Research on Dissections of a Net of a Cube into Nets of Cubes

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SUMMARY A rep-cube is a polyomino that is a net of a cube, and it can be divided into some polyominoes such that each of them can be folded into a cube. This notion was invented in 2017, which is inspired by the notions of polyomino and rep-tile, which were introduced by Solomon W. Golomb. A rep-cube is called regular if it can be divided into the nets of the same area. A regular rep-cube is of order $k$ if it is divided into $k$ nets. Moreover, it is called uniform if it can be divided into the congruent nets. In this paper, we focus on these special rep-cubes and solve several open problems.

key words: computational origami, polyomino, rep-cube, rep-tile

1. Introduction

A polyomino is a “simply connected” set of unit squares introduced by Solomon W. Golomb in 1954 [7]. Since then, sets of polyomino pieces have been playing an important role in recreational mathematics (see, e.g., [5]). In 1962, Golomb also proposed an interesting notion called rep-tile: a polygon is a rep-tile of order $k$ if it can be divided into $k$ replicas congruent to one another and similar to the original (see [6, Chap 19]). From these notions, Abel et al. introduced a new notion [1]; a polyomino is said to be a rep-cube of order $k$ if it is a net of a cube (or, it can fold into a cube), and it can be divided into $k$ polyominoes of which each can fold into a cube. If all $k$ polyominoes have the same size, we call the original polyomino a regular rep-cube of order $k$. Moreover, a regular rep-cube is a uniform rep-cube of order $k$ when all $k$ polyominoes are congruent. Simple examples of a regular rep-cube and a uniform rep-cube are shown in Fig. 1 (a) and (b), respectively. We note that crease lines are not necessarily along the edges of the polyomino as shown in the figure.

In [1], Abel et al. showed concrete regular rep-cubes of order $k$ for $k = 2, 4, 5, 8, 9, 36, 50, 64$. Later, in [16], Xu et al. also gave regular rep-cubes of order $k = 16, 18, 25$. In both papers, they showed some ways of construction of regular rep-cubes of order $k$ for infinitely many integers $k$. In these papers, the following two sets play important roles;

$$S = \{k \mid a^2 + b^2 = k \text{ for non-negative two integers } a, b\}$$

$$\bar{S} = \mathbb{Z} \setminus S,$$

where $\mathbb{Z}$ is the set of non-negative integers, namely, $\mathbb{Z} = \{1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, \ldots\}$ and $\bar{S} = \{3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, \ldots\}$. We can observe that all the integers where there exists a regular rep-cube of order $k$ in $S$. We note that both of $S$ and $\bar{S}$ are infinite sets by Dirichlet’s theorem on arithmetic progressions.

On the other hand, in [16], they showed that there are no regular rep-cube of order 3. They proved that if $k \in \bar{S}$, there does not exist a regular rep-cube of area $6k$ of order $k$. Intuitively speaking, they showed that $k$ copies of one net in Fig. 2 cannot cover a cube of area $6k$ if $k$ is in $\bar{S}$. However, they could not prove that it holds for general regular rep-cubes of order $k$ in $S$. We first solve this open problem. That is, we prove that there does not exist a regular rep-cube of order $k$ if $k$ is in $\bar{S}$. In other words, any set of $k$ (refined) polyominoes of the same area cannot cover a cube of area $6k$ if $k$ is in $\bar{S}$. (In [16], this claim was proved only for $k = 3$.) Oppositely, we conjecture that there exists a regular rep-cube of order $k$ if $k \in S$; however, we have to construct
one by one so far. In this paper, we give regular rep-cubes of order \( k = 10, 13, 17, 20 \), which did not appear in \([1], [16]\).

Next we focus on uniform rep-cubes of order \( k \), which consist of \( k \) copies of congruent nets. As a net of a cube, the eleven nets shown in Fig. 2 are quite popular since they are obtained by cutting along edges of a cube. (In the context of unfolding, they are sometimes called edge-unfolding of a cube.) Moreover, through the enumeration of regular rep-cubes of order \( k = 2 \) and \( k = 4 \) in \([16]\), we can observe that nine of eleven nets form uniform rep-cubes. For example, two copies of a net of T shape shown in Fig. 1 (b) cover a cube. In this context, it is natural to ask how many copies we need to cover a cube by each of eleven nets. Especially, can the last remaining two nets, indicated (a) and (b) in Fig. 2, form uniform rep-cubes? Our second results state that for both of two nets, we can cover a cube by eight copies of them, and we cannot cover by five copies as shown in Fig. 2.

Lastly, we consider a new notion of a universal rep-cube that contains all of eleven nets in Fig. 2. This notion itself first proposed in \([1]\) as an example of a regular rep-cube of order \( k = 50 \) with no special name. In \([16]\), the authors showed another one with \( k \) the minimum number of the universal rep-cube of order \( k \) greater than or equal to eleven, and it was first proposed in \([1]\) as an example of a regular rep-cube that contains all of eleven nets in Fig. 2. This notion and \( k \) should be in \( S \) and \( \bar{S} \) respectively. Let the length of an edge of \( Q \) be \( \ell \) and \( \ell \) be the length of an edge of \( P \). Then \( P_i \) is a \( 6\ell^2 \)-omino and \( \bar{P} \) is a \( 6k\ell^2 \)-omino. Here, we note that while \( 6k\ell^2 \) is an integer, \( \ell \) is not necessarily an integer.

Next we focus on uniform rep-cubes of order \( k \) with \( k \) uniform rep-cubes or not. Let name the nets Fig. 2 (a) and Fig. 2 (b) \( S \) and \( \bar{S} \) and \( P \) and \( P_i \) with no other factors. We can regard the first case (\( k \) is a prime) as the special case of the second case with \( p_i = k \) and \( d_i = 1 \) with no other factors. Thus we focus on the second case.

In the previous work, there exist uniform rep-cubes of order \( k \) for each \( k = 2, 4, 9 \) in \([1]\). In \([16]\), it is shown that how to construct infinitely many uniform rep-cubes recursively. Summarizing known uniform rep-cubes in Fig. 2, it is natural to ask if remaining two of eleven nets can form uniform rep-cubes or not. Let name the nets Fig. 2 (a) and Fig. 2 (b) \( P_w \) and \( P_z \) (from their shapes), respectively. For these two nets, we show the following theorem.

**Theorem 3:** Using \( P_w \) and \( P_z \), we can construct uniform rep-cubes of order \( k \) for \( k = 8 \). Moreover, we cannot construct uniform rep-cubes of order \( k \) with \( k = 2, 4, 5 \).

We have the following corollary.

**Corollary 4:** For each one of eleven nets in Fig. 2, \( k \) copies of one can cover a cube for some \( k = 2, 4, 8 \).

By enumerations in \([16]\) for \( k = 2, 4 \), Theorem 3 and Corollary 4 hold except \( P_w \) and \( P_z \). Thus we focus on \( P_w \) and \( P_z \).
Lemma 5: There exist uniform rep-cubes of order 8 by \( P_w \) and \( P_z \).

Proof. We prove the claim by construction. See Fig. 3.

Next we show the following lemma.

Lemma 6: There does not exist a uniform rep-cube of order 5 by \( P_w \) or \( P_z \).

Proof. The proof is done by case analysis. We first focus on \( P_w \). If five copies of \( P_w \) form a uniform rep-cube of order 5, the resulting cube \( Q \) with an edge of length \( \sqrt{5} \) is depicted in Fig. 4 (or its mirror image), since we have 5 nets each of which has 6 unit areas. Each face of \( Q \) is a square of area 5. On a rep-cube, each vertex of \( Q \) should coincide with a vertex of unit square of a net. (Otherwise, a vertex of \( Q \) will make a non-flat point inside of a net.) Therefore, on a surface of \( Q \), one unit square comes to the center, and the other four squares are surrounding it as shown in gray in Fig. 4. We call the square in the center of a face of \( Q \) a central square. That is, the cube \( Q \) has six central squares at each of six faces as shown in gray in Fig. 4. Then it is not difficult to see that \( P_w \) can put on \( Q \) with respect to the central squares in 5 different ways as shown in Fig. 5. To derive contradictions, we assume that five copies of \( P_w \) can cover on \( Q \) without any overlapping and any hole. We here note that \( Q \) has six central squares, and five copies of \( P_w \) cover them. Now each \( P_w \) can cover one or two central squares.

Therefore, one copy of \( P_w \) covers two central squares, and four copies of \( P_w \) cover one central squares. (We have no other combination.) Then we have two cases. The first case is that one copy of \( P_w \) is in the case Fig. 5 (b) and the other four copies are in the cases Fig. 5 (a)(c)(d), and the second case is that one copy of \( P_w \) is in the case Fig. 5 (e) and the others are in the cases Fig. 5 (a)(c)(d).

We first consider the first case; that is, a copy \( P_1 \) of \( P_w \) is in the case Fig. 5 (b). Without loss of generality, we assume that \( P_1 \) covers the central squares \( A \) and \( B \). Then, beside \( P_1 \), we have two central squares \( C \) and \( C' \) to be covered, where \( C' \) is the central square opposite with \( C \) in Fig. 4. We now focus on \( C \) and consider how we can cover it by a copy \( P_2 \) of \( P_w \). Then \( P_2 \) is in the case Fig. 5 (a)(c) or (d). For each of them, we have four orientations of \( P_2 \) on \( Q \) to cover \( C \). Then, in most cases, (1) \( P_1 \) and \( P_2 \) surround a unit square or a small rectangle of size \( 1 \times 2 \) or (2) \( P_2 \) overlaps with \( P_1 \). The only exception occurs one orientation in the case Fig. 5 (a). Thus we have only one way of attaching \( P_2 \) beside \( P_1 \) as shown in Fig. 6. Now we consider the next copy \( P_3 \) of \( P_w \) which covers the other neighbor central square \( C' \) opposite with \( C \). Then we can use the same argument since \( C' \) is symmetric to \( C \). Therefore, one copy of \( P_w \) covers two central squares, and four copies of \( P_w \) cover one central squares. (We have no other combination.) Then we have two cases. The first case is that one copy of \( P_w \) is in the case Fig. 5 (b) and the other four copies are in the cases Fig. 5 (a)(c)(d), and the second case is that one copy of \( P_w \) is in the case Fig. 5 (e) and the others are in the cases Fig. 5 (a)(c)(d).

We first consider the first case; that is, a copy \( P_1 \) of \( P_w \) is in the case Fig. 5 (b). Without loss of generality, we assume that \( P_1 \) covers the central squares \( A \) and \( B \). Then, beside \( P_1 \), we have two central squares \( C \) and \( C' \) to be covered, where \( C' \) is the central square opposite with \( C \) in Fig. 4. We now focus on \( C \) and consider how we can cover it by a copy \( P_2 \) of \( P_w \). Then \( P_2 \) is in the case Fig. 5 (a)(c) or (d). For each of them, we have four orientations of \( P_2 \) on \( Q \) to cover \( C \). Then, in most cases, (1) \( P_1 \) and \( P_2 \) surround a unit square or a small rectangle of size \( 1 \times 2 \) or (2) \( P_2 \) overlaps with \( P_1 \). The only exception occurs one orientation in the case Fig. 5 (a). Thus we have only one way of attaching \( P_2 \) beside \( P_1 \) as shown in Fig. 6. Now we consider the next copy \( P_3 \) of \( P_w \) which covers the other neighbor central square \( C' \) opposite with \( C \). Then we can use the same argument since \( C' \) is symmetric to \( C \). Therefore, one copy of \( P_w \) covers two central squares, and four copies of \( P_w \) cover one central squares. (We have no other combination.) Then we have two cases. The first case is that one copy of \( P_w \) is in the case Fig. 5 (b) and the other four copies are in the cases Fig. 5 (a)(c)(d), and the second case is that one copy of \( P_w \) is in the case Fig. 5 (e) and the others are in the cases Fig. 5 (a)(c)(d).

Next, we consider the second case; that is, a copy \( P_1 \) of \( P_w \) is in the case Fig. 5 (e). Then, beside \( P_1 \), we have two central squares \( C \) and \( C' \) to be covered again. Moreover, we can observe that Fig. 5 (b) and Fig. 5 (e) have almost the same shape except one white square. Thus we can use almost the same argument of the first case. We consider all ways of attaching of two neighbor central squares \( C \) and \( C' \) of \( P_1 \), and we have the same conclusion; we have an overlap or a hole when we attach \( P_1, P_2, \) and \( P_3 \) to cover them.

Thus, there does not exist a uniform rep-cube of order 5 by \( P_w \).

For the \( P_z \), we can have the similar case analysis, and confirm that there does not exist a uniform rep-cube of order 5 by \( P_z \). We note that the number of cases for \( P_z \) increases comparing to \( P_w \) because \( P_z \) can attach on \( Q \) without covering any central square. In such a case, two or three copies of \( P_z \) can cover two central squares simultaneously. These additional cases are also easy to check and the arguments are
essentially the same: we always have an overlap or a hole before completing the cover in every case.

By Lemmas 5 and 6 with known enumeration in [16], Theorem 3 immediately follows.

4. Universal Rep-Cubes

We say that a regular rep-cube of order \( k \) is universal if it can be divided into \( k \) polyominoes in Fig. 2 such that the set contains all of eleven nets. This notion was introduced in [1] without a name, and it was shown for \( k = 50 \). Later, it was improved to \( k = 25 \) as shown in [16]. It is a natural question for finding the minimum \( k \) such that a universal rep-cube exists. By definition, it is clear that \( k \geq 11 \). In this section, we prove that \( k = 13 \) by construction.

**Theorem 7:** The minimum number \( k \) such that there exists a universal rep-cube is \( k = 13 \).

**Proof.** By Theorem 1 and known result in [16], we can observe that \( k = 13, 16, 17, 18, 20, \) or 25. Since there is a universal rep-cube of order 13 as shown in Fig. 7, we have the claim.

In this context, Maekawa proposed an interesting puzzle for this problem [9]: We consider two polygons are different if they are mirror images with each other. The set of eleven nets of a unit cube contains two mirror symmetric shapes (T-shape and +-shape, which are rightmost in Fig. 2). Here, let \( S \) be the set of the nets of a unit cube, where mirror images are regarded as different with each other. Then \( S \) consists of 20 nets, and hence the nets in \( S \) are of area 120 in total. The Maekawa’s puzzle asks if you can make a rep-cube of area 120 or a cube of size \( 2 \sqrt{5} \times 2 \sqrt{5} \times 2 \sqrt{5} \) from this set \( S \) without flipping each net. We give an affirmative answer to this problem by construction.

**Theorem 8:** There exists a universal rep-cube of order \( k = 20 \) such that every different net (with respect to flip) appears exactly once.

**Proof.** A solution is shown in Fig. 8.

The reader may wonder how we can find them. In

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**Fig. 7** A minimum universal rep-cube of order \( k = 13 \).

**Fig. 8** A solution of Maekawa’s puzzle.
fact, one of the authors found the pattern of a universal rep-cube of order $k = 25$ in [16] by hand and it was quite tough. We found the patterns in Fig. 7 and Fig. 8 by using SCIP [17], which is one of the fastest non-commercial solvers for mixed integer programming. We give the formulation of our problem for solving by SCIP.

4.1 Integer Programming Formulation

We formulate the problem in terms of a 0-1 integer programming problem. Although we found the patterns in Fig. 3 by hand and proved Lemma 6 by case analysis, we use the case $k = 5$ for explanation.

We first number all unit squares on the target cube $Q$ (see Fig. 9 for a cube of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$; the ordering is arbitrary). We name each square $i$ for each $i = 1, 2, \ldots, 30$ for reference. Then, for each placement of $P_w$, we use a 0-1 integer variable. In Fig. 9, a placement of $P_w$ is indicated in gray. For this position, we define a 0-1 integer variable $P_w(3, 5, 8, 13, 17, 21)$. For each possible placement, we prepare one 0-1 integer variable $P_w(i_1, i_2, i_3, i_4, i_5, i_6)$, where $i_j$ indicates the name of the corresponding unit square. For each unit square $i$, there are four copies of $P_w$ that contain $i$ at the end of $P_w$. We have to consider the mirror image of $P_w$ in this case. We denote it by $P_w'(i_1, i_2, i_3, i_4, i_5, i_6)$.

However, we have duplicates; for example, two variables $P_w(3, 5, 8, 13, 17, 21)$ and $P_w'(21, 17, 13, 8, 5, 3)$ are essentially the same. Thus, we define the standard form that $i_1 < i_6$ for $P_w(i_1, i_2, i_3, i_4, i_5, i_6)$ and we only use the variables of the standard form. Therefore, we have $30 \times 4 \times 2/2 = 120$ 0-1 integer variables for this case.

Now we consider the constraints. For each square $i$, it should be covered by exactly once by a copy of $P_w$. In order to represent it, we have the following constraint for each $i$: $\sum_{i \in P_w} P_w(i_1, i_2, i_3, i_4, i_5, i_6) + \sum_{i \in P_w'} P_w'(i_1, i_2, i_3, i_4, i_5, i_6) = 1$. In total, we have 30 constraints.

The objective function is simply given by minimize $\sum_{i \in P_w} P_w(i_1, i_2, i_3, i_4, i_5, i_6) + \sum_{i \in P_w'} P_w'(i_1, i_2, i_3, i_4, i_5, i_6)$. The solution should be 5 in this case since we use five copies of $P_w$ or $P_w'$ to cover the cube $Q$.

In fact, the proof of Lemma 6 was double-checked by SCIP, and it confirmed that there is no solution for this case in 0.00 second. (We use SCIP version 7.0.0 on a laptop PC (AMD Ryzen 7, 2.30 GHz, 16 GB RAM, 64 bit Windows.).)

For finding the pattern in Fig. 7, we prepare 4960 variables for representing positions and 78 constraints for unit squares. We add eleven constraints so that each of eleven net appears at least once. In this case, the pattern in Fig. 7 was found in 17.00 seconds.

For finding the pattern in Fig. 8, we prepare 7680 variables and 140 constraints. Among them, 120 constraints are for unit squares and additional 20 constraints represent that each of 20 nets appears exactly once. SCIP found the pattern in Fig. 8 in 982.00 seconds.

5. Concluding Remarks

In this paper, we investigated uniform rep-cubes and universal rep-cubes. In general, we characterized the numbers that a regular rep-cube of order $k$ can exist if $k$ is in $S$, where $S = \{1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, \ldots\}$ and $\bar{S} = \{3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, \ldots\}$. Precisely, we can say that if $k$ is in $S$, we cannot find a regular rep-cube of order $k$. Even if $k$ is in $\bar{S}$, we have no idea whether it exists or not without explicit construction. In [1], [16], they explicitly gave a regular rep-cube of order $k$ for $k = (1), 2, 4, 5, 8, 9, 16, 18, 25, 36, 50, 64$. In this paper, we gave $k = 13$ (Fig. 7) and $k = 20$ (Fig. 8).

For $k = 10$, we found by hand as shown in Fig. 10. On the other hand, for $k = 17$, we used the same way for finding a universal rep-cube of order 13 in Fig. 7. In this case, we
have 6528 0-1 integer variables with 124 constraints, and SCIP found the solution shown in Fig. 11 in 11.00 seconds. In summary, we found regular rep-cubes of order $k$ with all possible $k \in S$ with $k \leq 25$. It seems that there exists a regular rep-cube of order $k$ for any $k \in S$. That is an open problem.

Acknowledgements

A part of this research is supported by JSPS KAKENHI Grant Number 20K11673, 20H05964, 18H04091, and 17H06287.

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