Explicit solution for Stefan problem with latent heat depending on the position and a convective boundary condition at the fixed face using Kummer functions.

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Abstract

An explicit solution of a similarity type is obtained for a one-phase Stefan problem in a semi-infinite material using Kummer functions. Motivated by [D.A. Tarzia, Relationship between Neumann solutions for two phase Lamé-Clapeyron-Stefan problems with convective and temperature boundary conditions, Thermal Sci. (2016) DOI 10.2298/TSCI140607003T, In press], and [Y. Zhou, L.J. Xia, Exact solution for Stefan problem with general power-type latent heat using Kummer function, Int. J. Heat Mass Transfer, 84 (2015) 114-118], we consider a phase-change problem with a latent heat defined as a power function of the position with a non-negative real exponent and a convective boundary condition at the fixed face \( x = 0 \). Existence and uniqueness of the solution is proved. Relationship between this problem and the problems already solved by Zhou and Xia with temperature and flux boundary condition is analysed. Furthermore it is studied the limit behaviour of the solution when the coefficient which characterizes the heat transfer at the fixed boundary tends to infinity. Numerical computation of the solution is done over certain examples, with a view to comparing this results with those obtained by general algorithms that solve Stefan problems.

Keywords: Stefan problem, Phase-change processes, Variable latent heat, Convective boundary condition, Kummer function, Explicit solution, Similarity solution.

Nomenclature

| Symbol | Description |
|--------|-------------|
| \( c \) | Coefficient that characterizes the heat flux at the fixed face, \([kg/s^{(5+\alpha)/2}]\). |
| \( d \) | Diffusivity coefficient, \([m^2/s]\). |
| \( h_0 \) | Coefficient that characterizes the heat transfer in condition (4), \([kg/(°C \cdot s^{\alpha/2})]\). |
| \( k \) | Thermal conductivity, \([W/(m \cdot °C)]\). |
| \( q, r, s, s_\infty \) | Position of the free front, \([m]\). |
| \( t \) | Time, \([s]\). |
| \( T \) | Temperature, \([°C]\). |
| \( T_0 \) | Coefficient that characterizes the temperature at the fixed face, \([°C/s^{\alpha/2}]\). |
| \( T_\infty \) | Coefficient that characterizes the bulk temperature, \([°C/s^{\alpha/2}]\). |
| \( x \) | Spatial coordinate, \([m]\). |

Greek symbols

| Symbol | Description |
|--------|-------------|
| \( \alpha \) | Power of the position that characterizes the latent heat per unit volume, dimensionless. |
| \( \gamma \) | Coefficient that characterizes the latent heat per unit volume, \([kg/(s^2 \cdot m^{\alpha+1})]\). |
| \( \lambda, \mu, \nu, \nu_\infty \) | Coefficient that characterizes the free interface, dimensionless. |
| \( \eta \) | Similarity variable in expression (6), dimensionless. |
| \( \Theta, \Psi, \Psi_\infty \) | Temperature, \([°C]\). |
1 Introduction.

The study of heat transfer problems with phase-change such as melting and freezing constitutes a broad field that has a wide engineering and industrial applications. Stefan problems can be formulated as models that represents thermal processes in phase transitions, where these phase transitions are characterized by heat diffusion and an exchange of latent heat. Due to their importance, they have been largely studied since the last century [1]-[5], [7], [12] and [16]. In [15] it was presented an extensive bibliography regarding this subject.

In the classical formulation of Stefan problems there are many assumptions on the physical factors involved in the phase-change that are taken into account in order to simplify the description of the process. One of this hypothesis, is to consider the latent heat as a constant. Although it is a reasonable assumption, it can be dropped in order to assume a variable latent heat. For example, it can be introduced a new kind of problems where the latent heat depends on the position. The physical bases of this particular problems can be found in the movement of a shoreline [18], in the ocean delta deformation [6] or in the cooling body of a magma [9].

In 1970, Primicerio [10] gave sufficient conditions for the existence and uniqueness of solution of a one-phase Stefan problem taking a latent heat as a general function of the position. Voller et al. [18] in 2004 found an exact solution for a one-phase Stefan problem considering the latent heat as a linear function of position.

On one hand, Salva and Tarzia [13] extended Voller’s work by considering the two-phase Stefan problem with a latent heat distributed linearly on the position. On the other hand, Zhou et al. in [19] generalized [18] by considering the one-phase Stefan problem with the latent heat as a power function of the position with an integer exponent. Recently Zhou and Xia [20] worked with the latter problem assuming a real non-negative exponent. They presented the explicit solution for two different problems defined according to the boundary conditions considered: temperature and flux. Explicit solutions for phase-change processes are given in [11] and [14]. A recent review on the subject can be find in [16].

Motivated by [17] and [20] we are going to analyse the existence and uniqueness of solution of a one-phase Stefan problem, considering an homogeneous semi-infinite material, with a latent heat as a power function of the position and a convective boundary condition at the fixed face \( x = 0 \). This problem can be formulated in the following way:

**Problem (P1):** Find the temperature \( \Psi(x,t) \) and the moving interface \( s(t) \) such that:

\[
\begin{align*}
\Psi_t(x,t) &= d\Psi_{xx}(x,t), \quad 0 < x < s(t), \quad t > 0, \\
\Psi(x,0) &= 0, \\
\Psi(s(t),t) &= 0, \quad t > 0, \\
k\Psi_x(0,t) &= h_0 t^{-1/2} \left[ \Psi(0,t) - T_\infty t^{\gamma/2} \right], \quad t > 0, \\
k\Psi_x(s(t),t) &= -\gamma s(t)^\alpha \dot{s}(t), \quad t > 0,
\end{align*}
\]

where \( \Psi \) is the temperature, \( s(t) \) is the moving interface, \( d \) is the thermal diffusion coefficient, \( k \) is the thermal conductivity, \( \gamma x^\alpha \) is the variable latent heat per unit of volume and the phase-transition temperature is zero. Condition (4) represents the convective boundary condition at the fixed face. \( T_\infty \) characterizes the bulk temperature at a large distance from the fixed face \( x = 0 \) and \( h_0 \) represents the heat transfer at the fixed face. Moreover \( \dot{s}(t) \) represents the velocity of the phase-change interface. We will work under the assumption that \( \gamma > 0, h_0 > 0 \) and \( T_\infty > 0 \) which corresponds to the melting case. In case of freezing it is sufficient to assume \( h_0 > 0, \gamma < 0 \) and \( T_\infty < 0 \).

The main objective of this article is to provide a detailed mathematical analysis of this heat transfer problem. In Section 2 we will use the similarity transformation technique in order to obtain an explicit solution for the problem governed by (1)−(5). In Section 3 we will present a relationship between the problem (P1) and the two related problems with temperature and heat flux boundary conditions on the fixed face \( x = 0 \) studied in [20]. Section 4 deals with the limit behaviour of the solution of (P1) when the coefficient that characterizes the heat transfer at the fixed face tends to infinity. Finally some computational examples will be shown in Section 5.
2 Explicit solution.

2.1 General case when $\alpha$ is a non-negative real exponent.

The following lemma have already been developed by Zhou-Xia in [20]. It is going to be useful in order to find solutions for the differential heat equation (1).

Lemma 2.1. [20]

a. Let
$$\Psi(x, t) = t^{\alpha/2} f(\eta), \quad \eta = \frac{x}{2\sqrt{sd}}$$
then $\Psi = \Psi(x, t)$ is a solution of the heat equation $\Psi_t(x, t) = d\Psi_{xx}(x, t)$, with $d > 0$ if and only if $f = f(\eta)$ satisfies the following ordinary differential equation:
$$\frac{d^2 f}{d\eta^2}(\eta) + 2\eta \frac{df}{d\eta}(\eta) - 2\alpha f(\eta) = 0.$$  
(7)

b. An equivalent formulation for equation (7), introducing the new variable $z = -\eta^2$, is given by:
$$z \frac{d^2 f}{dz^2}(z) + \left(\frac{1}{2} - z\right) \frac{df}{dz}(z) + \frac{\alpha}{2} f(z) = 0.$$  
(8)

c. The general solution of the ordinary differential equation (8), called Kummer’s equation, is given by:
$$f(z) = \hat{c}_{11} M\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right) + \hat{c}_{21} U\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right),$$
where $\hat{c}_{11}$ and $\hat{c}_{21}$ are arbitrary real constants and $M(a, b, z)$ and $U(a, b, z)$ are the Kummer functions defined by:
$$M(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s, \quad \text{where } b \text{ cannot be a non-positive integer},$$  
(10)
$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z).$$  
(11)

where $(a)_s$ is the pochhammer symbol defined by:
$$(a)_s = a(a+1)(a+2) \ldots (a+s-1), \quad (a)_0 = 1$$  
(12)

Remark 2.1. All the properties of Kummer’s functions to be used in the following arguments can be found in [5].

Remark 2.2. Taking into account definition (11) we can rewrite the general solution of (8) as:
$$f(z) = c_{11} M\left(-\frac{\alpha}{2}, \frac{1}{2}, z\right) + c_{21} z^{1/2} M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, z\right),$$
where $c_{11}$ and $c_{21}$ are real constants.

Our main outcome is given by the following theorem which ensures the existence and uniqueness of solution of the problem (P1) providing in addition, the corresponding explicit solution.

Theorem 2.1. There exists a unique solution of a similarity type for the one-phase Stefan problem (1)-(5) and it is given by:
$$\Psi(x, t) = t^{\alpha/2} \left[ c_{11} M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\eta^2\right) + c_{21} \eta M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\eta^2\right) \right]$$
(14)
$$s(t) = 2\nu \sqrt{dt}$$
(15)
where $\eta = \frac{x}{2\sqrt{dt}}$ and the constants $c_{11}$ and $c_{21}$ are given by:
\[ c_{11} = \frac{-\nu M \left( -\frac{\alpha}{2} + \frac{1}{2} \frac{3}{2} - \nu^2 \right)}{M \left( \frac{\alpha}{2} - \frac{1}{2} - \nu^2 \right)} c_{21}, \tag{16} \]
\[ c_{21} = \frac{-2 h_0 \sqrt{T_\infty} M \left( -\frac{\alpha}{2} - 1 \frac{1}{2} - \nu^2 \right)}{\left[ k M \left( -\frac{\alpha}{2} + 1 \frac{1}{2}, \frac{3}{2}, -\nu^2 \right) + 2 \sqrt{\tilde{d} h_0} \nu M \left( -\frac{\alpha}{2} + 1 \frac{3}{2} - \nu^2 \right) \right]} \tag{17} \]

and the dimensionless coefficient \(\nu\) is obtained as the unique positive solution of the following equation:

\[ \frac{h_0 T_\infty}{\gamma 2^{\alpha/(\alpha+1)} d^{1/2}} f_1(x) = x^{\alpha+1}, \quad x > 0. \tag{18} \]

in which:

\[ f_1(x) = \frac{1}{M \left( \frac{\alpha}{2} + 1 \frac{1}{2}, \frac{3}{2}, -\nu^2 \right) + 2 \sqrt{\tilde{d} h_0} \nu M \left( \frac{\alpha}{2} + 1 \frac{3}{2}, -\nu^2 \right)} \tag{19} \]

**Proof.** The general solution of equation (1), based on Kummer functions is given by the Lemma 2.1. According to Remark 2.2 we can write:

\[ \Psi(x, t) = t^{\alpha/2} \left[ c_{11} M \left( -\frac{\alpha}{2} - 1 \frac{1}{2}, -\nu^2 \right) + c_{21} \eta M \left( -\frac{\alpha}{2} + 1 \frac{3}{2} - \nu^2 \right) \right], \tag{20} \]

where \(\eta = \frac{x}{2 \sqrt{d} t}\) and where \(c_{11}\) and \(c_{21}\) are coefficients that must be determined in order to ensure that \(\Psi\) satisfies conditions (3)-(5).

Furthermore, condition (3) together with (20) implies that the free boundary should take the following form:

\[ s(t) = 2 \nu \sqrt{d} t. \tag{21} \]

where \(\nu\) is a constant that have to be determined.

From equations (3), (20) and (21) we obtain that:

\[ \Psi(s(t), t) = t^{\alpha/2} \left[ c_{11} M \left( -\frac{\alpha}{2} - 1 \frac{1}{2}, -\nu^2 \right) + c_{21} \nu M \left( \frac{\alpha}{2} + 1 \frac{3}{2}, -\nu^2 \right) \right] = 0, \tag{22} \]

and isolating \(c_{11}\) we arrive to (16).

On the other hand, we know that the derivative of the Kummer functions (see [8]) are:

\[ \frac{d}{dz} M(a, b, z) = \frac{a}{b} M(a + 1, b + 1, z), \tag{23} \]
\[ \frac{d}{dz} z^{b-1} M(a, b, z) = (b - 1) z^{b-2} M(a, b - 1, z), \tag{24} \]

and therefore by using (23) and (24) we get that:

\[ \Psi_x(x, t) = \frac{t^{(\alpha-1)/2}}{2 \sqrt{d}} \left[ c_{11} \eta M \left( \frac{\alpha}{2} + 1 \frac{3}{2} - \eta^2 \right) + \frac{c_{21}}{2} M \left( \frac{\alpha}{2} + 1 \frac{1}{2}, -\eta^2 \right) \right], \tag{25} \]

and in consequence, condition (4) is satisfied if and only if:

\[ k \frac{t^{(\alpha-1)/2}}{2 \sqrt{d}} c_{21} = h_0^{-1/2} \left[ t^{\alpha/2} c_{11} - T_\infty t^{\alpha/2} \right], \tag{26} \]

that is:

\[ k \frac{c_{21}}{2 \sqrt{d}} = h_0 \left[ c_{11} - T_\infty \right]. \tag{27} \]
Replacing $c_{11}$ given by (16) into (27) we find (17).

Until now we have obtained $c_{11}$ and $c_{21}$ as functions of $\nu$, arriving to the expressions (16)- (17). By combining equations (16), (17), (21) and (25) and using the following identities (20):

$$M(a, b, z) = e^z M(b - a, b, -z),$$  \tag{28}

$$e^{-\nu^2} = -2\alpha \nu^2 M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right) M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2\right) + M\left(-\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, -\nu^2\right) M\left(-\frac{\alpha}{2}, \frac{1}{2}, -\nu^2\right),$$  \tag{29}

we obtain that the Stefan condition (5) holds if and only if $\nu$ satisfies the equation:

$$kh_0 T_\infty \left[k M\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{1}{2}, x^2\right) + 2\sqrt{\alpha} h_0 x M\left(\frac{\alpha}{2} + 1, \frac{3}{2}, x^2\right)\right] = \gamma x^{\alpha+1/2} d^{(\alpha+1)/2}, \quad x > 0. \tag{30}$$

It means that $\Psi$ and $s$ defined in (14) and (15) constitute a solution of problem (1)-(5), with $c_{11}$ and $c_{22}$ given by equations (16) and (17) if and only if $\nu$, the unknown coefficient, verifies the equation (30).

Thus we have deduced an equality that must be satisfied by the positive coefficient $\nu$ and that can be written as:

$$\frac{h_0 T_\infty}{\gamma 2^\alpha d^{(\alpha+1)/2}} f_1(\nu) = \nu^{\alpha+1}, \tag{31}$$

where the real function $f_1$ is defined by (19).

The proof will be completed by showing the existence and uniqueness of solution to equation (31), i.e. equation (30), analysing the monotonicity of the left and the right hand side of this equality.

By using properties (23) and (24) of Kummer functions we can observe that:

$$f_1'(x) = - \left[2(\alpha + 1)x M\left(\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, x^2\right) + 2\sqrt{\alpha} h_0 M\left(\frac{\alpha}{2} + 1, \frac{1}{2}, x^2\right)\right] f_1^2(x) < 0, \quad \forall x > 0. \tag{32}$$

Therefore we can assure that $f_1(x)$ is a decreasing function of $x$. Consequently, the left hand side of (31), is also a decreasing function of $x$ that goes from $\frac{h_0 T_\infty}{\gamma 2^\alpha d^{(\alpha+1)/2}} > 0$ to 0 when $x$ increases from 0 to $+\infty$. Meanwhile the right hand side of (31) is an increasing function of $x$ that increases from 0 to $+\infty$, when $x$ goes from 0 to $+\infty$.

The above assertions allow us to conclude that there always exists a unique positive solution $\nu$ of (31) regardless of the data. Then we obtain that the problem (1)-(5) always has a unique solution given by (14)-(19).

\[\blacksquare\]

2.2 Special case when $\alpha$ is an integer.

In the special case that $\alpha$ is a positive integer, denoted by $n$, the Kummer functions are related with the iterated integral of the complementary error function and with the gamma function as follows (see [20], [8]):

$$M\left(-\frac{n}{2}, \frac{1}{2}, -z^2\right) = 2^n \Gamma\left(\frac{n}{2} + 1\right) E_n(z), \tag{33}$$

$$z M\left(-\frac{n}{2} + \frac{1}{2}, \frac{3}{2}, -z^2\right) = 2^{n-1} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) F_n(z), \tag{34}$$

where:

$$E_n(z) = \frac{[i^n erf(z) + i^n erf(-z)]}{2}, \tag{35}$$

$$F_n(z) = \frac{[i^n erf(-z) - i^n erf(z)]}{2}. \tag{36}$$

5
Such properties allow us to transform the solution of the problem (1)-(5), given by Theorem 2.1, in case that \( \alpha = n \in \mathbb{N} \) into:

\[
\Psi(x, t) = \frac{-t^{n/2}h_0T_\infty\sqrt{d}\Gamma\left(\frac{n}{2} + 1\right) \left[F_n(\eta)E_n(\nu) - F_n(\nu)E_n(\eta)\right]}{k\Gamma\left(\frac{n}{2} + 1\right)E_n(\nu) + \sqrt{d}h_0\Gamma\left(\frac{n}{2} + 1\right)F_n(\nu)},
\]

(37)

where \( \eta = \frac{x}{2\sqrt{d}t} \) and \( \nu \) is the unique positive solution of the following equation:

\[
\frac{h_0T_\infty}{\gamma d(n+1)/22n\left[\Gamma\left(\frac{n}{2} + 1\right)E_n(x) + \sqrt{d}h_0k\Gamma\left(\frac{n}{2} + 1\right)F_n(x)\right]} = x^{n+1}e^{x^2}.
\]

(39)

**Remark 2.3.** Taking into account that \( E_0(x) = 1 \) and \( F_0(x) = \text{erf}(x) \), in the case \( \alpha = 0 \), functions (37)-(38) and equation (39) reduce to:

\[
\Psi(x, t) = \frac{-h_0T_\infty\sqrt{d}\sqrt{\pi}\left[\text{erf}\left(-\frac{x}{2\sqrt{d}t}\right) - \text{erf}(\nu)\right]}{k\left[1 + \sqrt{d}\pi h_0k\text{erf}(\nu)\right]},
\]

(40)

\[
s(t) = 2\nu \sqrt{d}t,
\]

(41)

where \( \nu \) is the unique positive solution of:

\[
\gamma d^{(n+1)/2}2n\left[1 + \sqrt{d}\pi h_0k\text{erf}(x)\right] = xe^{x^2}, \quad x > 0.
\]

(42)

It can be noted that this solution is in accordance with the solution given by Tarzia [17] in case that initial temperature \( T_i = 0 \) (reducing the two-phase Stefan problem into a one-phase Stefan problem).

### 3 Equivalence between problems with temperature, flux and convective boundary conditions at the fixed face \( x = 0 \).

We denote by (P1) the problem governed by (1)-(5). If we change the convective condition (4) by a temperature boundary condition we obtain a problem that will be denote by (P2) whose explicit solution was presented in [20]. Similarly we can define the problem (P3) changing condition (4) by a flux boundary condition, whose exact solution was also presented by Zhou and Xia in [20]. It means that we have defined:

**Problem (P2):** Find the temperature \( \Theta(x,t) \) and the moving interface \( r(t) \) that satisfies:

\[
\Theta_t(x,t) = d\Theta_{xx}(x,t), \quad 0 < x < r(t), \quad t > 0,
\]

(43)

\[
r(0) = 0,
\]

(44)

\[
\Theta(r(t), t) = 0, \quad t > 0,
\]

(45)

\[
\Theta(0, t) = T_0t^{\alpha/2}, \quad t > 0,
\]

(46)

\[
k\Theta_x(r(t), t) = -\gamma r(t)^{\alpha} \dot{r}(t), \quad t > 0,
\]

(47)

where the solution according to [20] is given by:
\[ \Theta(x,t) = t^{\alpha/2} \left[ c_{12} M \left( -\frac{\alpha}{2} \frac{1}{2} - \eta^2 \right) + c_{22} \eta M \left( -\frac{\alpha}{2} + \frac{1}{2} \frac{3}{2} - \eta^2 \right) \right], \quad (48) \]

\[ r(t) = 2\mu \sqrt{dt}, \quad (49) \]

where \( \eta = \frac{x}{2\sqrt{dt}} \) and the constants \( c_{12} \) and \( c_{22} \) are given by:

\[ c_{12} = T_0, \quad c_{22} = \frac{-T_0 M \left( -\frac{\alpha}{2} \frac{1}{2} - \mu^2 \right)}{\mu M \left( -\frac{\alpha}{2} + \frac{1}{2} \frac{3}{2} - \mu^2 \right)} \quad (50) \]

and the parameter \( \mu \) is the unique positive solution of the following equation:

\[ \frac{kT_0}{2^{\alpha+1} \gamma^{\alpha+2+1}} f_2(x) = x^{\alpha+1}, \quad x > 0 \quad (51) \]

with:

\[ f_2(x) = \frac{1}{x M \left( \frac{\alpha}{2} + \frac{3}{2}, x^2 \right)} \quad (52) \]

**Problem (P3):** Find the temperature \( T(x,t) \) and the moving interface \( q(t) \) such as:

\[ T_t(x,t) = dT_{xx}(x,t), \quad 0 < x < q(t), \quad t > 0, \]

\[ q(0) = 0, \quad (53) \]

\[ T(q(t),t) = 0, \quad t > 0, \quad (54) \]

\[ kT_x(0,t) = -ct^{\left(\alpha-1\right)/2} \quad t > 0, \quad (55) \]

\[ kT_x(q(t),t) = -\gamma q(t)^\alpha \dot{q}(t), \quad t > 0, \quad (56) \]

where the solution according to [20] is given by:

\[ T(x,t) = t^{\alpha/2} \left[ c_{13} M \left( -\frac{\alpha}{2} \frac{1}{2} - \eta^2 \right) + c_{23} \eta M \left( -\frac{\alpha}{2} + \frac{1}{2} \frac{3}{2} - \eta^2 \right) \right] \quad (58) \]

\[ q(t) = 2\lambda \sqrt{dt} \quad (59) \]

where \( \eta = \frac{x}{2\sqrt{dt}} \) and the constants \( c_{13} \) and \( c_{23} \) are given by:

\[ c_{13} = \frac{-\lambda M \left( -\frac{\alpha}{2} + \frac{1}{2} \frac{3}{2} - \lambda^2 \right)}{M \left( -\frac{\alpha}{2}, \frac{1}{2} - \lambda^2 \right)} c_{23}, \quad c_{23} = \frac{-2c\sqrt{d}}{k} \quad (60) \]

and \( \lambda \) is the unique positive solution of the following equation:

\[ \frac{c}{\gamma^{2\alpha+2+1} d^{\alpha+1}/2} f_3(x) = x^{\alpha+1}, \quad x > 0 \quad (61) \]

where

\[ f_3(x) = \frac{1}{M \left( \frac{\alpha}{2} + \frac{1}{2} \frac{3}{2}, x^2 \right)} \quad (62) \]

Once we have defined our three problems, we are going to prove the equivalence between them. We refer to equivalence in the sense that if the data of both problems satisfy certain relationship then they have the same solution.
Theorem 3.1. The free boundary problems (P1) and (P2) are equivalents. Moreover we have:

a) the relationship between the datum $T_0$ of problem (P2) with the data $T_\infty$ and $h_0$ of the problem (P1) is given by:

$$T_0 = \frac{2\sqrt{dh_0 T_\infty} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)}{kM \left( -\frac{\alpha}{2}, -\nu^2 \right) + 2\sqrt{dh_0} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)}.$$  \hspace{1cm} (63)

where $\nu$ is the parameter that characterizes the moving interface in problem (P1) and it is given as the unique solution of the equation (68).

b) the relationship between the data $h_0$ and $T_\infty$ of problem (P1) with the datum $T_0$ of the problem (P2) is given by $T_\infty > T_0$ and:

$$h_0 = \frac{-kT_0 M \left( -\frac{\alpha}{2}, -\frac{1}{2}, -\mu^2 \right)}{2\sqrt{d(T_0 - T_\infty)} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)}.$$  \hspace{1cm} (64)

where $\mu$ is the parameter that characterizes the moving interface in problem (P2) and it is given as the unique solution of the equation (61).

Proof.

a) First, we solve the free boundary problem (P1) and we obtain $\Psi(x,t)$, and $s(t)$ through equations (13) - (19). If we compute the temperature of this problem at the fixed face $x = 0$ we get:

$$\Psi(0,t) = t^{\alpha/2}c_{11} = t^{\alpha/2} 2\sqrt{dh_0 T_\infty} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right) \frac{2\sqrt{dh_0 T_\infty} \nu M \left( -\frac{\alpha}{2}, -\nu^2 \right) + 2\sqrt{dh_0} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)}{kM \left( -\frac{\alpha}{2}, -\nu^2 \right) + 2\sqrt{dh_0} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)},$$  \hspace{1cm} (65)

so it leads us to define $T_0$ as $\frac{\Psi(0,t)}{t^{\alpha/2}}$ arriving to (63). Observe that $\nu$ is the parameter which defines $s(t)$ (the moving interface of problem (P1)) and it is the unique solution of (18).

Considering the problem (P2) with this particular $T_0$, defined by (63), we obtain that the temperature $\Theta(x,t)$ and the moving interface $r(t)$ are given by (15) - (22). From these equations we have that the parameter $\mu$ which characterizes $r(t)$ is the unique solution of:

$$k^{2\alpha + 1} d^{\alpha/2 + 1} \frac{2\sqrt{dh_0 T_\infty} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)}{kM \left( -\frac{\alpha}{2}, -\nu^2 \right) + 2\sqrt{dh_0} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)} f_2(x) = x^{\alpha + 1}, \quad x > 0.$$  \hspace{1cm} (66)

If we replace $x$ by $\nu$ in equation (66) we obtain equation (18) whose unique solution is $\nu$. So we can conclude that $\nu$ is a solution of (66). Therefore we get that $\nu = \mu$, and $r(t) = s(t)$. Working algebraically we obtain that the temperature of both problems are equal, i.e $\Theta(x,t) = \Psi(x,t)$. In other words, the problem (P1) has the same solution of problem (P2) when $T_0$ is defined in function of the data of (P1) as (63).

b) Conversely, we consider the problem (P2), and we solve it using equations (15) - (22), we obtain $\Theta(x,t)$ and $r(t)$. If we compute $\Theta(0,t)$ and $\Theta_s(0,t)$, the coefficient $h_0$ can be defined in order that convective condition (4) is satisfied. That is to say:

$$h_0 = \frac{k\Theta_s(0,t)}{t^{-1/2} \left[ T(0,t) - T_\infty \right]^{\alpha/2}}$$  \hspace{1cm} (67)

$$= \frac{-k t^{(\alpha - 1)/2} 2\sqrt{dh_0} \nu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)}{2\sqrt{d} \mu M \left( -\frac{\alpha}{2} + \frac{1}{2}, \frac{3}{2}, -\nu^2 \right)} t^{-1/2} \left[ t^{\alpha/2} T_0 - t^{\alpha/2} T_\infty \right]$$  \hspace{1cm} (68)
Theorem 3.2. The free boundary problems (P1) and (P3) are equivalents. Moreover we have:

a) the relationship between the datum \( c \) of problem (P2) with the data \( T_\infty \) and \( h_0 \) of the problem (P1) is given by:

\[
c = h_0T_\infty M \left( -\frac{\alpha}{2}, 1, -\nu^2 \right) \bigg[ M \left( -\frac{\alpha}{2}, 1, -\lambda^2 \right) + 2\sqrt{dh_0} \kappa \nu M \left( -\frac{\alpha}{2}, 1, 3, -\nu^2 \right) \bigg]^{-1}
\]

where \( \nu \) is the parameter that characterizes the moving interface in problem (P1).

b) the relationship between the data \( h_0 \) and \( T_\infty \) of problem (P1) with the datum \( c \) of the problem (P3) is given by :

\[
T_\infty = \frac{2c\sqrt{d} \lambda M \left( -\frac{\alpha}{2}, 1, 3, -\lambda^2 \right)}{M \left( -\frac{\alpha}{2}, 1, -\lambda^2 \right)}
\]

\[
h_0 = \frac{-cM \left( -\frac{\alpha}{2}, 1, -\lambda^2 \right)}{2c\sqrt{d} \lambda M \left( -\frac{\alpha}{2}, 1, 3, -\lambda^2 \right) - T_\infty M \left( -\frac{\alpha}{2}, 1, -\lambda^2 \right)}
\]

where \( \lambda \) is the parameter that characterizes the moving interface in problem (P3).

Proof.

a) First, we solve the free boundary problem (P1) and we obtain \( \Psi(x, t) \), and \( s(t) \) through equations (70)–(73). If we compute the flux \( \Psi \) at the fixed face \( x = 0 \) we get:

\[
\Psi_x(0, t) = \frac{-t^{(\alpha-1)/2} h_0 T_\infty M \left( -\frac{\alpha}{2}, 1, -\nu^2 \right)}{k \left[ M \left( -\frac{\alpha}{2}, 1, -\nu^2 \right) + 2\sqrt{dh_0} \kappa \nu M \left( -\frac{\alpha}{2}, 1, 3, -\nu^2 \right) \right]}
\]

so it leads us to define \( c = -\frac{k\Psi_x(0, t)}{t^{(\alpha-1)/2}} \) as in (73). Observe that \( \nu \) is the parameter which defines \( s(t) \) (the moving interface of problem (P1)) and it is the unique solution of (18).
If we consider the problem (P3) with this particular \( c \) defined by (70), we obtain that the solution, it means the temperature \( T(x,t) \) and the moving interface \( q(t) \) are given by (53)- (52). From these equations we have that the parameter \( \lambda \) which characterizes \( q(t) \) is the unique solution of:

\[
h_0 T_\infty M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\nu^2 \right) \left[ M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\nu^2 \right) + \frac{2\sqrt{d}h_0}{k} \nu M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\nu^2 \right) \right] \frac{1}{\gamma^{2\alpha d(\alpha+1)/2}} f_3(x) = x^{\alpha+1}, \quad x > 0. \tag{74}
\]

If we replace \( x \) by \( \nu \), we reduce equation (74) into (18), and as \( \nu \) is the unique solution of (18), we deduce that \( \nu \) is a solution of (74). Therefore we get that \( \lambda = \nu \), and \( q(t) = s(t) \). Working algebraically we obtain that the temperature of both problems are equal, i.e \( T(x,t) = \Psi(x,t) \). In other words, the problem (P1) has the same solution of problem (P3) considering a \( c \) defined by (70).

b) Conversely, if we take the problem (P3), and we solve it using equations (58)-(62), we obtain \( T(x,t) \) and \( q(t) \). For convective condition (3) to happen, we compute \( T(0,t) \) and \( T_s(0,t) \) and define \( h_0 \) as:

\[
h_0 = \frac{kT_s(0,t)}{\ell - 1/2 \left[ T(0,t) - T_\infty \ell^{\alpha/2} \right]}
\]

\[
= \frac{k}{2\sqrt{d}} \left( -\frac{2\sqrt{d}}{k} \right) \left[ \frac{2\sqrt{d}}{k} \lambda M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2 \right) - T_\infty M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2 \right) \right]
\]

arriving to an \( h_0 \) given by (72). Observe that \( \lambda \) is the parameter that characterizes the moving interface \( q(t) \), which is the unique solution of (61).

Prescribing a \( T_\infty \) as in (71), we are able to ensure that \( h_0 > 0 \). Hence we can pose the problem (P3) with \( h_0 \) defined by (72). By equations (14)- (19) we obtain the temperature \( \Psi(x,t) \) and the moving interface \( s(t) = 2\nu \sqrt{dt} \). From (18) and taking into account the form of \( h_0 \) we get that \( \nu \) is the unique solution of:

\[
-cM \left( -\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2 \right) \left[ \frac{2\sqrt{d}}{k} \lambda M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2 \right) - T_\infty M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\lambda^2 \right) \right] \frac{T_\infty}{\gamma^{2\alpha d(\alpha+1)/2}} f_1(x) = x^{\alpha+1}. \tag{77}
\]

If we replace \( x \) by \( \lambda \), equation (77) reduces to equation (61). As \( \lambda \) is the unique solution of (61), we obtain that \( \lambda \) is a solution of (77). By uniqueness of solution of equation (77) we get that \( \nu = \lambda \). In consequence, if follows that \( s(t) = q(t) \) and \( \Psi(x,t) = T(x,t) \). It yields that the problem (P3) has the same solution of the problem (P1) when \( h_0 \) and \( T_\infty \) are defined from the data of (P3) by equations (21)- (22).

Thus we can conclude that (P1) and (P3) are equivalents.

\[\blacksquare\]

4 Limit behaviour

In this section we are going to analyse the behaviour of the problem (P1) when the coefficient \( h_0 \) that characterizes the heat transfer at the fixed face \( x = 0 \) tends to infinity. Due to the fact that the solution of this problem, i.e the temperature and the free boundary depends on \( h_0 \), we will rename them. Thus, we will consider \( \Psi_{h_0}(x,t) := \Psi(x,t) \) and \( s_{h_0}(t) := s(t) \) defined by equations (14)- (15), where \( c_{11} = c_{11}(h_0) \), \( c_{21} = c_{21}(h_0) \) and \( \nu = \nu_{h_0} \) is the unique solution of the following equation:

\[
\frac{h_0 T_\infty}{\gamma^{2\alpha d(\alpha+1)/2}} f_1(x, h_0) = x^{\alpha+1}, \quad x > 0. \tag{78}
\]

in which:
\[ f_1(x, h_0) = \frac{1}{M \left( \frac{x}{2} + \frac{1}{2}, x^2 \right) + 2 \frac{\sqrt{dh_0}}{k} x M \left( \frac{x}{2} + 1, \frac{3}{2}, x^2 \right)} \]  

(79)

On the other hand, let us consider a new problem (P4) defined in the following way:

**Problem (P4):** Find the temperature \( \Psi_\infty(x, t) \) and the moving interface \( s_\infty(t) \) that satisfies:

\[
\Psi_\infty(x, t) = d \Psi_\infty(x, t), \quad 0 < x < s_\infty(t), \quad t > 0,
\]

(80)

\[
s_\infty(0) = 0,
\]

(81)

\[
\Psi_\infty(s_\infty(t), t) = 0, \quad t > 0,
\]

(82)

\[
\Psi_\infty(0, t) = T_\infty t^{\alpha/2} \quad t > 0,
\]

(83)

\[
k \Psi_\infty'(s_\infty(t), t) = -\gamma s_\infty(t)^\alpha s_\infty'(t), \quad t > 0,
\]

(84)

As we can observe, this problem corresponds to a problem where a temperature boundary condition is imposed at the fixed face \( x = 0 \). Thus the solution according to [20] can be obtained from equations (48)-(52):

\[
\Psi_\infty(x, t) = t^{\alpha/2} \left[ c_{11,\infty} M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\eta^2 \right) + c_{21,\infty} \eta M \left( -\frac{\alpha}{2} + 1, \frac{3}{2}, -\eta^2 \right) \right],
\]

(85)

\[
s_\infty(t) = 2 \nu_\infty \sqrt{dt},
\]

(86)

where \( \eta = \frac{x}{2\sqrt{dt}} \) and the constants \( c_{12,\infty} \) and \( c_{22,\infty} \) are given by:

\[
c_{11,\infty} = T_\infty, \quad c_{21,\infty} = \frac{-T_\infty M \left( -\frac{\alpha}{2}, \frac{1}{2}, -\nu_\infty^2 \right)}{\nu_\infty M \left( -\frac{\alpha}{2} + 1, \frac{3}{2}, -\nu_\infty^2 \right)}
\]

(87)

and the parameter \( \nu_\infty \) is the unique positive solution of the following equation:

\[
\frac{k T_\infty}{2^{n+1} d_{\infty} / 2 + 1} f_2(x) = x^{\alpha+1}, \quad x > 0
\]

(88)

with:

\[
f_2(x) = \frac{1}{x M \left( \frac{\alpha}{2} + 1, \frac{3}{2}, x^2 \right)}.
\]

(89)

Once we have introduced the problems (P1) and (P4) we are able to state the following convergence theorem.

**Theorem 4.1.** The problem (P1) converges to problem (P4) when \( h_0 \) tends to infinity, i.e:

\[
\lim_{h_0 \to +\infty} P1 = P4
\]

(90)

In this context the term “convergence” means that:

\[
\left\{ \begin{array}{ll}
\lim_{h_0 \to +\infty} \nu_{h_0} &= \nu_\infty, \\
\lim_{h_0 \to +\infty} s_{h_0}(t) &= s_\infty(t), \quad \forall t > 0 \\
\lim_{h_0 \to +\infty} \Psi_{h_0}(x, t) &= \Psi_\infty(x, t), \quad \forall t > 0, x > 0.
\end{array} \right.
\]

(91)

**Proof.** Let us consider the problem (P1). We know that the parameter that characterizes the free boundary, \( \nu_{h_0} \), is the unique solution of equation (78). In order to obtain the limit of \( \nu_{h_0} \) it is necessary to study the convergence of equation (78) when \( h_0 \) goes to infinity. The limit of the left hand side function of (78) is:
This implies that equation (78) converges to equation (88) when \( h_0 \to \infty \). On one hand, we have that the limit of \( \nu_{h_0} \) must be a solution of equation (88). On the other hand, (88) has a unique solution \( \nu_\infty \). Thus it turns out that \( \lim_{h_0 \to \infty} \nu_{h_0} = \nu_\infty \). Once obtained this convergence, it is immediately that \( \lim_{h_0 \to \infty} s_{h_0}(t) = s_\infty(t), \forall t > 0 \). For the convergence of the temperature \( \Psi_{h_0}(x,t) \) to \( \Psi_\infty(x,t) \) when \( h_0 \to \infty \), it can be easily proved that: \( \lim_{h_0 \to \infty} c_{21}(h_0) = c_{21\infty} \) and \( \lim_{h_0 \to \infty} c_{11}(h_0) = c_{11\infty} \).

5 Numerical Computation

From Theorem 2.1 the solution of the problem (P1) is characterized by a parameter \( \nu \) defined as the unique solution of equation (18). This equation can be rewritten into the following way:

\[
F(x) = \frac{h_0 T_\infty}{2} \left[ \frac{1}{h_0 M \left( \frac{1}{2} + \frac{1}{2} x^2 \right)} + 2 \frac{\sqrt{d}}{k} x M \left( \frac{1}{2} + \frac{1}{2} x^2 \right) \right] = 0, \quad x > 0.
\]

where \( f_1(x) \) is defined by (19).

In order to approximate the unique root of the nonlinear equation defined above we can apply Newton’s method. Beginning with an estimate \( \nu_0 \) of \( \nu \), we define inductively:

\[
\nu_{k+1} = \nu_k - \frac{F(\nu_k)}{F'(\nu_k)}
\]

where

\[
F'(x) = \frac{h_0 T_\infty}{2} \left[ 2(\alpha + 1) x M \left( \frac{\alpha}{2} + \frac{3}{2}, \frac{3}{2}, x^2 \right) + 2 \frac{\sqrt{d h_0}}{k} M \left( \frac{\alpha}{2} + 1, \frac{1}{2}, x^2 \right) \right].
\]

We have implemented Newton’s Method using Matlab software. The main reason for choosing this programming language is that the Kummer function \( M(a, b, z) \) can be represented by the command ‘hypergeom’. The stopping criterion used is the boundedness of the absolute error \( |\nu_k - \nu_{k+1}| < 10^{-15} \). Without loss of generality we assume \( \gamma = d = k = 1 \). The following Figures 1 to 4 present the computational values obtained for \( \nu \) versus \( h_0 \) corresponding to different values of \( T_\infty \) and \( \alpha \).
We can observe that, in all cases $\nu$ varies monotonically increasing with respect to $h_0$. In addition it can be appreciated that as $h_0$ increases, $\nu$ tends to stabilize. This behaviour is in accordance with Theorem 4.1 which ensures the existence of a limit for $\nu := \nu_{h_0}$ when $h_0$ goes to infinity. For this reason, we also applied Newton’s method to the problem (P4) taking into account equations (88)-(89), using the same stopping criterion as above and taking $\gamma = d = k = 1$. In the next Figures 5 to 8, we compare the coefficients $\nu_{h_0}$ and $\nu_\infty$ corresponding to problems (P1) and (P4) respectively for different input data $T_\infty$ and $\alpha$. 

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**Fig.1: Variation of $\nu$ with $h_0$ for $T_\infty=1$**

**Fig.2: Variation of $\nu$ with $h_0$ for $T_\infty=5$**

**Fig.3: Variation of $\nu$ with $h_0$ for $T_\infty=10$**

**Fig.4: Variation of $\nu$ with $h_0$ for $T_\infty=15$**

**Fig.5: Comparison between $\nu_{h_0}$ and $\nu_\infty$ with $T_\infty=1$**

**Fig.6: Comparison between $\nu_{h_0}$ and $\nu_\infty$ with $T_\infty=5$**
In Figure 9 we show the variation of the temperature $\Psi$ with respect to $x$ and $t$ taking the particular values of the data: $\gamma = k = d = 1$, $\alpha = 0.4$, $h_0 = 0.5$ and $T_\infty = 1$. As we are dealing with a melting problem, for every fixed value of the position ($x$) we can note when the phase-change takes place and observe how the temperature becomes greater over time once the phase-change have occurred.

6 Conclusions

In this article a closed analytical solution of a similarity type have been obtained for a one-dimensional one-phase Stefan problem in a semi-infinite material using Kummer functions. The novel feature in the problem studied concerns a variable latent heat that depends on the position as well as a convective boundary condition at the fixed face $x = 0$ of the material. On one hand, assuming a latent heat defined as a power function of the position allows the generalization of some previous theoretical results, finding its physical base in problems related to the movement of a shoreline or the cooling body of a magma. On the other hand, the fact of considering a convective condition at the fixed boundary reflects a more realistic way of heat input than an imposed temperature or flux, known as Dirichlet and Neumann conditions respectively.

The key contribution of this paper has been to present the exact solution of the problem which is worth finding not only to understand better the process involved but also to verify the consistency and estimate errors of numerical methods designed to solve Stefan problems. We have demonstrated also the equivalence between our problem and the problems defined by considering a temperature or a flux boundary condition instead of the convective one.

Besides, it has been analysed the limit behaviour of the solution when the coefficient $h_0$ that characterizes the heat transfer at the fixed face $x = 0$ tends to infinity. It can be said that our problem
(P1) converges pointwise to a problem (P4) where it is prescribed a temperature at the fixed boundary characterized by $T_\infty$.

Finally, we have applied Newton’s Method to the closed formula obtained for our problem (P1), in order to estimate the parameter that characterizes the free front numerically. In the same way we did to problem (P4). The computations obtained help us to validate our convergence result.

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