Effects of D-instantons in string amplitudes

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Abstract

We investigate the different energy regimes in the conjectured $SL(2, \mathbb{Z})$ invariant four graviton scattering amplitude that incorporates D-instanton contributions in 10d type IIB superstring theory. We show that the infinite product over $SL(2, \mathbb{Z})$ rotations is convergent in the whole complex plane $s, t$. For high energies $\alpha'/s \gg 1$, fixed scattering angle, and very weak coupling $g_s \ll 1/(\alpha'/s)$, the four-graviton amplitude exhibits the usual exponential suppression. As the energy approaches $1/g_s$, the suppression gradually diminishes until there appears a strong amplification near a new pole coming from the exchange of a $(p, q)$ string. At energies $\alpha' \ll 1/\sqrt{g_s}$, the pure D instanton contribution to the scattering amplitude is found to produce a factor $A_{4}^{\text{Dinst}} \approx \exp \left( cg_s^{3/2} e^{-\frac{2\pi}{g_s} s^3} \right)$. At energies $1/\sqrt{g_s} \ll \alpha' \ll 1/g_s$, the D-instanton factor becomes $A_{4}^{\text{Dinst}} \approx \exp \left( 2e^{-\frac{2\pi}{g_s} + \pi g_s s^2} \right)$, $\alpha' = 4$. At higher energies $\alpha' \gg 1/g_s$ the D-instanton contribution becomes very important, and one finds an oscillatory behavior which alternates suppression and amplification. This suggests that non-perturbative effects can lead to a high-energy behavior which is significantly different from the perturbative string behavior.
1. Introduction

A problem of interest is understanding what are the concrete effects that non-perturbative corrections can have in superstring theory, in particular, how they affect the high-energy behavior of string amplitudes. In ten-dimensional type IIB superstring theory, the source of non-perturbative corrections are the D instantons.

Computing the contribution of multiply-charged D instantons directly is complicated. However, combining different pieces of information, Green and Gutperle [1] conjectured the exact modular function that multiplies the $R^4$ term in the type IIB effective action, which exactly incorporates the infinite set of multiply-charged D instanton corrections.

One of the constraints on the effective action used by Green and Gutperle is precisely $SL(2, \mathbb{Z})$ invariance. The $SL(2, \mathbb{Z})$ symmetry of type IIB superstring theory requires that the effective action must be invariant under $SL(2, \mathbb{Z})$ transformations to all orders in the $\alpha'$ expansion. In particular, this implies that graviton scattering amplitudes must be $SL(2, \mathbb{Z})$ invariant, since there is a direct correspondence between the terms in the effective action and the momentum expansion of the scattering amplitude.

In [2,3] an $SL(2, \mathbb{Z})$ invariant four graviton amplitude was constructed by applying a simple $SL(2, \mathbb{Z})$ symmetrization of the tree-level string theory four graviton amplitude. The construction follows essentially the same rule used by Green and Gutperle to symmetrize the $R^4$ term. It was conjectured that this scattering amplitude incorporates the full series of D instanton corrections with the different D-instanton numbers. This symmetric amplitude satisfies a number of consistency conditions. In particular, corrections of perturbative origin appear with an integer power of $g_s^2$. This is non-trivial and does not hold for any symmetrization. It is also consistent with the conjecture that high derivative terms in the type II effective action of the form $H^{4k-4}R^4$ should not receive perturbative contributions beyond genus $k$ [4]. By construction, it reproduces the exact $R^4$ term proposed in [1], and it can be viewed as a tree-level amplitude that accounts for the exchange of $(p, q)$ string states [5].

These $(p, q)$ string states have a simple eleven-dimensional origin [6]. Type IIB superstring theory is obtained from M theory by compactification on a 2-torus and taking the zero area limit at fixed torus moduli. In this limit, most membrane states get an infinite mass, except a certain set of states that represent the $(p, q)$ strings of uncompactified 10d type IIB string theory. These states are precisely the states that contribute as simple poles in the $SL(2, \mathbb{Z})$ invariant amplitude of [2,3].

In this work we investigate the properties of the $SL(2, \mathbb{Z})$ invariant amplitude. In particular, we factorize the pure D instanton contribution and study the high energy limit.

The conjecture of [1] has withstood different tests and has been generalized in different directions [7-19]. The idea of organizing type IIB perturbation theory in $SL(2, \mathbb{Z})$ invariant way was also suggested by [20,21]. Scattering amplitudes at high energies incorporating higher genus effects were investigated by [22] and [23].
2. $SL(2, \mathbb{Z})$ invariant amplitude

The four-graviton scattering amplitude for 10d type IIB superstring introduced in [2,3] is given by the following formula:

$$A_4 = \kappa^2 K A_4^{sl(2)}(s, t) ,$$

(2.1)

$$A_4^{sl(2)}(s, t) = \frac{1}{stu} \prod_{(p,q)'} \frac{\Gamma(1 - s_{pq}) \Gamma(1 - t_{pq}) \Gamma(1 - u_{pq})}{\Gamma(1 + s_{pq}) \Gamma(1 + t_{pq}) \Gamma(1 + u_{pq})} ,$$

(2.2)

where $p$ and $q$ are relatively prime, $\tau = C^{(0)} + ig_s^{-1}$ is the usual coupling of type IIB superstring theory, and $K$ is the same kinematical factor depending on the momenta and polarization of the external states appearing in the tree-level Virasoro amplitude (see e.g. [24])

$$K = \zeta_1^{AA'} \zeta_2^{BB'} \zeta_3^{CC'} \zeta_4^{DD'} K_{ABCD}(k_i)K_{A'B'C'D'}(k_i) ,$$

$$K_{ABCD} = - \frac{1}{4stu} \eta_{AC} \eta_{BD} + ...$$

The scattering amplitude (2.2) can also be written as

$$A_4^{sl(2)}(s, t) = \frac{1}{stu} e^{\delta(s, t)} ,$$

(2.4)

with

$$\delta(s, t) = 2 \sum_{k=1}^{\infty} \frac{\zeta(2k + 1)g_s^{k+1/2}E_{k+1/2}(\tau)}{2k + 1} (\bar{s}^{2k+1} + \bar{t}^{2k+1} + \bar{u}^{2k+1}) ,$$

(2.5)

$$\bar{s} = \frac{1}{4}\alpha' s , \quad \bar{t} = \frac{1}{4}\alpha' t , \quad \bar{u} = \frac{1}{4}\alpha' u , \quad \bar{s} + \bar{t} + \bar{u} = 0 ,$$

and $E_r(\tau)$ is the non-holomorphic Eisenstein series, given by (Re $r > 1$)

$$E_r(\tau) = \sum_{(p,q),} \frac{\tau^r}{|p + q\tau|^{2r}} .$$

(2.6)

For very small coupling $g_s \ll 1$, the terms with $q \neq 0$ are negligible in the sum (2.5), so that $g_s^{k+1/2}E_{k+1/2}(\tau) \rightarrow 1$. One recovers the tree-level four-graviton Virasoro amplitude,

$$A_4(s, t) = \kappa^2 K A_4^0(s, t) , \quad A_4^0(s, t) = \frac{1}{stu} e^{\delta_0(s, t)} ,$$

(2.7)

$$\delta_0(s, t) = 2 \sum_{k=1}^{\infty} \frac{\zeta(2k + 1)}{2k + 1} (\bar{s}^{2k+1} + \bar{t}^{2k+1} + \bar{u}^{2k+1}) .$$

(2.8)
This gives

\[ A^0_4(s, t) = \frac{1}{stu \Gamma(1 + s) \Gamma(1 + t) \Gamma(1 + u)} \Gamma(1 - \bar{s}) \Gamma(1 - \bar{t}) \Gamma(1 - \bar{u}) . \]  

\[ (2.9) \]

The scattering amplitude \( A^{sl(2)}_4 \) adds to the Virasoro amplitude perturbative and non-perturbative contributions. They are seen explicitly by expanding the Eisenstein functions at large \( \tau_2 = g_s^{-1} \)

\[ E_r(\tau) = \tau^r + \gamma_r \tau^{1-r} + \frac{4r_2^{1/2}\pi^r}{\zeta(2r)\Gamma(r)} \sum_{n,w=1}^{\infty} \left( \frac{w}{n} \right)^{r-1/2} \cos(2\pi wn\tau_1) K_{r-1/2}(2\pi wn\tau_2) , \]  

\[ (2.10) \]

\[ \gamma_r = \frac{\sqrt{\pi} \Gamma(r - 1/2) \zeta(2r)}{\Gamma(r) \zeta(2r)} . \]

Using the asymptotic expansion for the Bessel function \( K_{r-1/2} \),

\[ K_{r-1/2}(2\pi wn\tau_2) = \frac{1}{\sqrt{4\pi wn\tau_2}} e^{-2\pi wn\tau_2} \sum_{m=0}^{\infty} \frac{1}{(4\pi wn\tau_2)^m} \frac{\Gamma(r + m)}{\Gamma(r - m) m!} , \]  

\[ (2.11) \]

we see that the \( E_{k+1/2}(\tau) \) terms in the amplitude are of the form

\[ g_B^{k+1/2} E_{k+1/2}(\tau) = 1 + \gamma_{k+1/2} g_B^{2k} + O(e^{-2\pi/g_B}) . \]  

\[ (2.12) \]

Note that the non-perturbative contributions are \( O(e^{-2\pi m/g_s}) \), where \( m = wn \) is an integer number. The coefficient \( 2\pi m \) is crucial in order to have a one-to-one correspondence between these terms and instanton contributions. It is a remarkable fact that the product over \( SL(2, \mathbb{Z}) \) rotations automatically generates the full series of D-instanton contributions.

We summarize the main properties of \( A^{sl(2)}_4 \):

1) It is \( SL(2, \mathbb{Z}) \) invariant. This is explicit in the Einstein frame, \( g_{\mu\nu}^E = g_B^{-1/2} g_{\mu\nu} \), so that \( s_E = g_B^{1/2} s, t_E = g_B^{1/2} t, u_E = g_B^{1/2} u \), and \( s_E, t_E, u_E \) remain fixed under \( SL(2, \mathbb{Z}) \) transformations.

2) It adds perturbative \( g_s^{2k} \) and non-perturbative \( O(e^{-2\pi m/g_s}) \) corrections to the Virasoro amplitude.

3) It has simple poles in the \( s-t-u \) channels at \( s_{pq} = n, t_{pq} = n, u_{pq} = n, n = 0, 1, 2, ... \) corresponding to a tree-level exchange of particles with masses

\[ \frac{1}{4} \alpha' M^2 = n|p + q\tau| . \]  

\[ (2.13) \]

4) It reproduces the exact (proportional to \( E_{3/2}(\tau) \)) \( R^4 \) term conjectured in [1], containing a one-loop correction and the full D-instanton contributions. It also reproduces the exact \( \zeta(5) E_{5/2}(\tau) \nabla^4 R^4 \) term conjectured in [18] (moreover, in [17] there was a calculation of a
genus one term in $\nabla^6 R^4$ which was found proportional to $2\zeta(3)\zeta(2)$, which differs from the prediction of the $A_4^{sl(2)}$ amplitude only by a factor of 2).

The spectrum (2.13) is the spectrum of $(p, q)$ string states [3]:

$$M^2 = 4\pi T_{pq}(N_R + N_L) = \frac{2}{\alpha'} |p + q\tau| (N_R + N_L), \quad N_R = N_L.$$ (2.14)

This spectrum corresponds to the zero winding sector of the spectrum studied in [3,21] for the nine-dimensional type IIB string theory. Setting $\tau_1 = C^{(0)} = 0$, the full spectrum in $D = 9$ is given by

$$M^2_9 = \frac{\eta^2}{R_{10}^2}(p^2 + \frac{q^2}{g_s^2}) + \frac{w_{10}R_{10}}{\alpha'^2} + \frac{2}{\alpha'} \sqrt{p^2 + \frac{q^2}{g_s^2}} (N_R + N_L), \quad N_R - N_L = nw_{10},$$ (2.15)

where $R_{10}$ is the radius of the compact tenth dimension. In the limit $R_{10} \to \infty$, one must set, as usual, the winding number $w_{10}$ to zero to have finite mass. The term proportional to $\frac{1}{R_{10}^2}$ becomes the continuous 10d component of the momentum $p_{10}$, so that $M^2 = M^2_9 - p_{10}^2$ and one gets eq. (2.14). It is important to note that a charged D string has mass $M = O(1/g_s)$, as seen from (2.15). The neutral $(p, q)$ strings of ten dimensions have masses given by (2.14) of order $M = O(1/\sqrt{g_s})$ for $q \neq 0$. This is why the product over $SL(2, \mathbb{Z})$ rotations produces poles at $\alpha'/s = O(1/g_s)$.

The collection of states (2.14) are the only quantum states of M-theory compactified on a 2-torus that remain of finite mass after taking the zero-area limit of the torus that leads to ten-dimensional type IIB string theory [3]. The scattering amplitude $A_4^{sl(2)}$ can thus be viewed as a tree-level scattering amplitude where all these states are exchanged.

The scattering amplitude $A_4^{sl(2)}$ does not describe loop effects such as discontinuity cuts (see [3,17] for discussions). In particular, it should not be a good approximation of the full scattering amplitude at $g_s = O(1)$ and $\alpha s$ large. It represents an improvement of the Virasoro amplitude at $g_s \ll 1$ (or at the S-dual situation, $g_s \gg 1$), where D-instanton (and some perturbative) contributions have been incorporated.

The exchange of $(p, q)$ string states is clear from the pole structure of (2.2). It becomes manifest by writing

$$\delta = \frac{1}{2} \sum_{(m,n)\neq(0,0)} \log \frac{M^2_{mn} + s}{M^2_{mn} - s} + (s \to t) + (s \to u),$$ (2.16)

where

$$\alpha' M^2_{mn} = 4|m + n\tau|.$$ 

This generalizes the analogous formula for the Virasoro amplitude, with $\delta_0$ written in the form

$$\delta_0 = \sum_{m=1}^\infty \delta_m, \quad \delta_m = \log \frac{M^2_m + s}{M^2_m - s} + (s \to t) + (s \to u), \quad \alpha' M^2_m = 4m.$$ (2.17)

Now the sum in (2.16) contains not only the terms $\alpha' M^2_m = 4m$, but all the terms $M^2_{mn}$, representing all $(p, q)$ string states.
3. Convergence properties

The scattering amplitude $A^{s_l(2)}_4$ is defined through an infinite product (2.2) over a pair $(p, q)$ of relatively prime integers, i.e. integers $(p, q)$ having greatest common divisor equal to one. An important issue is what are the convergence properties of this product.

To study the convergence, we write

$$\delta(s, t) = \sum_{(p, q)'} \log \frac{\Gamma(1-s_{pq})}{\Gamma(1+s_{pq})} \pm (s \to t) + (s \to u). \quad (3.1)$$

We have to look at the behavior of terms with large $p, q$. For any given $s, t, u$, there are positive integers $(p_0, q_0)$ such that all terms with $p > p_0, q > q_0$ have $|p + q\tau| \gg s, t, u$ and $s_{pq}, t_{pq}, u_{pq}$ small. These terms have the behavior

$$\log \frac{\Gamma(1-s_{pq})\Gamma(1-t_{pq})\Gamma(1-u_{pq})}{\Gamma(1+s_{pq})\Gamma(1+t_{pq})\Gamma(1+u_{pq})} \approx \frac{2\zeta(3)}{3|p + q\tau|^3}(s^3 + t^3 + u^3), \quad (3.2)$$

where we have used $s + t + u = 0$. The sum over $p, q$ of $|p + q\tau|^{-3}$ is known to be convergent [25] (in particular, the full sum over $p, q$ of $|p + q\tau|^{-3}$ defines $E_3^2(\tau)$, see (2.6)). Therefore the sum in (3.1) is convergent. Since we have made no assumption about $s, t$, the series (3.1) has infinite radius of convergence.

We have also investigated the convergence numerically, by explicit calculation of the infinite product in different sectors of the complex planes $s$ and $t$, for generic values of the coupling $\tau$. As an additional check, we have also computed the amplitude in the representation (2.16), obtaining the same (finite) numerical results.

Note that the series (2.5) defining the amplitude has a finite radius of convergence if we write $\zeta(2k + 1) = \sum_m m^{-2k-1}$ and perform first the sum over $k$. The sum over $k$ is the series of a logarithm and it diverges when $s, t$ or $u$ meet the first pole. The same of course applies for the Virasoro amplitude written in the form (2.8).

4. Approach to the different scales

Let us assume $g_s \ll 1$, and examine the different scales that appear as the center-of-mass energy is increased from zero. We set $C^{(0)} = \tau_1 = 0$, so that $\tau = i\tau_2 = ig_s^{-1}$. 

5
4.1. Region $\alpha'$'s $\ll \frac{1}{g_s}$

For $\alpha' \ll 1$, one has $A_{4}^{sll}(2)(s, t) \to \frac{1}{stu}$, and one recovers the supergravity tree-level four graviton amplitude.

In general, for any $\alpha' \ll \frac{1}{g_s}$, one has $A_{4}^{sll}(2)(s, t) \to A_{4}^{0}(s, t)$, one recovers the Virasoro amplitude (2.23) with its usual properties: simple poles at $\bar{s} = n$, $\bar{t} = n$, $\bar{u} = n$, with $n$ positive (recall that in the physical region of elastic scattering $s$ is positive, while $t, u$ are negative). The high energy behavior is as follows:

i) High $\alpha'$'s, fixed scattering angle $\varphi$:

$$A_{4}^{sll}(2)(s, t) \to A_{4}^{0}(s, t) \simeq \frac{1}{stu} e^{-\alpha' a_0 s} ,$$

$$a_0 = \frac{1}{2} \left| \sin^2 \frac{\varphi}{2} \log \sin^2 \frac{\varphi}{2} + \cos^2 \frac{\varphi}{2} \log \cos^2 \frac{\varphi}{2} \right| ,$$

$$t = -s \sin^2 \frac{\varphi}{2} , \quad u = -s \cos^2 \frac{\varphi}{2} .$$

ii) High $\alpha'$'s, fixed $t$:

$$A_{4}^{sll}(2)(s, t) \to A_{4}^{0}(s, t) \simeq \frac{1}{stu} (-1)^t s^{-\frac{\alpha'}{4}} |t| \frac{\Gamma(1 - \alpha't)}{\Gamma(1 + \alpha't)} .$$

It is useful to split the scattering amplitude $A_{4}^{sll}(2)$ in three factors:

$$A_{4}^{sll}(2)(s, t) = A_{4}^{0}(s, t) \times A_{4}^{pert} \times A_{4}^{Dinst} = \frac{1}{stu} e^{\delta_0 + \delta_{pert} + \delta_{Dinst}} .$$

Here $A_{4}^{pert}$ represents perturbative corrections to the Virasoro amplitude of the form $g_s^{2k}$ coming from the term $\gamma_r \tau_2^{1-r}$ in the expansion (2.10). The remaining factor $A_{4}^{Dinst}$ represents the pure D-instanton contribution, terms proportional to $e^{-2\pi wn/g_s}$ coming $K_{r-1/2}$ in (2.10). They are given by

$$\delta_{pert} = \sqrt{\pi} \sum_{k=1}^{\infty} \frac{(k-1)! \zeta(2k)}{\Gamma(k + \frac{3}{2})} g_s^{2k}(s^{2k+1} + t^{2k+1} + u^{2k+1}) ,$$

$$\delta_{Dinst} = 4\sqrt{\pi} \sum_{n, w, k=1}^{\infty} \left( \frac{w}{n} \right)^k \frac{\pi^k g_s^k}{\Gamma(k + \frac{3}{2})} K_k \left( \frac{2\pi wn}{g_s} (\bar{s}^{2k+1} + \bar{t}^{2k+1} + \bar{u}^{2k+1}) \right) .$$

These series converge for $\bar{s} < \frac{1}{g_s}$. In the region $\bar{s} \ll \frac{1}{g_s}$, the leading behavior of $\delta_{pert}$ is just given by the first term in the series (4.4), $\delta_{pert} \simeq \frac{2\pi^2}{9} (s^3 + t^3 + u^3)$. 

6
Assuming $g_s \ll 1$, one can use the asymptotic form of the Bessel function (2.11). Then one obtains that in this region $\alpha' s \ll \frac{1}{g_s}$, the D-instanton contribution is given by

$$
\delta_{\text{Dinst}} = 2\bar{s}\sqrt{\pi g_s}e^{-\frac{2\pi}{g_s}} \sum_{k=1}^{\infty} \frac{(\pi g_s \bar{s}^2)^k}{\Gamma(k + \frac{3}{2})} + (s \to t) + (s \to u)
$$

(4.6)

where Erf is the error function. There are two regimes, $\bar{s} \ll 1/\sqrt{g_s}$, so that $\pi g_s \bar{s}^2 \ll 1$, and $1/\sqrt{g_s} \ll \bar{s} \ll 1/g_s$. In the first case, we get

$$
\delta_{\text{Dinst}} \cong \frac{8\pi}{3} s^{3/2} e^{-\frac{2\pi}{g_s}} (\bar{s}^3 + \bar{t}^3 + \bar{u}^3), \quad \bar{s} \ll 1/\sqrt{g_s}.
$$

(4.7)

This is a positive contribution in the physical region of the Mandelstam parameters, but it is negligible compared to $\delta_0$ and $\delta_{\text{pert}}$. In the second case, we get

$$
\delta_{\text{Dinst}} \cong 2e^{-\frac{2\pi}{g_s}} (e^{\pi g_s \bar{s}^2} - e^{\pi g_s \bar{t}^2} - e^{\pi g_s \bar{u}^2}), \quad 1/\sqrt{g_s} \ll \bar{s} \ll 1/g_s.
$$

(4.8)

This is still tiny, since in this region $\pi g_s \bar{s}^2 \ll 2\pi/g_s$.

### 4.2. Region $\alpha' s = O(1/g_s)$

In this case one begins to see simple poles at $\alpha' s = \sqrt{m^2 + n^2/g_s^2}$ with $n \neq 0$. Since $g_s$ is small, there is an accumulation of poles with $m = 0, 1, 2, \ldots$ near the pole at $n = 1$, at $n = 2$, etc. These poles are not seen in a coarse grain plot of the amplitude, since they appear at special points.
Fig. 2: The separate contributions to $A_{4}^{sl(2)}$ for $g_s = 0.01$: a) The tree-level Virasoro part $\delta_0$ is a straight line. b) The perturbative part $\delta_{\text{pert}}$ has cusps and is positive, giving an amplification effect. c) The D instanton part $\delta_{\text{Dinst}}$ is still negligible at $\bar{s} < 320$.

Figure 1 shows $\delta(s, t)$ (the logarithm of the amplitude, see (2.4)) as a function of $s$ for large $s, t, u$ and fixed scattering angle $\varphi = \frac{\pi}{2}$ and for $g_s = 0.01$. One can see that at the beginning there is the straight line with negative slope as in (4.1), reproducing the usual suppression of the Virasoro amplitude. Then $\delta(s, t)$ becomes positive, producing an amplification of the amplitude near $\bar{s} = 1/g_s$ (see figure 2). As $s$ is further increased, the amplitude diminishes; then it is amplified again near $\bar{s} = 3/g_s$.

The behavior can be understood as a combination of the effects of $\delta_0$ and $\delta_{\text{pert}}$, since the D-instanton contribution $\delta_{\text{Dinst}}$ is still negligible in this region for $g_s = 0.01$. Figure 2 shows the two contributions separately.

4.3. Region $\alpha's \gg \frac{1}{g_s}$

To examine the behavior in this region, we again consider the three contributions $\delta_0$, $\delta_{\text{pert}}$, $\delta_{\text{Dinst}}$ separately.

The perturbative part (4.4) can be resummed explicitly, with the result [3]:

$$\delta_{\text{pert}} = -4 \sum_{m=1}^{\infty} \sqrt{\frac{m^2}{g_s^2} - \bar{s}^2 \arcsin \frac{\bar{s}g_s}{m} + (s \to t) + (s \to u)}.$$  \hfill (4.9)

Its high energy behavior is shown in Figure 3, which indicates a behavior $\delta_{\text{pert}} \cong \text{const. } s$. More precisely, it is bounded between two straight lines: $1.22 \bar{s} < \delta_{\text{pert}} < 1.69 \bar{s}$. On the other hand, we have from (4.1):

$$\delta_0 \cong -\alpha' a_0 \bar{s}.$$ \hfill (4.10)
Fig. 3: The perturbative part $\delta_{\text{pert}}$ computed at ultra high energies. It grows linearly with $s$.

The pure D instanton part $\delta_{\text{Dinst}}$ can be computed from $\delta_{\text{Dinst}} = \delta - \delta_0 - \delta_{\text{pert}}$, where $\delta(s, t)$ and $\delta_{\text{pert}}$ are computed from the convergent sums (3.1) and (4.9). Numerically, one finds that $\delta_{\text{Dinst}}$ oscillates between negative and positive values, which are of the same order of magnitude as $\delta_0, \delta_{\text{pert}}$. This gives rise to a behavior which alternates strong suppression and amplification of the amplitude as $s$ is increased.

The asymptotic behavior at very large $s$ is unclear since numerical precision is worst at high $s$. It is plausible that at $\alpha' s \gg 1/g_s$ there are higher genus corrections not contained in $A_{4}^{sl(2)}$ which become important. Among the different types of corrections, there are gravitational corrections corresponding to multiple exchange of gravitons [23,27]. In the present case of high energy and fixed scattering angle, the dominant genus $h$ contribution is known [22], though it is unclear how to resum the full series [28].

We find remarkable that the product over $SL(2, \mathbb{Z})$ rotations produces a convergent, mathematically well defined amplitude, and that the infinite D-instanton sum produces significant changes in the high energy behavior. It would be interesting to understand how to incorporate higher genus corrections to $A_{4}^{sl(2)}$ in an $SL(2, \mathbb{Z})$ invariant way.

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