprised to learn

\(^{18}\)That is, if $\rho$ is a zero of any Dirichlet $L$-function with $0 \leq \text{Re}(\rho) \leq 1$, then $\text{Re}(\rho) = \frac{1}{2}$.
that in both the classical theory, and the pretentious approach, the issue is how close $L(1, \chi)$ is to 0 for real characters $\chi$. A lower bound of the shape $L(1, \chi) > c/\sqrt{q}$ follows immediately from Dirichlet’s class number formula, and leads to the result that the primes are roughly equidistributed mod $q$ once $x > ce^{\sqrt{q}}$ for some constant $c > 0$ that one can determine. This lower bound for $x$ is far bigger than what we expect to be true.

In 1936 Siegel improved this lower bound to: For all $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that $L(1, \chi) > c_\epsilon/q^\epsilon$, and so the primes are roughly equidistributed mod $q$ once $x > \kappa_\epsilon e^{\sqrt{q}}$.

But there is a catch. The method of proof does not allow one to determine $c_\epsilon$ (note that we are not saying that it has not been computed, but rather that it cannot be computed). The proof is very surprising in that Siegel splits his considerations into two complementary cases:

Either, the Generalized Riemann Hypothesis is true, so it is easy to compute $c_\epsilon$.

Or, the Generalized Riemann Hypothesis is false, in which case it is easy to compute $c_\epsilon$ in terms of any given counterexample.

The problem with this dichotomy is the second case. We do not believe it holds, so we do not believe that it will give us any constant, but until someone proves the Generalized Riemann Hypothesis then we cannot discount that possibility! Think about it a bit and you will see why it is not possible to obtain an explicit constant from Siegel’s proof given our current state of knowledge.

All this talk of Riemann Hypotheses in the proof of Siegel’s theorem means that we are involving zeta-functions to the left of the 1-line, and so I had believed that this result could only be obtained by classical means. That was my prejudice, until my postdoc, Dimitris Koukoulopoulos,\textsuperscript{19} came up with a very subtle elementary argument that allowed him to completely replace Siegel’s argument by a purely pretentious one, with no analytic continuations in sight. To be sure it is not difficult, at a technical level, to see the links between his proof and that of Siegel’s (in fact he developed an idea of Pintz\textsuperscript{[P1]}), but rather amazingly we now have an “elementary proof” of this seemingly deep fact.

Koukoulopoulos\textsuperscript{[K1]} also showed that the primes are not only equidistributed mod $q$ once $x > \kappa_\epsilon e^{\sqrt{q}}$, but that the ratio is very close to 1 (as in the Siegel-Walfisz Theorem). This in turn allows one to use the large sieve to prove the Bombieri-Vinogradov Theorem. This says, roughly, that the primes are roughly equidistributed mod $q$ for almost all $q < x^{1/2-\epsilon}$. This is the consequence we expect from the Generalized Riemann Hypothesis, but we obtain it only for most $q$, not necessarily all $q$.

\textbf{13. Even deeper.} Seemingly, one of the deepest results about $L$-functions is that their zeros “repel” each other. That is, they do not like to be too close together. In particular one cannot have zeros of two Dirichlet $L$-function both close to 1, and this can be rephrased as saying that there is at most one real character $\chi$ (mod $q$), amongst all the real characters with $q$ in the range $Q < q \leq 2Q$, for which $L(1, \chi) < c/\log Q$. Hence $L(1, \chi) \geq c/\log q$ for all of the other real characters with $q$ in this range, and therefore one can state a strong prime number theorem for the arithmetic progressions for all these

\textsuperscript{19}And now my faculty colleague.
other moduli.\(^{20}\) In fact with such a strong lower bound on \(L(1, \chi)\) one can show that the
primes are roughly equidistributed mod \(q\) once \(x > q^A\) for some sufficiently large \(A\) (how
large depends on the constant \(c\), and how nearly you want the primes equidistributed).

Rather surprisingly, these repulsion results are much easier to prove in the pretentious
world. Basically \(L(1, \chi)\) being very small means that \(\mu\) is \(\chi(n) n^{it}\)-pretentious for some real
number \(t\), and so if \(L(1, \psi)\) is also small then \(\mu\) is \(\psi(n) n^{iu}\)-pretentious for some real number
\(u\). Now if \(\mu\) is very close to \(\chi(n) n^{it}\) as well as to \(\psi(n) n^{iu}\), then they are close to each other
(which formally follows from our triangle inequality), and therefore the Dirichlet character
\(\overline{\psi} \chi\) is \(n^{i(u-t)}\)-pretentious, which is easily shown to be impossible. This all goes to show
that pretentiousness is repulsive.

14. The pretentious large sieve. Perhaps the deepest proofs in the classical analytic
number theory approach to the distribution of prime numbers, are the proofs of Linnik’s
Theorem; that is that there exist constants \(c > 0\) and \(L > 0\) such that for any positive
integers \(a\) and \(q\), there is a prime \(\leq cq^L\) which is \(\equiv a\) (mod \(q\)). Linnik’s 1944 proof [L1]
has been improved many times (e.g. in Bombieri’s [Bo]) but remains delicate and subtle
and difficult. Inspired by a new, technically elementary proof in November 2009 given by
Friedlander and Iwaniec [F4], Soundararajan and I went on to develop an idea we had for
a pretentious large sieve [GS1], and we ended up giving what is surely the shortest and
technically easiest proof of Linnik’s Theorem (though bearing much in common with an
earlier proof of Elliott [E2]). I will say a little bit more about this in a moment. This
new technique has enormous potential, because it can replace some very subtle classical
techniques, and yet does not require the function involved to have an \(L\)-function that can
be analytically continued. We made one other application, with de la Bréteche [GS2], to
understand better the solutions to Pythagoras’ equation \(a^2 + b^2 = c^2\) when we are working
mod \(p\) (as well as to several other additive number theory problems).

15. From a collection of ad hoc results, to a new approach to prime num-
bers. Our easy proof of Linnik’s Theorem suggested to Soundararajan and me that we
should be able to prove all of the basic results of analytic number theory without ever
using analytic continuation. Since early 2010 we have been working on developing this
new approach. Our goal is to reprove all of the key results in the standard classical books
[Da] and [Bo] using only “pretentious methods”. Within a year we found that we could
prove some version of all of the results, perhaps not as strong, but much the same in prin-
ciple. In doing this we have stood firmly on the shoulders of giants. Many analytic number
theorists have developed ideas about multiplicative functions, over the last 50 years, that
have allowed them to prove results on different aspects of prime numbers. Those who
have been most central to our description of the subject are Erdös, Selberg, Wirsing, Del-
lange, Daboussi, Hildebrand, Hall, Tenenbaum, Pintz, Elliott, Montgomery and Vaughan,
Friedlander, Iwaniec and Kowalski, ...

Despite being able to prove some version of all of the principal results known on the
distribution of primes, we increasingly found ourselves frustrated for three reasons:

\(^{20}\)I have simplified here a little bit, rather than get in to the technicalities of primitive and induced
characters. For more on this, see Davenport’s book [Da].
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1. Although we could show the prime number theorem, we could not show that convergence is anywhere like as fast as had been shown by classical means. In fact we could not see how one might use pretentious methods to even prove something as (relatively) weak as $|\theta(x) - x| \leq x/\log x$ for $x$ sufficiently large.

2. There are many strong results in the subject that are proved assuming the Riemann Hypothesis. We could not conceive of proving analogous results since the Riemann Hypothesis is a conjecture about the zeros of the analytic continuation of $\zeta(s)$, something we are trying to avoid discussing at all.

3. All proofs of Halász’s Theorem (which lies at the center of the whole theory) were not only complicated, but also hard to motivate. We had modified this proof in several ways, for example in the proof of the pretentious large sieve, and this led to a lot of the theory seeming somewhat obscure (even if technically straightforward).

Fortunately several of the best young people in analytic number theory got interested in these issues, and they have satisfactorily resolved all of them, as I will now describe:

16. The strongest known form of the prime number theorem. The prime number theorem can be phrased as $\theta(x)/x \to 1$ as $x \to \infty$. The proofs of 1896 immediately yielded that for any fixed $A > 0$ we have

$$|\theta(x) - x| \leq \frac{x}{(\log x)^A}$$

for all sufficiently large $x$. In fact de la Vallée Poussin proved the much stronger result that there exists a constant $c > 0$ for which

$$|\theta(x) - x| \leq x/\exp \left( c\sqrt{\log x} \right)$$

for all sufficiently large $x$. The strongest version proved unconditionally is from 1959 (by Korobov and Viogradov), and gives

$$|\theta(x) - x| \leq x/\exp \left( c\frac{\log x^{3/5}}{(\log \log x)^{1/5}} \right).$$

(43)

There has been no improvement on this in over 50 years, yet it is so far from what we believe to be true (and can prove, assuming the Riemann Hypothesis).

In directly using Halász’s Theorem, applied to the multiplicative function $\mu(n)$, as described above, one can prove results like $|\theta(x) - x| \leq x/(\log x)^\tau$ for sufficiently large $x$, for some $\tau < 1$, but one can show that it is impossible to obtain larger $\tau$ as an immediate corollary. This is a lot weaker result than the simplest results that one obtains from classical methods.

Selberg showed that if $f(p) = \alpha$ for all primes $p$ (with $|\alpha| \leq 1$), then the bounds given by Halász’s Theorem are very near to the truth, unless $\alpha = 0$ or $-1$. Our favourite examples have this property: The mean value of $\mu(p)$ is $-1$, and the mean value of $\mu(p)\chi(p)$ is 0 for any Dirichlet character. The Koukoulopoulos converse theorem [K2] goes one big step forward, stating that if the mean value of $f$ is small then $f(p)$ must average 0 or $-1$.
over the primes. This opens the door to getting much stronger upper bounds for the mean value of $f$, via appropriate modifications of Halász’s Theorem. Indeed Koukoulopoulos was then able to prove the strongest versions of both the prime number theorem, and the prime number theorem for arithmetic progressions, using only pretentious methods, never venturing to the left of the 1-line.

At first sight it is surprising that he could not do better than the classical proofs. After all, if his proofs are so different from the classical proofs, then why would he also come up with such an unlikely bound as in (43)? The reason is that, despite appearances, the proofs are fundamentally the same. The classical proof uses deep tools of analysis, arguably artificially and certainly not necessarily, and these can be stripped away as in Koukoulopoulos’ proof. That is not to say that Koukoulopoulos’ proof is easy, but it does only use these simpler tools.

17. The Pretentious Riemann Hypothesis. The Riemann Hypothesis tells us about the zeros of (the analytic continuation of) $\zeta(s)$ being far into the domain of analytic continuation, that is they are all on the “$\frac{1}{2}$-line”. Can one feel the effect of this to the right of the “1-line”? This is certainly desirable since so many nice results are a corollary of the Riemann Hypothesis.

After some thought one might guess that the answer to this is “yes”: To count primes we looked at $\zeta'(s)/\zeta(s)$. The Riemann Hypothesis is equivalent to this not having any poles, other than at $s = 1$, to the right of the $\frac{1}{2}$-line; and one can remove the pole at $s = 1$ simply by working instead with $\zeta'(s)/\zeta(s) + 1/(s - 1)$. Now, if this function’s Taylor series around a point to the right of the 1-line, remains valid in a wide enough region, then we know there can be no poles in that region. To have a large radius of convergence for a Taylor series, its coefficients should not grow too fast, that is the derivatives of $\zeta'(s)/\zeta(s) + 1/(s - 1)$ should not grow too fast. This plan can be made to work, and allows us to state what we call:

The Pretentious Riemann Hypothesis. For all $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that for every integer $k \geq 1$ we have

$$\left| \left( \frac{\zeta'}{\zeta}(s) + \frac{1}{s - 1} \right)^{(k)} \right| \leq c_\epsilon k! 2^k (1 + t^\epsilon)$$

uniformly for $s = \sigma + it$ with $1 \leq \sigma < 2$ and $0 \leq t \leq e^k$.

Using Koukoulopoulos’ methods one can show that if the pretentious Riemann Hypothesis is true then $|\psi(x) - x| < \kappa_\epsilon x^{1/2 + \epsilon}$ for any fixed $\epsilon > 0$, which in turn implies the Riemann Hypothesis. On the other hand, the Riemann Hypothesis implies

$$\left| \left( \frac{\zeta'}{\zeta}(s) + \frac{1}{s - 1} \right)^{(k)} \right| \leq c k! 2^k \log t$$

for $s = \sigma + it$ with $\sigma \geq 1$ and $t \geq 1$, which is somewhat more than we asked for in the pretentious Riemann Hypothesis. Nonetheless these remarks imply that

The Riemann Hypothesis holds if and only if Pretentious Riemann Hypothesis holds.
18. A re-appraisal of the use of Perron’s formula. Soundararajan and I had written up as palatable a proof as we could of Halász’s Theorem for the first drafts of our book [GS1], but even we had to admit that it was difficult to motivate. So I was delighted when, in January 2013, my new postdoc, Adam Harper, suggested a new path, rather simpler proof to (a version of) Halász’s Theorem. Subsequently we have developed his idea with him, and find ourselves re-appraising the use of Perron’s formula when summing coefficients of Dirichlet series.

In Riemann’s approach one takes the formula (11), and shifts the line of integration far to the left side of the complex plane. In the pretentious approach one stays with the same line of integration. But then how can one get an accurate estimate, or even a decent upper bound, since the integral of the absolute value of the integrand, is usually much larger than the value of the integral? There are several important observations involved. First though, let’s look at this in more generality, with the identity

\[ \sum_{n \leq x} f(n) = \frac{1}{2i\pi} \int_{\text{Re}(s)=\sigma} F(s) \frac{x^s}{s} ds, \]

with \( F(s) := \sum_{n \geq 1} f(n)/n^s \), where \( \sigma = 1 + \frac{1}{\log x} \). One can give a version of this (like in the proof of the prime number theorem), with the values of \( s \) running over only those \( t \) where \( |t| \leq T \), for some suitably chosen \( t \) ("suitably chosen" in that the error in making this approximation is small). In fact taking \( T \) as a power of \( \log x \) will do. Then we can take absolute values in the integrand, noting that \( |x^s| = x^\sigma = ex \) to get an upper bound

\[ \frac{1}{2\pi} \int_{s=\sigma+it \mid |t| \leq T} |F(s)| \frac{|x^s|}{|s|} dt \leq 3x \cdot \max_{|t| \leq T} |F(\sigma + it)| \cdot \frac{1}{2\pi} \int_{|t| \leq T} \frac{1}{|\sigma + it|} dt \leq x \cdot \max_{|t| \leq T} |F(\sigma + it)| \cdot (\log T + 1). \]

This would more-or-less be Halász’s Theorem (via (31)) if the upper bound was divided through by \( \log x \). This is encouraging since our approach in getting this upper bound was very crude, and we can surely refine it a bit.

Studying the integrand \( F(s)x^s/s \), we might expect that \( F(s)x^\sigma/s \) does not change much while \( x^it \) rotates once around the unit circle (which requires an interval, for \( t \), of length \( 2\pi/\log x \)). The easiest way to pick up this cancelation is to integrate by parts, so that (47) becomes:

\[ \frac{1}{2\pi} \int_{\text{Re}(s)=\sigma} F(s) \frac{x^s}{s \log x} - \frac{1}{2\pi} \int_{\text{Re}(s)=\sigma} F'(s) \frac{x^s}{s \log x}. \]

The "\( \log x \)" in the denominator is the cancelation. It is not difficult to show that the first term, because of the \( s^2 \) in the denominator, is fairly small, and so we are left with

\[ -\frac{1}{2\pi \log x} \int_{\text{Re}(s)=\sigma} F'(s) \frac{x^s}{s} = -\frac{1}{2\pi \log x} \int_{\text{Re}(s)=\sigma} \frac{F'(s)}{F(s)} \cdot F(s) \frac{x^s}{s}. \]
If we take absolute values here, much as we did in (47), then we get the desired bound so long as $|F'(s)/F(s)|$ is “small” in a certain average sense. It is indeed this small for many multiplicative functions $f$ of interest to us, but not all, so another idea is needed.

This is where the key new idea of Harper comes in (though phrased very differently). Going from (47) to (53), we gained a factor of $\frac{1}{\log x} \cdot \frac{F'(s)}{F(s)}$, which does improve things and gaining another such factor would be enough to get us to our goal. We cannot quite get this, but an even more complicated formula gets us there, this one using the flexibility of slightly varying the (vertical) line of integration:

$$\int_0^\Delta \left( \frac{1}{i\pi} \int_{1-iT}^{1+iT} \frac{F'(s)}{F(s)} \cdot \frac{F'(s + \beta)}{F(s + \beta)} \cdot \frac{x^{s - \beta}}{s + \beta} ds \right) d\beta$$

where $\Delta$ is a suitably chosen multiple of $1/\log x$. There are similarities between this and the formula used in the usual proof of Halász’s Theorem, but it is now much clearer how we got here (which means that this new technique is much more flexible). In particular it allows us to obtain asymptotic formulae if the mean value is “reasonably well-behaved.”

One relatively easy application now is to prove Hoheisel’s deep theorem on primes in short intervals. That is, there exists a constant $\delta > 0$ and a constant $c_\delta > 0$ such that if $x^{1-\delta} < y \leq x$ then

$$\# \{ p \text{ prime : } x < p \leq x + y \} \geq c_\delta \frac{y}{\log x}.$$
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associated to elliptic curves. There is much to do to establish a pretentious theory here. The classical theory can prove much less with these L-functions, so we can hope that pretentious techniques might have significant impact here.

20. A proof and a challenge. The triangle inequality, (29), lies at the heart of this new theory. There are now several proofs, none of which are particularly elegant. The best was given by Eric Naslund, as part of an undergraduate research project in 2011, which I will reproduce now. To prove (29) it suffices to prove the simpler inequality

\[(59)\]
\[\eta(w, y) \leq \eta(w, z) + \eta(z, y).\]

where \(\eta(z, w)^2 := 1 - \text{Re}(zw)\), for any \(w, y, z \in \{z \in \mathbb{C} : |z| \leq 1\}\).

Proof. (Eric Naslund) Let \(r = |z|\). Then, since \(|wz| \leq r\) and \(|z\bar{y}| \leq r\), we can write \(w \bar{z} = r(a + bi)\), \(z \bar{y} = r(c + di)\) and \(w \bar{y} = (a + bi)(c + di)\) where \(a + bi\) and \(c + di\) lie in the unit disk.

Note that \(1 + ra, 1 + rc \leq 2\), so that \((1 + ra)(1 + rc) \leq 4\), and hence

\[2\sqrt{1-ra}\sqrt{1-rc} \geq \sqrt{1-r^2a^2}\sqrt{1-r^2c^2} \geq \sqrt{1-a^2}\sqrt{1-c^2} \geq bd.\]

Now, if \(a + c \geq 0\) then \(1-rc+ac \geq 1-a-c+ac = (1-a)(1-c) \geq 0\); and if \(a + c \leq 0\) then \(1-rc+ac \geq 1+ac \geq 0\). Either way \(1-rc+ac \geq 1-ac\). Adding this to the last displayed equation and taking square roots of both sides, we obtain

\[\sqrt{\eta(w, z) + \eta(z, y)} = \sqrt{1-ra} + \sqrt{1-rc} \geq \sqrt{1-ac} + bd = \eta(w, y).\]

We would still like to see a “proof from the book”\(^{21}\) of (59), a more natural and easy proof. I will leave this as a competition for our readers. Please email me your proof. The best one will appear in our book, with appropriate credit.

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\(^{21}\)The great Paul Erdős, used to say that the Supreme Being has a book of all of the best proofs, and just occasionally we are allowed to glimpse at a page. When you have such a proof, it is obvious that it is from “the book”! When he was still alive, there was no greater compliment than Erdős remarking, as he occasionally did, “I think that is from the book.”
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