THE GEOMETRY OF STABLE MINIMAL SURFACES IN METRIC LIE GROUPS

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Abstract. We study geometric properties of compact stable minimal surfaces with boundary in homogeneous 3-manifolds $X$ that can be expressed as a semidirect product of $\mathbb{R}^2$ with $\mathbb{R}$ endowed with a left invariant metric. For any such compact minimal surface $M$, we provide an a priori radius estimate which depends only on the maximum distance of points of the boundary $\partial M$ to a vertical geodesic of $X$. We also give a generalization of the classical Radó theorem in $\mathbb{R}^3$ to the context of compact minimal surfaces with graphical boundary over a convex horizontal domain in $X$, and we study the geometry, existence, and uniqueness of this type of Plateau problem.

1. Introduction

In this paper we study the geometry of compact minimal surfaces with boundary in homogeneous manifolds diffeomorphic to $\mathbb{R}^3$. By classification, each such homogeneous manifold $X$ is a metric Lie group, i.e., a simply connected 3-dimensional Lie group equipped with a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$. For such an $X$ there are two possibilities: either $X$ is isometric to the universal cover of the special linear group $\text{SL}(2, \mathbb{R})$ endowed with some left invariant metric or $X$ is a metric semidirect product. By definition, a metric semidirect product $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is given as a Lie group ($\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}, *$), together with a certain left invariant metric (its canonical metric; see Definition 2.1), where the product operation $*$ is expressed in terms of some real $2 \times 2$ matrix $A \in M_2(\mathbb{R})$ as

$$(p_1, z_1) * (p_2, z_2) = (p_1 + e^{z_1 A} \, p_2, z_1 + z_2);$$

see Subsection 2.1 for more details. When $\text{trace}(A) = 0$, $X$ is a unimodular semidirect product; typical examples of Riemannian manifolds in this situation are the Euclidean space $\mathbb{R}^3$, the Heisenberg space $\text{Nil}_3$, or the solvable Lie group $\text{Sol}_3$ with its usual Thurston geometry. When $\text{trace}(A) \neq 0$, we obtain the non-unimodular...
semidirect products, among which we highlight the hyperbolic space $\mathbb{H}^3$ and the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$.

The geometry of minimal surfaces in homogeneous 3-manifolds of non-constant sectional curvature has been deeply studied in the last decade, specially in the case that the isometry group of the homogeneous manifold has dimension 4. To indicate just a few relevant works in this area, we may cite [1–6,11,12,14,17,27,32,34,36]. An outline of the beginning of the theory of constant mean curvature surfaces in homogeneous 3-manifolds with a 4-dimensional isometry group can be consulted in [7,13].

For the generic, non-symmetric case of homogeneous 3-manifolds with an isometry group of dimension three, the theory of minimal surfaces is less developed. For some works dealing with this more general situation, see, e.g., [8–10,15,16,19–22,25,28,31]. For an introduction to the geometry of general simply connected homogeneous 3-manifolds, see [22].

In this paper we develop some aspects of the theory of compact minimal surfaces with boundary in metric semidirect products $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$. We shall be specially interested in the geometry, existence, and uniqueness of solutions to the Plateau problem for graphical boundaries on convex domains of $\mathbb{R}^2 \rtimes_A \{0\}$, and on estimating the radius of compact stable minimal surfaces with boundary in metric semidirect products. We recall that the radius of a compact Riemannian surface $M$ with boundary is the maximum distance of points in the surface to its boundary $\partial M$.

An important classical result of Radó [30] states that a simple closed curve $\Gamma$ in $\mathbb{R}^3$ that has a 1-1 orthogonal projection to a convex curve in a plane $P \subset \mathbb{R}^3$, is the boundary of a minimal disk of finite area that is a graph over its projection to $P$, and, furthermore, any branched minimal disk in $\mathbb{R}^3$ with boundary $\Gamma$ has similar properties. Other classical Radó type results for minimal surfaces in $\mathbb{R}^3$ were obtained in [18,29]. In Section 3 of this paper we will extend Radó’s theorem to the context of minimal surfaces in metric semidirect products; see Theorems 3.1 and 3.3.

In Section 4, Theorems 3.1 and 3.3 are used to study the geometry of compact minimal surfaces $\Sigma$ in a non-unimodular semidirect product, such that $\Sigma$ is the boundary of a round Euclidean circle in $\mathbb{R}^2 \rtimes_A \{0\}$. Namely we prove that as the radius of such a circle goes to infinity, the angles that $\Sigma$ makes with $\mathbb{R}^2 \rtimes_A \{0\}$ along its boundary circle converge uniformly to $\pi/2$. See Theorem 4.1 for a generalization of this result and also see the related application given in Corollary 4.3 to the existence of a minimal annulus bounded by two circles in $\mathbb{R}^2 \rtimes_A \{0\}$ of large radius, so that these circles can be taken arbitrarily far away from each other. All of these results are then applied in Section 5 to obtain radius estimates of compact minimal surfaces with boundary in metric semidirect products $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, as we explain next. Given $A \in \mathcal{M}_2(\mathbb{R})$, any vertical line $\Gamma = \{(x_0, y_0, z) \mid z \in \mathbb{R}\}$ in $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is a geodesic of $X$ (endowed with its canonical metric), which we call a vertical geodesic. By a metric solid cylinder of radius $r > 0$ in $X$ around $\Gamma$ we mean the set of points $W(\Gamma, r)$ in $X$ whose distance to $\Gamma$ is at most $r$. With these definitions in mind, the next theorem summarizes another main result of the paper.

**Theorem 1.1.** Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, and let $W(\Gamma, r)$ be a solid metric cylinder in $X$ of radius $r > 0$ around a vertical geodesic $\Gamma$. There exists some $R = R(r) > 0$ such that if $M$ is a compact, stable minimal surface in $W(\Gamma, r)$, then $M$ is contained in the metric solid cylinder $W(\Gamma, R)$.
X whose boundary $\partial M$ is contained in $W(\Gamma, r)$, then $M$ has radius at most $R$. In particular, there are no complete stable minimal surfaces contained in $W(\Gamma, r)$.

There is an application of Theorem 1.1 of special interest. In our paper [19] we give a classification of the constant mean curvature spheres immersed in any homogeneous manifold $X$, proving in particular that two spheres of the same constant mean curvature in such an $X$ differ by an ambient isometry. In order to prove this classification, a key step is to show that when $X$ is diffeomorphic to $\mathbb{R}^3$, if the areas of a sequence of immersed constant mean curvature spheres in $X$ diverge to infinity, then a subsequence of suitable translations in $X$ of these spheres converges to an entire Killing graph in $X$. The proof of this result in [19] follows from a case-by-case study. In one of the cases to be ruled out, the ambient space $X$ is a metric semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, and Theorem 1.1 is used to construct geodesic balls of a certain fixed radius in the abstract Riemannian three-balls bounded by the constant mean curvature spheres whose areas diverge to infinity and so that the volumes of these balls tend to infinity as $n \to \infty$. This unbounded volume result eventually provides a contradiction with Bishop’s theorem and rules out that situation, as desired.

Theorem 1.1 will be proved in Section 5; see Theorem 5.6. Another tool used to prove Theorem 5.6 is Proposition 5.4, where we will construct a certain family of mean convex solid cylinders over appropriately defined ellipses in non-unimodular metric semidirect products with positive Milnor $D$-invariant. For this and other purposes, we will prove in the Appendix a few additional technical results about the geometry of these metric semidirect products.

2. BACKGROUND MATERIAL ON 3-DIMENSIONAL METRIC LIE GROUPS

This preliminary section is devoted to stating some basic properties of 3-dimensional Lie groups endowed with a left invariant metric that will be used freely in later sections. For details of these basic properties, see the general reference [22].

Let $Y$ denote a simply connected, homogeneous Riemannian 3-manifold, and assume that it is not isometric to the Riemannian product of the 2-sphere $\mathbb{S}^2(\kappa)$ of constant curvature $\kappa > 0$ with the real line. Then $Y$ is isometric to a simply connected, 3-dimensional Lie group $G$ equipped with a left invariant metric $\langle \cdot, \cdot \rangle$; i.e., for every $p \in G$, the left translation $l_p: G \to G$, $l_p(q) = pq$, is an isometry of $\langle \cdot, \cdot \rangle$. We will call such a space a metric Lie group, $X = (G, \langle \cdot, \cdot \rangle)$. When $X$ is simply connected, there are three possibilities which are actually mutually disjoint in the case that the isometry group of $X$ has dimension 3:

- $X$ is isometric to the special unitary group $SU(2)$ with a left invariant metric. This is the only case in which $X$ is not diffeomorphic to $\mathbb{R}^3$ and the family of left invariant metrics is 3-dimensional.
- $X$ is isometric to the universal cover $\tilde{\text{SL}}(2, \mathbb{R})$ of the special linear group, equipped with a left invariant metric. Again, there is a 3-dimensional family of such metrics.
- $X$ is isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ equipped with its canonical metric, which is the left invariant metric introduced in Definition 2.1 below. In this third case, the underlying Lie group is $(\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}, \ast)$, where the group operation $\ast$ is expressed in terms of some real $2 \times 2$ matrix $A \in \mathcal{M}_2(\mathbb{R})$. 

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We define the canonical left invariant metric
\begin{equation}
\langle p_1, z_1 \rangle \ast \langle p_2, z_2 \rangle = \langle p_1 + e^{z_1A} p_2, z_1 + z_2 \rangle;
\end{equation}
here \( p_1, p_2 \in \mathbb{R}^2 \), \( z_1, z_2 \in \mathbb{R} \), and \( e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k \) denotes the usual exponentiation of a matrix \( B \in M_2(\mathbb{R}) \).

### 2.1. Semidirect products.

Consider the semidirect product \( \mathbb{R}^2 \rtimes_A \mathbb{R} \), where
\begin{equation}
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\end{equation}

Then, in terms of the coordinates \((x, y) \in \mathbb{R}^2, \ z \in \mathbb{R}\), we have the following basis \\( \{E_1, E_2, E_3\} \) of the linear space of right invariant vector fields on \( \mathbb{R}^2 \rtimes_A \mathbb{R} \):
\begin{equation}
F_1 = \partial_x, \quad F_2 = \partial_y, \quad F_3(x, y, z) = (ax + by) \partial_x + (cx + dy) \partial_y + \partial_z.
\end{equation}

In the same way, a left invariant frame \( \{E_1, E_2, E_3\} \) of \( X \) is given by
\begin{equation}
E_1(x, y, z) = a_{11}(z) \partial_x + a_{21}(z) \partial_y, \quad E_2(x, y, z) = a_{12}(z) \partial_x + a_{22}(z) \partial_y, \quad E_3 = \partial_z,
\end{equation}
where
\begin{equation}
e^{zA} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}.
\end{equation}

In terms of \( A \), the Lie bracket relations are:
\begin{equation}
[E_1, E_2] = 0, \quad [E_3, E_1] = aE_1 + cE_2, \quad [E_3, E_2] = bE_1 + dE_2.
\end{equation}

Observe that \( \text{Span}\{E_1, E_2\} \) is an integrable 2-dimensional distribution of \( \mathbb{R}^2 \rtimes_A \mathbb{R} \), whose integral surfaces are the leaves of the foliation \( F = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\} \) of \( \mathbb{R}^2 \rtimes_A \mathbb{R} \).

**Definition 2.1.** We define the canonical left invariant metric on the semidirect product \( \mathbb{R}^2 \rtimes_A \mathbb{R} \) to be that one for which the left invariant basis \( \{E_1, E_2, E_3\} \) given by \( (2.4) \) is orthonormal. Equivalently, it is the left invariant extension to \( \mathbb{R}^2 \rtimes_A \mathbb{R} \) of the inner product on the tangent space \( T_0(\mathbb{R}^2 \rtimes_A \mathbb{R}) \) at the identity element \( \tilde{0} = (0, 0, 0) \) that makes \( \{(\partial_x)_0, (\partial_y)_0, (\partial_z)_0\} \) an orthonormal basis.

We next emphasize some other metric properties of the canonical left invariant metric \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^2 \rtimes_A \mathbb{R} \):

- The mean curvature of each leaf of the foliation \( F = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\} \) with respect to the unit normal vector field \( E_3 \) is the constant \( H = \text{trace}(A)/2 \). All the leaves of the foliation \( F \) are intrinsically flat.
- The change from the orthonormal basis \( \{E_1, E_2, E_3\} \) to the basis \( \{\partial_x, \partial_y, \partial_z\} \) given by \( (2.4) \) produces the following expression for the metric \( \langle \cdot, \cdot \rangle \) in the \( x, y, z \) coordinates of \( X := (\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle \cdot, \cdot \rangle) \):
\begin{equation}
\langle \cdot, \cdot \rangle = \left[ a_{11}(-z)^2 + a_{21}(-z)^2 \right] dx^2 + \left[ a_{12}(-z)^2 + a_{22}(-z)^2 \right] dy^2 + dz^2
+ \left[ a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z) \right] (dx \otimes dy + dy \otimes dx)
= e^{-2 \text{trace}(A)z} \left[ [a_{21}(z)^2 + a_{22}(z)^2] dx^2 + [a_{11}(z)^2 + a_{12}(z)^2] dy^2 + dz^2
- e^{-2 \text{trace}(A)z} [a_{11}(z)a_{21}(z) + a_{12}(z)a_{22}(z)] (dx \otimes dy + dy \otimes dx) .
\end{equation}
The Levi-Civita connection associated to the canonical left invariant metric is easily deduced from the Koszul formula and (2.6) as follows:

\[
\begin{align*}
\nabla_{E_1} E_1 &= a \ E_3, & \nabla_{E_1} E_2 &= \frac{b+c}{2} \ E_3, & \nabla_{E_1} E_3 &= -a \ E_1 - \frac{b+c}{2} \ E_2 \\
\nabla_{E_2} E_1 &= \frac{b+c}{2} \ E_3, & \nabla_{E_2} E_2 &= d \ E_3, & \nabla_{E_2} E_3 &= -\frac{b+c}{2} \ E_1 - d \ E_2 \\
\nabla_{E_3} E_1 &= \frac{c-b}{2} \ E_2, & \nabla_{E_3} E_2 &= \frac{c-b}{2} \ E_1, & \nabla_{E_3} E_3 &= 0.
\end{align*}
\]

(2.8)

Remark 2.2. It follows from equation (2.7) that given \((x_0, y_0) \in \mathbb{R}^2\), the map

\[(x, y, z) \mapsto (-x + 2x_0, -y + 2y_0, z)\]

is an isometry of \(\mathbb{R}^2 \rtimes_A \mathbb{R} \), into itself. Note that \(\phi\) is the rotation by angle \(\pi\) around the line \(l = \{(x_0, y_0, z) \mid z \in \mathbb{R}\}\), and the fixed point set of \(\phi\) is the geodesic \(l\). In particular, vertical lines in the \(x, y, z\)-coordinates of \(\mathbb{R}^2 \rtimes_A \mathbb{R}\) are geodesics of its canonical metric, which are the axes or fixed point sets of the isometries corresponding to rotations by angle \(\pi\) around them. For any line \(L\) in \(\mathbb{R}^2 \rtimes_A \{0\}\), let \(P_L\) denote the vertical plane \(\{(x, y, z) \mid (x, y, 0) \in L, z \in \mathbb{R}\}\) containing the set of vertical lines passing through \(L\). It follows that the plane \(P_L\) is ruled by vertical geodesics, and furthermore, since the rotation by angle \(\pi\) around any vertical line in \(P_L\) is an isometry that leaves \(P_L\) invariant, that \(P_L\) has zero mean curvature. Thus, every metric Lie group that can be expressed as a semidirect product of the form \(\mathbb{R}^2 \rtimes_A \mathbb{R}\) with its canonical metric has many minimal foliations by parallel vertical planes, while by parallel we mean that the related lines in \(\mathbb{R}^2 \rtimes_A \{0\}\) for these planes are parallel in the intrinsic metric.

2.2. Unimodular groups. Among all simply connected, 3-dimensional Lie groups, the cases \(SU(2), \widetilde{SL}(2, \mathbb{R}), \text{Sol}_3\) (whose underlying group arises in the so-called Sol geometry), \(\widetilde{E}(2)\) (universal cover of the Euclidean group of orientation-preserving rigid motions of the plane), \(\text{Nil}_3\) (Heisenberg group), and \(\mathbb{R}^3\) comprise the unimodular Lie groups. The cases of \(\text{Sol}_3, \widetilde{E}(2), \text{Nil}_3,\) and \(\mathbb{R}^3\) with their left invariant metrics correspond to the metric semidirect products \(\mathbb{R}^2 \rtimes_A \mathbb{R}\), where the trace of \(A\) is zero. We refer the reader to [22] for further details.

2.3. Non-unimodular groups. The case \(X = \mathbb{R}^2 \rtimes_A \mathbb{R}\) with \(\text{trace}(A) \neq 0\) corresponds to the simply connected, 3-dimensional, non-unimodular metric Lie groups. In this case, up to the rescaling of the metric of \(X\), we may assume that \(\text{trace}(A) = 2\). This normalization in the non-unimodular case will be assumed from now on throughout the paper. After an appropriate orthogonal change of the left invariant frame that fixes the vertical field \(E_3\), we may express the matrix \(A\) uniquely as (see Section 2.5 in [22])

\[
A = A(\alpha, \beta) = \begin{pmatrix}
1 + \alpha & -(1 - \alpha)\beta \\
(1 + \alpha)\beta & 1 - \alpha
\end{pmatrix}, \quad \alpha, \beta \in [0, \infty).
\]

(2.9)

The canonical basis of the non-unimodular metric Lie group \(X\) is, by definition, the left invariant orthonormal frame \(\{E_1, E_2, E_3\}\) given in (2.4) by the matrix \(A\) in (2.9). In other words, every simply connected, non-unimodular metric Lie group is isomorphic and isometric (up to possibly rescaling the metric) to \(\mathbb{R}^2 \rtimes_A \mathbb{R}\) with its canonical metric, where \(A\) is given by (2.9). If \(A = I_2\) where \(I_2\) is the identity
matrix, we get a metric Lie group that we denote by $\mathbb{H}^3$, which is isometric to the hyperbolic 3-space with its standard metric of constant sectional curvature $-1$ and where the underlying Lie group structure is isomorphic to that of the set of similarities of $\mathbb{R}^2$. Under the assumption that $A \neq I_2$, the determinant of $A$ determines uniquely the Lie group structure.

**Definition 2.3.** The Milnor $D$-invariant of $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is the determinant of $A$:

$$D = (1 - \alpha^2)(1 + \beta^2) = \det(A).$$

Assuming $A \neq I_2$, given $D \in \mathbb{R}$, one can solve $\det(A) = D$ for $\alpha = \alpha(D, \beta)$, producing a related matrix $A(D, \beta)$ by equation (2.9), and the space of non-unimodular Lie group structures is parameterized by the values of $\beta \in [m(D), \infty)$, where

$$m(D) = \begin{cases} \sqrt{D - 1} & \text{if } D > 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, after scaling so that $\mathrm{trace}(A) = 2$ and assuming that $A \neq I_2$, the space of simply connected, 3-dimensional, non-unimodular metric Lie groups with a given $D$-invariant is 1-dimensional.

**Remark 2.4.** From now on, by a metric semidirect product $X$ we will mean (without loss of generality; see the explanation below) a semidirect product $\mathbb{R}^2 \rtimes_B \mathbb{R}$ endowed with its canonical left invariant metric $\langle \cdot, \cdot \rangle$ and such that the matrix $A = M_2(\mathbb{R})$ either has trace zero (unimodular case) or is given by expression (2.9) for some $\alpha, \beta \in [0, \infty)$ (non-unimodular case). We must observe that we do not lose any generality with this normalization, since by the previous discussion, every metric semidirect product $\mathbb{R}^2 \rtimes_B \mathbb{R}$ whose associated matrix $B$ has non-zero trace is both isomorphic and isometric (after an adequate rescaling) to a metric semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ where $A$ is given by (2.9). Moreover, the corresponding isomorphism takes the horizontal foliation $\{\mathbb{R}^2 \rtimes_B \{z\} \mid z \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_B \mathbb{R}$ to the horizontal foliation $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and also preserves the left invariant vertical vector fields $E_3$ of their respective canonical frames.

Throughout the paper, we will denote by $\Pi: \mathbb{R}^2 \rtimes_A \mathbb{R} \to \mathbb{R}^2 \rtimes_A \{0\}$ the projection $\Pi(x, y, z) = (x, y, 0)$.

### 3. Radó’s theorem in metric semidirect products

In this section we prove some results concerning the geometry of solutions to Plateau type problems in metric semidirect products $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ when there is some geometric constraint on the boundary values of the solution. The first of these results is Theorem 3.1 below. We remark that several versions of this theorem in the classical setting of $X = \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ were proved by Radó [30], Nitsche [29], and Meeks [18]. We point out that one of the difficulties in obtaining Radó type results in the situation below is that the vertical translation $(x, y, z) \mapsto (x, y, z + t)$ might not be an isometry of the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$. It is also convenient to recall here that $\mathbb{R}^2 \rtimes_A \{0\}$ is intrinsically flat and hence isometric to the Euclidean plane.

**Theorem 3.1** (Radó’s theorem in metric semidirect products). Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product. Suppose that $E$ is a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$,
Lemma 3.2. Suppose \( C = \partial E \), and \( \Gamma \subset \Pi^{-1}(C) \) is a continuous simple closed curve such that \( \Pi|_{\Gamma} : \Gamma \to C \) monotonically parameterizes \( C \). Then:

1. \( \Gamma \) is the boundary of a compact embedded disk \( D \) of finite least area.
2. The interior of \( D \) is a smooth \( \Pi \)-graph over the interior of \( E \).

Theorem 3.1 will be a direct consequence of Theorem 3.3 below, which actually gives a more complete statement. The proof of Theorem 3.3 also depends on the following Lemma 3.2; both of these results will also be used in the proof of Theorem 4.1 in Section 4.

\[ \text{Lemma 3.2. Suppose } X = \mathbb{R}^2 \rtimes_A \mathbb{R} \text{ is a metric semidirect product. Let } E \subset \mathbb{R}^2 \rtimes_A \{0\} \text{ be a compact convex disk with boundary curve } C. \text{ If } M \text{ is a compact branched minimal surface in } X \text{ with boundary contained in } \Pi^{-1}(C), \text{ then:} \]

1. \( \text{Int}(M) \) is contained in the interior of \( \Pi^{-1}(E) \).
2. If \( \partial M \) is of class \( C^2 \), then \( M \) is an immersion near its boundary and transverse to \( \Pi^{-1}(C) \) along \( \partial M \).

**Proof.** The proof of this lemma uses the fact stated in Remark 2.2 that for every line \( L \subset \mathbb{R}^2 \rtimes_A \{0\} \), the vertical plane \( \Pi^{-1}(L) \) has zero mean curvature. This implies, in particular, that \( \Pi^{-1}(C) \) is mean convex.

Suppose that \( M \) is a compact branched minimal surface with \( \partial M \subset \Pi^{-1}(C) \), and we will prove the first item in the lemma. Arguing by contradiction, assume there exists a point \( p \in \text{Int}(M) \) which is not contained in the interior of \( \Pi^{-1}(E) \).

Since \( E \) is convex, there exists a line \( L \subset \mathbb{R}^2 \rtimes_A \{0\} \) such that \( \Pi(p) \in L \) and \( L \) is disjoint from \( \text{Int}(E) \). Hence the vertical minimal plane \( \Pi^{-1}(L) \) intersects \( \text{Int}(M) \) at \( p \), and so, by the maximum principle, \( M \) contains interior points on both sides of \( \Pi^{-1}(L) \) near \( p \).

Consider the product foliation \( \mathcal{F}(L) = \{L_t\}_{t \in \mathbb{R}} \) of lines in \( \mathbb{R}^2 \rtimes_A \{0\} \) parallel to \( L = L_0 \) and parameterized so that \( E \subset \bigcup_{t \leq 0} L_t \). Let \( \{\Pi^{-1}(L_t)\}_{t \in \mathbb{R}} \) be the related foliation of \( X \) by minimal vertical planes. By compactness of \( M \), there is a largest value \( t_0 > 0 \) such that \( \Pi^{-1}(L_{t_0}) \cap M \neq \emptyset \). But at any point of this non-empty intersection, we obtain a contradiction to the maximum principle applied to the minimal surfaces \( \Pi^{-1}(L_{t_0}) \) and \( M \). This contradiction proves item (1) of the lemma. Item (2) of the lemma follows from Theorem 2 in [24].

**Theorem 3.3.** Let \( X = \mathbb{R}^2 \rtimes_A \mathbb{R} \) be a metric semidirect product, let \( E \) be a compact convex disk in \( \mathbb{R}^2 \rtimes_A \{0\} \), and let \( C = \partial E \). Suppose \( \Gamma \subset \Pi^{-1}(C) \) is a continuous simple closed curve such that the projection \( \Pi : \Gamma \to C \) monotonically parameterizes \( C \). Let \( W = \Pi^{-1}(E) \). Then:

1. If \( D \) is a compact, branched minimal disk in \( X \) with \( \partial D = \Gamma \), then the following properties hold:
   1a) \( D \) is an embedded disk.
   1b) The interior of \( D \) is a smooth \( \Pi \)-graph over the interior of \( E \); i.e., \( \Pi|_{\text{Int}(D)} : \text{Int}(D) \to \text{Int}(E) \) is a diffeomorphism.
   1c) If \( \Pi|_{\Gamma} : \Gamma \to C \) is a homeomorphism, then \( \Pi|_{D} : D \to E \) is a homeomorphism.
   1d) If \( \Gamma \) is of class \( C^2 \), then the inclusion map of \( D \) is an immersion along \( \partial D \) and \( D \) is transverse to \( \Pi^{-1}(C) \) along \( \Gamma \).

\[ \text{This means that for every point } p \in C, \Pi^{-1}(p) \cap \Gamma \text{ is a compact interval or a single point.} \]
(1e) If $\Pi|_{\Gamma}: \Gamma \to C$ is a diffeomorphism, then $\Pi|_{D}: D \to E$ is a diffeomorphism.

(2) There exist compact minimal disks $D_L, D_T, D_B$ in $W$ with boundary $\Gamma$ such that:

(2a) $D_L$ is an embedded disk of finite least area in $X$.

(2b) $D_T$ is an embedded disk of finite least area in the closed region of $W$ above the graph $D_T$.

(2c) $D_B$ is an embedded disk of finite least area in the closed region of $W$ below the graph $D_B$.

(2d) Any compact branched minimal surface $M$ in $X$ whose boundary lies in the compact set $W(D_T, D_B) \subset W$ between the graphs $D_T$ and $D_B$ satisfies $M \subset W(D_T, D_B)$. In particular, the disks $D_T$ and $D_B$ are uniquely defined by Properties (2b), (2c), and (2d); hence, $\Gamma$ is the boundary of a unique compact branched minimal surface if and only if $D_T = D_B$.

Proof. We first prove item (1) of the theorem. Let $D$ be a compact (possibly branched) minimal disk with boundary $\Gamma$. Consider $D$ to be the image of a conformal harmonic map $f: \mathbb{D} \to X$, where $\mathbb{D}$ is the closed unit disk in $\mathbb{C}$ and $f|_{\partial \mathbb{D}}$ is a homeomorphism to $\Gamma$. To prove that $(\Pi \circ f)|_{\text{Int}(\mathbb{D})}: \text{Int}(\mathbb{D}) \to \text{Int}(E)$ is a diffeomorphism, we will modify a classical argument of Rado [30], who proved a similar result for minimal surfaces in $\mathbb{R}^3$ whose boundaries have an orthogonal injective projection to a convex planar curve. Since $E$ is simply connected and $(\Pi \circ f)|_{\text{Int}(\mathbb{D})}: \text{Int}(\mathbb{D}) \to \text{Int}(E)$ is a proper map, to prove item (1b) it suffices to check that the differential of $\Pi \circ f$ has rank two at every point of $\text{Int}(\mathbb{D})$. By contradiction, suppose that $p \in \text{Int}(\mathbb{D})$ is a point where the differential of $\Pi \circ f$ has rank less than two. In this case, either $f$ is unbranched at $p$ and the tangent plane $T_pD$ is vertical or $p$ is a branch point for $f$. We first consider the special case that $f$ is unbranched at $p$ and the tangent plane $T_pD$ is vertical or, equivalently, there exists a line $L \subset \mathbb{R}^2 \times \{0\}$ passing through $(\Pi \circ f)(p) \in \text{Int}(E)$ such that the vertical plane $P = \Pi^{-1}(L)$ is tangent to $D$ at the point $f(p)$.

The set $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ is a 1-dimensional subset of $\mathbb{D}$ that contains no isolated points (at regular points of $f$, this is a consequence of the maximum principle, while at branch points of $f$ this follows from well-known properties of branched minimal surfaces). The $\Pi$-projection of the boundary of $f[f^{-1}(P)]$ consists of two points in $L \cap C$, and $f^{-1}(P)$ has locally around $p \in \text{Int}(\mathbb{D})$ the appearance of a system of at least two analytic segments crossing at $p$ (see, e.g., Lemma 2 in Meeks and Yau [29]). Since $f(\text{Int}(\mathbb{D}))$ is a proper analytic (possibly branched) surface in $\text{Int}(W)$, we conclude that $f^{-1}(P)$ contains the closure of the properly embedded analytic 1-complex $f^{-1}(P) \cap \text{Int}(\mathbb{D})$. Furthermore, $f^{-1}(P \cap \Gamma)$ consists of two components, each of which is a closed interval (possibly a point; this follows from the facts that $P \cap \Gamma$ consists of two components, $f(\text{Int}(\mathbb{D})) = \text{Int}(D) \subset \text{Int}(W)$ and $f|_{\partial \mathbb{D}}: \partial \mathbb{D} \to \Gamma$ is a homeomorphism), and the limit set of $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ intersects both of these components. Note that each vertex in $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ has a positive even number of associated edges with the number of edges at the vertex $p$ being at least 4. As the component $\Delta$ of $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ containing $p$ has at least 4 ends or it contains a simple closed curve, we conclude that either there is a simple closed curve $\alpha$ in $\Delta$ or there is a properly embedded arc $\alpha$ in $\Delta$ whose two ends are contained in the same component of $f^{-1}(P \cap \Gamma)$. In either case,
there is a compact subset $\mathbb{D}'$ of $\mathbb{D}$ with non-empty interior bounded by $\alpha$ together with some connected subset of $f^{-1}(P \cap \Gamma) \subset \partial \mathbb{D}$. Consider the product foliation \{L_t\}_{t \in \mathbb{R}} of lines in $\mathbb{R}^2 \times_A \{0\}$ parallel to $L = L_0$. Let \{$(\Pi^{-1}(L_t))_t$\} be the related foliation of $X$ by minimal vertical planes, and without loss of generality, suppose that $f(\mathbb{D}') \cap \bigcup_{t > 0} \Pi^{-1}(L_t) \neq \emptyset$. Note that $f(\partial \mathbb{D}') \subset \Pi^{-1}(L)$. By compactness of $\mathbb{D}'$, there is a largest value $t_0 > 0$ such that $\Pi^{-1}(L_{t_0}) \cap f(\mathbb{D}') \neq \emptyset$. But at any point of this non-empty intersection, we obtain a contradiction with the maximum principle applied to the minimal surfaces $\Pi^{-1}(L_{t_0})$ and $f(\mathbb{D}')$. This contradiction proves that if $p$ is not a branch point of $f$, then the differential of $\Pi \circ f$ has rank two at $p$.

On the other hand, if $p$ is a branch point of $f$, then choose a horizontal line $L \subset \mathbb{R}^2 \times_A \{0\}$ through $\Pi(p)$ so that the associated vertical plane $P = \Pi^{-1}(L)$ intersects $\Gamma$ in exactly two points. Then, the set $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ is a 1-dimensional subset of $\mathbb{D}$ that contains no isolated points, and $f^{-1}(P)$ has locally around $p \in \text{Int}(\mathbb{D})$ the appearance of a system of at least two analytic segments crossing at $p$. Arguing as in the previous case gives a contradiction, which proves that $f$ cannot have interior branch points; therefore, item (1b) holds. Clearly item (1b) implies (1a) and (1c).

Note that if $\Gamma$ is of class $C^2$ and $\Pi|\Gamma$: $\Gamma \to C$ is a $C^2$-immersion, then by the statement and proof of Lemma 3.2 $\Pi_D$ has rank two at every point of $\partial D$ and $D$ is an embedded disk transverse to $\partial W$ along $\Gamma$. The remaining items of item (1) of the theorem follow directly from item (1b) and this rank-two property of $\Pi|D$ along $\partial D$.

We next prove item (2) of the theorem. By Theorem 1 in [24], there exists a disk $D_L$ of finite least area in $W$ with boundary $\Gamma$, and every such least-area disk is an embedding. By Lemma 3.2 the interior of any compact branched minimal disk in $X$ with $\partial D = \Gamma$ must be contained in the interior of $W$, and so any least-area disk in $X$ with boundary $\Gamma$ is contained in $W$. The existence of $D_L$ proves item (2a) of the theorem.

The existence of $D_T$ can be found by constructing barriers. First suppose that $\Gamma$ is smooth. Consider a compact branched minimal surface $\Sigma$ in $X$ with $\partial \Sigma = \Gamma$. By Lemma 3.2 $\Sigma \subset W$. Consider the closure $C_\Sigma$ of the component of $W - \Sigma$ that contains a representative of the top end of $W$, by which we mean a closed region in $W$ above some horizontal plane $\mathbb{R}^2 \times_A \{t_0\}$. Then $\Gamma$ is homotopically trivial in $C_\Sigma$, and by the barrier results in [24], $\Gamma$ is the boundary of a finite least-area disk in $C_\Sigma$ which must be embedded (in fact, the interior of such a disk is a $\Pi$-graph over the interior of $E$ by item (1b) of this theorem). Furthermore, any two such least-area disks in $C_\Sigma$ intersect only along their common boundary $\Gamma$. Since this collection of ‘disjoint’ least-area embedded disks in $C_\Sigma$ with boundary $\Gamma$ forms a sequentially compact set (since they all have the same finite area in the homogeneously regular manifold $X$) and these disks are ordered by the relative heights of their graphing functions, there exists a unique highest least-area disk above $\Sigma$ that we denote by $D_T(\Sigma)$. Approximation results in [23][24] imply that when $\Gamma$ is only continuous, there also exists a similar embedded highest least-area disk $D_T(\Sigma)$ in $C_\Sigma$.

We claim that all the least-area disks $D_T(\Sigma)$ defined in the last paragraph lie in a compact region of $W$, independent of $\Sigma$. If $\text{trace}(A) = 0$ (equivalently, $X$ is unimodular), then $\mathcal{F} = \{\mathbb{R}^2 \times_A \{z\} \mid z \in \mathbb{R}\}$ is a minimal foliation of $X$, and a simple application of the maximum principle to any compact branched minimal surface $\Sigma'$ in $X$ with boundary $\Gamma$ and to the leaves of $\mathcal{F}$ gives that $\max(\left|z\right|_{\Sigma'}) \leq \max(\left|z\right|_{\Sigma'})$.

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max(\(z|_T\)), which proves the claim in this case. If \(X\) is non-unimodular, we can assume after scaling its metric that \(\text{trace}(A) = 2\). Suppose that the claim fails to hold. Then, there exists a sequence of least-area disks \(D_T(\Sigma_n) \subset W\) associated to compact branched minimal surfaces \(\Sigma_n\), with \(\partial \Sigma_n = \partial D_T(\Sigma_n) = \Gamma\) for all \(n\) and \(\max(z|_{\partial D_T(\Sigma_n)}) \to \infty\) as \(n \to \infty\) (observe that the mean curvature comparison principle applied to \(D_T(\Sigma_n)\) and to the leaves of \(\mathcal{F}\) ensures that \(\min(z|_{D_T(\Sigma_n)}) \geq \min(z|_T)\)). Given \(n \in \mathbb{N}\), let \(p_n \in D_T(\Sigma_n)\) be a point where \(z|_{D_T(\Sigma_n)}\) attains its maximum value. By taking \(n\) large enough, we can assume that the intrinsic distance from \(p_n\) to \(\Gamma\) is greater than 2. As the \(D_T(\Sigma_n)\) are stable, they have uniform curvature estimates away from their boundaries; in particular, the norm of the second fundamental form of \(D_T(\Sigma_n)\) in the intrinsic ball of radius 1 centered at \(p_n\) is less than some positive constant \(C\) independent of \(n\). This implies that there exists \(\varepsilon > 0\) such that, for \(n\) large enough, the intrinsic disk \(D(p_n, \varepsilon)\) of radius \(\varepsilon\) in \(\mathbb{R}^2 \times A\{z(p_n)\}\) centered at \(p_n\) satisfies the following property:

(P) Every vertical line passing through a point in \(D(p_n, \varepsilon)\) intersects \(D_T(\Sigma_n)\) near \(p_n\).

Property (P) follows from the following observation, whose proof we leave to the reader:

(O) Let \(\Sigma_1, \Sigma_2\) be two smooth surfaces in a homogeneous 3-manifold \(X\) that are tangent at a common point \(p\) such that the intrinsic distance from \(p\) to the boundaries of these surfaces is at least 1. If the norms of the second fundamental forms of these surfaces are less than some \(C > 0\), then there exists an \(\varepsilon = \varepsilon(C) \in (0, 1/2)\), less than the injectivity radius of \(X\), such that every point in the intrinsic ball \(B_{\Sigma_1}(p, \varepsilon) = \{q \in \Sigma_1 | d_{\Sigma_1}(p, q) < \varepsilon\}\) (here \(d_{\Sigma_1}\) denotes intrinsic distance in \(\Sigma_1\)) is a normal graph over a subdomain of the corresponding intrinsic ball \(B_{\Sigma_2}(p, 2\varepsilon)\) of absolute value less than \(\varepsilon\).

With property (P) at hand, we next find the desired contradiction that will prove our claim in the case \(X\) is non-unimodular. Since the area element \(dA_z\) for the restriction of the canonical metric to the plane \(\mathbb{R}^2 \times A\{z\}\) is \(dA_z = e^{-2z} dx \wedge dy\), we conclude that

\[
\text{Area}(E) \geq \text{Area}(\Pi(D(p_n, \varepsilon))) = e^{2z(p_n)} \text{Area}(D(p_n, \varepsilon)) = \pi \varepsilon^2 e^{2z(p_n)},
\]

which implies that \(z(p_n)\) is bounded from above, a contradiction. Now our claim is proved.

Once we know that all the least-area disks \(D_T(\Sigma)\) lie in a compact region of \(W\) independent of \(\Sigma\), we conclude that they also have uniformly bounded area by the following argument: consider the disk \(D_{z_0} := [\mathbb{R}^2 \times A\{z_0\}] \cap \Pi^{-1}(E)\) where \(z_0 \gg 1\) is chosen sufficiently large so that \(D_T(\Sigma)\) lies under \(D_{z_0}\) for every compact branched minimal surface \(\Sigma\) in \(X\) with boundary \(\Gamma\). Consider the union \(D'\) of \(D_{z_0}\) with the annular portion of \(\Pi^{-1}(\Gamma)\) below \(D_{z_0}\) and above \(\Gamma\). \(D'\) clearly has finite area. Since \(D_T(\Sigma)\) has least area among surfaces in the region of \(W\) above \(D_T(\Sigma)\) with boundary \(\Gamma\), the area of \(D'\) is greater than or equal to the area of \(D_T(\Sigma)\), as desired.

Given two of the disks, \(D_T(\Sigma_1), D_T(\Sigma_2)\), then using their union as a barrier, our previous arguments demonstrate that there is a least-area graphical disk \(D'\) with boundary \(\Gamma\) that lies in the region of \(\Pi^{-1}(E)\) above both of them; here “above” means in the sense that the II-graphing function \(h: \text{Int}(E) \to \mathbb{R}\) for \(\text{Int}(D')\) is greater than or equal to the graphing functions for the disks \(D_T(\Sigma_1), D_T(\Sigma_2)\). This notion of “above” induces a partial ordering on the set of disks of the form
A standard compactness argument using that the areas of these disks are uniformly bounded proves the existence of a minimal disk $D_T$ with boundary $\Gamma$ that is a maximal element in the partial ordering. By item (1a) of this theorem, $D_T$ is embedded, and by construction, $D_T$ has least area among all compact surfaces in the closed region of $W$ above $D_T$. This proves item (2b) of the theorem. Item (2c) about $D_B$ can be proven by similar reasoning as in the proof of item (2b) for $D_T$.

It remains to prove item (2d) of the theorem. Suppose that $M$ is a compact branched minimal surface in $X$ whose boundary lies in the closed set $W(D_T, D_B) \subset W$ between the graphs $D_T$ and $D_B$. Note that $M \subset W$ by the arguments in the proof of Lemma [3.2]. Suppose that some point $p$ of $M$ lies in $W - W(D_T, D_B)$. First suppose that $p$ lies in the closed region $D_T^+ \subset W$ that lies above $D_T$. Then using $D_T \cup (M \cap D_T^+)$ as a barrier, we obtain a minimal disk $D'_T$ of least area that lies above $D_T$, which contradicts that $D_T$ is the highest disk that bounds $\Gamma$. This contradiction implies $M$ does not intersect the interior of $D_T^+$; similar arguments imply that $M$ does not intersect the interior of the region of $W$ that lies below $D_B$. Hence, the main statement of item (2d) is proved. Elementary separation arguments now imply that $D_T$ and $D_B$ are uniquely defined, and clearly if $D_T = D_B$, then every compact branched minimal surface in $X$ with boundary $\Gamma$ is equal to $D_T$.

This completes the proof of Theorem [3.3].

By item (2d) of Theorem [3.3] if a curve $\Gamma$ satisfying the hypotheses of Theorem [3.3] is the boundary of a unique minimal disk, then it is also the boundary of a unique compact branched minimal surface. Our belief that graphical minimal disks bounding such a $\Gamma$ are unique leads us to the following conjecture.

**Conjecture 3.4.** Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product. Suppose $E$ is a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$, $C = \partial E$, and $\Gamma \subset \Pi^{-1}(C)$ is a continuous simple closed curve such that $\Pi: \Gamma \rightarrow C$ monotonically parameterizes $C$. Then the compact embedded disk $D_L$ of finite least area given in Theorem [3.1] is the unique compact branched minimal surface in $X$ with boundary $\Gamma$.

The uniqueness property stated in Conjecture [3.4] is clear in the particular case that $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is unimodular and $\Gamma$ is contained in the plane $\mathbb{R}^2 \times_A \{0\}$ by the maximum principle applied to $D_L = E$ and to the foliation of minimal planes $\{\mathbb{R}^2 \times_A \{z\} \mid z \in \mathbb{R}\}$.

Next we prove the following particular case of Conjecture [3.4] for the case that $X$ is non-unimodular.

**Proposition 3.5.** Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a non-unimodular metric semidirect product, and let $F_3(x, y, z) = F_3^H + \partial_z$ be the right invariant vector field in $X$ given by (2.3). Suppose that $E \subset \mathbb{R}^2 \rtimes_A \{0\}$ is a compact convex disk with $C^2$ boundary $\Gamma$ that is almost-transverse to $F_3^H$, in the sense that the inner product of the outward pointing unit conormal to $E$ along $\Gamma$ with $F_3^H$ is greater than or equal to zero. Then, $\Gamma$ is the boundary of a unique compact branched minimal surface which must therefore be the least-area, embedded minimal disk $D_L$ given in Theorem [3.1].

**Proof.** Arguing by contradiction, suppose that the proposition fails to hold. Then item (2) of Theorem [3.1] implies that the embedded minimal disks $D_T, D_B$ described there with boundary $\Gamma$ are not equal. Let $\eta_T, \eta_B$ denote the respective outward pointing unit conormals to these minimal disks along $\Gamma$. Since $F_3$ is a Killing...
vector field and $D_T, D_B$ are minimal, the divergence theorem gives
\begin{equation}
0 = \int_{D_i} \text{div}_{D_i}(F^T_{3i}) = \int_{\Gamma} \langle \eta_i, F_3 \rangle = \int_{\Gamma} \langle \eta_i, F^H_3 \rangle + \int_{\Gamma} \langle \eta_i, \partial_z \rangle,
\end{equation}
where $i = T, B$ and $\text{div}_{D_i}(F^T_{3i})$ denotes the intrinsic divergence in $D_i$ of the tangential component $F^T_{3i}$ of $F_3$ to $D_i$.

On the other hand, observe that by the boundary maximum principle, $D_B$ lies strictly below $D_T$ near their common boundary $\Gamma$.

As $X$ is non-unimodular, $\mathbb{R}^2 \rtimes_A \{0\}$ has mean curvature 1, and so both $D_T, D_B$ lie in $\mathbb{R}^2 \rtimes_A [0, \infty)$ (this follows from the interior maximum principle applied to $D_i$, $i = T, B$, and to $\mathbb{R}^2 \rtimes_A \{z_0\}$ for a suitable $z_0 < 0$) and $D_T, D_B$ are transverse to $\mathbb{R}^2 \rtimes_A \{0\}$ along $\Gamma$ (by the boundary maximum principle applied to $D_i$, $i = T, B$, and to $\mathbb{R}^2 \rtimes_A \{0\}$). Therefore, we deduce that
\begin{equation}
(\eta_T, \partial_z) < (\eta_B, \partial_z) < 0.
\end{equation}
Expressing $\eta_T$ as a sum of its horizontal and vertical components, we have
\[ \eta_T = \langle \eta_T, \eta_H \rangle \eta_H + \langle \eta_T, E_3 \rangle E_3, \]
where $\eta_H$ is the outward pointing unit conormal to $E$ along $\Gamma$. As $\langle F^H_3, E_3 \rangle = 0$,
\[ \langle \eta_T, F^H_3 \rangle = \langle \eta_T, \eta_H \rangle \langle \eta_H, F^H_3 \rangle. \]
Note that $\langle \eta_H, F^H_3 \rangle \geq 0$ since $\Gamma$ is almost transverse to $F^H_3$ and that $\langle \eta_T, \eta_H \rangle \geq 0$ as $D_T$ lies in $\Pi^{-1}(E)$. Thus $\langle \eta_T, F^H_3 \rangle \geq 0$. Arguing similarly with $D_B$ we will obtain $\langle \eta_B, F^H_3 \rangle = \langle \eta_B, \eta_H \rangle \langle \eta_H, F^H_3 \rangle$. As $D_B$ lies below $D_T$ near $\Gamma$, $\langle \eta_T, \eta_H \rangle \leq \langle \eta_B, \eta_H \rangle$. Altogether we deduce that
\begin{equation}
0 \leq \langle \eta_T, F^H_3 \rangle \leq \langle \eta_B, F^H_3 \rangle.
\end{equation}
The inequalities (3.2) and (3.3) imply that
\[ \int_{\Gamma} \langle \eta_T, F^H_3 \rangle + \int_{\Gamma} \langle \eta_T, \partial_z \rangle < \int_{\Gamma} \langle \eta_B, F^H_3 \rangle + \int_{\Gamma} \langle \eta_B, \partial_z \rangle, \]
which contradicts (3.1). This contradiction proves the proposition.

4. ASYMPTOTIC BEHAVIOR OF CERTAIN COMPACT MINIMAL SURFACES IN NON-UNIMODULAR METRIC LIE GROUPS

If $I_2$ is the identity matrix in $\mathcal{M}_2(\mathbb{R})$, then the metric Lie group $X = \mathbb{R}^2 \rtimes_{I_2} \mathbb{R}$ is isometric to hyperbolic 3-space, and the planes $\mathbb{R}^2 \rtimes_A \{t_0\}$ correspond to a family of horospheres with the same point at the ideal boundary of $X$. In this case, every circle $\Gamma_R$ of Euclidean radius $R > 0$ in the (flat) plane $\mathbb{R}^2 \rtimes_{I_2} \{0\}$ is the boundary of a least-area compact disk $\Sigma_R \subset \mathbb{R}^2 \rtimes_{I_2} [0, \infty)$, which is contained in a totally geodesic hyperbolic plane $H$ in $X$; in fact, $\Sigma_R$ is a geodesic disk of some hyperbolic radius in $H$. Furthermore, as the Euclidean radius of $\Gamma_R$ in $\mathbb{R}^2 \rtimes_{I_2} \{0\}$ goes to infinity, the Riemannian radius of $\Sigma_R$ also goes to infinity, and the constant angle that $\Sigma_R$ makes with $\mathbb{R}^2 \rtimes_{I_2} \{0\}$ approaches $\pi/2$; see Figure 1. The next theorem shows that a similar result holds in any non-unimodular metric Lie group, since every such non-unimodular metric Lie group is isomorphic and isometric to a non-unimodular semidirect product.

**Theorem 4.1.** Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, where $A \in \mathcal{M}_2(\mathbb{R})$ satisfies equation (2.4).
Proof. We will first prove a key assertion which, together with properties of convex simple closed curves in \(\mathbb{R}^2\), will lead to the proof of the theorem.

Assertion 4.2. Given \(\varepsilon \in (0, \frac{\pi}{2})\), there exists \(R = R(\varepsilon) > 0\) such that for any \(r > R\) the following property holds. Let \(C_r \subset \mathbb{R}^2 \times_A \{0\}\) be a circle of Euclidean radius \(r > 0\) and let \(D_B(r)\) be the minimal disk given by item (2c) of Theorem 3.3 with boundary \(C_r\). Then, the angle that the tangent plane to \(D_B(r)\) makes with \(\mathbb{R}^2 \times_A \{0\}\) is greater than \(\frac{\pi}{2} - \varepsilon\) at every point of \(C_r\).

Proof of the assertion. Arguing by contradiction, suppose there exist \(\varepsilon \in (0, \frac{\pi}{2})\), a sequence of circles \(C_{r(n)} \subset \mathbb{R}^2 \times_A \{0\}\) with Euclidean radii \(r(n) \nearrow \infty\), and points
whose boundary lies in given in item (2b) of Theorem 3.3. Note that \( \{ p \} \) converges to one of the two closed halfplanes bounded by \( p_0 \), and after choosing a subsequence, the \( C_r(n) \) converge as \( n \to \infty \) to a line \( L \subset \mathbb{R}^2 \times_A \{0\} \), and the closed disks \( E_r(n) \subset \mathbb{R}^2 \times_A \{0\} \) bounded by \( C_r(n) \) converge to one of the two closed halfplanes bounded by \( L \), which we denote by \( L^+ \).

By item (2c) of Theorem 3.3 given \( n \in \mathbb{N} \) the unique compact embedded, stable minimal disk \( D_B(r(n)) \) associated to the circle \( C_r(n) \) is a \( \Pi \)-graph over \( E_r(n) \), and item (2d) of the same theorem implies the following property:

\((\star)\) Let \( W_n \) be the closure of the non-compact complement of \( D_B(r(n)) \cup \mathbb{R}^2 \times_A \{0\} \) in \( \mathbb{R}^2 \times_A [0, \infty) \). If \( M \subset X \) is a compact, connected branched minimal surface whose boundary lies in \( W_n \), then \( M \subset W_n \).

To see why property \((\star)\) above holds, we only need to observe the following: let \( D_T(r(n)) \) denote the compact embedded stable minimal disk with boundary \( C_r(n) \) given in item (2b) of Theorem 3.3. Note that \( D_T(r(n)) \) lies in \( W_n \). Hence, if \( M \) is not contained in \( W_n \), there must exist a compact piece \( M' \) of \( M \) such that \( \partial M' \) lies in the region of \( \mathbb{R}^2 \times_A [0, \infty) \) bounded by \( D_T(r(n)) \) and \( D_B(r(n)) \), but \( M' \) has points outside that region. This would contradict item (2d) of Theorem 3.3 which proves property \((\star)\).

Since the circles \( C_r(n) \) converge on compact subsets to the horizontal straight line \( L \) as \( n \to \infty \), a standard argument shows that after choosing a subsequence, the \( D_B(r(n)) - C_r(n) \) converge to a minimal lamination \( \mathcal{L} \) of \( X - L \) (local uniform curvature estimates for the \( D_B(r(n)) \) hold because these minimal surfaces are stable; see [33,35]). \( \mathcal{L} \) is contained in \( \Pi^{-1}(L^+) \cap [\mathbb{R}^2 \times_A [0, \infty)] \), as \( D_B(r(n)) - C_r(n) \subset \Pi^{-1}(E_r(n)) \cap [\mathbb{R}^2 \times_A [0, \infty)] \) for every \( n \) and the \( E_r(n) \) converge to \( L^+ \) as \( n \to \infty \). It is clear that \( L \) is contained in the closure of \( \mathcal{L} \). We claim that \( \overline{\mathcal{L}} \cap L^+ = L \); if not, then there exists a point \( p \in (\overline{\mathcal{L}} \cap L^+) - L \), such that \( p \) is the limit of a sequence of points in \( D_B(r(n)) \). Consider a circle \( C(p) \) of radius \( \frac{1}{2}d(p, L) \) (\( d \) denotes Euclidean distance), and let \( D_B(p) \) be the compact embedded, stable minimal disk with boundary \( C(p) \) given by item (2c) of Theorem 3.3. A straightforward adaptation of property \((\star)\) shows that for \( n \) sufficiently large, \( D_B(r(n)) \) lies in the closure of the non-compact complement of \( D_B(p) \cup [\mathbb{R}^2 \times_A \{0\}] \) in \( \mathbb{R}^2 \times_A [0, \infty) \). This contradicts that \( p \) is the limit of a sequence of points in \( D_B(r(n)) \), thereby proving our claim.

Since the \( D_B(r(n)) \) are all \( \Pi \)-graphs, by standard arguments the leaves of \( \mathcal{L} \) are either \( \Pi \)-graphs over pairwise disjoint domains in \( L^+ \subset \mathbb{R}^2 \times_A \{0\} \) or they are contained in vertical halfplanes. In particular, there exists a unique leaf of \( \mathcal{L} \) whose closure contains \( L \). Since the disks \( D_B(r(n)) \) have uniform local ambient area bounds nearby their boundaries (this follows from the fact that \( D_B(r(n)) \) is least area in a certain region of \( \mathbb{R}^2 \times_A \mathbb{R} \); see the proof of Theorem 3.3) after choosing a subsequence the surfaces with boundary \( D_B(r(n)) \) converge smoothly to a surface with boundary near \( L \). Clearly, the interior of this limit surface is included in the unique leaf of \( \mathcal{L} \) that has \( L \) in its closure. Let \( D_\infty \) be the closure in \( \mathbb{R}^2 \times_A \mathbb{R} \) of such a leaf. The above arguments show that \( D_\infty \) is a smooth \( \Pi \)-graph with boundary \( L \) (smoothness of \( D_\infty \) holds up to its boundary). Observe that, by construction, the angle that the tangent plane to \( D_\infty \) at \( 0 \) makes with \( \mathbb{R}^2 \times_A \{0\} \) lies in \( (0, \frac{\pi}{2} - \varepsilon] \).

Consider the right invariant vector field \( V \) on \( X \) such that \( V(0) \) is unitary and tangent to \( L \). We claim that \( V \) is everywhere tangent to \( D_\infty \). Given \( \tau \in L \),
consider the angle \( \theta(\tau) \geq 0 \) that the \( \Pi \)-graph \( D_\infty \) makes with the plane \( \mathbb{R}^2 \times_A \{0\} \) at the point \( \tau \), viewed as a function \( \theta : L \rightarrow [0, \pi/2] \). Recall that \( \theta(\tilde{0}) \in (0, \frac{\pi}{2} - \varepsilon) \) and observe that \( \tau \in L \mapsto \theta(\tau) \) is analytic, as both \( X \) and \( D_\infty \) are analytic. If \( \theta = \theta(\tau) \) is constant along \( L \), then the unique continuation property of elliptic PDEs implies that for any \( \tau \in L \), the left translation \( \tau \ast D_\infty \) of \( D_\infty \) by \( \tau \) is equal to \( D_\infty \), which implies our claim. Hence for the remainder of the proof of our claim, we will assume that there is a \( \tau \in L - \{\tilde{0}\} \) such that \( \theta(\tilde{0}) \neq \theta(\tau) \).

(A) Suppose that \( \theta(\tilde{0}) > \theta(\tau) \). For \( n \in \mathbb{N} \), consider a point \( \sigma_n \) of \( \text{Int}(L^+) \) at Euclidean distance \( \frac{1}{n} \) from \( \tau \). Then, the left translate \( \sigma_n \ast D_B(r(n)) \) of \( D_B(r(n)) \) by \( \sigma_n \) has boundary \( \sigma_n \ast C_{r(n)} \). For \( n \in \mathbb{N} \) fixed, there exists an integer \( j(n) > n \) such that for all \( j \in \mathbb{N} \) with \( j \geq j(n) \), the boundary \( \partial(\sigma_n \ast D_B(r(n))) = \sigma_n \ast C_{r(n)} \) lies in the interior of the disk \( E_{r(j)} \). By property \((\ast)\) applied to \( \sigma_n \ast D_B(r(n)) \) and to \( M = D_B(r(j)) \), we deduce that \( D_B(r(j)) \) lies in the closure \( W_n \) of the non-compact complement of \( \{\sigma_n \ast D_B(r(n))\} \cup \mathbb{R}^2 \times_A \{0\} \) in \( \mathbb{R}^2 \times_A [0, \infty) \). Equivalently, the graphing functions \( v_n : \sigma_n \ast E_{r(n)} \rightarrow \mathbb{R} \), \( u_j : E_{r(j)} \rightarrow \mathbb{R} \) such that \( \sigma_n \ast D_B(r(n)) \) (resp. \( D_B(r(j)) \)) is the \( \Pi \)-graph of \( v_n \) (resp. of \( u_j \)) satisfy \( v_n \leq u_j \) in \( \sigma_n \ast E_{r(n)} \), for every \( j \geq j(n) \). After taking limits as \( j \rightarrow \infty \) (but letting \( n \) be fixed), we conclude that \( \sigma_n \ast D_B(r(n)) \) lies below \( D_\infty \). Taking limits as \( n \rightarrow \infty \), we have that \( \tau \ast D_\infty \) lies below \( D_\infty \). In particular, the angle that \( \tau \ast D_\infty \) makes with the plane \( \mathbb{R}^2 \times_A \{0\} \) at \( \tau \) (which equals \( \theta(\tilde{0}) \)) cannot be greater than the angle that \( D_\infty \) makes with \( \mathbb{R}^2 \times_A \{0\} \) at \( \tau \) (which is \( \theta(\tau) \)), a contradiction with our hypothesis in this case.

(B) We next perform slight modifications in the arguments in (A) to find a contradiction in the case that \( \theta(\tilde{0}) < \theta(\tau) \). Given \( n \in \mathbb{N} \) large, let \( \sigma_n \in \text{Int}(E_{r(n)}) \) be the point at distance \( 1/n \) from \( \tilde{0} \) so that the segment \( [\tilde{0}, \sigma_n] \) with end points \( \tilde{0} \) and \( \sigma_n \) is orthogonal to \( C_{r(n)} \) at \( \tilde{0} \). Arguing as in case (A), there exists an integer \( j(n) > n \) such that for every \( j \in \mathbb{N} \), \( j \geq j(n) \), the left translate of \( C_{r(n)} \) by \( \sigma_n \) lies in the interior of \( E_{r(j)} \). By property \((\ast)\) applied to \( \sigma_n \ast D_B(r(n)) \) and to \( M = \tau \ast D_B(r(j)) \), we deduce that \( \tau \ast D_B(r(j)) \) lies in the closure \( W_n \) of the non-compact complement of \( \{\sigma_n \ast D_B(r(n))\} \cup \mathbb{R}^2 \times_A \{0\} \) in \( \mathbb{R}^2 \times_A [0, \infty) \). Equivalently, the graphing functions \( v_n : \sigma_n \ast E_{r(n)} \rightarrow \mathbb{R} \), \( u_j : \tau \ast E_{r(j)} \rightarrow \mathbb{R} \) such that \( \sigma_n \ast D_B(r(n)) \) (resp. \( \tau \ast D_B(r(j)) \)) is the \( \Pi \)-graph of \( v_n \) (resp. of \( u_j \)) satisfy \( v_n \leq u_j \) in \( \sigma_n \ast E_{r(n)} \), for every \( j \geq j(n) \). Taking limits as \( j \rightarrow \infty \) with \( n \) fixed, we conclude that \( \sigma_n \ast D_B(r(n)) \) lies below \( \tau \ast D_\infty \). Taking limits as \( n \rightarrow \infty \), we have that \( D_\infty \) lies below \( \tau \ast D_\infty \). In particular, the angle that \( D_\infty \) makes with the plane \( \mathbb{R}^2 \times_A \{0\} \) at \( \tau \) (which equals \( \theta(\tilde{0}) \)) cannot be greater than the angle that \( \tau \ast D_\infty \) makes with \( \mathbb{R}^2 \times_A \{0\} \) at \( \tau \) (which is \( \theta(\tau) \)), that is, \( \theta(\tau) \leq \theta(\tilde{0}) \), which is contrary to our hypothesis.

From (A) and (B) we conclude the proof of our claim that \( V \) is everywhere tangent to \( D_\infty \).

Recall that \( V \) is horizontal and right invariant. It follows from equation \( [x, y, z] \)-coordinates in \( X = \mathbb{R}^2 \times_A \mathbb{R} \), \( V \) is a linear combination of \( \partial_x, \partial_y \) with constant coefficients, and thus its integral curves are horizontal lines, all parallel to \( L \) in the Euclidean sense. Since \( V \) is everywhere tangent to \( D_\infty \), the integral curves of \( V \) passing through points in \( D_\infty \) are completely contained in \( D_\infty \). Therefore,
$D_\infty$ is foliated by these horizontal lines. Let $L^\perp \subset \mathbb{R}^2 \times_A \{0\}$ be the straight line orthogonal to $L$ passing through the origin. Since $D_\infty$ is a minimal II-graph and it is foliated by straight lines parallel to $L$, the intersection of $D_\infty$ with the vertical plane $\Pi^{-1}(L^\perp)$ is a proper analytic arc $\gamma$ with $\bar{0} \in \gamma$, and $\gamma$ is a II-graph over its projection to $L^\perp$. Note that the $z$-coordinate restricted to $\gamma$ cannot have a local minimum value $z_0$, by the mean curvature comparison principle applied to the minimal surface $D_\infty$ and to the mean curvature one surface $\mathbb{R}^2 \times_A \{z_0\}$. Since $\gamma$ is analytic, if $\gamma$ is not parameterized by its $z$-coordinate, then $z|\gamma$ must have a first local maximum. As $\gamma$ lies above $\mathbb{R}^2 \times_A \{0\}$, $\gamma$ must be asymptotic to a horizontal line at height $z_1 \geq 0$, and thus $D_\infty$ is smoothly asymptotic to $\mathbb{R}^2 \times_A \{z_1\}$. This contradicts that $D_\infty$ is minimal and $\mathbb{R}^2 \times_A \{z_1\}$ has mean curvature one. This contradiction shows that $\gamma$ can be parameterized by its $z$-coordinate; in fact, the range of values of $z$ along $\gamma$ is $[0, \infty)$ since $D_\infty$ cannot be smoothly asymptotic to any horizontal plane $\mathbb{R}^2 \times_A \{z_2\}$ for any $z_2 > 0$. In what follows we will parameterize $\gamma$ by the height $z \in [0, \infty)$.

To obtain the contradiction that will prove Assertion 4.2, we apply a flux argument to an appropriate annular quotient of $D_\infty$. For any $t \in (0, \infty)$, consider the minimal strip

$$D_\infty(t) = D_\infty \cap (\mathbb{R}^2 \times_A [0, t]).$$

Fix $q \in L - \{\bar{0}\}$ and consider the infinite cyclic subgroup $\mathcal{I}$ of isometries of $X$ generated by the left translation by $q$. Then $X/\mathcal{I}$ is a homogeneous 3-manifold diffeomorphic to $S^1 \times \mathbb{R}^2$, the $z$-coordinate on $X$ descends to a well-defined function on $X/\mathcal{I}$ (which we will also call the height $z$), and every right invariant horizontal vector field $F$ on $X$ descends to a well-defined Killing field $\hat{F}$ on $X/\mathcal{I}$. Consider the quotient minimal annulus $\Omega(t) = D_\infty(t)/\mathcal{I}$ in $X/\mathcal{I}$.

First consider the case that the Milnor $D$-invariant of $X$ is positive. We next prove that the length in $X/\mathcal{I}$ of the boundary curve $c_t$ of $\Omega(t)$ at height $t$ converges to zero exponentially quickly as $t \to \infty$. To see this, without loss of generality we may assume that $L$ points in the $x$-direction (after a rotation in the $(x,y)$-plane), which does not change the ambient left invariant metric but does change the matrix $A$). Then, $q = (x(q), 0, 0)$ with $x(q) \in \mathbb{R} - \{0\}$, and equation (2.7) gives that

$$\text{length}(c_t) = \int_0^{x(q)} \|\partial_x\| \, dx = \int_0^{x(q)} \sqrt{a_{11}(-t)^2 + a_{21}(-t)^2} \, dx = \|\partial_x\| (0, 0, t) |x(q)|,$$

where $(a_{ij}(z))_{i,j}$ denotes the matrix $e^{zA} \in \mathcal{M}_2(\mathbb{R})$. Now we deduce that $\text{length}(c_t)$ converges to zero exponentially quickly as $t \to \infty$ from item (2a) of Proposition 6.1 in the Appendix, as the Milnor $D$-invariant is positive. The same item (2a) of Proposition 6.1 ensures that all horizontal right invariant vector fields in $X$ have lengths decaying exponentially as $z \to +\infty$. Pick a right invariant horizontal vector field $F$ in $X$ such that $F, V$ are linearly independent. Let $\hat{F}$ be the quotient Killing field of $F$ in $X/\mathcal{I}$. Consider for each $t \geq 0$ the flux of $\hat{F}$ across $c_t$, defined as

$$\text{Flux}(\hat{F}, c_t) = \int_0^{x(q)} \langle \hat{F}, \eta \rangle \, dx,$$

where $\eta$ is the unit vector field which is tangent to $D_\infty/\mathcal{I}$, orthogonal to $c_t$ with $\langle \eta, E_0 \rangle \geq 0$. Since $\hat{F}$ has constant length along $c_t$ with this constant being bounded as a function of $t > 0$, and the length of $c_t$ converges to zero as $t \to \infty$, $\text{Flux}(\hat{F}, c_t)$ also converges to zero as $t \to \infty$. As $\text{Flux}(\hat{F}, c_0)$ is non-zero and $\text{Flux}(\hat{F}, c_t)$ is
independent of $t$ (because the divergence of the tangential component of $\tilde{F}$ along $\Omega(t)$ is zero), we obtain a contradiction. This contradiction completes the proof of Assertion 4.2 in the case $D > 0$.

If $D \leq 0$, then item (2b) of Proposition 6.1 gives that $A$ is diagonalizable with one positive eigenvalue $\lambda$ and another non-positive eigenvalue $\mu$. In this case, there exists a horizontal right invariant vector field $F$ on $X$ with $\|F(0)\| = 1$ such that the norm of $F$ decreases exponentially quickly as $z \to +\infty$ (namely, $F$ is determined by $F(0)$ being the unitary eigenvector of the matrix $A$ associated to $\lambda$, since $\|F\|(z) = e^{-\lambda z}$ by item (2b) of Proposition 6.1). Let $\tilde{F}$ be the quotient Killing field of $F$ on $X/L$. Suppose for the moment that $F$ is not collinear with $V$. As before, we may assume that $L$ points in the $x$ direction and $q = (x(q), 0, 0)$. Then, the flux of $\tilde{F}$ along $c_t$ satisfies

$$|\text{Flux}(\tilde{F}, c_t)| \leq \int_0^{x(q)} \|F\|(x_0, y_0, t) \, dx.$$  

But the line element $dx$ grows at most exponentially as $z \to +\infty$, and in fact at most as the function $e^{-\mu z}$. Since $2 = \text{trace}(A) = \lambda + \mu$, then $\|F\| \, dx$ decays exponentially as $z \to +\infty$, and thus we arrive at a contradiction as in the former case $D > 0$.

The last case we must consider is $D \leq 0$ and $F$ is proportional to $V$. In this case, we still normalize $L$ to point in the direction of the $x$-axis and replace $F$ in the above computations by $\partial_y$. Then $dx$ decays like $e^{-\lambda z}$ (with the same notation as before), while $\|F\|$ increases at most like $e^{-\mu z}$ as $z \to +\infty$, and the conclusion is the same. Now Assertion 4.2 is proved. \hfill \square

We now prove item (1) in the statement of Theorem 4.1. Let $\Gamma$ be a simple closed convex curve contained in $\mathbb{R}^2 \rtimes_A \{0\}$, and let $M \subset X$ be a compact branched minimal surface with $\partial M = \Gamma$. By Lemma 3.2, $\text{Int}(M) \subset \Pi^{-1}(\text{Int}(E))$, where $E$ is the convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$ bounded by $\Gamma$. If $\text{Int}(M)$ is not contained in $\mathbb{R}^2 \rtimes_A (0, \infty)$, then there exists a point $p \in \text{Int}(M)$ with smallest non-positive $z$-coordinate $z_0$. An application of the mean curvature comparison principle gives a contradiction to the fact that the mean curvature of the plane $\mathbb{R}^2 \rtimes_A \{z_0\}$ is 1 and $M$ lies on the mean convex side of this plane at the point $p$. Therefore, conditions (a), (b) in item (1) of Theorem 4.1 hold.

Next we prove that condition (c) in item (1) holds. Arguing by contradiction, suppose that there exist an $\epsilon > 0$, a sequence $M(n)$ of compact branched minimal surfaces with $C^2$ simple closed convex boundary curves $\Gamma(n) \subset \mathbb{R}^2 \rtimes_A \{0\}$, and points $p_n \in \Gamma(n)$ such that

(C1) The geodesic curvature of $\Gamma(n)$ is uniformly going to zero as $n \to \infty$.
(C2) $\langle \eta_n(p_n), \partial_z(p_n) \rangle \geq -1 + \epsilon$ for all $n \in \mathbb{N}$, where $\eta_n$ is the exterior unit conormal vector to $M(n)$ along $\Gamma(n)$ (observe that in order to make sense of $\eta_n$ we are using that $M(n)$ is an immersion near $\Gamma(n)$ by item (2) of Lemma 3.2).

Let $R = R(\epsilon)$ be the positive number produced by Assertion 4.2 applied to the value $\epsilon > 0$ that appears in condition (C2) above. By condition (C1), for $n$ large enough we can assume that the geodesic curvature of $\Gamma(n)$ is less than $1/R$. This property allows us to find a round disk $\hat{E}_n \subset \mathbb{R}^2 \rtimes_A \{0\}$ of radius $R$ that is contained in the disk $E(n)$ bounded by $\Gamma(n)$ and such that $\hat{E}_n \cap E(n) = \{p_n\}$. Let $\hat{\Gamma}(n) = \partial \hat{E}_n$.
and let \( D_B(n) \) be the ‘lowest’ minimal disk bounded by \( \hat{\Gamma}(n) \) given by item (2c) of Theorem 3.3. Since \( D_B(n) \) lies below the branched minimal surface \( M(n) \) (this follows from property (\( * \)) in the proof of Assertion 4.2 applied to \( D_B(n) \) and \( M(n) \)) and \( D_B(n) \cap M(n) = \{ p_n \} \), the angle that the tangent plane \( T_{p_n} M(n) \) makes with \( \mathbb{R}^2 \times_A \{ 0 \} \) is greater than the angle \( \varphi_n \) that \( T_{p_n} D_B(n) \) makes with \( \mathbb{R}^2 \times_A \{ 0 \} \). Since condition (C1) and Assertion 4.2 allow us to take \( \varphi_n \) arbitrarily close to \( \pi/2 \) for \( n \) sufficiently large, we conclude that the angle that \( T_{p_n} M(n) \) makes with \( \mathbb{R}^2 \times_A \{ 0 \} \) converges to \( \pi/2 \) as \( n \to \infty \). This contradiction completes the proof of item (1c) of the theorem.

We next prove item (2) of the theorem. Suppose that \( \Gamma(n) \subset \mathbb{R}^2 \times_A \{ 0 \} \) is a sequence of \( C^2 \) simple closed convex curves with \( 0 \in \Gamma(n) \) having geodesic curvatures uniformly approaching 0 as \( n \to \infty \) and converging on compact subsets to a straight line \( L \) that contains \( \hat{0} \). Let \( M(n) \) be a sequence of compact branched minimal disks (or compact stable minimal surfaces) with \( \partial M(n) = \Gamma(n) \). Suppose for the moment that the \( M(n) \) are disks; we will discuss later the changes necessary to prove the case that the \( M(n) \) are stable. By item (1) of Theorem 3.3 the disks \( M(n) \) are unbranched and II-graphs over the compact convex disks \( E(n) \) bounded by \( \Gamma(n) \) in \( \mathbb{R}^2 \times_A \{ 0 \} \). We claim that the \( M(n) \) have uniformly bounded second fundamental forms up to their boundaries; to see this, suppose this property fails. Left translate the \( M(n) \) so that the norm of the second fundamental form is largest at the origin, and rescale the \((x, y, z)\)-coordinates by this maximum norm of the second fundamental form of the \( M(n) \), obtaining a new sequence of rescaled minimal II-graphs with uniformly bounded second fundamental form. After extracting a subsequence, these rescaled II-graphs converge to a non-flat minimal surface \( M_\infty \) in \( \mathbb{R}^3 \) possibly with boundary (if \( \partial M_\infty \) is non-empty, then \( \partial M_\infty \) is a horizontal straight line and \( M_\infty \) lies entirely above the horizontal plane that contains \( \partial M_\infty \)). Note that the Gaussian image of \( M_\infty \) is contained in the closed upper hemisphere, which is clearly impossible if some component of \( M_\infty \) has empty boundary (note that this component would be complete). This implies that \( M_\infty \) is connected and has non-empty boundary. It follows that \( M_\infty \) is a graphical stable minimal surface in the closed upper half-space of \( \mathbb{R}^3 \) bounded by the horizontal plane that contains \( \partial M_\infty \), which can also be easily ruled out, since \( M_\infty \) together with its image under the \( 180^\circ \)-rotation around \( \partial M_\infty \) is a complete, non-flat minimal graph. Therefore, the \( M(n) \) have uniformly bounded second fundamental forms up to their boundaries.

It follows that a subsequence of the \( M(n) \) (denoted in the same way) converges as \( n \to \infty \) on compact subsets of \( X \) to a minimal lamination \( \mathcal{L} \) of \( X - L \), and \( \mathcal{L} \) contains a leaf \( M_\infty \) with boundary the straight line \( L \). Since the geodesic curvatures of the curves \( \Gamma(n) \) converge uniformly to 0 as \( n \to \infty \), item (1) of this theorem implies that \( (\eta_n, \partial_\infty) \) is arbitrarily close to \(-1\) for \( n \) large enough (here \( \eta_n \) is the exterior conormal vector field to \( M(n) \) along \( \Gamma(n) \)). It follows that the limit surface \( M_\infty \) is tangent to the closed vertical halfplane \( \Pi^{-1}(L) \cap [\mathbb{R}^2 \times_A [0, \infty)] \) along \( L \). Since the \( M(n) \) all lie at one side of \( \Pi^{-1}(L) \cap [\mathbb{R}^2 \times_A [0, \infty)] \), \( M_\infty \) also lies at one side of \( \Pi^{-1}(L) \cap [\mathbb{R}^2 \times_A [0, \infty)] \), and thus the boundary maximum principle implies that \( M_\infty = \Pi^{-1}(L) \cap [\mathbb{R}^2 \times_A [0, \infty)] \). We now prove that \( \mathcal{L} \) contains no other leaves different from \( M_\infty \). Arguing by contradiction, any other leaf component \( \Sigma \) of \( \mathcal{L} \) must be a complete positive II-graph (without boundary) over its projection to \( \mathbb{R}^2 \times_A \{ 0 \} \), and \( \Sigma \) has bounded second fundamental form by arguments in the previous paragraph. But the existence of such a graphical leaf \( \Sigma \) in \( \mathbb{R}^2 \times_A (0, \infty) \)
is easily seen to be impossible by considering its behavior on a sequence of points $p_k = (x_k, y_k, z_k) \in \Sigma$ where $\lim_{k \to \infty} z_k$ is the infimum $z_0 \geq 0$ of the $z$-coordinate function of $\Sigma$ (recall that the minimal surface $\Sigma$ cannot be asymptotic to the mean curvature one surface $\mathbb{R}^2 \times_A \{z_0\}$). This contradiction proves that $\mathcal{L} = \{M_\infty\}$, and thus a subsequence of the $M(n)$ converges to the desired halfplane. Since every subsequence of the $M(n)$ has a convergent subsequence which equals this limit, the entire sequence $M(n)$ converges to $\Pi^{-1}(\mathcal{L}) \cap [\mathbb{R}^2 \times_A [0, \infty)]$. This completes the proof of item (2) of the theorem in the case that the $M(n)$ are disks.

If the $M(n)$ are compact stable surfaces (not disks), then the curvature estimates by Schoen [35] and Ros [33] give that the $M(n)$ have uniformly bounded second fundamental forms away from their boundaries. As previously, for each $n \in \mathbb{N}$ let $E(n)$ be the convex compact disk bounded by $\Gamma(n)$ in $\mathbb{R}^2 \times_A \{0\}$. Note that by barrier arguments as in the proof of items (2b) and (2c) in Theorem 3.10 for each $n \in \mathbb{N}$ there exists a least-area disk $D(n)$ with boundary $\Gamma(n)$ in the closure of the bounded region of $[\mathbb{R}^2 \times_A [0, \infty)] - M(n)$ that contains $E(n)$. Furthermore, $D(n) \subset \Pi^{-1}(E(n)) \cap [\mathbb{R}^2 \times_A [0, \infty)]$ is a $\Pi$-graph over $E(n)$. Also, $M(n)$ lies “above” the $\Pi$-graph $D(n)$. As the previously considered case of disks ensures that the $D(n)$ converge to $\Pi^{-1}(\mathcal{L}) \cap [\mathbb{R}^2 \times_A [0, \infty)]$ as $n \to \infty$, the $M(n)$ converge (as sets) to $\Pi^{-1}(\mathcal{L}) \cap [\mathbb{R}^2 \times_A [0, \infty)]$. We now check that the last convergence is of class $C^2$ by showing that the $M(n)$ have uniformly bounded second fundamental form up to their boundaries (this would finish the proof of item (2) of Theorem 4.1 in this case). If this is not the case, then the rescaling-by-curvature argument above produces a limit of a subsequence of the $M(n)$ which is a non-flat, stable minimal surface $M_\infty$ in $\mathbb{R}^3$ such that either has no boundary or $M_\infty$ has non-empty boundary given by a horizontal line and $M_\infty$ is contained in a quarter of space $Q \subset \mathbb{R}^3$ with $\partial M_\infty$ being the set of non-smooth points of $\partial Q$. If $\partial M_\infty = \emptyset$, then $M_\infty$ is complete, which contradicts that $M_\infty$ is non-flat and stable. Therefore, $\partial M_\infty \neq \emptyset$. In this case, the rescaled least-area disks $D(n)$ are all below the related rescaled $M(n)$. Since these rescaled images of $D(n)$ converge as $n \to \infty$ to a vertical halfplane in $Q$, $M_\infty$ must be equal to this vertical limit halfplane, which contradicts the non-flatness of $M_\infty$. Now the theorem is proved. 

**Corollary 4.3.** Let $X = \mathbb{R}^2 \times_A \mathbb{R}$ be a metric semidirect product, where $A \in \mathcal{M}_2(\mathbb{R})$ satisfies equation (2.9). Then, there exists a straight line $L \subset \mathbb{R}^2 \times_A \{0\}$ with $\bar{0} \in L$ such that the following property holds.

**(Q)** Let $p, q \in L$ be different points with $\bar{0} \in (p, q)$ (here $(p, q) \subset L$ is the open segment with extrema $p$, $q$), and let $C_p, C_q \subset \mathbb{R}^2 \times_A \{0\}$ be pairwise disjoint Euclidean circles centered at points in $L - \{p, q\}$, with $p \in C_p$, $q \in C_q$. If the Euclidean radii of $C_p, C_q$ are sufficiently large, then there exists an embedded least-area annulus $\Sigma \subset \mathbb{R}^2 \times_A [0, \infty)$ with boundary $\partial \Sigma = C_p \cup C_q$ (see Figure 2).

Furthermore:

1. If the Milnor $D$-invariant of $X$ is $D > 0$, then property (Q) holds for every line $L \subset \mathbb{R}^2 \times_A \{0\}$ with $\bar{0} \in L$.
2. If $D \leq 0$, then property (Q) holds for the line $L \subset \mathbb{R}^2 \times_A \{0\}$ with $\bar{0} \in L$ in the direction of the eigenvector of $A$ associated to a positive eigenvalue.
Proof. Suppose first that $D > 0$, and let $L \subset \mathbb{R}^2 \times_A \{0\}$ be any line with $\tilde{o} \in L$. By item (2a) in Proposition 6.1, the vertical lines $l_p = \{(p, z) \mid z \in \mathbb{R}\}$ and $l_q = \{(q, z) \mid z \in \mathbb{R}\}$ are both asymptotic to the $z$-axis as $z \to \infty$. Now consider pairwise disjoint Euclidean circles $C_p, C_q \subset \mathbb{R}^2 \times_A \{0\}$ centered at points in $L - [p, q]$, with $p \in C_p$, $q \in C_q$. Let $D_p, D_q$ be compact, embedded, least-area disks with boundaries $\partial D_p = C_p$, $\partial D_q = C_q$, which exist by Theorem 5. By Assertion 4.2 if the Euclidean radii of $C_p, C_q$ are sufficiently large, then $D_p, D_q$ are arbitrarily close to the vertical halfplanes $\Pi^{-1}(L_p) \cap [\mathbb{R}^2 \times_A [0, \infty))$, $\Pi^{-1}(L_q) \cap [\mathbb{R}^2 \times_A [0, \infty))$, where $L_p, L_q \subset \mathbb{R}^2 \times_A \{0\}$ are the lines orthogonal to $L$ that pass through $p, q$ respectively. Since $l_p \subset \Pi^{-1}(L_p)$ and $l_q \subset \Pi^{-1}(L_q)$ are asymptotic to the $z$-axis as $z \to \infty$, $l_p$ and $l_q$ are asymptotic to each other, and thus the distance between the disjoint area minimizing disks $D_p$ and $D_q$ is arbitrarily small if the Euclidean radii of $C_p, C_q$ are sufficiently large. Therefore, after replacing a pair of intrinsic geodesic disks $D_p' \subset D_p$, $D_q' \subset D_q$ of radius 1 centered at sufficiently close points of $D_p, D_q$ by an annulus of least area with boundary $\partial D_p' \cup \partial D_q'$, we obtain a piecewise smooth annulus with area less that the sum of the areas of the least-area disks $D_p', D_q'$. By the Douglas criterion (the area of some annulus bounding $C_p \cup C_q$ is less than the infimum of the areas of any two disks bounding $C_p \cup C_q$), there exists by Morrey an annulus $\Sigma$ of least area in $X$ with boundary $C_p \cup C_q$. Note that $\text{Int}(\Sigma) \subset \mathbb{R}^2 \times [0, \infty)$ by the maximum principle applied to $\Sigma$ and to planes $\mathbb{R}^2 \times_A \{z\}$ with $z < 0$. Then by the geometric Dehn’s lemma for planar domains given in Theorem 5 in 23, $\Sigma$ is a smooth embedded annulus (actually, Theorem 5 in 23 is stated for 3-manifolds with convex boundary, but the convex boundary is only used to obtain the existence of a least-area immersed annulus, which we already have in this case).

Suppose $D \leq 0$. As the eigenvalues of $A$ are the roots of the polynomial $\lambda^2 - 2\lambda + D = 0$, then $\lambda = 1 \pm \sqrt{1 - D}$. Hence, exactly one of these eigenvalues $\lambda_+$ is greater than or equal to 2, and the other one is non-positive. After an orthogonal change of basis (that does not change the metric Lie group structure of $X$), the matrix $A$ transforms to $A_1 = OAO^{-1}$ for some orthogonal matrix $O$, where $A_1$ has...
entries $a_{11} = \lambda_+$ and $a_{21} = 0$ and an associated eigenvector $(1, 0)$. Consider the line $L = \{(t, 0, 0)\} \subset \mathbb{R}^2 \times A$, \{0\}. In this case, equation (2.9) shows that $E_1 = e^{\lambda_+ t} \partial_2$, and so $\|\partial_2\|$ is exponentially decaying as $z \to \infty$. Hence, for any pair of different points $p, q \in L$, the vertical lines $l_p = \{(p, z) \mid z \in \mathbb{R}\}$ and $l_q = \{(q, z) \mid z \in \mathbb{R}\}$ are both asymptotic to the $z$-axis as $z \to \infty$. Now consider pairwise disjoint Euclidean circles $C_p, C_q \subset \mathbb{R}^2 \times A$, \{0\} centered at points in $L - [p, q]$, with $p \in C_p, q \in C_q$. Let $D_p, D_q$ be compact, embedded, least-area disks with boundaries $\partial D_p = C_p$, $\partial D_q = C_q$, which exist by Theorem 5.1. Arguing as in the previous paragraph, if the Euclidean radii of $C_p, C_q$ are sufficiently large, there exists an embedded least-area annulus $\Sigma \subset \mathbb{R}^2 \times A$ \{0, $\infty$\} with boundary $\partial \Sigma = C_p \cup C_q$, and the proof is complete. \hfill \square

5. Radius estimates for cylindrically bounded stable minimal surfaces

In this section we obtain radius estimates for compact stable minimal surfaces in semidirect products using the results from Section 4. Given $r > 0$ and a vertical geodesic $\Gamma$ in a metric semidirect product $X = \mathbb{R}^2 \rtimes A \mathbb{R}$, we will denote by $W(\Gamma, r) \subset X$ the closed solid metric cylinder of radius $r$ centered along $\Gamma$.

Proposition 5.1. Let $X = \mathbb{R}^2 \rtimes A \mathbb{R}$ be a metric semidirect product, where $A$ is either as in equation (2.9) with Milnor $D$-invariant less than 1 or where $\text{trace}(A) = 0$ (this is the case where $X$ is unimodular). For every vertical geodesic $\Gamma \subset X$ and $r > 0$, there exists a $j \in \mathbb{N}$ such that every compact immersed minimal surface $M$ in $X$ with $\partial M \subset W(\Gamma, r)$ satisfies $M \subset W(\Gamma, jr)$.

Proof. Without loss of generality, we will henceforth assume that $\Gamma$ is the $z$-axis. Recall that if $X$ is unimodular, then it is isomorphic to $\mathbb{R}^3$, $\tilde{E}(2)$, $\text{Nil}_3$, or $\text{Sol}_3$.

First suppose that $X$ is not isomorphic to $\tilde{E}(2)$ or $\text{Nil}_3$. By Theorem 3.6 in [22] (see also Examples 3.2–3.5 therein), there are two distinct vertical planes $P_1, P_2$ (in $(x, y, z)$-coordinates; in fact, $P_1$ can be taken as the $(x, z)$-plane and $P_2$ as the $(y, z)$-plane) that are Lie subgroups of $X$. For $i = 1, 2$, let $U_i(R)$ be the closed regular neighborhood of $P_i$ of radius $R > 0$. We claim that the boundary surfaces $\partial U_i(R) = P_i^+(R) \cup P_i^-(R)$ of $U_i(R)$ both have non-negative mean curvature in $X$ with respect to the inward pointing normal to $U_i(R)$. Since each of $P_i^+(R), P_i^-(R)$ are at constant distance from $P_i$, which is a connected, codimension-1 subgroup in $X$, Lemma 3.9 in [22] implies that $P_i^\pm(R)$ is a right coset of $P_i$ and is also a left coset of some 2-dimensional subgroup $\Sigma_i^\pm(R)$ of $X$. This last property implies that $P_i^\pm(R)$ has constant mean curvature, as every 2-dimensional subgroup in a metric Lie group has this property. Hence it remains to show that the mean curvature vector of $P_i^\pm(R)$ points towards $U_i(R)$. Note that $\Sigma_i^\pm(R)$ must be disjoint from $P_i^\pm(R)$ (otherwise $\Sigma_i^\pm(R) = P_i^\pm(R)$, which implies that $\tilde{U} \in P_i^\pm(R) \cap P_i$; hence $\tilde{R} = 0$, a contradiction). In this situation, the classification of codimension-1 subgroups in Theorem 3.6 in [22] implies that $\Sigma_i^\pm(R)$ is one of the elements in the 1-parameter family $A_i$ of 2-dimensional subgroups of $X$ that share the 1-dimensional subgroup $P_i \cap [\mathbb{R}^2 \times A$, \{0\}] (also called an algebraic open book decomposition of $X$). In the case that $X$ is unimodular (hence isomorphic to $\mathbb{R}^3$ or $\text{Sol}_3$), item (6) of Theorem 3.6 in [22] implies that all 2-dimensional subgroups of $X$ are minimal; hence $P_i^\pm(R)$ are minimal surfaces, and the claim is proved in this case. Next we will prove the desired mean convexity of $U_i(R)$ in the case that $X$ is non-unimodular.
and for \( i = 1 \) (for \( i = 2 \) the argument is similar and we leave it for the reader). Item (5) of Theorem 3.6 in [22] ensures that up to possibly rescaling the metric, \( X \) is isometric and isomorphic to \( A(b) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \), for some \( b \in \mathbb{R} \), \( b \neq -1 \). Furthermore, we can assume that \( P_1 = \{ y = 0 \} \), and thus the algebraic open book decomposition of \( \mathbb{R}^2 \rtimes A(b) \mathbb{R} \) that contains \( P_1 \) as one of its leaves is \( \mathcal{A}_1 = \{ H_1(\lambda) \mid \lambda \in \mathbb{R} \cup \{ \infty \} \} \), where

\[
H_1(\lambda) = \begin{cases} 
(x, \frac{\lambda}{b}(e^{bz} - 1), z) | x, z \in \mathbb{R} & \text{if } b \neq 0, \lambda \in \mathbb{R}, \\
(x, \lambda, z) | x, z \in \mathbb{R} & \text{if } b = 0, \lambda \in \mathbb{R}, \\
\mathbb{R}^2 \rtimes A(b) \{ 0 \} & \text{if } \lambda = \infty
\end{cases}
\]

(hence \( P_1 = H_1(0) \)). Observe that the 2-dimensional subgroups in \( \mathcal{A}_1 \) are products with the \( x \)-factor of proper graphs of the \( z \)-variable in the \( (y, z) \)-plane; this applies in particular to \( P_1 \) and to \( \Sigma_1^\pm(R) \). Therefore, \( \partial_x \) is everywhere tangent to \( P_1 \) and to \( \Sigma_1^\pm(R) \). As \( P_1^\pm(R) \) is a right coset of \( P_1 \) and \( F_1 = \partial_x \) is a right invariant vector field, then \( \partial_x \) is also everywhere tangent to \( P_1^\pm(R) \). In other words, \( P_1^\pm(R) \) is the product with the \( x \)-factor of a curve in the \( (y, z) \)-plane. In fact, this curve must be a proper graph of the \( z \)-variable (to see this, observe that every horizontal plane \( \mathbb{R}^2 \rtimes A(b) \{ z \} \) intersects \( P_1^\pm(R) \) in a line parallel to the \( x \)-axis which is the set of points of \( \mathbb{R}^2 \rtimes A(b) \{ z \} \) at distance \( R \) from \( P_1 \). Now the desired mean convexity of \( U_1(R) \) with respect to the inward normal vector can be understood by considering the related problem for \( z \)-graphs in the \( (y, z) \)-plane (i.e., after taking quotients in the \( x \)-factor): to do this, observe that \( P_1^\pm(R) \) lies entirely at one side of \( P_1 \), say its right side; see Figure 3. We can write \( P_1^+(R) = q * \Sigma_1^+(R) \), the left coset of \( \Sigma_1^+(R) \) obtained after left multiplication by an element \( q \in P_1^+(R) \). Observe that if \( \Sigma_1^+(R) = P_1 \), then \( \Sigma_1^+(R) \) is minimal (see Remark 2.2), and hence \( P_1^\pm(R) \) is minimal as well, which gives the desired mean convexity in this case. Thus, we can assume that \( \Sigma_1^+(R) \neq P_1 \). As \( P_1^+(R) \) lies at the right side of \( P_1 \) and is disjoint from \( \Sigma_1^+(R) \), then \( P_1^+(R) \) lies in the component \( \Omega \) of \( \mathbb{R}^2 \rtimes A(b) \mathbb{R} \) \( - \{ P_1 \cup \Sigma_1^+(R) \} \) that contains \( \mathbb{R}^2 \rtimes A(b) \{ 0 \} \) \( \cap \{ y > 0 \} \) (see Figure 3). As the mean curvature vector of \( P_1^+(R) \) at \( q \) equals the mean curvature vector of \( \Sigma_1^+(R) \) at \( \bar{0} \), a continuity argument in the variable \( R \) gives that the mean curvature vector of \( P_1^+(R) \) at \( q = q(R) \) points towards \( P_1 \), which finishes the proof of the claim.

By the last claim, the maximum principle (for the case \( X \) is unimodular) and the mean curvature comparison principle (when \( X \) is non-unimodular) applied to the foliation \( \{ P_1^+(R) \mid R > 0 \} \cup \{ P_1 \} \cup \{ P_2^+(R) \mid R > 0 \} \) gives that every compact minimal surface \( M \) with boundary in \( \mathcal{W}(\Gamma, r) \) lies in the domain \( U_1(r) \); hence, \( M \subset U_1(r) \cap U_2(r) \). If we prove that \( U_1(r) \cap U_2(r) \subset \mathcal{W}(\Gamma, 2r) \), then we would deduce that \( M \subset \mathcal{W}(\Gamma, 2r) \); i.e., the proposition holds with \( j = 2 \) in this case of \( X \) admitting two distinct vertical planes which are subgroups of \( X \). To check that \( U_1(r) \cap U_2(r) \subset \mathcal{W}(\Gamma, 2r) \), let \( p = (x, y, z) \in \mathcal{X} \); we will denote by \( p_1 = (x, 0, z) \in P_1 \), \( p_2 = (0, y, z) \in P_2 \) and \( p_3 = (0, 0, z) \in \Gamma \). Assume that the following properties concerning the extrinsic distance \( d \) in \( X \) hold (we will prove them later):

(A) \( d(p, p_i) = d(p, P_i) \), for \( i = 1, 2 \).
(B) \( d(p, p_3) = d(p, \Gamma) \).
Under these assumptions, we have

\[ d(p, \Gamma) = d(p, p_3) \leq d(p, p_1) + d(p_1, p_3) = d(p, P_1) + d(p_1, p_3) \]

where in (\*) we have left multiplied by \((0, y, 0)\) (which is an ambient isometry of \(X\), thus preserves distances) in the second summand. Now the inclusion \(U_1(r) \cap U_2(r) \subset \mathcal{W}(\Gamma, 2r)\) follows directly. We next prove (A) and (B). Given \(q \in X\) and \(A \subset X\), let \(C(p, q)\) (resp. \(C(p, A)\)) be the set of piecewise smooth curves \(\alpha : [0, 1] \to X\) such that \(\alpha(0) = p\) and \(\alpha(1) = q\) (resp. \(\alpha(1) \in A\)). Consider the maps

\[
\begin{align*}
\Theta_1 &: C(p, P_1) \to C(p, p_1), & \Theta_1(\alpha)(t) &= (x, y(t), z), \\
\Theta_2 &: C(p, P_2) \to C(p, p_2), & \Theta_2(\alpha)(t) &= (x(t), y, z), \\
\Theta_3 &: C(p, \Gamma) \to C(p, p_3), & \Theta_3(\alpha)(t) &= (x(t), y(t), z)
\end{align*}
\]

if \(\alpha(t) = (x(t), y(t), z(t)), t \in [0, 1]\). Since \(e^{zbA(b)} = \begin{pmatrix} e^z & 0 \\ 0 & e^{bz} \end{pmatrix}\), equation (2.7) implies that \(\Theta_i\) decreases lengths of curves, for each \(i = 1, 2, 3\). Thus, for \(i = 1, 2\)

\[ d(p, \Gamma) = d(p, p_3) = \min(d(p, p_1), d(p_1, p_3)) \]

\[ = d(p, P_1) + d(p_1, p_3) \]

FIGURE 3. The mean curvature vector of \(P_1^+(R) = q * \Sigma_1^+(R)\) (dotted) points towards the 2-dimensional subgroup \(\Sigma_1^+(R)\) (dashed), and hence towards the vertical plane \(P_1\). Above: the case \(b \neq 0\). Below: the case \(b = 0\). All graphics are representations in the \((y, z)\)-plane.
we have

\[
d(p, P_i) = \inf_{\alpha \in C(p, P_i)} \text{Length}(\alpha) \geq \inf_{\alpha \in C(p, P_i)} \text{Length}(\Theta_i(\alpha)) \geq \inf_{\beta \in C(p, P_i)} \text{Length}(\beta) = d(p, p_i).
\]

Since \( p_i \in P_i \), the last inequality is in fact an equality and (A) is proved. To prove (B) one uses the same argument, changing \( P_i \) by \( \Gamma, \Theta_i \) by \( \Theta_3 \), and \( p_i \) by \( p_3 \). This finishes the proof of the proposition when \( X \) is not isomorphic to \( \tilde{E}(2) \) or \( \text{Nil}_3 \).

Assume now that \( X \) is isomorphic to \( \tilde{E}(2) \). Thus, after scaling the metric of \( X \), it is isomorphic and isometric to \( \mathbb{R}^2 \rtimes_A \mathbb{R} \) with \( A(c) = \left( \begin{array}{cc} 0 & -c \\ 1/c & 0 \end{array} \right) \) for some \( c > 0 \) (see Section 2.7 of [22]). For \( t \in \mathbb{R} \), define the vertical planes \( P_1(t) = \{ x = t \} \), \( P_2(t) = \{ y = t \} \). For any \( t > 0 \) and \( i = 1, 2 \), let \( U_i(t) \) be the slab in \( X \) with boundary \( P_i(t) \). Equation (2.1) gives that the left translation in \( \Gamma \) finishes the proof of the proposition when \( \iota \) is unimodular, Theorem 3.6 in [22] implies that the foliation of surfaces at \( 3 \) consists of minimal surfaces in \( \mathcal{M}_2(\mathbb{R}) \). This finishes the proof of the proposition when \( \iota \) is isomorphic to \( \text{Nil}_3 \).

Given \( r > 0 \) fixed, the family of sets

\[
\{ U_1(nr) \cap U_2(nr) \cap \{ |z| \leq \pi \} \mid n \in \mathbb{N} \}
\]

forms a compact exhaustion for the horizontal slab \( \{ |z| \leq \pi \} \) and \( \mathcal{W}(\Gamma, r) \cap \{ |z| \leq \pi \} \) is compact, there exists a \( k \in \mathbb{N} \) such that \( \mathcal{W}(\Gamma, r) \cap \{ |z| \leq \pi \} \subset U_1(kr) \cap U_2(kr) \cap \{ |z| \leq \pi \} \). Furthermore, this integer \( k \) can be chosen independently from \( r \) (because the identity map from \( \{ |z| \leq \pi \} = \mathbb{R}^2 \rtimes_A [-\pi, \pi] \) into \( \mathbb{R}^2 \rtimes [-\pi, \pi] \) with the product metric is a quasi-isometry for every \( A \in \mathcal{M}_2(\mathbb{R}) \)). The left-invariance property of \( U_i(kr) \) and \( \mathcal{W}(\Gamma, r) \) by left translation by \( (0, 0, 2\pi) \) implies that \( \mathcal{W}(\Gamma, r) \) is contained in \( U_1(kr) \cap U_2(kr) \). This implies that if \( M \subset X \) is a compact minimal surface with \( \partial M \subset \mathcal{W}(\Gamma, r) \), then \( \partial M \subset U_1(kr) \cap U_2(kr) \), and by the maximum principle applied to the family of minimal surfaces \( \{ P_i(t) \mid t \in \mathbb{R} \} \), \( i = 1, 2 \), we deduce that \( M \subset U_1(kr) \cap U_2(kr) \). On the other hand, since the sets \( \mathcal{W}(\Gamma, r) \cap \{ |z| \leq \pi \} \) also form a compact exhaustion for the slab \( \{ |z| \leq \pi \} \), similar reasoning shows that there exists a \( j \in \mathbb{N} \) independent of \( r \) such that \( U_1(kr) \cap U_2(kr) \subset \mathcal{W}(\Gamma, jr) \), from which we conclude that \( M \subset \mathcal{W}(\Gamma, jr) \). This finishes the proof of the proposition in the case \( X \) is isomorphic to \( \tilde{E}(2) \).

Suppose now that \( X \) is isomorphic to \( \text{Nil}_3 \). After a scaling of the metric, we may assume that \( X \) is \( \mathbb{R}^2 \rtimes_A \mathbb{R} \) with \( A = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \). In this case there exists a unique vertical plane which is a subgroup of \( X \), namely \( P_1 = \{ y = 0 \} \). Since \( \text{Nil}_3 \) is unimodular, Theorem 3.6 in [22] implies that the foliation of surfaces at constant distance from \( P_1 \) consists of minimal surfaces in \( X \). Given \( r > 0 \), let \( P_1^\pm(r) = \{ y = \pm r \} \) be the boundary planes of the closed regular neighborhood \( U_1(r) \) of \( P_1 \) of radius \( r \) (one can check that the distance from \( 0 \) to \( (0, \pm r, 0) \) is \( r \)), and let \( S^\pm(r, R) \subset P_1^\pm(r) \) be the round circle of Euclidean radius \( R > 0 \) centered at the point \( (0, \pm r, 0) \). We claim that for \( R \) much larger than \( r \), there exists
an embedded minimal annulus $A(r, R) \subset X$ with boundary $\partial A(r, R) = S^+(r, R) \cup S^-(r, R)$. To see this, first note that

(I) $U_1(r)$ is quasi-isometric to the Riemannian product $\mathbb{R}^2 \times [-r, r]$ under the mapping arising from normal coordinates on $P_1$. This is because, in the fixed compact regular neighborhood of radius 1 of the segment $\{(0, t, 0) \mid t \in [-r, r]\}$ in the slab $U_1(r)$, the restricted mapping is a quasi-isometry with its image, and the differential of the normal coordinate map is invariant under left translations by elements in $P_1$.

(II) The Euclidean area of the cylinder $C(r, R) = \{(x, y, z) \mid (x, y, z) \in S^+(r, R), |y| \leq r\}$ is $4\pi r R$.

From (I), (II) we deduce that the area in $X$ of $C(r, R)$ is less than $4\pi cr R$, for some $c > 0$ that depends only on $r$. As the union of the two disks $D^\pm(r, R) \subset P_1^\pm(r)$ bounded by $S^\pm(r, R)$ has area $2\pi R^2$ and each of these disks is area-minimizing in $X$ (in fact, $D^\pm(r, R)$ is the unique solution of the Plateau problem for boundary $S^\pm(r, R)$ in $X$), then the Douglas criterion and the geometric Dehn’s lemma for planar domains in Theorem 5 in [23] (as adapted in the more general boundary setting of [21]) guarantee that for $R \gg r$, there exists an embedded least-area annulus $A(r, R)$ in $X$ with boundary $\partial A(r, R) = S^+(r, R) \cup S^-(r, R)$, and our claim is proved.

Consider the family of minimal annuli $F = \{p \ast A(r, R) \mid p \in P_1, [p \ast A(r, R)] \cap \mathcal{W}(\Gamma, r) = \emptyset\}$, where $p \ast A(r, R)$ denotes the left translation of $A(r, R)$ by the element $p$. Observe that $F$ satisfies the following properties.

(\mathcal{F}-I) $\mathcal{F}$ is non-empty. Furthermore, $\mathcal{F}$ is invariant under left translation by elements of $\Gamma$ and under the rotation $R_\Gamma$ by $\pi/2$ around $\Gamma$ (this follows from the invariance of $\mathcal{W}(\Gamma, r)$ and of $P_1$ under these ambient isometries of $X$).

(\mathcal{F}-II) There exists $k > r$ such that for all $t \in [k, \infty)$, we have

$$[(t, 0, 0) \ast (A(r, R))] \cap \mathcal{W}(\Gamma, r) = \emptyset$$

and

$$[(-t, 0, 0) \ast R_\Gamma(A(r, R))] \cap \mathcal{W}(\Gamma, r) = \emptyset$$

Let $\mathcal{F}'$ be the family of left translates of $A(r, R)$ and of $R_\Gamma(A(r, R))$ appearing in property (\mathcal{F}-II) together with their left translates by elements in $\Gamma$. It follows that there exists a $j \in \mathbb{N}$ such that

$$U_1(r) - \left( \bigcup_{F \in \mathcal{F}'} F \right) \subset \mathcal{W}(\Gamma, jr);$$

the existence of $j$ follows from similar arguments as those in the proof of property (I) above.

Finally, consider a compact minimal surface $M \subset X$ with boundary $\partial M \subset \mathcal{W}(\Gamma, r)$. Since $\mathcal{W}(\Gamma, r) \subset U_1(r)$ and the boundary planes of $U_1(r)$ are minimal, the maximum principle implies that $M \subset U_1(r)$. To finish the proof of Proposition 5.1 we will check that $M \subset \mathcal{W}(\Gamma, jr)$. Otherwise, as $M \subset U_1(r)$, 5.1 implies that $M$ intersects some annulus $F \in \mathcal{F}'$. But then by compactness of $M$ and $F$, there exists a largest $t > 0$ such that $M \cap [(t, 0, 0) \ast F] \neq \emptyset$, which gives a contradiction to the maximum principle since the boundary of $(t, 0, 0) \ast F$ is disjoint from $\partial M$. Now the proof of Proposition 5.1 is complete.

**Theorem 5.2.** Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, and let $\Gamma \subset X$ be a vertical geodesic. Then, given $r, C > 0$ there exists $R = R(r, C) > 0$ such that for
every compact immersed minimal surface \( M \subset \mathcal{W}(\Gamma, r) \) with the norm of its second fundamental form less than \( C \), the radius of \( M \) is less than \( R \).

**Proof.** After a fixed left translation of \( \Gamma \), we will assume that \( \Gamma \) is the \( z \)-axis in \( \mathbb{R}^2 \times_A \mathbb{R} \).

To prove the theorem we proceed by contradiction. Suppose that there exist \( r, C > 0 \) and a sequence of compact, immersed minimal surfaces \( h_n : M_n \rightarrow \mathcal{W}(\Gamma, r) \) with the norm of their second fundamental forms less than \( C \) and such that there exist points \( p_n \in M_n \) for which the intrinsic distances from \( p_n \) to the boundaries of the \( M_n \) satisfy \( d_{M_n}(p_n, \partial M_n) > n \) for all \( n \in \mathbb{N} \). Consider the compact domain \( Y = \mathcal{W}(\Gamma, r) \cap [\mathbb{R}^2 \times_A \{0\}] \). After left translating the immersions \( h_n \) appropriately by elements in the 1-parameter subgroup \( \Gamma = \{(0, 0, s) \in \mathbb{R}^2 \times_A \mathbb{R} \mid s \in \mathbb{R} \} \) and passing to a subsequence, we may assume that \( h_n(p_n) \in Y \) for all \( n \) and this sequence of points converges to a point \( q_\infty \in Y \).

Since the minimal immersions \( h_n \) have uniform curvature estimates, there exists a complete, connected immersed minimal surface \( h_\infty : M_\infty \rightarrow \mathcal{W}(\Gamma, r) \) of bounded second fundamental form that is a limit of the restriction of (a subsequence, denoted in the same way, of) the \( h_n \) to certain smooth compact domains \( \Omega_n \subset M_n \) with \( p_n \in \Omega_n \) and \( d_n(p_n, \partial \Omega_n) > n \) for all \( n \in \mathbb{N} \), and such that \( h_\infty(p_\infty) = q_\infty \) for some point \( p_\infty \in M_\infty \). Now consider the closure \( \mathcal{M} \) of the union of all left translations of \( h_\infty(M_\infty) \) by elements in \( \Gamma \), i.e.,

\[
\mathcal{M} = \{a \ast h_\infty(M_\infty) \mid a \in \Gamma\},
\]

which is a connected subset of \( \mathcal{W}(\Gamma, r) \). As \( h_\infty(M_\infty) \) has bounded second fundamental form, then, by the same compactness arguments, given any point \( q \in \mathcal{M} \), there exists a compact embedded minimal disk \( D(q) \subset M_\infty \) with \( q \in \text{Int}(D(q)) \).

We next consider the special case where \( X \) is non-unimodular, and so we assume that \( X = \mathbb{R}^2 \times_A \mathbb{R} \) with \( A \) satisfying (2.9); see Remark 2.1. Let \( L \) be the line given by Corollary 4.3. For \( p = (x, y, 0) \in L \) with \( x^2 + y^2 \) sufficiently large, the set \( Y \) lies in the interior of the strip \( S \subset \mathbb{R}^2 \times_A \{0\} \) bounded by the pair of Euclidean lines \( L_p, L_{-p} \subset \mathbb{R}^2 \times_A \{0\} \) that are orthogonal to \( L \) at the respective points \( p, -p \). By Corollary 4.3, there exist circles \( C_p, C_q = -p \) with \( p \in C_p, q \in C_q \) such that \( C_p \cup C_q \) does not intersect the interior of the strip \( S \) and \( C_p \cup C_q \) is the boundary of a least-area embedded annulus \( \Sigma \subset \mathbb{R}^2 \times_A \{0, \infty\} \). Let \( L^\perp \in \mathbb{R}^2 \times_A \{0\} \) be the line perpendicular to \( L \) at \( 0 \). For \( t \in L^\perp \), let \( t \ast \Sigma \) denote the left translation of \( \Sigma \) by \( t \). Note that for some \( t_0 \in \mathbb{R} \), \( (t_0 \ast \Sigma) \cap \mathcal{M} = \emptyset \) and that for \( |t| \) large, \( (t \ast \Sigma) \cap \mathcal{M} = \emptyset \). It follows that there exists a \( t_1 \in L^\perp \) with largest norm such that \( t_1 \ast \Sigma \) intersects \( \mathcal{M} \) at some point \( p_{t_1} \) and near \( p_{t_1} \) the set \( \mathcal{M} \) lies on one side of \( t_1 \ast \Sigma \). This implies that there exists an embedded minimal disk \( D(p_{t_1}) \subset \mathcal{M} \) containing \( p_{t_1} \) such that \( D(p_{t_1}) \) lies on one side of \( t_1 \ast \Sigma \). Now the maximum principle for minimal surfaces gives that \( t_1 \ast \Sigma \subset \mathcal{M} \), which is false since \( \partial(t_1 \ast \Sigma) \cap \mathcal{M} = \emptyset \). This contradiction proves the theorem in this special case where \( X \) is non-unimodular.

Finally, we consider the remaining case where \( X \) is unimodular (hence \( \text{tr}(A) = 0 \)). Consider the foliation of \( X \) by minimal planes produced in the proof of Proposition 5.1 i.e. (with the notation in that proposition), the planes \( P^\perp_1(R) \) at constant distance \( R > 0 \) from \( P_1 \) in the case that \( X \) is isomorphic to \( \text{Sol}_3, \text{Nil}_3 \) or \( \mathbb{R}^3 \), and the periodic planes \( P_1(t) = \{x = t\} \), in the case of \( X \) is isomorphic to \( \mathbb{E}(2) \). Since in all of these cases there exists one of these minimal planes \( P \) such that \( P \cap \mathcal{M} = \emptyset \) and \( \mathcal{M} \) lies on one side of \( P \), the argument in the previous paragraph with \( P \) in
place of Σ easily generalizes to give a contradiction. This contradiction completes
the proof of Theorem 5.2. □

As a direct consequence of Theorem 5.2 and the classical curvature estimates for
stable minimal surfaces [33, 35], we obtain:

\textbf{Corollary 5.3.} Let \( \Gamma = \mathbb{R}^2 \rtimes_A \mathbb{R} \) be a metric semidirect product, and let \( W(\Gamma, r) \subset X \) denote a solid metric cylinder in \( X \) of radius \( r > 0 \) around a vertical geodesic \( \Gamma \subset X \). Then, there are no complete stable minimal surfaces contained in \( W(\Gamma, r) \).

In order to prove Theorem 1.1 stated in the Introduction we will need an aux-
iliary construction for the case that \( X \) is non-unimodular with positive Milnor
\( D \)-invariant. To do this, in the remainder of this section we fix \( \alpha \in [0, 1) \)
and \( \beta \in [0, \infty) \), and we consider the matrix \( A = A(\alpha, \beta) \) given by (2.9) and the non-
unimodular metric Lie group \( X = X(\alpha, \beta) = \mathbb{R}^2 \rtimes_A \mathbb{R} \) with its usual left invariant
metric \( \langle \cdot, \cdot \rangle \) determined by \( A \) (see Definition 2.1). Under our hypotheses on \( \alpha, \beta \),
we have that the Milnor \( D \)-invariant of \( X \) is positive. Conversely, every non-
unimodular metric Lie group with positive Milnor \( D \)-invariant can be expressed
as \( X(\alpha, \beta) \) for some \( \alpha \in [0, 1) \) and \( \beta \geq 0 \); see Section 2.

The 1-parameter subgroup \( \{(0,0,s) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid s \in \mathbb{R}\} \) of \( X \) generates under
left multiplication a right invariant vector field \( F_3 \) of \( X \) (see equation (2.3), where
the notation for the matrix \( A \) is different from the one used here). Next we will
study the mean convexity of solid cylinders in \( X \) obtained after flowing the domains
enclosed by a family of homothetic ellipses in \( \mathbb{R}^2 \rtimes_A \{0\} \) through the 1-parameter
group of isometries \( \{\phi_s \mid s \in \mathbb{R}\} \) associated to \( F_3 \), namely the left translations by
elements in the \( z \)-axis of \( \mathbb{R}^2 \rtimes_A \mathbb{R} \). The technical property stated in Proposition 5.4
will be used in Theorem 5.6 below in order to obtain the desired radius estimate
for stable minimal surfaces in \( X \) that generalizes Corollary 5.3.

Consider an ellipse \( C_\mu = \{(x,y,0) \in \mathbb{R}^2 \rtimes_A \{0\} \mid x^2 + \frac{y^2}{\mu^2} = 1\} \), where \( \mu > 0 \) is
to be determined, and the family of homothetic ellipses
\begin{equation}
(5.2)
 rC_\mu = \{(rx,ry,0) \mid (x,y,0) \in C_\mu\}, \quad r > 0.
\end{equation}
Let \( rE_\mu \subset \mathbb{R}^2 \rtimes_A \{0\} \) denote the compact disk with boundary \( rC_\mu \), and let
\begin{equation}
(5.3)
\Omega(r) = \bigcup_{s \in \mathbb{R}} \phi_s(rE_\mu)
\end{equation}
be the \( F_3 \)-invariant closed solid cylinder obtained after flowing \( rE_\mu \) by the isometries
that generate \( F_3 \).

\textbf{Proposition 5.4.} Let \( X = \mathbb{R}^2 \rtimes_A \mathbb{R} \) be a metric semidirect product where \( A \) is as
in equation (2.9) with \( \alpha \in [0, 1) \) and \( \beta \in [0, \infty) \). Then, there exist \( \mu > 0 \) and \( r_0 > 0 \)
such that the \( F_3 \)-invariant solid cylinder \( \Omega(r) \) over \( rE_\mu \) defined in (5.3) is strictly
mean convex for every \( r \geq r_0 \).

\textbf{Proof.} Fix a positive \( \mu \) to be determined later. Given \( r > 0 \), parameterize \( rC_\mu \) by
\begin{equation}
(5.4)
x(t) = r \cos t, \quad y(t) = r \mu \sin t, \quad t \in [0, 2\pi].
\end{equation}
A parametrization \( \Phi \) of the \( F_3 \)-invariant cylinder given by the boundary \( \Sigma = \Sigma(r) \)
of \( \Omega(r) \) is obtained by flowing \( \gamma \) through the 1-parameter group \( \{\phi_s \mid s \in \mathbb{R}\} \), i.e.,
\begin{equation}
\Phi(t,s) = \phi_s(\gamma(t)), \quad (t,s) \in [0, 2\pi] \times \mathbb{R}.
\end{equation}
where \( \phi_s(p, z) = (e^{sA}p, s + z) \) for all \( s, z \in \mathbb{R} \) and \( p \in \mathbb{R}^2 \) (\( p \) is considered to be a column vector).

The mean curvature \( H = H(t, s) \) of \( \Sigma \) is given by the well-known formula

\[
2(EG - F^2)H = eG - 2fF + gE,
\]

where \( E, F, G \) and \( e, f, g \) are respectively the coefficients of the first and second fundamental forms of \( \Sigma \) (these coefficients are functions of \( (t, s) \)): \( E = \|\Phi_t\|^2 \), \( F = \langle \Phi_t, \Phi_s \rangle \), \( G = \|\Phi_s\|^2 \), \( e = \langle N, \nabla_{\Phi_t} \Phi_t \rangle \), \( f = \langle N, \nabla_{\Phi_s} \Phi_s \rangle \), \( g = \langle N, \nabla_{\Phi_s} \Phi_s \rangle \), where \( \Phi_t = \frac{\partial \Phi}{\partial t} \), \( \Phi_s = \frac{\partial \Phi}{\partial s} \), and \( N = \frac{\Phi_s \times \Phi_t}{\|\Phi_s \times \Phi_t\|} \) is the unit normal vector field to \( \Sigma \). Observe that \( \Phi_t(t, 0) \) defines the counterclockwise orientation on \( rC_\mu \), and that \( \Phi_s(t, 0) \) points upward. Therefore, \( N(t, 0) \) points outward \( \Omega(r) \) along \( \gamma \). Since \( \Sigma \) is \( F_3 \)-invariant, the strict mean convexity of \( \Omega(r) \) will follow from the existence of some \( \mu > 0 \) (depending solely on \( \alpha, \beta \)) such that the function \( t \in [0, 2\pi] \mapsto (eG - 2fF + gE)(t, 0) \) is strictly negative for \( r > 0 \) large enough.

Note that

\[
\Phi_t(t, 0) = \gamma'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ 0 \end{bmatrix} = \begin{bmatrix} x'(t) \\ y'(t) \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mu}y(t) \\ \mu x(t) \\ 0 \end{bmatrix},
\]

where the parentheses (resp. brackets) refer to coordinates with respect to the basis \( \{\partial_x, \partial_y, \partial_z\} \) (resp. to the usual orthonormal basis \( \{E_1, E_2, E_3\} \) of the Lie algebra of \( X \) given by (2.4)). In general, the change of coordinates between the two bases at a point \((x, y, z) \in \mathbb{R}^2 \times_A \mathbb{R} \) is

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{pmatrix} e^{sA} \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad a, b, c \in \mathbb{R}.
\]

Also, \( \Phi_s(t, s) = (F_3)_{\Phi(t, s)} \), and so the globally defined right invariant vector field \( F_3 \) extends \( \Phi_s \). Using (2.9) (recall that the entries of the matrix \( A \) are given by (2.10)), we have

\[
F_3(x, y, z) = \begin{pmatrix} \delta \\ \varepsilon \end{pmatrix} \begin{bmatrix} \delta a_{11}(-z) + \varepsilon a_{12}(-z) \\ \delta a_{21}(-z) + \varepsilon a_{22}(-z) \end{bmatrix},
\]

where

\[
\begin{align*}
\delta(x, y) &= (1 + \alpha)x - (1 - \alpha)\beta y, \\
\varepsilon(x, y) &= (1 + \alpha)\beta x + (1 - \alpha)y,
\end{align*}
\]

and \( a_{ij}(z) \) are the entries of the matrix \( e^{zA} \); see (2.7). In particular,

\[
\Phi_s(t, 0) = (F_3)_{\gamma(t)} = \begin{bmatrix} \delta(t) \\ \varepsilon(t) \end{bmatrix},
\]

where \( \delta(t) = \delta(\gamma(t)) \) and \( \varepsilon(t) = \varepsilon(\gamma(t)) \).

From (5.7) and (5.11) we can compute the coefficients of the first fundamental form at points of the form \( \Phi(t, 0) \):

\[
\begin{align*}
E(t, 0) &= x'(t)^2 + y'(t)^2, \\
F(t, 0) &= \delta(t)x'(t) + \varepsilon(t)y'(t), \\
G(t, 0) &= 1 + \delta(t)^2 + \varepsilon(t)^2.
\end{align*}
\]
The unit normal vector field at points of the form $\Phi(t, 0)$ is given by

\begin{equation}
(5.13) \quad N(t, 0) = \frac{1}{\Delta(t)} (\Phi_t \times \Phi_s) (t, 0) = \frac{1}{\Delta(t)} \begin{bmatrix}
y'(t) \\
-\delta(t)y'(t)
\end{bmatrix},
\end{equation}

where $\Delta(t) = \|\Phi_t \times \Phi_s\|(t, 0)$.

We next compute the coefficients of the second fundamental form of $\Sigma$. Using (5.7) and denoting by $D_W$ the covariant derivative of a vector field $W$ along $\gamma$, we have

\begin{equation}
(\nabla_{\Phi_t} \Phi_s)(t, 0) = D(t)(E_1)_{\alpha(t)} + y'(t)(E_2)_{\gamma(t)}
\end{equation}

\begin{equation}
= \begin{bmatrix}
x''(t) \\
y''(t)
\end{bmatrix} + x'(t)\nabla_{\gamma'(t)} E_1 + y'(t)\nabla_{\gamma'(t)} E_2
\end{equation}

\begin{equation}
(5.14)
\begin{bmatrix}
x''(t) \\
y''(t)
\end{bmatrix}
(1 + \alpha)x'(t) + 2\alpha\beta x'(t)y'(t) + (1 - \alpha)y'(t)^2.
\end{equation}

Analogously,

\begin{equation}
(\nabla_{\Phi_s} \Phi_s)(t, 0) = D(F_3 \circ \gamma) = D(t)(E_1)_{\alpha(t)} + (E_3)_{\gamma(t)}
\end{equation}

\begin{equation}
= \begin{bmatrix}
\delta'(t) \\
\varepsilon'(t)
\end{bmatrix} + \delta(t)\nabla_{\gamma'(t)} E_1 + \varepsilon(t)\nabla_{\gamma'(t)} E_2 + \nabla_{\gamma'(t)} E_3
\end{equation}

\begin{equation}
(5.15) \begin{bmatrix}
\delta'(t) - (1 + \alpha)x'(t) - \alpha\beta y'(t) \\
\varepsilon'(t) - \alpha\beta x'(t) - (1 - \alpha)y'(t)
\end{bmatrix}.
\end{equation}

To compute $g = \langle N, \nabla_{\Phi_s} \Phi_s \rangle = \langle N, \nabla_{\Phi_s} F_3 \rangle$ we use that $F_3$ is a Killing vector field that extends $\Phi_s$:

\begin{equation}
g(t, s) = -\langle \Phi_s, \nabla_N F_3 \rangle = -\frac{1}{2} N \left(\|F_3\|^2\right)
\end{equation}

\begin{equation}
-\langle \delta a_{11}(-z) + \varepsilon a_{12}(-z) \rangle N \left(\delta a_{11}(-z) + \varepsilon a_{12}(-z) \right)
-\langle \delta a_{21}(-z) + \varepsilon a_{22}(-z) \rangle N \left(\delta a_{21}(-z) + \varepsilon a_{22}(-z) \right).
\end{equation}

Hence,

\begin{equation}
g(t, 0) = -\delta(t) \left\{ N(\delta) + \delta(t) N(a_{11}(-z)) + \varepsilon(t) N(a_{12}(-z)) \right\}
-\varepsilon(t) \left\{ N(\varepsilon) + \delta(t) N(a_{21}(-z)) + \varepsilon(t) N(a_{22}(-z)) \right\},
\end{equation}

where we have simplified the notation $N(t, 0)$ by $N$. By using (5.10) and (5.13) one has (at $(t, 0)$):

\begin{equation}
(5.17) \begin{cases}
N(\delta) = \frac{1}{\Delta(t)} \left[ (1 + \alpha)y'(t) + (1 - \alpha)\beta x'(t) \right], \\
N(\varepsilon) = \frac{1}{\Delta(t)} \left[ (1 + \alpha)\beta y'(t) - (1 - \alpha)x'(t) \right], \\
N(a_{ij}(-z)) = -\frac{1}{\Delta(t)} \left[ \varepsilon(t)x'(t) - \delta(t)y'(t) \right] a'_{ij}(0), \quad \text{for } i, j = 1, 2.
\end{cases}
\end{equation}
Since \((a_{ij}(z))_{i,j} = e^{z^A}, \ (a'_{ij}(0))_{i,j} = A\). Now, \((5.16)\) and \((5.17)\) give \((5.18)\)
\[
\Delta(t)g(t, 0) = - \{(1 + \alpha)y' + (1 - \alpha)\beta x' - [(1 + \alpha)\delta - (1 - \alpha)\beta\varepsilon]\varepsilon x' - \delta y'\} \delta
\]
\[
- \{(1 + \alpha)\beta y' - (1 - \alpha)x' - [(1 + \alpha)\beta\varepsilon + (1 - \alpha)\delta]\varepsilon x' - \delta y'\} \varepsilon.
\]

A direct substitution from \((5.4), (5.6), (5.10), (5.14), (5.15),\) and \((5.18)\) gives that
\[
\Delta(t) (e^{G} - 2fF + gE)(t, 0)
\]
\[
= - \mu r^2 + r^4 h_{\alpha, \beta, \mu}(t)
\]
\[
- \frac{r^6}{4} \{2\mu + 2\alpha\mu \cos(2t) + \beta(1 + \alpha - (1 - \alpha)\mu^2) \sin(2t)\}^3,
\]
where \(h_{\alpha, \beta, \mu}(t)\) is a smooth, \(\pi\)-periodic function of \(t\) depending on the parameters \(\alpha, \beta, \mu\). From the last displayed expression we deduce that the mean curvature of \(\Sigma\) with respect to \(N\) is strictly negative for all \(r\) large enough provided that the expression
\[
G_{\alpha, \beta, \mu}(t) = 2\mu + 2\alpha\mu \cos(2t) + \beta(1 + \alpha - (1 - \alpha)\mu^2) \sin(2t)
\]
is positive as a function of \(t \in [0, 2\pi]\), for any given values \(\alpha \in [0, 1], \beta \in [0, \infty)\), and for some choice of \(\mu = \mu(\alpha, \beta) > 0\). Clearly,
\[
G_{\alpha, \beta, \mu}(t) = 2\mu + \langle u, v(t) \rangle \geq 2\mu - \|u\|,
\]
where \(u = u(\alpha, \beta, \mu) = (2\alpha u, \beta(1 + \alpha - (1 - \alpha)\mu^2))\), \(v(t) = (\cos(2t), \sin(2t)) \in \mathbb{R}^2\), and both the last inner product and norm refer to the usual flat metric in \(\mathbb{R}^2\). Therefore, the proposition will be proved if we show that the following elementary property holds:

(R) Given \((\alpha, \beta) \in [0, 1] \times [0, \infty)\), there exists \(\mu > 0\) such that
\[
4\mu^2 > \|u\|^2 = 4\alpha^2 \mu^2 + \beta^2(1 + \alpha - (1 - \alpha)\mu^2)^2.
\]

If \(\beta = 0\), then property (R) clearly holds as \(\alpha^2 < 1\). If \(\beta > 0\), then the proof of property (R) follows from an elementary analysis of the function \(\chi(\lambda) = 4(1 - \alpha^2)\lambda - \beta^2(1 + \alpha - (1 - \alpha)\lambda)^2\), which has a (unique) maximum at \(\lambda_0 = \frac{1 + \alpha}{1 - \alpha} (1 + 2\beta^2) > 0\), with value \(\chi(\lambda_0) = 4(1 + \alpha)^2 (1 + \beta^2) > 0\). Now the desired \(\mu > 0\) can be chosen as \(\mu = \sqrt{\lambda_0}\). This completes the proof of the proposition. \(\square\)

As a consequence of the mean convexity of \(\Omega(r)\), we have:

**Proposition 5.5.** Let \(X = \mathbb{R}^2 \times_{\lambda^A} \mathbb{R}\) be a non-unimodular metric semidirect product, where \(A\) is as in equation \((2.9)\) with \(\alpha \in [0, 1]\) and \(\beta \in [0, \infty)\). Let \(\mu, r_0 > 0\) and \(\Omega(r)\) be the numbers and related \(F_3\)-invariant, mean convex solid cylinder given in Proposition \((5.4)\). Suppose that \(r > r_0\). Then every compact immersed minimal surface \(M \subset X\) whose boundary lies in \(\Omega(r)\) satisfies that \(M \subset \Omega(r)\).

**Proof.** It is a consequence of a standard mean curvature comparison argument based on the following two facts:

- For every \(r \geq r_0\), \(\Omega(r)\) has mean convex boundary by Proposition \((5.4)\), and \(\Omega(r_0) \subset \Omega(r)\).
- The collection of boundaries \(\{\partial \Omega(r) \mid r \geq r_0\}\) forms a codimension-1 foliation of \(X - \Omega(r_0)\). \(\square\)
Finally, from Proposition 5.1, Theorem 5.2, and Proposition 5.5, we can conclude the desired radius estimate for compact stable minimal surfaces in metric semidirect products stated in Theorem 5.1.

**Theorem 5.6.** Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product. Given $r > 0$ and any vertical geodesic $\Gamma \subset X$, there exists a positive number $\Lambda(r) > 0$ such that the following property holds: for any compact stable minimal surface $M$ in $X$ such that all points of its boundary $\partial M$ are at distance at most $r$ from $\Gamma$, the radius of $M$ is at most $\Lambda(r)$.

**Proof.** We first consider the case where $A$ is as in equation (2.9) with $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$. Let $M$ be a compact stable minimal surface in $X$ whose boundary lies in the closed solid metric cylinder $\mathcal{W}(\Gamma, r)$ of radius $r > 0$ in $X$ around a vertical geodesic $\Gamma$. Note that there exists some $r' = r'(r) > 0$ such that $\mathcal{W}(\Gamma, r) \subset \Omega(r')$, where $\Omega(r')$ is the mean convex solid cylinder given in Proposition 5.4. Then, by Proposition 5.5 we deduce that $M \subset \Omega(r')$. As there exists some $r'' = r''(r) > 0$ such that $\Omega(r'') \subset \mathcal{W}(\Gamma, r'')$, we obtain from Theorem 5.2 and the Schoen-Ros curvature estimates for stable minimal surfaces [33,35] the desired radius estimate.

If $X$ is non-unimodular with $A$ given by (2.9) and non-positive Milnor $D$-invariant, or else trace$(A) = 0$, then we apply Proposition 5.1 to conclude that every compact immersed stable minimal surface in $X$ is contained in some metric cylinder $\mathcal{W}(\Gamma, r')$, where $r'$ depends only on $X$ and $r$. Then, as in the previous paragraph, Theorem 5.2 implies that $M$ has a radius estimate that depends only on $X$ and $r$. This last observation completes the proof. \hfill $\square$

6. Appendix

**Proposition 6.1.** Let $X$ be a non-unimodular semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric, where $A \in \mathcal{M}_2(\mathbb{R})$ is given by (2.9) for some constants $\alpha, \beta \geq 0$. Let $D = \det(A) = (1 - \alpha^2)(1 + \beta^2)$ be the Milnor $D$-invariant associated to the Lie group $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Then, the following properties hold:

1. Given $z \in \mathbb{R}$, the exponential of the matrix $zA$ is equal to

$$e^{zA} = e^{z} \left[ C_D(z) I_2 + S_D(z)(A - I_2) \right],$$

where $I_2 \in \mathcal{M}_2(\mathbb{R})$ is the identity matrix and

$$C_D(t) = \begin{cases} \cosh(\sqrt{1 - D} t) & \text{if } D < 1, \\ 1 & \text{if } D = 1, \\ \cos(\sqrt{D - 1} t) & \text{if } D > 1, \end{cases}$$

$$S_D(t) = \begin{cases} \frac{1}{\sqrt{1 - D}} \sinh(\sqrt{1 - D} t) & \text{if } D < 1, \\ t & \text{if } D = 1, \\ \frac{1}{\sqrt{D - 1}} \sin(\sqrt{D - 1} t) & \text{if } D > 1. \end{cases}$$

2. The norms of $\partial_x$, $\partial_y$, and their inner product with respect to the canonical metric are

$$\|\partial_x\|^2 = e^{-2z} \left\{ \beta^2 (1 + \alpha)^2 S_D(z)^2 + [C_D(z) - \alpha S_D(z)]^2 \right\},$$

$$\|\partial_y\|^2 = e^{-2z} \left\{ \beta^2 (1 - \alpha)^2 S_D(z)^2 + [C_D(z) + \alpha S_D(z)]^2 \right\},$$

$$\langle \partial_x, \partial_y \rangle = -2\alpha\beta e^{-2z} S_D(z) [S_D(z) + C_D(z)].$$
In particular:

(2a) If $D > 0$ then $\|\partial_x\|^2, |\partial_y\|^2, (\partial_x, \partial_y)$ decay exponentially as $z \to +\infty$, and so the norm of every horizontal right invariant vector field in $X$ decays exponentially as $z \to +\infty$ as well.

(2b) If $D < 1$, then $A$ is diagonalizable with distinct eigenvalues $\lambda_\pm = 1 \pm \sqrt{1 - D}$. Let $v_+, v_- \in \mathbb{R}^2$ be unitary eigenvectors of $A$ associated to $\lambda_+, \lambda_-$, and let $V_+, V_-$ be the horizontal right invariant vector fields in $X$ determined by $V_+(0) = v_+$ and $V_-(0) = v_-$. Then, $\|V_\pm\|(x,y,z) = e^{-\lambda_\pm z}$ for all $(x,y,z) \in X$.

Proof. Since $\text{trace}(A) = 2$ and $\text{det}(A) = D$, the characteristic equation for $A$ gives $A^2 - 2A + DI_2 = 0$. From here it is straightforward to show that if we define $f: \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$ by $f(z) = e^z [C_D(z)I_2 + S_D(z)(A - I_2)]$, then $f'(z) = Af(z)$ and $f(0) = I_2$ (for this, use that $S_D' = C_D$ and that $C_D' = (1 - D)S_D$), which gives item (1) of the proposition.

The three displayed equalities in item (2) of the proposition are also direct computations that only use (6.1) and the expression (2.7) of the canonical metric in terms of $x,y,z$. If $D > 0$, then (6.2) and the three displayed equalities in item (2) imply that $\|\partial_x\|^2, |\partial_y\|^2, (\partial_x, \partial_y)$ decay exponentially as $z \to +\infty$. If $D < 1$, then the characteristic equation of $A$ has two distinct real roots $\lambda_\pm = 1 \pm \sqrt{1 - D}$, which implies that $A$ is diagonalizable. After a fixed rotation in the $(x,y)$-plane around the origin (this change of coordinates does not affect either the Lie group structure in $\mathbb{R}^2 \times_A \mathbb{R}$ or its canonical metric), we can assume that $V_+ = \partial_x$, i.e.,

$$A = \begin{pmatrix} \lambda_+ & b \\ 0 & \lambda_- \end{pmatrix}$$

and thus $e^{zA} = \begin{pmatrix} e^{\lambda_+ z} & a_{12}(z) \\ 0 & e^{\lambda_- z} \end{pmatrix}$ for certain $b \in \mathbb{R}$ and $a_{12}(z)$ function of $z$. Thus (2.7) directly gives that $\|V_\pm\|(x,y,z) = e^{-\lambda_\pm z}$. The proof of $\|V_-\|(x,y,z) = e^{-\lambda_- z}$ is analogous. □

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