Toda molecule and Tomimatsu–Sato solution—towards the complete proof of Nakamura’s conjecture

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Abstract
We discuss the Nakamura’s conjecture stating that the Tomimatsu–Sato black hole solution with an integer deformation parameter \( n \) is composed of the special solutions of the Toda molecule equation at the \( n \)th lattice site. From the previous work, in which the conjecture was partly analytically proved, we go further towards the final full proof by rearranging the rotation parameter. The proof is explicitly performed for the highest and lowest orders. Though the proof for all orders still remains unsolved, the prospect to the full proof becomes transparent and workable by our method.

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1. Introduction
There are close and unexpected relations between a variety of integrable systems. The interplay between the black hole and the Toda molecule solutions is one of such examples. Nakamura found an important relation [1] between the Tomimatsu–Sato (hereafter we refer it as TS) solutions [2] and the Toda molecule solutions [3]. That is, he asserts that a series of the TS solutions with an integer deformation parameter \( n \) and rotation parameters \( p, q \) (\( p^2 + q^2 = 1 \)) are obtained from the special solutions of the Toda molecule equation at the \( n \)th lattice site (Nakamura’s conjecture). The Ernst equation is summarized as two sets of equations, (equations (14) and (15)) and (equations (16) and (17)). Nakamura checked that this conjecture is numerically satisfied for small \( n \). One of the present authors (TF) tried to prove this conjecture analytically for general \( n \) [4].

In [4], we proved for the general \( n \) case that the special solutions of Toda molecule satisfy the first set of equations without using the explicit form of the solution. However, the second set was proved to be satisfied for the restricted case of \( q = 0 \) by using the explicit form of the solution. The \( q = 0 \) black hole is corresponding to the (extended) Weyl solutions. (The solution in the case \( q = 0, \ n = 1 \) corresponds to the Schwarzschild solution.) Thus, the full proof has not been solved for the generic case, \( q \neq 0 \). For general \( n \), the two solutions of...
Toda molecule equation, \( g_n \) and \( f_n \) (see equation (4)), are given by the polynomials of \( p \) and \( q \) of homogeneous degree of \( n \) and \( n-1 \), respectively, that is, \( p^iq^{n-1} \), \( (i=0, \ldots, n) \) and \( p^i q^{n-i-1} \), \( (i=0, \ldots, n-1) \). If we change these parameters \( p, q \) (independent parameter is one) to \( t \) defined by equation (38), the special solutions of Toda molecule equation \( g_n \) and \( f_n \) are described by the Laurent polynomials whose highest (lowest) degrees are \( n \) \( (-n) \) and \( n-1 \) \( (-n+1) \), respectively. We give the explicit form of Nakamura’s conjecture order by order of \( t \). In this paper, we prove the second set of equations at the highest and lowest orders of \( t \). This may be the step towards the complete proof.

This paper is organized as follows. In the next section, we briefly review our previous work [4]. Section 3 is the central part of this paper, where the Toda molecule equations and Nakamura’s conjecture are expanded by the Laurent polynomials and this conjecture is proved at the highest and lowest orders of \( t \). Some detailed calculations are developed in the appendix. In section 4, we discuss on the implications of obtained results, future prospect and the relations with the other approaches.

2. Nakamura’s conjecture and our previous results

The two-dimensional Toda molecule equation is described by

\[
\frac{\partial^2}{\partial S \partial T} \log V_n \equiv (\log V_n)_{ST} = V_n - 2V_{n+1} + V_{n+1}.
\] (1)

Here \( S \) and \( T \) are the light cone coordinates related to the Cartesian coordinates, \( X \) and \( Y \), by

\[
S = \frac{1}{2} (X + Y), \quad T = \frac{1}{2} (X - Y),
\] (2)

and \( n \) indicates the \( n \)th lattice site. If we introduce the \( \tau \) function defined by

\[
V_n = (\log \tau_n)_{ST},
\] (3)

the Toda molecule equation is expressed in terms of Hirota’s bilinear forms [5] as

\[
D_D D_T \tau_n \cdot \tau_n - 2\tau_{n+1}\tau_{n-1} = 0,
\] (4)

where the Hirota derivative is defined by

\[
D_A (f \cdot g) = (\partial_A f)g - f(\partial_A g),
\]

\[
D_AB (f \cdot g) = (\partial_A \partial_B f)g - (\partial_A f)(\partial_B g) - (\partial_B f)(\partial_A g) + f(\partial_A \partial_B g),
\] (5)

and so on. The bilinear form is very useful for the integrable system [6], which is also the case in the present problem. \( n \) is a positive integer and the boundary condition is chosen as \( \tau_0 = 1 \) corresponding to the finite and semi-infinite lattices. The general solution of equation (4) is expressed in a form of the two-directional Wronskian [7]

\[
\tau_n = \det \begin{pmatrix}
\Psi & L_\Psi & \ldots & L^{n-1}_\Psi \\
L_\Psi & L_\Psi & \ldots & L^{n-1}_\Psi \\
\vdots & \vdots & \ddots & \vdots \\
L^{n-1}_\Psi & L^{n-1}_\Psi & \ldots & L^{n-1}_{n+1}_\Psi
\end{pmatrix}
\] (6)

with the boundary condition \( \tau_0 = 1 \) and initial condition \( \tau_1 = \Psi \). Here \( L_+ (L_-) \equiv \partial_+ (\partial_T) = \frac{\partial}{\partial X} + \frac{1}{2} \left( \frac{\partial}{\partial Y} - \frac{\partial}{\partial X} \right) \). For the finite lattice, the initial condition \( \Psi \) takes a form

\[
\Psi = \sum_{k=1}^{N+1} H_k(S) G_k(T).
\] (7)
So $V_n$ satisfies the boundary condition

$$V_0 = V_{N+1} = 0,$$

(8)

where $N$ stands for the total number of lattice sites. In this paper, the semi-infinite lattice corresponding to the $N \to \infty$ case is treated. We first show how the solution (6) of equation (4) appears as a result of a Pfaffian identity for the help of later discussions, though it is a known fact.

We introduce $D$ as a determinant of the $(n + 1) \times (n + 1)$ matrix $\tau_{n+1}$:

$$D \equiv \tau_{n+1}.$$

(9)

The minor $D_{ij}$ is defined by deleting the $i$th row and the $j$th column from $D$. Similarly $D_{ikl}$ is defined by deleting the $i$th and $k$th rows and the $j$th and $l$th columns from $D$ and so on. The Toda molecule equation (4) is now expressed as

$$D_{nn} D_{n+1 n+1} - D_{n+1 n} D_{nn+1} - D_{n+1 n+1} D_{nn+1} = 0.$$

(10)

It holds since it is nothing but Jacobi’s (Sylvester’s) formula for matrix minors, which is one of Pfaffian identities. Thus, the Toda molecule equation has been reduced to the Pfaffian identity in the direct method. It is also the case in the Einstein equation as will be shown in the following.

The Ernst equation for the axially symmetric metric of the Einstein equation is

$$(\xi \xi^* - 1) \nabla^2 \xi - 2 \xi^* \nabla \xi \cdot \nabla \xi = 0,$$

(11)

where the star superscript denotes the complex conjugate. It is shown that equation (11) is invariant under the global SU(1,1) transformation

$$\xi' = \frac{\alpha \xi + \beta^*}{\beta \xi + \alpha^*}, \quad (|\alpha|^2 - |\beta|^2 \neq 0).$$

(12)

Setting

$$\xi_n = \frac{g_n}{f_n},$$

(13)

Nakamura’s conjecture on the TS solutions consists of two ingredients [1].

(i) Equation (11) has a decomposition into two sets [1]

$$D_i (g_n \cdot f_n - g_n^* \cdot f_n^*) = 0,$$

(14)

$$D_j (g_n \cdot f_n + g_n^* \cdot f_n^*) = 0,$$

(15)

and

$$F (g_n^* \cdot f_n) = 0,$$

(16)

$$F (g_n^* \cdot g_n + f_n^* \cdot f_n) = 0.$$  

(17)

Here the bilinear operator $F$ is

$$F = (x^2 - 1) \partial_x^2 + 2x \partial_x + (y^2 - 1) \partial_y^2 + 2y \partial_y + c_n,$$

(18)

and $x$ and $y$ are the usual prolate spheroidal coordinates defined by

$$\partial_X = (x^2 - 1) \partial_x \quad \text{and} \quad \partial_Y = (y^2 - 1) \partial_y.$$  

(19)
(ii) From the solutions of the Toda molecule equation $\tau_n$, a set of TS solutions $\xi_n = \frac{g_n}{f_n}$ with the deformation parameter $n$ is obtained as

$$g_n = \tau_n = D \left[ \frac{n+1}{n+1} \right], \quad f_n = \tau_{n-1} \bigg|_{\varphi=t+1} = D \left[ 1, n+1 \right].$$

(20)

by choosing the arbitrary function $\Psi$ and the constant $c_n$ in (6) as

$$\Psi = \psi = px - i qy, \quad p^2 + q^2 = 1, \quad c_n = -2n^2.$$  

(21)

The first set equations (14) and (15) are proved in the general case whose detailed proof should be referred to [4]. While, the second set equations (16) and (17) are shown for a restricted case of $q = 0$ in [4]. The present paper is its extension to the generic case $q \neq 0$, and we review for $q = 0$ first.

To prove the second set equations (16) and (17), the explicit form of $\psi$ in equation (21) is required in contrast to the case of the first set. $g_n(f_n)$ is the determinant of the matrix whose $(i, j)$ ($(i - 1, j - 1)$) element is

$$L^{i-1}_+ L^{j-1}_- \psi = L^{i-1}_+ L^{j-1}_-(px - i qy) = pW_{i+j-1}(x) + (-1)^i i qW_{i+j-1}(y),$$

(22)

where

$$W_{n+1}(z) = (z^2 - 1) \frac{d}{dz} W_n(z) \quad \text{with} \quad W_1(z) = z.$$  

(23)

The polynomial expression for $W_n(z)$ is given by

$$W_n(z) = \sum_{m=0}^{n-2} \sum_{l=0}^{m} (-1)^l (m - l + 1)^{n-1} \left( \begin{array}{c} n \\ l \end{array} \right) (z + 1)^{m+1} (z - 1)^{l+1-1} \quad (n \geq 2),$$

(24)

where $\left( \begin{array}{c} n \\ l \end{array} \right)$ is the binomial coefficient.

In the case when $p = 1$ and $q = 0$, we have $\psi = x$. Then $g_n$ and $f_n$ are the real functions depending only on $x$. Explicit forms of $g_n$ and $f_n$ are

$$g_n = \det \begin{pmatrix} W_1(x) & W_2(x) & \cdots & W_n(x) \\ W_2(x) & W_3(x) & \cdots & W_{n+1}(x) \\ \vdots & \vdots & \vdots & \vdots \\ W_n(x) & W_{n+1}(x) & \cdots & W_{2n-1}(x) \end{pmatrix},$$

(25)

$$f_n = \det \begin{pmatrix} W_2(x) & \cdots & W_{n+1}(x) \\ \vdots & \vdots & \vdots \\ W_{n+1}(x) & \cdots & W_{2n-1}(x) \end{pmatrix}.$$  

Using equation (23), we can evaluate the determinant:

$$g_n = (x^2 - 1)^{\frac{n+1}{2}} \det \begin{pmatrix} x & 1 & 0 & 0 & \cdots & 0 \\ x^2 - 1 & 2x & 2 & 0 & \cdots & 0 \\ 2x(x^2 - 1) & 6x^2 - 2 & 12x & 12 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{n+1} & \cdots & \cdots & (x+1)^{n-1} & (x-1)^{n-1} \end{pmatrix}$$

$$= \frac{A_n}{2} (x^2 - 1)^{\frac{n+1}{2}} ((x+1)^n + (x-1)^n),$$

(26)

where the coefficient $A_n$ is

$$A_n = \left( \prod_{k=0}^{n-1} \Gamma(n - k) \right)^2 = (n-1)^2(n-2)^4 \cdots 2^{2(n-1)}.$$  

(27)
Equation (26) is proved by induction. For the \( n = 1 \) and \( n = 2 \) cases, equation (26) holds. Assuming it for the \( n = l - 1 \) and \( n = l \) cases, we prove it for the case of \( n = l + 1 \). Applying Jacobi’s formula (10) to \( g_n \), we obtain, corresponding to the Toda molecule equation in equation (6),

\[
g_{l-1}g_{l+1} = (L_X^2 g_l)g_l - (L_X g_l)^2.
\]

(28)

where \( L_X \equiv (x^2 - 1)\partial_x \). Using the assumed forms for \( g_l \) and \( g_{l-1} \), we find the expected form for \( g_{l+1} \):

\[
g_{l+1} = \frac{A_{l+1}}{2} (x^2 - 1)^{\frac{l+1}{l+2}} ((x + 1)^{l+1} + (x - 1)^{l+1}).
\]

(29)

Here we have used an equality obtained from the definition of \( A_n \) in equation (27):

\[
A_{n-1}A_{n+1} = n^2 A_n.
\]

(30)

Thus, equation (26) is proved for \( n = l + 1 \). Quite analogously \( f_n \) is shown to be

\[
f_n = \frac{A_n}{2} (x^2 - 1)^{\frac{2n-1}{2n}} ((x + 1)^n - (x - 1)^n).
\]

(31)

The second set equations (16) and (17) are

\[
F (g_n \cdot f_n) = 0,
\]

\[
F (g_n \cdot g_n + f_n \cdot f_n) = 0,
\]

(32)

and \( F \) becomes in the \( q = 0 \) and \( c_n = -2n^2 \) case

\[
F a \cdot b = \frac{1}{x^2 - 1} \left( (L_X^2 a)b + a(L_X^2 b) - 2(L_X a)(L_X b) - 2n^2 a b \right).
\]

(33)

It is straightforward to show that \( g_n \) and \( f_n \) in equations (26) and (31) satisfy equation (32).

3. Proof of Nakamura’s conjecture at the highest and lowest orders

In the paper [4], equations (14) and (15) are fully proved by reducing them to a Pfaffian identity. On the other hand, equations (16) and (17) are proved only in the restricted case \( p = 1, q = 0 \). It is due to the fact that the operator \( F \) of equation (33) is a rather complicated two-variable function and is not described by complete bilinear forms, which makes the use of algebraic relations very difficult.

In this section, we adopt a tedious but workable approach to go beyond the \( p = 1, q = 0 \) case, making use of already proved first set of equations. That is, rewriting \( p, q \) as equation (38) and expanding \( f_n, g_n \) as the powers of \( t \), we prove Nakamura’s conjecture order by order of \( t \). Unfortunately, the explicit proof is restricted in the case of the highest and lowest orders of \( t \), but it may give the route towards the final proof.

We will explain our proof step by step. In the first step, we observe how the SU(1,1) global symmetry of the Ernst equation reflects into the Toda molecule equation. We find a recursion formula which is a combination of the functions \( g_n \) and \( f_n \). The parameter \( t \) is introduced instead of \( p, q \) in the second step. The functions \( g_n \) and \( f_n \) are expressed in the form of the Laurent polynomials of \( t \), and the properties of their expansion coefficients are investigated.

In the third step, using the expressions for \( g_n \) and \( f_n \) of step 2, Toda molecule equations and the recursion formula obtained in step 1 are represented as the equations in every order of \( t \). In the last step, Nakamura’s conjecture at the highest and lowest orders of \( t \) is proved with the polynomial expressions for the highest and lowest terms of \( g_n \) and \( f_n \).
Step 1. If Nakamura’s conjecture is true, the global symmetry of equation (12) in the Ernst equation must be reflected in the Toda molecule equation. Equation (12) with equation (13) is rewritten as

\[
\frac{g_n'}{f_n'} = \frac{\alpha g_n + \beta^* f_n}{\beta g_n + \alpha^* f_n}.
\]

This may be divided into the following two transformations:

\[
g_n' = \alpha g_n + \beta^* f_n,
\]

\[
f_n' = \beta g_n + \alpha^* f_n.
\]

Indeed, it is easily shown that two sets of equations (14)–(17) in Nakamura’s conjecture remain invariant under the above transformations (35) and (36). Furthermore, it turns out that the transformed functions \(g_n'\) and \(f_n'\) are also solutions of the Toda molecule equations, i.e. any linear superposition of the solutions \(g_n\) and \(f_n\) given by equation (20) satisfies the Toda molecule equation. The proof for the linear property is confirmed by several ways. As the first proof, we consider that the transformation \(\psi \rightarrow \psi + \psi_0\) in equation (6) generates a different solution of the Toda molecule equation. Under the transformation, the function \(\tau_n\) becomes \(\tau_n + \frac{b}{a} \tau_{n-1} |_{\psi \rightarrow L \cdot L \cdot \psi}\). Taking the transformation and the rescaling of the tau function into account, one can show that the function \(a \tau_n + b \tau_{n-1} |_{\psi \rightarrow L \cdot L \cdot \psi}\) satisfies the Toda molecule equation, where \(a\) and \(b\) are any complex constants. Because \(\tau_n = g_n\) and \(\tau_{n-1} |_{\psi \rightarrow L \cdot L \cdot \psi} = f_n\) under the choices (21), it is shown that \(ag_n + bf_n\) is also a solution of the Toda molecule equation. As the second proof, it follows from this linear combination equation (35) (equation (36)) that

\[
D_2 D_2 D_2 f_n \cdot g_n = f_{n+1} g_{n-1} + f_{n-1} g_{n+1},
\]

which can be reduced to a Pfaffian identity. This equation is useful for obtaining the recursion formula (33) and proving the polynomial forms (58) and (59).

Step 2. Here let us consider the expansions of the functions \(g_n\) and \(f_n\) in terms of the parameters \(p\) and \(q\). For general \(n\), \(g_n\) and \(f_n\) are given by the polynomials of \(p\) and \(q\) of homogeneous degree of \(n\) and \(n-1\), respectively. However, they have rather complicated forms in terms of \(p\) and \(q\). So let us introduce one parameter \(t\) defined by

\[
p = \frac{t + t^{-1}}{2}, \quad q = \frac{t - t^{-1}}{2i}.
\]

Then \(\psi\) is rewritten as

\[
\psi = t \left(\frac{x - y}{2}\right) + t^{-1} \left(\frac{x - y}{2}\right) = tu + t^{-1} u, \quad u = \frac{x + y}{2}, \quad v = \frac{x - y}{2}.
\]

To show the \(t\) dependence of \(g_n\), \(f_n\) explicitly, we write them as

\[
g_n = g_n(x, y; t), \quad f_n = f_n(x, y; t).
\]

Then, \(g_n(x, y; t)\) and \(f_n(x, y; t)\) are expressed by the Laurent polynomials of \(n\) and \(n-1\) orders on \(t\), respectively:

\[
g_n(x, y; t) = \sum_{m=-n}^{n} g_n^{(m)}(x, y) t^m, \quad f_n(x, y; t) = \sum_{m=-n+1}^{n-1} \tilde{f}_n^{(m)}(x, y) t^m.
\]

Here, the functions \(g_n^{(m)}(x, y)\) and \(\tilde{f}_n^{(m)}(x, y)\) are real functions and have the following properties:

\[
g_n^{(m)} = g_n(x, y; t^{-1}), \quad f_n^{(m)} = f_n(x, y; t^{-1}).
\]
as shown from their definitions. It follows from these identities that
\[
\tilde{g}_n^{(n-2m-1)}(x, y) = 0, \quad \tilde{f}_n^{(n-2m+1)}(x, y) = f_n^{(n-2m-1)}(x, y) \quad (m = 0, \ldots, n - 1),
\]
\[
\tilde{g}_n^{(n-2m)}(x, y) = 0, \quad \tilde{g}_n^{(n-2m+2)}(x, y) = \tilde{g}_n^{(n-2m)}(x, y) \quad (m = 0, \ldots, n),
\]
and we obtain the expansions
\[
g_n(x, y; t) = \sum_{m=0}^{n} g_n^{(n-2m)}(x, y) t^{a-2m}, \quad f_n(x, y; t) = \sum_{m=0}^{n-1} f_n^{(n-2m-1)}(x, y) t^{a-2m-1}
\]
and
\[
g_n^x(x, y; t) = \sum_{m=0}^{n} g_n^{(n-2m+2)}(x, y) t^{a-2m-2}, \quad f_n^x(x, y; t) = \sum_{m=0}^{n-1} f_n^{(n-2m+1)}(x, y) t^{a-2m-1}
\]
with
\[
\tilde{g}_n^{(n)}(x, y) = \tilde{g}_n^{(m)}(x, y), \quad \tilde{f}_n^{(n)}(x, y) = \tilde{f}_n^{(m)}(x, y).
\]

**Step 3.** Because the Toda molecule equations for \(f_n\) and \(g_n\) and equation (37) hold for an arbitrary value of the parameter \(t\), we obtain order-by-order equations of the Laurent expansions of \(f_n\) and \(g_n\) (see appendix A.1). The highest order equations of them, corresponding to the \(l = 0\) case in appendix A.1, are given by
\[
D_n^3 T \tilde{g}_n^{(n)}(x, y) \cdot \tilde{g}_n^{(n)}(x, y) = 2 \tilde{g}_n^{(n+1)}(x, y) \tilde{g}_n^{(n-1)}(x, y),
\]
\[
D_n^3 T \tilde{f}_n^{(n-1)}(x, y) \cdot \tilde{f}_n^{(n-1)}(x, y) = 2 \tilde{f}_n^{(n+1)}(x, y) \tilde{f}_n^{(n-2)}(x, y),
\]
and
\[
D_n^3 T \tilde{g}_n^{(n-1)}(x, y) \cdot \tilde{g}_n^{(n-1)}(x, y) = \tilde{g}_n^{(n)}(x, y) \tilde{g}_n^{(n-1)}(x, y) + \tilde{g}_n^{(n-1)}(x, y) \tilde{g}_n^{(n+1)}(x, y).
\]
By definitions, the highest order terms \(\tilde{g}_n^{(n)}(x, y)\) and \(\tilde{f}_n^{(n-1)}(x, y)\) are written by the following determinants:
\[
\tilde{g}_n^{(n)}(x, y) = \left| \begin{array}{cccc}
v & L_{-1} & \cdots & L_{-n+1} \v \\
L_{+1} v & L_2 & \cdots & L_{-n+1} \v \\
\vdots & \vdots & \ddots & \vdots \\
L_{n-1} v & L_{n-2} & \cdots & L_{-n} \v \end{array} \right|, \quad (54)
\]
\[
\tilde{f}_n^{(n-1)}(x, y) = \left| \begin{array}{cccc}
v & L_{-1} & \cdots & L_{-n+1} \v \\
L_{+1} v & L_2 & \cdots & L_{-n+1} \v \\
\vdots & \vdots & \ddots & \vdots \\
L_{n-2} v & L_{n-3} & \cdots & L_{-n+1} \v \end{array} \right|.
\]

Then, all of these equations (51)--(53) reduce to Pfaffian identities.

In general, these functions \(g_n\) and \(f_n\) should be expressed as finite polynomials of the prolate spheroidal coordinates \(x, y\). After some numerical calculations, the highest term \(\tilde{g}_n^{(n)}(x, y)\) and the lowest term \(\tilde{g}_n^{(n-1)}(x, y)\) have the forms
\[
\tilde{g}_n^{(n)}(x, y) = \left| \begin{array}{cccc}
v & L_{-1} & \cdots & L_{-n+1} \v \\
L_{+1} v & L_2 & \cdots & L_{-n+1} \v \\
\vdots & \vdots & \ddots & \vdots \\
L_{n-1} v & L_{n-2} & \cdots & L_{-n+1} \v \end{array} \right| = 2^{n(n-1)} a_n \theta \frac{a_{n+1} \sin \theta}{v^{\frac{n+1}{2}}} \quad (56)
\]
and
\[
\tilde{g}^{\nu(n-1)}_n(x, y) = \begin{vmatrix}
\nu & L - u & \cdots & L^{-1}_{x=1} \nu \\
L + u & L + L - u & \cdots & L + L^{-1}_{x=1} \nu \\
\vdots & \vdots & \ddots & \vdots \\
L^{-1}_{x=1} \nu & L^{-1}_{x=1} \nu & \cdots & L^{-1}_{x=1} \nu
\end{vmatrix} = 2^{n(n-1)} A_n \frac{\nu_{x=1}^{\text{max} - 1} v_{x=1}^{\text{min} - 1}}{v_{x=1}^{\text{min} - 1}},
\] (57)

where \( A_n \) is given by equation (27). Also it turns out that the highest term \( \tilde{f}^{\nu(n-1)}_n(x, y) \) and the lowest term \( \tilde{f}^{\nu(n-1)}_n(x, y) \) are
\[
\tilde{f}^{\nu(n-1)}_n(x, y) = \begin{vmatrix}
L + L - u & \cdots & L + L^{-1}_{x=1} \nu \\
\vdots & \ddots & \vdots \\
L^{-1}_{x=1} L - u & \cdots & L^{-1}_{x=1} L^{-1}_{x=1} \nu
\end{vmatrix}
= \frac{\tilde{g}^{\nu(n-1)}_n(x, y)}{\sqrt{\pi}} \sum_{m=0}^{n-1} \sum_{l=0}^{m} (-1)^{l-m} \times \frac{\Gamma\left(\frac{2m+1}{2}\right) \Gamma\left(\frac{2(n-1)+1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}\right) \Gamma(l+1) \Gamma(m-l+1) \Gamma(n-m)} u^{2l} v^{2m-1}
\] (58)

and
\[
\tilde{f}^{\nu(n-1)}_n(x, y) = \begin{vmatrix}
L + L - u & \cdots & L + L^{-1}_{x=1} \nu \\
\vdots & \ddots & \vdots \\
L^{-1}_{x=1} L - u & \cdots & L^{-1}_{x=1} L^{-1}_{x=1} \nu
\end{vmatrix}
= \frac{\tilde{g}^{\nu(n-1)}_n(x, y)}{\sqrt{\pi}} \sum_{m=0}^{n-1} \sum_{l=0}^{m} (-1)^{l-m} \times \frac{\Gamma\left(\frac{2m+1}{2}\right) \Gamma\left(\frac{2(n-1)+1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}\right) \Gamma(l+1) \Gamma(m-l+1) \Gamma(n-m)} u^{2l} v^{2m-1}.
\] (59)

In deriving the above two equations, already proved first set of equations, equations (14) and (15) are very helpful. The explicit proof of equations (56)–(59) is given by applying the mathematical induction and using identities (50) as follows.

In the \( n = 1 \) and \( n = 2 \) cases, these expressions of equations (56)–(59) hold. Assuming it for the \( n = l - 1 \) and \( n = l \) cases, we prove them for the case of \( n = l + 1 \). Substituting the assumed polynomial forms \( \tilde{g}^{\nu(n-1)}_{l-1} \) and \( \tilde{g}^{\nu(n-1)}_l \) into the Toda molecule equation at the highest order,
\[
\tilde{g}^{\nu(n+1)}_{l+1}(x, y) = \frac{1}{2 \tilde{g}^{\nu(n-1)}_{l-1}(x, y)} \left(D_x D_y \tilde{g}^{\nu(n-1)}_l(x, y) \cdot \tilde{g}^{\nu(n-1)}_l(x, y)\right),
\] (60)

we obtain the expected polynomial form for \( \tilde{g}^{\nu(n+1)}_{l+1} \). By using identities (50), it is easily shown that equation (57) is correct. Equation (53) is used instead of equation (52) to prove the parts \( \tilde{f}^{\nu(n-1)}_n(x, y) \) and \( \tilde{f}^{\nu(n+1)}_n(x, y) \), with equations (56) and (57). It is also shown that the expressions for \( \tilde{f}^{\nu(n-1)}_n(x, y) \) and \( \tilde{f}^{\nu(n+1)}_n(x, y) \) are correct by substituting them into equation (53) and using identities (50). Thus, these polynomial expressions are correct. The linearity property for \( \tilde{g}_n \) and \( \tilde{f}_n \) ensures that the expression for \( \tilde{f}^{\nu(n-1)}_n(x, y) \) satisfies equation (52).

**Step 4.** Let us prove Nakamura’s conjecture at the highest order of \( t \). Substituting the Laurent expansions (48) and (49) into the conjecture (14)–(17), it reduces to the equations in every order of \( t \) (see appendix A.2). The highest orders of \( t \) for equations (14)–(16) and (17) are \( 2n - 1 \) and \( 2n \), respectively. They are described as the following equations at the highest order:
\[
D_x \left( \tilde{g}^{\nu(n-1)}_n(x, y) \cdot \tilde{f}^{\nu(n-1)}_n(x, y) - \tilde{g}^{\nu(n-1)}_n(x, y) \cdot \tilde{f}^{\nu(n-1)}_n(x, y) \right) = 0,
\] (61)
In the previous work [4], the first set equations (14) and (15) were proved completely for the first set equations (14) and (15), they are also reproduced from this approach.

For this case, it is known that the metric indicates an extremal TS solution, not asymptotically flat, and describes a region of the TS metric near its ergosphere [8].

In this paper, we have discussed a proof of Nakamura’s conjecture on the TS solutions.

4. Discussions

In this paper, we have discussed a proof of Nakamura’s conjecture on the TS solutions. In the previous work [4], the first set equations (14) and (15) were proved completely without using the explicit forms of $f_n$ and $g_n$. On the other hand, the proof of the second set equations (16) and (17) needed the explicit forms of $f_n$ and $g_n$ and was given for the restricted case $p = 1$, $q = 0$, deformed but non-rotating black hole, extended Weyl solution. In that case, the key point was that the functions $g_n$ and $f_n$ depend only on $x$ and their explicit polynomial expressions are required. This work is the extension of that work. That is, we have discussed the deformed and rotating case, we have rearranged the original parameters $p, q$ ($p^2 + q^2 = 1$) to $t$. Thanks to this arrangement, some properties of the functions $g_n$ and $f_n$ become transparent and workable (for instance equations (42)–(45)). Using property (45), the explicit forms of functions $g_n$ and $f_n$ are obtained for the $p = 0$, $q = 1$ ($t = \sqrt{-1}$) case and proved to satisfy Nakamura’s conjecture. For this case, it is known that the metric indicates an extremal TS solution, not asymptotically flat, and describes a region of the TS metric near its ergosphere [8].

For the most generic case, $pq \neq 0$, the functions $g_n$ and $f_n$ are embedded in two dimensions and the situation is drastically changed compared with the $pq = 0$ case. The introduction of $t$ enables us to prove the conjecture order by order of $t$. In fact, the Laurent expansions of the functions $g_n$ and $f_n$ lead us to the polynomial expressions of $x, y$ in equations (56)–(59) for the highest and lowest orders of $g_n$ and $f_n$. Consequently, it is proved that the highest and lowest orders of Nakamura’s conjecture are valid by the straightforward calculations. Unfortunately the proof of the conjecture has not been completed at all orders of $t$ though their explicit forms are described in the appendix.

Indeed, it is possible to recursively determine all order terms of $g_n$ and $f_n$ for general $n$ by using equations (A.2)–(A.10) though it is considerably complex. The next highest order equations are described in terms of the highest and next highest orders of $f_n$ and $g_n$ ($I = 1$ case in appendices A.1 and A.2). Substituting the polynomial forms of equations (56)–(59) into these equations, we have the implicit relations of the next highest functions $g_n^{(n-2)}$ and $f_n^{(n-3)}$. To obtain the explicit forms of them, we use equations (B1)–(B6) with $I = 1$.

Furthermore, we observe that the expression for the highest order of $f_n$ is connected with the gamma functions. It is strongly expected that the polynomial expressions for $g_n$ and $f_n$ are described by two-variable hypergeometric functions.
It is also worth noting that the SU(1,1) symmetry plays an important role for the proof of the polynomial expressions (58) and (59). Nakamura’s conjecture is invariant under this transformation. The physical meaning of its symmetry is that a NUT parameter is generated by the transformation (for instance, the Schwartzschild geometry becomes Taub-NUT geometry by the transformation). Therefore, if Nakamura’s conjecture is correct, its conjecture is also correct even for a generalization of the TS solutions with a NUT parameter.

Here some comments are in order on the relation between this theory and the other approaches, especially those using the Bäcklund transformations. Indeed, the Bäcklund transformation has played an important role to generate infinitely many solutions from an initial solution. There exist the well-known Bäcklund transformations in the stationary axisymmetric Einstein theory, the so-called discrete Neugebauer–Kramer mapping [9] and Ehlers transformations [10]. These transformations generate the Neugebauer–Kramer (NK) solution describing the nonlinear superposition of collinear $N(=1, 2, 3, \ldots)$ Kerr solitons, which is explicitly constructed for small $N$ [11]. The wholly analytic proof [12] of the NK solutions for general $N$ is given in terms of two series of the Nakamura solutions [13] called intermediate solutions (unphysical solutions). Then the Bäcklund transformations appeared in the NK solutions give the explicit link between the Nakamura solutions (see [12] for the details).

On the other hand, it is expected that the TS solutions are derived by taking a limit of NK solutions [11, 14] (TSNK conjecture), whose proof is left unsolved for the general $2N$-soliton case. Although one of the approaches to the TS solutions as the natural limit of the NK solutions is also discussed by Dale [15], it is not proved. Also, since its limitation changes the determinant form of NK solutions (indefinite form), the explicit connections between the TS solutions in the context of the Bäcklund transformations are not obtained. In contrast, Nakamura’s conjecture provides the generation mechanism of the TS solutions in terms of the Toda molecule equation. If one can explain the meaning of the generation mechanism by the Toda molecule equation as the Bäcklund transformation, then both of Nakamura’s conjecture and TSNK conjecture will be solved. The details of these connections will be considered elsewhere.

The above-mentioned relations are schematically written in figure 1.

| $\alpha, \beta, \varepsilon$ in the upper-left frame | $\delta$ in the upper-right one is a deformation parameter. |
|-----------------------------------------------|-----------------------------------------------|

Figure 1. The relations between ours and the other approaches. $\alpha, \beta, \varepsilon$ in the upper-left frame indicate the Bäcklund transformations. $\delta$ in the upper-right one is a deformation parameter.

| $\alpha, \beta, \varepsilon$ in the upper-left frame | $\delta$ in the upper-right one is a deformation parameter. |
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It is also worth noting that the SU(1,1) symmetry plays an important role for the proof of the polynomial expressions (58) and (59). Nakamura’s conjecture is invariant under this transformation. The physical meaning of its symmetry is that a NUT parameter is generated by the transformation (for instance, the Schwartzschild geometry becomes Taub-NUT geometry by the transformation). Therefore, if Nakamura’s conjecture is correct, its conjecture is also correct even for a generalization of the TS solutions with a NUT parameter.

Here some comments are in order on the relation between this theory and the other approaches, especially those using the Bäcklund transformations. Indeed, the Bäcklund transformation has played an important role to generate infinitely many solutions from an initial solution. There exist the well-known Bäcklund transformations in the stationary axisymmetric Einstein theory, the so-called discrete Neugebauer–Kramer mapping [9] and Ehlers transformations [10]. These transformations generate the Neugebauer–Kramer (NK) solution describing the nonlinear superposition of collinear $N(=1, 2, 3, \ldots)$ Kerr solitons, which is explicitly constructed for small $N$ [11]. The wholly analytic proof [12] of the NK solutions for general $N$ is given in terms of two series of the Nakamura solutions [13] called intermediate solutions (unphysical solutions). Then the Bäcklund transformations appeared in the NK solutions give the explicit link between the Nakamura solutions (see [12] for the details).

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|-----------------------------------------------|-----------------------------------------------|
Thus, the study of the relations between the Toda molecule equation and the Ernst equation is very important. It explains the reason why the deformation parameter \( n \) is limited as integers for the TS solutions, though generalized Tomimatsu–Sato solutions with non-integer deformation parameters were considered by Hori [16] and Cosgrove [17]. On the physical nature of the geometries of TS solution, for instance, geometrical differences in event horizons and causal structures of odd and even deformation parameters [18] will also be clarified from this approach [19].

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**Appendix**

We have seen that the functions \( g_n \) and \( f_n \) are expanded by the parameter \( t \). In the appendix, we extract order-by-order equations for the Toda molecule equations and Nakamura’s conjecture by substituting expressions (48) into them.

### A.1. Order-by-order equations for the Toda molecule equation

The Toda molecule equation for \( g_n \) is expressed by

\[
D_3 \frac{D_f t}{D_y} \left( \sum_{m=0}^{n} \frac{g_{(n-2m)}}{g_n} (x, y) t^{n-2m} \right) \cdot \left( \sum_{j=0}^{n} \frac{g_{(n-2j)}}{g_n} (x, y) t^{n-2j} \right)
= 2 \left( \sum_{m=0}^{n+1} \frac{g_{(n-2m+1)}}{g_n+1} (x, y) t^{n-2m+1} \right) \left( \sum_{l=0}^{n-1} \frac{g_{(n-2l-1)}}{g_{n-1}} (x, y) t^{n-2l-1} \right)
\]

(A.1)

in terms of the Laurent expansions by \( t \). The parameter \( t \) is an arbitrary constant, so that the above equation must hold at every order of \( t \). For the orders \( t^{2n-2l} \) \((I = 0, \ldots, 2n)\), we derive the following coupled equations:

(I) for \( 0 \leq I \leq n - 1 \),

\[
\sum_{J=0}^{I} D_3 \frac{D_f t}{D_y} \frac{g_{(n-2J)}}{g_n} (x, y) \cdot \frac{g_{(n-2J+2I)}}{g_n} (x, y) = 2 \sum_{J=0}^{I} \frac{g_{(n-2J+1)}}{g_{n+1}} (x, y) \frac{g_{(n-2J+2I-1)}}{g_{n-1}} (x, y),
\]

(A.2)

(II) for \( I = n \),

\[
\sum_{J=0}^{n} D_3 \frac{D_f t}{D_y} \frac{g_{(n-2J)}}{g_n} (x, y) \cdot \frac{g_{(-n+2J)}}{g_n} (x, y) = 2 \sum_{J=1}^{n} \frac{g_{(n-2J+1)}}{g_{n+1}} (x, y) \frac{g_{(-n+2J-1)}}{g_{n-1}} (x, y),
\]

(A.3)

(III) for \( n + 1 \leq I \leq 2n \), \( y \rightarrow -y \) in (I) and (II).

(A.4)

In the similar way, the Toda molecule equation for \( f_n \) becomes the following equations at the order \( t^{2n-2l} \) \((I = 0, \ldots, 2(n - 1))\):

(IV) for \( 0 \leq I \leq n - 2 \),

\[
\sum_{J=0}^{I} D_3 \frac{D_f t}{D_y} \frac{f_{(n-2J-1)}}{f_n} (x, y) \cdot \frac{f_{(n-2J+2I-1)}}{f_n} (x, y) = 2 \sum_{J=0}^{I} \frac{f_{(n-2J)}}{f_{n+1}} (x, y) \frac{f_{(n-2J+2I-2)}}{f_{n-1}} (x, y),
\]

(A.5)
Equation (15) reduces to the following equations at the order $12$

$$\sum_{J=0}^{n-1} D_D D_T J_n \tilde{g}^{(n-2J-1)}(x, y) \cdot J_n \tilde{g}^{(n-2J+1)}(x, y) = 2 \sum_{J=1}^{n-1} f^{(n-2J)}(x, y) \tilde{f}^{(n-2J+1)}(x, y), \quad (A.6)$$

(V) for $I = n - 1$,

$$(V) \quad \sum_{J=0}^{n-1} D_D D_T J_n \tilde{g}^{(n-2J-1)}(x, y) \cdot J_n \tilde{g}^{(n-2J+1)}(x, y) = 2 \sum_{J=1}^{n-1} f^{(n-2J)}(x, y) \tilde{f}^{(n-2J+1)}(x, y), \quad (A.6)$$

(VI) for $n \leq I \leq 2(n - 1)$, $y \rightarrow -y$ in (IV) and (V).

For $t^{2n-2I-1}$ ($I = 0, \ldots, 2n - 1$), equation (37) is reduced to

(VII) for $0 \leq I \leq n - 2$,

$$\sum_{J=0}^{I} D_D D_T J_n \tilde{g}^{(n-2J-1)}(x, y) \cdot J_n \tilde{g}^{(n-2J+2)}(x, y) = 2 \sum_{J=0}^{I} f^{(n-2J)}(x, y) \tilde{g}^{(n-2J+1)}(x, y) + f^{(n-2I-2J)}(x, y) \tilde{g}^{(n-2J+1)}(x, y) \quad (A.8)$$

(VIII) for $I = n - 1$,

$$\sum_{J=0}^{n-1} D_D D_T J_n \tilde{g}^{(n-2J-1)}(x, y) \cdot J_n \tilde{g}^{(n-2J+2)}(x, y) = 2 \sum_{J=0}^{n-1} f^{(n-2J)}(x, y) \tilde{g}^{(n-2J+1)}(x, y) + \sum_{J=0}^{n-2} f^{(n-2J-2)}(x, y) \tilde{g}^{(n-2J+3)}(x, y), \quad (A.9)$$

IX) for $n \leq I \leq 2n - 1$, $y \rightarrow -y$ in (VII) and (VIII).

The highest order equations correspond to the $I = 0$ case.

A.2. Order-by-order equations for Nakamura’s conjecture

Equations (14)–(17) of Nakamura’s conjecture are also rewritten by the Laurent expansions of $t$. Equation (14) becomes the following equations for the order $t^{2n-2I-1}$ ($I = 0, \ldots, 2n - 1$):

(I) for $0 \leq I \leq n - 1$,

$$\sum_{J=0}^{I} D_x \tilde{g}^{(n-2J)}(x, y) \cdot J_n \tilde{g}^{(n-2J+1)}(x, y) - \tilde{g}^{(n-2J)}(x, y) \cdot f^{(n-2J+1)}(x, y) = 0, \quad (A.11)$$

(II) for $I = n$,

$$\sum_{J=1}^{n} D_x \tilde{g}^{(n-2J)}(x, y) \cdot J_n \tilde{g}^{(n-2J+1)}(x, y) - \tilde{g}^{(n-2J)}(x, y) \cdot f^{(n-2J+1)}(x, y) = 0, \quad (A.12)$$

(III) for $n + 1 \leq I \leq 2n - 1$, $y \rightarrow -y$ in (I) and (II).

Equation (15) reduces to the following equations at the order $t^{2n-2I-1}$ ($I = 0, \ldots, 2n - 1$):

(IV) for $0 \leq I \leq n - 1$,

$$\sum_{J=0}^{I} D_x \tilde{g}^{(n-2J)}(x, y) \cdot J_n \tilde{g}^{(n-2J+1)}(x, y) + \tilde{g}^{(n-2J)}(x, y) \cdot f^{(n-2J+1)}(x, y) = 0, \quad (A.14)$$

(V) for $I = n - 1$,
(V) for \( I = n, \)
\[
\sum_{J=1}^{n} D_J \left( \tilde{g}^{(n-2J)}_n (x, y) \cdot \tilde{f}^{(-n+2J-1)}_n (x, y) + \tilde{g}^{(-n+2J)}_n (x, y) \cdot \tilde{f}^{(n-2J+1)}_n (x, y) \right) = 0, \tag{A.15}
\]

(VI) for \( n + 1 \leq I \leq 2n - 1, \) \( y \to -y \) in (IV) and (V).

For equation (16) at the order \( t^{2n-2I-1} (I = 0, \ldots, 2n - 1), \) it is,

(VII) for \( 0 \leq I \leq n - 1, \)
\[
\sum_{J=0}^{I} F \left( \tilde{g}^{(-n+2J)}_n (x, y) \cdot \tilde{f}^{(n-2J+1)}_n (x, y) \right) = 0, \tag{A.17}
\]

(VIII) for \( I = n, \)
\[
\sum_{J=1}^{n} F \left( \tilde{g}^{(-n+2J)}_n (x, y) \cdot \tilde{f}^{(n-2J+1)}_n (x, y) \right) = 0, \tag{A.18}
\]

(IX) for \( n + 1 \leq I \leq 2n - 1, \) \( y \to -y \) in (VII) and (VIII).

For equation (17) at the order \( t^{2n-2I} (I = 0, \ldots, 2n), \) it is,

(X) for \( 1 \leq I \leq n, \)
\[
\sum_{J=0}^{n-I} F \left( \tilde{g}^{(-n+2J)}_n (x, y) \cdot \tilde{f}^{(n-2J+2)}_n (x, y) + \tilde{g}^{(-n+2J)}_n (x, y) \cdot \tilde{f}^{(n-2J+1)}_n (x, y) \right) = 0, \tag{A.20}
\]

(XI) for \( I = 0, \)
\[
F \left( \tilde{g}^{(n-n)}_0 (x, y) \cdot \tilde{g}^{(n)}_n (x, y) \right) = 0, \tag{A.21}
\]

(XII) for \( n + 1 \leq I \leq 2n, \) \( y \to -y \) in (X) and (XI).

Equations (A.11)–(A.16) are corresponding to the first set equations (14) and (15) of Nakamura’s conjecture. As already mentioned, equations (14) and (15) are proved by using a Pfaffian identity [4]. Thus, equations (A.11)–(A.16) are correct. The highest order equations of Nakamura’s conjecture also correspond to the \( I = 0 \) case.

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