Link-cut trees have been introduced by D.D. Sleator and R.E. Tarjan (Journal of Computer and System Sciences, 1983) with the aim of efficiently maintaining a forest of vertex-disjoint dynamic rooted trees under cut and link operations. These operations respectively disconnect a subtree from a tree, and join two trees by an edge. Additionally, link-cut trees allow to change the root of a tree and to perform a number of updates and queries on cost values defined on the arcs of the trees. All these operations are performed in $O(\log n)$ amortized or worst-case time, depending on the implementation, where $n$ is the total size of the forest.

In this paper, we show that a list of elements implemented using link-cut trees (we call it a log-list) allows us to obtain a common running time of $O(\log n)$ for the classical operations on lists, but also for some other essential operations that usually take linear time on lists. Such operations require to find the minimum/maximum element in a sublist defined by its endpoints, the position of a given element in the list or the element placed at a given position in the list; or they require to add a value $a$, or to multiply by $-1$, all the elements in a sublist.

Furthermore, we use log-lists to implement several existing algorithms for sorting permutations by transpositions and/or reversals and/or block-interchanges, and obtain $O(n \log n)$ running time for all of them. In this way, the running time of several algorithms is improved, whereas in other cases our algorithms perform as well as the best existing implementations.

Keywords: efficient data structure; double-linked list; link-cut tree; dynamic list ranking; permutation sorting

1 Introduction

Many data structures have been defined up to now, allowing to store, modify and query a set of elements (see [2] for a non-exhaustive introduction). Each of these data structures has specificities related to the type of the set to be stored (disjoint elements or not, ordered elements or not), but overall the operations to be performed are as efficiently as possible: insert an element, delete an element, find an element, find the maximum/minimum element, find the predecessor/successor of an element (if it is defined), modify (the identifying key of) an element.

When the set is ordered (then we call it a list), an important number of applications exists where several consecutive elements must be inserted/deleted/modified simultaneously. The above-mentioned data structures applied to the ordered case allow only a sequential treatment of each element. Only linked-lists allow to insert or delete consecutive elements in/from the sets they represent, but the absence of a sublinear worst-case time access to an element given by its position makes that all the other operations on the linked-lists are too time-consuming.

In this paper, we use link-cut trees introduced in [12] to define a data structure, that we call a log-list, on which a significant number of useful operations take $O(\log n)$ time. These operations include the classical operations on lists (delete, insert a sublist), but also, for instance, simultaneously adding...
a real value to the values of a sublist, finding the minimum value in a sublist, or finding the element at the \(i\)-th position in the list (see the next section for more precisions). We subsequently use log-lists to improve the running time of several existing algorithms for sorting a permutation by (selected types of) transpositions and/or reversals and/or block-interchanges.

The paper is organized as follows. In Section 2 we present the operations to be performed on log-lists and the implementation of log-lists as link-cut trees. In Section 3 we recall the general features of link-cut trees, and propose additional features. In Section 4 we show that log-lists achieve \(O(\log n)\) running time for all the operations we propose for them, including when the elements have weights and the operations are performed on the weights rather than on the elements. In Section 5 we give the aforementioned applications to permutation sorting. Section 6 is the conclusion.

# Log-lists

Let \(L = (x_1, x_2, \ldots, x_n)\) be an ordered set (or list) of not necessarily distinct elements with values from a numerical set \(\Sigma\). We wish to perform in \(O(\log n)\) time (and less when possible) the operations below, termed list-operations, on the list \(L\). We assume each element is given by a pointer to it. An element is therefore seen as a cell containing the value of the element. Thus having (a pointer to) an element and getting the value of the element are two distinct requests. The list-operations are:

- \(\text{first}(L)\), which returns (a pointer to) the first element in (non-empty) \(L\), or null if \(L\) is empty
- \(\text{last}(L)\), which returns (a pointer to) the last element in (non-empty) \(L\), or null if \(L\) is empty
- \(\text{get-value}(\text{list } L, \text{ element } x)\), which returns the value of the element \(x\) of \(L\).
- \(\text{succ}(\text{list } L, \text{ element } x)\), which returns the value of the element immediately following \(x\) in \(L\) if \(x \neq \text{last}(L)\)
- \(\text{prec}(\text{list } L, \text{ element } x)\), which returns the value of the element immediately preceding \(x\) in \(L\), if \(x \neq \text{first}(L)\)
- \(\text{insert}(\text{list } L, \text{ list } L_1, \text{ element } x)\) which inserts a list \(L_1\) in (non-empty) \(L\) immediately after the element \(x\) (similarly: before \(x\)), and returns \(L\).
- \(\text{delete}(\text{list } L, \text{ element } x, \text{ element } y)\) which deletes the sublist \(L_1\) of \(L\) defined by its first element \(x\) and its last element \(y\), and returns \(L\) and \(L_1\).
- \(\text{reverse}(\text{list } L, \text{ element } x, \text{ element } y)\) which reverses the order of the elements in the sublist of \(L\) defined by its first element \(x\) and its last element \(y\), and returns the new list \(L\).
- \(\text{find-min}(\text{list } L, \text{ element } x, \text{ element } y)\) which returns an occurrence of the minimum value in the sublist of \(L\) defined by its first element \(x\) and its last element \(y\).
- \(\text{find-max}(\text{list } L, \text{ element } x, \text{ element } y)\) which is similar to \(\text{find-min}\) but requires the maximum element.
- \(\text{add}(\text{list } L, \text{ element } x, \text{ element } y, \text{ real } a)\) which adds a real value \(a\) to the value of each element in the sublist of \(L\) defined by its first element \(x\) and its last element \(y\), and returns the new list \(L\).
- \(\text{change-sign}(\text{list } L, \text{ element } x, \text{ element } y)\) which multiplies by \(-1\) the value of each element in the the sublist of \(L\) defined by its first element \(x\) and its last element \(y\), and returns \(L\).
- \(\text{find-rank}(\text{list } L, \text{ element } x)\) which returns the value \(i\) such that \(x\) is the \(i\)-th element of the list \(L\) (i.e. \(x = x_i\)).
- \(\text{find-element}(\text{list } L, \text{ integer } i)\) which returns a pointer to the \(i\)-th element of the list \(L\) (i.e. to \(x_i\)).

When the elements in \(L\) have weights (including keys), similar operations may be performed in \(O(\log n)\) on the weights. See Section 4.2.

In a classical double-linked list, the operations \text{first, last, get-value, succ, prec, insert, delete} and \text{reverse} need \(O(1)\) time, whereas the other operations need \(O(n)\) time. Our aim is to balance the running
Figure 1: Log-list for the list $L = \{8, 5, -4, 6\}$. a) link-cut tree $T(L)$ in standard form. b) link-cut tree $T(L)$ in non-standard form. The arc labels are the pairs $val; index$.

times of these operations, seeking $O(\log n)$ time or less for each of them. The powerful link-cut trees developed by Sleator and Tarjan in [12] in order to deal with dynamic trees apply here in the particular case where all trees are paths (as we show below), but need additional features that we provide in the next section.

We call support of a directed graph the undirected graph obtained by removing the orientations of the arcs. Consider the data structure, named a log-list for $L$ and denoted by $H(L)$, made of:

- a rooted tree $T(L)$ whose support is a path, built as follows (see Figure 1). It vertex set is $\{t_1, t_2, \ldots, t_{n+1}\}$ and its edges are given by the pairs $(t_i, t_{i+1})$, $1 \leq i \leq n$. In its standard form, the tree $T(L)$ is assumed have the root $t_{n+1}$, with arcs going towards the root. For each $i$, the arc $e_i$ with endpoints $t_i$ and $t_{i+1}$ has two costs, namely $val(e_i) := x_i$ and $index(e_i) := i$. If the list is empty, then the tree is empty.
- three pointers head($L$), tail($L$) and tail+($L$) which point respectively to the vertices $t_1$, $t_n$ and $t_{n+1}$ of $T(L)$. These pointers change only when the list $L$ changes, but are not modified when the root of the tree $T(L)$ changes.

Obviously, given a list $L$ we can build in $O(n)$ time the tree $T(L)$ and the three pointers head($L$), tail($L$), tail+($L$). The same data structure is used for all the lists and sublists handled during the aforementioned operations. Therefore, at each operation, one or several trees must be handled, and they undergo arc deletions and/or arc additions, that cut or link the existing trees. This forest of trees is thus implemented using link-cut trees [12], and therefore each tree of it is assumed to be a link-cut tree (see the next section for more details).

Remark 1. Notice that, in $T(L)$, the elements of the list are stored as costs on the arcs. The vertices in $T(L)$ are not identified with the elements in $L$. Therefore, when a pointer to an element $x_i$ of $L$ is given, we assume this means we are provided with a pointer to the source $t(x_i)$ of the arc with cost $x_i$ in the standard form of $T(L)$. See Figure 1.

Remark 2. Also note that other aggregate operations may be performed similarly to add, since link-cut tree support them, as observed in [13]. See Section 3 for details about the data structure used by link-cut trees.
3 Link-cut trees

3.1 General features

A link-cut tree is a rooted tree, whose arcs are supposed to be directed towards the root, so that if \((v, w)\)
is an arc then \(w\) is the parent of \(v\). The following dynamic tree operations may be performed on link-cut
trees, in any order (where \(cost\) is a cost with real values, on the arcs of the link-cut trees in the forest).
Each operation takes \(O(\log n)\) time [12].

1. \(d\text{parent}(\text{vertex } v)\), which returns the parent of \(v\) in the tree containing it, or null if \(v\) is the root.
2. \(d\text{root}(\text{vertex } v)\), which returns the root of the tree containing \(v\).
3. \(d\text{cost}(\text{vertex } v)\), which returns \(cost(v, d\text{parent}(v))\), provided \(v\) is not the root of the tree containing it.
4. \(d\text{mincost}(\text{vertex } v)\), which returns the vertex \(w\) closest to \(d\text{root}(v)\) such that \(cost(w, d\text{parent}(w))\)
is minimum among all vertices \(w'\) on the path from \(v\) to \(d\text{root}(v)\). Again, it is assumed that \(v \neq d\text{root}(v)\).
5. \(d\text{update}(\text{vertex } v, \text{real } a)\), which adds \(a\) to the cost of all edges on the path from \(v\) to \(d\text{root}(v)\).
6. \(d\text{link}(\text{vertex } v, \text{vertex } w, \text{real } a)\), which assumes that \(v = d\text{root}(v) \neq d\text{root}(w)\) and adds an arc \((w, v)\) with cost \(a\), thus combining the trees containing \(v\) and \(w\).
7. \(d\text{cut}(\text{vertex } v)\), which assumes that \(v \neq d\text{root}(v)\) and cuts the arc between \(v\) and \(d\text{parent}(v)\), thus dividing the tree initially containing \(v\) into two trees.
8. \(d\text{invert}(\text{vertex } v)\), which reverses the direction of all arcs on the path from \(v\) to \(d\text{root}(v)\), thus making \(v\) the root of the tree.

Remark 3. Given that the operations \(d\text{parent}\), \(d\text{root}\), \(d\text{cost}\), \(d\text{mincost}\) and \(d\text{update}\) are static (they do not change the forest of link-cut trees), several costs may be simultaneously defined and used in an arbitrary order. Naturally, \(d\text{cut}\) and \(d\text{link}\) have a number of cost parameters equal to the number of cost functions defined on the arcs of the tree.

Remark 4. Also note that \(d\text{cost}(v)\) takes \(O(\log n)\) time, and not \(O(1)\) time as we could expect. This is due to the fact that the cost values are not directly stored, but computed using additional information, in order to allow simultaneous modifications using \(d\text{update}\) (see Section 3.2).

To achieve \(O(\log n)\) running time for all these operations, in [12] the arc set of each link-cut tree in the forest is partitioned into solid arcs and dashed arcs. Each vertex has at most one ingoing solid arc and, since it has at most one outgoing solid arc. Thus the solid paths, which are all the maximal paths formed by solid arcs, partition the vertex set of the link-cut tree (assuming that a vertex belonging to no solid arc defines alone a trivial solid path). Each solid path is then represented as a binary tree of height \(O(\log n)\) whose internal nodes represent the arcs of the solid path (with their costs) and whose leaves represent the vertices of the path, in such a way that a symmetric traversal of the binary tree results into a “spelling” of the solid path from its head to its tail, including both its vertices and its arcs. The binary trees of all solid paths of a link-cut tree, which also contain a lot of additional information not described here, are then connected in order to depict the structure of the link-cut tree. Note that, following [12], we use the term vertex for the link-cut trees, and the term node for the binary trees.

This structure (see Figure 2 for a summary) has the twofold advantage of being highly parameterizable (the type of the binary tree, the definition of solid and dashed arcs) and of being able to reduce operations on link-cut trees to operations on paths, the later ones being themselves reduced to operations on binary trees. Then, when a dynamic operation on a tree has to be performed: either it is a basic operation that may be performed by querying the binary trees representing the tree without modifying them, and thus without modifying the solid paths (this is the case of \(d\text{parent}\) and \(d\text{cost}\)); or it is a complex
operation requiring first that a solid path be built from the vertex \( v \) to the root of its tree (this is the case of \( \text{droot}, \text{dmincost}, \text{dupdate}, \text{dcut}, \text{devert} \), with an exception for \( \text{dcut} \) where the path starts in \( w \) instead of \( v \)). Building the solid path from \( v \) to the root of the tree, and thus the binary tree associated with it, is done by an operation called \( \text{adexpose}(v) \), where \( v \) stands for auxiliary. Once \( \text{adexpose} \) is performed, finishing the treatment required by \( \text{droot}, \text{dmincost}, \text{dupdate} \) and \( \text{devert} \) needs only to move inside the binary tree, querying it or modifying values. However, the two remaining operations \( \text{dlink} \) and respectively \( \text{dcut} \) need to combine the binary tree obtained by \( \text{adexpose} \) with another one, and respectively to cut it into two trees. Overall, the topological modifications of binary trees are due to \( \text{adexpose}, \text{dlink} \) and \( \text{dcut} \), and are implemented using the four auxiliary operations below:

- \( \text{adconstruct}(\text{node } r, \text{node } s, \text{real } x) \), which combines two binary trees with roots \( r \) and \( s \) into another binary tree with root node having cost \( x \), left child \( r \) and right child \( s \).
- \( \text{addestroy}(\text{node } r) \), which splits the binary tree with root \( r \) into the two subtrees with roots given by its left and right child, and returns the two subtrees as well as the cost at node \( r \) before splitting.
- \( \text{adrotateleft}(\text{node } r) \), which assumes that \( r \) has a right child \( c \) and performs a left rotation on \( r \), i.e. \( r \) becomes the left child of \( c \), whose left child becomes the right child of \( r \). The operation returns the new root of the binary tree.
- \( \text{adrotateright}(\text{node } r) \), which is similar to \( \text{adrotateleft}(\text{node } r) \), with left and right sides exchanged.

The nodes of the binary tree store considerable information allowing to perform these four auxiliary operations in constant time, independently of the type of the binary tree. However, in order to perform the other dynamic tree operations in \( O(\log n) \) time (including \( \text{adexpose} \)), the type of the binary tree must be carefully chosen. With locally biased binary trees, amortized \( O(\log n) \) running time is achieved. With globally biased binary trees, worst-case \( O(\log n) \) running time is achieved if in addition the solid arcs are specifically defined as being the heavy arcs of the link-cut tree. An arc \((v, w)\) of a link-cut tree is heavy if \( 2\text{size}(v) > \text{size}(w) \), where \( \text{size}(u) \) denotes the number of vertices in the subtree of \( u \), including \( u \). The operations defined above remain valid, with the only difference that when an operation modifying the set of solid paths of a link-cut tree is performed (i.e. \( \text{droot}, \text{dmincost}, \text{dupdate}, \text{devert}, \text{dlink} \) and \( \text{dcut} \), which use \( \text{adexpose} \)), then it must be followed by a corrective procedure called \( \text{adconceal} \) that transforms the (possible temporarily non-heavy) solid paths into heavy paths. The efficient implementation of \( \text{adconceal} \) needs to augment again the data structure, with data whose update does not modify the running times of the other operations.

**Remark 5.** In our description, we assume link-cut trees use heavy paths and (locally or globally) biased binary trees, in order to achieve the \( O(\log n) \) amortized or worst-case running time. However, we do not have to go into these details to explain the additional features we add to the standard link-cut trees data structure, so that we only use the terms solid paths and binary trees to give our description.

### 3.2 Additional features

In this section, we propose several modifications of the data structure presented above, in order to allow the following additional operations on a link-cut tree:

9. \( \text{dsearchcost}(\text{vertex } v, \text{real } a) \), which searches for a vertex \( w \) on the path from \( v \) to \( \text{droot}(v) \) such that \( \text{cost}(w, \text{dparent}(w)) = a \), assuming the costs are strictly increasing as we go up from \( v \) to \( \text{droot}(v) \).
   The operation returns \( w \), if it exists, or the vertex \( w' \) with the largest value \( \text{cost}(w', \text{dparent}(w')) \) smaller than \( a \), if such a \( w' \) exists. Otherwise, it returns \( \text{droot}(v) \).

10. \( \text{dminuscost}(\text{vertex } v) \), which multiplies by \(-1\) all the costs on the path from \( v \) to \( \text{droot}(v) \).

As usual, both these operations start with a call to \( \text{adexpose}(v) \), which builds the solid path from \( v \) to \( \text{droot}(v) \) and the binary tree \( B_T \) associated with it. In \( B_T \) and as described in [12], each internal
node \( e \) (recall it is an arc of \( T \)) stores, among other information, pointers \( bparent(e) \) to the parent of \( e \) in \( B_T \) and \( bleft(e), bright(e) \) to respectively the left and right child of \( e \) in \( B_T \). Moreover (see Figure 3), it stores two values named \( netcost(e) \) and \( netmin(e) \), which are related to \( cost(e) \) and \( mintree(e) := \min \{ cost(f) \mid f \text{ belongs to the subtree rooted at } e \} \) by the following equations \cite{12}:

\[
netcost(e) = cost(e) - mintree(e)
\]

\[
netmin(e) = \begin{cases} 
mintree(e) & \text{if } e \text{ is the root of } B_T 
mintree(e) - mintree(bparent(e)) & \text{otherwise}
\end{cases}
\]

Then \( mintree(e) \) is equal to the sum of the \( netmin \) values on the path in \( B_T \) from \( e \) (included) to the root of \( B_T \) (included), and \( cost(e) \) is the sum of \( mintree(e) \) and \( netcost(e) \). Therefore \( mintree(e) \) and \( cost(e) \) are not stored in the tree, but only computed when needed. The values \( netcost(e) \) and \( netmin(e) \) are initialized when the forest of link-cut trees is initialized, in linear time; they are further updated in \( O(1) \) when the link-cut trees and/or their solid paths are modified, by the operations handling these modifications, namely \( adconstruct, addestroy, adrotateleft \) and \( adrotateright \). Note that updating the value of \( netmin \) at the root of \( B_T \) results into an update of \( mintree(e) \) and \( cost(e) \) in the entire tree, in \( O(1) \) time.

**Example.** The value \( mintree(c,d) \), which is 5, is computed as \( netmin(c,d) + netmin(d,h) + netmin(b,c) = 3 + 0 + 2 = 5 \). The value \( cost(c,d) \), which is also 5, is computed as \( netcost(c,d) + mintree(c,d) = 0 + 5 = 5 \).

We are now ready to prove the following claim.

**Claim 1.** Let \( T \) be a link-cut tree, and \( v \) one of its vertices. Assume the values of the function \( cost \) are strictly increasing when going from \( v \) to \( droot(v) \). Then \( dsearchcost(vertex v, real a) \) may be implemented in \( O(\log n) \) time.
Perform adexpose($v$) and let $v_1, v_2, \ldots, v_p$, with $v_1 = v$, $v_p = \text{droot}(v)$ and $p \geq 2$, be the vertices on the solid path going from $v$ to $\text{droot}(v)$, in this order. Then, by hypothesis, for each arc $(v_i, v_{i+1})$, $1 \leq i \leq p-2$, we have $\text{cost}(v_i, v_{i+1}) < \text{cost}(v_{i+1}, v_{i+2})$.

Let $B_T$ be the binary tree associated with the solid path, and recall that a symmetric traversal of $B_T$ allows us to “spell” the solid path from its head to its tail, including as well its vertices (which are the leaves of $B_T$) and its arcs (which are the internal nodes of $B_T$). Then, when the costs of the arcs are listed by the same symmetric traversal, they are in strictly increasing order, meaning that $B_T$ is a binary search tree. We deduce that a classical search for $a$ in $B_T$ allows us to find the arc $e$ with cost $a$ (if any), and thus the sought vertex $v_i$ (which is the rightmost leaf in the left subtree of $e$). If such an arc $e$ is not found, then again a classical search allows us to find the arc with largest cost lower than $a$, and the sought vertex $v_j$. This search takes a time proportional with the height of $B_T$, that is $O(\log n)$.

However, this approach assumes the cost of each arc in $T$ is known. Unfortunately, computing $\text{cost}(v, \text{bparent}(v))$ (which is $\text{droot}(v)$) takes $O(\log n)$ time, as indicated in Remark 4 and thus computing all the costs of the arcs would take $O(n \log n)$ time. We therefore need to go deeper into the representation of the costs in the binary tree $B_T$, in order to reduce the running time to $O(\log n)$. Recall that in a binary search tree we only need to compare the value we are looking for (here, $a$) with the values belonging to a unique branch of the tree, meaning that we only have to compute the costs of the arcs of $T$ encountered in $B_T$ during this branch traversal. If we show that all these $O(\log n)$ costs are computed in $O(\log n)$ time, then we are done.

Now, it is easy to see that the cost values of the arcs $e$ of $T$ encountered in $B_T$ during the search of $a$ may be computed in $O(1)$ time each, when we go down this branch. For this, it is sufficient to notice that $\text{mintree}(e) = \text{mintree}(\text{bparent}(e)) + \text{netmin}(e)$, except for the root, and that $\text{cost}(e)$ is computed in $O(1)$ time using $\text{mintree}$ and $\text{netcost}$. Then, all the cost values on the traversed branch of $B_T$ are computed in $O(\log n)$ time.

We focus now on the second operation we wish to add, dminuscost. Again, perform adexpose($v$) and build the binary tree $B_T$ corresponding to the solid path from $v$ to $\text{droot}(v)$. Notice that $\text{update}(v, x)$ updates the costs of all the nodes in $B_T$ (and thus of all the arcs on the solid path) in $O(1)$ time by adding $x$ to $\text{netmin}(r)$, where $r$ is the root node of $B_T$. Several $\text{update}$ operations may be performed.
consecutively, and each of them has an immediate effect on $netmin(r)$, implying that we do not have to store the real values involved in each such operation and, moreover, that we may perform up-down computations by accumulating $netmin$ values as in the proof of Claim 1. On the contrary, if one wants to introduce a multiplicative type of update, one has to store the multiplicative value $y$, since its effect cannot be reduced to a multiplication of $netmin(r)$ by $y$. If several successive updates hold, both additive and multiplicative, then all the real values involved in these updates must be stored. Moreover, the up-down computations become inefficient, since at each level one has to compute all the stored updates.

Therefore, $\text{dminuscost}$ is limited to multiplications by $-1$. In this case, we are able to ensure an immediate effect on the root of the tree.

**Claim 2.** The link-cut tree data structure may be modified such that, additionally to the other dynamic tree operations, $\text{dminuscost}(\text{vertex } v)$ takes $O(\log n)$ time, for each vertex $v$. The space requirements are still in $O(n)$.

**Proof.** As $\text{dupdate}$ does, $\text{dminuscost}$ calls $\text{adexpose}$ in order to compute the path from $v$ to $\text{droot}(v)$ and its associated binary tree. We modify the structure of the binary trees in order to enable efficient sign changes.

Consider the initial state of the forest of link-cut trees, in which the solid paths of each tree have been defined, and the binary trees to store them are about to be initialized. The idea of the proof is to store redundant information in each node $e$ of each binary tree, so that each of $\text{cost}(e)$ and $-\text{cost}(e)$ may be computed using its own series of $netmin$ values. The series computing $\text{cost}(e)$ is the positive series, whereas the series computing $-\text{cost}(e)$ is the negative series. They are disjoint, and the type (positive or negative) of each series is stored in the root $r$ of the tree. A multiplication by $-1$ of all the costs in $B_T$ then only requires to exchange the positive and negative series.

Formally, we define each node $e$ in the binary tree to have three parts: one of them, denoted $e^0$, contains the usual information stored in the node according to [12], except $\text{netcost}$ and $\text{netmin}$; another one, denoted $e^1$, contains two real variables $\text{netcost}(e^1)$ and $\text{netmin}(e^1)$ and a pointer $\text{up}(e^1)$; the third one, denoted $e^2$, contains two real variables $\text{netcost}(e^2)$ and $\text{netmin}(e^2)$, and a pointer $\text{up}(e^2)$. It is assumed that $e^0$, $e^1$ and $e^2$ may be pointed to separately. Pointers $\text{up}(e^1)$ and $\text{up}(e^2)$ point to $\text{bparent}(e)^1$ and to $\text{bparent}(e)^2$ respectively, or vice-versa, if $e$ is not the root. If $e$ is the root, then one of them points on its own source node (forming a loop) and the other one is null, according to rules that will be presented below.

Then the (initial) binary tree, whose root is denoted $r$, may be seen as composed of three binary trees (see Figure 4): the basic one given by the 0-parts of the nodes, and the arcs $(e, \text{bparent}(e))$; the 1-tree given by $r^1$, and the arcs $(e^1, \text{up}(e^1))$ such that $\text{up}(e^1) = r^1$ or there is a path from $\text{up}(e^1)$ to $r^1$; and the 2-tree given by $r^2$, and the arcs $(e^2, \text{up}(e^2))$ such that $\text{up}(e^2) = r^2$ or there is a path from $\text{up}(e^2)$ to $r^2$. These three trees are vertex- and arc-disjoint. The 1-tree and the 2-tree are, in some way, dual to each other, since one of them computes and updates $\text{cost}(e)$, for all $e$, whereas the other one computes and updates $-\text{cost}(e)$, for all $e$. It is understood that the one that computes $\text{cost}(e)$, that we call the positive tree, is the one whose $\text{up}$ pointer forms a loop (see above). The other one is then the negative tree. Its $\text{up}$ pointer is null.

**Example.** In Figure 4, the three trees are identified by their root (part $r^0$, $r^1$ or $r^2$ of the global root $r$) and by the arcs forming paths joining this root. Basically, the 1-tree uses the nodes and arcs on the left (double gray), the 0-tree the nodes and the arcs in the middle (black), and the 2-tree the nodes and the arcs on the right (simple gray) of the global tree (This left-middle-right partition changes when multiplications by $-1$ and topological changes occur, see below.). The 1-tree is the same as in Figure 5. The 2-tree has different values, but it allows us to compute, in the same way as the 1-tree, the $-\text{cost}(e)$ value for each edge $e$ in the solid path. For instance, recall that we computed $\text{cost}(c, d)$ as $\text{netcost}(c, d) + \text{mintree}(c, d) = 0 + (3 + 0 + 2) = 5$. Following the similar path from the node $(c, d)^2$ (i.e. the part 2 of the edge $(c, d)$) up to the root $r^2$ ($= (b, c)^2$) we compute the value $\text{netcost}((c, d)^2) + \text{mintree}((c, d)^2) = 0 + (0 + 2 + (-7)) = -5$, which is exactly $-\text{cost}(c, d)$. 

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The binary trees are initialized simultaneously for the entire forest in its initial state (whatever this state), as indicated below. Then, they are modified through the operations $\text{update}$, $\text{dminuscost}$, as well as by the operations $\text{adconstruct}$, $\text{addestroy}$, $\text{adrotateleft}$ and $\text{adrateright}$ we mentioned in Section 3.1, which must stay within $O(1)$ running time. Denote:

$$mintree(e^i) := \min\{\text{cost}(f^j) \mid f^j \text{ belongs to the subtree rooted at } e^i\}, \quad i = 1, 2$$

Then, throughout the modifications above, each binary tree is characterized by the following features:

A. Among the 1-tree and the 2-tree, the one with non-null $up$ pointer at its root is the positive tree, i.e. it computes $\text{cost}(e)$ for all nodes $e$ in the binary tree; the other one, the negative tree, computes $-\text{cost}(e)$ for all nodes $e$ in the binary tree.

B. The following equations hold ($i = 1, 2$):

$$\text{cost}(e^i) = -\text{cost}(e^2); \quad |\text{cost}(e^1)| = |\text{cost}(e^2)| = |\text{cost}(e)|$$

$$\text{netcost}(e^i) = \text{cost}(e^i) - mintree(e^i)$$

$$\text{netmin}(e^i) = \begin{cases} 
                          mintree(e^i) & \text{if } e \text{ is the root of the binary tree} \\
                          mintree(e^i) - mintree(up(e^i)) & \text{otherwise}
                           \end{cases}$$

C. Therefore we have:

$$mintree(e^i) = \sum_{f^j \text{ belongs to the path from } e^i \text{ to the root}} \text{netmin}(f^j)$$

$$\text{cost}(e^i) = \text{netcost}(e^i) + mintree(e^i).$$

In other words, each of the positive and negative trees has the same properties with respect to the costs as the binary tree used in [12]. The need to have two such trees come from the need to handle both the cost and its negation, which are made possible by the double choice for the $up$ values.
In the following, we present the initialization step and the aforementioned operations. Note that A, B and C below are satisfied by the initialization step.

**Initialization step.** See Figure 4a. Let $B$ be a binary tree with root $r$, corresponding to a solid path of a link-cut tree $T$. The binary tree $B$ is built such that $up(e^i) = bparent(e^i)$ for $i = 1, 2$, and $up(r^1) = r$, $up(r^2) = null$.

For each node $e$ of $B$, define:

$$\begin{align*}
\text{cost}(e^1) &= \text{cost}(e); \\
\text{cost}(e^2) &= -\text{cost}(e) \\
\text{mintree}(e^i) &= \min\{\text{cost}(f^i) \mid f^i \text{ belongs to the subtree rooted at } e^i\}, i = 1, 2.
\end{align*}$$

As in the initial structure in [12], none of these values is stored in $B$. Only the values $netcost(e^i)$ and $netmin(e^i)$ are stored, and allow to compute them. These values are defined similarly to [12]:

$$\begin{align*}
netcost(e^i) &= \text{cost}(e^i) - \text{mintree}(e^i), i = 1, 2 \\
netmin(e^i) &= \text{mintree}(e^i) \text{ if } e \text{ is the root of the binary tree, and } \\
&= \text{mintree}(bparent(e^i))^i, \text{ otherwise.}
\end{align*}$$

**Example.** In Figure 4a, the 1-tree (double gray arrows) contains exactly the same information as the unique tree in Figure 1 by the definitions above. It therefore computes the cost values. The 2-tree (simple gray arrows) similarly computes the $-\text{cost}$ values, as it uses the same definitions, but for the values $-\text{cost}(e)$ instead of $\text{cost}(e)$. For instance, let us see how the values $netcost$ and $netmin$ of the node $(d, h)^2$ are computed. The minimum value in the subtree with root $(d, h)^2$ of the 2-tree is $-5$ (given by $\text{cost}((c, d)^2)$, which is $-\text{cost}(c, d)$) and thus $\text{mintree}((d, h)^2) = -5$. Similarly, the minimum value in the subtree with root $(b, c)^2$ of the 2-tree is $-7$ (given by $\text{cost}((a, b)^2)$, which is $-\text{cost}(a, b)$) and thus $\text{mintree}((b, c)^2) = -7$. As $\text{cost}((d, h)^2) = -2$ we obtain that $netcost((d, h)^2) = \text{cost}((d, h)^2) - \text{mintree}((d, h)^2) = -2 - (-5) = 3$ and $netmin((d, h)^2) = \text{mintree}((d, h)^2) - \text{mintree}((b, c)^2) = -5 - (-7) = 2$.

Then we have:

$$\begin{align*}
\text{netcost}(e^i) &= \sum_{f^i \text{ belongs to the path from } e^i \text{ to the root of } \text{netmin}(f^i)} \text{cost}(e^i) \\
\text{cost}(e^i) &= \text{netcost}(e^i) + \text{netmin}(e^i).
\end{align*}$$

Consequently, the 1-tree computes the values $\text{cost}(e)$ for all nodes $e$ of $B$, whereas the 2-tree computes the costs $-\text{cost}(e)$. The pointers $up(r^1)$ and $up(r^2)$ are correctly initialized.

**Modifying the binary tree.** See Figure 4b. The binary tree, as defined above, is topologically modified by several dynamic tree operations, which call the four auxiliary operations $\text{adconstruct}$, $\text{addestroy}$, $\text{adrotateleft}$ and $\text{adrateright}$. In our version of the binary tree, each of them is implemented separately for the positive and the negative trees, using the same methods as in [12]. Therefore, the operation $\text{adconstruct}(r, s, x)$ then applies the (simple) $\text{adconstruct}$ operation in [12] once for the two positive trees and the cost value $x$ for $u^1$, and once for the two negative trees and the cost value $-x$ for $u^2$. The operations $\text{addestroy}(r)$, $\text{adrotateleft}(r)$ and $\text{adrateright}(r)$ also apply twice the same (simple) operations in [12]. Then, these operations have the same running time as those in [12], that is $O(1)$ time, and the $O(\log n)$ running time of the dynamic operations using them follows as in [12].

Additionally to the topological changes, the operations $\text{update}$ and $\text{dminuscost}$ modify some values of the binary tree. In our version of the binary tree, $\text{update}(v, x)$ first performs $\text{adexpose}(v)$ and, in the binary tree $B$ associated with the solid path from $v$ to $\text{droot}(v)$, adds $x$ to $\text{netmin}(r^1)$ and removes $x$ from $\text{netmin}(r^{3-i})$, where $r^i$ is the root of the positive tree of $B$. Operation $\text{dminuscost}(v)$ first performs $\text{adexpose}(v)$ and, in the binary tree $B$ associated with the solid path from $v$ to $\text{droot}(v)$, exchanges the positive and the negative trees by appropriately modifying the $up$ pointers of their roots.

Obviously, our version of the binary tree only duplicates the operations in the original version, and all the running times are unchanged.
4 List-operations on log-lists

Now, the tree $T(L)$ of any log-list $L$ is seen as a link-cut tree, whose support is a path (see Section 2) and which has two cost operations, $\text{val}$ and $\text{index}$, defined on each of its arcs.

4.1 Main result

We show that:

**Theorem 1.** In a log-list, all the list-operations take $O(\log n)$ time, except for first and last that take $O(1)$ time.

**Proof.** We assume that, before each operation, $T(L)$ is in standard form (otherwise, we apply $\text{devert}(\text{tail+}(L))$). We only have to show how dynamic tree operations allow to perform list-operations. Note that $L$ and $L_1$ are both represented as log-lists with cost values $\text{val}$ and $\text{index}$, and therefore the dynamic tree operations. Recall that when a pointer is given to an element $L$ in the list, this means a pointer is given to the vertex $t(x)$ of $T(L)$ which is the source of the arc recording $x$ in the standard form of $T(L)$ (according to Remark 1).

Operations $\text{first}(L)$ and $\text{last}(L)$ need only to return $\text{head}(L)$ and $\text{tail}(L)$ respectively, and take constant time.

Operation $\text{get-value}(L, x)$ may be realized by a simple call to $\text{val}(t(x))$, which is a call to $\text{decost}$ when the cost function is called $\text{val}$.

Operations $\text{succ}(L, x)$ and $\text{prec}(L, x)$ are easy to implement. For $\text{succ}(L, x)$, since $T(L)$ is in standard form, let $y = \text{dparent}(t(x))$. Then the value returned by $\text{val}(y)$ is the successor of $x$ in $L$, where $\text{dval}$ is the variant of the function $\text{decost}$ when the cost is given by the function $\text{val}$. For $\text{prec}(L, x)$, perform $\text{devert}(\text{head}(L))$ (which does not change the pointer $t(x)$) and return the value $\text{dval}(z)$, where $z = t(x)$. Note here that $t(\text{succ}(x))$ and $t(\text{prec}(x))$ are easy to compute in $O(\log n)$ by respectively returning $y$, and $\text{dparent}(z)$. These operations are used in Algorithms 1, 2 and 3 below.

**Figure 5:** Index correction procedures for insert and delete: a) insert($L, L_1, x$) for $L = \{8, 5, -4, 6\}$, $L_1 = \{3, 12\}$ and $x = x_3 = 5$, using Algorithm 1 b) delete($L, x, y$) for $L = \{8, 5, 3, 12, -4, 6\}$, $x = x_3 = 3$ and $y = x_4 = 12$, using Algorithm 2. Each arc indicates its $\text{val; index}$ pair.
The values \( v_T \) and \( d_{\text{link}} \) are parts of the forest, both of them are updated.

Reverse its reversal using \( d_{\text{index}} \) in Algorithm 3 where, for conciseness reasons, only the most general case is presented. The operation \( \text{update} \) allows it, assuming it is extended so as to have two cost parameters instead of one, according to Remark. Once this is done by the operation \( \text{insert}_{\text{topo}}(L, L_1, x) \) (not written here), the values \( v_L \) are in the right order. It remains to update the index values, so as to ensure that they correctly compute the position of each element of \( L \) in the list \( L \). This is done as in Algorithm 1. Note that a call to \( d_{\text{index}} \) means a call to the deepest dynamic tree operation, when cost is replaced by index.

Operation \( \text{insert}(L, L_1, x) \) (see Figure 5a) is written using classical deletions and insertions of arcs, except that each arc deletion is implemented using the \( \text{delete} \) dynamic tree operation, whereas each arc insertion uses the \( \text{link} \) dynamic tree operation. Notice that, since the elements in \( L_1 \) are on the arcs of \( T(L_1) \), \( tail(L_1) \) is cut from \( L_1 \) before inserting \( L_1 \), that is, before appropriately linking it. However, the values \( v_L \) and \( \text{index} \) of the former arcs \( (t(x), \text{parent}(t(x))) \) in \( T(L) \) and \( (tail(L_1), tail(L_1)) \) in \( T(L_1) \) are appropriately recorded on the two new linking arcs (the \( \text{link} \) operation allows it, assuming it is extended so as to have two cost parameters instead of one, according to Remark. Once this is done by the operation \( \text{insert}_{\text{topo}}(L, L_1, x) \) (not written here), the \( v_L \) values are in the right order. It remains to update the index values, so as to ensure that they correctly compute the position of each element of \( L \) in the list \( L \). This is done as in Algorithm 1. Note that a call to \( d_{\text{index}} \) means a call to the deepest dynamic tree operation, when cost is replaced by index.

Operation \( \text{delete}(L, x, y) \) (see Figure 5b) is written similarly, and outputs the list \( L \) as well as the list \( L_1 \) of deleted elements. Notice that a tail \( tail(L_1) \) is added to the subtree deleted from \( T(L) \) in order to form \( T(L_1) \), allowing its ingoing arc to receive the \( v_L \) cost equal to \( y \) and the corresponding index value (as they were on the arc \( (y, \text{parent}(y)) \) of \( L \)). Adding \( tail(L_1) \) and recording the costs as indicated is done using a \( \text{link} \) operation with a trivial tree containing only \( tail(L_1) \) (we assume a sufficient number of such trees, at most \( n \), is present in the forest, in order to allow such completion operations on log-lists). To ensure its conciseness, the index correcting Algorithm 2 is written in the most general case, where no list is empty. We assume that the (omitted) \( \text{delete}_{\text{topo}} \) operation performs the topological changes as described. As both lists \( L \) and \( L_1 \) must be output by the \( \text{delete} \) algorithm (since they are parts of the forest), both of them are updated.

Operation \( \text{reverse}(L, x, y) \) (see Figure 5c) consists basically in the deletion from \( L \) of the sublist with endpoints given by \( x \) and \( y \) (which is represented as a log-list and is thus a new tree in the forest), its reversal using \( \text{reverse}(\text{head}(L_1)) \) (where \( \text{head}(L_1) \) is \( t(x) \)), the update of the pointers head, tail and \( tail+ \), and the insertion of the resulting list in \( L \). As before, we assume that these changes are realized by the omitted operation \( \text{reverse}_{\text{topo}} \), and we only show how to update the values of \( \text{index} \). This is done in Algorithm 3 where, for conciseness reasons, only the most general case is presented. The operation \( d_{\text{index}} \) is the same as the operation \( d_{\text{index}} \) when the cost is the function \( \text{index} \).
Algorithm 3: reverse($L, x, y$)

//Note: we also update the values for the deleted sublist $L_1$

1. $H(L) ←$ log-list representing $L$; // assumes $tail+(L)$ is the root of $T(L)$
2. $w ← t(\text{proc}(x)); z ← t(\text{succ}(y))$;
3. $n_0 ← \text{dindex}(w); n_1 ← \text{dindex}(t(y))$; // number of elements in $L$, before $x$, and up to and including $y$
4. $H(L) ← \text{reverse}_\text{topo}(L, x, y)$; // assumes pointers to $w, z$ are still available
5. $\text{devert}(z); u ← \text{dparent}(w)$;
6. $\text{dminindex}(u)$; // let the indices in the sublist with endpoints $y$ and $x$ be negative but in increasing order
7. $\text{dupdate}(u, n_0 + n_1 + 1)$; // the indices in the sublist with endpoints $y$ and $x$ get the right values
8. $\text{devert}(\text{tail}+(L))$; return $H(L)$

![Diagram of Algorithm 3](image-url)

Figure 6: Index correction procedure for reverse($L, x, y$) with $L = \{8, 5, 3, 12, -4, 6\}$, $x = x_3 = 3$ and $y = x_4 = 12$, using Algorithm 3. Each arc indicates its $\text{val}; \text{index}$ pair.

Operation $\text{find-min}(L, x, y)$ first performs a call to $\text{devert}(t(\text{succ}(y)))$, in order to ensure that the arcs of the path going from $x$ to $\text{droot}(x)$ (which is now $t(\text{succ}(y))$) record the values $\text{val}$ corresponding to the sublist of $L$ between $x$ and $y$. Then $\text{dminval}(t(x))$, which is the dynamic tree operation corresponding to $\text{dmincost}$ when the cost is given by the function $\text{val}$, returns the vertex $v$ closest to the root that has minimum value $\text{val}(v, \text{dparent}(v))$.

Operation $\text{find-max}(L, x, y)$ applies $\text{find-min}(L, x, y)$ once the signs of the elements have been changed with $\text{devert}(y)$ followed by $\text{dminusval}(x)$ (the variant of $\text{dmincost}$ for the cost function $\text{val}$), and returns the opposite of the result.

Operations $\text{add}(L, x, y, a)$ and $\text{change-sign}(L, x, y)$ are done by simple calls to $\text{devert}(t(\text{succ}(y)))$ and then to $\text{dupdate}(t(x), a)$ and respectively to $\text{dminusval}(t(x))$.

Operation $\text{find-rank}(L, x)$ only needs to return $\text{dindex}(t(x))$ when $T(L)$ is in standard form, since the cost function $\text{index}$ is correctly updated. Again, $\text{dindex}$ is the $\text{dcost}$ operation when the cost function is $\text{index}$.

Operation $\text{find-element}(L, i)$ performs $\text{devert}(\text{tail}+(L))$ in order to ensure that $T(L)$ is in standard form, and applies the variant $\text{dsearchindex}(\text{head}(L), i)$ of $\text{dsearchcost}$ when the cost function is $\text{index}$. It is easy to check that the hypothesis of Claim 1 are verified.

Each list-operation uses a constant-bounded number of dynamic tree operations or computations of
Theorem 2. In a log-list with weights, the weighted list-operations above are performed in $O(\log n)$ time.

Proof. The weight of an element $x$ is another cost on the arc $(x, \text{dparent}(x))$ in the initial tree $T(L)$ for the list $L$. The dynamic tree operations $\text{dcost}$, $\text{dmincost}$ (as well as its variant $\text{dmaxcost}$), $\text{dupdate}$, $\text{dminuscost}$ and $\text{dsearchcost}$ applied to this new cost allow to perform the weighted-operations.
Figure 7: Improvements over existing algorithms, whose $O(n^2)$ running time was optimal before our implementation. One cell represents a variant of the permutation sorting, allowing all the operations indicated in the column header, in all the versions indicated in the line header. The Source, Result and Best lines respectively give the reference of the original algorithm, its approximation ratio, and its best current implementation. Note that in five cases over six, log-lists easily yield the best implementations, similarly to the implementation we proposed for Algorithm 4.

$$P_0 := (p_1 p_2 \ldots p_{i-1} p_i p_{j+1} \ldots p_{k-1} p_k p_{i+1} \ldots p_j p_k \ldots p_n).$$

In other words, the block of $P$ with endpoints $p_i$ and $p_{j-1}$ is moved between $p_{k-1}$ and $p_k$. The reversal $r(P, i, j)$, where $1 \leq i \leq j \leq n$, acts differently on unsigned and signed permutations. When $P$ is unsigned, the reversal is also unsigned and it transforms $P$ into the following permutation:

$$P_1 := (p_1 p_2 \ldots p_{i-1} p_i p_{j-1} \ldots p_{i+1} p_i p_{j+1} \ldots p_n).$$

Equivalently, the order of the elements in the block of $P$ with endpoints $p_i$ and $p_{j}$ is reversed. When $P$ is signed, the reversal is also signed and it transforms $P$ into the following permutation:

$$P_1 := (p_1 p_2 \ldots p_{i-1} - p_i - p_{j-1} \ldots - p_{i+1} - p_i p_{j+1} \ldots p_n).$$

Equivalently, the order of the elements in the block of $P$ with endpoints $p_i$ and $p_{j}$ is reversed, and the signs are changed. A transposition $t(P, i, k, l)$ or a reversal $r(P, i, j)$ is prefix if $i = 1$, in which case they are denoted $\text{prefix}(P, i, k, l)$ and $\text{reversal}(P, i, j)$ respectively. A suffix transposition or reversal is defined in a similar way, but involves a suffix of $P$ instead of a prefix.

The block interchange $b(P, i, j, k, l)$, where $1 \leq i < j \leq k < l \leq n + 1$, transforms $P$ into the following permutation:

$$P_2 := (p_1 p_2 \ldots p_{i-1} p_k p_{j+1} \ldots p_{j-1} p_j \ldots p_{i-1} p_i p_{i+1} \ldots p_{j-1} p_{j+1} \ldots p_k \ldots p_n).$$

Equivalently, the underlined blocks are switched.

### 5.2 Sorting by Prefix/Suffix Transpositions and Reversals in $O(n \log n)$ time

Several algorithms in the literature share similar principles for sorting signed or unsigned permutations either by transpositions only, or by transpositions and reversals, when all operations are assumed to be prefix or suffix. We present in detail one of them, which allows us to precisely state the bottlenecks of such an approach in terms of running time. Then we show how to address these bottlenecks with log-lists and we conclude on all the similar variants (given in Figure 7).

In [4], the asymptotic 2-approximation algorithm in Algorithm 4 is presented for sorting an unsigned permutation by prefix transpositions and prefix reversals. Its running time is of $O(n^2)$. Note that the element $n + 1$ is added at the end of the permutation. A strip of $P$ is a sequence $p_i, p_{i+1}, \ldots, p_j, j \geq i$, of consecutive integers either in increasing order (yielding an increasing strip) or in decreasing order (yielding a decreasing strip). A singleton is, by definition, both an increasing and a decreasing strip.

**Idea of Algorithm 4**. Given the initial set of strips in the permutation, the algorithm performs one prefix operation at each step, preferring an operation that performs two strip concatenations to an operation that performs only one concatenation. To this end, the position $i$ at the end of the first strip is identified (steps 4–5). If the first element in $P$ is 1, it is not very useful immediately, so the whole strip is
Algorithm 4: Sorting by Prefix Reversals and Prefix Transpositions [4]

**Input:** Permutation $P$, number $n$ of elements

**Output:** Number $d$ of prefix reversals and prefix transpositions needed to sort $P$

1. $d \leftarrow 0$;
2. while $P \neq I_d$ do
   3. $i \leftarrow 1$;
   4. while $|p_{i+1} - p_i| = 1$ do
      5. $i \leftarrow i + 1$;
   6. if $p_1 = 1$ then
      7. $P \leftarrow \text{preftr}(P, i + 1, n + 1)$;
   8. else
      9. // Try to find a prefix transposition extending the strips at both ends of the moved block:
         10. $a \leftarrow P_{p_{a-1}}^{-1} + 1$; $la \leftarrow P_{p_{a-1}}^{-1} + 1$; $ra \leftarrow P_{p_{a+1}}^{-1} + 1$;
         11. $b \leftarrow P_{p_{b-1}}^{-1} + 1$; $lb \leftarrow P_{p_{b-1}}^{-1} + 1$; $rb \leftarrow P_{p_{b+1}}^{-1} + 1$;
         12. if $|p_{a-1} - p_a| \neq 1$ then
            13. if $p_a \neq 1$ and $|p_{a-1} - p_a| \neq 1$ then
               14. $P \leftarrow \text{preftr}(P, la, a)$;
            15. else if $p_{ra} \neq 1$ and $|p_{ra-1} - p_{ra}| \neq 1$ then
               16. $P \leftarrow \text{preftr}(P, ra, a)$;
            17. else
               18. if $|p_{b-1} - p_b| \neq 1$ then
                  19. if $p_b \neq 1$ and $|p_{b-1} - p_b| \neq 1$ then
                     20. if $lb < b$ then
                        21. $P \leftarrow \text{preftr}(P, lb, b)$;
                     22. else if $p_{rb} \neq 1$ and $|p_{rb} - p_{rb}| \neq 1$ then
                        23. if $rb < b$ then
                           24. $P \leftarrow \text{preftr}(P, rb, b)$;
                     25. else
                        26. // Try to find a prefix reversal/transposition extending one strip of the moved block:
                           27. if $p_1 \leq p_i$ then
                              28. $x \leftarrow P_{p_i-1}^{-1}$;
                              29. if $p_x = p_{x+1} + 1$ then
                                 30. $P \leftarrow \text{prefrv}(P, x - 1)$;
                              31. else
                                 32. $P \leftarrow \text{preftr}(P, i + 1, x + 1)$;
                           33. else
                              34. $y \leftarrow P_{p_i+1}^{-1}$;
                              35. if $p_y = p_{y+1} - 1$ then
                                 36. $P \leftarrow \text{preftr}(P, i + 1, y + 1)$;
                              37. else
                                 38. $P \leftarrow \text{prefrv}(P, y - 1)$;
                           39. $d \leftarrow d + 1$;
      40. $P \leftarrow \text{preftr}(P, i + 1, n + 1)$;
   41. $d \leftarrow d + 1$;
42. return $d$;
sent at the end of the permutation (steps 6-7). If the first element \( p_1 \) is not 1, then the algorithm attempts to move a prefix \( (p_1, p_2 \ldots, p_j) \) with \( j \geq i \) between \( p_1 - 1 \) and its successor (whose positions are \( a - 1 \) and \( a \); steps 11-17) or between \( p_1 + 1 \) and its successor (whose positions are \( b - 1 \) and \( b \); steps 19-25). Such a move is performed only if it is possible to choose \( j \) such that, once the block \( (p_1 \ldots p_j) \) is moved, two strip concatenations are possible, at both its ends. If such a transposition is not possible, then either \( p_1 - 1 \) (steps 27-32) or \( p_1 + 1 \) (steps 34-38) allows us to concatenate two strips either by a reversal or by a transposition, depending on the type of the involved strips.

Our implementation. In our implementation of Algorithm\(^4\) the permutation \( P \) is stored as a list \( L = \{p_1, p_2, \ldots, p_n\} \) implemented as a log-list. In addition we need, for each element \( p_i \) in \( L \), two pointers \( b[p_i] \) and \( e[p_i] \), such that \( b[p_i] \) (respectively \( e[p_i] \)) is not null if and only if \( p_i \) is the last (respectively the first) element in its strip (in the order from \( \text{first}(L) \) to \( \text{last}(L) \)). In this case, \( b[p_i] \) (respectively \( e[p_i] \)) points to the first (respectively last) element of its strip. Obviously, these pointers of the abstract data structure \( L \) may be directly added to the underlying dynamic tree structure \( T(L) \). We note that:

1) the operations \( \text{find-element}(L, i) \) and \( \text{find-rank}(L, x) \) act similarly to \( P[i] = p_i \) and \( P^{-1}[x] \), except that they take \( O(\log n) \) time and that \( \text{find-element}(L, i) \) returns a pointer instead of a value. An operation \( \text{get-value} \) further allows us to obtain this value.

2) prefix transpositions and respectively prefix reversals are performed using a delete and an insert operation on \( L \), respectively using a reverse operation. Both assume the parameters are given as pointers instead of positions, and need \( O(\log n) \) time. Updating the strips, i.e. updating \( b[] \) and \( e[] \), when concatenations hold at the endpoints of the moved or reversed block is also done in \( O(\log n) \) time. Indeed, at most two concatenations hold once a block is moved/reversed, and - for each of them - updating needs only to cross the boundary between the concatenated strips (with \( \text{prec} \) and \( \text{succ} \)) and to follow the old \( b[] \) and \( e[] \) values in order to compute the new values. A prefix transposition or prefix reversal, together with the strips update, thus takes \( O(\log n) \) time.

With these remarks, it seems that we only succeeded to increase the running time of important operations instead of reducing it. However, we are able to show that:

**Theorem 3.** Algorithm\(^4\) implemented using log-lists performs in \( O(n \log n) \) time, thus improving the \( O(n^2) \) time needed by the algorithm in \( [17] \).

**Proof.** The initialization of the data structure \( H(L) \) is obviously done in \( O(n) \) time since the strips are identified by a simple traversal of the list. The while loop is executed \( O(n) \) times \([4]\). The comparison in line 2 is easily done in \( O(\log n) \) time by testing whether the index of \( e[\text{first}(L)] \) is \( n + 1 \) or not.

Steps 4-5 need \( O(1) \) time with our implementation, since \( i \) is the position of the last element in the first strip. A pointer \( iPtr \) to it is given by \( iPtr \leftarrow e[\text{first}(L)] \).

Steps 6-7 need \( O(\log n) \) time, as \( p_1 \) is obtained with \( \text{get-value}(\text{first}(L)) \) and we have the two pointers \( \text{first}(L) \) and \( iPtr \) needed by the transposition (as indicated in 2) above). Steps 9-10 also need \( O(\log n) \) time, according to 1 above.

Steps 11-17 (and similarly 19-25) are also immediate due to 1) and 2). We only have to call \( \text{find-element}(La) \) and \( \text{find-element}(a) \) before performing the transposition in step 14 (and similarly for the step 17), since in our version of the algorithm we need pointers instead of positions.

The same remarks hold for steps 27-32 (and similarly 34-38).

The maximum running time of a step is thus of \( O(\log n) \) implying that the overall running time is in \( O(n \log n) \). The theorem is proved. \( \blacksquare \)

In Figure\(^7\), the Unsigned Reversals and Transpositions column records on the first line the result of Theorem\(^3\). This result may be extended to the two other columns, and both their lines, since all these four algorithms are based on the same ideas and have the same bottlenecks as Algorithm\(^4\) find the last element of the first strip, compute \( P[i] = p_i \) and \( P^{-1}[x] \), and perform transpositions and reversals.
Each of these operations is performed in $O(\log n)$ with log-lists, as shown before. We only have to notice that a signed reversal combines an unsigned reversal and a call to change-sign.

The remaining cell in Figure 7, which concerns sorting by prefix and suffix variants of transpositions and of unsigned reversals, is particular. In this case, the algorithm is based on a graph representation of a permutation, requiring at each step to find a convenient edge, and this is not easier with log-lists then with classical data structures. The same reason explains the absence from Figure 7 of the columns Signed/Unsigned Reversals (only), for which the implementation with log-lists does not immediately provide an improvement.

5.3 Replacing permutation trees by log-lists in Sorting by Transpositions

In [7], Feng and Zhu introduce a new data structure, called permutation trees, and show that it allows us to improve the running time of the 1.5-approximation algorithm for sorting a permutation by transpositions [9] from $O(n^{3/2} \sqrt{\log n})$ time to $O(n \log n)$ time. Also, the improvement from $O(n^2)$ to $O(n \log n)$ time is achieved in [7] for the exact algorithm in [1] for sorting by block interchanges. Recently, the 1.375-approximation algorithm [6] for sorting by transpositions has also been improved from $O(n^2)$ time to $O(n \log n)$ time in [3], using permutation trees.

Given a permutation (or only a block of it) of size $n$, a permutation tree stores $O(n)$ space information about it and may be computed in $O(n)$ time. Moreover, the following operations are performed in $O(\log n)$ time on permutation trees: find the element at a given position in the permutation, find the position of a given element of the permutation, join the permutation trees of two neighboring blocks, split a permutation tree into two permutation trees corresponding to a given decomposition of the permutation into two blocks, and query a given block of the permutation represented by the tree looking for the maximum element in the block.

It is easy to see that log-lists also allow to perform all these operations in $O(\log n)$ time, using respectively the operations find-element, find-rank, insert, delete and find-max. Then we have:

**Theorem 4.** Log-lists successfully replace permutation trees in the $O(n \log n)$ time implementations of the 1.5- and 1.375-approximation algorithms [7, 3] for sorting by transpositions, as well as of the algorithm for sorting by block interchanges [7].

6 Conclusion

The data structure we proposed in this paper has significant interest when compared to existent data structures, as it combines the advantages of double-linked lists and those of binary search trees, and moreover adds some aggregate operations. Like a double-linked list, it allows us to insert/delete/reverse a sublist by cut and link operations. Like a binary search tree, it allows us to store elements according to their value (or key) order, and search for the element with a given value, or with the minimum/maximum value, using (basically) a binary search. In addition, log-lists allow us to keep trace of the rank of each element in its list, and search the element with a given rank.

We proposed here several applications to permutation sorting. They show that the optimal $O(n \log n)$ running time may be achieved for some algorithms whose main challenges are to handle the rank of the elements in the permutation, to perform a transposition or a reversal, and to merge two parts of the permutation. These operations are simple using log-lists, as all the difficulties are transferred to a lower abstraction level, handled using link-cut trees. Link-cut trees already had a lot of the functions needed by log-lists, but still were insufficient without the two supplementary operations we added in this paper.

Still, some permutation sorting algorithms use graph representations of the permutation, in which the search of the best move to perform is difficult mostly due to the graph complexity than to the data structure. In these cases, log-lists still allow to perform the transpositions and reversals on the permutation, but the running time is not easily improved with log-list. However, as it can be seen even in our applications, log-lists have a lot of operations, and we think that a more intensive and clever use of aggregate operations could allow further improvements.
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