THE CELLULAR SPECTRUM OF A POSET

RENAN M. MEZABARBA, LEANDRO F. AURICHI, AND LÚCIA R. JUNQUEIRA

Abstract. We investigate the notion of productive cellularity on arbitrary preorders by generalizing an intrinsic characterization of productively ccc preorders.

Introduction

Along this work, we shall be concerned with the behavior of the cellularity of preordered sets (posets for short) with respect to the product operation. Let us start by recalling some basic concepts regarding posets. First of all, a preorder \( \leq \) on a set \( P \) is just a reflexive and transitive binary relation – and it is called a partial order if in addition \( \leq \) is antisymmetric. Now, two elements \( p \) and \( q \) of a poset \((P, \leq)\) are said to be compatible if there exists an \( r \in P \) such that \( r \leq p, q \). Naturally, we say that \( p \) and \( q \) are incompatible if they are not compatible, what we abbreviate with \( p \perp q \). Finally, a subset \( A \subset P \) of pairwise incompatible elements is called an antichain of \( P \).

The cellularity of a poset \( P \), denoted by \( c(P) \), is the supremum of all infinite cardinals of the form \( |A| \) for some antichain \( A \subset P \). In this way, the so called countable chain condition (ccc for short) is achieved by the poset \( P \) precisely when the equality \( c(P) = \aleph_0 \) holds. Finally, for a poset \((Q, \preceq)\), we can consider the set \( P \times Q \) preordered by the relation \( \sqsubseteq \), which is defined by the rule

\[
(p, q) \sqsubseteq (p', q') \iff p \leq p' \text{ and } q \preceq q'.
\]

Here is where the story gets interesting, at least from a foundational perspective: it may be the case that \( P \times Q \) is ccc whenever \( P \) and \( Q \) are ccc posets, or it may not. Essentially, it all depends on the additional hypothesis we add to the ZFC axioms.

Indeed, in the realm of Martin’s Axiom (MA) plus the negation of the Continuum Hypothesis (CH), one can prove that every product of ccc posets is a ccc poset\(^1\) while a Suslin line turns out to be a ccc poset whose square is not ccc\(^2\). Since each of the statements “MA + ¬CH” and “there exists a Suslin line” are independent of ZFC\(^3\), it follows that productivity of ccc posets is itself independent of ZFC.

Since there are posets whose product with every ccc poset is ccc, e.g., the countable posets, it seems natural to ask, in ZFC, when a poset \( P \) is productively ccc, meaning that \( P \times Q \) is ccc whenever \( Q \) is ccc. As it was shown in \([6]\), a ZFC characterization of productively ccc posets can be obtained by analyzing the cellularity of some posets of antichains. The technique used there was an adaptation of the

\(^{1}\)Folklore, but possibly due to Juhász.

\(^{2}\)Attributed to Kurepa.

\(^{3}\)The reader may find the details in Kunen\([4]\).
methods applied by Aurichi and Zdomskyy [2] to characterize productively Lindelöf spaces.

Here, we generalize the results presented in [6], by characterizing what we call cellular spectrum\(^4\) of the poset \(P\), denoted \(\text{Sp}(P)\): the class of those infinite cardinals \(\kappa\) such that for all posets \(Q\),
\[
c(\kappa) \leq \kappa \Rightarrow c(P \times Q) \leq \kappa.
\]

The paper is organized as follows. In the first section, we state and prove our main results concerning an intrinsic characterization of \(\text{Sp}(P)\). In the second section, we investigate cardinals invariants that belongs to the spectrum, while in the third section we translate some classic theorems about ccc topological spaces to this spectral context. Finally, the last section is dedicated to discussing new perspectives for old (and open) problems concerning productively ccc posets. Along the text, \(\kappa\) and \(\lambda\) denote infinite cardinals.

1. The main theorem

Let us start by fixing some notations that will be useful. For a poset \(P\), we consider the cellular spectrum of \(P\), which is the class
\[
\text{Sp}(P) := \{\kappa \geq \aleph_0 : \forall Q (c(Q) \leq \kappa \Rightarrow c(P \times Q) \leq \kappa)\}.
\]

Our main goal in this section is to determine necessary and sufficient conditions in order to decide whether a given cardinal \(\kappa \geq \aleph_0\) belongs to \(\text{Sp}(P)\). To this end, we say that a family \(\mathcal{A}\) of antichains of \(P\) is a \(\kappa\)-large family if \(\bigcup \mathcal{A} \geq \kappa^+\), and we denote by \(\mathcal{L}_\kappa(P)\) the collection of all such families. Finally, for a \(\kappa\)-large family \(\mathcal{A} \in \mathcal{L}_\kappa(P)\), we set \(\mathcal{F}(\mathcal{A}) = \bigcup \mathcal{A} \subseteq [\kappa]^{<\aleph_0}\), partially ordered by the reverse inclusion relation.

We shall use posets of the form \(\mathcal{F}(\mathcal{A})\) in order to characterize the spectrum of \(P\). This can be done because incompatibility conditions in \(\mathcal{F}(\mathcal{A})\) translate to compatibility conditions of \(P\), as we show in the following lemma.

**Lemma 1.1.** Let \(\mathcal{A} \in \mathcal{L}_\kappa(P)\) and let \(P, Q \in \mathcal{F}(\mathcal{A})\). Then \(P \perp Q\) in \(\mathcal{F}(\mathcal{A})\) if, and only if, \(P \cup Q \notin \mathcal{F}(\mathcal{A})\).

**Proof.** Note that if \(P \cup Q \in \mathcal{F}(\mathcal{A})\), then \(P, Q \subseteq P \cup Q\), showing that \(P \perp Q\) does not hold. Conversely, if some \(R \in \mathcal{F}(\mathcal{A})\) contains both \(P\) and \(Q\), then there is an \(A \in \mathcal{A}\) such that \(R \subseteq A\), showing that \(P \cup Q\) is a finite subset of \(A\), i.e., \(P \cup Q \in \mathcal{F}(\mathcal{A})\). \(\square\)

**Theorem 1.2.** Let \(P\) be a poset. Then \(\kappa \in \text{Sp}(P)\) if and only if \(c(\mathcal{F}(\mathcal{A})) > \kappa\) for all \(\mathcal{A} \in \mathcal{L}_\kappa(P)\).

**Proof.** If \(\kappa \in \text{Sp}(P)\) and \(\mathcal{A} \in \mathcal{L}_\kappa(P)\) is such that \(c(\mathcal{F}(\mathcal{A})) \leq \kappa\), then we have that \(c(P \times \mathcal{F}(\mathcal{A})) \leq \kappa\). Now, let
\[
T := \{(p, \{p\}) : p \in \bigcup \mathcal{A}\}.
\]

Since \(\mathcal{A}\) is \(\kappa\)-large, the family \(T\) cannot be an antichain in \(P \times \mathcal{F}(\mathcal{A})\). Thus, there are \(p, p' \in \bigcup \mathcal{A}\) with \(p \neq p'\), \(r \in P\) and \(F \in \mathcal{F}(\mathcal{A})\) such that
\[
(r, F) \subseteq (p, \{p\}, (p', \{p'\})
\]

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\(^4\)The word “spectrum” references to the “frequency spectrum”, considered by Arhangel’skii in [1], a class of cardinals related to the tightness of products of topological spaces.
implying $p \not\perp p'$ and $\{p, p'\} \subseteq F \subseteq \mathcal{A}$ for some $\mathcal{A} \in \mathcal{A}$, showing that $\mathcal{A}$ is not an antichain, a contradiction.

Conversely, supposing $\kappa \not\in \mathrm{Sp}(\mathcal{P})$, we shall obtain a $\kappa$-large family $\mathcal{A}$ such that $c(\mathcal{F}(\mathcal{A})) \leq \kappa$. Let $\mathcal{Q}$ be a poset witnessing $\kappa \not\in \mathrm{Sp}(\mathcal{P})$, i.e., with $c(\mathcal{Q}) \leq \kappa$ and such that there exists an antichain $\mathcal{W} \subset \mathcal{P} \times \mathcal{Q}$ with $|\mathcal{W}| = \kappa^+$. For each $r \in \mathcal{Q}$, let $\mathcal{A}_r := \{p \in \mathcal{P} : \exists q \in \mathcal{Q}(r \leq q \text{ and } (p, q) \in \mathcal{W})\}$. We claim that $\mathcal{A} := \{\mathcal{A}_r : r \in \mathcal{Q}\}$ is the desired $\kappa$-large family.

Note that $\mathcal{A}_r$ is clearly an antichain for each $r \in \mathcal{Q}$, while $|\bigcup \mathcal{A}| = \kappa^+$ holds by the pigeonhole principle, showing that $\mathcal{A} \in \mathcal{L}_\kappa(\mathcal{P})$. It remains to show that $c(\mathcal{F}(\mathcal{A})) \leq \kappa$. Indeed, for if $\mathcal{F} \subset \mathcal{F}(\mathcal{A})$ is such that $|\mathcal{F}| = \kappa^+$, for each $F \in \mathcal{F}$ we take $r_F \in \mathcal{Q}$ with $F \subset \mathcal{A}_{r_F}$. Now we consider the set $\mathcal{R} := \{r_F : F \in \mathcal{F}\} \subseteq \mathcal{Q}$, that we shall use to obtain the desired inequality.

There are two cases:

1. if $|\mathcal{R}| \leq \kappa$, then the pigeonhole principle gives $F, G \in \mathcal{F}$ with $F \neq G$ and $r \in \mathcal{R}$ such that $F \cup G \subseteq \mathcal{A}_r$, showing that $F \cup G \in \mathcal{F}(\mathcal{A})$;

2. if $|\mathcal{R}| = \kappa^+$, then $c(\mathcal{Q}) \leq \kappa$ gives $F, G \in \mathcal{F}$ with $F \neq G$ and $r \in \mathcal{Q}$ such that $r \leq r_F, r_G$; showing that $\mathcal{A}_{r_F} \cup \mathcal{A}_{r_G} \subseteq \mathcal{A}_r$, from which it follows that $F \cup G \in \mathcal{F}(\mathcal{A})$.

In both cases, we obtain $F, G \in \mathcal{F}$ with $F \neq G$, such that $F \cup G \in \mathcal{F}(\mathcal{A})$, which is equivalent to say that $F \perp G$ by the previous lemma, showing that $\mathcal{F}$ is not an antichain of $\mathcal{F}(\mathcal{A})$, as desired. \hfill \square

**Corollary 1.3.** A poset $\mathcal{P}$ is productively ccc if, and only if, $\mathcal{F}(\mathcal{A})$ is not ccc for all $\mathcal{A} \in \mathcal{L}_\kappa(\mathcal{P})$.

The above characterizations become clearer in their contrapositive versions. For instance, Corollary 1.3 says that if a poset $\mathcal{P}$ is not productively ccc, then there is a witness of the form $\mathcal{F}(\mathcal{A})$ for some $\mathcal{A} \in \mathcal{L}_{\kappa_0}(\mathcal{P})$. Thus, if we have $\mathcal{L}_{\kappa_0}(\mathcal{P}) = \emptyset$, then $\mathcal{P}$ is vacuously productively ccc, since there are no witnesses to the contrary. This gives a very clean proof for the well known fact that countable posets are productively ccc. More generally, we have the following.

**Corollary 1.4.** If $|\mathcal{P}| < \kappa$, then $\kappa \in \mathrm{Sp}(\mathcal{P})$.

**Remark 1.** Although the presentation of Theorem 1.2 has been order-theoretic flavored, we originally found it out in a topological context, closer to [2]. In this case, the cellularity of a space $X$ is the cellularity of the poset $\mathcal{O}_X$ of nonempty open sets of $X$, while the spectral spectrum of $X$, also denoted by $\mathrm{Sp}(X)$, is the class of those infinite cardinals $\kappa$ such that $c(X \times Y) \leq \kappa$ holds for all topological spaces $Y$ with $c(Y) \leq \kappa$.

It seems quite clear that the arguments we used to settle Theorem 1.2 can be carried out to topological spaces. Still, this can be done indirectly, by using the characterization we already proved for posets.

**Theorem 1.5.** For a topological space $X$ one has $\mathrm{Sp}(X) = \mathrm{Sp}(\mathcal{O}_X)$.

**Proof.** Suppose $\kappa \in \mathrm{Sp}(X)$ and let $\mathcal{Q}$ be a poset with $c(\mathcal{Q}) \leq \kappa$. Denote by $Y$ the set $\mathcal{Q}$ endowed with the topology generated by the sets of the form $\{s \in \mathcal{Q} : s \leq q\}$, which clearly satisfies $c(Y) = c(\mathcal{Q})$. Now, the hypothesis gives $c(X \times Y) \leq \kappa$, while a straightforward calculation gives

$$c(X \times Y) = c(\mathcal{O}_X \times \mathcal{O}_Y) = c(\mathcal{O}_X \times \mathcal{Q}),$$
showing that $\kappa \in \text{Sp} (\mathcal{D}_X)$. The converse is trivial. \hfill \Box

2. A FEW INHABITANTS OF THE SPECTRUM

Besides showing that $\text{Sp} (\mathbb{P})$ is nonempty for all posets $\mathbb{P}$, Corollary 1.4 indicates that the possibly interesting cardinals in the cellular spectrum are smaller or equal to $|\mathbb{P}|$. In particular, it makes sense to define the productive cellularity of $\mathbb{P}$ to be the cardinal

$$\text{pc}(\mathbb{P}) := \min \text{Sp} (\mathbb{P}),$$

in reference to the fact that $\mathbb{P}$ is productively ccc if, and only if, $\text{pc}(\mathbb{P}) = \aleph_0$.

Since any poset $\mathbb{T}$ with a single element satisfies $c(\mathbb{T}) \leq \kappa$ for all $\kappa \geq \aleph_0$, it follows that for every poset $\mathbb{P}$ one has $c(\mathbb{P}) \leq \text{pc}(\mathbb{P})$, from which it follows that

$$(1) \quad c(\mathbb{P}) \leq \text{pc}(\mathbb{P}) \leq |\mathbb{P}|^+.$$

We shall explore the gap between the cardinals $\text{pc}(\mathbb{P})$ and $|\mathbb{P}|^+$ through the rest of this section.

Recall that a subset $D$ of a poset $\mathbb{P}$ is called dense if for all $p \in \mathbb{P}$ there is a $d \in D$ such that $d \leq p$. The density of $\mathbb{P}$, denoted by $d(\mathbb{P})$, is the least infinite cardinal of the form $|D|$ with $D \subset \mathbb{P}$ dense, which is a generalization of the separability in topological spaces.

Since the cardinality of an antichain of $\mathbb{P}$ is bounded by the cardinality of every dense subset of $\mathbb{P}$, it follows immediately that $c(\mathbb{P}) \leq d(\mathbb{P})$. This inequality can strengthen in the following way.

**Theorem 2.1.** If $\mathbb{P}$ is a poset, then $d(\mathbb{P}) \in \text{Sp} (\mathbb{P})$.

**Proof.** Let $D \subset \mathbb{P}$ be a dense subset and call $\kappa := |D|$. We shall prove that $\kappa$ belongs to the cellular spectrum of $\mathbb{P}$. By Theorem 1.2 we need to take $\mathcal{A} \in \mathcal{L}_\kappa (\mathbb{P})$ and show that $c(\mathcal{F}(\mathcal{A})) > \kappa$. Since $D$ is dense, it follows that for each $a \in \bigcup \mathcal{A}$ there exists a $\delta(a) \in D$ such that $\delta(a) \leq a$. Hence there exists a $d \in D$ such that the set $A := \{a \in \bigcup \mathcal{A} : \delta(a) = d\}$ has cardinality at least $\kappa^+$. Finally, since $d \leq a$ for all $a \in A$, one can readily see that the family $\{\{a\} : a \in A\}$ witnesses the inequality $c(\mathcal{F}(\mathcal{A})) > \kappa$, as desired. \hfill \Box

The arguments used above actually improve Corollary 1.4 allowing one to prove the following.

**Corollary 2.2.** If $\kappa \geq d(\mathbb{P})$, then $\kappa \in \text{Sp} (\mathbb{P})$.

In particular, (1) can be replaced by

$$(2) \quad c(\mathbb{P}) \leq \text{pc}(\mathbb{P}) \leq d(\mathbb{P}).$$

Note that in order to finish the proof of Theorem 2.1 we used a strong property of the family $A$, namely the existence of $d \in \mathbb{P}$ such that $d \leq a$ for all $a \in A$. As we shall see below, this condition can be relaxed.

For a natural number $n \geq 2$, a subset $A \subset \mathbb{P}$ is called $n$-linked if for all $F \in [A]^n$ there exists $p_A \in \mathbb{P}$ such that $p_A \leq p$ for each $p \in F$; $A$ is called centered if $A$ is $n$-linked for all $n \geq 2$. Then we have the following.

**Lemma 2.3.** Let $\mathbb{P}$ be a poset and $\mathcal{A} \in \mathcal{L}_\kappa (\mathbb{P})$. If a subset $A \subset \bigcup \mathcal{A}$ is $n$-linked for some natural number $n \geq 2$, then the family $\{\{a\} : a \in A\}$ is an antichain in $\mathcal{F}(\mathcal{A})$. 


Proof. For \( a, b \in A \) with \( a \neq b \), there is an \( r \in \mathbb{P} \) such that \( r \leq a, b \). Then we have \( \{a\}, \{b\} \in \mathcal{F}(\mathcal{A}) \) while \( \{a\} \cup \{b\} = \{a, b\} \notin \mathcal{F}(\mathcal{A}) \), yielding \( \{a\} \perp \{b\} \), by Lemma 1.1.

Although the above lemma seems to be innocuous, it has some interesting consequences. Let us recall a few more concepts in order to apply Lemma 2.3. Following [7], we say that a poset \( \mathbb{P} \) has the \( K_\alpha \)-property if for each \( A \in [\mathbb{P}]^{\aleph_1} \) there exists an \( n \)-linked subset \( B \in [A]^{\aleph_1} \). By replacing the occurrence of the term “\( n \)-linked” with “centered”, we obtain the property usually called \( \aleph_1 \)-precaliber, but for sake of brevity we shall refer to it simply by \( K_\omega \)-property. The letter “K” is a reference to Knaster, who first considered this type of property, for \( n = 2 \).

In the same way cellularity generalizes the countable chain condition, we define below the \( Knaster \) invariants of \( \mathbb{P} \) in order to generalize \( K_\alpha \) and \( K_\sigma \) properties.

More precisely, for each natural number \( n \geq 2 \) we define the cardinal

(3) \[ \mathcal{K}_n(\mathbb{P}) := \min \{ \kappa \geq \aleph_0 : \forall A \in [\mathbb{P}]^{\kappa^+} \exists B \in [A]^{\kappa^+} (B \text{ is } n \text{-linked}) \} , \]

and we let

(4) \[ \mathcal{K}_\omega(\mathbb{P}) := \min \{ \kappa \geq \aleph_0 : \forall A \in [\mathbb{P}]^{\kappa^+} \exists B \in [A]^{\kappa^+} (B \text{ is centered}) \} . \]

Note that for a poset \( \mathbb{P} \) and an ordinal \( \alpha \in [2, \omega) \), \( \mathbb{P} \) has the \( K_\alpha \)-property if, and only if, \( \mathcal{K}_\alpha(\mathbb{P}) = \aleph_0 \). The relations between the Knaster properties with the countable chain condition are in some sense preserved in the spectral context.

For a dense subset \( D \subset \mathbb{P} \) with \( |D| = \kappa \), the same reasoning applied in Theorem 2.1 allows one to prove that for every \( A \in [\mathbb{P}]^{\kappa^+} \) there is a centered subset \( B \in [A]^{\kappa^+} \), showing that \( \mathcal{K}_\omega(\mathbb{P}) \leq d(\mathbb{P}) \). Since we clearly have \( \mathcal{K}_\alpha(\mathbb{P}) \leq \mathcal{K}_\beta(\mathbb{P}) \) for \( \alpha \leq \beta \leq \omega \), it follows that

\[ \mathcal{K}_2(\mathbb{P}) \leq \mathcal{K}_n(\mathbb{P}) \leq \mathcal{K}_{n+1}(\mathbb{P}) \leq \mathcal{K}_\omega(\mathbb{P}) \leq d(\mathbb{P}) \]

holds for every poset \( \mathbb{P} \). We now put \( \text{pc}(\mathbb{P}) \) in one extreme of the above inequalities.

Theorem 2.4. If \( \mathbb{P} \) is a poset, then \( \mathcal{K}_2(\mathbb{P}) \in \text{Sp}(\mathbb{P}) \).

Proof. Let \( \kappa := \mathcal{K}_2(\mathbb{P}) \) and let \( \mathcal{A} \in \mathcal{L}_\alpha(\mathbb{P}) \). Since \( |\bigcup \mathcal{A}| > \kappa \), there exists an \( A \subseteq \bigcup \mathcal{A} \) such that \( |A| = \kappa^+ \). Now, there exists a 2-linked subset \( B \in [A]^{\kappa^+} \), so the conclusion follows from Lemma 2.3.

Differently of what happened in Theorem 2.1, we are not able to adapt the above argument to show that every \( \kappa \geq \mathcal{K}_2(\mathbb{P}) \) belongs to \( \text{Sp}(\mathbb{P}) \). Still, Lemma 2.3 can be used similarly to prove that the cardinal invariants \( \mathcal{K}_\alpha(\mathbb{P}) \) belongs to \( \text{Sp}(\mathbb{P}) \) for all \( \alpha \in [2, \omega) \). In summary, for every poset \( \mathbb{P} \) and every natural number \( n \geq 2 \), we have

(5) \[ c(\mathbb{P}) \leq \text{pc}(\mathbb{P}) \leq \mathcal{K}_2(\mathbb{P}) \leq \mathcal{K}_n(\mathbb{P}) \leq \mathcal{K}_{n+1}(\mathbb{P}) \leq \mathcal{K}_\sigma(\mathbb{P}) \leq d(\mathbb{P}). \]

3. The spectrum of products and some topological translations

Although the (productive) cellularity of posets may be interesting by itself, the topological interpretations of the previous results deserve some attention. For a warming up example, the topological counterpart of Theorem 2.1 says that separable spaces are productively ccc. Indeed, similar to what occurs with cellularity, the density of a topological space \( X \), denoted by \( d(X) \), is the density of the poset \( \mathcal{O}_X \). Thus, in this context, Theorems 1.5 and 2.1 together say that the density of a
space \( X \) belongs to \( \text{Sp}(X) \) and, since \( X \) is separable if and only if \( d(X) = \aleph_0 \), our claim follows. However, even more is known to be true:

**Proposition 3.1** (Fremlin [3], Corollary 12J). Every product of separable spaces is productively ccc.

The above proposition makes us wonder about the behavior of the cellular spectrum of a poset with respect to products. The very definition of the cellular spectrum implies that for posets \( P \) and \( Q \) one has
\[
\kappa \in \text{Sp}(P) \cap \text{Sp}(Q) \Rightarrow \kappa \in \text{Sp}(P \times Q),
\]
showing that \( \text{Sp}(P) \cap \text{Sp}(Q) \subset \text{Sp}(P \times Q) \). The reverse inclusion follows from the next easy lemma, whose proof we left for the reader.

**Lemma 3.2.** If \( \varphi : P \to Q \) is an increasing function from the poset \( P \) onto the poset \( Q \), then \( \text{Sp}(P) \subset \text{Sp}(Q) \).

**Theorem 3.3.** If \( P \) and \( Q \) are posets, then \( \text{Sp}(P \times Q) = \text{Sp}(P) \cap \text{Sp}(Q) \).

**Proof.** We already have \( \text{Sp}(P) \cap \text{Sp}(Q) \subset \text{Sp}(P \times Q) \). Now, since the projections \( P \times Q \to P \) and \( P \times Q \to Q \) are both increasing and surjective, the reverse inclusion follows from the previous lemma.

In order to extend this result for arbitrary products of posets, we need to consider a slightly different product, closer to the topological counterpart of arbitrary products. We follow the definitions presented by Kunen in [4], where the reader may find more details.

Let \( \{P_i : i \in I\} \) be a nonempty family of posets such that for each \( i \in I \) there is a largest element \( 1_i \in P_i \). Such posets are called forcing posets in [4]. The finite support product of the forcing posets \( P_i \), denoted by \( \prod_{i \in I} P_i \), is the subset of \( \prod_{i \in I} P_i \) whose elements are those \( I \)-tuples \( f \) such that \( |\{i \in I : f_i \neq 1_i\}| < \aleph_0 \), endowed with the coordinate-wise preordering. In some sense, this is the order-theoretic version of the standard topology on arbitrary products of topological spaces.

**Theorem 3.4.** For a nonempty family \( \{P_i : i \in I\} \) of forcing posets one has
\[
\text{Sp}\left(\prod_{i \in I} P_i\right) = \bigcap_{i \in I} \text{Sp}(P_i).
\]

**Proof.** The inclusion \( \text{Sp}\left(\prod_{i \in I} P_i\right) \subset \bigcap_{i \in I} \text{Sp}(P_i) \) follows from Lemma 3.2. On the other hand, the reverse inclusion can be proved with a straightforward application of the \( \Delta \)-system lemma.

With the previous theorem established, Fremlin’s result about separable spaces becomes the topological counterpart of the following.

**Corollary 3.5.** If \( \{P_i : i \in I\} \) is a nonempty family of forcing posets, then \( \sup_{i \in I} d(P_i) \in \text{Sp}\left(\prod_{i \in I} P_i\right) \).

**Proof.** Since \( d(P_j) \leq \sup_{i \in I} d(P_i) \) and \( d(P_j) \in \text{Sp}(P_j) \), it follows from Corollary 2.2 that \( \sup_{i \in I} d(P_i) \in \text{Sp}(P_j) \) for all \( j \in I \), showing that
\[
\sup_{i \in I} d(P_i) \in \bigcap_{i \in I} \text{Sp}(P_i) = \text{Sp}\left(\prod_{i \in I} P_i\right) \]
\( \square \).
4. Further questions and comments

Corollary 2.2 and the absence of a similar result for the Knaster invariants suggest a natural question about the behavior of the cardinals in the cellular spectrum. More precisely:

**Question 4.1.** Let $\mathbb{P}$ be a poset. Does every cardinal $\kappa$ such that $\text{pc}(\mathbb{P}) < \kappa < d(\mathbb{P})$ belongs to $\text{Sp}(\mathbb{P})$?

Concerning the Knaster invariants, we still do not know if they are consistently different from each other. On the other hand, they all coincide under a standard assumption.

**Example 4.2.** Assuming the existence of a Suslin Line $R$, one has $c(R) = \aleph_0$, while $\text{pc}(R) > \aleph_0$ since $R$ is not productively ccc. On the other hand, the inequality $d(X) \leq c(X)^+$ holds for every LOTS $X$. Thus, we have $d(R) = \aleph_1$, from which it follows that all the Knaster invariants of $R$ collapse to $\aleph_1$.

It may also be interesting to explore the connections of the cellular spectrum with Martin’s Axiom related topics. As we mentioned earlier, the standard strategy to show that $\text{MA+¬CH}$ implies that every ccc poset is productively ccc starts by showing that every ccc poset has the $K_\omega$-property. Note that we can restate both assertions, respectively, as the following implications:

(6) $\forall \mathbb{P} \ c(\mathbb{P}) = \aleph_0 \Rightarrow \text{pc}(\mathbb{P}) = \aleph_0$;  
(7) $\forall \mathbb{P} \ c(\mathbb{P}) = \aleph_0 \Rightarrow \mathcal{K}_\omega(\mathbb{P}) = \aleph_0$.

Now, since $\text{pc}(\mathbb{P}) \leq \mathcal{K}_\omega(\mathbb{P})$, it follows immediately that (7) $\Rightarrow$ (6). Although it is not completely well known, (7) is equivalent to $\text{MA}_{\aleph_1}$, thanks to the next proposition, due to Todorčević and Velicković.

**Proposition 4.3** (Todorčević and Velicković, Theorem 3.4). $\text{MA}_{\aleph_1}$ holds if, and only if, every uncountable ccc poset has an uncountable centered subset.

In [5], up to terminology, Larson and Todorčević asks whether any of the assumptions

(1) $\forall \mathbb{P} \ c(\mathbb{P}) = \aleph_0 \Rightarrow \mathcal{K}_2(\mathbb{P}) = \aleph_0$ or
(2) $\forall \mathbb{P} \ c(\mathbb{P}) = \aleph_0 \Rightarrow \text{pc}(\mathbb{P}) = \aleph_0$

imply $\text{MA}_{\aleph_1}$. Thus, after all we have done so far, Larson and Todorčević’s questions suggest the following.

**Question 4.4.** Does $\text{MA}_{\aleph_1}$ imply $c(\mathbb{P}) = \text{pc}(\mathbb{P})$ for every poset $\mathbb{P}$? Does the converse hold?

**Question 4.5.** Does $\text{MA}_{\aleph_1}$ imply $c(\mathbb{P}) = \mathcal{K}_2(\mathbb{P})$ for every poset $\mathbb{P}$? Does the converse hold?

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Centro de Ciências Exatas, Universidade Federal do Espírito Santo, Vitória, ES, 29075-910, Brazil

E-mail address: renan.mezabarba@ufes.br

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos, SP, 13560-970, Brazil

E-mail address: aurichi@icmc.usp.br

Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, SP, 05508-900, Brazil

E-mail address: lucia@ime.usp.br