GLOBAL REGULAR SOLUTIONS FOR 1-D DEGENERATE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH LARGE DATA AND FAR FIELD VACUUM

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Abstract. In this paper, the Cauchy problem for the one-dimensional (1-D) isentropic compressible Navier-Stokes equations (CNS) is considered. When the viscosity \( \mu(\rho) \) depends on the density \( \rho \) in a sublinear power law (\( \rho^{\delta} \) with \( 0 < \delta \leq 1 \)), based on an elaborate analysis of the intrinsic singular structure of this degenerate system, we prove the global-in-time well-posedness of regular solutions with conserved total mass, momentum, and finite total energy in some inhomogeneous Sobolev spaces. Moreover, the solutions we obtained satisfy that \( \rho \) keeps positive for all point \( x \in \mathbb{R} \) but decays to zero in the far field, which is consistent with the facts that the total mass of the whole space is conserved, and CNS is a model of non-dilute fluids where \( \rho \) is bounded below away from zero. The key to the proof is the introduction of a well-designed reformulated structure by introducing some new variables and initial compatibility conditions, which, actually, can transfer the degeneracies of the time evolution and the viscosity to the possible singularity of some special source terms. Then, combined with the BD entropy estimates and transport properties of the so-called effective velocity

\[ v = u + \varphi(\rho)_x \]  

\( u \) is the velocity of the fluid, and \( \varphi(\rho) \) is a function of \( \rho \) defined by \( \varphi'(\rho) = \mu(\rho)/\rho^2 \), one can obtain the required uniform a priori estimates of corresponding solutions. Moreover, in contrast to the classical theory in the case of the constant viscosity, one can show that the \( L^\infty \) norm of \( u \) of the global regular solution we obtained does not decay to zero as time \( t \) goes to infinity. It is worth pointing out that the well-posedness theory established here can be applied to the viscous Saint-Venant system for the motion of shallow water.

1. Introduction

The time evolution of the mass density \( \rho \geq 0 \) and the velocity \( u = (u^{(1)}, u^{(2)}, \ldots, u^{(d)})^\top \in \mathbb{R}^d \) of a general viscous isentropic compressible fluid occupying a spatial domain \( \Omega \subset \mathbb{R}^d \) is governed by the following isentropic compressible Navier-Stokes equations (ICNS):

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \text{div} T.
\end{cases}
\]  

(1.1)

Here, \( x = (x_1, x_2, \ldots, x_d)^\top \in \Omega, t \geq 0 \) are the space and time variables, respectively. For the polytropic gases, the constitutive relation is given by

\[ P = A \rho^\gamma, \quad A > 0, \quad \gamma > 1, \]  

(1.2)

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where $A$ is an entropy constant and $\gamma$ is the adiabatic exponent. $T$ denotes the viscous stress tensor with the form
\[
T = 2\mu(\rho)D(u) + \lambda(\rho)\text{div}uI_d,
\] (1.3)
where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ is the deformation tensor, $I_d$ is the $d \times d$ identity matrix,
\[
\mu(\rho) = \alpha\rho^{\delta}, \quad \lambda(\rho) = \beta\rho^{\delta},
\] (1.4)
for some constant $\delta \geq 0$, $\mu(\rho)$ is the shear viscosity coefficient, $\lambda(\rho) + 2\delta\mu(\rho)$ is the bulk viscosity coefficient, $\alpha$ and $\beta$ are both constants satisfying $\alpha > 0$ and $2\alpha + d\beta \geq 0$. (1.5)

In the theory of gas dynamics, the CNS can be derived from the Boltzmann equations through the Chapman-Enskog expansion, cf. Chapman-Cowling [9] and Li-Qin [30]. Under some proper physical assumptions, one can find that the viscosity coefficients and heat conductivity coefficient $\kappa$ are not constants but functions of the absolute temperature $\theta$ such as:
\[
\mu(\theta) = a_1\theta^{\frac{b}{2}}F(\theta), \quad \lambda(\theta) = a_2\theta^{\frac{b}{2}}F(\theta), \quad \kappa(\theta) = a_3\theta^{\frac{b}{2}}F(\theta),
\] (1.6)
for some constants $a_i$ ($i = 1, 2, 3$) (see [9]). Actually for the cut-off inverse power force models, if the intermolecular potential varies as $r^{-\nu}$, where $r$ is intermolecular distance, then in (1.6):
\[
F(\theta) = \theta^b \quad \text{with} \quad b = \frac{2}{\nu} \in [0, \infty).
\]
In particular (see §10 of [9]), for ionized gas, $\nu = 1$ and $b = 2$; for Maxwellian molecules, $\nu = 4$ and $b = \frac{1}{2}$; while for rigid elastic spherical molecules, $\nu = \infty$ and $b = 0$.

According to Liu-Xin-Yang [34], for isentropic and polytropic fluids, such a dependence is inherited through the laws of Boyle and Gay-Lussac:
\[
P = R\rho \theta = A\rho^\gamma \quad \text{for constant} \quad R > 0,
\]
i.e., $\theta = AR^{-1}\rho^{\gamma-1}$, and one can see that the viscosity coefficients are functions of $\rho$ of the form (1.4). Actually, there do exist some physical models that satisfy the density-dependent viscosities assumption (1.4), such as Korteweg system, Shallow water equations, lake equations and quantum Navier-Stokes system (see [2, 3, 4, 5, 6, 36, 46]).

In the current paper, the following 1-D degenerate ICNS is considered:
\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x + P(\rho)_x = (\mu(\rho)u_x)_x,
\end{cases}
\] (1.7)
where the pressure $P$ and the viscosity coefficient $\mu(\rho)$ are given by (1.2) and (1.4), respectively. We are concerned with the global well-posedness of regular solutions with large data and far field vacuum to the Cauchy problem of (1.7) with the following initial data and far field behavior:
\[
(\rho(0, x), u(0, x)) = (\rho_0(x), u_0(x)) \quad \text{for} \quad x \in \mathbb{R},
\]
\[
(\rho(t, x), u(t, x)) \to (0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0.
\] (1.8)
Throughout this paper, we adopt the following simplified notations, most of them are for the standard homogeneous and inhomogeneous Sobolev spaces:

\[ \|f\|_s = \|f\|_{H^s(\mathbb{R}^d)}, \quad \|f\|_p = \|f\|_{L^p(\mathbb{R}^d)}, \quad \|f\|_{m,p} = \|f\|_{W^{m,p}(\mathbb{R}^d)}, \]

\[ D^{k,r} = \{ f \in L^1_{loc}(\mathbb{R}^d) : |f|_{D^{k,r}} = |\nabla^k f| < \infty \}, \quad D^k = D^{k,2}, \]

\[ D^1_s = \{ f \in L^1(\mathbb{R}^3) : |f|_{D^1_s} = |\nabla f| < \infty \}, \]

\[ |f|_{D^1_s} = \|f\|_{D^1_s}, \quad \|f\|_{L^p(0,T;L^q)} = \|f\|_{L^p([0,T] ;L^q(\mathbb{R}^d))}, \]

\[ \|f\|_{X_1 \cap X_2} = \|f\|_{X_1} + \|f\|_{X_2}, \quad \int f \, dx = \int_{\mathbb{R}^d} f \, dx, \]

\[ X([0,T];Y) = X([0,T];Y(\mathbb{R}^d)), \quad \|(f,g)\|_X = \|f\|_X + \|g\|_X. \]

We will clearly indicate the value of \( d \) where the above notations are used. A detailed study of homogeneous Sobolev spaces can be found in [15].

For the flows of constant viscosities (\( \delta = 0 \) in (1.4)), there is a lot of literature on the well-posedness of strong/classical solutions to the system (1.1). When \( \inf_x \rho_0(x) > 0 \), the local well-posedness of three-dimensional (3-D) classical solutions to the Cauchy problem of (1.1) follows from the standard symmetric hyperbolic-parabolic structure satisfying the well-known Kawashima’s condition, cf. [25, 40, 42], which has been extended to be a compatibility condition:  

\[ -\text{div} \mathbb{T}_0 + \nabla P(\rho_0) = \sqrt{\rho_0} \mathbf{g} \quad \text{for some} \quad \mathbf{g} \in L^2(\mathbb{R}^3), \]

which leads to \( \sqrt{\rho_0} \mathbf{u}_0 \in L^\infty([0,T_*];L^2) \) for a short time \( T_* > 0 \). Then they established successfully the local well-posedness of smooth solutions with vacuum in \( \mathbb{R}^3 \) (see also Cho-Kim [12]), which, recently, has been shown to be a global one with small energy by Huang-Li-Xin [21] in \( \mathbb{R}^3 \). Later, Jiu-Li-Ye [22] proved that the 1-D Cauchy problem of (1.1) admits a unique global classical solution with arbitrarily large data and vacuum.

For density-dependent viscosities (\( \delta > 0 \) in (1.4)) which degenerate at the vacuum, system (1.1) has received extensive attentions in recent years. When \( \inf_x \rho_0(x) > 0 \), some important progresses on the global well-posedness of classical solutions to the Cauchy problem of (1.1) have been obtained, which include Burtea-Haspot [8], Constantin-Drivas-Nguyen-Pasqualotto [13], Haspot [20], Kang-Vasseur [24], Mellet-Vasseur [39] for 1-D flow with arbitrarily large data, and Sundbye [44] and Wang-Xu [47] for two-dimensional (2-D) flow with initial data close to a non-vacuum equilibrium in some Sobolev spaces. When vacuum appears, instead of the uniform elliptic structure, the
viscosity degenerates when density vanishes, which raises the difficulty of the problem to another level. Actually, in [11, 21, 22], the uniform ellipticity of the Lamé operator $L$ defined by

$$Lu = -\alpha \triangle u - (\alpha + \beta) \nabla \text{div} u$$

($Lu = -\alpha u_{xx}$ in $\mathbb{R}$) plays an essential role in the high order regularity estimates on $u$. One can use the standard elliptic theory to estimate $|u|_{D^{k+2}}$ by the $D^k$-norm of all other terms in momentum equations. However, for $\delta > 0$, viscosity coefficients vanish as the density function connects to vacuum continuously. This degeneracy makes it difficult to adapt the approach of the constant viscosity case in [11, 21, 22] to the current case.

A remarkable discovery of a new mathematical entropy function was made by Bresch-Desjardins [5] for $\lambda(\rho)$ and $\mu(\rho)$ satisfying the relation

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)),$$

which offers an estimate $\mu'(\rho)\sqrt{\rho} \in L^\infty([0,T]; L^2(\mathbb{R}^d))$ provided that $\mu'(\rho_0)\sqrt{\rho_0} \in L^2(\mathbb{R}^d)$ for any $d \geq 1$. This observation plays an important role in the development of the global existence of weak solutions with vacuum for system (1.1) and some related models, see Bresch-Desjardins [2], Bresch-Vasseur-Yu [7], Jiu-Xin [23], Li-Xin [29], Mellet-Vasseur [38], Vasseur-Yu [45] and so on. However, the regularities and uniqueness of such weak solutions with vacuum remain open.

Notice that, for the cases $\delta \in (0, \infty)$, if $\rho > 0$, (1.1) can be formally rewritten as

$$u_t + u \cdot \nabla u + \frac{A_\gamma}{\gamma - 1} \nabla \rho^{\gamma - 1} + \rho^{\delta - 1} Lu = \psi \cdot Q(u),$$

(1.10)

where the quantities $\psi$ and $Q(u)$ are given by

$$\psi \triangleq \nabla \ln \rho \quad \text{when} \quad \delta = 1;$$

$$\psi \triangleq \frac{\delta}{\delta - 1} \nabla \rho^{\delta - 1} \quad \text{when} \quad \delta \in (0,1) \cup (1,\infty);$$

$$Q(u) \triangleq \alpha(\nabla u + (\nabla u)^T) + \beta \text{div} u \mathbb{I}_d.$$  

When $\delta = 1$, from (1.10)-(1.11), the degeneracy of time evolution and elliptic structure in momentum equations caused by the vacuum has been transferred to the possible singularity of the term $\nabla \ln \rho$, which actually can be controlled by a symmetric hyperbolic system with a source term $\nabla \text{div} u$ in Li-Pan-Zhu [31]. Then via establishing the uniform a priori estimates in $L^6 \cap D^1 \cap D^2$ for $\nabla \ln \rho$, the existence of 2-D local classical solution with far field vacuum to (1.1) has been obtained in [31], which also applies to the 2-D Shallow water equations. When $\delta > 1$, (1.10)-(1.11) imply that actually the velocity $u$ can be governed by a nonlinear degenerate parabolic system without singularity near the vacuum region. Based on this observation, by using some hyperbolic approach which bridges the parabolic system (1.10) when $\rho > 0$ and the hyperbolic one

$$u_t + u \cdot \nabla u = 0 \quad \text{when} \quad \rho = 0,$$

the existence of 3-D local classical solutions with vacuum to (1.1) was established in Li-Pan-Zhu [32]. The corresponding global well-posedness in some homogeneous Sobolev spaces has been established by Xin-Zhu [48] under some initial smallness assumptions. Recently, for the case $0 < \delta < 1$, via introducing an elaborate elliptic approach on the operators $L(\rho^{\delta - 1} u)$ and some initial compatibility conditions, the existence of 3-D local
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regular solution with far field vacuum has been obtained by Xin-Zhu [49]. Some other interesting results and discussions can also be seen in Chen-Chen-Zhu [10], Ding-Zhu [14], Geng-Li-Zhu [16], Germain-Lefloch [18], Guo-Li-Xin [19], Lions [33], Luo-Xin-Zeng [35], Yang-Zhao [50], Yang-Zhu [51], Zhu [52] and so on.

However, due to the complicated mathematical structure of the degenerate ICNS and lack of smooth effect on solutions when vacuum appears, even many important progresses have been obtained, a lot of fundamental questions still remain open, such as which kind of initial data can cause the finite time blow-up of smooth solutions, or provide the global existence of smooth solutions with conserved total mass, momentum, and finite total energy. In the current paper, we identify a class of 1-D initial data, which admits a global classical solution with far field vacuum to the Cauchy problem for the cases $0 < \delta \leq 1$ in some inhomogeneous Sobolev spaces.

1.1. Global-in-time well-posedness of regular solutions. In order to present our results clearly, we first introduce the following two definitions for the case $0 < \delta < 1$ and the case $\delta = 1$ of regular solutions with far field vacuum to the Cauchy problem (1.7)-(1.8), respectively.

**Definition 1.1.** Assume $0 < \delta < 1$ and $T > 0$. The pair $(\rho, u)$ is called a regular solution to the Cauchy problem (1.7)-(1.8) in $[0, T] \times \mathbb{R}$, if $(\rho, u)$ satisfies this problem in the sense of distributions and:

1. $\inf_{(t,x)\in[0,T] \times \mathbb{R}} \rho(t,x) = 0$, $0 < \rho^{\gamma-1} \in C([0,T]; H^2)$,
   
   $(\rho^{\delta-1})_x \in C([0,T]; H^1)$;

2. $u \in C([0,T]; H^2) \cap L^2([0,T]; H^3)$, $\rho^{\frac{\delta-1}{\gamma-1}} u_x \in C([0,T]; L^2)$,
   
   $\rho^{\delta-1} u_{xx} \in C([0,T]; L^2) \cap L^2([0,T]; H^1)$.

**Definition 1.2.** Assume $\delta = 1$ and $T > 0$. The pair $(\rho, u)$ is called a regular solution to the Cauchy problem (1.7)-(1.8) in $[0, T] \times \mathbb{R}$, if $(\rho, u)$ satisfies this problem in the sense of distributions and:

1. $\inf_{(t,x)\in[0,T] \times \mathbb{R}} \rho(t,x) = 0$, $0 < \rho^{\gamma-1} \in C([0,T]; H^2)$,
   
   $\ln \rho \in C([0,T]; H^1)$;

2. $u \in C([0,T]; H^2) \cap L^2([0,T]; H^3)$, $u_t \in C([0,T]; L^2) \cap L^2([0,T]; H^1)$.

**Remark 1.1.** First, it follows from the Definitions 1.1-1.2 that $(\rho^{\delta-1})_x \in L^\infty$ for the case $0 < \delta < 1$, and $(\ln \rho)_x \in L^\infty$ for the case $\delta = 1$, which, along with $\rho^{\gamma-1} \in C([0,T]; H^2)$, means that the vacuum occurs if and only if in the far field.
Second, we introduce some physical quantities that will be used in this paper:

\[ m(t) = \int \rho(t,x)dx \quad \text{(total mass)}, \]
\[ \mathbb{P}(t) = \int \rho(t,x)u(t,x)dx \quad \text{(momentum)}, \]
\[ E_k(t) = \frac{1}{2} \int \rho(t,x)u^2(t,x)dx \quad \text{(total kinetic energy)}, \]
\[ E(t) = E_k(t) + \int \frac{P(t,x)}{\gamma - 1}dx \quad \text{(total energy)}. \]

Actually, it follows from the definitions that the regular solution satisfies the conservation of total mass (see Lemma 3.3) and momentum (see Lemma 4.1). Furthermore, it satisfies the energy equality (see Lemma 3.1 and its proof in Appendix B). Note that the conservation of momentum is not clear for the strong solution with vacuum to the flows of constant viscosities Jiu-Li-Ye [22] (see an example on non-conservation of momentum in [49]). In this sense, the definitions of regular solutions here are consistent with the physical background of CNS.

The regular solutions select velocity in a physically reasonable way when the density approaches the vacuum at far fields. Under the help of this notion of solutions, the momentum equations can be reformulated into a special quasi-linear parabolic system with some possible singular source terms near the vacuum, and the coefficients in front of \(u_{xx}\) will tend to \(\infty\) as \(\rho \to 0\) in the far field for \(0 < \delta < 1\). However, the problems become trackable through some elaborate arguments on the energy estimates.

For the case \(0 < \delta < 1\), the global-in-time well-posedness result reads as:

**Theorem 1.1.** Assume \(\delta\) and \(\gamma\) satisfy

\[ 0 < \delta < 1, \quad \gamma > 1, \quad \gamma \geq \delta + \frac{1}{2}. \quad (1.12) \]

If the initial data \((\rho_0, u_0)\) satisfy

\[ 0 < \rho_0 \in L^1, \quad (\rho_0^{\gamma-1}, u_0) \in H^2, \quad (\rho_0^{\delta-1})_x \in H^1, \quad (1.13) \]

and the initial compatibility conditions

\[ \partial_x u_0 = \rho_0^{\frac{1-\gamma}{\gamma}} g_1, \quad \alpha \partial_x^2 u_0 = \rho_0^{1-\delta} g_2, \quad (1.14) \]

for some \(g_i \in L^2(i = 1,2)\), then the Cauchy problem (1.7)-(1.8) admits a unique global classical solution \((\rho, u)\) in \((0, \infty) \times \mathbb{R}\) satisfying that, for any \(0 < T < \infty\), \((\rho, u)\) is a regular one in \([0, T] \times \mathbb{R}\) as defined in Definition 1.1, and

\[ \rho \in C([0, T]; L^1), \quad (\rho^{\gamma-1})_t \in C([0, T]; H^1), \quad (\rho^{\delta-1})_{tx} \in C([0, T]; L^2), \quad t^{\frac{1}{2}} \rho^{\frac{1-\gamma}{\gamma}} u_{tx} \in L^\infty([0, T]; L^2) \cap L^2([0, T]; D^1), \quad t^{\frac{1}{2}} u_{tt} \in L^2([0, T]; L^2). \quad (1.15) \]

**Remark 1.2.** First, observe that there are no smallness conditions imposed on \((\rho_0, u_0)\). Second, one can find the following class of initial data \((\rho_0, u_0)\) satisfying the conditions (1.13)-(1.14):

\[ \rho_0(x) = \frac{1}{1 + x^2}, \quad u_0(x) \in C^2_0(\mathbb{R}), \]
where \( \max \{ \frac{2}{\gamma}, \frac{1}{\gamma-1} \} \) < \( \sigma \) < \( \frac{1}{\gamma-1} \). Particularly, when \( \partial_x u_0 \) is compactly supported, the compatibility conditions (1.14) are satisfied automatically.

Moreover, the compatibility conditions (1.14) are also necessary for the existence of the unique regular solution \((\rho, u)\) obtained in Theorem 1.1. In particular, the one shown in (1.14) plays a key role in the derivation of \( u_t \in L^\infty([0, T_*]; L^2(\mathbb{R})) \), which will be used in the uniform estimates for \(|u|_{D^2} \).

For the case \( \delta = 1 \), the global-in-time well-posedness result reads as:

**Theorem 1.2.** Assume \( \delta = 1 \) and \( \gamma > \frac{3}{2} \). If the initial data \((\rho_0, u_0)\) satisfy

\[
0 < \rho_0 \in L^1, \quad (\rho^{-1}_0, u_0) \in H^2, \quad (\ln \rho_0)_x \in H^1,
\]

then the Cauchy problem (1.7)-(1.8) admits a unique global classical solution \((\rho, u)\) in \((0, \infty) \times \mathbb{R}\) satisfying that, for any \( 0 < T < \infty \), \((\rho, u)\) is a regular one in \([0, T] \times \mathbb{R}\) as defined in Definition 1.2, and

\[
\begin{align*}
\rho & \in C([0, T]; L^1), \quad (\rho^{-1})_t \in C([0, T]; H^1), \quad (\ln \rho)_t \in C([0, T]; L^2), \\
(\partial_t u)_x & \in L^\infty([0, T]; L^2) \cap L^2([0, T]; D^1), \quad |(\partial_t u)_x| \in L^2([0, T]; L^2).
\end{align*}
\]

**Remark 1.3.** First, observe that there are no smallness conditions imposed on \((\rho_0, u_0)\).

Second, one can find the following class of initial data \((\rho_0, u_0)\) satisfying (1.16):

\[
\rho_0(x) = \frac{1}{1 + x^{2\sigma}}, \quad u_0(x) \in C^0_\infty(\mathbb{R}),
\]

where \( \sigma > \max \{ \frac{2}{\gamma}, \frac{1}{\gamma-1} \} \).

Moreover, it should be pointed out that the theory established in Theorem 1.2 can be applied to the viscous *Saint-Venant* system for the motion of shallow water (i.e., \( \delta = 1 \), \( \gamma = 2 \), see Gerbeau-Perthame [17]).

**Remark 1.4.** It is worth pointing out that in the domain \( \Omega = [0, 1] \), Li-Li-Xin [28] proved that the entropy weak solutions for general large initial data satisfying finite entropy to the corresponding initial-boundary value problem of the system (1.7) with \( \delta > \frac{1}{2} \) exist globally in time. Moreover, the authors showed that for more regular initial data, there is a global entropy weak solution which is unique and regular with well-defined velocity field for short time, and the interface of initial vacuum propagates along the particle path during this time period. Furthermore, the dynamics of weak solutions and vacuum states are investigated rigorously. It is shown that for any global entropy weak solution, any (possibly existing) vacuum state must vanish within finite time, which means that there exist some time \( T_0 > 0 \) (depending on initial data) and a constant \( \rho_- \) such that

\[
\inf_{x \in \Omega} \rho(t, x) \geq \rho_- > 0 \quad \text{for} \quad t \geq T_0.
\]

The velocity (even if regular enough and well-defined) blows up in finite time as the vacuum states vanish, and after the vanishing of vacuum states, the global entropy weak solution becomes a strong solution and tends to the non-vacuum equilibrium state exponentially in time.

According to the conclusions obtained in Theorems 1.1-1.2, one finds that if we replaced the domain \( \Omega = [0, 1] \) by the whole space \( \mathbb{R} \), at least for sufficiently smooth initial values, the phenomena on vacuum states observed in (1.18) does not happen again. Actually,
for the global regular solutions obtained in Theorems 1.1-1.2 to the Cauchy problem (1.7)-(1.8), it is obvious that
\[ \inf_{x \in \mathbb{R}} \rho(t,x) = 0 \quad \text{for any} \quad t \geq 0. \]

1.2. Remarks on the asymptotic behavior of \( u \).
Moreover, in contrast to the classical theory in the case of the constant viscosity [21, 37], one can show that the \( L^\infty \) norm of \( u \) of the global regular solution we obtained does not decay to zero as time \( t \) goes to infinity. First, based on the physical quantities introduced in Remark 1.1, we define a solution class as follows:

**Definition 1.3.** Let \( T > 0 \) be any constant. For the Cauchy problem (1.7)-(1.8), a classical solution \((\rho, u)\) is said to be in \( D(T) \) if \((\rho, u)\) satisfies the following conditions:

(A) Conservation of total mass:
\[ 0 < m(0) = m(t) < \infty \quad \text{for any} \quad t \in [0, T]; \]

(B) Conservation of momentum:
\[ 0 < |P(0)| = |P(t)| < \infty \quad \text{for any} \quad t \in [0, T]; \]

(C) Finite kinetic energy:
\[ 0 < E_k(t) < \infty \quad \text{for any} \quad t \in [0, T]. \]

Then one has:

**Theorem 1.3.** Assume \( \gamma > 1 \) and \( \delta \geq 0 \). Then for the Cauchy problem (1.7)-(1.8), there is no classical solution \((\rho, u)\) in \( D(\infty) \) satisfying
\[ \limsup_{t \to \infty} |u(t, \cdot)|_{\infty} = 0. \] (1.19)

According to Theorems 1.1-1.3 and Remark 1.1, one shows that

**Corollary 1.1.** Assume that \( |P(0)| > 0 \). Then for the Cauchy problem (1.7)-(1.8), the global regular solutions \((\rho, u)\) obtained in Theorems 1.1-1.2 do not satisfy (1.19).

The rest of the paper is organized as follows: §2-§3 are devoted to establishing the global well-posedness of regular solutions with large data stated in Theorems 1.1-1.2. We start with the reformulation of the original problem (1.7)-(1.8) as (2.2) (resp. (2.14)) in terms of the new variables for the case \( 0 < \delta < 1 \) (resp. \( \delta = 1 \)), and establish the local-in-time well-posedness of smooth solutions to the reformulated problems under the assumption that the initial density keeps positive for all point \( x \in \mathbb{R} \) but decays to zero as \( |x| \to \infty \) in §2. The local smooth solutions to (2.2) and (2.14) are shown to be global ones in §3 by deriving the global-in-time a priori estimates through energy methods. Here, we have employed some arguments due to Bresch-Desjardins [2, 3] and Bresch-Desjardins-Lin [4] to deal with the strong degeneracy of the density-dependent viscosity \( \mu(\rho) = \alpha \rho^\delta \), which include the well-known B-D entropy estimates (see Lemma 3.2) and the effective velocity (see Lemma 3.7). Finally, in §4, we give the proofs for the non-existence theories of global regular solutions with \( L^\infty \) decay on \( u \) shown in Theorem 1.3 and Corollary 1.1. Furthermore, we give several appendixes to list some lemmas that are frequently used in our proof, and review some related local-in-time well-posedness theories for multi-dimensional flows of degenerate viscosities.
2. LOCAL-IN-TIME WELL-POSEDNESS

This section will be devoted to providing the local-in-time well-posedness of regular solutions with far field vacuum to the Cauchy problem (1.7)-(1.8) for the cases $0 < \delta \leq 1$.

2.1. Case 1: $0 < \delta < 1$. For the case $0 < \delta < 1$, the corresponding local-in-time well-posedness can be achieved by the following four steps.

2.1.1. Reformulation. Via introducing the following new variables

$$\phi = \frac{A\gamma}{\gamma - 1}\rho^{\gamma - 1}, \quad \psi = \frac{\delta}{\delta - 1}(\rho^{\delta - 1})_x,$$

the Cauchy problem (1.7)-(1.8) can be rewritten as

$$\begin{cases}
\phi_t + u\phi_x + (\gamma - 1)\phi u_x = 0, \\
u_t + uu_x + \phi_x - a\phi^2e u_{xx} = \alpha\psi u_x, \\
\psi_t + (u\psi)_x + (\delta - 1)\psi u_x + a\delta\phi^2e u_{xx} = 0, \\
(\phi, u, \psi)|_{t=0} = (\phi_0, u_0, \psi_0)
\end{cases}
$$

for $x \in \mathbb{R}$, where the constants $a$ and $e$ are given by

$$a = \left(\frac{A\gamma}{\gamma - 1}\right)^{\frac{\delta}{\gamma - 1}} \quad \text{and} \quad e = \frac{\delta - 1}{2(\gamma - 1)} < 0.$$

2.1.2. Discussion on $\nabla\rho_0^{\delta - 1} \in L^4$ in [49]. In [49], for the 3-D degenerate system (1.1)-(1.5), via introducing the following new variables

$$\phi = \frac{A\gamma}{\gamma - 1}\rho^{\gamma - 1}, \quad \psi = \frac{\delta}{\delta - 1}\nabla\rho^{\delta - 1},$$

Xin-Zhu studied the following reformulated problem:

$$\begin{cases}
\phi_t + u \cdot \nabla \phi + (\gamma - 1)\phi \text{div} u = 0, \\
u_t + u \cdot \nabla u + \nabla \phi + a\phi^2e L u = \psi \cdot Q(u), \\
\psi_t + \nabla(u \cdot \psi) + (\delta - 1)\psi \text{div} u + a\delta\phi^2e \nabla \text{div} u = 0, \\
(\phi, u, \psi)|_{t=0} = (\phi_0, u_0, \psi_0)
\end{cases}
$$

for $x \in \mathbb{R}^3$, where $|x| \to \infty$ for $t \geq 0$. 

$$a = \left(\frac{A\gamma}{\gamma - 1}\rho_0^{\gamma - 1}(x), u_0(x), \frac{\delta}{\delta - 1}\nabla\rho_0^{\delta - 1}(x)\right)$$

for $x \in \mathbb{R}$, where $|x| \to \infty$ for $t \geq 0$. 

$$(\phi, u, \psi) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0.$$
where assumptions of Lemma C.1 that, there exists a energy estimates process in the proof for the local existence of the regular solution shown which, actually, is only used in the initial data’s approximation process from the non-vacuum flow to the flow with far field vacuum, and does not appear in the corresponding energy estimates process in the proof for the local existence of the regular solution shown in [49]. Therefore, in Lemma C.1, the regularity of the quantity $\nabla \rho^{\frac{\delta-1}{\delta}}$ in positive time has not been mentioned.

Actually, in Section 3.6 (page 136) of [49], in order to approximate the flow with far field vacuum by the non-vacuum flow, for any $\eta \in (0, 1)$, one sets

$$\phi^\eta_0 = \phi_0 + \eta, \quad \psi^\eta_0 = \frac{a\delta}{\delta-1} \nabla (\phi_0 + \eta)^{2e}, \quad h^\eta_0 = (\phi_0 + \eta)^{2e}.$$

Then the initial compatibility conditions of the perturbed initial data can be given as

$$\begin{cases}
\nabla u_0 = (\phi_0 + \eta)^{-e} g_1^\eta, & aLu_0 = (\phi_0 + \eta)^{-2e} g_2^\eta, \\
\nabla (a(\phi_0 + \eta)^{2e} Lu_0) = (\phi_0 + \eta)^{-e} g_3^\eta,
\end{cases}$$

where $g_i^\eta$ (i = 1, 2, 3) are given as

$$\begin{cases}
g_1^\eta = \frac{\phi_0 - e}{(\phi_0 + \eta)^{-e}} g_1, & g_2^\eta = \frac{\phi_0 - 2e}{(\phi_0 + \eta)^{-2e}} g_2, \\
g_3^\eta = \frac{\phi_0 - 3e}{(\phi_0 + \eta)^{-3e}} (g_3 - \frac{a\nabla \phi_0}{\phi_0 + \eta} \phi_0^2 Lu_0).
\end{cases}$$

According to the definition of $\psi^\eta_0$, one has

$$\begin{align*}
\psi^\eta_0 &= \frac{a\delta}{\delta-1} \left( \frac{\phi_0}{\phi_0 + \eta} \right)^{2e} \nabla \phi_0^e, \\
\nabla \psi^\eta_0 &= \frac{a\delta}{\delta-1} \left( \frac{\phi_0}{\phi_0 + \eta} \right)^{2e} \nabla^2 \phi_0^e + (1 - 2e) \eta \nabla \phi_0 \otimes \nabla \phi_0^e \left( \frac{\phi_0}{\phi_0 + \eta} \right)^{-2e}, \\
\nabla^2 \psi^\eta_0 &= \frac{a\delta}{\delta-1} \left( \frac{\phi_0}{\phi_0 + \eta} \right)^{2e} \nabla^3 \phi_0^e + \frac{2(1 - 2e)}{e} \eta \phi_0^e \phi_0 \otimes \nabla \phi_0 \otimes \nabla \phi_0^e \left( \frac{\phi_0}{\phi_0 + \eta} \right)^{-2e} \\
&\quad + \frac{2(1 - 2e)}{e} \left( \frac{\phi_0}{\phi_0 + \eta} \right)^{-1} \left( \nabla \phi_0 \right)^3 + \frac{2\eta}{\phi_0 + \eta} \nabla \phi_0 \otimes \nabla^2 \phi_0^e.
\end{align*}$$

From the formula (2.6), one can find that the condition (2.4) is only used to guarantee the uniform boundedness of $|\psi^\eta_0|_{D^1}$ with respect to $\eta$. Then it follows from the initial assumptions of Lemma C.1 that, there exists a $\eta_1 > 0$ such that for any $0 < \eta \leq \eta_1$,

$$1 + \eta + ||\phi^\eta_0 - \eta||_3 + ||\psi^\eta_0||_{D^1\cap D^2} + ||u_0||_3 + ||g_1^\eta||_2 + ||g_2^\eta||_2 + ||g_3^\eta||_2 + ||(h^\eta_0)^{-1}||_{L^\infty\cap D^{1,6}\cap D^{2,3}} + ||\nabla h^\eta_0 / h^\eta_0||_{L^\infty\cap D^{1,3}\cap D^2} \leq c_0,$$
where \(c_0\) is a positive constant independent of \(\eta\). Based on such kind of initial approximation process, the Cauchy problem (2.3) can be solved locally in time by studying the corresponding Cauchy problem away from the vacuum and then passing to the limit as \(\eta \to 0\). The details can be found in [49].

For 1-D case, the corresponding initial data’s approximation process from the non-vacuum flow to the flow with far field vacuum can also be achieved from the above process with some minor modifications. Similarly, for any \(\eta \in (0, 1)\), one sets

\[
\phi_0^\eta = \phi_0 + \eta, \quad \psi_0^\eta = \frac{a\delta}{\delta - 1} \left( (\phi_0 + \eta)^{2\epsilon} \right)_x, \quad h_0^\eta = (\phi_0 + \eta)^{2\epsilon}.
\]

Then the initial compatibility conditions of the perturbed initial data can be given as

\[
\partial_x u_0 = (\phi_0 + \eta)^{-\epsilon} g_1^\eta, \quad a\partial_x^2 u_0 = (\phi_0 + \eta)^{-2\epsilon} g_2^\eta,
\]

where \(g_i^\eta (i = 1, 2)\) are given in the first line of (2.5). Then it follows from (1.13)-(1.14) that there also exists a \(\eta_1 > 0\) such that for any \(0 < \eta < \eta_1\),

\[
1 + \eta + \|\phi_0^\eta - \eta\|_2 + \|\psi_0^\eta\|_1 + \|u_0\|_2 + |g_1^\eta|_2 + |g_2^\eta|_2 + \|h_0^\eta\|_{L^\infty(\mathbb{D}^1 \cap \mathbb{D}^2)} + \|(\ln h_0^\eta)_x\|_1 \leq c_0.
\]

However, compared with the 3D case in [49], it is worth pointing out that the following initial assumption

\[
\partial_x \rho_0^{\delta^{-1}} \in L^4 \tag{2.7}
\]

is no longer required for the current 1D problem. Actually, one easily has

\[
\partial_x \rho_0^{\delta^{-1}} = \frac{1}{2} \rho_0^{\delta^{-1}} \partial_x \rho_0^{\delta^{-1}},
\]

which, along with the fact \(\partial_x \rho_0^{\delta^{-1}} \in H^1\), and the Sobolev embedding theorem, implies that (2.7) holds automatically.

### 2.1.3. Local-in-time well-posedness

Since the mathematical structure of the system considered in problem (2.2) is exactly the same as the one in problem (2.3), then based on the discussion in Section 2.1.2 and Lemma C.1, one can obtain the following local-in-time well-posedness theory in 1-D space.

**Lemma 2.1.** Assume that \(0 < \delta < 1\) and \(\gamma > 1\). If the initial data \((\rho_0, u_0)\) satisfy (1.13)-(1.14) except \(\rho_0 \in L^1\), then there exist a time \(T_* > 0\) and a unique regular solution \((\rho, u)\) in \([0, T_*] \times \mathbb{R}\) to the Cauchy problem (1.7)-(1.8) satisfying (1.15) with \(T\) replaced by \(T_*\) except \(\rho \in C([0, T_*]; L^1)\).

### 2.1.4. \(\rho \in C([0, T_*]; L^1)\)

It should be emphasized that the initial assumption \(\rho_0 \in L^1\) in Theorem 1.1 is only used in the proof of global-in-time existence of the unique regular solution satisfying (1.15), which is not required in the proof for the local-in-time existence of the general regular solution of [49]. Thus, in the following lemma, we show that actually, the solution obtained in Lemma 2.1 still satisfies:

**Lemma 2.2.** Let \((\rho, u)\) in \([0, T_*] \times \mathbb{R}\) be the solution to the Cauchy problem (1.7)-(1.8) obtained in Lemma 2.1. Then

\[
\rho \in C([0, T_*]; L^1) \text{ if } \rho_0 \in L^1 \text{ additionally.} \tag{2.8}
\]
Proof. First, one proves that

\[ \rho \in L^\infty([0, T_*]; L^1). \]  

(2.9)

For this purpose, let \( f : \mathbb{R}^+ \to \mathbb{R} \) satisfy

\[
\begin{cases}
1, & s \in [0, \frac{1}{2}] \\
\text{non-negative polynomial}, & s \in [\frac{1}{2}, 1] \\
e^{-s}, & s \in [1, \infty)
\end{cases}
\]

such that \( f \in C^2 \). Then there exists a generic constant \( C_* > 0 \) such that \( |f'(s)| \leq C_* f(s) \).

Define, for any \( R > 0 \),

\[ f_R(x) = f(|x|/R). \]

Then, according to Lemma 2.1, one can obtain that for any given \( R > 0 \),

\[
\int \left( (|\rho_t| + |\rho_x u| + |\rho u_x|) f_R(x) + \rho f_R(x) + |\rho u \partial_x f_R(x)| \right) dx < \infty.
\]

Multiplying (1.7) by \( f_R(x) \) and integrating over \( \mathbb{R} \), one has

\[
\frac{d}{dt} \int \rho f_R(x) dx = -\int (\rho_x u + \rho u_x) f_R(x) dx = \int \rho u f'_R(x) dx \leq C|u|_\infty \int \rho f_R(x) dx
\]

for some positive constant \( C \) depending only on the initial data, \( A, \gamma, \delta, C_* \) and \( T_* \) but independent of \( R \), which, along with the Gronwall inequality, implies that

\[
\int \rho f_R(x) dx \leq C \quad \text{for} \quad t \in [0, T_*].
\]  

(2.10)

Since \( \rho f_R(x) \to \rho \) almost everywhere as \( R \to \infty \), then it follows from the Fatou lemma (see Lemma A.2 in Appendix A) that

\[
\int \rho dx \leq \liminf_{R \to \infty} \int \rho f_R(x) dx \leq C \quad \text{for} \quad t \in [0, T_*].
\]  

(2.11)

Second, according to (1.7), one has

\[
|\rho_t| \leq C(|\rho_x u|_1 + |\rho u_x|_1) \leq C(|\psi|_2 |u|_2 |\rho|_\infty^{2-\delta} + |\rho|_2 |u_x|_2) \leq C,
\]

(2.12)

which, along with (2.9) and the Sobolev embedding theorem, implies that (2.8) holds.

The proof of Lemma 2.2 is complete. \( \square \)

2.2. Case 2: \( \delta = 1 \). For the case \( \delta = 1 \), the corresponding local-in-time well-posedness can be achieved by the following three steps.

2.2.1. Reformulation. Via introducing the following new variables

\[
\phi = \frac{A \gamma}{\gamma - 1} \rho^{\gamma - 1}, \quad \psi = \frac{1}{\gamma - 1} \frac{\phi_x}{\phi} = (\ln \rho)_x,
\]

(2.13)
the Cauchy problem (1.7)-(1.8) can be rewritten as
\[
\begin{cases}
\phi_t + u\phi_x + (\gamma - 1)\phi u_x = 0, \\
u_t + uu_x + \phi_x - \alpha u_{xx} = \alpha \psi u_x, \\
\psi_t + (u\psi)_x + u_{xx} = 0, \\
(\phi, u, \psi)|_{t=0} = (\phi_0, u_0, \psi_0)
\end{cases}
\] (2.14)

2.2.2. Local-in-time existence. In [31], for the 2-D degenerate system (1.1)-(1.5), via introducing the following new variables
\[
{\tilde{\phi}} = \rho^{\gamma - 1}, \quad \psi = \frac{2}{\gamma - 1} \frac{\nabla \tilde{\phi}}{\phi} = \nabla \ln \rho,
\]
Li-Pan-Zhu studied the following reformulated problem:
\[
\begin{cases}
{\tilde{\phi}}_t + u \cdot \nabla {\tilde{\phi}} + \frac{\gamma - 1}{2} {\tilde{\phi}} \text{div} u = 0, \\
u_t + u \cdot \nabla u + \frac{2A\gamma}{\gamma - 1} {\tilde{\phi}} \nabla u + Lu = \psi \cdot Q(u), \\
\psi_t + \nabla (u \cdot \psi) + \nabla \text{div} u = 0, \\
({\tilde{\phi}}, u, \psi)|_{t=0} = ({\tilde{\phi}}_0, u_0, \psi_0)
\end{cases}
\] (2.15)

Since the mathematical structure of the system considered in problem (2.14) is exactly the same as the one in problem (2.15), then based on Lemma C.2, one can obtain the following local-in-time well-posedness theory in 1-D space.

**Lemma 2.3.** Assume that \(\delta = 1\) and \(\gamma > 1\). If the initial data \((\rho_0, u_0)\) satisfy (1.16) except \(\rho_0 \in L^1\), then there exist a time \(T_* > 0\) and a unique regular solution \((\rho, u)\) in \([0, T_*] \times \mathbb{R}\) to the Cauchy problem (1.7)-(1.8) satisfying (1.17) with \(T\) replaced by \(T_*\) except \(\rho \in C([0, T_*]; L^1)\).

2.2.3. \(\rho \in C([0, T_*]; L^1)\). As mentioned in Section 2.1.4, one also needs to show that

**Lemma 2.4.** Let \((\rho, u)\) in \([0, T_*] \times \mathbb{R}\) be the solution to the Cauchy problem (1.7)-(1.8) obtained in Lemma 2.3. Then
\[
\rho \in C([0, T_*]; L^1) \quad \text{if} \quad \rho_0 \in L^1 \quad \text{additionally.}
\] (2.16)

The proof of this lemma is similar to that of Lemma 2.2. Here we omit it.
3. Global-in-time well-posedness

This section is devoted to proving Theorems 1.1-1.2, and we always assume $0 < \delta \leq 1$ and $\gamma > 1$. Let $T > 0$ be some time and $(\rho, u)$ be regular solutions to the problem (1.7)-(1.8) in $[0, T] \times \mathbb{R}$ obtained in Lemmas 2.1-2.2 and Lemmas 2.3-2.4. The main aim in the rest of this section is to establish the global-in-time a priori estimates for these solutions. Hereinafter, we denote $C_0$ (resp. $C$) a generic positive constant depending only on $(\rho_0, u_0, A, \gamma, \alpha, \delta)$ (resp. $(C_0, T)$), which may be different from line to line.

3.1. The $L^\infty$ estimate of $\rho$. We consider the upper bound of the mass density $\rho$ in $[0, T] \times \mathbb{R}$. First, the standard energy estimates yields that

Lemma 3.1. For any $T > 0$, it holds that

$$\int \left( \frac{1}{2} \rho u^2 + \frac{A}{\gamma - 1} \rho^\gamma \right) (t, \cdot) dx + \alpha \int_0^t \int \rho^\delta u_x^2 dx ds \leq C_0$$

for $0 \leq t \leq T$.

Second, we give the well-known B-D entropy estimate.

Lemma 3.2 ([5]). For any $T > 0$, it holds that

$$\int \left( \frac{1}{2} \rho |u + \varphi(\rho)x|^2 + \frac{A}{\gamma - 1} \rho^\gamma \right) (t, \cdot) dx + \alpha A \gamma \int_0^t \int \rho^\gamma + \frac{1}{3} \rho_x^2 dx ds \leq C_0$$

for $0 \leq t \leq T$, where $\varphi'(\rho) = \frac{\mu(\rho)}{\rho^\gamma}$.

The conclusions obtained in Lemmas 3.1-3.2 are classical, and the corresponding proofs can be found in Appendix B.

Next we show the regular solution $(\rho, u)$ keeps the conservation of total mass.

Lemma 3.3. For any $T > 0$, it holds that

$$m(t) = m(0) \quad \text{for} \quad 0 \leq t \leq T.$$

Proof. It follows from (1.7)$_1$ that

$$\frac{d}{dt} \int \rho(t, \cdot) dx = - \int (\rho u_x)(t, \cdot) dx = 0,$$

where one has used fact that $\rho \in C([0, T]; L^1)$ and $\rho u(t, \cdot) \in W^{1,1}(\mathbb{R})$.

The proof of Lemma 3.3 is complete. □

Now we are ready to give the uniform upper bound of the density.

Lemma 3.4. For any $T > 0$, it holds that

$$|\rho(t, \cdot)|_\infty \leq C_0 \quad \text{for} \quad 0 \leq t \leq T.$$

Proof. First, integrating the equation (1.7)$_2$ over $(-\infty, x]$ with respect to $x$, one has

$$\frac{d}{dt} \int_{-\infty}^x \rho u(t, z) dz + \rho u_x^2 + A \rho^\gamma = \alpha \rho^\delta u_x,$$

where one has used the far field behavior (1.8)$_2$. 

(3.1)
Denote \( \xi(t, x) = \int_{-\infty}^{x} \rho u(t, z)dz \). Then it follows from (3.1) that
\[
\xi_t + u\xi_x + \alpha \rho^{\delta-1}(\rho_t + u\rho_x) + A\rho^\gamma = 0. \tag{3.2}
\]
Second, let \( y(t; x) \) be the solution of the following problem
\[
\begin{cases}
\frac{dy(t; x)}{dt} = u(t, y(t; x)), \\
y(0; x) = x.
\end{cases} \tag{3.3}
\]
It follows from (3.2) that
\[
\frac{d}{dt} \left( \xi + \frac{\alpha}{\delta} \rho^\delta \right) (t, y(t; x)) \leq 0,
\]
which, along with the fact
\[
\xi(t, y(t; x)) \leq |\sqrt{\rho} u(t, \cdot)|_2 |\rho(t, \cdot)|^{\frac{1}{2}} \leq C_0 \quad \text{for } 0 \leq t \leq T, \tag{3.4}
\]
implies that
\[
\left( \xi + \frac{\alpha}{\delta} \rho^\delta \right) (t, y(t; x)) \leq \xi(0, y(0; x)) + \frac{\alpha}{\delta} \rho^\delta(x) \leq C_0. \tag{3.5}
\]
Finally, according to (3.4)-(3.5), one can obtain that
\[
|\rho^\delta(t, \cdot)|_\infty \leq C_0 \quad \text{for } 0 \leq t \leq T.
\]
The proof of Lemma 3.4 is complete. \( \square \)

3.2. The \( L^2 \) estimate of \( u \). We consider the \( L^2 \) estimate of \( u \). For this purpose, one first needs to show the following several auxiliary lemmas. The first one is on the \( L^q \) estimate of \( \rho^{\frac{1}{4}}u \) for any \( 2 \leq q < \infty \).

Lemma 3.5. Assume \( 2 \leq q < \infty \). Then for any \( T > 0 \), it holds that
\[
|\rho^{\frac{1}{4}}u(t, \cdot)|_q \leq C \quad \text{for } 0 \leq t \leq T,
\]
where the constant \( C \) depends on \( q \).

Proof. First, multiplying (1.7) by \( |u|^p u \) (\( p \geq 2 \)) and integrating over \( \mathbb{R} \), one has
\[
\frac{1}{p+2} \frac{d}{dt} \left| \rho^{\frac{1}{4}}u \right|_{p+2}^{p+2} + \alpha(p + 1) \left| \rho^{\frac{1}{4}}u^2 u_x \right|_2^2
= A(p + 1) \int \rho^\gamma |u|^p u_x dx
\leq \frac{\alpha(p + 1)}{2} \int \rho^\delta |u|^p u_x dx + \frac{A^2(p + 1)}{2\alpha} \int \rho^{2\gamma-\delta} |u|^p dx. \tag{3.6}
\]
When \( p = 2 \), it follows from (3.6), Lemmas 3.1 and 3.3-3.4 that
\[
\frac{d}{dt} \left| \rho^{\frac{1}{4}}u \right|_4^4 + \frac{3\alpha}{2} \left| \rho^{\frac{1}{4}}u u_x \right|_2^2
\leq C \left| \rho^\infty_{\infty} \right|_{\infty}^{2\gamma-\delta-1} \left| \rho^{\frac{1}{4}}u_4 \right|_4 \left| \rho^\frac{1}{2} u_2 \right|_2 \left| \rho^\gamma \right|_1 \leq C(1 + |\rho^{\frac{1}{4}}u_4|_4). \tag{3.7}
\]
When \( p > 2 \), it follows from (3.6), Lemmas 3.1 and 3.3-3.4 that
\[
\frac{1}{p + 2} \frac{d}{dt} \left| \rho^{\frac{1}{p+2}} u \right|^{p+2}_{p+2} + \frac{\alpha(p + 1)}{2} \left| \rho^{\frac{1}{2}} u x \right|^{2}_{2} \\
\leq C \left| \rho \right|^{2\gamma - \delta - 1}_{\infty} \left| \rho^{\frac{p-2}{p}} u \right|^{p-2(p+2)}_{p+2} \left| \rho^{\frac{1}{p}} u x \right|^{\frac{\delta}{2}}_{2} \\
\leq C \left| \rho \right|^{2\gamma - \delta - 1}_{\infty} \left| \rho^{\frac{1}{p+2}} u \right|^{(p-2)(p+2)}_{p+2} \left| \rho^{\frac{1}{p}} u x \right|^{\frac{\delta}{2}}_{2} \\
\leq C \left| \rho^{\frac{p-2}{p}} u \right|^{p+2}_{p+2} \leq C \left( 1 + | \rho^{\frac{1}{p+2}} u |^{p+2}_{p+2} \right).
\]

Second, integrating (3.7) and (3.8) over \([0, t]\), one can obtain that for any \( p \geq 2 \),
\[
\left| \rho^{\frac{1}{p+2}} u \right|^{p+2}_{p+2} + \int_{0}^{t} \left| \rho^{\frac{1}{2}} u x \right|^{2}_{2} ds \\
\leq \left| \rho_{0}^{\frac{1}{p+2}} u_{0} \right|^{p+2}_{p+2} + C \left( t + \int_{0}^{t} \left| \rho^{\frac{1}{p+2}} u \right|^{p+2}_{p+2} ds \right),
\]
which, along with the Gronwall inequality, yields that
\[
\left| \rho^{\frac{1}{p+2}} u (t, \cdot) \right|^{p+2}_{p+2} + \int_{0}^{t} \left| \rho^{\frac{1}{2}} u x (s, \cdot) \right|^{2}_{2} ds \\
\leq C \left( 1 + \left| \rho_{0}^{\frac{1}{p+2}} u_{0} \right|^{p+2}_{p+2} \right) \leq C \left( 1 + | \rho_{0}^{\frac{1}{p+2}} u |^{p+2}_{p+2} \right) \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]

Finally, the desired conclusion can be achieved from (3.9) and Lemma 3.1. The proof of Lemma 3.5 is complete. \( \square \)

The following lemma gives the estimate of \( \int_{0}^{t} \left| \rho \dot{u} (s, \cdot) \right|_{\infty} ds \) \((\frac{1}{2} \leq \delta < 1)\), which plays a crucial role in obtaining the \( L^{\infty} \) estimate of the so-called effective velocity \( v \) (see (3.13)).

**Lemma 3.6.** Assume
\[
\delta = \frac{1}{2} \quad \text{for} \quad 0 < \delta < 1; \quad \frac{1}{2} < \delta < 1 \quad \text{for} \quad \delta = 1.
\]

Then for any \( T > 0 \), it holds that
\[
\int_{0}^{t} \left| \rho \dot{u} (s, \cdot) \right|_{\infty} ds \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]

**Proof.** **Case 1:** \( 0 < \delta < 1 \). First, it follows direct calculations that
\[
\dot{\partial}_{x} (\rho^{\frac{1}{2}} u) = \frac{1}{2} \rho^{\frac{1}{2}} \left( \delta - \frac{\delta}{2} \right) \rho_{x} \rho^{\frac{\delta}{2}} \rho^{\frac{1}{2}} u + \rho^{\frac{1}{2}} \rho_{x}^{\frac{1}{2}} \rho^{\frac{\delta}{2}} \rho^{\frac{1}{2}} u x.
\]

According to Lemmas 3.1-3.3 and 3.5, one has
\[
\left| \rho^{\delta - \frac{\delta}{2}} \rho_{x} \right|_{2} + \left| \rho^{\frac{1}{2}} \dot{u} \right|_{\frac{2}{2 - \delta}} + \left| \rho^{\frac{1}{2}} \dot{u} \right|_{\frac{2}{2 - \delta}} + \int_{0}^{t} \left| \rho^{\frac{1}{2}} u x \right|^{2}_{2} ds \leq C,
\]
which, along with Lemma 3.4, yields that
\[
\int_{0}^{t} \left| \partial_{x} (\rho^{\frac{1}{2}} u) (s, \cdot) \right|^{2}_{2} ds \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]
Second, notice that $\frac{3}{2} \in (1, 2)$, then by the Sobolev embedding theorem (or see Lemma A.1), one has

$$|\rho^\frac{1}{2} u|_\infty \leq C |\rho^\frac{1}{2} u|_2^{1-\kappa} |\partial_x (\rho^\frac{1}{2} u)|_\infty^{\kappa}$$

with $\kappa = \frac{1}{1+\delta} \in (0, 1)$,

which, along with Lemma 3.1 and (3.11), yields that

$$\int_0^t |\rho^\frac{1}{2} u(s, \cdot)|_\infty ds \leq C \int_0^t \left(1 + |\partial_x (\rho^\frac{1}{2} u)|_2^{\frac{2}{1+\delta}}\right)(s, \cdot) ds \leq C$$

for $0 \leq t \leq T$.

**Case 2:** $\delta = 1$. First, it follows direct calculations that

$$\partial_x (\rho u) = \rho^{\frac{1}{2}} \rho_x \rho^{\frac{1}{2}} u + \rho^{\frac{1}{2}} \rho_x u_x.$$

According to Lemmas 3.1-3.3, 3.5 and $\frac{1}{2} < \iota < 1$, one has

$$|\rho^{\frac{1}{2}} \rho_x|_2 + |\rho^{\frac{1}{2}} \rho_x|^2_2 + |\rho^{\frac{1}{2}} \rho_x u_x|^2_2 \leq C,$$

which yields that

$$\int_0^t |\partial_x (\rho u)(s, \cdot)|_2^3 ds \leq C \text{ for } 0 \leq t \leq T. \tag{3.12}$$

Second, notice that $\frac{1}{2} \in (1, 2)$, then by Sobolev embedding theorem (or see Lemma A.1), one has

$$|\rho^{\frac{1}{2}} u|_\infty \leq C |\rho^{\frac{1}{2}} u|_2^{1-\Xi} |\partial_x (\rho^{\frac{1}{2}} u)|_\infty^{\Xi}$$

with $\Xi = \frac{1}{3-2\iota} \in (0, 1)$,

which, along with Lemmas 3.1, 3.4 and (3.12), yields that

$$\int_0^t |\rho^{\frac{1}{2}} u(s, \cdot)|_\infty ds \leq C \int_0^t \left(1 + |\partial_x (\rho^{\frac{1}{2}} u)|_2^{\frac{2}{1}}\right)(s, \cdot) ds \leq C$$

for $0 \leq t \leq T$.

The proof of Lemma 3.6 is complete. \qed

Next we show the $L^\infty$ estimate of the effective velocity:

$$v = u + \varphi(\rho) = u + \alpha \rho^{\delta-2} \rho_x. \tag{3.13}$$

**Lemma 3.7.** Assume additionally

$$\gamma \geq \delta + \frac{1}{2} \text{ for } 0 < \delta < 1; \quad \gamma > \frac{3}{2} \text{ for } \delta = 1. \tag{3.14}$$

Then for any $T > 0$, it holds that

$$|v(t, \cdot)|_\infty \leq C \text{ for } 0 \leq t \leq T.$$

**Proof.** First, it follows from the system (1.7) and the definition of $v$ that

$$v_t + uv_x + \frac{A_\gamma}{\alpha} \rho^{\gamma-\delta} u - \frac{A_\gamma}{\alpha} \rho^{\gamma-\delta} u = 0. \tag{3.15}$$

Here, the detail derivation of the above equation for $v$ can be found in Appendix B.

Via the standard characteristic method, one can obtain

$$v = \left(v_0 + \int_0^t \frac{A_\gamma}{\alpha} \rho^{\gamma-\delta} u \exp \left(\int_0^s \frac{A_\gamma}{\alpha} \rho^{\gamma-\delta} d\tau\right) ds\right) \exp \left(-\int_0^t \frac{A_\gamma}{\alpha} \rho^{\gamma-\delta} ds\right).$$
Then according to Lemmas 3.4 and 3.6, one has
\[ |v|_\infty \leq C \left( |v_0|_\infty + \int_0^t |\rho^{\gamma-\delta} u|_\infty ds \right) \]
\[ \leq C \left( 1 + \int_0^t |\rho^{\gamma-\delta-\frac{1}{2}} \rho^{\frac{1}{2}} u|_\infty ds \right) \]
\[ \leq C \quad \text{for} \quad 0 < \delta < 1, \quad \gamma \geq \delta + \frac{1}{2}, \]
\[ |v|_\infty \leq C \left( |v_0|_\infty + \int_0^t |\rho^{\gamma-1} u|_\infty ds \right) \]
\[ \leq C \left( 1 + \int_0^t |\rho|^{\gamma-1} |\rho^{\frac{1}{2}} u|_\infty ds \right) \]
\[ \leq C \quad \text{for} \quad \delta = 1, \quad \gamma \geq 1 + \epsilon > \frac{3}{2}. \quad (3.16) \]

The proof of Lemma 3.7 is complete. \( \square \)

Now we show the \( L^2 \) estimate of velocity \( u \).

**Lemma 3.8.** Assume (3.14) additionally. Then for any \( T > 0 \), it holds that
\[ |u(t, \cdot)|_2^2 + \int_0^t |\rho^{\frac{1}{2}} u_x(s, \cdot)|_2^2 ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \]

**Proof.** Case 1: \( 0 < \delta < 1 \). From (2.2) \( 2 \), one has
\[ u_t + uu_x + A\gamma \rho^{-2} \rho_x = \alpha (\rho^{\delta-1} u_x)_x + \alpha \rho^{\delta-2} \rho_x u_x. \quad (3.17) \]
Multiplying (3.17) by \( u \) and integrating over \( \mathbb{R} \), one has
\[ \frac{1}{2} \frac{d}{dt} |u|^2 + \alpha |\rho^{\frac{1}{2}} u_x|^2 = \int (-uu_x - A\gamma \rho^{-2} \rho_x + \alpha \rho^{\delta-2} \rho_x u_x) u dx \triangleq \sum_{i=1}^3 J_i. \quad (3.18) \]
According to Hölder’s inequality, Young’s inequality and Lemmas 3.1-3.7, one can obtain
\[ J_1 = - \int u^2 u_x dx = 0, \]
\[ J_2 = - A\gamma \int \rho^{-2} \rho_x u dx = - A\gamma \int \rho^{-\delta} \rho^{\delta-2} \rho_x u dx \]
\[ = - \frac{A\gamma}{\alpha} \int \rho^{-\delta} (v - u) u dx \]
\[ \leq C \left( |\sqrt{\rho}|_2 |u|_2 |v|_\infty |\rho|^{\gamma-\frac{\delta}{2}} + |\rho|^{|-\delta|} |u|^2_2 \right) \leq C |u|^2_2 + C, \quad (3.19) \]
\[ J_3 = \alpha \int \rho^{\delta-2} \rho_x u_x u dx = \int (v - u) u_x u dx \]
\[ \leq C |v|_\infty |\rho|_{\frac{1}{\infty}} |\rho^{\frac{1}{2}} u_x|_2 |u|_2 \leq C |u|^2_2 + \frac{\alpha}{2} |\rho^{\frac{1}{2}} u_x|^2. \]
It follows from (3.18)-(3.19) and the Gronwall inequality that
\[ |u(t, \cdot)|_2^2 + \int_0^t |\rho^{\frac{\delta-1}{2}} u_x(s, \cdot)|_2^2 ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \]

**Case 2:** $\delta = 1$. Multiplying (2.14) by $u$ and integrating over $\mathbb{R}$, one has
\[ \frac{1}{2} \frac{d}{dt} |u|^2 + \alpha |u_x|^2 = \int \left( -uu_x - A\gamma \rho^{\gamma-2} \rho_x + \alpha \psi u_x \right) u dx. \]

Via the similar argument for estimating $J_1$-$J_3$, one can obtain
\[ |u(t, \cdot)|_2^2 + \int_0^t |u_x(s, \cdot)|_2^2 ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \]

The proof of Lemma 3.8 is complete. \hfill $\square$

### 3.3. The first order estimates of $u$

Now we are ready to consider the first order estimates of $u$.

**Lemma 3.9.** Assume (3.14) additionally. Then for any $T > 0$, it holds that
\[ |\rho^{\frac{\delta-1}{2}} u_x(t, \cdot)|_2^2 + \int_0^t \left( |u|^2 + |\rho^{\delta-1} u_{xx}|_2^2 + |u_x|_\infty \right)(s, \cdot) ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \]

**Proof.** **Case 1:** $0 < \delta < 1$. Multiplying (3.17) by $u_t$ and integrating over $\mathbb{R}$, one has
\[ \frac{\alpha}{2} \frac{d}{dt} |\rho^{\frac{\delta-1}{2}} u_x|^2 + |u_t|^2 \]
\[ = \int \left( \frac{\alpha}{2} (\rho^{\delta-1})_t u_x^2 - (uu_x + A\gamma \rho^{\gamma-2} \rho_x - \alpha \rho^{\delta-2} \rho_x u_x) u_t \right) dx \leq \sum_{i=4} J_i. \quad (3.20) \]

According to the equation (1.7)$_1$, Hölder’s inequality, Lemma A.1 and Lemmas 3.1-3.8, one can obtain that
\[ J_4 = \frac{\alpha}{2} \int (\rho^{\delta-1}) u_x^2 dx = -\frac{\alpha}{2} \int ((\rho^{\delta-1})_x u + (\delta - 1) \rho^{\delta-1} u_x) u_x^2 dx \]
\[ = -\frac{\alpha(\delta - 1)}{2} \int \rho^{\delta-2} \rho_x uu_x^2 dx - \frac{\alpha(\delta - 1)}{2} \int \rho^{\delta-1} u_x^3 dx \]
\[ = -\frac{\delta - 1}{2} \int (v - u) uu_x^2 dx - \frac{\alpha(\delta - 1)}{2} \int \rho^{\delta-1} u_x^3 dx \]
\[ = -\frac{\delta - 1}{2} \int vuu_x^2 dx + \frac{\delta - 1}{2} \int u^2 u_x^2 dx - \frac{\alpha(\delta - 1)}{2} \int \rho^{\delta-1} u_x^3 dx \]
\[ \leq -\frac{\delta - 1}{2} \int vuu_x^2 dx - \frac{\alpha(\delta - 1)}{2} \int \rho^{\delta-1} u_x^3 dx \]
\[ \leq C(|v|_\infty |u|_\infty |\rho|^{1-\delta}_\infty + |u_x|_\infty |\rho^{\frac{\delta-1}{2}} u_x|_2^2) \]
\[ \leq C(1 + |u_{xx}|_1^\frac{1}{2}) |u_x|_2^\frac{1}{2} |\rho^{\frac{\delta-1}{2}} u_x|_2^\frac{1}{2}. \]
Then we need to consider $|u_{xx}|$. According to (3.17), one has

$$
|u_{xx}| \leq C|\rho|^1 s |\rho^{-1} u_{xx}|^2 \\
\leq C(|u_t|^2 + |u_x|^2 + |\rho^{-2} u_x|^2 + |\rho^{-2} u_x|) \\
\leq C(|u_t|^2 + |u_x|^2 + (v-u)u_x| + |\rho^{-2}(v-u)|) \\
\leq C(|u_t|^2 + |u_x|^2|u_x| + |(v - u)|u_x| + |\rho^{-2}(v - u)|) \\
+ |\rho^{-2} u_x|v| + |\rho|^{-2} u_x|) \\
\leq C(1 + |u_t|^2 + |u_x|^2) + \frac{1}{2}|u_{xx}|^2,
$$

which implies that

$$
|u_{xx}| \leq C \left(1 + |u_t|^2 + |u_x|^2\right). \tag{3.22}
$$

Consequently, according to (3.21)-(3.22), Lemma 3.4 and Young’s inequality, one can obtain

$$
J_4 \leq C \left(1 + |\frac{\delta}{2} u_{xx}|^\frac{10}{6} \right) + \frac{1}{8}|u_t|^2. \tag{3.23}
$$

For the terms $J_i$ ($i = 5, 6, 7$), one can similarly obtain

$$
\begin{align*}
J_5 &= -\int u_x u_t dx \leq C|u_x| |u_t|^2 \\
&\leq C|u|^2 |u_x|^2 |\rho|^\frac{1}{2} |u_x| |u_t|^2 \\
&\leq C|\rho^{-\frac{1}{2}} u_x|^2 + \frac{1}{8}|u_t|^2,
\end{align*}
$$

$$
\begin{align*}
J_6 &= -A\gamma \int \rho^{\gamma-2} \rho_x u_t dx = -\frac{A\gamma}{\alpha} \int \rho^{\gamma-2} (v-u)u_t dx \\
&\leq C\left(|\rho^{\gamma-2}|v| + |\rho|^{-\gamma}|u_t|\right) |u_t|^2 \leq C + \frac{1}{8}|u_t|^2, \\
J_7 &= \alpha \int \rho^{\frac{\gamma}{2}} \rho_x u_t dx = \int (v-u)u_x u_t dx \\
&\leq C \left(|v| + |u_t|\right) |u_x|^2 |u_t|^2 \\
&\leq C(1 + |u_x|^2) |\rho^{-\frac{1}{2}} u_x|^2 + \frac{1}{8}|u_t|^2.
\end{align*}
$$

Substituting (3.21)-(3.24) into (3.20), one gets

$$
\frac{d}{dt}|\rho^{\frac{\delta}{2}} u_{xx}(t, \cdot)|^2 + |u_{t\cdot}^2| \leq C \left(1 + |\rho^{\frac{\delta}{2}} u_{xx}(t, \cdot)|^\frac{10}{6}\right), \tag{3.25}
$$

which, along with the Gronwall inequality and (3.22), yields that

$$
|\rho^{\frac{\delta}{2}} u_{xx}(t, \cdot)|^2 + \int_0^t \left(|u_{t\cdot}^2| + |\rho\cdot u_x|^2\right) ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \tag{3.26}
$$

**Case 2:** $\delta = 1$. First, similarly to the derivation of (3.22), one still has

$$
|u_{xx}| \leq C \left(1 + |u_t|^2 + |u_x|^2\right). \tag{3.27}
$$
Second, multiplying (2.14) by $u_t$ and integrating over $\mathbb{R}$, one has
\[
\frac{\alpha}{2} \frac{d}{dt}|u_t|^2 + |u_t|^2 = -\int uu_xu_tdx - A_\gamma \int \rho^{-2}\rho_xu_tdx + \alpha \int \rho^{-1}\rho_xu_xu_tdx.
\]
Via the similar arguments for estimating $J_5-J_7$, one can get
\[
|u_x(t,\cdot)|^2 + \int_0^t(|u_t(s,\cdot)|^2 + |u_{xx}(s,\cdot)|^2)ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \tag{3.28}
\]
Finally, it follows from (3.26), (3.28) and Sobolev embedding theorem that
\[
\int_0^t|u_x(s,\cdot)|_{\infty}ds \leq C \int_0^t|u_x(s,\cdot)|_2^\frac{3}{2}|u_{xx}(s,\cdot)|_2^\frac{1}{2}ds \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]
The proof of Lemma 3.9 is complete. \hfill $\Box$

3.4. The second order estimates of $u$. We consider the second order estimates of $u$. For this purpose, we first give the $L^2$ and $L^4$ estimates for $\psi$.

**Lemma 3.10.** Assume (3.14) additionally. Then for any $T > 0$, it holds that
\[
|\psi(t,\cdot)|_2 + |\psi(t,\cdot)|_4 \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]

**Proof.** Multiplying (3.15) by $v$ and integrating the resulting equation over $\mathbb{R}$, one has
\[
\frac{d}{dt}|v|^2 \leq C \left( |u_x|_\infty|v|_2^2 + \rho^{-\delta}|u|_{\infty}^2 + \rho^{-\delta}|u_x|_2^2 \right) \leq C(1 + |u_x|_\infty)|v|_2^2 + C,
\]
which, along with Lemma 3.9 and the Gronwall inequality, yields that
\[
|v(t,\cdot)|_2 \leq C \quad \text{for} \quad 0 \leq t \leq T. \tag{3.29}
\]
It thus follows from (3.29), Lemmas 3.7-3.9 and the Sobolev embedding theorem that
\[
|\psi(t,\cdot)|_2 \leq C (|v(t,\cdot)|_2 + |u(t,\cdot)|_2) \leq C \quad \text{for} \quad 0 \leq t \leq T,
\]
\[
|\psi(t,\cdot)|_4 \leq C (|v(t,\cdot)|_4 + |u(t,\cdot)|_4) \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]
The proof of Lemma 3.10 is complete. \hfill $\Box$

Now we are ready to show the second order estimates of $u$.

**Lemma 3.11.** Assume (3.14) additionally. Then for any $T > 0$, it holds that
\[
|u_t(t,\cdot)|_2^2 + |u_{xx}(t,\cdot)|_2^2 + |\rho^{-\delta}u_{xx}(t,\cdot)|_2^2 + \int_0^t|\rho^{-\frac{\delta}{2}}u_{xx}(s,\cdot)|_2^2ds \leq C \quad \text{for} \quad 0 \leq t \leq T.
\]

**Proof.** **Case 1:** $0 < \delta < 1$. First, according to Lemmas 3.4, 3.8-3.10 and the Sobolev embedding theorem, one has
\[
|\phi_x|_4 = |A_\gamma \rho^{-\delta} \rho^{-\delta-2}\rho_x|_4 \\
\leq C|\rho|_\infty^{\gamma-\delta}|\rho^{-\delta-2}\rho_x|_4 \leq C(|v|_4 + |u|_4) \leq C.
\]
Then it follows from (2.2) that
\[
|\phi|_2 \leq C (|u|_4|\phi_x|_4 + |\phi|_\infty|u_x|_2) \leq C.
\]
According to (3.22) and Lemma 3.9, one has

\begin{align*}
|u_{xx}|_2 &\leq C(1 + |u_t|_2 + |u_x|_2^\frac{3}{2}) \leq C(1 + |u_t|_2), \\
|\rho^{-1}u_{xx}|_2 &\leq C(1 + |u_t|_2 + |u_x|_2^\frac{3}{2}) + \frac{1}{2}|u_{xx}|_2 \leq C(1 + |u_t|_2). 
\end{align*}

(3.30)

Second, differentiating (2.2) with respect to $t$, multiplying the resulting equation by $u_t$ and integrating over $\mathbb{R}$, one has

\begin{align*}
\frac{1}{2} \frac{d}{dt} |u_t|_2^2 + \alpha |\rho^{-\frac{1}{2}} u_{tx}|_2^2 \\
= \int \left( \alpha(\rho^{-1}) u_{xx} - \alpha(\rho^{-1}) u_t x + \alpha(\psi u_x)_t - (u u_x)_t - \phi_{tx} \right) u_t dx \pm \sum_{i=8}^{12} J_i. 
\end{align*}

(3.31)

According to the equations (1.7), Hölder’s inequality, Young’s inequality and Lemmas 3.4 and 3.8-3.10, one gets

\begin{align*}
J_8 &= \alpha \int (\rho^{-1}) u_{xx} u_t dx = \alpha(\delta - 1) \int \rho^{-\frac{1}{2}} \rho u_{xx} u_t dx \\
&= - \alpha(\delta - 1) \int \left( \rho^{-\frac{3}{2}} \rho_{xx} u + \rho^{-1} u_{xx} \right) u_t u_t dx \\
&\leq C \left( |\rho^{-\frac{3}{2}} \rho_{xx} u|_4 u_{xx}|u_t|_2 + |\rho^{-\frac{1}{2}} u_{xx}|_2 \rho^{-\frac{1}{2}} u_{xx}|u_t|_2 \right)|u_t|_\infty \\
&\leq C \left( |u_{xx}|_2 + |\rho^{-\frac{1}{2}} u_{xx}|_2 |u_t|_2^\frac{1}{2} |u_{tx}|_2^\frac{1}{2} \right) \\
&\leq C \left( |u_t|_2^2 + |\rho^{-\frac{1}{2}} u_{xx}|_2^2 \right) + \frac{\alpha}{8} |\rho^{-\frac{1}{2}} u_{xx}|_2^2, 
\end{align*}

(3.32)

\begin{align*}
J_9 &= - \alpha \int (\rho^{-1}) u_{tx} u_t dx = - \alpha(\delta - 1) \int \rho^{-\frac{3}{2}} \rho_{xx} u_{tx} u_t dx \\
&\leq C |\rho^{-\frac{3}{2}} \rho_{xx}|_4 |u_{tx}|_2 |u_t|_4 \\
&\leq C |u_{tx}|_2^\frac{5}{2} |u_t|_2^\frac{3}{2} \leq C |u_t|_2^\frac{3}{2} + \frac{\alpha}{8} |\rho^{-\frac{1}{2}} u_{xx}|_2^2, 
\end{align*}

where one has used the facts

\begin{align*}
|u_t|_\infty &\leq C |u_t|_2^\frac{3}{2} |u_{tx}|_2^\frac{1}{2} \quad \text{and} \quad |u_t|_4 \leq C |u_{tx}|_2^\frac{5}{2} |u_t|_2^\frac{3}{2}.
\end{align*}
Similarly, via the integration by parts, one has
\[ J_{10} = \alpha \int (\psi u_x)_t u_t \, dx \]
\[ = \alpha \int \left( \psi u_{tx} u_t - ((u\psi)_x + (\delta - 1)\psi u_x + \delta \rho^{\delta - 1} u_{xx}) u_x u_t \right) \, dx \]
\[ \leq C \left( |\psi|_4 (|u_{xx}|_2 |u_t|_4 + |u|_{\infty} |u_{xx}|_2 |u_t|_4 + |u|_4 |u_x|_{\infty} |u_t|_2 \right) \]
\[ + |u_x|_2 |u_t|_4 |u_x|_{\infty} + |\rho^{\delta - 1} u_{xx}|_2 |u_x|_2 |u_t|_2 \right) \]
\[ \leq C \left( 1 + |u_t|^2 + |\rho^{\delta - 1} u_{xx}|^2 \right) + \frac{\alpha}{8} |\rho^{\frac{\delta - 1}{2}} u_{tx}|^2, \quad (3.33) \]
\[ J_{11} = - \int (uu_{tx})_t u_t \, dx = - \int (uu_{tx} u_t + u_t^2 u_x) \, dx \]
\[ \leq C \left( |u|_{\infty} |u_{xx}|_2 |u_t|_2 + |u_x|_{\infty} |u_{tx}|^2 \right) \]
\[ \leq C \left( 1 + |u_{xx}|^\frac{1}{2} \right) |u_t|^2 + \frac{\alpha}{8} |\rho^{\frac{\delta - 1}{2}} u_{tx}|^2, \]
\[ J_{12} = - \int \phi_{tx} u_t \, dx = \int \phi_t u_{tx} \, dx \]
\[ \leq C |\phi_t|_2 |u_{tx}|_2 \leq C + \frac{\alpha}{8} |\rho^{\frac{\delta - 1}{2}} u_{tx}|^2. \]

Substituting (3.32)-(3.33) into (3.31), one can obtain
\[ \frac{d}{dt} |u_t|^2 + \rho^{\frac{\delta - 1}{2}} |u_{tx}|^2 \leq C \left( 1 + (1 + |u_{xx}|^\frac{1}{2}) |u_t|^2 + |\rho^{\delta - 1} u_{xx}|^2 \right). \quad (3.34) \]

Integrating (3.34) over \((\tau, t)(\tau \in (0, t))\), one has
\[ |u_t(\tau, \cdot)|^2_2 + \int_\tau^t \rho^{\frac{\delta - 1}{2}} u_{tx}(s, \cdot)|^2_2 \, ds \leq |u_t(\tau, \cdot)|^2_2 + C \int_\tau^t \left( 1 + (1 + |u_{xx}|^\frac{1}{2}) |u_t|^2 + |\rho^{\delta - 1} u_{xx}|^2 \right)(s, \cdot) \, ds. \quad (3.35) \]

It follows from (2.2)\textsubscript{2} that
\[ |u_t(\tau, \cdot)|_2 \leq C \left( |u|_{\infty} |u_x|_2 + |\phi_x|_2 + |\rho^{\delta - 1} u_{xx}|_2 + |\psi|_2 |u_x|_{\infty} \right)(\tau), \quad (3.36) \]
which, together with the time continuity of \((\rho, u)\) and (1.13)-(1.14), implies that
\[ \lim_{\tau \to 0} |u_t(\tau, \cdot)|_2 \leq C \left( |u_0|_{\infty} |\partial_x u_0|_2 + |\partial_x \phi_0|_2 + |g_2|_2 + |\psi_0|_2 |\partial_x u_0|_{\infty} \right) \leq C. \]

Letting \(\tau \to 0\) in (3.35) and using the Gronwall inequality, one can obtain
\[ |u_t(t, \cdot)|^2_2 + \int_0^t \rho^{\frac{\delta - 1}{2}} u_{tx}(s, \cdot)|^2_2 \, ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \]

It thus follows from (3.30) that
\[ |u_{xx}(t, \cdot)|_2 + |\rho^{\delta - 1} u_{xx}(t, \cdot)|_2 \leq C(1 + |u_t(t, \cdot)|_2) \leq C \quad \text{for} \quad 0 \leq t \leq T. \]
Case 2: $\delta = 1$. First, according to (2.14)$_2$, Hölder’s inequality, Young’s inequality and Lemmas 3.8-3.10, one has

\[
|u_{xx}|^2 = \frac{1}{\alpha} |u_t + uu_x + \phi_x - \alpha \psi u_x|^2 \\
\leq C\left(|u_t|^2 + |u_x| |u_{xx}|^2 + |\phi_x|^2 + |\psi|^4 |u_x|^4\right) \\
\leq C\left(1 + |u_t|^2 + \frac{3}{2} |u_{xx}|^2 \right) \\
\leq C\left(1 + |u_t|^2 + \frac{1}{2} |u_{xx}|^2, \right)
\]

which implies that

\[
|u_{xx}|^2 \leq C(1 + |u_t|^2). \tag{3.37}
\]

Second, differentiating (2.14)$_2$ with respect to $t$, multiplying the resulting equation by $u_t$ and integrating over $\mathbb{R}$, one has

\[
\frac{1}{2} \frac{d}{dt} |u_t|^2 + \alpha |u_x|^2 = - \int (uu_x + \phi_x - \alpha \psi u_x) u_t dx \triangleq \sum_{i=13}^{15} J_i. \tag{3.38}
\]

Then according to Hölder’s inequality, Young’s inequality, the Sobolev embedding theorem and Lemmas 3.1-3.10, one has

\[
J_{13} = - \int (uu_x) t u_t dx = - \int (|u_t|^2 u_x + uu_{tx} u_t) dx \\
\leq C(|u_x| |u_{xx}|^2 + |u_x| |u_{tx}| |u_t|^2) \\
\leq C\left(1 + |u_x|^2|u_{xx}|^2 + \alpha |u_{tx}|^2, \right)
\]

\[
J_{14} = - \int \phi_{tx} u_t dx = \int \phi_t u_{tx} dx \\
\leq C|\phi_t|^2|u_{tx}|^2 \leq C + \alpha |u_{tx}|^2,
\]

\[
J_{15} = \alpha \int (\psi u_x) t u_t dx = \alpha \int (\psi u_{tx} u_t + \psi_t u_x u_t) dx \\
= \alpha \int (\psi u_{tx} u_t - (u\psi)_x u_x u_t - u_{xx} u_x u_t) dx \\
\leq C\left(|\psi|^2 |u_{tx}| |u_t| + |u_x| |\psi|^2 \left(|u_{xx}| |u_t| + |u_x| |u_t| \right) \\
+ |u_x| |u_{xx}| |u_t|^2 \right) \\
\leq C\left(|u_t|^2 |u_{tx}|^2 + |u_{xx}|^2 |u_t|^2 + |u_{xx}|^2 |u_{tx}|^2 + |u_x|^2 |u_{xx}|^2 |u_{tx}|^2 \right) \\
\leq C\left(1 + |u_{xx}|^2 \right) \left(1 + |u_t|^2 \right) + \alpha |u_{xx}|^2,
\]

which, along with (3.37)-(3.38), yields that

\[
\frac{d}{dt} |u_t|^2 + |u_x|^2 \leq C\left(1 + |u_{xx}|^2 \right) \left(1 + |u_t|^2 \right) \leq C\left(1 + |u_t|^2 \right)^2. \tag{3.39}
\]
Integrating (3.39) over \((\tau, t) (\tau \in (0, t))\), one has
\[
|u_t(t, \cdot)|^2_2 + \int_\tau^t |u_{tx}(s, \cdot)|^2_2 ds \leq |u_t(\tau, \cdot)|^2_2 + C \int_\tau^t (1 + |u_t(s, \cdot)|^2_2) ds. \tag{3.40}
\]

According to (2.14)_2, one can obtain
\[
|u_t(\tau, \cdot)|_2 \leq C(|u_{xx}|_2 + |\phi_x|_2 + |u_{tx}|_2 + |\psi u_x|_2)(\tau)
\leq C(|u_\infty|_{x, |t|}^2 + |\phi_x|_2 + |u_{xx}|_2 + |\psi|_{t, \infty}(\tau),
\]
which, along with the time continuity of \((\rho, u)\) and (1.16), yields that
\[
\limsup_{\tau \to 0} |u_t(\tau, \cdot)|_2 \leq C\left(|u_0|_{\infty} + \|\phi_0\|_1 + \|u_0\|_2 + |\psi_0|_2\|u_0\|_2\right) \leq C.
\]

Letting \(\tau \to 0\) in (3.40), it follows from the Gronwall inequality that
\[
|u_t(t, \cdot)|^2_2 + |u_{xx}(t, \cdot)|^2_2 + \int_0^t |u_{tx}(s, \cdot)|^2_2 ds \leq C \quad \text{for } 0 \leq t \leq T.
\]

The proof of Lemma 3.11 is complete. \(\Box\)

3.5. The high order estimates of \(\phi\) and \(\psi\). The following lemma provides the high order estimates of \(\phi\) and \(\psi\).

**Lemma 3.12.** Assume (3.14) additionally. Then for any \(T > 0\), it holds that
\[
|\phi_{xx}(t, \cdot)|^2_2 + |\phi_{tx}(t, \cdot)|^2_2 + |\psi_x(t, \cdot)|^2_2 + |u_t(t, \cdot)|^2_2
+ \int_0^t (|u_{xxx}|^2_2 + |\rho^{-1} u_{xxx}|^2_2 + |\rho^{-1} u_{xx}|^2_2 + |\phi_t|^2_2)(s, \cdot) ds \leq C \quad \text{for } 0 \leq t \leq T.
\]

**Proof.** Case 1: \(0 < \delta < 1\). The proof will be divided into two steps.

**Step 1:** the estimate on \(u_{xxx}\). First, it follows from (2.2)_2, Hölder’s inequality, Young’s inequality and Lemmas 3.1-3.11 that
\[
|u_{xxx}|_2 \leq |\rho^{1-\delta} u_{xx}|^2_2 \leq C\left(\|u\|_{\infty}^{1-\delta} |\rho^{1-\delta} u_{xx}|^2_2 \leq C|\rho^{1-\delta} u_{xx}|^2_2
\leq C(|u_{tx}|_2 + |u_{xx}|_2 + |\phi_x|_2 + |(\psi u_x)|_2 + |(\rho^{-1})_{x, |t|} u_{xx}|_2)
\leq C\left(|u_{tx}|_2 + |u|_\infty |u_{xx}|_2 + |u_{xx}|_2 + |\phi_x|_2 + |\psi|_\infty |u_{xx}|_2 + |u_x|_2 \right)\right. \tag{3.41}
\]
\[
\leq C\left(1 + |u_{tx}|_2 + |\phi_x|_2 + |u_x|_2 \right).
\]

Second, we consider the last two terms on the right-hand side of (3.41). For \(|\psi_x|_2\), differentiating (2.2)_3 with respect to \(x\), multiplying the resulting equation by \(\psi_x\) and integrating over \(\mathbb{R}\), one can obtain
\[
\frac{d}{dx} |\psi_x|_2^2 \leq C \int \left(|u_x \psi_x \psi_x| + |u_{xx} \psi_x| + |\rho^{1-\delta} u_{xxx} \psi_x|\right) dx
\leq C\left(1 + |\psi_x|_2^2 + |\rho^{1-\delta} u_{xxx}|_2^2\right). \tag{3.42}
\]
For $|\phi_{xx}|^2$. Applying $\frac{\partial^2}{\partial t^2}$ to (2.2)$_1$, multiplying the resulting equation by $\phi_{xx}$ and integrating over $\mathbb{R}$, one has

$$\frac{d}{dt}|\phi_{xx}|^2 \leq C \int \left(|u_{xx}\phi_x| + |u_x\phi_{xx}| + |\phi u_{xxx}|\right)|\phi_{xx}| \, dx$$

$$\leq C\left(|u_{xx}|^2|\phi_x|_\infty + |u_x|\phi_{xx}|_2 + |\phi|_\infty u_{xxx}|_2\right)|\phi_{xx}|_2$$

$$\leq C\left(1 + |\phi_{xx}|^2 + |u_{xxx}|^2\right).$$

(3.43)

According to (3.41)-(3.43), one can obtain

$$\frac{d}{dt}(|\psi_x|^2 + |\phi_{xx}|^2) \leq C\left(1 + |u_{tx}|^2 + |\psi_x|^2 + |\phi_{xx}|^2\right),$$

which, along with the Gronwall inequality and Lemma 3.11, yields that

$$|\psi_x(t, \cdot)|^2 + |\phi_{xx}(t, \cdot)|^2 \leq C \quad \text{for } 0 \leq t \leq T.$$

At last, it thus follows from (3.41) and Lemma 3.11 that

$$\int_0^t \left(|u_{xxx}|^2 + |\rho^{\delta-1}u_{xxx}|^2 + |\rho^{\delta-1}u_{xxx}|^2\right)(s, \cdot) \, ds$$

$$\leq C \int_0^t \left(1 + |\psi_x|^2 + |u_{tx}|^2 + |\phi_{xx}|^2\right)(s, \cdot) \, ds \leq C \quad \text{for } 0 \leq t \leq T.$$

**Step 2:** estimates on time derivatives of the density. First, according to (2.2)$_3$, Hölder’s inequality and Lemma 3.11, one has

$$|\psi_t|^2 \leq C\left(|\rho^{\delta-1}u_{xx}|^2 + |u\psi_x|^2 + |u_x\psi|^2\right)$$

$$\leq C\left(1 + |u|_\infty|\psi_x|^2 + |u_x|_\infty|\psi|^2\right) \leq C.$$

Second, according to (2.2)$_1$ and Hölder’s inequality, one has

$$|\phi_{tx}| \leq C\left(|u\phi_{xx}|^2 + |u_x\phi_x|^2 + |\phi u_{xx}|^2\right)$$

$$\leq C\left(|u|_\infty|\phi_{xx}|^2 + |u_x|_\infty|\phi_x|^2 + |\phi|_\infty u_{xx}|_2\right) \leq C,$$

$$|\phi_{tt}| \leq C\left(|u_t\phi_x|^2 + |u_x\phi_x|^2 + |\phi_t u_x|^2 + |\phi u_{tx}|^2\right)$$

$$\leq C\left(|u_t|^2|\phi_x|_\infty + |u|_\infty|\phi_x|^2 + |\phi_t|^2 u_x|_\infty + |\phi|_\infty u_{tx}|_2\right)$$

$$\leq C\left(1 + |u_{tx}|_2\right),$$

which, along with Lemma 3.11, yields that

$$\int_0^t |\phi_{tt}(s, \cdot)|^2 \, ds \leq C \quad \text{for } 0 \leq t \leq T.$$

**Case 2:** $\delta = 1$. The proof will be divided into two steps.

**Step 1:** the estimate on $u_{xxx}$. First, it follows from (2.14)$_2$, Hölder’s inequality and Lemmas 3.1-3.11 that

$$|u_{xxx}| \leq C\left(|u_{tx}|^2 + |(u_{ux})_x|^2 + |\phi_{xx}|^2 + |(\psi u_x)_x|^2\right)$$

$$\leq C\left(|u_{tx}|^2 + |u|_\infty u_{xx}|_2 + |u_x|_\infty u_{x}|_2 + |\phi_{xx}|^2 + |\psi|_\infty u_{xx}|_2 + |\psi_x|_2 u_{xx}|_\infty\right)$$

$$\leq C\left(1 + |u_{tx}|^2 + |\psi_x|^2 + |\phi_{xx}|^2\right).$$

(3.45)
Second, for $|\psi_x|^2$, differentiating (2.14) with respect to $x$, multiplying the resulting equation by $\psi_x$ and using the integration by part, one can obtain

$$\frac{d}{dt}|\psi_x|^2 \leq C \int \left( |u_x \psi_x \psi_x| + |u_{xx} \psi \psi_x| + |u_{xxx} \psi_x| \right) dx$$

$$\leq C \left( |u_x|_\infty |\psi_x|^2 + |u_{xx}|_2 |\psi_x|_2 + |u_{xxx}|_2 |\psi_x|^2 \right)$$

$$\leq C \left( 1 + |\psi_x|^2 + |u_{xx}|_2^2 \right).$$

(3.46)

For $|\phi_{xx}|^2$, by the definition of $\phi$ and $\psi$, one has

$$|\phi_{xx}|^2 = (\gamma - 1)(|\psi\phi|_x)_2^2 \leq C \left( |\psi_x|^2 + |\phi|_\infty + |\psi_x|^2 \right) \leq C \left( 1 + |\psi_x|^2 \right).$$

(3.47)

Thus, combining (3.45)-(3.47) together, one deduces that

$$\frac{d}{dt}|\psi_x|^2 \leq C \left( 1 + |\psi_x|^2 + |u_{xx}|_2^2 \right),$$

(3.48)

which, together with the Gronwall inequality and Lemma 3.11, yields that

$$|\psi_x(t, \cdot)|_2 \leq C \quad \text{for} \quad 0 \leq t \leq T.$$  

(3.49)

It thus follows from (3.45), (3.47) and (3.49) that

$$\int_0^t |u_{xxx}(s, \cdot)|_2^2 ds \leq C \int_0^t \left( 1 + u_{xx}(s, \cdot) \right) ds \leq C \quad \text{for} \quad 0 \leq t \leq T.$$ 

Step 2: estimates on time derivatives of the density. The boundedness of $|\psi(t, \cdot)|_2$, $|\phi_{tx}(t, \cdot)|_2$ and $\int_0^t |\phi_t(s, \cdot)|_2^2 ds$ can be shown via similar argument as in the case $0 < \delta < 1$.

The proof of Lemma 3.12 is complete.  

3.6. Time-weighted energy estimates of $u$. At last, we establish the time-weighted energy estimates of $u$.

Lemma 3.13. Assume (3.14) additionally. Then for any $T > 0$, it holds that

$$t |\rho^{\delta-1} u_{tt}(t, \cdot)|_2^2 + \int_0^t \left( |u_{tt}|_2^2 + |u_{tx}|_2^2 \right) ds \leq C \quad \text{for} \quad 0 \leq t \leq T.$$ 

Proof. Case 1: $0 < \delta < 1$. First, differentiating (2.2) with respect to $t$, multiplying the resulting equation by $u_{tt}$ and integrating over $\mathbb{R}$, one arrives at

$$\frac{\alpha}{2} \frac{d}{dt} \rho^{\delta-1} u_{xx} + |u_{tx}|_2^2$$

$$= \int \left( \alpha (\rho^{\delta-1}) u_{xx} - \alpha (\rho^{\delta-1})_{xx} u_{tx} - (u u_x)_t \right)$$

$$- \phi_{tx} + \alpha (\psi u_x)_t \big|_{tt} dx + \frac{\alpha}{2} \int (\rho^{\delta-1})_{tt} u_{tx} dx \triangleq \sum_{j=16}^{21} J.$$ 

(3.50)
According to the equation (1.7), Hölder’s inequality, Lemma A.1 and Lemmas 3.1-3.12, one can obtain that
\[
J_{16} = \alpha \int \left( \rho^{\frac{\delta - 1}{2}} \right)_t u_{xx} u_t dx \\
= - \alpha \int \left( \left( \rho^{\frac{\delta - 1}{2}} \right)_x u + (\delta - 1) \rho^{\frac{\delta - 1}{2}} u_x \right) u_{xx} u_t dx \\
\leq C \left( |\psi|_\infty |u|_\infty |u_{xx}|_2 + \rho^{\frac{\delta - 1}{2}} |u_{xx}|_2 |u_t|_2 \right) \leq C + \frac{1}{8} |u_t|_2^2.
\]
and integrating with respect to \( s \) over \([\tau, t] \) for \( \tau \in (0, t) \), one has
\[
t |t^{\frac{\delta - 1}{2}} u_{tx} (t, \cdot) |_2^2 + \int_\tau^t s |u_{tt}(s, \cdot) |_2^2 ds \leq \tau |t^{\frac{\delta - 1}{2}} u_{tx} (\tau, \cdot) |_2^2 + C.
\]
which, along with (3.54), implies that
\[ \int_0^t s|u_{txx}(s,\cdot)|^2ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \]

**Case 2:** \( \delta = 1 \). Differentiating (2.14) with respect to \( t \), multiplying the resulting equation by \( u_{tt} \) and integrating over \( \mathbb{R} \), one has
\[ \frac{\alpha}{2} \frac{d}{dt}|u_{tx}|^2 + |u_{tt}|^2 = \int \left( -(uu_x)_t - \phi_{tx} + \alpha(\psi u_x)_t \right) u_{tt} dx. \]

Via the similar argument for estimating \( J_{18} - J_{20} \), one can get
\[ t|u_{tx}(t,\cdot)|^2 + \int_0^t s(|u_{tt}|^2 + |u_{txx}|^2)(s,\cdot)ds \leq C \quad \text{for} \quad 0 \leq t \leq T. \] (3.55)

The proof of Lemma 3.13 is complete. \( \Box \)

### 3.7. Proof of Theorem 1.1

Based on the local-in-time well-posedness obtained in Lemmas 2.1-2.2 and the global-in-time a priori estimates established in Lemmas 3.1-3.13, now we are ready to give the proof of Theorem 1.1.

**Step 1:** the global well-posedness of regular solutions. First, under the initial assumption (1.13)-(1.14), according to Lemmas 2.1-2.2, there exists a unique regular solution \((\rho, u)\) to the Cauchy problem (1.7)-(1.8) in \([0, T^*] \times \mathbb{R}\) for some \( T^* > 0 \), which satisfies (1.15) with \( T \) replaced by \( T^* \). Let \( T^* > 0 \) be the life span of the regular solution satisfying (1.15) obtained in Lemmas 2.1-2.2. It is obvious that \( T^* > T_0 \). Then we claim that \( T^* = \infty \). Otherwise, if \( T^* < \infty \), according to the uniform a priori estimates obtained in Lemmas 3.1-3.13 and the standard weak compactness theory, one can know that for any sequence \( \{t_k\}_{k=1}^\infty \) satisfying \( 0 < t_k < T^* \) and \( t_k \to T^* \) as \( k \to \infty \),

there exists a subsequence \( \{t_{1k}\}_{k=1}^\infty \) and functions \((\phi, u, \psi)(T^*, x)\) such that
\[ \phi(t_{1k}, x) \to \phi(T^*, x) \quad \text{in} \quad H^2 \quad \text{as} \quad k \to \infty, \]
\[ u(t_{1k}, x) \to u(T^*, x) \quad \text{in} \quad H^2 \quad \text{as} \quad k \to \infty, \]
\[ \psi(t_{1k}, x) \to \psi(T^*, x) \quad \text{in} \quad H^1 \quad \text{as} \quad k \to \infty. \] (3.56)

Second, we want to show that functions \((\phi, u, \psi)(T^*, x)\) satisfy all the initial assumptions shown in Lemma 2.1, which include (1.13)-(1.14) except \( \rho_0 \in L^1 \) and the following relationship between \( \phi \) and \( \psi \):
\[ \psi = \frac{a\delta}{\delta - 1}(\phi^{2\gamma})_x. \] (3.57)

It follows from (3.56) that (1.13) except \( \rho_0 \in L^1 \) still holds at the time \( t = T^* \).

Next for the relation in (3.57), we need to consider the following equation
\[ \phi_t + uu_x + (\gamma - 1)\phi u_x = 0, \] (3.58)

which holds in \([0, T^*) \times \mathbb{R}\) in the classical sense. Actually, if we regard \((\phi(T^*, x), u(T^*, x), \psi(T^*, x))\) and \((\phi_t(T^*, x), u(T^*, x), \psi(T^*, x))\),
\[ \phi_t(T^*, x) = -u(T^*, x)\phi_x(T^*, x) - (\gamma - 1)\phi(T^*, x)u_x(T^*, x), \]
as the extended definitions of the regular solution \((\phi(t, x), u(t, x), \psi(t, x))\) at the time \(t = T^*\), then one has

\[
\text{ess sup}_{0 \leq t \leq T^*} (\|\phi(t, \cdot)\|_2 + \|\phi_t(t, \cdot)\|_1) \leq C < \infty,
\]

which, together with the Sobolev embedding theorem, implies that

\[
\phi(t, x) \in C([0, T^*]; H^1).
\]

It follows from the first line in (3.56) and the consistency of weak convergence and strong convergence in \(H^1\) space that

\[
\phi(t_{1k}, x) \to \phi(T^*, x) \quad \text{in} \quad H^1 \quad \text{as} \quad k \to \infty.
\]  

(3.59)

Notice that for any \(0 < R < \infty\), there exists a generic constant \(C(T^*, R)\) such that

\[
\phi(t, x) \geq C(T^*, R) \quad \text{for any} \quad (t, x) \in [0, T^*] \times [-R, R],
\]  

(3.60)

which, along with the last line in (3.56) and (3.59), implies that (3.57) holds.

Moreover, it follows from Lemmas 3.1-3.13 and (3.56)-(3.57) that

\[
\sup_{\tau \leq t \leq T^*} \left( |\rho^\frac{\delta-1}{\gamma} u_x(t, \cdot)|_2 + |\rho^\frac{\delta-1}{\gamma} u_{xx}(t, \cdot)|_2 \right) \leq C,
\]  

(3.61)

for any \(\tau \in (0, T^*]\), which implies that \((\rho(T^*, x), u(T^*, x))\) satisfy all the initial assumptions on the initial data of Lemma 2.1.

Finally, if we solve the system (1.7) with the initial time \(T^*\), then Lemma 2.1 ensures that for some constant \(T_0 > 0\), \((\rho, u)(t, x)\) is the unique regular solution in \([T^*, T^* + T_0] \times \mathbb{R}\) to this problem, which satisfies (1.15) with \([0, T]\) replaced by \([T^*, T^* + T_0]\) except \(\rho \in C([T^*, T^* + T_0]; L^1)\). It follows from the boundedness of all required norms of the solution \((\rho, u)(t, x)\) in \([0, T^* + T_0] \times \mathbb{R}\) and the standard arguments for proving the time continuity of the regular solution (see Section 3.6 (pages 138-139) of [49]) that, \((\rho, u)(t, x)\) is actually the unique regular solution in \([0, T^* + T_0] \times \mathbb{R}\) to the Cauchy problem (1.7)-(1.8), which satisfies (1.15) with \([0, T_*]\) replaced by \([0, T^* + T_0]\) except \(\rho \in C([0, T^* + T_0]; L^1)\). Furthermore, according to Lemma 2.2, one has that the solution constructed above also satisfies \(\rho \in C([0, T^* + T_0]; L^1)\). Then the above conclusions contradict to the fact that \(0 < T^* < \infty\) is the maximal existence time of the unique regular solution \((\rho, u)(t, x)\) satisfying (1.15).

Remark 3.1. According to the proof in this subsection, the definitions of \((\phi(T^*, x), u(T^*, x), \psi(T^*, x))\) do not depend on the choice of the time sequences \(\{t_k\}_{k=1}^\infty\).

Step 2: the global well-posedness of classical solutions. Now we show that the regular solution obtained above is indeed a classical one in \((0, T] \times \mathbb{R}\) for any \(T > 0\).

First, due to \(\phi > 0\) and

\[
\rho = \left( \frac{\gamma - 1}{\gamma} \phi \right)^{\frac{1}{\gamma - 1}},
\]

one has

\[
(\rho, \rho_t, \rho_x, u, u_x) \in C([0, T] \times \mathbb{R}).
\]  

(3.62)
Second, for the term \(u_t\), according to Lemma 3.13, one has
\[
\begin{align*}
t^{1/2}u_{tx} & \in L^\infty([0,T];L^2), \\
t^{1/2}u_{tt} & \in L^2([0,T];L^2), \\
t^{3/2}u_{txx} & \in L^2([0,T];L^2), \\
t^{3/2}u_{ttt} & \in L^2([0,T];H^{-1}),
\end{align*}
\] (3.63)
which, along with the classical Sobolev embedding theorem:
\[
L^2([0,T];H^1) \cap W^{1,2}([0,T];H^{-1}) \hookrightarrow C([0,T];L^2),
\] (3.64)
yields that for any \(0 < \tau < T\),
\[
tu_t \in C([0,T];H^1) \quad \text{and} \quad u_t \in C([\tau,T] \times \mathbb{R}).
\] (3.65)

Finally, it remains to show that \(u_{xx} \in C([\tau,T] \times \mathbb{R})\)
for any \(0 < \tau < T\). Actually, it follows from the fact \(\rho > 0\), (3.62) and (3.65) that
\[
u_{xx} = \frac{1}{\alpha} \rho^{1-\delta} (u_t + uu_x + \phi_x - \alpha \psi u_x) \in C([\tau,T] \times \mathbb{R}).
\]

Until now, we have completed the proof of Theorem 1.1.

3.8. **Proof of Theorem 1.2.** The proof of Theorem 1.2 can be achieved via the similar argument used in Section 3.7.

4. **Non-existence of global solutions with \(L^\infty\) decay on \(u\)**

This section will be devoted to providing the proofs for the non-existence theories of global regular solutions with \(L^\infty\) decay on \(u\) shown in Theorem 1.3 and Corollary 1.1.

4.1. **Proof of Theorem 1.3.** Now we are ready to prove Theorem 1.3. Let \(T > 0\) be any constant, and \((\rho, u) \in D(T)\). It follows from the definitions of \(m(t)\), \(\mathbb{P}(t)\) and \(E_k(t)\) that
\[
|\mathbb{P}(t)| \leq \int \rho(t,x)|u(t,x)| \, dx \leq \sqrt{2m(t)E_k(t)},
\]
which, together with the definition of the solution class \(D(T)\), implies that
\[
0 < \frac{|\mathbb{P}(0)|^2}{2m(0)} \leq E_k(t) \leq \frac{1}{2} m(0) |u(t,\cdot)|_\infty^2 \quad \text{for} \quad t \in [0,T].
\]

Then one obtains that there exists a positive constant
\[
C_u = \frac{|\mathbb{P}(0)|}{m(0)}
\]
such that
\[
|u(t,\cdot)|_\infty \geq C_u \quad \text{for} \quad t \in [0,T].
\]
Thus one obtains the desired conclusion as shown in Theorem 1.3.
4.2. Proof of Corollary 1.1. Let \((\rho, u)(t, x)\) in \([0, T] \times \mathbb{R}\) be the global regular solution obtained in Theorem 1.1 or 1.2. Next we just need to show that \((\rho, u) \in D(T)\).

Actually, the conservation of the total mass and the energy equality have been shown in Lemma 3.3 and Appendix B, and the conservation of the total momentum can be verified as follows.

**Lemma 4.1.** For any \(T > 0\), it holds that
\[
\mathbb{P}(t) = \mathbb{P}(0) \quad \text{for} \quad t \in [0, T].
\]

**Proof.** First, one can obtain that
\[
\mathbb{P}(t) = \int \rho u \, dx \leq C|\rho|_1|u|_\infty < \infty. \tag{4.1}
\]

Second, the momentum equation (1.7) implies that
\[
\frac{d}{dt}(\rho u^2) = -\int (\rho u^2)_x \, dx - \int P_x \, dx + \int (\mu(\rho)u_x)_x \, dx = 0, \tag{4.2}
\]
where one has used the fact that \(\rho u^2(t, \cdot), \rho^\gamma(t, \cdot)\) and \(\rho^\delta u_x(t, \cdot) \in W^{1,1}(\mathbb{R})\).

According to Theorem 1.3 and Lemma 4.1, the proof of Corollary 1.1 is complete.

**Appendix A. Basic lemmas**

In this appendix, we list some useful lemmas which were used frequently in the previous sections. The first one is the well-known Gagliardo-Nirenberg inequality.

**Lemma A.1** ([41]). Let \(1 \leq q, r \leq \infty\), and \(p\) satisfies
\[
\frac{1}{p} = \frac{j}{d} + \zeta \left(\frac{1}{r} - \frac{m}{d}\right) + (1 - \zeta)\frac{1}{q}, \quad \text{and} \quad \frac{j}{m} \leq \zeta \leq 1.
\]
Then there exists a generic constant \(C > 0\) depends only on \(j, m, d, q, r\) and \(\zeta\) such that for \(f \in L^q \cap D^{m,r}(\mathbb{R}^d)\),
\[
|D^j f|_p \leq C|D^m f|^\zeta|f|_q^{1-\zeta}, \tag{A.1}
\]
with the exceptions that if \(j = 0\), \(rm < d, q = \infty\) we assume that \(f\) vanishes at infinity or \(f \in L^{|\tilde{q}|}\) for some finite \(\tilde{q} > 0\), while if \(1 < r < \infty\) and \(m - j - d/r\) is a non-negative integer we take \(j/m \leq \zeta < 1\).

In particular, letting \(d = 1\), \(j = 0\) and \(p = \infty\) in (A.1), one has
\[
|f|_\infty \leq C|f|^\frac{1}{2}|f_{x^2}|^\frac{1}{2}.
\]

The second one is Fatou’s lemma.

**Lemma A.2.** Given a measure space \((V, \mathcal{F}, \nu)\) and a set \(X \in \mathcal{F}\), let \(\{f_n\}\) be a sequence of \((\mathcal{F}, \mathcal{B}_{\mathbb{R}_{>0}})\)-measurable non-negative functions \(f_n : X \to [0, \infty]\). Define the function \(f : X \to [0, \infty]\) by setting
\[
f(x) = \liminf_{n \to \infty} f_n(x),
\]
for every $x \in X$. Then $f$ is $(\mathcal{F},\mathcal{B}_{\mathbb{R}^2_0})$-measurable, and
\[ \int_X f(x)\,d\nu \leq \liminf_{n \to \infty} \int_X f_n(x)\,d\nu. \]

The following lemma is used to obtain the time-weighted estimates of the velocity $u$.

**Lemma A.3.** [1] If $f(t,x) \in L^2([0,T];L^2)$, then there exists a sequence $s_k$ such that $s_k \to 0$ and $s_k|f(s_k,\cdot)|^2 \to 0$ as $k \to \infty$.

The last one shows some compactness results obtained via the Aubin-Lions Lemma.

**Lemma A.4 ([43]).** Let $X_0 \subset X \subset X_1$ be three Banach spaces. Suppose that $X_0$ is compactly embedded in $X$ and $X$ is continuously embedded in $X_1$. Then the following statements hold

(1) If $J$ is bounded in $L^p([0,T];X_0)$ for $1 \leq p < \infty$, and $\partial J/\partial t$ is bounded in $L^1([0,T];X_1)$, then $J$ is relatively compact in $L^p([0,T];X)$.

(2) If $J$ is bounded in $L^\infty([0,T];X_0)$ and $\partial J/\partial t$ is bounded in $L^p([0,T];X_1)$ for $p > 1$, then $J$ is relatively compact in $C([0,T];X)$.

**Appendix B. Proofs of Lemmas 3.1-3.2 and the Equation (3.15)**

In this appendix, for the sake of completeness, we will give the proof of Lemmas 3.1-3.2 and the derivation of the equation (3.15).

**Proof of Lemma 3.1.** It follows from (1.7) that
\[ \frac{P(\rho)_t}{\gamma - 1} + \frac{(P(\rho)u)_x}{\gamma - 1} + P(\rho) u_x = 0. \] (B.1)

Then multiplying (1.7) by $u$ and adding the resulting equation to (B.1), and integrating over $\mathbb{R}$, one arrives at
\[ \frac{d}{dt} \int \left( \frac{1}{2} \rho u^2 + \frac{P(\rho)}{\gamma - 1} \right) \,dx + \alpha \int \rho^\delta u_x^2 \,dx = 0, \] (B.2)
where one has used the fact that $\rho u^2(t,\cdot)$, $\rho^\gamma u(t,\cdot)$ and $\rho^\delta u_x u(t,\cdot) \in W^{1,1}(\mathbb{R})$.

Finally, integrating (B.2) over $[0,t]$, one can get the desired estimate.

The proof of Lemma 3.1 is complete.

**Proof of Lemma 3.2.** According to (1.7)$_1$ and (1.7)$_2$, one has
\[ \varphi(\rho)_{tx} + (\varphi(\rho)x)u_x + (\rho \varphi'(\rho)u)_x = 0, \] (B.3)
\[ \rho(u_t + uu_x) + P(\rho)_x = (\mu(\rho)u_x)_x. \] (B.4)

Multiplying (B.3) by $\rho$, one has
\[ \rho \varphi(\rho)_{tx} + \rho \varphi(\rho)_{xx} + (\rho^2 \varphi'(\rho)u)_x = 0. \] (B.5)

Then adding (B.5) to (B.4), by the definition of $v$, one can obtain
\[ \rho(v_t + uv_x) + P(\rho)_x - ((\mu(\rho) - \rho^2 \varphi'(\rho))u_x)_x = 0, \] (B.6)
which, along with
\[ \varphi'(\rho) = \mu(\rho) / \rho^2 \]
in (B.6), yields that
\[ \rho(v_t + uv_x) + P'(\rho) = 0. \]  
(B.7)
Multiplying (B.7) by \( v \) and integrating over \( \mathbb{R} \), one has
\[ \frac{d}{dt} \int \left( \frac{1}{2} \rho v^2 + \frac{A}{\gamma - 1} \rho^\gamma \right) dx + \int P'(\rho) \varphi' x dx = 0. \]  
(B.8)
Integrating (B.8) over \([0, T]\), one can obtain the desired conclusion.

**Proof of (3.15).** It follows from (B.7) that
\[
\begin{align*}
v_t + uv_x &= - \frac{P'(\rho)}{\rho} = -A\gamma \rho^{\gamma-2} \rho_x \\
&= -A\gamma \rho^{\gamma-\delta} \rho^{\delta-2} \rho_x = -\frac{A\gamma}{\alpha} \rho^{\gamma-\delta} (v - u),
\end{align*}
\]
which implies (3.15).

**Appendix C. Local-in-time well-posedness for multi-dimensional case**

In this appendix, for the sake of completeness, we will recall the local-in-time well-posedness theory (obtained in [31, 49]) of regular solutions with far field vacuum for the corresponding 3-D and 2-D Cauchy problems of the degenerate viscous flows.

**C.1. 0 < \delta < 1.** For the case \( 0 < \delta < 1 \), we first give the definition of regular solutions with far field vacuum to the 3-D Cauchy problem (1.1)-(1.5) with the following initial data and far field behavior:
\[
\begin{align*}
(\rho(0, x), u(0, x)) &= (\rho_0(x), u_0(x)) & &\text{for } x \in \mathbb{R}^3, \\
(\rho(t, x), u(t, x)) &\to (0, 0) & &\text{as } |x| \to \infty & &\text{for } t \geq 0.
\end{align*}
\]

**Definition C.1 ([49]).** Assume \( 0 < \delta < 1 \) and \( T > 0 \). The pair \((\rho, u)\) is called a regular solution to the Cauchy problem (1.1)-(1.5) with (C.1) in \([0, T] \times \mathbb{R}^3\), if \((\rho, u)\) satisfies this problem in the sense of distributions and:
\[
\begin{align*}
(1) & \quad \rho > 0, \quad \rho^{\gamma-1} \in C([0, T]; H^3), \quad \nabla \rho^{\delta-1} \in L^\infty([0, T]; L^\infty \cap D^2); \\
(2) & \quad u \in C([0, T]; H^3) \cap L^2([0, T]; H^4), \quad u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2), \quad \\
& \quad \rho^{\delta-1} \nabla u \in C([0, T]; L^2), \quad \rho^{\delta-1} \nabla u_t \in L^\infty([0, T]; L^2), \quad \\
& \quad \rho^{\delta-1} \nabla u \in L^\infty([0, T]; D^1), \quad \rho^{\delta-1} \nabla^2 u \in C([0, T]; H^1) \cap L^2([0, T]; D^2).
\end{align*}
\]

The corresponding local-in-time well-posedness can be stated as follows.

**Lemma C.1 ([49]).** Let parameters \( \gamma, \delta, \alpha \) and \( \beta \) satisfy
\[
\gamma > 1, \quad 0 < \delta < 1, \quad \alpha > 0, \quad 2\alpha + 3\beta \geq 0.
\]
Assume the initial data \((\rho_0, u_0)\) satisfy
\[
\begin{align*}
\rho_0 > 0, \quad (\rho_0^{\gamma-1}, u_0) &\in H^3, \quad \nabla \rho_0^{\delta-1} \in D^1 \cap D^2, \quad \nabla \rho_0^{\frac{\delta+1}{\alpha}} \in L^4,
\end{align*}
\]
and the initial compatibility conditions
\[ \nabla u_0 = \rho_0^{-\frac{\lambda}{2}} g_1, \quad Lu_0 = \rho_0^{-\frac{\delta}{2}} g_2, \quad \nabla \left( \rho_0^{\frac{\delta-1}{2}} Lu_0 \right) = \rho_0^{-\frac{\lambda}{2}} g_3, \]
for some \( g_i (i = 1, 2, 3) \in L^2 \), then there exist a time \( T_* > 0 \) and a unique regular solution \((\rho, u)\) in \([0, T_*) \times \mathbb{R}^2\) to the Cauchy problem (1.1)-(1.5) with (C.1) satisfying
\[ \rho^{\frac{\delta-1}{2}} u_t \in L^\infty([0, T_*]; L^2), \quad \rho^{\delta-1} \nabla^2 u_t \in L^2([0, T_*]; L^2), \]
\[ t^\frac{1}{2} u \in L^\infty([0, T_*]; D^4), \quad t^\frac{1}{2} u_t \in L^\infty([0, T_*]; D^2) \cap L^2([0, T_*]; D^3), \]
\[ u_{tt} \in L^2([0, T_*]; L^2), \quad t^\frac{3}{2} u_{tt} \in L^\infty([0, T_*]; L^2) \cap L^2([0, T_*]; D^4), \]
\[ \rho^{\delta-1} \in C([0, T_*]; L^\infty \cap D^{1.6} \cap D^{2.3} \cap D^3), \]
\[ \nabla \rho^{\delta-1} \in C([0, T_*]; D^1 \cap D^2), \quad \nabla \ln \rho \in L^\infty([0, T_*]; L^\infty \cap L^6 \cap D^{1.3} \cap D^2). \]

C.2. \( \delta = 1 \). For the case \( \delta = 1 \), we first give the definition of regular solutions with far field vacuum to the 2-D Cauchy problem of (1.1)-(1.5) with the following initial data and far field behavior:
\[ \begin{align*}
(\rho(0, x), u(0, x)) &= (\rho_0(x), u_0(x)) \quad \text{for} \quad x \in \mathbb{R}^2, \\
(\rho(t, x), u(t, x)) &\to (0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0.
\end{align*} \]

**Definition C.2** ([31]). Assume \( \delta = 1 \) and \( T > 0 \). The pair \((\rho, u)\) is called a regular solution to the Cauchy problem (1.1)-(1.5) with (C.2) in \([0, T] \times \mathbb{R}^2\), if \((\rho, u)\) satisfies this problem in the sense of distributions and:

1. \( 0 < \rho \in C^1([0, T] \times \mathbb{R}^2), \quad \rho^{\frac{1}{2}} \in C([0, T]; H^3), \quad \nabla \ln \rho \in C([0, T]; L^6 \cap D^1 \cap D^2); \)
2. \( u \in C([0, T]; H^3) \cap L^2([0, T]; H^1), \quad u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2); \)
3. \( \lim_{|x| \to \infty} (u_t + u \cdot \nabla u + Lu) = \lim_{|x| \to \infty} \left( \nabla \ln \rho \cdot Q(u) \right) \quad \text{for} \quad t > 0. \)

The corresponding local-in-time well-posedness can be stated as follows.

**Lemma C.2** ([31]). Let parameters \( \gamma, \delta, \alpha \) and \( \beta \) satisfy
\[ \gamma > 1, \quad \delta = 1, \quad \alpha > 0, \quad \alpha + \beta \geq 0. \]
Assume the initial data \((\rho_0, u_0)\) satisfy
\[ \rho_0 > 0, \quad (\rho_0^{-\frac{1}{2}}, u_0) \in H^3, \quad \nabla \ln \rho_0 \in L^6 \cap D^1 \cap D^2, \]
then there exist a time \( T_* > 0 \) and a unique regular solution \((\rho, u)\) in \([0, T_*) \times \mathbb{R}^2\) to the Cauchy problem (1.1)-(1.5) with (C.2) satisfying
\[ \begin{align*}
(\rho^{-\frac{1}{2}})_t &\in C([0, T_*]; H^3), \quad (\nabla \ln \rho)_t \in C([0, T_*]; H^1), \\
u_{tt} &\in L^2([0, T_*]; L^2), \quad t^{\frac{3}{2}} u_t \in L^\infty([0, T_*]; D^4), \\
u_{ttt} &\in L^\infty([0, T_*]; D^2) \cap L^2([0, T_*]; D^3), \quad t^{\frac{5}{2}} u_{tt} \in L^\infty([0, T_*]; L^2) \cap L^2([0, T_*]; D^4). 
\end{align*} \]

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