Anisotropic turbulence studies of liquid metal MHD flows using numerical simulations

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Abstract. Liquid metal flow at low magnetic Reynolds number is simulated. Direct numerical simulation using pseudo-spectral method in periodic box geometry has been used for this purpose. The statistical distribution of fluctuation energy in different Fourier modes of the velocity fields is calculated. Unlike the simple fluid, spectral distribution of energy in this situation does not follow Kolmogorov scaling law ($E(k) \sim k^{-5/3}$). Rather, our preliminary investigations suggest that it is steeper and follows $E(k) \sim k^{-3}$ scaling law for very strong magnetic field.

1. Introduction

Study of turbulent conducting fluid flow subject to an applied magnetic field is important in many practical applications [1, 2]. Most importantly, liquid metal is used as a coolant in nuclear reactors. In the proposed fusion reactor, ITER, cooling system will be based on the liquid metal flowing under the intense magnetic field [2]. The magnetic field influences the liquid metal flow via the Lorentz force [1, 3, 4, 5].

A relevant non-dimensional parameter of such flow is magnetic Reynolds number ($R_m$) [4, 6]. For low $R_m$, "quasi-static approximation" can be used to simplify the governing equations [7, 8]. Under this approximation apart from viscous dissipation, an extra damping term appears in the Navier-stokes (NS) equation (see Eq. 1). Damping of the flow depends on the strength of the applied magnetic field. Strength of the magnetic field against the non-linear term is characterized by a non-dimensional parameter known as magnetic interaction parameter $N$. In the absence of damping term, sufficiently far away from the solid boundaries, the turbulence is homogeneous and isotropic. However, the damping term due to magnetic field is direction dependent. This causes the energy transfer between the different length scales to be anisotropic [9].
Earlier researchers have studied the energy transfer between the different length scale in isotropic and homogeneous turbulence. For fluid turbulence Kolmogorov's theory is the pioneering work [10]. It predicts the distribution of energy in various length scales in the so called "inertial range". This has been followed by many experimental, analytical and computational studies in this area (see ref. [11] and references therein). Energy transfer in isotropic and homogeneous magnetohydrodynamic turbulence has been studied in great detail in recent past [12-16]. Our aim is to study the anisotropic energy transfer in liquid metal flow using the direct numerical simulation.

In this communication, we present the preliminary results of the direct numerical simulation of the low $R_m$ liquid metal flow for different values of $N$. The NS equation with magnetic dissipation term is solved using pseudo-spectral method [17] to study the effects of the mean magnetic field on the spectral distribution of the energy. It is observed that for non-zero $N$ the spectrum is steeper than the Kolmogorov spectra [3]. For $N = 9$, the energy spectrum $E(k) \sim k^{-3}.$

The paper is organized as follows: we write the basic equations and discuss the quasi-static approximation in section 2. This is followed by the simulation results in section 3.. Finally, conclusions are given in section 4.

2. Basic equations and quasi-static approximation

In this section, we shall write the equations governing the liquid metal flow in the presence of an external magnetic field. For the sake of completeness, we shall briefly discuss the quasi static approximation [8], which has been used in our simulation study. It should be noted that hereafter bold letters denote the vector quantities.

The relevant equations for an incompressible flow of the liquid metal in the presence of magnetic field are:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla p + \nu \nabla^2 \mathbf{u} + \frac{1}{\mu} [\mathbf{B}_0 + \mathbf{b}] \cdot \nabla \mathbf{b},$$  \hspace{1cm} (1)

$$\frac{\partial \mathbf{b}}{\partial t} = - (\mathbf{u} \cdot \nabla) \mathbf{b} + (\mathbf{B}_0 \cdot \nabla) \mathbf{u} + (\mathbf{b} \cdot \nabla) \mathbf{u} + \eta \nabla^2 \mathbf{b}.\hspace{1cm} (2)$$

$$\nabla \cdot \mathbf{u} = 0 \hspace{1cm} \nabla \cdot \mathbf{b} = 0$$  \hspace{1cm} (3)

In above equations $\mathbf{u}$ is the velocity of the fluid, $p$ is the sum of kinematic and the magnetic pressures and $\nu$ is the kinematic viscosity. $\eta = 1/\sigma \mu$, where $\sigma$ is electrical conductivity and $\mu$ is the magnetic permeability of the liquid metal. Here, magnetic field has been separated into external part $\mathbf{B}_0$ and the fluctuating part $\mathbf{b}$. The external part has been assumed to be uniform in space and constant in time. Note that density of the liquid metal in this case has been assumed to be constant and taken to be unity. Equation (1) is well-known Navier-Stoke's equation and Eq. (2) is referred to as the induction equation.

If we scale the velocity with root mean square velocity $\nu = \sqrt{\langle u_i u_i \rangle / 3}$, magnetic field with $b = \sqrt{\langle b_i b_i \rangle / 3}$ and length by the integral length scale $L$ (see ref. [8] for definition of integral length scale). It can be easily seen that

$$(\mathbf{u} \cdot \nabla) \mathbf{b} \approx \nu b / L, \hspace{0.5cm} (\mathbf{b} \cdot \nabla) \mathbf{u} \approx \nu b / L, \hspace{0.5cm} \eta \nabla^2 \mathbf{b} \approx \eta b / L^2.$$  \hspace{1cm} (4)

*Magnetic Reynolds number* $R_m$ which is an important parameter in MHD is given by

$$R_m = \frac{\nu L}{\eta} = \frac{\nu b / L}{\eta b / L^2}$$  \hspace{1cm} (5)
When $R_m \ll 1$, the first and the third terms on the right hand side of Eq. 2 can be neglected. The induction equation can be rewritten as

$$\frac{\partial \mathbf{b}}{\partial t} = (\mathbf{B}_0 \cdot \nabla) \mathbf{u} + \eta \nabla^2 \mathbf{b}. \quad (6)$$

It may be noted that the eddy turn-over time is $t_{\text{eddy}} = L/\nu$, while the magnetic diffusion time scale is $t_{\text{diff}} = L^2/\eta$. The magnetic Reynolds number $R_m = t_{\text{diff}}/t_{\text{eddy}}$. So, for $R_m \ll 1$, $t_{\text{diff}} \ll t_{\text{eddy}}$, i.e., magnetic diffusion process is much faster than the so called large-eddy turnover time. Thus, on the time scale of $t_{\text{eddy}}$, time dependence of fluctuating magnetic field can be ignored. As a result it may be assumed that $\partial \mathbf{b}/\partial t \approx 0$. This is known as quasi-static approximation. Under quasi-static approximation, Eq. 6 may be rewritten as,

$$(\mathbf{B}_0 \cdot \nabla) \mathbf{u} = -\eta \nabla^2 \mathbf{b}. \quad (7)$$

the solution of which in the Fourier space yields,

$$\hat{b}(k,t) = i \frac{(\mathbf{B}_0 \cdot \mathbf{k})^2}{\eta k^2} \hat{u}(k,t), \quad (8)$$

where

$$\hat{b}(x,t) = \frac{1}{2\pi^3} \int \hat{b}(k,t) e^{ik \cdot x} dk,$$

$$\hat{u}(x,t) = \frac{1}{2\pi^3} \int \hat{u}(k,t) e^{ik \cdot x} dk. \quad (9)$$

Thus, under the quasi-static approximation, the magnetic field can be expressed completely in terms of the velocity field. For incompressible flow, pressure can also be written in terms of velocity field [11]. Now, taking the spatial Fourier transform and using Eq. 8, the Navier stoke's equation (Eq. 1) may be rewritten as

$$\frac{\partial \hat{u}}{\partial t} + i\mathbf{k} \cdot \sum_{\mathbf{k} \cdot \mathbf{p} \cdot \mathbf{q}} \hat{u}(\mathbf{p}) \hat{u}(\mathbf{q}) = -i\mathbf{k} \hat{\rho} - \nu k^2 \hat{u} - \frac{1}{\mu \eta} \frac{(\mathbf{B}_0 \cdot \mathbf{k})^2}{k^2} \hat{u}. \quad (10)$$

Note that the last term in above equation is a damping term, but unlike the viscous damping it depends on the angle between the external magnetic field $\mathbf{B}_0$ and $\mathbf{k}$. Hence, the velocity component parallel to the magnetic field would be attenuated more compared to the components perpendicular to it, i.e., the magnetic field causes anisotropy in the liquid metal turbulence. An opposite tendency of restoring the isotropy comes from non-linear term $(\mathbf{u} \cdot \nabla) \mathbf{u}$. There is thus an interplay between the nonlinear term and the magnetic dissipation term. The ratio of magnetic field and the non-linear term is called the interaction parameter or Stuart number $N$, and it is a measure of the strength of the external magnetic field. The interaction parameter $N$ is given by
FIG. 1: Spectral energy distribution for different values of $N$. For $N = 0$, the slope of the spectrum is same as that predicted by the Kolmogorov theory. As $N$ increases, the slope also increases. In the inset, we show the spectra for $N = 9$ together with the curve $E(k) = Ck^{-3}$, where $C = 1000$.

$$N = \frac{\sigma B_0^2 L}{\nu \rho},$$

where, $\sigma = 1/\mu \eta$. Note that in this work $\rho$ has been assumed to be unity. The scaled form of magnetic dissipation term in Eq. 10 is $N \cos^2 \theta \hat{\mathbf{u}}(\mathbf{k})$ where $\theta$ is the angle between $\mathbf{B}_0$ and $\mathbf{k}$.

We have performed the numerical study of the anisotropic turbulence in liquid metal flow for different value of $N$ by solving equation 10. In the following section simulation results are presented.

3. Numerical simulation

We study the turbulence in liquid metal flow using pseudo-spectral code [17, 18]. Our code is fully dealiased using 2/3 rule. The geometry of the simulation is a periodic box of side length $2\pi$. We used $512^3$ grid for the results presented in this paper. It should be noted that the results given in this section are for the decaying turbulence. The fluid is given initial kinetic energy, and we let it evolve. The turbulence in the system becomes fully developed after around one eddy turn-over time. We study the energy spectrum of the fluid after it has reached the steady-state. The total energy decays, yet there is enough energy in the fluid to yield power law for the energy spectrum.

The initial total energy was taken to be 220.25 and the kinematic viscosity was taken to be $\nu = 0.003$. Reynolds number ($R_e = uL/\nu$) estimated using length of the periodic box ($L = 2\pi$) as the characteristic length and $\text{rms}$ velocity at the start of simulation as the characteristic velocity was
of the order of $4 \times 10^4$. Fourth-order Runge-Kutta scheme is used to time-advance the solution. The time step $dt$ was computed automatically using the Courant-Friedrichs-Lewy criterion and it was of the order of $1 \times 10^{-3}$.

In Fig. 2, we show the spectral energy distribution for four values of the interaction parameter $N$. The spectral energy density is defined as $E(k) = \sum_{i=1}^{3} E_i(k)$, where

$$E_i(k) = k^2 \int_{0}^{2\pi} \left| \hat{u}_i(k, \psi, \phi) \right|^2 d\phi \sin \theta d\theta,$$

where $k$ is the wave number. It is clear from the figure that for $N = 0$, the slope of the spectrum in the wave number range $k = 8$ to 40 is same as that given by the Kolmogorov theory for isotropic and homogeneous turbulence. But for higher $N$, the slope is steeper. In the inset of the same figure, it can be seen that for $N = 9$, the spectra $E(k) \sim k^{-3}$. This has been observed by earlier researchers as well [19].

4. Conclusions
The energy spectra for a turbulent liquid metal flow in the presence of external magnetic field has been calculated. We performed the direct numerical simulation using pseudo-spectral method in a periodic box geometry. Our preliminary investigation shows that the spectrum is steeper than the Kolmogorov spectrum.

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