Beyond Serre’s “Trees” in two directions: $\Lambda$–trees and products of trees

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Abstract

Serre [125] laid down the fundamentals of the theory of groups acting on simplicial trees. In particular, Bass-Serre theory makes it possible to extract information about the structure of a group from its action on a simplicial tree. Serre’s original motivation was to understand the structure of certain algebraic groups whose Bruhat–Tits buildings are trees. In this survey we will discuss the following generalizations of ideas from [125]: the theory of isometric group actions on $\Lambda$-trees and the theory of lattices in the product of trees where we describe in more detail results on arithmetic groups acting on a product of trees.

1 Introduction

Serre [125] laid down the fundamentals of the theory of groups acting on simplicial trees. The book [125] consists of two parts. The first part describes the basics of what is now called Bass-Serre theory. This theory makes it possible to extract information about the structure of a group from its action on a simplicial tree. Serre’s original motivation was to understand the structure of certain algebraic groups whose Bruhat–Tits buildings are trees. These groups are considered in the second part of the book.

Bass-Serre theory states that a group acting on a tree can be decomposed (splits) as a free product with amalgamation or an HNN extension. Such a group can be represented as a fundamental group of a graph of groups. This became a wonderful tool in geometric group theory and geometric topology, in particular in the study of 3-manifolds. The theory was further developing in the following directions:

1. Various accessibility results were proved for finitely presented groups that bound the complexity (that is, the number of edges) in a graph of groups
decomposition of a finitely presented group, where some algebraic or geometric restrictions on the types of groups were imposed [32, 16, 123, 33, 132].

2. The theory of JSJ-decompositions for finitely presented groups was developed [22, 115, 34, 124, 36].

3. The theory of lattices in automorphism groups of trees. The group of automorphisms of a locally finite tree $\text{Aut}(T)$ (equipped with the compact open topology, where open neighborhoods of $f \in \text{Aut}(T)$ consist of all automorphisms that agree with $f$ on a fixed finite subtree) is a locally compact group which behaves similarly to a rank one simple Lie group. This analogy has motivated many recent works in particular the study of lattices in $\text{Aut}(T)$ by Bass, Kulkarni, Lubotzky [18], [75] and others. A survey of results about tree lattices and methods is given in [6] as well as proofs of many results.

In this survey we will discuss the following generalizations of Bass-Serre theory and other Serre’s ideas from [125]:

1. The theory of isometric group actions on real trees (or $\mathbb{R}$-trees) which are metric spaces generalizing the graph-theoretic notion of a tree. This topic will be only discussed briefly, we refer the reader to the survey [14].

2. The theory of isometric group actions on $\Lambda$-trees, see Sections 2, 3. Alperin and Bass [1] developed the initial framework and stated the fundamental research goals: find the group theoretic information carried by an action (by isometries) on a $\Lambda$-tree; generalize Bass-Serre theory to actions on arbitrary $\Lambda$-trees. From the viewpoint of Bass-Serre theory, the question of free actions of finitely generated groups became very important. There is a book [23] on the subject and many new results were obtained in [65]. This is a topic of interest of the first author.

3. The theory of complexes of groups provides a higher-dimensional generalization of Bass-Serre theory. The methods developed for the study of lattices in $\text{Aut}(T)$ were extended to the study of (irreducible) lattices in a product of two trees, as a first step toward generalizing the theory of lattices in semisimple non-archimedean Lie groups. Irreducible lattices in higher rank semisimple Lie groups have a very rich structure theory and there are superrigidity and arithmeticity theorems by Margulis. The results of Burger, Moses, Zimmer [19, 20, 21] about cocompact lattices in the group of automorphisms of a product of trees or rather in groups of the form $\text{Aut}(T_1) \times \text{Aut}(T_2)$, where each of the trees is regular, are described in [90]. The results obtained concerning the structure of lattices in $\text{Aut}(T_1) \times \text{Aut}(T_2)$ enable them to construct the first examples of finitely presented torsion free simple groups. We will mention further results on simple and non-residually finite groups [133, 108, 107] and describe in more detail results on arithmetic groups acting on the product of trees [38, 127]. This is a topic of interest of the second author.
In [79] Lyndon introduced real-valued length functions as a tool to extend Nielsen cancelation theory from free groups over to a more general setting. Some results in this direction were obtained in [52, 53, 51, 103, 2]. The term $\mathbb{R}$-tree was coined by Morgan and Shalen [99] in 1984 to describe a type of space that was first defined by Tits [129]. In [23] Chiswell described a construction which shows that a group with a real-valued length function has an action on an $\mathbb{R}$-tree, and vice versa. Morgan and Shalen realized that a similar construction and results hold for an arbitrary group with a Lyndon length function which takes values in an arbitrary ordered abelian group $\Lambda$ (see [99]). In particular, they introduced $\Lambda$-trees as a natural generalization of $\mathbb{R}$-trees which they studied in relation with Thurston’s Geometrization Program. Thus, actions on $\Lambda$-trees and Lyndon length functions with values in $\Lambda$ are two equivalent languages describing the same class of groups. In the case when the action is free (the stabilizer of every point is trivial) we call groups in this class $\Lambda$-free or tree-free. We refer to the book [25] for a detailed discussion on the subject.

A joint effort of several researchers culminated in a description of finitely generated groups acting freely on $\mathbb{R}$-trees [15, 37], which is now known as Rips’ theorem: a finitely generated group acts freely on an $\mathbb{R}$-tree if and only if it is a free product of free abelian groups and surface groups (with an exception of non-orientable surfaces of genus 1, 2, and 3). The key ingredient of this theory is the so-called “Rips machine”, the idea of which comes from Makanin’s algorithm for solving equations in free groups (see [85]). The Rips machine appears in applications as a general tool that takes a sequence of isometric actions of a group $G$ on some “negatively curved spaces” and produces an isometric action of $G$ on an $\mathbb{R}$-tree as the Gromov-Hausdorff limit of the sequence of spaces. Free actions on $\mathbb{R}$-trees cover all Archimedean actions, since every group acting freely on a $\Lambda$-tree for an Archimedean ordered abelian group $\Lambda$ also acts freely on an $\mathbb{R}$-tree.

In the non-Archimedean case the following results were obtained. First of all, in [4] Bass studied finitely generated groups acting freely on $\Lambda_0 \oplus \mathbb{Z}$-trees with respect to the right lexicographic order on $\Lambda_0 \oplus \mathbb{Z}$, where $\Lambda_0$ is any ordered abelian group. In this case it was shown that the group acting freely on a $\Lambda_0 \oplus \mathbb{Z}$-tree splits into a graph of groups with $\Lambda_0$-free vertex groups and maximal abelian edge groups. Next, Guirardel (see [48]) obtained the structure of finitely generated groups acting freely on $\mathbb{R}^n$-trees (with the lexicographic order). In [70] the authors described the class of finitely generated groups acting freely and regularly on $\mathbb{Z}^n$-trees in terms of HNN-extensions of a very particular type. The action is regular if all branch points are in the same orbit. The importance of regular actions becomes clear from the results of [72], where it was proved that a finitely generated group acting freely on a $\mathbb{Z}^n$-tree is a subgroup of a finitely generated group acting freely and regularly on a $\mathbb{Z}^m$-tree for $m \geq n$, and the paper [26], where it was shown that a group acting freely on a $\Lambda$-tree (for arbitrary $\Lambda$) can always be embedded in a length-preserving way into a group acting freely and regularly on a $\Lambda$-tree (for the same $\Lambda$). The structure of finitely presented $\Lambda$-free groups was described in [69]. They all are $\mathbb{R}^n$-free.

Another natural generalization of Bass-Serre theory is considering group
actions on products of trees started in [19]. The structure of a group acting freely and cocompactly on a simplicial tree is well understood. Such a group is a finitely generated free group. By way of contrast, a group which acts similarly on a product of trees can have remarkably subtle properties.

Returning to the case of one tree, recall that there is a close relation between certain simple Lie groups and groups of tree automorphisms. The theory of tree lattices was developed in [18], [75] by analogy with the theory of lattices in Lie groups (that is discrete subgroups of Lie groups of finite co-volume). Let $G$ be a simple algebraic group of rank one over a non-archimedean local field $K$. Considering the action of $G$ on its associated Bruhat–Tits tree $T$ we have a continuous embedding of $G$ in $Aut(T)$ with co-compact image. In [130] Tits has shown that if $T$ is a locally finite tree and its automorphism group $Aut(T)$ acts minimally (i.e. without an invariant proper subtree and not fixing an end) on it, then the subgroup generated by edge stabilizers is a simple group. In particular the automorphism group of a regular tree is virtually simple. These results motivated the study of $Aut(T)$ looking at the analogy with rank one Lie groups.

When $T$ is a locally finite tree, $G = Aut(T)$ is locally compact. The vertex stabilizers $G_v$ are open and compact. A subgroup $\Gamma \leq G$ is discrete if $\Gamma_v$ is finite for some (and hence for every) vertex $v \in VT$, where $VT$ is the set of vertices of $T$. In this case we can define

$$Vol(\Gamma \backslash T) = \sum_{v \in \Gamma \backslash VT} 1/|\Gamma_v|.$$ 

We call $\Gamma$ a $T$-lattice if $Vol(\Gamma \backslash T) < \infty$. We call $\Gamma$ a uniform $T$-lattice if $\Gamma \backslash T$ is finite. In case $G \backslash T$ is finite, this is equivalent to $\Gamma$ being a lattice (resp., uniform lattice) in $G$. Uniform tree lattices correspond to finite graphs of groups in which all vertex and edge groups are finite. In the study of lattices in semisimple Lie groups an important role is played by their commensurators. Margulis has shown that an irreducible lattice $\Gamma < G$ in a semisimple Lie group is arithmetic if and only if its commensurator is dense in $G$. It was shown by Liu [74] that the commensurator of a uniform tree lattice is dense. All uniform tree lattices of a given tree are commensurable up to conjugation, the isomorphism class of the commensurator of a uniform tree lattice is determined by the tree. It was also shown [77] that for regular trees the commensurator determines the tree.

Consider now products of trees. The following results about lattices in semisimple Lie groups were established by Margulis: an irreducible lattice in a higher rank ($\geq 2$) semisimple Lie group is arithmetic; any linear representation of such a lattice with unbounded image essentially extends to a continuous representation of the ambient Lie group. Recall that a lattice $G$ in a semisimple Lie group is called reducible if $\Gamma$ contains a finite index subgroup of the form $\Gamma_1 \times \Gamma_2$ where $\Gamma_i < G_i$ is a lattice and $G = G_1 \times G_2$. A reducible torsion free group acting simply transitively on a product of two trees is virtually a direct product of two finitely generated free groups, in particular is residually finite. Burger, Moses, Zimmer [19] were studying a structure for lattices of groups of...
the form $\text{Aut}(T_1) \times \text{Aut}(T_2)$. For example, Burger and Mozes have proved rigidity and arithmeticity results analogous to the theorems of Margulis for lattices in semisimple Lie groups. We will describe their results about cocompact lattices in groups of the form $\text{Aut}(T_1) \times \text{Aut}(T_2)$, where each of the trees is regular. We will discuss this and similar topics in Section 4.

The smallest explicit example of a simple group, as an index 4 subgroup of a group presented by 10 generators and 24 short relations, was constructed by Rattaggi [108]. For comparison, the smallest virtually simple group of [19], Theorem 6.4, needs more than 18000 relations, and the smallest simple group constructed in [19], 6.5, needs even more than 360000 relations in any finite presentation.

If we restrict our attention to torsion free lattices that act simply transitively on the vertices of the product of trees (not interchanging the factors), then those lattices are fundamental groups of square complexes with just one vertex, complete bipartite link and a vertical/horizontal structure on edges (see 4.1.2 for precise definition). This is combinatorially well understood and there are plenty of such lattices, see [127] for a mass formula, but very rarely these lattices arise from an arithmetic context.

Let $T_n$ denote the tree of constant valency $n$. For example, there are 541 labelled candidate square complexes that after forgetting the label give rise to 43 torsion free and vertex transitive lattices acting on $T_4 \times T_4$. Among those only one lattice is arithmetic. For lattices acting on $T_6 \times T_6$ the number of labelled square complexes is $\approx 27 \cdot 10^9$, but only few thousands have finite abelianization (a necessary condition) and only 2 lattices are known to be arithmetic.

For different odd prime numbers $p \neq \ell$, Mozes [89], for $p$ and $\ell$ congruent to 1 mod 4, and later, for any two distinct odd primes, Rattaggi [107] found an arithmetic lattice acting on $T_{p+1} \times T_{\ell+1}$ with simply transitive action on the vertices.

For products of trees of the same valency arithmetic lattices were constructed in [127]. By means of a quaternion algebra over $\mathbb{F}_q(t)$, there is an explicit construction of an infinite series of torsion free, simply transitive, irreducible lattices in $\text{PGL}_2(\mathbb{F}_q((t))) \times \text{PGL}_2(\mathbb{F}_q((t)))$. The lattices depend on an odd prime power $q = p^r$ and a parameter $\tau \in \mathbb{F}_q^\times$, $\tau \neq 1$, and are the fundamental groups of a square complex with just one vertex and universal covering $T_{q+1} \times T_{q+1}$, a product of trees with constant valency $q + 1$. For $p = 2$ there are examples of arithmetic lattices acting on products of trees of valency 3 in [118].

The last subsection of our survey is dedicated to lattices in products of $n \geq 2$ trees.

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2 $\Lambda$-trees

The theory of $\Lambda$-trees (where $\Lambda = \mathbb{R}$) has its origins in the papers by Chiswell [23] and Tits [129]. The first paper contains a construction of an $\mathbb{R}$-tree starting
from a Lyndon length function on a group (see Section 2.6), an idea considered earlier by Lyndon in [81].

Later, in their very influential paper [84] Morgan and Shalen linked group actions on $\mathbb{R}$-trees with topology and generalized parts of Thurston’s Geometrization Theorem. Next, they introduced $\Lambda$-trees for an arbitrary ordered abelian group $\Lambda$ and the general form of Chiswell’s construction. Thus, it became clear that abstract length functions with values in $\Lambda$ and group actions on $\Lambda$-trees are just two equivalent approaches to the same realm of group theory questions. The unified theory was further developed in the important paper by Alperin and Bass [1], where authors state a fundamental problem in the theory of group actions on $\Lambda$-trees: find the group theoretic information carried by a $\Lambda$-tree action (analogous to Bass-Serre theory), in particular, describe finitely generated groups acting freely on $\Lambda$-trees ($\Lambda$-free groups).

Here we introduce basics of the theory of $\Lambda$-trees, which can be found in more detail in [1], [25] and [69].

### 2.1 Ordered abelian groups

In this section some well-known results on ordered abelian groups are collected. For proofs and details we refer to the books [39] and [73].

A set $A$ equipped with addition “+” and a partial order “$\leq$” is called a partially ordered abelian group if:

1. $\langle A, + \rangle$ is an abelian group,
2. $\langle A, \leq \rangle$ is a partially ordered set,
3. for all $a, b, c \in A$, $a \leq b$ implies $a + c \leq b + c$.

An abelian group $A$ is called orderable if there exists a linear order “$\leq$” on $A$, satisfying the condition (3) above. In general, the ordering on $A$ is not unique.

Let $A$ and $B$ be ordered abelian groups. Then the direct sum $A \oplus B$ is orderable with respect to the right lexicographic order, defined as follows:

$$(a_1, b_1) < (a_2, b_2) \iff b_1 < b_2 \text{ or } (b_1 = b_2 \text{ and } a_1 < a_2).$$

Similarly, one can define the right lexicographic order on finite direct sums of ordered abelian groups or even on infinite direct sums if the set of indices is linearly ordered.

For elements $a, b$ of an ordered group $A$ the closed segment $[a, b]$ is defined by

$$[a, b] = \{ c \in A \mid a \leq c \leq b \}.$$ 

A subset $C \subset A$ is called convex, if for every $a, b \in C$ the set $C$ contains $[a, b]$. In particular, a subgroup $B$ of $A$ is convex if $[0, b] \subset B$ for every positive $b \in B$. In this event, the quotient $A/B$ is an ordered abelian group with respect to the order induced from $A$. 
A group $A$ is called archimedean if it has no non-trivial proper convex subgroups. It is known that $A$ is archimedean if and only if $A$ can be embedded into the ordered abelian group of real numbers $\mathbb{R}_+$, or equivalently, for any $0 < a \in A$ and any $b \in A$ there exists an integer $n$ such that $na > b$.

It is not hard to see that the set of convex subgroups of an ordered abelian group $A$ is linearly ordered by inclusion (see, for example, [39]), it is called the complete chain of convex subgroups in $A$. Notice that

$$E_n = \{ f(t) \in \mathbb{Z}[t] \mid \deg(f(t)) \leq n \}$$

is a convex subgroup of $\mathbb{Z}[t]$ (here $\deg(f(t))$ is the degree of $f(t)$) and

$$0 < E_0 < E_1 < \cdots < E_n < \cdots$$

is the complete chain of convex subgroups of $\mathbb{Z}[t]$.

If $A$ is finitely generated then the complete chain of convex subgroups of $A$

$$0 = A_0 < A_1 < \cdots < A_n = A$$

is finite. The following result (see, for example, [25]) shows that this chain completely determines the order on $A$, as well as the structure of $A$. Namely, the groups $A_i/A_{i-1}$ are archimedean (with respect to the induced order) and $A$ is isomorphic (as an ordered group) to the direct sum

$$A_1 \oplus A_2/A_1 \oplus \cdots \oplus A_n/A_{n-1}$$

with the right lexicographic order.

An ordered abelian group $A$ is called discretely ordered if $A$ has a non-trivial minimal positive element (we denote it by $1_A$). In this event, for any $a \in A$ the following hold:

1. $a + 1_A = \min\{b \mid b > a\}$,
2. $a - 1_A = \max\{b \mid b < a\}$.

For example, $A = \mathbb{Z}^n$ with the right lexicographic order is discretely ordered with $1_{\mathbb{Z}^n} = (1,0,\ldots,0)$. The additive group of integer polynomials $\mathbb{Z}[t]$ is discretely ordered with $1_{\mathbb{Z}[t]} = 1$.

Recall that an ordered abelian group $A$ is hereditary discrete if for any convex subgroup $E \subseteq A$ the quotient $A/E$ is discrete with respect to the induced order. A finitely generated discretely ordered archimedean abelian group is infinite cyclic. A finitely generated hereditary discrete ordered abelian group is isomorphic to the direct product of finitely many copies of $\mathbb{Z}$ with the lexicographic order. [93]

2.2 $\Lambda$-metric spaces

Let $X$ be a non-empty set, and $\Lambda$ an ordered abelian group. A $\Lambda$-metric on $X$ is a mapping $d : X \times X \rightarrow \Lambda$ such that for all $x, y, z \in X$:
(M1) \( d(x,y) \geq 0 \),
(M2) \( d(x,y) = 0 \) if and only if \( x = y \),
(M3) \( d(x,y) = d(y,x) \),
(M4) \( d(x,y) \leq d(x,z) + d(y,z) \).

So a \( \Lambda \)-metric space is a pair \( (X,d) \), where \( X \) is a non-empty set and \( d \) is a \( \Lambda \)-metric on \( X \). If \( (X,d) \) and \( (X',d') \) are \( \Lambda \)-metric spaces, an isometry from \( (X,d) \) to \( (X',d') \) is a mapping \( f : X \to X' \) such that \( d(x,y) = d'(f(x),f(y)) \) for all \( x,y \in X \).

A segment in a \( \Lambda \)-metric space is the image of an isometry \( \alpha : [a,b]_\Lambda \to X \) for some \( a,b \in \Lambda \) and \( [a,b]_\Lambda \) is a segment in \( \Lambda \). The endpoints of the segment are \( \alpha(a), \alpha(b) \).

We call a \( \Lambda \)-metric space \( (X,d) \) geodesic if for all \( x,y \in X \), there is a segment in \( X \) with endpoints \( x,y \) and \( (X,d) \) is geodesically linear if for all \( x,y \in X \), there is a unique segment in \( X \) whose set of endpoints is \( \{x,y\} \).

It is not hard to see, for example, that \( (\Lambda,d) \) is a geodesically linear \( \Lambda \)-metric space, where \( d(a,b) = |a-b| \), and the segment with endpoints \( a,b \) is \( [a,b]_\Lambda \).

Let \( (X,d) \) be a \( \Lambda \)-metric space. Choose a point \( v \in X \), and for \( x,y \in X \), define
\[
(x \cdot y)_v = \frac{1}{2}(d(x,v) + d(y,v) - d(x,y)).
\]
Observe, that in general \( (x \cdot y)_v \in \frac{1}{2}\Lambda \).

The following simple result follows immediately

**Lemma 1.** \([25]\) If \( (X,d) \) is a \( \Lambda \)-metric space then the following are equivalent:

1. for some \( v \in X \) and all \( x,y \in X \), \( (x \cdot y)_v \in \Lambda \),
2. for all \( v,x,y \in X \), \( (x \cdot y)_v \in \Lambda \).

Let \( \delta \in \Lambda \) with \( \delta \geq 0 \). Then \( (X,p) \) is \( \delta \)-hyperbolic with respect to \( v \) if, for all \( x,y,z \in X \),
\[
(x \cdot y)_v \geq \min\{(x \cdot z)_v,(z \cdot y)_v\} - \delta.
\]

**Lemma 2.** \([25]\) If \( (X,d) \) is \( \delta \)-hyperbolic with respect to \( v \), and \( t \) is any other point of \( X \), then \( (X,d) \) is 2\( \delta \)-hyperbolic with respect to \( t \).

A \( \Lambda \)-tree is a \( \Lambda \)-metric space \( (X,d) \) such that:

(T1) \( (X,d) \) is geodesic,
(T2) if two segments of \( (X,d) \) intersect in a single point, which is an endpoint of both, then their union is a segment,
(T3) the intersection of two segments with a common endpoint is also a segment.

**Example 1.** \( \Lambda \) together with the usual metric \( d(a,b) = |a-b| \) is a \( \Lambda \)-tree. Moreover, any convex set of \( \Lambda \) is a \( \Lambda \)-tree.
Example 2. A $\mathbb{Z}$-metric space $(X,d)$ is a $\mathbb{Z}$-tree if and only if there is a simplicial tree $\Gamma$ such that $X = V(\Gamma)$ and $p$ is the path metric of $\Gamma$.

Observe that in general a $\Lambda$-tree can not be viewed as a simplicial tree with the path metric like in Example 2.

Lemma 3. [25] Let $(X,d)$ be a $\Lambda$-tree. Then $(X,d)$ is $0$-hyperbolic, and for all $x,y,v \in X$ we have $(x \cdot y)v \in \Lambda$.

Eventually, we say that a group $G$ acts on a $\Lambda$-tree $X$ if any element $g \in G$ defines an isometry $g : X \to X$. An action on $X$ is non-trivial if there is no point in $X$ fixed by all elements of $G$. Note, that every group has a trivial action on any $\Lambda$-tree, when all group elements act as identity. An action of $G$ on $X$ is minimal if $X$ does not contain a non-trivial $G$-invariant subtree $X_0$.

Let a group $G$ act as isometries on a $\Lambda$-tree $X$. $g \in G$ is called elliptic if it has a fixed point. $g \in G$ is called an inversion if it does not have a fixed point, but $g^2$ does. If $g$ is not elliptic and not an inversion then it is called hyperbolic.

A group $G$ acts freely and without inversions on a $\Lambda$-tree $X$ if for all $1 \neq g \in G$, $g$ acts as a hyperbolic isometry. In this case we also say that $G$ is $\Lambda$-free.

2.3 $\Lambda$-free groups

Recall that a group $G$ is called $\Lambda$-free if for all $1 \neq g \in G$, $g$ acts as a hyperbolic isometry. Here we list some known results about $\Lambda$-free groups for an arbitrary ordered abelian group $\Lambda$. For all these results the reader can be referred to [1, 4, 25, 83, 69].

Theorem 1. (a) The class of $\Lambda$-free groups is closed under taking subgroups.

(b) If $G$ is $\Lambda$-free and $\Lambda$ embeds (as an ordered abelian group) in $\Lambda'$ then $G$ is $\Lambda'$-free.

(c) Any $\Lambda$-free group is torsion-free.

(d) $\Lambda$-free groups have the CSA property. That is, every maximal abelian subgroup $A$ is malnormal: $A^g \cap A = 1$ for all $g \notin A$.

(e) Commutativity is a transitive relation on the set of non-trivial elements of a $\Lambda$-free group.

(f) Solvable subgroups of $\Lambda$-free groups are abelian.

(g) If $G$ is $\Lambda$-free then any abelian subgroup of $G$ can be embedded in $\Lambda$.

(h) $\Lambda$-free groups cannot contain Baumslag-Solitar subgroups other than $\mathbb{Z} \times \mathbb{Z}$. That is, no group of the form $\langle a,t \mid t^{-1}a^pt = a^q \rangle$ can be a subgroup of a $\Lambda$-free group unless $p = q = \pm 1$.

(i) Any two generator subgroup of a $\Lambda$-free group is either free, or free abelian.
(j) The class of $\Lambda$-free groups is closed under taking free products.

The following result was originally proved in [51] in the case of finitely many factors and $\Lambda = \mathbb{R}$. A proof of the result in the general formulation given below can be found in [25, Proposition 5.1.1].

**Theorem 2.** If $\{G_i \mid i \in I\}$ is a collection of $\Lambda$-free groups then the free product $\ast_{i \in I} G_i$ is $\Lambda$-free.

The following result gives a lot of information about the group structure in the case when $\Lambda = \mathbb{Z} \times \Lambda_0$ with the left lexicographic order.

**Theorem 3.** [4, Theorem 4.9] Let a group $G$ act freely and without inversions on a $\Lambda$-tree, where $\Lambda = \mathbb{Z} \times \Lambda_0$. Then there is a graph of groups $(\Gamma, Y^*)$ such that:

1. $G = \pi_1(\Gamma, Y^*)$,
2. for every vertex $x^* \in Y^*$, a vertex group $\Gamma_{x^*}$ acts freely and without inversions on a $\Lambda_0$-tree,
3. for every edge $e \in Y^*$ with an endpoint $x^*$ an edge group $\Gamma_e$ is either maximal abelian subgroup in $\Gamma_{x^*}$ or is trivial and $\Gamma_{x^*}$ is not abelian,
4. if $e_1, e_2, e_3 \in Y^*$ are edges with an endpoint $x^*$ then $\Gamma_{e_1}, \Gamma_{e_2}, \Gamma_{e_3}$ are not all conjugate in $\Gamma_{x^*}$.

Conversely, from the existence of a graph $(\Gamma, Y^*)$ satisfying conditions (1)–(4) it follows that $G$ acts freely and without inversions on a $\mathbb{Z} \times \Lambda_0$-tree in the following cases: $Y^*$ is a tree, $\Lambda_0 \subset Q$ and either $\Lambda_0 = Q$ or $Y^*$ is finite.

### 2.4 $\mathbb{R}$-trees

The case when $\Lambda = \mathbb{R}$ in the theory of groups acting on $\Lambda$-trees is the most well-studied (other than $\Lambda = \mathbb{Z}$, of course). $\mathbb{R}$-trees are usual metric spaces with nice properties which makes them very attractive from geometric point of view. The term $\mathbb{R}$-tree was introduced by Morgan and Shalen [99] to describe a type of space that was first defined by Tits [129]. In the last three decades $\mathbb{R}$-trees have played a prominent role in topology, geometry, and geometric group theory. They are the most simple of geodesic spaces, and yet by Theorem 7 below, every length space is an orbit space of an $\mathbb{R}$-tree. Lots of results were obtained in the last two decades about group actions on these objects. The most celebrated one is Rips’ Theorem about free actions and a more general result of Bestvina and Feighn about stable actions on $\mathbb{R}$-trees (see [37, 15]). In particular, the main result of Bestvina and Feighn together with the idea of obtaining a stable action on an $\mathbb{R}$-tree as a limit of actions on an infinite sequence of $\mathbb{Z}$-trees gives a very powerful tool in obtaining non-trivial decompositions of groups into fundamental groups of graphs of groups which is known as *Rips machine*. Such decompositions of groups as iterated applications of the operations of free product with amalgamation and HNN extension are called *splittings*. 
An $\mathbb{R}$-tree $(X, d)$ is a $\Lambda$-metric space which satisfies the axioms (T1) – (T3) listed in Subsection 2.2 for $\Lambda = \mathbb{R}$ with usual order. Hence, all the definitions and notions given in Section 2 hold for $\mathbb{R}$-trees.

**Proposition 1.** [25, Proposition 2.2.3] Let $(X, d)$ be an $\mathbb{R}$-metric space. Then the following are equivalent:

1. $(X, d)$ is an $\mathbb{R}$-tree,
2. given two points of $X$, there is a unique segment (with more than one point) having them as endpoints,
3. $(X, d)$ is geodesic and it contains no subspace homeomorphic to the circle.

**Example 3.** Let $Y = \mathbb{R}^2$ be the plane, but with metric $p$ defined by

$$p((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$

That is, to measure the distance between two points not on the same vertical line, we take their projections onto the horizontal axis, and add their distances to these projections and the distance between the projections (distance in the usual Euclidean sense).

**Example 4.** [25, Proposition 2.2.5] Given a simplicial tree $\Gamma$, one can construct its realization $\text{real}(\Gamma)$ by identifying each non-oriented edge of $\Gamma$ with the unit interval. The metric on $\text{real}(\Gamma)$ is induced from $\Gamma$.

**Example 5.** Let $G$ be a $\delta$-hyperbolic group. Then its Cayley graph with respect to any finite generating set $S$ is a $\delta$-hyperbolic metric space $(X, d)$ (where $d$ is a word metric) on which $G$ acts by isometries. Now, the asymptotic cone $\text{Cone}_\omega(X)$ of $G$ is a real tree (see [16, 30, 35]) on which $G$ acts by isometries.

An $\mathbb{R}$-tree is called polyhedral if the set of all branch points and endpoints is closed and discrete. Polyhedral $\mathbb{R}$-trees have strong connection with simplicial trees as shown below.

**Theorem 4.** [25, Theorem 2.2.10] An $\mathbb{R}$-tree $(X, d)$ is polyhedral if and only if it is homeomorphic to $\text{real}(\Gamma)$ (with metric topology) for some simplicial tree $\Gamma$.

Now we briefly recall some known results related to group actions on $\mathbb{R}$-trees. The first result shows that an action on a $\Lambda$-tree always implies an action on an $\mathbb{R}$-tree.

**Theorem 5.** [25, Theorem 4.1.2] If a finitely generated group $G$ has a non-trivial action on a $\Lambda$-tree for some ordered abelian group $\Lambda$ then it has a non-trivial action on some $\mathbb{R}$-tree.

Observe that in general nice properties of the action on a $\Lambda$-tree are not preserved when passing to the corresponding action on an $\mathbb{R}$-tree above.

The next result was one of the first in the theory of group actions on $\mathbb{R}$-trees. Later it was generalized to the case of an arbitrary $\Lambda$, see Theorem 4.
Theorem 6. [51] Let $G$ be a group acting freely and without inversions on an $\mathbb{R}$-tree $X$, and suppose $g, h \in G \setminus \{1\}$. Then $\langle g, h \rangle$ is either free of rank two or abelian.

Let $(X, d)$ be a metric space. Then $d$ is said to be a length metric if the distance between every pair of points $x, y \in X$ is equal to the infimum of the length of rectifiable curves joining them. (If there are no such curves then $d(x, y) = \infty$.) If $d$ is a length metric then $(X, d)$ is called a length space.

Theorem 7. [12] Every length space (resp. complete length space) $(X, d)$ is the metric quotient of a (resp. complete) $\mathbb{R}$-tree $(X, d)$ via the free isometric action of a locally free subgroup $\Gamma(X)$ of the isometry group $\text{Isom}(X)$.

It is not hard to define an action of a free abelian group on an $\mathbb{R}$-tree.

Example 6. Let $A = \langle a, b \rangle$ be a free abelian group. Define an action of $A$ on $\mathbb{R}$ (which is an $\mathbb{R}$-tree) by embedding $A$ into $\text{Isom}(\mathbb{R})$ as follows

$$a \rightarrow t_1, \quad b \rightarrow t_{\sqrt{2}},$$

where $t_\alpha(x) = x + \alpha$ is a translation. It is easy to see that

$$a^n b^m \rightarrow t_{n+m\sqrt{2}},$$

and since 1 and $\sqrt{2}$ are rationally independent it follows that the action is free.

The following result was very important in the direction of classifying finitely generated $\mathbb{R}$-free groups.

Theorem 8. [84] The fundamental group of a closed surface is $\mathbb{R}$-free, except for the non-orientable surfaces of genus 1, 2 and 3.

Then, in 1991 E. Rips completely classified finitely generated $\mathbb{R}$-free groups. The ideas outlined by Rips were further developed by Gaboriau, Levitt and Paulin who gave a complete proof of this classification in [37].

Theorem 9 (Rips’ Theorem). Let $G$ be a finitely generated group acting freely and without inversions on an $\mathbb{R}$-tree. Then $G$ can be written as a free product $G = G_1 * \cdots * G_n$ for some integer $n \geq 1$, where each $G_i$ is either a finitely generated free abelian group, or the fundamental group of a closed surface.

It is worth mentioning that there are examples by Dunwoody [17] and Zastrow [38] of infinitely generated groups that are not free products of fundamental groups of closed surfaces and abelian groups, but which act freely on an $\mathbb{R}$-tree. Zastrow’s group $G$ contains one of the two Dunwoody groups as a subgroup. The other group is a Kurosh group.

Berestovskii and Plaut proved the following result. We first introduce necessary notation. If $(X, d)$ is the length space, then $\mathbb{R}$-tree $\overline{X}$ is defined as the space of based “non-backtracking” rectifiable paths in $X$, where the distance between two paths is the sum of their lengths from the first bifurcation point to their endpoints. The group $\Gamma(X) \subset \overline{X}$ is the subset of loops with a natural group structure and the quotient mapping $\phi : \overline{X} \to X$ is the end-point map. We will refer to $\overline{X}$ as the covering $\mathbb{R}$-tree of $X$. 

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Theorem 10. Let $X$ be $S_c$ (Sierpinski carpet), $M$ a complete Riemannian manifold $M_n$ of dimension $n = 2$, or the Hawaiian earring $H$ with any compatible length metric $d$. Then $\Gamma(X)$ is an infinitely generated, locally free group that is not free and not a free product of surface groups and abelian groups, but acts freely on the $\mathbb{R}$-tree $\overline{X}$. Moreover, the $\mathbb{R}$-tree $\overline{X}$ is a minimal invariant subtree with respect to this action.

2.5 Rips-Bestvina-Feighn machine

Suppose $G$ is a finitely presented group acting isometrically on an $\mathbb{R}$-tree $\Gamma$. We assume the action to be non-trivial and minimal. Since $G$ is finitely presented there is a finite simplicial complex $K$ of dimension at most 2 such that $\pi_1(K) \simeq G$. Moreover, one can assume that $K$ is a band complex with underlying union of bands which is a finite simplicial $\mathbb{R}$-tree $X$ with finitely many bands of the type $[0,1] \times \alpha$, where $\alpha$ is an arc of the real line, glued to $X$ so that $\{0\} \times \alpha$ and $\{1\} \times \alpha$ are identified with sub-arcs of edges of $X$. Following [15] (the construction originally appears in [99]) one can construct a transversely measured lamination $L$ on $K$ and an equivariant map $\phi : \tilde{K} \to \Gamma$, where $\tilde{K}$ is the universal cover of $K$, which sends leaves of the induced lamination on $\tilde{K}$ to points in $\Gamma$. The complex $K$ together with the lamination $L$ is called a band complex with $\tilde{K}$ resolving the action of $G$ on $\Gamma$.

Now, Rips-Bestvina-Feighn machine is a procedure which given a band complex $K$, transforms it into another band complex $K'$ (we still have $\pi_1(K') \simeq G$), whose lamination splits into a disjoint union of finitely many sub-laminations of several types - simplicial, surface, toral, thin - and these sub-laminations induce a splitting of $K'$ into sub-complexes containing them. $K'$ can be thought of as the “normal form” of the band complex $K$. Analyzing the structure of $K'$ and its sub-complexes one can obtain some information about the structure of the group $G$.

In particular, in the case when the original action of $G$ on $\Gamma$ is stable one can obtain a splitting of $G$. Recall that a non-degenerate (that is, containing more than one point) subtree $S$ of $\Gamma$ is stable if for every non-degenerate subtree $S'$ of $S$, we have $\text{Fix}(S') = \text{Fix}(S)$ (here, $\text{Fix}(I) \subseteq G$ consists of all elements which fix $I$ point-wise). The action of $G$ on $\Gamma$ is stable if every non-degenerate subtree of $T$ contains a stable subtree. We say that a group $G$ splits over a subgroup $E$ is $G$ is either an amalgamated product or an HNN extension over $E$ (with the edge group $E$).

Theorem 11. [15] Theorem 9.5] Let $G$ be a finitely presented group with a nontrivial, stable, and minimal action on an $\mathbb{R}$-tree $\Gamma$. Then either

1. $G$ splits over an extension $E$-by-cyclic, where $E$ fixes a segment of $\Gamma$, or
2. $\Gamma$ is a line. In this case, $G$ splits over an extension of the kernel of the action by a finitely generated free abelian group.

The key ingredient of the Rips-Bestvina-Feighn machine is a set of particular operations, called moves, on band complexes applied in a certain order. These
operations originate from the work of Makanin [85] and Razborov [109] that ideas of Rips are built upon.

Observe that the group $G$ in Theorem 11 must be finitely presented. To obtain a similar result about finitely generated groups acting on $\mathbb{R}$-trees one has to further restrict the action. An action of a group $G$ on an $\mathbb{R}$-tree $\Gamma$ satisfies the ascending chain condition if for every decreasing sequence

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

of arcs in $\Gamma$ which converge into a single point, the corresponding sequence

$$\text{Fix}(I_1) \subset \text{Fix}(I_2) \subset \cdots \subset \text{Fix}(I_n) \subset \cdots$$

stabilizes.

**Theorem 12.** [17] Let $G$ be a finitely generated group with a nontrivial minimal action on an $\mathbb{R}$-tree $\Gamma$. If

1. $\Gamma$ satisfies the ascending chain condition,
2. for any unstable arc $J$ of $\Gamma$,
   - (a) $\text{Fix}(J)$ is finitely generated,
   - (b) $\text{Fix}(J)$ is not a proper subgroup of any conjugate of itself, that is, if $\text{Fix}(J)^g \subset \text{Fix}(J)$ for some $g \in G$ then $\text{Fix}(J)^g = \text{Fix}(J)$.

Then either

1. $G$ splits over a subgroup $H$ which is an extension of the stabilizer of an arc of $\Gamma$ by a cyclic group, or
2. $\Gamma$ is a line.

Now, we will discuss some applications of the above results which are based on the construction outlined in [13] and [104] making possible to obtain isometric group actions on $\mathbb{R}$-trees as Gromov-Hausdorff limits of actions on hyperbolic spaces. All the details can be found in [14].

Let $(X, d_X)$ be a metric space equipped with an isometric action of a group $G$ which can be viewed as a homomorphism $\rho : G \to \text{Isom}(X)$. Assume that $X$ contains a point $\varepsilon$ which is not fixed by $G$. In this case, we call the triple $(X, \varepsilon, \rho)$ a based $G$-space.

Observe that given a based $G$-space $(X, \varepsilon, \rho)$ one can define a pseudometric $d = d_{(X, \varepsilon, \rho)}$ on $G$ as follows

$$d(g, h) = d_X(\rho(g) \cdot \varepsilon, \rho(h) \cdot \varepsilon).$$

Now, the set $\mathcal{D}$ of all non-trivial pseudometrics on $G$ equipped with compact open topology and then taken up to rescaling by positive reals (projectivized),
forms a topological space and we say that a sequence \((X_i, \varepsilon_i, \rho_i), i \in \mathbb{N}\) of based \(G\)-spaces converges to the based \(G\)-space \((X, \varepsilon, \rho)\) and write
\[
\lim_{i \to \infty} (X_i, \varepsilon_i, \rho_i) = (X, \varepsilon, \rho)
\]
if we have \([d(X_i, \varepsilon_i, \rho_i)] \to [d(X, \varepsilon, \rho)]\) for the projectivized equivariant pseudometrics in \(D\). Now, the following result is the main tool in obtaining isometric group actions on \(\mathbb{R}\)-trees from actions on Gromov-hyperbolic spaces.

**Theorem 13.** [14, Theorem 3.3] Let \((X_i, \varepsilon_i, \rho_i), i \in \mathbb{N}\) be a convergent sequence of based \(G\)-spaces. Assume that
1. there exists \(\delta > 0\) such that every \(X_i\) is \(\delta\)-hyperbolic,
2. there exists \(g \in G\) such that the sequence \(d_{X_i}(\varepsilon_i, \rho_i(g) \cdot \varepsilon_i)\) is unbounded.

Then there is a based \(G\)-space \((\Gamma, \varepsilon)\) which is an \(\mathbb{R}\)-tree and an isometric action \(\rho : G \to \text{Isom}(\Gamma)\) such that \((X_i, \varepsilon_i, \rho_i) \to (\Gamma, \varepsilon, \rho)\).

In fact, the above theorem does not guarantee that the limiting action of \(G\) on \(\Gamma\) has no global fixed points. But in the case when \(G\) is finitely generated and each \(X_i\) is proper (closed metric balls are compact), it is possible to choose base-points in \(\varepsilon_i \in X_i\) to make the action of \(G\) on \(\Gamma\) non-trivial (see [14, Proposition 3.8, Theorem 3.9]). Moreover, one can retrieve some information about stabilizers of arcs in \(\Gamma\) (see [14, Proposition 3.10]).

Note that Theorem 13 can also be interpreted in terms of asymptotic cones (see [30, 31] for details).

The power of Theorem 13 becomes obvious in particular when a finitely generated group \(G\) has infinitely many pairwise non-conjugate homomorphisms \(\phi_i : G \to H\) into a word-hyperbolic group \(H\). In this case, each \(\phi_i\) defines an action of \(G\) on the Cayley graph \(X\) of \(H\) with respect to some finite generating set. Now, one can define \(X_i\) to be \(X\) with a word metric rescaled so that the sequence of \((X_i, \varepsilon_i, \rho_i), i \in \mathbb{N}\) satisfies the requirements of Theorem 13 and thus obtain a non-trivial isometric action of \(G\) on an \(\mathbb{R}\)-tree. Many results about word-hyperbolic groups were obtained according to this scheme, for example, the following classical result.

**Theorem 14.** [105] Let \(G\) be a word-hyperbolic group such that the group of its outer automorphisms \(\text{Out}(G)\) is infinite. Then \(G\) splits over a virtually cyclic group.

Combined with the shortening argument due to Rips and Sela [117], this scheme gives many other results about word-hyperbolic groups, for example, the theorems below.

**Theorem 15.** [117] Let \(G\) be a torsion-free freely indecomposable word-hyperbolic group. Then the internal automorphism group (that consists of automorphisms obtained by compositions of Dehn twists and inner automorphisms) of \(G\) has finite index in \(\text{Aut}(G)\).
Theorem 16. Let $G$ be a finitely presented torsion-free freely indecomposable group and let $H$ be a word-hyperbolic group. Then there are only finitely many conjugacy classes of subgroups of $G$ isomorphic to $H$.

There are similar recent results for relatively hyperbolic groups [42].

For more detailed account of applications of the Rips-Bestvina-Feighn machine please refer to [14].

2.6 Lyndon length functions

In 1963 Lyndon (see [81]) introduced a notion of \textit{length function on a group} in an attempt to axiomatize cancelation arguments in free groups as well as free products with amalgamation and HNN extensions, and to generalize them to a wider class of groups. The main idea was to measure the amount of cancellation in passing to the reduced form of a product of reduced words in a free group and free constructions, and it turned out that the cancelation process could be described by rather simple axioms. The idea of using length functions became quite popular (see, for example, [51, 23, 52]), and then it turned out that the language of length functions described the same class of groups as the language of actions on trees. Below we give the axioms of (Lyndon) length function and recall the main results in this field.

Let $G$ be a group and $\Lambda$ be an ordered abelian group. Then a function $l : G \to \Lambda$ is called a \textit{(Lyndon) length function} on $G$ if the following conditions hold:

(L1) $\forall g \in G : l(g) \geq 0$ and $l(1) = 0$,

(L2) $\forall g \in G : l(g) = l(g^{-1})$,

(L3) $\forall f,g,h \in G : c(f,g) > c(f,h)$ implies $c(f,h) = c(g,h)$,

where $c(f,g) = \frac{1}{2}(l(f) + l(g) - l(f^{-1}g))$.

Observe that in general $c(f,g) \notin \Lambda$, but $c(f,g) \in \Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$, where $\mathbb{Q}$ is the additive group of rational numbers, so, in the axiom (L3) we view $\Lambda$ as a subgroup of $\Lambda_{\mathbb{Q}}$. But in some cases the requirement $c(f,g) \in \Lambda$ is crucial so we state it as a separate axiom.

(L4) $\forall f,g \in G : c(f,g) \in \Lambda$.

It is not difficult to derive the following two properties of Lyndon length functions from the axioms (L1) – (L3):

- $\forall f,g \in G : l(fg) \leq l(f) + l(g)$,

- $\forall f,g \in G : 0 \leq c(f,g) \leq \min\{l(f), l(g)\}$.

The following examples motivated the whole theory of groups with length functions.
Example 7. Given a free group $F(X)$ on the set $X$ one can define a (Lyndon) length function on $F$ as follows

$$w(X) \to |w(X)|,$$

where $| \cdot |$ is the length of the reduced word in $X \cup X^{\pm 1}$ representing $w$.

Example 8. Given two groups $G_1$ and $G_2$ with length functions $L_1 : G_1 \to \Lambda$ and $L_2 : G_2 \to \Lambda$ for some ordered abelian group $\Lambda$ one can construct a length function on $G_1 * G_2$ as follows (see [25, Proposition 5.1.1]). For any $g \in G_1 * G_2$ such that

$$g = f_1 g_1 \cdots f_k g_k f_{k+1},$$

where $f_i \in G_2$, $i \in [1,k+1]$, $f_i \neq 1$, $i \in [2,k]$ and $1 \neq g_i \in G_2$, $i \in [1,k]$, define $L(1) = 0$ and if $g \neq 1$ then

$$L(g) = \sum_{i=1}^{k+1} L_1(f_i) + \sum_{j=1}^k L_2(g_j) \in \Lambda.$$

A length function $l : G \to \Lambda$ is called free, if it satisfies

(L5) $l(g^2) > l(g)$ for all non-trivial $g \in G$.

Obviously, the $\mathbb{Z}$-valued length function constructed in Example 7 is free. The converse is shown below (see also [52] for another proof of this result).

Theorem 17. [81] Any group $G$ with a length function $L : G \to \mathbb{Z}$ can be embedded into a free group $F$ of finite rank whose natural length function extends $L$.

Example 9. Given two groups $G_1$ and $G_2$ with free length functions $L_1 : G_1 \to \Lambda$ and $L_2 : G_2 \to \Lambda$ for some ordered abelian group $\Lambda$, the length function on $G_1 * G_2$ constructed in Example 8 is free.

Observe that if a group $G$ acts on a $\Lambda$-tree $(X,d)$ then we can fix a point $x \in X$ and consider a function $l_x : G \to \Lambda$ defined as $l_x(g) = d(x,gx)$. Such a function $l_x$ on $G$ we call a length function based at $x$. It is easy to check that $l_x$ satisfies all the axioms (L1) – (L4) of Lyndon length function. Now if $\| \cdot \|$ is the translation length function associated with the action of $G$ on $(X,d)$ (for $g \in G$, $\|g\| = \inf \{d(p,gp), p \in X\}$), then the following properties show the connection between $l_x$ and $\| \cdot \|$.

(i) $l_x(g) = \|g\| + 2d(x,A_g)$ (where $A_g$ is the axis of $g$), if $g$ is not an inversion.

(ii) $\|g\| = \max\{0, l_x(g^2) - l_x(g)\}$.

Here, it should be noted that for points $x \notin A_g$, there is a unique closest point of $A_g$ to $x$. The distance between these points is the one referred to in (i). While $A_g = A_g^r$ for all $n \neq 0$ in the case where $g$ is hyperbolic, if $g$ fixes a point, it is possible that $A_g \subset A_g^2$. We may have $l_x(g^2) - l_x(g) < 0$ in this case. Free
actions are characterized, in the language of length functions, by the facts (a) \( \|g\| > 0 \) for all \( g \neq 1 \), and (b) \( l_x(g^2) > l_x(g) \) for all \( g \neq 1 \). The latter follows from the fact that \( \|g^n\| = n\|g\| \) for all \( g \). We note that there are properties for the translation length function which were shown to essentially characterize actions on \( \Lambda \)-trees, up to equivariant isometry, by Parry, \[106]\.

The following theorem is one of the most important results in the theory of length functions.

**Theorem 18.** \[23\] Let \( G \) be a group and \( l : G \to \Lambda \) a Lyndon length function satisfying condition (L4). Then there are a \( \Lambda \)-tree \( (X,d) \), an action of \( G \) on \( X \), and a point \( x \in X \) such that \( l = l_x \).

The proof is constructive, that is, one can define a \( \Lambda \)-metric space out of \( G \) and \( l \), and then prove that this space is in fact a \( \Lambda \)-tree on which \( G \) acts by isometries (see \[25\], Subsection 2.4) for details).

A length function \( l : G \to \Lambda \) is called regular if it satisfies the regularity axiom:

\[(L6) \quad \forall g, f \in G, \exists u, g_1, f_1 \in G : g = u \circ g_1 \quad \& \quad f = u \circ f_1 \quad \& \quad l(u) = c(g,f) .\]

Observe that a regular length function need not be free, conversely freeness does not imply regularity.

### 2.7 Finitely generated \( \mathbb{R}^n \)-free groups

Guirardel proved the following result that describes the structure of finitely generated \( \mathbb{R}^n \)-free groups, which is reminiscent of the Bass’ structural theorem for \( \mathbb{Z}^n \)-free groups. This is not by chance, since every \( \mathbb{Z}^n \)-free group is also \( \mathbb{R}^n \)-free, and ordered abelian groups \( \mathbb{Z}^n \) and \( \mathbb{R}^n \) have a similar convex subgroup structure. However, it is worth to point out that the original Bass argument for \( \Lambda = \mathbb{Z} \oplus \Lambda_0 \) does not work in the case of \( \Lambda = \mathbb{R} \oplus \Lambda_0 \).

**Theorem 19.** \[48\] Let \( G \) be a finitely generated, freely indecomposable \( \mathbb{R}^n \)-free group. Then \( G \) can be represented as the fundamental group of a finite graph of groups, where edge groups are cyclic and each vertex group is a finitely generated \( \mathbb{R}^{n-1} \)-free.

In fact, there is a more detailed version of this result, Theorem 7.2 in \[18\], which is rather technical, but gives more for applications. Observe also that neither Theorem \[19\] nor the more detailed version of it, does not “characterize” finitely generated \( \mathbb{R}^n \)-free groups, i.e. the converse of the theorem does not hold. Nevertheless, the result is very powerful and gives several important corollaries.

**Corollary 1.** \[48\] Every finitely generated \( \mathbb{R}^n \)-free group is finitely presented.

This comes from Theorem \[19\] and elementary properties of free constructions by induction on \( n \).

Theorem \[19\] and the Combination Theorem for relatively hyperbolic groups proved by Dahmani in \[27\] imply the following.
Corollary 2. Every finitely generated $\mathbb{R}^n$-free group is hyperbolic relative to its non-cyclic abelian subgroups.

A lot is known about groups which are hyperbolic relative to its maximal abelian subgroups (toral relatively hyperbolic groups), so all of this applies to $\mathbb{R}^n$-free groups. We do not mention any of these results here, because we discuss their much more general versions in the next section in the context of $\Lambda$-free groups for arbitrary $\Lambda$.

3 Finitely presented $\Lambda$-free groups

In [69] the following main problem of the Alperin-Bass program was solved for finitely presented groups.

**Problem.** Describe finitely presented (finitely generated) $\Lambda$-free groups for an arbitrary ordered abelian group $\Lambda$.

The structure of finitely presented $\Lambda$-free groups is going to be described in Section 3.2.

3.1 Regular actions

In this section we give a geometric characterization of group actions that come from regular length functions.

Let $G$ act on a $\Lambda$-tree $\Gamma$. The action is *regular with respect to* $x \in \Gamma$ if for any $g, h \in G$ there exists $f \in G$ such that $[x, fx] = [x, gx] \cap [x, hx]$.

The next lemma shows that regular actions exactly correspond to regular length functions (hence the term).

**Lemma 4.** [70] Let $G$ act on a $\Lambda$-tree $\Gamma$. Then the action of $G$ is regular with respect to $x \in \Gamma$ if and only if the length function $\ell_x : G \to \Lambda$ based at $x$ is regular.

If the action of $G$ is regular with respect to $x \in \Gamma$ then it is regular with respect to any $y \in Gx$.

**Lemma 5.** [70] Let $G$ act freely on a $\Lambda$-tree $\Gamma$ so that all branch points of $\Gamma$ are $G$-equivalent. Then the action of $G$ is regular with respect to any branch point in $\Gamma$.

**Proof.** Let $x$ be a branch point in $\Gamma$ and $g, h \in G$. If $g = h$ then $[x, gx] \cap [x, hx] = [x, gx]$ and $g$ is the required element. Suppose $g \neq h$. Since the action is free then $gx \neq hx$ and we consider the tripod formed by $x, gx, hx$. Hence, the center of the tripod $y$ is a branch point in $\Gamma$ and by the assumption there exists $f \in G$ such that $y = fx$. \hfill $\Box$

**Example 10.** Let $\Gamma'$ be the Cayley graph of a free group $F(x, y)$ with the basepoint $\varepsilon$. Let $\Gamma$ be obtained from $\Gamma'$ by adding an edge labeled by $z \neq x^{\pm 1}, y^{\pm 1}$ at every vertex of $\Gamma'$. $F(x, y)$ has a natural action on $\Gamma'$ which we can extend
to the action on $\Gamma$. The edge at $\varepsilon$ labeled by $z$ has an endpoint not equal to $\varepsilon$ and we denote it by $\varepsilon'$. Observe that the action of $F(x,y)$ on $\Gamma$ is regular with respect to $\varepsilon$ but is not regular with respect to $\varepsilon'$.

### 3.2 Structure of finitely presented $\Lambda$-free groups

A group $G$ is called a regular $\Lambda$-free group if it acts freely and regularly on a $\Lambda$-tree. We proved the following results.

**Theorem 20.** [69] Any finitely presented regular $\Lambda$-free group $G$ can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. $G_1$ is a free group,
2. $G_{i+1}$ is obtained from $G_i$ by finitely many HNN-extensions in which associated subgroups are maximal abelian, finitely generated, and the associated isomorphisms preserve the length induced from $G_i$.

**Theorem 21.** [69] Any finitely presented $\Lambda$-free group can be isometrically embedded into a finitely presented regular $\Lambda$-free group.

**Theorem 22.** [69] Any finitely presented $\Lambda$-free group $G$ is $\mathbb{R}^n$-free for an appropriate $n \in \mathbb{N}$, where $\mathbb{R}^n$ is ordered lexicographically.

In his book [25] Chiswell (see also [112]) asked the following very important question (Question 1, p. 250): If $G$ is a finitely generated $\Lambda$-free group, is $G$ $\Lambda_0$-free for some finitely generated abelian ordered group $\Lambda_0$? The following result answers this question in the affirmative in the strongest form. It comes from the proof of Theorem 22 (not the statement of the theorem itself).

**Theorem 23.** [69] Let $G$ be a finitely presented group with a free Lyndon length function $l : G \to \Lambda$. Then the subgroup $\Lambda_0$ generated by $l(G)$ in $\Lambda$ is finitely generated.

The following result follows directly from Theorem 20 and Theorem 21 by a simple application of Bass-Serre Theory.

**Theorem 24.** [69] Any finitely presented $\Lambda$-free group $G$ can be obtained from free groups by a finite sequence of amalgamated free products and HNN extensions along maximal abelian subgroups, which are free abelian groups of finite rank.

The following result concerns abelian subgroups of $\Lambda$-free groups. For $\Lambda = \mathbb{Z}^n$ it follows from the main structural result for $\mathbb{Z}^n$-free groups and [71], for $\Lambda = \mathbb{R}^n$ it was proved in [48]. The statement 1) below answers Question 2 (page 250) from [25] in the affirmative for finitely presented $\Lambda$-free groups.
Theorem 25. [69] Let \(G\) be a finitely presented \(\Lambda\)-free group. Then:

1) every abelian subgroup of \(G\) is a free abelian group of finite rank, which is uniformly bounded from above by the rank of the abelianization of \(G\).

2) \(G\) has only finitely many conjugacy classes of maximal non-cyclic abelian subgroups,

3) \(G\) has a finite classifying space and the cohomological dimension of \(G\) is \(r\) where \(r\) is the maximal rank of an abelian subgroup of \(G\), except if \(r = 1\) when the cohomological dimension of \(G\) is 1 or 2.

Theorem 26. [69] Every finitely presented \(\Lambda\)-free group is hyperbolic relative to its non-cyclic abelian subgroups.

This follows from the structural Theorem 20 and the Combination Theorem for relatively hyperbolic groups [27].

The following results answers affirmatively the strongest form of the Problem (GO3) from the Magnus list of open problems [11], in the case of finitely presented groups.

Corollary 3. Every finitely presented \(\Lambda\)-free group is biautomatic.

Proof. This follows from Theorem 26 and Rebbecchi’s result [116].

Definition 1. A hierarchy for a group \(G\) is a way to construct it starting with trivial groups by repeatedly taking amalgamated products \(A \ast_C B\) and HNN extensions \(A \ast_{C'} D\) whose vertex groups have shorter length hierarchies. The hierarchy is quasi convex if the amalgamated subgroup \(C\) is a finitely generated subgroup that embeds by a quasi-isometric embedding, and if \(C\) is malnormal in \(A \ast_C B\) or \(A \ast_{C'} D\).

Theorem 27. Every finitely presented \(\Lambda\)-free group \(G\) has a quasi-convex hierarchy with abelian edge groups.

Theorem 28. [134] Suppose \(G\) is toral relatively hyperbolic and has a malnormal quasi convex hierarchy. Then \(G\) is virtually special (therefore has a finite index subgroup that is a subgroup of a right angled Artin group (RAAG)).

As a corollary one gets the following result.

Corollary 4. Every finitely presented \(\Lambda\)-free group \(G\) is locally undistorted, that is, every finitely generated subgroup of \(G\) is quasi-isometrically embedded into \(G\).

Corollary 5. Every finitely presented \(\Lambda\)-free group \(G\) is virtually special, that is, some subgroup of finite index in \(G\) embeds into a right-angled Artin group.

The following result answers in the affirmative to Question 3 (page 250) from [25] in the case of finitely presented groups.
Theorem 29. Every finitely presented $\Lambda$-free group is right orderable and virtually orderable.

Since right-angled Artin groups are linear and the class of linear groups is closed under finite extension we get the following

Theorem 30. Every finitely presented $\Lambda$-free group is linear and, therefore, residually finite, equationally noetherian. It also has decidable word, conjugacy, subgroup membership, and diophantine problems.

Indeed, decidability of equations follows from [27]. Results of Dahmani and Groves [28] imply the following two corollaries.

Corollary 6. Let $G$ be a finitely presented $\Lambda$-free group. Then:

- $G$ has a non-trivial abelian splitting and one can find such a splitting effectively,
- $G$ has a non-trivial abelian JSJ-decomposition and one can find such a decomposition effectively.

Corollary 7. The isomorphism problem is decidable in the class of finitely presented $\Lambda$-free groups.

3.3 Limit groups

Limit groups (or finitely generated fully residually free groups) play an important part in modern group theory. They appear in many different situations: in combinatorial group theory as groups discriminated by $G$ ($\omega$-residually $G$-groups or fully residually $G$-groups) [8, 9, 91, 7, 10], in the algebraic geometry over groups as the coordinate groups of irreducible varieties over $G$ [7, 57, 58, 60, 120], group universally equivalent to $G$ [112, 44, 91], limit groups of $G$ in the Gromov-Hausdorff topology [24], in the theory of equations in groups [79, 109, 111, 57, 58, 60, 45], in group actions [13, 57, 83, 10, 48], in the solutions of Tarski problems [62, 122], etc. Their numerous characterizations connect group theory, topology and logic.

Recall, that a group $G$ is called fully residually free if for any non-trivial $g_1, \ldots, g_n \in G$ there exists a homomorphism $\phi$ of $G$ into a free group such that $\phi(g_1), \ldots, \phi(g_n)$ are non-trivial.

It is a crucial result that every limit group admits a free action on a $\mathbb{Z}^n$-tree for an appropriate $n \in \mathbb{N}$, where $\mathbb{Z}^n$ is ordered lexicographically (see [58]). The proof comes in several steps. The initial breakthrough is due to Lyndon, who introduced a construction of the free $\mathbb{Z}[t]$-completion $F^{\mathbb{Z}[t]}$ of a free group $F$ (nowadays it is called Lyndon’s free $\mathbb{Z}[t]$-group) and showed that this group, as well as all its subgroups, is fully residually free [79]. Much later Remeslennikov proved that every finitely generated fully residually free group has a free Lyndon length function with values in $\mathbb{Z}^n$ (but not necessarily ordered lexicographically) [112]. That was a first link between limit groups and free actions on $\mathbb{Z}^n$-trees. In 1995 Myasnikov and Remeslennikov showed that Lyndon free exponential group
F^Z[t] has a free Lyndon length function with values in \( Z^n \) with lexicographical ordering and stated a conjecture that every limit group embeds into \( F^Z[t] \). Finally, Kharlampovich and Myasnikov proved that every limit group \( G \) embeds into \( F^Z[t] \) \[58\].

Below, following \[93\] we construct a free \( Z[t] \)-valued length function on \( F^Z[t] \) which combined with the result of Kharlampovich and Myasnikov mentioned above gives a free \( Z^n \)-valued length function on a given limit group \( G \). There are various algorithmic applications of these results which are based on the technique of infinite words and Stallings foldings techniques for subgroups of \( F^Z[t] \) (see \[63, 69\]).

### 3.4 Lyndon’s free group \( F^Z[t] \)

Let \( A \) be an associative unitary ring. A group \( G \) is termed an \( A \)-group if it is equipped with a function (exponentiation) \( G \times A \to G: (g, \alpha) \mapsto g^\alpha \) satisfying the following conditions for arbitrary \( g, h \in G \) and \( \alpha, \beta \in A \):

1. \((\text{Exp}1)\) \quad \( g^1 = g \), \quad \( g^{\alpha+\beta} = g^\alpha g^\beta \), \quad \( g^{\alpha \beta} = (g^\alpha)^\beta \),
2. \((\text{Exp}2)\) \quad \( g^{-1} h^\alpha g = (g^{-1} hg)^\alpha \),
3. \((\text{Exp}3)\) \quad if \( g \) and \( h \) commute, then \( (gh)^\alpha = g^\alpha h^\alpha \).

The axioms (Exp1) and (Exp2) were introduced originally by R. Lyndon in \[79\], the axiom (Exp3) was added later in \[98\]. A homomorphism \( \phi : G \to H \) between two \( A \)-groups is termed an \( A \)-homomorphism if \( \phi(g^\alpha) = \phi(g)^\alpha \) for every \( g \in G \) and \( \alpha \in A \). It is not hard to prove (see, \[98\]) that for every group \( G \) there exists an \( A \)-group \( H \) (which is unique up to an \( A \)-isomorphism) and a homomorphism \( \mu : G \to H \) such that for every \( A \)-group \( K \) and every \( A \)-homomorphism \( \theta : G \to K \), there exists a unique \( A \)-homomorphism \( \phi : H \to K \) such that \( \phi \mu = \theta \). We denote \( H \) by \( G^A \) and call it the \( A \)-completion of \( G \).

In \[72\] an effective construction of \( F^Z[t] \) was given in terms of extensions of centralizers. For a group \( G \) let \( S = \{ C_i \mid i \in I \} \) be a set of representatives of conjugacy classes of proper cyclic centralizers in \( G \), that is, every proper cyclic centralizer in \( G \) is conjugate to one from \( S \), and no two centralizers from \( S \) are conjugate. Then the HNN-extension

\[
H = \langle G, s_{i,j} \mid (i \in I, j \in \mathbb{N}) \mid [s_{i,j}, u_i] = [s_{i,j}, s_{i,k}] = 1 (u_i \in C_i, i \in I, j, k \in \mathbb{N}) \rangle,
\]

is termed an extension of cyclic centralizers in \( G \). Now the group \( F^Z[t] \) is isomorphic to the direct limit of the following infinite chain of groups:

\[
F = G_0 < G_1 < \cdots < G_n < \cdots < \cdots,
\]

where \( G_{i+1} \) is obtained from \( G_i \) by extension of all cyclic centralizers in \( G_i \).
3.5 Limit groups embed into $F_{\mathbb{Z}[t]}$

The following results illustrate the connection of limit groups and finitely generated subgroups of $F_{\mathbb{Z}[t]}$.

**Theorem 31.** [58] Given a finite presentation of a finitely generated fully residually free group $G$ one can effectively construct an embedding $\phi : G \to F_{\mathbb{Z}[t]}$ (by specifying the images of the generators of $G$).

Combining Theorem 31 with the result on the representation of $F_{\mathbb{Z}[t]}$ as a union of a sequence of extensions of centralizers one can get the following theorem.

**Theorem 32.** [60] Given a finite presentation of a finitely generated fully residually free group $G$ one can effectively construct a finite sequence of extensions of centralizers

$$ F < G_1 < \cdots < G_n, $$

where $G_{i+1}$ is an extension of the centralizer of some element $u_i \in G_i$ by an infinite cyclic group $\mathbb{Z}$, and an embedding $\psi^* : G \to G_n$ (by specifying the images of the generators of $G$).

Now Theorem 32 implies the following important corollaries.

**Corollary 8.** [60] For every freely indecomposable non-abelian finitely generated fully residually free group one can effectively find a non-trivial splitting (as an amalgamated product or HNN extension) over a cyclic subgroup.

**Corollary 9.** [60] Every finitely generated fully residually free group is finitely presented. There is an algorithm that, given a presentation of a finitely generated fully residually free group $G$ and generators of the subgroup $H$, finds a finite presentation for $H$.

**Corollary 10.** [60] Every finitely generated residually free group $G$ is a subgroup of a direct product of finitely many fully residually free groups; hence, $G$ is embeddable into $F_{\mathbb{Z}[t]} \times \cdots \times F_{\mathbb{Z}[t]}$. If $G$ is given as a coordinate group of a finite system of equations, then this embedding can be found effectively.

Let $K$ be an HNN-extension of a group $G$ with associated subgroups $A$ and $B$. Then $K$ is called a separated HNN-extension if for any $g \in G$, $A^g \cap B = 1$.

**Corollary 11.** [60] Let a group $G$ be obtained from a free group $F$ by finitely many centralizer extensions. Then every finitely generated subgroup $H$ of $G$ can be obtained from free abelian groups of finite rank by finitely many operations of the following type: free products, free products with abelian amalgamated subgroups at least one of which is a maximal abelian subgroup in its factor, free extensions of centralizers, separated HNN-extensions with abelian associated subgroups at least one of which is maximal.

**Corollary 12.** [60, 47] One can enumerate all finite presentations of finitely generated fully residually free groups.

**Corollary 13.** [58] Every finitely generated fully residually free group acts freely on some $\mathbb{Z}^n$-tree with lexicographic order for a suitable $n$.  

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3.6 Description of \( \mathbb{Z}^n \)-free groups

Given two \( \mathbb{Z}[t] \)-free groups \( G_1, G_2 \) (with free Lyndon lengths functions \( \ell_1 \) and \( \ell_2 \)) and maximal abelian subgroups \( A \leq G_1, B \leq G_2 \) such that

(a) \( A \) and \( B \) are cyclically reduced,

(b) there exists an isomorphism \( \phi : A \to B \) such that \( \ell_2(\phi(a)) = \ell_1(a) \) for any \( a \in A \).

Then we call the amalgamated free product

\[
\langle G_1, G_2 \mid A \overset{\phi}{=} B \rangle
\]

the length-preserving amalgam of \( G_1 \) and \( G_2 \).

Given a \( \mathbb{Z}[t] \)-free group \( H \) and non-conjugate maximal abelian subgroups \( A, B \leq H \) such that

(a) \( A \) and \( B \) are cyclically reduced,

(b) there exists an isomorphism \( \phi : A \to B \) such that \( \ell(\phi(a)) = \ell(a) \) and \( a \) is not conjugate to \( \phi(a)^{-1} \) in \( H \) for any \( a \in A \).

Then we call the HNN extension

\[
\langle H, t \mid t^{-1}At = B \rangle
\]

the length-preserving separated HNN extension of \( H \).

We now get the description of regular \( \mathbb{Z}^n \)-free groups in the following form.

Theorem 33. [63] A finitely generated group \( G \) is regular \( \mathbb{Z}^n \)-free if and only if it can be obtained from free groups by finitely many length-preserving separated HNN extensions and centralizer extensions.

Theorem 34. [69] A finitely generated group \( G \) is \( \mathbb{Z}^n \)-free if and only if it can be obtained from free groups by a finite sequence of length-preserving amalgams, length-preserving separated HNN extensions, and centralizer extensions.

Using this description it was proved in [18] that \( \mathbb{Z}^n \)-free groups are \( \text{CAT}(0) \).

4 Products of trees

4.1 Lattices from square complexes

Lattices in products of trees provide examples for many interesting group theoretic properties, for example there are finitely presented infinite simple groups [19], [108] and many are not residually finite [133]. For a great survey of results of Burger, Mozes, Zimmer on simple infinite groups acting on products of trees see [10].
Torsion free lattices that are acting simply transitively on the vertices of the product of trees (not interchanging the factors) are fundamental groups of square complexes with just one vertex, a complete bipartite link and a VH-structure. There are many of such lattices, see [127] for a mass formula, but very rarely these lattices arise from an arithmetic context. The main purpose of this chapter is to concentrate on the arithmetic case, mentioning other cases as well as a mass formula for the relevant square complexes.

In this section we give a quick introduction to the geometry of square complexes and fix the terminology.

### 4.1.1 Square complexes and products of trees

A square complex \( S \) is a 2-dimensional combinatorial cell complex: its 1-skeleton consists of a graph \( S_1 = (V(S), E(S)) \) with set of vertices \( V(S) \), and set of oriented edges \( E(S) \). The 2-cells of the square complex come from a set of squares \( S(S) \) that are combinatorially glued to the graph \( S_1 \) as explained below. Reversing the orientation of an edge \( e \in E(S) \) is written as \( e \mapsto e^{-1} \) and the set of unoriented edges is the quotient set
\[
\overline{E}(S) = E(S)/(e \sim e^{-1}).
\]

More precisely, a square \( \square \) is described by an equivalence class of 4-tuples of oriented edges \( e_i \in E(S) \)
\[
\square = (e_1, e_2, e_3, e_4)
\]
where the origin of \( e_{i+1} \) is the terminus of \( e_i \) (with \( i \) modulo 4). Such 4-tuples describe the same square if and only if they belong to the same orbit under the dihedral action generated by cyclically permuting the edges \( e_i \) and by the reverse orientation map
\[
(e_1, e_2, e_3, e_4) \mapsto (e_4^{-1}, e_3^{-1}, e_2^{-1}, e_1^{-1}).
\]

We think of a square shaped 2-cell glued to the (topological realization of the) respective edges of the graph \( S_1 \). For more details on square complexes we refer for example to [20]. Examples for square complexes are given by products of trees.

**Remark 1.** We note, that in our definition of a square complex each square is determined by its boundary. The group actions considered in the present survey are related only to such complexes.

Let \( T_n \) denote the \( n \)-valent tree. The product of trees
\[
M = T_m \times T_n
\]
is a Euclidean building of rank 2 and a square complex. Here we are interested in lattices, i.e., groups \( \Gamma \) acting discretely and cocompactly on \( M \) respecting
the structure of square complexes. The quotient $S = \Gamma \backslash M$ is a finite square complex, typically with orbifold structure coming from the stabilizers of cells.

We are especially interested in the case where $\Gamma$ is torsion free and acts simply transitively on the set of vertices of $M$. These yield the smallest quotients $S$ without non-trivial orbifold structure. Since $M$ is a CAT(0) space, any finite group stabilizes a cell of $M$ by the Bruhat–Tits fixed point lemma. Moreover, the stabilizer of a cell is pro-finite, hence compact, so that a discrete group $\Gamma$ acts with trivial stabilizers on $M$ if and only if $\Gamma$ is torsion free.

Let $S$ be a square complex. For $x \in V(S)$, let $E(x)$ denote the set of oriented edges originating in $x$. The link at the vertex $x$ in $S$ is the (undirected multi-)graph $Lk_x$ whose set of vertices is $E(x)$ and whose set of edges in $Lk_x$ joining vertices $a, b \in E(x)$ are given by corners $\gamma$ of squares in $S$ containing the respective edges of $S$, see [20].

A covering space of a square complex admits a natural structure as a square complex such that the covering map preserves the combinatorial structure. In this way, a connected square complex admits a universal covering space.

**Proposition 2.** The universal cover of a finite connected square complex is a product of trees if and only if the link $Lk_x$ at each vertex $x$ is a complete bipartite graph.

**Proof.** This is well known and follows, for example, from [3] Theorem C. \[\square\]

### 4.1.2 VH-structure

A vertical/horizontal structure, in short a VH-structure, on a square complex $S$ consists of a bipartite structure $\mathcal{E}(S) = E_v \sqcup E_h$ on the set of unoriented edges of $S$ such that for every vertex $x \in S$ the link $Lk_x$ at $x$ is the complete bipartite graph on the induced partition of $E(x) = E(x)_v \sqcup E(x)_h$. Edges in $E_v$ (resp. in $E_h$) are called vertical (resp. horizontal) edges. See [133] for general facts on VH-structures. The partition size of the VH-structure is the function $V(S) \to \mathbb{N} \times \mathbb{N}$ on the set of vertices

$$x \mapsto (\#E(x)_v, \#E(x)_h)$$

or just the corresponding tuple of integers if the function is constant. Here $\#(-)$ denotes the cardinality of a finite set.

We record the following basic fact, see [20] after Proposition 1.1:

**Proposition 3.** Let $S$ be a square complex. The following are equivalent.

1. The universal cover of $S$ is a product of trees $T_m \times T_n$ and the group of covering transformations does not interchange the factors.

2. There is a VH-structure on $S$ of constant partition size $(m, n)$.

**Corollary 14.** Torsion free cocompact lattices $\Gamma$ in $\text{Aut}(T_m) \times \text{Aut}(T_n)$ not interchanging the factors and up to conjugation correspond uniquely to finite square complexes with a VH-structure of partition size $(m, n)$ up to isomorphism.
Proof. A lattice $\Gamma$ yields a finite square complex $S = \Gamma \backslash T_m \times T_n$ of the desired type. Conversely, a finite square complex $S$ with VH-structure of constant partition size $(m, n)$ has universal covering space $M = T_m \times T_n$ by Proposition and the choice of a base point vertex $\hat{x} \in M$ above the vertex $x \in S$ identifies $\pi_1(S, x)$ with the lattice $\Gamma = Aut(M/S) \subset Aut(T_m) \times Aut(T_n)$. The lattice depends on the chosen base points only up to conjugation.

Simply transitive torsion free lattices not interchanging the factors as in Corollary [14] correspond to square complexes with only one vertex and a VH-structure, necessarily of constant partition size. These will be studied in the next section.

4.1.3 1-vertex square complexes

Let $S$ be a square complex with just one vertex $x \in S$ and a VH-structure $E(S) = E_v \sqcup E_h$. Passing from the origin to the terminus of an oriented edge induces a fixed point free involution on $E(x)_v$ and on $E(x)_h$. Therefore the partition size is necessarily a tuple of even integers.

Definition 2. A vertical/horizontal structure, in short VH-structure, in a group $G$ is an ordered pair $(A, B)$ of finite subsets $A, B \subseteq G$ such that the following holds.

1. Taking inverses induces fixed point free involutions on $A$ and $B$.
2. The union $A \cup B$ generates $G$.
3. The product sets $AB$ and $BA$ have size $\#A \cdot \#B$ and $AB = BA$.
4. The sets $AB$ and $BA$ do not contain 2-torsion.

The tuple $(\#A, \#B)$ is called the valency vector of the VH-structure in $G$.

Similar as in [20] §6.1, starting from a VH-structure the following construction

$$(A, B) \mapsto S_{A, B}$$

(3)

yields a square complex $S_{A, B}$ with one vertex and VH-structure starting from a VH-structure $(A, B)$ in a group $G$. The vertex set $V(S_{A, B})$ contains just one vertex $x$. The set of oriented edges of $S_{A, B}$ is the disjoint union

$E(S_{A, B}) = A \sqcup B$

with the orientation reversion map given by $e \mapsto e^{-1}$. Since $A$ and $B$ are preserved under taking inverses, there is a natural vertical/horizontal structure such that $E(x)_h = A$ and $E(x)_v = B$. The squares of $S_{A, B}$ are constructed as follows. Every relation in $G$

$ab = b'a'$

(4)

with $a, a' \in A$ and $b, b' \in B$ (not necessarily distinct) gives rise to a 4-tuple

$$(a, b, a^{-1}, b'^{-1}).$$

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The following relations are equivalent to \((4)\):

\[
\begin{align*}
a'b^{-1} &= b^{-1}a, \\
a^{-1}b' &= ba'^{-1}, \\
a'^{-1}b'^{-1} &= b^{-1}a^{-1}.
\end{align*}
\]

and we consider the four 4-tuples so obtained

\[
(a, b, a'^{-1}, b'^{-1}), \quad (a', b^{-1}, a^{-1}, b'), \quad (a^{-1}, b', a', b^{-1}), \quad (a'^{-1}, b'^{-1}, a, b)
\]
as equivalent. A square \(\square\) of \(S_{A,B}\) consists of an equivalence class of such 4-tuples arising from a relation of the form \((4)\).

**Lemma 6.** The link \(L_{k,x}\) of \(S_{A,B}\) in \(x\) is the complete bipartite graph \(L_{A,B}\) with vertical vertices labelled by \(A\) and horizontal vertices labelled by \(B\).

**Proof.** By \((3)\) of Definition 2 every pair \((a, b)\) \(\in\) \(A \times B\) occurs on the left hand side in a relation of the form \((4)\) and therefore the link \(L_{k,x}\) contains \(L_{A,B}\).

If \((4)\) holds, then the set of left hand sides of equivalent relations

\[
\{ab, a'b'^{-1}, a'^{-1}b', a'^{-1}b'^{-1}\}
\]
is a set of cardinality 4, because \(A\) and \(B\) and \(AB\) do not contain 2-torsion by Definition \((2)\) and the right hand sides of the equations are unique by Definition \((3)\) Therefore \(S_{A,B}\) only contains \((\#A \cdot \#B)/4\) squares. It follows that \(L_{k,x}\) has at most as many edges as \(L_{A,B}\), and, since it contains \(L_{A,B}\), must agree with it.

**Definition 3.** We will call the complex \(S_{A,B}\) a \((\#A, \#B)\)-complex, keeping in mind, that there are many complexes with the same valency vector. Also, if a group is a fundamental group of a \((\#A, \#B)\)-complex, we will call it a \((\#A, \#B)\)-group.

### 4.2 Mass formula for one vertex square complexes with VH-structure

In this section we present a mass formula for the number of one vertex square complexes with VH-structure up to isomorphism where each square complex is counted with the inverse order of its group of automorphisms as its weight \([127]\).

Let \(A\) (resp. \(B\)) be a set with fixed point free involution of size \(2m\) (resp. \(2n\)). In order to count one vertex square complexes \(S\) with VH-structure with vertical/horizontal partition \(A \cup B\) of oriented edges we introduce the generic matrix

\[
X = (x_{ab})_{a \in A, b \in B}
\]
with rows indexed by \(A\) and columns indexed by \(B\) and with \((a, b)\)-entry a formal variable \(x_{ab}\). Let \(X^t\) be the transpose of \(X\), let \(\tau_A\) (resp. \(\tau_B\)) be the permutation matrix for \(e \mapsto e^{-1}\) for \(A\) (resp. \(B\)). For a square \(\square\) we set

\[
x_{\square} = \prod_{e \in \square} x_e
\]
where the product ranges over the edges $e = (a, b)$ in the link of $S$ originating from □ and $x_e = x_{ab}$. Then the sum of the $x_{□}$, when □ ranges over all possible squares with edges from $A \sqcup B$, reads

$$
\sum □ x_{□} = \frac{1}{4} \text{tr} \left( \left( \tau_A X \tau_B X^t \right)^2 \right),
$$

and the number of one vertex square complexes $S$ with VH-structure of partition size $(2m, 2n)$ and edges labelled by $A$ and $B$ is given by

$$
\tilde{BM}_{m,n} = \frac{1}{(mn)!} \cdot \frac{\partial^{4mn}}{\prod_{a,b} \partial x_{ab}} \left( \frac{1}{4} \text{tr} \left( \left( \tau_A X \tau_B X^t \right)^2 \right) \right)^{mn}. \quad (5)
$$

Note that this is a constant polynomial.

We can turn this into a mass formula for the number of one vertex square complexes with VH-structure up to isomorphism where each square complex is counted with the inverse order of its group of automorphisms as its weight. We simply need to divide by the order of the universal relabelling

$$
\#(\text{Aut}(A, \tau_A) \times \text{Aut}(B, \tau_B)) = 2^n (n)! \cdot 2^m (m)!.
$$

Hence the mass of one vertex square complexes with VH-structure is given by

$$
BM_{m,n} = \frac{1}{2^{n+m+2mn}(n)! \cdot (m)! \cdot (mn)!} \cdot \frac{\partial^{4mn}}{\prod_{a,b} \partial x_{ab}} \left( \text{tr} \left( \left( \tau_A X \tau_B X^t \right)^2 \right) \right)^{mn}. \quad (6)
$$

The formula (5) reproduces the numerical values of $\tilde{BM}_{m,n}$ for small values $(2m, 2n)$ that were computed by Rattaggi in [107] table B.3. Here small means $mn \leq 10$.

### 4.3 Arithmetic groups acting on products of two trees

There is a deep connection between arithmetic lattices in products of trees and quaternion algebras. For background on quaternion algebras see [131].

For any ring $R$ we consider the $R$-algebra

$$
\mathbb{H}(R) = \{ a = a_0 + a_1 i + a_2 j + a_3 k; a_1, a_2, a_3, a_4 \in R \},
$$

with $R$-linear multiplication given by $ij = k = -ji$, $i^2 = j^2 = k^2 = -1$. An example of such an algebra are classical Hamiltonian quaternions $\mathbb{H}(R)$. Recall, that the (reduced) norm is a homomorphism

$$
\| \cdot \| : \mathbb{H}(R) \to R^\times, \| a \| = a_0^2 + a_1^2 + a_2^2 + a_3^2.
$$

Let $\mathbb{H}(\mathbb{Z})$ be the integer quaternions. Let $S_p$ be the set of integer quaternions

$$
a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}(\mathbb{Z})
$$
with \( a_0 > 0, a_0 \) odd, and \( |a|^2 = p \), so the reduced norm of \( a \) is \( p \). Then, by a result of Jacobi, \( \#S_p = p + 1 \).

If \( p \equiv 1(\text{mod } 4) \) is prime, then \( x^2 \equiv -1(\text{mod } p) \) has a solution in \( \mathbb{Z} \), so, by Hensel’s Lemma, \( x^2 = -1 \) has a solution \( i_p \) in \( \mathbb{Q}_p \), where \( \mathbb{Q}_p \) is the field of \( p \)-adic numbers.

Define the following homomorphism

\[
\psi_p : \mathbb{H}(\mathbb{Z}[\frac{1}{p}])^\times \rightarrow \text{PGL}_2(\mathbb{Q}_p).
\]

by

\[
\psi_p(a) = \begin{pmatrix}
a_0 + a_1 i_p & a_2 + a_3 i_p \\
-a_2 + a_3 i_p & a_0 - a_1 i_p
\end{pmatrix}
\]

**Theorem 35** \([56]\). \( \psi_p(S_p) \) contains \( p + 1 \) elements and generates the free group \( \Gamma_p \) of rank \( (p + 1)/2 \).

\( \Gamma_p \) acts freely and transitively on the vertices of the \((p+1)\)-regular tree \( T_{p+1} \).

\[
\psi_{p,l} : \mathbb{H}(\mathbb{Z}[\frac{1}{pl}])^\times \rightarrow \text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_l).
\]

by

\[
\psi_{p,l}(a) = \begin{pmatrix}
a_0 + a_1 i_p & a_2 + a_3 i_p \\
-a_2 + a_3 i_p & a_0 - a_1 i_p
\end{pmatrix}, \begin{pmatrix}
a_0 + a_1 i_l & a_2 + a_3 i_l \\
-a_2 + a_3 i_l & a_0 - a_1 i_l
\end{pmatrix}
\]

where

\[
a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}(\mathbb{Z}),
\]

\( p, l \equiv 1(\text{mod } 4) \) are two distinct primes and \( i_p \in \mathbb{Q}_p, i_l \in \mathbb{Q}_l \) satisfy the conditions \( i_p^2 + 1 = 0 \) and \( i_l^2 + 1 = 0 \), as above.

Let \( S_p \) be as above and \( S_l \) be the set of integer quaternions

\[
a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}(\mathbb{Z}[\frac{1}{pl}])
\]

with \( a_0 > 0, a_0 \) odd, and \( |a|^2 = l \). Then \( S_l = l + 1 \).

Let \( \Gamma_{p,l} \) be the subgroup of \( \text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_l) \) generated by \( \psi(S_p \cup S_l) \).

Mozes has proved the following result:

**Theorem 36** \([89]\). If \( p, l \equiv 1(\text{mod } 4) \) are two distinct prime numbers, then

\( \Gamma_{p,l} \leq \text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_l) \leq \text{Aut}(T_{p+1}) \times \text{Aut}(T_{l+1}) \)

is a \((p + 1, l + 1)\)-group (see definition 3 above).
Rattaggi [107] extended this construction to primes \( p \equiv 3(\text{mod} \ 4) \) using solutions in \( \mathbb{Z} \) of \( x^2 + y^2 \equiv -1(\text{mod} \ p) \).

Note, that the constructions of Mozes and Rattaggi can not be extended for products of trees of the same valency. However, some applications of arithmetic lattices, like constructions of fake quadrics in algebraic geometry, do require arithmetic lattices acting on products of trees of the same valency. They turn out to be related to quaternion algebras in finite characteristic and are arithmetic lattices in \( \text{PGL}_2(\mathbb{F}_q((t))) \times \text{PGL}_2(\mathbb{F}_q((t))) \).

Let \( D \) be a quaternion algebra over a global function field \( K/\mathbb{F}_q \) of a smooth curve over \( \mathbb{F}_q \). Let \( S \) consist of the ramified places of \( D \) together with two distinct unramified \( \mathbb{F}_q \)-rational places \( \tau \) and \( \infty \). An \( S \)-arithmetic subgroup \( \Gamma \) of the projective linear group \( G = \text{PGL}_{1,D} \) of \( D \) acts as a cocompact lattice on the product of trees \( T_{q+1} \times T_{q+1} \) that are the Bruhat–Tits buildings for \( G \) locally at \( \tau \) and \( \infty \). There are plenty of such arithmetic lattices, however, in general the action of \( \Gamma \) on the set of vertices will not be simply transitive.

Let \( q \) be an odd prime power and let \( c \in \mathbb{F}_q^* \) be a non-square. Consider the \( \mathbb{F}_q[t] \)-algebra with non-commuting variables \( Z,F,B = \mathbb{F}_q[t]\{Z,F\}/(Z^2 = c,F^2 = t(t - 1),ZF = -FZ) \), an \( \mathbb{F}_q[t] \)-order in the quaternion algebra \( D = B \otimes \mathbb{F}_q(t) \) over \( \mathbb{F}_q(t) \) ramified in \( t = 0,1 \). Let \( G = \text{PGL}_{1,B} \) be the algebraic group over \( \mathbb{F}_q[t] \) of units in \( B \) modulo the center.

The following results give first examples of lattices acting on products of trees in finite characteristic.

**Theorem 37 ([127]).** Let \( q \) be an odd prime power, and choose a generator \( \delta \) of the cyclic group \( \mathbb{F}_q^* \). Let \( \tau \neq 1 \) be an element of \( \mathbb{F}_q^* \), and \( \zeta \in \mathbb{F}_q^* \) be an element with norm \( \zeta^{1+q} \equiv (\tau - 1)/\tau \). Let \( G/\mathbb{F}_q[t] \) be the algebraic group of \( G \). The irreducible arithmetic lattice \( G(\mathbb{F}_q[t],\frac{1}{t(t-\tau)}) \) has the following presentation:

\[
G(\mathbb{F}_q[t],\frac{1}{t(t-\tau)}) \cong \left\{ d,a,b \left| \begin{array}{l}
d^{q+1} = a^2 = b^q = 1 \\
d^{l-ad^{-i}(l-bd^{-j})} = (d^{bd^{-l}})(d^{k-ad^{-k}})
\end{array} \right. \right\},
\]

where \((*)\) is the system of equations in the quotient group \( \mathbb{F}_q^*/\mathbb{F}_q^* \)

\[d^{q+1} = (1 - \zeta^{(i-1)(1-q)}) \cdot \delta^{(q+1)/2}\]

We are now prepared to establish presentations for the arithmetic lattices with VH-structures in finite characteristic.

**Theorem 38 ([127]).** Let \( 1 \neq \tau \in \mathbb{F}_q^* \), let \( c \in \mathbb{F}_q^* \) be a non-square and let \( \mathbb{F}_q[Z]/\mathbb{F}_q \) be the quadratic field extension with \( Z^2 = c \). The group \( \Gamma_\tau \) is a
torsion free arithmetic lattice in $G$ with finite presentation

$$\Gamma_r = \left\langle a_\xi, b_\eta \text{ for } \xi, \eta \in F_q[Z] \text{ with } \frac{N(\xi)}{\eta} = -c \text{ and } \frac{N(\eta)}{\eta} = \frac{c_r}{1-r} \left| \begin{array}{l}
\frac{a_\xi a_{-\xi}}{1} = 1, \frac{b_\eta b_{-\eta}}{1} = 1 \\
\text{and } a_\xi b_\eta = b_\lambda a_\mu \text{ if in } F_q[Z] : \eta = \lambda^q(\lambda + \xi)^1 - q \\
\frac{\lambda^q(\lambda + \xi)^1 - q}{\eta}
\end{array} \right. \right\rangle$$

which acts simply transitively via the Bruhat Tits action on the vertices of $T_{q+1} \times T_{q+1}$, so the $\Gamma_r$ groups are $(q + 1, q + 1)$-groups.

Corollary 15 ([127]). The number of commensurability classes of arithmetic lattices acting simply transitively on a product of two trees grows at least linearly on $q$.

The following group is the smallest example of a torsion free arithmetic lattice acting on a product of two trees.

$$\Gamma_2 = \langle g_0, g_1, g_2, g_3 \mid g_0 g_1^{-1} = g_1 g_2, g_0 g_2^{-1} = g_3 g_2^{-1}, g_0 g_3 = g_1^{-1} g_0^{-1}, g_2 g_3 = g_1 g_2^{-1} \rangle$$

where, in the notation of Theorem 37 with a generator $\delta$ of the cyclic group $F_3[Z]^\times$, we have $g_i = a_{\delta^i}$ for $i$ even and $g_i = b_{\delta^i}$ for $i$ odd.

4.4 Fibered products of square complexes

The fibered product of two 1-vertex square complexes was defined in [20] in the following way:

Let 1-vertex square complex $X$ be given by the data: $A_1, A_2$ and $S \subset A_1 \times A_2 \times A_1 \times A_2$ then the fibered product $Y$ is defined by the data: $A_1 \times A_1, A_2 \times A_2$ and $R = \{(a_1, a_2), (b_1, b_2), (a_1', a_2'), (b_1', b_2') : (a_i, b_i, a_i', b_i') \in S, i = 1, 2 \}$.

It was shown in [21] that if the fundamental group of $X$ is just infinite (no proper normal subgroups of infinite index), then the fundamental group of $Y$ is not residually finite.

This result creates a machinery to construct many non-residually finite groups using arithmetic lattices. In particular, all families of arithmetic groups from the previous section generate families of non-residually finite groups.

5 Lattices in products of $n \geq 3$ trees

5.1 Arithmetic higher-dimensional lattices

Similar to products of two trees, there are many arithmetics lattices in products of $n \geq 3$ trees, see, for example, [33], but it is difficult to get simply transitive actions.

We start with the following example of a group acting simply transitively on a product of three trees, with 9 generators and 26 relations.

$$G = \{a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, c_4 | a_1 * b_1 * a_2 * b_2, a_1 * b_2 * a_2 * b_1^{-1}, a_1 * b_3 * a_2^{-1} * b_1, a_1 * b_3^{-1} * a_1 * b_2^{-1}, a_1 * b_1^{-1} * a_2^{-1} * b_3, a_2 * b_3 * a_2 * b_2^{-1}, a_1 *$$
c1 * a2^{-1} * c2^{-1}, a1 * c2 * a1^{-1} * c3, a1 * c3 * a2^{-1} * c4^{-1}, a1 * c4 * a1 * c1^{-1}, a1 * c4^{-1} * a2 * c1, a1 * c3^{-1} * a2 * c1, a2 * c3 * a2 * c2^{-1}, a2 * c4 * a2^{-1} * c1, c1 * b1 * c3 * b3^{-1}, c1 * b2 * c4 * b2^{-1}, c1 * b3 * c4^{-1} * b2, c1 * b3^{-1} * c4 * b3, c1 * b2^{-1} * c2 * b1, c1 * b1^{-1} * c4 * b1^{-1}, c2 * b2 * c3^{-1} * b3^{-1}, c2 * b3 * c4 * b1, c2 * b3^{-1} * c3 * b3, c2 * b2^{-1} * c3 * b2, c2 * b1^{-1} * c3 * b1^{-1}, c3 * b1 * c4 * b2 \}.

The group $G$ is an arithmetic lattice acting simply transitively on a product of three trees, and a particular case of a $n$-dimensional construction of arithmetic lattices, described below.

**Theorem 39 ([128]).** If $p_1, \ldots, p_n$ are $n$ distinct prime numbers, then there is an arithmetic lattice

$$\Gamma_{p_1, \ldots, p_n} < PGL_2(\mathbb{Q}_{p_1}) \times \ldots \times PGL_2(\mathbb{Q}_{p_n}) < Aut(T_{p_1+1}) \times \ldots \times Aut(T_{p_n+1})$$

acting simply transitively on the product of $n$ trees.

**Proof.** The group of $S$-integral quaternions (in the example above $S = \{3, 5, 7\}$), let’s say with $n$ equals the cardinality of $S$, should have the following structure: there are sets of generators $S_p$ for every $p \in S$ and relations of type $ab = b'a'$ for every pair of distinct primes $p, \ell \in S$ with $a, a' \in S_p$ and $b, b' \in S_\ell$. That’s because for every subset $T \subseteq S$ the $T$-integral quaternions form a subgroup of the $S$-integral ones. By work of Mozes [39] and Rattaggi [107] we know that $S_p \cup S_\ell$ generates the $\{p, \ell\}$-integral quaternions and that it acts simply transitively on vertices of a product of trees $T_{p+1} \times T_{\ell+1}$. We denote this group $\Gamma_{p, \ell}$ for future references.

Since these quaternions are integral at all other primes, this subgroup fixes the standard vertex of the (Bruhat-Tits)-tree for that component.

This means that the group generated by all the $S_p$’s for $p \in S$ will map the standard vertex to all its neighbours, hence by homogeneity and connectedness the action is simply transitive on vertices also on the product $T_{p_1} \times \ldots \times T_{p_n}$ where $S = \{p_1, \ldots, p_n\}$. We then may conclude that the action is simple transitive on vertices. This identifies the arithmetic group with the combinatorial fundamental group of the cube complex, whose $\pi_1$ is generated by edges (these are the $S_p$’s for direction ”$p$”) and relations all come from the 2-dim cells, hence the squares, all of which are also squares in one of the subgroups with just two primes. Because we know all squares for these subgroups, we know all relations for the $S$-integral quaternions. These relations give the relations for the group $\Gamma_{p_1, \ldots, p_n}$. In the example above, the group $G$, acting on a product of three trees has three 2-dimensional subgroups $\Gamma_{3,5}, \Gamma_{5,7}, \Gamma_{3,7}$.

Another example of a group acting cocompactly on a product of more than two trees is the lattice of Glasner-Mozes [38], its construction is inspired by Mumford’s fake projective plane.

### 5.2 Non-residually finite higher-dimensional lattices

We give an elementary construction of a family of non-residually finite groups acting simply transitively on products of $n \geq 3$ trees (for more details see [128]).
Theorem 40 (128). There exist a family of non-residually finite groups acting simply transitively on a product of \( n \geq 3 \) trees.

Proof. This is just a sketch. We start with an arithmetic lattice \( \Gamma \) acting simply transitively on a product of \( n \) trees. There are \( n \) 2-dimensional groups induced by \( \Gamma \) in a unique way. For each of them we apply the fibered product construction and take the union of all generators and relations of each fibered product. The resulting group is non-residually finite by [21].

6 Open problems

We will mention here some open problems and conjectures.

\( \Lambda \)-free groups.

Conjecture 1. [69] Every finitely generated \( \Lambda \)-free group is finitely presented.

This would imply that all finitely generated \( \Lambda \)-free groups are \( \mathbb{R}^n \)-free.

Problem 1. [69] Is any finitely presented \( \Lambda \)-free group \( G \) also \( \mathbb{Z}^k \)-free for an appropriate \( k \in \mathbb{N} \) and lexicographically ordered \( \mathbb{Z}^k \)?

Arithmetic groups.

Conjecture 2. (Caprace) The number of non-commensurable arithmetic lattices acting simply transitively on product of two trees is bounded polynomially on \( q \).

Problem 2. What is the structure and properties of groups acting on products of three and more trees.

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