Reliably distinguishing states in qutrit channels using one-way LOCC

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July 15, 2018

Abstract

We present numerical evidence showing that any three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^n$ has an orthonormal basis which can be reliably distinguished using one-way LOCC, where a measurement is made first on the 3-dimensional part and the result used to select an optimal measurement on the $n$-dimensional part. This conjecture has implications for the LOCC-assisted capacity of certain quantum channels, where coordinated measurements are made on the system and environment. By measuring first in the environment, the conjecture would imply that the environment-assisted classical capacity of any rank three channel is at least $\log 3$. Similarly by measuring first on the system side, the conjecture would imply that the environment-assisting classical capacity of any qutrit channel is $\log 3$. We also show that one-way LOCC is not symmetric, by providing an example of a qutrit channel whose environment-assisted classical capacity is less than $\log 3$. 
1 Introduction and statement of results

The noise in a quantum channel arises from its interaction with the environment. This viewpoint is expressed concisely in the Lindblad-Stinespring representation [6, 8]:

$$\Phi(|\psi\rangle\langle\psi|) = \text{Tr}_E[U(|\psi\rangle\langle\psi| \otimes |\epsilon\rangle\langle\epsilon|)U^*]$$

(1)

Here $E$ is the state space of the environment, which is assumed to be initially prepared in a pure state $|\epsilon\rangle$. The unitary operator $U$ describes the interaction between the system state space $\mathcal{H}$ and the environment, and it maps product states to entangled states in $\mathcal{H} \otimes E$. Taking the trace over $E$ corresponds to ignoring the environment part of the entangled state. Since the reduced density matrix of a pure entangled state is a mixed state, it follows that $\Phi$ generally maps pure input states to mixed output states.

Because $U$ is unitary, orthogonal states in the input Hilbert space $\mathcal{H}$ are mapped to orthogonal states in $\mathcal{H} \otimes E$. However after the environment is traced out, the resulting output states of the channel are no longer orthogonal, and therefore cannot be reliably distinguished by making measurements of the channel. If the channel is used for information transmission this loss of distinguishability limits the rate at which information can be reliably transmitted.

It may be possible to more reliably distinguish output states by using measurements on the environment in addition to measurements on the system. This idea of using information from the environment to enhance channel capacity has been pursued in a number of settings [4, 5, 11]. Our purpose here is to investigate this question for some low-dimensional systems, in order to see how coordinated measurements on the system and the environment can be used to distinguish orthogonal input states.

We will say that a set of states $\{\rho_i\}_{i=1}^N$ is reliably distinguished by the POVM $\{E_b\}_{b=1}^M$ if there is a partition $\{1, \ldots, M\} = S_1 \cup \cdots \cup S_N$ where $S_i$ are disjoint sets, such that $\text{Tr} \rho_i E_b = 0$ for all $b \notin S_i$. Operationally, this means that if the system is prepared in one of the states $\{\rho_i\}$, but the precise identity of the state is unknown, then this secret identity can be determined by applying the measurement $\{E_b\}$. In this paper we address the question: what is the largest number of orthonormal input states which can be reliably distinguished using one-way LOCC between the system output and the environment?

In order to distinguish the two directions of one-way LOCC, we will use the notation introduced recently by Winter [11]: environment-assisted, meaning a measurement which is first performed on the environment, and where the
result is used to select an optimal measurement on the system; and environment-assisting, meaning a measurement which is first performed on the system, and where the result is used to select an optimal measurement on the environment. More elaborate combinations of measurements are also defined in [11], but we will be concerned only with these two basic cases.

**Definition 1** For a quantum channel $\Phi$, we denote by $N_{\text{env}\rightarrow\text{sys}}(\Phi)$ the maximal number of orthonormal input states which can be reliably distinguished using an environment-assisted measurement. Similarly we denote by $N_{\text{sys}\rightarrow\text{env}}(\Phi)$ the maximal number of orthonormal input states which can be reliably distinguished using an environment-assisting measurement.

Note that a given channel $\Phi$ has many possible Lindblad-Stinespring representations of the form (1). So to compute $N_{\text{env}\rightarrow\text{sys}}(\Phi)$ and $N_{\text{sys}\rightarrow\text{env}}(\Phi)$ it is necessary to consider all such representations, and find the one that allows the maximal number of distinguishable states. It is known that every $d$-dimensional channel has a representation where the environment has dimension at most $d^2$. It would be interesting to know if $N_{\text{env}\rightarrow\text{sys}}(\Phi)$ and $N_{\text{sys}\rightarrow\text{env}}(\Phi)$ are always achieved with environments that satisfy this same bound.

The question of determining $N_{\text{env}\rightarrow\text{sys}}(\Phi)$ and $N_{\text{sys}\rightarrow\text{env}}(\Phi)$ for a channel $\Phi$ with a given environment can be re-expressed as the problem of distinguishing orthogonal entangled bipartite states using one-way LOCC. Indeed, as (1) shows, one way to describe the channel $\Phi$ is to specify the subspace of entangled states $V \subset \mathcal{H} \otimes \mathcal{E}$ that are generated by the interaction, that is

$$V = \{U(|\psi\rangle \otimes |\epsilon\rangle) : |\psi\rangle \in \mathcal{H}\}$$

Clearly $\dim V = \dim \mathcal{H}$, and there is a 1–1 correspondence between orthonormal sets in $\mathcal{H}$ and in $V$. So $N_{\text{env}\rightarrow\text{sys}}(\Phi)$ is the maximal number of orthonormal states in $V$ that can be distinguished using one-way LOCC, with measurements first in $\mathcal{E}$ and then in $\mathcal{H}$, and vice versa for $N_{\text{sys}\rightarrow\text{env}}(\Phi)$. Our main result concerns orthonormal sets in subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$, with $n \geq 3$. Because we rely on numerical methods, we state our results as conjectures with supporting evidence.

**Conjecture 1** Let $V$ be a 3-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^n$, with $3 \leq n \leq 9$. Then there is an orthonormal basis of $V$ which can be reliably distinguished using one-way LOCC, where measurements are made first on $\mathbb{C}^3$, and the result used to select the optimal measurement on $\mathbb{C}^n$.  

3
We will present our numerical evidence for Conjecture 1 in Section 3. The evidence was acquired by sampling over a large number of randomly selected subspaces, and in each case finding an orthonormal basis and a suitable local measurement which reliably distinguished the basis. In Appendix A we prove the following Corollary, which extends the result of Conjecture 1 to any \( n \geq 3 \).

**Corollary 2** If Conjecture 1 holds, then the same result extends to any three-dimensional subspace of \( \mathbb{C}^3 \otimes \mathbb{C}^n \), for any \( n \geq 3 \).

Assuming that Conjecture 1 is true, we can use it to find the environment-assisting capacity of any qutrit channel. For a qutrit channel, \( \mathcal{E} \) is a subspace of \( \mathcal{H} \otimes \mathcal{E} \) where \( \mathcal{H} = \mathbb{C}^3 \). Hence it follows that there is an orthonormal basis of \( \mathcal{H} \) whose image in \( \mathcal{H} \otimes \mathcal{E} \) can be reliably distinguished by an environment-assisting measurement.

**Corollary 3** For any qutrit channel \( \Phi \),

\[
N_{\text{sys} \rightarrow \text{env}}(\Phi) = 3
\]  

(3)

We can also deduce a result whenever a channel can be implemented with a three-dimensional environment. The minimal dimension of the environment needed to implement a channel via the Lindblad-Stinespring representation is called the rank of the channel. Recall that the rank of a channel \( \Phi \) which acts on states over \( \mathbb{C}^d \) can be defined in several equivalent ways; it is (a) the rank of the Choi-Jamiołkowski matrix \((I \otimes \Phi)(|ME\rangle\langle ME|)\) where \( |ME\rangle \) is the maximally entangled state on \( \mathbb{C}^d \otimes \mathbb{C}^d \), (b) the minimal number of operators needed in a Kraus representation for \( \Phi \), and (c) the minimal dimension of the environment needed to implement \( \Phi \) via the Lindblad-Stinespring representation.

**Corollary 4** For any channel \( \Phi \) whose rank is three,

\[
N_{\text{env} \rightarrow \text{sys}}(\Phi) = 3
\]  

(4)

In general \( N_{\text{env} \rightarrow \text{sys}}(\Phi) \neq N_{\text{sys} \rightarrow \text{env}}(\Phi) \), and in Section 3.4 we present examples of qutrit channels for which these numbers are different. These channels all have ranks greater than three, as they should by Corollary 1. Interestingly we have no found no examples of rank four qutrit channels for which the numbers are different, and we conjecture that \( N_{\text{env} \rightarrow \text{sys}}(\Phi) = 3 \) for all rank four qutrit channels.
The paper is organized as follows. In Section 2 we recall some related work on the question of state discrimination using LOCC, and the recent notion of environment-assisted capacity. In Section 3 we describe our numerical work in support of Conjecture 1, and in Section 3.4 we present some examples of qutrit channels where it is not possible to reliably distinguish three input states using environment-assisted measurements. Finally we discuss some conclusions in Section 4 and the Appendix contains the proof of Corollary 2.

2 Related work

2.1 LOCC

There has been a lot of work on the general problem of using LOCC (local operations and classical communication) to distinguish bipartite states \[1, 3, 4\]. Regarding the question of finding bases of subspaces which can be reliably distinguished, the first result was by Walgate et al \[9\], who showed that any two orthogonal bipartite states can be reliably distinguished using LOCC. This implies in particular that every two-dimensional subspace of a bipartite space has a basis that can be reliably distinguished. It follows that \[N_{env\rightarrow sys}(\Phi) = 2\] for any qubit channel \(\Phi\) \[9, 3\], and also that \[N_{env\rightarrow sys}(\Phi) \geq 2\] for every channel \[5\].

Gregoratti and Werner \[3\] have proved the existence of bipartite subspaces which do not have any basis that can be reliably distinguished using LOCC, though they did not provide any explicit examples or any bounds on the dimensions of the subspaces. Recently Watrous \[10\] has constructed explicit examples of \((d^2 - 1)\)-dimensional subspaces in \(\mathbb{C}^d \otimes \mathbb{C}^d\) which have no basis that can be distinguished using LOCC, for \(d \geq 3\). In fact Watrous’ result is even stronger, because he proves that there is no separable POVM which can distinguish the basis.

More is known when constraints are put on the allowed basis vectors, for example Nathanson \[7\] has shown that any three maximally entangled states in \(\mathbb{C}^3 \otimes \mathbb{C}^3\) can be distinguished using a product measurement. Results are also known for bases composed of generalized Bell states \[2\].

2.2 Environment-assisted capacity

In the context of channel capacities, it is natural to consider the maximal rate at which information (classical or quantum) can be sent through a noisy channel, using additional information gained from coordinated measurements of the
system and environment. Recently Winter [10] has analyzed the environment-assisted capacity of a quantum channel, which is the capacity for transmission of classical information when measurements from the environment can be used to assist transmission. Using an asymptotic formulation of the problem allowing entangled inputs and measurements, he has derived a lower bound for the rate of transmission, namely half the logarithm of the dimension of the input state space. While the results in this paper provide low-dimensional examples where this bound is not sharp, Winter has found evidence that the lower bound is sharp for some high-dimensional channels.

3 Evidence for Conjecture [1]

3.1 Partial measurements

We implement a partial measurement of a pure bipartite state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^n$ by projecting onto an orthonormal basis in $\mathbb{C}^d$. If the outcome of the partial measurement is known, then the result is a pure state in $\mathbb{C}^n$. To be specific, suppose that $|v_1\rangle, \ldots, |v_d\rangle$ is an orthonormal basis of $\mathbb{C}^d$, then the state $|\psi\rangle$ can be written in bipartite form as

$$|\psi\rangle = \sum_{a=1}^{d} |v_a\rangle \otimes |\psi_a\rangle \quad (5)$$

where $\{|\psi_a\rangle\}$ are (unnormalized) states in $\mathbb{C}^n$, and where $\sum_{a=1}^{d} \langle \psi_a | \psi_a \rangle = 1$. Projecting onto the basis $\{|v_a\rangle\}$ produces the state $|\psi_a\rangle$ with probability $\langle \psi_a | \psi_a \rangle$.

We are interested in the case where $d = 3$, and $V$ is a subspace $\mathbb{C}^3 \otimes \mathbb{C}^n$. We wish to find a basis for $V$ and a partial measurement on $\mathbb{C}^3$ which will perfectly distinguish the basis states. This will be possible if and only if the projected states in $\mathbb{C}^n$ corresponding to each basis state are orthogonal for every outcome of the partial measurement. To be specific, let $|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle$ be an orthonormal basis of $V$, and let $|v_1\rangle, |v_2\rangle, |v_3\rangle$ be an orthonormal basis of $\mathbb{C}^3$. Denote by $|\phi(i,a)\rangle$ the state in $\mathbb{C}^n$ which results when the state $|\theta_i\rangle$ is partially projected onto $|v_a\rangle$, so that as in (5)

$$|\theta_i\rangle = \sum_{a=1}^{3} |v_a\rangle \otimes |\phi(i,a)\rangle \quad (6)$$

The partial measurement on $\mathbb{C}^3$ will produce the result $a \in \{1, 2, 3\}$. The projected state in $\mathbb{C}^n$ will then be one of the states $|\phi(1,a)\rangle, |\phi(2,a)\rangle, |\phi(3,a)\rangle$, depending on which input state was used. In order to determine the identity of
the input state, it must be possible to perfectly distinguish these three possibilities, for each measurement result. Therefore the basis $|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle$ can be reliably distinguished by this partial measurement if and only if the three states $|\phi(1, a)\rangle, |\phi(2, a)\rangle, |\phi(3, a)\rangle$ are orthogonal for each $a = 1, 2, 3$.

In order to test orthogonality of these states, define

$$B = \{|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle\}, \quad M = \{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$$

so that $B$ denotes the basis we wish to distinguish, and $M$ is the partial measurement in $\mathbb{C}^3$. We will use the following objective function, where the various quantities are defined in (6):

$$H(B, M) = \sum_{a=1}^{3} \sum_{i \neq j=1}^{3} |\langle \phi(i, a) | \phi(j, a) \rangle|^2$$

Clearly $H \geq 0$, and $H = 0$ if and only if the vectors $\{|\phi(j, a)\rangle\}_{j=1}^{3}$ are orthogonal for each outcome $a$.

Our goal is to find the minimal value of $H(B, M)$ for different bases in the subspace $V$ and partial measurements in $\mathbb{C}^3$. Therefore we define

$$H_{\text{min}}(V) = \min_{B, M} H(B, M)$$

### 3.2 Random subspaces and bases

In order to compute (9) for a subspace $V$ we minimize the function $H(B, M)$ over bases of $V$ and bases of $\mathbb{C}^3$. We sample subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ by randomly selecting three orthonormal vectors $|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle$ in $\mathbb{C}^3 \otimes \mathbb{C}^n$, and defining $V$ to be the span of these vectors. Then every other orthonormal basis of $V$ can be found by applying a unitary matrix to these three vectors, that is

$$|\psi_i\rangle = \sum_{j=1}^{3} w_{ij} |\theta_j\rangle, \quad i = 1, 2, 3$$

where $W = (w_{ij})$ is a $3 \times 3$ unitary matrix. Therefore we can search over bases $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$ of $V$ by searching over unitary matrices $W$. Similarly every orthonormal basis of $\mathbb{C}^3$ is defined by a $3 \times 3$ unitary matrix $U$, whose columns are the vectors in the partial measurement.
## Table 1: Numerical data for three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ with $n = 3, 4, \ldots, 9$, namely: the average value of the objective function $H$ for values less than the threshold value of $10^{-6}$, and the number of tested subspaces.

| $n$ | Average $H$ | # subspaces |
|-----|-------------|-------------|
| 3   | $2.816258 \times 10^{-8}$ | 138211 |
| 4   | $3.789893 \times 10^{-8}$ | 30271 |
| 5   | $3.789893 \times 10^{-8}$ | 32278 |
| 6   | $4.127063 \times 10^{-8}$ | 30000 |
| 7   | $4.394015 \times 10^{-8}$ | 30000 |
| 8   | $5.130496 \times 10^{-8}$ | 30216 |
| 9   | $5.594670 \times 10^{-8}$ | 30006 |

Table 1: Numerical data for three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ with $n = 3, 4, \ldots, 9$, namely: the average value of the objective function $H$ for values less than the threshold value of $10^{-6}$, and the number of tested subspaces.

### 3.3 Numerical results

The minimization problem described above was implemented using the package TOMLAB. Table 1 shows the results for qutrit channels organized according to rank. Based on the results, we conjecture that all three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ have a basis which can be perfectly distinguished using one-way LOCC, where the partial measurement is first performed on the $\mathbb{C}^3$.

Table 1 was obtained by randomly sampling three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$, and for each subspace searching for an orthonormal basis that could be reliably distinguished using one-way LOCC. Column 3 shows the number of subspaces sampled for each value of $n$. For each subspace, the objective function (8) was minimized over choices of orthonormal basis in $V$ and partial measurements in $\mathbb{C}^3$. The threshold value $10^{-6}$ was used to terminate the search for the minimum value. In every case this threshold was reached. Column 2 shows the average minimum value of the objective function when the search was terminated.
3.4 One-way LOCC is not symmetric: rank five example

A numerical search readily turns up examples of three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^5$ which have no basis that can be reliably distinguished using one-way LOCC with partial measurement first on the $\mathbb{C}^5$ factor. We present one of these examples in Appendix B. Interestingly, we have found no example of a three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^4$ which cannot be distinguished by one-way LOCC using a partial measurement on the $\mathbb{C}^4$ factor.

4 Conclusions

We have investigated the question of whether it is always possible to reliably distinguish three orthogonal input states for a qutrit channel, assuming that coordinated measurements between the system and the environment are available to assist in the state discrimination. Our numerical results indicate that the answer is ‘yes’, as long as measurements are first performed on the system, and the result then used to find the optimum measurement in the environment. The result is equivalent to the statement that every three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^n$ has a basis which can be distinguished by one-way LOCC, as long as partial measurements are first performed on the $\mathbb{C}^3$ part. This also implies that every rank three channel has a set of three orthonormal vectors that can be reliably distinguished. Our results complement existing analytical results for qubit channels, and also for the LOCC question in higher dimensional cases.

There are several further directions to pursue in this line of research. One direction is to look for an analytical proof of our results, possibly by extending work of Nathanson [7] and others on LOCC state discrimination. Another direction is to continue numerical investigations in higher dimensions. We plan to work along both of these lines of investigation.

Acknowledgements This research was supported in part by National Science Foundation Grant DMS-0400426. The authors are grateful to Javed Aslam for use of computing facilities.

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A Proof of Corollary 2

The result of Conjecture 1 can be formulated as a mathematical conjecture concerning positive semidefinite $9 \times 9$ matrices, whose partial trace is the identity matrix. When formulated in this way, the proof of Corollary 2 is almost immediate.
To set up the notation, let $M$ be a $9 \times 9$ positive semidefinite matrix, written in block form as

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \quad (11)$$

and satisfying

$$M_{11} + M_{22} + M_{33} = I \quad (12)$$

Letting $|\phi(i, a)\rangle \in \mathbb{C}^9 = \mathbb{C}^3 \otimes \mathbb{C}^3$ denote the columns of $M^{1/2}$, it follows that

$$(M_{ab})_{ij} = \langle \phi(i, a)|\phi(j, b)\rangle \quad (13)$$

Now let \{\ket{v_1}, \ket{v_2}, \ket{v_3}\} be an orthonormal basis of $\mathbb{C}^3$ and define

$$\ket{\theta_i} = \sum_{a=1}^3 \ket{v_a} \otimes \ket{\phi(i, a)} \quad (14)$$

for $i = 1, 2, 3$. Then it follows from (12) that $|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle$ are orthonormal. Hence every matrix of the form (11) satisfying (12) can be associated with a three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^9$ with a chosen orthonormal basis, and with a specified basis of $\mathbb{C}^3$. Changing the basis vectors $|\theta_i\rangle$ is equivalent to conjugating each matrix $M_{ab}$ by the same element of $SU(3)$. Similarly, changing measurement basis in $\mathbb{C}^3$ is equivalent to conjugating the blocks of $M$ by a matrix in $SU(3)$. So we have the following equivalent formulation of Conjecture 4:

**Conjecture 5** Let $M$ be a positive semidefinite $9 \times 9$ matrix of the form (11), and satisfying (12). Then Conjecture 4 is equivalent to the following statement: there are unitaries $W, U \in SU(3)$ such that the diagonal blocks of the conjugated matrix

$$(W \otimes U)M(W \otimes U)^* \quad (15)$$

commute.

We now note that any three orthonormal vectors in $\mathbb{C}^3 \otimes \mathbb{C}^n$ define a matrix of the form (11) satisfying (12), for any $n \geq 1$. This follows by defining the matrix elements of $M$ via (13), where $|\phi(i, a)\rangle \in \mathbb{C}^n$ are defined as in (14). Then applying (15) gives the pair of unitary matrices which select the basis of $V$ and the partial measurement in $\mathbb{C}^3$ that allow the basis to be perfectly distinguished.
Example of no one-way LOCC in $\mathbb{C}^3 \otimes \mathbb{C}^5$

Example of a subspace in $\mathbb{C}^3 \otimes \mathbb{C}^5$ which does not have a basis that can be reliably distinguished using one-way LOCC, with partial measurement first made on the $\mathbb{C}^5$ factor. The three states $\{|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle\}$ below generate the subspace.

$$|\theta_1\rangle = \begin{pmatrix} -0.2450 - 0.0054i \\ -0.1694 + 0.0815i \\ 0.1071 - 0.3191i \\ 0.0655 - 0.3190i \\ -0.1911 - 0.1862i \\ 0.1185 + 0.3259i \\ -0.2530 + 0.0480i \\ 0.1194 - 0.1987i \\ 0.1948 - 0.2106i \\ 0.0595 + 0.2934i \\ 0.1286 - 0.1427i \\ -0.1420 + 0.1308i \\ -0.2367 + 0.1399i \\ 0.1384 - 0.0264i \\ 0.0867 + 0.1573i \end{pmatrix}, \quad |\theta_2\rangle = \begin{pmatrix} 0.1438 + 0.2108i \\ -0.3214 + 0.1308i \\ 0.1229 + 0.0319i \\ 0.1775 - 0.1070i \\ 0.2091 - 0.1811i \\ -0.0937 + 0.1880i \\ 0.1609 + 0.0272i \\ 0.1705 + 0.0996i \\ -0.0630 + 0.0729i \\ 0.3389 - 0.1242i \\ 0.0201 - 0.2668i \\ 0.1127 - 0.3331i \\ 0.2338 + 0.3325i \\ -0.1798 - 0.0796i \\ -0.1097 + 0.1360i \end{pmatrix}$$

$$|\theta_3\rangle = \begin{pmatrix} 0.0390 - 0.0484i \\ 0.0405 - 0.2603i \\ 0.2206 + 0.2432i \\ -0.2843 - 0.0751i \\ -0.2416 - 0.1380i \\ 0.0510 + 0.3270i \\ 0.1691 + 0.0829i \\ -0.3761 - 0.1033i \\ -0.3138 + 0.1388i \\ 0.3138 + 0.2228i \\ 0.0553 + 0.2272i \\ 0.0468 - 0.0164i \\ 0.1966 - 0.1044i \\ -0.0147 + 0.1239i \\ -0.2313 + 0.0715i \end{pmatrix}$$