THE COMPARATIVE STATICS OF PERSUASION*

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Abstract

In the canonical persuasion model, comparative statics has been an open question. We answer it, delineating which shifts of the sender’s interim payoff lead her optimally to choose a more informative signal. Our first theorem identifies a coarse notion of ‘increased convexity’ that we show characterises those shifts of the sender’s interim payoff that lead her optimally to choose no less informative signals. To strengthen this conclusion to ‘more informative’ requires further assumptions: our second theorem identifies the necessary and sufficient condition on the sender’s interim payoff, which strictly generalises the ‘S’ shape commonly imposed in the literature. We identify conditions under which increased alignment of interests between sender and receiver lead to comparative statics, and study applications.

1 Introduction

The persuasion model of Kamenica and Gentzkow (2011) is by now canonical, and yet has proved difficult to solve. Little is known about the qualitative

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properties of optimal signals beyond very special cases, such as when the sender’s interim payoff is ‘S’-shaped or the state is binary.

In this paper, we advance our understanding of optimal signals in the canonical persuasion model by changing the question: rather than ask what optimal signals look like, we ask how they vary with economic primitives. Concretely, we pose and answer the comparative-statics question: what shifts of model primitives, specifically of the sender’s interim payoff, lead her optimally to choose a more informative signal?

Recall that the persuasion model features an uncertain state of the world, whose distribution is called the prior, and a character called the sender. The sender flexibly designs what will and won’t be revealed about the state, by choosing a signal. The model’s primitives are the prior and the sender’s interim payoff function, which maps each posterior belief into an expected payoff. (This interim payoff is a reduced-form description of a downstream interaction, typically involving other players called receivers.)

Motivated by applications, we focus on the case in which the sender’s interim payoff depends on only one moment of the posterior belief—without loss, the mean. This assumption is maintained in much of the recent literature, and yields sharp results.

Our first theorem shows that a coarse notion of ‘less convex than’ characterises ‘non-decreasing’ comparative statics: coarsely more convex interim payoffs are exactly those that lead no less informative signals to be chosen by the sender, whatever the prior.

Our main theorem characterises what more is needed to obtain increasing comparative statics: it identifies a property of interim payoffs that is necessary and sufficient for coarse-convexity shifts to cause more informative signals to be chosen, whatever the prior. This property, the crater property, is a simple geometric condition that strictly generalises the ‘S’ shape often assumed in the literature.

A string of further results shows that our main theorem is robust, extending even to models of constrained persuasion. We also show that shifts of the prior cannot yield robust comparative statics.

The crater property is demanding. Nevertheless, we show that it is satisfied in a number of applications, permitting new comparative-statics conclusions to be drawn about the problems of persuading a privately informed receiver, persuading an electorate, and marketing new goods.

Finally, we ask whether and when increased alignment of interests between the sender and a receiver (who takes an action) yields a coarse-convexity shift, and thus potentially greater information-provision. We identify a simple condition that is sufficient and almost necessary.
1.1 Related literature

The persuasion model was introduced by Kamenica and Gentzkow (2011), with precedents in Aumann and Maschler (1968/1995) and Rayo and Segal (2010). A great deal of effort has been devoted to characterising optimal signals, yielding sharp descriptions of information-provision in a few special cases as well as some high-level general insights.¹

Comparative statics has been an open question. To our knowledge, it has been answered only in three special cases, each involving a particular sort of shift from one ‘S’-shaped interim payoff to another (Kolotilin, Mylovanov & Zapechelnyuk, 2022; Gitmez & Molavi, 2023). Our theorems nest all three cases, as we explain in §4.6 below.

A fourth case is that of costly information acquisition, where the interim payoff separates into a benefit from taking an action and a cost of acquiring information. Whitmeyer (2022) applies our results to obtain comparative statics for this problem as the action set expands (‘increased flexibility’).

We are informed and inspired by the general theory of monotone comparative statics (e.g. Topkis, 1978; Milgrom & Shannon, 1994; Quah & Strulovici, 2009). The results of that literature turn out to be of limited use for obtaining our theorems, however—our proofs instead exploit the particular structure of the persuasion model. A detailed discussion of how our analysis relates to the comparative-statics literature is given in appendix J.

At a high level, our work bears a kinship with Anderson and Smith (2021). Like us, these authors consider a canonical model (Becker’s (1973) marriage model) in which optimal/equilibrium outcomes have proved difficult to characterise outside of very special cases, and make progress by instead posing and answering a comparative-statics question.

1.2 Roadmap

We describe the canonical persuasion model in the next section. In §3, we characterise ‘non-decreasing’ comparative statics in terms of a coarse notion of ‘less convex than’ (Theorem 1). We then (§4) give necessary and sufficient conditions for ‘increasing’ comparative statics (Theorem 2 and several propositions). In §5, we extend our analysis to constrained persuasion, and in §6 we consider shifts of the prior. We conclude in §7 by studying alignment and some applications.

¹See e.g. Kolotilin (2014, 2018), Dworczak and Martini (2019), Kleiner, Moldovanu and Strack (2021) and Arieli, Babichenko, Smorodinsky and Yamashita (2023).
2 The persuasion model

There is an uncertain state of the world, formally a random variable taking values in a bounded interval $[x, \bar{x}]$. We assume without loss of generality that $x = 0$ and $\bar{x} = 1$. We shall use the term distribution to refer to CDFs $[0, 1] \to [0, 1]$. We write $F_0$ for the distribution of the state, and refer to it as ‘the prior (distribution)’. For two distributions $F$ and $G$, recall that $F$ is a mean-preserving contraction of $G$ exactly if

$$\int_0^x F \leq \int_0^x G \quad \text{for every } x \in [0, 1], \text{ with equality at } x = 1,$$

or equivalently iff $\int \psi dF \leq \int \psi dG$ for every convex $\psi : [0, 1] \to \mathbb{R}$.\(^2\)

A sender chooses a signal, i.e. a random variable jointly distributed with the state.\(^3\) Given a signal, each signal realisation induces a posterior belief via Bayes’s rule, whose expectation we call the posterior mean. Each signal thus induces a random posterior mean, with some distribution. Call a distribution feasible (given $F_0$) iff it is the posterior-mean distribution induced by some signal. Kolotilin (2014, Proposition 1) showed that the feasible distributions are precisely the mean-preserving contractions of the prior $F_0$.\(^4\)

The sender’s (interim) payoff at a given realised posterior belief is assumed to depend only on its mean: her payoff at posterior mean $m \in [0, 1]$ is $u(m)$, where $u : [0, 1] \to \mathbb{R}$ is upper semi-continuous. Her problem is to choose among the feasible distributions $F$ to maximise her expected payoff $\int u dF$.

**Remark 1.** Our assumption that only the mean matters is motivated by applications, where it is common for payoffs to depend on a single moment of the posterior distribution—without loss, the mean.\(^5\) This assumption is maintained in much of the recent persuasion literature. Imposing it yields sharper comparative-statics results than would otherwise be available.

2.1 Informativeness

**Definition 1.** For distributions $F$ and $G$, we call $F$ less informative than $G$ exactly if $F$ is a mean-preserving contraction of $G$.

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\(^2\)See e.g. Shaked and Shanthikumar (2007, §3.A).

\(^3\)Formally, a signal is $(M, \pi)$, where $M$ is a compact set and $\pi$ is a map $[0, 1] \to \Delta(M)$, where $\Delta(M)$ denotes the Borel probabilities on $M$. The idea is that $M$ is a set of messages, and that $\pi(x) \in \Delta(M)$ is the distribution of messages sent if the state is $x \in [0, 1]$.

\(^4\)This result may be traced to Hardy, Littlewood and Polya (1929) and Blackwell (1951).

\(^5\)This is without loss because if payoffs depend on the interim expectation of $f(X)$, where $X$ is the state of the world, then we may re-define the state of the world to be $f(X)$. 

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This captures informativeness in the spirit of Blackwell: a less informative
distribution is precisely one that is preferred ex-ante by every expected-utility
decision-maker who cares about the state only through its mean.\(^6\)

Since there need not be a unique optimal posterior-mean distribution,
comparative statics requires comparing sets of distributions. We handle this
in standard fashion by using the weak set order: for two sets $S, S'$ of feasible
distributions, we call $S$ lower than $S'$ exactly if for any $F \in S$ and $G \in S'$,
there is a distribution in $S'$ that is more informative than $F$, and there is a
distribution in $S$ that is less informative than $G$. We say that $S$ is strictly
lower than $S'$ exactly if $S$ is lower than $S'$ and $S'$ is not lower than $S$. Finally,
we call $S'$ (strictly) higher than $S$ exactly if $S$ is (strictly) lower than $S'$.

2.2 Interpretation

The interim payoff $u : [0, 1] \to \mathbb{R}$ is a reduced-form object, capturing
the (expected) payoff consequences for the sender of whatever downstream
interaction takes place after her chosen signal realises.

In many applications, that interaction involves a receiver taking an
action. Formally, there is a non-empty set $A$ of actions, and the sender’s and
receiver’s interim payoffs $U_S(a, m)$ and $U_R(a, m)$ depend on the chosen action
$a \in A$ and on the mean $m \in [0, 1]$ of their (posterior) belief about the state.\(^7\)
When the posterior mean is $m \in [0, 1]$, the receiver chooses action $A(m) \in\arg\max_{a \in A} U_R(a, m)$, so the sender’s interim payoff is $u(m) := U_S(A(m), m)$.
We assume that ($A$ is such that) the receiver breaks ties in the sender’s
favour, so that the interim payoff $u : [0, 1] \to \mathbb{R}$ is upper semi-continuous.

Our analysis will be robust to the details of this downstream interaction,
giving conditions directly on the interim payoff $u$ that are necessary and
sufficient for comparative statics. These conditions may then be checked in
particular applications; we give several examples in §7 below.

3 'Non-decreasing' comparative statics

In this section, we ask a preliminary ‘non-decreasing’ comparative-statics
question: what shifts of the sender’s interim payoff $u$ ensure that she does not

\(^6\)Explicitly, $F$ is less informative than $G$ exactly if for any finite (action) set $A$ and
any (payoff) $u : A \times [0, 1] \to \mathbb{R}$ such that $u(a, \cdot)$ is affine for each $a \in A$, we have
$\int \max_{a \in A} u(a, m) F(dm) \leq \int \max_{a \in A} u(a, m) G(dm)$. This is because all and only convex
functions may be approximated by $m \mapsto \max_{a \in A} u(a, m)$ for some finite $A$ and some $u$.

\(^7\)Equivalently, ex-post payoffs $\pi_S(a, x)$ and $\pi_R(a, x)$ depend on the action $a \in A$ and
the state $x \in [0, 1]$, and $\pi_S(a, \cdot)$ and $\pi_R(a, \cdot)$ are affine for each $a \in A$. 
choose a strictly less informative distribution? Intuition suggests that (local) convexity should be decisive, since a ‘more convex’ \( u \) embodies a greater liking for informative distributions. We validate this intuition, by defining a new coarse notion of relative convexity and proving that it is the necessary and sufficient condition for ‘non-decreasing’ comparative statics.

The definition is as follows:

**Definition 2.** For functions \( u, v : [0, 1] \to \mathbb{R} \), we say that \( u \) is coarsely less convex than \( v \) exactly if for any \( x < y \) in \([0, 1]\) such that

\[
u(\alpha x + (1 - \alpha)y) \leq \alpha u(x) + (1 - \alpha)u(y) \quad (u : \alpha)\]

holds for every \( \alpha \in (0, 1) \), we also have

\[
v(\alpha x + (1 - \alpha)y) \leq \alpha v(x) + (1 - \alpha)v(y) \quad (v : \alpha)\]

for every \( \alpha \in (0, 1) \), and furthermore any \( \alpha \in (0, 1) \) at which the inequality \((u : \alpha)\) is strict is also one at which \((v : \alpha)\) is strict.

We call \( v \) coarsely more convex than \( u \) exactly if \( u \) is coarsely less convex than \( v \). By inspection, the relation ‘coarsely less convex than’ is transitive and reflexive, but not anti-symmetric.

There is a simple sufficient condition:

**Lemma 1.** For functions \( u, v : [0, 1] \to \mathbb{R} \), if \( v(x) = \Phi(u(x), x) \) for every \( x \in [0, 1] \), where \( \Phi : \mathbb{R} \times [0, 1] \to \mathbb{R} \cup \{\infty\} \) is convex with \( \Phi(\cdot, x) \) strictly increasing for every \( x \in (0, 1) \), then \( u \) is coarsely less convex than \( v \).

**Proof.** Fix \( x, y, \alpha \in [0, 1] \) with \( u(\alpha x + (1 - \alpha)y) \leq (\alpha u(x) + (1 - \alpha)u(y) \).

If \( \alpha x + (1 - \alpha)y \in (0, 1) \), then \( \Phi(\cdot, \alpha x + (1 - \alpha)y) \) is strictly increasing, so

\[
v(\alpha x + (1 - \alpha)y) \leq (\alpha u(x) + (1 - \alpha)u(y), \alpha x + (1 - \alpha)y \)
\]

\[
\leq \alpha v(x) + (1 - \alpha)v(y),
\]

where the latter inequality follows from the convexity of \( \Phi \). If instead \( \alpha x + (1 - \alpha)y \in \{0, 1\} \), then \( u(\alpha x + (1 - \alpha)y) = \alpha u(x) + (1 - \alpha)u(y) \) and \( v(\alpha x + (1 - \alpha)y) = \alpha v(x) + (1 - \alpha)v(y) \).

Thus \( u \) is coarsely less convex than \( v \) whenever \( u \) is less convex than \( v \) in the conventional sense: \( v = \phi \circ u \) for some convex and strictly increasing function \( \phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) (to see this, take \( \Phi(k, x) := \phi(k) \) in Lemma 1). A different sufficient condition for \( u \) to be coarsely less convex than \( v \), which
features in the literature on costly information acquisition, is that \( v = u + \psi \) for some convex \( \psi : [0, 1] \to \mathbb{R} \) (take \( \Phi(k, x) := k + \psi(x) \) in Lemma 1). In case \( u \) and \( v \) are twice continuously differentiable, the former sufficient condition is equivalent to \( u'' \cdot |v'| \leq v'' \cdot |u'| \), and the latter to \( u'' \leq v'' \). For later reference, we summarise these findings in a corollary:

**Corollary 1.** For \( u, v : [0, 1] \to \mathbb{R} \), \( u \) is coarsely less convex than \( v \) whenever either (i) \( v = \phi \circ u \) for some convex and strictly increasing \( \phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) or (ii) \( v = u + \psi \) for some convex \( \psi : [0, 1] \to \mathbb{R} \).

We show in appendix K that Lemma 1 is nearly tight, by giving a partial converse as well as an exact characterisation of coarse-convexity-increasing maps \( \Phi : \mathbb{R} \times [0, 1] \to \mathbb{R} \).

The following result characterises ‘non-decreasing’ comparative statics.

**Theorem 1.** Let \( u, v : [0, 1] \to \mathbb{R} \) be upper semi-continuous. If \( u \) is coarsely less convex than \( v \), then for any distribution \( F_0 \),

\[
\arg \max_{F \text{ feasible given } F_0} \int u \, dF \text{ is not strictly higher than } \arg \max_{F \text{ feasible given } F_0} \int v \, dF. \tag{\star}
\]

Conversely, if (\( \star \)) holds for every distribution \( F_0 \), then \( u \) must be coarsely less convex than \( v \).

The proof is in appendix B. The second half (necessity) is almost immediate. For sufficiency, we prove that if \( u \) is coarsely less convex than \( v \), then \( F \mapsto \int u \, dF \) is *interval-dominated* by \( F \mapsto \int v \, dF \); this requires a substantial argument. Given this, a standard comparative statics theorem due to Quah and Strulovici (2007) yields that (\( \star \)) must hold for every distribution \( F_0 \).

### 4 ‘Increasing’ comparative statics

In this section, we ask what is required for a shift of the sender’s interim payoff to lead her optimally to choose a more informative distribution. By Theorem 1, it is necessary that the payoff become coarsely more convex.

This condition is not sufficient if all upper semi-continuous interim payoffs \( u, v : [0, 1] \to \mathbb{R} \) and all prior distributions \( F_0 \) are considered. (We will see this explicitly §4.2 below, in a sketch proof.) Our question is thus: on what restricted domains of interim payoffs \( u, v \) and/or priors \( F_0 \) are coarse-convexity shifts sufficient for ‘increasing’ comparative statics?

\[8\]For example, Chambers, Liu and Rehbeck (2020), Denti (2022), and Whitmeyer (2022).
Our main result (Theorem 2) describes the maximal domain of interim payoffs on which ‘increasing’ comparative statics holds. Concretely, it identifies the condition on the interim payoff $u$ that is necessary and sufficient for ‘increasing’ comparative statics to hold under any prior $F_0$ between $u$ and any coarsely more convex $v$. This condition is called the crater property. (The result features some mild regularity conditions on payoffs and priors.)

We also exhibit a suitable domain of priors. A binary prior is one with a two-point support; under such a prior, the state is effectively binary. We show (Proposition 1) that for ‘increasing’ comparative statics between payoffs $u$ and $v$ to hold across all binary priors $F_0$, it is both necessary and sufficient that $u$ be coarsely less convex than $v$.

We then show that Theorem 2 is robust (so Proposition 1 is tight): the crater property remains necessary for ‘increasing’ comparative statics even when only a restricted domain of priors is considered, so long as that domain contains at least one non-binary prior (Proposition 2). In other words, binary priors are special: they are the only ones for which ‘increasing’ comparative statics can be obtained without the crater property.

The crater property is demanding. A key message of this section is therefore that comparative statics are often highly prior-sensitive. On the other hand, the crater property does often hold in applications.

We next ask the mirror image of the question answered by Theorem 2: what condition on an interim payoff $v$ is necessary and sufficient for ‘decreasing’ comparative statics to hold under any prior $F_0$ between $v$ and any coarsely less convex $u$? The answer (Proposition 3) is that $v$ must be trivial: either concave or strictly convex. This finding reinforces the key message that comparative statics are prior-sensitive in the persuasion model.

Finally, we consider the special case of S-shaped interim payoffs, showing how our results nest and generalise three known comparative-statics results (Kolotilin, Mylovanov & Zapechelnyuk, 2022; Gitmez & Molavi, 2023).

4.1 Regularity

We shall mostly restrict attention to moderately well-behaved payoffs:

**Definition 3.** Call a function $u : [0, 1] \to \mathbb{R}$ regular iff (i) $u$ is continuous and possesses a continuous and bounded derivative $u' : (0, 1) \to \mathbb{R}$, and (ii) $[0, 1]$ may be partitioned into finitely many intervals, on each of which $u$ is either affine, strictly convex, or strictly concave.

Part (ii) of regularity merely rules out pathological functions whose curvature switches sign infinitely often; the same condition is imposed by
Dworczak and Martini (2019). Part (i) is also mild.

For a regular function \( u : [0,1] \to \mathbb{R} \), we extend the derivative \( u' : (0,1) \to \mathbb{R} \) to a bounded and continuous map \([0,1] \to \mathbb{R}\) by letting \( u'(0) \) be the right-hand derivative of \( u \) at 0 and \( u'(1) \) the left-hand derivative at 1.

4.2 Maximal domain of interim payoffs

The following property will be the key to comparative statics.

**Definition 4.** A regular function \( u : [0,1] \to \mathbb{R} \) satisfies the crater property if any only if for any \( x < y < z < w \) in \([0,1]\) such that \( u \) is concave on \([x,y]\) and \([z,w]\) and strictly convex on \([y,z]\), we have \( u'(x) \neq u'(w) \), and the tangents to \( u \) at \( x \) and at \( w \) cross at coordinates \((X,Y) \in \mathbb{R}^2\) satisfying \( y \leq X \leq z \) and \( Y \leq u(X) \).

The property is illustrated in Figure 1. Intuitively, it requires that any ‘valley’ of \( u \) be sufficiently steep-walled, wide and shallow—like a crater.

The crater property is demanding. It rules out multiple interior local maxima, for example.

Nevertheless, there are important classes of interim payoffs which satisfy the crater property. Call a function S-shaped iff it is continuous and either strictly convex on \([0,x]\) and concave on \([x,1]\) or concave on \([0,x]\) and strictly convex on \([x,1]\), for some \( x \in [0,1] \). Examples include the logit and probit functions, and all unimodal CDFs. Much of the persuasion literature focusses on S-shaped interim payoffs \( u \), as this allows for a sharp characterisation of optimal distributions.\(^9\) All S-shaped functions satisfy the crater property.

More generally, the crater property is satisfied by all bell-shaped functions, meaning continuous functions that are strictly convex on \([0,x]\) and on \([y,1]\) and concave on \([x,y]\), for some \( x \leq y \) in \([0,1]\). Examples include the densities of the Beta, Normal, Laplace and Cauchy distributions.

**Theorem 2.** Let \( u : [0,1] \to \mathbb{R} \) be regular. If \( u \) satisfies the crater property, then for every regular \( v : [0,1] \to \mathbb{R} \) that is coarsely more convex than \( u \) and every atomless convex-support distribution \( F_0 \),

\[
\arg \max_{F \text{ feasible given } F_0} \int udF \text{ is lower than } \arg \max_{F \text{ feasible given } F_0} \int vdF. \quad (\star\star)
\]

Conversely, if \((\star\star)\) holds for every regular \( v \) that is coarsely more convex than \( u \) and every atomless convex-support distribution \( F_0 \), then \( u \) satisfies the crater property.

\(^9\)In particular, ‘one-sided censorship’ distributions are optimal in this case (Kolotilin, 2014, p. 14). See also Kolotilin, Mylovanov and Zapechelnyuk (2022).
(a) A violation.  
(b) Not a violation.  
(c) S-shapes satisfy the property.  
(d) A function satisfying the property.

Figure 1: Illustration of the crater property.
In short, the crater property is necessary and sufficient for coarse-convexity shifts to yield ‘increasing’ comparative statics. Since the crater property is demanding, this may be viewed as a negative result: comparative statics is prior-sensitive, so that conclusions often cannot be drawn robustly across all (atomless and convex-support) priors $F_0$. But there is a bright side: the crater property does hold in some applications, and in such cases Theorem 2 delivers comparative statics. We treat several such applications in §7 below.

We view the restriction to atomless and convex-support priors $F_0$ as a mild form of regularity. It important to note that although the theorem focusses on prior distributions that have neither atoms nor support gaps, it imposes no such restrictions on what distributions the sender may choose: any distribution that is less informative than the prior $F_0$ is permitted.

Theorem 2 is proved in appendix C. The proof of the first part (the sufficiency of the crater property) works with the dual of the persuasion problem (see Dworczak & Martini, 2019). To convey why the crater property is necessary for comparative statics, we now give a sketch of the proof of the converse part of Theorem 2.

**Sketch proof of the converse part.** Suppose that $u$ is regular and does not satisfy the crater property; we shall find a regular $v$ that is more convex than $u$ and a prior distribution $F_0$ such that (**) fails.

Since $u$ violates the crater property (refer to Figure 2), there are $x' < x < y < z < w < w'$ in $[0, 1]$ such that $u$ is concave on $[x', y]$ and $[z, w']$ and strictly convex on $[y, z]$, and (assuming that $u$ is affine on neither $[x', y]$ nor $[z, w']$, which is the interesting case) there is a function $p : [0, 1] \to \mathbb{R}$ and an $X \in (x, w)$ such that $p$ is affine on $[x', X]$ and on $[X, w']$, weakly exceeds $u$.
on \([x', w']\), strictly exceeds \(u\) at \(X\), and is tangent to \(u\) at \(x\) and at \(w\). Let \(F_0\) be a distribution that is atomless with support \([x', w']\), and

\[
\frac{1}{F_0(X)} \int_X^X \xi F_0(d\xi) = x \quad \text{and} \quad \frac{1}{1 - F_0(X)} \int_X^X \xi F_0(d\xi) = w.
\]

Since \(u'\) is bounded, we may choose a regular \(v : [0, 1] \to \mathbb{R}\) that coincides with \(u\) on \([X, 1]\) and that weakly exceeds \(u\) and is strictly convex on \([0, X]\) (refer to Figure 2). It is easily seen that \(v\) is coarsely more convex than \(u\).

As \(v\) is S-shaped, an 'upper censorship' distribution \(F\) is optimal by Kolotilin’s (2014, p. 14) well-known result: for \(a \in (0, 1)\) satisfying

\[
\frac{v(b) - v(a)}{b - a} = v'(b), \quad \text{where} \quad b := \frac{1}{1 - F_0(a)} \int_a^1 \xi F_0(d\xi),
\]

this distribution \(F\) fully reveals \([0, a)\) and pools \([a, 1]\).\(^{10}\) A simple graphical argument shows that \(a\) must be strictly smaller than \(X\).\(^{11}\) Thus the optimal distribution \(F\) pools some states to the left of \(X\) with states to its right.

For the payoff \(u\), however, it is strictly sub-optimal to pool states on either side of \(X\) together. This is a consequence of the fact that the convex support function \(p\) is kinked (only) at \(X\). We prove this claim in appendix C.1.

Thus (**) fails: no distribution optimal for \(u\) given \(F_0\) is less informative than \(F\), since the latter pools across \(X\) while the former do not. \(\blacksquare\)

Remark 2. The crater property is local in character: it can be checked by separately inspecting each maximal interval \([x, w]\) on which \(u\) is concave–strictly convex–concave. This is noteworthy since it contrasts with the global character of the persuasion problem, in which a change of \(u\) on an interval \(I \subseteq [0, 1]\) can impact optimal information-provision about states far from \(I\).

4.3 The domain of binary priors

Call a distribution \(F\) **binary** iff its support comprises at most two values: \(F = p1_{[x, 1]} + (1 - p)1_{[y, 1]}\) for some \(p, x, y \in [0, 1]\). When the prior distribution \(F_0\) is binary, the persuasion model is equivalent to a simpler model in which there are just two states, and the sender’s interim payoff at posterior belief \((q, 1 - q)\) is \(u(q)\), for some upper semi-continuous function \(u : [0, 1] \to \mathbb{R}\).

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\(^{10}\)Explicitly, \(F = F_0\) on \([0, a)\), \(F = F_0(a)\) on \([a, b)\) and \(F = 1\) on \([b, 1]\).

\(^{11}\)We have \(b < w\), since \(b \geq w\) would imply both \(a < X\) (for tangency, as \(p > u = v\) at \(X\)) and \(a \geq X\) (as \(b\) is the mean conditional on the event \([a, 1]\)). Then since \(b\) (\(w\)) equals the mean conditional on the event \([a, 1]\) ([\(X, 1]\]), and \([x, w] \subseteq \text{supp}(F_0)\), we must have \(a < X\).
Proposition 1. Let \( u, v : [0, 1] \to \mathbb{R} \) be upper semi-continuous. If \( u \) is coarsely less convex than \( v \), then for any binary distribution \( F_0 \),

\[
\operatorname{arg \ max}_{F \text{ feasible given } F_0} \int u \, dF \text{ is lower than } \operatorname{arg \ max}_{F \text{ feasible given } F_0} \int v \, dF. \quad (**) 
\]

Conversely, if \((**\) holds for every binary distribution \( F_0 \), then \( u \) must be coarsely less convex than \( v \).

Thus restricting attention to binary priors obviates the need for the crater property, or indeed for any condition at all on \( u \). The proof is in appendix D.

4.4 Robustness and tightness

While Theorem 2 asserts that \( u : [0, 1] \to \mathbb{R} \) must satisfy the crater property if coarse-convexity shifts are to lead to greater information-provision under any prior, Proposition 1 shows that no condition on \( u \) is required if attention is restricted to binary prior distributions. In this section, we show that Theorem 2 is robust (so Proposition 1 is tight): binary priors are the only priors under which the crater property can be dispensed with.

Call a function \( u : [0, 1] \to \mathbb{R} \) M-shaped iff it is continuous and is concave on \([0, x]\) and on \([y, 1]\) and strictly convex on \([x, y]\), for some \( x \leq y \in [0, 1] \).

Proposition 2. For any distribution \( F_0 \) that is not binary, there are regular \( u, v : [0, 1] \to \mathbb{R} \) such that \( u \) is coarsely less convex than \( v \), and yet \((**\) fails. These \( u \) and \( v \) may be chosen to be M- and S-shaped, respectively.

In other words, binary distributions \( F_0 \) are the only ones for which \((**\) holds between any \( u \) and any coarsely more convex \( v \), even if attention is restricted to very well-behaved \( u, v : [0, 1] \to \mathbb{R} \) (in particular regular and, respectively, M- and S-shaped). ‘Increasing’ comparative statics can thus be guaranteed only by either restricting attention to interim payoffs \( u \) that satisfy the crater property (as in Theorem 2) or by restricting attention to binary prior distributions \( F_0 \) (as in Proposition 1).

The proof of Proposition 2 is in appendix E. The logic is close to that of the sketch proof of the necessity part of Theorem 2 (§4.2 above).

4.5 ‘Decreasing’ comparative statics

The question answered by Theorem 2 has a symmetric counterpart: what is the necessary and sufficient condition on an interim payoff \( v \) for every coarse-convexity decrease (to some \( u \)) to yield a decrease of informativeness, regardless of the prior distribution \( F_0 \)? The answer is as follows.
Proposition 3. Let \( v : [0, 1] \to \mathbb{R} \) be regular. If \( v \) is either concave or strictly convex, then for every regular \( u : [0, 1] \to \mathbb{R} \) that is coarsely less convex than \( v \) and every atomless convex-support distribution \( F_0 \),

\[
\arg \max_{F \text{ feasible given } F_0} \int udF \text{ is lower than } \arg \max_{F \text{ feasible given } F_0} \int vdF. \tag{**}
\]

Conversely, if (**) holds for every regular \( u \) that is coarsely less convex than \( v \) and every atomless convex-support distribution \( F_0 \), then \( v \) is either concave or strictly convex.

In other words, ‘decreasing’ comparative statics are highly prior-sensitive: a coarse-convexity decrease from \( v \) yields decreased informativeness whatever the prior \( F_0 \) only in the trivial cases of a concave \( v \) (when full pooling is optimal) or a strictly convex \( v \) (when full revelation is uniquely optimal).

The proof is in appendix F. The first half is close to obvious, while the second half is proved using logic similar to the proof of the necessity part of Theorem 2 (sketched in §4.2 above).

4.6 Special cases

Our results generalise the comparative-statics results of Kolotilin, Mylovanov and Zapechelnyuk (2022) and Gitmez and Molavi (2023). The former paper’s Proposition 1 assumes that \( u \) is S-shaped and less convex than \( v \) in the conventional sense (\( v = \phi \circ u \) for some convex and strictly increasing \( \phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \)). Theorem 2 shows that it suffices for \( u \) to be bell-shaped, and more generally that the crater property is enough. Similarly, \( u \) need only be coarsely less convex than \( v \), which admits e.g. convexity of \( v - u \) as an alternative sufficient condition. Similarly for the authors’ Proposition 2.

Call an interim payoff \( u : [0, 1] \to \mathbb{R} \) strongly S-shaped iff it is regular, S-shaped and affine on no interval. Suppose that \( u \) is strongly S-shaped and that \( u' \) is more convex than \( v' \) in the conventional sense: \( u' = \phi \circ v' \) for some convex and strictly increasing \( \phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \). These hypotheses imply that \( u \) is coarsely less convex than \( v \).\(^{12}\) Thus by Theorem 2, less

\(^{12}\)Assume without loss that \( u \) is convex–concave; the hypotheses then imply that \( v \) is also convex–concave, with the same inflection point \( \pi \). For any \( x \in [0, 1) \), define \( R^u_x : [x, 1] \to \mathbb{R} \) by \( R^u_x(y) := [u(y) - u(x)]/(y - x) \) for each \( y \in (x, 1] \) and \( R^u_x(x) := \lim_{y \to x} R^u_x(y) \). For any \( y \in (x, 1] \), since \( u \) is strongly S-shaped (and convex–concave), \( R^u_x \) is increasing on \( [x, y] \) iff \( R^u_x \) is strictly increasing on \( [x, y] \) iff \( u(\alpha x + (1 - \alpha)y) \leq \alpha u(x) + (1 - \alpha)u(y) \) for every \( \alpha \in [0, 1] \) iff \( u(\alpha x + (1 - \alpha)y) < \alpha u(x) + (1 - \alpha)u(y) \) for every \( \alpha \in (0, 1) \). The same applies to \( R^v_y \), analogously defined. What must be shown is therefore that for any \( x < y \) in \( [0, 1] \), if \( R^u_x \) is increasing on \( [x, y] \), then so is \( R^v_y \). So fix any \( x < y \) in \( [0, 1] \). Since \( R^u_x \) and \( R^v_y \) are
information is provided under $u$ than under $v$, whatever the prior. This finding generalises the main result of Gitmez and Molavi (2023), which draws the same conclusion under the additional assumption that the prior is binary.

5 Extension: constrained persuasion

In this section, we extend our analysis to encompass constraints on the sender’s choice of signal, following the small but growing literature on constrained (or costly) persuasion. We focus on two important types of constraint: monotonicity and coarseness. In the former case, the sender can use only monotone partitional signals; in the latter, she can use only signals that send at most $K$ messages, for some $K \geq 2$.

Our focus is on whether adding constraints allows us to weaken the stringent conditions for comparative statics identified by our main results, in particular Theorem 2. We show this is not the case: the crater property remains necessary, so constraints do not simplify comparative statics.

5.1 Monotone partitional signals

In many applied settings, information is provided via scores: the state space $[0, 1]$ is partitioned into intervals, and all that is revealed about the realisation of the state is which interval is belongs to. Examples include ratings in online commerce, grades in academic settings, and credit scores. Such signals are called monotone partitional.

We call a distribution $F$ $m$-feasible (given $F_0$) iff it is the posterior-mean distribution induced by some monotone partitional signal. As is well-known, a distribution $F$ is $m$-feasible given an atomless $F_0$ iff it is feasible for $F_0$ and $[0, 1)$ may be partitioned into intervals $[x, y)$ such that either (a) $F = F_0$ on $[x, y)$ or (b) $F = F_0(x)$ on $[x, \mu)$ and $F = F_0(y)$ on $[\mu, y)$ where $\mu := \left[\int_x^y zF_0(dz)\right]/\left[F_0(y) - F_0(x)\right]$. In other words, states are either fully revealed (case (a)) or pooled with adjacent states (case (b)).

strictly quasi-concave, it suffices to show that their respective maximisers $z$ and $w$ satisfy $z \leq w$. This is immediate if $w = 1$, so assume that $w < 1$. The first-order conditions are $R_x^*(z) \leq u'(z)$, with equality if $z < 1$, and $R_x^*(w) = v'(w)$. Thus since $z, w \in [x, 1]$, $z \leq w$ holds iff $R_x^*(w) \geq u'(w)$. And indeed $R_x^*(w) = (w-x)^{-1} \int_x^w \phi \circ v' \geq \phi((w-x)^{-1} \int_x^w v') = \phi(R_x^*(w)) = \phi(v'(w)) = u'(w)$ by Jensen’s inequality, since $\phi$ is convex. This argument is adapted from Gitmez and Molavi (2023, pp. 19–20).

13See e.g. Gentzkow and Kamenica (2014), le Treust and Tomala (2019) and Doval and Skreta (2022). Some of this work is surveyed by Kamenica, Kim and Zapechelnyuk (2021).
Proposition 4. Let \( u : [0, 1] \rightarrow \mathbb{R} \) be regular. If

\[
\arg \max_{F \text{ m-feasible given } F_0} \int u \, dF \text{ is lower than } \arg \max_{F \text{ m-feasible given } F_0} \int v \, dF
\]

for every regular \( v : [0, 1] \rightarrow \mathbb{R} \) that is coarsely more convex than \( u \) and every atomless convex-support distribution \( F_0 \), then \( u \) satisfies the crater property.

Thus restricting the sender to using only m-feasible distributions does not permit comparative-statics conclusions to be drawn under any weaker assumptions on the interim payoff \( u \): the crater property remains necessary.

Proposition 4 follows directly from the proof of the necessity half of Theorem 2 (sketched in §4.2 above) since by inspection, the feasible distributions \( F \) and \( G \) which appear in that argument are in fact m-feasible.

5.2 Coarse signals

In practice, communication is often coarse, with only a finite number of messages in use. This may be due to bounded rationality or information-processing costs, for example. Such coarseness can be modelled by constraining the sender to use only signals that send at most \( K \) messages, for some exogenous \( K \) (Aybas & Turkel, 2022; Lyu, Suen & Zhang, 2023).

A distribution \( F \) is the posterior-mean distribution induced by a signal satisfying this constraint if and only if \( F \) is feasible given \( F_0 \) and has \(|\text{supp}(F)| \leq K\). We call such distributions \( K \)-feasible (given \( F_0 \)).

Proposition 5. Let \( u : [0, 1] \rightarrow \mathbb{R} \) be regular, and fix any \( K \geq 2 \). If

\[
\arg \max_{F \text{ K-feasible given } F_0} \int u \, dF \text{ is lower than } \arg \max_{F \text{ K-feasible given } F_0} \int v \, dF \quad (*)_K
\]

for every regular \( v : [0, 1] \rightarrow \mathbb{R} \) that is coarsely more convex than \( u \) and every atomless convex-support distribution \( F_0 \), then \( u \) satisfies the crater property.

Sketch proof. We focus on the generic case in which optimal distributions are unique. We will show that with a small addition, the proof of the necessity half of Theorem 2 (sketched in §4.2 above) remains applicable. The argument there shows that if a regular \( u : [0, 1] \rightarrow \mathbb{R} \) violates the crater property, then there is a prior distribution \( F_0 \) and a coarsely more convex, regular and S-shaped \( v : [0, 1] \rightarrow \mathbb{R} \) such that the distribution \( G \) that is uniquely optimal for \( u \) given \( F_0 \) is binary, and is not less informative than the distribution \( F \) that is uniquely optimal for \( v \) given \( F_0 \). Since \( G \) is binary, it is \( K \)-feasible, so

\[
\arg \max_{H \text{ K-feasible given } F_0} \int u \, dH = \{G\}.
\]
Since $v$ is S-shaped, we have by Proposition 8 in Lyu, Suen and Zhang (2023) that
\[
\arg \max_{H \in K \text{-feasible given } F_0} \int u dH = \{ F^\dagger \}
\]
for a distribution $F^\dagger$ that is less informative than $F$. Then $G$ is not less informative than $F^\dagger$, so ($\star_K$) fails.

6 Shifts of the prior distribution

Our main results concerned comparative statics with respect to shifts of the sender’s interim payoff $u$. In this section, we consider shifts of the other primitive of the persuasion model: the distribution $F_0$ of the state.

Shifts of $F_0$ may be interpreted as changes in the information available to the sender. In particular, if the sender becomes better-informed about the latent state of the world (whose distribution is fixed), this manifests precisely as increased informativeness of $F_0$.

**Proposition 6.** There are no atomless distributions $F_0 \neq G_0$ such that
\[
\arg \max_{F \in \text{feasible given } F_0} \int u dF \text{ is lower than } \arg \max_{F \in \text{feasible given } G_0} \int u dF \quad (\dagger)
\]
holds for every regular and S-shaped $u : [0, 1] \to \mathbb{R}$.

In other words, the effect on optimal information-provision of a shift of the prior distribution $F_0$ depends finely on the interim payoff $u$: there are no shifts which deliver ‘increasing’ comparative statics robustly across all possible interim payoffs, not even if attention is restricted to the (small and well-behaved) class of regular and S-shaped interim payoffs.

The proof of Proposition 6 is in appendix G. In the same appendix, we explain how the atomlessness hypothesis may be dropped.

7 Applications

In this section, we apply our general results to various economic environments. In all of these applications, the sender’s interim payoff $u$ arises from a receiver choosing an action at the interim stage, informed by the realisation of the signal chosen by the sender.

Formally, recall the sender–receiver interpretation from §2.2. There is a non-empty set $\mathcal{A}$ of actions, and the sender’s and receiver’s interim payoffs
\(U_S(a, m)\) and \(U_R(a, m)\) depend on the chosen action \(a \in A\) and on the mean \(m \in [0, 1]\) of their (posterior) belief about the state. When the posterior mean is \(m \in [0, 1]\), the receiver chooses action \(A(m) \in A\), so the sender’s reduced-form interim payoff is \(u(m) := U_S(A(m), m)\). We assume that \(A : [0, 1] \to A\) is \(U_R\)-optimal, i.e., a selection from the correspondence \(m \mapsto \arg \max_{a \in A} U_R(a, m)\). We further assume that \(A\) is such that the receiver breaks ties in the sender’s favour, so that the interim payoff \(u : [0, 1] \to \mathbb{R}\) is upper semi-continuous.

The shape of the reduced-form interim payoff \(u\) is determined by the nature of the conflict of interest between the sender and receiver. Motivated by this, we begin (in the next section) by identifying when a closer alignment of interests makes \(u\) coarsely more convex. That result is used in the remaining sections, in which we study comparative statics for the problems of persuading a privately informed receiver (Kolotilin, Mylovanov, Zapechelnyuk & Li, 2017), persuading an electorate, and marketing new goods.

### 7.1 Alignment and coarse convexity

In this section, we ask whether and when an increased alignment of interests between the sender and receiver translates into coarse-convexity shifts of the sender’s reduced-form interim payoff \(u\).

We consider shifts of the sender’s interim payoff from \((a, m) \mapsto U_S(a, m)\) to \((a, m) \mapsto \Phi(U_S(a, m), U_R(a, m), m)\), where \(\Phi : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}\) is strictly increasing in its first argument—that is, \(\Phi\) is a utility transformation. We are interested in alignment-increasing utility transformations \(\Phi\), meaning those that are increasing in their second argument (the receiver’s payoff).

**Proposition 7.** Let \(\Phi : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}\) be convex with \(\Phi(k, \ell, x)\) strictly increasing and \(\Phi(k, \cdot, x)\) increasing for all \(k, \ell \in \mathbb{R}\) and \(x \in [0, 1]\). Then for any action set \(A\), any sender’s and receiver’s payoffs \(U_S, U_R : A \times [0, 1] \to \mathbb{R}\), and any \(U_R\)-optimal \(A : [0, 1] \to \mathbb{R}\), the map \(x \mapsto U_S(A(x), x)\) is coarsely less convex than the map \(x \mapsto \Phi(U_S(A(x), x), U_R(A(x), x), x)\).

In words, applying a convex alignment-increasing transformation \(\Phi\) to the sender’s payoff \(U_S\) always makes her reduced-form interim payoff \(u\) coarsely more convex. Convexity is satisfied by many natural alignment-increasing transformations, such as \((k, \ell, x) \mapsto k + \rho \ell\) for \(\rho \geq 0\).

The proof is in appendix H. The convexity-of-\(\Phi\) hypothesis is essential, indeed nearly necessary: Proposition 7 has a partial converse similar to that of Lemma 1 (see appendix K). It is therefore not generally true that increased alignment of interests leads to coarse-convexity shifts.
**Example 1.** Consider the alignment-increasing transformation $\Phi$ defined by $\Phi(k, \ell, x) := k + \phi(\ell)$ for all $k, \ell \in \mathbb{R}$ and $x \in [0, 1]$, where $\phi : \mathbb{R} \to \mathbb{R}$ is strictly increasing. It is natural for $\phi$ to be concave, as this captures inequality-aversion in the sender’s evaluation of (distributions of interim) receiver welfare. But when $\phi$ is concave and not convex, $x \mapsto U_S(A(x), x)$ fails to be coarsely less convex than $x \mapsto \Phi(U_S(A(x), x), U_R(A(x), x), x)$ for some $U_S, U_R : A \times [0, 1] \to \mathbb{R}$ and some $U_R$-optimal $A : [0, 1] \to A$.\(^{14}\)

### 7.2 Persuading a privately informed receiver

In the model of Kolotilin, Mylovanov, Zapechelnyuk and Li (2017), the receiver chooses whether to participate ($a = 1$) or not ($a = 0$). Participation may mean purchasing a good (at a fixed price), for example.

The receiver’s inside option (i.e. her payoff from participating) is uncertain, with distribution $F_0$. Her outside option is privately known to her; from the sender’s perspective, it is a random variable that is statistically independent of the inside option, with a distribution denoted by $G$. The sender values participation: her payoff is 1 if the receiver participates, and 0 otherwise.

The sender chooses a signal. No generality is lost by ruling out screening mechanisms that offer a menu of signals, even though the receiver has private information (Kolotilin, Mylovanov, Zapechelnyuk & Li, 2017, Theorem 1).

At the interim stage, the receiver participates iff $r \leq m$, where $r$ is her outside option and $m \in [0, 1]$ is the mean of her posterior belief about the inside option. The sender’s interim expected payoff is thus $u(m) := G(m)$ when the posterior mean is $m \in [0, 1]$. The function $u : [0, 1] \to \mathbb{R}$ is S-shaped if the outside-option distribution $G$ is unimodal.

Since a monotone-likelihood-ratio-higher distribution is exactly one that is more convex, Theorem 2 implies that the sender optimally provides more information whenever the outside-option distribution shifts from a unimodal $G$ to a monotone-likelihood-ratio-higher distribution $H$. This result, due to Kolotilin, Mylovanov and Zapechelnyuk (2022, §4.2), may be refined using our theorems. The shift from $G$ to $H$ can be more general: assuming for simplicity that $G, H$ admit densities $g, h$, it suffices e.g. for $h - g$ to be increasing (by Corollary 1, p. 7) or for $G$ to be less diffuse than $H$ in the sense of having a more convex density (see §4.6). Furthermore, unimodality

\(^{14}\)In particular, for any $U_R$ and $U_R$-optimal $A$ such that the (convex) function $x \mapsto U_R(A(x), x)$ is less convex than and not more convex than $\phi^{-1}$ in the conventional sense, the map $x \mapsto U_S(A(x), x) - \Phi(U_S(A(x), x), U_R(A(x), x), x) = -\phi(U_R(A(x), x))$ is convex and not concave, so by Corollary 1 (p. 7), $x \mapsto U_S(A(x), x)$ is coarsely more convex than and not coarsely less convex than $x \mapsto \Phi(U_S(A(x), x), U_R(A(x), x), x)$.\)
Applying Proposition 7 and Theorem 2 yields that given unimodality, any convex increase of alignment leads the sender to provide more information. An example is when the sender’s interim payoff shifts from $G$ to $m \mapsto G(m) + \phi(R(m))$, where $\phi : \mathbb{R} \to \mathbb{R}$ is increasing and convex, and $R(m)$ denotes the receiver’s interim expected payoff (not conditioned on the realisation of her outside option). This example nests Kolotilin, Mylovanov and Zapechelnyuk’s Proposition 3(i), in which $\phi$ is assumed affine. Increases of alignment that are not convex may not produce comparative statics: if $\phi$ is concave and not convex, then increased alignment may lead to strictly less information-provision, by Example 1 and Theorem 1.

7.3 Persuading privately informed voters

Consider a generalisation of the model of the previous section featuring $n \in \mathbb{N}$ receivers, who each cast a vote (‘yes’ or ‘no’). The receivers (collectively) participate iff at least $k \in \mathbb{N}$ of them voted ‘yes’, where $k \leq n$. The inside option is the same for all receivers, but outside options differ: from the sender’s perspective, they are independent draws from a distribution $G$.

We restrict the sender to choosing a public signal, so that all receivers are symmetrically informed ex interim. It remains dominant for each receiver to vote for participation whenever her outside option is less than the mean $m$ of her posterior belief about the inside option. The sender’s interim payoff at posterior mean $m \in [0, 1]$ is therefore $u(m) := G_{k:n}(m)$, where $G_{k:n}$ denotes the distribution of the $k$th-lowest of $n$ independent draws from $G$.

This model is like that of Alonso and Cámara (2016), except that voters’ preferences are not observed by the sender, and depend only on the mean.

If $G$ admits a strictly log-concave and differentiable density, then the sender optimally provides more information (i) when the outside-option distribution improves in the monotone-likelihood-ratio sense, (ii) when the size $n$ of the electorate falls, (iii) when the voting threshold $k$ rises, and (iv) when both $n$ and $k$ increase by an equal amount. To see why, observe that in each of these cases, $G_{k:n}$ improves in the monotone-likelihood-ratio sense, which by Corollary 1 (p. 7) implies that the sender’s interim payoff $u$ becomes coarsely more convex. Furthermore, $G_{k:n}$ admits a strictly log-concave density since $G$ does; hence $G_{k:n}$ is unimodal, so $u$ is S-shaped and thus satisfies the crater property. Theorem 2 is therefore applicable.

---

15 It can be shown that $R(m) = c + \int_0^m G$ for every $m \in [0, 1]$, where $c \in \mathbb{R}$ is a constant.
16 By Corollary 1.C.34 and Theorem 1.C.31 in Shaked and Shanthikumar (2007).
These results may be generalised to allow ex-ante heterogeneity, so long as the receivers \( i \in \{1, \ldots, n\} \) are ordered: for all \( i < j \), \( i \)'s outside-option distribution \( G_i \) is worse in the monotone-likelihood-ratio sense than \( j \)'s distribution \( G_j \). The exact same argument applies.

### 7.4 Marketing new goods

Consider a variation of the model of §7.2 in which the receiver has three options: to purchase from firm 1 (the sender), from firm 2, or not at all.

The firms are vertically differentiated: their goods belong to the same product category, but firm 2’s is known to be better and more expensive. Specifically, the receiver’s payoff if she buys from firm \( i \) is \( q_i v - p_i \), where \( q_2 > q_1 > 0 \) measure firms’ quality and \( p_2 > p_1 > 0 \) are their prices, and where \( v \) is the consumer’s valuation for the product category. We assume that the qualities \( q_1, q_2 \) and prices \( p_1, p_2 \) are exogenous and commonly known, and that they satisfy \( p_1/q_1 < p_2/q_2 \), so that firm 1’s offering is not dominated.

The receiver’s payoff is zero if she makes no purchase.

The receiver’s product-category valuation is uncertain, being the sum of a common-value component and an idiosyncratic component. These are independent random variables with respective distributions \( F_0 \) and \( G \), where \( G \) is atomless. The realisation of the idiosyncratic component is privately observed by the receiver. The sender (firm 1) chooses what information is provided about the realisation of the common-value component, by choosing a signal.

At the interim stage (after the signal has realised), given the receiver’s privately-observed idiosyncratic component \( r \in [0, 1] \) and the mean \( m \in [0, 1] \) of her posterior belief about the common-value component, she buys from firm 1 iff \( v \leq m + r \leq \overline{v} \), where \( \overline{v} := p_1/q_1 \) and \( \overline{v} := (p_2 - p_1)/(q_2 - q_1) \). The sender’s interim payoff at posterior mean \( m \in [0, 1] \) is therefore \( u(m) := [p_1 - c_1][G(\overline{v} - m) - G(v - m)] \), where \( c_1 \in [0, p_1) \) is her marginal cost.

**Remark 3.** Nothing changes if there are further firms \( i \in \{3, \ldots, n\} \), so long as firm 1’s offering is not weakly dominated:

\[
q_1 v - p_1 > \max \left\{ 0, \max_{i \in \{2, \ldots, n\}} (q_i v - p_i) \right\} \quad \text{for all } v \in I
\]

for some interval \( I \subseteq [0, 1] \). Then letting \( (\underline{v}, \overline{v}) \) be the maximal such interval, firm 1’s interim payoff remains \( m \mapsto [p_1 - c_1][G(\overline{v} - m) - G(v - m)] \).

We interpret this as a model of new or unfamiliar goods, where consumers’
uncertainty is primarily about how valuable the product category is, rather than about how different firms’ offerings compare.

**Example 2.** When a category of consumer electronics is new, its value \((v)\) is often uncertain—much doubt was expressed about smartphones, tablets, smartwatches and smartglasses, for example. By contrast, consumers may have a clear sense of how much better a high-end firm is than a lower-end firm at executing the design and manufacture of a given type of gadget \((q_2/q_1)\).

**Example 3.** Major cinemas differ in some ways, but they all show the same films. Consumers are uncertain (and so need to be informed) about which films are currently showing \((v)\), since this changes frequently. By contrast, different cinemas’ amenities \((q_i)\) rarely change, so are not very uncertain.

Recall that given an interval \(I \subseteq \mathbb{R}\), a function \(I \rightarrow \mathbb{R}\) is bell-shaped iff it is strictly convex on \((-\infty, x] \cap I\) and on \([y, \infty) \cap I\) and concave on \([x, y]\), for some \(x \leq y\) in \(\mathbb{R} \cup \{-\infty, \infty\}\). We call it truly bell-shaped iff in addition it is increasing on \((-\infty, z] \cap I\) and decreasing on \([z, \infty) \cap I\), for some \(z \in [x, y]\).

**Lemma 2.** If a distribution \(F\) admits a truly bell-shaped density, then for any \(a < b\), the map \(x \mapsto F(x + b) - F(x + a)\) is truly bell-shaped.

The proof is in appendix I. Many common distributions have truly bell-shaped densities, including the Normal, Laplace, Cauchy, and Beta\((\alpha, \beta)\) for \(\alpha, \beta \in [1, \infty)\). If the distribution \(G\) of the idiosyncratic component admits a truly bell-shaped density, then \(x \mapsto u(-x)\) is bell-shaped by Lemma 2, so \(u\) is also bell-shaped, hence satisfies the crater property.

Hence the sender provides more information whenever the distribution \(G\) of the idiosyncratic component admits a truly bell-shaped density \(g\) and then shifts to a more diffuse distribution \(H\), meaning one whose density \(h\) satisfies \(h = g + \psi\) for a convex \(\psi : [0, 1] \rightarrow \mathbb{R}\). This is because such a shift causes a convex function to be added to \(u\), which by Corollary 1 (p. 7) makes \(u\) coarsely more convex; thus Theorem 2 is applicable.

Similarly, by Proposition 7 and Theorem 2, the sender optimally provides more information when interests become more aligned in a convex way, provided the distribution \(G\) admits a truly bell-shaped density. This remains true if the sender cares only about the welfare of her own customers.\(^{17}\) Such an increase of alignment may stem from an increase in the importance of online customer ratings, for example.

\(^{17}\)I.e. she cares about \(C(m) := [G(\bar{v} - m) - G(\underline{v} - m)]^{-1} \int_{\bar{v} - m}^{\bar{v}} \max\{0, q_1[m + r] - p_1\}G(dr)\), rather than about \(R(m) := \int \max\{0, q_1[m + r] - p_1, q_2[m + r] - p_2\}G(dr)\). The proof of Proposition 7 goes through since \(C\) is convex.
Appendix A  Product structure of ‘more informative than’

In this appendix, we characterise the ‘less informative than’ order on distributions in terms of the product order on convex functions \([0, 1] \rightarrow \mathbb{R}\). This result will be used in appendices B and C below.

Given a prior \(F_0\), we write \(F\) for the space of all feasible distributions. For each \(F \in F\), let \(C_F\) denote the function \([0, 1] \rightarrow \mathbb{R}\) given by \(C_F(x) := \int_0^x F\) for each \(x \in [0, 1]\). Let \(C\) be the space of all convex functions \(C : [0, 1] \rightarrow \mathbb{R}\) whose right-hand derivative \(C^+ : [0, 1) \rightarrow \mathbb{R}\) satisfies \(0 \leq C^+ \leq 1\) and which obey \(C(x) \leq \int_0^x F_0\) for every \(x \in [0, 1]\), with equality at \(x = 0\) and \(x = 1\). Given any \(C \in C\), define \(C^+(1) := 1\) by convention. The product order (or ‘pointwise order’) on \(C\) is the partial order in which \(C\) smaller than \(C'\) exactly if \(C(x) \leq C'(x)\) for every \(x \in [0, 1]\).

The following extends Gentzkow and Kamenica’s (2016) observation: not only do distributions \(F\) correspond one-to-one with convex functions \(C_F\), but greater informativeness of \(F\) is equivalent to \(C_F\) being pointwise higher.

**Lemma 3.** Fix a prior \(F_0\). The map \(F \mapsto C_F\) is a bijection \(F \rightarrow C\) (with inverse \(C \mapsto C^+\)), and is increasing when \(F\) is ordered by ‘less informative than’ and \(C\) has the product order. Thus \(F\) and \(C\) are order-isomorphic.

**Proof.** Clearly the map \(F \mapsto C_F\) carries \(F\) into \(C\), and is increasing. The map \(C \mapsto C^+\) similarly carries \(C\) into \(F\), and by inspection \(F = C_F^+\) for every \(F \in F\); so we’ve found an inverse of \(F \mapsto C_F\) defined on all of \(C\), meaning that \(F \mapsto C_F\) is bijective. 

**Corollary 2.** For any given prior \(F_0\), the set \(F\) of all feasible distributions ordered by ‘less informative than’ is a complete lattice.

**Proof.** By Lemma 3, we need only show that when \(C\) has the product order, it holds for any family \(C' \subseteq C\) that \(C^* := \sup_{C \in C'} C\) is its least upper bound, and that the convex envelope of \(\inf_{C \in C'} C\), which we’ll call \(C_*\), is its greatest lower bound. For the former, \(C^*\) clearly belongs to \(C\), is clearly an upper bound of \(C'\), and is clearly pointwise smaller than any other upper bound. For the latter, \(C_*\) is an element of \(C\), is clearly a lower bound of \(C'\), and exceeds every other lower bound by definition of the convex envelope.

Appendix B  Proof of Theorem 1 (p. 7)

We shall prove the following generalisation of Theorem 1. Recall that for two distributions \(F\) and \(G\), the order interval \([G, F]\) is the set of all distributions
that are more informative than \( G \) and less informative than \( F \).

**Theorem 1**: For upper semi-continuous \( u, v : [0, 1] \to \mathbb{R} \), the following are equivalent:

(i) \( u \) is coarsely less convex than \( v \).

(ii) For every distribution \( F_0 \), \((\ast)\) holds.

(iii) For all distributions \( G_0, F_0 \) such that \( G_0 \) is less informative than \( F_0 \) and \( \int u \, dF, \int v \, dG > -\infty \) for some \( F, G \in [G_0, F_0] \),

\[
\arg \max_{F \in [G_0, F_0]} \int u \, dF \text{ is not strictly higher than } \arg \max_{F \in [G_0, F_0]} \int v \, dF.
\]

**Remark 4.** Property (iii) may be interpreted as concerning an extended persuasion model in which there is an outside source of information beyond the sender’s control. On this interpretation, \( G_0 \) is the posterior-mean distribution induced by observing the outside signal alone.\(^{18}\) Evidently \( G_0 \) is less informative than the prior \( F_0 \), and (extending Proposition 1 in Koltilin (2014)) the feasible distributions are exactly those in the order interval \([G_0, F_0]\).

In proving Theorem 1\(^*\), we shall make use of the product structure of ‘less informative than’ described in appendix A. We shall write \( \mu_F \) for the mean of a distribution \( F \), and shall sometimes abbreviate ‘\( F \) is less informative than \( G \)’ to ‘\( F \preceq G \)’. For \( x, y \in \mathbb{R} \) and \( \alpha \in [0, 1] \), we shall write \( x_\alpha y := \alpha x + (1 - \alpha)y \). The following restatement of the definition of ‘coarsely less convex than’ (p. 6) will be useful.

**Definition 2’.** Given \( u, v : [0, 1] \to \mathbb{R} \), \( u \) is coarsely less convex than \( v \) iff for any \( x < z \) in \([0, 1]\) satisfying

\[
u(x_\alpha z) \leq u(x_\alpha u(z) \quad \text{for all } \alpha \in (0, 1), \quad (\triangle)
\]

it holds for each \( \alpha \in (0, 1) \) that

\[
u(x_\alpha z) \leq (\angle) u(x_\alpha u(z) \quad \text{implies} \quad v(x_\alpha z) \leq (\angle) v(x_\alpha v(z). \quad (\Rightarrow : \alpha)
\]

In Theorem 1\(^*\), property (iii) implies property (ii) because a distribution is feasible given prior \( F_0 \) exactly if it belongs to \([\delta, F_0]\), where \( \delta \) is the point mass concentrated on \( \mu_{F_0} \), and obviously \( \int u \, d\delta, \int v \, d\delta > -\infty \). We shall prove that (ii) implies (i) and that (i) implies (iii).

\(^{18}\)When there is no outside signal, \( G_0 \) is the point mass concentrated at the mean of \( F_0 \).
B.1 Proof that (ii) implies (i)

We prove the contra-positive. Assume that (i) fails, meaning there are \( x < z \) in \([0,1]\) and an \( \alpha \in (0,1) \) such that \((\Delta)\) holds and \((\Rightarrow : \alpha)\) fails; we seek a distribution \( F_0 \) such that

\[
M_{F_0}(u) := \operatorname{arg\,max}_{F \in [\delta, F_0]} \int u dF \quad \text{is strictly higher than} \quad M_{F_0}(v) := \operatorname{arg\,max}_{F \in [\delta, F_0]} \int v dF,
\]

where \( \delta \) denotes the point mass concentrated at \( \mu_{F_0} \). For this, it suffices that \( F_0 \in M_{F_0}(u) \) and \( \mu_{F_0} \in M_{F_0}(v) \) (so that \( M_{F_0}(u) \) is higher than \( M_{F_0}(v) \)) and that either \( F_0 \notin M_{F_0}(v) \) or \( \mu_{F_0} \notin M_{F_0}(u) \) (so that \( M_{F_0}(v) \) is not higher than \( M_{F_0}(u) \)).

We consider two cases.

Case 1: \( v(x_\alpha z) \leq v(x)_\alpha v(z) \). Let \( F_0 \) be the distribution assigning weight \( \alpha \) to \( x \) and \( 1 - \alpha \) to \( z \), so that \( \mu_{F_0} = x_\alpha z \). By \((\Delta)\), \( F_0 \) belongs to \( M_{F_0}(u) \). Since \( v(x_\alpha z) \leq v(x)_\alpha v(z) \) and \((\Rightarrow : \alpha)\) fails by hypothesis, it must be that \( u(x_\alpha z) < u(x)_\alpha u(z) \) and \( v(x_\alpha z) = v(x)_\alpha v(z) \), or equivalently \( u(\mu_{F_0}) < \int u dF_0 \) and \( v(\mu_{F_0}) = \int v dF_0 \). Then \( \mu_{F_0} \) belongs to \( M_{F_0}(v) \) but not to \( M_{F_0}(u) \).

Case 2: \( v(x_\alpha z) > v(x)_\alpha v(z) \). Let \( \tilde{v} \) be the concave envelope (i.e. pointwise least majorant) of the restriction of \( v \) to \([x, z] \), and note that \( \tilde{v}(x_\alpha z) \geq v(x_\alpha z) > v(x)_\alpha v(z) \) and (since \( v \) is upper semi-continuous) that \( \tilde{v}(x) = v(x) \) and \( \tilde{v}(z) = v(z) \). Then there is a \( \beta \in (0,1) \) such that \( \tilde{v} \) is not affine on any neighbourhood of \( x_\beta z \), and \( \tilde{v}(x_\beta z) = v(x_\beta z) \) since \( v \) is upper semi-continuous. Let \( F_0 \) be the distribution assigning weight \( \beta \) to \( x \) and \( 1 - \beta \) to \( z \), so that \( \mu_{F_0} = x_\beta z \). Then \( M_{F_0}(v) = \{\mu_{F_0}\} \). And \( F_0 \) belongs to \( M_{F_0}(u) \) by \((\Delta)\).

B.2 Lemmata for the proof that (i) implies (iii)

‘Measurable’ will always mean ‘Lebesgue measurable’. Recall the following standard embedding:

\[\text{a distribution } F \text{ is less informative than another distribution } G \text{ if and only if there are random variables } X \sim F \text{ and } Y \sim G \text{ which satisfy } E(Y|X) = X \text{ a.s.}\]

Our first observation asserts that intervals \([F, H]\) have a ‘product’ structure: if only some of the realisations of \( F \) (those outside some set \( \mathcal{X} \)) are ‘split’ according to \( H \), the resulting distribution \( G \) belongs the interval \([F, H]\).

**Observation 1.** Let \( X, Z \) be \([0,1]\)-valued random variables that satisfy \( E(Z|X) = X \) a.s., let \( \mathcal{X} \subseteq [0,1] \) be measurable, and define a random variable \( Y \) by

\[
Y := \begin{cases} 
X & \text{if } X \in \mathcal{X} \\
Z & \text{otherwise.}
\end{cases}
\]

\[\text{See e.g. Shaked and Shanthikumar (2007, p. 112).}\]
Then the respective laws $F, G, H$ of $X, Y, Z$ satisfy $F \preceq G \preceq H$.

Proof. We have $F \preceq G$ since
\[ \mathbb{E}(Y - X|X) \overset{a.s.}{=} 1_{[0,1]}(X)\mathbb{E}(Z - X|X) \overset{a.s.}{=} 0. \]
And we have $G \preceq H$ because for any $G$-non-null $Y \subseteq [0, 1]$,
\[ \mathbb{E}(Z - Y|Y \in \mathcal{Y}) \overset{a.s.}{=} \mathbb{P}(X \in \mathcal{X}|Y \in \mathcal{Y})\mathbb{E}(Z - X|X \in \mathcal{X} \cap Y) \overset{a.s.}{=} 0, \]
where the final inequality holds since $\mathbb{E}(Z|X) = X$ a.s. $\blacksquare$

Given distributions $F \preceq H$, define
\[ I_{F,H} := \left\{ [x,z] : 0 \leq x < z \leq 1, \int_0^y F < \int_0^y H \quad \text{for all } y \in (x,z) \right\} \]
and write $\bigcup I_{F,H} := \bigcup_{I \in I_{F,H}} I$.

Our first lemma asserts that if $F \preceq H$, then a realisation of $F$ is ‘split’ by $H$ only if it belongs to one of the intervals $I \in I_{F,H}$, and if so, then the ‘split’ is confined to that same interval $I$. This will let us work with each of the intervals in $I_{F,H}$ separately (e.g. Lemma 6 below).

Lemma 4. Let $X \sim F$ and $Z \sim H$ satisfy $\mathbb{E}(Z|X) = X$ a.s. Then, almost surely, either (a) $X = Z \notin \bigcup I_{F,H}$ or (b) $X$ and $Z$ belong to the same element of $I_{F,H}$.

Proof. We first show that if $S := [0,1] \setminus \bigcup I_{F,H}$ is $F$-non-null, then conditional on $X \in S$, we have $Z = X$ a.s. $F$ and $H$ coincide on $S$ since $y \mapsto \int_0^y F$ and $y \mapsto \int_0^y H$ do, and $S$ is an open set.\(^{20}\) Thus if $S$ is $F$-non-null, then
\[ \mathbb{E}\left(Z^2|Z \in S\right) = \mathbb{E}\left(X^2|X \in S\right), \]
so that (iterating expectations and using $\mathbb{E}(Z|X) \overset{a.s.}{=} X$)
\[ \mathbb{E}\left((Z - X)^2|X \in S\right) = \mathbb{E}\left(X^2 - 2XZ + Z^2|X \in S\right) \]
\[ = \mathbb{E}\left(X^2 - 2XE(Z|X) + Z^2|X \in S\right) = \mathbb{E}\left(X^2 - 2X^2 + X^2|X \in S\right) = 0, \]
\(^{20}\)Since $S$ is open, every $s \in S$ has an open neighbourhood $\mathcal{N}$ that is contained in $S$. Since $y \mapsto \int_0^y F$ and $y \mapsto \int_0^y H$ coincide on $\mathcal{N}$, we have $F = H$ a.e. on $\mathcal{N}$, implying that $F = H$ on $\mathcal{N}$ since $F$ and $H$ are right-continuous.
implying that $Z = X$ a.s. conditional on $X \in \mathcal{S}$.

It remains to show that, for any $F$-non-null $I \in I_{F,H}$, we have $Z \in I$ a.s. conditional on $X \in I$. Suppose toward a contradiction that $I \in I_{F,H}$ is $F$-non-null but that $P(Z \in I | X \in I) < 1$. By considering the random variables $1 - X$ and $1 - Z$ if necessary, we may assume without loss of generality that there is an interval $I \in I_{F,H}$ such that the event `$Z < \min_{I} \leq X$' has positive probability. Abbreviate $y' := \min_{I} > 0$. Define a random variable by

$$Y := \begin{cases} X & \text{if } X \geq y' \\ Z & \text{otherwise,} \end{cases}$$

and write $G$ for its distribution. For any $y \in [0, y')$, we have

$$G(y) = P(X < y' \text{ and } Z \leq y)$$
$$\leq P(X \geq y' \text{ and } Z \leq y) + P(X < y' \text{ and } Z \leq y) = H(y),$$

and by hypothesis the inequality is strict for some $y < y'$. Hence $\int_{0}^{y'} G < \int_{0}^{y'} H = \int_{0}^{y'} F$, contradicting the fact that $F \preceq G$ by Observation 1.

Given $u : [0, 1] \to \mathbb{R}$ and $\mathcal{X} \subseteq [0, 1]$, let $\phi^{u}_{\mathcal{X}} : [0, 1] \to \mathbb{R} \cup \{\infty\}$ be given by

$$\phi^{u}_{\mathcal{X}}(x) := \begin{cases} u(x) & \text{if } x \in \mathcal{X} \\ \infty & \text{otherwise,} \end{cases}$$

and let $\Phi^{u}_{\mathcal{X}} : [0, 1] \to \mathbb{R} \cup \{-\infty, \infty\}$ be the convex envelope (i.e. pointwise greatest convex minorant) of $\phi^{u}_{\mathcal{X}}$.

**Lemma 5.** For upper semi-continuous $u, v : [0, 1] \to \mathbb{R}$ with $u$ coarsely less convex than $v$, it holds for any $\mathcal{X} \subseteq [0, 1]$ that $\Phi^{u}_{\mathcal{X}} \geq u$ implies $\Phi^{v}_{\mathcal{X}} \geq v$.

**Proof.** Suppose toward a contradiction that there are $\mathcal{X} \subseteq [0, 1]$ and $y \in [0, 1]$ such that $\Phi^{u}_{\mathcal{X}} \geq u$ and $\Phi^{v}_{\mathcal{X}}(y) < v(y)$. Clearly $\Phi^{u}_{\mathcal{X}} \geq u$ implies that $(\triangle)$ holds for any $x < z$ in $\mathcal{X}$. It thus suffices to find $x \in [0, y)$ and $z \in (y, 1]$ such that

$$\frac{(z - y) v(x) + (y - x) v(z)}{z - x} < v(y),$$

since then $(\Rightarrow : \alpha)$ fails for $\alpha := (z - y)/(z - x)$, contradicting the fact that $u$ is coarsely less convex than $v$.

Note that $\Phi^{u}_{\mathcal{X}}(y) < v(y) \leq \phi^{u}_{\mathcal{X}}(y)$. Since $\Phi^{u}_{\mathcal{X}}$ is the convex envelope of $\phi^{u}_{\mathcal{X}}$, it follows that $y \in (0, 1)$, and that for any $\varepsilon > 0$, there are $x \in [0, y)$ and $z \in (y, 1]$ such that

$$\frac{(z - y) \phi^{u}_{\mathcal{X}}(x) + (y - x) \phi^{u}_{\mathcal{X}}(z)}{z - x} < \Phi^{u}_{\mathcal{X}}(y) + \varepsilon.$$
This implies that $\phi^v_X(x)$ and $\phi^v_X(z)$ are finite, so that ($x$ and $z$ belong to $X$ and) $\phi^v_X(x) = v(x)$ and $\phi^v_X(z) = v(z)$. Thus choosing $\epsilon := v(y) - \Phi^v_X(y)$ yields $x, z$ with the desired property.

Our final lemma characterises the property ‘$\int udG \leq \int udH$ for all $G \in [F, H]$’, which is key to comparative statics (in particular, to the interval dominance order of Quah and Strulovici (2007, 2009)). The characterisation decomposes this property into a condition that must hold on each of the intervals in $I_{F,H}$, separately. (The condition: joining all of the points $(z, u(z))$ for $z \in \text{supp}(H) \cap I$ should yield a convex curve lying above the graph of $u$.)

**Lemma 6.** Given an upper semi-continuous $u : [0, 1] \to \mathbb{R}$ and distributions $F$ and $H$ satisfying $F \preceq H$ and $\max_{G \in [F, H]} \int udG > -\infty$, the following are equivalent:

(a) $\int udG \leq \int udH$ for all $G \in [F, H]$.

(b) $\Phi^u_{\text{supp}(H) \cap I} \geq u$ for all $I \in I_{F,H}$.

The proof is long, but not difficult.

**Proof.** To prove that (b) implies (a), suppose that (b) holds and fix $G \in [F, H]$; we must show that $\int udG \leq \int udH$. Since $G \preceq H$, there are random variables $Y \sim G$ and $Z \sim H$ such that $\mathbb{E}(Z|Y) = Y$ a.s. Since $F \preceq G$, there is a (necessarily unique) map $\psi : I_{G,H} \to I_{F,H}$ such that $I \subseteq \psi(I)$ for all $I \in I_{G,H}$. Then, assuming without loss of generality that $\int udG > -\infty$, we have

$$
\int ud(H - G) = \mathbb{E}(\mathbb{E}[u(Z) - u(Y)|Y])
$$

$$
= \mathbb{E}\left(\sum_{I \in I_{G,H}} 1_I(Y)\mathbb{E}\left[\phi^u_{\text{supp}(H) \cap I}(Z) - u(Y)|Y\right]\right)
$$

$$
\geq \mathbb{E}\left(\sum_{I \in I_{G,H}} 1_I(Y)\mathbb{E}\left[\Phi^u_{\text{supp}(H) \cap \psi(I)}(Z) - u(Y)|Y\right]\right)
$$

$$
\geq \mathbb{E}\left(\sum_{I \in I_{G,H}} 1_I(Y)\mathbb{E}\left[\Phi^u_{\text{supp}(H) \cap \psi(I)}(Y) - u(Y)|Y\right]\right) \geq 0,
$$

where the second equality follows from Lemma 4, the first inequality holds since $X \subseteq X'$ implies $\phi^u_X \geq \phi^u_{X'} \geq \Phi^u_{X'}$, the second inequality follows from Jensen’s inequality, and the last inequality holds by (b).
To show that (a) implies (b), we prove the contra-positive. Assume that \( \Phi^u_{\supp(H) \cap I}(\tilde{y}) < u(\tilde{y}) \) for some \( \tilde{y} \in [0,1] \) and \( I \in \mathcal{I}_{F,H} \). We shall exhibit a \( G \in [F, H] \) such that \( \int u dG > \int u dH \). To that end, we shall construct a family of distributions \( (G^\varepsilon)_{\varepsilon \in (0, \varepsilon)} \) (where \( \varepsilon > 0 \)) such that we have \( \int u dG^\varepsilon > \int u dH \) and \( G^\varepsilon \in [F, H] \) whenever \( \varepsilon \in (0, \varepsilon) \) is sufficiently small.

Let \( \mathcal{Y} := \supp(H) \cap I \). Since \( \Phi^u_H(\tilde{y}) < u(\tilde{y}) < \infty \), \( \tilde{y} \) belongs to the interior of \( I \) and there are \( \hat{x}, \hat{z} \in \mathcal{Y} \) such that \( \hat{x} < \tilde{y} < \hat{z} \) and \( u(\hat{x}) \leq u(\tilde{y}) < u(\hat{z}) \), where \( \hat{\alpha} \in (0, 1) \) is given by \( \hat{\alpha} \hat{x} = \tilde{y} \). Moreover, since \( u \) is upper semi-continuous, we may choose \( \hat{x} > \min I \) unless \( \min I, \min I + \delta \) is \( H \)-null for some \( \delta > 0 \), and we may choose \( \hat{z} < \max I \) except if \( \max I - \delta, \max I \) is \( H \)-null for some \( \delta > 0 \).

Note that \( P(Z \neq \tilde{y}) > 0 \) since \( \mathcal{Y} \supseteq \{ \hat{x}, \tilde{y}, \hat{z} \} \neq \{ \tilde{y} \} \). Let

\[
\hat{\varepsilon} := \min \{ P(Z \neq \tilde{y}), \hat{z} - \tilde{y} \}, \tilde{y} - \hat{y} - \hat{x} \}.
\]

For each \( \varepsilon \in (0, \hat{\varepsilon}) \), define

\[
X^\varepsilon := \{ y \in I \cap [\hat{x} - \varepsilon, \hat{x} + \varepsilon] : u(y) \leq u(\hat{x}) + \varepsilon \},
\]

\[
Z^\varepsilon := \{ y \in I \cap [\hat{z} - \varepsilon, \hat{z} + \varepsilon] : u(y) \leq u(\hat{z}) + \varepsilon \},
\]

\[
p^\varepsilon := P(Z \in X^\varepsilon) \text{ and } r^\varepsilon := P(Z \in Z^\varepsilon). \]

Note that \( p^\varepsilon, r^\varepsilon > 0 \) since \( u \) is upper semi-continuous and \( \hat{x}, \hat{z} \in \mathcal{Y} \subseteq \supp(H) \). Let \( x^\varepsilon := E(Z | Z \in X^\varepsilon) \) and \( z^\varepsilon := E(Z | Z \in Z^\varepsilon) \), and note that \( x^\varepsilon < \hat{y} < z^\varepsilon \). We may thus choose \( \pi^\varepsilon \in (0, p^\varepsilon] \) and \( \rho^\varepsilon \in (0, r^\varepsilon] \) such that \( \pi^\varepsilon + \rho^\varepsilon < \varepsilon \) and

\[
\frac{\pi^\varepsilon x^\varepsilon + \rho^\varepsilon z^\varepsilon}{\pi^\varepsilon + \rho^\varepsilon} = \tilde{y}. \tag{1}
\]

Define \( G^\varepsilon : [0,1] \to \mathbb{R} \) by, for each \( y \in [0,1] \),

\[
G^\varepsilon(y) := H(y) + (\pi^\varepsilon + \rho^\varepsilon) 1_{[y,1]}(y)
\]

\[
- \pi^\varepsilon P(Z \leq y | Z \in X^\varepsilon) - \rho^\varepsilon P(Z \leq y | Z \in Z^\varepsilon).
\]

Fix any \( \varepsilon \in (0, \hat{\varepsilon}) \); we claim that \( G^\varepsilon \) is a distribution. It is clearly right-

continuous, since each of its terms are. Furthermore,

\[
G^\varepsilon(1) = 1 + (\pi^\varepsilon + \rho^\varepsilon) - \pi^\varepsilon - \rho^\varepsilon = 1.
\]

\[21\]If \( \hat{\varepsilon} = \min I \) and \( (\hat{x}, \hat{x} + \delta] \) is \( H \)-non-null for every \( \delta > 0 \), then \( (\hat{x}, \hat{x} + \delta] \cap \mathcal{Y} \neq (\hat{x}) \) for all \( \delta > 0 \). Then for any \( \delta > 0 \), we may choose an \( x \in (\hat{x}, \hat{x} + \delta] \cap \mathcal{Y} \) such that \( u(x) \leq u(\tilde{y}) + \delta \). Since \( u(\hat{x}) - u(\tilde{y}) < u(\tilde{y}) \), it follows that \( u(x) < u(\tilde{y}) \) for \( \delta > 0 \) sufficiently small, where \( \alpha \in (0, 1) \) is given by \( \alpha \hat{x} = \tilde{y} \). Thus we may replace \( \hat{x} \) by \( x \). A similar argument applies if \( \hat{z} = \max I \) and \( (\hat{z} - \delta, \hat{z}) \) is \( H \)-non-null for all \( \delta > 0 \).
To show that $G^\varepsilon$ is non-negative and increasing, compute

\[
\left[ H(y) - \pi^\varepsilon \mathbb{P}(Z \leq y|Z \in \mathcal{X}^\varepsilon) - \rho^\varepsilon \mathbb{P}(Z \leq y|Z \in \mathcal{Z}^\varepsilon) \right] \\
- \left[ (\pi^\varepsilon - \pi^0) \mathbb{P}(Z \leq y|Z \in \mathcal{X}^\varepsilon) + (r^\varepsilon - \rho^\varepsilon) \mathbb{P}(Z \leq y|Z \in \mathcal{Z}^\varepsilon) \right] \\
= H(y) - \pi^\varepsilon \mathbb{P}(Z \leq y|Z \in \mathcal{X}^\varepsilon) - r^\varepsilon \mathbb{P}(Z \leq y|Z \in \mathcal{Z}^\varepsilon) \\
= \mathbb{P}(Z \leq y) - \mathbb{P}(Z \leq y \text{ and } Z \in \mathcal{X}^\varepsilon) - \mathbb{P}(Z \leq y \text{ and } Z \in \mathcal{Z}^\varepsilon) \\
= \mathbb{P}(Z \leq y \text{ and } Z \notin \mathcal{X}^\varepsilon \cup \mathcal{Z}^\varepsilon),
\]

where the final inequality holds since $\mathcal{X}^\varepsilon$ and $\mathcal{Z}^\varepsilon$ are disjoint. Rearranging and substituting yields an expression for $G^\varepsilon$ as the sum of non-negative and increasing terms.

We now show that $\int udG^\varepsilon > \int udH$ whenever $\varepsilon \in (0, \bar{\varepsilon})$ is small enough. As $\varepsilon \downarrow 0$, we have $x^\varepsilon \to \bar{x}$ and $z^\varepsilon \to \bar{z}$, and thus

\[
\frac{\pi^\varepsilon}{\pi^\varepsilon + \rho^\varepsilon} \to \bar{\alpha}
\]

by (1), and furthermore

\[
\limsup_{\varepsilon \downarrow 0} \mathbb{E}[u(Z)|Z \in \mathcal{X}^\varepsilon] \leq u(\bar{x}) \quad \text{and} \quad \limsup_{\varepsilon \downarrow 0} \mathbb{E}[u(Z)|Z \in \mathcal{Z}^\varepsilon] \leq u(\bar{z}).
\]

Since $u(\bar{x}) = u(\bar{z}) < u(\bar{y})$, it follows that for all sufficiently small $\varepsilon \in (0, \bar{\varepsilon})$, we have

\[
\frac{\pi^\varepsilon}{\pi^\varepsilon + \rho^\varepsilon} \mathbb{E}[u(Z)|Z \in \mathcal{X}^\varepsilon] + \frac{\rho^\varepsilon}{\pi^\varepsilon + \rho^\varepsilon} \mathbb{E}[u(Z)|Z \in \mathcal{Z}^\varepsilon] < u(\bar{y}).
\]

Thus, assuming without loss of generality that $\int udH > -\infty$, we have

\[
\int ud(G^\varepsilon - H) \\
= (\pi^\varepsilon + \rho^\varepsilon)u(\bar{y}) - \pi^\varepsilon \mathbb{E}[u(Z)|Z \in \mathcal{X}^\varepsilon] - \rho^\varepsilon \mathbb{E}[u(Z)|Z \in \mathcal{Z}^\varepsilon] > 0.
\]

It remains only to show that $G^\varepsilon$ belongs to $[F, H]$ whenever $\varepsilon \in (0, \bar{\varepsilon})$ is small enough. For $G^\varepsilon \leq H$, fix any $\varepsilon \in (0, \bar{\varepsilon})$; we have $G^\varepsilon \leq H$ on $[0, \bar{y})$, $G^\varepsilon \geq H$ on $[\bar{y}, 1]$, and

\[
\mu_{G^\varepsilon} = \mu_H + (\pi^\varepsilon + \rho^\varepsilon)\bar{y} - \pi^\varepsilon x^\varepsilon - \rho^\varepsilon z^\varepsilon = \mu_H
\]

by (1). To show that $F \preceq G^\varepsilon$ for any small enough $\varepsilon \in (0, \bar{\varepsilon})$, note that as $\varepsilon \downarrow 0$, we have $G^\varepsilon \to H$ uniformly, and thus the map $y \mapsto \int_{0}^{y} G^\varepsilon$ converges
uniformly to \( y \mapsto \int_0^y H \). Note that (by definition of \( I_{F,H} \), of which \( I \) is an element,) we have \( \int_0^y F < \int_0^y H \) for every \( y \in \text{int } I \). Thus given any \( \gamma \in (0, [\max I - \min I]/2) \), we may choose \( \varepsilon \in (0, \bar{\varepsilon}) \) sufficiently small that \( \int_0^y G^\varepsilon > \int_0^y F \) for all \( y \in [\min I + \gamma, \max I - \gamma] \).

To conclude, we shall exhibit a \( \gamma \) and an \( \varepsilon \) such that \( \int_0^y G^\varepsilon \geq \int_0^y F \) for all \( \varepsilon \in (0, \varepsilon_1) \) and \( y \leq \min I + \gamma_1 \). A similar argument (which we omit) then yields a further \( \gamma_1 \) and \( \varepsilon_1 \) such that \( \int_0^y G^\varepsilon \geq \int_0^y F \) for all \( \varepsilon \in (0, \varepsilon_1) \) and \( y \geq \max I - \gamma_1 \), completing the proof.

Note that \( G^\varepsilon = H \) on \( J^\varepsilon := [0,1] \setminus (I \cap [\bar{x} - \varepsilon, \bar{x} + \varepsilon]) \), so that \( \int_0^y G^\varepsilon = \int_0^y H \geq \int_0^y F \) for all \( y \in J^\varepsilon \). Thus if \( \bar{x} = \min I \), we may take \( \gamma \) and \( \varepsilon \) such that \( \varepsilon \geq 0 \) and \( \gamma \geq 0 \). If instead \( \bar{x} = \min I \), then (as previously observed) we may assume without loss of generality that \( (\min I, \min I + \delta] \) is \( H \)-null for some \( \delta \in (0, \bar{\varepsilon}) \), so that \( y \mapsto \int_0^y H \) is affine on this interval. Since \( \int_0^{\min I} F = \int_0^{\min I} H \) and \( \int_0^y F < \int_0^y H \) for all \( y \in [\min I, \min I - \varepsilon_1] \), we have that \( F(\min I + \gamma_1) < H(\min I) \) and \( \varepsilon_1 < \min \{ \delta, H(\min I) - F(\min I + \gamma_1) \} \).

\[ \text{B.3 Proof that (i) implies (iii)} \]

Fix upper semi-continuous \( u, v : [0,1] \to \mathbb{R} \) such that \( u \) is coarsely less convex than \( v \). We shall show that for any distributions \( F \leq H \) satisfying

\[
\max_{G \in [F,H]} \int u dG > -\infty \quad \text{and} \quad \max_{G \in [F,H]} \int v dG > -\infty
\]

and \( \int u dG \leq \int u dH \) for all \( G \in [F,H] \), we have that

\[
\int u dF \leq (\cdot) \int u dH \quad \text{implies} \quad \int v dF \leq (\cdot) \int v dH.
\]

(Modulo the finiteness proviso, this says precisely that \( G \mapsto \int u dG \) is interval-dominated by \( G \mapsto \int v dG \), as defined by Quah and Strulovici (2007, 2009).) This suffices by Proposition 5 in Quah and Strulovici (2007).

So fix such a pair of distributions \( F \) and \( H \). To prove the ‘weak inequality’ part \( \int u dF \leq \int v dH \), begin by applying Lemma 6: since \( \int u dG \leq \int u dH \) for

\[ \text{Fix } \varepsilon \in (0, \varepsilon_1). \quad \text{Since } G^\varepsilon = H \text{ on } [0, \min I] \subseteq J^\varepsilon \text{ and } H \geq F, \text{ we have } \int_0^y G^\varepsilon = \int_0^y H \geq \int_0^y F \text{ for every } y \leq \min I. \text{ On } [\min I, \min I + \gamma_1], \text{ we have } \int_0^y G^\varepsilon \text{ has slope at least } H(\min I) - \varepsilon_1 \text{ (since } \pi^\varepsilon + \rho^\varepsilon < \varepsilon < \varepsilon_1) \text{ while } y \mapsto \int_0^y F \text{ has slope at most } F(\min I + \gamma_1) \leq H(\min I) - \varepsilon_1, \text{ and thus } \int_0^y G^\varepsilon \geq \int_0^y F \text{ also for } y \in [\min I, \min I + \gamma_1]. \]

Our finiteness assumption is weaker than their requirement that \( \int u dG \) and \( \int v dG \) be finite for all \( G \in [F,H] \); but their proof goes through with our weaker assumption.

---

\[ \text{31} \]
every $G \in [F,H]$, it must be that $u$ satisfies Lemma 6's property (b). As $u$

is coarsely less convex than $v$, it follows by Lemma 5 that $v$ also satisfies

property (b), which by Lemma 6 again yields that $\int vdG \leq \int vdH$ for every $G \in [F,H]$; in particular, $\int vdF \leq \int vdH$.

For the 'strict inequality' part, we prove the contra-positive: assume that $\int vdF = \int vdH$; we must establish that $\int udF = \int udH$. To this end, let $X \sim F$ and $Z \sim H$ be such that $E(Z|X) = X$ a.s. Assume without loss that the distribution of $Z$ conditional on $X$ is regular, so that for any $(Z,X)$-integrable $\psi : [0,1]^2 \to \mathbb{R}$ with $E(|\psi(Z,X)|) < \infty$ for each $x \in [0,1]$, the map $x \mapsto E(\psi(Z,x)|X = x)$ is measurable and satisfies $E(\psi(Z,X)) = \int E(\psi(Z,x)|X = x)F(dx)$. It suffices to show that $E[u(Z)|X = x] = u(x)$ for $F$-almost all $x \in X$.

Note that by hypothesis, $\int vdF = E[v(X)] = E[v(Z)] = \int vdH = \max_{G \subseteq [F,H]} \int v dG > -\infty$. We shall use these properties without comment.

We claim that $E[v(Z)|X] = v(X)$ a.s. Since $E[v(Z)] = E[v(X)]$, it suffices to show that $E[v(Z)|X] \geq v(X)$ a.s. To that end, suppose toward a contradiction that there is an $F$-non-null $X \subseteq [0,1]$ with $E[v(Z) - v(X)|X \in X] < 0$. Define a random variable $Y$ by $Y := X$ if $X \in X$ and $Y := Z$ otherwise, and let $G$ be its distribution. Then $G$ belongs to $[F,H]$ by Observation 1, and

$$\int vdH - \int vdG = P(X \in X)E[v(Z) - v(X)|X \in X] < 0,$$

contradicting our hypothesis that $\int vdG \leq \int vdH$ for all $G \in [F,H]$.

The claim implies that $E[v(Z)|X = x] = v(x)$ for $F$-almost all $x \in [0,1]$. Similarly, the fact that $E(Z|X) = X$ a.s. implies that $E(Z|X = x) = x$ for $F$-almost all $x \in [0,1]$. Finally, by Lemma 4, we have for $F$-almost all $x \in [0,1]$ that conditional on $X = x$, it holds almost surely that either (I) $X = x \notin \bigcup I_{F,H}$ or (II) $Z$ and $x$ belong to the same element of $I_{F,H}$.

---

24 See e.g. Dudley (2004, §10.2) for regularity and why it is without loss.

25 Then since $u$ is $H$-integrable (because $-\infty < \max_{G \subseteq [F,H]} \int u dG = \int udH < \infty$, where the last inequality holds by upper semi-continuity), taking $\psi(z,x) := u(z)$ yields $\int udH = E[u(Z)] = \int E[u(Z)|X = x]F(dx) = \int udF$, as desired.

26 In general, it holds for any $(Z,X)$-integrable $\psi : [0,1]^2 \to \mathbb{R}$, with $E[|\psi(Z,x)|] < \infty$ for each $x \in [0,1]$ that if $E(\psi(Z,X)|X) = 0$ a.s., then $E(\psi(Z,x)|X = x) = 0$ for $F$-almost all $x \in [0,1]$. (If there were an $F$-non-null $X \subseteq [0,1]$ with $E(\psi(Z,x)|X = x) > 0$ for each $x \in X$, then we would have $0 < \int 1_X(x)E[\psi(Z,x)|X = x]F(dx) = E[1_X(Z)\psi(Z,X)]$, so that $E(\psi(Z,X)|X) \neq 0$ with positive probability.) Here we have taken $\psi(z,x) := |\psi(z) - v(x)|$.

27 Apply the reasoning in footnote 26 with $\psi(z,x) := |z - x|$.

28 Let $\psi(z,x) = 0$ if either (I') $z = x \notin \bigcup I_{F,H}$ or (II') $z$ and $x$ belong to the same element of $I_{F,H}$, and let $\psi(z,x) = 1$ otherwise. We have $E(E[\psi(Z,X)|X]) = E[\psi(Z,X)] = 0$ by Lemma 4, and thus $E[\psi(Z,X)|X] = 0$ a.s. Now apply footnote 26.
Thus to show that $E[u(Z)|X = x] = u(x)$ for $F$-almost every $x \in [0, 1]$, it suffices to note that this holds for any $x \in [0, 1]$ such that (I) is satisfied a.s. conditional on $X = x$, and to prove that it holds for any $x \in [0, 1]$ such that $E[v(Z)|X = x] = v(x)$, $E(Z|X = x) = x$, and (II) is satisfied a.s. conditional on $X = x$.

So fix an $x$ of the latter kind. By hypothesis, $x$ belongs to an interval $I$ that in turn belongs to $I_{F,H}$, and $P(Z \in I|X = x) = 1$. Define $\mathcal{Y} := \text{supp}(H) \cap I$, and remark that $P(Z \in \mathcal{Y}|X = x) = 1$. Since $E(Z|X = x) = x$, it suffices to exhibit a measurable $Z \subseteq \mathcal{Y}$ containing $x$ such that $P(Z \in Z|X = x) = 1$ and $u$ is affine on $Z$.

Toward constructing $Z$, observe that

$$v(x) \leq \Phi^v_\alpha(x) \leq E[\Phi^v_\alpha(Z)|X = x] \leq E[v(Z)|X = x] = v(x),$$

where the first inequality holds since (as shown early in this proof) $v$ satisfies Lemma 5’s property (b), the second holds by Jensen’s inequality since $E(Z|X = x) = x$ and $\Phi^v_\alpha$ is convex, and the third holds since $P(Z \in \mathcal{Y}|X = x) = 1$ and $\Phi^v_\alpha \leq v$ on $\mathcal{Y}$. Thus all three inequalities are equalities, whence (by the converse of Jensen’s inequality) $\Phi^v_\alpha$ must be affine on some measurable $Z \subseteq \mathcal{Y}$ which contains $x$ and satisfies $P(Z \in Z|X = x) = 1$.

It remains to show that $u$ is affine on $Z$. Since $v \leq \Phi^v_\alpha$ (because $v$ satisfies Lemma 5’s property (b)) and $v \geq \Phi^v_\alpha$ on $\mathcal{Y}$ (by definition of $\Phi^v_\alpha$), we have $v = \Phi^v_\alpha$ on $\mathcal{Y}$, and thus $v$ is affine on $Z \subseteq \mathcal{Y}$ since $\Phi^v_\alpha$ is. We may assume without loss of generality that there are $\hat{x}, \hat{z} \in Z$ with $\hat{x} < x < \hat{z}$, since otherwise $P(Z = x|X = x) = 1$ (as $E(Z|X = x) = x$ and $P(Z \in Z|X = x) = 1$), in which case we may replace $Z$ by $\{x\}$, on which $u$ is trivially affine. It suffices to show that $u(\hat{x})_\alpha u(\hat{z}) = u(x)$ for any such $\hat{x}$ and $\hat{z}$, where $\alpha \in [0, 1]$ is given by $\hat{x}_\alpha \hat{z} = x$.

So fix such a pair $\hat{x}, \hat{z}$. Since (as shown early in this proof) $u$ satisfies Lemma 6’s property (b), we have $\Phi^u_\alpha \geq u$. It follows that $\hat{x}, \hat{z}$ satisfy $(\Delta)$. Since $u$ is coarsely less convex than $v$ and $v(\hat{x})_\alpha v(\hat{z}) = v(x)$ (because $v$ is affine on $Z$), we must then have $u(\hat{x})_\alpha u(\hat{z}) = u(x)$, as desired. ■

Appendix C  Proof of Theorem 2 (p. 9)

We prove the necessity of the crater property for comparative statics in §C.1, following the logic of the sketch proof in the text (§4.2), and then prove sufficiency in §C.2. The latter proof relies on a lemma whose proof is relegated to a separate section, §C.3.
C.1 Proof of necessity

We shall rely on the following lemma.

**Lemma 7.** Suppose there are \( x' < x < X < w < w' \) in \([0, 1]\) and a function \( p : [0, 1] \to \mathbb{R} \) that is affine on \([x', X]\) and on \([X, w]\), with \( p \geq u \) on \([x', w']\) with equality on \( \{x, w\} \), and \( p'(x) < p'(w) \). Let \( F_0 \) be any atomless distribution with support \([x', w']\) such that

\[
\frac{1}{F_0(X)} \int_0^X \xi F_0(d\xi) = x \quad \text{and} \quad \frac{1}{1 - F_0(X)} \int_X^1 \xi F_0(d\xi) = w.
\]

Then \( \int_0^X F = \int_0^X F_0 \) for any distribution \( F \) optimal for \( u \) given prior \( F_0 \).

See Figure 2 (p. 11) for a graphical illustration of the hypotheses. The lemma asserts that under these hypotheses, it is strictly sub-optimal to pool states on either side of the kink point \( X \): optimal distributions \( F \) may pool within \([0, X]\) and within \([X, 1]\), but not across.

**Proof.** Fix a distribution \( F \) that is feasible given \( F_0 \) and that satisfies \( \int_0^X F < \int_0^X F_0 \). Let \( G \) be the distribution with support \( \{x, w\} \) and mean \( \mu_{F_0} \), and note that \( G \) is less informative than \( F_0 \) and that \( \int_0^X G = \int_0^X F_0 \). We will show that \( G \) is strictly better than \( F \) for \( u \) given \( F_0 \). Note that \( F(x') = 0 \) and \( F(w') = 1 \) since \( F_0 \) is atomless with support \([x', w']\).\(^{29}\) We have

\[
\int udF \leq \int pdF = p(w') - \int_0^{w'} p' F
\]

\[
= p(w') - p'(w) \left( \int_0^{w'} F \right) + \left[ p'(w) - p'(x) \right] \int_0^X F
\]

\[
< p(w') - p'(w) \left( \int_0^{w'} G \right) + \left[ p'(w) - p'(x) \right] \int_0^X G
\]

\[
= \int pdG = \int udG,
\]

where the weak inequality holds since \( u \leq p \) on \([x', w']\), the first equality is obtained by integrating by parts,\(^{30}\) the second equality holds since \( p \) is affine on \([x', X]\) and on \([X, w']\), the strict inequality holds since \( \int_0^{w'} F =

\(^{29}\)For the former, using the notation from appendix A, we have \( C_F \leq C_{F_0} \) and \( C_{F_0}(x') = 0 \leq C_F(x') \), whence \( (C_F(x) - C_F(x'))/(x - x') \leq (C_{F_0}(x) - C_{F_0}(x'))/(x - x') \) for every \( x \in (x', w'] \), so that letting \( x \downarrow x' \) yields \( F(x') \leq F_0(x') = 0 \). The latter is analogous.

\(^{30}\)Invoking e.g. Theorem 18.4 in Billingsley (1995).
$w' - \mu_{F_0} = \int_0^{w'} G, \int_0^x F < \int_0^x F_0 = \int_0^x G$ and $p'(x) < p'(w)$, the penultimate equality holds for the same reasons as the first two equalities (recalling that $G(w') = 1$), and the final equality holds because $p = u G$-a.e. 

Proof of the converse (necessity) part of Theorem 2. Let $u : [0, 1] \to \mathbb{R}$ be regular, and suppose that it does not satisfy the crater property. That means that there are $x' < y < z < w'$ in $[0, 1]$ such that $u$ is concave on $[x', y]$ and $[z, w']$ and strictly convex on $[y, z]$, we have $u'(x') \neq u'(w')$, and the tangents to $u$ at $x'$ and at $w'$ cross at coordinates $(x', y'), (z', w') \in \mathbb{R}^2$ that either violate $y \leq X' \leq z$ or satisfy $Y' > u(X')$. It cannot be that $u$ is affine on both $[x', y]$ and $[z, w']$, since that would imply $y \leq X' \leq z$ and $Y' < u(X')$. Assume that $u$ is not affine on $[x', y]$; the other case is analogous.

We seek a regular $v : [0, 1] \to \mathbb{R}$ that is coarsely more convex than $u$ and an atomless convex-support distribution $F_0$ such that $(\ast \ast)$ fails. We shall first construct a distribution $F_0$ and $x < X < w$ in $(x', w')$ such that the hypotheses of Lemma 7 are satisfied, and then construct a regular $v : [0, 1] \to \mathbb{R}$ that is coarsely more convex than $u$ and a distribution $F$ that is optimal for $v$ given prior $F_0$ and has $\int_0^X F < \int_0^X F_0$. Then by Lemma 7, every distribution $G$ that is optimal for $u$ given $F_0$ satisfies $\int_0^X F < \int_0^X F_0 = \int_0^X G$, so fails to be less informative than $F$—thus $(\ast \ast)$ fails.

We consider separately the cases in which $u$ is not and $u$ is affine on $[z, w']$. (The sketch proof in the text corresponds to the first case.)

Case 1: $u$ is not affine on $[z, w']$. In this case, there are $x \in (x', y)$ and $w \in (z, w')$ such that $u'(x) < u'(w)$ and the tangents to $u$ at $x$ and at $w$ intersect at coordinates $(X, Y) \in \mathbb{R}^2$ satisfying $y \leq X \leq z$ and $Y > u(X)$.

Let $p : [0, 1] \to \mathbb{R}$ be the pointwise maximum of the two tangents (refer to Figure 2 on p. 11); it is affine on $[x', X]$ and on $[X, w']$, satisfies $p \geq u$ on $[x', w']$ with equality on $\{x, w\}$, and $p'(x) < p'(w)$. Let $F_0$ be a distribution that is atomless with support $[x', w']$,

$$\frac{1}{F_0(x)} \int_0^X \xi F_0(d\xi) = x \quad \text{and} \quad \frac{1}{1 - F_0(x)} \int_X^1 \xi F_0(d\xi) = w.$$ 

Observe that the hypotheses of Lemma 7 are satisfied.

Since $u'$ is bounded, we may choose a regular $v : [0, 1] \to \mathbb{R}$ that coincides with $u$ on $[X, 1]$ and that weakly exceeds $u$ and is strictly convex on $[0, X]$.

---

31This is without loss since if $u'(x') = u'(w')$ then we may choose $x'$ and $w'$ differently (in particular, closer together) such that $u'(x') \neq u'(w')$.

32If the tangent to $u$ at $x'$ (at $w'$) strictly exceeds $u$ at $z$ (at $y$), choose $x$ (w) such that the tangent to $u$ at $x$ (at $w$) crosses $u$ at $z$ (at $y$); if not, choose $x := x' + \varepsilon$ ($w := w' - \varepsilon$) for a sufficiently small $\varepsilon \in (0, (w' - x')/2)$. 

35
(refer to Figure 2 on p. 11). It is easily seen that \( v \) is coarsely more convex than \( u \). By Lemma 7, it suffices to exhibit a distribution \( F \) that is optimal for \( v \) given prior \( F_0 \) and that satisfies \( \int_0^X F < \int_0^X F_0 \).

It is easily verified (see footnote 11 on p. 12) that there are \( a \in [0, X) \) and \( b \in (z, w) \) which satisfy

\[
\frac{v(b) - v(a)}{b - a} = v'(b) \quad \text{and} \quad b := \frac{1}{1 - F_0(a)} \int_a^1 \xi F_0(\xi) d\xi.
\]

Define a distribution \( F \) by \( F := F_0 \) on \([0, a)\), \( F := F_0(a) \) on \([a, b)\) and \( F := 1 \) on \([b, 1]\). Clearly \( F \) is feasible given \( F_0 \), and \( \int_0^X F < \int_0^X F_0 \) since \( a < X \leq z < b \).

It remains to prove that \( F \) is optimal for \( v \) given \( F_0 \). To this end, let \( q : [0, 1] \to \mathbb{R} \) match \( v \) on \([0, a] \cup \{b\} \) and be affine on \([a, 1]\). By inspection, \( q \) exceeds \( v \) on \([x', w']\), is convex on \([x', w']\), has \( \int q dF_0 = \int q dF \), and satisfies \( q = v \) \( F \)-a.e. Using each of these facts in turn, we obtain for any distribution \( G \) that is feasible given \( F_0 \) (i.e. any \( G \) less informative than \( F_0 \)) that

\[
\int v dG \leq \int q dG \leq \int q dF_0 = \int q dF = \int v dF.
\]

**Case 2: \( u \) is affine on \([z, w']\).** In this case, there is an \( x \in (x', y) \) such that the tangent to \( u \) at \( x \) crosses \( u \) at some \( X \in (z, w') \). Fix any \( w \in (X, w') \), and let \( p : [0, 1] \to \mathbb{R} \) match the aforementioned tangent on \([x', X] \) and match \( u \) on \([X, w'] \); it is affine on \([x', X] \) and on \([X, w'] \), satisfies \( p \geq u \) on \([x', w'] \) with equality on \( \{x, w\} \), and \( p'(x) < p'(w) \). Let \( F_0 \) be a distribution that is atomless with support \([x', w'] \),

\[
\frac{1}{F_0(X)} \int_0^X \xi F_0(\xi) d\xi = x \quad \text{and} \quad \frac{1}{1 - F_0(X)} \int_X^1 \xi F_0(\xi) d\xi = w.
\]

The hypotheses of Lemma 7 are satisfied.

Let \( v : [0, 1] \to \mathbb{R} \) be regular, affine on \([z, w'] \), and strictly convex on \([0, z] \) and \([w', 1]\). Clearly \( v \) is coarsely more convex than \( u \). Thus by Lemma 7, it suffices to exhibit a distribution \( F \) that is optimal for \( v \) given prior \( F_0 \) and that satisfies \( \int_0^X F < \int_0^X F_0 \) (i.e. states on either side of \( X \) are pooled).

To that end, let

\[
b := \frac{1}{1 - F(z)} \int_z^1 \xi F_0(\xi) d\xi.
\]

\[\text{We could instead appeal to the duality theorem of Dworczak and Martini (2019).}\]
and define a distribution \( F \) by \( F := F_0 \) on \([0, z)\), \( F := F_0(z) \) on \([z, b)\) and \( F := 1 \) on \([b, 1]\). Then \( \int_0^X F < \int_0^X F_0 \),\(^3\) and \( F \) is optimal for \( v \) given \( F_0 \). □

### C.2 Proof of sufficiency

Given any distribution \( F \), let \( C_F : [0, 1] \to \mathbb{R} \) be given by \( C_F(x) := \int_0^x F \) for each \( x \in [0, 1] \). We shall make free use of the order isomorphism described in appendix A between distributions \( F \) ordered by informativeness and convex functions \( C_F \) ordered by pointwise inequality.

The sufficiency proof relies on three lemmata. The first is a version of Dworczak and Martini’s (2019) duality theorem. Given any regular \( u : [0, 1] \to \mathbb{R} \), let \( \mathcal{M}(u) \) denote the space of all convex and Lipschitz continuous functions \( p : [0, 1] \to \mathbb{R} \) satisfying \( p \geq u \).

**Lemma 8.** Let \( u : [0, 1] \to \mathbb{R} \) be regular, and let \( F_0 \) be an atomless distribution. Then

\[
\min_{p \in \mathcal{M}(u)} \int p dF_0 = \max_{\text{\(F\) feasible given \(F_0\)}} \int u dF,
\]

where both sides are well-defined. Moreover, for \( p \in \mathcal{M}(u) \) and a distribution \( F \) feasible given \( F_0 \) to solve (respectively) the minimisation and maximisation problems, it is necessary and sufficient that both

(a) \( p \) is affine on any interval on which \( C_F < C_{F_0} \), and

(b) \( p = u \) on \( \text{supp}(F) \).

**Proof of Lemma 8.** Fix a distribution \( F_0 \). The result is trivial if \( F_0 \) is degenerate, so suppose not. Since \( u \) is regular, for any convex and continuous \( q : [0, 1] \to \mathbb{R} \) such that \( q \geq u \), there is a \( p \in \mathcal{M}(u) \) such that \( p \leq q \). Thus the first part follows from Theorem 1(ii) in Dizdar and Kováč (2020) applied to the restriction of \( u \) to \( \text{supp}(F_0) \), since \( u \) is regular.

For the second part, fix any \( p \in \mathcal{M}(u) \) and any distribution \( F \) that is feasible given \( F_0 \). Since \( F_0 \) is atomless, we have \( F_0(0) = 0 \) and thus \( F(0) = 0 \).\(^3\) Because \( p \) is convex and Lipschitz, we may extend its derivative \( p' : (0, 1) \to \mathbb{R} \) continuously to \([0, 1] \) by letting \( p'(0) \) and \( p'(1) \) be the right-

\(^3\)If \( b \geq X \), then \( \int_z^X (F_0 - F) = \int_z^X [F_0 - F_0(z)] > 0 \) as \( F_0 > F_0(z) \) on \((z, 1]\), while if \( b \leq X \) we have \( \int_b^X (F_0 - F) = \int_b^X (F - F_0) = \int_b^X (1 - F_0) > 0 \) as \( F_0 < 1 \) on \([0, w')\).

\(^3\)We have \( C_F \leq C_{F_0} \) and \( C_{F_0}(0) = 0 \leq C_F(0) \), whence \[ C_F(x) - C_F(0) \leq C_{F_0}(x) - C_{F_0}(0) \] for every \( x \in (0, 1] \), so that letting \( x \downarrow 0 \) yields \( F(0) \leq F_0(0) = 0 \).
and left-hand derivatives at 0 and at 1, respectively. Then for any distribution \( G \) with \( G(0) = 0 \), integrating by parts twice,

\[
\int p \, dG = p(1) - \int p' G = p(1) - p'(1) C_G(1) + \int C_G dp',
\]

where the last term is to be understood in the Lebesgue–Stieltjes sense. Thus

\[
\int p \, dF \geq \int p' \, dG \geq \int u \, dF,
\]

where the first inequality is strict unless (a) holds, while the second is strict unless (b) holds since \( p \) and \( u \) are continuous. 

\[\blacksquare\]

**Lemma 9.** Let \( u : [0, 1] \to \mathbb{R} \) be regular and satisfy the crater property, and suppose there are \( x < z \) in \([0, 1]\) such that the tangent to \( u \) at \( x \) (at \( z \)) weakly exceeds \( u \) on \([x, z]\). Then there is a \( y \in (x, z) \) (a \( y \in [x, z] \)) such that \( u \) is concave on \([x, y]\) (on \([y, z]\)) and strictly convex on \([y, z]\) (on \([x, y]\)).

**Proof of Lemma 9.** Suppose that the tangent to \( u \) at \( x \) weakly exceeds \( u \) on \([x, z]\); the other case is analogous. Let \( y \) be the largest \( y' \in [x, z] \) such that \( u \) is concave on \([x, y']\) (on \([y', z]\)) and strictly convex on \([y', z]\) (on \([x, y]\)).

Let \( \tilde{z} \) be the largest \( w \in [y, 1] \) such that \( u \) is strictly convex on \([y, w]\); clearly \( \tilde{z} > y \) by the regularity of \( u \). We must show that \( \tilde{z} \geq z \), so suppose toward a contradiction that \( \tilde{z} < z \). Then by regularity, \( u \) is concave on \([\tilde{z}, w]\) for some \( w \in (\tilde{z}, z] \). But then \( u \) violates the crater property, since the tangent to \( u \) at \( x \) strictly exceeds \( u \) on \([y, \tilde{z}]\) (as \( u \) is strictly convex on \([y, \tilde{z}]\)). 

\[\blacksquare\]

**Lemma 10.** Let \( u, v : [0, 1] \to \mathbb{R} \) be regular, and suppose that \( u \) satisfies the crater property and is coarsely less convex than \( v \). Let \( F_0 \) be an atomless convex-support distribution. Then for any

\[
p \in \arg\min_{r \in \mathcal{M}(u)} \int r \, dF_0 \quad \text{and} \quad q \in \arg\min_{r \in \mathcal{M}(v)} \int r \, dF_0,
\]

if \( q \) is affine on an interval \([x, y] \subseteq \text{supp}(F_0)\), then so is \( p \).

Lemma 10 is proved in the next section.

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36This is licensed by e.g. Theorem 18.4 in Billingsley (1995).
Proof of the first (sufficiency) part of Theorem 2. Fix regular \( u, v : [0, 1] \to \mathbf{R} \) such that \( u \) satisfies the crater property and is coarsely less convex than \( v \), let \( F_0 \) be an atomless convex-support distribution, and fix

\[
G' \in \arg \max_{F \text{ feasible given } F_0} \int udF \quad \text{and} \quad H' \in \arg \max_{F \text{ feasible given } F_0} \int vdF.
\]

We shall construct

\[
G'' \in \arg \max_{F \text{ feasible given } F_0} \int udF \quad \text{and} \quad H'' \in \arg \max_{F \text{ feasible given } F_0} \int vdF
\]
such that \( G'' \) is less informative than \( H' \) and \( G' \) is less informative than \( H'' \).

We derive \( G'' \) from \( G' \) by fully pooling signal realisations over each concavity interval of \( u \), in the following precise sense. Assume without loss of generality that \( u \) is not strictly convex, and enumerate the maximal proper intervals on which \( u \) is concave as \( \{(x_k, z_k)\}_{k=1}^K \) (where \( K \in \mathbf{N} \)). For each \( k \), let \( y_k \) denote the mean of \( G' \) conditional on the event \([x_k, z_k] \). (In case \([x_k, z_k]\) is \( G' \)-null, let \( y_k \) be an arbitrary element of \([x_k, z_k]\).) Define a distribution \( G'' \) by

\[
G''(w) := \begin{cases} 
G'(x_k-) & \text{if } w \in [x_k, y_k) \text{ for some } k \in \{1, \ldots, K\} \\
G'(z_k) & \text{if } w \in [y_k, z_k] \text{ for some } k \in \{1, \ldots, K\} \\
G'(w) & \text{otherwise,}
\end{cases}
\]

where ‘\( G(x-) \)’ is shorthand for \( \lim_{y \downarrow x} G'(y) \). For any \( G' \)-non-null \([x_k, z_k]\), the distribution ‘\( G'' \)’ conditional on \([x_k, z_k]\)’ is less informative than the distribution ‘\( G' \)’ conditional on \([x_k, z_k]\)’,\(^{37}\) so \( \int_{[x_k, z_k]} udG'' \geq \int_{[x_k, z_k]} udG' \). And we have \( G'' = G' \) on \( \mathcal{X} := [0, 1] \setminus \bigcup_{k=1}^K [x_k, z_k] \), so that \( \int_{\mathcal{X}} udG'' = \int_{\mathcal{X}} udG' \) since \( \mathcal{X} \) is open. Thus \( \int udG'' \geq \int udG' \), which since \( G' \) optimal for \( u \) given \( F_0 \) implies that \( G'' \) is, too.

We similarly derive \( H'' \) from \( H' \) by spreading signal realisations over each convexity interval of \( v \) as much as possible subject keeping \( H'' \) less informative than the prior \( F_0 \). Formally, assume without loss of generality that \( v \) is not strictly concave, enumerate the maximal proper intervals on which \( v \) is convex as \((I_t)_{t=1}^L \) (where \( L \in \mathbf{N} \)), and define \( I := \bigcup_{t=1}^L I_t \). Let \( C \) be the convex envelope of \( \mathbf{1}_I C_{F_0} + \mathbf{1}_{[0, 1]} \setminus \mathcal{C}_{H'} \), and let the distribution \( H'' \) be be defined by \( C_{H''} = C \). We have \( H'' = H' \) off \( I \), and clearly ‘\( H'' \)’ conditional on \( I_t \)’ is more informative than ‘\( H' \)’ conditional on \( I_t \)’ for each \( H' \)-non-null

\(^{37}\)Explicitly: the distribution \( \chi_{(x_k, z_k]} + \chi_{[x_k, z_k]} \times \frac{|G''(z_k) - G''(x_k-)|/|G''(z_k) - G''(x_k-)|}{|G'(z_k) - G'(x_k-)|/|G'(z_k) - G'(x_k-)|} \) is less informative than the distribution \( \chi_{(x_k, z_k]} + \chi_{[x_k, z_k]} \times \frac{|G'(z_k) - G'(x_k-)|/|G'(z_k) - G'(x_k-)|}{|G''(z_k) - G''(x_k-)|/|G''(z_k) - G''(x_k-)|} \).
\( I_t, \) so \( \ell \sum_{t=1}^{T} I_t \| H'' - H' \| \geq 0, \) which since \( H' \) is optimal for \( v \) given prior \( F_0 \) implies that \( H'' \) is, too.

It remains to prove that \( G'' \) is less informative than \( H' \) and that \( G' \) is less informative than \( H'' \). We shall rely on the following claim, whose proof (relegated to the end) hinges on Lemmata 9 and 10.

**Claim.** Let \( G \) and \( H \) be optimal (given prior \( F_0 \)) for \( u \) and \( v \), respectively. Then for any \( a < b \) in \([0, 1]\) such that \( C_H \subset C_{F_0} \) on \( (a, b) \) and \( C_H = C_{F_0} \) on \( \{a, b\} \), there are \( c \leq d \) in \( \text{supp}(G) \) such that \( C_G \leq C_H \) on \( [a, b] \setminus (c, d) \) and \( u \) is affine on \([c, d]\).

To prove that \( G'' \) is less informative than \( H' \), it suffices to show that for any \( a < b \) in \([0, 1]\) such that \( C_{H'} \subset C_{F_0} \) on \( (a, b) \) and \( C_{H'} = C_{F_0} \) on \( \{a, b\} \), we have \( C_{G''} \leq C_{H'} \) on \( (a, b) \). So fix such a pair \( a < b \). By the claim, there are \( c \leq d \) in \( \text{supp}(G') \) such that \( C_{G''} \leq C_{H'} \) on \( [a, b] \setminus (c, d) \) and \( u \) is affine on \([c, d]\). And \( (c, d) \) is empty, since \( \text{supp}(G'') \cap [c, d] \) must be a singleton by definition of \( G'' \) and the fact that \( u \) is concave on \([c, d]\).

Similarly, to prove that \( G' \) is less informative than \( H'' \), it suffices to show that for any \( a < b \) in \([0, 1]\) such that \( C_{H''} \subset C_{F_0} \) on \( (a, b) \) and \( C_{H''} = C_{F_0} \) on \( \{a, b\} \), we have \( C_{G'} \leq C_{H''} \) on \( (a, b) \). So fix such a pair \( a < b \). By the claim, there are \( c \leq d \) in \( \text{supp}(G') \) such that \( C_{G'} \leq C_{H''} \) on \( [a, b] \setminus (c, d) \) and \( u \) is affine on \([c, d]\). If \([a, b] \) and \([c, d]\) are disjoint, then we are done. Suppose for the remainder that \( [a, b] \cap [c, d] \) is non-empty. We must show that \( C_{G'} \leq C_{H''} \) on \([a', b'] := [a, b] \cap [c, d]\).

\( v \) is convex on \([a', b'] \) since \([a', b'] \subset [c, d]\), so by definition of \( H'' \), the restriction of \( C_{H''} \) to \([a', b'] \) equals the convex envelope of \( 1_{(a', b')} F_0 + 1_{(a', b')} C_{H''} \). We have \( C_{G'} \leq 1_{(a', b')} F_0 + 1_{(a', b')} C_{H''} \) on \([a', b'] \) by hypothesis and the fact that \( G' \) is less informative than the prior \( F_0 \). Since \( C_{G'} \) is convex, \( G' \) must satisfy \( C_{G'} \leq C_{H''} \) on \([a', b'] \).

**Proof of the claim.** Fix \( a < b \) in \([0, 1]\) such that \( C_H \subset C_{F_0} \) on \((a, b)\) and \( C_H = C_{F_0} \) on \( \{a, b\} \). Note that \([a, b] \subset \text{supp}(F_0) \) since the latter is convex. Since \( u \) and \( v \) are regular, Lemma 8 provides that there exist

\[
p \in \arg \min_{r \in \mathcal{M}(u)} \int r dF_0 \quad \text{and} \quad q \in \arg \min_{r \in \mathcal{M}(v)} \int r dF_0,
\]

and that \( q \) is affine on \([a, b]\). By Lemma 10, it follows that \( p \) is also affine on \([a, b]\). Write \([a', b']\) for the maximal interval \( I \) such that \( p \) is affine on \( I \) and

---

\textsuperscript{38} At \( a' \), we have if \( a' = c \) that \( C_{G'}(a') \leq C_{H''}(a') \), and if not then \( a' = a \), in which case \( C_{G'}(a') \leq C_{F_0}(a') = C_{H''}(a') \) since \( G' \) is less informative than \( F_0 \). Similarly at \( b' \).
Theorem 1. Let $u : [0, 1] \to \mathbb{R}$ be regular, let $F_0$ be an atomless convex-support distribution, and let $p$ minimise $\int pdF_0$ over $M(u)$. Then

(i) for any $x < z$ such that $[x, z]$ is maximal among the intervals of affineness of $p$ within $\text{supp}(F_0)$, there are

$$x < y \leq \frac{\int_x^z \xi F_0(d\xi)}{F_0(z) - F_0(x)} \leq y' < z$$

such that $p(y) = u(y)$ and $p(y') = u(y')$, and

(ii) if $p(y) > u(y)$ for some $y \in \text{supp}(F_0)$ such that $F_0(y) > 0$ ($F_0(y) < 1$), then $y > 0$ and there is $x \in [0, y)$ ($y < 1$ and there is $z \in (y, 1]$) such that $p$ is affine on $[x, y]$ (on $[y, z]$).

C.3 Proof of Lemma 10

We rely on the following result, which follows from Lemmata 8 and 9.

Corollary 3. Let $u : [0, 1] \to \mathbb{R}$ be regular, let $F_0$ be an atomless convex-support distribution, and let $p$ minimise $\int pdF_0$ over $M(u)$. Then

(i) for any $x < z$ such that $[x, z]$ is maximal among the intervals of affineness of $p$ within $\text{supp}(F_0)$, there are

$$x < y \leq \frac{\int_x^z \xi F_0(d\xi)}{F_0(z) - F_0(x)} \leq y' < z$$

such that $p(y) = u(y)$ and $p(y') = u(y')$, and

(ii) if $p(y) > u(y)$ for some $y \in \text{supp}(F_0)$ such that $F_0(y) > 0$ ($F_0(y) < 1$), then $y > 0$ and there is $x \in [0, y)$ ($y < 1$ and there is $z \in (y, 1]$) such that $p$ is affine on $[x, y]$ (on $[y, z]$).
Moreover, if $u$ satisfies the crater property, then

(iii) given $x < y$ such that $[x, y]$ is maximal among the intervals of affineness of $p$ within supp$(F_0)$, and $F_0(x) > 0$ ($F_0(y) < 1$), it holds that $p(x) = u(x)$ ($p(y) = u(y)$), that $u$ is convex and not affine on some open interval $I$ containing $x$ ($y$), and that

$$ u' < (>) \frac{p(y) - p(x)}{y - x} \quad \text{on} \ (0, x) \cap I \ (\text{on} \ (y, 1) \cap I). $$

Proof of Corollary 3. Fix $F$ maximising $\int u dF$ among distributions feasible given $F_0$. For (i), fix $x < z$ such that $[x, z]$ is maximal among intervals of affineness of $p$ within supp$(F_0)$. Then $C_F = C_{F_0}$ on $\{x, z\}$ by Lemma 8.\(^{39}\) Then $(x, z)$ is $F$-non-null since $F_0$ has convex support,\(^{40}\) and thus there are $y, y' \in \text{supp}(F)$ such that

$$ x < y \leq \int_{(x,z)} \xi F(d\xi) \int_{(x,z)} dF \leq y' < z. $$

By (b), $p(y) = u(y)$ and $p(y') = u(y')$. Finally, since $C_F = C_{F_0}$ on $\{x, z\}$ and $F_0$ is atomless, $F = F_0$ on $\{x, z\}$ and $F$ is continuous at $x$ and $z$, so that

$$ \frac{\int_x^z \xi F(d\xi)}{\int_{(x,z)} dF} = \frac{z F(z) - x F(x) - [C_F(z) - C_F(x)]}{F(z) - F(x)} = \frac{\int_x^z \xi F_0(d\xi)}{F_0(z) - F_0(x)}. $$

This proves (i).

For (ii), suppose that $p(y) > u(y)$ for some $y \in \text{supp}(F_0)$ such that $F_0(y) > 0$ (the case $F_0(y) < 1$ is analogous). Then $y \notin \text{supp}(F)$ by (b), so that $C_F$ is affine on a neighbourhood of $y$. Moreover, $y > \min \text{supp}(F_0)$ since $F_0$ is atomless. Then, $y > 0$ and, since supp$(F_0)$ is convex and $C_{F_0}$ is strictly convex on supp$(F_0)$, there is $x \in [0, y)$ such that $C_F < C_{F_0}$ on $[x, y)$. Hence, $p$ is affine on $[x, y]$ by (a), as $p$ is continuous.

For (iii), fix $x < y$ such that $[x, y]$ is maximal among intervals of affineness of $p$ within supp$(F_0)$, and $F_0(x) > 0$ (the case $F_0(y) < 1$ is analogous). By (i), there is $w \in (x, y)$ such that $p(w) = u(w)$, so that $p$ is tangent to $u$ at $w$. Then, there is $z \in [x, w)$ such that $u$ is strictly convex on $[x, z]$ and

\(^{39}\)If e.g. $C_F(x) < C_{F_0}(x)$, then $x$ lies in the interior of supp$(F_0)$, $C_F < C_{F_0}$ on a neighbourhood of $x$, and $p$ is affine on this neighbourhood by (a), contradicting the definition of $[x, z]$.

\(^{40}\)Since $F_0$ has convex support, $C_{F_0}$ is not affine on $[x, z]$. Then, neither is $C_F$, and thus supp$(F) \cap (x, z)$ is not empty.
concave on \([z, w]\), by Lemma 9. Let \(b := \min \text{supp}(F_0)\) and \(a\) be the smallest \(a' \in [b, x]\) such that \(p\) is affine on \([a', x]\). We consider two cases.

Case 1. \(a = x\). Note that \(x > b\) since \(F_0\) is atomless and \(F_0(x) > 0\). Then, by the hypothesis of this case, there exists an increasing sequence \((x_k)_{k \in \mathbb{N}} \subseteq (b, x)\) such that \(\lim_k x_k = x\) and on which \(C_F = C_{F_0}\), by (a). Then, there exists an increasing sequence \((y_k)_{k \in \mathbb{N}} \subseteq (b, x) \cap \text{supp}(F)\) such that \(\lim_k y_k = x\), since \(C_{F_0}\) is strictly convex on \(\text{supp}(F_0)\). By (b), \(p(y_k) = u(y_k)\) for each \(k \in \mathbb{N}\). Then, since \(p\) is convex and \(u\) is regular, by the hypothesis of this case, \(u\) is convex and not affine on \([y_{k'}, x]\) for some \(k' \in \mathbb{N}\), and

\[
u' < \frac{p(y) - p(x)}{y - x} \quad \text{on } (y_{k'}, x).
\]

Moreover, \(p(x) = u(x)\) and thus \(u\) is affine on \([x, w]\) if \(z = x\), since \(p \geq u\) with equality on \(\{x, w\}\) and \(u\) is concave on \([z, w]\). The result follows by choosing \(I = (y_{k'}, z)\) if \(z > x\), and \(I = (y_{k'}, w)\) otherwise.

Case 2. \(a < x\). In this case, there is \(\hat{x} \in (a, x)\) such that \(p(\hat{x}) = u(\hat{x})\), by (i). Then, \(p\) is tangent to \(u\) at \(\hat{x}\), and thus there is \(\hat{y} \in (\hat{x}, x)\) such that \(u\) is concave on \([\hat{x}, \hat{y}]\), and strictly convex on \([\hat{y}, x]\), by Lemma 9. Define

\[
I := \begin{cases} 
(\hat{y}, z) & \text{if } \hat{y} < x < z \\
(\hat{x}, z) & \text{if } \hat{y} = x \\
(\hat{y}, w) & \text{if } x = \hat{y}.
\end{cases}
\]

Note that \(\hat{y} < z\), for otherwise \(u\) would be concave on \([\hat{x}, w]\) and thus \(p\) would be affine on \([\hat{x}, w]\) (since \(p = u\) on \([\hat{x}, w]\)) contradicting \(\hat{x} < x\). Then \(I\) contains \(x\), since \(\hat{x} < \hat{y} \leq x \leq z < w\).

To show that \(u\) is convex and not affine on \(I\), note that \(u\) is strictly convex on \([\hat{y}, z]\), as it is regular and strictly convex on \([\hat{y}, x]\) and \([x, z]\). Then \(p(x) = u(x)\), since \(u\) satisfies the crater property and, clearly, the tangents to \(u\) at \(\hat{x}\) and \(w\) intersect at \((x, p(x))\). Hence \(u\) is affine on \([\hat{x}, x]\) (on \([x, w]\)) if \(\hat{y} = x\) \((x = z)\), since \(u\) is concave on \([\hat{x}, \hat{y}]\) with \(u(\hat{x}) = p(\hat{x})\) (on \([z, w]\) with \(u(w) = p(w)\)). Since \(\hat{y} < z\) and \(u\) is strictly convex on \([\hat{y}, z]\), \(u\) is convex and not affine on \(I\).

It remains to show that

\[
u' < \frac{p(y) - p(x)}{y - x} \quad \text{on } (0, x) \cap I.
\]

\[41\]To see why this last property must hold, suppose it were to fail. Then there is a sequence \((z_k)_{k=1}^\infty \subseteq (b, x)\) with \(\lim_{k \to \infty} z_k = x\) such that \(u'(z_k) > |p(y) - p(x)|/(y - x)\). Since \(u\) is regular, it follows that \(u' \geq |p(y) - p(x)|/(y - x)\) on \([y_{k'}, x]\) for some \(k' \in \mathbb{N}\). But then \(p\) is affine on \([y_{k'}, x]\) since \(u(y_{k'}) = p(y_{k'})\), contradicting the hypothesis of this case.
To this end, since $u$ is convex on $I$, we may assume without loss of generality that
\[
    u'(x) \geq \frac{p(y) - p(x)}{y - x}.
\]
Then $x = z$ and equality holds, since $p \geq u$ with equality at $x$ and $u$ is strictly convex on $[x, z]$. The result follows since $\bar{y} < z$ and $u$ is strictly convex on $[\bar{y}, x]$.

**Proof of Lemma 10.** Fix $F_0$, $p$ and $q$. Suppose toward a contradiction that there exist $\bar{x} < \bar{z}$ in supp($F_0$) such that $q$ is affine on $[\bar{x}, \bar{z}]$, but $p$ is not. Assume without loss of generality that $[\bar{x}, \bar{z}]$ is maximal among the intervals of affineness of $q$ within supp($F_0$). We consider two cases.

**Case 1.** $u$ is convex on $[\bar{x}, \bar{z}]$. We shall construct $a \in [0, \bar{x}]$ such that $u$ is concave on $[a, \bar{z}]$ and $p(a) = u(a)$. A similar argument yields $b \in [\bar{z}, 1]$ such that $u$ is concave on $[\bar{x}, b]$ and $p(b) = u(b)$. Then $u$ is concave on $[a, b]$ and thus $p$ is affine on $[a, b]$; contradicting the fact that $p$ is not affine on $[\bar{x}, \bar{z}] \subseteq [a, b]$.

To construct $a$, note that $v$ is convex on $[\bar{x}, \bar{z}]$ by the hypothesis of this case, since $u$ is coarsely less convex than $v$. Then $v$ is affine on $[\bar{x}, \bar{z}]$ by (i) (since (i) implies that $q(y) = v(y)$ for some $y \in (\bar{x}, \bar{z})$). Then so is $u$, as it is coarsely less convex than $v$. Then, if $p(\bar{x}) = u(\bar{x})$, we may take $a = \bar{x}$. Hence, assume without loss of generality that $p(\bar{x}) > u(\bar{x})$.

Let $\bar{z}$ be the largest $z \in [\bar{x}, 1]$ such that $p$ is affine on $[\bar{x}, z]$. Then $\bar{z} < \bar{z}$ by hypothesis, and $\bar{z} < \bar{x}$ by (ii) (which is applicable since $F_0(\bar{x}) < 1$). Let $\bar{x}$ be the smallest $x \in [0, \bar{x}] \cap \text{supp}(F_0)$ such that $p$ is affine on $[x, \bar{z}]$. By (i), there is $a \in (\bar{x}, \bar{z})$ such that $p(a) = u(a)$. And $a$ belongs to $[0, \bar{x}]$ since $u$ and $p$ are affine on $[\bar{x}, \bar{z}]$ and since $p \geq u$, with strict inequality at $\bar{x}$.

It remains to prove that $u$ is concave on $[\bar{a}, \bar{z}]$. As $u$ is affine on $[\bar{x}, \bar{z}]$ and regular, and $\bar{x} < \bar{z} < \bar{z}$, it suffices to show that $u$ is concave on $[\bar{a}, \bar{z}]$. Note that $p$ is tangent to $u$ at $a$ as $\bar{x} < a < \bar{z}$ and $p(a) = u(a)$. Then $u$ is concave on $[\bar{a}, \bar{z}]$ by Lemma 9, as $p \geq u$ on $[\bar{a}, \bar{z}]$, and $u$ and $p$ are affine on $[\bar{x}, \bar{z}]$.

**Case 2.** $u$ is not convex on $[\bar{x}, \bar{z}]$. In this case, since $u$ is regular, there are $\bar{x} \leq c < d \leq \bar{z}$ such that $[c, d]$ is maximal among the intervals within $[\bar{x}, \bar{z}]$ on which $u$ is strictly concave. Then $p$ and $u$ differ somewhere in $(c, d)$ and thus, by (ii), $p$ is not strictly convex on $(c, d)$. Hence there are $\bar{x} < \bar{z}$ such that $[\bar{x}, \bar{z}]$ is maximal among the intervals of affineness of $p$ within supp($F_0$), and $[\bar{x}, \bar{z}] \cap (c, d)$ is not empty. Since $p$ is not affine on $[\bar{x}, \bar{z}]$, either $\bar{x} < \bar{x}$ or $\bar{z} < \bar{z}$. We consider the case $\bar{x} < \bar{x}$; the other is analogous.

\[42\text{Indeed, Lemma 9 yields } y \in (a, \bar{z}) \text{ such that } u \text{ is concave on } [a, y] \text{ and strictly convex on } [y, \bar{z}]. \text{ And } y = \bar{z} \text{ since } u \text{ is affine on } [\bar{x}, \bar{z}].\]
Note that $\bar{x} < d \leq \bar{z}$, where the strict inequality holds as $[\bar{x}, \bar{z}] \cap (c, d)$ is not empty. We shall exhibit a $w \in (\bar{x}, \bar{z})$ such that
\[ u(\bar{x}) u(w) \geq u(\bar{x}, w) \quad \text{for all } \alpha \in (0, 1), \]
(2) a $\bar{y} \in (\bar{x}, w)$ such that $q(\bar{y}) = v(\bar{y})$, and show that $v(\bar{x}) < q(\bar{x})$. To see why this suffices, note that it implies that given $\alpha \in (0, 1)$ such that $\bar{x}, w = \bar{y}$,
\[ v(\bar{x}) v(w) < q(\bar{x}) v(w) = q(\bar{y}) = v(\bar{y}), \]
where the strict inequality holds since $\alpha \in (0, 1)$, $v(\bar{x}) < q(\bar{x})$ and $v(w) \leq q(w)$, and the first equality holds as $q$ is affine on $[\bar{x}, \bar{z}] \supseteq [\bar{x}, w]$. Together with (2), this contradicts the fact that $u$ is coarsely less convex than $v$.

To construct $w$ note that, by (i), there is
\[ \bar{x} < \frac{\int_{\bar{x}}^{\bar{z}} \xi F_0(d\xi)}{F_0(\bar{z}) - F_0(\bar{x})} \leq \bar{y} < \bar{z} \]
such that $p(\bar{y}) = u(\bar{y})$. Define $w := \min\{\bar{y}, \bar{z}\}$ and note that $w \in (\bar{x}, \bar{z})$. To establish (2), note $p$ is tangent to $u$ at $\bar{y}$, so that there is $\gamma \in [\bar{x}, \bar{y}]$ such that $u$ is strictly convex on $[\bar{x}, \gamma]$ and concave on $[\gamma, \bar{y}]$, by Lemma 9. Then (2) holds since $p(\bar{x}) = u(\bar{x})$ by (iii) (which is applicable since $F_0(\bar{x}) > 0$ and $\bar{x} < \bar{x}$).43

To construct $\bar{y} \in (\bar{x}, w)$ such that $q(\bar{y}) = v(\bar{y})$, let $[a, b]$ be the maximal interval of convexity of $u$ containing $\bar{x}$. (This is well-defined since $u$ is regular). Note that if $\bar{x} \in (c, d)$ then $\gamma = \bar{x}$, as $u$ is concave on $(c, d)$ and on $[\gamma, \bar{y}]$, and strictly convex on $[\bar{x}, \gamma]$. But then $u$ would be affine on $[\bar{x}, \bar{y}]$ since $p = u$ on $\{\bar{x}, \bar{y}\}$, contradicting the fact that $u$ is strictly concave on $[c, d]$. Hence $\bar{x} < c$ as $\bar{x} < d$. Then $b \leq c$, and by (iii) (applicable since $F_0(\bar{x}) > 0$ and $\bar{x} < \bar{x}$) we have that $a < \bar{x} < b$, that $u$ is not affine on $[a, b]$, and that
\[ u' < \frac{p(\bar{z}) - p(\bar{x})}{\bar{z} - \bar{x}} \quad \text{on } (a, \bar{x}). \]

(3)

We rely on the following claim, proved at the end.

Claim. $a \leq \bar{x}$ and $\bar{z} \leq \bar{x}$.

43This is easily seen graphically. It follows from the facts that $p$ is affine on $[\bar{x}, \bar{y}]$, that $p \geq u$ on $[\bar{x}, \bar{y}]$ with equality on $\{\bar{x}, \bar{y}\}$, that $u$ is convex on $[\bar{x}, \bar{z}]$ and concave on $[\bar{z}, \bar{y}]$ for some $\bar{z} \in \bar{x}, \bar{y}$, and that $\bar{x} < w \leq \bar{y}$. 

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By (i), we may choose
\[ \bar{x} < y \leq \frac{\int \sigma F_0(d\xi)}{F_0(\bar{z}) - F_0(x)} < \bar{z} \]
such that \( q(y) = v(y) \). Note that \( y < \min\{\bar{y}, \bar{z}\} = w \) since \( y < \bar{z} \) and
\[ y \leq \frac{\int \sigma w dF_0(w)}{F_0(\bar{z}) - F_0(x)} < \frac{\int \sigma w dF_0(w)}{F_0(\bar{z}) - F_0(\bar{x})} \leq \bar{y}, \]
where the strict inequality holds as \( F_0 \) has convex support, \( \bar{x} < \bar{y} \) and, by the claim, \( \bar{z} \leq \bar{y} \). Thus we may take \( \bar{y} := y \) if \( y > \bar{x} \). If instead \( y \leq \bar{x} \), note that \( v \) is convex on \([a, b]\), as \( u \) is coarsely less convex than \( v \) and convex on \([a, b]\). Moreover, \( q \) is affine on \([\bar{x}, \bar{z}]\) and \( q \geq v \) with equality at \( y \). Since \( a \leq \bar{x} < y \leq \bar{x} < b \leq c \leq \bar{z} \), it follows that \( v = q \) on \([\bar{x}, b] = [a, b] \cap [\bar{x}, \bar{z}] \). As \( \bar{x} < w \), we may then choose any \( \bar{y} \in (\bar{x}, \min\{b, w\}) \).

It remains to prove that \( v(\bar{x}) < q(\bar{x}) \). Note that, by (2) and (3),
\[ u(\bar{x})_\alpha u(w) > u(\bar{x}_\alpha w) \quad \text{for all } \alpha \in (0, 1), \]
since \( a \leq \bar{x} < \bar{y} \), and \( u \) is convex on \([a, \bar{y}]) \). Hence, choosing \( \alpha \in (0, 1) \) such that \( \bar{x} = \bar{x}_\alpha w \),
\[ q(\bar{x}) = q(\bar{x})_\alpha q(w) \geq v(\bar{x})_\alpha v(w) > v(\bar{x}), \]
where the equality holds since \( q \) is affine on \([\bar{x}, \bar{z}] \supseteq [\bar{x}, \bar{w}] \), the weak inequality as \( q \geq v \), and the strict inequality holds since \( u \) is less convex than \( v \).

**Proof of the claim.** We begin by exhibiting \( \bar{x} \leq c' < d' \leq \bar{z} \) such that \( u \) is strictly convex on \([\bar{x}, c']\) and \([d', \bar{z}]\), and concave on \([c', d']\). By (i),
\[ v(\bar{x}_\alpha \bar{z}) = q(\bar{x}_\alpha \bar{z}) = q(\bar{x})_\alpha q(\bar{z}) \geq v(\bar{x})_\alpha v(\bar{z}) \quad \text{for some } \alpha \in (0, 1), \]
where the second equality holds since \( q \) is affine on \([\bar{x}, \bar{z}]\), and the inequality holds since \( q \geq v \). Then
\[ u(\bar{x}_\alpha \bar{z}) \geq u(\bar{x})_\alpha u(\bar{z}) \quad \text{for some } \alpha \in (0, 1), \]
(4)
since \( u \) is coarsely less convex than \( v \). Hence the tangent to \( u \) at some \( a_* \in (\bar{x}, \bar{z}) \) weakly exceeds \( u \) on \([\bar{x}, \bar{z}]\), as \( u \) is regular. Therefore, by Lemma 9,

\[ \int_{\bar{x}}^{\bar{z}} \psi F_0(d\xi) \leq \int_{\bar{x}}^{\bar{z}} \psi F_0(d\xi) \]

and thus the continuous map that matches \( u \) on \([\bar{x}, \bar{z}] \cup \{w\} \) and is affine on \([\bar{x}, \bar{w}]\), is convex and not affine on \([\bar{x}, \bar{w}]\). Then the result follows from (2).
there are $c' \in [\bar{x}, a_*)$ and $d' \in (a_*, \bar{z}]$ such that $u$ is strictly convex on $[\bar{x}, c']$ and $[d', \bar{z}]$, and concave on $[c', a_*)$ and $[a_*, d']$. As $u$ is regular, it is concave on $[c', d']$, as desired.

Note that $b \leq c < d \leq \bar{z}$. Then $a \leq \bar{x}$ since $u$ is regular. Indeed, if $\bar{x} < a$ then, by definition of $a$ and $b$, there would exist $\bar{x} \leq a' < a$ and $b < b' \leq \bar{z}$ such that $u$ is strictly concave on $[a', a]$ and $[b, b']$. But then $c' \leq a'$ and $b' \leq d'$, contradicting the fact that $u$ is convex and not affine on $[a, b]$.

It remains to show that $\bar{z} \leq \bar{z}$. Suppose this fails and seek a contradiction. Then $p(\bar{z}) = u(\bar{z})$ by (iii), and thus

$$u(\bar{x})_\alpha u(\bar{z}) = p(\bar{x})_\alpha p(\bar{z}) = p(\bar{x}, \bar{z}) = u(\bar{x}, \bar{z}) \quad \text{for all } \alpha \in (0, 1),$$

where the first equality holds since $p(\bar{x}) = u(\bar{x})$, and the second since $p$ is affine on $[\bar{x}, \bar{z}]$. Moreover, $u$ is convex and not affine on some open interval $I$ containing $\bar{z}$, by (iii). Then

$$c' \leq c < d \leq \bar{z},$$

where the first inequality holds since $\bar{x} \leq c < d$ and $u$ is strictly convex on $[\bar{x}, c']$ and strictly concave on $[c, d]$, and the last inequality holds since $[\bar{x}, \bar{z}] \cap (c, d) \neq \emptyset$ and $u$ is strictly concave on $[c, d]$ and convex on $I \ni \bar{z}$.

Then $u$ is convex on $[\bar{z}, \bar{z}]$, as it is convex and not affine on $I \ni \bar{z}$, concave on $[c', d']$, and strictly convex on $[d', \bar{z}]$. Then (3) and (5) contradict (4), since $a \leq \bar{x} < \bar{x}$ and $u$ is convex on $[a, \bar{x}]$.\footnote{To see why, note that the map $1_{[x,z]}u + 1_{[x,z]}p$ is convex and not affine on $[\bar{x}, \bar{z}]$.}

With the claim established, the proof is complete. □

**Appendix D  Proof of Proposition 1 (p. 13)**

The second (converse) part of Proposition 1 follows from the proof in appendix B.1 of the second (converse) part of Theorem 1\footnote{The argument there shows that if $u$ is not coarsely less convex than $v$, then we can construct a prior $F_0$ such that $\arg\max_{\mu} \int udF$ is strictly higher than (a fortiori not lower than) $\arg\max_{\mu} \int vdF$. And the constructed prior is, in fact, binary.}.\footnote{If $F_0$ is degenerate ($F_0 = 1_{[\mu, 1]}$) then the result is trivial. If not, then $F_0$ is supported on $(x, y]$ with $x < \mu < y$, all feasible distributions have support in $[x, y]$, and $u|[x, y]$ is coarsely less convex than $v|[x, y]$; so the interval $[x, y]$ may as well be $[0, 1]$.}

To prove the first part, let $u, v : [0, 1] \to \mathbb{R}$ be upper semi-continuous, assume that $u$ is coarsely less convex than $v$, and let $F_0$ be a binary distribution. Write $\mu$ for the mean of $F_0$. Assume without loss of generality that $F_0$ is supported on $\{0, 1\}$ (that is, $F_0 = 1 - \mu + \mu 1_{\{1\}}$).\footnote{Given $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$, let us write $x_\alpha y := \alpha x + (1 - \alpha)y$.} Given $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$, let us write $x_\alpha y := \alpha x + (1 - \alpha)y$.\footnote{To see why, note that the map $1_{[x,z]}u + 1_{[x,z]}p$ is convex and not affine on $[\bar{x}, \bar{z}]$.}
Write \( \text{cav} \ u \) for the concave envelope of \( u \). Let \([x, w]\) be the maximal interval containing \( \mu \) on which \( \text{cav} \ u \) is affine. Define
\[
\mathcal{U} := \{u = \text{cav} u\} \cap [x, w],
\]
and note that \( x, w \in \mathcal{U} \) since \( u \) is upper semi-continuous. Further define
\[
y := \sup(\mathcal{U} \cap [0, \mu]) \quad \text{and} \quad z := \inf(\mathcal{U} \cap [\mu, 1]),
\]
and note that \( y, z \in \mathcal{U} \) by upper semi-continuity. Clearly \( x \leq y \leq \mu \leq z \leq w \).

Let
\[
M(u) := \arg \max_{F \text{ feasible given } \mathcal{F}_0} \int udF.
\]
Kamenica and Gentzkow (2011) showed that \( M(u) \) is the set of all mean-\( \mu \) distributions \( F \) such that \( \int udF = (\text{cav} u)(\mu) \). Thus \( M(u) \) is the set of all mean-\( \mu \) distributions supported on \( \mathcal{U} \). It follows that the distribution \( G \) (\( H \)) with mean \( \mu \) and support \( \{y, z\}\) (\( \{x, w\}\)) is the least (most) informative distribution in \( M(u) \).

For the function \( v \), analogously define \( \mathcal{V} \subseteq [0, 1] \), \( x', y', z', w' \in \mathcal{V} \), and distributions \( G', H' \) in \( M(v) \). We must show that \( H \) is less informative than \( H' \) and that \( G \) is less informative than \( G' \). The former requires precisely that \( x' \leq x \) and \( w \leq w' \), while the latter requires that \( y' \leq y \) and \( z \leq z' \).

We first show that \( x' \leq x \) and \( w \leq w' \). Since \( x, w \in \mathcal{U} \), we have \( u(x_\alpha w) \leq u(x_\alpha)u(w) \) for every \( \alpha \in (0, 1) \). As \( u \) is coarsely less convex than \( v \), it follows that \( v(x_\alpha w) \leq v(x_\alpha)v(w) \) for each \( \alpha \in (0, 1) \), implying that \([x, w] \subseteq [x', w']\).

**Claim.** \( \mathcal{V} \cap [x, w] \subseteq \mathcal{U} \).

**Proof.** Take any \( \widehat{y} \in \mathcal{V} \cap [x, w] \). The result is trivial if \( \widehat{y} = x \) or \( \widehat{y} = w \), so suppose not: \( \widehat{y} = x_\alpha w \) for some \( \alpha \in (0, 1) \). Then
\[
v(x_\alpha w) = (\text{cav} v)(x_\alpha w) \geq (\text{cav} v)(x_\alpha)(\text{cav} v)(w) \geq v(x_\alpha)v(w)
\]

since \( x_\alpha w \in \mathcal{V} \) (the equality), \( \text{cav} v \) is concave (first inequality), and \( \text{cav} v \geq v \) (second inequality), whence \( u(x_\alpha w) \geq u(x_\alpha)v(w) \) because \( u \) is coarsely less convex than \( v \). So \( u(x_\alpha w) = u(x_\alpha)v(w) \), and thus \( \widehat{y} = x_\alpha w \in \mathcal{U} \). \( \square \)

We now show that \( y' \leq y \); the argument for \( z \leq z' \) is analogous. If \( y' < x \), then \( y' < x \leq y \) since \( y \in \mathcal{U} \subseteq [x, w] \). Suppose instead that \( x \leq y' \). Then since \( y' \leq \mu \leq w \), we have \( y' \in [x, w] \). As \( y' \in \mathcal{V} \), it follows from the claim that \( y' \) belongs to \( \mathcal{U} \). So \( y' \in \mathcal{U} \cap [0, \mu] \), and thus \( y' \leq \sup(\mathcal{U} \cap [0, \mu]) = y \). \( \blacksquare \)
Appendix E  Proof of Proposition 2 (p. 13)

The argument is close to the proof of the converse (necessity) half of Theorem 2, which we sketched in §4.2 and which is given in full in appendix C. Fix a distribution $F_0$ that is not binary. Choose an $X \in (0,1)$ such that $0 < \lim_{z \uparrow X} F_0(z) \leq F_0(X) < 1$. Define

$$x := \frac{1}{F_0(X)} \int_{[0,X]} \xi F_0(d\xi) \quad \text{and} \quad w := \frac{1}{1 - F_0(X)} \int_{(X,1]} \xi F_0(d\xi),$$

and note that $x < X < w$. Fix a convex $p : [0,1] \to \mathbb{R}$ that is affine on $[0,X]$ and on $[X,1]$, but not affine on $[0,1]$. Clearly we may choose a regular and M-shaped $u : [0,1] \to \mathbb{R}$ such that $p = u$ on $\{x,w\}$ and $p > u$ on $[0,1] \setminus \{x,w\}$, and such that $u$ is convex on $[X,y]$ and concave on $[y,1]$ for some $y \in (X,1)$. Let $G$ be the distribution supported on $\{x,w\}$ whose mean is the same as that of $F_0$. Then $G$ is uniquely optimal for $u$ given $F_0$, since any other feasible distribution $F$ has $\int u dF \leq \int pdF \leq \int pdF_0 = \int pdG = \int udG$, where the weak inequality holds since $p$ is convex and $F$ is feasible given $F_0$, the first equality holds since $p$ is affine on $[0,X]$ and on $[X,1]$, and the final equality holds since $p = u$ $G$-a.e.

Since $u'$ is bounded, we may choose a regular $v : [0,1] \to \mathbb{R}$ that coincides with $u$ on $[X,1]$ and that weakly exceeds $u$ and is strictly convex on $[0,X]$. Then $v$ is S-shaped and coarsely more convex than $u$. Let $\delta := F_0(a) - \lim_{z \uparrow a} F_0(z)$, and observe that there are $a \in [0,X]$ and $\pi \in [0,1]$ such that

$$\frac{v(b) - v(a)}{b - a} = v'(b), \quad \text{where} \quad b := \frac{\pi \delta a + \int_a^1 \xi F_0(d\xi)}{\pi \delta + 1 - F_0(a)} > 0.$$

Define $F$ by $F := F_0$ on $[0,a)$, $F := F_0(a) - \pi \delta$ on $[a,b)$, and $F := 1$ on $[b,1]$. (That is, $F$ reveals $[0,a)$, pools $(a,1]$, reveals $a$ with probability $1 - \pi$, and otherwise pools it with $(a,1]$.) Let $q : [0,1] \to \mathbb{R}$ be affine on $[X,1]$ and satisfy $q \geq v$, with equality on $[0,a] \cup \{b\}$. The distribution $F$ is optimal for $v$ given $F_0$ since for any (other) feasible distribution $H$, we have $\int v dH \leq \int q dH \leq \int q dF_0 = \int q dF = \int v dF$, where the second inequality holds since $q$ is convex and $H$ is feasible given $F_0$, the first equality holds since $q$ is affine on $[a,1]$, and the final equality holds since $q = v$ $F$-a.e.

Since $p(X) > u(X)$, it must be either that $a < X$ or that $a = X$ and $\pi \delta > 0$. Thus $F$ is not more informative than $G$, so (**) fails. \[\blacksquare\]
Appendix F  Proof of Proposition 3 (p. 14)

For the first half (sufficiency), fix an atomless convex-support prior distribution \( F_0 \); and let \( u,v : [0,1] \to \mathbb{R} \) be regular with \( u \) coarsely less convex than \( v \). If \( v \) is concave, then so is \( u \), and thus \( u \) satisfies the crater property, so that (**) holds by Theorem 2. If instead \( v \) is strictly convex, then \( F_0 \) is uniquely optimal for \( v \) given \( F_0 \), so (**) holds.

For the second half (necessity), fix a regular \( v : [0,1] \to \mathbb{R} \) that is neither strictly convex nor concave; we shall exhibit a regular \( u : [0,1] \to \mathbb{R} \) that is coarsely less convex than \( v \), an atomless convex-support prior distribution \( F_0 \), and a distribution \( F \) that is optimal for \( v \) given \( F_0 \) such that no distribution optimal for \( u \) given \( F_0 \) is less informative than \( F \). The argument will be similar to the proof in appendix C.1 of the converse (necessity) part of Theorem 2, which we sketched in §4.2.

By hypothesis (and using regularity), there are \( x' < z < w' \in [0,1] \) such that either \( v \) is strictly convex on \([x', z]\) and concave on \([z, w']\), or \( v \) is concave on \([x', z]\) and strictly convex on \([z, w']\). We consider the former case (the latter is analogous), and distinguish two (sub-)cases.

Case 1: \( v \) is not affine on \([z, w']\). In this case, we may choose \( w \in (z, w') \) such that the tangent to \( v \) at \( w \) crosses \( v \) on \([x', w]\) exactly once, at some \( a' \in (x', z) \). Since \( v' \) is bounded, we may choose a regular \( u : [0,1] \to \mathbb{R} \) such that \( u - v \) is concave (so \( u \) is coarsely less convex than \( v \)), \( u \) is strictly concave on \([0, a']\) and on \([w, 1]\), and \( u \leq v \) on \([x', w']\), with equality on \([a', w]\). Then since \( u \) is strictly concave on \([x', a']\) and strictly convex on \([a', z]\), we may choose an \( x \in (x', a') \) such that the tangent to \( u \) at \( x \) lies strictly above (below) \( u \) at \( a' \) (at \( z \)). It follows that there is a convex \( p : [0,1] \to \mathbb{R} \) and an \( X \in (a', z) \) such that \( p \) is affine on \([x', X]\) and on \([X, w']\), and \( u \geq p \) on \([x', w']\), with equality on \( \{x, w \} \) and with strict inequality at \( X \).

Let \( F_0 \) be a distribution that is atomless with support \([x', w']\),

\[
\frac{1}{F_0(X)} \int_0^X \xi F_0(d\xi) = x \quad \text{and} \quad \frac{1}{1 - F_0(X)} \int_X^1 \xi F_0(d\xi) = w.
\]

As \( v \) is S-shaped on \([x', w']\), an ‘upper censorship’ distribution \( F \) is optimal by Kolotilin’s (2014, p. 14) well-known result: for \( a \in (0,1) \) satisfying

\[
\frac{v(b) - v(a)}{b - a} = v'(b), \quad \text{where} \quad b := \frac{1}{1 - F_0(a)} \int_a^1 \xi F_0(d\xi),
\]

this distribution \( F \) fully reveals \([0, a)\) and pools \([a, 1)\). It is easy to see graphically (in Figure 2 on p. 11, paying attention to \( p \)) that \( a \) must be

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48Explicitly, \( F = F_0 \) on \([0, a)\), \( F = F_0(a) \) on \([a, b)\) and \( F = 1 \) on \([b, 1]\).
strictly smaller than $X$. Thus the optimal distribution $F$ pools some states to the left of $X$ with states to its right. For the payoff $u$, however, it is strictly sub-optimal to pool states on either side of $X$ together. This is reasonably intuitive given the shape of $u$; formally, it follows from Lemma 7 in appendix C.1. Thus $(\ast\ast)$ fails: no distribution optimal for $u$ given $F_0$ is less informative than $F$, since the latter pools across $X$ while the former do not.

Case 2: $v$ is affine on $[z, w']$. In this case, since $v'$ is bounded, we may choose $x < y$ in $(x', z)$ and a regular $u : [0, 1] \rightarrow \mathbb{R}$ such that $u = v$ on $[y, w']$, $u$ is strictly concave on $[0, y]$ and on $[w', 1]$, and the tangent to $u$ at $x$ crosses $u$ exactly once on $(z, w')$, at some $X \in (z, w')$. Fix some $w \in (X, w')$.

Let $p : [0, 1] \rightarrow \mathbb{R}$ match the aforementioned tangent on $[x', X]$ and match $u$ on $[X, w']$. Clearly $p$ is affine on $[x', X]$ and on $[X, w']$ with $p'(x) < p'(w)$ (so $p$ is convex). We furthermore have $p \geq u$ on $[x', w']$, with equality on $\{x, w\}$. Let $F_0$ be a distribution that is atomless with support $[x', w']$,

$$\frac{1}{F_0(X)} \int_0^X \xi F_0(d\xi) = x \quad \text{and} \quad \frac{1}{1 - F_0(X)} \int_x^1 \xi F_0(d\xi) = w.$$ 

The hypotheses of Lemma 7 in appendix C.1 are satisfied. Thus to show that $(\ast\ast)$ fails, it suffices to exhibit a distribution $F$ that is optimal for $v$ given $F_0$ and that satisfies $\int_0^X F < \int_0^X F_0$ (i.e. states on either side of $X$ are pooled). To that end, let

$$b := \frac{1}{1 - F(z)} \int_z^1 \xi F_0(d\xi),$$

and define a distribution $F$ by $F := F_0$ on $[0, z)$, $F := F_0(z)$ on $[z, b)$ and $F := 1$ on $[b, 1]$. Then $\int_0^X F < \int_0^X F_0$, and clearly $F$ is optimal for $v$ given $F_0$.

Appendix G  Proof of Proposition 6 (p. 17)

Fix any atomless $F_0 \neq G_0$; we shall find a regular and S-shaped $u : [0, 1] \rightarrow \mathbb{R}$ for which $(\dagger)$ fails. If $F_0$ is not less informative than $G_0$, then $(\dagger)$ fails for any strictly convex $u : [0, 1] \rightarrow \mathbb{R}$, since $F_0$ ($G_0$) is uniquely optimal for $u$ given $F_0$ ($G_0$). Assume for the remainder that $F_0$ is less informative than $G_0$.

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49 If $b \geq X$, then $\int_0^X (F_0 - F) = \int_0^X [F_0 - F_0(z)] > 0$ as $F_0 > F_0(z)$ on $(z, 1]$, while if $b \leq X$ we have $\int_0^X (F_0 - F) = \int_0^1 (F - F_0) = \int_1^X (1 - F_0) > 0$ as $F_0 < 1$ on $[0, w')$.
For any atomless distribution $F$, integration by parts\textsuperscript{50} yields

$$\frac{1}{1 - F(y)} \int_y^1 xF(dx) = \frac{1 - yF(y) - \int_y^1 F}{1 - F(y)} = 1 + \frac{(1 - y)F(y) - \int_y^1 F}{1 - F(y)}$$

for each $y \in (0, 1)$. We have $\int_y^1 F \geq \int_y^1 G_0$ for every $y \in (0, 1)$ since $F_0$ is less informative than $G_0$. Since in addition $F_0 \neq G_0$, it cannot be that $F_0$ is first-order stochastically dominated by $G_0$, and thus $F_0(a) < G_0(a)$ for some $a \in (0, 1)$. It follows that

$$b := \frac{1}{1 - F_0(a)} \int_a^1 xF_0(dx) < \frac{1}{1 - G_0(a)} \int_a^1 xG_0(dx). \quad \text{(6)}$$

Choose a regular and S-shaped $u : [0, 1] \to \mathbb{R}$ such that $(u(b) - u(a))/(b - a) = u'(b)$. Let $F$ be the distribution given by $F := F_0$ on $[0, a)$, $F := F_0(a)$ on $[a, b)$ and $F := 1$ on $[b, 1]$. Write $a'$ for the unique $y \in (0, 1)$ satisfying

$$\frac{u(\beta(y)) - u(y)}{\beta(y) - y} = u'(\beta(y)), \quad \text{where} \quad \beta(y) := \frac{1}{1 - G_0(y)} \int_y^1 xG_0(dx),$$

define $b' := \beta(a')$, and let $G$ be the distribution given by $G := G_0$ on $[0, a')$, $G := G_0(a')$ on $[a', b')$ and $G := 1$ on $[b', 1]$. By Kolotilin’s (2014, p. 14) well-known result, $F$ ($G$) is uniquely optimal for $u$ given $F$ ($G$). By (6), we have $a > a'$, so $F$ is not less informative than $G$. Thus (†) fails.

The atomlessness hypothesis in Proposition 6 can be dropped: it suffices to assume that $F_0$ is not degenerate. Then there are $a, \alpha \in [0, 1]$ such that

$$\lim_{x \to a} F_0(x) + \alpha \left[ F_0(a) - \lim_{x \to a} F_0(x) \right] < \lim_{x \to a} G_0(x) + \alpha \left[ G_0(a) - \lim_{x \to a} G_0(x) \right] < 1,$$

and thus the proof above remains applicable, with minor modifications along the lines of the proof of Proposition 2 (appendix E) to take care of atoms.

**Appendix H  Proof of Proposition 7 (p. 18)**

For $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$, write $x_\alpha := \alpha x + (1 - \alpha)y$. Define $u, v : [0, 1] \to \mathbb{R}$ by $u(x) := U_S(A(x), x)$ and $v(x) := U_R(A(x), x)$ for each $x \in [0, 1]$. Choose any $x < y$ in $[0, 1]$ such that $u(x_\beta y) \leq u(x) \beta u(y)$ for every $\beta \in (0, 1)$, and

\textsuperscript{50}Licensed by e.g. Theorem 18.4 in Billingsley (1995).
fix an $\alpha \in (0, 1)$. Note that $v(x_\alpha y) \leq v(x)_\alpha v(y)$ since $v$ is convex (as $A$ is $U_R$-optimal). Thus

$$\Phi(u(x_\alpha y), v(x_\alpha y), x_\alpha y) \leq \Phi(u(x_\alpha y), v(x)_\alpha v(y), x_\alpha y) \leq \Phi(u(x)_\alpha u(y), v(x)_\alpha v(y), x_\alpha y) \leq \Phi(u(x), v(x), x)_\alpha \Phi(u(y), v(y), y),$$

where the first inequality holds since $\Phi(u(x_\alpha y), \cdot, x_\alpha y)$ is increasing, the second holds since $\Phi(\cdot, v(x)_\alpha v(y), x_\alpha y)$ is (strictly) increasing, and the final inequality holds since $\Phi$ is convex. Moreover, the second inequality is strict if $u(x_\alpha y) < u(x)_\alpha u(y)$, as $\Phi(\cdot, v(x)_\alpha v(y), x_\alpha y)$ is strictly increasing.

\[\blacksquare\]

**Appendix I  Proof of Lemma 2 (p. 22)**

Let $F$ admit a truly bell-shaped density $f$, i.e. one that is convex on $(-\infty, x]$ and on $[y, \infty)$, concave on $[x, y]$, increasing on $(-\infty, z]$, and decreasing on $[z, \infty)$, where $x \leq z \leq y$. By an approximation argument, we may assume without loss that $f$ is continuously differentiable. Then $u$ is twice continuously differentiable, with $u'(m) = f(m + b) - f(m + a)$ and $u''(m) = f'(m + b) - f'(m + a)$ for each $m \in (0, 1)$. Adopt the convention that $[c, d] = (c, d) = \emptyset$ if $c > d$.

Observe that $u'$ is positive on $(0, z - b)$ (as $f$ is increasing on $[0, z]$), decreasing on $(z - b, z - a) \cap [0, 1)$ (as $m \mapsto f(m + b)$ and $m \mapsto -f(m + a)$ are decreasing on this interval), and negative on $(z - a, 1)$ (as $f$ is decreasing on $[z, 1]$). Thus there is a $z' \in [z - a, z - b] \cap [0, 1]$ such that $u$ is increasing on $[0, z']$ and decreasing on $[z', 1]$. Moreover, $u''$ is positive on $(0, x - b)$ (as $f'$ is increasing on $(0, x)$), decreasing on $[x - b, \min\{y - b, x - a\}] \cap [0, 1]$ (as $m \mapsto f'(m + b)$ and $m \mapsto -f'(m + a)$ are decreasing on this interval), negative on $[\min\{y - b, x - a\}, \max\{y - b, x - a\}] \cap [0, 1]$ (as $m \mapsto f'(m + b)$ and $m \mapsto -f'(m + a)$ are negative on this interval if it is non-empty, since otherwise $f'$ would be decreasing on $[x, y]$), increasing on $[\max\{y - b, x - a\}, y - a] \cap [0, 1]$ (as $m \mapsto f(m + b)$ and $m \mapsto -f(m + a)$ are increasing on this interval), and positive on $[y - a, 1)$ (as $f$ is increasing on $[y, 1]$). Thus there are $x' \in [x - b, \min\{y - b, x - a\}] \cap [0, 1]$ and $y' \in [\max\{y - b, x - a\}, y - a] \cap [0, 1]$ such that $u$ is convex on $[0, x']$ and on $[y', 1]$ and concave on $[x', y']$.

\[\text{If } F \text{ admits a truly bell-shaped density, then it is the pointwise limit of a sequence } (F_n)_{n \in \mathbb{N}} \text{ of distributions that each admit a bell-shaped and continuously differentiable density. Moreover, if } x \mapsto F_n(x + b) - F_n(x + a) \text{ is truly bell-shaped for each } n \in \mathbb{N}, \text{ then so is } u \text{ (consider a subsequence along which the thresholds converge).}\]
\(x' \leq x - a \leq z - a \leq z' \leq z - b \leq x - b \leq x',\) it follows that \(u\) is truly bell-shaped.

\[\blacksquare\]

Appendix J  Relation to the theory of monotone comparative statics

In this appendix, we discuss in detail how our results relate to the general theory of monotone comparative statics.

J.1 The literature

The general comparative-statics literature (e.g. Topkis, 1978; Milgrom & Shannon, 1994; Quah & Strulovici, 2009) asks, for any problem in which an agent chooses an action \(a\) from a partially ordered set \(\mathcal{A}\), what shifts of the agent’s objective function \(U : \mathcal{A} \to \mathbb{R}\) lead her optimally to choose a higher action.\(^{52}\) This framework nests the persuasion problem, in which the sender’s action is a distribution \(F\) drawn from the set of all distributions feasible given the prior \(F_0\), ordered by ‘less informative than’, and her objective function is \(U(F) := \int u \, dF\). When \(u\) shifts, the sender’s objective function \(U\) changes, and our question is which such shifts lead to more informative choices.

Topkis (1978) and Milgrom and Shannon (1994) ask what shifts of \(U\) yield an increase of the optimal choices \(\arg\max_{a \in \mathcal{A}} U(a)\) for any constraint set \(\mathcal{A} \subseteq \mathcal{A}\).\(^{53}\) Quah and Strulovici (2009, 2007) describe weaker conditions which ensure that \(\arg\max_{a \in \mathcal{A}} U(a)\) increases for any order interval constraint set \(\mathcal{A} = [a, \bar{a}] \subseteq \mathcal{A}\).\(^{54}\) The persuasion model has an order interval constraint set, namely \([\delta, F_0]\), where \(F_0\) is the prior distribution and \(\delta\) denotes the point mass concentrated at the mean of \(F_0\).

To allow for the possibility of multiple optimal actions, one must extend the notion of ‘lower than’ from actions to sets of actions. In this paper, we have used the natural extension (defined at the end of §2.1), which is called the weak set order (WSO) in the literature. Most of the literature concerns itself with the strong set order (SSO). The sense in which the SSO is stronger is difficult to interpret, suggesting that the gap between the two set orders is a technical artefact without economic substance.\(^{55}\) Our reading of the

\[^{52}\]A detail: the literature actually restricts attention to action sets \(\mathcal{A}\) whose partial order has a lattice structure. This proviso is satisfied in the persuasion model (see appendix A). This technicality has no bearing on the discussion in this appendix.

\[^{53}\]Actually, precisely: any non-empty sublattice \(A \subseteq \mathcal{A}\).

\[^{54}\]Recall that \([a, \bar{a}] := \{a \in \mathcal{A} : a \lesssim a \lesssim \bar{a}\}\), where \(\lesssim\) denotes the partial order on \(\mathcal{A}\).

\[^{55}\]See Theorem 1 in Che, Kim and Kojima (2021) for a characterisation of the gap.
literature is that the SSO is used *despite* its uninterpretable extra strength, on the (reasonable) grounds that it yields a fruitful theory.\textsuperscript{56} The WSO/SSO distinction will become important toward the end of our discussion below.

The general theory features two classes of results and corresponding properties. The first concern ‘encouragement’ properties such as increasing/single-crossing differences and interval dominance, which capture the idea that one objective function is relatively more keen on higher actions than another objective function. Such ‘encouragement’ properties characterise ‘non-decreasing’ comparative statics: ‘encouraging’ shifts of the objective function do not strictly decrease optimal choices, and there is a converse.\textsuperscript{57}

The second kind of result in the literature introduces ‘complementarity’ assumptions such as (quasi-)supermodularity and I-quasi-supermodularity. ‘Complementary’ objective functions $U$ are those such that increasing one ‘dimension’ of the action makes the agent keener to increase other ‘dimensions’.\textsuperscript{58} When the objective shifts in an ‘encouraging’ way and either the old or the new objective exhibits ‘complementarity’, optimal choices *increase*; and there is a converse.

### J.2 Our theorems in context

Our Theorem 1 is a result of the first kind: it identifies the correct ‘encouragement’ property for $u$, namely ‘coarsely less convex than’, which characterises ‘non-decreasing’ comparative statics for the persuasion model. The proof of the sufficiency half of Theorem 1 is the one place where we are able to use a result from the literature: we (i) show that if $u$ is coarsely less convex than $v$, then $U(F) = \int u\,dF$ is interval-dominated by $V(F) = \int v\,dF$, and then (ii) invoke Quah and Strulovici’s (2007) Proposition 5 to conclude that choices under $u$ are not strictly higher than choices under $v$. (It turns out, however, that most of the action is in part (i): the argument there is fairly intricate, and exploits the special structure of the persuasion problem.) The necessity half of Theorem 1 is straightforward.

Having obtained Theorem 1, we next seek a result of the second kind, which identifies a further condition on $u$ or $v$ under which if $u$ is coarsely less convex than $v$, then *less* informative choices are made under $u$ than under

\textsuperscript{56}In particular, it yields *necessity* as well as sufficiency in the comparative-statics theorems of Milgrom and Shannon (1994) and Quah and Strulovici (2009).

\textsuperscript{57}This fact is dimly known in the literature, but rarely written down. Exceptions include Quah and Strulovici (2007, Proposition 5) and Anderson and Smith (2021).

\textsuperscript{58}This gloss is exact if actions are ordered by a product order, such as the usual inequality on $\mathbb{R}^n$. Beyond product orders, the ‘dimensions’ language is purely analogical.
The literature is of no help here, because the objective \( U(F) = \int u dF \)
satisfies no ‘complementarity’ property except in trivial cases (e.g. if \( u \) is concave). In fact, the literature tells us that comparative statics must fail in the strong set order, since complementarity is a necessary condition!

We must therefore strike out on our own, by asking for comparative statics in the (more natural) weak set order. This turns out to be fruitful: it delivers our Theorem 2, which describes a non-trivial property whose satisfaction by \( u \) is necessary and sufficient for comparative statics to hold between \( u \) and any \( v \) that is coarsely more convex. Both the sufficiency and necessity parts of our proof rely on novel arguments that exploit the structure of the persuasion model.

From the perspective of the comparative-statics literature, Theorem 2 may be viewed as a proof of concept: ‘increasing’ comparative statics in the weak set order can sometimes be had even though the literature’s ‘complementarity’ assumptions fail. This matters because in our experience, ‘complementarity’ tends to fail in economic models, severely limiting the applicability of the existing theory. (There is one important exception: in applications with totally ordered actions (e.g. real numbers), ‘complementarity’ holds automatically.)

Appendix K  Tightness of Lemma 1 (p. 6)

Lemma 1 is nearly tight, in the following sense:

**Partial converse of Lemma 1.** If \( \Phi : \mathbb{R} \times [0,1] \to \mathbb{R} \) is such that for every upper semi-continuous \( u : [0,1] \to \mathbb{R} \), \( u \) is coarsely less convex than \( x \mapsto \Phi(u(x), x) \), then \( \Phi \) must be convex on \( \mathbb{R} \times (0,1) \) with \( \Phi(\cdot, x) \) increasing for every \( x \in (0,1) \).

This result is implied by the following proposition, which closes the small gap between Lemma 1 and its converse by giving an exact characterisation of coarse-convexity-increasing transformations \( \Phi \).

**Proposition 8.** For a map \( \Phi : \mathbb{R} \times [0,1] \to \mathbb{R} \), the following are equivalent:

(i) For every \( u : [0,1] \to \mathbb{R} \), \( u \) is coarsely less convex than \( x \mapsto \Phi(u(x), x) \).

(ii) For every upper semi-continuous \( u : [0,1] \to \mathbb{R} \), \( u \) is coarsely less convex than \( x \mapsto \Phi(u(x), x) \).

(iii) For any \( x < y \) in \( [0,1] \), \( \alpha \in (0,1) \) and \( a, b, c \in \mathbb{R} \) such that \( c \leq (<) \alpha a + (1-\alpha)b \), we have \( \Phi(c, \alpha x + (1-\alpha)y) \leq (<) \alpha \Phi(a, x) + (1-\alpha)\Phi(b, y) \).
For the proof, we write \( a_\alpha b := \alpha a + (1 - \alpha)b \) for \( a, b \in \mathbb{R} \) and \( \alpha \in [0, 1] \).

**Proof of the partial converse of Lemma 1.** By Proposition 8, it suffices to show that property (iii) implies that \( \Phi \) is convex on \( \mathbb{R} \times (0, 1) \) and that \( \Phi(x, \cdot) \) is increasing for each \( x \in (0, 1) \). So let \( \Phi \) satisfy (iii), and note that it follows that for each \( c \in \mathbb{R} \), \( \Phi(c, \cdot) \) is convex, hence continuous on \( (0, 1) \).

For convexity, property (iii) immediately implies that \( \Phi(\alpha(a, x) + (1 - \alpha)(b, y)) \leq \Phi(a, x)\alpha\Phi(b, y) \) for any \( \alpha \in (0, 1) \) and any \( (a, x), (b, y) \in \mathbb{R} \times [0, 1] \) such that \( x \neq y \). To show that the same holds when \( x = y = z \in (0, 1) \), (in other words, that \( \Phi(\cdot, z) \) is convex for each \( z \in (0, 1) \)) observe that for any \( x \in (0, z) \) and \( y \in (z, 1) \) such that \( x_\alpha y = z \), we have \( \Phi(a_\alpha b, z) \leq \Phi(a, x)\alpha\Phi(b, y) \), so letting \( x, y \to z \) yields \( \Phi(a_\alpha b, z) \leq \Phi(a, z)\alpha\Phi(b, z) \) by continuity.

For monotonocity, take any \( z \in (0, 1) \) and \( c < a \in \mathbb{R} \); we must show that \( \Phi(c, z) \leq \Phi(a, z) \). For any \( x \in (0, z) \) and \( y \in (z, 1) \) such that \( \frac{1}{2}x + \frac{1}{2}y = z \), property (iii) implies \( \Phi(c, z) \leq \frac{1}{2}\Phi(a, x) + \frac{1}{2}\Phi(a, y) \), which as \( x, y \to z \) yields \( \Phi(c, z) \leq \Phi(a, z) \) by continuity.

**Proof of Proposition 8.** (iii) implies (i) since for any \( u : [0, 1] \to \mathbb{R} \) and any \( x < y \) in \([0, 1]\) such that \( u(x_\beta y) \leq u(x)u(y) \) for every \( \beta \in (0, 1) \), property (iii) (with \( a := u(x) \), \( b := u(y) \) and \( c := u(x_\alpha y) \)) implies for each \( \alpha \in (0, 1) \) that \( \Phi(u(x_\alpha y), x_\alpha y) \leq \Phi(u(x), x)\alpha\Phi(u(y), y) \), with strict inequality if \( u(x_\alpha y) < u(x)u(y) \). (i) immediately implies (ii). To show that (ii) implies (iii), we prove the contra-positive: let \( \Phi \) violate (iii), meaning that there are \( x < y \) in \([0, 1]\), \( \alpha \in (0, 1) \) and \( a, b, c \in \mathbb{R} \) such that either

\[
\begin{align*}
(1) \quad & c \leq a_\alpha b \text{ and } \Phi(c, x_\alpha y) > \Phi(a, x)\alpha\Phi(b, y), \\
(2) \quad & c < a_\alpha b \text{ and } \Phi(c, x_\alpha y) \geq \Phi(a, x)\alpha\Phi(b, y).
\end{align*}
\]

To show that (ii) fails, define \( u : [0, 1] \to \mathbb{R} \) by \( u := a \) on \([0, x]\), \( u(x_\alpha y) := c \), \( u := b \) on \([y, 1]\) and \( u := \min\{a, b, c\} - 1 \) on \((x, x_\alpha y) \cup (x_\alpha y, y)\). Clearly \( u \) is upper semi-continuous. We have \( u(x_\beta y) \leq u(x)u(y) \) for every \( \beta \in (0, 1) \), with strict inequality at \( \beta = \alpha \) in case (2), and furthermore \( \Phi(u(x_\alpha y), x_\alpha y) \geq \Phi(u(x), x)\alpha\Phi(u(y), y) \), with strict inequality in case (1). Thus \( u \) is not coarsely less convex than \( x \mapsto \Phi(u(x), x) \).

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