Spectral Flexibility of Symplectic Manifolds $T^2 \times M$

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Abstract

We consider Riemannian metrics compatible with the natural symplectic structure on $T^2 \times M$, where $T^2$ is a symplectic 2-Torus and $M$ is a closed symplectic manifold. To each such metric we attach the corresponding Laplacian and consider its first positive eigenvalue $\lambda_1$. We show that $\lambda_1$ can be made arbitrarily large by deforming the metric structure, keeping the symplectic structure fixed. The conjecture is that the same is true for any symplectic manifold of dimension $\geq 4$. We reduce the general conjecture to a purely symplectic question.

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1 Introduction and Main Results

Our paper concerns a rigidity result for deformations of quasi-Kähler structures. We begin by giving a historical overview of the problem considered. The following theorem due to Joseph Hersch is a first one in a series of rigidity results concerning $\lambda_1$.

**Theorem 1.1** ([14]). Let $(S^2, g)$ be the 2-sphere equipped with a Riemannian metric $g$. Then,

$$\lambda_1(S^2, g) \text{Area}(S^2, g) \leq 8\pi,$$

where $\lambda_1(S^2, g)$ is the first positive eigenvalue of the Laplacian on $(S^2, g)$.

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It is also known that equality occurs if and only if \((S^2, g)\) is the standard round sphere. Hersch’s theorem was extended to any surface:

**Theorem 1.2** (21). Let \((\Sigma, g)\) be a closed Riemannian surface. Then

\[
\lambda_1(g) \text{Area}(\Sigma, g) \leq 8\pi(\text{genus}(\Sigma) + 1).
\]

Here, the constant on the right hand side is not optimal.

The next result shows non-rigidity in higher dimensions.

**Theorem 1.3** (9). Let \(M\) be a closed Riemannian manifold of dimension \(> 2\). Then one can find a Riemannian metric of total volume 1 and of arbitrarily large \(\lambda_1\).

If one fixes a conformal class on a manifold \(M\), then one recovers rigidity for \(\lambda_1\):

**Theorem 1.4** (12, 13). Let \((M, g)\) be a closed Riemannian manifold of dimension \(n\). Then

\[
\lambda_1(fg) \text{Vol}(M, fg)^{2/n} \leq C(g),
\]

where \(f\) is any positive function on \(M\) and \(C(g)\) is a constant independent of \(f\).

In particular, by the Uniformization Theorem for the Riemann sphere this theorem extends Theorem 1.1.

In [16] Polterovich considers Theorem 1.2 in the Kähler and quasi-Kähler categories. In order to explain this, we recall the definitions of Kähler and quasi-Kähler structures: Let \((M, \omega)\) be a closed symplectic manifold. Let \(J\) be an almost complex structure on \(M\) which is compatible with \(\omega\), i.e. \(\omega(Jv, Jw) = \omega(v, w)\) and \(\omega(v, Jv) > 0\) unless \(v = 0\). Let \(g\) be the corresponding Riemannian metric \(g(v, w) := \omega(v, Jw)\).

The quadruple \((M, \omega, J, g)\) is called a quasi-Kähler structure on \(M\). If \(J\) is a complex structure, it is called a Kähler structure.

Having these terms in mind we reformulate Theorem 1.2 as

**Theorem 1.2’** (21). Let \((\Sigma, \omega)\) be a closed symplectic surface. For any quasi-Kähler structure \((\Sigma, \omega, J, g)\), one has

\[
\lambda_1(g) \text{Area}(\Sigma, \omega) \leq 8\pi(\text{genus}(\Sigma) + 1).
\]
The equivalence of the two formulations follows from Moser’s argument [15, Th. 3.17]. This argument shows that fixing the total area of a closed surface is equivalent to fixing the symplectic structure $\omega$ on it. Indeed, according to this argument, if $(\omega_t), 0 \leq t \leq 1,$ is a smooth family of symplectic forms on a compact manifold, all in the same cohomology class, then there exists a flow $(\varphi_t)$ for which $\varphi_t^*(\omega_t) = \omega_0.$ Now, for any two symplectic forms $\omega_0$ and $\omega_1,$ we define $\omega_t = (1-t)\omega_0 + t\omega_1.$ In dimension two $(\omega_t)$ is a family of symplectic forms, since this is a family of area forms. In addition, if $\omega_0$ and $\omega_1$ have the same total area, then all the $\omega_t$’s are cohomologous. Thus, the conditions of Moser’s theorem are satisfied and we conclude that $\omega_0$ and $\omega_1$ are symplectomorphic.

Also, linear algebra in $\mathbb{R}^2$ shows that any Riemannian metric on a symplectic surface comes from a quasi-Kähler structure whose quasi-Kähler form is $\omega$ (moreover, any almost complex structure on a closed surface is integrable).

The following theorem is proved in [16]. Its proof is based on results from [7] and [10].

**Theorem 1.5 ([16]).** Let $(M,\omega)$ be a closed symplectic manifold. Suppose $\omega$ is a rational form. Let $g$ be a Kähler metric whose Kähler form is $\omega.$ Then

$$\lambda_1(g) \leq C(\omega),$$

where $C(\omega)$ is independent of $g.$

It is still an open question whether the same is true for any real symplectic form $\omega.$ On the other hand, if we consider quasi-Kähler metrics, then the conjecture is

**Conjecture 1.6.** Let $(M,\omega)$ be a closed symplectic manifold of dimension $\geq 4.$ Then, there exists a quasi-Kähler structure on it with arbitrarily large $\lambda_1.$

For $T^4 \times M$ we have

**Theorem 1.7 ([16]).** Let $(T^4,\sigma)$ be the standard symplectic 4-torus. Let $(M,\omega)$ be a closed symplectic manifold. Then, on $(T^4 \times M,\sigma \oplus \omega)$ there exists a quasi-Kähler structure with arbitrarily large $\lambda_1.$

In his proof Polterovich shows existence of certain plane distributions on $(T^4 \times M,\sigma \oplus \omega)$ along which one can deform an almost complex structure $J.$
The main novelty of our paper is the introduction of non-regular distributions in this procedure. These distributions do not have a constant dimension. Their dimension may drop on exceptional subsets. The precise definition is given in Section 2. The advantage of our method is in two aspects: First, singular distributions may exist on manifolds on which regular distributions do not exist simply due to topological restrictions. Thus, we extend the family of manifolds for which Conjecture 1.6 is true. In particular, we prove

**Theorem 1.8.** Let \((\mathbb{T}^2, \sigma)\) be the standard symplectic 2-torus. Let \((M, \omega)\) be a closed symplectic manifold. Then, on \((\mathbb{T}^2 \times M, \sigma \oplus \omega)\) there exists a quasi-Kähler structure with arbitrarily large \(\lambda_1\).

Second, our method shows that Conjecture 1.6 follows from a purely symplectic conjecture. Namely,

**Conjecture 1.9.** Let \((M, \omega)\) be a closed symplectic manifold of dimension \(n \geq 4\). Then one can find on \((M, \omega)\) an isotropic singular distribution \(L\) which satisfies Hörmander’s condition.

Singular distributions and Hörmander’s condition are discussed in Section 2. We prove

**Theorem 1.10.** Conjecture 1.6 follows from Conjecture 1.9.

We would like to emphasize that any answer to Conjecture 1.9 would be very interesting. As mentioned, a positive answer will resolve Conjecture 1.6, while a negative answer will presumably lead to a new type of symplectic rigidity.

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2 Overview of the Main Ideas

The basic object we use is a finite set \( X = \{X_k\}_{k=1}^{N} \) of smooth vector fields, which satisfy

1. \( \exists A \in \mathbb{N} \) such that the dimension of
   \[
   \mathcal{L}|_x = \text{span}(X_1(x), \ldots, X_N(x))
   \]
   is \( A \) almost everywhere. The complement of \( \{\dim \mathcal{L}|_x = A\} \) is called the singular set.

2. The iterated commutators of the \( X_k \)'s and which are of lengths \( \leq r \text{ span } T_x M \) for every \( x \).

A set \( X \) which satisfies property 1 will be called a singular distribution, and will be denoted by \( \mathcal{L} \). Property 2 is called the \( r \)-Hörmander condition.

The idea to use Hörmander regular distributions in the context of \( \lambda_1 \) can be found in [6] and [16]. The new idea we introduce is to allow also distributions which might not have constant dimension. The dimension of a singular distribution might drop on subsets of measure 0. Singular distributions are subject to less topological restrictions, and thus can be used where smooth distributions do not exist.

To a Hörmander distribution associated with \( X \), we can attach the hypoelliptic operator

\[
L = \sum_j X_j^* X_j,
\]

where \( X_j^* = -X_j - \text{div}(X_j) \). We start with an arbitrary compatible Riemannian metric \( g \) and we deform it in a way which is related to the distribution associated with \( X \). We would like to relate the spectral properties of \( L \) to the spectrum of the Laplacian of a deformed metric. The estimate of the eigenvalues of the deformed metric is done through the variational principle.

Unlike in [16], we can only deform the Riemannian metric structure on \( M \) away from the singular set of \( \mathcal{L} \). The new difficulty we face in our estimates is the fact that the test functions near the singular set do not feel the deformation, and the estimate of [16] does not apply anymore.

To overcome this problem we apply the theory of anisotropic Sobolev spaces as developed in [17], and the machinery of fractional Sobolev spaces also known as Bessel Potential Spaces.

Theorem [18] is a direct consequence of the following two theorems.
Theorem 2.1. Suppose $L$ is an isotropic singular distribution on $(M, \omega)$ which satisfies Hörmander’s condition. Then, one can find a compatible almost complex structure $J$ on $M$ with arbitrarily large $\lambda_1(M, \omega, J)$.

Theorem 2.2. Let $(M, \sigma_M)$ be a closed symplectic manifold. On $(T^2 \times M, \sigma_T \oplus \sigma_M)$ there exists an isotropic singular distribution which satisfies Hörmander’s condition.

The proofs are given in section 3 and section 4 below.

3 Proof of Theorem 2.1

We assume that $(M, \omega)$ is a closed symplectic manifold of dim $n$, and $L$ is a singular isotropic distribution which satisfies the $r$-Hörmander condition. We can find on $M$ an arbitrary quasi-Kähler structure $(M, \omega, J_1, g_1)$.

3.1 A Deformation of the Almost Complex Structure

We introduce a deformation of the almost complex structure $J_1$, which keeps it compatible with $\omega$. The deformation will be defined with the help of the distribution $L$.

Let $B$ be the singular set of $L$. Let $\phi$ be a positive function on $M$ which is identically 1 in a neighborhood of $B$.

Consider the $g_1$-orthogonal decomposition at a point $x \in M$.

$$T_xM = L|_x \oplus J_1L|_x \oplus V_x.$$  

Set

$$J_\phi = \begin{cases} (1/\phi)J_1 & \text{on } L, \\ \phi J_1 & \text{on } J_1L, \\ J_1 & \text{on } V. \end{cases}$$ \hspace{1cm} (1)$$

One can check that $J_\phi$ defines a quasi-Kähler structure $(M, \omega, J_\phi, g_\phi)$.

3.2 Estimation of $\lambda_1$

We estimate $\lambda_1(M, \omega, J_\phi, g_\phi)$ by using the variational formula.

Let $K$ be such that $g_1(X_j, X_j) \leq K$ at every point. Since $L$ satisfies Hörmander’s condition, we can apply the following version of the Poincaré–Sobolev inequality. Its proof is given in section 5.3.
Theorem 3.1. If $2 - \frac{4}{\tau n + 2} < p < 2$, then

$$
\| u - \bar{u} \|_{L^2} \leq C_p \sum_{j=1}^{N} \left( \int_M |X_j u|^p \, d(vol) \right)^{1/p},
$$

for all $u \in C^\infty(M)$. Here $\bar{u}$ is the average of $u$ on $M$, and $C_p$ is independent of $u$.

Fix $p$ as in the theorem. Let $\langle \cdot, \cdot \rangle_\phi$, $\| \cdot \|_\phi$, $\nabla_\phi$ denote the inner product, the norm and the gradient respectively of the Riemannian metric $g_\phi$. Note that the volume form $vol_\phi = vol$ for any $\phi$, since $g_\phi$ is compatible with $\omega$. For any function $u$ of zero average, we obtain

$$
\| u \|_{L^2} \leq C_p \sum_{j=1}^{N} \left( \int_M |\nabla_\phi u, X_j|_\phi^p \, d(vol) \right)^{1/p}
$$

$$
\leq C_p N \sqrt{R} \left( \int_M \| \nabla_\phi u \|_\phi^p \frac{1}{\phi^{p/2}} \, d(vol) \right)^{1/p}
$$

$$
\leq C_p N \sqrt{R} \left( \int_M \| \nabla_\phi u \|^2_\phi \, d(vol) \right)^{1/2} \left( \int_M \frac{1}{\phi^{p/(2-p)}} \, d(vol) \right)^{\frac{2-p}{2p}},
$$

where we used the fact that $\| X_j \|_\phi = \phi^{-1/2} \| X_j \|_1 \leq \phi^{-1/2} K^{1/2}$. Hence,

$$
\frac{\int_M \| \nabla_\phi u \|^2_\phi \, d(vol)}{\int_M |u|^2 \, d(vol)} \geq (C_p N)^{-2} K^{-1} \| 1/\phi \|_{L^{p/(2-p)}}^{-1}.
$$

We can choose $\phi$ such that $\| 1/\phi \|_{L^{p/(2-p)}}$ is very small. By the variational principle we conclude that $\lambda_1$ can be made arbitrarily large.

4 Proof of Theorem 2.2

Let us denote the coordinates on the torus $T^2 = \mathbb{R}^2/(2\pi \mathbb{Z})^2$ by $x, y$ modulo $2\pi$. Let $z$ denote a point on $M$.

4.1 Construction of a Distribution

Let $h(z)$ be a Morse function on $M$. Let $Z_1, \ldots, Z_{2d}$ be vector fields on $M$, which span $T_z M$ for every $z \in M$. Let

$$
\phi_{2j-1}(x) = \sin(jx), \ \phi_{2j}(x) = \cos(jx), \ (1 \leq j \leq d).
$$
Remark and Notation. If we let \( Q(t) = \prod_{j=1}^{d}(t^2 + j^2) \), then \( (\phi_l)_{l=1}^{2d} \) is a basis of solutions to the differential equation \( Q(\frac{\partial}{\partial x})(u) = 0 \). We will also write \( Q(t)(u) = 0 \), where \( t \) acts by \( \partial/\partial x \).

Define the following two vector fields on \( T^2 \times M \):

\[
V(x, y, z) := \frac{\partial}{\partial x} + h(z)\frac{\partial}{\partial y},
\]

\[
W(x, y, z) := \sum_{l=1}^{2d} \phi_l(x)Z_l(z).
\]

We claim that these two vector fields span a singular isotropic Hörmander distribution \( \mathcal{L} \). It is clear that \( \mathcal{L} \) is isotropic, so it remains to check that the Hörmander condition and the singularity property are satisfied for \( \mathcal{L} \).

4.2 \( \mathcal{L} \) is Hörmander

For two vector fields \( X, Y \), set \( D_X(Y) := [X, Y] \).

We will prove that the following iterated commutators

\[
V, W, D_V^\alpha(W), D_V^\beta D_W^\gamma D_V^\delta(W),
\]

where \( 0 \leq \alpha < 2d \), \( 0 \leq \beta < 4d \) and \( 0 \leq \gamma < 2d \) span the tangent space at every point.

**Step 1.** We compute by induction on \( \alpha \)

\[
D_V^\alpha(W)(x, y, z) = \sum_{l=1}^{2d} \phi_l^{(\alpha)}(x)Z_l(z) - \alpha \phi_l^{(\alpha-1)}(x)(Z_l h)(z)\frac{\partial}{\partial y}(y).
\]

**Step 2.** We prove that the vector fields \( \{D_V^\alpha(W)\}_{0 \leq \alpha < 2d} \) span \( Z_l \) and \( (Z_l h)\partial/\partial y \) at every point for \( 1 \leq l \leq 2d \).

By Step 1, for any polynomial \( F(t) = \sum a_k t^k \) we have

\[
F(D_V(W)) = \sum_{k} a_k D_V^k(W) = \sum_{l=1}^{2d} (F(t)\phi_l)Z_l - \sum_{l=1}^{2d} (F'(t)\phi_l)(Z_l h)\frac{\partial}{\partial y}.
\]

In particular, for \( F(t) = t^{r-1}Q(t) = \sum_{k} q_k t^{r+k-1} \) we have

\[
\xi_r := \sum_{k=0}^{2d} q_k D_V^{r+k-1}(W) = -\sum_{l=1}^{2d} (t^{r-1}Q'(t)\phi_l)(Z_l h)\frac{\partial}{\partial y}.
\]

8
Lemma 4.1. $Q'(t)\phi_t = b_t \phi_t'$ for some integer $b_t \neq 0$, where $1 \leq t \leq 2d$.

We give the proof below. Thus,

$$\xi_r = - \sum_{l=1}^{2d} b_t(\phi_t^{(r)})(Z_lh) \frac{\partial}{\partial y}.$$ 

Since the $(2d \times 2d)$ Wronskian matrix $(\phi_t^{(r)})_{rt}$ is non-degenerate and $b_t \neq 0$, it follows that the $\xi_t$'s $(1 \leq t \leq 2d)$ span the vector fields $(Z_lh)\partial/\partial y$ at every point. Finally, from Step 1 we see that the $Z_t$'s are spanned by vectors of the forms $D^\gamma_V(W)$ and $(Z_lh)\partial/\partial y$.

**Step 3.** We show that

$$\{D^\gamma_V D^\beta_W D^\rho_V(W), D^\sigma_V(W)\} \text{ span } \sum_{k,l=1}^{2d} \phi_k^{(i)} \phi_l^{(j)} (Z_k Z_l h) \frac{\partial}{\partial y},$$

for $0 \leq i, j < 2d$, where $0 \leq \alpha < 2d$, $0 \leq \beta < 4d$ and $0 \leq \gamma < 2d$.

We compute:

$$D^\rho_W (Z_lh) = \sum_i a_i^\rho Z_i - \beta \sum_{k,l} \phi_k^{(\beta-1)} (Z_k Z_l h) \frac{\partial}{\partial y},$$

for some $C^\infty$-functions $a_i^\rho$ on $T^2 \times M$.

Since we can span the $Z_t$'s at every point of $T^2 \times M$ (by Step 2), it follows that we can also span $\sum_{k,l} \phi_k^{(j)} (Z_k Z_l h) \partial/\partial y$ for $0 \leq j \leq 2d$. Let

$$Y = \sum_{k,l} \phi_k^{(a)} \phi_l^{(b)} (Z_k Z_l h) \partial/\partial y.$$ 

By repeated uses of the formula

$$D^\gamma_V(Y) = \sum_{k,l} \left( \phi_k^{(a+1)} \phi_l^{(b)} + \phi_k^{(a)} \phi_l^{(b+1)} \right) (Z_k Z_l h) \frac{\partial}{\partial y},$$

we see that the given vectors span $\sum_{k,l} \phi_k^{(i)} \phi_l^{(j)} (Z_k Z_l h) \partial/\partial y$.

**Step 4.**

$$\{D^\gamma_V D^\beta_W D^\rho_V(W), D^\sigma_V(W)\} \text{ span } (Z_k Z_l h) \frac{\partial}{\partial y},$$

for any $1 \leq k, l \leq 2d$, where $0 \leq \alpha < 2d$, $0 \leq \beta < 4d$ and $0 \leq \gamma < 2d$. 

9
The $(2d \times 2d)$ matrix $B = (\phi_k(i))_{ik}$ is non-degenerate at every point. Hence the matrix $B \otimes B = (\phi_k(i) \phi_l(j))_{ik,kl}$ is non-degenerate. It follows from Step 3 that we can span $(Z_kZ_lh)\partial/\partial y$.

**Step 5.** We finish by noting that since $h$ is a Morse function, we have $\partial/\partial y$ at every point by Step 4 and Step 2. We use $V$ to get also $\partial/\partial x$.

### 4.2.1 Proof of Lemma 4.1

Let $1 \leq j \leq d$. $Q(t) = P(t^2)$, where $P(t) = \prod_{k=1}^{d} (t + k^2)$. We can write $P'(t) = F_j(t)(t + j^2) + b_j$, for some polynomial $F_j$ and integer $b_j$. Moreover, $b_j \neq 0$, since $P'(-j^2) = \prod_{k \neq j} (-j^2 + k^2) \neq 0$. Hence, $Q'(t) = 2tP'(t^2) = 2tF_j(t^2)(t^2 + j^2) + 2b_j t$.

Since $\phi_{2j-1}$ and $\phi_{2j}$ are solutions for the second order differential equation $(t^2 + j^2)u = 0$, we conclude that $Q'(t)\phi_l = 2b_j[l\lfloor\frac{l}{2}\rfloor] \phi'_l$.

### 4.3 $\mathcal{L}$ is a Singular Distribution

We will show that $\dim(\text{span}(V, W)) = 2$ almost everywhere. It suffices to check that $W$ vanishes on a closed set of measure 0.

Suppose $W(p) = 0$. Choose coordinates $(x, y, z_1, \ldots, z_n)$ in a neighborhood of $p$. We can write

$$Z_l = \sum_{j=1}^{n} a_{jl} \frac{\partial}{\partial z_j}$$

for some $C^\infty$-functions $a_{jl}$. The $(n \times 2d)$ matrix $A = (a_{jl})$ is of full rank $n$. When written in coordinates, the equation $W(p) = 0$ becomes the following system of equations:

$$\sum_{l=1}^{2d} \phi_l(x)a_{jl}(z) = 0, \ (1 \leq j \leq n),$$

which can also be written as $A(z)\phi(x) = 0$, where $\phi$ is the column vector of the $\phi_l$’s. Let $E = \{(x, y, z) \in \mathbb{T}^2 \times M : A(z)\phi(x) = 0\}$. It is clear that $E$ is closed. For each pair $(y, z)$ let

$$E_{y,z} = \{x : (x, y, z) \in E\}$$

be a subset of the circle.
We show that $E_{y,z}$ is a discrete set, hence finite of measure 0. Let $x \in E_{y,z}$. The matrix $(\phi_{l}(m)(x))_{m}$, $(1 \leq l \leq 2d, 0 \leq m < 2d)$ is of full rank. Therefore, for some $0 < m < 2d$, $A(z)\phi^{(m)}(x) \not= 0$. From here it follows that $x$ is an isolated point in $E_{y,z}$.

We have shown that, for every fixed pair $(y, z)$, $E_{y,z}$ is of measure 0. By Fubini’s Theorem it follows that $E$ is of measure 0.

5 Sobolev spaces associated with Hörmander Distributions

Let $M$ be a closed manifold, equipped with a volume form. In this section we give an overview of a family of Sobolev spaces associated with a Hörmander set of vector fields $X$ (see section 2). These spaces were defined and studied in [17].

For $1 < p < \infty$, let

$W^{1,p}_{X}(M) = \{ f \in L^{p}(M) : \forall j \ X_{j}f \in L^{p}(M) \}$.

The norm on $W^{1,p}_{X}(M)$ is given by

$$\|u\|_{1,p,X} = \|u\|_{L^{p}} + \sum_{j=1}^{N} \|X_{j}u\|_{L^{p}}.$$  

5.1 Bessel Potential Spaces

In order to formulate embedding theorems for the spaces $W^{1,p}_{X}$ we review a family of fractional Sobolev spaces also known as Bessel potential spaces. For more details on these spaces see [18 ch. 5], [1], pp. 219–222), [5, 2, 3, 8, 4, 19, 20].

For $s \geq 0$ and $1 < p < \infty$, let

$W^{s,p}(\mathbb{R}^{n}) = \{ f \in L^{p}(\mathbb{R}^{n}) : \mathcal{F}^{-1}((1 + |x|^{2})^{s/2}\mathcal{F}(f)) \in L^{p}(\mathbb{R}^{n}) \},$

where $\mathcal{F}$ is the Fourier transform on tempered distributions. The norm

$$\|f\|_{s,p} := \|\mathcal{F}^{-1}((1 + |x|^{2})^{s/2}\mathcal{F}(f))\|_{L^{p}}$$

makes $W^{s,p}(\mathbb{R}^{n})$ into a Banach space. For an integer $s$, $W^{s,p}$ coincides with the classical Sobolev space (not true for $p = 1$).

11
Remark. If we set $G_s = F((1 + |x|^2)^{-s/2})$ (the Bessel potential), then $W^{s,p} = G_s * L^p$, and $\|f\|_{s,p} = \|G_{-s} f\|_{L^p}$, where * denotes convolution.

Next we define $W^{s,p}(\Omega)$ for a bounded $C^\infty$-domain $\Omega \subset \mathbb{R}^n$ as the space of restrictions from $W^{s,p}(\mathbb{R}^n)$. The natural norm here is the quotient norm

$$\|f\|_{s,p,\Omega} = \inf \|\tilde{u}\|_{s,p,\mathbb{R}^n},$$

where $\tilde{u}$ is an extension of $u$ to $\mathbb{R}^n$.

The space $W^{s,p}(\mathbb{R}^n)$ is invariant under $C^\infty$-diffeomorphisms of $\mathbb{R}^n$ and is a $C^\infty(M)$-module. Therefore we can define $W^{s,p}(M)$ on a closed manifold $M$ in the standard way ([3], [20, Ch. 7]). Namely, on $M$ choose a finite partition of unity $\psi_i$ subordinated to an atlas $(\Omega_i, h_i)$ of $M$. We define

$$W^{s,p}(M) = \{ f : M \to \mathbb{R} : (\psi_i u) \circ h_i \in W^{s,p}(\Omega_i) \},$$

where a norm is given by

$$\|u\|_{s,p,M} := \sum_i \| (\psi_i u) \circ h_i \|_{s,p,\Omega_i}.$$

### 5.2 Embedding Theorems

Suppose that $X$ satisfies the $r$-Hörmander condition. We have

**Theorem 5.1** ([17, Theorem 17]). The space $W_X^{1,p}(M)$ is continuously embedded in $W^{1/r,p}(M)$.

Let $\dim(M) = n$. The (fractional) Sobolev embedding theorem is (see e.g. [5, pp. 470–471], [1, Theorem 7.63], [19, sec. 2.7]):

**Theorem 5.2.** If $sp < n$ and $0 \leq 1/p - 1/q \leq s/n$, then the space $W^{s,p}(M)$ is continuously embedded in $L^q(M)$.

The (fractional) Rellich–Kondrachov Theorem for a closed manifold $M$ is

**Theorem 5.3.** If $sp < n$ and $1/p - 1/q < s/n$, then the space $W^{s,p}(M)$ is compactly embedded in $L^q(M)$.

**Remark.** This result is stated in [11, sec. 2.5.1] for bounded domains in $\mathbb{R}^n$. The same result for closed manifolds is deduced in a standard way, and we omit the proof.
Corollary 5.4. If \( p/r < n \) and \( 0 \leq 1/p - 1/q < 1/(rn) \), then the space \( W^{1,p}_{X}(M) \) is continuously embedded in \( L^{q}(M) \).

Corollary 5.5. If \( p/r < n \) and \( 1/p - 1/q < 1/(rn) \), then the space \( W^{1,p}_{X}(M) \) is compactly embedded in \( L^{q}(M) \).

5.3 Poincaré–Sobolev Inequalities

Define the energy corresponding to the anisotropic Sobolev space \( W^{1,p}_{X}(M) \) by

\[
E_{X,p}(u) = \sum_{j=1}^{N} \| X_{j}u \|_{L^{p}}.
\]

Theorem 5.6. Let \( p > 1, 1/p - 1/q < 1/(rn) \). Then

\[
\| u - \bar{u} \|_{L^{q}} \leq C_{p,q}E_{X,p}(u),
\]

for any \( u \in C^{\infty}(M) \), where \( C_{p,q} \) is independent of \( u \) and where \( \bar{u} \) is the average of \( u \) on \( M \).

Proof. Step 1. We can assume that \( q = p \). Indeed, if \( q > p \), then by Corollary 5.4 one has

\[
\| u \|_{L^{q}} \leq C_{p,q}(\| u \|_{L^{p}} + E_{X,p}(u)).
\]

Joining this with the result for \( q = p \) gives the desired inequality.

If \( 1 < q < p \), then we conclude the inequality from Hölder’s inequality and the result for \( q = p \).

Step 2. It is enough to prove the result for functions with average 0. Suppose there does not exist a constant \( C_{p,p} \) as above. Then we can find a sequence \( (u_{n}) \), \( u_{n} \in C^{\infty}(M) \), with average 0, such that \( \| u_{n} \|_{L^{p}} = 1 \) and for all \( j \) \( X_{j}u_{n} \rightarrow 0 \). Hence, the sequence \( (u_{n}) \) is bounded in \( W^{1,p}_{X} \). By Corollary 5.5 we may assume that \( (u_{n}) \) converges to \( u \) in \( L^{p} \). On the one hand \( X_{j}u_{n} \rightarrow X_{j}u \) in distribution sense, and on the other hand \( X_{j}u_{n} \rightarrow 0 \) by Hölder’s inequality, and therefore also in distribution sense. We obtain that for all \( j \) \( X_{j}u = 0 \) in distribution sense.

Step 3. Any \( Y \in T_{x}M \) can be expressed as a sum \( Y = \sum a_{j}Y_{j} \), where the \( Y_{j} \)'s are iterated commutators of the \( X_{j} \)'s. Hence, \( Y_{j}u = 0 \) for any smooth vector field \( Y \). Therefore, \( u \) is a constant function. Since it has average 0, we conclude \( u = 0 \). This is a contradiction to \( 1 = \| u_{n} \|_{L^{p}} \rightarrow \| u \|_{L^{p}}. \)
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