A NOTE ON THE LOCAL REGULARITY OF DISTRIBUTIONAL SOLUTIONS AND SUBSOLUTIONS OF SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. In this note we prove local regularity results for distributional solutions and subsolutions of semilinear elliptic systems such as

\[ L^m_k u_k = f_k(x, u_1, \ldots, u_N) \quad \text{in } \mathbb{R}^n \quad (k = 1, \ldots, N) \]

where \( L_1, \ldots, L_N \) are of divergence-form and \( n \geq 2m \). We show that distributional subsolutions are locally bounded from above if

\[ \frac{\text{div} f_k(x, z)}{\text{div} z} \leq C (1 + |z|^p) \quad \text{for } 1 \leq p < \frac{n}{n-2m}, k = 1, \ldots, N. \]

Furthermore, regularity properties of subsolutions and improved versions for bounded subsolutions are given. Even for \( f_1 = \ldots = f_N = 0 \) our results are new.

1. Introduction

The starting point of this paper is the following question: Being given a semilinear system of elliptic partial equations of the form

\[ L^m_k u_k = f_k(x, u_1, \ldots, u_N) \quad \text{in } \Omega \quad (k = 1, \ldots, N) \]

on some open set \( \Omega \subset \mathbb{R}^n \) what is the maximal regularity of an arbitrary distributional subsolution or solution of (1)? More specifically we are interested in assumptions on divergence-form operators \( L_1, \ldots, L_N \) of second order and nonlinearities \( f_1, \ldots, f_N \) which ensure that distributional solutions or subsolutions (see (3),(4) for a definition) are locally bounded or bounded from above, respectively.

The study of unbounded weak and distributional solutions was initiated about 50 years ago and it gave rise to several interesting methods and results. Let us try to give a short overview of the subject with a focus on second order equations (i.e. \( m = 1, N = 1 \)) such as

\[ -\text{div}(A \nabla u) = f(x, u) \quad \text{in } \Omega \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n, n \geq 3 \). We start with unbounded solutions that owe their existence to the roughness of the matrix function \( A \) which we will always assume to be positive definite and bounded on \( \Omega \). In a fundamental paper Serrin [17] provided explicit examples for matrix functions \( A \) and unbounded solutions of (2) for \( f \equiv 0 \) such that the solutions lie in Sobolev spaces \( W^{1,s}_{\text{loc}}(\Omega) \) with \( s < 2 \). Given the fact that local regularity results of De Giorgi-Nash-Moser type (see chapter 8 in [17]) are valid for weak solutions lying in \( W^{1,2}_{\text{loc}}(\Omega) \) we see that different a priori assumptions on the regularity of the solution may lead to different kinds of solutions. In the works of Brezis [1] and Jin, Maz’ya, van Schaftingen [10]...
this issue was analyzed further. For instance, they proved that if the entries of \( A \) are only assumed to be continuous then solutions lying in \( W^{1,p}_{loc}(\Omega) \), \( p > 1 \) lie in \( W^{1,q}_{loc}(\Omega) \) for all \( q \in [1,\infty) \).

In particular these solutions are always bounded (in contrast to their gradients, see [10]). Under the assumption of Hölder-continuous coefficients the latter result had previously been obtained by Hager and Ross [9]. For weak solutions in \( W^{1,1}_{loc}(\Omega) \) Brezis proved the \( W^{1,q}_{loc}(\Omega) \)-regularity for matrix functions \( A \) with Dini-continuous entries, see Theorem 2 in [4].

All these results concern weak solutions whereas Bao and Zhang [2, 3] studied regularity results for distributional solutions with Lipschitz continuous \( A \). They showed that in this case distributional solutions of \( -\text{div}(A\nabla u) = f \) have the ”typical” regularity properties of elliptic problems and no pathological solutions like the ones mentioned above can exist.

Another way of producing unbounded solutions of (2) is to consider nonlinearities \( f(x,z) \) that grow sufficiently fast to infinity as \( z \) tends to infinity. This can be illustrated via the model equation \( -\Delta u = u^p \) on a domain \( \Omega \subset \mathbb{R}^n \). For \( p > \frac{n}{n-2} \) unbounded distributional solutions are given by the explicit formula \( \hat{u}(x) = c_n,p |x-x_0|^{-2/(p-1)} \) for some \( c_n,p > 0, x_0 \in \Omega \). More sophisticated unbounded distributional solutions of this equation in suitable domains \( \Omega \subset \mathbb{R}^n \) with appropriate boundary conditions are due to Pacard [15, 16] and Mazzeo, Pacard [13] for exponents \( p \geq \frac{n}{n-2} \). Finally let us mention that similar constructions were performed in [12] in the context of nonlinear Schrödinger equations on \( \mathbb{R}^n \) for \( p \) slightly larger than \( \frac{n}{n-2} \).

Let us now describe in which way our results contribute to the issue. We deal with subsolutions (Theorem 1) and solutions (Theorem 2) of the \( 2m \)-th order semilinear elliptic system (1) on an open set \( \Omega \subset \mathbb{R}^n \). We restrict our attention to the case \( n \geq 2m \) which, from the point of view of regularity theory, is more interesting. We find regularity properties of subsolutions which will be shown to be optimal in a general setting. A new feature of our approach is that these results can however be improved once we add some integrability assumption on the negative parts of the subsolutions. Furthermore, even in the easiest case of linear problems of second order equations (\( m = 1, N = 1, f = 0 \)) our results are new since the involved linear divergence-form differential operators \( L_1, \ldots, L_N \) may have Lipschitz continuous but also less regular coefficient functions, see assumption (A1)_a below. In particular we can treat more general situations than in \( 2, 3 \) where distributional solutions of \( -\text{div}(A\nabla u) = f \) were investigated under the assumption that \( A \) is locally Lipschitz. Our assumptions on the coefficient functions will be shown to be sharp in the sense that for slightly less regular coefficients our regularity results cannot hold any more in view of the pathological solutions found by Jin, Maz’ya, van Schaftingen [10] and Serrin [17]. In Remark 1 (c) this aspect will be explained in detail.

Before coming to the statement of our main result let us provide the definitions of distributional solutions and subsolutions of (1). To this end we introduce a class of differential operators which is suitable for the definition of distributional solutions lying in \( L^\infty_{loc}(\Omega; \mathbb{R}^N) \). We will always assume that the following hypothesis is satisfied:
Theorem 1. Let $\Omega \subset \mathbb{R}^n$ be open, $n \geq 2m$ and assume $u \in L^\alpha_{loc}(\Omega; \mathbb{R}^N)$ solves
\[ L^m_k u_k \leq C(g + u_1^p + \ldots + u_N^p) \quad \text{in } \Omega \quad (k = 1, \ldots, N) \]
in the distributional sense with $u^* \in L^p_{loc}(\Omega; \mathbb{R}^N)$ and where $L_1, \ldots, L_N$ satisfy (A1)$_\alpha$ and $g \in L^r_{loc}(\Omega), r > \frac{n}{2m}$. Furthermore assume $1 \leq p < \frac{n}{n-2m}$ or $p \geq \frac{n}{n-2m}, u^* \in L^p_{loc}(\Omega; \mathbb{R}^N)$ for some $q > \frac{n(p-1)}{2m}$. Then we have $u^* \in L^\infty_{loc}(\Omega; \mathbb{R}^N)$ and $u \in W^{2m-1,t}_{loc}(\Omega; \mathbb{R}^N)$ for all $t \in [1, \frac{n}{n-1})$. In addition, the following implications hold true:

(i) If $n > 2m$ and $u_\ast \in L^{q}_{loc}(\Omega; \mathbb{R}^N), \tilde{q} > \frac{n}{n-2m}$, then $u \in W^{2m-1,\tilde{t}}_{loc}(\Omega; \mathbb{R}^N), 1 \leq \tilde{t} < \frac{2m\tilde{q}}{1+(2m-1)\tilde{q}}$.

(ii) If $n = 2m$ and $u_\ast \in L^{\infty}_{loc}(\Omega; \mathbb{R}^N)$, then $u \in W^{2m-1,\frac{2m}{2m-1}}_{loc}(\Omega; \mathbb{R}^N)$.

Here we used the notation $u^* := (u^*_1, \ldots, u^*_N)$ to denote the vector of the positive/negative parts of the component functions of $u$. The proof of Theorem 1 is based on a representation formula for subsolutions and a well-known bootstrap procedure that seems to go back to Stampacchia in the case $m = 1$. In the following remark we discuss extensions of Theorem 1 and why it cannot be essentially improved.

Remark 1.

(a) The iteration scheme from the proof may be slightly modified to prove local boundedness results for problems of the kind
\[ L^m_k u_k \leq C(g_1 + g_2 u_1^p + \ldots + g_2 u_N^p) \quad \text{in } \Omega \quad (k = 1, \ldots, N) \]
where $g_1 \in L^{r_1}_{loc}(\Omega), r_1 > \frac{n}{2m}$ and $g_2 \in L^{r_2}_{loc}(\Omega), r_2 > 1$. In this situation the critical exponent for local boundedness $\frac{n}{n-2m}$ changes to $\frac{n}{n-2m}(1 - \frac{1}{r_2})$. Also the differential operators $L^m_k = L_k \circ \ldots \circ L_k$ may be replaced by compositions of $m$ different divergence-form operators as in (A1)$_\alpha$ without changing the result.
(b) The restrictions on \( p, q \) from the theorem are optimal in view of several results for the model equation \(-\Delta u = u^p\). Unbounded distributional solutions in the case \( p = \frac{n}{n-2} \) were found by Pacard [10] and in the case \( p > \frac{n}{n-2} \) the function \( \tilde{u} \) from the introduction may be taken. Note that \( \tilde{u} \in L^q_{\text{loc}}(\mathbb{R}^n) \) only for \( 1 \leq q < \frac{n(p-1)}{2} \).

(c) The condition \( \alpha \in \left[1, \frac{n}{n-1}\right] \) from assumption \((A_1)_\alpha \) is sharp in the sense that Theorem [1] is not true for \( \alpha \in \left(\frac{n}{n-1}, \infty\right) \). Indeed, in [10] Proposition 1.2 the authors constructed a cotinuous matrix function \( A \) and a weak (and hence distributional) solution \( u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \subset L^{\frac{n}{n-1}}_{\text{loc}}(\mathbb{R}^n) \) of the equation \( \text{div}(A\nabla u) = 0 \) such that \( u \notin W^{1,p}_{\text{loc}}(\mathbb{R}^n) \) for any other \( p > 1 \). Having a look at the proof of this result (and in particular Lemma 2.1 and equation (8) in [10]) one realizes that the coefficient functions of \( A \) lie in \( W^{1,n}_{\text{loc}}(\mathbb{R}^n) \). Hence, this example shows that our theorem does not hold in the case \( \alpha = \frac{n}{n-1} \). For \( \alpha \in \left(\frac{n}{n-1}, \frac{n}{n-2}\right) \) Theorem [1] can not hold either since Serrin’s pathological solutions from [17] provide counterexamples. In the case \( \alpha \in \left(\frac{n}{n-2}, \infty\right) \) the nonlinearity allows us to take the solutions from (b) as counterexamples. It is unclear to the author, however, whether the linear problem \( \text{div}(A\nabla u) = 0 \) admits unbounded distributional solutions for such \( \alpha \). Finally, we think that the case \( \alpha = \infty \) has not been treated yet.

(d) Also the results from the parts are (i),(ii) are close to optimal. First of all one should notice that regularity results in Sobolev spaces of order 2m or higher can in general not hold since the fundamental solution of \((-\Delta)^m\) does not have such weak derivatives. Moreover, the functions \( u_\delta(x) := \sigma(x^2 + \ldots + x_{2m}^2)^{\delta/2} \) for small \( \delta > 0 \) and appropriate choice of \( \sigma \in (-1,1) \) defined bounded poly-is-subharmonic functions which do not lie in \( W^{2m-1,2m/(2m-1-\delta)}_{\text{loc}}(\mathbb{R}^n) \) so that our result for \( \tilde{q} = \infty \) may be considered as sharp. More generally, for suitable \( \sigma \in \{-1,1\} \) the functions \( x \mapsto \sigma(x^2 + \ldots + x_{2m}^2)^{\delta/2} \) define polysubharmonic functions in \( L^{\tilde{q}}_{\text{loc}}(\mathbb{R}^n) \), \( \frac{n}{n-2m} < \tilde{q} < \infty \) provided \( \delta > \max\{2m-k,-\frac{k}{q}\} \). These functions do not lie in \( W^{2m-1,k/(2m-1-\delta)}_{\text{loc}}(\mathbb{R}^n) \) and for certain \( \tilde{q} \) the exponent \( \frac{k}{2m-1-\delta} \) can be very close to \( \frac{2m\tilde{q}}{\tilde{q}+1} \). More precisely, if we could define these functions for all \( k := \frac{2m\tilde{q}}{\tilde{q}+1} \in (2m,n) \) (which, in general, is not a natural number) then we would obtain the optimality of the exponent. Unfortunately, we have to leave open whether non-formal examples exist or not.

Theorem [1] admits a refined version for distributional solutions of (11) which we will formulate below for the sake of completeness. It generalizes the results of Bao and Zhang [2,3] to nonlinear higher order problems and complements the existence and regularity results of Jin, Maz’ya, van Schaftingen [10] and Brezis [4] in the sense of Remark [1] (c). The assumptions on the right hand side \( f \) are the following:

(A2) \( f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a Carathéodory function satisfying

\[
|f(x, z)| \leq C(g(x) + |z|^p) \quad \text{on} \quad \Omega \times \mathbb{R}^N \quad \text{where} \quad g \in L^r_{\text{loc}}(\Omega), r \in (1, \infty).
\]

Under the assumptions (A1)_\alpha, (A2) we show that in the case \( 1 \leq p < \frac{n}{n-2m} \) every distributional solution \( u \in L^p_{\text{loc}}(\Omega; \mathbb{R}^N) \cap L^p_{\text{loc}}(\Omega; \mathbb{R}^N) \) is a bounded weak solution so that classical elliptic regularity results as in [7] are applicable. We will omit the proof since it results from discussing special cases in the proof of Theorem [2] see Remark [2] (b).
Theorem 2. Let $\Omega \subset \mathbb{R}^n$ be open, $n \geq 2m$ and let $u \in L^p_{\text{loc}}(\Omega; \mathbb{R}^N) \cap L^p_{\text{loc}}(\Omega; \mathbb{R}^N)$ solve

$$L^p_{\text{loc}} u_k = f_k(x, u_1, \ldots, u_N) \quad \text{in } \Omega \quad (k = 1, \ldots, N)$$

in the distributional sense where $L_k, f_k$ satisfy (A1), (A2). Moreover assume $1 \leq p < \frac{n}{n-2m} \quad \text{or} \quad p \geq \frac{n}{n-2m}, u \in L^q_{\text{loc}}(\Omega; \mathbb{R}^N)$ for some $q > \frac{n(p-1)}{2m}$. Then we have $u \in W^{2m,r}_{\text{loc}}(\Omega; \mathbb{R}^N)$. Moreover, $A \in C^\infty(\Omega; \mathbb{R}^{n \times n}), f \in C^\infty(\Omega \times \mathbb{R}^N; \mathbb{R}^N)$ implies $u \in C^\infty(\Omega; \mathbb{R}^N)$.

As a consequence the assumptions of the theorem guarantee that unbounded distributional solutions of (5) can only exist for $n > 2m, p \geq \frac{n}{n-2m}$ and that these solutions have to be searched in subspaces of $L^q_{\text{loc}}(\Omega)$ with $p \leq q \leq \frac{n(p-1)}{2m}$ while distributional solutions lying in higher Lebesgue spaces are automatically locally bounded.

In Section 2 we provide an auxiliary Lemma needed for the proof of the parts (i),(ii) of Theorem 1. In Section 3 we prove Theorem 1 using some results concerning Green’s functions which we provide in the Appendix.

2. An auxiliary lemma

For a given ball $B \subset \mathbb{R}^n$, $\gamma < 0$ and a Radon measure $\mu$ on $B$ with $\mu(B) < \infty$ we introduce the measurable functions $\phi_\gamma, \psi : B \to [0, \infty]$ as follows:

$$\phi_\gamma(x) := \int_B |x-y|^\gamma d\mu(y), \quad \psi(x) := \int_B \log(1/|x-y|) d\mu(y).$$

By definition, a Radon measure on $B$ is a Borel-regular measure on $B$ which is locally finite. In the proof of Theorem 1 we will need the following result.

Lemma 1. Let $n, k \in \mathbb{N}, \gamma, p, q \in \mathbb{R}$ satisfy $0 > \gamma > k - n, q > p \geq 1$ and let $\mu, B$ be given as above. Then the following implications hold true:

(i) If $\phi_\gamma \in L^q_{\text{loc}}(B)$ then $\phi_{\gamma-k} \in L^p_{\text{loc}}(B)$ provided $1 \leq p < \frac{(n+k)q}{n+\gamma+(q-1)k}$.

(ii) If $\psi \in L^p_{\text{loc}}(B)$ then $\phi_{-k} \in L^q_{\text{loc}}(B)$ provided $n > 2k$.

Proof. In order to prove part (i) we set $r := \frac{p(q-1)}{q(p-1)}$ so that the following is true:

$$\frac{qr}{p} > r > 1, \quad \gamma - \frac{kr}{r-1} > -n.$$

Using these inequalities and Hölder’s inequality we obtain for every $\tilde{B} \subset \subset B$

$$\int_{\tilde{B}} |\phi_{\gamma-k}(x)|^p \, dx = \int_{\tilde{B}} \left( \int_{\tilde{B}} |x-y|^{r} |x-y|^{\frac{q(p-1)}{p} - k} d\mu(y) \right)^p d\mu(y) \leq \int_{\tilde{B}} \left( \int_{\tilde{B}} |x-y|^\gamma d\mu(y) \right)^\frac{p}{r} \left( \int_{\tilde{B}} |x-y|^\gamma d\mu(y) \right)^\frac{q(p-1)}{q(p-1)-r} d\mu(y)$$

$$= \int_{\tilde{B}} \phi_\gamma(x)^{\frac{p}{r}} \left( \int_{\tilde{B}} |x-y|^\gamma d\mu(y) \right)^\frac{q(p-1)}{q(p-1)-r} d\mu(y)$$

$$\leq \int_{B} \left( \int_{B} |x-y|^{r} |x-y|^{\frac{q(p-1)}{p} - k} d\mu(y) \right)^p d\mu(y) \leq \int_{B} \phi_\gamma(x)^{\frac{p}{r}} \left( \int_{B} |x-y|^\gamma d\mu(y) \right)^\frac{q(p-1)}{q(p-1)-r} d\mu(y)$$
we will use the hypothesis improved in an essential way. This condition is satisfied if and only if
\[ h \gamma^N \text{ for some } C \]

**Proof.**

Lipschitz continuous matrix functions in the case \( m \) the regularity of subsolutions of linear equations. We use the following result which, for \( \phi \) Proposition 1 (Linear regularity I). Let \( h \in L^p_{\text{loc}}(\Omega) \) satisfy \( L^m h = 0 \) in \( \Omega \) in the distributional sense. Then \( h \in W^{2m,t}_{\text{loc}}(\Omega) \) for all \( t \in [1, \infty) \). In particular \( h \in L^\infty_{\text{loc}}(\Omega) \).

**Proof.** First we consider the case \( m = 1 \). So let \( \beta = \frac{\alpha \alpha}{\alpha - \alpha} > \alpha \) and our first aim is to show \( h \in L^2_{\text{loc}}(\Omega) \). To this end we proceed as in [2] and take an arbitrary function \( w \in C^\infty(\Omega) \), let \( B \subset \subset \Omega \) be a ball and let \( \phi \) be the uniquely determined function satisfying

\[
- \text{div}(A\nabla \phi) = w \quad \text{ in } B, \quad \phi \in W^{1, \frac{\alpha}{\alpha - \alpha}}(B) \cap W^{2, \frac{\beta}{\beta - \beta}}(B),
\]

see Theorem 9.15 in [7]. By Theorem 9.19 in [7] the function \( \phi \) is twice continuously differentiable with Hölder-continuous second order derivatives and the differential equation is satisfied almost everywhere. Moreover, we have

\[
\| \phi \|_{W^{1, \frac{\alpha}{\alpha - \alpha}}(B)} \leq C \| \phi \|_{W^{2, \frac{\beta}{\beta - \beta}}(B)} \leq C' \| w \|_{L^{\frac{\beta}{\beta - \beta}}(B)}
\]
for some $C, C' > 0$ by Sobolev’s embedding theorem and Lemma 9.17 [7]. Therefore we may
test the equation with $\eta \phi$ for some cut-off function $\eta \in C^\infty_c(B)$ and obtain

$$0 = \int_B h(-\text{div}(A\nabla(\eta \phi)))$$

$$= \int_B h(-\text{div}(A\nabla \phi))\eta - 2\nabla \phi^T A\nabla \eta - \text{div}(A\nabla \eta)\phi$$

$$\geq \int_B h \text{w} \eta - C \|h\|_{L^\infty(B)} \left(\|A\|_{L^\infty(B)} \|\nabla \phi\|_{L^\infty(B)} + \|A\|_{W^{1, \frac{n}{n-1}}(B)} \|\phi\|_{L^\infty(B)}\right)$$

$$\geq \int_B h \text{w} \eta - C'' \|h\|_{L^\infty(B)} \|A\|_{W^{1, \frac{n}{n-1}}(B)} \|\phi\|_{W^{1, \frac{n}{n-1}}(B)}$$

$$\geq \int_B h \text{w} \eta - C'' \|h\|_{L^\infty(B)} \|A\|_{W^{1, \frac{n}{n-1}}(B)} \|w\|_{L^\frac{n}{n-1}(B)}$$

by definition of $\beta$ for some $C, C', C'' > 0$ independent of $w$. As in [2] the dual characterization
of Lebesgue spaces gives $h \in L^\beta_{\text{loc}}(B)$ and iterating this procedure as in [2] yields $h \in L^1_{\text{loc}}(B)$
for all $t < \infty$.

In order to prove the existence of weak derivatives we can not proceed as in [2] since
there a difference quotient method is used which relies on the Lipschitz continuity of the
matrix function. Instead we continue with the duality argument. To this end we consider
$w_0, w_1, \ldots, w_n \in C^\infty(B)$ and choose $\phi$ to be the uniquely determined function satisfying

$$(8) \quad -\text{div}(A\nabla \phi) = w_0 + \sum_{j=1}^n \partial_j w_j \quad \text{in } B, \quad \phi \in W^{1,2}_0(B) \cap W^{2,2}(B).$$

As above, $\phi$ is twice continuously differentiable with Hölder-continuous second order derivatives and the differential equation is satisfied almost everywhere. Instead of Lemma 9.17 in [7] we use the estimate

$$\|\phi\|_{W^{1, \frac{n}{n-1}}(B)} \leq C \sum_{j=0}^n \|w_j\|_{L^\frac{n}{n-1}(B)}$$

from Lemma 2.2 (i) [5] for $t > \frac{n}{n-1} > \alpha$. Proceeding as above we find $C, C', C'' > 0$ such that

$$0 = \int_B h(-\text{div}(A\nabla(\eta \phi)))$$

$$\geq \int_B h \left( w_0 + \sum_{j=1}^n \partial_j w_j \right) \eta - C \|h\|_{L^\infty(B)} \left(\|A\|_{L^\infty(B)} \|\nabla \phi\|_{L^\infty(B)} + \|A\|_{W^{1, \frac{n}{n-1}}(B)} \|\phi\|_{W^{1, \frac{n}{n-1}}(B)}\right)$$

$$\geq \int_B h \left( w_0 + \sum_{j=1}^n \partial_j w_j \right) \eta - C'' \|h\|_{L^\infty(B)} \|A\|_{W^{1, \frac{n}{n-1}}(B)} \|\phi\|_{W^{1, \frac{n}{n-1}}(B)}$$

$$\geq \int_B h \left( w_0 + \sum_{j=1}^n \partial_j w_j \right) \eta - C'' \|h\|_{L^\infty(B)} \|A\|_{W^{1, \frac{n}{n-1}}(B)} \sum_{j=0}^n \|w_j\|_{L^\frac{n}{n-1}(B)}.$$
In the case $m \geq 2$ instead of \( \mathfrak{N} \) (i.e. $L^m \phi = w$ in $B$) one solves $L^m \phi = w$ in $B$ in some subspace of $W^{2m, \beta/((\beta-1))}(B)$ which can be done by induction over $m$ using the same theorems as above. Similar estimates then allow to conclude as in the case $m = 1$. \hfill \Box

With Proposition \[\mathbb{I}\] at hand, we may now discuss the regularity properties of subsolutions of linear problems. One main feature of our result is that integrability assumptions on the negative parts of subsolutions can be used to deduce slightly better regularity properties.

**Proposition 2** (Linear regularity II). Let $u \in L^\alpha_{\text{loc}}(\Omega)$ satisfy $L^m u \leq g$ in $\Omega$ in the distributional sense where $g \in L^r_{\text{loc}}(\Omega), g \geq 0$. Then the following implications hold true:

(i) If $r = 1$ then $u^+ \in L^s_{\text{loc}}(\Omega)$ for all $s \in \left[1, \frac{n}{n-2m}\right)$,

(ii) If $r \in \left[1, \frac{n}{2m}\right)$ then $u^+ \in L^{n r/(n-2 m r)}_{\text{loc}}(\Omega)$,

(iii) If $r > \frac{n}{2 m}$ then $u^+ \in L^\infty_{\text{loc}}(\Omega)$.

In each of these cases one has $u \in W^{2m-1,t}_{\text{loc}}(\Omega)$ for all $t \in \left(1, \frac{n}{n-2m}\right)$. Moreover, assuming $u^+ \in L^\tilde{q}_{\text{loc}}(\Omega)$ and $r \geq \frac{2m q}{2m q + q(n-2m)}$ the following implications hold true:

(iv) If $\tilde{q} > \frac{n}{n-2m}, n > 2m$ then $u \in W^{2m-1,t}_{\text{loc}}(\Omega)$ for all $t \in \left(1, \frac{2m q}{1+2m-1}\right)$,

(v) If $\tilde{q} = \infty, n = 2m$ then $u \in W^{2m-1,2m/(2m-1)}_{\text{loc}}(\Omega)$.

**Proof.** It suffices to verify the above-mentioned regularity properties in an arbitrary compactly contained ball $B \subset \subset \Omega$. For any given such ball let $G_1$ be the Green’s function of the operator $L$ on a slightly larger ball $B'$ associated to homogeneous Dirichlet boundary conditions on $\partial B'$. Under our regularity assumption $(A1)\alpha$ the existence of $G_1$ is guaranteed by Theorem 1.1 in \[\mathfrak{S} \]. We define the function $G_m : B' \times B' \to [0, \infty)$ inductively via

\[ G_k(x, y) = \int_{B} G(x, z) G_{k-1}(z, y) dy \quad (k = 2, \ldots, m). \]

Then one has $L^m G_m(\cdot, y) = \delta(\cdot - y)$ in $B$ in the distributional sense as well as

\[ |\partial^\alpha G_m(x, y)| \leq C|y|^{2m-n-|\alpha|} \quad \text{for } |\alpha| = 1, \ldots, 2m-1 \quad (x, y \in B), \]

\[ G_m(x, y) \leq \begin{cases} C|x - y|^{2m-n}, & \text{if } n > 2m, \\ C \log(1/|x - y|) + C', & \text{if } n = 2m \end{cases} \quad \text{for } (x, y \in B). \]

\[ G_m(x, y) \geq \begin{cases} c|x - y|^{2m-n}, & \text{if } n > 2m \\ c \log(1/|x - y|) - c', & \text{if } n = 2m \end{cases} \quad \text{for } (x, y \in B). \]

Here, $c, c', C, C'$ are positive numbers independent of $x, y \in B$, $\alpha \in \mathbb{N}_0^n$ is a multiindex and $\delta$ is the Dirac measure on $\mathbb{R}^n$ centered at 0. In the Appendix we provide the references for these estimates.

**Step 1: A representation formula.** The function $v_B := -u|_B + \int_B G_m(\cdot, y) g(y) dy$ satisfies $L^m v \geq 0$ in $B$ in the distributional sense. Using Theorem 2.17 in \[\mathbb{L} \] we obtain $L^m v_B = \mu_B$ in $B$ where $\mu_B$ is a Radon measure. Defining $h_B := -v_B + \int_B G_m(\cdot, y) d\mu_B(y)$ we arrive at

\[
 u|_B = -\int_B G_m(\cdot, y) d\mu_B(y) + \int_B G_m(\cdot, y) g(y) dy + h_B \quad \text{where } L^m h_B = 0, h_B \in L^1_{\text{loc}}(B).
\]
Moreover, \( \mu_B = (L^m u + g)|_B = ((L^m u + g)|_{\mathcal{B}^B})|_B \) implies \( \mu_B(B) < \infty \) since \( \mu'_B \) is a Radon measure and thus locally finite. This property will be used when we apply Lemma \( \text{[1]} \) in Step 3.

**Step 2: Proof of (i),(ii),(iii) – Integrability.** By the Hardy-Littlewood-Sobolev Theorem (see [1], Theorem 4.3) \( u_2 \) has the integrability properties which we claimed to hold for \( u \) in (i),(ii),(iii), respectively. By Proposition \( \text{[1]} \) we moreover have \( h_B \in L^\infty_{\text{loc}}(B) \). Hence, the inequality \( u|_B^2 \leq u_2 + |h_B| \) implies that \( u^+ \) lies in the same Lebesgue spaces as \( u_2 \) which is what we wanted to show.

**Step 3: Proof of (iv),(v) – Regularity.** From (10) we get \( u_1, u_2 \in W^{2m-1,t}(B) \) for all \( t \in \left[ 1, \frac{m}{m-1} \right] \). Hence, Proposition \( \text{[1]} \) tells us that \( u = -u_1 + u_2 + h_B \) lies in the same spaces. Now let us additionally assume \( u^- \in L^{\tilde{q}}(B), g \in L^r_{\text{loc}}(B) \) for \( \tilde{q}, r \) as in the statement of the theorem. The assumption on \( r \), the upper bounds for the derivatives of \( G_m \) from (10) and the Hardy-Littlewood-Sobolev Theorem imply

\[
(12) \quad u_2 \in W^{2m-1, \frac{m}{m-1}}_{\text{loc}}(B) \subset W^{2m-1, \frac{2m\tilde{q}}{2m-1\tilde{q}}}_{\text{loc}}(B).
\]

Furthermore, the assumption \( u^- \in L^{\tilde{q}}_{\text{loc}}(\Omega) \) implies \( u_1 + |h_B| \in L^{\tilde{q}}(B) \) via (11) and thus \( u_1 \in L^{\tilde{q}}_{\text{loc}}(B) \) by Proposition \( \text{[1]} \). In the case \( n > 2m \) this implies \( \phi_{2m-n} \in L^{\tilde{q}}_{\text{loc}}(B) \) where \( \phi_{2m-n} \) was defined in (3). Indeed, using the lower bound for \( G_m \) from (10) we get

\[
u_1(x) = \int_B G_m(x,y) \, d\mu_B(y) \geq c \int_B |x-y|^{2m-n} \, d\mu_B(y) = c \phi_{2m-n}(x).
\]

From Lemma \( \text{[1]} \) (i) we obtain \( \phi_{1-n} \in L^p(B) \) for \( p \in \left[ 1, \frac{2m\tilde{q}}{1+(2m-1)\tilde{q}} \right] \) so that the upper bound for the derivatives of \( G_m \) from (10) imply

\[
(13) \quad u_1 \in W^{2m-1,p}_{\text{loc}}(B) \quad \text{if } p \in \left[ 1, \frac{2m\tilde{q}}{1+(2m-1)\tilde{q}} \right], \quad n > 2m.
\]

Similarly, in the case \( n = 2m \) the assumption \( u^- \in L^\infty_{\text{loc}}(\Omega) \) implies \( u_1, \psi \in L^\infty(\Omega) \) so that Lemma \( \text{[1]} \) (ii) yields \( \phi_{1-n} = \phi_{1-2m} \in L^{2m/(2m-1)}_{\text{loc}}(B) \) and thus

\[
(14) \quad u_1 \in W^{2m-1, \frac{2m}{2m-1}}_{\text{loc}}(B) \quad \text{if } \tilde{q} = \infty, \quad n = 2m.
\]

From (12),(13),(14) and Proposition \( \text{[1]} \) we obtain the assertion of the theorem. \( \square \)

**Remark 2.**

(a) Every (nonnegative) Radon measure \( \mu \) on \( B \) defines a distributional subsolution via the formula (11).

(b) In the more special case \( L^m u = g \) we have \( d\mu_B(y) = g^-(y) \, dy \) which improves the regularity properties of \( u_1 \) according to the integrability properties of \( g^- \). This is responsible for the fact that solutions of \( L^m u = g \) with \( g \in L^r_{\text{loc}}(\Omega) \), \( r > 1 \) are more regular than subsolutions. For elliptic equations with measure-valued right hand sides (such as subsolutions) this is in general not true.

**Proof of Theorem 1**
Applying Proposition $[2]$ to $g := C(1 + u^p)$ we find that it is sufficient to prove $u^* \in L^\infty_{\text{loc}}(\Omega)$. In case $1 \leq p < \frac{n}{n-2m}$ the proposition yields $u^* \in L^{q_0}_{\text{loc}}(\Omega)$ and thus $g \in L^{q_0/p}_{\text{loc}}(\Omega)$ for some $q_0 \in (\frac{n}{n-2m}, \frac{n}{n-2m})$. Using again Proposition $[2]$ one inductively proves $u^* \in L^{q_k}_{\text{loc}}(\Omega)$ where

$$q_{k+1} := \begin{cases} \frac{nq_k}{np-2mq_k}, & \text{if } 1 < \frac{q_k}{p} < \frac{n}{2m}, \\ 2q_k, & \text{if } \frac{q_k}{p} = \frac{n}{2m}, \\ \infty, & \text{if } \frac{q_k}{p} > \frac{n}{2m}. \end{cases}$$

The sequence $(q_k)$ increases due to $q_0 > \frac{n(p-1)}{2m}$ (because of $q_0 > p$) and reaches $+\infty$ after finitely many steps so that the assertion is proved in the case $1 \leq p < \frac{n}{n-2m}$. In the case $n > 2m, p \geq \frac{n}{n-2m}$ the assumption $u^* \in L^q_{\text{loc}}(\Omega), q > \frac{n(p-1)}{2m}$ leads to the choice $q_0 := q$ and the same iteration as above gives the result. \hfill $\Box$

**Appendix – On Green’s function**

Let us briefly recall why the estimates from $[10]$ hold if the assumption $(A_1)_\alpha$ is satisfied. First we note that due to this assumption the $(2m−2)$th order derivatives of the matrix functions $A_1, \ldots, A_N$ from the theorem are Hölder-continuous by Sobolev’s embedding theorem so that, generally speaking, $L^p$-estimates and Schauder estimates for elliptic problems in divergence-form are applicable. In the following let $A$ be one of these matrices and let $G_m$ be the function defined in $[9]$.

We first analyze the properties of $G_1$ and we start with the exceptional case $n = 2$. The logarithmic bounds for $G_1$ are proved in section 6 in $[5]$. Notice that for the lower bounds one uses that $B$ is compactly contained in $B'$. For the upper bounds of the derivatives let us recall from $[6]$ and estimate (2) in $[5]$ that the following interior estimates hold for any weak solution $w$ of $\text{div}(Aw) = 0$ in $B'$:

$$\|\nabla^k w\|_{L^\infty(B_{\delta})} \leq Cd^{-k}\|\nabla w\|_{L^2(B_{2\delta})} \quad (1 \leq k \leq 2m - 1, n = 2) \tag{15}$$

Here, $B_{2\delta}$ is a ball of radius $2\delta$ contained in $B'$ and $L^2(B_{2\delta})$ denotes the Lorentz space (or weak-$L^2$ space). We refer the interested reader to the paper of Dolzmann and Müller $[5]$ where these spaces are used in the context of Green’s functions. From estimate (11) in $[9]$ we get $\|\nabla G_1(\cdot, y)\|_{L^2 \cap (B')} \leq C$ so that the interior estimates (15) applied to $G_1(\cdot, y)$ on the ball $B_{2\delta}(x)$ with $d = \frac{1}{4}\min\{|x-y|, \text{dist}(B, \partial B')\}$ yield

$$|\partial^\alpha_x G_1(x, y)| \leq C|x-y|^{2-n-|\alpha|} \quad (x, y \in B, 1 \leq |\alpha| \leq 2m - 1, n = 2),$$

see also $[5]$, Lemma 3 where the same reasoning was used. This proves the estimates for $G_1$ from $[10]$ for $n = 2$.

In the case $n \geq 3$ the bounds for $G_1(x, y)$ follow from $[9]$, Theorem 1.1. The bounds for the derivatives of $G_1(\cdot, y)$ follow from interior estimates in the same manner as above. Instead of (15), however, one uses

$$\|\nabla^k w\|_{L^\infty(B_{\delta})} \leq Cd^{-k}\|w\|_{L^\infty(B_{2\delta})} \quad (k \leq 2m - 1, n \geq 3).$$
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This Schauder type estimate follows inductively from [7], Corollary 6.3, see also Lemma 3.1 in [8] for the special case \( k = 1 \). Hence we obtain

\[ |\partial^\alpha_x G_1(x, y)| \leq C|x - y|^{2-n-|\alpha|} \quad (x, y \in B, \ |\alpha| \leq 2m - 1, \ n \geq 3) \]

so that the estimates for \( G_1 \) in the case \( n \geq 3 \) are proved, too.

Finally, using the estimates for \( G_1 \) one can inductively derive the corresponding estimates for \( G_m \) via the formula (9).

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