Oversampling and undersampling in de Branges spaces arising from regular Schrödinger operators

Luis O. Silva*
Departamento de Física Matemática
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
C.P. 04510, México D.F.
silva@iimas.unam.mx

Julio H. Toloza†
Instituto de Matemática de Bahía Blanca
Universidad Nacional del Sur
Consejo Nacional de Investigaciones Científicas y Técnicas
Departamento de Matemática
Av. Alem 1253, B8000CPB Bahía Blanca, Argentina
julio.toloza@uns.edu.ar

Alfredo Uribe
Departamento de Física Matemática
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
C.P. 04510, México D.F.
alfredo.uribe.83@ciencias.unam.mx

Abstract

The classical results on oversampling and undersampling (or aliasing), of functions in Paley-Wiener spaces, are generalized to de Branges spaces arising from regular Schrödinger operators with a wide range of potentials.

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1. Introduction

This paper deals with the subject of oversampling and undersampling—the latter also known as aliasing in the engineering and signal processing literature—in the context of de Branges Hilbert spaces of entire functions (dB spaces for short). These notions play a prominent role in the theory of Paley-Wiener spaces \[ \cite{13,21} \]. Since Paley-Wiener spaces are leading examples of dB spaces, questions related to oversampling and undersampling in dB spaces emerge naturally.

Paley-Wiener spaces stem from the Fourier transform of functions with given compact support centred at zero, \( \text{viz.} \)

\[
\mathcal{P}W_a := \left\{ f(z) = \int_{-a}^{a} e^{-ixz} \phi(x) dx : \phi \in L_2(-a,a) \right\}.
\]

By the Whittaker-Shannon-Kotel’nikov theorem, any function \( f(z) \in \mathcal{P}W_a \) is decomposed as follows.

\[
f(z) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{a}\right) \mathcal{G}_a\left(z,\frac{n\pi}{a}\right), \quad \mathcal{G}_a(z,t) := \frac{\sin[a(z-t)]}{a(z-t)},
\]

where the convergence of the series is uniform in any compact of \( \mathbb{C} \). The function \( \mathcal{G}_a(z,t) \) is referred to as the sampling kernel.

In oversampling, the starting point is a function \( f(z) \in \mathcal{P}W_a \subset \mathcal{P}W_b \). Then, in addition to (1.1), one has

\[
f(z) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{b}\right) \mathcal{G}_b\left(z,\frac{n\pi}{b}\right)
\]

Moreover, \( f(z) \) admits a different representation

\[
f(z) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{b}\right) \tilde{\mathcal{G}}_{ab}\left(z,\frac{n\pi}{b}\right),
\]

with a modified sampling kernel \( \tilde{\mathcal{G}}_{ab}(z,t) \) depending on \( a \) and \( b \) (see [13, Thm. 7.2.5]). While the convergence of the sampling formula (1.1) is unaffected by \( l_2 \) perturbations of the samples \( f\left(\frac{n\pi}{a}\right) \), formula (1.2) is more robust because it is convergent even under \( l_\infty \) perturbations of the samples. That is, if the sequence \( \{\epsilon_n\}_{n \in \mathbb{Z}} \) is bounded and one defines

\[
\tilde{f}(z) := \sum_{n \in \mathbb{Z}} \left[ f\left(\frac{n\pi}{b}\right) + \epsilon_n \right] \tilde{\mathcal{G}}_{ab}\left(z,\frac{n\pi}{b}\right),
\]

then \( |f(z) - \tilde{f}(z)| \) is uniformly bounded in compacts of \( \mathbb{C} \) [13, Thm. 7.2.5].
Undersampling, on the other hand, looks for the approximation of a function in $\mathcal{PW}_b \setminus \mathcal{PW}_a$ by another one formally constructed using the sampling formula (1.1), namely,

$$\hat{f}(z) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{a}\right) G_a\left(z, \frac{n\pi}{a}\right).$$

The series in (1.4) is indeed convergent and, moreover, $|f(z) - \hat{f}(z)|$ is uniformly bounded in compact subsets of $\mathbb{C}$. Formula (1.4) yields in fact an approximation not only for functions in $\mathcal{PW}_b \setminus \mathcal{PW}_a$ but for the Fourier transform of elements in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ [13, Thm. 7.2.9].

Oversampling and undersampling are, to some extent, consequences of the fact that the chain of Paley-Wiener spaces $\mathcal{PW}_s$, $s \in (0, \infty)$, is totally ordered by inclusion. As this is a property shared by all dB spaces in the precise sense of [4, Thm. 35], it is expected that analogous notions should make sense in this latter class of spaces. We note that sampling formulas generalizing (1.1) are known for arbitrary reproducing kernel Hilbert spaces (see e.g. Kramer-type formulas in [6, 7, 16, 18]), dB spaces among them. Analysis of error due to noisy samples and aliasing, among other sources, in Paley-Wiener spaces goes back at least to [12]. More recent literature on the subject is, for instance, [1–3, 10]. However, to the best of our knowledge, estimates for oversampling and undersampling are not known for dB spaces apart from the Paley-Wiener class.

A function $f(z)$ belonging to a dB space $\mathcal{B}$ obviously admits a representation in terms of an orthogonal basis. In particular,

$$f(z) = \sum_{t \in \text{spec} (S(\gamma))} f(t) \frac{k(z, t)}{k(t, t)},$$

where $k(z, w)$ is the reproducing kernel of $\mathcal{B}$ and $S(\gamma)$ is a canonical selfadjoint extension of the operator of multiplication by the independent variable in $\mathcal{B}$. The expansion (1.5) is a sampling formula with $k(z, t)/k(t, t)$ being its sampling kernel. Note that (1.1) is a particular realization of (1.5) for the dB space $\mathcal{PW}_a$.

In order to obtain oversampling and undersampling estimates in analogy to the Paley-Wiener case, we look into dB spaces of the form

$$\mathcal{B}_s = \left\{ f(z) = \int_0^s \xi(x, z) \phi(x) \, dx : \phi \in L_2(0, s) \right\},$$

where $\xi(x, z)$ solves $(-\partial_x^2 + V(x))\xi(x, z) = z\xi(x, z)$ on $(0, s)$, $s \in (0, \infty)$, for $z \in \mathbb{C}$, with Neumann boundary condition at $x = 0$ (see Section 2). Here $V \in L_1(0, s)$ is a real function. By construction $\mathcal{B}_s \subset \mathcal{B}_{s'}$ whenever $s < s'$ (for more on this, see [15]).

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Define
\[ K_s(z, t) := \frac{k_s(z, t)}{k_s(t, t)}, \]
where \( k_s(z, w) \) is the reproducing kernel of the space \( B_s \). If \( S_s(\gamma) \) is a selfadjoint extension of the multiplication operator in \( B_s \), then any \( f(z) \in B_s \) has the representation
\[ f(z) = \sum_{t \in \text{spec}(S(\gamma))} f(t)K_s(z, t). \]

Our main results are Theorems 3.6 and 4.7, which can be (somewhat imprecisely) summarized as follows:

Theorem (oversampling). Assume \( V \in AC[0, \pi] \) such that \( V' \in L_1(0, \pi) \). Fix \( a \in (0, \pi) \) and consider an arbitrary \( f(z) \in B_a \). For a given \( \{\epsilon_t\} \in l_\infty \) define
\[ \tilde{f}(z) := \sum_{t \in \text{spec}(S_\pi(\pi/2))} [f(t) + \epsilon_t] \tilde{K}_{a\pi}(z, t), \]
where \( \tilde{K}_{a\pi}(z, t) \) is a modified sampling kernel (compare with (1.3) and see the concrete expression in (3.6)). Then, for any \( z \) in a compact \( K \) of \( \mathbb{C} \), there is a positive constant \( C \) (depending on \( K \) but not on \( z \)) such that
\[ |f(z) - \tilde{f}(z)| \leq C (\|\epsilon\|_\infty). \]

Theorem (undersampling). Assume \( V \in AC[0, b] \) such that \( V' \in L_1(0, b) \), for \( b > \pi \). Given \( g(z) \in B_b \setminus B_\pi \), define
\[ \tilde{g}(z) := \sum_{t \in \text{spec}(S_\pi(\pi/2))} g(t)K_\pi(z, t). \]
Then, for each compact \( K \subset \mathbb{C} \), there is \( C > 0 \) such that
\[ |g(z) - \tilde{g}(z)| \leq C \left( \int_\pi^b |\psi(x)| \, dx \right) \]
uniformly on \( K \), where \( \psi \in L_2(0, b) \) obeys \( g(z) = \langle \xi(\cdot, \pi), \psi(\cdot) \rangle_{L_2(0, b)} \).

These results are somewhat limited in several respects. First, we show oversampling relative to the pair \( B_a \subset B_\pi \), and undersampling relative to the pair \( B_\pi \subset B_b \) (for dB spaces defined according to (1.6)). These particular choices are related to a convenient simplification in the proofs, but our results can be extended to an arbitrary pair \( B_a \subset B_b \) by a scaling argument. Second, the sampling formulae use the spectra of selfadjoint operators with Neumann boundary condition at the right endpoint. This choice simplifies the asymptotic formulae for eigenvalues of the associated Schrödinger operator; it can also be removed.
but at the expense of a somewhat clumsier analysis. In our opinion this extra workload would not add anything substantial to the results. Finally, and more importantly from our point of view, our assumption on the potential functions is a bit too restrictive. In view of [15], we believe that our results should be valid just requiring $V \in L_1(0, s)$, but relaxing our present assumption on $V$ would require some major changes in the details of our proofs. Further generalizations of the results presented here (in particular, involving a wider class of dB spaces) are the subject of a future work.

About the organization of this work: Section 2 recalls the necessary elements on de Branges spaces and regular Schrödinger operators. Section 3 deals with oversampling. Undersampling is treated in Section 4. The Appendix contains some technical results.

2. dB spaces and Schrödinger operators

There are various ways of defining a de Branges space (see [4, Sec. 19], [15, Sec. 2], [19]). We recall the following definition: a Hilbert space of entire functions $\mathcal{B}$ is a de Branges (dB space) when it has a reproducing kernel $k(z, w)$ and is isometrically invariant under the mappings $f(z) \mapsto f^*(z) := \overline{f(z)}$ and

$$f(z) \mapsto \left(\frac{z-w}{z-w}\right)^{\text{Ord}_w(f)} f(z), \quad w \in \mathbb{C},$$

where $\text{Ord}_w(f)$ is the order of $w$ as a zero of $f$. The class of dB spaces appearing in this work has the following additional properties:

(a1) Given any real point $x$, there is a function $f \in \mathcal{B}$ such that $f(x) \neq 0$.

(a2) $\mathcal{B}$ is regular, i.e., for any $w \in \mathbb{C}$ and $f \in \mathcal{B}$, $(z-w)^{-1} (f(z) - f(w)) \in \mathcal{B}$.

A distinctive structural property of dB spaces is that the set of dB subspaces of a given dB space is totally ordered by inclusion [4, Thm. 35]. For regular dB spaces (in the sense of (a2)) this means that, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are subspaces of a dB space that are themselves dB spaces, then either $\mathcal{B}_1 \subset \mathcal{B}_2$ or $\mathcal{B}_1 \supset \mathcal{B}_2$ [5, Sec. 6.5].

The operator $S$ of multiplication by the independent variable in a dB space $\mathcal{B}$ is defined by

$$(Sf)(z) = zf(z), \quad \text{dom}(S) := \{f \in \mathcal{B} : Sf \in \mathcal{B}\}.$$

This operator is closed, symmetric and has deficiency indices (1, 1).

In view of (a1), the spectral core of $S$ is empty (cf. [8, Sec. 4]), i.e., for any $z \in \mathbb{C}$, the operator $(S - zI)^{-1}$ is bounded although, as a consequence of the
indices being \((1,1)\), its domain has codimension one. We consider dB spaces such that \(S\) is densely defined and denote by \(S(\gamma), \gamma \in [0,\pi)\), the selfadjoint restrictions of \(S^*\).

Since \(\langle (S^* - w)k(\cdot, \overline{w}), f(\cdot) \rangle = \langle k(\cdot, \overline{w}), (S - \overline{w}f(\cdot) \rangle = 0\) for all \(f(z) \in \text{dom}(S)\), we have \(k(z, \overline{w}) \in \ker(S^* - wI)\) for any \(w \in \mathbb{C}\). Thus

\[
\{k(z,t) : t \in \text{spec}(S(\gamma))\} \text{ is an orthogonal basis,} \tag{2.1}
\]

where \(\text{spec}(S(\gamma))\) denotes the spectrum of \(S(\gamma)\). Hence, the sampling formula

\[
f(z) = \sum_{t \in \text{spec}(S(\gamma))} f(t) \frac{k(z,t)}{k(t,t)}, \quad f \in \mathcal{B}, \tag{2.2}
\]

holds true. The convergence of this series is in the dB space, which in turn implies uniform convergence in compact subsets of \(\mathbb{C}\).

The dB spaces under consideration in this work are related to symmetric operators arising from differential expressions of the form

\[
\tau := -\frac{d^2}{dx^2} + V(x),
\]

where we assume

\textbf{(v1)} \(V\) is real-valued and belongs to \(L_1(0,s)\) for arbitrary \(s > 0\).

For each \(s > 0\), \(\tau\) determines a closed symmetric operator \(H_s\) in \(L_2(0,s)\),

\[
\text{dom}(H_s) := \{\varphi \in L_2(0,s) : \tau \varphi \in L_2(0,s), \varphi'(0) = \varphi(s) = \varphi'(s) = 0\}
\]

\[
H_s \varphi := \tau \varphi. \tag{2.3}
\]

This operator is known to have deficiency indices \((1,1)\) and empty spectral core. The selfadjoint extensions of \(H_s\) are given by

\[
\text{dom}(H_s(\gamma)) := \left\{ \varphi \in L_2(0,s) : \tau \varphi \in L_2(0,s), \begin{cases} 
\varphi'(0) = 0, \\ \varphi(s) \cos \gamma + \varphi'(s) \sin \gamma = 0
\end{cases} \right\}
\]

\[
H_s(\gamma) \varphi := \tau \varphi, \tag{2.4}
\]

with \(\gamma \in [0,\pi)\). Finally, the adjoint operator of \(H_s\) is

\[
\text{dom}(H_s^*) := \{\varphi \in L_2(0,s) : \tau \varphi \in L_2(0,s), \varphi'(0) = 0\}, \quad H_s^* \varphi := \tau \varphi.
\]

Let \(\xi : \mathbb{R}_+ \times \mathbb{C} \to \mathbb{C}\) be the solution of the eigenvalue problem

\[
\tau \xi(x,z) = z \xi(x,z), \quad \xi(0,z) = 1, \quad \xi'(0,z) = 0. \tag{2.5}
\]
(The derivative is taken with respect to the first argument.) The function $\xi(x, z)$ is real entire for any fixed $x \in \mathbb{R}_+$ [11, Thm. 1.1.1], [20, Thm. 9.1]. Also, $\xi(\cdot, z) \in L_2(0, s)$ for any $z \in \mathbb{C}$. Using [19, Sec. 4] one then establishes that $\xi(\cdot, z)$ is entire as an $L_2(0, s)$-valued map. Note that $\xi(\cdot, z)$ depends on the potential $V$ but does not depend on the right endpoint $s$.

According to [17, Props. 2.12 and 2.14] [19, Thm. 16], the functions

$$f(z) = \langle \xi(\cdot, z), \phi(\cdot) \rangle_{L_2(0,s)};$$

with $\phi \in L_2(0, s)$, form a dB space $B_s$ with the norm given by

$$\|f\|_{B_s} = \|\phi\|_{L_2(0,s)}. \quad (2.7)$$

A straightforward computation shows that the reproducing kernel of $B_s$ is

$$k_s(z, w) = \langle \xi(\cdot, z), \xi(\cdot, w) \rangle_{L_2(0,s)}.$$

$$(2.8)$$

**Remark 1.** In view of (2.8), $k_s(z, w)$ and $\xi(\cdot, w)$ are related by the isometry (2.6). Hence, using (2.1) and expression (2.7) for the norm of $B_s$, one obtains

$$\varphi(x) = \sum_{t \in \text{spec}(H_s(\gamma))} \frac{1}{k_s(t, t)} \langle \xi(\cdot, t), \phi(\cdot) \rangle_{L_2(0,s)} \xi(x, t), \quad \varphi \in L_2(0, s),$$

(2.9)

where the series converges in the $L_2$-norm.

If $r < s$, then $B_r$ is a proper dB subspace of $B_s$. Indeed, $\{B_r : r \in (0, s)\}$ is a chain of dB subspaces of $B_s$ in accordance with [4, Thm. 35]. The isometry from $L_2(0, s)$ onto $B_s$ induced by (2.6) transforms $H_s$ into the operator of multiplication by the independent variable in $B_s$, the later subsequently denoted by $S_s$. Also, the selfadjoint extensions $H_s(\gamma)$ are transformed into the selfadjoint extensions $S_s(\gamma)$ of $S_s$. When referring to unitary invariants (such as the spectrum), we use interchangeably either $H_s(\gamma)$ or $S_s(\gamma)$ throughout this text.

**Remark 2.** The space $B_s$ constructed from $L_2(0, s)$ via (2.6) depends on the potential $V$, which is assumed to satisfy $$(v1).$$ However, as shown in [15, Thm. 4.1], the set of entire functions in $B_s$ is the same for all $V \in L_1(0, s)$; what changes with $V$ is the inner product in $B_s$. Noteworthy, the operator of multiplication by the independent variable $S_s$ is always the same; yet, by modifying the metric of the space, each $V \in L_1(0, s)$ gives rise to a different family of selfadjoint extensions of $S_s$. As a consequence, every function in $B_s$ can be sampled by (2.2) using any sequence $\{\lambda_n\}$ as sampling points, as long as there exists $V \in L_1(0, s)$ such that $\{\lambda_n\}$ is the spectrum of some selfadjoint extension of the corresponding operator $H_s$. This fact can be considered as a generalization of the notion of irregular sampling, quite well studied in Paley-Wiener spaces by means of
classical analysis; the Kadec’s 1/4 Theorem is a chief example of this kind of results.

3. Oversampling

The oversampling of a function in $B_a$ is related to the fact that it can be sampled as a function in $B_b$ and the sampling kernel can be modified in such a way that the sampling series is convergent under $l_\infty$ perturbations of the samples (see the Introduction).

Let $0 < a < b < \infty$ and $V$ be as in (v1). Any $\varphi \in L_2(0, a)$ can be identified with an element in $L_2(0, b)$ since

$$\varphi = \varphi \chi_{[0,a]} + 0 \chi_{(a,b]},$$

where $\chi_E$ denotes the characteristic function of a set $E$. Define

$$\mathcal{R}(x) = \mathcal{R}_{ab}(x) := \chi_{[0,a]}(x) + \frac{b-x}{b-a} \chi_{(a,b]}(x), \quad x \in [0,b].$$

(3.2)

Taking into account (2.9) with $s = b$, (3.1) and (3.2) imply

$$\varphi(x) = \sum_{t \in \text{spec}(H_b(\gamma))} \frac{1}{k_b(t,t)} \langle \xi(\cdot, t), \varphi(\cdot) \rangle_{L_2(0,b)} \mathcal{R}(x) \xi(x, t),$$

(3.3)

where the convergence is in $L_2(0, b)$. Plugging (3.3) into (2.6) with $s = b$, we obtain

$$f(z) = \sum_{t \in \text{spec}(H_b(\gamma))} \frac{1}{k_b(t,t)} \langle \xi(\cdot, z), \mathcal{R}(\cdot) \xi(\cdot, t) \rangle_{L_2(0,b)} f(t), \quad z \in \mathbb{C},$$

(3.4)

which converges uniformly in compacts of $\mathbb{C}$.

**Hypothesis 3.1.** Given $0 < a < b$, the series

$$\sum_{t \in \text{spec}(H_b(\gamma))} \frac{1}{k_b(t,t)} \left| \langle \xi(\cdot, z), \mathcal{R}_{ab}(\cdot) \xi(\cdot, t) \rangle_{L_2(0,b)} \right|$$

(3.5)

converges uniformly in compact subsets of $\mathbb{C}$.

Assume that Hypothesis 3.1 takes place. Enumerate any given sequence
\[ \epsilon \in l_\infty \] such that \( \epsilon = \{ \epsilon_t \}_{t \in \text{spec}(H_b(\gamma))} \). Thus, in view of (3.4), the function
\[
\bar{f}(z) := \sum_{t \in \text{spec}(H_b(\gamma))} \frac{1}{k_b(t, t)} \langle \xi(\cdot, \xi(t)), R_{ab}(\cdot) \xi(\cdot, t) \rangle_{L^2(0,b)} (f(t) + \epsilon_t), \quad z \in \mathbb{C},
\]
(3.6)
is well defined and the defining series converges uniformly in compacts of \( \mathbb{C} \). Moreover,
\[
\left| \bar{f}(z) - f(z) \right| \leq \| \epsilon \|_{l_\infty} \sum_{t \in \text{spec}(H_b(\gamma))} \frac{1}{k_b(t, t)} \left| \langle \xi(\cdot, \xi(t)), R(\cdot) \xi(\cdot, t) \rangle_{L^2(0,b)} \right|,
\]
(3.7)
for all \( z \in \mathbb{C} \). Thus, the difference \( |\bar{f}(z) - f(z)| \) is uniformly bounded in compacts of \( \mathbb{C} \). Below we prove that Hypothesis 3.1 holds true when

\((v2)\) \( V \) is absolutely continuous in \([0, b]\) (hence satisfies \((v1)\) for \( s = b \)) and \( V' \in L_1(0, b) \).

This is performed in two stages, the first one deals with the case \( V \equiv 0 \), the second one employs perturbative methods to consider the general case.

If \( V \equiv 0 \), the function \( \xi \) given in Section 2 is
\[
\xi(x, z) = \cos(\sqrt{z} x), \quad 0 < x \in \mathbb{R}_+.
\]
(3.8)
Whenever we refer to the function \( \xi \) corresponding to \( V \equiv 0 \), we write the right-hand-side of (3.8). We reserve the use of the symbol \( \xi \) only for the case \( V \neq 0 \). Also, throughout this paper we use the main branch of the square root function.

As mentioned in the Introduction, for the sake of simplicity we assume \( b = \pi \) and fix \( \gamma = \pi/2 \). A straightforward calculation yields
\[
\text{spec} \left( H_{\pi} (\pi/2) \right) = \{ n^2 : n \in \mathbb{N} \cup \{ 0 \} \}.
\]
(3.9)
Moreover, by substituting (3.8) into (2.8), we verify that the reproducing kernel \( \hat{k}_\pi(z, w) \) corresponding to the case \( V \equiv 0 \) satisfies
\[
\hat{k}_\pi(n^2, n^2) = \begin{cases} 
\pi & \text{if } n = 0, \\
\frac{\pi}{2} & \text{if } n \in \mathbb{N}.
\end{cases}
\]
(3.10)
In the remainder of this section, we denote \( \langle \cdot, \cdot \rangle_{L^2(0, \pi)} \) simply as \( \langle \cdot, \cdot \rangle \).

**Proposition 3.2.** Hypothesis 3.1 holds true under the assumption \( V \equiv 0, b = \pi, \) and \( \gamma = \pi/2 \).
Proof. Consider an arbitrary compact set $K$ in $\mathbb{C}$ and assume that $\text{spec}(H_\pi(\pi/2))$ intersects $K$ only at $n_0^2$ with $n_0 \in \mathbb{N}$. It will be clear at the end of the proof that there is no loss of generality in this assumption. First note that $\left|\langle \cos(\sqrt{z} \cdot), \mathcal{R}(\cdot) \cos(n_0 \cdot) \rangle\right|$ is uniformly bounded in $K$ (one can use the Cauchy-Schwarz inequality and note that the factor depending on $z$ is continuous in $K$). On the other hand, by Lemma A.5,

$$\sum_{n \neq n_0} \left|\langle \cos(\sqrt{z} \cdot), \mathcal{R}(\cdot) \cos(n \cdot) \rangle\right| = \frac{1}{2} \sum_{n \neq n_0} \left|\frac{\cos((\sqrt{z} + n)a) - (-1)^n \cos(\sqrt{z} \pi)}{(\pi - a)(\sqrt{z} + n)^2} + \frac{\cos((\sqrt{z} - n)a) - (-1)^n \cos(\sqrt{z} \pi)}{(\pi - a)(\sqrt{z} - n)^2}\right|$$

$$\leq \frac{e^{\pi |\text{Im} \sqrt{z}|}}{(\pi - a)} \sum_{n \neq n_0} \left(\frac{1}{|\sqrt{z} + n|^2} + \frac{1}{|\sqrt{z} - n|^2}\right).$$

(3.11)

Thus, taking into account (3.10), the series (3.5) converges uniformly in $K$. □

Now, let us address the case of non-zero $V$ satisfying (v2). As before we set $b = \pi$ and $\gamma = \pi/2$. Also, we assume $\text{spec}(H_\pi(\pi/2)) = \{\lambda_n\}_{n=0}^\infty$ ordered such that $\lambda_{n-1} \leq \lambda_n$ for all $n \in \mathbb{N}$. The subsequent analysis make use of the following auxiliary functions.

**Definition 3.3.** For each $x \in [0, \pi]$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$, consider

$$\rho(x) := \frac{1}{2} \int_0^x V(y)dy - \frac{x}{2\pi} \int_0^\pi V(y)dy,$$

$$T(x, n) := \xi(x, \lambda_n) - \cos(nx) - \frac{\rho(x)}{n} \sin(nx),$$

$$F(x, z) := \xi(x, z) - \cos(\sqrt{z} x).$$

**Lemma 3.4.** Let $V$ be as in (v2). There exists $N \in \mathbb{N}$ such that, if $n \geq N$, then

$$\left|\langle \xi(\cdot, z), \mathcal{R}(\cdot) \xi(\cdot, \lambda_n) \rangle - \langle \cos(\sqrt{z} \cdot), \mathcal{R}(\cdot) \cos(n \cdot) \rangle\right| \leq C_\pi \frac{e^{\text{Im} \sqrt{z} \pi}}{n^2} \left(1 + \frac{1 + |z|}{1 + \pi |z|^{1/2}}\right),$$

for every $z \in \mathbb{C}$. Here $C_\pi$ is a positive number depending on $V$. 

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Proof. In terms of the functions introduced in Definition 3.3, one writes
\[
\langle \xi(\cdot, \bar{z}), \mathcal{R}(\cdot) \xi(\cdot, \lambda_n) \rangle - \langle \cos(\sqrt{z} \cdot), \mathcal{R}(\cdot) \cos(n \cdot) \rangle
\]
\[
= \int_0^\pi \left[ \cos(\sqrt{z}x)\mathcal{R}(x)\frac{\rho(x)}{n} \sin(nx) + F(x, z)\mathcal{R}(x)\frac{\rho(x)}{n} \sin(nx) \right. \\
+ F(x, z)\mathcal{R}(x) \cos(nx) + \cos(\sqrt{z}x)\mathcal{R}(x)T(x, n) + F(x, z)\mathcal{R}(x)T(x, n) \right] dx.
\]
(3.12)

It will be shown that each term on the right-hand side of (3.12) is appropriately bounded. For the first term, one uses the inequality (A.12) of Lemma A.4 and the first inequality of Lemma A.7. The estimate of the second term is obtained by combining (A.13) of Lemma A.4 and the second inequality of Lemma A.7. The third term on the right-hand side of (3.12) is estimated in Lemma A.6.

As regards the fourth and fifth terms in (3.12), one proceeds as follows. According to Lemma A.3(ii), \(T(x, n) = \mathcal{O}(n^{-2})\) as \(n \to \infty\) uniformly with respect to \(x \in [0, \pi]\). We note that there exist \(C_1, C_2 > 0\), and \(N \in \mathbb{N}\) (all possibly depending on \(V\)) such that, if \(n \geq N\) then, for all \(z \in \mathbb{C}\),
\[
\left| \int_0^\pi T(x, n)\mathcal{R}(x) \cos(\sqrt{z}x)dx \right| \leq \frac{C_3}{n^2} e^{\text{Im}\sqrt{z}|x|},
\]
(3.13)
\[
\left| \int_0^\pi T(x, n)\mathcal{R}(x)F(x, z)dx \right| \leq \frac{C_4}{n^2} \frac{\pi}{1 + \pi |z|^{1/2}} e^{\text{Im}\sqrt{z}|x|}.
\]
(3.14)

By combining the estimates of the first three terms, together with (3.13) and (3.14), the bound of the statement is established. \(\square\)

**Proposition 3.5.** Let \(V\) be as in \((v2)\). For \(b = \pi\) and \(\gamma = \pi/2\), Hypothesis 3.1 holds true.

**Proof.** From Lemma A.3(iii) we know that \(k_\pi(\lambda_n, \lambda_n) - \hat{k}_\pi(n^2, n^2) = \mathcal{O}(n^{-2})\) as \(n \to \infty\). Hence, using the Taylor decomposition of \(f(t) = t^{-1}\) with the Lagrange form of the remainder, one has
\[
\frac{1}{k_\pi(\lambda_n, \lambda_n)} - \frac{1}{\hat{k}_\pi(n^2, n^2)} = \frac{1}{k_\pi(\lambda_n, \lambda_n)} - \frac{2}{\pi} = \mathcal{O}(n^{-2}), \quad n \to \infty.
\]
(3.15)

Due to Lemma 3.4 and (3.15) there exists \(N \in \mathbb{N}\) such that, if \(n \geq N\), then
\[
\left| \frac{\langle \xi(\cdot, \bar{z}), \mathcal{R}(\cdot) \xi(\cdot, \lambda_n) \rangle}{k_\pi(\lambda_n, \lambda_n)} - \frac{\langle \cos(\sqrt{z} \cdot), \mathcal{R}(\cdot) \cos(n \cdot) \rangle}{\hat{k}_\pi(n^2, n^2)} \right| \leq \frac{c_1(z)}{n^2},
\]

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for all $z \in \mathbb{C}$, and where $c_1 : \mathbb{C} \to \mathbb{R}$ is a positive continuous function. As a consequence of the previous inequality, there exists another positive continuous function $c_2 : \mathbb{C} \to \mathbb{R}$ such that

$$\sum_{n=0}^{\infty} \left| \frac{\langle \xi(\cdot, \bar{\pi}), \mathcal{R}(\cdot)\xi(\cdot, \lambda_n) \rangle}{k_\pi(\lambda_n, \lambda_n)} - \frac{\langle \cos(\sqrt{\pi} \cdot), \mathcal{R}(\cdot) \cos(n \cdot) \rangle}{k_n(n^2, n^2)} \right| \leq c_2(z).$$

Hence, by Proposition 3.2, the series (3.5) converges uniformly in compacts of $\mathbb{C}$.

Arguing as in the paragraph below Hypothesis 3.1, one arrives at the following assertion in which the oversampling procedure is established (see the Introduction).

**Theorem 3.6.** Suppose $V$ obeys (v2) with $b = \pi$. Consider $\mathcal{B}_a$ with $a \in (0, \pi)$. Then, for every compact set $K \subset \mathbb{C}$, there exist a constant $C(K, a, V) > 0$ such that

$$|f(z) - \tilde{f}(z)| \leq C(K, a, V) \|\epsilon\|_\infty, \quad z \in K,$$

for all $f(z) \in \mathcal{B}_a$, where $\epsilon = \{\epsilon_t\}$ is any bounded real sequence and $\tilde{f}(z)$ is given by (3.6) with $b = \pi$ and $\gamma = \pi/2$.

### 4. Undersampling

In this section, we treat undersampling of functions in $\mathcal{B}_b \setminus \mathcal{B}_a$ ($a < b$) with the sampling points given by the spectrum of $S_a(\gamma)$ as explained in the Introduction.

**Hypothesis 4.1.** For $a < b$ and each $z \in \mathbb{C}$, the series

$$\sum_{t \in \text{spec}(H_a(\gamma))} \frac{k_a(t, \pi)}{k_a(t, t)} \xi(x, t)$$

(4.1)

converges absolutely and uniformly with respect to $x \in [0, b]$.

**Remark 3.** Note that (2.8) and (2.9) imply that the series

$$\sum_{t \in \text{spec}(H_a(\gamma))} \frac{k_a(t, \pi)}{k_a(t, t)} \xi(\cdot, t)$$

(4.2)

converges to $\xi(\cdot, z)$ in $L_2(0, a)$ for each $z \in \mathbb{C}$. Due to (2.1), if $z = \lambda \in \text{spec}(H_a(\gamma))$, then, as well as the series (4.2), the series (4.1) has only one term, namely, $\xi(x, \lambda)$ with $x \in [0, b]$. 

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Lemma 4.2. Assume that Hypothesis 4.1 takes place. Define

\[ \xi_\alpha^{\text{ext}}(x,z) := \sum_{t \in \text{spec}(H_a(\gamma))} \frac{k_a(t,z)}{k_a(t,t)} \xi(x,t), \quad x \in [0,b], \quad z \in \mathbb{C}. \]

Then, for each \( z \in \mathbb{C} \),

(i) \( \xi_\alpha^{\text{ext}}(\cdot,z) \) is continuous in \([0,b]\), and

(ii) \( \xi_\alpha^{\text{ext}}(x,z) = \xi(x,z) \) for a.e. \( x \in [0,a] \).

Moreover,

(iii) the function \( h_\alpha(z) := \sup_{x \in [a,b]} |\xi_\alpha^{\text{ext}}(x,z) - \xi(x,z)| \) is continuous in \( \mathbb{C} \), and

(iv) if \( \psi \in L_2(0,b) \) and \( g(z) \in \mathcal{B}_b \) are related by the isometry (2.6), then

\[
\left\langle \xi_\alpha^{\text{ext}}(\cdot,z), \psi(\cdot) \right\rangle_{L_2(0,b)} = \sum_{t \in \text{spec}(H_a(\gamma))} \frac{k_a(t,z)}{k_a(t,t)} g(t), \quad z \in \mathbb{C}. \quad (4.3)
\]

Proof. Enumerate \( \text{spec}(H_a(\gamma)) = \{\lambda_n\}_{n=0}^{\infty} \) such that \( \lambda_{n-1} \leq \lambda_n \) for all \( n \in \mathbb{N} \). Then (i) is a straightforward consequence of Hypothesis 4.1. Due to (i), \( \xi_\alpha^{\text{ext}}(\cdot,z) \) is an element of \( L_2(0,a) \) for each \( z \in \mathbb{C} \). Thus, Hypothesis 4.1 implies

\[
\lim_{m \to \infty} \left\| \xi_\alpha^{\text{ext}}(\cdot,z) - \sum_{n=0}^{m} \frac{k_a(\lambda_n,z)}{k_a(\lambda_n,\lambda_n)} \xi(\cdot,\lambda_n) \right\|_{L_2(0,a)} = 0.
\]

This, along with Remark 3, yields (ii). Item (iii) follows from Lemma A.1. To prove (iv), apply the dominated convergence theorem, which holds because of Hypothesis 4.1,

\[
\left\langle \xi_\alpha^{\text{ext}}(\cdot,z), \psi(\cdot) \right\rangle_{L_2(0,b)} = \lim_{m \to \infty} \sum_{n=0}^{m} \frac{k_a(\lambda_n,z)}{k_a(\lambda_n,\lambda_n)} \int_{0}^{b} \xi(x,\lambda_n) \psi(x) \, dx. \quad \square
\]

Assume that Hypothesis 4.1 holds true. Suppose that \( \psi \in L_2(0,b) \) and \( g(z) \in \mathcal{B}_b \) are related by the isometry (2.6), that is,

\[
g(z) = \left\langle \xi(\cdot,z), \psi(\cdot) \right\rangle_{L_2(0,b)}, \quad z \in \mathbb{C}. \quad (4.4)
\]

Define

\[
\tilde{g}(z) := \left\langle \xi_\alpha^{\text{ext}}(\cdot,z), \psi(\cdot) \right\rangle_{L_2(0,b)}, \quad z \in \mathbb{C}. \quad (4.5)
\]
Then, due to Lemma 4.2(ii),
\[
|g(z) - \hat{g}(z)| = \left| \int_a^b \left( \xi(x, z) - \xi_a^{ext}(x, z) \right) \psi(x) \, dx \right| \leq h_a(z) \int_a^b |\psi(x)| \, dx,
\]
where the function \( h_a \) has been defined in Lemma 4.2(iii). Therefore, for each \( \psi \in L_2(0, b) \), the difference \(|g(z) - \hat{g}(z)|\) is uniformly bounded in compacts of \( \mathbb{C} \). Below we prove that Hypothesis 4.1 holds true when \( \bar{V} \) satisfies (v2). As in the previous section, this is performed in two stages, the first one deals with the particular case \( \bar{V} \equiv 0 \) and the second one treats the general case.

In keeping with the simplification made in the previous section, we consider only the case \( a = \pi \) and \( \gamma = \pi/2 \).

Using trigonometric identities and equations (2.8) and (3.8) one verifies that
\[
\hat{k}_\pi(n^2, \overline{z}) = \frac{(-1)^{n+1}}{n^2 - z} \sqrt{\pi} \sin(\sqrt{\pi} z), \quad n \in \mathbb{N} \cup \{0\}, \quad z \in \mathbb{C} \setminus \{n^2\}. \quad (4.6)
\]
Recall that \( \hat{k}_\pi \) denotes the reproducing kernel within \( \mathcal{B}_\pi \) associated with \( \bar{V} \equiv 0 \).

**Proposition 4.3.** Hypothesis 4.1 holds true under the assumption \( \bar{V} \equiv 0 \), \( a = \pi \), and \( \gamma = \pi/2 \).

**Proof.** Let \( K \) be a compact set of \( \mathbb{C} \). As in the proof of Proposition 3.2, assume without loss of generality that \( n_0^2 \) is the only point of \( \text{spec}(H_\pi(\pi/2)) \) in \( K \) \((n_0 \in \mathbb{N})\). Due to (3.8)–(3.10), it suffices to show the uniform convergence of the series \( \sum_{n \neq n_0} |\hat{k}_\pi(n^2, \overline{z})| \) in \( K \). By (4.6), one obtains
\[
\sum_{n \neq n_0} \left| \hat{k}_\pi(n^2, \overline{z}) \right| \leq \left| \sqrt{\pi} \sin(\sqrt{\pi} z) \right| \sum_{n \neq n_0} \frac{1}{|n^2 - z|}. \quad \square
\]

Now we address the case \( \bar{V} \) satisfying (v2). Let \( \text{spec}(H_\pi(\pi/2)) = \{\lambda_n\}_{n=0}^\infty \) such that \( \lambda_{n-1} \leq \lambda_n \) for all \( n \in \mathbb{N} \). We aim to study the difference
\[
\frac{k_\pi(\lambda_n, \overline{z})}{k_\pi(\lambda_n, \lambda_n)} \xi(x, \lambda_n) - \frac{\hat{k}_\pi(n^2, \overline{z})}{\hat{k}_\pi(n^2, n^2)} \cos(nx), \quad x \in [0, b], \quad z \in \mathbb{C},
\]
for any given \( b > \pi \) and all \( n \in \mathbb{N} \) large enough.

**Lemma 4.4.** For any \( \bar{V} \) satisfying (v2), there exists \( N \in \mathbb{N} \) such that, if \( n \geq N \), then
\[
|k_\pi(\lambda_n, \overline{z}) - \hat{k}_\pi(n^2, \overline{z})| \leq D_\pi \frac{\text{Im } \sqrt{\pi}|z|}{n^2} \left( 1 + \frac{1 + |z|}{1 + \pi |z|^{1/2}} \right),
\]
for every \( z \in \mathbb{C} \). Here \( D_\pi \) is a positive real number depending on \( \bar{V} \).
Proof. In view of (2.8) and Definition 3.3,

\[
\begin{align*}
  k_\pi(\lambda_n, \bar{z}) - \bar{k}_\pi(n^2, \bar{z}) &= \int_0^\pi \left[ \cos(nx)F(x, z) + \frac{\rho(x)}{n} \sin(nx) \cos(\sqrt{z} x) \\
  &+ \frac{\rho(x)}{n} \sin(n x) F(x, z) + T(x, n) \cos(\sqrt{zx}) + T(x, n) F(x, z) \right] dx.
\end{align*}
\]

We proceed as in the proof of Lemma 3.4. The first three terms on the right-hand side of the last equality are estimated by Lemma A.4. The remaining terms have estimates obtained in the same way as the estimates (3.13) and (3.14).

Lemma 4.5. Assume that V satisfies (v2). Then, given \(b > \pi\), the asymptotic formula

\[
\xi(x, \lambda_n) - \cos(nx) = O(n^{-1}), \quad n \to \infty,
\]

holds uniformly with respect to \(x \in [0, b]\).

Proof. Using Lemma A.3(i) and repeating the reasoning leading to (3.15), one arrives at

\[
\lambda_n^{-1/2} n^{-1} = O(n^{-1}), \quad n \to \infty.
\]

This asymptotic formula and (A.6) yield

\[
\xi(x, \lambda_n) - \cos(\sqrt{\lambda_n x}) = O(n^{-1}), \quad n \to \infty,
\]

which leads to the desired result after noticing that Lemma A.3(i) also implies

\[
\cos(\sqrt{\lambda_n x}) - \cos(nx) = O(n^{-1}), \quad n \to \infty,
\]

uniformly with respect to \(x \in [0, b]\). \qed

Proposition 4.6. Let V be as in (v2). Set \(a = \pi\) and \(\gamma = \pi/2\). Then, Hypothesis 4.1 holds true.

Proof. Due to Lemmas 4.4 and 4.5, along with (3.15), there exists \(N \in \mathbb{N}\) and a continuous positive function \(c_3 : \mathbb{C} \to \mathbb{R}\) such that

\[
\left| \frac{k_\pi(\lambda_n, \bar{z})}{k_\pi(\lambda_n, \lambda_n)} \xi(x, \lambda_n) - \frac{\bar{k}_\pi(n^2, \bar{z})}{\bar{k}_\pi(n^2, n^2)} \cos(nx) \right| \leq c_3(z), \quad z \in \mathbb{C}, \quad x \in [0, b]. \tag{4.7}
\]

for all \(n \geq N\); we note that \(c_3\) may depend on \(b\) and \(V\). The estimate (4.7) in turn implies

\[
\sum_{n=0}^{\infty} \left| \frac{k_\pi(\lambda_n, \bar{z})}{k_\pi(\lambda_n, \lambda_n)} \xi(x, \lambda_n) - \frac{\bar{k}_\pi(n^2, \bar{z})}{\bar{k}_\pi(n^2, n^2)} \cos(nx) \right| \leq c_4(z)
\]
uniformly with respect to $x \in [0, b]$, where $c_4 : \mathbb{C} \to \mathbb{R}$ is another continuous positive function that may also depend on $b$ and $V$. The claimed assertion now follows from Proposition 4.3.

The next assertion corresponds to our main result pertaining to undersampling (aliasing) as mentioned in the Introduction.

**Theorem 4.7.** Suppose $V$ obeys $(v2)$ for $b > \pi$. Assume that $\psi \in L_2(0, b)$ and $g(z) \in \mathcal{B}_b$ are related by (4.4). For every compact set $K \subset \mathbb{R}$, there exist a constant $D(K, b, V) > 0$ such that

$$|g(z) - \hat{g}(z)| \leq D(K, b, V) \int_{\pi}^{b} |\psi(x)| \, dx, \quad z \in K,$$

where $\hat{g}(z)$ is given by (4.5) with $a = \pi$, i.e., $\hat{g}(z)$ is given by the series (4.3) with $a = \pi$ and $\gamma = \pi/2$.

### A. Auxiliary results

**Lemma A.1.** Let $Y$ be a compact interval of $\mathbb{R}$. Suppose $\theta : \mathbb{C} \times Y \to [0, \infty)$ is continuous. Then, $\Theta : \mathbb{C} \to [0, \infty)$ given by $\Theta(z) := \sup\{\theta(z, y) : y \in Y\}$ is continuous.

**Proof.** For each $z \in \mathbb{C}$, fix $\vartheta(z) \in Y$ such that $\theta(z, \vartheta(z)) = \sup\{\theta(z, y) : y \in Y\} = \Theta(z).$  \hspace{1cm} (A.1)

Take an arbitrary $z_0 \in \mathbb{C}$. Fix $r_0 > 0$ and let $K := \{w \in \mathbb{C} : |z_0 - w| \leq r_0\}$. Due to the compactness of $K \times Y$, the map $\theta |_{K \times Y}$ is uniformly continuous. Hence, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|z - w| < \delta \quad \text{and} \quad |y - v| < \delta \quad \text{implies} \quad |	heta(z, y) - \theta(w, v)| < \frac{\epsilon}{2}, \hspace{1cm} (A.2)$$

for any $(z, y), (w, v) \in K \times Y$. Take $w \in K$ such that $|z_0 - w| < \delta$. If $v \in Y$ satisfies $|\vartheta(z_0) - v| < \delta$ then, in view of (A.2),

$$\left| \theta(z_0, \vartheta(z_0)) - \theta(w, v) \right| \leq \left| \theta(z_0, \vartheta(z_0)) - \theta(z_0, v) \right| + \left| \theta(z_0, v) - \theta(w, v) \right| < \frac{\epsilon}{2}.$$

Due to (A.1) and the fact that $\theta$ is non negative, $\Theta(z_0) - \Theta(w) \leq \Theta(z_0) - \theta(w, v) < \epsilon$. Now, let $v \in Y$ such that $|\vartheta(w) - v| < \delta$. According to (A.2),

$$\left| \theta(w, \vartheta(w)) - \theta(z_0, v) \right| \leq \left| \theta(w, \vartheta(w)) - \theta(z_0, \vartheta(w)) \right| + \left| \theta(z_0, \vartheta(w)) - \theta(z_0, v) \right| < \frac{\epsilon}{2}.$$
Hence, $\Theta(w) - \Theta(z_0) \leq \Theta(w) - \theta(z_0, v) < \epsilon$. Therefore, we have proven that $-\epsilon < \Theta(z_0) - \Theta(w) < \epsilon$ whenever $|z_0 - w| < \delta$. \hfill \Box

The following Lemma is the analogous of [9, Lemma 2.2] for Neumann-like boundary conditions.

**Lemma A.2.** Given $a > 0$, suppose that $V \in L_1(0, a)$. Then, for each $z \in \mathbb{C}$, the unique solution of the initial value problem

$$
-\phi''(x, z) + V(x)\phi(x, z) = z\phi(x, z), \quad 0 \leq x \leq a,
$$

$$
\phi(0, z) = 1, \quad \phi'(0, z) = 0,
$$

satisfies the integral equation

$$
\phi(x, z) = \cos \left(\sqrt{z}x\right) + \int_0^x G(z, x, y)V(y)\phi(y, z)dy,
$$

where

$$
G(z, x, y) = \frac{1}{\sqrt{z}} \sin \left(\sqrt{z} (x - y)\right)
$$

is the corresponding Green’s function. This solution satisfies the estimate

$$
\left|\phi(x, z) - \cos \left(\sqrt{z}x\right)\right| \leq C \frac{x}{1 + |z|^{1/2}} e^{\text{Im} \sqrt{z}|x|} \int_0^x \frac{y |V(y)|}{1 + |z|^{1/2}}dy
$$

for some constant $C = C(a, V) > 0$. Furthermore, the derivative obeys

$$
\phi'(x, z) = -\sqrt{z} \sin \left(\sqrt{z}x\right) + \int_0^x \frac{\partial}{\partial x} G(z, x, y)V(y)\phi(y, z)dy,
$$

and satisfies the estimate

$$
\left|\phi'(x, z) + \sqrt{z} \sin \left(\sqrt{z}x\right)\right| \leq C e^{\text{Im} \sqrt{z}|x|} \int_0^x |V(y)|dy.
$$

**Proof.** Define

$$
\phi_0(x, z) := \cos \left(\sqrt{z}x\right), \quad \phi_{n+1}(x, z) := \int_0^x G(z, x, y)V(y)\phi_n(y, z)dy, \quad n \in \mathbb{N}.
$$

Since $|\cos \left(\sqrt{z}x\right)| \leq \exp(|\text{Im} \sqrt{z}| x)$ and

$$
|G(z, x, y)| \leq C_0 \frac{x}{1 + |z|^{1/2}} e^{\text{Im} \sqrt{z}(x-y)}, \quad 0 \leq y \leq x,
$$

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for some constant $C_0 > 0$ (cf. [9, Lemma A.1]), one has

$$|\phi_1(x, z)| \leq C_0 \|V\|_{L_1} \frac{x}{1 + |z|^{1/2} x} e^{\text{Im} \sqrt{z} x}.$$  

An induction argument then shows

$$|\phi_{n+1}(x, z)| \leq \frac{\|V\|_{L_1} C_0^{n+1}}{(n+1)!} \frac{x}{1 + |z|^{1/2} x} e^{\text{Im} \sqrt{z} x} \left(\int_0^x \frac{y |V(y)|}{1 + |z|^{1/2} y} dy\right)^n$$  

(A.10)

for all $n \in \mathbb{N}$. It follows that

$$\phi(x, z) := \sum_{n=0}^{\infty} \phi_n(x, z)$$

converges uniformly with respect to $x \in [0, a]$ for all $z \in \mathbb{C}$ and satisfies (A.5). The estimate (A.6) readily follows from (A.10) after noticing that

$$\int_0^x \frac{y |V(y)|}{1 + |z|^{1/2} y} dy \leq a \|V\|_{L_1}.$$  

The assertions (A.7) and (A.8) are proved by similar arguments so we omit the details. \(\square\)

The next results refer to the functions $\rho$, $T$, and $F$ introduced in Definition 3.3, as well as the reproducing kernel $k_b(z, w)$ from (2.8) and the particular case $k_b^0(z, w)$ when $V \equiv 0$.

**Lemma A.3** (see [11, Sec.1.2.2]). Assume $V \in AC[0, \pi]$ is real-valued such that $V' \in L_1(0, \pi)$. Let $H_\pi(\pi/2)$ be the selfadjoint operator defined in accordance with (2.4). Enumerate $\text{spec}(H_\pi(\pi/2))$ in increasing order and denote $\text{spec}(H_\pi(\pi/2)) = \{\lambda_n\}_{n=0}^{\infty}$. Then, the following assertions hold true.

(i) $\sqrt{\lambda_n} = n + O(n^{-1})$ as $n \to \infty$,

(ii) $T(x, n) = O(n^{-2})$ as $n \to \infty$, uniformly with respect to $x \in [0, \pi]$,

(iii) $k_\pi(\lambda_n, \lambda_n) = \hat{k}_\pi(n^2, n^2) + O(n^{-2})$ as $n \to \infty$.

We note that, in [11, Sec.1.2.2], the asymptotic formulas above are shown assuming $V'$ bounded in $[0, \pi]$. However, it is easy to see that it suffices to require $V' \in L_1(0, \pi)$.

**Lemma A.4.** Assume $V$ satisfies the hypothesis of Lemma A.3. Consider an arbitrary $a \in (0, \pi]$. Then, for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$, the following inequalities
hold true:

\[
\left| \int_0^\pi F(x, z) \cos(nx) \, dx \right| \leq C_1 \frac{e^{\pi |\text{Im} \sqrt{z}|}}{n^2} \left( 1 + \frac{1 + |z|}{1 + \pi |z|^{1/2}} \right), \tag{A.11}
\]

\[
\left| \int_0^a \rho(x) \cos(\sqrt{z}x) \sin(nx) \, dx \right| \leq C_2 \frac{e^{\pi |\text{Im} \sqrt{z}|}}{n} \left( 1 + \frac{1 + |z|}{1 + \pi |z|^{1/2}} \right), \tag{A.12}
\]

\[
\left| \int_0^a \rho(x) F(x, z) \sin(nx) \, dx \right| \leq C_3 \frac{e^{\pi |\text{Im} \sqrt{z}|}}{n} \left( 1 + \frac{1}{1 + |z|^{1/2} \pi} \right). \tag{A.13}
\]

Here, \( C_1 > 0 \) depends on \( V \) while \( C_2 > 0 \) and \( C_3 > 0 \) may, in addition, depend on \( a \).

**Proof.** Integrating by parts one obtains,

\[
\left| \int_0^\pi F(x, z) \cos(nx) \, dx \right| \leq \frac{1}{n^2} \left( 2 \sup_{x \in [0, \pi]} |F'(x, z)| + \pi \sup_{x \in [0, \pi]} |F''(x, z)| \right).
\]

On one hand, due to (A.8),

\[
\sup_{x \in [0, \pi]} |F'(x, z)| \leq C_V \exp(\pi |\text{Im} \sqrt{z}|). \tag{A.14}
\]

On the other hand, since \( F''(x, z) = V(x) \xi(x, z) - zF(x, z) \), it follows from (A.6) that

\[
|F''(x, z)| \leq e^{\pi |\text{Im} \sqrt{z}|} \left( \frac{C_V x}{1 + |x|^{1/2}} \left( \|V\|_{L_1} + |z|\right) + \|V\|_{L_1} \right).
\]

This implies (A.11).

The proof of (A.12) repeats the argumentation above: integrate by parts and observe that

\[
\sup_{x \in [0, \alpha]} \left| \frac{d}{dx} \rho(x) \cos(\sqrt{z}x) \right| \leq \|V\|_{L_1} e^{\pi |\text{Im} \sqrt{z}|} \left( \frac{|z| C \pi}{1 + |z|^{1/2} \pi} + 1 \right).
\]

The proof of (A.13) follows a similar reasoning. \( \square \)

**Lemma A.5.** Set \( \alpha \in (0, \pi) \) and consider \( R_{\alpha \pi} \) given by (3.2). Then, for any
\[ n \in \mathbb{N} \cup \{0\} \text{ and } z \in \mathbb{C} \setminus \{n^2\}, \]
\[
\langle \cos(\sqrt{z} \cdot), \mathcal{R}_{\pi} (\cdot) \cos(n \cdot) \rangle = \frac{1}{2(\pi - a)} \left( \frac{\cos \left( (\sqrt{z} + n)a \right) - (-1)^n \cos(\sqrt{z} \pi)}{(\sqrt{z} + n)^2} \right.
\]
\[
+ \frac{\cos \left( (\sqrt{z} - n)a \right) - (-1)^n \cos(\sqrt{z} \pi)}{(\sqrt{z} - n)^2} \right) .
\]

**Proof.** On one hand, the identity
\[
\cos(\sqrt{z} x) \cos(nx) = 2^{-1} \left( \cos((\sqrt{z} + n)x) + \cos((\sqrt{z} - n)x) \right)
\]
leads to
\[
\int_0^a \cos(\sqrt{z} x) \cos(nx) \, dx = \frac{1}{2} \left( \frac{\sin \left( (\sqrt{z} + n)a \right)}{\sqrt{z} + n} + \frac{\sin \left( (\sqrt{z} - n)a \right)}{\sqrt{z} - n} \right) .
\]
On the other hand,
\[
\int_a^\pi \cos(\sqrt{z} x) \cos(nx) \left( \frac{\pi - x}{\pi - a} \right) \, dx
\]
\[
= -\frac{1}{2(\pi - a)} \left( \int_a^\pi x \cos \left( (\sqrt{z} + n)x \right) \, dx + \int_a^\pi x \cos \left( (\sqrt{z} - n)x \right) \, dx \right)
\]
\[
+ \frac{\pi}{2(\pi - a)} \left( \frac{\sin \left( (\sqrt{z} + n)x \right)}{\sqrt{z} + n} + \frac{\sin \left( (\sqrt{z} - n)x \right)}{\sqrt{z} - n} \right) \bigg|_{x=\pi} .
\]
Another integration by parts yields
\[
\int_a^\pi x \cos((\sqrt{z} \pm n)x) \, dx
\]
\[
= \frac{(-1)^n \cos(\sqrt{z} \pi) - \cos((\sqrt{z} \pm n)a)}{(\sqrt{z} \pm n)^2} + \frac{\pi (-1)^n \sin(\sqrt{z} \pi) - a \sin((\sqrt{z} \pm n)a)}{\sqrt{z} \pm n} .
\]
This completes the proof. \(\square\)

**Lemma A.6.** Set \(a \in (0, \pi)\) and consider \(\mathcal{R}_{\pi}\) given by (3.2). Then, for every \(z \in \mathbb{C}\) and \(n \in \mathbb{N}\),
\[
\left| \int_0^\pi F(x, z) \mathcal{R}_{\pi} (x) \cos(nx) \, dx \right| \leq C e^{\pi |\text{Im} \sqrt{z}|} \left( 1 + \frac{1 + |z|}{1 + \pi |z|^{1/2}} \right) ,
\]
where $C > 0$ may depend on $V$.

**Proof.** Integration by parts yields

\[
\int_0^a F(x, z) \cos(nx) \, dx = \frac{1}{n} \left( F(a, z) \sin(na) - \int_0^a F'(x, z) \sin(nx) \, dx \right), \tag{A.15}
\]

\[
\int_a^\pi F(x, z) \cos(nx) \, dx = -\frac{1}{n} \left( F(a, z) \sin(na) + \int_a^\pi F'(x, z) \sin(nx) \, dx \right), \tag{A.16}
\]

and

\[
\int_a^\pi x F(x, z) \cos(nx) \, dx = -\frac{1}{n} \left( a F(a, z) \sin(na) \\
+ \int_a^\pi x F'(x, z) \sin(nx) \, dx + \int_a^\pi F(x, z) \sin(nx) \, dx \right). \tag{A.17}
\]

Now, (A.16) and (A.17) imply

\[
\int_a^\pi F(x, z) \left( \frac{\pi - x}{\pi - a} \right) \cos(nx) \, dx \\
= -\frac{\pi}{(\pi - a)n} \left( F(a, z) \sin(na) + \int_a^\pi F'(x, z) \sin(nx) \, dx \right) \\
+ \frac{1}{(\pi - a)n} \left( a F(a, z) \sin(na) + \int_a^\pi x F'(x, z) \sin(nx) \, dx \right) \\
+ \int_a^\pi F(x, z) \sin(nx) \, dx). \tag{A.18}
\]

Then, (A.15) and (A.18) yield

\[
\int_0^\pi F(x, z) \Re(x) \cos(nx) \, dx \\
= -\frac{1}{n} \int_0^a F'(x, z) \sin(nx) \, dx - \frac{\pi}{(\pi - a)n} \int_a^\pi F'(x, z) \sin(nx) \, dx \\
+ \frac{1}{(\pi - a)n} \left( \int_a^\pi x F'(x, z) \sin(nx) \, dx + \int_a^\pi F(x, z) \sin(nx) \, dx \right).
\]

The claimed assertion now follows by an argument similar to the proof of Lemma A.4.

**Lemma A.7.** Let $V$ as in (v2). Fix $a \in (0, \pi)$. Then,

\[
\left| \int_a^\pi \cos(\sqrt{z} \, x) \left( \frac{\pi - x}{\pi - a} \right) \rho(x) \sin(nx) \, dx \right| \leq \frac{C}{n} e^{\pi |\Im(\sqrt{z})|} \left( 1 + \frac{|z|}{1 + \pi |z|^{1/2}} \right),
\]

20
and

\[ \left| \int_a^\pi F(x,z) \left( \frac{\pi - x}{\pi - a} \right) \rho(x) \sin(nx) \, dx \right| \leq \frac{C}{n} e^{x|\text{Im} \sqrt{z}|} \left( 1 + \frac{1}{1 + \pi |z|^{1/2}} \right), \]

for arbitrary \( z \in \mathbb{C} \) and \( n \in \mathbb{N} \).

**Proof.** We prove the first inequality. The second one is proved analogously. Arguing as in the beginning of the proof of Lemma A.4, one obtains

\[ \left| \int_a^\pi \cos(\sqrt{z} x) \left( \frac{\pi - x}{\pi - a} \right) \rho(x) \sin(nx) \, dx \right| \leq \frac{1}{n} (2M_1(z) + \pi M_2(z)), \]

where

\[ M_1(z) := \sup \left\{ \left| \cos(\sqrt{z} x) \frac{\pi - x}{\pi - a} \rho(x) \right| : x \in [a, \pi] \right\}, \]

and

\[ M_2(z) := \sup \left\{ \left| \frac{d}{dx} \left( \cos(\sqrt{z} x) \frac{\pi - x}{\pi - a} \rho(x) \right) \right| : x \in [a, \pi] \right\}. \]

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