Supersymmetric $\mathcal{N} = 1$ Spin(10) Gauge Theory with Two Spinors via $a$-Maximization

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We give a detailed analysis of the superconformal fixed points of four-dimensional $\mathcal{N} = 1$ supersymmetric Spin(10) gauge theory with two spinors and vectors by using $a$-maximization procedure.
1. Introduction

In the previous paper [1], we studied four-dimensional $\mathcal{N} = 1$ supersymmetric Spin(10) gauge theory with a single chiral superfield $\Psi$ in the spinor representation and $N_Q$ chiral superfields $Q^i$ ($i = 1, \cdots, N_Q$) in the vector representation and with no superpotential at the superconformal infrared (IR) fixed point. This theory is believed to have a non-trivial IR fixed point for $7 \leq N_Q \leq 21$, where the dual description is available [2,3].

At the IR fixed points, since the conformal dimension $D(\mathcal{O})$ of a gauge invariant chiral primary operator $\mathcal{O}$ can be determined by the superconformal $U(1)_R$ charge $R(\mathcal{O})$ [4] as

$$D(\mathcal{O}) = \frac{3}{2} R(\mathcal{O}), \quad (1.1)$$

the $U(1)_R$ symmetry in the superconformal algebra plays an important role.

The unitarity requires the conformal dimension of a gauge invariant Lorentz scalar to satisfy $D(\mathcal{O}) \geq 1$, where the equality is satisfied if and only if $\mathcal{O}$ is free [5]. With (1.1),

$$R(\mathcal{O}) \geq \frac{2}{3}. \quad (1.2)$$

However, a gauge invariant chiral primary operator sometimes appears to violate the inequality (1.2), when we na"ively assume that the global symmetry at the IR fixed point is the same as that in the ultraviolet (UV) region. It has been argued in [4,8] that the operator $\mathcal{O}$ decouples from the remaining interacting system to become free at the IR fixed point, where a new global $U(1)$ symmetry which transforms only $\mathcal{O}$ is enhanced and the real $U(1)_R$ charge of $\mathcal{O}$ becomes $2/3$.

The superconformal $U(1)_R$ symmetry can be expressed as a linear combination of anomaly-free $U(1)$ symmetries as

$$U(1)_R = U(1)_\lambda + \sum_i x_i U(1)_i, \quad (1.3)$$

where global $U(1)$ symmetries under which the gaugino has no charge are denoted by $U(1)_i$ ($i = 1, 2, \cdots$) and an anomaly-free $U(1)$ symmetry which transforms the gaugino with charge 1 by $U(1)_\lambda$. In order to determine the superconformal $U(1)_R$ symmetry, we have to determine the coefficients $x_i$ in (1.3). In fact, we may use $a$-maximization [9] for this purpose. Following this method, we regard $x_i$ in (1.3) as variables to be determined and construct the trial $a$-function

$$a_0(x_1, x_2, \cdots) = 3 \text{Tr} R^3 - \text{Tr} R. \quad (1.4)$$

\footnote{We omit the overall factor $3/32$ of the trial $a$-function in this paper, which does not affect the calculation of the $U(1)_R$ charges.}
Each term in the right hand side of (1.4) represents the ’t Hooft anomaly [10], where the charge \( R \) is the \( U(1)_R \) charge given in terms of \( x_i \) in (1.3), but they are not necessarily the superconformal \( U(1)_R \) charges at the IR fixed point. If there are no accidental symmetries at the IR fixed point, the ’t Hooft anomalies can be evaluated in the UV by using the ’t Hooft anomaly matching condition for asymptotically-free theories. Then, \( a \)-maximization tells us that the local maximum of the function (1.4) gives \( x_i \) for the superconformal \( U(1)_R \) symmetry in (1.3).

However, as mentioned above, the function (1.4) does not make sense in the range of \( x_i \) where gauge invariant chiral primary operators seem to violate the unitarity bounds (1.2). It was proposed in the paper [7] that, in the range where operators \( O_i \) seem to violate the unitarity bounds (1.2), the trial \( a \)-function should be modified into

\[
a(x_1, x_2, \cdots) = a_0 + \sum_i \left[ -a_{O_i} \left( R(O_i) \right) + a_{O_i} \left( 2/3 \right) \right].
\]

The function \( a_{O_i} \) represents the contribution from the operator \( O_i \) to the trial \( a \)-function and can be evaluated as

\[
a_{O_i} \left( R(O_i) \right) = d_{O_i} \left[ 3 \left( R(O_i) - 1 \right)^3 - (R(O_i) - 1) \right],
\]

where \( d_{O_i} \) is the number of the components of the operator \( O_i \), and \( R(O_i) \) is the \( U(1)_R \) charge of \( O_i \), as given in (1.3). The term \( a_{O_i} \left( 2/3 \right) \) is obtained by substituting the value \( R(O_i) = 2/3 \) of free fields into (1.6) to give \( 2d_{O_i}/9 \). The prescription (1.3) can be interpreted as subtracting the contribution which is evaluated under the assumption that the operator \( O_i \) is interacting and adding the contribution of the operators as free fields. Thus, by dividing the range of \( x_i \) according to which operators hit the unitarity bounds and by modifying the trial \( a \)-function as (1.3) for each range, we obtain the trial \( a \)-function in the whole range of the variables \( x_i \). The superconformal \( U(1)_R \) symmetry could be identified by the local maximum of this function.

By using the method discussed above, we showed in the previous paper [1] that the meson operator \( M^{ij} = Q^i Q^j \) hits the unitarity bound and becomes free for \( N_Q = 7, 8, 9 \). We also analyzed the IR fixed point by using the electric-magnetic duality and found that the decoupling of the meson operator can be seen more clearly in the magnetic theory. In the magnetic theory, since the meson operator is described by elementary fields, we do not need the prescription (1.3). We thus proved the validity of the prescription (1.5) in the theory.
The magnetic theory is $SU(N_Q-5)$ gauge theory with $N_Q$ antifundamentals $\bar{q}_i$, a single fundamental $q$, a symmetric tensor $s$, and singlets $M^{ij}$ and $Y^i$, and its superpotential is given by

\[ W_{mag} = M^{ij}\bar{q}_i s\bar{q}_j + Y^i q\bar{q}_i + \text{det}s. \tag{1.7} \]

where $M^{ij}$ and $Y^i$ correspond to the gauge invariant operators $M^{ij} = Q^i Q^j$ and $Y^i = Q^i \Psi^2$ in the electric theory, respectively. When $M^{ij}$ hits the unitarity bound, it decouples from the interacting system, and thus, the interaction $M^{ij}\bar{q}_i s\bar{q}_j$ in (1.7) becomes irrelevant at the IR fixed point. This can be checked by evaluating the $U(1)_R$ charge of this term. Thus, we may identify the remaining interacting system with the theory without the term $M^{ij}\bar{q}_i s\bar{q}_j$ in the superpotential so that we can construct the trial $a$-function of this interacting system together with the free meson without the prescription (1.5), but the resulting function is actually identical to (1.5).

We further discussed that, since the interaction $M^{ij}\bar{q}_i s\bar{q}_j$ in (1.7) vanishes at the IR fixed point, we do not have the $F$-term condition for $M^{ij}$, and new massless degrees of freedom corresponding to $\bar{q}_i s\bar{q}_j$ appear there. The dual of the magnetic theory without the interaction term $M^{ij}\bar{q}_i s\bar{q}_j$ is given by the original electric theory but with the superpotential

\[ W = N_{ij} Q^i Q^j, \tag{1.8} \]

where $N_{ij}$ are additional singlets and correspond to $\bar{q}_i s\bar{q}_j$. We found that the IR fixed point of the original theory is identical to that of this theory. This renormalization group flow can be seen in the original electric theory by introducing auxiliary fields $M^{ij}$ and the Lagrange multipliers $N_{ij}$ to give the superpotential

\[ W = N_{ij} (Q^i Q^j - M^{ij}). \]

We can see that it is the same theory as the original one by integrating out $M^{ij}$ and $N_{ij}$. The equations of motion give the constraints

\[ M^{ij} = Q^i Q^j, \quad N_{ij} = 0. \tag{1.9} \]

When $M^{ij}$ hits the unitarity bound, the interaction $N_{ij} M^{ij}$ becomes irrelevant, to give rise to the superpotential (1.8) at the IR fixed point, where the constraints (1.9) does not exist. In this way, we find that $a$-maximization and the electric-magnetic duality reveal the rich dynamics at the IR fixed point.
In this paper, we extend the analysis to the theory with two spinors and $N_Q$ vectors and show that the meson operator $M^{ij} = Q^i Q^j$ decouples from the interacting system to become free for $N_Q = 6, 7$.

This paper is organized as follows: In section 2, we briefly review the electric-magnetic duality in the theory with two spinors, especially about the matching of gauge invariant operators. In section 3, we study which operators become free by using $a$-maximization for both electric and magnetic theory. Section 4 is devoted to summary and discussion. In the appendices, we discuss the gauge invariant operators in both the electric and the magnetic theory.

2. The Electric-Magnetic Duality

We study four-dimensional $\mathcal{N} = 1$ supersymmetric $Spin(10)$ gauge theory with two chiral superfields $\Psi_I$ ($I = 1, 2$) in the spinor representation and $N_Q$ chiral superfields $Q^i$ ($i = 1, \cdots, N_Q$) in the vector representation and with no superpotential. From the 1-loop beta function, we find that it is asymptotically free for $N_Q \leq 19$. It is believed that the theory has a non-trivial superconformal IR fixed point for $6 \leq N_Q \leq 19$, where the magnetic dual description exists \cite{11}.

This theory has the anomaly-free global symmetry $SU(N_Q) \times SU(2) \times U(1)_F \times U(1)_\lambda$, and the fields $Q^i$ and $\Psi_I$ have charges $(N_Q, 1, -4, 1)$ and $(1, 2, N_Q, -1)$, respectively, under the symmetry. Here, $U(1)_F$ is a global symmetry under which the gaugino have no charge, while $U(1)_\lambda$ transforms it with charge 1. If there are no accidental symmetries at the IR fixed point, the $U(1)_R$ symmetry in the superconformal algebra should be given as a linear combination of these $U(1)$ symmetries as

$$U(1)_R = xU(1)_F + U(1)_\lambda$$  \hspace{1cm} (2.1)

with some real number $x$. Thus, the $U(1)_R$ charge of the matter fields can be expressed as

$$R(Q) = -4x + 1, \quad R(\Psi) = N_Q x - 1.$$  \hspace{1cm} (2.2)

We will determine the value of $x$ in the next section by using $a$-maximization.

As explained in the introduction, in order to construct the trial $a$-function in the whole range of $x$, we need to know gauge invariant chiral primary operators at the IR fixed
point. As discussed in appendix A, the gauge invariant generators of the classical chiral ring of this theory are given by

\[ M^{ij} = Q^{ai} Q^{aj}, \]

\[ Y_X^i = \Psi_T^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^a \Psi_J Q^{ai}, \]

\[ C^{a_1 \cdots a_3} = \Psi_I^T C(\sigma_2)^{IJ} \Gamma^a \Psi_J Q^{a_1 i_1} \cdots Q^{a_3 i_3}, \]

\[ B_X^{i_1 \cdots i_5} = \Psi_I^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^a \Psi_J Q^{a_1 i_1} \cdots Q^{a_5 i_5}, \]

\[ F^{i_1 \cdots i_7} = \Psi_I^T C(\sigma_2)^{IJ} \Gamma^a \Psi_J Q^{a_1 i_1} \cdots Q^{a_7 i_7}, \]

\[ E_X^{i_1 \cdots i_9} = \Psi_I^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^a \Psi_J Q^{a_1 i_1} \cdots Q^{a_9 i_9}, \]

\[ G = \Psi_I^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^a \Psi_J T_K C(\sigma_2 \sigma_X)^{KL} \Gamma^a \Psi_L, \]

\[ H^{i_1 \cdots i_4} = \Psi_I^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^a \Psi_J T_K C(\sigma_2 \sigma_X)^{KL} \Gamma^a \Psi_L Q^{a_1 i_1} \cdots Q^{a_5 i_5}, \]

\[ D_0^{i_1 \cdots i_6} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_6 i_6} W_{\alpha} a_7 a_8 W_{\alpha} a_9 a_{10}, \]

\[ D_1^{i_1 \cdots i_8} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_8 i_8} W_{\alpha} a_9 a_{10}, \]

\[ D_2^{i_1 \cdots i_{10}} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_{10} i_{10}}, \]

\[ S = \text{Tr} W^\alpha W_{\alpha}. \]

Table 1: The matter contents of the magnetic theory.

| SU(NQ - 3) | SU(NQ) | SU(2) | U(1)_F | U(1)_\lambda |
|------------|--------|--------|--------|-------------|
| a, b, ...  | a, \beta, ... | i, j, ... | 1 | 1 | 2^{NQ-6} | \frac{1}{NQ-3} |
| \bar{q}_{a}^{i} | 2 | 1 | 2 | \frac{2NQ}{NQ-2} | \frac{NQ-2}{NQ-3} |
| \bar{q}_{a}^{i} | 1 | 1 | 3 | -2NQ | \frac{3NQ-10}{NQ-3} |
| s^{ab} | 1 | 1 | 1 | \frac{4NQ}{NQ-3} | \frac{2}{NQ-3} |
| f^{a} | 1 | 1 | 1 | 1 | -8 | 2 |
| M^{ij} | 1 | 1 | 1 | 1 | 2NQ | -2 |
| Y_{X}^{i} | 1 | 1 | 1 | 3 | 2NQ - 4 | -1 |

Here, \( a \) and \( a_1, a_2, \cdots \) are the indices of the gauge group \( Spin(10) \), and the matrix \( C \) is the charge conjugation matrix of it. The matrices \( \sigma_X \) \( (X = 1, 2, 3) \) are the Pauli matrices for the flavor of the spinors. Taking account of the number of the antisymmetrized indices of the \( SU(NQ) \) global symmetry, we see that whether each operator exists depends on \( NQ \). For example, the operator \( D_2^{i_1 \cdots i_{10}} \) exists only for \( NQ \geq 10 \).
Now, we turn to the magnetic theory, which is believed to be equivalent as the original electric theory in the IR region. The magnetic theory is given by $SU(N_Q - 3) \times Sp(1)$ gauge theory with the matter content given by Table 1 and with the superpotential

$$W_{\text{mag}} = M^{ij} q_{ai} s^{a} b_{b} q_{bj} + Y q_{ai} q_{ai} + \varepsilon_{\alpha \beta} \varepsilon_{IJ} q'_{a} a^{I} s^{a} b_{b} q'_{b} + \varepsilon_{\alpha \beta} (\sigma_{X} \sigma_{2}) IJ q'_{a} a^{I} q'_{X} t^{J}. \quad (2.4)$$

This theory has the same anomaly-free global symmetry as the electric theory. Thus, the $U(1)_R$ symmetry should also be expressed as $(2.1)$ with the same value of $x$ as in the electric theory.

There exist gauge invariant operators in this theory which correspond to those of the electric theory. They are fundamental singlets $M^{ij}$ and $Y_{ij}$ and the following composite operators:

$$(C)_{i_1 \cdots i_{N_Q - 3}} \sim \varepsilon^{a_1 \cdots a_{N_Q - 3}} q_{a_1 i_1} \cdots q_{a_{N_Q - 3} i_{N_Q - 3}},$$

$$(B)_{X i_1 \cdots i_{N_Q - 5}} \sim \varepsilon^{a_1 \cdots a_{N_Q - 3}} \varepsilon_{\alpha \beta} q_{a_1 i_1} \cdots q_{a_{N_Q - 5} i_{N_Q - 3}} q'_{a} a^{I} q'_{X} t^{J},$$

$$(F)_{i_1 \cdots i_{N_Q - 7}} \sim \varepsilon^{a_1 \cdots a_{N_Q - 3}} \varepsilon_{\alpha \beta} \varepsilon_{\gamma \delta} q_{a_1 i_1} \cdots q_{a_{N_Q - 7} i_{N_Q - 3}} q'_{a} a^{I} q'_{X} t^{J} (\sigma_{2} \sigma_{X} IJ) q'_{a} a^{I} q'_{X} t^{J} (\sigma_{2} \sigma_{X} KL) q'_{a} a^{I} q'_{X} t^{J} (\sigma_{2} \sigma_{X} \delta L),$$

$$(E)_{X i_1 \cdots i_{N_Q - 9}} \sim \varepsilon^{a_1 \cdots a_{N_Q - 3}} \varepsilon_{\alpha \beta} \varepsilon_{\gamma \delta} q_{a_1 i_1} \cdots q_{a_{N_Q - 9} i_{N_Q - 3}} q'_{a} a^{I} q'_{X} t^{J} (\sigma_{2} \sigma_{X} IJ) q'_{a} a^{I} q'_{X} t^{J} (\sigma_{2} \sigma_{X} KL) q'_{a} a^{I} q'_{X} t^{J} (\sigma_{2} \sigma_{X} \delta L),$$

$$G \sim \varepsilon_{\alpha \beta} t^{aI} (\sigma_{2}) IJ t^{\beta J},$$

$$(H)_{i_1 \cdots i_{N_Q - 4}} \sim \varepsilon_{IJ} \varepsilon^{a_1 \cdots a_{N_Q - 3}} \varepsilon_{\alpha \beta} q_{a_1 i_1} \cdots q_{a_{N_Q - 4} i_{N_Q - 3}} q'_{a} a^{I} q'_{X} t^{J},$$

$$(D_0)_{i_1 \cdots i_{N_Q - 6}} \sim \varepsilon_{XYZ} \varepsilon^{a_1 \cdots a_{N_Q - 3}} (\varepsilon_{IJK} q_{a_1 i_1} \cdots q_{a_{N_Q - 6} i_{N_Q - 3}} q_{a} a^{I} q_{a} a^{J} q_{a} a^{K} q_{a} a^{L},$$

$$(D_1)_{\alpha_1 \cdots \alpha_{N_Q - 8}} \sim \varepsilon_{a_1 \cdots a_{N_Q - 3}} \varepsilon_{XYZ} (\varepsilon_{IJK} q_{a_1 i_1} \cdots q_{a_{N_Q - 8} i_{N_Q - 3}} q_{a} a^{I} q_{a} a^{J} q_{a} a^{K} q_{a} a^{L},$$

$$(D_2)_{i_1 \cdots i_{N_Q - 10}} \sim \varepsilon_{a_1 \cdots a_{N_Q - 3}} \varepsilon_{XYZ} (\varepsilon_{IJK} q_{a_1 i_1} \cdots q_{a_{N_Q - 10} i_{N_Q - 3}} q_{a} a^{I} q_{a} a^{J} q_{a} a^{K} q_{a} a^{L},$$

$$S \sim \text{Tr} w_{\alpha} w_{\alpha}, \quad S' \sim \text{Tr} \bar{w}_{\alpha} \bar{w}_{\alpha}. $$

Here, $w_{\alpha}$ and $\bar{w}_{\alpha}$ are the field strength of the $SU(N_Q - 3)$ and $Sp(1)$ gauge groups, respectively, and the operation $*$ represents the Hodge duality with respect to the flavor

\footnote{The index $\alpha$ of the field strength $w_{\alpha}$ and $\bar{w}_{\alpha}$ is that of Lorentz spinors, which would not cause any confusion.}
\[ SU(N_Q) \] indices. The magnetic theory has two kinds of glueball superfields corresponding to the two gauge group factors. We can check that every operator has the same charges as that of the electric theory.

Furthermore, it seems that more gauge invariant generators exist in the magnetic theory than in the electric one. They are given by

\[
U_0 = \text{dets},
\]
\[
U_{1XY} = \varepsilon_{a_1 \cdots a_{N_Q} - 3} \varepsilon_{b_1 \cdots b_{N_Q} - 3} s^{a_1 b_1} \cdots s^{a_{N_Q} - 4 b_{N_Q} - 4} a_{N_Q}^{-3} b_{N_Q}^{-3} q_X q_Y,
\]
\[
U_{2XY} = \varepsilon_{a_1 \cdots a_{N_Q} - 3} \varepsilon_{b_1 \cdots b_{N_Q} - 3} \times s^{a_1 b_1} \cdots s^{a_{N_Q} - 5 b_{N_Q} - 5} a_{N_Q}^{-4} a_{N_Q}^{-3} b_{N_Q}^{-4} b_{N_Q}^{-3} q_X q_Y q_{X_1} q_{X_2} q_{Y_1} q_{Y_2},
\]
\[
U_3 = \varepsilon_{a_1 b_1} \cdots s^{a_{N_Q} - 6 b_{N_Q} - 6} a_{N_Q}^{-5} a_{N_Q}^{-4} b_{N_Q}^{-5} b_{N_Q}^{-4} b_{N_Q}^{-3} q_X q_Y q_{X_1} q_{X_2} q_{Y_1} q_{Y_2} q_X q_Y q_{X_3} q_{X_4} q_{Y_3} q_{Y_4},
\]
\[
(*E_0)_{X_1 \cdots i_{N_Q} - 5} = \varepsilon_{X Y Z} \varepsilon_{a_1 \cdots a_{N_Q} - 3} (s_{q_i} a_1 \cdots (s-q_{i_{N_Q} - 5})^{a_{N_Q} - 5} a_{N_Q}^{-4} a_{N_Q}^{-3},
\]
\[
(*E_1)_{\alpha X_1 \cdots i_{N_Q} - 7} = \varepsilon_{a_1 \cdots a_{N_Q} - 3} \varepsilon_{X Y Z} (s_{q_i a_{N_Q} - 7})^{a_{N_Q} - 7} \times (s w^\alpha a_{N_Q}^{-6} a_{N_Q}^{-5} a_{N_Q}^{-4} a_{N_Q}^{-3},
\]
\[
(*I_0)_{X_1 \cdots i_{N_Q} - 4} = \varepsilon_{a_1 \cdots a_{N_Q} - 3} (s_{q_i a_{N_Q} - 4})^{a_{N_Q} - 4} a_{N_Q}^{-3},
\]
\[
(*I_1)_{\alpha X_1 \cdots i_{N_Q} - 6} = \varepsilon_{a_1 \cdots a_{N_Q} - 3} (s_{q_i a_{N_Q} - 6})^{a_{N_Q} - 6} (s w^\alpha a_{N_Q}^{-5} a_{N_Q}^{-4} a_{N_Q}^{-3},
\]
\[
(*I_2)_{X_1 \cdots i_{N_Q} - 8} = \varepsilon_{a_1 \cdots a_{N_Q} - 3} (s_{q_i a_{N_Q} - 8})^{a_{N_Q} - 8} \times (s w^\alpha a_{N_Q}^{-7} a_{N_Q}^{-6} (s w^\alpha a_{N_Q}^{-5} a_{N_Q}^{-4} a_{N_Q}^{-3},
\]
\[
(*J_1)_{\alpha_1 \cdots i_{N_Q} - 5} = \varepsilon_{a_1 \cdots a_{N_Q} - 3} (s_{q_i a_{N_Q} - 5})^{a_{N_Q} - 5} (s w^\alpha a_{N_Q}^{-4} a_{N_Q}^{-3},
\]
\[
(*J_2)_{i_1 \cdots i_{N_Q} - 7} = \varepsilon_{a_1 \cdots a_{N_Q} - 3} (s_{q_i a_{N_Q} - 7})^{a_{N_Q} - 7} \times (s w^\alpha a_{N_Q}^{-6} a_{N_Q}^{-5} (s w^\alpha a_{N_Q}^{-4} a_{N_Q}^{-3},
\]

In spite of our best effort, we have not succeeded to show that these operators are decomposed or vanish in the classical chiral ring, as discussed in Appendix B.

The discrepancy makes it difficult for us to understand what happens at the IR fixed point. Though these two theories might actually not be equivalent to each other at the IR fixed point, it is not plausible that all the other non-trivial checks discussed in [1] are only accidental. Thus, in this paper, we assume that the classical chiral ring is deformed by the quantum effects and that the quantum chiral rings of both the theories are identical. However, it is still unclear what is indeed happening quantum-mechanically at the IR fixed point. This issue affects the construction of the trial \( a \)-function. Therefore, we will consider
both the functions in the electric and the magnetic theory and compare the results. In the
next section, we will see that both the functions have the identical local maximum.

3. \(a\)-Maximization

In this section, we study \(\text{Spin}(10)\) gauge theory with two spinors and \(N_Q\) vectors at
the superconformal IR fixed point both in the electric and the magnetic theory by using
\(a\)-maximization. We calculate the local maximum of the trial \(a\)-function defined in the
whole range of the parameter \(x\) and determine which operators become free at the IR fixed
point.

3.1. \(a\)-Maximization in the Electric Theory

We begin with the electric theory. As the result depends on \(N_Q\), we first analyze the
case \(N_Q = 6\). Taking account of the number of the antisymmetrized indices of the global
symmetry \(SU(N_Q)\), we find that the gauge invariant operators in this case are \(M, Y, C, B, G, H, D_0,\) and \(S\) in (2.3). Since the \(U(1)_R\) charge of the glueball superfield \(S\) is always
2 and never hit the unitarity bound, we can concentrate on the other seven operators.
Using (2.2), the \(U(1)_R\) charges \(R(O)\) of the gauge invariant operators can be written in
terms of \(x\) as

\[
R(M) = -8x + 2, \quad R(Y) = 8x - 1, \quad R(C) = 1, \quad R(B) = -8x + 3
R(G) = 24x - 4, \quad R(H) = 8x, \quad R(D_0) = -24x + 8.
\]  

(3.1)

By solving \(R(O) < 2/3\) for each operator, we find that the ranges of \(x\) are given in figure
1. Since the operator \(C\) does not hit the unitarity bound for all the ranges of \(x\), it does
not appear in the figure.

![Figure 1: The ranges of \(x\) where each operator hits the unitarity bound for \(N_Q = 6\).](image)
Now, we construct the trial $a$-function in the whole range of the parameter $x$. The trial $a$-function in the region where no operators hit the unitarity bound is given by

$$a_0(x) = 90 + 32F[R(Ψ)] + 10N_QF[R(Q)],$$

where $F(y) = 3(y-1)^3 - (y-1)$. The first term of this function is the contribution from the gaugino. The $U(1)_R$ charges $R(Ψ)$ and $R(Q)$ may be rewritten in terms of $x$ as (2.2). We modify this function as (1.5) for each range according to which operators hit the unitarity bound. Writing each term in the summation of (1.3) as $f_O(x) = -a_O\left( R(Ω) \right) + a_O(2/3)$, the trial $a$-function for the whole range of $x$ is given by

$$a(x) = \begin{cases} 
  a_0(x) + f_Y(x) + f_G(x) + f_H(x), & \left( x \leq \frac{1}{12} \right) \\
  a_0(x) + f_Y(x) + f_G(x), & \left( \frac{1}{12} \leq x \leq \frac{1}{6} \right) \\
  a_0(x) + f_M(x) + f_Y(x) + f_G(x), & \left( \frac{1}{6} \leq x \leq \frac{7}{36} \right) \\
  a_0(x) + f_M(x), & \left( \frac{5}{24} \leq x \leq \frac{7}{24} \right) \\
  a_0(x) + f_M(x) + f_B(x), & \left( \frac{7}{24} \leq x \leq \frac{11}{36} \right) \\
  a_0(x) + f_M(x) + f_B(x) + f_{D_o}(R), & \left( \frac{11}{36} \leq x \right)
\end{cases} \tag{3.2}$$

More explicitly, the function $f_O$ is given by

$$f_O(x) = d_O \left\{ -\left[ 3\left( R(Ω) - 1 \right)^3 - (R(Ω) - 1) \right] + 2/9 \right\}, \tag{3.3}$$

where $d_O$ is the number of the components of the operator $O$ and is given by

$$d_M = \frac{N_Q(N_Q + 1)}{2}, \quad d_Y = 3N_Q, \quad d_B = \frac{3N_Q!}{5!(N_Q - 5)!}, \quad d_G = 1, \quad d_H = \frac{N_Q!}{4!(N_Q - 4)!}, \quad d_{D_o} = \frac{N_Q!}{6!(N_Q - 6)!}. \tag{3.4}$$

The $U(1)_R$ charge $R(Ω_i)$ for each operator $Ω_i$ is given in (3.1). We find that the function (3.2) has a unique local maximum at

$$x = \frac{18N_Q + 6 - \sqrt{-4N_Q^3 + 143N_Q^2 - 928N_Q + 1824}}{6(N_Q^2 + 8N_Q - 12)}, \tag{3.5}$$

9
or equivalently, substituting this to (2.2),

\[
R(Q) = \frac{3N_Q^2 - 12N_Q - 48 + 2\sqrt{-4N_Q^3 + 143N_Q^2 - 928N_Q + 1824}}{3(N_Q^2 + 8N_Q - 12)},
\]

\[
R(\Psi) = \frac{12N_Q^2 - 42N_Q + 72 - N_Q\sqrt{-4N_Q^3 + 143N_Q^2 - 928N_Q + 1824}}{6(N_Q^2 + 8N_Q - 12)},
\]  

which is in the range where only the meson operator \(M^{ij}\) hits the unitarity bound. Thus, we find that the meson operator \(M^{ij}\) decouples from the interacting system to become free at the IR fixed point for \(N_Q = 6\).

Also in the case of \(N_Q = 7\), we find that \(M^{ij}\) hits the unitarity bound and the \(U(1)_R\) charges are given by (3.6) in the same way as for \(N_Q = 6\), though the ranges of \(x\) are different from figure 1.

\[\text{Figure 2: The ranges of } x \text{ where each operator hits the unitarity bound for } N_Q=8.\]

We go on to the case \(N_Q = 8\). The ranges of \(x\) are divided as figure 2. In this case, we encounter a subtlety that we do not understand how to deal with the situation where gauge invariant Lorentz spinors like \(D_1\alpha\) hit the unitarity bound \(\mathbb{H}\). The best we can do at this stage is just to neglect them assuming that such operators are massive in this case as in the previous paper \(\mathbb{I}\). Even if they are actually massless, our analysis in the region where they do not hit the unitarity bound, which is \(x \leq 1/4\) for this case, is still valid. We find that the trial \(a\)-function have a unique local maximum at

\[x = \frac{12N_Q - \sqrt{2900 - N_Q^2}}{6(N_Q^2 - 20)},\]  

\(3\) The unitarity bound for gauge invariant Lorentz spinors is \(R(O) \geq 1\) \(\mathbb{J}\).
or equivalently,
\[
R(Q) = \frac{3N_Q^2 - 24N_Q - 60 + 2\sqrt{2900 - N_Q^2}}{3(N_Q^2 - 20)},
\]
\[
R(\Psi) = \frac{6N_Q^2 + 120 - N_Q\sqrt{2900 - N_Q^2}}{6(N_Q^2 - 20)}.
\]

This is in the range where no operators hit the unitarity bounds. Though the ranges of \(x\) depend on \(N_Q\), we obtain the similar result for \(9 \leq N_Q \leq 19\).

In summary, we find that for \(N_Q = 6, 7\), the \(U(1)_R\) charges are given by (3.6) and the meson operator \(M^{ij}\) becomes free, while for \(8 \leq N_Q \leq 19\), the \(U(1)_R\) charges are given by (3.8) and no operators become free.

3.2. \(\alpha\)-Maximization in the Magnetic Theory

We next study the magnetic theory. Though we expect the same results as that in the electric theory, it is non-trivial because of the extra operators (2.6). The trial \(\alpha\)-function of the magnetic theory is different from that of the electric theory in the region where such operators hit the unitarity bounds.

We begin with the case \(N_Q = 6\) and compare with the result of the previous subsection. The gauge invariant operators are \(U_0, U_1, U_2, E_0, I_0, I_1,\) and \(J_1\) in (2.6), which exist only in the magnetic theory, as well as \(M, Y, C, B, G, H, D_0,\) and \(S\) in (2.5), which have the counterpart in the electric theory. The charge of these operators can be written with \(x\) of (2.7) by using the charges of \(U(1)_F\) and \(U(1)_\lambda\) for each field given in Table 1. They are given by (3.1) and also by
\[
R(U_0) = 24x - 2, \quad R(U_1) = 4, \quad R(U_2) = -24x + 10, \quad R(E_0) = -8x + 5
\]
\[
R(I_0) = 8x + 2, \quad R(I_1) = 3, \quad R(J_1) = 16x.
\]

We thus, find that the ranges of \(x\) where each operator hits the unitarity bound is given by figure 3. Since the operators \(C, U_1\) and \(I_1\) do not hit the unitarity bounds for all the ranges of \(x\), they do not appear in the figure 3. The bold arrows correspond to the operators which exist only in the magnetic theory. The dotted arrows correspond to the Lorentz spinor operators, which we ignore as in the previous subsection.
As in figure 3, we find that the trial $a$-function is given by

$$
a(x) = \begin{cases} 
  a_0(x) + f_y(x) + f_G(x) + f_{U_0}(x) + f_{H}(x) + f_{I_0}(x), & (x \leq -\frac{1}{6}) \\
  a_0(x) + f_y(x) + f_G(x) + f_{U_0}(x) + f_{H}(x), & (-\frac{1}{6} \leq x \leq \frac{1}{12}) \\
  a_0(x) + f_y(x) + f_G(x) + f_{U_0}(x), & \left( \frac{1}{12} \leq x \leq \frac{1}{9} \right) \\
  a_0(x) + f_y(x) + f_G(x), & \left( \frac{1}{9} \leq x \leq \frac{1}{6} \right) \\
  a_0(x) + f_M(x) + f_y(x) + f_G(x), & \left( \frac{1}{6} \leq x \leq \frac{7}{36} \right) \\
  a_0(x) + f_M(x) + f_y(x), & \left( \frac{7}{36} \leq x \leq \frac{5}{24} \right) \\
  a_0(x) + f_M(x), & \left( \frac{5}{24} \leq x \leq \frac{7}{24} \right) \\
  a_0(x) + f_M(x) + f_B(x), & \left( \frac{7}{24} \leq x \leq \frac{11}{36} \right) \\
  a_0(x) + f_M(x) + f_B(x) + f_{D_0}(x), & \left( \frac{11}{36} \leq x \leq \frac{7}{18} \right) \\
  a_0(x) + f_M(x) + f_B(x) + f_{D_0}(x) + f_{U_2}(x), & \left( \frac{7}{18} \leq x \leq \frac{13}{24} \right) \\
  a_0(x) + f_M(x) + f_B(x) + f_{D_0}(x) + f_{U_2}(x) + f_{E_0}(x), & \left( \frac{13}{24} \leq x \right) 
\end{cases}$$

(3.10)

where $f_\mathcal{O}(x)$ is given by (3.3). The numbers of the components $d_\mathcal{O}$ which appear in (3.3) are given by (3.4) and also by

$$
d_{U_0} = 1, \quad d_{U_2} = 6, \quad d_{E_0} = \frac{3 N_Q!}{5!(N_Q - 5)!}, \quad d_{I_0} = \frac{3 N_Q!}{4!(N_Q - 4)!}.$$

(3.11)
For the range $1/9 \leq x \leq 7/18$, where the operators which exist only in the magnetic theory do not hit the unitarity bound, the trial $a$-function (3.10) have the same shape as that of the electric theory. As the trial $a$-function (3.2) of the electric theory have a local maximum in this range, this function also have the local maximum at the same value of $x$. We can also check that there are no other local maximum throughout the whole range of $x$, though the function itself is different from that in the electric theory. Also in the case of $N_Q = 7$, we can obtain the same result as in the electric theory.

We can also check that there are no other local maximum throughout the whole range of $x$, though the function itself is different from that in the electric theory. Also in the case of $N_Q = 7$, we can obtain the same result as in the electric theory.

In the case of $N_Q = 8$, the ranges of $x$ are given in figure 4. We find that the trial $a$-function has the same shape as that of the electric theory for $1/12 \leq x \leq 23/96$, which includes the local maximum given by (3.7). Since we can verify that there are no local maximum outside this range, we find that the trial $a$-function have the unique local maximum, and no operators become free. Though the ranges of $x$ depend on $N_Q$, we can find the same result as in the electric theory also for $9 \leq N_Q \leq 19$.

Thus, we obtain the same results about the value of the $U(1)_R$ charge in spite of the discrepancy of the gauge invariant operators.

4. Summary and Discussion

We have studies $Spin(10)$ gauge theory with $N_Q$ vectors and two spinors. We found that the meson operator $M^{ij} = Q^i Q^j$ decouples from the interacting system to become free for $N_Q = 6, 7$.

We have discussed the renormalization group flow for the single spinor case in the paper [1]. In particular, for $N_Q = 7, 8, 9$, we have seen the two electric theories flow into the same theory at the IR fixed point. In the present case, since the magnetic theory flows
into the theory without the term $M^{ij} \bar{q}_i s_q^j$ in the IR, the electric theory flows into the same theory as that with $N_{ij} Q^i Q^j$ in the superpotential, as discussed for the single spinors case in the introduction (see [1] for more details).

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Appendix A. Gauge Invariant Operator of The Electric Theory

In this appendix, we explain how to obtain the gauge invariant generators (2.3) of the classical chiral ring of the electric theory. In order to deal with the operators including the $Spin(10)$ spinors $\Psi_I$, let us recall that the product of the spinors can be decomposed into antisymmetric tensor representations as

$$16 \times 16 = [1] + [3] + [5]_+$$

where $[n]$ represents the rank $n$ antisymmetric tensor, and the rank 5 tensor is self-dual. They can be explicitly expressed as $\Psi_I^T C T^{a_1 \cdots a_n} \Psi_J$. These are symmetric under the exchange of $I$ and $J$ for $n = 1, 5$ and antisymmetric for $n = 3$. All gauge invariant operators can be obtained by contracting the $Spin(10)$ gauge indices $a_i (= 1, \cdots 10)$ of the antisymmetric tensors $\Psi_I^T C T^{a_1 \cdots a_n} \Psi_J (n = 1, 3, 5)$, the vectors $Q^{ai}$, the field strength $W^{a_1 a_2}$, and the antisymmetric invariant tensors $\varepsilon^{a_1 \cdots a_{10}}$. However, many of the operators constructed in this way are decomposed into the product of other gauge invariant operators or vanish up to the $\bar{D}^2$ exact term. In order to identify the independent gauge invariant operators, we discuss the constraints among the chiral fields $Q$, $\Psi$, $W_\alpha$, and the invariant tensors $\varepsilon^{a_1 \cdots a_{10}}$.

Since the invariant tensor $\varepsilon^{a_1 \cdots a_{10}}$ satisfies

$$\varepsilon^{a_1 \cdots a_{10}} \varepsilon_{b_1 \cdots b_{10}} = \delta^{a_1}_{[b_1} \cdots \delta^{a_{10}}_{b_{10}]}$$

(A.1)

4 The brackets $[\ ]$ denote the antisymmetrization of the indices.
we can see that a pair of the invariant tensors can annihilate. Therefore, all the gauge
invariant operators can be reduced into those with at most one of the invariant tensor
\( \varepsilon^{a_1 \cdots a_{10}} \).

It follows from (A.1) that
\[
\varepsilon^{a_1 \cdots a_{10}} \Psi^T_I C \Gamma^{b_1 \cdots b_n} \Psi_J \propto \delta^{[a_1}_b \cdots \delta^{a_n]}_{b_n} \Psi^T_I C \Gamma^{a_{n+1} \cdots a_{10}} \Psi_J. \tag{A.2}
\]

If we introduce the antisymmetric tensors of rank 7 and 9, as seen from (A.2), we do not
need operators with both of the invariant tensor \( \varepsilon^{a_1 \cdots a_{10}} \) and the antisymmetric tensor.
Thus, we find that all the invariants are classified into operators containing no spinors
with at most one of the invariant tensors \( \varepsilon^{a_1 \cdots a_{10}} \) and those with spinors and none of the
invariant tensors \( \varepsilon^{a_1 \cdots a_{10}} \).

We begin with the operators in the former class. A constraint between the field
strength \( W_\alpha \) and other fields \( q \) is given by
\[
W_\alpha^A (T^A)^a_b q^b \propto \bar{D}^2 [e^{-V} D_\alpha (e^V q)]^a \sim 0, \tag{A.3}
\]
where \( q \) is a field in a representation of \( Spin(10) \) and \( T^A \) is the generator in the represen-
tation. For example, when it is the field strength \( W_\alpha \), we obtain that \( \{W_\alpha, W_\beta\} \sim 0 \).
Thus, operators with more than two of the field strength in this class vanish by the an-
ticommutativity of them. Taking account of (A.3), we find that all the operators in this
class are given by
\[
M^{ij} = Q^a_i Q^a_j, \quad S = Tr W^a W_\alpha, \\
D_0^{i_1 \cdots i_6} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_6 i_6} W_\alpha^a a_7 a_8 W^a a_9 a_{10}, \\
D_1^{i_1 \cdots i_8} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_8 i_8} W_\alpha^{a_9 a_{10}}, \\
D_2^{i_1 \cdots i_{10}} = \varepsilon^{a_1 \cdots a_{10}} Q^{a_1 i_1} \cdots Q^{a_{10} i_{10}}. \tag{A.4}
\]

We go on to the latter class. We first consider the operators without the field strength.
Most of the constraints on the spinors can be obtained from the Fierz identities. After
repeat use of the Fierz identities and lengthy calculations, we find that the product
\[
\Psi^T_I C \Gamma^{a_1 \cdots a_4} c_1 \cdots c_n \Psi_J \Psi^T_K C \Gamma^{b_1 \cdots b_j, c_1 \cdots c_n} \Psi_L \tag{A.5}
\]
can in general be given by a linear combination of the products
\[
\Psi^T_I C (\sigma_2 \sigma_X)^{IJ} \Gamma^a \Psi_J \Psi^T_K C (\sigma_2 \sigma_X)^{KL} \Gamma^a \Psi_L, \\
\Psi^T_I C (\sigma_2 \sigma_X)^{IJ} \Gamma^{a_1 \cdots a_4 b} \Psi_J \Psi^T_K C (\sigma_2 \sigma_X)^{KL} \Gamma^b \Psi_L, \tag{A.6}
\]
and those where two antisymmetric tensors are not at all contracted with each other. The sum of the ranks of the two antisymmetric tensors in the third contribution is always less than that of the original product (A.3). By using this fact, it turns out that the third contribution is decomposed into other invariant operators. When we use the products of the antisymmetric tensors, they are thus given by (A.4). Therefore, we can see that all the operators with no field strength in this class contain at most two of the antisymmetric tensors. More explicitly, they are given by

\[ Y^i_X = \Psi^T_I C(\sigma_2 \sigma_X)^{IJ} \Gamma^a \Psi J Q^{a_i}, \]

\[ C^{i_1 \cdots i_3} = \Psi_I^T C(\sigma_2)^{IJ} \Gamma^{a_1 \cdots a_3} \Psi J Q^{a_1 i_1} \cdots Q^{a_3 i_3}, \]

\[ B^{i_1 \cdots i_5} = \Psi_I^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^{a_1 \cdots a_5} \Psi J Q^{a_1 i_1} \cdots Q^{a_5 i_5}, \]

\[ F^{i_1 \cdots i_7} = \Psi_I^T C(\sigma_2)^{IJ} \Gamma^{a_1 \cdots a_7} \Psi J Q^{a_1 i_1} \cdots Q^{a_7 i_7}, \]

\[ E^{i_1 \cdots i_9} = \Psi_I^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^{a_1 \cdots a_9} \Psi J Q^{a_1 i_1} \cdots Q^{a_9 i_9}, \]

\[ G = \Psi_I^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^a \Psi J \Psi K^C(\sigma_2 \sigma_X)^{KL} \Gamma^a \Psi L, \]

\[ H^{i_1 \cdots i_9} = \Psi_I^T C(\sigma_2 \sigma_X)^{IJ} \Gamma^{a_1 \cdots a_5} \Psi J \Psi K^C(\sigma_2 \sigma_X)^{KL} \Gamma^{a_1} \Psi L Q^{a_2 i_2} \cdots Q^{a_5 i_5}. \]

We next consider operators with the spinors and the field strength. The field strength \( W_\alpha \) in the operators of this class only connect to another one \( W_\beta \) or the antisymmetric tensors due to (A.3) and \( \{ W_\alpha, W_\beta \} \sim 0 \). By using the identity

\[ \Gamma^{a_1 \cdots a_m} \Gamma^{bc} = \Gamma^{a_1 \cdots a_m bc} + \delta^{[a_1}_{[a} \Gamma^{a_2 \cdots a_m] c] - \delta^{[a_1 a_2}_{[a} \Gamma^{a_3 \cdots a_m]. \]

and the relation (A.3) for the spinor representation, we obtain

\[ 0 \sim \Psi^T_I C \Gamma a_1 \cdots a_m b c \Psi J + 2W^{|a_1 b} \Psi^T_I C \Gamma a_2 \cdots a_m]^b \Psi J - 2W^{|a_1 a_2} \Psi^T_I C \Gamma a_3 \cdots a_m \Psi J. \]

By decomposing this equation into the symmetric and the antisymmetric part under the exchange of the flavor indices \( I \) and \( J \), we obtain the equations

\[ W_{bc} \Psi^T_I C \Gamma a_1 \cdots a_m b c \Psi J \sim 2W^{|a_1 a_2} \Psi^T_I C \Gamma a_3 \cdots a_m \Psi J, \]

\[ W^{|a_1 b} \Psi^T_I C \Gamma a_2 \cdots a_m]^b \Psi J \sim 0. \]

We can see from the first equation of (A.8) that the rank of the antisymmetric tensor connected to the field strength with two indices can be reduced by four. By using the
second equation of (A.8), we find that the operators including the field strength contracted with two antisymmetric tensors,

\[
\Psi_I^T \Gamma^{a_1 \cdots a_{m-1}} b \Psi_J W^{bc} \Psi_K^T \Gamma^{c} \Psi_L,
\]

\[
\Psi_I^T \Gamma^{a_1 \cdots a_{m-1}} b \Psi_J W^{bc} W^{cd} \Psi_K^T \Gamma^{a_{\ell-1} \cdots a_n d} \Psi_L,
\]

can be reorganized into the operator where the two antisymmetric tensors are directly contracted. Similarly to the previous discussion leading to (A.7), such products of the antisymmetric tensors can be rewritten, and if not vanish, the field strength is in turn connected to the antisymmetric tensor with the two indices or is decomposed with another field strength into the glueball \( S \). Thus, we find that operators with the spinors and the field strength finally vanish according to (A.3) or are decomposed into the product of the glueball superfield \( S \) and operators with the spinors.

To summarize, the operators in (A.4), and (A.7) are the gauge invariant generators of the classical chiral ring of the electric theory, as listed in (2.3).

Appendix B. Gauge Invariant Operators of The Magnetic Theory

In this appendix, we only discuss the outline on how to obtain the gauge invariant generators of the classical chiral ring of the magnetic theory. Similarly to the case of the electric theory, an identity about the antisymmetric invariant tensors \( \varepsilon^{a_1 \cdots a_{NQ-3}} \) of \( SU(NQ-3) \) gauge group is given by

\[
\varepsilon^{a_1 \cdots a_{NQ-3}} \varepsilon_{b_1 \cdots b_{NQ-3}} = \delta_{a_1}^{b_1} \cdots \delta_{a_{NQ-3}}^{b_{NQ-3}}.
\]  

(B.1)

Thus, all the gauge invariant operators can be classified into operators with none of the antisymmetric invariant tensors, those with the invariant tensors with the lower indices, and those with the invariant tensors with the upper indices.

We first consider the operators without the antisymmetric invariant tensors. The equation (A.3) is also valid for the field strength \( w_\alpha \) and \( \tilde{w}_\alpha \) of \( SU(NQ-3) \) and \( Sp(1) \), respectively. Taking (A.3) into account together with the \( F \)-term conditions, we can verify that operators without the invariant tensors are given by the gauge singlets \( M, Y \), and the composites

\[
G \sim \varepsilon_{\alpha \beta} t^{[\alpha I} (\sigma_2)_{I J} t^{\beta J]}, \quad S \sim \text{Tr} w_\alpha w^\alpha, \quad S' \sim \text{Tr} \tilde{w}_\alpha \tilde{w}^\alpha.
\]  

(B.2)
Here, we also have used

\[ s^{ab} w_b^c \sim -s^{cb} w_b^a, \]  

which follows from \((A.3)\) for the symmetric tensors \(s^{ab}\).

We next consider operators including the invariant tensors \(\varepsilon_{a_1 \cdots a_{NQ-3}}\). It turns out that all the operators in this class are given by the contraction of the invariant tensor \(\varepsilon_{a_1 \cdots a_{NQ-3}}\) with the four operators

\[ q_X^{a_1}, (s\bar{q})^{a_1 i}, (sw)^{a_1 a_2}, (sw^n)^{a_1 b_1} \varepsilon_{b_1 \cdots b_{NQ-3}}, \quad (n = 0, 1, 2) \]  

which are supposed so that the indices \(a_1, a_2\) of the third in \((B.4)\) are contracted with those of \(\varepsilon_{a_1 \cdots a_{NQ-3}}\), while the indices \(b_2, \cdots, b_{NQ-3}\) of the other invariant tensor \(\varepsilon_{b_1 \cdots b_{NQ-3}}\) in the fourth are contracted with another of \((B.4)\). Taking account of the index \(X = 1, 2, 3\) of the field \(q_X\) and the index \(\alpha = 1, 2\) of the field strength \(w_\alpha\), we notice that at most three of the first in \((B.4)\) and two of the third can be contracted with the same invariant tensor. Therefore, all the operators with the single \(\varepsilon_{a_1 \cdots a_{NQ-3}}\) are given by

\[ D_n, E_n, I_n, J_m, \quad (n = 0, 1, 2, \quad m = 1, 2), \]  

in \((2.3)\) and \((2.6)\). Note that the operator

\[ (*J_0)_{i_1 \cdots i_{NQ-3}} = \varepsilon_{a_1 \cdots a_{NQ-3}} (s\bar{q}_{i_1})^{a_1} \cdots (s\bar{q}_{i_{NQ-3}})^{a_{NQ-3}} \]  

can be decomposed into the product of the operator \(C\) in \((2.3)\) and \(U_0\) in \((2.6)\).

We turn to the gauge invariant operators with more than one \(\varepsilon_{a_1 \cdots a_{NQ-3}}\) and find that all the independent gauge invariant operators are given by

\[ U_0 = \text{dets}, \]

\[ U_{1XY} = \varepsilon_{a_1 \cdots a_{NQ-3}} \varepsilon_{b_1 \cdots b_{NQ-3}} s^{a_1 b_1} \cdots s^{a_{NQ-4} b_{NQ-4}} q_X^{a_{NQ-3}} q_Y^{b_{NQ-3}}, \]

\[ U_{2XY} = \varepsilon_X X_1 X_2 \varepsilon_Y Y_1 Y_2 \varepsilon_{a_1 \cdots a_{NQ-3}} \varepsilon_{b_1 \cdots b_{NQ-3}} \]
\[ \times s^{a_1 b_1} \cdots s^{a_{NQ-5} b_{NQ-5}} q_X^{a_{NQ-4}} q_X^{-1} q_Y^{a_{NQ-3}} q_Y^{b_{NQ-3}}, \]  

\[ U_3 = \varepsilon_X X_1 X_2 \varepsilon_Y Y_1 Y_2 \varepsilon_{a_1 \cdots a_{NQ-3}} \varepsilon_{b_1 \cdots b_{NQ-3}} \]
\[ \times s^{a_1 b_1} \cdots s^{a_{NQ-6} b_{NQ-6}} q_X^{a_{NQ-5}} q_X^{a_{NQ-4}} q_X^{a_{NQ-3}} q_Y^{b_{NQ-5}} q_Y^{b_{NQ-4}} q_Y^{b_{NQ-3}}. \]
Let us begin with one invariant tensor $\varepsilon_{a_1\cdots a_{NQ-3}}$ and all the symmetric tensor $s^{ab}$ contracted with it,

$$\varepsilon_{a_1\cdots a_k a_{k+1}\cdots a_{NQ-3}} s^{a_1 b_1} \cdots s^{a_k b_k} \equiv T_{a_{k+1}\cdots a_{NQ-3}} b_1 \cdots b_k,$$

in an operator of this class. The indices $a_{k+1}, \cdots, a_{NQ-3}$ are supposed to be contracted with those of the first in (B.4) or those of the field strength $w_\alpha$ in the third. As the indices $b_1, \cdots, b_k$ in $T$ are antisymmetric, by using (B.4), we can rewrite it as

$$T_{a_{k+1}\cdots a_{NQ-3}} b_1 \cdots b_k \propto T_{a_{k+1}\cdots a_{NQ-3}} d_1 \cdots d_k \varepsilon_{d_1 \cdots d_k e_{k+1} \cdots e_{NQ-3}} \varepsilon^{b_1 \cdots b_k e_{k+1} \cdots e_{NQ-3}}. \quad \text{(B.7)}$$

On the other hand, since we are considering the operators with more than one invariant tensors, the operators have another invariant tensor $\varepsilon_{c_1 \cdots c_{NQ-3}}$ other than those included in (B.7). Then, we apply (B.4) again to $\varepsilon^{b_1 \cdots b_k e_{k+1} \cdots e_{NQ-3}}$ in (B.7) and $\varepsilon_{c_1 \cdots c_{NQ-3}}$. We thus obtain

$$T_{a_{k+1}\cdots a_{NQ-3}} b_1 \cdots b_k \varepsilon_{c_1 \cdots c_{NQ-3}} \propto T_{a_{k+1}\cdots a_{NQ-3}} d_1 \cdots d_k \varepsilon_{d_1 \cdots d_k [c_{k+1} \cdots c_{NQ-3}] \delta^{b_1 \cdots b_k}.} \quad \text{(B.8)}$$

After this procedure, other $s^{ab}$ besides those in (B.8) may connect to the original $\varepsilon_{a_1\cdots a_{i} a_{i+1} \cdots a_{NQ-3}}$, upon the use of (B.4). Then, we can use (B.4) for all the symmetric tensors $s^{ab}$ contracted with the tensor $\varepsilon_{a_1 \cdots a_{i} a_{i+1} \cdots a_{NQ-3}}$ to annihilate the other $\varepsilon_{d_1 \cdots d_k e_{k+1} \cdots e_{NQ-3}}$ in (B.8) and the appearing invariant tensor of the upper indices. If the resulting operator does not vanish, we obtain the following form

$$\varepsilon_{a_1\cdots a_i a_{i+1} \cdots a_{NQ-3}} s^{a_1 b_1} \cdots s^{a_i b_i} q_{i+1} \cdots q_{NQ-3} \varepsilon_{b_1 \cdots b_i b_{i+1} \cdots b_{NQ-3}}, \quad \text{(B.9)}$$

where the remaining indices $b_{k+1} \cdots b_{NQ-3}$ are contracted with those in (B.4). Again, we apply (B.4) to all symmetric tensors contracted with $\varepsilon_{b_1 \cdots b_k b_{k+1} \cdots b_{NQ-3}}$ in (B.9) to eliminate the original invariant tensor $\varepsilon_{a_1 \cdots a_{NQ-3}}$ and the newly appearing invariant tensor. We find that all the gauge invariant operators with more than one $\varepsilon^{a_1 \cdots a_{NQ-3}}$ except for (B.6) vanish or are decomposed into the gauge invariant operators.

We next consider operators including the invariant tensors $\varepsilon^{a_1 \cdots a_{NQ-3}}$ with the upper indices. It turns out that all the operators in this class are given by the contraction of the invariant tensor $\varepsilon^{a_1 \cdots a_{NQ-3}}$ with the five operators \footnote{The parentheses ( ) denote the symmetrization of the indices, while [ ] does the antisymmetrization.}

$$\begin{aligned}
\bar{q}_{a_1 i}, & \quad \varepsilon_{\alpha \beta} \varepsilon^{I J} \bar{q}^J a_1 \alpha I \beta J, & \quad \varepsilon_{\alpha \beta} \bar{q}^J a_1 \alpha (I J | b | \rho | K L) \\
\varepsilon_{\alpha \beta} \bar{q}^J a_1 \alpha (I J | b | \rho | K L) & \equiv \varepsilon_{\alpha \beta} \bar{q}^J a_1 \alpha (I J | b | \rho | K L) \\
\varepsilon_{\alpha \beta} \bar{q}^J a_1 \alpha (I J | b | \rho | K L) & \equiv \varepsilon_{\alpha \beta} \bar{q}^J a_1 \alpha (I J | b | \rho | K L).
\end{aligned} \quad \text{(B.10)}$$
where \( q^{aIJ} \) is related to \( q^{aX} \) in Table 1 as

\[
q^{aIJ} \equiv q^a_X (\sigma X \sigma_2)^{IJ},
\]

and thus, it is symmetric under exchange of the indices \( I \) and \( J \). The indices \( a_1 \) and \( a_2 \) of the fifth operator in (B.10) are contracted with those of \( \varepsilon^{a_1 \cdots a_{NQ-3}} \), while \( \varepsilon^{b_1 \cdots b_{NQ-3}} \) in the fourth is contracted with the operators in (B.10).

Taking account of the indices of the local \( Sp(1) \) and those of the global \( SU(2) \), we find that at most four \( \bar{q}' \) can be contracted with \( \varepsilon^{a_1 \cdots a_{NQ-3}} \). The numbers of the second, the third, the fourth, and the fifth operators in (B.10) contracted with the invariant tensor \( \varepsilon^{a_1 \cdots a_{NQ-3}} \) are limited from this fact. Further, if two \( \bar{q}' \) from the second, the third, and the fourth are contracted with the invariant tensor, the symmetric part of the global \( SU(2) \) indices of them can be rewritten in terms of the fifth and some other parts. In fact, when the indices of the global \( SU(2) \) of these two \( \bar{q}' \) are symmetric, the local \( Sp(1) \) indices of those \( \bar{q}' \) should be antisymmetric. Then, by using the relation for the invariant tensor of \( Sp(1) \),

\[
\varepsilon^{a_1 a_2} \varepsilon_{\beta_1 \beta_2} = \delta^{a_1}_{\beta_1} \delta^{a_2}_{\beta_2},
\]

we can see that

\[
\varepsilon^{a_1 a_2 \cdots a_{NQ-3}} \bar{q}'_{\alpha_1} (I \bar{q}'_{\alpha_2} | a_2 | J) = \frac{1}{2} \varepsilon^{a_1 a_2 \cdots a_{NQ-3}} \bar{q}'_{\alpha_1} (I \bar{q}'_{\alpha_2} | \beta_2 | J) \varepsilon_{\beta_1, \beta_2} \varepsilon^{a_1 a_2},
\]

and it gives rise to the fifth. This is always possible when more than two \( \bar{q}' \) from the second, the third, and the fourth are contracted with the invariant tensor \( \varepsilon^{a_1 \cdots a_{NQ-3}} \) of \( SU(N_Q - 3) \), because the global \( SU(2) \) indices of two \( \bar{q}' \) of them must take the same value, thus symmetric. Thus, we find that the total number of the second, the third, and the fourth contracted with the same invariant tensor \( \varepsilon^{a_1 \cdots a_{NQ-3}} \) should be less than three.

When four of \( \bar{q}' \) are contracted with the invariant tensor, each two of them take the same value of the \( SU(2) \) indices, respectively, and can be rewritten in terms of two copies of the fifth and some other parts. Thus, when one of the fifth is contracted with the invariant tensor, the total number of the second, the third, and the fourth contracted with the same invariant tensor \( \varepsilon^{a_1 \cdots a_{NQ-3}} \) should be less than two.
Wrapping up these facts, together with the $F$-term conditions, (A.3), and (B.1), we can verify that all the operators with the single $\varepsilon^{a_1 \cdots a_{N_Q-3}}$ are given by

\begin{align}
(C)_{i_1 \cdots i_{N_Q-3}} & \sim \varepsilon^{a_1 \cdots a_{N_Q-3}} \tilde{q}_{a_1 i_1} \cdots \tilde{q}_{a_{N_Q-3} i_{N_Q-3}}, \\
(B)_{i_1 \cdots i_{N_Q-5}} & \sim \varepsilon^{a_1 \cdots a_{N_Q-3}} \varepsilon_{\alpha \beta} \tilde{q}_{a_1 i_1} \cdots \tilde{q}_{a_{N_Q-5} i_{N_Q-5}} \tilde{q}_{a_{N_Q-4} i_{N_Q-4}} \sigma_2 \sigma_X (I J) \tilde{q}_{a_{N_Q-3}}, \\
(F)_{i_1 \cdots i_{N_Q-7}} & \sim \varepsilon^{a_1 \cdots a_{N_Q-3}} \varepsilon_{\alpha \beta \gamma} \tilde{q}_{a_1 i_1} \cdots \tilde{q}_{a_{N_Q-7} i_{N_Q-7}} \times \tilde{q}_{a_{N_Q-6} (I J) \tilde{q}_{a_{N_Q-5}} \tilde{q}_{a_{N_Q-4}} \tilde{q}_{a_{N_Q-3}}}, \\
(H)_{i_1 \cdots i_{N_Q-4}} & \sim \varepsilon_{(I J) \varepsilon^{a_1 \cdots a_{N_Q-3}} \varepsilon_{\alpha \beta} \tilde{q}_{a_1 i_1} \cdots \tilde{q}_{a_{N_Q-4} i_{N_Q-4}} \tilde{q}_{a_{N_Q-3}} t_{a_{N_Q-3}}. 
\end{align}

We go on to the operators with more than two $\varepsilon^{a_1 \cdots a_{N_Q-3}}$ and skip those with two here. The latter will be explained later. We will see that all the operators in these classes do not give the independent gauge invariant operators. Since the only fourth operators in (B.10) can connect with two invariant tensors $\varepsilon^{a_1 \cdots a_{N_Q-3}}$, the operators with more than two $\varepsilon^{a_1 \cdots a_{N_Q-3}}$ should include at least one $\varepsilon^{a_1 \cdots a_{N_Q-3}}$ which are contracted with two of the fourth. Further, all the remaining indices of the same $\varepsilon^{a_1 \cdots a_{N_Q-3}}$ must be contracted with the first operator in (B.10), as

\begin{align}
\varepsilon^{a_1 \cdots a_{N_Q-3}} \tilde{q}_{a_1} \cdots \tilde{q}_{a_{N_Q-5}} \left( \varepsilon_{\alpha \beta} \tilde{q}_{a_{N_Q-4}} \sigma(I \tilde{q}_{b_1}) |\beta J \varepsilon^{b_1 \cdots b_{N_Q-3}} \right) \\
\times \left( \varepsilon_{\gamma \delta} \tilde{q}_{a_{N_Q-3}} \gamma(K \tilde{q}_{c_1}) |\delta L \varepsilon^{c_1 \cdots c_{N_Q-3}} \right), 
\end{align}

as we can see from the previous discussion. Here, we apply the identity

\begin{align}
\varepsilon^{a_1 \cdots a_{N_Q-3}} \varepsilon^{b_1 \cdots b_{N_Q-3}} = 0, 
\end{align}

for $\varepsilon^{a_1 \cdots a_{N_Q-3}}$ and $\varepsilon^{b_1 \cdots b_{N_Q-3}}$ in (B.12). Ignoring the terms decomposed into the products of gauge invariant operators, we find that the resulting operators are given by

\begin{align}
\sum_{k=1}^{N_Q-5} \frac{1}{\bar{5}} \varepsilon^{a_1 \cdots a_{k-1} b_1 a_{k+1} \cdots a_{N_Q-3}} \tilde{q}_{a_1} \cdots \tilde{q}_{a_{k-1}} \tilde{q}_{a_{k+1}} \cdots \tilde{q}_{a_{N_Q-5}} \left( \varepsilon_{\alpha \beta} \tilde{q}_{a_{N_Q-4}} \sigma(I \tilde{q}_{b_1}) |\beta J \varepsilon^{b_1 \cdots b_{N_Q-3}} \right) \\
\times \left( \varepsilon_{\gamma \delta} \tilde{q}_{a_{N_Q-3}} \gamma(K \tilde{q}_{c_1}) |\delta L \varepsilon^{c_1 \cdots c_{N_Q-3}} \right). 
\end{align}

If the resulting operator is not decomposed into gauge invariant operators, the last factor $\varepsilon^{a_1 b_2 \cdots b_{N_Q-3}} \tilde{q}_{a_k}$ in (B.14) are connected with the invariant tensor $\varepsilon^{c_1 \cdots c_{N_Q-3}}$ via other operators. This happens only when the invariant tensor $\varepsilon^{c_1 \cdots c_{N_Q-3}}$ in (B.14) is contracted with two of the fourth in (B.10). This is the same situation we previously have seen for
the invariant tensor $\varepsilon^{a_1 \cdots a_{NQ-3}}$ in (B.12), and thus, we can repeat the same procedure to show that the resulting operator is decomposed into gauge invariant operators.

We now turn to the operators with two $\varepsilon^{a_1 \cdots a_{NQ-3}}$. As discussed previously, the only fourth operator in (B.10) can be used to connect the two invariant tensors. In particular, they are connected by at most two of the operator. By using the identity (B.13), we can see that the invariant tensors connected by two of the fourth in (B.10) can be reduced to those by one. Thus, we only have to consider the latter operators. If either of the invariant tensors does not have the fifth operator of (B.10), we can use the identity (B.13) to show that they are decomposed into gauge invariant operators. If both of them have the fifth operators, a closer examination is needed on the symmetry of the global $SU(2)$ indices of $\bar{q}$’s. Taking account of this point and the identity (B.13), we can verify that they are also decomposed into gauge invariant operators.

To summarize, the singlets $M, Y$, and the operators listed in (B.3), (B.5), (B.6), and (B.11) are the gauge invariant generators of the classical chiral ring of the magnetic theory.

As discussed in section 2, to all the gauge invariant generators (2.3) of the classical chiral ring in the electric theory, there exist the counterparts (2.5) in the magnetic theory. However, the extra gauge invariant operators (2.6) seem to exist in the magnetic theory. If the electric-magnetic duality is true for this model, this discrepancy should disappear at the quantum level.
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