A DUAL FORMULATION FOR (p and D)-BRANES
VIA TARGET-SPACE FIELD THEORY

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Abstract
It is shown how some field theories in the target-space induce the splitting of the space-time into a continuous of branes, which can be p-branes or D-branes depending on what the field theory it is. The basic symmetry underlying the construction is used to build an invariant action, which is proved to be off-shell identical to the p-brane (D-brane) action. The coupling with the abelian \((p + 1)\)-form in this formulation it is also found. While the classical brane’s embedding couple to the field strenght, the classical fields couple with its dual (in the Hodge sense), therefore providing an explicit electric-magnetic duality. Finally, the generic role of the underlying symmetry in the connection between the target-space theory and the world-volume one, is completely elucidated.
1 Introduction

Strings are the natural extension of point particles and in the last twenty five years they have proved to be a privileged framework for broaching the unification of all known interactions, including specially the elusive gravity \[4\]. Despite of the apparent crucial feature of the bidimensionality, mainly conformal invariance, some people considered that other extended objects, called branes, could play a role in this history (for two excellent review see \[1\]). This attempt deflates by the belief that kappa symmetry, of decisive importance on building superparticles and superstrings a la Green-Schwarz \[6\], is not generalizable to higher dimensional branes. Even more, the supermembrane seems to be quantically unstable. However, some time later, kappa-symmetric actions appeared for some branes in different dimensions. Aside from thoses advances, string theory centers the main part of the researcher effort. But, fortunately, this is not the end of the brane’s story.

In the same way that particles couple to the electromagnetic field, which is a one form in the target-space, branes of dimension \(d\), called \((d-1)\)-branes, couple to \(d\)-forms. The effective action of the heterotic string presents a dual formulation (in the forms sense) in terms of a six form. This fact suggested the existence of an S-dual formulation of the heterotic string in terms of a fundamental five-brane. The strong coupling of the heterotic string would be better described in the weak regime of the five-brane. This was called the string/five-brane duality conjectured \[3\]. From now, the role of branes in string theory is just starting. The relation between the \(p\)-forms of the effective actions and the \((p-1)\)-branes has been supported by the obtention of the corresponding elementary and solitonic solutions in every supergravity in ten dimensions. In particular, in \[5\] was shown how the heterotic effective field theory admits the five-brane as a soliton solution, then backing the conjectured string/five-brane duality.

But extended objects burst into string theory in another very exotic and surprising way by means of the Dirichlet-branes \[7\]. These are the hyper-surfaces in which the strings with Dirichlet boundary conditions rest their endpoints. Consistency requires that these branes must be promoted to dynamical objects carrying the Ramond-Ramond charges needed by duality. Therefore they are automatically included in the spectrum.
The discovery of the connection between the different compactifications of sugras in ten dimensions and the supergravity in eleven dimensions supported the idea of an underlying fundamental 11-dimensional theory, called, the M-theory\(^2\). Its low energy regime, the eleven dimensional supergravity, contains a three-form. Maybe there is a unique fundamental brane, the membrane in eleven dimensions.

As we can see, although string theory seems to be the unique starting point of all resulting scenarios, it is not a crazy idea that the underlying model could have as elemental constituent(s) other brane(s) than strings.

The usual description of a d-dimensional extended object \(\Sigma\) starts with its embedding in the D-dimensional space-time, i.e., \(X^\mu(\varepsilon^a)\), where \(\{\varepsilon^a\}\) with \(a = 1\) to \(d\) parametrize the (d-1)-brane. The dynamic is a direct extension to the corresponding with the point particle one: the hypervolume swept up by the classical configuration in its time evolution is an stationary point of the action. Then, the coupling with the metric is the Nambu-Goto type action:

\[
S^G_p = \int_\Sigma d^d\varepsilon \sqrt{\gamma(1)}
\]

where \(\gamma_{ab} = G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu\) is the metric induced in the brane and \(\gamma\) its determinant\(^1\). The coupling with the abelian electromagnetic field, i.e., the d-form \(A(X)\nu_1,\ldots,\nu_d\), follows the pattern as in the point particle case:

\[
S^A_p = \int_\Sigma d^d\varepsilon A(X)_{\nu_1,\ldots,\nu_d} \prod_{a=1}^d \partial_a X^{\nu_a}
\]

Obviously, both actions are invariant under reparametrizations in the world-volume \(\varepsilon^a \rightarrow \varepsilon'^a(\varepsilon^b)\) and so it is the dynamic, obtained from the whole action \(S_p = S^G_p + S^A_p\):
where the laplacian $\Delta X^\rho$ is conveniently defined as:

$$\Delta X^\rho = \gamma^{ab}(\partial_a \partial_b X^\rho + \Gamma_{\mu \nu}^\rho \partial_a X^\mu \partial_b X^\nu) \quad (4)$$

which is manifestly covariant under diffeomorphisms in the space-time $X^\mu \to X'^\mu(X^\nu)$, but no under reparametrizations in the world-volume. It is just the presence of the ortogonal (to the brane) proyector $P^\Sigma \mu \nu \equiv G^\mu \nu - U^\Sigma \mu \nu$, with $U^\Sigma \mu \nu = \gamma^{ab} \partial_a X^\mu \partial_b X^\mu$, which guarantees the invariance of (3) under diffeomorphisms in the world-volume. The field strength for the p-form is $F(X)_{\mu \nu_1, \ldots, \nu_d} \equiv \partial_\mu A(X)_{\nu_1, \ldots, \nu_d} - \partial_{\nu_1} A(X)_{\mu \nu_2, \ldots, \nu_d} - \cdots - \partial_{\nu_d} A(X)_{\nu_1, \nu_2, \ldots, \mu}$.

The D-branes are extended objects of a different class than p-branes. They appear in the context of open strings with Dirichlet boundary conditions, but, through this paper, we are only interested in two features: firstly, the coupling with the metric is given by the substitution $G_{\mu \nu} \to G_{\mu \nu} + B_{\mu \nu}$ on (1), where $B_{\mu \nu} = -B_{\nu \mu}$ is the Neveu-Schwarz (NS-NS) two form recalling the stringy origin for this brane. This is called a DBI-(Dirichlet-Born-Infeld)-Action [11]. Second, the electromagnetic term is the same type as in (2), but the d-form is now a composite one, given by a combination of the Ramond-Ramond (R-R) forms and the NS-NS two form [12]. The presence of the two-form in the DBI-action makes the equations very involved, so we let them to the appendix.

The purpose of this work consists on showing the existence of a reformulation for the dynamics of p-branes, D-branes and any other branes in terms of scalar fields in the target-space. The idea comes from [3, 4] where this was made in a few very simple examples. Here we will give the basic symmetry which shall serve to guide us for building the field equations for our scalar fields, no matter the dimension D of the space-time, its geometry, and the dimension d of the brane is. This symmetry, which basically consists on arbitrary redefinitions of the scalar fields, will allow us to interpret those fields as defining a hypersurface which will result to be the p-brane, or D-brane, depending on the field theory we are dealing with. In this formulation, the coupling with the electromagnetic d-form is also given. This will serve to see that both formulations are classically related by electric-magnetic duality. Even more, certain ambiguities on defining an action for the field theory having the right symmetry, are solved in a natural way by demanding its
correspondence with branes. After that, the off-shell identity between this action and the p-brane (or D-brane) action is shown, totally supporting the construction. Finally, the power of the symmetry under field redefinitions is used to justify the intrinsic relation between the target-space degrees of freedom and the world-volume ones, no matter the specific form of the action is. The rule giving the target-space field theory once the brane one is known, is also found. In this general context, the classical equivalence it is shown in detail.

In section 2, the basic symmetry is introduced and the simplest covariant field equations are built. Section 3 shows that the branes induced by those equations are classical p-branes in absence of electromagnetic field. In section 4, the problems on building the right action are solved demanding its brane origin and the invariance under field redefinitions. Then, the off-shell duality with the p-brane action is shown. In section 5, the electromagnetic contribution is found in this formulation, giving the p-brane’s one after dualization. The classical electric-magnetic duality is manifest. A simple example is worked out. In section 6, the invariance under field redefinitions is picked out as the feature responsible for the exchange of degrees of freedom between both formulations. The rule assigning the target space lagrangian in terms of the world-volume brane one is explicitly given. The explicit classical equivalence it is shown. Section 7 summarize the D-brane case. We end with the conclusions.

2 The Classical Model

The idea consists [8, 9] on describing an extended object Σ by means of usual fields, i.e., fields defined in the target-space $\mathcal{M}$, but not in an auxiliar manifold (world-volume). We start with a set of $\tilde{d}$ scalar fields $\phi^i(X)$ $i=1$ to $\tilde{d}=D-d$. Our hypersurface Σ is defined as the collection of points in the space-time where the whole set of fields vanish:

$$\Sigma \equiv \{ X \in M \setminus \phi^i(X) = 0 \ \forall \ i = 1 \ to \ \tilde{d} \} \quad (5)$$

This is a very simple an usual way to represent hypersurfaces. Now, the main problem is to build a dynamic for those fields which allow us to interpret them as representing the extended object [8]. The key observation of this work is that arbitrary redefinition of fields
\[ \phi^i(X) \longrightarrow \phi'^i(\phi^j(X)) \] (6)

must be a symmetry of the field equations always than \( \det \frac{\partial \phi'^i}{\partial \phi^j} \neq 0 \). In that way the hypersurfaces \( \Sigma' \) and \( \Sigma \), defined as \( \phi^i(X) = 0 \) and \( \phi(X) = 0 \), are the same. In this formalism, the invariance under (6) will become equivalent to the invariance under reparametrizations in the world-volume in the \( X^\mu(\varepsilon^a) \) description. It is just the invariance under (6) which guide us to build the right equations. Let us see how this works.

The basic first-derivative target-space scalar we get is

\[ h^{ij}(X) = G(X)^{\mu\nu} \partial_\mu \phi^i(X) \partial_\nu \phi^j(X) \] (7)

Because the \( \partial \phi^i \) are ortogonal to the hypersurfaces \( \phi = constant \), \( h_{ij} \) can be viewed as a sort of transverse metric. Moreover, is a \((2,0)\) tensor under field redefinitions :

\[ h'^{ij} = \frac{\partial \phi'^i}{\partial \phi^k} \frac{\partial \phi'^j}{\partial \phi^l} h^{kl} \] (8)

Now, for building the dynamics compatible with (6), we just need to realize the existence of a natural connection for our symmetry transformation (6):

\[ (\Omega_\mu)^i_j \equiv \Omega^i_j_\mu = h_{ik} \partial^\nu \phi^k \nabla_\mu \partial_\nu \phi^j \] (9)

where \( \nabla \) is the target-space covariant derivation built with the corresponding connection \( \Gamma^\rho_{\mu\nu} \) and \( h_{ij} \) satisfies \( h_{ik}h^{kj} = \delta^i_j \).

In matrix notation, the \( \Omega_\mu \) transformation under field redefinitions is

\[ \Omega'_\mu = \frac{\partial \phi'}{\partial \phi} \Omega^i_j \frac{\partial \phi^i}{\partial \phi'} + \partial_\mu \left( \frac{\partial \phi'}{\partial \phi} \right) \frac{\partial \phi^i}{\partial \phi'} \] (10)

\(^3\)Through this work, \( \Gamma^\rho_{\mu\nu} \) stands for the Levi-Civita connection and \( \nabla_\mu \) its corresponding covariant derivation.
that is, $\Omega$ is a real connection. Therefore, we can give covariant derivatives under (6) using the basic ones for the $(1, 0)$ $(V^i)$ and $(0, 1)$ $(V_j)$ tensors:

$$V' = \frac{\partial \phi}{\partial \phi'} V^k \rightarrow D_\mu V^i \equiv \partial_\mu V^i - \Omega^i_{\mu j} V^j \rightarrow D'_\mu V'^i = \frac{\partial \phi'}{\partial \phi} D_\mu V^k$$

$$V'_i = \frac{\partial \phi'}{\partial \phi} V'_j \rightarrow D_\mu V_i \equiv \partial_\mu V_i + \Omega^j_{\mu i} V_j \rightarrow D'_\mu V'_i = \frac{\partial \phi'}{\partial \phi} D_\mu V_j$$

(11)

In particular, the “transverse metric” is covariantly constant:

$$D_\mu h^{ij} = \partial_\mu h^{ij} - \Omega^i_{\mu k} h^{kj} - \Omega^j_{\mu k} h^{ik} = 0.$$  

Once the connection has been established, the simplest dynamic we can build is:

$$D_\mu \partial_\mu \phi^i = G^{\mu \nu} D_\mu \partial_\nu \phi^i = G^{\mu \nu} (\partial_\mu \partial_\nu \phi^i - \Omega^i_{\mu j} \partial_\nu \phi^j - \Gamma^i_{\mu \nu} \partial_\sigma \phi^i) = 0$$

(12)

where the target-space connection is also present to guarantee the invariance under diffeomorphisms on M. It will be very useful for studying their relation with the p-branes to rewrite (12) as:

$$D_\mu \partial^\mu \phi^i = U^{\mu \nu} \nabla_\mu \partial_\nu \phi^i = U^{\mu \nu} (\partial_\mu \partial_\nu \phi^i - \Gamma^i_{\mu \nu} \partial_\sigma \phi^i) = 0$$

(13)

in terms of the projector $U_{\mu \nu} = G_{\mu \nu} - P_{\mu \nu}$ with $P_{\mu \nu} \equiv h_{ij} \partial_\mu \phi^i(X) \partial_\nu \phi^j(X)$, satisfying the properties $U^\mu_{\nu} \partial_\mu \phi^i(X) = 0$ and $U^\mu_{\nu} U^\nu_{\rho} = U^\mu_{\rho}$.

It is just the contraction with the projector $U$ instead of the metric $G$ in (13) which makes that laplacian invariant under field redefinitions, at the same time that the own dependence of $U$ on the fields makes the equations highly non-linear.

\footnote{The indices are lowered (raised ) with the metric $G_{\mu \nu}$ ($G^{\mu \nu}$)}
3 The Duality with p-branes

Now we want to see what are the conditions imposed by the field equations (13) in one embedding $X^\mu (\varepsilon^a)$ for the hypersurface $\Sigma$ defined in (8). Therefore, the fundamental relation between the scalar fields and the embedding is:

$$\phi^i (X^\mu (\varepsilon^a)) = 0$$  \hfill (14)

which must be understood as a set of equations for the embedding once the scalar fields are given.

Taking derivatives with respect to $\varepsilon^a$:

$$\frac{\partial X^\mu}{\partial \varepsilon^a} \partial_\mu \phi^i (X(\varepsilon)) = 0$$  \hfill (15)

which immediately implies that $U^\Sigma$ annihilates $\partial_\mu \phi^i (X)$ when the last one is evaluated on $\Sigma$, and, at the same time $U$ evaluated over $\Sigma$ is the identity for $\partial_a X^\mu$:

$$U^\Sigma_{\nu} \mu (\partial_\mu \phi^i (X))_{\Sigma} = 0$$  \hfill (16)

$$\nu (U^\mu)_{\Sigma} \partial_\nu X^\mu = \partial_a X^\mu$$  \hfill (17)

This allow us to identify $U$ evaluated on $\Sigma$ with $U^\Sigma$:

$$U^\Sigma \mu \nu = (U^{\mu \nu})_{\Sigma}$$  \hfill (18)

Therefore, we can relate the field equations (13) with the p-brane equations (6) for the embedding, taking account (13) and (18), in the following way:

---

5 A circle of radius 1 in two dimensions is described by means of the relation $\phi (X, Y) = X^2 + Y^2 - 1 = 0$. A solution for (14) is the embedding $X = \cos \theta$ and $Y = \sin \theta$, which parametrizes the surface (curve) $\phi (X, Y) = 0$ with $\theta$ running from 0 to $2\pi$, and both points identified.

6 From now, when an object, say $\eta (X^\mu)$, is evaluated on the hypersurface $\Sigma$, we write $\eta_{\Sigma}$.

7 Here without the d-form.
\[(D_\mu \partial^\mu \phi^j)_{\Sigma} = (U^{\mu\nu} \nabla_\mu \partial_\nu \phi^j)_{\Sigma} = \]
\[\gamma^{ab} \partial_a X^\mu \partial_b X^\nu (\partial_\mu \partial_\nu \phi^i - \Gamma^\sigma_{\mu\nu} \partial_\sigma \phi^i) = \]
\[\gamma^{ab} \frac{\partial}{\partial \varepsilon} \{ \partial^a X^\mu \partial^b \phi^i (X(\varepsilon)) \} - \partial_\mu \phi^i \gamma^{ab} \partial_a \partial_\varepsilon X^\mu - \gamma^{ab} \Gamma^\rho_{\mu\nu} \partial_\rho \phi^i \partial_a X^\mu \partial_b X^\nu = \]
\[-(\partial_\mu \phi^i (X))_{\Sigma} \Delta X^\mu \]
(19)

But using \( P_{\mu\nu} \equiv h_{ij} \partial_\mu \phi^i (X) \partial_\nu \phi^j (X) \), we finally obtain :
\[(h_{ij} \partial_\nu \phi^j (X) D_\mu \partial^\mu \phi^i)_{\Sigma} = -P^\Sigma_{\nu\mu} \Delta X^\mu = 0 \]
(20)

i.e., the equations for the scalar fields \( \phi^i (X) \) evaluated over the hypersurface \( \Sigma \) defined as \( \phi^i (X) = 0 \) for all \( i \), imply the p-brane equations \( \Box \) for any embedding \( X^\mu (\varepsilon^a) \) of \( \Sigma \). Following the same criteria it is easy to show that the converse is true : the p-brane equations for the embedding of some hypersurface \( \Sigma \) imply the field equations (13) for any set of scalar fields satisfying (14).

\[(D_\mu \partial^\mu \phi^i)_{\Sigma} = 0 \iff P^\Sigma_{\nu\mu} \Delta X^\mu = 0 \]
(21)

In addition to the symmetries already mentioned, the dynamic for the scalar fields presents a trivial one under global translations :
\[\phi^i (X^\mu) \longrightarrow \phi^i (X^\mu) + C^i \]
(22)

Therefore every solution is not representing a unique p-brane, but the splitting of the entire manifold \( \mathcal{M} \) into a continuous of p-branes \( \phi^i (X^\mu) = C^i \) every one of them parametrized by the set \( C^i \) from \( i = 1 \) to \( \tilde{d} \). It is a fluid of classical p-branes.

As a very simple example of this reformulation, it can be easily shown that, in flat space-time, the scalar fields
\[\phi^i (X) = \phi^i (A^i_\mu X^\mu) \]
(23)

8
are always a solution of the field equations (13) for any functional form \( \phi^i(Y^j) \), such that \( (\det \frac{\partial \phi^i}{\partial Y^j}) \neq 0 \) provided that the constant matrix \( A^i_\mu \) satisfies \( \det(A^i_\mu A^j_\mu) \neq 0 \). The last condition is equivalent to the requirement that \( A^i_\mu X^\mu = C^i \) define a \( d \)-hyperplane, which is the obvious \( d \)-brane in flat space.

4 The Elementary Action and the off-shell Duality

Up to now we have been able to build a dynamic for the scalar fields based upon the symmetry guaranteeing their interpretation as representing an extended object (or a continuous of them). Moreover we have proved that those extended objects naturally induced by the fields, are p-branes. Now we are in searching of the minimal action principle reproducing the dynamic we have found. Even more, we are interested in cheking the off-shell equivalence, i.e., the relation between that action and the p-brane action (1), using (14) as a kinematic relation but without the use of the equations of motion.

At first sight, we are in trouble because we find no first-derivative invariants under both, target-space diffeomorphisms and field redefinitions. However, the lagrangian density \( \mathcal{L} = \sqrt{G} h \), with \( G = \det(G_{\mu\nu}) \) and \( h \equiv \det(h^{ij}) \) has the right variational derivative\(^8\):

\[
\frac{\delta \sqrt{G} h}{\delta \phi^i} = \sqrt{G} h h_{ij} U^{\mu\nu} \nabla_\mu \partial_\nu \phi^j = \sqrt{G} h h_{ij} D_\mu \partial^\mu \phi^j
\] (24)

Although the corresponding action is invariant under target-space diffeomorphisms, it is not the case for the redefinition of fields :

\[
(\sqrt{G} h)' = | det(\frac{\partial \phi^i'}{\partial \phi}) | \sqrt{G} h
\] (25)

In [10] it is shown how this kind of transformation is intimately related with the non-uniqueness of the lagrangian giving rise to the field equations. In our case, the identity

\[
\delta S = \partial_\mu \left( \frac{\partial L}{\partial \phi^i} \right)
\]
\[ \partial_\mu \phi^i (X) \frac{\partial \sqrt{G h}}{\partial (\partial_\mu \phi^j)} = \delta^i_j \sqrt{G h} \] (26)

ensures that the lagrangian \( L_F = F(\phi) L = F(\phi) \sqrt{G h} \) produces the same field equations independent of the arbitrary function \( F(\phi^1, ..., \phi^{\tilde{d}}) \), which therefore acts as "constant" in the variational derivative:

\[ \frac{\delta (F(\phi) \sqrt{G h})}{\delta \phi^i} = F(\phi) \frac{\delta \sqrt{G h}}{\delta \phi^i} \] (27)

If we want to describe an elementary object, instead of a fluid of them, the solution to the lagrangian ambiguity comes from restricting the integration just to the elementary object \( \Sigma \). The action is fixed to be:

\[ S^G_{\phi} = \int_M d^D X \sqrt{G h} \prod_{i=1}^{\tilde{d}} \delta (\phi^i (X)) \] (28)

Therefore, the arbitrariness in the function \( F(\phi^1, ..., \phi^{\tilde{d}}) \) is trivial \((F(\phi^1, ..., \phi^{\tilde{d}}))|_{\Sigma} = F(0, 0, ..., 0))\) and moreover, the presence of the delta functions makes the action invariant under field redefinitions, due to the identity

\[ | det \left( \frac{\partial \phi^j}{\partial \phi^i} \right) | \prod_{i=1}^{\tilde{d}} \delta (\phi^i (\phi^j (X))) = \prod_{i=1}^{\tilde{d}} \delta (\phi^i (X)) \] (29)

that, of course, it must be understood within the integral. Then, the Dirac's deltas transformation compensates the one for \( \sqrt{G h} \), and the action \( S^G_{\phi} \) is now invariant under field redefinitions.

The next step consist on showing the off-shell identity between the action (28) and the p-brane action (1). For that, we solve the set of equations \( \phi^i (X^\mu) = 0 \) defining \( \Sigma \) splitting locally the coordinates \( X^\mu = \{ X^a, X^i \} \) \((i = 1 \text{ to } \tilde{d} \text{ and } a = \tilde{d} + 1 \text{ to } D)\) to write the solution in the way \( X^i = X^i (X^a) \). Of course, this splitting guarantees that we can choose locally the gauge \( X^a = \varepsilon^a \) in the world-volume. In this gauge (13) implies \((\partial_a \phi^i)_{\Sigma} = -\partial_a X^j (\partial_j \phi^i)_{\Sigma} \) which allow to get the fundamental relation :

\( ^9 det \left( \frac{\partial \phi^j}{\partial \phi^i} \right)_{\Sigma} \neq 0 \) in this splitting.
\[
\left( \det \left( \frac{\partial X^i}{\partial \phi^j} \right) \right)^2 G h)^\Sigma = \gamma
\]  
(30)

obtained with the help of the identity

\[
det \{ 1 + A_2 B_1 + B_2 A_1 + A_2 (1 + B_1 B_2) A_1 \} = \\
det \{ 1 + B_1 A_2 + A_1 B_2 + A_1 (1 + B_2 B_1) A_2 \} =
\]  
(31)

where \( A_1, B_1 \) are arbitrary \( d \times \tilde{d} \) matrices and \( A_2, B_2 \) are arbitrary \( \tilde{d} \times d \) matrices.\(^{10}\)

Now, using (29), the integration in the \( X^i \) can be made resulting in the \( X^i = X'^i(X^a) \) restriction and a jacobian \( | \det (\partial X^i/\partial \phi^j) | \):

\[
S^G_\phi = \int d^dX \left( \int \prod_{i=1}^{\tilde{d}} \delta(\phi^i(X)) dX^i \sqrt{G h} \right) = \\
\int d^dX \left( \left| \det (\partial X^i/\partial \phi^j) \right| \sqrt{G h} \right)_{X^i = X'^i(X^a)} = \int d^dX \sqrt{\gamma}
\]  
(32)

where \( d^dX = \prod_{a=1}^{d} dX^a \). But we are in the \( X^a = \varepsilon^a \) gauge, so both actions are off-shell the same:\(^{\text{11}}\)

\[
S^G_\phi = \int_\mathcal{M} d^D X \sqrt{G h} \prod_{i=1}^{\tilde{d}} \delta(\phi^i(X)) = \int \Sigma d^d\varepsilon \sqrt{\gamma} = S^G_p
\]  
(33)

This proof does not use any privileged choice of coordinates in the target-space but it uses a privileged one in the world-volume. However, any other choice in the world-volume can be reached doing the target-space diffeomorphism \( X^a \rightarrow X'^a(X^b) \) and \( X^i \rightarrow X^i \) before solving the \( \phi^i(X^\mu) = 0 \) equations. Then, the procedure applies again to get (33).

\(^{10}\)This identity can be obtained from the determinant of the auxiliar \( D \times D \) matrix

\[
\begin{pmatrix}
(1 + B_1 B_2)^{-1} & -(A_1 + (1 + B_1 B_2)^{-1} B_1) \\
(A_2 + B_2 (1 + B_1 B_2)^{-1}) & (1 + B_2 B_1)^{-1}
\end{pmatrix}
\]
5 The Electromagnetic Contribution

In this section we will see how represent, in this dual formulation, the electromagnetic contribution, i.e., the coupling with the d-form $A(X)_{\nu_1,...,\nu_d}$. In this case, the invariance under redefinitions of the fields, diffeomorphisms in the target-space, and the gauge invariance $A(X)_{\nu_1,...,\nu_d} \rightarrow A(X)_{\nu_1,...,\nu_d} + \partial_{[\nu_1} \lambda_{\nu_2,...,\nu_d]}$, provides the natural candidate for the action:

$$S^A_\phi = \frac{1}{d!} \int_\mathcal{M} d^DX \prod_{i=1}^{d} \delta(\phi^i(X)) \epsilon^{\nu_1,...,\nu_d\mu_1,...,\mu_d} A(X)_{\nu_1,...,\nu_d} \prod_{j=1}^{d} \partial_{\mu_j} \phi^j(X) \tag{34}$$

Now, the d-form’s contribution to the classical equations is

$$\frac{\delta S^A_\phi}{\delta \phi^i} = \frac{1}{(d+1)!} F_{\mu\nu_1,...,\nu_d} \epsilon^{\nu_1,...,\nu_d\mu_1,...,\mu_d} \prod_{j=1,j\neq i}^{d} \partial_{\mu_j} \phi^j(X) \tag{35}$$

To evaluate the implications of this contribution when restricted to the $\Sigma$ hypersurface, i.e., the implications for the embedding $X^{\mu}(\varepsilon^a)$, we must take account the closure relation derived from (18)

$$\delta \nu^\mu = P^\nu_\mu + U^{\Sigma \nu} = h_{ij} \partial_{\nu^i} \phi^i \partial^{\mu} \phi^j + \gamma^{ab} \partial_a X^\nu \partial_b X^\rho G_{\rho\mu} \tag{36}$$

over the free indices in $\epsilon^{\nu_1,...,\nu_d\mu_1,...,\mu_d} F_{\mu_1,...,\nu_d}$ and the kinematical relation $J^2 G = \gamma h$ on $\Sigma$, with $J = det{\{\partial_{\nu^a}, \partial^{\mu} \phi^j\}}$. Finally we get:

$$\left(\frac{\delta S^A_\phi}{\delta \phi^i}\right)_\Sigma = (-1)^J \left(\frac{G h}{\gamma}\right)^{\frac{1}{2}} h_{ij} \partial^{\mu} \phi^j F_{\mu\nu_1,...,\nu_d} \prod_{a=1}^{d} \partial_a X^{\nu_a} \tag{37}$$

where $(-1)^J$ is the sign of the $J$ determinant. Choosing the orientation of the $\partial \phi^i$ vectors in such a way that $(-1)^J = 1$ and, then, joining equations (19), (24) and (37) we obtain one of the main results:

$$\left(\frac{\delta S^A_\phi}{\delta \phi^i} \partial_\mu \phi^j\right)_\Sigma = \left(\frac{\delta(S^A_\phi + S^A_\phi)}{\delta \phi^i} \partial_\mu \phi^j\right)_\Sigma = \tag{38}$$
\[
\frac{G h}{\gamma} \left( -\sqrt{\gamma} P_{\mu} \Delta X^{\mu} + F_{\nu_1 \ldots \nu_d} \prod_{a=1}^{d} \partial_a X^{\nu_a} \right) = \quad (39)
\]

\[
= - \left( \frac{G h}{\gamma} \right)^{\frac{1}{2}} \frac{\delta S_p}{\delta X^\nu} \quad (40)
\]

Therefore, the field equations for the scalar fields, including the electromagnetic contribution, imply, when evaluated on the hypersurface where all the fields vanish, the p-brane equations for any embedding of that hypersurface. The converse is also true. It is relevant to note that the classical scalar fields couple with the Hodge dual of the electromagnetic field strength \( \mathcal{B} \). Then, the relation between both brane formulations is an electric-magnetic duality.

Following the same steps as in the preceding section, it can be shown that the electromagnetic contribution \( \mathcal{B} \) reduces off-shell to the p-brane electromagnetic action \( \mathcal{L} \): first, we solve the \( \phi^i(X^\mu) = 0 \) equations splitting the coordinates in such a way that the solution is \( X^i = X^i(X^a) \). Second, we choose the gauge \( X^a = \varepsilon^a \) in the world-volume, third, we integrate in the transverse coordinates with the net effect of giving a Jacobian \( | \det \left( \frac{\partial X^i}{\partial \phi^j} \right) | \) and the restriction \( X^i = X^i(X^a) \), and fourth, we use the closure relation \( \mathcal{G} \) :

\[
S_\phi^A = \frac{1}{\Gamma} \int d^d X \left( \int \prod_{i=1}^{\tilde{d}} \delta\left( \phi^i(X) \right) dX^i A(X)_{\nu_1 \ldots \nu_d} \varepsilon^{\nu_1 \ldots \nu_d \mu_1 \ldots \mu_\tilde{d}} \prod_{j=1}^{\tilde{d}} \partial_{\mu_j} \phi^j(X) \right) = \quad (41)
\]

and the kinematic identities, say, \( | \det \left( \frac{\partial X^i}{\partial \phi^j} \right) | \frac{\mathcal{G} J}{\gamma} = 1 \) \(^{11}\), providing the off-shell equality with the p-brane action :

\[
S_\phi^A = \int_M d^D X \prod_{i=1}^{\tilde{d}} \delta\left( \phi^i(X) \right) \varepsilon^{\nu_1 \ldots \nu_d \mu_1 \ldots \mu_\tilde{d}} A(X)_{\nu_1 \ldots \nu_d} \prod_{j=1}^{\tilde{d}} \partial_{\mu_j} \phi^j(X) = \int_{\Sigma} d^d \varepsilon A(X)_{\nu_1 \ldots \nu_d} \prod_{a=1}^{d} \partial_a X^{\nu_a} = S_p^A \quad (42)
\]

\(^{11}\)Choosing again the \((-1)^{p+1} = 1\) orientation.
As a simple example of this formulation, we get the existence of classical 
\((D - 2)\)-spherical-branes in D-dimensional euclidean flat space. Obviously, 
this is possible thanks to the presence of a certain electromagnetic field:

\[ A(X)_{\nu_1,\ldots,\nu_d} = \frac{1}{r (D - 1)!} \epsilon_{\nu_1,\ldots,\nu_d\mu} X^\mu \]  \hspace{1cm} (43)\]

with \(X^\mu\) cartesian coordinates and the radius \(r \equiv (X^\mu X^\mu)^{\frac{1}{2}}\). This \((D - 1)\)-
form allows the existence of the spherical solution \(\phi = \phi(X^\mu X^\mu)\) (here \(\tilde{d} = 1\)).
The electromagnetic contribution compensates the non null one coming from
the coupling with the flat metric:

\[ \frac{\delta \sqrt{G} \, h}{\delta \phi} = \frac{(D - 1)}{r} \]  \hspace{1cm} (44)\]

As we can see, (44) is independent of the functional form \(\phi(r^2)\). Finally, 
\(\phi(X^\mu X^\mu) = 0\) describes the \((D - 1)\)-sphere \(r^2 = r_0^2\), now becoming a classic 
\((D - 2)\)-brane in flat space.

\section{Redefinition invariance and branes}

The purpose of this section is to exhibit how the brane emerges in a natural
way from the field theory, just due to the invariance under the redefinition
of the fields (6). In this context, the reparametrization invariance in the
world-volume, exists as a result of the general covariance under target-space
diffeomorphisms.

Let us start with the most general action for the set of \(\tilde{d}\) scalar fields 
\(\phi^i(X)\), with invariance under field redefinitions \(\phi^i(X) \rightarrow \phi^i(\phi^j(X))\) and
being a candidate to describe an elementary extended object \(\phi^i(X) = 0\) :

\[ S_{\phi} = \int_M d^D X \prod_{i=1}^{\tilde{d}} \delta(\phi^i(X)) \mathcal{L}_\phi \]  \hspace{1cm} (45)\]

The restriction over \(\phi^i = 0\) reduces the \(\phi\)-dependence in the lagrangian
density to be \(\mathcal{L}_\phi = \mathcal{L}(\partial_\mu \phi^i)\). Moreover, the invariance of the action under
field redefinitions forces the transformation
\[ \mathcal{L}_{\phi'} = \text{det}(\frac{\partial \phi'}{\partial \phi})\mathcal{L}_{\phi} \] (46)

because of the delta functions transformation (27). As was proved in (10), is just this covariance which guarantees the covariance of the field equations derived from that lagrangian.

Let us now assume that we solve the algebraic equations \( \phi^i(X) = 0 \), defining the hypersurface \( \Sigma \), in arbitrary coordinates through the split \( \{X^i, X^a\} \) with \( a = \hat{d} + 1 \) to \( D \), in the way \( X^i = X^i(X^a) \). Choosing the gauge \( X^a = \varepsilon^a \) in the world-volume, we have seen that (13) explicitly gives \( (\partial_a \phi^i)_{\Sigma} = -\partial_a X^j(\partial_j \phi^i)_{\Sigma} \). Using that, we can rewrite \( (\partial^a \phi^i)_{\Sigma} = M^j_i \partial^a \phi^j \), where the barred field configuration is \( \bar{\phi}^i = X^i - X^i(X^a) \) and the matrix \( M^j_i = \frac{\partial \phi^j}{\partial X^i} \). The point is that (16) implies \( \mathcal{L}_{\phi}(M^j_i \partial^a \phi^j) = \text{det} M \mathcal{L}_{\phi}(\partial^a \phi^j) \), for arbitrary matrix \( M^j_i \), so we have:

\[ S_{\phi} = \int_M d^D X \prod_{i=1}^{\hat{d}} \delta(\phi^i(X)) (\text{det} \frac{\partial \phi^i}{\partial X^j}) \mathcal{L}_{\phi}(-\partial_a X^i, \delta^i_j) \] (47)

but \( | \text{det} \frac{\partial \phi^k}{\partial X^j} | \prod_{i=1}^{\hat{d}} \delta(\phi^i(X)) = \prod_{i=1}^{\hat{d}} \delta(X^i - X^i(X^a)) \), and we can integrate in the \( X^i \) transverse coordinates, letting an action in which the explicit dependence on the scalar fields has totally disappeared in favor of the embedding’s \( X^\mu(\varepsilon^a) \) dependence, in the gauge \( X^a = \varepsilon^a \). Therefore, the starting action \( S_{\phi} \) can be written, in a general way, only in terms of the embedding of the hypersurface

\[ S_{\phi} = \int_{\Sigma} d^d \mathcal{L}_{\phi}(-\partial_a X^i, \delta^i_j) \equiv \int_{\Sigma} d^d \varepsilon \mathcal{L}_X = S_X \] (48)

12 It must be stressed that, in general, this barred configuration cannot reached by field redefinitions, although it seems to be the simplest one describing the extended object. If we try to introduce the barred configuration into the field equations in, for example, (13), we see that the equations are inconsistent, except in the \( X^i = X^i(X^a) \) hypersurface in which case they become the p-brane equations in the \( X^a = \varepsilon^a \) gauge. As we will see in this section, this is no a coincidence, but a consequence of the invariance under field redefinitions.

13 We have seen in the last section that there is an orientation which must be choosen. This possibility is ruled out when the lagrangian transforms as \( \mathcal{L}_{\phi'} = | \text{det} \frac{\partial \phi'}{\partial \phi} | \mathcal{L}_{\phi} \), as in (23).
or, in other words, what we have done is to obtain the world-volume
gauge fixed brane lagrangian $L_X$ from the target-space one $L_\phi$, in the way:

\begin{equation}
L_X(X^a = \varepsilon^a, \partial_a X^i) = L_\phi(\phi^i = \tilde{\phi}^i) = \\
= L_\phi(\partial_a \phi^i = -\partial_a X^i, \partial_j \phi^i = \delta_j^i)
\end{equation}  \hfill (49)

and that works only with the help of the invariance under redefinitions,
it doesn’t matter the particular details of the $\phi$-dependence. Those details
are, of course, relevant if we are interested in knowing what brane theory we
are dealing with, as it is the case in this paper. Moreover, the invariance of
the starting action under target-space diffeomorphisms ensures that $L_X$ is
the gauged-fixed action of a truly target-space and world-volume invariant
one.

More important to us is the converse problem, i.e., the derivation of the
target-space field theory once the brane one is already given. From the above
considerations, it is easy to realize that the invariance under field redefinitions
solves again the problem. In our split \{\(X^i, X^a\)\}, we can write:

\begin{equation}
L_\phi(\partial_\mu \phi^i) = L_\phi(\partial_a \phi^i, \partial_j \phi^i) = \\
= (\det \frac{\partial \phi^j}{\partial X^i}) L_\phi(\frac{\partial X^i}{\partial \phi^j}, \frac{\partial \phi^i}{\partial X^a}, \delta^i_j)
\end{equation}  \hfill (50)

But using (49) we get the final result:

\begin{equation}
L_\phi(\partial_\mu \phi^i) = (\det \frac{\partial \phi^i}{\partial X^a}) L_X(\varepsilon^a = X^a, \partial_a X^i = -\frac{\partial X^i}{\partial \phi^j} \frac{\partial \phi^i}{\partial X^a})
\end{equation}  \hfill (51)

Of course, the split of coordinates \{\(X^a, X^i\)\} breaks the manifest co-
variance under target-space diffeomorphisms, but if the original brane la-
grangian is properly built, this breaking is only apparent. Let’s do a very sim-
ple example; the coupling between the 0-brane and the electromagnetic field
is given by the term $L_X A_\mu \dot{X}^\mu = A_0(\tau, X^i(\tau)) + A_i(\tau, X^i(\tau)) \dot{X}^i$,
which is already written in the gauge $X^0 = \tau$ and with $\dot{X}^\mu = \frac{dX^\mu}{d\tau}$. Following
the prescription (51) the corresponding term in the target-space field theory will be:

\[ L_A^\phi(\partial_\mu \phi^i) = (\det \frac{\partial \phi^i}{\partial X^j})(A_0(X^0, X^k) - A_i(X^0, X^j) \frac{\partial X^i}{\partial \phi^j} \frac{\partial \phi^j}{\partial X^0}) \] (52)

which, in fact, has the target-space manifest covariant form:

\[ L_A^\phi(\partial_\mu \phi^i) = A_\mu(X)\epsilon^{\mu,\nu_1...\nu_{D-1}} \partial_{\nu_1} \phi^1 \times ... \times \partial_{\nu_{D-1}} \phi^{D-1} \] (53)

Of course, this example agrees with (34) in the case \( d = 1 \). Even more, the formula (51) will be used in the next section to get the D-brane dual target-space lagrangian.

So far, we have elucidated the relation between the lagrangians for both formulations of the same extended object. To end this section, we give the explicit formula relating both dynamics, and again, obtained with the extensive help of the invariance under field redefinitions, i.e., via (46) and its implications.

Given the lagrangian for our field theory \( L_\phi \) satisfying the covariance transformation (46) under field redefinitions, and given the gauge fixed brane lagrangian \( L_X \) in terms of it via (49), it can be shown that the field equations for both theories are related by the expression:

\[ (\frac{\delta S_\phi}{\delta \phi^i})_\Sigma = -(\det(\frac{\partial \phi^i}{\partial X^j}) \frac{\delta S_X}{\delta X^j})_\Sigma \] (54)

If the target-space action is properly built (invariance under space-time diffeomorphisms and of course, field redefinitions), the brane theory built from it through (43) will have the correspondent invariance under world-volume reparametrizations. In that case, the \( \frac{\delta S_X}{\delta X^a} = 0 \) equations imply the remaining \( \frac{\delta S_X}{\delta X^a} = 0 \) ones, due to the identity \( \frac{\delta S_X}{\delta X^a} \frac{\partial X^a}{\partial \phi^i} = 0 \), forced by the reparametrization invariance. This identity allow us to write the on-shell equivalence relation (54) in the final full covariant form:

\[ (\frac{1}{L_\phi} \frac{\delta S_\phi}{\delta \phi^i} \partial_\mu \phi^i)_\Sigma = -\frac{1}{L_X} \frac{\delta S_X}{\delta X^a} \] (55)
This identity was explicitly obtained in the last sections (38) for the p-branes. This was made without fixing any gauge in the world-volume, but paying attention to the particular form of the actions we worked with. Now the formula (55) was derived just with the help of the invariance under field redefinitions, no matter the specific form of the action is. From the point of view of this general approach, the only work we should do in trying to get the dual version of a brane theory, is to apply the formula (51) and then (55) satisfies identically, providing the classical equivalence.

7 The D-brane’s dual model

After the detailed calculations for the p-branes have been reproduced in the previous sections, and the general features of the world-volume/target-space dualization procedure have been elucidated in the last section, a few remarks concern for the Dirichlet-branes. The formula (51) gives the target-space action in terms of the DBI one, which results to be the (28) action with the replacement \( G_{\mu\nu} \to (G + B)_{\mu\nu} \equiv Q_{\mu\nu} \). The dual version of the electromagnetic contribution is the same as in the fifth section. The only difference, irrelevant to this dualization procedure, is that the d-form is given by a certain combination of the R-R forms and the NS-NS two-form.

To organize our results in such a way that the covariance under field redefinitions is manifest, we follow the same line as in the p-brane case. The main difference is the technical complications just due to the NS-NS two form \( B_{\mu\nu} \) in the DBI-action and in the (28)-type action. Now, the natural first-derivative target-space scalar (7) has an antisymmetric part, which is encoded in:

\[
h_{ij}^{\pm}(X) = Q(X)_{\mu\nu}^{\pm} \partial_{\mu}\phi^{i}(X)\partial_{\nu}\phi^{j}(X)
\]

where we define \( Q_{\mu\nu}^{\pm} \equiv (G \pm B)_{\mu\nu} \), and \( Q_{\mu\nu}^{\pm}Q_{\nu\sigma}^{\pm} = \delta_{\mu}^{\sigma} \). (56) imply the existence of two new privileged connections under field redefinitions:\[\]

\[
(\Omega^{\pm}_{\mu})_{i}^{j} \equiv \Omega^{\pm}_{\nu i}^{j} = \nabla_{\nu} \partial_{\sigma} \phi^{j} Q_{\sigma \nu}^{\pm} \partial_{\rho} \phi^{i} h_{\nu i}^{\pm}
\]

\[14\] \( h_{ik}^{\pm}h_{kl}^{\pm} = \delta_{j}^{i} \) and the target-space covariant derivative \( \nabla_{\mu} \) continues being constructed with the Levi-Civita connection.
with the corresponding covariant derivations:

\[ V'^i = \frac{\partial \phi'}{\partial \phi} V^k \to D^\pm V^i \equiv \partial_\mu V^i - \Omega^\pm_{\mu j} V^j \to D^\pm V'^i = \frac{\partial \phi'}{\partial \phi} D^\pm V^k \]
\[ V'_i = \frac{\partial \phi'}{\partial \phi} V_j \to D^\pm V_i \equiv \partial_\mu V_i + \Omega^\pm_{\mu i} V_j \to D^\pm V'_i = \frac{\partial \phi'}{\partial \phi} D^\pm V_j \quad (58) \]

But this plurality of connections prevents the existence of a privileged dynamic. Therefore, the straight way is to use the formula (51) giving the lagrangian for the target-space scalar fields in terms of the D-brane one contained in the DBI-action:

\[ S_{DBI}^Q = \int_\Sigma d^d \varepsilon \sqrt{\gamma^+} \quad (59) \]

where \( \gamma^\pm_{ab} = Q^\pm_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \) is the induced tensor and \( \gamma^+ \) its determinant. The result is the generalization of the action (28),

\[ S_{\phi}^Q = \int_{\mathcal{M}} d^d X \sqrt{Q^+ h^+} \prod_{i=1}^d \delta(\phi^i(X)) \quad (60) \]

where \( Q^+ \equiv det Q^+_{\mu\nu} = det Q_{\mu\nu} \) and \( h^+ \equiv det h^+_{ij} \). The field equations are:

\[ \delta S_{\phi}^Q / \delta \phi^i = \sqrt{Q^+ h^+} \left( \partial_\mu \phi^i j \left( h^+ + h^- \right)_{ij} Q^\alpha_+ D^+_\alpha \partial_\beta \phi^j - \frac{1}{4} (U^\rho_\mu - U^\rho_\mu) \left( \Omega^\pm_{\rho k} - \Omega^\mp_{\rho k} \right) + U^{\lambda \rho} \left( -H_{\rho \lambda \mu} + \frac{1}{2} U^\sigma_\mu \left( \nabla_\rho B_{\sigma \lambda} - \frac{1}{2} \nabla_\sigma B_{\rho \lambda} \right) + \frac{1}{2} U^\mu_\sigma (\nabla_\lambda B_{\rho \sigma} - \frac{1}{2} \nabla_\sigma B_{\rho \lambda}) \right) \right) \]

\[ \stackrel{\delta S_{\phi}^Q}{\delta \phi^i} = \frac{1}{2} \left( \partial_\mu B_{\nu \rho} + \partial_\nu B_{\mu \rho} + \partial_\rho B_{\mu \nu} \right) \quad (61) \]

\footnote{H_{\mu\nu\rho} \equiv \frac{1}{2} \left( \partial_\mu B_{\nu \rho} + \partial_\nu B_{\mu \rho} + \partial_\rho B_{\mu \nu} \right).}
comparison with the equations for the D-brane’s embedding coming from the DBI-action \(^1\). The main result is again the identity:

\[
\left( \frac{1}{\sqrt{Q^+h^+}} \frac{\delta S_Q^\phi}{\delta \phi^i} \delta \phi^i \right)_\Sigma = - \frac{1}{\sqrt{\gamma^+}} \frac{\delta S_Q^{DBI}}{\delta X^\mu} \tag{62}
\]

on \(\Sigma\) providing the on-shell equivalence of both formulations. Finally, the construction via the formula (51) trivially implies the off-shell identity between the DBI-action and the (60) one.

\[
S_Q^\phi = S_Q^{DBI} \tag{63}
\]

Again, because of the invariance under global translations of the field equations (61), every solution \(\phi^i(X)\) splits the whole space-time into a continuous of D-branes, a fluid of classic D-branes.

8 Conclusions

In this work, the equations for a set of target-space scalar fields \(\phi^i(X^\mu)\) for \(i = 1\) to \(\tilde{d}\), are built in such a way that they have a direct interpretation as representing a continuous of \(d\)-dimensional \((d + \tilde{d} = D)\) extended objects \(\Sigma_{\phi^i}\), everyone of them implicitly defined as the collection of the target-space points where the scalar fields acquire a given constant value, i.e., through the condition \(\phi^i(X^\mu) = \phi^i_0\). The fundamental criteria for getting the dynamic and relating it with branes, is the invariance under field redefinitions of the scalar fields. Even more, it is shown in detail how these extended objects are classical p-branes or D-branes, depending on what the field theory it is, then, fully supporting the construction. After the proof of the on-shell equivalence with p(or D)-branes have been made, the right actions providing the field equations and transforming nicely under the field redefinitions are found. These actions correspond to an elementary extended object instead of a continuous of them, and it is shown how it reduces off-shell (kinematically) to the p-brane and D-brane actions respectively. Therefore they are dual formulations of p(or D)-branes. The classical equivalence is of the electromagnetic duality-type, because the dual fields couple classically to the dual

\[^1\text{See Appendix}\]
(in the Hodge sense) field strength. The elementary feature of the actions makes harmless the non-uniqueness of the lagrangian density, then giving a sense for this ambiguity, at the same time that it is the responsible for the invariance of the action.

Moreover, it is shown how the elementarity and the invariance under field redefinitions are the basic properties allowing the generic exchange between the target-space \(\phi\)-degree of freedom and the embedding ones in the world-volume. The explicit expressions relating the lagrangians (and the dynamics) of both, the target-space and the world volume theories are worked out just with the help of the invariance under field redefinitions, which results to be the basic ingredient of this dualization procedure.

All this questions are solved in arbitrary space-time geometry, dimension, electromagnetic d-form field and dimensionality of the brane.

Acknowledgements. I am grateful to Enrique Alvarez, César Gómez and Pedro Silva by their comments. This work was supported by a Comunidad Autónoma de Madrid grant.

A Appendix: The D-brane Equations

In this appendix we are interested on getting the equations for the D-brane’s embedding. The main difference with the p-brane case lies in the presence of the NS-NS two-form. Then, we will only evaluate the kinetic part coming from the DBI-action defined in the last section (59). The task is to put in order the equations following two criteria: the explicit world-volume and target-space covariance, and our intention to compare with the field equations (61).

In order to understand the covariance structure of the equations let us start looking over the p-brane ones. Given the embedding \(X^\mu(\varepsilon)\) of a hypersurface \(\Sigma\) defined in a Riemannian manifold \(M\), we can always define the pullback of the (compatible with the metric) connection\(^{17}\) as \(\Gamma^\nu_{a\mu} \equiv \partial_a X^\rho \Gamma^\nu_{\rho\mu}\), and the world-volume connection as \(\gamma^c_{ab} \equiv \gamma^{cd} \partial_d X^\nu G_{\nu\mu} \nabla_a \partial_b X^\mu\), where \(\gamma_{ab}\) is the induced metric and \(\nabla_a \partial_b X^\mu = \partial_a \partial_b X^\mu - \Gamma^\mu_{\alpha\beta} \partial_a X^\alpha \partial_b X^\beta\) is the target-space (although no world-volume) embedding’s covariant derivative\(^{18}\). Then, a

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\(^{17}\)The Levi-Civita one throughout this paper.

\(^{18}\)It can be checked that the world-volume connection coincides with the Levi-Civita one of the induced metric \(\gamma\) always than the target-space one be the Levi-Civita connection.
simple way to prescribe a target-space and world-volume covariant dynamic for the embedding is through:

$$\gamma^{ab} D_a \partial_b X^\mu \equiv \gamma^{ab} (\partial_a \partial_b X^\mu + \Gamma^\mu_{\alpha\sigma} \partial_\alpha X^\nu - \gamma^{c\nu} \partial_c X^\mu) = 0$$

(64)

which, after a simple calculation results to be the p-brane equation without electromagnetic field (3), i.e.,

$$\gamma^{ab} D_a \partial_b X^\mu = \gamma^{ab} P_{\mu}^\rho \nabla_\rho \partial_b X^\nu.$$ 

Now, the NS-NS two-form allows two new different possibilities for a world-volume connection:

$$\gamma^{\pm c} \equiv \gamma^{cd} \partial^d X^\nu Q^{\pm \nu} \nabla_a \partial_b X^\mu$$

(65)

and the corresponding target-space/world-volume covariant derivations:

$$D^\pm_a \partial_b X^\mu \equiv \partial_a \partial_b X^\mu + \Gamma^\mu_{\alpha\nu} \partial_\nu X^\alpha - \gamma^{\pm c} \partial_c X^\mu$$

(66)

Now we are ready to structurate the D-brane equations, which, moreover will be specially adapted to compare with the field equations (61) when evaluated on $\Sigma$:

$$\frac{\delta S_{DBI}}{\delta X^\mu} = \sqrt{\gamma} \left\{ \frac{1}{2} (\gamma^{\alpha b} Q^+_{\mu a} D^+ a \partial_b X^\alpha + \gamma^{\alpha b} Q^-_{\mu a} D^- a \partial_b X^\alpha) - \frac{1}{4} (\gamma^{\alpha b} \partial_b X^\lambda Q^-_{\lambda \mu} - \gamma^{\alpha b} \partial_b X^\lambda Q^+_{\lambda \mu}) (\gamma^{-c} - \gamma^{+c}) + U^\alpha_{\mu \nu} (H_{\rho \lambda \mu} - \frac{1}{2} U^\alpha_{\mu \sigma} (\nabla_\rho B_{\sigma \lambda} - \frac{1}{2} \nabla_\rho B_{\lambda \rho}) - \frac{1}{2} U^\sigma_{\mu \nu} (\nabla_\rho B_{\sigma \lambda} - \frac{1}{2} \nabla_\rho B_{\lambda \rho})) \right\} = 0$$

(67)

where now we have two projectors built from $U_{\mu \nu} \equiv \gamma^{ab} \partial_a X^\mu \partial_b X^\nu$ in the way $U_{\mu \nu} = U_{\mu \nu}^\rho Q_\rho^+ \partial_\rho$ and $U_{\mu \nu} = Q_\mu^+ U_{\nu \rho}^\rho$. They act as the identity over the brane directions, but they define two different ”orthogonal” directions, i.e., $\partial_\beta \phi^i Q^{\beta \mu}$ and $Q^{\beta \mu} \partial_\beta \phi^i$. Of course, the relation between the D-brane equations (67) and the field equations (64), uses again the identity on $\Sigma$, $U_{\mu \nu} = (U_{\mu \nu})|_\Sigma$ where $U_{\mu \nu} = Q_\mu^+ Q_{\nu}^\beta U_{\alpha \beta}$ getting the result (62).

As it happened in the target-space field theoretic dual version, (57).
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