FATOU-BIEBERBACH DOMAINS IN $\mathbb{C}^n \setminus \mathbb{R}^k$

FRANC FORSTNERIČ AND ERLEND FornaESS WOLD

Abstract. We construct Fatou-Bieberbach domains in $\mathbb{C}^n$ for $n > 1$ which contain a given compact set $K$ and at the same time avoid a totally real affine subspace $L$ of dimension $< n$, provided that $K \cup L$ is polynomially convex. By using this result, we show that the domain $\mathbb{C}^n \setminus \mathbb{R}^k$ for $1 \leq k < n$ enjoys the Oka property with approximation for maps from any Stein manifold of dimension $< n$.

1. Introduction

A proper subdomain $\Omega$ of a complex Euclidean space $\mathbb{C}^n$ is called a Fatou-Bieberbach domain if $\Omega$ is biholomorphic to $\mathbb{C}^n$. Such domains abound in $\mathbb{C}^n$ for any $n > 1$; a survey can be found in [6, Chap. 4]. For example, an attracting basin of a holomorphic automorphism of $\mathbb{C}^n$ is either $\mathbb{C}^n$ or a Fatou-Bieberbach domain (cf. [19, Appendix] or [6, Theorem 4.3.2]). Fatou-Bieberbach domains also arise as regions of convergence of sequences of compositions $\Phi_k = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_1$, where each $\phi_j \in \text{Aut} \mathbb{C}^n$ is sufficiently close to the identity map on a certain compact set $K_j \subset \mathbb{C}^n$ and the sets $K_j \subset \tilde{K}_{j+1}$ exhaust $\mathbb{C}^n$; this is the so called push-out method (cf. [6, Corollary 4.4.2, p. 115]).

Besides their intrinsic interest, Fatou-Bieberbach domains are very useful in constructions of holomorphic maps. An important question is which pairs of disjoint closed sets $K, L \subset \mathbb{C}^n$ can be separated by a Fatou-Bieberbach domain $\Omega$, in the sense that $\Omega$ contains one of the sets and is disjoint from the other one. Recently it was shown by Forstnerič and Ritter [10] that this holds if $K \cup L$ is compact polynomially convex and one of the sets is convex or, more generally, holomorphically contractible. They applied this to the construction of proper holomorphic maps of Stein manifolds of dimension $< n$ to $\mathbb{C}^n$ avoiding compact convex sets. In this paper we prove a similar result in the case when $L$ is an affine totally real submanifold of $\mathbb{C}^n$ of dimension $< n$.

Theorem 1.1. Let $n > 1$. Assume that $L$ is a totally real affine subspace of $\mathbb{C}^n$ of dimension $\dim_{\mathbb{R}} L < n$ and $K$ is a compact subset of $\mathbb{C}^n \setminus L$. If $K \cup L$ is polynomially convex, then there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n$ with $K \subset \Omega \subset \mathbb{C}^n \setminus L$.

If $L$ is a closed, unbounded, totally real submanifold of $\mathbb{C}^n$ (for example, a totally real affine subspace as in the above theorem) and $K$ is a compact subset of $\mathbb{C}^n$, we say that $K \cup L$ is polynomially convex if $K \cup L'$ is such for every compact subset $L'$ of $L$. For results on such sets see for example [7].

Let $z = (z_1, \ldots, z_n)$, with $z_j = x_j + iy_j$ and $i = \sqrt{-1}$, denote the complex coordinates on $\mathbb{C}^n$. Any totally real affine subspace $L \subset \mathbb{C}^n$ of dimension $k$ can be mapped by an
affine holomorphic automorphism of $\mathbb{C}^n$ onto the standard totally real subspace

$$\mathbb{R}^k = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : y_1 = \cdots = y_k = 0, \; z_{k+1} = \cdots = z_n = 0\}.$$  

The conclusion of Theorem 1.1 is false if $\text{dim } L = n$ (the maximal possible dimension of a totally real submanifold of $\mathbb{C}^n$); an example is obtained by taking $L = \mathbb{R}^2 \subset \mathbb{C}^2$ and $K$ the unit circle in the imaginary subspace $i\mathbb{R}^2 \subset \mathbb{C}^2$. The reason for the failure in this example is topological: the circle links $\mathbb{R}^2$, in the sense that it is a nontrivial element of the fundamental group $\pi_1(\mathbb{C}^2 \setminus \mathbb{R}^2) = \pi_1(S^1) = \mathbb{Z}$, but it is contractible to a point in any Fatou-Bieberbach domain containing it. Such linking is impossible if $\text{dim } L < n$ since the complement $\mathbb{C}^n \setminus K$ of any compact polynomially convex set $K \subset \mathbb{C}^n$ is topologically a CW complex containing only cells of dimension $\geq n$, and hence it has vanishing homotopy groups up to dimension $n - 1$ (cf. [5, 13] or [6, §3.11]).

In spite of this failure of Theorem 1.1 for $\text{dim } L < n$, $\mathbb{C}^n \setminus \mathbb{R}^n$ is known to be a union of Fatou-Bieberbach domains for any $n > 1$; see Rosay and Rudin [19] or [6, Example 4.3.10, p. 112]. Therefore, the following seems a natural question.

**Problem 1.2.** Assume that $n > 1$ and $K$ is a compact set in $\mathbb{C}^n \setminus \mathbb{R}^n$ such that $K \cup \mathbb{R}^n$ is polynomially convex and $K$ is contractible to a point in $\mathbb{C}^n \setminus \mathbb{R}^n$. Does there exist a Fatou-Bieberbach domain $\Omega$ in $\mathbb{C}^n$ satisfying $K \subset \Omega \subset \mathbb{C}^n \setminus \mathbb{R}^n$?

Theorem 1.1 is proved in §2. We shall apply it to prove the following result.

**Theorem 1.3.** Assume that $L$ is an affine totally real subspace of $\mathbb{C}^n$ of real dimension $1 \leq \text{dim } L < n$. Let $X$ be a Stein manifold of dimension $\text{dim}_\mathbb{C} X < n$, $E \subset X$ be a compact $\mathcal{O}(X)$-convex set, $U \subset X$ be an open set containing $E$, and $f : U \to \mathbb{C}^n$ be a holomorphic map such that $f(E) \cap L = \emptyset$. Then $f$ can be approximated uniformly on $E$ by holomorphic maps $F : X \to \mathbb{C}^n \setminus L$.

In the language of Oka theory, this means that maps $X \to \mathbb{C}^n \setminus L$ from Stein manifolds $X$ of dimension $< n$ enjoy the Oka property with approximation (cf. [6, §5.15]). (For Oka theory we refer to the monograph [3], the surveys [7, 9], and the introductory note by Lárusson [16].) The conclusion of Theorem 1.3 is obvious if $2 \text{dim}_\mathbb{C} X + k < 2n$ since in this case a generic holomorphic map $X \to \mathbb{C}^n$ avoids any smooth submanifold of real dimension $\leq k$ in $\mathbb{C}^n$ in view of the Thom transversality theorem. However, the transversality theorem by itself does not suffice if $2 \text{dim}_\mathbb{C} X + k \geq 2n$. The first nontrivial case is when $k = 2, n = 3$ and $\text{dim}_\mathbb{C} X = 2$.

Theorem 1.3 is an immediate corollary to the following proposition which is proved in §3. The proof of Proposition 1.4 uses Theorem 1.1 together with the main result of the paper [3] by Drinovec Drnovšek and Forstnerič.

**Proposition 1.4.** Assume that $L \subset \mathbb{C}^n$, $E \subset X$ and $f: U \to \mathbb{C}^n$ are as in Theorem 1.3 with $f(E) \cap L = \emptyset$. Then there exist an arbitrarily small holomorphic perturbation $\tilde{f}$ of $f$ on a neighborhood of $E$ and a Fatou-Bieberbach domain $\Omega = \Omega_f \subset \mathbb{C}^n$ such that

$$\tilde{f}(E) \subset \Omega \subset \mathbb{C}^n \setminus L.$$  

Theorem 1.3 clearly follows from Proposition 1.4 by applying the Oka-Weil approximation theorem for holomorphic maps $X \to \Omega \simeq \mathbb{C}^n$.  

2
In the setting of Theorem 1.3, the map $f$ extends (without changing its values on a neighborhood of $E$) to a continuous map $f_0 : X \to \mathbb{C}^n \setminus L$ by purely topological reasons. In fact, the complement $\mathbb{C}^n \setminus L$ of an affine subspace $L \subset \mathbb{C}^n$ of real dimension $k$ is homotopy equivalent to the sphere $S^{2n-k-1}$ and we have $2n - k - 1 \geq n$ by the hypotheses of the theorem, while the pair $(X, E)$ is a relative CW complex of dimension at most $\dim_X X < n$ (see [13] or [6, p. 96]).

Theorem 1.3 is a first step in understanding the following problem.

**Problem 1.5.** Is $\mathbb{C}^n \setminus \mathbb{R}^k$ an Oka manifold for some (or for all) pair of integers $1 \leq k \leq n$ with $n > 1$? Equivalently, does the conclusion of Theorem 1.3 hold for maps $X \to \mathbb{C}^n \setminus \mathbb{R}^k$ from all Stein manifolds $X$ irrespectively of their dimension?

Since $\mathbb{C}^n \setminus \mathbb{R}^k$ is a union of Fatou-Bieberbach domains (for $1 \leq k < n$ this follows from Theorem 1.1 while for $1 < k = n$ this observation is due to Rosay and Rudin [19], see also [6, Example 4.3.10, p. 112]), it is strongly dominable by $\mathbb{C}^n$ and hence is a natural candidate to be an Oka manifold.

The motivation for looking at this problem comes from two directions. The first one is that the only open subsets of complex Euclidean spaces $\mathbb{C}^n$ ($n > 1$) which are presently known to be Oka manifolds are the complements $\mathbb{C}^n \setminus A$ of tame (in particular, of algebraic) complex subvarieties $A \subset \mathbb{C}^n$ of dimension $\dim A \leq n - 2$ (cf. [6, Proposition 5.5.14, p. 205]). In particular, the only compact sets $A \subset \mathbb{C}^n$ whose complements $\mathbb{C}^n \setminus A$ are known to be Oka are the finite sets. Recently, Andrist and Wold [1] showed that the complement $\mathbb{C}^n \setminus B$ of a closed ball $B$ in $\mathbb{C}^n$ fails to be elliptic in the sense of Gromov if $n \geq 3$ (cf. [6, Def. 5.5.11, p. 203]), but it is unclear whether $\mathbb{C}^n \setminus B$ is Oka. (Ellipticity implies the Oka property, but the converse is not known.) A positive step in this direction has been obtained recently in [10] by showing that maps $X \to \mathbb{C}^n \setminus B$ from Stein manifolds $X$ of dimension $\dim_X X < n$ into the complement of a ball satisfy the Oka principle with approximation and interpolation.

Our second and more specific motivation was an attempt to understand whether the set of Oka fibers is open in any holomorphic family of compact complex manifolds. (It was recently shown in [8, Corollary 5] that the set of Oka fibers fails to be closed in such families; an explicit recent example is due to Dloussky [2].) To answer this question in the negative, one could look at the example of Nakamura [18] of a holomorphic family of compact three-folds such that the universal covering of the center fiber is $\mathbb{C}^3$, while the other fibers have coverings that are biholomorphic to $(\mathbb{C}^2 \setminus \mathbb{R}^2) \times \mathbb{C}$ (cf. page 98, Case 3 in [18]). If $\mathbb{C}^2 \setminus \mathbb{R}^2$ fails to be an Oka manifold then, since the class of Oka manifolds is closed under direct products and under holomorphic coverings and quotients, the corresponding 3-fold (which is an unramified quotient of $(\mathbb{C}^2 \setminus \mathbb{R}^2) \times \mathbb{C}$) also fails to be Oka, so we would have an example of an isolated Oka fiber.

2. A construction of Fatou-Bieberbach domains

In this section we prove Theorem 1.1. There is a remote analogy between this result and [10, Proposition 11]; the latter gives a similar separation of two compact disjoint sets in $\mathbb{C}^n$ whose union is polynomially convex and one of them is holomorphically contractible. However, the proof of Theorem 1.1 is completely different, and considerably...
more involved, than the proof of the cited result from [10]. It hinges upon recent results of Kutzschebauch and Wold [15] concerning Carleman approximation of certain isotopies of unbounded totally real submanifolds of $\mathbb{C}^n$ by holomorphic automorphisms of $\mathbb{C}^n$.

We begin with some preliminaries.

If $L$ is a closed, unbounded, totally real submanifold of $\mathbb{C}^n$ and $K$ is a compact subset of $\mathbb{C}^n$, we say that $K \cup L$ is polynomially convex if $L$ can be exhausted by compact sets $L_1 \subset L_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} L_j = L$ such that $K \cup L_j$ is polynomially convex for all $j \in \mathbb{N}$. If this is the case, then standard results imply that any function that is continuous on $L_j$ and holomorphic on a neighborhood of $K$ can be approximated, uniformly on $K \cup L_j$, by holomorphic polynomials on $\mathbb{C}^n$. It follows immediately that $K \cup L'$ is then polynomially convex for any compact subset $L'$ of $L$. (See [17] for these results.)

We shall also need the following stability result. If $K$ and $L$ are as above, with $K \cap L = \emptyset$ and $K \cup L$ polynomially convex, then $K \cup \tilde{L}$ is polynomially convex for any totally real submanifold $L \subset \mathbb{C}^n$ that is sufficiently close to $L$ in the fine $\mathcal{C}^1$ Whitney topology (Løw and Wold [17]; for small $\mathcal{C}^2$ perturbations this was proved earlier in [4]).

We may assume that $L$ is the standard subspace $\mathbb{R}^k \subset \mathbb{C}^n$ ([11]).

The properties of $K$ and $L$ imply that there exists a strongly plurisubharmonic Morse exhaustion function $\rho$ on $\mathbb{C}^n$ which is negative on $K$, positive on $L$, and equals $|z|^2$ near infinity. The first two properties are obtained in a standard way from polynomial convexity of $K$; for the last property see [5, p. 299], proof of Theorem 1.

Fix a closed ball $B \subset \mathbb{C}^n$ centered at the origin and containing the compact set $K$ in its interior. Let $V$ be the gradient vector field of $\rho$, multiplied by a smooth cutoff function of the form $h(|z|^2)$ that equals one near $B$ and equals zero near infinity. The flow $\phi_t$ of $V$ then exists for all $t \in \mathbb{R}$ and equals the identity map near infinity. Apply this flow to $L$. Since $\dim_{\mathbb{R}} L < n$ and the Morse indices of $\rho$ are at most $n$, a general position argument (deforming $V$ slightly if needed) shows that the trace of the isotopy $L_t := \phi_t(L)$ for $t \geq 0$ does not approach any of the (finitely many) critical points of $\rho$. It follows that $L$ is flown completely out of the ball $B$ in a finite time $t_0 > 0$, fixing $L$ all the time near infinity. Also, we have $K \cap L_t = \emptyset$ for all $t \in [0, 1]$ by the construction. Reparametrizing the time scale, we may assume that this happens at time $t_0 = 1$; so $L_1$ is a smooth submanifold of $\mathbb{C}^n \setminus B$ which agrees with $L = \mathbb{R}^k$ near infinity.

Since $L = L_0$ is totally real and the isotopy is fixed near infinity, Gromov’s h-principle for totally real immersions (cf. [11] [12]) implies that the isotopy $L_t = \phi_t(L)$ ($0 \leq t \leq 1$) can be $\mathcal{C}^0$ approximated by another isotopy $\tilde{L}_t \subset \mathbb{C}^n$ ($t \in [0, 1]$) consisting of totally real embeddings $\mathbb{R}^k \hookrightarrow \mathbb{C}^n$, with $\tilde{L}_0 = L$ and $\tilde{L}_t = L_t = \mathbb{R}^k$ near infinity for all $t \in [0, 1]$.

(A general position argument shows that a generic isotopy of immersions $\mathbb{R}^k \to \mathbb{C}^n$ for $k < n$ actually consists of embeddings.) In particular, we may assume that the new isotopy $\tilde{L}_t$ also avoids the set $K$ and that $\tilde{L}_1 \subset \mathbb{C}^n \setminus B$. To simplify the notation, we denote the new isotopy again by $L_t$.

By Kutzschebauch and Wold [15, Theorem 1.1] there is a small smooth perturbation $\hat{L}_t \subset \mathbb{C}^n$ of the isotopy $L_t$ from the previous step, with $\hat{L}_0 = L$ and $\hat{L}_t = L_t = \mathbb{R}^k$ near infinity for all $t \in [0, 1]$, such that $\hat{L}_t$ is still totally real (this follows from stability of total reality) and, what is the main point, the union $K \cup \hat{L}_t$ is polynomially convex for every $t \in [0, 1]$. (For $t = 0$ this holds by the assumption. See also Proposition 4.1 in
Kohomorphism \(\Phi\) of the \(\text{ric-Rosay theorem}\) (cf. \cite[Theorem 1.1]{15}). Explicitly, there exists a holomorphic automorphism \(\Phi\) of \(\mathbb{C}^n\) which is arbitrarily close to the identity map on a neighborhood of \(K\) (so that \(\Phi(K) \subset \hat{B}\)) and whose restriction to \(L\) is arbitrarily close to the diffeomorphism \(\phi_1: L \to L_1\) in the fine \(\mathcal{C}^1\) topology on \(L\). In particular, as \(B \cup L_1\) is polynomially convex, we can ensure (by using stability of polynomial convexity under small \(\mathcal{C}^1\) deformations, mentioned at the beginning of this section) that \(B \cup \Phi_1(L)\) is also polynomially convex.

To construct a Fatou-Bieberbach domain \(\Omega\) satisfying the conclusion of the lemma, we apply the above construction together with the standard push out method (cf. \cite[Corollary 4.4.2, p. 115]{6} for the latter). Let us briefly describe this procedure. Choose an increasing sequence of closed balls \(B = B_1 \subset B_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} B_j = \mathbb{C}^n\), with \(B_j \subset \hat{B}_{j+1}\) for every \(j \in \mathbb{N}\). Let \(\Phi_1 = \Phi\) be the automorphism of \(\mathbb{C}^n\) constructed above. By choosing \(\Phi_1\) sufficiently close to the identity map on \(L\) in the fine \(\mathcal{C}^1\) Whitney topology near infinity, we can ensure that the Kutzschebauch-Wold theorem \cite[Theorem 1.1]{15} applies to the totally real submanifold \(L_1 := \Phi_1(L)\) and the ball \(B_1\). Repeating the above argument, we find an automorphism \(\Phi_2 \in \text{Aut} \mathbb{C}^n\) satisfying the following conditions:

- \(\Phi_2\) is uniformly as close as desired to the identity map on \(B_1\),
- the totally real submanifold \(L_2 := \Phi_2(L_1)\) of \(\mathbb{C}^n\) is contained in \(\mathbb{C}^n \setminus B_2\) and the union \(B_2 \cup L_2\) is polynomially convex, and
- \(\Phi_2\) is sufficiently close to the identity map in the fine \(\mathcal{C}^1\) topology on \(L_1\) near infinity such that \(L_2\) again satisfies the conditions of the Kutzschebauch-Wold theorem \cite[Theorem 1.1]{15}.

Continuing inductively, we obtain a sequence of automorphisms \(\Phi_j \in \text{Aut} \mathbb{C}^n\) \((j = 1, 2, \ldots)\) such that the sequence of their compositions \(\tilde{\Phi}_j = \Phi_j \circ \Phi_{j-1} \circ \cdots \circ \Phi_1\) contains the set \(K\) in the domain of convergence

\[
\Omega = \{ z \in \mathbb{C}^n : \exists M_z > 0 \text{ such that } |\tilde{\Phi}_j(z)| \leq M_z \forall j \in \mathbb{N} \},
\]

but \(L \cap \Omega = \emptyset\) since \(\tilde{\Phi}_j(L) \cap B_j = \emptyset\) by the construction. If \(\Phi_j\) is chosen sufficiently close to the identity map on \(B_j\) for every \(j > 1\), then \(\Omega\) is a Fatou-Bieberbach domain (cf. \cite[Corollary 4.4.2]{6}). This completes the proof of Theorem \cite{15}.

**Remark 2.1.** The proof can also be completed as follows. Let \(\Phi = \Phi_1\) be the first automorphism in the above sequence. Consider a biholomorphic map on a neighborhood of \(B \cup \Phi(L)\) which equals the identity map near \(\Phi(L)\) and equals the contraction \(z \mapsto \frac{1}{2}z\) near the ball \(B\). Since \(B \cup \Phi(L)\) is polynomially convex, we can apply \cite[Theorem 1.1]{15} to approximate this map, uniformly on a neighborhood of \(B\) and in the fine \(\mathcal{C}^1\) Whitney topology on \(\Phi(L)\), by an automorphism \(\Psi\) of \(\mathbb{C}^n\). If the approximation is
sufficiently close, then the automorphism \( \theta = \Phi^{-1} \circ \Psi \circ \Phi \in \text{Aut } \mathbb{C}^n \) approximates the identity map on the subspace \( L = \mathbb{R}^k \) and is contracting on the set \( B' := \Phi^{-1}(B) \). Observe that \( B' \) contains \( K \) in its interior. Continuing inductively, we obtain a desired Fatou-Bieberbach domain \( \Omega \) as the basin of attraction of a sequence of automorphisms \( \Theta_k = \theta_k \circ \theta_{k-1} \circ \cdots \circ \theta_1 \), where each \( \theta_k \in \text{Aut } \mathbb{C}^n \) is attracting on \( B' \) and is sufficiently close to the identity on the totally real submanifold \( \Theta_{k-1}(L) \). The details (in a similar situation) can be found in Wold [21, p. 966].

3. Separation of varieties from totally real affine subspaces by Fatou-Bieberbach domains

In this section we prove Proposition 1.4.

We may assume that \( L = \mathbb{R}^k \) is the standard totally real subspace \( \mathbb{R}^k \). Choose a number \( R > 0 \) such that \( f(E) \) is contained in the ball \( B_R = \{ z \in \mathbb{C}^n : |z| < R \} \). Pick a smooth increasing convex function \( h : \mathbb{R} \rightarrow \mathbb{R}_+ \) such that \( h(t) = 0 \) for \( t \leq 0 \) and \( h \) is strongly convex and strongly increasing on \( t > 0 \). The nonnegative function

\[
\rho(z_1, \ldots, z_n) = \sum_{j=1}^k y_j^2 + \sum_{j=k+1}^n |z_j|^2 + h(|z|^2 - R^2)
\]

is then a strongly plurisubharmonic exhaustion on \( \mathbb{C}^n \) that vanishes precisely on the set \( \overline{B}_R \cap L \) and has no critical points on \( \mathbb{C}^n \setminus (\overline{B}_R \cap L) \). Note that \( f(E) \subset B_R \setminus L \).

Choose a smoothly bounded, strongly pseudoconvex domain \( D \subset U \) containing \( E \). By choosing \( D \) sufficiently small around \( E \), we may assume that \( f(D) \subset B_R \setminus L \), and hence \( c := \min_D \rho \circ f > 0 \). By [3, Theorem 1.1] (see Case (b) in the cited theorem) we can approximate \( f \), uniformly on \( E \), by a proper holomorphic map \( \tilde{f} : D \rightarrow \mathbb{C}^n \) satisfying \( \rho \circ \tilde{f} \geq c/2 > 0 \) on \( D \). (It is crucial that \( \rho \) is noncritical outside of its zero set.) Assuming that \( \tilde{f} \) if close enough to \( f \) on \( E \), we have \( \tilde{f}(E) \subset B_R \). The image \( A := \tilde{f}(D) \) is then a closed complex subvariety of \( \mathbb{C}^n \) which is disjoint from the set \( \overline{B}_R \cap L \).

Set \( K = A \cap \overline{B}_R \). We claim that \( K \cup L \) is polynomially convex. Since \( \overline{B}_R \cup L \) is polynomially convex (see [20, Theorem 8.1.26]), we only need to consider points \( p \in \overline{B}_R \setminus (A \cup L) \). Since \( A \) is a closed complex subvariety of \( \mathbb{C}^n \) and \( p \notin A \), there is a function \( g \in \mathcal{O}(\mathbb{C}^n) \) such that \( g(p) = 1 \) and \( g|_A = 0 \). Also, since \( p \notin L \), there exists a function \( h \in \mathcal{O}(\mathbb{C}^n) \) satisfying \( h(p) = 1 \) and \( |h| < 1/2 \) on \( L \) (we use the Carleman approximation of the zero function on \( L \)). The entire function \( \xi = gh^N \) \( (N \in \mathbb{N}) \) then satisfies \( \xi(p) = 1 \) and \( \xi = 0 \) on \( A \), and it also satisfies \( |\xi| < 1 \) on a given compact set \( L' \subset L \) provided that the integer \( N \) is chosen big enough (depending on \( L' \)). Hence \( p \) does not belong to the polynomial hull of \( K \cup L \), so this set is polynomially convex.

Since \( \tilde{f}(E) \subset K \) by the construction, the existence of a Fatou-Bieberbach domain \( \Omega \) satisfying (1.2) now follows from Theorem 1.1 applied to the sets \( K \) and \( L \).

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1. Andrist, R.B.; Wold, E.F.: The complement of the closed unit ball in $\mathbb{C}^3$ is not subelliptic. arXiv:1303.1804
2. Dloussky, G.: From non-Kählerian surfaces to Cremona group of $\mathbb{P}^2(\mathbb{C})$. arXiv:1206.2518
3. Drinovec Drnovšek, B.; Forstnerič, F.: Strongly pseudoconvex Stein domains as subvarieties of complex manifolds Amer. J. Math. 132, 331–360 (2010)
4. Forstnerič, F.: Stability of polynomial convexity of totally real sets. Proc. Amer. Math. Soc. 96, 489–494 (1986)
5. Forstnerič, F.: Complements of Runge domains and holomorphic hulls. Michigan Math. J. 41, 297–308 (1994)
6. Forstnerič, F.: Stein Manifolds and Holomorphic Mappings (The Homotopy Principle in Complex Analysis). Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 56. Springer-Verlag, Berlin-Heidelberg (2011)
7. Forstnerič, F.; Lárusson, F.: Survey of Oka theory. New York J. Math. 17a, 1–28 (2011) http://nyjm.albany.edu/j/2011/17a-2.html
8. Forstnerič, F.; Lárusson, F.: Holomorphic flexibility properties of compact complex surfaces. Int. Math. Res. Notices IMRN (2013). doi: 10.1093/imrn/rnt044
9. F. Forstnerič: Oka manifolds: from Oka to Stein and back. With an appendix by F. Lárusson. Ann. Fac. Sci. Toulouse Math. (6) 22, no. 4., 747–809 (2013)
10. Forstnerič, F.; Ritter, T.: Oka properties of ball complements. Math. Z., in press. http://link.springer.com/10.1007/s00209-013-1258-2.
11. Gromov, M.: Convex integration of differential relations, I. Izv. Akad. Nauk SSSR, Ser. Mat. 37, 329–343 (1973) (Russian). English transl.: Math. USSR Izv. 37 (1973)
12. Gromov, M.: Partial differential relations. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 9. Springer-Verlag, Berlin–New York (1986)
13. Hamm, H.: Zum Homotopietyp Steinscher Räume. J. Reine Ang. Math. 338, 121–135 (1983)
14. Hörmander, L.: An Introduction to Complex Analysis in Several Variables, 3rd edn. North-Holland Mathematical Library, vol. 7. North-Holland, Amsterdam (1990)
15. Kutzschebauch, F.; Wold, E.F.: Carleman approximation by holomorphic automorphisms. Preprint (2014)
16. Lárusson, F.: What is ... an Oka manifold? Notices Amer. Math. Soc. 57, no. 1, 50-52 (2010)
17. Low, E.; Wold, E.F.: Polynomial convexity and totally real manifolds. Complex Var. Elliptic Equ. 54, 265-281 (2009)
18. Nakamura, I.: Complex parallelisable manifolds and their small deformations. J. Differential Geom. 10, 85-112 (1975)
19. Rosay, J.-P.; Rudin, W.: Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$. Trans. Amer. Math. Soc. 310, 47–86 (1988)
20. Stout, E. L.: Polynomial Convexity. Birkhäuser, Boston (2007)
21. Wold, E.F.: Embedding Riemann surfaces properly into $\mathbb{C}^2$. Internat. J. Math. 17, 963974 (2006)

F. Forstnerič, Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia
E-mail address: franc.forstneric@fmf.uni-lj.si

E. F. Wold, Matematisk Institutt, Universitetet i Oslo, Postboks 1053 Blindern, 0316 Oslo, Norway
E-mail address: erlendfw@math.uio.no