Linear canonical wavelet transform and the associated uncertainty principles

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Abstract
We define a novel time-frequency analyzing tool, namely linear canonical wavelet transform (LCWT) and study some of its important properties like inner product relation, reconstruction formula and also characterize its range. We obtain Donoho-Stark’s and Lieb’s uncertainty principle for the LCWT and give a lower bound for the measure of its essential support. We also give Shapiro’s mean dispersion theorem for the proposed LCWT.

Keywords: Linear canonical transform; Linear canonical wavelet transform; Uncertainty principle; Shapiro’s theorem

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1 Introduction
We first mention below some important abbreviations that will be used throughout this paper.

List of Abbreviations
FT - Fourier transform
FrFT - Fractional Fourier transform
LCT - Linear canonical transform
WT - Wavelet transform
FrWT - Fractional wavelet transform
WLCT - Windowed linear canonical transform
LCWT - Linear canonical wavelet transform
ONS - Orthonormal sequence
RKHS - Reproducing kernel Hilbert Space
UP - Uncertainty principle

As a generalization of FT [1] and FrFT [2, 3, 4], LCT is a four parameter family of linear integral transform proposed by Mohiny and Quesne [5] and is considered as the important tool for non-stationary signal processing. Because of the extra degrees of freedom, as compared to the FT and FrFT, its application can be found in number of fields including signal separation [6], signal reconstruction [7], filter designing [8] and many more. Recently in [9] the authors studied octonion linear canonical transform. For more detail on LCT and its application we refer the reader to [10].

Even though the wavelet transform (WT) [11] is a potential tool for the analysis of non-stationary signals, it is incompetent for analyzing the signals with not well concentrated energy in the time-frequency plane, for example, the chirp like signal which is ubiquitous in nature [12]. On the other hand, for the signal whose energy in the frequency domain is not well concentrated, LCT is an appropriate tool. However, because of its global kernel it is not capable of indicating the time localization of the LCT spectral components, and thus LCT is not suitable for processing the

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non-stationary signal whose LCT spectral characteristics changes with time. The WLCT [13] is thus proposed to overcome this drawback. In this case, the original signal is first segmented with time localization window, followed by performing the LCT spectral analysis for these segment. WLCT is capable of offering a joint signal representation in both time and LCT domain, but its fixed window width limits the practical application, it is impossible to provide good time resolution and spectral resolution simultaneously.

Thus to circumvent these limitations of LCT, WT and WLCT we propose a novel LCWT. In fact, Wei et al. [14] and Guo et al. [15] generalized the FrWT, studied in [16], to the LCWT. Wei et al. [14] studied its resolution in time and linear canonical domains and Guo et al. [15] studied its properties on Sobolev spaces. Dai et al. [12] gave a new definition of the FrWT(also see [17]), which we generalize in the context of the LCT and study the associated UP.

Definition 2.1. The LCT of $f \in L^2(\mathbb{R})$, with respect to a matrix parameter

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad AD - BC = 1$$

is defined as

$$(\mathcal{L}^M f)(\xi) = \left\{ \begin{array}{ll} \int_{\mathbb{R}} f(t)K_M(t, \xi)dt, B \neq 0, \\
\sqrt{D}e^{CD}f(D\xi), & B = 0, \end{array} \right.$$
where $K_M(t, \xi)$ is a kernel given by

$$K_M(t, \xi) = \frac{1}{\sqrt{2\pi i b}} \hat{f} \left( \frac{\xi^2 - \xi \xi t + \xi^2}{2} \right), \xi \in \mathbb{R}. \tag{1}$$

Among several important properties of the LCT, the important among them, that will be used in the sequel, is the Parseval’s formula

$$\int_{\mathbb{R}} f(t)\overline{g(t)}dt = \int_{\mathbb{R}} (\mathcal{L}^M f)(\xi)(\mathcal{L}^M g)(\xi)d\xi, \text{ where } f, g \in L^2(\mathbb{R}). \tag{2}$$

Particularly, if $f = g$, then we have the Plancherel’s formula

$$\|f\|_{L^2(\mathbb{R})} = \|\mathcal{L}^M f\|_{L^2(\mathbb{R})}. \tag{3}$$

The LCTs satisfies the additive property, i.e.,

$$\mathcal{L}^M \mathcal{L}^N f = \mathcal{L}^{MN} f, \text{ where } f \in L^2(\mathbb{R}), \tag{4}$$

and the inversion property

$$\mathcal{L}^{M^{-1}}(\mathcal{L}^M f) = f, \tag{5}$$

where, $M^{-1}$ denotes the inverse of $M$. For convenient, we now denote the matrix $M$ by $(A, B; C, D)$.

### 3 LCWT

We propose a new integral transform namely the LCWT. This definition is mainly motivated from the definition of FrWT defined by Dai et al. [12]. We shall discuss some of its basic properties along with the inner product relation, reconstruction formula and also prove that its range is a RKHS.

Motivated by the definition of the admissible wavelet pair in [32], we first define it in the setting of LCT domain.

**Definition 3.1.** A pair $\{\psi, \phi\}$ of functions in $L^2(\mathbb{R})$ is said to be an admissible linear canonical wavelet pair (ALCWP) if they satisfy the following admissibility condition

$$C_{\psi, \phi, M} := \int_{\mathbb{R}^+} \left( \mathcal{L}^M \psi \left( \frac{\xi}{a} \right) \right) \left( \mathcal{L}^M \phi \left( \frac{\xi}{a} \right) \right) \frac{da}{a} \tag{6}$$

is a non-zero complex constant independent of $\xi \in \mathbb{R}$ satisfying $|\xi| = 1$. In case $\psi = \phi$, we denote $C_{\psi, \psi, M}$ by $C_{\psi, M}$ and the required admissibility condition reduces to

$$C_{\psi, M} := \int_{\mathbb{R}^+} \left( \mathcal{L}^M \psi \left( \frac{\xi}{a} \right) \right)^2 \frac{da}{a} \tag{7}$$

is a positive constant independent of $\xi$ satisfying $|\xi| = 1$. We call $\psi \in L^2(\mathbb{R})$, satisfying equation (7), the admissible linear canonical wavelet (ALCW).

We now give the definition of the novel LCWT.

**Definition 3.2.** Let $f \in L^2(\mathbb{R})$, $M = (A, B; C, D)$ be a matrix with $AD - BC = 1$ and $B \neq 0$ then we define the LCWT of $f$ with respect to $M$ and an ALCW $\psi$ by

$$(W_\psi^M f)(a, b) = e^{-\frac{4\pi b^2}{\rho}} \left\{ f(t)e^{\frac{\xi^2}{4\rho}t^2} \ast \sqrt{a} \psi(-at)e^{\frac{\xi^2}{4\rho}(-at)^2} \right\} (b), \quad a \in \mathbb{R}^+, b \in \mathbb{R},$$

where $\ast$ denotes the convolution given by

$$(f \ast g)(\nu) = \int_{\mathbb{R}} f(x)g(\nu - x)dx, \quad \nu \in \mathbb{R}.$$
Proposition 3.1. Let \( \psi_{a,b}(t) = \sqrt{a} \psi(a(t - b)) \)

\[
\psi_{a,b}^M(t) = e^{-\frac{i}{\sqrt{a}} (t^2 - b^2) - \frac{i}{a}(a(t-b))^2} \psi_{a,b}(t).
\]  

(8)

Thus, we have an equivalent definition of the LCWT as

\[
(W_M f)(a, b) = \langle f, \psi_{a,b}^M \rangle_{L^2(\mathbb{R})}.
\]

(9)

It is to be noted that depending on the different choice of the matrix \( M \), we have different integral transforms:

- For \( M = (\cos \alpha, \sin \alpha; -\sin \alpha, \cos \alpha) \), \( \alpha \neq n\pi \), we obtain the FrWT as discussed in [12].
- For \( M = (0, 1; -1, 0) \) we obtain the traditional WT [32].

We now establish a fundamental relation between LCWT and the LCT. This relation will be useful in obtaining the resolution of time and linear canonical spectrum in the time-LCT-frequency plane and inner product relation associated with the LCWT.

**Proposition 3.1.** If \( W_M^\psi f \) and \( L^M f \) are respectively the LCWT and the LCT of \( f \in L^2(\mathbb{R}) \). Then,

\[
L^M \left( (W_M^\psi f)(a, \cdot) \right)(\xi) = \sqrt{-2\pi i B \sqrt{a}} e^{\frac{i}{B}(\xi \Delta)^2} (L^M f)(\xi) (L^M \psi) \left( \frac{\xi}{a} \right).
\]

(10)

**Proof.** Form the definition of the LCT and \( \psi_{a,b}^M \), it follows that

\[
(L^M \psi_{a,b}^M)(\xi) = \int_{\mathbb{R}} \sqrt{a} \psi(a(t - b)) \sqrt{\frac{1}{2\pi i B}} e^{\frac{i}{B} \left( \frac{a^2}{2} \xi^2 - \frac{1}{a} \xi^2 (a(t-b))^2 \right)} dt
\]

\[
= \int_{\mathbb{R}} \sqrt{a} \psi(at) \sqrt{\frac{1}{2\pi i B}} e^{\frac{i}{B} \left( \frac{a^2}{2} \xi^2 - \frac{1}{a} \xi^2 (at)^2 \right)} dt
\]

\[
= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \sqrt{a} \psi(t) \sqrt{\frac{1}{2\pi i B}} e^{\frac{i}{B} \left( \frac{a^2}{2} \xi^2 - \frac{1}{a} \xi^2 (t)^2 \right)} e^{\frac{i}{2\pi B} \frac{a^2}{2} (\xi)^2} dt
\]

\[
= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} e^{-\frac{i}{2\pi B} (\xi)^2} \sqrt{2\pi i B} \int_{\mathbb{R}} \psi(t) K_M(b, \xi) K_M \left( \frac{i}{a}, \frac{\xi}{a} \right) dt.
\]

Therefore, we have

\[
(L^M \psi_{a,b}^M)(\xi) = \sqrt{-2\pi i B} \sqrt{a} K_M(b, \xi) (L^M \psi) \left( \frac{\xi}{a} \right).
\]

(11)

Using (2) in (11), we get

\[
(W_M^\psi f)(a, b) = \langle L^M f, L^M \psi_{a,b}^M \rangle_{L^2(\mathbb{R})}.
\]

Using equation (11), we have

\[
(W_M^\psi f)(a, b) = \sqrt{-2\pi i B \sqrt{a}} \int_{\mathbb{R}} e^{-\frac{i}{2\pi B} (\xi)^2} (L^M f)(\xi) (L^M \psi) \left( \frac{\xi}{a} \right) K_M^{-1}(b, \xi) d\xi.
\]

(12)

Therefore, it follows that

\[
L^M \left( (W_M^\psi f)(a, \cdot) \right)(\xi) = \sqrt{-2\pi i B \sqrt{a}} e^{\frac{i}{B}(\xi \Delta)^2} (L^M f)(\xi) (L^M \psi) \left( \frac{\xi}{a} \right).
\]

This completes the proof.

\[ \square \]

**3.1 Time-LCT frequency analysis**

From equation (9) it follows that if \( \psi_{a,b}^M \) is supported in the time domain, then so is \( (W_M^\psi f)(a, b) \). Also, from equation (12), it follows that the LCWT can provide the local property of \( f(t) \) in the linear canonical domain. Thus the LCWT is capable of producing simultaneously the time-LCT frequency information and represent the signal in the time-LCT frequency domain. More precisely, if \( \psi \) and \( L^M \psi \) are window functions in time and linear canonical domain respectively with \( E_{\psi} \) and \( E_{L^M \psi} \) as centres and \( \Delta_{\psi} \) and \( \Delta_{L^M \psi} \) are radii respectively. Then the centre and radius of \( \psi_{a,b}^M \) are given respectively by

\[
E_{[\psi_{a,b}^M]} = \frac{1}{a} E_{\psi} + b,
\]

\[
\Delta_{[\psi_{a,b}^M]} = \frac{1}{a} \Delta_{\psi}.
\]
and
\[ \Delta[\psi_{a,b}^M] = \frac{1}{a} \Delta_\psi. \]
Similarly, the centre and radius of window function \((L^M \psi) \left( \frac{\xi}{a} \right)\) are given by
\[ E \left[ (L^M \psi) \left( \frac{\xi}{a} \right) \right] = aE_{L^M \psi}, \]
and
\[ \Delta \left[ (L^M \psi) \left( \frac{\xi}{a} \right) \right] = a\Delta_{L^M \psi}. \]
Thus, the \(Q\)-factor of the window function of the linear canonical transform domain is
\[ Q = \frac{\Delta_{L^M \psi}}{E_{L^M \psi}}, \]
which is independent of the scaling parameter \(a\) for a given parameter \(M\). This is called the constant \(Q\)-property of the LCWT.

### 3.2 Time-LCT frequency resolution

The LCWT \((L^M \psi)(a,b)\) localizes the signal \(f\) in the time window
\[ \left[ \frac{1}{a} E_\psi + b - \frac{1}{a} \Delta_\psi, \frac{1}{a} E_\psi + b + \frac{1}{a} \Delta_\psi \right]. \]
Similarly, we get that the LCWT gives linear canonical spectrum content of \(f\) in the window
\[ \left[ aE_{L^M \psi} - a\Delta_{L^M \psi}, aE_{L^M \psi} + a\Delta_{L^M \psi} \right]. \]
Thus, the joint resolution of the LCWT in the time and linear canonical domain is given by the window
\[ \left[ \frac{1}{a} E_\psi + b - \frac{1}{a} \Delta_\psi, \frac{1}{a} E_\psi + b + \frac{1}{a} \Delta_\psi \right] \times \left[ aE_{L^M \psi} - a\Delta_{L^M \psi}, aE_{L^M \psi} + a\Delta_{L^M \psi} \right], \]
with constant area \(4\Delta_\psi \Delta_{L^M \psi}\) in the time-LCT-frequency plane. Thus it follows that for a given parameter \(M\), the window area depends on the linear canonical admissible wavelets and is independent of the parameters \(a\) and \(b\). But it is to be noted that the the window gets narrower for large value of \(a\) and wider for small value of \(a\). Thus the window given by the transform is flexible and hence, it is capable of simultaneously providing the time linear canonical domain information. This flexibility of the window makes the proposed LCWT more advantageous then the WLCT as in this case the window is rigid.

Some basic properties of LCWT is given below.

**Theorem 3.1.** Let \(g, h \in L^2(\mathbb{R})\), \(\psi \) and \(\phi \) are ALCWs, \(\alpha, \beta \in \mathbb{C}, \lambda > 0\) and \(y \in \mathbb{R}\). Then

1. \(W^M_\psi(\alpha g + \beta h) = \alpha(W^M_\psi g) + \beta(W^M_\psi h)\)
2. \(W^M_{\alpha \psi + \beta \phi}(g) = \bar{\alpha}(W^M_\psi g) + \bar{\beta}(W^M_\psi g)\)
3. \((W^M_\psi \delta_\lambda g)(a, b) = (W^M_\psi g)(\frac{a}{\lambda}, b\lambda), \text{ where } (\delta_\lambda g)(t) = \sqrt{\lambda} g(\lambda t) \text{ and } \bar{M} = (A, \lambda^2 B; \frac{C}{\lambda^2}, D)\)
4. \((W^M_\psi \tau_y g)(a, b) = e^{\frac{\lambda}{\lambda} y(a-b)}(W^M_\psi e^{\frac{\lambda}{\lambda} y} g)(a, b-y), \text{ where } (\tau_y g)(t) = g(t+y)\).

**Proof.** The proofs are immediate and can be omitted. \(\square\)

If \(\{\psi, \phi\}\) is admissible linear canonical wavelet pair such that each \(\psi\) and \(\phi\) are ALCWs and \(f, g \in L^2(\mathbb{R})\) and is such that they are orthogonal then \(W^M_\psi(f)\) and \(W^M_\phi(g)\) are orthogonal in \(L^2(\mathbb{R}^+ \times \mathbb{R}, dadb)\). This result is justified by the following theorem, which further gives the resolution of identity for the LCWT.

**Theorem 3.2. (Inner product relation for LCWT).** Let \(\{\psi, \phi\}\) be an ALCWP such that \(\psi\) and \(\phi\) are ALCWs and \(f, g \in L^2(\mathbb{R})\), then
\[ \langle W^M_\psi f, W^M_\phi g \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R})} = 2\pi |B| C_{\psi, \phi, M}(f, g)_{L^2(\mathbb{R})}, \] (13)
where \(C_{\psi, \phi, M}\) is provided in (11).
Proof. Using equation (10), we get
\[
\langle W^M_\psi f, W^M_\phi g \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R})} = \int_{\mathbb{R}^+ \times \mathbb{R}} (L^M(L^M_\psi f)(a,b))(L^M(L^M_\phi g)(a,b)) \, da \, db
\]
\[
= \int_{\mathbb{R}^+ \times \mathbb{R}} (L^M(L^M_\psi f)) \left( \frac{\xi}{a} \right) \left( L^M(L^M_\phi g) \right) \left( \frac{\xi}{a} \right) \, d\xi \, da
\]
\[
= 2\pi |B| \int_{\mathbb{R}} (L^M(L^M_\psi f)) \left( \frac{\xi}{a} \right) \left( L^M(L^M_\phi g) \right) \left( \frac{\xi}{a} \right) \, d\xi \, da
\]
\[
= 2\pi |B| \left| \psi,\phi,\cdot \right|_{L^2(\mathbb{R})}
\]
\[
= 2\pi |B| \left( \left| \psi,\phi,\cdot \right|_{L^2(\mathbb{R})} \right)^2 = 2\pi |B| \left| \psi,\phi,\cdot \right|_{L^2(\mathbb{R})}.
\]
Replacing \( \psi = \phi \) in equation (13), we have
\[
\langle W^M_\psi f, W^M_\phi g \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R})} = 2\pi |B| \left| \psi,\phi,\cdot \right|_{L^2(\mathbb{R})},
\]
where \( C_{\psi,M} \) is provided in (7).

Remark 3.2. (Plancherel’s theorem for \( W^M_\psi \)) Replacing \( f = g \) and \( \phi = \psi \) in equation (13) we have the Plancherel’s theorem for \( W^M_\psi \) given by
\[
\| W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})} = (2\pi |B| C_{\psi,M})^{\frac{1}{2}} \| f \|_{L^2(\mathbb{R})}.
\]
Thus, from equation (15), it follows that LCWT from \( L^2(\mathbb{R}) \) into \( L^2(\mathbb{R}^+ \times \mathbb{R}) \) is a continuous linear operator. If further ALCW \( \psi \) is such that \( C_{\psi,M} = \frac{1}{2\pi |B|} \), then the operator is an isometry.

Theorem 3.3. (Reconstruction formula). Let \( \{ \psi, \phi \} \) be an ALCWP such that \( \psi \) and \( \phi \) are ALCWs and \( f \in L^2(\mathbb{R}) \), then \( f \) can be given by the formula
\[
f(t) = \frac{1}{2\pi |B| C_{\psi,\phi,M}} \int_{\mathbb{R}^+ \times \mathbb{R}} (W^M_\psi f)(a,b)\phi^M_{a,b}(t) \, da \, db \quad \text{a.e. } t \in \mathbb{R}.
\]
Proof. From equation (13), we get
\[
2\pi |B| C_{\psi,\phi,M} \langle f, g \rangle_{L^2(\mathbb{R})} = \langle W^M_\psi f, W^M_\phi g \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R})}
\]
\[
= \int_{\mathbb{R}^+ \times \mathbb{R}} (W^M_\psi f)(a,b) \left( \int_{\mathbb{R}} g(t) \phi^M_{a,b}(t) \, dt \right) \, da \, db
\]
\[
= \int_{\mathbb{R}^+ \times \mathbb{R}} \left( W^M_\psi f \right)(a,b) \phi^M_{a,b}(t) \, da \, db \langle f, g \rangle_{L^2(\mathbb{R})}.
\]
Since \( g \in L^2(\mathbb{R}) \) is arbitrary, we have
\[
f(t) = \frac{1}{2\pi |B| C_{\psi,\phi,M}} \int_{\mathbb{R}^+ \times \mathbb{R}} (W^M_\psi f)(a,b)\phi^M_{a,b}(t) \, da \, db \quad \text{a.e.}
\]
The proof is complete.

In particular, if \( \psi = \phi \) then we have the following reconstruction formula
\[
f(t) = \frac{1}{2\pi |B| C_{\psi,\phi,M}} \int_{\mathbb{R}^+ \times \mathbb{R}} (W^M_\psi f)(a,b)\psi^M_{a,b}(t) \, da \, db \quad \text{a.e. } t \in \mathbb{R}
\]
The following theorem characterizes the range of the LCWT and proves that the range is a RKHS. It also gives the explicit expression for the reproducing kernel.
Theorem 3.4. For $\psi$ being ALCW, $W^M_\psi (L^2(\mathbb{R}))$ is a RKHS with the kernel
\[
K^M_\psi (x,y;a,b) = \frac{1}{2\pi |B| C_{\psi,M}} \langle \psi^M_{a,b}, \psi^M_{x,y} \rangle_{L^2(\mathbb{R})}, \ (x,y),(a,b) \in \mathbb{R}^+ \times \mathbb{R}.
\]
Moreover, the kernel is such that $|K^M_\psi (x,y;a,b)| \leq \frac{1}{2\pi |B| C_{\psi,M}} \|\psi\|^2_{L^2(\mathbb{R})}$.

Proof. For $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$, we see that
\[
K^M_\psi (x,y;a,b) = \frac{1}{2\pi |B| C_{\psi,M}} (W^M_\psi \psi^M_{a,b})(x,y) \text{ for all } (x,y) \in \mathbb{R}^+ \times \mathbb{R}.
\]
Now,
\[
\|K^M_\psi (\cdot, \cdot; a,b)\|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2 = \frac{1}{2\pi |B| C_{\psi,M}} \|W^M_\psi \psi^M_{a,b}\|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2
\]
\[
= \frac{1}{2\pi |B| C_{\psi,M}} \|\psi\|^2_{L^2(\mathbb{R})}.
\]
Therefore, for $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$, $K^M_\psi (x,y;a,b) \in L^2(\mathbb{R}^+ \times \mathbb{R})$. Now, let $f \in L^2(\mathbb{R})$
\[
(W^M_\psi f)(a,b) = \langle f, \psi^M_{a,b} \rangle_{L^2(\mathbb{R})}
\]
\[
= \frac{1}{2\pi |B| C_{\psi,M}} (W^M_\psi f, 2\pi |B| C_{\psi,M} K^M_\psi (\cdot, \cdot; a,b))_{L^2(\mathbb{R}^+ \times \mathbb{R})}
\]
\[
= \langle W^M_\psi f, K^M_\psi (\cdot, \cdot; a,b) \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R})}.
\]
Thus, it follows that
\[
K^M_\psi (x,y;a,b) = \frac{1}{2\pi |B| C_{\psi,M}} \langle \psi^M_{a,b}, \psi^M_{x,y} \rangle_{L^2(\mathbb{R})},
\]
is the reproducing kernel of $W^M_\psi (L^2(\mathbb{R}))$.
Again,
\[
|K^M_\psi (x,y;a,b)| = \frac{1}{2\pi |B| C_{\psi,M}} |\langle \psi^M_{a,b}, \psi^M_{x,y} \rangle_{L^2(\mathbb{R})}| \leq \frac{1}{2\pi |B| C_{\psi,M}} \|\psi^M_{a,b}\|_{L^2(\mathbb{R})} \|\psi^M_{x,y}\|_{L^2(\mathbb{R})}
\]
\[
= \frac{\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi |B| C_{\psi,M}}.
\]
This completes the proof. \qed

4 Uncertainty principle

We prove some UPs that limits the concentration of the LCWT in some subset in $\mathbb{R}^+ \times \mathbb{R}$ of small measure. For related results in case of Fourier transform and windowed Fourier transform we refer the reader to \[33,34]. Kou et al. \[35\] studied the same for the WLCT.

Definition 4.1. Let $0 \leq \epsilon < 1$, $f \in L^2(\mathbb{R})$ and $E \subset \mathbb{R}$ be measurable, then $f$ is $\epsilon-$concentrated on $E$ if
\[
\left( \int_{E^c} |f(x)|^2 dx \right)^{\frac{1}{2}} \leq \epsilon \|f\|_{L^2(\mathbb{R})}.
\]
If $0 \leq \epsilon \leq \frac{1}{2}$, then we say that most of the energy of $f$ is concentrated on $E$ and $E$ is called the essential support of $f$. If $\epsilon = 0$, then support of $f$ is contained in $E$.

Lemma 4.1. If $\psi$ is an ALCW and $f \in L^2(\mathbb{R})$. Then $W^M_\psi f \in L^p(\mathbb{R}^+ \times \mathbb{R})$, for all $p \in [2, \infty)$. Moreover,
\[
\|W^M_\psi f\|_{L^p(\mathbb{R}^+ \times \mathbb{R})} \leq (2\pi |B|)^{\frac{1}{2}} C_{\psi,M} \|f\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}^{1-\frac{1}{p}}, \ p \in [2, \infty) \quad (17)
\]
\[
\|W^M_\psi f\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \leq \|\psi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})} \quad (18)
\]
Proof. Since $\psi$ is an ALCW, it follows that $W^M_\psi f \in L^2(\mathbb{R}^+ \times \mathbb{R})$. Again
\[
\left| \langle W^M_\psi f, (a, b) \rangle \right| \leq \|\psi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.
\]
Thus, $W^M_\psi f \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$. Also, since $W^M_\psi f \in L^2(\mathbb{R}^+ \times \mathbb{R})$, we have $W^M_\psi f \in L^p(\mathbb{R}^+ \times \mathbb{R})$, $p \in [2, \infty)$. Moreover,
\[
\|W^M_\psi f\|_{L^p(\mathbb{R}^+ \times \mathbb{R})} \leq \|W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R})} \|W^M_\psi f\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})}^{-\frac{1}{2}}
\]
\[
\leq (2\pi |B|C_{\psi,M})\frac{1}{2} \|\psi\|_{L^2(\mathbb{R})}^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R})}^{-\frac{1}{2}} \|\psi\|_{L^2(\mathbb{R})}^{-\frac{1}{2}}.
\]
This proves the lemma.

Definition 4.2. Let $0 \leq \epsilon < 1$, $F \in L^2(\mathbb{R}^+ \times \mathbb{R})$ and $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$ be measurable, then $F$ is $\epsilon$–concentrated on $\Omega$ if
\[
\left( \int_{\Omega} |F(x,y)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq \epsilon \|F\|_{L^2(\mathbb{R}^+ \times \mathbb{R})}.
\]
If $0 \leq \epsilon \leq \frac{1}{2}$, then we say that most of the energy of $F$ is concentrated on $\Omega$ and $\Omega$ is called the essential support of $F$. If $\epsilon = 0$, then support of $F$ is contained in $\Omega$.

We now prove the Donoho-Stark’s UP for the propose LCWT.

Theorem 4.1. Let $0 \leq \epsilon < 1$, $\psi$ is an ALCW and a non-zero $f \in L^2(\mathbb{R})$. Also let $W^M_\psi f$ is $\epsilon$–concentrated on $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$ then
\[
|\Omega| \|\psi\|_{L^2(\mathbb{R})}^2 \geq 2\pi |B|C_{\psi,M}(1-\epsilon^2),
\]
where $|\Omega|$ denoted the measure of $\Omega$.

Proof. In equation (15), we have
\[
\|W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2 = 2\pi |B|C_{\psi,M} \|f\|_{L^2(\mathbb{R})}^2.
\]
Now,
\[
\int_{\mathbb{R}^+ \times \mathbb{R}} |(W^M_\psi f(a,b))|^2 \, da \, db \leq \int_{\mathbb{R}^+ \times \mathbb{R}} |(W^M_\psi f)(a,b)|^2 \, da \, db + \epsilon^2 \|W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R})}.
\]
This gives
\[
(1-\epsilon^2)\|W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R})} \leq |\Omega| \|W^M_\psi f\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})}.
\]
Thus, using (15), we get
\[
2\pi |B|C_{\psi,M} \|f\|_{L^2(\mathbb{R})}^2 \leq |\Omega| \|f\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}.
\]
The result follows, since $f \neq 0$.

Corollary 4.1. If $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, in $L^2(\mathbb{R})$–norm, is $\epsilon_E$–concentrated on $E \subset \mathbb{R}$ and $W^M_\psi f$ is $\epsilon_\Omega$–concentrated on $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$, then
\[
|\Omega|m(E) \|\psi\|_{L^2(\mathbb{R})}^2 \|f\|_{L^2(\mathbb{R})}^4 \geq 2\pi |B|C_{\psi,M}(1-\epsilon_\Omega^2)(1-\epsilon_E^2)^2 \|f\|_{L^2(\mathbb{R})}^4,
\]
where, $m(E)$ denotes the measure of $E$.

Proof. Since, $W^M_\psi f$ is $\epsilon_\Omega$–concentrated on $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$ in $L^2(\mathbb{R}^+ \times \mathbb{R})$–norm, so we have $|\Omega| \|\psi\|_{L^2(\mathbb{R})}^2 \geq 2\pi |B|C_{\psi,M}(1-\epsilon_\Omega^2)$. Again, since $f$ is $\epsilon_E$–concentrated, we have
\[
\left( \int_{E^c} |f(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \epsilon_E \|f\|_{L^2(\mathbb{R})},
\]
which further implies that
\[
\|f\|_{L^2(\mathbb{R})}(1-\epsilon^2) \leq \int_{E^c} \chi_E(x)|f(x)|^2 \, dx.
\]
We have by Holder’s inequality
\[
\int_{\mathbb{R}} \chi_E(x)|f(x)|^2 \, dx \leq \left( \int_{\mathbb{R}} |\chi_E(x)|^2 \, dx \right)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R})},
\]
Thus
\[(1 - \epsilon_E^2)\|f\|_{L^2(\mathbb{R})}^2 \leq (m(E))^2\|f\|_{L^4(\mathbb{R})}^4.\]  
Therefore,
\[|\Omega|m(E)\|\psi\|_{L^2(\Omega)}^2\|f\|_{L^4(\Omega)}^4 \geq 2\pi|B|C_{\psi,M}(1 - \epsilon^2)(1 - \epsilon_E^2)^2\|f\|_{L^2(\mathbb{R})}^4.\]  
The proof is complete.

Corollary 4.2. If \(f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\), in \(L^2(\mathbb{R})\)-norm, is \(\epsilon_E\)-concentrated on \(E \subset \mathbb{R}\) and \(W^M_\psi f\) is \(\epsilon_\Omega\)-concentrated on \(\Omega \subset \mathbb{R}^+ \times \mathbb{R}\), then
\[|\Omega|m(E)\|\psi\|_{L^2(\Omega)}^2\|f\|_{L^\infty(\mathbb{R})}^2 \geq 2\pi|B|C_{\psi,M}(1 - \epsilon^2)(1 - \epsilon_E^2)^2\|f\|_{L^2(\mathbb{R})}^2.\]  
Proof. Since \(W^M_\psi f\) is \(\epsilon_\Omega\)-concentrated on \(\Omega \subset \mathbb{R}^+ \times \mathbb{R}\) in \(L^2(\mathbb{R}^+ \times \mathbb{R})\)-norm, so we have \(|\Omega|\|\psi\|_{L^2(\mathbb{R})}^2 \geq 2\pi|B|C_{\psi,M}(1 - \epsilon^2)(1 - \epsilon_E^2)^2\). Again, since \(f\) is \(\epsilon_E\)-concentrated, we have
\[\left(\int_{E^c} |f(x)|^2 dx\right) \leq \epsilon_E\|f\|_{L^2(\mathbb{R})},\]  
which further implies that
\[\|f\|_{L^2(\mathbb{R})}(1 - \epsilon_E^2) \leq \left(\int_{E^c} |f(x)|^2 \chi_E(x) dx.\right.\]  
Since \(f \in L^\infty(\mathbb{R})\), so
\[\int_{E^c} \chi_E(x)|f(x)|^2 dx \leq m(E)\|f\|_{L^\infty(\mathbb{R})}^2.\]  
Thus
\[\|f\|_{L^\infty(\mathbb{R})}m(E) \geq (1 - \epsilon_E^2)\|f\|_{L^2(\mathbb{R})}^2.\]  
Therefore,
\[|\Omega|m(E)\|\psi\|_{L^2(\Omega)}^2\|f\|_{L^\infty(\mathbb{R})}^2 \geq 2\pi|B|C_{\psi,M}(1 - \epsilon^2)(1 - \epsilon_E^2)^2\|f\|_{L^2(\mathbb{R})}^2.\]  
The proof is complete.

Theorem 4.2. (Lieb’s uncertainty principle). Let \(0 \leq \epsilon < 1\), \(\psi\) is an ALCW and a non-zero \(f \in L^2(\mathbb{R})\). Also let \(W^M_\psi f\) is \(\epsilon\)-concentrated on \(\Omega \subset \mathbb{R}^+ \times \mathbb{R}\), then
\[|\Omega|\|\psi\|_{L^2(\mathbb{R})}^2 \geq 2\pi|B|C_{\psi,M}(1 - \epsilon^2)^{\frac{p}{2-p}}\rho,\quad p > 2.\]  
Proof. Since \(W^M_\psi f\) is \(\epsilon\)-concentrated on \(\Omega\), we have
\[\|\chi_\Omega W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R}, dadb)} \leq \epsilon^2\|W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R}, dadb)}\]  
Now,
\[\|\chi_\Omega W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R}, dadb)} \leq \|\chi_\Omega W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R}, dadb)} + \epsilon^2\|W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R}, dadb)}\]  
This implies
\[2\pi C_{\psi,M}\|f\|_{L^2(\mathbb{R})}(1 - \epsilon^2) \leq \int_{\mathbb{R}^+ \times \mathbb{R}} \chi_\Omega(a,b)|(W^M_\psi f)(a,b)|^2 dadb.\]  
By Holder’s inequality, we get
\[2\pi C_{\psi,M}\|f\|_{L^2(\mathbb{R})}(1 - \epsilon^2) \leq \left(\int_{\mathbb{R}^+ \times \mathbb{R}} (\chi_\Omega(a,b))^{\frac{p}{2-p}} dadb\right)^{\frac{2-p}{2}} \left(\int_{\mathbb{R}^+ \times \mathbb{R}} |(W^M_\psi f)(a,b)|^p dadb\right)^{\frac{2}{p}}\]  
Using equation (17), we get
\[(2\pi|B|)^{1-\frac{2}{p}}(1 - \epsilon^2) \leq |\Omega|^{\frac{p-2}{2p}}\|\psi\|_{L^2(\mathbb{R})}^{2-\frac{2}{p}}\]  
Therefore, we get
\[|\Omega|\|\psi\|_{L^2(\mathbb{R})} \geq 2\pi|B|C_{\psi,M}(1 - \epsilon^2)^{\frac{p}{2-p}}.\]  
The proof is complete.
Corollary 4.3. If \( f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}) \), in \( L^2(\mathbb{R}) \)-norm, is \( \epsilon_E \)-concentrated on \( E \subset \mathbb{R} \) and \( W^M_\psi f \) is \( \epsilon_\Omega \)-concentrated on \( \Omega \subset \mathbb{R}^+ \times \mathbb{R} \), then
\[
|\Omega|m(E)\|\psi\|^2_{L^2(\mathbb{R})}\|f\|^2_{L^4(\mathbb{R})} \geq 2\pi |B|C_{\psi,M}(1-\epsilon_\Omega^2)^{\frac{1}{2}}(1-\epsilon_E^2)^2\|f\|^2_{L^2(\mathbb{R})}, \ p > 2.
\]

Proof. The proof follows similarly as theorem [14] we use Lieb’s UP instead of the Donoho-Stark’s UP.

Corollary 4.4. If \( f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), in \( L^2(\mathbb{R}) \)-norm, is \( \epsilon_E \)-concentrated on \( E \subset \mathbb{R} \) and \( W^M_\psi f \) is \( \epsilon_\Omega \)-concentrated on \( \Omega \subset \mathbb{R}^+ \times \mathbb{R} \), then
\[
|\Omega|m(E)\|\psi\|^2_{L^2(\mathbb{R})}\|f\|^2_{L^\infty(\mathbb{R})} \geq 2\pi |B|C_{\psi,M}(1-\epsilon_\Omega^2)^{\frac{1}{2}}(1-\epsilon_E^2)^2\|f\|^2_{L^2(\mathbb{R})}, \ p > 2.
\]

Proof. The proof follows similarly as theorem [14] we use Lieb’s UP instead of the Donoho-Stark’s UP.

5 Orthonormal sequences and uncertainty principle

We now express the UP in term of the generalized dispersion of \( W^M_\psi \), which is defined by
\[
\rho_p \left( W^M_\psi \right) f = \left( \int_{\mathbb{R}^+ \times \mathbb{R}} |(a,b)|^p \| (W^M_\psi f)(a,b) \|^2 dab \right)^{\frac{1}{2}}, \tag{23}
\]
where \( |(a,b)| = \sqrt{a^2 + b^2} \), \( p > 0 \).

Definition 5.1. Let \( T \) be a bounded linear operator on a Hilbert space \( \mathbb{H} \) over the field \( \mathbb{F} \) (where \( \mathbb{F} \) is \( \mathbb{R} \) or \( \mathbb{C} \)) and \( \{u_n\}_{n \in \mathbb{N}} \) be an orthonormal basis of \( \mathbb{H} \), then \( T \) is called a Hilbert-Schmidt operator if
\[
\|T\|_{HS} = \left( \sum_{n=1}^{\infty} \|Tu_n\|^2 \right)^{\frac{1}{2}} < \infty.
\]

Before discussing the main result of this section, we estimate the Hilbert-Schmidt norm of the product of some orthogonal projection operators and use it to estimate the concentration of \( W^M_\psi \) on subset of \( \mathbb{R}^+ \times \mathbb{R} \). Similar results were first studied by Wilczok [27] in the case of windowed FT and WT.

Theorem 5.1. Let \( f \in L^2(\mathbb{R}) \), \( \psi \) is an ALCW and \( \Omega \subset \mathbb{R}^+ \times \mathbb{R} \) such that \( |\Omega| < \frac{2\pi |B|C_{\psi,M}}{\|\psi\|^2_{L^2(\mathbb{R})}} \). Then
\[
\|\chi_\Omega W^M_\psi f\|_{L^2(\mathbb{R}^+ \times \mathbb{R})} \geq \sqrt{2\pi |B|C_{\psi,M} - |\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}\|f\|_{L^2(\mathbb{R})}.
\]

Proof. We consider the orthogonal projections \( P_\Omega \) from \( L^2(\mathbb{R}_+ \times \mathbb{R}, \text{dadb}) \) on the RKHS \( W^M_\psi(L^2(\mathbb{R})) \) and \( P_D \) on \( L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb}) \) defined by \( P_\Omega F = \chi_\Omega F \) for all \( F \in L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb}) \). According to Saitoh [36], for every \( (a,b) \in \mathbb{R}^+ \times \mathbb{R} \) and \( F \in L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb}) \), we get
\[
(P_\Omega P_\Psi F)(a,b) = \chi_\Omega(a,b) \langle F, K^M_\psi(\cdot, a,b) \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})}
= \int_{\mathbb{R}^+ \times \mathbb{R}} \chi_\Omega(a,b) F(x,y) K^M_\psi(x,y; a,b) dxdy.
\]

Thus the integral operator \( P_\Omega P_\Psi \) has the kernel \( N^M_\psi,\Omega \) defined on \( (\mathbb{R}^+ \times \mathbb{R})^2 \) by
\[
N^M_\psi,\Omega(x,y; a,b) = F(x,y) K^M_\psi(x,y; a,b)
\]
such that
\[
\int_{\mathbb{R}^+ \times \mathbb{R}} \int_{\mathbb{R}^+ \times \mathbb{R}} |N^M_\psi,\Omega(x,y; a,b)|^2 dxdy dadb
= \int_{\mathbb{R}^+ \times \mathbb{R}} \left( \int_{\mathbb{R}^+ \times \mathbb{R}} |K^M_\psi(x,y; a,b)|^2 dxdy \right) |\chi_\Omega(a,b)|^2 dadb
= \int_{\mathbb{R}^+ \times \mathbb{R}} \chi_\Omega(a,b) |K^M_\psi(\cdot, a,b)|^2_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} dadb
= \frac{|\Omega|\|\psi\|^2_{L^2(\mathbb{R})}}{2\pi |B|C_{\psi,M}}.
\]
Now, 
\[ \chi_\Omega W^M_\psi f = P_\Omega P_\psi (W^M_\psi f) \]
implies
\[ \| \chi_\Omega W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2 \leq \| P_\Omega P_\psi \|_2^2 \| W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2. \]
Therefore
\[ \| W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2 \leq \| P_\Omega P_\psi \|_2^2 \| W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2 + \| \chi_\Omega W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2. \]
i.e., \[ \| \chi_\Omega W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2 \geq (1 - \| P_\Omega P_\psi \|_2^2) 2\pi |B| C_{\psi,M} \| f \|_{L^2(\mathbb{R})}^2. \]
Now using the fact that \[ \| P_\Omega P_\psi \|_2 \leq \| P_\Omega P_\psi \|_{HS}, \]
where \[ \| \cdot \| \]
denotes the operator norm, we obtain
\[ \| \chi_\Omega W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2 \geq (1 - \| P_\Omega P_\psi \|_2^2) 2\pi |B| C_{\psi,M} \| f \|_{L^2(\mathbb{R})}^2. \]
Hence, we obtain
\[ \| \chi_\Omega W^M_\psi f \|_{L^2(\mathbb{R}^+ \times \mathbb{R})}^2 \geq \sqrt{2\pi |B| C_{\psi,M} - \| \phi \|_{L^2(\mathbb{R})}^2} \| f \|_{L^2(\mathbb{R})}. \]
This proves the theorem. \( \square \)

**Theorem 5.2.** If \( \psi \) is an ALCW, \( \{ \phi_n \}_{n \in \mathbb{N}} \subset L^2(\mathbb{R}) \) be an ONS and \( \Omega \subset \mathbb{R}^+ \times \mathbb{R} \) be such that its measure \( |\Omega| < \infty \), then for any non-empty \( \Lambda \subset \mathbb{N} \),
\[ \sum_{n \in \Lambda} \left( 1 - \| \chi_\Omega W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \|_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} \right) \leq \frac{|\Omega| \| \psi \|_{L^2(\mathbb{R})}^2}{2\pi |B| C_{\psi,M}} \]  \( (24) \)

**Proof.** Consider the orthonormal basis \( \{ h_n \}_{n \in \mathbb{N}} \) of \( L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb}) \). It has been proved in the above theorem that \( P_\Omega P_\psi \) is a Hilbert-Schmidt operator such that \( \| P_\Omega P_\psi \|_{HS}^2 = \frac{|\Omega| \| \psi \|_{L^2(\mathbb{R})}^2}{2\pi |B| C_{\psi,M}} \).
Since \( P_\Omega^2 = P_\Omega \) and both \( P_\Omega \), \( P_\psi \) are self-adjoint, the operator \( T = (P_\Omega P_\psi)^* (P_\Omega P_\psi) = P_\Omega P_\psi P_\Omega \) is positive and is such that
\[ \sum_{n \in \mathbb{N}} \langle T h_n, h_n \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} = \sum_{n \in \mathbb{N}} \langle P_\Omega P_\psi h_n, P_\Omega P_\psi h_n \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} \]
\[ = \sum_{n \in \mathbb{N}} \| P_\Omega P_\psi h_n \|_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})}^2 \]
\[ = \| P_\Omega P_\psi \|_{HS}^2 \]
\[ = \frac{|\Omega| \| \psi \|_{L^2(\mathbb{R})}^2}{2\pi |B| C_{\psi,M}} < \infty. \]
Therefore, \( T \) is a trace class operator with \( Tr(T) = \frac{|\Omega| \| \psi \|_{L^2(\mathbb{R})}^2}{2\pi |B| C_{\psi,M}}. \)
Now as \( \{ \phi_n \}_{n \in \mathbb{N}} \) is an ONS, from equation (14), it follows that \( \{ W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \}_{n \in \mathbb{N}} \) is an ONS in \( L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb}). \)
Hence, we have
\[ \sum_{n \in \Lambda} \left( \langle P_\Omega P_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right), W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} \right) = \sum_{n \in \Lambda} \langle P_\psi P_\Omega P_\psi \left( W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \right), W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} \]
\[ = \sum_{n \in \Lambda} \langle T \left( W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \right), W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} \]
\[ \leq \sum_{n \in \Lambda} \langle T \left( W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \right), W^M_\psi \left( \frac{\phi_n}{\sqrt{2\pi |B| C_{\psi,M}}} \right) \rangle_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} \]
\[ = Tr(T) = \frac{|\Omega| \| \psi \|_{L^2(\mathbb{R})}^2}{2\pi |B| C_{\psi,M}}. \]
Thus, we have
\[
\sum_{n \in \Lambda} \left( 1 - \left\| \chi_{G_s} W_{\phi}^M \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right\|_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} \right) \leq \frac{|G_s| \| \psi \|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}.
\]
This proves the theorem.

The theorem below shows that, if the LCWT of each member of an ONS are \( \epsilon \)-concentrated in a set of finite measure then the sequence is necessarily finite. The theorem also gives an upper bound of the cardinality of the so proved finite sequence.

**Theorem 5.3.** Let \( s, \epsilon > 0 \) such that \( \epsilon < 1 \). Let \( G_s = \{(a, b) \in \mathbb{R}^+ \times \mathbb{R} : a^2 + b^2 \leq s^2\} \) and \( \psi \) is an ALCW. Also let \( \Lambda \subset \mathbb{N} \) be non-empty and \( \{\phi_n\}_{n \in \Lambda} \subset L^2(\mathbb{R}) \) be an ONS. If \( W_{\phi}^M \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \) is \( \epsilon \)-concentrated in \( G_s \) for all \( n \in \Lambda \), then \( \Lambda \) is finite and
\[
\text{Card}(\Lambda) \leq \frac{s^2 \| \psi \|^2_{L^2(\mathbb{R})}}{4|B|C_{\psi,M}(1 - \epsilon)},
\]
where \( \text{Card}(\Lambda) \) denotes the cardinality of \( \Lambda \).

**Proof.** Applying above theorem, we have
\[
\sum_{n \in \Lambda} \left( 1 - \left\| \chi_{G_s} W_{\phi}^M \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right\|_{L^2(\mathbb{R}^+ \times \mathbb{R}, \text{dadb})} \right) \leq \frac{|G_s| \| \psi \|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}.
\]
Again, since for each \( W_{\phi}^M \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \) is \( \epsilon \)-concentrated in \( G_s \), we have
\[
\left\| \chi_{G_s} W_{\phi}^M \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right\|_{L^2(\mathbb{R}^+ \times \mathbb{R})} \leq \epsilon.
\]

Therefore, it follows that
\[
\sum_{n \in \Lambda} (1 - \epsilon) \leq \frac{|G_s| \| \psi \|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}
\]
i.e., \( \text{Card}(\Lambda)(1 - \epsilon) \leq \frac{|G_s| \| \psi \|^2_{L^2(\mathbb{R})}}{2\pi|B|C_{\psi,M}}. \)

Thus \( \text{Card}(\Lambda) \) is finite and using \( |G_s| = \pi s^2 \), we obtain
\[
\text{Card}(\Lambda) \leq \frac{s^2 \| \psi \|^2_{L^2(\mathbb{R})}}{4|B|(1 - \epsilon)C_{\psi,M}}.
\]

The proof is complete.
Corollary 5.1. Let \( p > 0 \), \( R > 0 \) and \( \psi \) is an ALCW. Also let \( \wedge \subset \mathbb{N} \), be non-empty and \( \{ \phi_n \}_{n \in \wedge} \subset L^2(\mathbb{R}) \) be an ONS. Then \( \wedge \) is finite if \( \{ \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \}_{n \in \wedge} \) is uniformly bounded. Moreover, if it is uniformly bounded by \( R \), then

\[
\text{Card}(\wedge) \leq \frac{2^{\frac{p}{2}+1} R^2 \|\psi\|_{L^2(\mathbb{R})}^2}{4|B|C_{\psi,M}}.
\]

Proof. Since for each \( n \in \wedge \), \( \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \leq R \), and thus

\[
\int_{|(a,b)| \geq R^{2\frac{p}{2}}} \left| \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) (a,b) \right|^2 \, dadb = \int_{|(a,b)| \geq R^{2\frac{p}{2}}} \left| (a,b)^{-p} (a,b) \right| \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) (a,b) \right|^2 \, dadb \leq \frac{1}{R^{2\frac{p}{2}}} \int_{\mathbb{R}^+ \times \mathbb{R}} |(a,b)|^p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) (a,b) \right|^2 \, dadb \leq \frac{1}{4}.
\]

Thus it follows that, for each \( n \in \wedge \), \( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \) is \( \frac{1}{4} \)-concentrated in

\[
G_{R^{2\frac{p}{2}}} = \left\{ (a,b) \in \mathbb{R}^+ \times \mathbb{R} : |(a,b)| < R^{2\frac{p}{2}} \right\}.
\]

Thus, from theorem 5.3, it follows that \( \wedge \) is finite and

\[
\text{Card}(\wedge) \leq \frac{(R^{2\frac{p}{2}})^2 \|\psi\|_{L^2(\mathbb{R})}^2}{4|B| \left( 1 - \frac{1}{4} \right) C_{\psi,M}},
\]

i.e., \( \text{Card}(\wedge) \leq \frac{2^{\frac{p}{2}+1} R^2 \|\psi\|_{L^2(\mathbb{R})}^2}{4|B|C_{\psi,M}} \).

Thus the proof is complete. \( \square \)

Lemma 5.1. Let \( p > 0 \), \( \psi \) is an ALCW and \( \{ \phi_n \}_{n \in \mathbb{N}} \subset L^2(\mathbb{R}) \) be an ONS, then \( \exists m_0 \in \mathbb{Z} \) for which

\[
\rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \geq 2^{m_0}, \forall n \in \mathbb{N}.
\]

Proof. Define \( P_m = \left\{ n \in \mathbb{N} : \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \in [2^{m-1}, 2^m) \right\} \), for each \( m \in \mathbb{Z} \).

Then for each \( n \in P_m \), we get

\[
\int_{\mathbb{R}^+ \times \mathbb{R}} |(a,b)|^p \left| W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right|^2 \, dadb < 2^{mp}.
\]

Now,

\[
\int_{|(a,b)| \geq 2^{m+1}} \left| \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) (a,b) \right|^2 \, dadb \leq \frac{1}{2^{mp+2}} \int_{\mathbb{R}^+ \times \mathbb{R}} |(a,b)|^p \left| W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right|^2 \, dadb \leq \frac{1}{2^{mp+2}} \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \left\{ 2^{mp} \right\}^p.
\]
Hence \( \exists \) every \( p > m \).

Letting \( m < k \), let \( C \) be an integer defined in the above lemma. Let \( m < k \) for all \( m \in \mathbb{N} \).

**Theorem 5.4. (Shapiro’s Dispersion theorem)**. Let \( \psi \) be an ALCW and \( \{ \phi_n \}_{n \in \mathbb{N}} \subset L^2(\mathbb{R}) \) be an ONS, then for every \( p > 0 \) and non-empty finite \( \land \subset \mathbb{N} \),

\[
\sum_{n \in \land} \left\{ \left( \frac{\phi_n}{\sqrt{\pi |B|C_{\psi,M}}} \right)^p \right\} \geq \frac{(\text{Card}(\land))^{\frac{p}{2} + 1}}{2^{p+1}} \left( \frac{3 |B|C_{\psi,M}}{2^{\frac{p}{2} + 2} \| \psi \|^2_{L^2(\mathbb{R})}} \right)^{\frac{1}{2} + 1}.
\]

**Proof.** Let \( m_0 \) be an integer defined in the above lemma. Let \( k \in \mathbb{Z} \) such that \( k \geq m_0 \). Define \( Q_k = \bigcup_{m=m_0}^k P_m \). Then we have

\[
\text{Card}(Q_k) = \sum_{m=m_0}^k \text{Card}(P_m) \leq \sum_{m=m_0}^k \frac{2^{2m+\frac{1}{2} + 1} \| \psi \|^2_{L^2(\mathbb{R})}}{4 |B|C_{\psi,M}} \leq \frac{2^{k+1} \| \psi \|^2_{L^2(\mathbb{R})}}{4 |B|C_{\psi,M}} \sum_{m=m_0}^k 2^{2m} \leq \frac{2^{k+1} \| \psi \|^2_{L^2(\mathbb{R})} 2^{2k+2}}{3 |B|C_{\psi,M}}.
\]

i.e., \( \text{Card}(Q_k) \leq \frac{2^{k+1} \| \psi \|^2_{L^2(\mathbb{R})}}{3 |B|C_{\psi,M}} 2^{2k} \).

Let \( C = \frac{2^{k+1} \| \psi \|^2_{L^2(\mathbb{R})}}{3 |B|C_{\psi,M}} \). Then \( \text{Card}(Q_k) \leq C \cdot 2^{2k} \). If \( \text{Card}(\land) > 2^{(m_0+1)} \), then \( \frac{1}{2 \log 2} \log \left( \frac{\text{Card}(\land)}{C} \right) > m_0 + 1 \).

Let us choose an integer \( k > m_0 + 1 \) such that

\[
k - 1 < \frac{1}{2 \log 2} \log \left( \frac{\text{Card}(\land)}{C} \right) \leq k.
\]

Then it results in

\[
C \cdot 2^{2(k-1)} < \text{Card}(\land) \leq C \cdot 2^{2k}.
\]

Thus, we have

\[
\text{Card}(Q_{k-1}) = \frac{C}{2} \cdot 2^{2(k-1)} < \frac{\text{Card}(\land)}{2}.
\]
Therefore,
\[
\sum_{n \in \Lambda} \left\{ \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p \geq \sum_{n \notin Q_k-1} \left\{ \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p \geq \frac{\text{Card}(\Lambda) 2^{(k-1)p}}{2} = \frac{\text{Card}(\Lambda) 2^{kp}}{2^{2p}}.
\]

Since, \( \text{Card}(\Lambda) \leq C 2^k \), we have \( \left( \frac{\text{Card}(\Lambda)}{C} \right)^p \leq 2^{kp} \).

Therefore,
\[
\sum_{n \in \Lambda} \left\{ \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p \geq \left( \frac{\text{Card}(\Lambda)}{C} \right)^p \left( \frac{1}{C} \right)^{\frac{p}{2}}.
\]

Again, if \( \text{Card}(\Lambda) \leq 2^{2(m_0+1)} \), then
\[
\sum_{n \in \Lambda} \left\{ \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p \geq \text{Card}(\Lambda) 2^{m_0p}, \text{ (using lemma 5.1)}.
\]

Now, \( \text{Card}(\Lambda) \leq C 2^{2(m_0+1)} \) implies \( \frac{1}{2p} \left( \frac{\text{Card}(\Lambda)}{C} \right)^p \leq 2^{m_0p} \). Thus we have
\[
\sum_{n \in \Lambda} \left\{ \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p \geq \left( \frac{\text{Card}(\Lambda)}{2} \right)^{\frac{p}{2p+1}} \left( \frac{1}{C} \right)^{\frac{p}{2}}.
\]

Hence, for any non-empty finite \( \Lambda \subset \mathbb{N} \), we have
\[
\sum_{n \in \Lambda} \left\{ \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p \geq \left( \frac{\text{Card}(\Lambda)}{2} \right)^{\frac{p}{2p+1}} \left( \frac{1}{C} \right)^{\frac{p}{2}}.
\]

Therefore, putting the value of \( C \) we get
\[
\sum_{n \in \Lambda} \left\{ \rho_p \left( W^M_{\psi} \left( \frac{\phi_n}{\sqrt{2\pi|B|C_{\psi,M}}} \right) \right) \right\}^p \geq \left( \frac{3|B|C_{\psi,M}}{2^{p+2} \|\psi\|^2_{L^2(\mathbb{R})}} \right)^{\frac{p}{2}}.
\]

This completes the proof. \( \square \)

6 Conclusions

We have proposed a novel time-frequency analyzing tool, namely LCWT, which combines the advantages of the LCT and the WT and offers time and linear canonical domain spectral information simultaneously in the time-LCT-frequency plane. We have studied its properties like inner product relation, reconstruction formula and also characterized its range. We also gave a lower bound of the measure of essential support of the LCWT via UP of Donoho-Stark and Lieb. Finally, we have studied the Shapiro’s mean dispersion theorem associated with the LCWT.

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