SUPERCUSPIDAL REPRESENTATIONS OF GL_n(F) DISTINGUISHED
BY A GALOIS INVOLUTION

by

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Abstract. — Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields of residual characteristic p ≠ 2, and let σ denote its non-trivial automorphism. Let R be an algebraically closed field of characteristic different from p. To any cuspidal representation π of GL_n(F), with coefficients in R, such that π^σ ∼ π^⊥ (such a representation is said to be σ-selfdual) we associate a quadratic extension D/D_0, where D is a tamely ramified extension of F and D_0 is a tamely ramified extension of F_0, together with a quadratic character of D_0^\hat{=}^0. When π is supercuspidal, we give a necessary and sufficient condition, in terms of these data, for π to be GL_n(F_0)-distinguished. When the characteristic ℓ of R is not 2, denoting by ω the non-trivial R-character of F_0^\hat{=}^0 trivial on F/F_0-norms, we prove that any σ-selfdual supercuspidal R-representation is either distinguished or ω-distinguished, but not both. In the modular case, that is when ℓ > 0, we give examples of σ-selfdual cuspidal non-supercuspidal representations which are not distinguished nor ω-distinguished. In the particular case where R is the field of complex numbers, in which case all cuspidal representations are supercuspidal, this gives a complete distinction criterion for arbitrary complex cuspidal representations, as well as a purely local proof, for cuspidal representations, of the dichotomy and disjunction theorem due to Kable and Anandavardhanan–Kable–Tandon.

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1. Introduction

1.1.

Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields of residual characteristic p ≠ 2, and let σ denote its non-trivial automorphism. Let G be a connected reductive group defined over F_0, let G denote the locally profinite group G(F) equipped with the natural action of σ and G^σ be the fixed points subgroup G(F_0). The study of those irreducible (smooth) complex representations of G which are G^σ-distinguished, that is which carry a non-zero G^σ-invariant linear form, goes back to the 1980’s. We refer to [22, 28] for the initial motivation for distinguished representations in a global context, and to [23, 16] in a non-Archimedean context.
1.2. In this work, we will consider the case where $G$ is the general linear group $GL_n$ for $n \geq 1$. We thus have $G = GL_n(F)$ and $G^\sigma = GL_n(F_0)$. In this case, it is well-known \([16, 16]\) that any distinguished irreducible complex representation $\pi$ of $G$ is $\sigma$-selfdual, that is, the contragredient $\pi^\vee$ of $\pi$ is isomorphic to the representation $\pi^\sigma = \pi \circ \sigma$, and the space $\text{Hom}_{G^\sigma}(\pi, 1)$ of $G^\sigma$-invariant linear forms on $\pi$ has dimension 1. Also, the central character of $\pi$ is trivial on $F^\times$. This gives us two necessary conditions for an irreducible complex representation of $G$ to be distinguished, and it is natural to ask whether or not they are sufficient.

1.3. First, let us consider in this paragraph the case where $F$ is a quadratic extension of finite fields of characteristic $p$, denoted $k/k_0$. In this case, Gow \([18]\) proved that an irreducible complex representation of $GL_n(k)$ is $GL_n(k_0)$-distinguished if and only if $\pi$ is $\sigma$-selfdual. (Note that the latter condition automatically implies that the central character is trivial on $k^\times$.) Besides, the $\sigma$-selfdual irreducible representations of $GL_n(k)$ are exactly those which come from an irreducible representation of the unitary group $U_n(k/k_0)$ by base change: see Kawanaka \([31]\).

1.4. We now go back to the $p$-adic setting and consider a $\sigma$-selfdual irreducible complex representation $\pi$ of $G$ whose central character is trivial on $F^\times$. When $\pi$ is cuspidal and $F/F_0$ is unramified, Prasad \([42]\) proved that:

1. if $n$ is odd, then $\pi$ is distinguished;
2. if $n$ is even, and if $\omega = \omega_{F/F_0}$ denotes the non-trivial character of $F^\times$ trivial on $F/F_0$-norms, then $\pi$ is either distinguished or $\omega$-distinguished (the latter case meaning that the complex vector space $\text{Hom}_{G^\sigma}(\pi, \omega \circ \det)$ is non-zero).

When $\pi$ is an essentially tame cuspidal representation, that is, when the number of unramified characters $\chi$ of $G$ such that $\pi \chi \simeq \pi$ is prime to $p$, Hakim and Murnaghan \([24]\) gave sufficient conditions for $\pi$ to be distinguished. These conditions are stated in terms of admissible pairs \([27]\), which parametrize essentially tame cuspidal complex representations of $G$ – see \([27, 9]\). Note that they assume that $F$ has characteristic 0, but their approach also works in characteristic $p$.

When $\pi$ is a discrete series representation and $F$ has characteristic 0, Kable \([30]\) proved that if $\pi$ is $\sigma$-selfdual, then it is either distinguished or $\omega$-distinguished: this is the Dichotomy Theorem. In addition, Anandavardhanan, Kable and Tandon \([2]\) proved that $\pi$ can’t be both distinguished and $\omega$-distinguished: this is the Disjunction Theorem. The proofs use global arguments, which is why $F$ were assumed to have characteristic 0, and the Asai L-function of $\pi$. However, these results still hold when $F$ has characteristic $p$, as is explained in \([3]\) Appendix A. Also note that the disjunction theorem implies that the sufficient conditions given by Hakim and Murnaghan in \([24]\) in the essentially tame cuspidal case are also necessary conditions.

When $\pi$ is cuspidal of level 0 – in particular $\pi$ is essentially tame – Coniglio \([11]\) gave a necessary and sufficient condition of distinction in terms of admissible pairs. Her proof is purely local, with no assumption on the characteristic of $F$, and does not use the disjunction theorem. (In fact,
she even considers representations of level 0 of inner forms of G whose local Jacquet-Langlands transfer is cuspidal).

If one takes the classification of distinguished cuspidal representations of general linear groups for granted, Anandavardhanan \cite{1} classified distinguished discrete series representations of G in terms of the distinction of their cuspidal support (see also \cite{5} and \cite{32}).

Matringe \cite{33,35} classified distinguished generic, as well as distinguished unitary, representations of G in terms of the Langlands classification. This provides a class of representations for which the dichotomy and disjunction theorems do not hold: some $\sigma$-selfdual generic irreducible representations are neither distinguished nor $\omega$-distinguished, and some are both.

In \cite{20} Max Gurevich extended the dichotomy theorem to the class of ladder representations, which contains all discrete series representations: a $\sigma$-selfdual ladder representation of G is either distinguished or $\omega$-distinguished, but it may be both (see \cite{20} Theorem 4.6). Here F is assumed to have characteristic 0.

Finally, one can deduce from these works the connection between distinguished generic irreducible representations of G and base change from a quasi-split unitary group; see \cite{17} Theorem 6.2, \cite{4} Theorem 2.3 or \cite{21}.

1.5.

The discussion above leaves two questions open about cuspidal representations:

(1) When $n$ is odd, is a $\sigma$-selfdual cuspidal representation of G with central character trivial on $\hat{F}_0$ automatically distinguished?

(2) More generally, can one find a distinction criterion for an arbitrary $\sigma$-selfdual cuspidal representation $\pi$, with no assumption on the characteristic of F, on the quadratic extension $F/F_0$, on $n$ nor on the torsion number of $\pi$?

In this paper, we propose an approach which allows us to generalize both Hakim-Murnaghan’s and Coniglio’s distinction criterions to all cuspidal irreducible complex representations of G, and which works:

– with no assumption on the characteristic of F,
– with purely local methods,
– not only for complex representations, but more generally for representations with coefficients in an algebraically closed field of arbitrary characteristic $\ell \neq p$.

We thus give a complete answer to Questions 1 and 2 above for cuspidal complex representations. We actually do more: we answer these questions in the larger context of supercuspidal representations with coefficients in an algebraically closed field of arbitrary characteristic $\ell \neq p$.

1.6.

First, let us say a word about the third item above. The theory of smooth representations of $GL_n(F)$ with coefficients in an algebraically closed field of characteristic $\ell \neq p$ has been initiated by Vignéras \cite{46,17} in view to extend the local Langlands programme to representations with coefficients in a field – or a ring – as general as possible (see for instance \cite{48,26}). Inner forms
have also been taken into account ([37, 43]) and the congruence properties of the local Jacquet–Langlands correspondence have been studied in [13, 39]. It is thus natural to extend the study of distinguished representations to this wider context, where the field of complex numbers is replaced by a more general field. Very little has been done about distinction of modular representations so far: a first study can be found in [44].

From now on, we fix an algebraically closed field $R$ of arbitrary characteristic $\ell \not= p$, and consider irreducible smooth representations of $G$ with coefficients in $R$. Note that $\ell$ can be 0. We need to recall that, in the modular case, that is when $\ell$ is positive, a cuspidal representation $\pi$ may occur as a subquotient of a proper parabolically induced representation (see [45] Corollaire 5). When this is not the case, $\pi$ is said to be supercuspidal.

1.7.

We first notice that, as in the complex case, any distinguished irreducible representation of $G$ with coefficients in $R$ is $\sigma$-selfdual (see Theorem 3.1). We prove that, if $\ell \not= 2$, the dichotomy and disjunction theorems hold for all supercuspidal representations with coefficients in $R$, with no assumption on the characteristic of $F$. In particular, when $R$ is the field of complex numbers, in which case any cuspidal representation is supercuspidal, we get a purely local proof of the dichotomy and disjunction theorems for cuspidal representations. When $\ell = 2$, in which case there is no character of order 2 on $F_0^\times$, the dichotomy theorem takes a simplified form: any $\sigma$-selfdual supercuspidal representation is distinguished. Let us summarize this first series of results in the theorem below.

**Theorem 1.1** (Theorems 3.1 and 9.8). — Let $\pi$ be an irreducible $R$-representation of $G$.

1. If $\pi$ is distinguished, then $\pi$ is $\sigma$-selfdual and $\text{Hom}_{G^\sigma}(\pi, 1)$ has dimension 1.

2. Suppose that $\pi$ is supercuspidal and $\sigma$-selfdual.

   a. If $\ell = 2$, then $\pi$ is distinguished.

   b. If $\ell \not= 2$, then $\pi$ is either distinguished or $\omega$-distinguished, but not both.

In the modular case, for $\ell > 2$, we give examples of $\sigma$-selfdual cuspidal, non-supercuspidal representations which are not distinguished nor $\omega$-distinguished (see Remarks 6.5 and 8.5).

1.8.

These dichotomy and disjunction theorems for supercuspidal $R$-representations both rely on a distinction criterion, which we state in Theorem 1.2 below. The basic idea is that we canonically associate to any $\sigma$-selfdual supercuspidal representation $\pi$ of $G$ a finite extension $D$ of $F$ equipped with an $F_0$-involution extending $\sigma$ and a quadratic character $\delta_0$ of the fixed points of $D^\times$; it is these data which govern the distinction of $\pi$. The character $\delta_0$ refines the information given by the central character of $\pi$ in the sense that they coincide on $F_0^\times$, the latter one being not enough in general to determine whether $\pi$ is distinguished or not. To state our distinction criterion, let us write $D_0$ for the fixed points subfield of $D$.

**Theorem 1.2** (Theorem 9.9). — A $\sigma$-selfdual supercuspidal $R$-representation of $G$ is distinguished if and only if:
(1) either \( \ell = 2 \),
(2) or \( \ell \neq 2 \) and:

(a) if \( F/F_0 \) is ramified, \( D/D_0 \) is unramified and \( D_0/F_0 \) has odd ramification order, then the character \( \delta_0 \) is non-trivial,
(b) otherwise, \( \delta_0 \) is trivial.

Even in the complex case, this is the first time a necessary and sufficient distinction criterion is exhibited for an arbitrary cuspidal representation of \( \text{GL}_n(F) \) in the spirit of \([24, 25, 11]\).

In particular when \( n \) is odd, Case (2.a) cannot happen, thus Theorem \([12]\) answers affirmatively Question 1 of Paragraph \([1.5]\) (see the first item of Theorem \([9.9]\)).

1.9.

The starting point of our strategy for proving Theorem \([12]\) is to use Bushnell-Kutzko’s construction of cuspidal representations of \( G \) via compact induction. This construction, elaborated in the complex case by Bushnell and Kutzko \([10]\), has been extended to the modular case by Vigneras \([46]\) and Minguez–Sécherre \([36]\). There is a family of pairs \((J, \lambda)\), made of certain compact mod centre open subgroups \( J \) of \( G \) and certain irreducible representation \( \lambda \) of \( J \), such that:

– for any such pair \((J, \lambda)\), the compact induction of \( \lambda \) to \( G \) is irreducible and cuspidal;
– any irreducible cuspidal representation of \( G \) occurs in this way, for a pair \((J, \lambda)\) uniquely determined up to \( G \)-conjugacy.

Such pairs are called extended maximal simple types in \([10]\), which we will abbreviate to types for simplicity. We need to give more details about the structure of these types:

1. the group \( J \) has a unique maximal compact subgroup \( J \), and a unique maximal normal pro-p-group \( J^1 \);
2. there is a group isomorphism \( J/J^1 \simeq \text{GL}_m(l) \) for some divisor \( m \) of \( n \) and finite extension \( l \) of the residual field \( k \) of \( F \);
3. the restriction of \( \lambda \) to \( J^1 \) is made of copies of a single irreducible representation \( \eta \), which extends (not uniquely, nor canonically) to \( J \);
4. given a representation \( \kappa \) of \( J \) extending \( \eta \), there is a unique irreducible representation \( \rho \) of \( J \) trivial on \( J^1 \) such that \( \lambda \) is isomorphic to \( \kappa \otimes \rho \); such a \( \rho \) restricts irreducibly to \( J \);
5. the representation of \( \text{GL}_m(l) \) obtained by restricting \( \rho \) to \( J \) is cuspidal.

The integer \( m \), called the relative degree of \( \pi \), is uniquely determined by \( \pi \). There is another type-theoretical invariant called the tame parameter field of \( \pi \): this is a tamely ramified extension \( T \) of \( F \), uniquely determined up to \( F \)-isomorphism, whose degree divides \( n/m \) and whose residual field is \( l \) (see \([8]\) for more details). Note that \( \pi \) if essentially tame is and only if \([T : F] = n/m \).

1.10.

Now consider a \( \sigma \)-selfdual cuspidal \( R \)-representation \( \pi \) of \( G \). The starting point of all our work is \([3]\) Theorem 4.1, which asserts that among all the types contained in \( \pi \), there is a type \((J, \lambda)\) which is \( \sigma \)-selfdual, that is \( J \) is \( \sigma \)-stable and \( \lambda^\sigma \) is isomorphic to \( \lambda^\sigma \). Moreover, the tame parameter field \( T \) of \( \pi \) is equipped with an \( F_0 \)-involution. If \( T_0 \) denotes the fixed points subfield of \( T \),
then $T/T_0$ is a quadratic extension, uniquely determined up to $F_0$-isomorphism. The invariants $m$ and $T/T_0$ associated with $\pi$ will play a central role in what follows.

First, the following result says that the distinction of $\pi$ can be detected by a $\sigma$-selfdual type.

**Theorem 1.3** (Theorem 5.1). — Let $\pi$ be a $\sigma$-selfdual cuspidal representation of $G$. Then $\pi$ is distinguished if and only if it contains a distinguished $\sigma$-selfdual type, that is a $\sigma$-selfdual type $(J, \lambda)$ such that $\text{Hom}_{G^\sigma}(\lambda, 1)$ is non-zero.

The proof of this theorem – which occupies Section 5 – is the most technical part of the paper: starting with a $\sigma$-selfdual type $(J, \lambda)$ contained in $\pi$ and $g \in G$, one has to prove that, if the type $(J^g, \lambda^g)$ is distinguished, then it is $\sigma$-selfdual, that is $\sigma(g)g^{-1} \in J$. First, one determines the set of the $g \in G$ such that $\eta^g$ is distinguished, as well as the dimension of the space of invariant linear forms (Paragraphs 5.1 to 5.3); then, one analyzes the distinction of $\kappa^g$ (Paragraphs 5.4 and 5.5); one obtains the final statement by using the cuspidality of the representation of $G\text{L}_m(I)$ induced by $\rho$ (see Theorem 5.20).

1.11.

When $T$ is unramified over $T_0$, the $\sigma$-selfdual types contained in $\pi$ form a single $G^\sigma$-conjugacy class. When $T$ is ramified over $T_0$, the $\sigma$-selfdual types contained in $\pi$ form $2^m$ different $G^\sigma$-conjugacy classes, characterized by an integer $i \in \{0, \ldots, [m/2]\}$ called the index of the class. Since the space $\text{Hom}_{G^\sigma}(\pi, 1)$ has dimension 1, only one of these conjugacy classes can contribute to distinction: we prove that it is the one with maximal index. This gives us the following refinement of Theorem 1.3.

**Proposition 1.4** (Propositions 4.16 and 6.1). — Let $\pi$ be a $\sigma$-selfdual cuspidal representation of $G$. Let $m$ be its relative degree and $T/T_0$ be its associated quadratic extension.

1. If $T$ is unramified over $T_0$, then $\pi$ is distinguished if and only if any of its $\sigma$-selfdual types is distinguished.

2. If $T$ is ramified over $T_0$, then $\pi$ is distinguished if and only if any of its $\sigma$-selfdual types of index $\lfloor m/2 \rfloor$ is distinguished.

Note that, when $R$ is the field of complex numbers, this proposition is proved in [3] in a different manner, based on a result of Ok [40] which is not available in the modular case.

When $T/T_0$ is ramified, one can be more precise (see Proposition 6.1): if $\pi$ is distinguished, $m$ is either even or equal to 1. It is not difficult to construct $\sigma$-selfdual cuspidal representations of $G$ such that $T/T_0$ is ramified and $m > 1$ is odd: such cuspidal representations are not distinguished nor $\omega$-distinguished (see Remark 6.5).

Now consider the case where $T/T_0$ is unramified. If $\pi$ is supercuspidal, then $m$ is odd (see Proposition 8.14). If $\pi$ is cuspidal only, $m$ can be even. This leaves us with an open question: if $\pi$ is a distinguished cuspidal representation of $G$, is its relative degree $m$ odd?
1.12.

As in the previous paragraph, π is a σ-selfdual cuspidal R-representation of G. The following definition will be convenient to us (see Remark 9.2 for the connection with the usual notion of a generic representation).

**Definition 1.5 (Definition 9.1).** — A σ-selfdual type in π is said to be generic if either T/T₀ is unramified, or T/T₀ is ramified and this type has index \([m/2]\).

Proposition 1.4 thus says that, up to \(G^\sigma\)-conjugacy, a σ-selfdual cuspidal representation π contains a unique generic σ-selfdual type, and that π is distinguished if and only if such a type is distinguished (see Theorem 9.3). This uniqueness property is crucial to the proof of the disjunction theorem 1.1.

Let us fix a generic σ-selfdual type \((J, \lambda)\) in π. Recall that, by construction, \(\lambda\) can be decomposed (non canonically) as \(\kappa \otimes \rho\). However, not any of these decompositions are suitable for our purpose. It is not difficult to prove that \(\kappa\) can be chosen to be σ-selfdual, but this is not enough: we need to prove that \(\kappa\) can be chosen to be both σ-selfdual and distinguished. The strategy of the proof depends on the ramification of \(T\) over \(T₀\).

The easiest case is when \(T/T₀\) is ramified. Using the fact that \(m\) is either even or equal to 1, we prove that \(\kappa\) can be chosen to be distinguished by adapting an argument of Matringe [34].

When \(T/T₀\) is unramified, the existence of a distinguished \(\kappa\) is more difficult. Our proof requires \(\pi\) to be supercuspidal, since in that case \(m\) is known to be odd, in which case \(\text{GL}_m(l₀)\)-distinguished cuspidal (and even supercuspidal) representations, where \(l₀\) is the residual field of \(T₀\).

In both cases, a distinguished \(\kappa\) is automatically σ-selfdual, and π is distinguished if and only if \(\rho\) is distinguished. Considering \(\rho\) as a σ-selfdual level 0 type we are then reduced to the level 0 case, which has been treated by Coniglio in the complex case. We thus have to extend her results to the modular case, which we know how to do when \(\pi\) is supercuspidal only.

In summary, we need the assumption that \(\pi\) is supercuspidal in Theorems 1.1 and 1.2 for two reasons: a) for the existence of a distinguished \(\kappa\) when \(T/T₀\) is unramified, and b) for the level 0 case.

1.13.

To study the distinction of \(\rho\) when \(\pi\) is supercuspidal, we use admissible pairs of level 0 as in Coniglio [11]. We attach to \(\rho\) a pair \((D/T, \delta)\) made of an unramified extension of degree \(m\) equipped with an involutive \(T₀\)-algebra homomorphism, non-trivial on \(T\), denoted by \(\sigma\), together with a character \(\delta\) of \(D^\times\) such that \(\delta \circ \sigma = \delta^{-1}\). (See Paragraphs 4.3 and 4.5 although the result is presented in a different way there.)

However, the distinguished representation \(\kappa\) of Paragraph 1.12 is not unique in general, thus neither \(\rho\) nor \(\delta\) are. Write \(D₀\) for the \(\sigma\)-fixed points of \(D\), and \(\delta₀\) for the restriction of \(\delta\) to \(D₀^\times\). This is a quadratic character, trivial on \(D/D₀\)-norms. We then prove in Proposition 9.5 that the pair \((D/D₀, \delta₀)\) is uniquely determined by \(\pi\) up to \(F₀\)-isomorphism. This is the one occurring in our distinction criterion theorem 1.2.
It remains to explain our strategy to prove the distinction criterion for $\rho$, in the modular case, in terms of the character $\delta_0$, as well as the dimension of the space of invariant linear forms. This depends on the ramification of $T/T_0$.

The easiest case is when $T/T_0$ is unramified. In this case, we are reduced to studying the distinction of supercuspidal representations of $GL_m(I)$ by $GL_m(I_0)$. That any distinguished irreducible representation is $\sigma$-selfdual follows from a finite and $\ell$-modular version of Theorem 3.1 (see Remark 3.2). For the converse statement, we use a lifting argument to characteristic 0, based on the fact that any $\sigma$-selfdual supercuspidal $\ell$-modular representation admits a $\sigma$-selfdual $\ell$-adic lift. This latter fact does not hold for $\sigma$-selfdual cuspidal representations: there exist $\sigma$-selfdual cuspidal representations, with $m$ even, which are not distinguished (see Remark 8.5).

In the case where $T/T_0$ is ramified, we are reduced to studying the distinction of supercuspidal representations of $GL_m(I)$ by either $GL_1(I)$ if $m = 1$, or $GL_r(I) \times GL_r(I)$ if $m = 2r$ is even. It is more difficult, as we do not have an analogue of Theorem 3.1. Our proof relies on the structure of the projective envelope of a supercuspidal representation of $GL_m(I)$, as well as a lifting argument to characteristic 0. We prove that a supercuspidal representation is distinguished if and only if it is selfdual. Unlike the complex case, one can find $\sigma$-selfdual cuspidal representations, with $m > 1$ odd, which are not distinguished (see Remark 7.2).

In both cases, we prove that a $\sigma$-selfdual supercuspidal representation of $GL_m(I)$ is distinguished if any only if it admits a distinguished lift to characteristic 0. We conclude by the following theorem. Let $\overline{Q}_\ell$ be an algebraic closure of the field of $\ell$-adic numbers, and $\mathbb{F}_\ell$ its residual field.

**Theorem 1.6 (Theorem 9.11).** — Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of the group $G$ with coefficients in $\mathbb{F}_\ell$.

1. The representation $\pi$ admits a $\sigma$-selfdual supercuspidal lift to $\overline{Q}_\ell$.

2. Let $\tilde{\pi}$ be a $\sigma$-selfdual lift of $\pi$, and suppose that $\ell \neq 2$. Then $\tilde{\pi}$ is distinguished if and only if $\pi$ is distinguished.

**1.14.**

Let us end this introduction by a couple of remarks. First, the assumption that $p \neq 2$ is crucial throughout the paper: we use at many places the fact that the first cohomology set of the group $Gal(F/F_0)$ in a pro-$p$-group is trivial.

Let us also mention that the methods developed in this paper are expected to be generalizable to other groups $G$, such as unitary groups.

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2. Notation and basic definitions

Let $F/F_0$ be a quadratic extension of locally compact non-archimedean local fields of residual characteristic $p \neq 2$. Write $\sigma$ for the non-trivial $F_0$-automorphism of $F$. Write $\mathcal{O}$ for the ring of integers of $F$ and $\mathcal{O}_0$ for that of $F_0$. Write $k$ for the residue field of $F$ and $k_0$ for that of $F_0$. The involution $\sigma$ induces a $k_0$-automorphism of $k$, still denoted $\sigma$, which generates $\text{Gal}(k/k_0)$.

Let $R$ be an algebraically closed field of characteristic different from $p$, denoted $\ell$. (Note that $\ell$ can be 0.) We say we are in the “modular case” when we consider the case where $\ell \neq 0$.

We fix once and for all a character:

$$(2.1) \quad \psi_0 : F_0 \to R^\times$$

trivial on the maximal ideal of $\mathcal{O}_0$ but not on $\mathcal{O}_0$, and define $\psi = \psi_0 \circ \text{tr}_{F/F_0}$.

When $\ell \neq 2$, we write:

$$(2.2) \quad \omega = \omega_{F/F_0} : F_0^\times \to R^\times$$

for the character of $F_0^\times$ whose kernel is the subgroup of $F/F_0$-norms.

Let $G$ be the locally profinite group $G = \text{GL}_n(F)$, with $n \geq 1$, equipped with the involution $\sigma$ acting componentwise. Its $\sigma$-fixed points is the closed subgroup $G^\sigma = \text{GL}_n(F_0)$. We will identify the centre of $G$ with $F^\times$, and that of $G^\sigma$ with $F_0^\times$.

By representation of a locally profinite group, we always mean a smooth representation on an $R$-module. Given a representation $\pi$ of a closed subgroup $H$ of $G$, we write $\pi^\vee$ for the smooth contragredient of $\pi$, and $\pi^\sigma$ for the representation $\pi \circ \sigma$ of the subgroup $\sigma(H)$. We say that $\pi$ is $\sigma$-selfdual if $H$ is $\sigma$-stable and $\pi^\sigma$, $\pi^\vee$ are isomorphic. If $g \in G$, we write $H^g = \{g^{-1}hg \mid h \in H\}$ and $\pi^g$ for the representation $x \mapsto \pi(gxg^{-1})$ of $H^g$. If $\chi$ is a character of $H$, we write $\pi\chi$ for the representation $g \mapsto \chi(g)\pi(g)$.

If $\mu$ is a character of $H \cap G^\sigma$, we say that $\pi$ is $\mu$-distinguished if the space $\text{Hom}_{H \cap G^\sigma}(\pi, \mu)$ is non-zero. If $\mu$ is the trivial character, we will simply say that $\pi$ is $H \cap G^\sigma$-distinguished, or just distinguished. If $H = G$ and $\phi$ is a character of $F_0^\times$, we will abbreviate $\phi \circ \det$-distinguished to $\phi$-distinguished.

An irreducible representation of $G$ is said to be cuspidal if all its proper Jacquet modules are trivial or, equivalently, if it does not occur as a subrepresentation of a proper parabolically induced representation. It is said to be supercuspidal if it does not occur as a subquotient of a proper parabolically induced representation (by [12] Corollaire B.1.3, this is equivalent to not occurring as a subquotient of the parabolic induction of any irreducible representation of a proper Levi subgroup of $G$).

3. A modular version of theorems of Prasad and Flicker

Let us write $G = \text{GL}_n(F)$ and $G^\sigma = \text{GL}_n(F_0)$ for some $n \geq 1$. We prove the following theorem, which is well-known in the complex case.

**Theorem 3.1.** — Let $\pi$ be a distinguished irreducible representation of $G$. Then:

1. The central character of $\pi$ is trivial on $F_0^\times$. 
The contragredient representation $\pi^\vee$ is distinguished.

The space $\text{Hom}_{G^\sigma}(\pi, 1)$ has dimension 1.

The representations $\pi^\sigma$ and $\pi^\vee$ are isomorphic, that is, $\pi$ is $\sigma$-selfdual.

**Proof.** — Property 1 is straightforward. Property 2 follows from an argument of Gelfand-Kazhdan (see [44] Proposition 8.4 in the modular case). For Properties 3 and 4, we follow the proofs of Prasad [41] and Flicker [16] in the complex case.

Write $\mathcal{C}_c^\infty(G)$ for the space of locally constant, compactly supported $\mathbb{R}$-valued functions on $G$, and fix an $\mathbb{R}$-valued Haar measure on $G$, that is, a non-zero $\mathbb{R}$-linear form on $\mathcal{C}_c^\infty(G)$ invariant under left translation by $G$.

Let $W$ denote the vector space of $\pi$, and $l : W \to \mathbb{R}$ be a non-zero $G^\sigma$-invariant linear form. For any $f \in \mathcal{C}_c^\infty(G)$, define a linear form on $W$ by:

$$\pi(f)l : w \mapsto \int_G f(x)l(\pi(x)w) \, dx.$$ 

Since $f$ is smooth, the linear form $\pi(f)l$ on $W$ is smooth. This defines a non-zero homomorphism $L : \mathcal{C}_c^\infty(G) \to W^\vee$. It is $G$-equivariant under right translation and $G^\sigma$-invariant under left translation. Since $W$ is irreducible, it is surjective. Similarly, given a non-zero $G^\sigma$-invariant linear form $m : W^\vee \to \mathbb{R}$, we obtain a surjective right $G$-equivariant and left $G^\sigma$-invariant homomorphism $M$ from $\mathcal{C}_c^\infty(G)$ to $W^\vee \simeq W$. We now define:

$$B(f, g) = \langle M(f), L(g) \rangle \in \mathbb{R}$$

for all $f, g \in \mathcal{C}_c^\infty(G)$. This defines a right $G^\sigma$-invariant and left $G^\sigma \times G^\sigma$-invariant linear form $B$ on the space $\mathcal{C}_c^\infty(G) \otimes \mathcal{C}_c^\infty(G) \simeq \mathcal{C}_c^\infty(G \times G)$. As in [41] Lemma 4.2 (and with [16] Proposition 10) we have:

$$B(f, g) = B(g \circ \sigma, f \circ \sigma)$$

for all $f, g \in \mathcal{C}_c^\infty(G)$. It follows that the kernel of $L$ is equal to $\{ f \circ \sigma \mid f \in \text{Ker}(M) \}$. Thus, if $l'$ is any nonzero $G^\sigma$-invariant linear form on $W$, with associated homomorphism $L'$, then $L, L'$ have the same kernel. It follows that $l' = cl$ for some $c \in \mathbb{R}^\infty$ by Schur’s Lemma. Thus $\text{Hom}_{G^\sigma}(W, 1)$ has dimension 1. This proves Property 3.

As in [41] Lemma 4.2, $B$ corresponds to the $G^\sigma$-bi-invariant linear form $D$ on $\mathcal{C}_c^\infty(G)$ defined by:

$$D(h) = m(\pi(h)l)$$

for all $h \in \mathcal{C}_c^\infty(G)$. The correspondence between $B$ and $D$ is given by:

$$D(h) = B(k), \quad \text{with } k : (x, y) \mapsto h(xy^{-1}).$$

Note that (3.1) gives $D(h) = D(h \circ \sigma \circ i)$ with $i : x \mapsto x^{-1}$ on $G$. Replacing $\pi$ by $\pi^* = \pi^\vee \sigma$ and exchanging the roles played by $l, m$ we get a linear form:

$$D^* : h \mapsto l(\pi^*(h)m).$$
Since we have \( l(\pi^*(h)m) = m(\pi(h \circ \sigma \circ i)) \), it follows that \( D^* = D \). In order to deduce Property 4, it remains to prove that \( D \) determines \( \pi \) entirely. For any \( \xi \in W^\vee \) we define the function:

\[
c_\xi : x \mapsto m(\pi^\vee(x)\xi) = m(\xi \circ \pi(x^{-1}))
\]
on \( G \). Then \( \xi \mapsto c_\xi \) is an embedding of \( W^\vee \) in the space \( C^\infty(G^\sigma \setminus G) \) of smooth \( R \)-valued functions on \( G^\sigma \setminus G \). For \( y \in G \) and \( h \in C_c^\infty(G) \), let \( yh \) denote the function \( x \mapsto h(xy) \). Since \( L \) and \( M \) are surjective, there is a function \( h \) such that \( \pi(h)l \) is non-zero. Then \( y \mapsto D(\xi h) \) is a non-zero function in the space \( C^\infty(G^\sigma \setminus G) \), generating a subrepresentation isomorphic to \( W^\vee \). Indeed, it is equal to \( c_{\pi(h)} \). It thus follows from the equality \( D^* = D \) that we have \( \pi^\sigma \simeq \pi^\vee \), as expected. \( \square \)

**Remark 3.2.** — The same results hold — and the same argument works — when \( F/F_0 \) is a quadratic extension of finite fields.

### 4. Preliminaries on simple types

We assume the reader is familiar with the language of simple types. We recall the main results on simple strata, characters and types \([10, 6, 8, 36]\) that we will need. Part of these preliminaries can also be found in \([3]\).

#### 4.1. Simple strata and characters

Let \([a, \beta]\) be a simple stratum in the \( F \)-algebra \( M_n(F) \) of \( n \times n \) matrices with entries in \( F \) for some \( n \geq 1 \). Recall that \( a \) is a hereditary order in \( M_n(F) \) and \( \beta \) is a matrix in \( M_n(F) \) such that:

1. The \( F \)-algebra \( E = F[\beta] \) is a field, whose degree over \( F \) is denoted \( d \),
2. The multiplicative group \( E^\times \) normalizes \( a \).

The centralizer of \( E \) in \( M_n(F) \), denoted \( B \), is an \( E \)-algebra isomorphic to \( M_m(E) \), with \( n = md \). The intersection \( b = a \cap B \) is a hereditary order in \( B \).

Write \( p_a \) for the Jacobson radical of \( a \), and \( U^1(a) \) for the compact open pro-\( p \)-subgroup \( 1 + p_a \) of \( G = \text{GL}_n(F) \). We recall the following useful simple intersection property \([10\) Theorem 1.6.1]): for all \( x \in B^\times \), we have:

\[(4.1) \quad U^1(a)xU^1(a) \cap B^\times = U^1(b)xU^1(b) \]

Associated with \([a, \beta]\), there are compact open subgroups:

\[H^1(a, \beta) \subseteq J^1(a, \beta) \subseteq J(a, \beta)\]
of \( a^\times \) and a finite set \( \mathcal{C}(a, \beta) \) of characters of \( H^1(a, \beta) \) called simple characters, depending on the choice of the character \( \psi \) fixed in Section \([2]\). Write \( J(a, \beta) \) for the compact mod centre subgroup generated by \( J(a, \beta) \) and the normalizer of \( b \) in \( B^\times \).

**Proposition 4.1** \([8\) 2.1]. — We have the following properties:

1. The group \( J(a, \beta) \) is the unique maximal compact subgroup of \( J(a, \beta) \).
2. The group \( J^1(a, \beta) \) is the unique maximal normal pro-\( p \)-subgroup of \( J(a, \beta) \).
3. The group \( J(a, \beta) \) is generated by \( J^1(a, \beta) \) and \( b^\times \), and we have:

\[(4.2) \quad J(a, \beta) \cap B^\times = b^\times, \quad J^1(a, \beta) \cap B^\times = U^1(b) \].
(4) The normalizer of any simple character $\theta \in \mathcal{C}(a, \beta)$ in $G$ is equal to $J(a, \beta)$.

(5) The intertwining set of any $\theta \in \mathcal{C}(a, \beta)$ in $G$ is equal to $J^1(a, \beta)B^xJ^1(a, \beta)$.

By [10] Theorem 3.4.1, the quotient $J^1(a, \beta)/H^1(a, \beta)$ is a finite $k$-vector space, and the map:

\[(x, y) \mapsto \langle x, y \rangle = \theta(xyx^{-1}y^{-1})\]

makes it into a non-degenerate symplectic space. More precisely, if $h^1(a, \beta)$ and $j^1(a, \beta)$ are the sub-$\mathcal{O}$-lattices of $a$ such that $H^1(a, \beta) = 1 + h^1(a, \beta)$ and $J^1(a, \beta) = 1 + j^1(a, \beta)$, then we have:

\[(1 + u, 1 + v) = \psi \circ \text{tr}(\beta(uv - vu))\]

for all $u, v \in j^1(a, \beta)$ ([7] Proposition 6.1), where tr denotes the trace map of $M_n(F)$.

Let $[a', \beta']$ be another simple stratum in $M_{n'}(F)$ for some $n' \geq 1$, and suppose that there is an $F$-algebra isomorphism $\varphi : F[\beta] \to F[\beta']$ such that $\varphi(\beta) = \beta'$. Then there is a canonical bijective map:

\[\mathcal{C}(a, \beta) \to \mathcal{C}(a', \beta')\]

called the transfer map ([10] Theorem 3.6.14).

When the hereditary order $b = a \cap B$ is a maximal order in $B$, we say that the simple stratum $[a, \beta]$ and the simple characters in $\mathcal{C}(a, \beta)$ are maximal. When this is the case, then, given a homomorphism of $E$-algebras $B \simeq M_m(E)$ identifying $b$ with the standard maximal order, there are group isomorphisms:

\[J(a, \beta)/J^1(a, \beta) \simeq b^\times /U^1(b) \simeq \text{GL}_m(l)\]

where $l$ is the residue field of $E$.

### 4.2. Types and cuspidal representations

Let us write $G = \text{GL}_n(F)$ for some $n \geq 1$. A family of pairs $(J, \lambda)$ called extended maximal simple types, made of a compact mod centre, open subgroup $J$ of $G$ and an irreducible representation $\lambda$ of $J$, has been constructed in [10] (see also [36] in the modular case).

Given an extended maximal simple type $(J, \lambda)$ in $G$, there are a maximal simple stratum $[a, \beta]$ in $M_n(F)$ and a maximal simple character $\theta \in \mathcal{C}(a, \beta)$ such that $J(a, \beta) = J$ and $\theta$ is contained in the restriction of $\lambda$ to $H^1(a, \beta)$. Such a simple character is said to be attached to $\lambda$. By [10] Proposition 5.1.1 (or [36] Proposition 2.1 in the modular case), the group $J^1(a, \beta)$ carries, up to isomorphism, a unique irreducible representation $\eta$ whose restriction to $H^1(a, \beta)$ contains $\theta$. It is called the Heisenberg representation associated to $\theta$ and has the following properties:

1. The restriction of $\eta$ to $H^1(a, \beta)$ is made of $(J^1(a, \beta) : H^1(a, \beta))^{1/2}$ copies of $\theta$,
2. The representation $\eta$ extends to $J$.

For any representation $\kappa$ of $J$ extending $\eta$, there is, up to isomorphism, a unique irreducible representation $\rho$ of $J$ trivial on $J^1(a, \beta)$ such that $\lambda \simeq \kappa \otimes \rho$. Through [10], the restriction of $\rho$ to the maximal compact subgroup $J = J(a, \beta)$ identifies with a cuspidal representation of $\text{GL}_m(l)$.

**Remark 4.2.** — The reader familiar with the theory of simple types will have noticed that we did not introduce the notion of beta-extension. Since $\text{GL}_m(l)$ is not isomorphic to $\text{GL}_2(F_2)$ (as
p is not 2), any character of \( \text{GL}_m(l) \) factors through the determinant. It follows that, if \([a, \beta]\) is a maximal simple stratum, any representation of \( J \) extending \( \eta \) is a beta-extension.

We have the following additional property, which follows from \([36]\) Lemme 2.6.

**Proposition 4.3.** — Let \( \kappa \) be a representation of \( J \) extending \( \eta \), and write \( J^1 \) for the maximal normal pro-p-subgroup of \( J \). The map:

\[
\xi \mapsto \kappa \otimes \xi
\]

induces a bijection between isomorphism classes of irreducible representations \( \xi \) of \( J \) trivial on \( J^1 \) and isomorphism classes of irreducible representations of \( J \) whose restriction to \( J^1 \) contains \( \eta \).

We now give the classification of cuspidal (irreducible) representation of \( G \) in terms of extended maximal simple types (see \([10]\) 6.2, 8.4 and \([36]\) Section 3 in the modular case).

**Proposition 4.4 \((10, 36)\).** — Let \( \pi \) be a cuspidal representation of \( G \).

1. There is an extended maximal simple type \((J, \lambda)\) such that \( \lambda \) occurs as a subrepresentation of the restriction of \( \pi \) to \( J \). It is uniquely determined up to \( G \)-conjugacy.
2. Compact induction defines a bijection between the \( G \)-conjugacy classes of extended maximal simple types and the isomorphism classes of cuspidal representations of \( G \).

From now on, we will abbreviate extended maximal simple type to type.

### 4.3. Supercuspidal representations

Let \( \pi \) be a cuspidal representation of \( G \). By Proposition 4.4, it contains a type \((J, \lambda)\). Fix an irreducible representation \( \kappa \) as in Proposition 4.3 and let \( \rho \) be the corresponding representation of \( J \) trivial on its maximal normal pro-p-subgroup \( J^1 \).

Fix a maximal simple stratum \([a, \beta]\) such that \( J = J(a, \beta) \). Write \( E = F[\beta] \) and let \( \rho \) be the cuspidal representation of \( J/J^1 \cong \text{GL}_m(l) \) induced by \( \rho \). We record the following fact.

**Fact 4.5 \((37)\) Proposition 6.10).** — The representation \( \pi \) is supercuspidal if and only if \( \rho \) is supercuspidal.

Now suppose that \( \pi \) is supercuspidal, thus \( \rho \) is also supercuspidal. We show how to parameterize \( \rho \) by an “admissible pair of level zero”. This will be needed in Sections 7 and 8.

First, let \( t \) be an extension of degree \( m \) of \( l \), and identify \( t^x \) with a maximal torus in \( \text{GL}_m(l) \). By Green \([19]\) when \( R \) has characteristic 0 and James \([29]\) when \( R \) has positive characteristic \( \ell \) prime to the residual characteristic \( q \) of \( k \), there is a surjective map:

\[
\xi \mapsto \rho \xi
\]

between \( l \)-regular characters of \( t^x \) and isomorphism classes of supercuspidal irreducible representations of \( \text{GL}_m(l) \), whose fibers are \( \text{Gal}(t/l) \)-orbits.

**Definition 4.6 \((27, 9)\).** — An admissible pair of level 0 over \( E \) is a pair \((K/E, \xi)\) made of a finite unramified extension \( K \) of \( E \) and a tamely ramified character \( \xi : K^\times \to \mathbb{R}^\times \) which does not factor through \( N_{K/L} \) for any field \( L \) such that \( E \subseteq L \subseteq K \). Its degree is \([K : E]\).
If \((K'/E, \xi')\) is another admissible pair of level 0 over \(E\), it is said to be isomorphic to \((K/E, \xi)\) if there is an isomorphism of \(E\)-algebras \(\varphi : K' \to K\) such that \(\xi' = \xi \circ \varphi\).

**Definition 4.7.** — An admissible pair \((K/E, \xi)\) of level 0 and degree \(m\) is attached to \(\rho\) if:

1. writing \(t\) for the residue field of \(K\), the \(l\)-regular character of \(t^\times\) induced by the restriction of \(\xi\) to the units of the ring of integers of \(K\) corresponds to \(\rho\) via \((4.7)\).
2. the central character of \(\rho\) and the character \(\xi\) coincide on \(E^\times\).

The following proposition is a refinement of the property of the map \((4.7)\).

**Proposition 4.8.** — The attachment relation defines a bijection:

\[(K/E, \xi) \mapsto \rho(K/E, \xi)\]

between isomorphism classes of admissible pairs of level 0 over \(E\) and isomorphism classes of irreducible representations of \(J\) trivial on \(J^1\) whose restriction to \(J\) defines a supercuspidal representation of GL\(_n\)(I) through \((4.1)\).

### 4.4. The \(\sigma\)-selfdual type theorem

Let us fix an integer \(n \geq 1\) and write \(G = \text{GL}_n(F)\). We recall the first main result of [3].

**Theorem 4.9** ([3] Theorem 4.1). — Let \(\pi\) be a cuspidal representation of \(G\). The representation \(\pi\) is \(\sigma\)-selfdual if and only if it contains a type \((J, \lambda)\) such that \(J\) is \(\sigma\)-stable and \(\lambda^\sigma \simeq \lambda^\vee\).

**Remark 4.10.** — More precisely (see [3] Corollary 4.21), any \(\sigma\)-selfdual cuspidal representation contain a \(\sigma\)-selfdual type \((J, \lambda)\) with the additional property that \(J = J(a, \beta)\) for some maximal simple stratum \([a, \beta]\) in \(\text{M}_n(F)\) such that:

1. the hereditary order \(a\) is \(\sigma\)-stable and \(\sigma(\beta) = -\beta\);
2. the element \(\beta\) has the block diagonal form:

\[
\beta = \begin{pmatrix}
\beta_0 & & \\
& \ddots & \\
& & \beta_0
\end{pmatrix} = \beta_0 \otimes 1 \in \text{M}_d(F) \otimes_F \text{M}_n(F) = \text{M}_n(F)
\]

for some \(\beta_0 \in \text{M}_d(F)\), where \(d\) is the degree of \(\beta\) over \(F\) and \(n = md\); the centralizer \(B\) of \(E = F[\beta]\) in \(\text{M}_n(F)\) thus identifies with \(\text{M}_m(E)\), equipped with the involution \(\sigma\) acting componentwise;

3. the order \(b = a \cap B\) is the standard maximal order of \(\text{M}_m(E)\).

Such a type will be useful in the discussion following Proposition \((4.16)\).

**Remark 4.11.** — If \((J, \lambda)\) is any \(\sigma\)-selfdual type, then there is a maximal simple stratum \([a, \beta]\) in \(\text{M}_n(F)\) such that \(J = J(a, \beta)\), the order \(a\) is \(\sigma\)-stable and \(\sigma(\beta) = -\beta\) (see [3] Corollary 4.24). Such a maximal simple stratum will be said to be \(\sigma\)-selfdual.

**Remark 4.12.** — Let \(\pi\) be a \(\sigma\)-selfdual cuspidal representation of \(G\). Let \((J, \lambda)\) be a \(\sigma\)-selfdual type in \(\pi\), let \([a, \beta]\) be a simple stratum such that \(J = J(a, \beta)\) and let \(\theta \in \mathfrak{C}(a, \beta)\) be the maximal simple character attached to \(\lambda\). Then \(H^1(a, \beta)\) is \(\sigma\)-stable and \(\theta \circ \sigma = \theta^{-1}\).
Let $\pi$ be a $\sigma$-selfdual cuspidal representation of $G$. Let $(J, \chi)$ be a $\sigma$-selfdual type in $\pi$ and fix a $\sigma$-selfdual simple stratum $[a, \beta]$ as in Remark 4.11. Then $E = F[\beta]$ is $\sigma$-stable. We denote by $E_0$ the field of $\sigma$-fixed points in $E$, by $T$ the maximal tamely ramified sub-extension of $E$ over $F$ and by $T_0$ the intersection $T \cap E_0$. Also write $d = [E : F]$ and $n = md$.

**Proposition 4.13** (*[3] Proposition 4.30*). — The integer:

$$m(\pi) = m = n/d$$

and the $F_0$-isomorphism class of the quadratic extension $T/T_0$ only depend on $\pi$, and not on the choice of the $\sigma$-selfdual simple stratum $[a, \beta]$ as in Remark 4.11.

The integer $m$ defined by (4.9) is called the *relative degree* of $\pi$. We record a list of properties of the field extension $T/F$.

**Lemma 4.14.** —

1. The canonical homomorphism of $T_0 \otimes_{F_0} F$-modules:

$$T_0 \otimes_{F_0} F \to T$$

is an isomorphism.

2. If $F/F_0$ is unramified, then $T/T_0$ is unramified and $T/F$ has odd residual degree.

3. The extension $T/T_0$ is ramified if and only if $F/F_0$ is ramified and $T_0/F_0$ has odd ramification order.

**Proof.** — Assertion (1) is *[3] Lemma 4.10*. We now prove (2) and (3).

First, suppose that $F/F_0$ is unramified. We remark that:

$$f(T/F) \cdot f(F/F_0) = f(T/T_0) \cdot f(T_0/F_0)$$

is even. Since $F$ does not embed in $T_0$ as an $F_0$-algebra, $T_0$ has odd residue degree over $F_0$. It follows that $f(T/T_0) = 2$ and that $T$ has odd residue degree over $F$.

Suppose $F/F_0$ is ramified, and let $\varpi$ be a uniformizer of $F$ such that $\varpi_0 = \varpi^2$ is a uniformizer of $F_0$. Let $e_0$ be the ramification order of $T_0/F_0$, and let $t_0$ be a uniformizer of $T_0$ such that:

$$\varpi_0 = t_0^{e_0} \zeta_0$$

for some root of unity $\zeta_0 \in F_0^\times$ of order prime to $p$. Let $a$ be the greatest integer smaller than or equal to $e_0/2$, and write $x = \varpi t_0^{-a}$. We have $\sigma(x) = -x$, thus $x \not\in T_0$ and $x^2 \in T_0$.

If $e_0$ is odd, then $x^2 = \zeta_0 t_0$ is a uniformizer of $T_0$, whereas $x$ is a uniformizer of $T$, thus $T$ is ramified over $T_0$.

If $e_0$ is even, then $x^2 = \zeta_0$. It follows that $x$ is a root of unity of order prime to $p$ which is in $T$ but not in $T_0$, thus $T$ is unramified over $T_0$.

**Remark 4.15.** —

1. The extensions $E/E_0$ and $T/T_0$ have the same ramification order.

2. The extension $E/E_0$ is ramified if and only if $F/F_0$ is ramified and $E_0/F_0$ has odd ramification order.

The first property comes from *[3] Remark 4.22*, and the second one follows from the first one together with Lemma 4.14.
We now recall the classification of all \(\sigma\)-selfdual types contained in a given \(\sigma\)-selfdual cuspidal representation of \(G\) (see [3] Proposition 4.31).

**Proposition 4.16.** — Let \(\pi\) be a \(\sigma\)-selfdual cuspidal representation of \(G\), and let \(T/T_0\) denote the quadratic extension associated to it.

1. If \(T\) is unramified over \(T_0\), the \(\sigma\)-selfdual types contained in \(\pi\) form a single \(G^\sigma\)-conjugacy class.

2. If \(T\) is ramified over \(T_0\), the \(\sigma\)-selfdual types contained in \(\pi\) form exactly \([m/2]+1\) different \(G^\sigma\)-conjugacy classes.

One can give a more precise description in the ramified case. Suppose that \(T\) is ramified over \(T_0\), and let \((J_0, \lambda_0)\) be a \(\sigma\)-selfdual type in \(\pi\) satisfying the conditions of Remark [3.11]. Let us fix a uniformizer \(t\) of \(E\). For \(i = 0, \ldots, [m/2]\), let \(t_i\) denote the diagonal matrix:

\[
\text{diag}(t, \ldots, t, 1, \ldots, 1) \in B = \text{GL}_m(E)
\]

where \(t\) occurs \(i\) times. Then the pairs \((J_i, \lambda_i) = (J_0^i, \lambda_0^i)\), for \(i = 0, \ldots, [m/2]\), form a set of representatives of the \(G^\sigma\)-conjugacy classes of \(\sigma\)-selfdual types in \(\pi\).

**Definition 4.17.** — The integer \(i\) is called the index of the \(G^\sigma\)-conjugacy class of \((J_i, \lambda_i)\). It does not depend on the choice of \((J_0, \lambda_0)\), nor on that of \(t\).

Let \([a, \beta]\) be a simple stratum as in Remark [4.10] such that \(J_0 = J(a, \beta)\). If one identifies the quotient \(J(a, \beta)^{t_i}/J^1(a, \beta)^{t_i}\) with \(\text{GL}_m(l)\) via:

\[
J(a, \beta)^{t_i}/J^1(a, \beta)^{t_i} \simeq J(a, \beta)/J^1(a, \beta) \simeq U(b)/U^1(b) \simeq \text{GL}_m(l)
\]

then \(\sigma\) acts on \(\text{GL}_m(l)\) by conjugacy by the diagonal element:

\[
\delta_i = \text{diag}(-1, \ldots, -1, 1, \ldots, 1) \in \text{GL}_m(l)
\]

where \(-1\) occurs \(i\) times, and \((J(a, \beta)^{t_i}/G^\sigma)/(J^1(a, \beta)^{t_i} \cap G^\sigma)\) identifies with the Levi subgroup \((G_{\lambda} \times \text{GL}_{m-1})\) of \(\text{GL}_m(l)\).

### 4.5. Admissible pairs and \(\sigma\)-selfduality

Let \((J, \lambda)\) be a \(\sigma\)-selfdual type in \(G\). Fix a \(\sigma\)-selfdual maximal simple stratum \([a, \beta]\) such that \(J = J(a, \beta)\) as in Remark [4.11] and a decomposition of \(\lambda\) of the form \(\kappa \otimes \rho\) as in Paragraph [4.2]. Write \(E = F[\beta]\) as usual.

**Proposition 4.18.** — Suppose that the representation \(\rho\) is \(\sigma\)-selfdual, and let \((K/E, \xi)\) be an admissible pair of level 0 attached to it. Then there is a unique involutive \(E_0\)-automorphism \(\tau\) of \(K\) such that \(\xi \circ \tau = \xi^{-1}\) and \(\tau\) coincides with \(\sigma\) on \(E\).

**Proof.** — Let \(K'\) denote the extension of \(E\) given by the field \(K\) equipped with the map \(x \mapsto \sigma(x)\) from \(E\) to \(K\). Then the pair \((K'/E, \xi)\) is admissible of level 0, and it is attached to \(\rho^\sigma\). On the other hand, \((K/E, \xi^{-1})\) is admissible of level 0, attached to \(\rho^\sigma\). Since \(\rho\) is \(\sigma\)-selfdual, there is an \(E\)-algebra isomorphism \(\tau: K \to K'\) such that \(\xi \circ \tau = \xi^{-1}\). We thus have:

\[
\xi \circ \tau^2 = \xi^{-1} \circ \tau = \xi
\]
and \( \tau^2 \) is an \( E \)-algebra automorphism of \( K \). By admissibility of \((K/E, \xi)\), the latter automorphism is trivial, thus \( \tau \) satisfies the required conditions. Uniqueness follows by admissibility again. □

For simplicity, we will write \( \sigma \) for the involutive automorphism given by Proposition 4.18. Let \( K_0 \) be the field of \( \sigma \)-fixed points of \( K \). The following lemma will be useful in Section 9.

**Lemma 4.19.** — If \( E/E_0 \) is ramified and \( m \) is even, then \( K/K_0 \) is unramified.

**Proof.** — Write \( m = 2r \) for some \( r \geq 1 \). Let \( t \) be a uniformizer of \( E \) such that \( \sigma(t) = -t \) and let \( \zeta \in K \) be a root of unity of order \( Q^m - 1 \), where \( Q \) is the cardinality of \( l \). We thus have \( E = E_0[t] \) and \( K = E[\zeta] \). Since \( \sigma \) is involutive, it induces an involutive \( l \)-automorphism of \( t \), the residual field of \( K \). If the latter were trivial, the relation \( \zeta \circ \sigma = \zeta^{-1} \) would imply that the character \( \overline{\xi} \) of \( t^* \) induced by \( \xi \) is quadratic, contradicting the fact that it is \( l \)-regular. The automorphism of \( t \) induced by \( \sigma \) is thus the \( r \)th power of the Frobenius automorphism. Now consider the element:

\[
\alpha = \zeta^{(Q^r+1)/2}.
\]

It has order \( 2(Q^r - 1) \), thus \( \sigma(\alpha) = -\alpha \). Since \( \alpha^2 \) has order \( Q^r - 1 \), the extension of \( E_0 \) it generates is unramified and has degree \( r \). We thus have \( E_0[\alpha, t\alpha] \subseteq K_0 \) and their degrees are equal. Now we deduce that \( K = K_0[\alpha] = K_0[\zeta] \) is unramified over \( K_0 \). □

### 4.6.

The following lemma will be useful in Sections 6 and 8 when we investigate decompositions of \( \sigma \)-selfdual types of the form \( \kappa \otimes \rho \) which behave well under \( \sigma \).

Let \( \theta \in \mathfrak{C}(\alpha, \beta) \) be a maximal simple character such that \( H^1(\alpha, \beta) \) is \( \sigma \)-stable and \( \theta \circ \sigma = \theta^{-1} \). Let \( J \) be its normalizer in \( G \), let \( J^1 \) be the maximal normal \( \text{pro-}p \)-subgroup of \( J \) and \( \eta \) be the irreducible representation of \( J^1 \) containing \( \theta \).

**Lemma 4.20.** — Let \( \kappa \) be a representation of \( J \) extending \( \eta \). There is a unique character \( \mu \) of \( J \) trivial on \( J^1 \) such that \( \kappa^{\sigma^{-1}} \cong \kappa \mu \). It satisfies the identity \( \mu \circ \sigma = \mu \).

**Proof.** — Let \( \kappa \) be an irreducible representation of \( J \) extending \( \eta \). By uniqueness of the Heisenberg representation, the fact that \( \theta \circ \sigma = \theta^{-1} \) implies that \( \eta^{\sigma^{-1}} \) is isomorphic to \( \eta \). Thus \( \kappa \) and \( \kappa^{\sigma^{-1}} \) are representations of \( J \) extending \( \eta \). There is a character \( \mu \) of \( J \) trivial on \( J^1 \) such that we have \( \kappa^{\sigma^{-1}} \cong \kappa \mu \). It satisfies \( \mu \circ \sigma = \mu \). It is unique by Proposition 4.3. □

### 5. The distinguished type theorem

In this section we prove the following result, which is our first main theorem. It will be refined by Theorem 5.3 in Section 9.

**Theorem 5.1.** — Let \( \pi \) be a \( \sigma \)-selfdual cuspidal representation of \( G \). Then \( \pi \) is distinguished if and only if it contains a \( \sigma \)-selfdual type \((J, \lambda)\) such that \( \text{Hom}_{\text{Irreducible}}(\lambda, 1) \) is non-zero.

**Remark 5.2.** — If \( \pi \) is distinguished, it follows easily from the multiplicity 1 property in Theorem 3.1 that the distinguished \( \sigma \)-selfdual types \((J, \lambda)\) occurring in \( \pi \) form a single \( G^\sigma \)-conjugacy class (see Remark 5.22).
Remark 5.3. — When $\pi$ is a complex $\sigma$-selfdual cuspidal representation, Theorem 5.1 is proved in [3] in a different manner than the one given here, although both proofs use the $\sigma$-selfdual type theorem 4.9. The proof given in [3] is based on a result of Ok [40], which is available for complex representations only.

Let $\pi$ be a $\sigma$-selfdual cuspidal representation. Theorem 4.9 tells us that it contains a $\sigma$-selfdual type $p_J, \lambda q$, and Proposition 4.4 tells us that $\pi$ is compactly induced from $\lambda$. A simple application of the Mackey formula gives us:

$$\text{Hom}_{G^\sigma}(\pi, 1) \cong \bigoplus_g \text{Hom}_{J^\sigma \cap G^\sigma}(\lambda^g, 1)$$

where $g$ ranges over a set of representatives of $(J, G^\sigma)$-double cosets in $G$.

Remark 5.4. — It follows from Theorem 3.1 that there is at most one double coset $J g G^\sigma$ such that the space $\text{Hom}_{J^\sigma \cap G^\sigma}(\lambda^g, 1)$ is non-zero, and that this space has dimension at most 1. Thus the product in (5.1) is actually a direct sum.

In this section, our main task (achieved in Paragraph 5.5) is to prove that, if $\text{Hom}_{J^\sigma \cap G^\sigma}(\lambda^g, 1)$ is non-zero, then $\sigma(g) g^{-1} \in J$. Theorem 5.1 will follow easily from there (see Paragraph 5.6).

We may assume that $J = J(a, \beta)$ for a maximal simple stratum $[a, \beta]$ satisfying the conditions of Remark 4.10. The extension $E = F[\beta]$, its centralizer $B$ and the maximal order $b = a \cap B$ are thus stable by $\sigma$. We write $d = [E : F]$ and $n = md$. We identify $B$ with the $E$-algebra $M_m(E)$ equipped with the involution $\sigma$ acting componentwise, and $b$ with its standard maximal order.

We write $E_0 = E^\sigma$, the field of $\sigma$-invariant elements of $E$, and fix once and for all a uniformizer $t$ of $E$ such that:

$$\text{(5.2)} \quad \sigma(t) = \begin{cases} t & \text{if } E \text{ is unramified over } E_0, \\ -t & \text{if } E \text{ is ramified over } E_0. \end{cases}$$

We also write $J = J(a, \beta)$, $J^1 = J^1(a, \beta)$ and $H^1 = H^1(a, \beta)$. Recall that $J = E^\sigma J$.

We denote by $T$ the maximal tamely ramified sub-extension of $E$ over $F$, and set $T_0 = T \cap E_0$.

We insist on the fact that, throughout this section, we assume that the stratum $[a, \beta]$ satisfies the conditions of Remark 4.10.

5.1. Double cosets contributing to the distinction of $\theta$

Let $\theta \in \mathcal{C}(a, \beta)$ be the maximal simple character occurring in the restriction of $\lambda$ to $H^1$. Suppose that $\text{Hom}_{J^\sigma \cap G^\sigma}(\lambda^g, 1)$ is non-zero for some double coset $J g G^\sigma$. Restricting to $H^{1g} \cap G^\sigma$, we deduce that the character $\theta^g$ is trivial on $H^{1g} \cap G^\sigma$.

In this paragraph, we look for the double cosets $J g G^\sigma \subseteq G$ such that the character $\theta^g$ is trivial on $H^{1g} \cap G^\sigma$. For this, let us introduce the following general lemma.

Lemma 5.5. — Let $\tau$ be an involution of $G$, let $H$ be a $\tau$-stable open pro-$p$-subgroup of $G$ and let $\chi$ be a character of $H$ such that $\chi \circ \tau = \chi^{-1}$. Let $g \in G$. Then $\chi^g$ is trivial on $H^g \cap G^\tau$ if and only if $\tau(g) g^{-1}$ intertwines $\chi$. 
Supercuspidal representations of \( \text{GL}_n(F) \) distinguished by a Galois involution

**Proof.** Write \( K \) for the \( \tau \)-stable subgroup \( H^g \cap \tau(H^g) \), which contains \( H^g \cap G^\tau \). Let \( A \) be the quotient of \( K \) by \([K,K]\), the closure of the derived subgroup of \( K \). This is a \( \tau \)-stable commutative pro-\( p \)-group. Given \( x \in K \), write \( x' \) for its image in \( A \). For any \( b \in A \), we have:

\[
b = \sqrt{\sqrt{b}} \cdot \sqrt{\sqrt{b}}^{-1}
\]

where \( b \mapsto \sqrt{b} \) is the inverse of the automorphism \( b \mapsto b^2 \) of \( A \). Thus, for any \( x \in K \), there are \( y, z \in K \) such that \( x = yz \) and \( \tau(y') = y' \) and \( \tau(z') = z'^{-1} \).

Since \( \tau(z) = z^{-1}h \) for some \( h \in [K,K] \), we have:

\[
(5.3) \quad \chi^g(\tau(z)) = \chi^g(z^{-1}h) = \chi^g(z)^{-1}.
\]

On the other hand, since \( \tau(y) = yk \) for some \( k \in [K,K] \), the element \( y^{-1}\tau(y) \) defines a 1-cocycle in the \( \tau \)-stable pro-\( p \)-group \([K,K]\). Since \( p \neq 2 \), this cocycle is a coboundary, which implies:

\[
(5.4) \quad y \in (H^g \cap G^\tau)[K,K].
\]

Now suppose that \( \chi^g \) is trivial on \( H^g \cap G^\tau \). Then \( (5.3) \) and \( (5.4) \) imply that:

\[
(5.5) \quad \chi^g(\tau(x)) = \chi^g(z) = \chi^g(z)^{-1} = \chi^g(x)^{-1}, \quad \text{for all} \ x \in K.
\]

Besides, \( (5.5) \) is equivalent to \( \chi^g \) being trivial on \( H^g \cap G^\tau \). On the other hand, we have:

\[
(5.6) \quad \chi^g \circ \tau = (\chi \circ \tau)^\tau = (\chi^{-1})^\tau = (\chi^\tau)^{-1}
\]

on \( K \) by assumption on \( \chi \). If we set \( \gamma = \tau(g)^{-1} \), then \( (5.5) \) is equivalent to:

\[
\chi(h) = \chi^\gamma(h) \quad \text{for all} \ h \in H \cap \gamma^{-1}H\gamma.
\]

This amounts to saying that \( \gamma \) intertwines \( \chi \).

**Proposition 5.6.** Let \( g \in G \). Then the character \( \theta^g \) is trivial on \( H^{1g} \cap G^\tau \) if and only if we have \( \sigma(g)g^{-1} \in JB^\times J \).

**Proof.** This follows from Lemma 4.5 applied to the simple character \( \theta \) of \( H^1 \) and the involution \( \sigma \), together with the fact that the intertwining set of \( \theta \) is JB\(^\times\)J by Proposition 4.7. \( \square \)

### 5.2. The double coset lemma

We now prove the following fundamental lemma.

**Lemma 5.7.** Let \( g \in G \). Then \( \sigma(g)g^{-1} \in JB^\times J \) if and only if \( g \in JB^\times G^\tau \).

**Proof.** Write \( \gamma = \sigma(g)g^{-1} \). If \( g \in JB^\times G^\tau \), one verifies immediately that \( \gamma \in JB^\times J \). Conversely, suppose that \( \gamma \in JcJ \) for some \( c \in B^\times \). We will first show that the double coset representative \( c \) can be chosen nicely.

**Lemma 5.8.** There is a \( b \in B^\times \) such that \( \gamma \in JbJ \) and \( b\sigma(b) = 1 \).
Proof. — Recall that $B^\times$ has been identified with $\text{GL}_m(E)$ and $U = J \cap B^\times = b^\times$ is its standard maximal compact subgroup. By the Cartan decomposition, $B^\times$ decomposes as the disjoint union of the double cosets:

$$U \cdot \text{diag}(t^{a_1}, \ldots, t^{a_m}) \cdot U$$

where $a_1 \geq \ldots \geq a_m$ ranges over non-increasing sequences of $m$ integers, and $\text{diag}(\lambda_1, \ldots, \lambda_m)$ denotes the diagonal matrix of $B^\times$ with eigenvalues $\lambda_1, \ldots, \lambda_m \in E^\times$. We thus may assume that $c = \text{diag}(t^{a_1}, \ldots, t^{a_m})$ for a uniquely determined sequence of integers $a_1 \geq \ldots \geq a_m$.

The fact that $\sigma(\gamma) = \gamma^{-1}$ implies that we have $c \in Jc^{-1}J \cap B^\times$. Using the simple intersection property [1] and the fact that $J = UJ^1$ and $J^1 \subseteq U^1(a)$, we have $Jc^{-1}J \cap B^\times = Uc^{-1}U$. The uniqueness of the Cartan decomposition of $B^\times$ thus implies that the sequences $a_1 \geq \ldots \geq a_m$ and $-a_m \geq \ldots \geq -a_1$ are equal. We thus have $a_i + a_{m+1-i} = 0$ for all $i \in \{1, \ldots, m\}$. Now write $\kappa = \sigma(t)^{-1} \in \{-1, 1\}$ and choose signs $\kappa_1, \ldots, \kappa_m \in \{-1, 1\}$ such that $\kappa_i \kappa_{m+1-i} = \kappa^{a_i}$ for all $i$.

This is always possible since $a_{(m+1)/2} = 0$ when $m$ is odd. Then the antidiagonal element:

$$b = \begin{pmatrix} \kappa_1 t^{a_1} \\ \vdots \\ \kappa_m t^{a_m} \end{pmatrix} \in B^\times$$

satisfies the required conditions $b\sigma(b) = 1$ and $\gamma \in JbJ$.

Now write $\gamma = x'bx$ with $x, x' \in J$ and $b \in B^\times$. Replacing $g$ by $\sigma(x')^{-1}g$ does not change the double coset $JgG\gamma$ but changes $\gamma$ into $bx\sigma(x')$. From now on, we will thus assume that:

$$\gamma = bx, \quad b\sigma(b) = 1, \quad x \in J, \quad b \text{ is of the form } (5.7).$$

Write $K$ for the group $J \cap b^{-1}Jb$. Since $\sigma(b) = b^{-1}$ and $J$ is $\sigma$-stable, we have $x \in K$.

Lemma 5.9. — The map $\delta : k \mapsto b^{-1}\sigma(k)b$ is an involutive group automorphism of $K$.

Proof. — This follows from an easy calculation using the fact that $b\sigma(b) = 1$. 

Let $b_1 > \cdots > b_r$ be the unique decreasing sequence of integers such that:

$$\{a_1, \ldots, a_m\} = \{b_1, \ldots, b_r\}$$

and $m_j$ denote the multiplicity of $b_j$ in $(a_1, \ldots, a_m)$, for $j \in \{1, \ldots, r\}$. We have $m_j = m_{r+1-j}$ for all $j$, and $m_1 + \cdots + m_r = m$. These integers define a standard Levi subgroup:

$$M = \text{GL}_{m_1d}(F) \times \cdots \times \text{GL}_{m_rd}(F) \subseteq G.$$ 

Write $P$ for the standard parabolic subgroup of $G$ generated by $M$ and upper triangular matrices. Let $N$ and $N^-$ denote the unipotent radicals of $P$ and its opposite parabolic subgroup with respect to $M$, respectively. Since $b$ has the form $(5.7)$, it normalizes $M$ and we have:

$$K = (K \cap N^-) \cdot (K \cap M) \cdot (K \cap N),$$ 

$$K \cap P = J \cap P,$$

$$K \cap N^- \subseteq J^1 \cap N^-.$$
We have similar properties for the subgroup \( V = K \cap B^x = U \cap b^{-1}Ub \) of \( B^x \), that is:

\[
V = (V \cap N^{-}) \cdot (V \cap M) \cdot (V \cap N),
\]

\[
V \cap P = U \cap P,
\]

\[
V \cap N^{-} \subseteq U^1 \cap N^{-},
\]

where \( U^1 = J^1 \cap B^x = U^1(b) \). Note that this subgroup \( V \) is stable by \( \delta \).

**Lemma 5.10.** — The subset:

\[
K^1 = (K \cap N^{-}) \cdot (J^1 \cap M) \cdot (K \cap N)
\]

is a \( \delta \)-stable normal pro-\( p \)-subgroup of \( K \), and we have \( K = VK^1 \).

**Proof.** — To prove that \( K^1 \) is a subgroup of \( K \), it is enough to prove that one has the containment \( (K \cap N) \cdot (K \cap N^{-}) \subseteq K^1 \). Let \( j^1 \) be the sub-\( 0 \)-lattice of \( \mathfrak{a} \) such that \( J^1 = 1 + j^1 \) and let \( j = b + j^1 \), thus \( J = j^x \). A simple computation shows that \( K \cap N^{-} \subseteq (1 + tj) \cap N^{-} \) and:

\[
(K \cap N) \cdot (K \cap N^{-}) \subseteq (K \cap N^{-}) \cdot ((1 + tj) \cap M) \cdot (K \cap N).
\]

The expected result thus follows from the fact that \( tj \subseteq j^1 \). Besides, \( K^1 \) is a \( \delta \)-stable pro-\( p \)-group.

Since \( V \cap M \) normalizes \( K \cap N^{-} \), \( K \cap N \) and \( K^1 \cap M \), we have \((V \cap M)K^1 = K\) whence \( K^1 \) is normal in \( K \) and \( K = VK^1 \), as expected. \( \square \)

The subgroup \( K^1 \) is useful in the following lemma. Note that we have \( x\delta(x) = 1 \).

**Lemma 5.11.** — Let \( y \in K \) be such that \( y\delta(y) = 1 \). There are \( k \in K \) and \( v \in V \) such that:

1. The element \( v \) is diagonal in \( B^x \) with eigenvalues in \( \{-1,1\} \) and it satisfies \( v\delta(v) = 1 \),
2. one has \( \delta(k)yk^{-1} \in vK^1 \).

**Proof.** — Let \( V^1 = V \cap K^1 = K^1 \cap B^x \). We have:

\[
V^1 = (V \cap N^{-}) \cdot (U^1 \cap M) \cdot (U \cap N).
\]

We thus have canonical \( \delta \)-equivariant group isomorphisms:

\[
K/K^1 \simeq V/V^1 \simeq (U \cap M)/(U^1 \cap M).
\]

By (5.7), we have \( M \cap B^x = GL_{m_1}(E) \times \cdots \times GL_{m_r}(E) \), thus the right hand side of (5.10) identifies with \( M = GL_{m_1}(l) \times \cdots \times GL_{m_r}(l) \), where \( l \) denotes the residue field of \( E \). Besides, since \( b \) is given by (5.7), the involution \( \delta \) acts on \( M \) as:

\[
(g_1, \ldots, g_r) \mapsto (\sigma(g_r), \ldots, \sigma(g_1)).
\]

Write \( y = vy' \) for some \( v \in V \) and \( y' \in K^1 \). The simple intersection property (4.1) gives us:

\[
\delta(v)^{-1} = \delta(y')vy' \in V \cap K^1 \cdot vK^1 = V^1 \cdot vV^1.
\]

Thus there is \( u \in V^1 \) such that \( vu\delta(vu) \in V^1 \). Replacing \( (v, y') \) by \( (vu, u^{-1}y') \), we may and will assume that \( y = vy' \) with \( v\delta(v) \in V^1 \).
We now compute the first cohomology set of $\delta$ in $\mathcal{M}$. Let $w = (w_1, \ldots, w_r)$ denote the image of $y$ in $\mathcal{M}$. We have $w\delta(w) = 1$, that is:

$$\sigma(w_j) = w_{r+1-j}^{-1}, \quad \text{for all } j \in \{1, \ldots, r\}.$$  

If $r$ is even, one can find an element $a \in \mathcal{M}$ such that $w = \delta(a)a^{-1}$. If $r$ is odd, say $r = 2s - 1$, one can find an element $a \in \mathcal{M}$ such that:

$$\delta(a)w^{-1} = (1, \ldots, 1, w_s, 1, \ldots, 1)$$

and we have $w_s\sigma(w_s) = 1$. If $E/E_0$ is unramified, then $l$ is quadratic over the residue field of $E_0$, and it follows from the triviality of the first cohomology set of $\sigma$ in $GL_{m_s}(l)$ that $w = \sigma(c)c^{-1}$ for some $c \in \mathcal{M}$. In these two cases, we thus may find $k \in K$ such that $\delta(k)xk^{-1} \in K^1$.

It remains to treat the case where $r$ is odd and $E/E_0$ is ramified. In this case, we have $w_s^2 = 1$, thus $w_s$ is conjugate in $GL_{m_s}(l)$ to a diagonal element with eigenvalues 1 and $-1$. Let $i$ denote the multiplicity of $-1$. Let:

$$v \in U \cap \mathcal{M} = GL_{m_1}(\mathcal{O}_E) \times \cdots \times GL_{m_r}(\mathcal{O}_E)$$

(here $\mathcal{O}_E$ is the ring of integers of $E$) be a diagonal matrix with eigenvalues 1 and $-1$, such that $-1$ occurs with multiplicity $i$ and only in the $s$th block. Then $v\delta(v) = 1$ and there is $k \in K$ such that $\delta(k)yk^{-1} \in vK^1$.

Applying Lemma [5.11] to $x$ gives us $k \in K$, $v \in V$ such that $bv\sigma(bv) = 1$ and $\delta(k)xk^{-1} \in vK^1$.

Besides, $bv$ is antidiagonal of the form (5.7) and $\sigma(k)\gamma k^{-1} \in bvK^1$. Therefore, replacing $g$ by $kg$, which does not change the double coset $JgG^\sigma$, we will assume that $\gamma$ can be written:

(5.11) $\gamma = bx, \quad b\sigma(b) = 1, \quad x \in J^1, \quad b$ is of the form (5.7).

Comparing with (5.8), we now have a stronger condition on $x$, that is $x\delta(x) = 1$ and $x \in K^1$.

Since $K^1$ is a $\delta$-stable pro-$p$-group and $p$ is odd, the first cohomology set of $\delta$ in $K^1$ is trivial. Thus $x = \delta(y)y^{-1}$ for some $y \in K^1$, hence $\gamma = \sigma(y)by^{-1}$. Since $b\sigma(b) = 1$ and the first cohomology set of $\sigma$ in $B^\times$ is trivial, one has $b = \sigma(h)h^{-1}$ for some $h \in B^\times$. Thus $g \in yhG^\sigma \leq JB^\times G^\sigma$, and Lemma [5.7] is proved.

5.3. Contribution of the Heisenberg representation

Let $\eta$ be the Heisenberg representation of $J^1$ associated to $\theta$ (see §4.2). In this paragraph, we prove the following result.

**Proposition 5.12.** — Given $g \in G$, we have:

$$\dim \text{Hom}_{J^1 \cap G^\sigma}(\eta^g, 1) = \begin{cases} 1 & \text{if } g \in JB^\times G^\sigma, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** — Suppose that $\text{Hom}_{J^1 \cap G^\sigma}(\eta^g, 1)$ is non-zero. Restricting to $H^1g \cap G^\sigma$, the character $\theta^g$ is trivial on $H^1g \cap G^\sigma$, and Proposition [5.6] together with Lemma [5.7] give us $g \in JB^\times G^\sigma$. Conversely, assume that $g \in JB^\times G^\sigma$. Since the dimension of $\text{Hom}_{J^1 \cap G^\sigma}(\eta^g, 1)$ does not change when $g$ varies in a given $(J, G^\sigma)$-double coset, we may and will assume that we have $g \in B^\times$. Thus we have $\gamma = \sigma(g)g^{-1} \in B^\times$ as well.
Lemma 5.13. — The map \( \tau : x \mapsto \gamma^{-1}\sigma(x)\gamma \) is an involutive automorphism of \( G \) and, for any subgroup \( H \subseteq G \), we have \( H^\sigma \cap G^\sigma = (H \cap G)^\sigma \).

Proof. — This follows from an easy calculation using the fact that \( \sigma(\gamma) = \gamma^{-1} \).

Our goal is thus to prove that the space \( \text{Hom}_{J^1 \cap G^\tau} (\eta, 1) \) has dimension 1. By Paragraph 4.2, the representation of \( J^1 \) induced from \( \theta \) decomposes as the direct sum of \( (J^1 : H^1)^{1/2} \) copies of the representation \( \eta \). The space:

\[
\text{Hom}_{J^1 \cap G^\tau} \left( \text{Ind}_{H^1}^1 (\theta), 1 \right)
\]

thus decomposes as the direct sum of \( (J^1 : H^1)^{1/2} \) copies of \( \text{Hom}_{J^1 \cap G^\tau} (\eta, 1) \). Applying the Frobenius reciprocity and the Mackey formula, the space \( \text{Hom}_{J^1 \cap G^\tau} (\eta, 1) \) is isomorphic to:

\[
\text{Hom}_{J^1} \left( \text{Ind}_{H^1}^1 (\theta), \text{Ind}_{H^1 \cap G^\tau}^1 (1) \right) \cong \bigoplus_{x \in X} \text{Hom}_{H^1} \left( \theta, \text{Ind}_{H^1 \cap G^\tau}^1 (1) \right)
\]

where \( X \) is equal to \( J^1/(J^1 \cap G^\tau)H^1 \). Since \( J^1 \) normalizes \( \theta \), this is isomorphic to:

\[
\bigoplus_{x \in X} \text{Hom}_{H^1} \left( \theta, \text{Ind}_{H^1 \cap G^\tau}^1 (1) \right) \cong \bigoplus_{x \in X} \text{Hom}_{H^1 \cap G^\tau} (\theta, 1).
\]

Since \( \text{Hom}_{H^1 \cap G^\tau} (\theta, 1) \) has dimension 1, the right hand side of (5.13) has dimension the cardinality of \( X \). It thus remains to prove that \( X \) has cardinality \( (J^1 : H^1)^{1/2} \), or equivalently:

\[
(J^1 \cap G^\tau : H^1 \cap G^\tau) = (J^1 : H^1)^{1/2}.
\]

Now consider the groups \( J^1 \cap J^{1\gamma} \) and \( H^1 \cap H^{1\gamma} \), which are both stable by \( \tau \).

Lemma 5.14. — We have \( \theta(\tau(x)) = \theta(x)^{-1} \) for all \( x \in H^1 \cap H^{1\gamma} \).

Proof. — Given \( x \in H^1 \cap H^{1\gamma} \), and using the fact that \( \theta \circ \sigma = \theta^{-1} \) on \( H^1 \), we have:

\[
\theta(\tau(x))^{-1} = \theta \circ \sigma(\tau(x)) = \theta(\tau(x)) = \theta(x)
\]

since \( \gamma \in B^\times \) intertwines \( \theta \).

Let us write \( \mathbb{V} \) for the \( k \)-vector space \( (J^1 \cap J^{1\gamma})/(H^1 \cap H^{1\gamma}) \) equipped with both the involution \( \tau \) and the symplectic form \( (x, y) \mapsto \langle x, y \rangle \) induced by (4.12). We write \( \mathbb{V}^+ = \{ v \in \mathbb{V} \mid \tau(v) = v \} \) and \( \mathbb{V}^- = \{ v \in \mathbb{V} \mid \tau(v) = -v \} \). We have the decomposition \( \mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^- \) since \( p \neq 2 \).

Lemma 5.15. — The subspaces \( \mathbb{V}^+ \) and \( \mathbb{V}^- \) are totally isotropic.

Proof. — Indeed, thanks to Lemma 5.14 first note that:

\[
\langle \tau(x), y \rangle = \langle \tau(y), x \rangle, \quad x, y \in \mathbb{V}.
\]

If \( x, y \in \mathbb{V}^+ \), then we get \( \langle x, y \rangle = \langle x, y \rangle^{-1} \), thus \( \langle x, y \rangle = 1 \) since \( p \neq 2 \). If \( x, y \in \mathbb{V}^- \), then we get \( \langle x^{-1}, y \rangle = \langle x, y^{-1} \rangle^{-1} \). But \( \langle x^{-1}, y \rangle = \langle x, y^{-1} \rangle^{-1} = \langle x, y \rangle^{-1} \). It follows again that \( \langle x, y \rangle = 1 \).

Let \( \mathbb{W} \) denote the kernel of the symplectic form \( (x, y) \mapsto \langle x, y \rangle \) on \( \mathbb{V} \), that is:

\[
\mathbb{W} = \{ w \in \mathbb{V} \mid \langle w, v \rangle = 1 \text{ for all } v \in \mathbb{V} \}.
\]

Let \( \mathbb{Y} \) and \( \mathbb{Y}' \) denote the images of \( H^1 \cap J^{1\gamma} \) and \( J^1 \cap H^{1\gamma} \) in \( \mathbb{V} \), respectively.
Lemma 5.16. — The subspaces $\mathcal{Y}$ and $\mathcal{Y}'$ are both contained in $\mathcal{W}$, and we have $\mathcal{W} = \mathcal{Y} \oplus \mathcal{Y}'$.

Proof. — One easily verifies that $\tau$ stabilizes $\mathcal{W}$ and exchanges $\mathcal{Y}$ and $\mathcal{Y}'$. First note that $\mathcal{Y} \subseteq \mathcal{W}$, since $\langle x, y \rangle = 1$ for any $x \in \mathcal{H}$ and $y \in J^1$. By applying $\tau$, and thanks to (5.15), we deduce that $\mathcal{Y}'$ is also contained in $\mathcal{W}$. Now, thanks to (5.14), we have:

$$\langle 1 + x, 1 + y \rangle = \psi \circ \text{tr}(\beta(xy - yx))$$

for all $x, y \in J^1 \cap j^1 \gamma$, where $J^1$ is the sub-$\mathcal{O}$-lattice of $\mathcal{O}$ such that $J^1 = 1 + J^1$. Let $a\beta$ denote the endomorphism of $F$-algebras $x \mapsto \beta x - x\beta$ of $M_n(F)$. Given a subset $S \subseteq M_n(F)$, write $S^*$ for the set of $a \in M_n(F)$ such that $\psi(\text{tr}(as)) = 1$ for all $s \in S$. Then the set of $x \in J^1 \cap j^1 \gamma$ such that $\langle 1 + x, 1 + y \rangle = 1$ for all $y \in J^1 \cap j^1 \gamma$ is equal to:

$$j^1 \cap j^1 \gamma \cap a\beta(j^1 \cap j^1 \gamma)^* = j^1 \cap j^1 \gamma \cap (a\beta(j^1) \cap a\beta(j^1 \gamma))^*$$

$$= j^1 \cap j^1 \gamma \cap (a\beta(j^1))^* + a\beta(j^1 \gamma)^*$$

$$= j^1 \gamma \cap (j^1 \cap a\beta(j^1)^*) + j^1 \cap (j^1 \cap a\beta(j^1 \gamma)^*)$$

We now claim that:

$$j^1 \cap a\beta(j^1)^* = \mathfrak{h}^1.$$  

(5.16)

To see this, look at the case where $g = 1$. On the one hand, for $x \in J^1$, we have $\langle 1 + x, 1 + y \rangle = 1$ for all $y \in J^1$ if and only if $x \in J^1 \cap a\beta(j^1)^*$. On the other hand, the symplectic form (4.3) on the space $J^1 / \mathcal{H}$ is non-degenerate. We thus have $j^1 \cap a\beta(j^1)^* \subseteq \mathfrak{h}^1$ and the other containment follows from the fact that $\psi$ is trivial on the maximal ideal of $\mathcal{O}$.

We now go back to our general situation with $g \in \mathcal{B}^\times$. Applying (5.16) to $J^1$ and $j^1 \gamma$, we get:

$$j^1 \cap j^1 \gamma \cap a\beta(j^1 \cap j^1 \gamma)^* = j^1 \gamma \cap \mathfrak{h}^1 + \mathfrak{h}^1 \gamma \cap j^1.$$

The result follows.

Corollary 5.17. — The subspaces $\mathcal{W}^+ = \mathcal{W} \cap \mathcal{V}^+$ and $\mathcal{W}^- = \mathcal{W} \cap \mathcal{V}^-$ have the same dimension and we have $\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^-.$

Proof. — The map $x \mapsto x + \tau(x)$ is an isomorphism from $\mathcal{Y}$ to $\mathcal{W}^+$, and the map $x \mapsto x - \tau(x)$ is an isomorphism from $\mathcal{Y}$ to $\mathcal{W}^-$. Thanks to Lemma 5.10 and the fact that $\mathcal{Y}$, $\mathcal{Y}'$ have the same dimension, we thus have:

$$\dim \mathcal{W}^+ + \dim \mathcal{W}^- = 2 \cdot \dim \mathcal{Y} = \dim \mathcal{W},$$

which ends the proof of the corollary.

Now consider the non-degenerate symplectic space $\mathcal{V} / \mathcal{W}$. It decomposes into the direct sum of two totally isotropic subspaces $(\mathcal{V}^+ + \mathcal{W}) / \mathcal{W}$ and $(\mathcal{V}^- + \mathcal{W}) / \mathcal{W}$. We thus have:

$$\max(\dim((\mathcal{V}^+ + \mathcal{W}) / \mathcal{W}), \dim((\mathcal{V}^- + \mathcal{W}) / \mathcal{W})) \leq \frac{1}{2} \cdot \dim(\mathcal{V} / \mathcal{W}),$$

$$\dim((\mathcal{V}^+ + \mathcal{W}) / \mathcal{W}) + \dim((\mathcal{V}^- + \mathcal{W}) / \mathcal{W}) = \dim(\mathcal{V} / \mathcal{W}).$$

These spaces thus have the same dimension and are maximal totally isotropic. Corollary 5.17 now implies that $\mathcal{V}^+$ and $\mathcal{V}^-$ have the same dimension. This ends the proof of the proposition.
5.4. Contribution of the mixed Heisenberg representation

Let \( g \in J B^\times G^\sigma \). We have seen in Paragraph \([5.2]\) (see \([5.11]\)) that, changing \( g \) without changing the double coset \( J g G^\sigma \), we may assume that \( g \in B^\times \) and that \( \gamma = (g) g^{-1} \) can be written \( \gamma = bu \) with \( b \) of the form \([5.7]\) and \( u \in U^1 = J^1 \cap B^\times \). We write \( \tau \) for the involution defined by Lemma \([5.13]\), and \( U = J \cap B^\times \).

We have a standard Levi subgroup \( M \) of \( G \) defined by \([5.9]\) and parabolic subgroups \( P, P^- \) of \( G \) with Levi component \( M \), opposite to each other and with unipotent radicals \( N, N^- \) respectively. There is a unique standard hereditary order \( b_m \subseteq b \) such that:

\[
b_m^\times = (U^1 \cap N^-) \cdot (U \cap P).
\]

Since \( u \in U^1 \subseteq U^1(b_m) \) and thanks to the specific form of \( b \), one verifies that:

\[
(5.17) \quad U^1(b_m) = (U^1 \cap P^-) \cdot (U \cap N) = (U \cap U^1)U^1.
\]

Let \( a_m \subseteq a \) be the unique hereditary order of \( M_n(F) \) normalized by \( E^\times \) such that \( a_m \cap B = b_m \). This gives us a simple stratum \([a_m, \beta]\). Let \( \theta_m \in \mathfrak{C}(a_m, \beta) \) be the transfer of \( \theta \) (see \([4.3]\)) and \( \eta_m \) be the Heisenberg representation on \( J_m^1 = J^1(a_m, \beta) \) associated with \( \theta_m \) (by \([10]\) Proposition 5.1.1 and \([36]\) Proposition 2.1).

Let \( S^1 \) be the pro-\( p \)-subgroup \( U^1(b_m)J \subseteq J \). By \([10]\) Proposition 5.1.15, there is an irreducible representation \( \mu \) of the group \( S^1 \), unique up to isomorphism, extending \( \eta \) such that:

\[
(5.18) \quad \text{Ind}_{S^1}^{U^1(a_m)}(\mu) \cong \text{Ind}_{J_m^1}^{U^1(a_m)}(\eta_m).
\]

In this paragraph, we prove the following result.

**Proposition 5.18. —** We have \( \dim \text{Hom}_{S^1 \cap G^\sigma}(\mu^\theta, 1) = 1 \).

**Proof.** — Since \( \mu \) extends \( \eta \), the space \( \text{Hom}_{S^1 \cap G^\sigma}(\mu^\theta, 1) \) is contained in the 1-dimensional space \( \text{Hom}_{J_m^1 \cap G^\sigma}(\eta^\theta, 1) \). It is thus enough to prove that \( \text{Hom}_{S^1 \cap G^\sigma}(\mu^\theta, 1) \) is non-zero. Equivalently, by Lemma \([5.13]\) it is enough to prove that \( \text{Hom}_{S^1 \cap G^\sigma}(\mu, 1) \) is non-zero.

First note that, since \( b_m \) is \( \sigma \)-stable, \( a_m \) is \( \sigma \)-stable as well. We have:

\[
\sigma(H^1(a_m, \beta)) = H^1(a_m, \sigma(\beta)) = H^1(a_m, -\beta) = H^1(a_m, \beta)
\]

thus \( H^1_m = H^1(a_m, \beta) \) is \( \sigma \)-stable. By an argument similar to the one used in \([3]\) Paragraph 4.6, it then follows that \( \theta_m \circ \sigma = (\theta_m)^{-1} \).

Since \( \gamma \) intertwines \( \theta_m \) by Proposition \([4.4]\), it follows from Proposition \([5.6]\) that the character \( \theta_m^\sigma \) is trivial on \( H^1_m \cap G^\sigma \), thus \( \text{Hom}_{J_m^1 \cap G^\sigma}(\eta^\theta_m, 1) = \text{Hom}_{J_m^1 \cap G^\sigma}(\eta_m, 1) \) is non-zero. Inducing to \( U^1(a_m) \), we get:

\[
\text{Hom}_{U^1(a_m) \cap G^\sigma} \left( \text{Ind}_{J_m^1}^{U^1(a_m)}(\eta_m), 1 \right) \neq \{0\}.
\]

Applying the Frobenius reciprocity and the Mackey formula, it follows that there is a \( x \in U^1(a_m) \) such that:

\[
(5.19) \quad \text{Hom}_{S^1 \cap G^\sigma}(\mu^x, 1) \neq \{0\}.
\]
We claim that \( x \in \text{S}^1(J^1(a_m) \cap G^\sigma) \). Restricting \((5.19)\) to the subgroup \( H^{1x} \cap G^\sigma \) and applying Proposition \(5.6\) we get:

\[
\sigma(xg)g^{-1}x^{-1} = \sigma(x)\gamma x^{-1} \in J^1B^xJ^1 \cap U^1(a_m)\gamma U^1(a_m).
\]

Write \( \sigma(x)\gamma x^{-1} = jcb' \) for some \( j, j' \in J^1 \) and \( c \in B^x \). Since \( \gamma \in B^x \) and \( J^1 \subseteq U^1(a_m) \), the simple intersection property \((4.1)\) implies that \( c \in U^1(a_m)\gamma U^1(a_m) \cap B^x = U^1(b_m)\gamma U^1(b_m) \). Therefore we have \( \sigma(x)\gamma x^{-1} = \sigma(s)\gamma s' \) for some \( s, s' \in S^1 \). If we let \( y = s^{-1}x \), then we have \( \sigma(y)\gamma y^{-1} = \gamma l \) for some \( l \in S^1 \), that is \( \tau(y)y^{-1} = l \). Since the first cohomology set of \( \tau \) in \( S^1 \) is trivial, we get \( l = \tau(h)h^{-1} \) for some \( h \in S^1 \). This gives us:

\[
x \in U^1(a_m) \cap S^1(G^\sigma)\gamma^{-1}
\]

and the claim follows from the fact that \( S^1 \subseteq U^1(a_m) \).

Putting the claim and \((5.19)\) together, we deduce that \( \text{Hom}_{\text{S}^1 \cap G^\sigma} (\mu, 1) \) is non-zero.

\[\blacksquare\]

5.5. The double coset theorem

Let \( \kappa \) be an irreducible representation of \( J \) extending \( \eta \) as in Paragraph \((4.2)\). There is an irreducible representation \( \rho \) of \( J \), unique up to isomorphism, which is trivial on the subgroup \( J^1 \) and satisfies \( \lambda \approx \kappa \otimes \rho \). We have the following lemma.

**Lemma 5.19.** — Let \( g \in J^1B^xG^\sigma \).

(1) There is a unique character \( \chi \) of \( J^9 \cap G^\sigma \) trivial on \( J^1 \cap G^\sigma \) such that:

\[
\text{Hom}_{J^1 \cap G^\sigma} (\eta^9, 1) = \text{Hom}_{J^1 \cap G^\sigma} (\kappa^9, \chi^{-1}).
\]

(2) The canonical linear map:

\[
\text{Hom}_{J^1 \cap G^\sigma} (\eta^9, 1) \otimes \text{Hom}_{J^1 \cap G^\sigma} (\rho^9, \chi) \to \text{Hom}_{J^1 \cap G^\sigma} (\lambda^9, 1)
\]

is an isomorphism.

**Proof.** — Let us fix a non-zero linear form \( \mathcal{E} \in \text{Hom}_{J^1 \cap G^\sigma} (\eta^9, 1) \). The choice of \( \kappa \) defines an action of \( J^9 \cap G^\sigma \) on the space \( \text{Hom}_{J^1 \cap G^\sigma} (\eta^9, 1) \), which has dimension 1 by Proposition \(5.12\). This determines a unique character \( \chi \) of \( J^9 \cap G^\sigma \) trivial on \( J^1 \cap G^\sigma \) such that:

\[
\mathcal{E} \circ \kappa^9(x) = \chi(x^{-1}) \cdot \mathcal{E}
\]

for all \( x \in J^9 \cap G^\sigma \). This gives us the first part of the lemma.

Given \( \mathcal{L} \in \text{Hom}_{J^1 \cap G^\sigma} (\lambda^9, 1) \) and \( w \) in the space of \( \rho \), the linear form \( v \mapsto \mathcal{L}(v \otimes w) \) defined on the space of \( \eta \) is in \( \text{Hom}_{J^1 \cap G^\sigma} (\eta^9, 1) \). By Proposition \(5.12\) it is thus of the form \( \mathcal{F}(w)\mathcal{E} \) for a unique \( \mathcal{F}(w) \in R \). We have \( \mathcal{L} = \mathcal{E} \otimes \mathcal{F} \) and \( \mathcal{F} \in \text{Hom}_{J^1 \cap G^\sigma} (\rho^9, \chi) \).

\[\blacksquare\]

**Theorem 5.20.** — Let \( g \in G \) and suppose \( \text{Hom}_{J^1 \cap G^\sigma} (\lambda^9, 1) \) is non-zero. Then \( \sigma(g)g^{-1} \in J \).

**Proof.** — We know from Proposition \(5.6\) and Lemma \((5.7)\) that \( g \in J^{1x}G^\sigma \). We thus may assume that \( g \in B^x \) and \( \gamma = \sigma(g)g^{-1} \) is as in Paragraph \((5.4)\). In particular, we have a standard hereditary order \( b_m \subseteq b \) and an involution \( \tau \).

Let us fix an irreducible representation \( \kappa \) of \( J \) extending \( \eta \), and let \( \chi \) be the character given by Lemma \((5.19)\). The restriction of \( \kappa \) to \( J \), denoted \( \kappa \), is an irreducible representation of \( J \) extending...
η. It follows from Remark 4.2 that κ is a beta-extension of η, and from [10] Theorem 5.2.3 that κ extends μ. Proposition 5.18 thus implies:

$$\text{Hom}_{\mathcal{S}^1\mathbb{R}\cap G^*}(\mu^\lambda, 1) = \text{Hom}_{\mathcal{J}^g \cap G^*}(\kappa^\lambda, \chi^{-1})$$

and χ is trivial on U^1(b_m)^\# \cap G^\#. By Lemma 5.19 the space Hom_{\mathcal{J}^g \cap G^*}(\rho^\lambda, \chi) is non-zero.

Write U = J \cap B^\times and U^1 = J^1 \cap B^\times. Since g ∈ B^\times, we have J^g \cap G^\# = (U^g \cap G^\#)(J^g \cap G^\#). Let ρ be the restriction of ρ to J. Then Hom_{U^1 \cap G^*}(\rho^\lambda, \chi) is non-zero. Lemma 5.13 implies:

(5.20) $$\text{Hom}_{U^1(b_m) \cap G^*}(\rho, 1) = \text{Hom}_{U^1(b_m) \cap G^*}(\rho^\lambda, 1) \neq \{0\}.$$  

We now describe more carefully the subgroup U^1(b_m).

**Lemma 5.21.** — We have U^1(b_m) = (U^1(b_m) \cap G^\#)U^1.

**Proof.** — We follow the proof of Proposition 5.20. According to Lemma 5.17 it is enough to prove that U \cap U^{17} is contained in (U^1(b_m) \cap G^\#)U^1. Let x ∈ U \cap U^{17} and define y = x^{-1}τ(x)^{-1}xτ(x). Then y ∈ U^1 \cap U^{17} and yτ(y) = 1. Since the first cohomology set of τ in U^1 \cap U^{17} is trivial, we get y = zτ(z)^{-1} for some z ∈ U^1 \cap U^{17}. Define x' = xτ(x)τ(z). Then x' ∈ U^1(b_m) \cap G^\# and we have x ∈ x'U^1.

Since ρ is trivial on U^1, Lemma 5.21 and 5.20 together imply that Hom_{U^1(b_m)}(\rho, 1) is non-zero. Since U^1(b_m)/U^1 is a unipotent subgroup of U/U^1 ≃ GL_m(l), the fact that the representation ρ is cuspidal (see Paragraph 3.2) implies that b_m = b, that is γ ∈ U ⊆ J.

Lemma 4.25 of [3] gives a detailed account of the elements g ∈ G such that gσ(g)g^{-1} ∈ J.

**5.6. Proof of Theorem 5.1**

Let π be a σ-selfdual cuspidal representation of G, and (J, λ) be a σ-selfdual type in π given by Theorem 5.19. If the space Hom_{J^g \cap G^*}(\lambda, 1) is non-zero, then 5.11 implies that π is distinguished.

Conversely, suppose that π is distinguished and that (J, λ) has been chosen as in Remark 4.10 as it may be. Then the space Hom_{J^g \cap G^*}(\lambda^g, 1) is non-zero for some g ∈ G. By Theorem 5.20 one has g \sigma(g)g^{-1} ∈ J. Thus J^g is σ-stable, and:

$$\lambda^g = (\lambda^g)^{\sigma(g)} \simeq (\lambda^g)^g = (\lambda^g)^\gamma$$

thus the type (J^g, \lambda^g) is σ-selfdual.

**Remark 5.22.** — Let (J, λ) and (J', λ') be two distinguished σ-selfdual types in π. Since they both occur in π, there is a g ∈ G such that J' = J^g and λ' ∼ λ^g. Thanks to the multiplicity 1 property of Theorem 3.1 the formula 5.1 tells us that the double cosets JG^\# and JgG^\# are equal, which implies that g ∈ JG^\#. Thus a distinguished cuspidal representation π contains, up to G^\#-conjugacy, a unique distinguished σ-selfdual type.

Recall that Proposition 4.13 associates to any σ-selfdual cuspidal representation of G a quadratic extension T/T_0.

**Corollary 5.23.** — Let π be a σ-selfdual cuspidal representation of G, and suppose that T/T_0 is unramified. Then π is distinguished if and only if any σ-selfdual type in π is distinguished.
Proof. — This follows from Theorem 5.1 together with Proposition 4.16 which says that the representation $\pi$ contains, up to $G^\sigma$-conjugacy, a unique $\sigma$-selfdual type. □

When $T/T_0$ is ramified, Proposition 4.16 tells us that $\pi$ contains more than one $G^\sigma$-conjugacy class of $\sigma$-selfdual types as soon as its relative degree $m$ is at least 2. In the next section, we will see that the $G^\sigma$-conjugacy class of index $\lfloor m/2 \rfloor$ (see Definition 4.17) is the only one which may contribute to the distinction of $\pi$.

6. The cuspidal ramified case

As usual, write $G = \text{GL}_n(F)$ for some $n \geq 1$. To any $\sigma$-selfdual cuspidal representation of $G$, one can associate a quadratic extension $T/T_0$ and its relative degree $m$ (see Proposition 4.13). In this section, we will consider the case where $T/T_0$ is ramified.

6.1.

The first main result of this section is the following proposition, which we will prove in Paragraph 6.4.

Proposition 6.1. — Let $\pi$ be a $\sigma$-selfdual cuspidal representation of $G$ with quadratic extension $T/T_0$ and relative degree $m$. Suppose $T/T_0$ is ramified. Then $\pi$ is distinguished if and only if:

1. either $m = 1$ or $m$ is even, and
2. any $\sigma$-selfdual type of index $\lfloor m/2 \rfloor$ contained in $\pi$ is distinguished.

Remark 6.2. — Proposition 6.1 refines Theorem 5.1 by saying that, if $T/T_0$ is ramified, then the $G^\sigma$-conjugacy class of $\sigma$-selfdual types of index $\lfloor m/2 \rfloor$ contained in $\pi$ is the only one which may contribute to the distinction of $\pi$. See [3] Proposition 5.5 for a characterization of this class in terms of Whittaker data. See also Definition 9.1 and Remark 9.2 below.

Remark 6.3. — If we assume $\pi$ to be supercuspidal in Proposition 6.1, then $m$ is automatically either even or equal to 1, even if $\pi$ is not distinguished (see Proposition 7.8).

Remark 6.4. — In particular, if $\pi$ is a complex $\sigma$-selfdual cuspidal irreducible representation of $G$, then $m$ is either even or equal to 1. In this case, Proposition 6.1 is proved in [3] (see Remark 5.3 above and Corollary 6.11 and Remark 6.12).

Remark 6.5. — However, if $\pi$ is non-supercuspidal in Proposition 6.1, then its relative degree $m$ need not be either even nor equal to 1. Let $k$ be a divisor of $n$, and $\tau$ be a $\sigma$-selfdual supercuspidal representation of $\text{GL}_{n/k}(F)$. Assume $R$ has characteristic $\ell > 0$, let $\nu$ be the unramified character “absolute value of the determinant” and let $e(\tau)$ be the smallest integer $i \geq 1$ such that $\tau \nu^i \simeq \tau$. Suppose that $k = e(\tau)\ell^u$ for some $u \geq 0$. Then [37] Théorème 6.14 tells us that the unique generic irreducible subquotient $\pi$ of the normalized parabolically induced representation:

$$\tau \times \tau \nu \times \cdots \times \tau \nu^{k-1}$$

is cuspidal, and that it is $\sigma$-selfdual since $\tau$ is. If $k > 1$ and $m(\pi) = km(\tau)$ is odd, then $\pi$ is a $\sigma$-selfdual cuspidal representation which is not distinguished nor $\omega$-distinguished. (For instance,
let $\tau$ be the trivial character of $F^\times$ and choose $k = n = \ell$ with $\ell \neq 2$ dividing $q - 1$, which gives $m(\tau) = e(\tau) = 1$.

### 6.2. A mirabolic prelude

In this paragraph, the notation is independent from the rest of the article. This paragraph is inspired from Matringe [34]. As before, $k$ denotes the residue field of $F$.

Let $G$ be the group $GL_n(k)$ for some $n \geq 2$. Write $P$ for the mirabolic subgroup of $G$, made of all matrices in $G$ whose last row is $(0 \ldots 0 1)$. Let $U$ be the unipotent radical of $P$, and $G' = P''$ be the image of $GL_{n-1}(k)$ in $G$ under the group homomorphism:

\[
g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.
\]

We thus have $P = G'U$, and we write $P'$ for the mirabolic subgroup of $G'$. Let $N$ be the maximal unipotent subgroup of $G$ made of all upper triangular unipotent matrices, and $\psi$ be a non-trivial character of $k$. We still write $\psi$ for the non-degenerate character $x \mapsto \psi(x_{1,2} + \cdots + x_{n-1,n})$ of $N$. We have a functor:

\[
\pi \mapsto \text{Ind}_{P'U}^P(\pi \otimes \psi)
\]
denoted $\Phi^+$, from $R$-representations of $P'$ to $R$-representations of $P$, where $\pi \otimes \psi$ is the representation of $P'U$ defined by $xu \mapsto \pi(x)\psi(u)$ for all $x \in P'$ and $u \in U$.

Given integers $r, s \geq 0$ such that $r + s = n$, let $H_{r,s}$ be the subgroup of $G$ defined in Matringe [34]. This is the conjugate of the Levi subgroup $GL_r(k) \times GL_s(k)$ of $G$ under some permutation matrix $w_{r,s}$. If $s \geq 1$, let $H'_{r,s}$ be the subgroup $G' \cap H_{r,s}$ (denoted $H_{r,s-1}$ in [34]).

**Lemma 6.6.** Let $\pi$ be a representation of $P'$, and let $\chi$ be a character of $H_{r,s}$. Suppose that the space $\text{Hom}_{P \cap H_{r,s}}(\Phi^+(\pi), \chi)$ is non-zero. Then $\text{Hom}_{P' \cap H'_{r,s}}(\pi, \chi)$ is non-zero.

**Proof.** Applying the Mackey formula, and since $G'$ normalizes $U$, the restriction of $\Phi^+(\pi)$ to $P \cap H_{r,s}$ decomposes as the direct sum:

\[
\bigoplus_g \text{Ind}_{P' \cap H'_{r,s}}^P(\pi^g \otimes \psi^g)
\]

where $g$ ranges over a set of representatives of $(P'U, P \cap H_{r,s})$-double cosets in $P$. Since $P = G'U$, we may assume $g$ ranges over a set of representatives of $(P', H'_{r,s})$-double cosets in $G'$. For each $g$, let us write:

\[
\pi^g \otimes \text{Ind}_{P \cap H_{r,s}}^P(\psi^g) \cong \text{Ind}_{P' \cap H'_{r,s}}^P(\pi^g \otimes \psi^g) \cong \text{Ind}_{P' \cap H'_{r,s} \cap P''U}^P(\pi^g \otimes \psi^g) \otimes \text{Ind}_{P \cap H_{r,s} \cap P''U}^P (1)
\]

(6.1)

where $\pi^g$ is considered as a multiplicity space in the left hand side. Since the induced representation $\text{Ind}_{P \cap H_{r,s} \cap P''U}^P (1)$ canonically surjects onto the trivial character of $P \cap H_{r,s} \cap P''U$ by Frobenius reciprocity, the right hand side of (6.1) surjects onto:

\[
\text{Ind}_{P \cap H_{r,s} \cap P''U}^P(\pi^g \otimes \psi^g).
\]
It follows that, if \( \text{Hom}_{P \cap H_{r,s}}(\Phi^+(\pi), \chi) \) is non-zero, then there is a \( g \in G' \) such that:

\[
\text{Hom}_{P \cap H_{r,s}} \left( \text{Ind}_{U \cap H_{r,s}}^{P \cap H_{r,s}}(\psi^g), \chi \right) \neq \{0\}.
\]

By Frobenius reciprocity, this implies that \( \psi^g = \chi \) on \( U \cap H_{r,s} \). Since \( \chi \) extends to \( H_{r,s} \), which is \( G \)-conjugate to \( \text{GL}_r(k) \times \text{GL}_s(k) \), and since the cardinality of \( k \) is odd, \( \chi \) is trivial on unipotent elements of \( H_{r,s} \). We thus get \( \psi^g = 1 \) on \( U \cap H_{r,s} \). By \cite{Vincent1986} Lemma 3.1, this implies \( g \in P' H'_{r,s} \), that is, we may assume that \( g = 1 \). We thus have:

\[
\text{Hom}_{P \cap H_{r,s}}(\Phi^+(\pi), \chi) = \text{Hom}_{P \cap H_{r,s}} \left( \text{Ind}_{U \cap H_{r,s}}^{P \cap H_{r,s}}(\pi \otimes \psi), \chi \right) \cong \text{Hom}_{U \cap H_{r,s}}(\pi \otimes \psi, \chi)
\]

and the latter is contained in \( \text{Hom}_{P' \cap H'_{r,s}}(\pi, \chi) \).

**Remark 6.7.** — We have \( P' U \cap H_{r,s} = P' \cap H'_{r,s} \) thus the last containment is actually an equality, as has been pointed out to me by N. Matringe.

Now write \( G'' \) for the copy of \( \text{GL}_{n-2}(k) \) in the upper left block of \( G' \) and \( P'' \) for the mirabolic subgroup of \( G'' \), and \( H''_{r,s} = G'' \cap H_{r,s} \).

**Lemma 6.8.** — Let \( \pi' \) be a representation of \( P'' \) and \( \chi' \) be a character of \( H''_{r,s} \). Suppose that the space \( \text{Hom}_{P' \cap H'_{r,s}}(\Phi^+(\pi'), \chi') \) is non-zero. Then \( \text{Hom}_{P'' \cap H''_{r,s}}(\pi', \chi') \) is non-zero.

**Proof.** — Same proof, replacing \cite{Vincent1986} Lemma 3.1 by \cite{Vincent1986} Lemma 3.2.

Let \( \Gamma \) denote the mirabolic representation of \( P \). Recall that it is defined as the representation of \( P \) induced from the character \( \psi \) of \( N \).

**Lemma 6.9.** — Let \( n \geq 2 \) and \( r, s \geq 0 \) be such that \( r + s = n \). Let \( \chi \) be a character of \( H_{r,s} \). If \( \text{Hom}_{P \cap H_{r,s}}(\Gamma, \chi) \) is non-zero, then \( r = s \).

**Proof.** — If \( \min(r, s) = 0 \) then \( H_{r,s} = G \) and the result follows from the fact that \( \Gamma \) is irreducible of dimension greater than 1.

Suppose that \( r, s \geq 1 \) and that the space \( \text{Hom}_{P \cap H_{r,s}}(\Gamma, \chi) \) is non-zero. The mirabolic representation \( \Gamma \) is isomorphic to \( \Phi^+(\Gamma') \), where \( \Gamma' \) denotes the mirabolic representation of \( P' \). By Lemma 6.6 the space \( \text{Hom}_{P' \cap H'_{r,s}}(\Gamma', \chi') \) is non-zero, where \( \chi' \) is the restriction of \( \chi \) to \( H'_{r,s} \). Now identify \( \Gamma' \) with \( \Phi^+(\Gamma'') \), where \( \Gamma'' \) is the mirabolic representation of \( P'' \). By Lemma 6.8 the space \( \text{Hom}_{P'' \cap H''_{r,s}}(\Gamma'', \chi'') \) is non-zero, where \( \chi'' \) is the restriction of \( \chi' \) to \( H''_{r,s} = H_{r-1,s-1} \). By induction on \( n \), the fact that \( \text{Hom}_{P'' \cap H''_{r-1,s-1}}(\Gamma'', \chi'') \) is non-zero implies that \( r - 1 = s - 1 \), thus \( r = s \).

**Proposition 6.10.** — Let \( n \geq 2 \) and \( r, s \geq 0 \) be such that \( r + s = n \). Let \( \rho \) be a cuspidal representation of \( G \), and \( \chi \) be a character of \( M = (\text{GL}_r \times \text{GL}_s)(k) \). Suppose \( \text{Hom}_M(\rho, \chi) \) is non-zero. Then \( r = s \).

**Proof.** — This follows from Lemma 6.9 and the fact that the restriction of \( \rho \) to \( P \) is isomorphic to \( \Gamma \) (see \cite{Vincent1986} III.1 in the modular case).
Remark 6.11. — Suppose that $r = s \geq 1$. Putting Lemmas 6.6 and 6.8 together with Remark 6.7, we get

$$\text{Hom}_{P \cap H, r}((\Phi^+)^2(\pi), 1) \simeq \text{Hom}_{P \cap H - 1, r - 1}(\pi, 1)$$

for any representation $\pi$ of $P^\sigma$. By induction, we get an isomorphism $\text{Hom}_{P \cap H, r}(\rho, 1) \simeq R$ of vector spaces for all cuspidal representation $\rho$ of $G$. Thus the dimension of $\text{Hom}_{GL_r(k) \times GL_r(k)}(\rho, 1)$ is at most 1.

6.3. Existence of $\sigma$-selfdual extensions of the Heisenberg representation

We now go back to our usual notation. Let $[a, \beta]$ be a maximal simple stratum in $M_n(F)$ such that $a$ is $\sigma$-stable and $\sigma(\beta) = -\beta$. Write $E$ for the extension $F[\beta]$, and suppose that it is ramified over the field $E_0$ of $\sigma$-fixed points in $E$. Let $d$ be the degree $[E : F]$ and write $n = md$.

Let $l$ denote the residue field of $E$. Let us notice once and for all that, since $p \neq 2$, any character of $GL_m(l)$ is of the form $\alpha \circ \det$, for some character $\alpha$ of $l^\times$.

The following lemma generalizes [11] Lemma 3.4.6 (which is concerned with complex representations and $\chi$ trivial only).

Lemma 6.12. — Let $\chi$ be a character of $(GL_i \times GL_{m-i})(l)$ for some $i \in \{0, \cdots, [m/2]\}$. Suppose there is a $\chi$-distinguished cuspidal representation of $GL_m(l)$. Then either $m = 1$ or $m = 2i$.

Proof. — If $m = 1$, then $\chi$ is a $\chi$-distinguished and supercuspidal character of $GL_1(l)$. If $m \geq 2$, then the result follows from Proposition 6.10.

Let $\theta \in \mathbb{C}(a, \beta)$ be a maximal simple character such that $H^1(a, \beta)$ is $\sigma$-stable and $\theta \circ \sigma = \theta^{-1}$, and let $J = J(a, \beta)$ be its normalizer in $G$. Let $\eta$ be the Heisenberg representation of $J^1 = J^1(a, \beta)$ containing $\theta$, and write $J = J(a, \beta)$.

Lemma 6.13. — There is a $\sigma$-selfdual representation $\kappa$ of $J$ extending $\eta$.

Proof. — Conjugating by a suitable element in $G$, we may assume that the stratum $[a, \beta]$ satisfies the conditions of Remark 4.10. Indeed, if it doesn’t, there is a $g \in G$ such that $\theta^g$ is $\sigma$-selfdual and $[a^g, \beta^g]$ satisfies these conditions. This implies that $\gamma = \sigma(g)^{-1}$ normalizes $\theta$, that is $\gamma \in J$.

Now, assuming the lemma to be true for $[a^g, \beta^g]$, there exists a $\sigma$-selfdual representation $\kappa'$ of $J^g$ extending $\eta^g$. Define a representation $\kappa$ of $J$ by $\kappa^g = \kappa'$. Then $\kappa$ extends $\eta$, and it is $\sigma$-selfdual since $\gamma \in J$.

From now on, we will assume that $[a, \beta]$ satisfies the conditions of Remark 4.10 and will identify $J/J^1$ with $GL_n(l)$, on which $\sigma$ acts trivially. Let $\kappa$ be a representation of $J$ extending $\eta$ and $\mu$ be the character of $J$ trivial on $J^1$ such that $\kappa^\sigma \simeq \kappa \mu$ given by Lemma 4.20. We claim that there is a character $\nu$ of $J$ trivial on $J^1$ such that $(\nu \circ \sigma)\nu = \mu$. Indeed, the representation $\kappa \nu$ will then extend $\eta$ and be $\sigma$-selfdual.
Suppose first that \( R \) has characteristic 0, and let \( \chi \) be the character of \( J \cap G^\sigma \) associated with \( \kappa \) by Lemma 5.19. We have:
\[
\Hom_{J \cap G^\sigma} (\eta, 1) = \Hom_{J \cap G^\sigma} (\kappa, \chi^{-1}) \\
\cong \Hom_{J \cap G^\sigma} (\kappa^\sigma, \chi) \\
= \Hom_{J \cap G^\sigma} (\kappa, \chi\mu^{-1}).
\]
where the isomorphism in the middle follows from the fact that the restriction of \( \kappa \) to \( J \cap G^\sigma \) is semisimple and by duality. By uniqueness of \( \chi \), the restriction of \( \mu \) to \( J \cap G^\sigma \) is \( \chi \). Restricting to \( J \cap G^\sigma \) and writing \( \mu = \varphi \circ \det \) and \( \chi = \alpha \circ \det \) as characters of \( GL_m(l) \) for suitable characters \( \varphi, \alpha \) of \( l^\times \), we get \( \varphi = \alpha^2 \). Let \( \nu \) be the unique character of \( J \) which is trivial on \( J^t \) and equal to \( \alpha \circ \det \) as a character of \( GL_m(l) \). Since \( J \) is generated by \( t \) and \( J \), it remains to extend \( \nu \) to \( J \) by fixing a scalar \( \nu(t) \in R^\times \) such that \( \nu(t)^2 = \nu(-1)\mu(t) \).

Suppose now that \( R \) has characteristic \( \ell > 0 \) prime to \( q \). Let \( \tilde{\theta} \) be a lift of \( \theta \) to characteristic 0. We have \( \tilde{\theta} \circ \sigma = \tilde{\theta}^{-1} \). There is thus a \( \sigma \)-selfdual representation \( \tilde{\kappa} \) of \( J \) extending the irreducible representation \( \tilde{\eta} \) of \( J^t \) containing \( \tilde{\theta} \). It is integral if and only if its central character on \( F^\times \) is integral, which is the case since it is \( \sigma \)-selfdual. By \([36]\) Paragraph 2.11, the reduction of \( \tilde{\kappa} \) to \( R \), denoted \( \kappa \), is thus a \( \sigma \)-selfdual representation of \( J \) extending \( \eta \).

### 6.4. Proof of Proposition 6.1

Let \((J, \lambda)\) be a \( \sigma \)-selfdual type, with associated simple character the character \( \theta \) of \( J^t \).

**Lemma 6.14.** — If \((J, \lambda)\) is distinguished, then:

1. either \( m = 1 \),
2. or \( m = 2r \) for some \( r \geq 1 \), and \((J, \lambda)\) has index \( r \).

**Proof.** — Let \( \kappa \) be a \( \sigma \)-selfdual representation of \( J \) extending \( \eta \) provided by Lemma 6.13. Let \( \rho \) be the unique irreducible representation of \( J \) trivial on \( J^t \) such that \( \lambda \cong \kappa \otimes \rho \), and let \( i \) be the index of \((J, \lambda)\). Lemma 6.13 tells us that \( \rho \) is \( \chi \)-distinguished for some character \( \chi \) of \( J \cap G^\sigma \) trivial on \( J^t \cap G^\sigma \). Let \( \rho \) be the restriction of \( \rho \) to \( J \) and identify \( J/J^t \) with \( GL_m(l) \). Then we get a cuspidal representation \( \rho \) of \( GL_m(l) \) and a character \( \chi \) of \((GL_i \times GL_{m-i})(l)\) such that \( \rho \) is \( \chi \)-distinguished. The result follows from Lemma 6.12.

Let \( \pi \) be a \( \sigma \)-selfdual cuspidal representation of \( G \), and suppose that the quadratic extension \( T/T_0 \) associated with it by Proposition 4.13 is ramified. Let \((J, \lambda)\) be a \( \sigma \)-selfdual type contained in \( \pi \). By Remark 4.11 we may assume that it is defined with respect to a \( \sigma \)-selfdual simple stratum. By Remark 4.15 \( E \) is ramified over \( E_0 \). We can thus apply the results of Paragraph 6.3 and Lemma 6.14. Proposition 6.1 now follows from Theorem 5.1 together with Lemma 6.14.

### 6.5. Existence of distinguished extensions of the Heisenberg representation

The second main result of this section is the following proposition.

**Proposition 6.15.** — Let \( \pi \) be a \( \sigma \)-selfdual cuspidal representation of \( G \) with ramified quadratic extension \( T/T_0 \). Assume that \( m = 1 \) or \( m \) is even, and let \((J, \lambda)\) be a \( \sigma \)-selfdual type in \( \pi \) of index
Let $J^1$ be the maximal normal pro-$p$-subgroup of $J$ and $\eta$ be an irreducible component of the restriction of $\lambda$ to $J^1$.

1. There is a distinguished representation of $J$ extending $\eta$, and any such representation of $J$ is $\sigma$-selfdual.
2. Let $\kappa$ be a distinguished representation of $J$ extending $\eta$, and let $\rho$ be the unique representation of $J$ trivial on $J^1$ such that $\lambda \simeq \kappa \otimes \rho$. Then $\pi$ is distinguished if and only if $\rho$ is distinguished.

We start with the following lemma, which slightly refines part (1) of the proposition.

**Lemma 6.16.** — Let $(J, \lambda)$ be as in Proposition 6.15.

1. There is a distinguished representation $\kappa$ of $J$ extending $\eta$.
2. If $\ell = 2$ or if $m$ is even, such a distinguished representation $\kappa$ is unique.
3. Any distinguished representation $\kappa$ of $J$ extending $\eta$ is $\sigma$-selfdual.

**Remark 6.17.** — If $\ell \neq 2$ and $m = 1$, there are exactly two distinguished representations of $J$ extending $\eta$, twisted of each other by the unique non-trivial character of $J$ trivial on $(J \cap G^\sigma)J^1$. (See the proof below, which shows that $(J \cap G^\sigma)J^1$ has index 2 in $J$.)

**Proof.** — Let $J$ be the maximal compact subgroup of $J$, and $J^1$ be its maximal normal pro-$p$-subgroup. As usual, we fix a maximal simple stratum $[a, \beta]$ defining $(J, \lambda)$ such that $a$ is $\sigma$-stable and $\sigma(\beta) = -\beta$, and write $E = F[\beta]$ and $I$ for its residue field. We will identify $J/J^1$ with $GL_m(I)$ equipped with an involution whose fixed points is $(GL_\ell \times GL_{m-\ell})(I)$ where $\ell = \lfloor m/2 \rfloor$.

Let $\kappa$ be an irreducible representation of $J$ extending $\eta$. By Lemma 5.19, there is a character $\chi$ of $J \cap G^\sigma$ trivial on $J^1 \cap G^\sigma$ associated to $\kappa$. We claim that $\chi$ extends to a character $\phi$ of $J$ trivial on $J^1$. It will then follow that $\kappa \phi$ is distinguished and extends $\eta$.

Suppose first that $m = 1$. We then have canonical group isomorphisms:

$$\frac{(J \cap G^\sigma)}{(J^1 \cap G^\sigma)} \simeq J/J^1 \simeq I^\times.$$  

Thus there is a unique character $\phi$ of $J$ trivial on $J^1$ which coincides with $\chi$ on $J \cap G^\sigma$. Since $J$ is generated by $t$ and $J$, and since $t$ normalizes $\phi$, this character extends to a character of $J$ trivial on $J^1$.

**Lemma 6.18.** — Suppose that $m = 1$. Then $J \cap G^\sigma$ is generated by $J \cap G^\sigma$ and $t^2$.

**Proof.** — Since we have $J = E^\times J^1$ when $m = 1$, we may consider the exact sequence of $\sigma$-groups:

$$1 \rightarrow U^1_E \rightarrow E^\times \times J^1 \rightarrow J \rightarrow 1.$$  

Taking $\sigma$-invariants and since the first cohomology group $H^1(\sigma, U^1_E)$ is trivial, $J \cap G^\sigma$ is generated by $E^\times_0$ and $J^1 \cap G^\sigma$. The result follows from (6.2) and the fact that $t^2$ is a uniformizer of $E_0$.

It follows from Lemma 6.18 that $\chi$ can be extended to a character $\phi$ of $J$ trivial on $J^1$. Since we must have $\phi(t^2) = \chi(t^2)$ in the field $R$ of characteristic $\ell$, there are at most two such characters, with uniqueness if and only if $\ell = 2$.  


Suppose now that $m = 2r$ for some $r \geq 1$, and consider the element:

$$w = \begin{pmatrix} \text{id}_r \\ \text{id}_r \end{pmatrix} \in b^\times \subseteq \text{GL}_m(E)$$

where $\text{id}_r$ is the identity matrix in $\text{GL}_r(E)$.

**Lemma 6.19.** Suppose that $m = 2r$. The group $J \cap G^\sigma$ is generated by $J \cap G^\sigma$ and $tw$.

**Proof.** First, notice that $t' = tw$ is $\sigma$-invariant. Any $x \in J$ can be written $x = t^ky$ for unique $k \in \mathbb{Z}$ and $y \in J$. We thus have $x \in J \cap G^\sigma$ if and only if $y \in J \cap G^\sigma$. □

Since $\kappa$ and $J \cap G^\sigma$ are normalized by $w$, we have $\text{Hom}_{J \cap G^\sigma}(\kappa, \chi^{-1}) = \text{Hom}_{J \cap G^\sigma}(\kappa, (\chi^w)^{-1})$, and the uniqueness of $\chi$ implies that $\chi^w = \chi$. First, consider the character of:

$$(J \cap G^\sigma)/(J^1 \cap G^\sigma) \simeq (\text{GL}_r \times \text{GL}_r)(l)$$

defined by $\chi$ and write it $(\alpha_1 \circ \det) \otimes (\alpha_2 \circ \det)$ for some characters $\alpha_1, \alpha_2$ of $l^\times$. The identity $\chi^w = \chi$ implies that $\alpha_1 = \alpha_2$, thus there is a unique character $\phi$ of $J$ trivial on $J^1$ which coincides with $\chi$ on $J \cap G^\sigma$. By Lemma 6.19 there is a unique character $\phi$ of $J$ trivial on $J^1$ extending $\chi$. This proves (1) and (2).

Now let $\kappa$ be a distinguished representation of $J$ extending $\eta$. It satisfies $\kappa^\sigma \simeq \kappa\mu$ for some character $\mu$ of $J$ trivial on $J^1$ such that $\mu \circ \sigma = \mu$ (see Lemma 4.20). Since $\kappa$ is distinguished, $\mu$ is trivial on $J \cap G^\sigma$. We will prove that $\kappa$ is $\sigma$-selfdual, that is, that the character $\mu$ is trivial.

Suppose first that $m = 1$. Thus $(J, \kappa)$ is a distinguished type in $G$. Let $\pi$ denote the cuspidal irreducible representation of $G$ compactly induced from $\kappa$. It is distinguished, thus $\sigma$-selfdual by Theorem 3.1. It follows that $\kappa$ and $\kappa^\sigma \simeq \kappa\mu$ are both contained in $\pi$, thus $\mu$ is trivial.

Suppose now that $m = 2r$. Since $\mu$ is trivial on $(\text{GL}_r \times \text{GL}_r)(l)$, it must be trivial on $\text{GL}_m(l)$. Since $tw$ is $\sigma$-invariant, we have $\mu(tw) = 1$. Thus $\mu$ is trivial. This proves (3). □

For part (2) of Proposition 6.15 it suffices to fix a distinguished representation $\kappa$ of $J$ extending $\eta$ and to consider the canonical isomorphism:

$$\text{Hom}_{J \cap G^\sigma}(\kappa, 1) \otimes \text{Hom}_{J \cap G^\sigma}(\rho, 1) \to \text{Hom}_{J \cap G^\sigma}(\lambda, 1)$$

(compare with Lemma 5.19).

Proposition 6.15 reduces the problem of the distinction of $\pi$ to that of $\rho$. In the next section, we investigate the distinction of $\rho$ in the case where $\pi$ is supercuspidal.

### 7. The supercuspidal ramified case

In this section, we investigate the distinction of $\sigma$-selfdual supercuspidal representations of $G$ in the case where $T/T_0$ ramified.
7.1. The finite field case

In this paragraph, we assume that $F$ is ramified over $F_0$, thus $k$ identifies with $k_0$. We consider the supercuspidal irreducible representations of $GL_n(k)$ distinguished by some maximal Levi subgroup.

**Lemma 7.1.** — Let $n \geq 1$ be a positive integer. There exists a selfdual supercuspidal irreducible representation of $GL_n(k)$ if and only if either $n = 1$ or $n$ is even.

**Proof.** — If $n = 1$, the trivial character of $k^\times$ is selfdual and supercuspidal. Suppose that $n \geq 2$. Let us fix an extension $t$ of $k$ of degree $n$, and identify $t^\times$ with a maximal torus in $GL_n(k)$. We consider the Green-James parametrization (4.7) of isomorphism classes of supercuspidal irreducible representations of $GL_n(k)$ by $k$-regular characters of $t^\times$.

Given a $k$-regular character $\xi$ of $t^\times$, the representation $\rho_\xi$ is selfdual if and only if:

\[
(7.1) \quad \xi^{-1} = \xi^q^i, \quad \text{for some } i \in \{0, \ldots, n-1\}.
\]

Taking the contragredient again gives us the equality $\xi = \xi^{q^2i}$. The regularity assumption on $\xi$ implies that $n$ divides $2i$ and, since $0 \leq i \leq n-1$, we get $n = 2i$. It follows that:

\[
(7.2) \quad \rho_\xi \text{ is selfdual } \iff \xi^{-1} = \xi^{q^{n/2}}.
\]

This is also equivalent to $\xi$ being trivial on $t'^\times$, where $t'$ is the subfield of $t$ with cardinality $q^{n/2}$.

Conversely, suppose $n$ is even, $n = 2r$. We look for a $k$-regular character $\xi : t^\times \to \mathbb{R}^\times$ such that $\xi^{q^r+1} = 1$. In other words, we look for a character of $t^\times$ whose order $a$ satisfies:

1. $a$ divides $q^r + 1$,
2. the order of $q$ in $(\mathbb{Z}/a\mathbb{Z})^\times$ is equal to $n$.

Pick a character of order $q^r + 1$, which is possible since the group $t^\times$ has order $q^n - 1 = q^{2r} - 1$. Then $q$ has order $n = 2r \mod q^r + 1$.

**Remark 7.2.** — When $R$ has characteristic $\ell > 0$, the group $GL_n(k)$ may have selfdual cuspidal (non supercuspidal) representations even if $n$ is odd and $\ell > 1$. Indeed, let $c$ be the order of $q$ mod $\ell$, and suppose that $n = c\ell^m$ for some $m \geq 0$. Then the unique generic irreducible subquotient of the representation induced from the trivial character of a Borel subgroup of $GL_n(k)$ is both cuspidal and selfdual (38 Théorème 2.4). One then can choose $c$, $\ell$ and $u$ such that the integer $n$ is odd and $\ell > 1$. (For instance, $\ell$ divides $q-1$ and $n = \ell$ with $\ell \neq 2$.)

**Lemma 7.3.** — Suppose that $n = 2r$ with $r \geq 1$, and let $\rho$ be a supercuspidal representation of $GL_n(k)$. The following assertions are equivalent:

1. The representation $\rho$ is selfdual.
2. The representation $\rho$ is $(GL_r \times GL_r)(k)$-distinguished.
3. The $R$-vector space $\text{Hom}_{(GL_r \times GL_r)(k)}(\rho, 1)$ has dimension 1.

**Proof.** — When $R$ has characteristic 0, this is [24] Proposition 6.1 (see also [11] Lemme 3.4.10). Suppose now $R$ has prime characteristic $\ell$ not dividing $q$. Let us prove first that (1) implies (2).

Let $\xi$ be a $k$-regular character of $t^\times$ parametrizing some selfdual supercuspidal representation $\rho$ of $GL_n(k)$. It follows from (7.2) that $\xi^{-1} = \xi^q^r$. Let $\overline{Q_\ell}$ denote an algebraic closure of the field
of \( \ell \)-adic numbers, \( \mathbf{Z}_\ell \) be the ring of \( \ell \)-adic integers in \( \mathbf{Q}_\ell \) and \( \mathbf{F}_\ell \) be the residue field of \( \mathbf{Z}_\ell \). Since the image of \( \xi \) is contained in \( \overline{\mathbf{F}}_\ell \), it can be lifted to a \( k \)-regular character \( \tilde{\xi} \) of \( t^x \) with values in \( \mathbf{Z}_\ell \), which satisfies the relation \( \tilde{\xi}^{-1} = \tilde{\xi}^{q^r} \). This lift corresponds to a selfdual supercuspidal representation \( \tilde{\rho} \) of \( \text{GL}(k) \) on a vector space \( V \) over \( \mathbf{Q}_\ell \). It follows from the characteristic 0 case that there is a non-zero (\( \text{GL}_r \times \text{GL}_s)(k) \)-invariant linear form:

\[ \varphi : V \to \overline{\mathbf{Q}}_\ell. \]

Fix a \( \text{GL}_n(k) \)-stable \( \mathbf{Z}_\ell \)-lattice \( L \subseteq V \), which we can choose such that \( \varphi(L) \) is equal to \( \mathbf{Z}_\ell \). Then the reduction of \( \varphi \) mod the maximal ideal of \( \mathbf{Z}_\ell \) gives a non-zero \( (\text{GL}_r \times \text{GL}_s)(k) \)-invariant \( \overline{\mathbf{F}}_\ell \)-linear form on \( L \otimes \overline{\mathbf{F}}_\ell \). By [14] [29], the representation of \( \text{GL}_n(k) \) on \( (L \otimes \overline{\mathbf{F}}_\ell) \otimes \overline{\mathbf{F}}_\ell \mathbf{R} \) is isomorphic to \( \rho \). This implies that \( \rho \) is distinguished, thus (1) implies (2).

We now prove that (2) implies (1). Let us write \( G = \text{GL}_n(k) \) and \( H = (\text{GL}_r \times \text{GL}_s)(k) \). Let \( \rho \) be an \( H \)-distinguished supercuspidal representation of \( G \).

First, since \( H \) is stable by the involution \( x \mapsto \text{transpose of } x^{-1} \) of \( G \), we see, as in the proof of Theorem 3.1, that an irreducible representation of \( G \) has a non-zero \( H \)-invariant linear form if and only if it has a non-zero \( H \)-invariant vector. More precisely, the \( \mathbf{R} \)-vector spaces \( \text{Hom}_H(\rho, 1) \) and \( \text{End}_H(1, \rho) \) have the same dimension.

Since \( G \) is finite, there exists a supercuspidal irreducible \( \overline{\mathbf{F}}_\ell \)-representation \( \rho_0 \) of \( G \), unique up to isomorphism, such that \( \rho \) is isomorphic to \( \rho_0 \otimes \mathbf{R} \). This representation \( \rho_0 \) is \( H \)-distinguished. Let \( f : P_0 \to \rho_0 \) be a projective envelope of \( \rho_0 \) in the category of all \( \mathbf{Z}_\ell[G] \)-modules. Since \( \rho_0 \) is supercuspidal, it has the following properties (see for instance [46] III.2.9):

- the representation \( P_0 \otimes \mathbf{Q}_\ell \) is isomorphic to the direct sum of all the \( \mathbf{Q}_\ell \)-lifts of \( \rho_0 \);
- there are \( \ell^a \) such lifts, where \( a \) is the \( \ell \)-valuation of \( q^n - 1 \);
- the representation \( P_0 \otimes \mathbf{F}_\ell \) is indecomposable of length \( \ell^a \), it has a unique irreducible quotient, and all its irreducible components are isomorphic to \( \rho_0 \).

By projectivity of \( P_0 \), any non-zero homomorphism \( i : \mathbf{Z}_\ell[H \backslash G] \to \rho_0 \) defines a non-zero homomorphism:

\[ (7.3) \quad j : P_0 \to \mathbf{Z}_\ell[H \backslash G] \]

such that \( i \circ j = f \). Inverting \( \ell \), we get a non-zero homomorphism from \( P_0 \otimes \mathbf{Q}_\ell \) to \( \mathbf{Q}_\ell[H \backslash G] \). It follows that \( \rho_0 \) has at least one \( H \)-distinguished lift \( \tilde{\rho}_0 \). Thanks to the characteristic 0 case, this lift is selfdual. Reducing to \( \overline{\mathbf{F}}_\ell \), it follows that \( \rho_0 \) is selfdual, thus so is \( \rho \).

We finally prove that (2) implies (3). Let \( V \) denote the maximal direct summand of \( R[H \backslash G] \) in the block of \( \rho \). This means that \( R[H \backslash G] \) decomposes as a direct sum \( V \oplus V' \) where all irreducible subquotients of \( V \) are isomorphic to \( \rho \), and no irreducible subquotient of \( V' \) is isomorphic to \( \rho \). Besides, we have:

\[ \dim \text{Hom}_H(\rho, 1) = \dim \text{Hom}_H(1, \rho) = \dim \text{Hom}_G(R[H \backslash G], \rho) = \dim \text{Hom}_G(V, \rho). \]

We thus have to prove that the cosocle of \( V \) is isomorphic to \( \rho \).

**Lemma 7.4.** — The \( R \)-algebra \( A = \text{End}_G(V) \) is commutative.
Proof. — First, A is isomorphic to the convolution algebra \( R[H \setminus G/H] \). For \( x \in G \), let \( f_x \) be the characteristic function in \( R[H \setminus G/H] \) of the double coset \( HxH \). For \( x, y \in G \), one has:
\[
f_x \ast f_y = \sum_{z \in H \setminus G/H} a(x, y, z) f_z
\]
where \( a(x, y, z) \in R \) is the image of the cardinality of \((HxH \cap zHy^{-1}H)/H \) in \( R \). When \( R \) has characteristic 0, the algebra \( R[H \setminus G/H] \) is known to be commutative since \( R[H \setminus G] \) is multiplicity free as an \( R[G] \)-module, thus:
\[
\text{(7.4)} \quad \text{card } (HxH \cap zHy^{-1}H)/H = \text{card } (HyH \cap zHxH)/H
\]
for all \( x, y, z \in G \). Now if \( R \) has characteristic \( \ell > 0 \), reducing \text{(7.4)} mod \( \ell \) gives us a congruence relation which tells us that the algebra \( R[H \setminus G/H] \) is commutative. \( \square \)

It remains to prove that the cosocle of \( V \) is multiplicity free. Let \( m \geq 1 \) be the multiplicity of \( \rho \) in the cosocle of \( V \) and \( P \) be the projective indecomposable \( R[G] \)-module associated with \( \rho \). It has length \( \ell^m \), has a unique irreducible quotient, and all its irreducible components are isomorphic to \( \rho \). Write \( V = V_1 \oplus \cdots \oplus V_m \) where \( V_1, \ldots, V_m \) are indecomposable \( R[G] \)-modules with cosocle isomorphic to \( \rho \). There is a nilpotent endomorphism \( N \in \text{End}_G(P) \) such that:
\[
\text{End}_G(P) = R[N]
\]
with \( N^{\ell^m} = 0 \) and \( N^{\ell^m - 1} \neq 0 \). Therefore each \( V_i \) is isomorphic to the quotient of \( P \) by the image of \( N^{k_i} \) for some \( k_i \geq 0 \). Reordering if necessary, we may assume that \( \text{Hom}(V_1, V_i) \) is non-zero for all \( i \geq 1 \). Suppose that \( m \geq 2 \), and define two endomorphisms \( u, u' \in A \) by:

1. the endomorphisms \( u, u' \) are trivial on \( V_i \) for all \( i \geq 2 \),
2. the restriction of \( u \) to \( V_1 \) is the identity on \( V_1 \),
3. the restriction of \( u' \) to \( V_1 \) coincides with some non-zero homomorphism in \( \text{Hom}(V_1, V_2) \).  

Then \( uu' = 0 \) and \( u'u \neq 0 \), thus \( A \) is not commutative. Thus \( m = 1 \). \( \square \)

Remark 7.5. — Throughout the proof of Lemma 7.3 we proved that, if \( R \) is \( \overline{F_\ell} \), the representation \( \rho \) is distinguished if and only if it has a distinguished lift to \( \overline{Q_\ell} \).

Remark 7.6. — If we only assume \( \rho \) to be cuspidal in Lemmas 7.3 and 7.4 then the lifting argument may not work, that is, \( \xi \) may not have a \( \sigma \)-selfdual \( k \)-regular lift \( \tilde{\xi} \). Besides, the structure of the projective envelope of \( \rho \) is more complicated when \( \rho \) is cuspidal non-supercuspidal.

Remark 7.7. — For another proof of the fact that (2) implies (3), which works for all cuspidal representations \( \rho \) of \( \text{GL}_m(k) \) and not only for supercuspidal ones, see Remark 6.11.

7.2. The relative degree

Let \( \pi \) be a \( \sigma \)-selfdual cuspidal representation of \( G \) such that \( T/T_0 \) is ramified. Let \( (J, \lambda) \) be a \( \sigma \)-selfdual type contained in \( \pi \) and let \( \kappa \) be a \( \sigma \)-selfdual representation of \( J \) extending \( \eta \) given by Lemma 6.13. This defines a \( \sigma \)-selfdual irreducible representation \( \rho \) of \( J \) trivial on \( J^1 \). Let \( J \) denote the maximal compact subgroup of \( J \) and \( \rho \) denote the cuspidal representation of \( J/J^1 \cong \text{GL}_m(l) \) induced by \( \rho \).
Since $\rho$ is $\sigma$-selfdual, the representation $\rho$ is selfdual. Applying Fact 4.15 together with Lemma 7.1 we get the following lemma mentioned in Remark 6.3.

**Proposition 7.8.** — Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of $G$ such that $T/T_0$ is ramified. Then its relative degree $m$ is either even or equal to 1.

### 7.3. Distinction criterion in the ramified case

Let $(J, \lambda)$ be a $\sigma$-selfdual type of index $|m/2|$ contained in $\pi$. We fix a distinguished representation $\kappa$ of $J$ extending $\eta$ given by Proposition 6.15. It is $\sigma$-selfdual, thus the representation $\rho$ of $J$ trivial on $J^1$ which correspond to this choice is $\sigma$-selfdual. By Proposition 6.15 again, $\pi$ is distinguished if and only if $\rho$ is distinguished. We now investigate the distinction of $\rho$. For this, we will use the admissible pairs of level 0 introduced in Paragraphs 4.3 and 4.5.

Let us fix a $\sigma$-selfdual maximal simple stratum $[a, \beta]$ such that $J = J(a, \beta)$. Write $E = F[\beta]$. Let $(K/E, \xi)$ be an admissible pair of level 0 attached to $\rho$ in the sense of Definition 4.7. Since $\rho$ is $\sigma$-selfdual, Proposition 4.18 tells us that there is a unique involutive $E_0$-representation $\tilde{\rho}$ of $J$ trivial on $J^1$. One checks easily that its reduction mod $\ell$ is $\rho$, which comes from the congruence properties of the map (1.7) itself (see 29). Note that the restriction of $\xi$ to $K_0^\times$ is either trivial or (if $\ell \neq 2$) equal to $\omega_{K/K_0}$.

We consider the canonical $\mathbb{Q}_\ell$-lift $\tilde{\xi}$ of $\xi$, obtained via the canonical embedding of $\mathbb{F}_\ell^\times$ in $\mathbb{Z}_\ell^\times$. The pair $(K/E, \tilde{\xi})$ is admissible of level 0. Associated with it, there is thus a $\mathbb{Q}_\ell$-representation $\tilde{\rho}$ of $J$ trivial on $J^1$. One checks easily that its reduction mod $\ell$ is $\rho$, which comes from the congruence properties of the map (1.7) itself (see 29). Note that the restriction of $\xi$ to $K_0^\times$ is either trivial or (if $\ell \neq 2$) equal to $\omega_{K/K_0}$.

Suppose $\xi$ is non-trivial on $K_0^\times$. Then the same holds for $\tilde{\xi}$, and the characteristic 0 case tells us that $\tilde{\rho}$ is distinguished. As in the proof of Lemma 7.3 reducing mod $\ell$ a non-zero invariant form on $\tilde{\rho}$ gives us a non-zero invariant form on $\rho$, which is thus distinguished.

Suppose now that $\xi$ is trivial on $K_0^\times$. Then the same holds for $\tilde{\xi}$. Let $\tilde{\alpha}$ be the unramified $\ell$-adic character of $K^\times$ of order 2. Then $(K/E, \tilde{\xi} \tilde{\alpha})$ is an admissible pair of level 0 and it is attached to $\tilde{\rho} \tilde{\varphi}$ where $\tilde{\varphi}$ is the unramified $\ell$-adic character of $J$ of order 2. Since $\tilde{\xi} \tilde{\alpha}$ is non-trivial on $K_0^\times$, we get the following lemma mentioned in Remark 6.3.

**Lemma 7.9.** — The representation $\rho$ is distinguished if and only if at least one of the following conditions is fulfilled:

1. $\ell = 2$,
2. $m = 1$ and $\rho$ is trivial on $E_0^\times$,
3. $m$ is even and $\xi$ is non-trivial on $K_0^\times$.

**Remark 7.10.** — Note that Case (3) cannot happen when $\ell = 2$.

**Proof.** — The case $m = 1$ is clear. Let us suppose that $m = 2r$ for some $r \geq 1$. The case where the characteristic of $R$ is 0 is given by 24 Proposition 6.3. Suppose $R$ has characteristic $\ell > 0$. Since $\xi \circ \sigma = \xi^{-1}$, the image of $\xi$ is contained in $\mathbb{F}_\ell^\times \subseteq R$. It thus defines a $\sigma$-selfdual $\mathbb{F}_\ell$-representation $\rho_0$ such that $\rho$ is isomorphic to $\rho_0 \otimes R$ and $\rho$ is distinguished if and only if $\rho_0$ is. We thus may assume that $R$ is equal to $\mathbb{F}_\ell$.

We consider the canonical $\mathbb{Q}_\ell$-lift $\tilde{\xi}$ of $\xi$, obtained via the canonical embedding of $\mathbb{F}_\ell^\times$ in $\mathbb{Z}_\ell^\times$. The pair $(K/E, \tilde{\xi})$ is admissible of level 0. Associated with it, there is thus a $\mathbb{Q}_\ell$-representation $\tilde{\rho}$ of $J$ trivial on $J^1$. One checks easily that its reduction mod $\ell$ is $\rho$, which comes from the congruence properties of the map (1.7) itself (see 29). Note that the restriction of $\xi$ to $K_0^\times$ is either trivial or (if $\ell \neq 2$) equal to $\omega_{K/K_0}$.
the representation $\tilde{\rho}^\varphi$ is distinguished. Thus $\rho$ is $\varphi$-distinguished, where $\varphi$ is the reduction mod $\ell$ of $\varphi$.

If $\ell = 2$, then $\rho$ is distinguished. Suppose now that $\ell \neq 2$. If $\rho$ were both $\varphi$-distinguished and distinguished, one would have two linearly independent linear forms in $\text{Hom}_{J,G^\varphi}(\rho, 1)$, and this would contradict Lemma 7.3(3). The result follows.

The field extension $E$ of $F$ is not uniquely determined by $\pi$, unlike its maximal tamely ramified extension $T$. To remedy this, let $D$ be the maximal tamely ramified sub-extension of $K/F$. Write $D_0 = D \cap K_0$, and let $\delta_0$ be the restriction of $\xi$ to $D_0^\times$.

Since $\xi \circ \sigma = \xi^{-1}$ the character $\delta_0$ is quadratic, either trivial or (if $\ell \neq 2$) equal to $\omega_{D/D_0}$. We will see in Proposition 9.5 that, up to $F_0$-equivalence, $D/D_0$ and $\delta_0$ are determined by $\pi$.

**Theorem 7.11.** — Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of $G$. Suppose that $T/T_0$ is ramified. Let $m$ be its relative degree and $\delta_0$ be the quadratic character of $D_0^\times$ associated to it.

1. The representation $\pi$ is distinguished if and only if at least one of the following conditions is fulfilled:
   - (a) $\ell = 2$,
   - (b) $m = 1$ and $\delta_0$ is trivial,
   - (c) $m$ is even and $\delta_0$ is non-trivial.

2. Suppose that $\ell \neq 2$. Then $\pi$ is $\omega$-distinguished if and only if:
   - (a) either $m = 1$ and $\delta_0$ is non-trivial,
   - (b) or $m$ is even and $\delta_0$ is trivial.

**Remark 7.12.** — If $R$ has characteristic 2, then $\pi$ is always distinguished. If $R$ has characteristic not 2, then $\pi$ is either distinguished or $\omega$-distinguished, but not both.

**Proof.** — By Proposition 6.15 and Lemma 7.9 it suffices to compare the restriction of $\xi$ to $K_0^\times$ with $\delta_0$ when $\ell \neq 2$.

Suppose first that $m = 1$ and $\delta_0$ is trivial. Since the restriction of $\rho$ to $E_0^\times$ is equal to either 1 or $\omega_{E/E_0}$, its restriction to $T_0^\times$ is either 1 or $\omega_{T/T_0}$, respectively. Since $\delta_0$ is trivial, we are in the first case, that is, the restriction of $\rho$ to $E_0^\times$ is trivial.

Suppose now that $m \neq 1$ and $\xi$ is non-trivial on $K_0^\times$. We want to prove that $\delta_0$ is non-trivial. The restriction of $\xi$ to $K_0^\times$ is equal to $\omega_{K/K_0}$. Thus $\delta_0$ is equal to $\omega_{D/D_0}$.

Now suppose that $R$ has characteristic different from 2, and let $\chi$ be an unramified character of $F^\times$ extending $\omega$. Note that the twisted representation $\pi' = \pi(\chi^{-1} \circ \det)$ is supercuspidal and $\sigma$-selfdual, and that the character associated with $\pi'$ is $\delta_0' = \delta_0(\chi^{-1} \circ N_{K/F})|_{D_0^\times}$, where $N_{K/F}$ is the norm map from $K$ to $F$. Suppose first that $m = 1$. Then:

$$\pi \text{ is } \omega\text{-distinguished} \iff \pi' \text{ is distinguished}$$
$$\iff \text{the character } \delta_0' \text{ is trivial}$$
$$\iff \text{the character } \delta_0 \text{ coincides with } \chi \circ N_{E/F} \text{ on } T_0^\times.$$
Remark 7.14. — The finite field case

Suppose now that $m \neq 1$. Then:

- $\pi$ is $\omega$-distinguished $\iff \pi'$ is distinguished
- $\iff$ the character $\delta_0'$ is non-trivial
- $\iff$ the character $\delta_0$ coincides with $(\chi \circ N_{K/F})\omega_{D/D_0}$ on $D_0^\times$.

The restriction of $\chi \circ N_{K/F}$ to $D_0^\times$ is $\omega \circ N_{D_0/F_0} = \omega_{D/D_0}$ to the power of $[K_0 : D_0]$, which is a $p$-power with $p$ odd. This gives us the expected result. \qed

Remark 7.13. — If $m > 1$, the central character $\omega_\pi$ of $\pi$ is always trivial on $F_0^\times$. Indeed, since $\pi$ and $\lambda$ have the same central character, we can express $\omega_\pi$ as the product $\omega_\kappa \omega_\rho$, where $\omega_\kappa$ and $\omega_\rho$ are the central characters of $\kappa$ and $\rho$ on $F^\times$, respectively. Since $\kappa$ is distinguished its central character is trivial on $F_0^\times$, thus $\omega_\pi$ and $\delta_0$ coincide on $F_0^\times$. If $\delta_0$ is trivial then $\omega_\pi$ is trivial on $F_0^\times$.

Now assume that $\delta_0$ is equal to $\omega_{D/D_0}$. Since $D_0/D_0$ is unramified by Lemma 4.19, its restriction to $F_0^\times$ is trivial if and only if $e(D_0/F_0)$ is even, which is the case since $e(D_0/F_0) = 2$ when $m$ is even.

Remark 7.14. — On the other hand, since $T/T_0$ is ramified, the restriction of $\omega_{T/T_0}$ to $F_0^\times$ is trivial if and only if $k_0^\times$ is contained in the subgroup $I^\times$ of squares in $I^\times$, that is, if and only if $f(T_0/F_0)$ is even. It follows that, if $m = 1$ and $f(T_0/F_0)$ is odd (which is equivalent to $n$ being odd by Lemma 4.14 and since $m$ is either 1 or even) then $\pi$ is distinguished if and only if $\omega_\pi$ is trivial on $F_0^\times$.

8. The supercuspidal unramified case

Let $\pi$ be a $\sigma$-selfdual cuspidal representation of $G = \text{GL}_n(F)$ for $n \geq 1$. In this section, we investigate the case where the quadratic extension $T/T_0$ is unramified. By Corollary 5.23, the representation $\pi$ is distinguished if and only if any of the $\sigma$-selfdual types it contains is distinguished.

Unlike the ramified case, in order to investigate the distinction of the $\sigma$-selfdual types in $\pi$, we will have to do the level 0 case first, and assume that $\pi$ is supercuspidal from the beginning.

8.1. The finite field case

In this paragraph and the next one, we assume that $F$ is unramified over $F_0$. Thus $k$ is quadratic over $k_0$ and $\sigma$ induces the non-trivial $k_0$-automorphism of $k$.

Lemma 8.1. — Let $n \geq 1$ be a positive integer. There exists a $\sigma$-selfdual supercuspidal irreducible representation of $\text{GL}_n(k)$ if and only if $n$ is odd.

Proof. — We follow the same argument as in the proof of Lemma 7.1, replacing (7.1) by:

\begin{equation}
\xi^{-q_0} = \xi^{q_0^2}, \quad \text{for some } i \in \{0, \ldots, n - 1\},
\end{equation}

where $q_0$ is the cardinality of $k_0$. (We thus have $q_0^2 = q$.) Exponentiating to $-q_0$ again gives us the equality $\xi^q = \xi^{q^2_i}$. The regularity assumption on $\xi$ implies that $n$ divides $2i - 1$, thus $n$ is
odd. More precisely, since \( 0 \leq i \leq n - 1 \), we have \( n = 2i - 1 \). It follows that:
\[
(8.2) \quad \rho_\xi \text{ is } \sigma\text{-selfdual } \iff \xi^{-1} = \xi^{q_0^a}.
\]
(This is also equivalent to \( \xi \) being trivial on \( t_0^x \), where \( t_0 \) is the subfield of \( t \) with \( q_0^a \) elements.)

Conversely, suppose that \( n \) is odd. We look for a \( k\)-regular character \( \xi : t^x \to R^x \) such that \( \xi^{-1} = \xi^{q_0^a} \). In other words, we look for a character of \( t^x \) whose order \( a \) satisfies:

1. \( a \) divides \( q_0^n + 1 \),
2. the order of \( q \) in \((\mathbb{Z}/a\mathbb{Z})^x\) is equal to \( n \).

Pick a character of order \( q_0^n + 1 \), which is possible since the group \( t^x \) has order \( q^n - 1 = q_0^{2n} - 1 \). Then \( q_0 \) has order \( 2n \mod q_0^n + 1 \), thus \( q = q_0^2 \) has order \( n \mod q_0^n + 1 \).

**Remark 8.2.** — When \( R \) has characteristic \( \ell > 0 \), the group \( \text{GL}_n(k) \) may have \( \sigma\text{-selfdual cuspidal} \) (non supercuspidal) representations for \( n \) even. Indeed, let \( e \) be the order of \( q \) mod \( \ell \), and suppose that \( n = e\ell^u \) for some \( u \geq 0 \). Then the unique generic subquotient \( \pi \) of the representation induced from the trivial character of a Borel subgroup of \( \text{GL}_n(k) \) is cuspidal and \( \sigma\text{-selfdual} \). (Note that, by Remark 7.2, it is also selfdual.) One may choose \( q, \ell \) and \( u \) such that \( n \) is even.

**Lemma 8.3.** — Let \( \rho \) be a supercuspidal representation of \( \text{GL}_n(k) \) for some \( n \geq 1 \). The following assertions are equivalent:

1. The representation \( \rho \) is \( \sigma\text{-selfdual} \).
2. The representation \( \rho \) is distinguished.
3. The space \( \text{Hom}_{\text{GL}_n(k)}(\rho,1) \) has dimension 1.

**Proof.** — When \( R \) has characteristic 0, this is due to Gow [18]. Suppose that \( R \) has characteristic \( \ell > 0 \) prime to \( q \). By Remark 3.2 any distinguished irreducible representation of \( \text{GL}_n(k) \) is \( \sigma\text{-selfdual} \) and (2) is equivalent to (3). To prove that (1) implies (2), we follow the same lifting argument as in the proof of Lemma 7.3 replacing (7.2) by (8.2).

**Remark 8.4.** — As in Remark 7.3 we proved that, if \( R \) is the field \( \overline{F}_\ell \), the representation \( \rho \) is distinguished if and only if it has a distinguished lift to \( \overline{Q}_\ell \).

**Remark 8.5.** — We give an example of a \( \sigma\text{-selfdual cuspidal} \) non supercuspidal representation of \( \text{GL}_n(k) \) which is not distinguished. With the notation of Remark 8.2 assume that \( n = e = 2 \). Thus \( \pi \) is a \( \sigma\text{-selfdual cuspidal} \) (non supercuspidal) representation of \( \text{GL}_2(k) \). Let \( \tilde{\pi} \) be an \( \ell\)-adic lift of \( \pi \) (which exists by [14, 15]), and decompose its restriction to \( \text{GL}_2(k_0) \) as a direct sum:

\[
V_1 \oplus \cdots \oplus V_r
\]

of irreducible components. Since the order of \( \text{GL}_2(k_0) \) is prime to \( \ell \), reduction mod \( \ell \) preserves irreducibility, and the restriction of \( \pi \) to \( \text{GL}_2(k_0) \) is semisimple. It follows that \( \pi \) decomposes as \( W_1 \oplus \cdots \oplus W_r \), where \( W_i \) is irreducible and is the reduction mod \( \ell \) of \( V_i \) for each \( i = 1, \ldots, r \). Suppose that \( \pi \) is distinguished. Then \( W_i = 1 \) for some \( i \). Thus \( V_i \) is a character, and it must be trivial since \( \text{GL}_2(k_0) \) has no non-trivial character of order a power of \( \ell \), which implies that \( \tilde{\pi} \) is distinguished. This is impossible, since \( n = 2 \) is even.
Remark 8.6. — More generally, the argument of Remark 8.5 shows that, if $H$ is a subgroup of $\text{GL}_n(k)$ whose order is prime to $\ell$, then a cuspidal representation $\pi$ of $\text{GL}_n(k)$ is $H$-distinguished if and only if any $\ell$-adic lift of $\pi$ is $H$-distinguished.

8.2. Existence of $\sigma$-selfdual extensions of the Heisenberg representation

Let $[\alpha, \beta]$ be a maximal simple stratum as in Remark 4.10. Write $E = F[\beta]$ and suppose that it is unramified over $E_0 = E^\sigma$. Let us write $J = J(\alpha, \beta)$, $J = J(\alpha, \beta)$ and $J^1 = J^1(\alpha, \beta)$.

We may and will identify $J/J^1$ with the group $\text{GL}_m(t)$, denoted $G$, equipped with the residual involution $\sigma \in \text{Gal}(l/l_0)$, where $l$ and $l_0$ are the residue fields of $E$ and $E_0$, respectively.

Lemma 8.7. — The group $J \cap G^\sigma$ is generated by $t$ and $J \cap G^\sigma$.

Proof. — Any $x \in J$ can be written $x = t^m y$ for unique $m \in \mathbb{Z}$ and $y \in J$. Since $t$ is $\sigma$-invariant, we have $x \in J \cap G^\sigma$ if and only if $y \in J \cap G^\sigma$.

Lemma 8.8. — Any character of $J \cap G^\sigma$ trivial on $J^1 \cap G^\sigma$ extends to a character of $J$ trivial on $J^1$.

Proof. — Let $\chi$ be a character of $J \cap G^\sigma$ trivial on $J^1 \cap G^\sigma$. Since $J^1$ is a pro-$p$-group, the first cohomology group of $\sigma$ in $J^1$ is trivial. The subgroup $G^\sigma$ thus identifies with $(J \cap G^\sigma)/(J^1 \cap G^\sigma)$. The restriction of $\chi$ to $J \cap G^\sigma$ thus induces a character of $G^\sigma$, which can be written $\alpha_0 \circ \det$ for some character $\alpha_0$ of $l_0^\times$. Let $\alpha$ be a character of $l^\times$ extending $\alpha_0$, and let $\phi$ be the character of $J$ trivial on $J^1$ inducing the character $\alpha \circ \det$ of $G$. Since $J = b^*J^1$, the element $t$ acts trivially on $J/J^1$ by conjugacy, thus normalizes $\phi$. We thus may extend $\phi$ to $J$ by setting $\phi(t) = \chi(t)$.

Lemma 8.7 implies that $\phi$ extends $\chi$ and it is trivial on $J^1$.

Let $\theta \in \mathcal{C}(\alpha, \beta)$ be a maximal simple character such that $H^1(\alpha, \beta)$ is $\sigma$-stable and $\theta \circ \sigma = \theta^{-1}$. Let $\eta$ denote the Heisenberg representation of $\theta$ on the group $J^1$.

Lemma 8.9. — There is a $\sigma$-selfdual representation $\kappa$ of $J$ extending $\eta$.

Proof. — Let $\kappa$ be an irreducible representation of $J$ extending $\eta$. By Lemma 4.20, there is a character $\mu$ of $J$ trivial on $J^1$ such that $\kappa^\sigma \nu \approx \kappa \mu$ and $\mu \circ \sigma = \mu$. We claim that there is a character $\nu$ of $J$ trivial on $J^1$ such that $(\nu \circ \sigma) \nu = \mu$. Indeed, if this is the case, the representation $\kappa \nu$ extends $\eta$ and is $\sigma$-selfdual.

Consider first $\mu$ as a character of $G$ and write $\mu = \phi \circ \det$ for some character $\phi$ of $l^\times$. Then we have $\phi \circ \sigma = \phi$, thus there is a character $\alpha$ of $l^\times$ such that $(\alpha \circ \sigma) \alpha = \phi$. Choosing such an $\alpha$, there exists a unique character $\nu$ of $J$ inducing $\alpha \circ \det$ on $G$. Since $J$ is generated by $t$ and $J$, it remains to extend $\nu$ to $J$ by choosing a scalar $\nu(t) \in R^\times$ such that $\nu(t)^2 = \mu(t)$.

8.3. Existence of distinguished extensions of the Heisenberg representation

Let $(J, \lambda)$ be a $\sigma$-selfdual type, with associated simple character the character $\theta$ of $8.2$.

In this paragraph, we suppose that $m$ is odd.

Proposition 8.10. — Suppose that $m$ is odd. There is a $\sigma$-selfdual distinguished representation $\kappa$ of $J$ extending $\eta$. 

Proof. — By Lemma 8.9 there is a \( \sigma \)-selfdual representation \( \kappa \) of \( J \) extending \( \eta \). Let \( \chi \) denote the character of \( J \cap G^\sigma \) trivial on \( J^1 \cap G^\sigma \) associated to \( \kappa \) by Lemma 8.10. Since \( m \) is odd, Lemma 8.4 implies that \( \mathfrak{S} \) possesses a \( \sigma \)-selfdual supercuspidal representation \( \rho \). We will denote by \( \rho \) the unique representation of \( J \) trivial on \( J^1 \) such that \( t \in \text{Ker}(\rho) \) and whose restriction to \( J \) is the inflation of \( \rho \). This representation \( \rho \) is \( \sigma \)-selfdual. By Lemma 8.3 it is also distinguished.

Now let \( \lambda \) denote the \( \sigma \)-selfdual type \( \kappa \otimes \rho \) on \( J \). The natural isomorphism:

\[
\text{Hom}_{J \cap G^\sigma}(\kappa, \chi) \otimes \text{Hom}_{J \cap G^\sigma}(\rho, 1) \rightarrow \text{Hom}_{J \cap G^\sigma}(\lambda, \chi)
\]

thus shows that \( \lambda \) is \( \chi \)-distinguished.

By Lemma 8.8 there exists a character \( \phi \) of \( J \) trivial on \( J^1 \) extending \( \chi \). The representation \( \chi' = \lambda \phi^{-1} \) is thus a distinguished type. Let \( \pi' \) be the cuspidal representation of \( G \) compactly induced from \((J, \chi')\). It is distinguished, thus \( \sigma \)-selfdual by Theorem 5.1. Since \( \chi' = \lambda^{\sigma \nu} \simeq \chi' \phi(\phi \circ \sigma) \) are both contained in \( \pi' \), it follows that \( \phi(\phi \circ \sigma) \) is trivial. This implies that \( \kappa' = \kappa \phi^{-1} \) is both \( \sigma \)-selfdual and distinguished.

Remark 8.11. — I don’t know whether Proposition 8.10 holds if \( m \) is even, because in that case I don’t know whether \( \mathfrak{S} \) possesses a distinguished cuspidal representation. When \( m \) is even and \( \ell > 2 \), one can prove that \( \mathfrak{S} \) has a \( \sigma \)-selfdual cuspidal representation if and only if the order \( e \) of the cardinality of \( \ell \) mod \( \ell \) divides \( m \) and \( m/e \) is odd. Besides, when this is the case, Remarks 8.5 and 8.6 show that a \( \sigma \)-selfdual cuspidal representation of \( \mathfrak{S} \) cannot be distinguished when \( m/e = 1 \).

Corollary 8.12. —

1. Any distinguished representation of \( J \) extending \( \eta \) is \( \sigma \)-selfdual.

2. If \( \ell = 2 \), any \( \sigma \)-selfdual representation of \( J \) extending \( \eta \) is distinguished.

Proof. — Let us fix a \( \sigma \)-selfdual representation \( \kappa \) of \( J \) extending \( \eta \) given by Proposition 8.10. Let \( \kappa' \) be a distinguished representation of \( J \) extending \( \eta \). Then \( \kappa' = \kappa \phi \) for some character \( \phi \) of \( J \) trivial on \((J \cap G^\sigma)J^1 \). Thus \( \phi(t) = 1 \) and \( \phi \) induces the character \( \alpha \circ \text{det} \) on \( \mathfrak{S} \), where \( \alpha \) is a character of \( L^\times \) trivial on \( t_0^\times \), or equivalently \( \alpha^{\nu_0+1} = 1 \). Thus we have \( \phi(\phi \circ \sigma) = 1 \). This implies that \( \kappa' \) is \( \sigma \)-selfdual, which proves the first assertion.

Now suppose that \( \kappa' \) is a \( \sigma \)-selfdual representation of \( J \) extending \( \eta \). Then \( \kappa' = \kappa \xi \) for some character \( \xi \) of \( J \) such that \( \xi(\xi \circ \sigma) \) is trivial. Thus \( \xi(t) \in \{-1, 1\} \) and there is a character \( \nu \) of \( L^\times \) such that \( \xi \) induces \( \nu \circ \text{det} \) on \( \mathfrak{S} \) and \( \nu^{\nu_0+1} = 1 \). It follows that \( \xi \) is trivial on \((J \cap G^\sigma)J^1 \). Thus, if \( \ell = 2 \), the representation \( \kappa' \) is distinguished.

Remark 8.13. — Let \( \kappa \) be a \( \sigma \)-selfdual representation of \( J \) extending \( \eta \). Then the character \( \chi \) of \( J \cap G^\sigma \) associated to \( \kappa \) by Lemma 5.19 is quadratic and unramified.

8.4. Distinction criterion in the unramified case

Let \( \pi \) be a \( \sigma \)-selfdual supercuspidal representation of \( G \). Associated to it by Proposition 4.13 there is a quadratic extension \( T/T_0 \). We assume that \( T \) is unramified over \( T_0 \).

Recall that, by Theorem 5.1 and Proposition 4.10 the representation \( \pi \) is distinguished if and only any of its \( \sigma \)-selfdual types is distinguished. The following result is the analogue of Proposition 6.15.
Proposition 8.14. — Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of $G$, with unramified quadratic extension $T / T_0$ and relative degree $m$. Let $(J, \lambda)$ be a $\sigma$-selfdual type in $\pi$. Let $J^1$ be the maximal normal pro-$p$-subgroup of $J$ and $\eta$ be an irreducible component of the restriction of $\lambda$ to $J^1$.

1. The integer $m$ is odd.
2. There is a distinguished representation of $J$ extending $\eta$, and any such extension of $\eta$ is $\sigma$-selfdual.
3. Let $\kappa$ be a distinguished representation of $J$ extending $\eta$, and let $\rho$ be the unique representation of $J$ trivial on $J^1$ such that $\lambda \simeq \kappa \otimes \rho$. Then $\pi$ is distinguished if and only if $\rho$ is distinguished.

Proof. — We may and will assume that $J = J(a, \beta)$ for some maximal simple stratum $[a, \beta]$ as in Remark 4.10. Following Remark 4.15, the extension $E$ is unramified over $E_0$. We thus may apply the results of Paragraph 8.2.

Let $\kappa$ be a $\sigma$-selfdual representation of $J$ extending $\eta$, the existence of which is given by Lemma 8.9, and let $\rho$ be the irreducible representation of $J$ trivial on $J^1$ such that $\lambda \simeq \kappa \otimes \rho$. Since $\lambda$ and $\kappa$ are $\sigma$-selfdual, $\rho$ is $\sigma$-selfdual. Its restriction to $J$ induces a cuspidal irreducible representation of $GL_m(l)$, denoted $\rho$. Since $\pi$ is supercuspidal, $\rho$ is also supercuspidal by Fact 4.5. Lemma 8.1 implies that $m$ is odd. We thus apply Proposition 8.10, which gives us a $\sigma$-selfdual distinguished representation extending $\eta$.

Part (2) of the proposition is given by Corollary 8.12. For (3), it suffices to fix a distinguished representation $\kappa$ of $J$ extending $\eta$ and to consider the canonical isomorphism:

$$\text{Hom}_{J \otimes G^\sigma}(\kappa, 1) \otimes \text{Hom}_{J \otimes G^\sigma}(\rho, 1) \to \text{Hom}_{J \otimes G^\sigma}(\lambda, 1)$$

as in the ramified case. □

Remark 8.15. — If one relaxes the supercuspidality assumption on $\pi$ (that is, we only assume $\pi$ to be $\sigma$-selfdual cuspidal with $T / T_0$ unramified), then its relative degree $m$ need not be odd, in which case our proof of Proposition 8.14 doesn’t apply (see Remarks 8.2 and 8.11).

Remark 8.16. — In both ramified and unramified cases, the distinguished representation $\kappa$ of $J$ extending $\eta$ is not unique in general, so neither is $\rho$. If $\kappa$ is a distinguished representation of $J$ extending $\eta$, the other ones are exactly the $\kappa \phi$ where $\phi$ ranges over the set of characters of $J$ trivial on $(J \cap G^\sigma)J^1$.

From now, we will thus assume that $\kappa$ is a distinguished $\sigma$-selfdual representation of $J$ extending $\eta$. Proposition 8.14 reduces the problem of the distinction of $\pi$ to that of $\rho$. We now investigate the distinction of $\rho$.

Let $\rho$ be the representation of $GL_m(l)$ defined by restricting $\rho$ to $J$. It is $\sigma$-selfdual. By Fact 4.5, it is also supercuspidal.

Lemma 8.17. — The representation $\rho$ is distinguished if and only if its central character $\omega_\rho$ is trivial on $E_0^\times$. 

Lemma 8.18. — The character $\varepsilon_0$ is quadratic and unramified.

Proof. — Since $\rho$ is $\sigma$-selfdual, its central character is trivial on $E/E_0$-norms. Thus its restriction to $E_0^\times$ is either trivial or (if $\ell \not= 2$) equal to $\omega_{E/E_0}$. We get the expected result by restricting to $T_0^\times$, since $E$ is unramified over $E_0$ and $e(E_0/T_0)$ is a $p$-power with $p$ odd.

We will see below (Remark 9.7) that the character $\varepsilon_0$ is uniquely determined by $\pi$.

Theorem 8.19. — Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of $G$. Suppose that $T$ is unramified over $T_0$.

1. The representation $\pi$ is distinguished if and only if $\varepsilon_0$ is trivial.
2. Suppose that the characteristic of $R$ is not 2. Then $\pi$ is $\omega$-distinguished if and only if $\varepsilon_0$ is non-trivial.

Remark 8.20. — If $R$ has characteristic 2, then $\pi$ is always distinguished. If $R$ has characteristic not 2, then $\pi$ is either distinguished or $\omega$-distinguished, but not both.

Proof. — By Proposition 8.14, the representation $\pi$ is distinguished if and only if $\rho$ is distinguished. Lemma 8.18 tells us that it is distinguished if and only if $\omega_\rho(t) = 1$. The restriction of $\omega_\rho$ to $E_0^\times$ is a quadratic unramified character. Since the ramification index of $E_0$ over $T_0$ is odd (for it is a $p$-power), $\omega_\rho$ is trivial on $E_0^\times$ if and only if it is trivial on $T_0^\times$. The first assertion is proven.

Now suppose that $R$ has characteristic different from 2, and let $\chi$ be an unramified character of $F^\times$ extending $\omega$. Note that the twisted representation $\pi' = \pi(\chi^{-1} \circ \det)$ is supercuspidal and $\sigma$-selfdual, and that the character associated with $\pi'$ is $\varepsilon_0' = \varepsilon_0(\chi^{-m} \circ N_{E/F})|_{T_0^\times}$ where $N_{E/F}$ is the norm map from $E$ to $F$. Thus:

$\pi$ is $\omega$-distinguished \iff $\pi'$ is distinguished
\iff the character $\varepsilon_0'$ is trivial on $T_0^\times$
\iff the character $\varepsilon_0$ coincides with $\chi^m \circ N_{E/F}$ on $T_0^\times$.

The restriction of $\chi \circ N_{E/F}$ to $T_0^\times$ is equal to $\omega \circ N_{T_0/F_0} = \omega_{T/T_0}$ to the power of $[E_0 : T_0]$, which is a $p$-power. The second assertion then follows from the fact that $p$ and $m$ are odd.

Corollary 8.21. — Let $\pi$ be a supercuspidal representation of $G$. Suppose that $T/T_0$ is unramified, and that the ramification index of $T/F$ is odd. Then $\pi$ is distinguished if and only if it is $\sigma$-selfdual and its central character is trivial on $F_0^\times$. 

Proof. — Suppose that $\pi$ is $\sigma$-selfdual and that its central character $\omega_\pi$ is trivial on $F_0^\times$. By using a $\sigma$-selfdual type $(J, \lambda)$ contained in $\pi$ as above, we can express $\omega_\pi$ as the product $\omega_\kappa \omega_\rho$, where $\omega_\kappa$ and $\omega_\rho$ are the central characters of $\kappa$ and $\rho$ on $F^\times$, respectively. Since $\kappa$ is distinguished, its central character is trivial on $F_0^\times$, thus $\omega_\kappa$ and $\varepsilon_0$ coincide on $F_0^\times$. It remains to prove that $\varepsilon_0$ is trivial if and only if it is trivial on $F_0^\times$.

Suppose $\varepsilon_0$ is trivial on $F_0^\times$. By Lemma 8.18 it is unramified, thus $\varepsilon_0^{(T_0/F_0)}$ is trivial on $T_0^\times$. Since $e(T/F)$ is odd, $e(T_0/F_0)$ is odd too, and the expected result follows from the fact that $\varepsilon_0$ is quadratic.

Remark 8.22. — In particular, when $n$ is odd and $F$ is unramified over $F_0$, a supercuspidal representation of $G$ is distinguished if and only if it is $\sigma$-selfdual and its central character is trivial on $F_0^\times$. This has been proved by Prasad [42] when $R$ has characteristic 0. Note that, since $m$ and $p$ are odd here, $n$ is odd if and only if $[T:F]$ is odd.

Remark 8.23. — Note that, in the proof of Prasad [42] Theorem 4, the isomorphism of $\pi$ with $\pi^{\sigma_\psi}$ gives an element $g \in G$ which has the property that $g\sigma(g) \in J = J(a, \beta)$, but $g$ has a priori no reason to normalize $J$. Anyway, $g$ can be chosen in the maximal compact open subgroup $\pi^\times$ which contains $J$, thus the group generated by $g$ and $J$ will indeed be compact mod centre.

9. Statement of the final results

In this section we put together the main results of Sections 6 to 8. Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of $G$. Associated to it, there are its relative degree $m$ and the quadratic extension $T/T_0$. It is convenient to introduce the following definition, which comes from [3].

Definition 9.1. — A $\sigma$-selfdual type in $\pi$ is said to be generic if either $T/T_0$ is unramified, or $T/T_0$ is ramified and this type has index $[m/2]$.

Remark 9.2. — It is proved in [3] Proposition 5.5 that a $\sigma$-selfdual type is generic in the sense of Definition 9.1 if and only if there are a $\sigma$-stable maximal unipotent subgroup $N$ of $G$ and a $\sigma$-selfdual non-degenerate character $\psi_N$ of $N$ such that $\text{Hom}_{J\times N}(\lambda, \psi_N)$ is non-zero.

Definition 9.1 is convenient to us because of the following result, which subsumes Propositions 6.14 and 7.8 (compare with Theorem 5.1).

Theorem 9.3. — A $\sigma$-selfdual cuspidal representation of $G$ is distinguished if and only if any of its generic $\sigma$-selfdual types is distinguished.

Let $(J, \lambda)$ be a generic $\sigma$-selfdual type contained in $\pi$. Let $[a, \beta]$ be a $\sigma$-selfdual simple stratum such that $J = J(a, \beta)$. The restriction of $\lambda$ to the maximal normal pro-$p$-subgroup $J^1$ is made of copies of a single irreducible representation $\eta$. We fix a distinguished $\sigma$-selfdual representation $\kappa$ of $J$ extending $\eta$, the existence of which is given by Propositions 6.14 and 8.14. Let $\rho$ be the representation of $J$ trivial on $J^1$ such that $\lambda$ is isomorphic to $\kappa \otimes \rho$. Let $(K/E, \xi)$ be an admissible pair of level 0 attached to $\rho$ and $\sigma$ be the involution of $K$ given by Proposition 4.18. Let $K_0$ be the field of $\sigma$-fixed points of $K$. We thus have $K \simeq K_0 \otimes_{F_0} F$. 

**Definition 9.4.** — Let \( D \) be the maximal tamely ramified sub-extension of \( K/F \). Write \( D_0 = D \cap K_0 \), and let \( \delta_0 \) be the restriction of \( \xi \) to \( D_0^\times \).

It follows immediately from the definition that \( D_0/F_0 \) is tamely ramified and the character \( \delta_0 \) is quadratic, either trivial or (if \( \ell \neq 2 \)) equal to \( \omega_{D/D_0} \).

**Proposition 9.5.** — The \( F_0 \)-equivalence class of the quadratic extension \( D/D_0 \) is uniquely determined by \( \pi \), as well as \( \delta_0 \).

**Proof.** — Start with a generic \( \sigma \)-selfdual type contained in \( \pi \). Since it is unique up to \( G^\sigma \)-conjugacy, we may assume this is \( (J, \lambda) \). Fix a \( \sigma \)-selfdual stratum \([a', \beta']\) such that \( J = J(a', \beta') \). By [3] Lemma 4.29, we may assume that the maximal tamely ramified sub-extension of \( E' = F[\beta'] \) over \( F \) is equal to \( T \). Fix a distinguished \( \sigma \)-selfdual representation \( \kappa' \) of \( J \) extending \( \eta \), let \( \rho' \) be the representation of \( J \) trivial on \( J \) corresponding to this choice and \((K'/E', \xi')\) be an admissible pair of level 0 attached to \( \rho' \). This gives us a quadratic extension \( D'/D_0 \) and a character \( \delta_0' \) of \( D_0^\times \).

First, suppose that \([a', \beta'] = [a, \beta] \) and \( K' = K \). We have \( \kappa' = \kappa \phi \) for some character \( \phi \) of \( J \) trivial on \((J \cap G^\sigma)J\), thus \( \rho' \) is isomorphic to \( \rho \phi^{-1} \). Thus \( \xi' \) is \( E \)-isomorphic to \( \xi \alpha^{-1} \) for some tamely ramified character \( \alpha \) of \( K \) trivial on \( K_0^\times \). Restricting to \( D_0 \), we get \( \delta_0' = \delta_0 \).

We now go back to the general case. By the previous argument, we may assume that \( \kappa' = \kappa \), thus \( \rho' = \rho \). Since \( D \) and \( D' \) are both unramified of same degree \( m \) over \( T \), they are \( T \)-isomorphic. Let us fix a \( T \)-isomorphism \( f : D \rightarrow D' \). Write \( \sigma' \) for the involutive automorphism of \( K' \) given by Proposition 4.18.

**Lemma 9.6.** — We have \( \sigma' \circ f = f \circ \sigma \).

**Proof.** — Let us identify the residual fields of \( K \) and \( D \), denoted \( t \), and those of \( E \) and \( T \), denoted \( l \). Note that, if \( \varphi \) is any \( T \)-automorphism of \( D \), then it commutes with \( \sigma \) since \( \varphi \) and \( \sigma \circ \varphi \circ \sigma^{-1} \) are both in \( \text{Gal}(D/T) \) and have the same image in \( \text{Gal}(t/l) \).

We now consider the pair \((D/T, \xi|_{D^\times})\). Since \( D/T \) is unramified of degree \( m \), it is admissible of level 0. Moreover, the \( L \)-regular character of \( t^\times \) it induces is \( \text{Gal}(t/l) \)-conjugate to the one induced by \((K/E, \xi)\), which doesn’t depend on the identifications of residual fields we have made. We have a similar result for \((D/T, \xi' \circ f)\) and \((K'/E', \xi')\). Since \( \rho' = \rho \) we deduce that \( \xi' \circ f = \xi \circ \varphi \) for some \( \varphi \in \text{Gal}(D/T) \). Let \( \alpha \) be the \( T \)-automorphism \( \sigma' \circ f \circ \sigma^{-1} \circ f^{-1} \) of \( D' \). We have:

\[
\xi' \circ \alpha = \xi'^{-1} \circ f \circ \sigma^{-1} \circ f^{-1} = \xi^{-1} \circ \varphi \circ \sigma^{-1} \circ f^{-1} = \xi \circ \varphi \circ f^{-1} = \xi'.
\]

It follows from admissibility of \( \xi' \) that \( \alpha \) is trivial, as expected.

**Remark 9.7.** — In particular, the character \( \varepsilon_0 \) of Paragraph 8.4, which is the restriction of \( \delta_0 \) to \( T_0^\times \), is uniquely determined by \( \pi \).

We state the dichotomy and disjunction theorem.
Theorem 9.8. — Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of $G$. Let $\ell$ be the characteristic of $R$.

(1) If $\ell \neq 2$, then $\pi$ is either distinguished or $\omega$-distinguished, but not both.
(2) If $\ell = 2$, then $\pi$ is always distinguished.

Proof. — See Remarks 7.12 and 8.20.

We now state the distinction criterion theorem.

Theorem 9.9. — Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of $G$. Attached to it, there are the quadratic extensions $T/T_0$ and $D/D_0$ and the character $\delta_0$.

(1) Suppose that $n$ is odd. Then $\pi$ is distinguished if and only if its central character is trivial on $F_0^\times$.
(2) If $\ell \neq 2$, $T/T_0$ is ramified and $D/D_0$ is unramified, then $\pi$ is distinguished if and only if the character $\delta_0$ is non-trivial.
(3) Otherwise, $\pi$ is distinguished if and only if $\delta_0$ is trivial.

Proof. — For (1) see Remarks 7.14 and 8.22. For the remaining items, see Theorems 7.11 and 8.19: it suffices to check that, if $T/T_0$ is unramified, then $\delta_0$ is trivial if and only if its restriction $\varepsilon_0$ to $T_0^\times$ is trivial, which follows from the fact that $m$ is odd in that case.

Remark 9.10. — The following conditions are equivalent:

(1) $D/D_0$ is ramified;
(2) $T/T_0$ is ramified and $m = 1$.

Indeed this follows from Remark 4.15, Lemma 4.19 et Proposition 7.8. The following conditions are thus also equivalent:

(1) $T/T_0$ is ramified and $D/D_0$ is unramified;
(2) $F/F_0$ is ramified, $T_0/F_0$ has odd ramification order and $D/D_0$ is unramified;
(3) $F/F_0$ is ramified, $T_0/F_0$ has odd ramification order and $m \neq 1$.

We now state the distinguished lift theorem.

Theorem 9.11. — Let $\pi$ be a $\sigma$-selfdual supercuspidal representation of $G$ over $F_\ell$.

(1) The representation $\pi$ admits a $\sigma$-selfdual supercuspidal lift to $Q_\ell$.
(2) Let $\tilde{\pi}$ be a $\sigma$-selfdual lift of $\pi$, and suppose that $\ell \neq 2$. Then $\tilde{\pi}$ is distinguished if and only if $\pi$ is distinguished.

Proof. — Let $(J, \lambda)$ be a $\sigma$-selfdual type in $\pi$. Let $\eta$ be the Heisenberg representation contained in the restriction of $\lambda$ to $J^1$ and let $\tilde{\eta}$ be the $\ell$-adic lift of $\eta$, which is unique up to isomorphism. By Propositions 6.15 and 8.10 there is a distinguished $\sigma$-selfdual representation $\tilde{\kappa}$ of $J$ extending $\tilde{\eta}$. Its reduction mod $\ell$, denoted $\kappa$, is a distinguished $\sigma$-selfdual representation of $J$ extending $\eta$. Let $\rho$ be the irreducible representation of $J$ trivial on $J^1$ such that $\lambda \simeq \kappa \otimes \rho$. By Remarks 7.5 and 8.21 it can be lifted to a $\sigma$-selfdual representation $\tilde{\rho}$ trivial on $J^1$. We then define $\tilde{\lambda} = \tilde{\kappa} \otimes \tilde{\rho}$, which is a $\sigma$-selfdual $\ell$-adic type whose reduction mod $\ell$ is $\lambda$. Inducing $\tilde{\lambda}$ to $G$, we get a $\sigma$-selfdual supercuspidal $\ell$-adic lift $\tilde{\pi}$ of $\pi$. 
Suppose that $\ell \neq 2$ and let $\widehat{\omega}$ be the canonical $\ell$-adic lift of $\omega$. Using the reduction argument of invariant linear forms, we see that if $\widehat{\pi}$ is distinguished (respectively, $\widehat{\omega}$-distinguished), then $\pi$ is distinguished (respectively, $\omega$-distinguished). By dichotomy (see Theorem 9.8), this is an equivalence.

We end with the following result, which is useful in [3].

**Proposition 9.12.** — Suppose that $\pi$ is a distinguished supercuspidal representation of $G$ and that $\ell \neq 2$. Then $\pi$ has no $\omega$-distinguished unramified twist if and only if $D/D_0$ is ramified, that is, if and only if $T/T_0$ is ramified and $m = 1$.

**Proof.** — Consider an unramified twist $\pi' = \pi(\chi \circ \det)$ of $\pi$, where $\chi$ is an unramified character of $F^\times$. We are looking for a $\chi$ such that $\pi'$ is $\omega$-distinguished. First, $\pi'$ is $\sigma$-selfdual if and only if $\pi \chi = \pi \chi p q$ $\sigma q$, that is, if and only if:

$$t(p) = \chi 2t(\pi) = 1$$

where $t(\pi)$ denotes the torsion number of $\pi$, that is the number of unramified characters $\alpha$ of $G$ such that $\pi \alpha \simeq \pi$. By [36] §3.4, we have $t(\pi) = f(K/F) = f(D/F)$. Now the quadratic character associated with $\pi'$ is $\delta_0' = \delta_0(\chi \circ N_{K/F})|_{D_0^\times}$ and we have:

$$\left(\chi \circ N_{K/F}\right)|_{D_0^\times} = \left(\chi \circ N_{D_0/F_0}\right)|^{K,D}.$$

By Theorem 9.11, the representation $\pi'$ is $\omega$-distinguished if and only if (9.2) is equal to $\omega_{D/D_0}$. If $D$ is ramified over $D_0$, this is not possible since $\chi$ is unramified. If $D/D_0$ is unramified, choosing an unramified character $\chi$ of order $f(D_0/F_0)$ gives us an $\omega$-distinguished twist $\pi'$.

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