Realistic construction of split fermion models

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Abstract

The Standard Model flavor structure can be explained in theories where the fermions are localized on different points in a compact extra dimension. We show that models with two bulk scalars compactified on an orbifold can produce such separations in a natural way. We study the shapes and overlaps of the fermion wave functions. We show that, generically, realistic models of Gaussian overlaps are unnatural since they require very large Yukawa couplings between the fermions and the bulk scalars. We give an example of a five dimensional two scalar model that accounts naturally for the observed quark masses, mixing angles and CP violation.
I. INTRODUCTION

The Standard Model (SM) flavor puzzle is to understand the origin of the smallness and hierarchy of most of the fermion masses and mixing angles. The likely solution of this puzzle requires the existence of a fundamental theory that generates the observed patterns in a natural way, namely without small dimensionless parameters. One such framework is due to Arkani-Hamed and Schmaltz (AS) [1]. The idea is to separate the various SM fermion fields inside compact extra dimensions. In that case the four dimensional Yukawa couplings between two fermions are suppressed by the overlap of their zero mode wave functions. Once the widths of the wave functions are smaller than their separation in the extra dimensions, small and hierarchical Yukawa couplings in four dimensions are induced.

Various scenarios based on this idea had been constructed. In some models, only the fermion wave functions have non trivial shapes in the extra dimension [1, 2, 3, 4, 5, 6, 7]. In other cases, also the Higgs field is used to generate the flavor structure [4, 8, 9]. A specific model in five dimensions (5d) that generates the observed fermion masses and mixing angles was found by Mirabelli and Schmaltz (MS) [2]. This model, however, cannot accommodate the observed CP violation in the kaon and $B$ systems [10]. Experimental tests of the split fermion idea were studied in [1, 11].

While the AS framework is very attractive there is no complete realization of it. In [1] a model with one infinite extra dimension, $x_5$, was considered where domain wall fermions were used to localize the fermions and to separate them. The basic idea is that there is a scalar, $\Phi$, which develops an $x_5$ dependent vacuum expectation value (vev), $h(x_5)$. Then, the fermions have $x_5$ dependent effective masses $m_{\text{eff}} = \lambda h(x_5) - M$, where $M$ is a bare mass term and $\lambda$ is the Yukawa coupling between the fermion and $\Phi$. The minimum of the right handed (RH) zero mode and the maximum of the left handed (LH) zero modes is where the effective mass vanishes, namely at $x_5^m$ such that $\lambda h(x_5^m) = M$. The LH chirality state, with the maximum at $x_5^m$ is localized around $x_5^m$. The RH mode, however, is localized at infinity, namely it is non-normalizable and it drops out of the spectrum. The separation between the LH modes is achieved by introducing a different $M$ for the different fields.

The generalization of this original realization of the split fermions idea to the realistic case of a compact dimension is not trivial. The problem is that in a compact space both LH and RH zero modes are normalizable, and thus the model is not really chiral. Another problem, which is related to the first one, is that in a compact space the domain wall configuration is not stable.

Some ideas toward solving these problems had been proposed by Georgi, Grant and Hailu (GGH) [3] and by Kaplan and Tait (KT) [4]. They considered 5d scenarios with one extra dimension compactified on an $S_1/Z_2$ orbifold. The $Z_2$ orbifold symmetry projects out one chiral zero mode, making this a chiral theory in four dimensions (4d). A scalar, $\Phi$, which is odd under the $Z_2$ symmetry is also introduced. Since $\Phi$ is odd, its vev must vanish on the orbifold fixed points. When $\Phi$ develops a non zero vev in the bulk, an $x_5$ dependent
vev is generated. Fermion mass terms of the form $\Phi \bar{\Psi}_R \Psi_L$ are allowed while bare mass terms, $\bar{\Psi}_R \Psi_L$, are forbidden by the $Z_2$ symmetry. Therefore, the zero modes are localized around one of the boundaries since these are the only places where their effective masses vanish. The sign of the Yukawa coupling between the fermion and the scalar determines on which boundary the fermion is localized. The idea of split fermions where each fermion is stuck at a different point in the bulk cannot be achieved. The zero modes can be localized only around one of the orbifold fixed points. An additional flavor structure in the Yukawa couplings is needed in order to naturally generate the observed fermion parameters in such two location models \[4\]. More complicated models, using large non renormalizable\(^1\) terms \[3\] or boundary terms \[4\] can localize fermions in the bulk.

The problems of this setup can be traced down to two points. The first is that the scalar vev vanishes only on the boundaries. Thus, the fermions cannot be localized within the bulk. The second is that mass terms are absent. In the original AS model the scalar was used to localize the fermion zero modes, and different constant mass terms were used to split them. In the simple orbifold models, while the scalar can be used to localize the fermions, there is no simple mechanism that splits them.

We are therefore interested in improving upon the models of \[3, 4\]. We propose to use two scalar fields that couple to the fermions. This rather minor modification of the GGH and KT scenarios can produce localization in the bulk and at the same time does not affect the appealing features of these models. The advantage that two scalar models have over one scalar models is that the effective mass can vanish in the bulk. Intuitively the picture is as follows. For one scalar the sign of the Yukawa coupling determines the boundary where the fermion is localized \[3\]. Once a second scalar with opposite sign Yukawa coupling is introduced, the picture is more complicated. The second scalar tends to localize the fermion on the other boundary. In some cases one scalar is dominant and localization is at the boundary determined by the sign of its Yukawa coupling and the second scalar only modifies the shape of the wave function. In other cases, however, the tension between the two scalar results in a compromise: A configuration where the fermion is localized in the bulk.

In section II we introduce the two scalar model and investigate its scalar sector. In section III we study the shapes of the fermion wave functions in various limits of the model and their implications for model building. In section IV we present a two scalar model which accounts naturally for the quark masses, mixing angles and CP violation. We conclude in section V.

\(^1\) More precisely, we refer here to 5d terms which after dimensional reduction produce non renormalizable operators in the 4d effective theory.
II. TWO SCALAR MODEL

We now move to study the features of the scalar sector of our model. We explain how it can generate localization and separation of fermion wave functions in the extra dimension.

The space-time of our model is described by ordinary 4d space-time and an additional space dimension compactified on an $S_1/Z_2$ orbifold. The physical region is defined as $0 \leq x_5 \leq L$ and $x_5 = 0, L$ are the orbifold fixed points. The model includes two scalar fields, $\Phi_i$ ($i = 1, 2$), and a fermion field, $\Psi$. The scalars and $\Psi_R$ are odd while $\Psi_L$ is even under the $Z_2$ orbifold symmetry. For simplicity, we assume that there is no interaction between the two scalars. We further consider the case where the two scalars develop $x_5$ dependent vevs.

The conditions for this case to be realized can be found by a straightforward generalization of the single scalar case (see e.g. [7]). Here we only remark that having an $x_5$ dependent vev is a generic situation. Then, the action is

$$ S = \int dx_M \left\{ \bar{\Psi} \left( i \gamma^M \partial_M + \tilde{f}_1 \Phi_1 + \tilde{f}_2 \Phi_2 \right) \Psi + \sum_i \left[ \frac{1}{2} \partial^M \Phi_i \partial_M \Phi_i - \tilde{\lambda}_i (\Phi_i^2 - \tilde{v}_i^2)^2 \right] \right\}. \quad (1) $$

where $\tilde{f}_i$ and $\tilde{\lambda}_i$ are real with $\tilde{\lambda}_i > 0$. (Our notation is such that capital letters run over the five dimensions while Greek letters over the four dimensions.)

It is convenient to use dimensionless parameters. We therefore use rescaled fields

$$ \Psi \rightarrow \psi \sqrt{L}, \quad \Phi \rightarrow \varphi \tilde{v}, \quad (2) $$

and introduce the following dimensionless quantities

$$ u \equiv \frac{x_5}{L}, \quad a_i \equiv \sqrt{2 \tilde{\lambda}_i \tilde{v}_i L}, \quad f_i \equiv \frac{\tilde{f}_i}{\sqrt{2 \tilde{\lambda}_i}}, \quad X \equiv - \frac{a_2 f_2}{a_1 f_1}. \quad (3) $$

Note that $0 \leq u \leq 1$. We order the two scalars such that $a_1 < a_2$ and further define $f \equiv f_1$ and $a \equiv a_1$. Then, the action can be written as

$$ S = \int dx_5 \int_0^1 du \left\{ \bar{\psi} \left( i \gamma^\mu \partial_\mu - \frac{1}{L} \gamma_5 \partial_5 - \frac{f a}{L} (\varphi_1 - X \varphi_2) \right) \psi \right\} 
+ \sum_i \frac{\tilde{v}_i^2}{L} \left[ L^2 \frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i - \frac{1}{2} \partial_5 \varphi_i \partial_5 \varphi_i - \frac{a_i^2}{2} (\varphi_i^2 - 1)^2 \right], \quad (4) $$

where $\partial_5 \equiv \frac{\partial}{\partial u}$.

We move to investigate the functions $h_i(u) \equiv \langle \varphi_i \rangle(u)$. For one scalar they were investigated in detail in [3, 7]. Since we assume that there is no interaction between the scalars we can use the one scalar model results for each of the scalars. Neglecting the Yukawa interactions, and working in the limit where $[8]

$$ a_i \gg 1, \quad (5) $$
we have
\[ h_i(u) = \tanh [a_i u] \tanh [a_i (1 - u)]. \tag{6} \]
Since \( h_i(u) \) is symmetric under \( u \to 1 - u \) we consider only the \( 0 \leq u \leq 1/2 \) range. We will make use of the following approximations
\[ h_i(u) \approx \begin{cases} 1 & \text{for } u \gg 1/a_i , \\ a_i u & \text{for } u \ll 1/a_i . \end{cases} \tag{7} \]
Of particular interest is the following function
\[ g(u) = h_1(u) - X h_2(u). \tag{8} \]
Given the values of \( a_i \) and recalling that \( a_1 < a_2 \), the sign of \( g(u) \) is determined by \( X \) as follows:

(i) \( X > 1 \)
\[ \text{sign}[g(u)] = -1. \tag{9} \]

(ii) \( X < a_1/a_2 \)
\[ \text{sign}[g(u)] = 1. \tag{10} \]

(iii) \( 1 > X > a_1/a_2 \)
\[ \text{sign} \left[ g \left( u \ll \frac{1}{a_2} \right) \right] = -1, \quad \text{sign} \left[ g \left( u \approx \frac{1}{2} \right) \right] = 1. \tag{11} \]

Eq. (11) teaches us that when condition (iii) is satisfied \( g(u) \) vanishes not only on the boundaries. Namely, there exists a point \( 0 < u_{\text{max}} < 1/2 \) such that \( g(u_{\text{max}}) = 0 \). (Note that since \( h_i(u) \) is symmetric under \( u \to 1 - u \) if \( g(u) \) crosses zero in the bulk, it crosses it twice, at \( u_{\text{max}} \) and \( 1 - u_{\text{max}} \).)

While we cannot solve analytically for \( u_{\text{max}} \) in the general case, we can do it in what we denote as the constant mass approximation
\[ a_1/a_2 \ll 1. \tag{12} \]
In this approximation \( u_{\text{max}} \) is given by
\[ u_{\text{max}} \approx \frac{\arctanh(X)}{a} = \frac{X}{a} + O(X^3). \tag{13} \]

Figs. (1b) and (1c) show the function \( g(u) \) (the thin black curves) in the cases described by eq. (11) for typical values of parameters. The figures demonstrate the main point of this section: The two scalar model yields an effective scalar profile which has several zeros. In particular, one of them is not an orbifold fixed point. Thus, we can construct models in which the fermions are localized at various points in the fifth dimension and not only at the orbifold fixed points.
III. THE FERMION ZERO MODE

In the following we analyze the functional behavior of the fermion zero mode wave function and, once more fermions are introduced, the overlaps between their zero mode wave functions in the fifth dimension. Though we do it in the context of a two scalar model we shall see that some of our conclusions are rather generic. They apply to other realizations of the split fermions scenario.

Following the standard treatment (see e.g. [1]) we find the differential equation for $y(u)$, the fifth dimension wave function of $\Psi_L$:

$$\frac{1}{y(u)} \frac{\partial y(u)}{\partial u} = -fa \ g(u) , \quad (14)$$

where $g(u)$ is defined in (8). The solution is given by

$$y(u) = N \exp \left[ -fa \int_0^u g(w) \ dw \right] , \quad (15)$$

where $N$ is the normalization factor. The local maxima of $y(u)$ are at points in which $g(u)$ vanishes. Which of these maxima is the global maximum depends on the model parameters, and can be at any of these points.

A. A single scalar model

We begin our analysis by setting $X$ to zero in eq. (8) and returning to the single scalar case [1, 4]. Then, the fermion zero mode is given by

$$y(u) \simeq N \exp \left[ -f \ln \cosh(au) \right] \quad \text{for} \quad u \lesssim 1/2 . \quad (16)$$

We shall analyze $y(u)$ in two limits, large and small $f$. We show that these are related to the two scenarios studied by AS [1] and KT [4] respectively. We assume that $f$ is positive and thus the maximum of $y(u)$ is at $u = 0$. Then, we have

$$y(u) \approx N \exp \left[ -\frac{fa^2}{2} u^2 \right] \quad \text{for} \quad u \ll 1/a ; \quad (17)$$

$$y(u) \approx N \exp [-f(a u - \ln 2)] \quad \text{for} \quad 1/a \lesssim u \lesssim 1/2 . \quad (18)$$

When eq. (17) is applicable, the fermion wave function is approximately a Gaussian with a width

$$\Gamma \sim \frac{1}{a \sqrt{f}} . \quad (19)$$

When eq. (18) is applicable, the fermion wave function is approximately an exponential with a width

$$\Gamma \sim \frac{1}{af} . \quad (20)$$
Substituting \( u = 1/a \) in (17) we learn that when \( f \gg 1 \) the exponential part is negligible and then \( y(u) \) is of a Gaussian shape. Similarly, substituting \( u = 1/a \) in (18) we learn that for \( f \lesssim 1 \) the Gaussian part is negligible.

B. A two scalar model

We now move to the two scalar scenario. While the integral in (15) can be solved analytically, the result is not very illuminating. We therefore consider \( y(u) \) in various limits related to the three cases of eqs. (9), (10) and (11).

When \( X > 1 \) or \( X < a_1/a_2 \) the function \( g(u) \) vanishes only on the fixed points and \( y(u) \) is localized at either \( u = 0 \) or \( u = 1 \). This case is similar to the single scalar model (with a zero mass) discussed above. The corresponding fermion wave function will be of the form of either Gaussian or exponent as shown in eqs. (17) or (18). In the limit of large \( a_i \) and with \( f \) of order unity it reproduces the KT exponential model.

When \( 1 > X > a_1/a_2 \) the fermions can be localized in the bulk. It is useful to discuss this case in the constant mass approximation [see eq. (12)]. In this approximation the second term in \( g(u) \) can be treated as a constant and then

\[
y(u) \simeq N \exp \left\{ -f \left[ \ln \cosh(au) - X \right]\right\} \quad \text{for} \quad u \lesssim 1/2 .
\]

Substituting \( X = ML/af \) in (21), with \( M \) being the 5d fermion mass, the model reproduces the constant mass models of [1, 2]. Thus all our following analysis and conclusions apply to these models as well.

We like to approximate \( y(u) \) near its maxima. The situation in this case is more involved since \( y(u) \) has two maxima and which is the dominant one depends on the interplay between the values of \( X, a \) and \( f \). Inspecting Eq. (15) we see that this is determined by the relative size of \( I_p \) and \( I_n \) defined as

\[
I_p = \int_0^{u_{\text{max}}} g(u)du, \quad I_n = \int_{u_{\text{max}}}^{1-u_{\text{max}}} g(u)du .
\]

Once \( I_p > I_n \) the global maximum is at one of the fixed points, while in the opposite case it is at \( u_{\text{max}} \) or \( 1 - u_{\text{max}} \). We define \( X_b \) to be the transition point, namely, the point where \( I_p = I_n \). Using the constant mass approximation [eq. (21)] and the linear approximation for \( h_1 \) [eq. (7)] we find the following estimate for the transition point:

\[
X_b \sim \frac{2a - 4}{2a - 3} .
\]

We therefore consider three cases:

- \( X > X_b \). The global maximum is at one of the orbifold fixed points. This case is similar to the case of a function with only one maximum and does not lead to a new structure.
• $X < X_b$. The dominant maximum is in the bulk, $0 < u_{\text{max}} < 1$, and $y(u)$ is approximately given by

$$y(u) \approx N \exp \left[ -\frac{fa^2(1-X^2)}{2}(u-u_{\text{max}})^2 \right] \quad \text{for} \quad |u-u_{\text{max}}| \ll 1/a. \quad (24)$$

This is a Gaussian with a maximum at $u_{\text{max}}$ and a width

$$\Gamma = \frac{1}{a\sqrt{f(1-X^2)}}. \quad (25)$$

For $X \ll 1$ the width is independent of $X$ [as in eq. (19)], and the maximum location is linear in $X$. This is the case discussed in [1, 2] (with the substitution $X = ML/af$). The generic situation, however, is that the peak of the wave function is not linear with $X$ and that its width grows with the distance of the peak from the origin.

• $X \sim X_b$. The two maxima are significant. This case cannot be described analytically but it is a combination of the above two cases. We remark that this transition region between the other two cases considered above becomes sharper as the value of $f$ increases. Since we are working with moderate values of $f$ it does not require fine tuning of $X$ to have a situation where both peaks are important.

In figures (1a)-(1c) we plot the effective two scalar vev, $g(u)$, and the corresponding fermion wave function for these three cases.

### C. Wave functions overlaps

We extend the model by including the SM fermion fields and one Higgs field. After compactification, the zero modes of these fields consist of the 4d SM fields. In particular, we are interested in how the 4d Yukawa interactions are generated form the 5d theory. Consider, for example, the 5d Yukawa interaction between a quark doublet ($Q$), a down type quark singlet ($D$) and the Higgs field ($H$)

$$\mathcal{L} = \frac{Y_5}{\sqrt{M_*}} \bar{Q}HD + h.c., \quad (26)$$

where $M_*$ is the natural scale of the 5d theory. After compactification the 4d Yukawa interaction is given by

$$\mathcal{L} = Y_5 \bar{d} Lh d_R + h.c., \quad (27)$$

where lower case letters represent 4d zero mode fields. The overlap $K$ is defined as

$$K \equiv \int_0^1 y_D(u)y_Q(u)du, \quad (28)$$

where $y_D$ ($y_Q$) is the fifth dimension wave function of the singlet (doublet) field. We assumed here that the Higgs vev is flat in $u$. While this is not generally the case, we assume that its
shape is not far from flat, and thus approximating it by a flat profile is reasonable. Assuming $Y_5 \sim O(1)$ the flavor structure arises from the various overlap integrals [1].

Suppose that we consider a realistic 5d model which accounts for the quark masses and mixing angles by Gaussian like suppressions. (Such overlaps were assumed, for example, in constructing the MS model [2] and in many other papers see e.g. [3, 5].) The model is specified by the Yukawa couplings $f_i$ and effective masses $X^i$ of the nine SM quark fields. We ask what are the ranges of the 5d flavor parameters in order for the above model to yield the correct masses and CKM matrix elements.

In order to obtain Gaussian suppressed overlaps and to reproduce the MS model several conditions are required. First, the significant overlap between any two fermion wave function has to be in the region where both wave functions are, to a good approximation, Gaussian. Second, their maxima have to scale linearly with $X$. The first condition is satisfied in the large $f$ limit. The second condition requires small $X$.

To make our discussion concrete we assume that the most distant fermions have $X < \sim 0.3$ which implies $u_{\text{max}} \sim 0.3/a$. Using Eq. (24) we find that the overlap between the Gaussian wave functions, assuming universal couplings $f$, is given by $\exp(-fX^2)$. In [4] it was found that a viable model of quark masses and mixing requires a separation of roughly 18 standard deviations between the two most distant quarks. Then, for $X \lesssim 0.3$ we find

$$f \gtrsim 2 \left( \frac{18}{0.3} \right)^2 \sim 10^4.$$  

(29)

Assuming perturbativity, this implies that $\tilde{\lambda}_i$ [introduced in (1)] must be much smaller than their natural values [12]. Note that to a large extent, condition (29) is independent of the fundamental theory parameters such as $L$, $a$ and $M_*$. Thus, it is generic that models with Gaussian overlaps require very large couplings $f$.

While the model of pure Gaussian wave functions seems unnatural, using the analysis of subsection III B it is possible to construct models which account naturally for the quark flavor parameters. This is demonstrated in the next section.

IV. TOWARD A REALISTIC NATURAL MODEL

As already mentioned, several models that produce the observed fermion masses and mixing angles had been constructed. Some of them use Higgs wave functions that are not flat to generate the flavor structure. We do not discuss these models here. Instead we discuss models that assume, similar to our model, that the Higgs vev do not play a significant role in generating the observed flavor patterns. In the Appendix we discuss the GGH model [3] since our model is not directly related to it. Below we describe the KT model [4] and our improved version of it.

The KT setup is as follows. LH fermions are localized on one of the orbifold fixed points, and RH fermions on the other one. The different fermion wave functions have different
widths such that the lighter the fermion is, its LH and RH wave functions are narrower. This induces hierarchical structures for the corresponding mass matrices. Yet, in order to produce the observed quark masses and mixing angles, this structure is not enough and the 5d Yukawa couplings must also have some flavor structure [4].

One reason for that, for example, is related to the large top mass. In order to generate the $O(1)$ 4d Yukawa coupling, the overlap between $y^{Q_3}$ and $y^{u_3}$ has to be large. Since these wave functions are localized on different orbifold fixed points, it implies that at least one of them has to be very broad. Both possibilities, however are problematic. When $y^{Q_3}$ is broad, a large mass for the $b$ quark is generated, in contrast to observation. When $y^{u_3}$ is broad the overlaps between it and the other doublets are large. Then, large CKM mixing angles are generated, in disagreement with the data. Similar problems are encountered when one tries to account for the flavor parameters of the other generations.

The KT model is significantly improved once some of the fields are localized in the bulk. For example, the problem with the third generation is solved once $y^{Q_3}$ and $y^{u_3}$ are localized near each other far from any of the orbifold fixed points. Then, their overlap is large and large $m_t$ is produced. At the same time, the overlaps of $y^{u_3}$ with the other quark doublet wave functions are small, thus producing small CKM mixing angles. The $b$ quark mass is produced by constructing $y^{d_3}$ to have a double peak shape, such that it has moderate overlap with $y^{Q_3}$.

Based on these ideas we construct an example of a configuration which roughly reproduces the correct quark masses and mixing angles. We choose the following parameters:

\[
a_1 = 4, \quad a_2 = 12; \\
Q^1,2,3 \sim 30, 4, 40, \quad \Xi^{Q1,2,3} = -4, -1.9, 0.73; \\
Q^1,2,3 \sim -1, 1, 28, \quad \Xi^{u1,2,3} = -2.3, 2.3, 0.78; \\
Q^1,2,3 \sim -1, -1, 21, \quad \Xi^{d1,2,3} = -2, -1.4, 0.87. \quad (30)
\]

The resultant shapes of the various wave functions are shown in figs. (2a) and (2b) for the up and down type quarks respectively. The mass matrices of the up type and down type quarks are given, up to order one coefficients, by (in MeV units)

\[
M^u \sim \begin{pmatrix} 1.9 & 3.1 \cdot 10^2 & 6.4 \cdot 10^2 \\ 4.7 & 7.2 \cdot 10^2 & 4.2 \cdot 10^3 \\ 1.7 \cdot 10^2 & 3.8 \cdot 10^3 & 1.6 \cdot 10^5 \end{pmatrix}, \quad M^d \sim \begin{pmatrix} 5.3 & 4.0 \cdot 10^1 & 8.8 \\ 1.3 \cdot 10^1 & 9.3 \cdot 10^1 & 5.3 \cdot 10^1 \\ 2.4 \cdot 10^2 & 7.2 \cdot 10^2 & 3.4 \cdot 10^3 \end{pmatrix}. \quad (31)
\]

Their eigenvalues are given by

\[
m_u \sim 1.9 \text{ MeV}, \quad m_c \sim 7.2 \cdot 10^2 \text{ MeV}, \quad m_t \sim 1.6 \cdot 10^5 \text{ MeV}, \\
m_d \sim 5.3 \text{ MeV}, \quad m_s \sim 9.3 \cdot 10^1 \text{ MeV}, \quad m_b \sim 3.4 \cdot 10^3 \text{ MeV}. \quad (32)
\]

which agree with the known quark masses, calculated at the $M_Z$ scale [13, 14]. The rotation
matrices from the left that diagonalize the mass matrices are given by

\[ V_L^u \sim \begin{pmatrix} 1 & 2\lambda & 0.7 \cdot \lambda^3 \\ 2\lambda & 1 & 0.5 \cdot \lambda^2 \\ 0.4\lambda^3 & 0.6 \cdot \lambda^2 & 1 \end{pmatrix}, \quad V_L^d \sim \begin{pmatrix} 1 & 2\lambda & 0.4 \cdot \lambda^3 \\ 2\lambda & 1 & 0.4 \cdot \lambda^2 \\ 0.5 \cdot \lambda^3 & 0.4 \cdot \lambda^2 & 1 \end{pmatrix}, \quad (33) \]

where \( \lambda = 0.22 \). The absolute values of the elements of the CKM matrix, \( V_{CKM} = (V_L^u)^{\dagger} V_L^d \), are given to leading order in powers of \( \lambda \) by

\[ |V_{CKM}| \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}. \quad (34) \]

The values of the quark mixing angles agree with the experimental data \([13]\). Assuming arbitrary phases we find for the Jarlskog measure of CP violation \([13]\)

\[ J \sim 10^{-5}, \quad (35) \]

in agreement with the value obtained from global fit to the available data \([13]\). We conclude that the above model also reproduces the observed size of CP violation. (The same is true also in the original KT model \([4]\).) This is in contrast to the MS model, where the observed order of magnitude of \( J \) cannot be accommodated \([10]\). The reason is that in the MS model there are very small entries due to the strong Gaussian suppressions. Here this is not the case. The matrices \( M^u \) and \( M^d \) do not contain such tiny entries. Actually, the generic situation is that split fermion models in five dimensions that produce the correct masses and mixing angles also account for the observed CP violation.

The example above is only a demonstration that the observed fermion masses and mixing angles can be generated. The apparent small deviations from the experimental values are due to the unknown Yukawa couplings. The point we emphasize is that the flavor hierarchy arises only from the overlap between the different wave functions. The unknown \( O(1) \) Yukawa couplings do not carry any hierarchical flavor structure.

V. DISCUSSION AND CONCLUSION

There are several related issues that we did not study and seem to us worth further investigation. We did not discuss the lepton sector. It is easy to generate a model for the charged lepton masses. The issue of neutrino masses is more complicated and may require additional ingredients. We also did not investigate the Higgs vev shape. While we assumed that it is flat, generally this is not the case. It seems to us that it can be flat enough in order not to upset our general conclusions. However, it is interesting to fully understand the vev shape and its implications.

Our model can be extended, for example, by adding more scalars. Then, the number of the fermion wave functions maxima can be more then two, up to the number of scalars.
One of the maxima is an orbifold fixed point, and the others are in the bulk. Having more
than one maxima in the bulk can be used to obtain a richer structure. However, this extra
flexibility is not needed in order to generate the observed quark masses and mixing angles.

The framework we considered cannot be directly related to the fundamental theory of
nature. It must be a low energy limit of a more fundamental theory that should also explain
the compactification and stabilization of the orbifold setup. Therefore, in general, non
renormalizable terms and terms localized on the orbifold fixed points can be present in the
effective theory. Since such terms modify the shape of the fermion wave functions they were
used in [3] and [4] to generate the needed structure. These terms, however, have to be very
large in order to have significant effects. In our case, however, we do not need them. Thus,
assuming that they are at their natural size, they are not expected to significantly affect the
important ingredients of our model: The number of maxima, their widths and their rough
locations.

So far we considered only the zero modes. Heavy modes, namely, the Kaluza-Klein tower
of the SM fields and heavy SM singlets fields, cannot be ignored since they can significantly
contribute to SM rare processes. In particular, proton decay, $\nu-\bar{\nu}$ oscillation, meson mixing
and lepton number violating processes. The most severe bound on the properties of the heavy
fields is usually obtained form proton decay data. In models where the fundamental scale
is low, very small overlap between the leptons and quarks wave functions was proposed as
a way to avoid rapid proton decay [4]. The required tiny overlap can be achieved in models
with Gaussian wave functions. However, as we explained, such models require significant
fine tuning. Thus, proton stability can be explained by having a very high fundamental
scale or by adding new ingredients to the model. On the other hand, the data on other rare
processes can be accommodated in models with a relatively low fundamental scale (see e.g.
[4]).

To conclude we summarize our main results. We showed that realistic models of split
fermions can be constructed using two scalars in a world with one compact extra dimension
compactified on an orbifold. We found that models of split fermions with Gaussian overlaps
require large 5d Yukawa couplings, and are therefore unnatural. We construct a realistic
two scalar model where the overlap is not purely Gaussian which accounts naturally for the
quark flavor parameters.

Acknowledgments

It is a pleasure to acknowledge helpful discussions with David E. Kaplan and Martin
Schmaltz. Y.G. thanks the Aspen Center for Physics for hospitality while this work was
started. Y.G. is supported in part by the Israel Science Foundation (ISF) under Grant
No. 237/01, by the United States–Israel Binational Science Foundation (BSF) through Grant
No. 2000133, and by the fund for the promotion of research at the Technion.
APPENDIX A: THE GGH MODEL

Here we analyze the GGH mechanism of fermion separation [3]. We show that in this model it is rather unnatural to obtain sharp localization and separation of the fermions at the same time.

With our scaling [see eqs. (2) and (3)], the GGH model is described by

\[
S_\psi = \int dx_\mu \int_0^1 du \left\{ \bar{\psi} \left[ i \gamma^\mu \partial_\mu - \frac{f a}{L} \left( 1 - \frac{\kappa}{L^2 M_*^2} \left( L^2 \partial_\mu \partial^\mu - \partial_5^2 \right) \right) \right] \psi \right. \\
\left. + \frac{\tilde{v}^2}{L} \left[ L^2 \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi_i - \frac{1}{2} \partial_5 \varphi \partial_5 \varphi - \frac{a^2}{2} \left( \varphi^2 - 1 \right)^2 \right] \right\}. \tag{A1}
\]

When \( a \) is large, \( g(u) \) is approximated by

\[
g(u) \approx f a \tanh(au) \left[ 1 - \frac{2\kappa}{b^2} \text{sech}^2(au) \right] \quad \text{for} \quad u \lesssim 1/2, \tag{A2}
\]

with \( b \equiv L M_* / a \gg 1 \). The fermion wave function in the fifth dimension is given by [see eq. (15)]

\[
y(u \lesssim 1/2) \approx N \exp \left\{ -f \left[ \ln [\cosh (au)] + \frac{\kappa}{ab^2} \text{sech}^2 (au) \right] \right\}. \tag{A3}
\]

The maximum is found at

\[
u_{\text{max}} = \frac{1}{a} \arctanh \sqrt{1 - \frac{b^2}{2\kappa}}, \tag{A4}
\]

and the width is given approximately by

\[
\Gamma \approx \frac{1}{a \sqrt{f}} \cdot \frac{1}{\tanh(au_{\text{max}})}. \tag{A5}
\]

Note few problematic points:

- In order to have a maximum in the bulk, very large coupling is needed, \( \kappa > b^2/2 \) [3].

- When \( \kappa \gg b^2/2 \), the peak of the wave function is far from the origin. In that case, however, \( u_{\text{max}} \propto \log(\kappa) \). Consequently, to separate the fermions in the extra dimension, different flavors must have large hierarchy in their corresponding values of \( \kappa \).

- When \( u_{\text{max}} \) is small the width is proportional to \( 1/u_{\text{max}} \). Consequently, a very large \( f \) is needed in order to get a sharply localized wave functions.

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Fig. 1: The typical shapes of the scalar vevs and the fermion wave functions. The thin black curves correspond to the effective scalars vev, \( g(u) \) [eq. \((8)\)]. The thick gray curves correspond to the fermion wave functions, \( y(u) \) [eq. \((21)\)]. The relevant parameters are \( a_1 = 5, \ a_2 = 17 \), and \( X = 1.15, 0.75, 0.85 \) for figs. (1a), (1b) and (1c) respectively. In this example \( X_b \sim 0.87 \) [eq. \((23)\)].

Fig. 2: The configuration of the quark wave functions in the extra dimension. Figs (2a), (2b) are related to overlaps between the up and down type quarks respectively. \( y^{Q1,2,3} \) are the dotted, dashed, and dashed-dotted curves respectively (shown in both figs). \( y^{u1,2,3} \) and \( y^{d1,2,3} \) are the thin black, mid gray and thick light curves of figs. (2a) and (2b) respectively. The relevant parameters are given in [eq. \((30)\)].