Supersymmetric Quantization of Gauge Theories

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Abstract

We develop a new operator quantization scheme for gauge theories in which the dynamics of the ghost sector is described by a $N = 2$ supersymmetry. In this scheme no gauge condition is imposed on the gauge fields. The corresponding path integral is explicitly Lorentz invariant and, in contrast to the BRST-BFV path integral in the Lorentz gauge, it is free of the Gribov ambiguity, i.e., it is also valid in the non-perturbative domain. The formalism can therefore be used to study the non-perturbative properties of gauge theories in the infra-red region (gluon confinement).

1 Introduction.

Systems with first class constraints, and in particular relativistic gauge theories, posses unphysical degrees of freedom which have to be eliminated in the quantization process \cite{1}. It is, however, important to note that in general the process of gauge fixing and quantization does not commute as was exemplified in \cite{2}, and that the unphysical degrees of freedom have to be eliminated after quantization. The conventional procedure of the path integral quantization of gauge theories \cite{3} relies on the elimination of unphysical (gauge) degrees of freedom on the classical level via gauge fixing and, therefore, should be modified \cite{4, 5}.

In relativistic gauge theories the situation is slightly more complicated since one wants to build a Lorentz invariant theory. To this end one has to use Lorentz invariant gauges, e.g., the Lorentz gauge. This implies, however, that unphysical

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bosonic degrees of freedom are dynamically activated and have to be suppressed. A general procedure for doing this, based on the BRST symmetry, was given in [6].

A problem common to all quantization procedures based on imposing a gauge condition on the gauge fields, is the Gribov ambiguity [7]. This states that it is impossible (assuming certain boundary conditions at spatial infinity) to find a gauge condition that would eliminate all gauge equivalent configurations [8, 9]. This implies that even after gauge fixing there is still a set of residual gauge transformations under which the action will be invariant. Although these residual transformations do not lead to a further reduction in the number of physical degrees of freedom, they are important in determining the topological structure of physical configuration (phase) space and therefore they play a crucial role in the quantization process and the corresponding functional representation [5]. For normal Lorentz invariant gauges, such as the Lorentz gauge, this set of residual transformations is extremely complex [9] and the task of incorporating them in the quantization process seems hopeless.

There is good reason to believe that the Gribov ambiguity is irrelevant in the perturbative (asymptotic free) domain [7]. However, there is evidence that it plays an important role in the infra-red, non-perturbative behavior of the gluon propagator [7, 10, 11].

From the previous introductory remarks it should be clear that one needs to develop a quantization procedure which eliminates the unphysical degrees of freedom while it (a) avoids imposing a gauge condition on the gauge fields, i.e., the Gribov ambiguity and (b) leads to a Lorentz covariant theory. The basic ingredients for such a scheme were given in [12]. The idea of [12] is to extend the theory in a supersymmetric and gauge invariant way by introducing bosonic and fermionic ghosts transforming in a gauge multiplet. The supersymmetry insures that the contributions of the boson and fermion ghosts cancel in the partition function, while the gauge invariance makes it possible to avoid imposing a gauge condition on the gauge fields, but rather to eliminate the unphysical degrees of freedom by imposing a gauge condition on the bosonic ghost fields. Finally the whole procedure respects the Lorentz invariance of the theory. This program was, however, performed within the functional integral setting and, given the ambiguities that arise in the functional integral formalism, it is highly desirable to develop this program on the operator level and derive the corresponding functional integral representation from there.

Our aim with the present paper is to develop this operator quantization scheme. The steps we follow in doing this are essentially the same as outlined above. We show how any quantum mechanical system (regardless whether it has a gauge symmetry or not) can be extended by adding $N = 2$ supersymmetric ghosts. In the extended theory physical states are identified as those invariant under SUSY transformations. Matrix elements of any system operator calculated in the physical subspace coincide with those of the original system.

In the case of gauge theories this extension is done by putting the ghosts in a gauge multiplet. This implies (a) that we not only modify the Hamiltonian, but also the constraints and (b) that the extension respects the gauge symmetry, i.e,
commutators of the extended Hamiltonian with the extended constraints and the extended constraints with each other vanish weakly. The presence of scalar ghosts is then exploited to remove the gauge arbitrariness by imposing gauge fixing on them.

We organize the paper as follows: In section 2 we consider a 1-dimensional quantum system to illustrate the supersymmetric extension and to construct the functional integral representation of the system transition amplitude in the extended space. In section 3 a simple mechanical model with a gauge symmetry is considered to show how the ghosts can be added to the theory and how gauge fixing of the variables describing the original gauge system can be avoided. We emphasize that the choice of simple mechanical models to illustrate the procedure is only for the convenience of presentation since the generalization is straightforward. In section 4 the scheme is applied to Yang-Mills theories. The implementation of the scheme to evaluate Green’s functions is given in section 5. Section 6 discusses the relation to normal gauge fixing and section 7 contains our conclusions.

2 Ghost extension of a quantum system.

Consider a 1-dimensional quantum system with Hamiltonian

\[ \hat{H}_s = \frac{\hat{p}^2}{2} + V(\hat{x}) \, , \quad [\hat{x}, \hat{p}] = i \, . \] (2.1)

We denote by \( |s\rangle \) (or \( |\psi\rangle_s \)) vectors in the system Hilbert space.

Consider the Hamiltonian

\[ \hat{H}_{gh} = \hat{p}_\eta^\dagger \hat{p}_\eta + \hat{p}_z^\dagger \hat{p}_z + \omega^2(x)(\hat{z}^\dagger \hat{z} + \eta^\dagger \eta) \, . \] (2.2)

Here \( (\eta^\dagger)^2 = \eta^2 = (\hat{p}_\eta^\dagger)^2 = \hat{p}_\eta^2 = 0 \), i.e., they are Grassmann canonical operators, while \( \hat{z}, \hat{p}_z \) and their adjoints have boson statistics:

\[ [\hat{z}, \hat{p}_z^\dagger] = [\hat{z}^\dagger, \hat{p}_z] = i \, , \quad [\eta, \hat{p}_\eta^\dagger] = -[\eta^\dagger, \hat{p}_\eta] = i \, . \] (2.3)

Here \( [\, , \,] \) is the supercommutator \( [A, B] = AB - (-1)^{\epsilon_A\epsilon_B} BA \) where \( \epsilon_{A,B} \) are Grassmann parities of \( A \) and \( B \), i.e., \( \epsilon = 0 \) for bosons and even elements of the Grassmann algebra and \( \epsilon = 1 \) otherwise. We shall call these variables ghosts.

We denote by \( |gh\rangle^x \) (or \( |\psi\rangle_{gh}^x \)) vectors in the (indefinite) Hilbert space of system (2.2). Note that \( \hat{H}_{gh} \) depends on a real parameter \( x \), therefore the index \( x \) on the states.

We construct a coordinate (Schrödinger) representation of the algebra (2.3) on the space of functions \( \psi_{gh}(\kappa) = \langle \kappa | \psi \rangle_{gh}^x = \langle z, z^*, \eta, \eta^* | \psi \rangle_{gh}^x \) with the inner product

\[ x_{gh} \langle \phi | \psi \rangle_{gh}^x = \int d\kappa \, \phi_{gh}^*(\kappa) \psi_{gh}(\kappa) \, , \] (2.4)
where $d\kappa = dz^* dz d\eta^* d\eta$. It is easily verified that
\begin{equation}
\hat{\eta} = \eta, \quad \hat{\eta}^\dagger = \eta^*, \quad \hat{p}_\eta = -i \frac{\partial}{\partial \eta^*}, \quad \hat{p}_\eta^\dagger = i \frac{\partial}{\partial \eta}, \quad (2.5)
\end{equation}
in this representation. Note that the dagger denotes the adjoint with respect to the inner product (2.4) so that the Hamiltonian (2.2) is self-adjoint. The inner product (2.4) is indefinite and the space contains zero and negative norm states. We remark that negative norm states always occur when quantizing Grassmann Lagrangians bilinear in generalized velocities, which is the case in our fermionic ghost sector (see (2.28)) as well as in the conventional BRST-BFV formalism.

The ghost Hilbert space can also be constructed as a Fock space. Define annihilation operators
\begin{align}
\hat{b}_1 &= (\omega \hat{z} + i \hat{p}_z)/\sqrt{2\omega}, & \hat{b}_2 &= (\omega \hat{z}^\dagger + i \hat{p}_z^\dagger)/\sqrt{2\omega}, \\
\hat{c}_1 &= (\omega \hat{\eta} + i \hat{p}_\eta)/\sqrt{2\omega}, & \hat{c}_2 &= (\omega \hat{\eta}^\dagger + i \hat{p}_\eta^\dagger)/\sqrt{2\omega},
\end{align}
and creation operators through their adjoints. From (2.3) follows
\begin{equation}
[\hat{b}_1, \hat{\bar{b}}_1] = [\hat{b}_2, \hat{\bar{b}}_2] = [\hat{c}_1, \hat{\bar{c}}_1] = 1, \quad [\hat{c}_2, \hat{\bar{c}}_2] = -1. \quad (2.9)
\end{equation}

Defining the vacuum, $|0\rangle^x_{gh}$, as the state annihilated by all annihilation operators and with $x^x_{gh} \langle 0 | 0 \rangle^x_{gh} = 1$, one notes from (2.4) that states containing $\hat{c}_2^\dagger$ have negative norm. In Fock space the ghost Hamiltonian (2.2) is given by
\begin{equation}
\hat{H}_{gh} = \omega \hat{N}_{gh} = \omega (\hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2 + \hat{c}_1^\dagger \hat{c}_1 - \hat{c}_2^\dagger \hat{c}_2), \quad (2.10)
\end{equation}
where $\hat{N}_{gh}$ is the ghost number operator; it is non-negative, $\hat{N}_{gh} \geq 0$.

Define the fermion operators
\begin{equation}
\hat{Q} = \hat{c}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{c}_2, \quad \hat{R} = \hat{c}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{c}_2
\end{equation}
and their adjoints $\hat{Q}^\dagger$ and $\hat{R}^\dagger$. We have
\begin{equation}
\hat{N}_{gh} = [\hat{Q}, \hat{R}^\dagger] = [\hat{Q}^\dagger, \hat{R}]
\end{equation}
and
\begin{equation}
[\hat{N}_{gh}, \hat{Q}] = [\hat{N}_{gh}, \hat{Q}^\dagger] = [\hat{N}_{gh}, \hat{R}] = [\hat{N}_{gh}, \hat{R}^\dagger] = 0.
\end{equation}
The ghost Hamiltonian exhibits an $N = 2$ supersymmetry generated by $\hat{Q}, \hat{R}$ and their adjoints.

Introduce a system with the extended Hamiltonian
\begin{equation}
\hat{H} = \hat{H}_s + \hat{H}_{gh},
\end{equation}

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where $\hat{H}_{gh}$ was defined in (2.2) and $\omega^2 = \omega^2(\hat{x})$. Therefore states in the ghost Fock space depend on the operator $\hat{x}$ and accordingly act as operators (functions of $\hat{x}$) on the system Hilbert space. This motivates the following definition of the extended space:

$$|\psi\rangle = |gh\rangle^x \cdot |s\rangle,$$

where the dot stands to emphasize that $|gh\rangle^x$ is an operator acting on the system state $|s\rangle$.

In the coordinate representation $|\kappa\rangle \cdot |x\rangle$, $\hat{\kappa}|\kappa\rangle = \kappa|\kappa\rangle$ we have

$$\langle x| \cdot \langle \kappa|\psi\rangle = \langle x| \cdot \langle \kappa|gh\rangle^x \cdot |s\rangle = \langle \kappa|gh\rangle^x \langle x|s\rangle,$$

since $\hat{x}$ is diagonal.

We observe that

$$[\hat{H}, \hat{Q}] = [\hat{H}, \hat{Q}^\dagger] = 0 \quad (2.17)$$

since $[\hat{x}, \hat{Q}] = [\hat{p}, \hat{Q}] = 0$, but $[\hat{H}, \hat{R}] \neq 0$ as $[\hat{p}, \hat{R}] \neq 0$ (while $[\hat{x}, \hat{R}] = 0$). The $N = 2$ supersymmetry is therefore explicitly broken to $N = 1$. In the Appendix we discuss an $N = 2$ supersymmetric extension of the system dynamics.

Consider the subspace in the extended space defined by the conditions

$$\hat{Q}|ph\rangle = \hat{Q}^\dagger|ph\rangle = 0 \quad (2.18)$$

In the Fock space representation of the ghost Hilbert space, one can show that the solutions of (2.18) are

$$|ph\rangle = \lambda|0\rangle^x_{gh} \cdot |s\rangle + \sum_{n=1}^{\infty} |\phi_n^x_{gh} \cdot |s_n\rangle \quad (2.19)$$

Here $|s\rangle$ and $|s_n\rangle$ are system states and

$$|\phi_n^x_{gh} = \hat{\Gamma}^n|0\rangle^x_{gh} = \hat{Q}\hat{\Phi}\hat{\Gamma}^{n-1}|0\rangle^x_{gh} = \hat{Q}^\dagger\hat{\Phi}'\hat{\Gamma}^{n-1}|0\rangle^x_{gh} \quad (2.20)$$

with

$$\hat{\Gamma} = \hat{b}_1^e\hat{b}_2^e + \hat{c}_1^e\hat{c}_2^e, \quad \hat{\Phi} = \hat{b}_1^e\hat{c}_2^e, \quad \hat{\Phi}' = \hat{b}_2^e\hat{c}_1^e \quad (2.21)$$

Note that $\langle \langle ph|ph\rangle = 0$ if $\lambda = 0$ in (2.19) since vectors (2.20) have, by definition (2.18), zero norms.

We remark that the choice of Fock space to discuss the properties of the states (2.18) is only a matter of convenience. Indeed, it is easy to prove that these properties hold generally, i.e., a state satisfying (2.18) in an arbitrary Hilbert space is the sum of a closed state (belonging to $\text{ker}(\hat{Q}) / \text{im}(\hat{Q}) \cap \text{ker}((\hat{Q})^\dagger) / \text{im}((\hat{Q})^\dagger)$) and an exact state (expressible as $\hat{Q}$ or $\hat{Q}^\dagger$ on some state). By definition (2.18) the latter have zero norm. In Fock space the ghost vacuum is the only closed state.

For any operator $\hat{O}$ commuting with $\hat{Q}$ and $\hat{Q}^\dagger$, it follows from (2.20), (2.18) and (2.19) that

$$\langle \langle ph'|\hat{O}|ph\rangle = \langle s'| \cdot (0_{gh} |\hat{O}|0_{gh}\rangle \cdot |s\rangle \quad (2.22)$$
if $\lambda = \exp(i\alpha)$ and $\alpha$ is independent of $x$ (Even in the case where $\alpha$ is $x$-dependent this relation still holds, provided that $\hat{p}$ is replaced by $\hat{p} - d\alpha/dx$). For $\hat{O} = 1$ this yields $\langle\langle ph'| ph\rangle\rangle = \langle s'| s\rangle$.

If $\hat{O} = \hat{O}_s = p(\dot{x},\hat{p})$ equation (2.22) becomes

$$\langle\langle ph'| \hat{O}_s| ph\rangle\rangle = \langle s'| \hat{O}_s| s\rangle.$$  \hspace{1cm} (2.23)

To prove (2.23) we note that since $\hat{x}$ commutes with $|0\rangle_{gh}$, it acts directly on the system state. We only have to prove that the same holds for $\hat{p}$. From the relations (2.8), (2.9) one easily verifies that $\hat{p}$ is diagonal in $|0\rangle_{gh}$, where $\beta = -i\omega/(2\omega)$, and similar relations for the other creation and annihilation operators. From the definition of the physical vacuum we have $[\hat{p}, |0\rangle_{gh}] = 0$ and analogous relations for the other annihilation operators. As $\hat{p}$ commutes with $Q$ and $Q\dagger$, these relations imply that the physical vacuum $\langle\langle ph| |0\rangle_{gh}\rangle = 0$ and analogues for the other annihilation operators. Thus for $\hat{O}_s = \hat{p}^n$ we deduce

$$\langle\langle ph'| \hat{O}_s| ph\rangle\rangle = \langle s'| \hat{O}_s| s\rangle.$$  \hspace{1cm} (2.24)

Relations (2.23) as well as (2.10), (2.12) and (2.14) imply that $\langle\langle ph'| \hat{H}| ph\rangle\rangle = \langle\langle ph'| \hat{H}_s| ph\rangle\rangle = \langle s'| \hat{H}_s| s\rangle$. Since both $\hat{H}$ and $\hat{H}_s$ commute with $Q$ and $Q\dagger$, their application to a physical state leads to a physical state and a simple inductive argument then shows that

$$\langle\langle ph'| \hat{H}^n| ph\rangle\rangle = \langle\langle ph'| \hat{H}_s^n| ph\rangle\rangle = \langle s'| \hat{H}_s^n| s\rangle$$

where (2.23) was used in the latter equality. We conclude

$$\langle\langle ph'| e^{-it\hat{H}}| ph\rangle\rangle = \langle s'| e^{-it\hat{H}_s}| s\rangle.$$  \hspace{1cm} (2.25)

As an application of (2.23) we take $|s\rangle = |x\rangle$ and $|s'\rangle = |x'\rangle$. This yields

$$\langle\langle ph'| e^{-it\hat{H}}| ph\rangle\rangle = \int [dx] e^{iS_s(x)},$$  \hspace{1cm} (2.26)

where $S_s(x) = \int_0^t dt' (\dot{x}^2/2 - V(x))$ is the system action. The trajectories contributing in the functional integral obey the boundary conditions $x(0) = x$ and $x(t) = x'$.

The amplitude in (2.26) can be represented as a functional integral over the extended configuration space. Consider the functional integral representation of the transition amplitude in the extended configuration space

$$\langle\langle x'| e^{-it\hat{H}}| x, \kappa\rangle\rangle = \int [dx][dz^*][dz][d\eta^*][d\eta] e^{iS_e},$$  \hspace{1cm} (2.27)

where $|x, \kappa\rangle = |\kappa\rangle \cdot |x\rangle$ and

$$S_e = S_s + \int_0^t dt' (\dot{z}^2 + \dot{\eta}^2 - \omega^2(x)(z^2 + \eta^2)) \equiv S_s + S_{gh}. \hspace{1cm} (2.28)$$
The trajectories for the ghost variables obey the boundary conditions \( \kappa(0) = \kappa \) and \( \kappa(t) = \kappa' \).

The functional integral (2.27) coincides with (2.26) if we calculate its convolution with any two physical states (2.18). There is a particular choice of boundary conditions for the ghost fields in (2.27) such that (2.27) coincides with (2.26). This is achieved by choosing \( \kappa' = \kappa = 0 \). This implies that we choose in (2.26) \( |\phi\rangle = |0\rangle \cdot |x\rangle \) and \( |\phi'\rangle = |0\rangle \cdot |x'\rangle \) where \( \kappa(0) = 0 \) (not to be confused with the ghost vacuum). To verify that these states satisfy (2.18), one expresses the operators \( \hat{Q} \) and \( \hat{Q}^\dagger \) through relations (2.8), (2.9) in terms of the ghost canonical variables. Using the coordinate representation (2.5), (2.6) we obtain

\[
\langle \kappa | \hat{Q} | 0 \rangle = \langle \kappa | \hat{Q}^\dagger | 0 \rangle = 0 \quad (2.29)
\]

since \( \langle \kappa | 0 \rangle = \delta(\kappa) \). Note that (2.27) is a Gaussian integral for the ghost variables. Therefore

\[
\int [d\kappa] e^{iS_{gh}} = F(x, \kappa', \kappa) \Delta_F / \Delta_B = F(x, \kappa', \kappa) \quad (2.30)
\]

where \( \Delta_F = \Delta_B = \det(-\partial^2 - \omega^2) \) and \( F(x, \kappa', \kappa) = \exp(iS_{gh}(\kappa_{cl})) \) with \( \kappa_{cl} \) a classical solution obeying the boundary conditions \( \kappa(0) = \kappa \) and \( \kappa(t) = \kappa' \).

Relation (2.26) implies

\[
\int d\kappa d\kappa' \langle \kappa' | \hat{Q} \rangle \langle \kappa | \hat{Q}^\dagger \rangle = 1 \quad (2.31)
\]

For vanishing ghost boundary conditions, \( \langle \kappa | \hat{Q} \rangle = \delta(\kappa) \) and \( F[x, 0, 0] = 1 \).

3 A gauge model.

In this section we illustrate, by means of a simple model with a SU(2) gauge symmetry, how the method of the previous section can be used to construct a functional integral for a gauge theory without fixing a gauge.

Let the canonical variables \( \hat{p} \) and \( \hat{x} \) be elements of a linear unitary representation of SU(2). We choose the system Hamiltonian as in (2.1), but now \( \hat{p}^2 = (\hat{p}, \hat{p}) \) where \( (\ , \ ) \) is the invariant inner product. We denote by \( \lambda^a \) the SU(2) generators in the chosen representation. The Hamiltonian (2.1) describes a gauge theory if the physical states are gauge invariant \([1]\), i.e.,

\[
\hat{\sigma}^a |\phi\rangle_s = i\hat{x} \lambda^a \hat{p} |\phi\rangle_s \equiv i(\hat{x}, \lambda^a \hat{p}) |\phi\rangle_s = 0
\]

(3.1)

and \( [\hat{H}_s, \hat{\sigma}^a] = 0 \). The canonical commutation relation \( [\hat{x}, \hat{p}] = i \) implies \( [\hat{\sigma}^a, \hat{\sigma}^b] = 2i\epsilon^{abc} \hat{\sigma}^c \) so that the constraints generate SU(2). The Hamiltonian (2.1) is invariant under gauge transformations generated by \( \hat{\sigma}^a, \hat{\rho} \rightarrow T_g \hat{\rho}, \hat{x} \rightarrow T_g \hat{x} \) and \( T_g T_g^\dagger = T_g^\dagger T_g = 1 \).
As described in section 2, we extend the system Hamiltonian (2.1) by adding a ghost Hamiltonian $\hat{H} = \hat{H}_s + \hat{H}_{gh}$, where

$$\hat{H}_{gh} = (\hat{p}_z^\dagger, \hat{p}_z) + (\hat{p}_\eta^\dagger, \hat{p}_\eta) + (\hat{z}^\dagger, \Omega(\hat{x})\hat{z}) + (\hat{\eta}^\dagger, \Omega(\hat{x})\hat{\eta})$$

(3.2)

and $\Omega$ is a hermitian, strictly positive matrix. For simplicity we assume the ghost variables to realize the fundamental representation of $SU(2)$, i.e., they are complex isospinors.

To make (3.2) invariant under the $SU(2)$ transformations $\hat{z} \to \hat{T}_g \hat{z}$, $\hat{\eta} \to \hat{T}_g \hat{\eta}$, with $\hat{T}_g$ a $2 \times 2$ unitary matrix in the fundamental representation of $SU(2)$, we require the following transformation rule for $\Omega$: $\Omega(T_g \hat{x}) = \hat{T}_g \Omega(\hat{x}) \hat{T}_g^\dagger$, so that $\Omega$ transforms in the adjoint representation.

Introducing the fermionic operators

$$\hat{Q} = i[(\hat{\eta}^\dagger, \hat{p}_\eta) - (\hat{p}_\eta^\dagger, \hat{\eta})]$$

and their adjoints $\hat{Q}^\dagger$, $\hat{R}^\dagger(\hat{x})$ we find

$$\hat{H}_{gh} = [\hat{Q}, \hat{R}^\dagger(\hat{x})] = [\hat{Q}^\dagger, \hat{R}(\hat{x})].$$

(3.4)

Note that the operators $\hat{Q}$, $\hat{Q}^\dagger$ commute with the total Hamiltonian, while $\hat{R}(\hat{x})$ and $\hat{R}^\dagger(\hat{x})$ commute only with the ghost Hamiltonian.

As before the Hilbert space of the ghost sector can be build as a Fock space. Indeed, if $\omega_i^2(\hat{x})$ are the eigenvalues of $\Omega(\hat{x})$, then $\hat{Q} = \sum_i \hat{Q}_i$, and $\hat{R}(\hat{x}) = \sum_i \omega_i(\hat{x}) \hat{R}_i$ with $\hat{Q}_i$, and $\hat{R}_i$ determined by (2.11) for each SUSY oscillator labelled by $i = 1, 2$.

The constraints are also extended $\hat{\sigma}^a = \hat{\sigma}^a_s + \hat{\sigma}^a_{gh}$ with

$$\hat{\sigma}^a_{gh} = i\hat{z}^\dagger \tau^a \hat{p}_z - i\hat{p}_z^\dagger \tau^a \hat{z} + i\hat{\eta}^\dagger \tau^a \hat{p}_\eta - i\hat{p}_\eta^\dagger \tau^a \hat{\eta} = [\hat{Q}, i\hat{z}^\dagger \tau^a \hat{p}_z - i\hat{p}_z^\dagger \tau^a \hat{z} - i\hat{\eta}^\dagger \tau^a \hat{p}_\eta + i\hat{p}_\eta^\dagger \tau^a \hat{\eta}].$$

(3.5)

Here $\tau^a$ are the Pauli matrices. By construction $[\hat{\sigma}^a, \hat{H}] = 0$, $[\hat{\sigma}^a, \hat{Q}] = [\hat{\sigma}^a, \hat{Q}^\dagger] = 0$ and $\hat{\sigma}^a$ generate $SU(2)$.

We prove the relation

$$\langle \langle ph'|e^{-it\hat{H}}|ph\rangle \rangle_s = \langle \langle ph'|e^{-it\hat{H}_s}|ph\rangle \rangle_s,$$

(3.6)

where the physical states $|ph\rangle$ and $|ph\rangle'$ in the extended space have non-zero norm and satisfy the conditions

$$\hat{Q}|ph\rangle = \hat{Q}^\dagger|ph\rangle = \hat{\sigma}^a|ph\rangle = 0,$$

(3.7)

while the system states on the right of (3.6) obey (3.1). Relation (3.6) establishes a one-to-one correspondence between physical transition matrix elements of the original gauge theory and its supersymmetric extension.

From (2.23) we note that it is sufficient to prove that (3.7) imply (3.1). Consider the second order Casimir operator $\hat{C} = \hat{\sigma}^a \hat{\sigma}^a$. From (3.3) it is easily established
that $\hat{C} = \hat{C}_s + [\hat{Q}, \hat{\Lambda}]$ where $\hat{C}_s$ is the second order Casimir of the system, and the explicit form of $\hat{\Lambda}$ can be read of from (3.3). It follows
\[
\langle \langle \rho | \hat{C} | \rho \rangle \rangle = \langle \langle \rho | \hat{C}_s | \rho \rangle \rangle = \langle s | \hat{C}_s | s \rangle , \tag{3.8}
\]
where (2.23) was used. Since the system Hilbert space has a strictly positive norm, (3.7) implies (3.1). Thus relation (3.6) is proved.

Next we turn to the functional integral representation of (3.6). To avoid divergences in the measure of the functional integral caused by the gauge invariance, we have to solve the constraints $\hat{\sigma}^a | \rho \rangle = 0$ explicitly by choosing a parameterization (coordinates) in the physical configuration space, i.e., we should fix a gauge in the extended theory. In contrast with the BRST-BFV scheme we have boson ghosts so that we can impose the gauge condition on them, while the system degrees of freedom remain free of any gauge condition.

To fulfil this program, we choose the coordinate representation (2.16)
\[
\langle \langle x | \cdot | \rho \rangle \rangle = \psi_{\rho}^p(x, \kappa),
\]
where all states are functions of $x$ and $\kappa$. To project the total Hamiltonian on the subspace of gauge invariant functions $\psi_{\rho}^p(x, \kappa)$, we change variables in the superspace
\[
z = T_g \chi \rho / \sqrt{2}, \quad x \rightarrow T_g x, \quad \eta \rightarrow \tilde{T}_g \eta , \tag{3.9}
\]
with $\chi^\dagger = (1, 0)$, $\rho$ is real and $T_g$ is the group element $\tilde{T}_g$ in the representation of $x$. Denoting the group parameters by $\theta^a$, the constraint operators in the new coordinates assume the form $\hat{\sigma}^a \sim \partial / \partial \theta^a \[4\]$. Therefore physical states (gauge invariant states) are functions independent of $\theta^a$. We transform $\hat{H}$ to the curvilinear supercoordinates (3.9) and drop all terms containing $\partial / \partial \theta^a$ to obtain the physical Hamiltonian
\[
\hat{H}_{\rho}^p = \hat{H}_s + \frac{p_\rho^2}{2} + \frac{3}{8 \rho^2} + \hat{p}_\eta \hat{p}_\eta + \frac{1}{2 \rho^2} (\hat{\sigma}^a)^2 + \frac{\rho^2}{2} \chi^\dagger \Omega \chi + \eta^\dagger \Omega \eta , \tag{3.10}
\]
where $\hat{\sigma}^a = \hat{\sigma}^a_s + \hat{\sigma}^a_f$ with $\hat{\sigma}^a_f = i \eta^\dagger \tau^a \hat{p}_\eta - i \hat{p}_\eta^\dagger \tau^a \eta$ and $\hat{p}_\rho = -i \rho^{-3/2} \partial_\rho \circ \rho^{3/2}$ is the hermitian momentum conjugated to $\rho$. The third term in (3.10) is due to the operator ordering. Restoring Planck’s constant, this would be proportional to $\hbar^2$.

The above procedure to obtain the physical Hamiltonian also applies to the operators $\hat{Q}$ and $\hat{Q}^\dagger$ to derive the supersymmetry generators on the space of functions invariant under the gauge transformations generated by $\hat{\sigma}^a$. We find
\[
\hat{Q}_\rho = \frac{1}{\sqrt{2}} \eta^{\dagger} \chi \partial_\rho - \frac{i \rho}{\sqrt{2}} \hat{p}_\eta \chi - \frac{1}{\sqrt{2} \rho} \left[ \eta^\dagger \chi \hat{\sigma}^3 + \eta^\dagger \psi (\hat{\sigma}^2 - i \hat{\sigma}^1) \right] ,
\]
\[
\hat{Q}^\dagger_\rho = -\frac{1}{\sqrt{2}} \chi^\dagger \eta \partial_\rho + \frac{i \rho}{\sqrt{2}} \chi^\dagger \hat{p}_\eta - \frac{1}{\sqrt{2} \rho} \left[ \chi^\dagger \eta \hat{\sigma}^3 + \psi^\dagger \eta (\hat{\sigma}^2 + i \hat{\sigma}^1) \right] \tag{3.11}
\]
with $\psi^\dagger = (0, 1)$. Taking into account that $\eta$ and $\eta^\dagger$ do not commute with $\hat{\sigma}^a$, one can verify that these two operators are adjoints with respect to the measure $\rho^3 d \rho$. 


Indeed, the term proportional to $1/\rho$, which appears when the adjoint of $\partial_\rho$ is taken with respect to this measure, precisely cancels against a similar term originating from the reordering of $\eta$ or $\eta^\dagger$ and $\hat{\sigma}^a$ in the adjoint of the third term. It can also be checked explicitly that these operators commute with the physical Hamiltonian (3.10) as they should.

The physical states of the system are obtained as before by imposing

$$\hat{Q}_\rho |\text{ph}\rangle = \hat{Q}^\dagger_\rho |\text{ph}\rangle = 0. \quad (3.12)$$

We note that the system variables $\hat{x}$ and $\hat{p}$ do not commute with the supersymmetry generators $\hat{Q}_\rho$ and $\hat{Q}^\dagger_{\rho}$. Indeed, the operator $\xi^\dagger \hat{Q}_\rho + \hat{Q}^\dagger_\rho \xi$, with $\xi$ and $\xi^\dagger$ Grassmann variables, involves a linear combination of the system constraint operators $\sigma^a_{\rho}$. Therefore it generates an isotopic rotation of $\hat{x}$ and $\hat{p}$ of the same form as an infinitesimal gauge transformation, but with the parameters being even elements of a Grassmann algebra. Therefore the condition (3.12) ensures gauge invariance in the vacuum sector of the ghosts, i.e., a vector describing any system excitation will automatically satisfy the Dirac condition (3.1).

Note how the reduction in the degrees of freedom is obtained: we start with $n$ system degrees of freedom, $n$ being the real dimension of the representation in which the $\hat{x}$ transforms. Then we add 4 real bosonic and 4 real fermionic ghost degrees of freedom in a supersymmetric combination. Since the system variables realize a trivial representation of the supersymmetry an exact cancellation of the ghost degrees of freedom is ensured, leaving only the $n$ system degrees of freedom. Next we eliminate three real bosonic ghost degrees of freedom by projecting on the subspace of gauge invariant functions (see (3.9)). The projection (3.9) mixes the system and ghost degrees of freedom in a non-linear way. In particular the angular variables of the bosonic ghosts are absorbed in the new system variables so that three angular variables of $x$ are the supersymmetric partners of three fermionic ghost degrees of freedom. The condition (3.12) ensures that the 4 fermionic ghost degrees of freedom are balanced by the $\rho$ ghost degree of freedom and 3 system degrees of freedom. Hence, we are left with $n - 3$ degrees of freedom, which is correct as we have 3 independent constraints (3.1) in the original model.

In our arguments above we have assumed that the system constraints $\hat{\sigma}^a_{\rho}$ are irreducible, i.e., they are all independent, which would not be the case if, for example, we take $x$ to be in the adjoint representation. Nonetheless, in our approach the supersymmetry condition (3.12) will still provide the elimination of the right number of unphysical degrees of freedom in the system sector, provided the ghost sector is chosen so that the constraints $\hat{\sigma}^a_{\text{gh}}$ (3.5) are irreducible. The latter is always possible so that, in contrast to the conventional BRST-BFV formalism, we do not need ghosts for ghosts if the system exhibits reducible constraints.

Returning to the functional integral representation, let $\kappa_\rho$ denote a set $(\rho, \eta^\dagger, \eta)$. 

10
Using the formalism of [4] we find
\[ \langle x', \kappa'_\rho | e^{-it\hat{H}_{ph}} | x, \kappa_\rho \rangle = \int_{-\infty}^{\infty} d\kappa'_\rho d\kappa''_\rho (\rho^3 \rho'^3)^{-1/2} U^\text{eff}_t (x', \kappa'_\rho, x'', \kappa''_\rho) Q_S (x'', \kappa''_\rho, x, \kappa_\rho) ; \tag{3.13} \]
the amplitude \( U^\text{eff}_t \) is given by a functional integral for the extended theory in the unitary gauge \( z = \chi \rho \):
\[ U^\text{eff}_t (x, \kappa_\rho, x'', \kappa''_\rho) = \int [dp][dx][d\kappa_\rho][dp_{\kappa_\rho}] e^{iS^{ph}_H} \tag{3.14} \]
\[ = \int [dx][dy][d\kappa_\rho][\Pi_{t=0}]|\rho|^3 e^{iS^{ph}}. \tag{3.15} \]
Here
\[ S^{ph}_H = \int_0^t dt' \left[ \frac{1}{2} (D_t x)^2 + (\hat{D}_t \eta)^2 + \frac{1}{2} \rho^2 + \frac{1}{2} \rho^2 |y\chi|^2 \right. \]
\[ -V(x) - \frac{1}{2} \rho^2 \chi^2 \Omega \chi - \eta^2 \Omega \eta - \frac{3}{8} \rho^2 \right] . \tag{3.16} \]
and
\[ S^{ph} = \int_0^t dt' \left[ \frac{1}{2} (D_t x)^2 + (\hat{D}_t \eta)^2 + \frac{1}{2} \rho^2 + \frac{1}{2} \rho^2 |y\chi|^2 \right. \]
\[ -V(x) - \frac{1}{2} \rho^2 \chi^2 \Omega \chi - \eta^2 \Omega \eta - \frac{3}{8} \rho^2 \right] . \tag{3.17} \]

We denote by \( D_t = \partial_t + iy^a \lambda^a \) and \( \hat{D}_t \) the covariant derivatives in the representation of \( x \) and the fundamental representation \( (\lambda_a \rightarrow \tau_a) \), respectively.

To go from (3.14) to (3.13) we first replace the measure \([d\kappa_\rho][dp_{\kappa_\rho}]\) by the Faddeev-Popov measure in the unitary gauge \([d\kappa][dp_\kappa]\Pi_{t=0} \Delta_{FP} \delta (z - \chi \rho) \delta (\sigma^a)\). The Faddeev-Popov determinant for the unitary gauge \( z = \rho \chi \) is \( \Delta_{FP} \sim |\rho|^3 \). Correspondingly \( S^{ph}_H \) in (3.14) is replaced by the Hamiltonian action for the total extended Hamiltonian. Next the identity \( f [dy^a] \exp (iy^a \sigma^a) = \Pi_t \delta (\sigma^a) \) is used and the integrals over all momenta are done. Finally the integral over \( z \) and \( z^\dagger \) are done using the delta function and (3.15) is obtained. Note that apart from the operator ordering term \( (\sim \hbar^2 / \rho^2) \) in (3.17) all the other terms depending on \( \rho \) can be written as \( |\hat{D}_t z|^2 - z^\dagger \Omega z \) where \( z = \rho \chi \). Thus (3.17) is the action of the extended supersymmetric gauge theory in the unitary gauge imposed on the boson ghost \( z \).

The kernel \( Q_S \) in (3.13) takes care of the global properties of the change of variables (3.2), 4, 13
\[ Q_S (x'', \kappa''_\rho, x', \kappa'_\rho) = \sum_{\hat{s}} \delta (x'' - \hat{s} x') \delta_{gh}(\kappa''_\rho, \hat{s} \kappa'_\rho) \tag{3.18} \]
\( (\int d\kappa' \delta_{gh}(\kappa', \kappa) \psi(\kappa') = \psi(\kappa)) \). The sum is over all inequivalent elements \( \hat{s} \) acting on \( x \) and \( \kappa_\rho \) as follows: \( \hat{s} x = T_{s} x \) and \( \hat{s} \kappa_\rho = \hat{T}_{s} \kappa_\rho \) where, by definition, \( \hat{T}_{s} \chi = \pm \chi \) and \( T_{s} \) are corresponding \( SU(2) \) elements in the representation of \( x \). The \( \hat{s} \)-transformations form a \( \mathbb{Z}_2 \) subgroup of \( SU(2) \), \( \hat{T}_{\pm} = \pm 1 \).

We note that any physical state \( \psi(x_\rho, \kappa_\rho) \) is invariant under the \( \hat{s} \)-transformations. Here we denote the transformed system variables defined in (3.2) by \( x_\rho \).
to distinguish from the original system variables. We also define the action of $\hat{s}$ on $\theta^a$ by $\hat{T}_g(\hat{s}\theta) = \hat{T}_g(\theta)\hat{T}_s^{-1}$. Then $\hat{s}x = x$ and $\hat{s}\kappa = \kappa$ so that $\psi(\hat{s}x, \hat{s}\kappa) = \psi(x, \kappa)$ for any element of the extended Hilbert space. For a physical state we have $\psi_{ph}(x, \kappa) = \psi_{ph}(T_g(\theta)x, \kappa) = \psi_{ph}(x, \kappa)$. Since $\hat{s}$ leaves $x$ and $\kappa$ invariant, we can replace $\theta^a$, $x^\rho$ and $\kappa_\rho$ by their $\hat{s}$-transforms in the latter equality. Thus we conclude $\psi_{ph}(\hat{s}x_\rho, \hat{s}\kappa_\rho) = \psi_{ph}(x_\rho, \kappa_\rho)$. The time evolution must respect this symmetry. However, the functional integrals (3.14) or (3.15) do not respect this symmetry so that they cannot describe the correct time evolution. The particular linear combination of functional integrals (3.15) with different boundary conditions as prescribed by (3.13) and (3.18) has the desired invariance property [4, 13].

From (3.6) we know that (3.13) coincides with the physical system transition amplitude only after supersymmetric and gauge invariant boundary conditions have been imposed on the ghosts. In analogy with section 2, the state $\langle \langle x, \kappa|\psi \rangle \rangle = \delta_{gh}(\kappa)\psi_s(x)$ is supersymmetric, i.e., it is annihilated by both supersymmetric generators. However, it is not gauge invariant so that it cannot be used as boundary conditions in (3.13). To construct a gauge invariant supersymmetric state we average $\delta_{gh}(\kappa)\psi_s(x)$ over the gauge transformations. This yields

$$\langle \langle \kappa_\rho, x|ph \rangle \rangle = \frac{2}{|\rho|^3}\delta(\rho)(\eta^\dagger \eta)^2\psi^ph_s(x) = \frac{2}{z^\dagger z}\delta(z^\dagger z)(\eta^\dagger \eta)^2\psi^ph_s(x), \quad (3.19)$$

where $\psi^ph_s(x)$ is any gauge invariant function of $x$. The ghost part of (3.19) is just a delta function with respect to the scalar product measure $\int_0^\infty d\rho \rho^3 \int d\eta^1 d\eta^2$. Substituting (3.19) into (3.3) and using (3.13) we obtain

$$s\langle ph'|e^{-iH_s}|ph \rangle_s = \int dx dx' \langle ph'|x'|x\rangle \bar{U}^{eff}_t(x', x)\langle x|ph \rangle \quad (3.20)$$

with

$$\bar{U}^{eff}_t(x', x) = \lim_{\kappa_\rho, \kappa_\rho' \to 0}(\rho \rho')^{-3/2}U^{eff}_t(x', \kappa_\rho, x, \kappa_\rho'). \quad (3.21)$$

There is no problem to impose zero boundary conditions on the fermion ghosts in the functional integral (3.15). For the boson ghost $\rho$ one can only prove that the limit (3.21) exists. Indeed, in the neighborhood of $\rho = 0$ or $\rho' = 0$, the integral (3.13) satisfies the Schrödinger equation $(-\partial^2/\rho + 3/(8\rho^2))U^{eff}_t = 0 + O(1)$. Therefore $U^{eff}_t \sim (\rho \rho')^{3/2}$ as $\rho$ and $\rho'$ approach zero and the limit (3.21) exists.

### 4 Yang-Mills theories.

Here we briefly outline how the procedure of the previous sections can be generalized to Yang-Mills theories. A more detailed discussion can be found in [12].

One can go from the gauge model of section 3 to the Yang-Mills theory (with $SU(2)$ gauge symmetry) by simple analogy. We identify the vector gauge potentials $A_i$, ($i = 1, 2, 3$) with the system variables $x$, the Lagrange multiplier $y$ with $A_0$ and
κ becomes a set of ghost fields. One can convince oneself that the expressions (3.20) and (3.21) still hold with the replacements \( x \rightarrow A \) and \( dx \rightarrow \Pi_x dA_i(x) \), where the product is taken over all space points. Similarly the measure \( \rho^3 \) is replaced by the functional measure 

\[ \mu = \Pi_x \rho^3(x) \]

and

\[ U_{\text{eff}}[A', \kappa', A, \kappa] = \int [dA_\mu][d\rho][d\eta][d\eta^\dagger]e^{iS_{\text{eff}}} \]  

with

\[ S_{\text{eff}} = \int d^4x \left[ -\frac{1}{4}F^2_{\mu\nu} + (\tilde{D}_\mu \eta)(\tilde{D}^\mu \eta) + \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}A^2_\mu \rho^2 - m^2(\rho^2 + \eta^\dagger \eta) - V_q(\rho) \right]. \]  

(4.2)

Here \( \tilde{D}_\mu \) is the covariant derivative in the fundamental representation and a specific choice of the matrix \( \Omega \) (see (3.2)) was made, \( \Omega = -\tilde{D}_i^2(A) + m^2 \), (in order for \( \Omega \) to be strictly positive a mass term is added for the ghosts). The quantum corrections to the potential energy is proportional to a divergent factor \( V_q(\rho) = 3[\delta^3(0)]^2/(8\rho^2) \) as is usual for operator ordering terms in a field theory. Making use of a lattice regularization of space in (4.2) our earlier arguments can be repeated to prove the existence of the limit (3.21) for each degree of freedom of the field \( \rho(x) \). The coupling of the field oscillators due to the term \((\partial_\mu \rho)^2\) is of no consequence in the limit \( \rho \to 0 \) as it is negligible in comparison with the singular quantum potential \( V_q \). Therefore the limit (3.21) exists and supersymmetric gauge invariant boundary conditions can be imposed as before.

Due to the presence of the singular potential \( V_q(\rho) \) the limit (3.21) is highly undesirable and leads to technical difficulties. This limit originates from our choice of supersymmetric boundary conditions for the path integral. Fortunately, all physically relevant information about a quantum system can always be extracted from Green’s functions. In the next section we show that the restriction of the supersymmetric boundary conditions can be dropped in the path integral representation of Green’s function and, hence, the aforementioned technical difficulty can be avoided.

5 System Green’s functions.

The quantities one is normally interested in are the Green’s functions of the system from which the S-matrix elements and the excitation spectrum can be extracted. They are defined as the vacuum (groundstate) expectation values of time ordered products of the field operators. For complex time they are defined by the following quantities, taken in the limit \( \tau_f \to \infty, \tau_i \to -\infty \):

\[
\langle s'|e^{-\hat{H}_s(\tau_f - \tau_i)}\hat{O}_1^s(\tau_1)\hat{O}_2^s(\tau_2)\ldots\hat{O}_n^s(\tau_n)e^{\hat{H}_s(\tau_i)}|s\rangle /
\langle s'|e^{-\hat{H}_s(\tau_f - \tau_i)}|s\rangle .
\]  

(5.1)

Here \( \tau_f > \tau_1 > \ldots \tau_n > \tau_i \), \( |s\rangle \) and \( |s'| \) are arbitrary system states and \( \hat{O}_i^s(\tau_i) = e^{\hat{H}_s(\tau_i)}\hat{O}_i^se^{-\hat{H}_s(\tau_i)} \) are system operators in the Heisenberg picture. The real time Green’s
functions are obtained from the complex time Green’s functions by analytic continuation (Wick rotation). Following the standard procedure one can also express the Green’s functions as functional integrals.

When the system possesses a gauge symmetry, it is again necessary to eliminate the unphysical degrees of freedom to obtain well defined Green’s functions. Following the Faddeev-Popov approach, one is again faced by the difficulties associated with gauge fixing. Therefore we would like to extend our procedure to the evaluation of system Green’s functions. In principle this is straightforward, except for the following point which require special care: The groundstate of the supersymmetric extended Hamiltonian may turn out not to be supersymmetric, i.e., the supersymmetry is spontaneously broken and the groundstate does not satisfy (2.18) and is thus unphysical. If this happens the Green’s functions of the supersymmetric extended system do not coincide with those of the original system, as the expectation value of the system operators are evaluated with respect to an unphysical state. One way of avoiding this difficulty is to impose supersymmetric boundary conditions in the functional integral for the Green’s functions as relation (2.25) holds regardless whether the supersymmetry is spontaneously broken or not. From our previous discussion it is, however, clear that these boundary conditions are inconvenient from a calculational point of view. Instead we present here an alternative way of obtaining the system Green’s function which allows us to choose arbitrary boundary conditions.

For the purpose of illustration let us consider again the 1-dimensional quantum system of section 2. The complex time Green’s functions of the supersymmetric extended system is obtained from a simple generalization of (5.1)

\[
\frac{\langle\langle \psi' | e^{-\hat{H}_{\tau f}} \hat{O}_1(\tau_1) \hat{O}_2(\tau_2) \cdots \hat{O}_n(\tau_n) e^{\hat{H}_{\tau i}} | \psi \rangle\rangle}{\langle\langle \psi' | e^{-\hat{H}(\tau_f-\tau_i)} | \psi \rangle\rangle}.
\]

Here \( |\psi\rangle = |gh\rangle \cdot |s\rangle \), \( |\psi'\rangle = |gh'\rangle \cdot |s'\rangle \) are states in the extended Hilbert space and \( \hat{H} \) is the supersymmetric extended Hamiltonian

\[
\hat{H} = \hat{H}_s + \alpha \hat{H}_{gh}
\]

where \( \hat{H}_s, \hat{H}_{gh} \) were defined in (2.1, 2.2) and \( \alpha \) is an arbitrary real number. The operators \( \hat{O}_i(\tau_i) = e^{\hat{H}_{\tau i}} \hat{O}_i e^{-\hat{H}_{\tau i}} \) are supersymmetric extensions of the system operators in the Heisenberg picture defined by \( \hat{O}_i = \hat{O}_i^s + [\hat{Q}, (\hat{O}_i)_gh] \) with any desired choice of \( (\hat{O}_i)_gh \).

Note that the extended Hamiltonian above differs slightly from that of (2.14) in the presence of the arbitrary real number \( \alpha \). Since the ghost Hamiltonian is irrelevant on the physical subspace (see (2.23, 2.25)), physical quantities will be independent of \( \alpha \), while excitations in the ghost sector will be affected. We now exploit this fact to our advantage by considering the limit \( \alpha \to \infty \). From eq. (2.10) we note that this limit will push all states with non-zero ghost number to infinite energy, while the states with zero ghost number will be untouched. Thus in this limit the
groundstate of the extended Hamiltonian will be supersymmetric and assumes the form \( |0\rangle_{gh} \cdot |gs\rangle_s \), where \( |gs\rangle_s \) is the system groundstate (see (2.19)), so that the Green’s functions of the extended Hamiltonian coincide with those of the system Hamiltonian. We now prove that this is indeed the case.

Let \( |N, x\rangle \rangle = |N\rangle_{gh} \cdot |x\rangle \) be eigenstates of \( \hat{N}_{gh} \) and \( \hat{x} \) where we noted that \( \{\hat{N}_{gh}, \hat{x}\} = 0 \). The states \( |N, x\rangle \rangle \) form a complete basis in the extended Hilbert space \( (|N\rangle_{gh} \rangle \) is the Fock basis of the ghosts). Writing out the Heisenberg operators in (5.2) and inserting a complete set of states in the form

\[
1 = \sum_N \int dx\langle x, N | x\rangle \rangle \langle x, N | \langle x, N | \langle x, N | \langle x, N |
\]

before and after each time evolution operator, the Green’s function (5.2) reduces to the product of generic matrix elements of the extended system operators \( \langle x', N' | \hat{O}_i | N, x\rangle \rangle \) and the time evolution operator \( \langle x', N'|e^{-T(\hat{H}_s + \hat{H}_{gh})}|N, x\rangle \rangle \equiv \langle x', N'|\hat{U}_T | N, x\rangle \rangle \) where \( T = \tau_j - \tau_{j+1} > 0 \). Consider the latter matrix elements. From

\[
\hat{H}_{gh}|N, x\rangle = \omega(x)N|N, x\rangle \]

(5.5)

follows that in the limit \( \alpha \to \infty \) a perturbation expansion in \( \hat{H}_s \) can be developed, using \( \alpha \hat{H}_{gh} \) as the unperturbed Hamiltonian. Introducing an interaction picture we can express the time evolution operator as:

\[
\hat{U}_T = e^{-\alpha \hat{H}_{gh}T} \sum_{n=0}^{\infty} (-1)^n \int_0^T dt_1 \int_0^{\tau_1} dt_2 \ldots \int_0^{\tau_{n-1}} dt_n \hat{H}_s(\tau_1)\hat{H}_s(\tau_2)\ldots \hat{H}_s(\tau_n)
\equiv e^{-\alpha \hat{H}_{gh}T} \sum_{n=0}^{\infty} (-1)^n \hat{B}_n ,
\]

(5.6)

where \( T > \tau_1 > \tau_2 \ldots \tau_n > 0 \) and \( \hat{H}_s(\tau_i) = e^{\alpha \hat{H}_{gh}\tau_i} \hat{H}_s e^{-\alpha \hat{H}_{gh}\tau_i} \). To evaluate the matrix elements \( \langle x', N'|\hat{U}_T | N, x\rangle \rangle \), consider a typical term in the series (5.6) and insert the complete set of states (5.4) between the \( \hat{H}_s(\tau_i) \). Using (5.3) we find:

\[
\langle x', N'|\hat{B}_n | N, x\rangle \rangle = \sum_{N_1\ldots N_{n-1}} \int dx_1 \ldots dx_{n-1} d\tau_1 \int_0^{\tau_1} \int_0^{\tau_2} \ldots \int_0^{\tau_{n-1}} e^{-\omega N'(T-\tau)}\langle x', N' | \hat{H}_s | N_1, x_1\rangle \rangle \langle x_1, N_1 | \hat{H}_s | N_2, x_2\rangle \rangle \ldots \langle x_{n-1}, N_{n-1} | \hat{H}_s | N, x\rangle \rangle e^{-\omega N\tau_n}.
\]

(5.7)

Here the shorthand notation \( \omega' = \omega(x') \), \( \omega_i = \omega(x_i) \) and \( \omega = \omega(x) \) was used. Since \( T > \tau_1 > \tau_2 \ldots \tau_n > 0 \) we deduce that in the limit \( \alpha \to \infty \) only terms with \( N' = N_1 = \ldots = N = 0 \) survive. The ghost vacuum \( (N = 0) \) is a physical state and therefore \( \langle x', 0|\hat{H}_s|0, x\rangle \rangle = \langle x'|\hat{H}_s|x\rangle \rangle \) according to (2.23). Making use of these observations we find that eq. (5.7) becomes

\[
\langle x', N'|\hat{B}_n | N, x\rangle \rangle \to \frac{T^n}{n!} \langle x'|\hat{H}_s^n | x\rangle \rangle \delta_{N',0} \delta_{N,0} ,
\]

(5.8)
when $\alpha \to \infty$. From (5.8) we conclude

$$\langle \langle x', N' | \hat{U}_T | N, x \rangle \rangle \to \langle x' | e^{-\hat{H}_s T} | x \rangle \delta_{N',0} \delta_{N,0}, \quad \alpha \to \infty.$$ (5.9)

Substituting this result in (5.2) and using (2.23), we obtain in the limit $\alpha \to \infty$

$$\langle \langle \psi' | e^{-\hat{H}_f \tau_f} \hat{O}_1(\tau_1) \cdots \hat{O}_n(\tau_n) e^{\hat{H}_i \tau_i} | \psi \rangle \rangle \to \langle s' | e^{-\hat{H}_s \tau_f} \hat{O}_1(\tau_1) \cdots \hat{O}_n(\tau_n) e^{\hat{H}_s \tau_i} | s \rangle \rangle \langle s' | e^{-\hat{H}_s (\tau_f - \tau_i)} | s \rangle \rangle.$$ (5.10)

This establishes that by extending the Hamiltonian as in (5.3), and taking the limit $\alpha \to \infty$ before the limits $\tau_f \to \infty$, $\tau_i \to -\infty$, we indeed recover the system Green’s functions from the Green’s functions of the supersymmetric extended system. In this approach arbitrary boundary conditions can be imposed in the path integral for the Green’s function as the limit $\alpha \to \infty$ ensures that the supersymmetric groundstate is selected in the evaluation of the expectation value.

We conclude this section with a couple of general remarks. The first remark concerns degeneracies. We note that for some values of $x, x', N$ and $N'$ the following equality holds: $\omega(x) N = \omega(x') N'$ and we have to analyze what happens to the perturbation expansion (5.7) when this degeneracy occurs in the ghost energy levels (see (5.3)). This degeneracy does not affect our conclusions (5.8) and, hence, (5.10). Indeed, the operator $\hat{H}_s$ is local and therefore its matrix elements $\langle x' | \hat{H}_s | x \rangle$ have support only on $x = x'$. Consequently this equality only holds when $N = N'$ and it is necessary to consider the degeneracy of a ghost level with a fixed ghost number only. In this case some of the exponential factors disappear and the corresponding integration yields a typical factor $\tau_i g_N$, with $g_N$ the degeneracy of the state $| N \rangle_{gh}$. Since this is a finite factor, independent of $\alpha$, we conclude that in the limit $\alpha \to \infty$ the factor $e^{-\alpha \omega(x') N'/T}$ ensures that only contributions from the ghost vacuum survive.

The above argument will hold for any supersymmetric extension of the Hamiltonian, provided that the ghost Hamiltonian itself does not exhibit spontaneous breaking of the supersymmetry.

Clearly the above procedure would not be necessary if one can prove that the supersymmetry is not spontaneously broken for $\alpha = 1$. If the groundstate of the extended Hamiltonian has the form (2.19), it is not hard to see that the Green’s functions (5.2) coincide with the system Green’s functions (5.1) in the limit $\tau_f \to \infty$, $\tau_i \to -\infty$. However, a proof that the supersymmetry is not spontaneously broken does not seem easy.

We remark that the procedure as outlined above does not apply to relativistic systems as it would break Lorentz invariance. However, a relativistic invariant formulation of the above procedure can be obtained by taking the limit where the mass of the ghost fields go to infinity, $m \to \infty$ (see (4.2)). In this limit contributions of ghost excitations to the Green’s functions are exponentially suppressed because their energies grow as $m$. It is then not difficult to generalize the above argument to show that our conclusion (5.10) holds in this case.
In the case of a gauge system, one is interested in the Green’s functions of gauge invariant system operators. They are obtained from our prescription above by considering only physical operators, \( \hat{O}_i (\tau_i) \), in (5.2), i.e., operators that commute with the supersymmetry generators obtained after projecting onto the space of gauge invariant functions. For the gauge model of section 3 we would therefore consider operators that commute with the supersymmetry generators \( \hat{Q}_\rho \) and \( \hat{Q}_\rho^\dagger \) of (3.11). From the discussion below (3.12) it follows that any system operator which commutes with \( \hat{Q}_\rho \) and \( \hat{Q}_\rho^\dagger \) has to commute with the system constraint operators \( \hat{\sigma}_a \), and is thus gauge invariant. Therefore the Green’s functions obtained from our description above coincide with the system Green’s functions of gauge invariant observables.

Finally we remark that the evaluation of the Green’s functions in a gauge theory using the supersymmetric quantization is not an easy task. Though the problem of the supersymmetric boundary conditions has been avoided, which is a great simplification from the calculational point of view, there is still a lack of a well elaborated techniques in quantum field theories to treat, even in perturbation theory, local measures in the functional integral (see (3.13)) and singular quantum potentials (see (3.17)) that usually occur in unitary gauges. Nonetheless it is possible to calculate the renormalized Green’s function using a modified loop expansion [15].

6 The relation to gauge fixing.

Here we explain how the functional integral (4.1) is related to the functional integral of a gauge fixed Yang-Mills theory. For simplicity we consider the model of section 3. Let \( x = x_F \) be a generic configuration satisfying the gauge condition \( F(x) = 0 \). We assume that the gauge condition fixes all continuous gauge arbitrariness, i.e., the stationary group of \( x = x_F \) is trivial. However, the equation \( F(x) = 0 \) may have many solutions related by discrete gauge transformations (Gribov copies). This gauge arbitrariness does not decrease the number of physical degrees of freedom, but it does reduce the volume of the physical configuration space spanned by \( x_F \). If all configurations \( x_F \) form an Euclidean space \( \mathbb{R}^M \) and \( S \) is a set of residual discrete gauge transformations, then the physical configuration space (a fundamental modular domain) is isomorphic to \( \Lambda_F \sim \mathbb{R}^M / S \subset \mathbb{R}^M \).

Consider the change of variables \( x_\rho = T_g x_F, \ z = T_g \chi_\rho, \ \eta = T_g \eta_\rho, \) where the first relation determines the element \( T_g \) for a given \( x_F \) and \( x_\rho \), and \( T_g \) is this element in the fundamental representation. The latter two relations define the new variables \( z \) and \( \eta \). Introducing this change of variables in the Hamiltonian (3.10), the ghost part assumes the form (3.12), while in the system Hamiltonian the Laplace operator \( \hat{p}_F^2 \) must be written in the curvilinear coordinates \( x_\rho = T_g x_F \). If there were no ghosts in the theory, the Laplace-Beltrami operator \( \hat{p}_F^2 \) would contain operators \( \hat{\sigma}_a \), corresponding to momenta conjugate to the group parameters \( \hat{\theta}_a \). Since the above change of variables involves the ghost variables, all operators \( \hat{\sigma}_a \) get replaced by
just as the boson ghost constraints were replaced by $\hat{\sigma}^a_i + \hat{\sigma}^a_f$ in (3.10). The functional integral for this Hamiltonian takes the form (3.13) where the measure $\rho^3$ is replaced by $\mu(x_F) \left( \int_0^\infty d\kappa \rho^3 \int dx_x = \int d\kappa \int_{\Lambda_F} dx_F \mu(x_F) \right)$. The measure $\mu(x_F)$ is the Faddeev-Popov determinant in the gauge chosen. The kernel (3.18) contains a sum over all elements $T_s$ satisfying $F(T_s x_F) = 0$. Imposing supersymmetric boundary conditions for the ghosts by choosing $\langle\langle \kappa, x_F | \psi \rangle\rangle_{ph} = \delta_{gh}(\kappa) \psi_{ph}(x_F)$, the ghost integral is Gaussian and can be done explicitly. Boson and fermion determinants cancel and the prefactor is 1 due to the supersymmetric boundary conditions (see (2.31)). We therefore end up with the functional integral for the system action in the gauge $F(x) = 0$.

For the Yang-Mills theory we can choose $F(A) = \partial_i A_i = 0$ (Coulomb gauge). The above procedure yields the functional integral obtained in [5].

### 7 Conclusions.

The normal procedure of Lorentz covariant gauge fixing has the disadvantage that it leads to Gribov copying. The description of these copies and the fundamental modular domain is extremely difficult and in the continuum theory not well founded since it depends strongly on the functional space chosen for the gauge potentials (the normal choice of $L^2$ forms a set of zero measure in the functional integral) [4]. However, to calculate the non-perturbative Green’s functions in covariant gauges one needs all copies for a generic configuration satisfying the gauge condition [14]. Given the complexity of these copies [7], the non-perturbative evaluation of Green’s functions in covariant gauges seems to be a hopeless task.

In the procedure developed above we have solved these difficulties. Our effective theory is Lorentz invariant and at the same time the Gribov problem has been avoided. The price we pay is in the appearance of an additional scalar ghost field. Furthermore we have encountered a difficulty with the supersymmetric boundary conditions to be imposed on the functional integral. This difficulty can, however, be avoided in the path integral for the Green’s functions as we have shown.

We would like to stress that in the asymptotic free domain where perturbation theory is valid, the normal approach of Lorentz covariant gauge fixing [3] is more appropriate. However, in the infra-red limit where the non-perturbative aspects of the theory begin to dominate, the present formalism should come into its own right. One of its possible applications would therefore be to study the infra-red behavior of the theory. As the present approach is gauge independent it makes it possible to extract the infra-red properties of the theory due to the true dynamics of the system (self-interactions of the gluons) and free from any influences coming from the presence of the gauge dependent Gribov horizon in a gauge fixed approach. This would hopefully provide insight into the physical meaning of the (non-perturbative) dynamically generated mass scale found in [7, 10, 16].
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Appendix.

There exists an $N = 2$ supersymmetric extension of the system dynamics. To obtain it, one should find a ghost extension $\hat{P} = \hat{P}^\dagger$ of the momentum $\hat{p}$ such that

$$[\hat{P}, \hat{R}] = [\hat{P}, \hat{R}^\dagger] = [\hat{P}, \hat{Q}] = [\hat{P}, \hat{Q}^\dagger] = 0 \quad (A.1)$$

and replace the $\hat{p}$ by it in all system operators. One can convince oneself that the operator

$$\hat{P} = \hat{P}^\dagger = \hat{p} - i\beta(\hat{x})(\hat{c}_2\hat{c}_1^\dagger - \hat{c}_1\hat{c}_2^\dagger + \hat{b}_2\hat{b}_1 - \hat{b}_1\hat{b}_2^\dagger), \quad (A.2)$$

with $\beta = -\omega'/(2\omega)$, is the desired $N = 2$ supersymmetric extension of the momentum operator. An advantage of this extension is that we know the structure of the eigenstates of the extended Hamiltonian

$$|E\rangle = |N\rangle_{gh}^x \cdot |E^N\rangle_s, \quad (A.3)$$

where $|N\rangle_{gh}^x$ are the eigenstates of the ghost number operator $N_{gh}$ and

$$(\hat{H}_s + \omega(\hat{x})N)|E^N_s\rangle = E^N_s|E^N_s\rangle. \quad (A.4)$$

The $N = 2$ supersymmetry is not spontaneously broken in this case as the groundstate $|gs\rangle = |0\rangle_{gh}^x \cdot |gs\rangle_s$, with $|gs\rangle_s$ the system groundstate, is manifestly $N = 2$ supersymmetric.

A disadvantage of this extension is the non-locality of the extended system Hamiltonian that arises in the field theory case. Recall that $\omega^2(x)$ is replaced by $-\tilde{D}^2(A) + m^2$ and, hence, $\beta$ would involve the non-local operator $(-\tilde{D}^2(A) + m^2)^{-1}$. For this reason we did not use this extension to resolve the problem with the supersymmetric boundary conditions.

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