Characters on the Full Group of the Odometer.

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Abstract

Let $X$ be the space of all infinite $0,1$-sequences and $T$ be the odometer on $X$. In this paper we introduce a dense subgroup $S(2^\infty)$ of the full group $[T]$ and describe all indecomposable characters on $S(2^\infty)$. As result we obtain a description of indecomposable characters on $[T]$.

1 Introduction.

Let $X = \prod_1^{\infty} \{0,1\}$ be the space of all infinite $0,1$-sequences with the product topology and $T$ be the odometer on $X$: $Tx = x + 1$. By definition, the full group of the automorphism $T$ is the group $[T]$ of all Borel automorphisms $S$ of $X$, such that $Sx \subset O_T(x)$ for all $x \in X$, where $O_T(x)$ is the orbit of $x$. Let $\mu = \nu^\otimes \infty$ be the standard product measure on $X$, where $\nu(\{0\}) = \nu(\{1\}) = 1/2$. The group $[T]$ is a topological group with the uniform topology, given by the norm $\|S_1 - S_2\| = \mu(\{x : S_1x \neq S_2x\})$. In this paper we obtain a description of all indecomposable characters on $[T]$.

Let $X_n$ be the set of all 0,1-sequences of length $n$. Denote $S(2^n)$ the group of all permutations on $X_n$. Elements of $S(2^n)$ are arbitrary bijections $X_n \to X_n$. The group $S(2^n)$ acts naturally on $X$:

$$s \in S(2^n) : X \to X, \quad s((x, a)) = (s(x), a) \text{ for any } x \in X_n, a \in X.$$  

Denote $S(2^\infty) = \bigcup_{n \in \mathbb{N}} S(2^n)$. Then $S(2^\infty)$ is a dense subgroup in $[T]$. It follows, that for each continuous factor representation $\pi$ of $[T]$ the restriction of $\pi$ onto $S(2^\infty)$ generates the same $W^*$ algebra, therefore, also is a factor representation. In this paper we describe all indecomposable characters on $S(2^\infty)$. It turns out, that indecomposable characters on $S(2^\infty)$ have very
simple structure and each indecomposable character on $S(2^\infty)$ gives rise to an indecomposable character on $[T]$.

The group $S(2^\infty)$ is a parabolic analog of the infinite symmetric group $S(\infty)$. Another parabolic analog of $S(\infty)$ is the group $R$ of rational rearrangements of the segment (see [3]). In [3] E. Goryachko studied $K_0$-functor and characters of the group $R$. Indecomposable characters on the infinite symmetric group were described by E. Thoma in [7]. In [1] and [2] A. Vershik and S. Kerov developed the asymptotic theory of characters on $S(\infty)$. In [5] and [6] G. Olshanski developed the semigroup approach to representations of groups, connected to $S(\infty)$. Using the semigroup approach, A. Okounkov found a new proof of the Thoma’s result (see [4]). In this paper we use the approach of Olshanski and Okounkov.

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Now we remind some definitions from the representation theory.

**Definition 1.** A character on a group $G$ is a function $\chi : G \rightarrow \mathbb{C}$, satisfying the following properties:

1) $\chi(g_1g_2) = \chi(g_2g_1)$ for any $g_1, g_2 \in G$;

2) the matrix $\{\chi(g_i^{-1}g_j)\}_{i,j=1}^n$ is nonnegatively defined for any $n$ and $g_1, \ldots, g_n \in G$;

3) $\chi(e) = 1$.

A character $\chi$ is called indecomposable, if it can’t be represented in the form $\chi = \alpha \chi_1 + (1 - \alpha) \chi_2$, where $0 < \alpha < 1$ and $\chi_1, \chi_2$ are distinct characters.

For a unitary representation $\pi$ of a group $G$ denote $\mathcal{M}_\pi$ the $W^*$-algebra, generated by the operators of the representation $\pi$. By definition, the commutant $S'$ of a set $S$ of operators in a Hilbert space $\mathcal{H}$ is the algebra $S' = \{A \in B(\mathcal{H}) : AB = BA \text{ for any } B \in S\}$.

**Definition 2.** A representation $\pi$ of a group $G$ is called a factor representation, if the algebra $\mathcal{M}_\pi$ is a factor, that is $\mathcal{M}_\pi \cap \mathcal{M}'_\pi = \mathbb{C}I$.

The indecomposable characters on a group $G$ are in one to one correspondence with the finite type factor representations of $G$. Namely, starting with an indecomposable character $\chi$ on $G$ one can construct a triple $(\pi_\chi, \mathcal{H}_\chi, \xi_\chi)$,
called the Gelfand-Naimark-Siegel construction. Here $\pi_\chi$ is a finite type factor representation, acting in the space $\mathcal{H}_\chi$, and $\xi_\chi$ is a unit vector in $\mathcal{H}_\chi$, such that $\chi(g) = (\pi_\chi(g)\xi_\chi, \xi_\chi)$ for any $g \in G$. Note, that the vector $\xi_\chi$ is cyclic and separating for the algebra $\mathcal{M}_\pi$.

For $n \in \mathbb{N}$ denote the inclusion $i_n : S(2^\infty) \hookrightarrow S(2^\infty)$ as follows:

$$i_n(s)((x, y)) = (x, s(y)),$$

for any $x \in X_n, y \in X$. (1)

Put $S_n(2^\infty) = i_n(S(2^\infty))$. Note, that for any $s_1 \in S(2^n), s_2 \in S_n(2^\infty)$ one has $s_1s_2 = s_2s_1$. The following property is known as multiplicativity.

**Proposition 3.** A character $\chi$ on $S(2^\infty)$ is indecomposable iff $\chi(s_1s_2) = \chi(s_1)\chi(s_2)$ for any $n \in \mathbb{N}$ and $s_1 \in S(2^n), s_2 \in S_n(2^\infty)$.

The last proposition can be proven the same way, as the analogous statement for the indecomposable characters on the infinite symmetric group (see [4]). For $f \in [T]$ denote $\text{Fix}(f) = \{x \in X : f(x) = x\}$. The main results of this paper are the following two propositions:

**Theorem 4.** A function $\chi$ on $S(2^\infty)$ is an indecomposable character, if and only if there exists $\alpha \in \mathbb{Z}_+ \cup \{\infty\}$, such that $\chi(s) = \mu(\text{Fix}(s))^\alpha$ for any $s \in S(2^\infty)$.

In the last theorem we assume $0^0 = 1, x^\infty = 0$ for any $x \in [0, 1)$ and $1^\infty = 1$. Note, that $\alpha = 0$ and $\infty$ correspond to the trivial and the regular characters.

**Corollary 5.** A function $\chi$ on $[T]$ is an indecomposable character, if and only if there exists $\alpha \in \mathbb{Z}_+ \cup \{\infty\}$, such that $\chi(f) = \mu(\text{Fix}(f))^\alpha$ for any $f \in [T]$.

## 2 Construction of representations.

In this section we give a construction of $II_1$ factor-representations of $[T]$. Denote

$$Y_n = \{(x, y) \in X \times X : x_k = y_k \text{ for all } k > n\}, \ Y = \cup Y_n.$$  

For $a, b \in X_n$ introduce cylindrical set

$$Y_n^{a,b} = \{(x, y) \in Y_n : x_i = a_i, y_i = b_i \text{ for all } i \leqslant n\}.$$  

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Introduce the measure $\gamma$ on $Y$ by the formula
\[ \gamma(Y^{a,b}) = 2^{-n} \] for each $n \in \mathbb{N}$ and $a, b \in X_n$. \hfill (4)

Denote the unitary representation $\pi$ of $[T]$ in $\mathcal{H} = L^2(Y, \gamma)$ by the formula
\[ (\pi(s)f)(x, y) = f(s^{-1}(x), y) \] for each $f \in \mathcal{H}, s \in [T]$. \hfill (5)

Put $\xi(x, y) = \delta_{x,y}$ and $\chi(s) = (\pi(s)\xi, \xi), s \in [T]$. Then direct calculations show, that $\chi(s) = \mu(Fix(s))$ for all $s \in [T]$. In particular, $\chi$ is a central function on $[T]$. It follows, that $\chi$ is a character. By the proposition $3$, $\chi$ is an indecomposable character on $S(2^\infty)$. It follows, that $\chi$ is an indecomposable characters on $[T]$. Moreover, for any $k \in \mathbb{N}$, considering the triple $(\pi^{\otimes k}, \mathcal{H}^{\otimes k}, \xi^{\otimes k})$, we get, that $\chi^k$ is an indecomposable character.

3 System of orthogonal projections.

Let $\chi$ be an indecomposable character on the group $S(2^\infty)$. Denote $(\pi, \mathcal{H}, \xi)$ the corresponding GNS-construction. In this section we find a system of orthogonal projections in the algebra $\mathcal{M}_\pi^*$, satisfying remarkable properties.

First we describe the conjugate classes in $S(2^\infty)$. Let $g_1, g_2 \in S(2^\infty)$. Then there exists $n$, such that $g_1, g_2 \in S(2^n)$. The elements $g_1$ and $g_2$ are conjugate in $S(2^\infty)$, if and only if $g_1$ and $g_2$ are conjugate in $S(2^n)$. Remind, that conjugate classes in finite symmetric groups are parameterized by partitions, made from the lengths of the cycles.

For subsets $A \subset X_n, B \subset X$ denote
\[ A \times B = \{(a_1, \ldots, a_n, b_1, b_2, \ldots) : (a_1, \ldots, a_n) \in A, (b_1, b_2, \ldots) \in B\} \subset X. \]

We will call a subset $A \subset X$ nice, if $A = C \times X$ for some $k \in \mathbb{N}$ and $C \subset X_k$. Let $A$ be nice and $m > k$. Denote $s^A_m \in S(2^\infty)$ as follows:
\[ s^A_m(x) = \begin{cases} x, & \text{if } x \in A, \\ (x_1, \ldots, x_{m-1}, 1 - x_m, x_{m+1}, \ldots), & \text{if } x \notin A. \end{cases} \] \hfill (6)

Note, that $s^A_m$ affects only $m$-th coordinate of an element of $X$.

**Lemma 6.** For any nice $A \subset X$ there exists the weak limit $P^A = \lim_{m \to \infty} \pi(s^A_m)$. The operators $P^A$ are orthogonal projections.
Proof. Consider any elements $g_1, g_2 \in S(2^\infty)$. Fix $M$, such that $g_1, g_2 \in S(2^M)$. One can check, that for any $m > M$ the conjugate class of the element $g_1^{-1}s_m^Ag_2$ doesn’t depend on $m$. By the centrality of $\chi$, the value

$$\left(\pi \left(s_m^A\right) \pi(g_2)\xi, \pi(g_1)\xi\right) = \chi\left(g_1^{-1}s_m^Ag_2\right)$$  \hspace{1cm} (7)

doesn’t depend on the choice of $m > M$. By definition of the GNS-construction, the vectors $\pi(g)\xi, g \in S(2^\infty)$ are dense in $\mathcal{H}$. Therefore, there exists the weak limit $P^A = \lim_{m \to \infty} \pi \left(s_m^A\right)$.

Further, since $\pi \left(s_m^A\right)^* = \pi \left((s_m^A)^{-1}\right) = \pi \left(s_m^A\right)$ for any $m$, the operator $P^A$ is self-adjoint. As follows, $(P^A)^2$ is a positive operator. Obviously, $\|P^A\| \leq 1$. One can check, that for any $m_1 > m_2 > m_3$ the elements $s_{m_1}s_{m_2}s_{m_3}$ and $s_{m_1}s_{m_2}$ are conjugate. Therefore,

$$\left(\pi \left(s_{m_1}s_{m_2}s_{m_3}\right) \xi, \xi\right) = \left(\pi \left(s_{m_1}s_{m_2}\right) \xi, \xi\right).$$

In the limit we get:

$$\left((P^A)^3 \xi, \xi\right) = \left((P^A)^2 \xi, \xi\right) = \|P^A\xi\|^2.$$  \hspace{1cm} (8)

From the other hand, by the Cauchy-Schwartz inequality,

$$\left((P^A)^3 \xi, \xi\right) \leq \|P^A\xi\| \| (P^A)^2 \xi\| \leq \|P^A\xi\|^2.$$

The equality means, that $P^A\xi = c (P^A)^2\xi$ for some constant $c$. Since $\xi$ is separating, the latter means, that $P^A = c (P^A)^2$. From (8) we get, that $P^A = (P^A)^2$, which finishes the proof. \hfill \Box

Recall, that the unique normalized trace on the algebra $\mathcal{M}_\pi$ is given by the formula: $\text{tr}(T) = (T\xi, \xi)$.

**Proposition 7.** For any nice $A, B \subset X$ and $C \subset X_n, D \subset X_m, n, m \in \mathbb{N}$, the following is true:

1) if $s \in S(2^\infty)$, then $\pi(s)P^A\pi(s^{-1}) = P^{s(A)}$;

2) $P^AP^B = P^{A\cap B}$;

3) $\text{tr} \left(P^{C\times D\times X}\right) = \text{tr} \left(P^{C\times X}\right) \text{tr} \left(P^{D\times X}\right)$;
4) If \( \mu(A) \leq \mu(B) \), then \( \text{tr } (P^A) \leq \text{tr } (P^B) \).

In the item 3) by \( C \times D \times X \) we mean the set of sequences of the form
\[
(c_1, \ldots, c_n, d_1, \ldots, d_m, x_1, x_2, \ldots),
\]
where \((c_1, \ldots, c_n) \in C\), \((d_1, \ldots, d_m) \in D\) and \(x_i \in \{0, 1\}\).

**Proof.** 1) The first property follows immediately from the equation \( s_m s_m s^{-1} = s_m^{(A)} \) for large enough \( m \).

2) Let \( m_1 > m_2 > m_3 \). It follows from (6), that the elements \( s_m^{A} s_m^{B} s_m^{A \cap B} \), \( s_m^{m_1} s_m^{m_2} \) and \( s_m^{m_1} s_m^{m_2} \) are in the same conjugate class. Therefore
\[
\chi (s_m^{m_1} s_m^{m_2} s_m^{A \cap B}) = \chi (s_m^{m_1} s_m^{B}) = \chi (s_m^{A \cap B}).
\]

When \( m_1, m_2, m_3 \) go to infinity, we get
\[
\text{tr } (P^A P^B P^{A \cap B}) = \text{tr } (P^A P^B) = \text{tr } (P^A P^{A \cap B}). \tag{9}
\]

From the other hand, by the Cauchy-Schwartz inequality, centrality of \( \text{tr} \) and lemma 6
\[
\text{tr } (P^A P^B P^{A \cap B}) \leq \text{tr } (P^A P^B)^{\frac{1}{2}} \text{tr } (P^A P^{A \cap B})^{\frac{1}{2}}. \tag{10}
\]

By (9), the equality holds. Therefore, as in the proof, that \( P^A \) is an orthogonal projection, we get, that \( P^A P^B = P^A P^{A \cap B} \).

3) This property follows from the multiplicativity of \( \chi \) (see prop. 3). Indeed, by the definition of operators \( P^A \) and the proof of the lemma 6 one has
\[
\text{tr } (P^{C \times D \times X}) = \chi (s_{n+m+1}^{C \times D \times X}). \tag{11}
\]

The conjugate class of the element \( s_{n+m+1}^{C \times D \times X} \) contains \( s_{n+1}^{C \times X} s_{n+m+2}^{X \times D \times X} \). Thus,
\[
\chi (s_{n+1}^{C \times X} s_{n+m+2}^{X \times D \times X}) = \chi (s_{n+1}^{C \times X}) \chi (s_{n+m+2}^{X \times D \times X}). \tag{12}
\]

To finish the proof, we note, that by centrality of \( \chi \), \( \chi (s_{n+1}^{C \times X}) = \text{tr } (P^{C \times X}) \) and
\[
\chi (s_{n+m+2}^{X \times D \times X}) = \text{tr } (P^{D \times X}).
\]

4) By the property 1), without loss of generality we may assume, that \( A \subset B \). By the property 2), \( P^A \leq P^B \). Therefore, \( \text{tr } (P^A) \leq \text{tr } (P^B) \).
Corollary 8. There exists $\alpha \in \mathbb{R}_+ \cup \{\infty\}$, such that for any $n$ and finite union of cylinders $A \subset X$ one has $\text{tr} \left( P^A \right) = \mu(A)^\alpha$.

Proof. We split the proof into three cases, according to the possible values of $\alpha$ ($0$, $\infty$ or a positive number).

1. First assume, there exists $C \subset X$, $C \neq X$, such that $\text{tr} \left( P^{C \times X} \right) = 1$. Then for any $m$ and any $D \subset X_m$, $D \neq \emptyset$ one can find $k$, such that $\mu \left( C^k \times X \right) \leq \mu(D \times X)$. By the proposition[1] $\text{tr} \left( P^{D \times X} \right) \geq \text{tr} \left( P^{C^k \times X} \right) = 1$. Therefore, $\text{tr} \left( P^{D \times X} \right) = 1$. Now we only need to check, that $\text{tr} \left( P^\emptyset \right) = 1$. Since $\xi$ is separating for $M_\pi$, it follows, that $P^A = Id$ for any finite union of cylinders $A \neq \emptyset$. By the property 2) from the proposition[2] $P^\emptyset = Id$. Thus, in this case the corollary holds for $\alpha = 0$.

2. Now assume, that there exists $C \subset X$, $C \neq \emptyset$, such that $\text{tr} \left( P^{C \times X} \right) = 0$. Using the same ideas, as in the case 1, one can prove, that for any finite union of cylinders $B \subset X$, $B \neq X$, one has $\text{tr} \left( P^B \right) = 0$. Since $P^X = Id$, $\text{tr} \left( P^X \right) = 1$. In this case the corollary holds for $\alpha = \infty$.

3. Now assume, that $0 < \text{tr} \left( P^A \right) < 1$ for any $n$ and any $A \subset X$, such that $A \neq X$ and $A \neq \emptyset$. It follows from the property 4) of the previous proposition, that $\text{tr} \left( P^A \right)$ depends only on the measure of $A$. Therefore, there is a function $\varphi$ from the set $D = \left\{ \frac{k}{2^n} : p, q \in \mathbb{N}, p < 2^q \right\}$ of dyadic numbers to the set of positive numbers, such that

$$\text{tr} \left( P^A \right) = \varphi(\mu(A)) \text{ for any } A \subset X, A \neq X, A \neq \emptyset. \quad (13)$$

By the properties 3), 4) from the proposition[7] $\varphi$ is a monotone multiplicative homomorphism. It follows, that there exists $0 < \alpha < \infty$, such that $\phi(d) = d^\alpha$ for any $d \in D$.

\begin{proof}

4 The proof of the classification theorems.

Proposition 9. Let $A \subset X$ be nice and $s \in S(2^\omega)$, such that $A \subset \text{Fix}(s) = \{x \in X : s(x) = x\}$. Then $\pi(s)P^A = P^A$.

Proof. There exists $n$, such that $A = C \times X$ for some $C \subset X_n$ and $s \in S(2^n)$. Assume first, that $s$ contains only cycles of length 1 and 2. Than the permutations $ss_m^A$ and $s_m^A$ are conjugate for large $m$. Therefore, $(\pi(s)P^A, \xi) = (P^A, \xi) = \|P^A\|^2$. Using the Cauchy-Schwartz inequality, we get

$$(\pi(s)P^A, \xi) = (\pi(s)P^A, P^A, \xi) \leq \|\pi(s)P^A\| \|P^A\| \leq \|P^A\|^2 \quad (14).$$

\end{proof}
Since the equality holds, \( \pi(s)P^A \xi = P^A \xi \). Since \( \xi \) is separating, \( \pi(s)P^A = P^A \). Now notice, that permutations \( s \in S(2^n) \), such that \( A \subseteq Fix(s) \) and \( s \) has only cycles of length 1 and 2, generate all permutations \( w \in S(2^n) \), such that \( A \subseteq Fix(w) \). This finishes the proof.

**Corollary 10.** Let \( s \in S(2^\infty) \) have cycles of length 1 and only. Then \( \chi(s) = \mu(Fix(s))^\alpha \), where \( \alpha \) is from the corollary 8.

**Proof.** Denote \( A = Fix(s) \). Then for large \( m \) the elements \( s \) and \( ss_m^A \) are conjugate. Therefore, using the propositions 9 and corollary 8 we get

\[
\chi(s) = tr \left( \pi(s)P^A \right) = tr \left( P^A \right) = \mu(A)^\alpha.
\]  

(15)

**Proposition 11.** For any \( s \in S(2^\infty) \) one has \( \chi(s) = \mu(Fix(s))^\alpha \), where \( \alpha \) is the number from the corollary 8.

**Proof.** Let \( s \neq e \). Put \( A = Fix(s) \). Fix arbitrary \( k \in \mathbb{N} \). There exist permutations \( s_1, \ldots, s_k \in S(2^\infty) \), such that the following is true:

1) elements \( s_i \) are conjugate to \( s \);

2) \( Fix(s_is_j^{-1}) = Fix(s_i) = A \) for any \( i \neq j \);

3) for any \( i \neq j \) the element \( s_is_j^{-1} \) has cycles of length 1 and 2 only.

We postpone the proof of the existence of \( s_i \) to the appendix, since this statement is purely combinatoric. Consider the system of vectors \( \eta_i = (\pi(s_i) - P^A) \xi \). Put \( \eta = (\pi(s) - P^A) \xi \). It follows from 1), that \( (\eta_i, \xi) = (\eta, \xi) \) for any \( i \). By the proposition 9 and property 2) of \( s_i \), \( \pi(s_i)P^A = P^A \). Thus, by the corollaries 8 and 10 for any \( i \neq j \)

\[
(\eta_i, \eta_j) = ((\pi(s_i) - P^A) \xi, (\pi(s_j) - P^A) \xi) = \chi(s_is_j^{-1}) - tr \left( P^A \right) = 0. \quad (16)
\]

Note, that by the same reasons \( \|\eta_i\| = \sqrt{1 - tr \left( P^A \right)} = \|\eta\| \) for any \( i \). Therefore, one has:

\[
|\langle \eta, \xi \rangle| = \frac{1}{k} \left| \sum_{i=1}^{k} \eta_i, \xi \right| \leq \frac{1}{k} \left\| \sum_{i=1}^{k} \eta_i \right\| = \frac{\|\eta\|}{\sqrt{k}}. \quad (17)
\]

Since \( k \) is arbitrary, the last inequality means \( \langle \eta, \xi \rangle = 0 \). It follows, that \( \chi(s) = tr \left( P^A \right) = \mu(A)^\alpha \).
Proposition 12. Let $0 < \alpha < \infty$. Denote $\chi_\alpha(s) = \mu(Fix(s))^{\alpha}$, $s \in S(2^\infty)$. Assume, that $\chi_\alpha$ is a character. Then $\alpha \in \mathbb{N}$.

Proof. Note first, that $\chi_\alpha$ is indecomposable by the proposition 3. Let $(\pi_\alpha, \mathcal{H}_\alpha, \xi_\alpha)$ be the GNS-construction, corresponding to $\chi_\alpha$.

Let $n \in \mathbb{N}$. Following Okounkov [4], consider the orthogonal projection $\text{Alt}(n) = \frac{1}{2n!} \sum_{s \in S(2^n)} \sigma(s) \pi_\alpha(s),$ where $\sigma(s)$ is the sign of the permutation $s$. One has:

$$0 \leq (\text{Alt}(n)\xi_\alpha, \xi_\alpha) = \frac{1}{2n!} \sum_{s \in S(2^n)} \sigma(s) \chi_\alpha(s) =$$

$$= \frac{1}{2n!} \sum_{s \in S(2^n)} \sigma(s) \frac{\{x \in X_n : s(x) = x\}^{\alpha}}{2^{\alpha n}}. \quad (18)$$

Calculate the last sum. Denote $\Sigma_k = \sum_{E_k} \sigma(s)$, where $E_k$ is the set of permutations $s \in S(k)$, such that $s(j) \neq j$ for $1 \leq j \leq k$. We prove by induction, that $\Sigma_k = (-1)^{k-1}(k - 1)$. Base $k = 1, 2$ is obvious. Let $k \geq 2$. Each element of $E_{k+1}$ can be represented as $(i, k+1)s$, where $s$ is either any element of $E_k$, or a permutation from $S(k)$, such that $s(i) = i$ and $s(j) \neq j$ for $j \neq i, 1 \leq j \leq k$. Therefore, $\Sigma_{k+1} = -k(\Sigma_k + \Sigma_{k-1}) = (-1)^kk$. Let $m \in \mathbb{N}$. One has:

$$\sum_{s \in S(m)} \sigma(s) \{j : s(j) = j\}^{\alpha} = \sum_{A \subset \{1, \ldots, m\} \text{Fix}(s) = A} \sum_{s \in S(m)} \sigma(s) |A|^{\alpha}. \quad (19)$$

For any $j$ there are $C_m^j$ subsets $A$ of cardinality $m - j$, each of which makes contribution $\Sigma_j \cdot (m - j)^{\alpha} = (-1)^{j-1}(j - 1)(m - j)^{\alpha}$ to the sum (19). Thus,

$$\sum_{A \subset \{1, \ldots, m\} \text{Fix}(s) = A} \sum_{A \subset \{1, \ldots, m\} \text{Fix}(s) = A} \sigma(s) |A|^{\alpha} = \sum_{j=0}^{m} C_m^j (-1)^{j-1}(j - 1)(m - j)^{\alpha}. \quad (20)$$

From (18) – (20) for $m = 2^n$ one gets:

$$C_\alpha(m) = \sum_{j=0}^{m} C_m^j (-1)^{j-1}(j - 1)(m - j)^{\alpha} \geq 0. \quad (21)$$
We will show, that for any noninteger $\alpha > 0$ there exist $m \in \mathbb{N}$, such that $C_\alpha(m) < 0$.

Note, that for $\alpha = n \in \mathbb{N}$ the last sum can be written in terms of Stirling numbers of the second type:

$$C_n(m) = m!(S(n, m) + S(n, m - 1)),$$

where $S(n, m) = \frac{1}{m!} \sum_{j=0}^{m} C_j^m (-1)^j (m - j)^n$.

Remind, that $S(n, m) = 0$ for $n < m$ and $S(n, m) > 0$ for $n \geq m$. Further, using the binomial rule, we get:

$$C_\alpha(m) = m^\alpha \sum_{j=0}^{m} C_j^m (-1)^{j-1} (j - 1) \left(1 - \frac{j}{m}\right)^\alpha =$$

$$m^\alpha \sum_{j=0}^{m} C_j^m (-1)^{j-1} (j - 1) \sum_{k=0}^{\infty} \binom{m-1}{k} (-\frac{j}{m})^k$$

$$= \sum_{k=0}^{\infty} (-1)^{k} m^{\alpha-k} C_\alpha^k \sum_{j=0}^{m} C_j^m (-1)^{j-1} (j - 1) j^k.$$

Using change $r = m - j$, we get:

$$\sum_{j=0}^{m} C_j^m (-1)^{j-1} (j - 1) j^k = \sum_{r=0}^{m} C_r^m (-1)^{m-r-1} (m - r - 1) (m - r)^k =$$

$$(-1)^{m-1} \left((m - 1) \sum_{r=0}^{m} C_r^m (-1)^r (m - r)^k - \sum_{r=0}^{m} r C_r^m (-1)^r (m - r)^k\right) =$$

$$m! ((m - 1) S(k, m) + S(k, m - 1)).$$

Finally, we get

$$C_\alpha(m) = m! \sum_{k=m-1}^{\infty} (-1)^{k+m-1} m^{\alpha-k} C_\alpha^k ((m - 1) S(k, m) + S(k, m - 1)).$$

Let $\alpha$ be noninteger. The sign of the expression $(-1)^k C_\alpha^k$ doesn’t depend on $k$ for $k > [\alpha] + 1$. It follows, that $C_\alpha(m) < 0$ either for $m = [\alpha] + 3$ or for $m = [\alpha] + 4$. This finishes the proof. \qed
5 Appendix: existence of $s_i$.

Here we prove the next combinatorial statement.

**Proposition 13.** Let $s \in S(2^\infty)$. Then for any $r$ there exist permutations $s_1, \ldots, s_{2r} \in S(2^\infty)$, such that the following is true:

1) elements $s_i$ are conjugate to $s$;
2) $\text{Fix}(s_is_j^{-1}) = \text{Fix}(s_i) = A$ for any $i \neq j$;
3) for any $i \neq j$ the element $s_is_j^{-1}$ has cycles of length 1 and 2 only.

We will divide the proof of the last proposition into several lemmas. For pairwise distinct numbers $k_0, \ldots, k_{l-1}$ denote $(k_0, k_1, \ldots, k_{l-1})$ the cyclic permutation, sending $k_i$ to $k_i \mod l$. In particular, $(k, l)$ stands for the transposition of $k$ and $l$.

**Lemma 14.** Let $k > 4$ be an odd number. Then for $l \in \{2k - 2, 2k - 4\}$ there exist permutations $g_1, g_2 \in S(l)$, such that each of $g_1, g_2$ has only cycles of length $k$ or 1, and $g_1g_2^{-1}$ has only even cycles.

**Proof.** Let $l = 2k - 2$. Put

$$g_1 = (1, 2, \ldots, k) = (1, 2)(2, 3) \cdots (k - 1, k), \quad g_2 = (2k - 2, 2k - 3, \ldots, k - 1) = (2k - 2, 2k - 3)(2k - 3, 2k - 4) \cdots (k, k - 1).$$

Then $g_1$ and $g_2$ are cycles of length $k$ and

$$g_1g_2^{-1} = (1, 2)(2, 3) \cdots (k - 2, k - 1) \times (k, k + 1)(k + 1, k + 2) \cdots (2k - 3, 2k - 2) = (1, 2, \ldots, k - 1) \times (k, k + 1, \ldots, 2k - 2)$$

is a product of two cycles of length $k - 1$.

Let $l = 2k - 4$. Then denote $g_1 = (1, 2)(2, 3) \cdots (k - 1, k)$ as above and $g_2 = (2k - 4, 2k - 5)(2k - 5, 2k - 6) \cdots (k + 1, k) \times (k - 3, k - 2)(k - 2, k - 1)(k - 1, k)$. Again, $g_1$ and $g_2$ are cycles of length $k$. One has

$$g_1g_2^{-1} = (1, 2)(2, 3) \cdots (k - 4, k - 3) \times (k, k + 1)(k + 1, k + 2) \cdots (2k - 5, 2k - 4)$$

is a product of two cycles of length $k - 3$. \qed
Corollary 15. For any $k$ there exists $M = M(k)$, such that for any $m \geq M$
there exist $g_1, g_2 \in S(2^m)$, satisfying the following conditions:
1) $g_1$ and $g_2$ have cycles of length dividing $k$;
2) $g_1 g_2^{-1}$ has only even cycles.

Proof. If $k$ is even, put $M = 2$. Let $m \geq 2$. Consider $S(2^m)$ as the group of
permutations on $X_m$. Let $g_i, i = 1, 2$ be the permutation, arising from the
change $0 \leftrightarrow 1$ on the $i$-th coordinate of $X_m$. Obviously, $g_1, g_2$ satisfy to the
condition of the corollary.

Let $k = 3$. Denote $g_1 = (1, 2)(2, 3), g_2 = (4, 3)(3, 2) \in S(2^2)$. For any
$m \geq 2$ one can consider $g_1, g_2$ as elements of $S(2^m)$, acting on first two
coordinates only. Obviously, these elements satisfy to the conditions of the
corollary.

Let $k > 4$ be odd. Put $M = k$. For any $m \geq k$ the number $2^m$
can be represented as $2^m = (2k - 4)l + (2k - 2)r$ with $r, l$ nonnegative integer.
Divide the set of $2^m$ elements onto $l$ subsets of $2k - 4$ elements and $r$ subsets
of $2k - 2$ elements. Organize the permutations $g_1, g_2$ on each subset using
the lemma [13].

Proof of the proposition [13]. Let $s \in S(2^n)$. For $x \in X_n$ denote $ord(x)$ the
number of elements in the trajectory of $x$. That is, $ord(x)$ is the length of
the cycle of $s$, containing $x$. Let $m$ be the maximal of $M(ord(x)), x \in S(2^n)$
(see the corollary [15]). For any $x$, putting $k = ord(x)$, let $g_i^{(k)}$, $i = 1, 2$ be the
permutations from $S(2^m)$, which satisfy to the conditions of the corollary [15].
Let $r \in \mathbb{N}$. For $a = (a_1, a_2, \ldots, a_r), a_i \in \{1, 2\}, k \in \{ord(x) : x \in X_n\}$
denote $g_a^{(k)} \in S(2^{mr}), s_a \in S(2^{n+mr})$ as follows:

$$g_a^{(k)}(y_1, y_2, \ldots, y_r) = (g_{a_1}^{(k)}(y_1), g_{a_2}^{(k)}(y_2), \ldots, g_{a_r}^{(k)}(y_r)),$$

$$s_a(x, y) = \begin{cases} (x, y), & \text{if } x \in Fix(s), \\ (s(x), g_a^{(ord(x))}(y)), & \text{otherwise}, \end{cases}$$

where $x \in X_n, y_i \in X_m, y \in X_{mr}$. Note, that each $g_a^{(k)}$ has cycles of length $k$
and 1 only. It follows, that $s_a$ is conjugate to $s$ for each $a$. Moreover, for any
$a, b \in \{1, 2\}^r$ one has:

$$s_a s_b^{-1}(x, y) = \begin{cases} (x, y), & \text{if } x \in Fix(s), \\ (x, g_a^{(ord(x))}(g_b^{(ord(x))})^{-1}(y)), & \text{otherwise}. \end{cases}$$
Therefore, by the definition of $g^{(k)}_i$ (see the corollary [13]), $\text{Fix} \left( s_a s_b^{-1} \right) = \text{Fix}(s)$ and $s_a s_b^{-1}$ has only even cycles and fixed elements for $a \neq b$. Thus, $s_a$ satisfy to the conditions of the proposition [13].

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