LIE GROUPS OF DIMENSION 4
AND ALMOST HYPERCOMPLEX MANIFOLDS
WITH HERMITIAN-NORDEN METRICS

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Abstract. Object of investigation are almost hypercomplex manifolds with Hermitian-Norden metrics of the lowest dimension. The considered manifolds are constructed on 4-dimensional Lie groups. It is established a relation between the classes of a classification of 4-dimensional indecomposable real Lie algebras and the classification of the manifolds under study. The basic geometrical characteristics of the constructed manifolds are studied in the frame of the mentioned classification of the Lie algebras.

1. Introduction

A triad of anticommuting almost complex structures such that each of them is a composition of the other two structures is called an almost hypercomplex structure \( H \) on a \( 4n \)-dimensional smooth manifold \( M \). The structure \( H \) could be equipped with a metric structure of Hermitian-Norden type, generated by a pseudo-Riemannian metric \( g \) of neutral signature \([6, 7]\). In this case, in each tangent fibre, one of the almost complex structures of \( H \) acts as an isometry and the other two act as anti-isometries with respect to \( g \). The metric \( g \) is Hermitian with respect to one of almost complex structures of \( H \) and \( g \) is a Norden metric regarding the other two. Then, we have three associated \((0,2)\)-tensors to the metric \( g \) – a Kähler form and two Norden metrics.

The manifold \( M \), equipped with the considered structures, is called an almost hypercomplex manifold with Hermitian-Norden metrics. The same manifolds are investigated in \([6, 7]\) under the name almost hypercomplex pseudo-Hermitian manifolds and in \([10, 11]\) as almost hypercomplex manifolds with Hermitian and anti-Hermitian metrics.

Almost hypercomplex manifolds with Hermitian-Norden metrics can be constructed on Lie groups. In this work we use classification of four-dimensional indecomposable Lie algebras, known from \([4]\). The goal of this paper is to find a relation between the classes in this classification and the corresponding manifolds to the classifications given in \([5]\) and \([3]\), which are derived by the tensor structures and metrics of the respective manifolds. Moreover, the present work gives the basic geometrical characteristics of the considered manifolds in each case.

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The author’s intention with this article is to complete the considered problem for all classes of the mentioned classification and thus to generalize the results from \[8\] and \[9\]. Smooth manifolds with similar structures on Lie groups are studied in \[2\] \[12\] \[14\] \[17\].

2. ALMOST HYPERCOMPLEX MANIFOLDS WITH HERMITIAN-NORDEN METRICS

The subject of our study are almost hypercomplex manifolds with Hermitian-Norden metrics \([7]\). A differentiable manifold \(M\) of this type has dimension \(4n\) and it is denoted by \((M, H, G)\), where \((H, G)\) is an almost hypercomplex structure with Hermitian-Norden metrics. More precisely, the almost hypercomplex structure \(H = (J_1, J_2, J_3)\) has the following properties:

\[
J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \quad J_\alpha^2 = -I,
\]

for all cyclic permutations \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\) and the identity \(I\). The quadruplet \(G = (g, g_1, g_2, g_3)\) consists of a neutral metric \(g\), associated 2-form \(g_1\) and associated neutral metrics \(g_2\) and \(g_3\) on \((M, H)\) having the properties

\[
(2.1) \quad g(\cdot, \cdot) = \varepsilon_\alpha g(J_\alpha \cdot, J_\alpha \cdot),
\]

\[
(2.2) \quad g_\alpha(\cdot, \cdot) = g(J_\alpha \cdot, \cdot) = -\varepsilon_\alpha g(\cdot, J_\alpha \cdot).
\]

where

\[
\varepsilon_\alpha = \begin{cases} 1, & \alpha = 1; \\ -1, & \alpha = 2; 3. \end{cases}
\]

Here and further, \(\alpha\) will run over the range \(\{1, 2, 3\}\) unless otherwise is stated.

Let us remark that the considered type of manifolds is the only possible way to involve Norden-type metrics on almost hypercomplex manifolds. The following three tensors of type \((0, 3)\) are the fundamental tensors of the almost hypercomplex manifold with Hermitian-Norden metrics \([7]\)

\[
(2.3) \quad F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = (\nabla_x g_\alpha)(y, z),
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). These tensors have the properties

\[
(2.4) \quad F_\alpha(x, y, z) = -\varepsilon_\alpha F_\alpha(x, z, y) = -\varepsilon_\alpha F_\alpha(x, J_\alpha y, J_\alpha z)
\]

and they are related to each other as follows

\[
F_1(x, y, z) = F_2(x, J_3 y, z) + F_3(x, y, J_2z),
\]

\[
F_2(x, y, z) = F_3(x, J_1 y, z) + F_1(x, y, J_3z),
\]

\[
F_3(x, y, z) = F_1(x, J_2 y, z) - F_2(x, y, J_1z).
\]

The corresponding 1-forms \(\theta_\alpha\) of \(F_\alpha\), known as Lee forms, are determined by

\[
(2.5) \quad \theta_\alpha(\cdot) = g^{ij} F_\alpha(e_i, e_j, \cdot),
\]

where \(\{e_1, e_2, \ldots, e_{4n}\}\) is an arbitrary basis of \(T_p M, p \in M\) and \(g^{ij}\) are the corresponding components of the inverse matrix of \(g\).

According to \([7]\), \((M, J_1, g)\) is an almost Hermitian manifold whereas the manifolds \((M, J_2, g)\) and \((M, J_3, g)\) are almost complex manifolds with Norden metric. These two types of manifolds are classified in \([5]\) and \([5]\), respectively. In the case of
the lowest dimension 4, the four basic classes of almost Hermitian manifolds with respect to $J_1$ are restricted to two:

$$W_2(J_1): \mathcal{G}_{x,y,z} \{ F_1(x,y,z) \} = 0;$$

$$W_4(J_1): F_1(x,y,z) = \frac{1}{2} \{ g(x,y)\theta_1(z) - g(x,J_1y)\theta_1(J_1z) - g(x,z)\theta_1(y) + g(x,J_1z)\theta_1(J_1y) \},$$

where $\mathcal{G}$ is the cyclic sum by three arguments. In the 4-dimensional case, the basic classes of almost Norden manifolds ($\alpha = 2$ or $3$) are determined as follows:

$$W_1(J_\alpha): F_\alpha(x,y,z) = \frac{1}{4} \{ g(x,y)\theta_\alpha(z) + g(x,J_\alpha y)\theta_\alpha(J_\alpha z) + g(x,z)\theta_\alpha(y) + g(x,J_\alpha z)\theta_\alpha(J_\alpha y) \};$$

$$W_2(J_\alpha): \mathcal{G}_{x,y,z} \{ F_\alpha(x,y,J_\alpha z) \} = 0, \quad \theta_\alpha = 0;$$

$$W_3(J_\alpha): \mathcal{G}_{x,y,z} \{ F_\alpha(x,y,z) \} = 0.$$

The curvature (1,3)-tensor of $\nabla$ is defined as usual by $R = [\nabla, \nabla] - \nabla(\cdot, \cdot)$. The corresponding curvature (0,4)-tensor with respect to $g$ is denoted by the same letter, i.e.

$$R(x,y,z,w) = g(R(x,y)z,w),$$

and it has the following well-known properties:

$$R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z),$$

$$R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0.$$

The Ricci tensor $\rho$ and the scalar curvature $\tau$ for $R$ as well as their associated quantities $\rho^*_\alpha$, $\tau^*_\alpha$ and $\tau^{**}_\alpha$ are defined by:

$$\rho(y,z) = g^{ij} R(e_i,y,z,e_j), \quad \rho^*_\alpha(y,z) = g^{ij} R(e_i,y,z,J_\alpha e_j),$$

$$\tau = g^{ij} \rho(e_i,e_j), \quad \tau^*_\alpha = g^{ij} \rho^*_\alpha(e_i,e_j), \quad \tau^{**}_\alpha = g^{ij} \rho^{**}_\alpha(e_i,J_\alpha e_j).$$

The following properties for $\rho$ and $\rho^*_\alpha$ are valid:

$$\rho_{jk} = \rho_{kj}, \quad (\rho^*_\alpha)_{jk} = -\epsilon_\alpha (\rho^*_\alpha)_{kj},$$

where $\rho_{jk} = \rho(e_j,e_k)$ and $(\rho^*_\alpha)_{jk} = \rho^*_\alpha(e_j,e_k)$ are the basic components of $\rho$ and $\rho^*_\alpha$, respectively.

Let $\mu$ be a non-degenerate 2-plane with a basis $\{x,y\}$ in $T_p\mathcal{M}$, $p \in \mathcal{M}$. The sectional curvature of $\mu$ with respect to $g$ and $R$ is defined by

$$k(\mu; p) = \frac{R(x,y,y,x)}{g(x,y)g(y,y) - g(x,y)^2}.$$

A 2-plane $\mu$ is called holomorphic (resp., totally real) if the condition $\mu = J_\alpha \mu$ (resp., $\mu \perp J_\alpha \mu \neq \mu$ with respect to $g$) holds. The sectional curvature of a holomorphic (resp., totally real) 2-plane is called holomorphic (resp., totally real) sectional curvature. The 2-plane $\mu$ and its sectional curvature $k(\mu; p)$ are called a basic 2-plane and a basic sectional curvature, respectively, if $\mu$ has a basis $\{e_i, e_j\}$ ($i, j \in \{1, 2, \ldots, 4n\}, i \neq j$) for a basis $\{e_1, e_2, \ldots, e_{4n}\}$ of $T_p\mathcal{M}$. In the latter case we denote $k_{ij}$. 
Table 1. Correspondence between some classifications of Lie algebras

|    |    |    |    |
|----|----|----|----|
| $\mathfrak{g}_{4,1}$ | $\mathfrak{n}_4$ | $\mathfrak{g}_{4,1}$ | $A_{4,1}$ |
| $\mathfrak{g}_{4,2}$ | $\mathfrak{t}_{4,a}$ | $\mathfrak{g}_{4,2}$ | $A_{4,2}$ |
| $\mathfrak{g}_{4,3}$ | $\mathfrak{t}_{4,0}$ | $\mathfrak{g}_{4,3}$ | $A_{4,3}$ |
| $\mathfrak{g}_{4,4}$ | $\mathfrak{t}_{4}$ | $\mathfrak{g}_{4,4}$ | $A_{4,4}$ |
| $\mathfrak{g}_{4,5}$ | $\mathfrak{t}_{4,a,b}$ | $\mathfrak{g}_{4,5}$ | $A_{4,5}$ |
| $\mathfrak{g}_{4,6}$ | $\mathfrak{t}_{4,a,b}'$ | $\mathfrak{g}_{4,6}$ | $A_{4,6}$ |
| $\mathfrak{g}_{4,7}$ | $\mathfrak{h}_{4}$ | $\mathfrak{g}_{4,7}$ | $A_{4,7}$ |
| $\mathfrak{g}_{4,8}$ | $\mathfrak{d}_{4}$ | $\mathfrak{g}_{4,8}(-1)$ | $A_{4,8}$ |
| $\mathfrak{g}_{4,9}$ | $\mathfrak{d}_{4,1/1+b}$ | $\mathfrak{g}_{4,9}$ | $A_{4,9}$ |
| $\mathfrak{g}_{4,10}$ | $\mathfrak{t}_{4,0}'$ | $\mathfrak{g}_{4,10}$ | $A_{4,10}$ |
| $\mathfrak{g}_{4,11}$ | $\mathfrak{t}_{4,a}'$ | $\mathfrak{g}_{4,11}$ | $A_{4,11}$ |
| $\mathfrak{g}_{4,12}$ | $\operatorname{aff}(\mathbb{C})$ | $\mathfrak{g}_{4,12}$ | $A_{4,12}$ |

3. Four-dimensional indecomposable real Lie algebras

Different authors study real 4-dimensional indecomposable Lie algebras. Firstly, a classification is given in [13], which could be be found easily in [15] and [4]. The object of investigation in [1] are four-dimensional solvable real Lie algebras. In all of the cited works, the basic classes are described by the non-zero Lie brackets with respect to a basis \{e_1, e_2, e_3, e_4\}. In Table 1 it is shown the correspondence between the mentioned classifications.

In the present work, we use the notation of the classes from [4], namely

$$\begin{align*}
\mathfrak{g}_{4,1} : & [e_2, e_4] = e_1, & [e_3, e_4] = e_2; \\
\mathfrak{g}_{4,2} : & [e_1, e_4] = me_1, & [e_2, e_4] = e_2, & (m \neq 0); \\
\mathfrak{g}_{4,3} : & [e_1, e_4] = e_1, & [e_3, e_4] = e_2; \\
\mathfrak{g}_{4,4} : & [e_1, e_4] = e_1, & [e_2, e_4] = e_1 + e_2, & (a_1 \neq 0, a_2 \neq 0); \\
\mathfrak{g}_{4,5} : & [e_1, e_4] = e_1, & [e_2, e_4] = a_1 e_2, & (b_1 \neq 0, b_2 \geq 0); \\
\mathfrak{g}_{4,6} : & [e_1, e_4] = b_1 e_1, & [e_2, e_4] = b_2 e_2 - e_3, & (a_1 \neq 0, a_2 \neq 0); \\
\mathfrak{g}_{4,7} : & [e_1, e_4] = 2e_1, & [e_2, e_3] = e_1, & (b_1 \neq 0, b_2 \geq 0); \\
\mathfrak{g}_{4,8} : & [e_2, e_3] = e_1, & [e_2, e_4] = e_2, & (b_1 \neq 0, b_2 \geq 0); \\
\end{align*}$$

(3.1a)
A certain class regarding Lie algebra \( \mathfrak{g}_{4,i} \) is defined as in [16]:

\[
\begin{align*}
\mathfrak{g}_{4,9} & : [e_1, e_4] = (p + 1)e_1, & [e_2, e_4] &= e_2, & [e_3, e_4] &= pe_3, & (1 < p \leq 1); \\
\mathfrak{g}_{4,10} & : [e_2, e_3] = e_1, & [e_2, e_4] &= e_2; \\
\mathfrak{g}_{4,11} & : [e_1, e_4] = 2qe_4, & [e_2, e_3] &= e_1, & [e_3, e_4] &= e_2 - e_3, & (q > 0); \\
\mathfrak{g}_{4,12} & : [e_2, e_3] = e_2, & [e_2, e_4] &= e_1,
\end{align*}
\]

(3.1b)

where \( a_1, a_2, b_1, b_2, m, p, q \in \mathbb{R} \).

4. **Lie groups as almost hypercomplex manifolds with Hermitian-Norden metrics**

Let \( \mathcal{L} \) be a simply connected 4-dimensional real Lie group with corresponding Lie algebra \( \mathfrak{l} \). A standard hypercomplex structure on \( \mathfrak{l} \) for its basis \( \{e_1, e_2, e_3, e_4\} \) is defined as in [10]:

\[
\begin{align*}
J_1e_1 &= e_2, & J_1e_2 &= -e_1, & J_1e_3 &= -e_4, & J_1e_4 &= e_3; \\
J_2e_1 &= e_3, & J_2e_2 &= e_4, & J_2e_3 &= -e_1, & J_2e_4 &= -e_2; \\
J_3e_1 &= -e_4, & J_3e_2 &= e_3, & J_3e_3 &= -e_2, & J_3e_4 &= e_1.
\end{align*}
\]

(4.1)

Let \( g \) be a pseudo-Riemannian metric of neutral signature for \( x(x^1, x^2, x^3, x^4), y(y^1, y^2, y^3, y^4) \in \mathfrak{l} \) defined by:

\[
g(x, y) = x^1y^1 + x^2y^2 - x^3y^3 - x^4y^4.
\]

Bearing in mind the latter equality, it is valid that

\[
g(e_i, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = 1, \quad i \neq j \in \{1, 2, 3, 4\}.
\]

(4.2)

Let us note that further the indices \( i, j, k, l \) run over the range \( \{1, 2, 3, 4\} \). The metric \( g \) generates an almost hypercomplex structure with Hermitian-Norden metrics on \( \mathfrak{l} \), according to (2.1) and (2.2). Then, \( (\mathcal{L}, H, G) \) is an almost hypercomplex manifold with Hermitian-Norden metrics.

**Theorem 4.1.** Let \( (\mathcal{L}, H, G) \) be a 4-dimensional almost hypercomplex manifold with Hermitian-Norden metrics. Then, the manifold \( (\mathcal{L}, H, G) \), which is corresponding to the different classes of 4-dimensional Lie algebras \( \mathfrak{g}_{4,i} \), \( (i = 1, \ldots, 12) \), belongs to a certain class regarding \( J_\alpha \) given in Table 4, where we denote for brevity \( W_i \oplus W_j \) and \( W_i \oplus W_j \oplus W_k \) by \( W_{ij} \) and \( W_{ijk} \), respectively.

Moreover, we have:

- for each \( a_1 \neq 0 \) and \( a_2 \neq 0 \), \( (\mathcal{L}, H, G) \) does not belong to neither of \( W_0 \) for \( J_1; W_0, W_3, W_1 \oplus W_3 \) for \( J_2; W_0, W_1, W_3, W_1 \oplus W_3 \) for \( J_3; W_0, W_1, W_2 \) for \( J_3; \)
- for each \( b_1 \neq 0, b_2 \geq 0 \), \( (\mathcal{L}, H, G) \) does not belong to neither of \( W_0, W_2, W_3, W_1 \oplus W_2, W_1 \oplus W_3 \) for \( J_2; W_0, W_1, W_2, W_3, W_1 \oplus W_3 \) for \( J_3; \)
- for each \( m \neq 0 \), \( (\mathcal{L}, H, G) \) does not belong to neither of \( W_0, W_3, W_1 \oplus W_3 \) for \( J_1; W_0, W_1, W_2, W_3, W_1 \oplus W_3, W_2 \oplus W_3 \) for \( J_2; W_0, W_1, W_2, W_3, W_1 \oplus W_3, W_2 \oplus W_3 \) for \( J_3; \)
Table 2. Correspondence between different classes Lie algebras and the classes almost hypercomplex manifold with Hermitian-Norden metrics

| Lie algebra | Parameters | $J_1$ | $J_2$ | $J_3$ |
|-------------|------------|-------|-------|-------|
| $\mathfrak{g}_{4,1}$ | $m = 1$ | $W_{24}$ | $W_{123}$ | $W_{123}$ |
| $\mathfrak{g}_{4,2}$ | $m \neq 0; m \neq 1$ | $W_{24}$ | $W_{123}$ | $W_{123}$ |
| $\mathfrak{g}_{4,3}$ | $a_1 = -1, a_2 = 1$ | $W_2$ | $W_2$ | $W_{123}$ |
| $\mathfrak{g}_{4,4}$ | $a_1 = -1, a_2 = -1$ | $W_2$ | $W_{123}$ | $W_2$ |
| $\mathfrak{g}_{4,5}$ | $a_1 = -1, a_2 \neq \pm 1$ | $W_2$ | $W_{123}$ | $W_{123}$ |
| $\mathfrak{g}_{4,6}$ | $b_1 \neq 0, b_2 \geq 0$ | $W_{24}$ | $W_{123}$ | $W_{123}$ |
| $\mathfrak{g}_{4,7}$ | $a_1 = 1, a_2 = 1$ | $W_4$ | $W_1$ | $W_{12}$ |
| $\mathfrak{g}_{4,8}$ | $a_1 = 1, a_2 = -3$ | $W_4$ | $W_{23}$ | $W_{23}$ |
| $\mathfrak{g}_{4,9}$ | $a_1 = \pm 1, a_2 = 1$ | $W_{24}$ | $W_{12}$ | $W_{123}$ |
| $\mathfrak{g}_{4,10}$ | $a_1 = -\frac{1}{2} (a_2 + 1), a_2 \neq \{-3, -\frac{1}{2}, 1\}$ | $W_{24}$ | $W_{23}$ | $W_{23}$ |
| $\mathfrak{g}_{4,11}$ | $a_1 = a_2, a_2 \neq \pm 1, -\frac{1}{2}$ | $W_{24}$ | $W_{123}$ | $W_{12}$ |
| $\mathfrak{g}_{4,12}$ | $a_1 = -a_2 - 2, a_2 \neq -3, -1$ | $W_{24}$ | $W_{123}$ | $W_{23}$ |
| $\mathfrak{g}_{4,13}$ | $a_1 \neq 0, a_2 \neq 0$ | $W_{24}$ | $W_{23}$ | $W_{23}$ |

- for each $-1 < p \leq 1$, $(\mathcal{L}, H, G)$ does not belong to neither of $W_0, W_2$ for $J_1$; $W_0, W_1, W_2$ for $J_2$; $W_0, W_1, W_2, W_3, W_1 \oplus W_3, W_2 \oplus W_3$ for $J_3$;
- for each $q > 0$, $(\mathcal{L}, H, G)$ does not belong to neither of $W_0, W_2, W_4$ for $J_1$; $W_0, W_1, W_2, W_3, W_1 \oplus W_2, W_1 \oplus W_3, W_2 \oplus W_3$ for $J_2$; $W_0, W_1, W_2$ for $J_3$.

Proof. Now, we give our arguments for the case when the corresponding Lie algebra of $(\mathcal{L}, H, G)$ is from $\mathfrak{g}_{4,1}$. Then, using (2.1), (3.1), (4.1) and the well-known Koszul equality

$$2g(\nabla e_i, e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) + g([e_k, e_j], e_i),$$

for each $-1 < p \leq 1$, $(\mathcal{L}, H, G)$ does not belong to neither of $W_0, W_2$ for $J_1$; $W_0, W_1, W_2$ for $J_2$; $W_0, W_1, W_2, W_3, W_1 \oplus W_3, W_2 \oplus W_3$ for $J_3$.
we obtain the components of the Levi-Civita connection $\nabla$ for the considered basis. The non-zero of them are:

$$\nabla e_1 e_2 = \nabla e_2 e_1 = \nabla e_2 e_3 = \nabla e_3 e_2 = \frac{1}{2} e_4,$$

$$\nabla e_1 e_4 = \nabla e_3 e_4 = -\nabla e_4 e_3 = \frac{1}{2} e_2,$$

$$\nabla e_2 e_4 = \frac{1}{2} (e_1 - e_3), \quad \nabla e_4 e_2 = -\frac{1}{2} (e_1 + e_3).$$

Then, we obtain the basic components $(F_\alpha)_{ijk} = F_\alpha(e_i, e_j, e_k)$ of $F_\alpha$ by virtue of (2.3), (1.4), (1.2) and (4.3). The non-zero of them are determined by the following ones and properties (2.3)

$$(F_1)_{141} = (F_1)_{213} = (F_1)_{341} = (F_1)_{413} = (F_2)_{212} = (F_2)_{223} = (F_2)_{414}$$

$$= -(F_2)_{412} = \frac{1}{2} (F_2)_{122} = \frac{1}{2} (F_2)_{322} = (F_3)_{134} = -(F_3)_{213}$$

$$= (F_3)_{334} = (F_3)_{413} = -\frac{1}{2} (F_3)_{211} = -\frac{1}{2} (F_3)_{422} = \frac{1}{2}.$$

Using (2.3) and (4.3), we establish the basic components $(\theta_\alpha)_i = (\theta_\alpha)(e_i)$ of the corresponding Lee forms and the non-zero are

$$(\theta_1)_2 = (\theta_2)_3 = -(\theta_3)_2 = - (\theta_3)_4 = 1.$$

After that, bearing in mind the classification conditions (2.6) and (2.7) for dimension 4, we conclude that in this case the manifold $(L, H, G)$ belongs to

$$(W_2 \oplus W_4)(J_1) \cap (W_1 \oplus W_2 \oplus W_3)(J_2) \cap (W_1 \oplus W_2 \oplus W_3)(J_3).$$

The proofs for the cases of the classes $g_{4,2}$, $g_{4,5}$, $g_{4,6}$, $g_{4,9}$ and $g_{4,11}$ are given in [8] and [9].

In a similar way we prove the assertions for the other classes using the following results for each case:

$g_{4,3}$ :

$$\nabla e_1 e_1 = 2 \nabla e_2 e_3 = 2 \nabla e_3 e_2 = e_4, \quad \nabla e_1 e_4 = e_1, \quad \nabla e_2 e_4 = e_2 = \frac{1}{2} e_3,$$

$$\nabla e_3 e_2 = \frac{1}{2} e_1 - e_2, \quad \nabla e_3 e_4 = \frac{1}{2} e_1 + e_2 - \frac{1}{2} e_3, \quad \nabla e_4 e_1 = -\frac{1}{2} e_1 - \frac{1}{2} e_3;$$

$$\frac{1}{2} (F_1)_{113} = -(F_1)_{114} = (F_1)_{213} = -\frac{1}{2} (F_1)_{214} = -(F_1)_{314} = (F_1)_{413} = \frac{1}{2},$$

$$(F_2)_{112} = (F_2)_{212} = (F_2)_{223} = -2 (F_2)_{412} = 1, \quad (F_2)_{112} = (F_2)_{212} = (F_2)_{223} = -2 (F_2)_{412} = 1,$$

$$\frac{1}{2} (F_3)_{111} = 2 (F_3)_{213} = 2 (F_3)_{312} = (F_3)_{422} = -1;$$

$$(\theta_1)_2 = (\theta_2)_3 = (\theta_2)_3 = -\frac{1}{2} (\theta_3)_1 = - (\theta_3)_4 = 1;$$

$g_{4,4}$ :

$$\nabla e_1 e_1 = 2 \nabla e_1 e_2 = 2 \nabla e_2 e_2 = 2 \nabla e_2 e_3 = 2 \nabla e_3 e_2 = - \nabla e_3 e_3 = e_4,$$

$$\nabla e_1 e_4 = e_1 + \frac{1}{2} e_2, \quad \nabla e_2 e_4 = \frac{1}{2} e_1 + e_2 - \frac{1}{2} e_3, \quad \nabla e_3 e_4 = \frac{1}{2} e_2 + e_3,$$

$$\nabla e_4 e_1 = -\nabla e_4 e_3 = \frac{1}{2} e_2, \quad \nabla e_4 e_2 = -\frac{1}{2} e_1 - \frac{1}{2} e_3;$$

$$\frac{1}{2} (F_1)_{113} = -(F_1)_{114} = (F_1)_{213} = -\frac{1}{2} (F_1)_{214} = -(F_1)_{314} = (F_1)_{413} = \frac{1}{2},$$

$$(F_2)_{112} = (F_2)_{212} = (F_2)_{223} = -2 (F_2)_{412} = \frac{1}{2} (F_2)_{322} = (F_2)_{314} = (F_2)_{413} = -2 (F_2)_{412} = 1,$$

$$\frac{1}{2} (F_3)_{111} = 2 (F_3)_{213} = 2 (F_3)_{312} = (F_3)_{422} = -1;$$

$$2 (\theta_1)_2 = (\theta_1)_3 = \frac{1}{2} (\theta_2)_2 = 2 (\theta_2)_3 = -\frac{1}{2} (\theta_3)_1 = -2 (\theta_3)_2 = -2 (\theta_3)_4 = 2;$$

$g_{4,7}$ :

$$\frac{1}{2} \nabla e_1 e_1 = \nabla e_2 e_2 = - \nabla e_3 e_3 = e_4, \quad \nabla e_1 e_2 = \nabla e_2 e_1 = - \nabla e_4 e_2 = \frac{1}{2} e_3,$$
\[ \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = -\nabla_{e_4}e_3 = \frac{1}{5}e_2, \quad \nabla_{e_1}e_4 = 2e_1, \quad \nabla_{e_2}e_3 = \frac{1}{2}e_1 + \frac{1}{2}e_4, \\
abla_{e_2}e_4 = e_2 - \frac{1}{2}e_3, \quad \nabla_{e_3}e_2 = -\frac{1}{2}e_1 + \frac{1}{2}e_4, \quad \nabla_{e_3}e_4 = \frac{1}{2}e_2 + e_3; \\
(F_1)_{113} = -(F_1)_{214} = -3(F_1)_{314} = 3(F_1)_{413} = \frac{3}{2}, \\
\frac{1}{2}(F_2)_{112} = (F_2)_{211} = -2(F_2)_{214} = \frac{1}{2}(F_2)_{222} \\
= \frac{1}{2}(F_2)_{314} = (F_2)_{322} = -2(F_2)_{412} = 1, \\
\frac{1}{4}(F_3)_{111} = -(F_3)_{122} = 2(F_3)_{212} = 2(F_3)_{213} \\
= 2(F_3)_{312} = -\frac{1}{2}(F_3)_{313} = (F_3)_{422} = -1; \\
(\theta_1)_{2} = \frac{1}{3}(\theta_1)_{3} = \frac{1}{3}(\theta_2)_{2} = (\theta_2)_{3} = -\frac{1}{3}(\theta_3)_{1} = -(\theta_3)_{4} = 1; \]

\[ \mathfrak{g}_{4.8} : \\
\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\frac{1}{2}\nabla_{e_3}e_4 = \frac{1}{2}e_3, \quad \nabla_{e_1}e_3 = \frac{1}{2}\nabla_{e_3}e_4 = \nabla_{e_3}e_1 = \frac{1}{2}e_2, \\
\nabla_{e_2}e_3 = \nabla_{e_3}e_2 = e_4, \quad \nabla_{e_3}e_2 = -\frac{1}{2}e_1 + e_4; \\
(F_1)_{113} = (F_1)_{214} = \frac{1}{2}(F_1)_{314} = -\frac{1}{2}, \\
(F_2)_{112} = \frac{1}{2}(F_2)_{211} = -\frac{1}{2}(F_2)_{214} = (F_2)_{314} = \frac{1}{2}(F_2)_{322} = \frac{1}{2}, \\
(F_3)_{122} = 2(F_3)_{212} = -(F_3)_{213} = -(F_3)_{312} = 2(F_3)_{313} = 1; \\
(\theta_1)_{2} = (\theta_2)_{2} = (\theta_2)_{3} = -\frac{1}{2}(\theta_3)_{4} = 1; \]

\[ \mathfrak{g}_{4.12} : \\
\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_1}e_3 = -\nabla_{e_4}e_3 = e_1, \quad \nabla_{e_2}e_3 = \nabla_{e_3}e_1 = e_2, \\
(F_1)_{114} = \frac{3}{2}(F_1)_{213} = -\frac{3}{2}, \quad \frac{1}{2}(F_2)_{111} = (F_2)_{212} = (F_2)_{414} = 1, \\
(F_3)_{112} = \frac{1}{2}(F_3)_{222} = (F_3)_{3413} = 1, \quad (\theta_1)_{4} = -(\theta_2)_{1} = -\theta_{3/2} = -2. \]

\[ \square \]

5. Curvature properties of the manifolds under study

In this section we determine some geometric characteristics of the manifolds \((\mathcal{L}, H, G)\) in all the classes considered in the previous section. The focus of the considerations in \([8]\) and \([9]\) are the classes of the classification of 4-dimensional indecomposable real Lie algebras, given in \((5.1)\), depending on real parameters. Actually, these five classes are families of manifolds whose properties are functions of the parameters. The curvature properties of the considered manifolds are summarized in the following

**Theorem 5.1.** Let \((\mathcal{L}, H, G)\) be a 4-dimensional almost hypercomplex manifold with Hermitian-Norden metrics, as well as let the corresponding Lie algebra \(l\) of \(L\) be from the class \(\mathfrak{g}_i, (i = 1, \ldots, 12)\), given in \((5.1)\). Then the following propositions are valid:

1. Every \((\mathcal{L}, H, G)\) is non-flat;
2. An \((\mathcal{L}, H, G)\) is scalar flat if and only if \(l\) belongs to:
   (a) \(\mathfrak{g}_{4.1}, \)
(b) \( g_{4,6} (b_1 = -b_2 \pm \sqrt{1-2b_2}, 0 \leq b_2 \leq \frac{\sqrt{3}}{2}, b_2 \neq \frac{\sqrt{3}}{2}) \),
(c) \( g_{4,11} (q = \frac{\sqrt{3}}{6}) \); 
(3) An \((\mathcal{L}, H, G)\) has positive scalar curvature if and only if \( l \) belongs to:
(a) \( g_{4,1} (i = 2, 3, 4, 5, 7, 8, 9, 11, 12) \),
(b) \( g_{4,6} (b_1 \neq 0, b_2 > \frac{\sqrt{3}}{2}) \),
(c) \( g_{4,11} (q > \frac{\sqrt{3}}{6}) \);
(4) An \((\mathcal{L}, H, G)\) has negative scalar curvature if and only if \( l \) belongs to:
(a) \( g_{4,10} \),
(b) \( g_{4,11} (0 < q < \frac{\sqrt{3}}{6}) \);
(5) Every \((\mathcal{L}, H, G)\) is *-scalar flat w.r.t. \( J_1 \) and \( J_2 \);
(6) An \((\mathcal{L}, H, G)\) is *-scalar flat w.r.t. \( J_3 \) if and only if \( l \) belongs to:
(a) \( g_{4,1} (i = 1, 5, 8, 9, 10, 12) \),
(b) \( g_{4,2} (m = -2) \),
(c) \( g_{4,6} (b_1 = -2b_2) \);
(7) An \((\mathcal{L}, H, G)\) is **-scalar flat w.r.t. \( J_1 \) if and only if \( l \) belongs to:
(a) \( g_{4,2} (m = -\frac{4}{3}) \),
(b) \( g_{4,5} (a_1 = -a_2^2) \),
(c) \( g_{4,6} (b_1 = b_2^{-1} - b_2) \),
(d) \( g_{4,11} (q = \frac{\sqrt{3}}{6}) \);
(8) An \((\mathcal{L}, H, G)\) is **-scalar flat w.r.t. \( J_2 \) if and only if \( l \) belongs to:
(a) \( g_{4,2} (m = -\frac{4}{3}) \),
(b) \( g_{4,5} (a_2 = -a_2^2) \),
(c) \( g_{4,6} (b_1 = b_2^{-1} - b_2) \),
(d) \( g_{4,11} (q = \frac{\sqrt{3}}{6}) \);
(9) An \((\mathcal{L}, H, G)\) is **-scalar flat w.r.t. \( J_3 \) if and only if \( l \) belongs to:
(a) \( g_{4,1} \),
(b) \( g_{4,9} (p = \frac{\sqrt{3} - 3}{2}) \);
(10) An \((\mathcal{L}, H, G)\) has positive basic holomorphic sectional curvatures w.r.t. \( J_1 \)
(i.e. \( k_{12} \) and \( k_{34} \)) if and only if \( l \) belongs to:
(a) \( g_{4,1} (i = 4, 7) \),
(b) \( g_{4,2} (m > 0) \),
(c) \( g_{4,5} (a_1 > 0) \),
(d) \( g_{4,6} (b_1 > 0, b_2 > 1) \),
(e) \( g_{4,9} (-\frac{3}{4} < p \leq 1, p \neq 0) \),
(f) \( g_{4,11} (q > 1) \);
(11) An \((\mathcal{L}, H, G)\) has positive basic holomorphic sectional curvatures w.r.t. \( J_2 \)
(i.e. \( k_{13} \) and \( k_{24} \)) if and only if \( l \) belongs to:
(a) \( g_{4,1} (i = 4, 7) \),
(b) \( g_{4,2} (m > 0) \),
(c) \( g_{4,5} (a_2 > 0) \),
(d) \( g_{4,6} (b_1 > 0, b_2 > 1) \),
(e) \( g_{4,9} (\frac{\sqrt{3} - 1}{2} < p \leq 1) \),
(f) \( g_{4,11} (q > 1) \);
(12) An \((\mathcal{L}, H, G)\) has positive basic holomorphic sectional curvatures w.r.t. \( J_3 \)
(i.e. \( k_{14} \) and \( k_{23} \)) if and only if \( l \) belongs to:
(a) $g_{4,i}$ ($i = 2, 3, 4, 6, 7, 11$),
(b) $g_{4,5}$ ($a_1 a_2 > 0$),
(c) $g_{4,9}$ ($-\frac{4}{9} < p \leq 1$, $p \neq 0$);

(13) An $(L, H, G)$ has negative basic holomorphic sectional curvatures w.r.t. $J_1$ (i.e. $k_{12}$ and $k_{34}$) if and only if $1 \in J_1$
(a) $g_{4,1}$,
(b) $g_{4,6}$ ($b_1 < 0$, $0 < b_2 < 1$),
(c) $g_{4,11}$ ($0 < q < \frac{7}{5}$);

(14) An $(L, H, G)$ has negative basic holomorphic sectional curvatures w.r.t. $J_2$ (i.e. $k_{13}$ and $k_{24}$) if and only if $1 \in J_2$
(a) $g_{4,6}$ ($b_1 < 0$, $0 < b_2 < 1$),
(b) $g_{4,11}$ ($0 < q < \frac{7}{5}$);

(15) Every $(L, H, G)$ has non-negative basic holomorphic sectional curvatures w.r.t. $J_1$ (i.e. $k_{14}$ and $k_{23}$);

(16) An $(L, H, G)$ has positive basic totally real sectional curvatures w.r.t. $J_1$ (i.e. $k_{13}, k_{14}, k_{23}$, and $k_{24}$) if and only if $1 \in J_1$
(a) $g_{4,1}$ ($i = 4, 7$),
(b) $g_{4,2}$ ($m > 0$),
(c) $g_{4,5}$ ($a_1 > 0$, $a_2 > 0$),
(d) $g_{4,6}$ ($b_1 > 0$, $b_2 > 1$),
(e) $g_{4,9}$ ($\frac{\sqrt{7} - 1}{2} < p \leq 1$),
(f) $g_{4,11}$ ($q > 1$);

(17) An $(L, H, G)$ has positive basic totally real sectional curvatures w.r.t. $J_2$ (i.e. $k_{12}, k_{14}, k_{23}$, and $k_{34}$) if and only if $1 \in J_2$
(a) $g_{4,1}$ ($i = 4, 7$),
(b) $g_{4,2}$ ($m > 0$),
(c) $g_{4,5}$ ($a_1 > 0$, $a_2 > 0$),
(d) $g_{4,6}$ ($b_1 > 0$, $b_2 > 1$),
(e) $g_{4,9}$ ($-\frac{4}{9} < p \leq 1$, $p \neq 0$),
(f) $g_{4,11}$ ($q > 1$);

(18) An $(L, H, G)$ has positive basic totally real sectional curvatures w.r.t. $J_3$ (i.e. $k_{12}, k_{13}, k_{24}$, and $k_{34}$) if and only if $1 \in J_3$
(a) $g_{4,1}$ ($i = 4, 7$),
(b) $g_{4,2}$ ($m > 0$),
(c) $g_{4,5}$ ($a_1 > 0$, $a_2 > 0$),
(d) $g_{4,6}$ ($b_1 > 0$, $b_2 > 1$),
(e) $g_{4,9}$ ($\frac{\sqrt{7} - 1}{2} < p \leq 1$),
(f) $g_{4,11}$ ($q > 1$);

(19) Every $(L, H, G)$ has non-negative basic totally real sectional curvatures w.r.t. $J_1$ (i.e. $k_{13}, k_{14}, k_{23}$, and $k_{24}$) and $J_2$ (i.e. $k_{12}, k_{14}, k_{23}$, and $k_{34}$);

(20) An $(L, H, G)$ has negative basic totally real sectional curvatures w.r.t. $J_3$ (i.e. $k_{12}, k_{13}, k_{24}$, and $k_{34}$) if and only if $1 \in J_3$
(a) $g_{4,6}$ ($b_1 < 0$, $0 < b_2 < 1$),
(b) $g_{4,11}$ ($0 < q < \frac{7}{5}$).

Proof. Firstly, we present our proof for the case when the corresponding Lie algebra of $L$ belongs to $g_{4,1}$. 
Using (2.3), (2.4), (4.11) and the definition of \( \mathfrak{g}_{4,1} \) in (5.1), we calculate the basic components \( R_{ijkl} = R(e_i, e_j, e_k, e_l) \) of \( R \). The non-zero of them are determined by the following ones and properties (2.9)

\[
R_{1212} = -R_{2123} = -R_{1414} = R_{1443} = R_{2323} = \frac{1}{3} R_{2424} = \frac{1}{3} R_{3434} = \frac{1}{3}.
\]

Bearing in mind the latter equalities, (2.3), (4.11) and (4.12), we obtain the basic components \( \rho_{jk} = \rho(e_j, e_k) \), \( (\rho_{\ast})_{jk} = \rho_{\ast}(e_j, e_k) \), as well as the values of \( \tau, \tau_{\ast}, \tau_{\ast\ast} \) and \( k_{ij} = k(e_i, e_j) \). Having in mind properties (2.10), the non-zero of them are determined by

\[
\rho_{11} = -\frac{1}{3} \rho_{22} = -\rho_{33} = -\frac{1}{2}, \quad (\rho_{\ast})_{12} = (\rho_{\ast})_{14} = -(\rho_{\ast})_{23} = \frac{1}{3} (\rho_{\ast})_{34} = -\frac{1}{3},
\]

\[
(\rho_{\ast})_{22} = -\frac{1}{3} (\rho_{\ast})_{24} = (\rho_{\ast})_{44} = -\frac{1}{3}, \quad (\rho_{\ast})_{12} = (\rho_{\ast})_{14} = (\rho_{\ast})_{23} = \frac{1}{3},
\]

\[
\tau_{\ast} = -\tau_{\ast\ast} = -2, \quad k_{12} = k_{14} = -k_{23} = -\frac{1}{3} k_{24} = \frac{1}{3} k_{34} = -\frac{1}{3}.
\]

By virtue of (5.11) and (6.22), we establish the truthfulness of the statements for the case of \( \mathfrak{g}_{4,1} \).

The results for the cases of the classes \( \mathfrak{g}_{4,2}, \mathfrak{g}_{4,5}, \mathfrak{g}_{4,6}, \mathfrak{g}_{4,9} \) and \( \mathfrak{g}_{4,11} \), which are are summarized here, are given in [3] and [4].

In a similar way as for \( \mathfrak{g}_{4,1} \), we obtain the following results for \( (\mathcal{L}, H, G) \) in the other cases and we prove the respective assertions:

\[ \mathfrak{g}_{4.3} : \]

\[
\frac{1}{3} R_{1213} = \frac{1}{3} R_{1414} = R_{2323} = R_{2424} = \frac{1}{3} R_{3434} = \frac{1}{3};
\]

\[
\frac{1}{3} \rho_{11} = \rho_{22} = \rho_{33} = -\rho_{44} = \frac{1}{3},
\]

\[
-\frac{1}{3} (\rho_{\ast})_{34} = -\frac{1}{3} (\rho_{\ast})_{24} = (\rho_{\ast})_{13} = -\frac{1}{3} (\rho_{\ast})_{14} = -(\rho_{\ast})_{23} = \frac{1}{3},
\]

\[
\tau = \tau_{\ast\ast} = 3 \tau_{\ast\ast} = 5 \tau_{\ast\ast} = \frac{3}{2}, \quad k_{14} = k_{23} = -k_{24} = \frac{3}{2} k_{34} = \frac{3}{2};
\]

\[ \mathfrak{g}_{4.4} : \]

\[
-\frac{1}{3} R_{1212} = -2 R_{1213} = -4 R_{1223} = R_{1313} = 2 R_{1323} = \frac{3}{4} R_{1414}
\]

\[
= R_{1424} = 4 R_{1434} = \frac{1}{2} R_{2323} = \frac{1}{2} R_{2424} = R_{2434} = -4 R_{3434} = 1;
\]

\[
\frac{1}{3} \rho_{11} = \rho_{12} = \frac{1}{3} \rho_{22} = \rho_{23} = -\rho_{33} = -\rho_{44} = \frac{1}{2},
\]

\[
\frac{1}{3} (\rho_{\ast})_{12} = (\rho_{\ast})_{13} = -\frac{1}{3} (\rho_{\ast})_{14} = -(\rho_{\ast})_{23} = \frac{1}{3} (\rho_{\ast})_{24} = (\rho_{\ast})_{34} = \frac{1}{3},
\]

\[
(\rho_{\ast})_{22} = \frac{1}{3} (\rho_{\ast})_{24} = (\rho_{\ast})_{44} = -\frac{1}{3},
\]

\[
(\rho_{\ast})_{12} = (\rho_{\ast})_{14} = (\rho_{\ast})_{23} = \frac{1}{3},
\]

\[
\tau_{\ast} = 6 \tau_{\ast\ast} = 2 \tau_{\ast\ast} = 3 \tau_{\ast\ast} = 12,
\]

\[
k_{12} = \frac{5}{3} k_{13} = k_{14} = \frac{5}{3} k_{23} = \frac{5}{3} k_{24} = 3 k_{34} = \frac{3}{2};
\]

\[ \mathfrak{g}_{4.7} : \]

\[
-\frac{1}{3} R_{1212} = -R_{1213} = 4 R_{1224} = -2 R_{1234} = \frac{3}{4} R_{1313} = 2 R_{1323} = 4 R_{1334}
\]

\[
= R_{1424} = 4 R_{1434} = \frac{1}{2} R_{2323} = \frac{1}{2} R_{2424} = R_{2434} = -4 R_{3434} = 1;
\]

\[
\frac{1}{3} \rho_{11} = \frac{1}{3} \rho_{22} = \frac{1}{3} \rho_{23} = -\frac{1}{3} \rho_{33} = -\frac{1}{3} \rho_{44} = 1,
\]

\[
\frac{1}{3} (\rho_{\ast})_{12} = (\rho_{\ast})_{13} = -\frac{1}{3} (\rho_{\ast})_{14} = (\rho_{\ast})_{23} = \frac{1}{3} (\rho_{\ast})_{24} = (\rho_{\ast})_{34} = \frac{1}{3},
\]

\[
-\frac{1}{3} (\rho_{\ast})_{22} = \frac{1}{12} (\rho_{\ast})_{24} = \frac{1}{11} (\rho_{\ast})_{44} = \frac{1}{11} (\rho_{\ast})_{34} = 1,
\]
\[(\rho^*_1)_{11} = -\frac{1}{2}(\rho^*_1)_{14} = -4(\rho^*_1)_{22} = (\rho^*_1)_{23} = -4(\rho^*_1)_{33} = -(\rho^*_1)_{44} = 2,\]
\[\varphi_{11} = 3\tau_3^* = 3\tau_1^* = 2\tau_2^* = \tau_3^* = 12,\]
\[k_{12} = k_{13} = \frac{7}{16}k_{14} = \frac{7}{8}k_{23} = \frac{7}{8}k_{24} = 7k_{34} = \frac{7}{4};\]
\[g_{4,8} :\]
\[4R_{1212} = 2R_{1234} = -4R_{1313} = 2R_{1324} = -4R_{2323} = 2424 = -R_{3434} = 1;\]
\[\rho_{11} = -\rho_{22} = \rho_{33} = \frac{1}{3}\rho_{44} = -\frac{1}{2},\]
\[(\rho^*_1)_{12} = -\frac{3}{2}(\rho^*_1)_{13} = (\rho^*_1)_{14} = -\frac{3}{2}(\rho^*_1)_{24} = -\frac{3}{2}(\rho^*_1)_{23} = -\frac{3}{2}(\rho^*_1)_{21} = -\frac{3}{2},\]
\[\tau = \frac{3}{4}\tau_1^* = \frac{3}{4}\tau_2^* = \frac{3}{4}\tau_3^* = \frac{3}{4},\]
\[k_{12} = k_{13} = k_{23} = -\frac{1}{4}k_{24} = -\frac{1}{4}k_{34} = -\frac{1}{4};\]
\[g_{4,10} :\]
\[4R_{1212} = 2R_{1224} = -4R_{1313} = 2R_{1334} = \frac{1}{4}R_{2323} = -R_{2424} = R_{3434} = 1;\]
\[\rho_{11} = -\rho_{22} = \rho_{33} = \frac{1}{3}\rho_{44} = -\frac{1}{2},\]
\[4(\rho^*_1)_{12} = 2(\rho^*_1)_{13} = 2(\rho^*_1)_{14} = (\rho^*_1)_{24} = (\rho^*_1)_{34} = -1,\]
\[2(\rho^*_2)_{12} = 4(\rho^*_2)_{13} = (\rho^*_2)_{14} = 2(\rho^*_2)_{24} = 2(\rho^*_2)_{34} = -1,\]
\[\tau = \frac{7}{10}\tau_1^* = \frac{7}{10}\tau_2^* = \frac{7}{10}\tau_3^* = \frac{7}{10} = \frac{7}{10},\]
\[k_{12} = k_{13} = -\frac{1}{4}k_{23} = \frac{1}{4}k_{24} = \frac{1}{4}k_{34} = -\frac{1}{4};\]
\[g_{4,12} :\]
\[R_{1212} = -R_{1313} = -R_{2323} = 1;\]
\[\rho_{11} = \rho_{22} = \rho_{33} = 2,\]
\[(\rho^*_1)_{12} = (\rho^*_2)_{13} = (\rho^*_3)_{23} = 1,\]
\[\tau = \frac{7}{10}\tau_1^* = \tau_2^* = \tau_3^* = \frac{7}{10} = \frac{7}{10},\]
\[k_{12} = k_{13} = k_{23} = 1.\]

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