ABSTRACT. We discuss the behavior of the minimal eigenvalue \( \lambda \) of the Dirichlet Laplacian in the domain \( D_1 \setminus D_2 := D \) (an annulus) where \( D_1 \) is a circular disc and \( D_2 \subset D_1 \) is a smaller circular disc. It is conjectured that the minimal eigenvalue \( \lambda \) has a maximum value when \( D_2 \) is a concentric disc. If \( h \) is a displacement of the center of the disc \( D_2 \) and \( \lambda(h) \) is the corresponding minimal eigenvalue, then \( \frac{d\lambda(h)}{dh} < 0 \) so that \( \lambda(h) \) is minimal when \( \partial D_2 \) touches \( \partial D_1 \), where \( \partial D \) is the boundary of \( D \). Numerical results are given to back the conjecture. Upper and lower bounds are given for \( \lambda(h) \). The above conjecture is proved.

1. Introduction

Let \( D_1 \) be a disc on \( \mathbb{R}^2 \), centered at the origin, of radius 1, \( D_2 \subset D_1 \) be a disc of radius \( a < 1 \), the center \((h, 0)\) of which is at the distance \( h \) from the origin. Denote by \( \lambda(h) \) the minimal Dirichlet eigenvalue of the Laplacian in the annulus \( D := D_h := D_1 \setminus D_2 \).

In this paper the following conjecture is formulated and proved:

Conjecture C. The minimal eigenvalue \( \lambda(h) \) is a monotonically decreasing function of \( h \).
on the interval $0 \leq h \leq 1 - a$. In particular

\begin{equation}
\lambda(0) > \lambda(h), \quad h > 0.
\end{equation}

Let $\dot{\lambda} := \frac{d\lambda}{dh}$ and let $S$ denote $\partial D_2$, the boundary of $D_2$.

The following results are given to back this conjecture:

**Lemma 1.** One has

\begin{equation}
\dot{\lambda} = \int_S u_N^2 N_1 ds,
\end{equation}

where $N$ is the unit normal to $S = S_h$ pointing into the annulus $D_h$, $N_1$ is the projection of $N$ onto $x_1$-axis, $u_N$ is the normal derivative of $u$, and $u(x) = u(x_1, x_2)$ is the normalized in $L^2(D)$ eigenfunction corresponding to the first eigenvalue $\lambda$:

\begin{equation}
\Delta u + \lambda u = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D_1 \cup \partial D_2 := \partial D,
\end{equation}

\begin{equation}
\|u\|_{L^2(D)} = 1.
\end{equation}

It is argued at the end of Section 2 that

\begin{equation}
\dot{\lambda} < 0 \quad \text{if} \quad 0 < h < 1 - a.
\end{equation}

In Lemma 2 below we give upper and lower bounds (1.6) for $\lambda(h)$. These bounds are practically convenient, especially for small $h$.

Let $D(r)$ be the disc $|x| \leq r$, $\mu(r)$ be the first Dirichlet eigenvalue of the Laplacian in $D_1 \setminus D_1(r)$. In Section 3 inequality (1.5) is illustrated by the numerical results in $D_1 \setminus D(r)$.

**Lemma 2.** One has

\begin{equation}
\mu(a - h) < \lambda(h) < \mu(a + h), \quad 0 < h < 1 - a, \quad h < a.
\end{equation}

In section 2 proofs are given and the conjecture is proved.

2. Proofs.

**Proof of Lemma 2.** Lemma 2 is an immediate consequence of the variational principle for $\lambda$ since $D_1 \setminus D(a + h) \subset D_h \subset D_1 \setminus D(a - h)$. Note that $\mu(b), a \leq b < 1$, can be calculated
efficiently. Indeed, by symmetry the first eigenfunction \( \phi \) of the Dirichlet Laplacian in \( D_1 \setminus D(b) \) depends on the radial variable \( r = |x| \) only, and solves the problem

\[
(2.1) \quad \phi'' + \frac{1}{r} \phi' + \mu \phi = 0, \quad b \leq r \leq 1; \quad \phi(b) = \phi(1) = 0.
\]

Thus

\[
(2.2) \quad \phi = c_1 J_0(\sqrt{\mu} r) + c_2 N_0(\sqrt{\mu} r),
\]

where \( J_0 \) and \( N_0 \) are the Bessel functions, and \( c_1, c_2 \) are constants. The boundary conditions (2.1) are satisfied if \( \mu = \mu(b) > 0 \) is a positive root of the equation:

\[
(2.3) \quad J_0(\sqrt{\mu} b) N_0(\sqrt{\mu}) - J_0(\sqrt{\mu}) N_0(\sqrt{\mu} b) = 0.
\]

The smallest positive root \( \mu = \mu(b) \) of (2.3) is the desired first eigenvalue of the Dirichlet Laplacian in \( D_1 \setminus D(b) \). Equation (2.3) can be solved numerically. This makes (1.6) an efficient estimate of \( \lambda(h) \), especially for small \( h > 0 \).

**Proof of Lemma 1.** We use the known technique based on the domain derivative [1].

It is known that \( \lambda(h) \) is continuously differentiable with respect to \( h \) [2]. Let \( \dot{u} = \frac{du}{dh} \), where \( u \) solves (1.3)-(1.4). Differentiate the equation and the boundary condition (1.3) with respect to \( h \) and get

\[
(2.4) \quad \Delta \dot{u} + \lambda \dot{u} = -\dot{\lambda} u \quad \text{in} \quad D = D_h,
\]

\[
(2.5) \quad \dot{u} + u_N N_1 = 0 \quad \text{on} \quad S = S_h.
\]

Multiply (2.4) by \( u \), (1.3) by \( \dot{u} \), subtract, integrate over \( D = D_h \), use Green’s formula, and (2.5) and get:

\[
(2.6) \quad \dot{\lambda} \int_D u^2 dx = \int_S (u \dot{u}_N - \dot{u} u_N) ds = \int_S u_N^2 N_1 ds.
\]

From (2.6) and (1.4) one gets (1.2). Lemma 1 is proved.

It follows from (1.2) by symmetry that \( \dot{\lambda}(0) = 0 \). Indeed, if \( h = 0 \), then \( u_N^2|_{S_0} = \text{const} \) by symmetry, and \( \int_{S_0} N_1 ds = 0 \).

If \( h > 0 \), then \( u_N^2 \) on the half circle \( S_h^+ \), the part of the boundary of \( S_h \) which is closer to \( \partial D_1 \), is likely to be less than on the other half \( S_h^- \) of \( S_h \), while \( N_1 > 0 \) on \( S_h^+ \) and
\( N_1 < 0 \) on \( S_h^- \). Moreover, \(|N_1|\) is the same at the symmetric points of \( S_h^+ \) and \( S_h^- \), where the axis of symmetry is the vertical diameter of \( D_2 \). Therefore one expects \( \dot{\lambda}(h) < 0 \) for \( h > 0 \), which is the conjecture \( C \).

Let us prove that the above argument is indeed valid. What we wish to prove is the inequality for the normal derivative \( u_N \) mentioned above.

The following argument completes the proof of the Conjecture (C). This argument was communicated to AGR by Professor M. Ashbaugh. Consider the reflection of the part of the domain which is situated to the right of the vertical line passing through the center of the smaller disc with respect to this line. Let \( D_h \) denote the domain symmetric with respect to this line \( \ell \) and \( v \) denote the function equal to \( u \) to the right of \( \ell \), and equal to \( w \) to the left of \( \ell \). Here \( w(x, y) = u(x, -y) \), where the \( y \)-axis is the line \( \ell \). By the maximum principle one has \( u > v \) on the part of the boundary of \( D_h \) which lies to the left of \( \ell \) and, by the Hopf lemma (strong maximum principle), it follows that \( u_N > v_N \) on this part of the boundary of \( D_h \). This is the desired inequality since \( v = u \) to the right of \( \ell \).

3. Numerical Results

We use a finite element method to calculate \( u_N^2 \) at a number of nodal points \( \phi \) on \( \partial D_2 \), where \( \phi \) is the angle between the radial line at the positive x-axis. Due to symmetry, it is sufficient to consider \( 0 \leq \phi \leq \pi \). The following tables give values for \( u_N^2 \) for various values of \( h \) and \( \phi \). The last row gives \( \lambda(h) \) for different values of \( h \).

| Table 1 | Values for \( u_N^2 \) |
|---------|------------------------|
|         | \( \alpha = 0.1 \)     | \( \lambda(0) = 10.98324859 \) |
|         | \( h = 0.1 \)          | \( h = 0.3 \)          | \( h = 0.6 \)          | \( h = 0.8 \)          |
| \( \phi \) |                      |                       |                       |                       |
| 0°      | 0.18340156            | 0.08997194            | 0.03502936            | 0.00538128            |
| 15°     | 0.18586555            | 0.09354750            | 0.03875921            | 0.00792279            |
| 30°     | 0.19312533            | 0.10408993            | 0.04977909            | 0.01615017            |
| 45°     | 0.20478642            | 0.12105508            | 0.06745736            | 0.03118122            |
| 60°     | 0.22019869            | 0.14357017            | 0.09052611            | 0.05294918            |
| 75°     | 0.23846941            | 0.17048691            | 0.11728631            | 0.07901455            |
| 90°     | 0.25848609            | 0.20042583            | 0.14624640            | 0.10706183            |
| 105°    | 0.27895498            | 0.23176494            | 0.17645804            | 0.13678539            |
| 120°    | 0.29846292            | 0.26256868            | 0.20707716            | 0.16793719            |
| 135°    | 0.31556947            | 0.29053976            | 0.23653390            | 0.19964213            |
| 150°    | 0.32892971            | 0.31313057            | 0.26197403            | 0.22879649            |
\[
\begin{array}{cccc}
165^\circ & 0.33743644 & 0.32789921 & 0.27954557 & 0.24990529 \\
180^\circ & 0.34035750 & 0.33304454 & 0.28585725 & 0.25766770 \\
\lambda(h) & 10.51624800 & 8.76956649 & 6.91928150 & 6.21431318 \\
\end{array}
\]

Table 2
Values for \( u_N^2 \)

\[
a = 0.3 \quad \lambda(0) = 19.46950428 \\
h = 0.1 \quad h = 0.3 \quad h = 0.6 \\
\phi \\
\begin{array}{ccc}
0^\circ & 0.04651448 & 0.00601084 & 0.00006665 \\
15^\circ & 0.05078040 & 0.00792264 & 0.00029224 \\
30^\circ & 0.06389146 & 0.01432651 & 0.00162487 \\
45^\circ & 0.08665951 & 0.02711431 & 0.00616138 \\
60^\circ & 0.12001996 & 0.04901522 & 0.01734345 \\
75^\circ & 0.16444947 & 0.08285892 & 0.03916871 \\
90^\circ & 0.21927390 & 0.13049149 & 0.07481155 \\
105^\circ & 0.28204163 & 0.19150347 & 0.12521694 \\
120^\circ & 0.34820007 & 0.26211532 & 0.18784387 \\
135^\circ & 0.41130766 & 0.33475001 & 0.25580537 \\
150^\circ & 0.46389778 & 0.39885669 & 0.31827254 \\
165^\circ & 0.49888764 & 0.44319924 & 0.36272535 \\
180^\circ & 0.51117180 & 0.45907590 & 0.37887932 \\
\lambda(h) & 17.00607073 & 12.31240018 & 8.54494014 \\
\end{array}
\]
Table 3
Values for $u_N^2$

\[ a = 0.6 \quad \lambda(0) = 61.2854372 \]

\[ h = 0.1 \quad h = 0.3 \]

| $\phi$  | $\lambda(h)$ |
|--------|-------------|
| $0^\circ$ | 0.00010994  | 0.00000018 |
| $15^\circ$ | 0.00025775  | 0.00000144 |
| $30^\circ$ | 0.00101252  | 0.00002268 |
| $45^\circ$ | 0.00370221  | 0.00026580 |
| $60^\circ$ | 0.01190759  | 0.00195778 |
| $75^\circ$ | 0.03332159  | 0.00947178 |
| $90^\circ$ | 0.08086609  | 0.03287792 |
| $105^\circ$ | 0.17026477  | 0.08782665 |
| $120^\circ$ | 0.32267905  | 0.18896048 |
| $135^\circ$ | 0.49728793  | 0.33653240 |
| $150^\circ$ | 0.69311417  | 0.50402714 |
| $165^\circ$ | 0.84533543  | 0.64040281 |
| $180^\circ$ | 0.90307061  | 0.69330938 |

In all the cases above, $u_N^2$ increases in value as $\phi$ increases from zero to $\pi$, thereby confirming that $\dot{\lambda} < 0$ (see formula (2.6)). From the above tables we also note that for fixed $a$, $\lambda(h)$ is a decreasing function of $h$, and that $\lambda(h) < \lambda(0)$ for $h > 0$ thus confirming the Conjecture C.

REFERENCES

1. J. Sokolowski, J. Zolezio, *Introduction to shape optimization*, Springer Verlag, Berlin, 1992

2. T. Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin, 1966.