Covariant and Quasi-Covariant Quantum Dynamics in Robertson–Walker Space–Times

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Abstract

We propose a canonical description of the dynamics of quantum systems on a class of Robertson–Walker space–times. We show that the world-line of an observer in such space–times determines a unique orbit in the local conformal group $SO(4,1)$ of the space–time and that this orbit determines a unique transport on the space–time. For a quantum system on the space–time modeled by a net of local algebras, the associated dynamics is expressed via a suitable family of “propagators”. In the best of situations, this dynamics is covariant, but more typically the dynamics will be “quasi-covariant” in a sense we make precise. We then show by using our technique of “transplanting” states and nets of local algebras from de Sitter space to Robertson–Walker space that there exist quantum systems on Robertson–Walker spaces with quasi-covariant dynamics. The transplanted state is locally passive, in an appropriate sense, with respect to this dynamics.

1 Introduction

We address the question of how the dynamics of quantum systems on curved space–times can be described in a physically motivated and mathematically useful manner in the special case of a class of Robertson–Walker space–times. In the literature known to us, the general question is broached either for free fields or in some quasi-classical approximation. We would like to give a purely quantum field-theoretic treatment which is valid also for interacting quantum fields. We do so in the language of algebraic quantum field theory [19, 57], which, though not widely known, has the advantages of being conceptually broad enough to subsume within it most approaches to quantum field theory and yet being mathematically rigorous.
Corresponding to the operationally motivated nature of algebraic quantum field theory, we begin our considerations with the worldline of an observer. This moving observer is subject to various forces — if his worldline is a geodesic, these forces are purely gravitational; otherwise, he is subject to acceleration due to non-gravitational causes. The same is true if the worldline is that of a quantum system, subject now to forces on the quantum level. What would be a suitable, mathematically rigorous description of the attendant dynamics of this system?

We propose an answer to that question, at least for quantum systems on a large class of Robertson–Walker space–times, which includes de Sitter space. In Section 2, we shall specify the class of Robertson–Walker space–times we shall be considering. For all of these space–times, the local conformal group is $SO(4,1)$. We show that given an arbitrary worldline in such a space–time, two physically motivated assumptions allow us to find a unique curve in $SO(4,1)$ which generates the worldlines of the given system and neighboring systems. These considerations are purely geometrical in nature.

Quantum theory enters for the first time in Section 3, where we describe quantum systems on our class of Robertson–Walker space–times using nets of local observable algebras supplied with a state. We define there what we mean by covariant, respectively quasi-covariant, dynamics of such systems. This will involve continuous 2-parameter families of automorphisms of the global observable algebra satisfying canonical propagator identities, which implement the action of the curve in $SO(4,1)$ associated with the evolution of the initial worldline, and which act locally on the observables in a well motivated manner.

In Section 4, we shall use results from a previous paper to construct nets and states on the specified Robertson–Walker space–times which admit dynamics in the sense of Section 3. We provide examples which admit covariant dynamics, as well as examples which only admit quasi-covariant dynamics. The transplanted state will be shown to be passive in an appropriate sense for this dynamics, so that for this dynamics the state behaves like a local equilibrium state from which cyclic processes cannot extract energy.

In the final section, we shall discuss the relation between our work and an alternate proposal for dynamics of quantum systems on curved space–times made by Keyl. We shall also make some remarks about the further outlook of this program. An appendix contains proofs of a more technical nature.

## 2 Kinematics

To begin, we specify the class of Robertson–Walker space–times we shall be considering, simultaneously establishing notation we shall be using throughout this paper. Robertson–Walker space–times are Lorentzian warped products of a connected open subset $I$ of $\mathbb{R}$ with a Riemannian manifold of constant sectional curvature. We restrict our attention here to the subclass of those having

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1We shall, however, neglect the extreme situations where the back reaction of the quantum system on the space–time must be taken into account. Thus, for our purposes, it will suffice to consider quantum field theory on a fixed classical space–time background.
positive curvature, which may be assumed to be equal to +1. The corresponding
Robertson–Walker space–times are homeomorphic to $I \times S^3$, and one can choose
coordinates so that the metric has the form

$$ds^2 = dt^2 - S^2(t)d\sigma^2. \quad (1)$$

Here, $t$ denotes time, $S(t)$ is a strictly positive smooth function and $d\sigma^2$ is the
time-independent metric on the unit sphere $S^3$:

$$d\sigma^2 = d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2). \quad (2)$$

Following [20] we define a rescaled time parameter $\tau(t)$ by

$$d\tau/dt = 1/S(t). \quad (3)$$

In terms of this new variable, the metric takes the form

$$ds^2 = S^2(\tau)(d\tau^2 - d\sigma^2), \quad (4)$$

where $S(\tau)$ is shorthand for $S(t(\tau))$. Since $S$ is everywhere positive, $\tau$ is a con tinuous, strictly increasing function of $t$; its range is therefore an open interval in $\mathbb{R}$. In this paper we shall further restrict our attention to those Robertson–Walker space–times with functions $S(t)$ such that the range of values of $\tau$ is of the form $(-\alpha, \alpha)$, with $\alpha \leq \pi/2$. Henceforth, we shall denote by $RW$ any one of this class
of Robertson–Walker space–times.

As is well known, the four–dimensional de Sitter space–time $dS$ can be em bedded into five–dimensional Minkowski space $\mathbb{R}^5$ as follows:

$$dS = \{x \in \mathbb{R}^5 \mid x_0^2 - x_1^2 - \ldots - x_4^2 = -1\}. \quad (5)$$

This space–time is topologically equivalent to $\mathbb{R} \times S^3$, and in the natural coordinates the metric has the form

$$ds^2 = dt^2 - \cosh^2(t)d\sigma^2. \quad (6)$$

We recognize $dS$ as a special case of Robertson–Walker space–time with the choice $S(t) = \cosh(t)$. Once again, we change time variables by defining

$$\tau = \arcsin(\tanh(t)) \quad (7)$$

(so that $d\tau/dt = 1/\cosh(t)$), which takes values $-\pi/2 < \tau < \pi/2$. Then the metric becomes

$$ds^2 = \cos(\tau)^{-2}(d\tau^2 - d\sigma^2). \quad (8)$$

The isometry group, as well as the conformal group, of $dS$ is the de Sitter group
$O(4,1)$.

Comparing equations (4) and (8), it is now clear (and well known [20]) that
each of the Robertson–Walker space–times specified above can be conformally
embedded into de Sitter space–time, i.e. there exists a global conformal diffeomorphism \( \varphi \) (see Definition 9.16 in [1]) from \( RW \) onto a subset of \( dS \). How large the embedding in \( dS \) is depends on the range of the variable \( \tau \) in each case examined, which itself depends upon the function \( S(t) \). If \( \alpha = \pi/2 \), then the embedded \( RW \) coincides with \( dS \). In this case, the de Sitter group \( O(4, 1) \) is the conformal group of \( RW \). If \( \alpha < \pi/2 \), then \( O(4, 1) \) is only locally the conformal group of \( RW \), i.e. there is an action induced upon \( RW \) through the embedding, but some orbits of points under this action reach infinity in finite “time”. Henceforth, for the sake of brevity of expression, we shall identify \( RW \) with its embedding in \( dS \) whenever it is convenient.

We can now turn to the first step of our program: given a worldline of an observer, construct a curve in \( SO(4, 1) \) which generates a flow describing the worldlines of the observer and his neighbors. This construction will be based upon physically natural hypotheses, and the resultant curve will turn out to be unique.

Let \( t \mapsto x_t \) be the worldline in \( RW \) of the observer, parametrized by proper time \( t \). Let then \( t \mapsto \lambda_t \) be a differentiable curve in the group \( SO(4, 1) \) such that

\[
\lambda_t x_0 = x_t \quad \text{for all } t \in \mathbb{R}. \tag{9}
\]

Such a curve always exists, but it is only fixed up to elements of the stability group of \( x_0 \). In order to remove this ambiguity, we bring to bear physical considerations. We require that the curve generates, in the same sense as in equation (9), also the worldlines of neighboring material particles moving along with the observer without colliding with him or rotating with respect to him. These are interpreted as material points in the observer’s laboratory, which itself is assumed to be at rest relative to gyroscope axes carried by the observer. These considerations find their formal expression in the following conditions:

(a) For each \( y_0 \) in some neighborhood of \( x_0 \), the events \( \lambda_t y_0, t \in \mathbb{R} \), describe, potentially, the worldline of some material particle. This worldline is either disjoint from the observer’s worldline or coincides with it.

(b) For given \( y_0 \) spacelike to \( x_0 \), the axis of a gyroscope carried by the observer at the space-time point \( \lambda_t x_0 \) points towards the point \( \lambda_t y_0 \) at all times \( t \).

It turns out that these mild assumptions, in addition to equation (9), uniquely fix \( t \mapsto \lambda_t \). We sketch the elements of the argument and state the theorem here; the proof can be found in the appendix.

Since only the conformal structure is involved, we may equally well consider the (conformally equivalent) \( dS \) metric. Our assumptions imply that the curve \( t \mapsto \lambda_t \) generates a local flow, which is irrotational and rigid with respect to the \( dS \) metric. This implies that the differential map of \( \lambda_t \) coincides with the

\[^2\text{By a worldline we mean an inextendible timelike curve of class } C^2, \text{ not necessarily parametrized by proper time.}\]

\[^3t \text{ is in general not the proper time for these particles.}\]
Fermi–Walker transport along the worldline with respect to the $dS$ metric, cf. Definition 2 in the appendix. The desired solution is then obtained as follows.

Consider $(dS, g)$ as being embedded into ambient Minkowski space $(\mathbb{R}^5, \bar{g})$, cf. equation (5), and denote $x_t \in dS$ after this identification by $\tilde{x}_t$. Then the acceleration of the observer in ambient Minkowski space is given by

$$a_t = \frac{d^2}{ds^2} \tilde{x}_t,$$

where $s = s(t)$ is the $dS$ proper time. (10)

Define now, for each $t \in \mathbb{R}$, a linear transformation of $\mathbb{R}^5$ by

$$M_t = g(\dot{\tilde{x}}_t, \cdot) \tilde{a}_t - g(\tilde{a}_t, \cdot) \dot{\tilde{x}}_t.$$ (11)

Obviously, $M_t$ is skew-adjoint with respect to $\bar{g}$. More specifically, in the Lorentz frame of $\tilde{x}_t \in \mathbb{R}^5$, $M_t$ is an infinitesimal boost with direction $\tilde{a}_t$. With these definitions, we have:

**Proposition 2.1** Given a worldline $t \mapsto x_t$ in $RW$, there is precisely one curve $t \mapsto \lambda_t$ in $SO(4,1)$ satisfying equation (9) and the above assumptions (a) and (b). It is given by

$$\lambda_t = T \exp \int_0^t M_s ds,$$ (12)

where $T$ denotes time ordering.

**Remarks.**
1. Our assumptions, as well as the result, are independent of the parametrization of the worldline: If $\tilde{x}_t \doteq x_{h(t)}$ is a reparametrization of the worldline (where $h(0) = 0$), then $\tilde{\lambda}_t \doteq \lambda_{h(t)}$ is the corresponding solution in $SO(4,1)$.
2. One might impose, instead of (b), the weaker assumption that the measuring device be irrotational only in the sense of hydrodynamics [29]. But in the present case of a rigid motion (described by de Sitter isometries $\lambda_t$) this weaker notion is equivalent to our assumption (b).
3. If $t \mapsto x_t$ is a geodesic in $dS$, then our curve $t \mapsto \lambda_t$ coincides with the well known corresponding one-parameter group of boosts [4, 26].

We prove the Proposition and the first remark in the appendix.

3 Quantum Dynamics

We shall describe quantum systems in $RW$ by a net $\{A(W)\}_{W \in \mathcal{W}}$ of $C^*$-algebras indexed by a suitable set of open subregions $\mathcal{W}$ of $RW$ and a state $\omega$. The algebra $A(W)$ is interpreted as the algebra generated by all observables measurable in the region $W$, and the state $\omega$ models the preparation of the quantum system. We emphasize that these data contain the same physical information encoded in the standard formulation of quantum field theory in terms of fields (cf. Section 4 of [11] and further references there). The algebras are subalgebras of some
$C^*$-algebra $\mathcal{A}$ on which the state $\omega$ is defined. Since we shall only be concerned with a single representation of this net, it will be no loss of generality to consider the net $\{\mathcal{R}(W)\}_{W \in \mathcal{Y}}$ with $\mathcal{R}(W) = \mathcal{A}(W)''$ taken to be the von Neumann algebra generated in the GNS representation space $\mathcal{H}$ associated with $(\mathcal{A}, \omega)$. Hence, the unit vector $\Omega \in \mathcal{H}$ implementing the state $\omega$ on $\mathcal{A}$ is cyclic for $\mathcal{A}$.

A propagator family is a family $\{\alpha_{t,s}\}_{t, s \in \mathbb{R}}$ of automorphisms of the observable algebra $\mathcal{R} = \mathcal{A}''$ which satisfies the propagator identities

$$\alpha_{t,s} \alpha_{s,t'} = \alpha_{t,t'} \quad \text{and} \quad \alpha_{t,t} = \text{id}, \quad \text{for all } t, t', s \in \mathbb{R} \quad (13)$$

(id denotes the identity map on $\mathcal{R}$), and which is continuous in the following sense. Denoting by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on the representation space $\mathcal{H}$, we require that the maps $\mathcal{R} \ni t \mapsto \alpha_{t,s}(A) \in \mathcal{B}(\mathcal{H})$ be continuous for each fixed $s \in \mathbb{R}$ and $A \in \mathcal{R}$, with $\mathcal{B}(\mathcal{H})$ endowed with the weak operator topology (and similarly with the roles of $t$ and $s$ interchanged).

In order for an interpretation of this family as a dynamics for the observer to be physically reasonable, $t$ should be the observer’s proper time and $\alpha_{t,s}$ should map the observables measurable in the observer’s immediate neighborhood at his proper time $s$ onto those measurable at time $t$. More precisely, let $t \mapsto x_t$ be the worldline of our observer, and let, for each $t \in \mathbb{R}$, $\mathcal{O}(t)$ be a suitable neighborhood of $x_t$ (interpreted as the space-time region wherein the measurement process initiated at time $t$ takes place. We shall say that the propagator family $\{\alpha_{t,s}\}_{t, s \in \mathbb{R}}$ is a covariant dynamics with respect to the family of space-time regions $\{\mathcal{O}(t)\}_{t \in \mathbb{R}}$, if

$$\alpha_{t,s} \mathcal{R}(\mathcal{O}(s)) = \mathcal{R}(\mathcal{O}(t)), \quad (14)$$

for all $t, s \in \mathbb{R}$.

It cannot be expected that the covariance property $(14)$ will be strictly satisfied in all circumstances of physical interest. There are two directions in which this property could be relaxed. First of all, it could well happen that $(14)$ will only hold for all $t, s$ in some finite open interval. In addition, the dynamics may not maintain the localization of all of the observables localized in $\mathcal{O}(t)$ but will do so for “sufficiently many” such observables. For these reasons, if the propagator family satisfies

$$\alpha_{t,s} \mathcal{R}_0(\mathcal{O}(s)) = \mathcal{R}_0(\mathcal{O}(t)), \quad (15)$$

for all $t, s \in I$, for some open interval $I \subset \mathbb{R}$, and for subalgebras $\mathcal{R}_0(\mathcal{O}(t)) \subset \mathcal{R}(\mathcal{O}(t))$ which are sufficiently large that the distinguished vector $\Omega$ is still cyclic for each of them, we shall say that the propagator family is a quasi-covariant dynamics with respect to $\{\mathcal{O}(t)\}_{t \in I}$. Intuitively speaking, the forces described by such dynamics leave a large class of observables localized within the observer’s laboratory, at least for some large but finite time interval but can act upon some of the observables in such a way that their localization is moved out of the immediate neighborhood of the initial worldline, a situation which one envisages in the presence of certain external fields.
4 Examples Provided by Transplantation

In this section we provide specific examples fitting into the above setting by employing the novel technique of transplantation of theories from one space–time onto another [12]. We consider a fixed $RW$ space–time with embedding parameter $\alpha \leq \pi/2$, as described in Section 2. For such a space–time we have described in [12] how one can obtain a net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ on $RW$ from a covariant de Sitter theory via transplantation. We shall review that process below for the reader’s convenience. Given an arbitrarily accelerated observer in $RW$, we shall construct a propagator family acting on the transplanted $RW$ theory and describing the time evolution of certain distinguished observables in the observer’s neighborhood, in the sense of quasi-covariant dynamics. When $\alpha = \pi/2$, the constructed dynamics is even covariant.

It is necessary to summarize some of the results from our previous paper [12]. Consider the embedding (3) of $dS$ into Minkowski space and let $SO_{0}(4,1)$ denote the proper orthochronous Lorentz group in five dimensions. Let $\tilde{W}$ be the family of Minkowski space regions obtained by applying the elements of $SO_{0}(4,1)$ to a single wedge–shaped region of the form

$$\tilde{W}^{(1)} = \{x \in \mathbb{R}^{5} | x_{1} > |x_{0}|\},$$

(16)

i.e. this family of regions is $\tilde{W} = \{\lambda \tilde{W}^{(1)} | \lambda \in SO_{0}(4,1)\}$. Let then $W$ be the collection of intersections $\{\tilde{W} \cap dS | \tilde{W} \in \tilde{W}\}$. We call $W$ the set of de Sitter wedges. A wedge in de Sitter space is the causal completion of the worldline of a freely falling observer. We define wedges in $RW$ to be the intersections with $RW$ of those de Sitter wedges whose edges⁴ are contained in $RW$. They correspond to the causal completions of the union of worldlines of freely falling observers in $RW$. The set of these Robertson–Walker wedges will be denoted by $\mathcal{W}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Penrose diagram indicating a wedge in $dS$ (right) and in $RW$ (left).}
\end{figure}

We recall the following lemma [12, Lemma 2.1].

**Lemma 4.1** $\mathcal{W}$ is stable under the operation of taking causal complements and under the action of the isometry group of rotations $SO(4)$ on $RW$. Further, the elements of $\mathcal{W}$ separate spacelike separated points in $RW$ and $\mathcal{W}$ is a subbase for the topology in $RW$.

⁴See below for a precise definition.
One obtains a more intrinsic characterization of \( W \) by noticing that wedges in de Sitter space–time can be characterized by their edges. Let \( \hat{E}^{(1)} \) be the edge of \( \hat{W}^{(1)} \), i.e. the three–dimensional subspace \( \{ \tilde{x} \in \mathbb{R}^5 \mid \tilde{x}_0 = \tilde{x}_1 = 0 \} \). Applying the elements of \( SO_0(4,1) \) to \( \hat{E}^{(1)} \), one obtains all three–dimensional spacelike linear subspaces \( \hat{E} \) of \( \mathbb{R}^5 \). The intersections of these with \( dS \) are exactly the two–dimensional, spacelike, totally geodesic, complete, connected submanifolds of \( dS \). These submanifolds will be called de Sitter edges and are denoted by \( E \).

The causal complement of \( \hat{E}^{(1)} \) has two connected components, one being \( \hat{W}^{(1)} \) and the other one being its causal complement \( \hat{W}^{(1)}' \in \hat{W} \). Hence also the causal complement of any de Sitter edge has two connected components, each being a wedge, i.e. a Lorentz transform of \( \hat{W}^{(1)} \) intersected with \( dS \). So we conclude that the wedges in \( dS \) may be characterized as the connected components of the causal complements of de Sitter edges.

Based on this observation, we have given in \([12]\) an analogous, more geometric characterization of wedges in a Robertson–Walker space–time \( RW \). In fact, an \textit{ad hoc} condition which was still left in this characterization can be replaced by completely intrinsic geometric constraints. Thus the choice we made above of wedges in \( RW \) is well-motivated from an intrinsic point of view. The details of this characterization will not be necessary in this paper, however, and will be presented elsewhere.

In \([12]\) we defined a one–to–one map of the set \( \mathcal{W} \) of de Sitter wedges onto the set \( \mathcal{W}' \) of Robertson–Walker wedges, which we give again here. Recall that an element of the class of Robertson–Walker space–times considered in this paper is embedded into \( dS \) with a characteristic interval \( |\tau| < \alpha \leq \pi/2 \). If \( \alpha = \pi/2 \), then the embedded \( RW \) coincides with \( dS \). In this case a Robertson–Walker wedge is just a de Sitter wedge and the families \( \mathcal{W} \) and \( \mathcal{W}' \) are identified by the embedding. To cover the general case \( 0 < \alpha \leq \pi/2 \), we define a diffeomorphism \( \Phi \) from \( dS \) onto \( RW \) which bijectively maps the set of de Sitter edges onto the set of Robertson–Walker edges. It is given by

\[
\Phi (\tau, \chi, \theta, \phi) = (f(\tau), \chi, \theta, \phi), \tag{17}
\]

where

\[
f(\tau) = \arcsin \left( \sin(\alpha) \sin(\tau) \right). \tag{18}
\]

The map \( \Phi \) gives rise to a one–to–one correspondence

\[
\Xi : \mathcal{W} \rightarrow \mathcal{W}' \tag{19}
\]

as follows. Let \( W \) be a de Sitter wedge with edge \( E \). The causal complement \( \Phi(E)' \) of \( \Phi(E) \) in \( RW \) has two connected components, exactly one of which has nontrivial intersection with \( W \).

**Definition 1** Let \( W \) be a de Sitter wedge with edge \( E \). We define \( \Xi(W) \) to be the connected component of \( \Phi(E)' \) in \( RW \) which has nontrivial intersection with \( W \).
The map $\Xi$ thus maps the family of de Sitter wedges onto the family of Robertson–Walker wedges. It has the following specific properties which we recall from \cite[Prop. 2.3]{12} and which are crucial for our process of the transplantation of nets.

**Proposition 4.2** The map $\Xi : \mathcal{W} \to \mathcal{W}$ is a bijection. It commutes with the operation of taking causal complements in the respective spaces and intertwines the action of the isometry group of rotations $SO(4)$ on $\mathcal{W}$ with its action on $\mathcal{W}$. 

As shown in \cite{12}, if $\alpha < \pi/2$, then $\Xi$ is not induced by a bijective point transformation from $dS$ onto $RW$, i.e. there is no map $p : dS \to RW$ such that $\Xi(W) = \{ p(x) \mid x \in W \}$, for all $W \in \mathcal{W}$.

We now consider an arbitrary covariant de Sitter theory having the Reeh–Schlieder property. Such theories either can be constructed directly on de Sitter space or can themselves be transplanted from a covariant Minkowski space theory via a process akin to holography \cite{12,17}. Hence, we consider a net $\{ \mathcal{R}(W) \}_{W \in \mathcal{W}}$ of local algebras, a state vector $\Omega$ in the representation space $H$ and a strongly continuous unitary representation $U$ of the proper orthochronous de Sitter group $SO_p(4,1)$ satisfying

$$U(\lambda)\mathcal{R}(W)U(\lambda)^{-1} = \mathcal{R}(\lambda W), \quad (20)$$

as well as $U(\lambda)\Omega = \Omega$, for all $W \in \mathcal{W}$ and $\lambda \in SO_p(4,1)$. The Reeh–Schlieder property refers to the condition that the vector $\Omega$ is cyclic for any local algebra $\mathcal{R}(O)$ corresponding to any (sufficiently small) bounded open region $O$ — cf. \cite{4}.

We will also assume that the de Sitter theory satisfies the following physically motivated condition: Let $W \in \mathcal{W}$ be any de Sitter wedge and $\alpha^W_t$, $t \in \mathbb{R}$, the automorphism group induced on $\mathcal{R}(W)$ by the one-parameter subgroup of boosts which leaves $W$ invariant and induces a future directed Killing vector field in $W$. Then $(\mathcal{R}(W), \alpha^W_t, \Omega)$ satisfies the KMS–condition \cite{6}, i.e. there exists some $\beta > 0$ such that for any pair $A, B \in \mathcal{R}(W)$ there is an analytic function $F$ in the strip $\{ z \in \mathbb{C} \mid 0 < \text{Im}(z) < \beta \}$ with continuous boundary values at $\text{Im}(z) = 0$ and $\text{Im}(z) = \beta$, which are respectively given by

$$F(t) = \langle \Omega, A\alpha^W_t(B)\Omega \rangle, \quad F(t + i\beta) = \langle \Omega, \alpha^W_t(B)A\Omega \rangle, \quad (21)$$

for all $t \in \mathbb{R}$. As in \cite{4}, we shall refer to this condition as the geodesic KMS–property for brevity. The physical content of this assumption is that to any uniformly accelerated observer in $W$ the de Sitter vacuum must appear to be a thermal equilibrium state at inverse temperature $\beta$. In fact, it has been shown in \cite{4} that this temperature must be the Gibbons–Hawking temperature.

The geodesic KMS–property itself has been proven to hold in many situations. For example, if (1) the de Sitter theory satisfies the Condition of Geometric Modular Action and the Modular Stability Condition \cite{8,12}, or if (2) the de Sitter theory is obtained from a five-dimensional Minkowski space theory \cite{12,17} which itself is locally associated with a quantum field satisfying the Wightman axioms and...
then the geodesic KMS–property is fulfilled \cite{12}. Or, as a third alternative, if the $n$-point functions of the de Sitter theory satisfy a certain weak spectral condition \cite{7}, then the geodesic KMS–property is satisfied, as long as the de Sitter quantum field is locally affiliated with the corresponding de Sitter algebras $\mathcal{R}(W)$.

We proceed from the given net on $dS$ to a corresponding net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ on $RW$ by transplantation, setting

$$\mathcal{R}(W) = \mathcal{R}(\Xi(W)),$$

where $\Xi : \mathcal{W} \to \mathcal{W}$ is the bijection defined above. In addition, we proceed from $\Omega$ to a state vector $\Omega$ in the $RW$ theory by defining

$$\Omega = \Omega. \quad (23)$$

We thus obtain from the original net and state vector describing a theory on de Sitter space a net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ and state vector $\Omega$ describing a theory on $RW$ which now are re-interpreted in terms of Robertson–Walker space–time. The physical significance of the operators and state vector thereby changes.

For any (sufficiently small) bounded region $Q \subset RW$ we define

$$\mathcal{R}(Q) = \bigcap_{W \supset Q} \mathcal{R}(W). \quad (24)$$

We remark that if the “preimage” in $dS$ of $Q \subset RW$, defined as

$$\Xi^{-1}(Q) = \bigcap_{W \supset Q} \Xi^{-1}(W) \subset dS. \quad (25)$$

contains an open set, then $\Omega$ is cyclic for the algebra $\mathcal{R}(Q)$, since it contains $\mathcal{R}(\Xi^{-1}(Q))$. On the other hand, if $\alpha < \pi/2$ the set $\Xi^{-1}(Q)$ may be empty if $Q$ is too small \cite{12}, see below. In this case, and if the underlying de Sitter theory has the property that intersections of algebras corresponding to disjoint wedges consist only of scalar multiples of the identity\footnote{In \cite{33} it is shown that this condition is satisfied under very general circumstances.}, then $\mathcal{R}(Q)$ is trivial in the same sense. We shall consider, specifically, double cones; we remind the reader that a double cone $Q$ is a nonempty intersection of the interiors of a future lightcone and a past lightcone. A particularly convenient class is comprised of the “upright” double cones in $RW$. Let $x = (\tau, \chi, \theta, \phi) \in RW$ and let $\varepsilon$ be a positive number such that $|\tau \pm \varepsilon| < \alpha$. To these data we associate a double cone

$$Q_{x,\varepsilon} = I_+(x_{-\varepsilon}) \cap I_-(x_{+\varepsilon}), \quad \text{where } x_{\pm \varepsilon} = (\tau \pm \varepsilon, \chi, \theta, \phi). \quad (26)$$

Here, $I_\pm(p)$ denotes the future, respectively past, light cone of a point $p$. By Proposition 2.5 of \cite{12}, $\Xi^{-1}(Q_{x,\varepsilon})$ contains an open set if $\varepsilon > \pi/2 - \alpha$, but is empty if $\varepsilon < \pi/2 - \alpha$.

Given the curve $t \mapsto \lambda_t$ in $SO(4,1)$ associated to the observer’s worldline $t \mapsto x_t$ by Proposition 2.7, we define

$$\alpha_{t,s} = \text{Ad } U(\lambda_t \lambda_s^{-1}) \quad \text{for } s, t \in \mathbb{R}. \quad (27)$$
By the strong continuity and the representation property of $U$, this defines a propagator family. We shall see that it acts (quasi-)covariantly on certain observables in the observer’s neighborhood. As it turns out, this propagator family provides a covariant dynamics in the special case where the embedding parameter $\alpha$ of the RW space–time under consideration equals $\pi/2$, a particular instance of which is the de Sitter space–time itself. In fact, for geodesics in $dS$ this construction is well known [26] and is generalized here to the case of worldlines corresponding to an arbitrarily accelerated motion in any $RW$ space–time conformally equivalent to $dS$. In contrast, if $\alpha < \pi/2$, the propagator family is a quasi-covariant dynamics with respect to $\{O_{\sim t}\}_{t \in I}$ and cannot be covariant.

We recall that when $\alpha = \pi/2$, the de Sitter group $SO(4,1)$ coincides with the group of conformal, orientation preserving transformations of $RW$, and the net of local algebras $\{R(\mathcal{W})\}_{\mathcal{W} \in \mathcal{W}}$ is covariant under the adjoint action of the representation $U$ [12 Cor. 3.2]. If $O$ is a neighborhood of $x_0$, then

$$O(t) \doteq \lambda_t O$$

is a neighborhood of $x_t$, which we take as the space-time region representing the observer’s laboratory at time $t$. The above-mentioned covariance then implies that $\alpha_{t,s}$ satisfies equation (13) for this family of neighborhoods, thus complying with the conditions for a covariant dynamics. We therefore have proven the following.

**Proposition 4.3** Let $t \mapsto \lambda_t$ be the curve in $SO(4,1)$ assigned to the observer’s worldline according to Proposition [27], and let $\{\alpha_{t,s}\}_{t,s \in \mathbb{R}}$ be the corresponding propagator family as defined in equation (27). If the embedding parameter $\alpha = \pi/2$, then it is a covariant dynamics with respect to every family of space-time regions of the form $O(t) = \lambda_t O$, with $O$ a neighborhood of $x_0$. It describes the time evolution of observables measured in a non-rotating rest frame for the worldline.

We now turn to the case $\alpha < \pi/2$. As indicated above, in this case there is a minimal length scale below which no nontrivial observables can be localized [12 Cor. 3.3], provided the de Sitter theory has the trivial-intersection property mentioned above.

In light of this fact, we require the space-time region $O(t)$ representing the observer’s laboratory at time $t$ (in the sense indicated earlier) to contain the closure of the upright double cone $O_{x_t, \pi/2 - \alpha}$ centered at $x_t$ — see equation (28). We then say that $O(t)$ is sufficiently large for $x_t$, since $R(O(t))$ is large enough so that $\Omega$ is cyclic for it, cf. [12 Cor. 3.3]. We shall further require that each $O(t)$ is a double cone and that the family of double cones is continuous in the sense that the two curves described by the future and past apices of the elements of the family $\{O(t)\}$ are both continuous in $t$. If $\alpha = \pi/2$, the family of double cones considered above (see equation (28)) satisfies both of these requirements. If $\alpha < \pi/2$, a continuous family of sufficiently large double cones can only exist

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If $\alpha < \frac{\pi}{2}$, such sufficiently large double cones do not exist at all [12 Cor. 3.3]. As a matter of fact, in this case the smallest localization regions are wedges.
for \( t \) in some finite interval \( I \), since eventually the corresponding future and past apices will reach the future, respectively past, horizon of \( RW \). Taking these basic facts into account, we can establish the following result, whose proof is given below.

**Theorem 4.4** Let \( t \mapsto \lambda_t \) be the curve in \( SO(4,1) \) assigned to the observer’s worldline according to Proposition 2.7, and let \( \{\Omega_{t,s}\}_{t,s \in \mathbb{R}} \) be the corresponding propagator family as defined in equation (27). If \( \{\mathcal{O}(t)\}_{t \in I} \), with \( I \subset \mathbb{R} \) compact, is any continuous family of double cones such that \( \mathcal{O}(t) \) is sufficiently large for \( x_t, t \in I \), then \( \{\Omega_{t,s}\}_{t,s \in I} \) is a quasi-covariant dynamics for this family.

It is noteworthy that the subalgebras \( \mathcal{R}_0(\mathcal{O}(t)) \subset \mathcal{R}(\mathcal{O}(t)) \) of observables, which remain localized within the laboratory under the action of the dynamics (cf. equation (13)), can be intrinsically characterized as follows. Given the time interval \( I \) and the family of space-time regions \( \mathcal{O}(t) \), the desired algebras are given by

\[
\mathcal{R}_0(\mathcal{O}(t)) \doteq \bigcap_{s \in I} \Omega_{t,s} \mathcal{R}(\mathcal{O}(s)) , \quad t \in I .
\]  

(29)

This is a subalgebra of \( \mathcal{R}(\mathcal{O}(t)) \), and for each \( t, t' \in I \), the propagator \( \Omega_{t,t'} \) maps the algebra \( \mathcal{R}_0(\mathcal{O}(t')) \) onto \( \mathcal{R}_0(\mathcal{O}(t)) \) by virtue of the propagator identities (13).

What we claim is that these algebras are sufficiently large such that the distinguished vector \( \Omega \) is cyclic for each of them.

**Proof.** For fixed \( t \in I \), consider the intersection of the \( \tau \)-coordinate line through \( x_t \) with the boundary of \( \mathcal{O}(t) \). Since the latter consists of two achronal sets, this intersection consists of two points \( x_{\pm}(t) \). Let \( \varepsilon_t \) be the minimum of the two numbers \( |\tau(x_t) - \tau(x_{+}(t))| \) and \( |\tau(x_t) - \tau(x_{-}(t))| \). Then, by hypothesis, one has \( \mathcal{O}_{x_t,\varepsilon_t} \subset \mathcal{O}(t) \) and \( \varepsilon_t > \frac{\tau}{2} - \alpha \). With \( f \) the scaling function defined in (18), this implies that one has \( \varepsilon_t = \frac{\tau}{2} - f^{-1}(\frac{\tau}{2} - \varepsilon_t) > 0 \). Hence, Prop. 2.5 in [12] entails that the double cone \( \mathcal{O}_{x_t,\varepsilon_t} \) is an open neighbourhood of \( x_t \) contained in \( \Xi^{-1}(\mathcal{O}(t)) \).

Now define the regions

\[
\mathcal{O}_t \doteq \lambda_t^{-1} \mathcal{O}_{x_t,\varepsilon_t} \quad \text{and} \quad \mathcal{O} \doteq \bigcap_{t \in I} \mathcal{O}_t .
\]  

(30)

It will be shown that \( \mathcal{O} \) contains an open set. To this end denote by \( y_t \) the point determined by the intersection of the future boundary of \( \mathcal{O}_t \) with the \( \tau \)-coordinate line containing \( x_0 \). Since \( \mathcal{O}_t \) is an open neighbourhood of \( x_0 \), \( \tau(y_t) \) is strictly larger than \( \tau(x_0) \), for each \( t \in I \). Furthermore, \( t \mapsto \tau(y_t) \) is a continuous function due to the assumed continuity of the family \( \{\mathcal{O}(t)\}_{t \in I} \). Hence by continuity of the function and compactness of the interval \( I \), the quantity

\[
\delta \doteq \min_{t \in I} \{\tau(y_t)\}
\]  

(31)

is also strictly larger than \( \tau(x_0) \). Now, by construction, all points on the \( \tau \) coordinate line through \( x_0 \) with \( \tau \) coordinate in \([\tau(x_0), \delta]\) are contained in \( \mathcal{O} \).
Hence $O$ contains a timelike curve segment. Since $O$ is causally closed, this implies that $O$ contains an open set.

By construction, $O$ satisfies $\lambda_t O \subset \Xi^{-1}(Q(t))$, $t \in I$. Hence the local $dS$ algebra $\mathcal{R}(\lambda_t O)$ is contained in $\mathcal{R}(O(t))$, $t \in I$. Furthermore, the propagator $\varphi_{t,s}$ maps $\mathcal{R}(\lambda_s O)$ onto $\mathcal{R}(\lambda_t O)$, $s, t \in I$, by covariance of the underlying $dS$ theory. These two facts imply that $\mathcal{R}(\lambda_t O)$ is contained in the algebra $\mathcal{R}_0(\xi(t))$, for each $t \in I$ (see equation (29)). Since $O$ is sufficiently large, $\Omega = \Omega$ is cyclic for $\mathcal{R}(\lambda_t O)$, completing the proof that $\varphi_{t,s}$ is almost covariant. □

To further bolster our claim that the propagator family is describing a physically reasonable local dynamics, we recall the notion of a passive state of a $W^*$-dynamical system $(\mathcal{R}, \alpha_t)$. In [28] Pusz and Woronowicz give a definition which is motivated by an attempt to find an exact mathematical expression for the physical idea that systems in equilibrium cannot perform mechanical work in cyclic processes. We refer the reader to [28] for that definition. For our purposes here, what is more useful is the following condition, which Pusz and Woronowicz proved is actually equivalent to the physically motivated definition. If $\delta$ is the generator of the automorphism group $\{\alpha_t\}_{t \in \mathbb{R}}$ representing the dynamics on $\mathcal{R}$, i.e. if $\delta : \mathcal{R} \to \mathcal{R}$ is defined by

$$\delta(A) = \lim_{t \to 0} \frac{1}{t}(\alpha_t(A) - A),$$

then a state $\langle \Omega, \cdot \Omega \rangle$ on $\mathcal{R}$ is passive (with respect to $\alpha_t$) if and only if

$$-i \frac{d}{dt} \langle \Omega, U^* \alpha_t(U) \Omega \rangle |_{t=0} = -i \langle \Omega, U^* \delta(U) \Omega \rangle \geq 0,$$

for all $U$ in the connected component of the group of all unitary elements of $\mathcal{R}$ in the uniform topology. In the special case that $\alpha_t = \text{Ad} e^{itH}$, with $H$ self-adjoint, equation (33) becomes

$$\langle \Omega, U^*HU\Omega \rangle = \langle \Omega, U^*[H,U]\Omega \rangle = -i \langle \Omega, U^*\delta(U)\Omega \rangle \geq 0,$$

where in the first equality we have used the fact that a passive state is invariant under $\alpha_t$, for all $t \in \mathbb{R}$ [28]. Pusz and Woronowicz have revealed the close connection between the KMS–condition and passivity [28]. And recent work has indicated that passivity is closely related with other criteria for the stability of quantum theories on curved space–times [15, 30]. See also [23, 24].

Our dynamics is not expressed in terms of an automorphism group, so we cannot expect the Second Law of Thermodynamics to have quite the same translation into our setting as in that considered by Pusz and Woronowicz. However, we find the following analogous situation. For the specified worldline and the resultant curve $\lambda_t$ in $SO(4,1)$ from Proposition 2.1, one has for each fixed $s$

$$\varphi_{t,s} = \text{Ad} U(\lambda_t \lambda_s^{-1}) = \text{Ad} U(\lambda_t) U^{-1}(\lambda_s),$$

9In point of fact, Pusz and Woronowicz treat $C^*$-dynamical systems, but their work can be extended to $W^*$-dynamical systems in a manner which in all essential respects is identical — cf. [6].

10Of course, $U$ must also be in the domain of $\delta$. 
hence the analogue of the left hand sides of \((33)\) and \((34)\) is

\[-i \frac{d}{dt} \langle \Omega, U^{*} Q_{t,s}(U) \Omega \rangle |_{t=s} = \langle \Omega, U^{*} H_{s} U \Omega \rangle ,\]

where we have used Prop. 2.1 and \(H_{s}\) denotes the self-adjoint generator of the unitary one-parameter group \(t \mapsto U(\exp(tM_{s}))\). The quantity \((34)\) measures the energy transferred by the operation \(U\) to the state in the local reference frame of the observer. We claim that for each \(s \in \mathbb{R}\) this quantity is nonnegative for all unitaries \(U\) in the algebra \(\mathcal{R}(\Xi(W_{s}))\), where \(W_{s}\) is the unique de Sitter wedge containing the point \(x_{s}\) (viewed as an element of \(dS\) after the embedding) and left invariant by the boost subgroup generated by \(M_{s}\), cf. the proof of Theorem 4.5 below. To be able to interpret this statement in terms of the Second Law, the operation \(U\) should be performed in a neighborhood of \(x_{s}\), i.e. the localization region \(\Xi(W_{s})\) of \(U\) should contain \(x_{s}\). This certainly holds at the instant of time when \(x_{s}\) lies in the \(\tau = 0\) hypersurface (since then \(x_{s} \in W_{s}\) implies \(x_{s} \in \Xi(W_{s})\)) and, by continuity, for some open time interval around this instance. If \(\alpha = \pi/2\), then of course one has \(\Xi(W_{s}) = W_{s}\), so that \(x_{s} \in \Xi(W_{s})\) for all \(s \in \mathbb{R}\).

**Theorem 4.5** Assume that the underlying de Sitter theory \(\{\mathcal{R}(W)\}_{W \in \mathcal{W}}, \Omega\) satisfies the geodesic KMS–property, and let \(\{\mathcal{R}(W)\}_{W \in \mathcal{W}}, \Omega\) be obtained from such a theory by transplantation, as above. Let \(t \mapsto \lambda_{t}\) be the curve in \(SO(4,1)\) assigned to the worldline \(t \mapsto x_{t}\) by Proposition 2.3, and let \(\{\alpha_{t,s}\}_{t,s \in \mathbb{R}}\) be the corresponding propagator family as defined in equation (24).

For each \(s \in \mathbb{R}\), the state induced by \(\Omega\) on the net \(\{\mathcal{R}(W)\}_{W \in \mathcal{W}}\) is quasi-passive for \(\alpha_{t,s}\) on the algebra \(\mathcal{R}(W_{s})\), in the following sense: The quantity in \((34)\) is nonnegative for all unitaries in \(\mathcal{R}(W_{s})\) which are in the domain of the indicated derivation, where \(W_{s} = \Xi(W_{s})\) and \(W_{s}\) is the unique de Sitter wedge on which the boost-subgroup generated by \(M_{s}\) induces a future directed flow. The wedge \(W_{s}\) contains \(x_{s}\) for all \(s\) and \(W_{s}\) contains \(x_{s}\) for \(s\) in some open interval \(I\) around the instant of time when the observer passes the \(\tau = 0\) hypersurface. If \(\alpha = \pi/2\), then \(I = \mathbb{R}\).

**Proof.** Fix \(s \in \mathbb{R}\) and consider the one-parameter subgroup of \(SO(4,1)\) generated by the boost \(M_{s}\), cf. equation (14). Denote by \(W_{s}\) the unique wedge such that this subgroup leaves \(W_{s}\) invariant and induces a future directed timelike vector field on \(W_{s}\), and denote by \(\alpha_{t}^{W_{s}}, t \in \mathbb{R}\), the corresponding automorphism group on \(\mathcal{R}(W_{s})\). From Theorem 1.2 of [23] the geodesic KMS–property implies that the state induced on \(\mathcal{R}(W_{s})\) by \(\Omega\) is passive with respect to the group \(\alpha_{t}^{W_{s}}, t \in \mathbb{R}\). Hence, equation (34) and the \(SO(4,1)\)-invariance of \(\Omega\) imply the non-negativity of the right hand side of equation (34) for all unitaries in \(\mathcal{R}(W_{s}) = \mathcal{R}(W_{s})\) which are in the domain of the indicated derivation.

To show that \(x_{s} \in W_{s}\), it is convenient to consider the ambient Minkowski space, denoting the point \(x_{s}\) after the identification (3) by \(\tilde{x}_{s}\) and the Minkowski wedge corresponding to \(W_{s}\) by \(\tilde{W}_{s}\). Recall [8] that every wedge \(\tilde{W}\) in \(\mathbb{R}^{5}\) can be characterized by a pair of linearly independent lightlike vectors \(l^{+}, l^{-}\) as the set of all \(\tilde{x}\) satisfying \(g(\tilde{x}, l^{\pm}) < 0\). Consider now the two lightlike vectors \(\tilde{l}_{s}^{\pm} = \tilde{x}_{s} \pm \tau_{s}\tilde{\nu}_{s} = \tilde{x}_{s} \pm \tau_{s}(\ln(\cosh(\alpha_{s}) - \cosh(\alpha_{s}^{W_{s}})))^{1/2}\).


$|a_s|^{-1}a_s \pm |\dot{x}_s|^{-1}\dot{x}_s$, where $a_s$ denotes the acceleration in ambient Minkowski space as in equation (10), and $|\dot{x}| = |\hat{g}(\dot{x}, \ddot{x})|^{1/2}$. By relation (11) one checks that $M_s \hat{l}_s^= = \pm a |\ddot{x}_s|$ for $a = |a_s| |\dot{x}_s| > 0$. Hence the boost subgroup generated by $M_s$ leaves the lightlike rays corresponding to $\hat{l}_s^=$ invariant and generates a future-directed flow on these rays. It follows that the pair $\hat{l}_s^=$ characterizes $\hat{W}_s$ in the sense indicated above. Now $\hat{g}(\dot{x}_s, \ddot{x}_s) = 0$ and $\hat{g}(\dot{x}_s, \ddot{a}_s) = -1$, hence $\ddot{x}_s \in W_s$ as claimed.

As shown before the theorem, also $W_s = \Xi(W_s)$ contains $x_s$ for $s$ in some interval $I$. Note however that, if $\alpha \neq \pi/2$, this interval is in general not the entire real line, not even if the worldline is a $RW$ geodesic. □

5 Conclusion and Outlook

Given the worldline of any observer in $RW$, we have constructed a curve in $SO(4,1)$ which generates the worldline of the observer and of neighboring worldlines. Our curve is uniquely fixed by the requirement that the latter worldlines describe, potentially, material particles moving along with the observer without colliding with him or rotating with respect to him. With the same assumptions, we show that this curve determines a transport of vectors along the worldline which coincides with the Fermi–Walker transport. 11

We proposed to implement the dynamics associated with this transport using propagator families, and then we produced a large class of examples of nets and states admitting such dynamics. We also have shown why it is necessary to allow for dynamics of the type we call quasi-covariant. In these examples we have also demonstrated that the states are locally quasi-passive with respect to the constructed dynamics, so these states behave locally as if they were equilibrium states with respect to the quantum dynamics we have associated with arbitrarily accelerated observers on the class of Robertson–Walker space–times which can be conformally embedded into de Sitter space.

We anticipate that this process of transplantation of theories from one space–time to another can also be performed for those Robertson–Walker space–times which can be conformally embedded into Minkowski space, as well as for all the space–times which can be conformally embedded into Einstein space. We then expect that quasi-covariant dynamics can be constructed for such theories along the lines followed in this paper.

In [21,22] Keyl takes another approach to finding dynamics for quantum systems on curved space–times. Though his starting point is also the worldline of an observer, the “dynamical maps” he constructs are *-isomorphisms between pairs of local algebras associated with double cones centered on the worldline at different times and having the same extension in the time direction. They therefore must satisfy propagator identities which differ from ours. He only considers the analogue of what we call covariant dynamics. Although Keyl’s approach has the advantage over (the present state of) our proposal of being applicable to any

This last point is proven at the end of the appendix.
globally hyperbolic space–time, his construction of dynamics uses in an essential manner properties of free field theory.

We close with a final comment. It is well known that in the standard passage from classical Lagrangian field theory to quantized field theory, symmetries of the classical Lagrangian can be broken in the quantized theory. What is less well known is that a quantum field theory can have symmetries which have no counterpart in the classical theory — modular symmetries associated with the local algebras $\mathcal{R}(\mathcal{O})$ and a state vector $\Omega$ by the Tomita–Takesaki modular theory (cf. [3, 22]). These modular symmetries may be implemented by the modular unitaries or, more generally [8], by the modular conjugations. The consequences and uses of these symmetries are not yet well understood, though some indications are beginning to emerge [3, 8, 13, 31]. As was shown in [12], the representation $U(\lambda)$ of $SO(4,1)$ used in this paper to construct the quasi-dynamics on the Robertson–Walker net of observable algebras consists of modular symmetries of the $\text{RW}$ theory constructed above and can therefore be derived from the $\text{RW}$ theory without any reference to the $dS$ theory. Hence, we have demonstrated in this paper that such modular symmetries may be used to define a dynamics for a quantum system, even though they have no relation to the isometries of the underlying space–time, indeed though they are not even implemented by point transformations on the space–time.

A Proof of Proposition 2.1

In the following discussion we shall have recourse to the Fermi–Walker transport, which formalizes in classical relativity theory the notion of irrotational transport along an observer’s worldline. For the reader’s convenience, we recall its definition. We will denote the tangent space to a manifold $M$ at $x \in M$ by $T_x M$ and the differential map of a map $\phi : M \to N$ at $x \in M$ by $T_x \phi$.

**Definition 2** Let $t \mapsto x_t$ be a worldline in a space-time $(M, g)$. The Fermi–Walker derivative of a vector field $V$ along the worldline $t \mapsto x_t$ is defined by

$$F_{\dot{x}_t} V = \nabla_{\dot{x}_t} V + g(a_t, V)\dot{x}_t - g(\dot{x}_t, V)a_t,$$

(37)

where $a_t$ is the corresponding acceleration: $a_t = \nabla_{\dot{x}_t} x'_t$, with $x'_t$ denoting the normalized tangent vector, $x'_t = g(\dot{x}_t, \dot{x}_t)^{-1/2} \dot{x}_t$.

A vector $v_t \in T_{x_t} M$ is said to arise from $v_0 \in T_{x_0} M$ by Fermi–Walker transport along the worldline if there is a vector field $V$ on the worldline satisfying $V_{x_0} = v_0$, $V_{x_t} = v_t$ and $F_{\dot{x}_t} V = 0$ for all $s \in [0, t]$.

It follows from the definition that the Fermi–Walker transport is an isometry and preserves the normalized tangent vector to the curve (in the sense of equation (38) below). Further, the linearity property

$$F_{f(t)\dot{x}_t} V = f(t) F_{\dot{x}_t} V$$

(38)
implies that the Fermi–Walker transport is independent of the parametrization. That is to say, if $\bar{x}_t$ is a reparametrization of the worldline, then the Fermi–Walker transports from $x_0$ to $x_{t_0}$ along the curves $t \mapsto \bar{x}_t$ and $t \mapsto x_t$ coincide.

For the proof of Proposition 2.1 we proceed in two steps. First, we show that any differentiable curve $t \mapsto \lambda_t$ satisfying equation (9) and conditions (a) and (b) in Section 2 has to satisfy the initial condition $\lambda_0 = 1$ and the differential equation

$$\frac{d}{dt} \lambda_t = M_t \circ \lambda_t, \quad \text{for all } t \in \mathbb{R},$$

where $M_t$ is given by (11). It therefore is unique. Then we verify that the solution (12) of this equation indeed complies with the above constraints.

Thus, let $t \mapsto \lambda_t$ be a differentiable curve in $SO(4, 1)$ satisfying equation (9) and assumptions (a) and (b). Since these assumptions involve only the conformal, not the metric, structure, $t \mapsto x_t$ may be considered as a curve in $dS$. As is well known and will be shown subsequent to this proof for the convenience of the reader, the constraints (a) and (b) are equivalent to the condition that the differential map $T_{x_0} \lambda_t$ of $\lambda_t$ coincides with the Fermi–Walker transport, with respect to the $dS$ metric, from $x_0$ to $x_t$ along the worldline. Hence, one has

$$F_{\dot{x}_t}(T_{x_0} \lambda_t \cdot v) = 0 \quad \text{for all } v \in T_{x_0} dS, t \in \mathbb{R},$$

$$T_{x_0} \lambda_0 = 1 \quad \text{(initial condition)}.$$  

To evaluate these equations, we consider $dS$ as being embedded into ambient 5-dimensional Minkowski space $\mathbb{R}^5$, cf. equation (5), and denote $x \in dS$ after this identification by $\tilde{x}$. Then the tangent space at a point $\tilde{x} \in dS$ can be identified with the orthogonal complement $\tilde{x}^\perp$ of $\tilde{x}$. Further, the action of $SO(4, 1)$ on $dS$ is just the restriction of its linear action on $\mathbb{R}^5$, and the differential map $T_{x_0} \lambda_t$ is just the linear map $\lambda_t : \tilde{x}^\perp \rightarrow (\lambda_t \tilde{x}_0)^\perp$. Denote by $\tilde{g}$ and $\tilde{\nabla}$ the metric and covariant derivative in $\mathbb{R}^5$, respectively. For vector fields $X, Y$ on $dS$ one has [27, Lemma 4.27]

$$\tilde{\nabla}_X Y = \nabla_X Y + \tilde{g}(X, Y) P,$$

where $P$ denotes the position vector field $P_\tilde{x} = \tilde{x} \in \mathbb{R}^5$. Using this equation, one readily verifies that the (restriction of the) Fermi–Walker derivative of ambient $\mathbb{R}^5$ coincides with that of $dS$. Hence equation (14) is equivalent to

$$\tilde{\nabla}_{\tilde{x}_t} (\lambda_t \tilde{v}) = \tilde{g}(\tilde{x}_t, \lambda_t \tilde{v}) a_t - \tilde{g}(a_t, \lambda_t \tilde{v}) \tilde{x}_t \quad \text{for all } \tilde{v} \in \tilde{x}_0^\perp, \ t \in \mathbb{R},$$

where $a_t = \tilde{\nabla}_{\tilde{x}_t} \tilde{x}_t' = \frac{d^2}{ds^2} \tilde{x}_t$ as in equation (14). On the other hand, one has

$$\tilde{\nabla}_{\tilde{x}_t} (\lambda_t \tilde{v}) = \frac{d}{dt}(\lambda_t \tilde{v}) = (\frac{d}{dt} \lambda_t) \tilde{v}.$$  

Since $\tilde{v} \in \tilde{x}_0^\perp$ was arbitrary and $\lambda_t$ is fixed by its action on $\tilde{x}_0^\perp$, equation (13) can therefore be written as the differential equation (39) for $\lambda_t$.  

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The initial condition (11), translated to ambient $\mathbb{R}^5$, reads $\lambda_0 = 1$ on $\bar{x}_0^\perp$. Hence, in view of $\lambda_0 \bar{x}_0 = \bar{x}_0$, we find that $\lambda_0 = 1$. As $s \mapsto M_s$ is continuous, the unique solution to the differential equation (39), with initial condition $\lambda_0 = 1$, is given by equation (12). Moreover, since $M_t$ is skew-adjoint with respect to $\tilde{g}$ for each $t$, it lies in $SO(4,1)$. It remains to show that this solution indeed satisfies $\lambda_t \bar{x}_0 = \bar{x}_t$ for all $t \in \mathbb{R}$, as required by equation (9). To this end, let $\tilde{y} \in \mathbb{R}^5$. Then one has

$$
\frac{d}{dt} \tilde{g}(\bar{x}_t, \lambda_t \tilde{y}) = \tilde{g}(\bar{x}_t, \lambda_t \tilde{y}) + \tilde{g}(\bar{x}_t, M_t \lambda_t \tilde{y}) = \tilde{g}(\bar{x}_t, \lambda_t \tilde{y}) (1 + \tilde{g}(\bar{x}_t, \tilde{a}_t)),
$$

where relation (11) and $\tilde{g}(\bar{x}_t, \bar{x}_t) = 0$ was used in the second equation. By equation (12), the vector $\tilde{a}_t - \bar{x}_t$ is in $T_{x_0}dS = \bar{x}_0^\perp$, hence one sees that $\tilde{g}(\bar{x}_t, \tilde{a}_t) = \tilde{g}(\bar{x}_t, \bar{x}_t) = -1$. The right hand side of equation (45) is therefore zero, which implies that $\tilde{g}(\bar{x}_t, \lambda_t \tilde{y}) = \tilde{g}(\bar{x}_0, \tilde{y})$. It follows, in particular, that $\lambda_t$ maps $\bar{x}_0^\perp$ to $\bar{x}_t^\perp$. Since $\lambda_t$ is a time-orientation preserving isometry, this implies that $\lambda_t \bar{x}_0 = \bar{x}_t$, completing the proof of the Proposition.

We next prove our assertion that assumptions (a) and (b) are equivalent to the condition that $T_{x_0} \lambda_t$ coincides with the Fermi–Walker transport, which we denote by $T_{x_0} t \in \mathbb{R}$. The first step is to show that assumption (a) implies that the differential map preserves the normalized (with respect to the $dS$ metric) tangent vector to the world line, as does the Fermi–Walker transport:

$$
T_{x_0} \lambda_t \cdot g(\bar{x}_0, \bar{x}_0)^{-1} \dot{x}_0 = g(\bar{x}_t, \bar{x}_t)^{-1} \dot{x}_t.
$$

Note that $\lambda_t x_s$ must lie on the worldline of the observer for all $t \in \mathbb{R}$ and sufficiently small $s$. (Otherwise, $t \mapsto \lambda_t x_s$ would be a neighboring worldline colliding with the observer at $x_s$, contradicting assumption (a).) That is to say, $\lambda_t x_s = x_f(t,s)$ for some function $f$. Hence, $T_{x_0} \lambda_t \cdot \bar{x}_0$ is of the form $\partial_2 f(t,0) \bar{x}_t$, and is, in particular, collinear with $\dot{x}_t$. Since $\lambda_t$ preserves the metric and time-orientation, this implies equation (16).

Suppose now that the axis of a gyroscope carried by the observer points towards some neighboring worldline at time $t = 0$. The corresponding spatial direction is a unit vector in $\bar{x}_0^\perp$, which we denote by $e$. As is well known, the motion of the gyroscope axis corresponds to the Fermi–Walker transport along the worldline $[23,24]$, i.e. its direction at time $t$ then corresponds to $T_{x_0} t \cdot e \in \bar{x}_t^\perp$. On the other hand, the direction of our measurement device at time $t$ corresponds, by assumption (a), to the projection of $T_{x_0} \lambda_t \cdot e$ onto $\bar{x}_t^\perp$. But as a consequence of equation (16), the latter coincides with $T_{x_0} \lambda_t \cdot e$. Hence, assumption (b) implies that $T_{x_0} \lambda_t = T_{x_0} t \cdot e$ on $\bar{x}_0$. Since the Fermi–Walker transport also acts on $\bar{x}_0$ like $T_{x_0} \lambda_t$ (see equation (13), we have shown that (a) and (b) imply $T_{x_0} \lambda_t = T_{x_0} t \cdot e$, $t \in \mathbb{R}$. The converse implication is also clear from our discussion.

We now show that the result is independent of the parametrization, as remarked after the Proposition. To this end, let $\bar{x}_t \equiv x_{h(t)}$ be a reparametrization of the worldline (with $h(0) = 0$), and $\bar{\lambda}_t$ the corresponding solution of $\bar{\lambda}_t \bar{x}_0 = \bar{x}_t$ complying with assumptions (a) and (b). Since the Fermi–Walker transport is independent of the parametrization, (a) and (b) still imply that the differential map of $\bar{\lambda}_t$ coincides with the Fermi–Walker transport, with respect to the $dS$
metric, from $\vec{x}_0$ to $\vec{x}_t$ along $t \mapsto \vec{x}_t$. Hence, equations (40) and (41) must hold, with $x_t$ and $\lambda_t$ replaced by $\vec{x}_t$ and $\vec{\lambda}_t$. Since both the Fermi–Walker derivative and the covariant derivative are function-linear as in equation (38), the line of arguments leading to equation (39) now leads to

$$\frac{d}{dt}\vec{\lambda}_t = h'(t) M_{h(t)} \circ \vec{\lambda}_t \quad \text{for all } t \in \mathbb{R},$$

with $M_t$ as in equation (11). The unique solution with the correct initial condition $\vec{\lambda}_0 = 1$ is given by $\vec{\lambda}_t = T \exp \int_0^{h(t)} M_s ds = \lambda_{h(t)}$, as claimed.

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