Noise in BosonSampling and the threshold of efficient classical simulability

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We study the quantum to classical transition in BosonSampling by analysing how \(N\)-boson interference is affected by inevitable noise in an experimental setup. We adopt the Gaussian noise model of Kalai and Kindler and relate it to realistic experimental imperfections in BosonSampling. We reveal a connection between noise in BosonSampling and partial distinguishability of bosons, which allows us to prove efficient classical simulability of the noisy BosonSampling model, in the non-collision regime, with the noise amplitude \(\epsilon = \Theta(1)\) as \(N \to \infty\). On the other hand, using an equivalent representation of the noise as losses of bosons compensated by random (dark) counts of detectors, we prove that the noisy BosonSampling model with noise amplitude \(\epsilon = O(1/N)\) is as hard to simulate classically, in the no-collision regime, as the ideal BosonSampling. We find that the ratio of “noise clicks” (lost bosons compensated by dark counts) to the number of bosons \(N\) vanishes as \(N \to \infty\) in the intermediate regime of noise amplitudes \(\epsilon = \omega(1/N)\) and conjecture that such a noisy BosonSampling is also hard to simulate classically. An extension of the Gaussian noise model beyond the no-collision regime is given, with some of our results preserving their validity. In general, our results reveal how imperfections (noise) in an experimental setup cause transition from quantum to classical behaviour in \(N\)-boson interference on a linear network when \(N \gg 1\).

I. INTRODUCTION

Quantum supremacy [1], i.e., the computational advantage over digital computers in some specific tasks, promised by quantum mechanics to exist in mesoscopic-size devices, would be a significant step on the way to the universal quantum computer [2]. Several proposals were put forward to achieve this goal: BosonSampling model [3], the clean qubit model [4] and the commuting quantum circuits model [5]. Under some computational complexity conjectures, these models cannot be simulated efficiently on digital computers (i.e., with computations polynomial in the model size). The question is whether the quantum advantage can be maintained under unavoidable imperfections (i.e., noise) in any experimental realisation [6].

In this work we focus on BosonSampling [3], which realises quantum interference of \(N\) single bosons in a randomly chosen unitary linear network with \(M\) input and output ports, which could be random [8], but have to be known in each run [7]. The so-called “no-collision” regime is considered, when at least \(N \ll \sqrt{M}\) as \(N\) and \(M\) scale up, thus simple on-off detectors at a network output can be used, since the probability of two or more bosons per output port (the “collision” in terms of Ref. [3]) scales as \(O(N^2/M)\) [8]. Classical simulation of the output probability distribution of BosonSampling to a small total variation distance \(\epsilon\) polynomial in \(N\) and \(1/\epsilon\) is impossible under some plausible conjectures [3].

In comparison to the universal quantum computation with linear optics [11], BosonSampling seems to be simpler to realise experimentally [11,14], as it does not require neither interaction between photons nor error correction schemes. Some improvements are available for its experimental implementation, such as the BosonSampling from a Gaussian state [8], experimentally tested in Ref. [15], and with the time-bin network [16], also tested experimentally [17]. Spectacular advances in experimental BosonSampling with linear optics are being continuously reported [15,20]. Moreover, alternative platforms include ion traps [21], superconducting qubits [22], neutral atoms in optical lattices [23] and dynamic Casimir effect [24].

The classical hardness of BosonSampling stems from the fact that quantum amplitudes of many-boson interference are given by the matrix permanents of \(N\)-dimensional submatrices of the network matrix [25,26] classically hard to compute [27,28], with the fastest known algorithm, due to Ryser [29,30], requiring \(O(N2^N)\) operations.

Recently advanced classical algorithms were shown to be able to sample the output distribution of BosonSampling on the digital computers up to \(N \approx 50\) bosons [31,32], a scale-up of the previous estimate \(N \approx 30\) [3]. Besides this higher threshold for the quantum supremacy demonstration, imperfections in experimental setup can allow for efficient classical algorithms [33,38], complicating the prospects of quantum supremacy demonstration with BosonSampling at least with some of the considered implementations. Some of the efficient classical algorithms, due to imperfections/noise, are applicable generally, beyond the usual no-collision regime of BosonSampling [34,39,40].

Besides the experimental challenges on the way of quantum supremacy demonstration with BosonSampling, the exact location of the boundary between classical hardness and classical simulability is not currently known, even in the no-collision regime. It is known that an imperfect BosonSampling can be arbitrarily close in the total variation distance to the ideal one if error in the components of a network is inversely proportional to the network depth times the number of bosons [39,40], and distinguishability of bosons is inversely proportional to the total number of bosons [41,42]. Similarly, a BosonSampling device with loss of bosons and/or detector noise (dark counts) remains as classically hard as the
ideal BosonSampling if the rate of loss of bosons (dark counts) is inversely proportional to the number of bosons \[ o(N) \] . It was also suggested that the domain of noise amplitudes allowing quantum advantage in a noisy BosonSampling model is scale-dependent, so that even noise with a vanishing amplitude in the total number of bosons permits an efficient classical simulation \[ [33] \]. The aim of this work is to study the transition to classical simulability in a noisy BosonSampling. For this study, we adopt the noise model of Ref. \[ [33] \], though the results have general applicability, due to the revealed relations of the noise model of Ref. \[ [33] \] to other models of imperfections in BosonSampling, such as distinguishability \[ [41] \] and losses compensated with dark counts \[ [43] \].

In the next section we state the model of Gaussian noise in BosonSampling and the recall the main conclusions of Ref. \[ [33] \]. Section \[ III \] states our main results and provides an outline of the technical steps performed in our analysis of the noisy BosonSampling model. In section \[ V \] the output probability distribution is derived for the Gaussian noise model of Ref. \[ [33] \]. In section \[ V \] the noise model is extended beyond the no-collision regime. In section \[ VII \] a connection to BosonSampling with partially distinguishable bosons is discussed. In section \[ VII \] we prove classical simulability of noisy BosonSampling with the \( \Theta(1) \)-noise. In section \[ VIII \] we prove classical hardness of the noisy BosonSampling with \( \Theta(1/N) \)-noise.

The main results and open problems are summarised in the concluding section \[ IX \].

### II. THE GAUSSIAN NOISE MODEL

Let us recall the model of noise in BosonSampling proposed in Ref. \[ [33] \] and the main conclusions there. Some familiarity with the BosonSampling idea \[ [3] \] is assumed below.

Recall that the \( N \)-dimensional submatrices of a unitary network matrix \( U_{kl} \), where \( k, l = 1, \ldots, M \), the matrix permanents of which give the amplitudes of output probabilities in BosonSampling, are approximated in the no-collision regime by the \( N \)-dimensional matrices composed of i.i.d. complex Gaussians \[ [3] \], used to formulate a classically hard sampling problem. In the model of noise of Ref. \[ [33] \], each matrix element \( U_{kl} \) is modified also by an independent Gaussian noise, so that the resulting network matrix becomes

\[
U_{kl} = \sqrt{1-\epsilon} U_{kl} + \sqrt{\epsilon} Z_{kl}
\]

where \( Z_{kl} \) is a rescaled standard complex Gaussian, normalised as \( \langle |Z_{kl}|^2 \rangle = 1/\sqrt{M} \) and \( \epsilon \) is the noise amplitude \( 0 < \epsilon < 1 \). Under these conditions, it was shown in Ref. \[ [33] \] that, as \( N \to \infty \), for a finite noise amplitude \( \epsilon = \Theta(1) \) (meaning that \( \epsilon(N) \) is bounded away from zero, \( \delta < \epsilon \) for some fixed \( \delta > 0 \)) the noisy BosonSampling can be simulated classically with polynomial computations and that for the noise amplitude \( \epsilon = \omega(1/N) \) \( (N\epsilon(N) \) is unbounded from above, but \( \epsilon(N) \) still vanishes as \( N \to \infty \) the correlations between the noisy BosonSampling and ideal BosonSampling (for a Haar-random network \( U \)) tend to zero. Moreover, it was suggested that the noisy BosonSampling is at a finite total variation distance from the ideal one for the noise amplitude \( \Omega(1/N) \) (i.e., \( \epsilon(N) \leq C/N \) for a constant \( C \)) and that the noisy BosonSampling remains hard under the noise with amplitude \( \epsilon = o(1/N) \) (i.e., \( N\epsilon(N) \) is arbitrarily small). These conclusions were suggested to hold to other models of imperfections and beyond the no-collision regime.

Below a detailed analysis of the noisy BosonSampling model is attempted in order to establish the exact location of the boundary of the transition to efficient classical simulability. Moreover, we explore the relation of the Gaussian noise model to other models of imperfections.

### III. OUTLINE THE WORK AND MAIN RESULTS

We study the noisy BosonSampling model of section in the following series of steps.

We start by deriving the output probability distribution of the noisy BosonSampling (section \[ V \]). This result provides a new (discrete) representation of the (continuous) Gaussian noise by its effect on many-boson interference: the noise acts as uniform boson losses with the lost bosons compensated by random detector clicks at the network output. This new representation is used to extend the Gaussian noise model beyond the no-collision regime (section \[ V \]).

We present a model of BosonSampling with partially distinguishable bosons (and no noise) as an equivalent of the noisy BosonSampling model in terms of computational advantage over digital computers (section \[ VI \]). Given the latter equivalence holds, the closeness of the noisy BosonSampling model to the ideal BosonSampling in the total variation distance is found to require the noise amplitude to be \( o(1/N) \) (section \[ VIIA \]).

By further utilising the relation of the noisy BosonSampling to that with partially distinguishable bosons and the results of Refs. \[ [32, 33, 38, 41] \], we prove efficient classical simulability of the noisy BosonSampling with the noise amplitude \( \epsilon = \Theta(1) \) in the no-collision regime (section \[ VII \]).

Finally, we find an effective bound on the number of “noise clicks” in the discrete representation of noise in the BosonSampling model of Eq. \[ (1) \], such that BosonSampling with bounded noise clicks is arbitrarily close in the total variation distance to a given noisy one (section \[ VIII \]). This allows us to prove, by utilising the results of Ref. \[ [43] \], the classical computational hardness of the noisy BosonSampling in the no-collision regime under the noise amplitude \( \epsilon = \Theta(1/N) \).

The main results, proven in the present work in sections \[ VII \] and \[ VIII \] respectively, can be stated as follows.

**Theorem 1** Given \( 0 < \delta < 1 \) and \( \epsilon > 0 \), the noisy BosonSampling model of Eq. \[ (1) \] with the noise amplitu-
tude $\epsilon = \Theta(1)$, as the total number of bosons $N \to \infty$, can be simulated classically in the no-collision regime with the success probability at least $1 - \delta$ to the error $\epsilon$ in the total variation distance with the computations polynomial $(N, 1/\epsilon, 1/\delta)$.

Theorem 1 agrees with one of the conclusions of Ref. 33. Moreover, our proof of efficient classical simulability for noisy BosonSampling applies also for BosonSampling with a finite boson distinguishability (and no noise), previously considered in Ref. 33. On the other hand, we also prove the following.

**Theorem 2** The noisy BosonSampling model of Eq. (1) with the noise amplitude $\epsilon = O(1/N)$, as $N \to \infty$, is as hard to simulate classically in the no-collision regime as the ideal BosonSampling.

Our results also indicate that the output distribution of the noisy BosonSampling of theorem 2 is at a constant total variation distance to that of the ideal BosonSampling (as suggested in Ref. 33) for arbitrary relation between $N$ and $M$ under two plausible conjectures: (i) the noisy BosonSampling model is as hard to simulate classically as a model of BosonSampling with partially distinguishable bosons, introduced in section VII, and (ii) that a bound of Ref. 12 on the boson distinguishability limiting the total variation distance to the ideal Boson-Sampling is tight (see section VIII).

Our approach does not resolve if the noisy BosonSampling with the noise amplitude $\epsilon = \omega(1/N)$, intermediate between theorems 1 and 2, can be efficiently simulated classically. However, the ratio of the effective total number of “noise clicks” to the total number of bosons vanishes under such a noise as $N$ scales up (valid for arbitrary relation between $N$ and $M$), see section VIII.

Such a noise is therefore similar to the $O(1/N)$-noise by the fact that the dominating contribution to the output probability distribution comes from the quantum many-boson interferences. It is natural then to conjecture that such a noisy BosonSampling is still hard to simulate classically.

The proofs of theorems 1 and 2 are limited to the no-collision regime, due to the limited applicability of the previous results, Refs. 33, 41, 43, used in the proofs. The main technical challenges for extension beyond the no-collision regime are (i) evaluating the averages over the Haar-random network without using the Gaussian approximation (ii) extension of the quantum computational complexity results to many-boson interference beyond the no-collision regime.

### IV. OUTPUT PROBABILITY DISTRIBUTION OF NOISY BOSONSAMPLING

In this section we consider the no-collision regime and derive the output probability distribution of the noisy BosonSampling model of Ref. 33. We fix the input ports to be $k = 1, \ldots, N$. Assuming that the noise $Z$ of a noisy network (1) changes from run to run, let us derive the probability $p^{(\epsilon)}(l_1 \ldots l_N | 1 \ldots N)$ to detect $N$ input bosons at distinct output ports $l_1, \ldots, l_N$. Denoting by $M(k_1 \ldots k_n | l_1 \ldots l_n)$ the submatrix of a matrix $M$ on the rows $k_1, \ldots, k_n$ and columns $l_1, \ldots, l_n$ and by per$M(k_1 \ldots k_n | l_1 \ldots l_n)$ the matrix permanent of such a submatrix, we have

\[
p^{(\epsilon)}(l_1 \ldots l_N | 1 \ldots N) = \left< \text{per} U(1 \ldots N | l_1 \ldots l_N)^2 \right>
\]

\[
= \left< \sum_{n=0}^{N} \epsilon^n (1-\epsilon)^{N-n} \sum_{k_1 \ldots k_n, j_1 \ldots j_n} \text{per} Z(k_1 \ldots k_n | l_{j_1} \ldots l_{j_n}) \text{per} U(k_{n+1} \ldots k_N | l_{j_{n+1}} \ldots l_{j_N})^2 \right>,
\]

(2)

here (in this and the next section) the notation $\langle \ldots \rangle$ stands for the averaging over the noise $Z$, the matrix permanent of a sum of two matrices in Eq. (1) is expanded, using the permanent expansion formula (14), where we partition the set 1, \ldots, $N$ twice into two groups of subsets: (i) $k_1, \ldots, k_n$ and $k_{n+1}, \ldots, k_N$ and (ii) $j_1, \ldots, j_n$ and $j_{n+1}, \ldots, j_N$. The averaging over the noise is easily performed within the expansion in Eq. (2), where we need to consider the averaging of the products of the following type

\[
\text{per} Z(k_1 \ldots k_n | l_{j_1} \ldots l_{j_n}) \text{per} Z^*(k'_1 \ldots k'_n | l'_{j_1} \ldots l'_{j_n})
\]

\[
= \sum_{\sigma, \tau} \prod_{\alpha=1}^{m} \prod_{\beta=1}^{m} Z_{k_{\sigma}(\alpha) l_{j_{\sigma}(\alpha)}} Z_{k'_{\tau}(\beta) l'_{j_{\tau}(\beta)}}
\]

\[
= \frac{\delta_{n,m}}{MN} \sum_{\sigma, \tau} \prod_{\alpha=1}^{n} \delta_{k_{\sigma}(\alpha), k'_{\tau}(\alpha)} \delta_{l_{j_{\sigma}(\alpha)}, l'_{j_{\tau}(\beta)}}
\]

\[
= \frac{\delta_{n,m}}{MN} n! \prod_{\alpha=1}^{n} \delta_{k_{\sigma} k'_{\tau}} \delta_{l_{j_{\sigma}}} l'_{j_{\tau}},
\]

(3)
where we have used that $Z$ is a matrix of i.i.d. Gaussians with $\langle Z_{kl} \rangle = 0$ and $\langle |Z_{kl}|^2 \rangle = 1/M$. Inserting the result of Eq. (3) into Eq. (2), introducing the binomial distribution

$$B_n(x) \equiv \binom{N}{n} x^n (1-x)^{N-n}$$

and rearranging the summations (with $n$ and $N-n$ interchanged) we obtain the probability as follows

$$p(l_1 \ldots l_N | 1 \ldots N) = \sum_{n=0}^N B_n(1-\epsilon) \frac{(N-n)!}{M^{N-n}} \left( \begin{array}{c} N \\ n \end{array} \right)^{-1} \sum_{k_1 \ldots k_n} p_q^{(U)}(l_1, \ldots, l_n | k_1 \ldots k_n),$$

(5)

where

$$p_q^{(U)}(l_1, \ldots, l_n | k_1 \ldots k_n) = |\text{per} U(k_1 \ldots k_n | l_1 \ldots l_n)|^2,$$

(6)
i.e., the probability of $n$ indistinguishable bosons, sent to distinct input ports $k_1, \ldots, k_n$ of the unitary network $U$, to be detected at distinct output ports $l_1, \ldots, l_n$.

The factors on the right hand side of Eq. (5) have physical interpretation as some probabilities. The factor $B_n(1-\epsilon)$ is the probability that only $n$ of the total $N$ input bosons, from $n$ random input ports, remain indistinguishable during the propagation in the noisy network. These bosons contribute the probability factor

$$\frac{1}{p_q^{(U)}(l_1 \ldots l_n | k_1 \ldots k_n)} \equiv \left( \begin{array}{c} N \\ n \end{array} \right)^{-1} \sum_{k_1 \ldots k_n} p_q^{(U)}(l_1, \ldots, l_n | k_1 \ldots k_n).$$

(7)

The rest $N-n$ of the input bosons behave as distinguishable bosons (or classical particles), uniformly randomly populating the output ports, thus contributing the probability factor $(N-n)!/M^{N-n}$. Observe that the noise in BosonSampling is represented as uniform losses with the transmission $1-\epsilon$ (Eq. 4) is equivalent to that for a BosonSampling with a lossy input, with only $n$ out of $N$ bosons making to a network), compensated by dark counts (noise at output ports): there so many (uniformly distributed) dark counts over the output ports as lost bosons, so that the total number of detector clicks is equal to the total number of input bosons. The action of noise is therefore similar to the model of shuffled bosons discussed in Ref. [3] (note, however, that the probability distribution of our “dark counts” is different from that of physical dark counts of detectors, see more on this in section V). We will use this equivalence in section VIII.

V. EXTENDING THE GAUSSIAN NOISE MODEL BEYOND THE NO-COLLISION REGIME

The discrete representation of the Gaussian noise, derived in the previous section, allows one to extend the noisy BosonSampling model of Ref. [3] for arbitrary $N, M$, i.e., beyond the usual no-collision regime. Generally speaking, such an extension is not unique. We choose an extension that has a similar effect on the many-boson interference in the general case as the Gaussian noise has in the no-collision regime. Since all the factors in Eq. (5) are interpreted as some probabilities, such an extension can be easily obtained by generalising them and the summation over distinct output ports $l_1, \ldots, l_n$ in Eq. (5) for general $N, M$.

For arbitrary $N, M$ there are multiply occupied output ports, hence, we have a multi-set $l_1 \leq \ldots \leq l_N$. Let us fix the notations for below. We denote by $m = (m_1, \ldots, m_M)$, $|m| \equiv m_1 + \ldots + m_M = N$, the output configuration corresponding to a multi-set $l_1 \leq \ldots \leq l_N$ (where $m_l$ is the total number of bosons in output port $l$), whereas $s = (s_1, \ldots, s_M)$, $|s| = n$, and $r = (r_1, \ldots, r_M)$, $|r| = N - n$, will denote the output configurations corresponding, respectively, to the multi-subsets $l_1 \leq \ldots \leq l_n$ and $l_{n+1} \leq \ldots \leq l_j$ of the above multi-set (thus $m = r+s$). We also set $|m|! \equiv m_1! \ldots m_M!$.

First, the probability of an output configuration $r$ obtained by uniformly randomly distributing $N-n$ distinguishable bosons, or, equivalently [40], indistinguishable classical particles over the output ports reads

$$q(r) = \frac{(N-n)!}{r!M^{N-n}}.$$

(8)

Eq. (8) can be derived as follows. Assume that classical particles are enumerated (i.e., classical and distinguishable), then the probability to get any given output is $1/M^{N-n}$, since each time one chooses one of $M$ ports uniformly randomly. Ignoring the identities of the particles, we obtain for an output configuration $r$ exactly $(N-n)!/r!$ distributions of distinguishable classical particles, i.e., the probability in Eq. (8).

Second, the summation over the subsets of distinct output ports $l_1 < \ldots < l_j$, (equivalently, over the subindices $1 \leq j_1 < \ldots < j_n \leq N$) in Eq. (5) must be replaced for general relation between $N$ and $M$ by the summation over all sub-configurations $s \subset m$ (i.e., $s_\alpha \leq m_\alpha$ for all $\alpha = 1, \ldots, M$), each corresponding to a multi-set of the output ports $l_{j_1} \leq \ldots \leq l_{j_n}$, to ensure that we do not double count the same output configurations (since different subsets $1 \leq j_1 < \ldots < j_n \leq N$ correspond to the same output configuration $s$). There is a mathematical relation between the two types of summations valid for any symmetric function $f(l_1, \ldots, l_N)$ (e.g., Ref. [44])

$$\sum_{s \subseteq m, |s| = n} f(l_1, \ldots, l_N) = \sum_{j_1 \ldots j_n} f(l_1, \ldots, l_N) \prod_{\alpha=1}^M \left( \begin{array}{c} m_\alpha \\ s_\alpha \end{array} \right)^{-1}.$$

(9)

With the above two observations, replacing the multi-set output port indices by the corresponding occupations, i.e., $l_1 \leq \ldots \leq l_N$ by $m$ and $l_{j_1} \leq \ldots \leq l_{j_n}$ by $s$, the probability distribution of Eq. (5) generalised beyond
the no-collision regime becomes
\[
\langle p^{(U)}(m|1\ldots N) \rangle \equiv \sum_{n=0}^{N} B_n (1 - \epsilon) \times \binom{N-1}{n}^{-1} \sum_{k_1\ldots k_n} \sum_{s \in \mathbb{C}_m, |s| = n} q(r) p_q^{(U)}(s|k_1\ldots k_n),
\]
(10)

here \( p^{(U)}(s|k_1\ldots k_n) \) is the probability for indistinguishable bosons from input ports \( k_1,\ldots,k_n \) to be detected in output configuration \( s \) corresponding to multi-set of output ports \( l_1,\ldots,l_n \). \[^{22,26}\]

\[
p_q^{(U)}(s|k_1\ldots k_n) = \frac{1}{s!} |\text{per} U(k_1\ldots k_n|l_{j_1}\ldots l_{j_n})|^2,
\]
(11)

generalising that of Eq. \((9)\). One can verify by straightforward calculation with use of the summation identity \[^{14}\],

\[
\sum_{|m| = N} f(l_1,\ldots,l_N) = \sum_{l_1=1}^{M} \cdots \sum_{l_N=1}^{M} \frac{m!}{|N|!} f(l_1,\ldots,l_N),
\]
(12)

valid for any symmetric function \( f(l_1,\ldots,l_N) \), and unitarity of the network matrix \( U \), that the probabilities in Eq. \((10)\) sum to 1,

\[
\sum_{|m| = N} \langle p^{(U)}(m|1\ldots N) \rangle = 1.
\]

The equivalent representation of the Gaussian noise found in section \[^{IV}\] Eq. \((9)\), and extended here for arbitrary \( N \) and \( M \), Eq. \((10)\), is the basis for our analysis of the effect of noise on BosonSampling.

Let us return here to the equivalence of the noisy BosonSampling model Eq. \((10)\) to that with losses and dark counts of detectors, discussed at the end of section \[^{IV}\]. There is some difference between our model of dark counts and the dark counts of independent physical detectors, with each detector following a Poisson distribution \( p(n) = \frac{\sqrt{\nu n}}{n!} e^{-\nu} \), \( \nu > 0 \). The probability of having dark counts of physical detectors in an output configuration \( r \) would be in this case given by the distribution

\[
\tilde{q}(r) = \frac{\nu |r|}{r!} e^{-\nu |r|},
\]
(13)

which is different from the probability \( q(r) \) of Eq. \((8)\) describing a model of correlated dark counts at output ports. The output distribution in Eq. \((10)\) shows nevertheless equivalence to the shuffled bosons model of Ref. \[^{43}\], where the lost bosons are compensated exactly by the dark counts of detectors in uniformly random output ports. Moreover, as we discuss below, there is also an approximate equivalence of the noisy BosonSampling model to that with partially distinguishable bosons (and no noise), which we consider in the next section.

VI. NOISE IN BOSON SAMPLING AND PARTIAL DISTINGUISHABILITY OF BOSONS

The representation of the Gaussian noise by its action on many-boson interference, discussed in sections \[^{IV}\] and \[^{V}\] allows one to reinterpret the effect of noise on the classical hardness of BosonSampling to that of partial distinguishability of bosons. Such a relation will be useful in the proof of classical simulability of the noisy BosonSampling model (section \[^{VII}\]). Here we discuss such a connection in detail.

Let us distribute the distinguishable bosons in the discrete representation of the noisy BosonSampling model, Eq. \((10)\), not at the output of a network, but at the input. The probability of uniformly randomly distributing \( N - n \) particles over the output ports of Eq. \((8)\) must be replaced in this case by a classical probability of the respective output sub-configuration \( r \) of \( N - n \) distinguishable bosons (which depends on the absolute values squared of the network matrix elements). We conjecture that replacing in Eq. \((10)\) the uniform probability with a classical probability has no effect on the classical hardness of the output distribution. Indeed, classical probabilities are given by the matrix permanents of doubly-stochastic matrices (with the elements \( |U_{kl}|^2 \)), simulated by an efficient classical algorithm \[^{14}\]. The conjecture is also partially confirmed below: (i) by comparison of our condition on the noise amplitude, obtained under the conjecture in subsection \[^{VII}\]A with the previously established bounds \[^{39,40}\] and (ii) in section \[^{VII}\] where we find that conditions on classical simulability of the BosonSampling with partially distinguishable bosons and that on the noisy BosonSampling are the same for the same amplitude of respective imperfections.

Consider thus, instead of the output distribution of Eq. \((10)\), the following distribution

\[
\tilde{p}_{cl}^{(U)}(m|1\ldots N) = \frac{N}{1!} \sum_{n=0}^{N} B_n (1 - \epsilon) \binom{N-1}{n}^{-1} \times \sum_{k_1\ldots k_n} \sum_{s \in \mathbb{C}_m, |s| = n} \tilde{p}_cl^{(U)}(s|k_1\ldots k_n) \tilde{p}^{(U)}(r|k_{n+1}\ldots k_N),
\]
(14)

here the probability factor \((8)\) in Eq. \((10)\) is replaced by the classical probability of \( N - n \) distinguishable bosons, sent at the inputs \( k_{n+1},\ldots,k_N \), to be detected in the configuration \( r \) of output ports, i.e.,

\[
\tilde{p}_cl^{(U)}(r|k_{n+1}\ldots k_N) = \frac{1}{r!} \sum_{\sigma \in S_{N-n}, \alpha = n+1}^{N} \prod_{k} |U_{k_{\sigma(a)}k_{j_a}}|^2
\]

\[
= \frac{1}{r!} |\text{per} U|^2 (k_{n+1}\ldots k_N|l_{j_{n+1}}\ldots l_{j_N}),
\]
(15)

where \( l_{j_{n+1}} \leq \ldots \leq l_{j_N} \) are the output ports corresponding to \( r \). For instance, in a uniform network \( |U_{kl}| = \frac{1}{\sqrt{M}} \) the probabilities of Eqs. \((8)\) and \((15)\) coincide.

We now use the theory of partial distinguishability to present a model of distinguishability giving precisely the
output distribution of Eq. (14). Before attacking this goal, let us recall the basics of the theory (see Refs. [41, 46, 47] for details). In this theory, bosonic degrees of freedom are partitioned into two types: the operating modes and the internal states. A unitary network performs a unitary transformation $U$ of the operating modes. Assuming that bosons are in the same pure internal state (i.e., the state over all other degrees of freedom other than the operating modes [46]) one obtains the output probability formula of Eq. (11). The usual operating modes of single photons, for example, are the propagating modes of a spatial optical network. Another possibility is to use the temporal operating modes, i.e., the time-bins [16, 17]. However, $N$ single photons, usually coming from distinct sources or at different times, are not in the same internal pure state. Single photon generation is a probabilistic process (such the resolution only adds network-independent randomness to boson counts), the probability of an output configuration $m$ in the general case of partially distinguishable bosons becomes [41, 46, 47]

$$p_j^{(U)}(m|1...N) = \frac{1}{m!} \sum_{\sigma_1,\sigma_2} J(\sigma_1^{-1}\sigma_2) \prod_{k=1}^{N} U_{\sigma_1(k),k} U_{\sigma_2(k),k},$$

(16)

where, for simplicity, we use the same notation for probability as in Eq. (14), the double summation with $\sigma_1, \sigma_2$ runs over the symmetric group $S_N$ of $N$ objects and the complex-valued function $J(\sigma)$ depends on the internal state of $N$ bosons $\rho \in H^\otimes N$, where $H$ is the internal Hilbert space of single boson, and is defined as follows [41, 46]

$$J(\sigma) = \text{Tr}\{P_{\sigma} \rho\},$$

(17)

with $P_{\sigma}$ being the unitary operator representation of the symmetric group $S_N$ action in the tensor product $H^\otimes N$. In the case of completely indistinguishable bosons $J(\sigma) = 1$, for all $\sigma \in S_N$, we recover from Eqs. (16)–(17) the probability of Eq. (11).

Now, we can introduce a model of partial distinguishability which results precisely in the output distribution of Eq. (14) (another such model is presented in section VII). Consider $N$ single bosons, where boson at input port $k$ is in the following internal state

$$\rho_k = (1 - \epsilon) |\phi_0\rangle \langle \phi_0| + \epsilon |\phi_k\rangle \langle \phi_k|,$$

(18)

where $|\phi_i\rangle = |\delta_{i,j}\rangle$ for all $i, j \in \{0, 1, ..., N\}$. Eq. (18) means that with probability $1 - \epsilon$ boson $k$ is in the common pure state $|\phi_0\rangle$ and with probability $\epsilon$ in a unique orthogonal state $|\phi_k\rangle$. The internal state of $N$ such independent bosons reads $\rho = \rho_1 \otimes ... \otimes \rho_N$. By Eq. (18) and orthogonality of the specific states $|\phi_k\rangle$, the probability to have $n$ indistinguishable single bosons (at randomly chosen input ports) is $B_n(1 - \epsilon)$, where $B_n(x)$ is the binomial distribution [4]. The partial distinguishability function is straightforward to obtain from the definition (17). By expanding the tensor product $\rho_1 \otimes ... \otimes \rho_N$ using the expression of Eq. (18) and substituting the result into Eq. (17) we get

$$J(\sigma) = \sum_{n=0}^{N} (1 - \epsilon)^n \epsilon^{N-n} \sum_{k_1,...,k_n} \prod_{\alpha=n+1}^{N} \delta_{k_{\sigma(\alpha)},k_{\alpha}},$$

(19)

where the two subsets $k_1,...,k_n$ and $k_{n+1},...,k_N$ of $1,...,N$ give the contributions from the first and the second terms in Eq. (18), respectively, (the second subset contains only the fixed points of permutation $\sigma$ by orthogonality of the unique states $|\phi_k\rangle, k = 1,...,N$). Thus for each subset $k_1,...,k_n$ the nonzero terms in Eq. (19) correspond to permutations of the type $\sigma = \tau \otimes 1$, with $\tau \in S_n$ acting on this subset and $J$ being the identity (the permutation consisting of fixed points).

Using the partial distinguishability of Eq. (19) in the general formula (16) reproduces the result of Eq. (14). Indeed, modifying slightly Eq. (19), by setting $\sigma = \sigma_2^{-1}$ and $\sigma_R = \sigma_1 \sigma_2^{-1}$, reordering the factors in the product, and using the identity of Eq. (9) we get
\[
p_j^{(U)}(m|1\ldots N) = \frac{1}{m!} \sum_{\sigma_R, \pi \in S_N} J(\sigma_R) \prod_{k=1}^{N} U^*_{\sigma_R(k), l_\sigma(k) U_{k_\alpha, l_\sigma(k)}} \prod_{\alpha=n+1}^{N} \left| U_{k_\alpha, l_\sigma(k)} \right|^2
\]

where we have taken into account that by Eq. (19) \( \sigma_R = \tau_R \otimes I \), introduced \( \nu = \pi \tau_R^{-1} \) to factor the summations over the permutations, and used the underbraces to show the factorisation of permutation \( \sigma \in S_N \) as follows \( \sigma = (\pi \otimes \mu) \rho \) with arbitrary permutations \( \pi \in S_n \) and \( \mu \in S_{N-n} \) and \( \rho \in S_N/\left( S_n \otimes S_{N-n} \right) \) (i.e., from the factor group) selecting a subset \( j_1, \ldots, j_n \) from 1, \ldots, \( N \).

### A. Sufficient condition on noise amplitude for closeness of noisy BosonSampling and the ideal one

In Ref. [42] a plausibly tight bound was found for closeness in the total variation distance of the output distribution of an imperfect BosonSampling with partially distinguishable bosons to that of the ideal BosonSampling. Here we consider the bound in relation to the noisy BosonSampling model. The bound is as follows. The output probability distribution \( p^{(U)}_j \) of BosonSampling with \( N \) partially distinguishable bosons in an internal state given by \( J(\sigma) \) (and no other imperfections in the setup) is \( 1-d_J(N) \) close in the total variation distance to the ideal BosonSampling \( p^{(U)}_q \),

\[
\mathcal{D}(p^{(U)}_j, p^{(U)}_q) = \frac{1}{2} \sum_{|m| = N} |p^{(U)}_q(m) - p^{(U)}_j(m)| \leq 1 - d_J(N),
\]

where

\[
d_J(N) = \frac{1}{N!} \sum_{\sigma \in S_N} \text{Tr}\{P_\sigma \rho^{(\text{int})}\} = \frac{1}{N!} \sum_{\sigma \in S_N} J(\sigma).
\]

The bound of Eq. (20) can be easily understood: the quantity \( d_J(N) \) of Eq. (21) is the magnitude of projection of the subspace of the Hilbert space \( H^\otimes N \) consisting of the completely symmetric internal states (such an internal state corresponds to the completely indistinguishable bosons [44]). Indeed, the (non-normalised) projection reads

\[
\rho^{(S)} = \hat{S}_N \rho \hat{S}_N, \quad \hat{S}_N = \frac{1}{N!} \sum_{\sigma \in S_N} P_\sigma,
\]

giving \( d_J(N) = \text{Tr}\{\rho^{(S)}\} = \text{Tr}\{\hat{S}_N \rho\} \) by the fact that \( S_N = \hat{S}_N \). We obtain from Eqs. (19) and (22)

\[
d_J(N) = \frac{1}{N!} \sum_{n=0}^{N} \epsilon^n (1-\epsilon)^{N-n} \sum_{k_1 \ldots k_n} \sum_{\sigma \in S_N} \prod_{\alpha=n+1}^{N} \delta_{k_\sigma(k\alpha), k\alpha}
\]

\[
= \frac{1}{N!} \sum_{n=0}^{N} \epsilon^n (1-\epsilon)^{N-n} \sum_{k_1 \ldots k_n} \sum_{\sigma \in S_N} \prod_{\alpha=n+1}^{N} 1
\]

\[
= (1-\epsilon)^N \sum_{n=0}^{N} \frac{1}{n!} \left( \frac{\epsilon}{1-\epsilon} \right)^n \geq (1-\epsilon)^N,
\]

where for small \( \epsilon \) (our case) the lower bound is numerically very close to the exact value. Eqs. (20) and (22) tell us that if \( \epsilon \leq 1 - (1-\epsilon)^N = (1 + |O(\epsilon)|)\epsilon/N \), i.e., for \( \epsilon < 1 \) if

\[
\epsilon \leq \frac{\epsilon}{N},
\]

is satisfied, the BosonSampling with partially distinguishable bosons with the distinguishability function of Eq. (19) is \( \epsilon \)-close in the total variation distance to the ideal BosonSampling. Assuming the equivalence between the two models of imperfections in their effect on the
computational complexity of BosonSampling, Eq. (24) applies also to the noisy BosonSampling. Note that Eq. (24) agrees with the conclusion of Ref. 33 on noise stability of BosonSampling for noise amplitudes $\epsilon = o(1/N)$ and with the sufficient condition on network noise amplitude of Refs. 39, 40. This agreement supports our conjecture on the equivalence of the effect of noise to that of distinguishability on BosonSampling. However, in section VIII we will show that condition (24) is not necessary for classical hardness of noisy BosonSampling.

VII. EFFICIENT CLASSICAL SIMULATION OF THE NOisy BOSONSAMPLING WITH $\Theta(1)$-NOISE

In this section we further explore the connection of the noisy BosonSampling to that with partially distinguishable bosons, discussed in section VII. This connection allows us to prove theorem 1 of section III. It should be stressed, that our proof does not depend on the conjectured equivalence of the two models of BosonSampling. The main utility of the above connection is to simplify some technical calculations, that are much easier for the partial distinguishability model than for the noise model, but, as we show, the results apply to the latter model as well. Most of the technical calculations for the BosonSampling model with partially distinguishable bosons have been performed before in Ref. 41, in the no-collision regime. Below we consider this regime only and heavily rely on the results of Appendix B of Ref. 41. Moreover, at the end of this section, we provide an additional argument supporting the conjectured equivalence of the noisy BosonSampling model Eq. (10) and that with partially distinguishable bosons Eq. (11) in terms of the classical simulability.

Let us first derive an equivalent representation of the distinguishability function of Eq. (19). Rearranging the summation in Eq. (19), we get

$$J(\sigma) = \sum_{s=0}^{N} \epsilon^s (1-\epsilon)^{N-s} \sum_{k_1, \ldots, k_s} \prod_{\alpha=1}^{s} \delta_{k_{\sigma(\alpha)}, k_{\alpha}}$$

$$= \sum_{s=0}^{N} \epsilon^s (1-\epsilon)^{N-s} \sum_{k_1, \ldots, k_s} \sum_{m_1, \ldots, m_s} I_{D_{k_{m_1+1}, \ldots, k_N}} (\sigma)$$

$$= \sum_{m=0}^{N} \sum_{k_1, \ldots, k_m} m \sum_{s=0}^{m} \epsilon^s (1-\epsilon)^{N-s} I_{D_{k_{m+1}, \ldots, k_N}} (\sigma)$$

$$= \sum_{m=0}^{N} \sum_{k_1, \ldots, k_m} (1-\epsilon)^{N-m} I_{D_{k_{m+1}, \ldots, k_N}} (\sigma)$$

$$= (1-\epsilon)^{N-C_1(\sigma)},$$

(25)

where $k_1, \ldots, k_s$ and $k_{s+1}, \ldots, k_N$ correspond to the second and first terms in Eq. (13), respectively, we have rewritten the summation introducing the total number $(m)$ of fixed points of permutation $\sigma$, introduced the indicator function $I_{D_{k_1, \ldots, k_s}} (\sigma)$ of the derangements of $k_1, \ldots, k_s$, the total number of fixed points $C_1(\sigma)$ of $\sigma$ (see, e.g., Ref. 49) and observed that

$$\sum_{k_1, \ldots, k_n} I_{D_{k_1, \ldots, k_n}} (\sigma) = \delta_{C_1(\sigma), N-n}.$$

Eq. (25) means that in the model of Eq. (13), a permutation $\sigma$ is weighted according to the number derangements $N - C_1(\sigma)$, thus permutations with more fixed points $C_1(\sigma)$ contribute with a larger weight to the output probability.

There is an equivalent model of partial distinguishability of bosons, giving the same distinguishability function (25), which consists of bosons in pure internal states $|\psi_k\rangle$ with a uniform overlap $\langle\psi_k|\psi_l\rangle = 1-\epsilon$ for $k \neq l$. Precisely this model was used previously 35, 38 in the proof of classical simulability of BosonSampling with partially distinguishable bosons. This simplifies our task, as we can follow the previous approach. The main step is to consider the following auxiliary version of an imperfect BosonSampling with partially distinguishable bosons and reduced many-boson interference

$$\tilde{J}(\sigma) = (1-\epsilon)^{N-C_1(\sigma)} \theta(C_1(\sigma) - N + R),$$

$$\theta(m) = \begin{cases} 1, & m \geq 0, \\ 0, & m < 0 \end{cases}$$

(26)

Now we consider the output probability distributions of the above two models. BosonSampling with distinguishability of Eq. (25) in the no-collision regime has the output probability distribution given by Eq. (24), which we reproduce here (omitting the input ports 1, $\ldots$, $N$, for simplicity)

$$p(l_1, \ldots, l_N) = \sum_{n=0}^{N} B_n (1-\epsilon)^{(N-n-1)}$$

$$\times \sum_{j_1, \ldots, j_n} \sum_{k_1, \ldots, k_n} p_{\text{cl}}(l_1, \ldots, l_n|k_1, \ldots, k_n) \times p_{\text{cl}}(l_{n+1}, \ldots, l_N|k_{n+1}, \ldots, k_N).$$

To obtain the output probability distribution for the model of Eq. (26), we rewrite the $\tilde{J}$-function as the second line of Eq. (19) with the right hand side multiplied by $\theta(C_1(\sigma) - N + R)$, i.e.,

$$\tilde{J}(\sigma) = \sum_{n=0}^{N} (1-\epsilon)^n \epsilon^{N-n} \sum_{k_1, \ldots, k_n} \sum_{\tau \in S_n} \theta(C_1(\tau) - n + R) \delta_{\sigma, \tau \otimes I},$$

(28)

where we have used that $C_1(\sigma) = N - n + C_1(\tau)$ for $\sigma = \tau \otimes I$. Comparing with Eq. (19) we obtain from Eq. (27) the probability distribution for the auxiliary
BosonSampling model of Eq. (29)
\[
\bar{p}(l_1 \ldots l_N) = \sum_{n=0}^{N} B_n (1 - \epsilon)^{n} \binom{N}{n}^{-1} \times \sum_{j_1 \ldots j_n \in [N]} \sum_{k_1 \ldots k_n \in [N]} \bar{p}_q(U)(l_{j_1} \ldots l_{j_n}|k_1 \ldots k_n),
\]
\[
	imes p_d(U)(l_{k_n+1} \ldots l_{k_N}|k_{n+1} \ldots k_N)
\]
where now
\[
\bar{p}_q(U)(l_{j_1} \ldots l_{j_n}|k_1 \ldots k_n) = \sum_{\sigma_1, \sigma_2} \theta(C_1(\sigma_1 \sigma_2^{-1}) - n + R) \prod_{n=1}^{N} U_{k_{\sigma_1(n)}, l_{j_n}} U_{k_{\sigma_2(n)}, l_{j_n}},
\]
with the double summation over $\sigma_1, \sigma_2 \in S_n$. The $\theta$-function in Eq. (30) cuts-off the many-boson interferences at the order $R$ of the interaction in Eq. (30) to cut-off the many-boson interference at the order $R$ of the interaction in Eq. (30) to cut-off the many-boson interference.

For Eqs. (27) and (29), we replace the classical probability factors by the average in a random network $U$, i.e., $\langle p_d(U)(l_{j_1} \ldots l_{j_n}|k_1 \ldots k_n) \rangle = \frac{(N-n)!}{M^{N-n}}$ (in the no-collision regime), we obtain the noisy BosonSampling model of Eq. (5) and, what we will call below, the R-modified noisy BosonSampling model. The latter is our main auxiliary device for the proof of the classical simulability of the noisy BosonSampling with $\Theta(1)$-noise.

It is known that, on average in a random network $U$, the output probability distributions of Eqs. (27) and (29) are close in the total variation distance [35]. In our proof of efficient classical simulability of the noisy BosonSampling model, however, we will need some technical details of derivation of an upper bound on the total variation distance between two distributions of BosonSampling with different partial distinguishability functions [41].

Consider the variation distance $D$ between the output distributions of Eqs. (27) and (29) over the no-collision outputs $l_1 < \ldots < l_N$ averaged over the Haar random unitary $U$ (whereas the outputs with collisions have contribution vanishing as $O(N^2/M)$ [3, 9]). We obtain
\[
\langle D \rangle = \frac{1}{2} \sum_{l_1 \ldots l_N} \langle |p(l_1 \ldots l_N) - \bar{p}(l_1 \ldots l_N)| \rangle
\]
\[
\leq \frac{1}{2} \sum_{l_1 \ldots l_N} \sqrt{\langle (p(l_1 \ldots l_N) - \bar{p}(l_1 \ldots l_N))^2 \rangle}
\]
\[
= \frac{1}{2} \sum_{l_1 \ldots l_N} \frac{N!}{M^N} \left[ \frac{1}{N!} \sum_{\sigma \in S_N} \left( J(\sigma) - \bar{J}(\sigma) \right)^2 \chi(C_1(\sigma)) \right]^{1/2}
\]
\[
= \frac{1}{2} \left[ \frac{N!}{N!} \sum_{\sigma \in S_N} \left( J(\sigma) - \bar{J}(\sigma) \right)^2 \chi(C_1(\sigma)) \right]^{1/2},
\]
where we have used that for any real random variable $X$ $\langle |X| \rangle \leq \sqrt{\langle X^2 \rangle}$ [32], that the average probability is equal to the inverse of the number of no-collision output configurations (approximately $M^N/N!$) and that the average of the squared difference reads
\[
\langle (p(l_1 \ldots l_N) - \bar{p}(l_1 \ldots l_N))^2 \rangle
\]
\[
= \frac{N!}{M^N} \sum_{\sigma \in S_N} \left[ J(\sigma) - \bar{J}(\sigma) \right]^2 \chi(C_1(\sigma)),
\]
with
\[
\chi(n) = \sum_{\tau \in S_n} 2^{C_1(\tau)} = n! \sum_{k=0}^{n} \frac{1}{k!}.
\]
The derivation of Eq. (32) just repeats that of Appendix B in Ref. [41], where one only has to replace the distinguishability function of the ideal BosonSampling $\bar{J}(\sigma) = 1$ by that of the model (20).

The right hand side of Eq. (32) can be easily estimated. Introducing the variable $s = C_1(\sigma)$ and using the $\bar{J}$-functions from Eqs. (25) and (29) we obtain
\[
\frac{1}{N!} \sum_{\sigma \in S_N} \left[ J(\sigma) - \bar{J}(\sigma) \right]^2 \chi(C_1(\sigma))
\]
\[
= \frac{1}{N!} \sum_{s=0}^{N-R-1} \chi(s)(1 - \epsilon)^{2[N-s]} \sum_{\sigma \in S_N} \delta_{C_1(\sigma), s}
\]
\[
< \sum_{s=0}^{N-R-1} (1 - \epsilon)^{2[N-s]} \left( 1 + \frac{\epsilon}{N - s + 1} \right)
\]
\[
< \left( 1 + \frac{\epsilon}{R + 2} \right) \frac{N-R-1}{R+1} (1 - \epsilon)^{2[R-1]}
\]
\[
< \left( 1 + \frac{\epsilon}{R + 2} \right) \frac{1}{1 - (1 - \epsilon)^{2}},
\]
where we have used the bound $\chi(s) < s!e$ and bounded the expression for the number of derangements with $N-s$ elements [49].

\[
\sum_{\sigma \in S_N} \delta_{C_1(\sigma), s} = \frac{N!}{s!} \sum_{i=0}^{N-s} (-1)^i \frac{i!}{i!},
\]
as follows
\[
\sum_{i=0}^{N-s} (-1)^i \frac{i!}{i!} < \begin{cases} e^{-1}, & N - s = \text{odd}, \\ 1 + \frac{1}{(N-s+1)!}, & N - s = \text{even} \end{cases}
\]
(the sum of two consecutive terms is always positive in the odd case, the even case is reduced to the odd one by adding the term with $i = N - s + 1$). Substitution of the expression in Eq. (34) into Eq. (31) gives the upper bound for the average variation distance over the non-collision outputs between the distributions in Eqs. (27) and (29)
\[
\langle D \rangle < \frac{1}{2} \left( 1 + \frac{\epsilon}{R + 2} \right) \frac{1}{\sqrt{e(2 - \epsilon)}}.
\]
We obtain that for any $\varepsilon > 0$ and the cut-off order of the many-boson interference $R$ given by

$$R = \frac{\ln \left( \frac{\sqrt{2}}{\sqrt{\varepsilon}} \right)}{\ln \left( \frac{1}{\varepsilon} \right)}$$

the average variation distance \[^{[55]}\] between the probability distributions of Eqs. \[^{[27]}\] and \[^{[29]}\] is bounded as $\langle D \rangle < \varepsilon / 2$. It is easy to see that for any given error $\varepsilon$, to have $R = O(1)$, as $N \to \infty$, requires that $\varepsilon$ is bounded from below away from zero, i.e., $\varepsilon = \Theta(1)$.

The remaining step is to show that the bound of Eq. \[^{[55]}\] applies also to the variation distance between the output distributions of the noisy BosonSampling model and the $R$-modified noisy BosonSampling model both obtained from Eqs. \[^{[27]}\] and \[^{[29]}\], respectively, by replacing the classical probability factors in the respective output distributions by the average value in a random network $U$. To this goal, we need some facts established previously \[^{[41]}\]. The average in Eq. \[^{[55]}\] is equal to the variance of the difference, since in the no-collision regime the average probability is uniform over the output ports, independently of the distinguishability function \[^{[41]}\]. Replacing the classical probability factor by its average in the output distributions of the two BosonSampling models of Eqs. \[^{[27]}\] and \[^{[29]}\] (note that we replace the same classical factors in both models Eqs. \[^{[27]}\] and \[^{[29]}\] for each $N$, containing the same distinct matrix elements of $U$) can only decrease the variance of the difference in Eq. \[^{[55]}\] (due to mutual independence of the random unitary matrix elements in the no-collision regime \[^{[3]}\]). Therefore, the variation distance between the noisy BosonSampling of Eq. \[^{[5]}\] and the $R$-modified noisy BosonSampling is also bounded from above by the right hand side of Eq. \[^{[55]}\].

The final step is based on the fact that the $R$-modified noisy BosonSampling model, similar as its parent model Eq. \[^{[29]}\], allows only up to $R$-boson interferences. Therefore, this model is efficiently simulated classically for $R = O(1)$, with at most $O(R^2) + \text{poly}(M,N)$ classical computations \[^{[22]}\]. An explicit algorithm for the parent model of Eq. \[^{[29]}\], i.e., the BosonSampling with partially distinguishable bosons of Eq. \[^{[26]}\], was proposed before in Refs. \[^{[33]}\] \[^{[37]}\] \[^{[38]}\] (see the comment in the paragraph above Eq. \[^{[29]}\]). The $R$-modified noisy model differs from its parent model Eq. \[^{[29]}\] only by the classical probability factors of Eq. \[^{[29]}\] replaced by the average values, thus it also can be simulated classically by the algorithm of Refs. \[^{[33]}\] \[^{[37]}\]. By the the triangle inequality on the total variation distance and the bound in Eq. \[^{[55]}\], with the help of standard Markov’s inequality in the probability theory (see Refs. \[^{[33]}\] \[^{[37]}\]), the noisy BosonSampling Eq. \[^{[5]}\] can also be efficiently simulated classically to any small error $\varepsilon$ in the total variation distance, if the noise amplitude $\varepsilon = \Theta(1)$ (thus $R = O(1)$ by Eq. \[^{[55]}\]). The classical simulability result of the $R$ can be formulated as follows: the number of classical computations required to simulate the noisy BosonSampling model Eq. \[^{[27]}\] to a given error $\varepsilon$ and with the probability of success at least $1 - \delta$ for any $\delta > 0$ is a polynomial function of $(N, 1/\varepsilon, 1/\delta)$ for $\varepsilon = \Theta(1)$. Hence, we have proven theorem 1 of section \[^{[III]}\].

Now let us examine by how much the variance of the difference in Eq. \[^{[52]}\] decreases after replacing the classical probability factors in Eqs. \[^{[27]}\] and \[^{[29]}\] by the average values. The change of the variance in the Gaussian approximation comes from the fourth-order moments $\langle U_{kl} \rangle^4 = 2M^2$ in the average of a squared classical probability factor being replaced by $\langle (U_{kl}^2) \rangle^2 = 1/M^2$, thus, for each such replacement, the variance loses a factor 2 in Eq. \[^{[53]}\] (here we use that $\chi(n)$ Eq. \[^{[55]}\] accounts for the relative fourth-order moments $\langle U_{kl} \rangle^4 / \langle (U_{kl}^2) \rangle^2 = 2$, when evaluating the average in Eq. \[^{[52]}\] in the Gaussian approximation, see Appendix B in Ref. \[^{[11]}\]). But such fourth-order moments are multiplied by $\varepsilon^2$ each, since the number of $|U_{kl}|^2$ in the classical probability factor in Eqs. \[^{[27]}\] and \[^{[29]}\] $(N - n$ for each term in the summation over $n$) is precisely the number of noise clicks each having $\varepsilon$ as a factor. The decrease in the variance is therefore on the order $O(\varepsilon^3)$. Since the variance of Eq. \[^{[52]}\] gives the bound of Eq. \[^{[55]}\] on the total variation distance, the very small difference in the variance when switching between the noisy BosonSampling model and that with partially distinguishable bosons supports the conjecture of section \[^{[VII]}\] on their equivalence in terms of classical simulability.

**VIII. HOW MANY “NOISE CLICKS” IN NOISY BOSONSAMPLING?**

Let us return to the output probability distribution of Eq. \[^{[10]}\], i.e., the equivalent representation of the noisy BosonSampling model. It replaces the continuous model of noise by an equivalent one with discrete “noise clicks” (i.e., lost bosons compensated by random detector counts). Consider now the noisy BosonSampling model with $N$ input bosons, but with the number of noise clicks bounded by $R - 1$ (in terms of Eq. \[^{[10]}\] this amounts to imposing the cut-off on the summation index $N - R + 1 \leq n \leq N$). What is the scaling $R = R(N, \varepsilon)$ sufficient to approximate the output probability distribution of a given noisy BosonSampling \[^{[10]}\] to a total variation distance $D \leq \varepsilon$? The question is answered below and the answer allows to prove theorem 2 of section \[^{[III]}\] by using the results of Ref. \[^{[43]}\].

The difference of the output probabilities of the above discussed two models of the noisy BosonSampling reads

$$\Delta p(m) = \sum_{n=0}^{N-R} B_n (1 - \varepsilon) \times \binom{N}{n}^{-1} \sum_{k_1 \ldots k_n \subset m} \sum_{s \subset m} \frac{(N - n)!}{r!M^{N-n} p(U)(l_1, \ldots, l_n | k_1, \ldots, k_n)}.$$

(37)
Noticing that $\Delta p(m) \geq 0$ the calculation of the total variation distance amounts to summation of the result in Eq. (37) over all output configurations $m$ of $N$ bosons. Since, for a fixed $n$, the respective factor given by the second line in Eq. (38) is a probability (of some probability distribution over the output configurations), the summation over $m$ gives nothing else than the partial sum of the Binomial distribution

$$D = \frac{1}{2} \sum_{n=0}^{N-R} B_n(1-\epsilon) = \frac{1}{2} \sum_{s=R}^{N} B_s(\epsilon). \quad (38)$$

One immediate consequence of Eq. (38) is that for the noisy BosonSampling with the noise amplitude $\epsilon = O(1/N)$ for any given small error $\epsilon$ taking $R = O(1)$ is sufficient to bound the total variation distance $D \leq \epsilon$ between the two models of noise. Recall that for $\epsilon = O(1/N)$ the binomial distribution $B(\epsilon)$ is approximated by the Poisson distribution with the expected number of clicks $\langle R \rangle = \epsilon N = O(1)$, thus bounding the number of clicks by $R$, such that $R > \langle R \rangle$, is sufficient to approximate the distribution (i.e., make the tail in Eq. (38) arbitrarily small). Below this is proven with the use of Hoeffding’s bound in the probability theory [50].

Hoeffding’s bound (see theorem 1 in Ref. [50]), applied here for the partial sum of the Binomial distribution represented as $N$ i.i.d. trials $x_i \in \{0,1\}$, where $x_i = 1$ with probability $\epsilon$, states that for $\epsilon < R/N < 1$

$$\sum_{i=1}^{N} x_i \geq R \Rightarrow \left( \frac{N\epsilon}{R} \right)^R \left( \frac{1-\epsilon}{1-R/N} \right)^{N-R} \leq 2\epsilon, \quad R > \epsilon N, \quad (39)$$

We get that if there is such $R$ satisfying

$$\left( \frac{N\epsilon}{R} \right)^R \left( \frac{1-\epsilon}{1-R/N} \right)^{N-R} \leq 2\epsilon, \quad R > \epsilon N, \quad (40)$$

then the BosonSampling with the number of noise clicks bounded by $R - 1$ is within the total variation distance $\epsilon$ from the original noisy BosonSampling of Eq. (10). Consider the second factor in the first inequality in Eq. (10) and set $X \equiv (R/N - \epsilon)/(1-\epsilon)$ then

$$(1-\epsilon)^{N-R} = \exp\{- (N-R) \ln(1-X) \}$$

$$\leq \exp\left\{ \frac{(N-R)X}{1-X} \right\} = \exp\left\{ \frac{(N-R)(R/N-\epsilon)}{1-R/N} \right\}$$

$$= \exp\{R - N\epsilon\},$$

where we have used that $- \ln(1-X) \leq X/(1-X)$ (here $0 < X < 1$, under the second condition in Eq. (40)). Then, the first inequality in Eq. (10) follows from the following one

$$R - N\epsilon + R \ln \frac{N\epsilon}{R} \leq \ln(2\epsilon), \quad (41)$$

which evidently can be satisfied by choosing an appropriately large $R$, since the function on the left hand side of Eq. (11) decreases with $R$ and is negative for $R > N\epsilon$. Finally, $R = O(1)$ if $\epsilon = O(1/N)$.

The existence of a finite bound on the noise clicks and the equivalence of noise in BosonSampling to boson losses compensated by detector dark counts (see the comment at the end of section V) allows us to use the results of Ref. [12] on the classical hardness of BosonSampling under finite losses and/or dark counts. Reformulating the result, we get that the noisy BosonSampling with the noise amplitude $\epsilon = O(1/N)$ is as hard as the ideal BosonSampling for classical simulations, at least in the no-collision regime. Hence, we have proven theorem 2 of section III.

In Ref. [33] it was suggested that for $O(1/N)$-noise the noisy BosonSampling must be at a finite total variation distance from the ideal one. The same is implied also by Eqs. (20)-(24) of section VII under the conjectured equivalence of the noisy BosonSampling Eq. (10) and that with partially distinguishable bosons Eq. (14), if the bound in Eq. (20) is tight for the closeness to the ideal BosonSampling, as suggested in Ref. [12]. Nevertheless, a finite total variation distance from the ideal case caused by such a noise does not allow, by theorem 2, an efficient classical simulation of such a noisy BosonSampling model.

Therefore, at least in the no-collision regime, the boundary noise amplitude leading to efficient classical simulability of the noisy BosonSampling lies above $O(1/N)$. From section VII we know that for a finite noise $\epsilon = \Theta(1)$ the noisy BosonSampling can be efficiently simulated classically. Consider now the intermediate noise amplitudes: $\epsilon = \omega(1/N)$. An efficient classical simulation algorithm for such a noisy BosonSampling was suggested to exist in Ref. [33]. To satisfy Eq. (10) we have to use $R > \epsilon N$ which is unbounded in this case. Note, however, that the ratio of the number of noise clicks $R$ satisfying Eq. (11) to the total number of bosons $N$ in the setup can be made vanishing in this regime (e.g., take $R = \epsilon N + r$, where $r \leq \ln(2\epsilon)$), similar to the classically hard case of $\epsilon = O(1/N)$. We conjecture that no efficient classical simulation is possible in this case as well.

IX. CONCLUSION

We have analysed noisy BosonSampling with the Gaussian noise model of Ref. [33]. Using a relation between the noisy BosonSampling model and BosonSampling with partially distinguishable bosons, we have proven that there is an efficient classical simulation algorithm for $N$-boson noisy BosonSampling with a constant noise amplitude $\epsilon = \Theta(1)$ as $N \rightarrow \infty$. By using the equivalence between the noise and losses of bosons compensated by random counts of detectors, we have also proven that the noise amplitude $\epsilon = O(1/N)$ preserves
quantum advantage: such a noisy BosonSampling is as hard as the ideal one for classical simulations.

We have also analysed the intermediate regime of noise, when the noise amplitude scales as $\epsilon = \omega(1/N)$. Such a noise modifies the ideal BosonSampling by a vanishing fraction of “noise clicks”, as $N \to \infty$, in the equivalent operational representation as boson losses compensated by random counts of detectors (where the number of noise clicks is the number of lost bosons). Efficient classical simulation in this regime of noise, suggested to exist in Ref. [32], would mean an extreme sensitivity of BosonSampling to noise, or to any other imperfection affecting many-boson interference. Indeed, throughout the work we have exposed the equivalence of noise to other imperfections in the setup (boson distinguishability, losses with random counts) in terms of their effect on the quantum many-body interferences. Though our approach does resolve whether imperfections having amplitudes vanishing with the total number of bosons allow an efficient classical simulation of such imperfect BosonSampling, based on the fact that for the $\omega(1/N)$-noise the ratio of effective total number of noise clicks to the total number of bosons $N$ still vanishes as $N \to \infty$, similar as in the hard case of BosonSampling with $O(1/N)$-noise, we conjecture the persistence of the quantum advantage under $\omega(1/N)$-noise.

Moreover, an extension of the Gaussian noise model beyond the usual no-collision regime has been proposed in the present work. At least some of our conclusions are valid for general relation between total number of bosons and a network size, beyond the no-collision regime. For instance, our bound on the effective number of noise clicks in the noisy BosonSampling is applicable in the general case. However, due to technical reasons (we use previous results with restricted validity) our results on the computational complexity in theorems 1 and 2 apply to the no-collision regime only. Efficient classical algorithms due to imperfections in experimental setup exist also for BosonSampling with arbitrary relation between the total number of bosons and network size [34, 36, 37]. It is hoped that the equivalence relations between various models of imperfections/noise in BosonSampling revealed in the present work will be helpful also to understand the general case beyond the no-collision regime.

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[52] Take two mutually independent copies of \(X\), say \(X_1\) and \(X_2\), and expand \(\langle (|X_1| - |X_2|)^2 \rangle \geq 0\).