Evolution of linear perturbations in spherically symmetric dust spacetimes

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Abstract
We present results from a numerical code implementing a new method to solve the master equations describing the evolution of linear perturbations in a spherically symmetric but inhomogeneous background. This method can be used to simulate several configurations of physical interest, such as relativistic corrections to structure formation, the lensing of gravitational waves (GWs) and the evolution of perturbations in a cosmological void model. This paper focuses on the latter problem, i.e. structure formation in a Hubble scale void in the linear regime. This is considerably more complicated than linear perturbations of a homogeneous and isotropic background because the inhomogeneous background leads to coupling between density perturbations and rotational modes of the spacetime geometry, as well as GWs. Previous analyses of this problem ignored this coupling in the hope that the approximation does not affect the overall dynamics of structure formation in such models. We show that for a giga-parsec void, the evolution of the density contrast is well approximated by the previously studied decoupled evolution only for very large-scale modes. However, the evolution of the gravitational potentials within the void is inaccurate at more than the 10% level, and is even worse on small scales.

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1. Introduction

We present results from a numerical code implementing a new method that solves the first order gauge-invariant linear perturbation equations in a Lemaître–Tolman–Bondi (LTB)
background. LTB models are spherically symmetric but inhomogeneous dust solutions of the Einstein field equations. Compared to a Friedman–Lemaître–Robertson–Walker (FLRW) background, perturbations around a LTB background are complicated by the fact that they cannot be decomposed into the standard scalar vector and tensor modes, as the reduction of symmetry (through radial inhomogeneity) causes these modes to couple [1].

The method presented here can be used to model a variety of different astrophysical/cosmological scenarios. For example:

**Relativistic corrections for structure formation.** Currently structure formation in cosmology is modelled either using nonlinear models of Newtonian systems, or relativistically but only in the linear regime. This leaves an important area unexplored: nonlinear relativistic aspects of structure formation—see, e.g., [2] for a review. Certain aspects of this have started to be taken into account in N-body methods which make use of relativistic corrections to the potentials [3, 4]. Our model can be used to analyse the growth of structure on top of a strongly nonlinear background—either an over-density such as a cluster, or a large void, both of which generate large curvature and shear. The coupling of density perturbations to vector and tensor degrees of freedom can be explored and the errors induced by neglecting this coupling quantified.

**Evolution of perturbations in void models.** If we were to live in a large, underdense void of a few giga-parsecs in diameter, distant supernovae would appear fainter than expected in a FLRW model, and the dark energy phenomenon could be explained on purely relativistic terms without invoking any new physics (see, e.g., [5–9] for a comprehensive review). Clearly, structure formation places a key constraint on such models by probing their difference from the concordance model, and so serves as a test of the Copernican principle [10]. Structure formation in LTB models has only been quantified for the special case which neglects the coupling of the scalar gravitational potential to vector and tensor degrees of freedom [10–15]. While this seems reasonable, the accuracy has not been quantified. A recent alternative approach to this problem, based on second-order perturbation theory in FLRW, can be found here: [16].

**Weak lensing of gravitational waves.** Gravitational waves (GWs) from supermassive black hole mergers act as precise ‘standard sirens,’ promising to significantly improve upon standard(izable) candles such as Type-Ia Supernovae (SN1a) (see, e.g., [17]). However, weak lensing of the GWs by the intervening dark matter distribution distorts the signal, degrading their use as cosmological distance estimators [18]. A particular problem is that the GW wavelength is comparable to the size of the dark matter halos which produce a portion of the lensing effect. Thus the geometric optics approximation cannot be used to model the expected lensing. By modelling a dark matter halo using a LTB model, and scattering GWs off it using our method, we can hope to quantify the lensing of GWs more accurately.

In this paper, we focus on the linear evolution of perturbations in Gpc-void cosmological models, and present the results for this case (other scenarios shall be analysed elsewhere). To illustrate the performance of the method and the physics of the evolution of perturbations, we concentrate on polar perturbations in a large-scale cosmological void that is asymptotically Einstein–de Sitter (EdS) and that fits the distance–redshift relations given by SN1a observations as well as age data. We show results for several different spherical harmonic frequencies, considering a variety of initial conditions at different locations throughout the void. We find that couplings can only be neglected if one is interested in the matter density growth function on very large scales. Indeed, for this quantity, the evolution without couplings is accurate at the sub-percent level when the large-scale quadrupole is considered. However, the
metric perturbation that is the closest to the standard ‘Newtonian’ gravitational potential gets corrections of up to 10% even in this case (which implies percent-level inaccuracies to the two-point correlation function given in [10]). Furthermore, for higher frequency perturbations even the density contrast suffers more than 10% inaccuracies. This has significant impact on constraints on LTB models which use such approximations for analyses of structure formation.

2. Background evolution

The general unperturbed LTB line element may be written as
\[ ds^2 = -dt^2 + X^2(t, r)dr^2 + A^2(t, r)d\Omega^2, \]
where [5]
\[ X(t, r) = \frac{a_\perp(t, r)}{\sqrt{1 - \kappa(r)r^2}}, \quad A(t, r) = ra_\perp(t, r), \quad a_\parallel \equiv (ra_\perp)', \]
with a prime being shorthand for \( \partial_r \). The angular and radial scale factors, \( a_\perp \) and \( a_\parallel \), respectively, are associated with individual expansion rates
\[ H_\perp = \frac{\dot{a}_\perp}{a_\perp}, \quad H_\parallel = \frac{\dot{a}_\parallel}{a_\parallel}, \]
where an overdot denotes \( \partial_t \). While we only focus on a dust energy-density component in our analysis, we include the cosmological constant \( \Lambda \) in our equations for completeness. The angular component of the expansion then obeys the following Friedmann equation with different curvature, i.e. \( \kappa(r) \), on each radial shell:
\[ H_\perp^2(t, r) = H_{10}^2 \left[ \Omega_\perp a_\perp^{-3} + \Omega_\parallel a_\parallel^{-2} + \Omega_\Lambda \right], \]
where
\[ \Omega_m(r) \equiv \frac{M(r)}{H_{10}^2(r)}, \quad \Omega_\kappa(r) \equiv -\frac{\kappa(r)}{H_{10}^2(r)}, \quad \Omega_\Lambda(r) = \frac{\Lambda}{3H_{10}^2(r)}, \]
\[ \Omega_m(r) + \Omega_\kappa(r) + \Omega_\Lambda(r) = 1. \]

In these expressions, \( H_{10}(r) \) is the angular Hubble parameter today, \( M(r) \) a boundary condition related to the matter content within a comoving shell of radius \( r \), and we set \( a_\perp(r) = 1 \) by convention. The expansion and shear scalars are given by
\[ \Theta \equiv 2H_\perp + H_\parallel, \]
\[ \sigma^2 \equiv \frac{2}{3}(H_\parallel - H_\perp), \]
respectively. The total energy density (including dust and a cosmological constant) may be expressed as
\[ 8\pi G \left( \rho_m + \rho_\Lambda \right) = 3H_{10}^2 \left\{ \frac{\Omega_m}{\dot{a}_\perp^2 a_\perp^2} \left[ 1 + \frac{r}{3} \left( 2\frac{H_{10}^2}{H_\parallel} + \frac{\Omega_\Lambda}{\Omega_m} \right) \right] + \Omega_\Lambda \right\}. \]

We require the solution (i.e. \( a_\perp \)) to equations (4), from which everything else follows. Integrating equations (4) we find
In this work we consider a test case to demonstrate the method, and so focus on an open ($\Omega > \kappa$), dust-only background model ($\Lambda = 0$) that is asymptotically EdS and which is known to comfortably accommodate distances to SN1a (see e.g. [7]). Here we model the total (dimensionless) matter density profile today according to:

$$\Omega(t,r) = \Omega(t,\infty) - \left( \Omega(t,\infty) - \Omega(t,\infty) \right) \exp \left[ -r^2/L^2 \right],$$  

where $\Omega(t,r)$ is the bang time function.

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$$\Omega_m(r) = \Omega_m^{\text{out}} - \left( \Omega_m^{\text{out}} - \Omega_m^{\text{in}} \right) \exp \left[ -r^2/L^2 \right].$$  

where $\Omega_m^{\text{in}} = 0.2$, $\Omega_m^{\text{out}} = 1.0$, $L = 2.0$ Gpc is the length-scale of the inhomogeneity, and $H_0 \equiv H_{10}$ is $H_{10} = 70$ km s$^{-1}$ Mpc$^{-1}$ is the Hubble constant. The superscript ‘in’ denotes evaluation at $r = 0$. For consistency with an inflationary paradigm, we ignore decaying modes by forcing the bang time function to be uniform throughout space; setting $t_B(r) = 0$ is sufficient. In this case, we may write the solution to equations (9) in the following parametric form

$$a_{\perp}(t, r) = \frac{\Omega_m(r)}{2\Omega_{\perp}} \left[ \cosh 2\mu(t, r) - 1 \right],$$  

$$t = \frac{\Omega_m(r)}{2H_{10}} \frac{[\sinh 2\mu(t, r) - 2\mu(t, r)]}{[\Omega_{\perp}]^{3/2}},$$  

where $t_B(r)$ is the bang time function.

Figure 1. The spacetime evolution of selected contrasts in the background dynamics, illustrating the growth of the background void over time. Left: contrast in the energy density $\rho_\perp$. Centre: contrast in $H_\perp$. Right: contrast in $H_\parallel$. Note that scales of the vertical and horizontal axes apply to all such 2D plots in this paper. The values of $A$ and $B$ (respectively the maximum and minimum of the color scale) can be read at the top of each 2D plot.
where

\[ H_{\pm 0}(r) = \frac{\Omega_m(r)}{2t_0} \left[ \sinh 2u_0(r) - 2u_0(r) \right], \] (13)

\[ u_0(r) = \frac{1}{2} \cosh^{-1}\left[ \frac{2}{\Omega_m(r)} - 1 \right], \] (14)

\[ t_0 = \frac{\Omega_{m}^{\text{in}}}{2H_0} \left[ \sinh 2u_0^{\text{in}} - 2u_0^{\text{in}} \right] \left( \frac{\Omega_k}{\Omega_m} \right)^{3/2}. \] (15)

Figures 1 and 2 show the behaviour of various illustrative quantities in this model for the values of the constants quoted above.

3. Polar perturbations

Perturbations on a spherically symmetric background are decomposed into spherical harmonics, where tensorial quantities of degree 1 or 2 split into two parities—polar and axial (this is analogous, but not identical, to the usual scalar–vector–tensor split in a FLRW background). In this paper, we restrict our attention to the polar (even parity) sector since this is where the density perturbation is defined. Furthermore, we only consider modes with spherical harmonic index \( l > 1 \). The modes corresponding to \( l = 0 \) and \( l = 1 \) obey different
equations (see [1]) and should be treated separately, although in a similar fashion as far as the numerical integration is concerned. A similar treatment could be straightforwardly applied to the axial (odd parity) sector. Details of the derivation of the perturbation equations we present below can be found in [1]. Let us emphasize here that we are not trying to develop a full analysis of realistic structure formation in a LTB universe. Rather, we would like to demonstrate that the method we develop can integrate the perturbation equations, allowing us to study a few remarkable features of the evolution of perturbations in a large cosmic void. Note that we have adapted the equations to include the cosmological constant. We have also written them in terms of partial radial derivatives rather than frame derivatives, in readiness for numerical integration.

3.1. Formalism

The general form of polar perturbations to the background LTB metric, in the Regge–Wheeler (RW) gauge, is expanded in spherical harmonics as:

\[
dx^2 = -[1 + (2\eta - \chi - \varphi)Y]dr^2 - 2\zeta YXdtdr + [1 + (\chi + \varphi)Y]X^2d\Omega^2 + [1 + \varphi Y]A^2d\Omega^2,
\]

where \(\chi, \varphi, \zeta\) are functions of \((l, t, r)\) (independent of \(m\) due to the spherical symmetry of the background), \(Y = Y(lm)(\theta, \phi)\) are the scalar spherical harmonic functions. For notational convenience, an implicit sum over \((l, m)\) is implied whenever a quantity is multiplied by \(Y\).

The four-velocity field in the polar sector is given by (indices which are capitals run over \(t, r, \) and indices \(a, b, c, \ldots\) run over \(\theta, \phi\))

\[
u_\mu = \hat{u}_A + \left(\hat{w}_A + \frac{1}{2}k_{AB}\hat{v}_B\right)Y_{\theta A},
\]

where \(\hat{u}^A = (1, 0)\) and \(\hat{w}^A = (0, X^{-1})\), \(w\) and \(v\), both functions of \((l, t, r)\), are the radial and angular (peculiar) velocities, respectively, \(k_{AB}\) is the gauge-invariant metric perturbation, and \(Y_a \equiv V_a Y\). The energy–momentum tensor

\[
T^\mu_\nu = \rho_{\mu} (1 + \Delta Y)\nu^\mu u_\nu,
\]

defines the density contrast \(\Delta\). Note that all our perturbation variables, i.e. both metric and fluid perturbations, are automatically gauge-invariant. This is due to the fact that all the perturbations conveniently reduce, in the RW gauge, to the corresponding variables arising from a general gauge (coordinate) transformation (see [20] and references therein).

The first-order perturbed Einstein equations for the case \(l \geq 2\) reduce to:

\[
\ddot{\phi} = -4H_L \dot{\phi} - H_L \dot{\chi} + \frac{\dot{a}_L}{X^2a_L r} - \chi' + \left[\frac{2\kappa}{a_L} - \Lambda\right]\varphi + \frac{3a_0 \sigma^2}{Xa_L r^2} - 2H_L \dot{\sigma} - X^{-1}\chi',
\]

\[
\ddot{\zeta} = -2H_L \zeta - X^{-1}\chi',
\]

where
The three equations (19), (20) and (21) represent the master equations of our problem, and are the evolution equations for the metric perturbations $\varphi$, $\varsigma$ and $\chi$. Knowing these master variables, one can then obtain $\Delta$, $w$ and $v$, i.e. the behaviour of the matter perturbations. These are given by

\begin{align}
8\pi G \rho_a \Delta &= -X^{-2} \varphi'' + X^{-2} \left[ \frac{a_{||}'}{a_{||}} + \frac{kr + \frac{1}{2} r^2 \kappa'}{1 - kr^2} - 2 \frac{a_{\perp}}{a_{\perp} r} \right] \varphi' \\
&\quad + 2X^{-1} [H_1 \varsigma' - 3H_1 
\chi] + 2X^{-1} [H_1 \chi'] + \Theta \varphi + H_1 \chi' \\
&\quad + \left[ 3H_1 \left( a^2 + H_1 \right) - \left( 1 + 2 \frac{a_{||}}{a_{||}} \right) \frac{\kappa}{a_{\perp}^2} - \frac{r \kappa'}{a_{\perp} a_{\parallel}} \right] (\varphi + \chi) - \left[ \frac{l(l+1) - 2}{2 a_{\perp}^2 r^2} \right] \chi \\
&\quad + X^{-1} \frac{a_{||}}{a_{\perp} r} \left( 3 \sigma^2 + 4 H_1 \right) \varsigma, \quad \text{(21)}
\end{align}

\begin{align}
8\pi G \rho_a w &= X^{-1} \left[ \varphi' - \left( 3 \sigma^2 - H_1 \right) \varphi' - \frac{a_{||}}{a_{\perp} r} \chi' + H_1 \chi' \right] \\
&\quad + \left[ \frac{3}{2} H_1 \left( \sigma^2 + H_1 \right) - \left( \frac{a_{||}}{a_{||}} - \frac{1}{2} \right) \frac{\kappa}{a_{\perp}^2} \right. \\
&\quad - \left. \frac{r \kappa'}{2 a_{\perp} a_{||}} + \frac{l(l+1)}{2 a_{\perp}^2 r^2} - \frac{A}{2} \right] \varsigma, \quad \text{(22)}
\end{align}

\begin{align}
8\pi G \rho_a v &= \varphi + \frac{1}{2} \chi + \frac{1}{2} \varphi' + H_1 (\varphi + \chi). \quad \text{(23)}
\end{align}

The conservation of the perturbed energy–momentum, i.e. $V_{\mu} T_{\mu}^v = 0$, implies that our solutions must satisfy:

\begin{align}
C_\Delta &\equiv \dot{\Delta} + \frac{3}{2} \varphi + \frac{1}{2} \chi + X^{-1} (w + \varsigma/2) \chi' + X^{-1} \left[ \frac{\rho_{\omega}}{\rho_{\omega}} + 2 \frac{a_{||}}{a_{\perp} r} \right] (w + \varsigma/2) - \frac{l(l+1)}{a_{\perp} r^2} v \\
&= 0
\end{align}
These three equations can thus be seen as constraints that a solution to the previous system, i.e. equations (19)–(21) along with equations (22)–(24), must satisfy. These constraints can be used to test the accuracy of the numerical implementation.

3.2. Weyl curvature

In studying perturbations of a LTB background, it is interesting to consider the evolution of the Weyl curvature tensor, as it describes genuine relativistic effects. In particular the magnetic part of the Weyl tensor is zero in the background and is a gauge-invariant tensor sourced by purely relativistic effects—frame dragging (vector modes in FLRW perturbation theory, for example) and gravitational waves.

In the background $H_{\mu\nu}$ is zero, and the only non-zero background parts for $E_{\mu\nu}$ are:

$$
\delta E_{rr} = X^2 \frac{1}{3} \left[ \frac{a_{\parallel}'}{a_{\parallel}} + \frac{\kappa r + \frac{1}{2} r^2 \kappa'}{a_{\parallel}^2} + \frac{r k'}{a_{\parallel}} \right] \phi' - 2XH_\perp \zeta' - \frac{3}{2} \sigma^2 \phi - X^2 H_\perp \chi' + X^2 \left( \frac{a_{\parallel}'}{a_{\parallel}} - 1 \right) \frac{\kappa}{a_{\perp}^2} - 3H_\perp \sigma^2 + \frac{r k'}{a_{\perp} a_{\parallel}} + \frac{l(l + 1)}{2 a_{\perp}^2 r^2} \left( \phi + \chi \right) - 2X \frac{a_{\parallel}'}{a_{\parallel} r} \left( 3\sigma^2 - H_\parallel \right) \zeta - X^2 \left( \frac{l(l + 1) - 2}{a_{\perp}^2 r^2} \right) \chi',
$$

$$
\delta E_r = -\frac{1}{2} \left[ \phi' - \frac{a_{\parallel}'}{a_{\parallel} r} (\phi + \chi) - XH_\perp \zeta \right],
$$

$$
\delta E_{(T)} = -\frac{1}{2} \frac{A}{X} \delta E_{rr} + \frac{A_2}{2} \left[ 3H_\perp \sigma^2 + \left( \frac{a_{\parallel}'}{a_{\parallel}} - 1 \right) \frac{\kappa}{a_{\perp}^2} + \frac{r k'}{2 a_{\parallel} a_{\perp}} \right] \chi',
$$

$$
\delta E_{(T)} = -\frac{1}{2} (\phi + \chi),
$$

$$
C_v \equiv \dot{\varphi} - \frac{1}{2} \left[ \phi' - \frac{a_{\parallel}'}{a_{\parallel}} (\phi + \chi) - XH_\perp \zeta \right] = 0,
$$

$$
C_v \equiv \dot{\varphi} - \frac{1}{2} (\phi + \chi) = 0.
$$
and
\[ \delta H_t = -\frac{1}{4} \xi^2 - \frac{3}{4} X \sigma^2 (\varphi + \chi) + \frac{1}{2} a_{\perp} r \xi, \quad (34) \]
\[ \delta H_{TF} = -\frac{1}{2} \xi, \quad (35) \]
respectively. Here the trace (T) and trace free (TF) parts arise from a decomposition on the two-sphere. The magnetic part of the Weyl tensor has a parity opposite to the electric part, and carries a bar to denote that the axial part of it appears in the polar equations (and vice versa).

4. Numerical implementation

We use a method of lines approach [19] to solving the system equations (19)–(21), whereby the spatial domain is discretized by standard finite differences and integrated pointwise in time using a fourth-order Runge–Kutta solver. The system can be recast in terms of dimensionless variables through the following transformations:
\[ \tilde{\eta} \equiv H_0 \eta, \quad (36) \]
\[ \tilde{\eta} \equiv H_0 \eta = H_0 \int \frac{dt}{a_{\perp}^{\in\in}(t)}, \quad (37) \]
where the last equality, motivated by the standard form, defines the conformal time \( \eta \) for central observers. Then, using
\[ \partial_t = \frac{H_0}{a_{\perp}^{\in\in}} \partial_{\tilde{\eta}}, \quad (38) \]
\[ \partial_r = H_0 \partial_r, \quad (39) \]
we have
\[ H_{\perp/\parallel} = \frac{H_0}{a_{\perp}^{\in\in}} \tilde{\eta}_{\perp/\parallel}, \quad (40) \]
\[ \kappa = H_0^2 \tilde{k}, \quad (41) \]
\[ \rho_{\parallel} = \left( \frac{H_0}{a_{\perp}^{\in\in}} \right)^2 \tilde{\rho}_{\parallel}, \quad (42) \]
\[ \chi = a_{\perp}^{\in\in} \tilde{X}. \quad (43) \]
We also introduce the dimensionless angular peculiar velocity,
\[ \tilde{v} \equiv \frac{a_{\perp}^{\in\in}}{H_0} \tilde{v}. \quad (44) \]
and relate the cosmic time \( t \) to the dimensionless conformal time \( \tilde{\eta} \) by
\[ t = \frac{\Omega_{\xi}^{\in\in}}{2} \left[ \sinh \left( \tilde{\eta} \sqrt{\Omega_{\xi}^{\in\in}} \right) - \tilde{\eta} \sqrt{\Omega_{\xi}^{\in\in}} \right]. \quad (45) \]
Thus, to evolve the system from some initial time \( t_{\text{min}} \) until today \( t_0 \), we compute the corresponding initial and final values for \( \bar{\eta} \), i.e. \( \bar{\eta}_{\text{min}} \) and \( \bar{\eta}_0 \), respectively, using:

\[
\bar{\eta}(t) = \frac{2}{\sqrt{\Omega^m}} u^m(t).
\]  

(46)

4.1. Discretization

We introduce discretized time and space coordinates \( \bar{\eta}_i \) and \( \bar{r}_j \) given by

\[
\bar{\eta}_i = \bar{\eta}_{\text{min}} + i \Delta \bar{\eta} / \alpha,
\]

(47)

\[
\bar{r}_j = \bar{r}_{\text{min}} + j \Delta \bar{r} / \alpha,
\]

(48)

where \( \Delta \bar{\eta} \) and \( \Delta \bar{r} \) are grid spacings in \( \eta \) and \( r \), respectively, and \( i = 0 \ldots N_{\eta} \) and \( j = 0 \ldots N_r \). The factor \( \alpha \) determines the grid resolution relative to an \( \alpha = 1 \) baseline (with the number of grid points \( N_{\eta} \) and \( j = 0 \ldots N_r \) increased proportionally to cover the same domain). For our system of equations, the Courant–Friedricks–Lewy condition requires that

\[
\frac{\Delta \bar{\eta}}{\Delta \bar{r}} \leq \bar{X},
\]

(49)

for numerical stability, where \( \bar{X} \in \{1, 1.47\} \). For our simulations we use

\[
\frac{\Delta \bar{\eta}}{\Delta \bar{r}} = 0.98 \alpha^{-1}.
\]

(50)

Spatial derivatives are calculated using second-order finite difference operators. For some quantity \( Q_{i,j} \) evaluated at time \( \bar{\eta}_i \) and position \( \bar{r}_j \) at the baseline resolution

\[
H_0^{-1} Q'_{i,j} = \frac{Q_{i,j+1} - Q_{i,j-1}}{2 \Delta \bar{r}} + \mathcal{O} \left( \Delta \bar{r}^2 \right),
\]

(51)

\[
H_0^{-2} Q''_{i,j} = \frac{Q_{i,j+1} - 2Q_{i,j} + Q_{i,j-1}}{\Delta \bar{r}^2} + \mathcal{O} \left( \Delta \bar{r}^2 \right).
\]

(52)

We evaluate the rhs of equations (19)–(21) on a \( \bar{\eta} = \) constant slice and evolve forward in time using a standard fourth-order Runge–Kutta integrator. The overall scheme is second-order accurate due to the spatial finite differencing used.

4.2. Initial and boundary conditions

For this work, we have used a generic set of initial conditions for each of the three master variables:

\[
Q_{0,j} = \sum_{k=1}^{5} \exp \left( -\frac{(r_j - p_k)^2}{s^2} \right),
\]

(53)

\[
Q_{0,j} = 0,
\]

(54)

where \( p_k \equiv 0.99 \times \{1, 2, 3, 4, 5\} \) Gpc is an array of five equally spaced points between \( r_{\text{min}} \) and our desired region of interest \( r_0 \), and \( s \equiv 0.08 \) Gpc sets the width of each pulse—see the right panel of figure 3. The conditions are set at \( \bar{\eta}_{\text{min}} = 0.42 \) (corresponding to a time \( t_{\text{min}} = 0.018 \) Gyr, or a redshift \( z \approx 100 \) in a fiducial \( \Lambda \)CDM cosmology). These initial data are
not meant to represent a realistic physical state of perturbations in the early Universe; they simply allow us to test the method and to extract the physical behavior of perturbations.

Regularity conditions determine the variables in the neighbourhood of the origin according to the prescription of [20]. Near \( r = 0 \), we require (for all \( l \geq 2 \)):

\[
\chi, \phi, \varsigma \equiv \hat{\chi}, \hat{\phi}, \hat{\varsigma},
\]

where the hatted variables are all polynomials of even power in \( r \). Using

\[
\sum a_n r^{2n}, \quad \sum b_n r^{2n}, \quad \sum c_n r^{2n},
\]

we find that

\[
\chi_r = \sum (l + 2n + 2)a_n r^{l+n+1},
\]

\[
\chi_{rr} = \sum (l + 2n + 2)(l + 2n + 1)a_n r^{l+n},
\]

\[
\phi_r = \sum (l + 2n)b_n r^{l+2n-1},
\]

\[
\varsigma_r = \sum (l + 2n + 1)c_n r^{l+2n},
\]

which vanish at \( r = 0 = r_{\text{min}} \). Fixing the value of all variables to zero at the origin is sufficient for regularity\(^4\).

We require an additional boundary condition at the outer edge of the computational domain, \( r_{\text{max}} \). This boundary condition is necessarily artificial since we do not compactify the spatial coordinate. We place the boundary at a distance from the origin such that it is causally

\(^4\) Note that while our choice of initial conditions is not exactly zero at the origin, they are well below the machine precision.
disconnected from the region where the initial perturbations (53) and (54) are set for the duration of the evolution. This prevents artificially reflected signals from influencing the evolution of these perturbations. We can estimate an appropriate distance by tracing null geodesics inward from the outer boundary. Using the background LTB line element, radial null geodesics are given by

$$\frac{d\tilde{\eta}}{d\tilde{r}} = \tilde{X},$$

where our characteristics approach $45^\circ$ at large distances in our coordinates. An appropriate value for $\tilde{r}_{\text{max}}$ which is sufficiently removed from the measurement domain is

$$\tilde{r}_{\text{max}} = \tilde{r}_0 + \frac{1}{2} \int_{\tilde{r}_0}^{\tilde{r}_0} d\tilde{\eta} \tilde{X}^{-1},$$

where $r_0$ is the outer boundary of the domain in which we would like to analyse the behaviour of perturbations between times $\tilde{\eta}_{\text{min}}$ and $\tilde{\eta}_0$. Since we are working in a single spatial dimension, this grid extension to remove outer boundary effects is not overly costly in terms of memory or computation time, though in the future it may be useful to consider a logarithmic radial coordinate. Given that the spacetime in our model is effectively homogeneous above $r = 5\,\text{Gpc}$, we choose a conservative region of interest of $6\,\text{Gpc}$.

5. Convergence tests

To establish the accuracy of the discretisation, we carry out a standard convergence test. We check the second-order convergence empirically by carrying out a series of runs of the same initial data at successively doubled resolution, corresponding to $\alpha = n$, $\alpha = 2n$, and $\alpha = 4n$ in equations (47) and (48). The rate of convergence for a variable $Q$ is given by

$$\beta_Q^{(n)} = \log_2 \left| \frac{\|Q^{(n)}\| - \|Q^{(2n)}\|}{\|Q^{(2n)}\| - \|Q^{(4n)}\|} \right|,$$

where $\|Q^{(n)}\|$ is the $L_2$-norm on the fixed-resolution grid

$$\|Q^{(n)}\| = \frac{1}{N} \left( \sum_{j=1}^{N} (Q_{i,j}^{(n)})^2 \right)^{1/2},$$

with $N (< N_r)$ the number of spatial grid-points stored for analysis within the range $0 \leq \tilde{r} \leq \tilde{r}_0$. We use the following dimensionless measure to quantify how well the constraints are satisfied:

$$C_Q^{(n)}(\tilde{\eta}) = \frac{\|C_Q(\tilde{\eta})\|_2}{\|Q^{(n)}(\tilde{\eta})\|_2},$$

where $Q \in \{\Delta, w, \tilde{v}\}$, $C_Q$ is one of equations (25)–(27), and we estimate $\dot{Q}$ via a centered difference, i.e.

$$\frac{a_H w^{(\tilde{\eta})}}{H_0} \dot{Q}^{(\tilde{\eta})} = \frac{Q^{(\tilde{\eta} + \Delta \tilde{\eta})} - Q^{(\tilde{\eta} - \Delta \tilde{\eta})}}{2\Delta \tilde{\eta}}.$$

For all of our evolution variables and constraints, we observe the expected second-order convergence rate ($\beta = 2$). Examples are plotted in figure 4, which shows how well the
constraint equations perform for various multipole moments in the case of an initial $\phi$. Using a reference resolution of $n = 8$, we include curves of double ($n = 16$) and four times ($n = 32$) the resolution. When the curves line up, then $\beta = 2$, i.e. we obtain second-order convergence as expected. It is clear that, in the case $l = 1000$, one must go to a resolution of at least $n = 16$ for the curves to align satisfactorily. The sudden drop in error seen in some plots around $t = 4$ Gyr is associated with the exiting of the initial pulses from the measurement domain.

6. Results

6.1. Evolution of perturbations

To study the evolution of perturbations in a LTB cosmological void, we concentrate, for the sake of illustration, on the evolution of the perturbation variables for the spherical modes $l = 2$ and $l = 10$. For each $l$, we consider three distinct cases: we initialize the profile of any one of $\phi$, $\zeta$ and $\chi$ according to equations (53) while setting the others to zero, and apply equations (54) to all variables (except $\zeta$ since it satisfies a first-order partial differential
equation (PDE)). Here, cases 1, 3 and 5 correspond to \( l = 2 \), and cases 2, 4 and 6 correspond to \( l = 10 \).

The evolution of each of the variables is presented in figure 5 for each of the six corresponding cases. The resolutions used in all of the plots are typically in the range \( 32 \leq n \leq 128, \ 4 \leq \alpha \leq 16 \).

**Cases 1 and 2:** In these cases, we initialize \( \phi \), and set \( \varsigma = \chi = 0 \) initially. We clearly see the ‘bleeding’ of the modes due to the coupling. On the two-dimensional plot, figure 5, \( \chi \) behaves like a propagating degree of freedom, evolving along the characteristics of the spacetime, and radiating energy away from each pulse. The behaviour of \( \varsigma \) is more difficult to qualitatively describe because it is a mixture of frame dragging and gravitational wave degrees of freedom—the combination of non-propagating decay with some radiation can be seen in the figures. It is proportional to the TF part of the magnetic Weyl curvature equations (35), and thus represents a true relativistic degree of freedom. Then, \( \phi \) follows a standard evolution throughout the spacetime: staying constant around the EdS region, while decaying faster deep inside the void.

The top panel of figure 6 presents the profile of \( \phi \) today for these cases, as well as its time evolution along selected radii. As expected, \( \phi \) remains constant in the outer, quasi-FLRW regions of the void, given that it essentially satisfies the Bardeen equation there. Deep inside the void, \( \phi \) decreases for the most part as the usual Bardeen potential would in an open FLRW dust model, but there is evidence of influence from \( \varsigma \) and \( \chi \) at least at the sub-percent-level, as can be seen by the amplitudes of the latter in the middle and bottom panels of figure 6; see also section 6.2 for a discussion on the importance of the couplings.

**Cases 3 and 4:** Here, we initialize \( \varsigma \), and set \( \phi = \chi = 0 \) initially.

From the middle panels of figure 5 we see that \( \varsigma \) decays very quickly—in fact, roughly proportional to \( a^{-2} \)—along the peaks from where it is initially located, while sourcing \( \phi \) and \( \chi \). As expected, \( \chi \) is very well described as a propagating degree of freedom, but one also sees that the sourced \( \phi \) has a propagating component that follows the characteristics of the background spacetime and escapes the void.

The middle panel of figure 7 presents the profile of \( \varsigma \) today for these cases, including the time evolution along selected radii. It is clear that \( \varsigma \) decays, for the most part, approximately as \( a^{-2} \). (In the FLRW limit this would be a pure vector mode with this exact decay rate.) The greater decay in \( \varsigma \) seen in the central regions of the void can be attributed to the faster expansion rate there. The top and bottom panels of figure 7 show the profiles for the other variables, \( \phi \) and \( \chi \).

**Cases 5 and 6:** Here, we initialize \( \chi \), and set \( \phi = \varsigma = 0 \) initially.

According to the bottom panels of figure 5, \( \chi \) and the generated \( \varsigma \) propagate to the outskirts of the void along the characteristics of the background, resulting in the localized generation of the potential \( \phi \), and the associated growth of density perturbations, as is shown in figure 9.
Figure 5. Spacetime evolution of each of the master variables in the various cases considered. In cases 1 and 2, $\varphi$ excites both $\varsigma$ and $\chi$ to about the sub-percent level. The propagating modes resulting from $\chi$ are visible in $\varsigma$, or alternatively the trace-free part of the magnetic Weyl tensor $\delta H_{TF}$ equations (35), thus clearly showing relativistic degrees of freedom at work. In cases 3 and 4, while an initial $\varsigma$ decays away quickly due to the Hubble friction, it still manages to excite the other two variables, albeit to a low level. The final two cases, 5 and 6, are rather interesting: an initial $\chi$ generated inside a void excites a relatively significant amount of $\varphi$ and $\varsigma$. The presence of propagating modes is more apparent in all the variables here. The maximum and minimum values of the colour scale are indicated respectively in brackets above and below each 2D plot.
The bottom panel of figure 8 presents the profile of $\chi$ today for these cases, as well as the time evolution along selected radii, while the profiles for the other variables, $\phi$ and $\varsigma$, are shown in the top and middle panels.

All of these cases demonstrate that $\phi$, $\chi$, and $\varsigma$ are much more difficult to interpret than on a FLRW background. As emphasized in [1], they are mixtures of scalar, vector and tensor modes and therefore their coupling is an essential ingredient of first-order perturbation theory around a LTB background: in principle, they cannot be treated as separate, independent modes that describe different physical aspects of perturbations. In the next subsection, we compare the behaviour of the fully coupled perturbation system to cases where they are decoupled ‘by hand’ as has been done before in various ways to simplify the analysis [10, 12, 13]. In particular, we would like to analyse the errors in $\phi$ and $\Delta$ when $\chi$ and $\varsigma$ are neglected, not only for initial data but also during the evolution of the system.

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| CASE 1 | CASE 2 |
|--------|--------|
| **Time evolution** | **Profile today** | **Time evolution** | **Profile today** |
| ![Graph](image1) | ![Graph](image2) | ![Graph](image3) | ![Graph](image4) |

**Figure 6.** Temporal and spatial slices through the spacetime evolution observed in the top panel of the figure. The variable $\phi$ is clearly unaffected on the outskirts of the void in which the spacetime is almost Einstein–de Sitter. Since $\varsigma$ and $\chi$ are relatively sub-percent in amplitude, on each radial shell $\phi$ more or less behaves as expected in an open, dust-dominated FLRW model, i.e. decays with time.
6.2. Coupled versus uncoupled dynamics

In this section, we quantify the errors induced when assuming that the coupling of $\phi$ to $\chi$ and $\varsigma$ is negligible by considering models which initialize $\phi$ only. We compare these to cases where equations (19)–(21) are solved retaining terms with no coupling between $\phi$ and $\{\chi, \varsigma\}$, that is, by solving the reduced system:

$$\dot{\phi} = -4H_1 \phi + \left[\frac{2\kappa}{a_1^2} - \Lambda\right] \phi,$$

(67)
As it turns out, the full coupling is seen to be important for the dynamics of $\phi$, and also for the behaviour of $\Delta$ on small angular scales (large $l$)—see figures 10 and 11.

\begin{align*}
8\pi G \rho_{\text{DE}} \Delta &= -X^{-2} \varphi'' + X^{-2} \left[ \frac{a_{\parallel}'}{a_{\parallel}} + \frac{\kappa \varphi'}{1 - \kappa r^2} - \frac{2 a_{\parallel}'}{a_{\parallel} r} \right] \varphi' + \\
&+ \Theta \varphi + \left[ 3 H_\perp \left( \sigma^2 + H_\perp \right) - \left( 1 + 2 \frac{a_{\perp}}{a_{\parallel}} \right) \kappa \frac{r_{\perp}'}{a_{\perp} a_{\parallel}} \right] \varphi + \frac{a_{\perp}'}{a_{\parallel} r^2} \end{align*}

(68)
Figure 9. Spacetime evolution of selected quantities derived from those presented in figure 5. We show: $\Delta$ and $w$ normalized to their maximum values (along the radial dimension) at the initial time ($t_{\text{min}}$), as well as $\delta E_{(TF)}$, which describes the sum of $\phi$ and $\chi$. Note that, due to the initial amplitude of unity chosen throughout, $\Delta$ here eventually becomes less than $-1$; while $\Delta$ does not exactly reduce to the usual density contrast in the FLRW limit—it is sourced by propagating degrees of freedom—an appropriate rescaling of the initial amplitudes (fluctuations in the standard Newtonian potential $\Phi$ are $\sim 10^{-4}$ at $z = 100$) in any case is sufficient to avoid any issues regarding the physical interpretation of $\Delta$ as a density contrast. The maximum and minimum values of the colour scale are indicated respectively in brackets above and below each 2D plot.
From figure 10 we see that deep inside the void (first peak) the differences in $\varphi$ are already of order 15% for $l = 10$ and 8% for $l = 2$. This could have a major impact down the central observer’s past light-cone and therefore such couplings could be very important in determining observables accurately. On the other hand, $\Delta$ is well approximated by the uncoupled dynamics for large scales, with errors below 1% for $l = 2$; but, already for $l = 10$, we see errors of order 7–8%.

Including a few more angular scales, all the way up to $l = 1000$, as well as intermediate snapshots in time, an overall picture of the error in neglecting the couplings is captured in figure 11. Regardless of where (in radial distance) we choose to observe $\varphi$ and $\Delta$, as we go to smaller scales their expected errors approach some equivalent maximum value—equivalent due to their relation via the analogue of the Poisson equation; equation (22) which has $\Delta \propto l^2$ on small scales (large $l$).

As for observable quantities such as the two-point correlation function of the galaxy distribution, we should expect corrections of a few percent in the amplitude of the baryon acoustic oscillation (BAO) bump when including the full coupling (because this quantity is of the order of the square of $\Delta$)—see [10] for the particular case in which the coupling is neglected.

7. Conclusions and perspectives

We have developed a numerical scheme to solve the system of coupled, linear PDEs describing the evolution of (polar) perturbations on a background LTB spacetime. The
implementation is numerically consistent, attaining the expected second-order convergence with resolution over a wide range of scales.

To illustrate the nature of the coupling between the three master variables in the problem, in separate runs we initialized the data by several Gaussian peaks in each variable, spanning regions both inside and outside the void while setting the remaining two variables to zero initial amplitude. Initial pulses in $\phi$ result in growth of $\zeta$ and $\chi$ at the sub-percent level, implying that the variable $\phi$—commonly ascribed to the analogue of the Bardeen/Newtonian potential—nevertheless contains relativistic degrees of freedom. Initializing non-zero $\zeta$ induces a sub-percent signal in $\phi$ and $\chi$, all while decaying roughly as $a_l^{-2}$—analogous, but not equivalent, to the vector mode in a FLRW spacetime. Finally, a non-zero $\chi$ induces a $\phi$ to the level of nearly 50% today, while inducing only a sub-percent level of $\zeta$ (from a maximum level of $\sim 20\%$ at earlier times). The propagating nature of $\chi$ is clearly seen in this case.

We also investigated whether the coupling between the master variables may be safely ignored. In particular, we focused on the case of an initialized $\phi$, and considered how much error we expect to obtain on $\Delta$ and $\phi$ when neglecting the coupling of $\phi$ to $\zeta$ and $\chi$. Our results

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![Figure 11. Percentage errors acquired on $\phi$ (black squares with dashed line) and $\Delta$ (red squares with solid line) from neglecting the coupling between the modes in the case of an initial $\phi$, as a function of $l$ at selected times and radii. We see in general that the errors increase with time, as well as with increasing $l$, and are larger deep within the void than towards the outskirts. The errors in $\phi$ and $\Delta$ converge on smaller scales since the term in (22) proportional to $l(l + 1)\phi$ dominates. Note that at $t = t_0$ we reach errors of around 30% well within the void, and thus in general can expect percent-level corrections to, say, the amplitude of the baryon acoustic oscillation (BAO) bump in the two-point correlation function of the galaxy distribution.](image-url)
indicate that, well inside the void and on the largest scales, the errors picked up on $\Delta$ are at the sub-percent level, and so neglecting the coupling in that case is not an unreasonable assumption. However, the corresponding corrections to $\phi$ itself will be more important, and contributions from lensing and integrated Sachs–Wolfe effects may be enhanced at around the 10% level when taking the coupling into account. On smaller scales though, corrections to the assumption of negligible coupling can grow to a few tens of percent for both $\phi$ and $\Delta$ for regions well inside the void. For an observable such as the galaxy–galaxy correlation function, we estimate corrections to the amplitude of the BAO peak at the percent-level. Of course, since we have considered aspects of structure formation only valid in the linear regime, we expect that any nonlinear effects—the details of which are not clear at this point—will modify small-scale corrections in some non-trivial way. In any case, as we approach the outskirts of the void corrections are well below the percent-level on all scales, as expected in regions of spacetime close to FLRW.

Having performed such a calculation for the case of a cosmological-sized void, our analysis can be easily adapted to smaller astrophysical-sized voids, and even halos. This will be left for future work.

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Appendix. Runge–Kutta algorithm

Let us write our system of coupled PDEs in the following compact way

$$\dot{\phi} = F^\phi(t, \phi, \varphi),$$
$$\dot{\varphi} = F^\varphi(t, \varphi, \varphi),$$
$$\dot{\chi} = F^\chi(t, \chi),$$
$$\dot{\zeta} = F^\zeta(t, \zeta),$$

where $\phi$, $\varphi$ and $\chi$ are functions of time ($t$) and radial coordinate ($r$), and a prime denotes partial differentiation with respect to $r$. Splitting equations (A.1) and (A.3) into two pairs of first-order equations we get

$$\dot{\phi} = \phi',$$
$$\dot{\varphi} = \phi' + S^\phi(t, \chi', \chi, \chi),$$
$$\dot{\chi} = \chi',$$
$$\dot{\zeta} = \zeta',$$

Discretising equations (A.2), (A.4), (A.5), (A.6) and (A.7) on a spacetime grid in a Runge–Kutta fashion for the time dependence we have

$$q_{i+1,j} = q_{ij} + \frac{\Delta t}{6} \left( a_{1ij} + 2a_{2ij} + 2a_{3ij} + a_{4ij} \right),$$
\[ \bar{\varphi}_{i+1,j} = \bar{\varphi}_{i,j} + \frac{\Delta t}{6} \left( b_{i,j} + 2b_{2i,j} + 2b_{3i,j} + b_{4i,j} \right), \quad (A.9) \]
\[ \zeta_{i+1,j} = \zeta_{i,j} + \frac{\Delta t}{6} \left( c_{i,j} + 2c_{2i,j} + 2c_{3i,j} + c_{4i,j} \right), \quad (A.10) \]
\[ \chi_{i+1,j} = \chi_{i,j} + \frac{\Delta t}{6} \left( d_{i,j} + 2d_{2i,j} + 2d_{3i,j} + d_{4i,j} \right), \quad (A.11) \]
\[ \bar{\chi}_{i+1,j} = \bar{\chi}_{i,j} + \frac{\Delta t}{6} \left( e_{i,j} + 2e_{2i,j} + 2e_{3i,j} + e_{4i,j} \right), \quad (A.12) \]

where
\[ a_{1i,j} = \bar{\varphi}_{i,j}, \quad (A.13) \]
\[ b_{i,j} = F^\varphi \left( t_i, \varphi_{ij}, \varphi_{ij} \right) + S^\varphi \left( t_i, \chi_{i,j}, \bar{\chi}_{i,j}, \zeta_{i,j} \right), \quad (A.14) \]
\[ c_{i,j} = F^\zeta \left( t_i, \zeta_{ij} \right) + S^\zeta \left( t_i, \chi_{i,j} \right), \quad (A.15) \]
\[ d_{i,j} = \bar{\chi}_{i,j}, \quad (A.16) \]
\[ e_{i,j} = F^\chi \left( t_i, \chi_{i,j}, \bar{\chi}_{i,j}, \chi_{i,j} \right) + S^\chi \left( t_i, \varphi_{ij}, \varphi_{ij}, \zeta_{ij}, \zeta_{ij} \right), \quad (A.17) \]
\[ a_{2i,j} = a_{1i,j} + b_{1i,j} \Delta t/2, \quad (A.18) \]
\[ d_{2i,j} = d_{1i,j} + e_{1i,j} \Delta t/2, \quad (A.19) \]
\[ b_{2i,j} = F^\varphi \left( t_i + \Delta t/2, a_{2i,j}, \varphi_{ij} + a_{1i,j} \Delta t/2 \right) + S^\varphi \left( t_i + \Delta t/2, \right), \quad (A.20) \]
\[ c_{2i,j} = F^\zeta \left( t_i + \Delta t/2, \zeta_{ij} + c_{1i,j} \Delta t/2 \right) + S^\zeta \left( t_i + \Delta t/2, \right), \quad (A.21) \]
\[ e_{2i,j} = F^\chi \left( t_i + \Delta t/2, \chi_{i,j} + d_{1i,j} \Delta t/2, \bar{\chi}_{i,j} + d_{1i,j} \Delta t/2, \right), \quad (A.22) \]
\[ a_{3i,j} = a_{1i,j} + b_{2i,j} \Delta t/2, \quad (A.23) \]
\[ d_{3i,j} = d_{1i,j} + e_{2i,j} \Delta t/2, \quad (A.24) \]
\[ b_{3ij} = F^\varphi \left( t_i + \Delta t/2, a_{3i,j}, \varphi_{ij} + a_{2i,j} \Delta t/2 \right) + S^\varphi \left( t_i + \Delta t/2, \chi_{ij} + d_{3i,j} \Delta t/2, \varphi_{ij} + c_{2i,j} \Delta t/2 \right). \]  
(A.25)

\[ c_{3ij} = F^\xi \left( t_i + \Delta t/2, \varphi_{ij} + c_{2i,j} \Delta t/2 \right) + S^\xi \left( t_i + \Delta t/2, \chi_{ij} + d_{3i,j} \Delta t/2 \right), \]  
(A.26)

\[ e_{3ij} = F^\chi \left( t_i + \Delta t/2, \chi_{ij} + d_{3i,j} \Delta t/2, \varphi_{ij} + c_{2i,j} \Delta t/2, d_{3i,j}, \chi_{ij} + d_{2i,j} \Delta t/2 \right) \]  
(A.27)

\[ a_{4i,j} = a_{i,j} + b_{3i,j} \Delta t, \]  
(A.28)

\[ d_{4i,j} = d_{i,j} + c_{3i,j} \Delta t, \]  
(A.29)

\[ b_{4i,j} = F^\varphi \left( t_i + \Delta t, a_{4i,j}, \varphi_{ij} + a_{3i,j} \Delta t \right) + S^\varphi \left( t_i + \Delta t, \chi_{ij} + d_{4i,j} \Delta t/2, \varphi_{ij} + c_{3i,j} \Delta t \right). \]  
(A.30)

\[ c_{4i,j} = F^\xi \left( t_i + \Delta t, \varphi_{ij} + c_{3i,j} \Delta t \right) + S^\xi \left( t_i + \Delta t, \chi_{ij} + d_{4i,j} \Delta t \right), \]  
(A.31)

\[ e_{4i,j} = F^\chi \left( t_i + \Delta t, \chi_{ij} + d_{4i,j} \Delta t, \varphi_{ij} + c_{3i,j} \Delta t, d_{4i,j}, \chi_{ij} + d_{3i,j} \Delta t \right) \]  
(A.32)

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