A speed preserving Hilbert gradient flow for generalized integral Menger curvature

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Abstract

We establish long-time existence for a projected Sobolev gradient flow of generalized integral Menger curvature in the Hilbert case, and provide $C^{1,1}$-bounds in time for the solution that only depend on the initial curve. The self-avoidance property of integral Menger curvature guarantees that the knot class of the initial curve is preserved under the flow, and the projection ensures that each curve along the flow is parametrized with the same speed as the initial configuration. Finally, we describe how to simulate this flow numerically with substantially higher efficiency than in the corresponding numerical $L^2$ gradient descent or other optimization methods.

1 Introduction

Integral Menger curvature is one of several geometrically defined curvature functionals that are used in geometric knot theory to separate different knot classes by infinite energy barriers. It is defined as the triple integral

$$M_p(\gamma) := \iiint_{(\mathbb{R}/\mathbb{Z})^3} \frac{|\gamma'(u_1)||\gamma'(u_2)||\gamma'(u_3)|}{R_p(\gamma(u_1),\gamma(u_2),\gamma(u_3))} \, du_1 \, du_2 \, du_3$$

(1.1)

evaluated on absolutely continuous closed curves $\gamma \in AC(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, where $R(x,y,z)$ denotes the circumcircle radius of three points $x,y,z \in \mathbb{R}^n$. For exponents $p$ above scale-invariance, that is for $p > 3$, this geometric curvature energy has regularizing properties. It was shown in [34] for $n = 3$ that any locally homeomorphic arc length parametrization $\Gamma$ of $\gamma$ with $M_p(\gamma) < \infty$ is an embedding or a multiple cover of the image manifold of class $C^{1,1-3/p}$, which can be viewed as a geometric Morrey-Sobolev embedding theorem. This interpretation was later confirmed by S. Blatt’s characterization of finite energy curves as exactly those arc length parametrized embeddings that are of fractional Sobolev-Slobodeckii regularity $W^{2-(2/p)p}$, which precisely embeds into $C^{1,1-3/p}$ [8]. The combination of its self-avoidance and regularizing effects can be used to minimize integral Menger curvature within any prescribed given tame knot class, and to bound classic knot invariants and therefore the number of knot classes representable below given energy values. This makes $M_p$ a valuable instrument in geometric knot theory.

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1For the definition and several useful properties of periodic Sobolev-Slobodeckii spaces we refer to Appendix A.
1 Introduction

Figure 1: Select iterations of the projected Sobolev gradient flow with line search. (Iteration numbers are indicated in parentheses.) The discrete model is a polygonal line with 600 edges. The obtained minimizer (bottom right) is shown from a further angle in order to reveal its three-fold rotational symmetry.

We refer to the survey [35] for more information on the rôle of integral Menger curvature in the context of knot theory.

The Euler-Lagrange equations for integral Menger curvature were first derived by T. Hermes [20]. The principal part of this complicated integro-differential equation seems too degenerate to expect smoothness of critical points of \( M_p \). Therefore, Blatt and Ph. Reiter [11] came up with the idea of splitting the powers in the numerator and denominator of the fraction defining the circumcircle radius

\[
R(p,q)(x,y,z) = \frac{|z-x||y-x||z-y|}{|z-x\wedge(y-x)|^q}
\]

for \( x,y,z \in \mathbb{R}^n \), which led them to introduce the family of generalized integral Menger curvatures

\[
\text{int}M^{(p,q)}(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^3} \frac{|\gamma'(u_1)||\gamma'(u_2)||\gamma'(u_3)|}{R^{(p,q)}(\gamma(u_1),\gamma(u_2),\gamma(u_3))} \, du_1 \, du_2 \, du_3, \tag{1.2}
\]

where

\[
R^{(p,q)}(x,y,z) := \frac{|z-x||y-x||z-y|^p}{|(z-x)\wedge(y-x)|^q}
\]

generalizes the circumcircle’s diameter\(^2\) so that one has \( \mathcal{M}_p = 2^p \text{int}M^{(p,p)} \). It turns out that finite energy curves of class \( C^1 \) possess an arc length parametrization of class \( W^{(3p-2)/q-1,q} \) if \( q > 1 \) and \( p \in \left( \frac{2}{3}q + 1, q + \frac{2}{3} \right) \), which implies that also \( \text{int}M^{(p,q)} \) can be minimized in tame knot classes; see [11, Theorems 1 & 2]. In the more specific Hilbert space case, i.e. for \( q = 2 \) and \( p \in \left( \frac{7}{3}, \frac{2}{3} \right) \) (thus excluding the original integral Menger curvature, unfortunately), the variational equations become more accessible for regularity arguments, so that Blatt and Reiter could show \( C^\infty \)-smoothness for any critical point of \( \text{int}M^{(p,2)} \) [11, Theorem 4].

\(^2\)Throughout the paper \( a \wedge b \) denotes the exterior product of the two vectors \( a, b \in \mathbb{R}^n \), which reduces to the usual cross product \( a \times b \in \mathbb{R}^3 \) for \( n = 3 \).
It is natural to ask if one can set up a gradient flow

\[
\begin{aligned}
\frac{\partial}{\partial t} \gamma(\cdot,t) &= -\nabla \int \text{M}(p,2)(\gamma(\cdot,t)) \quad \text{on } \mathbb{R}/\mathbb{Z} \text{ for } t > 0, \\
\gamma(\cdot,0) &= \gamma_0.
\end{aligned}
\]  

(1.3)
in the appropriate Hilbert space \( H := W^{2,p-2,2}((\mathbb{R}/\mathbb{Z}),\mathbb{R}^n) \), deforming any given initial knotted curve \( \gamma_0 \) in that space to a critical point of \( \int \text{M}(p,2) \) within the knot class represented by \( \gamma_0 \). Note that we consider the full Sobolev-Slobodeckiǐ gradient here, in other words, (1.3) represents the \( W^{2,p-2,2} \) gradient flow for the generalized integral Menger curvature \( \int \text{M}(p,2) \).

For a visualization of this flow, see Figure 1. The energy is indeed continuously differentiable on regular embeddings of class \( W^{2,p-2,2}((\mathbb{R}/\mathbb{Z}),\mathbb{R}^n) \) according to [11, Theorem 3], and we show in Section 2 by analyzing its second variation that its differential \( D \int \text{M}(p,2) \) is even locally Lipschitz continuous. By the classic Picard-Lindelöf Theorem for ordinary differential equations in Banach spaces this can be turned into a short time existence result for (1.3), together with an explicit lower bound on the maximal time of existence, depending only on the Sobolev-Slobodeckiǐ seminorm of the tangent \( \gamma_0 \) and on the bilipschitz constant of the initial curve \( \gamma_0 \); see Theorem 3.2 of Section 3. For arc length parametrized curves both these quantities can be controlled in terms of the energy according to [11, Theorem 1 & Proposition 2.1]. Unfortunately, it seems hard to keep the velocities \( |\gamma'(\cdot,t)| \) of the curves \( \gamma(\cdot,t) \) bounded away from zero along the evolution (1.3), or to control the fractional Sobolev seminorms of the time-dependent reparametrizations to arc length, which would allow us to continue the flow up to infinite time. The right-hand side \( -\nabla \int \text{M}(p,2) \) is not explicit because of the underlying Riesz-isomorphism \( J: H \rightarrow H^* \) for the Hilbert space \( H = W^{2,p-2,2}((\mathbb{R}/\mathbb{Z}),\mathbb{R}^n) \). This makes it hard to derive evolution equations for the velocities of the evolving curves. Instead we overcome this difficulty of possibly degenerating parametrizations by projecting the energy gradient onto the null-space of a constraint’s gradient, thus preserving the initial velocity |\gamma_0(\cdot)| along this projected flow. This idea, which goes back to J. W. Neuberger [25, Chapter 6], was used by the second author in cooperation with S. Scholtes and M. Wardetzky in the context of discrete elastic curves.

Let us briefly describe Neuberger’s approach in an abstract setting first. For two real Hilbert spaces \( H_1, H_2 \), and an open set \( O \subseteq H_1 \), consider an energy functional \( E: O \rightarrow \mathbb{R} \) and a constraint mapping \( S: O \rightarrow H_2 \), both Fréchet differentiable. If we aim for a curve \( x: [0,T] \rightarrow O \) that chooses the direction of steepest descent for \( E \) under the condition that it conserves an initial value of \( S \), we may use the differential equation

\[
\frac{d}{dt} x(t) = -\nabla_S E(x(t)), \quad t > 0,
\]

(PrFlow)

where the projected gradient on the right-hand side is defined as

\[
\nabla_S E(y) := \Pi_{\mathcal{N}(DS(y))} \nabla E(y) \quad \text{for } y \in O.
\]  

(1.4)

Here, \( \Pi_{\mathcal{N}(DS(y))} \) denotes the orthogonal projection onto the (closed) null space of the differential \( DS(y) \) of the constraint mapping \( S \) at \( y \in O \). Indeed, if \( x \) satisfies the projected evolution equation (PrFlow), we simply differentiate the constraint with respect to time to obtain

\[
\frac{d}{dt} S(x(t)) = DS(x(t)) \frac{d}{dt} x(t) \overset{(\text{PrFlow})}{=} -DS(x(t)) \Pi_{\mathcal{N}(DS(x(t)))} \nabla E(x(t)) = 0
\]

and therefore,

\[
S(x(t)) = S(x(0)) \quad \text{for all } t \geq 0.
\]  

(1.5)
In Section 4 we analyze in the abstract Hilbert space setting under which circumstances the orthogonal projection $A \mapsto \Pi_{\mathcal{A}}(A)$ for a bounded linear operator $A: \mathcal{H}_1 \to \mathcal{H}_2$ is Lipschitz continuous in order to keep the projected flow (PrFlow) in the realm of the Picard-Lindelöf existence theory.

Our actual choice of constraint in the present context is, as in [32, Section 2], the logarithmic strain $S := \Sigma$ defined as

$$\Sigma(\gamma) := \log (|\gamma'(t)|) \in \mathcal{H}_2 := W^{2,p-3,2}(\mathbb{R}/\mathbb{Z})$$ (1.6)

for curves $\gamma$ contained in the open$^3$ subset

$$\mathcal{O} := W^{2,p-2,2}_0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \subset \mathcal{H}_1 := \mathcal{H} = W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$$ (1.7)

of injective regular curves in $\mathcal{H}$, i.e., for those $\gamma \in \mathcal{H}$ of which the restrictions $\gamma|_{[0,1)}$ are injective and the velocities $|\gamma'|$ are strictly positive on $\mathbb{R}/\mathbb{Z}$. With this choice we intend to guarantee the conservation of the initial velocity throughout the projected gradient flow for $\text{int}M^{(p,2)}$. Section 5 is therefore devoted to establishing the sufficient conditions of Section 4 for the logarithmic strain constraint $\Sigma$ to maintain uniform Lipschitz continuity. The main issue here is to prove the existence of a bounded right inverse for the constraint’s differential $D \Sigma(\gamma)$ for $\gamma \in W^{2,p-2,2}_0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. In Section 6 we combine the results of Sections 2 and 5 to obtain a locally Lipschitz continuous right-hand side of the projected evolution for the generalized integral Menger curvatures $\text{int}M^{(p,2)}$. In addition, we gain explicit quantitative control over the size of the right-hand side’s Lipschitz domain, which allows us to prove the following central long time existence result by means of standard continuation arguments.

**Theorem 1.1 (Long time existence).** Let $p \in (\frac{7}{3}, \frac{8}{3})$ and $\gamma_0 \in W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be injective on $[0,1)$ and suppose that $|\gamma_0'| > 0$ on $\mathbb{R}/\mathbb{Z}$. Then, there is a unique$^4$ map

$$t \mapsto \gamma(.,t) \in C^1([0,\infty), W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$$ (1.8)

satisfying

$$\begin{cases}
\frac{\partial}{\partial t} \gamma(.,t) = -\nabla_S \text{int}M^{(p,2)}(\gamma(.,t)) \quad \text{on } \mathbb{R}/\mathbb{Z} \text{ for all } t > 0, \\
\gamma(.,0) = \gamma_0,
\end{cases}$$ (1.9)

where $\Sigma$ is defined as in (1.6). Moreover, the energy is non-increasing in time, and the initial velocity and the barycenter are preserved, that is,

$$|\gamma'(.,t)| = |\gamma_0'(.)| \text{ on } \mathbb{R}/\mathbb{Z} \text{ and } \int_{\mathbb{R}/\mathbb{Z}} \gamma(u,t) \, du = \int_{\mathbb{R}/\mathbb{Z}} \gamma_0(u) \, du \text{ for all } t > 0.$$ (1.10)

In particular, the length of $\gamma(.,t)$ is constant in time and the curves $\gamma(.,t)$ remain uniformly bounded, that is, $L(\gamma(.,t)) = L(\gamma_0(\cdot))$ and $\|\gamma(.,t)\|_{W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)} \leq C$ for all $t > 0$.

The fractional Sobolev space $W^{2,p-2,2}$ continuously embeds into the Hölder space $C^{1,\alpha(p)}$ with Hölder exponent $\alpha(p) := \frac{2}{2p} - \frac{1}{p} - 3 \in (0,1)$ (see Theorem A.2 in Appendix A), which implies that all curves $\gamma(.,t)$ are also uniformly bounded in $C^{1,\alpha(p)}$. Moreover, the initial embedding $\gamma_0$ represents a tame knot class $K := \gamma_0$; see, e.g., [13, Appendix I]. Tame knot classes are stable with respect to $C^1$-perturbations (see [28, 14, 6]), so that the knot class is preserved along the flow.

$^3$In Corollary B.4 we provide a quantitative version of the fact that the regular embedded curves form an open subset in this fractional Sobolev space.

$^4$The solution $\gamma(.,t)$ is even unique among all $W^{2,p-2,2}$-valued mappings that are absolutely continuous in time for which the differential equation (1.8) holds only for a.e. $t > 0$. 


Corollary 1.2 (Flow within prescribed knot class). The solution curves $\gamma(\cdot, t)$ in Theorem 1.1 are injective on $[0, 1)$ for all $t \geq 0$. If the initial embedding $\gamma_0$ represents the tame knot class $\mathcal{K}$, i.e., $[\gamma_0] = \mathcal{K}$, then $[\gamma(\cdot, t)] = \mathcal{K}$ for all $t > 0$.

In addition, due to the embedding into $C^{1, \alpha}(\mathbb{R})$, the regularity assumption $|\gamma| > 0$ is meant to hold pointwise everywhere on $\mathbb{R}/\mathbb{Z}$, which by continuity and periodicity implies that

$$v_{\gamma_0} := \min_{\mathbb{R}/\mathbb{Z}} |\gamma| > 0, \quad (1.11)$$

and one can analyze how this minimal initial velocity $v_{\gamma_0}$ enters the $t$-dependence of the solution.

Theorem 1.3 (Regularity in time). The solution $t \mapsto \gamma(\cdot, t)$ of (1.9) in Theorem 1.1 is of class $C^{1, 1}([0, \infty), W^{2, p}_{2n-2, 2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$, and the Lipschitz constants for the mappings $t \mapsto \gamma(\cdot, t)$ and $t \mapsto \frac{\partial}{\partial t}\gamma(\cdot, t)$ depend only on $n$ and $p$, non-decreasingly on the initial energy $\int_\mathcal{M}^{(p, 2)}(\gamma_0)$ and initial fractional seminorm $|\gamma|^2_{p-2, 2}$, and non-increasingly on the minimal velocity $v_{\gamma_0}$.

At this point it is not clear yet whether the solution curves $\gamma(\cdot, t)$ converge to a projected critical point of $\int_\mathcal{M}^{(p, 2)}$ as $t \to \infty$. We do know, however, that we have subconvergence of $\gamma(\cdot, t_k)$ to some limiting knot $\gamma^*$ in the same fixed knot class for any sequence $t_k \to \infty$ as $k \to \infty$.

Corollary 1.4. For the solution $\gamma(\cdot, t)$ of (1.9) in Theorem 1.1 one has

$$E_\infty := \lim_{t \to \infty} \int_\mathcal{M}^{(p, 2)}(\gamma(\cdot, t)) \in [0, \int_\mathcal{M}^{(p, 2)}(\gamma_0)], \quad \lim_{t \to \infty} \nabla_\Sigma \int_\mathcal{M}^{(p, 2)}(\gamma(\cdot, t)) = 0, \quad (1.12)$$

and for every sequence $t_k \to \infty$ as $k \to \infty$ there is a subsequence $(t_k)_l \subset (t_k)_k$ and a curve $\gamma^* \in W^{2, p}_{2n-2, 2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $\int_\mathcal{M}^{(p, 2)}(\gamma^*) \leq E_\infty$ and with knot class $[\gamma^*] = [\gamma_0]$, such that $\gamma(\cdot, t_k)$ converges weakly in $W^{2, p}_{2n-2, 2}$ and strongly in $C^1$ to $\gamma^*$ as $l \to \infty$.

Analytical results regarding gradient flows for non-local self-avoidance energies seem scarce. The most significant contributions are the long time existence results of Blatt for the $L^2$-flow of O’Hara’s knot energies including the Möbius energy [7, 10, 9]. There is also work on the $L^\infty$-flow of linear combinations of the classic bending energy and a (non-local) self-avoidance term [24], [36], but there the crucial a priori estimates are obtained by virtue of the leading order curvature energy. For a linear combination of a discrete bending and self-repulsive tangent-point-type energy, however, numerical analysis has been performed recently by S. Bartels, Reiter, and J. Riege [5, 4]. In his Ph.D. thesis, Hermes [20] implemented a numerical scheme for an $L^2$-type gradient flow of integral Menger curvature $\mathcal{M}_p$ which was supplemented later by A. Gilsbach [17] to numerically minimize under symmetry constraints, but there is no analytical foundation yet for that flow with or without symmetry enforcement. The situation seems similar for the publicly available numerical flows such as R. Scharein’s KnotPlot [31] or Ridgerunner [3] that are widely used in the community of geometric or applied knot theory. Fractional Sobolev metrics for the Möbius energy have already been investigated by Reiter and the second author in [29]. However, because the attendant energy space is not a Hilbert space, long time existence could not be shown there. Similar metrics for tangent-point energies of curves have been discussed from a more experimental point of view by Ch. Yu, K. Crane, and the second author in [38]. There are also more advanced discretization and solving strategies being developed to make the evaluation of the discrete energy and the discrete gradient more efficient. Some of the methods discussed there may also be applied to the integral Menger energy, but this shall not be our concern here.
Figure 2: The Sobolev gradient flow has also regularizing properties. Here a noisy, polyhedral square knot (connected sum of a left-handed and a right-handed trefoil) of 600 edges is chosen as initial configuration. (Again, numbers in parentheses indicate iteration numbers of the projected gradient descent.)

Sobolev gradient flows are also easier to discretize than $L^2$ gradient flows. The problem with the $L^2$-flow of an energy $E$ is that its governing equation is a parabolic partial differential equation and that certain Courant–Friedrichs–Lewy conditions have to be fulfilled in order to make its explicit space-time discretization stable. Typically, these conditions are of the form $\tau \lesssim h^\alpha$ where $\tau > 0$ denotes the time step size and $h > 0$ denotes the spatial mesh size (here: the longest edge in a polyhedral discretization), and $\alpha > 0$ is the order of the differential operator $\gamma \mapsto DE(\gamma)$. In our case, we have $\alpha = 2s > 2$, hence a stable explicit time integration would require prohibitively small step size $\tau \lesssim h^{2s}$ for small $h > 0$. This can be mended by using implicit time integration, but this requires the second derivative of the energy and it introduces other costs like having to solve a nonlinear equation in each gradient step. Instead, the discrete Sobolev flow comes without dependence of the step size $\tau$ on the mesh size $h$, allowing us to use inexpensive explicit time integrators and adaptive time stepping strategies. The latter provides sufficient stability so that the discrete flow with oversized time steps can be used as fairly efficient optimization routine. Even with a naive all-pairs discretization of the energy and dense matrix arithmetic, this allowed us to compute local discrete minimizers in Figure 1, Figure 2 and Figure 3 within less than 20 minutes on a consumer laptop. For the algorithmic details see Section 7. There we also briefly explain how we employ refined time stepping techniques in order to guarantee that the discrete flow preserves the knot class. This is necessary because i) the discrete energy is not a knot energy (because we use inexact quadrature rules) and because ii) finite time steps may lead to pull-through even for true knot energies.

We close this introduction with a few remarks concerning notation. The inner product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$, inner products on other Hilbert spaces $\mathcal{H}$ usually carry an index, i.e., $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Quite often we need to know how constants depend on certain parameters, so constants $C = C(a, b, c, \ldots)$ are frequently interpreted as functions depending monotonically on the parameters $a, b, c, \ldots$. By $\text{Lip}_f$ we denote Lipschitz constants of mappings $f$. Closed curves are $\ell$-periodic vector-valued functions $\gamma$ on $\mathbb{R}$ for some $\ell > 0$, and we simply write
Figure 3: Spontaneous symmetry breaking: (a) Symmetric 5-3 torus knot. (b) Symmetric critical point after 40 projected Sobolev gradient iterations. (c) The eigenvector to the smallest negative(!) eigenvalue of the constraint Hessian. (d) Small, almost invisible perturbation of (b) in direction of this eigenvector. (e) Continuing the projected Sobolev gradient flow for 100, (f) 200, and (g) 300 iterations. From there on the energy landscape becomes rather shallow, so we applied Newton’s method to obtain the local minimizer (h).

\( \gamma : \mathbb{R}/\mathbb{L} \to \mathbb{R}^n \). The bilipschitz constant of \( \gamma \) defined as

\[
\text{BiLip}(\gamma) := \inf_{u_1, u_2 \in \mathbb{R}/\mathbb{L}} \frac{|\gamma(u_1) - \gamma(u_2)|}{|u_1 - u_2|_{\mathbb{R}/\mathbb{L}}} \tag{1.13}
\]

is essentially the inverse of Gromov’s distortion [19, Sect. 9] and describes the embeddedness of \( \gamma \) in a quantitative way. As in (1.11) we write \( v_\gamma \) for the minimal velocity of any absolutely continuous curve \( \gamma \). Finally, let \( W^{1,1}_{ir}(\mathbb{R}/\mathbb{L}) \) be the subset of closed \( W^{1,1} \)-curves \( \gamma \) that are injective on \([0, \ell)\) and regular, i.e. with \( v_\gamma > 0 \) a.e. on \( \mathbb{R}/\mathbb{L} \).

2 Lipschitz continuity of the energy gradient

We calculate and estimate the first and second variation of the integral Menger curvature on the space \( W^{3p-2,2}_{ir}(\mathbb{R}/\mathbb{L}, \mathbb{R}^n) \) of periodic regular Sobolev Slobodecki\( \acute{\text{i}} \) curves that are injective on the fundamental domain \([0, 1)\).

**Theorem 2.1.** For \( p \in \left( \frac{7}{3}, \frac{8}{3} \right) \) the first variation \( \delta \text{intM}^{(p,2)}(\gamma, \cdot) \) and the second variation \( \delta^2 \text{intM}^{(p,2)}(\gamma) \) of the generalized integral Menger curvature \( \text{intM}^{(p,2)} \) exist at every curve \( \gamma \in W^{3p-2,2}_{ir}(\mathbb{R}/\mathbb{L}, \mathbb{R}^n) \), satisfying the estimates

\[
|\delta \text{intM}^{(p,2)}(\gamma, h)| \leq C_1 |h|^\frac{3}{2p-3,2},
\]

\[
|\delta^2 \text{intM}^{(p,2)}(\gamma)[h, g]| \leq C_2 |h|^\frac{3}{2p-3,2} |g|^\frac{3}{2p-3,2},
\]
for all $h, g \in W^{2,2}_{p-2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, where the constants $C_i = C_i(n, p, \text{BiLip}(\gamma), [\gamma']^2_{p-2})$ for $i = 1, 2$ depend non-increasingly on the biLipschitz constant $\text{BiLip}(\gamma)$ of $\gamma$ and non-decreasingly on the seminorm $[\gamma']^2_{p-2}$ of the tangent $\gamma'$.

Proof. Since $\gamma$ is injective on $[0, 1)$ and a regular curve of class $W^{2,2}_{p-2}$ which by Morrey’s embedding, Theorem A.2, embeds into $C^1$, we have $0 < \text{BiLip}(\gamma) \leq \nu_\gamma \leq |\gamma'(u)|$ for all $u \in \mathbb{R}$; see Lemma B.3 and Corollary B.4. Using the explicit embedding inequality (A.4) we obtain $[\gamma']^2_{p-2} \geq C^{-1}_E |\gamma'|_{C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)} \geq C^{-1}_E \nu_\gamma > 0$. Therefore, we can choose $\varepsilon > 0$ sufficiently small such that

$$[\gamma']^2_{p-2} \leq 2 [\gamma']^2_{p-2} \quad \text{and} \quad v_\gamma \geq \text{BiLip}(\gamma) \geq \text{BiLip}(\gamma)/2 > 0$$

for all $\eta \in B_{\varepsilon}(\gamma) \subset W^{2,2}_{p-2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, where we used (B.3) of Lemma B.1 as well as (B.7) of Corollary B.4 in Appendix B. Fix such a curve $\eta \in B_{\varepsilon}(\gamma)$. Similarly as in [11], we use the difference $\Delta_{\nu,\nu} f(u) := f(u + \nu) - f(u - \nu)$ for a function $f: \mathbb{R} \rightarrow \mathbb{R}$ and arbitrary parameters $u, v, w \in \mathbb{R}$ to abbreviate

$$[f, g] := \Delta_{\nu,\nu} f(u) \wedge \Delta_{\nu,\nu} g(u) + \Delta_{\nu,0} g(u) \wedge \Delta_{\nu,0} f(u) \quad \text{and} \quad \hat{f} := [f, f/2] = \Delta_{\nu,0} f(u) \wedge \Delta_{0,0} f(u),$$

where also $g: \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary function. This allows us to rewrite the Lagrangian of $\int M f, g, h [x] := \langle f, x \rangle$ for differentiable functions $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$ we can express the integrand of the first variation of $\text{int} M f, g, h [x]$ for all $h \in W^{2,2}_{p-2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ in the direction of $h \in W^{2,2}_{p-2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ derived in [11, Section 3]

$$\delta L(\eta, h)(u, v, w) := L(\eta)(u, v, w) \cdot \left[ 2[\hat{\eta}, [\hat{\eta}, h]]/|\hat{\eta}|^2 + A(\eta, \eta, h)(0) + A(\eta, \eta, h)(v) + A(\eta, \eta, h)(w) - p \{ B[\eta, \eta, h](w, 0) + B[\eta, \eta, h](v, 0) + B[\eta, \eta, h](v, w) \} \right].$$

Denoting by $A[\eta, h]$ the sum of all the $A[\cdot, \cdot, \cdot](\cdot)$-terms, and by $B[\eta, h]$ the sum of all the $B[\cdot, \cdot, \cdot](\cdot)$-terms, we arrive at the shorter expression

$$\delta L(\eta, h)(u, v, w) := L(\eta)(u, v, w) \cdot \left[ A[\eta, h] - pB[\eta, h] + 2(\hat{\eta}, [\hat{\eta}, h])|\hat{\eta}|^2 \right] := L(\eta)(u, v, w) \cdot I(h).$$

After replacing $\eta$ with the perturbed curve $\eta_\tau := \eta + \tau g$ for some $g \in W^{2,2}_{p-2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and $0 < |\tau| \leq \tau_\eta \ll 1$, so that $\eta_\tau \in B_{\varepsilon}(\gamma)$ for all $|\tau| \leq \tau_\eta$, we compute the derivative with respect to $\tau$ and evaluate at $\tau = 0$ to obtain by means of the product rule

$$\delta^2 L(\eta, h)(u, v, w) := \left. \frac{d}{d\tau} \right|_{\tau=0} [\delta L(\eta_\tau, h)(u, v, w)] = L(\eta)(u, v, w) I(g) I(h)
+ L(\eta)(u, v, w) \cdot \left. \frac{d}{d\tau} \right|_{\tau=0} [A(\eta_\tau, h) - pB(\eta_\tau, h) + 2(\hat{\eta}_\tau, [\hat{\eta}_\tau, h])|\hat{\eta}_\tau|^2].$$

Using the observation $\left. \frac{d}{d\tau} \right|_{\tau=0} \langle b \tau + c \rangle = \langle b, c \rangle / |a|^2 - 2 \langle a, b \rangle \langle a, c \rangle / |a|^4$ for $a, b, c \in \mathbb{R}^n$ in terms $A[\eta, h]$ and $B[\eta, h]$ together with their very similar structure, we get for $A' := \left. \frac{d}{d\tau} \right|_{\tau=0} A(\eta_\tau, h)$ and $B' := \left. \frac{d}{d\tau} \right|_{\tau=0} B(\eta_\tau, h)$ the expressions

$$A' = \sum_{\sigma \in \{0, v, w\}} A[\eta, h, g](\sigma) - 2A[\eta, \eta, h](\sigma) A[\eta, \eta, g](\sigma)$$

and

Note that the last three terms ought to have the factor $\langle R^{\nu, \eta} \rangle^{-1}$ instead of $R^{\nu, \eta}$ in the expression for $\delta L$ preceding Lemma 3.1 in [11].
\[ B' = \sum_{(\sigma, \varrho) \in \{(v, 0), (w, 0), (v, w)\}} B[\eta, h, g(\sigma, \varrho) - 2B[\eta, \eta, h]B[\eta, \eta, g(\sigma, \varrho)]. \]

For the last term in (2.5) we obtain from quotient and product rule
\[
\frac{d}{d\tau} \bigg|_{\tau=0} \left( \hat{\eta}, [\hat{g}, h] \right) \bigg|_{[\hat{g}, h]} = \left\{ \left[ [h, \eta], [g, \eta] \right] + \left[ \hat{q}, [h, g] \right] \right\} \frac{1}{|\eta|^2} - \left( \hat{q}, [h, \eta] \right) \left( \hat{q}, [g, \eta] \right) \frac{2}{|\eta|^4}. \tag{2.6}
\]

In total, we have
\[
\delta^2 L(\eta, h, g)(u, v, w) = L(\eta)(u, v, w) \left[ I(\eta)I(h) + \mathcal{A}' - pB' \right] + \left\{ \left[ [h, \eta], [g, \eta] \right] + \left[ \hat{q}, [h, g] \right] \right\} 2|\eta|^{-2} - \left( \hat{q}, [h, \eta] \right) \left( \hat{q}, [g, \eta] \right) 4|\eta|^{-4}. \tag{2.7}
\]

Following the idea of Blatt and Reiter in [11, Lemma 1.1] we do not consider the integral of this formula over \( \mathbb{R}/\mathbb{Z} \times (-1/2, 1/2)^2 \), but instead compute six times the integral over \( \mathbb{R}/\mathbb{Z} \times D \) with
\[
D := \left\{ (v, w) \in (-\frac{1}{2}, 0) \times (0, \frac{1}{2}) : w \leq 1 + 2v, v \geq -1 + 2w \right\} \tag{2.8}
\]
which does not change the value of the functional but guarantees that \( |v| = |v|_{\mathbb{R}/\mathbb{Z}}, |w| = |w|_{\mathbb{R}/\mathbb{Z}} \), and
\[
|v - w|/2 \leq |v - w|_{\mathbb{R}/\mathbb{Z}} \leq |v - w| \quad \text{for all} \quad (v, w) \in D. \tag{2.9}
\]
For the non-obvious first inequality in (2.9) notice that \( |v - w| \leq \frac{2}{3} \) for all \( (v, w) \in D \), and therefore \( 1 - |v - w| = |v - w|/(|v - w|^{-1} - 1) \geq |v - w| \left((3/2) - 1\right) \), which yields (2.9). Combining this with (2.3) we have
\[
|\eta(x) - \eta(y)|^{-1} \leq (\text{BiLip}(\eta)) |x - y|_{\mathbb{R}/\mathbb{Z}} \leq 2/(\text{BiLip}(\eta)) |x - y| \leq 4/(\text{BiLip}(\gamma)) |x - y| \tag{2.10}
\]
for distinct parameters \( x \in \{u + w, u + v\} \), and \( y \in \{u + v, u + w\} \) with \((v, w) \in D\). Combining the Morrey embedding, Theorem A.2, in particular inequality (A.4) with Poincaré’s inequality (A.6) for \( \eta \) (which satisfies \( \int_0^\infty \eta'(\tau) \, d\tau = 0 \)) leads to the estimates
\[
\frac{|\eta(x) - \eta(y)|}{|x - y|} \leq \|\eta'\|_{C^{0}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)} \leq \|\eta'\|_{W^{\frac{1}{2}, p-3, 2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)} \leq C_{E} C_{P} |\eta'|^{\frac{1}{2}}_{p-3, 2} \tag{2.11}
\]
for all distinct \( x, y \in \mathbb{R} \). Using (2.10), (2.11), and (2.3) we can estimate the various terms in the integrands of the first and second variation.

\[
|L(\eta)(u, v, w)| \leq (\frac{4 \pi \mathcal{C}_{E} C_{E}}{\text{BiLip}(\gamma)} |\gamma'|^\frac{1}{2} - 3) \frac{|\eta|^2}{|\eta|^2 - |v - w|^2} =: C_{L} \frac{|\eta|^2}{|\eta|^2 - |v - w|^2},
\]
\[
|\mathcal{A}' - pB'| \leq \frac{4(1 + p) \mathcal{C}_{E} C_{E}}{\text{BiLip}(\gamma)} |\gamma'|^\frac{1}{2} - 3 |\gamma'| \frac{1}{2} - 3 \leq C_{A} |\gamma'|^\frac{1}{2} - 3 |\gamma'| \frac{1}{2} - 3,
\]
\[
|I(h)| \leq 2(1 + p) \mathcal{C}_{E} C_{E} |\gamma'| \frac{1}{2} - 3 |\gamma'| \frac{1}{2} - 3 + \frac{1}{|\eta|^2} \leq C_{I} |\gamma'|^\frac{1}{2} - 3 |\gamma'| \frac{1}{2} - 3. \tag{2.12}
\]
Here, the constants \( C_{L}, C_{A} \), and \( C_{I} \) depend only on \( n \) and \( p \), non-increasingly on \( \text{BiLip}(\gamma) \), and non-decreasingly on \( |\gamma'| \). The last inequality yields \( I(h)I(g) \leq C_{F} |\gamma'|^\frac{1}{2} - 3 |g'| \frac{1}{2} - 3 + \frac{1}{|\eta|^2} + \frac{1}{|\eta|^2} \leq C_{F} |\gamma'|^\frac{1}{2} - 3 |g'| \frac{1}{2} - 3 + \frac{1}{|\eta|^2} + \frac{1}{|\eta|^2} \leq C_{F} |\gamma'|^\frac{1}{2} - 3 |g'| \frac{1}{2} - 3 + \frac{1}{|\eta|^2} + \frac{1}{|\eta|^2} \). Now we estimate the terms in the
expression (2.7) for $\delta^2 L(\eta, h, g)$ term by term and obtain
\begin{align}
|L(\eta)(u, v, w)(A' - pB')| &\leq C_L C_{\text{AB}} [h']_{2p-3,2} [g']_{2p-3,2} |\hat{\eta}|^2 (|v| |w| |v - w|)^{-p}, \\
|L(\eta)(u, v, w)\{\langle \eta, h \rangle, [\eta, g] + \langle \eta, [h, g] \rangle\}\}_{\partial|\eta|} &\leq 2C_L \frac{|h, \eta|}{{|\eta|}} |g, \eta| + |\eta| [h, g], \\
|L(\eta)(u, v, w)(\langle \eta, [h, g] \rangle, \langle \eta, [g, h] \rangle)\}_{\partial|\eta|} &\leq 4C_L |h, \eta| |g, \eta| (|v| |w| |v - w|)^{-p}, \\
|L(\eta)(u, v, w)I(h)I(g) &\leq C_L [h']_{2p-3,2} [g']_{2p-3,2} |\hat{\eta}|^2 + 4[|h, \eta|] [\hat{\eta}, \eta] \\
&+ 2C_I \frac{|h'|_{2p-3,2} [\hat{\eta}, \eta] + [g']_{2p-3,2} [\hat{h}, \eta] [\hat{\eta}, \eta]}{(|v| |w| |v - w|)^{-p}.} 
\end{align}

We are going to apply these estimates to curves $\eta := \gamma + s\tau g \in B_{\delta}(\gamma)$ for $s \in [0,1]$ and $|\tau| \leq \tau_g \ll 1$. In order to sum up the terms on the right-hand sides of (2.13)–(2.16) to find an integrable majorant for $\delta^2 L(\gamma + s\tau g, h, g)$ independent of $s$ and $t$, we need to use the bilinearity and commutativity of $[\cdot, \cdot]$. Indeed, using the bounds
\begin{align}
|\hat{\eta}|^2 &\leq |\hat{\eta}, \eta|/2 |^2 \leq |[\gamma, \gamma]| + s\tau |[\gamma, \gamma]| + (s\tau)^2 |[\gamma, \gamma]|^2 \\
&\leq [\gamma + s\tau |g, \gamma] + (s\tau)^2 g \|^2 \leq 3(|\gamma|^2 + \tau_g^2 |\gamma, \gamma|^2 + \tau_g^2 |\gamma|^2), \\
|\eta|_{[\gamma, \gamma]} &\leq |[\gamma, \gamma]| + s\tau |g, \gamma| \leq \tau_g |g, \gamma|
\end{align}

as well as analogous bounds for $|\hat{\eta}, \eta|$, all independent of $s \in [0,1]$ and $\tau \in [-\tau_g, \tau_g]$ in (2.13)–(2.16) we obtain by means of (2.7)
\begin{align}
|\delta^2 L(\gamma + s\tau g)| &\leq \frac{K}{|\gamma| \tau_g} [\hat{h}]_{2p-3,2} [\hat{g}]_{2p-3,2} \left([\gamma']_{2p-3,2}^4 + [\tau_g]_{2p-3,2}^4 |\gamma|^2 \right) \\
&+ (|\gamma|_{[\gamma, \gamma]} + \tau_g |\gamma, \gamma|) \left[ [\hat{h}, \gamma]_{2p-3,2} + [\hat{g}, \gamma]_{2p-3,2} \right] \\
&+ [g']_{2p-3,2} \left([\gamma, \gamma]_{2p-3,2} + [\tau_g, \gamma]_{2p-3,2} \right) \left[ [\hat{h}, \gamma]_{2p-3,2} + [\hat{g}, \gamma]_{2p-3,2} \right] \left[ [\gamma, \gamma]_{2p-3,2} + [\tau_g, \gamma]_{2p-3,2} \right] =: G_{\tau_g},
\end{align}

where the right-hand side defines the desired majorant $G_{\tau_g}$ on $\mathbb{R}/\mathbb{Z} \times D$ independent of $s$ and $\tau$. Notice that we have subsumed all constants appearing in (2.13)–(2.16) under the constant $K = K(n, p, \text{Bilip}(\gamma), [\gamma']_{2p-3,2})$, still depending only on $n$ and $p$, non-increasingly on $\text{Bilip}(\gamma)$, and non-decreasingly on $[\gamma']_{2p-3,2}$. Applying Lemma 2.2 below separately to integrated products like $[\hat{h}, \gamma]|\gamma, \gamma]_2$ in (2.17) we obtain that $G_{\tau_g}$ is integrable on $\mathbb{R}/\mathbb{Z} \times D$ with the estimate
\begin{align}
\int \int_{\mathbb{R}/\mathbb{Z} \times D} G_{\tau_g} |u, v, w| \, dw \, dv \, du &\leq C^* K [h']_{2p-3,2} [g']_{2p-3,2} \left([\gamma']_{2p-3,2}^4 + [\gamma]_{2p-3,2}^3 \right) \\
&+ [\gamma']_{2p-3,2}^2 + \tau_g [g']_{2p-3,2} \left([\gamma']_{2p-3,2}^2 + [\gamma]_{2p-3,2}^2 \right) \\
&+ \tau_g^2 [g']_{2p-3,2}^2 \left([\gamma']_{2p-3,2}^2 + [\gamma]_{2p-3,2}^2 + 1 \right) + \tau_g^2 [g']_{2p-3,2},
\end{align}

where we have also used Young’s inequality, and where we have subsumed all numerical constants as well as the Morrey and Poincaré constants $C_E$ and $C_P$ under the new constant $C^* \geq 1$. We obtain for the difference quotients $\tau^{-1} (\delta \int \text{intM}(p, 2)(\gamma + \tau g, h) - \delta \int \text{intM}(p, 2)(\gamma, h))$ the expressions
\begin{align}
= \int \int_{\mathbb{R}/\mathbb{Z} \times D} \tau^{-1} (\delta L(\gamma + \tau g, h)(u, v, w) - \delta L(\gamma, h)(u, v, w) \tau) \, dw \, dv \, du \\
= \int \int_{\mathbb{R}/\mathbb{Z} \times D} \int_0^1 \delta^2 L(\gamma + \tau s g, h, g) |u, v, w| \, ds \, dw \, dv \, du,
\end{align}
converging to \(6 \int \int \int_{\mathbb{R}/\mathbb{Z} \times D} \delta^2 L(\gamma, h, g)(u, v, w) \, dw \, dv \, du = \delta^2 \text{intM}^{(p,2)}(\gamma)[h, g] \) as \( \tau \to 0 \) according to Lebesgue’s dominated convergence theorem, since the integrands \( \int_{\gamma} \delta^2 L(\gamma + \sigma \tau y, h, g) \, ds \) in (2.19) converge pointwise to \( \delta^2 L(\gamma, h, g) \) as \( \tau \to 0 \) (see (2.6)), and \( G_{\tau y} \) serves as an integrable majorant for these integrands for all \( \tau \in (-\tau_y, \tau_y) \) by virtue of (2.17) and (2.18). So, we established the existence of the second variation \( \delta^2 \text{intM}^{(p,2)}(\gamma)[h, g] \), and the bound (2.2) follows from (2.17) and (2.18) letting \( \tau_y \to 0 \), if we set \( C_2 := C^* K([\gamma]'_{2p-3,2} + [\gamma]'_{2p-3,2} + [\gamma]'_{2p-3,2}) \) and recalling that \( K = K(n, p, \text{BiLip}(\gamma), [\gamma]'_{2p-3,2}) \) depends non-increasingly on the bilipschitz constant \( \text{BiLip}(\gamma) \) and non-decreasingly on the seminorm \( [\gamma]'_{2p-3,2} \).

The terms \( |\delta L(\gamma, h)(u, v, w)| = |L(\gamma)(u, v, w)|I(h) \) constituting the integrand of the first variation (see (2.4)) are estimated in (2.12), so that we can again use Hölder’s inequality and the prototype estimate (2.20) of Lemma 2.2 to bound this integrand by

\[
\frac{C_G}{|v|^p |w| |v-w|} (C_1 [h]'_{2p-3,2} |\gamma|^2 + 2|\overline{h}| |\gamma|) \leq K [h]'_{2p-3,2} (\gamma'_{2p-3,2} + [\gamma]'_{2p-3,2}) + \text{const} \quad \text{for all} \quad h \in \mathbb{R}/\mathbb{Z} \times D \quad \text{where} \quad D \quad \text{is defined in} \quad (2.8).
\]

Then there is a constant \( C = C(C_E, C_P, p) \) such that

\[
\|f_1, f_2\|_{L^2_p(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R}^n)} \leq C [f_1]'_{2p-3,2} \cdot [f_2]'_{2p-3,2} \quad \text{for all} \quad f_1, f_2 \in W^{2p,2} \quad (\mathbb{R}/\mathbb{Z}, \mathbb{R}^n). \quad (2.20)
\]

\( C_E \) and \( C_P \) are the constants of the Morrey embedding (A.4) and Poincaré’s inequality (A.6).

**Proof.** Using the fact that \( |\xi \wedge \zeta| \leq |\xi| |\zeta| \) for all \( \xi, \zeta \in \mathbb{R}^n \), see [37, Chapter I, Section 12, Equation (13)], we can bound the numerator of the integrand as

\[
|f_1, f_2|^2 = |w|^2|v|^2 \left| \int_0^1 f_1(u + yw)dy \wedge f_2(u + yw)dy + \int_0^1 f_2(u + yw)dy \wedge f_1(u + yw)dy \right|^2
\]

\[
\leq 2|w|^2|v|^2 \left\{ \left| \int_0^1 f_1(u + yw)dy \right|^2 \left| \int_0^1 f_2(u + yw)dy \right|^2 + \left| \int_0^1 f_2(u + yw)dy \right|^2 \left| \int_0^1 f_1(u + yw)dy \right|^2 \right\}
\]

\[
\leq 2C_E^2 C_P^2 |w|^2|v|^2 \left\{ \left| \int_0^1 f_1(u + yw)dy \right|^2 \left| \int_0^1 f_2(u + yw)dy \right|^2 + \left| \int_0^1 f_2(u + yw)dy \right|^2 \left| \int_0^1 f_1(u + yw)dy \right|^2 \right\},
\]

where we used Morrey’s embedding (A.4) and the Poincaré inequality (A.6). It therefore suffices by Jensen’s inequality to give an upper bound for the triple integral

\[
\int \int \int_{\mathbb{R}/\mathbb{Z} \times D} \left| f_1(u + yw) - f_2(u + yw) \right|^2 \, dv \, dw \, du \quad \text{for} \quad i = 1, 2,
\]

which can be rewritten by means of Tonelli’s variant of Fubini’s theorem, the substitution \( z := u + yw \), periodicity of the resulting integrand in the \( z \)-variable, and a second application of Fubini’s theorem as \( \int_0^1 \int \int_{\mathbb{R}/\mathbb{Z} \times D} \left| f_1(z + \theta(z-w)) - f_2(z) \right|^2 \, dw \, dv \, dz \, d\theta \) for \( i = 1, 2 \). Following an idea in the proof of [1, Theorem 1, p. 7] we use the transformation \( \Phi : D \to \Phi(D) \) mapping a pair \((v, w) \in D \to (t, \theta) := (v/(v-w), y/(v-w)) \in \Phi(D) \subset (0, 1) \times (-1, 0) \) by definition of the parameter range \( D \). Observing that \( |\det D\Phi(v, w)| = y/|v-w| \) for all \((v, w) \in D \) we arrive at the transformed integral \( \int_0^1 \int \int_{\mathbb{R}/\mathbb{Z} \times D} \left| f_1(z + \theta(z-w)) - f_2(z) \right|^2 \, dw \, dv \, dz \, d\theta \) for \( i = 1, 2 \).
Regrouping one obtains finite integrals over the \( y \)-variable and over the \( t \)-variable due to the parameter range \( p \in (\frac{7}{3}, \frac{8}{3}) \), multiplied by the double integral

\[
\int_{\mathbb{R}/\mathbb{Z}} \int_{0}^{1} \frac{|f'(t+x)-f'(x)|^2}{|y|^p} \, dy \, dx \leq \int_{\mathbb{R}/\mathbb{Z}} \int_{0}^{1} \frac{|f'(t+x)-f'(x)|^2}{|y|^p} \, dy \, dx,
\]

which is bounded by \( |f'|_{L_p}^2 (\mathbb{R}/\mathbb{Z} \times \mathbb{R}^n) \leq (1 + 12C_E^2 C_P^2) |f'_{32}^2|_{L_p}^2 - 3,2 \) for \( i = 1, 2 \) by virtue of the Morrey embedding (A.4) and Poincaré’s inequality (A.6). In summary, we obtain

\[
\|f_1 f_2\|_{L_p^2(\mathbb{R}/\mathbb{Z} \times D, \mathbb{R}^n)} \leq 4C_E^2 C_P^2 (1 + 12C_E^2 C_P^2) C^2(p) |f'_{32}^2|_{L_p}^2 - 3,2 |f'_{32}^2|_{L_p}^2 - 3,2.
\]

As mentioned in the introduction it was already shown in [11, Theorem 3] that the general-ized integral Menger curvature is continuously differentiable on regular embeddings of class \( W^{2p-2,2} \). Now, Theorem 2.1 yields local Lipschitz continuity of the differential \( D \text{intM}^{(p,2)} \). To quantify this we define the radius of Lipschitz continuity \( R_M(\gamma) \) for a regular embedded curve \( \gamma \in W^{2p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) as

\[
R_M \equiv R_M(\gamma) := \min \{(2C_E C_P)^{-1} \text{BiLip}(\gamma), 1\} \leq 1,
\]

where \( C_E \) denotes the constant of Morrey’s embedding Theorem A.2, and \( C_P \) is the constant in Poincaré’s inequality (A.6). Exactly as in the beginning of the previous proof we notice that \( R_M > 0 \) since \( \gamma \) is a regular embedded fractional Sobolev curve. Corollary B.4 in Appendix B implies that \( \text{BiLip}(\eta) \geq \text{BiLip}(\gamma)/2 \) and \( |\eta'|_{L_2} - 3,2 < |\gamma'|_{L_2} - 3,2 + 1 \) for every \( \eta \in B_{R_M}(\gamma) \). Therefore, combining (2.2) with a suitable mean value inequality (see, e.g., [16, Theorem 3.2.7]) we immediately deduce how the Lipschitz constant of the differential \( D \text{intM}^{(p,2)} \) depends on the bilipschitz constant and the fractional Sobolev norm of the embedded curve \( \gamma \) within the ball \( B_{R_M}(\gamma) \).

**Corollary 2.3.** For \( p \in (\frac{7}{3}, \frac{8}{3}) \) and \( \gamma \in W^{2p-2,2} \) the differential \( D \text{intM}^{(p,2)}(\cdot) \) and hence also the gradient \( \nabla \text{intM}^{(p,2)}(\cdot) \) are Lipschitz continuous on the ball \( B_{R_M}(\gamma) \subset W^{2p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) with Lipschitz constant

\[
\text{Li}_{D \text{intM}^{(p,2)}} \equiv \text{Li}_{\nabla \text{intM}^{(p,2)}} (n, p, \text{BiLip}(\gamma), |\gamma'|_{L_2} - 3,2),
\]

depending non-increasingly on \( \text{BiLip}(\gamma) \) and non-decreasingly on \( |\gamma'|_{L_2} - 3,2 \).

### 3 Short Time Existence

We first state the well-known Picard-Lindelöf Theorem for ordinary differential equations in Banach spaces; see e.g. [12, Part II, Corollary 1.7.2].

**Theorem 3.1 (Picard-Lindelöf).** Let \( \mathcal{X} \) be a real Banach space with norm \( \| \cdot \|_\mathcal{X} \), \( V \) a neighbourhood of \((t_0, x_0) \in \mathbb{R} \times \mathcal{X} \) and \( f : V \to \mathcal{X} \) be a continuous function with \( \|f(t, y) - f(t, z)\|_\mathcal{X} \leq \text{Lip}_f \|y - z\|_\mathcal{X} \) for all \((t, y), (t, z) \in V\). Then, there is \( \alpha > 0 \) such that the differential equation

\[
\frac{dy}{dx} = f(t, y) \text{ has a unique solution } \varphi \in C^1(I, \mathcal{X}) \text{ for } I := [t_0 - \alpha, t_0 + \alpha] \text{ with } \varphi(t_0) = x_0.
\]

More precisely, if \( \tau > 0 \) and \( r > 0 \) are chosen to be small enough such that \([t_0 - \tau, t_0 + \tau] \setminus B_r(x_0) \text{ is contained in } V \) and \( \|f(t, x)\|_\mathcal{X} \leq F \) for all \([t - t_0, \leq \tau] \) and \( \|x - x_0\|_\mathcal{X} \leq r \), then \( \alpha \) may be chosen as \( \alpha := \min \{\tau, r/F\} \) and the image \( \varphi(I) \) is contained in the ball \( B_r(x_0) \subset \mathcal{X} \).

In the last section, we saw that the gradient of \( \text{intM}^{(p,2)} \) is locally Lipschitz continuous, so we have short time existence of a solution of (1.3).
Theorem 3.2 (Short time existence). Let \( p \in \left( \frac{2}{3}, \frac{8}{7} \right) \) and \( \gamma_0 \in W^{\frac{2}{3}p-2,2}_{\text{i}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \). Then, there exists a time \( T > 0 \) and a unique \( ^e \phi \) mapping \( t \mapsto \gamma(t) \in C^{1,1}(\mathbb{R}/\mathbb{Z}, W^{\frac{2}{3}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)) \) such that \( \gamma \) solves (1.3), with strict energy decay in time, i.e., such that the mapping \( t \mapsto \int M(\gamma) \) is strictly decreasing \([0, T]\). One can estimate the minimal time \( T \) of existence from below as

\[
T \geq T_{\min} := \frac{R_M(\gamma_0)}{C_1(n, p, B_{\text{Lip}}(\gamma_0)/2, [\gamma_0]_{\frac{2}{3}p-3,2} + 1)},
\]

where \( R_M \) is the Lipschitz radius of the initial curve \( \gamma_0 \) defined in (2.22) and \( C_1 \) is the constant bounding the first variation \( \delta \int M(\gamma) \) in (2.1) of Theorem 2.1. Moreover, the image \( \gamma([0, T_{\min}]) \) is contained in the ball \( B_{R_M}(\gamma_0) \subset W^{\frac{2}{3}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \), and the barycenter is conserved along the flow, i.e.

\[
\frac{1}{\mu} = \int_{\gamma([0, T_{\min}])} \gamma \, dt = \int_{\gamma([0, T_{\min}])} \gamma_0 \, dt = \bar{\gamma}_0 \quad \text{for all } t \in [0, T].
\]

Proof. Set \( \mathcal{H} := W^{\frac{2}{3}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \). According to Corollary 2.3 the right-hand side of (1.3) is Lipschitz continuous on the open neighbourhood \( V := R \times B_{R_M}(\gamma_0) \) of \( (0, \gamma_0) \in R \times \mathcal{H} \), so that we can apply Theorem 3.1 to conclude short time existence of a solution \( \gamma \in C^1([0, T], \mathcal{H}) \). Differentiating the energy \( E := \int M(\gamma) \) of \( \gamma(t) \) with respect to time we obtain

\[
\frac{d}{dt} E(\gamma(t)) = \langle \nabla E(\gamma(t)), \frac{\partial}{\partial t} \gamma(t) \rangle_H = -\|\nabla E(\gamma(t))\|_H^2 < 0
\]

for all \( t \in (0, T) \), which implies the strict energy decay in time. Notice for the strict inequality in (3.3) that \( E = \int M(\gamma) \) does not possess any critical point since it is not scale-invariant.

Indeed, every strictly positive and Fréchet-differentiable energy \( F \) on an open subset \( O \) of a Hilbert space with the homogeneity condition \( F(\mu x) = \mu^2 F(x) \) for all \( x \in O \) and \( |\mu - 1| \ll 1 \), for some fixed \( \beta \neq 0 \), satisfies \( D\mathcal{F}(x) = \beta \mathcal{F}(x) \neq 0 \) for all \( x \in O \setminus \{0\} \).

The minimal existence time \( T \) can be bounded from below according to the explicit choice of \( \alpha \) in Theorem 3.1, which leads to (3.1), since the upper bound (2.1) in Theorem 2.1 yields

\[
\|D\mathcal{F}(\gamma(t))\|_H = \|\delta E(\gamma, \cdot)\|_H \leq C_1(n, p, B_{\text{Lip}}(\gamma), [\gamma]_{\frac{2}{3}p-3,2}) \leq C_1(n, p, B_{\text{Lip}}(\gamma_0)/2, [\gamma_0]_{\frac{2}{3}p-3,2} + 1) := F \quad \text{for all } \eta \in B_{R_M}(\gamma_0),
\]

by choice of the radius \( R_M \) in (2.22) combined with Corollary B.4 in the appendix.

The time derivative \( \partial_t \gamma := \partial_t / \partial t \) is even Lipschitz continuous with respect to time since the right-hand side \( f(\eta) := \nabla \mathcal{E}(\eta) \) of (1.3), satisfying \( \|f(\eta)\|_H \leq F \) for all \( \eta \in B_{R_M}(\gamma_0) \), does not depend explicitly on \( t \). Therefore, we may estimate by means of Corollary 2.3 for \( 0 \leq \sigma < s \)

\[
\|\partial_t \gamma(s, \cdot) - \partial_t \gamma(\sigma, \cdot)\|_H \leq \int f(\gamma(s, \cdot)) - f(\gamma(s, \cdot)) \, \|\partial_t \gamma(s, \cdot) - \gamma(\sigma, \cdot)\|_H \leq \text{Lip}_{\mathcal{F}} \int_{\sigma}^s \|\nabla E(\gamma(t, \cdot))\|_H \, dt \leq \text{Lip}_{\mathcal{F}} \int_{\sigma}^s \|\nabla E(\gamma(t, \cdot))\|_H \, dt \leq F \cdot \text{Lip}_{\mathcal{F}} |s - \sigma|.
\]

The conservation of the barycenter follows from the first identity in (3.10) of Lemma 3.3 below, applied to the open subset \( O := W^{\frac{2}{3}p-2,2}_{\text{i}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) of the Hilbert space \( \mathcal{H}_1 := \mathcal{H} \) and the energy \( E = \int M(\gamma) \) yielding

\[
\nabla \mathcal{E}(\gamma(t)) := \int_{\gamma([0, T_{\min}])} \nabla \mathcal{E}(\gamma(t, \cdot)) \, dt = 0 \quad \text{for all } t \in [0, T].
\]

By the uniform bound \( F \) on the energy’s gradient (and therefore on \( \partial_t \gamma(t) \) according to (1.3)) presented in (3.4) we may interchange differentiation and integration to obtain

\[
\frac{d}{dt} \int_{\gamma([0, T_{\min}])} \gamma \, dt = \int_{\gamma([0, T_{\min}])} \partial_t \gamma \, dt = -\nabla \mathcal{E}(\gamma(t)) = 0 \quad \text{for all } t \in [0, T].
\]

---

\(^6\gamma \) is unique in the sense that any other \( C^{1,1} \)-mapping \( \eta \) solving (1.3) on a time interval \([0, T]\) coincides with \( \gamma \) on \([0, \text{min}(T, T)]\).
We finish this section with the above mentioned technical lemma on vanishing integral means of gradients in the following more abstract situation. Assume that $\mu$ is a measure on a set $K$ with $\mu(K) < \infty$ so that the constant vector-valued functions $\alpha: K \to \mathbb{R}^n$ are automatically contained in the space $L^2_\mu(K, \mathbb{R}^n)$ of square integrable functions on $K$ with respect to the measure $\mu$. We denote the integral mean of a function $f$ by $\overline{f} := \mu(K)^{-1} \int_K f \, d\mu$.

**Lemma 3.3 (Gradients with zero integral mean).** Let $\mathcal{H}_1$ be a real Hilbert space containing the constant functions $\alpha: K \to \mathbb{R}^n$, and let $\mathcal{O} \subset \mathcal{H}_1$ be an open subset of $\mathcal{H}_1$. Assume that $\mathcal{E}: \mathcal{O} \to \mathbb{R}$ is a Fréchet-differentiable, translationally invariant energy. Then

$$\langle \nabla \mathcal{E}(x), \alpha \rangle_{\mathcal{H}_1} = 0 \quad \text{for all } x \in \mathcal{O}, \alpha \in \mathbb{R}^n.$$  \hfill (3.6)

If, moreover, for a second real Hilbert space $\mathcal{H}_2$, there is a Fréchet-differentiable mapping $S: \mathcal{O} \to \mathcal{H}_2$ satisfying

$$DS(x)\alpha = 0 \quad \text{for all } x \in \mathcal{O}, \alpha \in \mathbb{R}^n,$$  \hfill (3.7)

then one has

$$\langle \nabla_S \mathcal{E}(x), \alpha \rangle_{\mathcal{H}_1} = 0 \quad \text{for all } x \in \mathcal{O}, \alpha \in \mathbb{R}^n,$$  \hfill (3.8)

where $\nabla_S \mathcal{E}(y) = \Pi_{N(DS(y))} \nabla \mathcal{E}(y)$ for $y \in \mathcal{O}$ denotes the projected gradient as defined in (1.4) in the introduction.

If, finally, $\mathcal{H}_1$ is contained in $L^2_\mu(K, \mathbb{R}^n)$ and if there is a constant $C > 0$ with

$$\langle x, \alpha \rangle_{\mathcal{H}_1} = C \langle x, \alpha \rangle_{L^2_\mu} \quad \text{for all } x \in \mathcal{H}_1, \alpha \in \mathbb{R}^n,$$  \hfill (3.9)

then (3.6) and (3.8) imply, respectively,

$$\nabla \mathcal{E}(x) = 0, \quad \text{and} \quad \nabla_S \mathcal{E}(x) = 0 \quad \text{for all } x \in \mathcal{O}.$$  \hfill (3.10)

**Proof.** With $\mathcal{E}(x + \epsilon \alpha) = \mathcal{E}(x)$ for all $x \in \mathcal{O}$, $\alpha \in \mathbb{R}^n$, and $\epsilon \in \mathbb{R}$ with $|\epsilon| \ll 1$, we find $\langle \nabla \mathcal{E}(x), \alpha \rangle_{\mathcal{H}_1} = D \mathcal{E}(x) \alpha = \delta \mathcal{E}(x, \alpha) = \lim_{\epsilon \to 0} \epsilon^{-1} \cdot (\mathcal{E}(x + \epsilon \alpha) - \mathcal{E}(x)) = 0$, which is (3.6). To prove (3.8) we use the assumption (3.7) as well as (3.6) to infer

$$\langle \nabla_S \mathcal{E}(x), \alpha \rangle_{\mathcal{H}_1} = \langle \Pi_{N(DS(x))} \nabla \mathcal{E}(x), \alpha \rangle_{\mathcal{H}_1} = \langle \nabla \mathcal{E}(x), \Pi_{N(DS(x))} \alpha \rangle_{\mathcal{H}_1} = \langle \nabla \mathcal{E}(x), \alpha \rangle_{\mathcal{H}_1} = 0.$$  

Choosing $\alpha := \mu(K)^{-1} \nabla \mathcal{E}(x) \in \mathbb{R}^n$ in (3.6) one obtains from assumption (3.9)

$$\begin{align*}
0 &= C^{-1} \langle \nabla \mathcal{E}(x), \frac{1}{\mu(K)} \nabla \mathcal{E}(x) \rangle_{\mathcal{H}_1} \\
&= \frac{1}{\mu(K)} \int_K \langle \nabla \mathcal{E}(x)(w), \nabla \mathcal{E}(x) \rangle_{\mathbb{R}^n} \, d\mu(w) \\
&= \left( \frac{1}{\mu(K)} \int_K \nabla \mathcal{E}(x)(w) \, d\mu(w), \nabla \mathcal{E}(x) \right)_{\mathbb{R}^n} = |\nabla \mathcal{E}(x)|^2,
\end{align*}$$

so that the first identity in (3.10) is established. Testing the identity $\langle \nabla_S \mathcal{E}(x), \alpha \rangle_{L^2_\mu} \overset{(3.9)}{=} C^{-1} \langle \nabla_S \mathcal{E}(x), \alpha \rangle_{\mathcal{H}_1} \overset{(3.8)}{=} 0$ with $\alpha := \mu(K)^{-1} \nabla_S \mathcal{E}(x)$ yields the second identity in (3.10). \hfill \square

For a proof of long time existence for (1.3) via a continuation argument replacing the initial data $\gamma_0$ by the configuration $\gamma(\cdot, T)$ as a new starting point, it would be helpful to control the essential quantities in (3.1) that bound $T$ from below, namely the bilipschitz constants of $\gamma(\cdot, t)$ and the seminorms $[\gamma(\cdot, t)]_{p-3,2}$. If we knew that the curves $\gamma(\cdot, t)$ had unit speed for all $t \in [0, T]$ we could use [11, Proposition 2.1 & Theorem 1] to bound $\text{BiLip}(\gamma(\cdot, t))$ from below and $[\gamma(\cdot, t)]_{p-3,2}$ from above in terms of the energies $\text{intM}^{(p,3)}(\gamma(\cdot, t))$ that do not increase in time as $t \to T$ along the gradient flow (1.3). However, even if the initial curve $\gamma_0$ is assumed to be arc length parametrized, we do not know in general how the parametrizations...
4 Projection onto the null space of a linear operator

Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two real Hilbert spaces with inner products \( \langle \cdot, \cdot \rangle_{\mathcal{H}_i} : \mathcal{H}_i \times \mathcal{H}_i \to \mathbb{R} \) for \( i = 1, 2 \). Throughout this section \( A : \mathcal{H}_1 \to \mathcal{H}_2 \) is a bounded linear operator, the set of all such operators is denoted by \( \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \). The adjoint \( A^* : \mathcal{H}_2 \to \mathcal{H}_1 \) of \( A \) satisfies by definition

\[
\langle Ax, y \rangle_{\mathcal{H}_2} = \langle x, A^* y \rangle_{\mathcal{H}_1}, \quad \text{for all } x \in \mathcal{H}_1, y \in \mathcal{H}_2.
\]

(4.1)

We start with three simple observations regarding the composition \( AA^* : \mathcal{H}_2 \to \mathcal{H}_2 \).

**Lemma 4.1.** The null spaces of \( A^* \) and \( AA^* \) are identical, that is, \( \mathcal{N}(A^*) = \mathcal{N}(AA^*) \subset \mathcal{H}_2 \).

**Proof.** It suffices to prove that \( \mathcal{N}(AA^*) \subset \mathcal{N}(A^*) \), the other inclusion is immediate. For \( y \in \mathcal{N}(AA^*) \) one has by means of (4.1) \( 0 = \langle Ax, y \rangle_{\mathcal{H}_2} = \langle A^* x, y \rangle_{\mathcal{H}_1} \) for all \( x \in \mathcal{H}_2 \), which can be used for \( x := y \) to obtain \( \|A^* y\|_{\mathcal{H}_1} = 0 \) so that \( y \in \mathcal{N}(A^*) \).

**Lemma 4.2.** Let \( A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) be surjective. Then \( AA^* \) is invertible and the inverse \( (AA^*)^{-1} \) is bounded.

**Proof.** By [30, Theorem 4.12], we have for any bounded linear operator \( T \) between two Hilbert spaces the identity \( \mathcal{N}(T^*) = \mathcal{R}(T)^\perp \). In particular, \( \mathcal{N}(A^*) = \{0\} \) because \( \mathcal{R}(A) = \mathcal{H}_2 \) by assumption. Lemma 4.1 yields \( \mathcal{N}(AA^*) = \{0\} \), i.e., \( AA^* \) is injective. To show that \( AA^* \) is also surjective, and therefore invertible, we use the fact that \( \mathcal{R}(A) = \mathcal{H}_2 \) as assumption and apply [30, Theorem 4.15] to the linear operator \( T := A \) to find a constant \( c > 0 \) such that \( \|A^* y\|_{\mathcal{H}_1} \geq c \|y\|_{\mathcal{H}_2} \) for all \( y \in \mathcal{H}_2 \). This implies by means of (4.1) and the Cauchy-Schwarz inequality \( c^2 \|y\|_{\mathcal{H}_2}^2 \leq \langle Ax, y \rangle_{\mathcal{H}_2} \) for all \( x \in \mathcal{H}_2 \), which simplifies to

\[
\|(AA^*)^* y\|_{\mathcal{H}_2} \geq c^2 \|y\|_{\mathcal{H}_2}, \quad \text{for all } y \in \mathcal{H}_2,
\]

(4.2)

so that again by [30, Theorem 4.15] now applied to the linear operator \( T := AA^* \) we obtain \( \mathcal{R}(AA^*) = \mathcal{H}_2 \). Now that we know that \( (AA^*)^{-1} \) exists we can evaluate (4.2) at \( y := (AA^*)^{-1} x \) for an arbitrary \( x \in \mathcal{H}_2 \), and use again that \( AA^* = (AA^*)^* \) to arrive at the estimate \( \|x\|_{\mathcal{H}_2} \geq c^2 \|(AA^*)^{-1} x\|_{\mathcal{H}_2} \) for all \( x \in \mathcal{H}_2 \), which implies \( \|(AA^*)^{-1}\|_{\mathcal{L}(\mathcal{H}_2)} \leq 1/c^2 \).

The bound on the norm of the operator \( (AA^*)^{-1} \) is rather implicit, since its origin in the proof of [30, Theorem 4.15] lies in the open mapping theorem. However, we do need more explicit
bounds in order to verify Lipschitz continuity of the projection involving the logarithmic strain constraint. Therefore, we add an extra assumption in the present abstract context, namely, that the operator $A$ possesses a bounded right inverse $Y$.

**Lemma 4.3.** Suppose that $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ possesses a right inverse, that is, there is a linear operator $Y \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that $AY = \text{Id}_{\mathcal{H}_2}$. Then,

$$
\|x\|_{\mathcal{H}_2} \leq \|Y\|_{\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)} \|A^*x\|_{\mathcal{H}_1} \quad \text{for all} \ x \in \mathcal{H}_2,
$$

(4.3)

the composition $AA^*$ is invertible, and its inverse $(AA^*)^{-1}$ satisfies the estimate

$$
\|(AA^*)^{-1}\|_{\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)} \leq \|Y\|_{\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)}^2.
$$

(4.4)

**Proof.** Because $A$ has a right-inverse, it is surjective, so that according to Lemma 4.2 $(AA^*)^{-1}$ exists and is bounded. For all $x \in \mathcal{H}_2$, we infer by (4.1) and the Cauchy-Schwarz inequality that $\|x\|_{\mathcal{H}_2}^2 = \langle Yx, x \rangle_{\mathcal{H}_2} = \|Yx\|_{\mathcal{H}_1} \|A^*x\|_{\mathcal{H}_1} \leq \|Y\|_{\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)} \|A^*x\|_{\mathcal{H}_1} \|x\|_{\mathcal{H}_2}$, which implies (4.3). Furthermore, we have by the Cauchy-Schwarz inequality, (4.1), and (4.3)

$$
\|(AA^*)^{-1}\|_{\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)} = \sup_{0 \neq x \in \mathcal{H}_2} \frac{\|(AA^*)^{-1}x\|_{\mathcal{H}_2}}{\|x\|_{\mathcal{H}_2}} = \sup_{0 \neq x \in \mathcal{H}_2} \frac{\|(AA^*)^{-1}A^*x\|_{\mathcal{H}_2}}{\|A^*x\|_{\mathcal{H}_2}} 
\leq \sup_{0 \neq x \in \mathcal{H}_2} \frac{\|A^*x\|_{\mathcal{H}_1}^2}{\|x\|_{\mathcal{H}_2}^2} = \sup_{0 \neq x \in \mathcal{H}_2} \langle A^*x, A^*x \rangle_{\mathcal{H}_1} \leq \|Y\|_{\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)}^2.
$$

Our interest in the invertibility of the mapping $AA^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_2)$ is motivated by the following characterization of orthogonal projections onto the null space of $A$, which can be found in [25, Lemma 6.2]. For the readers’ convenience we provide an alternative proof here.

**Lemma 4.4 (Orthogonal projection onto $\mathcal{N}(A)$).** Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ linear and bounded. Suppose $(AA^*)^{-1}$ exists and is linear and bounded. Then, the orthogonal projection $\Pi_{\mathcal{N}(A)}$ of $\mathcal{H}_1$ onto $\mathcal{N}(A)$ is given by

$$
\Pi_{\mathcal{N}(A)} = \text{Id}_{\mathcal{H}_1} - A^*(AA^*)^{-1}A.
$$

(4.5)

The operator $A^*(AA^*)^{-1}$ is one form of the Moore-Penrose pseudoinverse of $A$.

**Proof.** Denoting the right-hand side of (4.5) as $Z$ we observe that $Z : \mathcal{H}_1 \to \mathcal{H}_1$ is linear and bounded, $AZ(x) = 0$ for all $x \in \mathcal{H}_1$, and $Z$ restricted to $\mathcal{N}(A)$ coincides with the identity $\text{Id}_{\mathcal{H}_1}$ and therefore $R(Z) = \mathcal{N}(A)$. Finally, $x - Z(x) \perp \mathcal{N}(A)$ for all $x \in \mathcal{H}_1$, since by property (4.1) characterizing the adjoint $\langle x - Z(x), y \rangle_{\mathcal{H}_1} = \langle A^*(AA^*)^{-1}Ax, y \rangle_{\mathcal{H}_1} = \langle (AA^*)^{-1}Ax, Ay \rangle_{\mathcal{H}_1} = 0$ for all $y \in \mathcal{N}(A)$. This last property characterizes orthogonal projections onto closed subspaces in Hilbert spaces according to [2, 4.4.2], so that $Z = \Pi_{\mathcal{N}(A)}$.\hfill \Box

We are going to prove Lipschitz continuity of the mapping $A \mapsto \Pi_{\mathcal{N}(A)}$ by bounding its differential, similarly as in the proof of Corollary 2.3. In view of the explicit formula (4.5) we therefore need to estimate the differential of inverses of operators, for which we provide the preparation in the two following elementary lemmas.

For Banach spaces $\mathcal{B}_1, \mathcal{B}_2$, recall that $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is the space of linear and bounded operators from $\mathcal{B}_1$ to $\mathcal{B}_2$, $\text{epi}(\mathcal{B}_1, \mathcal{B}_2)$ denotes the subset of surjective linear bounded operators, whereas $\mathcal{G}\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is the subset of such operators that are bijective.

The following lemma is a quantified version of [1, Lemma 2.5.4].
Lemma 4.5. Let $B$ be a Banach space with operator norm $\|\cdot\| := \|\cdot\|_{\mathcal{L}(B,B)}$. Suppose that the two operators $f \in \mathcal{GL}(B,B)$ and $g \in \mathcal{L}(B,B)$ satisfy $\|f - g\| \leq \frac{1}{2}\|f^{-1}\|^{-1}$. Then, $g \in \mathcal{GL}(B,B)$ and $\|g^{-1}\| \leq 2\|f^{-1}\|$.

Proof. The operator $g = f \circ (\text{Id}_B - f^{-1} \circ (f - g))$ is invertible since $f$ is, and because $\|f^{-1} \circ (f - g)\| \leq \|f^{-1}\| \cdot \|f - g\| \leq 1/2$, and therefore the Neumann series $\sum_{k=0}^{\infty} (f^{-1} \circ (f - g))^k$ converges and equals $(\text{Id}_B - f^{-1} \circ (f - g))^{-1}$. Thus, $g^{-1}$ exists and equals $(\text{Id}_B - f^{-1} \circ (f - g))^{-1} \circ f^{-1}$, satisfying the estimate $\|g^{-1}\| \leq \|(\text{Id}_B - f^{-1} \circ (f - g))^{-1}\| \cdot \|f^{-1}\| \leq \sum_{k=0}^{\infty} \|f^{-1} \circ (f - g)\|^k \|f^{-1}\| \leq 2\|f^{-1}\|$.

Lemma 4.6 ([1, Lemma 2.5.5]). Let $B_1, B_2$ be Banach spaces and consider the mapping $\mathfrak{I} : \mathcal{GL}(B_1, B_2) \to \mathcal{GL}(B_2, B_1)$ defined as $\mathfrak{I}(f) := f^{-1}$ for $f \in \mathcal{GL}(B_1, B_2)$. Then, $\mathfrak{I}$ is of class $C^\infty$ and

$$\mathfrak{I}(f)g = -f^{-1} \circ g \circ f^{-1} \quad \text{for all} \quad g \in \mathcal{T}_f \mathcal{GL}(B_1, B_2) \simeq \mathcal{L}(B_1, B_2).$$  \hfill (4.6)

With this, we may differentiate $A \mapsto \Pi_{\mathcal{N}(A)}$ if $A$ is surjective, since then we may use the explicit representation (4.5) keeping in mind that $(AA^*)^{-1}$ exists and is a linear and bounded operator by virtue of Lemma 4.2.

Lemma 4.7. Let $H_1, H_2$ be Hilbert spaces. Then $\Pi_{\mathcal{N}(A)} : \text{epi}(H_1, H_2) \to \mathcal{L}(H_1, H_2)$ is Fréchet differentiable, and the differential $D(\Pi_{\mathcal{N}(A)})$ at $A \in \text{epi}(H_1, H_2)$ may be estimated as

$$\|D(\Pi_{\mathcal{N}(A)})B\|_{\mathcal{L}(H_1, H_2)} \leq 2 \|A\|_{\mathcal{L}(H_1, H_2)} \|B\|_{\mathcal{L}(H_1, H_2)} \|\mathcal{L}(H_1, H_2)\| (AA^*)^{-1} \|\mathcal{L}(H_1, H_2)\| \|B\|_{\mathcal{L}(H_1, H_2)} \|\mathcal{L}(H_1, H_2)\| \|B\|_{\mathcal{L}(H_1, H_2)}$$

for $B \in \mathcal{L}(H_1, H_2)$. \hfill (4.7)

Proof. Differentiating $\Pi_{\mathcal{N}(A+tB)} = \text{Id}_{H_1} - (A + tB)^*(A + tB)(A + tB)^*$ - $(A + tB)$ with respect to $t$ and evaluating at $t = 0$ we obtain by virtue of Lemma 4.6

$$D(\Pi_{\mathcal{N}(A)})B = -B^*(AA^*)^{-1}A - A^*(AA^*)^{-1}B + A^*(AA^*)^{-1}(AB^* + BA^*)(AA^*)^{-1}A$$

$$= -\Pi_{\mathcal{N}(A)}B^*(AA^*)^{-1}A - A^*(AA^*)^{-1}B\Pi_{\mathcal{N}(A)}.$$

Now (4.7) immediately follows since the orthogonal projection $\Pi_{\mathcal{N}(A)}$ has norm one, and the adjoint of a bounded linear operator has the same norm as the operator itself.

To bound the right-hand side of (4.7) in a controlled way, it is helpful to have the explicit bounds on $\|\mathcal{L}(H_1, H_2)\|$ that are available if $A$ has a right inverse according to Lemma 4.3.

Lemma 4.8. Let $A \in \mathcal{L}(H_1, H_2)$ and $Y \in \mathcal{L}(H_2, H_1)$ satisfy $AY = \text{Id}_{H_2}$. Then, $\Pi_{\mathcal{N}(A)}$ restricted to the ball $B_{R_H}(A) \subset \mathcal{L}(H_1, H_2)$ of radius

$$R_H := \min \left\{ \left[ 2 \left\| Y \right\|^2 \left( \left\| A \right\|_{\mathcal{L}(H_1, H_2)} + 1 \right) \right]^{-1}, 1 \right\}$$

is Lipschitz continuous with Lipschitz constant $\text{Lip}_H = \text{Lip}_H(\left\| A \right\|_{\mathcal{L}(H_1, H_2)}, \left\| Y \right\|_{\mathcal{L}(H_2, H_1)})$, which is non-decreasing in its arguments.

Proof. Within this proof we simply denote by $\|\cdot\|$ all operator norms disregarding the respective domains and target spaces of the operators involved. The convex combination of two operators $A_0, A_1 \in B_{R_H}(A)$ will be abbreviated by $A_t := (1 - t)A_0 + tA_1$ for $t \in [0, 1]$ so that we can estimate the difference of their projections as $\|\Pi_{\mathcal{N}(A_0)} - \Pi_{\mathcal{N}(A_1)}\| \leq \int_0^1 \|D(\Pi_{\mathcal{N}(A_t)})(A_t - A_0)\| \, dt$. With (4.7) in Lemma 4.7 we may bound the integrand by $2\|A_t\|\|A_1 - A_0\|\|A_tA_t^*\|^{-1}$. Since
\[ \| A_t \| \leq \| A \| + R_t \leq \| A \| + 1 \text{ for all } t \in [0, 1], \text{ the remaining task is to bound } \| (A_t A_t^*)^{-1} \|. \]

According to Lemma 4.3 the operator \( A A^* \) is invertible satisfying the estimate \( \| (A A^*)^{-1} \| \leq \| Y \|^{-2} \), and Lemma 4.5 gives that if

\[ \| A_t A_t^* - A A^* \| \leq (2 \| Y \|^{-2})^{-1} \leq (2 \| (A A^*)^{-1} \|)^{-1}, \]  \hspace{1cm} (4.9)

then \( A_t A_t^* \) is invertible and satisfies \( \| (A_t A_t^*)^{-1} \| \leq 2 \| (A A^*)^{-1} \| \leq 2 \| Y \|^{-2} \). In order to ensure condition (4.9) we observe by virtue of the norm identity for operators and their adjoints, and by (4.8)

\[ \| A_t A_t^* - A A^* \| \leq \| A_t A_t^* - A_t A^* + A_t A^* - A A^* \| \leq (\| A_t + \| A \|) \| A_t - A \| \leq (2 \| A \| + \| A \|) \| A_t - A \| \leq (2 \| A \| + \| A \| + 1) \| A_t - A \| \leq (2 \| Y \|^{-2})^{-1}, \]

\[ \text{which is the desired prerequisite (4.9) that we needed to bound } \| (A_t A_t^*)^{-1} \|. \]

In summary, \( \| \Pi_{\mathcal{X}(A_0)} - \Pi_{\mathcal{X}(A_1)} \| \leq 2 \| A_t + 1 \| Y \|^{-2} \| A_0 - A_1 \| =: \text{Lip}_\mathcal{X}(\| A \|, \| Y \|) \| A_0 - A_1 \|. \] \[ \square \]

5 The logarithmic strain constraint

Recall from (1.6) in the introduction that we chose the logarithmic strain \( \Sigma(\gamma) := \log(\| \gamma'(\cdot) \|) \) as the constraint determining the projected flow (1.9). Here, we consider \( \Sigma \) as a mapping defined on the space \( W^{2,p-2,2}_v(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) of regular and embedded fractional Sobolev curves \( \gamma \) whose minimal velocity \( v_\gamma = \min \| \gamma' \| \) is well-defined and strictly positive because of the Morrey embedding, Theorem A.2. The logarithmic strain \( \Sigma \) takes values in the space of scalar-valued functions of class \( W^{2,p-3,2}_v \), thus loosing one order of differentiability in the image space. We need differentiability and local Lipschitz continuity of the differential \( \Delta \Sigma \), which is provided by the following proposition for \( \varrho := 2 \) and \( s := \frac{3p}{2} - 3 \).

**Proposition 5.1.** Let \( s \in (0, 1) \) and \( \varrho > 1 \) satisfy \( s - 1/\varrho > 0 \). Then the logarithmic strain \( \Sigma: W^{1+s,\varrho}_v(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to W^{-s,\varrho}_v(\mathbb{R}/\mathbb{Z}) \) defined as \( \Sigma(\gamma) := \log(\| \gamma'(\cdot) \|) \) is Fréchet-differentiable and its differential is given by

\[ D\Sigma(\gamma): W^{1+s,\varrho}_v(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to W^{-s,\varrho}_v(\mathbb{R}/\mathbb{Z}), \quad D\Sigma(\gamma) h = \left( \frac{\gamma''(\cdot)}{\| \gamma''(\cdot) \|}, \frac{h'(\cdot)}{\| h'(\cdot) \|} \right), \]  \hspace{1cm} (5.1)

for \( \gamma \in W^{1+s,\varrho}_v(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) applied to \( h \in W^{1+s,\varrho}_v(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \). Moreover \( D\Sigma \) is bounded and locally Lipschitz-continuous with \( \| D\Sigma(\gamma) \|_{L(W^{1+s,\varrho}_v, W^{-s,\varrho}_v)} \leq C_\Sigma, \) and

\[ \| D\Sigma(\eta_1) - D\Sigma(\eta_2) \|_{L(W^{1+s,\varrho}_v, W^{-s,\varrho}_v)} \leq \text{Lip}_{D\Sigma} \| \eta_1 - \eta_2 \|_{s,\varrho} \]  \hspace{1cm} (5.2)

for all \( \eta_1, \eta_2 \in W^{1+s,\varrho}_v(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) with \( \| \gamma' - \eta'_1 \|_{s,\varrho} < \text{BiLip}(\gamma)/(2C_\Sigma C_P) \). The constant \( C_\Sigma \) and the Lipschitz constant \( \text{Lip}_{D\Sigma} \) depend only on \( n, p, \) non-increasingly on \( v_\gamma, \) and non-decreasingly on \( \| \gamma' \|_{s,\varrho} \).

**Proof.** Define \( \varphi: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \varphi(x) := \log(\| x \|) \) and observe that \( \varphi \) is smooth. Its first three partial derivatives are given by \( \delta_i \varphi(x) = x_i/\| x \|^2, \delta_i \delta_j \varphi(x) = (\delta_{ij} - 2x_i x_j/\| x \|^2)\| x \|^{-2}, \) and

\[ \delta_i \delta_j \delta_k \varphi(x) = -\frac{4}{\| x \|^4} \left( \delta_{ij} x_k + \delta_{jk} x_i + \delta_{ki} x_j + \frac{4x_i x_j x_k}{\| x \|^2} \right), \]

so that the operator norms \( \| D^k \varphi(x) \| \) are bounded from above by \( C(k) \| x \|^{-k} \) for\(^7 k = 1, 2, 3.\)

With our candidate expression for the Fréchet derivative from (5.1) and Taylor’s theorem, we

\[ \text{This is actually true for all orders of differentiability } k \in \mathbb{N}. \]
have pointwise for each $u \in \mathbb{R}/\mathbb{Z}$:
\[
(\Sigma(\gamma + h) - \Sigma(\gamma) - (D\Sigma(\gamma) h))(u) = \varphi((\gamma + h)'(u)) - \varphi'(\gamma(u)) - D\varphi(\gamma(u)) h'(u)
\]
\[
= \int_0^1 (1 - \theta) D^2 \varphi((\gamma + \theta h)'(u)) (h'(u), h'(u)) \, d\theta.
\]
Thus, in order to show that our candidate is indeed the Fréchet derivative, we merely have to show that the integral on the right is dominated by $\|h\|_{\mathcal{B}_{1+s,\varrho}}^2$ for sufficiently small $\|h\|_{\mathcal{B}_{1+s,\varrho}}$. Hence we may assume that $[h'_s]_{s,\varrho} \leq \text{BiLip}(\gamma)/(2C_{E,C})$, so that by means of Corollary B.4 $|\gamma'_0(u)| \geq v_\varphi/2$ for all $\theta \in [0,1]$ and $u \in \mathbb{R}/\mathbb{Z}$ where we have abbreviated $\gamma_\theta := \gamma + \theta h$. Therefore, by definition of the seminorm, boundedness and Lipschitz continuity of $D^2 \varphi$, we have for any $\theta \in [0,1]$
\[
\|D^2 \varphi(\gamma_\theta(\cdot))\|_{W^{s,\varrho}} \leq \|D^2 \varphi(\gamma_0(\cdot))\|_{L^\infty} + [D^2 \varphi(\gamma_0(\cdot))]_{s,\varrho}
\]
\[
\leq v_\varphi^{-2} (4C(2) + C(3)v_\varphi^{-1} [\gamma']_{s,\varrho}) \leq v_\varphi^{-2} (4C(2) + C(3)v_\varphi^{-1} [\gamma']_{s,\varrho} + [h']_{s,\varrho}). \tag{5.3}
\]
By the product rule from Proposition A.5 and by (5.3), we have
\[
\| \int_0^1 (1 - \theta) D^2 \varphi((\gamma_\theta(\cdot)) (h'(\cdot), h'(\cdot)) \, d\theta \|_{W^{s,\varrho}} \leq \int_0^1 \|D^2 \varphi((\gamma_\theta(\cdot)) (h'(\cdot), h'(\cdot)))\|_{W^{s,\varrho}} \, d\theta \leq C_{\Sigma}(n, s, \varrho)^2 v_\varphi^{-2} (4C(2) + C(3)v_\varphi^{-1} [\gamma']_{s,\varrho} + [h']_{s,\varrho}) \|h\|_{W^{1+s,\varrho}}^2.
\]
Using the bounds on $D^2 \varphi$ and $D^2 \varphi$ on $\mathbb{R}^n \setminus \{0\}$ we obtain similarly to (5.3)
\[
\|D\Sigma(\gamma)\| = \sup \|D\varphi(\gamma(\cdot)) h(\cdot)\|_{W^{s,\varrho}} : \|h\|_{W^{1+s,\varrho}} \leq 1 \} \leq C_{\Sigma}
\]
where $C_{\Sigma}$ depends only on $s, \varrho$, non-increasingly on $v_\varphi$, and non-decreasingly on $[\gamma']_{s,\varrho}$. Repeating the whole proof with $\varphi$ replaced by $D^2 \varphi$, one shows that $\Sigma$ is actually twice Fréchet-differentiable with uniformly bounded operator norm $\|D^2 \Sigma(\eta)\|$ for all $\eta$ in the ball of radius $r := \text{BiLip}(\gamma)/(2C_{E,C})$ around $\gamma$. Thus $D\Sigma$ is locally Lipschitz-continuous with Lipschitz constant $\text{Lip}_{D\Sigma} = \sup_{\eta \in B(\gamma)} \|D^2 \Sigma(\eta)\| < \infty$ depending non-increasingly on $v_\varphi$ and non-decreasingly on $[\gamma']_{s,\varrho}$.

In view of the results of Section 4 we need to show that the differential $D\Sigma(\gamma)$ possesses a uniformly bounded right inverse. The following lemma and its proof are simpler variants of [32, Lemma 2.4].

**Lemma 5.2.** Let $s \in (0,1)$ and $\varrho > 1$ satisfy $s - 1/\varrho > 0$. For all $\gamma \in W^{1+s,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ the differential $D\Sigma(\gamma)$ admits a right inverse $Y_\gamma : (W^{s,\varrho}(\mathbb{R}/\mathbb{Z}), W^{1+s,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ satisfying $\|Y_\gamma\|_{L(W^{s,\varrho}, W^{1+s,\varrho})} \leq K_{\gamma}$, where $K_{\gamma} = K_{\gamma}(n, s, \varrho, v_\varphi, [\gamma']_{s,\varrho}) > 0$ depends non-increasingly on the minimal velocity $v_\varphi$, and non-decreasingly on $[\gamma']_{s,\varrho}$.

**Proof.** Fix a curve $\gamma \in W^{1+s,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Our goal is to construct a bounded linear map $Y_\gamma : W^{s,\varrho}(\mathbb{R}/\mathbb{Z}) \to W^{1+s,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ by assigning to a given function $\lambda \in W^{s,\varrho}(\mathbb{R}/\mathbb{Z})$ the vector-valued function
\[
(Y_\gamma \lambda)(x) := \int_0^x \left( \lambda(y) \gamma'(y) + P_{\gamma'(y)} : V_{\lambda} \right) \, dy, \quad x \in \mathbb{R}. \tag{5.4}
\]
Here, $V_{\lambda} \in \mathbb{R}^n$ is a vector (depending on the given $\lambda$) to be determined later to secure that $Y_\gamma \lambda$ is 1-periodic, and $P_{\gamma'(y)} := \text{Id}_{\mathbb{R}^n} - \gamma'(y)\gamma'(y)^\top/|\gamma'(y)|^2$ is the orthogonal projection onto the orthogonal complement of $\gamma'(y)$. From the product rule Proposition A.5 we learn that $Y_\gamma \lambda$ is indeed a member of $W^{1+s,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$: The mappings $\lambda$, $\gamma'$ and $\gamma'/|\gamma'|$ are all of class $W^{s,\varrho}$,
so multilinear combinations of them are again of class $W^{s,\varphi}$; after integration, we end up with $Y_\gamma(\lambda) \in W^{1+s,\varphi}(\mathbb{R}^3, \mathbb{R}^n)$ where we assumed the 1-periodicity of $Y_\gamma, \lambda$ by a suitable choice of the vector $V_\lambda$ below. More precisely, we may use Lemma 5.6 in addition to obtain

$$[(Y_\gamma \lambda)]_{s,\varphi} \leq C [\gamma]_{s,\varphi} + C (1 + \|\gamma\|_4^4) \|V_\lambda\|.$$  

We conclude from (5.1) that $D\Sigma(\gamma) Y_\gamma \lambda = (\gamma/|\gamma|^2, \lambda \gamma' + P_{\gamma',\gamma} \cdot V_\lambda)_{\mathbb{R}^n} = \lambda$, no matter what choice we make for $V_\lambda \in \mathbb{R}^n$. Regarding that choice notice first that by the Morrey embedding Theorem A.2, $(Y_\gamma \lambda)$ is continuous and 1-periodic. Hence $Y_\gamma \lambda$ is 1-periodic if and only if 0 = $(Y_\gamma \lambda)(1) - (Y_\gamma \lambda)(0) = \int_0^1 \lambda(y) \gamma'(y) dy + \Theta_\gamma(V_\lambda)$, where $\Theta_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map defined by

$$\Theta_\gamma := \int_0^1 P_{\gamma(\langle u \rangle)} du = \int_0^1 (\text{Id}_{\mathbb{R}^n} - \gamma'(u)/|\gamma'(u)|^2) du.$$ 

Hence, assuming $\Theta_\gamma$ is boundedly invertible, we put $V_\lambda := -\Theta_\gamma^{-1} \int_0^1 \lambda(y) \gamma'(y) dy$ and observe that $|V_\lambda| \leq C \|\Theta_\gamma^{-1}\| \|\lambda\|_{W^{s,\varphi}}$. By the Morrey embedding, we have $\gamma'/|\gamma'| \in C^{0,\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $\alpha = s - 1/\varphi > 0$. So finally, Lemma 5.3 below applied to $\Theta := \Theta_\gamma$ with $\tau := \gamma'/|\gamma'|$ and $H := \mathbb{R}^n$ provides us with an explicit bound on the operator norm $\|\Theta_\gamma^{-1}\| = 1/\min \lambda_{\min}(\Theta_\gamma)$ and in doing so yields the lacking invertibility. Note that the unit tangent $\tau$ is indeed not contained in any hemisphere of $S^{n-1}$, since otherwise there existed a vector $v \in S^{n-1}$ such that $\langle v, \tau(t) \rangle > 0$ for all $t \in \mathbb{R}/\mathbb{Z}$. Then, also $0 < \langle v, \tau(t) \rangle |\gamma'(t)| = \langle v, \gamma'(t) \rangle$ and so $0 < \int_0^1 \langle v, \gamma'(t) \rangle dt = \langle v, \gamma(1) - \gamma(0) \rangle = 0$ which is a contradiction. The Hölder constant in Lemma 5.3 is compatible to the one defined via the Euclidean distance as $|x - y| = 2 \sin(\pi/2 < (x, y)) = 2 \sin(\pi/2 d_S(x, y))$ for $|x| = |y| = 1$ and as such we have $|x - y| \leq d_S(x, y) \leq \pi/2 |x - y|$. As $\lambda \mapsto V_\lambda$ is linear, one has linearity of $\lambda \mapsto Y_\gamma \lambda$ as well. Finally, Lemma 5.3 yields the estimate $|V_\lambda| \leq C(s, \varphi, \gamma) \|\gamma||_{W^{s,\varphi}}$ with a constant $C$ depending non-increasingly on $\varphi$ and non-decreasingly on $[\gamma]_{s,\varphi}$, and this inequality inserted in (5.5) together with a straightforward estimate of the $L^P$-norm of $Y_\gamma, \lambda$ implies the desired bound on the operator norm $\|Y_\gamma\|_{L^p(W^{s,\varphi}, W^{1+s,\varphi})}$.

**Lemma 5.3.** Let $\mathcal{H}$ be a real Hilbert space of dimension at least 2, let $0 < \alpha \leq 1$ and denote by $S \subset \mathcal{H}$ the unit sphere. Let $\tau : \mathbb{R}/\mathbb{Z} \to S$ be a closed $\alpha$-Hölder-continuous curve which is not contained in any hemisphere and has Hölder constant $C = \sup_{\rho \neq t} d_S(\tau(t), \tau(s)) |s - t|^{-\alpha}$. Here $d_S$ denotes the angular distance on the sphere. Then the smallest eigenvalue of the self-adjoint operator $\Theta := \int_0^1 (\text{Id}_{\mathcal{H}} - \tau(t)\tau(t)^\top) dt$ is bounded from below by

$$\lambda_{\min}(\Theta) \geq \frac{\pi^2}{(2\pi+1)\alpha} \left(\frac{\pi\lambda}{2\pi+1}\right)^{(1/\alpha)}.$$ 

**Proof.** Let $v \in S$ be an eigenvector of $\Theta$ corresponding to $\lambda_{\min}(\Theta)$. For $\delta \in (0, \pi/2]$ define the polar caps $\mathbb{B}_\delta \lambda := \{ e \in S \mid \langle e, \tau(\rho) \rangle > \cos(\delta) \}$ and the set $I_\delta := \{ t \in \mathbb{R}/\mathbb{Z} \mid \tau(t) \notin \mathbb{B}_\delta \lambda \cup \mathbb{B}_{-\delta} \}$. Observe that

$$\lambda_{\min}(\Theta) = \int_0^1 \left(1 - \langle v, \tau(t) \rangle^2\right) dt \geq \int_{I_\delta} \left(1 - \langle v, \tau(t) \rangle^2\right) dt \geq |I_\delta| \sin^2(\delta) \geq \frac{\pi}{2\pi+1} \delta^2.$$ 

By assumption, the curve cannot be contained in only one of the polar caps $\mathbb{B}_\delta \lambda$. So there must be at least one $t_0 \in \mathbb{R}/\mathbb{Z}$ so that $\tau(t_0)$ lies on the equator, i.e., $\langle v, \tau(t_0) \rangle = 0$. From there, the curve has to travel a distance at least $\pi/2 - \delta$ to reach $\mathbb{B}_\delta \lambda$; since $\tau$ is $\alpha$-Hölder continuous, it requires at least $(\pi/2 - \delta)^{1/\alpha}$ distance in the pre-image for that. The same applies for reaching $\mathbb{B}_{-\delta}$. Since $\tau$ is closed, it has to traverse the equator twice. This provides us with $|I_\delta| \geq \frac{4}{(\pi/2 - \delta)^{1/\alpha}}$ and thus with $\lambda_{\min}(\Theta) \geq \left(\pi/2 - \delta\right)^{-1/\alpha} \delta^2$. Maximizing the right-hand side for $0 < \delta < \pi/2$ leads to $\delta = \frac{2}{2\pi+1}$ and substitution leads to the stated lower bound for $\lambda_{\min}(\Theta)$.\[\Box\]
6 Long Time Existence

For a given curve $\gamma \in W^{2,p-2,2}_t(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ define the radius

$$R_{M,\Sigma} = R_{M,\Sigma}(\gamma) := \min \{ \text{BiLip}(\gamma), 1 \}$$

where $C_E$ and $C_P$ are the constants in the Morrey embedding (A.4) and in Poincaré’s inequality (A.6), and $C_\Sigma$ and $\text{Lip}_{\Sigma}$ are the bound on and the Lipschitz constant of the differential $D\Sigma(\gamma)$ in Proposition 5.1, and $K_Y$ is the bound on its right inverse $Y_\gamma$ in Lemma 5.2.

**Lemma 6.1.** For $\gamma \in W^{2,p-2,2}_t(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with $p \in (\frac{7}{3}, \frac{8}{3})$ the projected gradient $\nabla_\Sigma \int \mathbb{M}^{(p,2)}$ is bounded and Lipschitz continuous on the ball $B_{R_{M,\Sigma}}(\gamma) \subset W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with bound $\tilde{C}_1$ and Lipschitz constant $\text{Lip}_{\Sigma}^{\infty}$ depending on $n, p$, non-increasingly on $\text{BiLip}(\gamma)$ and $v_\gamma$, and non-decreasingly on $[\gamma']_{2p-3,2}^+$.

**Proof.** Comparing the definition of $R_{M,\Sigma}$ with that of the Lipschitz radius $R_M$ in (2.22) we find $R_{M,\Sigma} \leq R_M$, and therefore all previous results proved on $B_{R_M}(\gamma) \subset W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ hold on the smaller concentric ball $B_{R_{M,\Sigma}}(\gamma)$ as well. In particular, by Corollary B.4

$$\text{BiLip}(\eta) \geq \text{BiLip}(\gamma)/2 \quad \text{and} \quad [\eta']_{2p-3,2}^+ \leq [\gamma']_{2p-3,2}^+ + R_{M,\Sigma} \leq [\gamma']_{2p-3,2}^+ + 1 \quad (6.2)$$

for all $\eta \in B_{R_{M,\Sigma}}(\gamma)$. Therefore, we can use (2.1) in Theorem 2.1 to estimate for $E := \int \mathbb{M}^{(p,2)}$, $H := W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and each $\eta \in B_{R_{M,\Sigma}}(\gamma)$

$$\|\nabla E(\eta)\|_H \leq C_1(\text{BiLip}(\eta), [\eta']_{2p-2,2}^+) \leq C_1(\text{BiLip}(\gamma)/2, [\gamma']_{2p-2,2}^+ + 1) =: \tilde{C}_1, \quad (6.3)$$

where $\tilde{C}_1$ still depends non-increasingly on $\text{BiLip}(\gamma)$ and non-decreasingly on $[\gamma']_{2p-2,2}^+$ by the monotonicity properties of $C_1$ stated in Theorem 2.1 (suppressing its dependence on $n, p$). Moreover, recall from Corollary 2.3 that

$$\|\nabla E(\eta_1) - \nabla E(\eta_2)\|_H \leq \text{Lip}_{\Sigma} \|\eta_1 - \eta_2\|_H \quad (6.4)$$

for all $\eta_1, \eta_2 \in B_{R_{M,\Sigma}}(\gamma)$, where the Lipschitz constant $\text{Lip}_{\Sigma}$ depends non-increasingly on $\text{BiLip}(\gamma)$, and non-decreasingly on $[\gamma']_{2p-2,2}^+$. In view of the logarithmic constraint we infer from (5.2) in Proposition 5.1 for $\eta_1, \eta_2 \in B_{R_{M,\Sigma}}(\gamma)$ the inequality

$$\|D\Sigma(\eta_i) - D\Sigma(\gamma)\|_{\mathcal{L}(H, W^{2,p-3,2})} \leq \text{Lip}_{\Sigma} \cdot R_{M,\Sigma} \quad \text{for} \quad i = 1, 2, \quad (6.5)$$

where the Lipschitz constant $\text{Lip}_{\Sigma}$ depends non-increasingly on $v_\gamma$, and non-decreasingly on $[\gamma']_{2p-3,2}^+$. With $\|D\Sigma(\gamma)\|_{\mathcal{L}(H, W^{2,p-3,2})} \leq C_\Sigma$ by means of Proposition 5.1, and with $\|\gamma_\Sigma\|_{\mathcal{L}(H, W^{2,p-3,2})} \leq K_\gamma$ according to Lemma 5.2 we find for the radius $R_{\Pi}(D\Sigma(\gamma))$ of Lemma 4.8 for $A := D\Sigma(\gamma)$ the lower estimate (see (4.8))

$$R_{\Pi}(D\Sigma(\gamma)) \geq \min \{ [2K_\Sigma^2(2C_\Sigma + 1)]^{-1}, 1 \} \geq \text{Lip}_{D\Sigma} R_{M,\Sigma}. \quad (6.6)$$

Thus, (6.5) implies that for $i = 1, 2$, we have $D\Sigma(\eta_i) \in B_{R_M(D\Sigma(\gamma))}(D\Sigma(\gamma))$ in the space $\mathcal{L}(H, W^{2,p-3,2})$ of linear mappings, as long as $\eta_1, \eta_2 \in B_{R_{M,\Sigma}}(\gamma)$. This in turn allows us to
together with the abstract Lemma 4.8 for \( A := D\Sigma(\gamma) \) with the Lipschitz estimate (5.2) for the differential \( D\Sigma(\cdot) \) in Proposition 5.1 to conclude
\[
\|\Pi_N(D\Sigma(\eta_1)) - \Pi_N(D\Sigma(\eta_2))\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \leq \text{Lip}_\Pi \|D\Sigma(\eta_1) - D\Sigma(\eta_2)\|_{\mathcal{L}(\mathcal{H}, W^{2-p,2})} \leq \text{Lip}_\Pi \text{Lip}_{D\Sigma} [\eta'_1 - \eta'_2]_{p-3,2},
\]
where we took for simplicity \( \text{Lip}_\Pi = \text{Lip}_\Pi(C_\Sigma, K_Y) \) because those two arguments bound the norms of \( D\Sigma \) and \( Y_\gamma \), and \( \text{Lip}_\Pi \) is non-decreasing in both of its arguments; see Lemma 4.8. Finally, we infer from (6.2), (6.4), and (6.7) the estimate
\[
\|\nabla_\Sigma \mathcal{E}(\eta_1) - \nabla_\Sigma \mathcal{E}(\eta_2)\|_{\mathcal{H}} = \|\Pi_N(D\Sigma(\eta_1))(\nabla \mathcal{E}(\eta_1)) - \Pi_N(D\Sigma(\eta_2))(\nabla \mathcal{E}(\eta_2))\|_{\mathcal{H}} \\
\leq \|\Pi_N(D\Sigma(\eta_1)) - \Pi_N(D\Sigma(\eta_2))\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \|\nabla \mathcal{E}(\eta_1)\|_{\mathcal{H}} + \|\Pi_N(D\Sigma(\eta_2))\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})} \cdot \|\nabla \mathcal{E}(\eta_1) - \nabla \mathcal{E}(\eta_2)\|_{\mathcal{H}} \\
\leq (\tilde{C}_1 \text{Lip}_\Pi \text{Lip}_{D\Sigma} + \text{Lip}_{\nabla \mathcal{E}}) \|\eta_1 - \eta_2\|_{\mathcal{H}}
\]
for all \( \eta_1, \eta_2 \in B_{R_{M,S}(\cdot)} \) as \( \Pi \) has operator norm bounded by 1, which proves the claim with the Lipschitz constant \( \text{Lip}_{\nabla_\Sigma \mathcal{E}} := \tilde{C}_1 \cdot \text{Lip}_\Pi D\Sigma + \text{Lip}_{\nabla \mathcal{E}} \) depending non-increasingly on \( \text{BiLip}(\gamma) \) and \( v_\gamma \), and non-decreasingly on \( [\gamma'_1]_{p-3,2} \). Indeed, as mentioned above, \( \tilde{C}_1 \), \( \text{Lip}_{\nabla \mathcal{E}} \), and \( \text{Lip}_{D\Sigma} \) depend non-decreasingly on \( [\gamma'_1]_{p-3,2} \), and non-increasingly on \( \text{BiLip}(\gamma) \) or \( v_\gamma \), respectively. Both arguments \( C_\Sigma \) and \( K_Y \) of \( \text{Lip}_\Pi(\cdot, \cdot) \) are non-increasing in \( v_\gamma \) and non-decreasing in \( [\gamma'_1]_{p-3,2} \), which concludes the proof.

With the exact same proof as for Theorem 3.2 we can show the following short time existence for the projected gradient flow (1.9) for integral Menger curvature. Notice in that context that the right-hand side of (1.9) has the same upper bound as the gradient flow (1.3) since it differs from that only by an orthogonal projection with operator norm 1.

**Corollary 6.2 (Short time existence).** For \( p \in \left( \frac{7}{4}, \frac{3}{2} \right) \) and \( \gamma_0 \in W^{2-p,2}_{0,\mathbb{Z}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) there exists a time \( T > 0 \) and a unique mapping \( t \mapsto \gamma(\cdot, t) \in C^{1,1}(0, T]; W^{2-p,2}_{0,\mathbb{Z}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) such that \( \gamma \) solves (1.9) and satisfies
\[
|\gamma'(\cdot, t)| = |\gamma'_0(\cdot)| \quad \text{on } \mathbb{R}/\mathbb{Z} \quad \text{for all } t \in [0, T].
\]

The mapping \( t \mapsto \text{intM}^{(p,2)}(\gamma(t)) \) is non-increasing on \( [0, T] \), and the minimal time \( T \) of existence is estimated from below as
\[
T \geq T_{\min}(\gamma_0) := \frac{R_{M,S}(\gamma_0)}{C_1(n, p, \text{BiLip}(\gamma_0))/2, [\gamma'_0]_{p-3,2} + 1},
\]
where \( R_{M,S}(\gamma_0) \) is the radius defined in (6.1) for \( \gamma_0 \) instead of \( \gamma \), and \( C_1 \) is the bound for the differential of \( \text{intM}^{(p,2)} \) in (2.1) of Theorem 2.1. Moreover, the image \( \gamma(\cdot, [0, T_{\min}]) \) is contained in the ball \( B_{R_{M,S}(\gamma_0)} \subset W^{2-p,2}_{0,\mathbb{Z}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \).

**Proof.** According to our remark above referring to the proof of Theorem 3.2 it suffices to verify (6.9) and the energy decay in time. The conservation of velocities directly follows from the projected gradient flow (1.9), since this implies \( -v_\gamma \) as pointed out in the introduction in (1.5) – that the logarithmic constraint \( \Sigma \) is a conserved quantity in time, i.e., \( \Sigma(\gamma(\cdot, t)) = \Sigma(\gamma(0)) \) for all \( t \in [0, T] \), thus leading to (6.9) since the logarithm is injective on \( (0, \infty) \). For the energy decay we abbreviate \( \mathcal{E} := \text{intM}^{(p,2)}(\gamma) \), \( \mathcal{H} := W^{2-p,2}_{0,\mathbb{Z}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \), and differentiate the energy of \( \gamma(\cdot, t) \) with respect to time and obtain similarly as in (3.3) by means of (1.9)
\[
\frac{d}{dt} \mathcal{E}(\gamma(\cdot, t)) = \langle \nabla \mathcal{E}(\gamma(\cdot, t)), \frac{d\gamma}{dt}(\gamma(\cdot, t)) \rangle_\mathcal{H} = -\langle \nabla \mathcal{E}(\gamma(\cdot, t)), \nabla_\Sigma \mathcal{E}(\gamma(\cdot, t)) \rangle_\mathcal{H} = -\|\nabla_\Sigma \mathcal{E}(\gamma(\cdot, t))\|_{\mathcal{H}}^2 \leq 0
\]
for all \( t \in (0, T) \), which implies the energy decay in time. \( \square \)
Our goal is to extend the short time solution by a fixed time step depending only on the initial configuration. To that extent we first establish a priori estimates for the bilipschitz constant and the fractional seminorm along the flow \((1.9)\).

**Proposition 6.3 (A priori estimates).** Let \(\gamma_0 \in \mathcal{W}^{\frac{2}{p}-2,2}_{i\nu}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)\) for \(p \in \left(\frac{5}{4}, \frac{8}{5}\right)\). Then the energy \(\int \mathcal{M}^{(p, 2)}(\gamma(\cdot, t))\) of any solution \(t \mapsto \gamma(\cdot, t) \in C([0, T^*], \mathcal{W}^{\frac{2}{p}-2,2}_{i\nu}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))\) of \((1.9)\) on the time interval \([0, T^*]\) for some \(T^* > 0\) is non-increasing in time, and the velocities of \(\gamma(\cdot, t)\) satisfy

\[
|\gamma'(\cdot, t)| = |\gamma(0)| \quad \text{for all} \ t \in [0, T^*].
\]

Moreover, the bilipschitz constant \(\text{BiLip}(\gamma(\cdot, t))\) and the seminorm \(|\gamma'(\cdot, t)|_{\frac{2}{p}-2,2}\) of \(\gamma(\cdot, t)\) satisfy

\[
\text{BiLip}(\gamma(\cdot, t)) \geq v_0 b(\int \mathcal{M}^{(p, 2)}(\gamma_0)),
\]

\[
|\gamma'(\cdot, t)|_{\frac{2}{p}-2,2} \leq s(\int \mathcal{M}^{(p, 2)}(\gamma_0)) |\gamma(0)|_{\frac{2}{p}-2,2} (1 + v_0^{-1} |\gamma(0)|_{\frac{2}{p}-2,2})^{1/2}
\]

for all \(t \in [0, T^*]\), where the functions \(b(\cdot)\) and \(s(\cdot)\) are strictly positive, and \(b(\cdot)\) is non-increasing whereas \(s(\cdot)\) is non-decreasing on \((0, \infty)\). Finally, the time derivative \(t \mapsto \partial_t \gamma(\cdot, t) / \partial t\) is Lipschitz continuous with a uniform Lipschitz constant \(\text{Lip}_{\partial_t} \gamma\) depending non-decreasingly on the initial energy \(\int \mathcal{M}^{(p, 2)}(\gamma_0)\), on the seminorm \(|\gamma(0)|_{\frac{2}{p}-2,2}\), and non-increasingly on \(v_0\).

**Proof.** Abbreviate \(E := \int \mathcal{M}^{(p, 2)}\). The energy decay in time as well as the conservation of velocity \((6.11)\) can be shown by the exact same observations as in the previous proof. The energy dependent lower bound for the bilipschitz constant established in [11, Proposition 2.1] implies the non-increasing function \(b : (0, \infty) \to \mathbb{R}\) defined as

\[
b(E) := \inf \{ \text{BiLip}(\Gamma) : \Gamma \in \mathcal{W}^{\frac{2}{p}-2,2}_{i\nu}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), \|\Gamma\| = 1, E(\Gamma) \leq E \}
\]

is strictly positive for each \(E \in (0, \infty)\). Introducing the arc length parametrization \(\Gamma(\cdot, t)\) of the solution \(\gamma(\cdot, t)\) of \((1.9)\) as in Appendix B, we infer from Lemma B.1 that

\[
\text{BiLip}(\gamma(\cdot, t)) \geq v_0 \text{BiLip}(\Gamma(\cdot, t)) \overset{(6.11)}{=} v_0 \text{BiLip}(\Gamma(\cdot, t))
\]

for all \(t \in [0, T^*]\), where we also used the conservation of velocities \((6.11)\) implying identical minimal velocities \(v_0(\cdot, t)\) at all times \(t \in [0, T^*]\). Integral Menger curvature is a parameter-invariant energy so that the monotonicity of the mapping \(t \mapsto E(\gamma(\cdot, t)) = E(\Gamma(\cdot, t))\) implies for the energy values \(E(t) := E(\gamma(\cdot, t))\) that \(E(t) \leq E(0)\) for all \(t \in [0, T]\), and therefore by means of \((6.15)\) and \((6.14)\), \(\text{BiLip}(\gamma(\cdot, t)) \geq v_0 \text{BiLip}(\Gamma(\cdot, t)) \geq v_0 b(E(t)) \geq v_0 b(E(0)),\) which proves \((6.12)\). Setting \(\tilde{s}(E) := C_2 (p(E + E_{\delta}^{2q}))/2\) such that the constants \(C_2\) and \(\delta_2\) coincide with the corresponding constants \(C\) and \(\beta\) in [11, (0.8) in Theorem 1] for \(q = 2\), we find by the energy decay in time that

\[
|\Gamma'(\cdot, t)|_{\frac{2}{p}-2,2} \leq \tilde{s}(E(t)) \leq \tilde{s}(E(0)) \quad \text{for all} \ t \in [0, T].
\]

Now we use the arc length function \(\mathcal{L}(\gamma(\cdot, t), \cdot)\) of \(\gamma(\cdot, t)\) defined in (B.1) of Appendix B, which for every \(t \in [0, T]\) coincides with the arc length function \(\mathcal{L}(\gamma_0, \cdot)\) of the initial curve \(\gamma_0\) according to the conservation of velocities \((6.11)\), to rewrite \(\gamma(\cdot, t)\) as \(\gamma(\cdot, t) = \gamma(\cdot, t) \circ \mathcal{L}(\gamma(\cdot, t), \cdot)^{-1} \circ \mathcal{L}(\gamma(\cdot, t), \cdot) = \Gamma(\cdot, t) \circ \mathcal{L}(\gamma_0, \cdot)\) for all \(t \in [0, T]\). This can be combined with \((6.16)\)
and the chain rule inequality (A.7) in Lemma A.4 in Appendix A to obtain with \( L(\eta_0, \cdot)' = |\eta_0| \) and \( |\gamma_0'|_{p-2,3.2} \leq |\gamma_0'_{p-3.2} | \) the estimate

\[
[\gamma'_{p-3.2}] \leq C\delta(E(0))\left( [L(\gamma_0, \cdot)']_{p-3.2}^2 + v_{\gamma(\cdot, t)}^{-1} ||L(\gamma_0, \cdot)'||_{C^0}^{3-4} \right)^{1/2} \\
\leq C\delta(E(0))\left( |\gamma_0|_{p-3.2}^2 + v_{\gamma(\cdot, t)}^{-1} ||\eta_{0}'||_{C^0}^{3-4} \right)^{1/2}, \\
\leq C\delta(E(0)) |\gamma_0'_{p-3.2} \left( 1 + (C_E C_P)^{3-4} v_{\gamma_0}^{-1} |\gamma_{p-3.2}'| \right)^{1/2}
\]

by means of the Morrey embedding (A.4) and the Poincaré inequality (A.6). This proves (6.13) if we define the non-decreasing function \( \delta(-) := (1 + C_E C_P)^{3-4}/2 \cdot C\delta(-) \), where \( C = C(n, p) \) is the constant from Lemma A.4. Now let \( H := \mathbb{R}^{\frac{p}{2}-2,2} (\mathbb{R}, \mathbb{R}^n) \). To prove Lipschitz continuity of \( \gamma(t) \) we first note that the radius \( R_{M, \Sigma}(\gamma, t) \) as defined in (6.1) depends non-decreasingly on \( B_{\text{LiP}}(\gamma, t) \) and the minimal velocity \( v_{\gamma(\cdot, t)} \), and non-increasingly on the seminorm \( [\gamma'_{\cdot, t}] \). Combining this with the already established a priori estimates (6.12) and (6.13) we find a uniform radius \( R_0 \) depending only on the initial curve \( \gamma_0 \) such that \( R_{M, \Sigma}(\gamma, t) \geq R_0 > 0 \) for all \( t \in [0, T^*] \). Similarly, we find a uniform energy bound

\[
\|\nabla \Sigma E(\eta)\|_{2,1} \leq \|\nabla E(\eta)\|_{2,1} \leq C_1 \tag{6.17}
\]

for all \( \eta 
∈ \mathcal{B}_{R_0}(\gamma_0) \). By Lemma 6.1, in particular (6.3), in combination with the a priori estimates (6.12) and (6.13). The constant \( C_1 \) depends non-increasingly on \( v_{\gamma_0} \) and non-decreasingly on both \( E(\gamma_0) \) and \( |\gamma_0'_{p-3.2} | | \). Using (6.12) and (6.13) a third time, again in connection with Lemma 6.1, we obtain a uniform Lipschitz constant \( \overline{\text{Lip}}_{\Sigma E} \) for \( \nabla \Sigma E \) restricted to the ball \( \mathcal{B}_{R_0}(\gamma_0, t) \) for any time \( t \in [0, T^*] \). This uniform Lipschitz constant has the same dependencies and monotonicities as \( C_1 \). For any \( 0 \leq \sigma_1 < \sigma_2 \leq T^* \) one finds

\[
\|\gamma(\cdot, \sigma_2) - \gamma(\cdot, \sigma_1)\|_{2,1} \leq \int_{\sigma_1}^{\sigma_2} \|\frac{\partial}{\partial \tau} \gamma(\cdot, \tau)\|_{2,1} d\tau \leq C_1 \sigma_2 - \sigma_1. \tag{6.18}
\]

Now for any \( 0 \leq \sigma < \tau \leq T^* \) choose a partition \( \sigma = t_0 < \ldots < t_N = \tau \) such that \( \|\gamma(t_k) - \gamma(t_{k-1})\|_{2,1} < R_0 \) for all \( k = 1, \ldots, N \), to estimate by means of (6.18)

\[
\|\frac{\partial}{\partial \tau} \gamma(\cdot, \tau) - \frac{\partial}{\partial \tau} \gamma(\cdot, \sigma)\|_{2,1} \leq \sum_{k=1}^{N} \|\nabla \Sigma E(\gamma, t_k) - \nabla \Sigma E(\gamma, t_{k-1})\|_{2,1} \leq \overline{\text{Lip}}_{\Sigma E} \sum_{k=1}^{N} \|\gamma(\cdot, t_k) - \gamma(\cdot, t_{k-1})\|_{2,1} \leq C \overline{\text{Lip}}_{\Sigma E} \sum_{k=1}^{N} |\gamma(t_k) - \gamma(t_{k-1})| \tag{6.19}
\]

Notice that the newly defined Lipschitz constant \( \overline{\text{Lip}}_{\Sigma E} \) is non-decreasing in the first two arguments and non-increasing in its third argument as claimed, since \( C_1 \) and \( \overline{\text{Lip}}_{\Sigma E} \) have these monotonicity properties.

**Corollary 6.4.** Let \( \gamma_0 \in W_{p-2,2}^1 (\mathbb{R}^n, \mathbb{R}^n) \) for \( p \in \left( \frac{3}{2}, \frac{\infty}{2} \right) \) and suppose that the mapping \( t \mapsto \gamma(\cdot, t) \in C^1([0, T^*], \mathbb{R}^n) \) is a solution of (1.9) on the time interval \( [0, T^*] \) for some \( T^* > 0 \). Then there is a constant \( T_0 > 0 \) depending only on \( \gamma_0 \) such that for any \( t_0 \in [0, T^*] \), a unique solution \( \gamma(t) \) of (1.9) with initial point \( \gamma(t_0) \) exists and its minimal existence time \( T_0 \) from Corollary 6.2 is uniformly bounded below by \( T_0 > 0 \).

**Proof.** Set \( H := W_{p-2,2}^1 (\mathbb{R}^n, \mathbb{R}^n) \). According to (6.12) the evolving curves \( \gamma(\cdot, t) \) are regular and embedded for all \( t \in [0, T^*] \) so that each \( \gamma(\cdot, t) \) can serve as a new initial curve for the
evolution (1.9). Let \( \eta \in C^1([0,T_\eta], \mathcal{H}) \) be the unique solution of (1.9) with initial curve \( \gamma(\cdot,t_0) \), which exists by virtue of the short time existence result, Corollary 6.2, up to time
\[
T_\eta \geq T_{\text{min}}(\gamma(\cdot,t_0)) = \frac{R_{M,\Sigma}(\gamma(\cdot,t_0))}{C_2(\gamma(\cdot,t_0), \gamma(\cdot,t_0), \gamma(\cdot,t_0))_2^{\frac{3}{2}} + 1};
\]
(6.20)
see (6.10). To obtain a uniform lower bound \( T_0 \) independent of \( t_0 \) for the right-hand side of (6.20) notice that the denominator \( R_{M,\Sigma} \) yields a non-decreasing dependence on BiLip\((\gamma(\cdot,t_0))\) and on the minimal velocity \( v_{\gamma(\cdot,t_0)} \), as well as a non-increasing dependence on \( \|\gamma(\cdot,t_0)\|_2^{\frac{3}{2}} \). We can neglect the dependence on the minimal velocity, since the conservation of velocities (6.11) in Proposition 6.3 implies \( v_{\gamma(\cdot,t_0)} = v_{\gamma(\cdot,t_0)} \) for all \( t \in [0, T_\eta] \). To summarize, the right-hand side of (6.20) may be written as a strictly positive function \( \Upsilon = \Upsilon(\text{BiLip}(\gamma(\cdot,t_0)), \|\gamma(\cdot,t_0)\|_2^{\frac{3}{2}}) \) that depends non-decreasingly on BiLip\((\gamma(\cdot,t_0))\), and non-increasingly on \( \|\gamma(\cdot,t_0)\|_2^{\frac{3}{2}} \). We use (6.12) in Proposition 6.3 to estimate the bilipschitz constant from below as BiLip\((\gamma(\cdot,t_0)) \geq v_{\gamma(\cdot,t_0)} b(E(0)) \), where we denoted the energy by \( E(t) := \text{int}\{ M(p,\Sigma)(\gamma(\cdot,t)) \} \) for \( t \in [0, T^*] \) as in the previous proof. Moreover, (6.13) in Proposition 6.3 implies that \( F_0 := \Upsilon(0) \) is bounded from above. Consequently, by (6.20) \( T_0 \geq \Upsilon(\text{BiLip}(\gamma(\cdot,t_0)), \|\gamma(\cdot,t_0)\|_2^{\frac{3}{2}}) \geq \Upsilon(0, b(E(0)), F_0) =: T_0 > 0 \).}

**Proof of Theorem 1.1.** By Lemma 6.1, \( -\nabla \Sigma E \) is locally Lipschitz continuous, so standard ODE theory (cf. for example [12, Part II, Theorem 1.8.3]) yields existence of a unique solution \( \gamma \in C^1(J, W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^\eta)) \) such that \( \gamma(\cdot,t) \) is embedded and regular for all \( t \in J \) for some largest interval \( J \) containing 0. Let \( T_{\max} \) be a supp \( J \) which is bounded below by \( T_{\text{min}}(\gamma_0) \), see (6.10). Assume that \( T_{\max} < \infty \), then we can use Corollary 6.4 to find a solution \( \eta \) of (1.9) with initial value \( \gamma(\cdot,t_0) \) which exists at least up to time \( T_0 \). By the global uniqueness theorem for ODEs, see e.g. [12, Part II, Theorem 1.8.2], we can extend \( \gamma \) with \( t \mapsto \eta(\cdot,t) \) independent of \( t \) to \( (T_{\max} - \frac{1}{2} T_0, T_{\max} + \frac{1}{2} T_0) \) to a solution on the larger time interval \( [0,T_{\max} + \frac{1}{2} T_0] \) which is a contradiction to the maximality of \( J \) and \( T_{\max} \).

The energy decay in time as well as the conservation of velocities (1.10) was proven in Proposition 6.3; see (6.11). That the barycenter is preserved along the flow can be proven exactly as in the proof of the short time existence result for the gradient flow (1.3) in Theorem 3.2 using the second identity of (3.10) instead of the first in Lemma 3.3 for \( S := \Sigma \) and \( H_2 := W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}) \). The necessary uniform bound on \( \nabla_{\Sigma} \text{int}\{ M(p,\Sigma)(\gamma(\cdot,t)) \} \) is independent of \( t \) to interchange integration with differentiation with respect to \( t \) was established in (6.17) of Proposition 6.3 for an arbitrary solution of the projected flow (1.9). Now we use the conservation of the barycenter and of the velocity to estimate for arbitrary \( x \in \mathbb{R}/\mathbb{Z} \) and \( t \in (0, \infty) \) the absolute value as \( |\gamma(x,t)| \leq |\gamma(x,t) - \int_{\mathbb{R}/\mathbb{Z}} \gamma(u,t) du| + |\int_{\mathbb{R}/\mathbb{Z}} \gamma_0(u) du| \leq \|\gamma(\cdot,t)\|_{\mathcal{C}(\mathbb{R}/\mathbb{Z},\mathbb{R}^\eta)} + |\int_{\mathbb{R}/\mathbb{Z}} \gamma_0(u) du| = \|\gamma_0\|_{\mathcal{C}(\mathbb{R}/\mathbb{Z},\mathbb{R}^\eta)} + |\int_{\mathbb{R}/\mathbb{Z}} \gamma_0(u) du| \), so that we have a uniform \( L^\infty \)-bound on the solution \( \gamma(\cdot,t) \) independent of \( t \). The seminorms \( |\gamma(\cdot,t)|_{2}^{\frac{3}{2}} \) are uniformly bounded by means of the a priori estimate (6.13) of Proposition 6.3. The Poincaré inequality (A.6) implies that the fractional Sobolev norms \( \|\gamma(\cdot,t)\|_{W^{2,p-2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^\eta)} \) are uniformly bounded, which together with the \( L^\infty \)-bound on \( \gamma(\cdot,t) \) leads to a uniform bound on \( \|\gamma(\cdot,t)\|_{W^{2,p-2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^\eta)} \) independent of \( t \).
Proof of Corollary 1.4. Abbreviate $\mathcal{H} := W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $\mathcal{E} := \text{int} M^{(p,2)}$, and $E(t) := \mathcal{E}(\gamma(t), t)$ for $t \in [0, \infty)$, where $\gamma(t)$ is the solution of (1.9). For the proof of (1.12) we take a monotonically increasing sequence $(T_i)_i \subset (0, \infty)$ with $T_i \to \infty$ as $i \to \infty$ and notice that the non-negative sequence of energy values $E(T_i)$ is non-increasing according to Theorem 1.1 and hence convergent with limit $E_{\infty} := \lim_{i \to \infty} E(T_i) \in [0, E(0)]$. This limit does not depend on the choice of the $T_i$ by the subsequence principle. We can take the limit $i \to \infty$ in the identity $E(T_i) - E(0) = \int_0^{T_i} \langle \nabla \mathcal{E}(\gamma(t), t), \partial_t \gamma(t), t \rangle \rangle dt = -\int_0^{T_i} \langle \nabla \mathcal{E}(\gamma(t), t), \nabla_{\Sigma} \mathcal{E}(\gamma(t), t) \rangle \rangle dt = -\int_0^{T_i} \| \nabla_{\Sigma} \mathcal{E}(\gamma(t), t) \|_{\mathcal{H}}^2 dt = E(0) - E_{\infty} < \infty$, which implies (1.12). Along any sequence of times $t_k \to \infty$ the fractional Sobolev norms $\| \gamma(t_k) \|_{\mathcal{H}}$ are uniformly bounded by Theorem 1.1. This guarantees the existence of a subsequence $(t_{k_l})_l \subset (t_k)_k$ such that the $(\gamma(t_{k_l}), t_{k_l})$ converge weakly in $\mathcal{H}$ and strongly in $C^1$ (by Morrey’s embedding, Theorem A.2) to a limiting curve $\gamma^* \in \mathcal{H}$ as $l \to \infty$. By Fatou’s Lemma $\mathcal{E}$ is sequentially lower semicontinuous with respect to $C^1$-convergence, so that $\mathcal{E}(\gamma^*) \leq E_{\infty}$. The a priori estimate (6.12) of Proposition 6.3 is conserved in the $C^1$-limit, which implies that $\gamma^*$ is embedded and hence of the same knot class as $\gamma(t_{k_l})$ for $l \gg 1$, which proves $[\gamma_t^*] = [\gamma_0^*]$. □

We conclude with a simple stability estimate for solutions of (1.9).

Corollary 6.5. Let $\gamma(t)$ and $\eta(t)$ be solutions to (1.9) with initial curves $\gamma_0, \eta_0 \in \mathcal{H} := W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Then there are constants $K_{\gamma_0}$ and $K_{\eta_0}$, depending only on $\gamma_0$ and $\eta_0$, respectively, such that $\| \gamma(t) - \eta(t) \|_{\mathcal{H}} \leq \| \gamma_0 - \eta_0 \|_{\mathcal{H}} + (K_{\gamma_0} + K_{\eta_0}) t$ for all $t \in [0, \infty)$.

Proof. Set $\mathcal{E} := \text{int} M^{(p,2)}$ and use the Lipschitz constant $K_{\gamma_0} := \overline{C}_1 = \overline{C}_1(\mathcal{E}(\gamma_0), \gamma_0, [\gamma_0]^{2,p-3,2}, \nu_{\gamma_0})$ for $t \rightarrow \gamma(t)$ introduced in (6.17) and (6.18), and analogously for $\eta(t)$ setting $K_{\eta_0} := C_0(\mathcal{E}(\eta_0), [\eta_0]^{2,p-3,2}, \nu_{\eta_0})$, to estimate $\| \gamma(t) - \eta(t) \|_{\mathcal{H}} \leq \| \gamma(t) - \gamma_0 \|_{\mathcal{H}} + \| \gamma_0 - \eta_0 \|_{\mathcal{H}} + \| \eta_0 - \eta(t) \|_{\mathcal{H}} \leq \| \gamma_0 - \eta_0 \|_{\mathcal{H}} + (K_{\gamma_0} + K_{\eta_0}) t$. □

7 Numerical Experiments

7.1 Discretization

For a numerical treatment of the gradient flow, not only the energy $\mathcal{E} := \text{int} M^{(p,2)}$ and its derivative $D\mathcal{E}$ have to be discretized, but also the Riesz isomorphism $J$ and the constraints $\Sigma : \mathcal{O} \to \mathcal{H}_2$, where $\mathcal{H}_2 := W^{2,p-2,2}(\mathbb{R}/\mathbb{Z})$ and $\mathcal{O} := W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is an open subset of $\mathcal{H}_1 := W^{2,p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. In a nutshell, we do this by discretizing a curve $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ by a closed polygonal line $P$ and by replacing all occurring integrals by appropriate sums. To this end, we fix a partition $T$ of $\mathbb{R}/\mathbb{Z}$ into a set of disjoint intervals or edges $E(T)$. Denote the set of interval end points (vertices) of $E(T)$ by $V(T)$. By a polygonal line we mean a mapping $P : V(T) \to \mathbb{R}^n$ and identify it with periodic, piecewise-linear interpolation. Hence, our discrete configuration space is $\mathcal{H}_{T,1} := \{ \Phi : V(T) \to \mathbb{R}^n \}$. This is a vector space of dimension $n \cdot N$, where $N$ is the number of vertices (which is also equal to the number of edges). In the following, we will employ the natural basis $\Phi_{v,i}, v \in V(T), i \in \{1, \ldots, n\}$ where $\langle \Phi_{v,i}(v'), e_j \rangle_{\mathbb{R}^n} = \delta_{v,v'} \delta_{i,j}$.

Denoting the left and right endpoints of an edge $I \in E(T)$ by $I^1$ and $I^2$, respectively, we may write down the edge length $\ell_P(I)$ and the unit edge vector $\tau_P(I)$ of edge $I$ with respect to $P$ as follows:

$$\ell_P(I) := |P(I^1) - P(I^2)| \quad \text{and} \quad \tau_P(I) := \frac{P(I^1) - P(I^2)}{|P(I^1) - P(I^2)|}.$$
7.2 Constraints

With $\mathcal{H}_{T,2} := \{ \Phi : E(T) \to \mathbb{R} \} \cong \mathbb{R}^N$, we may discretize the constraint $\Sigma(\gamma) = \log(|\gamma'|)$ by

$$\Sigma_T : \mathcal{H}_{T,1} \to \mathcal{H}_{T,2}, \quad \Sigma_T(P) := \left( \log \left( \frac{\ell_P(I)}{|I|} \right) \right)_{I \in E(T)}.$$

In terms of the basis $\Phi_{b,i}$ on $\mathcal{H}_{T,1}$ and the standard basis on $\mathcal{H}_{T,2}$, the derivative $D\Sigma_T(P)$ can be represented by a sparse matrix of size $N \times (nN)$ with only $2nN$ nonzero entries. The assembly is straightforward once the derivatives of $\log(\ell_P(I)/|I|)$ with respect to the coordinates of $P(I^1)$ and $P(I^2)$ have been computed. These derivatives are given by

$$\frac{\partial}{\partial \ell_P(I)} \log \left( \frac{\ell_P(I)}{|I|} \right) = -\frac{\tau_P(I^\top)}{\ell_P(I)} \quad \text{and} \quad \frac{\partial}{\partial \ell_P(I)} \log \left( \frac{\ell_P(I)}{|I|} \right) = +\frac{\tau_P(I^\top)}{\ell_P(I)}.$$

7.3 Energy

In terms of the basis $\Phi_w$, we may discretize the constraint $\Sigma(\gamma) = \log(|\gamma'|)$ by $\Sigma_T(P)$.

By Proposition A.3, this is indeed an inner product when restricted to the subspace $H^{\gamma}$. Nonetheless, the computational complexity of evaluating $\Sigma_T(P)$ is $O(N^3)$.

The term $W_P(I_1, I_2, I_3)$ is symmetric in $I_1, I_2, I_3$, and exploiting this, one can reduce the number of summands by a factor of $\frac{1}{2}$. Nonetheless, the computational complexity of $E_T(P)$ is $O(N^3)$.

The term $W_P(I_1, I_2, I_3)$ is an analytical expression in the six points $P(I_1^1), P(I_1^2), P(I_2^1), P(I_2^2), P(I_3^1)$, and $P(I_3^2)$. Hence the derivative $W_P(I_1, I_2, I_3)$ with respect to these points can be computed symbolically (we strongly suggest to use a CAS for that) and compiled into a library for runtime performance. The derivative $DE_T(P) \in \mathcal{H}_{T,2}$ can be represented by a vector of size $nN$; it can then be assembled from the output of this library and from the vertex indices of the interval end points $I_1^1, I_1^2, I_2^1, I_2^2, I_3^1$, and $I_3^2$.

7.4 Metric

We define the Gagliardo product of $\varphi, \psi \in H_1$ by

$$\langle J \varphi, \psi \rangle := \int_{(\mathbb{R}/\mathbb{Z})^2} \frac{(\varphi'(u_1) - \varphi'(u_2))(\psi'(u_1) - \psi'(u_2))}{|u_1 - u_2|^{1+s}} \, du_1 \, du_2.$$

By Proposition A.3, this is indeed an inner product when restricted to the subspace $H_1 := \{ \varphi \in H_1 \mid \int_{\mathbb{R}/\mathbb{Z}} \varphi(u) \, du = 0 \}$. 


Analogously to the energy, we may discretize this metric by first decomposing the integral with respect to the partition \( T \):

\[
\langle J \varphi, \psi \rangle = \sum_{I_1, I_2 \in E(T)} \int_{I_1} \int_{I_2} \frac{\varphi'(u_1) - \varphi'(u_2) \psi'(u_1) - \psi'(u_2)}{|u_1 - u_2|^{\alpha-1}} \, du_1 \, du_2.
\]

We interpret \( \Phi, \Psi \in \mathcal{H}_{T,1} \) as a piecewise linear functions. Thus \( \Phi' \) and \( \Psi' \) are constant on (the interior of) each edge. This is why we may put \( \Phi'(I) := \frac{\Phi(I_1) - \Phi(I_2)}{I_1 - I_2} \), \( \Psi'(I) := \frac{\Psi(I_1) - \Psi(I_2)}{I_1 - I_2} \) and define the discrete Riesz isomorphism \( J_T : \mathcal{H}_{T,1} \to \mathcal{H}_{T,1}^* \) via the midpoint rule by

\[
\langle J_T \Phi, \Psi \rangle := \sum_{I_1, I_2 \in E(T), \, I_1 \neq I_2} \frac{\Phi'(I_1) - \Phi'(I_2), \Psi'(I_1) - \Psi'(I_2)}{|m(I_1) - m(I_2)|^{\alpha-1}} |I_1| |I_2|,
\]

where \( m(I) \) denotes the midpoint of the interval \( I \). In the following, we identify \( J_T \) with the Gram matrix with respect to the basis \( \Phi_{e,i} \) of the bilinear form \( G \) defined by \( G(\Phi, \Psi) := \langle J_T \Phi, \Psi \rangle \). This is a matrix of size \((n \cdot N) \times (n \cdot N)\). Since different spatial components of \( \Phi \) of \( \Psi \) do not interact, one may reorder the degrees of freedom such that the matrix \( J_T \) is block diagonal with \( n \) identical dense blocks of size \( N \times N \) on its diagonal; each of the blocks is a copy of \( J_T \) for \( n = 1 \).

We abbreviate \( B_T := D \Sigma_T(P) \) and denote the discrete barycenter constraint by \( C_T : \mathcal{H}_{T,1} \to \mathbb{R}^n \), \( C_T \Phi = \frac{1}{2} \sum_{I \in E(T)} (\Phi(I) + \Phi(I^T)) |I| \). With the symmetric saddle point matrix

\[
A_T(P) := \begin{pmatrix} J_T & B_T^T \\ B_T & 0 \\ C_T & 0 \\ \lambda & \mu \end{pmatrix},
\]

(7.1)

the discrete projected gradient \( \nabla_\Sigma E_T(P) \) of \( E_T \) can be computed by solving the linear system

\[
A_T(P) \begin{pmatrix} \nabla_\Sigma E_T(P) \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} D E_T(P) \\ 0 \\ 0 \end{pmatrix},
\]

(7.2)

where \( \lambda \in \mathcal{H}_{T,2}^* \) and \( \mu \in (\mathbb{R}^n)^* \) act as Lagrange multipliers. Note that we included \( C_T \) and \( C_T^T \) into the saddle point matrix (7.1) in order to obtain a symmetric matrix of full rank because otherwise constant functions would be in the kernel of the saddle point matrix.

Unfortunately, the matrix \( J_T \) is fully populated so that sparse matrix methods do not help in solving (7.2) numerically. In principle, one may enforce that all edges have the same length and then exploit that \( J_T \) is a Toeplitz matrix (e.g., via using the fast Fourier transform). However, in our experiments, we simply used the dense \( LU \)-factorization provided by \( \text{LAPACK} \). Although its costs are of order \(((n + 1)N + n)^3 = O((n + 1)^3 N^3)\), the factorization took up only a relatively small portion of the overall computation time for \( N \leq 600 \). This is because the timings of the naive \( O((nN)^3) \)-implementations we employed for computing \( E_T(P) \) and \( DE_T(P) \) were dominant.

### 7.5 Discrete Gradient Flow

The gradient flow equation for \( P(t) \) reads as follows:

\[
A_T(P(t)) \begin{pmatrix} \dot{P}(t) \\ \dot{\lambda}(t) \\ \dot{\mu}(t) \end{pmatrix} = \begin{pmatrix} -D E_T(P(t)) \\ 0 \\ 0 \end{pmatrix} .
\]
In our experiments, we simply used the local model \( P(t + \tau) = P(t) + \dot{P}(t) \tau + O(\tau^2) \) for the update step, which effectively leads to the explicit Euler method. Thus with

\[
Q_0(\tau) := P(t) + \dot{P}(t) \tau,
\]
we have \( Q_0(\tau) = P(t + \tau) + O(\tau^2) \). In particular, this guarantees that the constraint violation grows only slowly:

\[
\Sigma_T(Q_0(\tau)) = \Sigma_T(P(t)) + O(\tau^2).
\]

The remaining constraint violation can be reduced by applying a few iterations of the following modified Newton method involving the saddle point matrix \( \mathcal{A}_T \) from (7.1):

\[
Q_{k+1}(\tau) = Q_k(\tau) - v_k, \quad \text{where} \quad \mathcal{A}_T(P(t)) \begin{pmatrix} v_k \\ \lambda_k \\ \mu_k \end{pmatrix} = \begin{pmatrix} 0 \\ \Sigma_T(Q_k(\tau)) \\ 0 \end{pmatrix}. \quad (7.3)
\]

Then \( Q_\infty(\tau) := \lim_{k \to \infty} Q_k(\tau) \) may serve as next iterate of the discrete flow. In contrast to the classical Newton method, the matrix in the linear system is not updated; this allows us to reuse the already computed matrix \( \mathcal{A}_T(P(t)) \) and its factorization. The update vectors \( v_k \) are always perpendicular to the tangent space of the constraint manifold at \( P(t) \) (see Figure 4 (a)). The iteration in (7.3) converges because of the implicit function theorem, see [39, Theorem 4.B (b)].

By backtracking, i.e., by systematically decreasing \( \tau \) if necessary, we may enforce the following three conditions.

1. The point \( Q_\infty(\tau) \) is well-defined. In practice, we require that the constraint violation of \( Q_k(\tau) \) is below a prescribed threshold after a prescribed number \( k \) of modified Newton iterations (7.3) and we use \( Q_k(\tau) \) as approximation to \( Q_\infty(\tau) \); otherwise, we shrink \( \tau \) and restart (7.3).

2. For some parameter \( \sigma \in (0, 1) \) and for the function \( \varphi(\tau) := \mathcal{E}_T(Q_\infty(\tau)) \), the Armijo condition

\[
\varphi(\tau) \leq \varphi(0) + \sigma \tau \varphi'(0) < \varphi(0) = \mathcal{E}_T(P(t))
\]

is fulfilled (see Figure 4). This guarantees that the step size \( \tau \) is adapted to the “stiffness” of the ODE. Satisfying the Armijo condition is possible because \( \varphi'(0) \) is negative (unless \( P(t) \) is a critical point) and because we have

\[
\varphi(\tau) \leq \varphi(0) + \tau \varphi'(0) + \frac{1}{2} \text{Lip}(\varphi') \tau^2
\]

due to local Lipschitz continuity of \( \varphi' \). For more details on the Armijo condition and on line search methods that guarantee it, we refer the interested reader to [26, Chapter 3].

3. The curve \( Q_\infty(\tau) \) (or its proxy \( Q_k(\tau) \)) is ambient isotopic to \( P(t) \). Because \( P(t) \) is an embedded polygonal line, such a step size \( \tau > 0 \) must exist.

The last condition is a bit involved. In principle, there are various strategies to ensure it. What we do is the following: We consider the homotopy \( F(u, \lambda) := (1 - \lambda) P(t)(u) + \lambda Q_\infty(\tau)(u) \) (see Figure 5) and check the following:

\footnote{In Zeidler’s notation, set \( F = \Sigma_T, X = \mathcal{N}(D\Sigma_T(P(t))) \) and \( Y = X^\perp \) as subspaces of \( \mathcal{N}(C_T) \subset \mathcal{H}_{T,1} \). Then, for \( Z = \mathcal{H}_{T,1} \), one can show that \( F_y(P(t)) = D\Sigma_T(P(t))\Pi_Y \) is bijective by first constructing a right-inverse as in Section 5 and then projecting this to \( Y \). The equation \( v_k = F_y^{-1}(P(t))F_\lambda(Q_k(\tau)) \) is then equivalent to (7.3), because \( C_T^+(\mathcal{H}_{T,1}) = \mathcal{N}(C_T)^\perp \) (see e.g. [23]) and the same holds for \( B_T \).}
A Periodic Sobolev-Slobodeckiı spaces

For fixed \( \ell > 0, s \in (0, 1) \) and \( \varrho \in [1, \infty) \) define the seminorm

\[
[f]_{s, \varrho} := \left( \int_{\mathbb{R}/\ell \mathbb{Z}} \int_{-\ell/2}^{\ell/2} \frac{|f(u + w) - f(u)|^n}{|w|^{1+s\varrho}} \, dw \, du \right)^{1/\varrho}
\]  

(A.1)

for \( \ell \)-periodic functions \( f: \mathbb{R} \to \mathbb{R}^n \) that are locally \( \varrho \)-integrable on \( \mathbb{R} \), i.e., for \( f \in L^\varrho(\mathbb{R}/\ell \mathbb{Z}, \mathbb{R}^n) \).

**Definition A.1.** For \( k \in \mathbb{N} \cup \{0\} \) the function space

\[
W^{k+s, \varrho}(\mathbb{R}/\ell \mathbb{Z}, \mathbb{R}^n) := \{ f \in W^{k, \varrho}(\mathbb{R}/\ell \mathbb{Z}, \mathbb{R}^n) : \|f\|_{W^{k+s, \varrho}} < \infty \},
\]

Figure 4: (a) The modified Newton method starts at \( Q_0(\tau) \) and updates it along \( \mathcal{N}(D\Sigma T(P(t)))^\perp \) until it converges to \( Q_\infty(\tau) \). The orthogonality can be seen by computing \( (u, 0, 0) A_T(P(t))(v, \lambda, \mu)^T \) for \( u \in \mathcal{N}(D\Sigma T(P(t))) \cap \mathcal{N}(C_T) \) and keeping in mind that \( J_T \) is the Gram matrix of the discrete Gagliardo product on the admissible space \( \mathcal{N}(C_T) \). (b) Singularities are the defining properties of self-avoiding energies, so they are frequently encountered in the computational treatment of their flows. Thus, adaptive choice of step size is crucial for the stability. The interval of step sizes that satisfy the Armijo condition is highlighted in green.

\*\*\* Question \*\*\*  

- Are the edge lengths of all the polygonal lines defined by \( F(\cdot, \lambda) \), \( \lambda \in [0, 1] \) bounded away from 0?  
- Are all the turning angles of the polygonal lines \( F(\cdot, \lambda) \), \( \lambda \in [0, 1] \) bounded away from \( \pi \)?  
- Is the image surface of \( F \) in \( \mathbb{R}^3 \times [0, 1] \) free of self-intersections? (See [27, Section 4.2] for computational techniques to answer this question.)

If the answer to all these questions is affirmative, then \( F \) is a level preserving \( C^0 \)-isotopy. In general, the existence of a level preserving \( C^0 \)-isotopy \( F \) does not imply that \( F(\cdot, 0) \) and \( F(\cdot, 1) \) are ambient isotopic. But the special structure of \( F \) implies that it is a locally trivial isotopy and thus a locally unknotted isotopy in the sense of [21, p. 72]. Thus [21, Theorem 2] guarantees that \( F(\cdot, 0) = P(t) \) and \( F(\cdot, 1) = Q_\infty(\tau) \) are indeed ambient isotopic.

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where \( \| f \|_{W^{k+s,q}(\mathbb{R}/\ell\mathbb{Z},\mathbb{R}^n)} = (\| f \|_{W^{k,q}(0,\ell),\mathbb{R}^n}) + [f^{(k)}]^{\#})^{1/\theta} \), is called the periodic Sobolev-Slobodecki\u0111 space with fractional differentiability of order \( k + s \) and integrability \( q \). Here, \( W^{k,q}(\mathbb{R}/\ell\mathbb{Z},\mathbb{R}^n) \subset W^{k,q}(\mathbb{R},\mathbb{R}^n) \) denotes the usual Sobolev space of \( \ell \)-periodic functions whose weak derivatives \( f^{(i)} \) of order \( i \in \{0,\ldots,k\} \) are locally \( q \)-integrable, where we have set \( f^{(0)} := f \).

Since the restriction of any \( \ell \)-periodic Sobolev function \( f \in W^{k,q}(\mathbb{R}/\ell\mathbb{Z},\mathbb{R}^n) \) to its fundamental domain \( (0,\ell) \subset \mathbb{R} \) is contained in the Sobolev space \( W^{k,q}(0,\ell),\mathbb{R}^n) \) it suffices to compare the seminorm (A.1) to the standard Gagliardo seminorm (which we denote by \( \| \cdot \| \)) used in the literature on (non-periodic) Sobolev-Slobodecki\u0111 spaces, in order to transfer known results to the periodic setting. Indeed, for \( w \in (-\ell/2,\ell/2) \) the absolute value \( |w| \) in the integrand of (A.1) equals the \( \ell \)-periodic distance \( |w|_{\mathbb{R}/\ell\mathbb{Z}} = \min_{k\in\mathbb{Z}} |w+k\ell| \), so that we rewrite the seminorm (A.1) by substituting \( v(u) := u + w \), and estimate for \( f \in L^q(\mathbb{R}/\ell\mathbb{Z},\mathbb{R}^n) \)

\[
[f]_{k,q}^q = \int_{\mathbb{R}/\ell\mathbb{Z}} \int_{-\ell/2}^{\ell/2} \frac{|f(u)-f(u)|^q}{|w|^{1+q}} \, du \, dv = \int_{\mathbb{R}/\ell\mathbb{Z}} \int_{-\ell/2}^{\ell/2} \frac{|f(u)-f(u)|^q}{|w|^{1+q}} \, du \, dv
\]

\[
\geq \int_0^\ell \int_0^\ell \frac{|f(u)-f(u)|^q}{|w|^{1+q}} \, dv \, du = \| f \|_{k,q}^q.
\]

where we used the \( \ell \)-periodicity of the integrand to shift the domain of the inner integration by \(-u + \ell/2\).

As a first application of inequality (A.2) we show that the well-known Morrey embedding into classical Hölder spaces transfers to the periodic setting. To be more precise, the periodic Hölder seminorm for an \( \ell \)-periodic function \( f : \mathbb{R} \to \mathbb{R}^n \) and \( \alpha \in (0,1] \) is defined as (cf. [18, Def. 3.2.5])

\[
H\ddot{o}l_{\mathbb{R}/\ell\mathbb{Z},\alpha}(f) := \sup_{x,y \in [0,\ell]} \frac{|f(x)-f(y)|}{|x-y|^{1+\alpha}} = \sup_{x \in [0,\ell]} \sup_{0 < |u| \leq \ell/2} \frac{|f(x+u)-f(x)|}{|u|^{\alpha}},
\]

and the periodic Hölder space \( C^{k,\alpha}(\mathbb{R}/\ell\mathbb{Z},\mathbb{R}^n) \) for \( k \in \mathbb{N} \cup \{0\} \) consists of those functions \( f \in C^k(\mathbb{R}/\ell\mathbb{Z},\mathbb{R}^n) \) whose Hölder norm satisfies \( \| f \|_{C^{k,\alpha}(\mathbb{R}/\ell\mathbb{Z},\mathbb{R}^n)} := \| f \|_{C^0(\mathbb{R}/\ell\mathbb{Z},\mathbb{R}^n)} + H\ddot{o}l_{\mathbb{R}/\ell\mathbb{Z},\alpha}(f) < \infty \).
Notice that the second equality in (A.3) follows from the \( \ell \)-periodicity of \( f \), since for any distinct \( x, y \in [0, \ell] \) with, say, \( x < y \), we find \( 0 \neq w \in [-\ell/2, \ell/2] \) such that \( x + w = y \), or \( x + \ell = y + \ell \), namely \( w := y - x \) if \( |y - x| \leq \ell/2 \), or \( w := -y - x \) if \( \ell/2 < |y - x| \leq \ell \).

**Theorem A.2 (Morrey embedding).** Let \( k \in \mathbb{N} \cup \{0\} \) and \( \ell \in (0, \infty) \). If \( \varrho \in (1, \infty) \) and \( s \in (1/\varrho, 1) \), then there is a positive constant \( C_E = C_E(n, k, s, \varrho, \ell) \) such that

\[
\|f\|_{C^{k,s-1/\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)} \leq C_E \|f\|_{W^{k+s,\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)} \quad \text{for all } f \in W^{k+s,\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n). \tag{A.4}
\]

**Proof.** It suffices to treat the case \( k = 0 \), the full statement follows by induction. Moreover, \( \|f\|_{C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)} = \|f\|_{C^0([0,\ell], \mathbb{R}^n)} \) for any \( \ell \)-periodic function \( f : \mathbb{R} \to \mathbb{R}^n \), so that we can focus on the periodic Hölder seminorm. Rewrite the numerator in (A.3) as \( \|f \circ \tau_x - \frac{\ell}{2}(w + \frac{\ell}{2}) - f \circ \tau_x - \frac{\ell}{2}(\frac{\ell}{2})\| \) where we used the shift \( \tau_x(x) := x + a \) for some fixed \( a \in \mathbb{R} \). Hence the right-hand side of (A.3) for \( \alpha := s - (1/\varrho) \in (0, 1) \) may be rephrased by setting \( z := w + \ell/2 \), and estimated from above as

\[
\sup_{x \in [0, \ell]} \sup_{z \in [\ell/2]} \frac{|f \circ \tau_x - \ell/2(z) - f \circ \tau_x - \ell/2(\ell/2)|}{|z - (\ell/2)|^{s-1/\varrho}} \leq \sup_{x \in [0, \ell]} \text{Hölder}_{[0, \ell], s-1/\varrho}(f \circ \tau_x - \ell/2). \tag{A.5}
\]

By virtue of the non-periodic Morrey embedding [15, Theorem 8.2] and (A.2) the right-hand side may be bounded from above by the expression \( \sum_{x \in [0, \ell]} C_{\text{Hölder}}(f \circ \tau_x - \ell/2) \|W^{s,\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)\| \leq C_{\text{Hölder}}(f \circ \tau_x - \ell/2) \|W^{s,\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)\| \), for some constant \( C = C(n, s, \varrho, \ell) \). Changing variables via translations by \( x - (\ell/2) \) and using the \( \ell \)-periodicity of the respective integrands in the \( W^{s,\varrho} \)-norm one can easily check that \( \|f \circ \tau_x - \ell/2\|_{W^{s,\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)} = \|f\|_{W^{s,\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)} \) for each \( x \in \mathbb{R} \), which concludes the proof. \( \square \)

We frequently make use of the following Poincaré inequality.

**Proposition A.3 (Poincaré inequality).** For every \( \ell \in (0, \infty) \), \( s \in (0, 1) \), and \( \varrho \in [1, \infty) \) there is a constant \( C_P = C_P(s, \varrho, \ell) \) such that

\[
\|f\|_{L^s(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)} \leq C_P \|f\|_{s, \varrho} \quad \text{for all } f \in W^{1+s,\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \text{ with } \int_{\mathbb{R}/\ell\mathbb{Z}} f(t) \, dt = 0. \tag{A.6}
\]

**Proof.** By Jensen’s inequality, we have

\[
\int_{\mathbb{R}/\ell\mathbb{Z}} |f(u)|^\varrho \, du = \int_{\mathbb{R}/\ell\mathbb{Z}} |f(u) - f_{\mathbb{R}/\ell\mathbb{Z}} f(t)|^\varrho \, du \leq \int_{\mathbb{R}/\ell\mathbb{Z}} \int_{\mathbb{R}/\ell\mathbb{Z}} |f(u) - f(t)|^\varrho \, du \, du \leq \frac{1}{|\mathbb{R}/\ell\mathbb{Z}|} \left( \frac{\ell}{2} \right)^{1+s} \int_{\mathbb{R}/\ell\mathbb{Z}} \int_{\mathbb{R}/\ell\mathbb{Z}} \frac{|f(u) - f(t)|^\varrho}{|u - t|} \, du \, dt.
\]

\( \square \)

As a first application we deal with the composition of Sobolev-Slobodeckii functions.

**Lemma A.4.** Let \( \ell \in (0, \infty) \), \( \varrho \in (1, \infty) \), \( s \in (1/\varrho, 1) \), and \( g \in C^1([0, L]) \) with \( |g'| > 0 \) on \( [0, L] \), and \( \ell := g(L) - g(0) \), and \( f \in W^{1+s,\varrho}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \). Then \( g \) can be extended to a function \( G \in C^1(\mathbb{R}) \) such that \( f \circ G \in C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \) and \( G' \in C^0(\mathbb{R}/\ell\mathbb{Z}) \), satisfying

\[
[(f \circ G)']_{s, \varrho} \leq C \left( |G'|_{s, \varrho}^{\varrho} + v_g^{-1} \|G'\|_{C^0(\mathbb{R}/\ell\mathbb{Z})}^{1+s} \right)^{1/\varrho} [f']_{s, \varrho} \tag{A.7}
\]

for some constant \( C = C(n, s, \varrho, \ell) \), where \( v_g := \min_{[0, L]} |g'| > 0 \).

Notice that both sides in (A.7) may be infinite as we do not assume that \( g \in W^{1+s,\varrho}((0, L)) \). We should mention also that we have suppressed the respective domains \( [0, L] \), or \( \mathbb{R}/\ell\mathbb{Z} \), or \( \mathbb{R}/\ell\mathbb{Z} \), in our notation for the seminorms of \( G \) and \( f \circ G \), or of \( f \) in these inequalities.
Proof. In order to prove (A.7) we extend g onto all of \( \mathbb{R} \), first by setting \( G(x) := g(x) \) for \( x \in (0, L] \) and \( G(x) := g(x - L) + \ell \) consecutively for \( x \in (L, 2L], (2L, 3L], \ldots \), and then by setting \( G(x) := g(x + \ell - L) \) consecutively for \( x \in (-L, 0], (-2L, -L], (-3L, -2L], \ldots \), to find \( G(x + L) = G(x) + \ell \) for all \( x \in \mathbb{R} \). We then bound the numerator in the integrand of the seminorm \( \| (f \circ G)' \|_{s, \theta} \) for \( x \in \mathbb{R} \) and \( |w| \leq \frac{L}{2} \) from above as \( |(f \circ G)'(x + w) - (f \circ G)'(x)|^\theta \leq 2^{\theta - 1} |G'(x + w)|^\theta |f'(G(x + w)) - f'(G(x))|^\theta |G'(x + w) - G'(x)|^\theta \), which leads to

\[
[(f \circ G)']^\theta_{s, \theta} \leq 2^{\theta - 1} \int_{\mathbb{R}} \int_{-L/2}^{L/2} [w]^{-\alpha} |G'(x + w)|^\theta |f'(G(x + w)) - f'(G(x))|^\theta \, dw \, dx
\]

where we have set \( \alpha := 1 + gs \). Note that if \( f \) is of class \( C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) due to Theorem A.2. From now on we may assume without loss of generality that the seminorm \( [G']^\theta_{s, \theta} \) is finite, otherwise there is nothing to prove for (A.7). Substituting \( u(w) := x + w \) and using periodicity we can rewrite the double integral as

\[
\int_{\mathbb{R}/\mathbb{Z}} \int_{-L/2}^{L/2} |u - x|^{-\alpha} |G'(u)|^\theta |f'(G(u)) - f'(G(x))|^\theta \, du \, dx.
\]

We obtain for the inverse function \( G^{-1} : \mathbb{R} \to \mathbb{R} \) for all \( x, u \in \mathbb{R} \)

\[
\|G^{-1}(G(u)) - G^{-1}(G(x))\|_{\mathbb{R}/\mathbb{Z}} \leq \|u - x\|_{\mathbb{R}/\mathbb{Z}} \geq \|G^{-1}\|_{C^0(\mathbb{R}/\mathbb{Z})}\|G(u) - G(x)\|_{\mathbb{R}/\mathbb{Z}}.
\]

Now we use the transformation \( \Phi : (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2, (u, x) \to (\tilde{u}, \tilde{x}) := (G(u), G(x)) \) with \( |\det(D\Phi(u, x))| = G'(u)G'(x) \geq v_\theta > 0 \) for all \( u, v \in \mathbb{R} \) due to periodicity of \( G' \) and \( G'|_{[0, L]} = g' \), to rewrite and estimate (A.9) by means of (A.10) as

\[
\int_{(\mathbb{R}/\mathbb{Z})^2} |G^{-1}(\tilde{u}) - G^{-1}(\tilde{x})|_{\mathbb{R}/\mathbb{Z}}^\alpha |G'(G^{-1}(\tilde{u}))|^{\theta - 1} |G'(G^{-1}(\tilde{x}))|^{-1} |f'(\tilde{u}) - f'(\tilde{x})|^\theta \, d\tilde{u} \, d\tilde{x}
\]

\[
\leq v_\theta^{-1} \|G'\|^\theta_{C^0(\mathbb{R}/\mathbb{Z})} \|f'\|^\theta_{s, \theta} = v_\theta^{-1} \|G'\|^\theta_{C^0(\mathbb{R}/\mathbb{Z})} \|f'\|^\theta_{s, \theta}.
\]

The \( C^0 \)-norm of \( f' \) in (A.8) may be bounded by \( C_{E}C_P\|f'\|_{s, \theta} \) according to the Morrey embedding, Theorem A.2, and the Poincaré inequality, Proposition A.3, and combining this with (A.11) leads to (A.7) with the constant \( C := (2^{\theta - 1}((C_{E}C_P)\theta + 1))^{1/\theta} \).

The following result provides an estimate for the Sobolev-Slobodeckij norms of products of functions of fractional Sobolev regularity.

**Proposition A.5 (Bounds on inner product).** Let \( k \in \mathbb{N} \cup \{0\} \) and \( \ell \in (0, \infty) \), \( g \in (1, \infty) \), and \( s \in (1/\theta, 1) \). Then, there is a constant \( C = C(n, k, s, \theta, \ell) \in (0, \infty) \) such that

\[
\|f(\cdot, g(\cdot))\|_{W^{k+s, \theta}(\mathbb{R}/\mathbb{Z})} \leq C \|f\|_{W^{k+s, \theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)} \|g\|_{W^{k+s, \theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)} \forall f, g \in W^{k+s, \theta}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n).
\]

Proof. For brevity, we write \( \langle \xi, \zeta \rangle := \langle \xi, \zeta \rangle_{\mathbb{R}^n} \) for \( \xi, \zeta \in \mathbb{R}^n \), and then \( \langle F, G \rangle := \langle F(\cdot), G(\cdot) \rangle \) for any two functions \( F, G : \mathbb{R} \to \mathbb{R}^n \). By virtue of the Morrey embedding, Theorem A.2, the functions \( f, g \) are both of class \( C^k \). Then, according to the Leibniz rule, we have for every \( 0 \leq i \leq k \)

\[
\langle f, g \rangle^{(i)} = \sum_{j=0}^{i} \binom{i}{j} \langle f^{(j)}, g^{(i-j)} \rangle,
\]

of which the \( L^\theta \)-norm can be estimated as

\[
\|f \cdot g \|_{L^\theta(\mathbb{R}/\mathbb{Z})} = \int_{\mathbb{R}/\mathbb{Z}} \left[ \sum_{j=0}^{i} \binom{i}{j} (f^{(j)})(x), g^{(i-j)}(x) \right]^{\theta} dx \leq 2^{(2\theta - 1)i} \|f\|_{C^{-s}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)}^{\theta} \|g\|_{C^{-s}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)}^{\theta} \text{ for all } 0 \leq i \leq k.
\]
which implies by means of the Morrey embedding (A.4) that the Sobolev norm \( \| (f, g) \|^6_{W^k, \varrho} \)
may be bounded from above by
\[
C_E^g (k + 1) 2^{(2q - 1)k} \| f \|_{W^{k, \varrho} (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \| g \|_{W^{k+\epsilon, \varrho} (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)}.
\]  
(A.13)

The numerator of the integrand in the seminorm \( [(f, g)^{(k)}]_{s, \varrho} \) can be estimated from above by means of (A.12) similarly as before by
\[
2^{(2q - 1)k} \sum_{j=0}^{k} \left| \langle f(j), g^{(k-j)}(x + w) - f^{(k-j)}(x) \rangle \right|^q \\
\leq 2^{(2q - 1)(k+1)} \sum_{j=0}^{k} \left\{ \left| \langle f(j)(x + w), g^{(k-j)}(x + w) - g^{(k-j)}(x) \rangle \right|^q \\
+ \left| \langle f^{(k)}(x + w) - f^{(j)}(x), g^{(k-j)}(x) \rangle \right|^q \right\} \quad \text{for } x \in \mathbb{R}, |w| \leq \ell/2,
\]
which may be bounded by
\[
2^{(2q - 1)(k+1)} \sum_{j=0}^{k} \left\{ \| f \|_{C^k (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \| g^{(j)}(x + w) - g^{(j)}(x) \|_q \\
+ \| g \|_{C^k (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \| f^{(j)}(x + w) - f^{(j)}(x) \|_q \right\} \quad \text{for } x \in \mathbb{R}, |w| \leq \ell/2.
\]

Integrating this against \( |w|^{-(1+\varrho)} \) and using the Morrey embedding inequality (A.4) we find
\[
[(f, g)^{(k)}]_{s, \varrho} \leq 2^{(2q - 1)(k+1)} \left\{ \| f \|_{C^k (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \sum_{j=0}^{k} \| g^{(j)} \|_{C^k (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} + \| g \|_{C^k (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \sum_{j=0}^{k} \| f^{(j)} \|_{C^k (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \right\} \\
(A.4) \leq 2^{(2q - 1)(k+1)} C_E^g \left\{ \| f \|_{W^{k+\epsilon, \varrho} (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \sum_{j=0}^{k} \| g^{(j)} \|_{W^{k+\epsilon, \varrho} (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} + \| g \|_{W^{k+\epsilon, \varrho} (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \sum_{j=0}^{k} \| f^{(j)} \|_{W^{k+\epsilon, \varrho} (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \right\}.
\]

For \( i < k \) we observe that
\[
\| g^{(i)} \|_{W^{k+\epsilon, \varrho} (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} = \int_\mathbb{R} / \mathbb{Z} \int_{-\ell/2}^{\ell/2} |g^{(i)}(u + w)|^q |w|^{-(1+\varrho)} \, dw \, du \\
\leq \| g^{(i+1)} \|_{L^\infty (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \int_{-\ell/2}^{\ell/2} |w|^{q-1-\varrho} \, dw =: \| g^{(i+1)} \|_{L^\infty (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \epsilon(s, \varrho, \ell) \\
(A.4) \leq C_E^g \| g \|_{W^{k+\epsilon, \varrho} (\mathbb{R}^n / \mathbb{Z}, \mathbb{R}^n)} \epsilon(s, \varrho, \ell),
\]

since \( q - 1 - s \varrho > q - 2 > -1 \), and by Morrey’s embedding (A.4). Analogous estimates hold for the seminorms of \( f^{(j)} \), which combine with (A.13) to the inequality as claimed with constant \( C(n, k, s, \varrho, \ell) := 2^{(2q - 1)(k+1)/q} C_E^g (k+1)^{1/q} [1 + 2c(s, \varrho, \ell) C_E^g]^{1/q} \). Notice that target dimension \( n \) enters this constant through the Morrey embedding constant \( C_E^g \); cf. Theorem A.2.

We conclude this section with an estimate on the fractional Sobolev norm of the multiplicative inverse of a non-vanishing Sobolev-Slobodeckii function.

**Lemma A.6.** Let \( \ell \in (0, \infty) \), \( q \in (1, \infty) \), \( s \in (\frac{1}{q}, 1) \), \( c > 0 \) and \( h \in W^{s, q} (\mathbb{R} / \mathbb{Z}) \) with \( |h(x)| \geq c \) for almost all \( x \in \mathbb{R} \). Then the multiplicative inverse, \( 1/h: \mathbb{R} \to \mathbb{R}, x \mapsto 1/h(x) \) is in \( W^{s, q} (\mathbb{R} / \mathbb{Z}) \), and we have \( \| 1/h \|^q_{W^{s, q} (\mathbb{R} / \mathbb{Z})} \leq \| h \|^q_{W^{s, q} (\mathbb{R} / \mathbb{Z})} / c^2 \).

**Proof.** The estimate \( \| 1/h \|^q_{L^\infty(\mathbb{R} / \mathbb{Z})} \leq \ell \| 1/h \|^q_{L^\infty(\mathbb{R} / \mathbb{Z})} \leq \ell/c^2 \) follows immediately from the pointwise lower bound on \( h \). The integrand of the seminorm \( \| 1/h \|_{W^{s, q} (\mathbb{R} / \mathbb{Z})} \) may be estimated as \( |w|^{-q-1} |h(x + w)|^{-q} |h(x)|^{-q} |h(x + w) - h(x)|^q \leq e^{-q} |w|^{-q-1} |h(x + w) - h(x)|^q \), which implies \( \| 1/h \|^q_{W^{s, q} (\mathbb{R} / \mathbb{Z})} \leq e^{-2q} \| h \|^q_{L^\infty(\mathbb{R} / \mathbb{Z})} \), and together with \( \| h \|^q_{L^\infty(\mathbb{R} / \mathbb{Z})} \geq c^2 \ell \) we obtain the desired estimate for \( \| 1/h \|^q_{W^{s, q} (\mathbb{R} / \mathbb{Z})} \).

\[ \square \]
B Arc length and bilipschitz constants

Whenever we speak about the arc length parametrization of a curve \( \gamma \in W^{1,1}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \) with \( |\gamma'| > 0 \) a.e. on \( \mathbb{R} \) and with length \( L := \mathcal{L}(\gamma) \in (0, \infty) \) we refer to the mapping \( \Gamma := \gamma \circ \mathcal{L}(\gamma, \cdot)^{-1} \in C^{0,1}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n) \) obtained from the inverse \( \mathcal{L}(\gamma, \cdot)^{-1} \) of the strictly increasing arc length function \( \mathcal{L}(\gamma, \cdot) \) of class \( W^{1,1}_{loc}(\mathbb{R}) \) defined as

\[
\mathcal{L}(\gamma, x) := \int_0^x |\gamma'(y)| \, dy \quad \text{for} \quad x \in \mathbb{R}.
\] (B.1)

Notice that \( \mathcal{L}(\gamma, x + \ell) = \mathcal{L}(x) + L \) and \( (\mathcal{L}(\gamma, x + \ell))' = (\mathcal{L}(\gamma, x))' \) for all \( x \in \mathbb{R} \), as well as \( \mathcal{L}(\gamma, z + L)^{-1} = \mathcal{L}(\gamma, z)^{-1} + \ell \) for all \( z \in \mathbb{R} \), and therefore the composition \( \Gamma = \gamma \circ \mathcal{L}(\gamma, \cdot)^{-1} \) is \( L \)-periodic and \( 1 \)-Lipschitz because of

\[
|\frac{d}{ds} \Gamma(s)| = |\gamma'(\mathcal{L}(\gamma, s)^{-1})| \leq \frac{1}{|\gamma'(x)|} \quad \text{for a.e.} \quad s \in [0, L].
\] (B.2)

Recall from the introduction the definition of the bilipschitz constant \( (1.3) \) and the notation \( W^{1,1}_n(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \) for the subset of all regular \( W^{1,1} \)-curves such that their restriction to the fundamental interval \( [0, \ell) \) is injective.

**Lemma B.1.** Let \( \ell \in (0, \infty) \) and \( \gamma \in W^{1,1}_n(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \) with length \( L := \mathcal{L}(\gamma) \in (0, \infty) \) and with arc length parametrization \( \Gamma \in W^{1,\infty}(\mathbb{R}/L\mathbb{Z}, \mathbb{R}^n) \). Then the bilipschitz constants of \( \gamma \) and \( \Gamma \) are related by

\[
\text{BiLip}(\gamma) = [v_\gamma, \text{BiLip}(\Gamma), v_\gamma],
\] (B.3)

where \( v_\gamma = \text{ess inf}_{[0, \ell]} |\gamma'(\cdot)| \geq 0 \).

**Proof.** The upper bound on \( \text{BiLip}(\gamma) \) in (B.3) follows immediately from the estimate \( |\gamma'(x)| = \lim_{h \to 0} |h|^{1/L} \gamma(x + h) - \gamma(x) | \geq \text{BiLip}(\gamma) \) at points \( x \in \mathbb{R}/\ell\mathbb{Z} \), where \( \gamma'(x) \) exists. Taking the infimum over all such \( x \in \mathbb{R}/\ell\mathbb{Z} \) establishes the upper bound in (B.3). For the proof of the lower bound in (B.3) assume w.l.o.g. that \( 0 \leq y < x < \ell \), so that by monotonicity \( 0 \leq \mathcal{L}(\gamma, y) \leq \mathcal{L}(\gamma, x) < L \). Then \( |\gamma(x) - \gamma(y)| \) can be written as and estimated from below by

\[
|\Gamma \circ \mathcal{L}(\gamma, x) - \Gamma \circ \mathcal{L}(\gamma, y)| \geq \text{BiLip}(\Gamma)|\mathcal{L}(\gamma, x) - \mathcal{L}(\gamma, y)|_{\mathbb{R}/L\mathbb{Z}}.
\] (B.4)

If \( |\mathcal{L}(\gamma, x) - \mathcal{L}(\gamma, y)|_{\mathbb{R}/L\mathbb{Z}} = |\mathcal{L}(\gamma, x) - \mathcal{L}(\gamma, y)| = (\mathcal{L}(\gamma, x) - \mathcal{L}(\gamma, y)) \), then the distance on the right-hand side of (B.4) equals \( \int_y^x |\gamma'(\tau)| \, d\tau \geq v_\gamma |x - y| \geq v_\gamma |x - y|_{\mathbb{R}/\ell\mathbb{Z}} \), which proves the claim in this case. If, on the other hand, \( |\mathcal{L}(\gamma, x) - \mathcal{L}(\gamma, y)|_{\mathbb{R}/L\mathbb{Z}} = L - |\mathcal{L}(\gamma, x) - \mathcal{L}(\gamma, y)| \) the latter can be written as

\[
L - (\mathcal{L}(\gamma, x) - \mathcal{L}(\gamma, y)) = L - \int_y^x |\gamma'(\tau)| \, d\tau = \int_y^x |\gamma'(\tau)| \, d\tau + \int_{y}^{\ell/2} |\gamma'(\tau)| \, d\tau,
\]

which can be bounded from below by \( v_\gamma (y + \ell - x) = v_\gamma (\ell - |x - y|) \geq v_\gamma |x - y|_{\mathbb{R}/\ell\mathbb{Z}} \), and this again can be inserted into (B.4) to yield the desired inequality (B.3).

**Lemma B.2 (Seminorm of arc length).** Let \( s \in (0, 1) \), \( g \in (1, \infty) \), and \( \gamma \in W^{1,g}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \). Then, \( [\mathcal{L}(\gamma, \cdot)]_{s,g} \leq |\gamma|_{s,g} \).

Note that the right-hand side of this estimate might be infinite, since we did not assume that \( \gamma \) is of class \( W^{1+g,s}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \).

**Proof.** If the seminorm on the right-hand side is infinite there is nothing to prove. If, on the other hand \( \gamma \in W^{1+g,s}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n) \) then we estimate

\[
[\mathcal{L}(\gamma, \cdot)]_{s,g}^g = \int_{\mathbb{R}/\ell\mathbb{Z}} \int_0^{\ell/2} |w|^{1-s} |\gamma'(x + w) - |\gamma'(x)| |^s \, dw \, dx
\leq \int_{\mathbb{R}/\ell\mathbb{Z}} \int_0^{\ell/2} |w|^{1-s} |\gamma'(x + w) - |\gamma'(x)| |^g \, dw \, dx = |\gamma|_{s,g}^g.
\]

\[\square\]
The bilipschitz constant $\text{BiLip}(\gamma)$, defined in (1.13) of the introduction, of a closed embedded, i.e., immersed and injective $C^1$-curve is strictly positive, since under this regularity assumption the quotient in (1.13) converges to $|\gamma'(u_2)| \geq v_\gamma = \min_{[0,\ell]} |\gamma'| > 0$ as $u_1 \to u_2$. This means that this positive quotient possesses a continuous extension onto all of $\mathbb{R} \times \mathbb{R}$, which is bounded by a positive constant from below by periodicity and continuity. We now quantify a neighbourhood of such a curve $\gamma$, in which the bilipschitz constant remains under control.

**Lemma B.3.** Let $\ell \in (0, \infty)$ and let $\gamma \in C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)$ be injective on $[0, \ell]$ with $|\gamma'| > 0$ on $\mathbb{R}$. Then

$$\text{BiLip}(\eta) \geq \frac{\text{BiLip}(\gamma)}{2} > 0 \quad (B.5)$$

for all $\eta \in C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)$ with $\|\eta' - \gamma\|_{C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)} \leq \text{BiLip}(\gamma)/2$.

**Proof.** It suffices to prove the first inequality in (B.5), and for that assume w.l.o.g. that $0 \leq y \leq x < \ell$. We have

$$|\eta(x) - \eta(y)| \geq |\gamma(x) - \gamma(y)| - |(\eta - \gamma)(x) - (\eta - \gamma)(y)|. \quad (B.6)$$

If $|x - y|_{\mathbb{R}/\ell\mathbb{Z}} = |x - y| = (x - y)$, then we can estimate the right-hand side from below by $\text{BiLip}(\gamma)|x - y|_{\mathbb{R}/\ell\mathbb{Z}} - \int_y^x |(\eta - \gamma)'(t)| \, dt \geq \text{BiLip}(\gamma)|x - y|_{\mathbb{R}/\ell\mathbb{Z}} - \|\eta' - \gamma\|_{C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)}|x - y|_{\mathbb{R}/\ell\mathbb{Z}}$, which is bounded from below by $\text{BiLip}(\gamma)|x - y|_{\mathbb{R}/\ell\mathbb{Z}}/2$. If, on the other hand $|x - y|_{\mathbb{R}/\ell\mathbb{Z}} = \ell - |x - y| = \ell - (x - y)$ then we use the $\ell$-periodicity to replace the term $(\eta - \gamma)(y)$ in (B.6) by $(\eta - \gamma)(\ell + y)$ to estimate the right-hand side of (B.6) from below by $\text{BiLip}(\gamma)|x - y|_{\mathbb{R}/\ell\mathbb{Z}} - \int_y^{x+\ell} |(\eta - \gamma)'(t)| \, dt$, which can be bounded from below by $\text{BiLip}(\gamma)|x - y|_{\mathbb{R}/\ell\mathbb{Z}} - \|\eta' - \gamma\|_{C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)}|x - y|_{\mathbb{R}/\ell\mathbb{Z}}/2$, which concludes the proof.

The following corollary quantifies the fact that the set $W^{1+s,\phi}_{ir}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)$ of all regular embedded Sobolev-Slobodeckii loops is an open subset of that fractional Sobolev space if that spaces embeds into $C^1(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)$.

**Corollary B.4.** Let $\ell \in (0, \infty)$, $\phi \in (1, \infty)$, and $s \in (1/\phi, 1)$, and suppose $\gamma \in W^{1+s,\phi}_{ir}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)$. Then $\text{BiLip}(\gamma) > 0$, and one has for the bilipschitz constant $\text{BiLip}(\eta)$ and the minimal velocity $v_\eta := \min_{[0,\ell]} |\eta'|$

$$\text{BiLip}(\eta) \geq \frac{\text{BiLip}(\gamma)}{2} \quad \text{and} \quad v_\eta \geq v_\gamma/2 \quad (B.7)$$

for all $\eta \in W^{1+s,\phi}_{ir}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)$ satisfying $|\eta' - \gamma'|_{s,\phi} \leq (2C_E C_P)^{-1} \text{BiLip}(\gamma)$, where $C_E = C_E(n, s, \phi, \ell)$ denotes the constant in (A.4) of the Morrey embedding, Theorem A.2, and $C_P = C_P(n, s, \phi, \ell)$ is the constant in the Poincaré inequality (A.6).

Notice that neither in this corollary nor in the preceding Lemma B.3 anything is required about the $C^0$-distance between the curves, so in principle $\gamma$ and $\eta$ could be far apart in $\mathbb{R}^n$.

**Proof.** By the Morrey embedding, Theorem A.2, the curve $\gamma$ is of class $C^1$, so that $\text{BiLip}(\gamma) > 0$. Moreover, again by Theorem A.2, in combination with the Poincaré inequality, Proposition A.3, we estimate using our assumption on $\eta$

$$\|\eta' - \gamma'\|_{C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)}^{(A.4)} \leq C_E \|\eta' - \gamma'\|_{W^{s,\phi}(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)}^{(A.6)} \leq C_E C_P |\eta' - \gamma'|_{s,\phi} \leq \text{BiLip}(\gamma)/2,$$

so that we can apply Lemma B.3 to finish the proof. Similarly, again by Morrey’s embedding (A.4) and Poincaré’s inequality (A.6) one finds $v_\eta \geq v_\gamma - \|\eta' - \gamma'\|_{C^0(\mathbb{R}/\ell\mathbb{Z}, \mathbb{R}^n)} \geq v_\gamma - C_E C_P |\eta' - \gamma'|_{s,\phi} 2\gamma > v_\gamma - \text{BiLip}(\gamma)/2 \geq v_\gamma/2 > 0$. 

\[\Box\]
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