DISCRETIZATION OF SPRINGER FIBRES

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1. Let $\tilde{G}$ be an almost simple simply connected algebraic group over $\mathbb{C}$, let $G$ be the adjoint group of $\tilde{G}$ and let $\mathfrak{g}$ be the Lie algebra of $\tilde{G}$. Let $B$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let $e \in \mathfrak{g}$ be a nilpotent element and let $\zeta : SL_2(\mathbb{C}) \to \tilde{G}$ be a homomorphism of algebraic groups whose differential carries \[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\] to $e$. Let $B_e = \{b \in B; e \in b\}$. This variety (known as a Springer fibre) plays a very important role in representation theory. Following [L1] we note that $\lambda : b \mapsto Ad(\zeta(\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}))b$ is a well defined $\mathbb{C}^\times$-action on $B_e$. Let $N_e$ be the sum of all Betti numbers of $B_e$. (Recall that the Betti numbers of $B_e$ in odd degrees are zero.) Let $K_{C^\times}(B_e)$ be the $K$-theory of $C^\times$-equivariant coherent sheaves on $B_e$; this is regarded as a module over $\mathbb{Z}[v, v^{-1}]$ (the representation ring of $C^\times$) in the usual way. Here $v$ is an indeterminate representing the identity homomorphism $C^\times \to C^\times$. The $\mathbb{Z}[v, v^{-1}]$-module $K_{C^\times}(B_e)$ is free of finite rank $N_e$. (This is proved in [L5, 1.14(i), 1.19], based on results in [DLP].) In [L5, 5.15] the author has introduced a certain finite set $B_e^\pm$ (which we now denote by $B_e^\pm$) of the set of nonzero elements in $K_{C^\times}(B_e)$. (The definition of $B_e^\pm$ is somewhat analogous to the definition of canonical bases in the theory of quantum groups.) We have $-B_e^\pm = B_e^- \to$ and we denote by $B_e$ the set of orbits of multiplication by $\{1, -1\}$ on $B_e^\pm$. The set $B_e$ is a discretization (or discrete analogue) of $B_e$: according to [L5, 5.16], we have conjecturally:

(a) $B_e^\pm$ is a signed basis of the $\mathbb{Z}[v, v^{-1}]$-module $K_{C^\times}(B_e)$; hence $\text{card}(B_e) = N_e$.

Recently, Bezrukavnikov and Mirkovic [BM] have shown that (a) holds.

Let $\bar{F}$ be the centralizer in $\tilde{G}$ of the image of $\zeta$ (a reductive group). Let $F$ be the image of $\bar{F}$ in $G$. The set $B_e^\pm$ carries a natural action of $\bar{F}$ which factors through an action of the finite group $\bar{F} := F/F^0$ (for an algebraic group $G$ we denote by $G^0$ the identity component of $G$). For $b \in B_e$ let $\bar{F}_b \subset \bar{F}$ be the stabilizer of $b$ for the $\bar{F}$-action on $B_e$.

The set $B_e$ appears in representation theory in at least two ways. It indexes the simple objects in a certain block of unrestricted representations of the analogue of...
g over $\mathbf{F}_p$, for a large prime $p$ (this has been conjectured in [L4, §14] and proved in [BM]).

A second application of the set $\mathcal{B}_e$ is as follows. Let $W'$ be the affine Weyl group corresponding to the dual of $G$; let $W$ be the (extended) affine Weyl group corresponding to the dual of $G$. We have $W' \subset W$. Let $u = \exp(e) \in \tilde{G}$ and let $c$ be the two-sided cell of $W$ associated to the $\tilde{G}$-conjugacy class of $u$ in [L3, 4.8]. By [L5, 17.1], the $\tilde{F}$-set $\mathcal{B}_e$ should be the set $Y$ appearing in the conjecture [L3, 10.5] which provides

(b) a bijection between $c$ and the set of indecomposable $\tilde{F}$-equivariant vector bundles on $Y \times Y$ up to isomorphism.

In the case where $G = \text{PGL}_n(\mathbb{C})$, $n \geq 2$, (b) has been established in [Xi]; for general $G$ a weak form of (b) has been established in [BFO].

Note that the definition of the $\tilde{F}$-set $\mathcal{B}_e$ is not amenable to computation. In this paper we will give a definition of an $\tilde{F}$-set $Y'$ which is conjecturally isomorphic to $\mathcal{B}_e$, and which is amenable to computation. Substituting $Y$ by $Y'$ in the conjecture [L3, 10.5] makes that conjecture more concrete.

2. Assume now that $H$ is an $\mathbf{F}_2$-vector space with a fixed ordered basis $\{\xi_1, \xi_2, \ldots, \xi_d\}$ where $d \geq 0$. We define a set $\mathcal{F}_H$ of $\mathbf{F}_2$-subspaces of $H$ by induction on $d$ as follows. When $d = 0$ we have $\mathcal{F}_H = \{0\}$. Assume now that $d \geq 1$. For any $j \in [1, d]$ let $H_j$ be the subspace of $H$ with basis $\{\xi_1, \xi_2, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_d\}$. Then $\mathcal{F}_H$ is defined. For any $j \in [2, d]$ let $H_j'$ be the subspace of $H$ with basis $\{\xi_1, \xi_2, \ldots, \xi_{j-2}, \xi_{j-1} + \xi_j, \xi_{j+1}, \ldots, \xi_d\}$. Then $\mathcal{F}_H'$ is defined. Let $\mathcal{F}_H$ be the set of subspaces of $H$ of the form $L + F \xi_j$ where $L \in \mathcal{F}_{H_j}$, $j \in [1, d]$, or of the form $L'$ where $L' \in \mathcal{F}_{H_j'}$, $j \in [2, d]$. This completes the inductive definition of $\mathcal{F}_H$. For example, if $d \leq 2$, $\mathcal{F}_H$ consists of all $\mathbf{F}_2$-subspaces of $H$. If $d = 3$, $\mathcal{F}_H$ consists of all $\mathbf{F}_2$-subspaces of $H$ except for the subspace $\mathbf{F}_2(\xi_1 + \xi_3)$ and the subspace $\mathbf{F}_2(\xi_1 + \xi_2) + \mathbf{F}_2(\xi_2 + \xi_3)$.

We can find a finite parabolic subgroup $W''$ of $W'$ and a two-sided cell $c''$ of $W''$ such that $c'' \subset c$ (see [L3, 4.8(d)]); moreover, by [L8, 1.5(b2)], we can assume that the finite group $G_{c''}$ associated to $c''$ in [L2, 3.5] coincides with $\tilde{F}$. Let $\mathcal{F}_c$ be the set of subgroups of $\tilde{F} = G_{c''}$ attached in [L2, 3.8] to the various left cells of $W''$ contained in $c''$ (or rather one such subgroup in each $\tilde{F} = G_{c''}$-conjugacy class).

For any $n \geq 1$ we denote by $S_n$ the symmetric group in $n$ letters $1, 2, \ldots, n$. For $n' \geq 1$ such that $n' < n$ we view $S_{n'}$ as the subgroup of $S_n$ of permutations which keep $n'+1, n'+2, \ldots, n$ fixed.

The group $\tilde{F}$ is known to be either $S_n$ with $n \in \{3, 4, 5\}$ or an $\mathbf{F}_2$-vector space. If $\tilde{F} = S_3$ (with $G$ not of type $G_2$) then $\mathcal{F}_e$ consists of the subgroups $\{1\}, S_2, S_3$ of $S_3$.

If $\tilde{F} = S_3$ (with $G$ of type $G_2$) then $\mathcal{F}_e$ consists of the subgroups $S_2, S_3$ of $S_3$.

If $\tilde{F} = S_4$ then $\mathcal{F}_e$ consists of the subgroups $S_2, S_3, S_4, S_2 \times S_2, D_8$ of $S_4$. Here $S_2 \times S_2$ is viewed as the subgroup of $S_4$ of permutations which permute 1,2 and
permute 3, 4; $D_8$ is viewed as the subgroup of $S_4$ of permutations which commute with the permutation $1 \mapsto 4 \mapsto 1, 2 \mapsto 3 \mapsto 2$.

If $\tilde{F} = S_5$ then $\mathcal{F}_e$ consists of the subgroups $S_2, S_3, S_4, S_5, S_2 \times S_2, S_2 \times S_3, D_8$ of $S_5$. Here the subgroups $S_2, S_3, S_4, S_2 \times S_2, D_8$ of $S_5$ are viewed as subgroups of $S_5$ via the imbedding $S_4 \subset S_5$; $S_2 \times S_3$ is viewed as the subgroup of $S_5$ of permutations which permute 1, 2 and permute 3, 4, 5.

If $\tilde{F}$ is an $\mathbb{F}_2$-vector space $H$ of dimension $d$ then $H$ (in the description $\mathcal{G}_e$) has a canonical ordered $\mathbb{F}_2$-basis and we have $\mathcal{F}_e = \mathcal{F}_H$ except in the case where $d = 1$, $G$ is of type $E_7$ or $E_8$ and the Springer representation of the Weyl group associated to $e$ has dimension 512 or 4096, in which case $\mathcal{F}_e = \{\{1\}\}$.

It is likely that:

(a) a subgroup of $\tilde{F}$ is the stabilizer in $\tilde{F}$ of some point in $\mathcal{B}_\epsilon$ if and only if it is conjugate to a subgroup in $\mathcal{F}_e$.

3. Let $H$ be a finite group. Let $M(H)$ be the set of all pairs $(s, \rho)$ where $s \in H$ and $\rho$ is an irreducible representation over $\mathbb{C}$ (up to isomorphism) of the centralizer $Z_H(s)$ of $s$ in $H$; the pairs $(s, \rho)$ are taken modulo $H$-conjugacy. For any subgroup $H'$ of $H$ and any $s \in H$ we set $(H/H')^s = \{hH' \in H/H'; shH' = hH'\}$. Now $Z_H(s)$ acts on $(H/H')^s$ by left multiplication. This induces a $Z_H(s)$-module structure on the vector space of $\mathbb{C}$-valued functions on $(H/H')^s$. For any irreducible representation $\rho$ of $Z_H(s)$ we denote by $f_{H'}(s, \rho)$ the multiplicity of $\rho$ in this $Z_H(s)$-module. Thus $H'$ gives rise to a function $f_{H'} : M(H) \to \mathbb{C}$ with values in $\mathbb{N}$. Note that $f_{H'} \neq 0$; for example, for any $s \in H$ we have $f_{H'}(s, 1) > 0$.

4. We choose a Borel subgroup $B$ of $\tilde{F}^0$ and a maximal torus $T$ of $B$. Let $\tilde{F}' = \{g \in \tilde{F} ; gBg^{-1} = B, gTg^{-1} = T\}$. Then $\tilde{F}'^0 = T$ and the obvious map $\tilde{F}'/\tilde{F}'^0 \to \tilde{F}/\tilde{F}^0$ is an isomorphism. Let $\mathcal{B}_e^T = \{b \in \mathcal{B}_e ; \text{Ad}(t)b = b \text{ for all } t \in T\}$. Note that $\tilde{F}'$ acts on $\mathcal{B}_e^T$ by $g : b \mapsto \text{Ad}(g)b$. This action is trivial on $T$ hence it induces an action of $\tilde{F}'/T = \tilde{F}/\tilde{F}^0$ on $\mathcal{B}_e^T$. This last action must be trivial on the kernel of the obvious surjective map $\tilde{F}/\tilde{F}^0 \to \tilde{F}$ hence it induces an action of $\tilde{F}$ on $\mathcal{B}_e^T$. Let $s \in \tilde{F}$. Let $\mathcal{B}_e^{T,s}$ be the fixed point set of the action of $s$ on $\mathcal{B}_e^T$. Note that $Z_{\tilde{F}}(s)$ acts on $\mathcal{B}_e^{T,s}$ as the restriction of the $\tilde{F}$-action on $\mathcal{B}_e^T$. Hence for any $i$ there is an induced action of $Z_{\tilde{F}}(s)$ on $H^i(\mathcal{B}_e^{T,s}, \mathbb{C})$. We define a function $\phi : M(\tilde{F}) \to \mathbb{C}$ by

\[ \phi(s, \rho) = \sum_i (-1)^i \text{(multiplicity of } \rho \text{ in } Z_{\tilde{F}}(s) \text{-module } H^i(\mathcal{B}_e^{T,s}, \mathbb{C})). \]

We now state the following.

**Definition/Conjecture.** There are uniquely defined natural numbers $n_{H'}$ (for $H' \in \mathcal{F}_e$ so that $H' \subset \tilde{F}$) such that $\phi = \sum_{H' \in \mathcal{F}_e} n_{H'} f_{H'}$ as functions $M(\tilde{F}) \to \mathbb{C}$. We define a $\tilde{F}$-set $Y'$ as the disjoint union $\bigsqcup_{H' \in \mathcal{F}_e} Y'_H$, where $Y'_H$ consists on $n_{H'}$ copies of the transitive $\tilde{F}$-set with isotropy group $H'$. The $\tilde{F}$-set $Y'$ is isomorphic to the $\tilde{F}$-set $\mathcal{B}_e$.

One can easily check that the functions $f_{H'}$ (for $H' \in \mathcal{F}_e$ so that $H' \subset \tilde{F}$) are linearly independent so that $n_{H'}$ above are unique if they exist.
We note also that the function $\phi$ is in principle computable by making use of the known algorithms to compute Green functions and the explicit knowledge of the Springer correspondence. It follows that the coefficients $n_{H'}$ are computable.

5. Let $J_c$ be the $C$-vector space with basis $\{t_z; z \in c\}$. We regard $J_c$ as an associative $C$-algebra with 1 and with structure constants in $N$ as in [L3, 1.3]. We now reformulate the conjecture in [L3, 10.5]. (Note that the $F$-set $Y'$ can be regarded as a $F'$-set via the obvious map $\tilde{F} \to \tilde{F}$.)

**Conjecture.** There exists a bijection

$$\Pi : c \sim \{ \text{set of irreducible } F'-\text{vector bundles on } Y' \times Y' (\text{up to isomorphism}) \}$$

with the following properties:

(a) The $C$-linear map $J_c \to C \otimes K_{\tilde{F}}(Y' \times Y')$, $t_z \mapsto \Pi(z)$ is an algebra isomorphism preserving the unit element.

(b) For $z \in c$, $\Pi(z^{-1})$ is the inverse image under $(y'_1, y'_2) \mapsto (y'_2, y'_1)$ of the dual of the vector bundle $\Pi(z)$.

(c) Under $\Pi$, the simple $C \otimes K_{\tilde{F}}(Y' \times Y')$-module $E_{s, \rho}$ (see [L3, 10.3]) corresponds to the simple $J_c$-module $E(u, s, \rho)$ in [L3, 4.9].

Here $K_{\tilde{F}}(Y' \times Y')$ denotes the $K$-theory of $\tilde{F}$-equivariant complex vector bundles on $Y' \times Y'$ with the ring structure given by convolution, see [L3, 10.2].

6. For any $m \in Z$ let

$$g_m = \{ x \in g; \text{Ad}(\zeta \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right))x = \lambda^m x \ \forall \lambda \in C^* \}.$$ 

Then $p := \sum_{m \in \mathbb{N}} g_m$ is the Lie algebra of a parabolic subgroup $P$ of $\tilde{G}$ such that $F \subset P$. Let $M$ be the (finite) set of orbits of $P$ on $B$ (for the conjugation action). For any $\omega \in M$ let $B_{e, \omega} = B_e \cap \omega$. Let $M_e = \{ \omega \in M; B_{e, \omega} \neq 0 \}$. By [DLP, 3.4(d), 3.7(a), 2.2(i)], for $\omega \in M_e$, the variety $B_{e, \omega}$ is a vector bundle over a smooth projective variety. Moreover, $F$ acts on $B_{e, \omega}$ by conjugation and this induces an action of $F$ on the set of connected components of $B_{e, \omega}$ which by [DLP, 2.2(iii)] is transitive. Let $E_e$ be the set of subvarieties $X$ of $B_e$ such that $X$ is a connected component of $B_{e, \omega}$ for some $\omega \in M_e$. Note that each $X \in E_e$ is smooth (in fact a vector bundle over a smooth projective variety). Moreover, $F$ acts naturally on $E_e$ and the set of orbits is in bijection with $M_e$. For $X \in E_e$ let $F_X \subset F$ be the stabilizer of $X$ in this action.

We now describe a conjectural partition

(a) $B_e = \bigsqcup_{X \in E_e} B_{e, X}$.

Let $KC^+(B_e)$ be the subgroup of $KC^+(B_e)$ generated by $\cup_{n \in \mathbb{Z}_{>0}} v^n B_\pm$.

For $\omega \in M_e$ and $X \in E_e$ such that $X \subset \omega$, let $\tilde{X}$ be the union of all $X'$ where $X'$ runs though the elements of $E_e$ such that $X' \subset \omega'$ where $\omega' \in M_e$ is contained in the closure of $\omega$ in $B$ and $\omega' \neq \omega$. Note that $\tilde{X}$ is a closed subvariety of $B_e$. 


stable under the $\mathbb{C}^*$-action. The inclusion $j_X : \hat{X} \to B_e$ induces a homomorphism $j_X^* : K_{\mathbb{C}^*}(\hat{X}) \to K_{\mathbb{C}^*}(B_e)$. Let $\xi \in B^\pm$. It is likely that there is a unique $X \in \mathcal{E}_e$ of minimum dimension such that

(b) $\xi \in j_X^*(K_{\mathbb{C}^*}(\hat{X})) + K_{\mathbb{C}^*}(B_e)^{>0}$.

(This at least holds in the examples considered in [L6], [L7].) Assuming (b) we see that $\xi \mapsto X$ defines a map $\alpha : B_e \to \mathcal{E}_e$.

Let $\mathcal{B}_e, X = \alpha^{-1}(X)$. This defines the partition (a). From the definitions we see that $\alpha$ is compatible with the $\bar{F}$-actions on $B_e, \mathcal{E}_e$. Hence for $b \in B_e$ we have

(c) $\bar{F}_b \subset \bar{F}_{\alpha(b)}$.

Assume for example that $G$ is of type $E_8$ and $e$ is such that $\bar{F} = S_5$. In this case the subgroups $\{\bar{F}_X ; X \in \mathcal{E}_e\}$ of $\bar{F}$ are exactly the conjugates of the subgroups in $F_e$ (a result of [DLP]); we expect that in this case (c) is an equality.

Assume now that $G$ is of type $E_8$ and $e$ is of type $E_8(b_6)$ (notation as in [Ca, p.407]). In this case we have $\bar{F} = S_3$ and $\bar{F}_{\alpha(b)}$ is one of the subgroups $S_2, S_3$ or a cyclic group of order 3 of $S_3$ (this can be deduced from [DLP, 4.1]); if $\bar{F}_{\alpha(b)}$ is cyclic of order 3, we expect to have $\bar{F}_b = \{1\}$ so that (c) is not an equality.

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