An Interactive Simulation of the Nucleon-Core Interacting Using the Fourth-Order Runge-Kutta Method for Studying Nuclear Physics

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Abstract. Nuclear physics is one of the theories in the field of physics that aims to study the structure of atomic nuclei, the interactions between the compilers of atomic nuclei, and the underlying forces. This study aims to obtain a model of interaction that occurs between nucleons. The interactions between these nucleons are modeled by a binary center of mass system (two objects) and their potential experience is expressed by the Woods-Saxon potential with certain parameters. The obtained model equations are then solved using the fourth-order Runge Kutta numerical method that programmed in the Matlab programming language. In this study, this model used has been simulated on the interaction of nucleons in Deuteron, Triton, Helium-3 isotope, and Helium works well, but it did not work well in Lithium atom.

1. Introduction
For understanding atomic nuclei effectively is needed a describing the many-body system clearly. The dynamics system of the single nucleon is a starting step to understand all many-body methods practically. A good approach of a fundamental system in single-nucleon is usually a key to success to the understanding of any quantum many-body. The motion of non-interacting particles is generally not a real solution of the many-body problem. Therefore, residual interactions of the many-body system to represent these collective dynamics must be present. This method is hoped can demonstrate to be very successful. Nuclear physics currently is now expanding its reach, so the potential that goes far beyond the valley of stability to the spectrum potential of the continuum at nuclear areas is important. To overcome this situation in nuclear reaction physics, so the Woods-Saxon potential was chosen as a model of potential approaches in general [1, 2].

A number estimation of parameterizations of the Woods-Saxon potential has been obtained in previous published. This estimation has been created for different purposes which is relevant to different nuclear mass features. To describe the single-particle binding energies from proton and neutron orbitals around the lighter atom mass regions, so the parameterized characteristic is selected based on a different radius for the proton- and neutron potentials [1, 2].

The Woods-Saxon potential is a reasonable potential for describing nuclear models so that this potential draws much attention in the field of nuclear physics. The Woods-Saxon potential plays an important role in physics microscopically because it can be used to describe interactions between nucleons and the heavy nucleus [3].

The potential interaction between nuclei is generally used potential consisting of Coulomb potential, nuclear potential or the combined of both potentials. These potential formed is usually called
the Woods-Saxon potential. The Coulomb potential is combined with the Woods-Saxon potential is known to play a large role in nuclear physics [4].

The main role in nuclear physics is the depiction of the properties of atomic nuclei, especially in terms of interactions between nucleon pairs. The two basic questions of nuclear physics today are (i) How to understand the structure of atomic nuclei in terms of interactions between nucleons? (ii) How to connect strong nuclear interactions through core physics regulation. These two questions illustrate many sides of nucleon-nucleon interactions [5, 6].

There is no doubt that the Woods-Saxon potential is a fairly successful interaction in nuclear physics. As expected for the nuclear force, this potential tends to be fast towards zero for large radii and its mathematical structure has shown that great force is given to the nucleons near the surface [7].

The nucleon orbit of the parent nucleus which has a common mass center has been simulated by Hogan and Hasbun [8] to provide a pedagogical approach to understanding the structure of atomic nuclei. The potential governing the mean-field is modeled by the Woods-Saxon potential with parameters that have been obtained from experimental results. The Woods-Saxon potential is one of the good potential interactions for describing nuclear interactions, so the simulation of a nucleon-core interaction can be demonstrated in the classroom. Because of the motion of a single nucleon that interacted with the core can be approached with this potential interaction to perform the simulation of the simple motion [8].

The purpose of this study is to simulate nucleon interactions using the concept of center of mass and Woods-Saxon potential calculated by the fourth-order Runge-Kutta method. The simulation of nucleon interactions using the concept of center of mass and Woods-Saxon potential can be described by coupled ordinary differential equations. The solution for coupled ordinary differential equations has been commonly calculated using the Runge-Kutta method. In this study, the fourth-order Runge-Kutta method empirically can be used to calculate the number of steps to achieve the solutions of coupled ordinary differential equations. In the future, these results will be conducted to analyze the numerical solution and to develop an algorithm that determines the number of steps to optimize the application of the fourth-order Runge-Kutta method [9-11]. This implementation of the algorithm will be tested to simulate the nucleon interactions in several types of particles.

2. Method
2.1. The fourth-order Runge-Kutta method
In this paper, the numerical solution of the fourth-order Runge-Kutta method will be applied to the solution of the nucleon interactions equations. Even though the fourth-order Runge-Kutta method is stable, but this study has been identified that there is the stiffness of this numerical method when solving the parameter values of these equations. The parameter values of the numerical solution are used to solve this problem. Therefore, this research proposes an algorithm to determine the best parameter values in the fourth-order Runge-Kutta method to guarantee the reliability of the solutions. This algorithm will be implemented and used in different parameters.

Most of the physics simulations as like in this paper are commonly presented using an ordinary differential equation and solved numerically using the Runge-Kutta algorithm. This algorithm has been known to be a very accurate and well-behaved algorithm for a wide range of problems. To approximate the solution of a first-order ordinary differential equation is given by:

$$\frac{dy(t)}{dt} = y'(xt) = f(y(t), t),$$

with boundary condition:

$$y(t_0) = y_0.$$  

The functions $y(t)$ is a function of the independent variable $t$ which would be approximated and the derivate function $y'(t)$ is represented by $y'(t) = f(y(t), t)$ as the function of $t$ and $y(t)$ itself. At an initial value, $t_0$ is corresponding $y(t)$ with value is $y_0$. The function $f(y(t), t)$ and the data $t_0$, $y_0$ are given. The solution for $y(t)$ function can be represented in the equation:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$
where:

\begin{align*}
    k_1 &= hf(t_n, y_n), \\
    k_2 &= hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}), \\
    k_3 &= hf(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}), \\
    k_4 &= hf(t_n + h, y_n + k_3),
\end{align*}

The parameter value \(k_1\) is the same value as given by the Euler method that is the current point to the next predicted point along with the numerical solution. Parameter value \(k_2\) is calculated by the function \(f(t)\) for \(t_n + h/2\) with the assumption that the \(y(t)\) value will also be changed by \(y_n + k_1/2\). The parameter \(k_2\) is calculated at 2 points that lie halfway between the current point and the predicted next point. Parameter value \(k_3\) is calculated with the same formula as \(k_2\). The function value \(f\) at this point is estimated at the slope of the numerical solution at the midpoint of the approximation of solution interval. The midpoint value of the function \(y\) is always represented by the \(k_2\) value. Parameter value \(k_4\) is calculated by the \(f\) function at the \(x_n + h\) point while the value of function \(y\) is equal to \(y_n + k_3\). This method is commonly aimed to obtain numerical solutions of first-order ordinary differential equations.

To solve the higher-order ordinary differential equations, the equations will be reduced to first-order ordinary differential equations that assumed by the function value \(y'(t_0) = y'_0\):

\[ y''(t) = f(y(t), t), \]

then a new function is introduced in the equation:

\[ z(t) = y'(t). \]

From equation (8) and (9) can be defined in the equation:

\[ z'(t) = f(y(t), t). \]

To solve the final step of the numerical solution can be defined in the equation:

\[ y'(t) = z(t). \]

2.2. The Woods-Saxon potential equation

A Woods-Saxon potential is a simplification form of the nuclear potential produced by all nucleons (protons and neutrons) contained in the nucleus. The Woods-Saxon potential is used to describe the force experienced by each nucleon. The Woods-Saxon potential has the form of the Fermi equation:

\[ f(r, R, a) = \left[ 1 + \exp \left( \frac{r - R}{a} \right) \right]^{-1}, \]

where \(R\) is the radius of the nucleus and shows the thickness of the surface. The magnitude of \(R\) is expressed in the equation:

\[ R = R_0 A^{\frac{1}{3}}, \]
where $A$ is the atomic mass number. The nuclear potential at distance from the center of the nucleus is expressed in the equation:

$$V(r) = -V_0 f(r, R, a),$$

$$V(r) = -\int f(r) \cdot dr.$$  (15)

The parameters $R_0$, $a$, and $V_0$ are potential Woods-Saxon parameters that are important in this research. Parameterization for Woods-Saxon potential has been carried out by various researchers. In this study, the values of Woods-Saxon potential parameters that have been published by Schwierz et. al. in 2013 [1]. The values of parameters based on Schwierz et. al. is $R_0 = 1.26$ fm, $a = 0.662$ fm and $V_0 = 52.06$ MeV [1].

2.3. The concept of reduced mass the many-body system

In this study, the interaction of nucleons in the atomic nucleus was observed using a binary system approach. In the nucleus, nucleons are numbering $A$ (atomic mass) which is the number of protons and neutrons. These nucleons consist of $Z$ numbers (atomic numbers) and neutrons numbering $A-Z$. This study reviews the interaction of a nucleon with mass $m_1$ viewed separately from all other nucleons with mass $m_2$ consisting of $A-1$ nucleons. Illustration of the system reviewed can be seen in Figure 1 below.

The center of the mass system is an important concept in classical mechanics which makes it easier for us and it is more essential to review the motion of a system with more than one particle. The motion of a system consisting of more than one particle is expressed in a center of mass system. The motion of a system consisting of more than one particle is expressed in a center of mass system. Suppose there is a system consisting of particles $p_1$, $p_2$, $p_3$, ..., $p_N$ with the mass of each particle $m_1$, $m_2$, $m_3$, ..., $m_N$ and has the position vectors of each $r_1$, $r_2$, $r_3$, ..., $r_N$. The center of mass of the system is expressed by the $r_{CM}$ vector as follows:

$$r_{CM} = \frac{m_1 r_1 + m_2 r_2 + \cdots + m_N r_N}{m_1 + m_2 + \cdots + m_N} = \frac{\sum_{i=1}^{N} m_i r_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i r_i}{M},$$

where $M$ is the sum of all particle masses.

The first particle with mass $m_1$ has coordinates $r_1$ and the second particle with mass $m_2$ has coordinates $r_2$. These two particles interact by orbiting a center of mass which has $r_{CM}$ coordinates. Force in mass $m_1$ which is caused by mass $m_2$ is denoted by $f_{12}$ and reaction force in mass $m_2$ due to mass $m_1$ is denoted by $f_{21}$. The force on each particle is expressed in the equation below:

$$m_1 \frac{d^2 r_1}{dt^2} = \frac{f_{12}}{r_{12}},$$

$$m_2 \frac{d^2 r_2}{dt^2} = \frac{f_{21}}{r_{21}}.$$  (18)

If equation (17) is multiplied by mass $m_1$ and equation (18) is multiplied by mass $m_2$, then equation (17) is reduced by equation (18), equation final becomes:

$$m_1 m_2 \frac{d^2}{dt^2} (r_2 - r_1) = m_1 \frac{f_{21}}{r_{21}} r_2 - m_2 \frac{f_{12}}{r_{12}} r_1,$$

$$m_1 m_2 \frac{d^2 r}{dt^2} = \frac{f}{r} (m_1 + m_2),$$

$$\mu \frac{d^2 r}{dt^2} = \mu \mathbf{f}.$$  (21)
where the reduced mass is defined by \( \mu = \frac{m_1 m_2}{m_1 + m_2} \).

Using the Cartesian coordinate system, equation (21) is transformed into:

\[
\frac{d^2}{dt^2}(x\mathbf{i} + y\mathbf{j}) = \frac{f}{\mu} \left( \frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j} \right),
\]

then the equation (22) is separated for each coordinate as follow:

\[
\frac{d^2 x}{dt^2} = \frac{f}{\mu \sqrt{x^2 + y^2}} x,
\]

\[
\frac{d^2 y}{dt^2} = \frac{f}{\mu \sqrt{x^2 + y^2}} y.
\]

Based on equation (15), a Woods-Saxon nuclear force equation is obtained:

\[
f(r) = \frac{dV(r)}{dr},
\]

\[
f(r) = -\frac{V_0}{a} \left[ 1 + \exp \left( \frac{r - R}{a} \right) \right]^{-2} \exp \left( \frac{r - R}{a} \right).
\]

Equation (26) is substituted to equations (23) and (24) so that it is obtained:

\[
\frac{d^2 x}{dt^2} = -\frac{V_0}{\mu a} \left[ 1 + \exp \left( \frac{\sqrt{x^2 + y^2} - R}{a} \right) \right]^{-2} \exp \left( \frac{\sqrt{x^2 + y^2} - R}{a} \right) \frac{x}{\sqrt{x^2 + y^2}},
\]

\[
\frac{d^2 y}{dt^2} = -\frac{V_0}{\mu a} \left[ 1 + \exp \left( \frac{\sqrt{x^2 + y^2} - R}{a} \right) \right]^{-2} \exp \left( \frac{\sqrt{x^2 + y^2} - R}{a} \right) \frac{y}{\sqrt{x^2 + y^2}}.
\]

2.4. Numerical method of the many-body system
This research uses the fourth-order Runge-Kutta numerical method. Equation (27) and (28) will be calculated using the fourth-order Runge-Kutta method and then written in the form of a Matlab programming language. The example used to find the solution of equation (27) is as follows:
The fourth-order Runge-Kutta numerical integral method for \( \frac{dx}{dt} \) is:

\[
\frac{dx}{dt} = v_x; \\
\frac{dv_x}{dt} = \frac{d^2x}{dt^2}.
\] (29)

The fourth-order Runge-Kutta numerical integral method for \( \frac{dv_x}{dt} \) is:

\[
v_x(t_i + \Delta t) = v_x(t_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4);
\]

\[
k_1 = \Delta t \ f(t_i, v_x(t_i)) \\
k_2 = \Delta t \ f\left(t_i + \frac{\Delta t}{2}, v_x(t_i) + \frac{k_1}{2}\right) \\
k_3 = \Delta t \ f\left(t_i + \frac{\Delta t}{2}, v_x(t_i) + \frac{k_2}{2}\right) \\
k_4 = \Delta t \ f(t_i + \Delta t, v_x(t_i) + k_3).
\]

The example used to find the solution of equation (28) is as follows:

\[
\frac{dy}{dt} = v_y; \\
\frac{dv_y}{dt} = \frac{d^2y}{dt^2}.
\] (31)

The fourth-order Runge-Kutta numerical integral method for \( \frac{dv_y}{dt} \) is:

\[
v_y(t_i + \Delta t) = v_y(t_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
k_1 = \Delta t \ f(t_i, y(t_i)) \\
k_2 = \Delta t \ f\left(t_i + \frac{\Delta t}{2}, y(t_i) + \frac{k_1}{2}\right) \\
k_3 = \Delta t \ f\left(t_i + \frac{\Delta t}{2}, y(t_i) + \frac{k_2}{2}\right) \\
k_4 = \Delta t \ f(t_i + \Delta t, y(t_i) + k_3).
\]

3. Results and Discussion

In this study, nucleon interaction simulation was conducted on Deuteron (\(^2\text{H}\)), Triton (\(^3\text{H}\)), Helium-3 isotope (\(^3\text{He}\)), Helium (\(^4\text{He}\)) or alpha particles, and Lithium atom (\(^7\text{Li}\)). The characteristics of each particle are shown in Table 1.

| Particle type         | Notation | Mass Number (sma) | Atomic Mass (sma) |
|-----------------------|----------|-------------------|-------------------|
| Deuteron              | \(^2\text{H}\) or D | 2                  | 2.014             |
| Triton                | \(^3\text{H}\) or T | 3                  | 3.023             |
| Helium-3 isotope      | \(^3\text{He}\) | 3                  | 3.022             |
| Helium                | \(^4\text{He}\) or \(^4\alpha\) | 4                  | 4.002             |
| Lithium               | \(^7\text{Li}\) | 7                  | 6.939             |
The numerical solutions were performed as many as 2000 iterations using the Matlab programming language from 0 to 51-time units with a time interval for each iteration of 0.0255. This study uses the concept of the center of mass of a binary system (two objects) to simulate each particle according to Figure 1.

Interactions between nucleons that occur in each type of particle are modeled in a binary state where two particles interact. Particles are particles consisting of only one nucleon. This one particle can be a proton or a neutron. Particle two is a particle that has a mass greater than a particle one. Particle two has as many nucleons as atomic mass number (A) minus one. Based on the interaction of the first particle and the second particle is taken a point that becomes the center of mass of the interaction of the two particles. This center of mass is at point (0,0). Furthermore, from the interaction of the two mass particles taken a reduced mass whose magnitude depends on the first particle mass and the mass of the second particle. Each of these masses interacts with a center of mass (0,0).

The path taken by the first particle of small mass is expressed by a red line. The path traversed by the second particle with a greater mass is expressed by a black line. Furthermore, the black line is the path traversed by the reduced mass of the two particles. Based on the research conducted the interaction model of each particle is obtained to the center of mass of the system.

Deuteron \((^2\text{H})\) is a particle consisting of one proton and one neutron [12]. In the simulation model, particle \(m_1\) is one proton and \(m_2\) particle is one neutron. The distance between the \(m_1\) (red) and \(m_2\) (blue) particles to the center of mass \((0,0)\) are not much different because the difference in mass between the \(m_1\) and \(m_2\) particles is relatively small. The black path is the path of the reduced mass of the two particles towards the center of mass. The reduced mass value of the two particles is smaller than \(m_1\) and \(m_2\) so that the path distance is greater.
Triton ($^3\text{H}$) is a hydrogen isotope that has two neutrons and one proton [13]. In the simulation model, particle $m_1$ is one neutron and $m_2$ particles consist of one neutron and one proton. It is seen that the distance of the particle trajectory $m_2$ (blue) to the center of mass (0.0) is smaller than the path $m_1$ (blue) and the reduced mass trajectory (black). This is due to the model $m_2$ having a greater mass followed by $m_1$ and its reduced mass.
The Helium-3 isotope ($^3\text{He}$) is an isotope of helium which has one neutron and two protons [14]. In the simulation model above $m_1$ particle is one proton and $m_2$ particle consists of one neutron and one proton. It is seen that the distance of the particle trajectory $m_2$ (blue) to the center of mass (0.0) is smaller than that of $m_1$ (red) and the reduced mass (black). The simulation model for Triton and Helium-3 isotope is not much different. This is because the difference in mass between the two is indeed not too large.

Helium ($^2\text{He}$) is an atom that has two neutrons and two protons [15]. In the simulation model above particle $m_1$ is one proton and $m_2$ particle consists of two neutrons and one proton. It is seen that the distance of the particle trajectory $m_2$ (blue) to the center of mass (0.0) is smaller than that of $m_1$ (red) and the reduced mass (black).

Lithium ($^3\text{Li}$) is an atom that has four neutrons and three protons [16]. In the simulation model, particle $m_1$ is one proton and the $m_2$ particle consists of four neutrons and two protons. In the lithium simulation, the results obtained are distorted and very different from previous simulations. This distorted result might indicate that the simulation model used in the study cannot be applied to atoms with more nucleons than Helium. This might be due to the shortcomings of the binary mass center system used in this study. Further development of simulation models is needed to be able to simulate nucleon interactions that occur in particles that are more complex than Helium.

Figure 5. The positions $x$ and $y$ for $m_1$, $m_2$, and the Reduced Mass on the Helium.
4. Conclusion

The concept of reduced mass and Woods-Saxon potential with the fourth-order Runge-Kutta method in this research model can be used to model interactions that occur between nucleons (protons and neutrons). The model used in this study can be used for particles that have a small number of nucleons and a relatively small atomic mass of around 2-4 atomic mass units (sma), such as Deuteron, Triton, Helium-3 isotope, and Helium. Based on the simulation model of this research, it is found that when two-particle systems interact and their interactions form a certain center of mass, the smaller mass particles will have a greater range of motion than the larger mass particles. This is following the concept of the center of mass. The model in this study does not work for Lithium atom. This might be because the Lithium atom has more nucleons than Helium, so interactions between nucleons are more complex.

The simulation model in this study cannot be used for particles that have more nucleons and greater atomic mass than Helium. Further development is needed so that the concept of Woods-Saxon center of mass and potential can be used to model nucleon interactions in larger particles. The development that can be done for example is to modify the resultant force equation or further development can also be done in the form of the use of the concept of the center of mass with a minimum number of three particles. The addition of particle objects to the concept of the center of mass will certainly be useful to be able to model other particles that have more nucleon constituents compared to Helium.

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