Linear System Identification under Multiplicative Noise from Multiple Trajectory Data

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Abstract—The study of multiplicative noise models has a long history in control theory but is re-emerging in the context of complex networked systems and systems with learning-based control. We consider linear system identification with multiplicative noise from multiple state-input trajectory data. We propose exploratory input signals along with a least-squares algorithm to simultaneously estimate nominal system parameters and multiplicative noise covariance matrices. The asymptotic consistency of the least-squares estimator is demonstrated by analyzing first and second moment dynamics of the system. The results are illustrated by numerical simulations.

I. INTRODUCTION

The study of multiplicative noise models has a long history in control theory [1]–[3] but is re-emerging in the context of complex networked systems and systems with learning-based control. In contrast with the well-known additive noise setting, multiplicative noise has the ability to capture dependence of the noise on the state and/or control input. This situation occurs in modern control systems as diverse as robotics [4], networked systems with noisy communication channels [5], [6], modern power networks with high penetration of intermittent renewables [7]–[9], turbulent fluid flow [10], and neuronal brain networks [11]. Linear systems with multiplicative noise are particularly attractive as a stochastic modeling framework because they remain simple enough to admit closed-form expressions for stability and stabilization via generalized Lyapunov equations (e.g. [12]), optimal control via the solution of generalized Riccati equations [3], [13] and state estimation. Additionally, recent results show that the optimal control of this class of systems can be learned strictly from sample data without constructing a model via the reinforcement learning technique of policy gradient [14]. As a complementary perspective, here we tackle the problem from a model-based perspective where the goal is to learn and construct a model from sample data, which can then be used e.g. for optimal control design.

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The first issue that must be addressed is that a complete multiplicative noise system model requires accurate estimates not only of the nominal linear system matrices, but also the noise covariance structure. This stands in stark contrast to the additive noise case where the noise covariance structure has no bearing on the control design and can thus be ignored during system identification. For the identification of a nominal linear system, recursive algorithms have been developed in the control literature, such as the recursive least-squares algorithm [15]. These can be utilized for linear systems with multiplicative noise provided that certain conditions on the noise hold, but some stability assumptions for the system may be necessary. For the estimation of noise covariances, both recursive and batch estimation methods have been proposed over the last few decades (see [16] for a review), but these focus nearly exclusively on additive noise. In order to estimate multiplicative noise covariances, the maximum-likelihood approach was introduced in [17], [18], and the Bayesian framework was utilized in [19]. These methods, however, require prior assumptions on the noise statistics whose incorrectness may worsen the performance of the concerned algorithms for optimal control. Therefore, our paper concentrates on jointly estimating the nominal system parameters and the multiplicative noise covariances without imposing any prior assumptions on the noise, which complicates the problem. The presence of both state- and control-dependent noise in the system leads to coupling, which also makes the identification task more difficult.

The second issue we address is that of performing system identification based on multiple state-input trajectory data rather than a single trajectory. Multiple trajectory data arises in situations such as episodic tasks which end in finite time as encountered in iterative learning control and reinforcement learning problems [20]. For multiple trajectory data the duration of each trajectory sample may be rather small, but large sample size can often be obtained as a result of collecting data from multiple systems in parallel, whether physical or virtual. Thus, there is a growing interest in system identification based on multiple trajectory data, along with their applications in machine learning literature [21], [22].

In this paper we consider linear system identification with multiplicative noise from multiple trajectory data. Our contributions are two-fold:

- We propose a least-squares estimation algorithm to jointly estimate the nominal system matrices and multiplicative noise covariances from sample averages of multiple finite-horizon trajectory rollouts (Algorithm 1). A two-stage algorithm based on first and second
moment dynamics that separate the nominal parameters from the noise variances is utilized, where a stochastic input design, from Gaussian and Wishart distributions, is used for exciting the moment dynamics. The algorithm does not need prior knowledge for the multiplicative noise or stability conditions for the system, so it may be applied to a wide range of scenarios.

- The asymptotic consistency of our proposed algorithm is demonstrated. First, it is shown that deterministic dynamics defined by the first-order and second-order moments of the state can generate a well-defined closed-form expression of the parameters, provided sufficiently exciting input sequences and certain controllability conditions hold (Theorems 1 and 2). Then by assuming the multiple trajectory data are independent and identically distributed (i.i.d.), the consistency of the estimator, i.e., convergence to the true value as the number of trajectory samples grows to infinity, is obtained by combining the former result and the law of large numbers (Theorem 3).

The remainder of the paper is organized as follows: we formulate the problem in Section II then in Section III the algorithm is introduced and theoretical results are given, numerical simulation results are presented in Section IV and in Section V we conclude.

**Notation.** We denote the $n$-dimensional Euclidean space by $\mathbb{R}^n$, and the set of $n \times m$ real matrices by $\mathbb{R}^{n \times m}$. We use $\| \cdot \|$ to denote the Euclidean norm for vectors and the Frobenius norm for matrices. The expectation of a random vector $X$ is represented by $E\{X\}$. The Kronecker product of two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is represented by $A \otimes B$, and the vectorization of $A$ is represented by $\text{vec}(A) = (a_{11} \ a_{21} \cdots \ a_{m1} \ a_{12} \ a_{22} \cdots a_{mn})^\top$. For a block matrix

$$
B = \begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1n} \\
B_{21} & B_{22} & \cdots & B_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
B_{m1} & B_{m2} & \cdots & B_{mn}
\end{bmatrix} \in \mathbb{R}^{mp \times nq},
$$

where $B_{ij} \in \mathbb{R}^{p \times q}$, we define the following matrix reshaping operator $F : \mathbb{R}^{mp \times nq} \rightarrow \mathbb{R}^{mn \times pq}$:

$$
F(B, m, n, p, q) := \begin{bmatrix}
\text{vec}(B_{11}) & \text{vec}(B_{21}) & \cdots & \text{vec}(B_{m1}) \\
\text{vec}(B_{12}) & \text{vec}(B_{22}) & \cdots & \text{vec}(B_{mn})
\end{bmatrix}^\top.
$$

Then we have that $F(AX, A, m, n, n, n) = \text{vec}(A) (\text{vec}(A))^\top$ for $A \in \mathbb{R}^{m \times n}$, which demonstrates the relation between the entries of $A \otimes A$ and those of $\text{vec}(A) (\text{vec}(A))^\top$. Note when $p = q = 1$, $F(\cdot)$ degenerates to $\text{vec}(\cdot)$.

II. PROBLEM FORMULATION

We consider linear systems with multiplicative noise

$$
x_{t+1} = (A + \hat{A}_t)x_t + (B + \hat{B}_t)u_t + (A + \hat{A}_t)x_t + (B + \hat{B}_t)u_t \tag{1}
$$

where $x_t \in \mathbb{R}^n$ is the system state and $u_t \in \mathbb{R}^m$ is the control input to be designed. The dynamics are described by a nominal dynamics matrix $A \in \mathbb{R}^{n \times n}$ and nominal input matrix $B \in \mathbb{R}^{n \times m}$ and incorporate multiplicative noise terms modeled by the i.i.d. and mutually independent random matrices $\hat{A}_t$ and $\hat{B}_t$, which have zero mean and covariance matrices $\Sigma_A := E\{\text{vec}(\hat{A}_t) \text{vec}(\hat{A}_t)^\top\} \in \mathbb{R}^{n^2 \times n^2}$ and $\Sigma_B := E\{\text{vec}(\hat{B}_t) \text{vec}(\hat{B}_t)^\top\} \in \mathbb{R}^{nm \times nm}$.

As an example of the system (1), consider the following system studied in the optimal control literature [12], [14].

$$
x_{t+1} = (A + \sum_{i=1}^r A_i p_{i,t}) x_t + (B + \sum_{j=1}^s B_j q_{j,t}) u_t,
$$

where $\{p_{i,t}\}$ and $\{q_{j,t}\}$ are mutually independent i.i.d. scalar random variables, with $E\{p_{i,t}\} = E\{q_{j,t}\} = 0$, $E\{p_{i,t}^2\} = \sigma_i$, and $E\{q_{j,t}^2\} = \delta_j$, $\forall i \in [1, r], j \in [1, s], t \geq 0$. It can be seen that $\hat{A}_t = \sum_{i=1}^r A_i p_{i,t}$ and $\hat{B}_t = \sum_{j=1}^s B_j q_{j,t}$, where $\sigma_i$ and $\delta_j$ are the eigenvalues of $\Sigma_A$ and $\Sigma_B$, and $A_i$ and $B_i$ are the reshaped eigenvectors of $\Sigma_A$ and $\Sigma_B$.

**Problem.** Suppose that the system parameters $A, B, \Sigma_A$, and $\Sigma_B$ are unknown, but state-input trajectories are available for system identification. Our goal in this paper is to estimate $A, B, \Sigma_A$, and $\Sigma_B$ based on multiple trajectory data $\{x_t, 0 \leq t \leq \ell\}$, by appropriately designing the input sequence $\{u_t, 0 \leq t \leq \ell - 1\}$, where $\ell$ is the final time-step for every trajectory sample.

III. SYSTEM IDENTIFICATION VIA LEAST SQUARES WITH FIRST AND SECOND MOMENTS

A. Algorithm Design

In this section, we propose our exploratory input sequence design and least squares system identification algorithm to estimate the system parameters out from multiple trajectory data. We assume that the sampled trajectory data are collected independently, and refer to each trajectory sample as a rollout. Because every rollout is affected by the multiplicative noise, we will use least-squares on the first and second moment dynamics averaged over multiple trajectories to solve the system identification problem.

Taking the expectation of both sides of (1) we obtain the mean dynamics

$$
\mu_{t+1} = A \mu_t + B \nu_t, \tag{2}
$$

where $\mu_t := E\{x_t\}$ and $\nu_t := E\{u_t\}$. Likewise, denoting the vectorization of the instantaneous second moment matrices of state, state-input, and input at time $t$ by $X_t := E\{\text{vec}(x_t x_t^\top)\}$, $W_t := E\{\text{vec}(x_t u_t^\top)\}$, and $U_t := E\{\text{vec}(u_t u_t^\top)\}$, then from the independence of $\hat{A}_t$ and $\hat{B}_t$, and vectorization, the second moment dynamics of (1) is

$$
X_{t+1} = (A \otimes A)X_t + (B \otimes A)W_t + (A \otimes B)W_t^\top + (B \otimes B)U_t + E\left\{\text{vec}(\hat{A}_t \otimes \hat{A}_t) \text{vec}(x_t x_t^\top)\right\} + E\left\{(\hat{B}_t \otimes \hat{B}_t) \text{vec}(u_t u_t^\top)\right\}
$$

$$
= (A \otimes A)X_t + (B \otimes A)W_t + (A \otimes B)W_t^\top + (B \otimes B)U_t + \Sigma_A' X_t + \Sigma_B' U_t
$$

$$
= (A \otimes A + \Sigma_A) X_t + (B \otimes B + \Sigma_B)U_t + (B \otimes A)W_t + (A \otimes B)W_t^\top \tag{3}
$$

where we denote $\Sigma_A' = E\{\hat{A}_t \otimes \hat{A}_t\} \in \mathbb{R}^{n^2 \times n^2}$ and $\Sigma_B' = E\{\hat{B}_t \otimes \hat{B}_t\} \in \mathbb{R}^{nm \times nm}$. Note that $F(\Sigma_A, n, n, n, n) = \Sigma_A$.
and \( F(\Sigma_B', n, m, n, m) = \Sigma_B \), where the reshaping operator \( F(\cdot) \) is defined in the notation section.

The first and second moment dynamics \([2] \) and \([3] \) are deterministic and linear in the dynamic model parameters to be estimated. It is natural to consider a two-stage least squares procedure, where first the nominal system matrices \((A, B)\) are estimated from \([2] \), and then these estimates are plugged in to obtain estimates for the variances \((\Sigma_A, \Sigma_B)\) from \([3] \). If we had access to the exact first and second moments, this procedure would produce exact estimates provided a sufficient rollout length and persistent excitation of both first and second moment dynamics from the input mean \( \nu_t \) and second moment \( U_t \). However, we must estimate the first and second moments from rollout data, and we propose to take a sample average over multiple independent rollouts. To obtain persistently exciting inputs, we randomly generate the first and second moment of the input sequence distribution from standard Gaussian and Wishart \([23] \) distributions, respectively. The overall algorithm is shown in Algorithm 1 where the superscript \((k)\) represents the \(k\)-th rollout.

**Remark 1:** Since \( \Sigma_A \) and \( \Sigma_B \) are the covariance matrices of \( \tilde{A}_t \) and \( \tilde{B}_t \) they must be positive semidefinite. Hence a positive semidefinite constraint must be imposed on the optimization problem in line 14 of Algorithm 1 which can be easily achieved by generic convex optimization parser-solvers such as CVX in MATLAB \([24, 25] \). Note that \( \Sigma'_{A_t} \) and \( \Sigma'_{B_t} \) are related to \( \Sigma_A \) and \( \Sigma_B \) via one-to-one mappings by inverse of the \( F(\cdot) \) operator. However, if the estimator is consistent (as we will prove later it is), then as the amount of sample data increases the estimated covariances will become arbitrarily close to the true values and the semidefinite constraint will naturally become ineffective.

**B. Theoretical Consistency Analysis**

In this section we analyze the consistency of Algorithm 1 by investigating the moment dynamics \([2] \) and \([3] \), which motivated the least-squares approach in Algorithm 1.

1) **Moment Dynamics:** Note again that if we know the true values of \( \mu_t \) and \( X_t \), then it is possible to recover the parameters via least-squares as in lines 13 and 14 in Algorithm 1 because dynamics \([2] \) and \([3] \) are deterministic ones. Denote

\[
\begin{align*}
Y &:= \begin{bmatrix} \mu_\ell & \cdots & \mu_1 \end{bmatrix}, \quad Z := \begin{bmatrix} \mu_{\ell-1} & \cdots & \mu_0 \\ \mu_{\ell-1} & \cdots & \mu_0 \end{bmatrix}, \\
C &:= \begin{bmatrix} C_\ell & \cdots & C_1 \end{bmatrix}, \quad D := \begin{bmatrix} X_{\ell-1} & \cdots & X_0 \\ U_{\ell-1} & \cdots & U_0 \end{bmatrix},
\end{align*}
\]

where \( C_t = X_t - [(A \otimes A)X_{t-1} + (B \otimes A)W_{t-1} + (A \otimes B)W_{t-1}^T + (B \otimes B)U_{t-1}], \ 1 \leq t \leq \ell \). Then closed-form solutions of the least-squares problems are

\[
(\hat{A}, \hat{B}) = YZ^T(ZZ^T)^{\dagger},
\]

where \( C, D, Y, \) and \( Z \) are defined in \([3] \) above, and the sign \(^{\dagger}\) represents the pseudoinverse. Hence, the first question towards the consistency of the algorithm is whether the matrices \( ZZ^T \) and \( DD^T \) are invertible, which is necessary for the consistency of the algorithm. This property is determined by the input sequence and the controllability of the systems \((A, B)\) and \((A \otimes A + \Sigma'_A, B \otimes B + \Sigma'_B)\), including the value of the final time-step \( \ell \). We have the following results.

**Theorem 1:** Suppose that \( \ell \geq \frac{1}{2} mn^2 + \frac{1}{2} mn + m + 1 \) and \((A, B)\) is controllable. The matrix \( Z \) has full row rank with probability 1, and consequently \( ZZ^T \) is invertible, if the entries of \( \nu_t \), \( 0 \leq t \leq \ell - 1 \), are generated i.i.d. from a non-degenerate Gaussian distributions.

**Proof:** See Appendix A.

**Remark 2:** The above theorem shows that for large enough time step of each rollout, the full row rank condition of \( Z \) can be guaranteed with probability one if the mean of the input at each time step is generated randomly and independently. In the proof, the controllability of \((A, B)\)

\[
(\hat{\Sigma}'_A, \hat{\Sigma}'_B) = CD^TDD^T, \quad (6)
\]
plays a key role. In addition, although the lower bound in the theorem is relatively small, one may conjecture that $\ell \geq n + m$ is a sharper lower bound for the invertibility of $ZZ^T$, which will be a future work.

**Theorem 2:** Suppose that $\ell \geq \frac{1}{2}m^2n^4 + \frac{1}{2}m^2n^2 + m^2 + 1$ and $(A \otimes A + \Sigma_A^r, B \otimes B + \Sigma_B^r)$ is controllable. The matrix $D$ has full row rank with probability $1$, and consequently $DD^T$ is invertible, if $\nu_t$ have been fixed, and the entries of $U_t$ are generated i.i.d. from a non-degenerate Wishart distributions, $0 \leq t \leq \ell - 1$.

**Proof:** See Appendix B.

**Remark 3:** The controllability condition in Theorem 2 reflects the nature of the multiplicative noise, i.e., coupling between $A_t$ and $x_t$, and that between $B_t$ and $u_t$. It also indicates that a controllability condition on $A_t \otimes A_t$ and $B_t \otimes B_t$ of the input sequences have been generated in Algorithm 1. The estimates above depend on the first and second moments of the state and makes it possible to estimate all model parameters in the presence of multiplicative noise.

2) **Consistency:** After the discussion in the previous section, we now assume that the means and second moments of the input sequences have been generated in Algorithm 1 and both $ZZ^T$ and $DD^T$ are invertible. The closed-form estimates generated by Algorithm 1 are

$$\hat{A}, \hat{B} = YZ^T(ZZ^T)^{-1},$$

$$\hat{\Sigma}_A, \hat{\Sigma}_B = \hat{C}D^T(DD^T)^{-1},$$

where

$\hat{Y} := [\hat{\mu}_t \ldots \hat{\mu}_1], \quad \hat{Z} := [\hat{\mu}_{t-1} \ldots \hat{\mu}_0]$

$\hat{C} := \hat{C}_t \ldots \hat{C}_1, \quad \hat{D} := [\hat{X}_t \ldots \hat{X}_0]$

and $\hat{C}_t = \hat{X}_t - [(A \otimes A)\hat{X}_{t-1} + (B \otimes B)\hat{W}_{t-1} + (A \otimes B)\hat{W}_{t-1} + (B \otimes B)\hat{U}_{t-1}], 1 \leq t \leq \ell$. Here $A$ and $B$ are obtained by Algorithm 1.

**Assumption 1:** For all rollouts, (i) the final time-step is fixed to be $\ell \geq \frac{1}{2}m^2n^4 + \frac{1}{2}m^2n^2 + m^2 + 1$; (ii) the initial state $x_0^{(k)}, 1 \leq k \leq n_r$, is generated independently from the same distribution with $E[|x_0^{(k)}|^2] < \infty$, and is independent of the subsequent process; (iii) $\{A_t^{(k)}\}$ and $\{B_t^{(k)}\}, 0 \leq t \leq \ell, 1 \leq k \leq n_r$, have zero mean and finite second moments, i.e., $E[A_t^{(k)}] = E[B_t^{(k)}] = 0$ and $E[\Sigma_{A_t}||\Sigma_B] < \infty$. Also, they are mutually independent, and i.i.d. respectively. (iv) the input signals are generated according to Line 6 of Algorithm 1.

Under Assumption 1, the rollouts $x_0^{(k)}, \ldots, x_t^{(k)}, 1 \leq k \leq n_r$, are i.i.d., so we can establish consistency of the above estimators from Kolmogorov’s strong law of large numbers [26].

**Theorem 3:** (Consistency) Suppose that Assumption 1 holds, and both $ZZ^T$ and $DD^T$ are invertible. Then the estimators (7)-(8) are asymptotically consistent, i.e.,

$$(\hat{A}, \hat{B}) \rightarrow (A, B), \quad (\hat{\Sigma}_A, \hat{\Sigma}_B) \rightarrow (\Sigma_A, \Sigma_B),$$

with probability one as the number of rollouts $n_r \rightarrow \infty$.

**Proof:** See Appendix C.

**Remark 5:** This theorem indicates that despite the relatively small final time-step for each trajectory, an increasing number of rollouts compensates for this deficiency and guarantees asymptotic estimation performance.

IV. Numerical Simulations

To empirically validate our theoretical consistency result, we applied our least-squares estimator to an example system with $n = 2, m = 1, A = [-0.4 \ 0.3, B = [-1.8 \ -0.8]$, and noise covariances

$$\Sigma_A = \frac{1}{100} \begin{bmatrix} 8 & -2 & 0 & 0 \\ -2 & 16 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \Sigma_B = \frac{1}{100} \begin{bmatrix} 5 & -2 & 0 \\ -2 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We performed a simulated experiment where rollout data of length $\ell = \frac{1}{2}m^2n^4 + \frac{1}{2}m^2n^2 + m^2 + n = 12$, i.e., according to the bound prescribed by our theoretical result, was collected for $n_r = 10,000,000$. We used control inputs distributed as $u_t \sim N(\nu_t, \hat{U}_t)$, where $\nu_t$ and $\hat{U}_t$ are generated from $N(0, I)$ and $W(0.1r, n)$, respectively, and then are fixed. Model estimates were computed at 100 increasing logarithmically spaced numbers of rollouts between 1 and $n_r$. This experiment was repeated 50 times and the results are plotted in Fig. 1. Python code which implements the algorithms and performs the simulated experiment described here is available on Github at [https://github.com/ SummersLab/sysid-multinonse](https://github.com/ SummersLab/sysid-multinonse)

It is well known that least-squares estimation of finite-impulse response (FIR) models yields estimates whose error
parameters and multiplicative noise covariances. The asymptotic consistency of the algorithm was demonstrated by analyzing moment dynamics of the system, and illustrated by numerical simulations. Ongoing and future research directions include studying the asymptotic performance (convergence rate) and non-asymptotic behavior of the proposed algorithm, problems of optimal input design, identification from single-trajectory data, and sparsity-promoting regularization for identification of networked systems.

**APPENDIX A: THE PROOF OF THEOREM 1**

We begin by stating a standard result regarding the zero set of a polynomial which will be needed later:

**Lemma 1:** A polynomial function $\mathbb{R}^n$ to $\mathbb{R}$ is either identically 0 or non-zero almost everywhere.

**Proof:** The conclusion is a standard result from real analysis [28], [29]; we omit the proof due to space limitations.

Now we provide a lemma which will naturally lead to the conclusion of Theorem 1:

**Lemma 2:** Consider the moment dynamics (9). Suppose that $\ell > \frac{1}{2}mn^2 + \frac{1}{2}mn + m + 1$ and $(A, B)$ is controllable. The set of $\nu_t$ such that the rank of $Z$ is less than $n + m$,

$$V := \{ (\nu_0^T \cdots \nu_{t-1}^T)^T \in \mathbb{R}^{m\ell} : \text{rank}(Z) < n + m \}$$

is of Lebesgue measure zero.

**Proof:** Since $\text{rank}(Z) < n + m$ if and only if all $(n + m) \times (n + m)$ minors of $Z$ are 0,

$$V = \bigcap_{k=1}^{m+n+m} \{ (\nu_0^T \cdots \nu_{t-1}^T)^T \in \mathbb{R}^{m\ell} : [Z]_k = 0 \},$$

where $\{ [Z]_k, 1 \leq k \leq C_{t-1}^{m+n} \}$ contains all $(n + m)$-order minors of $Z$. It suffices to prove that, under the conditions of the lemma, there exists an $(n + m)$-order minor $[Z]_k$ such that the set of input expectations $\{ (\nu_0^T \cdots \nu_{t-1}^T)^T \in \mathbb{R}^{m\ell} : [Z]_k = 0 \}$ is of Lebesgue measure zero.

From the definition of $Z$, $(n + m)$-order minors of $Z$ are polynomials of $(\nu_0^T \cdots \nu_{t-1}^T)^T$ with coefficients being the entries of matrices $A^{1-B} B, \ldots, B$, as well as $\mu_0$. Thus, if we can show that $[Z]_k$ is not trivial (equal to zero almost surely), then its zero set is of Lebesgue measure zero by Lemma 1 which implies the conclusion.

It follows from the definition of controllability of $(A, B)$ that there exist $B_i$, $AB_i$, $\ldots$, $A^\ell B_i$, $\ldots$, $B_p$, $\ldots$, $A^\ell B_p$, $1 \leq i_1 < \cdots < i_p \leq m$, $0 \leq r_k \leq n - 1$ for $1 \leq k \leq p$, such that they form a basis of $\mathbb{R}^n$, where $B_i$ is the $i$-th column of $B$. We sort these $n$ vectors according to the ascending order of the power of $A$: $A^{s_0}B_{f_0(1)}$, $\ldots$, $A^{s_0}B_{f_0(m)}$, $A^{s_1}B_{f_1(1)}$, $\ldots$, $A^{s_1}B_{f_1(s_1)}$, $\ldots$, $A^{s_\ell}B_{f_\ell(s_\ell)}$, where $0 = s_0 < s_1 < s_2 < \cdots < s_\ell \leq n - 1$, $f_k(\cdot)$ is strictly increasing functions, $1 \leq q_k \leq m$, $0 \leq k \leq \ell$, $v(0 \leq \ell \leq s_\ell - 1)$ is the total number of different power of $A$ appearing in the above basis, and $\sum_{k=0}^{v_k} q_k = n$.

**V. CONCLUSIONS**

In this paper we developed and analyzed a system identification scheme for linear systems with multiplicative noise based on multiple trajectory data. By designing appropriate persistently exciting input signals, a least-squares algorithm was proposed for the joint estimation of nominal system parameters and multiplicative noise covariances. The asymptotic consistency of the algorithm was demonstrated by analyzing moment dynamics of the system, and illustrated by numerical simulations. Ongoing and future research directions include studying the asymptotic performance (convergence rate) and non-asymptotic behavior of the proposed algorithm, problems of optimal input design, identification from single-trajectory data, and sparsity-promoting regularization for identification of networked systems.
\[
Z = \begin{bmatrix}
A^{\ell-h_1} \mu_0 + \sum_{\ell-h_1}^t A^\ell \nu_{\ell-h_1} - t, & A^{\ell-h_2} \mu_0 + \sum_{\ell-h_2}^t A^\ell \nu_{\ell-h_2} - t, & \cdots & A^\ell \mu_0 + B \nu_0, \mu_0 \\
A^{\ell-h_{n+m+1}} \mu_0 + \sum_{\ell-h_{n+m+1}}^t A^\ell \nu_{\ell-h_{n+m+1}} - t, & \cdots & \cdots & \cdots
\end{bmatrix}
\]

where \( h_{n+m+1} := m + \sum_{p=0}^v (q_p - 1) + 1, \)
and \( v \) is the j-th component of \( v_i \).

As above, \( h_{m+n+1} \) is well defined because of the assumption for \( \ell: h_{m+n+1} = h_{n+m} + s_1 + 1 \leq h_{n+m} + n \leq \ell \). Now note for \( 1 \leq d \leq m \) that \( v_{1-h_d} \) in (14) no longer appears at columns on its right side in (11), which can be observed from (10).

So the absolute value of the coefficient of (14) in (11) is determined by the upper right \( n \times n \) determinant of (11), namely (12).

From observing (10), it follows that \( \nu_{\ell-h_d} \) only appears at the first \( h_d \) columns of (10), and moreover only appears at the first \( h_d - 1 \) columns of the first \( n \) rows in (10), for \( m+2 \leq d \leq m+n+1 \). Hence, \( v_{1-h_d} \) only appears once at the \( (d-m) \)-th column of (12), \( m+1 \leq d \leq m+n+1 \). Also note that the difference of \( h_{d+1} \) and \( h_d \) for \( m+1 \leq d \leq m+n \) is \( h_{d+1} - h_d = s_k + 1 \) for \( d = m + \sum_{p=0}^v q_p + 1, 1 \leq d \leq q_k, \) and \( 0 \leq k \leq v \), so the corresponding term of \( \nu_{\ell-h_{d+1}} \) in the summation at the \( h_d \)-th column of (10) is \( A^{s_k} B \nu_{\ell-h_d-s_k} = A^{s_k} B \nu_{\ell-h_{d+1}}. \)

Thus, the absolute value of the coefficient of (14) is identical to the determinant (13), from the selection of \( v \) in (14) and the fact that \( A^\ell B \nu = \sum_{j=1}^m A^\ell B_j v_{ij}, \) where \( B_j \) is the j-th column of \( B \).

Here the columns containing \( A^{s_k} \) can be selected because the assumption for \( l \) ensures that \( h_{n+m+1} = h_{n+m} + s_1 + 1 \leq \ell \). Therefore, we show that the polynomial of inputs (11) is non-zero almost everywhere, and consequently the conclusion of the theorem holds.

**Proof of Theorem 7.** The conclusion follows from Lemma 2 and the fact that the probability density function of a non-degenerate Gaussian is absolutely continuous with respect to the Lebesgue measure of corresponding dimension.
APPENDIX B: THE PROOF OF THEOREM 2

We write (3) as

\[ X_{t+1} = (A \otimes A + \Sigma_A^t)X_t + (B \otimes B + \Sigma_B^t)U_t + [(B \otimes A)W_t + (A \otimes B)W_t^\top] \]

\[ := \hat{A}X_t + \hat{B}U_t + \eta_t \]

\[ = \hat{A}^{t+1}X_0 + \sum_{k=0}^t \hat{A}^k \hat{B}U_{t-k} + \sum_{k=0}^t \hat{A}^k \eta_{t-k}, \]

so the conclusion follows from an argument essentially identical to that of Theorem 1 by noticing that \( \eta_t \), \( 0 \leq t \leq t - 1 \), are fixed, and by considering \( \hat{U}_t \) as an input.

APPENDIX C: THE PROOF OF THEOREM 3

Consider each rollout \( [(x_0^{(k)})^\top, \ldots, (x_t^{(k)})^\top]^\top \) as an independent sample of the random vector \( x_t := [x_0^\top, \ldots, x_t^\top]^\top \), and from Assumption 1(ii) and (iii) we know that the random vector \( x_t \) has finite first and second moments. So it follows from the Kolmogorov’s strong law of large numbers that \( Y \to Y \) a.s., and similarly \( Z \to Z \) a.s., as \( n_r \to \infty \). From the assumption that \( ZZ^\top \) is invertible and the continuous mapping theorem (Theorem 2.3 in [26]), it can be obtained that as \( n_r \to \infty \)

\[ \hat{Y}Z^\top(ZZ^\top)^{-1} \to YZ^\top(ZZ^\top)^{-1}, \text{ a.s.} \]

Here \((ZZ^\top)^{-1}\) exists because of assumption. Note that \( \hat{Z} \) is the average of trajectories, which depends on independent Gaussian inputs \( u_{t}^{(k)} \). So from Lemma 1 \((ZZ^\top)^{-1}\) exists with probability one. If it does not exist, then we can define it to be the zero matrix. Combining the above convergence with the Kolmogorov’s strong law of large numbers, the convergence of \( C \) and \( D \) follows. Therefore, applying the continuous mapping theorem again, we obtain the consistency of the estimator \( (\Sigma_A^t, \Sigma_B^t) \).

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