Information-theoretic Bounds on Matrix Completion under Union of Subspaces Model

Vaneet Aggarwal and Shuchin Aeron

Abstract

In this short note we extend some of the recent results on matrix completion under the assumption that the columns of the matrix can be grouped (clustered) into subspaces (not necessarily disjoint or independent). This model deviates from the typical assumption prevalent in the literature dealing with compression and recovery for big-data applications. The results have a direct bearing on the problem of subspace clustering under missing or incomplete information.

1 Introduction

Matrix completion refers to the recovery of a low-rank matrix from a (small) subset of its entries or a (small) number of linear combinations of its entries [1–4]. In essence, the methods are aimed at recovering the column/row subspaces from limited measurements. Even the sketching methods [8] aim to find the best column (or row) subspace of a matrix.

However, in many practical applications, the columns of the data matrix can belong to different low rank subspaces (or affine subspaces) [5–7, 9]. Motivated by this observation, in this paper we assume that the different columns in the data matrix of size $m \times n$ lie in one of the $K$ subspaces, where the dimension of these subspaces are $(r_1, \ldots, r_K)$. Now, suppose we have $k$ linear measurements of this matrix. The general question is how many linear measurements, satisfying certain properties are sufficient such that the data can be recovered from these linear measurements.

This problem has direct bearing on the problem of subspace clustering [9–11, 15, 16] under missing or incomplete data. Subspace clustering with missing data has been studied in [12–13]. Recently, the authors of [14] considered the number of samples needed for reconstruction of data, where the number of partially observed data vectors per subspace is a rank-degree power of the dimension. In contrast to these results, in this paper we show information-theoretically, number of linear measurements greater than $Kr(m + n/K - r)$ suffice for reconstruction of data when the columns are assumed to come from a union of $K$ subspaces each of dimension $r$. Further, we note that rank-1 measurement matrices are enough over the whole data matrix.

Our main tool to obtain the sufficiency result relies on recent information-theoretic results on matrix completion in [18], inspired by fundamental limits on analog source compression in [17]. In this paper, we specialize these results to the union of subspaces model.

The rest of the paper is organized as follows. Section II gives the system model, and Section III gives the sufficient number of linear measurements. Section IV concludes this paper.

*V. Aggarwal is with Purdue University, W. Lafayette, IN 47907, email: vaneet@purdue.edu. This work of V. Aggarwal was supported in part by Air Force Research Lab Visiting Faculty Research Program Award. S. Aeron is with Tufts University, Medford, MA 02155, email: shuchin@ece.tufts.edu. S. Aeron is supported in part by NSF grant 1319653.
2 Model and Preliminaries

For notations, let Roman letters \( A, B, \cdots \) designate deterministic matrices and \( a, b, \cdots \) stands for deterministic vectors. Bold-face letters \( \mathbf{A}, \mathbf{B}, \cdots \) and \( \mathbf{a}, \mathbf{b}, \cdots \) denote random matrices and vectors, respectively. Let \( \mathcal{M}^{m \times n}_{r} \) and \( \mathcal{N}^{m \times n}_{r} \) denote the set of matrices \( A \in \mathbb{R}^{m \times n} \) with \( \text{rank}(A) \leq r \) and \( \text{rank}(A) = r \), respectively. For a random matrix \( X \in \mathbb{R}^{m \times n} \) of arbitrary distribution \( \mu_X \), an \((m \times n, k)\) code consists of linear measurements \( \langle A_1, \cdots, A_k, X \rangle \) and a measurable decoder \( g : \mathbb{R}^k \rightarrow \mathbb{R}^{m \times n} \). For given measurement matrices \( A_i \), we say that a decoder \( g \) achieves error probability \( \epsilon \) if \( \Pr(g(\langle A_1, X \rangle, \cdots, < A_k, X \rangle^T) \neq X) \leq \epsilon \). For \( \epsilon > 0 \), we call a nonempty bounded set \( S \subseteq \mathbb{R}^{m \times n} \) an \( \epsilon \)-support set of the random matrix \( X \in \mathbb{R}^{m \times n} \) if \( \Pr[X \in S] \geq 1 - \epsilon \).

We next define Minkowski dimension.

**Definition 1** ([IN]). Let \( \mathcal{S} \) be a nonempty bounded set in \( \mathbb{R}^{m \times n} \). The lower Minkowski dimension of \( \mathcal{S} \) is defined as

\[
\underline{\dim}(\mathcal{S}) = \liminf_{\rho \to 0} \frac{\log N_{\mathcal{S}}(\rho)}{-\log \rho},
\]

and the upper Minkowski dimension of \( \mathcal{S} \) is defined as

\[
\overline{\dim}(\mathcal{S}) = \limsup_{\rho \to 0} \frac{\log N_{\mathcal{S}}(\rho)}{-\log \rho},
\]

where \( N_{\mathcal{S}} \) denotes the covering number of \( \mathcal{S} \) given by

\[
N_{\mathcal{S}}(\rho) = \min\{k \in \mathbb{N} : \mathcal{S} \subseteq \bigcup_{i \in \{1, \cdots, k\}} \mathcal{B}(M_i, \rho), M_i \in \mathbb{R}^{m \times n}\},
\]

and \( \mathcal{B}(\mu, s) \) denotes the open ball of radius \( s \) centered at \( \mu \in \mathbb{R}^k \).

We next give a bound of number of measurements needed to decode matrix from limited measurements.

**Lemma 1** ([IN]). Let \( \mathcal{S} \subseteq \mathbb{R}^{m \times n} \) be an \( \epsilon \)-support set of \( X \in \mathbb{R}^{m \times n} \). Then, for Lebesgue a.a. measurement matrices \( A_i, i = 1 \cdots, k \), there exists a decoder achieving error probability \( \epsilon \), provided that \( k > \dim(\mathcal{S}) \).

We will now describe the union of subspace model that is considered in this paper.

**Definition 2.** The union of subspace set \( \mathcal{US}^{m \times n}_{K,(r_1, \cdots, r_K)} \) is the of matrices \( X \) for which its columns can be divided among \( K \) groups to get \( X_1, \cdots, X_K \), where each column of \( X \) is in exactly one \( X_i \), and \( X_i \in \mathcal{M}^{m \times n}_{r_i} \), where \( n_i \geq 0, \sum_{i=1}^{K} n_i = n \).

3 Main Results

**Theorem 1.** Let \( \mathcal{S} \subseteq \mathcal{US}^{m \times n}_{K,(r_1, \cdots, r_K)} \) be a non-empty bounded set. Then,

\[
\overline{\dim}(\mathcal{S}) \leq m \sum_i r_i + n \max_i r_i - \sum_i r_i^2,
\]

We will now describe the union of subspace model that is considered in this paper.

**Definition 2.** The union of subspace set \( \mathcal{US}^{m \times n}_{K,(r_1, \cdots, r_K)} \) is the of matrices \( X \) for which its columns can be divided among \( K \) groups to get \( X_1, \cdots, X_K \), where each column of \( X \) is in exactly one \( X_i \), and \( X_i \in \mathcal{M}^{m \times n}_{r_i} \), where \( n_i \geq 0, \sum_{i=1}^{K} n_i = n \).
Proof. We can represent \( X \in \mathcal{US}_{K,(r_1,\ldots,r_K)}^{m \times n} \) with a set of columns \( C_i, i = 1, \ldots, K \) for the \( K \) subspaces with \( |C_i| = n_i \), and \( X(C_i) \in \mathcal{M}_{r_i}^{n_i} \) represents the \( X \) in those columns. This, \( \mathcal{US}_{K,(r_1,\ldots,r_K)}^{m \times n} \) is equivalent to
\[
\bigcup_{i=1}^{K} C_i \times K \subseteq \mathcal{M}_{r_i}^{n_i}
\]
where \( \times \) refers to the Cartesian product. Therefore the manifold of union of subspaces is a product manifold. Since upper Minkowski dimension for \( \mathcal{M}_{r_i}^{n_i} \) is at most \( r_i(m + |C_i| - r_i) \), the upper Minkowski dimension for \( \bigcup_{i=1}^{K} C_i \times K \) is at most \( \sum_i (r_i(m + |C_i| - r_i)) \).

Further, since upper Minkowski dimension of union is the max of the Minkowski dimension [Section 3.2, [19]], we have
\[
\overline{\dim}(S) \leq \max_{C_1,\ldots,C_K} \sum_i (r_i(m + |C_i| - r_i)) \tag{5}
\]
\[
= \sum_i r_i m + \max_{C_1,\ldots,C_K} \sum_i r_i |C_i| - \sum_i r_i^2 \tag{6}
\]
\[
\leq m \sum_i r_i + \max_{C_1,\ldots,C_K} \sum_i (\max_i r_i) |C_i| - \sum_i r_i^2 \tag{7}
\]
\[
= m \sum_i r_i + n(\max_i r_i) - \sum_i r_i^2 \tag{8}
\]

We note the following points.

1. Since \( \dim \leq \overline{\dim} \), from Theorem 1 and Lemma 1, we see that there exists a decoder that achieves error probability \( \epsilon \) for \( k \) Lebesgue a.a. measurement matrices, for \( k > m \sum_i r_i + n \max_i r_i - \sum_i r_i^2 \).

2. In the special case when the subspaces are independent and when \( r_i = r \) for all \( i \), we have the sufficient number of linear measurements for the UOS model as \( Kr(m - r) + nr + 1 \). Under a single subspace model, the number of measurements for a matrix with rank \( Kr \) (which will be total dimension of the space spanned by the columns under the independence assumption and assuming \( n \geq Kr \)) is \( (m + n - Kr)Kr + 1 \). The difference in the number of linear measurements is \( (n - Kr)(K - 1)r \). When \( n = Kr \) this tells us that there is no advantage in using the UOS model as compared to a single subspace model.

3. Note that there is no additional overhead in number of measurements for the knowledge of \( K \) sets of columns that make each subspace since the number of measurements are equivalent to measuring each of \( K \) subspaces knowing which columns make each subspace. We also note that with exact completion with these measurements, subspace clustering can be performed [11] to also get different subspace clusters.

4. We further note from Theorem 2 of [18] that rank one measurement matrices are sufficient, rather than general linear measurement matrices. Rank one measurement matrices are attractive as they require less storage space than general measurement matrices and can also be applied faster.

5. We note that rank one measurements are used over the whole data rather than performing subspace clustering with limited measurements, followed by performing measurements in each subspace.
4 Conclusion

This paper finds the number of linear measurements that are sufficient to estimate a matrix that is formed by a union of subspaces. The savings of measurements with the additional structure of union of subspace model depend on the product of dimension, number of subspaces, and the rank of subspace.

References Cited

[1] E. J. Candes and Y. Plan, “Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements,” IEEE Trans. Inf. Theory, vol. 4, no. 57, pp. 2342–2359, Apr. 2011.

[2] D. Gross, “Recovering low-rank matrices from few coefficients in any basis,” IEEE Trans. Inf. Theory, vol. 57, no. 3, pp. 1548–1566, Mar. 2011.

[3] B. Recht, “A simpler approach to matrix completion,” J. Mach. Learn. Res., vol. 12, pp. 341–3430, 2011.

[4] T. T. Cai and A. Zhang, “ROP: Matrix recovery via rank-one projections,” Ann. Stat., vol. 43, no. 1, pp. 102138, 2015.

[5] J. P. Costeira and T. Kanade, “A multi-body factorization method for independently moving objects,” International Journal of Computer Vision, vol. 29, 1998.

[6] K. Kanatani, “Motion Segmentation by Subspace Separation and Model Selection,” in Proc. IEEE International Conference on Computer Vision, 2001, vol. 2, pp. 586–591.

[7] B. Eriksson, P. Barford, J. Sommers, and R. Nowak, “DomainImpute: Inferring Unseen Components in the Internet,” in Proc. IEEE INFOCOM Mini-Conference, April 2011, pp. 171–175.

[8] Christos Boutsidis, Petros Drineas, and Malik Magdon-Ismail, “Near-Optimal Column-Based Matrix Reconstruction”, SIAM Journal on Computing 2014 43:2, 687-717.

[9] R. Vidal, “Subspace clustering,” IEEE Signal Processing Magazine, 28(2):52–68, 2011

[10] D. Park, C. Caramanis, and S. Sanghavi, “Greedy Subspace Clustering,” in Proc. NIPS 2014

[11] Y. Wang, Y. Wang, A. Singh, “Clustering Consistent Sparse Subspace Clustering,” arXiv:1504.01046v1, Apr. 2015

[12] B. Eriksson, L. Balzano, and R. Nowak, “High-Rank Matrix Completion and Subspace Clustering with Missing Data”, in Proc. Conference on Artificial Intelligence and Statistics (AISTATS), 2012.

[13] L. Balzano, R. Nowak, A. Szlam, and B. Recht, “k-Subspaces with missing data”, in Proc. Statistical Signal Processing Workshop, 2012.

[14] D. Pimentel, R. Nowak, and L. Balzano, “On the sample complexity of subspace clustering with missing data,” in Proc. IEEE Workshop on Statistical Signal Processing (SSP), vol., no., pp.280,283, June 29 2014-July 2 2014

[15] R. Heckel and H. Bolcskei, “Robust subspace clustering via thresholding,” arXiv:1307.4891v2, 2014.

[16] Y.-X. Wang, H. Xu, and C. Leng, “Provable subspace clustering: When LRR meets SSC,” in Proc. Advances in Neural Information Processing Systems (NIPS), December 2013.

[17] Y. Wu and S. Verdu, “Renyi information dimension: Fundamental limits of almost lossless analog compression,” IEEE Trans. Inf. Theory, vol. 56, no. 8, pp. 37213748, Aug. 2010.

[18] Erwin Riegler, David Stotz, Helmut Bolcskei, “Information-Theoretic Limits of Matrix Completion,” arXiv:1504.04970v2, Apr 2015.

[19] K. Falconer, Fractal Geometry, 1st ed. New York, NY: Wiley, 1990.