CP violation of quarks in $A_4$ modular invariance

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Abstract

We discuss the quark mass matrices in the $A_4$ modular symmetry, where the $A_4$ triplet of Higgs is introduced for each up-quark and down-quark sectors, respectively. The model has six real parameters and two complex parameters in addition to the modulus $\tau$. By inputting six quark masses and three CKM mixing angles, we can predict the CP violation phase $\delta$ and the Jarlskog invariant $J_{CP}$. The predicted ranges of $\delta$ and $J_{CP}$ are consistent with the observed values. The absolute value of $V_{ub}$ is smaller than 0.0043, while $V_{cb}$ is larger than 0.0436. In conclusion, our quark mass matrices with the $A_4$ modular symmetry can reproduce the CKM mixing matrix completely with observed quark masses.
1 Introduction

The origin of three families of quarks and leptons remains most important problems of Standard model (SM). In order to understand the flavor structure of quarks and leptons, considerable interests in the discrete flavor symmetry \[1\]–\[9\] have been developed by the early models of quark masses and mixing angles \[10, 11\], more recently, the large flavor mixing angles of the leptons.

Many models have been proposed by using \(S_3\), \(A_4\), \(S_4\), \(A_5\) and other groups with larger orders to explain the large neutrino mixing angles. Among them, the \(A_4\) flavor model is attractive one because the \(A_4\) group is the minimal one including a triplet irreducible representation, which allows for a natural explanation of the existence of three families of leptons \[12\]–\[17\]. However, variety of models is so wide that it is difficult to obtain clear clues of the \(A_4\) flavor symmetry. Indeed, symmetry breakings are required to reproduce realistic mixing angles \[18\]. The effective Lagrangian of a typical flavor model is given by introducing the gauge singlet scalars which are so-called flavons. Their vacuum expectation values (VEVs) determine the flavor structure of quarks and leptons. As a consequence, the breaking sector of flavor symmetry typically produces many unknown parameters.

Recently, new approach to the lepton flavor problem based on the invariance under the modular group \[19\], where the model of the finite modular group \(\Gamma_3 \simeq A_4\) has been presented. This work inspired further studies of the modular invariance approach to the lepton flavor problem. It should be emphasized that there is a significant difference between the models based on the \(A_4\) modular symmetry and those based on the usual non-Abelian discrete \(A_4\) flavor symmetry. Yukawa couplings transform non-trivially under the modular symmetry and are written in terms of modular forms which are holomorphic functions of a complex parameter, the modulus \(\tau\).

It is interesting that the modular group includes \(S_3\), \(A_4\), \(S_4\), and \(A_5\) as its finite subgroups \[20\]. Along the work of the \(A_4\) modular group \[19\], models of \(\Gamma_2 \simeq S_3\) \[21\], \(\Gamma_4 \simeq S_4\) \[22\] and \(\Gamma_5 \simeq A_5\) \[23\] have been proposed. Also numerical discussions of the neutrino flavor mixing have been done based on \(A_4\) \[24, 25\] and \(S_4\) \[26\] modular groups respectively. In particular, the comprehensive analysis of the \(A_4\) modular group has provided a clear prediction of the neutrino mixing angles and the CP violating phase \[25\]. On the other hand, the \(A_4\) modular symmetry has been applied to the \(SU(5)\) grand unified theory of quarks and leptons \[27\], and also the residual symmetry of the \(A_4\) modular symmetry has been investigated \[28\]. Furthermore, modular forms for \(\Delta(96)\) and \(\Delta(384)\) were constructed \[29\], and the extension of the traditional flavor group is discussed with modular symmetries \[30\].

In this work, we discuss the quark mixing angles and the CP violating phase, which were a main target of the early challenge for flavors \[10, 11\]. Since the quark masses and mixing angles are remarkably distinguished from the leptonic ones, that is the hierarchical structure of masses and mixing angles, it is challenging to reproduce observed hierarchical three CKM mixing angles and the CP violating phase in the \(A_4\) modular symmetry \[1\].

We can easily construct quark mass matrices by using the \(A_4\) modular symmetry. The up-quark and down-quark mass matrices have the same structure as the charged lepton mass matrix in Ref. \[25\]. Then, parameters, apart from the modulus \(\tau\), are determined by the observed quark masses. The remained parameter is only the modulus \(\tau\). However, it is very difficult to reproduce observed three CKM mixing angles by fixing \(\tau\) since the observed mixing angles are considerably hierarchical angles, and moreover, precisely measured.

\(^1\)Recently, the \(S_3\) modular symmetry is also applied to the quark sector \[31\].
Therefore, we extend the Higgs sector in the $A_4$ modular symmetry by introducing the $A_4$ triplet for Higgs doublets in up-quark and down-quark sectors, respectively. Then, one complex parameter related with the $A_4$ tensor product appears in each quark mass matrix of the up- and down-quarks. The model has six real parameters and two complex parameters in addition to the modulus $\tau$. It is remarked that those quark mass matrices can predict the magnitude of the CP violation of the CKM mixing by inputting quark masses and three mixing angles.

The paper is organized as follows. In section 2, we give a brief review on the modular symmetry. In section 3, we present the model for quark mass matrices. In section 4, we present numerical results. Section 5 is devoted to a summary. In Appendix A, the relevant multiplication rules of the $A_4$ group is presented. In Appendix B, we show how to determine the coupling coefficients of quarks. In Appendix C, we discuss the Higgs potential in our model.

## 2 Modular group and modular forms

The modular group $\Gamma$ is the group of linear fractional transformation $\gamma$ acting on the complex variable $\tau$, so called modulus, belonging to the upper-half complex plane as:

$$\tau \rightarrow \gamma \tau = \frac{a\tau + b}{c\tau + d}, \quad \text{where } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1, \quad \text{Im}[\tau] > 0,$$

which is isomorphic to $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{I, -I\}$ transformation. This modular transformation is generated by $S$ and $T$,

$$S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau + 1,$$

which satisfy the following algebraic relations,

$$S^2 = I, \quad (ST)^3 = I.$$

We introduce the series of groups $\Gamma(N) \ (N = 1, 2, 3, \ldots)$ defined by

$$\Gamma(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2,\mathbb{Z}), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{N} \right\}.$$  

For $N = 2$, we define $\Gamma(2) \equiv \Gamma(2)/\{I, -I\}$, while, since the element $-I$ does not belong to $\Gamma(N)$, for $N > 2$, we have $\bar{\Gamma}(N) = \Gamma(N)$, which are infinite normal subgroup of $\bar{\Gamma}$, called principal congruence subgroups. The quotient groups defined as $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$ are finite modular groups. In this finite groups $\Gamma_N$, $T^N = I$ is imposed. The groups $\Gamma_N$ with $N = 2, 3, 4, 5$ are isomorphic to $S_3, A_4, S_4$ and $A_5$, respectively [20].

Modular forms of level $N$ are holomorphic functions $f(\tau)$ transforming under the action of $\Gamma(N)$ as:

$$f(\gamma \tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma(N):$$

where $k$ is the so-called as the modular weight.

Superstring theory on the torus $T^2$ or orbifold $T^2/Z_N$ has the modular symmetry [32, 37]. Its low-energy effective field theory is described in terms of supergravity theory, and string-derived
supergravity theory has also the modular symmetry. Under the modular transformation of Eq. (1),
chiral superfields \( \phi^{(I)} \) transform as \([38]\),
\[
\phi^{(I)} \rightarrow (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi^{(I)},
\]
where \(-k_I\) is the modular weight and \(\rho^{(I)}(\gamma)\) denotes an unitary representation matrix of \(\gamma \in \Gamma(N)\).

The kinetic terms of their scalar components are written by
\[
\sum_I \frac{|\partial_{\dot{\mu}} \phi^{(I)}|^2}{(-i\tau + i\bar{\tau})^{k_I}},
\]
which is invariant under the modular transformation. Here, we use the convention that the superfield and its scalar component are denoted by the same letter. Also, the superpotential should be invariant under the modular symmetry. That is, the superpotential should have vanishing modular weight in global supersymmetric models, while the superpotential in supergravity should be invariant up to the Kähler transformation. In the following sections, we study global supersymmetric models, e.g. minimal supersymmetric standard model (MSSM) and its extension with Higgs \(A_4\) triplet. Thus, the superpotential has vanishing modular weight. The modular symmetry is broken by the vacuum expectation value of \(\tau\), i.e. at the compactification scale, which is of order of the planck scale or slightly lower scale.

For \(\Gamma_3 \simeq A_4\), the dimension of the linear space \(\mathcal{M}_k(\Gamma_3)\) of modular forms of weight \(k\) is \(k + 1\) \([39, 41]\), i.e., there are three linearly independent modular forms of the lowest non-trivial weight 2. These forms have been explicitly obtained \([19]\) in terms of the Dedekind eta-function \(\eta(\tau)\):
\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (8)
\]
where \(q = e^{2\pi i\tau}\) and \(\eta(\tau)\) is a modular form of weight 1/2. In what follows we will use the following basis of the \(A_4\) generators \(S\) and \(T\) in the triplet representation:
\[
S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (9)
\]
where \(\omega = e^{\frac{2\pi}{3} i}\). The modular forms of weight 2 \((Y_1(\tau), Y_2(\tau), Y_3(\tau))\) transforming as a triplet of \(A_4\) can be written in terms of \(\eta(\tau)\) and its derivative \([19]\):
\[
Y_1(\tau) = \frac{i}{2\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right),
\]
\[
Y_2(\tau) = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right),
\]
\[
Y_3(\tau) = -\frac{i}{\pi} \left( \frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} \right). \quad (10)
\]
The overall coefficient in Eq. (10) is one possible choice; it cannot be uniquely determined. The triplet modular forms of weight 2 have the following \(q\)-expansions:
\[
Y = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \end{pmatrix}. \quad (11)
\]
They satisfy also the constraint [19]:

\[(Y_2(\tau))^2 + 2Y_1(\tau)Y_3(\tau) = 0\].

\[12\]

3 Quark mass matrices in the $A_4$ triplet Higgs model

Let us consider a $A_4$ modular invariant flavor model for quarks. In order to construct models with minimal number of parameters, we introduce no flavons. There are freedoms for the assignments of irreducible representations and modular weights to quarks and Higgs doublets. We take similar assignments of the left-handed quarks and right-handed one as seen in the charged lepton sector [25]: that is, three left-handed quark doublets are of a triplet of $A_4$, and $(u^c, c^c, t^c)$ and $(d^c, s^c, b^c)$ are of three different singlets ($1, 1'', 1'$) of $A_4$, respectively. For both left-handed quarks and right-handed quarks, the modular weights are assigned to be $-1$, while the modular weight is 0 for Higgs doublets. Then, there appear three independent couplings in the superpotential of the up-quark sector and down-quark sector, respectively:

\[w_u = \alpha_u u^c H_u Y Q + \beta_u c^c H_u Y Q + \gamma_u t^c H_u Y Q\],

\[w_d = \alpha_d d^c H_d Y Q + \beta_d s^c H_d Y Q + \gamma_d b^c H_d Y Q\],

\[13\]

\[14\]

where $Q$ is the left-handed $A_4$ triplet quarks, and $H_q$ is the Higgs doublets. The parameters $\alpha_q, \beta_q, \gamma_q (q = u, d)$ are constant coefficients. If the Higgs doublets $H_q$ are singlet of $A_4$, the quark mass matrices are simple form. By using the decomposition of the $A_4$ tensor product in Appendix A, the supertoptential in Eqs.\[13\] and \[14\] gives the mass matrix of quarks, which is written in terms of modular forms of weight 2:

\[M_q = \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}\],

\[15\]

where $\tau$ in the modular forms $Y_i(\tau)$ is omitted. Unknown couplings $\alpha_q, \beta_q, \gamma_q$ can be adjusted to the observed quark masses. The remained parameter is only the modulus, $\tau$. The numerical study of the quark mass matrix in Eq.\[15\] is rather easy. However, it is very difficult to reproduce observed three CKM mixing angles by fixing one complex parameter $\tau$ because the CKM mixing angles are hierarchical ones and they have been precisely measured.

Therefore, we enlarge the Higgs sector. Let us consider the Higgs doublets to be one component of a $A_4$ triplet [42,46] for each up-quark and down-quark, respectively as follows: We introduce $A_4$ triplets Higgs $H_u$ and $H_d$, which are gauge doublets, as follows:

\[H_u = \begin{pmatrix} H_{u1} \\ H_{u2} \\ H_{u3} \end{pmatrix}, \quad H_d = \begin{pmatrix} H_{d1} \\ H_{d2} \\ H_{d3} \end{pmatrix}\].

\[16\]

Including these $A_4$ triplet Higgs, we summarize the assignments of representations and modular weights $-k_I$ to the relevant fields in Table
Now, the quark mass matrices are obtained by the tensor products among the $A_4$ singlet right-handed quarks, the $A_4$ triplet modular forms $Y(\tau)$, the $A_4$ triplet Higgs $H_\theta$ and the $A_4$ triplet left-handed quarks $Q$. Since the tensor product of $3 \otimes 3$ is decomposed into the symmetric triplet and the antisymmetric triplet as seen in Appendix A, the $A_4$ invariant superpotential in Eq.(13) is expressed by introducing additional two parameters $g_{u1}$ and $g_{u2}$ as:

$$w_u = (\alpha_u u^c(1) + \beta_u u^c(1') + \gamma_u e^c(1')) \otimes \left[ g_{u1} \begin{pmatrix} 2H_{u1}Y_1 - H_{u2}Y_3 - H_{u3}Y_2 \\ 2H_{u3}Y_3 - H_{u1}Y_2 - H_{u2}Y_1 \\ 2H_{u2}Y_2 - H_{u3}Y_1 - H_{u1}Y_3 \end{pmatrix} + g_{u2} \begin{pmatrix} H_{u2}Y_3 - H_{u3}Y_2 \\ H_{u1}Y_2 - H_{u2}Y_1 \\ H_{u3}Y_1 - H_{u1}Y_3 \end{pmatrix} \right] \otimes \begin{pmatrix} u \\ c \\ t \end{pmatrix},$$

(17)

where the neutral component of $H_{q_i}$ is taken, and the $A_4$ singlet component should be extracted in the tensor product. The up-quark mass matrix is given in terms of VEV’s of $H_{ui}, v_{ui}$ in Appendix C and modular forms $Y_i(i = 1, 2, 3)$ as follows:

$$M_u = \begin{pmatrix} \alpha_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \times \left[ \frac{g_{u1}}{\sqrt{2}} \begin{pmatrix} 2v_{u1}Y_1 - v_{u2}Y_3 - v_{u3}Y_2 \\ 2v_{u3}Y_3 - v_{u1}Y_2 - v_{u2}Y_1 \\ 2v_{u2}Y_2 - v_{u3}Y_1 - v_{u1}Y_3 \end{pmatrix} + \frac{g_{u2}}{\sqrt{2}} \begin{pmatrix} v_{u2}Y_3 - v_{u3}Y_1 - v_{u1}Y_2 - v_{u2}Y_1 \\ v_{u1}Y_2 - v_{u2}Y_1 \\ v_{u3}Y_1 - v_{u1}Y_2 - v_{u2}Y_1 \end{pmatrix} \right],$$

(18)

where $\alpha_u, \beta_u, \gamma_u$ are taken to be real positive by rephasing right-handed quark fields without loss of generality. The down-quark mass matrix is also given by replacing $u$ with $d$ in Eq.18.

The vacuum structure of our model is determined by the scalar potential $V(H_u, H_d)$, which is presented in Appendix C. Since the modular forms $Y_i$’s do not couple to the scalar potential due to the modular weight of 0 for the Higgs doublets, the vacuum structure of the scalar potential is independent of VEV of $\tau$. Therefore, the scalar potential is similar to the one in MSSM. As discussed in the non-SUSY model with the $A_4$ triplet Higgs, there are some choices of $v_{q_i}$’s to realize the vacuum, which is the global minimum. In our work, we take the simplest one of $\langle H_q \rangle$ of the global minima coexist and are degenerate. For example, $\langle H_d \rangle = \frac{1}{\sqrt{2}} (v_d, v_d, v_d)$ and $\langle H_u \rangle = \frac{1}{\sqrt{2}} (v_u, v_u, v_u)$ lead to the global minimum. Upon small variation of the parameters around this special point, one minimum point becomes the global minimum while the other turns into a local one, and it is clearly possible to make either of them the global minimum.

|       | $Q$ | $(u^c(d^c), c^c(s^c), t^c(b^c))$ | $H_u$ | $H_d$ | $Y$ |
|-------|-----|---------------------------------|------|------|-----|
| $SU(2)$ | 2   | 1                              | 2    | 2    | 1   |
| $A_4$  | 3   | $(1, 1', 1')$                  | 3    | 3    | 3   |
| $-k_I$ | -1  | $(-1, -1, -1)$                 | 0    | 0    | $k = 2$ |

Table 1: The assignments of representations and modular weights $-k_I$ to the MSSM fields, where Higgs sector is extended to the non-trivial representation of $A_4$, 3.
in our SUSY framework as follows:

\[
\langle H_u \rangle = \frac{1}{\sqrt{2}} (v_{u1}, 0, 0), \quad \langle H_d \rangle = \frac{1}{\sqrt{2}} (v_{d1}, 0, 0) ,
\]

in the basis of \( S \) and \( T \) in Eq. (19). Here \( v_{u1} \) and \( v_{d1} \) are taken to be real and \( v_{u1}^2 + v_{d1}^2 = 2v_H^2 \) where \( v_H = 174.1 \text{GeV} \). The vacuum alignment in Eq. (19) easily realizes the minimum of the scalar potential by taking the condition

\[
\frac{\partial V(H_u, H_d)}{\partial H_{qk}} = 0 , \quad (q = u, d ; \; k = 1, 2, 3) ,
\]

while the Hessian

\[
\frac{\partial^2 V(H_u, H_d)}{\partial H_{qk} \partial H_{qj}} , \quad (q = u, d ; \; k, j = 1, 2, 3) ,
\]

is required to have non-negative eigenvalues, which correspond to that all physical masses being positive except for vanishing masses of the Goldstone bosons as seen in Appendix C.

Indeed, we have checked numerically for \( \tan \beta = v_{u1}/v_{d1} \) = 10 that the extra scalars and pseudo-scalars could be \( \mathcal{O}(10)\text{TeV} \) keeping the light SM Higgs mass. This situation is achieved due to some fine-tuning and rather large scalar self-couplings by taking account of the radiative corrections of SUSY and \( \tilde{m}_{H_u} = \mathcal{O}(10)\text{TeV}, \; B = \mathcal{O}(10)\text{TeV} \) and \( \mu = \mathcal{O}(10)\text{TeV} \). However, loop corrections to the scalar masses become important as shown in two Higgs doublet model [47,48]. Therefore, such high splittings of scalar masses should be carefully examined in the context of the phenomenology. Moreover, there could be unsuppressed flavor changing neutral current (FCNC) of quarks, which was discussed in the \( A_4 \) triplet Higgs model [42]. Indeed, the study of FCNC in Kaon and B meson systems is important. However, we do not discussed the phenomenology, which is out of scope in the present work.

In our model, only \( H_{u1} \) and \( H_{d1} \) have VEVs, therefore, it is easy to find that the couplings to the observed 125GeV Higgs boson are expected to be proportional to quark masses. This situation is understandable since \( H_{q1} \) do not mix with \( H_{q2} \) and \( H_{q3} \) in the Higgs potential as seen in Appendix C. The electromagnetism is not broken: a minimum of the potential satisfying \( \partial V/\partial H_{qk}^\pm = 0 \) gives \( \langle H_{qk}^\pm \rangle = 0 \).

It is also noticed that the VEV in Eq. (19) has a residual \( Z_2 \) symmetry of \( A_4 \). However, this \( Z_2 \) symmetry of the Higgs sector is accidental since an obtained \( \tau \) of our result breaks completely \( A_4 \) symmetry. The choice of \( (v_q, 0, 0) \) should be considered to reduce the number of free parameters. Indeed, the numerical fit of experimental data of the CKM matrix is improved by using another alignment of \( (v_q, v'_q, 0) \), which has no the \( Z_2 \) symmetry.

Finally, we obtain the up-quark and down-quark mass matrices:

\[
M_q = \frac{1}{\sqrt{2}} v_{q1} \; g_q \begin{pmatrix}
\alpha_q & 0 & 0 \\
0 & \beta_q & 0 \\
0 & 0 & \gamma_q
\end{pmatrix}
\begin{pmatrix}
2Y_1 & -(1 + g_q)Y_3 & -(1 - g_q)Y_2 \\
-(1 - g_q)Y_2 & 2Y_1 & -(1 + g_q)Y_3 \\
-(1 + g_q)Y_3 & -(1 - g_q)Y_2 & 2Y_1
\end{pmatrix}_{RL} , \quad (q = u, d) ,
\]

where \( g_q \equiv g_{q2}/g_{q1} \) (\( q = u, d \)). There are six real parameters \( \alpha_q, \beta_q, \gamma_q \) (\( q = u, d \)), and the VEV of the modulus, \( \tau \). In addition, we have two complex parameters \( g_u \) and \( g_d \). It is noted that the factor \( v_{q1}g_{q1} \) in front of the right hand side of Eq. (22) is absorbed into \( \alpha_q, \beta_q \) and \( \gamma_q \). Thus, we have six
real parameters and three complex ones. That is to say, there are twelve free real parameters in our mass matrices. It is also noticed that \( v_u \) does not appear explicitly in our calculations because it is absorbed in \( \alpha_q, \beta_q \) and \( \gamma_q \). Therefore, our numerical result is independent of \( \tan \beta = v_u/v_d \).

The quark mass matrix in Eq.\((22)\) has a specific flavor structure due to the \( A_4 \) symmetry. It is easily found relations among matrix elements as follows:

\[
\frac{M_q(1,1)}{M_q(2,2)} = \frac{M_q(1,2)}{M_q(2,3)} = \frac{M_q(1,3)}{M_q(2,1)}, \quad \frac{M_q(2,2)}{M_q(3,3)} = \frac{M_q(2,1)}{M_q(3,2)} = \frac{M_q(2,3)}{M_q(3,1)}.
\]

Moreover, a constraint among \( Y_1, Y_2 \) and \( Y_3 \) in Eq.\((12)\) provide a relation

\[
\frac{M_q(2,1)}{M_q(2,2)} = \frac{(g_q - 1)^2 M_q(3,1)}{g_q + 1 M_q(3,2)}.
\]

These relations correlate CKM mixing angles each other. Thus, the three CKM mixing angles are not independent in our quark mass matrix. Indeed, parameter region of \( \tau \), \( g_u \) and \( g_d \) are restricted to be in rather narrow regions in order to reproduce the three CKM mixing angles, as seen in numerical result. Then, the CP violating phase is predicted in the restricted region in spite of the excess of parameters compared with observed ones.

## 4 Numerical results

Let us begin with explaining how to get our prediction of the CP violation in terms of twelve real parameters. At first, we take a random point of \( \tau \) and \( g_u, g_d \), which are scanned in the complex plane by generating random numbers. The scanned ranges of \( \text{Im}[\tau] \) is \([0.5, 10]\), in which the lower-cut 0.5 comes from the accuracy of calculating modular functions, and the upper-cut 10 is enough large for estimating \( Y_i \) in practice. On the other hand, \( \text{Re}[\tau] \) is scanned in the fundamental region of \([-3/2, 3/2]\) in Eq.\((10)\) because the modular function \( Y_i \) is given in terms of \( \eta(\tau/3) \). We also scan in \( |g_u| \in [0, 1000] \) and \( |g_d| \in [0, 1000] \) while these phases are scanned in \([-\pi, \pi]\).

Then, parameters \( \alpha_q, \beta_q, \gamma_q \) \( (q = u, d) \) are determined by computing functions \( C^q_i (i = 1 - 3) \) in Appendix B after inputting six quark masses (see Appendix B). We use the six quark masses at the \( M_Z \) scale \([19]\).

Finally, we can calculate three CKM mixing angles in terms of the model parameters \( \tau \), \( g_u \) and \( g_d \), while keeping the parameter sets leading to values allowed by the experimental data of the CKM mixing angles. We continue this procedure to obtain enough points for plotting allowed region.

We adopt the data of quark Yukawa couplings at the \( M_Z \) scale as input in order to constraint the model parameters \([19]\):

\[
y_d = (1.58^{+0.23}_{-0.10}) \times 10^{-5}, \quad y_s = (3.12^{+0.17}_{-0.16}) \times 10^{-4}, \quad y_b = (1.639 \pm 0.015) \times 10^{-2}, \\
y_u = (7.4^{+1.5}_{-3.6}) \times 10^{-6}, \quad y_c = (3.60 \pm 0.11) \times 10^{-3}, \quad y_t = 0.9861^{+0.0086}_{-0.0087},
\]

which give quark masses as \( m_q = y_q v_H \) with \( v_H = 174.1 \) GeV. We also take the absolute values of CKM elements \( V_{us}, V_{cb} \) and \( V_{ub} \) for input as follows \([50]\):

\[
|V_{us}| = 0.2243 \pm 0.0005, \quad |V_{cb}| = 0.0422 \pm 0.0008, \quad |V_{ub}| = (3.94 \pm 0.36) \times 10^{-3}.
\]

In Eqs.\((25)\) and \((26)\), the error-bars denote interval of 1\(\sigma\), and 3\(\sigma\) error-bars are used as input.
The obtained parameter region of $\tau$, $g_u$ and $g_d$ are as follows:

$$\text{Re}[\tau] = -(1.49 - 1.50) \ , \quad \text{Im}[\tau] = 2.01 - 2.02 \ ,$$
$$\text{Re}[g_u] = 0.70 - 0.93 \ , \quad \text{Im}[g_u] = \pm (0.002 - 0.022) \ ,$$
$$\text{Re}[\frac{1}{g_d}] \simeq -(0.99 - 1.03) \times 10^{-3} \ , \quad \text{Im}[\frac{1}{g_d}] = -(0.052 - 0.108) \ ,$$

(27)

where the modulus $\tau$ is almost fixed. By using these values, we can predict the CP violation phase $\delta$ and the Jarlskog invariant $J_{CP}$ \[51\]. Those are compared with the observed values at the electroweak scale \[50\]:

$$\delta = (73.5^{+1.2}_{-1.1})^\circ \ , \quad J_{CP} = (3.18 \pm 0.15) \times 10^{-5} .$$

(28)

Our predictions are presented in Figs.1–3. We show the predicted CP violating phase $\delta$ versus $J_{CP}$ in Fig.1. Here, the observed CKM mixing elements $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$ are input with $3\sigma$ error interval. The predicted ranges of $\delta$ and $J_{CP}$ is $(65^\circ - 140^\circ)$ and $(2 - 4) \times 10^{-5}$, respectively. Those include the allowed regions of the experimental data in Eq.(28), which are denoted by red dashed-lines with $3\sigma$ error interval. The predicted region of $\delta$ is still broad. It is remarked that $\delta$ is more restricted if error-bars of inputting quark masses are reduced, especially, the s-quark mass and the c-quark mass are important to predict $\delta$.

We show the $|V_{ub}|$ dependence of predicted $J_{CP}$ in Fig.2. Although observed $|V_{ub}| [0.0028, 0052]$ is
input, our model does not allow the region larger than 0.0043. The $|V_{ub}|$ is cut below the lower-bound of experimental data. The predicted $J_{CP}$ is approximately proportional to $|V_{ub}|$. The upper hard cut of $J_{CP}$ is due to the maximal value of $\sin \delta = 1$.

In Fig 3, we show the allowed region on $|V_{cb}|$–$|V_{ub}|$ plane. The $|V_{cb}|$ is restricted in the very narrow range, which is larger than 0.0436, close to the 3σ upper-bound of the observed one 0.0446. This prediction provides us a crucial test of our model.

We can also discuss the ratio of CKM matrix elements of $V_{ub}$ and $V_{cb}$, which is in the range of $[0.065, 0.098]$ from Fig. 3. It should be compared with the observed values [22]:

$$\frac{|V_{ub}|}{|V_{cb}|} = 0.083 \pm 0.006 . \quad (29)$$

Our prediction is inside of the observed 3σ interval in Eq. (29). This measurement was given in the semileptonic decays of $\Lambda_b$ at LHCb. This prediction provides another complementary test of our model.

Finally, we show a typical set with twelve parameters as one sample, which gives us successful CKM parameters as well as $J_{CP}$:

$$\begin{align*}
\tau &= -1.495 + i \ 2.011 , \\
g_u &= 0.918 + i \ 0.0116 , \\
g_d &= -980 - i \ 18.9 , \\
\alpha_u/\gamma_u &= 2.496 \times 10^{-5} , \\
\beta_u/\gamma_u &= 5.995 \times 10^{-3} , \\
\alpha_d/\gamma_d &= 2.855 \times 10^{-3} , \\
\beta_d/\gamma_d &= 3.812 \times 10^{-2}, \\
\tilde{\gamma}_u &\equiv \frac{1}{\sqrt{2}} v_u g_u \alpha_u = 85.85\text{GeV} , \\
\tilde{\gamma}_d &\equiv \frac{1}{\sqrt{2}} v_d g_d \alpha_d = 1.427\text{GeV}.
\end{align*} \quad (30)$$

This set gives

$$\begin{align*}
|V_{us}| &= 0.224 , \\
|V_{cb}| &= 0.0443 , \\
|V_{ub}| &= 3.20 \times 10^{-3} , \\
J_{CP} &= 2.98 \times 10^{-5} , \\
\delta &= 74.9^\circ ,
\end{align*} \quad (31)$$

which are remarkably consistent with the observed values. It is noticed that ratios of $\alpha_q/\gamma_q$ and $\beta_q/\gamma_q$ ($q = u, d$) in Eq. (30) correspond to the observed quark mass hierarchy.

In conclusion, our quark mass matrix with the $A_4$ modular symmetry can reproduce the CKM mixing matrix completely with observed quark masses.

## 5 Summary

We have discussed the quark mass matrices in the $A_4$ modular symmetry, where the $A_4$ triplet of Higgs doublets is introduced for each up-quark and down-quark sectors, respectively. The model has six real parameters and two complex parameters in addition to the modulus $\tau$. Then, we have constrained the model parameters by inputting six quark masses and three CKM mixing angles at the electroweak scale. We have predicted the CP violation phase $\delta$ and the Jarlskog invariant $J_{CP}$.

The predicted ranges of $\delta$ and $J_{CP}$ is $(65^\circ - 140^\circ)$ and $(2-4) \times 10^{-5}$, respectively. Those include the allowed regions of the experimental data. The absolute value of $V_{ub}$ is smaller than 0.0043. The magnitude of $V_{cb}$ is larger than 0.0436, which is close to the 3σ upper-bound of the observed one. Thus, our quark mass matrices with the $A_4$ modular symmetry can reproduce the CKM mixing matrix completely with observed quark masses.
Our mass matrices have been analyzed at the electroweak scale in this work. The renormalization-group evolution from the GUT scale to the electroweak scale have been examined in some textures of the quark mass matrix [53]. The textures of the quark mass matrix are essentially stable against the evolution. We expect that the conclusions derived in this paper do not change much even if we consider the mass matrix at the GUT scale.

We will also discuss the lepton mass matrices in the modular $A_4$ symmetry by introducing the $A_4$ triplet of Higgs doublets elsewhere.

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**Appendix**

### A Multiplication rule of $A_4$ group

We take

$$S = \frac{1}{3} \begin{pmatrix}
-1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1
\end{pmatrix}, \quad T = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}, \quad (32)$$

where $\omega = e^{i\frac{2}{3}\pi}$ for a triplet. In this base, the multiplication rule of the $A_4$ triplet is

$$\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}_3 \otimes \begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}_3 = (a_1b_1 + a_2b_3 + a_3b_2)_1 + (a_3b_3 + a_1b_2 + a_2b_1)_1'$$

$$\oplus (a_2b_2 + a_1b_3 + a_3b_1)_1''$$

$$\oplus \frac{1}{3} \left( \begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix} \right)_3 \oplus \frac{1}{2} \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_3$$

$$1 \otimes 1 = 1 , \quad 1' \otimes 1' = 1'' , \quad 1'' \otimes 1'' = 1' , \quad 1' \otimes 1'' = 1 . \quad (33)$$

More details are shown in the review [2][3].

### B $\alpha_q/\gamma_q$ and $\beta_q/\gamma_q$ in terms of quark masses

The coefficients $\alpha_q$, $\beta_q$, and $\gamma_q$ in Eq. (22) are taken to be real positive without loss of generality. These parameters are described in terms of the modulus $\tau$ and quark masses. The mass matrix is...
written as

\[ M_q = \frac{1}{\sqrt{2}} v_q g_q \gamma_q \begin{pmatrix} \hat{\alpha}_q & 0 & 0 \\ 0 & \hat{\beta}_q & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2Y_1 & -(1 + g_q)Y_3 & -(1 - g_q)Y_2 \\ -(1 - g_q)Y_2 & 2Y_1 & -(1 + g_q)Y_3 \\ -(1 + g_q)Y_3 & -(1 - g_q)Y_2 & 2Y_1 \end{pmatrix}_{RL}, \tag{34} \]

where \( \hat{\alpha}_q \equiv \alpha_q / \gamma_q \) and \( \hat{\beta}_q \equiv \beta_q / \gamma_q \). Then, we have three equations as:

\[ \sum_{i=1}^{3} m_{q_i}^2 = \text{Tr}[M_q^\dagger M_q] = \tilde{\gamma}_q^2 (1 + \hat{\alpha}_q^2 + \hat{\beta}_q^2) C_1^q, \tag{35} \]

\[ \prod_{i=1}^{3} m_{q_i}^2 = \det[M_q^\dagger M_q] = \tilde{\gamma}_q^6 \hat{\alpha}_q \hat{\beta}_q C_2^q, \tag{36} \]

\[ \chi = \frac{\text{Tr}[M_q^\dagger M_q]^2 - \text{Tr}[(M_q^\dagger M_q)^2]}{2} = \tilde{\gamma}_q^4(\hat{\alpha}_q + \hat{\beta}_q^2 + \hat{\beta}_q) C_3^q, \tag{37} \]

where \( \chi \equiv m_{q_1}^2 m_{q_2}^2 + m_{q_2}^2 m_{q_3}^2 + m_{q_3}^2 m_{q_1}^2 \) and \( \tilde{\gamma}_q = (v_q g_q \gamma_q) / \sqrt{2} \). The coefficients \( C_1^q, C_2^q \), and \( C_3^q \) depend only on \( Y_i \) and \( g_q \), where \( Y_i \)'s are determined if the value of modulus \( \tau \) is fixed, and \( g_q \) is an arbitrary complex coefficient. Those are given explicitly as follows:

\[ C_1^q = 4|Y_1|^2 + |g_q - 1|^2|Y_2|^2 + |g_q + 1|^2|Y_3|^2, \]

\[ C_2^q = 2 \text{Re} \left[ 8Y_1^2 + (g_q - 1)^2Y_2^2 + (g_q + 1)^2Y_3^2 + 6(g_q^2 - 1)Y_1Y_2Y_3 \right], \]

\[ C_3^q = 16|Y_1|^4 + |g_q - 1|^2|Y_2|^4 + |g_q + 1|^2|Y_3|^4 + 4|g_q - 1|^2|Y_1Y_2|^2 + 4|g_q + 1|^2|Y_1Y_3|^2 + |g_q^2 - 1|^2|Y_2Y_3|^2 \]

\[ + 4 \text{Re} \left[ (g_q - 1)^2(g_q^* + 1)Y_1^2Y_2^*Y_3^* + 2(g_q^2 - 1)Y_1^2Y_2^*Y_3^* - (g_q + 1)^2(g_q^* - 1)Y_1^*Y_2^*Y_3^* \right]. \]

Then, we obtain two equations which describe \( \hat{\alpha} \) and \( \hat{\beta} \) as functions of quark masses, \( \tau \) and \( g_q \):

\[ \frac{(1 + s)(s + t)}{t} = \frac{(\sum m_i^2 / C_1^q)(\chi / C_2^q)}{\prod m_i^2 / C_2^q}, \quad \frac{(1 + s)^2}{s + t} = \frac{(\sum m_i^2 / C_1^q)^2}{\chi / C_3^q}, \tag{38} \]

where we redefine the parameters \( \hat{\alpha}_q^2 + \hat{\beta}_q^2 = s \) and \( \hat{\alpha}_q \hat{\beta}_q = t \). They are related as follows,

\[ \hat{\alpha}_q^2 = \frac{s \pm \sqrt{s^2 - 4t}}{2}, \quad \hat{\beta}_q^2 = \frac{s + \sqrt{s^2 - 4t}}{2}. \tag{39} \]

## C Scalar potential of \( A_4 \) triplet Higgs

The \( A_4 \) triplets Higgs, which are SU(2) gauge doublets, \( H_u \) and \( H_d \) are expressed as:

\[ H_u = \begin{pmatrix} H_{u1} \\ H_{u2} \\ H_{u3} \end{pmatrix}, \quad H_d = \begin{pmatrix} H_{d1} \\ H_{d2} \\ H_{d3} \end{pmatrix}. \tag{40} \]

Since each component is SU(2) doublet, it is written as:

\[ H_{uk} = \left( \frac{1}{\sqrt{2}} (v_{uk} + r_{uk} + i z_{uk}) \right), \quad H_{dk} = \left( \frac{1}{\sqrt{2}} (v_{dk} + r_{dk} + i z_{dk}) \right), \tag{41} \]
where $v_{uk}$ and $v_{dk}$ are VEV's of $H_{uk}$ and $H_{dk}$, respectively.

The $A_4$ invariant superpotential of Higgs sector is written by

$$w_H = \mu (H_{u1}H_{d1} + H_{u2}H_{d2} + H_{u3}H_{d3}).$$

(42)

The scalar potential of the D-term is given as

$$V_D = \frac{g_2^2}{8} (H_{u1}^\dagger \sigma_a H_{u1} + H_{u2}^\dagger \sigma_a H_{u2} + H_{u3}^\dagger \sigma_a H_{u3} + H_{d1}^\dagger \sigma_a H_{d1} + H_{d2}^\dagger \sigma_a H_{d2} + H_{d3}^\dagger \sigma_a H_{d3})^2$$

$$+ \frac{g_Y^2}{8} (H_{u1}^\dagger H_{u1} + H_{u2}^\dagger H_{u2} + H_{u3}^\dagger H_{u3} - H_{d1}^\dagger H_{d1} - H_{d2}^\dagger H_{d2} - H_{d3}^\dagger H_{d3})^2,$$

(43)

where $g_2$ and $g_Y$ are gauge couplings of SU(2) and U(1), respectively, and $\sigma_a$ (a=1-3) denote the Pauli matrix.

On the other hand, the soft breaking term under $A_4$ invariance is also given by

$$V_{soft} = \tilde{m}_{Hu}^2 (H_{u1}^\dagger H_{u1} + H_{u2}^\dagger H_{u2} + H_{u3}^\dagger H_{u3}) + \tilde{m}_{Hd}^2 (H_{d1}^\dagger H_{d1} + H_{d2}^\dagger H_{d2} + H_{d3}^\dagger H_{d3})$$

$$+ B\mu (H_{u1}i\sigma_2 H_{d1} + H_{u2}i\sigma_2 H_{d2} + H_{u3}i\sigma_2 H_{d3} + h.c.).$$

(44)

The resulting Higgs potential is then given by:

$$V(H_u, H_d) = m_{Hu}^2 H_{u1}^\dagger H_{u1} + |\mu|^2 (|H_{u2}|^2 + |H_{u3}|^2) + \tilde{m}_{Hu}^2 (H_{u1}^\dagger H_{u1} + H_{u2}^\dagger H_{u2} + H_{u3}^\dagger H_{u3})$$

$$+ m_{Hd}^2 H_{d1}^\dagger H_{d1} + |\mu|^2 (|H_{d2}|^2 + |H_{d3}|^2) + \tilde{m}_{Hd}^2 (H_{d1}^\dagger H_{d1} + H_{d2}^\dagger H_{d2} + H_{d3}^\dagger H_{d3})$$

$$+ \frac{g_2^2}{8} (H_{u1}^\dagger \sigma_a H_{u1} + H_{u2}^\dagger \sigma_a H_{u2} + H_{u3}^\dagger \sigma_a H_{u3} + H_{d1}^\dagger \sigma_a H_{d1} + H_{d2}^\dagger \sigma_a H_{d2} + H_{d3}^\dagger \sigma_a H_{d3})^2$$

$$+ \frac{g_Y^2}{8} (H_{u1}^\dagger H_{u1} + H_{u2}^\dagger H_{u2} + H_{u3}^\dagger H_{u3} - H_{d1}^\dagger H_{d1} - H_{d2}^\dagger H_{d2} - H_{d3}^\dagger H_{d3})^2$$

$$+ B\mu (H_{u1}i\sigma_2 H_{d1} + H_{u2}i\sigma_2 H_{d2} + H_{u3}i\sigma_2 H_{d3} + h.c.),$$

(45)

where $m_{u}^2 \equiv |\mu|^2 + \tilde{m}_{Hu}^2$, $m_{d}^2 \equiv |\mu|^2 + \tilde{m}_{Hd}^2$.

We can study the minima in the potential $V(H_u, H_d)$ of Eq.(45) by taking the first derivative system

$$\frac{\partial V(H_u, H_d)}{\partial H_{qk}} = 0, \quad (q = u, d; \ k = 1, 2, 3)$$

(46)

where $H_{qk}$ is of the field $h_{uk}^+, h_{dk}^-, r_{uk}, z_{uk}, r_{dk}$ and $z_{dk}$. Here, the Hessian

$$\frac{\partial^2 V(H_u, H_d)}{\partial H_{qk} \partial H_{qj}}, \quad (q = u, d; \ k, j = 1, 2, 3)$$

(47)

is required to have non-negative eigenvalues, which correspond to that all physical masses being positive except for vanishing masses of the Goldstone bosons.

Our Higgs potential analysis is same as in MSSM. Indeed, we have checked numerically by taking $\tan \beta = v_{u1}/v_{d1} = 10$ that the extra scalar and pseudo-scalar masses are larger than in $O(1)$TeV keeping the light SM Higgs mass.

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