THE ACTION OF THE HECKE OPERATORS ON THE COMPONENT GROUPS OF MODULAR JACOBIAN VARIETIES

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ABSTRACT. For a prime number \( q \geq 5 \) and a positive integer \( N \) prime to \( q \), Ribet proved the action of the Hecke algebra on the component group of the Jacobian variety of the modular curve of level \( Nq \) at \( q \) is “Eisenstein”, which means the Hecke operator \( T_\ell \) acts by \( \ell + 1 \) when \( \ell \) is a prime number not dividing the level. In this paper, we completely compute the action of the Hecke algebra on this component group by a careful study of supersingular points with extra automorphisms.

1. Introduction

Let \( q \geq 5 \) be a prime number, and let \( N \) be a positive integer. Let \( X_0(Nq) \) denote the modular curve over \( \mathbb{Q} \) and \( J_0(Nq) \) its Jacobian variety. For any integer \( n \), there is the Hecke operator \( T_n \) acting on \( J_0(Nq) \). Let \( \Phi_q(Nq) \) denote the component group of the special fiber \( \mathcal{J} \) of the Néron model of \( J_0(Nq) \) at \( q \). According to the theorems of Ribet [12, 13] (when \( q \) does not divide \( N \)) and Edixhoven [2] (in general), the action of the Hecke algebra on \( \Phi_q(Nq) \) is “Eisenstein.” Here by “Eisenstein” we mean the Hecke operator \( T_\ell \) acts on \( \Phi_q(Nq) \) by \( \ell + 1 \) when a prime number \( \ell \) does not divide \( Nq \).\(^1\) In this article, we compute the action of the Hecke operators \( T_\ell \) on the component group \( \Phi_q(Nq) \) when \( \ell \) divides \( Nq \) and \( q \) does not divide \( N \).

Here is an exotic example\(^2\) which leads us to this study: Let \( N = \prod_{i=1}^{\nu} p_i \) be the product of distinct prime numbers with \( \nu \geq 1 \), and let \( q \equiv 2 \) or \( 5 \) (mod 9) be an odd prime number. Assume that \( p_i \equiv 4 \) or \( 7 \) (mod 9) for all \( 1 \leq i \leq \nu \). Let \( \mathbb{T}(Nq) \) (resp. \( \mathbb{T}(N) \)) denote the \( \mathbb{Z} \)-subalgebra of \( \text{End}(J_0(Nq)) \) (resp. \( \text{End}(J_0(N)) \)) generated by all the Hecke operators \( T_n \) for \( n \geq 1 \). Let

\[
\mathfrak{m} := (3, T_{p_i} - 1, T_q + 1, T\mathcal{E} - \ell - 1 : \text{for all } 1 \leq i \leq \nu, \text{and for primes } \ell \nmid Nq) \subset \mathbb{T}(Nq)
\]

and

\[
\mathfrak{n} := (3, T_{p_i} - 1, T\mathcal{E} - \ell - 1 : \text{for all } 1 \leq i \leq \nu, \text{and for primes } \ell \nmid N) \subset \mathbb{T}(N)
\]

be Eisenstein ideals. By [18, Theorem 1.4], \( \mathfrak{m} \) is maximal. Furthermore, \( \mathfrak{n} \) is maximal if and only if \( \nu \geq 2 \).

As observed by the second author [19], the dimension of \( J_0(N)[n] \) is \( \nu \) if \( n \) is maximal, i.e., \( \nu \geq 2 \). (Here \( J_0(N)[n] := \{ x \in J_0(N)(\overline{\mathbb{Q}}) : Tx = 0 \text{ for all } T \in \mathfrak{n} \} \).) It is an extension of \( \mu_3^{2\nu} \) by \( \mathbb{Z}/3\mathbb{Z} \), and it does not contain a submodule isomorphic to \( \mu_3 \). On the other hand, the dimension of \( J_0(Nq)[m] \) is either \( 2\nu \) or \( 2\nu + 1 \). Furthermore \( J_0(Nq)[m] \) contains a submodule \( \mathcal{N} \) isomorphic to \( J_0(N)[n] \), and it also contains \( \mu_3^{2\nu} \) (which is contributed from the Shimura subgroup). As \( \mathcal{N} \) is unramified at \( q \), by Serre-Tate [15], \( \mathcal{N} \) maps injectively into \( \mathcal{J}[m] \) and it turns out that its image is isomorphic to \( \mathcal{J}^0[m] \), where \( \mathcal{J}^0 \) is the identity component of \( \mathcal{J} \). (Note that \( \Phi_q(Nq) \) is the quotient of \( \mathcal{J} \) by \( \mathcal{J}^0 \).) Since \( \mu_3^{2\nu} \) is also unramified at \( q \), it maps into \( \mathcal{J}[m] \) and therefore its

\(^{1}\)On the other hand, Ribet and Edixhoven did not proceed to compute the action of the Hecke operator \( T_p \) on \( \Phi_q(Nq) \) for a prime divisor \( p \) of the level \( Nq \) because their results were enough for their applications.

\(^{2}\)This phenomenon cannot occur when the residual characteristic is greater than 3.
image maps injectively to $\Phi_q(Nq)[m]$. (This statement is also true when $\nu = 1$.) The structure of the component group $\Phi_q(Nq)$ is known by the work of Mazur and Rapoport [8]:

$$\Phi_q(Nq) = \Phi \oplus (\mathbb{Z}/3\mathbb{Z})^{2^\nu - 1},$$

where $\Phi$ is cyclic and generated by the image of the cuspidal divisor $(0) - (\infty)$. The action of the Hecke operators on $\Phi$ is well-known (e.g. [17, Appendix A1]), and so $\Phi[m] = 0$. Therefore $(\mathbb{Z}/3\mathbb{Z})^{2^\nu - 1}[m] \neq 0$ and its dimension is at least $\nu$. Indeed it is equal to $2^{\nu - 1}$, which can easily be computed by the theorems below.

Now, we introduce our results. The first one is as follows:

**Theorem 1.1.** For a prime divisor $p$ of $N$, the Hecke operator $T_p$ acts on $\Phi_q(Nq)$ by $p$.

The key idea of the proof is that the two degeneracy maps coincide on the component group (cf. [12], [2, Lemme 2 of §4.2]).

Now, the missing one is the action of the Hecke operator $T_q$ on $\Phi_q(Nq)$. Note that $T_q$ acts on $\Phi_q(Nq)$ by an involution because the action of the Hecke algebra on $\Phi_q(Nq)$ is “$q$-new.” To describe its action more precisely, we need notation: For $N = \prod p^m$ being the prime factorization of $N$ (i.e., $n_p > 0$), let $\nu := \#\{p : p \neq 2, 3\}$ and let

$$u := \begin{cases} 
0 & \text{if } q \equiv 1 \pmod{4} \text{ or } 4 \mid N \text{ or } \exists p \equiv -1 \pmod{4} \\
1 & \text{otherwise},
\end{cases}$$

$$v := \begin{cases} 
0 & \text{if } q \equiv 1 \pmod{3} \text{ or } 9 \mid N \text{ or } \exists p \equiv -1 \pmod{3} \\
1 & \text{otherwise}.
\end{cases}$$

Suppose that $(u, v) = (0, 0)$ or $\nu = 0$. Then $\Phi_q(Nq) = \Phi$ and $T_q$ acts on $\Phi$ by $1$, where $\Phi$ is the cyclic subgroup generated by the image of the cuspidal divisor $(0) - (\infty)$ (Proposition 4.1). If $\nu \geq 1$, $\Phi_q(Nq)$ becomes isomorphic to

$$\Phi' \oplus A \oplus B,$$

where $A \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus u(2^\nu - 2)}$, $B \simeq (\mathbb{Z}/3\mathbb{Z})^{\oplus v(2^\nu - 1)}$ and $\Phi'$ is a cyclic group containing $\Phi$ and $\Phi'/\Phi \simeq (\mathbb{Z}/2^\nu \mathbb{Z})$.

**Theorem 1.2.** Assume that $(u, v) \neq (0, 0)$ and $\nu \geq 1$.

1. Suppose that $\nu = 1$. Then there are distinct subgroups $B_i \simeq \mathbb{Z}/3\mathbb{Z}$ of $B$ so that $B = \oplus B_i$.

For any $1 \leq i \leq (2^\nu - 1)$, $T_q$ acts on $B_i$ by $(-1)^i$.

2. Suppose that $u = 1$. Then there are distinct subgroups $A_i \simeq \mathbb{Z}/2\mathbb{Z}$ of $A$ so that $A = \oplus A_i$.

For any $1 \leq k \leq (2^{\nu - 1} - 2)$, $T_q$ acts on $A_{2k-1} \oplus A_{2k}$ by the matrix $(1 \; 0 \; 0 \; 1)$. In other words, if $A_{2k-1} = \{u_{2k-1}\}$ and $A_{2k} = \{u_{2k}\}$, then

$$T_q(u_{2k-1}) = u_{2k-1} + u_{2k} \quad \text{and} \quad T_q(u_{2k}) = u_{2k}.$$

For a complete description of the action of $T_q$ on each subgroups, see Section 4.

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3There are some minor errors in the paper, which are corrected by Edixhoven [2, §4.4.1]

4The structure of $\Phi_q(Nq)$ is already known by Mazur and Rapoport [8] when $N$ is square-free and prime to 6, and by Edixhoven [3, §4.4.1] in general.

5This remind us the result by Mazur [7]: when $N$ is a prime number, the kernel of the Eisenstein prime of $J_0(N)$ containing a prime number $\ell$ is completely reducible when $\ell$ is odd, and is indecomposable when $\ell = 2$. 
2. Supersingular points of $X_0(N)$

From now on, we always assume that $q \geq 5$ is a prime number and $N$ is a positive integer which is prime to $q$. Let $p$ denote a prime divisor of $N$. Let $F$ be an algebraically closed field of characteristic $q$.

Let $\Sigma(N)$ denote the set of supersingular points of $X_0(N)(F)$. Since we assume that $q \geq 5$, the group of automorphisms of supersingular points is cyclic of order 2, 4 or 6. Let

$$\Sigma_n(N) := \{s \in \Sigma(N) : \#\text{Aut}(s) = n\} \quad \text{and} \quad s_n(N) := \#\Sigma_n(N).$$

Note that $s_4(N) = u \cdot 2^v$ and $s_6(N) = v \cdot 2^v$ (cf. [2, §4.2, Lemme 1]), where $u, v$ and $\nu$ are as in Section 1. Moreover $s_2(N)$ can be computed using Eichler’s mass formula [6, Theorem 12.4.5, Corollary 12.4.6]:

$$\frac{s_2(N)}{2} + \frac{s_4(N)}{4} + \frac{s_6(N)}{6} = \frac{(q-1)Q}{24},$$

where $Q := N \prod_{p \mid N} (1 + p^{-1})$ is the degree of the degeneracy map $X_0(N) \to X_0(1)$.

In the remainder of this section, we study $\Sigma_4(N)$ and $\Sigma_6(N)$ in detail. (See also [12, §2], [14, §4] or [2, §4.2].) In the section below, we always assume that $\nu \geq 1$, i.e., there is a prime divisor $p \geq 5$ of $N$. (If $\nu = 0$ then $s_2e(N) \leq 1$ for $e = 2$ or 3, and the description is very simple.)

Let $\mathcal{E}$ be a supersingular elliptic curve with $\text{Aut}(\mathcal{E}) = \langle \sigma \rangle$, and let $C$ be a cyclic subgroup of $\mathcal{E}$ of order $N$. Assume that $q \equiv 1 \pmod{4}$ (resp. $q \equiv 1 \pmod{3}$) if $\sigma = \sigma_4$ (resp. $\sigma = \sigma_6$), where $\sigma_k$ is a primitive $k$-th root of unity.

**Proposition 2.1.** Let $\mathcal{E} = \mathbb{Z}[\sigma]$ for some $n \geq 1$ with $p \geq 5$. Suppose $\text{Aut}(\mathcal{E}, C) = \langle \sigma \rangle$. Then, there exists another cyclic subgroup $D$ of order $N$ such that $\mathcal{E}[N] \simeq C \oplus D$. Moreover, $\text{Aut}(\mathcal{E}, D) = \langle \sigma \rangle$ and $\langle \mathcal{E}, C \rangle$ is not isomorphic to $\langle \mathcal{E}, D \rangle$.

**Proof.** Here, we closely follow the argument in the proof of Proposition 1 in [12, §2].

Let $R$ be the subring $\mathbb{Z}[\sigma]$ of $\text{End}(\mathcal{E}, C)$. Since $\text{Aut}(\mathcal{E}, C) = \langle \sigma \rangle$, $p \equiv 1 \pmod{4}$ (resp. $p \equiv 1 \pmod{3}$) if $\sigma = \sigma_4$ (resp. $\sigma = \sigma_6$). Therefore $p$ splits completely in $R$. Note that $R = \mathbb{Z}[\sigma]$ is a principal ideal domain and therefore

$$R/pR \simeq R/\gamma R \oplus R/\delta R \simeq \delta R/pR \oplus \gamma R/pR$$

with $p = \gamma \delta$. Moreover,

$$R/NR = R/p^nR \simeq R/\gamma^nR \oplus R/\delta^nR \simeq \delta^nR/NR \oplus \gamma^nR/NR.$$ 

Note that $\mathcal{E}[N]$ is a free of rank 1 module over $R/NR$ by the action of $R$ on $\mathcal{E}$. We may identify $C$ with the quotient $I/NR$ for some ideal $I$ of $R$ containing $N$ if we fix an $R$-isomorphism between $\mathcal{E}[N]$ and $R/NR$. Thus, $I = \delta^nR$ or $\gamma^nR$. Suppose that $I = \delta^nR$. Then, by the fixed isomorphism $C = \mathcal{E}[\gamma^n]$. Let $D := \mathcal{E}[\delta^n]$ so that its corresponding ideal is $\gamma^nR$. Then, $\mathcal{E}[N] \simeq C \oplus D$.

Moreover since $\gamma^nR$ is also an ideal of $R$, $D$ is also stable under the action of $\sigma$. In other words, $\text{Aut}(\mathcal{E}, D) = \langle \sigma \rangle$.

Since $\text{Aut}(\mathcal{E}) = \langle \sigma \rangle$ and $\sigma(C) = C$, $\langle \mathcal{E}, C \rangle$ cannot be isomorphic to $\langle \mathcal{E}, D \rangle$. \hfill \Box

From now on, we use the same notation as in the proof of Proposition 2.1.

**Definition 2.2.** By the above formulas for every $n \geq 1$ and $p \equiv 1 \pmod{4}$ (resp. $p \equiv 1 \pmod{3}$), there are precisely two cyclic subgroups $C, D$ of $\mathcal{E}$ of order $p^n$ such that $\text{Aut}(\mathcal{E}, C) = \text{Aut}(\mathcal{E}, D) = \langle \sigma \rangle$ (and $\mathcal{E}[p^n] \simeq C \oplus D$) if $\sigma = \sigma_4$ (resp. if $\sigma = \sigma_6$). Thus, for each $n \geq 1$ we define $C_{p^n}$ and $D_{p^n}$ by

$$C_{p^n} := \mathcal{E}[\gamma^n] \quad \text{and} \quad D_{p^n} := \mathcal{E}[\delta^n].$$
Proposition 2.3. For each $n \geq 1$, $\mathcal{C}_{p^n+1}[p^n] = \mathcal{C}_{p^n}$ and $\mathcal{D}_{p^n+1}[p^n] = \mathcal{D}_{p^n}$.

Proof. By the fixed $R$-isomorphism $\iota$ between $\mathcal{E}[p^{n+1}]$ and $R/p^{n+1}R$, we identify $\mathcal{C}_{p^n+1}$ with $I/p^{n+1}R$, where $I = \delta^{n+1}R$. As $I$ is an ideal of $R$, $\gamma I = p(\delta^n R) \subset I$ and $I/\gamma I \simeq R/\gamma R \simeq \mathbb{Z}/p\mathbb{Z}$. Therefore

$$\mathcal{C}_{p^n+1}[p^n] \xrightarrow{\iota} (I/p^{n+1}R)[p^n] = \gamma I/p^{n+1}R \cong (\delta^n R)/p^nR,$$

which corresponds to $\mathcal{C}_{p^n}$. Similarly, we prove that $\mathcal{D}_{p^n+1}[p^n] = \mathcal{D}_{p^n}$, and the proposition follows.

$\square$

Let $N = Mp^n$ with $(6M, p) = 1$ and $n \geq 1$. Let $L$ be a cyclic subgroup of $\mathcal{E}$ of order $M$.

Proposition 2.4. Suppose that $\text{Aut}(\mathcal{E}, \mathcal{C}_{p^n+1}, L) = \langle \sigma \rangle$. Then, there is an isomorphism between $(\mathcal{E}/\mathcal{C}_p, \mathcal{C}_{p^n+1}/\mathcal{C}_p, (L \oplus \mathcal{C}_p)/\mathcal{C}_p))$ and $(\mathcal{E}, \mathcal{C}_{p^n}, L)$.

Proof. We mostly follow the idea of the proof of Proposition 2 in [12, §2].

The endomorphism $\gamma$ sends $\mathcal{E}[\gamma^{n+1}] = \mathcal{C}_{p^n+1}$ to $\mathcal{E}[\gamma^n] = \mathcal{C}_{p^n}$, and $L$ to itself (because $L \cap \mathcal{E}[p] = 0$). Now we denote by $\overline{\gamma}$ the map $\mathcal{E}/\mathcal{C}_p \to \mathcal{E}$ induced by $\gamma$. Note that $\overline{\gamma}$ is an isomorphism because $\mathcal{C}_p$ is $\mathcal{E}[\gamma]$, the kernel of $\gamma$. By the above consideration, this isomorphism $\overline{\gamma}$ sends $(\mathcal{C}_{p^n+1}/\mathcal{C}_p, (L \oplus \mathcal{C}_p)/\mathcal{C}_p))$ to $(\mathcal{C}_{p^n}, L)$ because $\mathcal{C}_{p^n+1}/\mathcal{C}_p$ and $L \oplus \mathcal{C}_p)/\mathcal{C}_p$ are the images of $\mathcal{C}_{p^n+1}$ and $L$ by the quotient map $\mathcal{E} \to \mathcal{E}/\mathcal{C}[p]$, respectively. Therefore $\overline{\gamma}$ gives rise to the desired isomorphism between triples.

$\square$

Corollary 2.5. The map $(\mathcal{E}, C, L) \to (\mathcal{E}/C[p^n], L)$ induces a bijection between $\Sigma_{2e}(Np)$ and $\Sigma_{2e}(N)$, where $\sigma = \sigma_{2e}$. Moreover if $(\mathcal{E}, C, L) \in \Sigma_{2e}(Np)$, we have

$$(\mathcal{E}, C[p^n], L) \simeq (\mathcal{E}/C[p], C/C[p], (L \oplus C[p])/C[p]).$$

The corollary tells us that two degeneracy maps $\alpha_p$ and $\beta_p$ in Section 3 coincide on $\Sigma_{2e}(Np)$, which is a generalization of [2, §4.2, Lemme 2].

Proposition 2.6. Suppose that $\text{Aut}(\mathcal{E}, \mathcal{C}_p, L) = \langle \sigma \rangle$. Then, $\text{Frob}(\mathcal{E}) = \mathcal{E}$ and $\text{Frob}(\mathcal{C}_p) = \mathcal{D}_{p^n}$, where $\text{Frob}$ is the Frobenius morphism in characteristic $q$. Furthermore, $\text{Frob}^2(\mathcal{E}, \mathcal{C}_p, L) = (\mathcal{E}, \mathcal{C}_p, L)$.

Proof. Since $\mathcal{E}$ is isomorphic to the reduction of the elliptic curve with $j$-invariant 1728 (resp. 0) if $\sigma = \sigma_4$ (resp. $\sigma = \sigma_6$), the Frobenius morphism is an endomorphism of $\mathcal{E}$ (cf. [16, Chapter V, Examples 4.4 and 4.5]). Moreover, the Frobenius morphism and $\sigma$ generates $\text{End}(\mathcal{E})$, which is a quaternion algebra. (Note that the degree of the Frobenius morphism is $q$.) Since $\text{End}(\mathcal{E})$ is a quaternion algebra, we have

$$\sigma \circ \text{Frob} = \text{Frob} \circ \overline{\sigma} = \text{Frob} \circ \overline{\sigma}^{-1},$$

where $\overline{\sigma}$ denotes the complex conjugation in $R = \mathbb{Z}[\sigma]$. Analogously, we have

$$\gamma \circ \text{Frob} = \text{Frob} \circ \overline{\gamma} = \text{Frob} \circ \overline{\gamma}.$$

Since $\sigma(\text{Frob}(\mathcal{C}_p)) = \text{Frob}(\sigma^{-1}(\mathcal{C}_p)) = \text{Frob}(\mathcal{C}_p)$, $\text{Frob}(\mathcal{C}_p)$ is also stable under the action of $\sigma$. Moreover $\mathcal{C}_p$ does not intersect with the kernel of $\text{Frob}$. Thus, $\text{Frob}(\mathcal{C}_p)$ is either $\mathcal{C}_p$ or $\mathcal{D}_{p^n}$. As an endomorphism of $\mathcal{E}$, $\gamma$ sends $\mathcal{C}_p$ (resp. $\mathcal{D}_p$) to $\mathcal{C}_{p^n-1}$ (resp. $\mathcal{D}_{p^n}$). Similarly, $\delta$ maps $\mathcal{C}_p$ (resp. $\mathcal{D}_p$) to $\mathcal{C}_{p^n}$ (resp. $\mathcal{D}_{p^n-1}$). Therefore if $\text{Frob}(\mathcal{C}_p) = \mathcal{C}_p$, then

$$\gamma \circ \text{Frob}(\mathcal{C}_p) = \gamma(\mathcal{C}_p) = \mathcal{C}_{p^n-1} \quad \text{and} \quad \text{Frob} \circ \delta(\mathcal{C}_p) = \text{Frob}(\mathcal{C}_p) = \mathcal{C}_{p^n},$$

which is a contradiction. Thus, we get $\text{Frob}(\mathcal{C}_p) = \mathcal{D}_{p^n}$.

Since every supersingular point can be defined over $\mathbb{F}_{q^2}$, the quadratic extension of $\mathbb{F}_q$, $\text{Frob}^2$ acts trivially on $\Sigma(N)$ (cf. [13, Remark 3.5.b]), which proves the last claim.

$\square$
Remark 2.7. By taking $H = (\mathbb{Z}/N\mathbb{Z})^*$ in Lemma 1 of [14], we can obtain a similar result if we show that the Atkin-Lehner style involution in [14, §4] is equal to the Frobenius morphism.

3. The action of $T_p$ on the component group

Before discussing the action of the Hecke operators on the component group, we study it on the group of divisors supported on supersingular points, which we denote by $\text{Div}(\Sigma(N))$.

Let $N = Mp^n$ with $(M, p) = 1$ and $n \geq 1$, and assume that $(N, q) = 1$. Let $\alpha_p, \beta_p : X_0(Npq) \to X_0(Nq)$ denote two degeneracy maps of degree $p$, defined by

$$
\alpha_p(E, C, L) := (E, C[p^n], L) \quad \text{and} \quad \beta_p(E, C, L) := (E/C[p], C/C[p], (L + C[p])/C[p]),$

where $C$ (resp. $L$) denotes a cyclic subgroup of order $p^{n+1}$ (resp. $Mq$) in an elliptic curve $E$ (cf. [9, §13]). Let $T_p$ and $\xi_p$ be two Hecke correspondences defined by

$$
\xymatrix{ X_0(Npq) & X_0(Nq) \ar[l]_{\alpha_p} \ar[r]^{\beta_p} \ar@{<->}[d]_{\xi_p} & X_0(Nq), \ar[l]_{T_p} }
$$

By pullback, the Hecke correspondence $T_p$ (resp. $\xi_p$) induces the Hecke operator $T_p := \beta_{p,*} \circ \alpha_p^*$ (resp. $\xi_p := \alpha_{p,*} \circ \beta_p^*$) on $J_0(Nq)$.

The same description of the Hecke operator $T_p$ on $\text{Div}(\Sigma(N))$ as above works. In other words, we have two degeneracy maps of degree $p$, defined by

$$
\alpha_p(E, C, L) := (E, C[p^n], L) \quad \text{and} \quad \beta_p(E, C, L) := (E/C[p], C/C[p], (L + C[p])/C[p]),
$$

where $C$ (resp. $L$) denotes a cyclic subgroup of order $p^{n+1}$ (resp. $M$) in a supersingular elliptic curve $E$ over $\mathbb{F}$. These maps induce the maps on their divisor groups:

$$
\text{Div}(\Sigma(N)) \xrightarrow{\alpha_p^*} \text{Div}(\Sigma(Np)) \xrightarrow{\beta_{p,*}} \text{Div}(\Sigma(N))
$$

and the Hecke operator $T_p$ (resp. $\xi_p$) can be defined by $\beta_{p,*} \circ \alpha_p^*$ (resp. $\alpha_{p,*} \circ \beta_p^*$). (For the details when $n = 0$, see [13, §3], [11, p. 18–22], [2, §4.1] or [3, §7]. By the same method, we get the above description without further difficulties.)

Now, let $\Phi_q(Nq)$ denote the component group of the special fiber $\mathcal{J}$ of the Néron model of $J_0(Nq)$ at $q$. To compute the action of $T_p$ on it, we closely follow the method of Ribet (cf. [12], [13, §2, 3], [2, §1]). Since $N$ is not divisible by $q$, the identity component $\mathcal{J}^0$ of $\mathcal{J}$ is a semi-abelian variety by Deligne-Rapoport [1] and Raynaud [10]. Moreover, $\mathcal{J}^0$ is an extension of $J_0(Nq)_{\mathbb{F}} \times J_0(Nq)_\mathbb{F}$ by $\mathcal{T}$, the torus of $\mathcal{J}^0$. Let $\mathcal{X}$ be the character group of the torus $\mathcal{T}$. By Grothendieck, there is a (Hecke-equivariant) monodromy exact sequence [4] (see also [13, §2, 3], [11], or [5, §4]),

$$
0 \to \mathcal{X} \to \text{Hom}(\mathcal{X}^t, \mathbb{Z}) \to \Phi_q(Nq) \to 0.
$$

Here $\mathcal{X}^t$ denotes the character group corresponding to the dual abelian variety of $J_0(Nq)$, which is equal to $J_0(Nq)$. Namely, $\mathcal{X}^t = \mathcal{X}$ as sets, but the action of the Hecke operator $T_t$ on $\mathcal{X}^t$ is equal to the action of its dual $\xi_t$ on $\mathcal{X}$ (cf. [12], [13, §3] and [3, §7]). Note that $\mathcal{X}$ is the group of degree 0 elements in $\mathbb{Z}^{\Sigma(N)}$. For $s,t \in \Sigma(N)$, let $e(s) := \#\text{Aut}(s)$ and

$$
\phi_s(t) := \begin{cases} e(s) & \text{if } s = t, \\ 0 & \text{otherwise}, \end{cases}
$$

6every elliptic curve isogenous to a supersingular one is also supersingular
and extends via linearity, i.e., $\phi_s(\sum a_i t_i) = \sum a_i \phi_s(t_i)$. Then, $\iota(s-t) = \phi_s - \phi_t$. Note also that $\text{Hom}(\mathbb{Z}\Sigma(N), \mathbb{Z})$ is generated by $\psi_s := 1/e(s)\phi_s$, and $\text{Hom}(\mathcal{X}, \mathbb{Z})$ is its quotient by the relation:

$$\sum_{s \in \Sigma(N)} \psi_s = \sum_{s \in \Sigma(N)} e(s) \phi_s = 0.$$ 

(This is the minimal relation to make $\sum a_w \psi_w$ vanish for all the divisors of the form $s-t$, which are the generators of $\mathcal{X}$.) For more details, see [13, §2, 3] or [11].

In conclusion, the component group $\Phi_q(Nq)$ is isomorphic to

$$\text{Hom}(\mathbb{Z}\Sigma(N), \mathbb{Z})/R,$$

where $R$ is the set of relations:

$$(3.1) \quad R = \{e(s)\psi_s = e(t)\psi_t \text{ for any } s, t \in \Sigma(N), \sum_{t \in \Sigma(N)} \psi_t = 0\}.$$ 

Let $\Psi_s$ denote the image of $\psi_s$ by the natural projection $\text{Hom}(\mathbb{Z}\Sigma(N), \mathbb{Z}) \to \Phi_q(Nq)$. The Hecke operator $T_p$ acts on $\text{Hom}(\mathbb{Z}\Sigma(N), \mathbb{Z})$ via the action of $\xi_p$ on $\text{Div}(\Sigma(N))$, i.e.,

$$T_p(\psi_s)(t) := \psi_s(\xi_p(t)) = \psi_s(\alpha_{p,*} \circ \beta_p^*(t)).$$

For $s \in \Sigma(N)$, we temporarily denote $\alpha_{p,*}(s) = \sum_{i=1}^p A^i(s)$ and $\beta_p^*(s) = \sum_{i=1}^p B^i(s)$ (allowing repetition). We note that if $e(s) = 1$ then there is no repetition, i.e., $A^i(s) \not\approx A^j(s)$ and $B^i(s) \not\approx B^j(s)$ if $i \not= j$. If $e(s) = e > 1$, then after renumbering the index properly we have

$$e(A^i(s)) = 1 \text{ for } 1 \leq i \leq p-1 \quad \text{and} \quad e(A^p(s)) = e.$$ 

Moreover, we have

$$A^{e(k-1)+1}(s) \simeq \cdots \simeq A^{ek}(s) \text{ for } 1 \leq k \leq \frac{p-1}{e}, \text{ and } A^i(s) \not\approx A^j(s) \text{ if } \left\lfloor \frac{i}{e} \right\rfloor \neq \left\lfloor \frac{j}{e} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$. This can be seen as follows: Let $\sigma = \sigma_{2e}$, and let $s$ represent a pair $(\mathcal{E}, C)$, where $C$ is a cyclic subgroup of $E$ of order $N$. Since $e(s) = e$, $\sigma(C) = C$. Suppose that $s' \in \Sigma(Np)$ with $\alpha_{p,*}(s') = s$. Then $s'$ represents a pair $(\mathcal{E}, D)$ with $D[N] = C$. If $\sigma(D) = D$, then $\text{Aut}([[\mathcal{E}, D]]) = \langle \sigma \rangle$ and $(\mathcal{E}, D) \not\simeq (\mathcal{E}, D')$ if $D \not= D'$. (Note that there is a unique such $D$.) On the other hand, if $\sigma(D) \not= D$ then

$$(\mathcal{E}, D) \simeq (\mathcal{E}, \sigma(D)) \simeq \cdots \simeq (\mathcal{E}, \sigma^{e-1}(D)) \simeq (\mathcal{E}, \sigma^e(D)) = (\mathcal{E}, D)$$

and $\text{Aut}([[\mathcal{E}, D]]) = \{\pm 1\}$. Thus, we can rearrange $A^i(s)$ as above. (Note that this can only be possible when $p \equiv 1 \pmod{2e}$, which is indeed true because $e(s) = e$.)

Now, we claim that $\phi_s(\alpha_{p,*}(t)) = \phi_t(\alpha_{p,*}(s))$. Indeed, $\phi_s(\alpha_{p,*}(t))$ is nonzero if and only if $t \in \{A^1(s), \ldots, A^p(s)\}$. So, it suffices to show this equality when $t \in \{A^1(s), \ldots, A^p(s)\}$. If $e(s) = 1$, then there is no repetition and the claim follows clearly (both are 1). Now, let $e(s) = e > 1$. If $e(t) = 1$, then $t = A^i(s)$ for some $1 \leq i \leq p-1$. Since the number of repetition of $t = A^i(s)$ in $\{A^1(s), \ldots, A^p(s)\}$ is $e$, the above equality holds. If $e(t) = e$, then $t = A^p(s)$ and $\phi_s(\alpha_{p,*}(t)) = e = \phi_t(\alpha_{p,*}(s))$, as claimed. Analogously, we have

$$\phi_t(\beta_{p,*}(s)) = \phi_s(\beta_{p,*}(t)).$$

More generally, we get

$$\phi_s(\alpha_{p,*} \circ \beta_p^*(t)) = \sum_{i=1}^p \phi_s(\alpha_{p,*}(B^i(t))) = \sum_{i=1}^p \sum_{j=1}^p \phi_{B^i(t)}(A^j(s)) = \sum_{j=1}^p \sum_{i=1}^p \phi_{A^j(s)}(B^i(t)) = \sum_{j=1}^p \phi_{A^j(s)}(\beta_p^*(t)) = \sum_{j=1}^p \phi_t(\beta_{p,*}(A^j(s))) = \phi_t(\alpha_{p,*}(s)) = \phi_t(T_p(s)).$$
If we set $T_p(s) = \sum s_i$, then $\phi_t(T_p(s)) = \sum \phi(s_i(t) = \sum e(s_i))$ and hence for any $t \in \Sigma(N)$,

$$e(s)T_p(\psi_s)(t) = \phi(s)(\alpha_{p, *} \circ \beta_p^*(t)) = \phi_t(T_p(s)) = e(s_i)\psi_s(t).$$

In other words, we get

$$(3.2) \quad T_p(\Psi_s) = \frac{1}{e(s)} \sum e(s_i)\Psi_{s_i}.$$ 

We can also define the action of $T_p$ on the component group via functorialities. Namely, let

$$\Phi_q(Nq) \xrightarrow{\alpha_p} \Phi_q(Npq) \xrightarrow{\alpha_{p, *}} \Phi_q(Nq).$$

denote the maps functorially induced from the degeneracy maps\(^7\). Then, as before $T_p := \beta_{p, *} \circ \alpha_p^*$. Note that since the degrees of $\alpha_p$ and $\beta_p$ are $p$, we have $\alpha_{p, *} \circ \alpha_p^* = \beta_{p, *} \circ \beta_p^* = p$.

**Lemma 3.1.** $\alpha_{p, *} = \beta_{p, *}$ on $\Phi_q(Npq)$.

**Proof.** For $s \in \Sigma_2(Npq)$ with $e = 2$ or $3$, $\alpha_p(s) = \beta_p(s)$ by Remark 2.5, and hence $\alpha_{p, *}(\Psi_s) = \beta_{p, *}(\Psi_s)$. For $s \in \Sigma_2(Npq)$, let $\alpha_p(s) = t$ and $\beta_p(s) = w$. Then, $\alpha_{p, *}(\Psi_s) = e(t)\Psi_t = e(w)\Psi_w = \beta_{p, *}(\Psi_s)$. In other words, for any $s \in \Sigma(Npq)$, $\alpha_{p, *}(\Psi_s) = \beta_{p, *}(\Psi_s)$. Since $\Psi_s$’s generate $\Phi_q(Npq)$, the result follows.

In fact, Theorem 1.1 is an easy corollary of the above lemma.

**Proof of Theorem 1.1.** Since $\alpha_{p, *} = \beta_{p, *}$ on $\Phi_q(Npq)$, we have

$$T_p(\Psi_s) = \beta_{p, *}(\alpha_p(\Psi_s)) = \alpha_{p, *}(\Psi_s) = p\Psi_s,$$

which implies the result. \qed

4. THE ACTION OF $T_q$ ON THE COMPONENT GROUP

In this section, we provide a complete description of the action of $T_q$ on the component group $\Phi_q(Nq)$. See Propositions 4.2, 4.3 and 4.4, which imply Theorem 1.2.

Note that the Hecke operator $T_q$ acts on $\Sigma(N)$ by the Frobenius morphism [13, Proposition 3.8], and the same is true for $\xi_q$. Since the Frobenius morphism is an involution on $\Sigma(N)$ (cf. Proposition 2.6), we have

$$T_q(\psi_s)(t) = \psi_s(\xi_q(t)) = \psi_s(\text{Frob}(t)) = \psi_{\text{Frob}(s)}(t) \text{ for any } t \in \Sigma(N),$$

which implies that $T_q(\psi_s) = \psi_{\text{Frob}(s)}$.

From now on, if there is no confusion we remove $(N)$ from the notation for simplicity. Let $n := \frac{(q-1)q}{12}$ (which is not necessarily an integer), and let $\Phi$ denote the cyclic subgroup of $\Phi_q(Nq)$ generated by $\Psi_s$ for a fixed $s \in \Sigma_2$. (Note that this $\Phi$ is the same as that of Mazur and Rapoport [8], namely, $\Phi$ is equal to the cyclic subgroup generated by the image of the cuspidal divisor $(0) - (\infty)$.)

\(^7\)If $a_p^*(s) = \sum t_j$ then $\alpha_p^*(\Psi_s) = \sum \Psi_{t_j}$ and if $\alpha_p(t) = s$ then $\alpha_{p, *}(\Psi_t) = e(s)/e(t)\Psi_s$; and similarly for $\beta_p^*$ and $\beta_{p, *}$.
4.1. Case 1: \((u,v) = (0,0)\) or \(\nu = 0\).
Let \(e = 1\) if \((u,v) = (0,0)\) and \(e = 2u + 3v\) if \((u,v) \neq (0,0)\) and \(\nu = 0\). If \((u,v) = (0,0), s_2 = n\) and \(s_4 = s_6 = 0\). If \((u,v) \neq (0,0)\) and \(\nu = 0\), then \(s_{2e} = 1\) and \(s_2 = \frac{2n - 1}{e}\). (Note that \(s_2\) is an integer but \(n\) is not.)

**Proposition 4.1.** The component group \(\Phi_q(Nq)\) is equal to \(\Phi\), which is cyclic of order \(en\). The Hecke operator \(T_q\) acts on it by 1.

**Proof.** First, we assume that \((u,v) = (0,0)\). Then for any \(s \in \Sigma = \Sigma_2, \Psi_s = \Psi_s\). Therefore \(\Phi_q(Nq) = \Phi\) and \(n\Psi_s = \sum_{s \in \Sigma} \Psi_s = 0\). Moreover, \(T_q(\Psi_s) = \Psi_{s'} = \Psi_s\), where \(s' = \text{Frob}(q)\).

Now, we assume that \((u,v) \neq (0,0)\) and \(\nu = 0\). In this case, either \(N = 2q\) (with \((u,v) = (1,0)\) and \(e = 2\)) or \(N = 3q\) (with \((u,v) = (0,1)\) and \(e = 3\)). In each case, let \(z \in \Sigma_{2e}\). Then
\[
\sum_{s \in \Sigma_2} \Psi_s + \Psi_z = s_2 \Psi_s + \Psi_z = 0 \quad \text{and} \quad \Psi_s = e\Psi_z.
\]
Therefore the component group is generated by \(\Psi_z\), and its order is \((es_2 + 1) = en\). Since \(en = es_2 + 1\) is prime to \(e\), this group is also generated by \(\Psi_z = e\Psi_z\). (In fact, \(\Psi_z = -s_2 \Psi_z\).) Moreover we have \(T_q(\Psi_z) = \Psi_z\) as above. \(\square\)

4.2. Case 2: \((u,v) = (0,1)\) and \(\nu \geq 1\).
In this case, \(s_4 = 0, s_6 = 2^s,\) and \(s_2 = \frac{3n - 2^{2s}}{3}\). Let \(\Sigma_0 := \{t_1, t_2, \ldots, t_{2^v}\}\). Here we assume that \(\text{Frob}(t_{2k-1}) = t_{2k}\) for \(1 \leq k \leq 2^{v-1}\). \(^8\) Let \(t := t_{2^v-1}\) and \(t' := t_{2^v}\).

**Proposition 4.2.** The component group \(\Phi_q(Nq)\) decomposes as follows:
\[
\Phi_q(Nq) = \bigoplus_{i=0}^{2^{v-1}} B_i =: B_0 \oplus B,
\]
where \(B_0 = \Phi\) is cyclic of order \(3n\), and for \(1 \leq i \leq 2^v - 1, B_i\) is cyclic of order \(3\). For \(1 \leq k \leq 2^{v-1}, B_{2k-1}\) and \(B_{2k}\) are generated by
\[
\nu_{2k-1} := \Psi_{t_{2k-1}} - \Psi_{t_{2k}} \quad \text{and} \quad \nu_{2k} := \Psi_{t_{2k-1}} + \Psi_{t_{2k}} - \Psi_t - \Psi_{t'}, \text{ respectively.}
\]
The Hecke operator \(T_q\) acts on \(B_i\) by \((-1)^{k+1}\).

**Proof.** Note that \(\Psi_s = 3\Psi_{s_2} = 3\Psi_{s_3}\) for all \(i, j\) and \(\sum_{i=1}^{2^v} \Psi_{t_i} + s_2 \Psi_s = 0\). Therefore \(\Phi_q(Nq)\) is generated by \(\Psi_{t_i}\) for \(1 \leq i \leq 2^v - 1\). The order of each group \((\Psi_{t_i})\) is \(9n\) because
\[
9n \Psi_{t_i} = 3s_2 (3\Psi_{t_i}) + \sum_{i=1}^{2^v} 3\Psi_{t_i} = 3 \left( \sum_{s \in \Sigma_2} \Psi_s + \sum_{i=1}^{2^v} \Psi_{t_i} \right) = 0,
\]
and \(9n\) is the smallest positive integer to make this happen. Moreover \(\langle \Psi_{t_i} \rangle \cap \langle \Psi_{t_j} \rangle\) is of order \(3n\) for any \(i \neq j\). Since \(3n = 3s_2 + 2^v\) is prime to 3, we can decompose the component group into
\[
\langle 3\Psi_t \rangle \oplus \left( \langle (3s_2 + 2^v) \Psi_t \rangle \bigoplus_{i=1}^{2^{v-2}} \langle \Psi_{t_1} - \Psi_{t_i} \rangle \right).
\]
Since \(\Psi_s = 3\Psi_{t_i} = 3\Psi_{t_3}\) for any \(i\) and \(\sum_{i=1}^{2^v} \Psi_{t_i} = -3s_2 \Psi_t,\) we have
\[
\Psi_{2k-1} - \Psi_t = 2\nu_{2k-1} + 2\nu_{2k} + \nu_{2^{v-1}};
\]
\[
\Psi_{2k} - \Psi_t = \nu_{2k-1} + 2\nu_{2k} + \nu_{2^{v-1}};
\]
\[
(3s_2 + 2^v) \Psi_t = \sum_{i=1}^{2^{v-1}} (\Psi_t - \Psi_{t_i}) = -\sum_{k=1}^{2^{v-1}} \nu_{2k} - (-1)^v \nu_{2^{v-1}}.
\]
\(^8\)By Proposition 2.6, we know that Frob is an involution of \(\Sigma_0\) without fixed points.
Therefore the decomposition in the proposition is isomorphic to (4.2). The action of $T_q$ on each $B_i$ is obvious from its construction. □

4.3. Case 3: $(u, v) = (1, 0)$ and $\nu \geq 1$.

Note that $s_4 = 2^\nu$, $s_6 = 0$, and $s_2 = n - 2^\nu - 1$. Let $\Sigma_4 = \{w_1, w_2, \ldots, w_2^\nu\}$. As before, we assume that $\text{Frob}(w_2k-1) = w_2k$ for $1 \leq k \leq 2^\nu - 1$. Let $w := w_2v - 1$ and $w' := w_2v$.

**Proposition 4.3.** The component group $\Phi_q(Nq)$ decomposes as follows:

$$\Phi_q(Nq) = \bigoplus_{i=0}^{2^\nu-2} A_i = A_0 \oplus A,$$

where $A_0$ is cyclic of order $4n$ generated by $\Psi_w$, and for $1 \leq i \leq 2^\nu - 2$, $A_i$ is cyclic of order 2. For $1 \leq k \leq 2^\nu - 1 - 2$, $A_{2k-1}$ and $A_{2k}$ are generated by

$$u_{2k-1} := \Psi_{w_{2k-1}} - \Psi_w \text{ and } u_{2k} := \Psi_{w_{2k}} - \Psi_w - \Psi_{w'},$$

respectively.

And $A_{2^\nu-3}$ and $A_{2^\nu-2}$ are generated by

$$u_{2^\nu-3} := \Psi_{w_{2^\nu-3}} - \Psi_w \text{ and } u_{2^\nu-2} := \Psi_{w_{2^\nu-3}} - \Psi_{w_{2^\nu-2}},$$

respectively.

Moreover, the action of the Hecke operator $T_q$ on each group as follows:

$$T_q(\Psi_w) = (1 + 2n)\Psi_w + \sum_{i=1}^{2^\nu-1-1} u_{2i};$$

$$T_q(u_{2k-1}) = u_{2k-1} + u_{2k} \text{ and } T_q(u_{2k}) = u_{2k} \text{ for } 1 \leq k \leq 2^\nu - 1 - 2;$$

$$T_q(u_{2^\nu-3}) = 2n\Psi_w + u_{2^\nu-3} + \sum_{i=1}^{2^\nu-1-2} u_{2i} \text{ and } T_q(u_{2^\nu-2}) = u_{2^\nu-2}.$$

**Proof.** The argument in Proposition 4.2 applies mutatis mutandis. For instance, when $\nu \geq 2$ an isomorphism between $A_0 \bigoplus_{i=1}^{2^\nu-2} \langle \Psi_{w_i} - \Psi_w \rangle$ and $A_0 \oplus A$ can be given by the following data: for $1 \leq k \leq 2^\nu - 1 - 2$,

$$\Psi_{w_{2k}} - \Psi_w = u_{2k} + u_{2k-1} + (\Psi_{w'} - \Psi_w) \text{ and } \Psi_{w} - \Psi_{w'} = 2n\Psi_w + \sum_{i=1}^{2^\nu-1-1} u_{2i};$$

$$\Psi_{w_{2^\nu-3}} - \Psi_w = u_{2^\nu-3} + u_{2^\nu-2}.$$  

The action of Hecke operator $T_q$ on each $A_i$ is clear except

$$T_q(\Psi_w) = \Psi_{w'} = \Psi_w - (\Psi_w - \Psi_{w'}) = (1 + 2n)\Psi_w + \sum_{i=1}^{2^\nu-1-1} u_{2i};$$

$$T_q(u_{2^\nu-3}) = \Psi_{w_{2^\nu-3}} - \Psi_w = u_{2^\nu-3} + u_{2^\nu-2} + (\Psi_w - \Psi_{w'}) = 2n\Psi_w + u_{2^\nu-3} + \sum_{i=1}^{2^\nu-1-2} u_{2i}. $$
4.4. **Case 4:** \((u, v) = (1, 1)\) and \(\nu \geq 1\).

Note that \(s_4 = s_6 = 2^\nu\) and \(s_2 = \frac{6n - s_4}{2}\). Let \(\Sigma_4 = \{w_1, \ldots, w_{2^\nu}\}\) and \(\Sigma_6 := \{t_1, \ldots, t_{2^\nu}\}\). As before, we assume that \(\text{Frob}(w_{2k-1}) = w_{2k}\) and \(\text{Frob}(t_{2k-1}) = t_{2k}\) for \(1 \leq k \leq 2^\nu - 1\). Let \(w := w_{2^\nu - 1}\) and \(w' := w_{2^\nu}\). Also, let \(t := t_{2^\nu - 1}\) and \(t' := t_{2^\nu}\).

**Proposition 4.4.** The component group \(\Phi_q(Nq)\) decomposes as follows:

\[
\Phi_q(Nq) = A_0 \oplus A \oplus B,
\]

where \(A_0\) is cyclic of order \(12n\) generated by \(\Psi_w\). The structures of \(A\) and \(B\) are the same as those in Propositions 4.2 and 4.3. The actions of \(T_q\) on \(A\) and \(B\) are the same as before except on \(A_{2\nu - 3}\) (when \(\nu \geq 2\)), where \(T_q\) acts by

\[
T_q(u_{2^\nu - 3}) = 6n\Psi_w + u_{2^\nu - 3} + \sum_{i=1}^{2^\nu - 1} u_{2i}.
\]

Moreover, the action of \(T_q\) on \(A_0\) is analogous as before:

\[
T_q(\Psi_w) = (1 + 6n)\Psi_w + \sum_{i=1}^{2^\nu - 1} u_{2i}.
\]

**Proof.** Note that from (3.1) we have

\[
s_2\Psi_s + \Psi_{w_1} + \cdots + \Psi_{w^\nu} + \Psi_{t_1} + \cdots + \Psi_{t^\nu} = 0.
\]

Multiplying by 3, we have

\[
(3) \quad \Psi_{w_1} + \cdots + \Psi_{w^\nu} = -(3s_2 + 2 \cdot 2^\nu)\Psi_s = -(6s_2 + 4 \cdot 2^\nu)\Psi_w.
\]

Also, multiplying by 4, we have

\[
(4) \quad \Psi_{t_1} + \cdots + \Psi_{t^\nu} = -(4s_2 + 3 \cdot 2^\nu)\Psi_s = -(12s_2 + 9 \cdot 2^\nu)\Psi_t.
\]

Therefore \(\Psi_{w_1}, \ldots, \Psi_{w^\nu}, \Psi_{t_1}, \ldots, \Psi_{t^\nu}\) can generate the whole group. By the similar computation, the order of \(\langle \Psi_{w_1} \rangle\) is \(12n\) and the order of \(\langle \Psi_{t_1} \rangle\) is \(18n\). All of them contain \(\Phi\) as a subgroup, which is of order \(6n\). Here we note that \(\langle \Psi_{t_1} \rangle = \langle 3\Psi_{t_1} \rangle \oplus \langle 6n\Psi_{t_1} \rangle\) because \(6n = 6s_2 + 5 \cdot 2^\nu\) is prime to 3. Therefore we can decompose \(\Phi_q(Nq)\) into

\[
(5) \quad \langle \Psi_w \rangle \bigoplus \bigoplus_{i=1}^{2^\nu - 2} \langle \Psi_{w_i} \rangle \bigoplus \langle \Psi_{t_1} - \Psi_t \rangle \bigoplus \langle 6n\Psi_{t_1} \rangle.
\]

As in Propositions 4.2 and 4.3, we can find an isomorphism between (5) and \(A_0 \oplus A \oplus B\), which proves the first part. From (3.1) (and the previous discussions) we have

\[
\Psi_w - \Psi_{w^\nu} = (6s_2 + 5 \cdot 2^\nu)\Psi_w + \sum_{i=1}^{2^\nu - 1} u_{2i} = 6n\Psi_w + \sum_{i=1}^{2^\nu - 1} u_{2i}.
\]

The action of \(T_q\) on each components is also obvious except

\[
T_q(\Psi_w) = \Psi_{w^\nu} = \Psi_w - (\Psi_w - \Psi_{w^\nu}) = (1 + 6n)\Psi_w + \sum_{i=1}^{2^\nu - 1} u_{2i}
\]

and

\[
T_q(u_{2^\nu - 3}) = \Psi_{w_{2^\nu - 2}} - \Psi_{w^\nu} = u_{2^\nu - 3} + u_{2^\nu - 2} + (\Psi_w - \Psi_{w^\nu}) = 6n\Psi_w + u_{2^\nu - 3} + \sum_{i=1}^{2^\nu - 2} u_{2i}.
\]

\[\square\]
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