Leading twist distribution amplitudes of $P$-wave nonrelativistic mesons.

V.V. Braguta,1 A.K. Likhoded,1 and A.V. Luchinsky1

1Institute for High Energy Physics, Protvino, Russia

This paper is devoted to the study of the leading twist distribution amplitudes of $P$-wave nonrelativistic mesons. It is shown that at the leading order approximation in relative velocity of quark-antiquark pair inside the mesons these distribution amplitudes can be expressed through one universal function. As an example, the distribution amplitudes of $P$-wave charmonia mesons are considered. Within QCD sum rules the model for the universal function of $P$-wave charmonia mesons is built. In addition, it is found the relations between the moments of the universal function and the nonrelativistic QCD matrix elements that control relativistic corrections to any amplitude involving $P$-wave charmonia. Our calculation shows that characteristic size of these corrections is of order of $\sim 30\%$.

PACS numbers: 12.38.-t, 12.38.Bx, 13.66.Bc, 13.25.Gv

I. INTRODUCTION

Hard exclusive processes are very interesting both from theoretical and experimental points of view. Commonly, theoretical approach to the description of such processes is based on the factorization theorem [1, 2]. Within this theorem the amplitude of hard exclusive process can be separated into two parts. The first part is partons production at very small distances, which can be treated within perturbative QCD. The second part is hadronization of the partons at larger distances. This part contains information about nonperturbative dynamic of strong interaction. For hard exclusive processes it can be parameterized by process independent distribution amplitudes (DA), which can be considered as hadrons’ wave functions at light like separation between the partons in the hadron. It should stressed that DAs are very important for the calculation of the amplitude of any hard exclusive process.

Recently, the leading twist DAs of $S$-wave nonrelativistic mesons have become the object of intensive study [3, 4, 5, 6, 7, 8, 9, 10]. Knowledge about these DAs allowed one to build some models for $S$-wave charmonia DAs, that can be used in practical calculations. In this paper general properties of the leading twist DAs of $P$-wave nonrelativistic mesons will be studied. The results of this study will be used to build the model for the DAs of $P$-wave charmonia mesons, that can be used in calculations.

This paper is organized as follows. In the next section all definitions of the DAs of $P$-wave mesons will be given. These DAs will be studied in section III at the leading order approximation in relative velocity of quark-antiquark pair inside $P$-wave meson. In section IV QCD sum rules will be applied to the calculation of the moments of the $P$-wave charmonia DAs. Using the results of this study the model of the $P$-wave charmonia DAs will be build in section V. In the last section the results of this paper will be summarized.

II. DEFINITIONS OF THE DISTRIBUTION AMPLITUDES.

In this section the definitions of the leading twist distribution amplitudes (DA) of $P$-wave nonrelativistic mesons will be given. In the conventional quark model nonrelativistic mesons are quark-antiquark($Q\bar{Q}$) bound states. In these mesons the quark-antiquark pair can be in the spin singlet or spin triplet states. Since orbital momentum of the quark-antiquark pair is unity one can conclude that there are four $P$-wave mesons: $\chi_0(1^3P_0), \chi_1(1^3P_1), \chi_2(1^3P_2), h(1^1P_1)$. The leading twist DAs of these mesons can be defined as follows.

for the $\chi_0$ meson:

$$\langle \chi_0(p)|\bar{Q}(z)\gamma_{\mu}[z,-z]Q(-z)|0\rangle = f_{\chi_0P_\mu} \int_{-1}^{1} d\xi e^{i(pz)\xi} \phi_{\chi_0}(\xi,\mu),$$

(1)}
for the $\chi_1$ meson:

$$\langle \chi_1(p, \epsilon_\lambda=0) | \vec{Q}(z) | \gamma_{\mu 5} [z, -z] Q(-z) | 0 \rangle = f'_{\chi_1} p_\mu \int_{-1}^{1} d\xi \, e^{i(pz)\xi} \phi'_{\chi_1}(\xi, \mu),$$

$$\langle \chi_1(p, \epsilon_\lambda=\pm 1) | \vec{Q}(z) | \sigma_{\mu 5} [z, -z] Q(-z) | 0 \rangle = f_{\chi_1} \epsilon_{\mu \nu \alpha \beta} \epsilon^{\alpha \beta} \int_{-1}^{1} d\xi \, e^{i(pz)\xi} \phi_{\chi_1}(\xi, \mu),$$

(2)

for the $h$ meson:

$$\langle h(p, \epsilon_\lambda=0) | \vec{Q}(z) | \gamma_{\mu 5} [z, -z] Q(-z) | 0 \rangle = f_h p_\mu \int_{-1}^{1} d\xi \, e^{i(pz)\xi} \phi_h(\xi, \mu),$$

$$\langle h(p, \epsilon_\lambda=\pm 1) | \vec{Q}(z) | \sigma_{\mu 5} [z, -z] Q(-z) | 0 \rangle = f'_h \epsilon_{\mu \nu \alpha \beta} \epsilon^{\alpha \beta} \int_{-1}^{1} d\xi \, e^{i(pz)\xi} \phi'_h(\xi, \mu),$$

(3)

for the $\chi_2$ meson:

$$\langle \chi_2(p, \epsilon_\lambda=0) | \vec{Q}(z) | \gamma_{\mu 5} [z, -z] Q(-z) | 0 \rangle = f_{\chi_2} p_\mu \int_{-1}^{1} d\xi \, e^{i(pz)\xi} \phi_{\chi_2}(\xi, \mu),$$

$$\langle \chi_2(p, \epsilon_\lambda=\pm 1) | \vec{Q}(z) | \sigma_{\mu 5} [z, -z] Q(-z) | 0 \rangle = f_{\chi_2} M_{\chi_2}(\rho_\mu p_\nu - \rho_\nu p_\mu) \int_{-1}^{1} d\xi \, e^{i(pz)\xi} \phi_{\chi_2}(\xi, \mu), \quad \rho_\mu = \frac{\epsilon_{\mu \nu} z^\nu}{p_z},$$

(4)

where the following designations are used: $x_1, x_2$ are the fractions of momentum of meson carried by quark and antiquark correspondingly, $\xi = x_1 - x_2$, $p, \epsilon$ are the momentum and polarizations of $P$-wave mesons. For the mesons $\chi_1, h$ the polarization $\epsilon$ is described by the four vector $\epsilon_\mu$, for the $\chi_2$ meson the polarization $\epsilon$ is described by the tensor $\epsilon_{\mu \nu}$. The factor $[z, -z]$, that makes matrix elements (1-3) gauge invariant, is defined as

$$[z, -z] = P \exp[ig \int_{-z}^{z} dx^\mu A_\mu(x)].$$

(5)

In applications it is useful to rewrite the four-vector $\rho_\mu$ in the following way. Evidently, one can write the polarization of the $\chi_2$ meson in terms of the polarization of two vector mesons. Thus for the transverse polarization of the meson $\chi_2$ one has $\epsilon_{\mu \nu}^{\chi_2} = (\epsilon_{\mu \nu}^{\chi_1} + \epsilon_{\mu \nu}^{\chi_2})/\sqrt{2}$ ($\epsilon_{\mu \nu}^{\chi_2} = 1$). If we further contract the polarization tensor $\epsilon_{\mu \nu}$ with lightlike four-vector $z$, to the leading twist accuracy we will get $\epsilon_{\mu \nu} z_\nu = \epsilon_{\lambda \mu}^{\chi_2}(p_z)/(\sqrt{2} M_{\chi_2})$ or $\rho_\mu = \epsilon_{\lambda \mu}^{\chi_2}/(\sqrt{2} M_{\chi_2})$. This form of the vector $\rho$ can be used in the calculation with the leading twist accuracy. It should be noted that the states of the $\chi_2$ meson with the polarizations $\lambda = \pm 2$ give contribution only to higher twist DAs. Since, this paper is devoted to the study of the leading twist DAs we don’t consider these states.

The functions without primes $\phi_{\chi_1}(\xi), \phi_{h}(\xi), \phi_{\chi_2}(\xi), \phi_{h}(\xi)$ are $\xi$ odd and they are normalized as

$$\int_{-1}^{1} d\xi \, \phi(\xi) = 1.$$  

(6)

The functions with primes $\phi_{\chi_1}'(\xi), \phi_{h}'(\xi)$ are $\xi$ even and they are normalized as

$$\int_{-1}^{1} d\xi \, \phi'(\xi) = 1.$$  

(7)

In this paper all DAs will be parameterized by their moments

$$\langle \xi^n \rangle = \int_{-1}^{1} d\xi \, \xi^n \phi(\xi).$$  

(8)

Evidently, for the DAs with primes all odd moments are zero. For the DAs without primes all even moments are zero. To separate the moments of different DAs of the $\chi_2$ meson, below we are going to use the following designations: $\langle \xi^n \rangle$ for the moments of the function $\phi_{\chi_2}(\xi)$, $\langle \xi^3 \rangle$ for the moments of the function $\phi_{h}(\xi)$.

The distribution amplitudes $\phi(\xi)$ and the constants $f$ that parameterizes corresponding currents (1-4) are scale dependent objects. For applications it is useful to write how they depend on the scale $z$. To do this we expand DA in the series

$$\phi(\xi, \mu) = \frac{3}{4} (1 - \xi^2) \sum_{n=0}^{\infty} a_n(\mu) C_n^{3/2}(\xi),$$

(9)
where $C_{3/2}^3(\xi)$ are Gegenbauer polynomials. At the leading logarithmic accuracy the coefficients $a_n$ are renormalized multiplicatively

$$a_n(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma_n/b_0} a_n^{\bar{\mu}, T}(\mu_0),$$

where $\gamma_n$ are the anomalous dimensions. For the current $\bar{Q}\sigma_{\mu\nu}[z,-\bar{z}]Q$ the anomalous dimensions are

$$\gamma_n = C_f \left( 1 + 4 \sum_{j=2}^{n+1} \frac{1}{j} \right), \quad b_0 = 11 - \frac{2}{3} n_H, \quad C_f = \frac{4}{3},$$

for the other currents

$$\gamma_n = C_f \left( 1 - \frac{2}{(n+1)(n+2)} + 4 \sum_{j=2}^{n+1} \frac{1}{j} \right).$$

It is clear that the DAs without primes contain only n-odd terms in series [9]. DAs with primes contain n-even terms in the series.

The constants defined in equations [11]-[14] are multiplicatively renormalizable. Using formulas [11]-[12] one can determine the evolution of these constants

$$f(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma/b_0} f(\mu_0).$$

For the constants $f_{\chi_0}, f_h, f_{\chi_2}$ the anomalous dimensions $\gamma$ is equal to $8/3C_f$, for the constants $f_{\bar{\chi}_1}, \tilde{f}_{\chi_2}$ $\gamma = 3C_f$, for the constant $f_h^\prime$ $\gamma = C_f$, for the constant $f_{\chi_1}^\prime$ $\gamma = 0$. It should be noted that if the anomalous dimensions $\gamma$ of the constants $f$ are factored from sum [9], the anomalous dimensions of the remaining terms equal to the difference $\gamma_n - \gamma$.

### III. THE DISTRIBUTION AMPLITUDES AT THE LEADING ORDER APPROXIMATION IN RELATIVE VELOCITY.

In this section the DAs under study will be considered at the leading order approximation in relative velocity of quark-antiquark pairs inside the mesons. First let us consider the DA of the $\chi_0$ meson. The moments of this DA can be represented as follows

$$f_{\chi_0}(pz)^{n+1} < \xi^n >_{\chi_0} = \langle \chi_0 | \bar{Q} \hat{z} (-i\bar{z}D)^n Q | 0 \rangle.$$  

(14)

To get the expressions for the constant $f_{\bar{\chi}_0}$ and the moment $< \xi^n >_{\chi_0}$ one needs to calculate the matrix element in the right hand side. At the leading order approximation this calculation can be done using projector [11, 12]

$$\hat{Q}(\hat{p})Q(p) \rightarrow \int dq \frac{\varphi(-q^2)}{\sqrt{3m_Q}} \frac{1}{4\sqrt{2E(E+m_Q)}} (\hat{p} - m_Q) \Gamma(\hat{P} + 2E)(\hat{p} + m_Q),$$

(15)

where $P, q$ are the total and relative momentum of the $QQ$ pair, $m_Q$ is the mass of the quark $Q$, $p = P/2 + q, \hat{p} = P/2 - q$, $E^2 = P^2/4 = m_Q^2 - q^2$. The matrix $\Gamma = \gamma_5, \hat{e}_S$ for the spin singlet and spin triplet quark-antiquark pair correspondingly, where $e_S$ is the spin polarization of this pair. The scalar products $P \cdot e_S = 0, P \cdot q = 0$. In the center mass frame the $dq$ is reduced to the $d^3q/(2\pi)^3$ and the function $\varphi(-q^2)$ is reduced to the usual nonrelativistic wave function $\phi(q^2)$. For the $\chi_0$ meson the wave function $\varphi(q^2)$ can be written in the form

$$\varphi(q^2) = \frac{e_S \cdot q}{\sqrt{3}} \psi(q).$$

(16)

At the leading order approximation in relative velocity the function $\psi(q)$ is universal function for all $P$-wave mesons. It is normalized as

$$\int \frac{d^3q}{(2\pi)^3} q^i q^j |\psi(q)|^2 = \delta^{ij}.$$  

(17)
With this normalization of the function $\psi(q)$, the function $\varphi(q)$ is normalized as
\[
\int \frac{d^3q}{(2\pi)^3} |\varphi(q^2)|^2 = 1. \tag{18}
\]
The same normalization condition will be used for the wave functions of all mesons under consideration. Using equations (14), (15) and (16) one gets the result
\[
f_{\chi_0} < \xi^{n+1} >_{\chi_0} = -2^{n+1} \frac{A_{n+2}}{m_Q M_{\chi_0}^{n+1} n + 3}, \tag{19}
\]
where $M_{\chi_0}$ is the mass of the $\chi_0$ meson, $A_n$ equals to
\[
A_n = \int \frac{d^3q}{(2\pi)^3} |q|^n \psi(q). \tag{20}
\]
It should be noted that because of the Coulombic part of the nonrelativistic QCD potential the right hand side of equation (19) is ultraviolet divergent [13, 14]. The moments of the DA in the left hand side are QCD operators (see equation (14)). In full QCD the Coulombic part of the nonrelativistic potential corresponds to the rescattering of the quark-antiquark pair of the QCD operators which is also ultraviolet divergent [2]. To control the divergences in the right and left hand sides of equation (19) it is assumed that both sides are regularized within dimensional regularization.

It is interesting to note that relation (19) is closely connected with Brodsky-Huang-Lepage (BHL) [15] procedure. This fact can be seen as follows. Let us rewrite this relation as follows:
\[
\int d\xi \xi^{n+1} \varphi_{\chi_0}(\xi) \sim \frac{1}{n + 3} \frac{A_{n+2}}{m_Q^{n+1}} \sim \frac{1}{n + 3} \int q^2 dq \frac{q^{n+2}}{m_Q^{n+1}} \psi(q) \sim \int d^3q \left( \frac{q_z}{m_Q} \right)^{n+1} q_z \psi(q). \tag{21}
\]
Note that $q_z \psi(q)$ is the $L_z = 0$ component of the wave function $\varphi(q^2)$ which is the only component important for the leading twist DA. So, the last relation can be written as follows
\[
\int d\xi \xi^{n+1} \varphi_{\chi_0}(\xi) \sim \int dq \left( \frac{q_z}{m_Q} \right)^{n+1} \int d^2q_{\bot} \varphi_{L_z=0}(q^2). \tag{22}
\]
Further, we change the variables in the right side of this equation
\[
q_{\bot} \rightarrow q_{\bot}, \quad q_z \rightarrow \xi M_0, \quad M_0^2 = \frac{M_Q^2 + q_{\bot}^2}{1 - \xi^2}. \tag{23}
\]
Note also that $\xi \ll 1, q_{\bot} \ll M_Q$ and at the leading order approximation in relative velocity of the quark-antiquark pair in the meson relation (22) can be written as follows
\[
\int d\xi \xi^{n+1} \varphi_{\chi_0}(\xi) \sim \int d\xi \xi^{n+1} \times \int d^2q_{\bot} \varphi_{L_z=0} \left( \frac{M_Q^2 \xi^2 + q_{\bot}^2}{1 - \xi^2} \right). \tag{24}
\]
So, at this level of accuracy the DA is just
\[
\varphi_{\chi_0}(\xi) \sim \int d^2q_{\bot} \varphi_{L_z=0} \left( \frac{M_Q^2 \xi^2 + q_{\bot}^2}{1 - \xi^2} \right), \tag{25}
\]
what coincides with BHL procedure.

It is not difficult to get the relations for the other DAs and mesons
\[
\chi_1 \text{ meson: } \phi(q^2) = \frac{\epsilon_{ijk} e_i^* e_k^* q^j}{\sqrt{2}} \psi(q),
\]
\[
h \text{ meson: } \phi(q^2) = \frac{e \cdot q}{\sqrt{2}} \psi(q),
\]
\[
\chi_2 \text{ meson: } \phi(q^2) = e_{ij} e_s \hat{q}^i q^j \psi(q). \tag{26}
\]
The same is true for all DAs with primes, which are equal to one function which will be designated below as $\Psi(\xi)$. In addition, one can relate the moments of the $\Phi(\xi)$ function to the moments of the $\Psi(\xi)$ as follows:

$\Phi(\xi) = \phi_{\lambda_\chi}(\xi) = \phi_{\lambda_\chi 1}(\xi) = \phi_{\lambda h}(\xi) = \phi_{\lambda_\chi 2}(\xi) = \tilde{\phi}_{\lambda_\chi 2}(\xi).$ \hfill (29)

The same is true for all DAs with primes, which are equal to one function which will be designated below as $\Psi(\xi)$

$\Psi(\xi) = \phi'_{\lambda_\chi 1}(\xi) = \phi'_{\lambda h}(\xi).$ \hfill (30)

In addition, one can relate the moments of the $\Phi(\xi)$ to the moments of the $\Psi(\xi)$ as follows

$<\xi^n>_{\Phi} = \frac{<\xi^{n+1}>_{\Phi}}{n+1}. \hfill (31)$

It should be noted that the constants and DAs in relations (28)-(31) depend on scale in a different way. This means that relations (28)- (31), which are valid at not too large scale, will be violated at sufficiently large scale.

Recursive relation (31) determines the function $\Psi(\xi)$ through the function $\Phi(\xi)$. One can guess the solution of this relation:

$\Psi(\xi) = - \int_{-1}^{\xi} dt \Phi(t). \hfill (32)$

To prove that (32) is the solution of relations (31) one should put this function to the definition of the n-th moment and integrate the resulting expression by parts. It is seen from equations (28), (30) and (32) that all DAs of the P-wave mesons are defined through the universal function $\Psi(\xi)$. It should be noted that this fact results from the nonrelativistic spin-symmetry, which holds at leading order in the heavy-quark velocity. Below equations (28) and (32) will be used to build the models for the function $\Phi(\xi)$ and $\Psi(\xi)$ of P-wave charmonia.
At the end of this section it is interesting to discuss the question about relativistic corrections to the matrix elements involving $P$-wave charmonia. If we ignore the contribution coming from the higher Fock states, relativistic corrections to the matrix involving, for instance, the $\chi_0$ meson is given by the matrix element

$$\langle v^n \rangle_P = \frac{1}{(m^*_Q)^2} \frac{\langle \chi_0 | \gamma^+ (\sigma D) D^2 \psi | 0 \rangle}{\langle \chi_0 | \gamma^+ (\sigma D) \psi | 0 \rangle},$$  

where $\psi$ and $\chi^+$ are Pauli spinor fields that annihilate a quark and an antiquark respectively, $\sigma$ are Pauli matrices, $m^*_Q$ is the quark pole mass. Using equation 19 it is not difficult to obtain general formula that connects the moment $\langle x^{n+1} \rangle$ with the matrix element $\langle v^n \rangle_P$ at the leading order approximation in relative velocity

$$\langle v^n \rangle_P = \frac{A_{n+2}}{A_2} + O(v^{n+2}) = \frac{n + 3}{3} \langle x^{n+1} \rangle + O(v^{n+2}).$$  

Although the derivation was done for the $\chi_0$ meson, at the leading order approximation in relative velocity the matrix element $\langle v^n \rangle_P$ is universal for all $P$-wave mesons.

### IV. CHARMONIA DISTRIBUTION AMPLITUDES WITHIN QCD SUM RULES.

In this section QCD sum rules approach [16, 17] will be applied to the calculation of the moments of the $P$-wave charmonia DAs. The problem which can be met if one tries to apply QCD sum rules to the calculation of these moments is that it is not possible to write QCD sum rules for one DA. Commonly, the contribution of different mesons and different DAs mix in QCD sum rules, what does not allow us to calculate the constants and the moments of DAs [1]-[4] separately. Only some combinations of different constants and moments can be extracted from QCD sum rules. For instance, two point QCD sum rules for the currents $\bar{Q}(x)\gamma_\mu(z\bar{D})Q(x) \cdot \bar{Q}(0)\gamma_\mu(z\bar{D})^{n+1}Q(0)$ can be written as follows

$$\frac{f^2_{\chi_0}(\xi_{n+1})_{\chi_0}^{(n+1)}}{(m^2_{\chi_0} + Q^2)^{m+1}} + \frac{f^2_{\chi_2}(\xi_{n+1})_{\chi_2}^{(n+1)}}{(m^2_{\chi_2} + Q^2)^{m+1}} = \frac{1}{\pi} \int_{4m^2}^{\infty} ds \frac{\text{Im} \, \Pi_{\text{pert}}(s, n)}{(s + Q^2)^{m+1}} + \Pi_{\text{pert}}^{(m)}(Q^2, n).$$

The expressions for the $\text{Im} \, \Pi_{\text{pert}}(s, n)$, $\Pi_{\text{pert}}^{(m)}(Q^2, n)$ will be given below (equations 37, 38). It is seen from this example that it is not possible to extract the constants $f_{\chi_0}$, $f_{\chi_2}$ or the moments $\langle \xi_{n+1} \rangle_{\chi_0}$, $\langle \xi_{n+1} \rangle_{\chi_2}$ from (35) separately. Evidently, this strongly restricts the accuracy of the calculation. The only QCD sum rules which are free from this problem are two point sum rules for the currents $\bar{Q}(x)\gamma_\mu(z\bar{D})Q(x) \cdot \bar{Q}(0)\gamma_\mu(z\bar{D})^{n+1}Q(0)$

$$\frac{f^2_{\chi_0}(\xi_{n+1})_{\chi_0}^{(n+1)}}{(m^2_{\chi_0} + Q^2)^{m+1}} + \frac{f^2_{\chi_2}(\xi_{n+1})_{\chi_2}^{(n+1)}}{(m^2_{\chi_2} + Q^2)^{m+1}} = \frac{1}{\pi} \int_{4m^2}^{\infty} ds \frac{\text{Im} \, \Pi_{\text{pert}}(s, n)}{(s + Q^2)^{m+1}} + \Pi_{\text{pert}}^{(m)}(Q^2, n),$$

where $\text{Im} \, \Pi_{\text{pert}}(s, n)$, $\Pi_{\text{pert}}^{(m)}(Q^2, n)$ can be written as

$$\text{Im} \, \Pi_{\text{pert}}(s, n) = \frac{3}{8\pi} \left( \frac{1}{n + 3} - \frac{v^2}{n + 5} \right), \quad v^2 = 1 - \frac{4m^2}{s},$$

$$\Pi_{\text{pert}}^{(m)}(Q^2, n) = \frac{1}{24\pi} \left( \frac{\langle \alpha_s G^2 \rangle}{m_c} (m + 1) \int_{-1}^{1} d\xi \left( \xi^{n+2} + \frac{n(n + 1)}{4} \xi^n (1 - \xi^2) \right) \right) \frac{(1 - \xi^2)^{m+2}}{(4m^2_c + Q^2 (1 - \xi^2))^{m+2}} + \Pi_{\text{pert}}^{(m)}(Q^2, n),$$

$$\Pi_{\text{pert}}^{(m)}(Q^2, n) = -\frac{\langle \alpha_s G^2 \rangle}{6\pi} m_c (m^2 + 3m + 2) \int_{-1}^{1} d\xi \frac{\xi^{n+2} (1 + 3\xi^2)}{(4m^2_c + Q^2 (1 - \xi^2))^{m+3}}.$$

In the calculation we take $Q^2 = 4m^2_c$ [18]. In the numerical analysis of QCD sum rules the values of parameters $m_c$ and $\langle \alpha_s G^2 / \pi \rangle$ will be taken from paper [18]:

$$m_c = 1.24 \pm 0.02 \text{ GeV}, \quad \langle \frac{\alpha_s G^2}{\pi} \rangle = 0.012 \pm 30\% \text{ GeV}^4.$$  

(39)
First sum rules (36) will be applied to the calculation of the constant \( f_{h_c}^2 \). It is not difficult to express the constant \( f_{h_c}^2 \) from equation (36) at \( n = 0 \) as a function of \( m \). For too small values of \( m \) \((m < m_1)\) there is large contributions from higher resonances and continuum which spoil sum rules (36). Although for \( m > m_1 \) these contributions are strongly suppressed, it is not possible to apply sum rules for too large \( m \) \((m > m_2)\) since the contribution arising from higher dimensional vacuum condensates rapidly grows with \( m \) what invalidates our approximation. If \( m_1 < m_2 \) there is some region of applicability of sum rules (36) \([m_1, m_2]\) where the resonance and the higher dimensional vacuum condensates contributions are not too large. Within this region \( f_{h_c}^2 \) as a function of \( m \) varies slowly and one can determine the value of this constant. The value of the continuum threshold \( s_0 \) must be taken so that to appear stability region \([m_1, m_2]\). Our calculation shows that for the central values of parameters (39) there exists stability region for \( s_0 > (4.3 \text{ GeV})^2 \). If the value of the continuum threshold \( s_0 \) is varied in the region \( s_0 \in (4.3^2, \infty) \text{ GeV}^2 \), the value of the constant \( f_{h_c}^2 \) can be written as \( f_{h_c}^2 = (0.037 \pm 0.005) \text{ GeV}^2 \). In addition to the uncertainty due to the variation of the value of \( s_0 \), there are uncertainties due to the variation of the values of the \( m_c \) (which is \( \pm 0.004 \)) and \( \langle \alpha_s / \pi G^2 \rangle \) (which is \( \pm 0.001 \)). The last source of uncertainty is the radiative corrections to the perturbative density \( \text{Im} \Pi_{\text{pert}}(s, n) \), which will be estimated as \( \alpha_s(m_c) / \pi \sim 13\% \). Adding these uncertainties in quadrature, one gets

\[
 f_{h_c}^2 (\mu \sim m_c) = (0.037 \pm 0.007) \text{ GeV}^2. \tag{40}
\]

As it was noted above, the value of the constant \( f_{h_c}^2 \) is scale dependent quantity. The characteristic scale of QCD sum rules is \( \sim m_c \). This means that the value of the constant \( f_{h_c}^2 \) is determined at the scale \( \sim m_c \), as it is shown in (40).

Next let us consider the moments \( \langle \xi^3 \rangle_{h_c}, \langle \xi^5 \rangle_{h_c}, \langle \xi^7 \rangle_{h_c} \). However, instead of considering QCD sum rules for \( n = 2, 4, 6 \) we will consider the ratios of sum rules at \( n = 2, 4, 6 \) and the sum rules at \( n = 0 \). Such approach improves the accuracy of the calculation (see paper [9] for details). The analysis similar to that for the constant \( f_{h_c}^2 \) gives

\[
\begin{align*}
\langle \xi^3 \rangle &= 0.18 \pm 0.03, \\
\langle \xi^5 \rangle &= 0.050 \pm 0.010, \\
\langle \xi^7 \rangle &= 0.017 \pm 0.004.
\end{align*}
\tag{41}
\]

It should be noted that these moments are defined at the scale \( \sim m_c \). It should be also noted that the values of the moments (41) are in good agreement with potential model estimation (see paper [8]). For instance, within Buchmuller-Tye potential model (19) \( \langle \xi^3 \rangle = 0.18, \langle \xi^5 \rangle = 0.047, \langle \xi^7 \rangle = 0.016 \); within Cornell potential model (20) \( \langle \xi^3 \rangle = 0.16, \langle \xi^5 \rangle = 0.040, \langle \xi^7 \rangle = 0.013 \).

Using values (41) one can find the matrix elements that control relativistic corrections to any process with \( P \)-wave charmonia in the initial or final state. The relationships between these matrix elements and the moments are given in equation (44). These relations are valid up to the higher order relativistic corrections, which can be estimated as \( \langle v^2 \rangle \). Taking into account this additional source of uncertainty one gets

\[
\begin{align*}
\langle v^2 \rangle_P &= \frac{5}{3} \langle \xi^3 \rangle_P = 0.30 \pm 0.10, \\
\langle v^4 \rangle_P &= \frac{7}{3} \langle \xi^5 \rangle_P = 0.12 \pm 0.04, \\
\langle v^6 \rangle_P &= \frac{9}{3} \langle \xi^7 \rangle_P = 0.051 \pm 0.018.
\end{align*} \tag{42}
\]

In the next sections results (41) will be used to build the model for the DAs \( \Phi(\xi) \) and \( \Psi(\xi) \).

V. MODEL FOR CHARMONIA DISTRIBUTION AMPLITUDES.

To build the model of the function \( \Phi(\xi, \mu \sim m_c) \) we use Borel version (16, 17) of sum rules (36) but without continuum contribution and power corrections

\[
 f_{h_c}^2 \langle \xi^{n+1} \rangle e^{-m_{h_c}^2/M^2} = \frac{M^2}{4\pi^2} \int_1^\infty d\xi' \xi'^{n+2} \frac{3}{4} (1 - \xi'^2) \exp \left( -\frac{4m_{h_c}^2}{M^2} \frac{1}{1 - \xi'^2} \right). \tag{43}
\]

Evidently, within this approximation the function \( \Phi(\xi, \mu \sim m_c) \) can be written in the form

\[
 \Phi(\xi, \mu \sim m_c) = c(\beta_P)(1 - \xi^2) \xi \exp \left( -\beta_P \frac{1}{1 - \xi^2} \right), \tag{44}
\]
where \( c(\beta_P) \) is a normalization constant and \( \beta_P \) is some constant. We propose function (44) as the model for DAs \( \Phi(\xi, \mu \sim m_c) \). To fix the constant \( \beta_P \) the value of the moment \( \langle \xi^2 \rangle \) (41) will be used. Thus we get \( \beta_P = 3.4^{+1.5}_{-0.9} \). The constant \( c(\beta) \) can be determined from normalization condition (6). The moments of the function (44) are
\[
\langle \xi^3 \rangle = 0.18 \pm 0.03, \\
\langle \xi^5 \rangle = 0.047 \pm 0.014, \\
\langle \xi^7 \rangle = 0.015 \pm 0.006.
\]
(45)

Using the model for the function \( \Phi(\xi, \mu \sim m_c) \) and equation (32) one can get the model of the DA \( \Psi(\xi, \mu \sim m_c) \)
\[
\Psi(\xi, \mu \sim m_c) = - \int_1^\xi dt \Phi(t, \mu \sim m_c) = \frac{c(\beta_P)}{2} (1 - \xi^2)^2 E_3 \left( \frac{\beta_P}{1 - \xi^2} \right),
\]
(46)
where the function \( E_3(z) \) is the exponential integral function
\[
E_3(z) = \int_1^\infty \frac{dt}{t^3 e^{zt}}.
\]
(47)

To determine DAs (1)-(4) at a scale different from \( m_c \) one should use relations (29), (30) and than evolution equations for the corresponding DAs.

VI. CONCLUSION.

In this paper we have considered the leading twist distribution amplitudes (DA) of \( P \)-wave nonrelativistic mesons. At the leading order approximation in relative velocity of quark-antiquark pair inside the mesons these functions can be expressed through one universal DA. We have derived the relations between the DAs of \( P \)-wave nonrelativistic mesons and the universal DA.

As an example, we have considered the DAs of the \( P \)-wave charmonia mesons. Within QCD sum rules we found the moments of the leading twist DA of \( h_c \) meson, what allowed us to build the model of the universal DA for the \( P \)-wave charmonia mesons.

In addition, we have found the relations between the moments and the nonrelativistic QCD matrix elements that control relativistic corrections to any amplitude involving \( P \)-wave charmonia. The calculation shows that characteristic size of these corrections is \( \sim 30\% \).

This work was partially supported by Russian Foundation of Basic Research under grant 07-02-00417. The work of V. Braguta was partially supported by CRDF grant Y3-P-11-05 and president grant MK-2996.2007.2. The work of A. Luchinsky was partially supported by president grant MK-110.2008.2 and Russian Science Support Foundation.

[1] G. P. Lepage and S. J. Brodsky, Phys. Rev. D 22, 2157 (1980).
[2] V. L. Chernyak and A. R. Zhitnitsky, Phys. Rept. 112, 173 (1984).
[3] G. T. Bodwin, D. Kang and J. Lee, Phys. Rev. D 74, 114028 (2006) [arXiv:hep-ph/0603185].
[4] J. P. Ma and Z. G. Si, Phys. Lett. B 647, 419 (2007) [arXiv:hep-ph/0608221].
[5] V. V. Braguta, A. K. Likhoded and A. V. Luchinsky, Phys. Lett. B 646, 80 (2007) [arXiv:hep-ph/0611021].
[6] V. V. Braguta, Phys. Rev. D 75, 094016 (2007) [arXiv:hep-ph/0701234].
[7] V. V. Braguta, arXiv:0709.3885 [hep-ph].
[8] H. M. Choi and C. R. Ji, Phys. Rev. D 76, 094010 (2007) [arXiv:0707.1173 [hep-ph]].
[9] T. Feldmann and G. Bell, arXiv:0711.4014 [hep-ph].
[10] G. Bell and T. Feldmann, JHEP 0804, 061 (2008) [arXiv:0802.2221 [hep-ph]].
[11] G. T. Bodwin and A. Petrelli, Phys. Rev. D 66, 094011 (2002) [arXiv:hep-ph/0205210].
[12] E. Braaten and J. Lee, Phys. Rev. D 67, 054007 (2003) [arXiv:hep-ph/0211085];
[13] G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D 51, 1125 (1995) [Erratum-ibid. D 55, 5853 (1997)] [arXiv:hep-ph/9407339].
[14] G. T. Bodwin, D. Kang and J. Lee, Phys. Rev. D 74, 014014 (2006) arXiv:hep-ph/0603186.
[15] S. J. Brodsky, T. Huang and G. P. Lepage, In "Banff 1981, Proceedings, Particles and Fields 2", 143-199.
[16] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 147, 385 (1979).
[17] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 147, 448 (1979).
[18] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rept. 127, 1 (1985).
[19] W. Buchmuller and S. H. H. Tye, Phys. Rev. D 24, 132 (1981).
[20] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T. M. Yan, Phys. Rev. D 17, 3090 (1978) [Erratum-ibid. D 21, 313 (1980)].