Primordial Inflation and Present-Day Cosmological Constant from Extra Dimensions

Carl L. Gardner∗
gardner@asu.edu
Department of Mathematics
Arizona State University
Tempe AZ 85287

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Abstract

A semiclassical gravitation model is outlined which makes use of the Casimir energy density of vacuum fluctuations in extra compactified dimensions to produce the present-day cosmological constant as $\rho_\Lambda \sim M^8/M_P^4$, where $M_P$ is the Planck scale and $M$ is the weak interaction scale. The model is based on $(4 + D)$-dimensional gravity, with $D = 2$ extra dimensions with radius $b(t)$ curled up at the ADD length scale $b_0 = M_P/M^2 \sim 0.1$ mm. Vacuum fluctuations in the compactified space perturb $b_0$ very slightly, generating a small present-day cosmological constant.

The radius of the compactified dimensions is predicted to be $b_0 \approx k^{1/4}0.09$ mm (or equivalently $M \approx 2.4 \text{ TeV}/k^{1/8}$), where the Casimir energy density is $k/b^4$.

Primordial inflation of our three-dimensional space occurs as in the cosmology of the ADD model as the inflaton $b(t)$, which initially is on the order of $1/M \sim 10^{-17}$ cm, rolls down its potential to $b_0$.

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1 Introduction

Supernova data indicate that the energy density $\rho_\Lambda$ in a present-day cosmological constant is on the order of $0.7\rho_c$, where the current critical density $\rho_c \approx (2.5 \times 10^{-3} \text{ eV})^4$. It is intriguing that $\rho_\Lambda \sim b_0^4$ where $b_0 \sim 0.1 \text{ mm}$—just the length scale for compactified extra dimensions predicted by Arkani-Hamed-Dimopoulos-Dvali (ADD) type theories \[1\] with two extra spatial dimensions.

It is possible that this dark energy derives from vacuum fluctuations in extra compactified dimensions. We outline here a semiclassical gravitation model which makes use of this mechanism to produce the present-day cosmological constant. The model is based on $(4 + D)$-dimensional gravity, with $D = 2$ extra dimensions with radius $b(t)$ curled up at the ADD length scale $b_0$, where the subscript “0” denotes present-day values.

The ADD model can be realized \[2\] in type I ten-dimensional string theory, with standard model fields naturally restricted to a 3-brane \[3\], while gravitons propagate in the full higher dimensional space. For $D = 2$, two of the six compactified dimensions are curled up with radius $\sim b_0$, while the remaining four are curled up with radius $\sim 1/M_I$, with the type I string scale $M_I \sim 1 \text{ TeV}$. In this picture, the ADD model is formulated within a consistent quantum theory of gravity.

In addition, if supersymmetry is broken only on the 3-brane, then the bulk cosmological constant vanishes (see e.g. Ref. \[4\]). A single fine tuning of parameters in the potential for $b$ can then cancel the brane tension, setting the usual four-dimensional cosmological constant to zero.

Semiclassical $(4 + D)$-dimensional gravitation—with a potential for the scale $b$ of the extra compactified dimensions—rapidly becomes a good approximation to the string theory for energies below $M_I$ \[5\]. In the semiclassical gravitation model, we will assume a potential for $b(t)$ which stabilizes $b(t_0)$ at $b_0 = M_P/M^2$ and which vanishes\[6\] at $b_0$ in the absence of the Casimir effect, where the (reduced) Planck scale $M_P = 2.4 \times 10^{18} \text{ GeV}$ and the weak interaction scale $M \sim 1 \text{ TeV}$. Vacuum fluctuations in the compactified space will then perturb $b(t_0)$ very slightly away from $b_0$, generating a small present-day cosmological constant in our three-dimensional world. This mechanism differs from previous cosmological models incorporating the Casimir effect from

\[1\] In other words, we assume that the 3-brane tension is exactly cancelled in the stabilization potential at $b = b_0$. 

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vacuum fluctuations in extra compactified dimensions (see e.g. Ref. [3]), in which the Casimir energy density in our three-dimensional world is cancelled by a bulk cosmological constant.

Primordial inflation of our three-dimensional space will occur in the model as the inflaton $b(t)$, which initially is on the order of $1/M \sim 10^{-17}$ cm, rolls down its potential to $b_0$ [7, 8]. Many e-folds of inflation of our 3-space can occur for sufficiently flat potentials.

We will take the spacetime metric to be $R^1 \times S^3 \times T^2$ symmetric where $S^3$ is a 3-sphere and $T^2$ is a 2-torus:

$$g_{MN} = \text{diag}\{1, -a^2(t)\tilde{g}_{ij}, -b^2(t)\tilde{g}_{mn}\}$$ (1)

where $M, N$ run from 0 to 5; $i, j$ run from 1 to 3; and $m, n$ run from 4 to 5. $\tilde{g}_{ij}$ is the metric of a unit 3-sphere and $\tilde{g}_{mn}$ is the metric of a unit 2-torus, with $a(t)$ the radius of physical 3-space and $b(t)$ the radius of the compactified space.

The nonzero components of the $(4 + D)$-dimensional Ricci tensor are

$$R_{00} = -\left(3\ddot{a}/a + D\dot{b}/b\right)$$

$$R_{ij} = -\left(\ddot{a}/a + 2\dot{a}^2/a^2 + D\dot{a}/a + 2\dot{a}/a^2\right)\tilde{g}_{ij}$$

$$R_{mn} = -\left(\ddot{b}/b + (D - 1)\dot{b}^2/b^2 + 3\dot{a}/a\dot{b}/b\right)\tilde{g}_{mn}.$$ (2)

The generalized Einstein equations are

$$R_{MN} = 8\pi \bar{G} \left(T_{MN} - \frac{T_P}{D} g_{MN}\right)$$ (3)

where $8\pi \bar{G} = 8\pi G\nu_0 = \nu_0/M_D^2 = \bar{\Omega}_D/M^{D+2}$ is the $(4 + D)$-dimensional gravitational constant, $\nu_0 = \bar{\Omega}_2b_0^2$ is the volume of the compactified dimensions today, $\bar{\Omega}_D$ denotes the volume of the unit $D$-torus, and $T_{MN}$ is the energy-momentum tensor. The gravitational coupling $8\pi G = 1/(b_0^2 M^4)$ is weak in the ADD picture because $b_0$ is much greater than the $(4 + D)$-dimensional Planck length $1/M$.

\[^2\text{Our treatment through Eq. (11) parallels that of Kolb and Turner [9].}\]
The nonzero components of the energy-momentum tensor are given by:

\[
T_{00} = \rho \\
T_{ij} = -p_{a}g_{ij} \\
T_{mn} = -p_{b}g_{mn},
\]

Thus \( T_{PP} = \rho - 3p_{a} - Dp_{b} \). Expressed in terms of the radii \( a \) and \( b \), the energy density \( \rho \), and the pressures \( p_{a} \) and \( p_{b} \), the Einstein equations become:

\[
3\ddot{a} + D\ddot{b} = -\frac{8\pi G}{D+2} [(D+1)\rho + 3p_{a} + Dp_{b}]
\]

\[
\frac{\ddot{a}}{a} + 2\dddot{a} + D\ddot{b} \frac{b}{a} + \frac{2}{a^2} = \frac{8\pi G}{D+2} [\rho + (D-1)p_{a} - Dp_{b}]
\]

\[
\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\dddot{a} \frac{b}{a} = \frac{8\pi G}{D+2} [\rho - 3p_{a} + 2p_{b}].
\]

After a few e-folds of primordial inflation of our physical 3-space, the curvature term \( 2/a^2 \) on the left-hand side of Eq. (6) will be negligible, and we will henceforth set this term to zero.

We will be looking for solutions (neglecting matter) in which physical 3-space is inflating at the present epoch during which \( b(t) \) is fixed at \( b_{0} \), or in the primordial epoch just after the quantum birth of the universe during which \( b(t) \) is inflating to its present value. For an inflating 3-space (without matter), \( p_{a} = -\rho \) and the Einstein equations become:

\[
3\ddot{a} + D\ddot{b} = \frac{8\pi G}{D+2} [-(D-2)\rho - Dp_{b}]
\]

\[
\frac{\ddot{a}}{a} + 2\dddot{a} + D\ddot{b} \frac{b}{a} + \frac{2}{a^2} = \frac{8\pi G}{D+2} [-(D-2)\rho - Dp_{b}]
\]

\[
\frac{\ddot{b}}{b} + (D-1)\frac{\dot{b}^2}{b^2} + 3\dddot{a} \frac{b}{a} = \frac{8\pi G}{D+2} [4\rho + 2p_{b}].
\]

The energy density and pressures on the right-hand sides of Eqs. (8)–(10) are derivable from the internal energy \( U = U(a, b) \):

\[
\rho = \frac{U}{V}, \quad p_{a} = -\frac{a\partial U/\partial a}{3V}, \quad p_{b} = -\frac{b\partial U/\partial b}{DV}
\]
where \( \mathcal{V} = \Omega_3 a^3 \tilde{\Omega}_2 b^2 \) is the volume of \((3 + D)\)-space and \( \Omega_3 \) denotes the volume of the unit 3-sphere.

We will consider a potential \( V(b) \) for the radius \( b(t) \) in the internal energy

\[
U(a, b) = \Omega_3 a^3 M^4 V(b) \tag{12}
\]

(at zero temperature) which will produce sufficient primordial inflation to solve the horizon, flatness, homogeneity, isotropy, and monopole problems, and which will stabilize \( b \) at \( b_0 = \frac{M_P}{M^2} \sim 0.1 \text{ mm} \), with a vanishing cosmological constant. Note that if \( p_a \) is to equal \(-\rho\), then \( U \) must be proportional to \( a^3 \), and that \( V(b) \) is dimensionless.

The potential \( V(b) \) will generate a potential \( B(b) \) with the right-hand side of the Einstein equation (10) equal to \(-B'(b)/b\). If \( B(b) \) is sufficiently flat near \( b \sim 1/M \), then many e-folds of inflation will occur in our physical 3-space as \( b(t) \) rolls from \( 1/M \) to \( b_0 \).

Quantum fields will be periodic in the compactified space, producing a Casimir effect \[6\] in the compactified space and in our three-dimensional world. Adding a Casimir \((C)\) term to the internal energy

\[
U_C(a, b) = \Omega_3 a^3 \left( \frac{k}{b^4} + M^4 V(b) \right) \tag{13}
\]

from vacuum fluctuations in the compactified space will perturb \( b(t_0) \) very slightly away from \( b_0 \) and generate a residual present-day cosmological constant \( \rho_\Lambda = k/b_0^4 \). The sign and magnitude of the constant \( k \) depend on the particle content and structure of the underlying quantum gravity theory. The magnitude of \( k \) may be expected to be roughly in the range \( 10^{-7} - 10^{-3} \) based on the analysis of Candelas and Weinberg \[6\], who calculated the one-loop Casimir contribution from massless scalar and spin-\(\frac{1}{2}\) particles in \((4 + D)\)-dimensional gravitation with an odd number of extra dimensions \( D \) curled up near the Planck length. In their work, \( k \) is positive for a single massless real scalar field for odd dimensions \( 3 \leq D \leq 19 \), but may be positive or negative. For our model to produce a positive present-day cosmological constant, we will need \( k > 0 \).

\[3\] A logarithmic dependence \( \ln(M^2 b^2) \) can be absorbed into the definition of \( k \) without changing the conclusions below.
2 Primordial Inflation

In this section, we briefly review the cosmological results for primordial inflation of Refs. [7, 8] for the ADD model with internal energy $U$, and check that the Casimir terms in the Einstein equations when $U$ is replaced by $U_C$ do not qualitatively change the primordial cosmological picture.

The Einstein equations with the internal energy given by $U$ in Eq. (12) take the form

$$3\dddot{a}/a + 2\ddot{b}/b = 3\dot{H} + 3H^2 + 2\dot{H}_b + 2H_b^2 = V'(b)/4b$$ (14)

$$\dddot{a}/a + 2\ddot{a}/a^2 + 2\dot{a}\dot{b}/ab = \dddot{H} + 3H^2 + 2HH_b = V'(b)/4b$$ (15)

$$\ddot{b}/b + \dot{b}^2 + 3\dot{a}\dot{b}/ab = \ddot{H}_b + 2H_b^2 + 3HH_b = V(b)/b^2 - V'(b)/4b \equiv -B'(b)/b$$ (16)

where the Hubble parameters $H \equiv \dot{a}/a$ and $H_b \equiv \dot{b}/b$. For a vanishing present-day cosmological constant, $V'(b_0) = 0$ from Eq. (15). Eq. (16) then implies $V(b_0) = 0$ to stabilize $b(t_0)$ at $b_0$.

To summarize the successful phenomenology of Ref. [8]: The ADD model can produce sufficient inflation ($\gg 70$ e-folds) to solve the cosmological problems for a class of potentials $V(b)$ which satisfy

$$H^{-1} \sim H_b^{-1} \geq 1/M$$ (17)

at the beginning of inflation at the quantum birth of the universe when $a \sim b \sim 1/M$, and

$$H \gg H_b, \quad \dot{H}_b \ll H^2$$ (18)

during the initial stages of inflation. The correct magnitude and approximate scale invariance of density perturbations $\delta\rho/\rho = 2 \times 10^{-5}$ are created if at an intermediate stage of inflation when $b(t) \sim 10^{3/2}/M \ll b_0$, $H_b \approx H/100$. There may be a period of contraction (similar to the vacuum Kasner solutions) of our physical 3-space, but for $D = 2$, the amount of contraction of $a(t)$ is at most 7 e-folds, so the contraction phase does not invalidate the solution of the flatness problem.

Replacing $U$ by $U_C$ in Eq. (13) introduces Casimir terms into the Einstein equations:

$$3\dot{H} + 3H^2 + 2\dot{H}_b + 2H_b^2 = -k/M^4b^6 + V'(b)/4b$$ (19)
\[
\dot{H} + 3H^2 + 2HH_b = -\frac{k}{M^4 b^6} + \frac{V'(b)}{4b}
\]  
(20)

\[
\dot{H}_b + 2H_b^2 + 3HH_b = \frac{2k}{M^4 b^6} + \frac{V(b)}{b^2} + \frac{V'(b)}{4b} \equiv -\frac{B_C'(b)}{b}.
\]  
(21)

The Casimir terms do not qualitatively change the primordial inflationary period of the ADD model, since initially

\[
\frac{k}{M^4 b^6} \approx kM^2 \ll M^2 \sim H^2 \sim H_b^2
\]  
(22)

and in the intermediate stage of inflation

\[
\frac{k}{M^4 b^6} \approx 10^{-9}kM^2 \ll 10^{-11}M^2 \sim H_b^2 \sim 10^{-4}H^2
\]  
(23)

for \(k \lesssim 10^{-3}\), using the estimates in Ref. [8].

### 3 Present-Day Cosmological Constant

In the present epoch, the internal dimensions have a fixed radius \(b(t_0) \gg 1/M\) and \(H_b = 0\). Without the Casimir terms, the static solution for \(b(t_0)\) requires \(V(b_0) = 0 = V'(b_0)\). In our model, vacuum fluctuations in the compactified space perturb \(b_0\) very slightly to \(\tilde{b}_0\), producing a small cosmological constant in our three-dimensional world. We assume that the potential \(V(b)\) is independent of the Casimir effect, so that \(V(b_0)\) and \(V'(b_0)\) still equal zero.

The Einstein equations with Casimir contributions for an inflating 3-space now take the form

\[
3H_0^2 = -\frac{k}{M^4 b_0^6} + \frac{V''(\tilde{b}_0)}{4b_0^2}
\]  
(24)

\[
0 = \frac{2k}{M^4 b_0^6} + \frac{V(\tilde{b}_0)}{b_0^2} - \frac{V'(\tilde{b}_0)}{4b_0}.
\]  
(25)

Setting \(\tilde{b}_0 = (1 + \delta)b_0\) and solving Eq. (25) to order \(\delta \sim M^4/M_P^4\) yields

\[
\frac{\delta}{4}V''(b_0) + O(\delta^2) = \frac{2k}{M^4 b_0^6}
\]  
(26)

or

\[
\tilde{b}_0 \approx \left(1 + \frac{8k}{M^4 b_0^6 V''(b_0)}\right)b_0 = \left(1 + O\left(\frac{kM^4}{M_P^4}\right)\right)b_0
\]  
(27)
where $V''(b_0) \sim 1/b_0^5 = M^4/M_P^2$. Eq. (24) then predicts a present-day cosmological term

$$3H_0^2 = \frac{\delta}{4} V''(b_0) - \frac{k}{M^4 b_0^6} + O(\delta^2) = \frac{k}{M^4 b_0^6} + O(\delta^2)$$  \tag{28}$$
or, in other words,

$$H_0^2 = \frac{8\pi G}{3} \rho_\Lambda, \quad \rho_\Lambda = \frac{k}{b_0^6} = \frac{k M^8}{M_P^4}. \tag{29}$$

This cosmological term will $\approx 0.7 \rho_c$ if $b_0 \approx k^{1/4} 0.09$ mm, or equivalently if $M \approx 2.4$ TeV/$k^{1/8}$.

Note that the Casimir effect has caused the stabilized radius $b_0$ to increase slightly, yielding a positive present-day cosmological constant.

The canonically normalized “radion” field $\varphi(t) = 2M^2 b(t)$. The mass squared of the radion field is

$$m^2_\varphi = M^4 \left. \frac{d^2 V}{d\varphi^2} \right|_{\varphi_0} \sim \frac{M^4}{M_P^4}$$  \tag{30}$$

which must be positive at $\varphi_0 = 2M^2 b_0 = 2M_P$ to have a linearly stable $b_0$ solution [4].

The stability properties of $B_C(b)$ in Eq. (21) are the same as of $B(b)$ in Eq. (16): the respective solutions with $b(t_0) = b_0$ and $\tilde{b}_0$ are linearly stable if the radion mass squared is positive, since the radion mass squared including the Casimir contribution

$$m^2_{\varphi, C} = m^2_\varphi + \frac{5kM^8}{M_P^4} \sim \frac{M^4}{M_P^4} \left(1 + \frac{5kM^4}{M_P^4}\right)$$  \tag{31}$$
is positive if $m^2_\varphi$ is, and are globally stable if the respective potentials $B(b)$ and $B_C(b)$ are, for example, concave upward (the simplest case), since

$$B_C(b) = B(b) + \frac{k}{2M^4 b_0^4} + \text{const} \tag{32}$$
is concave upward if $B$ is.

If the number of extra dimensions $D$ is allowed to be greater than two, the Einstein equations (24) and (25) for an inflating 3-space with static $b(t)$ change to

$$3H_0^2 = -\frac{k}{M^{D+2}b_0^{D+4}} - \frac{D - 2}{D + 2} \frac{V(\tilde{b}_0)}{M^{D-2}b_0^D} + \frac{V'(\tilde{b}_0)}{(D + 2)M^{D-2}b_0^{D-1}} \tag{33}$$

8
\[ 0 = \frac{4k}{D M^{D+2} b_0^{D+1}} + \frac{4}{D + 2} \frac{V \left( b_0 \right)}{M^{D-2} b_0^D} - \frac{2}{D(D + 2)} \frac{V' \left( b_0 \right)}{M^{D-2} b_0^{D-1}} \]  

(34)

but the result for the present-day cosmological constant has the same form

\[ \rho_\Lambda = \frac{k}{b_0^4} = \frac{k M^{4+\frac{D}{2}}}{M_P^2} \]  

(35)

where now \( b_0 \) satisfies \( b_0^D M^{D+2} = M_P^2 \). Thus \( \rho_\Lambda \) has the right parametric dependence \( M^8/M_P^4 \) only for \( D = 2 \).

4 Conclusion

The cosmological picture presented here joins smoothly onto the primordial inflation and big-bang cosmological pictures: The quantum birth of the universe begins with \( a \) and \( b \sim 1/M \). Many \( (\gg 70) \) e-folds of primordial inflation occur as the inflaton \( b(t) \) rolls down its potential to \( b_0 \). \( b(t) \) then undergoes damped oscillations about \( \tilde{b}_0 \), heating the universe up to a temperature \( T \) above the temperature for big-bang nucleosynthesis (BBN) and creating essentially all the matter and energy we see today. (See Refs. [1] and [11] for two differing views on the maximum value of \( T \), above which the evolution of the universe in ADD-type theories cannot be described by the radiation-dominated Friedmann-Robertson-Walker model.) At this point, the universe evolves according to the standard big-bang picture, expanding and cooling, with a fixed small cosmological constant \( \rho_\Lambda = k/b_0^4 \approx (2.3 \times 10^{-3} \text{ eV})^4 \).

This dark energy density is much less than the BBN energy density \( \sim (1 \text{ MeV})^4 \) and plays a role in the evolution of the universe only recently, long after the equality of energy density \( \sim (1 \text{ eV})^4 \) in matter and radiation. The radius \( b(t) \) of the compactified space has not changed since well before BBN.

Finally we note that if the stabilization potential \( V(b) \) vanishes at its global minimum, the resolution of the cosmic coincidences of Ref. [11] is naturally realized in the Casimir effect since parametrically \( \rho_\Lambda \sim M^8/M_P^4 \).

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