Free differential Lie Rota-Baxter algebras and Gröbner-Shirshov bases

Jianjun Qiu
School of Mathematics and Statistics, Lingnan Normal University
Zhanjiang 524048, P. R. China
jianjunqiu@126.com

Yuqun Chen†
School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
yqchen@scnu.edu.cn

Abstract: We establish the Gröbner-Shirshov bases theory for differential Lie Ω-algebras. As an application, we give a linear basis of a free differential Lie Rota-Baxter algebra on a set.

Key words: Gröbner-Shirshov basis, Lyndon-Shirshov word, differential Lie Rota-Baxter algebra

AMS 2000 Subject Classification: 16S15, 13P10, 16W99, 17A50

1 Introduction

Let $k$ be a field and $\lambda \in k$. A differential algebra of weight $\lambda$ or a $\lambda$-differential algebra (\cite{19,23,29}) is a $k$-algebra $(R, \cdot)$ together with a differential operator (of weight $\lambda$) $D : R \to R$ satisfying

\[ D(x \cdot y) = D(x) \cdot y + x \cdot D(y) + \lambda D(x) \cdot D(y), \quad x, y \in R. \]

The differential algebras were first studied by J.F. Ritt \cite{29} and have developed to be an important branch of mathematics in both theory and applications (see for instance \cite{15,19,33}).

A Rota-Baxter algebra of weight $\lambda$ or $\lambda$-Rota-Baxter algebra (\cite{4,22,30}) is a $k$-algebra $(R, \cdot)$ together with a Rota-Baxter operator (of weight $\lambda$) $P : R \to R$ satisfying

\[ P(x) \cdot P(y) = P(x \cdot P(y)) + P(P(x) \cdot y) + \lambda P(x \cdot y), \quad x, y \in R. \]
The Rota-Baxter operator on an associative algebra initially appeared in probability [4] and then in combinatorics [30] and quantum field theory [14]. There are a number of studies on associative Rota-Baxter algebras on both commutative and noncommutative case. For more details we refer the reader to [22] and the references given there. The Rota-Baxter operator of weight 0 on a Lie algebra is also called the operator form of the classical Yang-Baxter equation [31]. The Lie Rota-Baxter algebras are closely related with the pre-Lie algebras. Recently, there are many results on Lie Rota-Baxter algebras and related topics (see for instance [2, 3, 21, 26, 28]).

Similarly to the relation between differential operator and integral operator as in the First Fundamental Theorem of Calculus, L. Guo and W. Keigher [23] introduced the notion of differential Rota-Baxter algebra which is a $k$-algebra $R$ together with a differential operator $D$ and a Rota-Baxter operator $P$ such that $DP = Id_R$.

As we known, the free objects of various varieties of linear algebras play an important role. Sometimes, it is difficult to give a linear basis of a free algebra, for example, it is an open problem to find a linear basis of a free Jordan algebra. A linear basis of the free differential associative (resp. commutative and associative) Rota-Baxter algebra on a set was given by L. Guo and W. Keigher [23]. In this paper, we apply the Gröbner-Shirshov bases method to construct a free differential Lie Rota-Baxter algebra. Especially, we give a linear basis of a free differential Lie Rota-Baxter algebra on a set.

Gröbner bases and Gröbner-Shirshov bases have been proved to be very useful in different branches of mathematics, which were invented independently by A.I. Shirshov [32], H. Hironaka [24] and B. Buchberger [13] on different types of algebras. For more details on the Gröbner-Shirshov bases and their applications, see for instance the surveys [8, 10], the books [1, 11, 16, 18] and the papers [9, 17, 20, 27, 28].

The $\Omega$-algebra was introduced by A.G. Kurosh [25]. A differential Lie $\Omega$-algebra over a field $k$ is a differential Lie algebra $L$ with a set of multilinear operators $\Omega$ on $L$. It is easy to see that a differential Lie Rota-Baxter algebra is a differential Lie $\Omega$-algebra with a single operator satisfying the Rota-Baxter relation.

The paper is organized as follows. In Section 2, we review the Gröbner-Shirshov bases theory for differential associative $\Omega$-algebras. In Section 3, we firstly construct a free differential Lie $\Omega$-algebra by the differential nonassociative Lyndon-Shirshov $\Omega$-words, which is a generalization of the classical nonassociative Lyndon-Shirshov words. Secondly, we establish the Gröbner-Shirshov bases theory for differential Lie $\Omega$-algebras. In Section 4, we obtain a Gröbner-Shirshov basis of a free $\lambda$-differential Lie Rota-Baxter algebra and then a linear basis of such an algebra is obtained by the Composition-Diamond lemma for differential Lie $\Omega$-algebras.
2 Gröbner-Shirshov bases for $\lambda$-differential associative $\Omega$-algebras

In this section, we briefly review the Gröbner-Shirshov bases theory for $\lambda$-differential associative $\Omega$-algebras, which can be found in [27].

2.1 Free $\lambda$-differential associative $\Omega$-algebras

Let $D$ be a 1-ary operator and $\Omega := \bigcup_{m=1}^{\infty} \Omega_m$, where $\Omega_m$ is a set of $m$-ary operators for any $m \geq 1$. For any set $Y$, we define the following notations:

- $S(Y)$: the set of all nonempty associative words on $Y$.
- $Y^*$: the set of all associative words on $Y$ including the empty word $1$.
- $Y^{**}$: the set of all nonassociative words on $Y$.
- $\Delta(Y) := \bigcup_{m=0}^{\infty} \{D^m(y) | y \in Y\}$, where $D^0(y) = y, y \in Y$.
- $\Omega(Y) := \bigcup_{m=1}^{\infty} \{\omega^{(m)}(y_1, y_2, \ldots, y_m) | y_i \in Y, 1 \leq i \leq m, \omega^{(m)} \in \Omega_m\}$.

Let $X$ be a set. Define the differential associative and nonassociative $\Omega$-words on $X$ as follows. For $n = 0$, define $(D, \Omega; X)_0 = S(\Delta(X))$, $(D, \Omega; X)_0 = (\Delta(X))^{**}$. For $n > 0$, define

$(D, \Omega; X)_n = S(\Delta(X \cup \Omega((D, \Omega; X)_{n-1})))$, 
$(D, \Omega; X)_n = (\Delta(X \cup \Omega((D, \Omega; X)_{n-1})))^{**}$.

Set

$(D, \Omega; X) = \bigcup_{n=0}^{\infty} (D, \Omega; X)_n$, 
$(D, \Omega; X) = \bigcup_{n=0}^{\infty} (D, \Omega; X)_n$.

The elements of $(D, \Omega; X)$ (resp. $(D, \Omega; X)$) are called differential associative (resp. nonassociative) $\Omega$-words on $X$. A differential associative $\Omega$-word $u$ is called prime if $u \in \Delta(X \cup \Omega((D, \Omega; X)))$.

Let $k$ be a field and $\lambda \in k$. A $\lambda$-differential associative $\Omega$-algebra over $k$ is a $\lambda$-differential associative $k$-algebra $R$ together with a set of multilinear operators $\Omega$ on $R$.

Let $DA(\Omega; X) = k(D, \Omega; X)$ be the semigroup algebra of $(D, \Omega; X)$. Let $u = u_1 u_2 \cdots u_t \in (D, \Omega; X)$, where each $u_i$ is prime. If $t = 1$, i.e. $u = D^i(u')$ for some $i \geq 0, u' \in X \cup \Omega((D, \Omega; X))$, then we define $D(u) = D^{i+1}(u')$. If $t > 1$, then we recursively define

$D(u) = D(u_1)(u_2 \cdots u_t) + u_1 D(u_2 \cdots u_t) + \lambda D(u_1)D(u_2 \cdots u_t)$. 

3
Extend linearly $D$ to $DA(\Omega; X)$. For any $\omega^{(m)} \in \Omega_m$, define

$$\omega^{(m)} : (D, \Omega; X)^m \rightarrow (D, \Omega; X), (u_1, u_2, \ldots, u_m) \mapsto \omega^{(m)}(u_1, u_2, \ldots, u_m)$$

and extend it linearly to $DA(\Omega; X)^m$.

**Theorem 2.1** ([27]) $(DA(\Omega; X), D, \Omega)$ is a free $\lambda$-differential associative $\Omega$-algebra on the set $X$.

### 2.2 Composition-Diamond lemma for $\lambda$-differential associative $\Omega$-algebras

Let $\ast$ is a symbol, which is not in $X$. By a differential $\ast$-$\Omega$-word we mean any expression in $(D, \Omega; X \cup \{\ast\})$ with only one occurrence of $\ast$. The set of all the differential $\ast$-$\Omega$-words on $X$ is denoted by $(D, \Omega; X)^\ast$. Let $\pi$ be a differential $\ast$-$\Omega$-word and $s \in DA(\Omega; X)$. Then we call $\pi|_{s} = \pi|_{s:s}$ a differential $s$-word.

Let $\deg(u)$ be the number of all occurrences of $x \in X$, $\omega \in \Omega$ and $D$ in $u$. If $u = u_1 u_2 \cdots u_m$, where $u_i$ is prime, then the breath of $u$, denoted by $\bre(u)$, is defined to be the number $m$. Define

$$\wt(u) = (\deg(u), \bre(u), u_1, u_2, \cdots, u_m).$$

Let $X$ and $\Omega$ be well-ordered sets and assume that $\omega > D$ for any $\omega \in \Omega$. We define the Deg-lex order $>_{\Omega}$ on $(D, \Omega; X)$ as follows. For any $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_m \in (D, \Omega; X)$, where $u_i, v_j$ are prime, define

$$u >_{\Omega} v \text{ if } \wt(u) > \wt(v) \text{ lexicographically,}$$

where if $u_i = \omega(u_{i1}, u_{i2}, \cdots, u_{id}), v_i = \theta(v_{i1}, v_{i2}, \cdots, v_{id}), \omega, \theta \in \{D\} \cup \Omega$ and $\deg(u_i) = \deg(v_i)$, then $u_i >_{\Omega} v_i$ if

$$(\omega, u_{i1}, u_{i2}, \cdots, u_{id}) > (\theta, v_{i1}, v_{i2}, \cdots, v_{id}) \text{ lexicographically.}$$

It is easy to check that $>_{\Omega}$ is a well order on $(D, \Omega; X)$. For any $0 \neq f \in DA(\Omega; X)$, let $\bar{f}$ be the leading term of $f$ with respect to the order $>_{\Omega}$. Let us denote $\lt(f)$ the coefficient of the leading term $\bar{f}$ of $f$.

For $1 \leq t \leq n$, define

$$I^t_n = \{(i_1, i_2, \cdots, i_n) \in \{0, 1\}^n | \sum i_j = t\}.$$

**Lemma 2.2** ([27]) If $u = u_1 u_2 \cdots u_n \in (D, \Omega; X)$, where each $u_i$ is prime, then

$$D(u) = \sum_{(i_1, i_2, \cdots, i_n) \in I^t_n} D^i(u_1)D^{i_2}(u_2) \cdots D^{i_n}(u_n)$$

$$+ \sum_{t=2}^n \sum_{(i_1, i_2, \cdots, i_n) \in I^t_n} \lambda^{t-1} D^i(u_1)D^{i_2}(u_2) \cdots D^{i_n}(u_n).$$
Lemma 2.3 \([27]\) Let \(u = u_1u_2 \cdots u_n \in \langle D, \Omega; X \rangle\), where each \(u_i\) is prime.

(a) If \(\lambda = 0\), then \(\overline{D^i(u)} = D^i(u_1)u_2 \cdots u_n\) and \(\text{lc}(D^i(u)) = 1\).

(b) If \(\lambda \neq 0\), then \(\overline{D^i(u)} = D^i(u_1)D^i(u_2) \cdots D^i(u_n)\) and \(\text{lc}(D^i(u)) = \lambda^{(n-1)i}\).

It follows that if \(u, v \in \langle D, \Omega; X \rangle\) and \(u >_{Dl} v\), then \(\overline{D(u)} > \overline{D(v)}\).

Proposition 2.4 \([27]\) For any \(u, v \in \langle D, \Omega; X \rangle, \pi \in \langle D, \Omega; X \rangle^*\), if \(u >_{Dl} v\), then \(\overline{\pi|u} >_{Dl} \overline{\pi|v}\).

If \(\overline{\pi|s} = \overline{\pi|s}_{\pi}\), where \(s \in DA(\Omega; X)\) and \(\pi \in \langle D, \Omega; X \rangle^*\), then \(\overline{\pi|s}\) is called a normal differential \(s\)-word. Note that not each differential \(s\)-word is a normal differential \(s\)-word, for example, if \(u = D(x)P(D^2(s))\) and \(s = xy\), where \(P \in \Omega\). \(x, y \in X\), then \(\overline{\pi|s}\) is not a normal differential \(s\)-word. However, if we take \(\pi' = D(x_1)P(s)\), then \(\overline{\pi|s} = \overline{\pi'|D^2(s)}\) and \(\overline{\pi'|D^2(s)}\) is a normal differential \(D^2(s)\)-word.

Lemma 2.5 \([27]\) For any differential \(s\)-word \(\overline{\pi|s}\), there exist \(i \geq 0\) and \(\pi'\) such that \(\overline{\pi|s} = \overline{\pi'|D^i(s)}\) and \(\overline{\pi'|D^i(s)}\) is a normal differential \(D^i(s)\)-word.

Let \(f, g \in DA(\Omega; X)\). There are two kinds of compositions.

(i) If there exists a \(w = \overline{D^i(f)a = bD^j(g)}\) for some \(a, b \in \langle D, \Omega; X \rangle\) such that \(\text{bre}(w) < \text{bre}(f) + \text{bre}(g)\), then we call

\[
(f, g)_w = \text{lc}(D^i(f))^{-1}D^i(f)a - \text{lc}(D^j(g))^{-1}bD^j(g)
\]

the intersection composition of \(f\) and \(g\) with respect to the ambiguity \(w\).

(ii) If there exists a \(\pi \in \langle D, \Omega; X \rangle^*\) such that \(w = \overline{D^i(f)} = \overline{\pi|D^i(g)}\), then we call

\[
(f, g)_w = \text{lc}(D^i(f))^{-1}D^i(f) - \text{lc}(D^j(g))^{-1}\pi|D^j(g)
\]

the inclusion composition of \(f\) and \(g\) with respect to the ambiguity \(w\).

Let \(S\) be a subset of \(DA(\Omega; X)\). Then the composition \((f, g)_w\) is called trivial modulo \((S, w)\) if

\[
(f, g)_w = \sum \alpha_i \pi_i|_{D^i(s_i)}
\]

where each \(\alpha_i \in k, \pi_i \in \langle D, \Omega; X \rangle^*, s_i \in S, \pi_i|_{D^i(s_i)}\) is a normal differential \(D^i(s_i)\)-word and \(\pi_i|_{D^i(s_i)} <_{Dl} w\). If this is the case, we write

\[
(f, g)_w \equiv_{ass} 0 \mod(S, w).
\]

In general, for any two polynomials \(p\) and \(q\), \(p \equiv_{ass} q \mod(S, w)\) means that \(p - q = \sum \alpha_i \pi_i|_{D^i(s_i)}\), where each \(\alpha_i \in k, \pi_i \in \langle D, \Omega; X \rangle^*, s_i \in S, \pi_i|_{D^i(s_i)}\) is a normal differential \(D^i(s_i)\)-word and \(\pi_i|_{D^i(s_i)} <_{Dl} w\).

A set \(S \subset DA(\Omega; X)\) is called a Gröbner-Shirshov basis in \(DA(\Omega; X)\) if any composition \((f, g)_w\) of \(f, g \in S\) is trivial modulo \((S, w)\).

5
Theorem 2.6 ([27], Composition-Diamond lemma for differential associative \(\Omega\)-algebras) Let \(S\) be a subset of \(DA(\Omega; X)\), \(Id_{DA}(S)\) the ideal of \(DA(\Omega; X)\) generated by \(S\) and \(\succ_{Dl}\) the Deg-lex order on \(D, \Omega; X\) defined as before. Then the following statements are equivalent:

(i) \(S\) is a Gröbner-Shirshov basis in \(DA(\Omega; X)\).

(ii) \(f \in Id_{DA}(S) \Rightarrow \bar{f} = \pi|_{D^{i}(s)}\) for some \(\pi \in \langle D, \Omega; X \rangle \ast, s \in S\) and \(i \geq 0\).

(iii) The set \(\operatorname{Irr}(S) = \{w \in \langle D, \Omega; X \rangle \mid w \neq \pi|_{D^{i}(s)}\), \(s \in S, i \geq 0, \pi|_{D^{i}(s)}\) is a normal differential \(D^{i}(s)\)-word\} is a linear basis of the differential associative \(\Omega\)-algebra \(DA(\Omega; X/S) := DA(\Omega; X)/Id_{DA}(S)\).

3 Gröbner-Shirshov bases for \(\lambda\)-differential Lie \(\Omega\)-algebras

3.1 Lyndon-Shirshov words

In this subsection, we review the concept and some properties of Lyndon-Shirshov words, which can be found in [7, 32].

For any \(u \in X^*\), let us denote by \(\deg(u)\) the degree (length) of \(u\). Let \(\succ\) be a well order on \(X\). Define the lex-order \(\succ_{lex}\) and the deg-lex order \(\succ_{deg-lex}\) on \(X^*\) with respect to \(\succ\) by:

(i) \(1 \succ_{lex} u\) for any nonempty word \(u\), and if \(u = x_i u'\) and \(v = x_j v'\), where \(x_i, x_j \in X\), then \(u \succ_{lex} v\) if \(x_i > x_j\), or \(x_i = x_j\) and \(u' \succ_{lex} v'\) by induction.

(ii) \(u \succ_{deg-lex} v\) if \(\deg(u) > \deg(v)\), or \(\deg(u) = \deg(v)\) and \(u \succ_{lex} v\).

A nonempty associative word \(w\) is called an associative Lyndon-Shirshov word on \(X^*\) if \(u \succ_{lex} v\) for any decomposition of \(u = uv\), where \(1 \neq u, v \in X^*\).

A nonassociative word \((u) \in X^{**}\) is said to be a nonassociative Lyndon-Shirshov word on \(X\) with respect to the lex-order \(\succ_{lex}\), if

(a) \(u\) is an associative Lyndon-Shirshov word on \(X\);

(b) if \((u) = ((v)(w))\), then both \((v)\) and \((w)\) are nonassociative Lyndon-Shirshov words on \(X\);

(c) if \((v) = ((v_1)(v_2))\), then \(v_2 \leq_{lex} w\).

Let \(ALSW(X)\) (resp. \(NLSW(X)\)) denote the set of all the associative (resp. nonassociative) Lyndon-Shirshov words on \(X\) with respect to the lex-order \(\succ_{lex}\). It is well known that for any \(u \in ALSW(X)\), there exists a unique
Shirshov standard bracketing way \([u]\) (see for instance [1]) on \(u\) such that \([u] \in NLSW(X)\). Then \(NLSW(X) = \{[u]|u \in ALSW(X)\}\).

Let \(k(X)\) be the free associative algebra on \(X\) over a field \(k\) and \(Lie(X)\) be the Lie subalgebra of \(k(X)\) generated by \(X\) under the Lie bracket \((uv) = uv - vu\).

It is well known that \(Lie(X)\) is a free Lie algebra on the set \(X\) and \(NLSW(X)\) is a linear basis of \(Lie(X)\).

### 3.2 Differential Lyndon-Shirshov \(Ω\)-words

Let \(>_{Dl}\) be the Deg-lex order on \((D, Ω; X)\) and \(\succ\) the restriction of \(>_{Dl}\) on \(Δ(X \cup Ω((D, Ω; X)))\). Define the differential Lyndon-Shirshov \(Ω\)-words on the set \(X\) as follows.

For \(n = 0\), let \(Z_0 := Δ(X)\). Define

\[
ALSW(D, Ω; X)_0 := ALSW(Z_0),
\]

\[
NLSW(D, Ω; X)_0 := NLSW(Z_0) = \{[u]|u \in ALSW(D, Ω; X)_0\}
\]

with respect to the lex-order \(\succ_{lex}\) on \((Z_0)^*\), where \([u]\) is the Shirshov standard bracketing way on \(u\).

Assume that we have defined

\[
ALSW(D, Ω; X)_{n-1},
\]

\[
NLSW(D, Ω; X)_n := NLSW(Z_n) := \{[u]|u \in ALSW(D, Ω; X)_{n-1}\}.
\]

Let \(Z_n := Δ(X \cup Ω(ALSW(D, Ω; X)_{n-1}))\). Define

\[
ALSW(D, Ω; X)_n := ALSW(Z_n).
\]

with respect to the lex-order \(\succ_{lex}\) on \(Z_n\). For any \(u \in Z_n\), define the bracketing way on \(u\) by

\[
[u] := \begin{cases} 
  u, & \text{if } u = D^i(x), x \in X, \\
  D^i(ω(m)([u_1], [u_2], \ldots, [u_m])), & \text{if } u = D^i(ω(m)(u_1, u_2, \ldots, u_m)).
\end{cases}
\]

Let \([Z_n] := \{[u]|u \in Z_n\}\). Thus, the order \(\succ\) on \(Z_n\) induces an order on \([Z_n]\) by \([u] \succ [v]\) if \(u \succ v\) for any \(u, v \in Z_n\). For any \(u = u_1u_2 \cdots u_t \in ALSW(D, Ω; X)_n\), where each \(u_i \in Z_n\), we define

\[
[u] := [[u_1][u_2] \cdots [u_t]]
\]

the Shirshov standard bracketing way on the word \([u_1][u_2] \cdots [u_t]\), which means that \([u]\) is a nonassociative Lyndon-Shirshov word on the set \{\([u_1][u_2] \cdots [u_t]\)\}. Define

\[
NLSW(D, Ω; X)_n := \{[u]|u \in ALSW(D, Ω; X)_n\}.
\]

It is easy to see that \(NLSW(D, Ω; X)_n = NLSW([Z_n])\) with respect to the lex-order \(\succ_{lex}\) on \([Z_n]^*\).
Set
\[ ALSW(D, \Omega; X) := \bigcup_{n=0}^{\infty} ALSW(D, \Omega; X)_n, \]
\[ NLSW(D, \Omega; X) := \bigcup_{n=0}^{\infty} NLSW(D, \Omega; X)_n. \]

Then, we have
\[ NLSW(D, \Omega; X) = \{ [u] | u \in ALSW(D, \Omega; X) \}. \]

The elements of \( ALSW(D, \Omega; X) \) (resp. \( NLSW(D, \Omega; X) \)) are called the differential associative (resp. nonassociative) Lyndon-Shirshov \( \Omega \)-words on the set \( X \).

### 3.3 Free \( \lambda \)-differential Lie \( \Omega \)-algebras

In this subsection, we prove that the set \( NLSW(D, \Omega; X) \) of all differential nonassociative Lyndon-Shirshov \( \Omega \)-words on \( X \) forms a linear basis of the free \( \lambda \)-differential Lie \( \Omega \)-algebra on \( X \).

A \( \lambda \)-differential Lie algebra is a Lie algebra \( L \) with a linear operator \( D : L \to L \) satisfying the differential relation
\[ D([xy]) = [D(x)y] + [xD(y)] + \lambda[D(x)D(y)], \quad x, y \in L. \]

A \( \lambda \)-differential Lie \( \Omega \)-algebra is a \( \lambda \)-differential Lie algebra \( L \) with a set of multilinear operators \( \Omega \) on \( L \).

Let \( (R, \cdot, D, \Omega) \) be a \( \lambda \)-differential associative \( \Omega \)-algebra. Then it is easy to check that \( (R, [\cdot, \cdot], D, \Omega) \) is a \( \lambda \)-differential Lie \( \Omega \)-algebra under the Lie bracket \( [a, a'] = a \cdot a' - a' \cdot a, \quad a, a' \in R \).

Let \( DLie(\Omega; X) \) be the \( \lambda \)-differential Lie \( \Omega \)-subalgebra of \( DA(\Omega; X) \) generated by \( X \) under the Lie bracket \( (uv) = uv - vu \).

Similar to the proofs of Lemma 2.6 and Theorem 2.8 in [28], we have the following results.

**Lemma 3.1** If \( u \in ALSW(D, \Omega; X) \), then \( [u] = u \) with respect to the order \( \succ_{Dl} \) on \( <D, \Omega; X> \).

**Theorem 3.2** \( DLie(\Omega; X) \) is a free \( \lambda \)-differential Lie \( \Omega \)-algebra on the set \( X \) and \( NLSW(D, \Omega; X) \) is a linear basis of \( DLie(\Omega; X) \).

### 3.4 Composition-Diamond lemma for differential Lie \( \Omega \)-algebras

In this subsection, we establish the Composition-Diamond lemma for differential Lie \( \Omega \)-algebras.
Lemma 3.3 Let $\pi \in \langle D, \Omega; X \rangle^*$ and $\pi|_v \in ALSW(D, \Omega; X)$. Then there is a $\pi' \in \langle D, \Omega; X \rangle^*$ and $c \in \langle D, \Omega; X \rangle$ such that

\[
[\pi|_v] = [\pi'|_{\ve v}],
\]

where $c$ may be empty. Let

\[
[\pi|_v]_v = [\pi'|_{\ve v}][\ve v_{c_1}] \cdots [\ve v_{c_m}]
\]

where $c = c_1c_2 \cdots c_m$ with each $c_i \in ALSW(D, \Omega; X)$ and $c_i \leq_{\text{lex}} c_{i+1}$. Then,

\[
[\pi|_v]_v = \pi|_v + \sum \alpha_i \pi_i|_v,
\]

where each $\alpha_i \in k$ and $\pi_i v <_{Di} \pi v$. It follows that $[\pi|_v]_v = \pi|_v$ with respect to the order $>_{Di}$.

Proof. The proof is the same as the one of Lemma 3.2 in [28].

Let $0 \neq f \in DLie(\Omega; X) \subseteq DA(\Omega; X)$. If $\pi|_f \in ALSW(D, \Omega; X)$, then we call

\[
[\pi|_f]_f = [\pi|_f][f]_{\pi|_f} = f
\]

a special normal differential $f$-word.

Corollary 3.4 Let $f \in DLie(\Omega; X)$ and $\pi|_f \in ALSW(D, \Omega; X)$. Then

\[
[\pi|_f]_f = \pi|_f + \sum \alpha_i \pi_i|_f,
\]

where each $\alpha_i \in k$ and $\pi_i|_f <_{Di} \pi|_f$.

Let $f, g \in DLie(\Omega; X)$. There are two kinds of compositions.

(i) If there exists a $w = D^i(f) a = bD^j(g)$ for some $a, b \in \langle D, \Omega; X \rangle$ such that $bre(w) < bre(f) + bre(g)$, then we call

\[
\langle f, g \rangle_w = lc(D^i(f))^{-1}[D^i(f) a]_{D^i(f)} - lc(D^j(g))^{-1}[bD^j(g)]_{D^j(g)}
\]

the intersection composition of $f$ and $g$ with respect to the ambiguity $w$.

(ii) If there exists a $\pi \in \langle D, \Omega; X \rangle^*$ such that $w = D^i(f) = \pi|_{D^i(g)}$, where $\pi|_{D^i(g)}$ is a normal differential $D^i(g)$-word, then we call

\[
\langle f, g \rangle_w = lc(D^i(f))^{-1}D^i(f) - lc(D^j(g))^{-1}[\pi|_{D^j(g)}]_{D^j(g)}
\]

the inclusion composition of $f$ and $g$ with respect to the ambiguity $w$.
If \( S \) is a subset of \( DLie(\Omega; X) \), then the composition \( (f, g)_w \) is called trivial modulo \( (S, w) \) if
\[
(f, g)_w = \sum \alpha_i [\pi_i]_{Dli(s_i)} \frac{D^i(s_i)}{\pi_i} \mod \pi_i <_{Dli} w.
\]
where each \( \alpha_i \in k \), \( s_i \in S \), \([\pi_i]_{Dli(s_i)} \frac{D^i(s_i)}{\pi_i} \) is a special normal differential \( D^i(s_i) \)-word and \( \pi_i \frac{D^i(s_i)}{\pi_i} <_{Di} w \). If this is the case, then we write
\[
(f, g)_w \equiv 0 \mod (S, w).
\]

In general, for any two polynomials \( p \) and \( q \), \( p \equiv q \mod (S, w) \) means that \( p - q = \sum \alpha_i [\pi_i]_{Dli(s_i)} \frac{D^i(s_i)}{\pi_i} \) where each \( \alpha_i \in k \), \( \pi_i \in \langle D, \Omega; X \rangle \), \( s_i \in S \), \([\pi_i]_{Dli(s_i)} \frac{D^i(s_i)}{\pi_i} \) is a normal differential \( D^i(s_i) \)-word and \( \pi_i \frac{D^i(s_i)}{\pi_i} <_{Di} w \).

**Definition 3.5** A set \( S \subset DLie(\Omega; X) \) is called a Gröbner-Shirshov basis in \( DLie(\Omega; X) \) if any composition \( (f, g)_w \) of \( f, g \in S \) is trivial modulo \( (S, w) \).

**Lemma 3.6** Let \( f, g \in DLie(\Omega; X) \). Then
\[
(f, g)_w - (f, g)_w \equiv_{ass} 0 \mod (\{ f, g \}, w).
\]

**Proof.** If \( (f, g)_w \) and \( (f, g)_w \) are compositions of intersection, where \( w = D^i(f)a = bD^i(g) \), then
\[
(f, g)_w = lc(D^i(f))^{-1}[D^i(f)\frac{D^i(f)}{a}]_{D^i(f)} - lc(D^i(g))^{-1}[bD^i(g)\frac{D^i(g)}{b}]_{D^i(g)}
\]
\[
= lc(D^i(f))^{-1}D^i(f)b + \sum \alpha_i\alpha_iD^i(f)a'_i - lc(D^i(g))^{-1}bD^i(g)j - \sum \beta_jb_jD^i(g)b'_j
\]
\[
= (f, g)_w + \sum \alpha_i\alpha_iD^i(f)a'_i - \sum \beta_jb_jD^i(g)b'_j,
\]
where \( \alpha_iD^i(f)a'_i, b_jD^i(g)b'_j <_{Dli} w \). It follows that
\[
(f, g)_w - (f, g)_w \equiv_{ass} 0 \mod (\{ f, g \}, w).
\]

If \( (f, g)_w \) and \( (f, g)_w \) are compositions of inclusion, where \( w = f = \pi \frac{D^i(g)}{\pi_i} \), then
\[
(f, g)_w = f - lc(D^i(g))^{-1}[\pi]_{D^i(g)} \frac{D^i(g)}{\pi_i} = f - lc(D^i(g))^{-1}[\pi]_{D^i(g)} - \sum \alpha_i\pi_i \frac{D^i(g)}{\pi_i},
\]
where \( \pi_i \frac{D^i(g)}{\pi_i} <_{Di} w \). It follows that
\[
(f, g)_w - (f, g)_w \equiv_{ass} 0 \mod (\{ f, g \}, w).
\]
The proof is complete.

**Lemma 3.7** Let \( S \subset DLie(\Omega; X) \subset DA(\Omega; X) \). Then the following two statements are equivalent:

10
(i) $S$ is a Gröbner-Shirshov basis in $DLie(\Omega; X)$,

(ii) $S$ is a Gröbner-Shirshov basis in $DA(\Omega; X)$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose that $S$ is a Gröbner-Shirshov basis in $DLie(\Omega; X)$. Then, for any composition $(f, g)_w$, we have

$$
(f, g)_w = \sum \alpha_i [\pi_i |_{D^{i_1(\epsilon_i)}}]_{D^{i_1(\epsilon_i)}},
$$

where each $\alpha_i \in k$, $s_i \in S$, $\pi_i |_{D^{i_1(\epsilon_i)}} < D_i w$. By Corollary 3.4, we have

$$
(f, g)_w = \sum \beta_i \pi_i |_{D^{i_1(\epsilon_i)}},
$$

where each $\beta_i \in k$, $s_i \in S$, $\pi_i |_{D^{i_1(\epsilon_i)}} < D_i w$. Therefore, by Lemma 3.6, we can obtain that

$$(f, g)_w \equiv \text{ass} 0 \mod (S, w).$$

Thus, $S$ is a Gröbner-Shirshov basis in $DA(\Omega; X)$.

(ii) $\Rightarrow$ (i). Assume that $S$ is a Gröbner-Shirshov basis in $DA(\Omega; X)$. Then, for any composition $(f, g)_w$ in $S$, we have $(f, g)_w \in DLie(\Omega; X)$ and $(f, g)_w \in Id_{DA}(S)$. By Theorem 2.6, $(f, g)_w = \pi_1 |_{D^{i_1(\epsilon_i)}} \in ALSW(D, \Omega; X)$. Let

$$
h_1 = (f, g)_w - \alpha_1 [\pi_1 |_{D^{i_1(\epsilon_i)}}]_{D^{i_1(\epsilon_i)}},
$$

where $\alpha_1$ is the coefficient of $(f, g)_w$. Then, $\overline{h_1} <_{D_i} \overline{(f, g)_w}$, $h_1 \in Id_{DA}(S)$ and $h_1 \in DLie(\Omega; X)$. Now, the result follows from induction on $(f, g)_w$. □

**Lemma 3.8** Let $S \subset DLie(\Omega; X)$ and

$Irr(S) = \{[w]|w \in ALSW(D, \Omega; X), w \neq \pi |_{D^{i_1(\epsilon_i)}} s \in S, \pi \in \langle D, \Omega; X \rangle^*, i \geq 0 \}.$

Then, for any $h \in DLie(\Omega; X)$, $h$ can be expressed by

$$
h = \sum \alpha_i [u_i] + \sum \beta_j [\pi_j |_{D^{i_j(\epsilon_j)}}]_{D^{i_j(\epsilon_j)}},
$$

where each $\alpha_i$, $\beta_j \in k$, $u_i \in ALSW(D, \Omega; X), u_i \leq_{D_i} \overline{h}$ and $s_j \in S$, $\pi_j |_{D^{i_j(\epsilon_j)}} \leq_{D_i} \overline{\pi_j}$. □

**Proof.** By induction on $\overline{h}$, we can obtain the result.

The following theorem is the Composition-Diamond lemma for differential Lie $\Omega$-algebras. It is a generalization of Shirshov’s Composition lemma for Lie algebras [32], which was specialized to associative algebras by L.A. Bokut [6], see also G.M. Bergman [5] and B. Buchberger [12, 13].

11
Theorem 3.9 (Composition-Diamond lemma for differential Lie $\Omega$-algebras)

Let $S \subset DLie(\Omega; X)$ be a nonempty set and $Id_{DLie}(S)$ the ideal of $DLie(\Omega; X)$ generated by $S$. Then the following statements are equivalent:

(I) $S$ is a Gröbner-Shirshov basis in $DLie(\Omega; X)$.

(II) $f \in Id_{DLie}(S) \Rightarrow \bar{f} = \pi_{\mathcal{D}_{\lambda}} \in ALSW(D, \Omega; X)$ for some $s \in S$, $\pi \in \langle D, \Omega; X \rangle^*$ and $i \geq 0$.

(III) The set

$Irr(S) = \{ [w] \in ALSW(D, \Omega; X), w \neq \pi_{\mathcal{D}_{\lambda}}, s \in S, \pi \in \langle D, \Omega; X \rangle^*, i \geq 0 \}$

is a linear basis of the $\lambda$-differential Lie $\Omega$-algebras $DLie(\Omega; X|S)$.

Proof. (I) $\Rightarrow$ (II). Since $f \in Id_{DLie}(S) \subseteq Id_{DA}(S)$, by Lemma 3.7 and Theorem 2.6 we have $\bar{f} = \pi_{\mathcal{D}_{\lambda}}$ for some $s \in S$, $\pi \in \langle D, \Omega; X \rangle^*$ and $i \geq 0$.

(II) $\Rightarrow$ (III). Suppose that $\sum \alpha_i [u_i] = 0$ in $DLie(\Omega; X|S)$, where each $[u_i] \in Irr(S)$ and $u_i >_{\mathcal{D}_{\lambda}} u_{i+1}$. That is, $\sum \alpha_i [u_i] \in Id_{DLie}(S)$. Then each $\alpha_i$ must be 0. Otherwise, say $\alpha_1 \neq 0$, since $\sum \alpha_i [u_i] = u_1$ and by (II), we have $[u_1] \in Irr(S)$, a contradiction. Therefore, $Irr(S)$ is linear independent. By Lemma 3.8 $Irr(S)$ is a linear basis of $DLie(\Omega; X|S) = DLie(\Omega; X)/Id_{DLie}(S)$.

(III) $\Rightarrow$ (I). For any composition $(f, g)_w$ with $f, g \in S$, we have $(f, g)_w \in Id_{DLie}(S)$. Then, by (III) and by Lemma 3.8

$$(f, g)_w = \sum \beta_j [\pi_{\mathcal{D}_{\lambda}}] <_{\mathcal{D}_{\lambda}}$$

where each $\beta_j \in k$, $\pi_{\mathcal{D}_{\lambda}} <_{\mathcal{D}_{\lambda}} w$. This proves that $S$ is a Gröbner-Shirshov basis in $DLie(\Omega; X)$.

\[\square\]

4 Free $\lambda$-differential Rota-Baxter Lie algebras

In this section, by using Theorem 3.9 we give a Gröbner-Shirshov basis of a free $\lambda$-differential Rota-Baxter Lie algebra on a set $X$ and then a linear basis of such an algebra is obtained.

4.1 Gröbner-Shirshov bases for free $\lambda$-differential Lie Rota-Baxter algebras

Let $k$ be a field and $\lambda \in k$. A differential Lie Rota-Baxter algebra of weight $\lambda$, called also $\lambda$-differential Lie Rota-Baxter algebra, is a Lie algebra $L$ with two linear operators $P, D : L \rightarrow L$ such that for any $x, y \in L$,

(a) (Rota-Baxter relation) $[P(x)P(y)] = P([xP(y)]) + P([P(x)y]) + \lambda P([xy])$:
(b) (differential relation) $D([xy]) = [D(x)y] + [xD(y)] + \lambda[D(x)D(y)];$

(c) (section relation) $D(P(x)) = x.$

It is easy to see that any $\lambda$-differential Lie Rota-Baxter algebra is a $\lambda$-
differential Lie $\{P\}$-algebra satisfying the relations (a) and (c).

Let $DLie(\{P\}; X)$ be the free $\lambda$-differential Lie $\{P\}$-algebra on the set $X$
and write

$$g(u) := D(P([u])) - [u],$$

$$f(u, v) := [P([u]P([v])) - P([u]P([v])) - P(P([u][v])) - \lambda P([u][v]), u >_{dl} v,$$

where $u, v \in ALSW(D, \{P\}; X)$. Set

$$S = \{f(u, v), g(w)|u, v, w \in ALSW(D, \{P\}; X), u >_{dl} v\}.$$ 

It is clear that $DRBL(X) := DLie(\{P\}; X|S)$ is a free $\lambda$-differential Lie 
Rota-Baxter algebra on $X$. For any $f \in DLie(\{P\}; X)$, let us denote $r(f) := f - lc(f)|f\rangle$.

**Lemma 4.1** The set $S_1 := \{D(P([u])) - [u]|u \in ALSW(D, \{P\}; X)\}$ is a Gröbner-Shirshov basis in $DLie(\{P\}; X)$.

**Proof.** It is easy to check that $S_1$ is a Gröbner-Shirshov basis in $DLie(\{P\}; X)$.

□

**Lemma 4.2** Let $u, v \in ALSW(D, \{P\}; X)$ and $u >_{dl} v$.

(a) If $\lambda \neq 0$ and $j > 0$, then

$$D^j(f(u, v)) = \lambda^j(D^j(f(u, v)) - (D^{j-1}(f(u, v))D^{j-1}([v]))) \mod(S_1, D^j(f(u, v))).$$

(b) If $\lambda = 0$ and $j > 0$, then

$$D^j(f(u, v)) = \lambda^j(D^j(f(u, v)) - (D^{j-1}(f(u, v))P([v])) \mod(S_1, D^j(f(u, v))).$$

**Proof.** (a) The proof is by induction on $j$. For $j = 1$, we have

$$D(f(u, v)) = D((P([u]P([v]))) - D(P([u]P([v]))) - D(P([u][v]))) - \lambda D(P([u][v])))$$

$$\equiv \lambda(D(P([u]))D(P([v]))) - \lambda D(P([u][v])))$$

$$\equiv \lambda(D(f(u, v))) - ([u][v])) \mod(S_1, D(f(u, v))).$$

Assume that the result is true for $j - 1, j \geq 2$, i.e.

$$D^{j-1}(f(u, v)) = \lambda^{j-1}(D^{j-1}(P([u]))D^{j-1}(P([v])))$$

$$- \lambda^{j-1}(D^{j-2}([u])D^{j-2}([v])) + \sum \alpha_i [\pi_i|_{D^{j-1}(f(u, v))}D^{j-1}(f(u, v))).$$
where each $\alpha_i \in k$, $s_i \in S_1$, $\pi_i \mid_{D^{i}(P(v))} <_{Di} D^{j-1}(P(u))D^{j-1}(P(v))$. Since $S_1$ is a Gröbner-Shirshov basis in $DLie(\{P\}; X)$,

$$D(\sum \alpha_i \pi_i \mid_{D^{i}(u)} \mid_{D^{i}(v)}) = \sum \beta_i \sigma_i \mid_{D^{i}(u)} \mid_{D^{i}(v)},$$

where each $\beta_i \in k$, $s_i \in S_1$, $\sigma_i \mid_{D^{i}(u)} \mid_{D^{i}(v)}$ is a special normal differential $D^k(s_i)$-word. By Lemma 2.2

$$\sigma_i \mid_{D^{i}(u)} \mid_{D^{i}(v)} \leq_{Di} D((D^{j-1}(P([u]))D^{j-1}(P([v]))) = D^j(P(u))D^j(P(v)).$$

Thus, we have

\[
\begin{align*}
D^j(f(u, v)) &= D(D^{j-1}(f(u, v)) \\
&= \lambda^{-1}(D^{j-1}(P([u]))D^{j-1}(P([v]))) - \lambda^{-1}D((D^{j-2}(P([u]))D^{j-2}(P([v]))) \\
&= \lambda^j(D^j(P([u]))D^j(P([v]))) + \lambda^{-1}(D^{j-1}(P([u]))D^{j-1}(P([v]))) \\
&+ \lambda^{-1}(D^{j-2}(P([u]))D^{j-2}(P([v]))) - \lambda^{j-1}(D^{j-1}(P([u]))D^{j-1}(P([v]))) \\
&= \lambda^j(D^j(P([u]))D^j(P([v]))) + \lambda^{-1}(D^{j-1}(P([u]))D^{j-1}(P([v]))) \\
&+ \lambda^{-1}(D^{j-2}(P([u]))D^{j-2}(P([v]))) - \lambda^{j-1}(D^{j-1}(P([u]))D^{j-1}(P([v]))) \\
&= \lambda^j(D^j(P([u]))D^j(P([v]))) - \lambda^{j-1}(D^{j-1}(P([u]))D^{j-1}(P([v]))) \\
&= \lambda^j((D^j(f(u, v)) - (D^{j-1}(P([u]))D^{j-1}(P([v]))) \mod(S_1, D^j(f(u, v))).
\end{align*}
\]

(b) The proof is similar to Case (a). \qed

**Theorem 4.3** With the order $\succ_{Di}$ on $\langle D, \{P\}; X \rangle$ defined as before, the set $S$ is a Gröbner-Shirshov basis in $DLie(\{P\}; X)$.

**Proof.** There are two cases $\lambda \neq 0$ and $\lambda = 0$ to consider.

Case 1. For $\lambda \neq 0$, all possible compositions of the polynomials in $S$ are list as below:

\[
\begin{align*}
\langle g(\pi \mid_{D^{j}(P(u))} \mid_{D^{j}(P(v))}), g(v) \rangle_{w_1}, & \quad w_1 = D^j(D(P(\pi \mid_{D^{j}(P(u))} \mid_{D^{j}(P(v))})), \\
\langle g(\pi \mid_{D^{j}(P(u))D^{j}(P(v))}, f(u, v) \rangle_{w_2}, & \quad w_2 = D^j(D(P(\pi \mid_{D^{j}(P(u))D^{j}(P(v))}))), \\
\langle f(u, v), g(v) \rangle_{w_3}, & \quad w_3 = D^j(P(u))D^j(P(v)), \quad l > 0, \\
\langle f(u, v), g(u) \rangle_{w_4}, & \quad w_4 = D^j(P(u))D^j(P(v)), \quad l > 0, \\
\langle f(\pi \mid_{D^{j}(P(u))}, v), g(u) \rangle_{w_5}, & \quad w_5 = D^j(P(\pi \mid_{D^{j}(P(u))} \mid_{D^{j}(P(v))})))D^j(P(v)), \\
\langle f(u, \pi \mid_{D^{j}(P(u))}), g(v) \rangle_{w_6}, & \quad w_6 = D^j(P(u))D^j(P(\pi \mid_{D^{j}(P(u))})), \\
\langle f(u, v), f(v, w) \rangle_{w_7}, & \quad w_7 = D^j(P(u))D^j(P(v))D^j(P(w)),
\end{align*}
\]
\langle f(\pi_{D_i(P(w))}^{D_j(P(w))}), w \rangle, f(u, v) \rangle_{w_8}, \quad w_8 = D^j(P(\pi_{D_i(P(w))}^{D_j(P(w))})),
\langle f(u, \pi_{D_i(P(w))}^{D_j(P(w))}), f(v, w) \rangle_{w_9}, \quad w_9 = D^j(P(u))D^j(P(\pi_{D_i(P(w))}^{D_j(P(w))}))
\rangle,

where \( i, j \geq 0 \).

We check that all the compositions in \( S \) are trivial. Here, we just check one composition as example.

If \( j > 0 \), then by Lemma 3.2 we have

\[ \langle f(\pi_{D_i(P(w))}^{D_j(P(w))}), w \rangle, f(u, v) \rangle_{w_8} = \lambda^{-i}D^i(f(\pi_{D_j(P(w))}^{D_j(P(w))}), w) - \lambda^{-i}[D^i(P(\pi_{D_j(P(w))}^{D_j(P(w))})D^j(P(w)))]_{D_i(f(w,v))} \]

\[ \equiv -D^{i-1}([\pi_{D_j(P(w))}^{D_j(P(w))}]D^{j-1}([w])) - \lambda^{-i}(D^i(P(r([\pi_{D_j(P(w))}^{D_j(P(w))}]))D^j(P([w]))) \]

\[ \equiv \lambda^{-i}(D^{i-1}(r([\pi_{D_j(P(w))}^{D_j(P(w))}]D^{j-1}([w]))) - \lambda^{-i}(D^{i-1}(r([\pi_{D_j(P(w))}^{D_j(P(w))}]D^{j-1}([w]))) \]

\[ \equiv 0 \mod(S, w_8). \]

If \( j = 0 \), then

\[ \langle f(\pi_{D_i(P(w))}^{D_j(P(w))}), w \rangle, f(u, v) \rangle_{w_8} = f(\pi_{D_i(P(w))}^{D_j(P(w))}), w) - \lambda^{-i}[P(\pi_{D_j(P(w))}^{D_j(P(w))})P(w)]_{D_i(f(w,v))} \]

\[ \equiv -P([\pi_{D_j(P(w))}^{D_j(P(w))}][w]) - P([\pi_{D_j(P(w))}^{D_j(P(w))}][w]) \]

\[ \equiv \lambda^{-i}P(r([\pi_{D_j(P(w))}^{D_j(P(w))}]D^{j-1}([w]))) + \lambda^{-i}P(r([\pi_{D_j(P(w))}^{D_j(P(w))}]D^{j-1}([w]))) \]

\[ \lambda^{-i}P([r([\pi_{D_j(P(w))}^{D_j(P(w))}]D^{j-1}([w]))]D^{j-1}([w])) - \lambda^{-i}P([r([\pi_{D_j(P(w))}^{D_j(P(w))}]D^{j-1}([w]))]D^{j-1}([w))) \]

\[ \equiv 0 \mod(S, w_8). \]

Case 2. For \( \lambda = 0 \), all possible compositions of the polynomials in \( S \) are list as below:

\[ \langle g(\pi_{D_i(P(w))}^{D_j(D(P(w)))}), g(v) \rangle_{w_1}, \quad w_1 = D^j(D(P(\pi_{D_j(D(P(w)))})) \rangle,
\langle g(\pi_{D_i(P(w))}^{D_j(D(P(w)))}), f(u, v) \rangle_{w_2}, \quad w_2 = D^j(D(P(\pi_{D_j(D(P(w)))}) \rangle),
\langle f(u, v), g(u) \rangle_{w_3}, \quad w_3 = D^j(P(u))P(v), \quad l > 0,
\langle f(\pi_{D_i(D(P(w)))}, v), g(u) \rangle_{w_4}, \quad w_4 = D^j(P(\pi_{D_j(D(P(w)))})P(v),
\langle f(u, \pi_{D_i(D(P(w)))}), g(v) \rangle_{w_5}, \quad w_5 = D^j(P(u))P(\pi_{D_j(D(P(w)))}),
\langle f(u, v), f(v, w) \rangle_{w_6}, \quad w_6 = D^j(P(u))P(v)P(w),
\langle f(\pi_{D_i(D(P(w)))}, w), f(u, v) \rangle_{w_7}, \quad w_7 = D^j(P(\pi_{D_j(D(P(w)))})P(w),
\langle f(u, \pi_{D_i(D(P(w)))}), f(v, w) \rangle_{w_8}, \quad w_8 = D^j(P(u))P(\pi_{D_j(D(P(w)))}) \rangle,

where \( i, j \geq 0 \). We check that all the compositions in \( S \) are trivial. The proof is similar to Case 1. \( \square \)
4.2 A linear basis of a free $\lambda$-differential Lie Rota-Baxter algebra

In this subsection, by Theorems 3.9 and 4.3, we obtain a linear basis of the free $\lambda$-differential Lie Rota-Baxter algebra on the set $X$.

For $n = 0$, define $\langle\{P\}; \Delta(X)\rangle_0 := S(\Delta(X))$ and $\langle\{P\}; \Delta(X)\rangle_0 := (\Delta(X))^{**}$. For $n > 0$, define

$$\langle\{P\}; \Delta(X)\rangle_n := S(\Delta(X) \cup P(\langle\{P\}; \Delta(X)\rangle_{n-1})),$$

$$\langle\{P\}; \Delta(X)\rangle_n := (\Delta(X) \cup P(\langle\{P\}; \Delta(X)\rangle_{n-1}))^{**}.$$

Set

$$\langle\{P\}; \Delta(X)\rangle := \bigcup_{n=0}^{\infty} \langle\{P\}; \Delta(X)\rangle_n, \quad \langle\{P\}; \Delta(X)\rangle := \bigcup_{n=0}^{\infty} \langle\{P\}; \Delta(X)\rangle_n.$$

Let $*$ be a symbol, which is not in $X$. By a $*$-symbol word on $\Delta(X)$, we mean any expression in $\langle\{P\}; \Delta(X) \cup \{*\}\rangle$ with only one occurrence of $*$. We will denote by $\langle\{P\}; \Delta(X)\rangle^*$ the set of all the $*$-symbol words on $\Delta(X)$. Let $\pi$ be a $*$-symbol word on $\Delta(X)$ and $u \in \langle\{P\}; \Delta(X)\rangle$. Let us denote $\pi|_u = \pi|_{\pi \rightarrow u}$, i.e. $*$ is replaced by $u$.

It is easy to see that $\langle\{P\}; \Delta(X)\rangle \subseteq (D; \langle\{P\}; X\rangle)$. We also use the order $\succ_{Dl}$ on $\langle\{P\}; \Delta(X)\rangle$ and $\succ$ on $\Delta(X) \cup P(\langle\{P\}; \Delta(X)\rangle)$.

For $n = 0$, let $Y_0 = \Delta(X)$. Define

$$ALSW(\langle\{P\}; \Delta(X)\rangle_0 := ALSW(Y_0),$$

$$NLSW(\langle\{P\}; \Delta(X)\rangle_0 := NLSW(Y_0) = \{[u]|u \in ALSW(\langle\{P\}; \Delta(X)\rangle_0\}$$

with respect to the lex-order $\succ_{lex}$.

Assume that we have defined

$$ALSW(\langle\{P\}; \Delta(X)\rangle_{n-1},$$

$$NLSW(\langle\{P\}; \Delta(X)\rangle_{n-1} = \{[u]|u \in ALSW(\langle\{P\}; \Delta(X)\rangle_{n-1}\}.$$

Let $Y_n := \Delta(X) \cup P(ALSW(\langle\{P\}; \Delta(X)\rangle_{n-1})$. Define

$$ALSW(\langle\{P\}; \Delta(X)\rangle_n := ALSW(Y_n).$$

For any $u \in Y_n$, define the bracketing way on $u$ by

$$[u] := \begin{cases} u, & \text{if } u \in \Delta(X), \\ P([u_1]), & \text{if } u = P(u_1). \end{cases}$$

Let $[Y_n] := \{[u]|u \in Y_n\}$. Therefore, the order $\succ$ on $Y_n$ induces an order on $[Y_n]$ by $[u] \succ [v]$ if $u \succ v$ for any $u, v \in Y_n$. For any $u = u_1u_2 \cdots u_m \in ALSW(\langle\{P\}; \Delta(X)\rangle_n)$, where each $u_i \in Y_n$, let us denote

$$[u] := [[u_1][u_2]\cdots[u_m]]$$
the nonassociative Lyndon-Shirshov word on \{[u_1], [u_2], \cdots, [u_m]\} with respect to the lex-order \succ_{\text{lex}}.

Define

\[ NLSW(P; \Delta(X)) = \{ [u] | u \in ALSW(P; \Delta(X)) \} \]

It is easy to see that \( NLSW(P; \Delta(X)) = NLSW([Y_n]) \). Define

\[ ALSW(P; \Delta(X)) = \bigcup_{n=0}^{\infty} ALSW(P; \Delta(X))_n, \]

\[ NLSW(P; \Delta(X)) = \bigcup_{n=0}^{\infty} NLSW(P; \Delta(X))_n. \]

Therefore,

\[ NLSW(P; \Delta(X)) = \{ [u] | u \in ALSW(P; \Delta(X)) \}. \]

By Theorems 3.9 and 4.3 we have the following theorem.

**Theorem 4.4** The set

\[ \text{Irr}(S) = \{ [w] \in NLSW(P; \Delta(X)) | \begin{array}{l} w \neq \pi|_{P(u)P(v)}, \pi \in \langle \{P; \Delta(X)\}^* \rangle, \\ u, v \in ALSW(P; \Delta(X)); u \succ_{\text{lex}} v \end{array} \} \]

is a linear basis of the free \( \lambda \)-differential Lie Rota-Baxter algebra \( DRBL(X) \) on \( X \).

**References**

[1] W.W. Adams, P. Loustaunau, An introduction to Gröbner bases, Graduate Studies in Mathematics, Vol. 3, American Mathematical Society (AMS), 1994.

[2] H. An, C. Bai, From Rota-Baxter algebras to pre-Lie algebras, *J. Phys. A: Math. Theor.*, 41 (2008), 015201.

[3] C. Bai, L. Guo, X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, *Commun. Math. Phys.*, 297 (2010), 553-596.

[4] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.*, 10(1960), 731-742.

[5] G.M. Bergman, The diamond lemma for ring theory, *Adv. in Math.*, 29(1978), 178-218.

[6] L.A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika*, 15(1976), 117-142.
[7] L.A. Bokut, Y. Chen, Gröbner-Shirshov bases: after A.I. Shirshov, Southeast Asian Bull. Math., 31(2007) 1057-1076.

[8] L.A. Bokut, Y. Chen, Gröbner-Shirshov bases and their calculation, Bull. Math. Sci., 4(2014), 325-395.

[9] L.A. Bokut, Y. Chen, J. Qiu, Gröbner-Shirshov bases for associative algebras with multiple operations and free Rota-Baxter algebras, J. Pure Appl. Alg., 214(2010), 89-100.

[10] L.A. Bokut, P.S. Kolesnikov, Gröbner-Shirshov bases: from their incipiency to the present, J. Math. Sci., 116(1)(2003), 2894-2916.

[11] L.A. Bokut, G. Kukin, Algorithmic and Combinatorial algebra, Kluwer Academic Publ., Dordrecht, 1994.

[12] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal, Ph.D. thesis, University of Innsbruck, Austria, 1965 (in German).

[13] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations, Aequationes Math., 4(1970), 374-383.

[14] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, Comm. Math. Phys., 210 (2000), 249-273.

[15] R.M. Cohn, Difference algebra, Interscience Publishers, 1965.

[16] D.A. Cox, J. Little, D. O’Shea, Ideals, varieties and algorithms: An introduction to computational algebraic geometry and commutative algebra, Undergraduate Texts in Mathematics, New York: Springer-Verlag, 1992.

[17] V. Dotsenko, A. Khoroshkin, Gröbner bases for operads, Duke Math. J., 153(2010), 363-396

[18] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Math., Vol.150, Berlin and New York: Springer-Verlag, 1995.

[19] E. Kolchin, Differential Algebra and Algebraic Groups, Academic Press, NewYork, 1973.

[20] X. Gao, L. Guo, M. Rosenkranz, Free integro-differential algebras and Gröbner-Shirshov bases, J. Alg., 442(2015), 354-396.

[21] V. Gubarev, P. Kolesnikov, Gröbner-Shirshov basis of the universal enveloping Rota-Baxter algebra of a Lie algebra, [arXiv:1602.07409 [math.RA]].

[22] L. Guo, An introduction to Rota-Baxter algebra, Beijing, Higher education press, 2012.
[23] L. Guo, W. Keigher, On differential Rota-Baxter algebras, *J. Pure Appl. Alg.*, 212(2008), 522-540.

[24] H. Hironaka, Resolution of singulatities of an algebraic variety over a field if characteristic zero, I, II, *Ann. Math.*, 79(1964), 109-203, 205-326.

[25] A.G. Kurosh, Free sums of multiple operator algebras, *Siberian. Math. J.*, 1(1960), 62-70 (in Russian).

[26] J. Pei, C. Bai, L. Guo, Rota-Baxter operators on $sl(2, C)$ and solutions of the classical Yang-Baxter equation, *J. Math. Phys.*, 55(2)(2014), 021701.

[27] J. Qiu, Y. Chen, Composition-Diamond lemma for $\lambda$-differential associative algebras with multiple operators, *J. Alg. Appl.*, 9 (2010), 223-239.

[28] J. Qiu, Y. Chen, Gröbner-Shirshov bases for Lie $\Omega$-algebras and free Rota-Baxter Lie algebras, *J. Alg. Appl.*, 16(2) (2017), 1750190.

[29] J.F. Ritt, Differential equations from the algebraic standpoint, Amer. Math. Soc. Colloq. Pub. 14. Amer. Math. Soc., New York, 1934.

[30] G.-C. Rota, Baxter algebras and combinatorial identities I, *Bull. Amer. Math. Soc.*, 5(1969), 325-329.

[31] M.A. Semenov-Tian-Shansky, What is a classical $R$-matrix? *Funct. Anal. Appl.*, 17(4) (1983), 259-272.

[32] A.I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.*, 3 (1962), 292-296 (in Russian); English translation in SIGSAM Bull., 33(2) (1999), 3-6.

[33] M. Singer, M. van der Put, Galois theory of linear differential equations, Springer, 2003.