Localization in lattice QCD (with emphasis on practical implications)

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When Anderson localization takes place in a quenched disordered system, a continuous symmetry can be broken spontaneously without accompanying Goldstone bosons. Elaborating on this observation we propose a unified, microscopic physical picture of the phase diagram of quenched and unquenched QCD with two flavors of Wilson fermions. The phase with Goldstone bosons—by definition the Aoki phase—is always identified as the region where the mobility edge of the (hermitian) Wilson operator is zero. We then discuss the implications for domain-wall and overlap fermions. We conclude that both formulations are valid only well outside the Aoki phase of the associated Wilson-operator kernel, because this is where locality and chirality can be both maintained.

1. Introduction

Long ago, a conjecture was made for the phase diagram of lattice QCD with two flavors of Wilson fermions and a standard plaquette action \cite{1}. (For earlier work see ref. \cite{2}.) This phase diagram is displayed in Fig. 1. In the Aoki phase (region B in the figure), parity and isospin undergo spontaneous symmetry breaking (SSB) by a pion condensate.\cite{1} According to the Banks-Casher relation \cite{4} for the case at hand, the pion condensate is the response to an (infinitesimal) applied twisted-mass \cite{1,5},

\[
\langle \pi \rangle = \frac{2\pi \rho(0)}.
\]

Here $\rho(\lambda)$ is the spectral density of the hermitian\cite{2} Wilson operator $H(m_0)$. The twisted-mass term is $m_1 i \gamma_5 \tau_3 \psi$. Inside the Aoki phase, the other two pions are Goldstone bosons associated with SSB of isospin down to a U(1) symmetry.\cite{1}

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\textsuperscript{1}For $g_0 > 0$, chiral symmetry is explicit broken by the Wilson term, and $\langle \bar{\psi} \psi \rangle$ is not an order parameter. In the continuum limit chiral symmetry is recovered, and one can rotate the pion condensate back to $\langle \bar{\psi} \psi \rangle$, see e.g. ref. \cite{6}.

\textsuperscript{2}Conventions: $H(m_0) = D(m_0) = D^{\text{naive}} - W - m_0$. The sum of the bare mass $m_0$ plus the Wilson term $W$ is a strictly positive operator for $m_0 > 0$. The super-critical region where zero modes may exist is $0 > a m_0 > -8$. Exceptional configurations \cite{11} – defined by the condition that $H(m_0)$ has a zero mode for some $m_0$ – were discarded in ref. \cite{10}. We return to this issue below.

Region C of the phase diagram is interesting because, as explained below, this is where one can use domain-wall \cite{6} and overlap \cite{8} fermions. According to the original conjecture \cite{1}, all correlation functions of Wilson fermions are short-ranged well inside region C. This was supported by quenched numerical results \cite{10}.\textsuperscript{3} Today, we have convincing evidence that, in region C of the quenched theory, there is a non-zero density of small-size near-zero modes of $H(m_0)$ for any $g_0 > 0$ \cite{12,13}.\textsuperscript{4} This raises the following puzzle. Given that $\rho(0) \neq 0$, the Banks-Casher relation \cite{4} implies that the pion condensate is non-zero; hence there is SSB of isospin and parity; but since isospin is a continuous symmetry, the Goldstone theorem requires the existence of massless Goldstone bosons. Thus, the absence of long-range correlations appears to contradict the existence of a density of near-zero modes in region C of the quenched theory!

Before tackling this puzzle, it is instructive to describe the above results using a language borrowed from the physics of disordered systems (for reviews see ref. \cite{14}). For this purpose we may interpret $H(m_0)$ as the hamiltonian of a

\[\text{Further evidence comes from the body of numerical work using domain-wall and/or overlap fermions.}\]
five-dimensional system, and thus its eigenvalues $\lambda_n$ as “energy” eigenvalues. The eigenfunctions $\Psi_n(x)$ may be thought of as the wave functions of “electrons” that reside on the sites $x$ of a four-dimensional spatial lattice. Disorder is provided by the ensemble of gauge-field configurations, in terms of which $H(m_0)$ is defined.

Fig. 2 is a cartoon aimed at explaining how the small-size, small-$\lambda$ eigenstates of $H(m_0)$ emerge with increasing disorder. An infinite volume is assumed. The first row describes the free operator $H_0(m_0)$, assuming for definiteness that $am_0 \approx -1$. The gap is $O(1)$ in lattice units. Next, consider one small dislocation (second row), by which we mean that the link variables inside a small-size hypercube may take any value, but all other links are still equal to one. It was shown that a bound state – whose eigenvalue is located anywhere we like inside the gap – can be produced by adjusting the links of the dislocation $[13]$. The next step is to consider a dilute gas of small dislocations, choosing the position and shape of the individual dislocations at random. A bound state produced by a particular dislocation is only negligibly affected by all other dislocations. Taken together, the (fairly) randomly distributed eigenvalues of all the bound states fill up the gap.

The last step is to consider realistic (quenched or dynamical) Monte-Carlo configurations. Unlike the previous nearly-free cases, now we cannot define an “asymptotic value of the potential,” nor what is the “binding energy.” The concept of bound states is inadequate and, instead, we have exponentially localized states. Likewise, we now have extended states instead of scattering states. Still, we hypothesize that there are distinct spectral intervals containing either extended or localized eigenstates, but not both.$^5$

Up to short-distance random fluctuations, the mode density of an exponentially localized eigenstate behaves like

$$|\Psi_n(x)|^2 \sim \frac{\exp \left( -|x - x_n^0|/l_n \right)}{l_n^4}.$$  \hspace{1cm} (2)

On average, the mode density decays exponentially away from some “center” $x_n^0$, where $l_n$ is the localization length.$^6$ As we increase $|\lambda_n|$ starting from $\lambda_n \sim 0$, tunneling becomes easier and, on average, $l_n$ increases. The localization length diverges when $|\lambda_n|$ reaches the mobility edge $\lambda_c$, and for $|\lambda_n| > \lambda_c$ the eigenstates become extended.$^7$

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$^5$We are not aware of any proof of this – widely used – assertion.

$^6$In four dimensions, typically $\Psi_n(x)$ will be localized inside a region whose volume is $O(l_n^4)$. This explains the factor of $1/l_n^4$ on the right-hand side of eq. (2), for a normalized eigenstate.

$^7$Since $H_0(m_0)$ is bounded, we expect another mobility edge $\overline{\lambda}_c$, such that, for $|\lambda_n| > \overline{\lambda}_c$, the eigenstates are again localized (lightly shaded area to the very right in Fig. 2).
The value of the mobility edge is a dynamical feature. For $g_0 \ll 1$ it will be close to the band edge of $H_0(m_0)$, but for larger $g_0$ it may go down and, eventually, even vanish (for $am_0 \sim -1$).

2. Localization, mobility edge and the phase diagram

Let us now reconsider the above puzzle. Assume that the condensate arises from extended states only ($\lambda = 0$). Since these eigenstates spread throughout the entire lattice, some long-range correlations may be expected, and the pion condensate should be accompanied by Goldstone bosons. In contrast, as expected, and the pion condensate should be accompanied by Goldstone bosons. In contrast, we are now closer to resolving the puzzle: if $\rho(0)$ arises from exponentially localized eigenstates only, indeed we should not expect any long-range correlations.

What about the Goldstone theorem? The relevant Ward identity is

\[ \sum_\mu (1 + O(ap)) p_\mu \tilde{\Gamma}_\mu(p) + 2m_1 \tilde{\Gamma}(p) = \langle \pi_3 \rangle, \tag{3} \]

where $\tilde{\Gamma}(p)$ and $\tilde{\Gamma}_\mu(p)$ are the Fourier transforms of $\langle \pi_+(x) \pi_-(y) \rangle$ and $\langle J^\mu_+(x) \pi_-(y) \rangle$ respectively. $J^\mu_+(x)$ is a conserved isospin current. (The $\pm$ refer to operators that raise/lower isospin by one unit.) As we explain below, when SSB takes place in a quenched theory, it is possible to saturate the Ward identity without Goldstone bosons. Instead, the two-point function of the would-be Goldstone bosons diverges like $1/m_1$ in the limit $m_1 \to 0$. In the unquenched theory, on the other hand, the product $m_1 \tilde{\Gamma}(p)$ vanishes in this limit for any $p \neq 0$, and the Goldstone theorem applies.

In the rest of this section we expand on the main physical issues. More results, as well as missing technical details, may be found in ref. [14], henceforth denoted I.

After a change of variables $\tilde{\psi} = \psi'\gamma_5$ (see also footnote 2), the fermion lagrangian reads

\[ \mathcal{L} = \overline{\psi} (H(m_0) - im_1 \tau_3) \psi. \]

Let us first consider the finite-volume quenched theory, and look for a possible $1/m_1$ divergence of the (position space) pion two-point function $\Gamma(x, y)$. In terms of the spectrum of $H(m_0)$ we have

\[ \Gamma(x, y) = \left( \sum_{kn} \frac{\Psi_n(x) \Psi_k(x) \Psi_k(y) \Psi_n(y)}{(\lambda_k + im_1)(\lambda_n - im_1)} \right). \tag{4} \]

The terms labeled by $k$ ($n$) refer to the up (down) quark. $\langle \cdot \rangle$ denotes the functional integration over the gauge field with the Boltzmann weight of the quenched theory. In the denominators, the $im_1$ terms provide the prescription for integrating around the poles at $\lambda_k = 0$ or $\lambda_n = 0$ for $m_1 \to 0$. The only terms which may diverge in this limit are those where $\lambda_k = \lambda_n$, i.e. those where $k = n$ (ignoring accidental degeneracies). Keeping these terms only we have, for $m_1 \to 0$,

\[ \Gamma(x, y) \approx \frac{1}{m_1} \sum_n |\Psi_n(x)|^2 |\Psi_n(y)|^2 \frac{m_1}{\lambda_n^2 + m_1^2} \]

\[ \sim \frac{\pi}{m_1} \sum_n |\Psi_n(x)|^2 |\Psi_n(y)|^2 \delta(\lambda_n) \]

In a (large but) finite volume, $H(m_0)$ has near-zero modes practically everywhere inside the super-critical region (see I for a more detailed discussion). Since the quenched-theory Boltzmann weight is strictly positive and independent of $m_1$, the pion two-point function indeed has a $1/m_1$ divergence! Clearly, (nearly) exceptional configurations are the ones responsible for the near-zero modes of $H(m_0)$ — and the pion two-point function is ill-defined for $m_1 = 0$, so one must keep $m_1 \neq 0$.

By summing the previous equation over $x$ and then using translation invariance to average over $y$ we find, for $m_1 \to 0$,

\[ m_1 a^3 \sum_x \Gamma(x, y) \to \frac{\pi}{V} \left( \sum_n \delta(\lambda_n) \right) = \pi \rho(0), \tag{5} \]

which verifies the Ward identity [13] for $p = 0$ in finite volume. The $1/m_1$ divergence allows for $\rho(0) > 0$ and, hence, SSB, in the finite-volume
quenched theory.\footnote{In the unquenched case one can prove that $\Gamma(x,y)$ is bounded. The $m_1 \to 0$ limit of eq. (4) then yields zero, which is the familiar result that there is no SSB in finite volume.}

While conceptually important, the previous discussion provides little insight into the origin of the $1/m_1$ divergence. In fact, we believe that in the infinite-volume limit this divergence arises only from exponentially localized eigenstates. We will illustrate the relevant physics via a heuristic argument: provided $\rho(0)$ arises from localized states only, there are no Goldstone poles. To this end, consider the Fourier transform of the mode density $\mathcal{H}_n(p) = a^4 \sum_x |\Psi_n(x)|^2 e^{-ipx}$. In view of eq. (2) we assume the ansatz

$$\mathcal{H}_n(p) \approx \frac{e^{-ipx_n^0}}{1 + p^2 l_n^2}.$$  

(6)

The essential point is that, for $p^2 l_n^2 \ll 1$, one should feel only the exponentially decaying envelope of the mode density (cf. eq. (2)) but not the short-distance fluctuations. The phase factor in the numerator reflects the location of the mode. The normalization implies $\mathcal{H}_n(0) = 1$. (Positivity of the mode density implies $|\mathcal{H}_n(p)| \leq 1$, so $p = 0$ is the (global) maximum of $|\mathcal{H}_n(p)|$. This forbids a linear term in $p$ in the denominator of eq. (6).) For $m_1 \to 0$ and $p^2 l_n^2 \ll 1$ we now obtain

$$2m_1 \bar{\Gamma}(p) = \frac{2}{V} \left( \sum_n |\mathcal{H}_n(p)|^2 \frac{m_1}{l_n^2 + m_1^2} \right) \sim \frac{2 \pi \rho(0)}{1 + O\left(p^2 l_n^2\right)},$$

(7)

where $\bar{l}$ is the (suitably defined) average localization length of the near-zero modes. Using eq. (6) we conclude that $\bar{\Gamma}_\mu(p) \sim i p_\mu \rho(0) O(\bar{l})$, implying that there are no Goldstone poles.

In conclusion, we arrive at the following physical conjecture for $m_1 \to 0$ in the super-critical region as the simplest one that fits all the existing evidence. We define the Aoki phase as the region where pseudo-scalar Goldstone bosons exist. This region coincides with the part of the phase diagram where $\lambda_c = 0$. Inside (and close to the boundaries of) the Aoki phase, the chiral lagrangian should provide a valid description of the non-perturbative physics.\footnote{Outside the Aoki phase $\lambda_c > 0$, and the spectrum of localized eigenstates extends down to $\lambda = 0$. So far, this picture applies to both the quenched and the unquenched theories. The difference is that, for $\lambda_c > 0$, the quenched theory has $\rho(0) > 0$ and a divergent pion two-point function; the unquenched theory has $\rho(\lambda) \sim \lambda^2$ for small $\lambda$ due to the fermion determinant, and a condensate can only be built from extended eigenstates (i.e. when $\lambda_c = 0$).

3. Domain-wall and overlap fermions

Domain-wall fermions (DWF) and overlap fermions are sophisticated descendants of Wilson fermions, that allow separation of the chiral and the continuum limits. They are recognized today as closely-related realizations of the Ginsparg-Wilson (GW) relation~\footnote{Here we assume a fifth-dimension lattice spacing $a_5 \leq 1$.}. They are both built around a Wilson-operator “kernel” $H(-M)$, where $0 < aM < 2$ is the domain-wall height.\footnote{DWF are five-dimensional Wilson fermions, in which only hopping terms in the four physical directions couple to four-dimensional link variables, which themselves are independent of the fifth coordinate~\footnote{5}. The latter takes values $s = 1, 2, \ldots, N_s$. The left- and right-handed components of the quark field are localized on opposite boundaries of the five-dimensional space, and chiral symmetry becomes exact in the limit $N_s \to \infty$.} DWF are five-dimensional Wilson fermions, in which only hopping terms in the four physical directions couple to four-dimensional link variables, which themselves are independent of the fifth coordinate~\footnote{5}. The latter takes values $s = 1, 2, \ldots, N_s$. The left- and right-handed components of the quark field are localized on opposite boundaries of the five-dimensional space, and chiral symmetry becomes exact in the limit $N_s \to \infty$~\footnote{5}. The left- and right-handed components of the quark field are localized on opposite boundaries of the five-dimensional space, and chiral symmetry becomes exact in the limit $N_s \to \infty$.}

Let $T(M, a_5)$ be the transfer matrix that hops the fermions in the fifth dimension. The “hamiltonian” $H(M, a_5) = - \log(T(M, a_5))/a_5$ is closely related to the Wilson operator. One has $H(M, 0) = H(-M)$. Also, the spectrum of exact zero modes is independent of $a_5$. We may thus study the approach to the chiral limit using the physical concepts of the previous section. We begin by considering the DWF’s PCAC relation (the superscript $a$ is an SU(N)-flavor index)

$$\sum_{\mu} \partial_{\mu} A_{\mu}^a(x) = 2m_q J_q^a(x) + 2J_b^a(x).$$

(8)

Here $A_{\mu}^a(x)$ is the partially-conserved DWF axial current, $\partial_{\mu}$ is the backward derivative, and $m_q$ is the bare quark mass. The pseudo-scalar density

$$\rho(\lambda) \sim \lambda^2$$

for small $\lambda$ due to the fermion determinant, and a condensate can only be built from extended eigenstates (i.e. when $\lambda_c = 0$).
\( J^5_\sigma(x) \) is localized on the boundaries of the fifth dimension and serves as an interpolating field for pions. \( J^5_\sigma(x) \) is another pseudo-scalar density located in the middle of the fifth dimension, which gives rise to finite-\( N_s \) chiral symmetry violations. A measure of these violations is the residual mass \( m_{\text{res}} = m_{\text{res}}(\tau, N_s) \), which may be defined as

\[
m_{\text{res}} = \frac{\sum \tilde{g}_\tau \left\langle J^5_{\sigma}(\vec{x}, \tau + \tau') J^5_{\sigma}(\vec{y}, \tau') \right\rangle}{\sum \tilde{g}_\tau \left\langle J^5_{\sigma}(\vec{x}, \tau + \tau') J^5_{\sigma}(\vec{y}, \tau') \right\rangle}.
\]

Here we singled out one lattice direction as euclidean time \( \tau \). For large \( \tau > 0 \) and large \( N_s \), we expect, ignoring power corrections,\(^\text{10}\)

\[
m_{\text{res}} \sim c_1 \exp(-\tau(1/\tilde{T} - m_\pi)) + c_2 q^{N_s}.
\]

This should be true provided \( \lambda_c > 0 \), where \( \lambda_c = -\log(q)/a_5 \) is the mobility edge of \( \hat{H}(M, a_5) \); equivalently, \( 0 < q < 1 \). Let us briefly explain this result (for the limitations of our analysis, see \( I \)). The denominator in eq. (9) is governed by a zero-momentum pion and decays like \( \exp(-m_\pi \tau) \). In the numerator, one should distinguish between the contributions of extended and of localized modes of \( \hat{H}(M, a_5) \). For large \( N_s \), modes with eigenvalue near \( \lambda_c \) dominate the extended modes’ contribution. This results in universal fifth-dimension wave functions for the left- and right-handed quarks, which decay exponentially with a \( \lambda_5 \). The extended modes’ contribution thus factorizes like \( q^{N_s} \exp(-m_\pi \tau) \). The contribution of localized modes to the numerator is not suppressed exponentially with \( N_s \), because the localized spectrum goes down to zero. This contribution can be estimated using the asymptotic form \( \tilde{I} \). It will be dominated by localized near-zero modes centered around the straight line connecting the two space-time points (see \( I \) for details) and is estimated to behave like \( \exp(-m_\pi \tilde{T}) \) where now \( \tilde{T} \) is, in effect, the maximal localization length of the near-zero modes.

According to eq. (10), \( m_{\text{res}} \) does not vanish exponentially with \( N_s \), for any fixed value of \( \tau \). However, let us imagine taking the infinite-volume limit while keeping the lattice spacing fixed. If we extract \( m_{\text{res}} \) using increasingly large \( \tau \), the first term on the right-hand side of eq. (10) will ultimately vanish even for fixed \( N_s \), provided \( 1/\tilde{T} > m_\pi \). Therefore, the localized modes’ contribution to \( m_{\text{res}} \) does not necessarily constitute a violation of chiral symmetry.

The overlap operator \( \ov = 1 - \gamma_5 \tilde{\gamma}_5 \), \( \tilde{\gamma}_5 \equiv \frac{H(m_0)}{|H(m_0)|} \).

This operator satisfies the GW relation \( \begin{array}{c} 18 \end{array} \), i.e. it possesses a modified chiral symmetry (with the same algebraic properties as ordinary chiral symmetry). The overlap operator cannot have a finite range \( \begin{array}{c} 20 \end{array} \), and the relevant question is what are its localization properties. Exponential locality of the overlap operator was proved in ref. \( \begin{array}{c} 21 \end{array} \), provided all the plaquette variables are uniformly bounded close to one ("admissibility condition"). In this case the spectrum of \( H(m_0) \) has a gap, as on the first row of Fig. 2.\(^\text{11}\) In realistic MC simulations, however, one cannot impose such a constraint on the plaquettes. Assuming the mobility edge satisfies \( \lambda_c > 0 \), the spectrum looks like the last row of Fig. 2. The effect of the localized spectrum can be analyzed along similar lines. Considering the restriction of \( \tilde{\gamma}_5 \) to (say) localized modes with \( |\lambda| \leq \lambda_c/2 \), denoted \( \tilde{\gamma}_5 \), we estimate

\[
\left| \tilde{\gamma}_5(x, y) \right| \leq \exp(-|x - y|/(2\tilde{I})),
\]

where now \( \tilde{I} \) is determined by all eigenmodes with \( |\lambda| \leq \lambda_c/2 \). Once the near-zero modes do not hamper locality, exponential locality of \( D_{ov} \) can be established as in ref. \( \begin{array}{c} 21 \end{array} \). Finally, a similar analysis may be applied to the \( N_s \rightarrow \infty \) limit of DWF with \( a_5 > 0 \), which is also described by eq. (11), except that \( H(m_0) \) is replaced by \( \hat{H}(M, a_5) \) (see also ref. \( \begin{array}{c} 23 \end{array} \)).

The main conclusion of the previous discussion is the following: DWF and overlap fermions can be used only \textit{well outside} the Aoki phase of their Wilson kernel. This situation is quenched-like:

\(^{10}\)In a numerical simulation, the signal of localized modes becomes more complicated when full translation invariance is not enforced.

\(^{11}\)Ref. \( \begin{array}{c} 21 \end{array} \) generalized the proof to the case depicted on the second row of Fig. 2.
(even) for (dynamical) DWF or overlap simulations, the Boltzmann weight does not contain the determinant of the Wilson operator itself. Nevertheless, the spectral properties of the Wilson-operator kernel are crucial, and, for any ensemble of gauge-field configurations, we may (formally) introduce two quenched Wilson flavors and look for their Aoki phase.

When the mobility edge of the Wilson kernel is \( O(1) \) in lattice units, chiral symmetry of DWF will be recovered exponentially with increasing \( N_s \), for large \( \tau \).\(^{12}\) Well outside the Aoki phase we also expect the average localization length of the near-zero modes to be \( O(1) \) in lattice units, and this guarantees the locality of the overlap operator, as well as of the generalized overlap operators obtained from DWF with \( a_5 > 0 \). In contrast, being inside the Aoki phase means that the mobility edges of both the Wilson kernel and \( H(M, a_5) \) are zero. This corresponds to taking the limits \( q \to 1 \) and \( \tilde{l} \to \infty \), where eqs. \((10)\) and \((12)\) cease to hold. Thus, DWF will develop long-range correlations in all five dimensions, and the (generalized) overlap operator defined by the \( N_s \to \infty \) limit will become non-local, including for \( a_5 \to 0 \).

Near the continuum limit, DWF and overlap fermions are local. The mobility edge and \( 1/\tilde{l} \) both scale like the lattice cutoff, and \( \rho(0) \) tends to zero rapidly for \( g_0 \to 0 \).\(^{12}\) But, for present-day simulations, it is unclear if \( 1/\tilde{l} \) is large compared to e.g. the rho-meson mass. How, then, can one tell if a given set of simulation parameters lies safely outside the Aoki phase? At the very least, one should require that \( 1/\tilde{l} \) will be bigger than the mass of all hadrons of interest. The issue definitely deserves further study.

For DWF, eq. \((10)\) shows that, by monitoring the dependence of \( m_{\text{res}} \) on both \( N_s \) and \( \tau \), one can extract the crucial spectral properties of \( H(M, a_5) \). The mobility edge is determined in terms of \( q \), and may be extracted from the \( N_s \) dependence. The (dominant) localization length of the near-zero modes may be extracted from the \( \tau \) dependence, and the constant \( c_1 \) provides information on the density of the near-zero modes. A well-behaved \( m_{\text{res}} \) (cf. eq. \((10)\)) also ensures the locality of the effective four-dimensional operator defined by the \( N_s \to \infty \) limit. \( m_{\text{res}} \) is routinely calculated in any new DWF simulation. For a lattice cutoff \( a^{-1} \sim 2 \text{ GeV} \), well-behaved \( m_{\text{res}} \) have been obtained in quenched simulations with Iwasaki \(^{26}\) and DBW2 \(^{27}\) gauge actions. Likewise, when overlap fermions are employed, it should become a routine practice to determine the localization properties of the overlap operator! In particular, the relation between the localization lengths of the overlap and of the near-zero modes should be studied in more detail.

Dynamical DWF \(^{28}\) and overlap simulations are, and will be, very expensive. In a dynamical simulation the danger is that one will end up too close to, or even inside, the Aoki phase. In this case, one cannot maintain both chirality and locality.\(^{13}\) Having to give up on something, we believe that one should insist on locality, at the price of doing worse on chirality. The reason is that, unlike with approximate chiral symmetry (see below), very little is known about how to monitor for the physical consequences of deteriorating locality. Any non-locality of an operator that satisfies the GW relation should be regarded as the source of an unknown systematic error.

In a situation where the overlap operator (or generalizations to \( a_5 > 0 \)) is non-local, locality could be maintained, for example, by using DWF with modest \( N_s \).\(^{14}\) For other approximate solutions of the GW relation see ref. \(^{29}\).\(^{15}\) While vio-

\(^{12}\) The situation is more complicated when considering the lattice renormalization of the effective Electro-Weak hamiltonian. Especially when power-divergent subtractions are involved, localized modes may still give rise to chiral symmetry violations which are not suppressed by any space-time separation. In this case, employing smeared links \(^{24}\) and/or the “projection method” of ref. \(^{25}\) may provide better control over the subtractions needed to recover chiral symmetry.

\(^{13}\) We believe that this applies to present-day thermodynamical simulations carried out with a lattice cutoff \( a^{-1} \sim 1 \text{ GeV} \).

\(^{14}\) If no pseudo-fermion fields are included, this is strictly local.

\(^{15}\) Local approximations of the overlap exist, too. For example, a rational polynomial approximation may be represented as a continued fraction which, in turn, may be cast in the form of a five-dimensional action with nearest-neighbor coupling \(^{29}\). This action is not yet local because it contains parameters which are functions of the minimal eigenvalue of \( H(m_0) \), which in turn is a global property of the gauge-field configuration. If one fixes these parame-
lations of chiral symmetry in Ward identities will be non-negligible in this case, the well-established renormalization program allows us to subtract them. In principle, this is the same situation as with ordinary Wilson fermions. In practice, however, unlike Wilson fermions, as long as $m_{res}$ is a few MeVs (or less), it may be possible to carry out the subtractions successfully for a wide range of weak matrix elements.

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