Threshold Perturbations in Current-Carrying Superconducting Bridges with a Finite Length near the Critical Temperature

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Near the critical temperature of a superconducting transition, the energy of the threshold perturbation $\delta F_{thr}$ that transfers a superconducting bridge to a resistive state at a current below the critical current $I_c$ has been determined. It has been shown that $\delta F_{thr}$ increases with a decrease in the length of a bridge for short bridges with lengths $L < \xi$ (where $\xi$ is the coherence length) and is saturated for long bridges with $L \gg \xi$. At certain geometrical parameters of banks and bridge, the function $\delta F_{thr}(L)$ at the current $I \to 0$ has a minimum at $L \sim 2 - 3\xi$. These results indicate that the effect of fluctuations on Josephson junctions made in the form of short superconducting bridges is reduced and that the effect of fluctuations on bridges with lengths $\sim 2 - 3\xi$ is enhanced.

It is known that a superconducting state becomes unstable with respect to infinitely small perturbations of the superconducting order parameter $\Delta$ when the current $I$ flowing in a superconductor is larger than a certain critical value, $I > I_c$. However, switching to a resistive state can occur at a lower current if the appearance of a finite perturbation in the system is possible. This effect is well known from the theory of Josephson junctions [1]. Such a switching in superconducting bridges/wires with a finite length $L$ was studied experimentally in [2, 3]. These perturbations are due to thermal or quantum fluctuations. If fluctuation-induced change in the order parameter $\Delta$ is small, the superconducting system returns to the equilibrium state without dissipation. However, if this change in $\Delta$ is sufficiently large, instability will be developed in the superconductor, leading to the appearance of a finite resistance and dissipation. As a result, in the presence of a sufficiently high current, the superconductor can be heated and switched to the normal state. If the energy of threshold perturbation $\delta F_{thr}$ is much higher than the thermal energy $k_BT$, the probability of the appearance of such a perturbation owing to thermal fluctuation is determined primarily by the Arrhenius factor $\exp(-\delta F_{thr}/k_BT)$.

It will be shown below that threshold perturbation $\delta F_{thr}$ in bridges with the length $L < \xi$ increases rapidly with a decrease in $L$ because of enhanced suppression of superconductivity in banks. For this reason, Josephson junctions based on short bridges/constrictions are more stable with respect to fluctuations at a decrease in the length of the bridge/constriction. At the same time, small $\delta F_{thr}$ is necessary in some other situations.

In particular, this is important when studying macroscopic quantum tunneling in superconducting systems [4]. As will be shown below, sufficiently narrow bridges with a length of about 2-3$\xi$ have the minimum $\delta F_{thr}$. Consequently, they are more preferable as compared to shorter or longer bridges for their use in devices based on quantum tunneling between different states (e.g., so-called flux qubits [5]).

To calculate threshold perturbation, it is necessary to determine a saddle-point state in the system nearest in energy to the ground state. For a long ($L \gg \xi$, where $\xi$ is the coherence length) one-dimensional (transverse dimensions smaller than $\xi$) superconducting bridge, such a problem was solved in well-known work [6]. It was found that threshold perturbation (saddle state) corresponds to a partial suppression of the superconducting order parameter in a finite segment of the bridge with dimensions of about $\xi$, and the amplitude of suppression increases with a decrease in the flowing current. Langer and Ambegaokar [6] obtained the dependence of the energy of threshold perturbation on the applied current. It is described well by the expression [7]

$$
\delta F_{LA} = \frac{4\sqrt{2}}{3} F_0 \left(1 - \frac{I}{I_{dep}}\right)^{5/4} \\
= \frac{\sqrt{6}}{2} I_{dep} \frac{h}{e} \left(1 - \frac{I}{I_{dep}}\right)^{5/4},
$$

where $F_0 = \Phi_0^2 S/32\pi^3 \lambda^2 \xi$, $\Phi_0$ is the magnetic flux quantum, $S = wd$ is the area of the cross section of the bridge with the width $w$ and thickness $d$, $\lambda$ is the London penetration depth of the magnetic field, and $I_{dep} = 2I_0/\sqrt{3}$ ($I_0 = c\Phi_0 S/8\pi^2 \lambda^2 \xi$) is the depairing current in the

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Ginzburg–Landau model, which coincides with the expected critical current of the long \((L > \xi)\) bridge.

We calculate the energy of threshold perturbation for the superconducting bridge with an arbitrary length \(L\), which can be both smaller and larger than \(\xi\). This problem is of interest because of the development of technologies and the appearance of superconducting bridges with a length of about the coherence length \(\xi\). As in \([10]\), we use the Ginzburg–Landau model; therefore, our results are applicable only near \(T_c\). We find that the current dependence of \(\delta F_{thr}\) varies smoothly from \(1\) for bridges with the length \(L \gg \xi\) to the expression \(\delta F_{thr} = \hbar I_c (1 - I/I_c)^{3/2}/e\) for bridges with the length \(L \ll \xi\), where \(I_c = I_0 \xi/L\) is the critical current of the short bridge \([9]\). In the latter case, the current dependence of \(F_{thr}(I)\) coincides with the known result for Josephson junctions with a sinusoidal current-phase relation \([11]\), where \(I_c\) is the critical current of the junction. Furthermore, we found that the suppression of the superconducting order parameter in banks of short bridges is of great importance: it is responsible for the dependence of \(\delta F_{thr}\) on the length of the bridge and the width of banks. In the onedimensional model, we obtained the dependence \(\delta F_{thr}(I = 0) \sim 1/L\) for a short bridge with \(L < \xi\). In the two-dimensional model, we found a region of the parameters where \(\delta F_{thr}(I = 0)\) depends nonmonotonically on the length of the bridge and reaches a minimum at \(L \sim 2 - 3\xi\). Our results can be used to analyze experimental data on the switching current of short superconducting bridges/wires and fluctuation resistance of bridges at temperatures near \(T_c\).

We consider a model system consisting of the superconducting bridge with the area of cross section \(S\) and length \(L\), which connects two superconducting banks (located at \(x = \pm L_{sys}/2\)).

![Superconducting bridge](image)

Fig. 1. Superconducting bridge with the area of cross section \(S\) and length \(L\) connecting superconducting banks with the area of cross section \(S_{pad}\).

dimensional and only the dependence on the longitudinal coordinate \(x\) is taken into account. In this case, the dimensionless Ginzburg – Landau equation has the form (the solution is sought in the form \(\Delta(x)/\Delta_{GL} = f(x)e^{|i\varphi(x)|}\))

\[
d^2f\quad dx^2 - \frac{j^2}{f} + f - f^3 = 0,
\]

where the condition of the constant current in the system, \(I = \text{const.}\), is used (here, \(j = f^2d\varphi/dx = I/S\) is the current density in the bridge and \(j = I/S_{pad} < I/S\) is the current density in banks). In \([5]\), the magnitude of the superconducting order parameter \(f\), length, and current density are measured in units of \(\Delta_{GL} = \Delta_{GL}(0) \sqrt{1 - T/T_c}, \xi = \xi_{GL}(0) \sqrt{T/T_c}\), and \(I_0/S\), respectively.

Equation \((3)\) should be supplemented with boundary conditions at the ends of the bridge:

\[
\frac{df}{dx} \left|_{x = \frac{L_{sys}}{2}} = \frac{S}{S_{pad}} \frac{df^C}{dx} \right|_{x = \frac{L_{sys}}{2}} = \frac{S}{S_{pad}} \frac{df^R}{dx} \left|_{x = \frac{L_{sys}}{2}} = \frac{df^L}{dx} \right|_{x = \frac{L_{sys}}{2}},\]

\(f^L \left|_{x = \frac{L_{sys}}{2}} = f^C \right|_{x = \frac{L_{sys}}{2}} = f^C \left|_{x = \frac{L_{sys}}{2}} = f^R \right|_{x = \frac{L_{sys}}{2}},\)

where \(f^L, f^C\) and \(f^R\) are the magnitudes of the order parameter in the left bank, bridge, and right bank, respectively.

Condition \((1a)\) appears from the variation of the Ginzburg – Landau functional for the superconductor with the cross section depending on \(x\) (which is responsible for the appearance of the derivative \(d/dx(S(x)df/dx)\) in the Ginzburg – Landau equation). This condition is exact in the case of a continuous change in \(S\) slow at the scale \(\xi\), where the dependence of \(f\) on the transverse coordinate can be neglected. In our model, this change is stepwise. Consequently, the ratio \(S/S_{pad}\) here is not the actual ratio of the areas of the cross sections but is a reference parameter characterizing a change in the derivative of the function \(f\) in the \(x\) direction at the transition through the bank-bridge interface. We also assume that the entire system is connected to wider banks (located at \(x = \pm L_{sys}/2\)).
where the current density is almost zero and the order parameter reaches its equilibrium value $f = 1$. In order to exclude the effect of these banks on the transport characteristics of the bridge, we set $L_{sys} - L = 20\xi$ in numerical calculations. The energy of threshold perturbation can be determined using the expression

$$\frac{\delta F_{thr}}{F_0} = F_{saddle} - F_{ground} - \frac{L}{L_0} \delta \varphi,$$  \hspace{1cm} (5)

where $\delta \varphi$ is the additional phase difference between the ends of the bridge appearing in the saddle-point state and $F_{saddle}$ and $F_{ground}$ are the dimensionless free energies of the saddle-point and ground states, respectively:

$$F_{saddle, ground} = -\frac{1}{2} \int f^4 dx.$$  \hspace{1cm} (6)

Equation (3) with boundary conditions (4) was solved numerically for arbitrary $L$ values and analytically in the limit $L \ll \xi$. In the numerical solution, we used the relaxation method: the time derivative $\partial f/\partial t$ was added to Ginzburg–Landau equation (3) and iterations were performed until the time derivative became zero within a given accuracy. To find the saddle-point state, we used the numerical method proposed in (11): at a given current, we fixed the magnitude of the order parameter $f(0)$ in the center of the bridge, allowing variations of $f$ at all other points. The state with the minimum fixed $f(0)$ value for which a steady-state solution exists is a saddle-point state. In the case of long bridges, this numerical method gives $\delta F_{thr}$ values coinciding with Eq. (11).

To analytically find the energy of the saddle-point state, we take into account that the order parameter varies rapidly at scales much smaller than $\xi$. Therefore, the linear and cubic terms can be neglected in Eq. (3) for a short bridge. In this case, we arrive at the equation

$$\frac{d^2 f}{dx^2} + \frac{j^2}{f^3} = 0,$$  \hspace{1cm} (7)

which has the first integral

$$\frac{1}{2} \left( \frac{df}{dx} \right)^2 + \frac{j^2}{2f^2} = E,$$  \hspace{1cm} (8)

and the solution

$$x = \frac{1}{2} \int_{u_1}^{u} \frac{du}{\sqrt{2Eu - j^2}} =$$  \hspace{1cm} (9)

$$= \frac{1}{\sqrt{2E}} \left( \sqrt{u - \frac{j^2}{2E}} - \sqrt{u(0) - \frac{j^2}{2E}} \right).$$

Here, $u(x) = f^2(x)$. Owing to the symmetry of the system, $\frac{du}{dx}_{x=0} = 0$. At this step, we assume that variations of $f$ are small in banks and use the boundary condition $u(L/2) = u(-L/2) = 1$ to find the constant $E$:

$$f = \sqrt{2E \pm x^2 + \frac{\hat{j}^2}{2E}},$$  \hspace{1cm} (10)

$$E_{\pm} = \frac{1 \pm \sqrt{1 - \left(\frac{\hat{j}}{E}\right)^2}}{L^2},$$  \hspace{1cm} (11)

where $E_+$ and $E_-$ correspond to the saddle-point and ground states, respectively, and $I_c$ is the critical current of the short bridge (9).

It is fundamentally important to take into account change in $\Delta$ in the banks when determining the energy of the saddle-point state in the case of short bridges. Otherwise, fixing $\Delta$ in the banks, as in the problem of the critical current of bridges (9), one can find (from solutions presented below; see Eq. (15)) that the energy of the saddle-point state is negative in a wide range of the current $I < I_c$.

We seek the solution in the banks in the form $f = 1 - f_1$, where $f_1 \ll 1$, and neglect the depairing effect of the current. Then, Eq. (3) for $f_1$ in the range $|x| > L/2$ becomes

$$\frac{d^2 f_1}{dx^2} - 2f_1 = 0.$$  \hspace{1cm} (12)

with the solution

$$f_1 = Ce^{\pm \sqrt{r}(x \pm L/2)},$$  \hspace{1cm} (13)
where the signs $+ \text{ and } -$ correspond to the left and right banks, respectively. The constant $C$ is determined from boundary conditions (14). When the current density in the banks is $j \ll 1$ and the ratio of the cross sections is $S/S_{pad} \ll 1$, Eq. (10) can be used for $f$ and the constant $C$ is determined as

$$C = \frac{S}{S_{pad}} \sqrt{\frac{E_+ - E_-}{2}} \quad (14)$$

Taking into account a decrease in the order parameter in the banks in Eq. (6), we obtain the following expression for the energy of threshold perturbation:

$$\frac{\delta F_{thr}}{F_0} = 2\sqrt{2} \xi \left( \frac{1}{L} \sqrt{1 - \frac{1}{\gamma^2} - \frac{\gamma^2}{2}} - \frac{2L}{5\xi} \sqrt{1 - \frac{1}{\gamma^2}} - 4\frac{\xi}{L} \arccos(\gamma) \right) \quad (15)$$

where $\gamma = I/L$. If a decrease in $\Delta$ in the banks is disregarded, the first term in Eq. (15) is absent and $\delta F_{thr} < 0$ is in a wide range of currents.

Figure 2 shows the results of the numerical calculation $\delta F_{thr}$ for bridges with different lengths in comparison with the results obtained by expressions (1) and (15). It is seen that $\delta F_{thr}$ for a bridge with the length $L = 4\xi$ is well reproduced by Eq. (1), whereas Eq. (15) almost exactly reproduces the numerical results for the bridge with the length $L = \xi$. Expression (15) for short bridges with lengths $L \ll \xi$ is close to the approximation expression

$$\delta F_{thr} = \frac{4L}{L} F_0 (1 - I/L_c)^{3/2} = \frac{L}{e} (1 - I/L_c)^{3/2} \quad (16)$$

which coincides with the known result (1) following from the theory of Josephson junctions with the sinusoidal current phase relation if $L$ is treated as the critical current of a Josephson junction.

Expression (15) was obtained under the assumption that $C \ll 1$ and $f_1 \ll 1$, which are ensured by the condition

$$\frac{S}{S_{pad}} \frac{\xi}{L} \ll 1. \quad (17)$$

Our numerical calculations show that the current dependence of the ratio $\delta F_{thr}(I/L_c)/\delta F_{thr}(0)$ varies slightly and is determined primarily by the length of the bridge even when condition (17) is invalid and $\delta F_{thr}(0)$ depends on the ratio $S_{pad}/S$ (see inset in Fig. 3).

If Eq. (15) is formally used for bridges with an arbitrary length, the dependence $\delta F_{thr}(L)$ at $I \to 0$ has the form

$$\frac{\delta F_{thr}}{F_0} (I \to 0) = \frac{4\xi}{L} + \frac{2L}{5\xi} \quad (18)$$

According to Eq. (15), $\delta F_{thr}$ should have a minimum at

$$L = \sqrt{10}\xi \approx 3\xi. \quad (19)$$

However, numerical calculations within the one-dimensional model do not confirm this result (see Fig. 3). With an increase in the length of the bridge, $\delta F_{thr}$ decreases monotonically, approaching the known value $\delta F_{thr}(0)/F_0 = 4\sqrt{2}/3 \approx 1.89$ at $L \gg \xi$ (see Eq. (1)).

However, the sizes of the bridge and banks at which the dependence $\delta F_{thr}(L)$ at $I \to 0$ is nonmonotonic can be found beyond the one-dimensional model. To this end, we considered a two-dimensional model system shown in Fig. 4. This model implies the numerical solution of the two-dimensional Ginzburg-Landau equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + f - f^3 = 0, \quad (19)$$

with fixed $f(x = \pm L_{sys}/2, y) = 1$, normal derivative $\partial f/\partial n = 0$ at the other edges of the superconducting system, and additional condition $f(x = 0, y) = 0$ (see Fig. 4) corresponding to the saddle-point state at $I \to 0$.

We considered various $w_{pad}$, $w$, and $L$. It is seen in Fig. 5 that the dependence $\delta F_{thr}(L)$ at a sufficiently small width of the bridge ($w = \xi/5$ for the parameters under consideration) has a minimum at $L \simeq 2 - 3\xi$.

We also analyzed the dependence of $\delta F_{thr}(0)$ on the width of the banks in the two-dimensional model (the data are shown in Fig. 6). As in the one-dimensional model, the energy of threshold perturbation becomes independent of the ratio $w_{pad}/w$ when the width of the bridge becomes much smaller than $w_{pad}$. However, for
The critical current depends also on the ratio $L/\xi$, which is due to the suppression of $\Delta$ in banks. It is worth noting that the fluctuation resistance of the bridge depends strongly not only on its length but also on the size of the banks in view of the dependence of $\delta F_{\text{thr}}(0)$ on $S_{\text{pad}}/S$ (see inset in Fig. 3 and Fig. 6). The critical current depends also on the ratio $S_{\text{pad}}/S$. In particular, within the one-dimensional model, it is easy to show that

$$I_c = I_0 \frac{\xi}{L} \left( 1 - \sqrt{2 \frac{S}{S_{\text{pad}}} \frac{\xi}{L}} \right)$$

under condition (17). However, since $\delta F_{\text{thr}}(0)$ appears in Eq. (20) in the exponential, variations of the size of the banks affect the fluctuation resistance $R$ more strongly than its critical current.

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