Fermionic path integrals and correlation dynamics in a 1D XY system

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Abstract

We derive time dependent correlation functions in an one dimensional XY spin model with the use of generating functionals, the latter being defined as path integrals over fermionic coherent states. We focus on the proper construction of the aforementioned integrals in order to avoid the inconsistencies accompanying the corresponding functional integrals over spin coherent states and we prove that, at the static limit, the known results are reproduced. In the same context, we examine the case of a driven transverse magnetic field and we confirm the consistency of our results with the Kibble-Zurek mechanism.
1 Introduction

The Feynman path integral formalism is the most powerful tool for taking into account quantum behaviour via classical computations [1,2]. Ideally suited for semiclassical calculations, the path integral machinery provides a variety of analytical methods for studying the dynamics of quantum correlations in closed and open quantum systems [1,3]. The extension of path integration into the ordinary complex plane \( \mathbb{C} \), through the Glauber coherent states [4] in the complex non-flat manifold \( \hat{\mathbb{C}} \) through the \( su(2) \) spin coherent states [5-7], and to fermionic systems, through the fermionic coherent states [1,8] has opened the road for applying path integral techniques to the study of many-body systems [9,10]. These systems are of great interest for both the condensed matter physics and the quantum information science due to the fact that they naturally support entangled states. Correlations in these states have a fundamental quantum character, they do not have a classical counterpart, and can serve both as the probe for understanding quantum phase transitions and as the main tool for quantum information processing [11-13]. During the last years, there have been considerable advances in the study of the static properties and the dynamics of closed
many-body quantum systems both at experimental and theoretical level \cite{14,15}. However, despite these advances, the usage of path integral techniques in the corresponding analysis is rather restricted. Among the reasons for this fact, is that the relevant path integrals, which must be defined on the complexified spaces spanned by the coherent state bases, were considered, up until recently, to suffer from inconsistencies. \cite{16-19}. When free from conceptual and defining problems, path integrals over coherent states can provide a wide palette of techniques, analytical and numerical, for the analysis of quantum systems either closed or open. In the current paper we make a step towards this direction by undertaking the task of examining an integrable, well-studied, spin-chain system through the use of the fermionic coherent state path integration language. In the first part of our study we discuss the construction of path integrals over anticommuting Grassmann variables. Although this is a known \cite{1} construction, the resulting formal expression inherits the problems already known from the bosonic case. To overcome these, we propose a definite procedure that leads to an inconsistency-free path integral expression capable to interpret fermionic quantum systems. In the second part of the present study we analyze a simple spin system, the XY model in one dimension, which is perfectly suited for checking the proposed formalism as it is exactly solvable and, yet, it has a quite rich structure that supports a quantum phase transition \cite{11,12,20,21}. We reproduce the known static results pertaining to the entanglement entropy and, at the same time, we take advantage of the path integral representation to calculate the time dependence of the ground state correlation functions. We end up by considering the case of a driven transverse magnetic field and we confirm that the Kibble-Zurek mechanism characterizes the passage of a driven system through a quantum critical point \cite{22,23}. The structure of the paper is the following: In Section II (a) we present the problems arising in the construction of path integrals over fermionic coherent states and examine the solution that already has been proposed for the confrontation of similar problems appearing in integrals over bosonic coherent states \cite{24,25}. In Section II (b) we discuss the recently proposed path integral representation for Majorana fermions \cite{26} proving that it can be used as an intermediate step towards an inconsistency-free path integral description of fermionic systems. In Section III we define the path integral representation of the 1D XY model to derive the time-depended ground state correlators needed for the calculation of the entanglement entropy and we compare with the known static result.
In Section IV we examine the case of a driven transverse magnetic field and we find the time dependence of the ground state correlators we are interested for. We also find that the driving of the system through the critical point is consistent with the Kibble-Zurek scaling mechanism. Finally, in Section V we present our conclusions and the perspectives of our work. The paper is accompanied by two appendices. In Appendix A we present the path integral calculation of the partition function pertaining to the toy-model $\hat{H} = -\omega \hat{S}_1 \cdot \hat{S}_2$. In Appendix B we present details of the calculations pertaining to the results that appear in Section III and IV.

2 Functional integration over Grassmann variables

2.1 Fermionic coherent state path integrals

In this section we shall discuss path integration in the space spanned by the fermionic coherent states, that is, the eigenstates of the fermionic annihilation operator $\hat{\psi}$, the eigenvalues of which are complex Grassmann variables:

$$\hat{\psi} | \zeta \rangle = \zeta | \zeta \rangle . \quad (2.1)$$

These states form an overcomplete basis

$$\int \frac{d \bar{\zeta} d \zeta}{\pi} | \zeta \rangle \langle \zeta | = | 0 \rangle \langle 0 | + | 1 \rangle \langle 1 | = \hat{I} \quad (2.2)$$

which can be used to transcribe fermionic amplitudes into the context of path integrals [1]. The need for a careful reconsideration of this transcription can be traced back in the issues occurring in both bosonic and spin coherent state path integrals, issues that question even their very meaning [16]. By referring to the Jordan-Wigner [27] transformation one expects similar problems to occur when path integration is performed in the basis of fermionic coherent states. In the present work we shall examine correlations in fermionic systems using their respective path integral representation, thus the use of a formulation free of inconsistencies is of vital importance.

To begin with, consider the partition function of a system, the Hamiltonian of which
is expressed in terms of fermionic creation and annihilation operators:

\[ Z = \text{Tr} e^{-\beta \hat{H}} = \int \frac{d\bar{\eta} d\eta}{\pi} \langle -\eta | e^{-\beta \hat{H}} | \bar{\eta} \rangle. \] (2.3)

By dividing \( \beta \) in \( N + 1 = \beta / \epsilon \) segments and inserting the completeness relation Eq. (2.2) in each division, the following formal result is achieved [1] at the limit \( \epsilon \to 0 \):

\[ Z = \int_{\text{AP}} D\bar{\zeta} D\zeta \exp \left\{ -\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \left[ \bar{\zeta} \frac{\partial}{\partial \tau} \zeta + H \left( \bar{\zeta}, \zeta \right) \right] \right\}. \] (2.4)

The last expression is formal in the sense that both the form of the function \( H \) that represents the quantum Hamiltonian \( \hat{H} \), and its discrete ancestor, must be carefully defined for inconsistencies not to appear. To give a very simple example, consider the case of the fermionic oscillator

\[ \hat{H} = \omega \left( \hat{\psi}^\dagger \hat{\psi} - \frac{1}{2} \right) \] (2.5)

which is connected to the spin Hamiltonian \( \hat{H} = -\omega \hat{S}_z \) via the Jordan-Wigner transformation. The partition function in this case can be trivially computed without any reference to path integration:

\[ \text{Tr} e^{-\beta \hat{H}} = e^{\omega \beta / 2} + e^{-\omega \beta / 2} = 2 \cosh \left( \omega \beta / 2 \right). \] (2.6)

Trying to obtain the same result by means of path integration, we follow the standard discretization procedure and adopt for the classical function \( H \) the expression

\[ \frac{H(\zeta_n, \zeta_{n-1})}{\langle \zeta_n \rvert \zeta_{n-1} \rangle} = \omega \left( \zeta_n \bar{\zeta}_{n-1} - \frac{1}{2} \right) \xrightarrow{N \to \infty} \omega \left( \bar{\zeta} \zeta - \frac{1}{2} \right). \] (2.7)

Thus, for the partition function we get the following result

\[ Z = e^{\omega \beta / 2} \int_{\text{AP}} D\bar{\zeta} D\zeta \exp \left\{ -\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \left[ \bar{\zeta} \frac{\partial}{\partial \tau} \zeta + (\partial + \omega) \right] \right\}. \] (2.8)

The integral involved in the above expression can be evaluated using the formula [1]:

\[ \text{Trln} (\partial + \omega) = \beta \int_{0}^{\omega} d\omega' C_{\omega'}^{(+)} (\tau, \tau) \] (2.9)
where \( G_{\omega}^{(+)}(\tau, \tau) \) is the Green’s function satisfying 
\[
(\partial_\tau + \omega) G_{\omega}^{(+)}(\tau, \tau') = \delta(\tau - \tau')
\]
with antiperiodic boundary conditions \( G_{\omega}^{(+)} \left(-\frac{\beta}{2}, \tau\right) = -G_{\omega}^{(+)} \left(\frac{\beta}{2}, \tau\right) \):

\[
G_{\omega}^{(+)}(\tau, \tau') = \left[ \Theta(\tau - \tau') - \left(1 + e^{\beta \omega}\right)^{-1} \right] e^{-\omega(\tau - \tau')}. \tag{2.10}
\]

The function \( \Theta(\tau - \tau') \) is usually chosen \([1]\) to be the Heaviside function for which \( \Theta(0) = \frac{1}{2} \). This choice is ultimately related to the requirement for path integrals to be defined through dimensional regularization \([1]\), which in turn gives a definite prescription for \( G^{(+)}(0) \). However, due to the unavoidable discontinuity of the Green’s function at \( \tau = \tau' \), questions arise \([24, 25]\), regarding the meaning of \( G^{(+)}(\tau, \tau) \) in Eq. (2.9), as different prescriptions yield different results. Indeed, by adopting the symmetric prescription indicated in Eq. (2.10), we find for the partition function the wrong result

\[
Z = 2e^{\omega \beta/2} \cosh \left(\omega \beta/2\right). \tag{2.11}
\]

For the cure of this awkward situation, it has been proposed \([24, 25]\) to take into account the discrete ancestor of the continuous Hamiltonian (see Eq. (2.7)) and to use the limit form \( G_{\omega}^{(+)} = (1 + e^{\beta \omega})^{-1} \) in Eq. (2.9). In this way, the path integral of Eq. (2.8) yields the result:

\[
\int_{AP} D\xi D\zeta \exp \left\{- \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \xi (\partial_\tau + \omega) \zeta \right\} = 1 + e^{-\beta \omega}. \tag{2.12}
\]

Thus, the correct answer for the partition function is recovered. Nevertheless, in the case of path integration in terms of bosonic and spin coherent states, this prescription is not enough \([16-19]\) for curing the inconsistencies appearing in less trivial systems, such as the Bose-Hubbard model, or the spin system \( \hat{H} = \omega \hat{S}_z^2 \), with \( s > \frac{1}{2} \).

In the fermionic case, the Grassmannian character of the fields does not permit non-linearities to appear, and thus the asymmetric prescription \( G(\tau, \tau + 0) \) can successfully cope with the calculation of the path integral in that specific case. However, this prescription is strongly tied to the asymmetric form of the underlying discrete action, an action that makes the discrete path integral not invariant under transformations that should leave the continuous path integral intact. For example, in the l.h.s. of Eq.
we can apply the transformation
\[
\zeta(\tau) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} e^{i \omega_n (\tau + \frac{\beta}{2})} a_n, \quad \omega_n = \frac{\pi (2n + 1)}{\beta}
\] (2.13)
to check that it is invariant \(1\) under backtracking \(\tau \rightarrow -\tau\). On the other hand, the antiperiodic Green’s function defined through \((-\partial_\tau + \omega)(G^{-}(\omega)(\tau, \tau') = \delta(\tau - \tau')\) assumes the form:
\[
G_{\omega}^{-}(\tau, \tau') = \left[ \left(1 + e^{-\beta \omega}\right)^{-1} - \Theta(\tau' - \tau) \right] e^{-\omega(\tau' - \tau)}.
\] (2.14)

By following the same reasoning as before, we must adopt the prescription \(G_{\omega}^{-}(\tau + 0, \tau)\) for the evaluation of the determinant, meaning that the r.h.s. of Eq. (2.12) changes as \(1 + e^{-\beta \omega} \rightarrow 1 + e^{\beta \omega}\). More than this, the discrete ancestor of the continuous path integral is not invariant under canonical transformations, a fact that contradicts the physical demand for the path integral to share this invariance with classical mechanics \(1\).

### 2.2 The Majorana path integral

To face the inconsistencies appearing in the standard formulation of path integration in the presence of interactions, we have proposed \(17, 19\) a simple approach which circumvents the direct construction, through the introduction of Hermitian “position” and “momentum” operators. The construction of the Feynman phase space path integral is then possible with the use of the previous operators’ eigenvalues. This integral can be then transcribed to the desired complex, flat or non-flat, manifold, through a canonical transformation. The extension of this approach to the fermionic case is much more involved, as the corresponding Hermitian operators are Majorana fermions, for which the canonical quantization is highly non-trivial and the formalism of coherent states does not exist. However, it has recently \(26\) been proved that the quantization of a Majorana system through the path integral representation is possible. In the relevant construction, the Majorana system is considered as a constrained system that is handled via the Faddeev-Jackiw method \(28\). In this approach one can directly define the path integral pertaining to the system at hand, after obtaining the classical action which is consistent with the corresponding quantum theory. It is on this approach that
we will rely to specify the structure of path integrals over fermionic coherent states, for systems like the one in Eq. (2.4). To be concrete, let us examine again the trivial case of the simple fermionic oscillator (2.5). By introducing the Majorana operators

\[ \hat{\psi} + \hat{\psi}^\dagger = \hat{\gamma}_1, \quad \hat{\psi}^\dagger - \hat{\psi} = -i \hat{\gamma}_2; \quad \{ \hat{\gamma}_a, \hat{\gamma}_b \} = 2 \delta_{ab} \]  

(2.15)

the Hamiltonian (2.5) assumes the form \( \hat{H}_M = -i \frac{\omega}{2} \hat{\gamma}_2 \hat{\gamma}_1 \). The construction of the relevant path integral proceeds then via the Faddeev-Jackiw method and dictates [26] the form \( H_M = -i \frac{\omega}{2} \gamma_2 \gamma_1 \), with \( \{ \gamma_a, \gamma_b \} = 0 \) for the classical function that weights path integration. The integral constructed in this way represents the partition function of the system as a path integral over real Majorana Grassmann variables. It is then an inevitable demand for this integral to be connected via the canonical transformation \( \gamma_1 = \zeta + \bar{\zeta}, i \gamma_2 = \zeta - \bar{\zeta} \) with the corresponding integral over the complex Grassmann variables. This approach yields the Hamiltonian \( H_M = \omega \bar{\zeta} \zeta \) as the proper weight for the integration over fermionic paths

\[ Z = \int_{AP} D\bar{\zeta} D\zeta \exp \left\{ - \int \frac{\partial}{\partial \tau} \bar{\zeta} \left( \partial_\tau + \omega \right) \zeta \right\}. \]  

(2.16)

It is worth noting that for the above-mentioned canonical transformation to be valid, the discretization prescription underlying the continuous form must be the symmetric one \( \bar{\zeta}_n \zeta_n \xrightarrow{N \to \infty} \bar{\zeta} \zeta \). Thus, for the calculation of the integral we must adopt as the Green’s function (2.10) the symmetric limit value \( G^{(+)}_\omega(0) = \frac{1}{2} - (1 + e^{\beta \omega})^{-1} \). In this way, the correct result is produced.

The calculation we presented can be summarized in a simple proposal: To use the path integral formalism for a system, the quantum Hamiltonian of which is given in terms of fermionic creation and annihilation operators, begin by rewriting it in terms of Majorana operators and continue by replacing these with the corresponding real Grassmann variables. Next, perform a canonical change of variables to get the classical Hamiltonian that must weight paths expressed in terms of fermionic coherent states. This whole construction fixes the discrete ancestor of the continuous expressions to be the symmetric one.

To demonstrate the general form of the quantum Hamiltonians we are going to deal
with, consider the spin Hamiltonian

\[ \hat{H} = - \sum_{j=1}^{N} \left[ a_j \sigma_j^x \sigma_{j+1}^x + b_j \sigma_j^y \sigma_{j+1}^y + c_j \sigma_j^z \sigma_{j+1}^z + h_j \sigma_j^z \right]. \tag{2.17} \]

Applying the Jordan-Wigner transformation, we can re-express it in terms of fermionic creation and annihilation operators as

\[ \hat{H} = - \sum_{j=1}^{N} \left[ a_j \left( \hat{\psi}_j^+ \hat{\psi}_{j+1} + \hat{\psi}_{j+1}^+ \hat{\psi}_j \right) + b_j \left( \hat{\psi}_j^+ \hat{\psi}_{j+1} - \hat{\psi}_{j+1}^+ \hat{\psi}_j \right) + c_j \left( 1 - 2 \hat{\psi}_j^+ \hat{\psi}_j \right) \left( 1 - 2 \hat{\psi}_{j+1}^+ \hat{\psi}_{j+1} \right) + h_j \left( 1 - 2 \hat{\psi}_j^+ \hat{\psi}_j \right) \right]. \tag{2.18} \]

Introducing the Majorana operators

\[ \hat{\gamma}_{2j-1} = \hat{\psi}_j^+ + \hat{\psi}_j, \quad -i \hat{\gamma}_{2j} = \hat{\psi}_j^+ - \hat{\psi}_j; \quad \{ \hat{\gamma}_j, \hat{\gamma}_k \} = \delta_{jk} \tag{2.19} \]

the Hamiltonian (2.17) assumes the form

\[ \hat{H} = i \sum_{j=1}^{N} \left( a_j \hat{\gamma}_{2j} \hat{\gamma}_{2j+1} + b_j \hat{\gamma}_{2j+2} \hat{\gamma}_{2j-1} + i c_j \hat{\gamma}_{2j-1} \hat{\gamma}_{2j+1} \hat{\gamma}_{2j} + h_j \hat{\gamma}_{2j-1} \hat{\gamma}_{2j} \right). \tag{2.20} \]

Replacing the Majorana operators with classical real Grassmann variables \( \gamma_j \) and changing these back to the complex Grassmann variables

\[ \gamma_{2j-1} = \bar{\gamma}_j + \gamma_j, \quad -i \gamma_{2j} = \bar{\gamma}_j - \gamma_j; \quad \{ \bar{\gamma}_j, \gamma_k \} = \{ \bar{\gamma}_j, \gamma_k \} = \{ \gamma_j, \gamma_k \} = 0 \tag{2.21} \]

we get the classical expression

\[ H_M = \sum_{j=1}^{N} \left[ a_j (\bar{\gamma}_{j+1} + \gamma_{j+1}) (\bar{\gamma}_j - \gamma_j) + b_j (\bar{\gamma}_j + \gamma_j) (\bar{\gamma}_{j+1} - \gamma_{j+1}) + \right. \]

\[ \left. - 4c_j |\bar{\gamma}_j|^2 |\gamma_{j+1}|^2 + 2h_j |\gamma_j|^2 \right]. \tag{2.22} \]

This defines the function \( H(\bar{\gamma}, \gamma) \) that represents the Hamiltonian appearing in path integrals, as the one in Eq. (2.4). It is worth noting that the consistent definition of the path integral representation of transition amplitudes, in the continuum limit, is not only an academic issue, since it makes available a variety of techniques - borrowed from the quantum field theory toolkit - to support the study of systems of interest in
the fields of condensed matter physics and quantum information science. The formalism is, for example, ideally suited for the calculation of time dependent correlation functions, either exactly or semiclassically. In the present paper, we shall focus on the dynamics of ground state correlators in a spin-chain system described by the XY model. Besides the exact evaluation of correlators’ time-dependence, we shall recover the correct equal time results, that have been evaluated by different means. As a concrete example we present, in Appendix A, a simple calculation pertaining to the two-spin system $\hat{H} = -\omega \hat{S}_1 \hat{S}_2$.

3 Correlator dynamics in the XY model

3.1 Time dependent correlations

The study of entanglement in 1D, spin-1/2 chain models is of great interest not only in the field of condensed matter physics, but also for the quantum information science, where entangled states are of fundamental importance for information processing. The XY model is a well-known and exactly solvable model that exhibits a quantum phase transition. This transition signals the onset of long-range correlations in the ground state of the system, and it is of purely quantum mechanical nature, as it is connected to the entanglement properties of the ground state [11–13]. Thus, the XY model constitutes the ideal stage for the application of the path integral formalism, since it provides the tools to explore the dynamics of the vacuum correlation functions.

The couplings in the anisotropic XY model are defined [11–13] to be $a_j = (1 + r)/2$, $b_j = (1 - r)/2$ and $h_j = h$, $c_j = 0 \forall j$. Thus, the Hamiltonian (2.22) reads as follows

$$H_{XYM} = \sum_{j=1}^{N} \left[ r \left( \zeta_j \bar{\zeta}_{j+1} - \bar{\zeta}_j \zeta_{j+1} \right) + \left( \zeta_j \bar{\zeta}_{j+1} - \bar{\zeta}_j \zeta_{j+1} \right) + 2h \bar{\zeta}_j \zeta_j \right]. \quad (3.1)$$

For the purposes of this chapter we introduce Grassmann sources, redefining the generating functional as:

$$Z[J] = \int_{AP} D\bar{\zeta} D\zeta \exp \left\{ - \int_{-\beta}^{\beta} d\tau \left[ \sum_{j=1}^{N} \bar{\zeta}_j \dot{\zeta}_j + H_{XYM}(\zeta, \bar{\zeta}) - i \sum_{j=1}^{N} (\bar{J}_j \zeta_j + \bar{\zeta}_j J_j) \right] \right\}. \quad (3.2)$$
The functional derivatives of this integral generate, at the limit $\beta \to \infty$, the vacuum (connected) expectation values:

$$\frac{\delta^2 \ln Z[J]}{\delta J_b(\tau_2) \delta J_a(\tau_1)} \bigg|_{J=0} = \langle \bar{\zeta}_b(\tau_2) \bar{\zeta}_a(\tau_1) \rangle, \ldots$$  \hspace{1cm} (3.3)

The study of the static entanglement entropy has been based on the equal time version of the above defined correlation functions. In the current section, we shall use the aforementioned path integral technique to confirm the known static results, while at the same time investigating the dynamics of the underlying entanglement. To deal with the path integration weighted by (2.22), we shall follow the usual tactic, which is based on the introduction of the Fourier transforms:

$$\zeta_i = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{i \frac{2\pi}{N} (m + \frac{1}{2})} c_m, \quad J_i = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{i \frac{2\pi}{N} (m + \frac{1}{2})} \lambda_m. \quad (3.4)$$

Note, at this point, that transformation (3.4) implies the anti-periodic condition $\zeta_{N+1} = -\zeta_1$. In turn, this is connected to a chain of even number of fermions. If this number is odd, the change $m + \frac{1}{2} \to m$ is required. Inserting (3.4) in (3.2) we are able to write the classical Hamiltonian in the form:

$$H_{XYM} = \sum_{m=0}^{N/2-1} H_m, \quad H_m = 2 \left( \begin{array}{cc} \bar{c}_m & c_{N-m-1} \\ c_m & \bar{c}_{N-m-1} \end{array} \right) \left( \begin{array}{cc} k_m - i l_m \\ i l_m - k_m \end{array} \right) \left( \begin{array}{c} c_m \\ \bar{c}_{N-m-1} \end{array} \right). \quad (3.5)$$

To arrive at the last expression we introduced the abbreviations:

$$k_m = h - \cos \frac{2\pi}{N} \left( m + \frac{1}{2} \right), \quad l_m = \sin \frac{2\pi}{N} \left( m + \frac{1}{2} \right). \quad (3.6)$$

The crucial observation here is that the interactions connect only the fields $\bar{c}_m$ with $c_m$ or $\bar{c}_{N-m-1}$ and the fields $c_m$ with $\bar{c}_m$ or $c_{N-m-1}$. Thus, the generating functional can be factorized as

$$Z[J] = \prod_{m=0}^{N/2-1} Z_m[J], \quad Z_m[J] = \int_{\mathcal{D}} \mathcal{D}c_m \mathcal{D}c_{N-m-1} \mathcal{D}c_{N-m-1} e^{-S_m[J]} \quad (3.7)$$
where
\[
S_m[J] = \int_{-\infty}^{+\infty} d\tau \left\{ (\dd{c}_m + \dd{c}_{N-m-1} + H_m - i (\dd{\lambda}_m + \dd{\lambda}_m + \dd{\lambda}_{N-m-1} + \dd{\lambda}_{N-m-1}) \right\}. \tag{3.8}
\]

The Hamiltonian $H_m$, defined in (3.5), can be easily diagonalized through a unitary Bogoliubov transformation:
\[
H_m = U_m^\dagger \begin{pmatrix} \varepsilon_m & 0 \\ 0 & -\varepsilon_m \end{pmatrix} U_m, \quad U_m = \begin{pmatrix} \cos \theta_m & i \sin \theta_m \\ i \sin \theta_m & \cos \theta_m \end{pmatrix}. \tag{3.9}
\]

In this expression
\[
\varepsilon_m = 2 \sqrt{\left( h - \cos \frac{2\pi}{N} \left( m + \frac{1}{2} \right) \right)^2 + \left( r \sin \frac{2\pi}{N} \left( m + \frac{1}{2} \right) \right)^2} \tag{3.10}
\]
and
\[
\tan(2\theta_m) = \frac{r \sin \frac{2\pi}{N} \left( m + \frac{1}{2} \right)}{h - \cos \frac{2\pi}{N} \left( m + \frac{1}{2} \right)}. \tag{3.11}
\]

By making the change of variables
\[
\begin{pmatrix} c_m \\ \dd{c}_{N-m-1} \end{pmatrix} = U_m \begin{pmatrix} \xi_m \\ \dd{\xi}_{N-m-1} \end{pmatrix} \tag{3.12}
\]
the action in Eq. (3.7) can be written in the following form
\[
S_m[J] = \int_{-\infty}^{+\infty} d\tau \left( \phi_m^\dagger D_m \phi_m - i \mu_m^\dagger \phi_m - i \phi_m^\dagger \mu_m \right). \tag{3.13}
\]

Here
\[
\phi_m = \begin{pmatrix} \xi_m \\ \xi_{N-m-1} \end{pmatrix}, \quad D_m = \begin{pmatrix} \partial_\tau + \varepsilon_m & 0 \\ 0 & \partial_\tau - \varepsilon_m \end{pmatrix} \tag{3.14}
\]
and
\[
\mu_m^\dagger = \begin{pmatrix} \dd{\lambda}_m & -\dd{\lambda}_{N-m-1} \end{pmatrix} U_m. \tag{3.15}
\]

Before proceeding, it is worth noting that the change of variables (3.12) and the subsequent diagonalization is permitted by the symmetric form of the discrete time lattice structure which defines the path integral. On the contrary, if the asymmetric discrete form had been kept, this change would not be possible. Written in this form, the inte-
The Green’s function $G_{m}^{(+)}(\tau - \tau')$, which propagates the $m$ modes, has been chosen to obey causality: $G_{m}^{(+)}(\tau - \tau') = 0$ for $\tau - \tau' < 0$. It is, in fact, the antiperiodic function $G_{m}^{(+)}(\tau - \tau') = \left[ \Theta(\tau - \tau') - (1 + e^{\beta\epsilon_m})^{-1} \right] e^{-(\tau - \tau')\epsilon_m}$ (see Eq. (2.10) at the limit $\beta \to \infty$. The advanced function $G_{m}^{(-)}(\tau - \tau')$ propagates backwards the $N - m - 1$ conjugate modes, and obeys the boundary condition $G_{m}^{(-)}(\tau - \tau') = 0$ for $\tau - \tau' > 0$. As expected, it is the $\beta \to \infty$ limit of the antiperiodic function $G_{m}^{(-)}(\tau - \tau') = \left[ (1 + e^{\beta\epsilon_m})^{-1} - \Theta(\tau' - \tau) \right] e^{-(\tau' - \tau)\epsilon_m}$ that coincides with $G_{-\omega}^{(-)}(\tau - \tau')$ in Eq. (2.14). Note that, according to our prescription, the $\Theta$ function appearing in (3.17) is the Heaviside step function, for which $\Theta(0) = 1/2$. In Eq. (3.16), the subsystem’s partition function $Z_{m}[0]$ appears as a normalization factor. The calculation of this factor and, consequently, the calculation of the partition function of the whole system can be easily performed yielding the result:

$$Z(\beta) = \prod_{m=0}^{N-1} 2 \cosh(\beta\epsilon_m/2).$$

(3.18)

By acting with the functional derivatives on the generating functional (3.7), it is an easy task to compute the following correlation functions:

$$\langle \tilde{\xi}_b(\tau_2) \tilde{\xi}_a(\tau_1) \rangle = \frac{1}{N} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}(m+\frac{1}{2})(b-a)} \left( \cos^2 \theta_m G_m^{(+)}(\tau_2 - \tau_1) + \sin^2 \theta_m G_m^{(-)}(\tau_2 - \tau_1) \right)$$

(3.19)

$$\langle \tilde{\xi}_b(\tau_2) \xi_a(\tau_1) \rangle = -\frac{1}{N} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}(m+\frac{1}{2})(b-a)} \left( \cos^2 \theta_m G_m^{(+)}(\tau_1 - \tau_2) + \sin^2 \theta_m G_m^{(-)}(\tau_1 - \tau_2) \right)$$

(3.20)
\[ \langle \zeta_b(\tau_2)\bar{\zeta}_a(\tau_1) \rangle = \frac{i}{2N} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N} (m+\frac{1}{2})(b-a)} \sin(2\theta_m) \left( G_m^{(+)}(\tau_1 - \tau_2) - G_m^{(-)}(\tau_1 - \tau_2) \right) \]  
\tag{3.21}

\[ \langle \bar{\zeta}_b(\tau_2)\zeta_a(\tau_1) \rangle = -\frac{i}{2N} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N} (m+\frac{1}{2})(b-a)} \sin(2\theta_m) \left( G_m^{(+)}(\tau_2 - \tau_1) - G_m^{(-)}(\tau_2 - \tau_1) \right) . \]  
\tag{3.22}

For comparison reasons we can compute the correlator of the Majorana variables \( (2.21) \):
\[ iB_{ba}(\tau_2 - \tau_1) \equiv \langle \gamma_{2b}(\tau_2)\gamma_{2a-1}(\tau_1) \rangle . \]  
\tag{3.23}

To find the real time result, one has to make the Wick rotation \( \tau \to it \) and take the real part of the emerging expression. This last step is connected to the symmetric rewriting of the kinetic term in Eq. \( (3.13) \): \( \phi_m^+D_m\phi_m \to \frac{1}{2}\phi^+D_m\phi + \frac{1}{2} (D_m\phi_m)^+\phi_m \).

Inserting the correlators \( (3.19)-(3.22) \) into Eq. \( (3.23) \) we get the real time correlator
\[ B_{ba}(t_2 - t_1) = -\frac{1}{N} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N} (m+\frac{1}{2})(b-a)} e^{-2i\theta_m} \cos \left[ (t_2 - t_1)\epsilon_m \right] . \]  
\tag{3.24}

At the thermodynamic limit \( N \to \infty \) we get
\[ B_1(t) = -\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i\phi\theta - 2i\epsilon(\phi)} \cos [t \epsilon(\phi)] . \]  
\tag{3.25}

in this expression, the function \( \theta(\phi) \) is defined as:
\[ 2\theta(\phi) = \begin{cases} \arctan \frac{\sin \phi}{h - \cos \phi} , & h - \cos \phi > 0, \\ \arctan \frac{\sin \phi}{h - \cos \phi} + \pi , & h - \cos \phi < 0. \end{cases} \]  
\tag{3.26}

and
\[ \epsilon(\phi) = 2\sqrt{(h - \cos \phi)^2 + (\sin \phi)^2} . \]  
\tag{3.27}

For the XX model \( (r \to 0) \) and \( |h| \leq 1 \) the integral \( (3.25) \) simplifies to:
\[ B_1(t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i\phi\theta} \text{sign}(\cos \phi - h) \cos (2t(\cos \phi - h)) = \] 
\[ = \frac{1}{\pi} \int_{\phi_h}^{\phi_h} d\phi \epsilon \epsilon e^{i\phi\theta} \cos (2t(\cos \phi - h)) \]  
\tag{3.28}
where $\phi_h$ is defined through the equation $\cos \phi_h = h$.

The evaluation of this integral yields the result:

$$B_l(t) = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} J_k(2t) \frac{\sin[(l+k)\phi_h]}{l+k} \cos \left(2ht - k\frac{\pi}{2}\right) \quad (3.29)$$

with $J_k$ being the Bessel functions. The static case is achieved at the limit $t \to 0$ where only the $k = 0$ term survives, giving:

$$B_l(0) = \frac{2 \sin(l\phi_h)}{l}. \quad (3.30)$$

At the vicinity of the critical value $h = 1$ we find that $\phi_h \simeq \sqrt{2\lambda}$ where $\lambda \equiv 1 - h \to +0$. This value sets the scale after which strong oscillations diminish correlations. Thus, the correlation length behaves as $\xi \sim \lambda^{-\frac{1}{2}} \sim \lambda^{-\nu}$ giving the corresponding critical exponent $\nu = 1/2$ [21]. For $t > 0$, expression (3.29), near the critical point simplifies as follows:

$$B_l(t) \simeq \frac{2}{\pi l} \sin(\phi_h) l \cos(2t\lambda). \quad (3.31)$$

The time scale after which correlations turn off due to the strong oscillations can easily be seen from the last equation being: $t \sim \lambda^{-1} \sim \xi^2$. Consequently, the dynamical critical exponent defined through equation $t \sim \xi^z$ is $z = 2$ [22]. Therefore, we expect [11–13] the entanglement entropy to scale as $S \sim \frac{1}{3} \log \xi \sim \frac{1}{6} \log t$ at the critical vicinity.

### 3.2 Driven correlations

In this section we examine the case of a time dependent transverse field $h = h(t)$ in the XY spin chain. For the quantum Ising model ($r = 1$) and for a field linearly dependent on time, the entanglement dynamics have been extensively studied [14, 15, 30–32], mainly by numerical methods. The present work is a contribution to the analytical methods available for the study of spin systems out of equilibrium. In the path integral representation, the exact calculation of the equal time vacuum state correlators of the XY model is possible, irrespectively of the form the function $h(t)$ has. Leaving the details of the calculation for Appendix B, it suffices here to note that in the presence of a time dependent magnetic field, both eigenvalues $\epsilon_m$, and the matrix $U_m$, which diagonalizes the Hamiltonian in Eq. (3.5), also become time dependent quantities. As
a consequence, the extra off-diagonal contribution

\[
U_m^\dagger \partial_t U_m = i \dot{\theta}_m \sigma^x, \quad \dot{\theta}_m = -\hbar \frac{2r}{\epsilon_m^2} \sin \left[ \frac{2\pi}{N} \left( m + \frac{1}{2} \right) \right] \equiv -\hbar \frac{2r}{\epsilon_m^2} \sin \phi_m \quad (r \neq 0)
\] (3.32)
appears in the action (3.13), a fact that makes the problem more involved. However, the calculation can be carried out without difficulty due to the quadratic nature of the Hamiltonian appearing in Eq. (3.1) and the application of the Cluster Expansion theorem [33]. The partition function of the driven system is found to be of the following form

\[
Z(\beta) = \prod_{m=0}^{N-1} \left( 2 \cosh \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \frac{\epsilon_m(\tau)}{2} \right) e^{-E_m}
\] (3.33)

where

\[
E_m = \frac{1}{2} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau_2 \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau_1 \dot{\theta}_m(\tau_2) \dot{\theta}_m(\tau_1) e^{-2 \int_{\tau_1}^{\tau_2} d\tau \epsilon_m(\tau)}
\] (3.34)

For the case of the XX model, for which \( r = 0 \), the angle \( 2 \theta_m \) is not constant, as it jumps between 0 and \( \pi \) according to the sign of the difference \( \cos \phi - h \) as indicated in Eq. (3.26). However, its time derivative is zero, and consequently we can repeat the steps we followed for the time-independent system. This way we get the result

\[
B_l(t) = \frac{1}{\pi} \int_{-\phi_h(t)}^{\phi_h(t)} d\phi e^{il\phi} = \frac{2}{\pi} \sin \left( \frac{l \phi_h(t)}{2} \right).
\] (3.35)

In what follows we shall assume that the field changes linearly with time. The rate of this change, \( |\dot{h}| = 1/\rho \), sets a characteristic time scale for the system. After the rescaling \( \tau \rightarrow \rho \sigma \), the exponent of the second factor in equation (3.33) reads at the zero temperature limit:

\[
E_m = \frac{1}{2} \int_{-\infty}^{\infty} d\sigma_2 \int_{-\infty}^{\sigma_2} d\sigma_1 \dot{\theta}_m(\sigma_2) \dot{\theta}_m(\sigma_1) e^{-2 \rho \int_{\sigma_1}^{\sigma_2} d\sigma \epsilon_m(\sigma)}
\] (3.36)

where

\[
\dot{\theta}_m(\sigma) = -\frac{2r}{\epsilon_m^2(\sigma)} \sin \phi_m \quad \epsilon_m(\sigma) = 2 \sqrt{(\sigma - \cos \phi_m)^2 + r^2 \sin^2 \phi_m}, \quad (r \neq 0)
\] (3.37)

Working with \( r = 1 \) for simplicity, we can easily confirm that \( E_m \) reduces, at the sudden limit \( \rho \rightarrow 0 \), to a constant:

\[
E_m \simeq \frac{\pi^2}{16} + O(\rho),
\] (3.38)
while at the adiabatic limit, $\rho \to \infty$ (see Appendix B), it reduces to:

$$E_m \simeq \frac{1}{4\rho} \int_{-\infty}^{\infty} \frac{d\sigma}{e_m(\sigma)} + \mathcal{O}\left(\frac{1}{\rho^2}\right) = \frac{1}{24\sin^2 \phi_0} \frac{1}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right). \tag{3.39}$$

Thus, at the adiabatic regime, the exponent function (3.36) is almost zero, unless the angle $\phi_m$ is sufficiently small: $|\phi_m| \sim \rho^{-1/2}$. For the correlation function in Eq. (3.23), we find (again in Appendix B) the following exact result:

$$B_l(\sigma) = -\frac{1}{N} \sum_{m=0}^{N-1} e^{i\phi_m - 2i\theta_m(\sigma)} [1 - ib_m(\sigma)]. \tag{3.40}$$

The function in the last factor is defined as:

$$b_m(\sigma) = \int_{-\infty}^{\infty} d\sigma' \frac{\partial}{\partial \sigma} \left[ \Theta(\sigma' - \sigma) e^{-2\rho \int_{\sigma'}^{\sigma} d\sigma_m} - \Theta(\sigma - \sigma') e^{-2\rho \int_{\sigma}^{\sigma'} d\sigma_m} \right]. \tag{3.41}$$

This is a bounded function, $|b_m(\sigma)| \leq \pi/2$, which attains its bounds at the sudden limit, and at the vicinity of the critical point $|h - 1| \simeq \phi \simeq 0$:

$$|b_m(\sigma)| \simeq \rho \to 0 |2\theta_m(\sigma)| \simeq \frac{\pi}{2} + \mathcal{O}(\rho). \tag{3.42}$$

At the adiabatic regime we easily find that

$$b_m(\sigma) \simeq \frac{1}{2\rho^2} e(\phi; \sigma) \frac{\partial}{\partial \sigma} \left[ \frac{\partial}{\partial \sigma} \left( \frac{\phi}{\rho} \right) \right] + \mathcal{O}(1/\rho^4). \tag{3.43}$$

At the thermodynamic limit, the angle $\phi_m$ changes continuously and the correlation function (3.40) reads as follows:

$$B_l(\sigma) = -\frac{1}{2\pi} \int_0^{2\pi} e^{i\phi - 2i\theta(\phi; \sigma)} [1 - ib(\phi; \sigma)] \equiv B_l^S + B_l^Q. \tag{3.44}$$

The first term in the last expression reproduces, for each instantaneous value $h(t)$, the corresponding static results [12]. The second term incorporates the changes induced to the correlator by the time dependent magnetic field. When the driving is slow enough, this term is, according to Eq. (3.43), negligible, when the magnetic field is away from its critical value. Thus, if the evolution is adiabatic, the "static" term, $B_l^S$, in Eq. (3.44) controls the correlator. However, as the system approaches the critical region, $|h - 1| = \lambda \to 0$, $B_l^Q$ becomes increasingly important, due to the leading term in the adiabatic asymptotic expansion (3.43) being of order unity, for $\phi \sim \lambda \sim \rho^{-1/2}$.
- followed by $O(\rho^{-1/2})$ terms - and $b_m(\sigma)$ approaches its sudden limit (see Appendix B). At the critical point $h = 1$ it is very simple to find that

$$B_1^Q \simeq \frac{i}{2\pi} \int_{-\phi_0}^{\phi_0} d\phi e^{i\phi - 2i\theta(\phi; \sigma)} b(\phi; \sigma)_{h=1} \frac{\sin[\phi_0(2l + 1)/2]}{2l + 1}, \quad \phi_0 \sim \rho^{-1/2}. \quad (3.45)$$

This behaviour sets the length $\xi \sim \rho^{1/2}$, as the scale that characterizes the passing of the system through the critical point. This, is in accordance with the so-called Kibble-Zurek mechanics (KZM) or the adiabatic-impulse-adiabatic approximation [22, 23], which is based on the fact that the evolution of a system driven through a second-order phase transition, cannot be adiabatic near the critical point, irrespective of how slow the driving is. In KZM time evolution is considered initially as adiabatic, becoming non-adiabatic near the critical point, where the energy gap changes with a rate comparable to the energy gap itself: $|\dot{h}|/|h - 1| \simeq |h - 1| \rightarrow \lambda \sim \rho^{-1/2}$. In such a case, the entanglement entropy is expected [15] to behave as $S \sim \frac{1}{12} \log_2 \rho$.

4 Conclusions-Perspectives

In the present paper we have used path integration over fermionic coherent states to analyze quantum correlations in a paradigmatic spin model. We discussed the construction of the relevant path integral and we adopted the Faddeev-Jackiw technique to avoid possible pitfalls. We calculated the time-dependent vacuum expectation values needed for the calculation of the entanglement entropy and we confirmed that the correct static limit is reproduced. In the last section we examined the case of a time-dependent magnetic field that drives the system through the critical point and we confirmed that our results are consistent with the Kibble-Zurek mechanism. The aim of the current work is not to present novel results but, rather, to present a novel way to analyze closed spin-chain systems. In a forthcoming work we shall use the path integral technique to derive the reduced dynamics of open fermionic systems.

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A Functional integration of $\hat{H} = -\omega \hat{S}_1 \cdot \hat{S}_2$

In this Appendix, we examine a simple system, the dynamics of which can be described by the Hamiltonian $\hat{H} = -\omega \hat{S}_1 \cdot \hat{S}_2$. The partition function of this system is known to be:

$$Z = \text{Tr} \left[ e^{-\beta \hat{H}} \right] = e^{-3\beta \omega / 4} + 3e^{\beta \omega / 4}. \quad (A.1)$$

Using the Jordan-Wigner transformation, the Hamiltonian in hand assumes the form:

$$\hat{H} = -\frac{\omega}{4} \left[ (\hat{\psi}_1^\dagger - \hat{\psi}_1) (\hat{\psi}_2^\dagger + \hat{\psi}_2) + (\hat{\psi}_2^\dagger - \hat{\psi}_2) (\hat{\psi}_1^\dagger + \hat{\psi}_1) + (1 - 2\hat{\psi}_1^\dagger \hat{\psi}_1) (1 - 2\hat{\psi}_2^\dagger \hat{\psi}_2) \right]. \quad (A.2)$$

In terms of Majorana operators, this reads:

$$\hat{H} = i\frac{\omega}{4} (\gamma_2 \gamma_3 + \gamma_4 \gamma_1 - i\gamma_1 \gamma_2 \gamma_3 \gamma_4). \quad (A.3)$$

According to the Faddeev-Jackiw quantization scheme, the classical function entering the Majorana path integral representation is the classical counterpart of (A.3):

$$H_m = i\frac{\omega}{4} (\gamma_2 \gamma_3 + \gamma_4 \gamma_1 - i\gamma_1 \gamma_2 \gamma_3 \gamma_4) \quad (A.4)$$

which in turn translates to the complex Grassmann variables as:

$$H_m = -\frac{\omega}{2} \left( \xi_1 \bar{\xi}_2 + \xi_2 \bar{\xi}_1 + 2|\xi_1|^2|\xi_2|^2 \right). \quad (A.5)$$

Thus, the integral to be evaluated is:

$$Z = \left( \prod_{j=1}^{2} \int_{AP} D\bar{\xi}_j D\xi_j e^{-\frac{\beta}{2} d\tau \bar{\xi}_j \xi_j} \right) \exp \left[ \frac{\omega}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau \left( \bar{\xi}_1 \xi_2 + \bar{\xi}_2 \xi_1 + 2|\xi_1|^2|\xi_2|^2 \right) \right]. \quad (A.6)$$
It is convenient to perform a change of variables, induced through the unitary transformation:

\[
\begin{pmatrix}
\zeta_1 \\
\zeta_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}.
\] (A.7)

After this change, the integral (A.6) is recasted to the form:

\[
Z = \left( \prod_{j=1}^{2} \int_{AP} D\eta_j D\bar{\eta}_j e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \eta_j \eta_j} \right) \exp \left[ -\frac{\omega}{2} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \left( |\eta_1|^2 - |\eta_2|^2 + 2|\eta_1|^2 |\eta_2|^2 \right) \right].
\] (A.8)

We proceed by performing the integration over the first Grassmann field

\[
Z_1 = \int_{AP} D\eta_1 D\bar{\eta}_1 \exp \left[ -\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \eta_1 \left( \partial_\tau - \frac{\omega}{2} - \omega |\eta_2|^2 \right) \eta_1 \right] = 2\cosh \left( \frac{\beta \omega}{4} + \frac{\omega}{2} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau |\eta_2|^2 \right).
\] (A.9)

To arrive to the last result, we have relied on the symmetric prescription for the underlying lattice structure. Inserting Eq. (A.9) into Eq. (A.8), we get the correct quantum result:

\[
Z = e^{\beta \omega / 4} \int_{AP} D\eta_2 D\bar{\eta}_2 e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \eta_2 \bar{\eta}_2} + e^{-\beta \omega / 4} \int_{AP} D\bar{\eta}_2 D\eta_2 e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau (\partial_\tau + \omega) \eta_2} = e^{-3\beta \omega / 4} + 3e^{\beta \omega / 4}
\] (A.10)

### B Calculation of equal time correlation function

In this Appendix, we calculate the equal time correlation function

\[
iB_{ba}(\tau) \equiv \langle \gamma_{2b}(\tau) \gamma_{2a-1}(\tau) \rangle
\] (B.1)

when the parameter $h$ is time dependent: $h = h(\tau)$.

We begin by using Eq. (2.21) to rewrite the correlator in the form

\[
B_{ba}(\tau) = \langle \bar{\zeta}_b \zeta_a - \zeta_b \bar{\zeta}_a + \bar{\eta}_b \zeta_a - \bar{\eta}_b \bar{\zeta}_a \rangle,
\] (B.2)
in which all fields are defined at the same moment $\tau$. Following the standard procedure, we use the Fourier transform of the Grassmann fields, and get the following expression for the correlator:

$$B_{ba}(\tau) = \frac{1}{N} e^{2\pi i (m+\frac{1}{2})(b-a)} \left( \begin{array}{c} \bar{c}_m \\ c_{N-m-1} \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \left( \begin{array}{c} c_m \\ \bar{c}_{N-m-1} \end{array} \right).$$  \hspace{1cm} (B.3)

By changing the variables as indicated in (3.12) and (3.14) we easily confirm that:

$$B_{ba}(\tau) = \frac{1}{N} e^{2\pi i (m+\frac{1}{2})(b-a)} \langle \phi_m^+(\tau) (\sigma^z - i\sigma^y) \phi_m(\tau) \rangle$$  \hspace{1cm} (B.4)

where terms of the form $c_m c_{N-m-1}$ and $\bar{c}_m \bar{c}_{N-m-1}$ were omitted due to these being decoupled in the action. To calculate the vacuum expectation value that appears in the last equation we introduce the generating functional

$$Z_m[g] = \int_{AP} D\bar{c}_m Dc_m D\bar{c}_{N-m-1} Dc_{N-m-1} e^{-S_m[g]}$$  \hspace{1cm} (B.5)

with the action being

$$S_m[g] = \int_{-\infty}^{\infty} d\tau \left( \phi_m^+ D_m \phi_m + i\dot{\theta}_m \phi_m^+ \sigma^x \phi_m + g(\tau) \phi_m^+ (\sigma^z - i\sigma^y) \phi_m \right).$$  \hspace{1cm} (B.6)

After the calculation of (B.5) the correlation function (B.4) can be computed by taking the functional derivative

$$B_{ba}(\tau) = -\frac{1}{N} \sum_{m=0}^{N-1} e^{2\pi i (m+\frac{1}{2})(b-a)} \left. \frac{\delta \ln Z_m[g]}{\delta g(\tau)} \right|_{g=0}.$$  \hspace{1cm} (B.7)

It will be useful to reorganize the action (B.6) as follows:

$$S_m[g] = \int_{-\infty}^{\infty} d\tau \left( \phi_m^+ \tilde{D}_m \phi_m + \phi_m^+ \Sigma_m \phi_m \right)$$  \hspace{1cm} (B.8)

where

$$\tilde{D}_m = \left( \begin{array}{cc} \partial_\tau + \bar{c}_m(\tau) & 0 \\ 0 & \partial_\tau - c_m(\tau) \end{array} \right), \quad \bar{c}_m(\tau) = c_m(\tau) + g(\tau)$$  \hspace{1cm} (B.9)

and

$$\Sigma_m = \left( \begin{array}{cc} 0 & i\dot{\theta}_m(\tau) - g(\tau) \\ i\dot{\theta}_m(\tau) + g(\tau) & 0 \end{array} \right).$$  \hspace{1cm} (B.10)
After these abbreviations, the generating functional in (B.5) can be recasted into the following compact expression:

\[ Z_{m[g]} = N_{m[g]} \langle \exp\left(-\int_{-\infty}^{\infty} d\tau \phi_m^+ \Sigma_m \phi_m \right) \rangle_g \]  \hspace{1cm} (B.11)

where the expectation value above is defined as follows

\[ \langle (...) \rangle_g \equiv N_{m^{-1}[g]} \int_{AP} D\bar{c}_m Dc_m D\bar{c}_{N_m-1} Dc_{N_m-1} e^{-\int_{-\infty}^{\infty} d\tau \phi^+_m \hat{D}_m \phi_m (\ldots)} \]  \hspace{1cm} (B.12)

The normalization factor in Eq. (B.11) and (B.12) is defined to be:

\[ N_{m[g]} = \int_{AP} D\bar{c}_m Dc_m D\bar{c}_{N_m-1} Dc_{N_m-1} e^{-\int_{-\infty}^{\infty} d\tau \phi^+_m \hat{D}_m \phi_m} = \left[ 2 \cosh \left( \int_{-\infty}^{\infty} d\tau \frac{\hat{G}_m}{2} \right) \right]^2 \]  \hspace{1cm} (B.13)

To continue, we can use the Cluster Expansion theorem [33] to express the expectation value in Eq. (B.11) in terms of connected correlation functions:

\[ \langle \exp\left(-\int_{-\infty}^{\infty} d\tau M_m \right) \rangle = \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} d\tau_1 \ldots \int_{-\infty}^{\infty} d\tau_n \langle M_m(\tau_1) \ldots M_m(\tau_n) \rangle_C \right] , \]  \hspace{1cm} (B.14)

where for the case in hand,

\[ M_m = \phi^+_m \Sigma_m \phi_m. \]  \hspace{1cm} (B.15)

The correlators in Eq. (B.14) are defined as follows

\[ \langle M_m(\tau_1) \rangle_C = \langle M_m(\tau_1) \rangle \]

\[ \langle M_m(\tau_1) M_m(\tau_2) \rangle_C = \langle M_m(\tau_1) M_m(\tau_2) \rangle - \langle M_m(\tau_1) \rangle \langle M_m(\tau_2) \rangle , \ldots \]  \hspace{1cm} (B.16)

For the quadratic action which weights the expectation values in Eq. (B.12), one can easily confirm that

\[ \langle \phi^+_m \Sigma_m \phi_m \rangle_g = -Tr \left[ \Sigma_m \hat{D}_m^{-1} \right] = 0. \]  \hspace{1cm} (B.17)

Here, we used

\[ \hat{D}_m^{-1} = \begin{pmatrix} \hat{G}_m^{(+)} & 0 \\ 0 & \hat{G}_m^{(-)} \end{pmatrix} \]  \hspace{1cm} (B.18)
\[ \tilde{G}_m^{(+)}(\tau, \tau') = \Theta(\tau - \tau') e^{-\int_\tau^{\tau'} d\tau'' \tilde{\epsilon}_m(\tau'')} , \quad \tilde{G}_m^{(-)}(\tau, \tau') = -\Theta(\tau' - \tau) e^{-\int_{\tau'}^{\tau''} d\tau'' \tilde{\epsilon}_m(\tau'')} . \]  
(B.19)

Due to the quadratic form of the action, Wick’s theorem is applicable and, consequently, the only surviving term in the Cluster Expansion (B.14) is the second one. Thus:

\[ Z_m[g] = N_m[g] \exp \left( -\frac{1}{2} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \langle M_m(\tau_1) M_m(\tau_2) \rangle_g \right) . \]  
(B.20)

A simple calculation yields the result

\[ \langle M_m(\tau_1) M_m(\tau_2) \rangle_g = \text{Tr} \left[ \Sigma_m(\tau_2) \hat{D}_m^{-1}(\tau_2, \tau_1) \Sigma_m(\tau_1) \hat{D}_m^{-1}(\tau_1, \tau_2) \right] . \]  
(B.21)

The \( g \to 0 \) limit of the above relations confirms the partition function appearing in Eq. (B.12). Combining Eqs. (B.20) and (B.21) we can easily find that:

\[ \frac{\delta \ln Z_m[g]}{\delta g(\tau)} \bigg|_{g=0} = \tanh \left( \int_{-\infty}^{\infty} d\tau \frac{\epsilon_m}{2} \right) - i \int_{-\infty}^{\infty} d\tau' \hat{\theta}_m(\tau') f(\tau', \tau) \]  
(B.22)

where

\[ f(\tau', \tau) = \Theta(\tau' - \tau) e^{-2 \int_\tau^{\tau'} d\tau' \epsilon_m} - \Theta(\tau - \tau') e^{-2 \int_{\tau'}^{\tau''} d\tau'' \epsilon_m} . \]  
(B.23)

In the following, we shall adopt a transverse field linearly dependent on time \([15, 22]\).

In this case, \( \int_{-\infty}^{\infty} d\tau \epsilon_m \to \infty \) and the correlation function (B.1) turns out to have the following form

\[ B_{ba}(\tau) = -\frac{1}{N} \sum_{m=0}^{N-1} e^{2i m (b-a) - 2i \tilde{\theta}_m(\tau)} [1 - i b_m(\tau)] \]  
(B.24)

where

\[ b_m(\tau) = \int_{-\infty}^{\infty} d\tau' \hat{\theta}_m(\tau') f(\tau', \tau) , \quad |b_m(\tau)| \leq \pi \hat{h}/2. \]  
(B.25)

At the thermodynamic limit, the last expression can be rewritten in the form

\[ B_{ba}(\tau) = -\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{i(b-a)\phi - 2i \tilde{\theta}(\phi; \tau)} [1 - i b(\phi; \tau)] . \]  
(B.26)
To find the adiabatic limit of $b_m$ we perform the rescaling $\tau = \rho \sigma$ and redefinitions (B.37), to rewrite (B.25) as

$$b(\phi; \sigma) = \int_{-\infty}^{\infty} d\sigma' \dot{\theta}(\phi; \sigma') \left[ \Theta(\sigma' - \sigma)e^{-2\rho \int_{\sigma'}^{\sigma} d\sigma e(\phi; \sigma')} - \Theta(\sigma - \sigma')e^{-2\rho \int_{\sigma}^{\sigma'} d\sigma e(\phi; \sigma')} \right].$$

(B.27)

At the sudden limit $\rho \to 0$ (for $r = 1$) we easily find the result indicated in Eq. (3.42)

$$|b(\phi; \sigma)| \xrightarrow{\rho \to 0} |2\theta(\phi; \sigma)| \approx \frac{\pi}{2} + O(\rho).$$

(B.28)

Using repeatedly the identities

$$e^{-2\rho \int_{\sigma}^{\sigma'} d\sigma e} = \frac{1}{2\rho e} \frac{\partial}{\partial \sigma'} e^{-2\rho \int_{\sigma'}^{\sigma} d\sigma e}, \quad e^{-2\rho \int_{\sigma'}^{\sigma} d\sigma e} = -\frac{1}{2\rho e} \frac{\partial}{\partial \sigma'} e^{-2\rho \int_{\sigma'}^{\sigma} d\sigma e}$$

(B.29)

we find the asymptotic expansion inq. (3.43):

$$b(\phi; \sigma) \simeq \frac{1}{2\rho^2} \frac{\partial}{\partial \sigma} \left[ \frac{\dot{\theta}(\phi; \sigma)}{e(\phi; \sigma)} \right] + O(1/\rho^4).$$

(B.30)

With the same method, one can produce the asymptotic expansion presented in Eq. (3.39). Thus, at the adiabatic limit, the function $b$ is almost negligible except for a small neighbourhood \(|h - 1| \approx \lambda \approx \phi \approx \rho^{-1/2}\) of the critical point, where the expression (B.28) reduces to a constant:

$$b(\phi; \sigma \approx 1 - \lambda) \approx \frac{1}{\lambda \to 0} \frac{1}{\rho^2\lambda^4}.$$

(B.31)

In this neighbourhood all the terms in the expansion (B.28) become the same order as the main contribution in the integral (B.27).

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