THE EUCLIDEAN DISTANCE DEGREE
OF FERMAT HYPERSURFACES

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Abstract. Finding the point in an algebraic variety that is closest to a given point is an optimization problem with many applications. We study the case when the variety is a Fermat hypersurface. Our formula for its Euclidean distance degree is a piecewise polynomial whose pieces are defined by subtle congruence conditions.

1. Introduction

Let \( X \in \mathbb{R}^n \) be an real affine algebraic variety, i.e. \( X \) is the common zero set of some polynomials \( f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n] \). We consider the following problem: given \( u \in \mathbb{R}^n \), compute \( u^* \in X \) that minimizes the squared Euclidean distance \( d_u(x) = \sum_{i=1}^{n} (u_i - x_i)^2 \) from the given point \( u \). This problem arises from best approximation problems. Once we have a mathematical model \( X \) to be satisfied by a data \( u \) obtained by, for example, an experiment or reception from someone’s transmission, usually \( u \) contains some error and hence we want to correct it. The nearest point \( u^* \) in \( X \) to \( u \) represents the original data suggested by \( u \).

In order to find \( u^* \) algebraically, we consider the zeroes in \( \mathbb{C}^n \) of the equations defining \( X \), and we examine all complex critical points of the squared distance function \( d_u(x) = \sum_{i=1}^{n} (u_i - x_i)^2 \) on \( X \setminus X_{\text{sing}} \) where \( X_{\text{sing}} \) is the singular locus of \( X \). If \( X \) has some singular locus, then there could be infinitely many critical points of \( d_u(x) \) on \( X \). Thus we remove the singular locus of \( X \). The number of such critical points is finite and constant on a dense open subset of data \( u \in \mathbb{R}^n \). That number of critical points was studied by J.Draisma et al [4]. It is called the Euclidean distance degree (ED-degree) of the variety \( X \), and denoted as \( \text{EDdeg}(X) \). From now on, all the objects will be considered as complex varieties, except in Section 2.3.

Sometimes, \( X \) is given by homogeneous polynomials. The set of \( m_1 \) by \( m_2 \) matrices of rank at most \( k \) is a typical example. Such a variety is called a projective algebraic variety in \( \mathbb{P}^n(\mathbb{C}) \). For the definition of \( \mathbb{P}^n(\mathbb{C}) \) and more informations, see Chapter 8 of the book by Cox, Little, and O’Shea [3]. For a projective \( X \subset \mathbb{P}^n(\mathbb{C}) \), we define \( \text{EDdeg}(X) \) to be the ED-degree of the affine cone of \( X \) in \( \mathbb{C}^{n+1} \). That is, just regard \( X \) as an affine variety and compute the ED-degree. The ED-degrees of determinantal varieties as above have been studied by G.Ottaviani et al [9].

This paper is motivated by following general upper bound on the ED-degree.

**Proposition 1.1.** [4, Corollary 2.9] Let \( X \) be a hypersurface in \( \mathbb{P}^n(\mathbb{C}) \) defined by a homogeneous polynomial \( f \) of degree \( d \). Then

\[
\text{EDdeg}(X) \leq d \sum_{i=0}^{n-1} (d-1)^i.
\]
and equality holds when $f$ is generic.

In this paper, we focus on Fermat hypersurfaces and its variations.

**Definition 1.2.**

- A Fermat hypersurface of degree $d$ in $\mathbb{P}^n(\mathbb{C})$, denoted by $F_{n,d}$, is the projective variety defined by the polynomial $x_0^d + \cdots + x_n^d$.
- An affine Fermat hypersurface of degree $d$ in $\mathbb{C}^n$, denoted by $AF_{n,d}$, is the affine variety defined by the polynomial $x_1^d + \cdots + x_n^d - 1$.
- A scaled Fermat hypersurface of degree $d$ in $\mathbb{P}^n(\mathbb{C})$ with scaling vector $a = (a_0, \ldots, a_n) \in (\mathbb{C}^*)^{n+1}$, denoted by $SF_a^{n,d}$, is the projective variety defined by the polynomial $x_0^d/a_0 + \cdots + x_n^d/a_n$.

In statistical optimization, maximum likelihood estimation (MLE) is an important tool. The generic number of the critical points of maximum likelihood function, called ML-degree, is a parallel concept to ED-degree. The ML-degrees of many statistically relevant varieties have been computed [5]. Recently, in particular, the ML-degree of $F_{n,d}$ is partially given by D.Agostini et al [2]. Their results, which we review in Example 2.6, serve as motivation our study of the ED-degree of $F_{n,d}$.

This paper is organized as follows. In Section 2, we will investigate the sharpness of the general bound (Proposition 1.1) for the Fermat hypersurfaces. We gives a formula for the ED-degree of $F_{n,d}$ (Theorem 2.2), and gives an explicit formula for $n \leq 3$ (Remark 2.3, Example 2.6). If we fix $n$ and consider the general bound as a function in $d$, it is the best possible polynomial bound (Lemma 2.4), while the gap can be arbitrary large (Remark 2.3). The main theorem can be used for an efficient algorithm which computes the ED-degree of Fermat hypersurfaces numerically (Example 2.11). The proof of Theorem 2.2 can be used similarly to evaluate the ED-degree for an affine Fermat hypersurface $AF_{n,d}$ (Corollary 2.12). After that, an open problem (Conjecture 2.14) about real Fermat hypersurfaces will be discussed.

In Section 3, we will consider the scaled Fermat hypersurfaces for fixed $n$ and $d$. We introduce the exponential cyclotomic polynomial $Q_{m,p}$ which has a special role for the scaling vector $a$ of $SF_a^{n,d}$ (Theorem 3.1). As a corollary, we will see that the ED-degree of scaled Fermat hypersurface usually achieve the general bound.

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2. ED-degree for Fermat hypersurfaces

2.1. Main theorem for Fermat hypersurfaces. In this section, we compute the ED-degree of $F_{n,d}$ for each $n, d$.

**Definition 2.1.** For a positive integer $p$, fix a $p$-th primitive root of unity $\zeta$. Define $\delta(m, p)$ to be the number of complex $m$-tuples $(t_1, \ldots, t_m)$, $1 \leq t_i \leq p$, satisfying

$$1 + \sum_{i=1}^{m} \zeta^{2t_i} = 0.$$
Note that it does not depend the choice of $\zeta$.

**Theorem 2.2.** The ED-degree of the Fermat hypersurface $F_{n,d}$ is given by

$$\text{EDdeg}(F_{n,d}) = d \sum_{i=0}^{n-1} (d-1)^i - \sum_{m=1}^{n} \left( \frac{n+1}{m+1} \right) \cdot \delta(m, d-2)$$

**Remark 2.3.** For small $m$, the following are derived easily from the definition:

(i) $\delta(1, p) = \begin{cases} 2 & \text{if } p \equiv 0 \mod 4 \\ 0 & \text{otherwise} \end{cases}$

(ii) $\delta(2, p) = \begin{cases} 8 & \text{if } p \equiv 0 \mod 6 \\ 2 & \text{if } p \equiv 3 \mod 6 \\ 0 & \text{otherwise} \end{cases}$

(iii) $\delta(3, p) = \begin{cases} 12p - 24 & \text{if } p \equiv 0 \mod 4 \\ 0 & \text{otherwise} \end{cases}$

In particular, (iii) implies that the difference between the general bound and ED-degree can be arbitrarily large. Although, following lemma shows that the general bound is the best possible polynomial bound.

**Lemma 2.4.** If $p$ is a prime bigger than $m+1$, then $\delta(m, p) = 0$.

**Proof.** Assume $1 + \sum_{i=1}^{m} \zeta^{2i} = 0$. Replacing $2t_i$ to $2t_i - p$ if $2t_i \geq p$ for each $i$, we have a polynomial in $\zeta$ whose degree is less than $p$. Since $p$ is a prime, it should be a scalar multiple of the cyclotomic polynomial $\Phi_p(\zeta) := 1 + \zeta + \cdots + \zeta^{p-1}$. It has $p$ terms, hence $m+1 \geq p$. \qed

No closed formula for $\delta(m, p)$ is known, but it has been studied in both algebraic geometry and number theory [1, 7, 8]. In particular, Theorem 2 in [7] implies that $\delta(m, p)$ is a polynomial periodic function in $d$.

**Corollary 2.5.** For fixed $n$, the ED-degree of $F_{n,d}$ is a polynomial periodic function in $d$.

**Example 2.6.** In [2], the ML-degree of the Fermat curves ($n = 2$) is given by

$$\text{MLdeg}(F_{2,d}) = \begin{cases} d^2 + d & \text{if } d \equiv 0, 2 \mod 6 \\ d^2 + d - 3 & \text{if } d \equiv 3, 5 \mod 6 \\ d^2 + d - 2 & \text{if } d \equiv 4 \mod 6 \\ d^2 + d - 5 & \text{if } d \equiv 1 \mod 6. \end{cases}$$

By the Theorem 2.2, we have

$$\text{EDdeg}(F_{2,d}) = \begin{cases} d^2 & \text{if } d \equiv 0, 1, 3, 4, 7, 9 \mod 12 \\ d^2 - 2 & \text{if } d \equiv 5, 11 \mod 12 \\ d^2 - 6 & \text{if } d \equiv 6, 10 \mod 12 \\ d^2 - 8 & \text{if } d \equiv 8 \mod 12 \\ d^2 - 14 & \text{if } d \equiv 2 \mod 12. \end{cases}$$

It is a polynomial periodic function in $d$, and the general bound $\text{EDdeg}(F_{2,d}) \leq d^2$ is the best possible polynomial bound. Comparing with $\text{MLdeg}$, both are periodic while there periods are different.
The system for critical points of the distance function is given by

\[
\begin{align*}
\{ & x_0^d + \cdots + x_n^d = 0, \\
& x_i^{d-1}(x_j - u_j) = x_j^{d-1}(x_i - u_i) \text{ for each } i \neq j
\}
\tag{2.1}
\end{align*}
\]

where the vector \( u = (u_0, \ldots, u_n) \in \mathbb{C}^n \) is sufficiently generic. The ED-degree is the number of solutions of (2.1) except \((0, \ldots, 0)\), which is a (unique) singular point of the cone over the Fermat hypersurface \( F_{n,d} \).

Introducing a new variable \( t \), we modify the system (2.1) into following homogeneous system in \( S[t] = \mathbb{C}[x_0, \ldots, x_n, t] \).

\[
\begin{align*}
\{ & x_0^d + \cdots + x_n^d = 0, \\
& x_i^{d-1}(x_j - u_j t) = x_j^{d-1}(x_i - u_i t) \text{ for each } i \neq j
\}
\tag{2.2}
\end{align*}
\]

Each solution of (2.2) of the form \( (c_0 : \cdots : c_n : 1) \) corresponds to the solution \( (c_0, \ldots, c_n) \) of (2.1). The system (2.2) has more solutions that we don’t want to count. Let \( \text{mult}(0) \) be the multiplicity of \((0 : \cdots : 0 : 1)\) for the system (2.2), and \( \epsilon(n, d) \) be the number of solutions of the form \( (a_0 : \cdots : a_n : 0) \) counting multiplicities. Then the ED-degree of \( F_{n,d} \) is given by

\[
\text{EDdeg}(F_{n,d}) = \text{deg}(2.2) - \text{mult}(0) - \epsilon(n, d)
\tag{2.3}
\]

where \( \text{deg}(2.2) \) is the degree of the projective scheme defined by the system (2.2).

Now, Theorem 2.2 is just a consequence of following lemmas.

**Lemma 2.7.** The multiplicity of \((0 : \cdots : 0 : 1)\) for the system (2.2), denoted by \( \text{mult}(0) \), is \( d(d-1)^n \).

**Proof.** Let \( I \) be the ideal in \( S[t] \) generated by equations in (2.2) and \( m = (x_0, \ldots, x_n) \) be the ideal corresponding the point \((0 : \cdots : 0 : 1)\). Then \( \text{mult}(0) \) is defined by the length of \( S[t]_m/I_m \) as an \( S[t]_m \)-module. In the local ring \( S[t]_m \), the factor \((x_i - u_i t)\) is a unit. (By the genericity of \( u_i \), we may assume \( u_i \neq 0 \) for all \( i \).) Writing \( \mu_i = (x_0 - u_0 t) \cdot (x_i - u_i t)^{-1} \), the localized ideal \( I_m \) is generated by

\[
\begin{align*}
\{ & x_0^d + \cdots + x_n^d = 0, \\
& x_i^{d-1} = x_j^{d-1} \mu_i \text{ for each } i.
\}
\tag{2.4}
\end{align*}
\]

Here, the length of \( S[t]_m/I_m \) is just the maximum size of a monomial set in \( S \) which are independent modulo \( I_m \). By direct counting, we see that \( \text{mult}(0) = \left| \{x_0^{\alpha_0} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_0 \leq d - 1, 0 \leq \alpha_1, \ldots, \alpha_n \leq d - 2 \} \right| = d(d-1)^n \). Alternatively, it is same as \( \dim_\mathbb{C}(S/\bar{I}) \) where \( \bar{I} \) is the ideal in \( S \) defined by (2.4) after changing each \( \mu_i \) into arbitrary nonzero value in \( \mathbb{C} \). Therefore, by Bézout theorem, we get the same answer. \( \square \)

To compute \( \epsilon(n, d) \) in (2.3), we want to put \( t = 0 \) in the system (2.2) to get

\[
\begin{align*}
\{ & x_0^d + \cdots + x_n^d = 0, \\
& x_i^{d-1} x_j = x_j^{d-1} x_i \text{ for each } i \neq j
\}
\tag{2.5}
\end{align*}
\]

This could give the wrong answer if (2.2) and the hyperplane \( t = 0 \) meet non-transversally. The next lemma shows that it is not the case.

**Lemma 2.8.** The system (2.2) and the hyperplane \( t = 0 \) meet transversally. Hence \( \epsilon(n, d) = \text{deg}(2.5) \).
The degree of the system \((2.2)\) is given by
\[
\text{deg}((2.2)) = d \sum_{i=0}^{n} (d-1)^i.
\]

**Lemma 2.9.** Let \(\delta(p, m)\) be the function defined in Theorem 2.2. Then we have
\[
\epsilon(n, d) = \sum_{m=1}^{n} \binom{n+1}{m+1} \cdot \delta(m, d-2).
\]

**Proof.** By Lemma 2.8, \(\epsilon(n, d) = \text{deg}((2.5))\). Let \((c_0 : \cdots : c_n)\) be a solution of \((2.5)\). Suppose that \(c_i \neq 0\) for all \(i\). Then \(x_0 = 1\) in the system \((2.5)\) implies that all \(c_i\)’s are some \((d-2)\)-nd roots of unity. Fix a \((d-2)\)-nd primitive root of unity \(\zeta\), and write \(c_i = \zeta_i\). Then the system \((2.5)\) has \(\delta(n, d-2)\) many solutions. If a solution has \(m + 1\) many nonzero coordinates, the number of such solutions is \(\binom{n+1}{m+1}\) (for the choices of nonzero coordinates) times \(\delta(m, d-2)\). □

**Lemma 2.10.** The degree of the system \((2.2)\) is given by
\[
\text{deg}((2.2)) = d \sum_{i=0}^{n} (d-1)^i.
\]

**Proof.** Let \(I\) be the ideal generated by
\[
x_i^{d-1}(x_j - u_j t) = x_j^{d-1}(x_i - u_i t) \text{ for each } i \neq j
\]
\[
= 2 \times 2 \text{ minors of } \begin{bmatrix} x_0^{d-1} & \cdots & x_n^{d-1} \\ x_0 - u_0 & \cdots & x_n - u_n \end{bmatrix}
\]
It defines a curve in \(\mathbb{C}^{n+1}\). By Lemma 2.8, \(\text{deg}(I) = \text{deg}(I + (t))\). If \(c = (c_0 : \cdots : c_n : 0)\) is a solution for \(I + (t)\), write \(c_m = 1\) where \(m\) is the first nonzero entry of \(c\). Then for each nonzero entry except \(c_m\), there are \(d-2\) choices to be a solution. Hence the total number of solutions is
\[
\sum_{i=1}^{n+1} \binom{n+1}{i} (d-2)^{i-1} = \sum_{i=0}^{n} (d-1)^i.
\]
Therefore \(\text{deg}((2.2)) = \text{deg}(F_{n, d}) \cdot \text{deg}(I) = d \sum_{i=0}^{n} (d-1)^i\) by Bézout. □

**Proof of Theorem 2.2.** We have
\[
\text{EDdeg}(F_{n, d}) = \text{deg}((2.2)) - \text{mult}(0) - \epsilon(n, d).
\]
Apply Lemma 2.10, 2.7, and 2.9 to each term in the right side. □

**Example 2.11.** We showed that the ED-degrees of the Fermat hypersurfaces can be computed by \(\delta(m, p)\) or \(\epsilon(m, p)\) without using the random data \(u\). The following Macaulay2 code computes the ED-degree of \(F_{n, d}\) efficiently.
The output reveals that the Fermat quintic cone $F_{2,5}$ has ED-degree 23.

2.2. Affine Fermat Hypersurfaces. Let $X$ be the affine Fermat hypersurfaces $AF_{n,d}$. The system for critical points of the distance function is given by

$$
\begin{align*}
\{ & x_0^d + \cdots + x_n^d = 1 \\
& x_i^{d-1}(x_j - u_j) = x_j^{d-1}(x_i - u_i) \text{ for each } i \neq j
\end{align*}
$$

and the homogenized system is

(2.6)

$$
\begin{align*}
\{ & x_0^d + \cdots + x_n^d = t^d \\
& x_i^{d-1}(x_j - u_jt) = x_j^{d-1}(x_i - u_it) \text{ for each } i \neq j
\end{align*}
$$

In this case, $(0 : \cdots : 0 : 1)$ is not a solution for (2.6) (see Lemma 2.7). Except that, the ED-degree of $AF_{n,d}$ can be computed in the same way as in the homogeneous cases.

**Corollary 2.12.** The ED-degree of the affine Fermat hypersurface $AF_{n,d}$ is given by

$$
EDdeg(AF_{n,d}) = d \sum_{i=0}^{n-1} (d-1)^i - \sum_{m=1}^{n-1} \binom{n}{m+1} \cdot \delta(m, d-2).
$$

Note that the summand is the general bound for affine varieties, given in [4, Corollary 2.5]

2.3. Real Critical Points. For odd $d$, the Fermat hypersurface $F_{n,d}$ can be considered as a nonempty real variety. In this case, the number of the real critical points of the squared distance function highly depends on the location of the given point $u \in \mathbb{R}_{n+1}$. Nonetheless, the next theorem gives an upper bound for the maximum possible (finite) number of the real critical points

**Theorem 2.13.** For the Fermat hypersurface $F_{n,d}$, the number of the nonzero real critical points of the squared distance function is bounded by

$$
\sqrt{2}^{25n^2-3n+2} \cdot (n+2)^{5n}
$$

**Proof.** Let $u \in \mathbb{R}_{n+1}^n$ be a point not in $F_{n,d}$, whose entries are all nonzero. Then the critical equation (2.1) can be written by

$$
\begin{align*}
\{ & x_0^d + \cdots + x_n^d = 0 \\
& x_i^{d-1}(x_0 - u_0) = x_0^{d-1}(x_i - u_i) \text{ for each } i = 1, \ldots, n
\end{align*}
$$

This system has $n+1$ polynomials in $n+1$ variables, and the number of monomials used in this system is $5n+1$, which does not depend on $d$. By Khovanskii’s fewnomial bound [6], this system has at most

$$
2^{5n} \cdot (n+2)^{5n}
$$
positive solutions. It is also an upper bound for the number of real solutions in any orthant, hence we can have at most
\[ 2^{n+1} \cdot 2^{\left(\frac{5n}{2}\right)} \cdot (n+2)^{5n} \]
in total.

Note that this bound does not depend on \( d \), hence we can ask for the sharp bound for each \( n \). For \( n = 1 \), the real cone of \( F_{1,d} \) is a straight line in \( \mathbb{R}^2 \), hence the critical equation has one real solution. For \( n = 2 \), the maximum possible number seems to be 3, but we don’t have any proof for this and higher dimensional cases.

**Conjecture 2.14.** The number of real critical points of \( (2.1) \) is at most \( 2^n - 1 \).

We note that Theorem 2.13 is also valid for the scaled Fermat hypersurface \( SF^n_{a,d} \) since the critical system contains the same number of monomials for all scaling vectors \( a \).

### 3. Scaled Fermat Hypersurfaces

#### 3.1. Genericity of scaled Fermat hypersurfaces

Recall the relation (2.3)
\[ F_{n,d} = \deg((2.2)) - \text{mult}(0) - \epsilon(n,d). \]

The first two terms in this expression are invariant under any \( GL(n+1, \mathbb{C}) \) action (acting on the variables), thus we only focus on the last term \( \epsilon(n,d) \), which is a sum of \( \delta(m,d-2) \) with binomial coefficients (See Lemma 2.9).

For a given scaling vector \( a \in (\mathbb{C}^*)^{m+1} \), define \( \delta(m,p,a) \) to be the number of solutions of
\[ \begin{cases} 1 + x_1^2 + \cdots + x_m^2 = 0 \\ x_i^p = a_i/a_0 \text{ for each } i = 1, \cdots, m \end{cases} \]
whose entries are all nonzero. Note that \( \delta(m,p,1) = \delta(m,p) \) where \( 1 = (1, \cdots, 1) \).

For \( I \subseteq \{0, \ldots, n\} \), let \( a_I = (a_{i_1}, \ldots, a_{i_{|I|}}) \) where \( a_{i_j} \) is the \( j \)-th entry of \( a \). Now the ED-degree of \( SF^n_{a,d} \) is given by
\[ \text{EDdeg}(SF^n_{a,d}) = d \sum_{i=0}^{n-1} (d-1)^i - \sum_{I \subseteq \{0, \ldots, n\}} \delta(|I| - 1, d-2, a_I). \]

Therefore the ED-degree of \( SF^n_{a,d} \) achieves the equality in the general bound (Proposition 1.1) if and only if the latter summands are all zero. To examine, we need to define the exponential cyclotomic polynomial \( Q_{m,p} \in \mathbb{Z}[x_0, \ldots, x_m] \).

For an integer \( p \) and a primitive \( p \)-th root of unity \( \zeta \), consider the polynomial
\[ P_{m,p}(A_0, \ldots, A_m) = \prod_{t_1, \ldots, t_m=1}^{p} \left( A_0 + \sum_{k=1}^{m} \zeta^{t_k} A_i \right). \]

One can easily see that \( P_{m,p}(A_0, \ldots, A_m) \in \mathbb{Z}[A_0^p, \ldots, A_m^p] \). Replace \( A_i^p \) by \( x_i \) to get a polynomial \( Q_{m,p}(x_0, \ldots, x_m) \), i.e., \( Q_{m,p} \) is the unique polynomial such that
\[ Q_{m,p}(A_0^p, \ldots, A_m^p) = P_{m,p}(A_0, \ldots, A_m). \]

**Theorem 3.1.** \( \delta(m,p,a) \neq 0 \) if and only if
\[ \begin{cases} Q_{m,p}(a_0^2, \ldots, a_m^2) = 0 \quad \text{for } p \text{ odd,} \\ Q_{m,p/2}(a_0, \ldots, a_m) = 0 \quad \text{for } p \text{ even.} \end{cases} \]
Proof. Let \( a \in (\mathbb{C}^*)^{n+1} \) be given. We may assume \( a_0 = 1 \). For each \( i \), choose a complex number \( b_i \) so that \( b_i^d = a_i \). Let \( \zeta \) be a primitive \( p \)-th root of unity. By definition, \( \delta(m,p,a) \neq 0 \) if and only if the system

\[
\begin{align*}
1 + x_1^2 + \cdots + x_m^2 &= 0 \\
x_i^d &= a_i \quad \text{for each } i = 1, \ldots, m
\end{align*}
\]

has a solution whose entries are all nonzero. Any solution of the second equations is of the form \( (b_1 \zeta^{t_1}, \ldots, b_m \zeta^{t_m}) \). It satisfies the first equation if and only if \( 1 + b_1^2 \zeta^{2t_1} + \cdots + b_m^2 \zeta^{2t_m} = 0 \). The image of the square map \( z \mapsto z^2 \) defined on the set of all \( p \)-th roots of unity is itself if \( p \) is odd, or is the set of all \((p/2)\)-nd roots of unity if \( p \) is even. Therefore \( \delta(m,p,a) \neq 0 \) if and only if

\[
\prod_{t_1,\ldots,t_m=1}^p \big(1 + b_1^2 \zeta^{t_1} + \cdots + b_m^2 \zeta^{t_m}\big) = 0
\]

for \( d \) odd,

\[
\prod_{t_1,\ldots,t_m=1}^{p/2} \big(1 + b_1^2 (\zeta^2)^{t_1} + \cdots + b_m^2 (\zeta^2)^{t_m}\big) = 0
\]

for even \( d \). Now the theorem follows by replacing \( b_i^d \) with \( a_i \) after expanding the product. \( \square \)

**Corollary 3.2.** For generic \( a \in (\mathbb{C}^*)^{n+1} \),

\[
\text{EDdeg}(SF_n^a) = d \sum_{i=0}^{n-1} (d-1)^i
\]

The exponential cyclotomic polynomial \( Q_{m,p} \) would be interesting itself. We close this section with a theorem showing that \( Q_{m,p} \) has a nice property as an algebraic object.

**Theorem 3.3.** For any integer \( m \) and \( p \), the exponential cyclotomic polynomial \( Q_{m,p} \) is irreducible over \( \mathbb{C} \).

Proof. Let \( f \in \mathbb{C}[x_0, \ldots, x_m] \) be an irreducible factor of \( Q_{m,p} \). Then \( f(A_0^p, \ldots, A_m^p) \) is a factor of \( P_{m,p}(A_0, \ldots, A_m) \), and we may assume that \( f(A_0^p, \ldots, A_m^p) \) is divisible by \( (A_0 + \cdots + A_m) \). For any \( (t_1, \ldots, t_m), t_i = 1, \ldots, p \), the polynomial \( f(A_0^p, \ldots, A_m^p) \) is stable under the action \( A_i \mapsto \zeta^{t_i} A_i \). Therefore \( f(A_0^p, \ldots, A_m^p) \) is divisible by \( (A_0 + \sum_i \zeta^{t_i} A_i) \). Since \( (t_1, \ldots, t_m) \) was arbitrary, \( f(A_0^p, \ldots, A_m^p) \) is divisible by every possible linear factor of \( P_{m,p}(A_0, \ldots, A_m) \). Therefore \( f(A_0^p, \ldots, A_m^p) = P_{m,p}(A_0, \ldots, A_m) \) and hence \( Q_{m,p} = f(x_0, \ldots, x_m) \) up to scalar multiplication. \( \square \)

**References**

[1] S. Adams and P. Sarnak: *Betti Numbers of Congruence Groups*, Israel J. of Math. **88** (1994), 31-72.
[2] D. Agostini, D. Alberelli, F. Grande, and P. Lella: *The maximum likelihood degree of fermat hypersurfaces*, arXiv:1404.5745.
[3] D. Cox, J. Little, and D. O'Shea: *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992.
[4] J. Draisma, E. Horobet, G. Ottaviani, B. Sturmfels, and R. Thomas: *The Euclidean Distance Degree of an Algebraic Variety*, arXiv:1309.0049.
[5] J. Huh, B. Sturmfels: *Likelihood Geometry*, in Combinatorial Algebraic Geometry (eds. A. Conca et al.), Lecture Note in Mathematics 2014, Springer (2014) 63-117
[6] A.G. Khovanskii: *A class of systems of transcendental equations*, Dokl. Akad. Nauk. SSSR 255 (1980), no. 4, 804-807.
[7] M. Laurent: *Equations diophantiennes exponentielles*, Invent. Math. 78 (1984), 299-327.
[8] H. Mann, *On linear relations between roots of unity*, Mathematika. 12 (1965), 107-117.
[9] G. Ottaviani, P.-J. Spaenlehauer, and B. Sturmfels: *Exact Solutions in Structured Low-Rank Approximation*, arXiv:1311.2376