THE HAMILTON-JACOBI EQUATION ON LIE AFFGEBROIDS

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Abstract. The Hamilton-Jacobi equation for a Hamiltonian section on a Lie affgebroid is introduced and some examples are discussed.

1. Introduction

Recently, in [7] (see also [9]) the authors developed a Hamiltonian description of Mechanics on Lie algebroids. If \( \tau_E : E \to M \) is a Lie algebroid on \( M \) then, in this description, the role of the cotangent bundle of the configuration manifold is played by the \( E \)-tangent bundle \( T^E E^* \) to \( E^* \) (the prolongation of \( E \) over \( \tau_E^* : E^* \to M \) in the terminology of [7]). One may construct the canonical symplectic 2-section associated with the Lie algebroid \( E \) as a closed non-degenerate section \( \Omega_E \) of the vector bundle \( \wedge^2(T^E E^*)^* \to E^* \). Then, given a Hamiltonian function \( H : E^* \to \mathbb{R} \), the Hamiltonian section associated with \( H \) is the section \( \xi_H \) of \( T^E E^* \to E^* \) characterized by the equation

\[
i_{\xi_H} \Omega_E = d_{T^E E^*} H,
\]

where \( d_{T^E E^*} \) is the differential of the Lie algebroid \( T^E E^* \to E^* \). The integral sections of \( \xi_H \) are the solutions of the Hamilton equations for \( H \). In fact, these solutions are just the integral curves of the Hamiltonian vector field of \( H \) with respect to the linear Poisson structure on \( E^* \) associated with the Lie algebroid \( E \).

Using the canonical symplectic section \( \Omega_E \), one may also give a description of the Hamiltonian Mechanics on the Lie algebroid \( E \) in terms of Lagrangian submanifolds of symplectic Lie algebroids (see [7]). An alternative approach, using the linear Poisson structure on \( E^* \) and the canonical isomorphism between \( T^* E \) and \( T^* E^* \) was discussed in [3].

In [7], the authors also introduced the Hamilton-Jacobi equation for a Hamiltonian function \( H : E^* \to \mathbb{R} \) and they proved that knowing one solution of the Hamilton-Jacobi equation simplifies the search of trajectories for the corresponding Hamiltonian vector field.

On the other hand, in [2, 11] a possible generalization of the notion of a Lie algebroid to affine bundles is introduced in order to create a geometric model which provides a natural framework for a time-dependent version of Lagrange equations on Lie algebroids (see also [3, 10, 12]). The resultant objects are called Lie affgebroid structures (in the terminology of [2]). If \( \tau_A : A \to M \) is an affine bundle modelled on the vector bundle \( \tau_V : V \to M \), \( \tau_{A^+} : A^+ = Aff(A, \mathbb{R}) \to M \) is the dual bundle to \( A \) and \( \tilde{A} = (A^+)^* \) is the bidual bundle, then a Lie affgebroid structure on \( A \) is equivalent to a Lie algebroid structure on \( \tilde{A} \) such that the distinguished section \( 1_A \) of \( \tau_{A^+} : A^+ \to M \) (corresponding to the constant function 1 on \( A \)) is a 1-cocycle in the Lie

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algebroid cohomology complex of $\tilde{A}$. Now, if $h : V^* \to A^+$ is a Hamiltonian section (that is, $h$ is a section of the canonical projection $\mu : A^+ \to V^*$) then one may construct a symplectic structure $(\Omega_h, \eta)$ on the prolongation of $\tilde{A}$ over the fibration $\tau^+_h : V^* \to M$ and one may consider the Reeb section $R_h$ of $(\Omega_h, \eta)$. The integral sections of $R_h$ are just the solutions of the Hamilton equations for $h$ (see [5, 10]). Alternatively, one may prove that the solutions of the Hamilton equations for $h$ are the integral curves of the Hamiltonian vector field of $h$ on $V^*$ with respect to the canonical aff-Poisson structure on the line affine bundle $\mu : A^+ \to V^*$ (see [6]). Aff-Poisson structures on line affine bundles were introduced in [2] (see also [3]) as the affine analogs of Poisson structures. The existence of an aff-Poisson structure on the affine bundle $\mu : A^+ \to V^*$ is a consequence of some general results proved in [5].

The aim of this paper is to introduce the Hamilton-Jacobi equation for a Hamiltonian section $h$ and, then, to prove that knowing one solution of the Hamilton-Jacobi equation simplifies the search of trajectories for the corresponding Hamiltonian section (see Theorem 4.1).

The paper is organized as follows. In Section 2, we recall some definitions and results about Lie algebroids and Lie affgebroids which will be used in the rest of the paper. The Hamiltonian formalism on Lie affgebroids is developed in Section 3. The Hamilton-Jacobi equation for a Hamiltonian section on a Lie affgebroid $A$ is introduced in Section 4 and, then, we prove the main result of the paper (see Theorem 4.1). Some examples are discussed in the last section (Section 5). In particular, if $A$ is a Lie algebroid then we recover the Hamilton-Jacobi equation considered in [2] (see Example 5.1). When $A$ is the 1-jet bundle of local sections of a trivial fibration $\tau : \mathbb{R} \times P \to \mathbb{R}$ then the Hamiltonian section $h$ may be considered as a time-dependent Hamiltonian function $H : \mathbb{R} \times T^*P \to \mathbb{R}$ and the resultant equation is just the classical time-dependent Hamilton-Jacobi equation (see Example 5.2). Finally, we obtain some results about the Hamilton-Jacobi equation on the so-called Atiyah affgebroid associated with a principal G-bundle $p : Q \to M$ and a fibration $\nu : M \to \mathbb{R}$ (see Example 5.3).

2. Lie algebroids and Lie affgebroids

2.1. Lie algebroids. Let $E$ be a vector bundle of rank $n$ over the manifold $M$ of dimension $m$ and $\tau_E : E \to M$ be the vector bundle projection. Denote by $\Gamma(\tau_E)$ the $C^\infty(M)$-module of sections of $\tau_E : E \to M$. A Lie algebroid structure $(\llbracket \cdot, \cdot \rrbracket_E, \rho_E)$ on $E$ is a Lie bracket $\llbracket \cdot, \cdot \rrbracket_E$ on the space $\Gamma(\tau_E)$ and a bundle map $\rho_E : E \to TM$, called the anchor map, such that if we also denote by $\rho_E : \Gamma(\tau_E) \to \mathfrak{x}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then $\llbracket X, JY \rrbracket_E = f \llbracket X, Y \rrbracket_E + \rho_E(X)(f)Y$, for $X, Y \in \Gamma(\tau_E)$ and $f \in C^\infty(M)$. The triple $(E, \llbracket \cdot, \cdot \rrbracket_E, \rho_E)$ is called a Lie algebroid over $M$ (see [5]). In such a case, the anchor map $\rho_E : \Gamma(\tau_E) \to \mathfrak{x}(M)$ is a homomorphism between the Lie algebras $(\Gamma(\tau_E), \llbracket \cdot, \cdot \rrbracket_E)$ and $(\mathfrak{x}(M), [\cdot, \cdot])$.

If $(E, \llbracket \cdot, \cdot \rrbracket_E, \rho_E)$ is a Lie algebroid, one may define a cohomology operator which is called the differential of $E$, $d^E : \Gamma(\wedge^k \tau_E^\ast) \to \Gamma(\wedge^{k+1} \tau_E^\ast)$, as follows

\[
(d^E \mu)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \rho_E(X_i)(\mu(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) \\
+ \sum_{i<j} (-1)^{i+j} \mu([X_i, X_j]_E, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k),
\]

(2.1)
for \( \mu \in \Gamma(\wedge^k \tau^*_E) \) and \( X_0, \ldots, X_k \in \Gamma(\tau_E) \). Moreover, if \( X \in \Gamma(\tau_E) \), one may introduce, in a natural way, the Lie derivative with respect to \( X \), as the operator \( L^E_X : \Gamma(\wedge^k \tau^*_E) \to \Gamma(\wedge^k \tau^*_E) \) given by \( L^E_X = i_X \circ d^E + d^E \circ i_X \).

If \( E \) is the standard Lie algebroid \( TM \) then the differential \( d^E = dTM \) is the usual exterior differential associated with \( M \), which we will denote by \( d \).

If we take local coordinates \((x^i)\) on \( M \) and a local basis \( \{e_\alpha\} \) of sections of \( E \), then we have the corresponding local coordinates \((x^i, y^\alpha)\) on \( E \), where \( y^\alpha(a) \) is the \( \alpha \)-th coordinate of \( a \in E \) in the given basis. Such coordinates determine local functions \( \rho_\alpha^i, C^\gamma_{\alpha\beta} \) on \( M \) which contain the local information of the Lie algebroid structure, and accordingly they are called the structure functions of the Lie algebroid. They are given by

\[
\rho^i_E(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \quad \text{and} \quad [e_\alpha, e_\beta]_E = C^\gamma_{\alpha\beta} e_\gamma.
\]

If \( f \in C^\infty(M) \), we have that

\[
d^E f = \frac{\partial f}{\partial x^i} \rho^i_\alpha e_\alpha,
\]

where \( \{e^\alpha\} \) is the dual basis of \( \{e_\alpha\} \). On the other hand, if \( \theta \in \Gamma(\tau^*_E) \) and \( \theta = \theta^\gamma e_\gamma \) it follows that

\[
d^E \theta = \frac{\partial \theta^\gamma}{\partial x^i} \rho^i_\alpha e_\alpha - \frac{1}{2} \theta^\gamma C^\gamma_{\alpha\beta} e_\beta \wedge e_\gamma.
\]

For a Lie algebroid \((E, \{\cdot, \cdot\}_E, \rho_E)\) over \( M \) we may consider the generalized distribution \( \mathcal{F}^E \) on \( M \) whose characteristic space at point \( x \in M \) is given by \( \mathcal{F}^E(x) = \rho_E(E_x) \), where \( E_x \) is the fibre of \( E \) over \( x \). The distribution \( \mathcal{F}^E \) is finitely generated and involutive. Thus, \( \mathcal{F}^E \) defines a generalized foliation on \( M \) in the sense of Sussmann [14]. \( \mathcal{F}^E \) is the Lie algebroid foliation on \( M \) associated with \( E \).

Note that if \( f \in C^\infty(M) \) then \( d^E f = 0 \) if and only if \( f \) is constant on the leaves of \( \mathcal{F}^E \).

Now, suppose that \((E, \{\cdot, \cdot\}_E, \rho_E)\) and \((E', \{\cdot, \cdot\}_{E'}, \rho_{E'})\) are Lie algebroids over \( M \) and \( M' \), respectively, and that \( F : E \to E' \) is a vector bundle morphism over the map \( f : M \to M' \).

Then \((F, f)\) is said to be a Lie algebroid morphism if

\[
d^E((F, f)^* \phi') = (F, f)^*(d^E \phi'), \quad \text{for } \phi' \in \Gamma(\wedge^k (\tau_{E'})^*) \quad \text{and for all } k.
\]

Note that \((F, f)^* \phi' \) is the section of the vector bundle \( \wedge^k E^* \to M \) defined by

\[
((F, f)^* \phi')_x(a_1, \ldots, a_k) = \phi'_f(x)(F(a_1), \ldots, F(a_k)),
\]

for \( x \in M \) and \( a_1, \ldots, a_k \in E_x \). If \((F, f)\) is a Lie algebroid morphism, \( f \) is an injective immersion and \( F|_{E_x} : E_x \to E'_{f(x)} \) is injective, for all \( x \in M \), then \((E, \{\cdot, \cdot\}_E, \rho_E)\) is said to be a Lie subalgebroid of \((E', \{\cdot, \cdot\}_{E'}, \rho_{E'})\).

2.1.1. The prolongation of a Lie algebroid over a fibration. In this section, we will recall the definition of the Lie algebroid structure on the prolongation of a Lie algebroid over a fibration (see [5, 7]).

Let \((E, \{\cdot, \cdot\}_E, \rho_E)\) be a Lie algebroid of rank \( n \) over a manifold \( M \) of dimension \( m \) with vector bundle projection \( \tau_E : E \to M \) and \( \pi : M' \to M \) be a fibration.

We consider the subset \( \tau^E \mathcal{M}' \) of \( E \times TM' \) and the map \( \tau^E_M : \tau^E \mathcal{M}' \to M' \) defined by

\[
\tau^E_M(b, v') = (T\pi)(v'), \quad \tau^E_M(b, v') = \pi_M(v'),
\]
where $T\pi: TM' \to TM$ is the tangent map to $\pi$ and $\pi_M : TM' \to M'$ is the canonical projection. Then, $\tau^-_E: T^E M' \to M'$ is a vector bundle over $M'$ of rank $n + \dim M' - m$ which admits a Lie algebroid structure $([\cdot, \cdot]_E, \rho^E)$ characterized by

$$\llbracket (X \circ \pi, U'), (Y \circ \pi, V') \rrbracket_E = \llbracket (X, Y)_E \circ \pi, [U', V'] \rrbracket, \quad \rho^E(X \circ \pi, U') = U',$$

for all $X, Y \in \Gamma(\tau^-_E)$ and $U', V'$ vector fields which are $\pi$-projectable to $\rho_E(X)$ and $\rho_E(Y)$, respectively. $(T^E M', [\cdot, \cdot]_E, \rho^E)$ is called the prolongation of the Lie algebroid $E$ over the fibration $\pi$ or the $E$-tangent bundle to $M'$ (for more details, see [5] [7]).

Next, we consider a particular case of the above construction. Let $E$ be a Lie algebroid over a manifold $M$ with vector bundle projection $\tau^-_E : E \to M$ and $T^E E^*$ be the prolongation of $E$ over the projection $\tau^+_E : E^* \to M$. $T^E E^*$ is a Lie algebroid over $E^*$ and we can define a canonical section $\lambda_E$ of the vector bundle $(T^E E^*)^* \to E^*$ as follows. If $a^* \in E^*$ and $(b, v) \in (T^E E^*)_{a^*}$ then

$$\lambda_E(a^*)(b, v) = a^*(b).$$

$\lambda_E$ is called the Liouville section associated with the Lie algebroid $E$.

Now, one may consider the nondegenerate $2$-section $\Omega_E = -dT^E E^* \cdot \lambda_E$ of $T^E E^* \to E^*$. It is clear that $dT^E E^* \cdot \Omega_E = 0$. In other words, $\Omega_E$ is a symplectic section. $\Omega_E$ is called the canonical symplectic section associated with the Lie algebroid $E$.

Suppose that $(x^i)$ are local coordinates on an open subset $U$ of $M$ and that $\{e_\alpha\}$ is a local basis of sections of the vector bundle $\tau^{-1}_E(U) \to U$ as above. Then, $\{\bar{e}_\alpha, \bar{e}_\alpha\}$ is a local basis of sections of the vector bundle $(\tau^{-1}_E(U)) = (\tau^{-1}_E(U)) (\tau^{-1}_E(U)) = (\tau^{-1}_E(U))$, where $\tau^{-1}_E : T^E E^* \to E^*$ is the vector bundle projection and

$$\bar{e}_\alpha(a^*) = (e_\alpha(\tau^-_E(a^*)), \rho^{\alpha}_{\alpha} \frac{\partial}{\partial x^i |a^*}), \quad \bar{e}_\alpha(a^*) = (0, \frac{\partial}{\partial y_\alpha |a^*}).$$

Here, $(x^i, y_\alpha)$ are the local coordinates on $E^*$ induced by the local coordinates $(x^i)$ and the dual basis $\{e^\alpha\}$ of $\{e_\alpha\}$. Moreover, we have that

$$\llbracket \bar{e}_\alpha, \bar{e}_\beta \rrbracket_E^E = C^{\varnothing}_{\alpha \beta} \bar{e}_\gamma, \quad \llbracket \bar{e}_\alpha, \bar{e}_\beta \rrbracket_E^E = \llbracket \bar{e}_\alpha, \bar{e}_\beta \rrbracket_E^E = 0, \quad \rho^E_{\alpha}(\bar{e}_\alpha) = \rho^E_{\alpha} \frac{\partial}{\partial x^i |a^*}, \quad \rho^E_{\alpha}(\bar{e}_\alpha) = \frac{\partial}{\partial y_\alpha},$$

and

$$\lambda_E(x^i, y_\alpha) = y_\alpha \bar{e}_\alpha, \quad \Omega_E(x^i, y_\alpha) = \bar{e}_\alpha \wedge \bar{e}_\alpha + \frac{1}{2} C^{\gamma}_{\alpha \beta} y_\gamma \bar{e}_\alpha \wedge \bar{e}_\beta,$$

(1) for more details, see [7] [10] [11].

### 2.2 Lie algebroids

Let $\tau_A : A \to M$ be an affine bundle with associated vector bundle $\tau_V : V \to M$. Denote by $\tau_A^+ : A^+ = Aff(A, \mathbb{R}) \to M$ the dual bundle whose fibre over $x \in M$ consists of affine functions on the fibre $A_x$. Note that this bundle has a distinguished section $1_A \in \Gamma(\tau_A^+)$ corresponding to the constant function $1$ on $A$. We also consider the bidual bundle $\tau^-_A : A \to M$ whose fibre at $x \in M$ is the vector space $\bar{A}_x = (A_x^+)^*$. Then, $A$ may be identified with an affine subbundle of $\bar{A}$ via the inclusion $i_A : A \to \bar{A}$ given by $i_A(a)(\varphi) = \varphi(a)$, which is injective affine map whose associated linear map is denoted by $i_V : V \to A$. Thus, $V$ may be identified with a vector subbundle of $\bar{A}$. Using these facts, one can prove that there is a one-to-one correspondence between affine functions on $A$ and linear functions on $\bar{A}$. On the
other hand, there is an obvious one-to-one correspondence between affine functions on $A$ and sections of $A^\cdot$.

A Lie affgebroid structure on $A$ consists of a Lie algebra structure $\lbrack \cdot, \cdot \rbrack_V$ on the space $\Gamma(\tau_V)$ of the sections of $\tau_V : V \to M$, a $\mathbb{R}$-linear action $D : \Gamma(\tau_A) \times \Gamma(\tau_V) \to \Gamma(\tau_V)$ of the sections of $A$ on $\Gamma(\tau_V)$ and an affine map $\rho_A : A \to TM$, the anchor map, satisfying the following conditions:

- $D_X[\tilde{Y}, \tilde{Z}]_V = [D_X \tilde{Y}, \tilde{Z}]_V + [\tilde{Y}, D_X \tilde{Z}]_V$,
- $D_{X + \tilde{Y}} \tilde{Z} = D_X \tilde{Z} + [\tilde{Y}, \tilde{Z}]_V$,
- $D_X(f \tilde{Y}) = f D_X \tilde{Y} + \rho_A(X)(f) \tilde{Y}$,

for $X \in \Gamma(\tau_A)$, $\tilde{Y}, \tilde{Z} \in \Gamma(\tau_V)$ and $f \in C^\infty(M)$ (see [2, 11]).

If $(\lbrack \cdot, \cdot \rbrack_V, D, \rho_A)$ is a Lie affgebroid structure on an affine bundle $A$ then $(V, \lbrack \cdot, \cdot \rbrack_V, \rho_V)$ is a Lie algebroid, where $\rho_V : V \to TM$ is the vector bundle map associated with the affine morphism $\rho_A : A \to TM$.

A Lie affgebroid structure on an affine bundle $\tau_A : A \to M$ induces a Lie algebroid structure $(\lbrack \cdot, \cdot \rbrack_A, \rho_A)$ on the bidual bundle $\tilde{A}$ such that $1_A \in \Gamma(\tau_A^\cdot)$ is a 1-cocycle in the corresponding Lie algebroid cohomology, that is, $d^3 1_A = 0$. Indeed, if $X_0 \in \Gamma(\tau_A)$ then for every section $\tilde{X}$ of $\tilde{A}$ there exists a unique function $f \in C^\infty(M)$ and a unique section $\tilde{X} \in \Gamma(\tau_V)$ such that $\tilde{X} = f X_0 + \tilde{X}$ and

$$\rho_A(f X_0 + \tilde{X}) = f \rho_A(X_0) + \rho_V(\tilde{X}),$$

$$[f X_0 + \tilde{X}, gX_0 + \tilde{Y}]_{\tilde{A}} = (\rho_V(\tilde{X})(g) - \rho_V(\tilde{Y})(f) + f \rho_A(X_0)(g) - g \rho_A(X_0)(f)) X_0 + [\tilde{X}, \tilde{Y}]_V + f D_{X_0} \tilde{Y} - g D_{X_0} \tilde{X}.$$

Conversely, let $(U, \lbrack \cdot, \cdot \rbrack_U, \rho_U)$ be a Lie algebroid over $M$ and $\phi : U \to \mathbb{R}$ be a 1-cocycle of $(U, \lbrack \cdot, \cdot \rbrack_U, \rho_U)$ such that $\phi|_{U_x} \neq 0$, for all $x \in M$. Then, $A = \phi^{-1}\{1\}$ is an affine bundle over $M$ which admits a Lie affgebroid structure in such a way that $(U, \lbrack \cdot, \cdot \rbrack_U, \rho_U)$ may be identified with the bidual Lie algebroid $(\tilde{A}, \lbrack \cdot, \cdot \rbrack_\tilde{A}, \rho_\tilde{A})$ to $A$ and, under this identification, the 1-cocycle $1_A : \tilde{A} \to \mathbb{R}$ is just $\phi$. The affine bundle $\tau_A : A \to M$ is modelled on the vector bundle $\tau_V : V = \phi^{-1}\{0\} \to M$. In fact, if $i_V : V \to U$ and $i_A : A \to U$ are the canonical inclusions, then

$$i_V \circ [\tilde{X}, \tilde{Y}]_U = [i_V \circ \tilde{X}, i_V \circ \tilde{Y}]_U, \quad i_V \circ D_{X} \tilde{Y} = [i_A \circ X, i_V \circ \tilde{Y}]_U,$$

$$\rho_A(X) = \rho_V(i_A \circ X),$$

for $\tilde{X}, \tilde{Y} \in \Gamma(\tau_V)$ and $X \in \Gamma(\tau_A)$ (for more details, see [2, 11]).

A trivial example of a Lie affgebroid may be constructed as follows. Let $\tau : M \to \mathbb{R}$ be a fibration and $\tau_{1,0} : J^1 \tau \to M$ be the 1-jet bundle of local sections of $\tau : M \to \mathbb{R}$. It is well known that $\tau_{1,0} : J^1 \tau \to M$ is an affine bundle modelled on the vector bundle $\pi = (\pi_M)|_{V_\tau} : V_\tau \to M$, where $V_\tau$ is the vertical bundle of $\tau : M \to \mathbb{R}$. Moreover, if $t$ is the usual coordinate on $\mathbb{R}$ and $\eta$ is the closed 1-form on $M$ given by $\eta = \pi^*(dt)$ then we have the following identification

$$J^1 \tau = \{v \in TM/\eta(v) = 1\}$$

(see, for instance, [13]). Note that $V_\tau = \{v \in TM/\eta(v) = 0\}$. Thus, the bidual bundle $\tilde{J^1_\tau}$ to the affine bundle $\tau_{1,0} : J^1 \tau \to M$ may be identified with the tangent bundle $TM$ to $M$ and, under this identification, the Lie algebroid structure on $\pi_M : TM \to M$ is the standard Lie algebroid structure and the 1-cocycle $1_{J^1_\tau}$ on $\pi_M : TM \to M$ is just the closed 1-form $\eta$. 

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3. Hamiltonian formalism on Lie affgebroids

3.1. A cosymplectic structure on $T^\mathcal{A}V^*$. Suppose that $(\tau_A : A \to M, \tau_V : V \to M, (\llbracket \cdot, \cdot \rrbracket, D, \rho_A))$ is a Lie affgebroid and let $(T^\mathcal{A}V^*, \llbracket \cdot, \cdot \rrbracket^\mathcal{A}, \rho^\mathcal{A}_V)$ be the prolongation of the bidual Lie algebroid $(\mathcal{A}, \llbracket \cdot, \cdot \rrbracket, \rho_A)$ over the fibration $\tau_V : V^* \to M$. In this section, we are going to construct a cosymplectic structure on $T^\mathcal{A}V^*$ using the canonical symplectic section associated with the Lie algebroid $\mathcal{A}$ and a Hamiltonian section (this construction was done in [6]).

Let $(x^i)$ be local coordinates on an open subset $U$ of $M$ and $\{e_0, e_\alpha\}$ be a local basis of sections of the vector bundle $\tau_1^{-1}(U) \to U$ adapted to the 1-cocycle $1_A$ (that is, $1_A(e_0) = 1$ and $1_A(e_\alpha) = 0$, for all $\alpha$) and such that

$$[e_0, e_\alpha]_\mathcal{A} = C^\gamma_{\alpha\beta} e_\gamma, \quad [e_\alpha, e_\beta]_\mathcal{A} = C^\gamma_{\alpha\beta} e_\gamma, \quad \rho^\mathcal{A}_0(e_0) = \rho^\mathcal{A}_\alpha(e_\alpha) = \frac{\partial}{\partial x^i}, \quad \rho^\mathcal{A}_0(e_\alpha) = \frac{\partial}{\partial y^\alpha}.$$

Denote by $(x^i, y^0, y^\alpha)$ the corresponding local coordinates on $\mathcal{A}$ and by $(x^i, y^0, y_\alpha)$ the dual coordinates on the dual vector bundle $\tau_{A^*}: A^* \to \mathcal{A}$. Then, $(x^i, y^0, y_\alpha)$ are local coordinates on $V^*$ and $\{\bar{e}_0, \bar{e}_\alpha, \bar{e}_\alpha\}$ is a local basis of sections of the vector bundle $\tau_V^{-1}: T^\mathcal{A}V^* \to V^*$, where

$$[\bar{e}_0, \bar{e}_\beta]_\mathcal{A}^V = C^\gamma_{0\beta} \bar{e}_\gamma, \quad [\bar{e}_\alpha, \bar{e}_\beta]_\mathcal{A}^V = C^\gamma_{\alpha\beta} \bar{e}_\gamma,$$

$$[\bar{e}_0, \bar{e}_\alpha]_\mathcal{A}^V = [\bar{e}_\alpha, \bar{e}_\beta]_\mathcal{A}^V = [\bar{e}_\alpha, \bar{e}_\alpha]_\mathcal{A}^V = 0,$$

for all $\alpha$ and $\beta$. Thus, if $\{\bar{e}^0, \bar{e}^\alpha, \bar{e}^\alpha\}$ is the dual basis of $\{\bar{e}_0, \bar{e}_\alpha, \bar{e}_\alpha\}$ then

$$d\tau^\mathcal{A}V^* f = \frac{\partial f}{\partial x^i} \bar{e}^0 + \frac{\partial f}{\partial y^0} \bar{e}^\alpha + \frac{\partial f}{\partial y^\alpha},$$

$$d\tau^\mathcal{A}V^* \bar{e}^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} \bar{e}^\alpha \wedge \bar{e}^\beta - \frac{1}{2} C^\gamma_{\alpha\beta} \bar{e}^\alpha \wedge \bar{e}^\beta,$$

for $f \in C^\infty(V^*)$.

Let $\mu : A^+ \to V^*$ be the canonical projection given by $\mu(\varphi) = \varphi^l$, for $\varphi \in A^+_x$, with $x \in M$, where $\varphi^l \in V^*_x$ is the linear map associated with the affine map $\varphi$ and $h : V^* \to A^+$ be a Hamiltonian section of $\mu$, that is, $\mu \circ h = Id$.

Now, we consider the Lie algebroid prolongation $T^\mathcal{A}A^+$ of the Lie algebroid $\mathcal{A}$ over $\tau_{A^+}: A^+ \to M$ with vector bundle projection $\tau^\mathcal{A}A^+ : T^\mathcal{A}A^+ \to A^+$ (see Section 2.1.1). Then, we may introduce the map $\mathcal{T}h : T^\mathcal{A}V^* \to T^\mathcal{A}A^+$ defined by $\mathcal{T}h(a, X_\alpha) = (\bar{a}, (\tau_{A^+})(h)(X_\alpha))$, for $(\bar{a}, X_\alpha) \in (T^\mathcal{A}V^*)_\alpha$, with $\alpha \in V^*$. It is easy to prove that the pair $(\mathcal{T}h, h)$ is a Lie algebroid morphism between the Lie algebroids $\tau_V^\mathcal{A} : T^\mathcal{A}V^* \to V^*$ and $\tau_{A^+}^\mathcal{A} : T^\mathcal{A}A^+ \to A^+$. 

Next, denote by $\lambda_h$ and $\Omega_h$ the sections of the vector bundles $(\tilde{T}^*A)^* \rightarrow V^*$ and $\Lambda^2(\tilde{T}^*A)^* \rightarrow V^*$ given by

$$
(3.4) \quad \lambda_h = (Th, h)^*(\lambda_{\tilde{A}}), \quad \Omega_h = (Th, h)^*(\Omega_{\tilde{A}}),
$$

where $\lambda_{\tilde{A}}$ and $\Omega_{\tilde{A}}$ are the Liouville section and the canonical symplectic section, respectively, associated with the Lie algebroid $\tilde{A}$. Note that $\Omega_h = -d\tilde{T}^*V^*\lambda_h$.

On the other hand, let $\eta : \tilde{T}^*A \rightarrow \mathbb{R}$ be the section of $(\tilde{T}^*A)^* \rightarrow V^*$ defined by

$$
(3.5) \quad \eta(\tilde{a}, X_\alpha) = 1_A(\tilde{a}),
$$

for $(\tilde{a}, X_\alpha) \in (\tilde{T}^*A)^*\alpha$, with $\alpha \in V^*$. Note that if $pr_1 : \tilde{T}^*A \rightarrow \tilde{A}$ is the canonical projection on the first factor then $(pr_1, \tau^\gamma_A)$ is a morphism between the Lie algebroids $\tau^\gamma_A : \tilde{T}^*A \rightarrow V^*$ and $\tau_{\tilde{A}} : \tilde{A} \rightarrow M$ and $(pr_1, \tau^\gamma_A)^*(1_A) = \eta$. Thus, since $1_A$ is a 1-cocycle of $\tau_{\tilde{A}} : \tilde{A} \rightarrow M$, we deduce that $\eta$ is a 1-cocycle of the Lie algebroid $\tau^\gamma_A : \tilde{T}^*A \rightarrow V^*$.

Suppose that $h(x^i, y_\alpha) = (x^i, -H(x^i, y_\beta), y_\alpha)$. Then $\eta = \tilde{\gamma}^0$ and, from (3.3), (3.4) and the definition of the map $Th$, it follows that

$$
(3.6) \quad \Omega_h = \tilde{\gamma}^\gamma \wedge \tilde{\gamma}^\beta + \frac{1}{2} C^\gamma_{\gamma\beta}a_y \tilde{\gamma}^\gamma \wedge \tilde{\gamma}^\beta + (\rho^i \frac{\partial H}{\partial x^i} - C^\gamma_{\gamma\beta}a_y) \tilde{\gamma}^\gamma \wedge \tilde{\gamma}^\beta + \frac{\partial H}{\partial y_\gamma} \tilde{\gamma}^\gamma \wedge \tilde{\gamma}^0.
$$

Thus, it is easy to prove that the pair $(\Omega_h, \eta)$ is a cosymplectic structure on the Lie algebroid $\tau^\gamma_A : \tilde{T}^*A \rightarrow V^*$, that is,

$$
\{\eta \wedge \Omega_h \wedge \ldots \wedge \Omega_h\}(\alpha) \neq 0, \quad \text{for all } \alpha \in V^*, \quad \text{and} \quad d\tilde{T}^*V^*\eta = 0, \quad d\tilde{T}^*V^*\Omega_h = 0.
$$

**Remark 3.1.** Let $T^*V^*$ be the prolongation of the Lie algebroid $V$ over the projection $\tau^V_A : V^* \rightarrow M$. Denote by $\lambda_V$ and $\Omega_V$ the Liouville section and the canonical symplectic section, respectively, of $V$ and by $(i_V, Id) : T^*V^* \rightarrow \tilde{T}^*A$ the canonical inclusion. Then, using (2.2), (3.4), (3.5) and the fact that $\mu \circ h = Id$, we obtain that

$$
(i_V, Id)^*(\lambda_h) = \lambda_V, \quad (i_V, Id)^*(\eta) = 0.
$$

Thus, since $(i_V, Id)$ is a Lie algebroid morphism over the identity of $V^*$, we also deduce that

$$
(i_V, Id)^*(\Omega_h) = \Omega_V.
$$

Now, given a section $\gamma$ of the dual bundle $V^*$ to $V$, we can consider the morphism $(T\gamma, \gamma)$ between the vector bundles $\tau_{\tilde{A}} : \tilde{A} \rightarrow M$ and $\tau^\gamma_A : \tilde{T}^*A \rightarrow V^*$

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{T\gamma} & \tilde{T}^*A \\
\tau_{\tilde{A}} & \searrow & \tau^\gamma_A \\
\downarrow & & \downarrow \\
M & \xrightarrow{\gamma} & V^*
\end{array}
\]

defined by $T\gamma(\tilde{a}) = (\tilde{a}, (T_x\gamma)((\rho_{\tilde{A}}(\tilde{a}))$, for $\tilde{a} \in \tilde{A}_x$ and $x \in M$. 

\end{document}
Theorem 3.2. If $\gamma$ is a section of the vector bundle $\tau^*_\gamma : V^* \to M$ then the pair $(\mathcal{T} \gamma, \gamma)$ is a morphism between the Lie algebroids $(\tilde{A}, [\cdot, \cdot]_{\tilde{A}}, \rho_{\tilde{A}})$ and $(\tilde{A}^V, [\cdot, \cdot]_{\tilde{A}^V}, \rho_{\tilde{A}^V})$. Moreover,

$$(\mathcal{T} \gamma, \gamma)^* \lambda_h = h \circ \gamma, \quad (\mathcal{T} \gamma, \gamma)^*(\Omega_h) = -d\bar{\lambda}(h \circ \gamma).$$

Proof. Suppose that $(x^i)$ are local coordinates on $M$, that $\{e_0, e_\alpha\}$ is a local basis of $\Gamma(\tau^*_\gamma)$ adapted to $1_A$ and that

$$\gamma = \gamma_\alpha e^\alpha,$$

with $\gamma_\alpha$ local real functions on $M$ and $\{e^0, e^\alpha\}$ the dual basis to $\{e_0, e_\alpha\}$. Denote by $\{\tilde{e}_0, \tilde{e}_\alpha, \bar{e}_\alpha\}$ the corresponding local basis of $\Gamma(\tau^*_\gamma)$. Then, using (3.2), it follows that

$$T \gamma \circ e_0 = (\tilde{e}_0 + \rho_0^\beta \frac{\partial \gamma_\alpha}{\partial x^\beta} \tilde{e}_\alpha) \circ \gamma, \quad T \gamma \circ e_\alpha = (\tilde{e}_\alpha + \rho_\alpha^\beta \frac{\partial \gamma_\alpha}{\partial x^\beta} \bar{e}_\beta) \circ \gamma,$$

for $\alpha \in \{1, \ldots, n\}$, $\rho_0^\beta, \rho_\alpha^\beta$ being the components of the anchor map of $\tilde{A}$ with respect to the local coordinates $(x^i)$ and to the basis $\{e_0, e_\alpha\}$. Thus,

$$(\mathcal{T} \gamma, \gamma)^*(e^0) = e^0, \quad (\mathcal{T} \gamma, \gamma)^*(e^\alpha) = e^\alpha,$$

$$(\mathcal{T} \gamma, \gamma)^*(\bar{e}_\alpha) = \rho_0^\beta \frac{\partial \gamma_\alpha}{\partial x^\beta} e^0 + \rho_\alpha^\beta \frac{\partial \gamma_\alpha}{\partial x^\beta} e^\beta = dA^\gamma_\alpha,$$

where $\{e^0, e^\alpha, \bar{e}_\alpha\}$ is the dual basis to $\{\tilde{e}_0, \tilde{e}_\alpha, \bar{e}_\alpha\}$. Therefore, from (2.1), (3.1) and (3.3), we obtain that the pair $(\mathcal{T} \gamma, \gamma)$ is a morphism between the Lie algebroids $\tilde{A} \to M$ and $\tilde{A}^V \to V^*$. On the other hand, if $x$ is a point of $M$ and $\tilde{a} \in \tilde{A}_x$ then, using (2.2), we have

$$((\mathcal{T} \gamma, \gamma)^* \lambda_h)(x)(\tilde{a}) = ((\mathcal{T} \gamma, \gamma)^* (T h, h)^* \lambda_{\tilde{A}})(x)(\tilde{a}) = ((T (h \circ \gamma), h \circ \gamma)^* \lambda_{\tilde{A}})(x)(\tilde{a}) = \lambda_{\tilde{A}}(h(\gamma(x)))(\tilde{a}, T(h \circ \gamma)(\rho_{\tilde{A}}(\tilde{a}))) = (h \circ \gamma)(x)(\tilde{a})$$

that is,

$$(\mathcal{T} \gamma, \gamma)^* \lambda_h = h \circ \gamma.$$

Consequently, since $\Omega_h = -d\tilde{A}^V \lambda\gamma_h$ and $(\mathcal{T} \gamma, \gamma)$ is a morphism between the Lie algebroids $\tilde{A}$ and $\tilde{A}^V$, we deduce that

$$(\mathcal{T} \gamma, \gamma)^* (\Omega_h) = -d\tilde{\lambda}(h \circ \gamma).$$

\[\Box\]

3.2. The Hamilton equations on a Lie affgebroid. Let $h : V^* \to A^+$ be a Hamiltonian section of $\mu : A^+ \to V^*$ and $R_h \in \Gamma(\tau^*_\gamma)$ be the Reeb section of the cosymplectic structure $(\Omega_h, \eta)$ on $\tilde{A}^V$. $R_h$ is characterized by the following conditions

$$i_{R_h} \Omega_h = 0 \quad \text{and} \quad i_{R_h} \eta = 1.$$

With respect to the basis $\{\tilde{e}_0, \tilde{e}_\alpha, \bar{e}_\alpha\}$ of $\Gamma(\tau^*_\gamma)$, $R_h$ is locally expressed as follows:

$$R_h = \tilde{e}_0 + \frac{\partial H}{\partial y_\alpha} \bar{e}_\alpha - (C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \rho_\alpha^\beta \frac{\partial H}{\partial x^\beta} - C^\gamma_{0\alpha} y_\gamma) \bar{e}_\alpha.$$
Thus, the integral sections of $R_h$ (i.e., the integral curves of the vector field $\rho_{\tilde{A}}^* (R_h)$) satisfy the following equations

$$\frac{dx^i}{dt} = \rho_0^i + \frac{\partial H}{\partial y_\alpha} \rho_\alpha^i, \quad \frac{dy_\alpha}{dt} = -\rho_0^\alpha \frac{\partial H}{\partial x^i} + y_\gamma (C_{0\alpha}^\gamma + C_{\beta\alpha}^\gamma \frac{\partial H}{\partial y_\beta}),$$

for $i \in \{1, \ldots, m\}$ and $\alpha \in \{1, \ldots, n\}$. These equations are called the Hamilton equations for $h$ (see [11]) and the section $R_h$ is called the Hamiltonian section associated with $h$ (see [6]).

4. The Hamilton-Jacobi Equation on Lie Affgebroids

In this section, we will prove the main result of the paper.

Let $\tau_A : A \rightarrow M$ be an affine bundle with associated vector bundle $\tau_V : V \rightarrow M$ and suppose that $h : V^* \rightarrow A^+$ is a section of the canonical projection $\mu : A^+ \rightarrow V^*$ and that $\alpha : M \rightarrow A^+$ is a section of the vector bundle $\tau_{A^+} : A^+ \rightarrow M$. Then, $h \circ \mu \circ \alpha : M \rightarrow A^+$ is also a section of the vector bundle $\tau_{A^+} : A^+ \rightarrow M$ and there exists a unique real function on $M$, which we will denote by $f(h, \alpha)$, such that

$$\alpha - h \circ \mu \circ \alpha = f(h, \alpha) 1_A.$$

Now, we will prove the result announced above.

**Theorem 4.1.** Let $\tau_A : A \rightarrow M$ be a Lie affgebroid modelled over the vector bundle $\tau_V : V \rightarrow M$ with Lie affgebroid structure $([\cdot, \cdot]_V, D, \rho_A)$ and $(T^A V^*, [\cdot, \cdot]_A^*, \rho_A^*)$ be the prolongation of the bidual Lie algebroid $(\tilde{A}, [\cdot, \cdot]_{\tilde{A}}, \rho_{\tilde{A}})$ over the fibration $\tau_V^* : V^* \rightarrow M$. Suppose that $h : V^* \rightarrow A^+$ is a Hamiltonian section of the canonical projection $\mu : A^+ \rightarrow V^*$ and that $R_h \in \Gamma(\tau_{A^+}^*)$ is the corresponding Hamiltonian section. Let $\alpha \in \Gamma(\tau_{A^+})$ be a 1-cocycle, $d^A \alpha = 0$, and denote by $R_h^\alpha \in \Gamma(\tau_A^*)$ the section of $\tau_A : A \rightarrow M$ given by $R_h^\alpha = pr_1 \circ R_h \circ \mu \circ \alpha$, where $pr_1 : T^A V^* \rightarrow \tilde{A}$ is the canonical projection over the first factor. Then, the two following conditions are equivalent:

(i) For every integral curve $c$ of the vector field $\rho_{\tilde{A}}^* (R_h^\alpha)$, that is, $c$ is a curve on $M$ such that

$$\rho_{\tilde{A}}^* (R_h^\alpha) (c(t)) = \dot{c}(t), \text{ for all } t,$$

the curve $t \rightarrow (\mu \circ \alpha \circ c)(t)$ on $V^*$ satisfies the Hamilton equations for $h$.

(ii) $\alpha$ satisfies the Hamilton-Jacobi equation $d^V f(h, \alpha) = 0$, that is, $f(h, \alpha)$ is constant on the leaves of the Lie algebroid foliation of $V$.

**Proof.** For a curve $c : I = (-\epsilon, \epsilon) \subset \mathbb{R} \rightarrow M$ on the base we define the curves $\beta : I \rightarrow V^*$ and $\gamma : I \rightarrow A$ by

$$\beta(t) = \mu (\alpha(c(t))) \quad \text{and} \quad \gamma(t) = R_h^\alpha (c(t)).$$

Since $\mu \circ \alpha$ and $R_h^\alpha$ are sections of $\tau_V^* : V^* \rightarrow M$ and $\tau_A : A \rightarrow M$, respectively, it follows that both curves project to $c$.

We consider the curve $v = (\gamma, \dot{\gamma})$ in $\tilde{A} \times TV^*$ and notice the following important facts about $v$:

- $v(t)$ is in $T^A V^*$, for every $t \in I$, if and only if $c$ satisfies (4.2). Indeed $\rho_{\tilde{A}} \circ \gamma = \rho_{\tilde{A}} \circ R_h^\alpha \circ c$ while $T_{\tau_V^*} \circ \dot{\gamma} = \dot{c}$.
In such a case, $\beta$ is a solution of the Hamilton equations for $h$ if and only if $v(t) = R_h(\beta(t))$, for every $t \in I$. Indeed, the first components coincide $pr_1(v(t)) = \gamma(t)$ and $pr_1(R_h(\beta(t))) = pr_1(R_h(\mu(\alpha(c(t))))) = R_h(\mu(\alpha(c(t)))) = \gamma(t)$, and the equality of the second components is just $\beta(t) = \rho_{\hat{A}}^\ast(R_h(\beta(t)))$.

We denote by $\alpha_\mu = \mu \circ \alpha$ which is a section of the image of $\hat{A}$ over $\tilde{A}$ under $\alpha$ over $\tilde{A}$. Indeed, the first components coincide $pr_1(v(t)) = \gamma(t)$ and $pr_1(R_h(\beta(t))) = pr_1(R_h(\mu(\alpha(c(t))))) = R_h(\mu(\alpha(c(t)))) = \gamma(t)$, and the equality of the second components is just $\beta(t) = \rho_{\hat{A}}^\ast(R_h(\beta(t)))$.

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Remark 4.2. Obviously, we can consider as a cocycle $\alpha$ a 1-coboundary $\alpha = d^{A}S$, for some function $S$ on $M$. Nevertheless, it should be noticed that on a Lie algebroid there exist, in general, 1-cocycles that are not locally 1-coboundaries.

5. Examples

5.1. The Hamilton-Jacobi equation on Lie algebroids. Let $\tau_{E} : E \to M$ be a Lie algebroid over a manifold $M$. Then, $\tau_{E} : E \to M$ is an affine bundle and the Lie algebroid structure induces a Lie affgebroid structure on $\tau_{E} : E \to M$. In fact, the dual bundle to $E$, as an affine bundle, is the vector bundle $\tau_{E}^{+} : E^{*} \to M$, the bidual bundle is the vector bundle $\tau_{E}^{+} : E^{*} \times \mathbb{R} \to M$ and the Lie algebroid structure $([\cdot, \cdot]_{E}, \rho_{E})$ on $\tau_{E} : E = E \times \mathbb{R} \to M$ is given by

\[
[(X, f), (Y, g)]_{E} = ([X, Y]_{E}, \rho_{E}(X)(g) - \rho_{E}(Y)(f)), \quad \rho_{E}(X, f) = \rho_{E}(X),
\]

for $(X, f), (Y, g) \in \Gamma(\tau_{E}^{+}) \cong \Gamma(\tau_{E}) \times C^{\infty}(M)$, where $([\cdot, \cdot]_{E}, \rho_{E})$ is the Lie algebroid structure on $E$. The 1-cocycle $1_{E}$ on $E$ is the section $(0, 1)$ of $\tau_{E}^{+} : E^{*} \to M$. The map $\mu : E^{*} \to M$ is the canonical projection over the first factor and a Hamiltonian section $h : E^{*} \to E^{*} \times \mathbb{R}$ may be identified with a Hamiltonian function $H$ on $E^{*}$ in such a way that

\[
h(\beta_{x}) = (\beta_{x}, -H(\beta_{x})), \quad \text{for } \beta_{x} \in E_{x}^{*} \text{ and } x \in M.
\]

Now, if $\alpha$ is a 1-cocycle of $\tau_{E} : E \to M$ then $\alpha$ may be considered as the section $(\alpha, 0)$ of $\tau_{E}^{+} : E^{*} \times \mathbb{R} \to M$ and it is clear that $(\alpha, 0)$ is also a 1-cocycle of $\tau_{E} : E = E \times \mathbb{R} \to M$. In addition, if $f(h, \alpha)$ is the real function on $M$ characterized by

\[
\alpha - h \circ \mu \circ \alpha = f(h, \alpha)1_{E},
\]

it is easy to prove that $f(h, \alpha) = H \circ \alpha$. Thus, the equation

\[
d^{E}(f(h, \alpha)) = 0
\]

is just the Hamilton-Jacobi equation considered in [7].

5.2. The classical Hamilton-Jacobi equation for time-dependent Mechanics. Let $\tau : M \to \mathbb{R}$ be a fibration and $\tau_{1,0} : J^{1}\tau \to M$ be the associated Lie affgebroid modelled on the vector bundle $\pi = (\pi_{M})|_{V\tau} : V\tau \to M$. As we know, the bidual vector bundle $\hat{J}^{1}\tau$ to the affine bundle $\tau_{1,0} : J^{1}\tau \to M$ may be identified with the tangent bundle $TM$ to $M$ and, under this identification, the Lie algebroid structure on $\pi_{M} : TM \to M$ is the standard Lie algebroid structure and the 1-cocycle $1_{J^{1}\tau}$ on $\pi_{M} : TM \to M$ is just the 1-form $\eta = \tau^{*}(dt)$, $t$ being the coordinate on $\mathbb{R}$ (see Section 2.2). If $(t, q^{i})$ are local fibred coordinates on $M$ then $\{\frac{\partial}{\partial q^{j}}\}$ (respectively, $\{\frac{\partial}{\partial \dot{q}^{i}}, \frac{\partial}{\partial q^{i}}\}$) is a local basis of sections of $\pi : V\tau \to M$ (respectively, $\pi_{M} : TM \to M$). Denote by $(t, q^{i}, \dot{q}^{i})$ (respectively, $(t, q^{i}, \dot{q}^{i}, \dot{q}^{i})$) the corresponding local coordinates on $V\tau$ (respectively, $TM$). Then, the (local) structure functions of $TM$ with respect to this local trivialization are given by

\[
\begin{align*}
C_{ij}^{k} &= 0 \quad \text{and} \quad \rho_{j}^{l} = \delta_{ij}, \quad \text{for } i, j, k \in \{0, 1, \ldots, n\}.
\end{align*}
\]
Now, let $\pi^*: V^*\tau \to M$ be the dual vector bundle to $\pi: V\tau \to M$ and $(J^1\tau)^+ \cong T^*M$ be the cotangent bundle to $M$. Denote by $(t, q^i, p_i)$ (resp., $(t, q^i, p_i, p_i)$) the dual coordinates on $V^*\tau$ (resp., $T^*M$) to $(t, q^i, \dot{q}^i)$ (resp., $(t, q^i, \dot{l}, \dot{q}^i)$). Then, since the anchor map of $\pi_M: TM \to M$ is the identity of $TM$, it follows that the Lie algebroids $\pi_M^*: T(T^*M) \to V^*\tau$ and $\pi_{V*:}^*: T(V^*\tau) \to V^*\tau$ are isomorphic.

Next, let $h$ be a Hamiltonian section, that is, $h: V^*\tau \to (J^1\tau)^+ \cong T^*M$ is a section of the canonical projection $\mu: (J^1\tau)^+ \cong T^*M \to V^*\tau$. $h$ is locally given by

$$h(t, q^i, p_i) = (t, q^i, -H(t, q^j, p_i)).$$

Moreover, the cosymplectic structure $(\Omega_h, \eta)$ on the Lie algebroid $\pi_M^*: T(T^*M) \cong T(V^*\tau) \to V^*\tau$ is, in this case, the standard cosymplectic structure $(\Omega_h, \eta)$ on the manifold $V^*\tau$ locally given by (see [60] and [61])

$$\Omega_h = dq^i \wedge dp_i + \frac{\partial H}{\partial q^i} dq^i \wedge dt + \frac{\partial H}{\partial p_i} dp_i \wedge dt, \quad \eta = dt.$$

Thus, the Reeb section of $(\Omega_h, \eta)$ is the vector field $R_h$ on $V^*\tau$ defined by

$$R_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}.$$

It is clear that the integral curves of $R_h$

$$t \mapsto (t, \dot{q}^i(t), p_i(t))$$

are just the solutions of the classical time-dependent Hamilton equations for $h$

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

Furthermore, using Theorem 4.1 we deduce the following result.

**Corollary 5.1.** Let $\tau: M \to \mathbb{R}$ be a fibration and $\tau_{1,0}: J^1\tau \to M$ be the associated Lie affgebroid modelled on the vector bundle $\pi = (\pi_M)|_{V\tau}: V\tau \to M$. Let $h: V^*\tau \to T^*M$ be a Hamiltonian section and $R_h$ be the Reeb vector field of the corresponding cosymplectic structure $(\Omega_h, \eta)$ on $V^*\tau$. Suppose that $\alpha$ is a closed 1-form on $M$ and denote by $R_h^\alpha$ the vector field on $M$ given by $R_h^\alpha = T\pi^* \circ R_h \circ \mu \circ \alpha$. Then, the following conditions are equivalent:

(i) For every integral curve $t \mapsto c(t)$ of $R_h^\alpha$, the curve $t \mapsto \mu(\alpha(c(t)))$ on $V^*\tau$ satisfies the Hamilton equations for $h$.

(ii) $\alpha$ satisfies the Hamilton-Jacobi equation $d^{V^*\tau}(f(h, \alpha)) = 0$, that is, the function $f(h, \alpha)$ is constant on the leaves of the vertical bundle to $\tau$.

**Remark 5.2.** i) We recall that the function $f(h, \alpha)$ on $M$ is characterized by the following condition

$$\alpha - h \circ \mu \circ \alpha = f(h, \alpha)\eta.$$

ii) If the fibers of $\tau$ are connected then the equation $d^{V^*\tau}(f(h, \alpha)) = 0$ holds if and only if the function $f(h, \alpha)$ is constant on the fibers of $\tau$. 

$\diamondsuit$
Now, suppose that the fibration \( \tau \) is trivial, that is, \( M = \mathbb{R} \times P \) and \( \tau \) is the canonical projection on the first factor. Then, the vector bundle \( \pi^* : V^* \tau \rightarrow M \) is isomorphic to the product \( \mathbb{R} \times T^* P \) (as a vector bundle over \( M = \mathbb{R} \times P \)). Thus, a Hamiltonian section \( h : V^* \tau \cong \mathbb{R} \times T^* P \rightarrow T^* M \cong (\mathbb{R} \times \mathbb{R}) \times T^* P \) may be identified with a time-dependent Hamiltonian function \( H : \mathbb{R} \times T^* P \rightarrow \mathbb{R} \) in such a way that

\[
h(t, \beta) = (t, -H(t, \beta), \beta), \quad \text{for} \quad (t, \beta) \in \mathbb{R} \times T^* P.
\]

Moreover, if \( \alpha \) is an exact 1-form on \( M \), that is, \( \alpha = dW \) with \( W \) a real function on \( M = \mathbb{R} \times P \), then one may prove that the function \( f(h, dW) \) is given by

\[
f(h, dW)(t, q) = \frac{\partial W}{\partial t} |_{(t, q)} + H(t, dW_1(q)), \quad \text{for} \quad (t, q) \in M = \mathbb{R} \times P,
\]

where \( W_t : P \rightarrow \mathbb{R} \) is the real function on \( P \) defined by

\[
W_t(q) = W(t, q).
\]

Thus, if \( P \) is connected the equation

\[
dV^* \tau (f(h, dW)) = 0
\]

holds if and only if the function

\[
(t, q) \in \mathbb{R} \times P \mapsto \frac{\partial W}{\partial t} |_{(t, q)} + H(t, dW_1(q)) \in \mathbb{R}
\]

doesn’t depend on \( q \).

This last condition may be locally expressed as follows:

\[
\frac{\partial W}{\partial t} + H(t, q^i \frac{\partial W}{\partial q^i}) = \text{constant on } P
\]

which is the classical time-dependent Hamilton-Jacobi equation for the function \( W \) (see [3]).

5.3. The Hamilton-Jacobi equation on the Atiyah algebroid. Let \( p : Q \rightarrow M \) be a principal \( G \)-bundle. Denote by \( \Phi : G \times Q \rightarrow Q \) the free action of \( G \) on \( Q \) and by \( T\Phi : G \times TQ \rightarrow TQ \) the tangent action of \( G \) on \( TQ \). Then, one may consider the quotient vector bundle \( \pi_Q|G : TQ/G \rightarrow M = Q/G \) and the sections of this vector bundle may be identified with the vector fields on \( Q \) which are invariant under the action \( \Phi \). Using that every \( G \)-invariant vector field on \( Q \) is \( p \)-projectable and that the usual Lie bracket on vector fields is closed with respect to \( G \)-invariant vector fields, we can induce a Lie algebroid structure on \( TQ/G \). This Lie algebroid is called the Atiyah algebroid associated with the principal \( G \)-bundle \( p : Q \rightarrow M \) (see [2] [5]).

Now, suppose that \( \nu : M \rightarrow \mathbb{R} \) is a fibration of \( M \) on \( \mathbb{R} \). Denote by \( \tau : Q \rightarrow \mathbb{R} \) the composition \( \tau = \nu \circ p \). Then, \( \Phi \) induces an action \( J^1\Phi : G \times J^1\tau \rightarrow J^1\tau \) of \( G \) on \( J^1\tau \) such that

\[
J^1\Phi(g, j^1 \gamma) = j^1_\tau(\Phi_g \circ \gamma),
\]

for all \( g \in G \) and \( \gamma : I \subset \mathbb{R} \rightarrow Q \) a local section of \( \tau \) with \( t \in I \). Moreover, the projection

\[
\tau_{i,0}|G : J^1\tau/G \rightarrow M, \quad [j^1_i \gamma] \mapsto p(\tau_{i,0}(j^1_i \gamma)) = p(\gamma(t))
\]

defines an affine bundle on \( M \) which is modelled on the quotient vector bundle

\[
\pi|G : V\tau/G \rightarrow M, \quad [u_q] \mapsto p(q), \quad \text{for} \quad u_q \in V_q \tau,
\]
In addition, the bidual vector bundle of $J\pi_{14}$ is $J\pi_{14}$.

On the other hand, if $G$ is the Lie group $\mathfrak{g}$ acts on the vector bundles $TQ/G$ and $V^*TQ/G$ with the principal $TQ/G$-equivariant section $\phi$ (see [11]). $\mathfrak{g}$ is endowed with this structure is called the Atiyah algebroid associated with the principal $G$-bundle $p: Q \to M$ and the fibration $\nu: M \to \mathbb{R}$ (see [11]).

The Lie group $G$ acts on the vector bundles $T^*Q$ and $V^*TQ$ in such a way that the dual vector bundles to $TQ/G$ and $V^*TQ/G$ may be identified with the quotient vector bundles $T^*Q/G$ and $V^*TQ/G$, respectively (see [11]). Moreover, the canonical projection between the vector bundles $(J^1\tau/G)^+ \cong T^*Q/G$ and $V^*\tau/G$ is the map $\mu|G$ given by

$$(\mu|G)[\alpha_q] = [\mu(\alpha_q)], \quad \text{for } \alpha_q \in T_q^*Q \text{ and } q \in Q,$$

where $\mu: T^*Q \to V^*\tau$ is the projection between $T^*Q$ and $V^*\tau$. Thus, if $\hat{h}: V^*\tau/G \to (J^1\tau/G)^+ \cong T^*Q/G$ is a Hamiltonian section of $\mu|G$ then $\hat{h}$ induces a $G$-equivariant Hamiltonian section $\hat{h}: V^*\tau \to T^*Q$ such that $\hat{h} = h|G$, that is,

$$\hat{h}[\alpha_q] = [h(\alpha_q)], \quad \text{for } \alpha_q \in V^*_q\tau \text{ and } q \in Q.$$

Conversely, if $h: V^*\tau \to T^*Q$ is a $G$-equivariant Hamiltonian section of $\mu: T^*Q \to V^*\tau$ then $h$ induces a Hamiltonian section $\hat{h}: V^*\tau/G \to T^*Q/G$ such that $\hat{h} = h|G$, that is,

$$\hat{h}[\alpha_q] = [h(\alpha_q)], \quad \text{for } \alpha_q \in V^*_q\tau \text{ and } q \in Q.$$

Next, we will discuss the relation between the solutions of the Hamilton-Jacobi equation for the Hamiltonians $h$ and $\hat{h}$. In fact, we will prove the following result.

**Proposition 5.3.** There exists a one-to-one correspondence between the solutions of the Hamilton-Jacobi equation for $h|G$ and the $G$-invariant solutions of the Hamilton-Jacobi equation for $h$.

**Proof.** If $p_{TQ}: TQ \to TQ/G$ is the canonical projection then $p_{TQ}$ is a fiberwise bijective Lie algebroid morphism over $p: Q \to M = Q/G$. Thus, there exists a one-to-one correspondence between the 1-cocycles of the Atiyah algebroid $\tau_Q|G: TQ/G \to M = Q/G$ and the $G$-invariant closed 1-forms on $Q$. Indeed, if $\alpha: Q \to T^*Q$ is a $G$-invariant closed 1-form on $Q$ then $\alpha|G: M = Q/G \to T^*Q/G$ defined by

$$(\alpha|G)([q]) = [\alpha(q)], \quad \text{for } q \in Q,$$

is a 1-cocycle of the Atiyah algebroid (note that $\alpha|G = (p_{TQ} \circ p)^*(\alpha)$). Moreover, if

$$\alpha - h \circ \mu \circ \alpha = f(h, \alpha)\eta$$

then

$$\alpha|G - (h|G) \circ (\mu|G) \circ (\alpha|G) = (f(h, \alpha)|G)\phi,$$

where $f(h, \alpha)|G: M = Q/G \to \mathbb{R}$ is the real function on $M$ which is characterized by the condition

$$f(h, \alpha)|G \circ p = f(h, \alpha).$$
Therefore, $f(h|G, \alpha|G) = f(h, \alpha)|G$.

On the other hand, if $p_{V\tau}: V\tau \rightarrow V\tau/G$ is the canonical projection then $p_{V\tau}$ is a Lie algebroid morphism over $p: Q \rightarrow M = Q/G$ and, from (5.2), it follows that

$$(p_{V\tau}, p)^*(d^{V\tau/G}(f(h, \alpha)|G)) = d^{V\tau}(f(h, \alpha)).$$

Finally, using that $p_{V\tau}$ is a fiberwise bijective morphism, we conclude that

$$d^{V\tau/G}(f(h, \alpha)|G) = 0 \iff d^{V\tau}(f(h, \alpha)) = 0,$$

which proves the result. 

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