Pre-threshold fractional susceptibility functions at Misiurewicz parameters

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Abstract

We show that the response, frozen and semifreddo fractional susceptibility functions of certain real-analytic unimodal families, at Misiurewicz parameters and for fractional differentiation index $0 \leq \eta < \frac{1}{2}$, are holomorphic on a disk of radius greater than one. This is a step towards solving a conjecture of Baladi and Smania, in the case of the aforementioned susceptibility functions.

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1. Introduction

We consider in this paper the quadratic family $(f_t)_{t \in (1,2)}$, classically defined by

$$f_t(x) := t - x^2$$

on the interval $I_t = [-\beta_t, \beta_t]$ with $\beta_t = \frac{1 + \sqrt{1 + 4t^2}}{2t} \in (0, 2)$. This map admits $c = 0$ as its unique critical point; we will denote by $c_{k,t} = f^k_t(0), \ k \geq 1$ the post-critical orbit.

It is well-known since the work of Lyubich [12] that for almost every parameter $t \in (1,2)$, one of two behavior occurs: either there is an attracting periodic cycle (at so called regular parameters; this concerns a dense subset $\mathcal{R}$ of $(1,2)$), or there exists an absolutely continuous invariant probability measure $d\mu_t = \rho_t \, dx$ for $f_t$ (at so called stochastic parameters, which form a positive Lebesgue measure subset $\mathcal{S}$ of $(1,2)$).

In the setting of unimodal interval maps, Collet and Eckmann [8] famously introduced the following condition: the quadratic map $f_t$ satisfies the Collet–Eckmann condition if there is
\( \lambda_c > 1 \) and \( N_c > 0 \) such that for any \( k \geq N_c \)
\[
| (f^k_t)'(c_{1,t}) | \geq \lambda^k,
\] (CE)
i.e. one requires the Lyapunov exponent computed along the post-critical orbit to be positive. This condition implies the existence of an absolutely continuous invariant measure, i.e. parameters \( t \) satisfying Collet–Eckmann (CE) are in \( S \). From now on, we will call them (CE) parameters.

Let \( \mathcal{M} \subset CE \) be the set of Misiurewicz parameters, that is the set of parameters \( t \) for which the critical point \( c = 0 \) is not an accumulation point of the post-critical set. Let \( MT \subset \mathcal{M} \) be the set of Misiurewicz–Thurston parameters, for which the post-critical orbit is preperiodic and hyperbolic, i.e. such that there exists \( \ell, p \in \mathbb{N} \), for which \( c_{1,t} = f^\ell_t(0) \) is periodic of period \( p \), and \( |Df^\ell_t(c_{1,t})| > 1 \). \( \mathcal{M} \) has zero Lebesgue measure [18], and \( MT \) is a countable subset of CE.

The question we want to study here is connected to fractional response of the invariant measure, that is the Hölder regularity of the map \( t \mapsto \mu_t \), restricted to a suitable subset\(^1\) of \( S \), at a point \( t \in \mathcal{M} \), where \( \mu_t \) is seen as a Radon measure or a distribution of some finite order. The question of regularity of \( t \mapsto \mu_t \) has already received a lot of attention, and partial answers. For the quadratic family (1), Thunberg [19, corollaries 1 and 2] showed that the map \( t \mapsto \mu_t \) is discontinuous at every point in \( S \), and cannot be continuous on any full-measure subset of parameters. However, restricted to a suitable subset, this map is continuous: Tsujii [20] showed that, restricted to a positive measure parameter subset \( S' \neq S \), \( t \mapsto \mu_t \) is weak-* continuous at Misiurewicz points. Rychlik and Sorets [16] showed that for yet another positive measure subset \( S'' \) of parameters, the invariant density \( \rho_t \in L^p \) for \( 1 < p < 2 \) and that \( t \in S'' \mapsto \rho_t \in L^p \), \( 1 \leq p < 2 \) is continuous at Misiurewicz parameters, via a Hölder estimate.

More recently, and for more general smooth unimodal families, Baladi et al [4] proved that at almost every CE parameter \( t_0 \), for any \( \frac{1}{2} \)-Hölder observable \( \phi \), the map \( R_{\phi} : t \mapsto \int \phi \, d\mu_t \) is \( C^0 \) at \( t = t_0 \) for any \( \eta < \frac{1}{2} \), in the sense of Whitney, on a set of CE parameters having \( t_0 \) as a density point. Furthermore, they show that at any mixing MT parameter, there exists \( \phi \in C^\infty \), \( C > 1 \), and \( t_n \in MT \) with \( t_n \to t \), such that
\[
C^{-1} | t - t_n |^{1/2} \leq | R_{\phi}(t_n) - R_{\phi}(t) | \leq C | t - t_n |^{1/2} .
\]
Interestingly, those results seem to contradict earlier ones obtained by Ruelle [11, 14], who suggested that, for the quadratic family, the map \( R_{\phi} \) had a well-defined derivative at \( t = t_0 \), for \( t_0 \) a MT parameter, which raised the hope that linear response (i.e. differentiability of the map \( t \mapsto \mu_t \)) holds in this setting. Note that this hope was already diminished by a series of paper [2, 6, 9], which exhibited smooth families of piecewise expanding unimodal maps for which linear response fails, and highlighted a sufficient condition, called horizontality of the perturbation, for linear response to hold. For a full account of this intricate story, we refer to the introduction of [1, 5].

In a recent preprint, Aspenberg et al [1] introduced several fractional susceptibility functions, whose connection to fractional response is similar to the one between classical susceptibility function and linear response (i.e., the value at 1 of the fractional susceptibility function, if it is well-defined, is the (fractional) derivative of \( R_{\phi} \) at \( t = t_0 \)). We recall here their definitions: given \( \phi \in L^\infty(t_0) \), and denoting \( \mathcal{L}_t \) the (Ruelle–Perron–Frobenius) transfer operator associated to \( f_t \), we define, as formal power series

\(^1\)In particular, regularity should be understood in the sense of Whitney [21].
• The response fractional susceptibility function:

$$
\Psi_{\phi}^{\text{fr}}(\eta, z) := - \sum_{j=0}^{\infty} z^j \int_{I_0} \phi \circ f_{t_0}^j(x) M^0[\rho_{t_0}](x) \, dx.
$$

(2)

where we denoted $M^0$ the two-sided Marchaud derivative (see definition 3 and (11)).

• The frozen fractional susceptibility function:

$$
\Psi_{\phi}^{\text{fr}}(\eta, z) := \sum_{j=0}^{\infty} z^j \int_{I_0} \phi \circ f_{t_0}^j(x) M^0[\mathcal{L}_{t_0} \rho_{t_0}][x] \, dx.
$$

(3)

where we denoted $M^0_t$ the two-sided Marchaud derivative w.r.t. the parameter $t$.

To define a fractional susceptibility function in the spirit of [1], more care is needed. Indeed, as the invariant density $\rho_{t_0}$ is not defined for every parameter $t$, one has to consider integrals over some positive measure subset $\Omega \subset S$. Then one sets

$$
\Psi_{\phi}^{\Omega}(\eta, z) := \frac{\eta}{2 \Gamma(1 - \eta)} \sum_{j=0}^{\infty} z^j \int_{-2}^{2} \int_{-t_0}^{t_0} \phi \circ f_{t_0}^j(x) \frac{(\mathcal{L}_{t_0 + t} - \mathcal{L}_{t_0}) \rho_{t_0}}{|t|^{1+\eta}} \, \text{sgn}(t) \, dt \, dx.
$$

(4)

As an intermediary object between the frozen and real fractional susceptibility, one may introduce the semifreddo susceptibility function (see [5, section 7.2]), defined by

$$
\Psi_{\phi}^{\text{sf}}(\eta, z) := \frac{\eta}{2 \Gamma(1 - \eta)} \sum_{j=0}^{\infty} z^j \int_{-2}^{2} \int_{-t_0}^{t_0} \phi \circ f_{t_0}^j(x) \frac{(\mathcal{L}_{t_0 + t} - \mathcal{L}_{t_0}) \rho_{t_0}}{|t|^{1+\eta}} \, \text{sgn}(t) \, dt \, dx.
$$

(5)

In the context of the quadratic family, Baladi and Smania [5, conjecture A] formulate a set of conjectures for those fractional susceptibility functions associated with compactly supported $C^1$ observables $\phi$: holomorphy in $z$ in a disk of radius greater than one and fractional response formula for $0 < \eta < 1/2$, existence of a certain decomposition at $\eta = 1/2$, holomorphy in a disk of radius smaller than one for $1/2 < \eta < 1$. Note the qualitative change of behavior when $\eta$ passes the value $1/2$: this is the reason of our ‘threshold’ terminology.

In particular, [5, theorem C] establishes the existence of some decomposition, related to the presence of poles on the unit circle, for the response and frozen fractional susceptibilities at the threshold value $\eta = 1/2$.

As a step towards proving those conjectures, we establish in theorem 13, for mixing Misiurewicz parameters, holomorphy of (2), (3) and (5) in a disk of radius greater than one, for $0 \leq \eta < 1/2$. In light of the previous discussion, this result is the best one can expect in this setting.

Given the global nature of the Marchaud fractional derivative and (3), it will be useful to extend the range of parameters $t \in (1, 2)$ of the quadratic family to $\mathbb{R}$, as follows: for a

\[ \text{In [5, definition 2.3], another definition of the response fractional susceptibility is given. We note that the two definitions coincide for } 0 < \eta < 1/2 \text{ and } \phi \in C^1 \text{ compactly supported (see also [5, lemma 5.2]).} \]

\[ \text{Marchaud fractional derivatives were introduced in his PhD thesis [13]; for a quick introduction to the subject, we refer to [10].} \]

\[ \text{This property is only valid for the ‘real’ fractional susceptibility function (4).} \]
t_0 \in (1, 2)$, fix a $t_{\text{min}} < 0$ (resp. a $t_{\text{max}} > 0$) such that $t_0 + t_{\text{min}} > 1$ (resp. $t_0 + t_{\text{max}} < 2$), and set $f_t := f_{t_0 + t_{\text{min}}}$ for $t \leq t_0 + t_{\text{min}}$ and $f_t := f_{t_0 + t_{\text{max}}}$ for all $t \geq t_0 + t_{\text{max}}$.

**Remark 1.** We point out that our results are stated and proven in a more general setting than the quadratic family: let $(f_t)_{t \in [t_{\text{min}}, t_{\text{max}}]}$ be a family of real-analytic, unimodal maps, with negative Schwarzian derivative, satisfying at a Misiurewicz parameter $t_0 \in [t_{\text{min}}, t_{\text{max}}]_1$:

\[ f_{t_0 + t} = f_{t_0} + tX_{t_0} \circ f_{t_0}, \tag{6} \]

with $X_{t_0}$ real-analytic. Note that for the quadratic family, (6) holds with $X_{t_0} \equiv 1$.

## 2. Preliminaries

In this section, we recall useful definitions and results. We start with:

**Definition 2.** Let $0 < \eta < 1$ and $1 \leq p < \frac{1}{\eta}$. For $g \in L^p(\mathbb{R})$, we define its fractional integrals $I^\eta_\pm[g]$ by:

\[ I^\eta_- [g](x) := \frac{1}{\Gamma(\eta)} \int_{-\infty}^x \frac{g(t)}{(t-x)^{1-\eta}} \, dt \tag{7} \]

\[ I^\eta_+ [g](x) := \frac{1}{\Gamma(\eta)} \int_x^\infty \frac{g(t)}{(t-x)^{1+\eta}} \, dt \tag{8} \]

Parallel to those fractional integrals, we introduce the (Marchaud) fractional derivatives:

**Definition 3.** Given $0 < \eta < 1$ and a bounded, $\alpha$-Hölder ($\alpha > \eta$) $g$ on the real line$^5$, we define its (left and right) $\eta$-Marchaud fractional derivative by

\[ M^\eta_- [g](x) := \frac{\eta}{\Gamma(1-\eta)} \int_{-\infty}^x \frac{g(x) - g(t)}{(x-t)^{1+\eta}} \, dt \tag{9} \]

\[ M^\eta_+ [g](x) := \frac{\eta}{\Gamma(1-\eta)} \int_x^\infty \frac{g(x) - g(t)}{(t-x)^{1+\eta}} \, dt. \tag{10} \]

We may then define the two-sided Marchaud derivative $M^\eta$ as

\[ M^\eta[g] := \frac{M^\eta_- [g] - M^\eta_+ [g]}{2}. \tag{11} \]

As in the case of classical differentiation and integration, fractional differentiation is the left inverse of fractional integration (see [17, section 5.4 and theorem 6.1]).

We also introduce fractional integrals on an interval $(a, b)$, $I^\eta_{a_+} [g]$ defined for $0 < \eta < 1$ and (say) bounded $g : (a, b) \to \mathbb{R}$ by

\[ I^\eta_- [g](x) := \frac{1}{\Gamma(\eta)} \int_{a}^{x} \frac{g(t)}{(x-t)^{1-\eta}} \, dt \tag{12} \]

\[ I^\eta_+ [g](x) := \frac{1}{\Gamma(\eta)} \int_a^x \frac{g(t)}{(t-x)^{1+\eta}} \, dt \tag{13} \]

$^5$The fractional derivative operators $M^\eta_\pm$ may be defined for a larger class of functions. For more details, we refer to [17, section 5.4].
Definition 4. Given $0 \leq s < 1$ and $p > 1$, denoting by $\mathcal{F}$ the Fourier transform, we consider $\phi \in C^\infty_c(\mathbb{R})$ and set
\[
\| \phi \|_{H^s_p} := \left\| \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{s/2} \mathcal{F}(\phi) \right) \right\|_{L^p}.
\]
We then define $H^s_p(\mathbb{R}) := C^\infty_c(\mathbb{R}) \| \cdot \|_{H^s_p}$.

Given an interval $I \subset \mathbb{R}$, we will also consider the space $H^s_p(I)$ of functions $f \in H^s_p(\mathbb{R})$ that are supported in $I$.

Definition 5. Let $-\infty < a < b < +\infty$. We say that $f \in I^b_+(L^p(a, b))$ (resp. $f \in I^b_-(L^p(a, b))$) if there exists some $\phi \in L^p(a, b)$ such that $f = I^b_+ [\phi]$ (resp. $f = I^b_- [\phi]$).

We define similarly $I^a_+ (L^p(\mathbb{R}))$ and $I^a_- (L^p(\mathbb{R}))$.

One has [17, corollary to theorems 11.4 and 11.5], for $1 < p < 1$:
\[
I^b_+ (L^p(a, b)) = I^b_- (L^p(a, b)) =: I^b (L^p(a, b))
\]
(resp. $I^a_+ (L^p(\mathbb{R})) = I^a_- (L^p(\mathbb{R})) =: I^a (L^p(\mathbb{R}))$).

We will make extensive use of the following fact: if a $L^p$ function is representable as the (left or right) $s$-fractional integral of some $\phi \in L^p$, $1 < p < \frac{1}{s}$, then it belongs to $H^s_p$. More precisely (see [17, corollary to theorem 18.2]):

Theorem 6. Let $0 \leq s < 1$ and $1 < p < \frac{1}{s}$. One has
\[
H^s_p(\mathbb{R}) = L^p(\mathbb{R}) \cap I^s (L^p(\mathbb{R}))
\]

We state a useful lemma, taken from [3, lemma 2.39, p 56]

Lemma 7. Let $0 \leq \tilde{s} \leq s \leq 1$, and $1 < p < \infty$. For $g \in H^s_p(\mathbb{R})$, and $T \in C^1$, one has
\[
\| g \circ T - g \|_{H^\tilde{s}_p} \leq C d_{C^1}(T, Id) (s - \tilde{s}) \| g \|_{H^s_p},
\]
with $C$ depending boundedly on $\| T \|_{C^1}$.

Finally, we recall how the Marchaud derivative acts on the scale of Sobolev spaces: this is a consequence of theorem 6 and [17, theorem 5.3].

Proposition 8. Let $s > 0$, $0 \leq \eta < s$ and $1 < p < \frac{1}{\eta}$. Then
\[
M^\eta_p (H^s_p(\mathbb{R})) \subset L^q(\mathbb{R}) \cap I^{s-\eta} (L^p(\mathbb{R})),
\]
with $q$ such that $\frac{1}{q} - \frac{1}{p} = \eta$.

Remark 9. Let us mention that there exists several other notions of fractional derivatives (Bessel fractional potential or the more classical Riemann–Liouville derivative, to name a few), that may appear more natural in different contexts. However, we believe that the result one would obtain using those notions of fractional derivatives are qualitatively similar to ours. Indeed, proposition 8, a key ingredient in our approach, also holds for other types of fractional derivatives (by construction in the case of the Bessel fractional potential and fractional Sobolev spaces).
3. Marchaud derivative of the invariant density at Misiurewicz parameters

Recall that at a stochastic parameter $t_0$, the quadratic map $f_{t_0}$ (or more generally, a unimodal real-analytic map with negative Schwarzian derivative) admits a unique absolutely continuous invariant probability measure $d\mu_{t_0} := \rho_{t_0} \, dx$.

Denoting by $(c_k)_{k \geq 0}$ the critical orbit, Ruelle proved [15, theorem 9] the following decomposition for the invariant density $\rho_{t_0}$ of a Misiurewicz real-analytic unimodal map:

$$\rho_{t_0}(x) = \psi_0(x) + \sum_{k=1}^{\infty} C_{k,0} \frac{\delta_{w_0}(x-c_k)}{\sqrt{\sigma_k(x-c_k)}} + C_{k,1} \delta_{w_1}(x-c_k),$$  \hspace{1cm} (14)

where $\psi_0$ is a $C^1$ function, $C_{k,0} := \rho_{t_0} / |f_{t_0}'(c_k)|^{1/2}$, $|C_{k,1}| \leq \frac{w_0}{|f_{t_0}'(c_k)|^{1/2}}$, $\sigma_k := \text{sgn}(Df_{t_0}^{-1}(c_k)) \in \{\pm\}$ and $w_0, w_1 > 0$.

Our goal here is to study the regularity of the $\eta$-Marchaud derivative, $0 \leq \eta < \frac{1}{2}$, of $\rho_{t_0}$, for $t_0$ a Misiurewicz parameter. The main result of this section is:

**Theorem 10.** Let $f_{t_0}$ be a Misiurewicz real-analytic unimodal map as in remark 1, with invariant density $\rho_{t_0}$.

For $0 \leq \eta < \frac{1}{2}$, the Marchaud derivatives $M^f_\eta[\rho_{t_0}] \in H^f_\eta(\mathbb{R})$, for any $0 \leq s < \frac{1}{2} - \eta$, $1 < p < \frac{1}{1/2 + \eta/\pi}$.

We introduce the following notation, for $-1 \leq \beta \leq \frac{1}{2}$:

$$\begin{cases}
  f_{\beta,a,+}(x) := (x-a)\beta \\
  f_{\beta,a,-}(x) := (a-x)\beta \\
  \tilde{f}_{\beta,a,+}(x) := 1_{a<x<a+1}(x-a)\beta \\
  \tilde{f}_{\beta,a,-}(x) := 1_{a-1<x<a}(a-x)\beta
\end{cases} \hspace{1cm} (15)$$

We begin by computing relevant fractional integrals of the previously defined functions:

**Lemma 11.** For any $0 \leq \eta \leq 1$, and $\sigma \in \{\pm\}$ one has

$$I^\eta_{\beta,a,\sigma} (f_{\beta,a,\sigma}) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 + \eta)} f_{\eta,\beta,a,\sigma}.$$  \hspace{1cm} (16)

In particular, for $0 \leq \eta < \frac{1}{2}$,

$$f_{-1/2,a,\sigma} = I^\eta_{-1/2,a,\sigma} \left( \frac{\Gamma(1/2)}{\Gamma(1/2 - \eta)} f_{-1/2,a,\sigma} \right)$$

$$f_{1/2,a,\sigma} = I^\eta_{1/2,a,\sigma} \left( \frac{\Gamma(3/2)}{\Gamma(3/2 - \eta)} f_{1/2,a,\sigma} \right)$$

so that $f_{-1/2,a,+} \in P(\mathbb{L}^p(a,a+1))$, resp. $f_{1/2,a,-} \in P(\mathbb{L}^p(a-1,a))$, for $0 \leq s < \frac{1}{2}$, $1 < p < \frac{1}{1/2 + \eta/\pi}$, and for any $A > 0$.

**Proof.** We focus on the case $\sigma = +$, the other case being similar. By (12), for $x > a$ one has, by the change of variables $t = a + (x-a)s$,

$$I^\eta_{\beta,a,+} (f_{\beta,a,+})(x) = \frac{1}{\Gamma(\eta)} \int_a^x (t-a)^{\beta}(x-t)^{-\eta-1} \, dt$$

$$= \frac{1}{\Gamma(\eta)} (x-a)^{\beta+\eta} \int_0^1 s^{\beta}(1-s)^{-\eta-1} \, ds = \frac{\Gamma(\beta + 1, \eta)}{\Gamma(\eta)} (x-a)^{\beta+\eta},$$

where $\Gamma(\beta + 1, \eta)$ is the incomplete gamma function. Similarly for $\sigma = -$.
Lemma 12. Let $A > 0$. One has $\tilde{f}_{\pm 1/2,a,A,σ} \in H^s_p(\mathbb{R})$, for any $0 \leq s < \frac{1}{2}$, $1 < p < \frac{1}{1-2\sigma}$. Furthermore, $\|\tilde{f}_{\pm 1/2,a,A,σ}\|_{L^p} = \|\tilde{f}_{\pm 1/2,0,A,σ}\|_{L^p}$ is independent of $a$.

Proof. We treat only the case $σ = +$, the other case being similar. Recall that $\tilde{f}_{\pm 1/2,a,A,+} = 1_{a < x < a + A}\tilde{f}_{\pm 1/2,a,+}$, hence it is clear that it belongs to $L^p(\mathbb{R})$ for $1 \leq p < 2$.

By the previous lemma, $f_{\pm 1/2,a,+} \in L^p(a,a + A)$, for any $0 \leq s < \frac{1}{2}$, $1 < p < \frac{1}{1-2\sigma}$, so that by [17, theorem 13.10], $\tilde{f}_{\pm 1/2,a,A,+} \in L^p(\mathbb{R})$ for the same range of $s$, $p$. The first part of the lemma then follows from theorem 6.

The second part of the lemma is easily seen from the definition of the $H^s_p$ norm and invariance of Lebesgue measure by translation.

Proof of Theorem 10. Functions appearing in the series in Ruelle’s decomposition (14) are of the form $\tilde{f}_{\pm 1/2,a,A,σ}$. Hence, lemma 12 entails that their $H^s_p$ norm are uniformly bounded, and the CE condition ensures that the series in (14) converges exponentially fast in $H^s_p$ norm. Thus, $\rho_{0} \in H^s_p(\mathbb{R})$ for all $0 \leq s < \frac{1}{2}$, $1 < p < \frac{1}{1-2\sigma}$, and proposition 8 yields $M^s_p[\rho_{0}] \in L^p(\mathbb{R}) \cap L^{p,q}(\mathbb{R})$ for all $0 \leq s < \frac{1}{2}$, $1 < p < \frac{1}{1-2\sigma}$, and $q \in \left(\frac{1}{1+\eta},\frac{1}{1-2\sigma}\right)$. Taking $p = q$, which is possible in the range $\left(1,\frac{1}{1-2\sigma}\right)$, gives the wanted result.

4. The fractional susceptibility functions

In this section, we show our main result: the formal series defining the fractional susceptibility functions (2), (3) and (5) are holomorphic on a disk of radius greater than one.

To alleviate notation we will denote by $I = I_{0}$.

Theorem 13. Let $f_{0}$ be a mixing, Misiurewicz real-analytic unimodal map as in remark 1, $φ \in L^∞(I)$, $0 \leq \eta < 1/2$.

Then the response, frozen and semifreddo fractional susceptibility functions $Ψ^\eta_{φ}(η, \cdot), Ψ^\eta_{φ}(\eta, \cdot)$ and $Ψ^\eta_{φ}(η, \cdot)$ are well-defined and holomorphic in a complex disk $\mathbb{D}(0, θ^{-1})$ with $0 < θ < 1$.

Our starting point in this study is the following lemma:

Lemma 14. For a mixing Misiurewicz real-analytic unimodal map $f_{0}$ as in remark 1, a $L^∞(I)$ observable $φ$ and a $H^s_p(I)$ observable $ψ(s > 0, p > 1)$, there exists $C = C(φ_{0}, s, p)$ and $0 < θ < 1$ such that

$\left|\int_{I} φL^\prime_{0}(ψ) dx - \int_{I} φ\rho_{0} dx \int_{I} ψ dx\right| \leq C\theta \|φ\|_{L^∞}\|ψ\|_{H^s_p}$.

Proof. It is shown in [7, propositions 4.10 and 4.11] that for a Misiurewicz parameter $t_{0}$, there is a tower extension $f_{0}^{t_{0}} : I \cap$ with associated transfer operator $L_{0}$ such that there exists a

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6 In fact, the results of [7] hold in the more general setting of topologically slow recurrent (TSR) parameters. Those are measure-theoretic generic among stochastic parameters [7, remark 2.3]. We refer to [7, equation (5), proposition 2.2] for a definition and characterisation of TSR parameters.
Banach space $B$, a fixed point $\hat{\rho}_0 \in B$ and a measure $\nu$, constants $C > 0$ and $\kappa < 1$ satisfying:

for any $\hat{\psi} \in B$

$$\left\| \hat{L}_n^a(\hat{\psi}) - \hat{\rho}_0 \int I \psi d\nu \right\|_B \leq C \kappa^n \| \hat{\psi} \|_B.$$ 

Furthermore, there is a bounded operator $\Pi : B \to L^1(I)$, such that $\Pi \circ \hat{L}_n = L_n \circ \Pi$ (see [7, definition 4.5 and p 34]). It follows that for $\psi \in C^1$, supported in $I$, we may construct $\hat{\psi} = (\psi, 0, \ldots, 0, \ldots) \in B$, that trivially satisfies $\| \hat{\psi} \|_B = \| \psi \|_{W^{1,1}}$, $\int I \hat{\psi} d\nu = \int I \psi dx$ and $\Pi(\hat{\psi}) = \psi$. Then one has

$$\left\| \hat{L}_n^a \psi - \rho_0 \int I \psi dx \right\|_{L^1} \leq C \kappa^n \| \psi \|_{W^{1,1}}.$$ 

for any $\psi \in C^1$ supported in $I$. By duality we have, for any $\phi \in L^\infty(I)$

$$\left| \int \phi L_n^a \psi dx - \int \phi \rho_0 dx \right| \leq C \kappa^n \| \phi \|_{L^\infty} \| \psi \|_{W^{1,1}}. \quad (17)$$

To extend this estimate to $\psi \in H^s_p(I)$, $s > 0$, $p > 1$, we notice that since $p > 1$, the Sobolev embeddings imply that, for any $\tilde{s} > 2$ (we may choose $\tilde{s} < 2 + s$), there exists $\tilde{C}$ such that for any compactly supported $g \in H^s_p$

$$\| g \|_{C^1} \leq \tilde{C} \| g \|_{H^s_p}.$$ 

Since $s > 0$, using mollification, we can approach $\psi$ by $C^1$ functions $\psi_\varepsilon$ with

$$\| \psi_\varepsilon \|_{C^1} \leq \tilde{C} \| \psi_\varepsilon \|_{H^s_p} \leq \tilde{C} \| \psi \|_{H^s_p} \varepsilon^{-2}, \quad \| \psi - \psi_\varepsilon \|_{L^p} \leq \tilde{C} \varepsilon \| \psi \|_{H^s_p},$$

for every $\varepsilon > 0$.

Note that, still by Sobolev embedding, $\| \psi_\varepsilon \|_{W^{1,1}} \leq C \| \psi_\varepsilon \|_{C^1}$.

The lemma then follows from the following facts.

First,

$$\int \phi(\varepsilon x) L_n^a(\psi_\varepsilon)(\varepsilon x) dx = \int (\phi \circ f_\varepsilon^n) \psi_\varepsilon dx.$$

Second,

$$\left| \int \phi d\mu_0 \int (\psi_\varepsilon - \psi) dx \right| \leq \sup \| \phi \| \| \psi - \psi_\varepsilon \|_{L^p}.$$

To conclude, for each $j$, choose $\varepsilon = \kappa^j/(s+2)$, so that

$$\frac{\kappa^j}{\varepsilon^2} = \kappa^j = \kappa^j/(s+2) =: \theta^j.$$ 

(18)
\textbf{Remark 15.} Looking at the regularity obtained for the Marchaud derivative in theorem 10 and (18) we see that
\[ \frac{1}{2} \leq \frac{1}{\kappa} \leq 1. \]
In particular, when \( \eta \to \frac{1}{2} \), we get \( \theta \to 1 \).

\textbf{Proof of Theorem 13.} Applying lemma 14 to \( \psi = 1_{I} \mathcal{L}_{(g)}(\rho_{0}) \in H^{p}_{\eta}(I) \) for \( 0 \leq \eta < 1/2 \), we obtain easily that \( \mathcal{P}_{\eta}^{\mathcal{L}_{h}}(\eta, \cdot) \) is holomorphic in a disk \( \mathbb{D}(0, \theta^{-1}) \). To extend this result to the frozen fractional susceptibility function (3), one may proceed as in [5, proposition 2.6], showing that the response and frozen susceptibilities differ by a function that is holomorphic in a disk of radius strictly greater than one.

Instead, we will rely on another method, closer to the one presented in [1], which apply to both the frozen and semifreddo fractional susceptibilities. Furthermore, it allows to treat the more general setting (6) described at the end of the introduction.

We remark that the relation \( f_{s+t} = f_{s} + tX_{s} \circ f_{0} \) implies that for any \( H_{s}^{p}(I) \) (\( s > 0 \)), \( p > 1 \) observable \( g \), any \( t \in [t_{\text{min}}, t_{\text{max}}] \), any \( x \in I \) we have
\[ (\mathcal{L}_{s}g)(x) = (\mathcal{L}_{s+s}g)(x + tX_{s}) \left( 1 + tX_{s}(x + tX_{s}) \right) \]
Note that in the case of the quadratic family (1), this last equality simply reads \( \mathcal{L}_{s}g(x) = \mathcal{L}_{s+s}g(x + t) \). Up to reducing the interval \([t_{\text{min}}, t_{\text{max}}]\), we may assume that \( \text{Id} + tX_{s} \) is a \( C^{1} \) diffeomorphism.

One then has:
\[
[\mathcal{L}_{s+s} - \mathcal{L}_{s}]\rho_{0}(x) = \frac{1}{1 + tX_{s}} \rho_{0} \circ (\text{Id} + tX_{s})^{-1} - \rho_{0}
\]
\[
= \frac{1}{1 + tX_{s}} \left[ \rho_{0} \circ (\text{Id} + tX_{s})^{-1} - \rho_{0} - tX_{s} \rho_{0} \right].
\]
By theorem 10, \( \rho_{0} \in H^{p}_{\eta} \) for \( 0 \leq s < \frac{1}{2} \) and \( 1 < p < \frac{1}{\gamma - 1} \), thus, by lemma 7, it follows easily that
\[\| [\mathcal{L}_{s+s} - \mathcal{L}_{s}]\rho_{0} \|_{p} \leq C |x|^{s-\gamma} \| \rho_{0} \|_{p}, \tag{19}\]
with \( C \) independent on \( t \).

For \( 0 < s < \frac{1}{2} \), \( 1 < p < \frac{1}{\gamma - 1} \), fix \( 0 < \tilde{s} < s \) such that \( 0 < \eta < s - \tilde{s} \). Applying lemma 14 to
\[
\psi = \frac{1}{t} [\mathcal{L}_{s+t} - \mathcal{L}_{s}]\rho_{0},
\]
which is in \( H^{p}_{\eta}(I) \) for \( 0 < s < \frac{1}{2} \), \( 1 < p < \frac{1}{\gamma - 1} \), and using (19), we get (note that \( \int_{I} \psi \, dx = 0 \))
\[\left| \int_{I} \psi \mathcal{L}_{s} \rho_{0} \left( \frac{[\mathcal{L}_{s+t} - \mathcal{L}_{s}]\rho_{0}}{|t|^{1+\eta}} \right) \, dx \right| \leq C \theta^{\| \phi \|_{L^{\infty}} \| \rho_{0} \|_{p}} \tag{20}\]
For our choice of \( s, \tilde{s} \), one may integrate this last bound for \( t \in [t_{\text{min}}, t_{\text{max}}] \), to obtain
\[\left| \int_{t_{\text{min}}}^{t_{\text{max}}} \int_{I} \psi \mathcal{L}_{s} \rho_{0} \left( \frac{[\mathcal{L}_{s+t} - \mathcal{L}_{s}]\rho_{0}}{|t|^{1+\eta}} \right) \, dx \, dt \right| \leq C \theta^{\| \phi \|_{L^{\infty}} \| \rho_{0} \|_{p}} \]
For $t > t_{\text{max}}$ (the case $t < t_{\text{min}}$ is similar), one gets

\[
\left| \int_{t > t_{\text{max}}} \int \phi \mathcal{L}_0 \left( \frac{L_{0+t} - L_{0}}{|t|^{1+\eta}} \right) \, dx \, dt \right| \leq \frac{1}{t^{1+\eta}} \left| \int_{t > t_{\text{max}}} \phi \mathcal{L}_0 \left( \mathcal{L}_{0+t_{\text{max}}} - L_{0} \right) \rho_0 \, dx \right|
\leq CB^\eta L_{\text{max}}^{-\frac{\eta}{1+\eta}} \| \phi \|_{L^\infty} \| \rho_0 \|_{H^s_{\eta}}
\]

Hence, by Fubini, the last two bounds implies that for any fixed $0 < \eta < \frac{1}{2}$ and $\phi \in L^\infty(I)$, the formal series defining $\Psi^M(\eta, \cdot)$ converges on a disk of radius $\theta^{-1} > 1$.

For the semifreddo susceptibility function (5), one may integrate (20) for $t \in \Omega \cap [t_{\text{min}}, t_{\text{max}}]$, to obtain

\[
\left| \int_{\Omega \cap [t_{\text{min}}, t_{\text{max}}]} \int \phi \mathcal{L}_0 \left( \frac{L_{0+t} - L_{0}}{|t|^{1+\eta}} \right) \, dx \, dt \right| \leq CB^\eta \| \phi \|_{L^\infty} \| \rho_0 \|_{H^s_{\eta}} \int_{t_{\text{min}}}^{t_{\text{max}}} \delta \Omega |t|^{s-\tilde{s}-1-\eta} \, dt
\leq CB^\eta \| \phi \|_{L^\infty} \| \rho_0 \|_{H^s_{\eta}} \int_{t_{\text{min}}}^{t_{\text{max}}} |t|^{s-\tilde{s}-1-\eta} \, dt
\]

and thus we may conclude as in the previous case. For $t > t_{\text{max}}$, resp. $t < t_{\text{min}}$, we proceed similarly. \hfill \Box

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References

[1] Aspenberg M, Baladi V, Leppänen J and Persson T 2019 On the fractional susceptibility function of piecewise expanding maps (arXiv:1910.00369)
[2] Baladi V 2007 On the susceptibility function of piecewise expanding interval maps Commun. Math. Phys. 275 839–89
[3] Baladi V 2018 Dynamical zeta functions and dynamical determinants for hyperbolic maps: A Functional Approach (Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics vol 68) (Berlin: Springer)
[4] Baladi V, Benedicks M and Schnellmann D 2015 Whitney–Hölder continuity of the SRB measure for transversal families of smooth unimodal maps Invent. Math. 201 773–844
[5] Baladi V and Smania D Fractional susceptibility functions for the quadratic family: Misiurewicz–Thurston parameters Commun. Math. Phys. 385 1957–2007
[6] Baladi V and Smania D 2008 Linear response formula for piecewise expanding unimodal maps Nonlinearity 21 677–711
[7] Baladi V and Smania D 2013 Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps Ann. Sci. Éc. Norm. Supér. 45 861–926
[8] Collet P and Eckmann J-P 1983 Positive Liapunov exponents and absolute continuity for maps of the interval Ergod. Theor. Dynam. Syst. 3 13–46
[9] de Lima A and Smania D 2018 Central limit theorem for the modulus of continuity of averages of observables on transversal families of piecewise expanding unimodal maps J. Inst. Math. Jussieu 17 673–733
[10] Ferrari F 2018 Weyl and Marchaud derivatives: a forgotten history Mathematics 6 6
[11] Jiang Y and Ruelle D 2005 Analyticity of the susceptibility function for unimodal Markovian maps of the interval Nonlinearity 18 2447–53
[12] Lyubich M 2002 Almost every real quadratic map is either regular or stochastic Ann. Math. 156 1–78
[13] Marchaud M A 1927 Sur les dérivées et sur les différences des fonctions de variables réelles Doctorat d'état
[14] Ruelle D 2005 Differentiating the absolutely continuous invariant measure of an interval map $f$ with respect to $f$ Commun. Math. Phys. 258 445–53
[15] Ruelle D 2009 Structure and $f$-dependence of the A.C.I.M. for a unimodal map $f$ is Misiurewicz type Commun. Math. Phys. 287 1039–70
[16] Rychlik M and Sorets E 1992 Regularity and other properties of absolutely continuous invariant measures for the quadratic family Commun. Math. Phys. 150 217–36
[17] Samko S G, Kilbas A A and Marichev O I 1993 Fractional Integrals and derivatives, Theory and Applications (London: Gordon and Breach)
[18] Duncan S 1998 Misiurewicz maps are rare Commun. Math. Phys. 197 109–29
[19] Thunberg H 2001 Unfolding of chaotic unimodal maps and the parameter dependence of natural measures Nonlinearity 14 323–37
[20] Tsujii M 1996 On continuity of Bowen–Ruelle–Sinai measures in families of one dimensional maps Commun. Math. Phys. 177 1–11
[21] Whitney H 1934 Analytic extensions of differentiable functions defined in closed sets Trans. Am. Math. Soc. 36 63