Arithmetic Operations Beyond Floating Point Number Precision

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Abstract: In basic computational physics classes, students often raise the question of how to compute a number that exceeds the numerical limit of the machine. While technique of avoiding overflow/underflow has practical application in the electrical and electronics engineering industries, it is not commonly utilized in scientific computing, because scientific notation is adequate in most cases. We present an undergraduate project that deals with such calculations beyond a machine’s numerical limit, known as arbitrary precision arithmetic. The assignment asks students to investigate the approach of calculating the exact value of a large number beyond the floating point number precision, using the basic scientific programming language Fortran. The basic concept is to utilize arrays to decompose the number and allocate finite memory. Examples of the successive multiplication of even number and the multiplication and division of two overflowing floats are presented. The multiple precision scheme has been applied to hardware and firmware design for digital signal processing (DSP) systems, and is gaining importance to scientific computing. Such basic arithmetic operations can be integrated to solve advanced mathematical problems to almost arbitrarily-high precision that is limited by the memory of the host machine.

Keywords: computational physics education; floating point number; overflow/underflow; arbitrary precision; Fortran; GPU.

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1 Introduction

The concept of number of significant figures is important in various scientific and engineering disciplines. Fewer significant figures represents less precision, which can severely damage the result [1]. When calculations are made using computers, such that random and human errors do not apply, the highest and lowest values that can be represented and the precision of any number are limited by the finite number of digits that are used to
store a floating point number, which is a binary version of a number in scientific notation, on the computer.

In the fixed point (integer) representation, a single precision integer is stored as 32 bits of 1s and 0s. Converted to a decimal value, such a number may be as low as \(-2^{31} = -2,147,483,648\) or as high as \(2^{31} - 1 = +2,147,483,647\). The precision of the significand (number of digits in the significand) is 32 bit, or ten digits in the decimal system. In the IEEE 754 standard for floating point arithmetic, a single-precision real number is also stored using 32 bits binaried, but one bit is used for the sign, eight bits store the exponent, and 23 bits store the significand of the number in scientific notation (called the fractional mantissa). The precision is 24-bit, and a decimal value between \(\pm 1.4 \times 10^{-45}\) and \(\pm 3.4 \times 10^{38}\) is allowed, with a significand precision of \(\log(2^{23}) \approx 7\). Similarly, the IEEE 64-bit floating-point arithmetic standard provides 52 mantissa bits and 12 bits for the exponent, including the sign, giving approximately a 16 decimal digit accuracy and a decimal range of \([\pm 4.94 \times 10^{-324}, \pm 1.70 \times 10^{308}]\). The length of the significand determines the precision with which numbers can be represented.

Since the number of binary bits that are used to express a number is finite, the set of numbers that can be stored by a computer, called machine numbers, does not cover all numbers in the physical world. Single-precision fixed point integers are precisely stored integers with a 32-bit resolution. Single-precision floats, however, are not exact representations with a resolution of 24 bits; their value is truncated at the lower end, although numbers around \(2^{227}\), much larger than the maximum of 32-bit integers, \(2^{32}\), can be expressed. An overflow (\(< -3.4 \times 10^{-38}\) or \(> 3.4 \times 10^{38}\)) or underflow (between \(-1.4 \times 10^{-45}\) and \(+1.4 \times 10^{-45}\)) error condition occurs when the number exceeds the maximum or falls below the minimum of machine numbers.

In some computations, even though the final number can be represented within machine limits, the intermediate results may overflow. For instance, the number of ways to select ten objects from a set of \(200\) is \(200!/(190!10!)≈2.2 \times 10^{16}\); the answer fits in the double precision range, but both 200! and 190! overflow. Although large factorials can be evaluated via Stirling’s formula, which is used in formula calculators, such as the scientific calculator in Windows XP, in personal computers, the largest factorials that can be calculated as 32-bit and 64-bit integers are 12! and 20!, respectively. Additionally, the largest number that most electronic calculators using two-digit exponents can handle is 69! (\(1.71 \times 10^{69} < 69! < 10^{100} < 70!\)). Therefore, given available tools, a simple combinatorial problem may become computationally formidable.

This study presents an undergraduate project on the methods for computing the exact value of a number beyond machine limits, known as arbitrary precision arithmetic or bignum arithmetic, using the high-level scientific programming language Fortran. While the approach has practical applications to the electric/electronic firmware design industry, it is not commonly adopted by physical scientists. The basic idea of this method is to use arrays to store intermediate and final results, preventing numerical overflow or underflow. This investigation explores methods for multiplication and division. Such basic arithmetic operations can be integrated to solve more complicated mathematical problems to a high precision that is limited by the memory of the host machine.

2 Arbitrary Precision Arithmetic using Arrays

In a computer program, an array is an ensemble of numbers that represent data of the same class. The size of the dataset (number of elements) can vary upon declaring the array. If a large or small number with many digits is divided into several segments, which are assigned to various elements of an array, then the number can be expressed without overflowing or underflowing the limits of the declared data type. For example, the 11-digit number ‘10123456789’ can be stored in a single-precision integer array of 11 elements, with the ones digit ‘9’ assigned to the first element of the array, the tens digit ‘8’ to the second element, and so on. Each digit in the number is allocated to a single array element such that the whole number is represented by the array.

A single array element can also accommodate several digits, reducing computing time. If a 32-bit integer is to be represented, since the overflow threshold is \(2 \times 10^9\), one array element can accommodate as many as ten digits. An array of 1000 elements can therefore represent a number of \(\approx 10000\) digits, and if a 1000-digit number is to be stored, then an array with only 100 elements is needed. However, to prevent overflow in a subsequent computation, the maximum number of digits could be optimized to be four or five, depending on the problem. The size of an array can be increased until the memory limit of the host computer is reached. Hence, the represented numbers are restricted only by the computer memory.

3 Multiplication

Consider the successive multiplication \(2 \times 4 \times 6 \times 8 \times 10\ldots\) with a total of \(N\) items, such that the answer swiftly exceeds the upper limit of floating point numbers. A general approach for avoiding overflow is to take the logarithm of the expression and then compute the exponent of the result. However, logarithms and exponentiation introduce rounding errors, so this method can not yield the exact answer.

The unknown is here split into the multiplication of two terms: a power of two \(2^N\) and a factorial \(N!\). In the program “successive_multiply”, \(M^N\), where \(M = 2\), is first calculated. The subroutine “multiparameterreset” assigns initial values to the array elements \(A(i)\); all
elements are zero except that $A(1) = 1$. In the subroutine “multipl”, the initial multiplicand $A(1) = 1$ is multiplied by the multiplier $M$ ($M = 2$ for computing $2^N$), and the result is stored back to $A(1)$. Parameter $C$ denotes whether extra digits are produced. If the result has more than 1 digit ($C \neq 0$), then the extra digit is carried to (added to) the next product and array element $A(2)$. Take $5 \times 3$ for example, $8 \times 3 = 24$ is first calculated, and the ones digit ‘4’ is assigned to $A(1)$. The tens digit ‘2’ is carried to the tens place, such that $A(2)$ is assigned with ‘7’, which is the ones digit of the 2nd ($i=2$) operation, $5 \times 3 = 17$. In the 3rd ($i=3$) operation, the tens digit ‘1’ of the 2nd operation is passed over to the hundreds place, such that $A(3)$ stores the ones digit of $0 \times 3 + 1 = 1$. The same procedure is repeated for each digit (the $i$ loop in “multipl”) and each multiplication (the $i$ loop in the main program). In program “successive_multipl”, each element of the array $X$ stores one digit of the final result of $2^N$. The operation is similar to written multiplication based on decomposing the multiplicand. The $k$ loop serves to carry a multi-digit number. It can be omitted in the calculation of $2^N$, because only a single-digit number has to be carried forward.

The method to compute the factorial of $N$ is the same, except that the multiplier is replaced by $M = M + 1$. The array $Y$ stores the final result of $N!$. In program “successive_multiplier” (program 1 and 2), the power term and the factorial are multiplied together to attain the answer in the subroutine “multiply_two_ans”. Each digit of the multiplicand $X(i)$ is multiplied by the that of the multiplier $Y(i)$. The intermediate products $P(i + j - 1) = X(i) \times Y(j)$ are added up and accumulated in columns. Then in the subroutine “multipl”, extra digits in a product are carried to next higher places and the numbers are rearranged, which computes the product of $P$ and 1 ($M = 1$). Figure 1(a) illustrates the decomposed method used in “multiply_two_ans”. The total number of digits of the multiplied result is either $N x + N y - 1$ or $N x + N y$, where $N x$ and $N y$ are the numbers of digits in $X$ and $Y$. Notably, the answer to $2 \times 4 \times 6 \times 8 \ldots$ can also be computed directly via the subroutine “multipl”, with $M = M + 2$ being the multiplier.

The algorithm program 2 and Fig. 1(b) present the second algorithm for multiplication. This method differs from the previous one in that a double loop is used, such that the product $X(i - 1)Y(j)$ is carried to the next one $X(i)Y(j)$ immediately after each element is attained. Intermediate products, i.e., remainders, are attained in a row. The same operations are applied for various $j$ to yield another row of remainders, which values are added to previous ones in columns once they are obtained. Iterations of the above steps yield the final product and remainders (denoted by $P(i + j)$ or $P(k)$). The multiplication of overflowing or underflowing fixed numbers can be extended straightforwardly to that of floating-point numbers provided the exponents of the fixed numbers are specified. Program “multi2” (program 4) shows the multiplication of two floats.

Figure 1(c) and program “multi3_reverseorder” suggest a similar method to method 2. Each digit of the number is reversely ordered such that the highest place of a number is stored in the first array element. In each $i$ loop the $X(i)Y(j)$ products are similarly carried leftward in a row to compute the intermediate product (remainder). Contrary to method 2, remainders in various rows (varying $i$) are summed in columns; only when all of the remainders have been attained. In the computer program the summation is accomplished via the use of a separate double do-loop.

The memory required for such arithmetic operations is well below the typical size of memory adopted in home computers. Notably, a single-precision integer occupies four bytes of memory. If each single-precision array element stores one digit, then such an array of 200,000 elements can represent 0.2 million digits, whose text output is about $\sim 600$ pages long, at the expense of only 1MB of memory. Both the long program, invoking “multiply_two_ans”, and the direct method, using only “multipl”, take $\sim 150$ s to compute the $228288$ digit-long value of $50000! \times 2^{50000}$.

4 Division

4.1 Simple Division

Division may yield an infinite decimal without recurrence. The precision of division may therefore depend on the declared array size of the quotient. Double precision floats allow a maximum of 15-16 digits in the significand. The algorithm program 6 illustrates how to acquire a higher precision, say 30 digits, to the right of the decimal point, for a simple division of two numbers within the machine limits, $Q = X/Y$.

The integer quotient $Q0$ is obtained first. The remainder is stored in the variable $R$. The remainder is multiplied by 10, which result is taken as the new dividend and is divided by the divisor. The first decimal place of the quotient is then obtained, and assigned to the first element of array $Q(i)$. The same operations are performed for each digit until the required digits have been computed and stored. The idea is the same as that of division by hand in which the dividend is decomposed.

4.2 Successive Division

Division of an arithmetic progression of numbers, $1/2/4/6/8/10\ldots$ with a total of $N$ items is considered in program “sd03”. All of the temporary dividend and quotient $T(i)$ are initially ‘0’, except that $T(1) = X = 1$. When the $n$th decimal place of the quotient is attained, it is stored in the (n+1)th element of $T$. The final decimal quotient $Q(1:s)$ is extracted from $T(2:s+1)$. Since the value that is assigned to each array element is restricted by the machine number, in any step, the new dividend $10 \times R$ cannot be larger than the limit of a 32-bit fixed point number, 2,147,483,647, and the divisor cannot
Program 1 Successive Multiplication of Even Numbers

module global
implicit none
integer, parameter :: s=230000 ! array size
integer :: A(s),X(s),Y(s),P(s),M,C,Ntot,N=50000
end module

PROGRAM SUCCESSIVE_MULTIPL
use global
implicit none
integer :: i, Nx, Ny
real :: ti, tf
CALL CPU_TIME(ti)
!
-----calculate $2^N$-----
CALL multipl_parameter_reset
do i = 1, N
  M = 2
  CALL multipl ! $2^N$
end do
X = A
Nx = Ntot
write(*,"(6H2^N = ,230000I1)") (X(i), i=s,1,-1)
print*,"Number of digits of $2^N$ =", Nx
!
-----calculate $N!$-----
CALL multipl_parameter_reset
do i = 1, N
  M = M + 1
  CALL multipl ! $N!$
end do
Y = A
Ny = Ntot
write(*,"(6HN! = ,230000I1)") (Y(i), i=s,1,-1)
print*,"Number of digits of $N!$ =", Ny
!
-----calculate $2^N \times N!$-----
CALL multiply_two_Ans(Nx, Ny)
P = A
write(*,"(13H(2^N)*(N!) = ,230000I1)") (P(i), i=s,1,-1)
print*,"Number of total digits =", Ntot
CALL CPU_TIME(tf)
print *, 'computing time (sec) = ', tf-ti
END PROGRAM successive_multipl

SUBROUTINE multipl_parameter_reset
use global
implicit none
M = 0
C = 0
Ntot = 1
A(1) = 1
end Subroutine

SUBROUTINE multipl
use global
implicit none
integer :: i, k
do i = 1, Ntot
  A(i) = A(i)*M + C
  C = A(i)/10
  A(i) = mod(A(i),10)
end do
if (C /= 0) then
  Ntot = log10(dble(C)) + Ntot + 1
  do k = i,Ntot
    i = Ntot
    A(k) = A(k)+ mod(C,10)
    C = C/10
  enddo
end if
end Subroutine

end module

end program

Figure 1 Three decomposed methods for multiplication.
Program 2 Successive Multiplication of Even Numbers - Continued

SUBROUTINE multiply_two_Ans(Nx,Ny)
use global
implicit none
integer :: i, j, Nx, Ny
do j = 1, Ny
  do i = 1, Nx
    P(i+j-1) = P(i+j-1) + X(i)*Y(j)
  end do
end do
CALL multipl_parameter_reset
A = P
M = 1
Ntot = Nx + Ny - 2
CALL multipl
END SUBROUTINE

Program 3 Algorithm: Multiplication $X \cdot Y$ - Method 2

! initialization
K = 4; // number of digits for an array element
L = 10 ** K; // number limit for an array element
for i = 0 to Nx+Ny
  P(i) = 0;
end;

! multiplication
for j = 1 to Ny
  for i = 1 to Nx
    T = X(i) * Y(j) + P(i+j-1);
    C = T / L
    P(i+j-1) = mod(T,L)
    P(i+j) = P(i+j) + C
  enddo
endo

Etot = Ex + Ey
print"(1I1, 1H. 8I1, 3H E+, I3)", (P(k), k=9,1,-1), Etot+9-1
END PROGRAM

$ a.out
6.08431634 E+7

Program 4 Multiplication of Two Floats - Method 2

PROGRAM multi2
implicit none
integer :: X(5)=0, Y(4)=0, P(9)=0
integer :: T, C, i, j, k, L, Ex, Ey, Etot
X(5:1:-1) = (/ 8,5,3,4,6 /)
Ex = -3
Y(4:1:-1) = (/ 7,1,2,9 /)
Ey = +2
L = 10

do j = 1, 4
  do i = 1, 5
    T = X(i) * Y(j) + P(i+j-1)
    C = T / L
    P(i+j-1) = mod(T,L)
    P(i+j) = P(i+j) + C
  enddo
endo

Etot = Ex + Ey
print"(1I1, 1H. 8I1, 3H E+, I3)", (P(k), k=9,1,-1), Etot+9-1
END PROGRAM

$ a.out
6.08431634 E+7

place. Accordingly, the number 3 is stored as the first element of the quotient: Q(1)=3.

To compute the next digit of the quotient, take one or more digits starting from the remainders places in the dividend to form a new partial dividend that is larger than the divisor. Subtract the new partial dividend repeatedly until a partial quotient is yielded. Here, two more digits are included to form the new partial dividend [1,7,5], so the next partial quotient Q(2)=0. [1,7,5] is subtracted by [1,2,9] once and Q(3)=1 is yielded. The remainder is 46.

In the third cycle, [1,2,9] is subtracted from [4,6,6] four times; therefore, the partial quotient is Q(4)=4. In the fourth cycle, since the last digit of the dividend has been reached, ‘0’ is appended to the remainder 79 to form a partial dividend [7,9,0]. The operations are repeated until all of the digits of the quotient have been computed.

Program “division01” performs the division of two floating-point numbers partitioned into arrays. The values expressed by the arrays can be viewed either as integers of floats bearing the same number of digits to the left of the decimal point, while Ex and Ey denote the exponents. The zeroth elements are initialized to zero. Parameters ‘Li’ and ‘Lf’ refer to the initial and final places of the dividend that is used to form a partial dividend. Initially, Li=1 and Lf=Ny=3. In the second cycle, since the partial dividend begins from two digits to the right, ‘Li=3’ and ‘Lf=6’. The operations terminate when the quotient is expressed to the $s-1$
Program 5 Multiplication of Two Floats - Method 3

PROGRAM multi3_reverseorder

implicit none
integer :: i, j, k, T, L=10, m
integer :: C, C2 ! numbers to be carried when summing ! remainders in rows/columns
integer, parameter :: Nx= 5, Ny= 4, Ntot=Nx+Ny ! array size
integer, parameter :: Ex=-3, Ey=+2, Etot=Ex+Ey ! exponents
integer :: X(Nx)=0, Y(Ny)=0, R(Ny,Ntot)=0, P(Ntot)=0
X(1:Nx) = (/8, 5, 3, 4, 6/)
Y(1:Ny) = (/7, 1, 2, 9/)
m = Ny
AA: do j = Ny, 1, -1
   C=0
   BB: do i = Nx, 1, -1
      T = X(i)*Y(j) + C
      C = T / 10
      R(j, i+m) = mod(T,L)
      if (i==1) then
         R(j, m) = C
      endif
   enddo BB
   m = m - 1
enddo AA
C2=0
CC: do k = Ntot, 1, -1
   P(k) = P(k) + C2
enddo CC
EE: do j = Ny, 1, -1
   P(k) = R(j,k) + P(k)
enddo EE
C2 = P(k) / 10
P(k) = mod(P(k),L)
enddo CC
write(*, '(9I1,1X,4H*10^,1X,I3)') P, Etot
print'(1I1,1H. 8I1,3H E+,I3)',(P(k),k=1,Ntot),Etot+Ntot-1
END PROGRAM

Program 6 Algorithm: Simple Division

Q0 = X / Y // integer quotient
R = X % Y // remainder

for i = 1, array_size
   Q(i) = R * 10 / Y ; // decimal quotient
   R = R % Y
end ;

Program 7 Successive Divisions of Even Numbers

PROGRAM sd03

implicit none
integer, parameter :: s = 1000 ! array size
integer :: X = 1, Y = 1, R
integer :: Q(s), T(s+1)
intrinsic mod
integer :: i, j, N=99
T = 0
T(1) = X
do j = 1, N
   Y = Y+1 ! arithmetic progression
   ! Y = Y*2 ! geometric progression
   do i = 1, s
      R = T(i)*10 + T(i+1)
      Q(i) = R/Y 
      R = mod(R,Y)
      T(i+1) = R
   end do
   T(1) = 0
   T(2:s+1)=Q(1:s)
end do
write(*, "(0.,1000I1)") Q
END PROGRAM sd03
Program 8 Division of Two Floats - Method 1

PROGRAM division01
  implicit none
  integer :: i, j
  integer, parameter :: ss=200, s=30
  integer, dimension(0:ss) :: R=0, X=0, Y=0
  integer, dimension(1:s) :: Q=0

  integer, parameter :: Nx=6, Ny=3, Ex=2, Ey=-4, Etot=Ex-Ey
  integer :: Li=1, Lf=Ny

  X(0:Nx) = (/0,3,8,8,7,5,6/) ! Ex=+2, X = 3.88758E+2
  Y(0:Ny) = (/0,1,2,9/) ! Ey=-4, Y = 1.29E-4

  99 R = X
  ! Array subtraction: subtract divisor from partial dividend.
  AA: do i = Lf, Li, -1
    j = i - Lf + Ny
    if (X(i) < Y(j)) then
      X(i) = X(i) + 10 ! borrow 10 from left
      X(i) = X(i) - Y(j) ! subtract divisor
      X(i-1) = X(i-1)-1 ! subtract 1 from left
    else if (X(i) >= Y(j)) then
      X(i) = X(i) - Y(j) ! subtract directly
    end if
  end do AA
  ! Add numbers of subtraction to partial quotient.
  Q(Lf-Ny+1)=Q(Lf-Ny+1)+1
  ! Restore subtraction when remainder becomes negative.
  if( X(Li-1) < 0 ) then
    Q(Lf-Ny+1) = Q(Lf-Ny+1) - 1
    Li = Li + 1
    Lf = Lf + 1
    X = R
  end if
  ! Repeat operations.
  if( Lf-Ny+1 /= s ) goto 99

  print*, " quoitent:"
  write(*, *(1I1, 1H., 29I1, 7H * 10^ , 1I5)) &
  Q(1), (Q(i), i =2, s), Etot

END PROGRAM

$ a.out

quotient:
3.0136124031007751937984961240 * 10^-6

5 Validity of Stirling Approximation

The factorial function is related to Stirling’s formula [2],
\[
\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} n^n e^{-n}} = 1,
\]
and its logarithmic form – Stirling approximation:
\[
\ln(n!) = \sum_{i=1}^{n} \ln(i) \approx \int_{1}^{n} \ln(x) dx = n \ln(n) - n + 1
\]
Eq (2) is a simple and direct proof of Stirling’s approximation without invoking Stirling’s formula, whose associated error can be obtained using the gamma function \(\Gamma(n) = n! \approx \int_{0}^{\infty} e^{-x} x^{n-1} dx\), for both integer and non-integer. Since \(n!\) are very large even for relatively small values of \(n\), Stirling approximation is more often used. References states that the equation for sufficiently large \(n\) from Stirling formula, \(n! \approx \sqrt{2\pi n} n^n e^{-n}\), works well for small \(n\); at \(n = 10\) the error is \(\sim 1\%\). However, the acceptable limit of \(n\) to Stirling approximation is not clear.

We have computed the number of configurations \(W\) for \(n=100\) distinguishable objects being distributed into 4 identical states, with each state containing \(\{n_1, n_2, n_3, n_4\} = \{10, 20, 30, 40\}\) objects. If Stirling approximation is applied to the solution \(W = n! / \prod_{i=1}^{n} (n_i! n_i)\), such that \(\ln W = \ln(n!) - \sum_{i=1}^{n} \ln(n_i)\), then the error will exceed 100%. While the large error is due to misuse of the approximation equation, the precise value of \(W\) can be computed using the aforementioned methods. We found that \(n \geq 100\) is needed for \(\ln(n!)/(n \ln(n) - n)\) to attain an error of \(\sim 1\%\).

6 Application of High Precision Technique to the Engineering Industry and Scientific Computing

Technique to achieve high precision arithmetic is commonly used in public-key cryptography, which requires the manipulation of integers of thousands
Program 9 Division of Two Floats - Method 2

PROGRAM division02
implicit none
integer :: ii, jj, k, kk, C, T, L=10, m=1
integer, parameter :: ss=200, s=30
integer :: Xori(ss)=0, X(ss)=0, Y(ss)=0, P(ss)=0, Q(s)=0
integer, parameter :: Nx=7, Ny=4, Ex=2, Ey=-4, Etot=Ex-Ey

Xori(1:Nx) = (/0, 3, 8, 8, 7, 5, 6/)
Y (1:Ny) = (/0, 1, 2, 9/)  

do ii = 1, Ny
  X(ii) = Xori(ii)
enddo

k = 1 ! k=1/2 denotes single integer/1st decimal place
100 ii = 1

! Compare each digit of dividend X and divisor Y and
! calculate partial quotient when X(ii)>Y(ii).
! E.g.: X/Y=45316/742, X(ii)=453 < Y(ii)=742;
! set Xori(4)=1 such that X(ii)=4531 > Y(ii)=742.
Loop1: DO WHILE (.true.)
  if (X(ii)>Y(ii)) then
    Q(k) = ( X(1)*10+X(2) ) / Y(2) ! Q(1)=45/7
    exit
  else if (X(ii)==Y(ii)) then
    ii = ii + 1
  else if (X(ii)<Y(ii)) then
    do kk = 1, Ny
      X(kk) = X(kk+1)
    enddo
    X(Ny) = Xori(Ny+m) ! 4530 -> 4531
    Q(k) = 0
    k = k + 1
    m = m + 1
    ii = 1 ! Reset ii to re-compare X and Y.
  endif
END DO Loop1

! Multiply estimated partial quotient by divisor.
! 742*6 -> [4,4,5,2], P(Ny)=2, P(Ny-1)=5.
C = 0
Loop2: do ii = Ny, 1, -1
  T = Y(ii)*Q(k) + C
  C = T / 10
  P(ii) = mod(T,L)
enddo Loop2

! Re-calculate partial quotient.
ii=1
Loop3: DO WHILE(.true.)
  if (P(ii)>X(ii)) then ! E.g.,5194>4531,Q=Q-1
    Q(k) = Q(k)-1
    Loop4: do ii = Ny, 1, -1 ! Recalculate P.
      T = Y(ii)*Q(k) + C
      C = T / 10
      P(ii) = mod(T,L)
    enddo Loop4
  else if (P(ii)==X(ii)) then
    ii = ii+1
  else if (P(ii)<X(ii)) then ! E.g.,4452<4531,exit.
    exit
  endif
END DO Loop3

Figure 2 Two Repeated Subtraction Methods for Division of Floats.
of digits, and also experimental mathematics that involves the computation of fundamental mathematical constants, such as pi. The high precision scheme is also essential to the digital design of firmware in digital signal processing systems, including cell phone, PDAs, and GPS, which use a specialized microprocessor, the digital signal processor (DSP) [3], to process input signals with very limited memory or cache. When DSPs receive too many input signals exceeding the limit of cache or memory, the two’s complement method was adopted to store a number of up to $2^n$ bits. Most DSPs are based on fixed point arithmetic because the dynamical range provided by the floating point is not needed for signal processing. In the future DSPs may be adopted to reduce the cost and complexity of software development, at the expense of more expensive hardware, to attain a wider dynamic range like the floating point DSPs which are often used in military signal applications involving Radar, Sonar, and communications.

There is a rapidly growing need for a higher level of numeric precision in scientific computing. Recently, Bailey and Borwein (2008) conducted a survey of the high-precision package applied to scientific computing [4]. One of the application was supernovae with the QD package, which provides double-double (128-bit or 31-digit) and quad-double (256-bit or 62-digit) datatypes to solve for the non-local thermodynamic equilibrium populations of iron and other chemical elements [5]. Iron may be present in the outer envelope of a supernova as Fe II (Fe+1), but in the inner parts Fe IV (Fe+3) or Fe V (Fe+4) could be dominant. Since the relative population of any state is proportional to the exponential of the ionization energy, the dynamic range of these numerical values can be very large. Introducing artificial cutoff for the elemental distribution leads to spurious numerical glitches. In order to solve for all of the populations and yet reduce the computing time, a scheme was employed to determine whether to use 64-bit, 128-bit or 256-bit arithmetic in constructing and solving the linear system.

In addition to supernova simulations, Bailey & Borwein’s survey shows that the applications comprise climate modeling, planetary orbit calculations, Coulomb n-body atomic systems, scattering amplitudes of elementary particles, nonlinear oscillator theory, Ising theory, quantum field theory, as well as experimental mathematics. High-precision arithmetic techniques are becoming indispensable to modern large-scale computational science.

### 7 Discussion and Conclusions

The approach of calculating exact numbers that exceed a machine’s limit has been utilized in the electronic industry, and is becoming important to scientific computing. However, such a scheme is much less well recognized by researchers and instructors of scientific disciplines. Despite the fact that various arithmetical packages and stand-alone software are available for performing high-precision calculations, one of the most popular is the GMP (Gnu Multiple Precision Arithmetic) library, which is written in C, and Mathematica, which along with several machine-code languages adopts GMP, documents for the underlying algorithms have been scarce until only recently [6]. Motivated by the need to improve the educational value of computational physics courses, we performed an independent study on basic arithmetic operations to arbitrary precision. Various computing algorithms for the multiplication and division of overflowing/underflowing numbers are explored, which can be integrated to solve complicated algebraic equations.

Notably, software packages to solve mathematical functions as fundamental as the natural logarithm are still either under development or unavailable (as in GMP). Complicated computations using the multiple precision routines may suffer precision loss during intermediate calculations, depending on the problems and the size of computer memory, making the last few digits of the result unreliable. Understanding the approach and thus the imposed limitation is important to assure accurate solutions.

The algorithms for multiplication and division to an arbitrary precision are summarized as follows.

1. Given two numbers $X$ and $Y$, to calculate the product $P = X \times Y$ or the quotient $Q = X / Y$, where $X$ and $Y$ could exceed the machine number limit, the digits of $X$ and $Y$ are arranged into an array of elements: $X = [X(Nx), X(2), X(1)]$ and $Y = [Y(Ny), Y(2), Y(1)]$; and $P = [P(Nx + Ny), P(2), P(1)]$. The multiplication of an $Nx$-digit number and an $Ny$-digit number can generate a number with
either \( N_x + N_y \) or \( N_x + N_y - 1 \) digits. The precision achieved for division is determined by the declared array size of the quotient.

2. If a 32-bit fix-point number is used, then one array element can accommodate ten digits. To prevent overflow during the computation (such as multiplication), each element can hold a number of four digits. Allocating more digits to an element consumes less memory and improves performance. However, the costs in terms of memory and CPU time of our problem are not excessive, so one digit can be allocated to each element.

3. The algorithm for multiplication is based on decomposing the multiplicand, while the algorithm for division is based on decomposing the dividend. For multiplication, extra digits of intermediate products are carried to higher places and accumulated to yield the final product. The algorithm for division is much more difficult: a set of numbers is extracted to form a partial dividend, from which the divisor is repeatedly subtracted. These subtractions yield a partial quotient. This procedure continues to lower places and the total quotient is identical to that obtained by dividing the dividend by the divisor once.

4. Given the exponents to the fixed numbers, the multiplication and division of these fixed numbers can be extended straightforwardly to floating-point arithmetic.

5. The method can be regarded as an extension of the binary system in which 1 bit has one of two possible values and the decimal system in which 1 digit has one of ten values to a large-number system such as in which an element may comprise 10000 to 1000000000 values.

Given the subtraction method of division, numerous mathematical functions as simple as \( x^{3.0} = 100 \) and algebraic equations of the form \( f(x) = 0 \) can be solved to a high precision via Newton-Raphson iteration, \( x_{i+1} = x_i - f(x_i)/f'(x_i) \), provided the derivative of the function is known. However, for mathematical functions such as logarithms, series approximations must be employed to find the roots [7]. Several computer algebra systems that can express almost any number, including Mathematica and Maple, combine the arbitrary-precision scheme with symbolic computation, which transforms mathematical expressions into symbolic form before calculations are made, unlike truly numerical and, therefore, limited-precision computation.

For memory-intensive computations, memory can be allocated and released via the allocatable array, which can be controlled more effectively by C (using the malloc and memset functions) than by Fortran. Such a case may involve a large linearized system. For example, the implicit scheme may be invoked to solve a set of Navier-Stokes equations in multi dimensions. Methods for constructing and solving such a complex system of equations with a high precision are still being developed [4].

Studies of the newly emerged General-Purpose computing on a Graphical Processing Unit (GUGPU), which uses a multi-thread GPU as a co-processor with the CPU, demonstrate that PDE problems are solved much faster using GPUs than using CPUs. GPUs had been restricted to single precision floating point arithmetic until double precision GPU architecture (Nvidia Fermi GPU) became available in 2010. Before then, mixed precision iterative solvers for finite element simulations had been implemented by iteratively computing the residuals of a single precision approximate solution to acquire a higher precision, or defect correction, assigning the computationally intensive part to the less precise GPU and the precise correction part to the CPU [8]. However, for small- to medium-size problems, the GPU-CPU approach to increasing precision is not always faster than the pure CPU approach, because the CPU with a large cache outperforms the GPU. Emulating double-precision arithmetic on the single-precision GPU typically increases tenfold the operation count [9]. GPU programming requires complicated memory management which involves shared and global memories between threads and registers to implement effectively parallel algorithms on various hardware levels. The implementation of arithmetic to arbitrary precision on a GPU has not been documented, to the best of our knowledge.

The arbitrary precision algorithms for multiplication and division that are presented herein can be integrated to solve advanced mathematical problems to multiple precision or almost arbitrarily-high precision, limited only by the memory of the host machine. Notably, the success of the arbitrary precision implementation depends on the smart use of arrays and careful programming, and the details of the implements can vary greatly from problem to problem, or even within one problem. Finally, as ill-conditioned matrices are inevitably present in a system of implicitly discretized PDEs, where a small error in the matrix leads to a large error in the solution, the arbitrary precision technique, even for the GUGPU, may become an important future exploration in computing.

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