Speed Scaling with Multiple Servers Under A Sum Power Constraint

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ABSTRACT
The problem of scheduling jobs and choosing their respective speeds with multiple servers under a sum power constraint to minimize the flow time + energy is considered. This problem is a generalization of the flow time minimization problem with multiple unit-speed servers, when jobs can be parallelized, however, with a sub-linear, concave speedup function $k^{1/\alpha}, \alpha > 1$ when allocated $k$ servers, i.e., jobs experience diminishing returns from being allocated additional servers. When all jobs are available at time 0, we show that a very simple algorithm EQUI, that processes all available jobs at the same speed is $(2 - \frac{1}{\alpha}) \left(1 - \frac{1}{\alpha} \right)$ competitive, while in the general case, when jobs arrive over time, an LCFS based algorithm is shown to have a constant (dependent only on $\alpha$) competitive ratio.

1. INTRODUCTION
Scheduling jobs with multiple servers to minimize the sum of their response times (called the flow time) is an important practical problem, and finding optimal algorithms remains challenging. An added feature in modern servers is their ability to work at different speeds. This paradigm is called speed scaling \cite{2,6,16,17}, where one or more servers with tuneable speed are available, and operating any server at speed $s$ consumes energy at rate $P(s)$, a non-decreasing convex function of $s$. With speed scaling, the problem is to choose speed of operation so as to minimize the sum of the flow time and energy.

In prior work, speed scaling problem with multiple servers has been considered \cite{2,6,16,17}, however, with a fixed number of servers and without any upper bound on the power consumption of any server. For a single server, the flow time + energy problem under a power constraint or upper limit on speed has been solved in \cite{2}. In this paper, we consider the speed scaling problem to minimize the sum of the flow time plus energy with infinite servers under a sum-power constraint across all servers. We refer to this as the \textit{flow time + energy} problem. Even though there are unlimited number of servers, each job can only be processed by one server at any time. A special case of this problem is to minimize just the flow time, called the \textit{flow time} problem.

The flow time problem is also equivalent to the problem of scheduling parallelizable jobs \cite{10} with sub-linear speedup called \textbf{sub-linear-sched} problem described as follows. Let there be $N$ servers with unit speed, and jobs arriving over time with different sizes have to be assigned a set of servers, so as to minimize the flow time. Jobs receive a concave, sub-linear speedup from parallelization: decreasing marginal benefit from being allocated additional servers. In particular, if $k \leq N$ is the number of servers assigned to a job, then the resulting speed obtained is $k^{1/\alpha}$ for $\alpha > 1$. When jobs can be completely parallelizable $\alpha = 1$, processing the job with shortest remaining processing time (SRPT) on all servers is known to be optimal.

In this paper, we consider online algorithms (that have only causal job arrival information) for solving the flow time + energy problem. To quantify the performance of an online algorithm, we consider the metric of competitive ratio, that is defined as the ratio of the flow time of the online algorithm and the optimal offline algorithm OPT maximized over all possible inputs (worst case).

Prior Work The \textbf{sub-linear-sched} problem has been an object of immense interest \cite{3,4,11,7,8,1}, where practical algorithms include packing based \cite{18}, and resource reservation algorithms \cite{14}. Heuristic policies with only numerical performance analysis can be found in \cite{13}. In past, this problem has been considered for the combinatorial discrete allocation model \cite{11}, where an integer number of servers are assigned to any job, as well as the continuous allocation model \cite{7,8,1,3,4}, that treats the $N$ servers as a single resource block which can be partitioned into any size and assigned to any job.

For the discrete allocation model, in \cite{11}, a variant of the SRPT algorithm is shown to be $4^{1/(1-1/\alpha)} \log W$ competitive, where $W = w_{\text{max}}/w_{\text{min}}$ is the ratio of the largest and the smallest job size. For the continuous allocation model, this problem has been considered in \cite{7,8,1,3,4}, where with resource augmentation, i.e., the online algorithm is allowed more resources, e.g., faster machines, than the OPT, algorithms with constant competitive ratios have been derived as a function of the resource augmentation factor.

A special case of the problem in the continuous allocation model has been considered in \cite{4} recently, where all jobs arrive together/are available at time 0. In this simpler setting, \cite{4} derived an optimal algorithm, called heSRPT, that gave an explicit expression for the number of servers to be dedicated for each job, which prefers smaller jobs, but unlike SRPT, all jobs are given non-zero speed. In the stochastic setting, where jobs arrive over time that have exponentially distributed job sizes, \cite{5} showed that an algorithm called EQUI that dedicates equal number of resources to all out-
subject to \( \sum_{i: s_i(t)>0} P(s_i(t)) \leq p \).

**Remark 1.** Note that even under the sum power constraint, the metric of flow time + energy is meaningful, since it is not necessary that the energy used by an optimal algorithm is equal to the maximum possible allowed by the constraint. For example, when the number of outstanding jobs is small, an optimal algorithm may choose a small speed such that the total power consumed is less than the sum-power constraint.

Next, we show that problem \( \text{II} \) is equivalent to the sub-linear-sched problem, that has \( N \) parallel and identical servers. Similar to \( \text{II} [2, 8, 1] \), we consider the continuous allocation model, where \( N \) is treated as a single resource block which can be divided into chunks of arbitrary sizes and allocated to different jobs. Any job is parallelizable with concave speedup, i.e., if job \( j \) is allotted \( k_j(t) \) number of servers at time \( t \), then the service rate experienced by job \( j \) at time \( t \) is \( s_j(t) = S(k_j(t)) = k_j(t)^{1/\alpha} \), where \( \alpha > 1 \). Here, \( S(\cdot) \) denotes the speedup function, that is concave. The parameter \( \alpha \) controls the parallelizability of any job, and depending on \( \alpha \), jobs experience appropriate diminishing returns from being allocated additional servers. Note that

\[
A(t) \sum_{j=1}^N k_j(t) \leq N, \tag{3}
\]

where \( A(t) \) is the set of jobs that are given non-zero service rate at time \( t \). Note that the objective is to minimize the flow time of all jobs.

To cast the sub-linear-sched problem as a flow time problem \( \text{II} \), suppose that for each job we can ‘create’ its own dedicated server, and a job is processed on only one server, and cannot be parallelized. Let \( s_j(t) \) denote the speed allocated to the server processing job \( j \) at time \( t \). Let \( k_j(t) = P(s_j(t)) = S^{-1}(s_j(t)) \) be the power consumption of job \( j \) on its own server if it is processed at speed \( s_j(t) \), where \( P(s) = s^\alpha, \alpha > 1 \). Then, \( \text{II} \) is equivalent to

\[
\sum_{j \in A(t)} P(s_j(t)) \leq N, \tag{4}
\]

the total power used across all the active servers is at most \( N \).

In both these models, the service speed of job \( j \) is \( s_j \), so the flow times across both the models would be identical. Letting \( N = p \), we see that sub-linear-sched is equivalent to the flow time problem \( \text{II} \).

In the following, we will consider Problem \( \text{III} \), and propose algorithms and bound their competitive ratios \( \text{III} \). Minimal changes to be made for algorithms to be feasible, and analysis to be applicable for Problem \( \text{III} \) are mentioned in Remark \( \text{III} \). Consequently, we will only indicate the corresponding competitive ratio results for Problem \( \text{III} \).

**Metric** We represent the optimal offline algorithm (that knows the entire job arrival sequence in advance) as \( \text{OPT} \). Let \( n(t) (n_0(t)) \) and \( P_{\text{sum}}(t) (P_{\text{sum}}(t)) \) be the number of outstanding jobs with an online algorithm \( A \) (OPT), and the sum of the power used by an online algorithm \( A \) (OPT) across all servers at time \( t \), respectively. For Problem \( \text{III} \), we will consider the metric of competitive ratio which for an algorithm \( A \) is defined as

\[
\mu_A = \max_{\sigma} \frac{\int (n(t) + P_{\text{sum}}(t)) dt}{\int (n_0(t) + P_{\text{sum}}(t)) dt}, \tag{4}
\]

where \( \sigma \) is the input sequence consisting of jobs set \( J \). Since \( \sigma \) is arbitrary, we are not making any assumption on the job
algorithm (called heSRPT) has been derived. In particular,  
considering has been considered recently [4], and an optimal al-

gumentation has been shown to be {\it \Phi(\kappa(n))}.  

3. All Jobs Available at Time 0 

We first consider the simplest setting where all jobs of  
\tt arrive at time 0. For Problem 1, this setting has been considered recently [4], and an optimal al-

3.1 Algorithm EQUI 

At time 0, if the outstanding number of jobs in the system is  
\tt, then all \tt jobs are processed parallelly on \tt servers, each with identical speed  

do not require the knowledge of remaining job sizes, has a con-

3.2 Potential Function 

At time 0, let \tt be the set of unfinished jobs with EQUI  
n with \tt, and for the \ith job, i ∈ \tt, let \tt be its remaining size. Then  

Similarly, let \tt be the number of unfinished jobs with  
the OPT, and the corresponding quantity to \tt for the  
\ith job with the OPT, be denoted by \tt.  

Consider the potential function  

PROOF. The drift of EQUI satisfies the first boundary condition. Since all jobs are available at time 0, to check whether \Phi E is satisfying the second boundary condition, we only need to check  
whether \Phi E increases on a departure of a job with either the EQUI or the OPT.  

Lemma 4. Potential function \Phi E does not increase on a departure of a job with either the EQUI or the OPT.  

Next, we characterize the drift \pt/\pt.  

LEMMA 5. \pt/\pt ≤ -ct1(-max{\tt,0})  

\mu(\alpha) = \frac{2 - \frac{1}{\alpha}}{2(1 - \frac{\alpha}{\alpha})}.  

For \alpha = 2, \mu(2) = 6. Moreover, \mu(\alpha) is a decreasing function of \alpha > 1.  

PROOF. Case I: max{\tt - \tt,0} = 0. In this case, from  
Lemma 5 we can write 5, as  

where (a) follows since \tt ≤ \tt.  

Case II: \tt ≥ 0, and max{\tt - \tt,0} = \tt. Using Lemma 5 we can write 5, as  

\mu(\alpha) = \frac{2 - \frac{1}{\alpha}}{2(1 - \frac{\alpha}{\alpha})}.  

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\mu(\alpha) = \frac{2 - \frac{1}{\alpha}}{2(1 - \frac{\alpha}{\alpha})}.  

For \alpha = 2, \mu(2) = 6. Moreover, \mu(\alpha) is a decreasing function of \alpha > 1.
\[ + c_1 \left(1 - \frac{1}{\alpha}\right) n_o(t), \]
\[ \leq n(t) \left(2 - 2c + c_1 \left(\frac{1}{\alpha}\right)\right) + n_o(t) \left(c_1 \left(1 + \left(1 - \frac{1}{\alpha}\right)\right)\right) \]
\[ + c_1 \left(\frac{1}{\alpha}\right) P_{\text{sum}}^o(t), \]
\[ \leq \left(c_1 \left(1 - \frac{2}{\alpha}\right)\right) (n_o(t) + P_{\text{sum}}^o(t)) \]
\[ \text{(11)} \]
for \( c_1 \geq 2/\left(1 - \left(\frac{1}{\alpha}\right)\right) \). When \( n_o(t) = 0 \), then we do not have to add the contribution of the OPT from Lemma 5 and we get \( n(t) + P_{\text{sum}}(t) + d\Phi(t)/dt \)
\[ = n(t) + n(t) \left(\frac{\min\{n(t), p\}}{n(t)}\right) - c_1 n(t), \]
\[ \leq n(t)(2 - c_1) \leq 0, \]
\text{(12)}
for \( c_1 = 2 \). Thus, choosing \( c_1 = 2/\left(1 - \left(\frac{1}{\alpha}\right)\right) \), from (10), (11), and (12), (10) holds for \( \kappa = (2 - \frac{1}{\alpha}) \left(\frac{\alpha - 1}{\alpha}\right) \). \( \square \)

Identical proof follows for Problem 11, since essentially, the only difference when considering Problem 11 is that we can remove the energy term corresponding to \( P_{\text{sum}}(t) \), and choose \( c_1 = 1/\left(1 - \left(\frac{1}{\alpha}\right)\right) \), and show that (11) holds for \( \kappa = (2 - \frac{1}{\alpha}) \left(\frac{\alpha - 1}{\alpha}\right) \).

**Theorem 7.** The competitive ratio of EQU1 for Problem 11 when all jobs are available at time 0, is at most \( (2 - \frac{1}{\alpha}) \left(\frac{\alpha - 1}{\alpha}\right) \). For \( \alpha = 2 \), the upper bound is at most 3, and decreases to 2 as \( \alpha \) increases.

**Discussion:** In our approach, we showed that for minimizing flow time (Problem 11), a simpler algorithm (EQU1) than the optimal heSRPT algorithm [3], that processes all available jobs with the same speed, is constant (depending only on \( \alpha \)) competitive. Moreover, as \( \alpha \) increases, speeds chosen by EQU1 and heSRPT converge, and that is also reflected in the competitive ratio bound that improves as \( \alpha \) increases. Thus, knowing job sizes and using job dependent speed is not critical for staying close to the optimal performance. The utility of EQU1 is that it does not need to know the exact remaining sizes of the jobs, thus making it applicable for more wider network setting where pipelining [12, 5, 15] is implemented, and jobs on arrival do not reveal their true sizes.

4. ONLINE JOB ARRIVALS

In this section, we consider Problem 2 in the online case, where jobs arrive over time with arbitrary sizes and at arbitrary time instants.

4.1 Algorithm FRACTIONAL-LCFS-EQU1

At time \( t \), let the outstanding number of jobs in the system be \( n(t) \). **Scheduling:** Process all the \( \beta n(t) \), \( \beta < 1 \), jobs that have arrived most recently in their respective \( \beta n(t) \) servers. **Speed:** Use EQU1, to process all the \( \beta n(t) \) jobs at equal speed \( s(t) = P^{-1} \left(\frac{\min\{n(t), p\}}{\beta n(t)}\right) \).

By its very definition, FRACTIONAL-LCFS-EQU1 satisfies the total power constraint as follows \( P_{\text{sum}}(t) \)
\[ \leq \beta n(t) \left( P^{-1} \left(\frac{\min\{n(t), p\}}{\beta n(t)}\right) \right) \leq \min\{n(t), p\} \leq p. \]

\[ 1 \text{if } \beta n(t) \text{ is fractional, then we mean } [\beta n(t)]. \]

**Remark 8.** Choosing \( \beta = 1 \), FRACTIONAL-LCFS-EQU1 is identical to EQU1. Intuitively following heSRPT algorithm \( \text{(6)} \) at each time \( t \) in the online case appears better than FRACTIONAL-LCFS-EQU1, since it is locally optimal, however, analyzing the heSRPT algorithm in the online case appears challenging.

The main result of this section is as follows.

**Theorem 9.** For any \( \alpha > 1 \), there exists a \( \beta < 1 \), such that the competitive ratio of algorithm FRACTIONAL-LCFS-EQU1 for Problem 2 is a constant (depends only on \( \alpha \)) and is independent of the number of jobs, and their sizes. For example, for \( 2 \leq \alpha \leq 3 \), with \( \beta = \frac{7}{8} \), the competitive ratio is at most 693.

We get the result for Problem 11 as a corollary as follows.

**Corollary 10.** For any \( \alpha > 1 \), there exists a \( \beta < 1 \), such that the competitive ratio of algorithm FRACTIONAL-LCFS-EQU1 for Problem 11 is a constant (depends only on \( \alpha \)) and is independent of the number of jobs, and their sizes. In particular, the competitive ratio will be at most half of the competitive ratio for Problem 2. For example, for \( 2 \leq \alpha \leq 3 \), with \( \beta = \frac{7}{8} \), the competitive ratio is at most 345.

**Discussion:** Similar to EQU1, algorithm FRACTIONAL-LCFS-EQU1 is also a non-clairvoyant algorithm, i.e., it does not need to know the remaining size of any outstanding job. The main novelty of Theorem 9 over previous such results [7, 8, 9], is that it is proven without needing resource augmentation. With resource augmentation, an online algorithm is given servers that are allowed to operate at speed \( s(1 + \theta) \), \( \theta > 0 \) while consuming only power \( P(s) \), but the OPT’s consumption is kept intact at \( P(s) \) with speed \( s \). Thus, an online algorithm is given extra/faster resources. In [7, 8, 9], algorithms with competitive ratio as a function of \( \alpha \) and \( \theta \) have been derived for a similar but more complicated non-clairvoyant setting.

4.2 Proof of Theorem 9

From here on we refer to algorithm FRACTIONAL-LCFS-EQU1 as just algorithm. Let at time \( t \), the set of outstanding (unfinished) number of jobs with the algorithm be \( A(t) \) with \( n(t) = |A(t)| \). Let at time \( t \), the rank \( r_j(t) \) of a job \( j \in A(t) \) be equal to the number of outstanding jobs of \( A(t) \) with the algorithm that have arrived before job \( j \). Note that the rank of a job does not change on arrival of a new job, but can change if a job departs that had arrived earlier.

As before, \( Q(x) = \sum_{x_j} x^j \), which specializes to \( Q(x) = x^{1+\theta} \) for \( P(x) = x^\alpha \). Then we consider the following potential function
\[ \Phi(t) = c \sum_{j \in A(t)} r_j(t) \left(\frac{w_j^A(t) - w_j^o(t)}{P^{-1}(\min\{r_j(t), p\})Q(r_j(t))}\right)^+, \]
\[ \text{(13)} \]
where \( w_j^A(t) \) (\( w_j^o(t) \)) is the remaining size of job \( j \) with the algorithm (OPT) at time \( t \), and \( c \) is a constant to be chosen later.

**Remark 11.** The potential function (13) is very similar to the one used in [2] for a very different problem. The main novelty of our result is that we avoid resource augmentation.
When unlike\(7\). Moreover, we would like to point out that for Problem (2), the most popular potential functions used in \(2\) \(10\) cannot be used since they need job processing speed to be a function of \(P^{-1}(n(t))\) which is not possible because of the sum-power constraint.

We next show that the potential function \(\Phi(t)\) satisfies the second boundary condition. The fact that the first boundary condition is satisfied is trivial.

**Lemma 12.** Potential function \(\Phi(t)\) \(13\) does not change on arrival of any new job. Moreover, on a departure of a job with the algorithm or the OPT, the potential function \(\Phi(t)\) \(1\) does not increase.

We next bound the drift \(d\Phi(t)/dt\) because of the processing by the OPT, and the algorithm, respectively. To avoid cumbersome notation, we write \(\beta n(t)\) instead of \([\beta n(t)]\) everywhere.

**Lemma 13.** The change in the potential function \(\Phi(t)\) because of the OPT’s contribution is

\[
d\Phi(t)/dt \leq \frac{cP_{\text{sum}}^o(t)}{Q_n(n(t))} \frac{Q_0(n(t))}{Q(n(t))} \quad \text{if } n(t) \leq p,
\]

\[
d\Phi(t)/dt \leq \frac{(1 - \beta)(\beta - \gamma)n(t)}{P^{-1}(\beta)} \quad \text{if } n(t) > p.
\]

**Proof:** [Proof of Theorem 14] To prove the Theorem, we check the running condition \(5\) for the two cases separately: i) \(n_0(t) > \gamma n(t)\) and then ii) \(n_0(t) \leq \gamma n(t)\), and show that it holds for a constant \(\kappa\).

Case i) \(n_0(t) > \gamma n(t)\). In this case, we only count the OPT’s contribution to \(d\Phi(t)/dt\), which is sufficient since the algorithm’s contribution to \(d\Phi(t)/dt\) is always non-positive. When \(n(t) > p\), from Lemma 13 we have that \(5\), \(n(t) + P_{\text{sum}}(t) + d\Phi(t)/dt \leq 2 + c/\gamma(n_0(t) + P_{\text{sum}}'(t)).\) Combining, \(17\) and \(15\), when \(n_0(t) > \gamma n(t)\)

\[
n(t) + P_{\text{sum}}(t) + d\Phi(t)/dt \leq \frac{1}{\gamma}(2 + c)(n_0(t) + P_{\text{sum}}'(t)).
\]

Case ii) \(n_0(t) \leq \gamma n(t)\). Let \(n_0(t) > 0\). When \(n(t) > p\), From Lemma 13 and Lemma 14 \(5\) can be bounded as

\[
n(t) + P_{\text{sum}}(t) + d\Phi(t)/dt \leq n(t) + \min\{n(t), p\} + cn(t)\frac{Q(n_0(t))}{Q(n(t))} - \frac{(1 - \beta)(\beta - \gamma)n(t)}{P^{-1}(\beta)}.
\]

\[
\leq n(t)(2 + c)\left(\gamma^{1-\alpha} - \frac{(1 - \beta)(\beta - \gamma)}{P^{-1}(\beta)}\right) \leq 0,
\]

\[
\text{where (a) follows since } n_0(t) \leq \gamma n(t), \text{ while (b) follows for choice of } \gamma, \beta, c \text{ that satisfy}
\]

\[
\frac{(1 - \beta)(\beta - \gamma)}{P^{-1}(\beta)} > \gamma^{1-\alpha} \text{ and } c \geq \frac{2n(t)}{P^{-1}(\beta)}
\]

\[
\text{When } n_0(t) = 0, \text{ the OPT’s contribution is zero, and}
\]

\[
\text{where (a) follows as long as } c\left(\frac{(1 - \beta)(\beta - \gamma)}{P^{-1}(\beta)}\right) > 1. \text{ Combining 19, 21 and 23, using 5, the competitive ratio of the proposed algorithm is}
\]

\[
\frac{2 + c}{\gamma}
\]

where \(c, \beta, \gamma \text{ satisfy } 22\). Note that the bound \(24\) can be optimized by choosing the optimal value of \(\beta\) and \(\gamma\) satisfying \(22\). It is easy to see that depending on \(\alpha\), there exists a \(\beta\) satisfying \(22\).

For example, for \(\alpha = 2.3\), let \(\beta = 1/6\) and \(\gamma = 2^2\), we get a competitive ratio bound of 693 and 680, respectively, as follows. In fact for \(2 \leq \alpha \leq 3\), choosing \(\beta = 1/6\) and \(\gamma = 2^2\), the competitive ratio is at most 693. For \(\alpha = 2\), let \(\beta = 1/6\), and \(\gamma = 2^2\). Then \(\frac{(1 - \beta)(\beta - \gamma)}{P^{-1}(\beta)} = 5/6(5/36)/\sqrt{1/6} = 2.455/6(5/36) = .283\) while \(\gamma^{1-\alpha} = .167\). Thus, we have \((1 - \beta)(2^2 - \gamma) > \gamma^{1-\alpha}\) and \(c = 2/(283 - .167) = 17.24\). Thus, the competitive ratio when \(\alpha = 2\) is \(\frac{2 + c}{\gamma} = 36 \times (2 + 17.24) < 693\).

Similarly for \(\alpha = 3\), with \(\beta = 1/6\), and \(\gamma = 2^2\), \(c = 16.66\) and the competitive ratio upper bound is \(\frac{2 + c}{\gamma} = 36 \times (18.66) < 680\).

**Proof:** [Proof of Corollary 14] Proof is immediate by noting that with Problem 14, we do not have to add the energy consumption term \(P_{\text{sum}}(t)\) for checking the running condition 5, thus resulting in a two-fold decrease in the \(\kappa\) needed to satisfy 5.
5. CONCLUSIONS

In this paper, we considered an important problem of flow time minimization in data centers, where jobs have limited parallelizability, and they experience diminishing returns in time minimization in data centers, where jobs have limited parallelizability, and they experience diminishing returns when allocated additional servers. When all jobs are available at time 0, a very simple algorithm called EQUI that processes all outstanding jobs at the same speed is shown to have a constant competitive ratio that only depends on the speed-up exponent α. For the most relevant speed-up exponents of 2 and 3, the competitive ratio is at most 3. Thus, even without knowing job-sizes, and processing all of them at the same speed, EQUI is not too sub-optimal.

For the general online setting, where jobs arrive over time, we propose a LCFS type algorithm for scheduling and EQUI for speed selection, and show that its competitive ratio is a constant that only depends on the speedup exponent α. Our result overcomes fundamental difficulty found in literature where similar results were shown only in the presence of resource augmentation, where an online algorithm is allowed more resources than the optimal offline algorithm.

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APPENDIX

A.

Proof: [Proof of Lemma 4] On a departure of a job with the algorithm or the OPT, n(t,q) or n(t,q) changes for only q = 0, and since there is an integral outside, \( \int_0^\infty (n(t,q) - n(t,q))^\alpha dq \) remains the same on a departure of a job with either the algorithm or the OPT.

The pre-factor term \( P^{-1}(\frac{n(t)}{\min(n(t),p)}) \) changes though, when there is a departure of a job with the algorithm, on account of \( n(t) \rightarrow n(t) - 1 \). Consider time \( t^- \), just before a departure at time \( t \), where \( n(t) = n(t^-) - 1 \).

Case Ia: \( \min\{n(t^-),p\} = p \) and \( \min\{n(t),p\} = n(t) \). In this case, \( P^{-1}(\frac{n(t^-)}{\min(n(t^-),p)}) - P^{-1}(\frac{n(t)}{\min(n(t),p)}) = 0 \).

Case Ib: \( \min\{n(t^-),p\} = n(t^-) \) and \( \min\{n(t),p\} = n(t) \). In this case, \( P^{-1}(\frac{n(t^-)}{\min(n(t^-),p)}) = P^{-1}(1) = 0 \).

Thus, the pre-factor \( P^{-1}(\frac{n(t)}{\min(n(t),p)}) \) does not increase at any job departure. Moreover, the departure of any job with the OPT does not change the pre-factor. Since the integral is always non-negative, the assertion of the Lemma follows. \( \square \)
B.

Proof: [Proof of Lemma 12] On an arrival of a new job $j$, the ranks of all the existing jobs do not change, while for the newly arrived job $j$, $w_j(t) - w_j(t) = 0$. Hence the potential function $\Phi(t)$ does not change on arrival of any new job.

On a departure of a job with the algorithm, rank of any remaining job can only decrease, in particular by 1. Thus, if at time $t$ when job $k$ departs with the algorithm, job $j$’s ($j \in A(t^*)$) rank at time $t^*$ is either $r_j(t^*) = r_j(t)$ or $r_j(t^*) = r_j(t) - 1$. In the first case, there is no change in the potential function.

In the second case, when $r_j(t^*) = r_j(t) - 1$, first consider the case that $r_j(t) < p$ which implies that $r_j(t^*) < p$. In this subcase

$$r_j(t^*) = \frac{r_j(t)}{P^{-1}(\min\{r_j(t^*), p\}) Q(r_j(t^*))} = 1,$$

Thus, there is no change in $\Phi(t)$ in this subcase.

If instead $r_j(t) \geq p$ but $r_j(t^*) = r_j(t) - 1 < p$, then

$$r_j(t^*) = \frac{r_j(t)}{P^{-1}(\min\{r_j(t^*), p\}) Q(r_j(t^*))} = 1,$$

while $\frac{r_j(t)}{P^{-1}(\min\{r_j(t), p\}) Q(r_j(t))} = P^{-1}(r_j(t)/p) \geq 1$, since $p \leq r_j(t)$. Thus, in this case also, the potential function $\Phi$ does not increase on departure of a job with the algorithm. Similar conclusion follows if $r_j(t) > p$ and $r_j(t^*) > p$.

Since there is no discontinuity when a job departs with the OPT, hence $\Phi(t)$ does not change when a job departs with the OPT. □

Proof: [Proof of Lemma 13] From the definition of $\Phi(t)$ OPT can increase $\Phi(t)$ at time $t$ only if it processes jobs that also belong to the set $A(t)$ (outstanding jobs with the algorithm).

From Lemma 2 with $P_{\text{sum}}(t)$ being the total power used by OPT at time $t$, the sum of the speeds devoted to the set of $A(t)$ jobs of the algorithms with $n(t) = |A(t)|$ by the OPT is at most $Q(n(t))P^{-1}(P_{\text{sum}}(t))$. Moreover, since OPT contains only $n_o(t)$ jobs, sum of the speeds devoted to the $n(t)$ jobs of the algorithm is at most

$$Q(\min\{n(t), n_o(t)\})P^{-1}(P_{\text{sum}}(t)).$$

Moreover, from the definition of $\Phi(t)$, the maximum increase in $\Phi(t)$ is possible if the total speed of the OPT that it can dedicate to jobs belonging to $A(t)$ is dedicated to the single job with the largest rank among $A(t)$, i.e., the job with rank equal to $n(t)$. Thus, because of processing by the OPT, $d\Phi(t)/dt$

$$\leq cP^{-1}(\min\{n(t), p\})Q(n(t)) \times Q(\min\{n(t), n_o(t)\})P^{-1}(P_{\text{sum}}(t)).$$

If $n(t) \leq p$, then

$$d\Phi(t)/dt \leq cQ(n(t))P^{-1}(P_{\text{sum}}(t)),$$

$$\leq cQ(p)P^{-1}(P_{\text{sum}}(t)),$$

$$\leq cQ(p)P^{-1}(p),$$

$$= cp. \quad (25)$$

Otherwise, if $n(t) > p$, then

$$d\Phi(t)/dt \leq cn(t) \frac{Q(n_o(t))}{Q(n(t))}, \quad (26)$$

since $P_{\text{sum}}(t) \leq p$.

□

C.

Proof: [Proof of Lemma 5] Note that for at least max $\{n(t) - n_o(t), 0\}$ jobs belonging to $A(t)$, the corresponding terms $n(t, n_o(t), q) > 0$ in $\Phi(t)$. Thus, algorithm EQUI is decreasing work at speed $s_i(t)$ for at least max $\{n(t) - n_o(t), 0\}$ jobs. Hence, the drift $d\Phi_{\text{sum}}(t)/dt$ with respect to processing by the algorithm EQUI is $d\Phi_{\text{sum}}(t)/dt$

$$\geq c_1 P^{-1}\left(\min\{n(t), p\}\right) \left(\max\{n(t) - n_o(t), 0\}\right) s_i(t),$$

$$\geq c_1 \left(\max\{n(t) - n_o(t), 0\}\right)$$

where (a) follows since for at least max $\{n(t) - n_o(t), 0\}$ jobs, the EQUI algorithm is decreasing work at speed $s_i(t)$, while (b) follows since $s_i(t) = P^{-1}(\min\{n(t), p\})$ for all jobs i being processed by the EQUI algorithm.

From Lemma 2 we know that the sum of the speeds used by the OPT over its $n_o(t)$ jobs is at most

$$\sum_{i=1}^{n_o(t)} s_i(t) \leq Q(n_o(t))P^{-1}(P_{\text{sum}}(t)). \quad (28)$$

Using this, we bound the $d\Phi_{\text{sum}}(t)/dt$ with respect to processing by the OPT as follows

$$d\Phi_{\text{sum}}(t)/dt \leq c_1 P^{-1}\left(\min\{n(t), p\}\right) \left(\sum_{i=1}^{n_o(t)} s_i(t)\right),$$

$$\leq c_1 P^{-1}\left(\min\{n(t), p\}\right) Q(n_o(t))P^{-1}(P_{\text{sum}}(t)),$$

$$\leq c_1 P^{-1}(n(t))Q(n_o(t)),$$

$$= c_1 n(t)^{1\alpha}n_o(t)^{1-1/\alpha},$$

$$\leq c_1 \left(\frac{1}{\alpha} n(t) + c_1 \left(1 - \frac{1}{\alpha}\right) n_o(t). \quad (29)$$

where for (a) we assume that all the $n_o(t)$ jobs of the OPT are getting processed at non-zero speed (best case in terms of increasing $d\Phi_{\text{sum}}(t)/dt$), while (b) follows from (28), (c) holds when $\min\{n(t), p\} = p$ and since $P_{\text{sum}}(t) \leq p$, and (d) follows from the generalized AM-GM inequality.

For the case, when $\min\{n(t), p\} = n(t)$, following similar steps to reach (29), we get

$$d\Phi_{\text{sum}}(t)/dt \leq c_1 P^{-1}(P_{\text{sum}}(t))Q(n_o(t)),$$

$$\leq c_1 \left(\frac{1}{\alpha}\right) P_{\text{sum}}(t) + c_1 \left(1 - \frac{1}{\alpha}\right) n_o(t). \quad (30)$$

Combining (27), (29), and (30), the proof is complete. □

\footnote{For $a_i \geq 0$ and $\lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, then $\prod_{i=1}^{n} c_i \lambda_i \leq \sum_{i=1}^{n} \lambda_i c_i$.}
D. Proof: [Proof of Lemma 14] Since the algorithm executes the βn(t) jobs that have arrived most recently, the rank of job i that is being processed by the algorithm is ri(t) = n(t) − i + 1 for i = 1, . . . , βn(t). Since n0(t) ≤ γn(t), and γ < β,

\[ w_j^A(t) - w_j^B(t) > 0 \]

for at least (β − γ)n(t) jobs with the algorithm. In the worst case, the ranks of these (β − γ)n(t) jobs are (1 − β)n(t) + i − 1 for i = 1, . . . , (β − γ)n(t).

Since the speed for any of the job executed by the algorithm is s(t) = P^{-1} \left( \frac{\min(n(t), P)}{\beta n(t)} \right). Thus, the change in the potential function because of the algorithm’s dΦ(t)/dt processing is

\[
\begin{align*}
&\leq - c \sum_{i=(1-\beta)n(t)}^{(1-\beta)n(t)+1-\gamma n(t)} \frac{r_i(t)}{P^{-1}(\min(r_i(t), p))Q(r_i(t))} \\
&\quad \cdot P^{-1} \left( \frac{\min(n(t), p)}{\beta n(t)} \right),
\end{align*}
\]

(a)

\[
\begin{align*}
&\leq - c \sum_{i=(1-\beta)n(t)}^{(1-\beta)n(t)+1-\gamma n(t)} \frac{r_i(t)}{Q(n(t))P^{-1}(\beta n(t))},
\end{align*}
\]

(b)

\[
\begin{align*}
&\leq - c \sum_{i=(1-\beta)n(t)}^{(1-\beta)n(t)+1-\gamma n(t)} \frac{r_i(t)}{Q(n(t))P^{-1}(\beta n(t))},
\end{align*}
\]

(c)

\[
\begin{align*}
&\leq - c \frac{(1-\beta)(\beta-\gamma)n(t)n(t)}{\beta n(t)P^{-1}(\beta)}.
\end{align*}
\]

where (a) follows since ri(t) ≤ n(t) and

\[
\begin{align*}
&\frac{P^{-1} \left( \frac{\min(n(t), p)}{\beta n(t)} \right)}{P^{-1}(\min(r_i(t), p))} \geq \frac{1}{P^{-1}(\beta n(t))},
\end{align*}
\]

while (b) follows since Q(x)P^{-1}(x) = x, and finally (c) follows since there are (β − γ)n(t) jobs that are being executed each with rank at least (1 − β)n(t).

E. NUMERICAL RESULTS

In this section, we present simulation results for the mean flow time (per job). We first consider the special case when all jobs are available at time 0 and compare the performance of the optimal algorithm, heSRPT, and the EQUI in Fig. 1 for different values of α. We use the number of servers N = 1000, and consider 1000 jobs with size that is exponentially distributed with mean 20. For all the results, we compare the two algorithms for the same realization of random variables, and then average it out. As predicted by our results, the performance of EQUI improves compared to heSRPT as α increases. This has also been observed in [4].

Next, for the online setting, we compare the performance of the proposed algorithm with other known algorithms such as heSRPT and EQUI. For all the plots in the online setting, we use the number of servers N = 1000, and consider
Figure 2: Comparison of flow time for different algorithms with \( \alpha = 2 \).

Figure 3: Comparison of flow time for different algorithms with \( \alpha = 2.5 \).

Figure 4: Comparison of flow time for different algorithms with \( \alpha = 3 \).