Bucklin Voting is Broadly Resistant to Control*

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Abstract

Electoral control models ways of changing the outcome of an election via such actions as adding/deleting/partitioning either candidates or voters. These actions modify an election’s participation structure and aim at either making a favorite candidate win (“constructive control”) or prevent a despised candidate from winning (“destructive control”), which yields a total of 22 standard control scenarios. To protect elections from such control attempts, computational complexity has been used to show that electoral control, though not impossible, is computationally prohibitive. Among natural voting systems with a polynomial-time winner problem, the two systems with the highest number of proven resistances to control types (namely 19 out of 22 [ENR09, ER10, EPR10]) are “sincere-strategy preference-based approval voting” (SP-AV, a modification [ENR09] of a system proposed by Brams and Sanver [BS06]) and fallback voting [BS09]. Both are hybrid systems; e.g., fallback voting combines approval with Bucklin voting. In this paper, we study the control complexity of Bucklin voting itself and show that it behaves equally well in terms of control resistance for the 20 cases investigated so far. As Bucklin voting is a special case of fallback voting, all resistances shown for Bucklin voting in this paper strengthen the corresponding resistance for fallback voting shown in [EPR10].

1 Introduction

Since the seminal paper of Bartholdi et al. [BTT92], the complexity of electoral control—changing the outcome of an election via such actions as adding/deleting/partitioning either candidates or voters—has been studied for a variety of voting systems. Unlike manipulation [BTT89, BO91].

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CSL07, FHHR09a], which models attempts of strategic voters to influence the outcome of an election via casting insincere votes, control models ways of an external actor, the “chair,” to tamper with an election’s participation structure so as to alter its outcome. Another way of tampering with the outcome of elections is bribery [FH09, FHHR09a], which shares with manipulation the feature that votes are being changed, and with control the aspect that an external actor tries to change the outcome of the election. Faliszewski et al. [FHHR09b] survey known complexity results for control, manipulation, and bribery.

Elections have been used for preference aggregation not only in the context of politics and human societies, but also in artificial intelligence, especially in multiagent systems, and other topics in computer science (see, e.g., [ER97, GMHS99, DKNS01]). That is why it is important to study the computational properties of voting systems. In particular, complexity can be used to protect elections against tampering attempts in control, manipulation, and bribery attacks by showing that such attacks, though not impossible in principle, are computationally prohibitive.

Regarding control, a central question is to find voting systems that are computationally resistant to as many of the common 22 control types as possible, where resistance means the corresponding control problem is NP-hard. Each control type is either constructive (the chair seeking to make some candidate win) or destructive (the chair seeking to make some candidate not end up winning). Erdélyi and Rothe [ER10] proved that fallback voting [BS09], a hybrid voting system combining Bucklin with approval voting, is resistant to each of these 22 standard control types except five types of voter control. They proved that fallback voting is vulnerable to two of those control types (i.e., these control problems are polynomial-time solvable), leaving the other three cases open. Erdélyi, Piras and Rothe [EPR10] recently proved that fallback voting is resistant to constructive and destructive control by partition of voters in the tie-handling model “ties promote.”

Thus fallback voting is not only fully resistant to candidate control [ER10] but also fully resistant to constructive control. In terms of the total number of proven resistances it draws level with “sincere-strategy preference-based approval voting” (SP-AV, another hybrid system proposed by Brams and Sanver [BS06]): Both have the most (19 out of 22) proven resistances to control among natural voting systems with a polynomial-time winner problem. Among such systems, only plurality and SP-AV were previously known to be fully resistant to candidate control [BT92, HHR07, ENR09], and only Copeland voting and SP-AV were previously known to be fully resistant to constructive control [FHHR09a, ENR09]. However, plurality has fewer resistances to voter control, Copeland voting has fewer resistances to destructive control, and SP-AV is arguably less natural a system than fallback voting [ENR09, BEH+09].

Even more natural than fallback voting, however, is Bucklin voting itself, one of its two constituent systems. After all, fallback voting is a hybrid system in which voters are required to provide two types of preference (a ranking of candidates and an approval vector), whereas in Bucklin voting it is enough for the voters to rank the candidates. Moreover, Bucklin voting has the important property that it is majority-consistent, which means that whenever a majority candidate exists he or she is the (unique) Bucklin winner. In contrast, fallback voting is not majority-consistent.

In this paper, we study the control complexity of Bucklin voting and show that it has as many control resistances as fallback voting in the 20 cases investigated so far. In particular, Bucklin voting

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1A majority candidate is a candidate that is ranked at top position in more than half of the votes.
is also fully resistant to both candidate control and constructive control.

This paper is organized as follows. In Section 2 some basic notions from social choice theory, and in particular Bucklin voting, as well as the 22 standard types of control are defined. Our results on the control complexity of Bucklin voting are presented in Section 3. Finally, Section 4 provides some conclusions and open questions.

2 Preliminaries

2.1 Elections and Voting Systems

An election \((C, V)\) is given by a finite set \(C\) of candidates and a finite list \(V\) of votes over \(C\). A voting system is a rule that specifies how to determine the winner(s) of any given election. The two voting systems considered in this paper are Bucklin voting and fallback voting.

In Bucklin voting, votes are represented as linear orders over \(C\), i.e., each voter ranks all candidates according to his or her preferences. For example, if \(C = \{a, b, c, d\}\) then a vote might look like \(c d a b\), i.e., this voter (strictly) prefers \(c\) to \(d\), \(d\) to \(a\), and \(a\) to \(b\). Given an election \((C, V)\) and a candidate \(c \in C\), define the level \(i\) score of \(c\) in \((C, V)\) (denoted by \(\text{score}_i(C, V)(c)\)) as the number of votes in \(V\) that rank \(c\) among their top \(i\) positions. Denoting the strict majority threshold for a list \(V\) of voters by \(\text{maj}(V) = \lfloor \|V\|/2 \rfloor + 1\), the Bucklin score of \(c\) in \((C, V)\) is the smallest \(i\) such that \(\text{score}_i(C, V)(c) \geq \text{maj}(V)\). All candidates with a smallest Bucklin score, say \(k\), and a largest level \(k\) score are the Bucklin winners (BV winners, for short) in \((C, V)\). If some candidate becomes a Bucklin winner on level \(k\), we call him or her a level \(k\) BV winner in \((C, V)\). Note that a level 1 BV winner must be unique, but there may be more level \(k\) BV winners than one for \(k > 1\), i.e., an election may have more than one Bucklin winner in general.

As a notation, when a vote contains a subset of the candidate set, such as \(c D a\) for a subset \(D \subseteq C\), this is a shorthand for \(c d_1 \cdots d_\ell a\), where the elements of \(D = \{d_1, \ldots, d_\ell\}\) are ranked with respect to some (tacitly assumed) fixed ordering of all candidates in \(C\). For example, if \(C = \{a, b, c, d\}\) and \(D = \{b, d\}\) then “\(c D a\)” is a shorthand for the vote \(c b d a\).

2.2 Types of Electoral Control

There are 11 types of electoral control, each coming in two variants. In constructive control \[\text{BTT92}\], the chair tries to make his or her favorite candidate win; in destructive control \[\text{HHR07}\], the chair tries to prevent a despised candidate’s victory. We refrain from giving a detailed discussion of real-life scenarios for each of these 22 standard control types that motivate them; these can be found in, e.g., \[\text{BTT92}\, \text{HHR07}\, \text{FHHR09a}\, \text{HHR09}\, \text{ENR09}\]. However, we stress that every control type is motivated by an appropriate real-life scenario.

We start with partition of voters with the tie-handling rule “ties promote” (TP), see Hemaspaandra et al. \[\text{HHR07}\]. This control type produces a two-stage election with two first-stage and one final-stage subelections. The constructive variant of this problem is:

**Name:** Constructive Control by Partition of Voters in TP.

**Instance:** A set \(C\) of candidates, a list \(V\) of votes over \(C\), and a designated candidate \(c \in C\).
**Question:** Can \( V \) be partitioned into \( V_1 \) and \( V_2 \) such that \( c \) is the unique winner of the two-stage election in which the winners of the two first-stage subelections, \( (C,V_1) \) and \( (C,V_2) \), run against each other in the final stage?

The destructive variant of this problem is defined analogously, except it asks whether \( c \) is not a unique winner of this two-stage election. In both variants, if one uses the tie-handling model TE ("ties eliminate," see [HHR07]) instead of TP in the two first-stage subelections, a winner \( w \) of \( (C,V_1) \) or \( (C,V_2) \) proceeds to the final stage if and only if \( w \) is the only winner of his or her subelection. Each of the four problems just defined models "two-district gerrymandering."

There are many ways of introducing new voters into an election—think, e.g., of "get-out-the-vote" drives, or of lowering the age-limit for the right to vote, or of attracting new voters with certain promises or even small gifts), and such scenarios are modeled as CONSTRUCTIVE/DESTRUCTIVE CONTROL BY ADDING VOTERS: Given a set \( C \) of candidates, two disjoint lists of votes over \( C \) (one list, \( V \), corresponding to the already registered voters and the other list, \( W \), corresponding to the as yet unregistered voters whose votes may be added), a designated candidate \( c \in C \), and a nonnegative integer \( k \), is there a subset \( W' \subseteq W \) such that \( ||W'|| \leq k \) and \( c \) is (is not) the unique winner in \( (C,V \cup W') \)?

Disenfranchisement and other means of voter suppression is modeled as CONSTRUCTIVE/DESTRUCTIVE CONTROL BY DELETING VOTERS: Given a set \( C \) of candidates, a list \( V \) of votes over \( C \), a designated candidate \( c \in C \), and a nonnegative integer \( k \), can one make \( c \) the unique winner (not a unique winner) of the election resulting from deleting at most \( k \) votes from \( V \)?

Having defined these eight standard types of voter control, we now turn to the 14 types of candidate control. Now, the control action seeks to influence the outcome of an election by either adding, deleting, or partitioning the candidates, again for both the constructive and the destructive variant.

In the adding candidates cases, we distinguish between adding, from a given pool of spoiler candidates, an unlimited number of such candidates (as originally defined by Bartholdi et al. [BTT92]) and adding a limited number of spoiler candidates (as defined by Faliszewski et al. [FHHR09a], to stay in sync with the problem format of control by deleting candidates and by adding/deleting voters). CONSTRUCTIVE/DESTRUCTIVE CONTROL BY ADDING (A LIMITED NUMBER OF) CANDIDATES, is defined as follows: Given two disjoint candidate sets, \( C \) and \( D \), a list \( V \) of votes over \( C \cup D \), a designated candidate \( c \in C \), and a nonnegative integer \( k \), can one find a subset \( D' \subseteq D \) such that \( ||D'|| \leq k \) and \( c \) is (is not) the unique winner in \( (C \cup D',V) \)? The “unlimited” version of the problem is the same, except that the addition limit \( k \) and the requirement “\( ||D'|| \leq k \)” are being dropped, so any subset of the spoiler candidates may be added.

**CONSTRUCTIVE/DESTRUCTIVE CONTROL BY DELETING CANDIDATES** is defined by: Given a set \( C \) of candidates, a list \( V \) of votes over \( C \), a designated candidate \( c \in C \), and a nonnegative integer \( k \), can one make \( c \) the unique winner (not a unique winner) of the election resulting from deleting at most \( k \) candidates (other than \( c \) in the destructive case) from \( C \)?

Finally, we define the partition-of-candidate cases, again using either of the two tie-handling models, TP and TE, but now we define these scenarios with and without a run-off. The variant with run-off, CONSTRUCTIVE/DESTRUCTIVE CONTROL BY RUN-OFF PARTITION OF CANDIDATES, is analogous to the partition-of-voters control type: Given a set \( C \) of candidates, a list \( V \) of votes
over $C$, and a designated candidate $c \in C$, can $C$ be partitioned into $C_1$ and $C_2$ such that $c$ is (is not) the unique winner of the two-stage election in which the winners of the two first-stage subelections, $(C_1, V)$ and $(C_2, V)$, who survive the tie-handling rule run against each other in the final stage? The variant without run-off is the same, except that the winners of first-stage subelection $(C_1, V)$ who survive the tie-handling rule run against $(C_2, V)$ in the final round (and not against the winners of $(C_2, V)$ surviving the tie-handling rule).\[\]2.3 Immunity, Susceptibility, Resistance, and Vulnerability

Let $\mathcal{C}\Sigma$ be a control type. We say a voting system is immune to $\mathcal{C}\Sigma$ if it is impossible for the chair to make the given candidate the unique winner in the constructive case (not a unique winner in the destructive case) via exerting control of type $\mathcal{C}\Sigma$. We say a voting system is susceptible to $\mathcal{C}\Sigma$ if it is not immune to $\mathcal{C}\Sigma$. A voting system that is susceptible to $\mathcal{C}\Sigma$ is said to be vulnerable to $\mathcal{C}\Sigma$ if the control problem corresponding to $\mathcal{C}\Sigma$ can be solved in polynomial time, and is said to be resistant to $\mathcal{C}\Sigma$ if the control problem corresponding to $\mathcal{C}\Sigma$ is NP-hard. These notions are due to Bartholdi et al. [BTT92] (except that we follow the now more common approach of Hemaspaandra et al. [HHR09] who define resistant to mean “susceptible and NP-hard” rather than “susceptible and NP-complete”).

3 Results

3.1 Overview

Table 1 shows in boldface our results on the control complexity of Bucklin voting. For comparison, this table also shows the results for fallback voting that are due to Erdélyi et al. [ER10, EPR10], for approval voting that are due to Hemaspaandra et al. [HHR07], and for SP-AV that are due to Erdélyi et al. [ENR09].

**Theorem 3.1.** Bucklin voting is resistant, vulnerable, and susceptible to the 22 types of control defined in Section 2 as shown in Table 1.

Since Bucklin voting is the special case of fallback voting where each voter approves of every candidate, we have the following corollary. Note that, by the first item of Corollary 3.2 the resistances for Bucklin voting shown in the present paper imply all resistances for fallback voting shown in [ERT0, EPR10] except one: the destructive case of partition of voters in the tie-handling model TP.

**Corollary 3.2.**

1. Fallback voting inherits all the NP-hardness lower bounds from Bucklin voting (i.e., if Bucklin voting is resistant to a control type $\mathcal{C}\Sigma$ then fallback voting is also resistant to $\mathcal{C}\Sigma$).

2. Bucklin voting inherits all the P membership upper bounds from fallback voting (i.e., if fallback voting is vulnerable to a control type $\mathcal{C}\Sigma$ then Bucklin voting is also vulnerable to $\mathcal{C}\Sigma$).

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\[\]

2For example, think of a sports tournament in which certain teams (such as last year’s champion and the team hosting this year’s championship) are given an exemption from qualification.
3.2 Susceptibility

If an election system \( \mathcal{E} \) satisfies the “unique” variant of the Weak Axiom of Revealed Preference (Unique-WARP, for short), then \( \mathcal{E} \) is immune to constructive control by adding candidates (by adding a limited number of candidates), and this observation has been applied to approval voting [BTT92, HHR07]. Unlike approval voting but just as fallback voting, Bucklin voting does not satisfy Unique-WARP.

**Proposition 3.3.** Bucklin voting does not satisfy Unique-WARP.

**Proof.** Consider the election \((C,V)\) with candidate set \(C = \{a,b,c,d\}\) and voter collection \(V = \{v_1,v_2,\ldots,v_6\}\):

\[
\begin{align*}
v_1 &= v_2 = v_3 : & (C,V) \\
v_4 &= v_5 : & a \quad c \quad b \quad d \\
v_6 : & a \quad c \quad b \\
\end{align*}
\]

Candidate \(a\) is the unique Bucklin winner of the election \((C,V)\), reaching the strict majority threshold on level 2 with \(\text{score}^2_{(C,V)}(a) = 4\). By removing candidate \(b\) from the election, we get the subelection \((C',V)\) with \(C' = \{a,c,d\}\). There is no candidate on level 1 who passes the strict majority threshold. However, there are two candidates on the second level with a strict majority, namely candidates \(a\) and \(c\). Since \(\text{score}^2_{(C',V)}(c) = 5 > 4 = \text{score}^2_{(C',V)}(a)\), the unique Bucklin winner of the subelection \((C',V)\) is candidate \(c\). Thus, Bucklin voting does not satisfy Unique-WARP. \(\square\)

Indeed, as we will now show, Bucklin voting is susceptible to each of our 22 control types. Our proofs make use of the results of [HHR07] that provide general proofs of and links between certain...
susceptibility cases. For the sake of self-containment, we state their results, as Theorems A.1, A.2, and A.3 in the appendix.

We start with susceptibility to candidate control.

**Lemma 3.4.** Bucklin voting is susceptible to constructive and destructive control by adding candidates (in both the “limited” and the “unlimited” case), by deleting candidates, and by partition of candidates (with or without run-off and for each in both model TE and model TP).

**Proof.** From Theorem A.1 and the fact that Bucklin voting is a voiced voting system, it follows that Bucklin voting is susceptible to constructive control by deleting candidates, and to destructive control by adding candidates (in both the “limited” and the “unlimited” case).

Now, consider the election \((C, V)\) given in the proof of Proposition 3.3. The unique Bucklin winner of the election is candidate \(a\). Partition \(C\) into \(C_1 = \{a, c, d\}\) and \(C_2 = \{b\}\). The unique Bucklin winner of subelection \((C_1, V)\) is candidate \(c\), as shown in the proof of Proposition 3.3. In both partition and run-off partition of candidates and for each in both tie-handling models, TE and TP, candidate \(b\) runs against candidate \(c\) in the final stage of the election. The unique Bucklin winner is in each case candidate \(c\). Thus, Bucklin voting is susceptible to destructive control by partition of candidates (with or without run-off and for each in both model TE and model TP).

By Theorem A.3 Bucklin voting is also susceptible to destructive control by deleting candidates. By Theorem A.2 Bucklin voting is also susceptible to constructive control by adding candidates (in both the “limited” and the “unlimited” case).

Now, changing the roles of \(a\) and \(c\) makes \(c\) our distinguished candidate. In election \((C, V)\), \(c\) loses against candidate \(a\). By partitioning the candidates as described above, \(c\) becomes the unique Bucklin winner of the election. Thus, Bucklin voting is susceptible to constructive control by partition of candidates (with or without run-off and for each in both tie-handling models, TE and TP).

We now turn to susceptibility to voter control.

**Lemma 3.5.** Bucklin voting is susceptible to constructive and destructive control by adding voters, by deleting voters, and by partition of voters (in both model TE and model TP).

**Proof.** Consider the election \((C, V)\), where \(C = \{a, b, c, d\}\) is the set of candidates and \(V = \{v_1, v_2, v_3, v_4\}\) is the collection of voters with the following preferences:

\[
\begin{align*}
(C, V) \\
v_1 & : a \ b \ c \ d \\
v_2 & : d \ c \ a \ b \\
v_3 & : b \ a \ c \ d \\
v_4 & : b \ a \ c \ d
\end{align*}
\]

An election system is said to be **voiced** if the single candidate in any one-candidate election always wins.
We partition $V$ into $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4\}$. Thus we split $(C, V)$ into two subelections:

\[
\begin{align*}
\text{v}_1 : & & a & & c & & b & & d \\
\text{v}_2 : & & d & & c & & a & & b \\
\text{v}_3 : & & b & & a & & c & & d \\
\text{v}_4 : & & b & & a & & c & & d
\end{align*}
\]

Clearly, candidate $a$ is the unique Bucklin winner of $(C, V)$. However, $c$ is the unique Bucklin winner of $(C, V_1)$ and $b$ is the unique Bucklin winner of $(C, V_2)$, and so $a$ is not promoted to the final stage. Thus, Bucklin voting is susceptible to destructive control by partition of voters in both tie-handling models, TE and TP.

By Theorem \[A.1\] and the fact that Bucklin voting is a voiced voting system, Bucklin voting is susceptible to destructive control by deleting voters. By Theorem \[A.2\] Bucklin voting is also susceptible to constructive control by adding voters.

By changing the roles of $a$ and $c$ again, we can see that Bucklin voting is susceptible to constructive control by partition of voters in both model TE and model TP. By Theorem \[A.3\] Bucklin voting is also susceptible to constructive control by deleting voters. Finally, again by Theorem \[A.2\] Bucklin voting is susceptible to destructive control by adding voters. 

\[\blacksquare\]

### 3.3 Candidate Control

Fallback voting is a hybrid system combining Bucklin voting with approval voting. While fallback and approval voting behave quite differently with respect to immunity/vulnerability/resistance to control (contrast the results of Hemaspaandra et al. on approval voting [HHR07] with those of Erdélyi et al. [ER10, EPR10] on fallback voting, see Table 1), Bucklin voting seems to behave equally well as fallback voting in terms of control resistance. In particular, like fallback voting, Bucklin voting is also fully resistant to candidate control.

**Theorem 3.6.** Bucklin voting is resistant to each of the 14 standard types of candidate control.

All reductions except one (namely that for constructive control by deleting candidates, see Lemma \[3.13\]) apply Construction \[3.8\] below. This construction is based on that for fallback voting [ER10]; however, there are significant differences. In fallback voting, the disapproved candidates need not be ranked and can safely be ignored, since they cannot score points. In Bucklin voting, however, there are no disapproved candidates, so every candidate has to be placed at a suitable position in each vote to make the reduction work. Thus the reductions for Bucklin voting will be more specific and the arguments more involved. Also, since every candidate can potentially score points in Bucklin voting, no matter what his or her position in a vote is, we have to use a restricted version of HITTING SET, which by Lemma \[3.7\] is also NP-complete:

**Name:** Restricted Hitting Set.

**Instance:** A set $B = \{b_1, b_2, \ldots, b_n\}$, a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of nonempty subsets $S_i \subseteq B$ such that $n > m$, and a positive integer $k < m$. 


**Question:** Does $\mathcal{S}$ have a hitting set of size at most $k$, i.e., is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each $i$, $S_i \cap B' \neq \emptyset$?

We first need to show that **Restricted Hitting Set** is NP-complete in order to apply Construction 3.8 in the proof of Theorem 3.6.

**Lemma 3.7.** **Restricted Hitting Set** is NP-complete.

**Proof.** It is immediate that **Restricted Hitting Set** is in NP. To show NP-hardness, we reduce the (general) **Hitting Set** problem to **Restricted Hitting Set**. Given an instance $(\hat{B}, \hat{\mathcal{S}}, \hat{k})$ of **Hitting Set**, where $\hat{B} = \{b_1, b_2, \ldots, b_{\hat{m}}\}$ is a set, $\hat{\mathcal{S}} = \{S_1, S_2, \ldots, S_{\hat{n}}\}$ is a collection of nonempty subsets of $\hat{B}$, and $\hat{k} \leq \hat{m}$ is a positive integer, define the following instance $(B, \mathcal{S}, k)$ of **Restricted Hitting Set**:

$$
B = \begin{cases} 
\hat{B} \cup \{a\} & \text{if } \hat{n} \leq \hat{m} \\
\hat{B} & \text{if } \hat{n} > \hat{m}
\end{cases}
$$

$$
\mathcal{S} = \begin{cases} 
\hat{\mathcal{S}} \cup \{S_{\hat{n}+1}, S_{\hat{n}+2}, \ldots, S_{\hat{m}+2}\} & \text{if } \hat{n} \leq \hat{m} \\
\hat{\mathcal{S}} & \text{if } \hat{n} > \hat{m}
\end{cases}
$$

$$
k = \begin{cases} 
\hat{k} + 1 & \text{if } \hat{n} \leq \hat{m} \\
\hat{k} & \text{if } \hat{n} > \hat{m}
\end{cases}
$$

where

$$S_{\hat{n}+1} = S_{\hat{n}+2} = \cdots = S_{\hat{m}+2} = \{a\}.$$

Let $n$ be the number of members of $\mathcal{S}$ and $m$ be the number of elements of $B$. Note that if $\hat{n} > \hat{m}$ then $(B, \mathcal{S}, k) = (\hat{B}, \hat{\mathcal{S}}, \hat{k})$, so $n = \hat{n} > \hat{m} = m$; and if $\hat{n} \leq \hat{m}$ then $n = \hat{m} + 2 > \hat{m} + 1 = m$. Thus, in both cases $(B, \mathcal{S}, k)$ fulfills the restriction of **Restricted Hitting Set**.

It is easy to see that $\hat{\mathcal{S}}$ has a hitting set of size at most $\hat{k}$ if and only if $\mathcal{S}$ has a hitting set of size at most $k$. In particular, assuming $\hat{n} \leq \hat{m}$, if $\hat{\mathcal{S}}$ has a hitting set $B'$ of size at most $\hat{k}$ then $B' \cup \{a\}$ is a hitting set of size at most $k = \hat{k} + 1$ for $\mathcal{S}$; and if $\hat{\mathcal{S}}$ has no hitting set of size at most $\hat{k}$ then $\mathcal{S}$ can have no hitting set of size at most $k = \hat{k} + 1$ (because $a \not\in \hat{B}$, so $\{a\} \cap S_i = \emptyset$ for each $i$, $1 \leq i \leq \hat{n}$).

**Construction 3.8.** Let $(B, \mathcal{S}, k)$ be a given instance of **Restricted Hitting Set**, where $B = \{b_1, b_2, \ldots, b_m\}$ is a set, $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$ such that $n > m$, and $k < m$ is a positive integer. (Thus, $n > m > k \geq 1$.)

Define the election $(C, V)$, where $C = B \cup \{c, d, w\}$ is the candidate set and where $V$ consists of the following 6n(k + 1) + 4m + 11 voters:
For each . . .

| #  | For each . . . | number of voters | ranking of candidates |
|----|----------------|------------------|-----------------------|
| 1  |                | 2m + 1           | c d B w               |
| 2  |                | 2n + 2k(n − 1) + 3 | c w d B               |
| 3  |                | 2n(k + 1) + 5    | w c d B               |
| 4  | i ∈ {1, . . . , n} | 2(k + 1) | d S_i c w (B − S_i)  |
| 5  | j ∈ {1, . . . , m} | 2(k + 1) | d b_j w c (B − {b_j}) |
| 6  |                | 2(k + 1) | d w c B               |

We now give a detailed proof of Theorem 3.6 except for the case of constructive control by deleting candidates, which will be handled separately in Lemma 3.13 via Construction 3.8 in Lemmas 3.10, 3.11, and 3.12. The proofs of these lemmas in turn will make use of Lemma 3.9 below.

**Lemma 3.9.** Consider the election (C, V) constructed according to Construction 3.8 from a Restricted Hitting Set instance (B, S, k).

1. c is the unique level 2 BV winner of (c, d, w), V).

2. If S has a hitting set B' of size k, then w is the unique BV winner of election (B' ∪ {c, d, w}, V).

3. Let D ⊆ B ∪ {d, w}. If c is not a unique BV winner of election (D ∪ {c}, V), then there exists a set B' ⊆ B such that

   (a) D = B' ∪ {d, w},

   (b) w is the unique level 2 BV winner of election (B' ∪ {c, d, w}, V), and

   (c) B' is a hitting set for S of size at most k.

**Proof.** For the first part, note that there is no level 1 BV winner in election (c, d, w), V) and we have the following level 2 scores in this election:

\[
score^2_{(c, d, w), V}(c) = 6n(k + 1) + 2(m − k) + 9,
\]

\[
score^2_{(c, d, w), V}(d) = 2n(k + 1) + 4m + 2k + 3,
\]

\[
score^2_{(c, d, w), V}(w) = 4n(k + 1) + 2m + 10.
\]

Since n > m (which implies n > k), we have:

\[
score^2_{(c, d, w), V}(c) − score^2_{(c, d, w), V}(d) = 4n(k + 1) − (2m + 4k) + 6 > 0,
\]

\[
score^2_{(c, d, w), V}(c) − score^2_{(c, d, w), V}(w) = 2n(k + 1) − (2k + 1) > 0.
\]

Thus, c is the unique level 2 BV winner of (c, d, w), V).

For the second part, suppose that B' is a hitting set for S of size k. Then there is no level 1 BV winner in election (B' ∪ {c, d, w}, V), and we have the following level 2 scores:

\[
score^2_{(B ∪ (c, d, w), V)}(c) = 4n(k + 1) + 2(m − k) + 9,
\]

\[
score^2_{(B ∪ (c, d, w), V)}(d) = 2n(k + 1) + 4m + 2k + 3,
\]

\[
score^2_{(B ∪ (c, d, w), V)}(w) = 4n(k + 1) + 2(m − k) + 10,
\]

\[
score^2_{(B ∪ (c, d, w), V)}(b_j) ≤ 2n(k + 1) + 2 \text{ for all } b_j ∈ B'.
\]
It follows that $w$ is the unique level 2 BV winner of election $(B' \cup \{c,d,w\}, V)$.  

For the third part, let $D \subseteq B \cup \{d,w\}$. Suppose $c$ is not a unique BV winner of election $(D \cup \{c\}, V)$.  

(3a) Other than $c$, only $w$ has a strict majority of votes on the second level and only $w$ can tie or beat $c$ in $(D \cup \{c\}, V)$. Thus, since $c$ is not a unique BV winner of election $(D \cup \{c\}, V)$, $w$ is clearly in $D$. In $(D \cup \{c\}, V)$, candidate $w$ has no level 1 strict majority, and candidate $c$ has already on level 2 a strict majority. Thus, $w$ must tie or beat $c$ on level 2. For a contradiction, suppose $d \notin D$. Then

$$\text{score}^2_{(D \cup \{c\}, V)}(c) \geq 4n(k + 1) + 2m + 11.$$  

The level 2 score of $w$ is

$$\text{score}^2_{(D \cup \{c\}, V)}(w) = 4n(k + 1) + 2m + 10,$$  

which contradicts our assumption, that $w$ ties or beats $c$ on level 2. Thus, $D = B' \cup \{d,w\}$, where $B' \subseteq B$.  

(3b) This part follows immediately from part (3a).  

(3c) Let $\ell$ be the number of sets in $\mathcal{S}$ not hit by $B'$. We have that

$$\text{score}^2_{(B' \cup \{c,d,w\}, V)}(w) = 4n(k + 1) + 10 + 2(m – \|B'\|),$$  

$$\text{score}^2_{(B' \cup \{c,d,w\}, V)}(c) = 2(m – k) + 4n(k + 1) + 9 + 2(k + 1)\ell.$$  

From part (3a) we know that

$$\text{score}^2_{(B' \cup \{c,d,w\}, V)}(w) \geq \text{score}^2_{(B' \cup \{c,d,w\}, V)}(c),$$  

so

$$4n(k + 1) + 10 + 2(m – \|B'\|) \geq 2(m – k) + 4n(k + 1) + 9 + 2(k + 1)\ell.$$  

The above inequality implies

$$1 > \frac{1}{2} \geq \|B'\| – k + (k + 1)\ell \geq 0,$$

so $\|B'\| – k + (k + 1)\ell = 0$. Thus $\ell = 0$, and it follows that $B'$ is a hitting set for $\mathcal{S}$ of size $k$. This completes the proof of Lemma 3.9.  

**Lemma 3.10.** Bucklin voting is resistant to constructive and destructive control by adding candidates (both in the limited and the unlimited version of the problem).  

**Proof.** Susceptibility holds by Lemma 3.4. NP-hardness follows immediately from Lemmas 3.7 and 3.9 via mapping the RESTRICTED HITTING SET instance $(B, \mathcal{S}, k)$ to the instance
1. \(((\{c,d,w\} \cup B,V),w,k)\) of Constructive Control by Adding a Limited Number of Candidates,
2. \(((\{c,d,w\} \cup B,V),c,k)\) of Destructive Control by Adding a Limited Number of Candidates,
3. \(((\{c,d,w\} \cup B,V),w)\) of Constructive Control by Adding an Unlimited Number of Candidates, and
4. \(((\{c,d,w\} \cup B,V),c)\) of Destructive Control by Adding an Unlimited Number of Candidates.

where in each case \(c, d,\) and \(w\) are the qualified candidates and \(B\) is the set of spoiler candidates.  

**Lemma 3.11.** Bucklin voting is resistant to destructive control by deleting candidates.

**Proof.** Susceptibility holds by Lemma 3.4. To show the problem NP-hard, let \((C,V)\) be the election resulting from a RESTRICTED HITTING SET instance \((B,\mathcal{S},k)\) according to Construction 3.8 and let \(c\) be the distinguished candidate.

We claim that \(\mathcal{S}\) has a hitting set of size at most \(k\) if and only if \(c\) can be prevented from being a unique BV winner by deleting at most \(m - k\) candidates.

From left to right: Suppose \(\mathcal{S}\) has a hitting set \(B'\) of size \(k\). Delete the \(m - k\) candidates \(B - B'\).

Now, both candidates \(c\) and \(w\) have a strict majority on level 2, but
\[
\text{score}_{(\{c,d,w\} \cup B',V)}^2(c) = 4n(k + 1) + 2(m - k) + 9,
\]
\[
\text{score}_{(\{c,d,w\} \cup B',V)}^2(w) = 4n(k + 1) + 2(m - k) + 10,
\]
so \(w\) is the unique level 2 BV winner of this election.

From right to left: Suppose that \(c\) can be prevented from being a unique BV winner by deleting at most \(m - k\) candidates. Let \(D' \subseteq B \cup \{d,w\}\) be the set of deleted candidates (so \(c \notin D'\)) and \(D = (C - D') - \{c\}\). It follows immediately from Lemma 3.9 that \(D = B' \cup \{d,w\}\), where \(B'\) is a hitting set for \(\mathcal{S}\) of size at most \(k\).

**Lemma 3.12.** Bucklin voting is resistant to constructive and destructive control by partition of candidates and by run-off partition of candidates (for each in both tie-handling models, TE and TP).

**Proof.** Susceptibility holds by Lemma 3.4 so it remains to show NP-hardness. For the constructive cases, map the given RESTRICTED HITTING SET instance \((B,\mathcal{S},k)\) to the election \((C,V)\) from Construction 3.8 with distinguished candidate \(w\).

We claim that \(\mathcal{S}\) has a hitting set of size at most \(k\) if and only if \(w\) can be made the unique BV winner by exerting control via any of our four control scenarios (partition of candidates with or without run-off, and for each in either tie-handling model, TE and TP).

From left to right: Suppose \(\mathcal{S}\) has a hitting set \(B' \subseteq B\) of size \(k\). Partition the set of candidates into the two subsets \(C_1 = B' \cup \{c,d,w\}\) and \(C_2 = C - C_1\). According to Lemma 3.9 \(w\) is the unique level 2 BV winner of subelection \((C_1,V) = (B' \cup \{c,d,w\},V)\). Since (no matter whether we have a run-off or not, and regardless of the tie-handling rule used) the opponents of \(w\) in the final stage (if there are any opponents at all) each are candidates from \(B\). Since \(n > m\), \(w\) has a majority in the
For each . . .

E

n

k

1

C

j

w

F

C

S

C

For each $2 = D$ the number of candidates in $\mathcal{S}$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ a collection of nonempty subsets $S_i \subseteq B$, and a positive integer $k \leq m$.

Question: Does $\mathcal{S}$ have a hitting set of size at most $k$, i.e., is there a set $B' \subseteq B$ with $|B'| \leq k$ such that for each $i$, $S_i \cap B' \neq \emptyset$?

Lemma 3.13. Bucklin voting is resistant to constructive control by deleting candidates.

Proof. Susceptibility holds by Lemma 3.4. To prove NP-hardness we give a reduction from Hitting Set. Let $(B, \mathcal{S}, k)$ be a Hitting Set instance with $B = \{b_1, b_2, \ldots, b_n\}$ a set, $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ a collection of nonempty subsets $S_i \subseteq B$, and $k \leq m$ a positive integer. Define the election $(C, V)$ with candidate set $C = B \cup C' \cup D \cup E \cup F \cup \{w\}$, where $C' = \{c_1, c_2, \ldots, c_k\}$, $D = \{d_1, d_2, \ldots, d_k\}$, $E = \{e_1, e_2, \ldots, e_n\}$, $F = \{f_1, \ldots, f_{n+k}\}$, $w$ is the distinguished candidate, and the number of candidates in $D$ is $s = \sum_{i=1}^{n} s_i$ with $s_i = n + k - |S_i|$, so $s = n^2 + kn - \sum_{i=1}^{n} |S_i|$. For each $i$, $1 \leq i \leq n$, let $D_i = \{d_{1+\sum_{j=1}^{i-1} s_j}, \ldots, d_{\sum_{j=1}^{i} s_j}\}$. Define $V$ to be the following collection of $2(n + k + 1) + 1$ voters:

| # | For each . . . | number of voters | ranking of candidates |
|---|---|---|---|
| 1 | $i \in \{1, \ldots, n\}$ | 1 | $S_i \ D_i \ w \ C' \ E \ (D-D_i) \ (B-S_i) \ F$ |
| 2 | $j \in \{1, \ldots, k+1\}$ | 1 | $E \ (C' - \{c_j\}) \ c_j \ B \ D \ w \ F$ |
| 3 | | $k+1$ | $w \ F \ C' \ E \ B \ D$ |
| 4 | | $n$ | $C' \ D \ F \ B \ w \ E$ |
| 5 | | 1 | $C' \ w \ D \ F \ E \ B$ |

There is no unique BV winner in election $(C, V)$, since the candidates in $C'$ and candidate $w$ are level $n + k + 1$ BV winners.

We claim that $\mathcal{S}$ has a hitting set of size $k$ if and only if $w$ can be made the unique BV winner by deleting at most $k$ candidates.

From left to right: Suppose $\mathcal{S}$ has a hitting set $B'$ of size $k$. Delete the corresponding candidates. Now, $w$ is the unique level $n + k$ BV winner of the resulting election.

From right to left: Suppose $w$ can be made the unique BV winner by deleting at most $k$ candidates. Since $k + 1$ candidates other than $w$ have a strict majority on level $n + k + 1$ in election $(C, V)$, after deleting at most $k$ candidates, there is still at least one candidate other than $w$ with a
strict majority of approvals on level \( n + k + 1 \). However, since \( w \) was made the unique BV winner by deleting at most \( k \) candidates, \( w \) must be the unique BV winner on a level lower than or equal to \( n + k \). This is possible only if in all \( n \) votes of the first voter group \( w \) moves forward by at least one position. This, however, is possible only if \( \mathcal{S} \) has a hitting set \( B' \) of size \( k \).

\[ \square \]

### 3.4 Voter Control

We now turn to voter control for Bucklin voting. Our reductions are from the NP-complete problem \textsc{Exact Cover by Three-Sets} (X3C, for short), which is defined as follows (see [GJ79]):

**Name:** \textsc{Exact Cover by Three-Sets} (X3C).

**Instance:** A set \( B = \{b_1, b_2, \ldots, b_{3m}\} \), \( m \geq 1 \), and a collection \( \mathcal{S} = \{S_1, S_2, \ldots, S_n\} \) of subsets \( S_i \subseteq B \) with \( \|S_i\| = 3 \) for each \( i, 1 \leq i \leq n \).

**Question:** Is there a subcollection \( \mathcal{S}' \subseteq \mathcal{S} \) such that each element of \( B \) occurs in exactly one set in \( \mathcal{S}' \)?

**Theorem 3.14.** Bucklin voting is resistant to constructive control by adding voters, by deleting voters, and by partition of voters in model TE and model TP, but is vulnerable to destructive control by adding and by deleting voters.

**Lemma 3.15.** Bucklin voting is resistant to constructive control by adding voters.

**Proof.** Susceptibility holds by Lemma 3.5. Let \((B, \mathcal{S})\) be an X3C instance, where \( B = \{b_1, b_2, \ldots, b_{3m}\} \) is a set with \( m > 1 \) and \( \mathcal{S} = \{S_1, S_2, \ldots, S_n\} \) is a collection of subsets \( S_i \subseteq B \) with \( \|S_i\| = 3 \) for each \( i, 1 \leq i \leq n \). (Note that X3C is trivial to solve for \( m = 1 \).)

Define the election \((C, V \cup V')\), where \( C = B \cup \{w\} \cup D \) with \( D = \{d_1, \ldots, d_{m(3m-4)}\} \) is the set of candidates, \( w \) is the distinguished candidate, and \( V \cup V' \) is the following collection of \( n + m - 2 \) voters:

1. \( V \) is the collection of \( m - 2 \) registered voters of the form: \( B D w \).
2. \( V' \) is the collection of unregistered voters, where for each \( i, 1 \leq i \leq n \), there is one voter of the form: \( D_i S_i (D - D_i) (B - S_i) \), where \( D_i = \{d_{(i-1)(3m-4)+1}, \ldots, d_{i(3m-4)}\} \).

Since \( b_1 \in B \) has a majority already on the first level, \( w \) is not a unique BV winner in \((C, V)\).

We claim that \( \mathcal{S} \) has an exact cover for \( B \) if and only if \( w \) can be made a unique BV winner by adding at most \( m \) voters from \( V' \).

From left to right: Suppose \( \mathcal{S} \) contains an exact cover for \( B \). Let \( V'' \) contain the corresponding voters from \( V' \) (i.e., voters of the form \( D_i S_i (D - D_i) (B - S_i) \), for each \( S_i \) in the exact cover) and add \( V'' \) to the election. It follows that \( \text{score}_{(C, V \cup V'')}^m(d_j) = m - 1 \) for all \( d_j \in D \), \( \text{score}_{(C, V \cup V'')}^m(b_j) = m - 1 \) for all \( b_j \in B \), and \( \text{score}_{(C, V \cup V'')}^m(w) = m \). Thus, only \( w \) has a strict majority up to the \((3m+1)\)st level and so \( w \) is the unique level \( 3m + 1 \) BV winner of the election.

From right to left: Let \( V'' \subseteq V' \) be such that \( \|V''\| \leq m \) and \( w \) is the unique winner of election \((C, V \cup V'')\). Since \( w \) must in particular beat every \( b_j \in B \) up to the \((3m+1)\)st level, it follows that
For each $i$, $B_i$ is the number of voters in $B$ who prefer candidate $b_i$ to any other candidate. For each collection of $2$ voters, $D_i$ is the number of voters who prefer any other candidate to $b_i$. Also, for each $i$, $F_i$ is the number of voters who prefer $b_i$ to any other candidate.

Lemma 3.16. Bucklin voting is resistant to constructive control by deleting voters.

Proof. Susceptibility holds by Lemma 3.5. Let $(B, S)$ be an X3C instance as above. Define the election $(C, V)$, where $C = B \cup \{c, w\} \cup D \cup F \cup G$ is the set of candidates with $D = \{d_1, d_2, \ldots, d_{3m}\}$, $F = \{f_1, f_2, \ldots, f_{3n(m-1)}\}$, and $G = \{g_1, g_2, \ldots, g_{3m(m-1)}\}$, and where $w$ is the distinguished candidate. For each $j, 1 \leq j \leq 3m$, define $\ell_j = \|\{S_i \in S \mid b_j \in S_i\}\|$, and for each $i, 1 \leq i \leq n$, define $B_i = \{b_j \in B \mid i \leq n - \ell_j\}$, $D_i = \{d_{(i-1)3m+1}, \ldots, d_{3m-\|B_i\|}\}$, and $F_i = \{f_{(i-1)(3m-3)+1}, \ldots, f_{3(m-3)}\}$.

Also, for each $k, 1 \leq k \leq m-1$, define $G_k = \{g_{3m(k-1)+1}, \ldots, g_{3mk}\}$. Let $V$ consist of the following collection of $2n + m - 1$ voters:

| # | For each ... | number of voters | ranking of candidates |
|---|-------------|-----------------|-----------------------|
| 1 | $i \in \{1, \ldots, n\}$ | 1 | $S_i \hspace{0.1cm} c \hspace{0.1cm} F_i \hspace{0.1cm} D \hspace{0.1cm} (B - S_i) \hspace{0.1cm} G \hspace{0.1cm} (F - F_i) \hspace{0.1cm} w$ |
| 2 | $i \in \{1, \ldots, n\}$ | 1 | $b_i \hspace{0.1cm} D_i \hspace{0.1cm} w \hspace{0.1cm} F \hspace{0.1cm} (D - D_i) \hspace{0.1cm} (B - B_i) \hspace{0.1cm} G \hspace{0.1cm} c$ |
| 3 | $k \in \{1, \ldots, m-1\}$ | 1 | $c \hspace{0.1cm} G_k \hspace{0.1cm} F \hspace{0.1cm} D \hspace{0.1cm} (G - G_k) \hspace{0.1cm} B \hspace{0.1cm} w$ |

Candidate $c$ is the unique level 4 BV winner in the election $(C, V)$.

We claim that $S$ has an exact cover for $B$ if and only if $w$ can be made the unique BV winner by deleting at most $m$ voters.

From left to right: Suppose $S$ contains an exact cover for $B$. By deleting the corresponding voters from the first voter group, we have the following level $3m+1$ scores in the resulting election $(C, V')$:

\[
\text{score}_{(C, V')}^{3m+1}(w) = n,
\]

\[
\text{score}_{(C, V')}^{3m+1}(b_i) = \text{score}_{(C, V')}^{3m+1}(c) = n - 1 \quad \text{for all} \; i, 1 \leq i \leq 3m,
\]

\[
\text{score}_{(C, V')}^{3m+1}(d_j) = 1 \quad \text{or} \quad \text{score}_{(C, V')}^{3m+1}(d_j) = 0 \quad \text{for all} \; d_j \in D, \text{and}
\]

\[
\text{score}_{(C, V')}^{3m+1}(f_j) = \text{score}_{(C, V')}^{3m+1}(g_k) = 1 \quad \text{for all} \; f_j \in F \text{ and all} \; g_k \in G.
\]

Since now there are $2n - 1$ voters in the election, candidate $w$ is the first candidate having a strict majority, so $w$ is the unique BV winner of election $(C, V')$.

From right to left: Suppose $w$ can be made the unique BV winner by deleting at most $m$ voters. Since $w$ doesn’t score any points on any of the first $3m$ levels (see Footnote 5), neither $c$ nor any of the $b_i$ can have a strict majority on any of these levels. In particular, candidate $c$ must have lost exactly $m$ points (up to the $(3m+1)\text{st}$ level) after deletion, and this is possible only if the $m$ deleted voters.

---

5Note that $D_i = 0$ if $\|B_i\| = 3m$ and that $w$ is always ranked at or later than the $(3m+1)\text{st}$ position.
voters are all from the first or third voter group. On the other hand, each \( b_i \in B \) must have lost at least one point (up to the \((3m+1)\)st level) after deletion, and this is possible only if exactly \( m \) voters were deleted from the first voter group. These \( m \) voters correspond to an exact cover for \( B \).

**Lemma 3.17.** Bucklin voting is vulnerable to destructive control by adding and deleting voters.

**Proof.** Susceptibility holds by Lemma 3.5. The polynomial-time algorithms that solve the two control problems for fallback voting [ER10] can easily be adapted (e.g., by adjusting the input format from fallback elections to that for Bucklin elections) to work for Bucklin voting as well, as Bucklin voting is the special case of fallback voting where each voter approves of all candidates.

**Lemma 3.18.** Bucklin voting is resistant to constructive control by partition of voters in both tie-handling models, TE and TP.

**Proof.** Susceptibility holds by Lemma 3.5. To show NP-hardness we reduce X3C to our control problems. Let \((B, \mathcal{S})\) be an X3C instance with \( B = \{b_1, b_2, \ldots, b_{3m}\}, m \geq 1\), and a collection \( \mathcal{S} = \{S_1, S_2, \ldots, S_n\} \) of subsets \( S_i \subseteq B \) with \( |S_i| = 3 \) for each \( i, 1 \leq i \leq n \). We define the election \((C, V)\), where \( C = B \cup \{c, w, x\} \cup D \cup E \cup F \cup G \) is the set of candidates with \( D = \{d_1, \ldots, d_{3m}\} \), \( E = \{e_1, \ldots, e_{(3m-1)(m+1)}\} \), \( F = \{f_1, \ldots, f_{(3m+1)(m-1)}\} \), and \( G = \{g_1, \ldots, g_{m(3m-3)}\} \), and where \( w \) is the distinguished candidate. For each \( j, 1 \leq j \leq 3m \), define \( \ell_j = |\{S_i \in \mathcal{S} \mid b_j \in S_i\}| \), and for each \( i, 1 \leq i \leq n \), define

\[
B_i = \{b_j \in B \mid i \leq n - \ell_j\},
\]

\[
D_i = \{d_{(i-1)3m+1}, \ldots, d_{3m-|B_i|}\}, \quad \text{and}
\]

\[
G_i = \{g_{(i-1)(3m-3)+1}, \ldots, g_{i(3m-3)}\}.
\]

Also, for each \( k, 1 \leq k \leq m+1 \), define \( E_k = \{e_{(3m-1)(k-1)+1}, \ldots, e_{(3m-1)k}\} \), and for each \( l, 1 \leq l \leq m-1 \), define \( F_l = \{f_{(3m+1)(l-1)+1}, \ldots, f_{(3m+1)l}\} \). Let \( V \) consist of the following \( 2n + 2m \) voters:

| # | For each ... | number of voters | ranking of candidates |
|---|---|---|---|
| 1 | \( i \in \{1, \ldots, n\} \) | 1 | \( c \quad S_i \quad G_i \quad (G - G_i) \quad F \quad D \quad E \quad (B - S_i) \quad w \quad x \) |
| 2 | \( i \in \{1, \ldots, n\} \) | 1 | \( B_i \quad D_i \quad w \quad G \quad E \quad (D - D_i) \quad F \quad (B - B_i) \quad c \quad x \) |
| 3 | \( k \in \{1, \ldots, m+1\} \) | 1 | \( x \quad c \quad E_k \quad F \quad (E - E_k) \quad G \quad D \quad B \quad w \) |
| 4 | \( l \in \{1, \ldots, m-1\} \) | 1 | \( F_i \quad c \quad (F - F_i) \quad G \quad D \quad E \quad B \quad w \quad x \) |

In this election, candidate \( c \) is the unique level 2 BV winner with a level 2 score of \( n + m + 1 \).

We claim \( \mathcal{S} \) has an exact cover \( \mathcal{S}' \) for \( B \) if and only if \( w \) can be made the unique BV winner of the resulting election by partition of voters in both tie-handling models TE and TP.

From left to right: Suppose \( \mathcal{S} \) has an exact cover \( \mathcal{S}' \) for \( B \). Partition \( V \) the following way. Let \( V_1 \) consist of:
• the $m$ voters of the first group that correspond to the exact cover (i.e., those $m$ voters of the form $c S_i G_i (G - G_i) F D E (B - S_i) w x$ for which $S_i \in \mathcal{S}')$ and

• the $m + 1$ voters of the third group (i.e., all voters of the form $x c E_k F (E - E_k) G D B w$)

Let $V_2 = V - V_1$. In subelection $(C, V_1)$, candidate $x$ is the unique level 1 BV winner. In subelection $(C, V_2)$, candidate $w$ is the first candidate who has a strict majority and moves on to the final round of the election. Thus there are $w$ and $x$ in the final run-off, which $w$ wins with a strict majority on the first level. Since both subelections, $(C, V_1)$ and $(C, V_2)$, have unique BV winners, candidate $w$ can be made the unique BV winner by partition of voters in both tie-handling models, TE and TP.

From right to left: Suppose that $w$ can be made the unique BV winner by exerting control by partition of voters. Let $(V_1, V_2)$ be such a successful partition. Since $w$ wins the resulting two-stage election, $w$ has to win at least one of the subelections (say, $w$ wins $(C, V_1)$). If candidate $c$ participates in the final round, he or she wins the election with a strict majority no later than on the second level, no matter which other candidates move forward to the final election. That means that in both subelections, $(C, V_1)$ and $(C, V_2)$, candidate $w$ must not be a BV winner. Only in the second voter group candidate $w$ (who has to be a BV winner in $(C, V_1)$) gets points higher than on the second-to-last level. So $w$ has to be a level $3m + 1$ BV winner in $(C, V_1)$, which implies that there have to be voters from the second voter group in $V_1$. Therefore, in subelection $(C, V_2)$ only candidate $x$ can prevent $c$ from moving forward to the final round. Since $x$ is always placed behind $c$ in all votes except those votes from the third voter group, $x$ has to be a level 1 BV winner in $(C, V_2)$. Since in $(C, V_1)$ it is not possible that a candidate can tie with $w$ on the $(3m + 1)$st level, $w$ has to be the unique level $3m + 1$ BV winner in $(C, V_1)$. Thus both elections $(C, V_1)$ and $(C, V_2)$ have unique BV winners and so the construction works for both tie-handling models, TE and TP.

It remains to show that $\mathcal{S}$ has an exact cover $\mathcal{S}'$ for $B$. Since $w$ has to win $(C, V_1)$ with the votes from the second voter group, not all voters from the first voter group can be in $V_1$ (otherwise $c$ would have $n$ points already on the first level). On the other hand, there can be at most $m$ voters from the first voter group in $V_2$ because otherwise $x$ would not be a level 1 BV winner in $(C, V_2)$. To ensure that no candidate contained in $B$ has the same score as $w$, namely $n$ points, and gets these points on an earlier level than $w$ in $(C, V_1)$, there have to be exactly $m$ voters from the first group in $V_2$ and these voters correspond to an exact cover for $B$. $\square$

4 Conclusions and Open Questions

We have shown that Bucklin voting is resistant to all standard types of candidate control and all standard types of constructive control. In total, it possesses at least 18 resistances to the 22 commonly studied control types, it has at least two (and can have no more than four) vulnerabilities, and two cases remain open: destructive control by partition of voters in both tie-handling models, TE and TP. For comparison, recall from Table 1 that, for destructive control by partition of voters, approval voting is vulnerable both in model TE and TP, SP-AV is vulnerable in model TE but resistant.
in model TP, and fallback voting is resistant in model TP and it is open whether fallback voting is vulnerable or resistant to this control type in model TE.

Only SP-AV and fallback voting are currently known to be resistant to one more control type than Bucklin voting. However, Bucklin voting is arguably a simpler and more natural voting system; for example, unlike SP-AV and fallback voting, it is a majority-consistent voting rule.

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A Some Results of [HHR07] Used in Section 3.2

Theorem A.1 (HHR07).
1. If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.

2. Each voiced voting system is susceptible to constructive control by deleting candidates.

3. Each voiced voting system is susceptible to destructive control by adding candidates.

Theorem A.2 (HHR07).
1. A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.

2. A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.

3. A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.

4. A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.

Theorem A.3 (HHR07).
1. If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.

2. If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.

3. If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.

4. If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.

Following Bartholdi et al. [BTT92], Hemaspaandra et al. [HHR07] considered only the case of control by adding a limited number of candidates—the “unlimited” case was introduced only in (the conference precursors of) [FHHR09a]. However, it is easy to see that all results about control by adding candidates stated in Theorems A.1, A.2, and A.3 hold true in both the limited and the unlimited case.