On the Fairlie’s Moyal formulation of M(atrix)-theory

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Abstract
Starting from the Moyal formulation of M-theory in the large N-limit, we propose to reexamine the associated membrane equations of motion in 10 dimensions formulated in terms of Poisson bracket. Among the results obtained, we rewrite the coupled first order Nahm’s equations into a simple form leading in turn to their systematic relation with $SU(\infty)$ Yang Mills equations of motion. The former are interpreted as the vanishing condition of some conserved currents which we propose. We develop also an algebraic analysis in which an ansatz is considered and find an explicit form for the membrane solution of our problem. Typical solutions known in literature can also emerge as special cases of the proposed solution.

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Matrix model formulation of M-theory, introduced three years ago by Banks et al. [1], and studied later by many authors [2-3], have shown to be an important area of research. In this model, recall for instance that the fundamental degrees of freedom of the theory are the D0-branes whose interaction processes are described by space-time matrix coordinates [1]. Actually, we know that this matrix model is described by the maximally supersymmetric U(N) Yang-Mills gauge theory, where N is the light-like momentum or the number of D0-branes interpreted in [2] as the number of Green Schwartz strings (gas of N strings) in the light cone gauge. Its also known that these strings have different lengths depending on the winding number $m_k$ around the compact direction, a fact which can be described by the following expression

$$N = \sum m_k N_k.$$  

The large N limit is then shown to correspond to long strings in the infinite momentum frame (IFM). Several ways to investigate this large N limit have been followed. Among these issues one focus on the work of Fairlie [4] which purposes to clarify the meaning of this infinite limit in 4 and 10 dimensions in terms of Moyal bracket formalism. As claimed by this author, this Moyal bracket description is motivated by the fact that the matrix theory takes on an aspect analogous to a Moyal formulation of quantum mechanics [4]. Note by the way, that the Moyal bracket is defined via the star product known to be an essential ingredient towards setting a non commutative geometry framework [5-6]. Recently this old mathematical idea have been a subject of a revived interest in the study of string and matrix theory [7-8-9-10].

Before going into the description of the main lines of the present work, we will try to summarise the Fairlie’s Moyal bracket formulation in M-theory as done in [4]. As signalled above and in accordance with [4], the author examine the Moyal version of matrix theory in the belief that this is the most appropriate for the discussion of the large N limit, and for the investigation of parallels with quantum mechanics. Indeed, it is consistently shown that the Moyal bracket description of infinite limit of matrix theory in 2 and 8 transverse dimensions leads to a class of solutions for the second order matrix theory equations obtained from the first order ones.

Using this correspondence between the second and the first order equations, the author of [4] proceed then by solving the original theory by inverting the problem which means finding a solution to first order equations which guarantees the integrability of the second order ones. This remarkable feature is due to the existence of the Bogomol’nyi bound to the Euclideanised version of the above equations in 4 and 10 dimensions. The general construction of the obtained solutions is also given in terms of a representation of the target space coordinates as non local spinor bilinears, which are generalisations of the standard wigner functions on phase space.

In the present work, we consider an alternative way to reexamine the matrix the-
ory equations of motion in 10 dimensions in the infinite limit, formulated in terms of Poisson bracket instead of Moyal bracket. Among our results we rewrite in a first step the Nahm’s equations into a simple form. This consist precisely in reducing the 14 coupled Nahm’s equations (11,12) leading just to five ones (13,14). These reduced equations are convenient in the sense that their derivative with respect to $\sigma$ gives exactly the second order equations of motion (16) which we derive starting from the $SU(\infty)$ pure Yang Mills lagrangian density. We show also that the second order equations of motion exhibit a special invariance property as given in (21).

The next step consist to understand the meaning of the first order Nahm’s equations and their relation with the equations of motion. For this, we give an interpretation of the former as the vanishing condition of some conserved currents that we consider and interpret the above correspondence as the conservation property of these currents. Finally, we propose to solve the derived second order equations of motion. We have accomplish this task by considering the Fourier modes decomposition, using some algebraic manipulations and setting the important ansatz (48) leading to build our membrane solution.

Next, we try to look for the possible connection of our solution with other solutions known in literature. We start first from the important remark that our membrane solution (71) exhibits an oscillation behaviour which can be reduced to give rise to other solutions (see for example [15]) once particular choices of our parameters are performed.

We organise this paper as follows. In section 2, we review the large N limit in terms of Moyal bracket as done in [4] and present our alternative approach in section 3 to derive the second order membrane equations of motion in 10 dimensions. In section 4, we give a breif comment about Nahm’s equations and present in section 5 the analysis leading to construct a membrane solution in 10 dimensions. Section 6 is devoted to our conclusion.

2 Moyal bracket and large N limit

Since matrix model formulation of M-theory was introduced by Banks et al [1], a lot of stimulated papers elaborating the issue was done [2-3]. Focusing for instance one of these papers [4], when the author describes the large N limit of matrix theory in 4 and 10 dimensions using the Moyal bracket formalism. The starting point is the matrix theory with an action of a two dimensional $N = 8$ supersymmetric U(N) Yang-Mills theory namely [2]

$$ S = \frac{1}{2\pi\alpha'} \int Tr((D_{\mu}X)^2 + \theta^T \theta^T D\theta + g_s^2 F_{\mu\nu}^2 - \frac{1}{g_s^2} [X^i, X^j]^2 + \frac{1}{g_s} \theta^T \gamma_i [X^i, \theta] d\sigma d\tau, \quad (2) $$

where $X^i$ and $(\theta^a_L, \theta^a_R)$ are the 8 scalar and 8 fermionic fields respectively, which are $N \times N$ hermetian matrices transforming as the 8v vector and $(8s, 8c)$ spinor
representations of the SO(8) R-symmetry group of transversal rotations.

The passage to the large N limit consist in a first step in considering the matrices $X^i$ as functions of two phase space variables $\alpha, \beta$ as well as $\sigma, \tau$ such that $X^i(\alpha, \beta, \sigma, \tau)$. The associated matrix elements may be regarded as the Fourier components of $X^i$. The second step consist in substituting commutators of the $\{X^i, X^j\}$ in the following way

$$\int Tr[X^i, X^j]d\sigma \rightarrow \frac{1}{\lambda^2} \int d\sigma d\alpha d\beta \{X^i, X^j\}_{MB}^2,$$  \hspace{1cm} (3)

where $\{X^i, X^j\}_{MB} = \sin\{X^i, X^j\}$ is the sine or Moyal bracket, with deformation parameter $\lambda$ defined as the imaginary part of the star product $\ast$. Recall that the star product of two functions $X^i$ and $X^j$ is defined as:

$$X^i \ast X^j = \lim_{\lambda \rightarrow 0} \exp^{\lambda(\partial_\alpha \partial_{\alpha'} - \partial_\beta \partial_{\beta'})} X^i(\alpha, \beta, \sigma) X^j(\alpha', \beta', \sigma).$$  \hspace{1cm} (4)

The Moyal bracket is then defined as the antisymmetric part of the star product such that

$$\{X^i, X^j\}_{MB} = \lim_{\lambda \rightarrow 0} \sin \lambda(\partial_\alpha \partial_{\alpha'} - \partial_\beta \partial_{\beta'}) X^i(\alpha, \beta, \sigma) X^j(\alpha', \beta', \sigma).$$  \hspace{1cm} (5)

As quoted in [4], the point of this construction is that in the limiting points $\lambda \rightarrow \frac{2\pi}{N}$, the Moyal bracket (5) reproduces the commutators of $N \times N$ matrices $X^i$ (bosonic coordinates) through the association of the components $X^i_{mn}$ of $X^i$ with the Fourier modes of a function $X^i(\alpha, \beta, \sigma)$ periodic in $\alpha, \beta$. The remaining terms in the action (2) involving fermionic coordinates are replaced by

$$\{X^\mu, \theta\}_{MB} = \lim_{\lambda \rightarrow 0} \sin \lambda(\partial_\alpha \partial_{\alpha'} - \partial_\beta \partial_{\beta'}) X^\mu(\alpha, \beta, \sigma) \theta(\alpha', \beta', \sigma)$$

$$\{\theta^T, D\theta\}_{MB} = \lim_{\lambda \rightarrow 0} \cos \lambda(\partial_\alpha \partial_{\alpha'} - \partial_\beta \partial_{\beta'}) \theta^T(\alpha, \beta, \sigma) D\theta(\alpha', \beta', \sigma).$$  \hspace{1cm} (6)

Thus the action becomes

$$S_{MB} = \frac{1}{2\pi\sigma} \int ((D_\mu X)^2 + \cos\{\theta^T, D\theta\} + g_s^2 Tr F_{\mu\nu}^2) d\alpha d\beta d\sigma d\tau - \int ((\frac{\lambda}{\lambda g_s^2} \sin\{X^i, X^j\})^2 - \frac{1}{g_s^2} \cos\{\theta^T\gamma_i, \frac{1}{\lambda} \sin\{X^i, \theta\}\}) d\alpha d\beta d\sigma d\tau.$$  \hspace{1cm} (7)

Moreover, once the following correspondence $\lambda \rightarrow \frac{2N}{\pi}$ is performed, the final form of the action, in the large N-limit ($\lambda \rightarrow 0$), is then expressed in terms of ordinary Poisson brackets as

$$S_{PB} = \frac{1}{2\pi\sigma} \int ((D_\mu X)^2 + \theta^T D\theta + g_s^2 Tr F_{\mu\nu}^2) d\alpha d\beta d\sigma d\tau - \int ((\frac{1}{g_s^2} \{X^i, X^j\})^2 - \frac{1}{g_s^2} \theta^T\gamma_i \{X^i, \theta\}) d\alpha d\beta d\sigma d\tau.$$  \hspace{1cm} (8)

The obtained action defines a membrane SU($\infty$) pure Yang Mills theory. Later on, we will consider the longitudinal as well as the timelike coordinates in order to treat the system dynamically [4].
3 Membranes in 10 dimensions: an alternative approach

The aim of this section is to derive the equations of motion associated with the SU(\(\infty\)) Yang-Mills theory describing the membrane in 8 transverse directions and show how these second order equations are connected to the first order Nahm’s equations. The latter’s are shown to play a central role as they provide a way to emphasise the similarity to the phase space formulation of quantum mechanics [4].

A remarkable feature of the situation with 8 transverse directions is that, due to the existence of a self-dual (antisymmetric) 4-tensor \(T_{\mu\nu\rho\sigma}\) in 10 dimensions, the theory admits a class of solutions which we can obtain from a first order formulation (the Nahm equations). This is a set of 9 first order differential equations with 6 constraint equations given by
\[
\{X^\mu, X^\nu\} = \partial_\alpha X^\mu \partial_\beta X^\nu - \partial_\beta X^\mu \partial_\alpha X^\nu. \tag{10}
\]

Note by the way that (9) describes a 10-dimensional membrane with \(X^0 = \text{constant}\), living inside 8 transverse directions. This situation is motivated by the fact that the matrix theory admits a class of solutions obtainable from a first order formulation (the Nahm equations). This is a set of 9 first order differential equations with 6 constraint equations given by
\[
\begin{align*}
\partial_\sigma X^1 + \{X^2, X^9\} &= 0 \\
\partial_\sigma X^2 + \{X^9, X^1\} &= 0 \\
\partial_\sigma X^3 + \{X^4, X^9\} &= 0 \\
\partial_\sigma X^4 + \{X^9, X^3\} &= 0 \\
\partial_\sigma X^5 + \{X^6, X^9\} &= 0 \\
\partial_\sigma X^6 + \{X^9, X^5\} &= 0 \\
\partial_\sigma X^7 + \{X^8, X^9\} &= 0 \\
\partial_\sigma X^8 + \{X^9, X^7\} &= 0 \\
\partial_\sigma X^9 + \{X^1, X^2\} + \{X^3, X^4\} + \{X^5, X^6\} + \{X^7, X^8\} &= 0,
\end{align*}
\tag{11}
\]

with
\[
\begin{align*}
\{X^1, X^3\} + \{X^4, X^2\} + \{X^5, X^7\} + \{X^8, X^6\} &= 0 \\
\{X^1, X^4\} + \{X^2, X^3\} + \{X^8, X^5\} + \{X^7, X^6\} &= 0 \\
\{X^1, X^5\} + \{X^4, X^8\} + \{X^7, X^3\} + \{X^6, X^2\} &= 0 \\
\{X^1, X^6\} + \{X^2, X^5\} + \{X^3, X^8\} + \{X^4, X^7\} &= 0 \\
\{X^1, X^7\} + \{X^3, X^5\} + \{X^8, X^2\} + \{X^6, X^4\} &= 0 \\
\{X^1, X^8\} + \{X^5, X^4\} + \{X^2, X^7\} + \{X^6, X^3\} &= 0.
\end{align*}
\tag{12}
\]

\[
L = \frac{1}{2}(\partial_\alpha X^\mu)^2 + \frac{1}{4}\{X^\mu, X^\nu\}^2, \mu = 1, \ldots, 9, \tag{9}
\]
where \(\partial_\alpha = \partial/\partial \sigma\) and where the dependence upon \(\tau\) is ignored. The symbol \(\{X^\mu, X^\nu\}\) deals with the Poisson bracket given by
(\[\{X^\mu, X^\nu\} = \partial_\alpha X^\mu \partial_\beta X^\nu - \partial_\beta X^\mu \partial_\alpha X^\nu. \tag{10}\]
Note by the way that (9) describes a 10-dimensional membrane with \(X^0 = \text{constant}\), living inside 8 transverse directions. This situation is motivated by the fact that the matrix theory admits a class of solutions obtainable from a first order formulation (the Nahm equations). This is a set of 9 first order differential equations with 6 constraint equations given by
\[
\begin{align*}
\partial_\sigma X^1 + \{X^2, X^9\} &= 0 \\
\partial_\sigma X^2 + \{X^9, X^1\} &= 0 \\
\partial_\sigma X^3 + \{X^4, X^9\} &= 0 \\
\partial_\sigma X^4 + \{X^9, X^3\} &= 0 \\
\partial_\sigma X^5 + \{X^6, X^9\} &= 0 \\
\partial_\sigma X^6 + \{X^9, X^5\} &= 0 \\
\partial_\sigma X^7 + \{X^8, X^9\} &= 0 \\
\partial_\sigma X^8 + \{X^9, X^7\} &= 0 \\
\partial_\sigma X^9 + \{X^1, X^2\} + \{X^3, X^4\} + \{X^5, X^6\} + \{X^7, X^8\} &= 0,
\end{align*}
\tag{11}
\]

with
\[
\begin{align*}
\{X^1, X^3\} + \{X^4, X^2\} + \{X^5, X^7\} + \{X^8, X^6\} &= 0 \\
\{X^1, X^4\} + \{X^2, X^3\} + \{X^8, X^5\} + \{X^7, X^6\} &= 0 \\
\{X^1, X^5\} + \{X^4, X^8\} + \{X^7, X^3\} + \{X^6, X^2\} &= 0 \\
\{X^1, X^6\} + \{X^2, X^5\} + \{X^3, X^8\} + \{X^4, X^7\} &= 0 \\
\{X^1, X^7\} + \{X^3, X^5\} + \{X^8, X^2\} + \{X^6, X^4\} &= 0 \\
\{X^1, X^8\} + \{X^5, X^4\} + \{X^2, X^7\} + \{X^6, X^3\} &= 0.
\end{align*}
\tag{12}
\]
Focusing for the moment to reduce much more these equations, we have used some algebraic manipulations and showed that (11-12) can be simply written in the following way

\[
\partial_\sigma X^\mu + (-)^{\mu+1}\{X^\mu, X^{\mu+(-)^{\mu+1}}, X^9\} = 0, \mu = 1, ..., 8 \quad (a) \\
\partial_\sigma X^9 + \sum_{\mu=1}^8 \frac{1}{2}(-)^{\mu+1}\{X^\mu, X^{\mu+(-)^{\mu+1}}\} = 0, \quad (b)
\]

and

\[
\{X^1, X^j\} + (-)^{j+1}\{X^j, X^{(-)^{j+1}}\}, X^2 = \{X^7, X^{j+2(-)^{j+1}}\}, j = 3, 6 \\
\{X^1, X^j\} + (-)^{j+1}\{X^j, X^{(-)^{j+1}}\}, X^2 = \{X^5, X^{j-4(-)^{j+1}}\}, j = 4, 7 \\
\{X^1, X^j\} + (-)^{j+1}\{X^j, X^{(-)^{j+1}}\}, X^2 = \{X^3, X^{j+2(-)^{j+1}}\}, j = 5, 8
\]

Indeed, setting for example \(\mu = 1, 2\), we recover respectively from (13-a) the first two equations of (11) namely

\[
\partial_\sigma X^1 + \{X^2, X^9\} = 0 \\
\partial_\sigma X^2 + \{X^9, X^1\} = 0.
\]

Next note the important remark that there exist an intriguing correspondence between the first order Nahm’s equations (13) and the second order derived equations of motion. Indeed consider the lagrangian (9); and applying the Euler Lagrange equations we find the equations of motion for the membrane, namely:

\[
\partial_\sigma^2 X^\mu + \sum_{\nu=1}^8 \{X^\nu, \{X^\nu, X^\mu\}\} = 0,
\]

with \(\mu = 1, ..., 9\) and \(X_0 = constant\). The above correspondence is then established by derivating the Nahm’s equations with respect to the coordinate \(\sigma\) and using the constraints (14). As an example consider (13.b) such that

\[
\partial_\sigma^2 X^9 + \sum_{\nu=1}^8 \frac{1}{2}(-)^{\nu+1}\{\partial_\sigma X^\nu, X^{\nu+(-)^{\nu+1}}\} + \{X^\nu, \partial_\sigma X^{\nu+(-)^{\nu+1}}\} = 0, \quad (17)
\]

which becomes upon using (13.a)

\[
\partial_\sigma^2 X^9 + \sum_{\nu=1}^8 \frac{1}{2}\{\{X^{\nu+(-)^{\nu+1}}, X^{\nu+(-)^{\nu+1}}\}, X^9\} + \{X^\nu, \{X^\nu, X^9\}\} = 0, \quad (18)
\]

and reproducing then exactly the equations of motion (16) for \(\mu = 9\) with the following algebraic property

\[
\sum_{\nu=1}^8 \{X^{\nu+(-)^{\nu+1}}, X^{\nu+(-)^{\nu+1}}\}, X^9\} = \sum_{\nu=1}^8 \{X^\nu, \{X^\nu, X^9\}\}.
\]

Note by the way that by virtue of this equality, the equations of motion (16) can be equivalently written as:

\[
\partial_\sigma^2 X^\mu + \sum_{\nu=1}^8 \{X^{\nu+(-)^{\nu+1}}, X^{\nu+(-)^{\nu+1}}\}, X^{\nu+(-)^{\nu+1}}\} = 0, \quad (20)
\]
which shows an invariance property with respect to the following transformation

\[ \nu \rightarrow \nu + (-)^{\nu+1} \equiv \nu + \exp^{i\pi(\nu+1)}, \quad (21) \]
affecting the repeated index \( \nu \).

Consequently we note that the equation of motion concerning the longitudinal coordinates \( X^0 \) is simply obtained when derivating with respect to \( \sigma \). The remaining second order equations of motion, for transverse directions \( \mu = 1, \ldots, 8 \), arise thanks to the existence of constraint equations.

4 About the Nahm’s equations

Having shown explicitly how the second order equations of motion are derived from the Nahm’s first order ones, we will try now to search for the meaning of the relation between these two kind of equations. For this task, we will not ignore for instance about the coordinate \( \tau \) and assume that our fields \( X^\mu \) depend on the full set of coordinates \( \alpha, \beta, \sigma \) and \( \tau \) defining a space containing the world volume of the membrane. This space can be structured as follows. Let \( \sigma \) and \( \tau \) define a complex two-dimensional world-sheet \( \Sigma \) with local coordinates \( Z = \sigma + i\tau \) and \( \bar{Z} = \sigma - i\tau \). We use \( \partial \) and \( \bar{\partial} \) to denote \( \partial/\partial z \) and \( \partial/\partial \bar{z} \) respectively. The coordinates \( \alpha, \beta \) parametrise a phase space which we denote by \( P(\alpha, \beta) \) such that the fields \( X^\mu \) are shown to live inside the space \( \Sigma \otimes P(\alpha, \beta) \).

The lagrangian describing the SU(\( \infty \)) Yang-Mills theory of the membrane can be written formally as

\[ L = \frac{1}{2}(\partial X^\mu)(\bar{\partial} X^\mu) + \frac{1}{4}\{X^\mu, X^\nu\}^2, \mu = 1,..9, \quad (22) \]
in such a way that for \( \tau = 0 \), one recover directly the standard theory \( (9) \) for which \( \partial = \bar{\partial} = \partial_\sigma \).

The equation of motion associated to the lagrangian \( (22) \) is given by

\[ \partial \bar{\partial} X^\mu + \sum_{\nu=1}^{8}\{X^\nu, \{X^\nu, X^\mu\}\} = 0. \quad (23) \]

This equation looks like a standard 2d conformal field theory equation of motion, for which we are usually interested in deriving the conserved currents, discussing the underlying conformal symmetries and integrability. A part from being interesting for the above reasons, our equations contains further informations concerning the structure of the membrane in 8 transverse directions and the ”matrix” behaviour of the fields \( X^\mu \).

Using our knowledge on conformal fields theories and integrable systems for which 2d conserved currents are defined such that their conserved law reproduces in some sense the equation of motion, we are for instance interested in checking what happens for our equation \( (23) \). In fact, we remark from our previous analysis
that its possible to associate conserved currents to (23). Denoting these objects by $J^\mu \equiv (J^i, J^9)$, $i = 1, \ldots, 8$; we can write

\[ J^i = \partial X^i + (-)^{i+1} \{ X^{i+(-)^i+1}, X^9 \}, \quad i = 1, \ldots, 8 \quad (a) \]

\[ J^9 = \partial X^9 + \sum_{\nu=1}^8 (-)^{\nu+1} \{ X^\nu, X^{\nu+(-)^{\nu+1}} \}, \quad (b) \]

and their conservation properties are

\[ \bar{\partial} J^i = \bar{\partial} \partial X^i + \sum_{\nu=1}^8 \{ X^\nu, \{ X^\nu, X^i \} \} = 0, \quad i = 1, \ldots, 8 \quad (25) \]

and

\[ \bar{\partial} J^9 = \bar{\partial} \partial X^9 + \sum_{\nu=1}^8 \{ X^\nu, \{ X^\nu, X^9 \} \} = 0, \quad (26) \]

by virtue of (23).

Now, the point is that the conserved currents $J^\mu, \mu = 1, \ldots, 9$ given by (24) are nothing but the objects defining the first order Nahm’s equations (13) which we can rewrite as follows

\[ J^\mu = 0, \quad \mu = 1, \ldots, 8 \]

\[ J^9 = 0. \quad (27) \]

This property can be traced to the fact that when the coordinate $\tau$ is ignored, which means setting simply $\tau = 0$, the currents $J^\mu(\mu = 1, \ldots, 9)$ become then vanishing objects and give rise then to the first order Nahm’s equations (13). We can then interpret these kind of equations, as the vanishing property of the conserved currents of the theory (22) and interpret the conservation law equations (25, 26) as the property giving the correspondence between the first and the second order equations as discussed previously.

5 Solving the equations of motion

In this section we will return back to our discussion of section 3 in which we ignore the parameter $\tau$ and consider the following equations of motion (16)

\[ \partial_\sigma^2 X^\mu + \sum_{\nu=1}^8 \{ X^\nu, \{ X^\nu, X^\mu \} \} = 0. \quad (28) \]

Now, having shown how these equations describe the membrane in 8 transverse directions, we are now in position to look for the explicit solution of the model. To start, we assume that the coordinates $X^\mu(\alpha, \beta, \sigma)$ can be written in terms of the Fourier modes as follows

\[ X^\mu(\alpha, \beta, \sigma) = \sum_{mn \in \mathbb{Z}} X^\mu_{mn}(\sigma) L_{mn}(\alpha, \beta), \quad (29) \]

which suppose the periodicity of $X^\mu(\alpha, \beta, \sigma)$ in $\alpha, \beta$

\[ X^\mu(\alpha + 2\pi, \beta + 2\pi, \sigma) = X^\mu(\alpha, \beta, \sigma), \quad (30) \]
with
\[ L_{mn}(\alpha, \beta) = \exp i(m\alpha + n\beta). \]  

Furthermore, the modes \( X^\mu_{mn} \) as we will show below, can be considered as operator entries of the coordinates \( X^\mu \) satisfying
\[ \{X^\mu, X^\nu\} = \sum_{m_1, n_1 \atop m_2, n_2} X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma)\{L_{m_1n_1}, L_{m_2n_2}\}, \]  
where
\[ \{L_{m_1n_1}, L_{m_2n_2}\} = (m_2n_1 - m_1n_2)L_{m_1 + m_2, n_1 + n_2}, \]  
which reproduces in some sense the Poisson Bracket algebra in the large N limit of the area preserving diffeomorphism on the torus [12]. Note by the way that (33) share a striking resemblance with the Antoniadis et al Lie algebra [13]. This is an infinite dimensional algebra which was generalised to include the Virasoro algebra, the Frappat et al symmetries [12] as well as their W and central charges extensions [14].

An important question about (33) is to look for the meaning of the corresponding generalisations in our context. Moreover, the appearance of this algebra indicates also how the SU(N) symmetry (Moyal) of the supersymmetric Yang-Mills matrix theory becomes the area preserving diffeomorphism group namely \( SU(\infty) \) describing the membrane.

Now, before solving explicitly our problem, we will present some algebraic properties related to (28). To start, consider the antisymmetry property of the Poisson bracket (32) namely
\[ \{X^\mu, X^\nu\} = -\{X^\nu, X^\mu\}, \]  
from which we can write
\[ \{X^\mu, X^\nu\} = \sum_{m_1, n_1 \atop m_2, n_2} X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma)\{L_{m_1n_1}, L_{m_2n_2}\} \]
\[ = -\sum_{m_1, n_1 \atop m_2, n_2} X^\nu_{m_1n_1}(\sigma)X^\mu_{m_2n_2}(\sigma)\{L_{m_1n_1}, L_{m_2n_2}\}. \]  
On the other hand
\[ \{X^\mu, X^\nu\} = \sum_{m_1, n_1 \atop m_2, n_2} X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma)\{L_{m_1n_1}, L_{m_2n_2}\} \]
\[ = -\sum_{m_1, n_1 \atop m_2, n_2} X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma)\{L_{m_2n_2}, L_{m_1n_1}\} \]  
\[ = -\sum_{m_1, n_1 \atop m_2, n_2} X^\mu_{m_2n_2}(\sigma)X^\nu_{m_1n_1}(\sigma)\{L_{m_1n_1}, L_{m_2n_2}\}. \]  
The above formulas, lead then to set the following results
\[ X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma) = -X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma), \]  
\[ X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma) = -X^\mu_{m_2n_2}(\sigma)X^\nu_{m_1n_1}(\sigma), \]
showing the antisymmetry feature of the components \(X^\nu_{mn}\) with respect to both the indices \(\mu, \nu\) and the combined index \((m,n_i), i=1,2\) once the following mapping are performed
\[
(m_1n_1) \leftrightarrow (m_2n_2), \quad \mu \leftrightarrow \nu. \tag{39}
\]
Indeed, combining (37) and (38) we obtain
\[
X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma) = X^\nu_{m_2n_2}(\sigma)X^\mu_{m_1n_1}(\sigma). \tag{40}
\]
This result shows explicitly the bosonic behaviour of the coordinates \(X^\nu_{mn}\), a natural property which is suspected already at the level of the Fourier modes decomposition (29). Indeed, the latter’s suppose from the beginning that the coordinates \(X^\nu\) are "functions" of two phase space variables \(\alpha, \beta\) as well as \(\sigma\).

In the spirit to understand much more the equations of motion and their underlying symmetries, we remark that a realisation of the component \(X^\nu_{mn}\) is provided by the following expression
\[
X^\mu_{mn}(\sigma) = \gamma^\mu X_{mn}(\sigma), \tag{41}
\]
where \(\gamma^\mu, \mu = 1, ..., 9\) are the gamma matrices satisfying the Clifford algebra
\[
\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu, \quad \gamma^\mu 2 = -1. \tag{42}
\]
Injecting for example (41) into (40), one find by virtue of (42) a non commutative property of the component \(X_{mn}\) namely:
\[
X_{m_1n_1}(\sigma)X_{m_2n_2}(\sigma) = -X_{m_2n_2}(\sigma)X_{m_1n_1}(\sigma). \tag{43}
\]
In summary we can write (29) as follows
\[
X^\mu(\alpha, \beta, \sigma) = \sum_{m, n \mathbb{Z}} \gamma^\mu X_{mn}(\sigma)L_{mn}(\alpha, \beta), \tag{44}
\]
which gives after some computations
\[
\sum_{\nu=1}^{8} \{\{X^\mu, X^\nu\}, X^\nu\} = \sum_{\nu=1}^{8} \sum_{m_1, n_1, i=1,2,3} X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma)X^\nu_{m_3n_3}(\sigma)\{\{L_{m_1n_1}, L_{m_2n_2}\}, L_{m_3n_3}\}, \tag{45}
\]
with
\[
\{\{L_{m_1n_1}, L_{m_2n_2}\}, L_{m_3n_3}\} = (m_2n_1 - m_1n_2)[m_3(n_1 + n_2) - n_3(m_1 + m_2)]L \sum_{i=1,2,3} m_i \sum_{i=1,2,3} n_i, \tag{46}
\]
and
\[
\sum_{\nu=1}^{8} X^\mu_{m_1n_1}(\sigma)X^\nu_{m_2n_2}(\sigma)X^\nu_{m_3n_3}(\sigma) = -8\gamma^\mu X_{m_1n_1}(\sigma)X_{m_2n_2}(\sigma)X_{m_3n_3}(\sigma). \tag{47}
\]
Recall that our aim is to solve the membrane equations of motion which consist to give an explicit solution for the coordinates $X_{mn}$ satisfying the non commutative rule (43). In order to solve the above problem we propose the following ansatz.

$$X_{m_1 n_1}(\sigma) X_{m_2 n_2}(\sigma) X_{m_3 n_3}(\sigma) = \epsilon_{123} F_{\text{sym}}(\{m_i\}, \{n_i\}) X_{m_1+m_2+m_3, n_1+n_2+n_3}(\sigma),$$

(48)

where

$$\epsilon_{123} = \epsilon(m_1 n_1)(m_2 n_2)(m_3 n_3),$$

(49)

stands for the Levi-Cevita tensor given by

$$\epsilon_{123} = \begin{cases} -1 & \text{for odd permutation} \\ +1 & \text{for even permutation} \end{cases}$$

(50)

and where $F_{\text{sym}}(\{m_i\}, \{n_i\})$ is for the moment an arbitrary function required to be symmetric with respect to the following permutation of integer values

$$(m_i n_i) \leftrightarrow (m_j n_j), \ i, j = 1, 2, 3.$$  

(51)

Later on, we will denote simply this symmetric function as $F_{(123)}$.

The principal idea in setting the ansatz (48) is based on the fact that we need to write the bi-Poisson bracket $\{\{X^\mu, X^\nu\}, X^\nu\}$ as a simple function of $X_{mn}$ a fact which means that we should linearise the cubic matrix product $X_{m_1 n_1} X_{m_2 n_2} X_{m_3 n_3}$ to give rise to (48). The apparition of the fully antisymmetric tensor $\epsilon_{123}$ in this ansatz is justified by the non commutative behaviour of the components $X_{mn}$ as given by (43). On the other hand, the function $F_{\text{sym}}$ introduced in this ansatz and which we will discuss later, is chosen to be symmetric in agreement with the above non commutativity property.

Actually with this ansatz; one can easily identify both the left and hand sides of (28). In order to do this, remark first that we have

$$\sum_{\nu=1}^{8} \{\{X^\mu, X^\nu\}, X^\nu\} = \sum_{m_i, n_i} (-\gamma^\mu) 8 \epsilon_{123} F_{(123)} \omega(\{m_i\}, \{n_i\}) X \sum_{i=1,2,3} m_i \sum_{i=1,2,3} n_i (\sigma) L \sum_{i=1,2,3} m_i \sum_{i=1,2,3} n_i,$$

(52)

with

$$\omega(\{m_i\}, \{n_i\}) = (m_2 n_1 - m_1 n_2)[m_3 \sum_{i=1}^{3} n_i - n_3 \sum_{i=1}^{3} m_i].$$

(53)

which leads then to write the membrane equations of motion as follows

$$\sum_{p, q} L_{p, q} \partial^2 \sigma X_{p, q} = \sum_{m_i, n_i} 8 \epsilon_{123} F_{(123)} \omega(\{m_i\}, \{n_i\}) X \sum_{m_i} \sum_{n_i} (\sigma) L \sum_{m_i} \sum_{n_i}.$$  

(54)

Now, in order to do identification in (54) in a consistent way, one should sum over the same indices $p, q$ in both the sides of this equation. This is possible, since we can set $p = \sum m_i$ and $q = \sum n_i$ or equivalently

$$m_3 = p - (m_1 + m_2)$$

$$n_3 = q - (n_1 + n_2).$$

(55)
This leads to consider the following transformations

\[
\begin{align*}
\sum_{i=1,2,3} m_i \rightarrow \sum_{pq} \sum_{i=1,2,3} m_i \omega(m_i, \{n_i\}, \{n_i\}, p, q) \equiv \tilde{\epsilon} \\
\epsilon_{123} \rightarrow \epsilon_{123} \equiv \tilde{\epsilon} \\
F_{(123)} \rightarrow \tilde{F}_{sym},
\end{align*}
\]

(56)

where \(\epsilon_{123}\) is the Levi-Cevita tensor introduced previously and which is given by

\[
\epsilon_{123} = \epsilon(m_1n_1)(m_2n_2)(pq).
\]

(57)

With these simple transformations, the equations of motion (28) become

\[
\partial_\sigma^2 X_{pq} = \sum_{i=1,2,3} 8 \tilde{\epsilon} \tilde{\omega} \tilde{F}_{sym} X_{pq}(\sigma),
\]

(58)

with

\[
\tilde{\omega} = (m_2n_1 - m_1n_2)[p(n_1 + n_2) - q(m_1 + m_2)].
\]

(59)

Note by the way that, as \(\tilde{\omega}\) is antisymmetric with respect to the following mapping

\[
(m_1, n_1) \rightarrow (m_2, n_2),
\]

(60)

\(\epsilon_{123}\tilde{\omega}\) remains invariant with respect to the above change of integers values \((m_i, n_i), i = 1, 2\). We have

\[
\epsilon_{123}\tilde{\omega}(m_1, \{n_i\}, p, q) \equiv \epsilon_{213}\tilde{\omega}(m_2 \rightarrow m_1, n_1 \rightarrow n_2, p, q)
\]

(61)

since

\[
\epsilon_{123} = -\epsilon_{213} \quad \omega(m_i, n_i, p, q) = -\omega(m_1 \rightarrow m_2, n_1 \rightarrow n_2, p, q)
\]

(62)

Next, denoting by \(\Omega\) the following function

\[
\Omega(p, q) = \sum_{i=1,2,3} 8 \tilde{\epsilon} \tilde{\omega} \tilde{F}_{sym},
\]

(63)

whose convergence is closely connected to how we can choose the function \(\tilde{F}_{sym}\). In fact the introduction of this symmetric function in our ansatz is really a crucial step as we can easily check that for arbitrary values of \(p, q\) the following function

\[
\Omega'(p, q) = \sum_{i=1,2} 8 \tilde{\epsilon} \tilde{\omega},
\]

(64)

always diverge. To avoid this divergency problem one should then consider \(\tilde{F}_{sym}\) to play the role of the regularisation function which justify in some sense the introduction of this function in our ansatz (48). Following this discussion, a natural choice of this regularisation function is given by

\[
\tilde{F}_{sym} = F_{(123)} = exp(-|f_{123}|),
\]

(65)
where \( f_{12\bar{3}} = f(\{m_i\}, \{n_i\}, p, q) \) is some function of the integer values \( m_i, n_i, p, q \) required to be either symmetric or antisymmetric with respect to permutations of indices \( 1 \equiv (m_1 n_1), 2 \equiv (m_2 n_2) \) and \( 3 \equiv (pq) \).

The previous choice of \( \tilde{F} \) as given in (65) is motivated by our requirement that the function \( \Omega(p, q) \) (63) should converge in the infinite limit of the integer values \( m_i, n_i, i = 1, 2 \).

Several realisations of the function \( f_{(12\bar{3})} \) can occur. We give herebellow a typical example namely:

a) The symmetric choice

\[
f_{(12\bar{3})} = \frac{2}{i=1} (m_i + n_i) + p + q, \tag{66}
\]

b) The antisymmetric choice

\[
f_{(12\bar{3})} = \epsilon_{12\bar{3}} (\frac{2}{i=1} (m_i + n_i) + p + q) \tag{67}
\]

The equation of motion (58) reads

\[\partial_\sigma^2 X_{pq} - \Omega X_{pq} = 0, \tag{68}\]

whose solution is shown to be

\[X_{pq}(\sigma) = A_{pq}e^{\sqrt{\Omega}\sigma} + B_{pq}e^{-\sqrt{\Omega}\sigma}, \tag{69}\]

where \( A_{pq}, B_{pq} \) are arbitrary constants which should satisfy the following relations

\[
A_{pq}A_{rs} = -A_{rs}A_{pq} \\
B_{pq}B_{rs} = -B_{rs}B_{pq} \\
A_{pq}B_{rs} + B_{rs}A_{pq} = -(A_{rs}B_{pq} + B_{pq}A_{rs}) \tag{70}
\]

originated from the non commutativity properties of \( X_{pq} \) (43). Finally the matrix model membrane solution read as

\[X^\mu(\alpha, \beta, \sigma) = \sum_{pq} \gamma^\mu(A_{pq}e^{\sqrt{\Omega}\sigma} + B_{pq}e^{-\sqrt{\Omega}\sigma})L_{pq}, \tag{71}\]

The following significant question is in order: how one can compare or connect our solution with those already found in the same context in [15-16]?

One way to do this, is to discuss at the level of the derived solution (71) some particular cases related to the constants \( A_{pq}, B_{pq} \) and the regularised number \( \Omega(p, q) \).

**1. \( \Omega > 0 \)**

a) \( A_{pq} = B_{pq} \)

\[X^\mu(\alpha, \beta, \sigma) = \sum_{pq} 2\gamma^\mu A_{pq}ch(\sqrt{\Omega}\sigma)e^{i(\alpha q + q^2)}. \tag{72}\]
b) \( A_{pq} = -B_{pq} \)

\[
X^\mu(\alpha, \beta, \sigma) = \sum_{pq} 2\gamma^\mu A_{pq} \text{sh}(\sqrt{\Omega} \sigma) e^{i(p\alpha + q\beta)}. \quad (73)
\]

2. \( \Omega < 0 \)

a) \( A_{pq} = B_{pq} \)

\[
X^\mu(\alpha, \beta, \sigma) = \sum_{pq} 2\gamma^\mu A_{pq} \cos(\sqrt{\Omega} \sigma) e^{i(p\alpha + q\beta)}. \quad (74)
\]

b) \( A_{pq} = -B_{pq} \)

\[
X^\mu(\alpha, \beta, \sigma) = \sum_{pq} 2i\gamma^\mu A_{pq} \sin(\sqrt{\Omega} \sigma) e^{i(p\alpha + q\beta)}. \quad (75)
\]

The above examples correspond then to typical membrane solutions whose oscillation behaviour is the same for all the values of the space time index \( \mu \), with \( \mu = 1, \ldots, 9 \) once the signe of \( \Omega \) as well as the values of \( A_{pq} \) and \( B_{pq} \) are fixed.

Note also that we can consider from our solution (71), other examples for which the oscillation behaviour change when changing the index \( \mu \) as done by the author of [15]. We can then conclude that the solution we have derived is important in the sense that it exhibits among others an oscillation behaviour having a striking resemblance with the solution presented for example in [15]. The claim is to remark that the coordinates \( p, q \) and \( t \) used for example in the Kim’s work, coincide respectively with \( \alpha, \beta \) and \( \sigma \) in our construction.

6. Conclusion

This paper focus to solve some non-linear differential equations describing the membrane in 10 dimensions by means of Poisson bracket formalism.

Among the results obtained, we derive the equations of motion associated with \( SU(\infty) \) Yang Mills theory describing the membrane in 8 transverse directions and show how the second order equations of motion are related to the first order ones (the Nahm’s equations). This provides a way to emphasise the similarity to the phase space formulation of quantum mechanics as signalled by the author of [4].

We rewrite the Nahm’s equations in a simple form reducing then their number from 14 to 5. This is convenient in the sense that their derivative with respect to \( \sigma \) gives in a natural way the equations of motion.

We interpret the Nahm’s equations as the vanishing property of some conserved currents associated with the \( SU(\infty) \) Yang Mills theory and the correspondence with the equations of motion as the conservation property of these currents. The parameter \( \tau \) is shown to play an important role in this sense.
To solve our problem, which means find an explicit expression for the fields $X^\mu$, we develop an algebraic analysis and propose an ansatz leading to construct the membrane solution given in (71).

In the spirit to compare our solution to some of the well known ones, one performe special choices on the constants $A_{pq}, B_{pq}$ as well as on the regularised number $\Omega(p, q)$ (63) a fact which leads to recover a general oscillation behaviour (see our previous examples) shared by our solution and the other ones already established in literature.

Moreover, we guess that other non trivial solutions can be obtained if one forget about the ansatz (48) and know how to solve in general the non-linear differential equation (28).

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