Convergence properties and compactifications

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Abstract

In this paper, we will use investigate the existence of compactifications with particular convergence properties - pseudoradial, radial, sequential and Fréchet-Urysohn - through the use of spoke systems.

Keywords: compactification, Fréchet-Urysohn, one-point compactification, pseudoradial, radial, sequential, small cardinals, spoke, spoke system

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1 Introduction

In [Lee14], we exhibited a local characterisation of radiality using spoke systems, which are collections of subspaces with neighbourhood bases well-ordered by reverse inclusion at a specified point, that together reconstruct the original neighbourhood filter at that point. Using this characterisation, we can investigate how to compactify a locally compact space whilst preserving it’s convergence properties - the main properties under consideration will be radiality and Fréchet-Urysohn, although we will exhibit some results for pseudoradiality and sequentiality too. We will also demonstrate an alternative characterisation of radiality, using cofinal collections of spokes under local containment.

We shall first investigate one-point compactifications, since if we can compactify and preserve radiality or the Fréchet-Urysohn property then the one-point compactification will suffice. From this, we will see how to improve this result to finite and countable compactifications. Finally, we use small cardinals to find conditions for uncountable sequential and Fréchet-Urysohn compactifications.

For the first two sections of this article, we will not be assuming any separation conditions on our topological spaces. We recall the following definitions and theorems from [Lee14]:

Definition 1.1.

• A transfinite sequence is a net with well-ordered domain, typically indexed by an ordinal with the $\in$-ordering. If $x$ is a point in a topological space, then a transfinite sequence in $X$ is said to converge strictly to a point $x$ in a space if it converges to $x$ and $x$ is not in the closure of any of the proper initial segments of the transfinite sequence.

• We say that a topological space $X$ is radial at a point $x$ if for every subset $A$ of $X$ that contains $x$ in its closure, there is a transfinite sequence converging to $x$ whose range lies in $A$. If a space is radial everywhere then we call it a radial space.

By replacing the transfinite sequences with ordinary $\omega$-indexed sequences above, we arrive at the definition of the Fréchet-Urysohn property.

• A space $X$ is said to be well-based at $x$ if $x$ has a neighbourhood base well-ordered by $\subseteq$. Such a neighbourhood base is said to be well-ordered neighbourhood base.

A subspace of $X$ that contains $x$ and is well-based at $x$ is called a spoke at $x$. We will denote the set of (closed) spokes at $x$ by $\text{Sp}(x, X)$ ($\overline{\text{Sp}}(x, X)$).
• For a point \( x \) in a space \( X \), we denote its neighbourhood filter by \( \mathcal{N}_x^X \), or \( \mathcal{N}_x \) when the space is unambiguous. We define its \textit{neighbourhood core} to be the intersection of all neighbourhoods of \( x \). This will be denoted by \( N^X_x \), or \( N_x \) again if \( X \) is unambiguous. Note that in a \( T_1 \)-space, \( N_x = \{ x \} \).

The reason to introduce these strictly convergent sequences is because they allow us to construct spokes:

**Lemma 1.2.** [Lee14, Claim in Theorem 4.1, pg. 16] Let \( X \) be a topological space, \( x \in X \) be given and let \( f : \lambda \to X \) be an injective transfinite sequence that converges strictly to \( x \). Then \( N_x \cup \text{ran}(f) \) is a spoke at \( x \).

We refer to spokes of this form as \textit{basic spokes}. In [Lee14], we used spoke systems consisting of basic spokes in our proof characterising radiality.

**Definition 1.3** (Spoke system). Let \( X \) be a topological space, \( x \in X \) be given and let \( \mathcal{S} \) be a collection of spokes at \( x \). Then we say that \( \mathcal{S} \) is a \textit{spoke system} at \( x \) if

\[
\bigcup_{S \in \mathcal{S}} U_S : \forall S \in \mathcal{S}, U_S \in \mathcal{N}_x^S
\]

is a neighbourhood base for \( x \) with respect to \( X \). Note that this collection will always form a network at \( x \).

**Definition 1.4** (Almost-independent). Let \( X \) be a topological space, \( x \in X \) be given and let \( \mathcal{S} \) be a collection of spokes at \( x \) such that \( x \notin (S \cap T) \setminus N_x \) for all distinct \( S, T \in \mathcal{S} \). Then we say that \( \mathcal{S} \) is \textit{almost-independent}.

**Theorem 1.5.** [Lee14, Theorem 4.1, pg. 16] Let \( X \) be a topological space, \( x \in X \) be given. Then the following are equivalent:

1. \( X \) is radial at \( x \).
2. \( x \) has a spoke system.
3. \( x \) has an almost-independent, basic spoke system.

If we assume some extra separation axioms, we can thicken our spokes. This process will be useful when investigating radiality in compact spaces, and in particular compactifications. Recall:

**Definition 1.6** (Lindelöf degree). Let \( X \) be a topological space. Then we define the (\textit{strict}) \textit{Lindelöf degree} of \( X \) to be the least cardinal \( \kappa \) such that every open cover of \( X \) has a subcover of size strictly less than \( \kappa \). We denote this \( \kappa \) by \( L^*(X) \).

Thus, a space is compact if it has countable Lindelöf degree. Observe that \( L^*(Y) \leq L^*(X) \) for all closed subspaces \( Y \subseteq X \). It is easy to see that the definition of Lindelöf degree is absolute - we can take the covers to be open in a larger superspace.

**Lemma 1.7.** Let \( X \) be a regular space\(^1\), \( x \in X \) be given. Let \( f : \lambda \to X \) be an injective transfinite sequence that converges strictly to \( x \). Then \( S(f) := N_x \cup \bigcup_{\alpha < \lambda} \overline{f[\alpha]} \) is a spoke for \( x \) and \( L^*(S(f)) \leq L^*(X) + 1 \).

\textit{Proof.} Let \( \mathcal{U} \) be an open cover for \( S(f) \). Then there exists a \( U \in \mathcal{U} \) such that \( x \in U \) and thus there exists a closed neighbourhood \( V \) of \( x \) contained in \( U \). As \( f \to x \), there exists an \( \alpha < \lambda \) such that \( f[\alpha] \subseteq V \). By the observation above, there exists a \( V \subseteq \mathcal{U} \) that covers \( f[\alpha] \) such that \( |V| < L^*(f[\alpha]) \). Since for all \( \beta < \lambda, f[\beta] = f[\beta \cap \alpha] \cup f[\beta \setminus \alpha] \subseteq (\bigcup V) \cup \{ U \} \), it follows that \( V \cup \{ U \} \) is a subcover for \( S(f) \) of size at most \( L^*(X) \). Therefore \( L^*(S(f)) \leq L^*(X) + 1 \).

Now define for all \( \alpha < \lambda, B_\alpha := N_x \cup (S(f) \setminus \overline{f[\alpha]}), \) which is an open neighbourhood of \( x \) with respect to \( S(f) \). Note that \( \mathcal{B} = \{ B_\alpha : \alpha < \lambda \} \) is well-ordered with respect to \( \subseteq \). Let \( C \subseteq X \) be a closed neighbourhood of \( x \), so there exists an \( \alpha < \lambda \) such that \( f[\alpha] \subseteq C \) and thus ran(f) \subseteq f[\alpha]. Assume \( S(f) \setminus C \subseteq \overline{f[\alpha]} \), so there exists a \( y \in S(f) \setminus (C \cup \overline{f[\alpha]}) \). Define \( U := X \setminus (C \cup \overline{f[\alpha]}) \), which is an open

\(^1\)We are not assuming the \( T_1 \)-condition in our definition of regularity.
neighbourhood of \( y \). Then there exists a \( \beta < \lambda \) such that \( y \in \overline{\{\beta\}} \) and so there exists a \( \gamma < \beta \) such that \( f(\gamma) \in U \). As \( \text{ran}(f) \cap C \subseteq f(\alpha) \), it follows that \( \gamma < \alpha \) by injectivity. However \( f(\gamma) \in U \) and so \( \gamma \geq \alpha \), which is a contradiction. Therefore \( S(f) \cap C \subseteq \overline{\{\alpha\}} \) and so \( B_\alpha \subseteq C \). Thus \( \mathcal{S} \) is a neighbourhood base for \( x \) in \( S(f) \) and hence \( S(f) \) is a spoke for \( x \).

In particular, if \( X \) is compact and Hausdorff then \( S(f) \) is a closed spoke. Consequently, for these spaces we have another characterisation of radiality:

**Theorem 1.8.** Let \( X \) be a compact Hausdorff space and let \( x \in X \) be given. Then \( X \) is radial at \( x \) if and only if \( x \) has a closed spoke system.

**Proof.** By Theorem 1.5, it suffices to assume \( X \) is radial at \( x \), so there exists a collection \( \mathcal{F} \) of injective transfinite sequences that converge strictly to \( x \) such that \( \text{ran}(f) \cap \{x\} \subseteq \overline{\{\alpha\}} \) is a spoke system for \( x \). Then \( (S(f))_{f \in \mathcal{F}} \) is a collection of closed spokes and for all \( f \in \mathcal{F} \) and \( U_f \subseteq X \) open such that \( \{x\} \cap U_f \in \{x\} \), then \( \text{ran}(f) \cap U_f \subseteq \bigcup_{f \in \mathcal{F}} (S(f) \cap U_f) \) (note that if \( f \in \mathcal{F} \) has domain \( \alpha + 1 \) for some ordinal \( \alpha \), then \( f(\alpha) \in N_\lambda \subseteq S(f) \)). Since \( U \) is a neighbourhood of \( x \), it follows that \( (S(f))_{f \in \mathcal{F}} \) is a closed spoke system for \( x \).

However, we don’t necessarily have a spoke system that is both closed and almost-independent. We need to introduce some more notation: we will denote the one-point compactification of a space \( X \) by \( \alpha X \), with its point-at-infinity denoted by \( * \). Also, let \( \mathcal{K}(X) \) denote the set of compact subsets of a topological space \( X \).

**Theorem 1.9.** There exists a compact Hausdorff space \( X \) and a radial point \( x \in X \) with no closed, almost-independent spoke system.

**Proof.** Define \( X := A(\omega_1 \times \omega_2) \) and note that for all \( K \in \mathcal{K}(\omega_1 \times \omega_2) \), \( \pi_{\omega_1}[K], \pi_{\omega_2}[K] \) are bounded in \( \omega_1, \omega_2 \) respectively and hence \( K \subseteq \alpha \times \beta \) for some \( \alpha < \omega_1 \) and \( \beta < \omega_2 \). In particular, every \( \sigma \)-compact subset of \( \omega_1 \times \omega_2 \) has compact closure; i.e., \( * \) is a p-point.\(^2\)

Let \( A \subseteq \omega_1 \times \omega_2 \) be given such that \( * \in A \). Then \( A := \text{ran}(f) \) is not compact, so there exists an \( i = 1, 2 \) such that \( \pi_{\omega_i}[\lambda] \) is unbounded in \( \omega_i \). Then for all \( \alpha < \omega_i \), there exists an \( a_\alpha \in A \) such that \( \pi_{\omega_i}(a_\alpha) > \alpha \).

Let \( K \in \mathcal{K}(\omega_1 \times \omega_2) \), be given, so there exists an \( \alpha < \omega_1 \) and \( \beta < \omega_2 \) such that \( K \subseteq \alpha_1 \times \alpha_2 \). Then \( a_\beta \notin K \) for all \( \beta \in [\alpha_1, \alpha_2) \), so \( (a_\beta)_{\beta \in \omega_1} \to * \). Therefore \( * \) is radial in \( X \).

Now suppose there exists a closed, almost-independent spoke system \( \mathcal{F} \) for \( * \) and define \( \Lambda := \{ \lambda < \omega_2 : \text{cf}(\lambda) = \omega_1 \} \). We claim that for all \( \lambda \in \Lambda \), there exists an \( \alpha_\lambda < \omega_1 \), a \( \beta_\lambda < \lambda \) and an \( S_\lambda \in \mathcal{F} \) such that \( (\alpha_\lambda, \omega_1) \times (\beta_\lambda, \lambda) \subseteq S_\lambda \). Before proving this claim, we will show how it will allow us to derive a contradiction.

Suppose that \( \{S_\lambda : \lambda \in \Lambda \} \) is uncountable and pick \( f : \omega_1 \to \Lambda \) strictly increasing such that for all distinct \( \alpha, \beta < \omega_1, S_f(\alpha) \neq S_f(\beta) \). Define \( \Lambda := \sup(\text{ran}(f)) \in \Lambda \). Then \( f \) is cofinal in \( \lambda \), so there exists a \( \gamma < \omega_1 \) such that \( f(\gamma) > \beta_\lambda \). Thus for all \( \delta \in [\gamma, \omega_1) \):

\[
\{\max(\alpha_\delta, \alpha_\lambda, \omega_1) \times \max(\beta_\delta, \beta_\lambda) \} \subseteq S_f(\delta) \cap S_\lambda
\]

Hence \( * \in \bigcup_{\gamma} (S_f(\gamma) \cap S_\lambda) \setminus \{\star\} \). Since \( \mathcal{F} \) is almost-independent, it follows that \( S_f(\delta) = S_\lambda \) and in particular \( S_f(\gamma) = S_f(\gamma + 1) \), which is a contradiction. Therefore \( \{S_\lambda : \lambda \in \Lambda \} \) is countable and so there exists an \( L \subseteq \Lambda \) of cardinality \( \aleph_2 \) such that \( S_\lambda = S_\mu \) for all \( \lambda, \mu \in L \). As each \( S_\lambda \) contains a non-trivial \( \omega_1 \)-sequence converging to \( * \), it follows that \( \chi(\star, S_\lambda) = \aleph_1 \). However, \( (\alpha_\lambda, \beta_\lambda)_{\lambda \in L} \) is an \( \omega_2 \)-sequence in \( \mathcal{K}(\omega_1) \) that converges to \( * \), which is a contradiction. Therefore \( * \) doesn’t have a closed, almost-independent spoke system.

We will now prove our claim. Let \( f : \omega_1 \to \omega_1 \times \lambda \) be given such that \( f \to * \). Suppose for all \( S \in \mathcal{F} \), there exists a \( U_S \in \mathcal{N}_S^X \) such that \( \text{ran}(f) \cap U_S = \emptyset \). Then \( U := \bigcup_{S \in \mathcal{F}} U_S \in \mathcal{N}_X^X \) and \( \text{ran}(f) \cap U = \emptyset \), which is a contradiction. Thus there exists an \( S \in \mathcal{F} \) such that \( * \in \text{ran}(f) \cap S \). Since \( * \) is a p-point, it follows that \( \text{ran}(f) \cap S \) is uncountable. Now let \( h : \omega_1 \to \lambda \) be cofinal and strictly increasing and continuous. Define for all \( \alpha < \omega_1 \), \( f(\alpha) := (\alpha, h(\alpha)) \). Then \( f \to * \), so by the work above there exists an \( S_\lambda \in \mathcal{F} \) such that \( \text{ran}(f) \cap S_\lambda \) is uncountable. Define \( A := \pi_{\omega_1}(\text{ran}(f) \cap S_\lambda) \).

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\(^2\)A point \( x \) in a topological space is a \textit{p-point} if countable intersections of neighbourhoods of \( x \) are again a neighbourhood.
Suppose for all \( \alpha < \omega_1 \), there exists an \( x_\alpha \in ([\alpha, \omega_1] \times [h(\alpha), \lambda]) \setminus S_\lambda \). Then \( (x_\alpha)_{\alpha < \omega_1} \to \star \), so again there exists a \( T \in \mathcal{S} \) such that \( B := \pi_{\omega_1}(\{x_\alpha : \alpha < \omega_1 \} \cap T) \) is uncountable. Since \( (x_\alpha : \alpha < \omega_1) \cap S_\lambda = \emptyset \), it follows that \( T \) is distinct from \( S_\lambda \), so \( (S_\lambda \cap T) \setminus \{\star\} \) has compact closure in \( \omega_1 \times \omega_2 \). In particular, its projection onto \( \omega_1 \) is bounded.

Let \( \beta < \omega_1 \) be given. As \( \lambda \) has uncountable cofinality, \( A' \cap B' \) is a club. Let \( \gamma \in [\beta, \omega_1) \cap A' \cap B' \) be given, so there exist strictly increasing sequences \( \langle \delta_n \rangle_{n < \omega} \subseteq A_\lambda \setminus \{\epsilon_0\} \subseteq \omega \) with supremum \( \gamma \). Then by continuity of \( h, (\gamma, h(\gamma)) \in [(\delta_n, h(\delta_n)) : n < \omega] \subseteq \text{ran}(f) \cap S_\lambda \subseteq S_\lambda \). Moreover, for each \( n < \omega \), there exists an \( \alpha_n < \omega_1 \) such that \( x_{\alpha_n} \in T \) and \( \pi_{\omega_1}(x_{\alpha_n}) = \epsilon_n \). Therefore, since \( \lambda \) is sequentially compact, by virtue of being an ordinal with uncountable cofinality, there exists a subsequence of \( \langle \pi_{\omega_1}(x_{\alpha_n}) \rangle_{n < \omega_1} \) that converges to some ordinal \( \theta < \lambda \) and so \( (\gamma, \theta) \in [x_\alpha : \alpha < B \cap T \subseteq T \). Hence \( \gamma \in \pi_{\omega_1}(\{S_\lambda \cap T\} \setminus \{\star\}) \).

But this then shows that \( \pi_{\omega_1}(\{S_\lambda \cap T\} \setminus \{\star\}) \) is unbounded, which is a contradiction. Thus there exists an \( \alpha_1 < \omega_1 \) such that \( (\alpha_1, \omega_1) \cap [h(\alpha_1), \lambda) \subseteq S_\lambda \). By defining \( \beta_1 := h(\alpha_1) \), we conclude the proof of our claim and the theorem.

\[ \square \]

To finish this section, we will exhibit an alternative characterisation of radiality by ordering our spokes by local containment. This has the added advantage of characterising the subspaces which are radial at a specified point.

**Definition 1.10** (Locally contained). Let \( X \) be a topological space, \( x \in X, A, B \subseteq X \) be given. Then we say \( x \in A \) is contained in \( B \), written \( A \subseteq^x B \), if there exists a \( U \in \mathcal{N}_x \) such that \( A \cap U \subseteq B \), or equivalently, \( x \in A \setminus B \). If the ambient space \( X \) is unambiguous, we will drop the superscript in \( \subseteq^x \).

We will endow \( \text{Sp}(x, X) \) and \( \overline{\text{Sp}}(x, X) \) with this ordering and consider cofinal subsets of this quasi-ordered set.

**Lemma 1.11.** Let \( X \) be a topological space and let \( Y \subseteq X, x \in Y, S \subseteq \text{Sp}(x, Y) \) be given. Then \( S \cup N^X_x \subseteq \text{Sp}(x, X) \).

**Proof.** Let \( \{B_\alpha : \alpha < \lambda\} \) be a well-ordered neighbourhood base for \( x \) with respect to \( Y \), where for all \( \alpha, \beta < \lambda \) with \( \alpha < \beta \), \( B_\beta \subseteq B_\alpha \). Let \( U \in \mathcal{N}_x^Y \) be given, so there exists an \( \alpha < \lambda \) such that \( B_\alpha \subseteq U \cap S \). Then \( B_\alpha \cup N^X_x \subseteq U \cap (S \cup N^X_x) \). Moreover, for all \( \alpha < \lambda \), there exists a \( V \in \mathcal{N}_x \) such that \( B_\alpha = V \cap S \) and so \( B_\alpha \cup N^X_x = V \cap (S \cup N^X_x) \in \mathcal{N}_x^{S \cup N^X_x} \). Therefore \( \{B_\alpha \cup N^X_x : \alpha < \lambda\} \) is a well-ordered neighbourhood base for \( x \) with respect to \( S \cup N^X_x \). Hence \( S \cup N^X_x \subseteq \text{Sp}(x, X) \).

\[ \square \]

**Theorem 1.12.** Let \( X \) be a topological space, \( Y \subseteq X, x \in Y \) be given. Then the following are equivalent:

1. \( Y \) is radial at \( x \).
2. For all \( \mathcal{C} \subseteq \text{Sp}(x, X) \) cofinal, \( Y \subseteq^{x} \cap \mathcal{C} \).

Moreover, if \( X \) is compact and Hausdorff, then the two conditions above are equivalent to:

3. For all \( \mathcal{C} \subseteq \overline{\text{Sp}}(x, X) \) cofinal, \( Y \subseteq^{x} \cup \mathcal{C} \).

**Proof.** Suppose \( Y \) is radial at \( x \), so there exists a spoke system \( \mathcal{S} \) for \( x \) with respect to \( Y \). Let \( \mathcal{C} \subseteq \text{Sp}(x, X) \) be cofinal, so by the previous lemma for all \( S \in \mathcal{S} \), there exists a \( C_S \in \mathcal{C} \) such that \( S \cup N^X_x \subseteq C_S \) and thus there exists a \( U_S \in \mathcal{N}_x^Y \) such that \( (S \cap U_S) \cup N^X_x = (S \cup N^X_x) \cap U_S \subseteq C_S \). Define:

\[ U := S \cap U_S \in \mathcal{N}_x^Y \]

Then there exists a \( V \in \mathcal{N}_x^Y \) such that \( U = V \cap Y \). As \( U \subseteq \mathcal{C} \), it follows that \( Y \subseteq^{x} \cap \mathcal{C} \). Thus (1) implies (2). Moreover, if \( X \) is compact and Hausdorff, then we can take \( \mathcal{S} \) to consist of closed spokes by Theorem 1.8 and \( \mathcal{C} \subseteq \text{Sp}(x, X) \). Therefore (1) implies (3) too.

Now suppose that for all \( \mathcal{C} \subseteq \text{Sp}(x, X) \) cofinal, \( Y \subseteq^{x} \cup \mathcal{C} \). We will show that \( \{S \cap Y : S \in \text{Sp}(x, X)\} \) is a spoke system for \( x \) with respect to \( Y \). For all \( S \in \text{Sp}(x, X) \), let \( U_S \in \mathcal{N}_x^Y \) be given, so \( U_S \cap Y \in \mathcal{N}_x^Y \). Note that \( (S \cap U_S) \in \text{Sp}(x, X) \) is vacuously cofinal in \( \text{Sp}(x, X) \), so there exists a \( V \in \mathcal{N}_x^Y \) such that \( V \cap Y \subseteq \bigcup_{S \in \mathcal{S}} (S \cap U_S) \). Then \( V \cap Y \subseteq \bigcup_{S \in \mathcal{S}} ((S \cap Y) \cap (U_S \cap Y)) \), so the latter is a neighbourhood of \( x \) with respect to \( Y \). Therefore \( \{S \cap Y : S \in \text{Sp}(x, X)\} \) is a spoke system for \( x \) with respect to \( Y \) and thus \( Y \) is radial at \( x \) by Theorem 1.5. Hence (2) implies (1). Finally, note that if \( X \) is compact and Hausdorff then by replacing \( \text{Sp}(x, X) \) with \( \overline{\text{Sp}}(x, X) \), we see that (3) implies (1), concluding our proof.

\[ \square \]
Corollary 1.13. Let $X$ be a topological space, $x \in X$ be given. Then $X$ is radial at $x$ if and only if for all cofinal collections of spokes $\mathcal{C}, \cup \mathcal{C}$ is a neighbourhood of $x$.

Proof. By the previous theorem, $X$ is radial at $x$ if and only if $X \subseteq x \cup \mathcal{C}$ for all cofinal collections $\mathcal{C} \subseteq \text{Sp}(x, X)$, which is equivalent $\cup \mathcal{C} \in \mathcal{N}_x$. \qed

2 One-point compactifications

For the rest of this article, unless otherwise stated, we will assume that $X$ is a locally compact, non-compact Hausdorff space.

In this section, we will use our spoke characterisations to characterise being radial at $\star \in \alpha X$. Spoke systems and cofinal collections of spokes allow us to reflect these properties from compactifications down to the structure of the compact subsets of $X$. We will first show that it suffices to consider the points in the remainder.

Lemma 2.1. Let $X$ be a topological space, $U \subseteq X$ be open. If $U$ is radial and $X$ is radial at every point outside $U$ then $X$ is radial.

Proof. Let $u \in U, A \subseteq X$ be given such that $u \in \overline{A}$. Then for each $V \subseteq X$ open, if $u \in V$ then $V \cap A \neq \emptyset$. In particular, for each $V \subseteq U$ open, $V \cap A \neq \emptyset$ and so $u \in A \cap U$. Thus there exists a transfinite sequence contained in $A \cap U$ that converges to $u$ and therefore $X$ is radial at $u$. \qed

Definition 2.2 (Spoke at infinity). Let $S \subseteq X$ be given such that $S \cup \{\star\}$ is a spoke at $\star$ in $\alpha X$. Then we say that $S$ is a spoke at infinity of $X$. We will denote the set of (closed) spokes at infinity by $\text{Sp}^\infty(X) (\overline{\text{Sp}}^\infty(X))$.

Lemma 2.3. Let $S \subseteq X$ be closed. Then $S$ is a spoke at infinity if and only if there exists a cofinal chain in $(\mathcal{K}(S), \subseteq)$.

Proof. Assume $S$ is a spoke at infinity, so there exists a well-ordered neighbourhood base $\mathcal{B}$ of $\star$ in $S \cup \{\star\}$. By taking interiors, we can assume that $\mathcal{B}$ consists of open sets. Then $(S \setminus B : B \in \mathcal{B})$ is a chain in $(\mathcal{K}(S), \subseteq)$. Moreover, for all $K \in \mathcal{K}(S)$, there exists a $B \in \mathcal{B}$ such that $B \subseteq \alpha X \setminus K$ and so $K \subseteq S \setminus B$. Therefore $(S \setminus B : B \in \mathcal{B})$ is cofinal in $(\mathcal{K}(S), \subseteq)$.

Now assume that there exists a cofinal chain in $(\mathcal{K}(S), \subseteq)$, so by considering its cofinality, there exists an increasing, cofinal, transfinite sequence $(K_{\alpha})_{\alpha \prec \lambda} \subseteq \mathcal{K}(S)$. Then $(S \cup \{\star\}) \setminus K_{\alpha}$ is a neighbourhood of $\star$ in $S$ for each $\alpha \prec \lambda$. Let $V \subseteq \alpha X$ be open with $\star \in V$, so $X \setminus V$ is compact and hence $S \setminus V$ has compact closure in $S$. Thus there exists an $\alpha < \lambda$ such that $S \setminus V \subseteq K_{\alpha}$ and so $(S \cup \{\star\}) \setminus K_{\alpha} \subseteq S \setminus V$. Therefore $((S \cup \{\star\}) \setminus K_{\alpha})_{\alpha \prec \lambda}$ is a well-ordered neighbourhood base for $\star$ in $S \cup \{\star\}$ and hence $S$ is a spoke at infinity. \qed

The following theorem demonstrates an internal characterisation for radially at infinity, using the spoke system criterion.

Theorem 2.4. $\alpha X$ is radial at $\star$ if and only if there exists a collection $\mathcal{C} \subseteq \overline{\text{Sp}}^\infty(X)$ such that for all $C \in \prod_{\mathcal{C} \subseteq \mathcal{K}(S), \cup \mathcal{C} \subseteq C(S)}$ has co-compact interior.

Proof. Let $\mathcal{C} \subseteq \overline{\text{Sp}}^\infty(\alpha X)$ be given. Then $\mathcal{C}$ is a spoke system at $\star$ if and only if for all $C \in \prod_{\mathcal{C} \subseteq \mathcal{K}(S), \cup \mathcal{C} \subseteq C(S)} \mathcal{N}_x$, $\cup \mathcal{C} \subseteq C(S) \subseteq \mathcal{N}_x$. Since $\alpha X$ is compact, this is equivalent to $\cup \mathcal{C} \subseteq (C(S) \setminus \{\star\})$ having co-compact interior in $X$ for all $C \in \prod_{\mathcal{C} \subseteq \mathcal{K}(S \setminus \{\star\})}$. Thus by Theorem 1.8, the proof is complete. \qed

We also have the following characterisation in terms of spokes at infinity, purely from the radiality property itself.

Theorem 2.5. $\alpha X$ is radial at $\star$ if and only if for all $Y \subseteq X$ with non-compact closure in $X$, there exists a non-compact $Z \in \overline{\text{Sp}}^\infty(X)$ such that $K_{\alpha} = K_{\alpha} \setminus Y$ for all $\alpha < \lambda$, where $(K_{\alpha})_{\alpha < \lambda}$ is a cofinal chain in $(\mathcal{K}(Z), \subseteq)$.

\footnote{A subset is co-compact if its complement is compact.}
Proof. Suppose that \( \alpha X \) is radial at \( \star \) and let \( Y \subseteq X \) have non-compact closure in \( X \), so \( \star \in \overline{Y} \). Then by radiality there exists an injective transfinite sequence \( f : \lambda \rightarrow Y \) that converges strictly to \( \star \) (see [Lee14, Lemma 2.2, pg. 12]), so \( S(f) \in \overline{Sp}^\infty(X) \) and \( (f(\beta))_{\beta \in \lambda} \) is a cofinal chain in \( (\mathcal{K}(S(f)), \subseteq) \). Let \( \beta = \lambda \) be given. Then:

\[
f(\beta) \subseteq f(\beta) \cap Y \subseteq f(\beta) \cap Y \subseteq f(\beta)
\]

Hence \( f(\beta) = f(\beta) \cap Y \). Also, \( S(f) \) is non-compact, since \( \star \in \overline{S(f)}^{\alpha X} \).

Now suppose the converse holds and let \( A \subseteq X \) be given such that \( \star \in A \), so \( A \) has non-compact closure. Then there exists a non-compact \( Y \in \overline{Sp}^\infty(X) \), with \( (K_\beta)_{\beta \in \lambda} \) a strictly increasing, cofinal chain in \( \mathcal{K}(Y) \), such that \( K_\beta = K_\beta \cap A \) for all \( \beta < \lambda \). Since \( Y \) is non-compact, \( A \) must be a limit ordinal. Let \( \beta < \lambda \) be given, so \( K_\beta \cap A = K_\beta \subseteq K_{\beta+1} = K_{\beta+1} \cap A \) and hence there exists an \( x_\beta \in (K_{\beta+1} \setminus K_\beta) \cap A \). Now since \( ((\star) \cup (Y \setminus K_\beta))_{\beta \in \lambda} \) is a neighbourhood base for \( \star \) with respect to \( Y \cup \{ \star \} \) (by the proof of Lemma 2.3), it follows that \( (x_\beta)_{\beta \in \lambda} \) converges to \( \star \) and is contained in \( A \). Therefore \( \alpha X \) is radial at \( \star \).

Corollary 2.6. Suppose \( \alpha X \) is radial at \( \star \). Then for all \( A \subseteq X \) closed and non-compact, there exists a non-compact \( S \in \overline{Sp}^\infty(X) \) contained in \( A \).

Proof. By picking \( S \) and \( (K_\alpha)_{\alpha < \lambda} \) from the previous theorem, it follows that \( K_\alpha = K_\alpha \cap A \subseteq A \) for all \( \alpha < \lambda \) and so \( S = \bigcup_{\alpha < \lambda} K_\alpha \subseteq A \).}

Of course, the preceding corollary is not surprising when our spokes at infinity are \( \sigma \)-compact, for we can then take an \( \omega \)-sequence converging to \( \star \). However, this is more an artefact of \( T_1 \) implying finite subsets are closed. If \( \star \) is a p-point, then the spokes will contain closures of countably-infinite subsets, which could potentially be large. Unfortunately, even though this corollary is a more natural condition, it is not equivalent to radiality at \( \star \):

Theorem 2.7. There exists a non-compact, locally compact Hausdorff space such that for all \( A \subseteq X \) closed and non-compact, there exists a non-compact \( S \in \overline{Sp}^\infty(X) \) with \( S \subseteq A \), yet \( \alpha X \) is not radial at \( \star \).

Proof. Define the deleted Tychonoff plank to be \( X := ((\omega + 1) \times (\omega_1 + 1)) \setminus ((\omega, \omega_1)) \) and observe that \( \alpha X \equiv (\omega + 1) \times (\omega_1 + 1) \), so \( \alpha X \) is not radial at \( \star = (\omega, \omega_1) \) (as noted in [Lee14, pg. 12-13]). Let \( A \subseteq X \) be closed and non-compact and suppose \( \pi_\omega[A \cap (\omega \times (\omega_1))] \) and \( \pi_{\omega_1}[A \cap (\omega \times (\omega_1))] \) are bounded in \( \omega \) and \( \omega_1 \) respectively. Then there exists an \( n \in \omega \) such that \( A \subseteq ((\omega + 1) \times (\omega_1 + 1)) \setminus ((n, \omega) \times (\omega, \omega_1)) \). Thus \( A \subseteq (n \times (\omega + 1)) \cup ((\omega + 1) \times (\alpha + 1)) \), which is compact and hence a contradiction. Therefore either \( \pi_\omega[A \cap (\omega \times (\omega_1))] \) is unbounded in \( \omega \) or \( \pi_{\omega_1}[A \cap (\omega \times (\omega_1))] \) is unbounded in \( \omega_1 \). As \((\omega, \omega_1) \) and \((\omega \times \omega_1) \) are easily seen to be spokes at infinity, it follows that \( A \cap (\omega \times (\omega_1)) \in \overline{Sp}^\infty(X) \), \( A \cap (\omega \times (\alpha + 1)) \in \overline{Sp}^\infty(X) \) and one of these is non-compact. This completes the proof.

We will now present the third characterisation using cofinal spoke collections. We first need to translate the ordering on spokes of \( \star \) to spokes at infinity.

Definition 2.8. Let \( X \) be a non-compact, locally compact Hausdorff space. For all \( S, T \in \overline{Sp}^\infty(X) \), we define \( S \leq T \) if \( ST \) has compact closure. Observe that for \( S, T \in \overline{Sp}^\infty(X) \), \( S \cup \{ \star \} = \overline{X} \ T \cup \{ \star \} \) if and only if \( S \leq T \). We will endow \( \overline{Sp}^\infty(X) \) and \( \overline{Sp}^\infty(X) \) with this quasi-order.

Theorem 2.9. \( \alpha X \) is radial at \( \star \) if and only if for all \( C \subseteq \overline{Sp}^\infty(X) \) cofinal, \( \bigcup C \) has co-compact interior.

Proof. Assume \( \alpha X \) is radial at \( \star \) and let \( \mathcal{C} \subseteq \overline{Sp}^\infty(X) \) be cofinal. Then for all \( S \in \overline{Sp}^\infty(X) \), there exists a \( T_\mathcal{C} \in \mathcal{C} \) such that \( C_\mathcal{C} := ST_\mathcal{C} \) is compact. As \( \alpha X \) is radial at \( \star \), it follows by Theorem 2.4 that \( \bigcup\overline{Sp}^\infty(X)(S \cap C_\mathcal{C}) \) has co-compact interior. Note that for all \( S \in \overline{Sp}^\infty(X) \), \( S \cap C_\mathcal{C} \subseteq S \setminus (S \setminus T_\mathcal{C}) \subseteq T_\mathcal{C} \). Thus \( \bigcup\overline{Sp}^\infty(X)(S \cap C_\mathcal{C}) \subseteq \bigcup C \), so \( \bigcup C \) also has co-compact interior.

Now assume that for all \( C \subseteq \overline{Sp}^\infty(X) \) cofinal, \( \bigcup C \) has co-compact interior. Let \( C \in \prod_{S \in \overline{Sp}^\infty(X)} \mathcal{K}(S) \) be given and define \( \mathcal{C} := \{ S \cap C(S) : S \in \overline{Sp}^\infty(X) \} \). Then \( \mathcal{C} \) is cofinal in \( \overline{Sp}^\infty(X) \) and hence \( \bigcup \mathcal{C} \) has co-compact interior. Therefore by Theorem 2.4 again, \( \alpha X \) is radial at \( \star \).

We will now analyse two spaces, which are known to not be radial at infinity, and proving this fact using these theorems.
2.1 Deleted Tychonoff plank

Let $P$ denote the deleted Tychonoff plank and as before we may take $\alpha P = (\omega + 1) \times (\omega + 1)$ and $\star = (\omega, \omega_1)$. Define $S_0 := \omega \times \omega$, $S_1 := \omega_1 \times \omega_1$. Note that in an ordinal space, every compact subset is bounded and $S_0 = \bigcup_{n \in \omega} ((n \times \omega_1), S_1 = \bigcup_{n < \omega} ((\omega) \times (\alpha + 1)$, so $S_0, S_1 \in \mathbf{Sp}^\infty(X)$. We will show that $\{S_0, S_1\}$ is a cofinal collection of closed spokes at infinity.

Let $S \in \mathbf{Sp}^\infty(P)$ be non-compact and let $(K_\alpha)_{\alpha < \lambda} \subseteq \mathcal{K}(S)$ be a cofinal chain with $\lambda$ infinite. Without loss of generality, assume $\lambda$ is regular and for all $\alpha < \lambda, K_\alpha \subseteq K_{\alpha+1}$. Since $|P| = \aleph_1$, either $\lambda = \omega$ or $\lambda = \omega_1$.

Case 1: Suppose $\lambda = \omega$ and consider $S \cap S_1$. Then $(\mathcal{K}(S \cap S_1), \subseteq)$ has cofinal chains of lengths $\omega$ and $\omega_1$, so $S \cap S_1$ must be compact and hence there exists a $\beta < \omega_1$ such that $S \cap S_1 \subseteq (\omega) \times \beta$. Assume $S \not\subseteq S_0$, so for all $n < \omega$, there exists an $x_n \in S \cap ((n \times (\omega_1 + 1))) \cup (\omega_1 + 1) \cup S_0) = S \cap ((\omega_1 + 1) \setminus (\omega_1 \setminus \beta))$.

Since $\omega_1$ is sequentially compact, there exists a strictly increasing sequence $(r_n)_{n < \omega}$ in $\omega$ and $\gamma \in \omega_1\beta$ such that $(\pi_{\omega_1}(x_{r_n}))_{n < \omega} \rightarrow \gamma$ and hence $(x_{r_n})_{n < \omega} \rightarrow (\omega, \gamma)$. Therefore, since $S$ is closed, $(\omega, \gamma) \in S \cap S_1$, which is a contradiction. Therefore $S \subseteq S_0$.

Case 2: Suppose $\lambda = \omega_1$ and consider $S \cap S_0$. Again, $(\mathcal{K}(S \cap S_0), \subseteq)$ has cofinal chains of lengths $\omega$ and $\omega_1$, so $S \cap S_0$ must be compact and hence there exists an $n < \omega$ such that $S \cap S_0 \subseteq n \times \omega_1$. Since $S$ is closed, for all $m \geq n$ there exists a $\beta_m < \omega_1$ such that $S \cap \{(m \times ((\omega_1 + 1)) \setminus (\beta_m) = \emptyset$. Define $\beta := \sup(|\beta_m : m \geq n|) < \omega_1$, so $S \cap ((\omega_1 \times ((\omega_1 + 1) \setminus (\beta)) = \emptyset$. Then:

$$S \cap ((n \times (\omega_1 + 1)) \cup (\omega_1 + 1) \cup (\beta + 1) \cup S_1) = (S \setminus S_1) \cap (((\omega_1 + 1) \setminus (\omega_1 + 1) \setminus (\beta + 1)) = \emptyset$$

Therefore $S \subseteq S_1$.

Vacuously, every compact spoke at infinity is bounded above by $S_0$, so it follows that $\{S_0, S_1\}$ is a cofinal collection of paths to infinity. However, $S_0 \cup S_1$ has empty, and hence non-co-compact, interior in $P$, so $\alpha P$ is not radial at $\star$.

We also obtain a local result from Theorem 1.12: since $\{S_0, S_1\}$ is a cofinal collection of closed spokes at infinity, any subspace of $((\omega) \times (\omega_1 + 1)$ that is radial at $((\omega, \omega_1)$ must be locally contained at $((\omega, \omega_1)$ in $S_0 \cup S_1 \cup \{(\omega, \omega_1)\}$; indeed, as $S_0 \cup S_1 \cup \{(\omega, \omega_1)\}$ is a finite union of spokes at $((\omega, \omega_1)$, it is radial at $((\omega, \omega_1)$ and even a radial space.

2.2 Mrówka spaces

Let $\mathcal{A}$ be a maximal, almost-disjoint (m.a.d.) family of subsets of $\omega$; that is, a maximal collection of infinite subsets of $\omega$ such that any two distinct elements intersect finitely. We will define a topology on $\omega \cup \mathcal{A}$ as follows: let each $n \in \omega$ be isolated and for all $A \in \mathcal{A}$, let $\{\{A\} \cup (A \setminus F) : F \subseteq \omega$ is finite$\}$ be a neighbourhood base for $A$. We denote this space by $\Psi$ and call it a Mrówka space. By [Fra67, Example 7.1, pg. 54-55], it is non-compact, locally compact and Hausdorff. Moreover, it's one-point compactification is not radial at $\star$ since there is no (transfinite) sequence in $\omega$ converging to $\star$. We will now show that the countably infinite subsets of $\mathcal{A}$ form a cofinal collection of closed spokes at infinity, witnessing this fact.

First note that $\mathcal{A}$ is closed and discrete in $\Psi$, so it easily follows that every countably infinite subset of $\mathcal{A}$ is a $\sigma$-compact spoke at infinity. Let $S \in \mathbf{Sp}^\infty(\Psi)$ be non-compact. Then since no (transfinite) sequence in $\omega$ converges to $\star$, it follows that $S \cap \mathcal{A}$ must be a non-compact, closed spoke at infinity, since $\mathcal{A}$ is closed. Since $\mathcal{A}$ is closed, there are no infinite compact subsets of $\mathcal{A}$, so by Lemma 2.3 $S \cap \mathcal{A}$ is countably infinite. Thus $S = (S \cap \mathcal{A}) \cup (S \cap \omega)$ is countably infinite also.

Now suppose for all $\mathcal{F} \subseteq S \cap \mathcal{A}$ finite, $(S \cap \omega) \setminus (\cup \mathcal{F})$ is infinite and define

$$\mathcal{B} := \{A \in S \cap \mathcal{A} : A \subseteq S \text{ is infinite}\}.$$

Assume $\mathcal{B}$ is finite. Then by maximality, there exists an $A \in \mathcal{A}$ such that $(A \cap S) \setminus (\cup \mathcal{B})$ is infinite and thus $A \subseteq S = S$, which is a contradiction. Therefore $\mathcal{B}$ is infinite, so we can pick an enumeration
$\mathcal{B} = \{B_n : n < \omega\}$. Then for all $n < \omega$, there exists an $x_n \in (S \cap B_n) \setminus \{(x_m : m < n) \cup \bigcup_{m<n} B_m\}$ and so by maximality there is an $A \in \mathcal{A}$ such that $A \cap \{x_n : n < \omega\}$ is infinite. Then we get a contradiction, since $A \in \mathcal{S} = S$ and $A \cap S$ is infinite, but $A \cap B_n$ is finite for all $n < \omega$. Thus there exists a finite $F \subseteq S \cap \mathcal{A}$ such that $C := (S \cap \omega) \setminus (\cup F)$ is finite. Then $S \cap \omega \subseteq C \cup \bigcup_{A \in \mathcal{A}} ((A) \cup A)$ and the latter is compact, so $S \subseteq S \cap \mathcal{A}$. Therefore $|\mathcal{A}|^\omega := |\mathcal{A}' | \subseteq |\mathcal{A}' | = \aleph_0$ is a cofinal collection of paths to infinity. Note that $\bigcup |\mathcal{A}|^\omega = \mathcal{A}$ has empty, and hence non-co-compact, interior in $\Psi$, so $\alpha \Psi$ is not radial at $\ast$.

3 Beyond the one-point compactification

We will now investigate larger compactifications, assuming that $\alpha X$ is radial at $\ast$. We will start with finite and countable compactifications; in fact, we will demonstrate results for ordinal compactifications - those which have remainder homeomorphic to some ordinal. We also obtain conditions for the existence of sequential / pseudoradial compactifications.

Recall that we can obtain a one-point compactification of $X$ by identifying the remainder to a single point (see [Eng89, Theorems 3.5.12 & 3.5.13, pg. 170]). We will be implicitly using this identification from now on.

3.1 Finite, countable and ordinal compactifications

The following lemma demonstrates the usefulness of a space $X$ with the property that $\alpha X$ is radial at $\ast$.

**Lemma 3.1.** Let $\gamma X$ be an ordinal compactification of $X$; that is, $\gamma X \setminus X$ is homeomorphic to some ordinal and suppose $\alpha X$ is radial at $\ast$. Let $f : \lambda \to X$ be a transfinite sequence converges to $\ast$ in $\alpha X$. Then there is a subsequence$^4$ of $f$ that converges to some point in $\gamma X \setminus X$.

**Proof.** Assume not and pick an ordinal $\alpha$ such that $\gamma X \setminus X \equiv \alpha + 1$ (since $\gamma X \setminus X$ is compact and non-empty). We identify $\gamma X \setminus X$ with $\alpha + 1$. By assumption, $\alpha$ must be non-zero.

Define $\lambda := \text{dom}(f)$ and suppose there exists an $m < \omega$ and a strictly decreasing sequence of ordinals $(\beta_n : n \leq m)$ in $\alpha + 1$ and $(U_n : n < m)$ a sequence of open subsets of $\gamma X$ such that:

- $\beta_0 = \alpha$,
- $\beta_m > 0$,
- $U_n \cap X = (\beta_{n+1}, \beta_n]$ for all $n < m$,
- $D := \lambda \setminus f^{-1}[\bigcup_{n < m} U_n]$ is unbounded.

Then since no subsequence of $f$ converges to $\beta_m$, there exists an open subset $V \subseteq \gamma X$ such that $\beta_m \in V$ and $D \setminus f^{-1}[V] = \lambda \setminus f^{-1}[V \cup \bigcup_{n < m} U_n]$ is unbounded. Assume $[0, \beta_m] \subseteq V$. Then $U := V \cup \bigcup_{n < m} U_n$ is a neighbourhood of $\gamma X \setminus X$ and $\lambda \setminus f^{-1}[U]$ is unbounded, which is a contradiction since $f \to \ast$ in $\alpha X$. Thus there exists a $\beta_{m+1} \in (0, \beta_m)$ such that $[\beta_{m+1}, \beta_m] \subseteq V$ and furthermore there exists an open subset $W \subseteq \gamma X$ such that $[\beta_{m+1}, \beta_m] = W \setminus X$. Define $U_m := V \cap W$ and note that $\lambda \setminus f^{-1}[\bigcup_{n < m} U_m]$ is unbounded and $\beta_{m+1}, \beta_m] = U_m \setminus X$.

Therefore by recursion, we find a descending sequence in $\alpha + 1$, which is a contradiction. Hence there is a subsequence of $f$ that converges to some point in $\gamma X \setminus X$.

$^4$A subsequence of a transfinite sequence $f : \lambda \to X$ is a transfinite sequence of the form $f \circ g : \mu \to X$, where $g : \mu \to \lambda$ is strictly increasing.
In this section, we will use small cardinals to improve the results from the last section - these are uncountable cardinals bounded above by \( \epsilon := 2^{\aleph_0} \). The small cardinals we are using are defined below.

**Definition 3.8 (Small cardinals).**

- For \( f, g : \omega \to \omega \), we say that \( f \) is eventually bounded by \( g \) if \( \{ n \in \omega : f(n) > g(n) \} \) is finite. We denote this relation by \( f \preceq^* g \). Observe that \( \preceq^* \) is a quasi-order on \( \omega^\omega \).
• The bounding number, denoted by $b$, is the smallest cardinality of an unbounded subset of \((^\omega \omega, \leq^*)\).

• For $A, B \subseteq \omega$, we say that $A$ is almost-contained in $B$, written $A \subseteq^* B$, if $A \setminus B$ is finite.

• A pseudointersection of a family $\mathcal{F}$ of subsets of $\omega$ is a subset $P \subseteq \omega$ such that $P \subseteq^* F$ for all $F \in \mathcal{F}$.

• A family $\mathcal{P}$ of infinite subsets of $\omega$ has the strong finite intersection property if $\bigcap \mathcal{P}$ is infinite for all finite and non-empty $\mathcal{F} \subsetneq \mathcal{P}$. The pseudointersection number, denoted by $p$, is the smallest cardinality of a family of subsets of $\omega$ with no infinite pseudointersection.

• A tower is a transfinite sequence $(T_\alpha)_{\alpha<\omega_1}$ of infinite subsets of $\omega$ such that $T_\beta \subseteq^* T_\alpha \not\subseteq^* T_\beta$ for all $\alpha < \beta < \omega_1$. The tower number, denoted by $t$, is the smallest cardinality of a tower with no infinite pseudointersection.

All three cardinals $b, p, t$ are well-defined small cardinals - see [vD84]. We will also use a ‘not so small’ cardinal.

**Definition 3.9 (Novak number).** We define the Novak number, denoted by $n$, to be the smallest cardinality of a nowhere-dense cover of $\omega^*$.

We recall some facts about the Novak number from [BN10]: $t < n \leq 2^t$ and it is independent of ZFC whether $2^t$ or $n$ is bounded by the other. Moreover, every compact Hausdorff space of cardinality less than $\max(2^t, n)$ is sequentially compact.

We will now use these cardinals to obtain Fréchet-Urysohn and sequential compactifications, provided we know that our remainder is already Fréchet-Urysohn and sequential respectively. The following theorems are similar in spirit and can be summarised by the following meta-theorem: “Any theorem that implies certain convergence properties (e.g. sequentiality implying Fréchet-Urysohn, subsequentiality, sequential compactness) can be used to obtain compactification results.”

**Theorem 3.10.** Assume $\alpha X$ is Fréchet-Urysohn at $\star$. Then every compactification of $X$ with sequential remainder of cardinality strictly less than $\max(2^t, n)$ is sequential.

**Proof.** Let $\gamma X$ be a compactification of $X$ such that $\gamma X \setminus X$ is sequential and $|\gamma X \setminus X| < 2^t$. Note that if $A \subseteq X$ is a sequence that converges to $\star$ in $\alpha X$, then $\overline{A}^{\gamma X} = A \cup (\overline{A}^{\gamma X} \setminus X)$, so $|\overline{A}^{\gamma X}| < 2^t$ and hence is sequentially compact by [BN10]. Therefore by Lemma 3.5 $\gamma X$ is sequential. $\square$

The following theorem is an adaptation of [vD84, Theorem 6.2, pg. 129].

**Theorem 3.11.** Let $X$ be a topological space, $x \in X$ be given such that $\chi(x, X) < p$. Then $X$ is subsequential at $x$, that is, for all $A \in |X|^N$, if $x \in \overline{A}$ then there is a sequence of $A$ that converges to $x$.

**Proof.** Let $A \in |X|^N$ be given such that $x \in \overline{A}$. If there exists a $y \in A \cap N_x$ then $(y)_{n<\omega} \to x$, so suppose $N_x \cap A = \emptyset$. Let $\mathcal{B}$ be a neighbourhood base for $x$ with $|\mathcal{B}| < p$. Then $|A \cap B : B \in \mathcal{B}|$ has the strong finite intersection property, so there exists an infinite subset $C \subseteq A$ such that $C \subseteq^* B$ for all $B \in \mathcal{B}$. Hence $C \to x$ and so $x$ is subsequential. $\square$

**Theorem 3.12.** Suppose $\alpha X$ is Fréchet-Urysohn at $\star$ and $\chi(x, X) < b$ for all $x \in X$. Let $\gamma X$ be a compactification of $X$ such that:

• $\gamma X \setminus X$ is sequential.
• $|\gamma X \setminus X| < \max(2^t, n)$.
• $\chi(y, \gamma X) < b$ for all $y \in \gamma X \setminus X$.

Then $\gamma X$ is Fréchet-Urysohn.

**Proof.** By Theorem 3.10, $\gamma X$ is sequential, so by [BBM13, Proposition 3.4, pg. 534] it follows that $\gamma X$ is Fréchet-Urysohn, since $\chi(x, X) = \chi(x, \gamma X)$ for all $x \in X$ as $X$ is locally compact and thus open in $\gamma X$. $\square$
Lemma 3.13. Assume $X$ is countable and let $\gamma X$ be a compactification of $X$ such that $\chi(x, \gamma X) < t$ for all $x \in \gamma X \setminus X$. Then for all $A \subseteq X$ and $x \in \overline{A}^X$, there exists a sequence in $A$ that converges to $x$. In particular, $\gamma X$ is sequentially separable$^5$.

Proof. First, note that $\alpha X$ is a countable, compact Hausdorff space so is homeomorphic to an ordinal; in particular, $\alpha X$, and hence $X$, is Fréchet-Urysohn. Let $A \subseteq X, x \in \overline{A}^X$ be given. If $x \in X$ then $x \in \overline{A}^X$ and so there exists a sequence in $A$ that converges to $x$. Now suppose that $x \notin X$. By [MS13] and Theorem 3.11, there exists a sequence in $A$ that converges to $x$.

As $X$ is dense in $\gamma X$, it follows that $\gamma X$ is sequentially separable. □

Theorem 3.14. Suppose $X$ is countable and let $\gamma X$ be a compactification of $X$ such that $\gamma X \setminus X$ is Fréchet-Urysohn and $\chi(x, \gamma X) < t$ for all $x \in \gamma X \setminus X$. Then $\gamma X$ is Fréchet-Urysohn.

Proof. Let $A \subseteq \gamma X, x \in \overline{A} \setminus A$ be given. Notice that $A X$ is a countable, compact Hausdorff space so is homeomorphic to an ordinal and hence Fréchet-Urysohn. Thus $X$ is also Fréchet-Urysohn.

If $x \in X$ then there exists a sequence contained in $A \cap X$ that converges to $x$. Suppose $x \notin X$. Then $x \in \overline{A} \setminus X$ since $\gamma X \setminus X$ is Fréchet-Urysohn, there exists a sequence in $A$ that converges to $x$. Otherwise, by the previous lemma there also exists a sequence in $A$ that converges to $x$. Therefore $\gamma X$ is Fréchet-Urysohn. □

Question 3.15. Are any of the bounds in this section strict?

4 Open questions

The focus of this paper has been on building up compactifications from below, starting with the one-point compactification and extending results beyond that. The author believes that the existence of a maximal radial / Fréchet-Urysohn compactification should be a fruitful line of investigation. However, not every space has such a compactification. To show this, we first need the following lemma:

Lemma 4.1. The infinite continuous images of $\omega + 1$ are homeomorphic to itself.

Proof. Let $f : \omega + 1 \to X$ be continuous and surjective and suppose that $X$ is infinite. Then $X$ is homeomorphic to an infinite successor ordinal. Without loss of generality, suppose $X = \alpha + 1$, where $\alpha$ is a countable ordinal and suppose $\alpha \geq 2 - \omega$. Then $f^{-1}([0, \omega])$ and $f^{-1}([\omega + 1, 2 - \omega + 1])$ are infinite and closed, so intersect, which is a contradiction. Thus $\alpha = \omega + n$ for some $n < \omega$ and hence $X \cong \omega + 1$. □

Theorem 4.2. $X := (\omega + 1) \times \omega_1$ is a locally compact, non-compact, first-countable Hausdorff space with no maximal radial compactification, yet $\alpha X$ is radial.

Proof. First note that both $\omega + 1$ and $\omega_1$ are first-countable, so $X$ is Fréchet-Urysohn and hence radial. Let $A \subseteq X$ be given such that $\star \in \overline{A}^X$. Then for all $\beta < \omega_1, A \not\subseteq (\omega + 1) \times (\beta + 1)$, so there exists an $x_\beta \in A \setminus (\omega + 1) \times (\beta + 1)$. By regularity, there exists an uncountable $B \subseteq \omega_1$ such that $\pi_{\omega + 1}(x_\beta) = \pi_{\omega + 1}(x_\gamma)$ for all $\beta, \gamma \in B$. Now let $K \subseteq X$ be compact, so $\pi_{\omega_1}(K)$ is bounded and hence there exists a $\beta < \omega_1$ such that $K \subseteq (\omega + 1) \times (\beta + 1)$. Then for all $\gamma \in B, x_\gamma \notin K$, so $x_\beta \notin K$ and thus by Lemma 2.1, $\alpha X$ is radial. Furthermore, by Corollary 3.3, every finite compactification of $X$ is radial.

By [Eng89, Problem 3.12.20(c), pg. 237] the Tychonoff plank $P := (\omega + 1) \times (\omega_1 + 1)$ is the Stone-Cech compactification of $X$ with remainder homeomorphic to $\omega + 1$, so $X$ has no maximal finite compactifications - any compactification of $X$ is obtained by forming a closed partition of $\omega + 1$ and any finite, closed partition can be refined to a larger, finite closed partition. Thus if $X$ has a maximal radial compactification, it must have infinite remainder.

Let $\gamma X$ be a compactification of $X$ with infinite remainder, so there exists a continuous surjection $f : P \to \gamma X$ that extends $\text{id}_X$. Then $f(P \setminus X) = \gamma X \setminus X$. As $P \setminus X \cong \omega + 1$, it follows from the previous lemma that $\gamma X \setminus X \cong \omega + 1$. Let $\delta \in \gamma X \setminus X$ be the unique non-isolated point and define $A :=$
\[\pi_{n+1}[f^{-1}(\{\delta\})], B \supseteq X \setminus (A \times \omega_1).\] We claim that \(\delta \in \overline{B}^X\) but no transfinite sequence in \(B\) converges to \(\delta\), thus showing that there is no maximal radial compactification of \(X\).

Let \(U \subseteq \gamma X\) be an open neighbourhood of \(\delta\). If \(f(\omega, \omega_1) \neq \delta\) then \(f^{-1}[f(\omega, \omega_1)]\) is an open neighbourhood of \((\omega, \omega_1)\) and so contains \([n, \omega] \times \{\omega_1\}\) for some \(n < \omega\). But then \(f[P \setminus X] = \gamma X \setminus X\) is finite, which is a contradiction. Therefore \(f(\omega, \omega_1) = \delta\) and so there exists an \(n < \omega\) and an \(\epsilon < \omega_1\) such that \([n, \omega] \times [\epsilon, \omega_1] \subseteq f^{-1}[U]\). However \(\gamma X \setminus X\) is infinite, so there exists an \(m \in [n, \omega]\) such that \(f((m, \omega_1)) \neq \delta\) and hence \((m, \epsilon) \in U\). Thus \(\delta \in \overline{B}^X\).

Now suppose that there exists a transfinite sequence \(g\) in \(B\) that converges to \(\delta\). Then \(g\) converges to \(*\) in \(\alpha X\), so by Lemma 3.1 there exists an \(m \leq \omega + 1\) and a subsequence \(h\) of \(g\) that converges to \((m, \omega_1)\). However, as noted in section 2.1, no transfinite sequence in \(\omega \times \omega_1\) converges to \((\omega, \omega_1)\) in \(P\). Since \(f(\omega, \omega_1) = \delta\), it follows that \(m < \omega\). Moreover, \(f \circ g\) converges to \(\delta\), so \(m \in A\). But \((m) \times (\omega_1 + 1)\) is an open neighbourhood of \((m, \omega_1)\) disjoint from \(B\), which is a contradiction. Therefore no transfinite sequence in \(B\) converges to \(\delta\), so \(\gamma X\) is not radial, concluding our proof. \(\square\)

Regarding the structures of compactifications, the author believes the following two questions are of particular interest.

**Question 4.3.** When does a space have a maximal, or even greatest, radial / Fréchet-Urysohn compactification? When is \(\beta X\) radial / Fréchet-Urysohn?

**Question 4.4.** When do the radial / Fréchet-Urysohn compactifications form an ideal in the join-semilattice of compactifications?

Finally, there are still some basic questions regarding the existence of spoke systems with particular properties, most prominent is the following.

**Question 4.5.** When does a space have a closed, (almost-\(\)independent\(^6\)) spoke system?

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\(^6\)See [Lee14] for the definition of an independent spoke system.