On pair correlation and discrepancy

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Abstract. We say that a sequence \((x_n)_{n \geq 1}\) in \([0,1)\) has Poissonian pair correlations if
\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{s}{N} \right\} = 2s
\]
for all \(s > 0\). In this note we show that if the convergence in the above expression is—in a certain sense—fast, then this implies a small discrepancy for the sequence \((x_n)_{n \geq 1}\). As an easy consequence it follows that every sequence with Poissonian pair correlations is uniformly distributed in \([0,1)\).

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1. Introduction. The concept of Poissonian pair correlations for a sequence \((x_n)_{n \geq 1}\) in \([0,1)\) was introduced by Rudnick and Sarnak [5], and has been intensively studied by several authors over the last years (see, for instance, [2, 3, 6–8]). Let \(\|\cdot\|\) denote distance to the nearest integer. We say that a sequence \((x_n)_{n \geq 1}\) of real numbers in \([0,1)\) has Poissonian pair correlations if
\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{s}{N} \right\} = 2s
\]
for every \(s > 0\).

In this note we are concerned with the relation between the Poissonian pair correlation property and the notion of uniform distribution. We say that the sequence \((x_n)_{n \geq 1}\) is uniformly distributed, or equidistributed, in \([0,1)\) if

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for all $0 \leq a \leq b \leq 1$. It is well known that uniform distribution does not necessarily imply Poissonian pair correlations. One example confirming this is the Kronecker sequence $\{n\alpha\}_{n \geq 1}$, which is uniformly distributed for every irrational $\alpha$, but does not have Poissonian pair correlations for any value of $\alpha$. Whether the converse implication holds has until recently remained an open question: is every sequence in $[0,1)$ with Poissonian pair correlations uniformly distributed? We answer this question in the affirmative by establishing a quantitative result connecting the speed of convergence in (1.1) to the star-discrepancy $D^*_N$ of the sequence. We recall that the star-discrepancy $D^*_N$ of $(x_n)_{n \geq 1}$ is defined as

$$D^*_N = \sup_{0 \leq a \leq 1} \left| \frac{1}{N} \cdot A_N([0,a)) - a \right|,$$

where $A_N([0,a)) := \#\{1 \leq n \leq N : x_n \in [0,a)\}$, and that $(x_n)_{n \geq 1}$ is uniformly distributed in $[0,1)$ if and only if $\lim_{N \to \infty} D^*_N = 0$ (see, for example, [4]).

The main result of this paper is the following.

**Theorem 1.1.** Let $(x_n)_{n \geq 1}$ be a sequence in $[0,1)$, and suppose that there exists a function $F: \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ which is monotonically increasing in its first argument, and which satisfies

$$\max_{s=1,\ldots,K} \left| \frac{1}{2s} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{s}{N} \right\} - N \right| \leq F(K,N) \quad (1.2)$$

for all $N \in \mathbb{N}$ and all $K \leq N/2$. One can then find an integer $N_0 > 0$ such that for $N \in \mathbb{N}$, $N \geq N_0$, and arbitrary $K$ satisfying

$$\min \left( \frac{1}{2} N^{2/5}, \frac{N}{F(K^2,N)} \right) \leq K \leq N^{2/5}, \quad (1.3)$$

we have

$$ND^*_N \leq 5 \cdot \max \left( N^{4/5}, \sqrt{N} \cdot F(K^2,N) \right)$$

where $D^*_N$ is the star-discrepancy of $(x_n)_{n \geq 1}$.

The next result is an easy consequence of Theorem 1.1.

**Corollary 1.2.** If the sequence $(x_n)_{n \geq 1}$ in $[0,1)$ has Poissonian pair correlations, then it is uniformly distributed.\(^1\)

**Proof.** Suppose that $(x_n)_{n \geq 1}$ has Poissonian pair correlations, and fix any $\varepsilon > 0$. We then have

$$\max_{s=1,\ldots,\lfloor 1/\varepsilon^5 \rfloor} \left| \frac{1}{2s} \# \left\{ 1 \leq l \neq m \leq N : \|x_l - x_m\| < \frac{s}{N} \right\} - N \right| \leq \varepsilon N,$$

for all sufficiently large $N \geq N(\varepsilon)$. Hence, we may construct a function $F$ satisfying (1.2) where $F(L,N) = \varepsilon N$ for $N \geq N(\varepsilon)$ and $L \leq 1/\varepsilon^5$. Without

\(^1\)Simultaneously with our proof, another elegant proof of this result was given by Aistleitner et al. [1]. However, their approach is less elementary and does not provide the quantitative bound on the star discrepancy given by Theorem 1.1.
loss of generality, we may assume that $N(\varepsilon) \geq 1/\varepsilon^5$. If we fix $K := \lceil 1/\varepsilon^2 \rceil$, then for $N \geq N(\varepsilon)$ we have
\[
\frac{N}{F(K^2, N)} = \frac{N}{\varepsilon N} = \frac{1}{\varepsilon} \leq K \leq N^{2/5},
\]
and accordingly $K$ satisfies (1.3). By Theorem 1.1 it thus follows that
\[
D_N^* \leq \frac{5}{N} \cdot \max \left( N^{4/5}, N\varepsilon \right) = 5\sqrt[5]{\varepsilon}
\]
for $N \geq N_0$ (where in particular $N_0 \geq N(\varepsilon)$). □

2. Proof of Theorem 1.1. For a fixed pair of integers $(N, K)$, where $K$ satisfies (1.3), we introduce the notation
\[
H(N, K) := 5 \cdot \max \left( N^{4/5}, \sqrt{N \cdot F(K^2, N)} \right).
\]
Aiming for a proof by contradiction, we assume that $ND_N^* > H(N, K)$ for infinitely many pairs $(N, K)$. That is, there exist integers $1 < N_1 < N_2 < \cdots$ and corresponding integers $K_1, K_2, \ldots$ satisfying (1.3), as well as real numbers $B_1, B_2, \ldots \in (0, 1)$, such that either
\[
\# \left\{ 1 \leq n \leq N_j: x_n \in [0, B_j) \right\} - N_j B_j > H(N_j, K_j) \tag{2.1}
\]
for every $j$, or
\[
\# \left\{ 1 \leq n \leq N_j: x_n \in [0, B_j) \right\} - N_j B_j < -H(N_j, K_j) \tag{2.2}
\]
for every $j$. We assume in what follows that (2.1) holds (the case when (2.2) holds is treated analogously). Note that (2.1) implies
\[
N_j - N_j B_j - H(N_j, K_j) > 0. \tag{2.3}
\]
Let $N := N_j$, $K := K_j$, $B := B_j$, and $H := H(N_j, K_j)$ for some fixed $j$. We now consider the distribution of the points $x_n$ into subintervals of $[0, 1)$ of length $K/N$. Let
\[
A_i := \# \left\{ 1 \leq n \leq N: x_n \in \left[ i \cdot \frac{K}{N}, (i + 1) \cdot \frac{K}{N} \right) \right\}
\]
for $i = 0, 1, \ldots, \lfloor N/K \rfloor - 1$, and let
\[
A_{\lfloor N/K \rfloor} := \# \left\{ 1 \leq n \leq N: x_n \in \left[ \frac{N}{K}, 1 \right) \right\}.
\]
Moreover, for arbitrary positive integers $l$, let
\[
A_l := A_l \mod(\lfloor N/K \rfloor + 1).
\]
If we introduce the notation
\[
\mathcal{H}_L := \# \left\{ 1 \leq l \neq m \leq N: \|x_l - x_m\| < \frac{L K}{N} \right\}
\]
for $L = 1, 2, \ldots, K$, then
\[
\left| \frac{1}{2LK} \mathcal{H}_L - N \right| \leq F(K^2, N). \tag{2.4}
\]
We have that
\[ H_L \geq \sum_{i=0}^{\lfloor N/K \rfloor} (A_i(A_i - 1) + 2A_i(A_{i+1} + \cdots + A_{i+L-1})) \]
\[ = \sum_{i=0}^{\lfloor N/K \rfloor} \left( (A_i + \cdots + A_{i+L-1})^2 - (A_{i+1} + \cdots + A_{i+L-1})^2 \right) - N \]
\[ =: 2LKN \cdot \gamma_L - N, \]
where
\[ \gamma_L = \frac{1}{2LKN} \sum_{i=0}^{\lfloor N/K \rfloor} \left( (A_i + \cdots + A_{i+L-1})^2 - (A_{i+1} + \cdots + A_{i+L-1})^2 \right). \]

Thus, we get
\[ \frac{1}{2LKN} \cdot H_L \geq \gamma_L - \frac{1}{2L} \cdot \gamma_L. \tag{2.5} \]

Now consider
\[ \Gamma_K := \min_{x_1, \ldots, x_N} \max_{L=1,2,\ldots,K} \gamma_L, \tag{2.6} \]
where by \( \min_{x_1, \ldots, x_N} \) we mean the minimum over all configurations of the points \( x_1, \ldots, x_N \) satisfying (2.1). If we define
\[ Z_L := \frac{1}{2LKN} \sum_{i=0}^{\lfloor N/K \rfloor} (A_i + A_{i+1} + \cdots + A_{i+L-1})^2, \]
then
\[ \gamma_L = Z_L - \frac{L-1}{L} \cdot Z_{L-1}, \]
and thus
\[ \Gamma_K = \min_{x_1, \ldots, x_N} \max \left( Z_1, Z_2 - \frac{1}{2}Z_1, \ldots, Z_K - \frac{K-1}{K}Z_{K-1} \right). \]

We have
\[ \max \left( Z_1, Z_2 - \frac{1}{2}Z_1, \ldots, Z_K - \frac{K-1}{K}Z_{K-1} \right) \geq \frac{2}{K+1} Z_K. \]

To see this, assume to the contrary that \( Z_1 \) and \( Z_L - (L-1)Z_{L-1}/L \) are all less than \( 2Z_K/(K+1) \). Then by successive insertions we get the contradiction \( Z_K < Z_K \). Hence, we have
\[ \Gamma_K \geq \min_{x_1, \ldots, x_N} \frac{2}{K+1} \cdot Z_K. \tag{2.7} \]

Let us now estimate
\[ \min_{x_1, \ldots, x_N} Z_K = \frac{1}{2K^2N} \min_{A_0,A_1,\ldots,A_{\lfloor N/K \rfloor}} \sum_{i=0}^{\lfloor N/K \rfloor} (A_i + A_{i+1} + \cdots + A_{i+K-1})^2, \]
where the minimum on the right-hand side is taken over all possible values of \( A_0, A_1, \ldots, A_{\lfloor N/K \rfloor} \) provided that the points \( x_1, \ldots, x_N \) satisfy (2.1). By
definition, we have \( A_0 + \cdots + A_{\lfloor N/K \rfloor} = N \). Introducing the notation \( G_i = A_i + A_{i+1} + \cdots + A_{i+K-1} \), we thus get
\[
\sum_{i=0}^{\lfloor N/K \rfloor} G_i = K \cdot \sum_{i=0}^{\lfloor N/K \rfloor} A_i = KN. \tag{2.8}
\]
Moreover, by invoking condition (2.1) on the distribution of \( x_1, \ldots, x_N \), we have
\[
\sum_{i=-K+1}^{\lfloor NB/K \rfloor} G_i \geq K \sum_{i=0}^{\lfloor NB/K \rfloor} A_i \geq K(NB + H), \tag{2.9}
\]
and consequently
\[
\sum_{i=\lfloor NB/K \rfloor + 1}^{\lfloor N/K \rfloor - K} G_i \leq K(N(1 - B) - H). \tag{2.10}
\]
We get
\[
\min_{x_1, \ldots, x_N} Z_K \geq \frac{1}{2K^2N} \min_{G_0, G_1, \ldots, G_{\lfloor N/K \rfloor}} \sum_{i=0}^{\lfloor N/K \rfloor} G_i^2, \tag{2.11}
\]
where the minimum on the right-hand side is taken over all positive reals \( G_0, G_1, \ldots, G_{\lfloor N/K \rfloor} \) satisfying (2.8)–(2.10). It is an easy exercise to verify that this minimum is attained when
\[
G_i = \frac{K(NB + H)}{K + \lfloor NB/K \rfloor} \quad \text{for} \quad i = -K + 1, \ldots, \lfloor NB/K \rfloor,
\]
and
\[
G_i = \frac{K(N(1 - B) - H)}{\lfloor N/K \rfloor - K - \lfloor NB/K \rfloor} \quad \text{for} \quad i = \lfloor NB/K \rfloor + 1, \ldots, \lfloor N/K \rfloor - K.
\]
Note that since \( K \leq N^{2/5} \) and \( H \geq 5N^{4/5} \), we have \( K^2 \leq H/5 \), and hence by (2.3) both the numerator and the denominator of these \( G_i \) are positive. Thus, we get
\[
\frac{1}{2K^2N} \min_{G_0, G_1, \ldots, G_{\lfloor N/K \rfloor}} \sum_{i=0}^{\lfloor N/K \rfloor} G_i^2 \geq \frac{1}{2K^2N} \left( \frac{K^2(NB + H)^2}{K + \lfloor NB/K \rfloor} + \frac{K^2(N(1 - B) - H)^2}{\lfloor N/K \rfloor - K - \lfloor NB/K \rfloor} \right) \tag{2.12}
\]
for all \( N > N_0 \). For the final inequality in (2.12), we have again used that \( H \geq 5N^{5/4} \) and \( K^2 \leq H/5 \).

Finally, by combining (2.12), (2.11), and (2.7), we find the lower bound
\[
\Gamma_K \geq \frac{K}{K + 1} \left( 1 + \frac{H^2}{2N^2} \right).
\]
From the definition (2.6) of $\Gamma_K$ and (2.5), it follows that
\[
\max_{L=1,\ldots,K} \frac{1}{2LN} \mathcal{H}_L > \Gamma_K - \frac{1}{2K} \geq 1 + \frac{H^2}{4N^2} - \frac{2}{K},
\]
and recalling (2.4), we get
\[
\frac{1}{N} F(K^2, N) + 1 \geq \max_{L=1,\ldots,K} \frac{1}{2LN} \mathcal{H}_L > 1 + \frac{H^2}{4N^2} - \frac{2}{K}.
\]
This implies that
\[
H^2 < \frac{8N^2}{K} + 4NF(K^2, N)
\]
\[
\leq 12 \max \left( \frac{N^2}{K}, NF(K^2, N) \right)
\]
\[
< 25 \max \left( N^{8/5}, NF(K^2, N) \right) = H^2,
\]
which is a contradiction. Thus, our assumption (2.1) must be incorrect, and the proof of Theorem 1.1 is complete. (Note that the last inequality above is trivially true if $N^2/K \leq NF(K^2, N)$; in the opposite case we have $K < N/F(K^2, N)$, and by the condition (1.3) imposed on $K$, we then get $K \geq N^{2/5}/2$, and consequently $N^2/K \leq 2N^{8/5}$.)

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