An extension of the Unified Skew-Normal family of distributions and application to Bayesian binary regression

Paolo Onorati  
Sapienza University of Rome, p.onorati@uniroma1.it  
Brunero Liseo  
Sapienza University of Rome, brunero.liseo@uniroma1.it

Abstract

We consider the general problem of Bayesian binary regression and we introduce a new class of distributions, the Perturbed Unified Skew Normal (pSUN), which generalizes the SUN class. We show that the new class is conjugate to any binary regression model, provided that the link function may be expressed as a scale mixture of Gaussian densities. We discuss in detail the popular logit case, and we show that, when a logistic regression model is combined with a Gaussian prior, posterior summaries such as cumulants and normalizing constant, can be easily obtained, opening the way to straightforward variable selection procedures. For more general priors, the proposed methodology is based on a straightforward Gibbs sampler algorithm. We also claim that, in the $p > n$ case, it shows better performances both in terms of mixing and accuracy, compared to the existing methods. We illustrate the performance of the proposal through a simulation study and two real datasets, one covering the standard case with $n >> p$ and the other related to the $p >> n$ situation.

Keywords: Importance Sampling; Kolmogorov distribution; Logistic Regression, Scale Mixture of Gaussian Densities.

1 Introduction

Binary regression is among the most popular and routinely used statistical methods in applied science. Standard Bayesian approaches and off-the-shelf packages are today available, see for example the R suites brms or rstanarms. However, there is no general consensus on the choice of prior distributions and on how to select among different link functions.

A remarkable contribution in this direction has been provided by Durante [2019] who shows that the SUN family of densities [Arellano-Valle and Azzalini, 2006] can be used as a conjugate prior for the the probit regression model, that is the posterior distribution of the regression coefficients in a probit setting still belongs to the SUN family. Durante [2019] himself provides methods for efficiently sampling from the SUN distribution. His methodology is particularly useful in the case when $p >> n$.

The SUN (Unified Skew-Normal) family of densities has been introduced by Arellano-Valle and Azzalini [2006], who review and classify the various generalization of the
multivariate normal distribution appeared in the literature.

Following this line of research, we introduce a larger class of distributions, namely the perturbed SUN ($pSUN$ hereafter) family, which is obtained by replacing the two Gaussian laws appearing in the SUN definition with two scale mixtures of Gaussian distributions. This double scaling produces a larger class of densities which can be particularly useful in Bayesian binary regression.

Even though our results are quite general, at least in theory, here we concentrate on the logistic regression, which is by far the most popular regression model for binary outcomes in applied science. The mathematical background of our paper is based on old results obtained by Andrews and Mallows [1974] and Stefanski [1991] who both showed that the logistic distribution can be represented as a scale mixture of Gaussian distributions, when the mixing density is a particular transformation of a Kolmogorov distribution.

There have been already many attempts to apply a missing-data strategy - similar to the probit model - in the logit case (e.g. Holmes and Held [2006], Fruhwirth-Schnatter and Fruhwirth [2010], Gramacy and Polson [2012]). Polson et al. [2013] exploit the use of the representation of the logistic function in terms of a mixture of normals with a Polya-Gamma mixing density. Their approach produces a useful method for generating values from the posterior distribution of the parameters in a logistic regression set up, via a Gibbs sampler.

In this paper we will introduce a new algorithm for simulating values from a $pSUN$ distribution. The algorithm is based on a block Gibbs sampler, although, under special circumstances, it is also possible to obtain independent draws from the posterior distribution, via an importance sampling approach which dramatically shortens the computing time.

Results are presented in terms of an extensive simulation study and the use of two real data sets, already discussed in related literature. Simulations and real data problems are presented using different priors within the same large $pSUN$ family.

The gist of the our paper is to provide a general tool for Bayesian binary regression, able to work with a large family of link functions. In terms of comparative efficiency with alternative approaches we claim that

1. In standard logit models, with $n > p$, our Gibbs algorithm performance is comparable with the one proposed by Holmes and Held [2006] and it is second to the Polya-Gamma method described in Polson et al. [2013], although the distance decreases as $p$ gets large. In any case, inspection of the autocorrelation functions of the posterior chains always reveals a better mixing of our method, as we show in the Examples section. When $n << p$, our method is extremely efficient and often superior both in terms of speed, mixing and potential applicability. Moreover we will show that, when the prior distribution is Gaussian we are able to replace the Gibbs approach with an importance sampling strategy which is by far faster than any other competitors, up to our knowledge.

2. The $pSUN$ class inherits some good properties of the SUN family; for example, under very mild assumptions, the moment generating function is available in closed form. Due to conjugacy property, this allows a straightforward computation of the posterior moments of the regression coefficients for a large class of binary regression models. With closely similar calculations, we also show how to compute the marginal distribution of the data in order to perform model selection.
The rest of the paper is organized as follows. § 2 introduces the pSUN family and describes its properties. § 3 illustrates the use of the pSUN density as a conjugate prior for a large class of binary regression models. § 4 discusses in detail the logistic regression setup and introduces both the Gibbs and the importance sampling algorithms. The last section is devoted to a simulation study and empirical applications. In one of the examples, in the case \( n > p \), we also perform model and variable selection. More mathematical details are provided in the Appendix.

2 The perturbed SUN distribution

Arellano-Valle and Azzalini [2006] introduce the Unified Skew-Normal (SUN) class of densities which includes many of the several proposals appeared in the literature. It is based on the introduction of a given number \( m \) of latent variables; a \( d \)-dimensional random vector \( Y \) is said to have a SUN distribution, i.e.

\[
Y \sim \text{SUN}_{d,m}(\tau, \Delta, \xi, \Omega)
\]

if its density function is

\[
f_Y(y) = \phi_{\Omega}(y - \xi) \frac{\Phi_{\Gamma - \Delta\bar{\Omega}^{-1}\Delta}(\tau + \Delta'\bar{\Omega}^{-1}\text{diag}^{-\frac{1}{2}}(\Omega)(y - \xi))}{\Phi_{\Gamma}(\tau)},
\]

where \( \xi \in \mathbb{R}^d, \tau \in \mathbb{R}^m, \Gamma \) is a \( m \)-correlation matrix, \( \Omega \) is a \( d \)-covariance matrix, \( \Delta \) is \( d \times m \) matrix and \( \bar{\Omega} = \text{diag}^{-\frac{1}{2}}(\Omega) \Omega \text{diag}^{-\frac{1}{2}}(\Omega) \). It is also useful to provide a stochastic representation of a SUN random vector as \( Y = \xi + \text{diag}^{-\frac{1}{2}}(\Omega)Z \mid (U + \tau > 0) \), where

\[
\begin{bmatrix}
Z \\
U
\end{bmatrix} \sim \mathcal{N}_{d+m}(0, \Theta)
\]

We introduce a generalization of the above family by assuming that \( Z \) is a scale mixtures of Gaussian distributions and and \( U \) is a linear combination of \( Z \) and another random vector with a scale mixture of Gaussians density. For the sake of clarity we start from a slightly different parametrization of the SUN family, namely

\[
Y = \xi + \text{diag}^{-\frac{1}{2}}(\Omega)Z \mid (T \leq AZ + b),
\]

where \( A \in \mathbb{R}^{d \times m}, b \in \mathbb{R}^m \). This way, the conditioning latent vector \( T \) has a distribution independent of \( Z \) and \( T \sim \mathcal{N}_m(0, \Theta) \), with

\[
\Theta = \text{diag}^{-\frac{1}{2}}(\Gamma - \Delta'\bar{\Omega}^{-1}\Delta) \left( \Gamma - \Delta'\bar{\Omega}^{-1}\Delta \right) \text{diag}^{-\frac{1}{2}}(\Gamma - \Delta'\bar{\Omega}^{-1}\Delta),
\]

\[
A = \text{diag}^{-\frac{1}{2}}(\Gamma - \Delta'\bar{\Omega}^{-1}\Delta) \Delta'\bar{\Omega}^{-1}
\]

and

\[
b = \text{diag}^{-\frac{1}{2}}(\Gamma - \Delta'\bar{\Omega}^{-1}\Delta) \tau.
\]

Let \( V \) and \( W \) be an \( m \)-dimensional and a \( d \)-dimensional random vectors, respectively, both defined on their positive orthant. With a little abuse of notation, for a generic \( d \)-dimensional vector \( H \), let \( \text{diag}(H) \) be the \( d \)-dimensional diagonal matrix with same entries as \( H \). Assume that \( Z = \text{diag}^{1/2}(W)R \) and \( T = \text{diag}^{1/2}(V)S \), where

\[
\begin{align*}
V & \sim Q_V(\cdot) & S & \sim \mathcal{N}_m(0, \Theta) \\
W & \sim Q_W(\cdot) & R & \sim \mathcal{N}_d(0, \bar{\Omega}),
\end{align*}
\]
The pSUN class of distributions can then be defined as the expression \( \mathbf{1} \) with the above assumptions on \( Z \) and \( T \). We will denote the pSUN family with the following notation

\[
pSUN_{d,m} \left( Q_V, \Theta, A, b, Q_W, \Omega, \xi \right).
\]

Although a closed form expression of a pSUN density is available (see expression \( \mathbf{2} \)), its direct use is often computational expensive. Moreover, from a statistical perspective, the pSUN family is not identifiable, without imposing restrictions on the nature of \( Q_V(\cdot) \) and \( Q_W(\cdot) \). The next theorem provides the density of a general pSUN random vector.

**Theorem 1.** Let \( Y \sim \text{pSUN}_{d,m} \left( Q_V, \Theta, A, b, Q_W, \Omega, \xi \right) \). Let \( \phi_\Sigma \) and \( \Phi_\Sigma \) be the density and the CDF of a centred Gaussian random vector with covariance matrix \( \Sigma \). Then the density of \( Y \) can be written as

\[
f_Y(y) = \phi_{\Omega,Q_W} (y-\xi) \frac{\Phi_{\Theta,Q_V} \left( A \, \text{diag}^{-\frac{1}{2}}(\Omega)(y-\xi) + b \right)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)}, \tag{2}
\]

where, for a generic covariance matrix \( \Sigma \) and a cdf \( Q \) we have set

\[
\phi_{\Sigma,Q}(u) = \int_{\mathbb{R}^d} \prod_{i=1}^d \left( W_i^{-\frac{1}{2}} \right) \phi_\Sigma \left( \text{diag}^{-\frac{1}{2}}(W) \, u \right) dQ(W),
\]

\[
\Phi_{\Sigma,Q}(u) = \int_{\mathbb{R}^d} \Phi_\Sigma \left( \text{diag}^{-\frac{1}{2}}(W) \, u \right) dQ(W),
\]

and

\[
\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b) = \mathbb{P}(T - AZ \leq b), \quad T \sim \Phi_{\Theta,Q_V}(\cdot) \perp \perp Z \sim \Phi_{\Omega,Q_W}(\cdot). \tag{3}
\]

**Proof.** It is straightforward to see that the densities of \( Z = \text{diag}^{1/2}(W)R \) and \( T = \text{diag}^{1/2}(V)S \) are, respectively,

\[
f_Z(u) = \phi_{\Omega,Q_W}(u); \quad f_T(u) = \phi_{\Theta,Q_V}(u).
\]

Let \( Y_0 = Z | (T \leq AZ + b) \); the density of \( Y_0 \) can be written as

\[
f_{Y_0}(y) = f_Z(y | T \leq AZ + b) = f_Z(y) \frac{\mathbb{P}(T \leq AZ + b | Z = y)}{\mathbb{P}(T \leq AZ + b)} = \phi_{\Omega,Q_W}(y) \frac{\Phi_{\Theta,Q_V}(Ay + b)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)}. \]

However \( Y = \xi + \text{diag}^{1/2}(\Omega)Y_0 \), then

\[
f_Y(y) = \det \left( \text{diag}^{-\frac{1}{2}}(\Omega) \right) f_{Y_0} \left( \text{diag}^{-\frac{1}{2}}(\Omega)(y-\xi) \right) = \prod_{i=1}^d \left( \Omega_{ii}^{-\frac{1}{2}} \right) \phi_{\Omega,Q_W} \left( \text{diag}^{-\frac{1}{2}}(\Omega)(y-\xi) \right) \frac{\Phi_{\Theta,Q_V} \left( A \text{diag}^{-\frac{1}{2}}(\Omega)(y-\xi) + b \right)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)} = \phi_{\Omega,Q_W}(y-\xi) \frac{\Phi_{\Theta,Q_V} \left( A \text{diag}^{-\frac{1}{2}}(\Omega)(y-\xi) + b \right)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)}. \]

\[\Box\]
The denominator of expression (2) is expensive to compute for moderately large values of $m$. In Section 4 we describe a practical solution to this problem, at least in the special case when $Q_W(\cdot)$ is a point-mass distribution.

One can also obtain a simple expression for the moment generating function (MGF) of a pSUN distribution, providing that there exists the MGF of $Z$.

**Theorem 2.** Let $Y \sim \text{pSUN}_{d,m} (Q_V, \Theta, A, b, Q_W, \Omega, \xi)$. Let $M_Z(u)$ be MGF of $Z \sim \Phi_{\Omega,Q_W}(\cdot)$. Then the MGF of $Y$ can be written as

$$M_Y(u) = e^{u\xi}M_Z \left( \text{diag}^\frac{1}{2}(\Omega)u \right) \frac{\tilde{\Psi}_{Q_V,\Theta,A,Q_W,\Omega}(b, \text{diag}^\frac{1}{2}(\Omega)u)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)} ,$$

where we have denoted

$$\tilde{\Psi}_{Q_V,\Theta,A,Q_W,\Omega}(b, k) = P(T - A\bar{Z}_k \leq b)$$

with $T \sim \Phi_{\Theta,Q_V}(\cdot)$ \| $\bar{Z}_k$ and $\bar{Z}_k$ is the $k$-tilted distribution [Siegmund 1976] of $Z \sim \Phi_{\Omega,Q_W}(\cdot)$ that is

$$f_{\bar{Z}_k}(x) = \frac{e^{k\xi}f_Z(x)}{M_Z(k)} .$$

**Proof.** Let $Y_0 = Z|T \leq AZ + b$. Then

$$M_{Y_0}(u) = E(e^{uY_0}) = \int_{\text{R}^d} e^{u\phi_{\Omega,Q_W}(x)} \frac{\Phi_{\Theta,Q_V}(Ax + b)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)} dx_1dx_2\cdots dx_d$$

$$= \frac{M_Z(u)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)} \int_{\text{R}^d} e^{u\phi_{Q_W}(x)} \Phi_{\Theta,Q_V}(Ax + b) dx_1dx_2\cdots dx_d$$

$$= \frac{M_Z(u)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)} \int_{\text{R}^d} e^{u\phi_{Q_W}(x)} \Phi_{\Theta,Q_V}(Ax + b) dx_1dx_2\cdots dx_d$$

$$= M_Z(u) \frac{\tilde{\Psi}_{Q_V,\Theta,A,Q_W,\Omega}(b, u)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)} .$$

Since $Y = \xi + \text{diag}^\frac{1}{2}(\Omega)Y_0$,

$$M_Y(u) = e^{u\xi}M_Z \left( \text{diag}^\frac{1}{2}(\Omega)u \right) \frac{\tilde{\Psi}_{Q_V,\Theta,A,Q_W,\Omega}(b, \text{diag}^\frac{1}{2}(\Omega)u)}{\Psi_{Q_V,\Theta,A,Q_W,\Omega}(b)} .$$

\[\square\]

### 2.1 Sampling from a pSUN distribution

From the definition of pSUN, it is apparent that if $Y \sim \text{pSUN}_{d,m} (Q_V, \Theta, A, b, Q_W, \Omega, \xi)$ then $Y|(W, V) \sim \text{SUN}_{d,m}(\tau_0, \Delta_0, \Gamma_0, \xi, \Omega_0)$, where

$$\tau_0 = \text{diag}^\frac{1}{2}(\Theta + A_0\Omega A_0^T)b_0 ,$$

$$\Delta_0 = \Omega A_0^T \text{diag}^\frac{1}{2}(\Theta + A_0\Omega A_0^T) ,$$

$$\Gamma_0 = \text{diag}^\frac{1}{2}(\Theta + A_0\Omega A_0^T)(\Theta + A_0\Omega A_0^T) \text{ diag}^\frac{1}{2}(\Theta + A_0\Omega A_0^T) .$$

5
and $A_0, b_0, \Omega_0$ depend on $W$ and $V$, so that

$$A_0 = \text{diag}^{-\frac{1}{2}}(V) A \text{diag}^{\frac{1}{2}}(W),$$

$$b_0 = \text{diag}^{-\frac{1}{2}}(V) b,$$

$$\Omega_0 = \text{diag}^{-\frac{1}{2}}(W) \Omega \text{diag}^{\frac{1}{2}}(W).$$

In order to draw values from a pSUN distribution, one must be able to sample from the conditional distribution of $(W, V)| (T \leq AZ + b)$, which is not trivial; in fact, let

$$\Theta_V = \text{diag}^{1/2}(V) \Theta \text{diag}^{1/2}(V)$$

and

$$\Omega_V = \text{diag}^{1/2}(W) \Omega \text{diag}^{1/2}(W);$$

then use a rejection algorithm with auxiliary density given by

$$f_W(w) f_V(v).$$

In some special cases, one could numerically compute $\max [\Phi_{\Theta_V + A \Omega_W A'}(b)]$ and then use a rejection algorithm with auxiliary density given by $f_W(w) f_V(v)$. In these cases the acceptance rate would be given by $\Psi_{Q_V, \Theta, A, Q_W, \Omega}(b)$ divided by the above max; however, this rate could be very low, depending on the specific parameter values, with no theoretical guarantee. For this reason, in this paper, we adopt a more general technique based on the Gibbs Sampler, which we now describe. Let $TN_d(\ell, u, \mu, \Sigma)$ denote a $d$-variate Gaussian random vector truncated at $\ell$ and $u$, with mean $\mu$ and covariance matrix $\Sigma$. It is easy to simulate values from it using the R package `TruncatedNormal`, which is based on the algorithm of Botev [2017]. Suppose that, at iteration $t$, one has $Y_t, Z_t, T_t, W_t, V_t$; the update step follows Algorithm [1]. A key aspect of the algorithm is that one must be able to sample from the full conditional distributions $W|Z$ and $V|T$.

This task is not always easy, since it depends on the specific values of $\Theta, \Omega$, and the form of $Q_W(\cdot)$ and $Q_V(\cdot)$. However, as we will show in the next section, it is relatively simple in the most popular versions of the Bayesian binary regression.

---

Sample $V_{t+1} \sim V| T = T_t$

Sample $W_{t+1} \sim W| Z = Z_t$

In order to sample $Z_{t+1}, T_{t+1} \sim Z, T| T \leq AZ + b, W_{t+1}, V_{t+1}$ do the following steps:

Set $\Sigma_\varepsilon = \Theta_{W_{t+1}} + A \Omega_{W_{t+1}} A'$

Sample $\varepsilon \sim TN_m(-\infty, -b, 0, \Sigma_\varepsilon)$

Set $H_\mu = \Omega_{W_{t+1}} A' \Sigma^{-1}_\varepsilon$

Set $H_\Sigma = (I - H_\mu A) \Omega_{W_{t+1}}$

Sample $Z_{t+1} \sim N_d(H_\mu \varepsilon, H_\Sigma)$

Set $T_{t+1} = AZ_{t+1} - \varepsilon$

$\Rightarrow Y_{t+1} = \xi + \text{diag}^{1/2}(\Omega) Z_{t+1}$

---

Algorithm 1: Sampling from a pSUN distribution

3 Bayesian Linear Symmetric Binary Regression

In this section we will describe the use of the pSUN family of distributions as a conjugate prior for a large class of binary regression models. Consider a general version of
the model as
\[ Y_i | p_i \sim B_{\Lambda}(p_i), \quad \forall i = 1, 2, \ldots, n \]
\[ p_i = \Lambda(\eta(X_i)) \]
where \( \Lambda : \mathbb{R} \to [0, 1] \) is a known link function, \( \eta(\cdot) \) is a calibration function, and \( X_i \in \mathbb{R}^p \) is the \( i \)-th row of the design matrix \( X \). Typically \( \Lambda(\cdot) \) is a univariate CDF of some random variable, symmetric about 0, and \( \eta(x) \) takes the simple linear form, \( x^{\beta} \); we refer to this case as the linear symmetric binary regression model (LSBR).

Let \( \Lambda_n(x) = \prod_{i=1}^n \Lambda(x_i), \ x \in \mathbb{R}^n \) and \( B_x = [2 \operatorname{diag}(x) - I_n] \) for \( x \in \{0, 1\}^n \), where \( I_n \) is the identity matrix of size \( n \); the likelihood function of a LSBR model can be written as
\[ L(\beta; y) = \Lambda_n(B_y X \beta) \]
Before introducing the pSUN family as a conjugate prior for the LSBR model, we first consider an even more general prior family, which turns out to be conjugate in the LSBR case. As a prior density for \( \beta \), we adopt a latent linear selective sampling distribution (LLSS), in the spirit of Albert and Chib [1993] [but see also Azzalini and Capitanio [2014], expression (2.12)]
\[ \beta \overset{\text{d}}{=} U_0|U_1 \leq MU_0, \quad U_0 \in \mathbb{R}^p \sim G_0(\cdot), \ U_1 \in \mathbb{R}^m \sim G_1(\cdot), \ M \in \mathbb{R}^{m \times p} \]
with \( U_0 \perp U_1 \). This stochastic representation holds, of course, when \( P(U_1 \leq MU_0) > 0 \). We denote it as \( \beta \sim \text{LLSS}_{p,m}(G_0, M, G_1) \); it is straightforward to see that
\[ \beta \sim \text{LLSS}_{p,m}(G_0, 0_{m \times p}, G_1) \implies \beta \sim G_0(\cdot), \quad (4) \]
where \( 0_{m \times p} \) is a matrix of zeros. The LLSS class is not identifiable without further restrictions. The next theorem shows that the LLSS prior is conjugate with the LSBR model.

**Theorem 3.** In a Bayesian LSBR framework, the use of a prior \( \beta \sim \text{LLSS}_{p,m}(G_0, M, G_1) \) implies that the resulting posterior distribution is again LLSS. More precisely,
\[ \beta|Y = y \sim \text{LLSS}_{p,m+n}(G_0, \begin{bmatrix} M \\ B_y X \end{bmatrix}, G_1 \Lambda_n \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}) \]
where we have denoted
\[ G_1 \Lambda_n \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = G_1(x_1)\Lambda_n(x_2) \]
for \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{m+n} \).

**Proof.** The prior density of \( \beta \) is:
\[ f_\beta(\beta) = f_{U_0}(\beta|U_1 \leq MU_0) \]
\[ \propto f_{U_0}(\beta)P(U_1 \leq MU_0|U_0 = \beta) \]
\[ = g_0(\beta)G_1(M\beta) \]
and the corresponding posterior is
\[
f_\beta(\beta|Y = y) \propto g_0(\beta) G_1(M\beta) \Lambda_n(B_X X \beta)
\]
\[
= g_0(\beta) G_1 \Lambda_n \left( \begin{bmatrix} M \\ B_Y X \end{bmatrix} \beta \right).
\]

Because of (4), Theorem 3 provides a stochastic representation of the posterior distribution of \( \beta \) for any proper prior. However, sampling from the resulting distribution might not be simple when
\[
\int_\beta g_0(z) G_1 \Lambda_n \left( \begin{bmatrix} M \\ B_Y X \end{bmatrix} z \right) dz_1 dz_2 \cdots dz_p
\]
is small, as it is typically the case. On the other hand, specific choices of \( G_0 \) lead to a relatively easy sampling algorithm for \( \beta|Y \); for example, when \( XX' \) is invertible, one can set
\[
\beta \overset{d}{=} X'(XX')^{-1} U_2,
\]
where \( U_2 \sim G_2(\cdot) \) and \( G_2 \) is a \( n \)-dimensional distribution with independent and symmetric components, i.e.
\[
G_2(x) = \prod_{i=1}^n \mathbb{G}_2^* (x_i), \quad x_i \in \mathbb{R}, \ i = 1, \ldots n.
\]
Then it is immediate to see that
\[
\beta|Y = y \overset{d}{=} X'(XX')^{-1} U_2 | U_1 \leq B_Y U_2.
\]
In this case it is easy to obtain i.i.d. draws from \( \beta|Y = y \) by noting that \( U_1 \) and \( B_Y U_2 \) are independent with symmetric and mutually independent components. This is illustrated in Algorithm 2.

---

**Algorithm 2:** Sampling from the posterior when \((XX')^{-1}\) exists.

This general approach is very efficient as long as \( p > n \). In this case, in fact, one can elicit a uni-dimensional prior on each \( p_i^* \) and still produce exact posterior sampling. However, in the common case \( n > p \), this route is not viable and one needs to consider the class of restricted pSUN priors. In the next section we will restrict to the pSUN class and will show how to use it as the conjugate class for LSBR models.
3.1 A special case including logit and probit models: the pSUN family.

A pSUN distribution is a proper subset of LLSS family. In fact, if $X \sim \text{pSUN}_{d,m}(QV, \Theta, A, b, QW, \Omega, \xi)$ then $X \sim \text{LLSS}_{d,m}(G_0, A \text{ diag}^{-1/2}(\Omega), G_1)$, where

$$
G_0(x) = \Phi_{\Omega, QW}(x - \xi) \\
G_1(x) = \Phi_{\Theta, QV}(x - A\xi + b).
$$

The next theorem shows that, if one restricts the CDF $\Lambda(\cdot)$ to be a scale mixtures of Gaussian distributions, then the pSUN class of priors can be suitably used in Bayesian LSBR models.

**Theorem 4.** Consider a Bayesian LSBR model and assume that the prior distribution for $\beta$ is

$$
\beta \sim \text{pSUN}_{p,m}(QV_0, \Theta, A, b, QW, \xi, \Omega).
$$

Assume, in addition, that the link function $\Lambda(\cdot)$ has the following representation

$$
\Lambda(x) = \int_{0}^{+\infty} \Phi \left( \frac{x}{\sqrt{v}} \right) dQV_1(v).
$$

Then, the posterior distribution of $\beta$ belongs to the pSUN family. More precisely

$$
\beta|Y = y \sim \text{pSUN}_{p,m+n}(QV_0, \Theta, A, b, QW, \xi, \Omega),
$$

where we have denoted

$$
QV_0QV_1^{n-1}([x_1, x_2]) = QV_0(x_1) \prod_{i=1}^{n} QV_1(x_{i,i})
$$

for $[x_1, x_2] \in \mathbb{R}^{m+n}$.

**Proof.** The prior density of $\beta$ is:

$$
f_{\beta}(\beta) \propto \phi_{\alpha, QW}(y - \xi)\Phi_{\Theta, QV}(A \text{ diag}^{-\frac{1}{2}}(\Omega)(y - \xi) + b).
$$

The corresponding posterior density is then

$$
f_{\beta}(\beta|Y = y) \propto \Lambda_n(BYX\beta)\phi_{\alpha, QW}(y - \xi)\Phi_{\Theta, QV}(A \text{ diag}^{-\frac{1}{2}}(\Omega)(y - \xi) + b)
$$

$$
= \phi_{\alpha, QW}(y - \xi)\Phi_{\Theta, QV}(B_{YX} \text{ diag}^{\frac{1}{2}}(\Omega) \text{ diag}^{-\frac{1}{2}}(\Omega)(\beta - \xi) + B_{YX}\xi)
$$

$$
= \phi_{\alpha, QW}(y - \xi)\Phi_{\Theta, QV}\left( A \text{ diag}^{-\frac{1}{2}}(\Omega)(y - \xi) + b \right)
$$

$$
\quad \Phi_{\Theta, QV}\left( B_{YX} \text{ diag}^{\frac{1}{2}}(\Omega) \text{ diag}^{-\frac{1}{2}}(\Omega)(\beta - \xi) + b \right)
$$

$$
= \phi_{\alpha, QW}(y - \xi)\Phi_{\Theta, QV}\left( A \text{ diag}^{-\frac{1}{2}}(\Omega) \text{ diag}^{-\frac{1}{2}}(\Omega)(\beta - \xi) + b \right)
$$

$$
\quad \text{where } \Theta^* = \begin{bmatrix} \Theta & 0_{m,n} \\ 0_{n,m} & I_n \end{bmatrix}.
$$
Theorem 4 shows the pSUN family is a conjugate class for a large subset of LSBR, including probit and logit models, as we illustrate in the next sections. In this perspective, the previous theorem can be considered a generalization of the results of Durante [2019].

From a computational point of view, we notice that, in order to perform the Gibbs algorithm to produce a posterior sample from pSUN distribution, one must be able to sample from the full conditional distributions of $V$ and $W$. Regarding the latter, this is relatively simple when the prior for $\beta$ either has an elliptical structure or it has independent components. For example, the Symmetric Generalized Hyperbolic [Barndorff-Nielsen 1977] class of priors satisfies the elliptical constraint and corresponds to the $m = 0$ case. When $m = 1$ one obtains a new version of the skew Generalized Hyperbolic class.

To sample from the full conditional of $V$ may be difficult, depending on the nature of the link function $\Lambda(\cdot)$. Computation can made simpler when $\Theta$ is diagonal; in fact, in this case one can independently sample $V_i|T_i$ $i = 1, 2, \ldots, n + m$. This can be automatically obtained, for example, setting $m = 0$ or $m = 1$.

As a final note, expression (3) provides a way of computing the marginal probability of the observed $Y_i$’s for a given LSBR model. It would be enough to replace the quantity $Z$ in expression (3) with its analogue, that is

$$P(Y = y) = P(T^* \leq A^*Z^* + b^*),$$

where, using results in Theorem 4, $T^*$ is a scale mixture of a centred multivariate Gaussian density with correlation matrix $\Theta^*$ and mixture density $Q_{V_0}Q_{V^*}$, and

$$A^* = \begin{bmatrix} A & 0_{m \times p} \\ 0_{n \times p} & B g X \text{diag}^{-\frac{1}{2}}(\Omega) \end{bmatrix}, \quad Z^* = \text{diag}^{-\frac{1}{2}}(\Omega)(\beta - \xi), \quad b^* = \begin{bmatrix} b \\ B g X \xi \end{bmatrix}.$$  

The practical evaluation of (5) is however cumbersome and it typically requires additional simulation, as illustrated in the Examples section. However, in statistical practice, the most used link function are the probit and logit ones; while the former is deeply considered in Durante [2019], in the next section we show how to deal with the logistic regression model. In particular we show that, assuming a multivariate Gaussian prior on the regression coefficients, the normalizing constant in expression (3) and posterior mean and variance can be easily obtained via a straightforward importance sampling.

4 Bayesian Logistic Regression

In this section we consider the popular logistic regression model and show that the pSUN class of priors is conjugate to this model. In Section 4.1 we describe in detail a Gibbs sample algorithm for producing a posterior sample from the distribution of the regression coefficients. In Section 4.2, we claim that computation are dramatically easier to perform when the $\beta$ prior is Gaussian: in fact, under this assumption, the normalizing constant can be evaluated through an importance sampling algorithm: this opens the way to easy calculation of posterior means and variances and facilitates model selection procedures based on the Bayes factor.

The logistic distribution admits a representation in terms of a scale mixture of Gaussian distributions; see Andrews and Mallows [1974] and Stefanski [1991]. In detail, if $T_i|V_{0,i}$ is a centred Gaussian random variable with standard deviation $2V_{0,i}$ and $V_{0,i}$
follows a Kolmogorov distribution, say $K(\cdot)$, then $T_i$ has a marginal logistic distribution, that is 

$$ T_i | V_{0,i} \sim N(0, 4V_{0,i}^2) \quad \text{and} \quad V_{0,i} \sim K(\cdot) \implies T_i \sim \text{Logis}(0, 1) $$

that is 

$$ f_{T_i}(t) = \frac{\exp(-t)}{(1 + \exp(-t))^2} \quad t \in \mathbb{R}. $$

Holmes and Held [2006] have already used this representation in order to propose a data-augmentation Gibbs algorithm for several models including logistic regression. The approach described here and the one proposed by Holmes and Held [2006] share some characteristics in the binary logistic case although we introduced some improvements in terms of mixing and computational speed.

For the sake of clarity, we say that $V_i$ follows a logistic Kolmogorov distribution if 

$$ V_i = 4V_{0,i}^2, \quad V_{0,i} \sim K(\cdot) $$

and we denote it by $V_i \sim \text{Logist}(\cdot)$; the corresponding density function can be written as 

$$ \text{lk}(v) = \begin{cases} 
    v^{-2} \sqrt{2\pi} \sum_{j=1}^{+\infty} \left( (2j - 1)^2 \pi^2 - v \right) \exp \left( -\frac{(2j - 1)^2 \pi^2}{2v} \right) & \text{if } 0 < v \leq v^* \\
    \sum_{j=1}^{+\infty} (-1)^{j-1} j^2 \exp \left( -\frac{j^2 v^2}{2} \right) & \text{if } v > v^* 
\end{cases} 
$$

for some $v^* > 0$; see Onorati and Liseo [2022] for details. For numerical reasons, we set $v^* = 1.9834$.

4.1 Gibbs Algorithm for logistic regression.

Theorem [3] guarantees that the resulting marginal posterior of $\beta$ still belongs to the pSUN family. Here we only describe how to sample from the conditional distributions of $V | T, \beta, W, Y$ which actually reduces to $V | T$: this task is in fact particularly challenging in the logistic regression framework.

First, notice that the first $m$ components of $V | T$ are independent of the last $n$ ones, and they only depend on the prior distribution. So we focus on the last $n$ components of $V | T$, which involve the logistic Kolmogorov distribution. Also the last $n$ components of $V | T$ are independent and one only needs to sample from $V_i | T_i, i = m+1, m+2, \ldots, m+n$. In order to do that, we adopt an acceptance-rejection algorithm. Notice that 

$$ f_{V_i}(v | T_i = t) = \frac{f_{T_i}(t | V_i = v) f_{V_i}(v)}{f_{T_i}(t)} = \frac{\varphi(t/\sqrt{v}) \text{lk}(v)}{\sqrt{v} f_{T_i}(t)}, $$

where $f_{T_i}(\cdot)$ is the density of the standard logistic distribution. As the auxiliary density we consider $\tilde{V} | \tilde{T}$, that is 

$$ f_{\tilde{V}}(v | \tilde{T} = t) = \frac{f_{\tilde{T}}(t | \tilde{V} = v) f_{\tilde{V}}(v)}{f_{\tilde{T}}(t)} = \frac{\varphi(t/\sqrt{v}) f_{\tilde{V}}(v)}{\sqrt{v} f_{\tilde{T}}(t)}, $$

with $\tilde{V} \sim \text{Inv.Gamma}(\alpha, \gamma)$ so $\tilde{V} | \tilde{T} = t \sim \text{Inv.Gamma}(\alpha + 1/2, \gamma + t^2/2)$, and 

$$ f_{\tilde{T}}(t) = \frac{\Gamma(\frac{\alpha+1}{2})}{\Gamma(\alpha) \sqrt{2\pi\gamma}} \left( 1 + \frac{t^2}{2\gamma} \right)^{-\frac{\alpha+1}{2}}. $$

11
Then
\[ \frac{f_{V_i}(v|T_i = t)}{f_{\tilde{V}}(v|T = t)} = \frac{\text{lk}(v)}{\tilde{f}_V(v)} \frac{f_{\tilde{T}}(t)}{f_{T_i}(t)}; \]
then, only the first ratio depends on \( v \), and we need to find an upper bound \( M^* \) such that
\[ \frac{\text{lk}(v)}{\tilde{f}_V(v)} \leq M^*, \quad \forall v > 0. \]

Note that
\[ \text{lk}(v) \tilde{f}_V(v) = \begin{cases} \sqrt{2\pi \Gamma(\alpha)} \gamma^\alpha v^{\alpha - \frac{3}{2}} \sum_{j=1}^{+\infty} (2j - 1)^2 \pi^2 - v \exp \left( \frac{\gamma}{v} - \frac{(2j - 1)^2 \pi^2}{2v} \right) & \text{if } 0 < v \leq v^* \\ \Gamma(\alpha) \gamma^\alpha v^{\alpha + 1} \sum_{j=1}^{+\infty} (-1)^j j^2 \exp \left( \frac{\gamma}{v} - \frac{j^2 \gamma^2}{2v} \right) & \text{if } v > v^* \end{cases} \] (7)

Setting \( \gamma = \pi^2/2 \), expression (7) is bounded above by
\[ \delta_1(v) \delta_2(v) = \begin{cases} \sqrt{2\pi \Gamma(\alpha)} \gamma^\alpha v^{\alpha - \frac{3}{2}} & \text{if } 0 < v \leq v^* \\ \Gamma(\alpha) \gamma^\alpha v^{\alpha + 1} \exp \left( \frac{\pi^2}{2v} - \frac{v}{2} \right) & \text{if } v > v^* \end{cases}. \] (8)

We impose \( \alpha \geq 3/2 \) so one can obtain
\[ \arg \sup_{0 < v \leq v^*} \delta_1(v) = v^*, \]
and
\[ \frac{d \log \delta_2(v)}{dv} = \frac{1 + \alpha}{v} - \frac{\pi^2}{2v^2} - \frac{1}{2}, \]
so it is easy to show
\[ \arg \sup_{v > v^*} \delta_2(v) = \begin{cases} 1 + \alpha + \sqrt{(1 + \alpha)^2 - \pi^2} & \text{if } \alpha \geq \pi - 1 \\ v^* & \text{otherwise} \end{cases}. \]

Hence we obtain
\[ M^* = \max \left( \sup_{0 < v \leq v^*} \delta_1(v), \sup_{v > v^*} \delta_2(v) \right). \] (9)

Suitable values of \( \alpha \) in the auxiliary density are reported in the Appendix.

4.2 The case of Gaussian prior

In this section we consider a special, although important, case where the prior distribution on \( \beta \) is Gaussian, say \( \beta \sim N_p(\xi, \Omega) \). We propose an importance sampling approach which works differently in the \( n >> p \) case and in the rest of cases. In both scenarios the importance density function will be based on a SUN density, obtained by setting the latent variables \( V_i \) equal to their expected values \( E(V_i) = \pi^2/3 \). Also notice that \( \pi^2/3 \) is the \( \sigma^2 \) value that minimizes the KL divergence of a centered Gaussian with variance \( \sigma^2 \) from a standard Logistic density, as we show in the Appendix 5.3.1.
Consider first the former case $n \gg p$. Here our importance density $g(\cdot)$ is a multivariate Student $t$ distribution with a fixed and large number of degrees of freedom, say $50$. This choice guarantees that importance weights do not explode. The mean vector and variance-covariance matrix of the density $g(\cdot)$ are set equal to those of the above default SUN density. Details of this derivation are given in Appendix 5.3.1.

Consider now the other situations. In this case the Student $t$ distribution is not a valid approximation of a SUN density. Our proposal here is to adopt, as an importance density, a scale mixture of SUN densities; draws from this distribution are obtained by first (i) generating values from a SUN density using the Botev algorithm, and then (ii) multiplying each values by the square root of a random draw from an Inv.Gamma density with parameter $(\nu/2, \nu/2)$, where $\nu$ is, as before, a fixed and large number, say $50$. The expression of the importance density is given in Appendix 5.3.1.

After selecting the importance density $g(\cdot)$, it is possible to make inference completely avoiding MCMC simulation; indeed let $$w(\beta) = \frac{\phi^T(\beta - \xi)\Lambda_w (B_Y X \beta)}{g(\beta)}$$ be the importance weights of the simulated values. Then the marginal distribution of the data $p(y)$ can be simply estimated by $P_Y = \sum_{i=1}^n w(\beta_i)$. Similarly, the posterior mean of an arbitrary function $h(\cdot)$ can be estimated by $$E(h(\beta)|y) = \frac{\sum_{i=1}^n w(\beta_i)h(\beta_i)}{\sum_{i=1}^n w(\beta_i)},$$ with $\beta_i \overset{i.i.d.}{\sim} g(\cdot)$. Efficient estimates of the above quantities are particularly useful in a model selection scenario, as we will explore in §5.3. Also posterior moments of the $\beta$ components will allow easy derivation of uncertainty quantification.

5 Examples

In this section we first perform a simulation study to illustrate the finite sample performance of the algorithm. Then, we consider two different data sets, which are useful for highlighting different aspects of the proposed method and comparing it with the popular approach based on Polya-Gamma density described in Polson et al. [2013].

Regarding the prior distributions, we have adopted a pSUN prior with weakly informative hyper-parameters in the spirit of Gelman et al. [2008]. More precisely, we set $m = 0, \xi = 0_p$ and a diagonal matrix for $\Omega$. This amounts to say that $\pi(\beta)$ will be unimodal and symmetric about the origin. We first implement the probit model, which implies $V_1 = V_2 = \cdots = V_n = 1$. We have used four different priors for the $\beta$ vector, namely i) a Gaussian prior, which corresponds to setting $W_1 = W_2 = \cdots = W_p = 1$ [Durante 2019]; ii) a multivariate elliptical Cauchy prior, obtained by setting $W_1 = W_2 = \cdots = W_p = W^* = W^* \sim \text{Inv.Gamma}(0.5, 0.5)$; iii) a multivariate Laplace with independent components, that is $W_1, W_2, \ldots, W_p \overset{iid}{\sim} \text{Exp}(1/2)$; iv) a Dirichlet-Laplace prior [Bhattacharya et al., 2015], with a discrete uniform prior on the Dirichlet parameter, with support $\{1/300 \times j, j = 1, 2, \ldots, 300\}$. Finally, the diagonal elements of $\Omega$, appearing in the first three priors, were obtained, adapting a suggestion
in Gelman et al. [2008] for the probit case as follows:

- **Gaussian:** $\omega_{11} = 100; \quad \omega_{22} = \cdots = \omega_{pp} = 42.25$
- **Elliptical Cauchy:** $\omega_{11} = 56.25; \quad \omega_{22} = \cdots = \omega_{pp} = 3.0625$
- **Laplace with indep. components:** $\omega_{11} = 100; \quad \omega_{22} = \cdots = \omega_{pp} = 6.25$

Notice that the value of $\omega_{11}$ is much larger than the others because it refers to the intercept.

In the logit model, the $V_i$’s are i.i.d. with LK distribution. We have studied the performance of the same priors for $\beta$ as in the probit case. In the logit set up, the diagonal components of $\Omega$, appearing in the first three priors, were fixed to the following values:

- **Gaussian:** $\omega_{11} = 256; \quad \omega_{22} = \cdots = \omega_{pp} = 25$
- **Elliptical Cauchy:** $\omega_{11} = 100; \quad \omega_{22} = \cdots = \omega_{pp} = 6.25$
- **Laplace with indep. components:** $\omega_{11} = 210.25; \quad \omega_{22} = \cdots = \omega_{pp} = 14.0625$

### 5.1 Simulation Study

We show the results of a simulation study in order to comparatively discuss alternative priors within the pSUN family, to analyse the speed of convergence of the Gibbs algorithm and to verify the computational efficiency of the proposal.

Let $G = 2500$ be the number of simulations. For $g = 1, 2, \ldots, G$, we implement 7 different combinations of model and priors:

| Model | Prior for $\beta$ |
|-------|-------------------|
| $h = 1$ | Logit | Multiv. Gaussian |
| $h = 2$ | Logit | Elliptical Cauchy |
| $h = 3$ | Logit | Multiv. Laplace with indep. components |
| $h = 4$ | Logit | Dirichlet Laplace |
| $h = 5$ | Probit | Elliptical Cauchy |
| $h = 6$ | Probit | Multiv. Laplace with indep. components |
| $h = 7$ | Probit | Dirichlet Laplace |

The location parameter is always set equal to zero and the scale matrix is randomly generated by a Wishart distribution with expected value equal to the scale matrix described above, and the degrees of freedom parameter set at the smallest possible integer value, i.e. $p + 1$. Let fix the sample size at $n = 25$ and let $p = 10$ be the number of covariates. We consider the following scheme of simulation:

We have actually repeated the simulation study with two different numbers of iterations in the Gibbs sampler, $(N = 10^4$ and $N = 2 \times 10^4$). Since results are perfectly comparable, we only report the former. Figure 1 shows the frequentist coverage of the one-sided credible sets for the intercept in the logit model for the various prior distribution adopted.

All priors behave quite satisfactorily, the Gaussian prior, already considered in Durante [2019], is omitted.
for $g = 1, 2, \ldots, G$

- sample each single covariate value independently $X_{ij}^{(g)} \sim N(0, 1)$ and transform each column of $X^{(g)}$ in order to have a standard deviation equal to 0.5 for $h = 1, 2, \ldots, 7$
  - if $h \in \{1, 2, 5\}$ sample $\Sigma^{(g,h)} \sim \text{Wishart}$
  - if $h \in \{3, 6\}$ sample $\Sigma^{(g,h)} \sim \text{Wishart}$, and only take the diagonal elements
  - if $h \in \{4, 7\}$ set $\Sigma^{(g,h)} = I_p$ and $\alpha \sim \pi(\alpha)$
  - sample $\beta^{*}_{(g,h)} \sim \pi_h(\beta|\Sigma^{(g,h)})$
  - sample $Y_{i}^{(g,h)} \overset{ind}{\sim} B e(\Lambda_{h}(X_{i}^{(g)} \beta^{*}_{(g,h)}))$
  - draw $N$ values from the posterior distribution of $\beta$
  - compute the empirical quantiles of level $\gamma \in \{5/100 \times j, j = 1, 2, \ldots, 19\} \Rightarrow$ evaluate the frequentist coverage comparing the quantiles with $\beta^{*}_{(g,h)}$

**Algorithm 3:** Algorithm for the simulation study

Figures 3 and 4 report the average frequentist coverages of the other nine $\beta$ coefficients. We only present this average for the sake of brevity and because the 9 coefficients play a similar role in the simulation study. Comments on the behavior of the different priors are basically the same as for the intercept.

### 5.2 Cancer SAGE data set

The cancer SAGE data set has been discussed in [Durante 2019] and represents a situation where the number of coefficients is larger than the sample size. It is available online and consists of the gene expressions of $n = 74$ normal and cancerous biological tissues at 516 different tags [Durante, 2019]. It is of interest to quantify the effects of gene expressions on the probability of a cancerous tissue and predicting the status of new tissues as a function of the gene expression. We have standardized the gene expressions to have mean 0 and standard deviation 0.5.

For this data set we consider all the alternative priors defined above combined with the logistic and probit regression models. In addition, in the probit case, we also compare their behaviour with the Gaussian prior proposed in [Durante 2019], where $\omega_{11} = \cdots = \omega_{pp} = 16$. We report the posterior means for the vector of coefficients. Also, in the logit case we compare the performance of our approach with the Polya-Gamma algorithm. Comparisons were made using the prior adopted in [Polson et al. 2013], namely a multivariate Gaussian with independent components with mean 0 and standard deviation 10. While the final inference are identical, the computing time is quite different in this example: approximately 103 minutes for the Polya-Gamma approach, and 26 for the pSUN approach on a standard laptop with 16 Gb of RAM and, as CPU, an i7-8750H. Also the autocorrelation functions, shown in Figure 7, show a better mixing for the pSUN approach.
Figure 1: Frequentist coverage of the intercept parameter for different choices of the prior distribution in the logit case: Gaussian prior (top-left), Cauchy prior (top-right), Laplace with independent components (bottom-left), Dirichlet-Laplace (bottom-right)

Figure 2: Frequentist coverage of the intercept parameter for different choices of the prior distribution in the probit case: Cauchy prior (top-left), Laplace with independent components (top-right), Dirichlet-Laplace (bottom)
Figure 3: Average frequentist coverage of the other nine parameters for different choices of the prior distribution in the logit case: Gaussian prior (top-left), Cauchy prior (top-right), Laplace with independent components (bottom-left), Dirichlet-Laplace (bottom-right)

Figure 4: Average frequentist coverage of the other nine parameters for different choices of the prior distribution in the probit case: Cauchy prior (top-left), Laplace with independent components (top-right), Dirichlet-Laplace (bottom)
Aware that the computing time was still too high, we have implemented the much faster approach based on importance sampling described in § 4.2; in this case the computing time dramatically decreases to 33 seconds!

Figure 5 reports the posterior means of the \( \beta \)'s using the above described different priors in the probit scenario. Figure 6 does the same in the logit case. One can immediately notice that, being \( n << p \), final results are driven by the prior assumptions. For example, the prior adopted by Durante [2019] assumed a prior standard deviation for the \( \beta \)'s equal to 4, while our weakly informative prior set those values equal to 10, and this clarifies the larger variability of the posterior means of the \( \beta \) coefficients with the weakly informative prior. Also, most of the variability among estimates can be justified by the differences in tail thickness and in the choices of \( \Omega \) values.

Regarding the importance approach, the dramatic improvements in computing pays does not imply any detriment in precision of the estimates. Table 1 reports a summary of the differences between point estimates of the 517 regression coefficients using the Gibbs algorithm and the importance sampling approach: half of the differences are less than 0.0508.

Table 1: Differences between point estimates

| Min       | 1st Qu. | Median | 3rd Qu. | Max     | Mean     | Var     |
|-----------|---------|--------|---------|---------|----------|---------|
| -0.1909   | -0.0496 | -0.0028| 0.0514  | 0.1736  | 0.0012   | 0.0047  |

5.3 Pima Indians

This data set represents a benchmark for binary regression [Chopin and Ridgway 2017, Sabanés Bové and Held 2011, Bhattacharyya et al. 2022] and it is available on the R package mlbench. There are two versions of the data set: we have used the one named PimaIndiansDiabetes2, where implausible values of the original data set were replaced by NA's. We have also deleted the two variables triceps, insulin, containing too many NA's. The actual data set consisted of \( n = 724 \) observations on 6 covariates, while the response variable was the diabetes diagnosis. The main interest for this data set is in the variable selection problem, an important issue that we will tackle elsewhere; we have considered this data set as a test example in the case of \( n >> p \) in order to verify the performance of the proposed methodology in terms of accuracy of estimates. As in the Cancer SAGE example, we have considered both the logit and the probit models with different prior and hyperparameters choices. Figures 8 and 9 report point estimates and 95% credible sets for four different priors, namely Gaussian, Cauchy, Laplace with independent components and Dirichlet-Laplace with the same hyper-parameters used in § 5.2. As expected, in the \( n >> p \) scenario, the impact of the prior is almost negligible, and posterior credible intervals for the \( \beta \) coefficients are very similar across different prior choices. When \( n >> p \) the Gibbs algorithm proposed in this paper is much slower than the more specific algorithm described
Figure 5: SAGE example, probit model: Posterior means of the 516 $\beta$ coefficients plus the intercept $\beta_1$. Left: Durante’s prior (grey) and Gaussian prior (blue). Right: Cauchy prior (red), Laplace with independent components (black), and Dirichlet-Laplace (green).

Figure 6: SAGE example, logit model: Posterior means of the 516 $\beta$ coefficients plus the intercept $\beta_1$. Left: Gaussian prior (blue) and Cauchy prior (red); Right: Laplace with independent components (black) and Dirichlet-Laplace (green).
in Polson et al. [2013]; however, if the prior is Gaussian, the comparison between the Polya-Gamma algorithm of Polson et al. [2013] and our our importance sampling techniques is much more interesting, since the former requires approximately 130 secs while the latter takes about 100 secs; The little gain becomes however significant in a variable selection set up, where many normalizing constants need to be evaluated. The dramatic improvement in computing time with respect to the Gibbs approach, does not imply any loss in efficiency: Figure 10 compares the posterior means and 95% credible intervals adopting a Gaussian prior, and obtained using the Gibbs sampler algorithm and the importance sampling approach: in the latter case, bounds of credible interval are computed using a Gaussian approximation to the posterior.

5.3.1 Variable Selection

Here we implement a variable selection procedure based on the posterior probabilities of all possible subsets of covariates, adopting a uniform prior over the set of models and compatible Gaussian priors within each model, both in the logit and in the probit case. Using the importance sampling algorithm described in § 4.2, we are able to quickly compute, for each model, the normalizing constant. Assuming that the intercept is always included, one has $2^{p-1} = 2^6 = 64$ possible combinations of covariates. Let $M_L$ and $M_G$ be the model subspaces in the the logit and probit cases respectively. Let $M_{L,j}$ and $M_{G,j}$ be the $j$-th logit and probit model for $j = 1, 2, \ldots, 64$ respectively. Finally, let $S_L(\beta_i)$ and $S_G(\beta_i)$ be the subset of models such that the $i$-th covariate is included. For fixed $K = G, L$ we compute the posterior probability of inclusion of each single covariate as

$$P(S_K(\beta_i)|Y) = \sum_{j \in S_K(\beta_i)} P(Y|M_{K,j})/\sum_{j=1}^{64} P(Y|M_{K,j}); \quad (10)$$
Figure 8: Pima Indians, logit model: Posterior means and 95% credible sets of the 6 $\beta$ coefficients plus the intercept $\beta_1$: Gaussian prior (blue); Cauchy (red), Laplace with independent components (black) and Dirichlet-Laplace (green).

Figure 9: Pima Indians, probit model: Posterior means and 95% credible sets of the 6 $\beta$ coefficients plus the intercept $\beta_1$: Gaussian prior (blue); Cauchy (red), Laplace with independent components (black) and Dirichlet-Laplace (green).
an interesting by-product is that one can also easily compute the marginal likelihood
for the logit and probit families of models

\[ P(Y|\mathbf{M}_K) = \sum_{j=1}^{64} P(Y|\mathbf{M}_{K,j})/64, \]

and then the corresponding posterior probability as

\[ P(\mathbf{M}_K|Y) = \frac{P(Y|\mathbf{M}_K)}{P(Y|\mathbf{M}_L) + P(Y|\mathbf{M}_G)}. \]  (11)

Table 2: Posterior probabilities

|                 | Logit   | Probit  |
|-----------------|---------|---------|
| Pregnant        | 0.9945  | 0.9939  |
| Glucose         | \approx 1 | \approx 1 |
| Pressure        | 0.05    | 0.023   |
| Mass            | 0.9999  | 0.9999  |
| Pedigree        | 0.8924  | 0.5861  |
| Age             | 0.1323  | 0.077   |
| Subspace        | 0.9436  | 0.0564  |

Table 2 reports the results of (10) and (11). We highlight that logit and probit quite
agree; in fact, by assuming that a covariate is selected if its posterior inclusion prob-
ability is larger than 0.5 as in Barbieri and Berger [2004], the two families of models
select the same variables; however we note that the probit model is slightly more con-
servative. There is a bit of disagreement only for the pedigree variable. However,
the global evidence in favour of the logit model is overwhelming.
Acknowledgement

Brunero Liseo’s research has been supported by Sapienza Università di Roma, grant Progetti H2020 - Collaborativi, n.PH11916B88B59064.

Appendix 1

Sampling from $V_i | T_i$

From equation (9), we compute the expected number $M$ of proposals before accepting a value,

$$ M = M^* \frac{f_T(t)}{f_{\tilde{T}}(t)} . $$

Then the acceptance probability $M^{-1}$ depends on the values of $|t|$ and $\alpha$. Efficient values for the hyperparameter $\alpha$ of the proposal were obtained numerically

$$ \alpha = \begin{cases} 
1.99 & \text{if } 0 \leq |t| \leq 2.2878 \\
2.17 & \text{if } 2.2878 < |t| \leq 3.1572 \\
1.8982 + 0.0156|t| + 0.0349t^2 & \text{if } 3.1572 < |t| \leq 6.50 \\
0.4982 + 0.4376|t| + 0.0012t^2 & \text{if } 6.50 < |t| \leq 29.33 \\
-0.3201 + 0.4986|t| & \text{if } |t| > 29.33 
\end{cases} , $$

The above algorithm is quite efficient since drawing values from the Inv.Gamma is cheap and the acceptance probability is always greater than 0.7 as long as $|t| \leq 2750$. Even beyond that threshold, the algorithm performs relatively well; if $|t| = 10^6$, the acceptance probability is approximately 0.014 but in this case we are able to draw a sample of size $10^5$ in about 4.5 seconds with a laptop of 16 Gb of RAM and, as CPU, an i7-8750H.

Gaussian Prior: Importance Sampling

We first consider the case when $n >> p$, the other situations will be discussed later. In both cases the first step consists in approximating the pSUN density with a SUN distribution. Notice that the Kullback-Leibler divergence between a Logistic distribution and a $N(0,\sigma^2)$ density is

$$ KL(Logis(0, 1)||N(0, \sigma^2)) = \frac{1}{2} \log(2\pi\sigma^2) + \frac{\pi^2}{6\sigma^2} - 2 $$

and the value of $\sigma^2$ which minimizes expression (12) is the expected value of the logistic Kolmogorov density $V_i$, that is $E(V_i) = \pi^2/3$. Then we fix the latent variable $V_i$ of a pSUN distribution at its expected value to obtain a SUN density, when the prior is Gaussian.

We propose to adopt an importance sampling approach with two different importance densities according whether $n >> p$ or not.
Case I

\(n >> p\), We use a Student \(t\) distribution with location vector and scale matrix inherited by the above SUN density and a large number of degrees of freedom, say 50. The first two cumulants of \(Y \sim SU_{p,n}(\Theta, A, b, \xi, \Omega)\) are

\[
E(Y) = \xi + \text{diag}^{1/2}(\Omega)[E(Z)|T \leq AZ + b],
\]

\[
\text{Var}(Y) = \text{diag}^{1/2}(\Omega)[\text{Var}(Z)|T \leq AZ + b] \text{diag}^{1/2}(\Omega).
\]

Now we show how to obtain the conditional mean and variance of \(Z\). Let \(U = T - AZ\); then \(U \sim N_n(0, \Theta + A\Omega A')\) and \(\text{Cov}(Z, U) = -\Omega A'\), therefore

\[
Z|U \sim N_p\left(-\Omega A'\Theta + A\Omega A'\right)^{-1}U, \quad \Omega - \Omega A'\Theta + A\Omega A'\right)^{-1}A\Omega.
\]

Then we obtain

\[
[E(Z)|U \leq b] = E_{U|U \leq b}\left(E(Z)|U\right) = -\Omega A'\Theta + A\Omega A'\right)^{-1}[E(U)|U \leq b],
\]

\[
[\text{Var}(Z)|U \leq b] = \text{Var}_{U|U \leq b}\left(\text{Var}(Z)|U\right) = \Omega - \Omega A'\Theta + A\Omega A'\right)^{-1}(\Theta + A\Omega A')^{-1}A\Omega.
\]

The last expression is

\[
[\text{Var}(Z)|U \leq b] = \text{Var}(Z) - H(\text{Var}(U) - [\text{Var}(U)|U \leq b])H',
\]

where \(H = \Omega A'\Theta + A\Omega A'\right)^{-1}\). The quantities \(E(U|U \leq b)\) and \(\text{Var}(U|U \leq b)\) can be easily computed using the importance sampling algorithm in [Botev, 2017].

Case II

When the sample size is not so larger than \(p\), the approximation of a pSUN with a Student \(t\) distribution does not hold and we adopt a scale mixture of SUN densities. More in detail,

\[
\zeta_1 = \xi + \sqrt{S}\text{diag}^{\frac{1}{2}}(\Omega)\zeta_0,
\]

\[
S \sim \text{Inv.Gamma} \left(\frac{\nu}{2}, \frac{\nu}{2}\right) \perp \zeta_0 \sim \text{SUN}_{p,n}(\Theta, A, b, 0, \Omega),
\]

that is

\[
\zeta_1 = \xi + \sqrt{S}\text{diag}^{\frac{1}{2}}(\Omega)Z|T \leq AZ + b;
\]

notice that \((T \leq AZ + b) \equiv (\sqrt{ST} \leq \sqrt{S AZ} + \sqrt{b})\), so we define

\[
\tilde{T} = \sqrt{ST}, \quad \tilde{Z} = \sqrt{S}Z, \quad \tilde{b} = \sqrt{b}
\]

and we obtain

\[
\zeta_1 = \xi + \text{diag}^{\frac{1}{2}}(\Omega)\tilde{Z} | \tilde{T} \leq A\tilde{Z} + \tilde{b}.
\]

Then we compute the density of \(\zeta_1\) through the above representation:

\[
f_{\zeta_1}(x) = t_{\nu, \Omega}(x - \xi) \frac{P(\tilde{T} \leq A\tilde{Z} + \tilde{b} | \tilde{Z} = \text{diag}^{\frac{1}{2}}(\Omega)(x - \xi))}{\Phi_{\Theta + A\tilde{Z}|\tilde{Z} = A\tilde{Z} + \tilde{b}}(\tilde{b})},
\]

24
where $t_{\nu,\Omega_0}(\cdot)$ is the density function of a Student $t$ random variable with $\nu_0$ degrees of freedom and scale matrix $\Omega_0$. It is worth notice that the normalizing constant of $\zeta_1$ is the same as the one of the associated SUN. Then we only need to compute the probability at the numerator. In this respect, note that

$$S|\hat{Z} = \hat{z} \sim \text{Inv.Gamma} \left( \frac{\nu + p}{2}, \frac{\nu + p}{2} \right),$$

$$\frac{\nu + p}{\nu + \hat{z}'\hat{\Omega}^{-1}\hat{z}} S|\hat{Z} = \hat{z} \sim \text{Inv.Gamma} \left( \frac{\nu + p}{2}, \frac{\nu + p}{2} \right).$$

Then

$$P \left( \hat{T} \leq A\hat{Z} + \hat{b} \big| \hat{Z} \right) = T_{\nu+p,-b,\Theta} \left( \frac{\nu + p}{\nu + (x - \xi)'\hat{\Omega}^{-1}(x - \xi)} \text{Adiag}^{-\frac{1}{2}}(\Omega)(x - \xi) \right),$$

where $T_{\nu_0,\Theta_0}(\cdot)$ is the CDF of a random variable $R_1$ with the following stochastic representation:

$$R_1 = \sqrt{S}(R_0 + b),$$

where $S \sim \text{Inv.Gamma} \left( \frac{\nu_0}{2}, \frac{\nu_0}{2} \right)$ and $R_0 \sim N_n(0, \Theta_0).$

When $p = 1$ then (13) reduces to a non-central $t$ distribution. Therefore

$$f_{\zeta_1}(x) = t_{\nu,\hat{\Omega}}(x - \xi) T_{\nu+p,-b,\Theta} \left( \frac{\nu + p}{\nu + (x - \xi)'\hat{\Omega}^{-1}(x - \xi)} \text{Adiag}^{-\frac{1}{2}}(\Omega)(x - \xi) \right).$$

In general, the computation of $T_{\nu+p,-b,\Theta}$ is not simple; however, in the logistic regression with a Gaussian prior, $\Theta$ is a diagonal matrix, say $\Theta = D$. Then

$$T_{\nu+p,-b,D}(x) = \int_{0}^{\infty} \prod_{i=1}^{n} \Phi \left( \frac{x_i}{\sqrt{S}D_{i,i}} + b_i \right) f_{\bar{S}}(\bar{\tilde{s}}) d\bar{\tilde{s}},$$

where $f_{\bar{S}}(\cdot)$ is the density function of an inverse Gamma random variable with parameters $((\nu + p)/2, (\nu + p)/2)$. Let $\bar{u} = F_{\bar{S}}(\bar{\tilde{s}})$, we obtain

$$T_{\nu+p,-b,D}(x) = \int_{0}^{1} \prod_{i=1}^{n} \Phi \left( \frac{x_i}{\sqrt{F_{\bar{S}}^{-1}(\bar{u})D_{i,i}}} + b_i \right) d\bar{u},$$

and this expression suggests the use of a Quasi Monte Carlo algorithm, that is

$$T_{\nu+p,-b,D}(x) \approx \sum_{j=1}^{k} \prod_{i=1}^{n} \Phi \left( \frac{x_i}{\sqrt{F_{\bar{S}}^{-1}(u_j)D_{i,i}}} + b_i \right),$$

where $u_j = j/(k + 1)$ and we have set $k = 100.$
References

James H. Albert and Siddhartha Chib. Bayesian analysis of binary and polychotomous response data. *J. Amer. Statist. Assoc.*, 88(422):669–679, 1993. ISSN 0162-1459.

D. F. Andrews and C. L. Mallows. Scale mixtures of normal distributions. *J. Roy. Statist. Soc. Ser. B*, 36:99–102, 1974. ISSN 0035-9246.

Reinaldo B. Arellano-Valle and Adelchi Azzalini. On the unification of families of skew-normal distributions. *Scand. J. Statist.*, 33(3):561–574, 2006. ISSN 0303-6898. doi: 10.1111/j.1467-9469.2006.00503.x. URL https://doi.org/10.1111/j.1467-9469.2006.00503.x

A. Azzalini and A. Capitanio. *The skew-normal and related families*. IMS Monograph. Cambridge University Press, Cambridge., 2014.

M. M. Barbieri and J. O. Berger. Optimal predictive model selection. *The Annals of Statistics*, 32:870–897, 2004.

O. Barndorff-Nielsen. Exponentially decreasing distributions for the logarithm of particle size. In *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 401–409. The Royal Society, London, 1977.

Anirban Bhattacharya, Debdeep Pati, Natesh S. Pillai, and David B. Dunson. Dirichlet-Laplace priors for optimal shrinkage. *J. Amer. Statist. Assoc.*, 110(512):1479–1490, 2015. ISSN 0162-1459. doi: 10.1080/01621459.2014.960967. URL https://doi.org/10.1080/01621459.2014.960967

A. Bhattacharyya, S. Pal, R. Mitra, and S. Rai. Applications of bayesian shrinkage prior models in clinical research with categorical responses. *BMC Medical Research Methodology*, 22(1):1–19, 2022.

Z. I. Botev. The normal law under linear restrictions: simulation and estimation via minimax tilting. *J. R. Stat. Soc. Ser. B. Stat. Methodol.*, 79(1):125–148, 2017. ISSN 1369-7412. doi: 10.1111/rssb.12162. URL https://doi.org/10.1111/rssb.12162

Nicolas Chopin and James Ridgway. Leave Pima Indians alone: binary regression as a benchmark for Bayesian computation. *Statist. Sci.*, 32(1):64–87, 2017. ISSN 0883-4237. doi: 10.1214/16-STS581. URL https://doi.org/10.1214/16-STS581

Daniele Durante. Conjugate Bayes for probit regression via unified skew-normal distributions. *Biometrika*, 106(4):765–779, 2019. ISSN 0006-3444. doi: 10.1093/biomet/asz034. URL https://doi.org/10.1093/biomet/asz034

S. Fruhwirth-Schnatter and R. Fruhwirth. Data augmentation and MCMC for binary and multinomial logit models. In *Statistical Modelling and Regression Structures*, pages 111–132. Springer-Verlag, Berlin, 2010.

Andrew Gelman, Aleks Jakulin, Maria Grazia Pittau, and Yu-Sung Su. A weakly informative default prior distribution for logistic and other regression models. *Ann. Appl. Stat.*, 2(4):1360–1383, 2008. ISSN 1932-6157. doi: 10.1214/08-AOAS191. URL https://doi.org/10.1214/08-AOAS191
Robert B. Gramacy and Nicholas G. Polson. Simulation-based regularized logistic regression. *Bayesian Anal.*, 7(3):567–589, 2012. ISSN 1936-0975. doi: 10.1214/12-BA719. URL [https://doi.org/10.1214/12-BA719](https://doi.org/10.1214/12-BA719).

Chris C. Holmes and Leonhard Held. Bayesian auxiliary variable models for binary and multinomial regression. *Bayesian Anal.*, 1(1):145–168, 2006. ISSN 1936-0975. doi: 10.1214/06-BA105. URL [https://doi.org/10.1214/06-BA105](https://doi.org/10.1214/06-BA105).

Paolo Onorati and Brunero Liseo. Random Number Generator for the Kolmogorov Distribution. *arXiv*2208.13598, 2022. URL [https://arxiv.org/abs/2208.13598](https://arxiv.org/abs/2208.13598).

Nicholas G. Polson, James G. Scott, and Jesse Windle. Bayesian inference for logistic models using Pólya-Gamma latent variables. *J. Amer. Statist. Assoc.*, 108(504):1339–1349, 2013. ISSN 0162-1459. doi: 10.1080/01621459.2013.829001. URL [https://doi.org/10.1080/01621459.2013.829001](https://doi.org/10.1080/01621459.2013.829001).

Daniel Sabanés Bove and Leonhard Held. Hyper-g priors for generalized linear models. *Bayesian Anal.*, 6(3):387–410, 2011. ISSN 1936-0975. doi: 10.1214/10-BA1649. URL [https://doi.org/10.1214/10-BA1649](https://doi.org/10.1214/10-BA1649).

D. Siegmund. Importance sampling in the Monte Carlo study of sequential tests. *Ann. Statist.*, 4(4):673–684, 1976. ISSN 0090-5364.

Leonard A. Stefanski. A normal scale mixture representation of the logistic distribution. *Statist. Probab. Lett.*, 11(1):69–70, 1991. ISSN 0167-7152. doi: 10.1016/0167-7152(91)90181-P. URL [https://doi.org/10.1016/0167-7152(91)90181-P](https://doi.org/10.1016/0167-7152(91)90181-P).