EULER MEASURE AS GENERALIZED CARDINALITY

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Abstract. Schanuel has pointed out that there are mathematically interesting categories whose relationship to the ring of integers is analogous to the relationship between the category of finite sets and the semi-ring of non-negative integers. Such categories are inherently geometrical or topological, in that the mapping to the ring of integers is a variant of Euler characteristic. In these notes, I sketch some ideas that might be used in further development of a theory along lines suggested by Schanuel.

Foreword

In this informal article I have gathered together three memos I wrote in the mid-90s, based on conversations with Scott Axelrod, John Baez, Beifang Chen, Timothy Chow, Ezra Getzler, Greg Kuperberg, Michael Larsen, Ayelet Lindenstrauss, Haynes Miller, Lauren Rose, and Gian-Carlo Rota, and intended as prologues to further work. In the intervening five or six years my interests have taken me elsewhere, and I do not expect to return to these topics anytime soon. At the same time, I cannot help thinking that other people might be able to push these ideas further, and/or discover that they are more important (that is, that they are more relevant to other mathematics) than they currently appear to be.

The first section of this article, “A proposal for generalizing the Euler characteristic,” was written in April of 1995. (Where I’ve written “Euler characteristic” in this section, the reader should pretend I’ve written “Euler measure”. There are two different ways to generalize Euler’s $V - E + F$, one of which has the nice property of being a homotopy-invariant and the other of which has the property of being a valuation, and it seems reasonable to distinguish between them by using this terminology.) The second section, “Negative and fractional cardinalities via generalized polytopes,” was written in December of 1995, as an extended abstract for a conference on Formal Power Series and Algebraic Combinatorics. The final section, “Polyhedral sets and combinatorics” (which should be parsed as “polyhedral sets-and-combinatorics”, i.e.,

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polyhedral sets and polyhedral combinatorics), was written in October of 1996, to accompany a talk of the same title that I presented at the Mathematical Sciences Research Institute. This article concludes with a chapter containing additional comments, which I have refrained from inserting into the material that I wrote in the mid-90s.

Another unpublished memo in this vein, which I wrote at roughly the same time, has been newly revised by me and will be published in a special issue of *Algebra Universalis* being edited by Joseph Kung to honor the memory of Gian-Carlo Rota. It is available over the Web as an accompaniment to this article.

Rota always encouraged me to pursue my work on Euler measure, but I was never able to make the sorts of connections between this work and the broader world of mathematics that would justify the undertaking. It’s one thing to aspire to do foundational work, and another thing to have deep insights! I have often whimsically hoped that someone would create a journal called “Definitiones Mathematicae” that would serve as a haven for interesting definitions in search of serious theorems that would retroactively justify them. Lacking such an outlet for my musings, I have settled for self-publishing these memos (first on my home-page, and now in the arXiv). I have made no attempt to remove redundancies between the three sections of the article. Also, I have not always included the sort of bibliographic information that a good scholar should provide if only for politeness’ sake. If readers of this article have questions, I’ll be happy to try to answer them (and perhaps include the answers in later versions of the article).

I. A proposal for generalizing the Euler characteristic (1995)

A combinatorialist’s fundamental model of a non-negative integer \( n \) is a set of \( n \) points. Adding two positive integers corresponds to taking the disjoint union of two sets; multiplying corresponds to forming the Cartesian product.

To bring negative numbers into the game, we can follow a suggestion made by Stephen Schanuel, and replace cardinality by Euler characteristic. (Note that for finite point-sets, the two notions coincide.) Thus, one combinatorial model for the number \(-1\) would be a single open interval (0 vertices and 1 edge yields Euler characteristic \( 0 - 1 = -1 \)), and a model for the negative integer \(-n\) would be a disjoint union of \( n \) open intervals. Note that our notion of Euler characteristic is purely combinatorial, and that the sets in question are in general non-convex
and often non-connected; I will call them *objects* so as not to conflict with established geometric terminology.

In what respect does an open interval $I$ have the properties we expect of $-1$? In the first place, we have relations like $-1 + 1 = 0$; but this is boring. More interestingly, we have relations like $(-1) \times (-1) = 1$: the Cartesian square of an open interval is an open square, with Euler characteristic 1. Even more interestingly, if we define $\binom{X}{k}$ (where $X$ is a topological space and $k$ is a positive integer) to be the quotient of $X^k$ by the action of the symmetric group $S_k$, with the part on which $S_k$ does not act freely removed, then the Euler characteristic of $\binom{X}{k}$ is $\binom{n}{k}$, where $n$ is the Euler characteristic of $X$. For example, if $X$ is the interval $I$, then $\binom{X}{2}$ is the square $I \times I$ with the diagonal removed and with the two resulting pieces identified by reflection; this new space has Euler characteristic 1, which is indeed $\binom{-1}{2}$.

Most intriguing, however, is the prospect of exponentiation. In the case where $Y$ is a finite set of points, we can define $X^Y$ as the set of all functions from $Y$ to $X$, and it will indeed be the case that $\chi(X^Y) = \chi(X)^{\chi(Y)}$, where $\chi(\cdot)$ denotes the Euler characteristic. When $Y$ isn’t a finite set of points, but an interval or something even more complicated, then we clearly won’t want to define $X^Y$ as the set of all maps from $Y$ to $X$. But we might define $X^Y$ as the set of all “nice” maps from $Y$ to $X$, where niceness is some property or other that possesses the meta-property that every nice map can be specified by a finite number of real-valued parameters (possibly along with some additional combinatorial data). For instance, the set of nice maps from $[0, 1]$ to itself could be the set of piecewise-constant maps from $[0, 1]$ to itself, or the set of all piecewise-linear continuous maps from $[0, 1]$ to $[0, 1]$, or the set of all polynomial maps from $[0, 1]$ to $[0, 1]$. All of these have the finiteness meta-property mentioned above.

Given two objects $X$ and $Y$, and any notion of nice maps from $Y$ to $X$, we might hope to define a generalized Euler characteristic as the limit, as $n$ goes to infinity, of the (standard) Euler characteristic of a sequence of objects $P_n$, where the successive $P_n$’s correspond to sets of nice maps from $Y$ to $X$ with increasingly many (but, for each $n$, only boundedly many) parameters. Alternatively, one might wish to think of the limit-object $P_\infty$ directly as an infinite-dimensional complex having vertices, edges, faces, etc.

**Example 1:** Consider the the set of piecewise-constant maps from $(0, 1)$ to the two-point set $\{a, b\}$. We can define the object $P_n$ as the set of all such maps that are discontinuous at $n$ or fewer points in their
domain, and we can define $P_\infty$ as the direct limit of these objects under
the natural inclusion maps. We can then inquire into the behavior of
$\chi(P_n)$ as $n \to \infty$; we can think of this limit as either $\lim_{n \to \infty} \chi(P_n)$ or
$\chi(P_0) + \sum_{n=1}^{\infty} \chi(P_n - P_{n-1})$. I will adopt the latter point of view, and
think of there being vertices, edges, faces, etc.

There are just two vertices (the constant functions).

What about edges? These correspond to maps $f$ with a single dis-
continuity. If we have a single discontinuity, say at a point $x$, there are
three things we need to know in order to specify the function $f$: its
value to the left of $x$, its value to the right of $x$, and its value at $x$. This
would seem to give us 8 possibilities, but in fact we should take only
6 (the other 2 correspond to the already-counted constant functions).
Each of these 6 combinatorial possibilities ($aab$, $aba$, $abb$, $baa$, $bab$, $bba$)
yields an edge of $P_\infty$.

Faces correspond to functions with two discontinuities, $x$ and $y$. The
set of faces fibers over $\binom{I_2}{2}$, and each fiber is just a string of five letters
($a$’s and $b$’s), in which the first three can’t all be the same and the last
three can’t all be the same. There are 18 such strings. (Each of the 6
allowed strings of length three extends to 3 strings of length five.)

Similarly, there are 54 3-cells.

And so on. Thus the “Euler characteristic” is $2 - 6 + 18 - 54 + \ldots$.
This looks like nonsense, but we can apply Abel summation (or Euler’s
trust-your-pen principle) and assert that this geometric series has the
value $2/(1 - (-3)) = 1/2$. So $\chi(\{a, b\}^{(0,1)}) = 2^{-1} = \chi(\{a, b\})\chi(\{0, 1\})$.

Note, incidentally, that if we had decided to work in a category in
which our allowed maps were the continuous maps from $(0, 1)$ to $\{a, b\}$,
or less trivially the left-continuous maps, we would not get the answer
1/2. So the answer we get seems to be sensitive to what category we’re
in. Still, $2^{-1}$ seems like it should be the right answer, especially since we
can try other experiments in the category of piecewise-constant maps
and ascertain that in many other cases as well, $\chi(X^Y) = \chi(X)^\chi(Y)$.
For instance, I leave it to you to consider the case $X = Y = I$.

(Fractional Euler characteristics are not in and of themselves novel.
For instance, the infinite-dimensional projective plane $\mathbb{R}P^\infty$ is a 2-to-1
quotient of the infinite-dimensional sphere, which is contractible, so it
would make sense to define $\chi(\mathbb{R}P^\infty)$ to be 1/2. Indeed, $\mathbb{R}P^\infty$ has 1
vertex, 1 edge, 1 face, etc., so that the preceding “Eulerian” method
yields Euler characteristic $1 - 1 + 1 - 1 + \cdots = 1/(1 - (-1)) = 1/2$.
I’m fairly sure that people have pointed this out before — though I’m
not sure who. Fractional Euler characteristics also arise in the theory
of group cohomology, and I think they show up in orbifold theory as
well. But what’s novel here is the way in which exponentiation is seen to enter the story.

Let’s try out the piecewise-linear category next.

**Example 2:** A linear map $f$ from $[0, 1]$ to $(0, 1)$ is specified by $f(0)$ and $f(1)$, which are arbitrary numbers in $(0, 1)$. So the set of piecewise-linear continuous maps from $[0, 1]$ to $(0, 1)$ with no juncture-points is equivalent to an open square: the Euler characteristic is $0 - 0 + 1 = 1$.

What about piecewise-linear continuous maps with a single juncture-point? Each such map is determined by four numbers: the juncture point $x$ and the values $f(0)$, $f(1)$, and $f(x)$. Think of the set of such maps as being fibered over $(0, 1) \times (0, 1) \times (0, 1)$, corresponding to the choices we make for $x$, $f(0)$, and $f(1)$. Within each fiber, there is a single forbidden value for $f(x)$, since we don’t want the points $(0, f(0))$, $(x, f(x))$, and $(1, f(1))$ to be collinear (we’ve already counted the maps that are actually linear). So each fiber is equivalent to $(0, 1)$ with a single interior point removed, and thus has Euler characteristic $-2$. Multiplying this through by $\chi((0, 1) \times (0, 1) \times (0, 1)) = -1$, we find that the difference-object $P_1 - P_0$ has Euler characteristic $2$.

(Note that I’m using facts about Euler characteristic of fiber products; all this will of course have to be justified, once I figure out exactly what category I’m in! For now, though, I’m just trying to get a sense of what the theory could be like.)

What about piecewise-linear continuous maps with two juncture-points? Each such map $f$ is determined by the set of juncture points and by the values of $f$ at 0 and 1, together with its values at the juncture points themselves. Think of the first four numbers as determining a base-space, and the last two as determining a point within a fiber over that base. The base space is $\binom{(0, 1)}{2} \times (0, 1)$, which has Euler characteristic 1. Each fiber is equivalent (by inclusion-exclusion) to $I \times I - I - I + 1$, which has Euler characteristic 4. Hence $P_2 - P_1$ has Euler characteristic $1 \times 4 = 4$.

Continuing in this fashion, one finds that the quasi-Euler characteristic of the limit-object is $1 + 2 + 4 + 8 + \ldots$, whose Eulerian value is $-1$. And sure enough, this is what we should have expected: $\chi((0, 1)^{[0, 1]}) = \chi((0, 1))^{[0, 1]} = (-1)^1 = -1$. (Thanks to Lauren Rose for suggesting that I try this example.)

**Example 3:** Finally, let’s consider the set of polynomial maps from $[0, 1]$ to itself. For any fixed degree $d$, we can view the set of polynomials that take $[0, 1]$ into itself as a semialgebraic subset of the $d+1$-cube.
This subset is closed, being determined by (uncountably many) conditions of the form $0 \leq a_0 + a_1 t + \cdots + a_d t^d \leq 1$. It’s also bounded (though this requires proof). Finally, it’s contractible, because we can take such a polynomial function and multiply it by a constant $t$ and send $t$ to zero. So $P_n$, being a contractible compact set, has Euler characteristic 1 for all $n$, which would lead us to think that the limit-object $P_\infty$ has quasi-Euler characteristic 1 as well. And this, too, agrees with our prejudices, since $1 = 1^1 = \chi([0, 1])^\chi([0, 1])$.

This has been a very strange article: no definitions and no theorems! But I hope the heuristic calculations I’ve presented are suggestive of an interesting general theory that might exist.

II. Negative and fractional cardinalities via generalized polytopes (1995)

1. Introduction.

For a combinatorialist, the fundamental significance of the expressions $\binom{n}{k}$ and $k^n$ lies in their interpretation as cardinalities of sets, specifically, the set of all $k$-element subsets of an $n$-element set and the set of $n$-tuples of elements of a $k$-element set. However, these interpretations are only valid when $n$ and $k$ are non-negative integers. In this paper I will describe an extension of this standard interpretation that makes sense when $n$ is negative. Even formulas like “$\frac{1}{2}$ choose 2 equals $-\frac{1}{4}$” can in some sense be interpreted.

In this enlarged theory, sets and their cardinalities are replaced by polytopes and their Euler measures. The study of Euler measure has its root in Euler’s work on what is now called Euler characteristic, though the formulation of Euler measure as an additive set-function and an explication of its properties are due largely to Rota, Schanuel, and Chen. When a polytope is simply a finite collection of points, its Euler measure is simply its cardinality, and the standard combinatorial interpretation is recovered as a special case.

For the present purpose, I will need to extend the notions of polytope and Euler measure in two unrelated but compatible directions. The first extension, and one that will come as no surprise to those who know the orbifold notion of Euler characteristic, is to quotients of polytopes under the free action of a finite group. The more novel extension is to “infinite-dimensional polytopes” of a certain sort. To assign Euler measure to such objects, divergent sums must be assigned a notional
value via the physicist’s trick of regularization; more specifically, an infinite Eulerian sum $V - E + F - \ldots$ is interpreted as the value at $t = -1$ of the holomorphic function whose Taylor expansion in the vicinity of $t = 0$ is $V + Et + Ft^2 + \ldots$.

Section 2 of this article lays groundwork by reviewing basic properties of the polyhedral category and Euler measure. Section 3 discusses quotient polytopes and their role in providing an interpretation of $\binom{n}{k}$ with $n, k \in \mathbb{Z}$, $k \geq 0$. In Section 4, $\sigma$-polytopes are introduced and used to interpret $k^n$ with $n, k \in \mathbb{Z}$, $k \geq 1$. This section also makes it clear why the polyhedral category is a better setting for generalized combinatorics than the more familiar topological category. Section 5 makes a speculative survey of directions in which the polyhedral approach to combinatorial foundations can and should be extended, and Section 6 offers a summary and conclusions.

This article is a preliminary version of a longer article that I plan to complete before the summer of 1996. Conversations with Scott Axelrod, John Baez, Beifang Chen, Ezra Getzler, Greg Kuperberg, Michael Larsen, Ayelet Lindenstrauss, Haynes Miller, Lauren Rose, and Gian-Carlo Rota have helped me clarify my ideas.

2. The polyhedral category and Euler measure.

A polyhedron is any subset of a Euclidean space $\mathbb{R}^n$ that can be defined through conjunction and disjunction of a finite number of linear equations and inequalities involving the $n$ coordinates. Equivalently, the collection of polyhedra in $\mathbb{R}^n$ is the algebra of sets generated by the (open or closed) half-spaces of $\mathbb{R}^n$ under union, intersection, and complementation. For the most part I will focus on bounded polyhedra, or polytopes, though much of the theory carries over to the unbounded case (with complications).

The sum of two disjoint polyhedra in $\mathbb{R}^n$ is defined here as their union (not their Minkowski sum); more generally, the sum of two polyhedra $P, Q \subseteq \mathbb{R}^n$ is $P + Q = P \times \{1\} \cup Q \times \{2\} \subseteq \mathbb{R}^{n+1}$. The product of two polyhedra $P \subseteq \mathbb{R}^m, Q \subseteq \mathbb{R}^n$ is their usual Cartesian product $P \times Q \subseteq \mathbb{R}^{m+n}$. A function $f : P \rightarrow Q$ is a polyhedral map if its graph (a subset of the polyhedron $P \times Q$) is also a polyhedron. Two polyhedra are polyhedrally isomorphic if there is a bijective polyhedral map from one to the other.

Examples: The open interval $I = (0, 1)$ is a polyhedron in $\mathbb{R}$. A function from $I$ to $I$ is a polyhedral map iff it has finitely many “break-points” (points of discontinuity or non-differentiability) and it is linear.
on the interval between two consecutive break-points. \( I + I \) is polyhedrally isomorphic to \((0, 1) \cup (2, 3)\) (and to \((0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\)). \( I \times I \) is the open unit square.

Every polytope in \( \mathbb{R}^n \) can be written as a union of finitely many (relative-open) 0-cells, 1-cells, \ldots, and \( n \)-cells. The \( f \)-polynomial of such a decomposition is defined as
\[
f(t) = \sum_{i=0}^{n} f_i t^i \in \mathbb{N}[t],
\]
where \( f_i \) is the number of \( i \)-cells in the decomposition. Two polytopes are polyhedrally isomorphic iff they have the same dimension and Euler measure.

The \textit{dimension} of a polytope \( P \) is the largest \( d \) for which there exists an injective polyhedral map from the bounded \( d \)-dimensional cube into \( P \). The \textit{Euler measure} of a \( d \)-dimensional polytope \( P \) is the value of any associated \( f \)-polynomial at \( t = -1 \) (the value of this alternating sum is independent of the decomposition chosen). We denote the dimension and Euler measure of \( P \) by \( d(P) \) and \( \chi(P) \), respectively. Isomorphism classes of polytopes correspond to ordered pairs \((d, \chi)\) of integers, with \( d \geq -1 \) (as usual, we think of the empty set as being \(-1\)-dimensional); if \( d = -1 \), \( \chi \) must be 0, and if \( d = 0 \), \( \chi \) must be positive, while for \( d > 0 \), \( \chi \) may be any integer.

If \( f_P(t) \) and \( f_Q(t) \) are \( f \)-polynomials for polytopes \( P \) and \( Q \), arising from some specific decompositions, then these decompositions give rise to decompositions of \( P + Q \) and \( P \times Q \) with \( f \)-polynomials \( f_P(t) + f_Q(t) \) and \( f_P(t)f_Q(t) \), respectively. It follows from this that \( \chi(\cdot) \), in addition to being invariant under polyhedral isomorphism, is finitely additive:
\[
\chi(P + Q) = \chi(P) + \chi(Q).
\]
It also follows that \( \chi(\cdot) \) is multiplicative:
\[
\chi(P \times Q) = \chi(P)\chi(Q).
\]
If the polytope \( P \) is compact, then \( \chi(P) \) coincides with the Euler characteristic of \( P \). However, unlike Euler characteristic, Euler measure is not a homotopy invariant; for instance, the intervals \([0, 1]\), \([0, 1)\), and \((0, 1)\) have Euler measure 1, 0, and \(-1\), respectively, even though they are homotopy-equivalent.

If \( f : P \to \mathbb{R} \) is a piecewise-constant polyhedral map, with \( P \) a polytope, the \textit{Euler integral} \( \int_P f \, d\chi \) is defined as
\[
\sum_r r \cdot \chi(\{x \in P : f(x) = r\}),
\]
where \( r \) ranges over the finitely many real numbers in the range of \( f \). (More general versions of the integral can be defined, but will not be needed here.) If \( f \) is integer-valued, then \( \int f \, d\chi \) is an integer; in particular, for \( f \) equal to the indicator function of the polytope \( Q \subseteq P \), \( \int_P f \, d\chi = \chi(Q) \). The main utility of the Euler integral in this article is that it facilitates calculation of Euler measure by way of a Fubini
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theorem: for \( f : P \times Q \rightarrow \mathbb{R} \),

\[
\int_P \left( \int_Q f \, d\chi \right) \, d\chi = \int_{P \times Q} f \, d\chi = \int_Q \left( \int_P f \, d\chi \right) \, d\chi.
\]

3. Binomial coefficients and quotient polytopes.

In laying out some basic notions of quotient polytopes, I will sidestep the more interesting case of non-free actions, since they are not germane to my purpose. Furthermore, I will focus on actions of \( G = S_k \), the symmetric group on \( k \) letters, though everything I say applies to more general free actions of finite groups. If a polytope \( Q \) is acted on freely by the group \( G \), we can define the associated quotient polytope \( G/Q \) as the set of orbits of \( Q \) under the action of \( G \). For present intents, it suffices to take \( Q \) equal to the set of \( k \)-tuples consisting of \( k \) distinct points belonging to some fixed polytope \( P \), with \( S \) permuting the \( k \) entries; in this case, we let \( \left( \begin{array}{c} P \\ k \end{array} \right) \) denote the quotient polytope \( Q/S_k \).

One can always find a polytope in \( Q \) containing exactly one point in each orbit. For instance, define lexicographic ordering on \( P \subseteq \mathbb{R}^n \) in the usual way (with reference to its \( n \) coordinates); then we can represent each orbit in \( Q \) by the unique point in that orbit whose \( k \) components are arranged in lexicographically ascending order. Let us call this the lexicographic representation of \( Q/S_k \). (E.g., when \( P \) is the open interval \( I \), we can represent \( \left( \begin{array}{c} I \\ k \end{array} \right) \) by the set \( \{(x_1, x_2, \ldots, x_k) \in I^k : x_1 < x_2 < \cdots < x_k\} \), which, viewed as a subset of \( \mathbb{R}^k \), is just a \( k \)-dimensional simplex.) Each polytope that represents \( Q/S_k \) must have Euler measure \( \chi(Q)/k! \). \( \chi(Q) \) itself can be shown to equal

\[
\chi(P) \cdot (\chi(P) - 1) \cdot (\chi(P) - 2) \cdots (\chi(P) - k + 1)
\]

by repeated application of the Fubini theorem. Hence every polytope representing the quotient polytope \( \left( \begin{array}{c} P \\ k \end{array} \right) \) has as its Euler measure the integer \( \left( \chi(P)^k \right) \) (as given by the standard algebraic definition).

If \( P \) is a convex \( m \)-cell, then the lexicographic representation of \( \left( \begin{array}{c} P \\ k \end{array} \right) \) can be decomposed in a predictable way into cells of various dimensions; for instance, in the case \( k = 2 \), we get an \( m+1 \)-cell, an \( m+2 \)-cell, \ldots , and a \( 2m \)-cell. A cellular decomposition of the polytope \( P \) into convex cells, with \( f \)-polynomial \( f(t) \), gives rise to a cellular decomposition of \( \left( \begin{array}{c} P \\ 2 \end{array} \right) \) with \( f \)-polynomial

\[
\frac{1}{2} \left( [f(t)]^2 - f(t^2) \right) + \frac{t}{1-t} \left( f(t) - f(t^2) \right).
\]
Substituting \( t = -1 \) yields \((f^{(-1)})_2\). A similar situation prevails for \( \binom{P}{k} \) with \( k > 2 \), except that the formulas are more complicated.

4. Exponentiation and \( \sigma \)-polytopes.

Given polytopes \( P \) and \( Q \), define \( P^Q \) as the set of polyhedral maps \( Q \to P \). There are natural identifications that can be made purely at the functional level:

\[
\begin{align*}
P^{Q+R} &\equiv P^Q \times P^R \\
P^{Q \times R} &\equiv (P^Q)^R \\
(P \times Q)^R &\equiv P^R \times Q^R
\end{align*}
\]

If \( Q \) is a finite set, then every function from \( Q \) to \( P \) is polyhedral, so that the polyhedral definition of exponentiation coincides with the set-theoretic definition in this case; \( P^Q \) is effectively the \( |Q| \)th Cartesian power of \( P \).

However, when \( P \) is finite and \( Q \) is infinite, things get more complicated. Consider, for instance, the case \( |P| = 2, Q = (0,1) \). There are only two continuous maps from \( Q \) to \( P \); this accounts for the unsuitability of the category of topological spaces and continuous maps for the purpose of developing a "generalized combinatoric" that features exponentiation. However, there are infinitely many polyhedral maps.

For any sequence \( 0 = x_0 < x_1 < \cdots < x_k < x_{k+1} = 1 \) we obtain a polyhedral map \( f : (0,1) \to P \) by choosing values \( a_0, b_1, a_1, b_2, a_2, \ldots, b_k, a_k \) in \( P \) and defining \( f(x) = a_i \) for \( x_i < x < x_{i+1} \) and \( f(x_i) = b_i \); indeed, this representation is unique if one stipulates that the \( x_i \)'s are genuine points of discontinuity for \( f \), i.e., for every \( 1 \leq i \leq k \), the values \( a_{i-1}, b_i, a_i \) are not all equal to one another.

If \( P \) is finite (say \( |P| = m \)) and \( Q = (0,1) \), the set of polyhedral maps \( f : Q \to P \) with exactly \( k \) discontinuities has a natural realization in \( \mathbb{R}^k \times P^{2k+1} \) as \( S \times F \), where the simplex \( S = \{(x_1, x_2, \ldots, x_k) : 0 < x_1 < x_2 < \cdots < x_k \} \subset \mathbb{R}^k \) parametrizes the locations of the discontinuities of \( f \) and \( F \subset P^{2k+1} \) is the set of sequences \( (a_0, b_1, \ldots, b_k, a_k) \) for which one never has both \( b_i \) and \( a_i \) equal to \( a_{i-1} \). It is easily seen that \( |F| = m(m^2-1)^k \). Thus the set of all polyhedral maps \( Q \to P \) can be realized as a union of \( m \) 0-cells, \( m(m^2-1) \) 1-cells, \( m(m^2-1)^2 \) 2-cells, etc.

We cannot evaluate the divergent alternating series \( m - m(m^2-1) + m(m^2-1)^2 - \ldots \), but we can assign it a value through "regularization". If we define the \( f \)-series of this infinite collection of cells in the obvious
way, we get
\[ \sum_{k=0}^{\infty} m(m^2 - 1)^k t^k = \frac{m}{1 - (m^2 - 1)t}. \]

Evaluating this at \( t = -1 \) yields \( m/(1 + (m^2 - 1)) = m^{-1} = \chi(P)\chi(Q) \).

More generally, suppose \( P \) is a zero-dimensional polytope consisting of \( m \) points and \( Q \) is a one-dimensional polytope composed of \( f_0 \) vertices and \( f_1 \) 1-cells (where \( f_0 \) and \( f_1 \) are not determined by \( Q \) but \( f_0 - f_1 = \chi(Q) \) is). Every polyhedral map from \( Q \) to \( P \) is determined by \( f_0 \) polyhedral maps of a single point into \( P \) and \( f_1 \) polyhedral maps from a 1-cell into \( P \). Thus, our cell-stratification for the set of polyhedral maps from a 1-cell into \( P \), along with the obvious stratification for the set of polyhedral maps from a 0-cell into \( P \), yield a cell-stratification for \( PQ \) whose generating series is the series expansion of the rational function
\[ (m)^{f_0} \left( \frac{m}{1 - (m^2 - 1)t} \right)^{f_1}; \]
evaluating this function at \( t = -1 \) yields \( m^{f_0-f_1} = \chi(P)\chi(Q) \).

The preceding calculation is related to an alternative way of seeing that the set of polyhedral maps from \((0,1)\) to an \( m \)-point set \( P \) “ought” to be assigned Euler measure \( \frac{1}{m} \), without explicit recourse to regularization. On a functional level, \( P^{(0,1)} = P^{(0,\frac{1}{2})} \times P^{(\frac{1}{2})} \times P^{(\frac{1}{2},1)} \); so a desire for functoriality would lead us to want \( \chi(P^{(0,1)}) = \chi(P^{(0,\frac{1}{2})})\chi(P^{(\frac{1}{2})})\chi(P^{(\frac{1}{2},1)}) \). On the other hand, the polyhedral equivalence of \((0,\frac{1}{2}), (\frac{1}{2},1), \) and \((0,1)\) would lead us to expect \( \chi(P^{(0,\frac{1}{2})}) = \chi(P^{(\frac{1}{2},1)}) = \chi(P^{(0,1)}) \). Combining, we get \( \chi(P^{(0,1)}) = \chi(P^{(0,1)})^2 \chi(P^{(\frac{1}{2})}) \), so that either \( \chi(P^{(0,1)}) = 0 \) or else \( \chi(P^{(0,1)}) = 1/\chi(P^{(\frac{1}{2})}) = m^{-1} \).

Putting this differently: If we let \( X \) denote \( PQ \), then there is a nice way to decompose \( X \) into \( m \) copies of \( X \times X \), so any functor from generalized polytopes to real numbers that respects \( + \) and \( \times \) would have to take \( X \) either to 0 or to \( m^{-1} \).

Objects like \( PQ \) can be construed as special cases of \( \sigma \)-polytopes. I define a \( \sigma \)-polytope as a formal disjoint union of finite-dimensional cells, involving only finitely many \( k \)-dimensional cells for any particular \( k \). It may seem that these objects have too little structure — for instance, unlike CW-complexes they carry no information about which cells are parts of the boundary of which higher-dimensional cells — but this extra information is superfluous in the polyhedral category, since
polyhedral maps (unlike continuous maps) need not respect boundary-relationships between cells. Isomorphism classes of σ-polytopes correspond to elements of the semi-ring

\[ \mathbb{N}[[t]]/(t \sim 2t + 1, t^2 \sim 2t^2 + t, t^3 \sim 2t^3 + t^2, \ldots); \]

the elements of this semi-ring are equivalence classes of power series with non-negative integer coefficients, where two such series are equivalent if each can be obtained from the other by means of a finite sequence of moves, each of which replaces a monomial \( t^k \) by a sum \( 2t^k + t^{k-1} \) or vice versa. The geometric significance of the relations \( t^k \sim 2t^k + t^{k-1} \) is simple: every \( k \)-cell can be divided into two \( k \)-cells along with a \( k-1 \)-cell separating them. Two terminating series (i.e., polynomials in \( t \)) are equivalent iff they have the same degree and the same value at \( t = -1 \). Two non-terminating series are equivalent iff they differ by a polynomial that vanishes at \( t = -1 \); in this case, they have the same regularized value at \( t = -1 \) (assuming that they have a regularized value at \( t = -1 \) in the first place, which is not always the case).

It is important to note that the equivalence classes in \( \mathbb{N}[[t]] \) that constitute the elements of our semi-ring are not closed in \( \mathbb{N}[[t]] \) relative to the usual “\( t \)-adic” topology on formal power series in \( t \). Thus, the series

\[ 2 + 6t + 18t^2 + 54t^3 + 162t^4 + \ldots \]

is equivalent to the series

\[ 4 + 12t + 36t^2 + 108t^3 + 162t^4 + \ldots, \]

which is equivalent to the series

\[ 4 + 12t + 36t^2 + 108t^3 + 162t^4 + \ldots, \]

and so on; but none of these series is equivalent to the limit series

\[ 4 + 12t + 36t^2 + 108t^3 + 384t^4 + \ldots, \]

which is in fact double the original series (and has regularized value 1, rather than \( \frac{1}{2} \), at \( t = -1 \)).

It should also be noted that the various power series of the form

\[ (m)^{f_0} \left( \frac{m}{1 - (m^2 - 1)t} \right)^{f_1}, \]

with \( f_0 - f_1 \) fixed (but \( f_0, f_1 \) themselves varying) are typically inequivalent to each other in the semi-ring, even though the different polynomials \( f_0 + f_1t \) are equivalent. Thus when we raise one element of our semi-ring to the power of another, we should not expect to get a single element but rather a set of elements. We may nevertheless hope
that all the elements that we obtain are equivalent in the weaker sense that they have the same regularized value at \( t = -1 \).

As an indication of the compatibility between the ideas sketched in Sections 3 and 4, we note that if \( P \) is a \( \sigma \)-polytope with a generating series \( f(t) \) (relative to one particular decomposition), then there is a natural way to build a \( \sigma \)-polytope \( \left( \begin{array}{c} P \\ 2 \end{array} \right) \) whose elements are unordered pairs of points in \( P \); this \( \sigma \)-polytope acquires a cellular decomposition with generating series

\[
\frac{1}{2} ( [f(t)]^2 - f(t^2) ) + \frac{t}{1-t} (f(t) - f(t^2)) ;
\]

as long as \( f(t) \) has finite regularized value at \( t = +1 \), the above expression has regularized value \( \left( \frac{f(-1)}{2} \right) \) at \( t = -1 \). Thus, for instance, if \( P \) has regularized Euler measure \( \frac{1}{2} \), \( \left( \begin{array}{c} P \\ 2 \end{array} \right) \) will have regularized Euler measure "\( \frac{1}{2} \) choose 2", or \( -\frac{1}{8} \).

5. Broadening the scope.

One direction in which I am currently extending these ideas is providing analogous interpretations for \( \binom{n}{k} \) or \( k^n \) in the case where \( k \), as well as \( n \), is permitted to be negative.

In the case of \( \binom{n}{k} \), a natural approach to take is to define \( \left( \begin{array}{c} P \\ Q \end{array} \right) \) (for \( P, Q \) polytopes) as the set of polyhedral maps from \( P \) to \( Q \), modulo polyhedral bijections of \( Q \) with itself. This is equivalent to the set of polyhedral subsets of \( P \) that are polyhedrally equivalent to \( Q \), i.e., that have the same dimension and Euler measure as \( Q \). As a variant, one may consider the set of all polyhedral subsets of \( P \) that have Euler measure \( k \), for some fixed \( k \) (with no constraint on the dimension of the subset).

In the case of \( k^n \), the road to take is even clearer: one should try to find some natural stratification of the set of polyhedral maps from \( Q \) to \( P \), and then verify that the regularized value of the \( f \)-series at \( t = -1 \) is \( \chi(P) \chi(Q) \).

Another thing to try is to move both \( P \) and \( Q \) beyond the domain of 1-dimensional polytopes. Here we quickly encounter the problem that, although polyhedral dissections of a 1-dimensional polytope can be parametrized polyhedrally (by the locations of the breakpoints), polyhedral dissections of a 2-dimensional polytope cannot be so parametrized. Indeed, to parametrize all the ways of splitting a 2-dimensional polytope into three pieces by cutting it along a line (yielding one piece on each side of the line and one piece on the line itself), we really need to look in the Grassmannian that parametrizes lines in
2-space. No doubt recent theories of Euler measure on Grassmannians will be helpful in this endeavor.

Finally, it would be interesting to try to develop a notion of generalized Euler measure in a setting more central to modern mathematics. Specifically, we could look at the set of continuous polyhedral (i.e., piecewise-linear) maps from one polytope to another, and use the same method of decomposition and regularization to assign this set of maps an Euler measure. Piecewise-linear maps, which can be used to approximate continuous maps arbitrarily closely, so in some sense the set of continuous polyhedral maps might serve as a computational surrogate for the set of all continuous maps. This would give us a way to define an Euler characteristic for the set of continuous mappings from one topological space to another. While there is no inherent virtue in making a mere definition, it seems plausible that the “combinatorial Euler characteristic” arising under this approach might coincide with the “analytic Euler characteristic” obtained from other, more sophisticated approaches, such as Morse theory. Loop spaces are just one example of a setting in which this approach might bear fruit.

6. Conclusion.

There are clearly limits to what one should expect from a theory that purports to “combinatorify” exponentiation. After all,

\[ 2^{2^{-1}} \]

is transcendental, while

\[ (-1)^{1/2} \]

is complex (and double-valued to boot); worse still, once \( i \) gets admitted to one’s domain of discussion, the expression

\[ i^i \]

arises, taking on countably many values. So we should not expect our system to have good closure properties under exponentiation.

On the other hand, it is clear that one can go at least some distance towards the goal of interpreting exponentiation in a quasi-combinatorial way. The main problem with the current state of the theory, in my opinion, is that I cannot neither give a recipe for a canonical decomposition of a \( \sigma \)-polytope \( P^Q \) nor prove that the regularized Euler characteristic is independent of decomposition over a broad class of decompositions. Nevertheless, I have observed that different ways of trying to calculate regularized Euler characteristics of various \( \sigma \)-polytopes lead to the same answer — sometimes for trivial reasons but oftentimes not. In
trying to explain why these different “meaningless” calculations give rise to the same answer, we may be able to build the substratum of meaning on which they rest.

III. Polyhedral sets and combinatorics (1996)

A closed convex polytope \( P \subset \mathbb{R}^n \) is a set that can be written as the intersection of finitely many closed half-spaces. Given \( x \in P \), the local dimension of \( P \) at \( x \) is the maximal \( k \geq 0 \) for which \( \mathbb{R}^n \) contains a \( k \)-dimensional (affine) subspace \( W \) such that \( x \) is in the \( W \)-interior of \( P \cap W \). A \( k \)-face of \( P \) is a connected component of \( \{ x \in P : \text{the local dimension of } P \text{ at } x \text{ is } k \} \). More generally, a pure \( k \)-cell in \( \mathbb{R}^n \) (\( 0 \leq k \leq n \)) is the non-empty intersection of a finite number of open half-spaces within a \( k \)-dimensional (affine) subspace of \( \mathbb{R}^n \). Every \( k \)-face is a pure \( k \)-cell.

Euler-Poincaré Theorem: If \( P \subset \mathbb{R}^n \) is an \( n \)-dimensional non-empty compact convex polytope, \( F_0 - F_1 + F_2 - F_3 + \cdots + (-1)^n F_n = 1 \), where \( F_i \) is the number of \( i \)-faces of \( P \). We can prove this by defining a suitable valuation (additive function) on a large class of subsets of \( \mathbb{R}^n \).

A polyhedral set in \( \mathbb{R}^n \) is (1) a union of finitely many pure cells; or, equivalently, (2) a subset of \( \mathbb{R}^n \) described by a finite Boolean formula involving linear equations and inequalities. (Schanuel calls it a polyhedral set; Morelli calls it a hedral set.)

Hadwiger-Lenz lemma: There exists a function \( \chi(\cdot) \) (Euler measure or combinatorial Euler characteristic) mapping polyhedral sets to integers, such that: (1) \( \chi(A \cup B) = \chi(A) + \chi(B) \) for \( A \cap B = \emptyset \); (2) \( \chi(A) = 1 \) if \( A \) is a non-empty compact convex polytope; (3) \( \chi(A) = (-1)^k \) if \( A \) is a bounded pure \( k \)-cell. (This approach appears earlier in work of Jim Lawrence, Peter McMullen, and Alexander Barvinok; see also the exposition by Grünbaum and Shephard.)

The Euler-Poincaré Theorem is an immediate consequence of the lemma.

The \( \chi(A) \) constructed below is invariant under homeomorphisms, and in the case where \( A \) is a PL-manifold in \( \mathbb{R}^n \), \( \chi(A) \) coincides with the ordinary Euler characteristic; however, \( \chi((0,1)) = -1 \neq +1 = \chi([0,1]) \), so \( \chi \) does not coincide with the standard (homotopy-invariant) Euler characteristic.

Check:

\[
\chi(([0,3] \times [0,3]) \setminus ((1,2) \times (1,2))) = 0.
\]
χ((interior of triangle abc) \cup \{a, b, c\}) = 4.

(Note that the latter set is not locally compact, so ordinary homological approaches to Euler characteristic do not apply.)

Rota and Schanuel’s proof of the Hadwiger-Lenz lemma uses Euler integration: If \( f : \mathbb{R}^n \to \mathbb{Z} \) has the property that \( f^{-1}(k) \) is polyhedral for all \( k \) and empty for all but finitely many \( k \neq 0 \), put \( \int f \, d\chi = \sum_k k\chi(f^{-1}(k)) \). E.g., \( \int 1_A \, d\chi = \chi(A) \) if \( A \) is polyhedral. Less trivially, if \( f : \mathbb{R} \to \mathbb{R} \) with

\[
  f(x) = \begin{cases} 
    0 & \text{if } x < 0 \\
    2 & \text{if } x = 0 \\
    -1 & \text{if } 0 < x < 1 \\
    1 & \text{if } x = 1 \\
    0 & \text{if } x > 1 
  \end{cases}
\]

then

\[
\int f \, d\chi = (2)\chi(\{0\}) + (-1)\chi((0,1)) + (1)\chi(\{1\}) = (2)(1) + (-1)(-1) + (1)(1) = 4.
\]

Strategy of proof: Integrate \( n \)-dimensional cross-sectional Euler measure \( \chi_n \) with respect to \( 1 \)-dimensional Euler measure \( \chi_1 \) to define \( (n+1) \)-dimensional Euler measure \( \chi_{n+1} \). E.g.: If \( a, b, c \) are the points \((0,0), (0,1), \) and \((1,0)\) in \( \mathbb{R}^2 \), and \( A = (\text{interior of triangle abc}) \cup \{a, b, c\} \), then we put \( \chi_2(A) = \int f \, d\chi_1 \) where \( f(x) = \chi_1(\{y : (x, y) \in A\}) \) is the function just discussed; hence we get \( \chi_2(A) = 4 \), as before.

Outline of the Rota/Schanuel proof: Step 1: For all \( a \) in \( \mathbb{R} \), define \( \chi_1(\{a\}) = 1 \) and \( \chi_1((a, \infty)) = \chi_1((-\infty, a)) = -1 \), and for all \( a < b \) in \( \mathbb{R} \) define \( \chi_1((a,b)) = -1 \). Extend \( \chi_1 \) by finite additivity to all polyhedral subsets of \( \mathbb{R} \). Step \( n' \) (\( n \geq 1 \)): Define \( \int f \, d\chi_n = \sum_k k\chi_n(f^{-1}(k)) \). Step \( n + 1 \) (\( n \geq 1 \)): Define \( \chi_{n+1}(A) = \int \chi_n(\pi^{-1}(x)) \, d\chi_1 \), where \( \pi(x_1, x_2, \ldots, x_{n+1}) = x_{n+1} \). Then verify that properties (1),(2),(3) hold by induction for all \( n \).

Finite combinatorics is the study of the category of finite sets, with regard to the cardinality functor \( \# : \text{FinSet} \to \mathbb{N} \). (E.g.: \( \#(A \cup B) = \#(A) + \#(B) - \#(A \cap B) \), \( \#(A \times B) = \#(A) \#(B) \), \( \#(A^B) = \#(A)^\#(B) \).)

“Polyhedral combinatorics” is the study of the category of polyhedral sets, with regard to the Euler functor \( \chi : \text{PolySet} \to \mathbb{Z} \). (E.g.: \( \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \), \( \chi(A \times B) = \chi(A) \chi(B) \).)
A polyhedral map \( f : A \to B \) (\( A \subset \mathbb{R}^m, B \subset \mathbb{R}^n \) polyhedral sets) is a function whose graph is a polyhedral set in \( \mathbb{R}^{m+n} \). E.g.: the piecewise-linear discontinuous function \( f : \mathbb{R} \to \mathbb{R} \) introduced earlier, or the function \( g(x) = |x| \), or the function \( 1_A \) for any polyhedral set \( A \).

Just as two bijectively equivalent finite sets have the same cardinality, two polyhedrally equivalent sets have the same Euler characteristic.

A polyhedral permutation is an invertible polyhedral map \( f : A \to A \) all of whose orbits have finite cardinality; we define its trace \( \text{Tr}_A f \) as \( \chi(\text{Fix}_A f) \) where \( \text{Fix}_A f = \{ x \in A : f(x) = x \} \), and its parity \( (-1)^{f} \) as \( (-1)^{\chi(D)} = \prod_{k=1}^{\infty} (-1)^{\chi(O_{2k})} \), where \( D = \{ (x, y) \in A \times A : x < y \text{ but } f(x) > f(y) \} \) under any linear ordering of \( A \) whose graph in \( A \times A \) is polyhedral, and \( O_m \) is any polyhedral set containing one representative from each orbit of size \( m \). (Note: \( (-1)^{f \circ g} = (-1)^{f}(-1)^{g} \).)

Example: Let \( P = \) the union of the three edges of an equilateral triangle, \( G = \) the dihedral group of the triangle = \{id, flip, flip, flip, rot, rot\}. \( \text{Tr}_P(\text{id}) = -3, \text{Tr}_P(\text{flip}) = 1, \text{Tr}_P(\text{rot}) = 0. \)

Conjecture: Let \( P \subset \mathbb{R}^n \) be a union of \( k \)-faces, and let \( G \) be a group of isometries of \( \mathbb{R}^n \) sending \( P \) to \( P \). Define \( \rho : G \to \mathbb{Z} \) by \( \rho(g) = (-1)^{k} \text{Tr}_P g \). Then \( \rho \) is a linear character of \( G \). (Stronger conjecture: This is true for any finite group of polyhedral permutations.) Can we prove this by “finding the \( G \)-module”? [Note: The weaker form of the conjecture was proved independently by Miller Maley, Bruce Sagan, Richard Stanley, John Stembridge, and Dylan Thurston; see the Postscript at the end of these notes for Stembridge’s version.]

Conjecture (duality): Let \( G \) be a group of isometries of \( \mathbb{R}^n \) sending the \( n \)-dimensional compact convex polytope \( P \) to itself. Let \( P_k \) denote the union of the \( k \)-faces of \( P \) (\( 0 \leq k \leq n \)), determining the character \( \rho_k \) as above, and let \( \rho_{-1} \) be the trivial character. Also define characters \( \rho_k^{\circ} \) associated with the polar polytope \( P^\circ \) of \( P \). Then there exists a dimension-preserving involution on the set of characters of irreducible representations of \( G \) whose extension by linearity to an involution on the set of all characters of \( G \) exchanges \( \rho_k \) and \( \rho_{-1-k}^{\circ} \) for all \( -1 \leq k \leq n. \)

For \( P \) a polyhedral set and \( k \geq 0 \), consider the action of the symmetric group \( S_k \) on \( P^k \). A free orbit is one of cardinality \( k! \). Define \( \binom{P}{k} \) as any polyhedral set in \( P^k \) that contains exactly one point in each free orbit and no other points. (Such a polyhedral set exists, and all such sets are polyhedrally isomorphic.) Cf. Morelli’s \( \lambda \)-ring structure on the set of polytopes.
Example: \( P = (0, 1) \subset \mathbb{R} \), \((P^2) = \{(x, y) : 0 < x < y < 1\}\) is a pure 2-cell, \( \chi((P^2)) = +1 = \binom{1}{2} = \chi(P^2) \). Example: \( Q = (0, 1) \cup (2, 3) \),

\[
\binom{Q}{3} = \{ (x, y, z) : 0 < x < y < z < 1 \} \\
\cup \{ (x, y, z) : 0 < x < y < 1, 2 < z < 3 \} \\
\cup \{ (x, y, z) : 0 < x < 1, 2 < y < z < 3 \} \\
\cup \{ (x, y, z) : 2 < x < y < z < 3 \},
\]

\( \chi(\binom{Q}{3}) = -4 = \binom{-2}{3} = \chi(Q) \).

Theorem: For any polyhedral set \( P \), \( \chi((P^k)) = \binom{\chi(P)}{k} \).

Given a graph \( G = (V, E) \) and a polyhedral set \( P \), a \( P \)-coloring of \( G \) is a map \( f : V \to P \) such that \( f(x) \neq f(y) \) for all \( \{ x, y \} \in E \).

Theorem: The Euler characteristic of the set of \( P \)-colorings of \( G \) equals the chromatic polynomial of \( G \) evaluated at \( \chi(P) \). E.g.: If \( P = \mathbb{R} \), with \( \chi(P) = -1 \), \( P \)-colorings of \( G \) can be interpreted directly as points in the complement of the graphical sub-arrangement of the braid arrangement determined by \( G \). Since every component of the complement of this central hyperplane arrangement has \( \chi = (-1)^{\#(V)} \), this is Zaslavsky’s theorem.

Fix a polyhedral set \( P \subset \mathbb{R} \). A finite subset \( S \subset P \) is \textit{fabulous} if for all \( t, t' \in S \cup \{+\infty, -\infty\} \), \( \chi((P \setminus S) \cap (t, t')) \) is even. (Motivation: If \( P = \{1, 2, \ldots, n\} \), a subset of \( P \) is fabulous if its complement in \( P \) can be written as a disjoint union of pairs \( \{k, k + 1\} \), and the number of fabulous subsets is the \( n + 1 \)st Fibonacci number.) E.g.: If \( P = (0, 1) \cup (2, 3) \cup (4, 5) \cup (6, 7) \), the fabulous subsets of \( P \) are \( \emptyset \) and all sets \( \{ x, y \} \) with \( 2 < x < 3, 4 < y < 5 \).

Theorem: If \( \chi(P) = n \), the set of fabulous subsets of \( P \) has Euler characteristic \( (\varphi^{n+1} - \varphi^{-n-1})/\sqrt{5} \) (with \( \varphi = (1 + \sqrt{5})/2 \)). (Generalization to other sequences satisfying linear recurrence relations?)

If \( \Phi \) is a collection of finite subsets of \( \mathbb{R}^n \), let \( \Phi_k = \{ \phi \in \Phi : \#(\phi) = k \} \), where we identify a \( k \)-element subset of \( \mathbb{R}^n \) with a point in \( \mathbb{R}^{kn} \) as before. We define the “Euler series” \( \sum_{k=0}^{\infty} \chi(\Phi_k)t^k \). If this series converges in a neighborhood of \( t = 0 \) so as to give unique analytic continuation in a neighborhood of \( t = 1 \), we call the value at \( t = 1 \) the \textit{(regularized) Euler characteristic} of \( \Phi \).

Example 1: \( \Phi = \) the collection of all finite subsets of \( P \), where \( \chi(P) = n \). Then the Euler series is \( 1 + nt + \binom{n}{2}t^2 + \cdots = (1 + t)^n \to 2^n \) as \( t \to 1 \). (E.g., if \( P = (0, 1) \) with Euler characteristic \(-1 \), our \( \Phi \) has regularized Euler characteristic \(-1^{-1} = 1/2 \).)
More generally, if \( \Phi \) is a “colored” collection of finite subsets of \( \mathbb{R}^n \), where each element of \( \Phi \) has combinatorial as well as geometric data, define \( \Phi_k \) as the union of the \( k \)-sets in \( \Phi \), where \( k \)-sets of distinct combinatorial type are regarded as distinct.

Example 2: \( \Phi = \) the collection of all finite subsets of \( (0,1) \), \( \Phi' = \) the collection of all 2-element subsets \( \{A, B\} \) of \( \Phi \). We can view \( \{A, B\} \) as \( A \cup B \) equipped with a distinguished non-empty subset (the symmetric difference of \( A \) and \( B \)) along with a partition of this set into two subsets \( (A \setminus B) \) and \( (B \setminus A) \). The Euler series is \( -t + 4t^2 - 13t^3 + 40t^4 - \cdots = -t/(1 + t)(1 + 3t) \to -1/8 = (1/2)^2 \). (This generalizes to evaluation of the chromatic polynomial of a graph at any rational number.) Note that this is not the same approach as I used in my memo “Negative and fractional cardinalities via generalized polytopes,” which fails for this case.

Example 3: \( \Phi = \) the collection of all polyhedral subsets of \( (0,1) \). Every polyhedral \( P \subset (0,1) \) determines a finite set in \( (0,1) \), namely its set of break-points (i.e., points of discontinuity of the indicator function of the set); represent \( P \) by its set of break-points, along with combinatorial information concerning what happens at and between break-points and at the left and right ends of \( (0,1) \), vis-a-vis membership in \( P \). The Euler series is \( 2 - 6t + 18t^2 - 54t^3 + \cdots = 2/(1 + 3t) \to 1/2 \). (More generally, if \( P \) is 1-dimensional, the collection of polyhedral subsets of \( P \) has regularized Euler characteristic \( 2\chi(P) \).)

Example 4: \( \Phi = \) the collection of all polyhedral subsets of \( [0,1) \) of Euler characteristic 0. The coefficient of \( (-t)^n \) in the Euler series is the central coefficient of \( (x+1+x^{-1})^n \), so the Euler series is \( 1-t+3t^2-7t^3+19t^4-\cdots = \frac{1}{\sqrt{1+2t-3t^2}} \), which blows up near \( t = 1 \). (Generalization?)

Let \( \text{Map}(P,Q) \) be the set of polyhedral maps \( P \to Q \). When \( P \) is finite, \( \chi(\text{Map}(P,Q)) = \chi(Q)\chi(P) \). When \( P \) is 1-dimensional, one can still stratify \( \text{Map}(P,Q) \) by number-of-break-points, and if moreover \( Q \) is finite, \( \chi(\text{Map}(P,Q)) = \chi(Q)^{\chi(P)} \). But what about \( \chi(\text{Map}(P,Q)) \) when \( P \) and \( Q \) are genuinely 1-dimensional (e.g., \( P = Q = (0,1) \))? What about \( \chi(\text{Map}(P,Q)) \) when \( P \) is \( \geq 2 \)-dimensional?

What is the right framework for looking at these infinite-dimensional polyhedral sets? (Homology theory for non-locally-finite spaces?)

What are the connections between the present theory and the algebraic enumerative approach to Euler characteristic (counting points on varieties over finite fields)?
Postscript

John Stembridge writes:

I have a proof of Jim’s conjecture about group actions on polyhedral sets.
More specifically, let $G$ be a finite group of isometries of $\mathbb{R}^n$ that permutes a disjoint set of $k$-cells. (A $k$-cell is by definition a $k$-dimensional intersection of open half spaces.) For $g \in G$, define $f(g)$ = the Euler-measure of the polyhedral set that is fixed pointwise by $g$.

(Recall that the Euler measure of a $j$-cell is $(-1)^j$.)

CLAIM: $(-1)^k \ast f$ is the character of a representation of $G$.

BTW: We must insist that the cells are disjoint, or there exist counterexamples.

PROOF. Wlog, we can assume that there is just one orbit of $k$-cells. Fix a $k$-cell $C$, and let $H$ be the subgroup of $G$ that preserves $C$. Each $g$ in $H$ permutes the vertices of (the closure of) $C$, so in particular $H$ fixes the centroid of $C$’s vertices. Taking this centroid as our origin, let $V_C$ denote the vector space spanned by $C$. $H$ acts as a group of isometries of $V_C$. If some $g$ in $H$ has a $j$-dimensional space of fixed points, then the portion of $C$ that is fixed pointwise by $g$ is a $j$-cell, and hence has Euler measure $(-1)^j$. On the other hand, the determinant of a (real) orthogonal transformation of a $k$-dimensional space has determinant $(-1)^l$, where $l$ denotes the multiplicity of the eigenvalue $-1$. Since the complex eigenvalues occur in conjugate pairs, we have $l + j \equiv k \mod 2$, so $\det(g) = (-1)^{(k-j)}$. Using standard rules for inducing representations, it follows that $(-1)^k \ast f$ is the character obtained by inducing $\det$ from $H$ to $G$. QED.

Afterword (2002)

One direction that might be interesting to explore is the study of “polyhedral vector spaces”, as a generalization of the notion of finite-dimensional vector spaces. An example of such a vector space would be the space of polyhedral real-valued functions on the polyhedral set $A$. Such spaces have bases, and in every case I’ve looked at, there is a natural way to view the set of basis vectors as a polyhedral set, and what is more, the Euler measure of the basis turns out to be equal to the Euler measure of $A$. Is there a general theorem here?

Secondly, as an historical aside, I mention that the surprising formula $\chi(A^{-1}) = 0$ that holds when $A$ is a 1-dimensional polyhedral set satisfying $\chi(A) = 0$ and $A^{-1}$ is interpreted as the set of maps from
an open interval into $A$, and which is proved in the companion article “Exponentiation and Euler measure,” is reminiscent of an interesting “mistake” made by Brahmagupta of Multan in his 6th century treatise Brahmasphutasiddantha. In that work, Brahmagupta stated rules for manipulating zero in combination with ordinary numbers: $A + 0 = A$, $A - 0 = A$, $Ax0 = 0$, and $A/0 = 0$. Of course the last of these is wrong under the usual understanding of division. But it is amusing to find a context in which Brahmagupta’s postulate makes sense and is correct.

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