The properties of the Cosmic Microwave Background (CMB) radiation must be different in flat, positively and negatively curved universes. This fact leads to a direct way of determining the geometry of the universe. The signature of the predicted effect of geodesic mixing, i.e., of the ‘chaotic’ behavior of photon beams in negatively curved spaces peculiar to Anosov systems with strong statistical properties, has been detected while studying the COBE-DMR 4-year data [1]. This possible observation of the negative curvature of the universe suggests the need to search for more effective ways to analyze the CMB data expected from forthcoming high-precision experiments. Information theory offers such a descriptor for the CMB sky maps — the Kolmogorov complexity — as well as provides novel insight into the problem of the loss of information and time asymmetry in a hyperbolic universe.

I. INTRODUCTION

Cosmologists believe that CMB photons have been moving freely during most of the lifetime of the universe. If so, then reaching us without any modification since the epoch of last scattering, as believed, the CMB photons must contain information about the early phases of the universe. However, given the enormous distances in space-time over which the photons have traveled, they must also carry certain imprints of the expansion, geometry and even of the topology of the universe.

The expansion of the universe is reflected in the reddening of the CMB photons, which concerns individual photons. On the other hand a bundle of photons in a beam can also contain information about the geometry of 3-space, i.e., can possess different properties depending on the curvature of the universe. Obviously, these properties have to be statistical ones if they concern beams and not individual photons. The present technique of CMB measurements also deals with statistical characteristics, like the temperature anisotropy amplitude $\Delta T(\theta)/T$, the angular autocorrelation function, the anisotropy distribution over the sky, etc., and one can expect that this data must contain information about the geometry of space.

Here we review several CMB effects which will occur in a negatively curved $k = -1$ Friedmann-Robertson-Walker (FRW) universe. This curvature choice is motivated not only by the fact that the low density and negatively curved FRW universe is consistent with or favored by various observational data; see e.g. [2, 3]. It appears likely that it can also provide alternative insight into such basic problems as the second law and the arrow of time.

The free motion of photons is described by null geodesics. The motion of photon beams is therefore
given by a geodesic flow. The remarkable properties of geodesics in hyperbolic spaces have been continually attracting attention over many decades. In particular, it was shown that a geodesic flow on compact manifolds with constant negative curvature possesses strong statistical properties, i.e., according to the classification of the theory of dynamical systems it possesses the spectral properties of ergodicity, mixing of all degrees and K-mixing. Moreover, it was proven to be isomorphic to the Bernoulli shift, and to have positive Kolmogorov-Sinai (KS) entropy, exponential decay of the time correlation functions, and countable Lebesgue spectrum of conjugated group of unitary operators, and is also a structurally stable (coarse) system, i.e., is an Anosov system.

Considering the behavior of geodesics in a hyperbolic FRW universe, i.e., the effect of geodesic mixing, one arrives at the following consequences for the observable statistical parameters of the CMB:

1. decrease of the amplitude of temperature anisotropy in time;
2. flattening of the angular autocorrelation function, i.e., independence of the sky angle;
3. distortion of anisotropy sky maps.

The rate of the damping, flattening and distortion is determined by the KS-entropy, which itself depends on the curvature or the diameter of the universe — the only scale parameter in a homogeneous isotropic space. Evidently nothing is happening to any individual photon during free propagation after the last scattering epoch, and these effects are purely statistical ones and are determined by the principal limitations of obtaining information during measurements, i.e., by the impossibility of reconstructing the trajectory of any given photon while observing within a finite smoothing angle due to the overlapping of exponentially deviating geodesics in any cut of phase space.

Consider the third effect. Actually, it provides a way to trace the geometry using the CMB sky maps, since the effect is absent for $k = 0, +1$ curvatures. The predicted distortion can be roughly attributed to an elongation of anisotropy spots depending on the value of the curvature of the universe, and hence on the density parameter, as well as on the redshift of the last scattering epoch, i.e., the distance covered by the photons while moving along geodesics. Such a study has been performed using the COBE-DMR 4 year data and special pattern recognition codes, and a statistically significant signal of the sought after effect has been detected. If it is due to the effect of geodesic mixing, as predicted, then one has a direct indication of the negative curvature. Forthcoming more accurate observations will be helpful for confirmation of that result. Moreover this effect may also provide a way to determine the rate of expansion; this has become an interesting topic due to the recent claims on its possible accelerating expansion rate.

We have to note, however, that talking about the geometry of the 3-space as determined by geodesic mix-
ing and CMB data by no means implies that one can also have unambiguous information about the topology of the universe. Indeed, the Einstein equations define the geometry but not the topology of the space. The same geometry can correspond to different topologies, e.g., the flat $k = -1$ space can have $R^3, R^1 \times T^2, R^3 \times S^1, T^3$ and other topologies. The same is true for $k = -1, +1$ geometries. This includes both closed spaces, i.e., compact and without boundary, like $S^3, T^3, k$-handled tori, etc., as well as non-compact spaces. Therefore the study of the topology of the universe using CMB data is another intriguing problem, enabling one to obtain constraints and make efficient predictions [13]. The study of CMB properties in open models started in [14] is complicated due to the difficult problem of eigenfunctions of differential operators (Sobolev problem) in hyperbolic spaces when their topology cannot be defined a priori.

The geodesic mixing and CMB in a hyperbolic universe thus reflects the inevitable loss of initial information peculiar to chaotic dynamical systems. Therefore, this problem can be studied also from the point of view of information theory. Within that approach we show the efficiency of a new tool, the Kolmogorov (algorithmic) complexity [15]–[17], for the study of the CMB data, since this complexity is related to the curvature of the universe [18].

II. GEODESICS IN (3+1) AND (3)-SPACES

First let us consider the null geodesics, that is the free motion of photons. We will show how the null geodesics in a (3+1)-manifold can be projected into its 3-hypersurface, so that the projection will be a geodesic of the latter. More details can be found in [11].

Consider a 3-dimensional Riemannian manifold $U$ and a (3+1)-dimensional Lorentzian manifold $W$. Let the metric of the latter be $\mathbf{g}$ and let $W$ be oriented and time-oriented. Let

$$\iota : U \rightarrow W,$$

(1)

be an embedding of $U$, such that the embedded manifold

$$\Sigma_t = \iota_t(U)$$

(2)

is a space-like hypersurface in $W$, and the induced metric $\iota^* (\mathbf{g}) = \mathbf{g}$ is also a Riemannian metric on $U$.

A one-parameter family of vector fields $\mathbf{Y}_{t}$ on the embedded hypersurfaces $U_t \equiv \iota_t(M)$ is defined via the derivative

$$\mathbf{Y}(t) \equiv \frac{d\iota_t}{dt}. $$

(3)

It can be split into normal and tangential components to $\Sigma_t$

$$\mathbf{Y} = N\mathbf{Z}_{t} + \mathbf{X},$$

(4)
where \( Z_U \) is a time-like normal to \( \Sigma_t \).

The Riemannian metric \( ^4 \! g \) of \( W \) can be represented in the form

\[
g = -N^2 dt \otimes dt + ^3 \! g_{ab} (dx^a + X^a dt) \otimes (dx^b + X^b dt), \tag{5}
\]

where \( ^3 \! g_{ab} = (g_t)_{ab} \) is the metric of \( U \) and

\[
g_i = i_* ^4 \! g. \tag{6}\]

By means of the vector field \( Y \) one can define uniquely the projection

\[\pi : W \to U,\]

which obviously depends on \( N \) and \( X \). As a result any curve \( \gamma \) in \( W \) can be projected on a curve \( c \) in \( U \).

\[
\begin{array}{c}
W \\
\gamma \\
\uparrow \\
\rightarrow \\
\Sigma_t \otimes \Sigma_t \\
\gamma \mapsto c \\
\pi \circ \lambda \\
\end{array}
\]

Now can one find out what conditions \( ^4 \! g, N, X, \phi \) should satisfy in order that the projection of any null geodesic on \( W \) be a geodesic on \( U \) with respect to its metric \( ^3 \! g \)?

The necessary conditions can be shown to be \[13\]

\[
\begin{align*}
N &= 1, \tag{8} \\
X &= 0, \tag{9} \\
g &= a^2(t) \cdot h, \tag{10} \\
\lambda(t) &= A \int^t a^{-1}(s) ds. \tag{11}
\end{align*}
\]

where \( h \) is one of the metrics of the maximally symmetric homogeneous-isotropic 3-manifold. We are interested in such metrics since are considering the FRW universe; for non-maximally symmetric metrics the problem of obtaining analytical relations for necessary and sufficient conditions is more complicated.

For time-like trajectories of non-zero mass particles, compare the corresponding relations for isotropic and time-like trajectories:

\[
\lambda(t) = \begin{cases} 
A \int^t a^{-1}(s) ds & \text{isotropic} \\
B \int^t a^{-1}(s)[C^2 + a^2(s)]^{-1/2} ds & \text{time-like,}
\end{cases} \tag{12}
\]

where \( A, B, C \) are constants. Their difference is responsible for the different efficiencies of the geodesic mixing for photons and non-zero mass particles. Namely, in the former case the characteristic time scale of the effect is much smaller, so that photons have time to mix, while the matter does not.

\[\text{III. THE EFFECT OF GEODESIC MIXING}\]

A geodesic flow on the \((3 + 1)\)-manifold \( W = U \times R \) with Robertson–Walker metric thus can be reduced to a
geodesic flow on the (3)-manifold $U$ with metric $a_0^2 h$ and affine parameter $\lambda$.

The behavior of close geodesics is determined by the equation of geodesic deviation — the Jacobi equation

$$\frac{d^2 n}{d\lambda^2} + kn = 0,$$

where $n$ is the deviation vector and $k = 0, -1, +1$ is the normalized curvature. This equation has the following solutions depending on the three values of the curvature $k$

$$n(\lambda) = \begin{cases} n(0) + \dot{n}(0)\lambda & k = 0, \\ n(0) \cos \lambda + \dot{n}(0) \sin \lambda & k = +1, \\ n(0) \cosh \lambda + \dot{n}(0) \sinh \lambda & k = -1. \end{cases}$$

Thus in the case of negative constant curvature, nearby geodesics in $U$ deviate by an exponential law

$$l(\lambda) = l(0) \exp(h\lambda),$$

where $h$ is the KS-entropy and in accordance with the Pesin theorem equals the sum of the positive Lyapunov numbers

$$h = 2a^{-1},$$

where $a = \sqrt{-R}$ is the diameter of the universe, $R$ is the curvature, and $h = 0$ when $k = 0$ or $k = +1$ (cf. [20]); this is also valid for non-compact spaces. Taking into account the expansion $a(t)$ (from its initial value $a(t_0)$), the true deviation in $W$ will be

$$L(t) = L(t_0) \frac{a(t)}{a(t_0)} \exp(h\lambda(t)).$$

The next crucial point concerns the behavior of the time correlation functions for a geodesic flow. As it was proved by Pollicott [21], for 3-spaces the time correlation function of a geodesic flow $f^t$ decreases by an exponential law, i.e., $\exists c > 0$ such that for all $A_1, A_2 \in L^2(SM)$

$$\bar{b}(t) = |\int A_1(f^tu)A_2(u)d\mu - \int A_1(u)d\mu(u) \int A_2(u)d\mu(u)| =$$

$$const \| A_1 \| \| A_2 \| (1 + t)e^{-h(f)t} + O(e^{-ht}),$$

where $\mu(SM) = 1$ is the Liouville measure and

$$\| A \| = \left[ \int A(u)^2 d\mu(u) \right]^{1/2},$$

and $h$ is again the KS-entropy.

Two properties of Anosov systems have particular importance for our problem. First, the Anosov systems are coarse or structurally stable systems, i.e., they are topologically equivalent to any sufficiently close dynamical system [7]. In other words a perturbed Anosov system
also has the properties of an Anosov system. Although we live in a perturbed and not exact FRW universe, in view of the structural stability of Anosov systems, the geodesic flows in homogeneous-isotropic 3-hypersurface with small perturbations of the curvature have to possess the properties of an Anosov system as well.

The second property is the homogeneous mixing of Anosov systems. This means that in time the exponential stretching and contracting in equal numbers of dimensions (due to the Liouville theorem) of the initial configuration in the course of evolution will tend to become distributed homogeneously over all coordinates of the phase space. This property is responsible for the appearance of a complex shape of the anisotropies [1] (Figure 1).

Moreover, this property has to be independent of the temperature threshold, thus enabling it to be distinguished from similar effects caused by noise [22]. Just the threshold independence of the elongated structures has been detected in [1], indicating the reality of the sought after effect of the curvature [41].

For the last scattering redshift $z$ and the present value of the density parameter $\Omega$ the exponential factor can be written in the form [11]

$$e^{ht} = (1 + z)^2[1 + (\sqrt{1 - \Omega}) / (\sqrt{1 + z\Omega} + \sqrt{1 - \Omega})]^4.$$  

(21)

In particular the measured CMB temperature will tend exponentially to a constant mean temperature by time: the isotropy is the limiting state

$$\lim_{t \to \infty} T_\lambda = \bar{T}.$$  

(22)

For the normalized temperature autocorrelation function

$$C_\lambda(\theta, \beta) = \langle T_\lambda(u)T_\lambda(v)\rangle_{g(u,v)=\cos \theta},$$

(23)

the following inequality holds

$$|C_\lambda(\theta, \beta) - 1| \leq \text{const} |C_0(\theta, \beta) - 1| \frac{1}{(1 + z)^2} \left[ \frac{\sqrt{1 + z\Omega} + \sqrt{1 - \Omega}}{1 + \sqrt{1 - \Omega}} \right]^4$$

(24)

where $\theta$ and $\beta$ are the sky (separation) and observing beam angles, respectively. Thus the autocorrelation function $C(\theta, \beta)$ tends to become constant with respect to the sky angle $\theta$ in time, regardless of its form at the last scattering surface.

Numerically, the exponential factor, for say $\Omega \simeq 0.2$ and expansion rate $t^\alpha$, $\alpha = 2/3$, is

$$\exp(ht) \simeq 10^{-3}.$$  

(25)

Using the fact that the geodesic flow is an Anosov system, it can also be shown [10] that there exists an angle $\phi$ such that the smoothing factor $s$ of $\delta T/T(\beta)$ is almost constant if either $\beta \gg \phi$ ($s \sim e^{-h\lambda}$) or $\beta \ll \phi$ ($s \sim 1$), and it increases as $\beta$ decreases at $\beta \sim \phi$ by the law $s \sim \text{const} / \beta \cdot e^{-h\lambda}$. The appearance of the smoothing
angle φ is due to the fact that CMB measurements include averaging within some beam angle and time period, i.e., statistical smoothing with inevitable loss of information. Therefore, the more narrow is the beam size, the less information is lost in the smoothing in terms of the temperature autocorrelation function. Maximal information, e.g., the anisotropy corresponding to exactly the last scattering surface, should be obtained while measuring by beams within some limiting angle. Measurements on smaller angles should not influence the autocorrelation function which should be determined by physical conditions at the last scattering surface (its thickness, the Silk effect, etc). The predicted increase of the anisotropy at small beam angles for the parameters mentioned above is close to the angular region of the Doppler peak, therefore the separation of these effects can be of particular importance.

The value of the KS-entropy is therefore determining the observational effects of the loss of information. If for CMB photons the exponential factor takes values up to $10^3$, for photons arriving from redshifts of quasars or galaxies, it is sufficiently smaller: $(1 + z) \cdot 10^{-3} \sim 5 \cdot 10^{-3}$, i.e., the latter photons have little time to feel the geometry.

The effect of geodesic mixing with positive KS-entropy will also mean that certain properties of the CMB can seem similar on scales larger than the scale of the horizon not because of exchange of information but due to a corresponding loss of information about the initial conditions. In the next section we will deal with the information theory approach to the CMB problem.

IV. KOLMOGOROV COMPLEXITY

Various descriptors have been proposed to extract the cosmological information from the CMB data. Particularly for CMB maps those descriptors include, for example, the hot spot number density, genus, correlation function of local maxima, Euler-Poincare characteristic, percolation, wavelets etc. (see e.g. [24], [25]). Most of these descriptors have been already applied to the analysis of COBE-DMR sky maps [26], [27].

As discussed above, the geodesic mixing will lead to complex structures on the CMB sky maps [1]. To describe quantitatively the latter we suggest using the invariant definition of complexity — Kolmogorov or algorithmic complexity — introduced in the 1960s by Kolmogorov, [14], and independently by Solomonoff and Chaitin (see [16], [17]). The efficiency of the concept of complexity for the basics of classical and quantum physics and the second law of thermodynamics have been studied by Zurek, Caves, Bennett and others [28], [29].

The Kolmogorov complexity $K_u$ is defined as the minimal length of the binary coded program (in bits) which is required to describe the system completely, i.e., that will enable one to recover the initial system via a given com-
puter. The complexity of an object $y$ at a given object $x$ is defined as

$$K_{\phi(p)}(x) = \min_{p: \phi(p) = x} l(p),$$

(26)

where $l(p)$ is the length of the program $p$ with respect to the computer $\phi(p)$ describing the object completely, i.e., at the $0 - 1$ representation: $l(0) = 0$. The fundamental feature of Kolmogorov’s formulation is the independence of complexity on the computer which has to halt upon running the program. A computer is considered ‘universal’ if for any computer $\phi_C$ there exists a constant $S_C$ which can be added to any program $p$, so that $\phi_C p$ should execute the same operation on computer $\psi$ as the program $p$ on computer $\phi$. Therefore the Turing machine can be considered as a universal computer while computing the complexity.

The complexity is closely related with another basic concept — random sequences. The most general definition by Martin-Löf [30] formalizes the idea of Kolmogorov that random sequences have a very small number of rules compared to their lengths; the rule is defined as an algorithmically testable and rare property of a sequence. Indeed, the properties of complexity and randomness are not completely the same, although they are closely related for typical sequences [31]. Therefore in our problem, in principle, the estimation of the randomness of the data string (digitized figure) has to correlate with the estimation of the complexity; we will not discuss the randomness here but it could be an interesting problem for future studies.

It has been proven that no shortest algorithm exists enabling one to decide whether a given complex-looking sequence is really complex [17][31]. The complexity is the amount of information which is required to determine uniquely the object $x$. Kolmogorov had proved that the amount of information of $x$ with respect to $y$ is given by the formula [15]

$$I(y : x) = K(x) - K(x | y),$$

(27)

where the conditional complexity

$$K(x | y) = \min l(p),$$

(28)

is the minimal length of the program required to describe the object $x$ when the shortest program for $y$ is known. Note, that $K(x | x) = 0$ and $I(x : x) = K(x)$ and $I(x : x) \geq 0$; where $A \geq B$ denotes $A \leq B + \text{const}$.

The complexity measured in bits is related to KS-entropy via the relation [32]

$$\Delta I = \log_2(2^h(f)(t - t_0)) = h(f)(t - t_0),$$

(29)

where the loss of information $\Delta I$ during the time interval $t - t_0$ is

$$\Delta I = K_u(t) - K_u(t_0),$$

(30)

i.e., the information corresponding to the distortion of the pattern from the initial state $t_0$ (the last scattering
epoch) up to the observer at \( t \) i.e., at \( z = 0 \). This is guaranteed by the Shannon-McMillan-Breiman theorem stating the uniform exponential rate of loss of information in Anosov systems.

If the length of a sequence \( x \) is \( N \) then the obvious upper limit can be established

\[
K_U(x) < N. \tag{31}
\]

Let us estimate the fraction of the sequences among all \( N \)-bit sequences for which

\[
K_U(x) < N - m. \tag{32}
\]

This means that there exists a program of length \( N - m \) which computes \( x \). The total number of such programs of such a length cannot be larger than \( 2^{N-m+1} \); this is the upper limit without taking into account the prefix-free condition, i.e., the programs cannot be a prefix for some other program. [43] Thus, we have the following upper limit

\[
(2^{N-m+1} - 1)/2^N \approx 2^{-m+1}. \tag{33}
\]

This value is small if \( m \) is sufficiently large. Thus a more general relation than (31) can be established

\[
K_U(x) \approx c(x) N, \quad c(x) \approx 1. \tag{34}
\]

Return again to the cosmological problem. In a \( k = -1 \) FRW universe any initial CMB structure observed at redshift \( z_{\text{obs}} \) should have a more complex shape than at \( z < z_{\text{obs}} \)

\[
K(\text{CMB spot} \mid z = z_{\text{obs}}) > K(\text{circle} \mid z < z_{\text{obs}}); \tag{35}
\]

here ‘circle’ denotes a circle with Gaussian or other fluctuations. The complexity estimated for the anisotropy spot observed now should exceed the complexity of the primordial spot (in addition to the scale expansion by the factor \( 1 + z \)) by an amount depending on the KS-entropy and, hence on the curvature and the distance of the last scattering surface.

From Eqs. (29) and (30) we come to a simple equation linking the geometry with the complexity

\[
\Delta K = (2/a)\Delta T. \tag{36}
\]

i.e., the relative complexity of the observed spot with respect to a circular spot (with Gaussian or so fluctuations which one can expect for flat or positively curved spaces) will determine the curvature of the hyperbolic universe, where \( \Delta T \) is the time elapsed since photons started to move freely and thus tracing how curved the 3-D space is.

Although in general the shortest program cannot be reached, i.e., the exact complexity cannot be calculated, in certain problems the results obtained cannot be too far from that value. The calculation of relative complexity of a perturbed and non-perturbed object by means
of a given computer and developed code (although the latter cannot be proved to be the shortest possible), has to reflect the complexity introduced by the perturbation. In our problem the complexity is the ‘measure the perturbation’ caused by the curvature.

The cosmological information should be extracted from the CMB maps in the following way. First, certain criterion should define the spots represented via given configurations of pixels (e.g. [1]). Then, the computation of the complexity should be performed by means of the length of a special compressed code (string) completely defining the spots. Since the basic program will be the same, the only changes will be due to different data files, i.e., the coordinates of the pixels of various spots. This problem was technically solved in [34] and is discussed in the next section.

V. COMPLEXITY: ALGORITHM FOR CMB MAPS

We shall now describe an algorithm for estimating the complexity of spots given by certain pixel configuration on a grid and present the results of computations for a series of structures of different complexity, i.e., demonstrate the calculability of such an abstract descriptor as the Kolmogorov complexity for the CMB digitized maps. The correlation of complexity $K$ of the anisotropy spots with their Hausdorff (fractal) dimension $d$ will be shown as well. We observe the correlated growth of the complexity $K$ and $d$ with the increase of the complexity of the geometrical shape of the spots, starting from the simplest case — the circle.

To develop the algorithm of estimation of complexity one should clearly describe in which manner the objects, namely the anisotropy spots, are defined. The COBE-DMR CMB sky maps have the following structure [24, 27]. They represent a $M \times N$ grid with pixels determined by the beam angle of the observational device; more precisely the pixel’s size defines the scale within which the temperature is smoothed, so that each pixel is assigned a certain value of the temperature (a number). For example, COBE’s grid had 6144 pixels of about 2.9° size each, although they do not uniformly contain information about the CMB photons. By ‘anisotropy spots’ we understand the sets of pixels at a given temperature threshold [35].

Our problem is to estimate the complexity of the anisotropy spots, i.e., of various configurations of pixels on the given grid: the size of the grid, and both the size and the number of pixels are crucial for the result.

We proceed as follows. Each row of the grid is considered as an integer of $M$ digits in its binary representation, ‘0’ corresponding to the pixels not belonging to the spot, and ‘1’ to those of the spot. Considering all $N$ rows of the grid in one sequence (the second row added to the first one from the right, etc.) we have a string of length $N \times M$ in binary form with complexity $K$.
Strictly speaking we can estimate only the upper limit of $K$ corresponding to a given algorithm. By algorithm we understand the computer program (in PASCAL in our case), along with the data file, describing the coordinates of the pixel of the spot. Namely the data file includes compressed information about the string of digits. The program is a sequence of commands performing reconstruction of the string and calculations of the corresponding lengths. The complexity of the figure will be attributed to the file containing the information about the position of the pixels.

The code describing the spot works as follows. As an initial pixel we fix the upper left pixel of the spot and move clockwise along its boundary. Each step — a ‘local step’ — is a movement from a current pixel to the next one in the direction given above. This procedure is rigorously defining the ‘previous’ and ‘next’ pixels. Two cases are possible. First, when the next pixel (or several pixels) after the initial one is in the same row: we write down the number of pixels in such a ‘horizontal step’. The second case is, when the next pixel is in the vertical direction; then we perform the local steps in the vertical direction (‘vertical step’) and record the number of corresponding pixels. Via a sequence of horizontal and vertical steps we obviously return to the initial pixel, thus defining the entire figure via the resulting data file.

Obviously, the length of the horizontal step cannot exceed the number of columns, i.e., $N$, while the vertical step cannot exceed $M$, requiring $\log_2 M$ and $\log_2 N$ bits of information, correspondingly. For the configurations we are interested in, the lengths of the horizontal and vertical steps, however, are much less than $\log_2 M$ and $\log_2 N$ and therefore we need a convenient code for defining the length of these steps. Our code is realized for $M = N = 256$; apparently for each value of $M$ and $N$ one has to choose the most efficient code.

After each step, either horizontal or vertical, a certain number of bits of information is stored. The first two bits will contain information on the following bits defining the length of the given step in a manner given in the Table 1. The case when the first two bits are zero, denotes: if the following digit is zero than the length of the step is $l_s = 0$, and hence no digits of the same step do exist; if the next digit is 1, then 8 bits are following, thus defining the length of the step. If $l_s = 1$, then after the combination 01 the following digit will be either 0 or 1 depending whether the step is continued to the left or to the right with respect to the direction of the previous step. When $l_s = 2$ or 3, after the combination 01 the file records 0 in the first case, i.e., $l_s = 2$, and 1 in the second. When

### Table I:

| first 2 bits | next bits |
|--------------|-----------|
| 0 1          | 1         |
| 1 0          | 2         |
| 1 1          | 3         |
TABLE II:

| step length bits | 0, 1 | 3 |
|------------------|------|---|
| 2, 3             | 4    |
| 4-7              | 5    |
| 8                | 11   |

$l_s = 4, \ldots, 7$, then after the combination 1 1 the file records 0 and 1, at the left and right steps, and after two digits in binary form of the step length $l_s = 3$. Finally, when $l_s > 8$, the combination 0 0 1 is recorded, followed by the 8 bits of the step length $l_s$ in binary representation.

Thus, all possible values of the step length $l_s$ (they are limited by $M = N = 256$) are taken into account and the number of bits attributed to the length in the file depends on $l_s$ in the manner shown in Table 2. The figure recorded in the data file via the described code can be recovered unambiguously without difficulties.

Obviously one cannot exclude the existence of a code compressing more densely the information about the pixelized spots, however, even these codes appear to be rather efficient. Namely, the length of the program recovering the initial figure from the stored data file is 4908 bits, and it remains almost constant at the increase of $N$ and $M$.

VI. HAUSDORFF DIMENSION OF CMB SPOTS

The association of local exponential instability and deterministic chaos with fractals is well known (see e.g. [36]). Hence the idea to estimate the Hausdorff dimension of the spots is natural. We recall that the Hausdorff dimension is defined as the limit

$$d_H = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)},$$

where $N(\varepsilon)$ are circles of radius $\varepsilon$ covering at least one point of the set. By the definition of Mandelbrot the set is fractal if the Hausdorff dimension exceeds the topological dimension.

To compute the Hausdorff dimension we used the code Fractal by V. Nams [37], but its application was not so straightforward. The main problem to be solved was the approximation of the boundary of the pixelized figure via a smooth curve, so that its Hausdorff dimension can be determined by the above mentioned code. The trivial consideration of the profile of the pixels, obviously would introduce artificial fractal properties to the spot as a result of instrumental nature of pixel sizes. We used the following procedure: the centers of three or more neighbor pixels were connected by a line and its distance $h$ from the centers of the intermediate pixels has been calculated (it is obviously zero if the pixels are in one row). If $h$ exceeds some chosen value, namely 0.5 of the size of
the pixel, then the line was adopted as a good approximation of the boundary curve of the pixels. Otherwise, the centers of the next pixels are involved, etc. The runs of test (trivial) figures with various values of $h$ showed the validity of this procedure.

Figure 2 shows the results of computations of complexity and Hausdorff dimension for a sequence of toy spots, starting from a circle.

VII. DISCUSSION

Thus the hyperbolicity of the universe has to be reflected in the properties of CMB. The motion of photons in a $k = -1$ FRW universe via the geodesic flows/Anosov systems is linked with deterministic chaos and loss of information occurring for negative curvature.

Then the information theory approach can be applied. Kolmogorov (algorithmic) complexity can be an efficient descriptor of CMB sky maps: we presented an algorithm for its computation for a given configuration of pixels on a grid. We showed that it is calculable for CMB maps, if we are interested in the relative complexity $K_i - K_1$ (or $(K_i - K_1)/K_1$); for details see [34].

Together with the previous results on the rate of exponential mixing of geodesics determined by the Kolmogorov-Sinai (KS) entropy, which itself is related by the diameter (curvature) of the universe, this provides a new informative way of analyzing the CMB data.

Although the COBE-DMR data reveal certain genuine spots [38] [39], they were not sensitive enough for reliable evaluation of the shape of the spots. The next generation of observations, such as the Planck Surveyor and MAP, should allow one to apply the technique described here.

The chaos/information approach to the CMB problem briefly discussed above can open ways for even more general insights. Namely, the effect of geodesic mixing can be only one of the manifestations of a much deeper link with the fundamental physical laws — negative curvature — mixing — second law of thermodynamics — arrow of time. Thus, we observe the time asymmetry and the second law because we live in a universe with negative curvature, and those laws may not be the same in a flat or positively curved space. These aspects, including the relations between thermodynamic and cosmological arrows of time, are discussed in a separate paper [40].

The complexity and information way of thinking can be valuable also in other cosmological problems.

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[41] Note, that the map distortion due to geodesic mixing has no direct relation with the effects discussed in [23], and exists whatever the initial shapes of spots at the last scattering surface are.
[42] If $x$ is the binary representation of some integer $N_0$, then $N \approx \log_2 N_0$.
[43] A word $a$ is called prefix for a word $b$ if $b = ac$ with some other word $c$. 