Reentrant Phase Diagram of Two-Dimensional Electron Gas

Jukka A. Ketoja
Department of Mathematics, P. O. Box 4, FIN-00014 University of Helsinki, Finland

Indubala I. Satija
Department of Physics and
Institute of Computational Sciences and Informatics,
George Mason University,
Fairfax, VA 22030
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We study the phase diagram of an electron on a square lattice in a transverse magnetic field. Using a renormalization scheme, we show that the inequality of the two next-nearest neighbor couplings destroys the fat critical regime above the bicritical line and replaces it with another reentrant extended phase. Furthermore, the universal strange attractor describing self-similar fluctuations of the localized phase is replaced by a new fixed point. The reentrant extended phase and the new fixed point of the localized phase belong to the universality class of an anisotropic triangular lattice.

As the properties of the Harper equation [3], which describes an electron on a square lattice in transverse magnetic field, are fairly well understood [3], much less is known about more general quasiperiodic tight binding models (TBM). In this paper, we study the generalization which results from taking into account both the nearest-neighbor (NN) and next-nearest-neighbor (NNN) interaction in the electron problem. The associated TBM has the form [3]

\[ t_a (\psi_{k+1} + \psi_{k-1}) + 2t_b \cos(2\pi(k\sigma + \phi))\psi_k + \exp[i2\pi(k\sigma + \phi)] \{ t_{ab}e^{i\pi\sigma}\psi_{k+1} + t_{ab}e^{-i\pi\sigma}\psi_{k-1} \} + \exp[-i2\pi(k\sigma + \phi)] \{ t_{ab}e^{i\pi\sigma}\psi_{k-1} + t_{ab}e^{-i\pi\sigma}\psi_{k+1} \} = E\psi_k \]  

(1)

Here \( t_a \) and \( t_b \) are the NN couplings while \( t_{ab} \) are the diagonal NNN couplings. The quasiperiodicity results from the irrational parameter \( \sigma \) describing the magnetic flux. The Harper equation corresponds to vanishing of the NNN couplings. Furthermore, in the limit \( t_{ab} \) equal to zero, the model

Recently, the above model was studied in the isotropic limit where \( t_{ab} = t_{ab} \). Although the wave function \( \psi_k \) in general is a complex function of the lattice index \( k \), \( \psi_k \) can be taken to be real in the case. These studies revealed a very interesting phase diagram: For \( 2t_{ab} < t_a \), the model belonged to the universality class of the Harper model with both extended (E) \( (t_b < t_a) \) and localized (L) \( (t_b > t_a) \) phases and a critical (C) point at \( t_a = t_b \) where the system had the full square symmetry. On the other hand, for \( 2t_{ab} \geq t_a \) the model was found to belong to a new universality class where there was no E phase but instead the C phase existed in a finite parameter interval \( t_b \leq 2t_{ab} \). For \( t_b > 2t_{ab} \), the states were exponentially localized. The E and the C phases were separated by a bicritical line \( 2t_{ab} = t_a \).

Detailed decimation studies [3] showed that the wave functions within the fat C phase above the bicritical line were self-similar (at the band edges) only at certain special values of the parameters. These special points corresponded to universal limit cycles of the renormalization. However, for generic parameter values, the fractal characteristics of the critical wave functions did not exhibit self-similarity and were conjectured to be described by a strange attractor of the renormalization flow.

In another interesting study of this model, it was shown that the fluctuations of the wave functions in the L phase mimiced the behavior in the C phase [3]. The L phase of the Harper universality class was described by a renormalization fixed point of the strong coupling limit \( t_b/t_a \to \infty \) while the L phase above the bicritical line was described by a strange attractor of the associated renormalization [3].

The general case of the model where \( t_{ab} \) and \( t_{ab} \) are not equal and the resulting TBM is complex has not been fully investigated. In the three parameter space, \( \lambda = t_b/t_a, \alpha = 2t_{ab}/t_a, \) and \( \beta = 2t_{ab}/t_a \), using the duality property of the model, Han et al. [3] calculated the Lyapunov exponent of the model analytically and concluded that the system is localized for \( \lambda > 1 \) if \( (\alpha + \beta)/2 < 1 \) and for \( \lambda > (\alpha + \beta)/2 \) otherwise. Apart from the existence of metal-insulator transition, nothing is known about the scaling properties of the complex model.

In this paper, we study the complex model using our recently developed decimation scheme. We will confine ourselves to the case where \( \sigma = (\sqrt{3} - 1)/2 \). It is shown that the phase diagram changes discontinuously as \( \alpha - \beta \) becomes different from zero. The C phase of the real TBM above the bicritical line is replaced by another E phase. This complex reentrant E phase is described by...
a strong NNN coupling of the triangular lattice and is shown to be related to the weak coupling limit of the Harper equation. Furthermore, the invariant strange set of the renormalization describing fractal characteristics of the fluctuations in the L phase degenerates to a fixed point of the renormalization. These fluctuations are defined by the equation

\[ \psi_k = e^{-\gamma |k|} \eta_k \]  

where \( \gamma \) is the Lyapunov exponent which vanishes in the E and C phases and is positive in the L phase. In other words, \( \eta_k \) is equivalent to the original wave function \( \psi_k \) in the E and C phases whereas in the L phase \( \eta_k \) describes the fluctuations around the exponentially decaying wave function. Knowing the analytic formula for the Lyapunov exponent, it is easy to write a TBM for \( \eta_k \), resembling Eq. (1), in the L phase: 

\[ \eta_k \text{ at an arbitrary site } k \text{ with two neighboring Fibonacci sites } k + F_{n+1} \text{ and } k + F_n \text{ where } F_{n+1} = F_n + F_{n-1}: \]

\[ f_n(k) \eta(k + F_{n+1}) = \eta(k + F_n) + e_n(k) \eta(k). \]  

The way how the decimation functions \( f_n \) and \( e_n \) are placed in the decimation equation is somewhat arbitrary but here we choose the form which causes the asymptotic limits of the decimation functions \( e_n \) and \( f_n \) as \( n \to \infty \) to be bounded in all three phases. The additive property of the Fibonacci numbers provides exact recursion relations for the decimation functions \( e_n \) and \( f_n \):

\[ e_{n+1}(k) = -\frac{Ae_n(k)}{1 + Af_n(k)} \]  

\[ f_{n+1}(k) = \frac{f_{n-1}(k + F_n)f_n(k + F_n)}{1 + Af_n(k)} \]

\[ A = e_{n-1}(k + F_n) + f_{n-1}(k + F_n)e_n(k + F_n). \]

For fixed \( k \), the above coupled equations for the decimation functions define a RG flow which asymptotically \( (n \to \infty) \) converges on an attractor. The C phase is distinguished from the E phase by the existence of nontrivial limiting behavior. With anisotropic NNN coupling, the attractor is a \( p \)-cycle in all three phases for \( E = E_{\text{min}} \). The asymptotic \( p \)-cycle for \( e_n(0) \) and \( f_n(0) \) determines the universal scaling ratios

\[ \zeta_j = \lim_{n \to \infty} \eta(F_{pn+j})/\eta(0); \quad j = 0, ..., p - 1. \]  

whose absolute values are equal to unity in the E phase and less than unity in the C phase.

In Fig. 1 we show the phase diagram obtained by analyzing the asymptotic behavior of the decimation functions in different parts of the parameter space. As soon as \( \alpha \) and \( \beta \) differ, \( \alpha - \beta \) is an irrelevant parameter and the phase diagram is determined solely by two parameters: \( \lambda \) and \( (\alpha + \beta)/2 \). In the parameter range \( (\alpha + \beta)/2 < 1 \), the decimation functions become asymptotically real approaching the same universal cycles as for the Harper equation \( \alpha = \beta = 0 \). For \( (\alpha + \beta)/2 \geq 1 \), \( \alpha \neq \beta \), the decimation functions stay complex also asymptotically. The cycle length \( p \) in this case is six in all three phases. However, considering the absolute value of the decimation functions (or the scaling ratio \( \zeta \)), one observes a 3-cycle on the line \( (\alpha + \beta)/2 = 1 \) and a fixed point for \( (\alpha + \beta)/2 > 1 \). The 3-cycle observed at the critical point C is different from the universal 3-cycle observed along the rest of the (bi)critical line AC. Moreover, these fixed points are all universal and do not depend on the actual parameter values. Table I summarizes various universality classes of the model. It is interesting to note that the line CL divides the localized phase into two different universality classes. That is, the scaling properties of the self-similar fluctuations in the region BCL are different from those of the region LCE. Furthermore, the boundary line CL has its scaling characteristics different from the two regions that it separates.

The reentrant E phase described by a complex 6-cycle with complex scaling ratios is not as trivial as the real fixed point of the E phase below the bicritical line describing the weak coupling limit of the Harper model. The latter depends neither on the phase \( \phi \) nor on the lattice index \( k \) and can be easily solved from a fixed point equation: \( f_n(k) \equiv \sigma \) and \( e_n(k) \equiv -\sigma^2 \) resulting in \( \zeta = 1 \). In the complex E phase, complications arise from the fact that the decimation functions do depend both on \( \phi \) and \( k \). But the absolute value of a decimation function has the same constant value as in the real case. Noting the fact that the decimation functions and the scaling ratios above the line ACL remain the same, the reentrant E phase can be understood in the limit of highly anisotropic triangular lattice \( \alpha \to \infty \), \( \beta/\alpha \to 0 \), and \( \lambda/\alpha \to 0 \). In this limit, Eq. (1) reduces to the TBM describing the weak coupling limit of the Harper model,

\[ C_{k+1} + C_{k-1} = E \frac{E}{t_{ab}} C_k, \]  

where \( C_k \) is related to \( \psi_k \) via

\[ C_k = \exp(i2\pi \phi k) \exp(i\pi \sigma k^2) \psi_k. \]
Hence, the wave function of the infinitely anisotropic triangular lattice is related by a phase factor to the extended wave function of the similar NN square lattice. The above equation shows that the scaling factors $\zeta_j$ exist only for special (i.e. rational or those related to the golden mean) values of the phase $\phi$. The relation $F_n \sigma = F_{n-1} - (-\sigma)^n$ implies that

$$\sigma F^2_n = F_{n-1} F_n + \frac{(-1)^{n-1} + \sigma^{2n}}{1 + 2\sigma}. \quad (9)$$

From this and Eq. (8) it follows that for $\phi = 0, 1/2$

$$\zeta_j = \pm \exp[\pm i\pi/(1 + 2\sigma)] \quad (10)$$

The above scaling is relevant also in other parts of the complex E phase because the RG flow is attracted by the same 6-cycle in the whole region. Taking advantage of the above limiting solution, it is possible to derive an explicit expression for the 6-cycle. Substituting Eq. (8) into the decimation equation (3), we obtain

$$e_n(k) = e^h_n(k) \exp[-i2\pi\phi F_n] \exp[-i\pi\sigma(F^2_n + 2kF_n)]$$

$$f_n(k) = f^h_n(k) \exp[i2\pi\phi F_{n-1}] \times \exp[i\pi\sigma(F^2_{n-1} - F^2_n + 2kF_{n-1})], \quad (11)$$

where $e^h$ and $f^h$ are the decimation functions corresponding to the TBM (7) describing the weak coupling limit of the Harper equation. From these equations we see that $e_n$ and $f_n$ are functions of the fractional part of $k\sigma$, denoted by $< k\sigma >$, only. Therefore, we can write the decimation functions in terms of the renormalized variable $x = (-\sigma)^{-n} < k\sigma >$. For simplicity, let us assume that $\phi = 0, 1/2$. Applying the relation (9) and the fact that

$$(-\sigma)^n F_n = \frac{(-1)^n - \sigma^{2n}}{1 + 2\sigma} \quad (12)$$

we obtain six different limiting function pairs of the form

$$e^n(x) = \pm \sigma^2 \exp[\pm i\pi(2x - 1)/(1 + 2\sigma)]$$

$$f^n(x) = \pm \sigma \exp[\pm i\pi(2x + 1)/(1 + 2\sigma)] \quad (13)$$

as $n$ tends to infinity. These pairs form a 6-cycle of the recursion (4-5), written in the continuous variable $x$:

$$e_{n+1}(x) = -\frac{Ae_n(-\sigma x)}{1 + Af_n(-\sigma x)} \quad (14)$$

$$f_{n+1}(x) = \frac{f_{n-1}(\sigma^2 x + \sigma)f_n(-\sigma x - 1)}{1 + Af_n(-\sigma x)} \quad (15)$$

$$A = e_{n-1}(\sigma^2 x + \sigma) + f_{n-1}(\sigma^2 x + \sigma)e_n(-\sigma x - 1).$$

For the above 6-cycle, $1 + Af^n(-\sigma x) \equiv \sigma$.

Therefore, the characteristic feature of the reentrant E phase is the fact that the decimation functions depend explicitly on $x$. These functions are complex and consequently the universal scaling ratio has both real and imaginary parts but the absolute value of $\zeta$ is unity. This is unlike the Harper E phase where the real universal functions are site independent and are given in terms of the powers of the golden mean.

In summary, we have shown that with anisotropic NNN couplings, the renormalization behavior for the TBM describing an electron on a square lattice is a lot simpler than in the isotropic case. Firstly, the renormalization strange set corresponding to the fat C phase in the isotropic case is replaced by an attracting cycle associated with trivial scaling properties (i.e. E phase). Secondly, the bicritical lines of the isotropic case remain critical also when the NNN couplings are not equal but the renormalization attractor is again simpler (i.e. a cycle). Thirdly, the fluctuations of the exponentially localized wave functions are described by a universal fixed point and not by an infinite strange set as in the isotropic case.

The novel feature of our model is the existence of two extended phases separated by a critical line signaling a transition from real scaling ratios to complex ones. The two extended phases respectively fall into the universality classes of the weak coupling limit of the NN square lattice and the strong NNN coupling limit of the triangular lattice. It is interesting that also on the other side of the localization border (i.e. in the L phase), the scaling behavior is divided into three different universality classes. The Harper L phase corresponding to the strong NN coupling is separated from the strong NNN coupling phase by the CL line with its own scaling properties. Therefore, for a fixed value of NNN coupling, as the NN coupling $\lambda$ is varied, the behavior beyond the localization transition appears to be shadowed by the behavior before the localization transition: the existence of E phase before localization results in a nontrivial fixed point in the L phase while the existence of a period $p$ limit cycle describing self-similar critical states before the localization onset leads to the appearance of period $p$ limit cycle of the renormalization after localization describing self-similar fluctuations.

The phase diagram discussed here describes the universal properties of square and triangular lattices. Therefore, we believe that our results can be experimentally realized on two-dimensional mesoscopic systems. 

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FIG. 1. The reentrant phase diagram of the TBM (1) for the case $\alpha \neq \beta$. The (bi)critical line AC separates the two extended phases while the critical line CE describes the onset to localization. The extension of the AC line (line CL) also separates the localized phase into two different regions with different scaling properties. For the corresponding phase diagram in the isotropic case $\alpha = \beta$, see Fig. 1 of ref. [5].

TABLE I. The universal scaling ratios for various universality classes at the band edge. In order to avoid showing all $p$ scaling ratios, some of which differ only by signs of the real or imaginary part, we show all different ($|\text{Re} (\zeta_j)|, |\text{Im} (\zeta_j)|$). $\phi_c = 1/2$ for all other parts of the parameter space except the line AC where $\phi_c$ varies as a function of $\lambda$. In the reentrant E region ACE, the absolute value of $\zeta$ is unity. Note that the last three lines show the scaling properties of the fluctuations in an exponentially localized wave function.
