We present two complementary strategies for modeling nonlinear quantum optics in realistic resonant integrated optical devices, where scattering loss is present. In the first strategy, we model scattering loss as an effective absorption; in the second, we employ a Hamiltonian treatment, with the loss modelled as a coupling to a ‘phantom’ channel. As an example, we use these two approaches to model spontaneous four-wave mixing in (i) a ring-channel system and (ii) an add-drop system. We present the rates of photon pairs, broken pairs, and lost pairs for both systems, as well as the full biphoton wavefunction (BWF) including the effects of scattering. We find that for a high-finesse resonator coupled to an arbitrary number of channels, the full BWF can be extracted from the BWF associated with photon pairs in an experimentally accessible output channel.

I. INTRODUCTION

With recent advances in fabrication, integrated optical structures are of interest both for fundamental research in nonlinear quantum optics, and for the development of platforms for quantum information processing. The generation of quantum correlated pairs of photons (“signal” and “idler”) by spontaneous parametric down-conversion (SPDC) or spontaneous four-wave mixing (SFWM), as well as the use of those processes to generate squeezed states familiar from scattering theory. In the quantum context, asy-in and asy-out fields only differ by a frequency dependent phase \(\delta\). However, if scattering is included, then an asy-in field will contain out-going fields not only propagating in the channel to the right, but as well light propagating away from the chip. A similar construction holds for the asy-out fields. If these full asy-in and asy-out fields would be used to describe the pump and generated photons, then the resulting calculation of photon generation would include the effects of scattering.

Yet, since even the field profile for light propagating in a scattering-free channel or ring must generally be found numerically, the full asy-in and asy-out fields would be very difficult to construct, in that they would require a full calculation of the scattering. In this paper we offer two strategies to avoid this requirement.

The first strategy relies on the fact that the calculation of photon generation involves the overlap of pump, signal, and idler fields in the nonlinear medium. Although generalizations are possible, for a microring resonator as in Fig. 2(a) this is often taken just to be the ring, where the field strength is concentrated. So one needs the asy-in and asy-out fields only in the ring, and the details of how the scattered light propagates off the chip is irrelevant; only the attenuation constant that characterizes the loss of energy as light propagates in the ring is needed to provide a good approximation of the asy-in fields where those fields are needed. Correspondingly, the asy-out fields in the ring are characterized by an enhancement constant equal in magnitude to the attenuation constant of the asy-in fields. In this strategy we construct only the asy-in and asy-out fields in the ring and channels, in a standard phenomenological approach that treats the coupling between the channel and the ring by
introducing self- and cross-coupling coefficients (see Fig. 2) [6]. We can then calculate the rate at which pairs of photons will appear in the output channel, and the biphoton wave function that will characterize those pairs of photons. However, this approach does not allow us to calculate how often a pair of photons is lost to scattering or, perhaps even more importantly, how often only one photon of a generated pair is lost.

The second strategy involves modeling the scattering losses by the interaction of the ring with a “phantom channel” (see Fig. 5). Focusing only on a few ring resonances, we adopt a different coupling model for channel-ring interactions, which admits of a Hamiltonian formulation and leads to an analytic solution for the asym-in and asym-out fields for an arbitrary number of channels coupled to the ring; we use the coupling of light into the phantom channel to characterize the loss [7]. Since the phantom channel “keeps track” of the lost photons, we can use this approach to calculate the rate at which both pairs of photons and single photons will appear in the physical channel, at least within the approximation – implicit in the model – that the scattering losses are characterized by Lorentzian line shapes. Unlike the first strategy, however, the model used here is only a good approximation for high finesse systems [5]. For many systems of interest, of course, this is not a serious limitation.

In Section II we present the first strategy and in Section III we present the second. In these sections we focus on a single ring coupled to a channel, choosing such a resonant structure because of the importance of scattering losses. For example calculations we consider the generation of entangled pairs of photons by SFWM. An interesting deviation from the usual “critical coupling” condition assumed to maximize the photon pair generation rate is found and discussed. We also compare the results of the two strategies, finding good agreement for high finesse systems. In Section IV we use both strategies to calculate photon pair production in an add-drop structure involving a ring and two physical channels. We use the second strategy to calculate the generation rates for the appearance of signal and idler photons in different physical (and phantom) channels, and we calculate the biphoton wave function describing the output states. Our conclusions are presented in Section V.

II. PAIR PRODUCTION RATE

We use the results of a quantization approach for generic integrated structures that involves the displacement field as a fundamental field operator [5-9]. It will be convenient to break up the displacement operator into its contributions from different frequency bands $J$,

$$ D(r) = \sum_J D_J(r), $$

where each frequency band is centered at a frequency $\omega_J$, which will typically be associated with a resonance in the interaction region (see Fig. 1). Before detailing the form of the asymptotic-in and -out fields in the full structure, we first consider the field in an isolated waveguide of infinite length. In that case we write the $D_J(r)$ of Eq. (1) as $D_J^{\text{wg}}(r)$, and we have [8]

$$ D_J^{\text{wg}}(r) = \int dk D_{jk}^{\text{wg}}(r)a_J(k) + H.c., $$

where the $a_J(k)$ and their adjoints are ladder operators that obey the usual bosonic commutation relations,

$$ [a_J(k), a_J^\dagger(k')] = \delta(k - k')\delta_{JJ'}, $$

provided that the light associated with each band is far from any waveguide cut-off, and is well-localized in frequency such that there is no overlap between $k$ components in distinct bands [8]. We consider the $k$ of interest to range over positive values, and

$$ D_{jk}^{\text{wg}}(r) = \sqrt{\frac{\hbar\omega_{jk}}{4\pi}} d_{jk}(r_\perp) e^{iks}. $$

Here, increasing $s$ indicates the direction in which the field is propagating, and $r_\perp$ indicates the two Cartesian components perpendicular to that direction. We assume that all our frequency ranges are close enough that material dispersion can be neglected; then the $d_{jk}(r_\perp)$ are normalized according to

$$ \int \frac{d_{jk}(r_\perp) \cdot d_{jk}(r_\perp)}{\varepsilon_0 \varepsilon_1(r_\perp)} dr_\perp = 1, $$

where $\varepsilon_1(r_\perp)$ is the square of the local index of refraction. Finally, $\omega_{jk}$ is the frequency of a field at $k$ in frequency range $J$; we neglect group velocity dispersion over each frequency range, and write

$$ \omega_{jk} = \omega_J + v_J(k - K_J), $$

where $K_J$ is the value of $k$ at frequency $\omega_J$. Despite our neglect of material dispersion we allow the different $v_J$ to be different, both because of modal dispersion and because the different frequency ranges identified by $\omega_J$
could also be associated with different waveguide mode
profiles. We also allow \(v_j\) and \(K_j\) to depend on the
waveguide constituting each channel, denoting the de-
pendence by a superscript, \(v_j^{(X)}\) and \(K_j^{(X)}\).

We now turn to the asymptotic-in and -out fields as-
associated with a general structure such as that in Fig. 1.
These are stationary solutions of Maxwell’s equations for
the entire structure \[3, 10\]. An asymptotic-in wavepacket
consists in general of an incoming wavepacket at \(t \to -\infty\)
in the waveguide of one channel, and outgoing fields in
the waveguide of every channel at \(t \to \infty\). Similarly,
an asymptotic-out wavepacket consists of a single out-
going wavepacket at \(t \to \infty\), and fields incoming from
the waveguide of every channel at \(t \to -\infty\). Neglecting
the presence of any optical modes that are truly bound to
the interaction region, the displacement field \(D(r)\) can be
expanded in terms of either asymptotic-in or -out fields.
That is, for the full structure we can use Eq. (1) with
the \(D_j(r)\) equal to either \(D_j^{(i)}(r)\) or \(D_j^{(o)}(r)\), indicating
respectively either an asymptotic-in or -out expansion,
with

\[
D_{j}^{(i/o)}(r) = \sum_{X} \int dk D_{jk}^{(i/o)(X)}(r) a_{j}^{(i/o)(X)}(k) + H.c. \quad (7)
\]

where \((X)\) denote the different channels, and the \(a_{j}^{(i/o)(X)}(k)\) the associated ladder operators.

Note that for a given channel label \(X\) both \(D_{jk}^{(i)(X)}(r)\) and \(D_{jk}^{(o)(X)}(r)\) are in general nonvanishing for \(r\) in all
channel waveguides. For \(r\) in the waveguide of channel \(X\),
\(D_{jk}^{(i)(X)}(r)\) consists of the sum of the waveguide field Eq.
(4) for channel \(X\) with \(s\) increasing towards the interac-
tion region (an “incoming” field) together with an “out-
going” field; for \(r\) in all other waveguides there are only
“outgoing” fields. And for \(r\) in channel \(X\), \(D_{jk}^{(o)(X)}(r)\) consists of the sum of the waveguide field Eq. (4)
for channel \(X\) with \(s\) increasing away from the interaction
region (an “outgoing” field) together with an “incoming”
field; for \(r\) in all other channel waveguides there are only
“incoming” fields.

Due to their asymptotic behaviour, the asymptotic-in
expansion is the correct physical choice for any incom-
ing pump or seed fields, for \(t \to -\infty\) the \(a_{j}^{(i)(X)}\) can be
identified with the lowering operators in an infinite ver-
sion of the waveguide defining channel \(X\). Similarly, the
asymptotic-out expansion is suitable for fields that are
sought at the system’s output, such as fields generated
by nonlinear processes, for as \(t \to \infty\) the \(a_{j}^{(o)(X)}\) can be
identified with the lowering operators in an infinite
version of the waveguide defining channel \(X\).

We now consider the generation of such fields, through
the effect of the Hamiltonian for a third-order nonlinear
interaction,

\[
H_{NL} = -\frac{1}{4\epsilon_0} \int dr \Gamma_{3}^{ijkl}(r) D^{i}(r) D^{j}(r) D^{k}(r) D^{l}(r) ,
\quad (8)
\]

where \(i, j, k, l\) are Cartesian components, summed over
when repeated. The nonlinear parameter \(\Gamma_{3}^{ijkl}(r)\) is
related to the more familiar element of the third-order
nonlinear tensor \(\chi(3)\) by

\[
\Gamma_{3}^{ijkl}(r) = \frac{\chi^{ijkl}(r)}{\epsilon_0^2 c^4 |(r)|} ,
\quad (9)
\]

where \(\varepsilon_1(r)\) is the square of the local index of refraction.
If we consider only terms responsible for SFWM [11], the
nonlinear Hamiltonian becomes

\[
H_{SFWM} = -\frac{1}{4\epsilon_0} \frac{4!}{2!!1!!} \times \sum_{X,X',i} \int dk_1 dk_2 dk_3 dk_4 K^{XX'}(k_1, k_2, k_3, k_4)
\times a_{S}^{(i)(X)}(k_1) a_{D}^{(i)(X')}\ast(k_2)
\times a_{P}^{(o)(X_1)}(k_3) a_{P}^{(o)(X_2)}(k_4) + H.c. \quad (10)
\]

where

\[
K^{XX'}(k_1, k_2, k_3, k_4) = \int dr \Gamma_{3}^{ijkl}(r)
\times [D_{S}^{i}(r)]^{\ast} [D_{S}^{j}(r)]^{\ast} [D_{S}^{k}(r)]^{\ast} [D_{S}^{l}(r)]^{\ast} \times D_{P}^{k}(r) D_{P}^{l}(r).
\quad (11)
\]

We take the integral to range over the interaction region,
or at least the part of it where fields can be concentrated
and the nonlinear interaction is significant. Here we use
\(S, I,\) and \(P\) to denote respectively the signal, idler, and
pump frequency ranges and modes; the combinatorial
factor \(4!/2!!1!!)\) takes into account that the signal and idler
ranges are assumed distinct. We have assumed that the
pump fields are injected into a single channel which we
label “\(X_1\),” and in general we allow the generated
photons to exit the system via different channels, using
the first superscript in \(K^{XX'}(k_1, k_2, k_3, k_4)\) to denote an
output channel for the signal photon, and the second an
output channel for the idler photon.

Moving into an interaction picture, where Eq. (10)
is the perturbation, the linear Hamiltonian leads to the
raising and lowering operators in Eq. (11) all acquir-
ing the usual frequency dependences. Then treating the
strong pump field classically, we can put \(a_{P}^{(o)(X_1)}(k) \to \alpha_{P}(k)\), where \(\alpha_{P}(k)\) is a complex function. If we as-
sume a CW pump at a frequency \(\omega_{0}\) that could be
slightly detuned from the center frequency of the pump
range, \(\omega_{0} = \omega_{P} + \delta \omega_{P}\), with an associated \(k\) given by
\(k_{0} = K_{P}^{(X_1)} + (\omega_{0} - \omega_{P})/v_{P}^{(X_1)}\) (recall Eq. (9)), we show
in Appendix [A] that
\[ \alpha_p(k) = \sqrt{\frac{2\pi P_p}{\hbar \omega_0 v_p^3}} \delta(k - k_o). \] (12)

where \( P_p \) is the pump power. Then we can write the interaction picture version of Eq. (10) as
\[ H_{SFWM}^I(t) = -\sum_{X',X} \int dk_1 dk_2 M_{XX'}(k_1, k_2)e^{-i\Omega(k_1, k_2) t} \]
\[ \times a^\dagger_{S}(X') \alpha(k_1) a_{I}^\dagger(k_2), \] (13)

where
\[ M_{XX'}(k_1, k_2) = \frac{6\pi P_p}{\epsilon_0 \hbar \omega_0 v_p} K_{XX'}(k_1, k_2, k_o, k_o), \] (14)

and
\[ \Omega(k_1, k_2) = 2\omega_o - \omega_{Sk_1} - \omega_{Ik_2}. \] (15)

A standard Fermi’s Golden Rule calculation, detailed in the Appendix, then leads to the rate of pair production as
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \left| R_{XX'}(k_1, k_2)\right|^2 dk_1 dk_2 (2\omega_o - \omega_{Sk_1} - \omega_{Ik_2}) |M_{XX'}(k_1, k_2)|^2. \]

Moving from wave number \((k_1, k_2)\) to frequency variables \((\omega_1 = \omega_{Sk_1}, \omega_2 = \omega_{Ik_2})\), and putting
\[ J_{XX'}(\omega_1, \omega_2, \omega_3, \omega_4) \equiv \]
\[ K_{XX'}(k_1(\omega_1), k_2(\omega_2), k_3(\omega_3), k_4(\omega_4)) \] (17)

where on the right-hand side
\[ k_1(\omega) = \frac{\omega - \omega_S}{v_S}, \] (18)
\[ k_2(\omega) = \frac{\omega - \omega_I}{v_I}, \]
\[ k_3(\omega) = \frac{\omega - \omega_p}{v_p}, \]

using Eq. (14) in (16), and implementing the Dirac delta function, we find
\[ R_{XX'} = \frac{72\pi^3}{\epsilon_0 \hbar \omega_0 v_S^3} \frac{P_p^2}{v_I^2 v_p^2} \]
\[ \times \int d\omega_1 J_{XX'}(\omega_1, 2\omega_o - \omega_1, \omega_o, \omega_o) \] (19)

The two strategies we introduce below differ only in the way the asymptotic-in and asymptotic-out fields that appear in \( J_{XX'}(\omega_1, \omega_2, \omega_3, \omega_4) \) are constructed. As an example, we consider a system with multiple waveguides coupled to a single resonant element; the simplest example is shown in Fig. 2(a), where there is a single waveguide coupled to a ring resonator through a point coupler.

### III. A FIRST STRATEGY

If backscattering at the coupling point can be neglected, which is typical in well-designed systems, the ring resonator system shown in Fig. 2(a) acts as an “all pass” filter: light incident from the left can be coupled into the ring but is eventually coupled out again into the waveguide. Hence for light incident from the left the asymptotic-in and -out fields only contain light in the waveguide propagating to the right. To treat the coupling to the ring, in the first strategy we use a standard point coupling model in which the two input field amplitudes \((f_1, f_4)\) are connected to the two output field amplitudes \((f_2, f_3)\) by the linear system of equations
\[
\begin{align*}
f_2 &= \sigma f_1 + i\kappa f_4 \quad \text{if } z < 0, \\
f_3 &= i\kappa f_1 + \sigma f_4 \quad \text{if } z > 0,
\end{align*}
\] (20)

where \( \sigma \) and \( \kappa \) are the self-coupling and cross-coupling coefficients of the point coupler respectively [6], for convenience they are assumed to be real, with
\[ \kappa^2 + \sigma^2 = 1. \] (21)

The asymptotic-in or -out field in channel \((X)\) is given by
\[ D_{\text{chan}, jk}^{\text{in(out)}(X)}(r) = \frac{\hbar \omega_{jk}}{4\pi} d_{jk}^{\text{in(out)}(X)}(x, y) f_{jk}^{\text{in(out)}(X)}(z) e^{ikz}, \]
\[ (\text{cf. Eq. (1)}), \]

where the amplitude \( f_{jk}^{X}(z) \) takes into account the field distribution along \( z \),
\[ f_{jk}^{\text{in(out)}(X)}(z) = \begin{cases} f_1^{\text{in(out)}(X)} & \text{if } z < 0, \\
                        f_2^{\text{in(out)}(X)} & \text{if } z > 0,
\end{cases} \] (23)

and will be different depending on whether we are specifying an asymptotic-in or asymptotic-out field. For our problem, where the pump is incident from the left, we will need asymptotic-in fields for the left \((z < 0)\) channel \((X = L)\) and asymptotic-out fields for the right \((z > 0)\) channel \((X = R)\). Here both channels involve the same physical waveguide, so we can put \( d_{jk}^{L}(x, y) = e^{ik(x, y)} \), \( d_{jk}^{R}(x, y) = d_{jk}(x, y) \), \( f_{jk}^{\text{in(out)}(X)}(z) \) for the ring is
\[ D_{\text{ring}, jk}^{\text{in(out)}(X)}(r) = \frac{\hbar \omega_{jk}}{4\pi} d_{jk}(r_\perp; \zeta) f_{jk}^{\text{in(out)}(X)} e^{ik\zeta}, \] (24)

where \( \zeta \) is the coordinate in the direction of propagation around the ring, ranging from 0 (at the position identified by \( f_3 \)) to \( L \) (at the position identified by \( f_4 \)), and \( r_\perp \) refers to components in the plane perpendicular to the direction indicated by increasing \( \zeta \). Thus \( d_{jk}(r_\perp; \zeta) \) plays the role for the ring that \( d_{jk}(x, y) \) does for the
channels. In general it depends on all three coordinates because the direction in which the field is polarized can change with the angle \( \zeta \). However, \( \mathbf{d}_{jk}(r_\perp; \zeta) \cdot \mathbf{d}_{jk}(r_\perp; \zeta) \) will be independent of \( \zeta \), and if the ring width and the width of the waveguide are taken to be the same, \( \mathbf{d}_{jk}(x, y) \) and \( \mathbf{d}_{jk}(r_\perp; \zeta) \) are equal to very good approximation at \( \zeta = 0 \).

Here \( f_{jk}^{\text{in(out)}} \) can be identified with \( f_3 \), and we clearly have

\[
f_4 = f_3 e^{ikL}.
\]

Combining Eq. (20) with (25) we can then identify the asymptotic-in fields for the left channel by setting \( f_1 = 1 \), and the asymptotic-out fields for the right channel by setting \( f_2 = 1 \). We find

\[
f_{jk}^{\text{in(L)}}(z) = \begin{cases} 1 & \text{if } z < 0 \\ \sigma^{-1} e^{ikL} & \text{if } z > 0 \end{cases},
\]

and

\[
f_{jk}^{\text{in}} = \frac{iK}{1 - \sigma e^{ikL}},
\]

the field enhancement factor inside the ring for incident fields, while

\[
f_{jk}^{\text{out(R)}}(z) = \begin{cases} 1 - \sigma e^{ikL} & \text{if } z < 0 \\ 1 & \text{if } z > 0 \end{cases},
\]

and

\[
f_{jk}^{\text{out}} = \frac{iK}{\sigma - e^{ikL}}.
\]

If we use \( f_{jk}^{\text{in}}(\zeta) \) and \( f_{jk}^{\text{out}}(\zeta) \) to denote respectively the field amplitudes as they vary with \( \zeta \) inside the ring (see Eq. (24)), we have simply

\[
f_{jk}^{\text{in(out)}}(\zeta) = f_{jk}^{\text{in(out)}} e^{ik\zeta}.
\]

A similar approach is applied in more complicated structures; one may not be capable of finding an analytic expression for the field amplitude, but the problem can be solved numerically in such cases.

Up to this point, we have done nothing to address the presence of loss in the system; even if absorption can be neglected, scattering losses should be included in a description of practical systems. Here the problem is how to calculate the asymptotic fields in the presence of such losses. A possible approach would be modelling the system by using finite-difference-time-domain (FDTD) or finite-elements numerical tools, in which one could, in principle, take into account the structure imperfections and/or disorder that lead to scattering. However, this solution is difficult even for small systems, and impractical for large systems, for it would require a detailed description of the structure at the nanometer level, resulting in demanding calculations.

A different approach can be envisioned by noting that the description of the nonlinear interaction using Eq. 8 requires the asymptotic fields to be determined only in the region of space where the contribution to the nonlinear overlap integral is significant. For instance, in systems composed of waveguides and resonators, this is typically the region of space in which most of the electromagnetic field is confined and/or the strength of the nonlinear interaction is the largest.

If we consider a system in which light is guided in a waveguide, the effect of scattering is a decreasing of the field intensity as light propagates; for a structure of the type considered here, that effect will be most significant inside the ring resonator. From a phenomenological point of view, this attenuation can be modelled as an effective absorption in the ring region. A common way to describe this kind of loss consists of introducing a complex propagation wavevector

\[
\tilde{k}_n(\omega) = k_n(\omega) + i\frac{\xi_n(\omega)}{2},
\]

where \( k_n(\omega) \) is the usual wavevector (see Eq. (18)), and \( \xi_n(\omega) \) brings into effect the field intensity decay due to the propagation losses in frequency range \( n (\xi_1(\omega) = \xi_4(\omega)) \).

For asymptotic-out states, we have \( f_2 = 1 \) instead of \( f_1 = 1 \), and in the solution of Maxwell’s equations we seek there is light exiting in no other direction; the propagation in the ring is then characterized by a complex...
wave vector
\[ \tilde{k}^*_n(\omega) = k_n(\omega) - i \frac{\xi_n(\omega)}{2}. \] (32)

It should be stressed that in this way one can hope to obtain the asymptotic-in and -out fields in the ring region, but without any information about the field distribution outside the structure nor about where light is scattered. Nonetheless, following this approach and using Eq. (31) and Eq. (32) in Eqs. (22) and (24), we can use Eq. (19) to calculate the generation rate \( R_{RR} \) by applying the asymptotic fields we have identified in Eqs. (11) and (17). Here there is only one output channel for both the signal and the idler, so we can drop the superscripts \( RR \) on \( R_{RR}, J_{RR}, \) and \( K_{RR} \). Similarly, since only one waveguide is involved, we can drop the superscripts on \( v_p, v_S, \) and \( v_I \).

To evaluate \( J(\omega_1, 2\omega_0 - \omega_1, \omega_0, \omega_o) \) we restrict the integration to the ring, where the fields will be strongest and thus the effect of the nonlinearity the greatest. A benign approximation can be made immediately, since the mode profiles \( d_{jk}(r_{\perp}; \zeta) \) are typically weak functions of \( k \); we take
\[ d_{jk}(r_{\perp}; \zeta) \rightarrow d_j(r_{\perp}; \zeta), \] (33)

But to do a serious simplification we must assume that the \( d_j(r_{\perp}; \zeta) \) do not depend on \( \zeta \), \( d_j(r_{\perp}; \zeta) \rightarrow d_j(r_{\perp}). \) This will only happen if, to good approximation, the direction of the vector field \( d_j(r_{\perp}; \zeta) \) points everywhere "normal to the chip." Only in this limit do the integrations over \( r_{\perp} \) and \( \zeta \) in \( J(\omega_1, 2\omega_0 - \omega_1, \omega_0, \omega_o) \) factor to two separate integrals. Introducing a coefficient characterizing the nonlinearity,
\[ \gamma_{NL} = \frac{3\omega_p}{4\nu_p v_p} \int dL_3 r_3^{ijkl} (r_{\perp}) d_{S}^*(r_{\perp}) d_j^*(r_{\perp}) d_p^*(r_{\perp}) d_p(r_{\perp}), \] (34)

which involves an integral only over \( r_{\perp} \), from Eq. (19) we find a generation rate of photon pairs given by
\[ R = \frac{1}{2\pi} \left( \frac{\gamma_{NL}\nu_p}{\omega_p} \right)^2 \frac{v_p^2}{\nu_S v_I} \int d\omega_1 \omega_1 (2\omega_0 - \omega_1) \]
\[ \times |J(\omega_1, 2\omega_0 - \omega_1, \omega_0, \omega_o)|^2. \] (35)

Here \( J(\omega_1, 2\omega_0 - \omega_1, \omega_0, \omega_o) \) captures the effect of the variation over the ring of the fields in Eq. (30); namely,
\[ J(\omega_1, 2\omega_0 - \omega_1, \omega_0, \omega_o) = \int (f_{S} f_{k0}(\omega_1)(\zeta) f_{k0}(\omega_0))^{*} f_{S}^{*} f_{k0}(\omega_0) \] (36)
\[ \times \frac{\epsilon_i(\Delta k) - 1}{i(\Delta k)}, \]

since the integral over \( \zeta \) runs from 0 to \( L \), with
\[ \Delta k = \tilde{k}_3(\omega_3) + \tilde{k}_4(\omega_4) - \tilde{k}_1(\omega_1) - \tilde{k}_2(\omega_2). \] (37)

In Fig. 3 we plot the generation rate of photon pairs as a function of the coupling constant \( \sigma \), assuming the \( \xi_j(\omega) \) are the same for all \( j \) and independent of frequency, and assuming that indeed the vector field \( d_j(r_{\perp}; \zeta) \) points everywhere normal to the chip so that the reductions made above are valid. We find that although the maximum of the intensity inside a ring resonator is known to be reached at critical coupling, the maximum of the generation rate occurs when the system is slightly over coupled.

This first approach offers a practical way to analyze nonlinear interactions in complex geometries, allowing for the calculation of quantities such as the biphoton wavefunction and the pair generation rate. However, because we treat the scattering loss as an effective absorption, this approach can only describe photons that couple into physical channels. With this strategy, we cannot calculate the rate at which photons are scattered, or calculate the biphoton wavefunction of a broken photon pair. To resolve this issue, we have developed a complementary method to study this type of structure, which relies on the derivation of full asymptotic field expansions derived from a Hamiltonian treatment of the system, where we model scattering losses by introducing a 'phantom waveguide' to describe the scattering loss.

**IV. SECOND STRATEGY**

We now consider a ring resonator coupled to an arbitrary number of waveguides, one of which is a 'phantom waveguide' introduced to describe scattering loss. Because the asymptotic field amplitudes are defined throughout the entire structure, it is straightforward to
consider all of the possible sets of channels through which photons can exit. In this way, this approach enables us to calculate not only the pair generation rate, but also quantities such as rates of broken pairs. We derive a general biphoton wavefunction for the system, which accounts for all of the trajectories for the photons, rather than considering only the pairs which couple into detection channels.

### A. Fields and Hamiltonian

We begin by outlining the linear behaviour of the system, describing the free propagation of fields in the waveguides and in the ring, and the method used to describe the coupling between the ring and a given waveguide.

For each waveguide field, as introduced in Section II, we take the coordinate $z$ to label the direction of propagation, with the components of $\mathbf{r}_L$ then being $x$ and $y$. As indicated in Fig 4, for each waveguide the coupling point with the ring is located at $z = 0$ for that particular waveguide. We introduce the terms ‘input region’ and ‘output region’ to refer to the parts of the waveguide before and after the coupling point, respectively. The ‘input region’ is then defined by $z < 0$, and the ‘output region’ by $z > 0$. We introduce

$$\psi_J(z) = \int \frac{dk}{\sqrt{2\pi}} a_J(k) e^{ikz}, \quad (38)$$

and with the dispersion relation given in Eq. [6] the linear Hamiltonian for an isolated waveguide,

$$H_{Lw}^w = \sum_J \int d\omega J_k a_J(k) a_J(k), \quad (39)$$

can be written as

$$H_{Lw}^w = \sum_J \hbar \omega J_k \int \psi_J(z) \psi_J(z) dz \quad (40)$$

$$- \frac{1}{2} i\hbar v_J \int \left( \psi_J(z) \frac{\partial \psi_J(z)}{\partial z} - \frac{\partial \psi_J(z)}{\partial z} \psi_J(z) \right) dz,$$

provided the frequency bands are narrow enough that to good approximation we can put $\hbar \omega J_k \approx \hbar \omega_J$, and $d_{jk}(x,y) \approx d_J(x,y) \equiv d_{JK}(x,y)$; under this approximation, we also have from Eq. [2],

$$D_J(r) = \sqrt{\frac{\hbar \omega_J}{2}} d_J(x,y) e^{ikJz} \psi_J(z). \quad (41)$$

Using our earlier definition of $d_J(r_{\perp}; \zeta)$ in Eq. [33], the field in an isolated ring can be written as $[8]$

$$D(r) = \sum_J \sqrt{\frac{\hbar \omega_J}{2}} d_J(r_{\perp}; \zeta) b_J e^{iK_J \zeta} + H.c., \quad (42)$$

where

$$[b_J, b^\dagger_{J'}] = \delta_{JJ'}. \quad (43)$$

Here $\kappa_J = 2\pi m_J/L$, where $m_J$ is the index of the mode. The linear Hamiltonian for the ring modes is

$$H_{Lring}^w = \sum_J \hbar \omega_J b_J^\dagger b_J. \quad (44)$$

Finally, we treat the coupling between the ring and a waveguide by introducing different coupling coefficients associated with each mode of the isolated ring. Unlike in the point-coupling model described in Section III here we implicitly assume that the ring resonances are well separated so that these distinct modes $J$ are well-defined and can be identified. The coupling Hamiltonian is

$$H_{L}^{\text{coupling}} = \sum_J \left( \hbar \gamma_J b_J^\dagger \psi_J(0) + H.c. \right). \quad (45)$$

The coupling constant introduced in Eq. [45] can be related to the self-coupling constant $\sigma$ used in Section III by

$$\frac{|\gamma_J|^2}{2v_J} = \frac{(1 - \sigma)v_J}{L} \quad (46)$$

under the assumption that the ring and the waveguide are made of the same material, have the same cross-section, and thus in particular the same $v_J$; but this can be generalized. We emphasize that this Hamiltonian point-coupling model is valid only in the high-finesse limit, where the ring resonances are well-separated and can be treated as distinct modes; the phenomenological point-coupling model used in Section III is not constrained by this assumption, and can describe low-finesse structures. In practice, the high-finesse regime is usually of interest, and the Hamiltonian model is appropriate.

### B. Asymptotic fields

We begin with some preliminaries. As in the simpler scenario of Fig. 2(a), in the scenario of Fig. [3] there are twice as many channels as waveguides. For the direction of light propagation indicated, half of the channels will have asymptotic-in states of interest associated with them (those in the region of their waveguide with $z < 0$), and we call them “in-channels.” The other half will have asymptotic-out states of interest associated with them (those in the region of their waveguide with $z > 0$), and we call them “out-channels.” We denote the waveguide with which channel $X$ is associated by $X$.

Generalizing the approach used in Section III, for an asymptotic-in field associated with an in-channel $X$ we
can identify

\[ D^{\text{in}}_j(r) = D^{\text{wg}}_j(r) \]  
\[ D^{\text{in}}_j(r) = 0 \]  
\[ D^{\text{out}}_j(r) = D^{\text{wg}}_j(r) \]  
\[ D^{\text{out}}_j(r) = 0 \]

where \( D^{\text{in}}_j(r) \) is the asymptotic-in field amplitude, as introduced in Eq. (47). Similarly, for the asymptotic-out field associated with an out-channel \( X \), we can identify

\[ D^{\text{out}}_j(r) = D^{\text{wg}}_j(r) \]  
\[ D^{\text{out}}_j(r) = 0 \]

We can now outline the derivation of the asymptotic field amplitudes through the entire structure, using the Hamiltonian describing the system and the behavior in each region of the system, identified in Eqs. (47), (48), (49), (50). Because of the symmetry in the boundary conditions used in this approach, modeling a ring coupled to \( N \) waveguides is no more complicated than modeling a ring coupled to two waveguides, the latter being the simplest case one can consider while taking scattering loss into account.

With an arbitrary number of waveguides, the linear Hamiltonian is given by

\[ H_L = H^{\text{ring}}_L + \sum_X H^{\text{wg}}_L(X) + \sum_X H^{\text{coupling}}_L(X), \]  

where the prime on the sums indicates a sum over in-channels, resulting in a sum over waveguides, and the terms in the Hamiltonian are given by (40), (44), and (45) respectively for each waveguide; for the waveguide \( X \) associated with an in-channel \( X \) we introduce a set of operators \( \psi^{(X)}_j(z) \), group velocities \( v_j^{(X)} \), and coupling constants \( \gamma_j^{(X)} \).

Working in the Heisenberg picture with (51), and taking care that the resulting \( \psi^{(X)}_j(z,t) \) suffer a discontinuity across \( z = 0 \), for each \( \psi^{(X)}_j(z,t) \) we find that away from \( z = 0 \) we have

\[ \frac{\partial \psi^{(X)}_j(z,t)}{\partial t} + v_j^{(X)} \frac{\partial \psi^{(X)}_j(z,t)}{\partial z} + i\omega_j \psi^{(X)}_j(z,t) = 0, \]

and at the coupling point we have

\[ \psi^{(X)}_{j>}(0,t) = \psi^{(X)}_{j<}(0,t) - \frac{i\gamma_j^{(X)}}{v_j^{(X)}} b_j(t). \]

We also have

\[ \begin{pmatrix} d \frac{dt}{dt} + \Gamma_j + i\omega_j \end{pmatrix} b_j(t) = \sum_X -i \left( \frac{\gamma_j^{(X)}}{v_j^{(X)}} \right) \psi^{(X)}_{j<}(0,t), \]

with

\[ \Gamma_j = \sum_X \frac{\gamma_j^{(X)}}{v_j^{(X)}}, \]  
\[ \Gamma^{(X)}_j = \frac{\left| \gamma_j^{(X)} \right|^2}{2v_j^{(X)}}. \]

The \( \psi^{(X)}_{j<}(0,t) \) in Eqs. (54) and (55) are introduced to treat the discontinuity at the coupling point; \( \psi^{(X)}_{j<}(0,t) \) is the field \( \psi^{(X)}_j(z,t) \) for \( z < 0 \), extended to all \( z \) via (52). Likewise, \( \psi^{(X)}_{j>}(0,t) \) is the field \( \psi^{(X)}_j(z,t) \) for \( z > 0 \), extended to all \( z \) via (52).

For a general asymptotic-in field associated with in-channel \( X \), Eq. (47) requires that the field in the input region of channel \( X \) has the form

\[ \begin{aligned} D_j(r,t) &= \int dk \sqrt{\frac{\hbar\omega_j}{2}} d_j^{(X)}(x,y) \psi^{(X)}_{j<}(z,t)e^{iK_j^{(X)}z} \\ &+ \text{H.c.}, \end{aligned} \]

with

\[ \psi^{(X)}_{j<}(z,t) = \frac{a_j^{(X)}(k)}{\sqrt{2\pi}} e^{i(k-K_j^{(X)})z} e^{-i\omega_j t}, \]
we have the profile \( \psi_j^<(X)(z,t) \) under- 
stood, and we do not mark it explicitly with a label; the profile \( \mathbf{d}_j(X,y) \) is the waveguide profile \( \mathbf{d}_j(x,y) \) (see Eq. (41)) in waveguide \( \vec{X} \). Eq. (48) then requires that we have

\[
\psi_{j<}(Y \neq X)(z,t) = 0,
\]

so that the field amplitudes are zero in all other input channels. The fields in the output regions of each waveguide have a form similar to Eq. (56), where the dependence of \( \psi_{j>}(X)(z,t) \) on \( a_{j_{in}}^{(X)}(k) \) must be determined. Taking these and seeking a corresponding waveguide have a form similar to Eq. (56), where

\[
\begin{align*}
\tilde{b}_j(t) &= \tilde{b}_j \exp^{-i \omega_j k t}, \\
\psi_{j>}(X)(0,t) &= \frac{a_{j_{in}}^{(X)}(k)}{\sqrt{2 \pi}} e^{-i \omega_j k t} - \frac{i \gamma_j^{(X)}}{\nu_j} \tilde{b}_j \exp^{-i \omega_j k t}, \\
\psi_{j>}(Y \neq X)(0,t) &= -\frac{i \gamma_j^{(Y \neq X)}}{\nu_j} \tilde{b}_j e^{-i \omega_j k t}.
\end{align*}
\]

Carrying out the algebra given in Appendix [B] we find that for an in-channel \( X \) the associated asymptotic-in field amplitudes are given by

\[
D_{jk}^{in(X)}(r) = \sqrt{\frac{\hbar \omega_j}{4 \pi}} \mathbf{d}_j^{(X)}(r, y) e^{i k z}, \tag{62}
\]

where \( r \in \text{input region of waveguide } \vec{X} \),

\[
\begin{align*}
\tilde{b}_j(t) &= \tilde{b}_j \exp^{-i \omega_j k t}, \\
\psi_{j>}(X)(0,t) &= \frac{a_{j_{in}}^{(X)}(k)}{\sqrt{2 \pi}} e^{-i \omega_j k t} - \frac{i \gamma_j^{(X)}}{\nu_j} \tilde{b}_j \exp^{-i \omega_j k t}, \\
\psi_{j>}(Y \neq X)(0,t) &= -\frac{i \gamma_j^{(Y \neq X)}}{\nu_j} \tilde{b}_j e^{-i \omega_j k t}.
\end{align*}
\]

\[
\begin{align*}
\psi_{j<}(Y \neq X)(z,t) &= 0,
\end{align*}
\]

the complex field enhancement factors that arise in this resonant structure, linking the field in the ring to the input in asymptotic-in fields (see Eq. (62) above), and the field in the ring to the output in asymptotic-out fields (see Eq. (64) below).

To derive the asymptotic-out amplitudes, we proceed in the same manner, now applying the conditions outlined in Eqs. (49) and (50). Following the steps in Appendix [B] we find that for an out-channel \( X \) the associated asymptotic-out field amplitudes are given by

\[
D_{jk}^{out(X)}(r) = \sqrt{\frac{\hbar \omega_j}{4 \pi}} \mathbf{d}_j^{(X)}(x, y) e^{i k z}, \tag{64}
\]

where \( r \in \text{output region of waveguide } \vec{X} \),

\[
\begin{align*}
\psi_{j>}(X)(0,t) &= \frac{a_{j_{in}}^{(X)}(k)}{\sqrt{2 \pi}} e^{-i \omega_j k t} - \frac{i \gamma_j^{(X)}}{\nu_j} \tilde{b}_j \exp^{-i \omega_j k t}, \\
\psi_{j>}(Y \neq X)(0,t) &= -\frac{i \gamma_j^{(Y \neq X)}}{\nu_j} \tilde{b}_j e^{-i \omega_j k t}.
\end{align*}
\]

\[
\begin{align*}
\psi_{j<}(Y \neq X)(z,t) &= 0,
\end{align*}
\]

\[
\begin{align*}
\psi_{j<}(X)(z,t) &= 0,
\end{align*}
\]

\[
\begin{align*}
\psi_{j>}(X)(0,t) &= \frac{a_{j_{in}}^{(X)}(k)}{\sqrt{2 \pi}} e^{-i \omega_j k t} - \frac{i \gamma_j^{(X)}}{\nu_j} \tilde{b}_j \exp^{-i \omega_j k t}, \\
\psi_{j>}(Y \neq X)(0,t) &= -\frac{i \gamma_j^{(Y \neq X)}}{\nu_j} \tilde{b}_j e^{-i \omega_j k t}.
\end{align*}
\]

\[
\begin{align*}
\psi_{j<}(Y \neq X)(z,t) &= 0,
\end{align*}
\]

\[
\begin{align*}
\psi_{j<}(X)(z,t) &= 0,
\end{align*}
\]

the complex field enhancement factors that arise in this resonant structure, linking the field in the ring to the input in asymptotic-in fields (see Eq. (62) above), and the field in the ring to the output in asymptotic-out fields (see Eq. (64) below).

C. Spontaneous four-wave mixing

With the asymptotic-in and-out fields in hand, from Eq. (14), and restricting as usual the integral in Eq. (11) to range over the ring, we find

\[
M^{X,Y'}(k_1, k_2) = \frac{\hbar v_j^{(X_{in})}}{2 \pi} \sqrt{\omega_{S_{+}} - \omega_{P}} \int d \mathbf{r} \mathcal{L} \mathcal{P}_{P}^{(X)} \mathcal{L} \mathcal{P}_{P}^{(X')} \tag{65}
\]

where we have made the approximation \( \omega_p + \delta \omega_p = \omega_p \) in the denominator, we have introduced the nonlinear parameter

\[
\tau_{NL} = \frac{3 \omega_p}{4 \epsilon_0 v_j^{(in)}} \frac{2 \mathcal{L}}{\mathcal{P}_{P}^{(X)}} \int \mathbf{d} \mathbf{r} \mathcal{L} \mathcal{P}_{P}^{(X')} \mathcal{L} \mathcal{P}_{P}^{(X')} \mathcal{P}_{P}^{(X)} \mathcal{P}_{P}^{(X')}, \tag{66}
\]

\[
\begin{align*}
\mathcal{P}_{P}^{(X)}(r_{\perp}; \zeta) \mathcal{P}_{P}^{(X)}(r_{\perp}; \zeta) = \mathcal{P}_{P}^{(X)}(r_{\perp}; \zeta) \mathcal{P}_{P}^{(X)}(r_{\perp}; \zeta) e^{i \Delta k \zeta},
\end{align*}
\]

and we have introduced \( \Delta k = 2 k_p - k_S - k_I \), the wavenumber mismatch in the ring. Note that if the polarization of the profiles \( \mathbf{d}_P(r_{\perp}; \zeta) \) are everywhere "normal to the chip," and if \( (\Delta k) \mathcal{L} \ll 1 \) the latter can be expected if the signal, idler, and pump resonances are closely spaced – then we have \( \tau_{NL} \approx \gamma_{NL} \) (see Eq. (41)).
Putting Eq. (65) in (16), we find

\[ R_{XX'}^{XY} = \frac{\sqrt{\omega_S\omega_I}}{\omega_P} \left( \frac{v(P)}{v(P')} \right)^2 (\gamma_{NL}L)^2 \frac{1}{h\omega_P} P_P^2 P_{vac} \]

\[ \times |F_{P-}^{(X)} (K_P + \delta K_P)|^4 |F_{S+}^{(X)} (K_S)|^2 |F_{I+}^{(X')} (K_I)|^2, \]

where we have identified

\[ P_{vac} = \frac{\hbar}{2} \frac{\sqrt{\omega_S\omega_I}}{2(\omega_P + \delta\omega_P) - \omega_S - \omega_I} + (\Gamma_S + \Gamma_I)^2, \]

the fluctuating vacuum power \[2\]. As expected, this quantity is independent of the channels through which the photons exit; the vacuum power only relates to the generation of the photons in the resonator.

By inspecting Eq. (67) and recalling the definition of \( F_j^{(X)} (k) \) in Eq. (63), we see that the rate of generation of signal and idler photons in any pair of channels \( X \) and \( X' \) can be related to the rate of generation of signal and idler photons in another pair of channels \( Y \) and \( Y' \) by

\[ R_{XX'}^{XY} = \frac{\Gamma_S^{(X)} \Gamma_I^{(X')}}{\Gamma_S^{(Y)} \Gamma_I^{(Y')}}. \]

One can thus infer the rates in all sets of channels, given a 'reference rate' of photons in an arbitrary pair of channels, provided that the coupling constants between the channels and the ring are known. For example, one could take \( Y = Y' \) for the reference rate, so that it could be approximated by coincidence measurements in channel \( Y \).

If we consider an incident pump pulse sufficiently weak that at most a pair of photons are generated, we can calculate the biphoton wave function (the joint spectral amplitude (JSA)) that characterizes the pair; we again use the interaction Hamiltonian defined in Eq. (10), but in taking the limit of a classical pump we use \( a_P^{(X)} (k) \rightarrow \alpha \phi (k) \), where \( \alpha \) is the field amplitude and the pulse function \( \phi (k) \) is normalized according to

\[ \int |\phi (k)|^2 dk = 1. \]

Eq. (10), in the interaction picture, then becomes

\[ H_{SFWM}^{(I)} (t) = -\hbar^2 \frac{\omega_P}{4\pi^2} \left( \frac{v(P)}{v(P')} \right)^2 \sqrt{\omega_S\omega_I} \gamma_{NL}L \alpha^2 \sum_{XX'} \]

\[ \times \int dk_1 dk_2 dk_3 dk_4 e^{-i(\omega_P(k_4) + \omega_P(k_3) - \omega_I(k_2) - \omega_S(k_1))t} \]

\[ \times F_{S+}^{(X)} (k_1) F_{S+}^{(X')} (k_2) F_{P-}^{(X)} (k_3) F_{P-}^{(X')} (k_4) \phi (k_3) \]

\[ \times \phi (k_4) a_S^{(Y)} (k_1) a_P^{(Y')} (k_2) + H.c. \]

To first order, the ket that results after the pump pulse passes is

\[ |\Psi \rangle \approx |\text{vac} \rangle - \frac{i}{\hbar} \int_{-\infty}^{\infty} dt' H_{SFWM}^{(I)} (t') |\text{vac} \rangle + \ldots \]

\[ = |\text{vac} \rangle + \beta \sum_{XX'} |XX' \rangle + \ldots \]

(see Appendix A), with

\[ |XX' \rangle = \int dk_1 dk_2 \phi^{XX'} (k_1, k_2) \]

\[ \times a_S^{(X)} (k_1) a_P^{(X')} (k_2) |\text{vac} \rangle, \]

where again the first superscript \( X \) referring to the signal output channel and the second superscript \( X' \) to the idler output channel; \( \phi^{XX'} (k_1, k_2) \) is a biphoton wave function, normalized according to

\[ \sum_{XX'} \int dk_1 dk_2 |\phi^{XX'} (k_1, k_2)|^2 = 1. \]

We assume that \( \beta \ll 1 \) such that higher order terms are negligible, and so \( |\beta|^2 \) is approximately the probability that a pair of photons is generated. We have

\[ |\beta|^2 = \frac{\hbar^2}{4\pi^2 \omega_S\omega_I} v(P)^4 \frac{\gamma_{NL}L^2}{2} \]

\[ \times \sum_{XX'} \int dk_1 dk_2 |F_{S+}^{(X)} (k_1)|^2 |F_{I+}^{(X')} (k_2)|^2 |g(k_1, k_2)|^2, \]

and

\[ \phi^{XX'} (k_1, k_2) = \frac{\alpha^2 \hbar^2}{\beta} \sqrt{\omega_S\omega_I} \left( \frac{v(P)}{v(P')} \right)^2 \gamma_{NL} \]

\[ \times F_{S+}^{(X)} (k_1) F_{S+}^{(X')} (k_2) g(k_1, k_2), \]

where

\[ g(k_1, k_2) = \int dk_3 dk_4 F_{P-}^{(X)} (k_3) F_{P-}^{(X')} (k_4) \phi (k_3) \phi (k_4) \]

\[ \times \delta (\omega_P (k_4) + \omega_P (k_3) - \omega_I (k_2) - \omega_S (k_1)), \]

is determined by the shape of the pump pulse.

Using Eq. (77), we can relate the joint spectral amplitude associated with a pair of channels \( X, X' \) to the JSA associated with another pair of channels \( Y, Y' \); recognizing that the field enhancement factor in Eq. (63) can be rewritten using

\[ v_j^{(X)} (K_j - k) = \omega_j - \omega(k), \]

we can see that

\[ \phi^{XX'} (k_1, k_2) = \frac{\gamma_{NL}L^2}{\gamma_{NL}L^2} \phi^{YY'} (k_1, k_2). \]

Thus one could again define a 'reference' pair of channels for which the biphoton wave function is known, and
extract all the other biphoton wave functions from this and the coupling constants \( \gamma_j(x) \); the biphoton wave function associated with lost pairs and broken pairs can be inferred from the JSA associated with photons pairs in a particular channel, a more accessible quantity.

From Eq. \([80]\) we see that the shape of the biphoton wave function associated with each pair of channels is the same; this in turn implies that for this type of system, scattering has no effect on the system’s full biphoton wavefunction beyond its contribution to the total linewidth \( \bar{\Gamma} \). This is easily understood: since the nonlinear effects are confined to the ring, and the ring-channel coupling is frequency independent, the spectral properties of the photon pairs do not depend on which channels they couple into. The amplitude associated with each pair of channels depends on the coupling constants \( \gamma_j(x) \), as expected.

### D. Ring-channel system

We now apply these general results to the simple case of a lossy ring coupled to a bus waveguide. Since we model the ring’s scattering loss as a coupling to a phantom waveguide, here the labels \( X \) and \( X' \) range over two channels \([7]\).

We use the label \( O \) to denote the out-channel of the bus waveguide, and \( P \) to denote the out-channel of the phantom waveguide, as indicated in Fig. 5(a); photons that exit via channel \( O \) can be detected, whereas photons that exit via channel \( P \) are scattered and lost. We then have four scenarios, each with an associated rate: both photons can appear at the output (\( R^{OO} \), the signal can appear at the output while the idler is lost (\( R^{OP} \)), the idler can appear at the output while the signal is lost (\( R^{PO} \)), or both photons can be lost (\( R^{PP} \)). We predict the rate of pairs arriving at the output to be

\[
R^{OO} = \frac{\sqrt{\omega_S \omega_I}}{\omega_P} \left( \frac{v_s(x_{in})}{v_s(x)} \right)^2 (\gamma_{NL} \mathcal{L})^2 \frac{1}{\hbar \omega_P} P_P^2 P_{vac} \tag{81}
\]

\[
\times |F_{P-}^{(X_{in})}(K_P + \delta K_P)|^4 |F_S^{(O)}(K_S)|^2 |F_{I+}^{(O)}(K_I)|^2,
\]

Using Eq. \([69]\) we have

\[
R^{OP} = \frac{\Gamma^{(P)}}{\Gamma^{(O)}} R^{OO}, \tag{82}
\]

\[
R^{PO} = \frac{\Gamma^{(P)}}{\Gamma^{(O)}} R^{OO},
\]

\[
R^{PP} = \frac{\Gamma^{(P)}}{\Gamma^{(O)}} \frac{\Gamma^{(P)}}{\Gamma^{(O)}} R^{OO},
\]

\[
= \frac{\Gamma^{(P)}}{\Gamma^{(O)}} \frac{\Gamma^{(P)}}{\Gamma^{(O)}} R^{OO}.
\]

Using Eq. \([69]\) we have

\[
R^{OP} = \frac{1 - \eta_I}{\eta_I} R^{OO}, \tag{83}
\]

\[
R^{PO} = \frac{1 - \eta_S}{\eta_S} R^{OO},
\]

\[
R^{PP} = \frac{1 - \eta_S}{\eta_S} \left( \frac{1 - \eta_I}{\eta_I} \right) R^{OO}.
\]

In both forms, one can see that these rates are equal at critical coupling, where \( \Gamma^{(P)} = \Gamma^{(O)} \), so \( \eta_J = 0.5 \).

We now provide a sample calculation, assuming system parameters that are compatible with current fabrication technology, and consistent with those used in Section III. We assume the pump to be on resonance, and we assume that \( 2\omega_P \approx (\omega_S + \omega_I) \). Then we have

\[
R^{OO} = \frac{\sqrt{\omega_S \omega_I}}{\omega_P} \left( \frac{v_s(x_{in})}{v_s(x)} \right)^2 (\gamma_{NL} \mathcal{L})^2 \frac{1}{\hbar \omega_P} P_P^2 P_{vac} \]

\[
\times |F_{P-}^{(X_{in})}(K_P)|^4 |F_S^{(O)}(K_S)|^2 |F_{I+}^{(O)}(K_I)|^2, \tag{84}
\]

\[
= \frac{\Gamma^{(P)}}{\Gamma^{(O)}} \frac{\Gamma^{(P)}}{\Gamma^{(O)}} R^{OO},
\]

\[
= \frac{\Gamma^{(P)}}{\Gamma^{(O)}} \frac{\Gamma^{(P)}}{\Gamma^{(O)}} R^{OO},
\]

\[
= \frac{\Gamma^{(P)}}{\Gamma^{(O)}} \frac{\Gamma^{(P)}}{\Gamma^{(O)}} R^{OO}.
\]
The rates are equal at the critical coupling point. As each pair of channels as the ring-channel coupling is in-
at the bus waveguide's output. \(|\omega_S(1-\eta_T)Q_S(\text{int}) + \omega_I(1-\eta_S)Q_I(\text{int})|\),

where in Eq. (85) we write the vacuum power in terms of the ring’s intrinsic quality factors; we have used \(Q_I(\text{load}) = \omega_I/2\Gamma_I\). For example, with \(Q_I(\text{int}) \approx Q_S(\text{int}) = 2 \times 10^4\), \(\eta_T \approx \eta_S = 0.5\), and \(\lambda_I \approx \lambda_S = 1550\text{ nm}\), we have \(P_{\text{vac}} = 1.9\text{ nW}\).

For the field enhancement factors we have

\[
|F_{J\pm}^{(X)}(K_J)|^2 = \frac{1}{\mathcal{C}} \frac{|\gamma_J^{(X)}|^2}{\Gamma_J} = \frac{2\nu_J^{(X)}\eta_J}{\mathcal{C} \Gamma_J} = \frac{4\nu_J^{(X)}(1-\eta_J)\eta_J Q_I(\text{int})}{2\pi R \omega_J},
\]

where \(\mathcal{R}\) is the radius of the ring. Taking \(\nu_J^{(X)} = 10^8\) m/s and \(\mathcal{R} = 10\mu\text{m}\), we have \(|F_{J\pm}^{(X)}(K_J)|^2 = 26.2\). We assume that the resonances in question are spectrally close, such that these parameters are representative of each resonance, so that \(|F_{J\pm}^{(X)}(K_P)|^2 \approx |F_{S\pm}^{(X)}(K_S)|^2 \approx |F_{I\pm}^{(X)}(K_I)|^2 \approx 26.2\). Taking \(\gamma_{NL} = \gamma_{NL} = 100\text{ (Wm)}^{-1}\) and \(P_P = 1\text{ mW}\), we estimate \(1.08 \times 10^4\) pairs per second at the bus waveguide's output.

In Fig. 5, we plot the rate of photon pairs exiting from each pair of channels as the ring-channel coupling is increased, assuming the system parameters listed above. The rates are equal at the critical coupling point. As noted in Section IV, the rate of photon pairs at the output is maximized with the system slightly overcoupled (\(\eta = 0.6\)); here we see that the other rates decrease as the coupling is increased past the critical coupling point. A slightly overcoupled system is thus favourable in two ways: first, the rate of pairs at the output is maximized, and second, the ratio of pairs to broken pairs is larger.

In Fig. 7 we compare these analytic results to numerical results obtained as described in Section III. As expected, the two methods agree well when the ring’s finesse is sufficiently large. When the ring’s finesse is small, the resonances are no longer well separated which invalidates the assumptions upon which the analytic approach of this section is built. Of course, this high finesse regime is preferred in practice, because it enables higher pair generation rates.

Finally, in Fig. 8 we plot the JSA for the ring-channel system. We note that the JSA components associated with different pairs of channels differ only by a constant amplitude and a global phase, as discussed in Section IV.C. Even when the terms associated with broken and lost pairs are included in the full biphoton wavefunction, its shape has the form familiar from earlier treatments and experiments which address only the photon pairs at the output [13, 14].

V. SAMPLE CALCULATION: ADD-DROP SYSTEM

We now use the two strategies to model pair generation via SFWM in an add-drop system, sketched in Fig. 5(b). We take the input fields to be entering by the channel labeled ‘In’, although the channel labeled ‘Add’ could also be used as an input channel. The generated photons can couple from the ring into the ‘Through’ channel (\(T\)), the ‘Drop’ channel (\(D\)), or they can be scattered; in the second strategy, this is modelled as a coupling into the
FIG. 8. Plots of the modulus squared and phase of the biphoton JSA for each configuration of photon pairs, plotted in terms of dimensionless variables $\kappa_{1(2)} = v_{S(I)}(k_{1(2)} - K_{S(I)})/\Gamma_{S(I)}$. As indicated in Eq. (80), the JSAs corresponding to different pairs of channels are identical up to a constant factor. These constant factors simply amount to a rescaling, as indicated on the plots’ colourbars. Here we assume a 10 ps Gaussian pulse, and we have set $\eta = 0$. The ring parameters are consistent with those used in Fig. 3.

There are thus nine trajectories that the generated pairs can take; the first strategy can describe the four trajectories that involve neither photon being scattered, while the second strategy can describe all of them. As an example, we use Eq. (67) to evaluate the rates for these nine scenarios with an increasing coupling to the drop channel, starting from the ring-channel limit where $\Gamma_{j(D)}^{(D)} = 0$. We fix the through channel coupling such that $\Gamma_{j(T)}^{(T)} = 1.5\Gamma_{j(P)}^{(P)}$; the results are plotted in Fig. 9.

The behaviour shown in Fig. 9 aligns well with intuition: the increased coupling to the drop channel is initially accompanied by increasing rates of photons coupling into the drop channel. As the coupling is increased past a critical point, the rates decrease; increasing the coupling to the drop channel increases the total linewidth of the resonator. This results in a lower field enhancement, and the total pair generation rate falls. We again compare the results of the first and second strategies in Fig. 10 and find that the second strategy is valid provided the ring’s finesse is sufficiently large.

We can also consider the richer scenario in which both $\Gamma_{j(T)}^{(T)}$ and $\Gamma_{j(D)}^{(D)}$ are varied to yield a particular distribution of photon pairs among the possible channels. The dependence of the rates on these two parameters is not obvious; on top of the trade-off between coupling and field enhancement seen in Fig. 9 we now have a variable coupling to two channels which affect the rates in different ways, since only the through channel carries the pump field. In Fig. 11 we plot the rates associated with a subset of the configurations; the full set of plots can be

FIG. 9. Rates associated with each photon pair trajectory with increasing drop channel coupling. Since the signal and idler frequencies are similar, we have $\Gamma_{S(I)}^{(X)} \approx \Gamma_{j(S)}^{(X)}$ and $R_{XX}^{(X)} = R_{XX}^{(X)}$, so three of the nine rates are omitted from the legend and can be inferred from the others.

FIG. 10. Pair generation rate using the first strategy (black solid line) and the second strategy (red dashed line) for an add-drop system, as a function of resonator finesse. The parameters summarized in Fig. 3 are again used here. We fix the coupling between the ring and the in/through channel ($\sigma_1 = 0.9814$) and vary the coupling with the add/drop channel ($\sigma_2 \in [0.3, 1]$). Here the pair generation rate refers to the number of photon pairs per unit time collected in the through channel.
It is straightforward to plot the JSA for the add-drop system; we omit the JSA here since it is qualitatively identical to the ring-channel JSA given in Fig. 8, the two systems differ only in the coupling constants $\gamma_{J}^{(X)}$, which affects the scaling, and in their full linewidths $\bar{\Gamma}_{J}$, which affects the shape of the JSA in a trivial way.

VI. CONCLUSION

We have developed two strategies for realistic modeling of integrated photonic systems under the influence of scattering loss. Our first strategy is easily adapted to different geometries, even in the low-finesse limit, but it precludes a description of the scattered photons; our second strategy can fill this gap for high-finesse structures coupled to an arbitrary number of channels.

The first strategy is sufficient for describing photon pairs which couple into physical, experimentally accessible channels. Our second strategy shows us that for a high-finesse ring coupled to an arbitrary number of channels, the properties of the pairs which are scattered or broken can be related to the properties of the ‘experimentally accessible’ pairs. These theoretical results can be applied both in the interpretation of experimental results in realistic systems, and in the design of new systems.

Extensions focusing on generalizing the second strategy to other systems, particularly those where the coupling cannot be taken to be frequency independent, should provide insight to the effect of loss in less trivial systems, and may expose opportunities for engineering the linear system dynamics to provide a useful distribution of entangled photons among multiple channels.

ACKNOWLEDGMENTS

M.B. acknowledges support from the University of Toronto Faculty of Arts & Science Top Doctoral Fellowship. J.E.S. and M.B. acknowledge support from the Natural Sciences and Engineering Research Council of Canada. M.L. and L.Z. acknowledge support by Ministero dell’Istruzione, dell’ Università e della Ricerca (Dipartimenti di Eccellenza Program (2018–2022)).

FIG. 11. The dependence of pair rates on the through and drop channel coupling ($\Gamma^{(T)}$ and $\Gamma^{(D)}$, respectively). Of the nine configurations, here we plot the subset in which the signal photon exits via the drop channel; the idler photon couples into the drop, through, and phantom channel in panels (a), (b), and (c) respectively.
Appendix A: Technical details

Here we present some technical details. We begin with Eq. (12). It follows from noting first that in the asymptotic fields the form of the incident field for channel $X_{in}$ is the same as it would be for an infinite channel, and so we can identify $\alpha_P(k)$ by considering an infinite channel where the energy would be

$$H = \int \frac{d\omega P_k}{\sqrt{2\pi}} \alpha_P^*(k) \alpha_P(k),$$  \hspace{1cm} (A1)

where we take the integral over all $k$, although for our applications $\alpha_P(k)$ will be nonzero only for positive $k$. Moving to position space and taking

$$\phi_P(z) = \int \frac{dk}{\sqrt{2\pi}} \alpha_P(k) e^{i(k-k_o)z},$$  \hspace{1cm} (A2)

for a wave packet with components $k$ only very near $k_o$ we have

$$H \approx \hbar \omega_o \int dk \alpha_P^*(k) \alpha_P(k)$$

$$= \hbar \omega_o \int dz \phi_P^*(z) \phi_P(z)$$

Thus the local energy density is $\hbar \omega_o \phi_P^*(z) \phi_P(z)$, and since the group velocity is $v_P$ the local power is $\hbar \omega_o v_P |\phi_P(z)|^2 \phi_P(z)$. To set this equal to $P_P$ we see from Eq. (A2) that we should indeed set $\alpha_P(k)$ according to Eq. (12).

We next turn to the derivation of Eq. (16). We begin with Schrödinger’s equation in the interaction picture,

$$i\hbar \frac{d}{dt} \langle \Psi(t) \rangle = H^{(I)}_{SFWM}(t) |\Psi(t)\rangle,$$  \hspace{1cm} (A4)

where as usual $H^{(I)}_{SFWM}(t) = e^{iH_{SFWM}t/\hbar} H^{SFWM} e^{-iH_{SFWM}t/\hbar}$ and $H_0$ is the full linear Hamiltonian. If we assume that the classical pump field is on from $-T/2$ to $T/2$, and that the initial state has no signal or idler photons, the iterative solution of Eq. (A4) is

$$|\Psi(t > \frac{T}{2})\rangle = |\text{vac}\rangle - \frac{i}{\hbar} \int_{-\frac{T}{2}}^{\frac{T}{2}} H^{(I)}_{SFWM}(t) |\text{vac}\rangle dt + ...$$  \hspace{1cm} (A5)

The second term in this iteration is a two-photon state, and so to first order, the probability of generating a photon pair is

$$\mathcal{P} = \frac{1}{\hbar^2} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_{-\frac{T}{2}}^{\frac{T}{2}} dt' \langle \text{vac} | H^{(I)}_{SFWM}(t') H^{(I)}_{SFWM}(t) |\text{vac}\rangle,$$  \hspace{1cm} (A6)

and from Eq. (14) we have

$$\langle \text{vac} | H^{(I)}_{SFWM}(t') H^{(I)}_{SFWM}(t) |\text{vac}\rangle = \sum_{X,X'} \int dk_1 dk_2 |M^{XX'}(k_1,k_2)|^2 e^{-i\Omega(k_1,k_2)(t-t')} ,$$  \hspace{1cm} (A7)

where we have used $[a_{j}^{(X)}(k), a_{j'}^{(Y)}(k')] = \delta_{X,Y} \delta(k-k')$, and $\Omega(k_1, k_2)$ is given by Eq. (15). Putting this into Eq. (A6) and evaluating the integrals over time, we have

$$\mathcal{P} = \frac{1}{\hbar^2} \sum_{X,X'} \int dk_1 dk_2 \frac{4 \sin^2(\Omega(k_1,k_2)T/2)}{\Omega^2(k_1,k_2)} |M^{XX'}(k_1,k_2)|^2.$$  \hspace{1cm} (A8)

We then assume that $T$ is sufficiently long that we can use
\[
\frac{4\sin^2(\Omega(k_1, k_2)T/2)}{\Omega^2(k_1, k_2)} \to 2\pi T\delta(\Omega(k_1, k_2)), \tag{A9}
\]
so that
\[
P = \frac{2\pi T}{\hbar^2} \sum_{X, X'} \int dk_1dk_2\delta(\Omega(k_1, k_2))|M^{XX'}(k_1, k_2)|^2. \tag{A10}
\]
The total pair generation rate is then
\[
R = \frac{P}{T} = \frac{2\pi}{\hbar^2} \sum_{X, X'} \int dk_1dk_2\delta(\Omega(k_1, k_2))|M^{XX'}(k_1, k_2)|^2 = \sum_{X, X'} R^{XX'}, \tag{A11}
\]
where the rate of pairs coupling into each set of channels \(X, X'\) is given by
\[
R^{XX'} = \frac{2\pi}{\hbar^2} \int dk_1dk_2\delta(\Omega(k_1, k_2))|M^{XX'}(k_1, k_2)|^2, \tag{A12}
\]
which is Eq. \[16\].
Appendix B: Deriving asymptotic field amplitudes

1. Asymptotic-in

From Eq. \(59\), we have

\[
(i(\omega - \omega_{jk}) + \Gamma_j) \hat{b}_j = -i \left( \gamma_j^{(X)} \right)^* \frac{a_j^{in(X)}(k)}{\sqrt{2\pi}}, \tag{B1}
\]

\[
\hat{b}_j = \left( \frac{-i \left( \gamma_j^{(X)} \right)^*}{i(\omega - \omega_{jk}) + \Gamma_j} \right) \frac{a_j^{asy-in(X)}(k)}{\sqrt{2\pi}} \tag{B2}
\]

\[
= - \sqrt{\frac{E}{2\pi}} \frac{1}{\sqrt{L}} \left( \frac{\left( \gamma_j^{(X)} \right)^*}{\omega - \omega_{jk} - i\Gamma_j} \right) a_j^{in(X)}(k) \tag{B3}
\]

\[
= - \sqrt{\frac{E}{2\pi}} \frac{1}{\sqrt{L}} \left( \frac{\left( \gamma_j^{(X)} \right)^*}{v_j \left( \dot{K}_j^{(X)} - k \right) - i\Gamma_j} \right) a_j^{in(X)}(k), \tag{B4}
\]

\[
\hat{b}_j = - \sqrt{\frac{E}{2\pi}} F_{j-}^{(X)}(k) a_j^{in(X)}(k), \tag{B5}
\]

with the field enhancement factor \(F_j^{(X)}(k)\) defined in Eq. \(63\). In Eq. \(B4\), we have neglected the group velocity dispersion across the resonance; for this to be valid, the group velocity dispersion \(\beta_2\) must be small enough that \(\frac{1}{v_j (\omega - \omega)} \gg \beta_2 (\omega - \omega)^2\). For example, for a 1 GHz resonance linewidth, this would be a good approximation for \(\beta_2 < 10^{-20} s^2/m\).

Putting this into Eqs. \(60\) and \(59\), we have

\[
\tilde{\psi}_j^{(X)}(0, t) = \frac{a_j^{in(X)}(k)}{\sqrt{2\pi}} e^{-i\omega_{jk}t} + \frac{i \gamma_j^{(X)}}{v_j^{(X)}} \sqrt{\frac{E}{2\pi}} F_{j-}^{(X)}(k) a_j^{in(X)}(k) e^{-i\omega_{jk}t}, \tag{B6}
\]

\[
\tilde{\psi}_j^{(X)}(0, t) = \left( 1 + \frac{i \gamma_j^{(X)}}{v_j^{(X)}} \sqrt{\frac{E}{2\pi}} F_{j-}^{(X)}(k) \right) \frac{a_j^{in(X)}(k)}{\sqrt{2\pi}} e^{-i\omega_{jk}t}, \tag{B7}
\]

and

\[
\tilde{\psi}_j^{(Y \neq X)}(0, t) = \frac{i \gamma_j^{(Y \neq X)}}{v_j^{(Y \neq X)}} \sqrt{\frac{E}{2\pi}} F_{j-}^{(X)}(k) \frac{a_j^{in(X)}(k)}{\sqrt{2\pi}} e^{-i\omega_{jk}t}. \tag{B8}
\]

From Eq. \(52\) we have \(\tilde{\psi}_j^{(Y)}(z, t) = \tilde{\psi}_j^{(X)}(0, t) e^{i(k - \dot{K}_j^{(Y)})z}\) for all waveguides \(Y\), including \(\bar{X}\).

We now consider separately each \(k\) component of an asymptotic-in field mode \(J\). That is, we introduce \(D_j^{in(X)}(r, t)\) such that

\[
D_j^{in(X)}(r, t) = \sum_X \int dk \left( D^{in(X)}_{jk}(r, t) + H.c. \right). \tag{B9}
\]

The component \(D_j^{in(X)}(r, t)\) is a piecewise function with the form

\[
D_j^{in(X)}(r, t) = \sqrt{\frac{\hbar \omega_j}{2}} d_j^{(X)}(r, t) \tilde{\psi}_j^{(X)}(z, t) e^{iK_j^{(Y)}z}, \tag{B10}
\]

\[
r \in \text{input (output) region of all waveguides} \ Y, \tag{B11}
\]

\[
= \sqrt{\frac{\hbar \omega_j}{2}} d_j(r_x; \zeta) e^{-i\omega_{jk}t} e^{iK_j^{(Y)}z} \tag{B11}
\]

\[
r \in \text{ring.}
\]
Using Eqs. (57), (B5), (B7), and (B8), we have

\[ D_{jk}^{in(X)}(r, t) = \sqrt{\frac{\hbar \omega_j}{4\pi}} d_j^{(X)}(x, y) e^{ik_z a_j^{in(X)}(k)} e^{-i\omega_j k t} \]  \hspace{1cm} (B12)

where \( r \) ∈ input region of waveguide \( \tilde{X} \),

\[ D_{jk}^{out(X)}(r, t) = \sqrt{\frac{\hbar \omega_j}{4\pi}} d_j^{(Y)}(x, y) \left( 1 + \frac{i\gamma_j^{(X)}}{v_j^{(X)}} \sqrt{\mathcal{E}} F_j^{(X)}(k) \right) e^{ik_z a_j^{in(X)}(k)} e^{-i\omega_j k t} \]  \hspace{1cm} (B13)

where \( r \) ∈ output region of waveguide \( \tilde{X} \),

\[ D_{jk}^{out(X)}(r, \tilde{X}) = -\sqrt{\frac{\hbar \omega_j}{4\pi}} d_j^{(Y)}(r, \tilde{X}) e^{i\gamma_j^{(X)}} e^{ik_z a_j^{in(X)}(k)} e^{-i\omega_j k t} \]  \hspace{1cm} (B14)

where \( r \) ∈ output region of all other waveguides \( \tilde{Y} \neq \tilde{X} \),

\[ D_{jk}^{in(X)}(r, \text{ring}) = 0 \]  \hspace{1cm} (B15)

Finally, recalling the general form of the asymptotic-in field in Eq. (7), one has

\[ D_{jk}^{in(X)}(r, t) = D_{jk}^{in(X)}(r) a_j^{in(X)}(k) e^{-i\omega_j k t}, \]  \hspace{1cm} (B16)

and the asymptotic-in field amplitudes listed in Eq. (62) can be read directly off of Eqs. (B12), (B13), (B14), and (B15).

### 2. Asymptotic-out

We begin by imposing the appropriate asymptotic behaviour, namely that for an asymptotic-out field associated with an out-channel \( X \), we have

\[ \psi_j^{(X)}(z, t) = \frac{a_j^{out(X)}(k)}{\sqrt{2\pi}} e^{i(k - K_j^{(X)}) z} e^{-i\omega_j k t}, \]  \hspace{1cm} (B17)

\[ \psi_j^{(Y \neq X)}(z, t) = 0. \]  \hspace{1cm} (B18)

This ensures that the outgoing displacement field in out-channel \( X \) has the form of a field propagating through an isolated waveguide, and that this is the system’s only outgoing field. These boundary conditions along with Eqs. (53) and (54) give

\[ \frac{a_j^{out(X)}(k)}{\sqrt{2\pi}} e^{-i\omega_j k t} = \psi_j^{(X)}(0, t) - \frac{i\gamma_j^{(X)}}{v_j^{(X)}} \tilde{b}_j(t) \]  \hspace{1cm} (B19)

\[ 0 = \psi_j^{(Y \neq X)}(0, t) - \frac{i\gamma_j^{(Y \neq X)}}{v_j^{(Y \neq X)}} \tilde{b}_j(t) \]  \hspace{1cm} (B20)

\[ \left( \frac{d}{dt} + \Gamma_j + i\omega_j \right) \tilde{b}_j(t) = -i \left( \gamma_j^{(X)} \right)^*_j \psi_j^{(X)}(0, t) - i \sum_{Y \neq X} \left( \gamma_j^{(Y \neq X)} \right)^*_j \psi_j^{(Y \neq X)}(0, t), \]  \hspace{1cm} (B21)

Putting \( \tilde{b}_j(t) = \tilde{b}_j e^{-i\omega_j k t} \) and rearranging Eqs. (B19) and (B20), we have

\[ \psi_j^{(X)}(0, t) = \frac{a_j^{out(X)}(k)}{\sqrt{2\pi}} e^{-i\omega_j k t} + \frac{i\gamma_j^{(X)}}{v_j^{(X)}} \tilde{b}_j e^{-i\omega_j k t} \]  \hspace{1cm} (B22)

\[ \psi_j^{(Y \neq X)}(0, t) = \frac{i\gamma_j^{(Y \neq X)}}{v_j^{(Y \neq X)}} \tilde{b}_j e^{-i\omega_j k t}. \]  \hspace{1cm} (B23)
Putting these into Eq. (B21) and using $\Gamma_j^{(Y)} = \frac{|\gamma_j^{(Y)}|^2}{2e_j^{(Y)}}$, we have

\begin{equation}
-i(\omega_{jk} - \omega_j) + \Gamma_j \hat{b}_j e^{-i\omega_{jk}t} = -i \left( \gamma_j^{(X)} \right)^* \frac{a_j^{\text{out}(X)}(k)}{\sqrt{2\pi}} e^{-i\omega_{jk}t} + 2\Gamma_j^{(X)} \hat{b}_j e^{-i\omega_{jk}t} + \sum_{Y \neq X} 2\Gamma_j^{(Y)} \hat{b}_j e^{-i\omega_{jk}t}, \tag{B24}
\end{equation}

and since $\Gamma_j = \sum_X \Gamma_j^{(X)}$, this rearranges to

\begin{equation}
-i(\omega_{jk} - \omega_j) - \Gamma_j \hat{b}_j = -i \left( \gamma_j^{(X)} \right)^* \frac{a_j^{\text{out}(X)}(k)}{\sqrt{2\pi}}, \tag{B25}
\end{equation}

so

\begin{equation}
\hat{b}_j = \left( \frac{-i \left( \gamma_j^{(X)} \right)^*}{i(\omega_{jk} - \omega_j) - \Gamma_j} \right) \frac{a_j^{\text{out}(X)}(k)}{\sqrt{2\pi}}, \tag{B26}
\end{equation}

\begin{equation}
= -\sqrt{\frac{\mathcal{L}}{2\pi \sqrt{\mathcal{E}}}} \left( \frac{\left( \gamma_j^{(X)} \right)^*}{v_j \left( K_j^{(X)} - k \right) + i\Gamma_j} \right) a_j^{\text{out}(X)}(k), \tag{B27}
\end{equation}

\begin{equation}
\hat{b}_j = -\sqrt{\frac{\mathcal{L}}{2\pi \sqrt{\mathcal{E}}}} K_j^{(X)}(k) a_j^{\text{out}(X)}(k). \tag{B28}
\end{equation}

Using this in Eqs. (B22) and (B23), we have

\begin{equation}
\tilde{\psi}_{J<}^{(X)}(0,t) = \left( 1 - \frac{i\gamma_j^{(X)}}{v_j^{(X)}} \sqrt{\mathcal{L}} F_j^{(X)}(k) \right) \frac{a_j^{\text{out}(X)}(k)}{\sqrt{2\pi}} e^{-i\omega_{jk}t} \tag{B29}
\end{equation}

\begin{equation}
\tilde{\psi}_{J<}^{(Y \neq X)}(0,t) = -\frac{i\gamma_j^{(Y \neq X)}}{v_j^{(Y \neq X)}} \sqrt{\mathcal{L}} F_j^{(X)}(k) \frac{a_j^{\text{out}(X)}(k)}{\sqrt{2\pi}} e^{-i\omega_{jk}t}. \tag{B30}
\end{equation}

From here the derivation of the field amplitudes follows the asymptotic-in case; one introduces the components $\mathcal{P}_j^{\text{out}(X)}(r,t)$ of the full asymptotic-out field, and finds that the amplitudes given in Eq. (64) follow from Eqs. (B29), (B30), (B28), and (B17).
Appendix C: Supplementary figures

FIG. 12. Rate of signal and idler photons in channels $X$ and $X'$ respectively for the add-drop system, with variable through and drop channel coupling ($\Gamma^{(T)}$ and $\Gamma^{(D)}$) for a) $X = D$, $X' = D$; b) $X = D$, $X' = T$ or $X = T$, $X' = D$; c) $X = D$, $X' = P$ or $X = P$, $X' = D$; d) $X = T$, $X' = T$; e) $X = P$, $X' = P$; f) $X = P$, $X' = T$ or $X = T$, $X' = P$. 
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