ERGODIC THEORY FOR RIEMANN SURFACE LAMINATIONS: A SURVEY

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Dedicated to Kang-Tae Kim for his sixtieth birthday

Abstract. We survey some recent developments in the ergodic theory for hyperbolic Riemann surface laminations. The emphasis is on singular holomorphic foliations. These results not only illustrate the strong similarity between the ergodic theory of maps and that of Riemann surface laminations, but also indicate the fundamental differences between these two theories.

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1. Introduction

These notes are based on a series of lectures given by the author at KIAS and at the KSCV Symposium 11 in Gyeongju in 2016. The purpose is to review some developments in the ergodic theory of laminations by hyperbolic Riemann surfaces. In particular, we focus on the ergodic theory of singular holomorphic foliations. The emphasis is on recent results, but we also include some classical ones for the sake of completeness and historical perspective.

There is a well-known connection between Riemann surface laminations and the dynamics of iterations of continuous maps. In the meromorphic context, this becomes a link between singular holomorphic foliations by Riemann surfaces in dimension $k \geq 2$ and the dynamics of iterations of meromorphic maps in dimension $k-1$. On the one side, the abstract ergodic theory of maps has reached maturity with remarkable achievements like the Oseledec-Pesin theory. The ergodic theory of meromorphic maps is, however, much less developed. Indeed, it has only been studied intensively during the last three decades, see the survey of Dinh-Sibony [20]. On the other side, the ergodic theory for hyperbolic Riemann surface laminations, and in particular, for the subclass consisting of singular holomorphic foliations by Riemann surfaces, is only in the early stages of development and faces a range of challenges in finding its own way. In this paper we describe some recent approaches to this new theory. We hope that the ideas reviewed in this notes will be developed and expanded in the future. In writing these notes, we are inspired by the surveys and lecture notes of Deroin [14], Fornæss-Sibony [26], Ghys [30], Hurder [34], Zakeri [54] etc. In particular, we are largely influenced by the survey of Fornæss-Sibony [26] which gives an introduction to harmonic currents on singular foliations as developed by themselves. Harmonic currents are the analog of invariant measures in discrete dynamics. Their approach opens new avenues in studying the interplay between geometry, topology and dynamics in the theory of hyperbolic Riemann surface laminations.

In Section 2 we will recall basic facts on Riemann surface laminations (without and with singularities), singular holomorphic foliations. The hyperbolicity and the leafwise Poincaré metric will be introduced. As consequences, we will develop the heat diffusions and define the notion of harmonic measures for hyperbolic Riemann surface laminations. We also recall from [26] the notion of positive harmonic currents directed by a Riemann surface lamination (possibly with singularities), and compare it with the notion of positive harmonic currents on complex manifolds. We give a short digression to the isolated singularities for singular holomorphic foliations. Singular holomorphic foliations by Riemann surfaces in $\mathbb{P}^k \,(k > 1)$ provides a large family of examples where all the above notions apply. In the light of recent results of Jouanolou [37], Lins Neto-Soares [45], Glutsyuk [32], Lins Neto [44], Brunella [4], and Loray-Rebelo [42], we will describe the properties of a generic holomorphic foliation in $\mathbb{P}^k$ with a given degree $d > 1$.

In Section 3 we will introduce a function $\eta$ which measures the ratio between the ambient metric and the leafwise Poincaré metric of a lamination. This function plays an important role in the study of laminations by hyperbolic Riemann surfaces. We also
introduce the class of Brody hyperbolic laminations. This class contains not only all compact laminations by hyperbolic Riemann surfaces, it also includes many singular holomorphic foliations. We then state some recent results on the regularity of Brody hyperbolic laminations which arise from our joint-works with Dinh and Sibony in [17, 18].

In Section 4 we study the mass-distribution for directed positive harmonic currents in the local and global settings. Applications to the recurrence phenomenon of a generic leaf will be considered. The material for this section is mainly taken from [16, 48].

In Section 5 we introduce the diffusions of the heat equation for laminations (possibly with singularities) with respect to a positive harmonic current directed by a lamination. This approach allows us in [16] to extend the classical theory of L. Garnett [29] and A. Candel [7] to Riemann surface laminations with singularities or to foliations with not necessarily bounded geometry. We present two kinds of ergodic theorems for such currents: one associated to the heat diffusions and one of geometric nature close to Birkhoff’s averaging on orbits of a dynamical system.

In Section 6 we present a notion of hyperbolic entropy, using hyperbolic time, for laminations by hyperbolic Riemann surfaces. When the lamination is compact and transversally smooth, we state some theorems on the finiteness of the hyperbolic entropy. A notion of metric entropy is also introduced for directed positive harmonic measures. This section is based on our joint-works with Dinh and Sibony in [17, 18].

Section 7 is devoted to the Lyapunov theory for hyperbolic Riemann surface laminations. The central objects of this theory are the cocycles which are modelled on the holonomy cocycle of a foliation. We state the Oseledec multiplicative ergodic theorem for laminations. Next, we apply it to smooth compact laminations by hyperbolic Riemann surfaces and to compact singular holomorphic foliations by Riemann surfaces. After all, we characterize geometrically the Lyapunov exponents of a smooth cocycle with respect to a harmonic measure. This section is a synthesis of our several works in [46, 47, 49].

Several open problems develop in the course of the exposition. Finally, since the choice of the material reflects the limited knowledge of the author on the ergodic theory of hyperbolic Riemann surface laminations, we note that many topics are not included here. The author apologizes in advance for omissions or undue biases, and will welcome comments of suggested inclusions.

Main notation. Throughout the paper, \( \mathbb{D} \) denotes the unit disc in \( \mathbb{C} \), \( r\mathbb{D} \) denotes the disc of center 0 and of radius \( r \), and \( D_R \subset \mathbb{D} \) is the disc of center 0 and of radius \( R \) with respect to the Poincaré metric on \( \mathbb{D} \), i.e. \( D_R = r\mathbb{D} \) with \( R := \log[(1 + r)/(1 - r)] \). Poincaré metric on a hyperbolic Riemann surface, in particular on \( \mathbb{D} \) and on the leaves of a hyperbolic Riemann surface lamination, is given by a positive \((1, 1)\)-form that we denote by \( g_P \). The associated distance is denoted by \( \text{dist}_P \). Given a Riemann surface lamination \( (X, \mathcal{L}) \), a leaf through a point \( x \in X \) is often denoted by \( L_x \). Recall that \( d^c := \frac{i}{2\pi}(\overline{\partial} - \partial) \) and \( dd^c = \frac{i}{4}\partial\overline{\partial} \).

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2. BASIC RESULTS

2.1. Riemann surface laminations. Let $X$ be a locally compact space. A Riemann surface lamination $(X, \mathcal{L})$ is the data of a (lamination) atlas $\mathcal{L}$ of $X$ with (laminated) charts

$$\Phi_p : U_p \to \mathbb{B}_p \times \mathbb{T}_p.$$ 

Here, $\mathbb{T}_p$ is a locally compact metric space, $\mathbb{B}_p$ is a domain in $\mathbb{C}$, $U_p$ is an open set in $X$, and $\Phi_p$ is a homeomorphism, and all the changes of coordinates $\Phi_p \circ \Phi_q^{-1}$ are of the form

$$x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t),$$

where $\Psi, \Lambda$ are continuous functions and $\Psi$ is holomorphic in $y$.

The open set $U_p$ is called a flow box and the Riemann surface $\Phi_p^{-1}\{t = c\}$ in $U_p$ with $c \in \mathbb{T}_p$ is a plaque. The property of the above coordinate changes insures that the plaques in different flow boxes are compatible in the intersection of the boxes. Two plaques are adjacent if they have non-empty intersection.

A leaf $L$ is a minimal connected subset of $X$ such that if $L$ intersects a plaque, it contains that plaque. So a leaf $L$ is a Riemann surface immersed in $X$ which is a union of plaques. For every point $x \in X$, denote by $L_x$ the leaf passing through $x$. A subset $M \subset X$ is called leafwise saturated if $x \in M$ implies $L_x \subset M$.

We say that a Riemann surface lamination $(X, \mathcal{L})$ is smooth if each map $\Psi$ above is smooth with respect to $y$, and its partial derivatives of any order with respect to $y$ and $\bar{y}$ are jointly continuous with respect to $(y, t)$.

We are mostly interested in the case where the $\mathbb{T}_i$ are closed subsets of smooth real manifolds and the functions $\Psi, \Lambda$ are smooth in all variables. In this case, we say that the lamination $(X, \mathcal{L})$ is transversally smooth. If, moreover, $X$ is compact, we can embed it in an $\mathbb{R}^N$ in order to use the distance induced by a Riemannian metric on $\mathbb{R}^N$.

We say that a transversally smooth Riemann surface lamination $(X, \mathcal{L})$ is a smooth foliation if $X$ is a manifold and all leaves of $\mathcal{L}$ are Riemann surfaces immersed in $X$.

We say that a Riemann surface lamination $(X, \mathcal{L})$ is a holomorphic foliation if $X$ is a complex manifold (of dimension $k$) and there is an atlas $\mathcal{L}$ of $X$ with charts

$$\Phi_i : U_i \to \mathbb{B}_i \times \mathbb{T}_i,$$

where the $\mathbb{T}_i$’s are open sets of $\mathbb{C}^{k-1}$ and all above maps $\Psi, \Lambda$ are holomorphic.

Many examples of abstract compact Riemann surface laminations are constructed in [9] and [30]. Suspension of a group action gives already a large class of laminations without singularities.

2.2. Hyperbolicity and leafwise Poincaré metric. Consider now a Riemann surface lamination $(X, \mathcal{L})$.

Definition 2.1. A leaf $L$ of $(X, \mathcal{L})$ is said to be hyperbolic if it is a hyperbolic Riemann surface, i.e., it is uniformized by $\mathbb{D}$. $(X, \mathcal{L})$ is said to be hyperbolic if its leaves are all hyperbolic.
For every \( x \in X \) such that \( L_x \) is hyperbolic, consider a universal covering map
\[
(2.1) \quad \phi_x : D \to L_x \quad \text{such that } \phi_x(0) = x.
\]
This map is uniquely defined by \( x \) up to a rotation on \( D \). Then, by pushing forward the Poincaré metric \( g_P \) on \( D \) via \( \phi_x \), we obtain the so-called Poincaré metric on \( L_x \) which depends only on the leaf. The latter metric is given by a positive \((1, 1)\)-form on \( L_x \) that we also denote by \( g_P \) for the sake of simplicity.

2.3. Heat diffusions and harmonic measures. Let \((X, \mathcal{L})\) be a hyperbolic Riemann surface lamination. The leafwise Poincaré metric \( g_P \) induces the corresponding Laplacian \( \Delta \) on leaves (see formula (5.1) for \( \beta := g_P \) below). For every point \( x \in X \), consider the heat equation on \( L_x \)
\[
\frac{\partial p(x, y, t)}{\partial t} = \Delta_y p(x, y, t), \quad \lim_{t \to 0^+} p(x, y, t) = \delta_x(y), \quad y \in L_x, \ t \in \mathbb{R}^+.
\]
Here \( \delta_x \) denotes the Dirac mass at \( x \), \( \Delta_y \) denotes the Laplacian \( \Delta \) with respect to the variable \( y \), and the limit is taken in the sense of distribution, that is,
\[
\lim_{t \to 0^+} \int_{L_x} p(x, y, t) f(y) g_P(y) = f(x)
\]
for every smooth function \( f \) compactly supported in \( L_x \).

The smallest positive solution of the above equation, denoted by \( p(x, y, t) \), is called the heat kernel. Such a solution exists because \((L_x, g_P)\) is complete and of bounded geometry (see, for example, [9, 11]). The heat kernel \( p(x, y, t) \) gives rise to a one parameter family \( \{D_t : t \geq 0\} \) of diffusion operators defined on bounded measurable functions on \( X \) by
\[
(2.2) \quad D_tf(x) := \int_{L_x} p(x, y, t) f(y) g_P(y), \quad x \in X.
\]
This family is a semi-group, that is,
\[
(2.3) \quad D_0 = \text{id} \quad \text{and} \quad D_t1 = 1 \quad \text{and} \quad D_{t+s} = D_t \circ D_s \quad \text{for } t, s \geq 0,
\]
where 1 denotes the function which is identically equal to 1.

We denote by \( \mathcal{C}(X, \mathcal{L}) \) the space of all functions \( f \) defined and compactly supported on \( X \) which are leafwise \( C^2 \)-smooth and transversally continuous, that is, for each laminated chart \( \Phi_p : U_p \to \mathbb{R}_p \times T_p \) and all \( r, s \in \mathbb{N} \) with \( r + s \leq 2 \), the derivatives \( \frac{\partial^{r+s}(f \circ \Phi_p^{-1})}{\partial y^r \partial y^s} \) exist and are jointly continuous in \((y, t)\).

**Definition 2.2.** Let \( \Delta \) be the Laplacian on \( \Delta \), that is, the aggregate of the leafwise Laplacians \( \{\Delta_x\}_{x \in X} \).

A positive Borel measure \( \mu \) on \( X \) is said to be quasi-harmonic if
\[
\int_X \Delta u \, d\mu = 0
\]
for all functions \( u \in \mathcal{C}(X, \mathcal{L}) \).

A quasi-harmonic measure \( \mu \) is said to be harmonic if \( \mu \) is finite and \( \mu \) is \( D_t \)-invariant for all \( t \in \mathbb{R}^+ \), i.e,
\[
\int_X D_t f d\mu = \int_X f d\mu, \quad f \in \mathcal{C}(X, \mathcal{L}), \ t \in \mathbb{R}^+.
\]
2.4. Positive harmonic currents on complex manifolds. Let $M$ be a complex manifold of dimension $k$. A $(p, p)$-form on $M$ is positive if it can be written at every point as a combination with positive coefficients of forms of type

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p$$

where the $\alpha_j$ are $(1, 0)$-forms. A $(p, p)$-current or a $(p, p)$-form $T$ on $M$ is weakly positive if $T \wedge \varphi$ is a positive measure for any smooth positive $(k - p, k - p)$-form $\varphi$. A $(p, p)$-current $T$ is positive if $T \wedge \varphi$ is a positive measure for any smooth weakly positive $(k - p, k - p)$-form $\varphi$. If $M$ is given with a Hermitian metric $\beta$ and $T$ is a positive $(p, p)$-current on $M$, $T \wedge \beta^{k-p}$ is a positive measure on $M$. The mass of $T \wedge \beta^{k-p}$ on a measurable set $E$ is denoted by $\|T\|_E$ and is called the mass of $T$ on $E$. The mass $\|T\|$ of $T$ is the total mass of $T \wedge \beta^{k-p}$ on $M$.

A $(p, p)$-current on $M$ is harmonic if $dd^c T = 0$ in the weak sense (namely, $T(dd^c f) = 0$ for all compactly smooth $(k - p - 1, k - p - 1)$-forms $f$ on $M$).

In this article, for every $r > 0$ let $B_r$ denote the ball of center 0 and of radius $r$ in $\mathbb{C}^k$. The following local property of positive harmonic currents is discovered by Skoda [51].

**Proposition 2.3.** (Skoda [51]). Let $T$ be a positive harmonic current in a ball $B_{r_0}$. Define $\beta := dd^c|z|^2$ the standard Kähler form where $z$ is the canonical coordinates on $\mathbb{C}^n$. Then the function $r \mapsto \pi^{-(k-p)} r^{-2(k-p)} \|T \wedge \beta^{k-p}\|_{B_r}$ is increasing on $0 < r \leq r_0$. In particular, it is bounded on $[0, r_1]$ for any $0 < r_1 < r_0$.

The limit of the above function when $r \rightarrow 0$ is called the Lelong number of $T$ at 0. The above proposition shows that Lelong number always exists and is finite positive.

The next simple result allows for extending positive harmonic currents through isolated points.

**Proposition 2.4.** (Dinh-Nguyen-Sibony [16, Lemma 2.5]) Let $T$ be a positive current of bidimension $(1, 1)$ with compact support on a complex manifold $M$. Assume that $dd^c T$ is a negative measure on $M \setminus E$ where $E$ is a finite set. Then $T$ is a positive harmonic current.

2.5. Directed positive harmonic currents. Let $(X, \mathcal{L})$ be a Riemann surface lamination. Let $\mathcal{C}^1(X, \mathcal{L})$ denote the space of all forms $f$ of bidegree $(1, 1)$ defined on leaves of the lamination and compactly supported on $X$ such that $f$ is transversally continuous. The last continuity condition means that for each laminated chart $\Phi_p : U_p \rightarrow \mathbb{B}_p \times T_p$, the form $f \circ \Phi_p^{-1}$ is jointly continuous in $(y, t)$. For each chart $\Phi_p : U_p \rightarrow \mathbb{B}_p \times T_p$, the complex structure on $\mathbb{B}_p$ induces a complex structure on the leaves of $X$. Therefore, the operator $d$ and $dd^c$ can be defined so that they act leafwise on forms as in the case of complex manifolds. So we get easily that $dd^c : \mathcal{C}(X, \mathcal{L}) \rightarrow \mathcal{C}^1(X, \mathcal{L})$. A form $f \in \mathcal{C}^1(X, \mathcal{L})$ is said to be positive if its restriction to every plaque is a positive $(1, 1)$-form in the usual sense.

**Definition 2.5.** (Garnett [29], see also Sullivan [52]). A directed current on $(X, \mathcal{L})$ (or equivalently, a current directed by the lamination $(X, \mathcal{L})$) is a linear continuous form on $\mathcal{C}^1(X, \mathcal{L})$. Let $T$ be a directed current.

- $T$ is said to be positive if $T(f) \geq 0$ for all positive forms $f \in \mathcal{C}^1(X, \mathcal{L})$.
- $T$ is said to be harmonic current $T$ if $dd^c T = 0$ in the weak sense (namely, $T(dd^c g) = 0$ for all functions $g \in \mathcal{C}(X, \mathcal{L})$).

We have the following decomposition.
Proposition 2.6. Let $T$ be a directed positive harmonic current on $(X,\mathcal{L})$. Let $U \simeq \mathbb{B} \setminus \mathbb{T}$ be a flow box which is relatively compact in $X$. Then, there is a positive Radon measure $\nu$ on $T$ and for $\nu$-almost every $t \in T$, there is a positive harmonic function $h_t$ on $B$ such that if $K$ is compact in $B$, the integral $\int_K \|h_t\|_{L^1(K)} d\nu(t)$ is finite and
\[
T(f) = \int_T \left( \int_B h_t(g) f(y, t) \right) d\nu(t)
\]
for every form $f \in \mathcal{C}^1(X,\mathcal{L})$ compactly supported on $U$.

2.6. Directed positive harmonic currents vs harmonic measures. Recall that a positive finite measure $\mu$ on the $\sigma$-algebra of Borel sets in $X$ is said to be ergodic if for every leafwise saturated Borel measurable set $Z \subset X$, $\mu(Z)$ is equal to either $\mu(X)$ or 0. A directed positive harmonic current $T$ is said to be extremal if $T = T_1 + T_2$ for directed positive harmonic current $T_1, T_2$ implies that $T_1 = \lambda T$ for some $\lambda \in [0, 1]$. The following result relates the notions of harmonic measures and directed positive harmonic currents (see [16, 47]).

Theorem 2.7. Let $(X,\mathcal{L})$ be a hyperbolic Riemann surface lamination.
(i) If $X$ is compact, then each quasi-harmonic measure is harmonic.
(ii) The map $T \mapsto \mu = T \wedge g_P$ which is defined on the convex cone of all directed positive harmonic currents is one-to-one and its image is contained in the convex cone of all quasi-harmonic measures $\mu$. If, moreover, $X$ is compact, then this map is a bijection from the convex cone of all directed positive harmonic currents $T$ onto the convex cone of all harmonic measures $\mu$.
(iii) If $T$ is an extremal directed positive harmonic current and $\mu := T \wedge g_P$ is finite, then $\mu$ is ergodic.

2.7. Riemann surface laminations with singularities, singular holomorphic foliations and examples. We call Riemann surface lamination with singularities the data $(X,\mathcal{L},E)$ where $X$ is a locally compact space, $E$ a closed subset of $X$ and $(X \setminus E,\mathcal{L})$ is a Riemann surface lamination. The set $E$ is the singularity set of the lamination. In order to simplify the presentation, we will mostly consider the case where $X$ is a closed subset of a complex manifold $M$ of dimension $k \geq 1$ and $E$ is a locally finite subset of $X$. We assume that $M$ is endowed with a Hermitian metric $g_M$. We also assume that the complex structures on the leaves of the lamination coincide with the ones induced by $M$, that is, the leaves of $(X \setminus E,\mathcal{L})$ are Riemann surfaces holomorphically immersed in $M$.

We say that $\mathcal{F} := (X,\mathcal{L},E)$ is a singular foliation (resp. singular holomorphic foliation) if $X$ is a complex manifold and $E \subset X$ is a closed subset such that $X \setminus E = X$ and $(X \setminus E,\mathcal{L})$ is a foliation (resp. a holomorphic foliation). $E$ is said to be the set of singularities of the foliation $\mathcal{F}$. We say that $\mathcal{F}$ is compact if $X$ is compact.

Definition 2.8. Let $Z = \sum_{j=1}^k F_j(z) \frac{\partial}{\partial z_j}$ be a holomorphic vector field defined in a neighborhood $U$ of $0 \in \mathbb{C}^k$. Consider the holomorphic map $F := (F_1, \ldots, F_k) : U \to \mathbb{C}^k$. We say that $Z$ is
(1) singular at $0$ if $F(0) = 0$.
(2) generic linear if it can be written as
\[
Z(z) = \sum_{j=1}^k \lambda_j z_j \frac{\partial}{\partial z_j}
\]
where \( \lambda_j \) are non-zero complex numbers.

(3) is with non-degenerate singularity at 0 if \( Z \) is singular at 0 and the eigenvalues \( \lambda_1, \ldots, \lambda_k \) of the Jacobian matrix \( DF(0) \) are all non-zero.

(4) is with hyperbolic singularity at 0 if \( Z \) is singular at 0 and the eigenvalues \( \lambda_1, \ldots, \lambda_k \) of the Jacobian matrix \( DF(0) \) satisfy \( \lambda_j \neq 0 \) and \( \lambda_i/\lambda_j \notin \mathbb{R} \) for all \( 1 \leq i \neq j \leq k \).

The integral curves of \( Z \) defines a singular holomorphic foliation on \( U \). The condition \( \lambda_j \neq 0 \) implies that the foliation has an isolated singularity at 0.

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a singular holomorphic foliation such that \( E \) is an analytic subset of \( X \) with \( \text{codim}(E) \geq 2 \). Then \( \mathcal{F} \) is given locally by holomorphic vector fields and its leaves are locally, integral curves of these vector fields, and the singularities of \( \mathcal{F} \) coincide with the singular set of these vector fields. We say that a singular point \( a \in E \) is linearizable (resp. hyperbolic) if there is a local holomorphic coordinates system of \( X \) near \( a \) on which the leaves of \( \mathcal{F} \) are integral curves of a generic linear vector field (resp. of a holomorphic vector field admitting 0 as a hyperbolic singularity). In dimension 2 (i.e. \( \text{dim} X = 2 \)), if \( a \) is a hyperbolic singularity, then there is a local holomorphic coordinates system of \( X \) near \( a \) on which the leaves of \( \mathcal{F} \) are integral curves of a vector field \( Z(z_1, z_2) = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2} \), where \( \lambda_1, \lambda_2 \) are some nonzero complex numbers with \( \lambda_1/\lambda_2 \notin \mathbb{R} \). In particular, \( a \) is a linearizable singularity. The analytic curves \( \{z_1 = 0\} \) and \( \{z_2 = 0\} \) are called separatrices at \( a \).

Now we discuss singular holomorphic foliations on \( \mathbb{P}^k \) with \( k \geq 2 \). Let \( \pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k \) denote the canonical projection. Let \( \mathcal{F} \) be a singular holomorphic foliations on \( \mathbb{P}^k \). It can be shown that \( \pi^* \mathcal{F} \) is a singular foliation on \( \mathbb{C}^{k+1} \) associated to a vector field \( Z \) of the form

\[
Z = \sum_{j=0}^{k} F_j(z) \frac{\partial}{\partial z_j},
\]

where the \( F_j \) are homogeneous polynomials of degree \( d \geq 1 \). We call \( d \) the degree of the foliation. A point \( x \in \mathbb{P}^k \) is a singularity of \( \mathcal{F} \) if \( F(x) \) is colinear with \( x \), i.e., if \( x \) is either an indeterminacy point or a fixed point of \( f = [F_0 : \ldots : F_k] \) as a meromorphic map in \( \mathbb{P}^k \). For \( d \geq 2 \), let \( \mathcal{F}_d(\mathbb{P}^k) \) be the space of singular holomorphic foliations of degree \( d \) in \( \mathbb{P}^k \). Using the above form of \( Z \) we can show that \( \mathcal{F}_d(\mathbb{P}^k) \) can be canonically identified with a Zariski open subset of \( \mathbb{P}^N \), where \( N := (d + k + 1) \frac{(d+k-1)!}{(k-1)!} - 1 \) (see [4]). The next result describes the typical properties of a generic foliation \( \mathcal{F} \in \mathcal{F}_d(\mathbb{P}^k) \).

**Theorem 2.9.** Let \( d, k > 1 \).

1. (Jouanolou [37], Lins Neto-Soares [45]). There is a real Zariski dense open set \( \mathcal{H}(d) \subset \mathcal{F}_d(\mathbb{P}^k) \) such that for every \( \mathcal{F} \in \mathcal{H}(d) \), all the singularities of \( \mathcal{F} \) are hyperbolic and \( \mathcal{F} \) do not possess any invariant algebraic curve.

2. (Glutsyuk [32], Lins Neto [44]). If all the singularities of a foliation \( \mathcal{F} \in \mathcal{F}_d(\mathbb{P}^k) \) are non-degenerate, then \( \mathcal{F} \) is hyperbolic.

3. (Brunella [4]). If all the singularities of a foliation \( \mathcal{F} \in \mathcal{F}_d(\mathbb{P}^k) \) are hyperbolic and \( \mathcal{F} \) do not possess any invariant algebraic curve, then \( \mathcal{F} \) admits no nontrivial directed positive closed current.

Moreover, Loray-Rebelo [42] constructed a nonempty open set \( \mathcal{U}(d) \) of \( \mathcal{F}_d(\mathbb{P}^k) \) such that every leaf of \( \mathcal{F} \in \mathcal{U}(d) \) is dense.
Now we come to the notion of directed positive harmonic currents on singular Riemann surface laminations.

**Definition 2.10.** (Berndtsson-Sibony [1], Fornæss-Sibony [25, 26]). Let \((X, \mathcal{L}, E)\) be a Riemann surface lamination with singularities, where \(X\) is a closed subset of a complex manifold \(M\) and the leaves of \((X \setminus E, \mathcal{L})\) are Riemann surfaces holomorphically immersed in \(M\). A directed harmonic current on \((X, \mathcal{L}, E)\) is a positive harmonic current \(T\) of bidimension \((1, 1)\) on \(M\) such that the support of \(T\) is contained in \(X\) and that the restriction of \(T\) on \(X \setminus E\) is a directed harmonic current on the Riemann surface lamination \((X, \mathcal{L})\) in the sense of Definition 2.5.

The existence of directed positive harmonic currents for compact (nonsingular) laminations was proved by Garnett [29]. The case of compact singular Riemann surfaces laminations was proved by Berndtsson-Sibony. When a leaf \(L_x\) is hyperbolic, an average on \(L_x\) was introduced by Fornæss-Sibony. It allows another construction of directed positive harmonic currents.

**Theorem 2.11.** (Berndtsson-Sibony [1], Fornæss-Sibony [25], see also [26, Theorem 23 and Corollary 3]). Let \((X, \mathcal{L}, E)\) be a singular Riemann surface lamination as in the assumption of Definition 2.10. Assume moreover that \(X\) is compact. Then there is a nonzero directed positive harmonic current \(T\). In particular, if the set \(E\) does not support any nonzero positive harmonic current (e.g. \(E\) is a finite set), then the restriction of such a current \(T\) on \(X \setminus E\) induces a nonzero directed positive harmonic current on \((X \setminus E, \mathcal{L})\).

By Theorem 2.9, Theorem 2.11 applies to every generic foliation in \(\mathbb{P}^k\) with a given degree \(d > 1\).

### 2.8. General laminations/foliations

We formulate some general notions of laminations and foliations. Although they are not the main topic of this article, some results presented here could be extended to these general objects. Let \(l \geq 1\) be an integer.

A \(l\)-dimensional lamination \((X, \mathcal{L})\) is the data of a locally compact space \(X\) and a (lamination) atlas \(\mathcal{L}\) of with ( laminated) charts

\[
\Phi_p : U_p \to \mathbb{B}_p \times \mathbb{T}_p.
\]

Here, \(\mathbb{T}_p\) is a locally compact metric space, \(\mathbb{B}_p\) is a domain in \(\mathbb{R}^l\), \(U_p\) is an open set in \(X\), and \(\Phi_p\) is a homeomorphism, and all the changes of coordinates \(\Phi_p \circ \Phi_q^{-1}\) are of the form

\[
x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t),
\]

where \(\Psi, \Lambda\) are continuous functions. We call \(l\)-dimensional lamination with singularities the data \((X, \mathcal{L}, E)\) where \(X\) is a locally compact space, \(E\) is a closed subset of \(X\) and \((X \setminus E, \mathcal{L})\) is a \(l\)-dimensional lamination. The set \(E\) is the singularity set of the lamination.

A \(l\)-dimensional Riemannian foliation \((X, \mathcal{L})\) is the data of a (lamination) atlas \(\mathcal{L}\) of a Riemannian manifold \(X\) of dimension \(k \geq l\) with ( laminated) charts

\[
\Phi_p : U_p \to \mathbb{B}_p \times \mathbb{T}_p.
\]

Here, \(\mathbb{T}_p\) is a domain in \(\mathbb{R}^{k-l}\), \(\mathbb{B}_p\) is a domain in \(\mathbb{R}^l\), \(U_p\) is an open set in \(X\), and \(\Phi_p\) is a homeomorphism, and all the changes of coordinates \(\Phi_p \circ \Phi_q^{-1}\) are of the form

\[
x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t),
\]
where $\Psi, \Lambda$ are smooth functions. We call $l$-dimensional Riemannian foliation with singularities the data $(X, \mathcal{L}, E)$ where $X$ is a Riemannian manifold $X$ of dimension $k \geq l$, $E$ is a closed subset of $X$ and $(X \setminus E, \mathcal{L})$ is a $l$-dimensional Riemannian foliation. The set $E$ is the singularity set of the foliation.

By considering classical constructions such as projective limits and suspensions, Fornaess-Sibony-Wold obtain in [28] examples of laminations possessing many, or few, directed positive closed or directed positive harmonic currents. They also give examples of laminations by complex manifolds of dimension $\geq 2$ with no directed positive harmonic current. This is in contrast with the Riemann surface case, where such a current always exists.

3. Regularity of the leafwise Poincaré metric

Let $(X, \mathcal{L}, E)$ be a hyperbolic Riemann surface lamination with singularities. Let $g_P$ be the leafwise Poincaré metric for the lamination $(X \setminus E, \mathcal{L})$ given in Subsection 2.2. Let $g_X$ be a Hermitian metric on the leaves which is transversally smooth. We can construct such a metric on flow boxes and glue them using a partition of unity. We have

$$g_X = \eta^2 g_P \quad \text{where} \quad \eta(x) := \|D\phi_x(0)\|.$$  

Here, $\phi_x$ is defined in (2.1), and for the norm of the differential $D\phi_x$ we use the Poincaré metric on $D$ and the Hermitian metric $g_X$ on $L_x$.

The extremal property of the Poincaré metric implies that

$$\eta(x) = \sup \{\|D\phi(0)\|, \ \phi : D \to L \text{ holomorphic such that } \phi(0) = x\}.$$  

Using a map sending $D$ to a plaque, we see that the function $\eta$ is locally bounded from below on $X \setminus E$ by a strictly positive constant. When $X$ is compact and $E = \emptyset$, the classical Brody lemma (see [40, p.100]) implies that $\eta$ is also bounded from above.

The continuity of the function $\eta$ was studied by Candel, Ghys, Verjovsky, see [6, 30, 53]. The survey [26] establishes the result as a consequence of Royden’s lemma. Indeed with his lemma, Royden proved the upper-semicontinuity of the infinitesimal Kobayashi metric in a Kobayashi hyperbolic manifold (see [40, p.91 and p.153]). The following theorem gives refinements of the previous results.

**Theorem 3.1.** (Dinh-Nguyen-Sibony [17]). Let $(X, \mathcal{L})$ be a transversally smooth compact lamination by hyperbolic Riemann surfaces. Then the Poincaré metric on the leaves is Hölder continuous, that is, the function $\eta$ defined in (3.1) is Hölder continuous on $X$. Moreover, the exponent of Hölder continuity can be estimated in geometric terms.

The main tool of the proof of Theorem 3.1 is to use Beltrami’s equation in order to compare universal covering maps of any leaf $L_y$ near a given leaf $L_x$. More precisely, for $R > 0$ let $D_R$ be the disc of center 0 with radius $R$ with respect to the Poincaré metric on $D$ (see Main Notations in 1]. We first construct a non-holomorphic parametrization $\psi$ from $D_R$ to $L_y$ which is close to a universal covering map $\phi_x : D \to L_x$ for all $R$ large enough. Next, precise geometric estimates on $\psi$ allow us to modify it, using Beltrami’s equation. We then obtain a holomorphic map that we can explicitly compare with a universal covering map $\phi_y : D \to L_y$.

Next, we investigate the regularity of the leafwise Poincaré metric $g_P$ of a compact singular holomorphic foliation. Here we will meet an important difficulty for our study:
a leaf of the foliation may visit singular flow boxes without any obvious rule. We are interested in the following class of laminations.

**Definition 3.2.** (Dinh-Nguyen-Sibony [18]). A hyperbolic Riemann surface lamination with singularities \((X, \mathcal{L}, E)\) with \(X\) compact is said to be *Brody hyperbolic* if there is a constant \(c_0 > 0\) such that

\[
\|D\phi(0)\| \leq c_0
\]

for all holomorphic maps \(\phi\) from \(\mathbb{D}\) into a leaf.

**Remark 3.3.** It is clear that if the lamination is Brody hyperbolic then its leaves are hyperbolic in the sense of Kobayashi. Conversely, the Brody hyperbolicity is a consequence of the non-existence of holomorphic non-constant maps \(\mathbb{C} \to X\) such that out of \(E\) the image of \(\mathbb{C}\) is locally contained in leaves, see [26, Theorem 15].

On the other hand, Lins Neto proved in [44] that for every holomorphic foliation of degree larger than 1 in \(\mathbb{P}^k\), with non-degenerate singularities, there is a smooth metric with negative curvature on its tangent bundle, see also Glutsyuk [32]. Hence, these foliations are Brody hyperbolic. Consequently, holomorphic foliations in \(\mathbb{P}^k\) are generically Brody hyperbolic, see Theorem 2.9 (1).

Denote by \(\log^*(\cdot) := 1 + |\log(\cdot)|\) a log-type function, and by \(\text{dist}\) the distance on \(X\) induced by the Hermitian metric \(g_X\). The following result is a counterpart of Theorem 3.1 in the context of singular holomorphic foliations.

**Theorem 3.4.** (Dinh-Nguyen-Sibony [18]). Let \((X, \mathcal{L}, E)\) be a Brody hyperbolic singular holomorphic foliation on a Hermitian compact complex manifold \(X\). Assume that the singular set \(E\) is finite and that all points of \(E\) are linearizable. Then, there are constants \(c > 0\) and \(0 < \alpha < 1\) such that

\[
|\eta(x) - \eta(y)| \leq c \left( \max \{ \log^* \text{dist}(x, E), \log^* \text{dist}(y, E) \} \right)^\alpha
\]

for all \(x, y\) in \(X \setminus E\).

To prove this theorem, we analyze the behavior and get an explicit estimate on the modulus of continuity of the Poincaré metric on leaves. The following estimates are crucial in our method. They are also useful in other problems.

**Proposition 3.5.** (Dinh-Nguyen-Sibony [18]). Under the hypotheses of Theorem 3.4 there exists a constant \(c_1 > 1\) such that

\[
c_1^{-1} s \log^* s \leq \eta(x) \leq c_1 s \log^* s
\]

for \(x \in X \setminus E\) and \(s := \text{dist}(x, E)\).

We end the section with the following open question.

**Problem 3.6.** Let \((X, \mathcal{L}, E)\) be a compact singular holomorphic foliation by hyperbolic Riemann surfaces. Assume that every point \(a \in E\) is a non-degenerate singularity. Adding more assumptions on the nature of such a point \(a\) if necessary, study the function \(\eta\).
4. Mass-distribution of directed positive harmonic currents

Let \((X, \mathcal{L}, E)\) be a singular holomorphic foliation and let \(T\) be a positive harmonic current on \(X \setminus E\). By Proposition \ref{prop:mass} its mass with respect to any Hermitian metric on \(X\) is finite. We call Poincaré mass of \(T\) the mass of \(T\) with respect to Poincaré metric \(g_p\) on \(X \setminus E\), i.e. the mass of the positive measure \(m_p := T \wedge g_p\). A priori, Poincaré mass may be infinite near the singular points. The following proposition gives us a criterion for the finiteness of this mass. It can be applied to generic foliations in \(\mathbb{P}^k\) (see Theorem \ref{thm:generic}).

**Proposition 4.1.** (Dinh-Nguyen-Sibony \cite{DinhNguyenSibony16}). Let \((X, \mathcal{L}, E)\) be a singular holomorphic foliation. If \(a \in E\) is a linearizable singularity, then any positive harmonic current on \(X\) has locally finite Poincaré mass near \(a\).

The proof of this result is based on the finiteness of the Lelong number of \(T\) at \(a\) (see Proposition \ref{prop:lelong}). In dimension 2 we have a more precise result when the current \(T\) is directed and the singular point is hyperbolic.

**Theorem 4.2.** (Nguyen \cite{Nguyen48}). Let \((X, \mathcal{L}, E)\) be a singular holomorphic foliation with \(\dim X = 2\). If \(a \in E\) is a hyperbolic singularity, then for any directed positive harmonic current \(T\) on \(X\) which does not give mass to any of the two separatrices at \(a\), the Lelong number of \(T\) at \(a\) vanishes.

An immediate consequence of Theorem 4.2 is the following result on the Lelong numbers of a directed harmonic currents.

**Corollary 4.3.** Let \(\mathcal{F} = (X, \mathcal{F}, E)\) be a singular holomorphic foliation with \(X\) a compact complex surface. Assume that all the singularities are hyperbolic and that the foliation has no invariant analytic curve. Then for every harmonic current \(T\) directed by \(\mathcal{F}\), the Lelong number of \(T\) vanishes everywhere in \(X\).

The above corollary can be applied to every generic foliation in \(\mathbb{P}^2\) with a given degree \(d > 1\) (see Theorem \ref{thm:generic}).

We can apply Corollary 4.3 to study the the recurrence of a generic leaf. More specifically, let \(T\) be a positive harmonic current directed by a singular holomorphic foliation \((X, \mathcal{L}, E)\) with \(E\) a compact complex surface. Assume that all the singularities are hyperbolic and that the foliation has no invariant analytic curve. Consider the positive measure \(m_p := T \wedge g_p\). We know by Proposition 4.1 that \(m_p\) is a finite measure. Given a point \(x \in X\) and a \(\mu\)-generic point \(a \in X \setminus E\), we want to know how often the leaf \(L_a\) visits the ball \(B(x, r)\) as \(r \searrow 0\). Here \(B(x, r)\) denotes the open ball with center \(x\) and radius \(r\) with respect to a fixed metric on \(X\).

Let us introduce some more notation and terminology. Denote by \(r \mathbb{D}\) the disc of center 0 and of radius \(r\) with \(0 < r < 1\). In the Poincaré disc \((\mathbb{D}, \omega_p)\), \(r \mathbb{D}\) is also the disc of center 0 and of radius

\[
R := \log \frac{1 + r}{1 - r}.
\]

So, we will also denote by \(\mathbb{D}_R\) this disc, and by \(\partial \mathbb{D}_R\) its boundary which is also the Poincaré circle of center 0 and radius \(R\).

Together with Dinh and Sibony, we introduce the following indicator.
**Definition 4.4.** For each \( r > 0 \), the visibility of a point \( a \in X \setminus E \) within distance \( r \) from a point \( x \in M \) is the number

\[
N(a, x, r) = \limsup_{R \to \infty} \frac{1}{R} \int_0^R \left( \int_{\theta=0}^1 1_{B(x,r)}(\phi_a(s_t e^{2\pi i \theta})) d\theta \right) dt \in [0, 1],
\]

where \( 1_{B(x,r)} \) is the characteristic function associated to the set \( B(x,r) \), and \( s_t \) is defined by the relation \( t = \log \frac{1+s_t}{1-s_t} \), that is, \( s_t \mathbb{D} = \mathbb{D}_t \).

Geometrically, \( N(a, x, r) \) is the average, as \( R \to \infty \), over the hyperbolic time \( t \in [0, R] \) of the Lebesgue measure of the set \( \{ \theta \in [0, 1] : \phi_a(s_t e^{2\pi i \theta}) \in B(x,r) \} \). The last quantity may be interpreted as the portion which hits \( B(x,r) \) of the Poincaré circle of radius \( t \) with center \( a \) spanned on the leaf \( L_a \).

We combine Corollary 4.3 and the so-called geometric ergodic theorem (Theorem 5.4) which will be presented in the next section. Consequently, we obtain the following upper bound on the visibility of a generic point.

**Theorem 4.5.** We keep the above hypothesis and notation. Then for \( m_P \)-almost every point \( a \in X \setminus E \) and for every point \( x \in X \), we have that

\[
N(a, x, r) = \begin{cases} \alpha(r^2), & x \in X \setminus E; \\ \alpha(|\log r|^{-1}), & x \in E. \end{cases}
\]

Here are some open questions.

**Problem 4.6.** Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a compact singular holomorphic foliation by hyperbolic Riemann surfaces. Let \( a \in E \) be a non-degenerate singularity. Find sufficient conditions on \( \mathcal{F} \) and \( a \) so that any positive harmonic current on \( X \) has locally finite Poincaré mass near \( a \).

**Problem 4.7.** Can one generalize Theorem 4.2 to higher dimensions?

### 5. Heat Equation and Ergodic Theorems

In this section we will report some recent techniques used to obtain ergodic theorems for Riemann surface laminations.

#### 5.1. Ergodic theorems associated with the heat diffusions.

In collaboration with Dinh and Sibony [16], we introduce the heat equation relative to a positive harmonic closed current and apply it to the directed positive harmonic currents of a Riemann surface laminations with singularities. This permits to construct the heat diffusion with respect to various Laplacians that could be defined almost everywhere with respect to the positive harmonic current.

More concretely, let \( (X, \mathcal{L}, E) \) be a compact Riemann surface lamination with singularities, where \( X \) is a (compact) subset of a complex manifold \( M \) and the leaves of \( (X \setminus E, \mathcal{L}) \) are Riemann surfaces holomorphically immersed in \( M \). For simplicity, fix a Hermitian form \( g_M \) on \( M \). Let \( T \) be a directed positive harmonic current. So, \( T \wedge g_M \) is a positive measure. Consider a positive \((1,1)\)-form \( \beta \) which is defined almost everywhere on \( M \) with respect to \( T \wedge g_M \). We say that \( T \) is \( \beta \)-regular if \( T \wedge \beta \) is of finite mass and \( i\tau \wedge \overline{\tau} \wedge T \leq T \wedge \beta \), where \( \tau \) is a \((1,0)\)-form defined almost everywhere with respect to
$T \land g_p$ such that $\partial T = \tau \land T$. Under the notation of Proposition 2.6, we see that $\tau = h^{-1}_t \partial h_t$ on the plaque passing through $t \in \mathbb{T}$ for $\nu$-almost every $t \in \mathbb{T}$. The following result gives a typical example of $\beta$-regularity with $\beta := g_p$.

**Proposition 5.1.** Let $(X, \mathcal{L}, E)$ be a lamination as above. Let $\mathcal{F}$ be a singular holomorphic foliation on $M$ such that the restriction of $\mathcal{F}$ on $X \setminus E$ induces $\mathcal{L}$ and that all points of $E$ are linearizable singularities of $\mathcal{F}$. Then every directed harmonic current $T$ on $(X, \mathcal{L}, E)$ is $g_p$-regular.

This result is essentially proved in Proposition 3 and Theorem 9 in Fornæss-Sibony [27]. The proof makes use of Proposition 4.1.

We define the Laplacian $\Delta_\beta$ by

\[(\Delta_\beta u)T \land \beta := dd^c u \land T \text{ for } u \in \mathcal{C}_0^\infty(M).\]

We will extend the definition of $\Delta_\beta$ to larger spaces, suitable for developing $L^2$-techniques.

To this end, let $m_\beta$ denote the measure $T \land \beta$ and consider the Hilbert space $L := L^2(m_\beta)$.

We also introduce the Hilbert space $H = H^1_\beta(T) \subset L^2(m_\beta)$ associated with $T$ and $\beta$ as the completion of $\mathcal{C}_0^\infty(M)$ with respect to the norm

\[\|u\|_{H^1_\beta}^2 := \int |u|^2 T \land \beta + i \int \partial u \land \bar{\partial} u \land T.\]

Using the assumption that $T$ is $\beta$-regular, we can show that there exists a semi-group of contractions $S(t) : L \to L$, $t \in \mathbb{R}^+$ such that for every function $u_0 \in H$, $u(t, \cdot) := S(t)u_0$ satisfies

\[\frac{\partial u(t, \cdot)}{\partial t} = \Delta_\beta u(t, \cdot) \text{ and } u(0, \cdot) = u_0.\]

Recall that a family $S(t) : L \to L$, $t \in \mathbb{R}_+$, is a semi-group of contractions if $S(t + t') = S(t) \circ S(t')$ and if $\|S(t)\| \leq 1$ for all $t, t' \geq 0$.

To prove this result we use functional analysis (Hille-Yosida theorem, Lax-Milgram theorem etc.). We also use Stokes’ theorem on $M$. It is worthy noting that Garnett [29] and Candel [7] also solve the heat equation. But they consider the case without singularities. Moreover, they solve the equation pointwise, that is, in the space of smooth functions. So their approaches are totally different from ours. Indeed, we solve the equation with respect to a harmonic current, in a suitable $L^2$-space.

The following result is an ergodic theorem associated to the heat diffusions.

**Theorem 5.2.** (Dinh-Nguyen-Sibony [18]). We keep the above hypothesis and notation. Then

1. the measure $m_\beta$ is $S(t)$-invariant (i.e. $\langle S(t)u, m_\beta \rangle = \langle u, m_\beta \rangle$ for every $u \in L$), and

   $S(t)$ is a positive contraction in $L^p(m_\beta)$ (i.e. $\|S(t)u\|_{L^p(m_\beta)} \leq \|u\|_{L^p(m_\beta)}$ for every $u \in L$) for all $1 \leq p \leq \infty$;

2. for all $u_0 \in L^p(m_\beta)$, $1 \leq p < \infty$, the average

   \[\frac{1}{R} \int_0^R S(t)u_0 dt\]

converges pointwise $m_\beta$-almost everywhere and also in $L^p(m)$ to an $S(t)$-invariant function $u_0^*$ when $R$ goes to infinity. Moreover, $u_0^*$ is constant on the leaf $L_a$ for $m_\beta$-almost every $a$. If $m_\beta$ is an extremal harmonic measure, then $u$ is constant $m_\beta$-almost everywhere.
Combining Proposition 5.1 and Theorem 5.2, we obtain the following relation between harmonic measures and directed positive harmonic currents which is a complement to Theorem 2.7 (ii).

**Proposition 5.3.** Let \((\mathcal{X}, \mathcal{L}, E)\) be a lamination as above. Let \(\mathcal{F}\) be a singular holomorphic foliation on \(M\) such that the restriction of \(\mathcal{F}\) on \(X \setminus E\) induces \(\mathcal{L}\) and that all points of \(E\) are linearizable singularities of \(\mathcal{F}\). Then the map \(T \mapsto \mu = T \wedge g_P\) is a bijection from the convex cone of all directed positive harmonic currents \(T\) onto the convex cone of all harmonic measures \(\mu\).

### 5.2. Geometric ergodic theorems

In this subsection, we will give an analogue of Birkhoff’s ergodic theorem in the context of a compact Riemann surface lamination \((\mathcal{X}, \mathcal{L}, E)\) with singularities. Our ergodic theorem is of geometric nature and it is close to Birkhoff’s averaging on orbits of a dynamical system. Here the averaging is on hyperbolic leaves and the time is the hyperbolic time.

Let \((\mathcal{X}, \mathcal{L}, E)\) be a Riemann surface lamination with singularities which is embedded in a complex manifold \(M\) as in Subsection 5.1. Let \(T\) be a directed positive harmonic current on \((\mathcal{X}, \mathcal{L}, E)\) such that \(T\) is \(g_P\)-regular. A leaf \(L_x\) is called parabolic if it is not hyperbolic. We assume that \(T\) has no mass on the union of parabolic leaves and that \(m_P := T \wedge g_P\) is a probability measure. So by Proposition 2.7, \(m_P\) is a quasi-harmonic measure on \(X\) with respect to \(g_P\).

For any point \(x \in X \setminus E\) such that the corresponding leaf \(L_x\) is hyperbolic, let \(\phi_x : \mathbb{D} \to L_x\) be given by (2.1). Denote by \(r\mathbb{D}\) the disc of center 0 and of radius \(r\) with \(0 < r < 1\). Recall from (4.1) that in the Poincaré metric, this is also the disc of center 0 and of radius \(\mathcal{R} := \log \frac{1+r}{1-r}\), and we will also denote by \(D_\mathcal{R}\) this disc. For all \(0 < \mathcal{R} < \infty\), consider

\[
\begin{align*}
    m_{x, \mathcal{R}} &:= \frac{1}{M_\mathcal{R}} (\phi_x)_* (\log^+ \frac{r}{|\zeta|} g_P), \\
    \tau_{x, \mathcal{R}} &:= \frac{1}{M_\mathcal{R}} (\phi_x)_* (\log^+ \frac{r}{|\zeta|}).
\end{align*}
\]

(5.2)

where \(\log^+ := \max\{\log, 0\}\), \(g_P\) denotes also the Poincaré metric on \(\mathbb{D}\) and

\[
M_\mathcal{R} := \int \log^+ \frac{r}{|\zeta|} g_P = \int \log^+ \frac{r}{|\zeta|} \frac{2}{|\zeta| (1 - |\zeta|^2)} id\zeta \wedge d\zeta.
\]

So, \(m_{x, \mathcal{R}}\) (resp. \(\tau_{x, \mathcal{R}}\)) is a probability measure (resp. a directed positive current of bidermension \((1, 1)\)) which depends on \(x, \mathcal{R}\) but does not depend on the choice of \(\phi_x\).

**Theorem 5.4.** (Dinh-Nguyen-Sibony [16]). We keep the above hypothesis and notation. Assume in addition that the current \(T\) is extremal. Then for almost every point \(x \in X\) with respect to the measure \(m_P := T \wedge g_P\), the measure \(m_{x, \mathcal{R}}\) defined above converges to \(m_P\) when \(\mathcal{R} \to \infty\).

To prove the theorem, our main ingredient is a delicate estimate on the heat kernel of the Poincaré disc. This estimate allows us to deduce the desired result from the ergodic theorem associated to the heat diffusions (Theorem 5.2).

**Remark 5.5.** Let \((\mathcal{X}, \mathcal{L})\) be a compact lamination by Riemann surfaces without singularities. Let \(T\) be a positive harmonic current directed by the lamination which is extremal, with full mass on hyperbolic leaves and with Poincaré mass 1. Then, the conclusions of
Theorem 5.2 and Theorem 5.4 are still valid. The proofs are essentially the same but we need to use a finite partition of unity for $X$ instead of applying Stokes’ theorem for $M$. Moreover, Theorem 5.2 still holds for compact smooth $p$-dimensional laminations in the sense of Subsection 2.8.

5.3. Unique ergodicity theorems. In [25] Fornæss and Sibony develop the theory of harmonic currents of finite energy. They introduce a notion of energy for positive harmonic currents of bidegree $(1, 1)$ on a compact Kähler manifold $(M, \omega)$ of dimension $k \geq 2$. This allows to define $\int_X T \wedge T \wedge \omega^{k-2}$ for every positive harmonic current $T$ of bidegree $(1, 1)$ on $X$. This theory applies to directed positive harmonic current on singular holomorphic foliations on compact Kähler surfaces.

In [25, 27] Fornæss and Sibony also develop a geometric intersection theory for directed positive harmonic current on singular holomorphic foliations on $\mathbb{P}^2$.

Combining these two theories, they obtain the following remarkable unique ergodicity result for singular holomorphic foliations without invariant algebraic curves.

**Theorem 5.6.** (Fornæss-Sibony [27]). Let $\mathcal{F}$ be a singular holomorphic foliation in $\mathbb{P}^2$ whose singularities are all hyperbolic. Assume that $\mathcal{F}$ has no invariant algebraic curve. Then $\mathcal{F}$ has a unique directed positive harmonic current of mass 1. Moreover, this unique current $T$ is not closed. In particular, for every point $x$ outside the singularity set of $\mathcal{F}$, the current $\tau_{x,R}$ defined in (5.2) converges to $T$ when $R \to \infty$.

The case where $\mathcal{F}$ possesses invariant algebraic curves has recently been answered.

**Theorem 5.7.** (Dinh-Sibony [23]). Let $\mathcal{F}$ be a singular holomorphic foliation in $\mathbb{P}^2$ whose singularities are all hyperbolic. Assume that $\mathcal{F}$ admits a finite number of invariant algebraic curves. Then any directed positive harmonic current is a linear combination of currents of integration on these curves. In particular, all directed positive harmonic currents are closed.

To prove Theorem 5.7 we need to to show that if $T$ is a positive harmonic current directed by $\mathcal{F}$ having no mass on any leaf, then $T$ is zero. For this purpose, Dinh and Sibony [23] develop a theory of densities of positive $dd^c$-closed currents in a compact Kähler manifold with respect to a submanifold. The theory was developed by these authors in [21] for positive closed currents and numerous applications in complex dynamics of higher dimension could be found in [19, 22, 24] etc.

**Problem 5.8.** Can one generalize Theorem 5.6 and Theorem 5.7 to the case where $\mathbb{P}^2$ is replaced with a compact Kähler surface $X$?

6. Topological and metric entropies for hyperbolic Riemann surface laminations

A notion of geometric entropy for compact Riemannian foliations was introduced by Ghys-Langevin-Walczak [31], see also Candel-Conlon [8] and Walczak [55] for recent expositions. It is related to the entropy of the holonomy pseudogroup, which depends on the chosen generators. The basic idea is to quantify how much leaves get far apart transversally. The transverse regularity of the metric on leaves and the lack of singularities play a role in the finiteness of the entropy.
In [17] Dinh-Nguyen-Sibony introduce a general notion of entropy, which permits to describe some natural situations in dynamics and in foliation theory. This new notion of entropy contains a large number of classical situations. An interesting fact is that this entropy is related to an increasing family of distances as in Bowen’s point of view [2]. This allows, for example, for introducing other dynamical notions like metric entropy, local entropies etc.

Let $X$ be a metric space endowed with a distance $\text{dist}_X$. Consider a family $\mathcal{D} = \{\text{dist}_t\}$ of distances on $X$ indexed by $t \in \mathbb{R}^+$. We can also replace $\mathbb{R}^+$ by $\mathbb{N}$ and in practice we often have that $\text{dist}_0 = \text{dist}_X$ and that $\text{dist}_t$ is increasing with respect to $t \geq 0$.

Let $Y$ be a non-empty subset of $X$. Denote by $N(Y, t, \epsilon)$ the minimal number of balls of radius $\epsilon$ with respect to the distance $\text{dist}_t$ needed to cover $Y$. Define the entropy of $Y$ with respect to $\mathcal{D}$ by

$$h_{\mathcal{D}}(Y) := \sup_{\epsilon > 0} \limsup_{t \to \infty} \frac{1}{t} \log N(Y, t, \epsilon).$$

When $Y = X$ we will denote by $h_{\mathcal{D}}$ this entropy.

Observe that when $\text{dist}_t$ is increasing, $N(Y, t, \epsilon)$ is increasing with respect to $t \geq 0$. Moreover,

$$\limsup_{t \to \infty} \frac{1}{t} \log N(Y, t, \epsilon)$$

is increasing when $\epsilon$ decreases. So, in the above definition, we can replace $\sup_{\epsilon > 0}$ by $\lim_{\epsilon \to 0^+}$. If $\mathcal{D} = \{\text{dist}_t\}$ and $\mathcal{D}' = \{\text{dist}'_t\}$ are two families of distances on $X$ such that $\text{dist}'_t \geq A \text{dist}_t$ for all $t$ with a fixed constant $A > 0$, then $h_{\mathcal{D}'} \geq h_{\mathcal{D}}$.

A subset $F \subset X$ is said to be $(t, \epsilon)$-separated if for all distinct points $x, y$ in $F$ we have $\text{dist}_t(x, y) > \epsilon$. Let $M(Y, t, \epsilon)$ denote the maximal number of points in a $(t, \epsilon)$-separated family $F \subset Y$. The proof of the following lemma is immediate.

**Lemma 6.1.** We have

$$N(Y, t, \epsilon) \leq M(Y, t, \epsilon) \leq N(Y, t, \epsilon/2).$$

An important consequence of Lemma [6.1] is that we can formulate the entropy of a subset $Y \subset X$ using $M(Y, t, \epsilon)$ instead of $N(Y, t, \epsilon)$:

$$h_{\mathcal{D}}(Y) = \sup_{\epsilon > 0} \limsup_{t \to \infty} \frac{1}{t} \log M(Y, t, \epsilon).$$

Let $(X, \mathcal{L})$ be a hyperbolic Riemann surface lamination, where $X$ is a metric space endowed with a distance $\text{dist}_X$. Recall that for every $x \in X$, a universal covering map $\phi_x$ of the leaf $L_x$ with $\phi_x(0) = x$ is given in (2.1). For every universal covering map $\psi$ of the leaf $L_x$ with $\psi(x) = x$, there is $\theta \in \mathbb{R}$ such that $\psi$ is equal to the map $\mathbb{D} \ni \xi \mapsto \phi_x(e^{i\theta} \xi)$. In other words, those maps $\psi$ are unique up to a rotation on $\mathbb{D}$. Define the family of distances $\mathcal{D} := \{\text{dist}_t\}$:

$$\text{dist}_t(x, y) := \inf_{\theta \in \mathbb{R}} \sup_{\xi \in \mathbb{D}_t} \text{dist}_X(\phi_x(e^{i\theta} \xi), \phi_y(\xi)).$$

The metric $\text{dist}_t$ measures how far two leaves get apart before the hyperbolic time $t$. It takes into account the time parametrization like in the classical case where one measures the distance of two orbits before time $n$, by measuring the distance at each time $i < n$. So, we are not just concerned with geometric proximity.
Definition 6.2. (Dinh-Nguyen-Sibony [17]). The hyperbolic entropy of a hyperbolic Riemann surface \((X, \mathcal{L})\), denoted by \(h(\mathcal{L})\), is the entropy, computed by (6.1), of \(X\) with respect to the family \(\mathcal{D} := \{\text{dist}_t\}\) which is given by (6.2).

So, the value of the entropy is unchanged under homeomorphisms between laminations which are holomorphic along leaves. The advantage here is that the hyperbolic time we choose is canonical. The notion of hyperbolic entropy can be extended to \((l\text{-dimensional})\) Riemannian foliations with singularities, or more generally \((l\text{-dimensional})\) laminations with singularities, and a priori it is bigger than or equal to the geometric entropy introduced by Ghys, Langevin and Walczak.

Theorem 6.3. (Dinh-Nguyen-Sibony [17]). Let \((X, \mathcal{L})\) be a transversally smooth compact lamination by hyperbolic Riemann surfaces. Embed the lamination in \(\mathbb{R}^N\) in order to use the distance \(\text{dist}_X\) induced by a Riemannian metric on \(\mathbb{R}^N\). Then, \(2 \leq h(\mathcal{L}) < \infty\).

The following proposition gives a simple criterion for the finiteness of entropy. We will need it for the proof of Theorem 6.3.

Proposition 6.4. Assume that there are positive constants \(A\) and \(m\) such that for every \(\varepsilon > 0\) small enough \(X\) admits a covering by less than \(A\varepsilon^{-m}\) balls of radius \(\varepsilon\) for the distance \(\text{dist}_X\). Assume also that

\[
\text{dist}_t \leq e^{ct+d} \text{dist}_X + \varphi(t)
\]

for some constants \(c, d \geq 0\) and a function \(\varphi\) with \(\varphi(t) \to 0\) as \(t \to \infty\). Then, the entropy \(h_\mathcal{D}\) is at most equal to \(mc\).

As for the tranverse regularity of the Poincaré metric (see Theorem 3.1), the main tool to estimate the distance \(\text{dist}_t\) between leaves is to use the Beltrami equation.

To study the finiteness of the entropy for singular holomorphic foliations is a very hard matter. A satisfactory answer is only obtained for complex surfaces.

Theorem 6.5. (Dinh-Nguyen-Sibony [18]) Let \((X, \mathcal{L}, E)\) be a singular foliation by Riemann surfaces on a compact Hermitien complex surface \(X\). Assume that the singularities are linearizable and that the foliation is Brody hyperbolic. Then, its hyperbolic entropy \(h(\mathcal{L})\) is finite.

The proof of this theorem is quite delicate and requires a careful analysis of the dynamics around the singularities. We deduce from the above theorem and Theorem 2.9 the following corollary. It can be applied to foliations of degree at least 2 with hyperbolic singularities.

Corollary 6.6. (Dinh-Nguyen-Sibony [18]). Let \((\mathbb{P}^2, \mathcal{L}, E)\) be a singular foliation by Riemann surfaces on the complex projective plane \(\mathbb{P}^2\) endowed with the Fubini-Study metric. Assume that the singularities are linearizable. Then, the hyperbolic entropy \(h(\mathcal{L})\) of \((\mathbb{P}^2, \mathcal{L}, E)\) is finite.

Consider an abstract setting of a metric space \((X, \text{dist}_X)\) endowed with a family \(\mathcal{D} := \{\text{dist}_t\}_{t \geq 0}\) of distances. Let \(m\) be a probability measure on \(X\). For positive constants \(\varepsilon, \delta\) and \(t\), let \(N_m(t, \varepsilon, \delta)\) be the minimal number of balls of radius \(\varepsilon\) relative to the metric \(\text{dist}_t\) whose union has at least \(m\)-measure \(1 - \delta\). The (metric) entropy of \(m\) is defined by the following formula

\[
h_\mathcal{D}(m) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log N_m(t, \varepsilon, \delta).
\]
We have the following general property.

**Lemma 6.7.** For any probability measure \( m \) on \( X \), we have

\[
h_D(m) \leq h_D(\text{supp}(m)),
\]
where \( h_D(\text{supp}(m)) \) is computed using formula (6.1). In particular, the metric entropy of a probability measure is dominated by the entropy of the whole space.

As in Brin-Katok’s theorem [3], we can introduce the local entropies of \( m \) at \( x \in X \) by

\[
h_+^D(m, x, \epsilon) := \limsup_{t \to \infty} -\frac{1}{t} \log m(B_t(x, \epsilon)), \quad h_+^D(m, x) := \sup_{\epsilon > 0} h_+^D(m, x, \epsilon),
\]
and

\[
h_-^D(m, x, \epsilon) := \liminf_{t \to \infty} -\frac{1}{t} \log m(B_t(x, \epsilon)), \quad h_-^D(m, x) := \sup_{\epsilon > 0} h_-^D(m, x, \epsilon),
\]
where \( B_t(x, \epsilon) \) denotes the ball centered at \( x \) of radius \( \epsilon \) with respect to the distance \( \text{dist}_t \).

Note that in the case of an ergodic invariant measure associated with a continuous map on a metric compact space, the above notions of entropies coincide with the classical entropy of \( m \), see Brin-Katok [3].

Recall that in (6.2) we have associated to \((X, \mathcal{L})\) a special family of distances \( \mathcal{D} := \{\text{dist}_t\}_{t \geq 0} \). Therefore, we can associate to \( m \) a metric entropy and local entropies defined as above in the abstract setting. Recall that a harmonic probability measure \( m \) is extremal if all harmonic probability measures \( m_1, m_2 \) satisfying \( m_1 + m_2 = 2m \) are equal to \( m \). We have the following result.

**Theorem 6.8.** (Dinh-Nguyen-Sibony [17]). Let \((X, \mathcal{L})\) be a compact transversally smooth lamination by hyperbolic Riemann surfaces. Let \( m \) be a harmonic probability measure. Then, the local entropies \( h^\pm \) of \( m \) are constant on leaves. In particular, if \( m \) is extremal, then \( h^\pm \) are constant \( m \)-almost everywhere.

We reproduce from [17] some fundamental problems concerning metric entropies for Riemann surface laminations. Assume here that \((X, \mathcal{L})\) is a compact transversally smooth lamination by hyperbolic Riemann surfaces but the problems can be stated in a more general setting.

**Problem 6.9.** Consider extremal harmonic probability measures \( m \). Is the following variational principle always true

\[
h(\mathcal{L}) = \sup_m h(m) ?
\]

Even when this principle does not hold, it is interesting to consider the invariant

\[
h(\mathcal{L}) - \sup_m h(m)
\]
and to clarify the role of the hyperbolic time in this number.

**Problem 6.10.** If \( m \) is as above, is the identity \( h^+(m) = h^-(m) \) always true ?

We believe that the answer is affirmative and gives an analog of the Brin-Katok theorem.

Notice that there is a notion of entropy for harmonic measures introduced by Kaimanovich [38]. Consider a metric \( g \) of bounded geometry on the leaves of the lamination. Then,
we can consider the heat kernel $p(t, \cdot, \cdot)$ associated to the Laplacian determined by this
metric. If $m$ is a harmonic probability measure on $X$, Kaimanovich defines the entropy
of $m$ as
$$h_K(m) := \int dm(x) \left( \lim_{t \to \infty} \frac{1}{t} \int p(t, x, y) \log p(t, x, y) g(y) \right).$$
He shows that the limit exists and is constant $m$-almost everywhere when $m$ is extremal.

This notion of entropy has been extensively studied for the universal covering of a
compact Riemannian manifold, see e.g. Ledrappier [41].

**Problem 6.11.** It would be of interest to find relations between Kaimanovich entropy
and our notions of entropy. Moreover, study these relations in the context of singular
holomorphic foliations.

In Kaimanovich’s entropy, the transverse spreading is present through the variation of
the heat kernel from leaf to leaf. It would be also interesting to make this dependence
more explicit.

Here is an open problem from [18].

**Problem 6.12.** If $\mathcal{F}$ is a generic element in $\mathcal{F}_d(\mathbb{P}^k)$ with $d, k > 1$, is the hyperbolic
entropy of $\mathcal{F}$ finite? The same question is asked for a singular holomorphic foliation
on a compact Hermitian complex manifold. This is a generalization of Theorem 6.5 to
higher dimensions.

### 7. Lyapunov theory for hyperbolic Riemann surface laminations

The purpose of this section is to present some recent results obtained in our works
[46, 47].

#### 7.1. Brownian motion and Wiener measures.

We start with Garnett’s theory of leafwise Brownian motion in [29] (see also [7, 9]). Our presentation follows [47]. We first recall the construction of the Wiener measure $W_0$ on the Poincaré disc $(\mathbb{D}, g_P)$. Let $\Omega_0$ be
the space consisting of all continuous paths $\omega : [0, \infty) \to \mathbb{D}$ with $\omega(0) = 0$. A cylinder set (in $\Omega_0$) is a set of the form
$$C = C(\{t_i, B_i\} : 1 \leq i \leq m) := \{\omega \in \Omega_0 : \omega(t_i) \in B_i, \quad 1 \leq i \leq m\},$$
where $m$ is a positive integer and the $B_i$’s are Borel subsets of $\mathbb{D}$, and $0 < t_1 < t_2 < \ldots < t_m$ is a set of increasing times. In other words, $C$ consists of all paths $\omega \in \Omega_0$ which can be found within $B_i$ at time $t_i$. Let $\mathcal{A}_0$ be the $\sigma$-algebra on $\Omega_0$ generated by all cylinder sets. For each cylinder set $C := C(\{t_i, B_i\} : 1 \leq i \leq m)$ as above, define

$$W_x(C) := \left( D_{t_1} \left( 1_{B_1} D_{t_2 - t_1} \left( 1_{B_2} \cdots 1_{B_{m-1}} D_{t_m - t_{m-1}} \left( 1_{B_m} \cdots \right) \right) \right) \right)(x),$$

where, $1_{B_i}$ is the characteristic function of $B_i$ and $D_i$ is the diffusion operator given by
(2.2) where $p(x, y, t)$ therein is the heat kernel of the Poincaré disc. It is well-known that
$W_0$ can be extended to a unique probability measure on $(\Omega_0, \mathcal{A}_0)$. This is the canonical
Wiener measure at 0 on the Poincaré disc.

Let $(X, \mathcal{L})$ be a hyperbolic Riemann surface lamination endowed with the leafwise
Poincaré metric $g_P$. Let $\Omega := \Omega(X, \mathcal{L})$ be the space consisting of all continuous paths
$\omega : [0, \infty) \to X$ with image fully contained in a single leaf. This space is called the
sample-path space associated to $(X, \mathcal{L})$. Observe that $\Omega$ can be thought of as the set of
all possible paths that a Brownian particle, located at $\omega(0)$ at time $t = 0$, might follow as time progresses. For each $x \in X$, let $\Omega_x = \Omega_x(X, \mathcal{L})$ be the space of all continuous leafwise paths starting at $x$ in $(X, \mathcal{L})$, that is,
\[ \Omega_x := \{ \omega \in \Omega : \omega(0) = x \}. \]

For each $x \in X$, the following mapping
\[ \Omega_0 \ni \omega \mapsto \phi_x \circ \omega \mapsto \Omega_x, \]
where $\phi_x : D \to L_x$ is given in (2.1). Using this bijection we obtain a natural $\sigma$-algebra $\mathcal{A}_x$ on the space $\Omega_x$, and a natural probability (Wiener) measure $W_x$ on $\mathcal{A}_x$ as follows:
\[ \mathcal{A}_x := \{ \phi_x \circ A : A \in \mathcal{A}_0 \} \quad \text{and} \quad W_x(\phi_x \circ A) := W_0(A), \quad A \in \mathcal{A}_0, \]
where $\phi_x \circ A := \{ \phi_x \circ \omega : \omega \in A \} \subset \Omega_x$.

7.2. Ccocycles. The notion of (multiplicative) cocycles have been introduced in [46] for ($l$-dimensional) laminations. For the sake of simplicity we only formulate this notion for Riemann surface laminations in this article. In the rest of the section we make the following convention: $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Moreover, given any integer $d \geq 1$, $\text{GL}(d, \mathbb{K})$ denotes the general linear group of degree $d$ over $\mathbb{K}$ and $\mathbb{P}^d(\mathbb{K})$ denotes the $\mathbb{K}$-projective space of dimension $d$.

Definition 7.1. (Nguyen [46], Definition 3.2)] A $\mathbb{K}$-valued cocycle (of rank $d$) is a map $A : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{K})$ such that
(1) (identity law) $A(\omega, 0) = \text{id}$ for all $\omega \in \Omega$;
(2) (homotopy law) if $\omega_1, \omega_2 \in \Omega_x$ and $t_1, t_2 \in \mathbb{R}^+$ such that $\omega_1(t_1) = \omega_2(t_2)$ and $\omega_1|_{[0,t_1]}$ is homotopic to $\omega_2|_{[0,t_2]}$ (that is, the path $\omega_1|_{[0,t_1]}$ can be deformed continuously on $L_x$ to the path $\omega_2|_{[0,t_2]}$; the two endpoints of $\omega_1|_{[0,t_1]}$ being kept fixed during the deformation), then
\[ A(\omega_1, t_1) = A(\omega_2, t_2); \]
(3) (multiplicative law) $A(\omega, s + t) = A(\sigma_t(\omega), s)A(\omega, t)$ for all $s, t \in \mathbb{R}^+$ and $\omega \in \Omega$;
(4) (measurable law) the local expression of $A$ on each laminated chart is Borel measurable. Here, the local expression of $A$ on the laminated chart $\Phi : U \to D \times \mathbb{T}$, is the map $A : D \times D \times \mathbb{T} \to \text{GL}(d, \mathbb{K})$ defined by
\[ A(y, z, t) := A(\omega, 1), \]
where $\omega$ is any leafwise path such that $\omega(0) = \Phi^{-1}(y, t)$, $\omega(1) = \Phi^{-1}(z, t)$ and $\omega[0, 1]$ is contained in the simply connected plaque $\Phi^{-1}(\cdot, t)$.

A cocycle $A$ on a smooth Riemann surface lamination $(X, \mathcal{L})$ is called a smooth if, for each laminated chart $\Phi$, the local expression $A$ is smooth with respect to $(y, z)$ and its partial derivatives of any total order with respect to $(y, z)$ are jointly continuous in $(y, z, t)$.

It is worthy noting that the cocycles of rank 1 have been investigated by several authors (see, for example, Candel [7], Deroin [13], etc). The holonomy cocycle (or equivalently the normal derivative cocycle) of the regular part of a $n$-dimensional (possibly singular) holomorphic foliation by hyperbolic Riemann surfaces provides a typical example of $\mathbb{C}$-valued cocycles of rank $n - 1$. Another source of cocycles are those associated with foliations which are obtained from suspensions. These cocycles capture the topological aspect of the considered foliations. Moreover, we can produce new cocycles from the
old ones by performing some basic operations such as the wedge product and the tensor product (see [46, Section 3.1]).

7.3. Oseledec multiplicative ergodic theorem. Now we are in the position to state the Oseledec multiplicative ergodic theorem for hyperbolic Riemann surface laminations.

Theorem 7.2. (Nguyen [46, Theorem 3.11]). Let \((X, L)\) be a hyperbolic Riemann surface lamination. Let \(\mu\) be a harmonic measure which is also ergodic. Consider a cocycle \(A : \Omega \times \mathbb{R}^+ \rightarrow \text{GL}(d, \mathbb{K})\). Assume that the following integrability condition is satisfied for some real number \(t_0 > 0\):

\[
\int_{x \in X} \left( \int_{\Omega_x} \sup_{t \in [0, t_0]} \log^+ \|A(\omega, t)\| dW_x(\omega) \right) d\mu(x) < \infty,
\]

where \(\log^+ := \max(0, \log)\). Then there exist a leafwise saturated Borel set \(Y \subset X\) of total \(\mu\)-measure and a number \(m \in \mathbb{N}\) together with \(m\) integers \(d_1, \ldots, d_m \in \mathbb{N}\) such that the following properties hold:

(i) For each \(x \in Y\) there exists a decomposition of \(\mathbb{K}^d\) as a direct sum of \(\mathbb{K}\)-linear subspaces

\[
\mathbb{K}^d = \bigoplus_{i=1}^m H_i(x),
\]

such that \(\dim H_i(x) = d_i\) and \(A(\omega, t)H_i(x) = H_i(\omega(t))\) for all \(\omega \in \Omega_x\) and \(t \in \mathbb{R}^+\). Moreover, \(x \mapsto H_i(x)\) is a measurable map from \(Y\) into the Grassmannian of \(\mathbb{K}^d\).

For each \(1 \leq i \leq m\) and each \(x \in Y\), let \(V_i(x) := \bigoplus_{j=i}^m H_j(x)\). Set \(V_{m+1}(x) \equiv \{0\}\).

(ii) There are real numbers

\[
\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1,
\]

and for each \(x \in Y\), there is a set \(F_x \subset \Omega_x\) of total \(W_x\)-measure such that for every \(1 \leq i \leq m\) and every \(v \in V_i(x) \setminus V_{i+1}(x)\) and every \(\omega \in F_x\),

\[
\lim_{t \to \infty, t \in \mathbb{R}^+} \frac{1}{t} \log \frac{\|A(\omega, t)v\|}{\|v\|} = \chi_i.
\]

Moreover,

\[
\lim_{t \to \infty, t \in \mathbb{R}^+} \frac{1}{t} \log \|A(\omega, t)\| = \chi_1
\]

for each \(x \in Y\) and for every \(\omega \in F_x\).

Here \(\| \cdot \|\) denotes the standard Euclidean norm of \(\mathbb{K}^d\).

The above result is the counterpart, in the context of hyperbolic Riemann surface laminations, of the classical Oseledec multiplicative ergodic theorem for maps (see [39, 50]). In fact, Theorem 3.11 in [46] is much more general than Theorem 7.2. Indeed, the former is formulated for \(l\)-dimensional laminations and for leafwise Riemannian metrics which satisfy some reasonable geometric conditions.

Assertion (i) above says that the Oseledec decomposition exists for all points \(x\) in a leafwise saturated Borel set of total \(\mu\)-measure and that this decomposition is holonomy invariant. It is worthy noting that the Oseledec decomposition in (i) depends only on \(x \in Y\), in particular, it does not depend on paths \(\omega \in \Omega_x\).

The decreasing sequence of subspaces of \(\mathbb{K}^d\) given by assertion (i):

\[
\{0\} \equiv V_{m+1}(x) \subset V_m(x) \subset \cdots \subset V_1(x) = \mathbb{K}^d
\]
is called the Lyapunov filtration associated to $A$ at a given point $x \in Y$.

The numbers $\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1$ given by assertion (ii) above are called the Lyapunov exponents of the cocycle $A$ with respect to the harmonic measure $\mu$. Moreover, we infer from formulas (7.5) and (7.6) above that these characteristic numbers measure heuristically the expansion rate of $A$ along different vector-directions $v$ and along leafwise Brownian trajectories. In other words, the stochastic formulas (7.5)-(7.6) express the dynamical character of the Lyapunov exponents.

### 7.4. Applications to compact smooth laminations and compact singular foliations.

Let $A : \Omega \times \mathbb{R}^+ \rightarrow \text{GL}(d, \mathbb{K})$ be a smooth cocycle defined on a smooth hyperbolic Riemann surface lamination $(X, \mathcal{L})$. Observe that the map $A^{-1} \circ \mathcal{L} : \Omega \times \mathbb{R}^+ \rightarrow \text{GL}(d, \mathbb{K})$, defined by $A^{-1}(\omega, t) := (A(\omega, t))^{-1}$, is also a cocycle, where $A^*$ (resp. $A^{-1}$) denotes as usual the transpose (resp. the inverse) of a square matrix $A$.

We define two functions $\delta(A), \delta(A) : X \rightarrow \mathbb{R}$ as well as four quantities $\overline{\chi}_{\max}(A), \overline{\chi}_{\min}(A), \underline{\chi}_{\max}(A), \underline{\chi}_{\min}(A)$ as follows. Fix a point $x \in X$, an element $u \in \mathbb{R}^d \setminus \{0\}$ and a simply connected plaque $K$ of $(X, \mathcal{L})$ passing through $x$. Consider the function $f_{u,x} : K \rightarrow \mathbb{R}$ defined by

$$f_{u,x}(y) := \log \frac{\|A(\omega, 1)u\|}{\|u\|}, \quad y \in K, \quad u \in \mathbb{R}^d \setminus \{0\},$$

where $\omega \in \Omega$ is any path such that $\omega(0) = x, \omega(1) = y$ and that $\omega[0, 1]$ is contained in $K$.

Then define

$$\delta(A)(x) := \sup_{u \in \mathbb{R}^d : \|u\| = 1} (\Delta f_{u,x})(x) \quad \text{and} \quad \overline{\delta}(A)(x) := \inf_{u \in \mathbb{R}^d : \|u\| = 1} (\Delta f_{u,x})(x),$$

where $\Delta$ is, as usual, the Laplacian on the leaf $L_x$ induced by the leafwise Poincaré metric $g_P$ on $(X, \mathcal{L})$ (see formula (5.1) for $\beta := g_P$). We also define

$$\overline{\chi}_{\max} = \overline{\chi}_{\max}(A) := \int_X \delta(A)(x) d\mu(x),$$

$$\underline{\chi}_{\max} = \underline{\chi}_{\max}(A) := \int_X \overline{\delta}(A)(x) d\mu(x);$$

$$\overline{\chi}_{\min} = \overline{\chi}_{\min}(A) := -\overline{\chi}_{\max}(A^{-1}),$$

$$\underline{\chi}_{\min} = \underline{\chi}_{\min}(A) := -\underline{\chi}_{\max}(A^{-1}).$$

Note that our functions $\delta, \overline{\delta}$ are the multi-dimensional generalizations of the operator $\delta$ introduced by Candel [7].

We are in the position to state effective integral estimates on the Lyapunov exponents.

**Theorem 7.3.** (Nguyen [46, Theorem 3.12]). Let $(X, \mathcal{L})$ be a compact smooth lamination by hyperbolic Riemann surfaces. Let $\mu$ be a harmonic probability measure which is ergodic. Let $A : \Omega \times \mathbb{R}^+ \rightarrow \text{GL}(d, \mathbb{K})$ be a smooth cocycle. Let

$$\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1$$

be the Lyapunov exponents of the cocycle $A$ with respect to $\mu$, given by Theorem 7.2. Then following inequalities hold

$$\chi_{\max} \leq \chi_1 \leq \overline{\chi}_{\max} \quad \text{and} \quad \underline{\chi}_{\min} \leq \chi_m \leq \underline{\chi}_{\min}. \quad \text{(7.7)}$$
This theorem generalizes some results of Candel [6] and Deroin [13] who treat the case $d = 1$. Under the assumption of Theorem [7.3], the integrability condition (7.4) follows from some well-known estimates of the heat kernels of the Poincaré disc and the fact that the lamination is compact and is without singularities. In fact, we improve the method of Candel in [7].

The holonomy cocycle (or equivalently, the normal derivative cocycle) of a foliation is a very important object which reflects dynamical as well as geometric and analytic aspects of the foliation. Exploring this object allows us to understand more about the foliation itself. Since the main examples of holomorphic foliations by curves are those in the complex projective space $\mathbb{P}^k$ of arbitrary dimension (in which case there are always singularities) or in algebraic manifolds, the following fundamental question arises naturally:

**Question.** Can one define the Lyapunov exponent functions of a positive harmonic current $T$ directed by a compact singular holomorphic hyperbolic foliation $\mathcal{F} = (X, \mathcal{L}, E)$?

We have recently obtained an affirmative answer to this question for generic foliations in dimension two, that is, when the ambient manifold $X$ is a compact complex projective surface, the foliation enjoys Brody hyperbolicity, and $E$ is the set of singularities which are of hyperbolic type.

**Theorem 7.4.** (Nguyen [49, Theorem 1.1]). Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a holomorphic Brody hyperbolic foliation with hyperbolic singularities $E$ in a Hermitian compact complex projective surface $X$. Let $A$ be the holonomy cocycle of the foliation. Let $T$ be a positive harmonic current directed by $\mathcal{F}$ which does not give mass to any invariant analytic curve. Consider the corresponding harmonic measure $\mu := T \wedge g_P$, where $g_P$ is as usual the leafwise Poincaré metric. Then the integrability condition (7.4) is satisfied for all $t_0 > 0$.

Here is an immediate consequence of this theorem.

**Corollary 7.5.** Under the hypotheses and notation of Theorem 7.4, assume in addition that the measure $\mu$ is ergodic. Then $T$ admits the (unique) Lyapunov exponent $\lambda(T)$ given by the formula

$$\lambda(T) := \int_X \left( \int_{\Omega_x} \log \| A(\omega, 1) \| dW_x(\omega) \right) d\mu(x).$$

Moreover, for $\mu$-almost every $x \in X$, we have

$$\lim_{t \to \infty} \frac{1}{t} \log \| A(\omega, t) \| = \lambda(T)$$

for almost every path $\omega \in \Omega$ with respect to the Wiener measure at $x$ which lives on the leaf passing through $x$.

Consider a singular foliation by curves $\mathcal{F} = (\mathbb{P}^2, \mathcal{L}, E)$ on the complex projective plane $\mathbb{P}^2$ such that all the singularities of $\mathcal{F}$ are hyperbolic and that $\mathcal{F}$ has no invariant algebraic curve. By Remark 3.3 we know that $\mathcal{F}$ is Brody hyperbolic. Moreover, the unique ergodicity theorem of Fornæss-Sibony [27] (see Theorem 5.6) says that the harmonic current $T$ is unique up to a multiplicative constant. In particular, the convex cone of all harmonic currents of $\mathcal{F}$ is just a real half-line, and hence all these currents are extremal. Therefore, by Proposition 5.3 the measure $T \wedge g_P$ is ergodic. Consequently, Corollary 7.5 applies and gives us the following result. It can be applied to every generic foliation in $\mathbb{P}^2$ with a given degree $d > 1$. 
Corollary 7.6. Let $\mathcal{F} = (\mathbb{P}^2, \mathcal{L}, E)$ be a singular foliation by curves on the complex projective plane $\mathbb{P}^2$. Assume that all the singularities are hyperbolic and that $\mathcal{F}$ has no invariant algebraic curve. Let $T$ be the unique harmonic current tangent to $\mathcal{F}$ such that $\mu := T \wedge g_P$ is a probability measure. Let $\mathcal{A}$, be as in the statement of Theorem 7.4. Then the conclusion of this theorem as well as that of Corollary 7.5 hold. In particular, $\mathcal{F}$ admits a unique Lyapunov exponent.

The novelty of the last corollary is that the (unique) Lyapunov exponent of such a foliation $\mathcal{F}$ is intrinsic and canonical.

The proof of Theorem 7.4 consists of two steps. Let $g_X$ be a Hermitian metric on $X$ and let $\text{dist}$ denote the distance on $X$ induced by $g_X$. In the first step we show that Theorem 7.4 follows from the new integrability condition (7.10).

(7.10) (new integrability condition): \[
\int_X |\log \text{dist}(x, E)| \cdot (T \wedge g_P)(x) < \infty.
\]

This new condition has the advantage over the old one (7.4) since the former does not involve the somewhat complicating Wiener measures.

To this end we study the behavior of the holonomy cocycle near the singularities with respect to the leafwise Poincaré metric. Roughly speaking, this step quantifies the expansion speed of the hololomy cocycle in terms of the ambient metric $g_X$ when one travels along unit-speed geodesic rays. The main ingredients are in our joint-works with Dinh and Sibony in [16, 17, 18].

The second main step is then devoted to the proof of inequality (7.4). The main difficulty is that known estimates (see, for example, [16]) on the behavior of $T$ near linearizable singularities, only give a weaker inequality

(7.11) \[
\int_X |\log \text{dist}(x, E)|^{1-\delta} \cdot (T \wedge g_P)(x) < \infty, \quad \forall \delta > 0.
\]

So (7.10) is the limiting case of (7.11). The proof of (7.11) relies on the finiteness of the Lelong number of $T$ at every point established in Proposition 2.3. Moreover, Theorem 4.2 (see also a recent result of Dinh-Sibony [24]) sharpens the last estimate by showing that the Lelong number of $T$ vanishes at every hyperbolic singular point $x \in E$. Nevertheless, even this better estimate does not suffice to prove (7.10).

To overcome this obstacle, we use a cohomological idea which exploits fully the assumption that $X$ is projective. This assumption imposes a stronger mass-clustering condition on harmonic currents.

The condition of Brody hyperbolicity seems to be indispensable for the integrability of the holonomy cocycle. Indeed, in a very recent work [35, Theorem A] Hussenot discovers the following remarkable property for a class of Ricatti foliations $\mathcal{F}$ on $\mathbb{P}^2$. For every $x \in \mathbb{P}^2$ outside invariant curves of every foliation in this class, it holds that

$$\limsup_{t \to \infty} \frac{1}{t} \log \|\mathcal{A}(\omega, t)\| = \infty$$

for almost every path $\omega \in \Omega_x$ with respect to the Wiener measure at $x$ which lives on the leaf passing through $x$. By Theorem 2.9 these foliations are hyperbolic since all their singular points have nondegenerate linear part. Nevertheless, neither of them is Brody hyperbolic because they all contain integral curves which are some images of $\mathbb{P}^1$ (see Remark 3.3).
Problem 7.7. (see also [15, 35]). Is the Lyapunov exponent of a generic foliation with a given degree \( d > 1 \) in \( \mathbb{P} \) positive/negative/zero?

Problem 7.8. Does Theorem 7.4 still hold if the ambient compact projective manifold \( X \) is of dimension \( > 2 \)?

7.5. Geometric characterization of Lyapunov exponents. To find a geometric interpretation of these characteristic quantities, our idea consists in replacing the Brownian trajectories by the more appealing objects, namely, the unit-speed geodesic rays. These paths are parameterized by their length (with respect to the leafwise Poincaré metric). Therefore, we characterize the Lyapunov exponents in terms of the expansion rates of \( \mathcal{A} \) along the geodesic rays.

Recall from (2.1) that \( \{\phi_x\}_{x \in X} \) is a given family of universal covering maps \( \phi_x : \mathbb{D} \rightarrow L_x \) with \( \phi_x(0) = x \). For every \( x \in X \), the set of all unit-speed geodesic rays \( \omega : [0, \infty) \rightarrow L_x \) starting at \( x \) (that is, \( \omega(0) = x \)), can be described by the family \( (\gamma_{x, \theta})_{\theta \in [0, 1)} \), where

\[
\gamma_{x, \theta}(R) := \phi_x(e^{2\pi i \theta R}), \quad R \in \mathbb{R}^+,
\]

and \( r_R \) is uniquely determined by the equation \( r_R \mathbb{D} = \mathbb{D}_R \) (see (4.1)). The path \( \gamma_{x, \theta} \) is called the unit-speed geodesic ray at \( x \) with the leaf-direction \( \theta \). Unless otherwise specified, the space of leaf-directions \([0, 1)\) is endowed with the Lebesgue measure. The space of leaf-directions is visibly identified, via the map \( [0, 1) \ni \theta \mapsto e^{2\pi i \theta} \), with the unit circle \( \partial \mathbb{D} \) endowed with the normalized rotation measure.

The following notions of expansion rates for cocycles are needed.

Definition 7.9. Let \( \mathcal{A} \) be a \( \mathbb{K} \)-valued cocycle and \( R > 0 \) a time.

The expansion rate of \( \mathcal{A} \) at a point \( x \in X \) in the leaf-direction \( \theta \) at time \( R \) along the vector \( v \in \mathbb{K}^d \setminus \{0\} \) is the number

\[
\mathcal{E}(x, \theta, v, R) := \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}(R), R)v\|}{\|v\|}.
\]

The expansion rate of \( \mathcal{A} \) at a point \( x \in X \) in the leaf-direction \( \theta \) at time \( R \) is

\[
\mathcal{E}(x, \theta, R) := \sup_{v \in \mathbb{K}^d \setminus \{0\}} \mathcal{E}(x, \theta, v, R) = \sup_{v \in \mathbb{K}^d \setminus \{0\}} \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}(R), R)v\|}{\|v\|} = \frac{1}{R} \log \|A(\gamma_{x, \theta}, R)\|.
\]

Given a \( \mathbb{K} \)-vector subspace \( \{0\} \neq H \subset \mathbb{K}^d \), the expansion rate of \( \mathcal{A} \) at a point \( x \in X \) at time \( R \) along the vector space \( H \) is the interval \( \mathcal{E}(x, H, R) := [a, b] \), where

\[
a := \inf_{v \in H \setminus \{0\}} \int_0^1 \left( \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}(R), R)v\|}{\|v\|} \right) d\theta \quad \text{and} \quad b := \sup_{v \in H \setminus \{0\}} \int_0^1 \left( \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}, R)v\|}{\|v\|} \right) d\theta.
\]

Notice that \( \mathcal{E}(x, \theta, v, R) \) (resp. \( \mathcal{E}(x, \theta, R) \)) expresses geometrically the expansion rate (resp. the maximal expansion rate) of the cocycle when one travels along the unit-speed geodesic ray \( \gamma_{x, \theta} \) up to time \( R \). On the other hand, \( \mathcal{E}(x, H, R) \) represents the smallest closed interval which contains all numbers

\[
\int_0^1 \left( \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}(R), R)v\|}{\|v\|} \right) d\theta,
\]
where \( v \) ranges over \( H \setminus \{0\} \). Note that the above integral is the average of the expansion rate of the cocycle when one travels along the unit-speed geodesic rays along the vector-direction \( v \in H \) from \( x \) to the Poincaré circle with radius \( R \) and center \( x \) spanned on \( L_x \).

We say that a sequence of intervals \([a(R), b(R)] \subset \mathbb{R}\) indexed by \( R \in \mathbb{R}^+ \) converges to a number \( \chi \in \mathbb{R} \) and write \( \lim_{R \to \infty} [a(R), b(R)] = \chi \), if \( \lim_{R \to \infty} a(R) = \lim_{R \to \infty} b(R) = \chi \).

Now we are able to state the result.

**Theorem 7.10.** (Nguyen [47].) Let \((X, \mathcal{L})\) be a compact smooth hyperbolic Riemann surface lamination and \( T \) a directed positive harmonic current which is also extremal. Let \( \mu := T \wedge g_p \) be the measure associated to \( T \). Consider a smooth cocycle \( A : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{K}) \). Then there is a leafwise saturated Borel set \( Y \) of total \( \mu \)-measure which satisfies the conclusion of Theorem 7.2 and the following additional properties:

(i) For each \( 1 \leq i \leq m \) and for each \( x \in Y \), there is a set \( G_x \subset [0, 1) \) of total Lebesgue measure such that for each \( v \in V_i(x) \setminus V_{i+1}(x) \),

\[
\lim_{R \to \infty} \mathcal{E}(x, \theta, v, R) = \chi_i, \quad \theta \in G_x.
\]

Moreover, the maximal Lyapunov exponent \( \chi_1 \) satisfies

\[
\lim_{R \to \infty} \mathcal{E}(x, \theta, R) = \chi_1, \quad \theta \in G_x.
\]

(ii) For each \( 1 \leq i \leq m \) and each \( x \in Y \),

\[
\lim_{R \to \infty} \mathcal{E}(x, H_i(x), R) = \chi_i.
\]

Here \( \mathbb{K}^d = \bigoplus_{i=1}^m H_i(x) \), \( x \in Y \), is the Oseledec decomposition given by Theorem 7.2 and \( \chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1 \) are the corresponding Lyapunov exponents.

Theorem 7.10 gives a geometric meaning to the stochastic formulas (7.5)–(7.6).

Let \((M, \mathcal{L}, E)\) be a transversally smooth (resp. holomorphic) singular foliation by Riemann surfaces in a Riemannian manifold (resp. Hermitian complex manifold) \( M \). Consider a leafwise saturated, compact set \( X \subset M \setminus E \) whose leaves are all hyperbolic. So the restriction of the foliation \((M \setminus E, \mathcal{L})\) to \( X \) gives an inherited compact smooth hyperbolic Riemann lamination \((X, \mathcal{L})\). Moreover, the holonomy cocycle of \((M \setminus E, \mathcal{L})\) induces, by restriction, an inherited smooth cocycle on \((X, \mathcal{L})\). Hence, Theorem 7.10 applies to the latter cocycle. In particular, when \((M, \mathcal{L}, E)\) is a singular holomorphic foliation on a compact Hermitian complex manifold \( M \) of dimension \( n \), the last theorem applies to the induced holonomy cocycle of rank \( n - 1 \) associated with every minimal set \( X \) whose leaves are all hyperbolic. Here a minimal set is a leafwise saturated closed subset of \( M \) which contains no proper subset with this property.

The proof of Theorem 7.10 (i) relies on the theory of Brownian trajectories on hyperbolic spaces. More concretely, some quantitative results on the boundary behavior of Brownian trajectories by Lyons [43] and Cranston [12] and on the shadow of Brownian trajectories by geodesic rays are our main ingredients. This allows us to replace a Brownian trajectory by a unit-speed geodesic ray with uniformly distributed leaf-direction. Hence, Part (i) of Theorem 7.10 will follow from Theorem 7.3.

To establish Part (ii) of Theorem 7.10 we need two steps. In the first step we adapt to our context the so-called Ledrappier type characterization of Lyapunov spectrum which was introduced in [46]. This allows us to show that a similar version of Part (ii) of
Theorem 7.10 holds when the expansion rates in terms of geodesic rays are replaced by some heat diffusions associated with the cocycle. The second step shows that the above heat diffusions can be approximated by the expansion rates. To do this we establish a new geometric estimate on the heat diffusions, which relies on the proof of the geometric Birkhoff ergodic theorem (Theorem 5.4).

**Problem 7.11.** Is Theorem 7.10 still true if \((X, \mathcal{L})\) is the whole regular part of a singular holomorphic foliation \(\mathcal{F}\) by hyperbolic Riemann surfaces on a compact complex manifold \(M\) and \(A\) is the holonomy cocycle?

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