ON THE TORELLI GROUP ACTION ON COMPACT CHARACTER VARIETIES

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ABSTRACT. The aim of this article is to prove that the Torelli group action on some $G$-character varieties is ergodic. The case $G = SU(2)$ was obtained by Funar–Marché. We propose another proof of it based on independent methods which extend to a proof of the ergodicity of the Torelli group action on the $G$-character variety for $G$ a connected, semi-simple and compact Lie group.

1. INTRODUCTION

Let $\Sigma$ be a compact, connected, oriented and closed surface of genus $g \geq 2$. Denote $\Gamma$ its fundamental group and let $G$ be a semi-simple, connected and compact Lie group with Lie algebra $\mathfrak{g}$ and adjoint representation $\text{Ad} : G \to \text{GL}(\mathfrak{g})$. We define $\text{Hom}(\Gamma, G)$ to be the set of homomorphisms $\rho : \Gamma \to G$, on which the group $G$ acts by conjugation. We denote by $Z_G(\rho)$ the centralizer of $\rho$, i.e the set of elements of $G$ which commute with all the $\rho(\gamma)$, $\gamma \in \Gamma$. This centralizer $Z_G(\rho)$ is the stabilizer of $\rho$ for the conjugation action of $G$.

Definition 1.1. The $G$-character variety $\mathcal{X}(\Gamma, G)$ is the GIT-quotient

$$\text{Hom}(\Gamma, G)/\!/G.$$  

In the cases we consider, the set $\mathcal{X}(\Gamma, G)$ contains a dense open set, which is the set of classes of representations which have a discrete centralizer. Since a semi-simple Lie group has discrete center the previous condition makes sense. This set of regular points is denoted by $\mathcal{M}(\Gamma, G)$. Goldman proved in [10] that it carries a symplectic measure. The symplectic form, the centralizer $Z_G(\rho)$ of $\rho$ is identified with the first group cohomology:

$$H^1(\Gamma, \mathfrak{g}, \rho) = \frac{Z^1(\Gamma, \mathfrak{g}, \rho)}{B^1(\Gamma, \mathfrak{g}, \rho)},$$

where $Z^1(\Gamma, \mathfrak{g}, \rho)$ is the set of maps $u : \Gamma \to \mathfrak{g}$ such that for all $\gamma_1, \gamma_2 \in \Gamma$,

$$u(\gamma_1 \gamma_2) = u(\gamma_1) + \text{Ad}(\rho(\gamma_1))u(\gamma_2)$$

and $B^1(\Gamma, \mathfrak{g}, \rho)$ be the set of maps of the form $\gamma \mapsto x - \text{Ad}(\rho(\gamma))x$ for $x \in \mathfrak{g}$.

We define the second cohomology group $H^2(\Gamma, \mathbb{R})$ as the quotient :

$$\frac{Z^2(\Gamma, \mathbb{R})}{B^2(\Gamma, \mathbb{R})},$$

where $Z^2(\Gamma, \mathbb{R})$ is the set of maps $u : \Gamma^2 \to \mathbb{R}$ which verify the cocycle relation

$$u(\gamma_2, \gamma_3) - u(\gamma_1 \gamma_2, \gamma_3) + u(\gamma_1, \gamma_2 \gamma_3) - u(\gamma_1, \gamma_2) = 0$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ and $B^2(\Gamma, \mathbb{R})$ is the set of maps $\Gamma^2 \to \mathbb{R}$ of the form

$$\gamma_1, \gamma_2 \mapsto v(\gamma_2) + v(\gamma_1) - u(\gamma_1, \gamma_2).$$

It is well known that the second cohomology group $H^2(\Gamma, \mathbb{R})$ is isomorphic to $\mathbb{R}$. We hence define $\omega_G$ by the formula :

$$\omega_G[\rho] : H^1(\Gamma, \mathfrak{g}, \rho)^2 \to H^2(\Gamma, \mathbb{R}) \cong \mathbb{R}$$

$$(u, v) \mapsto \langle u(\gamma_1), \text{Ad}(\rho(\gamma_1))v(\gamma_2) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{g}$. It defines a symplectic measure $\mu$ on the character variety.
The mapping class group $\Mod^+(\Sigma)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma$. It acts on the $G$-character variety via:

$$[\psi] \cdot [\rho] = [\gamma \mapsto \rho(\psi^{-1}_* \gamma)]$$

and preserves $\mathcal{M}(\Gamma, G)$ and its symplectic structure. Naturally $\Mod^+(\Sigma)$ acts on the first homology group $H_1(\Sigma, \mathbb{Z})$ of $\Sigma$. The kernel of this action is called the Torelli group and denoted by $\Tor(\Sigma)$.

William Goldman and Eugene Xia proved in [11] the following theorem:

**Theorem 1.1.** The mapping class group acts ergodically on the character variety $\mathcal{M}(\Gamma, \text{SU}(2))$ with respect to the symplectic measure.

There are however stronger dynamical properties: Louis Funar and Julien Marché proved recently in [8] that the actions on $\mathcal{M}(\Gamma, \text{SU}(2))$ of the Johnson groups $\mathcal{K}_n$, defined by $\mathcal{K}_0 = \Mod^+(\Sigma)$ and $\mathcal{K}_{n+1} = [\mathcal{K}_n, \mathcal{K}_n]$, are ergodic.

As a consequence:

**Corollary 1.1.** The Torelli group acts ergodically on the SU(2)-character variety with respect to the symplectic measure.

Their proof use the local geometry of the character variety at the trivial representation and Taylor expansions of the trace functions.

The aim of this paper is to propose a new proof of the ergodicity of the Torelli group action on $\mathcal{M}(\Gamma, \text{SU}(2))$. The tools and the strategy we use will be adapted for the cases $G = \text{SU}(n)$, for $n \geq 2$:

**Theorem A.** The Torelli group acts ergodically on the SU(n)-character variety $\mathcal{M}(\Gamma, \text{SU}(n))$ with respect to the symplectic measure.

Doug Pickrell and Eugene Xia generalized the result of [11] in [14]:

**Theorem 1.2.** Let $G$ be a connected and compact Lie group. Then the mapping class group acts ergodically on each connected component of the $G$-character variety $\mathcal{M}(\Gamma, G)$ with respect to the measure on $\mathcal{M}(\Gamma, G)$ induced by the Haar measure on $G$.

We so generalize:

**Theorem B.** Let $G$ be a connected, semi-simple and compact Lie group. Then the Torelli group $\Tor(\Sigma)$ acts ergodically on each connected component of $\mathcal{M}(\Gamma, G)$ with respect to the Goldman symplectic measure.

As corollary, replacing $G$ by a finite product $G \times \cdots \times G$, we obtain:

**Theorem C.** Let $G$ be a group verifying the hypothesis of theorem B. Then, for all $k \geq 1$, the Torelli group $\Tor(\Sigma)$ acts ergodically on each connected components of the product $\mathcal{M}(\Gamma, G)^k$. In particular on these components, the action of the Torelli group is weakly mixing.

**Opening**

Following the works of Funar-Marché, the natural continuation of this article is about the ergodic action of the Johnson subgroups on the character varieties we consider:

**Question 1.** Let $\mathcal{M}(\Gamma, G)$ be a character variety considered in this article. Is the action of the Johnson subgroups ergodic on it? Is there a index $n > 0$ such that for all $i \leq n$ the action of $\mathcal{K}_i$ is ergodic and not for $i \geq n + 1$?

Julien Marché and Maxime Wolff proved in [13] that the mapping class group acts ergodically on some subspaces of the exotic components of the PSL$_2(\mathbb{R})$-character variety. The following question is a natural problem about the non-compact cases:

**Question 2.** Is the Torelli group action on the subspaces introduced by Marché and Wolff ergodic?
In order to give a complete description of the situation, we may wonder if theorem B holds by dropping the semi-simple condition. Namely, we ask:

**Question 3.** Is the theorem B true if $G$ is not assumed to be semi-simple?

In the abelian case, it is observed in [2], that the action is trivial.

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2. Background

2.1. The Torelli group. For more details on this part, see [6]. The *mapping class group* $\text{Mod}^+(\Sigma)$ of the surface $\Sigma$ is the quotient of the set of positive diffeomorphisms of $\Sigma$ quotiented by isotopy relation. It means that a mapping class is a class $[f] = \{ g \in \text{Diff}^+(\Sigma) \mid g \text{ and } f \text{ are isotopic}\}$.

It is generated by the $3g - 1$ Dehn twists $T_{a_1}, \ldots, T_{b_g}, T_{d_1}, \ldots, T_{d_{g-1}}$ where the curves $a_i, b_i$ are given by the presentation:

\[ \Gamma = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle \]

and the curves $d_i$ are the products $a_i^{-1}d_{i+1}$. The first homology group $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is freely generated by the curves $[a_1], \ldots, [b_g]$ and is a lattice in the real homology group $H_1(\Sigma, \mathbb{R})$.

**Definition 2.1.** The Torelli group $\text{Tor}(\Sigma)$ is the kernel of the action of the mapping class group on $H_1(\Sigma, \mathbb{Z})$.

An explicit generating set of the Torelli group is the set:

\[ \{ T_c, T_a T_b^{-1} \mid c \text{ separating curve and } a, b \text{ are cohomologuous curves} \} \]

We make an important remark for the following.

**Remark 1.** If $c_1, \ldots, c_t$ are the boundary components of a subsurface of $\Sigma$, then the product of the Dehn twists $T_{c_1}, \ldots, T_{c_t}$ acts trivially on $H_1(\Sigma, \mathbb{R})$ and hence is in the Torelli group.

2.2. Borel cross sections. Let $X$ be a topological set on which a topological group $H$ acts. The set $X$ is endowed with its Borel $\sigma$-algebra and a Borel measure $\mu$. A function $X_1 \to X_2$ between two measured sets is bimeasurable if it is, measurable, invertible and has a measurable inverse. If such a function exists, we say that $X_1$ and $X_2$ are bimeasurable. The quotient set $X/H$ carries the quotient topology such that the canonical projection $X \to X/H$ is continuous. A *Borel cross section* is a subset $S \subset X$ which intersects the orbits $H.x$, for each element $x \in X$, exactly once. Edward Effros gives a proof of the following theorem in [5]. We state a more restrictive result but sufficient for our purpose.

**Theorem 2.1.** If $X$ and $H$ are separable, complete, metrizable and locally compact and if $H$ acts continuously on $X$, then the following condition are equivalent:

- Each orbit is locally closed.
- Each orbit is locally compact.
- There exists a Borel cross section $S$ for the orbits of $H$ in $X$.

James Bondar gives this key corollary in [1]:

**Theorem 2.2.** If $X$ and $H$ verify the hypothesis of the previous theorem and if furthermore the action of $H$ on $X$ is free and one of the condition of the previous statement holds, then the Borel cross section $S$ is bimeasurable to the quotient space $X/H$ and for every measurable function $f : X \to \mathbb{R}$:

\[ \int_X f d\mu = \int_S \left( \int_H f(h.s) d\mu_H(h) \right) d\mu_S(s) \]

where $\mu_S$ is some Borel measure on $S$ and $\mu_H$ is the Haar-measure of $H$. 
3. Ergodicity for the case of SU(2)

The strategy in proving the ergodicity of the Torelli group action is to find a full measure subset of the character variety on which every measurable and Tor(Σ)-invariant function is invariant by the mapping class group action.

3.1. The mapping class group action on the SU(2)-characters. In the rank 1 case we consider, the Lie algebra \( \mathfrak{su}(2) \) of SU(2) is the Lie algebra of traceless skew-Hermitian complex \( 2 \times 2 \)-matrices. Let \( f : \text{SU}(2) \to [-2, 2] \) denote the trace function. Its variation function \( F \) is defined as the unique function \( F : \text{SU}(2) \to \mathfrak{su}(2) \) such that, for all \( x \in \text{SU}(2) \) and \( X \in \mathfrak{su}(2) \),

\[
\frac{d}{dt}|_{t=0} f(x, \exp(tX)) = \langle F(x), X \rangle.
\]

Goldman proved, in [9], that :

\[
F(x) = x - \frac{\text{tr}(x)}{2} \text{id}.
\]

Following [11], we let :

\[
\zeta^t : \text{SU}(2) \to \text{SU}(2)
\]

\[
x \longmapsto \exp(tF(x)).
\]

For \( x \in \text{SU}(2) \), the map \( t \mapsto \zeta^t(x) \) is a one-parameter subgroup of SU(2).

3.1.1. Separating curve. If a curve \( \alpha \) is separating (i.e \( \Sigma \setminus \alpha \) isn’t connected), then \( \Sigma \setminus \alpha \) is the disjoint union \( \Sigma_1 \sqcup \Sigma_2 \) and the fundamental group \( \Gamma \) is the amalgamated product :

\[
\pi_1 \Sigma_1 \ast_{\langle \alpha \rangle} \pi_1 \Sigma_2.
\]

The data of two representations \( \rho_1 : \pi_1 \Sigma_1 \to \text{SU}(2) \) and \( \rho_2 : \pi_1 \Sigma_2 \to \text{SU}(2) \) such that

\[
\rho_1(\alpha) = \rho_2(\alpha)
\]

allows to construct a unique representation \( \rho : \Gamma \to \text{SU}(2) \) defined by \( \rho_{|\pi_1 \Sigma_1} = \rho_1 \) and \( \rho_{|\pi_1 \Sigma_2} = \rho_2 \).

We let the twist flow \( \xi^t_\alpha : \mathcal{M}(\Gamma, \text{SU}(2)) \to \mathcal{M}(\Gamma, \text{SU}(2)) \) be defined by :

\[
\xi^t_\alpha \rho(\gamma) = \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\
\zeta^t(\rho(\alpha)) \rho(\gamma) \zeta^{-t}(\rho(\alpha)) & \text{if } \gamma \in \pi_1(\Sigma_2) 
\end{cases}
\]

This flow is well defined because \( \zeta^t(\rho(\alpha)) \) is the exponential of a polynomial in \( \rho(\alpha) \) and thus commutes with \( \rho(\alpha) \).

3.1.2. Non-separating curve. If \( \alpha \) is non-separating (i.e if \( \Sigma|\alpha \) is connected), the fundamental group \( \Gamma \) is the HNN-extension :

\[
\left( \pi_1(\Sigma|\alpha) \ast \langle \beta \rangle \right)/\langle \beta \alpha_- \beta^{-1} \alpha_+^{-1} \rangle
\]

where \( \alpha_\pm \) represent the boundary components of \( \Sigma|\alpha \). Hence the data of a representation

\[
\rho_0 : \pi_1(\Sigma|\alpha) \to \text{SU}(2)
\]

and a matrix \( B \in \text{SU}(2) \) such that

\[
B \rho_0(\alpha_-) B^{-1} = \rho_0(\alpha_+)
\]

defines a unique representation \( \rho : \Gamma \to \text{SU}(2) \) such that \( \rho_{|\pi_1(\Sigma|\alpha)} = \rho_0 \) and \( \rho(\beta) = B \).

We define the flow \( \xi^t_\alpha : \mathcal{M}(\Gamma, \text{SU}(2)) \to \mathcal{M}(\Gamma, \text{SU}(2)) \) be defined by :

\[
\xi^t_\alpha \rho(\gamma) = \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\
\zeta^t(\rho(\alpha)) \rho(\beta) & \text{if } \gamma = \beta 
\end{cases}
\]

This flow is well defined since it verifies the relation :

\[
\xi^t_\alpha \rho(\beta) \xi^t_\alpha \rho(\alpha_-) \xi^t_\alpha \rho(\beta)^{-1} = \xi^t_\alpha \rho(\alpha_+).
\]
3.1.3. The Dehn twists and the flows. For a simple and closed curve $\alpha$, let the trace function $f_\alpha: \mathcal{M}(\Gamma, \text{SU}(2)) \to [-2,2]$ associated with the curve $\alpha$, i.e.

$$f_\alpha(\rho) = \text{tr}(\rho(\alpha)).$$

In [9], Goldman proved that these flows are the Hamiltonian flows of the trace functions. That mean:

$$df_\alpha X = \omega_G \left( \frac{d}{dt}\big|_{t=0} \xi^t_{\alpha}, X \right)$$

for all $X \in T_\mathcal{M}(\Gamma, \text{SU}(2))$.

If $x \in \text{SU}(2)$, then there exists $g \in \text{SU}(2)$ such that $x = g \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}^{-1}$ with $\theta = \cos^{-1} \left( \frac{f(\rho(\alpha))}{2} \right)$. We then compute $F(x) = x - \frac{f(x)}{\theta} \text{id}$ which we can write:

$$F(x) = g \begin{pmatrix} 2i \sin(\theta) & 0 \\ 0 & -2i \sin(\theta) \end{pmatrix} g^{-1}$$

and so, by definition:

$$\zeta^t(x) = g \begin{pmatrix} e^{2it\sin(\theta)} & 0 \\ 0 & e^{-2it\sin(\theta)} \end{pmatrix} g^{-1}.$$

In particular, if $x = \pm \text{id}$, then for all $t \in \mathbb{R}$ we have $\zeta^t(x) = \text{id}$ and for $x \neq \pm \text{id}$, we have the equality $\zeta^t(x) = \text{id}$ if and only if $t \in \frac{2\pi}{\sin(\theta)} \mathbb{Z}$, and:

**Lemma 3.1.** If $x \neq \pm \text{id}$, then $x$ belongs to the one-parameter subgroup $\{\zeta^t(x)\}_{t \in \mathbb{R}}$ and more precisely $x = \zeta^{s(x)}(x)$ for:

$$s(x) = \frac{\theta}{2 \sin(\theta)}.$$

Moreover, for $x \in \text{SU}(2)$ such that $\theta \notin \pi \mathbb{Q}$, the subgroup $\langle x \rangle$ is dense in the circle $\{\zeta^t(x) | t \in \mathbb{R}\} \cong S^1$ and acts ergodically on it with respect to the Lebesgue measure.

We hence remark that the Dehn twist $T_\alpha$ acts on $\rho$ with the relation:

$$T_\alpha \cdot \rho = \xi^{s(\rho(\alpha))}_\alpha \rho,$$

and if $\theta$ is irrational, then the orbit $\langle T_\alpha \rangle \cdot \rho$ is dense in the circle defined by the orbit $\{\xi^t_\alpha \rho\}_{t \in \mathbb{R}}$. The Hamiltonian flow gives an action of the circle $U_\alpha := S^1$ on the subspace of the character variety consisting of classes of representations $[\rho]$ such that $\rho(\alpha) \neq \pm \text{id}$.

We will propose another version of these flows in the case of $G = \text{SU}(n)$, for $n \geq 3$, in section 5 and the more general case is treated in section 6.

3.2. Ergodicity of translation actions. We give here key results whose statements allow to find a condition on a representation and some curves to have a property of density and ergodicity of the $\mathbb{Z}$-action of a Dehn twists composition, similarly to the lemma 3.1.

We remark that if two simple curves $c_1$ and $c_2$ are disjoint, then the flows $\xi^t_{c_1}$ and $\xi^s_{c_2}$ commute. Hence, under this assumption, the actions of these flows on a representation $\rho$ give a topological torus orbit $\{\xi^t_{c_1}, \xi^s_{c_2}\}_{t, s \in \mathbb{R}}$ obtained by the action of $U_{c_1} \times \cdots \times U_{c_\ell}$ on the characters which are not $\pm \text{id}$ evaluated in $c_1, c_2$.

**Lemma 3.2.** Let $[\rho]$ be a class of representations in $\mathcal{M}(\Gamma, \text{SU}(2))$ and suppose that there exist $c_1, \ldots, c_\ell$ be simple closed curves of $\Sigma$ which are pairwise disjoint and such that

$$\pi, \theta_1 = \cos^{-1} \left( \frac{f(\rho(c_1))}{2} \right), \ldots, \theta_\ell = \cos^{-1} \left( \frac{f(\rho(c_\ell))}{2} \right)$$

are linearly independent over $\mathbb{Q}$. If we denote $h = T_{c_1} \cdots T_{c_\ell}$, then the action of $h$ on the orbit $U_{c_1} \times \cdots \times U_{c_\ell} \cdot [\rho]$ is ergodic with respect to the Lebesgue measure on this torus orbit.

In particular, almost every orbit for the action of $h$ is dense in the topological torus $U_{c_1} \times \cdots \times U_{c_\ell} \cdot [\rho]$. 
Remark 2. With the notations of the lemma 3.2 and if $[\rho]$ verifies the hypothesis of it, then for all $i = 1, \ldots, \ell$, the matrix $\rho(c_i)$ is different to $\pm \text{id}$, then the torus orbit $U_{c_1} \times \cdots \times U_{c\ell}.[\rho]$ is the torus obtained as the quotient:

$$R^\ell / \Lambda$$

where $\Lambda$ is the lattice $\frac{2\pi}{\sin \theta_1} \mathbb{Z} \oplus \cdots \oplus \frac{2\pi}{\sin \theta_\ell} \mathbb{Z}$. By definition of the torus $R^\ell / \Lambda$, the action of $U_{c_1} \times \cdots \times U_{c\ell}$ on the character variety is free.

The following is a classical result we need to prove the lemma 3.2 and whose no proof will be given (see [12] for more details). We denote by $T$ the torus $R/2\pi \mathbb{Z}$. This notation is justified by the fact that

$$T \cong \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in R \right\}$$

is a maximal torus of $SU(2)$.

Lemma 3.3. Let $t = (t_1, \ldots, t_\ell) \in R^\ell$ such that $t_1, \ldots, t_\ell, \pi$ are linearly independent over $Q$ and let $f_t : T^\ell \to T^\ell$ be the translation of vector $t$. Then the action of $\langle f_t \rangle$ on $T^\ell$ is ergodic with respect to the Lebesgue measure.

Proof of the lemma 3.2. We remark that the Dehn twists $T_{c_k}$ act as a translation of the torus orbit $U_{c_1} \times \cdots \times U_{c\ell}.[\rho]$. The orbit $h^Z.[\rho]$ is the orbit $\langle \xi_{c_1}^{t_1}, \ldots, \xi_{c_\ell}^{t_\ell} \rangle.[\rho]$ with for all $k \in \{1, \ldots, \ell\}$:

$$t_k = \frac{\theta_k}{2 \sin(\theta_k)}.$$

For such a $t_k$, the action of $c_{c_k}^{t_k}$ is given by the multiplication by a matrix conjugated to

$$\begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & e^{-i\theta_k} \end{pmatrix}.$$

We deduce from this that the action of $h$ on $U_{c_1} \times \cdots \times U_{c\ell}.[\rho]$ is given by the translation of the vector $(\theta_1, \ldots, \theta_\ell)$. Then the lemma 3.3 shows that this action is ergodic with respect to the Lebesgue measure since $\pi, \theta_1, \ldots, \theta_\ell$ are linearly independent over $Q$. □

3.3. A full measure set. A multicurve $m$ is the union of a finite number of simple, closed and pairwise disjoint curves. Let denote by $MC(\Sigma)$ the set of multicurves $m$ such that all the curves of $m$ are simple, closed and non-separating and such that $m$ is the boundary of a subsurface of $\Sigma$ and by $MC_0(\Sigma)$ its subset of multicurves bounding a pair of pant in the surface $\Sigma$.

Definition 3.1. Let $m = c_1 \cup \cdots \cup c_\ell \in MC(\Sigma)$ be a multicurve. A class $[\rho] \in M(\Gamma, SU(2))$ verifies the condition $(M_m)$ if the real numbers :

$$\pi, \theta_1 = \cos^{-1}\left(\frac{\text{tr}(\rho(c_1))}{2}\right), \ldots, \theta_\ell = \cos^{-1}\left(\frac{\text{tr}(\rho(c_\ell))}{2}\right)$$

are linearly independant over $Q$.

Following the previous definition, for $m \in MC_0(\Sigma)$, we let the set :

$$M_m(\Gamma, SU(2)) = \{ [\rho] \in M(\Gamma, SU(2)) \mid [\rho] \text{ satisfies the condition } (M_m)\}$$

The aim of this section is to prove the proposition :

Proposition 3.1. For $m \in MC_0(\Sigma)$, the set $M_m(\Gamma, SU(2))$ has full measure in $M(\Gamma, SU(2))$.

For a curve $\gamma$, the angle of $\rho(\gamma)$, expressed with the formula

$$\cos^{-1}\left(\frac{f_\gamma}{2}\right),$$

defines a function $\theta_\gamma : M(\Gamma, SU(2)) \to S^1$. To simplify the notations, as previously, for a multicurve $m = c_1 \cup c_2 \cup c_3 \in MC_0(\Sigma)$, we will denote by $\theta_1, \theta_2$ and $\theta_3$ the functions $\theta_{c_1}, \theta_{c_2}$ and $\theta_{c_3}$.

We will need the following lemma :
Lemma 3.4. Let $[\rho]$ be a class of representations in $\mathcal{M}(\Gamma, SU(2))$, i.e. $\rho$ has discrete centralizer, $m = c_1 \cup c_2 \cup c_3$ be a pair of pant such that $\rho(c_i) \neq \pm id$ and let $i_0 \in \{1, 2, 3\}$. Then there exists a vector $X \in T_{[\rho]} \mathcal{M}(\Gamma, SU(2))$ such that for $i = 1, 2, 3$

$$d_{[\rho]} f_{c_i} X = \delta_{i_0}^i.$$ 

The proof of the lemma 3.4 uses the Fox calculus, a notion of differential calculus on groups. We refer to the annex in section 6 for the background on the Fox calculus and its use in the computational proof of the lemma 3.4. We hence start by the proof of the proposition.

Proof of the proposition 3.1. The complement of $\mathcal{M}_m(\Gamma, SU(2))$ is the set :

$$\bigcup_{(q_0, q_1, q_2, q_3) \in \mathbb{Z}^4 \setminus \{0\}} \left\{ [\rho] \in \mathcal{M}(\Gamma, SU(2)) \mid q_1 \theta_1(\rho) + q_2 \theta_2(\rho) + q_3 \theta_3(\rho) = q_0 \pi \right\}.$$

Let $q$ be the vector $(q_1, q_2, q_3)$. If $q_0 \neq 0$ and $q = 0$, the relation $q_1 \theta_1(\rho) + q_2 \theta_2(\rho) + q_3 \theta_3(\rho) = q_0 \pi$ is empty. We then only have the case $q \neq 0$ to consider. The proposition will be hence proved if for every $(q_0, q_1, q_2, q_3) \in \mathbb{Z}^4 \setminus \{0\}$ with $q \neq 0 \mathbb{Z}^4$, the set :

$$\left\{ [\rho] \in \mathcal{M}(\Gamma, SU(2)) \mid q_1 \theta_1(\rho) + q_2 \theta_2(\rho) + q_3 \theta_3(\rho) = q_0 \pi \right\} = \psi_{m,q}^{-1}(q_0 \pi)$$

has null measure.

We will prove that the map $\psi_{m,q}$ is a submersion on $\mathcal{M}(\Gamma, SU(2))$. We compute that the differential of $\psi_{m,q}$ at a class $[\rho]$ is :

$$d_{[\rho]} \psi_{m,q} = \sum_{i=1}^3 \frac{-q_i}{\sin(\theta_i(\rho))} d_{[\rho]} f_{c_i}.$$

Let $i_0$ the first index for which $q_{i_0} \neq 0$. Then for all $[\rho] \in \mathcal{M}(\Gamma, SU(2))$, the lemma 3.4 gives a vector $X \in T_{[\rho]} \mathcal{M}(\Gamma, SU(2))$ such that for $i = 1, 2, 3$

$$d_{[\rho]} f_{c_i} X = \delta_{i_0}^i.$$

Hence, we compute :

$$d_{[\rho]} \psi_{m,q} X = \frac{-q_{i_0}}{\sin(\theta_{i_0}(\rho))} \neq 0.$$

We so conclude that the map $\psi_{m,q}$ is a submersion and then that the set:

$$\left\{ [\rho] \in \mathcal{M}(\Gamma, SU(2)) \mid q_1 \theta_1(\rho) + q_2 \theta_2(\rho) + q_3 \theta_3(\rho) = q_0 \pi \right\}$$

is a submanifold of the character variety with codimension 1 and hence has null measure. It follows that $\mathcal{M}_m(\Gamma, SU(2))$ has full measure as the complement of a countable union of codimension 1 submanifolds. 

Proof of the Lemma 3.4. Up to applying an element of the mapping class group, we can assume that $c_1 = a_1$, $c_2 = a_1 a_2$, $c_3 = a_2$ and that $i_0 = 1$.

The point is to construct a smooth path of representations $\rho_t$ such that $\rho_0 = \rho$ and such that :

$$\frac{d}{dt} \bigg|_{t=0} \text{tr}(\rho_t(a_1)) \neq 0, \frac{d}{dt} \bigg|_{t=0} \text{tr}(\rho_t(a_1 a_2)) = 0, \frac{d}{dt} \bigg|_{t=0} \text{tr}(\rho_t(a_2)) = 0.$$

Up to changing the representative of $\rho$ in its conjugacy class, we assume that $\rho(a_1)$ is the diagonal matrix 

$$\begin{pmatrix} e^{i \theta_1} & 0 \\ 0 & e^{-i \theta_1} \end{pmatrix}.$$

In order to construct the representation $\rho_t$, we only need to specify the paths of matrices $\rho_t(a_i)$ and $\rho_t(b_i)$, for $i = 1, \ldots, g$, such that the equation :

$$\prod_{i=1}^g [\rho_t(a_i), \rho_t(b_i)] = \text{id}$$

in
holds. We let:

\[ \rho_t(a_1) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \rho(a_1) \]

Hence we compute

\[ \frac{d}{dt}|_{t=0} \text{tr}(\rho_t(a_1)) = -2 \sin(\theta_1) \neq 0 \]

because \( \rho(a_1) \neq \pm \text{id} \).

We impose \( \rho_t(a_i) = \rho(a_i) \) and \( \rho_t(b_i) = \rho(b_i) \) for \( i > 2 \) (if \( g > 2 \)). To obtain the other conditions on the differentials of traces, we set:

\[ \rho_t(a_2) = g(t)\rho(a_2)g(t)^{-1}, \rho_t(a_1a_2) = h(t)\rho(a_1)\rho(a_2)h(t)^{-1}, \]

\[ \rho_t(b_1) = B_1(t) \text{ and } \rho_t(b_2) = B_2(t) \]

where \((g(t))_t, (h(t))_t, B_1(t) \text{ and } B_2(t)\) are smooth paths in \( SU(2) \) such that:

\[ \rho_t(a_1a_2) = \rho_t(a_1)\rho_t(a_2) \text{ and } [\rho_t(a_1), B_1(t)][\rho_t(a_2), B_2(t)] \prod_{i=3}^g [\rho_t(a_i), \rho_t(b_i)] = \text{id}. \]

With these conditions, the map \( \rho_t : \{a_1, \ldots, b_g\} \to SU(2) \) extends to a morphism \( \rho_t : \Gamma \to SU(2) \) which is unique. We have to prove the existence of such paths. Let \( K : SU(2)^2 \times SU(2)^2 \times \mathbb{R} \to SU(2) \times SU(2) \) defined by \( K(h, B_1, B_2, t) = \)

\[ \left( h \rho(a_1a_2)^{-1}h^{-1} \rho_t(a_1)g \rho(a_2)g^{-1}, [\rho_t(a_1), B_1][\rho_t(a_2), B_2] \prod_{i=3}^g [\rho_t(a_i), \rho_t(b_i)] \right). \]

Claim 1. If \( \rho : \Gamma \to SU(2) \) has a discrete centralizer, then the map \( K \) is a submersion at the point \((\text{id}, \text{id}, \rho(b_1), \rho(b_2), 0)\).

In the appendix, we prove the claim 1 and we continue the proof of Lemma 3.4 assuming that it is true. Hence the preimage \( K^{-1}(\text{id}, \text{id}) \) is a submanifold of codimension 1 of \( SU(2)^2 \times SU(2)^2 \times \mathbb{R} \). We so find \( \rho_t : \Gamma \to SU(2) \) with the conditions we hoped on the traces. It gives, up to multiply by a constant, a vector \( X \in T_{[\rho]} \mathcal{M}(\Gamma, SU(2)) \) which verifies the conclusion of the lemma 3.4.

### 3.4 Proof of the ergodicity

In order to prove the theorem \( A \) for \( n = 2 \), we will consider a measurable function which is \( \text{Tor}(\Sigma) \)-invariant and proved that, up to restrict it to a full measure subset, it is invariant under the action of enough Dehn twists to be \( \text{Mod}^+(\Sigma) \)-invariant.

Let \( F : \mathcal{M}(\Gamma, SU(2)) \to \mathbb{R} \) be a measurable function and assume that \( F \) is \( \text{Tor}(\Sigma) \)-invariant. Let \( x \in \{a_1, \ldots, b_g, d_1, \ldots, d_{g-1}\} \) be a point of the generating set the mapping class group, see section 2, fix a multicurve \( m_x = x \cup c_2 \cup c_3 \) in \( MC_0(\Sigma) \) and denote \( h \) the product \( T_x T_{c_2} T_{c_3} \).

The set:

\[ \mathcal{M}_{m_x}(\Gamma, SU(2)) \]

has full measure by the proposition 3.1. As the orbits \( U_x \times U_{c_2} \times U_{c_3}, [\rho] \) are tori and then are compact in \( \mathcal{M}(\Gamma, SU(2)) \), the theorem 2.1 insures the existence of a Borel cross section, we will denote by \( \mathcal{S}, \) of \( \mathcal{M}(\Gamma, SU(2)) \) for the action of \( U_x \times U_{c_2} \times U_{c_3} \) we will denote \( T^3 \) to simplify the notations.

Since \( U_x \times U_{c_2} \times U_{c_3} \) and \( \mathcal{M}(\Gamma, SU(2)) \) verify the assumption of the theorem 2.2, then the section \( \mathcal{S} \) is bimeasurable to the quotient \( \mathcal{M}(\Gamma, SU(2))/T^3 \) and the restriction \( \mu \) decompose itself, for all function \( f : \mathcal{M}(\Gamma, SU(2)) \to \mathbb{R}^+ \), by the formula:

\[ \int_{\mathcal{M}(\Gamma, SU(2))} f d\mu = \int_{\mathcal{S}} \left( \int_{T^3} f(t,s)\nu_{T^3}(t)\right) d\nu_{\mathcal{S}}(s) \]

where \( \nu_{T^3} \) is the Haar measure on the tori \( T^3 \) given by theorem 2.2 and \( \nu_{\mathcal{S}} \) a measure on \( \mathcal{S} \) given by the same theorem.
The function $F$ induces a measurable function $\tilde{F}: S \times T^3 \to \mathbb{R}$ defined by:

$$\tilde{F}(s, t) = F(t.s).$$

Fix $[\rho] \in \mathcal{M}_{m_{x}}(\Gamma, SU(2))$ and let $\tilde{F}_{[\rho]}: T^3 \to \mathbb{R}$. Such a function is measurable and invariant by the action of $\langle h \rangle$ since $F$ is. Since $[\rho] \in \mathcal{M}_{m_{x}}(\Gamma, SU(2))$, the action of the translation $\langle h \rangle$ on $T^3$ is ergodic with respect to $\nu_{T^3}$. It implies that $\tilde{F}_{[\rho]}$ is almost everywhere invariant by the Dehn twist $T_x$. Since the Dehn twist $T_x$ acts as a translation of the torus on $U_x \times U_{c_2} \times U_{c_3}.[\rho]$, we deduce that on a full measure subset of $U_x \times U_{c_2} \times U_{c_3}.[\rho]$ the function $F$ and $F \circ T_x$ are equal. This fact is true for almost every $[\rho] \in \mathcal{M}_{m_{x}}(\Gamma, SU(2))$ which has full measure by 3.1. It follows that $\tilde{F}_{[\rho]}$ is almost everywhere invariant by the Dehn twist $T_x$.

We then deduce that on the space

$$\bigcap_{x \in \{a_1, \ldots, b_y, d_1, \ldots, d_{g-1}\}} \mathcal{M}_{m_{x}}(\Gamma, SU(2)),$$

which has full measure in $\mathcal{M}(\Gamma, SU(2))$, the function $F$ is almost everywhere invariant by the Dehn twists $T_x$, for all $x \in \{a_1, \ldots, b_y, d_1, \ldots, d_{g-1}\}$. It implies that $F$ is almost everywhere invariant under the action of $\text{Mod}^+(\Sigma)$, which is known to be ergodic since the theorem 1.1. Hence $F$ is almost everywhere constant and this proves the ergodicity of the Torelli group action on $\mathcal{M}(\Gamma, SU(2))$.

4. Ergodicity for $G = SU(n)$

This part is devoted to the proof of the theorem A in the general case. We will use the same strategy than the previous section, adapting the tools. Let us introduce some notations we will use.

4.1. Torus actions on $\mathcal{M}^{\alpha\text{-reg}}(\Gamma, SU(n))$. For a matrix $A \in SU(n)$ with distinct eigenvalues, denote $\lambda_1(A), \ldots, \lambda_n(A)$ its eigenvalues and $\theta_1(A), \ldots, \theta_n(A)$ their arguments we can express by up to the sign:

$$\cos^{-1}\left(\frac{\lambda_i(A) + \lambda_i(A)^{-1}}{2}\right)$$

with the normalisation $0 \leq \theta_1(A) \leq \ldots \leq \theta_n(A) < 2\pi$. The group $SU(n)$ has rank $n - 1$ and every maximal torus is conjugated to the group:

$$\left\{ \begin{pmatrix} e^{i\theta_1} & 0 & \ldots & 0 \\ 0 & e^{i\theta_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{i\theta_n} \end{pmatrix} : (\theta_1, \ldots, \theta_n) \in [0, 2\pi[ \text{ and } \prod_{k=1}^{n} e^{i\theta_k} = 1 \right\}$$

which is isomorphic to $T^{n-1}$.

A matrix $A \in SU(n)$ is regular if its eigenvalues are simple. Similarly, for a curve $\alpha$, a character $[\rho] \in \mathcal{M}(\Gamma, SU(n))$ is $\alpha$-regular if the matrix $\rho(\alpha)$ is regular. We will denote $\mathcal{M}^{\alpha\text{-reg}}(\Gamma, SU(n))$ the subsets of $\alpha$-regular $SU(n)$-characters. For all curve $\alpha$, the subset $\mathcal{M}^{\alpha\text{-reg}}(\Gamma, SU(n))$ is an open subset of $\mathcal{M}(\Gamma, SU(n))$ and has full measure in it (see subsection 4.2).

We will define actions of a $(n-1)$-torus $T^{n-1} = (S^1)^{n-1}$ on the character variety $\mathcal{M}(\Gamma, SU(n))$. Let $z = (z_1, \ldots, z_{n-1}) \in T^{n-1}$ and $h_z$, the associated diagonal matrix:

$$\text{diag}(z_1, \ldots, z_{n-1}, \frac{1}{z_1 \cdots z_{n-1}})$$

in $SU(n)$.

Let $A \in SU(n)$ be a regular matrix. There exists a unique decomposition $[e_1] \oplus \cdots \oplus [e_n]$ of $\mathbb{C}^n$ in lines such that:

$$Ae_i = \lambda_i(A)e_i$$

for all $i \in \{1, \ldots, n\}$. 


Let \( \alpha \) be a simple and closed curve and let \([\rho] \in \mathcal{M}^{\alpha\text{-reg}}(\Gamma, \text{SU}(n))\). With the same notations than the section 3 and in a basis of \( \mathbb{C}^n \) in which \( \rho(\alpha) = h(\lambda_1(\rho(\alpha)), \ldots, \lambda_{n-1}(\rho(\alpha))) \), we define for \( z \in \mathbb{T}^{n-1} \), if \( \alpha \) is non-separating, the representation \( z \cdot \rho \) by:

\[
z \cdot \rho(\gamma) = \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma) \alpha \\
 h_z \rho(\beta) & \text{if } \gamma = \beta 
\end{cases}
\]

and if \( \alpha \) is separating by:

\[
z \cdot \rho(\gamma) = \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_1) \\
h_z \rho(\gamma) h_z^{-1} & \text{if } \gamma \in \pi_1(\Sigma_2) 
\end{cases}.
\]

It then defines an action, which depends hence of \( \alpha \), of the torus \( U_\alpha := \mathbb{T}^{n-1} \) on the subspace of \( \alpha \)-regular characters. We hence remark that the action of Dehn twists along \( \alpha \) on \( \mathcal{M}^{\alpha\text{-reg}}(\Gamma, \text{SU}(n)) \) is given by:

\[
T_\alpha[\rho] = \lambda(\rho(\alpha)) \cdot [\rho]
\]

where \( \lambda(\rho(\alpha)) = (\lambda_1(\rho(\alpha)), \ldots, \lambda_{n-1}(\rho(\alpha))) \).

A direct computation shows the essential fact that if \( \alpha \) and \( \beta \) are disjoint curves, then the actions of the tori \( U_\alpha \) and \( U_{\beta} \) on \( \mathcal{M}^{\alpha\text{-reg}}(\Gamma, \text{SU}(n)) \cap \mathcal{M}^{\beta\text{-reg}}(\Gamma, \text{SU}(n)) \) commute. It hence defines an action of \( U_\alpha \times U_{\beta} \) on the space of \( \alpha \)-regular and \( \beta \)-regular characters.

We so can state an analogue of the lemma 3.2:

**Lemma 4.1.** Let \([\rho] \in \mathcal{M}(\Gamma, \text{SU}(n))\) and suppose that there exist \( c_1, \ldots, c_\ell \) be pairwise disjoints, simple and closed curves of \( \Sigma \) such that:

\[
\theta_1(\rho(c_1)), \ldots, \theta_{n-1}(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_{n-1}(\rho(c_\ell)), \pi
\]

are linearly independent over \( \mathbb{Q} \). If we denote \( h = T_{c_1} \cdots T_{c_\ell} \), then the action of \( h \) on the orbit \( U_{c_1} \times \cdots \times U_{c_\ell}[\rho] \) is ergodic with respect to the Lebesgue measure on this torus orbit.

**Proof.** As for the proof of the lemma 3.2, the action of the Dehn twists \( T_{c_i} \) is an action by translation on the torus. The flows commute on the character variety because the curves are disjoint and the orbit is given by the formula:

\[
h^k.[\rho] = (\lambda_1(\rho(c_1))^k, \ldots, \lambda_{n-1}(\rho(c_\ell))^k, \ldots, \lambda_1(\rho(c_\ell))^k, \ldots, \lambda_{n-1}(\rho(c_\ell))^k).[\rho].
\]

Hence the condition on the \( \theta_i(\rho(c_\ell)) \) implies the expected ergodicity by the lemma 3.3. \( \Box \)

### 4.2. A full measure set and the ergodicty

We define in this section a full measure subspace of the character variety with the conditions of the previous lemma and conclude with the ergodicity of the Torelli group action.

**Definition 4.1.** Let \( m = c_1 \cup \cdots \cup c_\ell \) be a multicurve of simple closed and non-separating curves. A class of representation \([\rho] \in \mathcal{M}(\Gamma, \text{SU}(n))\) verifies the condition \((M_m)\) if:

\[
\theta_1(\rho(c_1)), \ldots, \theta_{n-1}(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_{n-1}(\rho(c_\ell)), \pi
\]

are linearly independent over \( \mathbb{Q} \).

**Remark 3.** If a class of representation \([\rho]\) verifies the condition \((M_m)\) for some \( m = c_1 \cup \cdots \cup c_\ell \in MC(\Sigma) \) then \([\rho]\) is \( c_i \)-regular for all \( i \in \{1, \ldots, \ell\} \).

To simplify the notations, for a multicurve \( m = c_1 \cup \cdots \cup c_\ell \), we will denote by \( \mathcal{M}^{m-\text{reg}}(\Gamma, \text{SU}(n)) \) the intersection:

\[
\bigcap_{i=1}^\ell \mathcal{M}^{c_i-\text{reg}}(\Gamma, \text{SU}(n))
\]

which has full measure.

We define, for \( m \in MC_0(\Sigma) \), the set:

\[
\mathcal{M}_m(\Gamma, \text{SU}(n)) = \{ [\rho] \in \mathcal{M}(\Gamma, \text{SU}(n)) \mid [\rho] \text { satisfies the condition } (M_m) \}.
\]

Remark 3 assures that \( \mathcal{M}_m(\Gamma, \text{SU}(n)) \) is contained in \( \mathcal{M}^{m-\text{reg}}(\Gamma, \text{SU}(n)) \).

**Proposition 4.1.** For all \( m \in MC_0(\Sigma) \), the set \( \mathcal{M}_m(\Gamma, \text{SU}(n)) \) has full measure in the character variety.
We can write its complement as the set:

\[ \bigcup_{q=(q_1',\ldots,q_{n-1}',q_1',\ldots,q_{n-1})' \in \mathbb{Z}^{(n-1)} \setminus \{0\}, q_0 \in \mathbb{Z}} \left\{ [\rho] \in \mathcal{M}(\Gamma, \text{SU}(n)) \mid \sum_{k=1}^{3} \sum_{i=1}^{n-1} q_k^i \theta_i(\rho(c_k)) = q_0 \pi \right\}. \]

We will show that each set of the previous union is a codimension 1 submanifold and hence show that the union has null measure.

Let \( \psi_{m,q} \) be the function \( \mathcal{M}^{m-\text{reg}}(\Gamma, \text{SU}(n)) \to \mathbb{R} \) defined by the formula:

\[ \psi_{m,q}([\rho]) = \sum_{k=1}^{3} \sum_{i=1}^{n-1} q_k^i \theta_i(\rho(c_k)). \]

**Lemma 4.2.** The function \( \psi_{m,q} \) is a submersion.

Denote \( \pi_k^i : [\rho] \mapsto \lambda_i(\rho(c_k)) + \lambda_i(\rho(c_k))^{-1} \) so that \( \theta_i(\rho(c_k)) = \arccos(\frac{\pi_k^i([\rho])}{2}) \) up to the sign. We then compute the differential of \( \psi_{m,q} \):

\[ d_{[\rho]} \psi_{m,q} = \sum_{k=1}^{3} \sum_{i=1}^{n-1} \frac{q_k^i}{2 \sin(\theta_i(\rho(c_k)))} d_{[\rho]} \pi_k^i. \]

**Lemma 4.3.** Let \([\rho]\) be a class of representations in \( \mathcal{M}(\Gamma, \text{SU}(n)) \), \( m = c_1 \cup c_2 \cup c_3 \) be a pant such that \( \rho(c_i) \) does not have \( \pm 1 \) as eigenvalues and let \( i_0 \in \{1,\ldots,n-1\} \), \( k_0 \in \{1,2,3\} \). Then there exists a vector \( X \in T_{[\rho]} \mathcal{M}(\Gamma, \text{SU}(n)) \) such that for \((i,k) \in \{1,\ldots,n-1\} \times \{1,2,3\} \),

\[ d_{[\rho]} \pi_{i_0}^{c_k} X = \delta_{i_0}^{i_1} \delta_k^{k_0}. \]

**Proof of lemma 4.3.** We are then looking for \( \rho_t \) approaching \( \rho \) such that for all \((j,k) \neq (i_0,k_0) \):

\[ \frac{d}{dt} \big|_{t=0} \pi_j^k(\rho_t) = 0 \]

and

\[ \frac{d}{dt} \big|_{t=0} \pi_{i_0}^{c_k}(\rho_t) \neq 0. \]

The strategy to find this family of representations is the same than in \( \text{SU}(2) \). Up to applying an element of the mapping class group, we assume \( c_1 = a_1, c_2 = a_1a_2, c_3 = a_2 \) and \( k_0 = 1 \). Up to changing the representative of \( \rho \) in its conjugacy class, we assume that

\[ \rho(a_1) = \begin{pmatrix} e^{i\theta_1} & 0 & \ldots & 0 \\ 0 & e^{i\theta_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{i\theta_n} \end{pmatrix} \text{ with } \prod_{k=1}^{n} e^{i\theta_k} = 1. \]

Multiply \( \rho(a_1) \) by the diagonal matrix \( h_{z_1} \) with \( z_i = (z_{i,i})_{i=1,\ldots,n-1} \) for \( z_{i,i_0} = e^{it} \) and \( z_{i,k} = 1 \) otherwise. As previously, we impose \( \rho_t(a_1) = h_t \rho(a_1), \rho_t(a_2) = \rho(a_1) \rho(a_2) h(t)^{-1} \) and \( \rho_t(b_1) = B_1(t) \) and \( \rho_t(b_2) = B_2(t) \) for paths \( g(t), h(t), B_1(t), B_2(t) \in \text{SU}(n) \) which verify:

\[ \rho_t(a_1 a_2) = \rho_t(a_1) \rho_t(a_2) \text{ and } \prod_{i=1}^{g} [\rho_t(a_i), \rho_t(b_i)] = \text{id}. \]

With such conditions, the map \( \rho_t : \{a_1, \ldots, b_g\} \to \text{SU}(n) \) extends to a unique morphism \( \rho_t : \Gamma \to \text{SU}(n) \). In particular, if such paths exist, we compute:

\[ \frac{d}{dt} \big|_{t=0} \pi_{i_0}^{c_k}(\rho_t) = -2 \sin(\theta_{i_0}(\rho(a_1))) \neq 0 \]
since $\rho(a_1)$ does not have $\pm 1$ as eigenvalue. For such paths the conditions \( \frac{d}{dt}|_{t=0} \pi_j^k(\rho_t) = 0 \) are clearly verified since we conjugate $\rho(a_2), \rho(a_1a_2)$ by matrices, which does not change the eigenvalues.

To find such paths, we let the map $K : SU(n)^2 \times SU(n)^2 \times \mathbb{R} \to SU(n) \times SU(n)$ defined by $K(g, h, B_1, B_2, t) =$

\[
(h \rho(a_1a_2)^{-1}h^{-1}\rho_i(a_1)g\rho(a_2)g^{-1}, \rho(a_1), B_1)[g\rho(a_2)g^{-1}, B_2] \prod_{i=3}^g \rho(a_i), \rho(b_i)) \).
\]

**Claim 2.** If $\rho : \Gamma \to SU(n)$ has a discrete centralizer, then the map $K$ is a submersion at the point $(id, id, \rho(b_1), \rho(b_2), 0)$.

Assuming the claim 2 which is proved in appendix in section 6, we hence conclude the existence of the path $\rho_t$ and then to the existence of a vector $X \in T_{(\rho)}\mathcal{M}(\Gamma, SU(n))$ which verifies:

\[
d_{(\rho)}\pi_i^kX = \delta_i^{k_0} \delta_{k_0}^0.
\]

We now prove the Lemma 4.2. Let $i_0 \in \{1, \ldots, n-1\}$ and $k_0 \in \{1, 2, 3\}$ the first index such that $q_{i_0}^{k_0}$ is non-zero. Let $X$ be the vector field associated to the index $i_0$ and $k_0$ in the lemma 4.3. We then have that

\[
d_{(\rho)}\psi_{m,q}X = -\frac{q_{i_0}^{k_0}}{2\sin(\theta_i(\rho(c_k))))} d_{(\rho)}\pi_i^{k_0}X \neq 0
\]

which proves that $\psi_{m,q}$ is a submersion. It proves the lemma 4.2 and then the proposition 4.1.

The proof of the ergodicity use the same arguments than the case of SU(2). Let $F : \mathcal{M}(\Gamma, SU(n)) \to \mathbb{R}$ be a measurable and Tor($\Sigma$)-invariant function. For each curve $x$ in the set \( \{a_1, \ldots, b_g, d_1, \ldots, d_{g-1}\} \), we fix a multicurve $m_x = x \cup c_2 \cup c_3$.

Replacing the torus $T^3$ used for the case SU(2) by the torus $T^{3(n-1)}$, we conclude by the same methods that on the space

\[
\bigcap_{x \in \{a_1, \ldots, b_g, d_1, \ldots, d_{g-1}\}} \mathcal{M}_{m_x}(\Gamma, SU(n)),
\]

which has full measure the proposition 4.1, the function $F$ is almost everywhere invariant by the Dehn twists $T_x$ for all $x \in \{a_1, \ldots, b_g, d_1, \ldots, d_{g-1}\}$. It implies that it is almost everywhere invariant by the mapping class group. The theorem 1.2 shows that $F$ is constant on a full measure subset of $\mathcal{M}(\Gamma, SU(n))$. This proves the ergodicity of the Torelli group action on $\mathcal{M}(\Gamma, SU(n))$.

5. ERGODICITY FOR THE GENERAL CASES OF SEMI-SIMPLE, CONNECTED AND COMPACT LIE GROUPS

In this section we generalize the proofs of the ergodicity of the Torelli group on character varieties with values in a semi-simple, connected and compact Lie group. We will use the same strategy than the compact Lie groups SU(n) but need to replace the tools we used by their appropriate analogues in a more general case.

5.1. Preliminaries on compact Lie group theory. Let $G$ be a semi-simple, connected and compact Lie group with Lie algebra $\mathfrak{g}$. A maximal torus is a connected and abelian subgroup of $G$ which is maximal for these properties. Such a subgroup exists and fix $T < G$ be a maximal torus. Let $\mathfrak{t}$ its Lie algebra. It is an abelian subalgebra of $\mathfrak{g}$. The subgroup $T$ is isomorphic to a $r$-dimensional torus $T^r$ and its Lie algebra $\mathfrak{t}$ is isomorphic to the commutative Lie algebra $\mathbb{R}^r$. We will so use the existence of coordinates on $\mathfrak{t}$ via this isomorphism. Precisely, for $i \in \{1, \ldots, r\}$ and $t \in T$, we denote by $\lambda_i(t)$ the projection on the $i$-th factor of $t \in T \cong T^r$. 
It is well known that every element of $G$ is contained in a maximal torus. We have however the more precise result (see [4] for more details):

**Theorem 5.1.** Every $k \in G$ is conjugated to an element of $T$. Moreover, all the maximal tori are conjugated and hence are isomorphic to $T^r$.

Remark that a maximal torus can contain two conjugated elements. The integer $r$ is called the rank of the group $G$. The Weyl group associated to $T$ is the group $N_G(T)/T$, where the subgroup $N_G(T)$ is the normalizer of $T$ in $G$.

**Proposition 5.1.** ([3]) The Weyl group is finite.

A weight of $T$ is a real and irreducible representation. Let $\omega$ be a weight of $T$ and $\sigma : G \rightarrow \text{Aut}(V)$ be a representation. The sum of all invariant subspaces of $\sigma|_T$ isomorphic to $\omega$ is called the weight space associated to $\omega$ of $\sigma$. Define, for $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r$, the linear form:

$$\Theta_n^* : t \mapsto \mathbb{R} (x_1, \ldots, x_r) \mapsto n_1x_1 + \cdots + n_rx_r$$

where we use the coordinates on $t$ given by the isomorphism $t \cong \mathbb{R}^r$ coming from $T \cong (\mathbb{R}/\mathbb{Z})^r$.

It is then well known that the weights of $T$ are either the trivial one-dimensional representation or the representations $\Theta_n : T \cong \mathbb{R}^r \rightarrow SO_2(\mathbb{R})$, for $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r \setminus \{0\}$, defined by:

$$\Theta_n([x_1, \ldots, x_r]) = \left(\cos(2\pi \Theta_n^*(x_1, \ldots, x_r)), \sin(2\pi \Theta_n^*(x_1, \ldots, x_r))\right) - \sin(2\pi \Theta_n^*(x_1, \ldots, x_r)), \cos(2\pi \Theta_n^*(x_1, \ldots, x_r))\right).$$

**Definition 5.1.** A linear form $\alpha \in \mathfrak{t}^*$ is a root of $G$ if there exists $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r \setminus \{0\}$ such that $\alpha = \Theta_n^*$ and that the weight space of the adjoint representation $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ associated to $\Theta_n$ is non-trivial.

We denote by $\Delta$ the set of roots of $G$.

Since $G$ is a semi-simple Lie group, the Killing form $\langle ., . \rangle$ is a scalar product and the subspace $\mathfrak{t} < \mathfrak{g}$ becomes a Euclidean space. Using the induced isomorphism $t \cong t^*$, we can see $\Delta$ as a subset of $t$ and for $\alpha \in \Delta$, we define the reflection:

$$r_\alpha : \beta \mapsto \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

It is well known that the Weyl group associated to $T$ is isomorphic to the subgroup of $\text{GL}(t)$:

$$\langle r_\alpha | \alpha \in \Delta \rangle.$$

The alcoves of $t$ are the connected components of

$$t \setminus \bigcup_{\alpha \in \Delta, n \in \mathbb{N}} \ker(r_\alpha - n \text{id}).$$

The Weyl group acts simply transitively on the images in $T$ of alcoves.

**Definition 5.2.** An element $k \in G$ is regular if it is contained in a unique maximal torus.

Let $M$ be the image by the exponential map $t \rightarrow T$ of an alcove of $t$, we will say such a $M$ is an alcove of $T$ and let $k \in G$ be a regular element. There exists a unique class $\overline{g_k}$ of $G/Z_G(k)$, with $Z_G(k)$ the centralizer of $k$ in $G$, such that:

$$g_kkg_k^{-1} \in M.$$

**Example 1.** For $G = \text{SU}(n)$ and $T$ the set of diagonal matrices, the roots are given by $\lambda_i - \lambda_k$ for $i, k \in \{1, \ldots, n\}$ such that $i \neq k$ and where the $\lambda_i$ are the eigenvalues. The Weyl group is then the symmetric group $\mathfrak{S}_n$. 
5.2. Density of some orbits. Let $G$ be a semi-simple, connected and compact Lie group of rank $r$, let $T$ be a maximal torus and $M$ be an alcove of $T$.

Let $\alpha \in \Gamma$ be a simple curve. A character $[\rho] \in \mathcal{M}(\Gamma, G)$ is $\alpha$-regular if $\rho(\alpha)$ is regular. Then there exists a unique class $\overline{g_{\rho(\alpha)}}$ in the quotient $G/Z_G(\rho(\alpha))$, such that:

$$g_{\rho(\alpha)}\rho(\alpha)g_{\rho(\alpha)}^{-1} \in M.$$  

The set of $\alpha$-regular characters is an open subset of $\mathcal{M}(\Gamma, G)$ and has full measure. The maximal torus $T$ acts on the space $\text{Hom}^{\alpha-\text{reg}}(\Gamma, G)$ of $\alpha$-regular representations via the action, defined if $\alpha$ is non-separating, for $t \in T$ and $\rho \in \text{Hom}^{\alpha-\text{reg}}(\Gamma, G)$, by:

$$t \cdot \rho(\gamma) = \begin{cases} g_{\rho(\alpha)}\rho(\gamma)g_{\rho(\alpha)}^{-1} & \text{if } \gamma \in \pi_1(\Sigma_{\alpha}) \\ tg_{\rho(\alpha)}\rho(\beta)g_{\rho(\alpha)}^{-1} & \text{if } \gamma = \beta \end{cases}$$

and if $\alpha$ is separating by:

$$t \cdot \rho(\gamma) = \begin{cases} g_{\rho(\alpha)}\rho(\gamma)g_{\rho(\alpha)}^{-1} & \text{if } \gamma \in \pi_1(\Sigma_1) \\ tg_{\rho(\alpha)}\rho(\gamma)g_{\rho(\alpha)}^{-1}t^{-1} & \text{if } \gamma \in \pi_1(\Sigma_2). \end{cases}$$

we use the same notations than the sections 3 and 4. Since this action commutes with the conjugation action of $G$ on the representations, we defined an action, which only depends of the curve $\alpha$, of the maximal torus $U_{\alpha} := T$ on the $\alpha$-regular characters. As for the case of $\text{SU}(n)$, we have an expression of the Dehn twist $T_{\alpha}$, along $\alpha$, action on the subspace of $\alpha$-regular characters via the formula:

$$T_{\alpha} \cdot [\rho] = (g_{\rho(\alpha)}\rho(\alpha)g_{\rho(\alpha)}^{-1}) \cdot [\rho].$$

We then precise the important fact that for two disjoint curves $\alpha$ and $\beta$, the actions of the maximal tori $U_{\alpha}$ and $U_{\beta}$ on $\mathcal{M}^{\alpha-\text{reg}}(\Gamma, G) \cap \mathcal{M}^{\beta-\text{reg}}(\Gamma, G)$ commute. It hence implies an action of the product $U_{\alpha} \times U_{\beta}$ on the previous intersection. We define the map $t_{\alpha} : \text{Hom}^{\alpha-\text{reg}}(\Gamma, G) \rightarrow M$ by:

$$t_{\alpha}(\rho) = g_{\rho(\alpha)}\rho(\alpha)g_{\rho(\alpha)}^{-1}.$$  

For every $i \in \{1,\ldots,r\}$, the projection $\lambda_i$ induces a function, we denote by $\lambda_{i,\alpha}$, on the space $\text{Hom}^{\alpha-\text{reg}}(\Gamma, G)$.

If we conjugate $\rho$ by $g \in G$, we obtain by uniqueness of $g_{\rho(\alpha)}$ up to the centralizer of $\rho(\alpha)$, that there exists $z \in Z_G(\rho(\alpha))$ such that:

$$g_{g\rho(\alpha)}g^{-1} = g_{\rho(\alpha)}z$$

and hence we obtain that $t_{\alpha}$ is invariant under the conjugation action of $G$ on the representation variety $\text{Hom}^{\alpha-\text{reg}}(\Gamma, G)$ and descends to a map

$$\mathcal{M}^{\alpha-\text{reg}}(\Gamma, G) \rightarrow M,$$

we will denote $t_{\alpha}$ again.

An element $k \in G$ is said generic if $\langle g_k kg_k^{-1} \rangle$ is dense in $T$.

Claim 3. An element $k \in G$ is generic if and only if for all non-trivial character $\chi : T \rightarrow S^1$, $\chi(g_k kg_k^{-1}) \neq 1$.

Proof. Let $\phi : T \rightarrow T^r$ be the isomorphism we mentioned. Then the map $\chi(\phi^{-1})$ is a non-trivial character of the torus $T^r$. Since a character of $T^r$ is induced by a linear form of $\mathbb{R}^r$ which preserved $2\pi \mathbb{Z}^r$, it as the form $(x_1, \ldots, x_r) \mapsto n_1 x_1 + \cdots + n_r x_r$, with the $n_i \in \mathbb{Z}$ are not all zero. An element $k \in T$ is generic if and only if $\phi(k)$ is generic, that mean if it generates a dense subgroup in $T^r$. We then have that $\phi(k)$ is generic if and only if $\chi(\phi^{-1})(\phi(k)) \neq 1$. □

In particular, a generic element of $k$ is regular.

Lemma 5.1. Let $\rho : \Gamma \rightarrow G$ be a representation and $\alpha$ be a simple close curve such that $\rho(\alpha)$ is generic. Then the orbit

$$\langle T_{\alpha} \rangle \cdot [\rho]$$

is dense in the torus orbit $U_{\alpha} \cdot [\rho]$. Moreover the action of $T_{\alpha}$ on $U_{\alpha} \cdot [\rho]$ is ergodic with respect to the Lebesgue measure.
Definition 5.3. Let \( m = c_1 \cup \cdots \cup c_\ell \) be a multicurve. A class of representation \([\rho] \in \mathcal{M}(\Gamma, G)\) is \( m \)-regular if the elements \( \rho(c_1), \ldots, \rho(c_\ell) \) are regular and we denote by \( \mathcal{M}^{m-\text{reg}}(\Gamma, G) \) the set of \( m \)-regular characters. Precisely we have:

\[
\mathcal{M}^{m-\text{reg}}(\Gamma, G) = \bigcap_{i=1}^\ell \mathcal{M}^{c_i-\text{reg}}(\Gamma, G).
\]

Since the curves \( c_1, \ldots, c_\ell \) are disjoint, the action of the tori \( U_{c_1}, \ldots, U_{c_\ell} \) on commute and the orbits are the torus orbits \( U_{c_1} \times \cdots \times U_{c_\ell} \cdot [\rho] \).

In this general setting and similarly to the lemmas 3.2 and 4.1, we state:

Lemma 5.2. Let \([\rho] \in \mathcal{M}(\Gamma, G)\) and suppose that there exist \( c_1, \ldots, c_\ell \) be pairwise disjoint, simple and closed curves of \( \Sigma \) such that for all non-trivial character \( \chi : T^\ell \to S^1 \):

\[
\chi(t_{c_1}([\rho]), \ldots, t_{c_\ell}([\rho])) \neq 1.
\]

Then, if we denote \( h = T_{c_1} \cdots T_{c_\ell} \), then the action of \( \langle h \rangle \) on \( U_{c_1} \times \cdots \times U_{c_\ell} \cdot [\rho] \) is ergodic with respect to the Lebesgue measure.

To simplify the notations we denote by \( T^\ell \) the product \( U_{c_1} \times \cdots \times U_{c_\ell} \).

Proof of the lemma 5.2. Let \( \phi \) be the isomorphism \( T \cong T^\ell \) and let \( \chi : T^\ell \to S^1 \) be a non-trivial character. Then the composition \( \chi \circ (\phi^{-1}, \ldots, \phi^{-1}) \) is a non-trivial character of \( T^\ell \), we identify with \( \mathbb{R}^\ell / \mathbb{Z}^\ell \). The action of \( h \) is then given by the translation of vector

\[
(\theta_1(\rho(c_1)), \ldots, \theta_\ell(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_\ell(\rho(c_\ell)))
\]

where \( \theta_i(\rho(c_k)) \) is the argument of \( \lambda_i(\rho(c_k)) \). Then, by the lemma 3.3, the action of \( h \) is ergodic on the torus orbit \( U_{c_1} \times \cdots \times U_{c_\ell} \cdot [\rho] \) with respect to the Lebesgue measure if and only if

\[
\theta_1(\rho(c_1)), \ldots, \theta_\ell(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_\ell(\rho(c_\ell)) \text{ and } 1
\]

are linearly independent over \( \mathbb{Q} \). As a character of a torus is given by a linear form of \( \mathbb{R}^\ell \) with integer coefficient, this condition is equivalent that for all non-trivial character \( \chi' \) of \( T^\ell \),

\[
\chi'(\theta_1(\rho(c_1)), \ldots, \theta_\ell(\rho(c_1)), \ldots, \theta_1(\rho(c_\ell)), \ldots, \theta_\ell(\rho(c_\ell))) \neq 1.
\]

The hypothesis allows then to conclude. \( \square \)

5.3. Proof of the ergodicity. We will adapt the previous proofs of ergodicity of sections 3 and 4 with the condition of the lemma 5.2.

Definition 5.4. A class of representation \([\rho] \in \mathcal{M}(\Gamma, G)\) verifies the condition \((M_m)\) if for all non-trivial character \( \chi : T^\ell \to S^1 \),

\[
\chi(t_{c_1}([\rho]), \ldots, t_{c_\ell}([\rho])) \neq 1.
\]

We introduce:

\[
\mathcal{M}_m(\Gamma, G) = \{ [\rho] \in \mathcal{M}(\Gamma, G) \mid [\rho] \text{ satisfies the condition } (M_m) \}.
\]

Remark 4. The set \( \mathcal{M}_m(\Gamma, G) \) is contained in \( \mathcal{M}^{m-\text{reg}}(\Gamma, G) \).

We hence prove the following:

Proposition 5.2. For all \( m \in MC_0(\Sigma) \), the space \( \mathcal{M}_m(\Gamma, G) \) has full measure in the character variety.

As in the previous cases, we prove that the set we introduced is the complement of a countable union of submanifold of codimension 1.

The strategy we use is the same than the propositions 3.1 and 4.1. Let \( m = c_1 \cup c_2 \cup c_3 \in MC_0(\Sigma) \), we write the complement of the set of characters which verify the condition \((M_m)\) by the union:
We will hence prove that for all non-trivial character $\chi : T^3 \to S^1$, the set

$$\left\{ [\rho] \in \mathcal{M}(\Gamma, G) \mid \chi(t_{c_1}([\rho]), t_{c_2}([\rho]), t_{c_3}([\rho])) = 1 \right\}$$

has null measure as a preimage of 1 by the map $\psi_{\chi,m} = \chi(t_{c_1}(\cdot), t_{c_2}(\cdot), t_{c_3}(\cdot))$, defined on $\mathcal{M}^{m-\text{reg}}(\Gamma, G)$, we will prove to be a submersion. It is the goal of the following lemma.

**Lemma 5.3.** For all non-trivial character $\chi : T^3 \to S^1$, the map $\psi_{\chi,m}$ is a submersion.

**Proof.** It suffices, for $[\rho] \in \mathcal{M}(\Gamma, G)$, to find a vector $X \in T_{[\rho]} \mathcal{M}(\Gamma, G)$ such that:

$$d_{[\rho]} \psi_{\chi,m} X \neq 0.$$  

Write:

$$d\psi_{\chi,m} X = d\chi(dt_{c_1} X, dt_{c_2} X, dt_{c_3} X)$$

where, for $\phi$ be the isomorphism $T \cong T^\tau$ we use in the proof of the lemma 5.2 and $d_{[\rho]}t_{c_k} X$ be:

$$d\phi^{-1}_{\rho(t_{c_k}([\rho]))} \left( \frac{d}{dt}_{|t=0} \lambda_1(\rho_t(c_k)), \ldots, \frac{d}{dt}_{|t=0} \lambda_r(\rho_t(c_k)) \right),$$

where $(\rho_t)_t$ is the path tangent to $X$.

Let $k_0 \in \{1,2,3\}$ and $i_0 \in \{1, \ldots, r\}$ such that $d\chi e_{i_0} \neq 0$, with $(e_i)_i$ denote the canonical basis of the $k_0$-th copy of $\mathbb{R}^r$. Such integers exist because $\chi$ is not the trivial character.

We are looking for $\rho_t$ approaching $\rho$ such that for all $(j,k) \neq (i_0, k_0)$:

$$\frac{d}{dt}_{|t=0} \lambda_j(\rho_t(c_k)) = 0 \quad \text{and} \quad \frac{d}{dt}_{|t=0} \lambda_{i_0}(\rho_t(c_{k_0})) \neq 0.$$

We assume that $c_1 = a_1, c_2 = a_1 a_2, c_3 = a_2$ and $k_0 = 1$. We multiply $\rho(a_1)$ by the element $u_t \in M$ which corresponds, by $T \cong T^\tau$, to the vector $(1, \ldots, e^t, \ldots, 1)$ of $T^\tau$ (with $e^t$ in $i_0$-th position) and we impose $\rho_t(a_1) = \rho(a_1)$ and $\rho_t(b_i) = \rho(b_i)$ when $i > 2$ (in genus $g \geq 2$) and define $\rho_t(a_1) = u_t \rho(a_1)$, $\rho_t(a_2) = g(t) \rho(a_2) g(t)^{-1}$, $\rho_t(b_1) = B_1(t)$, $\rho_t(b_2) = B_2(t)$ and $\rho_t(a_1 a_2) = h(t) \rho(a_1 a_2) h(t)^{-1}$ for smooth paths $g(t), h(t), B_1(t), B_2(t) \in G$ such that:

$$\rho_t(a_1 a_2) = \rho(a_1) \rho(a_2) \text{ and } \prod_{i=1}^{g} [\rho_t(a_i), \rho_t(b_i)] = 1.$$

Let the map $K : G^2 \times G^2 \times \mathbb{R} \to G \times G$ defined by $K(g, h, B_1, B_2, t) =$

$$\left( h \rho(a_1 a_2)^{-1} h^{-1} \rho(a_1) [g \rho(a_2) g^{-1}, [\rho(a_1), B_1] g \rho(a_2) g^{-1}], B_2 \prod_{i=3}^{g} [\rho(a_i), \rho(b_i)] \right).$$

**Claim 4.** If $\rho : \Gamma \to G$ has a discrete centralizer, then the map $K$ is a submersion at the point $(\text{id}, \text{id}, \rho(b_1), \rho(b_2), 0)$.

The claim 4 allows to find a path $(\rho_t)_t$ of representations and then a vector $X \in T_{[\rho]} \mathcal{M}(\Gamma, G)$ such that:

$$d_{[\rho]} \psi_{\chi,m} X \neq 0. \quad \square$$

The set of characters $T^3 \to S^1$ is countable because such a character is given by a linear form:

$$\tilde{\chi} : \mathbb{R}^{3r} \to \mathbb{R},$$

such that $\tilde{\chi}(\mathbb{Z}^{3r}) \subset \mathbb{Z}$. Hence there is a countable number of possibilities to obtain characters of the $3r$-torus, looking the image by $\tilde{\chi}$ of the canonical basis.

Since the complement of $\mathcal{M}_m(\Gamma, G)$ is a countable union of codimension 1 submanifolds and hence a countable union of null measure sets, we conclude to the proposition 5.2.
We conclude the proof of the ergodicity with the same arguments than the previous cases. We prove then that all Tor(Σ)-invariant and measurable function $\mathcal{M}(\Gamma, G) \to \mathbb{R}$ can be restricted to a full measure set on which it will be invariant under the generators of the mapping class group and then the theorem 1.2 allows to conclude that such a function is almost everywhere constant, that is the Torelli group action on $\mathcal{M}(\Gamma, G)$ is ergodic.

Appendix A. Appendix : Fox calculus and proof of the main claims

A.1. Fox Calculus. This appendix consists in giving the tools to do differential calculus on words of free groups. A derivation of a finitely-generated free group $A$. Fox Calculus.

Almost everywhere constant, that is the Torelli group action on mapping class group and then the theorem 1.2 allows to conclude that such a function is restricted to a full measure set on which it will be invariant under the generators of the Torelli group action on $\mathcal{M}(\Gamma, G)$ is ergodic.

Example 2. $[7]$ The set of derivations of $\mathcal{M}(\Gamma, G)$ is a submersion at the point $\rho \in \mathcal{M}(\Gamma, G)$ defined by $\mathcal{M}(\Gamma, G)$ is a submersion at the point $\rho \in \mathcal{M}(\Gamma, G)$.

We prove then that all elements of $G$ we constructed in section 5. We have to prove that the map $K : G^4 \times \mathbb{R} \to G \times G$ defined by $K(g, h, B_1, B_2, t) = \left(h\rho(a_1a_2)^{-1}h^{-1}\rho_t(a_1)g \rho(a_2)g^{-1}, [\rho_t(a_1), B_1][g \rho(a_2)g^{-1}, B_2] \prod_{i=3}^{g} [\rho_t(a_i), \rho(b_i)] \right)$, is a submersion at the point $\rho \in \mathcal{M}(\Gamma, G)$.

Its differential at the point $\rho \in \mathcal{M}(\Gamma, G)$ has the form :

$$
\begin{pmatrix}
D_{(\rho(a_1a_2)^{-1}h^{-1}\rho_t(a_1)g \rho(a_2)g^{-1}, [\rho_t(a_1), B_1][g \rho(a_2)g^{-1}, B_2] \prod_{i=3}^{g} [\rho_t(a_i), \rho(b_i)] \right)}
K_{1,0}

D_{(\rho(a_1a_2)^{-1}h^{-1}\rho_t(a_1)g \rho(a_2)g^{-1}, [\rho_t(a_1), B_1][g \rho(a_2)g^{-1}, B_2] \prod_{i=3}^{g} [\rho_t(a_i), \rho(b_i)] \right)}K_{2,0}
\end{pmatrix}
$$
where $D_{(\text{id},id,\rho(b_1),\rho(b_2))}K_{i,0} : \mathfrak{g}^4 \to \mathfrak{g}$ is the tangent map of $K_{i,0}(.,.,.,0) = K_i(.,.,.,0) : G^4 \to G$ at the point $(\text{id},id,\rho(b_1),\rho(b_2))$.

We hence compute that:

$$D_{g,h,B_1,B_2}K_{1,0} = Ad\left(\frac{\partial}{\partial g}(K_{1,0})\right)dg + Ad\left(\frac{\partial}{\partial h}(K_{1,0})\right)dh$$

with

$$\frac{\partial}{\partial g}(K_{1,0})(g,h) = h\rho(a_1a_2)^{-1}h^{-1}\rho(a_1)(\text{id} - gp(a_2)g^{-1})$$

and

$$\frac{\partial}{\partial h}(K_{1,0})(g,h) = \text{id} - h\rho(a_1a_2)^{-1}h^{-1}.$$ 

Since for two orthogonal transformations $T$ and $S$ of a Euclidean vector space, the orthogonal subspace to $\text{Im}(S(\text{id} - T))$ is the kernel of $\text{id} - STS^{-1}$, we have that the space

$$Ad\left(\frac{\partial}{\partial g}(K_{1,0})(\text{id},\text{id})\right)(\mathfrak{g}^4)^\perp$$

is the kernel of the operator $Ad(\text{id} - \rho(a_2)^{-1})$. Similarly the subspace

$$Ad\left(\frac{\partial}{\partial h}(K_{1,0})(\text{id},\text{id})\right)(\mathfrak{g}^4)^\perp$$

is the kernel of the operator $Ad(\text{id} - \rho(a_2)^{-1}\rho(a_1)^{-1})$.

Since we can write the space $D_{(\text{id},\text{id},B_1,B_2)K_{1,0}}(\mathfrak{g}^4)^\perp$ as the intersection

$$Ad\left(\frac{\partial}{\partial g}(K_{1,0})(\text{id},\text{id})\right)(\mathfrak{g}^4)^\perp \cap Ad\left(\frac{\partial}{\partial h}(K_{1,0})(\text{id},\text{id})\right)(\mathfrak{g}^4)^\perp,$$

as in [10], we deduce that the rank of $D_{(\text{id},\text{id},B_1,B_2)K_{1,0}}$ is the codimension of the centralizer of the set $\{\rho(a_1),\rho(a_2)\}$.

In the same way, we compute:

$$D_{g,h,B_1,B_2}K_{2,0} = Ad\left(\frac{\partial}{\partial g}(K_{2,0})\right)dg + Ad\left(\frac{\partial}{\partial B_1}(K_{2,0})\right)dB_1 + Ad\left(\frac{\partial}{\partial B_2}(K_{2,0})\right)dB_2$$

with

$$\frac{\partial}{\partial B_1}(K_{2,0}) = \rho(a_1)(\text{id} - B_1\rho(a_1)B_1^{-1})$$

and

$$\frac{\partial}{\partial B_2}(K_{2,0}) = [\rho(a_1),B_1]g\rho(a_2)g^{-1}(\text{id} - B_2g\rho(a_2)g^{-1}B_2^{-1}).$$

As in [10], the same computations for $K_{2,0}$ allow to deduce that the rank of the map $K$ at the point $(\text{id},\text{id},\rho(b_1),\rho(b_2))$ is the codimension of the centralizer $Z_G(\rho)$. Since $\mathcal{M}(\Gamma,G)$ is the set of classes of representations with discrete centralizer in $G$, we obtain that $K$ is a submersion at $(\text{id},\text{id},\rho(b_1),\rho(b_2))$. It proves completely the Lemmas 3.4, 4.3 and 5.3.

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