THE GAP IN PURE TRACTION PROBLEMS BETWEEN LINEAR ELASTICITY AND VARIATIONAL LIMIT OF FINITE ELASTICITY

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Abstract. A limit elastic energy for pure traction problem is derived from re-scaled nonlinear energy of an hyperelastic material body subject to an equilibrated force field.

We show that the strains of minimizing sequences associated to re-scaled non linear energies weakly converge, up to subsequences, to the strains of minimizers of a limit energy, provided an additional compatibility condition is fulfilled by the force field.

The limit energy is different from classical energy of linear elasticity; nevertheless the compatibility condition entails the coincidence of related minima and minimizers.

A strong violation of this condition provides a limit energy which is unbounded from below, while a mild violation may produce a limit energy with infinitely many extra minimizers which are not minimizers of standard linear elastic energy and whose strains are not uniformly bounded.

A relevant consequence of this analysis is that a rigorous validation of linear elasticity fails for compressive force fields that do not fulfil such compatibility condition.

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1. Introduction

Linear Theory of Elasticity ([18]) plays a relevant role among Mathematical-Physics theories for clearness, rigorous mathematical status and persistence. It was a great achievement of last centuries, which inspired many other theories of Continuum Mechanics and led to the formulation of a more general theory named Nonlinear Elasticity ([21], [31]), also known as Finite Elasticity to underline that no smallness assumptions is required.

There was always agreement in scholars community that the relation between linear and nonlinear theory amounts to the linearization of strain measure under the assumption of small displacement gradients: this is the precondition advocated in almost all the texts on elasticity. Nevertheless only at the beginning of present century, when appropriated tools of mathematical analysis were suitably tuned, the problem of a rigorous deduction of any particular theory based on some approximation hypotheses from a more general exact theory, became a scientific issue related to general problem of validation of a theory, as explained in [30].

In this conceptual framework, G. Dal Maso, M. Negri and D. Percivale in [12] proved that problems ruled by linear elastic energies can be rigorously deduced from problems ruled by non-linear energies in the case of Dirichlet and mixed boundary conditions, by exploiting De Giorgi Π-convergence theory ([11], [13]). This result clarified the mathematical consistency of the linear boundary value problems, under displacements and forces.

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prescribed on the boundary of a three-dimensional material body, via a rigorous deduction from the nonlinear elasticity theory. We mention several papers facing issues in elasticity which are connected with the context of our paper: [1], [2], [3], [4], [5], [6], [7], [8], [9], [20], [22], [23], [24], [26], [27], [28], [29].

The present paper focus on the same general question studied in [12], but here we deal with the pure traction problem, i.e. the case where the elastic body is subject to a system of equilibrated forces and no Dirichlet condition is imposed on the boundary.

Let an open set $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be the reference configuration of an hyperelastic material body, then the stored energy due to a deformation $y$ can be expressed as a functional of the deformation gradient $\nabla y$

$$\int_\Omega W(x, \nabla y) \, dx$$

where $W : \Omega \times \mathcal{M}^{N \times N} \to [0, +\infty]$ is a frame indifferent function, $\mathcal{M}^{N \times N}$ is the set of real $N \times N$ matrices and $W(x, F) < +\infty$ if and only if $\det F > 0$.

Then due to frame indifference there exists a function $\mathcal{V}$ such that

$$W(x, F) = \mathcal{V}(x, \frac{1}{2}(F^T F - I)), \quad \forall F \in \mathcal{M}^{N \times N}, \quad \text{a.e. } x \in \Omega.$$ 

We set $F = I + hB$, where $h > 0$ is an adimensional small parameter and

$$\mathcal{V}_h(x, B) := h^{-2} \mathcal{V}(x, I + hB).$$

We assume that the reference configuration has zero energy and is stress free, i.e.

$$W(x, I) = 0, \quad DW(x, I) = 0 \quad \text{for a.e. } x \in \Omega,$$

and that $W$ is regular enough in the second variable. Then Taylor’s formula entails

$$\mathcal{V}_h(x, B) = \mathcal{V}_0(x, \text{sym } B) + o(1) \quad \text{as } h \to 0_+$$

where $\text{sym } B := \frac{1}{2}(B^T + B)$ and

$$\mathcal{V}_0(x, \text{sym } B) := \frac{1}{2} \text{sym } B \, D^2 \mathcal{V}(x, 0) \, \text{sym } B.$$ 

If the deformation $y$ is close to the identity up to a small displacement, say $y(x) = x + h\nu(x)$ with bounded $\nabla \nu$ then, by setting $E(\nu) := \frac{1}{2}(\nabla \nu^T + \nabla \nu)$, one plainly obtains

$$\lim_{h \to 0} \int_\Omega \mathcal{V}_h(x, \nabla \nu) \, dx = \int_\Omega \mathcal{V}_0(x, E(\nu)) \, dx$$

The relationship (1.1) was considered as the main justification of the linearized theory of elasticity, but such point-wise convergence does not even entail that minimizers fulfilling a given fixed Dirichlet boundary condition actually converge to the minimizers of the corresponding limit boundary value problem: this phenomenon is made explicit by the Example 3.5 in [12] which exhibits a lack of compactness when $\mathcal{V}$ has several minima.

To set the Dirichlet problem in a variational perspective, referring to a prescribed vector field $\nu_0 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ as the boundary condition on a given closed subset $\Sigma$ of $\partial \Omega$ with $H^{N-1}(\Sigma) > 0$ and to a given load $g \in L^2(\Omega, \mathbb{R}^N)$, one has to study the asymptotic behavior of the sequence of functionals $I_h$, which is defined as

$$I_h(\nu) = \begin{cases} \int_\Omega \mathcal{V}_h(x, \nabla \nu) \, dx - \int_\Omega g \cdot \nu \, dx & \text{if } \nu \in H^1_{\nu_0, \Sigma} \\ +\infty & \text{else in } H^1(\Omega, \mathbb{R}^N), \end{cases}$$

where $H^1_{\nu_0, \Sigma}$ denotes the closure in $H^1(\Omega, \mathbb{R}^N)$ of the space of displacements $\nu \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ such that $\nu = \nu_0$ on $\Sigma$: it was proved in [12] that (under natural growth conditions and suitable regularity hypotheses on $W$) every sequence $\nu_h$ fulfilling

$$I_h(\nu_h) = \inf I_h + o(1)$$
has a subsequence converging weakly in $H^1(\Omega, \mathbb{R}^N)$ to the (unique) minimizer $v_*$ of the functional $I$ representing the total energy in linear elasticity, e.g.

$$I(v) = \begin{cases} \int_{\Omega} V_0(x, E(v)) \, dx - \int_{\partial\Omega} g \cdot v \, dH^{n-1}(x) & \text{if } v \in H^1_{\Omega, \Sigma} \\ +\infty & \text{else in } H^1(\Omega; \mathbb{R}^N) \end{cases},$$

and that the re-scaled energies converge, say

$$\lim_{h \to 0} I_h(v_h) = I(v_*) = \int_{\Omega} V_0(x, E(v_*)) \, dx - \int_{\Omega} g \cdot v_* \, dx.$$ 

Such result represents a complete variational justification of linearized elasticity, at least as far as Dirichlet and mixed boundary value problems are concerned. So it is natural to ask whether a similar result holds true also for pure traction problems whose variational formulation is described below.

In the present paper we focus our analysis on Neumann boundary conditions, say the pure traction problem in elasticity. Precisely we assume that $f \in L^2(\partial\Omega; \mathbb{R}^N)$, $g \in L^2(\Omega; \mathbb{R}^N)$ are respectively the prescribed boundary and body force fields, such that the whole system of forces is equilibrated, namely

$$\mathcal{L}(v) := \int_{\partial\Omega} f \cdot z \, dH^{N-1} + \int_{\Omega} g \cdot z \, dx = 0 \quad \forall z : E(z) \equiv 0$$

and we consider the sequence of energy functionals

$$\mathcal{F}_h(v) = \int_{\Omega} V_h(x, \nabla v) \, dx - \mathcal{L}(v).$$

We inquire whether the asymptotic relationship $\mathcal{F}_h(v_h) = \inf \mathcal{F}_h + o(1)$ as $h \to 0^+$ implies, up to subsequences, some kind of weak convergence of $v_h$ to a minimizer $v_0$ of a suitable limit functional in $H^1(\Omega; \mathbb{R}^N)$.

We emphasize that in the case of Neumann condition on the whole boundary things are not so plain: indeed even by choosing $\Omega$ Lipschitz and assuming the simplest dependance of the stored energy density $W$ on the deformation gradient $F$, say (see [10])

$$W(x, F) = \begin{cases} \|F^T F - I\|^2 & \text{if } \det F > 0 \\ +\infty & \text{otherwise,} \end{cases}$$

if $g \equiv 0$, $f = f n$, $f < 0$ and $n$ denotes the outward normal to $\partial\Omega$ (so that the global condition (1.2) holds true) then by the same techniques of [12] one can exhibit the $\Gamma$-limit of $\mathcal{F}_h$ with respect to weak $H^1$ topology:

$$\Gamma(w H^1) \lim_{h \to 0} \mathcal{F}_h(v) = \mathcal{E}(v),$$

where

$$\mathcal{E}(v) = 4 \int_{\Omega} |E(v)|^2 \, dx - f \int_{\partial\Omega} v \cdot n \, dH^{N-1}(x),$$

e.g. the classical linear elasticity formulation which achieves a finite minimum over $H^1(\Omega, \mathbb{R}^N)$ since the condition of equilibrated loads is fulfilled; nevertheless with exactly the same choices there is a sequence $w_h$ in $H^1(\Omega, \mathbb{R}^N)$ such that $\mathcal{F}_h(w_h) \to -\infty$ as $h \to 0^+$ (see Remark [2.3]). Although minimizers of $\mathcal{E}$ over $H^1(\Omega; \mathbb{R}^N)$ exist, functionals $\mathcal{F}_h$ are not uniformly bounded from below. These facts seem to suggest that, in presence of compressive forces acting on the boundary, the pure traction problem of linear elasticity cannot be deduced via the $\Gamma$-convergence from the nonlinear energy.

Moreover it is worth noting that if $W$ fulfills (1.4) and $g \equiv f \equiv 0$, hence $\inf \mathcal{F}_h = 0$ for every $h > 0$, then by choosing a fixed nontrivial $N \times N$ skew-symmetric matrix $W$, a real number $0 < 2\alpha < 1$ and setting

$$z_h := h^{-\alpha} W x,$$

we get $\mathcal{F}_h(z_h) = \inf \mathcal{F}_h + o(1)$, nevertheless $z_h$ has no subsequence weakly converging in $H^1(\Omega; \mathbb{R}^N)$, see Remark [2.3].

Therefore here, in contrast to [12], we cannot expect weak $H^1(\Omega; \mathbb{R}^N)$ compactness of minimizing sequences,
not even in the simplest case of null external forces: although this fact is common to pure traction problems in linear elasticity, we emphasize that in nonlinear elasticity this difficulty cannot be easily circumvented in general by standard translations since $\mathcal{F}_h(v_h) \neq \mathcal{F}_h(v_h - \mathbb{P}v_h)$, with $\mathbb{P}$ projection on infinitesimal rigid displacements. We deal this issue in the paper [25], showing nonetheless that at least for some special $\mathcal{W}$, if $\mathcal{F}_h(v_h) = \inf \mathcal{F}_h + o(1)$ then up to subsequences $\mathcal{F}_h(v_h - \mathbb{P}v_h) = \inf \mathcal{F}_h + o(1)$.

In order to have in general some kind of precompactness for sequences $v_h$ fulfilling $\mathcal{F}_h(v_h) = \inf \mathcal{F}_h + o(1)$, our approach consist in working with a very weak notion, say weak $L^2(\Omega; \mathbb{R}^N)$ convergence of linear strains: therefore the variational limit of $\mathcal{F}_h$ with respect to this convergence has to be investigated. Since w-$L^2$ convergence of linear strains does not imply an analogous convergence of the skew symmetric part of the gradient of displacements, it can be expected that the $\Gamma$ limit functional is different from the point-wise limit of $\mathcal{F}_h$.

Indeed under some natural assumptions on $\mathcal{W}$, a careful application of the Rigidity Lemma of [17] shows that if $E(v_h)$ are bounded in $L^2$ then, up to subsequences, $\sqrt{h} \nabla v_h$ converges strongly in $L^2$ to a constant skew symmetric matrix and the variational limit of the sequence $\mathcal{F}_h$, with respect to the w-$L^2$ convergence of linear strains, turns out to be a new functional

\begin{equation}
\mathcal{F}(v) := \min_{\mathcal{W}} \int_{\Omega} V_0 \left( x, E(v) - \frac{1}{2} \mathbf{W}^2 \right) \, dx - \mathcal{L}(v),
\end{equation}

where the minimum is evaluated over skew symmetric $N \times N$ matrices $\mathbf{W}$ and

\begin{equation}
V_0(x, B) := \frac{1}{2} B^T D^2 \mathbb{V}(x, 0) B \quad \forall B \in \mathcal{M}^{N \times N}_{sym}.
\end{equation}

We emphasize that the functional $\mathcal{F}$ in (1.8) is different from the functional $\mathcal{E}$ of linearized elasticity defined as

\[ \mathcal{E}(v) := \int_{\Omega} V_0(x, E(v)) \, dx - \mathcal{L}(v) \]

since if $v(x) = \frac{1}{2} \mathbf{W}^2 x$ with $\mathbf{W} \neq 0$ skew symmetric matrix, then $\mathcal{F}(v) = -\mathcal{L}(v) < \mathcal{E}(v)$. Nevertheless if $N = 2$ then (see Remark [2.5]).

\[ \mathcal{F}(v) = \mathcal{E}(v) - \frac{1}{4} \left( \int_{\Omega} V_0(x, \mathbf{I}) \, dx \right)^{-1} \left[ \left( \int_{\Omega} D V_0(x, \mathbf{I}) \cdot E(v) \, dx \right) \right]^2, \]

hence $\mathcal{F}(v) = \mathcal{E}(v)$ if

\[ N = 2, \quad \text{and} \quad \int_{\Omega} D V_0(x, \mathbf{I}) \cdot E(v) \, dx \geq 0. \]

In particular if $N = 2$ and $\mathcal{W}$ is the Green-St.Venant energy density then the previous inequality reduces to

\[ \int_{\Omega} \text{div} \, v \, dx \geq 0 \]

which means, roughly speaking, that the area of $\Omega$ is less than the area of the related deformed configuration $y(\Omega)$, where $y(x) = x + h v(x)$ and $h > 0$.

The main results of this paper are stated in Theorems [2.2] and [4.1], showing that under a suitable compatibility condition on the forces (subsequent formula (1.10)) the pure traction problem in linear elasticity is deduced via $\Gamma$-convergence from pure traction problem formulated in nonlinear elasticity, referring to weak $L^2$ convergence of the linear strains.

Precisely Theorem [2.2] states that, if the loads $\mathbf{f}$, $\mathbf{g}$ fulfill (1.2) together with the next compatibility condition

\begin{equation}
\int_{\partial \Omega} \mathbf{f} \cdot \mathbf{W}^2 x \, dH^{N-1} + \int_{\Omega} \mathbf{g} \cdot \mathbf{W}^2 x \, dx < 0 \quad \forall \text{ skew symmetric matrix } \mathbf{W} \neq 0,
\end{equation}

then every sequence $v_h$ with $\mathcal{F}(v_h) = \inf \mathcal{F}_h + o(1)$ has a subsequence such that the corresponding linear strains weakly converge in $L^2$ to the linear strain of a minimizer of $\mathcal{F}$, together with convergence (without relabeling) of energies $\mathcal{F}_h(v_h)$ to $\min \mathcal{F}$. Under the same assumptions Theorem [4.1] states that minimizers of $\mathcal{F}$ coincide with the ones of of linearized elasticity functional $\mathcal{E}$, thus providing a full justification of pure traction problems in linear elasticity at least if (1.10) is satisfied. In particular, as it is shown in Remark 2.8, this is true when
g \equiv 0, \ f = fn \text{ with } f > 0 \text{ and } n \text{ is the outer unit normal vector to } \partial \Omega, \text{ that is when we are in presence of tension-like surface forces. Moreover, if there exists an } N \times N \text{ skew symmetric matrix such that the strict inequality is reversed in (1.10), then functional } F \text{ is unbounded from below: see Remark } 4.5 \text{ and Example } 4.6. \text{ On the other hand if inequality in (1.10) is satisfied in a weak sense by every skew symmetric matrix, then argmin } F \text{ contains argmin } E, \min F = \min E \text{ but } F \text{ may have infinitely many minimizing critical points which are not minimizers of } E \text{ (see Proposition } 4.3). \text{ Summarizing, only two cases are allowed: either } \min F = \min E \text{ or } \inf F = -\infty; \text{ actually the second case arises in presence of compressive surface load. By oversimplifying we could say that } F \text{ somehow preserves memory of instabilities which are typical of finite elasticity, while they disappear in the linearized model described by } E. \text{ In the light of Theorem } 2.2 \text{ and of remarks and examples of Section 4, it seems reasonable that, as far as it concerns pure traction problems, the range of validity of linear elasticity should be restricted to a certain class of external loads, explicitly those verifying (1.10), a remarkable example in such class is a uniform normal tension load at the boundary as in Remark (2.7); while in the other cases equilibria of a linearly elastic body could be better described through critical points of } F, \text{ whose existence in general seems to be an interesting and open problem. Concerning the structure of the new functional, we emphasize that actually } F \text{ is different from the classical linear elasticity energy functional } E, \text{ though there are many relations between their minimizers (see Theorem } 4.1). \text{ Further and more detailed information about functional } F \text{ (a suitable property of weak lower semicontinuity, lack of subadditivity, convexity in } 2D, \text{ nonconvexity in } 3D) \text{ are described and proved in the paper } [25].

2. Notation and main result

Assume that the reference configuration of an elastic body is a

(2.1) \text{ bounded, connected open set } \Omega \subset \mathbb{R}^N \text{ with Lipschitz boundary, } N = 2, 3.

The generic point } x \in \Omega \text{ has components } x_j \text{ referring to the standard basis vectors } e_j \in \mathbb{R}^N; L^N \text{ and } B^N \text{ denote respectively the } \sigma\text{-algebras of Lebesgue measurable and Borel measurable subsets of } \mathbb{R}^N. \text{ For every } \alpha \in \mathbb{R} \text{ we set } \alpha^+ = \alpha \lor 0, \alpha^- = -\alpha \lor 0.

The notation for vectors } a, b \in \mathbb{R}^N \text{ and } N \times N \text{ real matrices } A, B, F \text{ are as follows: } a \cdot b = \sum_j a_j b_j; A \cdot B = \sum_{i,j} A_{i,j} B_{i,j}; |AB|_{i,j} = \sum_k A_{i,k} B_{k,j}; |F|^2 = \text{Tr}(F^T F) = \sum_{i,j} F_{i,j}^2 \text{ denotes the squared Euclidean norm of } F \text{ in the space } \mathcal{M}^{N \times N} \text{ of } N \times N \text{ real matrices; } I \in \mathcal{M}^{N \times N} \text{ denotes the identity matrix, } SO(N) \text{ denotes the group of rotation matrices, } \mathcal{M}^{N \times N}_{\text{sym}} \text{ and } \mathcal{M}^{N \times N}_{\text{skew}} \text{ denote respectively the sets of symmetric and skew-symmetric matrices. For every } B \in \mathcal{M}^{N \times N} \text{ we define } \text{sym } B := \frac{1}{2}(B + B^T) \text{ and skew } B := \frac{1}{2}(B - B^T).

It is well known that the matrix exponential maps } \mathcal{M}^{N \times N}_{\text{skew}} \text{ to } SO(N) \text{ and is surjective on } SO(N) \text{ (see [19]). Therefore for every } R \in SO(N) \text{ there exist } \vartheta \in \mathbb{R} \text{ and } W \in \mathcal{M}^{N \times N}_{\text{skew}}, |W|^2 = 2 \text{ such that } \exp(\vartheta W) = R. \text{ By taking into account that } W^3 = -W, \text{ the Taylor’s series expansion of } \vartheta \to \exp(\vartheta W) = \sum_{k=0}^{\infty} \vartheta^k W^k / k! \text{ yields the Euler-Rodrigues formula:}

(2.2) \text{exp}(\vartheta W) = R = I + \sin \vartheta W + (1 - \cos \vartheta) W^2.

In particular from (2.2) it follows that if we set

(2.3) \mathcal{K} := \{ \tau (R - I) : \tau > 0, R \in SO(N) \}

then we obtain

(2.4) \mathcal{K} = \mathcal{K} \cup \mathcal{M}^{N \times N}_{\text{skew}}.

For every } U : \Omega \times \mathcal{M}^{N \times N} \to \mathbb{R}, \text{ with } U(x, \cdot) \in C^2 \text{ a.e. } x \in \Omega, \text{ we denote by } DU(x, \cdot) \text{ and } D^2U(x, \cdot) \text{ respectively the gradient and the hessian of } g \text{ with respect to the second variable. For every displacements field } v \in H^1(\Omega; \mathbb{R}^N), \mathcal{E}(v) := \text{sym } \nabla v \text{ denotes the infinitesimal strain tensor field, } \mathcal{R} := \{ v \in H^1(\Omega; \mathbb{R}^N) : \mathcal{E}(v) = 0 \} \text{ denotes the space spanned by set of the infinitesimal rigid displacements and}
We set \( f_0 = \int_{\Omega} v dx \).

We consider a body made of an hyperelastic material, say there exists a \( \mathcal{L}^N \times \mathcal{B}^N \)-measurable \( W : \Omega \times \mathcal{M}^{N \times N} \to [0, +\infty) \) such that, for a.e. \( x \in \Omega \), \( W(x, \nabla y(x)) \) represents the stored energy density, when \( y(x) \) is the deformation and \( \nabla y(x) \) is the deformation gradient.

Moreover we assume that for a.e. \( x \in \Omega \)
\[
W(x, F) = +\infty \quad \text{if } \det F \leq 0 \quad \text{(orientation preserving condition)},
\]
\[
W(x, RF) = W(x, F) \quad \forall R \in SO(N) \quad \forall F \in \mathcal{M}^{N \times N} \quad \text{(frame indiffernce)},
\]
\[
\exists \text{ a neighborhood } \mathcal{A} \text{ of } SO(N) \text{ s.t. } W(x, \cdot) \in C^2(\mathcal{A}),
\]
\[
\exists C > 0 \text{ independent of } x : \ W(x, F) \geq C|F^T F - I|^2 \forall F \in \mathcal{M}^{N \times N} \quad \text{(coerciveness)},
\]
\[
W(x, I) = 0, \quad D W(x, I) = 0, \quad \text{ for a.e. } x \in \Omega,
\]

that is the reference configuration has zero energy and is stress free, so by (2.6) we get also
\[
W(x, R) = 0, \quad D W(x, R) = 0 \quad \forall R \in SO(N).
\]

By frame indifferece there exists a \( \mathcal{L}^N \times \mathcal{B}^N \)-measurable \( V : \Omega \times \mathcal{M}^{N \times N} \to [0, +\infty] \) such that for every \( F \in \mathcal{M}^{N \times N} \)
\[
W(x, F) = V(x, \frac{1}{2} (F^T F - I))
\]

and by (2.7)
\[
\exists \text{ a neighborhood } \mathcal{O} \text{ of } 0 \text{ such that } V(x, \cdot) \in C^2(\mathcal{O}), \text{ a.e. } x \in \Omega.
\]

In addition we assume that there exists \( \gamma > 0 \) independent of \( x \) such that
\[
|B^T D^2V(x, D) B| \leq 2 \gamma |B|^2 \quad \forall D \in \mathcal{O}, \forall B \in \mathcal{M}^{N \times N}.
\]

By (2.10) and Taylor expansion with Lagrange reminder we get, for a.e. \( x \in \Omega \) and suitable \( t \in (0, 1) \) depending on \( x \) and on \( B \):
\[
V(x, B) = \frac{1}{2} B^T D^2 V(x, tB) B.
\]

Hence by (2.12)
\[
V(x, B) \leq \gamma |B|^2 \quad \forall B \in \mathcal{M}^{N \times N} \cap \mathcal{O}.
\]

According to (2.10) for a.e. \( x \in \Omega \), \( h > 0 \) and every \( B \in \mathcal{M}^{N \times N} \) we set
\[
V_h(x, B) := \frac{1}{h^2} W(x, I + hB) = \frac{1}{h^2} \ V(x, h \text{ sym } B + \frac{1}{2} h^2 B^T B).
\]

Taylor’s formula with (2.9), (2.15) entails \( V_h(x, B) = \frac{1}{2} (\text{sym } B) D^2V(x, 0) (\text{sym } B) + o(1) \), so
\[
V_h(x, B) \to V_0(x, \text{sym } B) \quad \text{as } h \to 0_+,
\]

where the point-wise limit of integrands is the quadratic form \( V_0 \) defined by
\[
V_0(x, B) := \frac{1}{2} B^T D^2V(x, 0) B \quad \text{a.e. } x \in \Omega, \ B \in \mathcal{M}^{N \times N}.
\]

The symmetric fourth order tensor \( D^2V(x, 0) \) in (2.17) plays the role of the classical elasticity tensor.

By (2.8) we get
\[
V_h(x, B) = \frac{1}{h^2} W(x, I + hB) \geq C |2 \text{sym } B + h B^T B|^2
\]
so that (2.17) and (2.18) imply the ellipticity of \( V_0 \):
\[
V_0(x, \text{sym } B) \geq 4 C |\text{sym } B|^2 \quad \text{a.e. } x \in \Omega, \ B \in \mathcal{M}^{N \times N}.
\]
Let \( f \in L^2(\partial \Omega; \mathbb{R}^N) \), \( g \in L^2(\Omega; \mathbb{R}^N) \) be a pair of surface and body force fields respectively. For a suitable choice of the adimensional parameter \( h > 0 \), the functional representing the total energy is labeled by \( \mathcal{F}_h : H^1(\Omega; \mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\} \) and defined as follows

\[
\mathcal{F}_h(v) := \int_{\Omega} V_h(x, \nabla v) \, dx - \mathcal{L}(v),
\]

where

\[
\mathcal{L}(v) := \int_{\partial \Omega} f \cdot v \, d\mathcal{H}^{n-1} + \int_{\Omega} g \cdot v \, dx.
\]

In this paper we are interested in the asymptotic behavior as \( h \to 0^+ \) of functionals \( \mathcal{F}_h \) and to this aim we introduce the limit energy functional \( \mathcal{F} : H^1(\Omega; \mathbb{R}^N) \to \mathbb{R} \) defined by

\[
\mathcal{F}(v) = \min_{W \in \mathcal{M}^{N \times N}_{skew}} \int_{\Omega} V_0(x, \mathbb{E}(v) - \frac{1}{2} W^2) \, dx - \mathcal{L}(v).
\]

We emphasize that the minimum in right-hand side of definition (2.22) exists: precisely the finite dimensional minimization problem has exactly two solutions which differ only by a sign, since by (2.19),

\[
\lim_{|W| \to +\infty, W \in \mathcal{M}^{N \times N}_{skew}} \int_{\Omega} V_0(x, \mathbb{E}(v) - \frac{1}{2} W^2) \, dx = +\infty
\]

and \( V_0(x, \cdot) \) is strictly convex by (2.17), (2.19). All along this paper we assume (2.1) together with the **standard structural conditions** (2.5)-(2.9), (2.12) as usual in scientific literature concerning elasticity theory and we refer to the notations (2.10), (2.15), (2.17), (2.20)-(2.22).

The pair \( f, g \) describing the loads is said to be **equilibrated** if

\[
\int_{\partial \Omega} f \cdot z \, d\mathcal{H}^{N-1} + \int_{\Omega} g \cdot z \, dx = 0 \quad \forall \ z \in \mathcal{R},
\]

and it is said to be **compatible** if

\[
\int_{\partial \Omega} f \cdot W^2 \, d\mathcal{H}^{N-1} + \int_{\Omega} g \cdot W^2 \, dx < 0 \quad \forall \ W \in \mathcal{M}^{N \times N}_{skew} \text{ s.t. } W \neq 0.
\]

**Definition 2.1.** Given an infinitesimal sequence \( h_j \) of positive real numbers, we say that \( v_j \in H^1(\Omega; \mathbb{R}^N) \) is a minimizing sequence of the sequence of functionals \( \mathcal{F}_{h_j} \) if

\[
(\mathcal{F}_{h_j}(v_j) - \inf \mathcal{F}_{h_j}) \to 0 \quad \text{as} \quad h_j \to 0^+.
\]

We will show (see Lemma 3.1) that for every infinitesimal sequence \( h_j \) the minimizing sequences of the sequence of functionals \( \mathcal{F}_{h_j} \) do exist.

Now we can state the main result, whose proof is postponed.

**Theorem 2.2.** Assume that the standard structural conditions and (2.24), (2.25) hold true. Then for every sequence of strictly positive real numbers \( h_j \to 0 \) there are minimizing sequences of the sequence of functionals \( \mathcal{F}_{h_j} \).

Moreover for every minimizing sequence \( v_j \in H^1(\Omega; \mathbb{R}^N) \) of \( \mathcal{F}_{h_j} \) there exist: a subsequence, a displacement \( v_0 \in H^1(\Omega; \mathbb{R}^N) \) and a constant matrix \( W_0 \in \mathcal{M}^{N \times N}_{skew} \) such that, without relabeling,

\[
\mathbb{E}(v_j) \rightharpoonup \mathbb{E}(v_0) \quad \text{weakly in } L^2(\Omega; \mathcal{M}^{N \times N}),
\]

\[
\sqrt{h_j} \nabla v_j \rightharpoonup W_0 \quad \text{strongly in } L^2(\Omega; \mathcal{M}^{N \times N}),
\]

\[
\lim_{j \to +\infty} \mathcal{F}_{h_j}(v_j) = \mathcal{F}(v_0) = \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}(v),
\]

\[
\mathcal{F}(v_0) = \int_{\Omega} V_0(x, \mathbb{E}(v_0) - \frac{1}{2} W_0^2) \, dx - \mathcal{L}(v_0).
\]
Remark 2.3. It is worth to underline that in contrast to the case of Dirichlet problem faced in [12], here in pure traction problem we cannot expect even weak \( H^1(\Omega; \mathbb{R}^N) \) convergence of minimizing sequences. Indeed choose: \( f = g \equiv 0 \) and

\[
W(x, F) = \begin{cases} 
|F^T F - I|^2 & \text{if } \det F > 0 , \\
+\infty & \text{otherwise ,}
\end{cases}
\]

(2.30)

\[
v_j := h_j^{-\alpha} W x \quad \text{with } W \in \mathcal{M}^N_{\text{skew}}, \quad 0 < 2\alpha < 1, \quad h_j \to 0_+. 
\]

(2.31)

Then \( \mathcal{F}_{h_j}(v_j) = o(1) \) and, due to \( \inf \mathcal{F}_{h_j} = 0 \), the sequence \( v_j \) is a minimizing sequence which has no subsequence weakly converging in \( H^1(\Omega; \mathbb{R}^N) \). It is well known that such phenomenon takes place for pure traction problems in linear elasticity too, but in nonlinear elasticity this difficulty cannot be easily circumvented in general, since the fact that \( v_j \) is a minimizing sequence does not entail that also \( v_j - \mathbb{P} v_j \) is minimizing sequence. In [25] we show that for some special integrand \( \mathcal{W} \), as in the case of Green-St-Venant energy density, if \( v_j \) is a minimizing sequence then \( w_j := v_j - \mathbb{P} v_j \) is a minimizing sequence too and there exist a (not relabeled) subsequence of functionals \( \mathcal{F}_{h_j} \) such that the related \textit{minimizing subsequence} \( w_j \) converges weakly in \( H^1(\Omega; \mathbb{R}^N) \) to a minimizer \( v_0 \) of \( \mathcal{F} \), provided (2.24) and (2.25) hold true.

Remark 2.4. A careful inspection of the proof shows that Theorem 2.2 remains true if hypothesis (2.8) is weakened by assuming

\[
\inf_{|B| \geq \rho} \inf_{x \in \Omega} \mathcal{V}(x, B) > 0 \quad \forall \rho > 0,
\]

(2.32)

\[
\exists \alpha > 0, \rho > 0 \quad \text{such that } \inf_{x \in \Omega} \mathcal{V}(x, B) \geq \alpha |B|^2 \quad \forall \ |B| \leq \rho
\]

(2.33)

and

\[
\lim_{|B| \to +\infty} \inf_{x \in \Omega} \frac{1}{|B|} \mathcal{V}(x, B) > 0.
\]

(2.34)

This can be shown by exploiting Lemma 3.1 in [12]; notice that under such weaker assumption the constant appearing on the right-hand side of inequality (2.4) must be modified accordingly in Lemma 3.1.

It is worth noting that (2.8) implies (2.32), (2.33) and (2.34).

Here we show some preliminary remarks about the main result and the limit functional \( \mathcal{F} \).

Remark 2.5. If \( N = 2 \), then for every \( W \in \mathcal{M}^N_{\text{skew}} \) there is \( a \in \mathbb{R} \) s.t. \( W^2 = -a^2 I \), hence (2.22) reads

\[
\mathcal{F}(v) = \min_{a \in \mathbb{R}} \int_{\Omega} \mathcal{V}_0(x, E(v) + \frac{a^2}{2} I) \, dx - \mathcal{L}(v),
\]

(2.35)

therefore, a minimizer \( a_*(v) \) of functional (2.35) (with respect to \( a \in \mathbb{R} \) with fixed \( v \)) fulfills

\[
a^2_*(v) \int_{\Omega} \mathcal{V}_0(x, I) \, dx + a_*(v) \int_{\Omega} D \mathcal{V}_0(x, I) \cdot E(v) \, dx = 0
\]

that is

\[
a^2_*(v) = \left( \int_{\Omega} \mathcal{V}_0(x, I) \, dx \right)^{-1} \left( \int_{\Omega} D \mathcal{V}_0(x, I) \cdot E(v) \, dx \right)^{-}\n\]

(2.36)

and

\[
\mathcal{F}(v) = \int_{\Omega} \mathcal{V}_0(x, E(v) + \frac{a^2_*(v)}{2} I) \, dx - \mathcal{L}(v).
\]

(2.37)
By taking into account that $V_0$ is a quadratic form, we can make explicit the gap between the new functional $\mathcal{F}$ and the classical linear elasticity functional $\mathcal{E}$, when $N = 2$:

$$\mathcal{F}(v) = \int_{\Omega} V_0(x, E(v)) \, dx - \frac{1}{4} \left( \int_{\Omega} V_0(x, I) \, dx \right)^{-1} \left[ \left( \int_{\Omega} D V_0(x, I) \cdot E(v) \, dx \right)^{-1} - L(v) \right]^2$$

(2.38)

$$= \mathcal{E}(v) - \frac{1}{4} \left( \int_{\Omega} V_0(x, I) \, dx \right)^{-1} \left[ \left( \int_{\Omega} D V_0(x, I) \cdot E(v) \, dx \right)^{-1} \right]^2.$$

In particular, if $N = 2$, $\lambda, \mu > 0$ and

$$W(x, F) = \begin{cases} \mu |F^T F - I|^2 + \frac{\lambda}{2} |\text{Tr} (F^T F - I)|^2 & \text{if } \det F > 0 \\ +\infty & \text{otherwise,} \end{cases}$$

then $V_0(x, B) = 4\mu|B|^2 + 2\lambda|\text{Tr} B|^2$ and we get

$$a^2(v) = |\Omega|^{-1} \left( \int_{\Omega} \text{div } v \, dx \right)^2.$$

Roughly speaking, this means that in 2D the global energy $\mathcal{F}(v)$ of a displacement $v$ is the same of linearized elasticity if the area of the associated deformed configuration $y(\Omega)$ is greater than the area of $\Omega$.

**Remark 2.6.** The compatibility condition (2.25) cannot be dropped in Theorem 2.2 even if the (necessary) condition (2.24) holds true. Moreover plain substitution of strong with weak inequality in (2.25) leads to a lack of compactness for minimizing sequences.

Indeed, if $n$ denotes the outer unit normal vector to $\partial \Omega$ and we choose $f = fn$ with $f < 0$, $g \equiv 0$ then

$$\int_{\partial \Omega} f \cdot W^2 x \, dH^{N-1} = 2f(\text{Tr} W^2)|\Omega| > 0 \quad \forall \ W \in \mathcal{M}_{\text{skew}}^{N \times N} \setminus \{0\},$$

say, the strict inequality in (2.25) is reversed in a strong sense by any $W \in \mathcal{M}_{\text{skew}}^{N \times N} \setminus \{0\}$; fix a sequence of positive real numbers such that $h_j \to 0$, $W \in \mathcal{M}_{\text{skew}}^{N \times N}$, $W \neq 0$, and set $v_j = h_j^{-1} \left( \frac{1}{2} W^2 + \sqrt[3]{2} W \right) x$; then $I + \left( \frac{1}{2} W^2 + \sqrt[3]{2} W \right) \in SO(N)$ and, by frame indifference,

$$\mathcal{F}_{h_j}(v_j) = -\frac{f}{2h_j} \int_{\partial \Omega} W^2 x \cdot n \, dH^{n-1} = -\frac{f}{2h_j} (\text{Tr} W^2)|\Omega| \to -\infty.$$

On the other hand, assume $W$ as in (2.30) and $f = g \equiv 0$, so that the compatibility inequality is substituted by the weak inequality; if $v_j$ are defined as above then, hence

$$\mathcal{F}_{h_j}(v_j) = 0 = \inf \mathcal{F}_{h_j}$$

but $E(v_j)$ has no weakly convergent subsequences in $L^2(\Omega; \mathcal{M}_{\text{skew}}^{N \times N})$.

**Remark 2.7.** It is worth noticing that the compatibility condition (2.25) holds true when $g \equiv 0$, $f = fn$ with $f > 0$ and $n$ the outer unit normal vector to $\partial \Omega$.

Indeed let $W \in \mathcal{M}_{\text{skew}}^{N \times N}$, $W \neq 0$: hence by (2.24) and the Divergence Theorem we get

$$\int_{\partial \Omega} f \cdot W^2 x \, dH^{N-1} = 2f(\text{Tr} W^2)|\Omega| < 0$$

thus proving (2.25) in this case. Roughly speaking this means that in presence of tension-like surface forces and of null body forces the compatibility condition holds true.
Remark 2.8. It is possible to observe some analogy between the energy functional \( F \) and the results in [14, 15] where the approximate theory of small strain accompanied by moderate rotations is discussed under suitable kinematical assumptions. More precisely, if \( F = I + h \nabla v \) is the deformation gradient, \( F = RU \) the polar decomposition, [14] shows that the assumptions \( (2.25) \)

\[
(2.45) \quad R = I + O(\sqrt{h}), \quad U = I + O(h)
\]
as \( h \to 0 \) (in the sense of pointwise convergence) are equivalent to \( (2.26) \)

\[
(2.46) \quad \mathbb{E}(v) = O(1), \quad h(\text{skew} \nabla v) = O(\sqrt{h})
\]
as \( h \to 0 \) again in the sense of pointwise convergence. Therefore \( (2.47) \)

\[
(2.47) \quad U = I + h(\mathbb{E}(v) - \frac{1}{2}(\text{skew} \nabla v)^2) + o(h)
\]
and the point-wise limit of \( F_h \) (not the \( \Gamma \)-limit!) becomes \( \int_{\Omega} \mathcal{B}_0(x, \mathbb{E}(v) - \frac{1}{2}(\text{skew} \nabla v)^2) \, dx - L(v), \)
which is quite similar to \( (2.22) \).
We highlight the fact that \( (2.45) \) cannot be understood in the sense of \( L^2(\Omega; M^{N \times N}) \) whenever \( v \equiv v_* \) on a closed subset \( \Sigma \) of \( \partial \Omega \) with \( H^{N-1}(\Sigma) > 0 \), since by Korn and Poincaré inequalities we get
\[
\int_{\Omega} |\nabla|^2 \, dx \leq C \left( \int_{\Omega} |\mathbb{E}(v)|^2 \, dx + \int_{\Sigma} |v_*|^2 \, dH^{N-1} \right),
\]
therefore if \( \mathbb{E}(v) = O(1) \) then \( h \nabla v = O(h) \), thus contradicting the second of \( (2.45) \). On the other hand a careful application of the rigidity Lemma of [17] show that if \( \mathbb{E}(v) = O(1) \) and \( U = I + O(h) \) in the sense of \( L^2(\Omega; M^{N \times N}) \), then there exists a constant skew symmetric matrix \( W \) such that \( h \nabla v^T \nabla v = -W^2 + o(1) \) in the sense of \( L^1(\Omega; M^{N \times N}) \) (see the proof of Lemma 3.3 below). Therefore
\[
(2.48) \quad U = I + h(\mathbb{E}(v) - W^2/2) + o(h)
\]
where equality is understood in the sense of \( L^1(\Omega; M^{N \times N}) \) and \( W \) a constant skew symmetric matrix.

3. Proofs

We recall three basic inequalities exploited in the sequel, for reader’s convenience and in order to label the related constants.

**Poincaré Inequality.** There exists a constant \( C_P = C_P(\Omega) \) such that
\[
(3.1) \quad \| v - f_\Omega v \|_{L^2(\Omega; \mathbb{R}^N)} \leq C_P \| \nabla w \|_{L^2(\Omega; M^{N \times N})} \quad \forall v \in H^1(\Omega; \mathbb{R}^N).
\]

**Korn-Poincaré Inequality.** There exists a constant \( C_K = C_K(\Omega) \) such that
\[
(3.2) \quad \| v - P_v \|_{L^2(\Omega; \mathbb{R}^N)} + \| v - P_v \|_{L^2(\partial \Omega; \mathbb{R}^N)} \leq C_K \| \mathbb{E}(v) \|_{L^2(\Omega; M^{N \times N})} \quad \forall v \in H^1(\Omega; \mathbb{R}^N).
\]

**Geometric Rigidity Inequality (17).** There exists a constant \( C_G = C_G(\Omega) \) such that for every \( y \in H^1(\Omega; \mathbb{R}^N) \) there is an associated rotation \( R \in SO(N) \) such that we have
\[
(3.3) \quad \int_\Omega |\nabla y - R|^2 \, dx \leq C_G \int_\Omega \text{dist}^2(\nabla y; SO(N)) \, dx.
\]

The first step in our analysis is the next lemma showing that if \( (2.24), (2.25) \) hold true then the functionals \( F_h \) are bounded from below uniformly with respect to \( h \in \mathbb{N} \): this implies the existence of minimizing sequences of the sequence of functionals \( F_{h_j} \) (see Definition 2.1).
Lemma 3.1. Assume \((2.24)\) and \((2.25)\). Then

\[
(3.4) \quad \inf_{h > 0} \inf_{v \in H^1} \mathcal{F}_h(v) > -\frac{C_P^2 C_G}{C} \left( \|f\|_{L^2}^2 + \|g\|_{L^2}^2 \right),
\]

where \(C\) is the coercivity constant in \((2.18)\) and \(C_P, C_G\) are the constants related to the basic inequalities above. Actually \((3.4)\) holds true even if strict inequality is replaced by weak inequality in \((2.25)\).

Proof. Let \(v \in H^1(\Omega; \mathbb{R}^N)\) and \(y = x + hv\). Since \(\nabla y > 0\) a.e., by polar decomposition for a.e. \(x\) there exist a rotation \(R_h(x)\) and a symmetric positive definite matrix \(U_h(x)\) such that \(\nabla y(x) = R_h(x)U_h(x)\), hence \(\nabla y^T \nabla y = U_h^2\), so that for a.e. \(x\)

\[
|\nabla y^T \nabla y - I|^2 = |U_h^2 - I|^2 = |(U_h - I)(U_h + I)|^2 \geq |U_h - I|^2 = \|(\nabla y - R_h)^2 \geq \text{dist}^2(\nabla y, \text{SO}(N)).
\]

By \((2.8),(3.5)\) and the Geometric Rigidity Inequality \((3.3)\) there exists a constant rotation \(R\) such that

\[
\mathcal{F}_h(v) \geq Ch^{-2} \int_{\Omega} |\nabla y^T \nabla y - I|^2 \, dx - h^{-1} \mathcal{L}(y - x) \geq \frac{C}{C_G} h^{-2} \int_{\Omega} |\nabla y - R|^2 \, dx - h^{-1} \mathcal{L}(y - x).
\]

If now

\[c := |\Omega|^{-1} \int_{\Omega} (y - Rx) \, dx\]

by \((2.24)\), by Poincaré and Young inequality we get, for every \(\alpha > 0\),

\[
\|y - Rx - c\|_{L^2} \leq C_P \|\nabla (y - Rx)\|_{L^2} = C_P \|\nabla y - R\|_{L^2},
\]

\[
\mathcal{L}(y - Rx - c) \leq C_P \|\nabla y - R\|_{L^2} \left( \|f\|_{L^2} + \|g\|_{L^2} \right) \leq \alpha^{-1} \frac{C_P}{2} \|\nabla y - R\|_{L^2}^2 + \alpha \frac{C_P}{2} \left( \|f\|_{L^2} + \|g\|_{L^2} \right)^2\]

\[
\leq \alpha^{-1} \frac{C_P}{2} \|\nabla y - R\|_{L^2}^2 + \alpha C_P \left( \|f\|_{L^2}^2 + \|g\|_{L^2}^2 \right).
\]

By choosing \(\alpha = h C_P C_G / C\)

\[
\mathcal{L}(y - x) = \mathcal{L}(y - Rx - c) + \mathcal{L}(Rx - x) \leq \alpha^{-1} \frac{C_P}{2} \|\nabla y - R\|_{L^2}^2 + \alpha C_P \left( \|f\|_{L^2} + \|g\|_{L^2} \right)^2 + \mathcal{L}(Rx - x) = h^{-1} \frac{C/C_G}{2} \|\nabla y - R\|_{L^2}^2 + \frac{C_P^2}{C/C_G} h \left( \|f\|_{L^2}^2 + \|g\|_{L^2}^2 \right) + \mathcal{L}(Rx - x).
\]

Exploiting the standard representation \((2.22)\) of the rotation \(R = I + W \sin \vartheta + (1 - \cos \vartheta)W^2\) for suitable \(\vartheta \in \mathbb{R}\) and \(W \in \mathcal{M}_{skew}^{N \times N}\), by \((2.24),(2.25),(3.6)\) and \((3.7)\) we get

\[
\mathcal{F}_h(v) \geq \frac{C/C_G}{2} h^{-2} \int_{\Omega} |\nabla y - R|^2 \, dx - \frac{C_P^2}{C/C_G} \left( \|f\|_{L^2}^2 + \|g\|_{L^2}^2 \right) - h^{-1} \mathcal{L}((R - I)x) \geq \frac{C_P^2 C_G}{C} \left( \|f\|_{L^2}^2 + \|g\|_{L^2}^2 \right) \quad \forall v \in H^1(\Omega; \mathbb{R}^N), \forall h > 0.
\]
Lemma 3.2. Let $v_n \in H^1(\Omega; \mathbb{R}^N)$ be a sequence such that $\mathbb{E}(v_n) \to T$ in $L^2(\Omega; \mathcal{M}^{N\times N})$. Then there exists $w \in H^1(\Omega; \mathbb{R}^N)$ such that $T = \mathbb{E}(w)$. If in addition $\nabla v_n \rightharpoonup G$ in $L^2(\Omega; \mathbb{R}^N)$ then there exists a constant matrix $W \in \mathcal{M}^{N\times N}$ such that $\nabla w = G - W$.

Proof. Since $\mathbb{E}(v_n) \to T$ in $L^2(\Omega; \mathcal{M}^{N\times N})$ then $v_n - \mathbb{P}v_n$ are equibounded in $H^1(\Omega; \mathbb{R}^N)$, where $\mathbb{P}$ the projection on the set $\mathcal{R}$ of infinitesimal rigid displacements. Therefore, up to subsequence, we can assume that $w_n := v_n - \mathbb{P}v_n \to w$ in $H^1(\Omega; \mathbb{R}^N)$ and we get

$$\nabla w_n = \mathbb{E}(w_n) + \text{skew}\nabla w_n = \mathbb{E}(v_n) + \text{skew}\nabla w_n$$

hence there exists $S \in L^2(\Omega; \mathcal{M}^{N\times N})$ such that $\text{skew}\nabla w_n \rightharpoonup S$ in $L^2(\Omega; \mathcal{M}^{N\times N})$ and by letting $n \to +\infty$ into (3.9) we have $T + S = \nabla w$. Since $S \in L^2(\Omega; \mathcal{M}^{N\times N})$ we get $\mathbb{E}(w) = T$ and if in addition $\nabla v_n \rightharpoonup G$ in $L^2(\Omega; \mathcal{M}^{N\times N})$ then there exists a constant $W \in \mathcal{M}^{N\times N}$ such that $\nabla \mathbb{P}v_n \to W$ in $L^2(\Omega; \mathcal{M}^{N\times N})$, actually converging in the finite dimensional space of constant skew-symmetric matrices, thus proving the Lemma. ■

Remark 3.3. It is worth noting that if $\mathbb{E}(v_j) \to T$ in $L^2(\Omega; \mathcal{M}^{N\times N})$ then by Lemma 3.2 there exists $v \in H^1(\Omega; \mathbb{R}^N)$ such that $T = \mathbb{E}(v)$ and if $T = \mathbb{E}(w)$ for some $w \in H^1(\Omega; \mathbb{R}^N)$, then $v - w$ is an infinitesimal rigid displacement in $\Omega$, i.e. $\mathbb{E}(v - w) = 0$.

Next we show a preliminary convergence property: we compute a kind of Gamma limit of the sequence of functional $F_h$ with respect to weak $L^2$ convergence of linearized strains.

Lemma 3.4. (energy convergence) Assume that (2.24) holds true and let $h_j \to 0$ be a decreasing sequence. Then

i) For every $v_j, v \in H^1(\Omega; \mathbb{R}^N)$ such that $\mathbb{E}(v_j) \to \mathbb{E}(v)$ in $L^2(\Omega; \mathcal{M}^{N\times N})$ we have

$$\liminf_{j \to +\infty} F_h_j(v_j) \geq F(v).$$

ii) For every $v \in H^1(\Omega; \mathbb{R}^N)$ there exists a sequence $v_j \in H^1(\Omega; \mathbb{R}^N)$ such that $\mathbb{E}(v_j) \to \mathbb{E}(v)$ in $L^2(\Omega; \mathcal{M}^{N\times N})$ and

$$\limsup_{j \to +\infty} F_h_j(v_j) \leq F(v).$$

Proof. First we prove i). We set $y_j = x + hv_j$ and denote various positive constants by $C, C', C'', ..., L', L''$. We may assume without restriction that $F_h_j(v_j) \leq C$; by taking into account (2.8) we get

$$Ch^{-2} \int_{\Omega} |\nabla y_j \nabla y_j - I|^2 dx - L(v_j) \leq F_h_j(v_j)$$

and by (2.24)

$$h^{-2} \int_{\Omega} |\nabla y_j \nabla y_j - I|^2 dx \leq C' + L(v_j) = C' + L(v_j - \mathbb{P}v_j),$$

where $\mathbb{P}v_j$ is the projection of $v_j$ onto the set of infinitesimal rigid displacements. Hence by Korn-Poincaré inequality we have

$$h^{-2} \int_{\Omega} |\nabla y_j \nabla y_j - I|^2 dx \leq C' + C'' \left( \int_{\Omega} |\mathbb{E}(v_j)|^2 dx \right)^{\frac{1}{2}} \leq C'''.$$

Inequality (3.10) together with the Rigidity Lemma of [17] imply, by the same computations at the beginning of proof of Lemma 3.1 that for every $h_j$ there exists a constant rotation $R_j \in SO(N)$ and a constant $C'''$, dependent only on $\Omega$, such that

$$\int_{\Omega} |\nabla y_j - R_j|^2 dx \leq C'''h_j^2$$

that is

$$\int_{\Omega} |I + h\nabla v_j - R_j|^2 dx \leq C'''h_j^2.$$
Due to the representation (2.2) of rotations for every \( j \in \mathbb{N} \) there exist \( \vartheta_j \in \mathbb{R} \) and \( W_j \in \mathcal{M}_{skew}^{N\times N}, \|W_j\|^2 = 2 \) such that \( R_j = \exp(\vartheta_j W_j) \) and

(3.12) \[ R_j = \exp(\vartheta_j W_j) = I + \sin \vartheta_j W_j + (1 - \cos \vartheta_j) W_j^2 \]

hence by (3.11)

(3.13) \[ \int_{\Omega} |h_j \nabla v_j - \sin \vartheta_j W_j - (1 - \cos \vartheta_j) W_j^2|^2 \, dx \leq C''' h_j^2. \]

Since

\[ \text{sym} \left( h_j \nabla v_j - \sin \vartheta_j W_j - (1 - \cos \vartheta_j) W_j^2 \right) = h_j \mathbb{E}(v_j) - (1 - \cos \vartheta_j) W_j^2 \]

we get

\[ \int_{\Omega} \| \mathbb{E}(v_j) - (1 - \cos \vartheta_j) h_j^{-1} W_j^2 \|^2 \, dx \leq C''''. \]

By recalling that \( \mathbb{E}(v_j) \to \mathbb{E}(v) \) in \( L^2(\Omega; \mathcal{M}^{N\times N}) \), we deduce for suitable \( L > 0 \)

(3.14) \[ |1 - \cos \vartheta_j| = \frac{1}{2} \| (1 - \cos \vartheta_j) W_j^2 \| \leq L h_j \]

that is

(3.15) \[ |\sin \vartheta_j| \leq \sqrt{2Lh_j}. \]

By (3.13), (3.14) and (3.15) we obtain

(3.16) \[ \frac{1}{2} \int_{\Omega} |\sqrt{h_j} \nabla v_j|^2 \, dx \leq \]

\[ \leq \int_{\Omega} |\sqrt{h_j} \nabla v_j - h_j^{-1/2} \sin \vartheta_j W_j|^2 \, dx + \int_{\Omega} |h_j^{-1/2} \sin \vartheta_j W_j|^2 \, dx \leq \]

\[ \leq (C''' + 2L|\Omega|) h_j, \]

hence, up to subsequences, by (3.15) there exists a constant matrix \( W \in \mathcal{M}_{skew}^{N\times N} \) such that

(3.17) \[ \sqrt{h_j} \nabla v_j \to W \quad \text{strongly in } L^2(\Omega; \mathcal{M}^{N\times N}) \]

and therefore

(3.18) \[ h_j \nabla v_j^T \nabla v_j \to W^T W = -W^2 \quad \text{strongly in } L^1(\Omega; \mathcal{M}^{N\times N}). \]

By Lemmas 4.2 and 4.3 of [12] for every \( k \in \mathbb{N} \) there exist an increasing sequence of Caratheodory functions \( \Psi^k_j : \Omega \times \mathcal{M}_{skew}^{N\times N} \to [0, +\infty) \) and a measurable function \( \mu^k : \Omega \to (0, +\infty) \) such that \( \Psi^k_j(x, \cdot) \) is convex for a.e. \( x \in \Omega \) and satisfies

(3.19) \[ \Psi^k_j(x, B) \leq \mathcal{V}(x, h_j B)/h_j^2 = \mathcal{V}_{h_j}(x, h_j B) \quad \forall B \in \mathcal{M}_{sym}^{N\times N}, \]

(3.20) \[ \Psi^k_j(x, B) = \left( 1 - \frac{1}{k} \right) \mathcal{V}_0(x, B) \quad \text{for } \mathcal{V}_0(x, B) \leq \mu^k(x)/h_j^2. \]

Property (3.20) entails

(3.21) \[ \lim_{j \to +\infty} \Psi^k_j(x, B) = \left( 1 - \frac{1}{k} \right) \mathcal{V}_0(x, B) \quad \text{a.e. } x \in \Omega, \forall B \in \mathcal{M}_{sym}^{N\times N} \]

then, by exploiting (3.21), Lemma 4.3 of [12], and taking into account that

\[ B_j := \mathbb{E}(v_j) + \frac{1}{2} h_j \nabla v_j^T \nabla v_j \to \mathbb{E}(v) - \frac{1}{2} W^2 \] in \( L^1(\Omega; \mathcal{M}^{N\times N}) \),

by (3.19) and (3.21) we get

\[ \liminf_{j \to 0} \int_{\Omega} \mathcal{V}_{h_j}(x, \nabla v_{h_j}) \, dx \geq \liminf_{j \to +\infty} \int_{\Omega} \Psi^k_j(x, B_j) \, dx \]

\[ \geq \int_{\Omega} \left( 1 - \frac{1}{k} \right) \mathcal{V}_0 \left( x, \mathbb{E}(v) - \frac{1}{2} W^2 \right) \, dx \quad \forall k \in \mathbb{N}. \]
Up to subsequences $v_j - P v_j \to w$ in $H^1(\Omega; \mathbb{R}^N)$, moreover $E(v) = E(w)$. Then by (3.24) for every $k \in \mathbb{N}$ we obtain
\[
\liminf_{j \to +\infty} F_{h_j}(v_j) \geq \int_\Omega \left( 1 - \frac{1}{k} \right) \nu_0(x, E(v) - \frac{1}{2} W^2) \, dx - L(w) = \int_\Omega \left( 1 - \frac{1}{k} \right) \nu_0(x, E(v) - \frac{1}{2} W^2) \, dx - L(v)
\]
Taking the supremum as $k \to \infty$ we deduce
\[
\liminf_{j \to +\infty} F_{h_j}(v_j) \geq \int_\Omega \nu_0(x, E(v) - \frac{1}{2} W^2) \, dx - L(v) \geq F(v)
\]
which proves i).
We are left to prove claim ii). To this aim, we set for every $v \in H^1(\Omega; \mathbb{R}^N)$:
\[(3.23)\]
\[W_v \in \arg\min \left\{ \int_\Omega \nu_0(x, E(v) - \frac{1}{2} W^2) \, dx : W \in M_{skew}^{N \times N} \right\}.
\]
Without relabeling, $v$ denotes also a fixed compactly supported extension in $H^1(\mathbb{R}^N; \mathbb{R}^N)$ of the given $v$ (such extension exists since $\Omega$ is Lipschitz due to (2.1)).
We may define a recovery sequence $w_j \in C^1(\Omega; \mathbb{R}^N)$ for every $j$, as follows. Set
\[(3.24)\]
\[w_j = h_j^{-1/2} W_v x + v \ast \phi_j \]
where the sequence $\phi_j$ is chosen in such a way that $h_j \phi_j^{-3} \to 0$ holds true and $\phi_j(x) = \phi_j^N(x/\epsilon_j)$ is a mollifier supported in $B_{\epsilon_j}(0)$. Sobolev embedding entails $v \in L^6(\mathbb{R}^N; \mathbb{R}^N)$, since $v \in H^1(\mathbb{R}^N; \mathbb{R}^N)$ and $N = 2, 3$; then by Young Theorem and $0 < \epsilon_j \leq 1$ we have
\[(3.25)\]
\[
\|\nabla(v \ast \phi_j)\|_{L^6} \leq \|v\|_{L^6} \|\nabla \phi_j\|_{L^{6/5}} \leq \epsilon_j^{-N/6-1} \|\nabla \phi\|_{L^{6/5}} \|v\|_{L^6} \leq \epsilon_j^{-3/2} \|\nabla \phi\|_{L^{6/5}} \|v\|_{L^6}.
\]
By $\nabla w_j = h^{-1/2} W_v + \nabla(v \ast \phi_j)$ and $W_v^T = -W_v$ we get
\[
E(w_j) = E(v) + \phi_j,
\]
\[h_j \nabla w_j^T \nabla w_j = -W_v^2 + h_j \nabla(v \ast \phi_j)^T \nabla(v \ast \phi_j) + h_j^{1/2} (\nabla(v \ast \phi_j)^T W_v - W_v \nabla(v \ast \phi_j)),
\]
hence, by taking into account (3.25) and $h_j \phi_j^{-3} \to 0$, we get
\[(3.26)\]
\[
E(w_j) + \frac{1}{2} h_j \nabla w_j^T \nabla w_j \to E(v) - \frac{1}{2} W_v^2 \quad \text{in} \quad L^2(\Omega, \mathbb{R}^N),
\]
\[(3.27)\]
\[
h_j \left( E(w_j) + \frac{1}{2} h_j \nabla w_j^T \nabla w_j \right) \to 0 \quad \text{in} \quad L^\infty(\Omega, \mathbb{R}^N).
\]
Therefore Taylor’s expansion of $V$ entails
\[(3.28)\]
\[
\lim_{j \to +\infty} \nu_{h_j}(x, E(w_j) + \frac{1}{2} h_j \nabla w_j^T \nabla w_j) = \nu_0(x, E(v) - \frac{1}{2} W_v^2) \quad \text{for a.e. } x \in \Omega,
\]
and taking into account (2.14), (2.15), (2.16), (3.27) we have
\[(3.29)\]
\[
\nu_{h_j}(x, E(w_j) + \frac{1}{2} h_j \nabla w_j^T \nabla w_j) \leq C \|E(w_j) + \frac{1}{2} h_j \nabla w_j^T \nabla w_j\|,
\]
hence the Lebesgue dominated convergence theorem yields
\[
F_{h_j}(w_j) \to \min_{W \in M_{skew}^{N \times N}} \int_\Omega \nu_0(x, E(v) - \frac{1}{2} W_v^2) \, dx - L(v),
\]
thus proving ii).

\[\]

**Remark 3.5.** If $W$ is a convex function of $F^T F - I$ then (3.22) is a straightforward consequence of weak $L^1(\Omega; M_{skew}^{N \times N})$ convergence of $B_j$, hence introduction and use of integrands $\nu^k_j$ in the proof is unnecessary in such case. So restriction to decreasing sequences $h_j$ (used to apply Lemmas 4.2, 4.3 of [12]) can be deleted in the assumptions of Lemma 3.4 if $W$ is convex.
Lemma 3.1 entails existence of minimizing sequences for the sequence of functionals $\mathcal{F}_h$. Next Proposition entails a (very weak) relative compactness property of these sequences; in its proof we will consider the cone $\mathbb{K} = \{ \tau (\mathbf{R} - \mathbf{I}) : \tau > 0, \mathbf{R} \in SO(N) \}$ which fulfills, thanks to (2.3) and (2.4):

$$\mathbb{K} = \mathbb{K} \cup \mathcal{M}^{N \times N}_{\text{skew}}.$$

(3.30)

$$\mathbb{K} + \mathcal{M}^{N \times N}_{\text{skew}} = \mathbb{K} + \mathcal{M}^{N \times N}_{\text{skew}} = \{ \mathbf{W} + \mathbf{Z}^2 : \mathbf{W}, \mathbf{Z} \in \mathcal{M}^{N \times N}_{\text{skew}} \}.$$

(3.31)

**Lemma 3.6. Compactness of minimizing sequences** Assume that (2.24), (2.25) hold true, $h_j \to 0$ is a sequence of strictly positive real numbers and the sequence of displacements $\mathbf{v}_j \in H^1(\Omega; \mathbb{R}^N)$ fulfill $(\mathcal{F}_{h_j}(\mathbf{v}_j) - \inf \mathcal{F}_{h_j}) \to 0$.

Then there exists $C > 0$ such that $\| \mathbb{E}(\mathbf{v}_j) \|_{L^2} \leq C$.

**Proof.** By Lemma 3.1 there exists $c$ such that

$$-\infty < c \leq \inf \mathcal{F}_{h_j} \leq \mathcal{F}_{h_j}(\mathbf{0}) = 0.$$ (3.32)

Assume by contradiction that $t_j := \| \mathbb{E}(\mathbf{v}_j) \|_{L^2} \to +\infty$ and set $\mathbf{w}_j = t_j^{-1} \mathbf{v}_j$. It is readily seen that by Lemma 3.2 there exists $\mathbf{w} \in H^1(\Omega; \mathbb{R}^N)$ and a subsequence such that without relabeling $\mathbb{E}(\mathbf{w}_j) \to \mathbb{E}(\mathbf{w})$ in $L^2(\Omega; \mathcal{M}^{N \times N})$.

By (3.32) we can assume up to subsequences that $\mathcal{F}_{h_j}(\mathbf{v}_j) \leq h_j^2$.

Moreover by setting $\mathbf{y}_j = \mathbf{x} + h_j \mathbf{v}_j = \mathbf{x} + h_j t_j \mathbf{w}_j$, arguing as at the beginning of Lemma 3.1 proof and exploiting Korn-Poincaré inequality (3.22), we obtain that for every $j \in \mathbb{N}$ there exists a constant rotation $\mathbf{R}_j \in SO(N)$ such that

$$\int_\Omega |\nabla \mathbf{y}_j - \mathbf{R}_j|^2 d\mathbf{x} \leq h_j^2 + C_K(\|f\|_{L^2(\partial \Omega)} + \|g\|_{L^2(\Omega)}) t_j h_j^2;$$

that is, by setting $C' = C_K(\|f\|_{L^2(\partial \Omega)} + \|g\|_{L^2(\Omega)})$,

$$\int_\Omega (\mathbf{I} + h_j t_j \nabla \mathbf{w}_j - \mathbf{R}_j)^2 d\mathbf{x} \leq h_j^2 (1 + C' t_j).$$ (3.33)

By possible further extraction of subsequences one among the three alternatives take place:

a) $h_j t_j \to \lambda > 0$,  

b) $h_j t_j \to 0$,  

c) $h_j t_j \to +\infty$.

If condition a) holds true we have

$$\int_\Omega \left| \nabla \mathbf{w}_j - \frac{\mathbf{R}_j - \mathbf{I}}{h_j t_j} \right|^2 d\mathbf{x} \leq \frac{1}{h_j^2} + \frac{C'}{t_j},$$

hence, up to subsequences,

(3.34)

$$\nabla \mathbf{w}_j \to \frac{\mathbf{R} - \mathbf{I}}{\lambda}$$

strongly in $L^2(\Omega; \mathcal{M}^{N \times N})$ for a suitable constant matrix $\mathbf{R} \in SO(N)$ and by Lemma 3.2 we get $\nabla \mathbf{w} \in \mathbb{K} + \mathcal{M}^{N \times N}_{\text{skew}}$.

If condition b) holds true, then by using formula (3.12) and (3.33) there exist $\vartheta_{h_j} \in [0, 2\pi]$ and a constant $\mathbf{W}_j \in \mathcal{M}^{N \times N}_{\text{skew}}$ with $|\mathbf{W}_j|^2 = |\mathbf{W}_j^2|^2 = 2$ such that

(3.35)

$$\int_\Omega |h_j t_j \nabla \mathbf{w}_j - \sin \vartheta_{h_j} \mathbf{W}_j - (1 - \cos \vartheta_{h_j}) \mathbf{W}_j^2|^2 d\mathbf{x} \leq h_j^2 (1 + C' t_j).$$

Since $\mathbf{W}_j$ and $\mathbf{W}_j^2$ are respectively skew-symmetric and symmetric, (3.35) yields

(3.36)

$$\int_\Omega \left| \mathbb{E}(\mathbf{w}_j) - \frac{1 - \cos \vartheta_{h_j}}{h_j t_j} \mathbf{W}_j^2 \right|^2 d\mathbf{x} \leq t_j^{-2} + C' t_j^{-1}$$

and bearing in mind that $\int_\Omega |\mathbb{E}(\mathbf{w}_j)|^2 d\mathbf{x} = 1$ we get

(3.37)

$$\left| \frac{1 - \cos \vartheta_{h_j}}{h_j t_j} \mathbf{W}_j \right| = \frac{1}{2} \left| \frac{1 - \cos \vartheta_{h_j}}{h_j t_j} \mathbf{W}_j \right| \leq C''.$$
Since (3.32) entails

\[ |\sin \vartheta_j| \leq \sqrt{2(1 - \cos \vartheta_j)} \leq \sqrt{2C'' h_j t_j} \]

and by (3.35)

\[ \int_{\Omega} \left| \frac{h_j t_j}{\sqrt{h_j t_j}} \nabla w_j - \frac{\sin \vartheta_j}{\sqrt{h_j t_j}} \mathbf{W}_j - \frac{(1 - \cos \vartheta_j)}{\sqrt{h_j t_j}} \mathbf{W}_j^2 \right|^2 \, dx \leq C'' h_j. \]

By (3.37) we know that \( 1 - \cos \vartheta_j = o(\sqrt{h_j t_j}) \), hence (3.35), (3.39) entail, up to subsequences, \( \sqrt{h_j t_j} \nabla w_j \to \mathbf{W} \in \mathcal{M}^{N \times N} \) strongly in \( L^2(\Omega; \mathcal{M}^{N \times N}) \) and since by (2.8) and Poincaré-Korn inequality

\[
t_j \int_{\Omega} \left[ E(w_j) + \frac{1}{2} h_j t_j \nabla w_j^T \nabla w_j \right] \, dx \leq C^{IV} + \mathcal{L}(w_j)
= C^{IV} + \mathcal{L}(w_j - P w_j)
\leq C^{V} \left( \int_{\Omega} |E(w_j)|^2 \, dx \right)^{\frac{1}{2}},
\]

we deduce

\[
\int_{\Omega} |E(w_j) + \frac{1}{2} h_j t_j \nabla w_j^T \nabla w_j|^2 \, dx \to 0.
\]

On the other hand, since

\[
\liminf_{j \to +\infty} \int_{\Omega} |2E(w_j) + h_j t_j \nabla w_j^T \nabla w_j|^2 \, dx \geq \int_{\Omega} |2E(v) - \mathbf{W}| \, dx,
\]

we get \( 2E(v) = \mathbf{W}^2 \) which implies \( \nabla w = \text{skew} \nabla w + \frac{1}{2} \mathbf{W}^2 \), hence \( \text{skew} \nabla w \) is a gradient field, that is a constant skew-symmetric matrix. By taking

\[
\mathbf{R} := I + \frac{1}{2} \mathbf{W}^2 + \frac{\sqrt{3}}{2} \mathbf{W}
\]

and by applying formula (3.12) we get \( \mathbf{R} \in SO(N) \) that is \( \nabla w - (\mathbf{R} - I) \in \mathcal{M}^{N \times N} \) which implies \( \nabla w \in \mathbb{K} + \mathcal{M}_{\text{skew}}^{N \times N} \) whenever condition b) holds.

Eventually if condition c) holds true, by (3.33) we get

\[
\int_{\Omega} \left| \nabla w_j - \frac{\mathbf{R}_j - I}{h_j t_j} \right|^2 \, dx \leq \frac{1}{t_j^2} + \frac{C'}{t_j},
\]

and by taking into account that \( |\mathbf{R}_j - I| = o(h_j t_j) \) due to (3.33), we have \( \nabla w_j \to 0 \) strongly in \( L^2(\Omega; M^{N \times N}) \), hence \( \nabla w \in \mathbb{K} + \mathcal{M}_{\text{skew}}^{N \times N} \) still by Lemma 3.2.

By summarizing, in all three cases if \( t_j := \| E(v_j) \|_{L^2} \to +\infty \) and \( F_{h_j}(t_j v_j) \leq C \) then \( \nabla(t_j^{-1} v_j) = \nabla w_j \to \nabla w \) in \( L^2(\Omega; \mathcal{M}^{N \times N}) \) and \( \nabla w \in \mathbb{K} + \mathcal{M}_{\text{skew}}^{N \times N} \). Therefore \( \mathbb{E}(w_j) \to \mathbb{E}(w) \) in \( L^2(\Omega; \mathcal{M}^{N \times N}) \).

Since \( \tilde{w}_j := w_j - P w_j \) are equiibrated in \( H^1(\Omega; \mathbb{R}^N) \), every subsequence of \( \tilde{w}_j \) has a weakly convergent subsequence and if \( \tilde{w} \) is one of the limits we get \( \mathbb{E}(\tilde{w}) = \mathbb{E}(w) \) hence by (2.24) \( \mathcal{L}(\tilde{w}) = \mathcal{L}(w) \). Therefore every subsequence of \( \mathcal{L}(w_j) \) has a subsequence which converges to \( \mathcal{L}(w) \) that is the whole sequence \( \mathcal{L}(\tilde{w}_j) \) converges to \( \mathcal{L}(w) \), hence

\[
- \mathcal{L}(w) = - \limsup_{j \to +\infty} \mathcal{L}(\tilde{w}_j) = - \limsup_{j \to +\infty} \mathcal{L}(w_j) \leq \liminf_{j \to +\infty} t_j^{-1} F_{h_j}(v_j).
\]

Since (3.32) entails \( \limsup_{j \to +\infty} t_j^{-1} F_{h_j}(v_j) \leq 0 \), by (3.41) we get \( \mathcal{L}(w) \geq 0 \).

By taking into account that \( \nabla w \in \mathbb{K} + \mathcal{M}_{\text{skew}}^{N \times N} \) then, either \( \nabla w \in \mathcal{M}_{\text{skew}}^{N \times N} \) or

\[ w(x) = \tau (\mathbf{R} - I)x + \mathbf{A} x + \mathbf{c}, \text{ for some } \tau > 0, \mathbf{R} \in SO(N), \mathbf{R} \neq I, \mathbf{A} \in \mathcal{M}_{\text{skew}}^{N \times N}, \mathbf{c} \in \mathbb{R}^N. \]
The second case cannot occur since in such case by (2.22) there would exist \( \vartheta \in \mathbb{R} \) with \( \cos \vartheta < 1 \) and \( \mathbf{W} \in \mathcal{M}_{\text{sym}}^{N \times N} \), \( \mathbf{W} \neq 0 \) such that \( \mathbf{R} = \mathbf{I} + (1 - \cos \vartheta)\mathbf{W}^2 + (\sin \vartheta)\mathbf{W} \in SO(N) \) hence (2.24), (2.25) would entail the contradiction below

\[
\mathcal{L}(\mathbf{w}) = \tau \int_{\partial \Omega} \mathbf{f} \cdot (\mathbf{R} - \mathbf{I}) \mathbf{x} d\mathcal{H}^{N-1} + \tau \int_{\Omega} \mathbf{g} \cdot (\mathbf{R} - \mathbf{I}) \mathbf{x} =
\]

(3.42)

\[
= \tau (1 - \cos \vartheta) \int_{\partial \Omega} \mathbf{f} \cdot \mathbf{W}^2 \mathbf{x} d\mathcal{H}^{N-1} + \tau (1 - \cos \vartheta) \int_{\Omega} \mathbf{g} \cdot \mathbf{W}^2 \mathbf{x} < 0.
\]

Hence \( \nabla \mathbf{w} \in \mathcal{M}_{\text{skew}}^{N \times N} \) that is \( E(\mathbf{w}) = 0 \) which is again a contradiction since \( ||E(\mathbf{w}_j)||_{L^2} = 1 \) and \( E(\mathbf{w}_j) \to E(\mathbf{w}) \) in \( L^2(\Omega; \mathcal{M}^{N \times N}) \).

**Proof of Theorem 2.2** - First we notice that minimizing sequences for \( \mathcal{F}_{h_j} \) do exist \( h_j \) for every sequence of positive real numbers converging to 0, thank to Lemma 3.4.

Fix a sequence of real numbers \( h_j > 0 \) converging to 0 and a minimizing sequence \( \mathbf{v}_j \) for \( \mathcal{F}_{h_j} \).

Up to a preliminary extraction of a subsequence we can assume that \( h_j \) is decreasing.

Since \( -\infty < \inf \mathcal{F}_{h_j} \leq 0 \) there is \( C > 0 \) such that \( \mathcal{F}_{h_j}(\mathbf{v}_j) \leq C \), hence by Lemmas 3.6 and 3.2

\[
||E(\mathbf{v}_j)||_{L^2} \leq C
\]

and there exists \( \mathbf{v}_0 \in H^1(\Omega; \mathbb{R}^N) \) such that, up to subsequences, \( E(\mathbf{v}_j) \rightharpoonup E(\mathbf{v}_0) \) in \( L^2(\Omega; \mathcal{M}^{N \times N}) \), thus proving (2.26). By Lemma 3.4 we get

\[
\liminf_{j \to +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \mathcal{F}(\mathbf{v}_0)
\]

and again by Lemma 3.4 for every \( \mathbf{v} \in H^1(\Omega; \mathbb{R}^N) \) there exists \( \mathbf{v}_j \in H^1(\Omega; \mathbb{R}^N) \) such that \( E(\mathbf{v}_j) \rightharpoonup E(\mathbf{v}) \) in \( L^2(\Omega; \mathcal{M}^{N \times N}) \) and

\[
\limsup_{j \to +\infty} \mathcal{F}_j(\mathbf{v}_j) \leq \mathcal{F}(\mathbf{v}).
\]

Hence

(3.43)

\[
\mathcal{F}(\mathbf{v}_0) \leq \liminf_{n \to +\infty} \mathcal{F}_j(\mathbf{v}_j) \leq \liminf_{n \to +\infty} (\inf \mathcal{F}_{h_j} + o(1)) \leq \limsup_{j \to +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \leq \mathcal{F}(\mathbf{v})
\]

and (2.28) is proven.

Eventually we notice that (2.27) is a straightforward consequence of (3.13), (3.14), (3.15) and (3.17) in the proof of Lemma 3.4 so we are left only to prove (2.29).

To this aim it will be enough to notice that by (2.26) and (2.27) we get

\[
E(\mathbf{v}_j) + \frac{1}{2} h_j \nabla \mathbf{v}_j^T \nabla \mathbf{v}_j \rightharpoonup E(\mathbf{v}_0) - \frac{1}{2} \mathbf{W}_0^2 \quad \text{in} \quad L^1(\Omega; \mathcal{M}^{N \times N})
\]

and by recalling (2.28) and (3.22) we get

\[
\mathcal{F}(\mathbf{v}_0) = \liminf_{j \to +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \int_\Omega \mathbf{V}_0(\mathbf{x}, E(\mathbf{v}_0) - \frac{1}{2} \mathbf{W}_0^2) \, d\mathbf{x} \geq \mathcal{F}(\mathbf{v}_0)
\]

thus proving (2.29).

**4. Limit Problem and Linear Elasticity**

We denote by \( \mathcal{E} : H^1(\Omega; \mathbb{R}^N) \to \mathbb{R} \) the energy functional of classical linear elasticity

(4.1)

\[
\mathcal{E}(\mathbf{v}) := \int_\Omega \mathcal{V}_0(\mathbf{x}, \mathcal{E}(\mathbf{v})) \, d\mathbf{x} - \mathcal{L}(\mathbf{v}).
\]

Notice that (1.6) is just a particular model case of (4.1) corresponding to (1.4).

As it was already emphasized the inequality \( \mathcal{F} \leq \mathcal{E} \) always holds true. Moreover the two functionals cannot coincide: indeed \( \mathcal{F}(\mathbf{v}) < \mathcal{E}(\mathbf{v}) \) whenever \( \mathbf{v}(\mathbf{x}) = \frac{1}{2} \mathbf{W}^2 \mathbf{x} \) with \( \mathbf{W} \in \mathcal{M}_{\text{skew}}^{N \times N} \). However we can show that the two functionals \( \mathcal{F} \) and \( \mathcal{E} \), notwithstanding their differences, have the same minimum and same set of minimizers when the loads are equilibrated and compatible, say when the load fulfills both (2.24) and (2.25).

Next results clarify the relationship between the minimizers of classical linear elasticity functional \( \mathcal{E} \) and the
minimizers of functional $\mathcal{F}$ defined in (1.3), which is the variational limit of nonlinear energies $\mathcal{F}_h$ in the sense shown by Theorem 2.2.

**Theorem 4.1.** Assume that (2.24) and (2.25) hold true. Then

\begin{equation}
\min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}(v) = \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}(v),
\end{equation}

and

\begin{equation}
\arg \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F} = \arg \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}.
\end{equation}

**Proof.** Both functionals $\mathcal{F}, \mathcal{E}$ do have minimizers under conditions (2.24), (2.25): $\mathcal{E}$ by classical results and $\mathcal{F}$ by Theorem 2.2. Taking into account that $\mathcal{F}(v) \leq \mathcal{E}(v)$ for every $v \in H^1(\Omega; \mathbb{R}^N)$, and setting $z_W(x) := \frac{1}{2} W^2 x$ for every $W \in \mathcal{M}_{skew}^{N \times N}$, we get $E(z_W) = \frac{1}{2} W^2$ and

\begin{equation}
\min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}(v) \geq \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}(v) =
\end{equation}

\begin{equation}
\min_{W \in \mathcal{M}_{skew}^{N \times N}} \left\{ \min_{v \in H^1(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} V_0(x, E(v) - \frac{1}{2} W^2) \, dx - L(v) \right\} \right\} =
\end{equation}

\begin{equation}
\min_{W \in \mathcal{M}_{skew}^{N \times N}} \left\{ \min_{v \in H^1(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} V_0(x, E(v - z_W)) \, dx - L(v - z_W) - L(z_W) \right\} \right\} =
\end{equation}

\begin{equation}
\min_{z \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}(z) - \max_{W \in \mathcal{M}_{skew}^{N \times N}} L(z_W) \geq \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}.
\end{equation}

where last inequality follows by $L(z_W) \leq 0$, due to (2.25). Therefore (4.2) is proved and we are left to show (1.3).

First assume $v \in \arg \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}$ and let

\begin{equation}
W = \arg \min \left\{ \int_{\Omega} V_0(x, E(v) - \frac{1}{2} W^2) \, dx : W \in \mathcal{M}_{skew}^{N \times N} \right\}.
\end{equation}

If $W \neq 0$ then, by setting $z_W(x) = \frac{1}{2} W^2 x$ we get $E(z_W) = \nabla z_W = \frac{1}{2} W^2$ and, by compatibility (2.25) we obtain

\begin{equation}
\min \mathcal{F} = \mathcal{F}(v) = \int_{\Omega} V_0(x, E(v - z_W)) \, dx - L(v - z_W) - L(z_W) =
\end{equation}

say a contradiction. Therefore $W = 0$, $z_W = 0$, and all the inequalities in (4.6) turn out to be equalities, hence we get $\mathcal{F}(v) = \mathcal{E}(v) = \min \mathcal{E} = \min \mathcal{F}$, say $v \in \arg \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}$ and $\arg \min_{H^1(\Omega; \mathbb{R}^N)} \mathcal{F} \subset \arg \min_{H^1(\Omega; \mathbb{R}^N)} \mathcal{E}$.

In order to show the opposite inclusion, we assume $v \in \arg \min_{v \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}$ and still referring to the choice (4.5) we set $z_W = \frac{1}{2} W^2 x$. Then

\begin{equation}
\mathcal{F}(v) = \int_{\Omega} V_0(x, E(v - z_W)) \, dx - L(v - z_W) - L(z_W) =
\end{equation}

\begin{equation}
\mathcal{E}(v - z_W) - L(z_W) \geq \min \mathcal{E} - L(z_W) > \min \mathcal{E},
\end{equation}

This leads to the contradiction $\mathcal{F}(v) > \mathcal{F}(v - z_W)$ if $z_W \neq 0$, due to (2.25); therefore $z_W = 0$ and we have equalities in place of inequalities in (4.7): therefore $\mathcal{E}(v) = \mathcal{F}(v)$ and $v \in \arg \min_{H^1(\Omega; \mathbb{R}^N)} \mathcal{F}$. □

**Corollary 4.2.** Assume the standard structural assumptions, (2.24), (2.25) and $W_0 \in \mathcal{M}_{skew}^{N \times N}$ is the matrix whose existence is warranted by Theorem 2.2. Then $W_0 = 0$. 


Proof. Let $v_0$ be in $\arg\min \mathcal{F}$, $W_0$ be the skew symmetric matrix in the claim of Theorem 2.2 and assume by contradiction that $W_0 \neq 0$. By (1.2) and (1.3) we get
\[ \int_\Omega V_0(x, E(v_0) - \frac{1}{2} W_0^2) \, dx = \int_\Omega V_0(x, E(v_0)) \, dx, \]
so by taking into account that $V_0$ is a positive definite quadratic form
\[ \int_\Omega V_0(x, \frac{1}{2} W_0^2) \, dx - \frac{1}{2} \int_\Omega D V_0(x, E(v_0)) \cdot W_0^2 = 0. \]
Since by (4.3) $v_0 \in \arg\min E$, the Euler-Lagrange equation yields
\[ \int_\Omega D V_0(x, E(v_0)) \cdot W_0^2 = \mathcal{L}(W_0^2x), \]
hence
\[ 0 \leq \int_\Omega V_0(x, \frac{1}{2} W_0^2) \, dx = \frac{1}{2} \mathcal{L}(W_0^2x) < 0 \]
which is a contradiction by (2.25).
Hence $W_0 = 0$. □

If strong inequality in (2.25) is replaced by a weak inequality, then Theorem 4.1 cannot hold as it is, nevertheless a weaker claim still holds true as it is shown by the following general result.

Proposition 4.3. If the structural assumptions together with (2.21) are fulfilled, but (2.25) is replaced by

(4.8) \[ \mathcal{L}(W^2x) \leq 0 \quad \forall W \in \mathcal{M}^{N \times N}_{\text{skew}} \]
then $\arg\min \mathcal{F}$ is still nonempty and

(4.9) \[ \min \mathcal{F} = \min \mathcal{E}, \]
but the coincidence of minimizers sets is replaced by the inclusion

(4.10) \[ \arg\min \mathcal{E} \subset \arg\min \mathcal{F}. \]

If (4.8) holds true and there exists $U \in \mathcal{M}^{N \times N}_{\text{skew}}$, $U \neq 0$ such that $\mathcal{L}(U^2x) = 0$, then $\mathcal{F}$ admits infinitely many minimizers which are not minimizers of $\mathcal{E}$, precisely

(4.11) \[ \arg\min \mathcal{E} \subsetneq \arg\min \mathcal{E} + \left\{ U^2x : U \in \mathcal{M}^{N \times N}_{\text{skew}}, \mathcal{L}(U^2x) = 0 \right\} \subset \arg\min \mathcal{F}, \]
where the last inclusion is an equality in 2D:

(4.12) \[ \arg\min \mathcal{E} \subsetneq \arg\min \mathcal{E} + \left\{ -t x : t \geq 0 \right\} = \arg\min \mathcal{F}, \quad \text{if } N = 2. \]

Proof. The set $\arg\min \mathcal{E}$ is nonempty by classical arguments. Fix $v_\ast \in \arg\min \mathcal{E}$. Then for every $v \in H^1(\Omega; \mathbb{R}^N)$ and for every $W \in \mathcal{M}^{N \times N}_{\text{skew}}$, by setting $z_W = \frac{1}{2} W^2 x$, we get
\[ \mathcal{F}(v_\ast) \leq \mathcal{E}(v_\ast) \leq \mathcal{E}(v - z_W) = \int_\Omega V_0(x, E(v) - \frac{1}{2} W^2) \, dx - \mathcal{L}(v - z_W) \leq \]
\[ \leq \int_\Omega V_0(x, E(v) - \frac{1}{2} W^2) \, dx - \mathcal{L}(v) \]
hence for every $v \in H^1(\Omega; \mathbb{R}^N)$

(4.14) \[ \mathcal{F}(v_\ast) \leq \min_{W \in \mathcal{M}^{N \times N}_{\text{skew}}} \int_\Omega V_0(x, E(v) - \frac{1}{2} W^2) \, dx - \mathcal{L}(v) = \mathcal{F}(v) \]
thus proving that \( \argmin \mathcal{F} \) is nonempty and (4.10).

Moreover by setting

\[
(4.15) \quad \mathbf{W}_v \in \argmin \left\{ \int_{\Omega} \nu_0(x, E(v) - \frac{1}{2} W^2) \, dx : \mathbf{W} \in \mathcal{M}_{skew}^{N \times N} \right\}, \quad \forall \, v \in H^1(\Omega; \mathbb{R}^N)
\]

condition (4.8) entails

\[
(4.16) \quad \mathcal{F}(v_*) = \int_{\Omega} \nu_0(x, E(v_* - z_{W_*})) \, dx - \mathcal{L}(v_*) = \mathcal{E}(v_* - z_{W_*}) = \mathcal{L}(z_{W_*}) \geq \mathcal{E}(v_*)
\]

hence (4.9) follows by (4.16).

If (4.8) holds true, \( \mathcal{L}(z_{U}) = 0 \) for some \( 0 \neq U \in \mathcal{M}_{skew}^{N \times N} \) and \( v^* \in \argmin \mathcal{E} \) then, by comparing the finite dimensional minimization over \( \mathbf{W} \) with evaluation at \( \mathbf{W} = \mathbf{U} \) and exploiting (4.9), we get

\[
(4.17) \quad \mathcal{F}(v_* + z_{U}) = \min_{\mathbf{W}} \int_{\Omega} \nu_0(x, E(v_* - \mathbf{W})) \, dx - \mathcal{L}(v_* + z_{U}) \leq \int_{\Omega} \nu_0(x, E(v_*)) \, dx - \mathcal{L}(v_*) = \mathcal{E}(v_*) = \min \mathcal{E} = \min \mathcal{F},
\]

that is \( v_* + z_{U} \in \argmin \mathcal{F} \). Since \( \nu_0 \) is strictly convex we get \( \argmin \mathcal{E} = \{ v_* + z : z \in \mathbb{R} \} \) hence \( \mathcal{E}(v_* + z_{U}) > \mathcal{E}(v_*) \) thus proving the strict inclusion in (4.11).

Concerning last claim, if (4.8) holds true, \( \mathcal{L}(z_{U}) = 0 \) for some \( 0 \neq U \in \mathcal{M}_{skew}^{2 \times 2} \), \( v^* \in \argmin \mathcal{F} \) and \( N = 2 \), then \( \mathcal{M}_{skew}^{2 \times 2} \) is a 1D space, therefore we can assume \( U = (e_1 \otimes e_2 - e_2 \otimes e_1) \), \( U^2 = -I \), \( \mathcal{M}_{skew}^{2 \times 2} = \text{span} U \) and \( W_{v_*} = \lambda U \) for some \( \lambda \in \mathbb{R} \), and by (4.9)

\[
\min \mathcal{E} = \min \mathcal{F} = \mathcal{F}(v^*) = \int_{\Omega} \nu_0(E(v^*) - \frac{1}{2} W_{v_*}^2) \, dx - \mathcal{L}(v^*) = \int_{\Omega} \nu_0(E(v^*) - \frac{\lambda^2}{2} U^2) \, dx - \mathcal{L}(v^* - z_{\lambda U}) = \mathcal{E}(v^* - z_{\lambda U}),
\]

that is \( (v_* - z_{\lambda U}) \in \argmin \mathcal{E} \) for every \( v_* \in \argmin \mathcal{F} \), therefore we get

\[
\argmin \mathcal{F} \subseteq \{ z_{\lambda U} : \lambda \in \mathbb{R} \} \subseteq \argmin \mathcal{E}, \quad \argmin \mathcal{F} \subseteq \argmin \mathcal{E} + \{ z_{\lambda U} : \lambda \in \mathbb{R} \},
\]

hence by \( z_{\lambda U} = \frac{\lambda^2}{2} U^2 x = -\frac{\lambda^2}{2} x \) we obtain the equality in place of the last inclusion in (4.11), hence (4.12). \( \blacksquare \)

Next example depicts the above Proposition in a simple explicit case.

**Example 4.4.** Let \( \Omega = (-1/2, 1/2)^2 \), \( g \equiv 0 \), \( f = (1_{S_+} - 1_{S_-})e_2 + (1_{T_+} - 1_{T_-})e_1 \) where \( S_{\pm} \) denote respectively the right and the left side, and \( T_{\pm} \) the upper and the lower side of the square (see Fig. 1).

![Figure 1](image_url)

**Figure 1.** Example 4.4

A straightforward computation gives, for suitable \( \lambda \in \mathbb{R} \),

\[
(4.18) \quad \int_{\partial \Omega} f \cdot W^2 x \, d\mathcal{H}^{N-1} = -\lambda^2 \int_{\partial \Omega} f \cdot x \, d\mathcal{H}^{N-1} = 0 \quad \forall \, W \in \mathcal{M}_{skew}^{2 \times 2},
\]
Then, since (2.24) and (4.8) are fulfilled, by (4.12) in Proposition 4.3 we know that, for every choice of \( V_0 \) satisfying the standard structural hypotheses, \( F \) has infinitely many minimizers \( v \) which are not minimizers of \( E \), given by

\[
v = (v^* - tx) \in \arg\min F \setminus \arg\min E \quad \text{if } v^* \in \arg\min E, \quad t > 0.
\]

It is quite natural to ask whether condition (2.25), which is essential in the proof of Theorem 2.2, may be dropped in order to obtain at least existence of \( \min F \): the answer is negative.

Indeed the next remark shows that, when compatibility inequality in (2.25) is reversed for at least one choice of the skew-symmetric matrix \( W \), then \( F \) is unbounded from below.

**Remark 4.5.** If

\[
\exists W_* \in \mathcal{M}^{N \times N}_{\text{skew}} : \quad \mathcal{L}(z_{W_*}) > 0, \quad \text{where } z_{W_*} = \frac{1}{2} W_*^2 x,
\]

then

\[
\inf_{v \in H^1(\Omega; \mathbb{R}^N)} F(v) = -\infty.
\]

Indeed, by arguing as in (4.4) we get

\[
\inf_{H^1(\Omega; \mathbb{R}^N)} F = \min_{H^1(\Omega; \mathbb{R}^N)} E - \sup_{W \in \mathcal{M}^{N \times N}_{\text{skew}}} \mathcal{L}(z_W) \quad \text{where } z_W = \frac{1}{2} W^2 x.
\]

Hence

\[
\inf_{H^1(\Omega; \mathbb{R}^N)} F \leq \min_{H^1(\Omega; \mathbb{R}^N)} E - \tau \mathcal{L}(z_{W_*}) \quad \forall \tau > 0,
\]

which entails (4.20).

Next example shows that in case of uniform compression along the whole boundary functional \( F \) is unbounded from below.

**Example 4.6.** Assume \( \Omega \subset \mathbb{R}^N \) is a Lipschitz, connected open set, \( N = 2, 3 \), \( g \equiv 0 \), \( f = -n \), where \( n \) denotes the outer unit normal vector to \( \partial \Omega \) (see Fig.2).

![Figure 2. Example 4.6](image)

Then (4.19) holds true hence, by Remark 4.5, \( \inf_{v \in H^1(\Omega; \mathbb{R}^N)} F(v) = -\infty. \)

Indeed, for every \( W \in \mathcal{M}^{N \times N}_{\text{skew}} \) such that \( |W|^2 = 2 \) we obtain

\[
\int_{\partial \Omega} f \cdot W^2 x \, d\mathcal{H}^{N-1} = -\int_{\partial \Omega} n \cdot W^2 x \, d\mathcal{H}^{N-1} = -\int_{\Omega} \text{div}(W^2 x) \, dx = -|\Omega| \text{Tr} W^2 = 2 |\Omega| > 0.
\]

Therefore any Lipschitz open set turns out to be always unstable when uniformly compressed in the direction of the inward normal vector along its boundary and the linearized model proves inadequate for this case even for small load.
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