Eigenfrequency spectrum analysis of bending vibrations for naturally swirled rod

V P Pavlov, V M Kudoyarova* and L R Nusratullina

Ufa State Aviation Technical University, st. K. Marcza,12, Ufa, 450000, Russian Federation
* Kudoyarova@inbox.ru

Abstract. The paper presents a numerical method for calculation of eigenfrequency vibration for naturally twisted rod considered as a first approximation model of a real blade of a gas turbine engine (GTE). The proposed method is based on the method of a fifth-degree spline developed by the authors. This method makes possible to solve fourth-order differential equations very accurately that arising in problems of transverse vibrations of rods. These research are focused on studying the more complex vibration process of a rod's spatial bending unlike the known works where bending vibrations are considered only in a single plane.

1. Introduction
To ensure the fatigue strength of GTE blades an alternating stress level should be sufficiently low in them. To eliminate the coincidence of an eigenfrequency of the GTE blade oscillations with the disturbing force frequency at a certain rotor rotation frequency, it is necessary correctly determine an eigenfrequency of GTE blade oscillation already at the design stage, especially a lowest flexural vibration frequency [1-3].

GTE blade is a naturally twisted rod with a variable cross-sectional length. In this case, for determining the frequency spectrum of blade natural oscillations we cannot use the traditional analytical formulas that are developed only for rod with a constant cross section [3, 4]. Therefore, currently the most acceptable approach is a use of numerical methods.

To date, a number of numerical methods have been developed for calculating the eigenfrequency oscillation of a rod with variable cross section [1-14], but all of these solutions considered a flat bending of rod during vibration.

In this connection, it seems very relevant to develop a numerical method that allows one to determine an eigenfrequency of oscillation for a naturally twisted rod with its actual spatial bending. Thus, this is paper solves this problem by using the fifth degree spline method developed by the authors [14-19].

2. Problem definition
A naturally twisted rod is considered as an approximate blade model of GTE axial compressor or turbine (Figure 1).
In Figure 1 Z-axis is directed in radial direction perpendicular to a turbine wheel axis of rotation. The rod’s geometry (Figure 1) is determined by its cross-sections defined in the coordinate \( XY \)-plane and by the spin function \( \alpha = \alpha(z) \) for the rod, which determines the rod’s cross-section rotation angle relatively to the static coordinate system while moving along the \( Z \)-axis.

Thus, the stated task is:

- To form differential equations describing spatial bending vibrations for naturally twisted rod with variable cross-section along the length,
- To develop a numerical method for solving these equations,
- To create software that implements the numerical method,
- To analyse the influence of the rod's twist degree on the frequencies of its own oscillations.

3. Differential equations describing transverse oscillations of a naturally twisted rod

A rod bending deformation under oscillations, we define the displacements by the \( u, v \) points of its axis, which are functions of \( z \)-coordinate and time \( t \):

\[
\begin{align*}
  u &= u(z, t), \\
  v &= v(z, t).
\end{align*}
\]

A moving of arbitrary cross-section point (Figure 1) along the axis is determined by the expression [1-3]:

\[
\begin{align*}
  w &= -\varphi_x x + \varphi_y y,
\end{align*}
\]

where \( \varphi_x \) and \( \varphi_y \) – the angles of cross-section rotation relative to the \( X \) - and \( Y \)-axes, respectively.

Taking into account (1) and Figure 1 we write:

\[
\varphi_x = \frac{\partial u}{\partial z}, \quad \varphi_y = -\frac{\partial v}{\partial z}.
\]

Linear deformation of the material \( \varepsilon_z \) along \( Z \)-axis at the point \( B \) (Figure 1) is determined by the formula [1-3]:

\[
\varepsilon_z(x,y,z) = \frac{\partial w}{\partial z} = -\frac{\partial \varphi_x}{\partial z} x + \frac{\partial \varphi_y}{\partial z} y = -\frac{\partial^2 u}{\partial z^2} x - \frac{\partial^2 v}{\partial z^2} y.
\]
Normal stress at the point $B$ is determined by known deformation $\varepsilon_z$ on the basis of Hooke's law and relations (4):
\[
\sigma_z = E\varepsilon_z = E\left(-\frac{\partial^2 u}{\partial z^2} x - \frac{\partial^2 v}{\partial z^2} y\right),
\]
where $E$ – elastic modulus of the material.

Next, the internal bending moments is being defined $M_x, M_y [3]$:
\[
\begin{align*}
M_x &= \int_A \sigma_x y dA = -\frac{\partial^2 u}{\partial z^2} E \int_A x y dA - \frac{\partial^2 v}{\partial z^2} E \int_A y^2 dA, \\
M_y &= -\int_A \sigma_x x dA = \frac{\partial^2 u}{\partial z^2} E \int_A x^2 dA + \frac{\partial^2 v}{\partial z^2} E \int_A x y dA,
\end{align*}
\]
where integration is made over the entire cross-sectional area $A$.

The differential equations of interaction between internal force factors in a rod [3] are being written:
\[
\begin{cases}
\frac{\partial Q_x}{\partial z} = -q_x, & \frac{\partial Q_y}{\partial z} = -q_y, \\
\frac{\partial M_y}{\partial z} = Q_y, & \frac{\partial M_x}{\partial z} = -Q_x.
\end{cases}
\]
Removing from (7) the transverse forces $Q_x$ and $Q_y$, we write down:
\[
\begin{align*}
\frac{\partial^2 M_x}{\partial z^2} &= -q_y, & \frac{\partial^2 M_y}{\partial z^2} &= q_x.
\end{align*}
\]

The projections of rod’s axis point acceleration on $X$- and $Y$-axis $a_x$ and $a_y$ are being defined:
\[
a_x = \frac{\partial^2 u(z,t)}{\partial t^2}, \quad a_y = \frac{\partial^2 v(z,t)}{\partial t^2}.
\]

A rod with variable mass per unit length along $Z$-axis is under consideration:
\[
\mu = \mu(z).
\]

The projections on the $X$- and $Y$-axis of inertia force per length $q_x^{\infty}$ and $q_y^{\infty}$ that acting on a rod are being defined by d'Alembert principle:
\[
q_x^{\infty} = -\mu a_x = -\mu \frac{\partial^2 u}{\partial t^2}, \quad q_y^{\infty} = -\mu a_y = -\mu \frac{\partial^2 v}{\partial t^2}.
\]

Let’s substitute (6) and (11) in (8) under the notation
\[
I_x = \int_A x^2 dA, \quad I_y = \int_A y^2 dA, \quad I_{xy} = \int_A x y dA,
\]
system of two differential equations describing own bending vibrations of a rod with varying mass per length and moments of inertia along $Z$-axis is being obtained:
\[
\begin{cases}
EI_{xy} \frac{\partial^4 u}{\partial z^4} + 2E \frac{\partial I_y}{\partial z} \frac{\partial^3 u}{\partial z^3} + E \frac{\partial^3 I_y}{\partial z^3} \frac{\partial^2 u}{\partial z^2} + E \frac{\partial I_y}{\partial z} \frac{\partial^2 I_y}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + E \frac{\partial^2 I_y}{\partial z^2} \frac{\partial^2 I_y}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + E \frac{\partial^3 I_y}{\partial z^2} \frac{\partial^3 I_y}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + E \frac{\partial^4 I_y}{\partial z^2} \frac{\partial^4 I_y}{\partial z^2} \frac{\partial^2 u}{\partial z^2} + \mu \frac{\partial^2 u}{\partial t^2} = 0, \\
EI_{xy} \frac{\partial^4 v}{\partial z^4} + 2E \frac{\partial I_y}{\partial z} \frac{\partial^3 v}{\partial z^3} + E \frac{\partial^3 I_y}{\partial z^3} \frac{\partial^2 v}{\partial z^2} + E \frac{\partial I_y}{\partial z} \frac{\partial^2 I_y}{\partial z^2} \frac{\partial^2 v}{\partial z^2} + E \frac{\partial^2 I_y}{\partial z^2} \frac{\partial^2 I_y}{\partial z^2} \frac{\partial^2 v}{\partial z^2} + E \frac{\partial^3 I_y}{\partial z^2} \frac{\partial^3 I_y}{\partial z^2} \frac{\partial^2 v}{\partial z^2} + E \frac{\partial^4 I_y}{\partial z^2} \frac{\partial^4 I_y}{\partial z^2} \frac{\partial^2 v}{\partial z^2} + \mu \frac{\partial^2 v}{\partial t^2} = 0.
\end{cases}
\]
The solution of system (13) is being looking for in a form of [4]:

\[ u = U(z) \sin \omega t, \quad v = V(z) \sin \omega t, \quad (14) \]

where \( U = U(z) \) – the form of oscillations in \( XY \)-plane; \( V = V(z) \) – the form of oscillations in \( YZ \)-plane.

The system of two differential equations that describing forms of own transverse vibrations of a rod is being obtained after substituting (14) into (13) and after following \( \sin(\omega t) \) divisions of the obtained equations:

\[
\begin{align*}
2 \frac{\partial^2 U}{\partial z^2} + 2E \frac{\partial^3 U}{\partial z \partial^2 y} + E \frac{\partial^2 I_y}{\partial z \partial^2 y} \frac{\partial^2 U}{\partial z^2} + EI_y \frac{\partial^4 U}{\partial z^4} + E \frac{\partial^2 I_y}{\partial z^2} \frac{\partial^3 U}{\partial z^3} + \mu \omega^2 U &= 0, \\
2 \frac{\partial^2 V}{\partial z^2} + 2E \frac{\partial^3 V}{\partial z \partial^2 y} + E \frac{\partial^2 I_y}{\partial z \partial^2 y} \frac{\partial^2 V}{\partial z^2} + EI_y \frac{\partial^4 V}{\partial z^4} + E \frac{\partial^2 I_y}{\partial z^2} \frac{\partial^3 V}{\partial z^3} + \mu \omega^2 V &= 0,
\end{align*}
\]

(15)

where \( \omega \) – is a circular frequency of natural rod’s oscillations, which must be determined from (15); \( U = U(z) \) and \( V = V(z) \) – the own modes of oscillation, respectively, in the \( ZX \)- and \( ZY \)-planes.

4. Fifth-degree spline method

Method based on fifth-degree spline of defect 1 is effectively used for various problems in [14-19] has been applied to solve the created system of differential equations.

The mesh \( \Delta: 0 = z_1 < z_2 < \ldots < z_N = l \) is formed on the segment \( [0, l] \) for spline of fifth-degree defect 1 [14-19], which has \( N \) nodes. On the mesh base, a spline -function of 5th degree defect 1 \( W_{5,1}(x) \) is constructed with \( N_s = N + 4 \) degrees of freedom.

In the limits of each segment \( [z_i, z_{i+1}], \ i = 1, N-1 \) a spline-function \( W_{5,1}(x) \) is polynomial of fifth degree.

\[
W_{5,1}(z) = \sum_{i=0}^{5} a_i(z-z_i)^i, \quad z \in [z_i, z_{i+1}], \quad i = 1, N-1.
\]

(16)

Parameters determining a spline are accumulated in column vector \( Q \), which consists of \( N_s = N + 4 \) spline-parameters \( Q = (q_k, \ k = 1, N+1)^T \), where:

\[
\begin{align*}
q_1 &= W_{5,1}(z_i), & q_2 &= \frac{dW_{5,1}(z_i)}{dz}, & q_3 &= \frac{d^2W_{5,1}(z_i)}{dz^2}, \\
q_4 &= \frac{d^3W_{5,1}(z_i)}{dz^3}, & q_{i+4} &= \frac{d^iW_{5,1}(z_i)}{dz^i}, & i = 1, 2, \ldots, N.
\end{align*}
\]

(17)

The spline-function \( W_{5,1}(x) \) values and its derivatives up to the fourth order inclusive are considered at the nodes of mesh \( \Delta \):

\[
f_i^{(s)} = \frac{d^iW_{5,1}(x)}{dz^i}, \quad i = 1, N, \quad s = 0, 4.
\]

(18)

Then let’s form them as column vectors:

\[
V_{df} = (f_i^{(s)}, \ i = 1, N)^T, \quad s = 0, 4.
\]

(19)

The vectors of nodal values for spline-function \( W_{5,1}(x) \) and its derivatives are determined by matrix representations with accordance to [14]:

\[
V_{df} = M_{df} Q, \quad s = 0, 4,
\]

(20)

where \( M_{df} \), \( s = 0, 4 \) – rectangular matrices of size \( N_s = N + 4 \)
which formed on the base of methodology and depending only on the mesh $\Delta$ of the spline nodes, given in [14].

5. Discrete analogue of equations describing shape of oscillations

For constructing a discrete analogue of system of differential equations (15), in accordance with (16) for each of functions $U(z)$ and $V(z)$ we introduce corresponding two splines: $W_{s,j}^{(U)}(z)$ and $W_{s,j}^{(V)}(z)$ that determined by corresponding parameter vectors $Q_U$ and $Q_V$ in the form of (17).

\[
Q_U = (q_{1i}^U, i = 1, N+4)^T, \quad Q_V = (q_{1i}^V, i = 1, N+4)^T.
\]

The values of desired functions $U(z)$, $V(z)$ and their derivatives at the spline nodes $z_i$, $i = 1, N$ are being reduced into vectors

\[
U_{dV} = \left( \frac{d^2U(z_i)}{dz^2}, \quad i = \overline{1, N} \right)^T, \quad V_{dV} = \left( \frac{d^2V(z_i)}{dz^2}, \quad i = \overline{1, N} \right)^T, \quad s = \overline{0, 4},
\]

which are being defined by the matrix relations along the analogy of (20)

\[
U_{dU} = M_{dU}Q_U, \quad U_{dV} = M_{dV}Q_V, \quad s = \overline{0, 4}.
\]

The vectors $Q_U$ and $Q_V$ are being combined into a single vector of parameters $Q_{\Sigma}$:

\[
Q_{\Sigma} = (q_{1i}^U, i = 1, 2N+8)^T
\]

at $q_{i}^U = q_{i}, \quad q_{i+N+4}^V = q_{i}^V, \quad i = 1, N+4$.

The system of linear algebraic equations is being written in a matrix form based on the above relations:

\[
AQ_{\Sigma} = 0,
\]

where $A = (A_{ij}, \quad i = \overline{1, 2N+8}, \quad j = \overline{1, 2N+8})$ – the square matrix in which the $2N$-first lines are being formed as discrete analogs of the system of differential equations (15):

\[
\begin{align*}
A_{i,j} &= EI_{ij}(z_i)M_{i,j}^{(d_4)} + 2E\frac{\partial I_{ij}(z_i)}{\partial z}M_{i,j}^{(d_2)} + E\frac{\partial^3 I_{ij}(z_i)}{\partial z^3}M_{i,j}^{(d_3)}, \\
A_{i,k} &= EI_{ij}(z_i)M_{i,j}^{(d_3)} + 2E\frac{\partial I_{ij}(z_i)}{\partial z}M_{i,j}^{(d_2)} + E\frac{\partial^3 I_{ij}(z_i)}{\partial z^3}M_{i,j}^{(d_3)} - \mu(z_i)M_{i,j}^{(d_0)}\omega^2, \\
& \quad \text{at} \quad k = j + N + 4, \quad i = 1, N, \quad j = 1, N+4, \\
A_{i,N,j} &= EI_{ij}(z_i)M_{i,j}^{(d_4)} + 2E\frac{\partial I_{ij}(z_i)}{\partial z}M_{i,j}^{(d_2)} + E\frac{\partial^3 I_{ij}(z_i)}{\partial z^3}M_{i,j}^{(d_3)} - \mu(z_i)M_{i,j}^{(d_0)}\omega^2, \\
A_{i,N,k} &= EI_{ij}(z_i)M_{i,j}^{(d_3)} + 2E\frac{\partial I_{ij}(z_i)}{\partial z}M_{i,j}^{(d_2)} + E\frac{\partial^3 I_{ij}(z_i)}{\partial z^3}M_{i,j}^{(d_3)}, \\
& \quad \text{at} \quad k = j + N + 4, \quad i = 1, N, \quad j = 1, N+4.
\end{align*}
\]

The last equations of system (17) are being formed as discrete analogue of the boundary conditions of a rod under consideration:
\[
\begin{aligned}
&U(z_i) = 0, \quad \frac{\partial U(z_i)}{\partial z} = 0, \quad M_y(z_i) = 0, \quad Q_y(z_i) = 0, \\
&V(z_i) = 0, \quad \frac{\partial V(z_i)}{\partial z} = 0, \quad M_y(z_i) = 0, \quad Q_y(z_i) = 0.
\end{aligned}
\] (29)

The system of equations (26) has a nonzero solution only if the determinant of the matrix is equal to zero:
\[
\det[A_y] = 0. \tag{30}
\]

Solving (30), we determine the eigenfrequencies of oscillation for a naturally twisted rod.

6. Results

6.1. Test problem

The rod fixed at the end \( z = 0 \), having a rectangular cross-section with dimensions \( b = 2 \cdot 10^{-2} \text{ m} \), \( h = 3 \cdot 10^{-3} \text{ m} \) with a length \( l = 0.2 \text{ m} \), shown in Figure 1 is considered. Young's modulus \( E = 11 \cdot 10^{11} \text{ Pa} \) and density \( \rho = 7.85 \cdot 10^{-3} \text{ kg/m}^3 \). The rod’s mass per unit length is \( \mu = \rho bh = 0.471 \text{ kg/m} \). The moments of inertia for a given cross-section relative to its main central axes \( \xi, \eta \) are:
\[
I_\xi = \frac{bh^3}{2} = 4.5 \cdot 10^{-11} \text{ m}^4, \quad I_\eta = \frac{hb^3}{2} = 2 \cdot 10^{-9} \text{ m}^3, \quad I_{\xi\eta} = 0. \tag{31}
\]

An orientation angle \( \alpha \) of the cross-section varies linearly along \( Z \)-axis.
\[
\alpha = \gamma z \quad \text{at} \quad \gamma = \Delta \alpha / l, \tag{32}
\]

where \( \Delta \alpha \) – is the angle of rod twisting at its full length in Figure 2.

The moments of inertia for the rod cross-sections relative to the global coordinate \( X \)- and \( Y \)-axes are being determined by the expressions [3]:
\[
\begin{aligned}
&I_x = I_\xi \cos^2 \alpha + I_\eta \sin 2\alpha + I_\eta \sin^2 \alpha, \quad I_\eta = I_\xi \sin^2 \alpha - I_\eta \sin 2\alpha + I_\eta \cos^2 \alpha, \\
&I_{\xi\eta} = I_{\xi\eta} \cos 2\alpha + \frac{I_\xi - I_\eta}{2} \sin 2\alpha.
\end{aligned} \tag{33}
\]

The expressions have been defined for calculating the values of the derivatives \( \partial I_x/\partial z, \partial I_y/\partial z, \partial I_x/\partial z, \partial I_y/\partial z, \partial^2 I_x/\partial z^2, \partial^2 I_y/\partial z^2, \partial^2 I_{\xi\eta}/\partial z^2 \) necessary for equation (15) by differentiating (33) according to (32).

Thus, there is all the necessary information to specify the system of two differential equations (15) and it remains only to use the proposed numerical method to calculate the eigenfrequencies of oscillation at various angles of twist \( \Delta \alpha \) for various oscillation modes.

6.2. Results of numerical calculation

The calculations have been made for the first five eigenfrequencies at the angles of twist \( \alpha = 0^\circ; 15^\circ; 30^\circ; 45^\circ; 60^\circ; 75^\circ; 90^\circ \).

The results are presented in the form of graph for eigenfrequency of vibrations on the angle of twist for the first five modes of oscillations in Figure 2.
Figure 2 shows that the rod's twist significantly affects the frequency spectrum of the natural bending vibrations of the rod and therefore, it can be argued that the proposed calculation method may be useful for design of real GTE compressor and turbine blades.

7. Conclusion
This paper proposed the method for calculating the natural bending vibration frequencies of naturally twisted rod that include:

- creating the system of differential equations describing spatial bending vibrations of a naturally twisted rod,
- development of the numerical method for solving the system of differential equations of spatial oscillations for a naturally twisted rod based on the fifth degree spline method,
- Computer implementation of the method for solving the problem of spatial oscillations for the naturally twisted rod with obtaining specific results showing a significant dependence of eigenfrequency vibration on rod's angle of twist.

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