Abstract. In [11] and [13] we showed that a loop in a simply connected compact Lie group \( \hat{U} \) has a unique Birkhoff (or triangular) factorization if and only if the loop has a unique (what we propose to call) “root subgroup factorization” (relative to a choice of a reduced sequence of simple reflections in the affine Weyl group). In this sequel our main purpose is to investigate Birkhoff and root subgroup factorization for loops in a noncompact type semisimple Lie group \( \hat{G}_0 \) of inner type.

In the first part of this paper, we consider the finite dimensional situation (i.e. constant loops in \( \hat{G}_0 \)), and we show that a group element has a Birkhoff factorization if and only if it has a root subgroup factorization.

In the second part of this paper we consider loops in \( \hat{G}_0 \). In this case both Birkhoff and root subgroup factorization are far more complicated than for loops in \( \hat{U} \). In this noncompact context a root subgroup factorization implies a unique Birkhoff factorization, but there are several obstacles for the converse. As in the compact case, root subgroup factorization is intimately related to factorization of Toeplitz determinants.

{2000 Mathematics Subject Classifications: 22E67}

0. Introduction

Finite dimensional Riemannian symmetric spaces come in dual pairs, one of compact type and one of noncompact type. Given such a pair, there is a diagram of finite dimensional groups

\[
\begin{array}{ccc}
\hat{G} & \rightarrow & G \\
\downarrow & & \downarrow \\
\hat{G}_0 & \rightarrow & \hat{U} \\
\downarrow & & \downarrow \\
\hat{K} & \rightarrow & \hat{U}
\end{array}
\]

where \( \hat{U} \) is the universal covering of the identity component of the isometry group of the compact type symmetric space \( \hat{X} \cong \hat{U}/\hat{K} \), \( \hat{G} \) is the complexification of \( \hat{U} \), and \( \hat{G}_0 \) is a covering of the isometry group for the dual noncompact symmetric space \( \hat{X}_0 = \hat{G}_0/\hat{K} \).

The main purpose of this paper is to investigate Birkhoff (or triangular) factorization and “root subgroup factorization” for \( \hat{G}_0 \), and for the loop group of \( \hat{G}_0 \), assuming \( \hat{G}_0 \) is of inner type. Birkhoff factorization is investigated in [4] and Chapter 8 of [15], from various points of view. In particular Birkhoff factorization for \( L\hat{U} := C^\infty(S^1, \hat{U}) \) is developed in Chapter 8 of [15], using the Grassmannian model for the homogeneous space \( L\hat{U}/\hat{U} \). Root subgroup factorization for generic loops
The Birkhoff decomposition for $L\hat{G}_0 := C^\infty(S^1, \hat{G}_0)$, i.e. the intersection of the Birkhoff decomposition for $L\hat{G}$ with $L\hat{G}_0$, is far more complicated than for $L\hat{U}$. With respect to root subgroup factorization, beyond loops in a torus (corresponding to imaginary roots), in the compact context the basic building blocks are exclusively spheres (corresponding to real roots), and in the inner noncompact context the building blocks are a combination of spheres and disks. This introduces additional analytic complications, and perhaps the main point of this paper is to communicate the problems that arise from noncompactness.

For $g \in L\hat{U}$, the basic fact is that $g$ has a unique triangular factorization if and only if $g$ has a unique “root subgroup factorization” (relative to the choice of a reduced sequence of simple reflections in the affine Weyl group). We will show that this is also true for $\hat{G}_0$ (constant loops). However this is far from true for loops in $\hat{G}_0$.

Relatively little sophistication is required to state the basic results, and identify the basic obstacles, in the rank one noncompact case. This is essentially because (in addition to loops in a torus) the basic building blocks are exclusively disks, and there is an essentially unique way to choose a reduced sequence of simple reflections in the affine Weyl group, so that this can be suppressed.

0.1. The Rank 1 Case. We consider the data determined by the Riemann sphere and the Poincaré disk. For this pair, the diagram (0.1) becomes

\[
\begin{array}{ccc}
\text{SL}(2, \mathbb{C}) & \rightarrow & \text{SU}(1, 1) \\
\downarrow & & \downarrow \\
\text{SU}(2) & \rightarrow & S(\mathbb{U}(1) \times \mathbb{U}(1))
\end{array}
\]

Let $L_{fin}\text{SL}(2, \mathbb{C})$ denote the group consisting of maps $S^1 \rightarrow \text{SL}(2, \mathbb{C})$ having finite Fourier series, with pointwise multiplication. The subset of those functions having values in $\text{SU}(1, 1)$ is then a subgroup, denoted $L_{fin}\text{SU}(1, 1)$.

Example 0.1. For each $\zeta \in \Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and $n \in \mathbb{Z}$, the function $S^1 \rightarrow \text{SU}(1, 1)$ defined by

\[
z \mapsto a(\zeta) \begin{pmatrix} 1 & \zeta z^{-n} \\ \bar{\zeta} z^n & 1 \end{pmatrix}, \quad \text{where } a(\zeta) = (1 - |\zeta|^2)^{-1/2},
\]

is in $L_{fin}\text{SU}(1, 1)$.

$L_{fin}\text{SU}(2)$ and $L_{fin}\text{SU}(1, 1)$ are dense in the smooth loop groups $L\text{SU}(2) := C^\infty(S^1, \text{SU}(2))$ and $L\text{SU}(1, 1) := C^\infty(S^1, \text{SU}(1, 1))$, respectively. This is proven in the compact case in Proposition 3.5.3 of [15], and the argument applies also for $\text{SU}(1, 1)$, taking into account the obvious modifications.

For a Laurent series $f(z) = \sum f_n z^n$, let $f^*(z) = \sum \bar{f}_n z^{-n}$. If $\Omega$ is a domain on the Riemann sphere, we write $H^0(\Omega)$ for the vector space of holomorphic scalar valued functions on $\Omega$. If $f \in H^0(\Delta)$, then $f^* \in H^0(\Delta^*)$, where $\Delta^*$ denotes the open unit disk at $\infty$. 
Theorem 0.1. Suppose that \( g_1 \in L_{\text{fin}}SU(1,1) \) and fix \( n > 0 \). Consider the following three statements:

(I.1) \( g_1 \) is of the form
\[
g_1(z) = \begin{pmatrix} a(z) & b(z) \\ b^*(z) & a^*(z) \end{pmatrix}, \quad z \in S^1,
\]
where \( a \) and \( b \) are polynomials in \( z \) of order \( n - 1 \) and \( n \), respectively, with \( a(0) > 0 \).

(I.2) \( g_1 \) has a “root subgroup factorization” of the form
\[
g_1(z) = a(\eta_n) \begin{pmatrix} 1 & \tilde{\eta}_nz^n \\ \tilde{\eta}_n z^{-n} & 1 \end{pmatrix} \cdots a(\eta_0) \begin{pmatrix} 1 & \tilde{\eta}_0 \\ \eta_0 & 1 \end{pmatrix},
\]
for some sequence \((\eta_i)_{i=1}^n\) in \( \Delta \) and \( a : \Delta \to \mathbb{R} \) is the function in Example 0.1.

(I.3) \( g_1 \) has a triangular factorization of the form
\[
\begin{pmatrix} 1 & \sum_{j=0}^{n-1} \tilde{y}_jz^{-j} \\ \sum_{j=0}^{n-1} \tilde{y}_jz^{-j} & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix},
\]
where \( a_1 > 0 \), the third factor is a matrix valued polynomial in \( z \) which is unipotent upper triangular at \( z = 0 \).

Statements (I.1) and (I.3) are equivalent. (I.2) implies (I.1) and (I.3). If \( g_1 \) is in the identity connected component of the sets in (I.1) and (I.3), then the converse holds, i.e. \( g_1 \) has a root subgroup factorization as in (I.2).

There is a similar set of implications for \( g_2 \in L_{\text{fin}}SU(1,1) \) and the following statements:

(II.1) \( g_2 \) is of the form
\[
g_2(z) = \begin{pmatrix} c^*(z) & d^*(z) \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1,
\]
where \( c \) and \( d \) are polynomials in \( z \) of order \( n \) and \( n-1 \), respectively, with \( c(0) = 0 \) and \( d(0) > 0 \).

(II.2) \( g_2 \) has a “root subgroup factorization” of the form
\[
g_2(z) = a(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ \zeta_n z^n & 1 \end{pmatrix} \cdots a(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ \zeta_1 z & 1 \end{pmatrix},
\]
for some sequence \((\zeta_i)_{i=1}^n\) in \( \Delta \) and \( a : \Delta \to \mathbb{R} \) is the function in Example 0.1.

(II.3) \( g_2 \) has a triangular factorization of the form
\[
\begin{pmatrix} 1 & \sum_{j=0}^{n-1} \tilde{x}_jz^{-j} \\ \sum_{j=0}^{n-1} \tilde{x}_jz^{-j} & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix},
\]
where \( a_2 > 0 \), and the third factor is a matrix valued polynomial in \( z \) which is unipotent upper triangular at \( z = 0 \).

When \( g_1 \) and \( g_2 \) have root subgroup factorizations, the scalar entries determining the diagonal factor have the product form
\[
a_1 = \prod_{i=0}^{\infty} a(\eta_i) \quad \text{and} \quad a_2^{-1} = \prod_{k=1}^{\infty} a(\zeta_k), \quad \text{respectively}.
\]
In general we do not know how to describe the connected component that arises in the first and third conditions.

**Example 0.2.** Consider the case $n = 2$ and $g_2$ is as in II.3 with $x = x_1 z + x_2 z^2$, $1 - x_2 \bar{x}_2 \neq 0$,

$$
\begin{align*}
\alpha_2 &= 1 - a_2^{-2} \bar{x}_1 x_2 z, \\
\beta_2 &= -\frac{x_2^{-2} \bar{x}_1^2 x_2}{1 - x_2 \bar{x}_2} \\
\gamma_2 &= \frac{x_1}{1 - x_2 \bar{x}_2} z + x_2 z^2, \\
\delta_2 &= 1 + \frac{\bar{x}_1 x_2}{1 - x_2 \bar{x}_2} z
\end{align*}
$$

and

$$a_2^2 = \frac{(1 - x_2 \bar{x}_2)^2 - x_1 \bar{x}_1}{1 - x_2 \bar{x}_2}.$$

It is straightforward to check that this $g_2$ does indeed have values in $SU(1, 1)$. In order for $a_2^2 > 0$, there are two possibilities: the first is that both the numerator and denominator are positive, in which case there is a root subgroup factorization, and the second is that both the top and bottom are negative, in which case root subgroup factorization fails (because when there is a root subgroup factorization, $\zeta_2 = x_2$, and we must have $|\zeta_2| < 1$).

In order to formulate a general factorization result, we need a $C^\infty$ version of Theorem 0.1.

**Theorem 0.2.** Suppose that $g_1 \in \text{LSU}(1, 1)$. The following conditions are equivalent:

(I.1) $g_1$ is of the form

$$g_1(z) = \begin{pmatrix} a(z) & b(z) \\ b^*(z) & a^*(z) \end{pmatrix}, \quad z \in S^1,$$

where $a$ and $b$ are holomorphic in $\Delta$ and have $C^\infty$ boundary values, with $a(0) > 0$.

(I.3) $g_1$ has triangular factorization of the form

$$
\begin{pmatrix} 1 & 0 \\ y^* & 1 \end{pmatrix}
\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}
\begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix},
$$

where $y$ is holomorphic in $\Delta$ with $C^\infty$ boundary values, $a_1 > 0$, and the third factor is a matrix valued polynomial in $z$ which is unipotent upper triangular at $z = 0$.

Similarly if $g_2 \in \text{LSU}(1, 1)$, the following statements are equivalent:

(II.1) $g_2$ is of the form

$$g_2(z) = \begin{pmatrix} d^*(z) & c^*(z) \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1,$$

where $c$ and $d$ are holomorphic in $\Delta$ and have $C^\infty$ boundary values, with $c(0) = 0$ and $d(0) > 0$.

(II.3) $g_2$ has a triangular factorization of the form

$$
\begin{pmatrix} 1 & x^* \\ 1 & 1 \end{pmatrix}
\begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix}
\begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix},
$$

where $a_2 > 0$, $x$ is holomorphic in $\Delta$ and has $C^\infty$ boundary values, $x(0) = 0$, and the third factor is a matrix valued function which is holomorphic in $\Delta$ and has $C^\infty$ boundary values, and is unipotent upper triangular at $z = 0$. 
Let $\sigma : \text{SL}(2, \mathbb{C}) \to \text{SL}(2, \mathbb{C})$ denote the anti-holomorphic involution of $\text{SL}(2, \mathbb{C})$ which fixes $\text{SU}(1, 1)$; explicitly

$$\sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} d^* & c^* \\ b^* & a^* \end{pmatrix}.$$ 

The following theorem is the analogue of Theorem 0.2 of [11] (the notation in part (b) is taken from Section 1 of [11], and reviewed below the statement of the theorem).

**Theorem 0.3.** Suppose $g \in \text{LSU}(1, 1)_{(0)}$, the identity component. Then $g$ has a unique “partial root subgroup factorization” of the form

$$g(z) = \sigma(g_1^{-1}(z)) \begin{pmatrix} e^{\chi(z)} & 0 \\ 0 & e^{-\chi(z)} \end{pmatrix} g_2(z)$$

where $\chi \in C^\infty(S^1, i\mathbb{R})/2\pi i\mathbb{Z}$ and $g_1$ and $g_2$ are as in Theorem 0.2, if and only if $g$ has a triangular factorization $g = lm au$ (see (0.5) below) such that the boundary values of $l_{21}/l_{11}$ and $u_{21}/u_{22}$ are $< 1$ in magnitude on $S^1$.

The following example shows that the unaesthetic condition on the boundary values in part (b) is essential.

**Example 0.3.** Consider $g_2$ as in Theorem 0.1. The loop $g = g_2^*$ (the Hermitian conjugate of $g_2$ around the circle) has triangular factorization

$$g = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sum_{j=1}^n x_j z^j & 1 \end{pmatrix}.$$ 

If $n = 2$, then $x_1 = \bar{\zeta}_1 (1 - |\zeta_2|^2)$ and $x_2 = \bar{\zeta}_2$, and this loop will often not satisfy the condition $|x_1 z + x_2 z^2| < 1$ on $S^1$. In this case $g$ will not have a partial root subgroup factorization in the sense of Theorem 0.3.

The group $\text{LSL}(2, \mathbb{C})$ has a Birkhoff decomposition

$$\text{LSL}(2, \mathbb{C}) = \bigsqcup_{w \in W} \Sigma_w \text{LSL}(2, \mathbb{C})$$

where $W$ (an affine Weyl group, and in this case the infinite dihedral group) is a quotient of a discrete group of unitary loops

$$W = \{ w = \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : n \in \mathbb{Z}, \epsilon \in \mathbb{Z}_4 \}/\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}$$

where

$$r_0 = \begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix}, r_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(the reflections corresponding to the two simple roots for the Kac-Moody extension of $\mathfrak{sl}(2, \mathbb{C})$). The set $\Sigma_w \text{LSL}(2, \mathbb{C})$ consists of loops which have a (Birkhoff) factorization of the form

$$g = l \cdot w \cdot m \cdot a \cdot u,$$

where $w = [w]$, $l = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \in H^0(\Delta^*, G)$, $l(\infty) = \begin{pmatrix} 1 & 0 \\ l_{21}(\infty) & 1 \end{pmatrix}$,
has smooth boundary values on $S^1$, $m = \left( \begin{array}{cc} m_0 & 0 \\ 0 & m_0^{-1} \end{array} \right)$, $m_0 \in S^1$, $a = \left( \begin{array}{cc} a_0 & 0 \\ 0 & a_0^{-1} \end{array} \right)$, $a_0 > 0$,

$$u = \left( \begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array} \right) \in H^0(\Delta, G), \quad u(0) = \left( \begin{array}{cc} 1 & u_{12}(0) \\ 0 & 1 \end{array} \right),$$

and $u$ has smooth boundary values on $S^1$. If $w = 1$, the generic case, then we say (as in Section 1 of [11]) that $g$ has a triangular factorization, and in this case the factors are unique.

Next, let $LSU(1, 1)_n$ denote the connected component containing $\left( \begin{array}{cc} z^n & 0 \\ 0 & z^{-n} \end{array} \right) \in Hom(S^1; \hat{T})$, and let

$$\Sigma^{LSU(1,1)}_w := \Sigma^{LSL(2,\mathbb{C})}_w \cap LSU(1, 1)$$

and

$$\Sigma^{LSU(1,1)(n)}_w := \Sigma^{LSL(2,\mathbb{C})}_w \cap LSU(1, 1)_n.$$

Based on finite dimensional intuition (the first part of this paper), and the compact case, one might expect the following to be true:

1. Modulo $\hat{T}$, the circle subgroup, it should be possible to contract $\Sigma^{LSU(1,1)(n)}_w$ down to $w$; in particular $\Sigma^{LSU(1,1)}_w$ should be empty unless $w$ is represented by a loop in $SU(1, 1)$, i.e. $w = \left[ \begin{array}{cc} z^n & 0 \\ 0 & z^{-n} \end{array} \right]$ for some $n$.

2. $\Sigma^{LSU(1,1)}_1 = LSU(1, 1)_0$.

3. Each $\Sigma^{LSU(1,1)(n)}_w$ should admit a relatively explicit parameterization.

Statements (1) and (2) are definitely false; statement (3) is very elusive, if not doubtful.

**Proposition 0.1.**

(a) $\Sigma^{LSU(1,1)(n)}_w$ nonempty does not imply that $w$ is represented by a loop in $SU(1, 1)$. For example, $\Sigma^{LSU(1,1)}_1$ is nonempty.

(b) $\Sigma^{LSU(1,1)}_0$ is properly contained in $LSU(1, 1)_0$.

To summarize, the set of loops having a root subgroup factorization is properly contained in the set of loops in the identity component which have a triangular factorization which, in turn, is a proper subset of the identity component of $LSU(1, 1)$. The first set has an explicit parametrization. The second does not, a severe flaw, and the third is very simple topologically. This stands in contrast to the compact case of $LSU(2)$ where there is only one connected component and every loop admitting a triangular factorization admitted a root subgroup factorization. This begs the question of why anyone should care about root subgroup factorization in the noncompact case.

To close, we mention an application. The group $LSU(1, 1)$ acts by bounded multiplication operators on the Hilbert space $H := L^2(S^1; \mathbb{C}^2)$. As in chapter 6 of [15], this defines a homomorphism of $LSU(1, 1)$ into the restricted general linear group of $H$ defined relative to the Hardy polarization $H = H_+ \oplus H_-$, where $H_+$ is the subspace of boundary values of functions in $H^0(\Delta^*, \mathbb{C}^2)$ and $H_-$ is the subspace of boundary values of functions in $H^0(\Delta, \mathbb{C}^2)$. For a loop $g$, let $A(g)$ (respectively,
$A_1(g)$ denote the corresponding Toeplitz operator, i.e., the compression of multiplication by $g$ to $H_+$ (resp., the shifted Toeplitz operator, i.e. the compression to $H_+ \ominus C \begin{pmatrix} 0 \\ 1 \end{pmatrix}$). It is well known that $A(g)A(g^{-1})$ and $A_1(g)A_1(g^{-1})$ are determinant class operators (i.e., of the form $1 + \text{trace class}$).

**Theorem 0.4.** Suppose that $g \in \text{LSU}(1,1)_{(0)}$ has a root subgroup factorization as in part (b) of Theorem 0.3. Then

$$
\det(A(g)A(g^{-1})) = \left( \prod_{i=0}^{\infty} \frac{1}{(1 - |\eta_i|^2)^i} \right) \times \left( \prod_{j=1}^{\infty} e^{-2j|\chi_j|^2} \right) \times \left( \prod_{k=1}^{\infty} \frac{1}{(1 - |\zeta_k|^2)^k} \right),
$$

$$
\det(A_1(g)A_1(g^{-1})) = \left( \prod_{i=0}^{\infty} \frac{1}{(1 - |\eta_i|^2)^{i+1}} \right) \times \left( \prod_{j=1}^{\infty} e^{-2j|\chi_j|^2} \right) \times \left( \prod_{k=1}^{\infty} \frac{1}{(1 - |\zeta_k|^2)^{k-1}} \right)
$$

and if $g = \text{lmau}$ is the triangular factorization as in (0.5) (with $w = 1$), then

$$a_0^2 = \frac{\det(A_1(g)A_1(g^{-1}))}{\det(A(g)A(g^{-1}))} = \prod_{k=1}^{\infty} (1 - |\zeta_k|^2) \prod_{i=0}^{\infty} (1 - |\eta_i|^2).$$

When $(\eta_i)_{i=0}^{\infty}$ and $(\zeta_k)_{k=1}^{\infty}$ are the zero sequences (the abelian case), the first formula specializes to a result of Szego and Widom (see Theorem 7.1 of [19]). Estelle Basor pointed out to us that this result, for $g$ as in (0.3), can be deduced from Theorem 5.1 of [18]. More recently Basor and Torsten Ehrhardt have discovered a more elementary proof of Theorem 0.4 (which does not make explicit use of Kac-Moody extensions).

### 0.2. The General Setting of This Paper.

The fundamental assumption of this paper is that

$$\text{rank}(\dot{K}) = \text{rank}(\dot{G}_0).$$

It is well known that this condition is equivalent to a number of other conditions:

- the Cartan involution $\sigma$ for the pair $(\dot{g}_0, \dot{t})$ is inner;
- $\dot{G}_0$ has discrete series unitary representations;
- $C(\dot{K}) = S^1$;
- the quotients $\dot{U}/\dot{K}$ and $\dot{G}_0/\dot{K}$ are Hermitian symmetric.

This equal rank condition implies the existence of a (symmetric space compatible) triangular decomposition of $\dot{g}$ such that each positive root is either of compact or noncompact type. In the compact case the corresponding root homomorphism induces an embedding $SU(2) \to \dot{K}$, and complementing the torus we obtain a sphere. In the noncompact case the corresponding root homomorphism induces an embedding of the rank one diagram (0.2) into the diagram (0.1), and complementing the torus we obtain a disk. This is the origin for the diversity of basic building blocks for the factorization in the inner noncompact case.

### 0.3. Plan of the Paper.

This paper consists of two parts. The first part of the paper concerns Birkhoff and root subgroup factorization for the finite dimensional groups (the constant loops) appearing in (0.1). The second part concerns Birkhoff and root subgroup factorization for the corresponding loop groups.

Section 1 is on background for finite dimensional groups.
Section 2 concerns factorization for the finite dimensional groups ˙\(U\) and ˙\(G_0\). The compact case is relatively well-understood, thanks in large part to Lu (see especially [8]). We review this, with emphasis on the algorithm for root subgroup factorization (which depends on an ordering of noninverted roots), because this is an important guide in the loop cases. In finite dimensions the noncompact inner case largely reduces to the compact case, because of the existence of a “block (or coarse) triangular decomposition”. But there is one part of the argument which is indirect, i.e., not algebraic: this is in showing that everything in a component of the Birkhoff decomposition, \(\Sigma^{G_0}_w\), has a root subgroup factorization. This concludes part one.

Section 3 is more background, needed for loop groups. The last subsection describes the basic framework for the remainder of the paper.

In section 4 we consider the intersection of the Birkhoff decomposition for \(L\hat{G}\) with \(L\hat{G}_0\). Unfortunately for loops in \(\hat{G}_0\), there does not exist an analogue of “block (coarse) triangular decomposition”. Consequently there does not exist a reduction to the compact type case, as in finite dimensions. One might still naively expect that there could be a relatively transparent way to parameterize the Birkhoff components intersected with \(L\hat{G}_0\) (as in the finite dimensional case, and in the case of loops into compact groups, e.g. using root subgroup factorization). But these intersections turn out to not be so simple topologically. Most of the section is devoted to rank one examples which illustrate the basic complications.

In Section 5 we consider root subgroup factorization for generic loops in \(\hat{G}_0\). Our objective in this section is to prove partial analogues of Theorems 4.1, 4.2, and 5.1 of [13], for generic loops in the identity component of (the Kac-Moody central extension of) \(L\hat{G}_0\) (when \(\hat{G}_0\) is of inner type). As in the rank one case above, all of the statements have to be severely modified. The structures of the arguments in this noncompact context are roughly the same as in [13]. But there are obviously important differences, and in this paper we will present all of the details in this generic context.

0.4. Acknowledgement. The second author thanks Hermann Flaschka, whose questions motivated us to consider loops in noncompact groups. We also thank Estelle Basor for many useful conversations.

1. Notation and Background, I: Finite Dimensional Algebras and Groups

In this paper we make use of the fact that finite dimensional simple Lie algebras and loop algebras of finite dimensional simple Lie algebras can be put in the common framework of Kac-Moody Lie algebras. To separate the data of finite dimensional algebras and groups from the analogous data for the corresponding loop algebras and groups, we will follow a convention of Kac and decorate the symbols for finite dimensional algebras and groups with an overhead dot.

Let \(\hat{m}\) be a simple Lie algebra over \(\mathbb{R}\) and write \(\hat{g}\) for the complexification \(\hat{m}^\mathbb{C}\) of \(\hat{m}\). Let \(\hat{G}\) be the simply connected complex Lie group with Lie algebra \(\hat{g}\) and let \(\hat{M}\) denote the connected real subgroup of \(\hat{G}\) with Lie algebra \(\hat{m}\subset\hat{g}\). In this section, we will establish notation for studying factorization of elements of \(\hat{M}\) relative to a Birkhoff decomposition of \(\hat{G}\).
The Lie algebra \( \mathfrak{m} \) is of inner type if all Cartan involutions of \( \mathfrak{m} \) are inner automorphisms. By definition, the subalgebra of \( \mathfrak{m} \) fixed by a Cartan involution is a maximal compact subalgebra. Thus, if \( \mathfrak{m} \) is a compact simple Lie algebra then the only Cartan involution it admits is the identity. Hence compact simple Lie algebras are of inner type. The compact case has been considered extensively. Our focus is on the case when \( \mathfrak{m} = \mathfrak{g}_0 \) is a simple noncompact Lie algebra over \( \mathbb{R} \) of inner type.

1.1. **Data determined by the choice of a Cartan involution.** The choice of a Cartan involution \( \Theta \) on \( \mathfrak{g}_0 \) determines a maximal compact Lie subalgebra \( \mathfrak{k} \) of \( \mathfrak{g}_0 \). Let \( \sigma \) denote the canonical complex conjugation on \( \mathfrak{g}_0 \) fixing \( \mathfrak{g}_0 \). If we extend \( \Theta \) to \( \mathfrak{g} \) in a complex linear fashion then the composition \( \tau = \sigma \circ \Theta \) is a complex conjugation on \( \mathfrak{g} \) fixing a compact real form \( \mathfrak{u} \) of \( \mathfrak{g} \). The extended involution \( \Theta \) on \( \mathfrak{g} \) stabilizes \( \mathfrak{u} \) and fixes \( \mathfrak{k} \) inside of \( \mathfrak{u} \). Thus, \( \mathfrak{g}_0 \cap \mathfrak{u} = \mathfrak{k} \). The assumption that \( \mathfrak{g}_0 \) is of inner type is equivalent to the condition that

\[
\text{rank}(\mathfrak{g}_0) = \text{rank}(\mathfrak{k}) = \text{rank}(\mathfrak{u}).
\]

We write \( \mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p} \) for the decomposition of \( \mathfrak{g}_0 \) into the eigenspaces of \( \Theta \) on \( \mathfrak{g}_0 \). Then \( \mathfrak{u} = \mathfrak{k} + i\mathfrak{p} \) where multiplication by \( i \) denotes the canonical complex structure on \( \mathfrak{g} \), and this is the decomposition of \( \mathfrak{u} \) into the eigenspaces of the extension of \( \Theta \) restricted to \( \mathfrak{u} \).

Let \( \dot{U} \) and \( \dot{K} \) denote the connected subgroups of \( \dot{G} \) having Lie algebras \( \mathfrak{u} \) and \( \mathfrak{k} \). Then we obtain a diagram of Lie algebras and corresponding connected Lie groups.

We will also use \( \Theta \) to denote the corresponding holomorphic involution of \( \dot{G} \), and its restrictions to \( \dot{G}_0 \) and \( \dot{U} \), which fixes \( \dot{K} \) in \( \dot{G}_0 \) and \( \dot{U} \), respectively.

1.2. **Data determined by the choice of a \( \Theta \)-stable Cartan subalgebra and a Weyl chamber in the Inner Case.** Fix a Cartan subalgebra \( \mathfrak{t} \subset \mathfrak{k} \). Because of our rank assumption (1.1), \( \mathfrak{t} \) is a \( \Theta \)-stable Cartan subalgebra of \( \mathfrak{g}_0 \) and every \( \Theta \)-stable Cartan subalgebra of \( \mathfrak{g}_0 \) is of this form. In addition, \( \mathfrak{t} \) is a \( \Theta \)-stable Cartan subalgebra of \( \mathfrak{u} \), and its centralizer \( \mathfrak{h} \) in \( \mathfrak{g} \) is a \( \Theta \)-stable Cartan subalgebra of \( \mathfrak{g} \). We write \( \hbar = \mathfrak{t} + \mathfrak{a} \), where \( \mathfrak{a} = it \), for the eigenspace decomposition of \( \mathfrak{h} \) under \( \Theta \) and let \( \mathcal{H} = \exp(\mathfrak{h}) \), \( \mathcal{T} = \exp(\mathfrak{t}) \), and \( \mathcal{A} = \exp(\mathfrak{a}) \), respectively.

We will use \( W := N_{\mathcal{G}}(\mathcal{T})/\mathcal{T} \) as a model for the Weyl group of \( (\mathfrak{g}, \mathfrak{h}) \). The choice of a Weyl chamber \( C \) in \( \mathfrak{a} \) determines a choice of positive roots for the action of \( \hbar \) on \( \mathfrak{g} \). Let \( \mathfrak{n}^+ \) denote the sum of the root spaces indexed by positive roots and \( \mathfrak{n}^- \) denote the sum of the root spaces indexed by negative roots. In this way, the choice of a \( \Theta \)-stable Cartan subalgebra \( \mathfrak{t} \) of \( \mathfrak{g}_0 \) and a Weyl chamber \( C \) determines a triangular decomposition

\[
\mathfrak{g} = \mathfrak{n}^- + \hbar + \mathfrak{n}^+.
\]
Set $\hat{N}^{\pm} = \exp(n^{\pm})$. Then $\hat{B}^+ = \hat{H}\hat{N}^+$ and $\hat{B}^- = \hat{N}^-\hat{H}$ are a pair of opposite Borel subgroups of $\hat{G}$.

Remarks.

(a) In the case that $\hat{m}$ is compact, $\Theta$ is the identity, so $\Theta$-stability of the triangular decomposition (1.2) is automatic.

(b) A consequence of the stability of $\hat{h}$ under $\sigma$ and $\tau$ is the fact that $\sigma(\hat{n}^{\pm}) = \hat{n}^{\mp}$ and $\tau(\hat{n}^{\mp}) = \hat{n}^{\pm}$.

Example 1.1. In this paper, a special role is played by the rank 1 example of $\hat{g}$.

$\hat{su}(1,1) = \left\{ \begin{pmatrix} i z & x + iy \\ x - iy & -iz \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

with Cartan involution given by

$$\text{Ad}_g \text{ where } g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$ 

The effect of this involution is to negate the off-diagonal entries. In this case, the maximal compact subalgebra fixed by the involution is the one dimensional subalgebra $\hat{su}(1,1)$ of diagonal matrices in $\hat{su}(1,1)$, which is abelian and hence a Cartan subalgebra. The complexification is $\mathfrak{sl}(2,\mathbb{C})$ and the associated compact real form $\mathfrak{su}(2)$. The involution fixing $\mathfrak{su}(2)$ is $X \mapsto -X^*$ (opposite conjugate transpose). We will work with the standard triangular decomposition

$$\mathfrak{sl}(2,\mathbb{C}) = \text{span}_\mathbb{C} \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} + \text{span}_\mathbb{C} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} + \text{span}_\mathbb{C} \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$ 

1.3. Root Data. Let $\hat{\theta}$ denote the highest root and normalize the Killing form so that $(\theta, \theta) = 2$. For each root $\alpha$ let $h_\alpha \in \mathfrak{a}$ denote the associated coroot defined by $\alpha(H) = \langle H, h_\alpha \rangle$ for each $H \in \mathfrak{h}$. The inner type assumption, together with the $\Theta$-stability of $\mathfrak{h}$, implies that each root space $\mathfrak{g}_\alpha$ is contained in either $\mathfrak{t}^\mathbb{C}$ or in $\mathfrak{p}^\mathbb{C}$ and thus the roots can be sorted into two types. A root $\alpha$ is of compact type if the root space $\mathfrak{g}_\alpha$ is a subset of $\mathfrak{t}^\mathbb{C} \subset \mathfrak{g}$ and of noncompact type otherwise, i.e., when $\mathfrak{g}_\alpha \subset \mathfrak{p}^\mathbb{C}$. The following proposition is a standard fact.

Proposition 1.1. For each simple positive root $\gamma$ there exists a Lie algebra homomorphism $\iota_\gamma: \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ which carries the standard triangular decomposition of $\mathfrak{sl}(2,\mathbb{C})$ (1.3) into the triangular decomposition $\hat{g} = \hat{n}^- + \hat{h} + \hat{n}^+$ and:

(a) in any case $\iota_\gamma$ restricts to a homomorphism $\iota_\gamma: \mathfrak{su}(2) \to \hat{\mathfrak{u}}$;

(b) when $\gamma$ is of compact type then $\iota_\gamma$ restricts to $\iota_\gamma: \mathfrak{su}(2) \to \hat{\mathfrak{u}}$;

(c) when $\gamma$ is of noncompact type then $\iota_\gamma$ restricts to $\iota_\gamma: \mathfrak{su}(1,1) \to \hat{\mathfrak{g}}_0$.

We denote the corresponding group homomorphism by the same symbol. Note that if $\gamma$ is of noncompact type, then $\iota_\gamma$ induces an embedding of the rank one diagram (0.2) into the finite dimensional group diagram (0.1). For each simple positive root $\gamma$, we use the group homomorphism to set

$$r_\gamma = \iota_\gamma \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in N_U(\hat{T})$$

and obtain a specific representative for the associated simple reflection $r_\gamma \in W = N_U(\hat{T})/\hat{T}$ corresponding to $\gamma$. (We will adhere to the convention of using boldface letters to denote representatives of Weyl group elements.)
The coroots \( h_{\tilde{\alpha}_1}, h_{\tilde{\alpha}_2}, \ldots, h_{\tilde{\alpha}_r} \) form a basis for \( \tilde{\mathfrak{a}} \) and the dual basis consists of the fundamental weights \( \Lambda_1, \Lambda_2, \ldots, \Lambda_r \). Introduce the lattices

\[
\widehat{T} = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \Lambda_i \quad \text{(weight lattice), and} \quad \check{T} = \bigoplus_{1 \leq i \leq r} \mathbb{Z} h_i \quad \text{(coroot lattice).}
\]

Recall that the kernel of \( \exp : \mathfrak{t} \to T \) is \( 2\pi i \check{T} \). Consequently there is a natural identification \( \widehat{T} \simeq \text{Hom}(\check{T}, T) \), where a weight \( \check{\Lambda} \) corresponds to the character \( \exp(2\pi i x) \to \exp(2\pi i \check{\Lambda}(x)), \) for \( x \in \mathfrak{t} \). Likewise, there is a natural identification \( \check{T} \to \text{Hom}(T, \check{T}) \), where an element \( h \) of the coroot lattice corresponds to the homomorphism \( T \to \check{T} \) given by \( \exp(2\pi i x) \to \exp(2\pi i x h), \) for \( x \in \mathbb{R} \).

In addition, let

\[
\widehat{R} = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \check{\alpha}_i \quad \text{(root lattice), and} \quad \check{R} = \bigoplus_{1 \leq i \leq r} \mathbb{Z} \check{\Theta}_i \quad \text{(coweight lattice).}
\]

Then these lattices and bases are also in duality. The elements \( \check{\Theta}_1, \ldots, \check{\Theta}_r \) are the fundamental coweights.

The affine Weyl group for \( \mathfrak{g} \) is the semidirect product \( \check{W} \rtimes \check{T} \). For the action of \( \check{W} \) on \( \mathfrak{h}_\mathbb{R} \), a fundamental domain is the positive Weyl chamber \( C \). For the natural affine action

\[
(W \rtimes \check{T}) \times \check{a} \to \check{a}
\]
a fundamental domain is the convex set

\[
C_0 = \{ x \in C : \check{\theta}(x) < 1 \} \quad \text{(fundamental alcove)}.
\]

(See page 72 of [15] for \( A_2 \) and Figure 1 of [13] for \( G_2 \).) The set of extreme points for the closure of \( C_0 \) is \( \{ 0 \} \cup \{ \frac{1}{m} \check{\Theta}_i \}, \) where \( \check{\theta} = \sum n_i \check{\alpha}_i \) (these coefficients are compiled in Section 1.1 of [7]). It is standard to set \( h_\check{\beta} = \sum_{i=1}^r \check{\Theta}_i \). Then \( 2h_\check{\beta} \in \check{T} \) and for each positive root \( \check{\alpha} \), one has \( \check{\alpha}(h_\check{\beta}) = \text{height}(\check{\alpha}) \).

2. Factorization for Finite Dimensional Groups

By definition, the Birkhoff decomposition of \( \hat{G} \) relative to the triangular decomposition \( \hat{g} = \check{a}^{-} + \check{h} + \check{a}^{+} \) is

\[
\hat{G} = \bigsqcup \Sigma^\hat{G}_w \quad \text{where} \quad \Sigma^\hat{G}_w = \check{N}^{-} w \check{B}^{+}.
\]

If we fix a representative \( w \in N_G(\check{T}) \) for \( w \in \check{W} \), then each \( g \in \Sigma^\hat{G}_w \) can be factored uniquely as

\[
g = lwmau, \quad \text{with} \quad l \in \check{N}^{-} \cap w\check{N}^{-}w^{-1}, \quad ma \in \check{T} \check{A}, \quad \text{and} \quad u \in \check{N}^{+}.
\]

This defines functions \( l: \Sigma^\hat{G}_w \to \check{N}^{-} \cap w\check{N}^{-}w^{-1}, \) \( m: \Sigma^\hat{G}_w \to \check{T}, \) \( a: \Sigma^\hat{G}_w \to \check{A}, \) and \( u: \Sigma^\hat{G}_w \to \check{U}. \) For fixed \( m_0 \in \check{A}, \) the subset \( \{ g \in \Sigma^\hat{G}_w : m(g) = m_0 \} \) is a stratum (topologically an affine space). It is therefore sensible and appropriate to refer to \( \Sigma^\hat{G}_w \) as the “isotypic component of the Birkhoff decomposition of \( \hat{G} \) corresponding to \( w \in \check{W} \).” However we may occasionally lapse into referring to \( \Sigma^\hat{G}_w \) as the “Birkhoff stratum corresponding to \( w \).” We are interested in describing the induced decomposition of \( \hat{G}_0 \subset \hat{G} \).
We say that the elements of $\Sigma_w^G$ have a \textit{triangular factorization} since then (2.2) reduces to
\[ g = l(g)d(g)u(g) \text{ where } d(g) = ma \in T\hat{A} = \hat{H} \]
and $l(g) \in \hat{N}^-$. The factor $d(g)$ can be explicitly computed in terms of root data by the formula
\[ d(g) = \prod_{j=1}^{r} \hat{\sigma}_j(g)^{h_{\alpha_j}} \]
where $\hat{\sigma}_j(g) = \phi_{\Lambda_j}(\pi_{\Lambda_j}(g)v_{\Lambda_j})$ is the fundamental matrix coefficient for the highest weight vector corresponding to $\Lambda_j$.

\section*{2.1. Factorization in the Compact Case.}
When $\mathfrak{m}$ is a compact Lie algebra, then we will write $\mathfrak{u}$ for $\mathfrak{m}$ and $\mathcal{U}$ for the group $\mathcal{M}$ inside of $\mathcal{G}$. Given $w \in W$, define
\[ \Sigma_w^G := \Sigma_w \cap \mathcal{U}. \]

\textbf{Theorem 2.1.} Fix a representative $w \in N_\mathcal{U}(\hat{T})$ for $w$. For $g \in \Sigma_w^G$ the unique factorization (2.2) induces a bijective correspondence
\[ \Sigma_w^G \leftrightarrow \left( \hat{N}^- \cap w\hat{N}^- w^{-1} \right) \times \hat{T} \text{ given by } g \mapsto (l, m). \]

\textbf{Remark 2.1.} For fixed $m_0 \in \hat{T}$, the set \{ $g \in \Sigma_w^G$ : $m(g) = m_0$ \} is a stratum, and we will refer to $\Sigma_w^G$ as the “isotypic component of the Birkhoff decomposition for $\mathcal{U}$ corresponding to $w \in W$.” The quotient of $\Sigma_w^G$ by $\hat{T}$ is the usual Birkhoff stratum for the flag space $\mathcal{U}/\mathcal{T} = \hat{G}/\hat{B}^+$ corresponding to $w$.

We now briefly recall Lu’s approach to root subgroup factorization from [8]. This involves the Bruhat decomposition $G = \sqcup_{w} \hat{B}^+ w \hat{B}^+$. A translation of Lu’s results over to the Birkhoff decomposition will be given below.

For $\zeta \in \mathbb{C}$, we define a function $k : \mathbb{C} \to SU(2)$ by
\[ k(\zeta) = a_+ (\zeta) \begin{pmatrix} 1 & -\zeta \\ \zeta & 1 \end{pmatrix} \in SU(2), \text{ where } a_+(\zeta) = (1 + |\zeta|^2)^{-1/2}. \]

\textbf{Theorem 2.2.} Fix $w' \in W$. Choose a minimal factorization $w' = r_l(w') \cdots r_1$, where each $r_j$ is a reflection and write $\gamma_j$ for the corresponding simple positive root. Then the map
\[ \mathcal{L}^{(w')} \times \hat{T} \to \hat{U} \cap \mathcal{U} \hat{B}^+ w' \hat{B}^+ \text{ given by } ((\zeta_j), t) \mapsto r_n i_{\gamma_n} (k(\zeta_n)) \cdots r_1 i_{\gamma_1} (k(\zeta_1)) t \]
is a bijection.

\textbf{Remarks.}
(a) The algorithm for choosing a factorization for $w'$ is the following: choose (a simple positive root) $\gamma_1$ such that $w' \cdot \gamma_1 < 0$, determining $r_1$; choose $\gamma_2$ such that $w' r_1 \cdot \gamma_2 < 0$, determining $r_2$; choose $\gamma_3$ such that $w' r_1 r_2 \cdot \gamma_3 < 0$, determining $r_3$; and so on. The positive roots flipped to negative roots are $\bar{\gamma}_j = r_j \cdots r_{j-1} \cdot \gamma_j$, for $j = 1, \ldots, l(w')$.
(b) A choice of factorization $w' = r_l(w') \cdots r_1$ determines a non-repeating sequence of adjacent Weyl chambers
\[ C, (w_1')^{-1} C, \ldots, (w_j')^{-1} C, \ldots, (w_1')^{-1} C \]
where \( w'_j := r_j . r_1 \) and the step from \((w'_{j-1})^{-1}C\) to \((w'_j)^{-1}C\) is implemented by the reflection \((w'_{j-1})^{-1}r_j w'_{j-1}^{-1}\) associated to \(r_j\). In particular the wall between \((w'_{j-1})^{-1}C\) and \((w'_j)^{-1}C\) is fixed by \((w'_{j-1})^{-1}r_j w'_{j-1}^{-1}\). Conversely, given a sequence of length \( l(w') \) of adjacent chambers \( C_1, \ldots, C_{l(w')} \) from \( C_1 = C \) to \( C_{l(w')} = (w')^{-1}C \) then there is a corresponding minimal factorization.

(c) A basic example of a factorization of the longest element of the Weyl group for \( \mathfrak{sl}(n, \mathbb{C}) \) is the lexicographic factorization

\[
\tau_1 = \lambda_1 - \lambda_2, \tau_2 = \lambda_1 - \lambda_3, \ldots, \tau_n = \lambda_1 - \lambda_n,
\]

The Bruhat and Birkhoff decompositions (in this finite dimensional context) are related by translation by \( w_0 \), the unique longest Weyl group element, i.e.

\[
\hat{N}^{-} w \hat{B}^+ = w_0 \hat{N}^+ w' \hat{B}^+ = w_0 \hat{B}^+ w' \hat{B}^+
\]

where \( w = w_0 w' \). Since we can choose representatives for \( w, w_0, \) and \( w' \) in \( \hat{U} \), the same relationship holds on the induced decompositions of \( \hat{U} \). In terms of this translation, the following lemma describes how to select the sequence of simple positive roots intrinsically in terms of \( w \) (without reference to \( w_0 \) and \( w' \)).

**Lemma 2.1.** Fix \( w \in \hat{W} \).

(a) Choose a sequence of simple positive roots \( \gamma_j \) in the following way: (1) choose \( \gamma_1 \) such that \( w \cdot \gamma_1 > 0 \); (2) choose \( \gamma_2 \) such that \( w r_1 \cdot \gamma_2 > 0 \); (3) choose \( \gamma_3 \) such that \( w r_1 r_2 \cdot \gamma_3 > 0 \), and so on, where \( r_j \) is the simple reflection corresponding to \( \gamma_j \). Let \( \hat{r}_j = r_1 \cdots r_{j-1} \cdot \gamma_j \). Then the \( \hat{r}_j \) are the positive roots which are mapped to positive roots by \( w \).

(b) This choice of positive roots determines a non-repeating sequence of adjacent Weyl chambers

\[
w^{-1}C, \ r_1 w^{-1}C, \ldots, \ r_{j-1} \cdots r_1 w^{-1}C, \ldots, \ -C.
\]

(2.5)

If \( w'_j = r_j \cdots r_1 \) then the step from \((w'_{j-1})^{-1}w^{-1}C\) to \((w'_j)^{-1}w^{-1}C\) is implemented by the reflection \( r_j \). Conversely, given a sequence \((C_j)\) of length \( n = l(w_0) - l(w) \) consisting of adjacent chambers from \( C_1 = w^{-1}C \) to \( C_n = -C \) which is minimal, there is a corresponding minimal factorization of \( w = w_0^{-1}w \).

**Proof.** As we noted above, this is equivalent to the more standard procedure of setting \( w' = w_0^{-1}w \) and choosing a reduced factorization \( w' = r_n \cdots r_1 \) where \( n = l(w_0) - l(w) = l(w') \).

**Theorem 2.3.** Fix \( w \in W(\hat{K}) \) and a representative \( w \in N_{\hat{K}}(T) \) for \( w \), then determine positive simple roots \( \gamma_1, \ldots, \gamma_n \) with associated simple reflections \( r_1, \ldots, r_n \),
and positive roots $\tau_1, \ldots, \tau_n$ as in Lemma 2.1. Set $w'_j = r_j \ldots r_1$ and $\iota_\tau(g) = w'_{j-1} \tau_j \iota_\tau(g)(w'_{j-1})^{-1}$ for each $g \in \text{SL}(2, \mathbb{C})$.

(a) Each $g \in \Sigma^\wedge_w$ has a unique factorization of the form

$g = w \iota_\tau(k(\zeta_n)) \ldots \iota_\tau(k(\zeta_1)) t$

for some $t \in T$ and some $(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$.

(b) The map

$\mathbb{C}^n \to N^- \cap wN^- w^{-1} : \zeta \to l(w \iota_\tau(k(\zeta_n)) \ldots \iota_\tau(k(\zeta_1)))$

is a diffeomorphism.

(c) If $g \in \Sigma^\wedge_w$ has the factorization in part (a) then the factor $a(g)$ from (2.2) has the product form

$a(g) = \prod_{j=1}^n a_+(\zeta_j)^{h_j}$

where $a_+$ is the function from (2.3).

Proof. This is a translation of Lu’s results. We will essentially reproduce the proof in the next subsection. □

Remark 2.2. Lu’s approach to factorization, for the Bruhat decomposition, extends in a relatively direct way to Kac-Moody groups; see the appendix to [10]. It is algebraic. The translation to the Birkhoff point of view, when we pass to the loop context, injects elements of analysis into the theory.

2.2. Factorization in the Noncompact Inner Case. Now we return to the case where $\mathfrak{m} = \dot{\mathfrak{g}}_0$ is a noncompact Lie algebra over $\mathbb{R}$ of inner type. Then each root space for $\mathfrak{h}$ on $\dot{\mathfrak{g}}$ is contained either in $\mathfrak{t}^C$ or in $\mathfrak{p}^C$. This yields a vector space decomposition

$\dot{\mathfrak{n}}^+ = \dot{\mathfrak{n}}^+_t + \dot{\mathfrak{n}}^+_p$

where $\dot{\mathfrak{n}}^+_t$ is spanned by root vectors corresponding to compact type positive roots, and $\dot{\mathfrak{n}}^+_p$ is spanned by root vectors corresponding to noncompact type positive roots; moreover, $\dot{\mathfrak{n}}^+_p$ is an abelian ideal of $\dot{\mathfrak{n}}^+$. Likewise, $\dot{\mathfrak{n}}^- = \dot{\mathfrak{n}}^-_t + \dot{\mathfrak{n}}^-_p$. Note that

(2.6) $\dot{\mathfrak{t}}^C = \dot{\mathfrak{n}}^+_t + \dot{\mathfrak{h}} + \dot{\mathfrak{n}}^+_t$ and $\dot{\mathfrak{p}}^C = \dot{\mathfrak{n}}^-_p + \dot{\mathfrak{n}}^+_p$

as vector spaces. The sum $\dot{\mathfrak{t}}^C + \dot{\mathfrak{n}}^+_p$ is the parabolic subalgebra of $\dot{\mathfrak{g}}$ corresponding to the set of simple positive roots of compact type. We denote the corresponding parabolic subgroup of $\dot{G}$ by $\dot{P} = K^C \dot{N}_p^+$. We refer to the decomposition

$\dot{\mathfrak{g}} = \dot{\mathfrak{n}}^- + \dot{\mathfrak{t}}^C + \dot{\mathfrak{n}}^+_p$

as a block triangular decomposition of $\dot{\mathfrak{g}}$. For $\dot{G}$, the corresponding group-level “block Birkhoff decomposition” is well-known. We will use $W(\dot{K}) = N_{\dot{K}}(\dot{T})/\dot{T}$ as a model for the Weyl group of $(\dot{t}, \dot{t}, \dot{t})$. Then there is a faithful embedding

$W(\dot{K}) := N_{\dot{K}}(\dot{T})/\dot{T} \to \dot{W} := N_{\dot{U}}(\dot{T})/\dot{T}$.

The components of the “block Birkhoff decomposition” are then indexed by the elements of quotient $\dot{W}/W(\dot{K})$. 
Lemma 2.2.

\[ \hat{G} = \bigsqcup_{w \in W/W(K)} \hat{N}_p^- w \hat{P} \]

and the natural map

\[ \hat{N}_p^- \cap w\hat{N}_p^- w^{-1} \to \hat{N}_p^- w \hat{P} \]

given by \( l \mapsto lw\hat{P} \)

is a diffeomorphism.

Proof. This follows from standard facts about Birkhoff stratification for a generalized flag space; see, for example, the Appendix to [10]. \( \square \)

Theorem 2.4.

(a) Each \( g \in \hat{G}_0 \) has a unique “block triangular factorization”

\[ g = l_pg_k u_p \]

where \( l_p \in \hat{N}_p^- , \ g_k \in \hat{K}^C , \) and \( u_p \in \hat{N}_p^+ . \)

(b) The set \( D(G_0, K) = \{ l_p : g = l_pg_k u_p \in \hat{G}_0 \} \) is a contractible bounded complex domain in \( \hat{N}_p^- . \)

(c) Each \( g \in \hat{G}_0 \) has a factorization of the form

\[ g = l_pl_k mw_k u_p \]

where \( l_p \) and \( u_p \) are the same factors occurring in part (a), \( w \) is a representative for some \( w \in W(K) , \ ma \in \hat{T}\hat{A} , \ l_k \in \hat{N}_k^- \cap w\hat{N}_k^- w^{-1} , \) and \( u_k \in \hat{N}_k^+ . \)

(d) Furthermore,

\[ \hat{G}_0 = \bigsqcup_{W(K)} \Sigma_{\hat{G}_0} , \] where \( \Sigma_{\hat{G}_0} : = \Sigma_{\hat{G}} \cap \hat{G}_0 . \)

(e) For each \( w \in W(K) , \) if a representative \( w \) for \( w \) is fixed then the factorization in part (c) is unique and defines functions \( l_p : \Sigma_{\hat{G}_0} \to D(G_0, K) , \)

\( l_k : \Sigma_{\hat{G}_0} \to \hat{N}_k^- \cap w\hat{N}_k^- w^{-1} , \) \( m : \Sigma_{\hat{G}_0} \to \hat{T}, \) and induces a diffeomorphism

\[ \Sigma_{\hat{G}_0} \to \{ l_p : g = l_pg_k u_p \in \hat{G}_0 \} \times \{ l_k \in \hat{N}_k^- \cap w\hat{N}_k^- w^{-1} \} \times \hat{T} \]

given by \( g \mapsto (l_p(g), l_k(g), m(g)) . \)

Proof. There is a natural map \( \hat{G}_0/\hat{K} \to \hat{G}/\hat{P} \) and by Lemma 2.2, \( \hat{G}/\hat{P} \) is a union of \( \hat{N}_p^- \)-orbits indexed by \( w \in W/W(K) . \) The top stratum corresponds to \( w = 1 \)
in \( W/W(\hat{K}) \) and is parameterized by \( \hat{N}_p^- . \) Theorem 5 of [9] shows that this image is contained in the top stratum. This can also be deduced from Lemma 7.9 on page 388 of [6]. Parts (a) and (b) of the theorem now follow from this observation. This also implies that \( \Sigma_{\hat{G}_0} \) is empty unless \( w \in W(K) \) (since \( W(\hat{K}) \) is the 1 in \( W/W(\hat{K}) \)). Hence part (d) follows from this observation as well.

Part (c) is a consequence of the triangular factorization for \( K^C \) (this is the sense in which triangular factorization for the noncompact inner case reduces to the compact case).

Because of the uniqueness of the factorization in (c), the map in (e) is 1-1. It remains to show the map is onto. Suppose that we are given \( (l_p, l_k, t) \) in the codomain of the map. By definition there exists \( g_0 \in \hat{G}_0 \) such that \( g_0 = l_pg_k u_p . \) Given \( k \in \hat{K}, \)

\[ g_0k = l_pg_k u_pk = l_pg_k' u_p' \]

where \( g_k' = g_k \in \hat{K}^C \) and \( u_p' = k^{-1}uPk \in \hat{N}_p^+ . \)
because $\hat{K}$ normalizes $N_p^+$. Since $\hat{K}$ acts transitively on the flag space $(B^- \cap K^C)\backslash \hat{K}^C$, we can choose $k$ such that $g_k^c$ has triangular factorization of the form \( l_k w \text{mau}_k \). We can always multiply on the right by a $t'$ to obtain the desired $t$ without affecting the other factors. This shows the map is onto. \( \square \)

**Corollary 2.1.**

(a) The map \( G_0 \to \hat{K}^C \) given by \( g \to g_k \) is a homotopy equivalence.

(b) For each \( w \in W(K) \) the map
\[
\Sigma^G_0 \to \Sigma^{K^C} \text{ given by } g \mapsto g_k
\]
induced by part (c) of Theorem 2.4 is a homotopy equivalence.

We now turn to root subgroup factorization. In the compact case, a special role was played by the function \( k : \mathbb{C} \to SU(2) \) defined by
\[
(2.8) \quad k(\zeta) = a_+(\zeta) \begin{pmatrix} 1 & -\bar{\zeta} \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} a_+(\zeta) & 0 \\ 0 & a_+^{-1}(\zeta) \end{pmatrix} \begin{pmatrix} 1 & -\bar{\zeta} \\ 0 & 1 \end{pmatrix} \in SU(2)
\]
where \( a_+(\zeta) = (1 + |\zeta|^2)^{-1/2} \). By composing copies of this function with root homomorphisms, interleaving the compositions into a minimal sequence of simple reflections factoring \( w \), and multiplying out the results we were able to parameterize \( \Sigma^G_0 \). To accommodate the new situation where the simple reflections may be associated to noncompact type roots, and to parameterize \( \Sigma'^G_0 \), we define a function \( q : \Delta \to SU(1,1) \) by
\[
(2.9) \quad q(\zeta) = a_-(\zeta) \begin{pmatrix} 1 & \bar{\zeta} \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} a_-^{-1}(\zeta) & 0 \\ 0 & a_-(\zeta) \end{pmatrix} \begin{pmatrix} 1 & \bar{\zeta} \\ 0 & 1 \end{pmatrix} \in SU(1,1)
\]
where \( a_-(\zeta) = (1 - |\zeta|^2)^{-1/2} \).

**Theorem 2.5.** Fix \( w \in W(\hat{K}) \) and a representative \( w \in N_K(T) \) for \( w \), then determine positive simple roots \( \gamma_1, \ldots, \gamma_n \) with associated simple reflections \( r_1, \ldots, r_n \), and positive roots \( \tau_1, \ldots, \tau_n \) as in Lemma 2.1. Set \( w_j' = r_j \cdot r_1 \) and \( t, \tau_j(g) = w_{j-1} t \tau_j(g)(w_j')^{-1} \) for each \( g \in SL(2, \mathbb{C}) \).

(a) Each \( g \in \Sigma'^G_0 \) has a unique factorization
\[
g = w t_{\tau_1}(g(\zeta_1)) \cdots t_{\tau_n}(g(\zeta_n)) t
\]
for some \( t \in T \), where if \( \tau_j \) is a noncompact type root, then \( |\zeta_j| < 1 \) and \( g(\zeta_j) = q(\zeta_j) \) from (2.9), and if \( \tau_j \) is a compact type root, then \( \zeta_j \) is unrestricted in \( \mathbb{C} \) and \( g(\zeta_j) = k(\zeta_j) \) as in (2.8).

(b) If \( g \in \Sigma'^G_0 \) has the factorization in part (a) then the factor \( a(g) \) of \( g \) from part (c) of Theorem 2.4 has the product form
\[
a(g) = \prod_{j=1}^n a(\zeta_j)^{h_{\tau_j}}
\]
where \( a(\zeta_j) = a_-^{-1}(\zeta_j) \) if \( \tau_j \) is a noncompact type root and \( a(\zeta_j) = a_+^{-1}(\zeta_j) \) if \( \tau_j \) is a compact type root.

**Proof.** We must show that the map
\[
(2.10) \quad \{ (\zeta_1, \ldots, \zeta_n) \} \times T \to \Sigma'^G_0 \text{ given by } ((\zeta_j), t) \mapsto g
\]
where \( g \) is defined as in part (a), is a diffeomorphism. Here it is understood that if the \( j \)th root is of noncompact type, then \( g(\xi_j) = g(\zeta_j) \) and \( |\zeta_j| < 1 \), and if the \( j \)th root is of compact type, then \( g(\xi_j) = k(\zeta_j) \) and \( \zeta_j \) is unrestricted in \( \mathbb{C} \).

We first calculate the triangular decomposition for
\[
g^{(n)} := \iota_{\tau_n}(g(\xi_n)) \ldots \iota_{\tau_1}(g(\xi_1))
\]
where \( g(\xi_j) = k(\xi_j) \) when \( \tau_j \) is a compact by induction on \( n \). In the process we will prove part (b), which will be used in the proof of part (a). First note that since \( \tau_j = (w_{j-1}^2)^{-1} \cdot \gamma_j \) and \( \iota_{\tau_j} \) preserves triangular factorizations,
\[
\iota_{\tau_j}(g(\xi_j)) = \iota_{\tau_j}\left( \left( \begin{array}{cc} 1 & 0 \\ \zeta_j & 1 \end{array} \right) a_\pm(\xi_j)^{h_{\tau_j}} \iota_{\tau_j}\left( \left( \begin{array}{cc} 1 & \pm \zeta_j \\ 0 & 1 \end{array} \right) \right) \right) 
\]
\[
= \exp(\zeta_j f_{\tau_j}) a_\pm(\xi_j)^{h_{\tau_j}} (w_{j-1}^2)^{-1} \exp(\pm \zeta_j e_{\gamma_j}) w_{j-1}'
\]
is a triangular factorization (the plus/minus case is used for the compact/noncompact root case, respectively). In what follows, we will simply write \( a(\xi_j) \) for \( a_\pm(\xi_j) \) since the appropriate sign can be inferred from the type of the corresponding root \( \tau_j \).

Suppose that \( n = 2 \). Then
\[
\tag{2.11}
g^{(2)} = \exp(\zeta_2 f_{\tau_2}) a(\xi_2)^{h_{\tau_2}} r_1^{-1} \exp(\pm \zeta_2 e_{\gamma_2}) r_1 \exp(\zeta_1 f_{\gamma_1}) a(\xi_1)^{h_{\gamma_1}} \exp(\pm \zeta_1 e_{\gamma_1})
\]
where \( \pm \zeta_j \) occurs according to whether \( \tau_j \) is a compact/noncompact type root, respectively. The key point is that
\[
r_1^{-1} \exp(\pm \zeta_2 e_{\gamma_2}) r_1 \exp(\zeta_1 f_{\gamma_1}) \\
= r_1^{-1} \exp(\pm \zeta_2 e_{\gamma_2}) \exp(\zeta_1 e_{\gamma_1}) r_1 \\
= r_1^{-1} \exp(\zeta_1 e_{\gamma_1}) \bar{u} r_1, \quad (\text{for some } \bar{u} \in N^+ \cap r_1^{-1} N^+ r_1) \\
= \exp(\zeta_1 f_{\gamma_1}) u_1 \quad (\text{for some } u_1 \in N^+).
\]
Insert this calculation into (2.11). We then see that \( g^{(2)} \) has a triangular factorization \( g^{(2)} = l^{(2)} a^{(2)} u^{(2)} \), where
\[
a^{(2)} = a(\xi_1)^{h_{\gamma_1}} a(\xi_2)^{h_{\tau_2}}
\]
and
\[
\tag{2.12}
l^{(2)} = \exp(\zeta_2 f_{\tau_2}) \exp(\zeta_1 a(\xi_2)^{-\tau_1(h_{\tau_2})} f_{\tau_1}) \\
= \exp(\zeta_2 f_{\tau_2} + \zeta_1 a(\xi_2)^{-\tau_1(h_{\tau_2})} f_{\tau_1}).
\]
Note that with \( l \)-factor, the sign difference between the calculations in the compact and noncompact case is only present in the corresponding type of \( a(\xi_2) \).

To apply induction, we assume that \( g^{(n-1)} \) has a triangular factorization \( g^{(n-1)} = l^{(n-1)} a^{(n-1)} u^{(n-1)} \) with
\[
\tag{2.13}
l^{(n-1)} = \exp(\zeta_{n-1} f_{\tau_{n-1}}) \bar{l} \in N^- \cap (w_{n-1}'^2)^{-1} N^+ w_{n-1}' = \exp(\sum_{j=1}^{n-1} C f_{\tau_j}),
\]
for some \( \bar{l} \in N^- \cap (w_{n-2}'^2)^{-1} N^+ w_{n-2}' = \exp(\sum_{j=1}^{n-2} C f_{\tau_j}) \), and
\[
a^{(n-1)} = \prod_{j=1}^{n-1} a(\xi_j)^{h_{\tau_j}}
\]

(2.14)
We have established this for \( n - 1 = 1, 2 \). For \( n \geq 2 \)
\[
g^{(n)} = \exp(\zeta_n f_{\tau_n}) a(\zeta_n)^{h_{\tau_n}}(w_{n-1})^{-1} \exp(\pm \zeta_n e_{\gamma_n}) w_{n-1} \exp(\pm \zeta_n f_{\tau_n}) \bar{u}^{(n-1)} a^{(n-1)} u^{(n-1)}
\]
where \( \bar{u} = w_{n-1} \exp(\zeta_n f_{\tau_n}) \bar{u} w_{n-1}^{-1} \exp(\zeta_n e_{\gamma_n}) \bar{u}^{(n-1)} a^{(n-1)} u^{(n-1)}, \)

where \( \bar{u} = w_{n-1} \exp(\zeta_n f_{\tau_n}) \bar{u} w_{n-1}^{-1} \exp(\zeta_n e_{\gamma_n}) \bar{u}^{(n-1)} a^{(n-1)} u^{(n-1)}, \)

where \( \bar{u} = w_{n-1} \exp(\zeta_n f_{\tau_n}) \bar{u} w_{n-1}^{-1} \exp(\zeta_n e_{\gamma_n}) \bar{u}^{(n-1)} a^{(n-1)} u^{(n-1)}, \)

Now we want to draw some conclusions. First note that the inductive calculation
\[
N^+ = (N^+ \cap w'_{n-1} N^- (w'_{n-1})^{-1}) (N^+ \cap w'_{n-1} N^+ (w'_{n-1})^{-1})
\]
and let
\[
l = a(\zeta_n)^{h_{\tau_n}} (w'_{n-1})^{-1} \bar{u}_1 w_{n-1} a(\zeta_n)^{-h_{\tau_n}} \in N^- \cap (w'_{n-1})^{-1} N^+ w_{n-1}.
\]

Then \( g^{(n)} \) has triangular decomposition
\[
g^{(n)} = (\exp(\zeta_n f_{\tau_n}) l) \left( a(\zeta_n)^{h_{\tau_n}} a^{(n-1)} \right) \left( (a^{(n-1)})^{-1} w'_{n-1} \bar{u}_2 (w'_{n-1})^{-1} a^{(n-1)} u^{(n-1)} \right)
\]
by (2.14).

Now suppose that we multiply this triangular decomposition on the left by \( w \) (as in part (a)). Because the \( \tau_j, j = 1, ..., n \), are the positive roots which are mapped to positive roots by \( w \), it follows that \( l^{(n)} \) will be conjugated by \( w \) into another element in \( \hat{N}^- \). It follows that \( g_n \), as defined in part (a), is in \( \Sigma_w^G \).

Now we want to draw some conclusions. First note that the inductive calculation of the triangular decomposition implies part (b) of the Theorem. We can also see that the map (2.10), which has domain a product of disks and affine planes, is 1-1 and open. Because \( \Sigma_w^G \) is connected (by Corollary 2.1), to conclude that the map (2.10) is a diffeomorphism, it suffices to show that the map (2.10) has a closed image in \( \Sigma_w^G \). Suppose that \( (g_n)_{n=1}^\infty \) is a sequence of elements in the image of (2.10) (these are elements that have root subgroup factorizations), and suppose this sequence converges to \( g' \in \Sigma_w^G \). We must show that \( g' \) has a root subgroup factorization. For each \( n \), consider the unique triangular factorization
\[
g_n = l_n w m_n a_n u_n, \quad l_n \in \hat{N}^- \cap w \hat{N}^- w^{-1}.
\]

Since \( g' \in \Sigma_w^G \) with triangular factorization \( g' = l' w m' a' u' \), \( l' \in \hat{N}^- \cap w \hat{N}^- w^{-1} \), we know that \( a_n \to a' \) as \( n \to \infty \), and hence the sequence \( (a_n)_{n=1}^\infty \) is bounded in \( \hat{A} \). The formula in part (b) for the \( a \)-component, applied to each \( g_n \), now implies that associated sequence of parameters in the domain must remain bounded as \( n \to \infty \). We can therefore find a subsequence of the sequence of parameters which converges to an element of the domain. The sequence \( g_n \) will then converge to the group element corresponding to this limiting parameter by continuity. This limit must be \( g' \), and hence we obtain a root subgroup factorization for \( g' \). This completes the proof.

### 2.3. Haar Measure in Root Subgroup Coordinates

In closing this first part, we mention one striking feature of root subgroup factorization, the fact that Haar measure is a product in these coordinates. The analogue of this in the compact case is due to Lu in [8], where she obtains product formulas for Kostant’s harmonic forms on \( \hat{U}/\hat{T} \), one of which is the invariant volume on \( \hat{U}/\hat{T} \). The argument here is more direct.
Theorem 2.6. With \( w = 1 \) and notation as in Theorem 2.5 we obtain a parametrization of the open dense subset \( \Sigma_{1}^{G_{0}} \subset \hat{G}_{0} \) by a product of copies of complex planes and disks, together with the torus \( T \). In terms of the complex parameters \((\zeta_{j})\) and \( t \in \hat{T} \) for
\[
g = i\tau_{a}(g(\zeta_{a}))...i\tau_{1}(g(\zeta_{1}))t \in \Sigma_{1}^{\hat{G}_{0}}
\]
where \( g(\zeta_{j}) = k(\zeta_{j}) \) when \( \tau_{j} \) is of compact type (resp. \( g(\zeta_{j}) = q(\zeta_{j}) \) when \( \tau_{j} \) is of noncompact type), then Haar measure for \( \hat{G}_{0} \) is (up to a constant)
\[
d\lambda_{\hat{G}_{0}}(g) = \left( \prod_{j=1}^{n} a(\zeta_{j})^{2\delta(h_{\tau_{j}})+2|d\zeta_{j}|} \right) d\lambda_{\hat{T}}(t)
\]
where \( a(\zeta_{j}) = a_{\pm}(\zeta_{j}) \) according to whether \( \tau_{j} \) is of compact/noncompact type, and where \( d\lambda_{\hat{T}}(t) \) denotes Haar measure for \( \hat{T} \).

Remark 2.3. Using the fact that \( \delta(h_{\tau}) \) is the height of the positive root \( \tau \), this formula for Haar measure can be alternatively written as
\[
d\lambda_{\hat{G}_{0}}(g) = d\lambda_{\hat{T}}(t) \prod_{\tau > 0} \frac{|d\zeta_{\tau}|}{(1 \pm |\zeta_{\tau}|^{2})^{1+\text{height}(\tau)}}
\]
where we choose the plus sign for compact roots, the negative sign for noncompact roots, \( \zeta_{\tau} \) is understood to be bounded by one when \( \tau \) is noncompact, and \( \zeta_{\tau} = \zeta_{j} \) when \( \tau = \tau_{j} \). This shows that the formula we are obtaining for Haar measure does not depend on the choice of ordering of the positive roots \( \tau_{j} \) (induced by the factorization of \( w_{0} \)).

Denote the triangular decomposition for \( g \in \Sigma_{1}^{\hat{G}_{0}} \) by
\[
g = l(g)m(g)a(g)u(g)
\]
Recall that \( g \) is uniquely determined by \( l(g) \in \hat{N}^{-} \) and \( m(g)\hat{T} \). The following formula should be attributed to Harish-Chandra:

Lemma 2.3. Up to a normalization
\[
d\lambda_{\hat{G}_{0}}(g) = a(g)^{4d}d\lambda_{\hat{N}^{-}}(l(g))d\lambda_{\hat{T}}(m(g))
\]
where (by slight abuse of notation) it is understood that we are restricting Haar measure for \( \hat{N}^{-} \) to the intersection of \( \hat{N}^{-} \) with the image of \( \Sigma_{1}^{\hat{G}_{0}} \).

Proof. This is equivalent to proving the coordinate expression
\[
d\lambda_{\hat{G}_{0}/\hat{T}}(g\hat{T}) = a(g)^{4d}d\lambda_{\hat{N}^{-}}(l(g))
\]
for the invariant measure on the quotient \( \hat{G}_{0}/\hat{T} \). The value of the density of \( d\lambda_{\hat{G}_{0}/\hat{T}} \) with respect to \( d\lambda_{\hat{N}^{-}}(l(g)) \) at \( l(g) \) can be computed as follows. Identify the tangent space to \( \hat{N}^{-} \) at \( l(g) \) with \( \hat{n}^{-} \) by left translation. The derivative at \( 1 \in l(\hat{\Sigma}_{1}^{\hat{G}_{0}}) \subset \hat{N}^{-} \) of left translation by \( g \in \hat{G}_{0} \) is then identified with a linear map from \( \hat{n}^{-} \) (viewed as the tangent space to \( \hat{N}^{-} \) at \( 1 \)) to \( \hat{n}^{-} \) (viewed as the tangent space at \( l(g) \)). The reciprocal of the determinant of this map is the value of the density at \( l(g) \).

Given \( X \in \hat{n}^{-} \), the curve \( \varepsilon \mapsto \exp \varepsilon X \) represents the corresponding tangent vector at \( 1 \in \hat{N}^{-} \). Let \( \varepsilon \mapsto g_{0}(\varepsilon) \) denote a lift of this curve to \( \hat{G}_{0} \), i.e., \( l(g_{0}(\varepsilon)) = \exp \varepsilon X \). We can arrange for this lift to have \( m(g_{0}(\varepsilon)) = 1 \) for \( \varepsilon \) small. Then
\[
\varepsilon \mapsto l(g)^{-1}l(gg_{0}(\varepsilon))
\]
represents the image of \( \varepsilon \mapsto \exp(\varepsilon X) \) under left translation by \( g \) through these identifications. Let \( g_0(\varepsilon) = l(\varepsilon)a(\varepsilon)u(\varepsilon) \) denote the triangular factorization of \( g_0(\varepsilon) \). Then

\[
\begin{align*}
    l(g)^{-1}l(gg_0(\varepsilon)) &= l(g)^{-1}l(l(g)a(g)u(g))l(\varepsilon)a(\varepsilon)u(\varepsilon) \\
    &= l(Ad_{a(g)u(g)}(l(\varepsilon))a(g)u(g)a(\varepsilon)u(\varepsilon)) \\
    &= l(\exp(\varepsilon Ad_{a(g)u(g)}(X)))
\end{align*}
\]

so the derivative of (2.15) at \( \varepsilon = 0 \) is the linear map

\[
(2.16) \quad X \mapsto (Ad_{a(g)u(g)}(X))_-
\]

where (\( \cdot \)_-) denotes the projection to \( n^- \) along the triangular decomposition \( g = n^- + b + n^+ \). We claim that the matrix representing (2.16) in terms of the basis of negative roots is triangular. Indeed, if \( X \in n^- \) is homogeneous of a given height then \( Ad_{a(g)}(X) = X + X'(g) \) where \( X'(g) \) is a sum of terms of strictly greater height than that of \( X \) because \( u(g) \in N^+ \) and thus \( Ad_{a(g)} \) is unipotent. Therefore, \( Ad_{a(g)u(g)}(X) = Ad_{a(g)}(X) + X''(g) \) where again \( X''(g) \) is a sum of terms of height strictly greater than height \( (X) \) since \( a(g) \in A \). Thus, the determinant of (2.16) as a real linear transformation is \( a(g) \) raised to twice the sum of the negative roots, i.e., \( a(g)^{-4\delta} \). Taking the reciprocal gives the desired formula for the density. \( \square \)

**Lemma 2.4.** In the \( \zeta \) coordinates

\[
d\lambda_{N^-}(l(g)) = \prod_{k=1}^{n} \frac{|d\zeta_k|}{(1 + |\zeta_k|^2)^{1 - \delta(k_\tau)}}
\]

where a sign is positive if and only if the corresponding root is of compact type.

**Remark 2.4.** With the same conventions as in Remark 2.3, the formula in Lemma 2.4 can be alternatively written as

\[
d\lambda_{N^-}(l(g)) = \prod_{\tau > 0} (1 + |\zeta_\tau|^2)^{\text{height}(\tau) - 1} |d\zeta_\tau|
\]

which makes it clear that this expression does not depend upon the ordering of the positive roots \( \tau \). In a similar way

\[
a^{-4\delta} = \prod_{\tau > 0} (1 + |\zeta_\tau|^2)^{-2\text{height}(\tau)}.
\]

Together with Lemma 2.3, these formulas immediately imply Theorem 2.6 (as formulated in Remark 2.3).

The basic idea of the proof of Lemma 2.4 is the following. If we write \( l(g) = \exp(\sum x_j f_{\tau_j}) \), then there is a triangular relationship between the \( \zeta \) variables and the \( x \) variables which is implicit in the proof of Theorem 2.5. To carefully prove this, we need to go back through the induction argument in that proof. This involves a more technical statement, and an algebraic lemma, which is probably well-known.

**Lemma 2.5.** For \( n = 1, \ldots, n \),

\[
\text{height}(\tau_n) - 1 = \sum_{k=1}^{n-1} \tau_k(h_{\tau_n})
\]
Lemma 2.6. Suppose that \( n \leq n \). As in the statement and proof of Theorem 2.5 (with \( w = 1 \)), let \( g^{(n)} = i_{\tau_n} g(\zeta_n) \cdot 1_{\tau_n} (g(\zeta_1)) \). Then \( l(g^{(n)}) \in \hat{N}^{-} \cap (w'_{n-1})^{-1} \hat{N}^{+} w'_{n} \) and the expression for Haar measure of this nilpotent group, in the \( \zeta \) coordinates, is given by

\[
d\lambda_{\hat{N}^{-} \cap (w'_{n-1})^{-1} \hat{N}^{+} w'_{n}}(l(g^{(n)})) = \prod_{k \leq n} a(\zeta_k)^{-2(\delta(h_{\tau_n})^{-1})} |d\zeta_k| = \prod_{k \leq n} (1 \pm |\zeta_k|^2)^{\text{height}(\tau_n)^{-1}} |d\zeta_k|
\]

up to a normalization, where a sign is positive if and only if the corresponding root is of compact type.

Proof. If \( n = 2 \), then by (2.12),

\[
l(g^{(2)}) = \exp(\zeta_2 f_{\tau_n} + a(\zeta_2)^{-\tau_n} \zeta_1 f_{\tau_n}).
\]

Together with Lemma 2.5, this completes the proof for \( n = 1, 2 \).

Now suppose \( n > 2 \) and the result holds for \( n - 1 \). We need to revisit how we obtained \( l(g^{(n)}) \), beginning after line (2.13) in the induction step for the proof of Theorem 2.5. Recall that

\[
g^{(n)} = \exp(\zeta_n f_{\tau_n} a(\zeta_n)^{h_{\tau_n}} (w'_{n-1})^{-1} \exp(\pm \zeta_n e_{\gamma_n}) w'_{n-1} \exp(\zeta_n^{-1} f_{\tau_n}^{-1}) \tilde{I}(n-1) u^{n-1}) = \exp(\zeta_n f_{\tau_n} a(\zeta_n)^{h_{\tau_n}} (w'_{n-1})^{-1} \exp(\pm \zeta_n e_{\gamma_n}) \tilde{u} w'_{n-1} a(n-1) u^{n-1},
\]

where \( \tilde{u} = w'_{n-1} \exp(\zeta_n^{-1} f_{\tau_n}^{-1}) \tilde{I}(w'_{n-1})^{-1} \in w'_{n-1} \hat{N}^{-} (w'_{n-1})^{-1} \cap N^{+} \), and \( \tilde{I} \in \hat{N}^{-} \cap (w'_{n-2})^{-1} \hat{N}^{+} w'_{n-2} \). The first term will be the first factor in the ultimate expression for \( l(g^{(n)}) \). The conjugation by \( a(\zeta_n)^{h_{\tau_n}} \) will affect volume, and we will consider this below. The last term does not affect \( l(g^{(n)}) \). Consider the product of the other terms, which we rewrite as

\[
(w'_{n-1})^{-1} \exp(\pm \zeta_n e_{\gamma_n}) \tilde{u} w'_{n-1} = \exp(\sum_{j<n} C f_{\tau_j})
\]

by inserting the identity. Since conjugation by \( w'_{n-1} \) is a group isomorphism from the lower triangular nilpotent group

\[
\hat{N}^{-} \cap (w'_{n-1})^{-1} \hat{N}^{+} w'_{n-1} = \exp(\sum_{j<n} C f_{\tau_j})
\]

to the upper triangular nilpotent group \( \hat{N}^{+} \cap w'_{n-1} \hat{N}^{-} (w'_{n-1})^{-1} \) (where \( \tilde{u} \) lives) the Haar measure for the first is pushed to the Haar measure for the second. We now consider the decomposition

\[
\hat{N}^{+} = \left( \hat{N}^{+} \cap w'_{n-1} \hat{N}^{-} (w'_{n-1})^{-1} \right) \left( \hat{N}^{+} \cap w'_{n-1} \hat{N}^{+} (w'_{n-1})^{-1} \right)
\]

and we write \( u = (u)_1 (u)_2 \) for the corresponding factorization of elements \( u \in \hat{N}^{+} \).

The key fact is that, if we set \( u_0 = \exp(\pm \zeta_n e_{\gamma_n}) \), then the map

\[
(2.18) \quad \hat{N}^{+} \cap w'_{n-1} \hat{N}^{-} (w'_{n-1})^{-1} \rightarrow \hat{N}^{+} \cap w'_{n-1} \hat{N}^{-} (w'_{n-1})^{-1} \text{ defined by } u \mapsto (u_0 w_{n-1} u_0^{-1})
\]

preserves the invariant volume. To see this, trivialize the tangent bundle for the nilpotent group \( \hat{N}^{+} \cap w'_{n-1} \hat{N}^{-} (w'_{n-1})^{-1} \) using left translation, and fix \( u \). Then derivative for the map (2.18) at \( u \) is identified with the linear transformation

\[
\hat{u}^{+} \cap w'_{n-1} \hat{N}^{-} (w'_{n-1})^{-1} \rightarrow \hat{u}^{+} \cap w'_{n-1} \hat{u}^{-} (w'_{n-1})^{-1}
\]
Hence, its determinant is 1 at each point \(u\) where the statement of the Lemma is of the form \(u\) where (2.5) Remark □ Applying Lemma 2.5 to the first product completes the proof.

We finally have

\[
\frac{d}{dt} \bigg|_{t=0} (u_0 u_0^{-1})^{-1} (u_0 u^t X u_0^{-1})_1.
\]

Now,

\[
(u_0 u^t X u_0^{-1})_1 = (u_0 u_0^{-1} u_0 e^{tX} u_0^{-1})_1
\]

\[
= ((u_0 u_0^{-1})_1 (u_0 u_0^{-1})_2 u_0 e^{tX} u_0^{-1})_1
\]

\[
= (u_0 u_0^{-1})_1 ((u_0 u_0^{-1})_2 u_0 e^{tX} u_0^{-1}(u_0 u_0^{-1})_2^{-1})_1
\]

because the factor map \((\cdot)_1\) is equivariant for left multiplication by the first factor group \(N^+ \cap w_{n-1}^+ N^- (w_{n-1}^-)^{-1}\) and invariant for right multiplication by the second factor group \(N^+ \cap w_{n-1}^+ N^- (w_{n-1}^-)^{-1}\). Consequently, (2.19) becomes

\[
\frac{d}{dt} \bigg|_{t=0} (u_0 u_0^{-1})^{-1} (u_0 u^t X u_0^{-1})_1
\]

\[
= \frac{d}{dt} \bigg|_{t=0} (Ad((u_0 u_0^{-1})_2 u_0)(e^{tX}))_1
\]

\[
= (Ad((u_0 u_0^{-1})_2 u_0)(X))_1
\]

where \((\cdot)_1\) now denotes the infinitesimal projection to \(\hat{n}^+ \cap w_{n-1}^+ \hat{n}^- (w_{n-1}^-)^{-1}\) in \(\hat{n}^+\). Since \((u_0 u_0^{-1})_2 u_0 \in \hat{n}^+\), this is clearly the compression of a unipotent map on \(\hat{n}^+\). Hence, its determinant is 1 at each point \(u\).

Note that

\[
\exp(\pm \bar{\zeta}_n e_{\gamma_n}) \tilde{u}_1 = \exp(\pm \bar{\zeta}_n e_{\gamma_n}) \tilde{u} \exp(\mp \bar{\zeta}_n e_{\gamma_n}) \exp(\pm \bar{\zeta}_n e_{\gamma_n})
\]

because \(\exp(\pm \bar{\zeta}_n e_{\gamma_n})\) is in the second factor of the nilpotent group decomposition. We finally have

\[
l(g^{(n)}) = \exp(\zeta_n f_{\tau_n}) l
\]

where

\[
l = a(\zeta_n)^{h_{n-1}} w_{n-1} (\exp(\pm \zeta_n e_{\gamma_n}) \tilde{w}_1) (w_{n-1}^{-1})^{-1} a(\zeta_n)^{-h_{n-1}} \in \hat{N}^-(w_{n-1}^-)^{-1} N^+ w_{n-1}^-.
\]

When we conjugate by the factor \(a(\zeta_n)^{h_{n-1}}\), we are multiplying the coefficient of \(f_{\tau_j}\) by a factor \(a(\zeta_n)^{-\gamma_j(h_{n-1})}\). Using the induction step, this implies that the Haar measure in the statement of the Lemma is of the form

\[
\left| \prod_{j=1}^{n-1} a(\zeta_n)^{-2\gamma_j^{(h_{n-1})}} \right| \left| \prod_{k<n} (1 \pm |\zeta_k|^2)^{\text{height}(\gamma_k)-1} |d\zeta_k| \right|
\]

Applying Lemma 2.5 to the first product completes the proof. □

Remark 2.5. In this paper we have avoided Poisson geometry. But we remark that the Liouville measure corresponding to the Evens-Lu Poisson Hamiltonian system on the top stratum \(\Sigma_1^\infty\) is given by

\[
\frac{1}{dl} \omega^d = \text{constant} \cdot a(g)^{2\delta} d\lambda_{\infty} (l(g)) d\lambda_{\infty} (m(g))
\]
3. Notation and Background, II: Loop Structures

3.1. Affine Lie Algebras. Let \( L\hat{\mathfrak{g}} = C^\infty(S^1, \hat{\mathfrak{g}}) \), viewed as a Lie algebra with pointwise bracket. There is a universal central extension

\[
0 \to \mathbb{C} c \to \tilde{L}\hat{\mathfrak{g}} \to L\hat{\mathfrak{g}} \to 0,
\]

where \( \tilde{L}\hat{\mathfrak{g}} = L\hat{\mathfrak{g}} \oplus \mathbb{C} c \) as a vector space, and in these coordinates

\[
[X + \lambda c, Y + \lambda' c]|_{L\hat{\mathfrak{g}}} = [X, Y]|_{L\hat{\mathfrak{g}}} + \frac{i}{2\pi} \int_{S^1} (X \wedge dY)c.
\]

The smooth completion of the untwisted affine Kac-Moody Lie algebra corresponding to \( \hat{\mathfrak{g}} \) is

\[
\tilde{L}\hat{\mathfrak{g}} = \mathbb{C} d \ltimes \tilde{L}\hat{\mathfrak{g}} \quad \text{(the semidirect sum),}
\]

where the derivation \( d \) acts by \( d(X + \lambda c) = \frac{1}{1 - \lambda} \frac{d\lambda}{d\theta} X \), for \( X \in L\hat{\mathfrak{g}} \), and \([d,c] = 0\).

**Proposition 3.1.** For both \( L\hat{\mathfrak{u}} \) and \( L\hat{\mathfrak{g}}_0 \), the cocycle \( (X,Y) \to \int_{S^1} (X \wedge dY) \) is real-valued. In particular the affine extension induces a unitary central extension

\[
0 \to i\mathbb{R} c \to \tilde{L}\hat{\mathfrak{g}}_0 \to L\hat{\mathfrak{g}}_0 \to 0
\]

and a real form \( \tilde{L}\hat{\mathfrak{g}}_0 = i\mathbb{R} d \ltimes \tilde{L}\hat{\mathfrak{g}}_0 \) for \( \tilde{L}\hat{\mathfrak{g}} \) (and similarly for unitary loops as in [13]).

We identify \( \hat{\mathfrak{g}} \) with the constant loops in \( L\hat{\mathfrak{g}} \). Because the extension is trivial over \( \hat{\mathfrak{g}} \), there are embeddings of Lie algebras

\[
\mathfrak{g} \to \tilde{L}\hat{\mathfrak{g}} \to \tilde{L}\hat{\mathfrak{g}}.
\]

There are triangular decompositions

\[
\tilde{L}\hat{\mathfrak{g}} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+ \quad \text{and} \quad \tilde{L}\hat{\mathfrak{g}} = \mathfrak{n}^- + (\mathbb{C} d + \mathfrak{h}) + \mathfrak{n}^+,
\]

where \( \mathfrak{h} = \hat{\mathfrak{h}} \oplus \mathbb{C} c \) and \( \mathfrak{n}^\pm \) is the smooth completion of \( \hat{\mathfrak{n}}^\pm + \hat{\mathfrak{g}}(z^\pm 1 \mathbb{C}[z^\pm 1]) \), respectively. The simple roots for \( (\tilde{L}_{fin}\hat{\mathfrak{g}}, \mathbb{C} d + \mathfrak{h}) \) are \( \{\alpha_j : 0 \leq j \leq r\} \), where

\[
\alpha_0 = d^* - \bar{\theta}, \quad \alpha_j = \bar{\alpha}_j, \quad j > 0,
\]

\( d^*(d) = 1, d^*(c) = 0, d^*(\bar{h}) = 0 \), and the \( \bar{\alpha}_j \) are extended to \( \mathbb{C} d + \mathfrak{h} \) by requiring \( \bar{\alpha}_j(c) = \bar{\alpha}_j(d) = 0 \). The simple coroots are \( \{h_j : 0 \leq j \leq rk\hat{\mathfrak{g}}\} \), where

\[
h_0 = c - \bar{h}_\theta, \quad h_j = \bar{h}_j, \quad j > 0.
\]

For \( i > 0 \), the root homomorphism \( \iota_{\alpha_i} \) is \( \iota_{\bar{\alpha}_i} \), followed by the inclusion \( \hat{\mathfrak{g}} \subset \tilde{L}\hat{\mathfrak{g}} \). For \( i = 0 \)

\[
\iota_{\alpha_0}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = e_\theta z^{-1}, \quad \iota_{\alpha_0}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = f_\theta z,
\]

where \( \{f_\theta, \bar{h}_\theta, e_\theta\} \) satisfy the \( \mathfrak{sl}(2, \mathbb{C}) \)-commutation relations, and \( e_\theta \) is a highest root vector for \( \hat{\mathfrak{g}} \). The fundamental dominant integral functionals on \( \mathfrak{h} \) are \( \Lambda_j \), \( j = 0, \ldots, r \).

We set \( t = i\mathbb{R} c \oplus i \) and \( a = \mathfrak{h}_\mathbb{R} = \mathbb{R} c \oplus \tilde{\mathfrak{h}}_\mathbb{R} \).
3.2. Loop Groups and Extensions. Let $\Pi : \tilde{L}G \to L\tilde{G}$ ($\Pi : \tilde{L}G_0 \to L\tilde{G}_0$) denote the universal central $\mathbb{C}^*$ (resp., $T$) extension of the smooth loop group $L\tilde{G}$ (resp. $L\tilde{G}_0$).

**Proposition 3.2.** $\Pi$ induces a central circle extension

$$1 \to \mathbb{T} \to \tilde{L}G_0 \to L\tilde{G}_0 \to 1$$

(and similarly for unitary loops as in [13]).

**Proof.** This follows from Proposition 3.1.

Let $N^\pm$ denote the subgroups corresponding to $\mathfrak{n}^\pm$. Since the restriction of $\Pi$ to $N^\pm$ is an isomorphism, we will always identify $N^\pm$ with its image, e.g. $l \in N^+$ is identified with a smooth loop having a holomorphic extension to $\Delta$ satisfying $l(0) \in \tilde{N}^+$. Also, set $T = \exp(t)$ and $A = \exp(a)$.

As in the finite dimensional case, for $\tilde{g} \in N^- \cdot T \cdot A \cdot N^+ \subset \tilde{L}G$, there is a unique triangular decomposition

$$\tilde{g} = l \cdot d \cdot u,$$

where $d = ma = \prod_{j=0}^{r} \sigma_j(\tilde{g})^{\tilde{h}_j}$, and $\sigma_j = \sigma_{\Lambda_j}$ is the fundamental matrix coefficient for the highest weight vector corresponding to $\Lambda_j$. If $\Pi(\tilde{g}) = g$, then because $\sigma_0^{h_0} = \sigma_0^{-h_0}$ projects to $\sigma_0^{-h_0}$, $g = l \cdot \Pi(d) \cdot u$, where

$$\Pi(d)(g) = \sigma_0(\tilde{g})^{-h_0} \prod_{j=1}^{r} \sigma_j(\tilde{g})^{\tilde{h}_j} = \prod_{j=1}^{r} \left( \frac{\sigma_j(\tilde{g})}{\sigma_0(\tilde{g})^{a_j}} \right)^{\tilde{h}_j},$$

and the $a_j$ are positive integers such that $\tilde{h}_j = \sum a_j \cdot h_j$ (these numbers are also compiled in Section 1.1 of [7]).

**Proposition 3.3.**

(a) $N^\pm$ are stable with respect to $\Theta$, whereas $N^\pm$ are interchanged by $(\cdot)^*$. If $\tilde{g}$ has triangular factorization $\tilde{g} = l \cdot m(\tilde{g}) a(\tilde{g}) \cdot u$ as in (3.4), then

$$\Theta(\tilde{g}) = \Theta(l) \cdot m(\tilde{g}) a(\tilde{g}) \cdot \Theta(u)$$

and

$$\tilde{g}^* = u^* \cdot m(\tilde{g})^* a(\tilde{g})^* \cdot l^*$$

are triangular factorizations.

(b) If $\tilde{g} \in \tilde{L}G$, then $\sigma_j(\Theta(\tilde{g})) = \sigma_j(\tilde{g})$ and $\sigma_j(\tilde{g}^*) = \sigma_j(\tilde{g})^*$.

(c) If $\tilde{g} \in \tilde{L}G_0$, then $|\sigma_j(\tilde{g})|$ depends only on $g = \Pi(\tilde{g})$, and

$$|\sigma_j(g)| = |\sigma_j(\tilde{g})| = (\sigma_j(\tilde{g})\sigma_j(\tilde{g}^{-1}))^{1/2}.$$  

(d) For $\tilde{g} \in \tilde{L}G_0$ and $g = \Pi(\tilde{g})$, $\tilde{g}$ has a triangular factorization if and only if $g$ has a triangular factorization. The restriction of the projection $\tilde{L}G_0 \to L\tilde{G}_0$ to elements with $m(\tilde{g}) = 1$ is injective.

**Proof.** (a) and (b) follow from the compatibility of the triangular factorization with respect to $\Theta$ and $u$. 

Proof of (d):
The first part of (c) follows from the fact that the induced extension \( \tilde{L}G_0 \) is unitary. The formula 3.6 in (c) follows from the fact that if \( \lambda \in \mathbb{T} \), then
\[
\sigma_j(\tilde{g}\lambda) = \lambda^l \sigma_j(\tilde{g})
\]
where \( l \) is the level.

3.3. A Note on the Rank One Case. In this subsection we will freely use the notation in Section 1 of [11] and [15] (as in section 1 of [11], we denote the Toeplitz and shifted Toeplitz operators by \( A \) and \( A_1 \), respectively).

In the rank one case \( \sigma_0 \) and \( \sigma_1 \) can be concretely realized as “regularized Toeplitz determinants.” In the notation of section 6.6 of [15], a concrete model for the central extension is
\[
\tilde{L}G = \{(g, q) : (g, q) \in \hat{G} \times GL(H_+), A(g)q^{-1} = 1 + \text{trace class}\}
\]
(here \( \hat{G} = SL(2, \mathbb{C}) \), \( H = L^2(S^1, \mathbb{C}^2) \), and \( H_+ \) is the subspace of boundary values of holomorphic functions on the disk). In this realization
\[
\sigma_0([g, q]) = \det(A(g)q^{-1})
\]

Proposition 3.4. For \( g \in \hat{L}G_0 \), using the notation in Proposition 3.3,
\[
|\sigma_0|^2(g) = \det(A(g)A(g^{-1})) \quad \text{and} \quad |\sigma_1|^2(g) \det(A_1(g)A_1(g^{-1})).
\]

Proof. This follows from (c) of Proposition 3.3.

3.4. Reduced Sequences in the Affine Weyl Group. The Weyl group \( W \) for \((\hat{L}G, Cd + \mathfrak{h})\) acts by isometries of \((\mathbb{R}d + \mathfrak{h}_\mathbb{R}, \langle \cdot, \cdot \rangle)\). The action of \( W \) on \( \mathbb{R}c \) is trivial. The affine plane \( d + \mathfrak{h}_\mathbb{R} \) is \( W \)-stable, and this action identifies \( W \) with the affine Weyl group \( \hat{W} \propto \hat{T} \) and its affine action (1.5) on \( \mathfrak{h}_\mathbb{R} \) (see Chapter 5 of [15]). In this realization
\[
r_{\alpha_0} = \hat{h}_\theta \circ r_{\theta}, \quad \text{and} \quad r_{\alpha_i} = r_{\alpha_i}, \quad i > 0.
\]

In general, given \( w \in W \), we write
\[
w = h_w \circ \hat{w}, \quad h \in \hat{T}, \quad \hat{w} \in \hat{W}
\]
and we let \( \text{Inv}(w) \) denote the inversion set of \( w \), i.e. the set of positive roots which are mapped to negative roots by \( w \).

Definition 1. A sequence of simple reflections \( r_1, r_2, \ldots \) in \( W \) is called reduced if \( w_n = r_n r_{n-1} \ldots r_1 \) is a reduced expression for each \( n \).

Lemma 3.1. Given a reduced sequence of simple reflections \( \{r_j\} \), corresponding to simple positive roots \( \gamma_j \),
(a) the inversion set
\[
\text{Inv}(w_n) = \{r_j = w_{j-1}^{-1} \cdot \gamma_j = r_1 \ldots r_{j-1} \cdot \gamma_j, \quad j = 1, \ldots, n\}
\]
(b) \( w_k \tau_n > 0 \), \( k < n \).

A reduced sequence of simple reflections determines a non-repeating sequence of adjacent alcoves
\[
C_0, w_1^{-1}C_0, \ldots, w_n^{-1}C_0 = r_1 \ldots r_{n-1}C_0, \ldots
\]
where the step from $w_{n-1}^{-1}C_0$ to $w_n^{-1}C_0$ is implemented by the reflection $r_{\tau_n} = w_{n-1}^{-1}r_n w_{n-1}$ (in particular the wall between $C_{n-1}$ and $C_n$ is fixed by $r_{\tau_n}$). Conversely, given a sequence of adjacent alcoves $(C_j)$ which is minimal in the sense that the minimal number of steps to go from $C_0$ to $C_j$ is $j$, there is a corresponding reduced sequence of reflections.

**Definition 2.** A reduced sequence of simple reflections $\{r_j\}$ is affine periodic if, in terms of the identification of $W$ with the affine Weyl group, (1) there exists $l$ such that $w_l \in \tilde{T}$ and (2) $w_{s+l} = w_s \circ w_l$, for all $s$. We will refer to $w^{-1}_l$ as the period ($l$ is the length of the period).

**Remarks.**

(a) The second condition is equivalent to periodicity of the associated sequence of simple roots $\{\gamma_j\}$, i.e. $\gamma_{s+l} = \gamma_s$. (b) In terms of the associated walk through alcoves, affine periodicity means that the walk from step $l+1$ onward is the original walk translated by $w^{-1}_l$.

We now recall Theorem 3.5 of [13] (this is what we will need in Section 5 for root subgroup factorization of generic loops in $\hat{G}_0$).

**Theorem 3.1.**

(a) There exists an affine periodic reduced sequence $\{r_j\}_{j=1}^{\infty}$ of simple reflections such that, in the notation of Lemma 3.1,

\[ \{\tau_j : 1 \leq j < \infty\} = \{qd^* - \alpha : \alpha > 0, q = 1, 2, \ldots\}, \]

i.e. such that the span of the corresponding root spaces is $\tilde{n}^-(z\mathbb{C}[z])$. The period can be chosen to be any point in $C \cap \tilde{T}$.

(b) Given a reduced sequence as in (a), and a reduced expression for $w_0 = r_{-N} \cdots r_0$ (where $w_0$ is the longest element of $\hat{W}$), the sequence $r_{-N}, \ldots, r_0, r_1, \ldots$ is another reduced sequence. The corresponding set of positive roots mapped to negative roots is

\[ \{qd^* + \alpha : \alpha > 0, q = 0, 1, \ldots\}, \]

i.e. the span of the corresponding root spaces is $\tilde{n}^+(\mathbb{C}[z])$.

### 3.5. The Basic Framework and Notation.

In the remainder of the paper we will mainly be concerned with a slightly restricted loop analogue of (0.1):

\[ (3.10) \]

where $U := \hat{L}U$, the (simply connected) central circle extension of $L\hat{U}$, $G := \hat{L}G$, the (simply connected) central $\mathbb{C}^*$ extension of $L\hat{G}$, $G_0 := (\hat{L}G_0)_0$, the identity component of the central circle extension of $L\hat{G}_0$, and $K := (\hat{L}K)_0$, the identity
component of the central circle extension of $\hat{L}K$. There is a corresponding diagram of Lie algebras, where the Lie algebra of $G$ is $\mathfrak{g} = \hat{\mathfrak{g}}$, and so on.

It will often happen that we can more simply work at the level of loops, rather than at the level of central extensions. We will often state results, for example, in terms of $G$, but in proving results it is often possible and easier to work with $\hat{L}G$.

3.6. **Contrast with Finite Dimensions.** In Part I of this paper (the finite dimensional case), the key fact (depending on the inner assumption) is that

$$\hat{n}^\pm = \hat{n}_t^\pm + \hat{n}_p^\pm$$

where the latter summand, $\hat{n}_p^\pm = \hat{n}^\pm \cap p^C$, is an abelian ideal in the parabolic subalgebra $\hat{\mathfrak{t}}^C + \hat{n}_p^\pm$ of $\hat{\mathfrak{g}}$. This leads to a block (coarse) triangular factorization, which largely reduces the (finite dimensional) inner noncompact case to the compact case.

In the present context there is an analogous decomposition

$$n^\pm = n_t^\pm + n_p^\pm.$$

In this case

$$n_p^\pm = (n^+ \cap L^+ n_p^-) + (n^- \cap L^- n_p^+)$$

where each of the two summands is a subalgebra, but the sum is not a Lie algebra, let alone an abelian ideal in a parabolic subalgebra. This is one reason why Birkhoff factorization and root subgroup factorization are so much more subtle for loops into $\hat{G}_0$ than for $G_0$ itself.

3.7. **Compact vs Noncompact type roots in $\mathfrak{g}$.** As before, a root of $\mathfrak{h}$ on $\mathfrak{g}$ is said to be of compact type if the corresponding root space belongs to $k^C$, and said to be of non-compact type if the corresponding root space belongs to $p^C$. Here $\mathfrak{t}^C = \hat{\mathfrak{t}}^C$ and $\mathfrak{p}^C = n^- + n_p^+$ (so this terminology is perhaps less than ideal).

**Example 3.1.** In rank one, the compact type roots are the imaginary roots and the noncompact type roots are the real roots. This is yet another special feature of the rank one case.

4. **Birkhoff Decomposition for Loops**

By definition the Birkhoff decomposition of $G = \hat{L}G$ is

$$G = \bigsqcup W \Sigma^G_w \text{ where } \Sigma^G_w := N^- wB^+.\quad (4.1)$$

If we fix a representative $w \in N_U(T)$ for $w \in W$, then each $g \in \Sigma^G_w$ has a unique Birkhoff factorization

$$g = lwmau, \quad l \in N^- \cap wN^-w^{-1}, \quad ma \in TA, u \in N^+.\quad (4.2)$$

As in the finite dimensional case, for fixed $m_0 \in T$, \{ $g \in \Sigma^G_w : m(g) = m_0$ \} is a stratum (diffeomorphic to the product of the Birkhoff stratum for the flag space $G/B^+$ corresponding to $w$ with $N^+$); see Theorem 8.7.2 of [15]. We will refer to $\Sigma^G_w$ as the “(isotypic) component of the Birkhoff decomposition of $G$ corresponding to $w \in W$.”

One virtue of root subgroup factorization is that it generates many explicit examples of Birkhoff factorizations.
4.1. Birkhoff Decomposition for $L^U$. Given $w \in W$, define
$$\Sigma^U_w := \Sigma^G_w \cap U.$$ 

**Theorem 4.1.** Fix a representative $w \in N_U(T)$ for $w$. For $g \in \Sigma^G_w$ the unique factorization (4.2) induces a bijective correspondence
$$\Sigma^U_w \leftrightarrow (N^- \cap wN^- w^{-1}) \times T \text{ given by } g \mapsto (l,m).$$

We refer to $\Sigma^U_w$ as the isotypic component of the Birkhoff decomposition for $U$; each component consists of a union of strata permuted by the action of $T$. The theorem provides an explicit parameterization for these strata. We have recalled this result simply for the sake of comparison. Our primary objective is to investigate the Birkhoff decomposition for $L^G_0$.

4.2. Birkhoff Decomposition for $G_0 := (\tilde{L}^G_0)_0$, the Identity Component.

Given $w \in W$, define
$$\Sigma^G_{G_0} := \Sigma^G_w \cap G_0$$
$$\Sigma^{L^G_0} := \Sigma^L_w \cap L^G_0$$
and so on.

As we stated in the introduction (where we focused on the rank one case), our original expectation was that each of these components would be (modulo a torus) contractible to $w$. Our main objective in this subsection is to provide examples in the rank one case, for the identity component, which illustrate why this is not true.

**Proposition 4.1.** $\Sigma^{LSU(1,1)(0)}$ is properly contained in $LSU(1,1)(0)$.

**Proof.** For any $g \in LSU(1,1)$ there is a pointwise polar decomposition
$$g = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right) \left( \begin{array}{cc} a & b \\ b^* & a \end{array} \right)$$
where $a = \sqrt{1 + |b|^2}$, and $\lambda : S^1 \to S^1$.

If $g \in LSU(1,1)(0)$, then $\lambda$ has degree zero, and thus $\lambda$ has a triangular factorization
$$\lambda = e^{\psi_0} e^{\psi}$$
where $\psi_+ = -\psi_0^*$ and $\psi_0 \in i\mathbb{R}$. Because $a$ is a positive periodic function, it will have a triangular factorization
$$a = e^{\chi_-} e^{\chi_0} e^{\chi_+}$$
where $\chi_- = \chi_0^*$ and $\chi_0 \in \mathbb{R}$.

We can always multiply $g$ on the left (right) by something in $B^- (B^+$, respectively) without affecting the question of whether $g$ has a triangular factorization. For example in determining whether $g$ has a triangular factorization, we can ignore the $exp(\psi_- + \psi_0)$ factor in $\lambda$, because this can be factored out on the left. We will use this observation repeatedly (note that we can recover $\psi_-$ from $\psi_+$, and the zero mode is inconsequential).

There is a factorization of
$$\left( \begin{array}{cc} a & b \\ b & a \end{array} \right)$$
as the product
$$\left( e^{\chi_-} \quad 0 \\ 0 \quad e^{-\chi_-} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & e^{\chi_0} & 0 \\ 0 & 0 & e^{-\chi_+} \end{array} \right) \left( e^{\chi_0} \quad 0 \\ 0 \quad e^{-\chi_+} \right)$$
To obtain $g$ we have to multiply this on the left by $\lambda$. It follows after some calculation that $g$ will have a triangular factorization if and only if

$$
\begin{pmatrix} 1 & 0 \\ e^x & 0 \end{pmatrix} \begin{pmatrix} 1 & b e^{(-x-x_0+x_+)} e^{2 \psi_1} \\ 0 & 1 \end{pmatrix}
$$

has a triangular factorization.

At this point, to simplify notation, we let $b_1 := b e^{(-x-x_0+x_+)}$. Note that $b_1 b_1^* = b b^* e^{(-2x_0)}$. Thus $g$ has a triangular factorization if and only if the loop

$$
\begin{pmatrix} 1 & b_1 e^{2 \psi_1} \\ b_1^* e^{-2 \psi_1} & 1 \end{pmatrix} \begin{pmatrix} e^{x_0} & 0 \\ 0 & e^{-x_0} \end{pmatrix} = e^{x_0} \begin{pmatrix} 1 & b_1 e^{2 \psi_1} \\ b_1^* e^{-2 \psi_1} & b_1 b_1^* + e^{-2x_0} \end{pmatrix}
$$

has a triangular factorization. Note that the $(2,2)$ entry of the right hand side equals $a a^* e^{-2x_0}$.

We directly calculate the kernel of the Toeplitz operator associated to this loop. We obtain (the equations (for $f_1, f_2 \in H^0(D)$)

$$
f_1 + (b_1 e^{2 \psi_1} f_2)_+ = 0, \text{ and } (b_1^* e^{-2 \psi_1} f_1 + (b_1 b_1^* + e^{-2x_0}) f_2)_+ = 0.
$$

We can solve the first equation for $f_1$. The second equation becomes

$$
((e^{-2x_0} + b_1 b_1^*) f_2 - b_1^* e^{-2 \psi_1} (b_1 e^{2 \psi_1} f_2)_+) = 0.
$$

If we set $b_2 = b_1 e^{x_0} = b e^{-x_+ + x_+}$, then this can be rewritten as

$$
(f_2 + b_2^* e^{-2 \psi_1} (b_2 e^{2 \psi_1} f_2)_-) = 0.
$$

If we set $F = e^{2 \psi_1} f_2$, then we see that there exists a nontrivial kernel if and only if there exists nonzero $F \in H_+$ such that

$$
(e^{-2 \psi_1} (F + b_2^* (b_2 F)_-)) = 0.
$$

It is easy to find $\psi_+$ and $b_2$ such that there exists a nonzero $F$ satisfying this condition.

**Example 4.1.** $b_2 = \frac{1}{z} - 1$, $e^{-2 \psi_1} = e^{2z}$, and $F = \frac{1}{z} e^{-z} - 1$. In other words if

$$
g = \begin{pmatrix} e^{\frac{1}{z} - z} & 0 \\ 0 & e^{-\frac{1}{z} + z} \end{pmatrix} \begin{pmatrix} (3 - \frac{1}{z} - z)^{1/2} & e^{-x_+ + x_+} (\frac{1}{z} - 1) \\ e^{x_+ - x_+} (z - 1) & (3 - \frac{1}{z} - z)^{1/2} \end{pmatrix}
$$

where

$$
(3 - \frac{1}{z} - z)^{1/2} = e^{x_0} e^{x_+}
$$

then $g$ is a loop in the identity component of $LSU(1,1)$ and does not have a Riemann-Hilbert factorization, hence also does not have a triangular factorization.

**Remark 4.1.** There is a naive idea which would seem to suggest that triangular factorization should always hold - the example above shows this idea is flawed. Consider the symplectic form on $\Omega^0(S^1; \mathbb{R}^2)$ given by

$$
\omega(f \wedge g) = \int_{S^1} \omega_0(f(\theta) \wedge g(\theta)) d\theta
$$

This is $LSU(1,1)$ invariant. The complex extension is given by the same formula, and the shifted Hardy polarization, where the standard ordered basis is split in the following way

$$
..\epsilon_1 z, \epsilon_{-1} z, \epsilon_1 |\epsilon_{-1}, \epsilon_1 z^{-1}, \epsilon_{-1} z^{-1}, ..
$$
is a Lagrangian splitting. Unfortunately this standard ordered basis, with the given splitting, is not standard in the sense of symplectic geometry, because the pairings of vectors on the left and their conjugates on the right alternate between ±1. If this idea had any validity, it would imply that the shifted Toeplitz operator $A_1$ would be invertible for all loops, and this is false (for example for diagonal loops which are not in the identity component). The problem is that we need something that is special to the identity component. This sort of argument applies to the entire loop group.

4.3. **Birkhoff Decomposition for nonidentity components of $L\hat{G}_0$.** Consider the rank one case and the problem of finding the Birkhoff factorization for $g$ which is of the form $g = \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} g_0$, where $g_0$ is in the identity component and has a known triangular factorization (as for example in Theorem 0.1), and $n > 0$. Write

$$g = \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} m_0 a_0 & 0 \\ 0 & (m_0 a_0)^{-1} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

Factor $l$ as

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} \sum_{k=-\infty}^{-2n} \alpha_k z^k & L_{12} \\ \sum_{k=-2n+1}^{-1} \beta_k z^k & L_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $g$ will have the form

$$g = L' \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \sum_{k=-2n+1}^{-1} x_k z^k & 1 \end{pmatrix} \begin{pmatrix} m_0 a_0 & 0 \\ 0 & (m_0 a_0)^{-1} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

where $L' \in N^-$. Consequently to find the Birkhoff factorization for $g$, it suffices to find the factorization for the triangular matrix valued function

$$(4.4) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \sum_{k=-2n+1}^{-1} x_k z^k & 1 \end{pmatrix} \begin{pmatrix} m_0 a_0 & 0 \\ 0 & (m_0 a_0)^{-1} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

Remark 4.2. What we are doing here is factoring $N^-$ as $N^- \cap \omega N^- \omega^{-1}$ and $N \cap \omega N^+ \omega^{-1}$. So this is very general. The problem of understanding Birkhoff factorization for triangular matrix valued functions is considered in [4].

Example 4.2. When $n = 1$, we could take $g_0 = g_1$ in Theorem 0.1. Then

$$g = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} a(z) \begin{pmatrix} \eta_1 & 1 \\ 1 & \eta_1 \end{pmatrix} a(z) \begin{pmatrix} 1 & \eta_0 z^{-1} \\ \eta_0 z & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & \eta_1 z^{-1} \\ \eta_0 + \bar{y}_0 z^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix}$$

where $y_1 = -\eta_1$ and $y_0 = -\eta_0 (1 - \eta_1 \eta_1)$ (note $|y_0|, |y_1| < 1$).

Lemma 4.1. Fix $n > 0$. For a triangular matrix valued function as in (4.4),

(a) the Toeplitz operator $A$ is invertible if and only if the Toeplitz matrix

$$A' = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n+1} \\ c_1 & c_0 & \cdots & c_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & \cdots & \cdots & c_0 \end{pmatrix}$$

is invertible, and
(b) the shifted Toeplitz operator $A_1$ is invertible if and only if the Toeplitz matrix

$$A'' = \begin{pmatrix} c_1 & c_0 & \cdots & c_{-n+2} \\ c_2 & c_1 & \cdots & c_{-n+3} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & \cdots & \cdots & c_1 \end{pmatrix}$$

is invertible.

Proof. The Fredholm indices for both operators are zero, so we need to check the kernels.

Part (a): Suppose that $$(f, h) = \left( \sum_{k=0}^{\infty} f_k z^k, \sum_{k=0}^{\infty} h_k z^k \right)$$ is in the kernel of $A$. Then $(z^{-n} f)_+ = 0$, implying $f = \sum_{k=0}^{n-1} f_k z^k$, and

$$\left( \sum_{k=-n+1}^{n} c_k z^k \left( \sum_{k=0}^{n-1} f_k z^k \right) + \sum_{k=0}^{\infty} h_k z^{k+n} \right) = 0.$$ 

This equation implies $h_k = 0$ for $k \geq n$. These equations have the matrix form

$$\begin{pmatrix} A' & 0 \\ C' & 1_{n \times n} \end{pmatrix} \begin{pmatrix} \vec{f} \\ \vec{h} \end{pmatrix} = \vec{0}$$

where $\vec{f}$ (resp. $\vec{h}$) is the vector of coefficients of $f$ (resp. $h$) and $A'$ is the $n \times n$ Toeplitz matrix

$$A' = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n+1} \\ c_1 & c_0 & \cdots & c_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & \cdots & \cdots & c_0 \end{pmatrix}.$$ 

This implies part (a).

Part (b): Suppose that $$(f, h) = \left( \sum_{k=0}^{\infty} f_k z^k, \sum_{k=1}^{\infty} h_k z^k \right)$$ is in the kernel of $A_1$. Then $(z^{-n} f)_+ = 0$, implying $f = \sum_{k=0}^{n-1} f_k z^k$, and

$$\left( \sum_{k=-n+1}^{n} c_k z^k \left( \sum_{k=0}^{n-1} f_k z^k \right) + \sum_{k=1}^{\infty} h_k z^{k+n} \right) = 0.$$ 

These equations have the matrix form

$$\begin{pmatrix} A'' & 0 \\ C'' & 1_{n \times n} \end{pmatrix} \begin{pmatrix} \vec{f} \\ \vec{h} \end{pmatrix} = \vec{0}$$

where $A''$ is the $n \times n$ Toeplitz matrix

$$A'' = \begin{pmatrix} c_1 & c_0 & \cdots & c_{-n+2} \\ c_2 & c_1 & \cdots & c_{-n+3} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & \cdots & \cdots & c_1 \end{pmatrix}.$$ 

This implies part (b). □
Example 4.3. Suppose \( n = 1 \).

When \( c_0 \neq 0 \) there is a Riemann-Hilbert factorization (because \( A \) is invertible)
\[
\left( \begin{array}{cc} \frac{1}{c_0} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \frac{1}{c_0} z^{-1}/c_0 & -c_1/c_0 & -1/c_0 \\ -c_1/c_0 & 0 & 0 \\ -c_1/c_0 z & 1 - c_1/c_0 z \end{array} \right)
\]
When \( c_0, c_1 \neq 0 \), there is a triangular factorization (because \( A \) and \( A_1 \) are invertible)
\[
\left( \begin{array}{cc} \frac{1}{c_0} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1/c_0 & 0 \\ 0 & 1 \end{array} \right)
\]

In this case \( g \in \Sigma^{\operatorname{LSU}(1,1)(-1)}_{r_1} \).

When \( c_1 \to 0 \) this “degenerates” to a Birkhoff factorization
\[
\left( \begin{array}{cc} \frac{1}{c_0} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1/c_0 & 0 \\ 0 & 1 \end{array} \right)
\]
In this case \( g \in \Sigma^{\operatorname{LSU}(1,1)(-1)}_{r_1} \).

When \( c_0 \to 0 \) this “degenerates” to a Birkhoff factorization
\[
\left( \begin{array}{cc} \frac{1}{c_0} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1/c_0 & 0 \\ 0 & 1 \end{array} \right)
\]
In this case \( g \in \Sigma^{\operatorname{LSU}(1,1)(-1)}_{r_0} \).

When both \( c_0, c_1 \to 0 \) this goes to \( z^{-1} \begin{array}{cc} 0 \\ 0 \end{array} \). In this case \( g \in \Sigma^{\operatorname{LSU}(1,1)(-1)}_{r_0 r_1} \),

where in the Weyl group \( \left( \begin{array}{cc} z^{-1} & 0 \\ 0 & z \end{array} \right) = r_0 r_1 \).

These calculations show that we are obtaining loops in the corresponding strata, despite the fact that neither \( r_0 \) nor \( r_1 \) are represented by loops in \( K = S^1 \). Moreover the conditions on \( c_0, c_1 \) above show that there is something topologically nontrivial about the intersection of the \( \Sigma_{r_1} \) component with the \( n = -1 \) connected component.

5. Root subgroup factorization for generic loops in \( \hat{G}_0 \)

Our objective in this section is to prove analogues of Theorems 4.1, 4.2, and 5.1 of [13], for generic loops in \( \hat{G}_0 \) (which is always assumed to be of inner type). The structure of the proofs in this noncompact context is basically the same as in [13]. But there are differences, and in this paper we will present all of the details in this generic context. In order to obtain formulas for determinants of Toeplitz operators, as in Theorem 0.4, we have to work with the central extension \( \hat{G}_0 \).

Throughout this section we choose a reduced sequence \( \{r_j\}_{j=1}^\infty \) as in Theorem 3.1, part (a). We set \( w_j = r_j \ldots r_1 \) and
\[
\begin{align*}
\dot{i}_{\gamma_n} &= w_{n-1}^{-1} w_{n-1}^{-1} \\
\dot{i}_{\tau_{(N-1)}} &= r_{-(N-1)}^{-1} \dot{i}_{\gamma_{(N-1)}}^{-1} \dot{i}_{\tau_{(N-1)}}^{-1} \ldots \\
\dot{i}_{\tau_0} &= w_0^{-1} w_0^{-1}
\end{align*}
\]
and for \( n > 0 \)
\[
\dot{i}_{\gamma_n} = w_n^{-1} w_n^{-1} w_n^{-1} w_n^{-1}.
\]
Remark 5.1. Note that \( \tilde{w}_0 \) is a representative for the longest element of the Weyl group \( \hat{W} \) of \( \hat{G} \). The affine Weyl group \( W \) does not have such an element. In Section 2 the finite dimensional expression corresponding to \( w_j \) was denoted \( w'_j \) in an effort to separate the equivalent Birkhoff and Bruhat viewpoints. Here, these viewpoints are not equivalent, and we are only concerned with the Birkhoff point of view, so we suppress the primes from the notation.

As before, for \( \zeta \in \mathbb{C} \), let \( a_+(\zeta) = (1 + |\zeta|^2)^{-1/2} \) and

\[
(5.1) \quad k(\zeta) = a_+(\zeta) \begin{pmatrix} 1 & -\bar{\zeta} \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} a_+(\zeta) & 0 \\ 0 & a_+(\zeta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\bar{\zeta} \\ 0 & 1 \end{pmatrix} \in SU(2).
\]

For \( |\zeta| < 1 \), let \( a_-(\zeta) = (1 - |\zeta|^2)^{-1/2} \) and

\[
(5.2) \quad q(\zeta) = a_-(\zeta) \begin{pmatrix} 1 & \bar{\zeta} \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} a_-(-\zeta) & 0 \\ 0 & a_-(\zeta)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \bar{\zeta} \\ 0 & 1 \end{pmatrix} \in SU(1,1).
\]

5.1. Generalizations of Theorem 0.1.

Theorem 5.1. Suppose that \( \tilde{g}_1 \in \tilde{L}_{\text{fin}} \hat{G}_0 \) and \( \Pi(\tilde{g}_1) = g_1 \). Consider the following three statements:

(I.1) \( m(\tilde{g}_1) = 1 \), and for each complex irreducible representation \( V(\pi) \) for \( \hat{G} \), with lowest weight vector \( \phi \in V(\pi) \), \( \pi(g_1)^{-1}(\phi) \) is a polynomial in \( z \) (with values in \( V \)), and is a positive multiple of \( \phi \) at \( z = 0 \).

(I.2) \( \tilde{g}_1 \) has a factorization of the form

\[
\tilde{g}_1 = i_{\tau_+}(g(\eta_0)) \ldots i_{\tau_+}(g(\eta_{-N})) \in \tilde{L}_{\text{fin}} \hat{G}_0
\]

where \( g(\eta_j) = k(\eta_j) \) for some \( \eta_j \in \mathbb{C} \) (resp. \( g(\eta_j) = q(\eta_j) \) for some \( \eta_j \in \Delta \)) when \( \tau_j \) is a compact type (resp. non-compact type) root.

(I.3) \( \tilde{g}_1 \) has triangular factorization of the form \( \tilde{g}_1 = l_1 a_1 u_1 \) where \( l_1 \in \hat{N}^- (\mathbb{C}[z^{-1}]) \).

Then statements (I.1) and (I.3) are equivalent. (I.2) implies (I.1) and (I.3).

Moreover, in the notation of (I.2),

\[
a_1 = \prod_{j=-N}^{n} a(\eta_j)^{h_{\tau_j}}.
\]

Similarly, suppose that \( \tilde{g}_2 \in \tilde{L}_{\text{fin}} \hat{G}_0 \) and \( \Pi(\tilde{g}_2) = g_2 \). Consider the following three statements:

(II.1) \( m(\tilde{g}_2) = 1 \), and for each complex irreducible representation \( V(\pi) \) for \( \hat{G} \), with highest weight vector \( \upsilon \in V(\pi) \), \( \pi(g_2)^{-1}(\upsilon) \) is a polynomial in \( z \) (with values in \( V \)), and is a positive multiple of \( \upsilon \) at \( z = 0 \).

(II.2) \( \tilde{g}_2 \) has a factorization of the form

\[
\tilde{g}_2 = i_{\tau_+}(g(\zeta_0)) \ldots i_{\tau_+}(g(\zeta_1))
\]

for some \( \zeta_j \in \Delta \).

(II.3) \( \tilde{g}_2 \) has triangular factorization of the form \( \tilde{g}_2 = l_2 a_2 u_2 \), where \( l_2 \in \hat{N}^+ (z^{-1} \mathbb{C}[z^{-1}]) \).

Then statements (II.1) and (II.3) are equivalent. (II.2) implies (II.1) and (II.3).

Also, in the notation of (II.2),

\[
a_2 = \prod_{j=1}^{n} a(\zeta_j)^{h_{\tau_j}}.
\]
Remark 5.2. Note that we are not making any attempt to characterize the set of \( l_1 \) that arise in (I.3) (and similarly for the set of \( l_2 \) in (II.3)).

Conjecture 5.1. If \( g_1 \) is in the identity connected component of the sets in (I.1) and (I.3), then the converse holds, i.e. \( g_1 \) has a root subgroup factorization as in (I.2). If \( g_2 \) is in the identity connected component of the sets in (II.1) and (II.3), then the converse holds, i.e., \( g_2 \) has a root subgroup factorization as in (II.2).

In the course of the following proof of Theorem 5.1, we will prove a version of this conjecture, in the rank case, which completes the proof of Theorem 0.1 (see Remark 5.3 below).

Proof. The two sets of implications are proven in the same way. We consider the second set.

We first want to argue that (II.2) implies (II.3). We recall that the subalgebra \( \mathfrak{n}^- \cap w_{n-1}^{-1} \mathfrak{n}^+ w_{n-1} \) is spanned by the root spaces corresponding to negative roots \( -\tau_j, j = 1, \ldots, n \). The calculation is the same as in the proof of Theorem 2.5. In the process we will also prove the product formula for \( a_2 \).

The equation (5.1) implies that
\[
\tau_j (g(\zeta_j)) = \tau_j (\begin{pmatrix} 1 & 0 \\ \zeta_j & 1 \end{pmatrix} a(\zeta_j)^{h_{\tau_j}} \tau_j (\begin{pmatrix} 1 & \pm \zeta_j \\ 0 & 1 \end{pmatrix})) = \exp(\zeta_j f_{\tau_j}) a(\zeta_j)^{h_{\tau_j}} w_{j-1}^{-1} \exp(\pm \zeta_j e_{\tau_j}) w_{j-1}
\]
is a triangular factorization. Here, \( a(\zeta_j) = a_{\pm}(\zeta_j) \) and the plus/minus case is used when \( \tau_j \) is a compact/noncompact type root, respectively.

Let \( g^{(n)} = \tau_{\zeta_n}(g(\zeta_n)) \cdots \tau_{\zeta_1}(g(\zeta_1)) \). First suppose that \( n = 2 \). Then
\[
(5.4) \quad g^{(2)} = \exp(\zeta_2 f_{\tau_2}) a(\zeta_2)^{h_{\tau_2}} r_1 \exp(\pm \zeta_2 e_{\tau_2}) r_1^{-1} \exp(\zeta_1 f_{\tau_1}) a(\zeta_1)^{h_{\tau_1}} \exp(\pm \zeta_1 e_{\tau_1}).
\]
The key point is that
\[
\begin{align*}
 r_1 \exp(\pm \zeta_2 e_{\tau_2}) r_1^{-1} \exp(\zeta_1 f_{\tau_1}) & = r_1 \exp(\pm \zeta_2 e_{\tau_2}) \exp(\zeta_1 e_{\tau_1}) r_1^{-1} \\
 & = r_1 \exp(\zeta_1 e_{\tau_1}) u r_1^{-1}, \quad (\text{for some } u \in N^+ \cap r_1 N^+ r_1^{-1}) \\
 & = \exp(\zeta_1 f_{\tau_1}) u, \quad (\text{for some } u \in N^+).
\end{align*}
\]
Insert this calculation into (5.4). We then see that \( g^{(2)} \) has a triangular factorization \( g^{(2)} = l^{(2)} a^{(2)} u^{(2)} \), where
\[
a^{(2)} = a(\zeta_1)^{h_{\tau_1}} a(\zeta_2)^{h_{\tau_2}}
\]
and
\[
(5.5) \quad l^{(2)} = \exp(\zeta_2 f_{\tau_2}) \exp(\zeta_1 a(\zeta_2)^{-\tau_1(h_{\tau_2})} f_{\tau_1})
\]
\[
= \exp(\zeta_2 f_{\tau_2} + \zeta_1 a(\zeta_2)^{-\tau_1(h_{\tau_2})} f_{\tau_1})
\]
(the last equality holds because a two dimensional nilpotent algebra is necessarily commutative).

To apply induction, we assume that \( g^{(n-1)} \) has a triangular factorization \( g^{(n-1)} = l^{(n-1)} a^{(n-1)} u^{(n-1)} \) with
\[
(5.6) \quad l^{(n-1)} = \exp(\zeta_{n-1} f_{\tau_{n-1}}) u \in N^- \cap w_{n-1}^{-1} N^+ w_{n-1} = \exp(\sum_{j=1}^{n-1} w_{\tau_j}),
\]
for some \( \bar{l} \in N^\gamma \cap w_{n-1}^{-1}N^+w_{n-2} = \exp(\sum_{j=1}^{n-2} \mathbb{C} f_{r_j}) \), and
\[
a^{(n-1)} = \prod_{j=1}^{n-1} a(\zeta_j)^{h_{r_j}}.
\]
We have established this for \( n - 1 = 1, 2 \). For \( n \geq 2 \)
\[
g^{(n)} = \exp(\zeta_n f_{r_n}) a(\zeta_n)^{h_{r_n}} w_{n-1}^{-1} \exp(\pm \bar{\zeta}_n e_{r_n}) w_{n-1} \exp(\zeta_{n-1} f_{r_{n-1}}) \bar{t}_a(g^{(n-1)}) u(g^{(n-1)})
\]
\[
= \exp(\zeta_n f_{r_n}) a(\zeta_n)^{h_{r_n}} w_{n-1}^{-1} \exp(\pm \bar{\zeta}_n e_{r_n}) \bar{u} w_{n-1} a(g^{(n-1)}) u(g^{(n-1)}),
\]
where \( \bar{u} = w_{n-1} \exp(\zeta_{n-1} f_{r_{n-1}}) \bar{t} w_{n-1}^{-1} \in w_{n-1} N^- w_{n-1}^{-1} \cap N^+ \). Now write
\[
\exp(\pm \bar{\zeta}_n e_{r_n}) \bar{u} = \bar{u}_1 \bar{u}_2,
\]
relative to the decomposition
\[
N^+ = \left( N^+ \cap w_{n-1} N^- w_{n-1}^{-1} \right) \left( N^+ \cap w_{n-1} N^+ w_{n-1}^{-1} \right).
\]
Let
\[
l = a(\zeta_n)^{h_{r_n}} w_{n-1}^{-1} \bar{u}_1 w_{n-1} a(\zeta_n)^{-h_{r_n}} \in N^- \cap w_{n-1}^{-1} N^+ w_{n-1}.
\]
Then \( g^{(n)} \) has triangular decomposition
\[
g^{(n)} = (\exp(\zeta_n f_{r_n}) l) \left( a(\zeta_n)^{h_{r_n}} a^{(n-1)} \right) \left( (a^{(n-1)})^{-1} u_2 a^{(n-1)} u^{(n-1)} \right).
\]
This implies the induction step.

This calculation shows that (II.2) implies (II.3). It also implies the product formula for (5.3) \( a_2 \).

Remark 5.3. In reference to Conjecture 5.1, we observe that the preceding calculation shows that we have a map (using the notation we have established above)
\[
\{ (\zeta_j) : j = 1, \ldots, n \} \rightarrow \exp(\oplus_{j=1}^n \mathbb{C} f_{r_j}) : (\zeta_j) \rightarrow l(g^{(n)})
\]
where \( \zeta_j \) ranges over either the complex plane or a disk, depending on whether the \( j \)th root is of compact or noncompact type. The calculation also show that the map is 1-1 and open. We claim that the image of this map is closed in
\[
\{ l_2 \in \exp(\oplus_{j=1}^n \mathbb{C} f_{r_j}) : \exists \quad \tilde{g}_2 \text{ having triangular factorization } \tilde{g}_2 = l_2 a_2 u_2 \}.
\]
This follows from the product formula for \( a_2 \), which shows that as the parameters tend to the boundary, the triangular factorization fails. This implies that the image of the map is the connected component which contains \( l_2 = 1 \). This does prove the implication (II.2) implies (II.3) in Theorem 0.1, because \( n \) is fixed in the statement of that theorem, but this does not complete the proof of Conjecture 5.1. The difficulty is that we do not know how to formulate statements (I.1) and (II.1) in the general case in a way that regards \( n \) as fixed.

It is obvious that (II.3) implies (I.1). In fact (II.3) implies a stronger condition. If (II.3) holds, then given a highest weight vector \( \nu \) as in (II.1), corresponding to highest weight \( \Lambda \), then
\[
\pi(g_2^{-1}) \nu = \pi(u_2^{-1} a_2^{-1} t^{-1}) \nu = a_2^{-\Lambda} \pi(u_2^{-1}) \nu,
\]
implies that \( \pi(g_2^{-1}) \nu \) is holomorphic in \( \Delta \) and nonvanishing at all points. However we do not need to include this nonvanishing condition in (II.1), in this finite case.

It remains to prove that (II.1) implies (II.3). Because \( \tilde{g}_2 \) is determined by \( g_2 \), as in Lemma 3.3, it suffices to show that \( g_2 \) has a triangular factorization (with trivial
component). Hence we will slightly abuse notation and work at the level of loops in the remainder of this proof.

To motivate the argument, suppose that \( g_2 \) has triangular factorization as in (II.3). Because \( u_2(0) \in \mathbb{N}^+ \), there exists a pointwise \( \mathcal{G} \)-triangular factorization

\[
(5.9) \quad u_2(z)^{-1} = \hat{l}(u_2(z)^{-1}) \hat{d}(u_2(z)^{-1}) \hat{u}(u_2(z)^{-1})
\]

which is certainly valid in a neighborhood of \( z = 0 \); more precisely, (5.9) exists at a point \( z \in \mathbb{C} \) if and only if

\[
\hat{\sigma}_i(u_2(z)^{-1}) \neq 0, \quad i = 1, \ldots, r.
\]

When (5.9) exists (and using the fact that \( g_2 \) is defined on \( \mathbb{C}^* \) in this algebraic context),

\[
g_2(z) = (l_2(z)a_2 \hat{u}(u_2(z)^{-1})^{-1} a_2^{-1}) \left( a_2 \hat{d}(u_2(z)^{-1})^{-1} \right) \hat{l}(u_2(z)^{-1})^{-1}.
\]

This implies

\[
(5.10) \quad g_2(z)^{-1} = \hat{l}(u_2(z)^{-1}) \left( \hat{d}(u_2(z)^{-1}) a_2^{-1} \right) (a_2 \hat{u}(u_2(z)^{-1}) a_2^{-1} l_2(z)^{-1}).
\]

This is a pointwise \( \hat{\mathcal{G}} \)-triangular factorization of \( g_2^{-1} \), which is certainly valid in a punctured neighborhood of \( z = 0 \). The important facts are that (1) the first factor in (5.10)

\[
(5.11) \quad \hat{l}(g_2^{-1}) = \hat{l}(u_2(z)^{-1})
\]

does not have a pole at \( z = 0 \); (2) for the third (upper triangular) factor in (5.10), the factorization

\[
(5.12) \quad \hat{u}(g_2^{-1})^{-1} = l_2(z) (a_2 \hat{u}(u_2(z)^{-1}) a_2^{-1})
\]

is a \( L\hat{\mathcal{G}} \)-triangular factorization of \( \hat{u}(g_2^{-1})^{-1} \in L\hat{\mathbb{N}}^+ \), where we view \( \hat{u}(g_2^{-1})^{-1} \) as a loop by restricting to a small circle surrounding \( z = 0 \); and (3) because there is an a priori formula for \( a_2 \) in terms of \( g_2 \) (see (3.5)), we can recover \( l_2 \) and (the pointwise triangular factorization for) \( u_2^{-1} \) from (5.10)-(5.12): \( l_2 = \hat{l}(\hat{u}(g_2^{-1})^{-1}) \) (by (5.12)), and

\[
(5.13) \quad \hat{l}(u_2(z)^{-1}) = \hat{l}(g_2(z)^{-1}), \quad \hat{d}(u_2(z)^{-1}) = \hat{d}(g_2(z)^{-1}) a_2, \quad \text{and} \quad \hat{u}(u_2(z)^{-1}) = a_2^{-1} u(\hat{u}(g_2(z)^{-1})) a_2.
\]

We remark that this uses the fact that \( g_2 \) is defined in \( \mathbb{C}^* \) in an essential way.

Now suppose that (II.1) holds. In particular (II.1) implies that \( \hat{\sigma}_i(g_2^{-1}) \) has a removable singularity at \( z = 0 \) and is positive at \( z = 0 \), for \( i = 1, \ldots, r \). Thus \( g_2^{-1} \) has a pointwise \( \mathcal{G} \)-triangular factorization as in (5.10), for all \( z \) in some punctured neighborhood of \( z = 0 \).

We claim that (5.11) does not have at pole at \( z = 0 \). To see this, recall that for an \( n \times n \) matrix \( g = (g_{ij}) \) having an LDU factorization, the entries of the factors can be written explicitly as ratios of determinants:

\[
\hat{d}(g) = \text{diag}(\sigma_1, \sigma_2/\sigma_1, \sigma_3/\sigma_2, \ldots, \sigma_n/\sigma_{n-1})
\]
where \( \sigma_k \) is the determinant of the \( k^{th} \) principal submatrix, \( \sigma_k = \det((g_{ij})_{1 \leq i,j \leq k}) \); for \( i > j \),

\[
(5.14) \quad l_{ij} = \det \begin{pmatrix} g_{11} & g_{12} & \ldots & g_{1j} \\ g_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & g_{j-1,j} \\ g_{i1} & \cdots & g_{i,j-1} & g_{ij} \end{pmatrix} / \sigma_j = \frac{\langle g \epsilon_1 \wedge \ldots \wedge \epsilon_j, \epsilon_1 \wedge \ldots \wedge \epsilon_{j-1} \wedge \epsilon_i \rangle}{\langle g \epsilon_1 \wedge \ldots \wedge \epsilon_j, \epsilon_1 \wedge \ldots \wedge \epsilon_i \rangle}
\]

and for \( i < j \),

\[
u_{ij} = \det \begin{pmatrix} g_{11} & g_{12} & \ldots & g_{1j} \\ g_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & g_{i,j-1} \\ g_{i1} & \cdots & g_{i,j-1} & g_{ij} \end{pmatrix} / \sigma_i.
\]

Apply this to \( g = g_2^{-1} \) in a highest weight representation. Then (5.14), together with (II.1), implies the claim.

The factorization (5.12) is unobstructed. Thus it exists. We can now read the calculation backwards, as in (5.13), and obtain a triangular factorization for \( g_2 \) as in (II.3) (initially for the restriction to a small circle about 0; but because \( g_2 \) is of finite type, this is valid also for the standard circle). This completes the proof.

In the \( C^\infty \) analogue of Theorem 5.1, it is necessary to add further hypotheses in parts I.1 and II.1; see (5.8). To reiterate, we are now assuming that the sequence \( \{r_j\}_{j=1}^\infty \) is affine periodic.

**Theorem 5.2.** Suppose that \( g_1 \in \tilde{L}G_0 \) and \( \Pi(g_1) = g_1 \). Consider the following three statements:

1. \( m(g_1) = 1 \), and for each complex irreducible representation \( V(\pi) \) for \( \tilde{G} \), with lowest weight vector \( \phi \in V(\pi) \), \( \pi(g_1)^{-1}(\phi) \) has holomorphic extension to \( \Delta \), is nonzero at all \( z \in \Delta \), and is a positive multiple of \( v \) at \( z = 0 \).
2. \( g_1 \) has a factorization of the form

\[
\tilde{g}_1 = \lim_{n \to \infty} i_{\tau}^n \circ g(\eta_n),
\]

where \( g(\eta_n) = k(\eta_n) \) for some \( \eta_n \in \mathbb{C} \) (resp. \( g(\eta_j) = q(\eta_j) \) for some \( \eta_j \in \Delta \)) when \( \tau_j \) is a compact type (resp. non-compact type) root and the sequence \( (\eta_j)_{j=-N}^\infty \) is rapidly decreasing.

3. \( g_1 \) has triangular factorization of the form \( \tilde{g}_1 = l_1a_1u_1 \) where \( l_1 \in H^0(\Delta^*, N^-) \) has smooth boundary values.

Then statements (I.1) and (I.3) are equivalent. (I.2) implies (I.1) and (I.3).

Moreover, in the notation of (I.2),

\[
(5.15) \quad a_1 = \prod_{j=-N}^\infty a(\eta_j)^{h(\tau_j)}.
\]

Similarly, suppose that \( g_2 \in \tilde{L}G_0 \) and \( \Pi(g_2) = g_2 \). Consider the following three statements:

1. \( m(g_2) = 1 \); and for each complex irreducible representation \( V(\pi) \) for \( \tilde{G} \), with highest weight vector \( v \in V(\pi) \), \( \pi(g_2)^{-1}(v) \in H^0(\Delta; V) \) has holomorphic
extension to $\Delta$, is nonzero at all $z \in \Delta$, and is a positive multiple of $v$ at $z = 0$.

(II.2) $\tilde{g}_2$ has a factorization of the form

$$\tilde{g}_2 = \lim_{n \to \infty} \tau_n(g(\zeta_n))\cdots \tau_1(g(\zeta_1))$$

where $g(\zeta_j) = k(\zeta_j)$ for some $\zeta_j \in \mathbb{C}$ (resp. $g(\zeta_j) = q(\zeta_j)$ for some $\zeta_j \in \Delta$) when $\tau_j$ is a compact type (resp. non-compact type) root and the sequence $(\zeta_j)_{j=1}^\infty$ is rapidly decreasing.

(II.3) $\tilde{g}_2$ has triangular factorization of the form $\tilde{g}_2 = l_2a_2u_2$, where $l_2 \in H^0(\Delta^*, \infty; \hat{N}^+, 1)$ has smooth boundary values.

Then statements (II.1) and (II.3) are equivalent. (II.2) implies (II.1) and (II.3).

Also, in the notation of (II.2),

$$(5.16) \quad a_2 = \prod_{j=1}^\infty a(\zeta_j)^{h_{\tau_j}}.$$  

**Conjecture 5.2.** If $g_1$ is in the identity connected component of the sets in (I.1) and (I.3), then the converse holds, i.e. $g_1$ has a root subgroup factorization as in (I.2). If $g_2$ is in the identity connected component of the sets in (II.1) and (II.3), then the converse holds, i.e. $g_2$ has a root subgroup factorization as in (II.2).

In Remark 5.4, at the end of the following proof, we will indicate how we envision proving this conjecture. The issue in this $C^\infty$ context involves analysis, and we are not as confident in the truth of this Conjecture 5.2.

**Proof.** The two sets of equivalences and implications are proven in the same way. We consider the second set.

Suppose that (II.1) holds. To show that (II.3) holds, it suffices to prove that $g_2$ has a triangular factorization with $l_2$ of the prescribed form (see Lemma 3.3). By working in a fixed faithful highest weight representation for $\hat{g}$, without loss of generality, we can suppose $\hat{G}_0$ is a matrix subgroup of SL$(n, \mathbb{C})$ (where $\hat{N}^+$ consists of upper triangular matrices). We will assume that this representation is the complexified adjoint representation, or some subrepresentation of the exterior algebra of the adjoint representation, so that we can suppose that $\hat{G}_0$ fixes a (indefinite) Hermitian form (in the case of the adjoint representation, this is derived from the Killing form).

For the purposes of this proof, we will use the terminology in Section 1 of [11]. We view $g_2 \in L\hat{G}_0$ as a multiplication operator on the Hilbert space $\mathcal{H} = L^2(S^1; \mathbb{C}^n)$, and we write

$$M_{g_2} = \begin{pmatrix} A(g_2) & B(g_2) \\ C(g_2) & D(g_2) \end{pmatrix}$$

relative to the Hardy polarization $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $A(g_2)$ is the compression of $M_{g_2}$ to $\mathcal{H}^+$, the subspace of functions in $\mathcal{H}$ with holomorphic extension to $\Delta$. To show that $g_2$ has a Birkhoff factorization, we must show that $A(g_2)$ is invertible (see Theorem 1.1 of [11]).

Let $C_1, \ldots, C_n$ denote the columns of $g_2^{-1}$, and let $C_1^*, \ldots, C_n^*$ denote the rows of $g_2$. We can regard these as dual bases with respect to the pairing given by matrix multiplication, i.e., $C_j^*C_i = \delta_{ij}$.
The hypothesis of (II.1) implies that both $C_1$ and $C_1^*$ have holomorphic extensions to $\Delta$ (in the latter case, by considering the dual representation). Now suppose that $f \in \mathcal{H}^+$ is in the kernel of $A(g_2)$. Then

$$(C_n^* f)_+ = 0, \quad j = 1, \ldots, n,$$

where $(\cdot)_+$ denotes projection to $\mathcal{H}^+$. Since $C_n^*$ has holomorphic extension to $\Delta$, $(C_n^* f)_+ = C_n f$ and therefore $C_n^* f$ is identically zero on $S^1$ by (5.17). This implies that for $z \in S^1$, $f(z)$ is a linear combination of the $n-1$ columns $C_j(z)$, $j < n$. We write

$$f = \lambda_1 C_1 + \ldots + \lambda_{n-1} C_{n-1}$$

where the coefficients are functions on the circle (defined a.e.). Now consider the pointwise wedge product of $C^n$ vectors

$$f \wedge C_1 \wedge \ldots \wedge C_{n-2} = \pm \lambda_{n-1} C_1 \wedge \ldots \wedge C_{n-1}.$$

The vectors $C_1 \wedge \ldots \wedge C_j$ extend holomorphically to $\Delta$, and never vanish, for any $j$, by (II.1) (by considering the representation $\bigwedge^j(\mathbb{C}^n)$). Since $f$ also extends holomorphically, this implies that $\lambda_{n-1}$ has holomorphic extension to $\Delta$. Now

$$C_{n-1}^* f = \lambda_{n-1} C_{n-1}^* C_{n-1} = \lambda_{n-1}$$

by (5.17) and duality.

Since the right hand side is holomorphic in $\Delta$, by (5.17) (for $j = n-1$) $\lambda_{n-1}$ vanishes identically. This implies that in fact $f$ is a (pointwise) linear combination of the first $n-2$ columns of $g_2^{-1}$. Continuing the argument in the obvious way (by next wedging $f$ with $C_1 \wedge \ldots \wedge C_{n-3}$ to conclude that $\lambda_{n-2}$ must vanish), we conclude that $f$ is zero. This implies that $\ker(A(g_2)) = 0$. Since $\hat{G}$ is simply connected, $A(g_2)$ has index zero. Hence $A(g_2)$ is invertible. This implies (II.3).

It is obvious that (II.3) implies (II.1); see (5.8). Thus (II.1) and (II.3) are equivalent.

Before showing that (II.2) implies (II.1) and (II.3), we need to explain why the $C^\infty$ limit in (II.2) exists. We first consider the projection of the product in $L \hat{K}$. Because $g(\zeta_j) = 1 + O(|\zeta_j|)$ as $\zeta_j \to 0$, the condition for the product in (II.2) to converge absolutely is that $\sum \zeta_n$ converges absolutely. So $g_2$ certainly represents a continuous loop.

We will now calculate the derivative formally. In this calculation, we let $g_2^{(n)}$ denote the product up to $n$, and $\tau_n = q(n) d^* - \hat{\alpha}(n)$ ($q(n) > 0$, and $\hat{\alpha}(n) > 0$). Then

$$g_2^{-1} \frac{\partial g_2}{\partial \theta} = \Pi \left( \sum_{n=1}^{\infty} \text{Ad}(g^{(n-1)})^{-1} \left( i_{\tau_n}(g(\zeta_n))^{-1} \frac{\partial}{\partial \theta} i_{\tau_n}(g(\zeta_n)) \right) \right)$$

$$= \sum_{n=1}^{\infty} \text{Ad}(g^{(n-1)})^{-1} \left( \sqrt{-1} \frac{q(n)}{1 \pm |\zeta_n|^2} (\mp \zeta_n^2 h_{\hat{\alpha}(n)} - \zeta_n^2 e_{\hat{\alpha}(n)} z^{-q(n)} \mp \zeta_n f_{\hat{\alpha}(n)} z^{q(n)}) \right).$$

Because we are using an affine periodic sequence of simple reflections (with period $w_i^{-1} \in C \subset \hat{h}_R$), $\tau_{n+1} = w_i^{-1} \cdot \tau_1$, $\tau_{n+2} = w_i^{-1} \tau_2$, and so on. In general, writing $\tau_j = k(j) d^* - \hat{\alpha}(j)$ as above, and using Proposition (4.9.5) of [15] to calculate the coadjoint action,

$$\tau_{nl+j} = w_i^{-n} \cdot \tau_j = (q(j) + n\hat{\alpha}(j)(w_l)) d^* - \hat{\alpha}(j).$$
Because $\dot{\alpha}(w_l) > 0$, for all $\dot{\alpha} > 0$, it follows that $q(n)$ is asymptotically $n$. Because $\text{Ad}(g^{(n-1)})$ is orthogonal, (5.18) implies that

$$
\int |g_2^{-1}(\frac{\partial g_2}{\partial \theta})|^2 d\theta \leq \sum_{n=1}^{\infty} \|\text{Ad}(g^{(n-1)})\|^2 \frac{q(n)^2}{(1 + |\zeta_n|^2)^4}(|\zeta_n|^4 + |\zeta_n|^2)|h_{\dot{\alpha}(n)}|^2
$$

by Bessel’s inequality. This is comparable to $\sum_{n=1}^{\infty} n^2 |\zeta_n|^2$ because $\|\text{Ad}(g^{(n-1)})\|^2$ is uniformly bounded in $n$. Thus $g_2$ is $W^1$ (the $L^2$ Sobolev space) whenever $(\zeta_j) \in w^1$. Higher derivatives can be similarly calculated. This shows that if $\zeta \in w^n$, then $g_2 \in C^\infty$.

Together with Proposition 3.3, this does imply that the product in (II.2) converges in $\hat{\text{L}}G_0$. But to explain this further, note that

$$|\sigma_0|^2(g_2) = a_2^{\lambda_0} = \prod_{j=1}^{\infty} (1 + |\zeta_j|)^{-\lambda_0(h_{\tau_j})}$$

converges, because $\Lambda_0(h_{\tau_j})$ is asymptotically $j$. It then follows clearly from (3.4) and (3.5) that the lifted product in (II.2) converges.

Now suppose that (II.2) holds. The map from $\zeta$ to $\tilde{g}_2$ is continuous, with respect to the standard Frechet topologies for rapidly decreasing sequences and smooth functions. The product (5.16) is also a continuous function of $\zeta$, and hence is nonzero. This implies that $\tilde{g}_2$ has a triangular factorization which is the limit of the triangular factorizations of the finite products $\tilde{g}_2^{(n)}$. By Theorem 5.1 and continuity, this factorization will have the special form in (II.3). Thus (II.2) implies (II.1) and (II.3).

**Remark 5.4.** We now want to give a naive argument for Conjecture 5.2. Suppose that we are given $g_2$ as in (II.1) and (II.3). Recall that $l_2$ has values in $\hat{N}^\tau$. We can therefore write

$$l_2 = \exp\left(\sum_{j=1}^{\infty} x_j^\ast f_{\tau_j}\right), \quad x_j^\ast \in \mathbb{C}. \tag{5.20}$$

(the use of $x^\ast$ for the coefficients is consistent with our notation in the $SU(1,1)$ case, see (II.3) of Theorem 0.1).

As a temporary notation, let $X$ denote the set of $g_2$ as in (II.1) and (II.3); $x^\ast$ is a global linear coordinate for this space. We consider the map

$$c^\infty \to X \text{ given by } \zeta \mapsto g_2. \tag{5.21}$$

This map induces bijective correspondences among finite sequences $\zeta$, $g_2 \in X \cap L_{f_{\text{fin}}}K$ and finite sequences $x^\ast$, and the maps $\zeta$ to $x^\ast$ and $x^\ast \to \zeta$ are given by rational maps; however (although it seems likely) it is not known that the limits of these rational maps actually make sense even for rapidly decreasing sequences (see the Appendix of [11] for the $SU(2)$ case). We will use an inverse function argument to show that the map (5.21) has a global inverse (technically, to apply the inverse function theorem, we should consider the maps of Sobolev spaces $w^n \to X^n$, where $X^n$ is the $W^n$ completion of $X$, but we will suppress this).
Given a variation of $\zeta$, denoted $\zeta'$, we can formally calculate the derivative of this map,
\[
g_2^{-1}g'_2 = \sum_{n=1}^\infty \text{Ad}(g_2^{(n-1)})^{-1}(i_{\tau_n}(a(\zeta_n)) \left( \begin{array}{c} -1 \zeta_n \\ 1 \end{array} \right) \{a(\zeta_n)' \left( \begin{array}{c} 1 \zeta_n \\ 1 \end{array} \right) + a(\zeta_n) \left( \begin{array}{c} 0 \zeta_n \\ 0 \end{array} \right) \}))
\]

\[(5.22) = \sum_{n=1}^\infty \text{Ad}(g_2^{(n-1)})^{-1}(i_{\tau_n}(a(\zeta_n))^{-1}a(\zeta_n)' \left( \begin{array}{c} 1 \zeta_n \\ 0 \end{array} \right) + a(\zeta_n)^2 \left( \begin{array}{c} \zeta_n' \zeta_n \\ \zeta_n' \zeta_n \end{array} \right) ))
\]

\[
= \sum_{n=1}^\infty \text{Ad}(g_2^{(n-1)})^{-1}(i_{\tau_n}(a(\zeta_n)^2 \left( \begin{array}{c} 1/2 \zeta_n' \zeta_n - \zeta_n' \zeta_n \\ \zeta_n' \zeta_n \end{array} \right) - 1/2(\zeta_n' \zeta_n - \zeta_n' \zeta_n) ))
\]

As before it is clear that this is convergent, so that (5.21) is smooth. At $\zeta = 0$ this is clearly injective with closed image, so that there is a local inverse. Consider more generally a fixed $g_2 \in X \cap L_{fin} \hat{G}_0$, so that $g_2^{(n-1)} = g_0$ for large $n$. Recall that the root spaces for the $\tau_n$ are independent and fill out $n^\ast(\mathbb{C}[z])$. Given a variation such that $g_2^{-1}g'_2 = 0$, the terms in the last sum in the derivative formula (5.22) must be zero for large $n$. But we know that the map (5.21) is a bijection on finite $\zeta$. Thus for a variation of a finite number of $\zeta$ which maps to zero, the variation vanishes. It is clear that the image of the derivative (5.22) is closed. The image is therefore the tangent space to $X$ (because we know that finite variations will fill out a dense subspace of the tangent space). This implies there is a local inverse. This local inverse is determined by its values on finite $x^\ast$, and hence there is a uniquely determined global inverse. This shows that (II.1) and (II.3) imply (II.2).

Finally (5.16) follows by continuity from (5.3).

5.2. Generalization of Theorem 0.3.

**Theorem 5.3.** Suppose $\bar{g} \in L\hat{G}_0$ and $\Pi(\bar{g}) = g$.

(a) The following are equivalent:

(i) $\bar{g}$ has a triangular factorization $\bar{g} = lmau$, where $l$ and $u$ have $C^\infty$ boundary values, and satisfy the conditions $l(z), u^{-1}(z) \in \hat{G}_0B^+$ for all $z \in S^1$.

(ii) $\bar{g}$ has a (partial root subgroup) factorization of the form

\[
\bar{g} = \Theta(\bar{g}_1) \exp(\chi)\bar{g}_2,
\]

where $\chi \in \bar{L}_1$, and $\bar{g}_1$ and $\bar{g}_2$ are as in (I.3) and (II.3) of Theorem 5.2, respectively.

(b) In reference to (ii) of part (a),

\[(5.23) \quad a(\bar{g}) = a(g) = a(g_1)a(\exp(\chi))a(g_2), \quad \Pi(a(g)) = \Pi(a(g_1))\Pi(a(g_2))
\]

and

\[(5.24) \quad a(\exp(\chi)) = |\sigma_0|^{(\exp(\chi))} \prod_{j=1}^\infty |\sigma_0|^{(\exp(\chi))^{h_j}}.
\]

**Remarks.** Suppose that $\hat{G}_0 = SU(1,1)$. In this case the last condition in (i) in Theorem 5.3, that $l(z), u^{-1}(z) \in \hat{G}_0B^+$, is equivalent to the condition in Theorem 0.3 that the boundary values $l_{21}/l_{11}$ and $u_{21}/u_{22}$ are $< 1$ in magnitude on $S^1$, and part (b) specializes to the statement of Theorem 0.4.
Proof. Our strategy of proof is the following. We will first show that in part (a), (ii) implies (i). In the process we will prove part (b). We will then show that (i) implies (ii).

Suppose that we are given $\tilde{g}$ as in (ii). Both $\tilde{g}_1$ and $\tilde{g}_2$ have triangular factorizations by Theorem 5.2. In the notation of Theorem 5.2,
\begin{equation}
\tilde{g} = \Theta(l_1 a_1 u_1^*) \exp(\chi)(l_2 a_2 u_2) = \Theta(u'_2) a_1(\Theta(l'_1) \exp(\chi) l_2 a_2 u_2
\end{equation}
since $\Theta$ preserves the $A$ factor. The basic observation is that
\begin{equation}
b = \Theta(l'_1) \exp(\chi) l_2 \in (\tilde{L}B^+)_{0}
\end{equation}
(the inverse image in the affine extension for the identity component of loops in $B^+$), and $b$ will have a triangular factorization which we can compute. To do this requires some care with the central extension, and this involves some preparation.

Because $\tilde{B}^+$ is the semidirect product of $\tilde{H}$ and $\tilde{N}^+$, there is an isomorphism of loop groups
\[ L\tilde{B}^+ = L\tilde{H} \ltimes L\tilde{N}^+ \]
The central extension is trivial for $L\tilde{N}^+$, and hence there is an isomorphism
\[ \tilde{L}\tilde{B}^+ = \tilde{L}\tilde{H} \ltimes L\tilde{N}^+ \]
where the action of $\tilde{L}\tilde{H}$ on $L\tilde{N}^+$ is the same as the conjugation action of $L\tilde{H}$ on $L\tilde{N}^+$, and $\tilde{L}\tilde{H}$ is a Heisenberg extension determined by the bracket (3.1).

Given $\chi \in \tilde{L}t$ as above, let $\chi = \chi_+^* + \chi_0 + \chi_-$ denote the linear triangular decomposition, where $\chi_0 \in t$, $\chi_+ \in H^0(\Delta; 0; \mathfrak{h}, 0)$ and $\chi_- = -\chi_+^*$. Then (calculating in terms of the Heisenberg extension)
\[
\exp(\chi) = \exp(\chi_-) \exp(\chi_0) \exp(-[\chi_-, \chi_+]) \exp(\chi+) = \exp(\chi_-) \exp(\chi_0) \exp(\sum_{j=1}^{\infty} j \langle \chi_j, \chi_j \rangle c) \exp(\chi+).
\]
Substituting this into (5.26) we find
\[
b = \exp(\chi_-) b_1 \exp(\chi+)
\]
where
\[
b_1 = \exp(-\chi_-) \Theta(l'_1) \exp(\chi_-) \exp(\chi_0) \exp(\sum_{j=1}^{\infty} j \langle \chi_j, \chi_j \rangle c) \exp(\chi_+) l_2 \exp(-\chi_+).
\]
Thus, $b$ has a triangular factorization
\[
b = (\exp(\chi_-) L)(m(b) a(b)) (U \exp(\chi_+)),
\]
where $m(b) = m(b_1) = \exp(\chi_0)$, $a(b) = a(b_1) = \exp(\sum_{j=1}^{\infty} j \langle \chi_j, \chi_j \rangle c)$,
\[
L = l(\exp(-\chi_-) \Theta(l'_1) \exp(\chi_-) \exp(\chi_0) \exp(\chi_+) l_2 \exp(-\chi_+) \in H^0(\Delta^*, \infty; \tilde{N}^+, 1),
\]
and
\[
U = u(\exp(-\chi_-) \Theta(l'_1) \exp(\chi_-) \exp(\chi_0) \exp(\chi_+) l_2 \exp(-\chi_+) \in H^0(\Delta; \tilde{N}^+).
\]
Thus, from (5.25), $\tilde{g}$ will have a triangular factorization $l(\tilde{g}) m(\tilde{g}) a(\tilde{g}) u(\tilde{g})$, with
\begin{equation}
l(\tilde{g}) = \Theta(u'_2) \exp(\chi_-) a_1 l a_1^{-1}, \quad m(\tilde{g}) = m(b) = \exp(\chi_0),
\end{equation}
a(\tilde{g}) = a_1 a_2 \exp(\sum_{j=1}^{\infty} j \langle \chi_j, \chi_j \rangle c), \quad u(\tilde{g}) = a_2^{-1} U a_2 \exp(\chi_+) u_2.
Thus, (ii) implies (i) in part (a). At the same time this also implies part (b).

Now we need to show that (i) implies (ii). For this direction, there is not any need to consider the central extension, so we will no longer use tildes for group elements.

Suppose \( g = lmau \), as in (i). At each point of the circle there exist \( \tilde{N}^+ \tilde{A} \tilde{G}_0 \) decompositions

\[
(5.28) \quad l^{-1} = \tilde{a}_1 \tilde{a}_2 \tilde{g}_1, \quad u = \tilde{a}_2 \tilde{g}_2.
\]

This is a consequence of the somewhat bizarre hypotheses in (i). Then \( \tilde{g}_1 = \Theta(\tilde{g}_1^{-1})^* = \tilde{a}_1^{-1} \Theta(\tilde{n}_1) \Theta(\ell^*) \) since \( \tilde{g}_2 \mapsto \Theta(\tilde{g}_2^{-1})^* \) is the involution fixing \( \tilde{G}_0 \) in \( \tilde{G} \), and \( \Theta \) acts as the inverse on \( A \) under the inner type assumption.

In turn, there are Birkhoff decompositions

\[
\tilde{a}_i^{-1} = \exp(\chi_i^* + \chi_{i,0} + \chi_i), \quad \chi_i \in H^0(\Delta, \tilde{\mathfrak{h}}), \quad \chi_{i,0} \in \tilde{\mathfrak{h}}_\mathbb{R}
\]

for \( i = 1, 2 \). Define

\[
g_i = \exp(-\chi_i^* + \chi_i) \tilde{g}_i
\]

for \( i = 1, 2 \). Then

\[
g_1 = \exp(-\chi_{1,0} - 2\chi_1^*) \Theta(\tilde{n}_1) \Theta(\ell^*)
\]

has triangular factorization with

\[
l(g_1) = l(\exp(-\chi_{1,0} - 2\chi_1^*) \Theta(\tilde{n}_1) \exp(\chi_{1,0} + 2\chi_1^*)) \in H^0(\Delta^*, \infty; \tilde{N}^+, 1),
\]

\[
\Pi(a(g_1)) = \exp(\chi_{1,0}),
\]

and similarly

\[
g_2 = \exp(2\chi_2 + \chi_{2,0}) \tilde{n}_2^{-1} u
\]

has triangular factorization with

\[
l(g_2) = l(\exp(2\chi_2 + \chi_{2,0}) \tilde{n}_2^{-1} \exp(-2\chi_2 - \chi_{2,0})) \in H^0(\Delta^*, \infty; \tilde{N}^+, 1),
\]

\[
\Pi(a(g_2)) = \exp(\chi_{2,0}).
\]

The conclusion is somewhat miraculous. On the one hand \( \Theta(g_1^*)^{-1} g_2^{-1} \) has values in \( \tilde{G}_0 \) because \( g_1 \mapsto \Theta(g_1^*)^{-1} \) is the pointwise involution fixing \( \tilde{G}_0 \) in \( \tilde{G} \). On the other hand

\[
\Theta(g_1^*)^{-1} g_2^{-1} = \Theta(g_1^*)^{-1} \text{mau} (\exp(2\chi_2 + \chi_{2,0}) \tilde{n}_2^{-1} u)^{-1}
\]

\[
= \Theta(g_1^*)^{-1} \text{mau} \exp(-2\chi_2 - \chi_{2,0})
\]

\[
(5.29)
\]

has values in \( \tilde{B}^+ \). Therefore \( \Theta(g_1^*)^{-1} g_2^{-1} \) has values in \( \tilde{G}_0 \cap \tilde{B}^+ = \tilde{T} \). It is also clear that (5.29) is connected to the identity, and hence \( \Theta(g_1^*)^{-1} g_2^{-1} \in (\tilde{L}\tilde{T})_0 \) and thus equals \( \exp(\chi) \). Hence, \( g = \Theta(g_1^*) \exp(\chi) g_2 \). Thus (i) implies (ii).

6. Comments on the Rank One Case

In this section we revisit the assertions in the introduction concerning the rank one case, \( \tilde{G}_0 = SU(1,1) \). The various theorems in the text reduce to the assertions in the Introduction, e.g. the last condition in (i) in Theorem 5.3, that \( l(z), u^{-1}(z) \in \tilde{G}_0 \tilde{B}^+ \), is equivalent to the condition in Theorem 0.3 that the boundary values \( l_{21}/l_{11} \) and \( u_{21}/u_{22} \) are < 1 in magnitude on \( S^1 \), and part (b) of Theorem 5.3, together (5.15) and (5.16), specialize to the statements of Theorem 0.4. However our intent is to actually spell out the content of the assertions from scratch, especially the algorithm for factoring a loop into \( SU(1,1) \).
For reference
\[ \Theta\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \]
and
\[ r_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad r_0 = \begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix}. \]

The affine Weyl decomposition of \( h_8 \) looks like a decomposition of the line into uniform intervals indexed as follows.

\[ \ldots r_1 r_0 r_1 C_0 \ldots r_1 r_0 C_0 r_1 C_0 \ldots C_0 \ldots r_0 C_0 \ldots \]

### 6.1. From Partial Root Subgroup Factorization to Triangular Factorization

Suppose that \( g \) has a root subgroup factorization \( g = \Theta(g_1^*) \exp(\chi) g_2 \). Then

\[ g_1 = \begin{pmatrix} 1 & 0 \\ y^* & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \]

and

\[ g_2 = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}. \]

It is convenient to slightly rewrite these factorizations as

\[ g_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y^* & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \]

where \( Y = a_1^2 y \); and

\[ g_2 = \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & X^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}, \]

where \( X = a_2^{-2} x \).

As in [11], given these triangular factorizations for \( g_1 \) and \( g_2 \), we can derive the triangular factorization for \( g \) as follows:

\[ g = \Theta\left( \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \right)^* \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 a_2 e^{-Y^*+\chi} & 0 \\ 0 & (a_1 a_2 e^{-Y^*+\chi})^{-1} \end{pmatrix} \begin{pmatrix} 1 & X^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \]

\[ = \left( \begin{pmatrix} \alpha_1^* & \beta_1^* \\ -\gamma_1^* & -\delta_1^* \end{pmatrix} \right) \begin{pmatrix} e^{-Y^*} & 0 \\ 0 & e^{X^*} \end{pmatrix} \begin{pmatrix} 1 & -e^{2Y^*} \gamma_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 a_2 e^{\chi} & 0 \\ 0 & (a_1 a_2 e^{\chi})^{-1} \end{pmatrix} \begin{pmatrix} 1 & e^{2Y^*} X^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^X & 0 \\ 0 & e^{-X} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \]

The product of the middle three factors is upper triangular, and it is easy to find its triangular factorization. Thus \( g = l(g)m(g)a(g)u(g) \), where

\[ l(g) = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1^* & -\gamma_1^* \\ -\beta_1^* & -\delta_1^* \end{pmatrix} \begin{pmatrix} e^{-Y^*} & 0 \\ 0 & e^{X^*} \end{pmatrix} \begin{pmatrix} 1 & (-e^{2Y^*} \gamma_1 + (a_1 a_2)^2 e^{2(\chi Y^*)}) \gamma_1 \\ 0 & 1 \end{pmatrix} \]

\[ m(g) = \begin{pmatrix} e^{\chi Y^*} & 0 \\ 0 & e^{-\chi Y^*} \end{pmatrix}, \quad a(g) = \begin{pmatrix} a_1 a_2 & 0 \\ 0 & (a_1 a_2)^{-1} \end{pmatrix} \]

\[ u(g) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 1 & (-a_1 a_2)^{-2} e^{-2(-Y^*+\chi Y^*)} + e^{2Y^* X^*} \gamma_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^X & 0 \\ 0 & e^{-X} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \]
6.2. From a Loop to a (Partial) Root Subgroup Factorization. Suppose that \( g : S^1 \to SU(1,1) \) (not necessarily in the identity component). There are various obstacles to obtaining a root subgroup factorization. We first want to find a partial root subgroup factorization, as in Section 5.2.

1. Find a unique Riemann-Hilbert factorization:
\[
g = g_+ g_0 g_-
\]
This is equivalent to the invertibility of \( A(g) \). We have seen that \( g \) in the identity component does not guarantee invertibility, and even for \( g \) in nonidentity components, this is a generic condition.

2. Obtain a triangular factorization for \( g_0 \), and hence a triangular factorization for \( g \):
\[
g = lmau
\]
This is possible if and only if \( A(g) \) and \( A_1(g) \) are invertible, and this is a generic condition.

3. At each point of the circle, there are \( \dot{N}_+ \dot{A} \dot{G}_0 \) decompositions
\[
l^{-1} = \dot{n}_1^{-1} \dot{a}_1^{-1} \dot{g}_1, \quad u = \dot{n}_2 \dot{a}_2 \dot{g}_2
\]
These exist if and only if \( |l_{21}/l_{11}| \) and \( |u_{21}/u_{22}| \) are strictly less than 1 on the circle. These are admittedly rather incomprehensible conditions. In turn there are Birkhoff decompositions
\[
\dot{a}^{-1}_i = \exp(\chi_i^* + \chi_{i,0} + \chi_i), \quad \chi_i \in H^0(\Delta, \dot{h}), \quad \chi_{i,0} \in \dot{h}_R
\]
for \( i = 1, 2 \). Define
\[
g_i = \exp(-\chi_i^* + \chi_i) \dot{g}_i
\]
Then we obtain a partial root subgroup factorization
\[
g = g_1^* \exp((\chi_- + \chi_0 + \chi_+) \dot{h}_1) g_2
\]
where
\[
g_2 = \begin{pmatrix} d^* & c^* \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}
\]
where \( c(z) = O(z) \), \( d = 1 + O(z) \), \( x(z) = O(z) \), \( a_2^{-2} = \prod_{n>0}(1 + |\zeta_n|^2) \), and \( \gamma_2(z) = a_2 c(z) \), \( \delta_2(z) = a_2 d(z) \). \( g_1 \) is obtained in a similar way.

4. To obtain a full root subgroup factorization for \( g \) now reduces to finding root subgroup factorizations for \( g_1 \) and \( g_2 \). For \( g_2 \) for example, \( g_2 \) has to be in the identity component of the the set of all \( g_2 \) having triangular factorizations of a given form. In particular \( g \) will now need to be in the identity component. This again is not a comprehensible criterion.
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E-mail address: jacaine@cpp.edu

E-mail address: pickrell@math.arizona.edu