THE ONE-DIMENSIONAL MODEL FOR D-CONES REVISITED

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ABSTRACT. A d-cone is the shape one obtains when pushing an elastic sheet at its center into a hollow cylinder. In a simple model, one can treat the elastic sheet in the deformed configuration as a developable surface with a singularity at the “tip” of the cone. In this approximation, the renormalized elastic energy is given by the bending energy density integrated over some annulus in the reference configuration. The thus defined variational problem depends on the indentation \( h \) of the sheet into the cylinder. This model has been investigated before in the physics literature; the main motivation for the present paper is to give a rigorous version of some of the results achieved there via formal arguments. We derive the Gamma-limit of the energy functional as \( h \) is sent to 0. Further, we analyze the minimizers of the limiting functional, and list a number of necessary conditions that they have to fulfill.

1. INTRODUCTION

Since the late ’90s, there has been a lot of interest in the crumpling of thin elastic sheets in the physics community. These works mainly treat what may be thought of as “building blocks” of the more complex folding patterns one obtains when crushing an elastic sheet into a container whose size is smaller than the diameter of the sheet. In other words, these contributions analyze single ridges or single vertices, where the elastic energy focuses. In the mathematics literature, ridges have been investigated in \([6]\), and vertices in \([1,13]\). More precisely, these latter works considered the so-called d-cone, the shape that one obtains when pushing a thin elastic sheet at its center into a hollow cylinder, which in the physics literature has been treated in \([2,4,11,16]\). This is the physical setup we will be interested in here.

In \([1,13]\), the d-cone has been modeled by the following variational problem. Let \( \gamma \in W^{2,2}(S^1, S^2) \) be a unit speed curve that is not contained in a plane, denote the unit ball in \( \mathbb{R}^2 \) by \( B \), identify its boundary with \( S^1 \), and set

\[
\mathcal{Y} = \{ y \in W^{2,2}(B; \mathbb{R}^3) : y|_{\partial B} = \gamma, y(0) = 0 \}.
\]

The elastic energy of \( y \in \mathcal{Y} \) is given by \( I_H(y) = \int_B |Dy^T D y - \text{Id}_{2 \times 2}|^2 + H^2 |D^2 y|^2 dx \), where \( H \) is a parameter that can be thought of as the thickness of the sheet. (This is a typical model energy for thin elastic sheets; for a justification see e.g. \([6]\).) The result of \([1,13]\) is that \( \inf_{y \in \mathcal{Y}} I_H(y) \) is equal to \( C(\gamma) H^2 |\log H| \) as \( H \to 0 \) in the leading order of \( H \), with an explicit constant \( C(\gamma) \).

In \([3,4]\), the d-cone has been modeled as a developable surface with a singularity at its tip. The connection to \([1,13]\) is that the shape of the d-cone here is entirely determined by the
boundary curve $\gamma$ from above. The energy in the present model is (up to numerical constants) given by $C(\gamma)$, and we look for configurations $\gamma$ with minimal energy. This model is one-dimensional, and can be treated with ODE methods. In [3, 4], quantitative results are given for the regime of “small deflections”, in which nonlinear terms are dropped. The resulting equation is a one-dimensional obstacle problem with an additional constraint. It is argued that solutions of this problem should consist of a finite number of “folds”, i.e., regions where the sheet lifts off the edge of the cylinder. The elastic energy of such configurations is computed numerically, and the numerical evidence clearly suggests that the solution consisting of a single fold (without any “sub-folds”) is the configuration of lowest energy. Since the small deflection regime is independent of the indentation $h$, the conclusion is that the shape of this minimizer is universal. This means in particular that the angle subtended by the region where the sheet lifts off the cylinder is independent of the indentation or any other parameter such as elastic moduli of the sheet or the radius of the cylinder. The value of this angle is roughly $140^\circ$, in good agreement with experimental observations.

Here, we give a rigorous derivation of the small deflection regime in the sense of $\Gamma$-convergence. Additionally, we reconsider the limiting functional and give a list of properties that have to be satisfied by its minimizers. This second part is quite similar to the analysis in [3]. However, we carefully derive the necessary conditions for minimizers with purely variational tools, and our results are slightly different, in that the necessary conditions we find are not quite strong enough to exclude a certain set of configurations that has been missed in [3, 4].

The plan of the present paper is as follows: In Section 2, we define our model and state the $\Gamma$-convergence result. We also give the proofs of the “compactness” and “lower bound” parts of this statement, which are straightforward. In Section 3, we prove the “upper bound” part, which is somewhat more complicated. The main difficulty is to make sure the various constraints are satisfied by the recovery sequence. In Section 4, we state and prove a number of necessary conditions for minimizers of the limiting functional. The proof relies on a generalized Lagrange multiplier rule from [9], valid for variations in a convex cone.

**Notation.** Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and let $\iota : \mathbb{R} \to S^1$ be the quotient map. When we write $(a, b) \subset S^1$ for $a, b \in \mathbb{R}$, it is understood that we are speaking of the image of the interval under the quotient map $\iota$. The function space $W^{k,p}(S^1)$ is given by

$$\left\{ f : S^1 \to \mathbb{R} : \exists \tilde{f} \in W^{k,p}_{\text{loc}}(\mathbb{R}), \tilde{f}(x) = \tilde{f}(x + 2\pi) = f(\iota(x)) \text{ for all } x \in \mathbb{R} \right\}.$$  

The spaces $C^k(S^1)$ are defined analogously. Letting $I = [-\pi, \pi]$ and using the above identification of $S^1 = \iota(I)$ with $I$, we define the $W^{k,p}$ norm on $S^1$ by

$$\|f\|_{W^{k,p}(S^1)} = \|f\|_{W^{k,p}(I)}.$$  

For the derivative of a function $f \in W^{1,1}(S^1)$, we use both the notation $f'$ and $\frac{df}{d\iota}$. For the use of the symbols $C$ and $O$ in Section 3 see the explanations at the beginning of that section.
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2. **Derivation of the Small Deflection Regime by $\Gamma$-Convergence**

The starting point is the variational problem given by the elastic energy

$$E_{\text{bending}} : W^{2,2}(S^1; \mathbb{R}^3) \to \mathbb{R},$$

$$\gamma \mapsto \begin{cases} \int_0^{2\pi} |\gamma''|^2 dt & \text{if } |\gamma| = |\gamma'| = 1 \text{ a.e.} \\ +\infty & \text{else.} \end{cases}$$

This is (up to a constant) the bending energy $\int |\nabla^2 y|^2$ of an elastic sheet $\Omega = B(0, 1) \setminus B(0, \varepsilon) \subset \mathbb{R}^2$ under the deformation $y : \Omega \to \mathbb{R}^3$, $y(r, t) = r\gamma(t)$ (the latter of course in polar coordinates). The constraints $|\gamma| = |\gamma'| = 1$ assure that $y$ is an isometry away from the origin. The energy $E_{\text{bending}}$ is the leading term in the energy scaling result for d-cones with boundary conditions given by $\gamma$, see [13].

Now we define the constrained functional

$$E_h : W^{2,2}(S^1; \mathbb{R}^3) \to \mathbb{R},$$

$$\gamma \mapsto \begin{cases} E_{\text{bending}}(\gamma) & \text{if } \gamma \cdot e_z \geq h \\ \infty & \text{else.} \end{cases}$$

This models the d-cone being pushed into a cylinder of height $h$ and radius $\sqrt{1-h^2}$. The limit functional for $h \to 0$ will also be defined on the space $W^{2,2}(S^1; \mathbb{R}^3)$. There will be constraints for allowed configurations, and we define the space of admissible deformations:

$$\mathcal{A} = \{(u, v, w) \in W^{1,\infty}(S^1; \mathbb{R}^2) \times W^{2,2}(S^1) : w \geq 1, u = -w^2/2, u + v = -w''/2\}.$$ 

The constraints captured in the definition of $\mathcal{A}$ are the remnants of the constraints $\gamma_h \cdot e_z \geq h$, $|\gamma_h| = |\gamma_h'| = 1$ for finite $h$. We define the limit functional

$$E_0 : W^{1,\infty}(S^1; \mathbb{R}^2) \times W^{2,2}(S^1) \to \mathbb{R},$$

$$(u, v, w) \mapsto \begin{cases} \int_0^{2\pi} |w'' + w|^2 dt & \text{if } (u, v, w) \in \mathcal{A} \\ +\infty & \text{else.} \end{cases}$$

In the following, we write

$$u_h(t) = \gamma_h(t) \cdot e_r(t) - 1 \quad v_h(t) = \gamma_h(t) \cdot e_\varphi(t) \quad w_h(t) = \gamma_h(t) \cdot e_z$$

(1)

for $t \in S^1$, where we have introduced the $t$-dependent orthonormal frame

$$e_r(t) = (\cos t, \sin t, 0), \quad e_\varphi(t) = (-\sin t, \cos t, 0), \quad e_z = (0, 0, 1).$$

We will prove the following $\Gamma$-convergence result:

**Theorem 1.** Compactness: If $\gamma_h$ is a sequence in $W^{2,2}(S^1; \mathbb{R}^3)$ with $\limsup_{h \to 0} h^{-2} E_h(\gamma_h) < \infty$, then there exists a subsequence (no relabeling) and $(u, v, w) \in \mathcal{A}$ such that

$$h^{-1} u_h \rightharpoonup u \text{ in } W^{2,2}(S^1)$$

$$h^{-2} (u_h, v_h) \rightharpoonup (u, v) \text{ in } W^{1,\infty}(S^1; \mathbb{R}^2)$$
Lower bound: Let \( \gamma_h \) be a sequence in \( W^{2,2}(S^1; \mathbb{R}^3) \) such that for \( u_h, v_h, w_h \) defined as in [1], we have \( h^{-1}w_h \rightharpoonup w \) in \( W^{2,2}(S^1) \) and \( h^{-2}(u_h, v_h) \rightharpoonup (u, v) \) in \( W^{1,\infty}(S^1; \mathbb{R}^2) \). Then
\[
\liminf_{h \to 0} h^{-2}E_h(\gamma_h) \geq E^0(u, v, w).
\]

Upper bound: Let \( (u, v, w) \in W^{1,\infty}(S^1; \mathbb{R}^2) \times W^{2,2}(S^1) \). Then there exists a sequence \( \gamma_h \) in \( W^{2,2}(S^1; \mathbb{R}^3) \) such that \( h^{-1}w_h \rightharpoonup w \) in \( W^{2,2}(S^1) \) and \( h^{-2}(u_h, v_h) \rightharpoonup (u, v) \) in \( W^{1,\infty}(S^1; \mathbb{R}^2) \), and additionally
\[
\lim_{h \to 0} h^{-2}E_h(\gamma_h) = E^0(u, v, w).
\]

Proof of compactness and lower bound. Using the notation from [1], we have
\[
E_h(\gamma_h) = \int \left(2v'_h - u'_h\right)^2 + \left(2u'_h - v'_h\right)^2 + \left(w''_h + w_h\right)^2.
\] (2)

By the coercivity of \( E_h \) in \( W^{2,2}(S^1; \mathbb{R}^3) \) and \( \limsup_{h \to 0} h^{-2}E^h(\gamma_h) < \infty \),
\[
h^{-1}(u_h, v_h, w_h) \text{ is bounded in } W^{2,2}(S^1; \mathbb{R}^3),
\] (3)

and hence a subsequence converges to some \( (U, V, w) \in W^{2,2}(S^1; \mathbb{R}^3) \). By \( h^{-1}w_h \geq 1 \), we have \( w \geq 1 \). By the constraints \( |\gamma_h| = 1 \), \( |\gamma'_h| = 1 \),
\[
|\gamma_h|^2 = (1 + u_h)^2 + v_h^2 + w_h^2 = 1
\] (4a)

\[
|\gamma'_h|^2 = (1 + u_h + v'_h)^2 + (u'_h - v_h)^2 + w_h^2 = 1.
\] (4b)

Differentiating (4a) twice, we get
\[
-2u''_h = 2u_hu''_h + u_h^2 + 2v_hv''_h + v_h^2 + 2w_hw''_h + w_h^2.
\]

Multiplying this equality with \( h^{-2} \), and using (3) and Hölder’s inequality on the right hand side, we get boundedness of \( h^{-2}u_h \) in \( W^{2,1}(S^1) \) and hence in \( W^{1,\infty}(S_1) \). In the same way, by differentiating (4b) twice, we get
\[
-2(u'_h + v'_h) = 2(u'_h + v'_h)(u_h + v_h) + 2(u''_h - v'_h)(u'_h - v_h) + 2w_hw'_h.
\]

Again multiplying by \( h^{-2} \), using (3) and the fact that \( h^{-2}u_h \) is bounded in \( W^{1,\infty}(S^1) \), we get boundedness of \( h^{-2}v_h \) in \( W^{2,1}(S_1) \). Choose a convergent subsequence such that \( h^{-2}(u_h, v_h) \rightharpoonup (u, v) \) in \( W^{1,\infty}(S_1) \).

Multiplying (4a) and (4b) by \( h^{-2} \) and taking the limit \( h \to 0 \) (say, the weak-* limit in \( L^\infty \)), we get
\[
-2u = w^2
\]
\[
-2(u + v') = w^2.
\]

We conclude that \( (u, v, w) \in A \). This proves the compactness part. The lower bound follows immediately from formula (2) for \( E_h \) and the weak lower semi-continuity of
\[
w \mapsto \int_{S^1} (w'' + w)^2 dt
\]
in \( W^{2,2}(S_1) \).

\[\square\]
3. CONSTRUCTION OF THE RECOVERY SEQUENCE IN THEOREM 1

We will construct a recovery sequence \( \gamma_h \in W^{2,2}(S^1, S^2) \) that meets the constraints \( |\gamma_h| = |\gamma'_h| = 1, \gamma \cdot e_z \geq h \) in several steps. We start off with some sequence \( \gamma_h^{(1)} : S^1 \to \mathbb{R}^3 \), and each step \( \gamma_h^{(i)} \to \gamma_h^{(i+1)} \) shall assure that one additional constraint is met. At first, we will give this sequence of modifications for \((u, v, w) \in \mathcal{A} \cap W^{2,\infty}(S^1; \mathbb{R}^3)\). The proof will be completed by an approximation argument.

In Lemma 1 and Lemma 2 below, \((u, v, w) \in \mathcal{A} \cap W^{2,\infty}(S^1; \mathbb{R}^3)\) will be fixed, and we will use the following notational convention:

A statement such as “\( f \leq Cg \)” will be shorthand for the statement “There exists a constant \( C > 0 \) that only depends on \( \|u\|_{W^{2,\infty}}, \|v\|_{W^{2,\infty}} \) and \( \|w\|_{W^{2,\infty}} \), such that \( f \leq Cg \).” Similarly, if \( f \) and \( g \) depend on \( h \), we will write \( f = g + O(h^k) \) if there exists a constant \( C \) that only depends on \( \|u\|_{W^{2,\infty}}, \|v\|_{W^{2,\infty}} \) and \( \|w\|_{W^{2,\infty}} \), such that \( |f - g| \leq Ch^k \) for all \( h \).

**Lemma 1.** Let \((u, v, w) \in \mathcal{A} \cap W^{2,\infty}(S^1; \mathbb{R}^3)\). Then there exists a sequence of curves \( \gamma_h^{(4)} : \mathbb{R} \supset [0, 2\pi] \to S^2 \) with the following properties for \( h \) small enough:

\[
|\frac{d\gamma_h^{(4)}}{dt}(2\pi) - \frac{d\gamma_h^{(4)}}{dt}(0)| \leq Ch^4, 
\]

\[
\frac{d^2\gamma_h^{(4)}}{dt^2}(2\pi) - \frac{d^2\gamma_h^{(4)}}{dt^2}(0) \leq Ch^4.
\]

**Proof.** The initial ansatz is to define \( \gamma_h^{(1)} : S^1 \to \mathbb{R}^3 \) by

\[
\gamma_h^{(1)} = (1 + h^2 u)e_r + h^2 ve_\varphi + hwe_z.
\]

To make sure that the constraint \( \gamma_h \cdot e_z \geq h \) holds even after the modifications we are going to perform in the sequel, we set

\[
\gamma_h^{(2)} = \gamma_h^{(1)} + h^{5/2}e_z.
\]

By a computation using \((u, v, w) \in \mathcal{A}\), we have

\[
|\gamma_h^{(2)}|^2 = 1 + h^4 (u^2 + v^2 + 2w) + h^5 
\]

\[
\frac{d\gamma_h^{(2)}}{dt} = 1 + h^4 ((u + v')^2 + (u' - v)^2). 
\]
For future reference, we also make the following computations,

\[
\left| \frac{d^2 \gamma_h^{(2)}}{dt^2} \right| = \left| (-1 + h^2(u'' - u - v'))e_\varphi + h^2(u' + v' - v)e_\varphi + hw''e_z \right|
\leq 1 + Ch
\]

\[
\frac{d}{dt} \left( |\gamma_h^{(2)}|^2 \right) = \frac{d}{dt} \left( 1 + h^2 (2u + w^2) + h^4 (u^2 + v^2 + 2w) + h^5 \right)
\]

\[
= 2h^4 (uu' + vv' + w')
\leq Ch^4
\]

\[
\frac{d^2}{dt^2} \left( |\gamma_h^{(2)}|^2 \right) \leq Ch^4
\]

Our next modification assures the constraint $|\gamma_h| = 1$. Namely, we define $\gamma_h^{(3)} : S^1 \to \mathbb{R}^3$ by

\[
\gamma_h^{(3)} = \frac{\gamma_h^{(2)}}{|\gamma_h^{(2)}|}.
\]

Note that $\gamma_h^{(3)} \cdot e_z \geq h + \frac{1}{2} h^5/2$ by (9) and (10a) for $h$ small enough. However, $\gamma_h^{(3)}$ does not fulfill the constraint $|\gamma_h'| = 1$:

\[
\left| \frac{d\gamma_h^{(3)}}{dt} \right| = \frac{d}{dt} \left( |\gamma_h^{(2)}|^2 \right) - \frac{1}{2} \left( \frac{d}{dt} |\gamma_h^{(2)}|^2 \right) \gamma_h^{(3)} \gamma_h^{(2)} = 1 + O(h^4).
\]

Hence we get for the length $L_h$ of $\gamma_h^{(3)}$,

\[
L_h = \int_0^{2\pi} \left| \frac{d\gamma_h^{(3)}}{dt} \right| dt
\]

\[
= 2\pi + O(h^4).
\]

The next step is a re-parametrization of $\gamma_h^{(3)}$, which will assure the condition $|\gamma_h'| = 1$, at the expense of the curve being closed. We define $\tau_h : [0, \infty) \to \mathbb{R}$ by

\[
\tau_h(s) = \int_0^s \left| \frac{d\gamma_h^{(3)}}{dt} \right|^{-1} dt.
\]

(Recall that by our notational convention, we do not distinguish on the right hand side between $\gamma_h^{(3)}$ and $\gamma_h^{(3)} \circ \iota$, where $\iota : S^1 \to S^2$ is the canonical projection.) Next we define $\gamma_h^{(4)} : [0, \infty) \to S^2$ by

\[
\gamma_h^{(4)} = \gamma_h^{(3)} \circ \tau_h.
\]
Note that \( \gamma_h^{(4)} \) automatically satisfies (5). Moreover, (6) is satisfied since \( \gamma_h^{(3)} \) fulfilled that property too. Further, by \( \gamma_h^{(4)}(L_h) = \gamma_h^{(4)}(0) \), \( \frac{d\gamma_h^{(4)}}{dt}(L_h) = \frac{d\gamma_h^{(4)}}{dt}(0) \) and (13),

\[
|\gamma_h^{(4)}(2\pi) - \gamma_h^{(4)}(0)| \leq C h^4 \sup \left| \frac{d\gamma_h^{(4)}}{dt} \right| \quad (14a)
\]

\[
\left| \frac{d\gamma_h^{(4)}}{dt}(2\pi) - \frac{d\gamma_h^{(4)}}{dt}(0) \right| \leq C h^4 \sup \left| \frac{d^2\gamma_h^{(4)}}{dt^2} \right| . \quad (14b)
\]

We estimate the suprema on the right hand sides,

\[
\sup \left| \frac{d\gamma_h^{(4)}}{dt} \right| \leq (\sup |\tau_h^\prime|) \sup \left| \frac{d\gamma_h^{(3)}}{dt} \right| \leq \left( \sup \left| \frac{d\gamma_h^{(3)}}{dt} \right|^{-1} \right) \sup \left| \frac{d\gamma_h^{(2)}}{dt} \right| = 1 + O(h^4) \quad (15a)
\]

\[
\sup \left| \frac{d^2\gamma_h^{(4)}}{dt^2} \right| \leq \sup |\tau_h''| \sup \left| \frac{d\gamma_h^{(3)}}{dt} \right| + \sup |\tau_h'| \sup \left| \frac{d^2\gamma_h^{(3)}}{dt^2} \right| . \quad (15b)
\]

The estimate (15a) in combination with (14a) proves (7), and it remains to prove (8). We first compute \( \frac{d^2\gamma_h^{(3)}}{dt^2} \), using the notation \( f = |\gamma_h^{(2)}|^{-1} \),

\[
\frac{d^2\gamma_h^{(3)}}{dt^2} = \frac{d^2}{dt^2} \left( f\gamma_h^{(2)} \right) = \left( \frac{d^2 f}{dt^2} \right) \gamma_h^{(2)} + 2 \left( \frac{df}{dt} \right) \left( \frac{d\gamma_h^{(2)}}{dt} \right) + f \frac{d^2\gamma_h^{(2)}}{dt^2} . \quad (16)
\]

The derivatives of \( f \) are estimated as follows,

\[
\frac{df}{dt} = - \frac{1}{2} |\gamma_h^{(2)}|^{-3} \frac{d}{dt} |\gamma_h^{(2)}|^2 \leq C h^4,
\]

\[
\frac{d^2 f}{dt^2} = - \frac{1}{2} |\gamma_h^{(2)}|^{-3} \frac{d^2}{dt^2} |\gamma_h^{(2)}|^2 + \frac{3}{4} |\gamma_h^{(2)}|^{-5} \left( \frac{d}{dt} |\gamma_h^{(2)}|^2 \right)^2 \leq C h^4 ,
\]

where we have used (11). Inserting into (16), and using again (11), we get

\[
\left| \frac{d^2\gamma_h^{(3)}}{dt^2} \right| \leq 1 + C h . \quad (17)
\]
Next we compute $\tau''_h$,

$$|\tau''_h| = \frac{1}{2} \left| \frac{d\gamma_h^{(3)}}{dt} \right|^3 \left| \frac{d}{dt} \left| \frac{d\gamma_h^{(3)}}{dt} \right|^2 \right| \leq \left| \frac{d\gamma_h^{(3)}}{dt} \right|^{-2} \left| \frac{d^2\gamma_h^{(3)}}{dt^2} \right| \leq 1 + C h,$$

where we have used (12) and (17). Inserting (18) into (15b), we get

$$\sup \left| \frac{d^2\gamma_h^{(4)}}{dt^2} \right| \leq C.$$

Thus by (14b), we have proved (8). Finally, we reduce the domain of $\gamma^{(4)}_h$ from $[0, \infty)$ to $[0, 2\pi]$. This completes the proof of the lemma.

The final step in the construction of the recovery sequence is gluing the ends of the non-closed curve $\gamma^{(4)}_h$ back together. For the sake of brevity, let us write $\gamma^{(4)}_h = \bar{\gamma}$. Further, we introduce an orthonormal-frame-valued map $F$ by $U = \bar{\gamma}$, $T = \bar{\gamma}'$, $N = T \wedge U$ and $F = (T, N, U)^T$. Finally, let $\kappa = N \cdot \gamma''$. Then $F$ (and in particular, $\bar{\gamma}$) is determined by initial conditions and the ODE

$$F'(t) = \begin{pmatrix} 0 & \kappa & -1 \\ -\kappa & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} F(t).$$

We will have to modify $F$ such that $F(0) = F(2\pi)$.

We introduce the following notation for modifications of curves:

**Definition 1.** For curves $\gamma \in W^{2,2}([0, 2\pi]; S^2)$ with $|\gamma'| = 1$, and $\psi \in C_0^\infty((0, 2\pi))$, we define $\kappa = \gamma'' \cdot (\gamma' \wedge \gamma)$ and let $F_\gamma[\psi] = (T, N, U)^T$ be the unique solution of the initial value problem

$$\begin{cases} F_\gamma[\psi](0) = \left( \frac{d\gamma}{dt}(0), \frac{d\gamma}{dt}(0) \wedge \gamma(0), \gamma(0) \right)^T \\ \frac{d}{dt} F_\gamma[\psi](t) = \begin{pmatrix} 0 & \kappa(t) + \psi(t) & -1 \\ -\kappa(t) + \psi(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} F_\gamma[\psi](t) \quad \text{for all } t \in [0, 2\pi]. \end{cases}$$

Furthermore, we set $\gamma[\psi](t) = U(t)$ and $\frac{d\gamma}{dt}[\psi](t) = T(t)$ for all $t \in [0, 2\pi]$. (Of course, this definition satisfies $\frac{d}{dt} (\gamma[\psi](t)) = \frac{d\gamma}{dt}[\psi](t)$.)

Another tool in the modification process will be the following standard implicit function theorem (see e.g. [7]):
**Theorem 2.** Let \( x_0 \in \mathbb{R}^n \), \( f : B_r(x_0) \to \mathbb{R}^n \) continuously differentiable. Further let \( \alpha, \beta, k \in \mathbb{R}^+ \) such that

\[
\begin{align*}
\text{a) } & \quad Df(x_0) \text{ is invertible, } |Df(x_0)^{-1}f(x_0)| \leq \alpha, \quad |Df(x_0)^{-1}| \leq \beta \\
\text{b) } & \quad |Df(x_1) - Df(x_2)| \leq k|x_1 - x_2| \quad \text{for all } x_1, x_2 \in B_r(x_0) \\
\text{c) } & \quad 2\alpha \beta k < 1 \quad \text{and } 2\alpha < r.
\end{align*}
\]

Then there exists a unique solution \( y \in B_r(x_0) \) to \( f(y) = 0 \).

For \( t \in S^1 \), we have \( \bar{\gamma}(t) \in S^2 \) and \( \frac{d\bar{x}}{dt}(t) \in S^2 \cap T_{\gamma(2\pi)}S^2 \simeq S^1 \), where \( T_pS^2 \) denotes the tangent space of \( S^2 \) at \( p \). I.e., the range of the map \( t \mapsto \langle \bar{\gamma}(t), \frac{d\bar{x}}{dt}(t) \rangle \) is the bundle \( M = \{ (x, T) \in S^2 \times S^2 : T \in T_xS^2 \} \).

This is a 3-dimensional manifold. In a small enough neighborhood \( U \) of a point \( (x, T) \in M \), a chart is given by

\[
\begin{align*}
\zeta : U & \to \mathbb{R}^3 \\
(x, T) & \mapsto (\bar{x} - x(x \cdot \bar{x}), (T \wedge x) \cdot T),
\end{align*}
\]

Here, in the first two components, we made the identification \( \{ y \in \mathbb{R}^3 : y \cdot x = 0 \} \simeq \mathbb{R}^2 \).

Below, we will choose the chart \( \zeta \) defined as above with \( x = \bar{\gamma}(2\pi) \) and \( T = \frac{d\bar{x}}{dt}(2\pi) \).

Now, for \( \bar{\psi} \in C_0^\infty((0, 2\pi); \mathbb{R}^3) \), we set

\[
\begin{align*}
f_h : B(0, \epsilon) & \subset \mathbb{R}^3 \to \mathbb{R}^3 \\
a & \mapsto \zeta \circ \left( \bar{\gamma}[a \cdot \bar{\psi}](2\pi), \frac{d\bar{\gamma}}{dt}[a \cdot \bar{\psi}](2\pi) \right) - \zeta \circ \left( \bar{\gamma}(0), \frac{d\bar{\gamma}}{dt}(0) \right), \quad (20)
\end{align*}
\]

and we want to get existence of \( a \in B_r(0) \) such that \( f_h(a) = 0 \). The heart of the matter will be the application of Theorem 2. The upcoming lemma assures that its conditions are met for \( \bar{\gamma} = \bar{\gamma}_h^{(4)} \) as in the conclusion of Lemma 1 for \( h \) small enough.

**Lemma 2.** Let \( u, v, w \in W^{2,\infty}(S^1) \) \( \cap \mathcal{A} \) and \( \bar{\gamma} = \bar{\gamma}_h^{(4)} \) as in the conclusion of Lemma 1. There exists \( \bar{\psi} \in C_0^\infty((0, 2\pi); \mathbb{R}^3) \) such that for every \( h \) small enough,

- The derivative \( Df_h \) of the function \( f_h \) defined in \( (20) \) has full rank at \( a = 0 \).
- There exist constants \( \alpha, \beta > 0 \) that do not depend on \( h \) such that
  \[
  |Df_h(0)| \leq \beta h^{-1}, \quad (21)
  \\
  |Df_h(0)^{-1}f_h(0)| \leq \alpha h^3. \quad (22)
  \]
- There exist \( r_0, k > 0 \) that do not depend on \( h \) such that \( f_h \) is \( C^2 \) on \( B(0, r_0) \), and
  \[
  \sup_{a \in B(0, r_0)} |D^2f_h(a)| \leq k. \quad (23)
  \]

**Proof.** Let \( \psi \in C_0^\infty(0, 2\pi) \). In this proof, we will write \( F_h = F \). We start with the computation of the derivative of \( \mathbb{R} \to W^{2,2}(S^1, S^2), \varepsilon \mapsto F[\varepsilon\psi](2\pi) \).

Set

\[
E = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
We have
\[
\frac{d}{d\varepsilon} F[\varepsilon \psi](t)|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_0^t \begin{pmatrix}
-\kappa + \varepsilon \psi & \varepsilon & -1 \\
\kappa + \varepsilon \psi & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} (s) F[\varepsilon \psi](s) ds
= \int_0^t \psi(s) EF(s) + \begin{pmatrix}
0 & \kappa & -1 \\
-\kappa & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} (s) \frac{d}{d\varepsilon} F[\varepsilon \psi](s)|_{\varepsilon=0} ds
\]

By the variation of constants formula, we get
\[
\frac{d}{d\varepsilon} F[\varepsilon \psi](t)|_{\varepsilon=0} = F(t) \int_0^t \psi(s) \frac{F^{-1}(s) EF(s)}{s} ds
= F(t) \int_0^t \psi(s)(-N, T, 0) \begin{pmatrix}
T^T \\
N^T \\
U^T
\end{pmatrix} ds
= F(t) \int_0^t \psi(s) \begin{pmatrix}
0 & -U_3 & U_2 \\
U_3 & 0 & -U_1 \\
-U_2 & U_1 & 0
\end{pmatrix} ds
\]

In particular, this yields
\[
\frac{d}{d\varepsilon} \gamma[\varepsilon \psi](2\pi)|_{\varepsilon=0} = \int_0^{2\pi} \gamma(2\pi) \wedge \gamma(s) \psi(s) ds
\]
\[
\frac{d}{d\varepsilon} \left( \frac{d\gamma}{dt}[\varepsilon \psi](2\pi) \right)|_{\varepsilon=0} = \int_0^{2\pi} \frac{d\gamma}{dt}(2\pi) \wedge \gamma(s) \psi(s) ds
\] (24)

Repeating these arguments, we can compute the second derivative of \((\varepsilon_1, \varepsilon_2) \mapsto \gamma[\varepsilon_1 \psi_1 + \varepsilon_2 \psi_2](2\pi), \)
\[
\frac{d}{d\varepsilon_2} \frac{d}{d\varepsilon_1} (\gamma[\varepsilon_1 \psi_1 + \varepsilon_2 \psi_2](2\pi))|_{\varepsilon_1, \varepsilon_2=0}
= \int_0^{2\pi} \left( \left( \int_0^{2\pi} \gamma(2\pi) \wedge \gamma(t) \psi_2(t) dt \right) \wedge \gamma(s) \psi_1(s) ds \\
+ \int_0^{2\pi} \gamma(2\pi) \wedge \left( \int_0^t dt \psi_2(t) \gamma(s) \wedge \gamma(t) \right) \psi_1(s) ds \right)
= \int_0^{2\pi} \psi_1(s) \psi_2(t) \gamma(2\pi) \wedge \left( \gamma(t) \wedge \gamma(s) + \gamma(s) \wedge \gamma(t) \right) dt ds
\]
\[
+ \int_0^{2\pi} \left( \int_s^{2\pi} \gamma(2\pi) \wedge \gamma(t) \psi_2(t) dt \right) \wedge \gamma(s) \psi_1(s) ds
\]

Similarly, we get
\[
\frac{d}{d\varepsilon_2} \frac{d}{d\varepsilon_1} \left( \frac{d\gamma}{dt}[\varepsilon_1 \psi_1 + \varepsilon_2 \psi_2](2\pi) \right)|_{\varepsilon_1, \varepsilon_2=0}
= \int_0^{2\pi} \left( \left( \int_0^{2\pi} \frac{d\gamma}{dt}(2\pi) \wedge \gamma(t) \psi_2(t) dt \right) \wedge \gamma(s) \psi_1(s) ds \right)
\]
Now, for arbitrarily chosen $\tilde{\psi} \in C_0^\infty((0,2\pi) ; \mathbb{R}^3)$ and arbitrary $a \in \mathbb{R}^3$ let $\tilde{\gamma} := \tilde{\gamma}[a \cdot \tilde{\psi}]$. We can repeat the above computations to obtain

$$
\frac{d}{d\epsilon_2} \frac{d}{d\epsilon_1} \left( \tilde{\gamma}[\epsilon_1 \psi_1 + \epsilon_2 \psi_2](2\pi) \right) \Big|_{\epsilon_1,\epsilon_2=0} = \int_0^{2\pi} \left( \int_s^{2\pi} \tilde{\gamma}(2\pi) \wedge \tilde{\gamma}(t) \psi_2(t) dt \right) \wedge \tilde{\gamma}(s) \psi_1(s) ds \quad (25a)
$$

$$
= \int_0^{2\pi} \left( \int_s^{2\pi} \frac{d\tilde{\gamma}}{dt} \left[ \epsilon_1 \psi_1 + \epsilon_2 \psi_2 \right](2\pi) \right) \Big|_{\epsilon_1,\epsilon_2=0} = \int_0^{2\pi} \left( \int_s^{2\pi} \frac{d\tilde{\gamma}}{dt} \left( 2\pi \right) \wedge \tilde{\gamma}(t) \psi_2(t) dt \right) \wedge \tilde{\gamma}(s) \psi_1(s) ds . \quad (25b)
$$

It is easily seen from the definition of $f_h$ that

$$
|\partial_i \partial_j f_h(a)|^2 \leq \left( \frac{d}{d\epsilon_2} \frac{d}{d\epsilon_1} \left( \tilde{\gamma}[\epsilon_1 \psi_1 + \epsilon_2 \psi_2](2\pi) \right) \Big|_{\epsilon_1,\epsilon_2=0} \right)^2 + \left( \frac{d}{d\epsilon_2} \frac{d}{d\epsilon_1} \left( \frac{d\tilde{\gamma}}{dt} \left[ \epsilon_1 \psi_1 + \epsilon_2 \psi_2 \right](2\pi) \right) \Big|_{\epsilon_1,\epsilon_2=0} \right)^2.
$$

Hence, (23) follows from the observation that the right hand sides in (25a) and (25b) are bounded by a constant $h$ that only depends on $\psi_1 = \psi_i$ and $\psi_2 = \psi_j$, for any choice of $\tilde{\psi}$. Next we want to compute the determinant of $Df_h(0)$. Denoting by $M_{ij}$ the 2 by 2 minor of $Df_h$ that is obtained by deleting the $i$th row and the $j$th column, we have

$$
det Df_h = \sum_{i=1}^3 (-1)^{i+1} M_{3i} \partial_i(f_h)_3 . \quad (26)
$$

We recall that the first two components of $f_h(a)$ are the projection of $\tilde{\gamma}[a \cdot \tilde{\psi}](2\pi) - \tilde{\gamma}(0)$ to $T_{\tilde{\gamma}(2\pi)} S^2 \simeq \mathbb{R}^2$. Assume $\{i,j,k\} = \{1,2,3\}$. By the remark we just made, the minor $M_{3i}$ is (up to a sign) the determinant of the 2 by 2 matrix formed by the partial derivatives of $\tilde{\gamma}[a \cdot \tilde{\psi}](2\pi)$ with respect to $a_j$ and $a_k$ in some orthonormal basis of $T_{\tilde{\gamma}(2\pi)} S^2$. In other words,

$$
M_{3i} = \epsilon_{jk} \tilde{\gamma}(2\pi) \cdot \left( \frac{d}{d\epsilon} \tilde{\gamma}[\epsilon \psi_j](2\pi) \big|_{\epsilon=0} \right) \wedge \left( \frac{d}{d\epsilon} \tilde{\gamma}[\epsilon \psi_k](2\pi) \big|_{\epsilon=0} \right).
$$

Here, $\epsilon_{jk} = 1$ if $j < k$, and $\epsilon_{jk} = -1$ if $j > k$. Using (24), we get

$$
M_{3i} = \epsilon_{jk} \int_0^{2\pi} ds dt \tilde{\psi}_j(s) \tilde{\psi}_k(t) \tilde{\gamma}(2\pi) \cdot (\tilde{\gamma}(2\pi) \wedge \tilde{\gamma}(s) \wedge \tilde{\gamma}(t))
$$

$$
= \epsilon_{jk} \int_0^{2\pi} ds dt \tilde{\psi}_j(s) \tilde{\psi}_k(t) (\tilde{\gamma}(2\pi) \cdot (\tilde{\gamma}(s) \wedge \tilde{\gamma}(t)))
$$

$$
= \epsilon_{jk} h \int_0^{2\pi} ds dt \tilde{\psi}_j(s) \tilde{\psi}_k(t) b(s,t) + O(h^2) , \quad (27)
$$

where

$$
b(s,t) = (\sin(t-s)w(2\pi) - \sin(t)w(s) + \sin(s)w(t)) .
$$
Next we compute the partial derivatives of the third component of \( f_h \),

\[
\frac{\partial (f_h)_3}{\partial a_i} \bigg|_{a=0} = \left( \frac{d\gamma}{dt} (2\pi) \wedge \bar{\gamma}(2\pi) \right) \cdot \frac{d}{d\varepsilon} \left( \frac{d\gamma}{dt} [\varepsilon \bar{\psi}_i](2\pi) \right) \bigg|_{\varepsilon=0} \\
= \left( \frac{d\gamma}{dt} (2\pi) \wedge \bar{\gamma}(2\pi) \right) \cdot \int_0^{2\pi} ds \bar{\psi}_i(s) \frac{d\gamma}{dt} (2\pi) \wedge \bar{\gamma}(s) \\
= \int_0^{2\pi} ds \bar{\psi}_i(s) \bar{\gamma}(s) \cdot \bar{\gamma}(2\pi) \\
= \cos(s) + O(h),
\]

where we have used (24) in the second equation.

\[
\det Df_h = h \int \left( \prod_{i=1}^3 \bar{\psi}_i(s_i) ds_i \right) \cos(s_1)b(s_2, s_3) - \cos(s_2)b(s_1, s_3) + \cos(s_3)b(s_1, s_2) + O(h^2) \\
= h \int \left( \prod_{i=1}^3 \bar{\psi}_i(s_i) ds_i \right) \tilde{b}(s_1, s_2, s_3) + O(h^2),
\]

where

\[
\tilde{b}(s_1, s_2, s_3) = (\sin(s_2 - s_1)w(s_3) + \sin(s_1 - s_3)w(s_2) + \sin(s_3 - s_2)w(s_1)).
\]

The only solutions of \( \bar{b}(s_1, s_2, s_3) = 0 \) for \( w \) are \( w(s) = A \cos(s), A \in \mathbb{R} \). Since this is not possible by \( (u, v, w) \in \mathcal{A}, \bar{\psi} \) can be chosen such that

\[
\det Df_h \geq Ch. \tag{29}
\]

Next, we estimate \( |Df_h^{-1}(0)| \) using the formula

\[
Df_h^{-1} = \frac{1}{\det Df_h \text{ cof } Df_h}, \tag{30}
\]

where

\[
(\text{cof } Df)_{ij} = (-1)^{i+j} M_{ij}
\]

is the cofactor matrix of \( Df \). Since by (24), \( |Df_h(0)| \leq C \), where \( C \) is independent of \( h \), the same holds true for \( \text{cof } Df_h \). Hence, by (29) and (30), we have shown (21). The property (22) follows from

\[
\gamma(2\pi) - \gamma(0) = O(h^4) \\
\frac{d\gamma}{dt}(2\pi) - \frac{d\gamma}{dt}(0) = O(h^4), \tag{31}
\]

which holds by Lemma 1. \( \square \)

The approximation of \( (u, v, w) \in \mathcal{A} \) by \( W^{2,\infty} \) functions is the content of the following lemma.
Lemma 3. Let \( w \in W^{2,2}(S^1) \) with \( w \geq 1 \), \( \int_{S^1} w^2 - w'^2 dt = 0 \). There exists a sequence \( w_\varepsilon \in W^{2,\infty}(S^1) \) such that
\[
\begin{align*}
\int_{S^1} w_\varepsilon^2 - w'^2 dt &= 0 \\
\varepsilon &
\end{align*}
\]
\( w_\varepsilon \to w \) in \( W^{2,2}(S^1) \) as \( \varepsilon \to 0 \).

Proof. Let \( \eta \) be a standard mollifier, i.e., \( \eta \in C_0^\infty((-1,1)) \), \( \eta(t) = 0 \) for \( |t| > 1 \), \( \int \eta(t) dt = 1 \). Moreover let \( \eta_\varepsilon(\cdot) = \varepsilon^{-1} \eta(\cdot/\varepsilon) \) and \( \tilde{w}_\varepsilon = \eta_\varepsilon * w \). By \( w \geq 1 \), \( \int \eta_\varepsilon = 1 \) and \( \eta_\varepsilon \geq 0 \), we have \( \tilde{w}_\varepsilon \geq 1 \). By standard properties of the convolution with mollifiers,
\[
\begin{align*}
\eta_\varepsilon * w &\to w \quad \text{in} \quad L^2(S^1) \quad \text{as} \quad \varepsilon \to 0, \\
\eta_\varepsilon * w' &\to w' \quad \text{in} \quad L^2(S^1) \quad \text{as} \quad \varepsilon \to 0.
\end{align*}
\]
In particular,
\[
\int \tilde{w}_\varepsilon^2 - \tilde{w}_\varepsilon'^2 dt \to 0. \tag{32}
\]
For \( \psi \in C^\infty(S^1) \), let
\[
G_\psi : \mathbb{R} \to \mathbb{R} 
\]
\[
\lambda \mapsto \int_{S^1} (w + \lambda \psi)^2 - (w' + \lambda \psi')^2.
\]
The derivative of \( G_\psi \) at 0 is given by
\[
DG_\psi(0) = \left. \frac{\partial}{\partial \lambda} \int_{S^1} (w + \lambda \psi)^2 - (w' + \lambda \psi')^2 \right|_{\lambda=0} = 2 \int_{S^1} (w + w')\psi.
\]
We claim that it is possible to choose \( \psi \) with \( \text{supp} \psi \subset U := \{ t : w(t) > 1 \} \) such that
\( DG_\psi(0) \neq 0 \). To see this, note first that by the continuity of \( w \), \( U \) is open. By \( w \geq 1 \) and \( \int_{S^1} w^2 - w'^2 = 0 \), \( U \) is non-empty. Assuming \( \int_U (w'' + w)\psi = 0 \) for all \( \psi \in C_0^\infty(U) \), we have
\[
||w'' + w||_{L^2(U)} = 0. \tag{33}
\]
Let \( U_0 \) be a connected component of \( U \). By \( \tag{33} \), \( w(t) = A \sin(t + \alpha) \) for \( t \in U_0 \) for some \( A \in \mathbb{R}, \alpha \in S^1 \). Let \( t_0 \in \partial U_0 \). Then \( 1 = w(t_0) = \lim_{t \to t_0} A \sin(t + \alpha) \), and \( t_0 \) is not a local maximum of the latter function. By the embedding \( W^{2,2}(S^1) \subset C^1(S^1) \), \( w' \) is continuous. Hence we have \( w'(t_0) = \lim_{t \to t_0} A \cos(t + \alpha) \neq 0 \). On the other hand, again by the continuity of \( w' \), we must have \( w'(t_0) = 0 \) (since there is no \( t \) in any neighborhood of \( t_0 \) with \( w(t) < w(t_0) \)). This contradiction proves that it is possible to choose \( \psi \in C_0^\infty(U) \) such that \( DG_\psi(0) = 2 \int_{S^1} (w + w'')\psi \neq 0 \).
Choose such a \( \psi \), and let \( \delta_1 \) be such that \( w \geq 1 + 2\delta_1 \) on \( \text{supp} \psi \). For \( \varepsilon \) small enough, we have \( \text{supp} \psi \subset \{ t : \tilde{w}_\varepsilon(t) \geq 1 + \delta_1 \} \). Again by standard properties of approximation by mollifiers, we have
\[
\int_{S^1} (\tilde{w}_\varepsilon'' + \tilde{w}_\varepsilon)\psi \to \int_{S^1} (w'' + w)\psi
\]
as \( \varepsilon \to 0 \). Hence there exists \( \delta_2 > 0 \) such that
\[
|DG_\psi^\varepsilon(0)| \geq \delta_2 \quad \text{uniformly in} \quad \varepsilon,
\]
where $G^c_\psi$ is defined by $\lambda \mapsto \int_{S^1} (\bar{w}_\epsilon + \lambda \psi)^2 - (\bar{w}'_\epsilon + \lambda \psi')^2$.

Now we are going to apply the implicit function theorem, Theorem 2, with $f = G^c_\psi$, $x_0 = 0$. Condition a) from that theorem can be fulfilled with $\alpha$ arbitrarily small, if we choose $\epsilon$ small enough. Condition b) is easily verified by direct computation,

$$|DG^c_\psi(\lambda_1) - DG^c_\psi(\lambda_2)| = 2|\lambda_1 - \lambda_2| \left| \int_{S^1} (\psi'' + \psi) \psi \right|.$$ 

Finally, property c) holds since $\alpha$ can be chosen arbitrarily small. Hence, we get the existence of $\lambda_\epsilon$ such that

$$G^c_\psi(\lambda_\epsilon) = 0,$$

with $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. Thus, again by choosing $\epsilon$ small enough, we get

$$w_\epsilon := \bar{w}_\epsilon + \lambda_\epsilon \psi \geq 1 \text{ on } S^1.$$ 

The sequence $w_\epsilon$ fulfills all the required properties. This proves the lemma. □

**Corollary 1.** Let $(u, v, w) \in A$. Then there exists a sequence $(u_\epsilon, v_\epsilon, w_\epsilon) \in A \cap W^{2, \infty}(S^1; \mathbb{R}^3)$ with $(u_\epsilon, v_\epsilon, w_\epsilon) \to (u, v, w)$ in $W^{2,2}(S^1; \mathbb{R}^3)$.

**Proof.** Let $w_\epsilon$ be the approximation of $w$ from Lemma 3. We set

$$u_\epsilon := -w_\epsilon^2/2,$$

$$v_\epsilon(t) := v(0) - \int_0^t u_\epsilon(t) + \frac{w_\epsilon'(t)^2}{2} \, dt.$$ 

This sequence has all required properties. □

**Proof of Theorem 4, upper bound.** By Corollary 1 and a standard diagonal sequence argument, it suffices to construct the recovery sequence for the case $(u, v, w) \in A \cap W^{2, \infty}(S^1; \mathbb{R}^3)$. Let $\gamma = \gamma_h^{(4)}$ be as in the conclusion of Lemma 1. By Lemma 2 and Theorem 2, there exists $\bar{\psi} \in C^\infty_0(0, 2\pi)$ (independent of $h$) and $a_h \in \mathcal{B}_{C^1}(\bar{0}) \subset \mathbb{R}^4$ such that $\bar{\gamma}[a_h \cdot \bar{\psi}](2\pi) = \bar{\gamma}[a_h \cdot \bar{\psi}](0)$ and $\frac{d\bar{\omega}}{dt}[a_h \cdot \bar{\psi}](2\pi) = \frac{d\bar{\omega}}{dt}[a_h \cdot \bar{\psi}](0)$. Set $\gamma_h = \bar{\gamma}[a_h \cdot \bar{\psi}]$. By the boundary values of $\gamma_h, \gamma_h'$ at 0 and $2\pi$, we may view $\gamma_h$ as a function in $W^{2,2}(S^1; \mathbb{R}^3)$. By $a_h = O(h^3)$ and $\bar{\gamma} \cdot e_z \geq h + \frac{1}{2} h^{3/2}$, we have $\gamma_h \cdot e_z \geq h$ for $h$ small enough. Thus, $\gamma_h$ fulfills the constraints $|\gamma_h| = |\gamma_h'| = 1$, and $\gamma_h \cdot e_z \geq h$. Finally, $\bar{E}_h(\gamma_h) \to \bar{E}^0(u, v, w)$ follows from the convergence $h^{-1}(u_h, v_h, w_h) \to (0, 0, 0)$ and 2. Hence $\gamma_h$ is the desired recovery sequence. □

4. Minimizers of the limit functional

To analyze the minimizers of the limiting functional $\bar{E}^0$, we introduce

$$\bar{E}^0 : W^{2,2}(S^1) \to \mathbb{R},$$

$$w \mapsto \begin{cases} \int_0^{2\pi} (w'' + w)^2 \, dt & \text{if } \int_0^{2\pi} (w^2 - w'^2) \, dt = 0 \\ +\infty & \text{else.} \end{cases}$$

It is easily seen that $E^0(u, v, w) < \infty$ only if $\bar{E}^0(w) < \infty$, and in that case $u$ and $v$ are (up to a constant) uniquely determined by $w$. Thus the study of minimizers of $E^0$ reduces to the study of minimizers of $\bar{E}^0$.

The existence of minimizers of $\bar{E}^0$ follows in an obvious way by an application of the
direct method. It is possible to compute rather explicitly the minimizers, provided one knows that they are in $C^2(S^1)$. The proof of this fact is one the main points of the following theorem. It will turn out that the following functions $(0, \pi] \to \mathbb{R} \cup \{\pm \infty\}$ play an important role for the characterization of $w$:

$$g_\alpha(z) := \frac{\alpha^2 \sin(z) \cos(\alpha z) - \alpha \sin(\alpha z) \cos(z)}{\sin(z) \cos(\alpha z) - \alpha \sin(\alpha z) \cos(z)}$$

$$\tilde{g}_\alpha(z) := \frac{\alpha^2 \sin(z) \cosh(\alpha z) - \alpha \sinh(\alpha z) \cos(z)}{\sin(z) \cosh(\alpha z) + \alpha \sinh(\alpha z) \cos(z)}.$$  

For plots of $g_\alpha$, $\tilde{g}_\alpha$ for $\alpha = 7$, see Figures 1 and 2.

**Remark 1.** If $\alpha \in (0, \infty) \setminus \{1\}$, then $g_\alpha^{-1}(k)$ is finite for every $k \in [0, \infty)$. This follows easily from the fact that $g_\alpha$ is a quotient of linearly independent trigonometric polynomials. Analogously, if $\alpha > 0$, then $\tilde{g}_\alpha^{-1}(k)$ is finite for every $k \in [0, \infty)$.

**Theorem 3.** Let $w$ be a local minimizer of the functional $\mathcal{E}^0$. Then there exist $\lambda \in \mathbb{R} \setminus \{0, -1\}$, $k \in [0, \infty)$ and finitely many mutually disjoint open intervals $I_i \subset S^1$, $i = 1, \ldots, m$ such that
Theorem 4. Let 

(i) \( w > 1 \) on \( I := \cup_i I_i \), \( w = 1 \) on \( S^1 \setminus I \), and \( w \in C^{2,1}(S^1) \).

(ii) \( w'' = k \) on \( S^1 \setminus I \)

(iii) For \( i = 1, \ldots, m \), writing \( I_i = (t_i - z_i, t_i + z_i) \) and \( \alpha = \sqrt{1 + \lambda} \), we have:

(a) If \( \lambda \geq -1 \), then \( z_i \in g^{-1}_\alpha(k) \) and
\[
    w(t) = \frac{\sin(z_i) \cos(\alpha(t - t_i)) - \alpha \sin(\alpha z_i) \cos(t - t_i)}{\sin(z_i) \cos(\alpha z_i) - \alpha \sin(\alpha z_i) \cos(z_i)} \quad \text{for } t \in I_i
\]

(b) If \( \lambda < -1 \), then \( z_i \in \tilde{g}^{-1}_\alpha(k) \) and
\[
    w(t) = \frac{\sin(z_i) \cos(\alpha(t - t_i)) + \alpha \sinh(\alpha z_i) \cos(t - t_i)}{\sin(z_i) \cosh(\alpha z_i) + \alpha \sinh(\alpha z_i) \cosh(z_i)} \quad \text{for } t \in I_i
\]

The proof of the theorem relies on (a special case of) Theorem 3.1 from [9], which we cite now.

Theorem 4. Let \( X \) be a normed space, \( K \subseteq X \) a closed convex cone with \( 0 \in K \) such that \( K - K = X \), \( f, g : X \rightarrow \mathbb{R} \) Fréchet differentiable, \( u_0 \in X \) a local minimum of the variational problem

\[
    \begin{aligned}
    f(u) &\rightarrow \min \\
    g(u) &= 0 \\
    u &\in x_0 + K.
    \end{aligned}
\]

Then the following Lagrange multiplier rule holds: there exists \( \lambda \in \mathbb{R} \) such that for every \( \varphi \in K \),

\[
    Df(u_0)\varphi + \lambda Dg(u_0)\varphi \geq 0.
\]

Proof of Theorem 3. Denote by \( I_i \) the connected components of \( I := \{ x \in S^1 : w(x) > 1 \} \). There are at most countably many of them, and we may write \( I = \cup_{i=1}^\infty I_i \). We are going to prove the statements (i), (ii) and (iii) for every \( i = 1, \ldots, \infty \), and conclude in the end that there is only a finite number of the \( I_i \)'s.

We apply Theorem 4 with \( X = W^{2,2}(S^1) \),
\[
    f(\bar{w}) = \int_{S^1} (w''(t) + \bar{w}(t))^2 \, dt \\
    g(\bar{w}) = \int_{S^1} (\bar{w}^2(t) - \bar{w}'^2(t)) \, dt,
\]

taking \( u_0 = w \) as the minimizer from the statement of the present theorem, and

\( K = \{ \varphi \in W^{2,2}(S^1) : \varphi(t) \geq 0 \text{ for } t \in S^1 \setminus I \} \).

The conditions from Theorem 4 are fulfilled and hence we obtain the existence of some \( \lambda \in \mathbb{R} \) such that
\[
    \int_{S^1} \left( \frac{d^4w}{dt^4} + (2 + \lambda)w'' + (1 + \lambda)w \right) \varphi \, dt \geq 0,
\]
for all \( \varphi \in K \), where the integral is defined by integration by parts. In particular, this last inequality holds true for every \( \varphi \in W^{2,2}(S^1) \) with \( \varphi \geq 0 \). By the Riesz-Schwartz Theorem (see e.g. [14]), the linear map
\[
    \bar{\mu} : W^{2,2}(S^1) \rightarrow \mathbb{R}
\]

\[
    \varphi \mapsto \int_{S^1} \left( \frac{d^4w}{dt^4} + (2 + \lambda)w'' + (1 + \lambda)w \right) \varphi \, dt
\]

There are at most countably many of them, and we may write \( I = \cup_{i=1}^\infty I_i \).
defines a non-negative Radon measure $\mu$ on $S^1$. In particular, it follows that $\frac{d^3w}{dt^3} \in L^\infty(S^1)$ and hence $w \in C^{2,1}(S^1)$. This proves (i).

For any $\varphi \in W^{2,2}(S^1)$ with $\text{supp} \varphi \subset I$, we have $\varphi \in K$, $-\varphi \in K$, and hence by (36),

$$\int_{S^1} \left( \frac{d^4w}{dt^4} + (2 + \lambda)w'' + (1 + \lambda)w \right) \varphi dt = 0.$$ 

This implies, in the sense of Radon measures,

$$\frac{d^4w}{dt^4} + (2 + \lambda)w'' + (1 + \lambda)w = 0 \quad \text{on } I.$$ 

We claim that

$$S^1 \neq I. \quad (37)$$

Indeed, assume the contrary were the case. Then $w$ is a local minimizer of the variational problem

$$\begin{cases}
\int_{S^1} (w'' + w)^2 dt \to \min \\
\int_{S^1} (w^2 - w'^2) dt = 0.
\end{cases}$$

By the standard Lagrange multiplier formalism, there exists $\tilde{\lambda} \in \mathbb{R}$ such that

$$\frac{d^4w}{dt^4} + (2 + \tilde{\lambda})w'' + (1 + \tilde{\lambda})w = 0 \quad \text{on } S^1. \quad (38)$$

Identifying $S^1$ with the interval $(-\pi, \pi)$, we must have

$$\begin{align*}
w(-\pi) &= w(\pi) \\
w''(-\pi) &= w''(\pi) \\
\text{and } \frac{d^3w}{dt^3}(-\pi) &= \frac{d^3w}{dt^3}(\pi)
\end{align*} \quad (39)$$

We claim that the dimension of the solution space of the boundary value problem defined by (38) and (39) is zero for $\tilde{\lambda} \neq -1$. Indeed, rewriting (38) as $x' = Ax$ with

$$x = \begin{pmatrix} \frac{d^3w}{dt^3}(t), w''(t), w'(t), w(t) \end{pmatrix}^T, \quad A = \begin{pmatrix} 0 & 2 + \tilde{\lambda} & 0 & 1 + \tilde{\lambda} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \end{pmatrix}$$

and (39) as $Ux = Mx(-\pi) + Nx(\pi) = 0$ with $M = -N = \text{Id}_{4 \times 4}$, we get that the “boundary form” $U$ applied to the fundamental matrix $\exp(A \cdot)$ is given by $U \exp(A \cdot) = \exp(-A\pi) - \exp(A\pi)$, which has full rank unless $\tilde{\lambda} = -1$. Hence, for $\tilde{\lambda} \neq -1$, the claim that the dimension of the solution space is zero follows from a standard result in ODE theory (see e.g. [5], Chapter 11, Theorem 3.3). Hence, $w(t) = 0$ is the unique solution to the boundary value problem above, which is a contradiction to $w \geq 1$.

If $\tilde{\lambda} = -1$, then the solutions to (38) are given by $w''(t) = -a \cos(t + t_0)$, where $a, t_0 \in \mathbb{R}$ are integration constants. This implies $w(t) = a \cos(t + t_0) + b$, where $b \in \mathbb{R}$ is yet another integration constant. From the constraint $\int_{-\pi}^{\pi} (w^2 - w'^2) = 0$, it follows $b = 0$, which again produces a contradiction to $w \geq 1$. This proves (37).

Now fix some $I_i$. After a translation, we may write $I_i = (-z_i, z_i)$ for some $z_i \in (0, \pi]$ by
By the regularity of \( w \), we have \( w(\pm z_i) = 1, w'(\pm z_i) = 0 \). Hence, \( w|_{\partial I} \) has to be a solution of the boundary value problem

\[
\frac{d^2w}{dt^2} + (2 + \lambda)w'' + (1 + \lambda)w = 0 \quad \text{on} \ (-z_i, z_i) \\
w(-z_i) = w(z_i) = 1 \\
w'(-z_i) = w'(z_i) = 0.
\]

If \( \lambda = 0 \), there do not exist any solutions to \( (38) \). This would imply \( I = \emptyset \), which cannot be by the constraint \( \int w^2 - w'^2 = 0 \). Hence, we conclude

\[
\lambda \neq 0.
\]

If \( \lambda \neq 0 \), there exists a unique solution to \( (40) \). We recall the notation \( \alpha = \sqrt{|1 + \lambda|} \). The solution of \( (40) \) is given by

\[
w(t) = \frac{\sin(z_i) \cos(\alpha t) - \alpha \sin(\alpha z_i) \cos(t)}{\sin(z_i) \cos(\alpha z_i) - \alpha \sin(\alpha z_i) \cos(z_i)} \quad \text{if} \ \lambda \geq -1,
\]

and

\[
w(t) = \frac{\sin(z_i) \cosh(\alpha t) + \alpha \sinh(\alpha z_i) \cos(t)}{\sin(z_i) \cosh(\alpha z_i) + \alpha \sinh(\alpha z_i) \cos(z_i)} \quad \text{if} \ \lambda < -1.
\]

Next, we prove (ii). By the explicit formulas \( (42), (43) \), we see that \( w'' \) is constant on the boundary of every \( I_i \) (which of course just consists of up to two points). Let \( k_i \) denote the value of \( w'' \) on \( \partial I_i \). Then we set

\[
v(t) = \begin{cases} k_i & \text{if} \ t \in I_i \\ w''(t) & \text{if} \ t \in S^1 \setminus I. \end{cases}
\]

By the regularity of \( w, v \) is Lipschitz. Further, \( v' = 0 \) on \( I \) and on \( S^1 \setminus I = \{w = 1\} \), we have \( v' = \frac{d^2w}{dt^2} = 0 \) almost everywhere. Hence \( v' = 0 \) almost everywhere in \( S^1 \) and \( v \) is constant. This proves (ii).

Let us consider some \( I_i \). Again, after translation, we may write \( I_i = (-z_i, z_i) \). Note that \( g_\alpha(z_i) = w''(z_i) \) if \( \lambda \geq -1 \) and \( \tilde{g}_\alpha(z_i) = w''(z_i) \) if \( \lambda < -1 \). By (ii) and the continuity of \( w'' \), we have \( z_i \in g_\alpha^{-1}(k) \) if \( \lambda \geq -1 \) and \( z_i \in \tilde{g}_\alpha^{-1}(k) \) if \( \lambda < -1 \). This holds true for any \( i \). In combination with \( (42) \) and \( (43) \), this shows (iii).

It remains to show that there is only a finite number of connected components of \( I \). First assume \( \lambda \geq -1 \). We may restate the relation \( z_i \in g_\alpha^{-1}(k) \) as

\[
\frac{1}{2} \mathcal{L}^1(I_i) \in \tilde{g}_\alpha^{-1}(k) \quad \text{for all} \ i,
\]

where \( \mathcal{L}^1 \) denotes the one-dimensional Lebesgue measure. We claim that \( \alpha \notin \{0, 1\} \). We have already seen \( \lambda \neq 0 \) in \( (41) \), and hence \( \alpha \neq 1 \). Further, if \( \alpha = 0 \), then \( w = 1 \) everywhere by \( (42) \). This is a contradiction to \( \int w^2 - w'^2 = 0 \). Hence, as we have noted in Remark \( \square\) above, \( \tilde{g}_\alpha^{-1}(k) \) is a finite set. In particular, there is a certain minimal length that any \( I_i \) can have. This implies that there are only finitely many connected components of \( I \):

\[
I = \bigcup_{i=1}^m I_i.
\]

If \( \lambda < -1 \), one argues in exactly the same way (using the finiteness of \( \tilde{g}_\alpha^{-1}(k) \)) to conclude that there are finitely many connected components of \( I \).
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