A comment on a paper by Carot et al.

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Abstract. In a recent paper Carot et al. considered carefully the definition of cylindrical symmetry as a specialisation of the case of axial symmetry. One of their propositions states that if there is a second Killing vector, which together with the one generating the axial symmetry, forms the basis of a two-dimensional Lie algebra, then the two Killing vectors must commute, thus generating an Abelian group. In this comment a similar result, valid under considerably weaker assumptions, is recalled: any two-dimensional Lie transformation group which contains a one-dimensional subgroup whose orbits are circles, must be Abelian. The method used to prove this result is extended to apply to three-dimensional Lie transformation groups. It is shown that the existence of a one-dimensional subgroup with closed orbits restricts the Bianchi type of the associated Lie algebra to be I (Abelian), II, III, VII$_g=0$, VIII or IX. The relationship between the present approach and that of the original paper is discussed.
1. Introduction

We begin by recalling a few relevant definitions and theorems. Following Carter (1970) a spacetime $\mathcal{M}$ is said to have cyclical symmetry if and only if the metric is invariant under the effective smooth action $SO(2) \times \mathcal{M} \to \mathcal{M}$ of the one-parameter cyclic group $SO(2)$. If, in a cyclically symmetric spacetime, the set of fixed points of the isometry is non-empty then the set of fixed points is an autoparallel time-like two-surface (Mars and Senovilla, 1993) and the spacetime is said to be axially symmetric. Mars and Senovilla also show that the Killing vector is spacelike in the neighbourhood of the axis and satisfies the elementary flatness condition on the axis.

In a recent paper Carot, Senovilla and Vera (1999) consider in detail the definition of cylindrical symmetry; this paper will be referred to below as CSV. Proposition 2 of CSV states that in an axially symmetric spacetime, if there is another Killing vector which, with the axial Killing vector generates a two-dimensional isometry group, then the two Killing vectors commute. Thus the isometry group is Abelian. A similar result for stationary axisymmetric spacetimes was proved by Carter (1970). Both proofs rely heavily on the existence of an axis where the axial Killing vector vanishes. CSV then use their result as the basis of a straightforward definition of a cylindrical symmetry without the additional assumptions that are often made namely: that the group is Abelian and that an elementary flatness condition holds at the axis. Thus a spacetime is cylindrically symmetric if it admits a two-dimensional isometry group with spacelike orbits (i.e. a $G_2$ acting on $S_2$) containing an axial symmetry.

Although the assumption of the existence of an axis is reasonable in many circumstances, there are physically realistic situations where an axis may not exist: the axis may be singular and so not part of the manifold, or the topology of the manifold may be such that no axis exists. An example of the latter situation is a torus embedded in a 3-dimensional flat space; the torus is cylindrically symmetric but the Killing vector field generating the rotation does not vanish anywhere on the surface. It is interesting therefore to consider whether a result of the same ilk as Proposition 2 of CSV is valid for cyclically symmetric spacetimes.

2. Cyclically symmetric manifolds admitting a $G_2$

Suppose $\mathcal{M}$ is cyclically symmetric and suppose $X_0$ is the Killing vector field generating the isometry. We restrict attention to the open submanifold $\mathcal{N}$ of $\mathcal{M}$ on which $X_0$ is non-zero. The orbit each point of $\mathcal{N}$ under the cyclic symmetry is a circle and these circles are the integral curves of the Killing vector field $X_0$. Let $\phi$ be a circular coordinate running from 0 to $2\pi$ which labels the elements of $SO(2)$ in the usual way. Then we can introduce a system of coordinates $x^i$ with $i = 1 \ldots n$ and $x^1 = \phi$ adapted to $X_0$ such that $X_0 = \partial_\phi$. These coordinates are determined only up to transformations of the form

$$\tilde{\phi} = \phi + f(x^\nu) \quad \tilde{x}^\mu = g^\mu(x^\nu)$$
where $f$ and $g^\mu$ are smooth functions and where Greek indices take values in the range $2 \ldots n$.

Suppose now that the isometry group of $M$ contains a two-dimensional subgroup $G_2$ and that the Killing vector field $X_1$ together with $X_0$ forms a basis of the associated Lie algebra with commutation relations $[X_0, X_1] = aX_0 + bX_1$, say, where $a$ and $b$ are constants. In an adapted coordinate system the commutation relations reduce to

$$\frac{\partial X^\mu}{\partial \phi} = bX^\mu_i$$
$$\frac{\partial X^1}{\partial \phi} = a + bX^1_1$$

A straightforward integration gives

$$X^\mu_1 = B^\mu(x^\nu) e^{b\phi}$$
$$X^1 = A(x^\nu) e^{b\phi} - a/b$$

for $b \neq 0$

$$X^\mu_1 = B^\mu(x^\nu)$$
$$X^1 = A(x^\nu) + a\phi$$

for $b = 0$

where $A$ and $B^\mu$ are arbitrary functions of integration. The solutions of these equations must be periodic in $\phi$ with period $2\pi$ if $X_1$ is to be single-valued on $N$. Thus both $a$ and $b$ must vanish and the subgroup $G_2$ must be Abelian.

Note that the dimensionality of the manifold, the existence of a metric and the fact that transformation group is an isometry group are not used in the proof; thus it has been proved that any two-dimensional Lie transformation group which acts on an $n$-dimensional manifold $M$ and which contains a one-dimensional subgroup acting cyclically on $M$ must be Abelian. This remarkably simple and general result is not new, but is perhaps not widely known.

### 3. Cyclically symmetric manifolds admitting a $G_3$

Suppose now that the manifold $M$ admits a three-dimensional Lie group $G_3$ of transformations which contains a one-dimensional subgroup, generated by the vector field $X_0$, acting cyclically on $M$. Let $X_1$ and $X_2$ be vector fields on $M$ which together with $X_0$ form a basis of the associated Lie algebra. Two cases arise: either the Lie algebra admits a two-dimensional subalgebra containing $X_0$ or there is no such subalgebra.

In the former case, by the result of the previous section the subalgebra is Abelian. We assume without loss of generality that $X_0$ and $X_1$ form a basis of this subalgebra. Hence the commutation relations can be written in the form

$$[X_0, X_1] = 0$$
$$[X_0, X_2] = aX_0 + bX_1 + cX_2$$

where $a$, $b$ and $c$ are constants. In terms of a coordinate system adapted to $X_0$ these commutators reduce to the differential equations

$$\frac{\partial X^i_1}{\partial \phi} = 0$$
$$\frac{\partial X^i_2}{\partial \phi} = a\delta^i_1 + bX^i_1 + cX^i_2$$

‡ The author remembers seeing this result presented as a footnote in a paper around 30 years ago, but is unable to locate the precise reference.
A straightforward argument similar to that in the previous section shows that if the solutions are to be periodic in $\phi$, we must have $a = b = c = 0$. Thus $X_0$ commutes with the other two basis vectors $X_1$ and $X_2$. The remaining commutator takes the form

$$[X_1 X_2] = dX_0 + eX_1 + fX_2$$

where $d$, $e$ and $f$ are constants. Three algebraically distinct cases arise. If $d = e = f = 0$ the group is Abelian (Bianchi type I). If $e = f = 0$ but $d \neq 0$, after a rescaling of $X_1$ and/or $X_2$, we may set $d = 1$. The commutation relations now are

$$[X_0 X_1] = 0 \quad [X_0 X_2] = 0 \quad [X_1 X_2] = X_0$$

which is the canonical form for an algebra of Bianchi type II (see for example Petrov, 1969). If $e$ and $f$ are not both zero, by a change of basis which leaves $X_0$ fixed, we can set $d = e = 0$ and $f = 1$. Thus the commutators are

$$[X_0 X_1] = 0 \quad [X_0 X_2] = 0 \quad [X_1 X_2] = X_2$$

which is (essentially) the canonical form for an algebra of Bianchi type III.

If the Lie algebra has no two-dimensional subalgebra containing $X_0$ then, without loss of generality, we may choose the basis vectors $X_1$ and $X_2$ such that the commutation relations become

$$[X_0 X_1] = X_2 \quad [X_0 X_2] = aX_0 + bX_1 + cX_2$$

where $a$, $b$ and $c$ are constants. In terms of a coordinate system adapted to $X_0$ these reduce to

$$\frac{\partial X_1^i}{\partial \phi} = X_2^i \quad \frac{\partial X_2^i}{\partial \phi} = a\delta^i_1 + bX_1^i + cX_2^i$$

From these equations it is easy to deduce that

$$\frac{\partial^2 X_2^i}{\partial \phi^2} = bX_2^i + c\frac{\partial X_2^i}{\partial \phi}$$

Thus for solutions periodic in $\phi$ with period $2\pi$, we must have $c = 0$ and $b = -n^2$ for some positive integer $n$. By a redefinition of the basis vector $\tilde{X}_1 = X_1 + a/bX_0$, we can also set $a = 0$ so that the commutation relations become

$$[X_0 X_1] = X_2 \quad [X_0 X_2] = -n^2X_1 \quad [X_1 X_2] = dX_0 + eX_1 + fX_2$$

where $d$, $e$ and $f$ are constants. In a coordinate system adapted to $X_0$ the basis vectors are given by

$$X_0^i = \delta^i_0$$

$$X_1^i = 1/n(A^i(x^\nu) \sin n\phi - B^i(x^\nu) \cos n\phi)$$

$$X_2^i = A^i(x^\nu) \cos n\phi + B^i(x^\nu) \sin n\phi$$
where $A^i$ and $B^i$ are functions of integration.

The commutators must, of course, satisfy the Jacobi identity

$$[X_0 [X_1 X_2]] + [X_1 [X_2 X_0]] + [X_2 [X_0 X_1]] = 0$$

which implies that

$$[X_0 (dX_0 + eX_1 + fX_2)] = eX_2 - n^2 fX_1 = 0$$

Thus $e = f = 0$. Three algebraically distinct cases arise. If $d = 0$ the group is of Bianchi type VII$_{q=0}$ which is, of course, the isometry group of the Euclidean plane. Note that the standard form of the commutation relations for Bianchi type VII$_{q=0}$ is obtained by means of the change of basis

$$\tilde{X}_0 = 1/nX_0 \quad \tilde{X}_1 = nX_1 \quad \tilde{X}_2 = X_2$$

For $d < 0$ and $d > 0$ the Bianchi type is VIII and IX respectively; which are of course the symmetry groups of Lobachevskii plane and two-sphere respectively. The change of basis required to reduce the commutation relations to the canonical form for Bianchi types VIII and IX is

$$\tilde{X}_0 = 1/nX_0 \quad \tilde{X}_1 = n/\epsilon X_1 \quad \tilde{X}_2 = 1/\epsilon X_2$$

where $\epsilon = n\sqrt{|d|}$.

The methods above can be applied to the case of four (and higher) dimensional Lie groups containing a one-dimensional cyclic subgroup. Again the allowed forms of the commutation relations are considerably restricted. These results will be presented elsewhere.

4. A correction

Senovilla (1999) has pointed out an error in the statement of Theorem 3 of CSV. At his request a corrected statement of the theorem is presented here.

Given a $G_3$ on $T_3$ group that contains an orthogonally transitive Abelian $G_2$ subgroup generated by an axial $\vec{\xi}$ (such that its set of fixed points is non-empty) and $\vec{\eta}$, and a third integrable timelike Killing vector field $\vec{\zeta}$, then it follows that: if $G_3$ is the maximal isometry group, then it must be Abelian; if $G_3$ is non-Abelian, then it is (locally) contained in a $G_4$ on $T_3$, and $\vec{\xi}$ and $\vec{\zeta}$ generate an orthogonally transitive subgroup $G_2$ on $T_2$.

The proof follows easily from the remarks immediately preceding the statement of the theorem in the original paper.
5. Summary

Low-dimensional Lie transformation groups which act on an \( n \)-dimensional manifold \( \mathcal{M} \) and which contain a one-dimensional subgroup acting cyclically on \( \mathcal{M} \) have been considered. For the two-dimensional case the group must be Abelian and for the three-dimensional case the Bianchi type of the group is restricted: types IV, V and VI cannot occur and only the subclass VII\(_{q=0} \) of type VII is allowed. As far as the author is aware the results for the three-dimensional case are new.

At first sight the restrictions in the three-dimensional case may not seem too severe: only three of the nine Bianchi types are excluded whilst a fourth type is restricted. However, it should be pointed out that Bianchi types VI and VII both involve an arbitrary parameter and so each contain an infinite number of algebraically distinct types. Those in type VI are excluded completely and for type VII only a single case survives. Moreover for most of the types which are permitted the structure of the Lie algebra is ‘aligned’ in some way relative to \( X_0 \). For example, for Bianchi type II, \( X_0 \) is also the generator of the first derived subalgebra and for Bianchi types II and III, \( X_0 \) lies in the centre of the algebra.

At first sight too, the result presented in section 2 seems considerably stronger than Proposition 2 of CSV; however this is only partially true. Certainly the results presented do not require the existence of an axis and they apply to any Lie transformation group. So \textit{a fortiori} the results apply to isometries, conformal motions, affine and projective collineations etc. whereas CSV’s results apply only to isometries (although some of the results, at least, may be extended to conformal motions). However the results presented here require that the transformations are defined everywhere on the manifold and so they say nothing about the structure of groups of \textit{local} transformations. CSV’s results apply to local isometries in the neighbourhood of the axis and so in this sense are stronger than the results presented above. Of course, if we assume that the isometries are globally well-defined and that an axis of fixed points exists, it may well be possible to derive stronger restrictions on the allowed group structure using a synthesis of the two methods. This is currently under investigation.

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