E21
ON A QUESTION OF GRINBLAT

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Abstract. We prove the consistency of: there is a $\kappa$-complete ideal on $\kappa$ for some $\kappa < 2^{\aleph_0}$ such that the Boolean algebra $\mathcal{P}(\kappa)/I$ is $\sigma$-centered and there are $Q$-sets of reals.

I would like to thank Alice Leonhardt for the beautiful typing.
Latest Revision - 99/Aug/17
In set theoretic language, Grinblat has been asking for some time

1.1 Problem: Is it consistent with ZFC that:

(a) there is an $\aleph_1$-complete ideal $I$ on some $\kappa < 2^{\aleph_0}$ such that $\mathcal{P}(\kappa)/I$ is centered

(b) there is a $Q$-set.

We answer positively.

1.2 Claim. Assume that $\kappa < \chi = \chi^\kappa$ and $\kappa$ is measurable, say $D$ is a normal ideal on $\kappa$ and $I$ is the dual ideal.

Then for some c.c.c. forcing notion $P$ of cardinality $\chi$ we have in $V^P$:

(*) (i) $2^{\aleph_0} = \chi$

(ii) MA$_{<\kappa, < \text{cf}(\chi)}$ (σ-centered) holds (i.e. MA for forcing notions of cardinality $< \kappa$ and $< \text{cf}(\chi)$ dense open subsets hence every set of reals of cardinality $< \kappa$ is a $Q$-set

(iii) the Boolean algebra $\mathcal{P}(\kappa)/I$ is centered, i.e. $\mathcal{P}(\kappa)\setminus I^V$ is the union of countably many directed sets (where $I^V = \{A \in V^P : A \subseteq \kappa$ and $A$ is included in some member of $I\}$).

1.3 Remark. 1) Why the “hence” in (ii)? As for $X \subseteq Y \subseteq \omega^2$ the natural forcing $Q$ of adding subtrees $T_n \subseteq \omega^>2$ for $n < \omega$ such that $\bigcup_{n<\omega} \text{lim}(T_n) \cap Y = X$ is σ-centered of cardinality $\leq |Y| + \aleph_0$ and it is enough to find a directed $G \subseteq Q$ intersecting $|Y| + \aleph_0$ dense subsets. E.g. we can use:

$$Q = \left\{ p = (\bar{t}, \bar{u}) : \text{for some } n = n(p) < \omega \text{ we have :} \right\}

(a) \quad \bar{t} = \langle t_\ell : \ell < n \rangle, \text{ each } t_\ell \text{ has the form } T \cap m_\ell(p) \geq 2, T
\text{ a perfect subtree of } \omega^>2, m_\ell(p) < \omega,$$

(b) \quad \bar{u} = \langle u_\ell^* : \ell < n(p) \rangle, u_\ell^* \subseteq X \text{ is finite}

(d) \quad \text{if } \ell < n(p), n \in u_\ell \text{ then } \eta \upharpoonright m_\ell(p) \subseteq t_\ell$$

The order is natural $p \leq q$ iff $n(p) \leq n(q) \& \bigwedge_{\ell < n(p) \wedge m_\ell(p) \geq 2} [t_\ell^p \subseteq t_\ell^q, u_\ell^p \subseteq u_\ell^q]$. $m_\ell(p) \leq m_\ell(q) \& t_\ell^p = t_\ell^q \cap m_\ell(p) \geq 2 \& u_\ell^p \subseteq u_\ell^q$.

Let for $\eta \in X, \mathcal{I}_\eta$ be $\{p \in Q : \eta \in u_\ell^p \text{ for some } \ell < n(p)\}$ and for $x \in Y \setminus X, n < \omega$ let $\mathcal{I}_x = \{p \in Q : n < n(p) \text{ and } n \upharpoonright m_\ell(p) \not\subseteq t_\ell^p\}$. 
Proof of 1.2. Let $Q = \langle P_i, Q_j : i \leq \chi, j < \chi \rangle$ be a FS iteration, in $V^{P_i}, Q_j$ is a $\sigma$-centered forcing notion of cardinality $< \kappa$ and its set of elements is an ordinal $< \kappa$, and any such forcing notion appear unboundedly often, more exactly, if $i_0 < \chi, Q$ is a $P_{i_0}$-name of a forcing notion with a set of elements (forced to be) $\alpha_Q < \kappa$, then for $\chi$ many (hence unboundedly many) $j \in (i, \chi)$ we have: $\models_{P_j}$ “if $Q$ is $\sigma$-centered then $Q \cong Q_j$”.

As each $Q_j$ is $\sigma$-center (in $V^{P_i}$) there is $\tilde{f} = \langle f_j : j < \chi \rangle$ such that $\models_{P_j}$ “$f_j$ is a function from $Q_j$ to $\omega$ such that each $\{ p \in Q_j : f_j(p) = n \}$ is directed”.

So $P_{1,i*} = \{ p \in P_i : j \in \text{Dom}(p) \text{ then } p \upharpoonright j \text{ forces a value to } f_j(p(j)) \text{ and a value to } p(j) \}$ is a dense subset of $P_i$. Now clearly clauses (i), (ii) of (*) holds in $V^P$. As $D$ is a normal ultrafilter on $\kappa$ there is a transitive class $M$ such that $M^* \subseteq M$ and there is an elementary embedding $j$ from $V$ to $M$ with critical ordinal $\kappa$ such that $D = \{ A : A \subseteq V, A \subseteq \chi \text{ and } \kappa \in j(A) \}$. Let $j(Q)$ be $Q' = \langle P_i', Q_j' : i \leq j(\kappa), j < j(\chi) \rangle$ and $\tilde{f}' = j(\tilde{f}) = \langle f_j' : j < j(\kappa) \rangle$, so $M$ “thinks” that $(Q', \tilde{f}')$ satisfies all the properties listed above, but in $V$ it relates all of those properties, though not $j(\kappa)$.

Let $P^* = \{ j(p) : p \in P^*_\chi \}$, so it is well known that $P^* \prec P^*_\chi$ and the completion of the Boolean algebra corresponding to $P^*_\chi/P^*$ is isomorphic to $\mathcal{P}(\kappa)/I^V$, so it is enough to prove that $\models_{P^*}$ “$P^*_\chi/P^*$ is $\sigma$-centered” (in $V^P$, which is the same as $V^{P_\chi}$). Note also the $P^*_\chi$ hence $P^*_\chi/P^*$ has cardinality $|P^*_\chi| = \kappa^\kappa = \chi$.

Now the point is that we can reorder the iteration $Q'$: first do $\langle Q_{j(i)} : j < \chi \rangle$ and then the rest, as each $Q_j$ depends on $< \kappa$, $j' < j$ and this set is not extended by $j$.

Note first that this suffices as the limit of FS iteration of $\sigma$-centered forcing notion each of cardinality $\leq 2^{\aleph_0}$ and length $< (2^{\aleph_0})^+$ (in $V^{P_{1,\chi}}$) is $\sigma$-centered.

Second, this reordering is possible. [Why? The set of elements of $Q_j$ is $\alpha_Q$ and $\alpha_Q$ is a $P_j'$-name of an ordinal $< \kappa$ and $P_j'$ satisfies the c.c.c. hence for some $\alpha^*_j < \kappa$ we have $\models_{P^*_j} \alpha^*_j \leq \alpha^*_j$” so $Q_j$ is a subset of $\alpha^*_j \times \alpha^*_j$. For any $\beta, \gamma < \alpha^*$, there is a maximal antichain $\mathcal{I}_{j, \beta, \gamma}$ of $P^*_j$ of conditions forcing “$Q_j \models \beta \leq \gamma$” so $\beta \in Q_j, \gamma \in Q_j$ or forcing its negation.

We choose $\tilde{I} = \langle \mathcal{I}_{j, \beta, \gamma}, j < \chi, \beta, \gamma < \alpha^*_j \rangle$. Let $A_j = \bigcup_{\beta, \gamma < \alpha^*_j} \bigcup_{p \in \mathcal{I}_{j, \beta, \gamma}} \text{Dom}(p)$, so $|A_j| < \kappa$ and call $A \subseteq \chi$ $\tilde{Q}$-closed if $(\forall j \in A)(A_j \subseteq A)$. Now in $M$ we can compute $j(\tilde{I})$ hence $A_j$ is $\mathcal{I}_{j, \beta, \gamma}$-closed if $j < j(\chi)$, now easily $A^M_{\tilde{Q}(j)} = \{ j(i) : i \in A_j \}$ as $|A_j| < \kappa$, so this reordering is O.K.]

So we are done.

\[ \Box_{1,2} \]