About Fibonacci trees.

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Abstract

In this first paper, we look at the following question: are the properties of the Fibonacci tree still true if we consider a finitely generated tree by the same rules but rooted at a black node? The direct answer is no, but new properties arise, a bit more complex than in the case of a tree rooted at a white node, but still of interest.

1 Introduction

This paper investigates a question the author raised to himself a long time ago but he had no time to look at it. When he established the properties of the finitely generated tree he called the Fibonacci tree, he wandered whether the properties still hold if, keeping the generating rules, we apply them to a tree rooted at a black node.

In section 2, we remind the reader the definitions about the Fibonacci tree and its connection with two tilings of the hyperbolic plane we call pentagrid and heptagrid. In section 3, we consider the properties which rely on the representations of the positive integers as sums of distinct Fibonacci numbers and we look at the question raised in the abstract. We show that the properties attached to the Fibonacci tree rooted at a white node are no more true for the similar tree rooted at a black node. However, other properties can be established, more complicated in that setting. We also show the connection of the Fibonacci tree rooted at a black node with the pentagrid and the heptagrid. In section 4, we consider another increasing sequence of positive numbers which we call the golden sequence which can also be attached to the Fibonacci trees, both the tree rooted at a white node and the other rooted at a black one. Again, the nice properties which connect the golden sequence with the Fibonacci tree rooted at a white node are no more true when it is rooted at a black node. However, other properties occur, more complicated than the previous ones, but still worth of interest.

Section 5 concludes the paper with open questions regarding the generalization of these results. It might be the goal of other papers.
2 The Fibonacci trees

In sub section 2.1, we remind the reader the definition of the Fibonacci tree as well as different variations about it which were investigated by the author in [1]. In sub section 2.2, we remind him/her the numbering of the nodes and the important properties of their Fibonacci representations, in particular the connections of the representation of the sons of a node with the representation of that node. In sub section 2.3, we remind the connection of the Fibonacci tree with the pentagrid and with the heptagrid.

2.1 Fibonacci trees rooted at a white node

We call Fibonacci tree the finitely generated tree with two kinds of nodes, black nodes and white ones whose generating rules are:

\[ B \rightarrow BW \quad \text{and} \quad W \rightarrow BWW. \]  

When the root of the tree is a white, black node, we call such a Fibonacci tree a white, black respectively, Fibonacci tree. In this whole section, we consider white Fibonacci trees only. Black Fibonacci trees will be studied in Sections 3 and 4. The status of a node says whether it is black or white.

The connection with Fibonacci numbers first appear when we count the number of nodes which lay at the same level of the tree. For a white Fibonacci tree, we have the following property:

**Theorem 1** [1] In a white Fibonacci tree, the level of the root being 0, the number of nodes on the level \( k \) of the tree is \( f_{2k+1} \) where \( \{f_n\}_{n \in \mathbb{N}} \) is the Fibonacci sequence with initial conditions \( f_0 = f_1 = 1 \).

As in [1], let us number the nodes starting from 1 given to the root and then, from level to level and, on each level, from left to right. From now on, we identify a node with the number it receives in the just described way. It is known that any positive integer \( n \) can be written as a sum of distinct Fibonacci numbers, the terms of the Fibonacci sequence considered in Theorem 1.

\[ n = \sum_{i=1}^{k} a_i f_i \quad \text{with} \quad a_i \in \{0, 1\}. \]  

2.2 The preferred son property

The \( a_i \) digits which occur in (1) are not necessarily unique for a given \( n \). They can be made unique by adding the following condition: for \( 0 < i < k \), if \( a_i = 1 \), then \( a_{i-1} = 0 \). If we write the \( a_i \)'s of (1) as a word \( a_1..a_k \), the condition for uniqueness of the representation says that in that word, the pattern 11 never occurs. We call code of \( n \) the unique word attached to \( n \) by (1) when the pattern 11 is ruled out. We also say code of \( \nu \) for the node of the white Fibonacci tree whose number is \( \nu \), and we write \( \lfloor \nu \rfloor \) for the code of \( \nu \). Formula (2) allows us to restore \( \nu \) from its code \( \lfloor \nu \rfloor \). We shall write \( \nu = (\lfloor \nu \rfloor) \). From Theorem 1, we get:
We can state the following property:

**Theorem 2** In a white Fibonacci tree, for any node \( \nu \) we have that among its sons a single one has \( [\nu]00 \) as its code. That son is called the preferred son of \( \nu \). If the node is black, its preferred son is its black son, if the node is white, its preferred son is its white son in between its black son and the other white one.

Figure 1 illustrates the properties stated in the theorem. The reader may easily check them.

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**2.3 Connection of the Fibonacci tree with the pentagrid and with the heptagrid**

As mentioned in the introduction, the white Fibonacci tree is connected with the pentagrid and the heptagrid, two tilings of the hyperbolic plane. The pentagrid is the tessellation \( \{5, 4\} \), which means that the tiling is generated by the reflection of a basic polygon in its sides and the recursive reflections of the images in their sides, where the basic polygon is the regular convex pentagon with right angles. That polygon lives in the hyperbolic plane, not in the Euclidean one. Similarly, the heptagrid is the tessellation \( \{7, 3\} \) which is generated in a
similar way where the basic polygon is the regular convex heptagon with \frac{2\pi}{3} as its vertex angle. Again, such a polygon lives in the hyperbolic plane and not in the Euclidean plane. Figure 2 illustrates the pentagrid, left hand side, and the heptagrid, right hand side.

**Figure 2** The tilings generated by the white Fibonacci tree. To left, the pentagrid, to right, the heptagrid.

Figure 3 illustrates how the white Fibonacci tree generates the considered tilings. In both tilings, the tiles of a sector, can be put in bijection with the nodes of the tree. In the case of the pentagrid, such a sector is a quarter of the plane: it is delimited by two perpendicular half-lines stemming from the same vertex \( V \) of a tile \( \tau \) and passing through the other ends of the edges of \( \tau \) sharing \( V \).

**Figure 3** How the white Fibonacci trees generate the pentagrid and the heptagrid: each isolated sectors in the above figures is spanned by the white Fibonacci tree. The colours of the tile show the Fibonacci structure: blue tiles are the black nodes, green and yellow tiles are the white ones. Yellow tiles are the second white son of a white node.
The definition of the sector is more delicate in the case of the heptagrid. The sector is also defined by half-lines which are, this time, issued from the mid-point of an edge $\eta$ and those half-lines pass through the mid-points of two consecutive sides of a tile sharing $V$ as a vertex, $V$ being also an end of $\eta$. The reader is referred to [2] for proofs of the just mentioned properties. Accordingly, as shown on the figure, five sectors allow us to locate tiles in the pentagrid and seven sectors allows us to perform the same thing in the heptagrid. From now on, we call tile $\nu$ the tile attached to the node $\nu$ of such a Fibonacci tree we assume to be fixed once and for all. We also say that $[\nu]$ is the code of the tile $\nu$.

The preferred son property allows us to compute in linear time with respect to the code of a node $\nu$ the codes of the nodes attached to the tiles which share a side with the tile $\nu$. Such tiles are called the neighbours of $\nu$. Theorem 2 also allows us to compute in linear time with respect to $[\nu]$ a shortest path in the tiling, leading from the tile $\nu$ to tile 1.

3 When Fibonacci numbers are used

So far, we mentioned the properties of the white Fibonacci tree. Let us look at the following problem with the black Fibonacci tree. In Subsection 3.1, we look at the analog of Theorem 1 and its Corollary 1. In Subsection 3.2, we investigate the analog of Theorem 2. In Subsection 3.3, we look at the connection of the black Fibonacci tree with the pentagrid and with the heptagrid.

3.1 The black Fibonacci tree and Fibonacci numbers

We can prove an analog of Theorem 1.

**Theorem 3** In a black Fibonacci tree, the level of the root being 0, the number of nodes on the level $k$ of the tree is $f_{2k}$ where $\{f_n\}_{n \in \mathbb{N}}$ is the Fibonacci sequence with initial conditions $f_0 = f_1 = 1$.

**Corollary 2** In a black Fibonacci tree, the rightmost node of the level $k$ has the number $f_{2k+1}$ and has the code $10^{2k}$ for non negative $k$.

The property stated in Theorem 3 was noted in a paper by Kenichi Morita, but we shall see in Section 4 that both Theorems 1 and 3 simply come from the fact that the same generating rules are applied to the trees and that the initial levels contain 1 and 3 nodes for the white tree while they contain 1 and 2 nodes for the black one. The latter properties are in some sense symmetric: in the white Fibonacci tree the levels are Fibonacci numbers with odd index so that the cumulative sum up to the current level plus one is a Fibonacci number with even index and in the black Fibonacci tree the situation is opposite: the levels give rise to Fibonacci numbers with even index whose cumulative sum is precisely a Fibonacci number with odd index. It is the reason why in the white tree, the Fibonacci numbers with even index occur at the head of a level why in the black one the Fibonacci numbers with odd index occur at the tail of a level.
3.2 The black Fibonacci tree and a successor property

The next question is: is Theorem 2 true for the black node if we number its nodes in the same way as in the white tree and if we attach to the nodes the codes of their numbers? The answer is no if we stick to the same statement and it is yes if we change the question by asking whether there is a rule to define the connection of a node $\nu$ with the one whose code is $[\nu]00$. Such a situation can be suspected by comparing Theorem 1 with Theorem 3 as well as Corollary 1 with Corollary 2.

Figure 4 The black Fibonacci tree. The same convention about colours of the nodes and of the edges between nodes as in Figure 1 is used. We can see that the preferred son property as stated in Theorem 2 is not true in the present setting.

Figure 4 shows us a first fact: the preferred son property is not observed in the black Fibonacci tree. We can see that for any black node $\nu$, none of its sons has the code $[\nu]00$. More other, for the white nodes $\nu$ whose code can be written $[\mu]01$, the code $[\mu]0100$ is that of the leftmost son of the node $\nu + 1$.

We can now state the property which holds in the black Fibonacci tree. As the preferred son property is no more observed with the exception of a few white nodes, we say that the successor of the node $\nu$ is the node whose code is $[\nu]00$. Theorem 2 says that in a white node, the successor of a node occurs among its sons with a precise rule depending on the colour of the node. In order to formulate the property, we need to define the end of the code of a node. The last two digits of the code of a node is either 00, 01 or 10. In that latter case, we know that in fact we can write the last three digits as 010 as the pattern 11 is ruled out. With two kinds of nodes, this would give us six classes of nodes. In fact we have five of them only, which can be written b00, b01, w00, w01 and w10, which we call the types of the nodes. The type of a node mixes its status with the ending of its code. Accordingly, we can see that the type b01 defines an empty class, a property we shall check. For a node $\nu$, $\text{succ}(\nu)$ is the number of its successor and $s_r(\nu)$, $s_l(\nu)$ is the number of its rightmost, leftmost
son respectively. Note that we always have:
\[ s_r(\nu)+1 = s_\ell(\nu+1). \] (3)

**Theorem 4** In the black Fibonacci tree, we have \( \text{succ}(\nu) = s_r(\nu)+1 \) if \( \nu \) is a black node or a white node of type \( \text{w01} \). For the types of white node \( \text{w00} \) and \( \text{w10} \), we have \( \text{succ}(\nu) = s_r(\nu) \). We have six rules giving the types of the sons of a node according to its type:

- \( \text{b00} \rightarrow \text{b01} - \text{w10} \)
- \( \text{b01} \rightarrow \text{b01} - \text{w10} \)
- \( \text{w00} \rightarrow \text{b00} - \text{w01} - \text{w00} \)
- \( \text{w00*} \rightarrow \text{b01} - \text{w10} - \text{w00}^* \)
- \( \text{w01} \rightarrow \text{b00} - \text{w01} - \text{w10} \)
- \( \text{w10} \rightarrow \text{b00} - \text{w01} - \text{w00} \)

where the type \( \text{w00*} \) indicates a node of the form \( f_{2k+1} \).

**Proof.** We prove the theorem by induction on all the properties listed in it. We note that the properties are trivially observed on Figure 4 for the root and its sons. Accordingly, we assume the property to be true for all nodes \( \mu < \nu \) and we check that it is true for at least \( \nu \) and \( \nu+1 \). Let \( k+1 \) be the level of \( \nu \), with \( k \geq 1 \) and let \( \mu \) be its father which is thus on the level \( k \).

From the statement of the theorem, namely by the rules induced on the types, we can deduce the following succession for two nodes \( \kappa \) and \( \kappa+1 \) lying on the same level of the tree.

\[
\begin{align*}
\text{b00}, \text{w01} & \rightarrow \text{b01} - \text{w10} \\
\text{b01} & \rightarrow \text{b01} - \text{w10} \\
\text{w00} & \rightarrow \text{b00} - \text{w01} - \text{w00} \\
\text{w00*} & \rightarrow \text{b01} - \text{w10} - \text{w00*} \\
\text{w01} & \rightarrow \text{b00} - \text{w01} - \text{w10} \\
\text{w10} & \rightarrow \text{b00} - \text{w01} - \text{w00} 
\end{align*}
\] (4)

Below, Tables 1 and 2 allow us to check the correctness of the statement of Theorem 4. In both tables, the following notations are used for nodes connected with \( \mu \) or \( \nu \). The node \( \mu_1 \) is \( \mu - 1 \). The node \( \nu_1 \) is \( \nu - 1 \). The node \( \lambda \) is \( s_\ell(\nu) \) and \( \lambda_1 \) is the node \( \lambda - 1 \). According to (3), \( \lambda_1 = s_r(\nu_1) \). In both table, we indicate the black nodes by writing their number in blue.

In both tables, we apply the induction hypothesis for all nodes \( \kappa \) with \( \kappa < \nu \). In particular, that applies to \( \nu_1 \) too. Accordingly, \( \lambda_1 \), which is always a white node is known. If we know the code of \( \nu_1 \), we know that of \( \lambda_1 \), of \( \lambda \) and then those of \( \lambda+1, \lambda+2 \) and \( \lambda+3 \).

Indeed, if we know the code of \( \kappa \) we can easily compute the node of \( \kappa - 1 \) and of \( \kappa + 1 \). In [\( \kappa \)], say that the letter \( \text{b} \) is before the letter \( \text{a} \) if \( \text{b} \) is on the left hand side of \( \text{a} \). Say that a \( \text{0} \) is safe if the letter before it is a \( \text{0} \) too. Note that according to (2), the leftmost letter of a code is \( \text{1} \). We assume that there is a safe \( \text{0} \) before the leading \( \text{1} \) of the code.

**Lemma 1** Let \( [\kappa] \) be the code of a node of a Fibonacci tree. Let \( [\kappa] = [\xi](01)^n \). Then, \( [\kappa+1] = [\xi]01(00)^{n-1} \). Let \( [\kappa] = [\xi](00)^n(10) \). Then, \( [\kappa+1] = [\xi]01(00)^n \). We can say that a pattern \( \text{11} \) generates a carry which propagates until a safe \( \text{0} \) which it replaces by a \( \text{1} \).
Lemma 2 Let $[\kappa]$ be the code of a node of a Fibonacci tree. If $[\kappa]$ ends with a 1, $[\kappa]-1$ can be written by replacing the rightmost 1 by a 0. If $[\kappa] = [\xi](00)^n$. Then, $[\kappa]-1 = [\xi]01(01)^{n-1}$. If $[\kappa] = [\xi](00)^n$. Then, $[\kappa]+1 = [\xi](010)^n$. We can say that a pattern 00 generates 10 or 01 depending on the parity of the repetitions of the pattern 00 before the rightmost 1. If that number is even it generates 01 for each pattern 00; otherwise, it generates the pattern 10.

Table 1 Table of the computation of the codes of the following nodes when $\nu$ is a black node: $\mu$ is the father of $\nu$, $\mu_1 = \mu-1$, $\nu_1 = \nu-1$, $\lambda$, the leftmost son of $\nu$, $\lambda_1 = \lambda-1$, as well as the nodes $\lambda+1$ and $\lambda+2$. We mark the black nodes in blue, except when $\lambda+2$ is the successor of $\nu$.

| $\mu_1$ | $\mu$ | $\nu$ | $\nu_1$ | $\lambda_1$ | $\lambda$ | $\lambda+1$ | $\lambda+2$ |
|--------|-------|-------|---------|-------------|-----------|-------------|-------------|
| $\alpha_0$ | $\alpha_01$ | $\beta_1010$ | $\alpha_0000$ | $\beta_100100$ | $\beta_100101$ | $\beta_101010$ | $\alpha_000000$ | $01,10$ |
| $\beta_01$ | $\alpha_10$ | $\beta_0010$ | $\beta_0100$ | $\beta_001000$ | $\beta_001001$ | $\beta_001010$ | $\beta_010000$ | $01,10$ |

$\mu$ is a black node, so that $\mu_1$ is white:

| $\nu$ | $\lambda$ | $\lambda_1$ | $\lambda+1$ | $\lambda+2$ |
|-------|-----------|-------------|-------------|-------------|
| $\beta_10$ | $\alpha_00$ | $\beta_1000$ | $\beta_100100$ | $\beta_100101$ | $\beta_101010$ | $\beta_100100$ | $01,10$ |
| $\beta_00$ | $\alpha_01$ | $\beta_0000$ | $\beta_0001$ | $\beta_000000$ | $\beta_000001$ | $\beta_000010$ | $\beta_000100$ | $01,10$ |

$\mu_1$ and $\mu$ are both white nodes:

| $\mu_1$ | $\nu$ | $\nu_1$ | $\lambda_1$ | $\lambda$ | $\lambda_1$ | $\lambda+1$ | $\lambda+2$ |
|-------|-------|---------|-------------|-----------|-------------|-------------|-------------|
| $\alpha_01$ | $\alpha_10$ | $\beta_0010$ | $\alpha_0100$ | $\alpha_001000$ | $\alpha_001001$ | $\alpha_001010$ | $\alpha_010000$ | $01,10$ |
| $\alpha_00$ | $\alpha_01$ | $\beta_0001$ | $\alpha_0100$ | $\alpha_001000$ | $\alpha_001001$ | $\alpha_001010$ | $\alpha_010000$ | $01,10$ |

Our proof consists in carefully looking at every possible case. We first assume that $\nu$ is a black node. Its father $\mu$ may be white or black. Assume that $\mu-1$ is black. Necessarily, $\mu$ is white. Now, from the hypothesis, the possible types for $\mu_1$ and $\mu$, taking into account that $\mu = \mu_1+1$, are: $b00, w01$ or $b01, w10$. As $\mu < \nu$, the induction hypothesis applies to $\mu$ and to $\mu_1$. From (4), we have that $\mu_1-1$ has the type $w10$. Accordingly, its rightmost son ends in $1000$, so that as $\nu_1$ is a black node, it is easy to check that $\nu_1$ ends in $1010$. If $\mu$ is black, so that $\mu_1$ is white, the possible succession of the types are $w00, b01$ and $w10, b00$: from the assumption, the rightmost son of a node is either $w10$ or $w00$. If $\mu_1$ and $\mu$ are both white, then the only possibilities for the succession of types are $w01, w00$ and $w10, w01$. Note that when $\mu$ is black or has the type $w01$, its successor is $\nu$, otherwise it is $\nu_1$. Similarly, if $\nu_1$, which is always white has the type $w01$, then its successor is $\lambda$, otherwise it is $\lambda_1$. From these features and the help of Lemmas 1 and 2 as well as (4), we obtain the computations of Table 1. Note that in the application of 2 we have to take into account on the assumption hypothesis and on the change from $b01$ to $\alpha_00$, for instance, in order to compute the code of $\kappa-1$ when the code of $\kappa$ ends with several contiguous 0.

Let us apply the similar arguments for the case when $\nu$ is a white node. This time, the computation is also based on the position of $\nu$ among the other sons of its father $\mu$. Consider the case when $\mu$ is black. Necessarily, $\mu_1$ is white and $\nu_1$ is black. The possible types for $\mu_1$, $\mu$ and $\nu_1$ are $w10, b00, b01$ and $w00, b01, b01$. In those cases, the successor of $\mu_1$ is $\nu_2 = \nu_1-1$. The computation from $\nu_2$ to $\nu_1$
is straightforward from Lemma 1, so that we do not mention the code of $\nu_2$ in Table 2. Also, as $\nu_1$ has the type $\text{b01}$, its successor is $\lambda$.

**Table 2** Table of the computation of the codes of the son of $\nu$ when it is a white node whose father is $\mu$. The same conventions as in Table 1 are used here too. In the upper part of the table, $\mu$ is a black node, so that $\nu_1$ is white. In the lower part of the table, $\mu$ is a white node and we assume that $\nu_1$ too is a white node. In the upper part of the table, $\nu_1$ is black. That node may be also white in the lower part of the table.

| $\nu_1$ is black, $\mu_1$ is white and is black: |  |
|---|---|
| $\mu_1$ | $\mu$ | $\nu_1$ | $\nu$ | $\lambda$ | $\lambda+1$ | $\lambda+2$ | $\lambda+3$ |
| $\beta 10$ | $\alpha 00$ | $\beta 1010$ | $\beta 10100$ | $\beta 101001$ | $\beta 1010010$ | $\beta 10100100$ | $00,01,10$ |
| $\alpha 00$ | $\alpha 01$ | $\alpha 0000$ | $\alpha 000000$ | $\beta 000001$ | $\beta 0000010$ | $\beta 0000100$ | $00,01,10$ |

$\nu_1$ and $\mu_1$ are black, $\mu$ is white:

| $\nu_1$ is black, $\mu_1$ and $\mu$ are both white: |  |
|---|---|
| $\mu_1$ | $\mu$ | $\nu_1$ | $\nu$ | $\lambda$ | $\lambda+1$ | $\lambda+2$ | $\lambda+3$ |
| $\alpha 01$ | $\alpha 10$ | $\alpha 0100$ | $\alpha 010000$ | $\alpha 0100001$ | $\alpha 0100100$ | $\alpha 0101000$ | $00,01,10$ |
| $\alpha 00$ | $\alpha 01$ | $\alpha 0000$ | $\alpha 000000$ | $\alpha 0000001$ | $\alpha 0000010$ | $\alpha 0001000$ | $00,01,10$ |

$\nu_1$ is black, $\mu_1$ and $\mu$ are both white:

| $\nu_1$ is a white node, so that $\mu$ is white too: |  |
|---|---|
| $\mu_1$ | $\mu$ | $\nu_1$ | $\nu$ | $\lambda$ | $\lambda+1$ | $\lambda+2$ | $\lambda+3$ |
| $\alpha 01$ | $\alpha 10$ | $\alpha 0100$ | $\alpha 010000$ | $\alpha 0100001$ | $\alpha 0100100$ | $\alpha 0101000$ | $00,01,10$ |
| $\alpha 00$ | $\alpha 01$ | $\alpha 0000$ | $\alpha 000000$ | $\alpha 0000001$ | $\alpha 0000010$ | $\alpha 0001000$ | $00,01,10$ |

Consider still the case when $\nu_1$ is black. We have the case when $\mu$ is white and both $\mu_1$ is black. The possible types are $\text{b01,w10,b00}$ and $\text{b00,w01,b00}$ for the nodes $\mu_1$, $\mu$ and $\nu_1$ in that order. Next, we have the case when both $\mu_1$ and $\mu$ are white nodes. For $\mu_1$ and $\mu$ the succession of types is $\text{w01,w10}$ or $\text{w01,w00}$ or also $\text{w10,w00}$. In the first two cases, the type of $\nu_1$ is $\text{b00}$, while in the exceptional case $\text{w10,w00}$, it is $\text{b01}$.

Now, consider the case when $\nu_1$ is white. Necessarily, $\mu$ is also a white node. When $\mu_1$ is black, we have the possibilities $\text{b01,w10}$ and $\text{b00,w01}$. When $\mu_1$ is white, we have three possibilities: $\text{w01,w00}$ or $\text{w01,w10}$ or also $\text{w01,w00}$. This latter case happens when $\mu$ and $\nu$ are the last nodes of two consecutive levels. We have already met this case when $\mu$ is the last node of a level and $\nu$ is the penultimate node on the next level. In that case $\nu_1$ is black: see Table 2.

The last column of the tables shows the types of the sons of $\nu$ with the type
written in blue for the black nodes, which allows us to drop the letter of the status. We can see that the hypothesis is checked for the sons of the node \( \nu \) and their types. This completes the proof of the theorem. \( \square \)

### 3.3 The black Fibonacci tree in the pentagrid and in the heptagrid

It is time to indicate which place a black Fibonacci tree takes in the pentagrid and in the heptagrid.

As illustrated by Figure 5, the sectors defined by Figure 3 in Sub section 2.3 can be split with the help of regions of the tiling generated by the white Fibonacci tree and by the black one.

![Figure 5](image)

*Figure 5* The decomposition of a sector spanned by the white Fibonacci tree into a tile, then two copies of the same sector and a strip spanned by the black Fibonacci tree. To left: the decomposition in the pentagrid; to right, the decomposition in the heptagrid. In both cases, the lines which delimit a sector spanned by each kind of tree.

In the figure, the sector is split into a tile, we call it the leading tile, and a complement which can be split into two copies of the sector and a region spanned by the black Fibonacci tree which we call a strip.

In both tilings, the strip appears as a region delimited by two lines \( \ell_1 \) and \( \ell_2 \) which are non-secant. It means that they never meet and that they also are not parallel, a property which is specific of the hyperbolic plane. There is a third line which supports the side of the tile \( \tau \) which is associated with the root of the black Fibonacci tree. That line is the common perpendicular to \( \ell_1 \) and \( \ell_2 \). The tile \( \tau \) is called the leading tile of the strip. It is worth noticing that the way we used to split the sector can be recursively repeated in each sector generated by the process of splitting. We can note that the strip itself can be exactly split into a tile, a sector and a strip. This process is closely related with the generating rules of the Fibonacci trees. At this point, it can be noticed that there are several ways to split a sector and a strip again into strips ans sectors. This can be associated with other rules for generating a tree which we again call a Fibonacci tree. There are still two kinds of nodes, white and black ones.
But the rules are different by the order in which the black son occurs among the sons of a node. There are two choices for black nodes and three choices for white ones. Accordingly, there are six possible definitions of Fibonacci tree. We can also decide to choose which rule is applied each time a node is met. In [1] those possibilities are investigated. We refer the interested reader to that paper.

Figure 6 The decomposition of a sector spanned by the white Fibonacci tree into a sequence of pairwise adjacent strips spanned by the black Fibonacci tree. To left: the decomposition in the pentagrid; to right, the decomposition in the heptagrid. In both cases, the lines which delimit the strips spanned by the tree.

Figure 7 Another look on the decomposition of the sector given by Figure 6. The structure of the Fibonacci is erased in order to highlight the decomposition into pairwise adjacent strips.

But a sector can be split in another way which is illustrated by figure 6. Consider a sector $S_0$. Consider its leading tile $T$. That tile is associated with the root of the white Fibonacci tree. Assume that we associate it with the black Fibonacci tree in such a way that in the association the leftmost son of $T$ is again the black son of the root in both trees. What remains in the sector? It remains a node which we can associate with the root of the white Fibonacci tree. A simple counting argument, taking into account that the levels are different...
by one step from the white tree to the black one in that construction, shows us that in this way we define an exact splitting of the sector. And so, there is another way to split the sector: into a strip $B_0$ and a sector again, $S_1$. Now, what was performed for $S$ can be repeated for $S_1$ which generates a strip $B_1$ and a new sector $S_2$. Accordingly, we proved:

**Theorem 5** The sector associated to the white Fibonacci tree can be split into a sequence of pairwise adjacent strips $B_n$, $n \in \mathbb{N}$, associated to the black Fibonacci tree. Equivalently, the white Fibonacci tree can be split into the union of a sequence of copies of the black Fibonacci tree. The leading tiles of the $B_n$'s are associated with the nodes $f_{2n+1}-1$ of the white Fibonacci tree, i.e. the nodes which are on the rightmost branch of the white Fibonacci tree.

### 4 When the golden sequence is used

Let us go back to the generating rules of the white Fibonacci tree. From the rules defined by (1) in Subsection 2.1, we can easily count the number of nodes which lie at the same level of the tree. Denote by $u_n$, $v_n$ the number of white, black nodes respectively lying on the level $n$. Denote the total number of nodes on that level by $w_n$. Clearly:

\[
\begin{align*}
    u_{n+1} &= 2u_n + v_n \\
    v_{n+1} &= u_n
\end{align*}
\]

from which we easily get:

\[
w_{n+2} = 3w_{n+1} - w_n. \tag{5}
\]

Note that (5) can be found directly: white nodes generate three nodes and black ones two nodes only, but the number of black nodes of the considered level is the number of nodes of the previous level as for each node there is a single black son. Now, equation (5) defines a polynomial $P(X) = X^2 - 3X + 1$ whose roots are the numbers $\frac{3 + \sqrt{5}}{2}$ and $\frac{3 - \sqrt{5}}{2}$. Now, $\frac{3 + \sqrt{5}}{2} = (\frac{1 + \sqrt{5}}{2})^2$ which explains the link with the Fibonacci sequence and why the number of nodes in the white Fibonacci tree are connected with the Fibonacci numbers whose index is odd.

We can define codes with the sequence $w_n$. Indeed, it is not difficult to prove that any positive number $n$ is a sum of distinct terms of the sequence $\{w_n\}_{n \in \mathbb{N}^+}$, where $\mathbb{N}^+$ is the set of positive integers. We have:

\[
n = \sum_{j=1}^{k} a_j w_j \text{ with } a_j \in \{0, 1, 2\}, \tag{6}
\]

where the sequence $\{w_n\}_{n \in \mathbb{N}^+}$ is defined by (5) and the initial conditions $w_1 = 1$ with $w_2 = 3$ for the white Fibonacci tree and $w_1 = 1$ with $w_2 = 2$ for the black Fibonacci tree. As in the case of the representation of positive numbers Fibonacci, this representation is not unique. Also, we can associate to the $a_j$'s of
formula (6) a word in the alphabet \(\{0, 1, 2\}\) which we write \(a_1\ldots a_k\) and which we call the **golden code** of \(n\) and we write again \([n]\) when there is no ambiguity with the code defined with the Fibonacci numbers, otherwise we write \([n]_0\) for the golden code. We can make the golden code of \(n\) unique by requiring that the pattern \(21^*2\) is ruled out, where \(1^*\) is either the empty word or a word consisting of \(1\)'s only. The code associated with the Fibonacci numbers will here be called **Fibonacci code**.

We can also associate the golden code to a tile in a fixed sector of the pentagrid or of the heptagrid by giving to the tile the golden code of the number associated to the tile.

In Subsection 4.1, we look at the properties of the golden code in the white Fibonacci tree. In Subsection 4.2, we look at the same issue in the black Fibonacci tree. We shall see an analogous phenomenon with what we observed in the previous sections.

### 4.1 The golden codes in the white Fibonacci trees

The golden codes in the white Fibonacci tree have properties which are similar to those of the Fibonacci codes we have depicted in Subsection 2.2 in the same tree.

The author and its co-author, Gencho Skordev, proved that the preferred son property is true with the golden codes. However, due to the different alphabet used for writing the codes, the definition of the preferred son in this context is a bit different. The definition comes with the following result:

**Theorem 6** (see [3, 2]) In the white Fibonacci tree fitted with the golden codes, each node has exactly one node among its sons whose code ends with \(0\). That son is called the **preferred son** of the node. Moreover, if \([\nu]\) is the golden code of \(\nu\), the golden code of its preferred son is \([\nu]_00\). In all nodes, the preferred son is the leftmost white son.

![Figure 8](image-url) *The golden codes in the white Fibonacci tree: we can check the properties stated by Theorem 6.*

We refer the reader to \[3, 2\] for the proof of the result which is stated for the general case in the quoted references.

4.2 The golden codes in the black Fibonacci trees

In this context too, when we look at the golden codes in the black Fibonacci tree, we can see that the preferred son property is no more true, as it can easily be checked on Figure 9.

However, there is a kind of regularity, more regular than in the case of the Fibonacci codes, which are indicated in Theorem 7. As in Sub section 3.2, we call successor of the node \(\nu\), the node whose golden code is \([\nu]_g 0\), the node being again denoted by \(\text{succ}(\nu)\).

**Theorem 7** In the black Fibonacci tree, for all nodes \(\nu\), we have that, except for the white nodes whose golden code ends in 0, \(\text{succ}(\nu) = s_r(\nu) + 1\). For the white nodes whose golden code ends in 0, we have \(\text{succ}(\nu) = s_r(\nu)\). Moreover, the last two letters of a golden codes combined with the statuses of the nodes give rise to five combinations: \(b0, b1, w0, w1\) and \(w2\). Each type give rises to a specific rule determining the types of the sons:

- \(b0 \rightarrow b0, w1\)
- \(b1 \rightarrow b1, w2\)
- \(w0 \rightarrow b0, w1, w0\)
- \(w1 \rightarrow b0, w1, w2\)
- \(w2 \rightarrow b0, w1, w2\)

The proof is very similar in its principle to the proof of Theorem 4. This is why, here, it will boil down to Tables 4 and 3. We just append the following remark: the successive types of two consecutive nodes are the following ones, which is a consequence of the rules and which are used also among the induction hypothesis:

\[
\begin{align*}
\text{b0, w1} &\rightarrow \text{b1, w2} \\
\text{w0} &\rightarrow \text{b0, w1, w0} \\
\text{w1} &\rightarrow \text{b0, w1, w2} \\
\text{w2} &\rightarrow \text{b0, w1, w2}
\end{align*}
\]

Although the general splitting of the proof is the same as in the case of the Fibonacci codes, there are in this case specificities connected with the special forbidden pattern \(2^*2\) and with the fact that a node of type \(w0\) has its rightmost son as its successor. We also have to check that such a node occurs as the last node of the level only. Also, the succession of types for two nodes is ruled by (7), which is different from (4) in the codes based on Fibonacci numbers.
The situation of the white node ending with 0 appears in Table 4 as the third line of the case when $\nu_1$ is also white. We can see that a code $\beta 11$ is followed by the code $\alpha 00$, which means that $\beta$ ends with a 2 followed by 1's only and so that $\alpha$ is the successor of $\beta$. This explains why the code of $\lambda +1$, $\beta 111$ is followed by the code $\alpha 000$ which is that of $\lambda +2$ as expected from the statement of Theorem 7 as in this case, the type of $\nu$ is $w_0$. From this and from the table, we can see that a rightmost son of type $w_0$ occurs at this line only, so that the assumption that the last node on a level is of that type is checked.

Table 3 Table of the computation of the golden codes of the following nodes when $\nu$ is a black node. We mark the black nodes in blue, except when $\lambda +2$ is the successor of $\nu$.

| $\mu_1$ | $\mu$ | $\nu_1$ | $\nu$ | $\lambda_1$ | $\lambda$ | $\lambda +1$ | $\lambda +2$ |
|---------|-------|---------|-------|-------------|----------|-------------|-------------|
| $\alpha 0$ | $\alpha 1$ | $\beta 12$ | $\alpha 00$ | $\beta 112$ | $\beta 120$ | $\beta 121$ | $\alpha 000$ | 0, 1 |
| $\alpha 0$ | $\alpha 1$ | $\beta 11$ | $\alpha 00$ | $\beta 102$ | $\beta 110$ | $\beta 111$ | $\alpha 000$ | 0, 1 |
| $\alpha 1$ | $\alpha 2$ | $\alpha 02$ | $\alpha 10$ | $\alpha 012$ | $\alpha 020$ | $\alpha 021$ | $\alpha 100$ | 0, 1 |

$\mu_1$ is a black node, so that $\mu$ is white:

| $\beta 1$ | $\alpha 0$ | $\beta 02$ | $\beta 10$ | $\beta 012$ | $\beta 020$ | $\beta 021$ | $\beta 100$ | 0, 1 |
| $\beta 2$ | $\alpha 0$ | $\beta 12$ | $\beta 20$ | $\beta 112$ | $\beta 120$ | $\beta 121$ | $\beta 200$ | 0, 1 |

$\mu_1$ is a white node and $\mu$ is black:

| $\beta 1$ | $\alpha 0$ | $\beta 02$ | $\beta 10$ | $\beta 012$ | $\beta 020$ | $\beta 021$ | $\beta 100$ | 0, 1 |
| $\alpha 1$ | $\alpha 2$ | $\alpha 02$ | $\alpha 10$ | $\alpha 012$ | $\alpha 020$ | $\alpha 021$ | $\alpha 100$ | 0, 1 |

$\mu_1$ and $\mu$ are white:

| $\beta 1$ | $\alpha 0$ | $\beta 02$ | $\beta 10$ | $\beta 012$ | $\beta 020$ | $\beta 021$ | $\beta 100$ | 0, 1 |

Figure 9 The golden codes in the black Fibonacci tree: we can check the properties stated by Theorem 7.

Also note that in Table 3 the first two lines of the table indicates two different situations with the predecessor of a code $\alpha 0$. In the first line, we have the standard situation where the predecessor is $\beta 2$. In the second line, the predecessor is $\beta 1$. This means that in that case, $\beta$ ends in a pattern $21^*$, so
that the next node on the level has the code \( \alpha 0^* \). Also note that when the code of the node \( \kappa \) is \( \beta 1 \) and that of \( \kappa +1 \) is \( \alpha 0 \), it means that \( \beta \) ends with the pattern \( 21^* \), so that the code \( \beta 1^k \) is followed by \( \alpha 0^* \). In the same line, if the code of \( \kappa \) is \( \alpha 1 \) and that of \( \kappa +1 \) is \( \alpha 2 \), it means that \( \alpha \) does not end with \( 21^* \), so that the code \( \alpha 1^{k+1} \) is followed by \( \alpha 1^2 \).

Table 4 Table of the computation of the golden codes of the following nodes when \( \nu \) is a white node. We mark the black nodes in blue, except when \( \lambda +2 \) is the successor of \( \nu \).

\( \nu_1 \) and \( \mu \) are black nodes, so that \( \mu_1 \) is white:

\[
\begin{array}{cccccccc}
\mu_1 & \mu & \nu_1 & \nu & \lambda & \lambda+1 & \lambda+2 & \lambda+3 \\
\beta 1 & \alpha 0 & \beta 10 & \beta 11 & \beta 100 & \beta 101 & \beta 102 & \beta 110 & 0,1 \\
\beta 2 & \alpha 0 & \beta 20 & \beta 21 & \beta 200 & \beta 201 & \beta 202 & \beta 210 & 0,1 \\
\end{array}
\]

\( \nu_1 \) is a black node, and \( \mu \) is white:

\[
\begin{array}{cccccccc}
\mu_1 & \mu & \nu_1 & \nu & \lambda & \lambda+1 & \lambda+2 & \lambda+3 \\
\alpha 0 & \alpha 1 & \alpha 00 & \alpha 01 & \alpha 000 & \alpha 001 & \alpha 002 & \alpha 010 & 0,1,2 \\
\alpha 1 & \alpha 2 & \alpha 10 & \alpha 11 & \alpha 100 & \alpha 101 & \alpha 102 & \alpha 110 & 0,1,2 \\
\beta 1 & \alpha 0 & \beta 10 & \beta 11 & \beta 100 & \beta 101 & \beta 102 & \beta 110 & 0,1,2 \\
\alpha 1 & \alpha 2 & \alpha 10 & \alpha 11 & \alpha 100 & \alpha 101 & \alpha 102 & \alpha 110 & 0,1,2 \\
\end{array}
\]

\( \nu_1 \) is a white node, and so \( \mu \) is white too:

\[
\begin{array}{cccccccc}
\mu_1 & \mu & \nu_1 & \nu & \lambda & \lambda+1 & \lambda+2 & \lambda+3 \\
\alpha 0 & \alpha 1 & \alpha 00 & \alpha 01 & \alpha 000 & \alpha 001 & \alpha 012 & \alpha 020 & 0,1,2 \\
\alpha 1 & \alpha 2 & \alpha 10 & \alpha 11 & \alpha 110 & \alpha 111 & \alpha 112 & \alpha 120 & 0,1,2 \\
\beta 1 & \alpha 0 & \beta 11 & \beta 10 & \beta 110 & \beta 111 & \beta 000 & \beta 001 & 0,1,0 \\
\alpha 1 & \alpha 2 & \alpha 11 & \alpha 12 & \alpha 110 & \alpha 111 & \alpha 112 & \alpha 120 & 0,1,2 \\
\end{array}
\]

With those last remarks and Table 4, we completed the proof of Theorem 7.

5 Conclusion

Conclusion

We can conclude the paper with several remarks.

The first one is the interest of the golden representation, which is not used as intensively as it should be by the author itself, although he found out the property stated by Theorem 6 a long time ago. The property stated in Theorem 7 is rather unexpected. It is a new one and it explains the weak use of that encoding.

The second remark, connected with both the Fibonacci and the golden representations is that the white Fibonacci tree is the best tree for navigation purpose in the pentagrid and in the heptagrid, those tessellation that live in the
A third remark is that it was proved in [2], that the properties found out in the pentagrid and in the heptagrid can be generalized to the tessellations \{p, 4\} and \{p+2, 3\}, still in the hyperbolic plane, where \( p \geq 5 \). As in the case of the pentagrid and of the heptagrid which corresponds to the case \( p = 5 \), for each \( p \), a specific tree generates the tiling in both \{p, 4\} and \{p+2, 3\}. Interestingly, the rules associated with such a tree extend in some sense the rules used to construct the Fibonacci trees. And so, in that generalized context, the white and the black trees also exist. Now, as the preferred son property is also true in the white tree, this time for an extension of the golden encoding, we can wonder whether an extension of Theorem 7 is true for the black tree.

References

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