A Note on the Algebraic Evaluation of Correlators in Local Chiral Conformal Field Theory

A. Honecker

Abstract

We comment on a program designed for the study of local chiral algebras and their representations in 2D conformal field theory. Based on the algebraic approach described by W. Nahm, this program efficiently calculates arbitrary $n$-point functions of these algebras. The program is designed such that calculations involving e.g. current algebras, $W$-algebras and $N$–Superconformal algebras can be performed. As a non-trivial application we construct an extension of the Virasoro algebra by two fields with spin four and six using the $N = 1$–Super-Virasoro algebra.
1. Introduction

There are many different approaches to conformal field theory (CFT). One can study the field algebra itself using e.g. the operator product expansion (OPE) [1], one can examine the analyticity properties of the correlation functions [2][3], explore relations to different fields in mathematics (e.g. modular invariance [4]), consider the conserved currents of Toda field theories [5][6][7] or one can – among others – focus on the algebraic aspects encoded in the mode algebra of the fields. In the latter approach W. Nahm was able to derive a general formula for the commutator of two quasiprimary local chiral fields [8]. Even more, he introduced a quasiprimary normal ordering prescription \( N \) which in addition to the derivative enables one to construct all fields in the local chiral CFT from a few ‘simple’ ones (for more details see [9]). This approach lends itself for the implementation in some computer algebra system. Very soon, it turned out that one of the major computational tasks is the calculation of structure constants, or more precisely the correlators involved [10][11].

In order to make this more transparent let us give a very simple example. Consider the ‘Virasoro algebra’ which is defined in terms of generators \( L_n \) satisfying the following commutation relations:

\[
[L_m, L_n] = (n - m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},
\]

(1.1)

\( c \) is a central element in the algebra and in an irreducible representation is equal to \( C \) with \( C \) a number. It is convenient to identify the central element \( c \) with the number \( C \). In physics, one very important representation of this algebra is the ‘vacuum representation’. The vacuum representation is defined via the existence of a cyclic vector \( |v\rangle \) that satisfies:

\[
L_n |v\rangle = 0 \quad \forall n < 2.
\]

(1.2)

Correlators of the Virasoro algebra are now defined in terms of vectors \( L_{i_1} \ldots L_{i_r} |v\rangle \) and elements of the dual space. The dual space can be characterized by a linear form \( \langle v| \) dual to \( |v\rangle \):

\[
\langle v| = (|v\rangle)^+ \quad \text{(1.3)}
\]

and an involution:

\[
(L_n)^+ = L_{-n}.
\]

(1.4)

A trivial consequence of (1.2) – (1.4) is:

\[
\langle v| L_n = 0 \quad \forall n > -2.
\]

(1.5)

Correlators of these states will now look as follows:

\[
\langle v| L_{i_1} \ldots L_{i_r} |v\rangle = p(c)
\]

(1.6)

and will be evaluable as a (possibly trivial) polynomial in \( c \).

It is not difficult to implement the Virasoro algebra (1.1) and its vacuum representation (1.2) – (1.5) in REDUCE. The following few lines will do the job:
operator bra, ket, l;
noncom bra, ket, l;
operator delta;

% the delta-operator
for all m such that numberp m let delta(m) =
    begin
        if m=0 then return 1
        else return 0
    end;

% the Virasoro algebra itself
for all m, n such that numberp m and numberp n and m<n let l(m)*l(n) =
    1(n)*l(m)+(n-m)*l(n+m)+c/12*(n*n*n-n)*delta(n+m);

% the vacuum-representation of the Virasoro algebra
for all n such that numberp n and n<2 let l(n)*ket(0,0) = 0;
for all n such that numberp n and n>-2 let bra(0,0)*l(n) = 0;
let bra(0,0)*ket(0,0) = 1;

In order to test the performance of this program we would like to calculate to following correlator:

\[ \langle v | L_{-N} \cdots L_{-2} L_{-1} L_{1} L_{2} \cdots L_{N} | v \rangle \]  \quad N \geq 1. \quad (1.7)

The appropriate REDUCE-input for such a correlator reads – specializing to \( N = 5 \):

\[ \text{bra}(0,0) * l(-5) * l(-4) * l(-3) * l(-2) * l(-1) * l(1) * l(2) * l(3) * l(4) * l(5) * \text{ket}(0,0); \]

We have evaluated the first few correlators of this form using REDUCE 3.3 on a IBM 3084 mainframe. The CPU-time needed (in seconds) is listed in the first column of the following table:

| \( N \) | REDUCE 3.3 IBM 3084 | COMMUTE IBM XT286 | REDUCE 3.4 DEC 5000/120 | COMMUTE DEC 5000/120 |
|-------|------------------|-------------------|--------------------------|--------------------------|
| 1     | 0.43             | < 1               | 0.1                      | 0.0                      |
| 2     | 0.77             | < 1               | 0.1                      | 0.0                      |
| 3     | 1.34             | < 1               | 1.0                      | 0.0                      |
| 4     | 9.34             | 1                 | 8.4                      | 0.0                      |
| 5     | 364.42           | 16                | 185.2                    | 0.7                      |
| 6     | 18114            | 448               | 14000.9                  | 21.8                     |
| 7     | 18114            | 448               | 14000.9                  | 21.8                     |
| 8     | 18114            | 448               | 14000.9                  | 21.8                     |
Missing entries in the table indicate that completion of the program could not sensibly achieved with the existing resources, e.g. a CPU-time limit of 30min on the mainframe. Obviously, performance of the above program is poor and calculation can be much more efficiently performed on a machine as small as a IBM XT286 running at 6MHz CPU clock. We will refer to this program written by the author as ‘COMMUT E’. A better comparison of both programs can be derived from CPU time needed by both programs on a DEC workstation \( ^1 \). Note that we have chosen the correlator (1.7) because it is worst case for COMMUTE (see also chapter 3).

This letter is to a great deal dedicated to the algorithmic considerations implemented in COMMUTE. Since they have been developed while performing calculations the next section intends to give a motivation for some of the implementation’s peculiarities through a short historic review. We will further outline the capabilities of COMMUTE and mention some possible applications. Finally, as a non-trivial application of the program we will explicitly show that the structure constants in the bosonic sector of the \( N = 1 \)–Super Virasoro algebra coincide with those of \( \mathcal{W}(2, 4, 6) \).

From now on we will use notations and formulae introduced in chapter 2 of [11] as well as chapter 3 of [12] and assume that the reader is familiar with them.

2. History of the program

In [11] a PASCAL program was used for the evaluation of the most complicated structure constants. First calculations for representations of \( \mathcal{W} \)-algebras [13][14] had also reached the limits of REDUCE. The analogue of the REDUCE program presented in chapter 1 implemented in MATHEMATICA was even slower. Thus it was clear that for explicit calculations with \( \mathcal{W} \)-algebras a much more efficient program was needed. Before going on with representation theory of \( \mathcal{W} \)-algebras [12] we therefore decided to develop a special purpose program in C. This had the disadvantage that even basic math routines for large integers had to be written again but optimal control on rule application was guaranteed.

The first version of COMMUTE was designed only to implement evaluation of correlators as presented in chapter 1. Soon, it turned out that expansion of normal ordered products produced so many summands that even their evaluated form could not be read any more by the computer algebra systems we were using. This lead to the implementation of mode’s and expansion of naive normal ordered products in COMMUTE. Not much later also expansion of quasiprimary normal ordered products into naive normal ordered products blew up expressions so much that it had to be implemented in COMMUTE. As only for \( \mathcal{N}(X, \partial^n L) \) a simple formula exists [12] it was necessary to tell COMMUTE how to evaluate the other \( \mathcal{N} \)’s. This was done introducing a file containing definitions. Now, only a few minor changes were necessary to also make calculations with twisted representations of bosonic \( \mathcal{W} \)-algebras (where some of the bosonic fields have half-integral modes [15]) possible.

Next, we turned to \( N = 1 - \mathcal{SW} \)-algebras [16]. Now the \( N = 1 \)-Super-Virasoro algebra had to be implemented in COMMUTE. Only a few minor improvements were necessary until

\( ^1 \) A SUN 4/75 sparc station at 40MHz is about 15% faster.
representations of higher spin $\mathcal{SW}$-algebras in the Ramond-sector needed special attention [17]. At this stage COMMUTE was already much more powerful than originally intended and performed most of the tasks earlier performed in REDUCE or MATHEMATICA. This motivated us to completely make it a stand-alone system. To this end the user had to be enabled to introduce new fields and their commutators. Therefore Nahm’s universal polynomials had to be implemented at last and it seemed prudent to implement replacement rules for structure constants and insertion of the central charge $c$.

It also turned out while developing large programs in REDUCE that its tendency to accept almost any input and almost never to produce any error messages made debugging of these programs difficult. Even more, the error condition is almost impossible to locate due to a missing trace-option. Therefore we implemented error checking in COMMUTE from the beginning. Even in an error condition COMMUTE continues and tries to make sense of what follows. Additionally, COMMUTE also supports a debugging mode in which it displays all steps it performs. This enables the user to control proper performance of COMMUTE and facilitates debugging.

This historical review should motivate some of the peculiarities of COMMUTE. Though COMMUTE can do much on its own, the use of some computer algebra system is recommended e.g. for factorization of polynomials and evaluation of the explicit form of complicated $\mathcal{N}$$\text{opp}$’s.

3. Algorithms used

First, we would like to make some comments that should be generally valid for computer algebra systems when applied to noncommutative objects. The first observation is that commutative and noncommutative objects should be stored in different ways because their treatment is completely different. We have chosen to store correlators as a vector of structures describing the modes of fields. This facilitates identification of locations where commutation should take place. Storing the polynomial in front of it is conceivable in many different ways. For our purposes it seemed sensible to store it as an array with integer coefficients of monoms in $\hbar$ and $c$ with a common denominator and a common monom in structure constants.

It is obvious that rules which define vanishing terms simplify expressions and should be applied as often as possible. Therefore, only one commutator resp. one field inserted in a step. Afterwards, those correlators are eliminated that vanish because the mode of the field annihilates the highest weight. Term collection also simplifies expressions but for general correlators is very time consuming and therefore seldom performed. Still, summing up polynomials is a comparably simple task and performed as often as possible. It should be possible to implement some heuristic method to keep expressions small in most computer algebra systems. The author does in fact not understand why (as far as he knows) no one of the computer algebra systems mentioned above tries to collect terms as soon as memory becomes rare. Such a simple improvement would certainly make tasks manageable that are currently impossible.
Let us now focus on algorithms that are more specific to conformal field theory. In our notation for correlators all vectors are $L_0$-eigenvectors:

$$L_0 (\phi_{1,i_1} \ldots \phi_{r,i_r} | h, w \rangle) = (h + i_1 + \ldots + i_r) \phi_{1,i_1} \ldots \phi_{r,i_r} | h, w \rangle.$$  \hfill (3.1)

Thus it is very simple to eliminate any $L_0$ and COMMUTE performs this at the same time when checking for vanishing correlators.

If a product of two same modes of the same fermion is encountered (this does generally not vanish) it can be replaced by one half of the anticommutator. This also is a considerable simplification and therefore performed next.

Even if we really have to evaluate a correlator there is a strategy that performs much better than the REDUCE-program described in chapter 1. First note that if you can evaluate $[L_{-r}, L_r]$, it is almost as simple to evaluate $[L_{-r}, L_r^n]$. COMMUTE indeed does this and therefore a correlator of the form (1.7) really is the worst case for COMMUTE – e.g. computation of $\langle h, w | L_{10}^{10} L_{-2}^{10} | h, w \rangle$ is performed instantly.

As we would like to perform no term collection it seems prudent to finish any summand currently being processed before proceeding with the next one. In most cases only a few steps are necessary to produce a scalar expression. In the meanwhile a few additional summands are produced inserting commutators. These are processed next. For a few exceptional cases, this strategy may be far from optimal as far as CPU-time is concerned. Definitely, it will require a minimum of memory. Within this strategy it is obviously most efficient to produce as few additional summands as possible before arriving at a scalar expression. This is ensured by searching the place most close to either $\langle h, w |$ or $| h, w \rangle$ for possible insertion of a commutator and performing it there instead of doing so e.g. at the leftmost possible position.

The remaining features of COMMUTE are implemented rather heuristically. Expansion of modes of fields is performed before any commutation is performed with the restriction that only one naive normal ordered product is expanded at a time because this blows expressions up. Externally defined commutators are inserted if no one of the internally defined ones applies any more. Replacement rules on structure constants are applied at the same time when term collection takes place. Other features of COMMUTE like replacement of quadratic fields in the Ramond-sector are also performed with low priority.

4. Availability, use and applications

COMMUTE is publically available together with a detailed technical description which we omit here. It may be redistributed and changed freely if the conditions supplied with the program are respected.

COMMUTE is written entirely in ANSI-C. Therefore it should run on almost every PC or workstation. Before compilation, compatibility to either REDUCE- or MATHEMATICA-notation can be chosen. Additionally, some limits (especially numerical range) may be specified before compilation so that many different versions of COMMUTE are conceivable. Therefore we have decided to distribute appropriate versions on request via e-mail to the author rather than to put the program on an internet-server.
Anybody interested in COMMUTE should send an e-mail to unp06b@ibm.rhrz.uni-bonn.de specifying the computer being used and the version of COMMUTE needed. You should also specify whether you would like a compiled version and/or the sources. The desired medium (ftp / disc / tape) should also be specified.

COMMUTE is invoked from the command line and replaces any correlator in an input file by its value producing an output file. This procedure makes it possible to call COMMUTE from most high level languages.

Some example files for what COMMUTE can do without any other computer algebra system (except for factorization of polynomials) are included with it. They cover e.g.:

- The associativity check for four point correlators of the algebra $\mathcal{W}(2, \frac{5}{2})$ reproducing the result of R. Varnhagen that only finitely many HWRs are permitted [13].
- Evaluation of correlators involving the simplest null fields in the Super-Virasoro algebra at $c = \frac{7}{10}$ yielding the permitted $h$-values of the minimal series in the Neveu-Schwarz- and Ramond-sector. Special attention is paid to the quadratic fermionic normal ordered product in the Ramond-sector where the projection on the quasiprimary part according to W. Nahm [8] is impossible due to the non-local effect of the boundary conditions.
- Definition of Zamolodchikov’s $\mathcal{W}(2, 3)$ [18]. This definition needs only very few lines, but enables one to calculate e.g. the contravariant form on the Verma module explicitly, thus demonstrating the efficiency of COMMUTE.
- Definitions for the $N = 2$-Super-Virasoro-Algebra.

5. An application: The bosonic sector of the $N = 1$–Super-Virasoro algebra

P. Bouwknegt [19] has argued some time ago that the bosonic sector of the $N = 1$–Super-Virasoro algebra should be an extension of the Virasoro algebra with one field of spin 4 and one field of spin 6 – called a $\mathcal{W}(2, 4, 6)$. We shall now verify this and calculate the structure constants.

The $N = 1$–Super-Virasoro algebra is given by the following commutation relations:

\[
\begin{align*}
[L_m, L_n] &= (n - m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
[L_m, G_n] &= (n - \frac{1}{2}m)G_{m+n} \\
[G_m, G_n]_+ &= 2L_{m+n} + \frac{c}{3}(m^2 - \frac{1}{4})\delta_{m+n,0}.
\end{align*}
\] (5.1)

This algebra is already implemented in COMMUTE.

We are now interested in bosonic normal ordered products built up from the field $G$ only. For dimension up to 6 there are only two of them:

\[
\begin{align*}
\mathcal{N}(G, \partial G) &= \mathcal{N}(G, \partial G) - \frac{1}{2} \partial \mathcal{N}(G, G) + \frac{1}{5} \partial^2 L \\
\mathcal{N}(G, \partial^3 G) &= \mathcal{N}(G, \partial^3 G) - \frac{3}{2} \partial \mathcal{N}(G, \partial^2 G) + \frac{2}{3} \partial^2 \mathcal{N}(G, \partial G) - \frac{1}{12} \partial^3 \mathcal{N}(G, G) + \frac{1}{84} \partial^4 L
\end{align*}
\] (5.2)
where we have already expressed the quasiprimary normal ordered product \( \mathcal{N} \) in terms of naive normal ordered products \( \mathcal{N} \). Note that the normal ordered products (5.2) are not implemented in COMMUTE and therefore have to be defined in a definition file.

In order to construct primary fields of dimension 4 resp. 6 one now takes arbitrary linear combinations of all quasiprimary fields with this dimension and adjusts the parameters such that structure constants become zero.

For a primary spin four field \( U \) the most general ansatz is:

\[
U := \alpha \mathcal{N}(L,L) + \beta \mathcal{N}(G,\partial G). \tag{5.3}
\]

The free parameters \( \alpha \) and \( \beta \) have to be adjusted such that \( C^L_{UL} = 0 \) holds. This yields the equation:

\[
0 = C^L_{UL} = \alpha C^L_{\mathcal{N}(L,L)L} + \beta C^L_{\mathcal{N}(G,\partial G)L} = \alpha \frac{5c + 22}{5} + \beta \frac{17}{5}. \tag{5.4}
\]

The structure constants \( C^Z_{XY} \) are defined as a solution of the system of equations:

\[
C^W_{XY}d_{WZ} = C_{XYZ}. \tag{5.5}
\]

with

\[
d_{XY} = \langle v | X_{-d(X)} Y_{d(Y)} | v \rangle \quad \text{and} \quad C_{XYZ} = \langle v | Z_{-d(Z)} X_{d(Z)-d(Y)} Y_{d(Y)} | v \rangle. \tag{5.6}
\]

The evaluation of the correlation functions in (5.6) is performed using COMMUTE while the equations (5.5) are afterwards easily solved using e.g. REDUCE.

In (5.4) the result of the corresponding calculations has already been inserted.

The general solution of (5.4) is \( \alpha = \gamma \frac{17}{5} \) and \( \beta = -\gamma \frac{5c + 22}{5} \) with a free parameter \( \gamma \) which we can use for normalization. We choose \( \gamma \) such that

\[
\frac{c}{4} = \langle v | U_{-4} U_4 | v \rangle = \alpha^2 \langle v | \mathcal{N}(L,L)_{-4} \mathcal{N}(L,L)_4 | v \rangle + 2\alpha \beta \langle v | \mathcal{N}(G,\partial G)_{-4} \mathcal{N}(L,L)_4 | v \rangle + \beta^2 \langle v | \mathcal{N}(G,\partial G)_{-4} \mathcal{N}(G,\partial G)_4 | v \rangle. \tag{5.7}
\]

If we now insert the two-point-functions that we have already evaluated using COMMUTE we arrive at:

\[
\gamma^2 = \frac{75}{2(10c - 7)(5c + 22)(4c + 21)}. \tag{5.8}
\]

The most general ansatz for a primary field \( V \) with dimension six is:

\[
V := \kappa \mathcal{N}(G\partial^3 G) + \lambda \mathcal{N}(\mathcal{N}(G,\partial G), L) + \mu \mathcal{N}(\mathcal{N}(L,L), L) + \nu \mathcal{N}(L,\partial^2 L). \tag{5.9}
\]
Again one calculates structure constants using COMMUTE. The first observation is that 
\[ C_{VL}^{N(G, \partial G), \partial L} = C_{VL}^{N(L, L)} = C_{VL}^{N(G, \partial G)} = 0. \]  
(5.10)

Similar arguments as for the spin four field lead to the general solution:

\[ \begin{align*}
\mu &= 10(218c - 293)\rho \\
\nu &= -3(11c - 86)(c + 24)\rho \\
\lambda &= -\frac{5}{15}\left(\frac{15c + 164}{10}\mu + \frac{16}{3}\nu\right) \\
\kappa &= -\frac{c + 24}{26}\lambda
\end{align*} \]  
(5.11)

with a free parameter \( \rho \). Imposing the normalization condition

\[ \frac{c}{6} = \langle v \mid V_{-6} V_6 \mid v \rangle, \]  
(5.12)

fixes \( \rho^2 \) to the following value:

\[ \rho^2 = \frac{1}{25(14c + 11)(10c - 7)(7c + 68)(4c + 21)(2c - 1)(c + 24)(c + 11)}. \]  
(5.13)

So far, no time intensive calculations need to be done. The next step is to replace the 
field \( N(G, \partial G) \) by the field \( U \) and the field \( N(G, \partial^3 G) \) with \( V \) and calculate again the 
\( d \)-matrix and structure constants. Especially the evaluation of the structure constants 
involving fields of dimension up to seven took about a day’s CPU-time on a SUN-SPARC 
at the Max-Planck-Institut für Mathematik in Bonn-Beuel. The matrix-inversion is easily 
performed using REDUCE.

The result shows that \( U \) and \( V \) are not only indeed primary but orthogonal to all normal 
ordered products not involving \( G \). This means that in the bosonic sector of the Super
Virasoro algebra the fields \( U \) and \( V \) are simple (i.e. non-composite). The non-trivial 
structure constants connecting three simple fields are evaluated as:

\[ \begin{align*}
(C_{UU})^2 &= \frac{54(10c^2 + 47c - 82)^2}{(10c - 7)(5c + 22)(4c + 21)} \\
(C_{UV})^2 &= \frac{144(14c + 11)(5c + 22)^2(c + 11)(c - 1)^2}{(10c - 7)(7c + 68)(4c + 21)(2c - 1)(c + 24)} \\
(C_{VU})^2 &= \frac{50(7c + 68)^2(3c + 20)^2(2c - 1)^2}{3(10c - 7)(5c + 22)(4c + 21)(c + 24)^2} \\
(C_{VV})^2 &= \frac{400(616c^4 + 19106c^3 + 183931c^2 + 574876c + 724096)^2(c - 1)^2}{9(14c + 11)(10c - 7)(7c + 68)(4c + 21)(2c - 1)(c + 24)^3(c + 11)}. \\
\end{align*} \]  
(5.14)
We have given only expressions for the squares in order to make it possible to insert the normalization constants $\gamma^2$ and $\rho^2$. One obtains two additional relations which fix signs among the coupling constants

$$C_{UU}^U C_{VU}^U = \frac{30(10c^2 + 47c - 82)(7c + 68)(3c + 20)(2c - 1)}{(10c - 7)(5c + 22)(4c + 21)(c + 24)}$$

$$C_{UU}^V C_{VU}^V = \frac{80(616c^4 + 19106c^3 + 18391c^2 + 574876c + 724096)(5c + 22)(c - 1)^2}{(10c - 7)(7c + 68)(4c + 21)(2c - 1)(c + 24)^2}$$

so that we are left with two free signs which reflects the freedom of choice of signs for $\rho$ and $\gamma$.

Equalities like $C_{UV}^L = \frac{2}{3} C_{LU}^V$ enable one to calculate all remaining structure constants connecting three simple fields. If one now calculates the dimension of the space of bosonic fields in the $N = 1$–Super-Virasoro algebra and for a $\mathcal{W}(2, 4)$-algebra one observes equality for integer dimension up to 9. At dimension 10 there is one field less in the bosonic projection of the $N = 1$–Super-Virasoro algebra compared to a general $\mathcal{W}(2, 4, 6)$-algebra. This shows that the fields $L$, $U$ and $V$ generate a closed $\mathcal{W}(2, 4, 6)$ subalgebra in the bosonic sector of the Super-Virasoro algebra. Since the $N = 1$–Super-Virasoro algebra satisfies Jacobi identities, so does this algebra, too.

There have been earlier constructions of $\mathcal{W}(2, 4, 6)$-algebras starting with simple fields of dimension two, four and six. Imposing some Jacobi identities H.G. Kausch and G.M.T. Watts have found two general solutions [10], but have not been able to check all relevant Jacobi identities. The structure constants (5.14), (5.15) of the algebra $\mathcal{W}(2, 4, 6)$ we have constructed here coincide with those of one of the two general solutions.

The cases where the normalization constants as well as the structure constants become singular deserve special attention. For $c \in \{-14/11, -7/68, 1/2, -24, -11\}$ the field $V$ is a null field before normalization and the procedure here only leads to a $\mathcal{W}(2, 4)$-algebra. For $c \in \{7/10, -21/4\}$ also the field $U$ is a null field before normalization and the bosonic sector of the Super-Virasoro algebra coincides with the Virasoro algebra. It might seem that for $c = -22/5$ we obtain a $\mathcal{W}(2, 6)$ which is not the case. Here, singularities in the structure constants $C_{VV}^X$ force us to also normalize $V$ to zero. This shows that there is no consistent $\mathcal{W}$-algebra in the bosonic sector at $c = -22/5$ and one is left with the Virasoro algebra again.

It has been observed in [12] that the representation theory of $\mathcal{W}(2, 4)$ and the $N = 1$–Virasoro algebra at $c = -11$ is much the same. This observation obviously should generalize to the bosonic sector of the $N = 1$–Virasoro algebra at arbitrary central charge.

6. Conclusions

We have described a program for the evaluation of correlators in local chiral conformal field theory. It is fast, easy to use and pays attention to wrong input. This program has already been used for representation theory of $\mathcal{W}(2, \delta)$-algebras [12], the construction of $N = 1$–$\mathcal{SW}^{(\frac{3}{2}, d)}$- and $\mathcal{SW}^{(\frac{3}{2}, d_1, d_2)}$-algebras [16] and finally for the representation
theory of $SW(\frac{3}{2}, d)$ [17]. Now it has been extended in order to make calculations involving even more general local chiral algebras possible. This includes $N$-extended superconformal algebras as well as current algebras. The next application will be $N = 2$–Super-\(W\)-algebras because of their important applications in String theory [20][21][22].

We have demonstrated the power of the program by constructing a $W(2, 4, 6)$ starting from the $N = 1$–Super-Virasoro algebra. Recently, H. Lu and C.N. Pope have observed that any free field construction may start with specific $W$-algebras as input [23]. COMMUTE would be especially well suited to implement this for more algebras.

It is a pleasure to thank everybody working in the group of W. Nahm at the ‘Physikalisches Institut’ in Bonn for numerous valuable discussions. I am especially grateful to W. Eholzer and R. Hübzel because the original concept of COMMUTE was developed together with them. Furthermore, I am indebted to the ‘Max-Planck-Institut für Mathematik’ in Bonn-Beuel because without their computers complicated applications for COMMUTE (like the one in this letter) would not have been possible.

Note added: I would like to thank H.G. Kausch for pointing out to me that he has also performed an explicit construction of $W(2, 4, 6)$ in terms of the $N = 1$–Super Virasoro algebra using a slightly different method [24].

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