Sudden birth of maximal hidden quantum correlations

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We report on the existence of the phenomenon of sudden birth of maximal hidden quantum correlations in open quantum systems. Specifically, we consider the CHSH-inequality for Bell-nonlocality, the $F_3$-inequality for EPR-steering, and usefulness for teleportation as quantum correlations in a bipartite qubit system under collective decoherence due to an environment. We show that, even though the system may undergo a sudden birth of entanglement, neither of the aforementioned correlations are present and yet, these can still be revealed by means of local filtering operations, thus evidencing the presence of hidden quantum correlations. Furthermore, there are extreme versions of this phenomenon for which the revealed correlations achieve their maximal amount allowed by quantum theory. The phenomenon of maximal hidden correlations relies on the qubits collective damping, and may take place even in long-distance separated qubits. These results prove that apparently-useless physical systems as quantum registers could still exhibit maximal correlations, which are hidden from us, but that can still be accessed by means of local filtering operations.

INTRODUCTION

The study of non-classical correlations has remained an intriguing aspect of the theory of quantum mechanics almost since the formalisation of the theory itself. Even though it initially started with the exploration of properties like entanglement\textsuperscript{1} and Bell-nonlocality\textsuperscript{2, 3}, it is nowadays clear that there is actually a plethora of different non-classical phenomena including: EPR-steering\textsuperscript{4, 5}, non-contextuality\textsuperscript{6}, discord\textsuperscript{7}, coherence\textsuperscript{8}, amongst many others\textsuperscript{9}. Recently, the study of these properties has found a renewed interest under the unifying umbrella of quantum resource theories\textsuperscript{10}.

One of the appealing characteristics of these counter-intuitive quantum properties is that they can be exploited for implementing operational tasks like device-independent and semi-device-independent information-processing protocols\textsuperscript{11}. This realisation was precisely a decisive point that helped establish the so-called second quantum revolution\textsuperscript{12, 13}. As a consequence of this, the identification and classification of quantum properties of physical systems, in addition to being of a purely theoretical interest, it also became of crucial importance for the development of quantum technologies\textsuperscript{14}. These quantum properties are fragile when considered for actual physical systems due to their inevitable interaction with a surrounding environment, a fact that translates into a loss of vital information (e.g., quantum correlations) stored in the relative phases (coherence) of their quantum interference dynamics, in a process known as decoherence\textsuperscript{15–17}. This usually occurs as an exponential decay in time, with it sometimes even undergoing sudden death\textsuperscript{16, 17}. It also sometimes happen however, that more interesting dynamics take place in the form of death and revival or sudden birth of quantum correlations\textsuperscript{18–20}. These different types of dynamics of correlations have been verified experimentally\textsuperscript{21, 22}.

In order to counteract the effects of decoherence, one could think of implementing strategies to preserve, or more interestingly, to recover quantum correlations. It is well-known that this is possible by using Stochastic Local Operations with Classical Communication (SLOCC) or, more colloquially, local filtering operations. SLOCCs were first used by Popescu\textsuperscript{23} as a technique for extracting quantum correlations and particularly, for proving the existence of the phenomenon of hidden Bell-nonlocality. Since then, the technique has been further explored for additional properties\textsuperscript{23–32}, and it has successfully been implemented experimentally as well\textsuperscript{33–38}. In the particular case of entanglement, it has been proven that there are physical scenarios in which local filtering can mitigate the effect of the environment\textsuperscript{39}, and sometimes even help to recover the initial amount of entanglement\textsuperscript{40}. Despite this progress, the behaviour of additional properties besides entanglement, and whether it is possible for physical systems to evidence hidden quantum correlations, has remained unexplored.

In this work, we show that dissipative quantum systems can reveal hidden quantum correlations for which the maximal values allowed by quantum theory can be reached; in particular, we explicitly show that an open bipartite qubit system under collective decoherence can exhibit the phenomenon of maximal hidden quantum correlations. To the best of our knowledge, this is the first time that a physical two-qubit state is shown to exhibit this type of maximal hidden correlations (the only
known example of such a nature is the qubit-qutrit erasure state [41, 42]). We focus on entanglement, and three sub-entanglement correlations which respect the strict hierarchy: Bell-nonlocality → EPR-steering → usefulness for teleportation → entanglement [43]. Specifically, we provide a scenario of sudden birth of entanglement for which the aforementioned sub-entanglement properties are not present in the bare system Markovian dynamics, but that can still be revealed by means of local filtering operations. Furthermore, that these hidden quantum correlations are in fact maximal. In this way, the physical system in consideration displays a type of all from nothing behaviour, both in terms of the correlations it can generate via local filtering and from the temporal evolution it undergoes.

OPEN SYSTEM DYNAMICS

We start by addressing the physical scenario which concerns two atom-like qubits in a common reservoir. Allowing the interaction of the two qubits with this bath, coherent and incoherent effects can be covered in a Markovian description of the quantum dynamics for the system Hamiltonian $H (\hbar = 1)$ [44]:

$$\dot{\rho}(t) = i[\rho(t), H] - \sum_{i,j=1}^{2} \frac{\Gamma_{ij}}{2} \left( \rho(t), \sigma_{+}^{(i)} \sigma_{-}^{(j)} \right) - 2 \sigma_{-}^{(i)} \rho(t) \sigma_{+}^{(j)},$$

with $H = -\frac{1}{2} \omega_{1} \sigma_{z} \otimes \mathbb{1} - \frac{1}{2} \omega_{2} \mathbb{1} \otimes \sigma_{z} + \frac{\mathcal{V}}{2} \left( \sigma_{x} \otimes \sigma_{x} + \sigma_{y} \otimes \sigma_{y} \right)$, $\mathcal{V}$ is the dipole-dipole interaction; the spontaneous emission $\Gamma_{ij} \equiv \gamma$. $\sigma_{i}$ are the Pauli matrices, and $\sigma_{+}^{(i)} = |1\rangle \langle 0| \ (\sigma_{-} = |0\rangle \langle 1|)$ the raising (lowering) operator in the computational basis $\{|0\rangle, |1\rangle\}$ for $i$-th qubit. This choice is motivated by the fact that, as we shall see, our main result holds even in the absence of a direct dipole-dipole interaction between the atoms or quantum emitters, and that structured environments (e.g., nanoplasmonic waveguides) can be used for the results to be applied in the “long-distance” separated qubits regime.

We now consider the initial state of the system in an X-form:

$$\rho(0) = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}.$$  

By making $b = c, z \in \mathbb{R} \ (\Im(z) = 0)$, and $w = 0$ (for the sake of simplicity), the solution of Eq. (1) is independent of the dipolar interaction $\mathcal{V}$, and the relevant parameter becomes the collective damping $\gamma$. Making $\gamma = k\tau$, $k \in (0, 1]$, we find the following analytical solution (see Appendix A for a more general analytical solution of the master equation):

$$a(t) = 1 - 2b(t) - d(t), \quad c(t) = b(t),$$

$$d(t) = \frac{1}{4} (1 - p) e^{-2\Gamma t},$$

$$b(t) = \frac{e^{-2\Gamma t}}{4(1 - k^2)} \left\{ 2 e^{i\Gamma t} \left[ (p(k^2 + k - 1) - k) \sinh(\Gamma kt) \\
+ \left( 1 - k^2 p \right) \cosh(\Gamma kt) \right] - (1 + k^2) (1 - p) \right\},$$

$$z(t) = \frac{e^{-2\Gamma t}}{2(1 - k^2)} \left\{ e^{i\Gamma t} \left[ (p(1 - k^2) + (1 - p) k) \cosh(\Gamma kt) \\
- \left( 1 - k^2 p \right) \sinh(\Gamma kt) \right] + kp - k \right\}.$$  

STANDARD QUANTUM CORRELATIONS

We now explore the quantum correlations of the physical system by employing analytical functions as follows: for entanglement we consider the concurrence [45], depicted as the function $C(\rho)$. For Bell-nonlocality we consider the violation of the Clauser-Horn-Shimony-Holt (CHSH) inequality by means of a renormalisation of the Horodecki criterion [46], as the function $B(\rho)$. For EPR-steering we consider the violation of the so-called $F_3$ inequality by means of a renormalisation of the Costa-Angelo criterion [47], as the function $D(\rho)$. The explicit definitions of these functions can be found in Appendix B. In short, the criterion for each property has been renormalised into a function $f$ that satisfies the properties: i) $0 \leq f(\rho) \leq 1$, ii) a state not displaying the property achieves $f(\rho) = 0$, and iii) a state achieving maximal correlations, which for two qubits is the singlet state (up to local unitaries), gets $f(\psi^-) = 1$.

The quantum correlations of the open system of interest (1) can then explicitly be calculated using these functions. It turns out that although there is a sudden birth of entanglement, this entanglement is weak, as it does not allow the sudden birth of additional sub-entanglement correlations. Explicitly, this means that during the whole temporal evolution of the system we have: $B(\rho(t)) = 0$ (Bell-nonlocality), $B_{F_3}(\rho(t)) = 0$ (EPR-steering), and $D(\rho(t)) = 0$ (usefulness for teleportation). We now address the effect on the system’s correlations when two observers, say Alice and Bob, implement local filtering operations, with the goal of trying to reveal hidden quantum correlations, if present at all.

HIDDEN QUANTUM CORRELATIONS

We now explore the effect of local filtering operations when potentially being implemented at each point of the dynamics. Observers Alice and Bob are interested in
maximising the quantum correlations of the state $\rho(t)$ ($\rho$ from now on) by means of local filtering operations which effectively transform the state $\rho$ into a state $\rho'$, in a procedure that works as follows. Alice and Bob share the two-qubit state $\rho$ and they have access to local binary measurements $E_W = \{E_W^{0}, E_W^{1}\}$, $W \in \{A, B\}$ where $E_W^{0} = 1 - E_W^{1}$, $E_W^{1} = f_W^† f_W$, with the complex matrices $f_W$ satisfying $f_W^† f_W \leq 1$. We can check that the $E_W$’s are valid local binary POVMs; $E_W^{1} \geq 0$, $\sum_W E_W = 1$, $W \in \{A, B\}$. The matrices $f_W$ can be thought as the associated Kraus operators so that after measuring the system, one of the possible four post-measured states is the unnormalised state $(f_A \otimes f_B)\rho(f_A \otimes f_B)^†$. The realisation of this particular final state occurs with probability $p = \text{Tr}[(f_A^† f_A \otimes f_B^† f_B)\rho]$ and therefore, Alice and Bob would repeat the experiment enough times so that to keep the post-measured state only when it has ended in the target state, and thus effectively implementing a local filtering process. A general local filtering operation can then be written as $\rho' = \frac{1}{p}(f_A \otimes f_B)\rho(f_A \otimes f_B)^†$, with $f_A$ and $f_B$ arbitrary positive semidefinite operators with $f_W^† f_W \leq 1$. The operators $f_A, f_B$ are termed local filters and the procedure as a whole is an SLOCC.

Alice and Bob now want to find appropriate local filters such that they increase the different measures of quantum correlations. In particular, it turns out that non-invertible operations always decrease entanglement [28] and therefore, we further constrain the set of local filters (in addition to $f_W^† f_W \leq 1$) to satisfy $f_A, f_B \in \text{GL}(2, \mathbb{C})$, the group of invertible 2 × 2 complex matrices. Furthermore, amongst all possible local filtering operations, here we address the Kent-Linden-Massar (KLM) SLOCC transformation [27], the KLM-SLOCC from now on. This particular SLOCC has the crucial property of transforming any state $\rho$ into its Bell-diagonal unique normal form [27], which we address as $\rho_{\text{BD}}^{\text{SLOCC}}$ [28, 29]. The KLM-SLOCC is the optimal local filtering transformation for simultaneously maximising the quantum correlations of: concurrence [28], usefulness for teleportation [29], and the violation of the CHSH-inequality for Bell-nonlocality [30]. The KLM-SLOCC then effectively acts as $\rho \rightarrow \rho_{\text{BD}}^{\text{SLOCC}}$.

Given an arbitrary $\rho$, it is then possible to derive analytic functions for exploring the quantum correlations on $\rho_{\text{BD}}^{\text{SLOCC}}$ in terms of the original state $\rho$. These correlations are known as the hidden quantum correlations of the original $\rho$, and we now define functions describing each property as follows: HB($\rho$) (hidden CHSH-nonlocality) [48], HF$_3$($\rho$) (hidden F$_3$-steering) [49], HD($\rho$) (hidden usefulness for teleportation) [50], and MEC($\rho$) (maximum extractable entanglement) [27][51]. Furthermore, it turns out that these last two quantities are in fact equal, HD($\rho$) = MEC($\rho$) [50] and therefore, we will only address HD($\rho$). Further details and the explicit definitions of these functions are shown in Appendix C.

### MAIN RESULT

The exact formulae for concurrence, the sub-entanglement correlations, and their respective hidden counterparts can then be calculated for arbitrary two-qubit states and particularly, for the two-qubit scenario given by the analytical dynamics in Eq. (7) ($w = 0$). The standard correlations and hidden correlations of interest are then explicitly given as shown in Table I. In spite of the fact that the considered dynamics can exhibit the well known sudden death, death and revival, and sudden birth of entanglement, we present our main result on the latter phenomenon. In short, after a sudden birth of entanglement by a small amount, all the sub-entanglement hidden quantum correlations do also exhibit birth and, more interestingly, they all achieve their maximum values, here represented by the value of 1. It is worth noting, as mentioned before, that all sub-entanglement standard correlations are identically zero during the whole dynamics.

In Fig. 1 we show the locally-filtering maximisation scenario for usefulness for teleportation, F$_3$-steering and CHSH-nonlocality at a collective damping $\gamma = \Gamma / 2$. The inset shows the sudden birth of concurrence. It has been done for ten steps of the initial (separable) state parameter $0 \leq p \leq 1 / 3$. The entanglement sudden birth phenomenon takes place (even though $p = 0$; the maximally mixed state). Concurrence achieves a small maximum value (up to $\sim 2.5 \times 10^{-2}$) and decays asymptotically to zero. This weak entanglement behaviour however, allows the maximisation of the sub-entanglement properties in the system by means of local filtering; hidden usefulness for teleportation, hidden F$_3$-steering and hidden CHSH-nonlocality suddenly appear and go to their maximum value (HB $\rightarrow$ HBF$_3$ $\rightarrow$ HD $\rightarrow$ 1) even though they all are completely zero along the whole dynamics before filtering. This result holds for all the spectrum of $p \in [0, 1]$. In particular, for $p > 1 / 3$, after concurrence exhibits the sudden death and revival phenomena, all the sub-entanglement hidden quantum correlations get their maximum. As shown in Fig. 1, the sudden birth takes place at different times for distinct correlations, in complete agreement with the well known hierarchy amongst these correlations [43], with hidden CHSH-nonlocality exhibiting the more delayed sudden birth phenomenon.

It is worth noting that this extreme hidden phenomenon holds for any $\gamma = k \Gamma$ with $k \neq 0$ (similar results are obtained for negative values of $\gamma$). The particular case $\gamma = \Gamma$ corresponds to qubits very close to each other and is not a desirable physical scenario for exploring spatially-separated correlations. The case $k \rightarrow 0$ on the other hand, is a very interesting scenario allowing enhanced correlations in long distance-separated qubits. An important fact in the range $k \in (0, 1)$ where the maximisation takes place, is that the time of the sudden birth phe-
nomenon for the hidden correlations strongly depends on the value of $\gamma$. Hence, the stronger the collective damping is the earlier the correlations appear in the dynamics (not shown). The specific case $k = 0$, which can be thought of as two particles interacting with independent reservoirs, is not considered here because the only way to get maximal correlations by filtering occurs for initial maximally entangled states as previously shown in [39]: the reason being that the Bell-diagonal state recovered after filtering coincides with the same initial entangled state.

PHIG. 1. Hidden quantum correlations: hidden CHSH-nonlocality (HB), hidden $F_3$-steering (HB$F_3$), and hidden usefulness for teleportation (HD). The dynamics portray $\gamma = \Gamma/2$ and initial separable states $\rho \in [0, 1/3]$. The inset shows the sudden birth of entanglement (concurrency) achieving values of order $10^{-2}$, and the unfiltered sub-entanglement quantum correlations not being present (red dashed line at zero), meaning that, for all $t$: $B(\rho(t)) = 0$ (CHSH-nonlocality), $BF_3(\rho(t)) = 0$ ($F_3$-steering), and $D(\rho(t)) = 0$ (usefulness for teleportation). Notwithstanding this, the filtered correlations achieve a sudden birth of maximal hidden quantum correlations (see the horizontal blue dashed line at 1, as a guide to the eye).

CONCLUSIONS

We have proven that physical systems undergoing a Markovian dissipative quantum dynamics can exhibit the phenomenon of sudden birth of maximal hidden quantum correlations. We have shown that atoms or quantum emitters undergoing a sudden birth of entanglement (of order $10^{-2}$, measured in terms of the concurrence), do not allow the birth of sub-entanglement correlations in the form of: CHSH-inequality violation for Bell-nonlocality, $F_3$-inequality violation for EPR-steering and usefulness for teleportation. Notwithstanding this, we have proven that this apparently not-so-useful dynamics is actually in possession of strong quantum correlations, hidden from us, that can be revealed by means of local filtering operations. Specifically, that at any point of the dynamics, there exist local filters that transform the state to a new state exhibiting these three forms of correlations. These hidden quantum correlations are, in actuality, the maximal possible allowed by quantum theory, i.e., those of the maximally entangled pure two-qubit state (the singlet, up to local unitaries).

Our results have potential application in real quantum registers as they could possibly be tested, for example, with artificial atoms [52], single molecules in dispersive media [19, 53–57], in tailored nanostructures such as plasmonic waveguides [55, 58–61], or in individual self-assembled supramolecular nanofibres [62], to cite but a few. The presence of the collective damping parameter is crucial for the quantum correlations maximisation to occur. It is also worth noting, that scenarios with long-distance separated quantum emitters are allowed as well, due to it is not mandatory to invoke the dipolar interaction. For example, in the case of plasmonic waveguides, inter-emitter distances equal to or greater than the operational wavelength ($\sim 1$ $\mu$m) [59, 60] should allow for the observation of the phenomenon here reported.

Another relevant scenario that deserves special attention to explore this maximal phenomenon is the case of a quantum system undergoing a non-Markovian dynamics. As it is well-known, death and revival behaviour of entanglement naturally arise as a consequence of non-Markovianity [63, 64]. A discussion of hidden quantum correlations within a non-Markovian evolution (under both collective and independent qubit coupling to a bath), as well as the introduction of a unified framework for quantifying both standard and hidden quantum cor-

| TABLE I. Analytical functions for standard quantum correlations (left column) and hidden quantum correlations (right column) in terms of the density matrix elements, where $I_{\pm}(t) = \sqrt{b(t)c(t)} \pm \sqrt{a(t)d(t)}$, $c(t) = a(t) + d(t) - b(t) - c(t)$, and $a(t), b(t), c(t), d(t), z(t)$ are as defined in the main text (see appendix A for the solution to more general initial states, $\omega \neq 0$). |
|-----------------------------|-----------------------------|
| **Standard quantum correlations** | **Hidden quantum correlations** |
| CHSH-nonlocality $B(\rho(t)) = \max \left\{ 0, (z(t))^2 + 4|z(t)|^2 - 1 \right\}$ | HB$F_3(\rho(t)) = \max \left\{ 0, \frac{(I_{+}(t))^2 + 2|z(t)|^2}{(I_{+}(t))^2} - 1 \right\}$ |
| $F_3$-steering $BF_3(\rho(t)) = \max \left\{ 0, \frac{(z(t))^2 + 8|z(t)|^2}{2} - 1 \right\}$ | HBF$F_3(\rho(t)) = \max \left\{ 0, \frac{(I_{+}(t))^2 + 2|z(t)|^2}{(I_{+}(t))^2} - 1 \right\}$ |
| Teleportation $D(\rho(t)) = 2 \max \left\{ 0, |z(t)| - \frac{b(t)+c(t)}{2} \right\}$ | HD$F(\rho(t)) = \frac{1}{r_{\pm}(t)} \max \left\{ 0, |z(t)| - \sqrt{a(t)d(t)} \right\}$ |
| Entanglement $C(\rho(t)) = 2 \max \left\{ 0, |z(t)| - \sqrt{a(t)d(t)} \right\}$ | MEC$F(\rho(t)) = \text{HD}(\rho(t))$ |
relations is part of an upcoming work [65].

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Appendix A: Full analytical solution of the master equation

The fully analytical solution of the master equation (1) for initial X-form states ($w \neq 0$) reads:

$$
\rho(t) = \begin{pmatrix}
    a(t) & 0 & 0 & w(t) \\
    0 & b(t) & z(t) & 0 \\
    0 & z^*(t) & b(t) & 0 \\
    w^*(t) & 0 & 0 & d(t)
\end{pmatrix}
$$

$$
a(t) = \frac{e^{-t(\gamma + 3\Gamma)}}{2(\gamma^2 - \Gamma^2)} \left[ 2(a + d)(\gamma^2 - \Gamma^2)e^{t(\gamma + 3\Gamma)} + c(e^{2t(\gamma + \Gamma)} - e^{t(2\gamma + 3\Gamma)} + e^{2\Gamma t}) + 2d(e^{(\gamma - \Gamma)^2}e^{2t(\gamma + \Gamma)} - (3\gamma^2 + \Gamma^2)e^{t(\gamma + \Gamma)} + (\gamma + \Gamma)^2e^{2\Gamma t}) - 2(\Gamma^2 - \gamma^2) \Re(z)(e^{2\Gamma t} - 1)e^{2\Gamma t} \right]
$$

$$
\begin{pmatrix}
    b(t) \\
    c(t) \\
    z(t) \\
    d(t) \\
    w(t)
\end{pmatrix} = \frac{1}{4(\gamma^2 - \Gamma^2)} \left[ 2de^{-t(\gamma + 2\Gamma)} \left( -2(\gamma^2 + \Gamma^2)e^{t(2\gamma + \Gamma)} + (\gamma - \Gamma)^2e^{t(2\gamma + \Gamma)} + (\gamma + \Gamma)^2e^{\Gamma t} \right) + 2(\gamma^2 - \Gamma^2)e^{t(\gamma - \Gamma)^2} + 2\gamma \Gamma e^{\gamma \Gamma t} \right]
$$

where $\omega_0 = \frac{\omega_1 + \omega_2}{2}$. 

Appendix B: Standard quantum correlations

An arbitrary two-qubit state $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ can be written as $\rho = \frac{1}{4} \sum_{i,j=0}^{1} R_{ij} \sigma_i \otimes \sigma_j$, with the real matrix $R_{ij} = \text{Tr}[(\sigma_i \otimes \sigma_j)\rho]$, $\sigma_0 = \mathbb{1}$ and $\{\sigma_i\}, i = 1,2,3$, the Pauli matrices. $R$ can be written as [28]:

$$R = \begin{pmatrix} 1 & b^T \\ a & T \end{pmatrix},$$

with the Bloch vectors $a = [a_i]$, $a_i = \text{Tr}[(\sigma_i \otimes \mathbb{1})\rho]$; $b = [b_i]$, $b_i = \text{Tr}[(\mathbb{1} \otimes \sigma_i)\rho]$, and correlation matrix $T = [T_{ij}], T_{ij} = \text{Tr}[\rho(\sigma_i \otimes \sigma_j)]$. We now consider some quantities of interest: the singular values of $T$, the eigenvalues of the product matrix $\rho \hat{\rho}$, in decreasing order, with $\hat{\rho} = (\sigma_y \otimes \sigma_y)\rho^* (\sigma_y \otimes \sigma_y)$, $\rho^*$ the complex conjugate of $\rho$ and, $\sigma_y$ the Pauli matrix. We now explicitly establish the quantum correlations considered in this work:

$$B(\rho) = \max \left\{ 0, s_1^2 + s_2^2 - 1 \right\},$$

$$\text{BF}_3(\rho) = \max \left\{ 0, \frac{s_1^2 + s_2^2 + s_3^2 - 1}{2} \right\},$$

$$D(\rho) = \max \left\{ 0, \frac{|s_1| + |s_2| - \chi|s_3| - 1}{2} \right\},$$

$$C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}.$$  

The maximal violation of the CHSH inequality by an arbitrary two-qubit state can be fully characterised by the Horodecki criterion [46], which establishes that $\rho$ violates the CHSH-inequality if and only if $M(\rho) = s_1^2 + s_2^2 > 1$. Similarly, the Costa-Angelo criterion [47] establishes the maximal violation of the so-called $F_3$-inequality for EPR-steering by arbitrary two-qubit states and reads $F_3(\rho) = s_1^2 + s_2^2 + s_3^2 > 1$. Considering the standard teleportation protocol [67] and the characterisation for arbitrary two-qubit states [43], the usual functions for exploring usefulness for teleportation include: the fidelity of teleportation $F(\rho) = \frac{1}{2} [\frac{1}{4} N(\rho) + 1]$, the singlet fraction $f(\rho) = \frac{1}{2} [1 + N(\rho)]$, with the function $N(\rho) = |s_1| + |s_2| - \chi|s_3|$. A state is said to be useful for teleportation when $N(\rho) > 1$ or $f(\rho) > \frac{1}{2}$ or $F(\rho) > \frac{2}{3}$. The metrics that we use in this work are renormalisations of these criteria.

Appendix C: Hidden quantum correlations

The KLM-SLOCC acts as $\rho \rightarrow \rho_{\text{UNF}}^{\text{BD}}$, which in terms of the R-picture this reads as $R \rightarrow R^{\text{BD}}_{\text{UNF}} = \text{diag}(1, -\sqrt{\nu_1/\nu_0}, -\sqrt{\nu_2/\nu_0}, -\sqrt{\nu_3/\nu_0})$, with $\nu_{i=0,1,2,3}$. The eigenvalues of the operator $\eta R_0 R^T$ in decreasing order and $\eta = \text{diag}(1, -1, -1, -1)$ [28, 48]. If we now replace the Bell-diagonal unique normal form of an arbitrary two-qubit state in the previous functions: Eq. 9, Eq. 10, Eq. 11, and Eq. 12, we obtain explicit measures for hidden quantum correlations as:

$$\text{HB}(\rho) = \max \left\{ 0, \frac{1}{2\nu_0} (\nu_1 + \nu_2 - 1) \right\}$$

$$\text{HB}_3(\rho) = \max \left\{ 0, \frac{1}{2\nu_0} (\nu_1 + \nu_2 + \nu_3 - 1) \right\},$$

$$\text{HD}(\rho) = \max \left\{ 0, \frac{1}{2\nu_0} (\sqrt{\nu_1} + \sqrt{\nu_2} + \sqrt{\nu_3} - 1) \right\}.$$  

Maximum extractable concurrence becomes equal to hidden usefulness for teleportation $\text{MEC}(\rho) = \text{HD}(\rho)$ [50]. A unified framework for quantifiers of both standard and hidden quantum correlations is addressed in an upcoming work [65].