Uniform shrinking and expansion
under isotropic Brownian flows

by

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Summary. We study some finite time transport properties of isotropic Brownian flows. Under a certain nondegeneracy condition on the potential spectral measure, we prove that uniform shrinking or expansion of balls under the flow over some bounded time interval can happen with positive probability. We also provide a control theorem for isotropic Brownian flows with drift. Finally, we apply the above results to show that under the nondegeneracy condition the length of a rectifiable curve evolving in an isotropic Brownian flow with strictly negative top Lyapunov exponent converges to zero as $t \to \infty$ with positive probability.

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1 Introduction

Stochastic flows generalize the notion of stochastic process and it has been suggested that they are a natural probabilistic model for the evolution of passive tracers within a turbulent fluid. The one point motions (trajectories of individual particles) are diffusions and the motions of adjacent points are correlated and form a stochastic flow of homeomorphisms. The flow is generated by a random field of continuous semimartingales $F(t, x)$ in the sense that loosely speaking for a short time increment $t \to t + \Delta t$ the passive tracer starting in $x \in \mathbb{R}^d$ at time $t$ moves along the increment of the field $F$, i.e. we have

$$
\phi_{t,t+\Delta t}(x) - x \approx F(t + \Delta t, x) - F(t, x).
$$

Infinitesimally, the above dynamics is achieved by the family of solutions (for different initial conditions) of the stochastic differential equation of Kunita type

$$
\phi_{s,t}(x) = x + \int_s^t F(du, \phi_{s,u}(x)).
$$

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We consider isotropic Brownian flows (IBF) on $\mathbb{R}^d$ for $d \geq 2$. These are a special class of stochastic flows characterized (modulo regularity conditions) by their spatial translation and rotational invariance, temporal homogeneity and independence of their increments. They have been extensively studied by many authors, including Baxendale and Harris [2], Le Jan [10], Le Jan and Darling [11], and Cranston, Scheutzow and Steinsaltz [4]. In Section 2 we will give a short introduction to IBF with emphasis on properties and facts needed for our purposes later on.

Often when speaking about macroscopic properties of stochastic flows one illustrates the action of the flow by the evolution of an oil spill on the surface of an ocean, where the oil spill is a passive tracer following the turbulence of the ocean’s surface. In Scheutzow and Steinsaltz [14] (see also [4] and [5]) it has been shown that the diameter of the oil spill grows linearly in time as $t \to \infty$, almost surely if the top Lyapunov exponent $\lambda$ is positive, and with strictly positive probability if $\lambda \leq 0$. In this paper, instead of looking at asymptotic behavior of the IBF as $t \to \infty$, we will study the behavior at a fixed time $T$ or, more generally, during a fixed time interval $[T_1, T_2]$. We consider the following question: is there a time interval $[T_1, T_2]$ such that a circular oil spill is uniformly squeezed with positive probability by the action of the ocean during the interval $[T_1, T_2]$? More precisely, we will be discussing the following question: Does an isotropic Brownian flow squeeze a ball of radius $R$ into a ball of radius $r < R$ with positive probability during some fixed time interval $[T_1, T_2]$?

Since the radial component of the motion of every point on the boundary of the ball of radius $R$ is a time changed (scalar) Brownian motion, the motion of every boundary point will almost surely cross the boundary infinitely many times during any time interval $(0, t]$. Therefore the ball cannot be mapped into itself throughout any interval $(0, t]$, and we will restrict attention to intervals $[T_1, T_2]$ with $T_1 > 0$.

Obviously the answer to our question is “No” if the flow is volume-preserving. However it turns out that apart from the volume-preserving case, the answer is usually “Yes”, subject to a particular non-degeneracy condition on the potential spectral measure $M_P$ associated with the flow. A brief survey of isotropic Brownian flows, including the definition of $M_P$, is given in Section 2. The non-degeneracy condition (condition $(\text{C})_\rho$) is given in Section 3 along with the main results (Theorems 3.2 and 3.3 and Corollary 3.4) on the squeezing of balls. Moreover a simple adaptation of the proofs shows that, under exactly the same condition, the flow can also expand balls (see Remark 3.5).

The proofs remain valid if we add a deterministic drift. For the sake of generality we do this and think of the drift part as the deterministic current in the ocean and the random isotropic Brownian part as the unpredictable turbulent movement on the surface. The drift is assumed to be time homogeneous for sake of simplicity.

Section 4 contains material on the reproducing kernel Hilbert space associated with the Brownian field $M(t, x)$, and Section 5 contains a control theorem for stochastic flows. Both of these sections may be of independent interest. The proofs of the main theorems are given in Section 6. Finally, Section 7 considers how the length of a curve evolves under an isotropic Brownian flow, and extends a result of Baxendale and Harris [2].
2 Isotropic Brownian flows

Here we provide a short introduction to IBF following mainly Baxendale and Harris [2], Le Jan [10] and Yaglom [16].

A (forward) stochastic flow of homeomorphisms on $\mathbb{R}^d$ is a family of random homeomorphisms \( \{ \phi_{s,t} : 0 \leq s \leq t < \infty \} \) of $\mathbb{R}^d$ into itself, such that almost surely $\phi_{u,t} \circ \phi_{s,u} = \phi_{s,t}$ for $s \leq u \leq t$ and $\phi_{t,t} = \text{Id}_{\mathbb{R}^d}$. If the increments $\phi_{s,t}$ on disjoint intervals are independent and time homogeneous then the flow is said to be a Brownian flow.

According to Kunita [9, Theorem 4.2.8], under suitable regularity conditions, Brownian flows of homeomorphisms can be realized as solutions of Kunita-type SDEs

\[ \phi_{s,t}(x) = x + \int_s^t M(du, \phi_{s,u}(x)) + \int_0^t v(\phi_{s,u})du. \]  

(2.1)

Here $v : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field and $M : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a mean-zero Gaussian random field. $M$ is called the generating Brownian field and its distribution is determined by the covariances

\[ \mathbb{E} \left[ \langle M(t,x), \xi \rangle \langle M(s,y), \eta \rangle \right] = (s \wedge t) \langle b(x,y)\xi, \eta \rangle, \quad \xi, \eta \in \mathbb{R}^d, \]

for some covariance tensor $b : \mathbb{R}^d \times \mathbb{R}^d \to L(\mathbb{R}^d)$. The function $b$ is positive semi-definite: for all $n \geq 1$, all $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \mathbb{R}^d$ and all $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)} \in \mathbb{R}^d$ we have

\[ \sum_{k,\ell=1}^n \left\langle b(x^{(k)}, x^{(\ell)})\xi^{(k)}, \xi^{(\ell)} \right\rangle \geq 0. \]  

(2.2)

The law of the stochastic flow \( \{ \phi_{s,t} : 0 \leq s \leq t < \infty \} \) is determined by the functions $b(x,y)$ and $v(x)$. The differentiability properties of the mappings $\phi_{s,t}$ depend on the differentiability of the functions $b(x,y)$ and $v(x)$.

An isotropic Brownian flow (IBF) on $\mathbb{R}^d$ is a Brownian flow of diffeomorphisms of $\mathbb{R}^d$ for which the distribution of each $\phi_{s,t}$ is invariant under rigid transformations of $\mathbb{R}^d$. IBF have been extensively studied by Baxendale and Harris [2] and Le Jan [10]. The invariance in distribution of $\phi_{s,t}$ under rigid motions implies the invariance in distribution of the generating Brownian field $M(t,x)$; we say that $M$ is an isotropic Brownian field. The invariance under translations implies that $b(x,y) = b(x-y,0) \equiv b(x-y)$ and then the invariance under rotations and reflections implies that

\[ b(x) = O^T b(Ox)O \]  

(2.3)

for all $O$ in the orthogonal group $O^d$. Moreover for IBF we have $v(x) \equiv 0$. In this paper we will assume that $b(x)$ is $C^4$. Then $b(x-y)$ will be $C^{2,2}$ as a function of $x$ and $y$, and so the resulting IBF will consist of $C^1$ diffeomorphisms, see Kunita [9, Theorem 4.6.5]. Moreover, the isotropy property (2.3) implies that $b(0) = cI$ for some constant $c$. At the cost of rescaling time by a constant factor we can and will assume that $b(0) = I$. In order to avoid the trivial case where the flow consists of translations, we assume also that $b(x) \neq 1$. 

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According to Yaglom [16, Section 4] (and as described in [2]) a positive semi-definite tensor \( b(x) \) with the above properties can be written in the form

\[
b_{ij}(x) = (B_L(|x|) - B_N(|x|)) \frac{x_ix_j}{|x|^2} + \delta_{ij} B_N(|x|)
\]

(2.4)

for \( x \neq 0 \), where \( B_L \) and \( B_N \) are the so-called longitudinal and transverse covariance functions defined by

\[
B_L(r) = b_{ii}(re_i), \quad B_N(r) = b_{ij}(re_j)
\]

for \( r \geq 0 \) and any \( i \neq j \). As usual \( e_i \) denotes the \( i \)-th standard basis vector in \( \mathbb{R}^d \). \( B_L \) and \( B_N \) are bounded \( C^4 \) functions with bounded derivatives. Further, the isotropic covariance tensor \( b \) can be decomposed

\[
b(x) = \mu_0 I + \mu_1 b_P(x) + \mu_2 b_S(x)
\]

(2.5)

with \( \mu_i \geq 0 \) and \( \mu_0 + \mu_1 + \mu_2 = 1 \), where \( b_P(x) \) is the covariance tensor for an isotropic Brownian field consisting of gradient vector fields, and \( b_S(x) \) is the covariance tensor for an isotropic Brownian field consisting of divergence-free vector fields. The labels \( P \) and \( S \) stand for “potential” and “solenoidal” respectively. With the normalizing conditions \( b_P(0) = b_S(0) = I \) and \( \lim_{|x| \to \infty} b_P(x) = \lim_{|x| \to \infty} b_S(x) = 0 \) the decomposition is unique (except trivially when \( \mu_1 = 0 \) or \( \mu_2 = 0 \)). The functions \( b_P \) and \( b_S \) can each be written in the form (2.4) and the corresponding longitudinal and transverse covariance functions \( B_{PL}, B_{PN}, B_{SL} \) and \( B_{SN} \) are uniquely determined by two finite spectral measures \( M_P \) and \( M_S \) on the positive real line \((0, \infty)\) through the expressions

\[
B_{PL}(s) = 2^{\frac{d-2}{2}} \Gamma \left( \frac{d}{2} \right) \int_{(0,\infty)} \left[ \frac{J_\nu(sr)}{(sr)^{\frac{d}{2}}} - \frac{J_{\nu+2}(sr)}{(sr)^{\frac{d}{2}}} \right] M_P(dr),
\]

\[
B_{PN}(s) = 2^{\frac{d-2}{2}} \Gamma \left( \frac{d}{2} \right) \int_{(0,\infty)} \frac{J_\nu(sr)}{(sr)^{\frac{d}{2}}} M_P(dr),
\]

\[
B_{SL}(s) = 2^{\frac{d-2}{2}} \Gamma \left( \frac{d}{2} \right) (d-1) \int_{(0,\infty)} \frac{J_\nu(sr)}{(sr)^{\frac{d}{2}}} M_S(dr),
\]

\[
B_{SN}(s) = 2^{\frac{d-2}{2}} \Gamma \left( \frac{d}{2} \right) \int_{(0,\infty)} \left[ \frac{J_{\nu+2}(sr)}{(sr)^{\frac{d}{2}}} - \frac{J_\nu(sr)}{(sr)^{\frac{d}{2}}} \right] M_S(dr),
\]

(2.6)

where \( J_\nu \) denotes the Bessel function (of the first kind) of order \( \nu \). \( M_P \) and \( M_S \) are the potential spectral measure and the solenoidal spectral measure respectively for the isotropic covariance function \( b \). The normalization \( b_P(0) = b_S(0) = I \) gives \( M_P((0, \infty)) = d \) and \( M_S((0, \infty)) = \frac{d}{d+2} \), and the assumption that \( b \) is \( C^4 \) implies that \( M_P \) and \( M_S \) both have finite 4th moments. For future reference, write

\[
\beta_L := -\frac{\partial^2 b_{ii}}{\partial x_i^2}(0) = -B''_L(0) = \frac{3\mu_1}{d(d+2)} \int s^2 M_P(ds) + \frac{(d-1)\mu_2}{d(d+2)} \int s^2 M_S(ds) > 0
\]

(2.7)
and
\[ \beta_N := -\frac{\partial^2 b_{ij}}{\partial x_j^2}(0) = -\mathcal{B}_N^0(0) = -\frac{\mu_1}{d(d+2)} \int s^2 M_F(ds) + \frac{(d+1)\mu_2}{d(d+2)} \int s^2 M_S(ds) > 0 \quad (2.8) \]
for \( j \neq i \). Conversely, to every pair of suitably normalized \( M_F \) and \( M_S \) with finite 4th moments and non-negative constants \( \mu_1 + \mu_2 \leq 1 \) we can construct a \( C^4 \) isotropic covariance tensor \( b \) and consequently an isotropic Brownian field and (via a Kunita-type SDE) also an isotropic Brownian flow.

3 Uniform shrinking and expansion

As mentioned in the introduction we will give the main result in the more general setting of IBF with drift rather than for pure IBF. Let \( M(t, x, \omega) : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d \) be an isotropic Brownian generating field with a \( C^4 \) covariance tensor \( b \) satisfying \( b(0) = I \) and \( b(x) \neq I \). Let \( v(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be a deterministic \( C^2 \) function (derivatives of order up to 2 exist and are bounded and continuous). Consider the semimartingale field
\[ F(t, x) = M(t, x) + tv(x) \]
and the stochastic flow \( \phi \) generated via the SDE:
\[ \phi_{s,t}(x) = x + \int_s^t F(du, \phi_{s,u}(x)) = x + \int_s^t M(du, \phi_{s,u}(x)) + \int_s^t v(\phi_{s,u}(x))du. \]
We will call \( \phi \) an isotropic Brownian flow with drift. Of course, in case \( v \equiv 0 \), \( \phi \) is simply an IBF.

Before we proceed with the main result we shall state the condition on the isotropic covariance tensor \( b(x) \) which will be needed below. Suppose \( \rho > 0 \)

**Condition (C)_\rho:** \( \mu_1 > 0 \) in the decomposition \([2.5]\) and the potential spectral measure \( M_F \) is not supported on the set of all zeros of the mapping \( s \mapsto \mathcal{J}_d^2(\rho s) \), i.e. \( \text{supp}(M_F) \not\subset \{ s \geq 0 : \mathcal{J}_d^2(\rho s) = 0 \} \).

**Remark 3.1.** (i) The condition \( \mu_1 > 0 \) says that the isotropic Brownian field \( M(t, x) \) is not divergence-free. From \([2.7]\) and \([2.8]\) we have \((d+1)\beta_L - (d-1)\beta_N \geq 0\), with equality if and only \( \mu_1 = 0 \). Recall that the flow is divergence-free (incompressible) if and only if \((d+1)\beta_L - (d-1)\beta_N = 0\).

(ii) If \( \mu_1 > 0 \) and the potential spectral measure \( M_F \) has a density then \((C)_\rho \) is satisfied for all \( \rho > 0 \). Using Fourier transform theory, the covariance function \( b \) can be written
\[ b(x) = \int_{\mathbb{R}^d} e^{i(x, \lambda)} F(d\lambda) \]
where \( F \) is a \( L(\mathbb{R}^d) \) valued measure with the property that \( F(A) \) is non-negative definite for every Borel subset \( A \subset \mathbb{R}^d \). The isotropy property \([2.3]\) of \( b \) implies that \( F \) can be written with respect to polar coordinates \( r = |\lambda| \) and \( \theta = \lambda/|\lambda| \) in the form
\[ F(d\theta, dr) = \mu_1 \theta_j \theta_k \sigma(d\theta) M_P(dr) + \mu_2 (\delta_{jk} - \theta_j \theta_k) \sigma(d\theta) M_S(dr) \]
for $r > 0$, where $\sigma$ denotes the uniform probability measure on the unit sphere $S^{d-1}$. (This is the argument used by Yaglom \[10\] to derive the formulas \[2.6\].) Therefore the integrability condition
\[
\int_{\mathbb{R}^d} \|b(x)\| \, dx < \infty
\]
implies the existence of a density for $F$, which in turn implies the existence of a density for $M_P$ and hence condition (C)$_{\rho}$ for all $\rho > 0$, so long as $\mu_1 > 0$.

The first main result deals with the squeezing, or contraction, of a ball of some fixed radius $R$. Since the distribution of the IBF is invariant under translations, it is enough to consider the ball $B(0, R)$ centered at 0.

**Theorem 3.2.** Let $\{\phi_{s,t}(x) : 0 \leq s \leq t < \infty, x \in \mathbb{R}^d\}$ be an isotropic Brownian flow with drift with generating field $\{F(t,x) : t \geq 0, x \in \mathbb{R}^d\}$. For $R > 0$, assume that the covariance tensor $b$ satisfies condition (C)$_R$. Then there exists $\delta > 0$ such that
\[
P(\phi_t(B(0, R + \delta)) \subset B(0, R - \delta) \text{ for all } t \in [T_1, T_2]) > 0
\]
for all $0 < T_1 < T_2$.

The second main result extends this result to the squeezing of the closed ball $\overline{B}(0, R)$ inside the open ball $B(0, r)$ for arbitrary $R$ and $r$. Clearly it is enough to consider the case where $r < R$.

**Theorem 3.3.** Let $\{\phi_{s,t}(x) : 0 \leq s \leq t < \infty, x \in \mathbb{R}^d\}$ be an isotropic Brownian flow with drift with generating field $\{F(t,x) : t \geq 0, x \in \mathbb{R}^d\}$. Suppose $0 < r < R$. Assume that the covariance tensor $b$ satisfies condition (C)$_{\rho}$ for all $\rho \in [r, R]$. Then
\[
P(\phi_t(B(0, r)) \subset B(0, R) \text{ for all } t \in [T_1, T_2]) > 0
\]
for all $0 < T_1 < T_2$.

**Corollary 3.4.** Let $\{\phi_{s,t}(x) : 0 \leq s \leq t < \infty, x \in \mathbb{R}^d\}$ be an isotropic Brownian flow with drift with generating field $\{F(t,x) : t \geq 0, x \in \mathbb{R}^d\}$. Assume that the covariance tensor satisfies condition (C)$_{\rho}$ for all $\rho > 0$. Then for arbitrary $R, r > 0$ the inequality \[3.10\] holds for all $0 < T_1 < T_2$.

Clearly Corollary 3.4 is an immediate consequence of Theorem 3.3. The proofs of Theorems 3.2 and 3.3 will be given in Section 6.

**Remark 3.5.** There are versions of these three results asserting expansion instead of expansion. Under exactly the same conditions, the equations \[3.9\] and \[3.10\] can be replaced by
\[
P(\phi_t(B(0, R - \delta)) \supset B(0, R + \delta) \text{ for all } t \in [T_1, T_2]) > 0
\]
and
\[
P(\phi_t(B(0, r)) \supset \overline{B}(0, R) \text{ for all } t \in [T_1, T_2]) > 0
\]
respectively.
4 RKHS and the generating Brownian field

Recall the covariance function $b(x, y) \in L(\mathbb{R}^d)$ of the generating Brownian field $M(t, x)$ is given by

$$\langle b(x, y)\xi, \eta \rangle = \mathbb{E}[\langle M(1, x), \xi \rangle \langle M(1, y), \eta \rangle] \quad \xi, \eta \in \mathbb{R}^d.$$ 

Assume that $b$ is continuous in both variables and bounded. Associated to the positive semi-definite function $b$ is a Hilbert space $H$ consisting of vector fields on $\mathbb{R}^d$, that is $H \subset C(\mathbb{R}^d : \mathbb{R}^d)$. The space $H$ is characterized by the properties

(i) for each $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ the vector field $b_{x, \xi} : y \rightarrow b(x, y)\xi$ is an element of $H$; and

(ii) $\langle f, b_{x, \xi} \rangle_H = \langle f(x), \xi \rangle$ for all $f \in H$ and $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$.

The property (ii) is called the reproducing property, and says that taking an inner product with $b_{x, \xi}$ acts like an evaluation map on $H$. In this setting the function $b$ is called a reproducing kernel and $H$ is the associated reproducing kernel Hilbert space (RKHS). For details of the theory of reproducing kernel Hilbert spaces for vector valued functions see Baxendale [1, Section 5].

The reproducing property (ii) shows how the Hilbert space $H$ determines the functions $b_{x, \xi}$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$. These in turn determine the function $b$ and hence the distribution of the generating Brownian field $M(t, x)$. In particular $H$ is the Cameron Martin space for the distribution $\gamma = p \circ M(1, \cdot)^{-1}$, see Bogachev [3].

The main result in this section is Theorem 4.2, which shows that the condition $(C)_\rho$ is equivalent to the existence of a particular sort of vector field in the RKHS $H$.

**Proposition 4.1.** Let $H$ be a reproducing kernel Hilbert space of functions $\mathbb{R}^d \rightarrow \mathbb{R}^d$, and suppose $\mu$ is a measure on $H$ such that

$$\int \|U\|_H^2 \, d\mu(U) < \infty$$

Then there exists $D \in H$ such that

$$\int \langle V, U \rangle_H \, d\mu(U) = \langle V, D \rangle_H \quad \text{for all } V \in H. \quad (4.13)$$

Moreover

$$D(x) = \int U(x) \, d\mu(U) \quad \text{for all } x \in \mathbb{R}^d$$

and

$$\|D\|_H^2 = \int \int \langle U_1, U_2 \rangle_H \, d\mu(U_1) d\mu(U_2).$$

**Proof.** Consider the linear mapping $H \rightarrow \mathbb{R}$ given by

$$V \mapsto \int \langle V, U \rangle_H \, d\mu(U).$$

Since

$$\int |\langle V, U \rangle_H| \, d\mu(U) \leq \int \|V\|_H \|U\|_H \, d\mu(U) = \|V\|_H \int \|U\|_H \, d\mu(U)$$

the mapping is well-defined and continuous and so there exists $D \in H$ satisfying (4.13). Taking $V = b_{x, \xi}$ we get

$$\langle D(x), \xi \rangle = \langle D, b_{x, \xi} \rangle_H = \int \langle U, b_{x, \xi} \rangle_H \, d\mu(U) = \int \langle U(x), \xi \rangle \, d\mu(U) = \left\langle \int U(x) \, d\mu(U), \xi \right\rangle.$$
so that
\[ D(x) = \int U(x) \, d\mu(U). \]

Finally
\[
\|D\|_H^2 = \langle D, D \rangle_H = \int \langle D, U_2 \rangle_H \, d\mu(U_2) = \int \left( \int \langle U_1, U_2 \rangle_H \, d\mu(U_1) \right) \, d\mu(U_2) \\
= \int \int \langle U_1, U_2 \rangle_H \, d\mu(U_1) \, d\mu(U_2)
\]

and the proof is complete. \( \square \)

The result above does not use isotropy or even homogeneity. For the remainder of this subsection we restrict to the isotropic situation.

**Theorem 4.2.** Let \( H \) be the reproducing kernel Hilbert space corresponding to the covariance tensor \( b(x) \) for an isotropic Brownian field \( M(t, x) \), and let \( \rho > 0 \). There exists a vector field \( V \in H \) which points inwards on the boundary of the ball with radius \( \rho \) and centered at the origin, that is,
\[
\langle V(\rho \theta), \theta \rangle < 0 \quad \text{whenever} \quad \|\theta\| = 1,
\]
if and only if the covariance tensor \( b(x) \) satisfies condition \((C)_\rho\).

**Proof.** Denote by \( \sigma \) the uniform probability measure on the unit sphere \( S^{d-1} \), and let \( \mu \) be the probability measure which is the image of \( \sigma \) under the mapping \( \phi \mapsto b_{\rho \phi, \phi} \) of \( S^{d-1} \) into \( H \).

Since \( \phi \mapsto \|b_{\rho \phi, \phi}\|_H = \sqrt{\langle b(0) \phi, \phi \rangle} \) is bounded on \( S^{d-1} \) we can apply Proposition 4.1 to obtain a vector field \( \tilde{V} \in H \) given by
\[
\tilde{V}(x) = \int_{S^{d-1}} b_{\rho \phi, \phi}(x) \, d\sigma(\phi).
\]

Using the isotropy of \( b \) and the fact that \( \sigma \) is invariant under the action of \( R \in SO(d) \) we obtain for \( \theta \in S^{s-1} \)
\[
\langle \tilde{V}(\rho \theta), \theta \rangle \quad = \quad \int_{S^{d-1}} \langle b_{\rho \phi, \phi}(\rho \theta), \theta \rangle \, d\sigma(\phi) \\
= \quad \int_{S^{d-1}} \langle b(\rho \phi - \rho \theta) \phi, \theta \rangle \, d\sigma(\phi) \\
= \quad \int_{S^{d-1}} \langle b(\rho R \phi - \rho R \theta) R \phi, R \theta \rangle \, d\sigma(\phi) \\
= \quad \int_{S^{d-1}} \langle b(\rho \phi - \rho R \theta) \phi, R \theta \rangle \, d\sigma(\phi) \\
= \quad \langle \tilde{V}(\rho R \theta), R \theta \rangle,
\]
so that \( \theta \mapsto \langle \tilde{V}(\rho \theta), \theta \rangle \) is constant. Moreover, for all \( U \in H \) we have
\[
\int_{S^{d-1}} \langle U(\rho \phi), \phi \rangle \, d\sigma(\phi) = \int_{S^{d-1}} \langle U, b_{\rho \phi, \phi} \rangle_H \, d\sigma(\phi) = \langle U, \tilde{V} \rangle_H
\]
and in particular
\[ 0 \leq \|\hat{V}\|_H^2 = \int_{S^{d-1}} \langle \hat{V}(\rho\phi), \phi \rangle \, d\sigma(\phi) = \int_{S^{d-1}} \int_{S^{d-1}} \langle b(\rho\phi - \rho\phi)\theta, \phi \rangle \, d\sigma(\theta)d\sigma(\phi). \tag{4.16} \]

If the right side of (4.16) is non-zero then taking \( V = -\hat{V} \) gives
\[ \langle V(\rho\theta), \theta \rangle = -\langle \hat{V}(\rho\theta), \theta \rangle = -\int_{S^{d-1}} \langle \hat{V}(\rho\theta), \theta \rangle \, d\sigma(\theta) = -\|\hat{V}\|_H^2 < 0, \]
so that (4.14) is satisfied. Conversely, if the right side of (4.16) is zero then \( \|\hat{V}\|_H = 0 \) and so
\[ \int_{S^{d-1}} \langle V(\rho\theta), \theta \rangle \, d\sigma(\theta) = \langle V, \hat{V} \rangle_H = 0 \]
for all \( V \in H \), so that \( V \) cannot satisfy (4.14). Therefore, in order to prove Theorem 4.2 it remains to show that the right side of (4.16) is positive if and only if \((C)_\rho \) is satisfied. The proof of Theorem 4.2 is completed using the following result.

**Proposition 4.3.** For \( \rho > 0 \)
\[ \int_{S^{d-1}} \int_{S^{d-1}} \langle b(\rho\phi - \rho\phi)\theta, \phi \rangle \, d\sigma(\theta)d\sigma(\phi) = \mu_1 2^{d-2} \left[ \Gamma \left( \frac{d}{2} \right) \right]^2 \int_{(0,\infty)} \left( \frac{J_{d/2}(\rho)}{\rho^{(d-2)/2}} \right)^2 dM_P(s). \tag{4.17} \]
In particular
\[ \int_{S^{d-1}} \int_{S^{d-1}} \langle b(\rho\phi - \rho\phi)\theta, \phi \rangle \, d\sigma(\theta)d\sigma(\phi) > 0 \]
if and only if condition \((C)_\rho \) is satisfied.

**Proof.** We consider three special cases corresponding to the three terms in the decomposition (2.5). First, if \( b_{jk}(x) = \delta_{jk} \), then
\[ \int_{S^{d-1}} \int_{S^{d-1}} \langle b(\rho\phi - \rho\phi)\theta, \phi \rangle \, d\sigma(\theta)d\sigma(\phi) = \int_{S^{d-1}} \int_{S^{d-1}} \langle \theta, \phi \rangle \, d\sigma(\theta)d\sigma(\phi) = 0. \]
For the second case we suppose that \( b_{jk}(x) \) is isotropic potential. In this case equation (4.25) of Yaglom [16] gives
\[ b_{jk}(x) = \int_{(0,\infty)} \int_{S^{d-1}} e^{i(x,s\psi)} \psi_j \psi_k d\sigma(\psi) dM_P(s) \]
and so
\[ \langle b(\rho\phi - \rho\phi)\theta, \phi \rangle = \int_{0}^{\infty} \int_{S^{d-1}} \langle \theta, \psi \rangle \langle \phi, \psi \rangle e^{i(\rho\phi - \rho\phi, s\psi)} d\sigma(\psi) dM_P(s). \]
Applying Fubini’s theorem to the bounded function \( \langle \theta, \psi \rangle \langle \phi, \psi \rangle e^{i(\rho\phi - \rho\phi, s\psi)} \) we get
\[ \int_{S^{d-1}} \int_{S^{d-1}} \langle b(\rho\phi - \rho\phi)\theta, \phi \rangle \, d\sigma(\theta)d\sigma(\phi) \]
\[ = \int_{S^{d-1}} \int_{S^{d-1}} \left( \int_{0}^{\infty} \int_{S^{d-1}} \langle \theta, \psi \rangle \langle \phi, \psi \rangle e^{i(\rho\phi - \rho\phi, s\psi)} d\sigma(\psi) dM_P(s) \right) d\sigma(\theta)d\sigma(\phi) \]
\[ = \int_{0}^{\infty} \int_{S^{d-1}} \left( \int_{S^{d-1}} \langle \theta, \psi \rangle \langle \phi, \psi \rangle e^{i(\rho\phi - \rho\phi, s\psi)} d\sigma(\theta) d\sigma(\phi) \right) \langle \phi, \psi \rangle dM_P(s) \]
\[ = \int_{0}^{\infty} \int_{S^{d-1}} \langle \theta, \psi \rangle e^{i(\rho\phi, s\psi)} d\sigma(\theta) \left( \int_{S^{d-1}} \langle \phi, \psi \rangle e^{-i(\rho\phi, s\psi)} d\sigma(\phi) \right) dM_P(s). \]
Now
\[
\int_{\mathbb{S}^{d-1}} \langle \theta, \psi \rangle e^{i(\rho \theta, s \psi)} d\sigma(\theta) = \frac{1}{is} \frac{\partial}{\partial \rho} \int_{\mathbb{S}^{d-1}} e^{i(\rho \theta, s \psi)} d\sigma(\theta)
\]
\[= \frac{(i)}{is} \frac{2^{(d-2)/2} \Gamma(d/2)}{\Gamma((d-2)/2)} \left( \frac{J_{(d-2)/2}(\rho s)}{(\rho s)^{(d-2)/2}} \right)\]
\[= \frac{(ii)}{i} \frac{-2^{(d-2)/2} \Gamma(d/2)}{\Gamma((d-2)/2)} \left( \frac{J_{d/2}(\rho s)}{(\rho s)^{(d-2)/2}} \right).
\]

and similarly
\[
\int_{\mathbb{S}^{d-1}} \langle \phi, \psi \rangle e^{-i(\rho \phi, s \psi)} d\sigma(\phi) = \frac{2^{(d-2)/2} \Gamma(d/2)}{i} \left( \frac{J_{d/2}(\rho s)}{(\rho s)^{(d-2)/2}} \right).\]

The equality (i) uses the fact that
\[
\int_{\mathbb{S}^{d-1}} e^{i(\rho \theta, \psi)} d\sigma(\theta) = \frac{\Gamma(d/2) J_{(d-2)/2}(|x|)}{|x|^{(d-2)/2}},
\]
see equations (4.9) through (4.11) of Yaglom [16]. For (ii) see, for example, Watson [15, page 45]. Therefore
\[
\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \langle b(\rho \theta - \rho \phi) \theta, \phi \rangle d\sigma(\theta) d\sigma(\phi)
\]
\[= \int_{0}^{\infty} \int_{\mathbb{S}^{d-1}} \left( \int_{\mathbb{S}^{d-1}} \langle \theta, \psi \rangle e^{i(\rho \theta, s \psi)} d\sigma(\theta) \right) \left( \int_{\mathbb{S}^{d-1}} \langle \phi, \psi \rangle e^{-i(\rho \psi, s \psi)} d\sigma(\phi) \right) d\sigma(\psi) dM_{P}(s)
\]
\[= 2^{d-2} \left[ \Gamma(d/2) \right]^{2} \int_{0}^{\infty} \left( \frac{J_{d/2}(\rho s)}{(\rho s)^{(d-2)/2}} \right)^{2} dM_{P}(s).
\]

For the third case we suppose that \( b_{jk}(x) \) is isotropic solenoidal, so that \( \text{div} V = 0 \) for every \( V \in H \). For the vector field \( \tilde{V} \) defined in \([4,15]\) the divergence theorem applied to the ball of radius \( \rho \) gives
\[
\langle V, \tilde{V} \rangle = \int_{\mathbb{S}^{d-1}} \langle V(\rho \theta), \theta \rangle dm(\theta) = 0
\]
for all \( V \in H \) because \( \text{div} \tilde{V} = 0 \). Taking \( V = \tilde{V} \) we get
\[
\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \langle b(\rho \theta - \rho \phi) \theta, \phi \rangle d\sigma(\theta) d\sigma(\phi) = \| \tilde{V} \|^{2} = 0.
\]

The first assertion is now obtained by summing the contributions of the three terms in the decomposition \([2.5]\). The second assertion follows because the right side of \([4.16]\) is strictly positive unless \( J_{d/2}(\rho s) = 0 \) for \( M_{P} \)-almost all \( s \in (0, \infty) \). □

5 A control theorem for stochastic flows

The following theorem is based on a result of Dolgopyat, Kaloshin and Koralov [7, Section 2.4]. It does not use the isotropy condition.
Theorem 5.1. Assume that \( v(t,x) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d \) is a deterministic function which is \( C^2 \) with bounded derivatives in the space variable \( x \) for every \( t \) and continuous in \( t \) for every \( x \). Let \( M(t,x) \) be a generating Brownian field with covariance tensor \( b(x,y) \) which is \( C^{2,2} \) with bounded derivatives in the variables \( x \) and \( y \). Let \( \{ \phi_{s,t}(x) : 0 \leq s \leq t < \infty, x \in \mathbb{R}^d \} \) be the flow generated by the semimartingale field \( F(t,x) = M(t,x) + \int_0^t v(s,x)ds \), i.e.

\[
\phi_{s,t}(x) = x + \int_s^t F(du, \phi_{u,s}(x)) \quad \text{for all } x \in \mathbb{R}^d, \ 0 \leq s \leq t.
\]

Suppose that \( V \) is a vector field in the reproducing kernel Hilbert space of \( M \), and let \( \xi_t(x) \) denote the solution of the ordinary differential equation

\[
\begin{aligned}
\dot{\xi}_t(x) &= V(\xi_t(x)), \\
\xi_0(x) &= x.
\end{aligned}
\]

Then for all compact subsets \( K \subset \mathbb{R}^d \), time instants \( T > 0 \) and all \( \delta > 0 \) there exist \( c_0 > 0 \) such that

\[
P(\sup_{t \leq T} \sup_{x \in K} |\phi_{t/c}(x) - \xi_t(x)| < \delta) > 0
\]

for all \( c \geq c_0 \).

Proof: Choose a compact set \( K \subset \mathbb{R}^d \), a time \( T > 0 \) and \( \delta > 0 \). As usual we will write \( \phi_t \) for \( \phi_{0,t} \). Let \( \{V_i : i \geq 1\} \) be a complete orthonormal set of vector fields for the reproducing kernel Hilbert space \( H \). Then as in [2] (see also Bogachev [3, Theorem 3.5.1]), the generating Brownian field \( M(t,x) \) can be written in distribution as

\[
M(t,x) = \sum_{i=1}^{\infty} B^i(t)V_i(x)
\]

where \( \{B^i(t) : t \geq 0\}_{i \geq 1} \) are independent standard scalar Brownian motions, and the Kunita stochastic differential equation for \( \phi \) can be written

\[
\phi_t(x) = x + \sum_{i=1}^{\infty} \int_0^t V_i(\phi_s(x))dB^i_s + \int_0^t v(s,\phi_s(x))ds.
\] (5.18)

We will work with a complete orthonormal set chosen so that \( V = \lambda V_1 \) for some \( \lambda > 0 \). For any \( c > 0 \) we will consider the SDE (5.18) for \( 0 \leq t \leq T/c \). Without loss of generality we can assume that the underlying probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) is the infinite product of the path spaces of the standard Brownian motions \( B^i \), that is

\[
(\Omega, \mathcal{F}, \mathbb{P}) = \left( \times_{i \in \mathbb{N}} C([0,T/c] : \mathbb{R}), \otimes B(C([0,T/c] : \mathbb{R})), \otimes \mathbb{P}_i \right),
\]

where \( \mathbb{P}_i \) denotes the Wiener measure on \( (C([0,T/c] : \mathbb{R}), B(C([0,T/c] : \mathbb{R}))) \) for all \( i \in \mathbb{N} \). On the canonical path space we have \( B^j_t(\omega) = \omega^j(t) \) for all \( \omega \in \Omega \) and \( t \leq T/c \). Define
\[ B_t^c = (B_t^{c,1}, B_t^{c,2}, \ldots) := (B_t^1 - \lambda ct, B_t^2, \ldots). \]
Then the SDE (5.18) becomes
\[
\phi_t(x) = x + c \int_0^t V(\phi_s(x))ds + \sum_{i=1}^\infty \int_0^t V_i(\phi_s(x))dB^{c,i}_s + \int_0^t v(s, \phi_s(x))ds
\]
for \( t \in [0, T/c] \). Let \( P_t^c \) be the measure on \( (C([0, T/c] : \mathbb{R}), \mathcal{B}(C([0, T/c] : \mathbb{R})) \) under which \( B_t^{c,1} \)

is a standard Brownian motion. It is well known that \( P^c \)

is equivalent to \( P_1 \) with
\[
P_1^c(d\omega^1) = \exp \left[ \lambda c \omega^1(T/c) - \frac{\lambda^2 c T}{2} \right] P_1(d\omega^1).
\]
Since the transformation of \( B \) into \( B^c \) involves only the first coordinate, it follows that \( B^c \) is a standard infinite-dimensional Brownian motion under \( P^c \). (For the time change in the stochastic integral see for example Øksendal [13, Theorem 8.20].) Under \( P^c \), the process \( \hat{B} = \{ \hat{B}_t^i : 0 \leq t \leq T \}_{i \geq 1} \) is a standard infinite-dimensional Brownian motion. It follows that the distribution of \( \{ \phi_{t/c}(x) : t \in [0, T], x \in \mathbb{R}^d \} \) under the probability measure \( P^c \) is the same as the distribution of \( \{ Y_t^c(x) : t \in [0, T], x \in \mathbb{R}^d \} \) generated by the SDE
\[
Y_t^c(x) = x + \int_0^t V(Y_s^c(x))ds + \frac{1}{\sqrt{c}} \sum_{i=1}^\infty \int_0^t V_i(Y_s^c(x))d\hat{B}_s^i + \frac{1}{c} \int_0^t v(s/c, Y_s^c(x))ds, \quad (5.19)
\]
for \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), where \( \hat{B} = \{ \hat{B}_t^i : 0 \leq t \leq T \}_{i \geq 1} \) is a standard infinite dimensional Brownian motion on some probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\). Since
\[
Y_t^c(x) - \xi_t(x) = \int_0^t (V(Y_s^c(x)) - V(\xi_s(x)))ds + \frac{1}{\sqrt{c}} \sum_{i=1}^\infty \int_0^t V_i(Y_s^c(x))d\hat{B}_s^i + \frac{1}{c} \int_0^t v(s/c, Y_s^c(x))ds
\]
for any \( p \geq 1 \) there exists \( L_p \) such that
\[
\mathbb{E} \sup_{t \leq T} |Y_t^c(x) - \xi_t(x)|^p \leq L_p \left( \frac{1}{c^p} + \frac{1}{c^{p/2}} \right)
\]
for all \( x \in \mathbb{R}^d \). Similarly, taking derivatives with respect to \( x \) in (5.20), there exists \( M_p \) such that
\[
\mathbb{E} \sup_{t \leq T} \|DY_t^c(x) - D\xi_t(x)\|^p \leq M_p \left( \frac{1}{c^p} + \frac{1}{c^{p/2}} \right)
\]
for all $x \in \mathbb{R}^d$. Taking $p > d$ and using the Sobolev embedding theorem, for any compact set $K$ there exists a constant $N_p$ such that

$$
\mathbb{E} \left( \sup_{t \leq T} \sup_{x \in K} |Y^c_t(x) - \xi_t(x)|^p \right) \leq N_p \left( \frac{1}{c^p} + \frac{1}{c^{p/2}} \right).
$$

(5.21)

Details of similar calculations can be found in Kunita [9, Section 5.4] and Ikeda and Watanabe [8, Section V.2]. Therefore there is $c_0 > 0$ such that if $c \geq c_0$ then

$$
\mathbb{P} \left( \sup_{t \leq T} \sup_{x \in K} |Y^c_t(x) - \xi_t(x)| < \delta \right) \geq \frac{1}{2},
$$

and since the distribution of $\{Y^c_t(x) : x \in \mathbb{R}^d, t \in [0, T] \}$ under $\mathbb{P}$ coincides with the distribution of $\{\phi_{t/c}(x) : x \in \mathbb{R}^d, t \in [0, T] \}$ under $\mathbb{P}^c$, we obtain

$$
\mathbb{P}^c \left( \sup_{t \leq T} \sup_{x \in K} |\phi_{t/c}(x) - \xi_t(x)| < \delta \right) \geq \frac{1}{2}.
$$

Since $\mathbb{P}^c \sim \mathbb{P}$ we obtain

$$
\mathbb{P} \left( \sup_{t \leq T} \sup_{x \in K} |\phi_{t/c}(x) - \xi_t(x)| < \delta \right) > 0,
$$

which is the statement of the theorem. \qed

6 Proofs for Section 3

Proof of Theorem 3.2: According to Theorem 1.2 under condition (C)\(_R\) the reproducing kernel Hilbert space $H$ contains a rotation invariant vector field $V$ pointing inwards everywhere on the surface of the ball $B(0, R)$, that is, there is a constant $\alpha > 0$ such that

$$
\langle V(x), x \rangle \leq -\alpha < 0 \quad \text{for all } x \in \partial B(0, R).
$$

The continuity of $V$ implies the existence of a $\delta > 0$ such that

$$
\langle V(x), x \rangle \leq -\frac{\alpha}{2} < 0 \quad \text{for all } x \in B(0, R + \delta) \setminus B(0, R - 3\delta).
$$

(6.22)

Let $\xi_t$ denote the time $t$ flow along the vector field $V$, as in Theorem 5.1 Using the condition (6.22), there is a time $T_3 > 0$ such that

$$
\begin{align*}
\xi_t(B(0, R - 2\delta)) &\subset B(0, R - 2\delta) \quad \text{for } 0 \leq t \leq T_3, \\
\xi_t(B(0, R + \delta)) &\subset B(0, R - 3\delta) \quad \text{for } t \geq T_3.
\end{align*}
$$

(6.23)

Now apply the Theorem 5.1 with $T = 2T_3$ and some compact set $K$ containing $B(0, R + \delta)$, so that there exists $c \geq T_3/T_1$ for which

$$
\mathbb{P} \left( \sup_{0 \leq t \leq 2T_3} \sup_{x \in B(0, R + \delta)} |\phi_{t/c}(x) - \xi_t(x)| < \delta \right) > 0.
$$

(6.24)
Using (6.23) we have

\[ \left\{ \sup_{0 \leq t \leq 2T_3} \sup_{x \in B(0, R+\delta)} \left| \phi_{t/c}(x) - \xi_t(x) \right| < \delta \right\} \]

\[ \subset \left\{ \phi_{t/c}(B(0, R - 2\delta)) \subset B(0, R - \delta) \text{ for } 0 \leq t \leq T_3 \right\}
\cap \left\{ \phi_{t/c}(B(0, R + \delta)) \subset B(0, R - 2\delta) \text{ for } T_3 \leq t \leq 2T_3 \right\}
\]

\[ = \left\{ \phi_t(B(0, R - 2\delta)) \subset B(0, R - \delta) \text{ for } 0 \leq t \leq T_3/c \right\}
\cap \left\{ \phi_t(B(0, R + \delta)) \subset B(0, R - 2\delta) \text{ for } T_3/c \leq t \leq 2T_3/c \right\}
\]

\[ \equiv A_0, \]

say, and then (6.24) gives:

\[ \mathbb{P}(A_0) \geq \mathbb{P} \left( \sup_{t \leq 2T_3} \sup_{x \in K} \left| \phi_{t/c}(x) - \xi_t(x) \right| < \delta \right) > 0. \] (6.25)

Let \( A_k \) denote the time \( 2kT_3/c \) shift of the event \( A_0 \), so that

\[ A_k = \left\{ \phi_{2kT_3/c,t}(B(0, R - 2\delta)) \subset B(0, R - \delta) \text{ for } 2kT_3/c \leq t \leq (2k + 1)T_3/c \right\}
\cap \left\{ \phi_{2kT_3/c,t}(B(0, R + \delta)) \subset B(0, R - 2\delta) \text{ for } (2k + 1)T_3/c \leq t \leq (2k + 2)T_3/c \right\}. \]

The events \( \{A_k : k \geq 0\} \) are independent with \( P(A_k) = P(A_0) \) for all \( k \geq 1 \), and

\[ \bigcap_{k=0}^{n-1} A_k \subset \left\{ \phi_t(B(0, R + \delta)) \subset B(0, R - \delta) \text{ for } T_3/c \leq t \leq 2nT_3/c \right\}. \]

Therefore

\[ \mathbb{P} \left( \phi_t(B(0, R + \delta)) \subset B(0, R - \delta) \text{ for } T_3/c \leq t \leq 2nT_3/c \right)
\geq \mathbb{P} \left( \bigcap_{k=0}^{n-1} A_k \right) = \prod_{k=0}^{n-1} P(A_k) = (P(A_0))^n > 0. \]

Since \( T_3/c \leq T_1 \) and \( n \) can be chosen so that \( 2nT_3/c \geq T_2 \), the proof of the squeezing result (3.3) is complete. The corresponding expansion result (3.11) can be proved using the same method. Simply replace \( \xi_t \) by the time \( t \) flow \( \hat{\xi}_t \) along \(-V\), so that (6.23) can be replaced by

\[ \left\{ \hat{\xi}_t(B(0, R + 2\delta)) \supset B(0, R + 2\delta) \text{ for } 0 \leq t \leq T_3, \right\}
\left\{ \hat{\xi}_t(B(0, R - \delta)) \supset B(0, R + 3\delta) \text{ for } t \geq T_3, \right\}
\]

and then \( \mathbb{P}(\hat{A}_0) > 0 \) where

\[ \hat{A}_0 = \left\{ \phi_t(B(0, R + 2\delta)) \supset B(0, R + \delta) \text{ for } 0 \leq t \leq T_3/c \right\}
\cap \left\{ \phi_t(B(0, R - \delta)) \supset B(0, R + 2\delta) \text{ for } T_3/c \leq t \leq 2T_3/c \right\}. \]
Proof of Theorem 3.3. According to (3.9) of Theorem 3.2 for each \( u \in [r, R] \) there exists \( \delta(u) > 0 \) such that
\[
\mathbb{P}[\phi_t(B(0, u + \delta(u))) \subset B(0, u - \delta(u)) \text{ for } t_1 \leq t \leq t_2] > 0
\]
for any \( 0 < t_1 < t_2 \). Consider the open cover
\[
\{(u - \delta(u), u + \delta(u)) : u \in [r, R]\}
\]
of the compact interval \([r, R]\), and choose a minimal finite subcover
\[
\{(u_i - \delta(u_i), u_i + \delta(u_i)) : i = 1, \ldots, N\}.
\]
Under the condition of minimality, the intervals can be labelled so that \( u_1 + \delta(u_1) > R, u_N - \delta(u_N) < r \) and \( u_i - \delta(u_i) < u_{i+1} + \delta(u_{i+1}) < u_i + \delta(u_i) \) for all \( i \in \{1, \ldots, N-1\} \). Set \( \pi_i := u_i + \delta(u_i) \) and \( \underline{u}_i = u_i - \delta(u_i) \). With this notation, the conditions on the subcover become: \( \pi_1 \geq R, \underline{u}_N \leq r \) and \( \underline{u}_i < \pi_{i+1} < \pi_i \) for all \( i \in \{1, \ldots, N-1\} \). Choose times \( 0 < t_1 < \cdots < t_{N-1}, T_1 \) and consider the events
\[
A_i = \{\phi_{t_{i-1}, t_i}(B(0, \pi_i)) \subset B(0, \underline{u}_i)\}
\]
for \( 1 \leq i \leq N-1 \), and
\[
A_N = \{\phi_{t_{N-1}, t}(B(0, \pi_N)) \subset B(0, \underline{u}_N) \text{ for } T_1 \leq t \leq T_2\}
\]
The events \( \{A_k : 1 \leq k \leq N\} \) are independent, and by Theorem 3.2 we have \( \mathbb{P}(A_k) > 0 \) for \( 1 \leq k \leq N \). Moreover
\[
\bigcap_{i=1}^{N} A_k \subset \bigcap_{i=1}^{N-1} \{\phi_{t_{i-1}, t_i}(B(0, \pi_i)) \subset B(0, \pi_{i+1})\}
\]
\[
\cap \{\phi_{r_{N-1}, r}(B(0, \pi_N)) \subset B(0, \underline{u}_N) \text{ for } T_1 \leq t \leq T_2\}
\]
\[
\subset \{\phi_{r, r}(B(0, \pi_1)) \subset B(0, \underline{u}_N) \text{ for } T_1 \leq t \leq T_2\}
\]
\[
\subset \{\phi_{r, r}(B(0, R)) \subset B(0, r) \text{ for } T_1 \leq t \leq T_2\}.
\]
Therefore
\[
\mathbb{P}(\phi_{r, r}(B(0, R)) \subset B(0, r) \text{ for } T_1 \leq t \leq T_2) \geq \mathbb{P}\left(\bigcap_{k=1}^{N} A_k\right) = \prod_{k=1}^{N} \mathbb{P}(A_k) > 0
\]
and the proof of the squeezing result is complete. The proof of the expanding result uses the same method with (3.11) in place of (3.9). \( \square \)

Remark 6.1. According to Theorem 4.2, if the condition \((C)_\rho\) is not satisfied there is no vector field \( V \) in the RKHS pointing strictly inwards on the boundary \( \partial B(0, \rho) \) and therefore we cannot apply our technique to show the ball of radius \( \rho \) can be squeezed. If \( \mu_1 > 0 \) it is an open question whether there exist two (or more) vector fields \( V_1, V_2 \) from the RKHS such that when applying them consecutively to \( B(0, \rho) \) the resulting image is strictly contained in \( B(0, \rho) \). However if \( \mu_1 = 0 \) then the isotropic Brownian field is almost surely divergence-free, so that the resulting IBF is volume preserving and the conclusion of Theorem 3.2 is definitely false.
7 Lengths of Curves

For fixed \( x \in \mathbb{R}^d \) and \( v \in \mathbb{R}^d \) the asymptotic rate of growth or decay of \( v_t = D\phi_t(x)v \) is given by the top Lyapunov exponent

\[
\lambda = \lim_{t \to \infty} \frac{1}{t} \log |v_t| \quad \mathbb{P} \text{ almost surely.}
\]

For an IBF the limit exists and takes the same value for all \( x \) and \( v \). It is shown in [2] and [10] that

\[
\lambda = (d - 1) \frac{\beta_N}{2} - \frac{\beta_L}{2}
\]

where \( \beta_L \) and \( \beta_N \) are given in (2.7) and (2.8).

The top Lyapunov exponent describes the rate of growth or decay of a single tangent vector. Here we will consider the more complicated situation concerning the growth or decay of the length of a differentiable curve under an IBF. Throughout this section we will assume that \( \Gamma \) is a piecewise \( C^1 \) curve in \( \mathbb{R}^d \) parameterized by \( \gamma(u) : [0, 1] \to \mathbb{R}^d \) with \( \gamma'(u) \neq 0 \) at all \( C^1 \) points \( u \in [0, 1] \). The image \( \phi_t(\Gamma) \) of \( \Gamma \) under the action of the flow will be denoted by \( \Gamma_t \). It is a piecewise \( C^1 \) curve parameterized by \( \gamma_t = \phi_t \circ \gamma : [0, 1] \to \mathbb{R}^d \). The length of \( \Gamma_t \) will be denoted by \( L_t \) and is given by

\[
L_t = \int_0^1 |\gamma'_t(u)| du = \int_0^1 |D\phi_t(\gamma(u))\gamma'(u)| du.
\]

The diameter of the curve \( \Gamma_t \) is given by \( \text{diam}(\Gamma_t) = \sup\{|x - y| : x, y \in \Gamma_t\} \).

It has been shown, see Dimitroff [6, Proposition 3.1.2], that the evolving length of a differentiable curve in an IBF satisfies the following bounds

\[
\lambda \leq \liminf_{t \to \infty} \frac{1}{t} \ln L_t \leq \limsup_{t \to \infty} \frac{1}{t} \ln L_t \leq \lambda + \frac{\beta_L}{2}.
\]

It follows that if the top Lyapunov exponent \( \lambda \) is positive then the length exhibits exponential growth. However if \( \lambda < 0 \) the above bounds are not very informative because \( \lambda + \frac{\beta_L}{2} = (d - 1) \frac{\beta_N}{2} \) is always strictly positive. Here we will be dealing with the case \( \lambda < 0 \). Baxendale and Harris have shown in [2] that if \( d = 2 \) and \( \frac{\beta_L}{\beta_N} > \frac{5}{3} \) then the length of a curve decreases to 0 on the set where the diameter of the curve decreases to 0, provided of course this happens with positive probability, i.e.

\[
\mathbb{P}(L_t \to 0 \text{ as } t \to \infty \mid \text{diam}(\Gamma_t) \to 0 \text{ as } t \to \infty) = 1
\]

provided \( \mathbb{P}(\text{diam}(\Gamma_t) \to 0 \text{ as } t \to \infty) > 0 \). The following result of Dimitroff [6, Theorem 3.2.1] strengthens (7.26) by giving an exponential decay rate under more general conditions. (In dimension \( d = 2 \) the condition \( \lambda < 0 \) is equivalent to \( \frac{\beta_L}{\beta_N} > 1 \).)

**Theorem 7.1.** Let \( \{\phi_{s,t}(x) : 0 \leq s \leq t < \infty, x \in \mathbb{R}^d\} \) be a \( d \)-dimensional isotropic Brownian flow with top Lyapunov exponent \( \lambda < 0 \). Then

\[
\mathbb{P}
\left(
\lim_{t \to \infty} \frac{1}{t} \log L_t = \lambda \mid \text{diam}(\Gamma_t) \to 0 \text{ as } t \to \infty
\right) = 1
\]

provided \( \mathbb{P}(\text{diam}(\Gamma_t) \to 0 \text{ as } t \to \infty) > 0 \).
We can apply our Corollary 3.4 to show that \( P(\text{diam}(\Gamma_t) \to 0 \text{ as } t \to \infty) > 0 \) provided that condition \((C)_\rho\) holds for all \( \rho > 0 \). This will be done in Proposition 7.2 below. Before doing so we first sketch the proof of the above theorem. Concepts and notation from this proof will then be used in the proof of Proposition 7.2.

**Sketch of the proof:** The detailed proof of Theorem 7.1 can be found in [6, Theorem 3.2.1]. Since the distribution of the IBF is translation invariant we can assume without loss of generality that \( 0 \in \Gamma \). We first consider the flow \( \psi_{s,t}(x) := \phi_{s,t}(x + \phi_{0,s}(0)) - \phi_{0,t}(0) \), which is simply the original flow viewed from the point of view of the moving particle started in the origin. Then \( \psi \) and \( \phi \) have the same Lyapunov spectrum and the length of \( \phi_t(\Gamma) = \Gamma_t \) equals the length of \( \psi_t(\Gamma) \equiv \hat{\Gamma}_t \). The flow \( \psi \) can be extended to a double sided random dynamical system with fixed point at the origin over a measurable ergodic shift \( \{\theta_t : t \in \mathbb{R}\} \), and then we can write \( \psi_{s,t}(x, \omega) = \psi(t-s, x, \theta_s \omega) \), see [6, Lemma 3.2.1]. This allows us to use the local stable manifold theorem, see Mohammed and Scheutzow [12]. One of the characterizations of the local stable manifold is

\[
S(\omega) = \left\{ x \in B(0, \rho(\omega)) \mid |\psi(n, x, \omega)| \leq \beta(\omega) \exp(\lambda + \epsilon)n \text{ for all } n \in \mathbb{N} \right\},
\]

where \( 1 > \beta(\omega) > \rho(\omega) > 0 \) and \( \epsilon \) is some fixed positive number between 0 and \(-\lambda\). Since \( \lambda < 0 \) we have \( S(\omega) = \overline{B}(0, \rho(\omega)) \). Since \( \{\rho \circ \theta_n\}_{n \in \mathbb{N}} \) is a stationary sequence of positive functions, for each \( \omega \) such that \( \text{diam}(\hat{\Gamma}_t(\omega)) \to 0 \text{ as } t \to \infty \) there exists \( n = n(\omega) \in \mathbb{N} \) such that \( \text{diam}(\hat{\Gamma}_n(\omega)) \leq \rho(\theta_n \omega) \) and so \( \hat{\Gamma}_n(\omega) \subset S(\theta_n \omega) \). Since

\[
L_{t+n}(\omega) \leq L_n(\omega) \sup \left\{ \frac{|\psi(t, x_1, \theta_n \omega) - \psi(t, x_2, \theta_n \omega)|}{|x_1 - x_2|} : x_1 \neq x_2, x_1, x_2 \in \hat{\Gamma}_n(\omega) \right\}
\]

the result now follows from the estimate

\[
\limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup \left\{ \frac{|\psi(t, x_1, \omega) - \psi(t, x_2, \omega)|}{|x_1 - x_2|} : x_1 \neq x_2, x_1, x_2 \in S(\omega) \right\} \right] \leq \lambda,
\]

see [12, Theorem 3.1(ii)(b)], with \( \omega \) replaced by \( \theta_n \omega \).

The next proposition concerns the probability of the event \( \{\text{diam}(\Gamma_t) \to 0\} \). It is a direct consequence of our main result (Theorem 3.3).

**Proposition 7.2.** Let \( \{\phi_{s,t}(x) : 0 \leq s \leq t < \infty, x \in \mathbb{R}^d\} \) be a \( d \)-dimensional isotropic Brownian flow with top Lyapunov exponent \( \lambda < 0 \). Assume that condition \((C)_\rho\) is satisfied for all \( \rho > 0 \). Then for all initial curves \( \Gamma \)

\[
P(\text{diam}(\Gamma_t) \to 0 \text{ as } t \to \infty) > 0.
\]

**Proof:** Let \( r, R > 0 \) be such that \( P(\rho(\omega) > r) > 0 \), where \( \rho(\omega) \) is the radius of the local stable manifold \( S(\omega) \) as in (7.27), and \( \Gamma \subset B(0, R) \). According to Theorem 3.3 there exists \( T > 0 \), such that \( P(\phi_T(\overline{B}(0, R)) \subset B(0, r)) > 0 \). Rewriting (7.28) in terms of the IBF \( \phi \) we have

\[
\limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} : x_1 \neq x_2, x_1, x_2 \in S(\omega) \right\} \right] \leq \lambda
\]
so that \( \text{diam}(\phi_t(S(\omega), \omega)) \to 0 \) as \( t \to \infty \). Using this result with \( \omega \) replaced by \( \theta_t \omega \) we obtain

\[
\mathbb{P}(\text{diam}(\Gamma_t(\omega)) \to 0 \text{ as } t \to \infty) \geq \mathbb{P}(\Gamma_t(\omega) \in S(\theta_t \omega)) \\
\geq \mathbb{P}(\Gamma_t(\omega) \subset B(0, r) \text{ and } B(0, r) \subset S(\theta_t \omega)) \\
\geq \mathbb{P}(\phi_T(B(0, R), \omega) \subset B(0, r) \text{ and } \rho(\theta_t \omega) > r)
\]

Observe that \( \rho(\theta_t \omega) \) is measurable with respect to the \( \sigma \)-algebra generated by the increments \( \{\phi_T, \cdot, \omega\} : t \geq T\} \) and hence it is independent of \( \phi_T(\cdot, \omega) \). Therefore

\[
\mathbb{P}(\text{diam}(\Gamma_t(\omega)) \to 0 \text{ as } t \to \infty) \geq \mathbb{P}(\phi_T(B(0, R) \subset B(0, r)) \mathbb{P}(\rho(\theta_t \omega) > r) \\
\geq \mathbb{P}(\phi_T(B(0, R) \subset B(0, r)) \mathbb{P}(\rho(\omega) > r) \geq 0
\]

which completes the proof. \qed

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