COHEN-MACAULAYNESS OF TRIANGULAR GRAPHS

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Abstract. We study the Cohen-Macaulay property of triangular graphs $T_n$. We show that $T_2$, $T_3$ and $T_5$ are Cohen-Macaulay graphs, and that $T_4$, $T_6$, $T_8$ and $T_n$ are not Cohen-Macaulay graphs, for $n \geq 10$. Finally, we prove that over fields of characteristic zero $T_7$ and $T_9$ are Cohen-Macaulay.

1. Introduction

Let $R = \mathbb{K}[x_1, \ldots, x_N]$ be the polynomial ring over $\mathbb{K}$, where $\mathbb{K}$ is any field. Let $G$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_N\}$ and edge set $E(G)$. We identify the vertex $v_i$ with the variable $x_i$. The edge ideal $I(G)$ of $G$ is the ideal $\langle x_ix_j : \{v_i, v_j\} \in E(G) \rangle$. The graph $G$ is called Cohen-Macaulay over $\mathbb{K}$ if $R/I(G)$ is a Cohen-Macaulay ring. According to [11], it is unlikely to have a general classification of Cohen-Macaulay graphs. This situation has led to an extensive study of the Cohen-Macaulay property of particular families of graphs (see, for instance, [5, 7, 9, 10, 11, 15, 17]).

In this note we study the Cohen-Macaulayness of triangular graphs. The triangular graph $T_n$ is the simple graph whose vertices are the 2-subsets of an $n$-set, $n \geq 2$, and two vertices are adjacent if and only if their intersection is nonempty. It is known that $T_n$ is isomorphic to the Johnson graph $J(n, 2)$, which is in turn the 2-token graph of the complete graph $K_n$ (see, for instance, [2, 6, 12]). In addition, the complement of $T_n$ is isomorphic to the Kneser graph $K(n, 2)$ and the complement of $T_5$ is isomorphic to the Petersen graph.

Our main theorem (Theorem 5.4) states that $T_2$, $T_3$ and $T_5$ are Cohen-Macaulay, and that $T_4$, $T_6$, $T_8$ and $T_n$ are not Cohen-Macaulay graphs, for $n \geq 10$. In addition, it is proved that over fields of characteristic zero $T_7$ and $T_9$ are Cohen-Macaulay.

This note is organized as follows. We start by recalling the basic definitions and results regarding Cohen-Macaulay graphs that we need. Next, in Section 3 we first prove that $T_n$ is unmixed for every $n \in \mathbb{N}$. Later, we give a characterization for the Cohen-Macaulay property of $T_n$ that follows from Reisner criterion (Proposition 3.3). In Section
we first prove that $T_3$ and $T_5$ are Cohen-Macaulay. Next, using a computer algebra system, we compute explicit regular sequences to show that $T_7$ and $T_9$ are Cohen-Macaulay over fields of characteristic zero. Finally, in Section 5 we show that $T_4$, $T_6$, $T_8$ and $T_n$ are not Cohen-Macaulay graphs, for $n \geq 10$.

When investigating about the Cohen-Macaulayness of $T_n$, we computed several regular sequences using symmetric polynomials and we noticed that there was a certain pattern on how these sequences behave as $n$ increases. We later realized that those patterns also appeared for the edge subring of any simple graph. We conclude this note with an appendix in which we present an explicit regular sequence of a particularly nice shape for Cohen-Macaulay graphs. To that end, we first prove the existence of an explicit homogeneous system of parameters using elementary symmetric polynomials.

2. Cohen-Macaulay graphs and Cohen-Macaulay simplicial complexes

Let $R = K[x_1, \ldots, x_N]$ be the polynomial ring over the field $K$. Let $\mathfrak{m} = \langle x_1, \ldots, x_N \rangle$ and let $I$ be a graded ideal of $R$. The depth of $R/I$ is defined as the largest integer $r$ such that there is a homogeneous sequence $\{h_1, \ldots, h_r\} \subset \mathfrak{m}$, such that $h_1$ is not a zero divisor of $R/I$ and $h_i$ is not a zero divisor of $R/(I, h_1, \ldots, h_{i-1})$, for every $i \geq 2$.

**Definition 2.1.** We say that $R/I$ is a Cohen-Macaulay ring (CM ring for short) if $\text{depth}(R/I) = \text{dim}(R/I)$, where $\text{dim}(R/I)$ denotes the Krull dimension of $R/I$.

Let $G$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_N\}$ and edge set $E(G)$. We identify each vertex $v_i$ with the variable $x_i$ in $R$. The edge ideal $I(G)$ of $G$ is the ideal $\langle x_i x_j : \{v_i, v_j\} \in E(G) \rangle$. The ring $R/I(G)$ is called the edge subring of $G$. We say that $G$ is a Cohen-Macaulay graph over $K$ if $R/I(G)$ is CM. We say that $G$ is a Cohen-Macaulay graph if $G$ is CM over any field.

A set $U$ of vertices in a graph $G$ is an independent set of vertices if no two vertices in $U$ are adjacent; a maximal independent set is an independent set which is not a proper subset of any independent set in $G$. The independence number of $G$ is the number of vertices in a largest independent set in $G$. It is well known that the Krull dimension of $R/I(G)$ is equal to the independence number of $G$ (see [8, 16]).

Let $\Delta$ be a simplicial complex on the vertex set $V = \{v_1, \ldots, v_N\}$, i.e., $\Delta$ is a family of subsets of $V$ closed under taking subsets and such that $\{v_i\} \in \Delta$, for every $i$. The elements of $\Delta$ are called faces of $\Delta$. The dimension of a face $F \in \Delta$ is $|F| - 1$. The dimension of $\Delta$ is the
largest dimension of its faces. As before, we identify $v_i$ with $x_i$. The Stanley-Reisner ideal $I_\Delta$ of $\Delta$ is the ideal generated by all monomials $x_{i_1} \cdots x_{i_r}$ such that \{$v_{i_1}, \ldots, v_{i_r}$\} $\notin \Delta$. We say that $\Delta$ is a Cohen-Macaulay simplicial complex over $\mathbb{K}$ if $R/I_\Delta$ is a CM ring. We say that $\Delta$ is a Cohen-Macaulay simplicial complex if $\Delta$ is CM over any field.

Remark 2.2. Let $G$ be a simple graph.

(1) Let $\Delta_G$ be the simplicial complex formed by the independent sets of $G$ (this is a simplicial complex since every subset of an independent set is also independent). Hence $I(G) = I_{\Delta_G}$. Therefore, $G$ is a CM graph if and only if $\Delta_G$ is a CM simplicial complex.

(2) A clique of a graph $G$ is a subset $S \subseteq V(G)$ such that the graph induced by $S$ is a complete graph. Let $\Delta(G)$ be the simplicial complex formed by all cliques of $G$ and let $\overline{G}$ be the complement graph of $G$. Notice that $\Delta(\overline{G}) = \Delta_G$: every clique of $\overline{G}$ is an independent set of $G$ and vice versa.

Definition 2.3. Let $\Delta$ be a simplicial complex and $F \in \Delta$. The link of $F$ in $\Delta$, denoted $lk_\Delta(F)$, is the simplicial complex \{$H \in \Delta$: $H \cap F = \emptyset$ and $H \cup F \in \Delta$\}. We will often denote the link of $F$ in $\Delta$ just as $lk(F)$ if there is no risk of confusion.

The CM property of a graph can be determined by the following homological criterion (see [13]).

Theorem 2.4. (Reisner’s criterion) Let $\Delta$ be a simplicial complex. The following conditions are equivalent:

(a) $\Delta$ is Cohen-Macaulay over $\mathbb{K}$.

(b) $\widetilde{H}_i(lk(F); \mathbb{K}) = 0$, with $F \in \Delta$ and $i < \dim lk(F)$.

Corollary 2.5. If $\Delta$ is a 1-dimensional simplicial complex, then $\Delta$ is CM if and only if $\Delta$ is connected.

We will also need a result relating the CM property of a simplicial complex to some property of the $h$–vector of the simplicial complex.

Definition 2.6. Let $\Delta$ be a simplicial complex of dimension $d$.

i. The $f$–vector of $\Delta$ is defined as $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$, where $f_{-1} = 1$ and $f_i$ denotes the number of faces of dimension $i$ of $\Delta$, for $i \geq 0$.

ii. The $h$–vector of $\Delta$ is defined as $h(\Delta) = (h_0, \ldots, h_{d+1})$, where

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1},$$
and $0 \leq k \leq d + 1$.

**Theorem 2.7.** [14, Chapter II, Corollary 3.2] Let $\Delta$ be a simplicial complex of dimension $d$. If $\Delta$ is CM, then $h_i(\Delta) \geq 0$, for $0 \leq i \leq d + 1$.

3. A caracterización for the CM property of $T_n$

The triangular graph $T_n$ is the simple graph having as vertices the 2-subsets of a $n$-set, $n \geq 2$, and two vertices are adjacent if and only if their intersection is nonempty. The triangular graph $T_4$ is shown in Figure 1.

![Figure 1. $T_4$.](image1)

![Figure 2. $\overline{T_4}$.](image2)

We denote by $(ij)$ the vertices of $T_n$, where $1 \leq i < j \leq n$, and by $\Delta(n)$ the simplicial complex of independent sets of $T_n$. If $n < 2$ we define $\Delta(n) = \emptyset$.

A graph $G$ is unmixed if any two maximal independent sets of $G$ have the same cardinality. Since every CM graph is unmixed, the following proposition is relevant.

**Proposition 3.1.** Every triangular graph $T_n$ is unmixed.

**Proof.** It is well known that the independence number of $T_n$ is $\lfloor n/2 \rfloor$. We prove, by contradiction, that any maximal independent set in $T_n$ has $\lfloor n/2 \rfloor$ vertices. Let $A$ be any maximal independent set of $T_n$ and suppose that $|A| < \lfloor n/2 \rfloor$. Notice that there are $n - 2|A|$ elements in $\{1, \ldots, n\} \setminus \cup A$, with $n - 2|A| > 1$. Therefore, we can take a 2-set, say $z$, from $\{1, \ldots, n\} \setminus \cup A$ to construct the independent set $A' = A \cup \{z\}$, which is a contradiction. \qed
We need the following lemma to give a characterization for the Cohen-Macaulay property of the triangular graph $T_n$.

**Lemma 3.2.** Let $F \in \Delta(n)$ be any face such that $|F| = m$, where $n \geq 2$ and $m \geq 0$. Then we have the following identification of simplicial complexes:

$$l_{k_{\Delta(n)}}(F) \cong \Delta(n-2m).$$

**Proof.** If $F = \emptyset$ the statement holds by definition of $l_{k_{\Delta(n)}}(F)$. If $m = \lceil n/2 \rceil$, then $n = 2m$ or $n = 2m+1$ which implies that $l_{k_{\Delta(n)}}(F) = \emptyset = \Delta(n-2m)$ in both cases. Suppose $1 \leq m < \lfloor n/2 \rfloor$. Assume $F = \{(i_1j_1), \ldots, (i_mj_m)\}$. Let $A = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m, j_1, \ldots, j_m\}$. Since $F$ is an independent set of $T_n$ we have that $|A| = n-2m \geq 2$. Notice that $l_{k_{\Delta(n)}}(F)$ consists of every independent set of $T_n$ formed by elements $(ij)$ such that $i, j \in A$, and $i \neq j$. Now observe that the set of independent sets formed with couples $(ij)$ with $i, j \in A$, $i \neq j$ can be identified with $\Delta(n-2m)$.

**Proposition 3.3.** Let $n \geq 2$. Assume $n$ is odd (resp. even). The simplicial complex $\Delta(n)$ is CM if and only if $\tilde{H}_i(\Delta(l); \mathbb{K}) = 0$ for every $l \leq n$, with $l$ odd (resp. even), and for every $i < \dim(\Delta(l))$.

**Proof.** Assume that $n$ is odd. Suppose that $\Delta(n)$ is CM. Choose any odd number $l$ such that $3 \leq l \leq n$. By Lemma 3.2, $\Delta(l) \cong l_{k_{\Delta(n)}}(F)$ for any face $F \in \Delta(n)$ such that $|F| = (n-2m)$. Thus, $\tilde{H}_i(\Delta(l); \mathbb{K}) = \tilde{H}_i(l_{k_{\Delta(n)}}(F); \mathbb{K}) = 0$, for every $i < \dim l_{k_{\Delta(n)}}(F) = \dim(\Delta(l))$, according to Reisner’s criterion (Theorem 2.4).

To prove the other implication, let $F \in \Delta(n)$ be such that $|F| = m$, with $m \geq 0$. By Lemma 3.2, $l_{k_{\Delta(n)}}(F) \cong \Delta(n-2m)$. Since $n$ is odd, $n-2m$ is also odd and $n-2m \leq n$. By the hypothesis, $\tilde{H}_i(l_{k_{\Delta(n)}}(F); \mathbb{K}) = \tilde{H}_i(\Delta(n-2m); \mathbb{K}) = 0$, for every $i < \dim(\Delta(n-2m)) = \dim l_{k_{\Delta(n)}}(F)$. By Reisner’s criterion, $\Delta(n)$ is CM. The proof is completely analogous for $n$ even.

**Corollary 3.4.** Suppose that there exists an odd (resp. even) integer $n_0$ such that $T_{n_0}$ is not CM. Then $T_n$ is not CM for every odd (resp. even) $n \geq n_0$.

4. **The Cohen-Macaulay property of $T_3$, $T_5$, $T_7$ and $T_9$**

**Proposition 4.1.** $T_3$ and $T_5$ are CM graphs.

**Proof.** Since $T_3$ is a complete graph, by Example 3.2 below, $T_3$ is CM. Now consider the following path in $\Delta(5)$:

$$(12), (34), (25), (14), (35), (24), (13), (45), (23), (15).$$
This path passes through all vertices in $\Delta(5)$, hence it is connected. Since the independence number of $T_5$ is 2, the simplicial complex $\Delta(5)$ is 1-dimensional. By Corollary 2.5 we conclude that $\Delta(5)$ is CM, that is, $T_5$ is CM.

To verify that $T_7$ and $T_9$ are CM we used the computer algebra system SINGULAR 4-0-2 [3]. One minor difficulty here is to effectively compute the edge ideal of $T_n$. To that end we use the following remark.

**Remark 4.2.** The graph $T_n$ can be obtained from $T_{n-1}$ and the complete graph $K_{n-1}$ on the vertices $(1,n), (2,n), \ldots, (n-1,n)$ by joining the vertex $(i,j) \in V(T_{n-1})$ with the vertices $(i,n)$ and $(j,n)$ of $K_{n-1}$. Then we can compute recursively the edge ideal $I(T_n)$: if the edge ideal $I(T_{n-1})$ has been computed, we only need to add the monomials corresponding to $(i,j) \sim (i,n)$, $(i,j) \sim (j,n)$, and all the monomials coming from $K_{n-1}$.

Using the previous procedure we computed $I(T_7) \subset R_1 = \mathbb{Q}[z_1, z_2, \ldots, z_{21}]$ and $I(T_9) \subset R_2 = \mathbb{Q}[z_1, z_2, \ldots, z_{36}]$. Using the library primdec.lib [4], we compute primary decomposition of ideals and we found that the sequence

$$\left\{ \sum_{i=1}^{21} z_i, \sum_{i=1}^{21} z_i^2, \sum_{i=1}^{21} z_i^3 \right\},$$

is a regular sequence of $R_1/I(T_7)$. Similarly, the sequence

$$\left\{ \sum_{i=1}^{36} z_i, \sum_{i=1}^{36} z_i^2, \sum_{i=1}^{36} z_i^3, \sum_{i=1}^{36} z_i^4 \right\},$$

is a regular sequence of $R_2/I(T_9)$ (see the appendix for a discussion on homogeneous system of parameters for edge ideals using symmetric polynomials). Since the independence number of $T_7$ and $T_9$ are 3 and 4, respectively, we conclude that

**Proposition 4.3.** $T_7$ and $T_9$ are CM graphs over any field of characteristic zero.

5. **Non-Cohen-Macaulayness of $T_4$, $T_6$, $T_8$ and $T_n$ for $n \geq 10$**

In this section we show that $T_n$ is not CM for $n$ even, $n \geq 4$. We also show that $T_n$ is not CM for $n$ odd, $n \geq 11$.

**Proposition 5.1.** The triangular graph $T_n$ is not CM if $n$ is even, except for $n = 2$. 
Proof. If $n = 2$, $T_n$ is a single vertex and so it is CM. Let $n = 4$. The simplicial complex $\Delta(4)$ is 1-dimensional and non-connected, actually $\Delta(4) = T_4$ (see Figure 2). By Corollary 2.5, $\Delta(4)$ is not CM. Now, Corollary 3.4 implies that $T_n$ is not CM for every $n \geq 4$ with $n$ even. □

Now we turn our attention to $T_n$ for $n \geq 11$, $n$ odd.

Lemma 5.2. [1, Theorem 6.9.1] The number of faces of dimension $i$ of $\Delta(n)$ is given by the following formula:

$$f_i = \frac{1}{2i+1} \cdot \frac{n!}{(i+1)!(n-2(i+1))!}.$$  

Proposition 5.3. $T_n$ is not CM for every $n \geq 11$, $n$ odd.

Proof. Using the formula of lemma 5.2, we find that

$$f(\Delta(11)) = (1, 55, 990, 6930, 17325, 10395)$$
$$h(\Delta(11)) = (1, 50, 780, 4280, 6220, -936)$$

Since there is a negative entry in $h(\Delta(11))$, Theorem 2.7 implies that $T_{11}$ is not CM. Corollary 3.4 implies that $T_n$ is not CM for every odd $n$, $n \geq 11$. □

Putting together the results of the previous sections we obtain the following classification of $T_n$ in terms of the CM property:

Theorem 5.4. For triangular graphs $T_n$, the following holds:

(i) $T_2$, $T_3$ and $T_5$ are CM graphs.
(ii) $T_7$ and $T_9$ are CM graphs over any field of characteristic zero.
(iii) $T_4$, $T_6$, $T_8$ and $T_n$, for $n \geq 10$, are not CM graphs.

Proof. The theorem follows from propositions 4.1, 4.3, 5.1 and 5.3. □

Remark 5.5. Using SINGULAR 4-0-2, we verified that $T_7$ is CM over some fields of positive characteristic, such as $F_2$, $F_3$ and $F_5$. In every case, we found explicit regular sequences using symmetric polynomials (see appendix). This fact suggests that $T_7$ and $T_9$ might be CM over any field, giving a complete classification of $T_n$ in terms of the CM property.

Appendix A. An explicit regular sequence for CM graphs

It is well known that for any graded ideal $I \subset \mathbb{K}[x_1, \ldots, x_N]$, there exists a homogeneous system of parameters (h.s.o.p. for short) for $\mathbb{K}[x_1, \ldots, x_N]/I$ (see, for instance, [16, Proposition 2.2.10]). In this appendix we revisit this result for edge ideals by showing the existence of an explicit h.s.o.p. of a particularly nice shape.
This study was motivated by the following fact. When investigating about the Cohen-Macaulayness of $T_n$, experimental computation showed that for small odd values of $n$, the sequence
\[
\left\{ \sum_{x_i \in A_1} x_i, \sum_{x_{i_1}, x_{i_2} \in A_2} x_{i_1}x_{i_2}, \ldots, \sum_{x_{i_1}, \ldots, x_{i_d} \in A_d} x_{i_1} \cdots x_{i_d} \right\}
\]
is a regular sequence of the edge subring of $T_n$, where $A_j$ is the set of independent sets of size $j$ in $T_n$ and $d$ is the Krull dimension of the edge subring. Inspired by this fact, we show that for every simple graph there is a h.s.o.p. for its edge subring having this shape.

Let $G$ be a simple graph on the set of vertices $\{x_1, \ldots, x_N\}$ and let $I(G) \subset \mathbb{K}[x_1, \ldots, x_N]$ be its edge ideal. Let us recall the correspondence between minimal vertex covers of $G$, i.e., complements of maximal independent sets of $V(G)$, and minimal primes of $I(G)$.

**Proposition A.1.** ([10] Proposition 6.1.16) If $p$ is an ideal of $\mathbb{K}[x_1, \ldots, x_N]$ generated by $C = \{x_{i_1}, \ldots, x_{i_r}\}$, then $p$ is a minimal prime of $I(G)$ if and only if $C$ is a minimal vertex cover of $G$.

**Example A.2.** Let $K_N$ be the complete graph on the vertices $\{x_1, \ldots, x_N\}$. Every minimal vertex cover of $K_N$ has the form $K_N \setminus \{x_i\}$, for some $i$. By the previous correspondence, every minimal prime of $I(K_N)$ is generated by all variables except $x_i$. It follows that $\sum_{i=1}^N x_i$ is not a zero divisor in $R/I(K_N)$, that is, $1 \leq \text{depth}(R/I(K_N)) \leq \dim(R/I(K_N)) = 1$. Thus, $K_N$ is CM.

**Example A.3.** The edge ideal of $T_4$ (Figure 1) is given by
\[I(T_4) = \langle x_{12}x_{13}, x_{12}x_{14}, x_{12}x_{23}, x_{12}x_{24}, x_{13}x_{34}, x_{14}x_{34}, x_{23}x_{34}, x_{24}x_{34} \rangle.\]
Let $R = \mathbb{K}[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$. Let $\sigma_1, \ldots, \sigma_6 \in R$ be the elementary symmetric polynomials. Since the independence number of $G$ is 2, we have that $[\sigma_i] = [0]$ in $R/I(T_4)$, for $i \in \{3, 4, 5, 6\}$. In addition,
\[
[\sigma_1] = [x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34}],
[\sigma_2] = [x_{12}x_{34} + x_{13}x_{24} + x_{14}x_{23}].
\]
Using lemma A.4 below, we conclude that
\[
\sqrt{\langle [\sigma_1], [\sigma_2] \rangle} = \sqrt{\langle [\sigma_1], \ldots, [\sigma_6] \rangle} = \langle [x_{12}], [x_{13}], [x_{14}], [x_{23}], [x_{24}], [x_{34}] \rangle.
\]

Since $\dim R/I(T_4) = 2$, we conclude that $\{[\sigma_1], [\sigma_2]\}$ is a h.s.o.p. for $R/I(T_4)$.

**Lemma A.4.** Let $\sigma_1, \ldots, \sigma_m \in \mathbb{K}[z_1, \ldots, z_m]$ be the elementary symmetric polynomials. Then $\sqrt{\langle \sigma_1, \ldots, \sigma_m \rangle} = \langle z_1, \ldots, z_m \rangle$. 


Proof. It is enough to consider the following telescopic sum:
\[ z_i^m = z_i^{m-1} \sigma_1 - z_i^{m-2} \sigma_2 + z_i^{m-3} \sigma_3 - \cdots + (-1)^{m+1} \sigma_m. \]

Proposition A.5. Let \( G \) be a simple graph on \( N \) vertices, \( S = \mathbb{K}[x_1, \ldots, x_N]/I(G) \), and \( d = \dim S \). Let \( A_j \) denote the set of independent sets of size \( j \) in \( G \). Then the sequence
\[
\left\{ \sum_{\{x_i\} \in A_1} x_i, \sum_{\{x_i, x_j\} \in A_2} x_i x_j, \ldots, \sum_{\{x_i, \ldots, x_d\} \in A_d} x_i \cdots x_d \right\}
\]
is a homogeneous system of parameters for \( S \). In particular, if \( G \) is CM then this sequence is a regular sequence for \( S \).

Proof. Let \( F_k = \sum_{\{x_{i_1}, \ldots, x_{i_k}\} \in A_k} x_{i_1} \cdots x_{i_k} \), for \( 1 \leq k \leq d \). Let \( \sigma_1, \ldots, \sigma_N \in \mathbb{K}[x_1, \ldots, x_N] \) be the elementary symmetric polynomials. Since the independence number of \( G \) is \( d \), we have \( [\sigma_k] = [0] \) in \( S \) for every \( k = d+1, \ldots, N \). In addition, \( [\sigma_k] = [F_k] \) for every \( k \in \{1, \ldots, d\} \). The proposition then follows from the previous lemma:
\[
\sqrt{\langle [F_1], \ldots, [F_d] \rangle} = \sqrt{\langle [\sigma_1], [\sigma_2], \ldots, [\sigma_N] \rangle} = \langle [x_1], \ldots, [x_N] \rangle.
\]

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