SEMIGROUPOID, GROUPOID AND GROUP ACTIONS ON LIMITS FOR THE GROMOV-HAUSDORFF PROPINQUITY

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Abstract. The Gromov-Hausdorff propinquity provides an analytical framework motivated by mathematical physics where quasi-Leibniz quantum compact metric spaces may be studied by means of metric approximations. A natural question in this setting, answered in this paper, is whether group actions pass to the limit for this new geometry: if a sequence of quasi-Leibniz quantum compact metric spaces is Cauchy for the propinquity and each entry carries a non-trivial compact group action, and if the resulting sequence of groups converges, does the Gromov-Hausdorff limit carry a non-trivial action of the limit group? What about actions from groupoids, or inductive limits of groups? We establish a general result addressing all these matters. Our result provides a first example of a structure which passes to the limit of quantum metric spaces for the propinquity, as well as a new method to construct group actions, including from non-locally compact groups seen as inductive limits of compact groups, on unital C*-algebras. We apply our techniques to obtain some properties of closure of certain classes of quasi-Leibniz quantum compact metric spaces for the propinquity.

The noncommutative Gromov-Hausdorff propinquity [15, 12, 18, 14, 17] is a fundamental component of an analytic framework for the study of the geometry of the hyperspace of quantum metric spaces, with sights on applications to mathematical physics.

A natural yet delicate question is to determine which properties do pass to the limit of a sequence of quantum metric spaces. Partial answers to this question can for instance help us determine the closure of a class of quantum metric spaces. Most importantly, the propinquity is intended to be used to construct models for physics from finite dimensional [10, 1, 31] or other form of approximations [11, 20], with the models obtained as limits, possibly using the fact that the propinquity is complete. Thus, we would like to know which properties of the approximate finite models would indeed carry to the eventual limit.

The existence of group actions on limits of sequences of quantum compact metric spaces is thus of prime interest. The question can be informally asked: if a sequence of metrized compact groups act on a Cauchy sequence of quasi-Leibniz quantum compact metric spaces, and if these groups converge in some appropriate sense, then does the limit carry an action of the limit group? Asked in such a manner, the answer is trivially yes — one may involve trivial actions — but it is easy and

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rather natural to impose a form of nontriviality using the underlying quantum metrics. Indeed, we can control the dilation of the automorphisms \[13\] arising from the actions at each level to ensure that any sort of limit would also be non-trivial. We also note that much efforts have been dedicated to constructing full matrix approximations of certain symmetric spaces \[27, 29, 30, 31\] with the idea that finite dimensional quantum metric spaces are the noncommutative analogues of finite spaces, though they can carry actions of compact Lie groups such as \(SU(n)\), and therefore have highly non-trivial symmetries — thus, offering an option to work with “finite spaces” with continuous symmetries. From this perspective, our present work addresses a sort of dual question: if we do use approximations of a physical model using, say, finite dimensional algebras with some specified symmetries, what are sufficient conditions for these symmetries to still be present at the limit for the propinquity?

We establish a formal answer to this problem, in a more general context: rather than only working with groups, we prove the result for an appropriate notion of action of semigroupoids — thus, our result applies to groups actions, groupoid actions on class of C*-algebras, and semigroups actions, among others. We also note that our result can be applied to prove the existence of non-trivial actions of inductive limits of groups, even if this inductive limit is not locally compact. While left for further research, the results of this paper hints at a possible application of techniques from noncommutative metric geometry to the study of actions of non-locally compact groups on C*-algebras.

As an application of our work, we exhibit closed classes of quasi-Leibniz quantum compact metric spaces, which is in general a very difficult problem to solve. The key observation is that with our method, the limit of ergodic actions is ergodic, in the sense described in this paper — an observation which is quite powerful. For instance, we can deduce from our work and known facts on ergodic actions of \(SU(2)\) that the closure of many classes of quasi-Leibniz quantum compact metric spaces whose quantum metrics are induced by ergodic actions of \(SU(2)\) consists only of type I C*-algebras. In contrast, the closure of classes of fuzzy tori whose quantum metric arises from a fixed length function of the tori, and ergodic actions of closed subgroups of the tori, consists only of fuzzy and quantum tori (and from our own prior work, include all of the quantum tori). More generally, our work opens a new direction for finding unital C*-algebras on which compact groups act ergodically, by investigating the closure for the propinquity of finite dimensional C*-algebras on which a given compact group acts ergodically. The study of this problem from a metric property will be the subject of subsequent papers, and represents an exiting potential development for our project.

We begin this paper with a review of our theory for the hyperspace of quantum metric spaces. We then prove our main result in the context of actions of countable semigroupoids, appropriately defined. Our result does not per say require an action of a semigroupoid, but rather an approximate form of it. We then turn to various applications of our main result, concerning actions of groups, groupoids, and semigroups.

The following notations will be used throughout this paper.

**Notation.** If \((E, d)\) is a metric space, the diameter \(\sup \{d(x, y) : x, y \in E\}\) of \(E\) is denoted by \(\text{diam}\ (E, d)\).
The norm of a normed vector space $X$ is denoted by $\| \cdot \|_X$ by default. The unit of a unital $C^*$-algebra $A$ is denoted by $1_A$. The state space of a $C^*$-algebra $A$ is denoted by $\mathcal{S}(A)$ while the real subspace of self-adjoint elements of $A$ is denoted by $\mathfrak{sa}(A)$.

Given any two self-adjoint elements $a, b \in \mathfrak{sa}(A)$ of a $C^*$-algebra $A$, the Jordan product $\frac{ab + ba}{2}$, i.e. the real part of $ab$, is denoted by $a \circ b$, while the Lie product $\frac{ab - ba}{2i}$ — the imaginary part of $ab$ — is denoted by $\{a, b\}$. The notation $\circ$ will often be used for composition of various maps, but will always mean the Jordan product when applied to elements of a $C^*$-algebra, as is customary. The triple $(\mathfrak{sa}(A), \circ, \{\cdot, \cdot\})$ is a Jordan-Lie algebra [9].

Last, for any continuous linear map $\varphi : E \to F$ between two normed vector spaces $E$ and $F$, the norm of $\varphi$ is denoted by $\|\varphi\|_F^E$ — or simply $\|\varphi\|_F$ if $E = F$.

1. Background

Compact quantum metric spaces were introduced by Rieffel [24, 25], inspired by the work of Connes [4], as a noncommutative generalization of algebras of Lipschitz functions over compact metric space [34]. We added to this original approach the requirement that a form of the Leibniz inequality holds for the generalized Lipschitz seminorm in a quantum compact metric space; the upper bound of this inequality is given using a particular type of functions:

Definition 1.1. A function $F : [0, \infty] \to [0, \infty]$ is admisible when:

1. $F$ is weakly increasing if $[0, \infty]^2$ is equipped with the product order,
2. for all $x, y, l, x, y \geq 0$ we have $xl + yl \leq F(x, y, l, l)$.

The second condition in the above definition of an admissible function ensures that the usual Leibniz inequality is the “best” upper bound one might expect for our results to hold in general.

A quasi-Leibniz quantum compact metric space is thus defined as follows:

Definition 1.2 ([4, 24, 25, 16]). A $F$-quasi-Leibniz quantum compact metric space $(\mathfrak{A}, L)$, where $F$ is an admisible function, is an ordered pair of a unital $C^*$-algebra $\mathfrak{A}$ and a seminorm $L$ defined on a dense Jordan-Lie subalgebra dom $(L)$ of $\mathfrak{sa}(\mathfrak{A})$ such that:

1. $\{a \in \text{dom} (L) : L(a) = 0\} = \mathbb{R}1_\mathfrak{A}$,
2. the Monge-Kantorovich metric $\mathfrak{mk}_L$ defined between any two states $\varphi, \psi \in \mathcal{S}(\mathfrak{A})$ of $\mathfrak{A}$ by:
   $\mathfrak{mk}_L(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)| : a \in \text{dom} (L), L(a) \leq 1 \}$
   metrizes the weak* topology of $\mathcal{S}(\mathfrak{A})$,
3. for all $a, b \in \text{dom} (L)$ we have:
   $\max \{ L(a \circ b), L(\{a, b\})\} \leq F (\|a\|_\mathfrak{A}, \|b\|_\mathfrak{A}, L(a), L(b))$,
4. $L$ is lower semi-continuous with respect to $\| \cdot \|_\mathfrak{A}$.

When $(\mathfrak{A}, L)$ is a quasi-Leibniz quantum compact metric space, the seminorm $L$ is called an $L$-seminorm.

Notation 1.3. If $(\mathfrak{A}, L)$ is a quasi-Leibniz quantum compact metric space, then $(\mathcal{S}(\mathfrak{A}), \mathfrak{mk}_L)$ is a compact metric space, so it has finite diameter, which we denote by $\text{diam} (\mathfrak{A}, L)$. 

Example 1.4 (Classical Metric Spaces). Let \((X, d)\) be a compact metric space, and for all \(f : X \to \mathbb{R}\), define:

\[
L(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}.
\]

The pair \((C(X), L)\) of the unital \(C^*\)-algebra of \(C\)-valued continuous functions on \(X\) and the seminorm \(L\) is a Leibniz quantum compact metric space.

Example 1.5 (Noncommutative Examples). Many examples of noncommutative quasi-Leibniz quantum compact metric spaces are known to exist, including quantum tori \([24, 26]\), curved quantum tori \([11]\), AF algebras \([1]\), noncommutative solenoids \([13]\), hyperbolic groups \(C^*\)-algebras \([23]\), nilpotent group \(C^*\)-algebras \([3]\), among others.

Quasi-Leibniz quantum compact metric spaces are objects for several natural categories. For this paper, we will use three different notions of morphisms. We begin with the weakest of these notions, which we will use to state our main theorem at a hopefully reasonable level of generality.

Definition 1.6. Let \((\mathfrak{A}, L_{\mathfrak{A}})\) and \((\mathfrak{B}, L_{\mathfrak{B}})\) be two quasi-Leibniz quantum compact metric spaces. A continuous linear map \(\varphi : \mathfrak{A} \to \mathfrak{B}\) is Lipschitz when:

\[
\forall a \in \text{dom}(L_{\mathfrak{A}}) \quad \varphi(a) \in \text{dom}(L_{\mathfrak{B}})
\]

and \(\varphi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}\).

Remark 1.7. A linear Lipschitz map, in particular, maps self-adjoint elements to self-adjoint elements, since it is continuous and maps the domain of an \(L\)-seminorm to self-adjoint elements — while domains of \(L\)-seminorms are dense in the self-adjoint part of \(C^*\)-algebras.

The terminology of Definition (1.6) is explained by the following corollary of our work in [20]:

Proposition 1.8. If \((\mathfrak{A}, L_{\mathfrak{A}})\) and \((\mathfrak{B}, L_{\mathfrak{B}})\) are quasi-Leibniz quantum compact metric spaces, and if \(\varphi : \mathfrak{A} \to \mathfrak{B}\) is Lipschitz when:

\[
\forall a \in \text{dom}(L_{\mathfrak{A}}) \quad \varphi(a) \in \text{dom}(L_{\mathfrak{B}})
\]

and \(\varphi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}\), then the following are equivalent:

1. \(\varphi\) is Lipschitz,
2. there exists \(k > 0\) such that \(L_{\mathfrak{B}} \circ \varphi \leq k L_{\mathfrak{A}}\).

Proof. This result follows by [20, Theorem 2.1].

It is straightforward to check that the identity of any quasi-Leibniz quantum compact metric spaces is a Lipschitz linear map, and that the composition of Lipschitz linear maps is itself Lipschitz, so we have defined a category, as announced.

Among all the Lipschitz linear functions, we find the Lipschitz morphisms [20]:

Definition 1.9. Let \((\mathfrak{A}, L_{\mathfrak{A}})\) and \((\mathfrak{B}, L_{\mathfrak{B}})\) be quasi-Leibniz quantum compact metric spaces. A Lipschitz morphism \(\varphi : (\mathfrak{A}, L_{\mathfrak{A}}) \to (\mathfrak{B}, L_{\mathfrak{B}})\) is a unital \(*\)-morphism which is also Lipschitz linear.

Using Lipschitz morphisms as arrows does form a category over the class of quasi-Leibniz quantum compact metric spaces, which is in fact the natural, default category for our theory. To be more explicit, one may argue that going from just linear maps to actual \(*\)-morphisms is the key reason to introduce the propinquity.
versus other versions of the noncommutative Gromov-Hausdorff distance [32, 8, 28]: the propinquity was introduced specifically to allow us to work within the category of C*-algebras with their *-morphisms. Moreover, the dual maps induced by unital *-morphisms restricts to functions between state spaces, and these functions are Lipschitz for the Monge-Kantorovich metric if and only if the *-morphism is a Lipschitz morphism, thus completing the geometric picture.

The only reason for our digression via the Lipschitz linear maps is to be able to phrase our main theorem at a slightly higher level of generality, as certain examples, such as semigroups of positive linear maps, have appeared in the literature and present potential examples for applications of our present work.

There is a natural distance on the space of unital continuous linear maps between quasi-Leibniz quantum compact metric spaces, which generalizes the Monge-Kantorovich length of *-automorphisms discussed in [20]. We will use this distance for two purposes. It will quantify the idea that the composition of two Lipschitz linear maps almost equal to some other Lipschitz linear map — a tool used in the hypothesis of our main theorem — and it will also give us a non-triviality test, in the conclusion of our main theorem.

Definition 1.10. Let \((\mathfrak{A}, L_\mathfrak{A})\) be a quasi-Leibniz quantum compact metric spaces and \(\mathfrak{B}\) be a unital C*-algebra. For any two linear maps \(\varphi, \psi\) from \(\mathfrak{A}\) to \(\mathfrak{B}\) which map \(1_\mathfrak{A}\) to \(1_\mathfrak{B}\), we define:

\[
\text{mkD}^{\mathfrak{A}}_{\mathfrak{B}}(\varphi, \psi) = \sup \{ \| \varphi(a) - \psi(a) \|_\mathfrak{B} : a \in \text{dom}(L_\mathfrak{A}), L_\mathfrak{A}(a) \leq 1 \}.
\]

In particular, if \((\mathfrak{A}, L_\mathfrak{A}) = (\mathfrak{B}, L_\mathfrak{B})\) then we write \(\text{mk} \ell^{\mathfrak{A}}_{\mathfrak{B}}(\varphi)\) for \(\text{mkD}^{\mathfrak{A}}_{\mathfrak{B}}(\text{id}, \varphi)\) where id is the identity automorphism of \(\mathfrak{A}\).

In [20, Proposition 5.2], we proved that for any quasi-Leibniz quantum compact metric space \((\mathfrak{A}, L)\), the function \(\text{mk} \ell_{\mathfrak{A}}\) is a length function on the group of *-automorphisms of \(\mathfrak{A}\) which metrizes the pointwise convergence topology. This result and its proof extends readily to the setting of Definition (1.10), so we sketch the proof rather briefly:

Proposition 1.11. Let \((\mathfrak{A}, L_\mathfrak{A})\) be a quasi-Leibniz quantum compact metric spaces and \(\mathfrak{B}\) be a unital C*-algebra. The function \(\text{mkD}^{\mathfrak{A}}_{\mathfrak{B}}(\cdot, \cdot)\) defines a distance on the set of all continuous linear functions from \(\mathfrak{A}\) to \(\mathfrak{B}\) which map \(1_\mathfrak{A}\) to \(1_\mathfrak{B}\).

Proof. Let \(\mu \in \mathcal{S}(\mathfrak{A})\). By definition of the Monge-Kantorovich metric, for all \(a \in \text{dom}(L_\mathfrak{A})\):

\[
\| a - \mu(a) 1_\mathfrak{A} \|_\mathfrak{A} \leq \text{diam} (\mathfrak{A}, L_\mathfrak{A}) L_\mathfrak{A}(a).
\]

Thus for any two continuous linear maps \(\varphi, \psi: \mathfrak{A} \to \mathfrak{B}\) mapping \(1_\mathfrak{A}\) to \(1_\mathfrak{B}\), and for all \(a \in \text{dom}(L_\mathfrak{A})\) with \(L_\mathfrak{A}(a) \leq 1\):

\[
\| \varphi(a) - \psi(a) \|_\mathfrak{B} = \| \varphi(a - \mu(a) 1_\mathfrak{A}) - \psi(a - \mu(a) 1_\mathfrak{A}) \|_\mathfrak{B} \leq \text{diam} (\mathfrak{A}, L_\mathfrak{A}) \left( \| \varphi \|_\mathfrak{B} + \| \psi \|_\mathfrak{B} \right).
\]

Thus \(\text{mkD}^{\mathfrak{A}}_{\mathfrak{B}}(\cdot, \cdot)\) is bounded. The triangle inequality and the symmetry properties of \(\text{mkD}^{\mathfrak{A}}_{\mathfrak{B}}(\cdot, \cdot)\) are obvious.

If \(\text{mkD}^{\mathfrak{A}}_{\mathfrak{B}}(\varphi, \psi) = 0\) then \(\varphi\) and \(\psi\) agree on the total set \(\{ a \in \mathfrak{A}(\mathfrak{A}) : L_\mathfrak{A}(a) \leq 1 \}\), and thus by linearity and continuity, they agree on \(\mathfrak{A}\). Thus \(\text{mkD}^{\mathfrak{A}}_{\mathfrak{B}}(\cdot, \cdot)\) is a distance as claimed.
The proof that \( \text{mkD}^L \) induces the topology of pointwise convergence follows the same argument as [20, Claim 5.4, Proposition 5.2], as the reader can easily checked (where pointwise convergence topology means the locally convex topology induced by the family of seminorms defined over the space of linear maps from \( \mathcal{A} \) to \( \mathcal{B} \) by \( \varphi \mapsto \| \varphi(a) \|_\mathcal{B} \) where \( a \) ranges over all of \( \mathcal{A} \)).

\[ \square \]

Remark 1.12. Definition (1.10) and Proposition (1.11) would make sense and hold if the codomain was some normed vector space, as long as we work on the set of continuous linear maps which map the unit to some fixed vector. This level of generality is not however needed here.

Now, among all the Lipschitz morphisms, the quantum isometries form a subcategory, and are essential tools in the construction of the propinquity. Moreover, our chosen notion of isomorphism between quasi-Leibniz quantum compact metric spaces will be given by the full quantum isometries — an important observation to understand the geometry of the propinquity. The notion of a quantum isometry is largely due to Rieffel, and essentially expresses that McShane theorem [21] on extension of real-valued Lipschitz functions holds in our noncommutative context. This justifies, a posteriori, the emphasis on self-adjoint elements in Definition (1.2).

Our modification of Rieffel’s notion consists of the requirement that quantum isometries be \(*\)-morphisms, and in particular, full quantum isometries be \(*\)-isomorphisms, rather than positive unital linear maps.

Definition 1.13. Let \((\mathcal{A}, L_\mathcal{A})\) and \((\mathcal{B}, L_\mathcal{B})\) be two quasi-Leibniz quantum compact metric spaces. A quantum isometry \( \pi : (\mathcal{A}, L_\mathcal{A}) \rightarrow (\mathcal{B}, L_\mathcal{B}) \) is a \(*\)-epimorphism \( \pi : \mathcal{A} \rightarrow \mathcal{B} \) such that for all \( b \in sa(\mathcal{B}) \):

\[ L_\mathcal{B}(b) = \inf \{ L_\mathcal{A}(a) : \pi(a) = b \}. \]

A full quantum isometry \( \pi : (\mathcal{A}, L_\mathcal{A}) \rightarrow (\mathcal{B}, L_\mathcal{B}) \) is a \(*\)-isomorphism \( \pi : \mathcal{A} \rightarrow \mathcal{B} \) such that \( L_\mathcal{B} \circ \pi = L_\mathcal{A} \).

A full quantum isometry is a quantum isometry which is also a bijection and whose inverse map is a quantum isometry. Quantum isometries are in particular 1-Lipschitz morphisms between quasi-Leibniz quantum compact metric spaces [14] and they are morphisms for a category whose objects are quasi-Leibniz quantum compact metric spaces [32].

Our work focuses on the construction of geometries on the hyperspace of all quasi-Leibniz quantum compact metric spaces, or phrased a little more formally, on defining metrics on the class of all quasi-Leibniz quantum compact metric spaces, or some subclass therein. There are many subtleties involved in these constructions; our focus will be on the dual Gromov-Hausdorff propinquity as introduced in [12, 18].

The dual Gromov-Hausdorff propinquity is a complete distance, up to full quantum isometry, on the class of \( F \)-quasi-Leibniz quantum compact metric spaces, which induces the topology of the Gromov-Hausdorff distance [5, 6] on the subclass of all classical metric spaces. Its construction is summarized as follows.

We first generalize the idea of an isometric embedding of two quasi-Leibniz quantum compact metric spaces into a third one.
Definition 1.14 ([12]). Let \((\mathfrak{A}, L_{\mathfrak{A}})\) and \((\mathfrak{B}, L_{\mathfrak{B}})\) be two \(F\)-quasi-Leibniz quantum compact metric spaces. An \(F\)-tunnel \(\tau = (\mathcal{D}, L_{\mathcal{D}}, \pi_\mathcal{D}, \pi_\mathfrak{B})\) is given by a \(F\)-quasi-Leibniz quantum compact metric space \((\mathcal{D}, L_{\mathcal{D}})\) as well as two quantum isometries \(\pi_\mathfrak{A} : (\mathcal{D}, L_{\mathcal{D}}) \to (\mathfrak{A}, L_{\mathfrak{A}})\) and \(\pi_{\mathfrak{B}} : (\mathcal{D}, L_{\mathcal{D}}) \to (\mathfrak{B}, L_{\mathfrak{B}})\).

The domain of \(\tau\) is \((\mathfrak{A}, L_{\mathfrak{A}})\) and the codomain of \(\tau\) is \((\mathfrak{B}, L_{\mathfrak{B}})\).

We then associate a numerical value to every tunnel.

Notation 1.15. If \(\pi : \mathfrak{A} \to \mathfrak{B}\) is a positive unital linear map from a unital C*-algebra \(\mathfrak{A}\) to a unital C*-algebra \(\mathfrak{B}\), then the map \(\varphi \in \mathcal{S}(\mathfrak{B}) \to \varphi \circ \pi \in \mathcal{S}(\mathfrak{A})\) is denoted by \(\pi^*\).

Notation 1.16. The Hausdorff distance ([7, p. 293]) induced on the hyperspace of compact subsets of a metric space \((X, d)\) is denoted by \(\text{Haus}_d\); if \(X\) is in fact a vector space and \(d\) is induced by some norm \(\|\cdot\|_X\), then \(\text{Haus}_d\) is denoted by \(\text{Haus}_{\|\cdot\|_X}\).

Definition 1.17 ([18]). The extent \(\chi(\tau)\) of a tunnel \(\tau = (\mathcal{D}, L_{\mathcal{D}}, \pi_1, \pi_2)\) is:

\[
\max \left\{ \text{Haus}_{\text{smq}, \varphi} (\mathcal{S}(\mathcal{D}), \pi_j^* (\mathcal{S}(\mathfrak{A}_j))) : j = 1, 2 \right\}.
\]

Now, the dual propinquity can in fact be “specialized”, by which we mean that we may consider only certain tunnels with additional properties. For instance, one might want to work with tunnels involving only strongly Leibniz \(L\)-seminorms [29]. The following definition summarizes the properties needed on a class of tunnels to allow for the definition of a version of the dual propinquity. The reader may skip this matter, except for the definition of the inverse of a tunnel in item (2), and simply assume that we will work with all \(F\)-tunnels for a given choice of an admissible function \(F\) in this paper; however, it is worth to point out that our main result applies equally well to any specialization of the propinquity.

Definition 1.18 ([18]). Let \(F\) be an admissible function. A class \(T\) of tunnels is appropriate for a nonempty class \(\mathcal{C}\) of \(F\)-quasi-Leibniz quantum compact metric spaces when the following properties hold.

1. If \(\tau \in T\) then \(\text{dom}(\tau), \text{codom}(\tau) \in \mathcal{C}\).
2. If \((\mathfrak{A}, L_{\mathfrak{A}})\) and \((\mathfrak{B}, L_{\mathfrak{B}})\) are in \(\mathcal{C}\), then there exists \(\tau \in T\) such that \(\text{dom}(\tau) = (\mathfrak{A}, L_{\mathfrak{A}})\) and \(\text{codom}(\tau) = (\mathfrak{B}, L_{\mathfrak{B}})\).
3. If \(\tau = (\mathcal{D}, L_{\mathcal{D}}, \pi, \rho) \in T\) then \(\tau^{-1} = (\mathcal{D}, L_{\mathcal{D}}, \rho, \pi) \in T\).
4. If there exists a full quantum isometry \(h : (\mathfrak{A}, L_{\mathfrak{A}}) \to (\mathfrak{B}, L_{\mathfrak{B}})\) for any \((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})\) in \(\mathcal{C}\), then the tunnels \((\mathfrak{A}, L_{\mathfrak{A}}, \text{id}_{\mathfrak{A}}, h)\) and \((\mathfrak{B}, L_{\mathfrak{B}}, h^{-1}, \text{id}_{\mathfrak{B}})\) are in \(T\), where \(\text{id}_E\) is the identity automorphism of any C*-algebra \(E\).
5. If \(\tau, \gamma \in T\) with \(\text{dom}(\tau) = \text{dom}(\gamma)\) then, for all \(\varepsilon > 0\), there exists \(\tau \circ \varepsilon \gamma \in T\) such that \(\text{dom}(\tau) = \text{dom}(\tau \circ \varepsilon \gamma)\), \(\text{codom}(\gamma) = \text{codom}(\tau \circ \varepsilon \gamma)\) and:

\[\chi(\tau \circ \varepsilon \gamma) < \chi(\tau) + \chi(\gamma) + \varepsilon.\]

The most important example of an appropriate class of tunnels is given by:

Proposition 1.19 ([18]). If \(F\) is an admissible function, then the class of all \(F\)-tunnels is appropriate for the class of all \(F\)-quasi-Leibniz quantum compact metric spaces.

We now have all the ingredients to define our propinquity.
Notation 1.20. Let $\mathcal{T}$ be a class of tunnels appropriate for a nonempty class $\mathcal{C}$ of $F$–quasi-Leibniz quantum compact metric spaces for some admissible function $F$. The class of all tunnels in $\mathcal{T}$ from $\mathcal{C}$ to $\mathcal{C}$ is denoted by:

$$\text{Tunnels} \left[ (\mathcal{A}, L_{a\mathcal{A}}) \xrightarrow{T} (\mathcal{B}, L_{a\mathcal{B}}) \right].$$

**Definition 1.21.** Let $\mathcal{T}$ be a class of tunnels appropriate for a nonempty class $\mathcal{C}$ of $F$–quasi-Leibniz quantum compact metric spaces for some admissible function $F$. Let $\mathcal{C} = (\mathcal{A}, L_{a\mathcal{A}})$ and $\mathcal{C} = (\mathcal{B}, L_{a\mathcal{B}})$ in $\mathcal{C}$.

The dual $\mathcal{T}$-propinquity $\Lambda^*_\mathcal{T}((\mathcal{A}, L_{a\mathcal{A}}), (\mathcal{B}, L_{a\mathcal{B}}))$ between $(\mathcal{A}, L_{a\mathcal{A}})$ and $(\mathcal{B}, L_{a\mathcal{B}})$ is the real number:

$$\inf \left\{ \chi(\tau) : \tau \in \text{Tunnels} \left[ (\mathcal{A}, L_{a\mathcal{A}}) \xrightarrow{T} (\mathcal{B}, L_{a\mathcal{B}}) \right] \right\}.$$

**Notation 1.22.** Let $F$ be some admissible function. If $\mathcal{C}$ is the class of all $F$–quasi-Leibniz quantum compact metric spaces and $\mathcal{T}$ is the class of all $F$ tunnels. The dual $\mathcal{T}$-propinquity is simply denoted by $\Lambda^*_\mathcal{T}$.

If, moreover, $F(x, y, z, w) = xw + yz$ for all $x, y, z, w \geq 0$, i.e. if we work within the class of Leibniz quantum compact metric spaces, then we denote $\Lambda^*_\mathcal{T}$ simply as $\Lambda^*$.

The main properties of our metric are summarized in the following:

**Theorem 1.23** ([10, 18]). Let $F$ be an admissible function. If $\mathcal{T}$ is a class of tunnels admissible for a nonempty class $\mathcal{C}$ of $F$–quasi-Leibniz quantum compact metric spaces for some admissible function $F$, then the dual $\mathcal{T}$-propinquity $\Lambda^*_\mathcal{T}$ is a metric up to full quantum isometry on $\mathcal{C}$.

Moreover the dual propinquity $\Lambda^*_\mathcal{T}$ is complete.

Last, when restricted to the class of classical metric spaces, through the identification given in Example (1.4), the propinquity $\Lambda^*_\mathcal{T}$ induces the same topology as the Gromov-Hausdorff distance [5, 6].

Tunnels behave, in some ways, like morphisms, albeit their composition is only defined up to an arbitrary choice of a positive constant. We will use the morphism-like properties of tunnels in a fundamental way in this paper and so we now recall them. Lipschitz elements of a quasi-Leibniz quantum compact metric space have images by tunnels, although these images are themselves subsets of the codomain.

**Definition 1.24.** Let $F$ be an admissible function, and let $(\mathcal{A}, L_{a\mathcal{A}})$ and $(\mathcal{B}, L_{a\mathcal{B}})$ be two $F$–quasi-Leibniz quantum compact metric spaces. Let $\tau = (\mathcal{D}, L_{\mathcal{D}}, \pi_{\mathcal{D}}, \pi_{\mathcal{B}})$ be an $F$-tunnel from $(\mathcal{A}, L_{a\mathcal{A}})$ to $(\mathcal{B}, L_{a\mathcal{B}})$. The $l$-target set of $a \in \text{dom}(L_{\mathcal{A}})$ for $l \geq L_{\mathcal{A}}(a)$ is:

$$t_{\tau}(a|l) = \left\{ \pi_{\mathcal{B}}(d) \in \text{sa}(\mathcal{B}) \mid \begin{array}{l}
  d \in \text{sa}(\mathcal{D}) \\
  L_{\mathcal{D}}(d) \leq l \\
  \pi_{\mathcal{A}}(d) = a
\end{array} \right\}.$$

Target sets are a form of set-valued morphisms:

**Proposition 1.25** ([12, 16]). Let $F$ be an admissible function, and let $(\mathcal{A}, L_{a\mathcal{A}})$ and $(\mathcal{B}, L_{a\mathcal{B}})$ be $F$–quasi-Leibniz quantum compact metric spaces. Let $\tau$ be an $F$-tunnel from $(\mathcal{A}, L_{a\mathcal{A}})$ to $(\mathcal{B}, L_{a\mathcal{B}})$. For all $a, a' \in \text{dom}(L_{a\mathcal{A}})$ and $l \geq \max \{ L_{\mathcal{A}}(a), L_{\mathcal{A}}(a') \}$, if $b \in t_{\tau}(a|l)$ and $b' \in t_{\tau}(a'|l)$ then:

1. for all $t \in \mathbb{R}$:

$$b + t b' \in t_{\tau}(a + t a'|(1 + |t|)l),$$
(2) if $K(a,a',l,\tau) = F([|a|_A + 2\chi(\tau), |a'|_A + 2\chi(\tau), l, l])$ then:

$$
\{a,a'\} \cap \kappa \subseteq t_\tau (a \circ a'[K(a,a',l,\tau)]) \text{ and } \{b,b'\} \subseteq t_\tau (\{a,a'\}[K(a,a',l,\tau)]).
$$

The observation on which many proofs, including proofs in this paper, rely, is that target sets size is controlled by the extend of the tunnel.

**Proposition 1.26 ([12]).** Let $F$ be an admissible function and let $(\mathfrak{A}, L_\mathfrak{A})$ and $(\mathfrak{B}, L_\mathfrak{B})$ be $F$-quasi-Leibniz quantum compact metric spaces. Let $\tau$ be an $F$-tunnel from $(\mathfrak{A}, L_\mathfrak{A})$ to $(\mathfrak{B}, L_\mathfrak{B})$. For all $a \in \dom (L_\mathfrak{A})$ and $l \geq L_\mathfrak{A}(a)$, the target set $t_\tau (a|l)$ is a nonempty compact subset of $sa(\mathfrak{B})$, and moreover if $d \in sa(\mathfrak{D})$ is chosen so that $\pi_\mathfrak{D}(d) = a$ and $L_\mathfrak{D}(d) \leq l$ then:

$$
||d||_\mathfrak{D} \leq ||a||_\mathfrak{A} + l\chi(\tau),
$$

so in particular, if $b \in t_\tau (a|l)$ then:

$$
||b||_\mathfrak{B} \leq ||a||_\mathfrak{A} + 2l\chi(\tau).
$$

As a corollary, if $a,a' \in \dom (L_\mathfrak{A})$ and $l \geq \max \{L_\mathfrak{A}(a),L_\mathfrak{A}(a')\}$ then:

$$
\forall b \in t_\tau (a|l) \forall b' \in t_\tau (a'|l) \quad ||b - b'||_\mathfrak{B} \leq ||a - a'||_\mathfrak{A} + 2l\chi(\tau)
$$

and therefore:

$$
\text{diam}(t_\tau (a|l), ||\cdot||_\mathfrak{B}) \leq 4l\chi(\tau).
$$

We conclude this section with a new lemma. A tunnel enables us to pick corresponding $\varepsilon$-dense finite subsets of its domain and codomain. This lemma will prove helpful in checking that the morphisms which our main theorems builds will be non-trivial, under appropriate conditions.

**Lemma 1.27.** Let $F$ be an admissible function. Let $(\mathfrak{A}, L_\mathfrak{A})$ and $(\mathfrak{B}, L_\mathfrak{B})$ be two $F$-quasi-Leibniz quantum compact metric spaces. Let $\varepsilon > 0$ and let $\tau$ be an $F$-tunnel from $(\mathfrak{A}, L_\mathfrak{A})$ to $(\mathfrak{B}, L_\mathfrak{B})$ such that $\chi(\tau) < \varepsilon$. Let $E$ be a finite subset of $\{a \in sa(\mathfrak{A}) : L_\mathfrak{A}(a) \leq 1\}$ such that for all $a \in sa(\mathfrak{A})$ with $L_\mathfrak{A}(a) \leq 1$, there exists $a' \in E$ and $t \in \mathbb{R}$ such that:

$$
||a - (a' + t1_\mathfrak{A})||_\mathfrak{A} < \varepsilon.
$$

There exists a finite subset $G \subseteq \{a \in sa(\mathfrak{B}) : L_\mathfrak{B}(a) \leq 1\}$ such that:

(1) for all $a \in sa(\mathfrak{B})$ with $L_\mathfrak{B}(a) \leq 1$, there exists $t \in \mathbb{R}$ and $a' \in G$ such that $||a - (a' + t1_\mathfrak{A})||_\mathfrak{B} \leq 3 \varepsilon$,

(2) for all $a \in G$ there exists $a' \in E$ such that $a \in t_\tau (a'|1)$, and conversely, for all $a \in E$ there exists $a' \in G$ such that $a \in t_{\tau^{-1}} (a'|1)$.

**Proof.** We write $\tau = (\mathfrak{D}, L_\mathfrak{D}, \pi_\mathfrak{A}, \pi_\mathfrak{B})$.

For all $a \in E$, choose $f(a) \in t_\tau (a|1)$. Note that $L_\mathfrak{B}(f(a)) \leq 1$ by construction. Moreover, note that $a \in t_{\tau^{-1}} (f(a)|1)$.

Let $G = \{f(a) : a \in E\}$. Now, let $b \in sa(\mathfrak{B})$ with $L_\mathfrak{B}(b) \leq 1$. Let $a \in t_{\tau^{-1}} (b|1)$. Since $L_\mathfrak{A}(a) \leq 1$, there exists $a' \in E$ and $t \in \mathbb{R}$ such that $||a - (a' + t1_\mathfrak{A})||_\mathfrak{B} \leq \varepsilon$.

By Definition (1.24), there exists $d \in \dom (L_\mathfrak{D})$ with $\pi_\mathfrak{A}(d) = a'$, $\pi_\mathfrak{B}(d) = f(a')$, and $L_\mathfrak{D}(d) \leq 1$. Now, $\pi_\mathfrak{A}(d + t1_\mathfrak{D}) = a' + t1_\mathfrak{A}$, while $\pi_\mathfrak{B}(d + t1_\mathfrak{D}) = f(a') + t1_\mathfrak{B}$, and moreover:

$$
L_\mathfrak{D}(d + t1_\mathfrak{D}) = L_\mathfrak{D}(d) \leq 1.
$$

Therefore $f(a') + t1_\mathfrak{B} \in t_\tau (a' + t1_\mathfrak{A}|1)$, again by Definition (1.24). It follows by Proposition (1.26) that:

$$
||b - (f(a') + t1_\mathfrak{B})||_\mathfrak{B} \leq ||a - (a' + t1_\mathfrak{A})||_\mathfrak{A} + 2\chi(\tau) \leq 3\varepsilon.
$$
This concludes our proof. □

We now turn to our main theorem in the next section.

2. Actions of Semigroupoids on Limits

We expand on the idea that tunnels are sort of set-valued morphisms by constructing a structure similar to a commutative diagram at the core of our main theorem. Thus, for the next few results, we will work with the following ingredients.

**Hypothesis 2.1.** Let \( F \) be an admissible function. Let \((A, L_A), (B, L_B), (E, L_E)\), and \((\mathfrak{F}, L_{\mathfrak{F}})\) be \(F\)-quasi-Leibniz quantum compact metric spaces. Let \( \tau \) be an \( F \)-tunnel from \((A, L_A)\) to \((E, L_E)\), and let \( \gamma \) be an \( F \)-tunnel from \((\mathfrak{F}, L_{\mathfrak{F}})\) to \((B, L_B)\). Let \( \varphi : \mathfrak{E} \to \mathfrak{F} \) be a Lipschitz linear map and let \( D > 0 \) such that \( L_{\mathfrak{F}} \circ \varphi \leq DL_E \) and \( \|\varphi\|_{\mathfrak{E}} \leq D \).

As we look at tunnels as a form of morphism and target sets as the mean to get an image for elements by such morphisms, it is natural to define the image of a set by a tunnel by simply setting, for a tunnel \( \tau \) from a quasi-Leibniz quantum compact metric space \((A, L_A)\), and for any nonempty subset \( A \subseteq \text{dom}(L_A) \) and for any real \( l > 0 \):

\[
t_{\tau}(A|l) = \bigcup_{a \in A} t_{\tau}(a|l).
\]

This image of a set is not empty as long as there exists an element \( a \in A \) such that \( L_A(a) \leq l \).

For our purpose, the following specific application of this idea will be central.

**Definition 2.2.** Given Hypothesis (2.1), for \( a \in \text{dom}(L_{\mathfrak{A}}) \) and \( l \geq L_{\mathfrak{A}}(a) \), the \( l \)-image-target set of \( a \) is defined by:

\[
i_{\varphi, \tau}(a|l) = \varphi(t_{\tau}(a|l))
\]

and the \( l \)-forward-target set of \( a \) is defined by:

\[
f_{\tau, \varphi, \gamma}(a|l, D) = t_{\gamma}(i_{\varphi, \tau}(a|l)|DL).
\]

**Remark 2.3.** Using Hypothesis (2.1), we thus have, for \( a \in \text{dom}(L_{\mathfrak{A}}) \) and \( l \geq L_{\mathfrak{A}}(a) \):

\[
f_{\tau, \varphi, \gamma}(a|l, D) = \bigcup_{c \in L_{\varphi, \tau}(a|l)} t_{\gamma}(c|DL) = \bigcup_{b \in t_{\tau}(a|l)} t_{\gamma}(\varphi(b)|DL).
\]

We thus form a sort of commutative diagram, though using set-valued functions:

\[
\begin{array}{ccc}
a \in \text{sa}(\mathfrak{A}) & \xrightarrow{f_{\tau, \varphi, \gamma}(a|l, D)} & \text{sa}(\mathfrak{B}) \\
\text{t}_{\tau}(.|l) & & \text{t}_{\gamma}(.|DL) \\
\text{t}_{\tau}(a|l) & \subseteq \text{sa}(\mathfrak{E}) & \xrightarrow{\varphi} \text{sa}(\mathfrak{F})
\end{array}
\]

By Definition (2.2), if \( l' \geq l \) and \( D' \geq D \), then with the notations of Hypothesis (2.1), we have:

\[
f_{\tau}(a|l, D) \subseteq f_{\tau}(a|l', D').
\]

The forward target sets enjoy properties akin to target sets, which we now establish. We begin with the linearity property.
Lemma 2.4. Assume Hypothesis (2.1). If \( a, a' \in \text{dom}\, (L_{\mathfrak{A}}) \) and \( l \geq \max\{L_{\mathfrak{A}}(a), L_{\mathfrak{A}}(a')\} \), then for all \( t \in \mathbb{R} \), \( f \in f_{\tau,\varphi,\gamma}(a|l, D) \) and \( f' \in f_{\tau,\varphi,\gamma}(a'|l, D) \):

\[
f + tf' \in f_{\tau,\varphi,\gamma}(a + ta'|\langle 1 + |t| \rangle l, D).
\]

Proof. Let \( f \in f_{\tau,\varphi,\gamma}(a|l, D) \) and \( f' \in f_{\tau,\varphi,\gamma}(a'|l, D) \). By Definition (2.2), there exists \( b \in t_{\tau}(a|l) \) and \( b' \in t_{\tau}(a'|l) \) such that, if \( c = \varphi(b) \) and \( c' = \varphi(b') \), then \( f \in t_{\tau}(c|Dl) \) and \( f' \in t_{\tau}(c'|Dl) \).

Now, \( b + tb' \in t_{\tau}(a + ta'|\langle 1 + |t| \rangle l) \) by Proposition (1.25). So:

\[
c + tc' = \varphi(b + tb') \in \varphi(t_{\tau}(a + ta'|\langle 1 + |t| \rangle l)).
\]

In particular, \( L_{\mathfrak{A}}(c + tc') \leq DL_{\mathfrak{A}}(b + tb') \leq DL(1 + |t|) \).

On the other hand, again by Proposition (1.25):

\[
f + tf' \in t_{\tau}(c + tc'|\langle 1 + |t| \rangle Dl).
\]

Thus \( f + tf' \in f_{\tau,\varphi,\gamma}(a + ta'|\langle 1 + |t| \rangle l, D) \). \( \square \)

We now establish the main geometric property of forward-target sets — namely, their diameters in norm is controlled by the extent of the tunnels from which they are defined.

Lemma 2.5. Assume Hypothesis (2.1). Let \( a, a' \in \text{dom}\, (L_{\mathfrak{A}}) \) and \( l \geq \max\{L_{\mathfrak{A}}(a), L_{\mathfrak{A}}(a')\} \). If \( c \in i_{\tau,\varphi}(a|l) \) then:

\[
\|c\|_{\mathfrak{A}} \leq D\|a\|_{\mathfrak{A}} + 2D\chi(\tau)
\]

and therefore for all \( c' \in i_{\tau,\varphi}(a'|l) \):

\[
\|c - c'\|_{\mathfrak{A}} \leq D\|a - a'\|_{\mathfrak{A}} + 4D\chi(\gamma).
\]

If moreover \( f \in f_{\tau,\varphi,\gamma}(a|l, D) \), then:

\[
\|f\|_{\mathfrak{A}} \leq D\|a\|_{\mathfrak{A}} + 2D(\chi(\tau) + \chi(\gamma)).
\]

On therefore for all \( f' \in f_{\tau,\varphi,\gamma}(a'|l, D) \):

\[
\|f - f'\|_{\mathfrak{A}} \leq D\|a - a'\|_{\mathfrak{A}} + 4D(\chi(\tau) + \chi(\gamma)).
\]

so that, in particular:

\[
\text{diam}(f_{\tau,\varphi,\gamma}(a|l, D), \|\cdot\|_{\mathfrak{A}}) \leq 4DD(\chi(\tau) + \chi(\gamma)).
\]

Proof. Let \( c \in i_{\tau,\varphi}(a|l) \). There exists, by Definition (2.2), an element \( b \in t_{\tau}(a|l) \) such that \( c = \varphi(b) \). Now by Proposition (1.26), we have:

\[
(2.1) \quad \|c\|_{\mathfrak{A}} \leq D\|b\|_{\mathfrak{E}} \leq D\|a\|_{\mathfrak{A}} + 2D\chi(\tau),
\]

as desired. If \( c' \in i_{\tau,\varphi}(a'|l) \) then \( c' = \varphi(b') \) with \( b' \in t_{\tau}(a|l) \), and thus \( c - c' = \varphi(b - b') \) with \( b - b' \in t_{\tau}(a - a'|2l) \) by Proposition (1.25). We conclude:

\[
\|c - c'\|_{\mathfrak{A}} \leq D\|a - a'\|_{\mathfrak{A}} + 4D\chi(\tau)
\]

using Expression (2.1).

Let now \( f \in f_{\tau,\varphi,\gamma}(a|l, D) \). By Definition (2.2), there exists \( c \in i_{\tau,\varphi}(a|l) \) such that \( f \in t_{\tau}(c|Dl) \). By Proposition (1.26) and Expression (2.1), we thus have:

\[
(2.2) \quad \|f\|_{\mathfrak{A}} \leq \|c\|_{\mathfrak{A}} + 2D(\chi(\tau) + \chi(\gamma))
\]

Let \( f' \in f_{\tau,\varphi,\gamma}(a'|l, D) \). By Lemma (2.4), we have:

\[
f - f' \in f_{\tau,\varphi,\gamma}(a - a'|2l, D)
\]
and therefore, by Expression (2.2):

\[ \|f - f'\|_\mathfrak{M} \leq D (\|a - a'\|_\mathfrak{M} + 4l(\chi (\tau) + \chi (\gamma))) \]

as needed.

Setting \( a = a' \) in Expression (2.3), we then get:

\[ \text{diam} (f_{\tau,\varphi,\gamma} (a|l, D), \| \cdot \|_\mathfrak{M}) \leq 4D l(\chi (\tau) + \chi (\gamma)), \]

thus concluding our lemma.

We now complete the algebraic properties of the forward target sets by proving the following lemma.

**Lemma 2.6.** Assume Hypothesis (2.1). If \( a, a' \in \text{dom} (L_\mathfrak{M}) \) and \( l \geq \max \{L_\mathfrak{M}(a), L_\mathfrak{M}(a') \} \), if \( f \in f_{\tau,\varphi,\gamma} (a|l, D) \) and \( f' \in f_{\tau,\varphi,\gamma} (a'|l, D) \), and if moreover \( \varphi \) is a Jordan-Lie morphism, then, setting:

\[
M_{\tau,\varphi,\gamma} (\|a\|_\mathfrak{M}, \|a'\|_\mathfrak{M}, l, D) = \max \left\{ F (\|a\|_\mathfrak{M} + 2l \chi (\tau), \|a'\|_\mathfrak{M} + 2l \chi (\tau), l, l), \right. \\
\left. \frac{1}{D} F (D (\|a\|_\mathfrak{M} + 2l \chi (\tau) + 2l \chi (\gamma)), D (\|a'\|_\mathfrak{M} + 2l \chi (\tau) + 2l \chi (\gamma)), Dl, Dl) \right\}
\]

we conclude:

\[ f \circ f' \in f_{\tau,\varphi,\gamma} (a \circ a'|M_{\tau,\varphi,\gamma} (\|a\|_\mathfrak{M}, \|a'\|_\mathfrak{M}, l, D), D) \]

and

\[ \{ f, f' \} \in f_{\tau,\varphi,\gamma} (\{ a, a' \}|M_{\tau,\varphi,\gamma} (\|a\|_\mathfrak{M}, \|a'\|_\mathfrak{M}, l, D), D). \]

**Proof.** By Definition (2.2), there exists \( b \in t_\tau (a|l) \) and \( b' \in t_\tau (a'|l) \) such that, if \( c = \varphi(b) \) and \( c' = \varphi(b') \), then \( f \in t_\gamma (c|Dl) \) and \( f' \in f_\gamma (a'|Dl) \).

Assume \( \varphi \) is a Lie-Jordan morphism. We then have by Proposition (1.25):

\[ b \circ b' \in t_\tau (a \circ a'|F (\|a\|_\mathfrak{M} + 2l \chi (\tau), \|a'\|_\mathfrak{M} + 2l \chi (\tau), l, l)) \]

and \( c \circ c' = \varphi(b) \circ \varphi(b') = \varphi(b \circ b') \) so:

\[ L_\mathfrak{M} (c \circ c') \leq DF (\|a\|_\mathfrak{M} + 2l \chi (\tau), \|a'\|_\mathfrak{M} + 2l \chi (\tau), l, l) \leq M_{\tau,\varphi,\gamma} (\|a\|_\mathfrak{M}, \|a'\|_\mathfrak{M}, l, D). \]

We also have, again by Proposition (1.25):

\[ f \circ f' \in t_\gamma (c \circ c'|F (\|c\|_\mathfrak{M} + 2Dl \chi (\gamma), \|c'\|_\mathfrak{M} + 2Dl \chi (\gamma), Dl, Dl)) \]

with:

\[ \|c\|_\mathfrak{M} \leq D \|b\|_\mathfrak{M} \leq D (\|a\|_\mathfrak{M} + 2l \chi (\tau)) \]

and

\[ \|c'\|_\mathfrak{M} \leq D \|b'\|_\mathfrak{M} \leq D (\|a'\|_\mathfrak{M} + 2l \chi (\tau)), \]

by Lemma (2.5), so:

\[ F (\|c\|_\mathfrak{M} + 2Dl \chi (\tau), \|c'\|_\mathfrak{M} + 2Dl \chi (\tau), Dl, Dl) \leq F (DF (\|a\|_\mathfrak{M} + 2l \chi (\tau))) + 2Dl \chi (\tau)) + 2Dl \chi (\tau)), Dl, Dl) \]

\[ \leq D M_{\tau,\varphi,\gamma} (\|a\|_\mathfrak{M}, \|a'\|_\mathfrak{M}, l, D). \]

By Definition, we thus have: \( f \circ f' \in f_{\tau,\varphi,\gamma} (a \circ a'|M_{\tau,\varphi,\gamma} (\|a\|_\mathfrak{M}, \|a'\|_\mathfrak{M}, l, D), D) \). The same proof holds for the Lie product.

Last, we identify the topological nature of forward target sets.
Lemma 2.7. Assume Hypothesis (2.1). For all $a \in \text{dom}(L_{a})$ and $l \geq L_{a}(a)$, the sets $i_{r,\varphi}(a|l)$ and $f_{r,\varphi,\gamma}(a|l, D)$ are nonempty and compact, respectively, in $sa(\mathfrak{F})$ and $sa(\mathfrak{B})$.

Proof. By [12, Lemma 4.2], the set $t_{r} (a|l)$ is a nonempty compact subset of $sa(\mathfrak{E})$. Since $\varphi$ is continuous, $i_{r,\varphi}(a|l)$ is not empty and compact in $sa(\mathfrak{F})$ as well.

Let now $b \in t_{r} (a|l)$. By construction, $L_{D} \circ \varphi (b) \subseteq DL_{a} (b) \subseteq DL$. Hence again by [12, Lemma 4.2], the set $t_{r} (b | DL)$ is a nonempty compact set. In particular, $f_{r,\varphi,\gamma}(a|l, D)$ is not empty as well. Moreover, by Lemma (2.5), the following inclusion holds:

$$f_{r,\varphi,\gamma}(a|l, D) \subseteq \{ b \in sa(\mathfrak{B}) : L_{B} (b) \subseteq DL, \| b \|_{a} \leq D (\| a \|_{a} + 2l (\chi (\tau) + \chi (\gamma))) \} ,$$

and the set on the right hand-side is compact since $L$ is an L-seminorm. So $f_{r} (a|l, D)$ is totally bounded in $sa(\mathfrak{B})$.

To prove that $f_{r,\varphi,\gamma}(a|l, D)$ is compact, it is thus sufficient to show that $a$ is is closed, since it is totally bounded and $sa(\mathfrak{B})$ is complete. To this end, we write the tunnel $\gamma$ as $(\mathfrak{D}, L_{D}, \tau, \varphi, \gamma)$.

Let $(f_{k})_{k \in \mathbb{N}}$ be a sequence in $f_{r,\varphi,\gamma}(a|l, D)$, converging in $\mathfrak{B}$ to some $f$. By Definition (2.2), for each $k \in \mathbb{N}$, there exists $c_{k} = \varphi (t_{r} (a|l))$ such that $f_{k} \in t_{r} (c_{k} | DL)$. There exists $b_{k} \in t_{r} (a|l)$ such that $c_{k} = \varphi (b_{k})$. Now, $t_{r} (a|l)$ is compact, so there exists a convergent subsequence $(b_{j(k)})_{k \in \mathbb{N}}$ converging in norm to some $b \in t_{r} (a|l)$. Therefore, $(\varphi (b_{j(k)}))_{k \in \mathbb{N}}$ converges to $\varphi (b)$, which we denote by $c$. Thus $c \in i_{r,\varphi}(a|l)$.

Now, for each $k \in \mathbb{N}$, since $f_{j(k)} \in t_{r} (c_{j(k)} | DL)$, there exists $d_{k} \in sa(\mathfrak{D})$ such that $L_{D} (d_{k}) \subseteq DL$ while $\pi (d_{k}) = c_{j(k)}$ and $\rho (d_{k}) = f_{j(k)}$.

Once more, $(d_{k})_{k \in \mathbb{N}}$ lies in the compact set:

$$\{ w \in sa(\mathfrak{D}) : L_{D} (w) \subseteq DL, \| w \|_{D} \leq D (\| a \|_{a} + 2l (\chi (\tau) + \chi (\gamma))) \}$$

and thus we extract a convergent subsequence $(d_{m(k)})_{k \in \mathbb{N}}$ with limit $d$ with $L_{D} (d) \subseteq DL$.

Moreover by continuity:

$$\lim_{k \to \infty} \rho (d_{m(k)}) = \lim_{k \to \infty} f_{j_{m}(k)} = f \text{ and } \lim_{k \to \infty} \pi (d_{m(k)}) = \lim_{k \to \infty} c_{j_{m}(k)} = c.$$

By Definition, we thus conclude that $f \in t_{r} (c | DL)$ and therefore $f \in f_{r,\varphi,\gamma}(a|l, D)$. Thus $f_{r,\varphi,\gamma}(a|l, D)$ is closed. This concludes our lemma. □

We now provide the setting for our main theorem. We will prove not only that families of morphisms pass to the limit, informally speaking, for the propinquity, but also that composition of morphisms also passes to the limit. To encode composition, we will use the structure of semigroupoid. A semigroupoid is, informally, a category without the requirement that is contains identities: it is a set endowed with a partially defined associative operation. Formally:

Definition 2.8. A semigroupoid $(\mathcal{S}, \mathcal{S}^{(0)}, d, c, \circ)$ is a pair of classes $\mathcal{S}$ and $\mathcal{S}^{(0)}$, two maps $c, d : \mathcal{S} \to \mathcal{S}^{(0)}$ and a map $\circ : \mathcal{S}^{(2)} \to \mathcal{S}$ where:

$$\mathcal{S}^{(2)} = \{ (s, s') \in \mathcal{S}^{2} : d(s) = c(s') \} ,$$

such that:

1. for all $(s, s') \in \mathcal{S}^{(2)}$, we have $d(s \circ s') = d(s')$ and $c(s \circ s') = c(s)$,
2. if $s, s', s'' \in \mathcal{S}$ with $(s, s'), (s', s'') \in \mathcal{S}^{(2)}$, then $(s \circ s') \circ s'' = s \circ (s' \circ s'')$. 

Categories are examples of semigroupoids, and a semigroupoid can be turned into a category by adding an element for each object which behaves as an identity:

**Definition 2.9.** Let \((S, S^{(0)}, d, c, o)\) be a semigroupoid. An element \(e \in S\) is a unit when \(d(e) = c(e)\) and \(s \circ e = s, e \circ t = t\) for all \(s, t \in S\) with \(d(s) = c(t) = d(e)\).

Given these similarities, a morphism of semigroupoid will be called a functoid if it is not required to preserve units, and a functor if it must.

**Definition 2.10.** A functoid \(F\) from a semigroupoid \((S, O, d, c, o)\) to a semigroupoid \((C, U, s, t, \Box)\) is a pair of maps (which by abuse of notations, we still write \(F\)), one from \(O\) to \(U\), and one from \(S\) to \(C\), such that:

\[
\forall \varphi \in S \quad F(d(\varphi)) = s(F(\varphi)) \quad F(c(\varphi)) = t(F(\varphi))
\]

and

\[
\forall (\varphi, \varphi') \in S^{(2)} \quad F(\varphi \circ \varphi') = F(\varphi) \Box F(\varphi').
\]

If moreover, for any unit \(u \in S\), the element \(F(u)\) is a unit of \(C\), then \(F\) is called a functor.

We note that Definition (2.10) is meaningful, as we easily check, using the notations of this definition, that if \(\varphi, \varphi' \in S^{(2)}\) then \((F(s), F(s')) \in C^{(2)}\).

An important example of semigroupoid is a groupoid, which is a semigroupoid for which all elements are invertible:

**Definition 2.11.** Let \((S, S^{(0)}, d, c, o)\) be a semigroupoid. An element \(u \in S\) is invertible when there exists \(u' \in S\), such that:

1. \(d(u) = c(u')\) and \(d(u') = c(u)\),
2. \(u \circ u'\) is a unit,
3. \(u' \circ u\) is a unit.

The element \(u'\) is called an inverse of \(u\).

We note that a trivial argument shows that units in a semigroupoid with a given domain are unique, and that the inverse of an element of a semigroupoid is unique as well. Functors preserve inverse while functoids do not in general.

We have now set the framework for our main theorem. We will use repeatedly in its proof the following simple result, which we record as a lemma to help clarify our main argument.

**Lemma 2.12.** If \((A_n)_{n \in \mathbb{N}}\) and \((B_n)_{n \in \mathbb{N}}\) are two sequences of nonempty compact subsets of a complete metric space \((X, d)\) such that:

1. \(\lim_{n \to \infty} \text{diam} (B_n, d) = 0\),
2. for all \(n \in \mathbb{N}\), we have \(A_n \subseteq B_n\),

then the sequence \((A_n)_{n \in \mathbb{N}}\) converges for \(\text{Haus}_d\) if and only if \((B_n)_{n \in \mathbb{N}}\) does, in which case they have the same limit, a singleton of \(X\).

**Proof.** We begin by proving that if a sequence \((C_n)_{n \in \mathbb{N}}\) of nonempty closed subsets of \(X\) with \(\lim_{n \to \infty} \text{diam} (C_n, d) = 0\) converges for \(\text{Haus}_d\) then its limit \(C\) is a singleton. Since \(\text{diam}(\cdot, d)\) is a continuous function for \(\text{Haus}_d\), it is immediate that \(\text{diam}(C, d) = 0\). On the other hand, for each \(n \in \mathbb{N}\), let \(c_n \in C_n\). Let \(\varepsilon > 0\). There exists \(N \in \mathbb{N}\) such that for all \(p, q \geq N\), we have \(\text{Haus}_d(C_p, C_q) < \frac{\varepsilon}{3}\) and \(\text{diam}(C_n, d) < \frac{\varepsilon}{3}\), so:

\[
d(c_p, c_q) \leq \text{diam}(C_p, d) + \text{Haus}_d(C_p, C_q) + \text{diam}(C_q, d) < \varepsilon,
\]
and thus \((c_n)_{n \in \mathbb{N}}\) is a Cauchy sequence; since \((X, d)\) is complete, \((c_n)_{n \in \mathbb{N}}\) converges to some \(c \in X\). Moreover, \(d(c, C_n) \leq \varepsilon\) for \(n \geq N\) by continuity. On the other hand, if \(x \in C_n\) then \(d(x, c) < \frac{\varepsilon}{3}\) for \(n \geq N\). Hence \(\text{Haus}_d(C_n, \{c\}) < \frac{\varepsilon}{3}\) for \(n \geq N\) and thus \(C = \{c\}\).

We now prove the rest of our lemma.

Let \(\varepsilon > 0\). There exists \(N \in \mathbb{N}\) such that for all \(n \geq N\):
\[
diam(B_n, d) < \frac{\varepsilon}{2}.
\]

Since \(A_n \subseteq B_n\) for all \(n \in \mathbb{N}\), we conclude that \(\text{Haus}_d(A_n, B_n) < \frac{\varepsilon}{2}\) for all \(n \geq N\). Now, if \((A_n)_{n \in \mathbb{N}}\) converges for \(\text{Haus}_d\), and \(\{a\}\) is its limit, then there exists \(M \in \mathbb{N}\) such that for all \(n \geq M\), we have:
\[
\text{Haus}_d(A_n, \{a\}) < \frac{\varepsilon}{2}.
\]

Therefore, for all \(n \geq \max\{N, M\}\):
\[
\text{Haus}_d(B_n, \{a\}) \leq \text{Haus}_d(B_n, A_n) + \text{Haus}_d(A_n, \{a\}) < \varepsilon,
\]
and thus \((B_n)_{n \in \mathbb{N}}\) converges for \(\text{Haus}_d\) to the same limit \(\{a\}\).

If instead, \((B_n)_{n \in \mathbb{N}}\) converges to \(\{b\}\) for \(\text{Haus}_d\), then similarly, there exists \(M \in \mathbb{N}\) such that \(\text{Haus}_d(B_n, \{b\}) < \frac{\varepsilon}{2}\) and thus for all \(n \geq \max\{N, M\}\), we conclude:
\[
\text{Haus}_d(A_n, \{b\}) < \varepsilon
\]
thus concluding our lemma. 

We now state and prove our main theorem.

**Theorem 2.13.** Let \(F\) be an admissible function. Let \(T\) be a class of tunnels appropriate for a nonempty class \(C\) of \(F\)-quasi-Leibniz quantum compact metric spaces.

Let \((S, S^{(0)}, d, c, o)\) be a semigroupoid, where \(S\) and \(S^{(0)}\) are both countable sets.

For all \(s \in S^{(0)}\), let \((\mathfrak{A}^s, L^s) \in \mathcal{C}\), and let \((\mathfrak{A}_n^s, L_n^s)_{n \in \mathbb{N}}\) be a sequence in \(\mathcal{C}\) such that:
\[
\lim_{n \to \infty} \Lambda_T^+(\mathfrak{A}^s_n, L^s_n, (\mathfrak{A}^s, L^s)) = 0.
\]
Let \(D : S \to [0, \infty)\).

If, for all \(s \in S\) and for all \(n \in \mathbb{N}\), there exists a Lipschitz linear map:
\[
\varphi_n^s : (\mathfrak{A}(d_s), L_n^{d(s)}) \to (\mathfrak{A}^{c(s)}_n, L_n^{c(s)})
\]
with the property:
\[
\|\varphi_n^s\|_{\mathfrak{A}(d_s)} \leq D(s) \text{ and } L_n^{d(s)} \circ \varphi_n^s \leq D(s) L_n^{d(s)},
\]
while for all \((s, t) \in S^{(2)}\):
\[
\lim_{n \to \infty} mK\mathcal{T}_n^{d(s)} (\varphi_n^s \circ \varphi_n^t) = 0,
\]
then there exists:
- a functor \(\Psi\) from \(\mathcal{S}\) to the category whose objects are the elements of \(\mathcal{C}\), and whose morphisms are the Lipschitz linear maps,
- a strictly increasing \(j : \mathbb{N} \to \mathbb{N}\)
- for all \(s \in S^{(0)}\) and \(n \in \mathbb{N}\), a tunnel \(\tau_n^s\) from \((\mathfrak{A}^s, L^s)\) to \((\mathfrak{A}_n^s, L_n^s)\) such that \(\lim_{n \to \infty} \chi (\tau_n^s) = 0\),
such that:
\[ \forall s \in S^{(0)} \quad \Psi(s) = (\mathfrak{A}^s, \mathcal{L}^s) \]

while
\[ \forall s \in S \quad \|\Psi(s)\|_{\mathcal{M}(s)}^{\mathcal{D}(s)} \leq \sqrt{2}D(s) \quad \text{and} \quad L^{c(s)} \circ \Psi(s) \leq D(s)\mathcal{L}^{d(s)}, \]

and, writing \( \gamma_n^s = \tau_n^s \) for all \( n \in \mathbb{N}, \ s \in S^{(0)} \):
\[ \forall s \in S \quad \forall a \in \text{dom} \left( \mathcal{L}^{d(s)} \right) \quad \forall l \geq \mathcal{L}^{d(s)}(a), D \geq D(s) \]
\[ \text{Haus}_{\|\cdot\|_{\mathcal{M}(s)}} \left( \mathbf{f}_{l,s}^{d(s)}(\gamma_n^s, \gamma_n^s)(a(l, D), \{\Psi(s)(a)\}) \right) \xrightarrow{n \rightarrow \infty} 0. \]

Furthermore:

1. If, for some \( s \in S \) and for all \( n \in \mathbb{N} \), the map \( \varphi_n^s \) is positive, and \( \{a \in \text{dom}(\mathcal{L}^{d(s)}): a \geq 0\} \) is dense in \( \{a \in \mathfrak{sa}(\mathfrak{A}^{d(s)}): a \geq 0\} \), then \( \Psi(s) \) is positive as well.
2. If, for some \( s \in S \), we can choose \( (\varphi_n^s)_{n \in \mathbb{N}} \) such that \( \varphi_n^s \) is a Lipschitz morphism for all \( n \in \mathbb{N} \), then \( \Psi(s) \) is a Lipschitz morphism — in particular,
\[ \|\Psi(s)\|_{\mathcal{M}(s)}^{\mathcal{D}(s)} = 1. \]
3. If, for some \( s, s' \in S \), we have \( c(s) = c(s') \) and \( d(s) = d(s') \), then \( \text{mkD}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\Psi(s), \Psi(s')) \) is a limit point of the sequence \( \left( \text{mkD}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^s, \varphi_n^{s'}) \right)_{n \in \mathbb{N}} \), and so in particular:
\[ \lim_{n \rightarrow \infty} \text{mkD}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^s, \varphi_n^{s'}) = \sup_{n \rightarrow \infty} \text{mkD}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^s, \varphi_n^{s'}). \]

Moreover, if for some \( s \in S \), we have \( d(s) = c(s) \), then:
\[ \lim_{n \rightarrow \infty} \text{mkf}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^s) = \text{sup}_{n \rightarrow \infty} \text{mkf}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^s). \]

In particular, if \( u \in S \) with \( c(u) = d(u) \) and if \( \lim_{n \rightarrow \infty} \text{mkf}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^u) = 0 \), then \( \Psi(u) \) is the identity of \( \mathfrak{A}^{d(u)} \).
4. If for any unit \( u \in S \) we have:
\[ \lim_{n \rightarrow \infty} \text{mkf}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^u) = 0 \]
then \( \Psi \) is a functor.
5. If some \( s \in S \) is invertible, with inverse \( s' \), and if for all \( n \in \mathbb{N} \), we have:
\[ \lim_{n \rightarrow \infty} \text{mkf}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^s) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{mkf}_{\mathcal{M}(s)}^{\mathcal{D}(s)}(\varphi_n^{s'}) = 0, \]
then \( \Psi(s) \) is a bijection from \( \mathfrak{A}^{d(s)} \) to \( \mathfrak{A}^{d(s)} \) with inverse \( \Psi(s') \).

Proof. We will employ all the notations given in the assumption of our theorem.

Up to extracting subsequences, a standard diagonal argument shows that since \( S^{(0)} \) is countable, we may as well assume that:
\[ \forall s \in S^{(0)} \forall n \in \mathbb{N} \quad \Lambda^s_T((\mathfrak{A}_n^s, \mathcal{L}_n^s), (\mathfrak{A}^s, \mathcal{L}^s)) \leq \frac{1}{n+2}. \]

For each \( s \in S^{(0)}, n \in \mathbb{N} \), let \( \tau_n^s \in \mathcal{T} \) be a tunnel from \( (\mathfrak{A}^s, \mathcal{L}^s) \) to \( (\mathfrak{A}_n^s, \mathcal{L}_n^s) \) with \( \chi(\tau_n^s) \leq \frac{1}{n+2} \). Let \( \gamma_n \in \mathcal{T} \) be the inverse tunnel \( (\tau_n^s)^{-1} \).

For any \( s \in S^{(0)} \), let \( \mathfrak{G}^s \) be a countable, dense subset of \( \mathfrak{sa}(\mathfrak{A}^s) \) with \( \mathfrak{G}^s \subseteq \text{dom}(\mathcal{L}^s) \).
For each $s \in \mathcal{S}$, $n \in \mathbb{N}$, we denote the forward-image set (see Definition (2.2)):
\[
\{ f_{s,n}(a), \text{dom} f_{s,n} \}
\]
simply as $f_{s,n}(\cdot)$. 

We now prove our theorem in several steps. The first few steps prove the existence of a linear map which will eventually be denoted by $\Psi(s)$ for each $s \in \mathcal{S}$. The construction is of intrinsic interest, as it may be possible to use it to extract additional information on these maps; some of the later steps will be examples of this idea.

**Step 1.** Let $s \in \mathcal{S}$. Let $j_0 : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function. Let $a \in \text{dom} (L^d(s))$. There exists $j_1 : \mathbb{N} \rightarrow \mathbb{N}$, strictly increasing, and $f \in \mathfrak{a} (\mathfrak{R}^d(s))$, such that:
\[
L(s)(f) \leq D(s)L^d(s)(a) \quad \text{and} \quad \|f\|_{\mathfrak{R}^d(s)} \leq D(s)\|a\|_{\mathfrak{R}^d(s)},
\]
and for all $l \geq L^d(s)(a)$, the sequence $(f_{s,j_0 \circ j_1(n)}(a|l, D(s)))_{n \in \mathbb{N}}$ converges, for $\text{Haus}_{\| \cdot \|_{\mathfrak{R}^d(s)}}$, to the singleton $\{f\}$.

Write $D = D(s)$ for this step. Let $a \in \text{dom} (L^d(s))$. By Lemmas (2.5) and (2.7), the sequence:
\[
(f_{s,j_0(n)}(a|L^d(s)(a), D))_{n \in \mathbb{N}}
\]
is a sequence of nonempty compact subsets of:
\[
\{ b \in \mathfrak{a} (\mathfrak{R}^d(s)) : L(s)(b) \leq DL^d(s)(a) \quad \text{and} \quad \|b\|_{\mathfrak{R}^d(s)} \leq D(\|a\|_{\mathfrak{R}^d(s)} + 2L^d(s)(a)) \}
\]
which is itself compact in $\mathfrak{a} (\mathfrak{R}^d(s))$ since $L(s)$ is an $L$-seminorm. Thus, there exists a strictly increasing function $j_1$ such that $(f_{s,j_0 \circ j_1(n)}(a|L^d(s)(a), D))_{n \in \mathbb{N}}$ converges for the Hausdorff distance $\text{Haus}_{\| \cdot \|_{\mathfrak{R}^d(s)}}$, since the Hausdorff distance induces a compact topology on the hyperspace of closed subsets of a compact metric spaces (Blaschke’s theorem [2, Theorem 7.2.8]). Let $f(a)$ be the limit of $(f_{s,j_0 \circ j_1(n)}(a|L^d(s)(a), D))_{n \in \mathbb{N}}$ for $\text{Haus}_{\| \cdot \|_{\mathfrak{R}^d(s)}}$. By Lemma (2.5), we have:
\[
\lim_{n \rightarrow \infty} \text{diam} \left( f_{s,j_0 \circ j_1(n)}(a|L^d(s)(a), D), \| \cdot \|_{\mathfrak{R}^d(s)} \right) = 0
\]
and thus $f(a)$ is a singleton, as desired. We write $f(a) = \{f\}$.

In particular, we thus note that $f$ is the limit of any sequence $(b_n)_{n \in \mathbb{N}}$ with $b_n \in f_{s,j_0 \circ j_1(n)}(a|L^d(s)(a), D)$ for all $n \in \mathbb{N}$. Thus $\|f\|_{\mathfrak{R}^d(s)} = \lim_{n \rightarrow \infty} \|b_n\|_{\mathfrak{R}^d(s)} \leq D\|a\|_{\mathfrak{R}^d(s)}$; moreover since $L(s)$ is lower semicontinuous, we also have:
\[
L(s)(f) \leq \liminf_{n \rightarrow \infty} L(s)_{j_0 \circ j_1(n)}(b_n) \leq DL^d(s)(a).
\]

To conclude this step, let $l \geq L^d(s)(a)$. Since for all $n \in \mathbb{N}$, we have:
\[
f_{s,n}(a|L^d(s)(a), D) \subseteq f_{s,n}(a|l, D),
\]
so by Lemma (2.12), since by Lemma (2.5) we have:
\[
\lim_{n \rightarrow \infty} \text{diam} \left( f_{s,n}(a|l, D), \| \cdot \|_{\mathfrak{R}^d_n} \right) = 0,
\]
we conclude that:
\[
(f_{s,j_0 \circ j_1(n)}(a|l, D))_{n \in \mathbb{N}}
\]
converges to $\{f\}$ as well.
**Step 2.** Let \( s \in \mathcal{S} \) and \( J : \mathbb{N} \to \mathbb{N} \) be a strictly increasing function. There exists a strictly increasing function \( j : \mathbb{N} \to \mathbb{N} \) such that, for all \( a \in \mathcal{G}^{(s)} \), there exists \( \varphi^s(a) \in \mathcal{A}_c(s) \), with:

\[
\|\varphi^s(a)\|_{\mathcal{A}_c(s)} \leq D(s)\|a\|_{\mathcal{G}^{(s)}} \quad \text{and} \quad \mathcal{L}^s \circ \varphi^s(a) \leq D(s)\mathcal{L}^d(a),
\]

such that for all \( l \geq \mathcal{L}^d(a) \), the sequence:

\[
(f_{s,j_0j_1(n)}(a[l,D(s)]))_{n \in \mathbb{N}}
\]

converges for \( \text{Haus}_{\|\cdot\|_{\mathcal{A}_c(s)}} \) to \( \{\varphi^s(a)\} \).

We now apply Step (1) repeatedly, using a diagonal argument. Fix \( s \in \mathcal{S} \) and again, just write \( D = D(s) \). Write \( \mathcal{G}^{(s)} = \{a_n : n \in \mathbb{N}\} \). By Step (1), there exists \( j_0 : \mathbb{N} \to \mathbb{N} \) strictly increasing and \( \varphi^s(a_0) \in \mathcal{A}(\mathcal{A}_c(s)) \), such that for all \( l \geq \mathcal{L}^d(a_0) \), the sequence \( (f_{s,j_0j_1(n)}(a_0[l,D]))_{n \in \mathbb{N}} \) converges to \( \{\varphi^s(a_0)\} \) for \( \text{Haus}_{\|\cdot\|_{\mathcal{A}_c(s)}} \), and \( \|\varphi^s(a_0)\|_{\mathcal{A}_c(s)} \leq D(a_0)\|\mathcal{L}^d(a)\|_{\mathcal{A}_c(s)} \) while \( \mathcal{L}^s \circ \varphi^s(a_0) \leq D\mathcal{L}^d(a_0) \).

Assume now that for some \( k \in \mathbb{N} \), there exist strictly increasing functions \( j_0, \ldots, j_k \) from \( \mathbb{N} \) to \( \mathbb{N} \) and \( \varphi^s(a_0), \ldots, \varphi^s(a_k) \in \mathcal{A}(\mathcal{A}_c(s)) \) such that for \( m \in \{0, \ldots, k\} \) and for all \( l \geq \mathcal{L}^d(a_j) \), the sequence:

\[
(f_{s,j_0j_1 \cdots j_{k+1}(n)}(a_j[l,D]))_{n \in \mathbb{N}}
\]

converges to the singleton \( \{\varphi^s(a_j)\} \) for \( \text{Haus}_{\|\cdot\|_{\mathcal{A}_c(s)}} \), with \( \|\varphi^s(a_j)\|_{\mathcal{A}_c(s)} \leq D\|a_j\|_{\mathcal{G}^{(s)}} \) and \( \mathcal{L}^s \circ \varphi^s(a_j) \leq D\mathcal{L}^d(a_j) \).

By Step (1), there exists a strictly increasing function \( j_{k+1} : \mathbb{N} \to \mathbb{N} \) such that, for all \( l \geq \mathcal{L}^d(a_{k+1}) \), the sequence \( (f_{s,j_0j_1 \cdots j_{k+1}(n)}(a_{k+1}[l,D]))_{n \in \mathbb{N}} \) converges, for \( \text{Haus}_{\|\cdot\|_{\mathcal{A}_c(s)}} \), to some singleton denoted by \( \{\varphi^s(a_{k+1})\} \) with \( \|\varphi^s(a_{k+1})\|_{\mathcal{A}_c(s)} \leq D\|a_{k+1}\|_{\mathcal{G}^{(s)}} \) and \( \mathcal{L}^s \circ \varphi^s(a_{k+1}) \leq D\mathcal{L}^d(a_{k+1}) \). This completes our induction.

Now, let \( j : m \in \mathbb{N} \to J \circ j_0 \circ \cdots \circ j_m(m) \). By construction, for all \( k \in \mathbb{N} \) and for all \( l \geq \mathcal{L}^d(a_k) \), the sequence \( (f_{s,j(n)}(a_k[l,D]))_{n \in \mathbb{N}} \) converges to \( \{\varphi^s(a_k)\} \) for \( \text{Haus}_{\|\cdot\|_{\mathcal{A}_c(s)}} \).

**Step 3.** There exists a strictly increasing function \( j : \mathbb{N} \to \mathbb{N} \) such that, for all \( s \in \mathcal{S} \) and for all \( a \in \mathcal{G}^{(s)} \), there exists \( \varphi^s(a) \in \mathcal{A}(\mathcal{A}_c(s)) \), with:

\[
\|\varphi^s(a)\|_{\mathcal{A}_c(s)} \leq D(s)\|a\|_{\mathcal{G}^{(s)}} \quad \text{and} \quad \mathcal{L}^s \circ \varphi^s(a) \leq D(s)\mathcal{L}^d(a),
\]

such that for all \( l \geq \mathcal{L}^d(a) \), the sequence:

\[
(f_{s,j(n)}(a[l,D(s)]))_{n \in \mathbb{N}}
\]

converges for \( \text{Haus}_{\|\cdot\|_{\mathcal{A}_c(s)}} \) to \( \{\varphi^s(a)\} \).

Since \( \mathcal{S} \) is countable, we write it as \( \mathcal{S} = \{s_m : m \in \mathbb{N}\} \) for this step. We now apply Step (2) repeatedly. First, let \( j_0 : \mathbb{N} \to \mathbb{N} \) and \( \varphi^{s_0} \) be given by Step (2) for \( s_0 \) and \( J : n \in \mathbb{N} \mapsto n \).

Assume we have constructed, for some \( k \in \mathbb{N} \), strictly increasing maps \( j_0, \ldots, j_k \) from \( \mathbb{N} \) to \( \mathbb{N} \), and functions \( \varphi^{s_0} : \mathcal{G}^{(s_0)} \to \mathcal{A}(\mathcal{A}_c(s_0)), \ldots, \varphi^{s_k} : \mathcal{G}^{(s_k)} \to \mathcal{A}(\mathcal{A}_c(s_k)) \) such that for all \( m \in \{0, \ldots, k\} \), \( a \in \mathcal{G}^{(s_m)} \), and \( l \geq \mathcal{L}^{s_m}(a) \), we have:

\[
\mathcal{L}^m \circ \varphi^{s_m} \leq D(s_m)\mathcal{L}^d(a) \quad \text{and} \quad \|\varphi^{s_m}(a)\|_{\mathcal{A}_c(m)} \leq D(s_m)\|a\|_{\mathcal{G}^{(s_m)}},
\]

and the sequence \( (f_{s_m,j_0 \circ \cdots \circ j_m(n)}(a[l,D(s_m)]))_{n \in \mathbb{N}} \) converges to \( \{\varphi^{s_m}(a)\} \) for \( \text{Haus}_{\|\cdot\|_{\mathcal{A}_c(m)}} \).
We apply Step (2) with $J : n \in \mathbb{N} \rightarrow j_0 \circ \ldots \circ j_k$ and $s_{k+1}$: thus there exists $j_{k+1} : \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing and $\varphi^{s_{k+1}} : \mathcal{S}^{d(k+1)} \rightarrow \mathfrak{A} (\mathfrak{C}^{(k+1)})$ satisfying the conclusions of Step (2). This completes our induction.

We conclude this step by setting $j : m \in \mathbb{N} \rightarrow j_0 \circ \ldots \circ j_m(m)$.

**Step 4.** There exists a strictly increasing function $j : \mathbb{N} \rightarrow \mathbb{N}$ and, for all $s \in S$, a function $\varphi^s : \text{dom}(L^d(s)) \rightarrow \mathfrak{A} (\mathfrak{C}^s)$ such that for all $a \in \text{dom}(L^d(s))$ and for all $l \geq L^d(s)(a)$, and $D \geq D(s)$, the sequence:

$$\left( f_{s,j(n)}(a,l,D) \right)_{n \in \mathbb{N}}$$

converges to $\{ \varphi^s(a) \}$ for $\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}}$; moreover:

$$\| \varphi^s(a) \|_{\mathfrak{C}^s} \leq D(s)\|a\|_{\mathfrak{C}^s} \text{ and } L^c(s)(\varphi^s(a)) \leq D(s)L^d(s)(a).$$

Let $j$ and, for all $s \in S$, let $\varphi^s : \mathcal{S}^d(s) \rightarrow \mathfrak{A} (\mathfrak{C}^s)$ be constructed by Step (3). Fix $s \in S$ and write $D \geq D(s)$.

Let $a \in \text{dom}(L^d(s))$, and $\varepsilon > 0$. There exists $a' \in \mathcal{S}^d(s)$ such that $\|a - a'\|_{\mathfrak{C}^d(s)} < \frac{\varepsilon}{12}$. Let $l = \max\{L^d(s)(a), L^d(s)(a')\}$. Since by Lemma (2.5):

$$\lim_{n \to \infty} \text{diam}(f_{s,n}(a'|l,D), \| \cdot \|_{\mathfrak{C}^s}) = 0$$

and since $(f_{s,j(n)}(a'|l,D(s)))_{n \in \mathbb{N}}$ converges to a singleton for $\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}}$, by Lemma (2.12), the sequence $(f_{s,j(n)}(a'|l,D))_{n \in \mathbb{N}}$ also converges to the same singleton.

Since $(f_{s,j(n)}(a'|l,D))_{n \in \mathbb{N}}$ is convergent for the Hausdorff distance $\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}}$ by Step (3), it is Cauchy for $\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}}$. Let $N \in \mathbb{N}$ such that, for all $p, q \geq N$, we have:

$$\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}} \left( f_{s,j(p)}(a'|l,D), f_{s,j(q)}(a'|l,D) \right) \leq \frac{\varepsilon}{3}.$$ 

By Lemma (2.5), we have, for all $l \geq \max\{L^d(s)(a), L^d(s)(a')\}$, for all $n \in \mathbb{N}$, for all $b \in f_{s,j(n)}(a,l,D)$ and for all $b' \in f_{s,j(n)}(a'|l,D)$:

$$\|b - b'|_{\mathfrak{C}^s} \leq D \left( \|a - a'\|_{\mathfrak{C}^d(s)} + 2l \left( \frac{1}{j(n) + 1} + \frac{1}{j(n) + 1} \right) \right) \leq \frac{\varepsilon}{4} + \frac{4l}{j(n) + 1}.$$ 

Let $M \in \mathbb{N}$ such that for all $n \geq M$, we have $\frac{4l}{j(n) + 1} \leq \frac{\varepsilon}{12}$, so that:

$$\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}} \left( f_{s,j(n)}(a,l,D), f_{s,j(n)}(a'|l,D) \right) \leq \frac{\varepsilon}{3}.$$ 

Hence for all $p, q \geq \max\{N,M\}$:

$$\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}} \left( f_{s,j(p)}(a,l,D), f_{s,j(q)}(a,l,D) \right) \leq \text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}} \left( f_{s,j(p)}(a,l,D), f_{s,j(p)}(a'|l,D) \right) + \text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}} \left( f_{s,j(p)}(a'|l,D), f_{s,j(q)}(a'|l,D) \right) + \text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}} \left( f_{s,j(q)}(a'|l,D), f_{j(q)}(a,l,D) \right) \leq \varepsilon.$$ 

Hence $(f_{s,j(n)}(a,l,D))_{n \in \mathbb{N}}$ is a Cauchy sequence for $\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}}$. Since $(\mathfrak{A} (\mathfrak{C}^s), \| \cdot \|_{\mathfrak{C}^s})$ is complete, so is $\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}}$ and thus $(f_{s,j(n)}(a,l,D))_{n \in \mathbb{N}}$ converges for $\text{Haus}_{\| \cdot \|_{\mathfrak{C}^s}}$ to some set $f(a)$; again since $\lim_{n \to \infty} \text{diam}(f_{s,j(n)}(a,l,D), \| \cdot \|_{\mathfrak{C}^s}) = 0$, the set $f(a)$ is a singleton which we denote by $\{ \varphi^s(a) \}$. We observe that if in fact
$a \in \mathcal{G}^{d(s)}$, then we introduce no confusion with this notation since by construction, 
$(f_{s,j(n)}(a|l,D))_{n \in \mathbb{N}}$ converges to $(\varphi^s(a))$ as defined by Step (3).

Using Lemma (2.12) and the observation that for all $n \in \mathbb{N}$, and for all $l'' \geq l' \geq L^{d(s)}(a)$, we have:

$$f_{s,j(n)}(a|l',D) \subseteq f_{s,j(n)}(a|l'',D)$$

we conclude that for all $l \geq L^{d(s)}(a)$, the sequence:

$$(f_{s,j(n)}(a|l,D))_{n \in \mathbb{N}}$$

converges for $\text{Haus}|||_{\mathcal{A}^{d(s)}}$ to $(\varphi^s(a))$.

In particular, if $a \in \text{dom}(L^{d(s)})$, then $(f_{s,j(n)}(a|L^{d(s)}(a),D(s)))_{n \in \mathbb{N}}$ converges to $(\varphi^s(a))$, and thus as before, since $L^{c(s)}$ is lower semi-continuous, we conclude that $L^{c(s)} \circ \varphi^s(a) \leq D(s)L^{d(s)}(a)$ for all $a \in \text{dom}(L^{d(s)})$; Similarly $\|\varphi(a)\|_{\mathcal{A}^{c(s)}} \leq D(s)||a||_{\mathcal{A}^{d(s)}}$.

**Step 5.** For all $s \in \mathcal{S}$, the map $\varphi^s : \text{dom}(L^{d(s)}) \to \text{dom}(L^{c(s)})$ is linear and $\varphi^s(1_{\mathcal{A}^{d(s)}}) = 1_{\mathcal{A}^{c(s)}}$.

Fix $s \in \mathcal{S}$ and write $D = D(s)$. Let $a,a' \in \text{dom}(L^{d(s)})$, $t \in \mathbb{R}$, and $l \geq \max\{L^{d(s)}(a),L^{d(s)}(a')\}$. For each $n \in \mathbb{N}$, let $b_n \in f_{s,j(n)}(a|l,D)$ and $b'_n \in f_{s,j(n)}(a'|l,D)$.

By construction, $(b_n)_{n \in \mathbb{N}}$ converges to $\varphi^s(a)$ while $(b'_n)_{n \in \mathbb{N}}$ converges to $\varphi^s(a')$ in $\text{sa}(\mathcal{A}^{c(s)})$. On the other hand, by Lemma (2.4):

$$\forall n \in \mathbb{N} \quad b_n + tb'_n \in f_{s,j(n)}((a + ta')(1 + |t|)l,D(s)),$$

so $(b_n + tb'_n)_{n \in \mathbb{N}}$ converges to $\varphi^s(a + ta')$ in $\text{sa}(\mathcal{A}^{c(s)})$. By uniqueness of the limit, we conclude:

$$\varphi^s(a + ta') = \varphi^s(a) + t\varphi^s(a').$$

We also note that since $\varphi^s$ maps unit to unit, we have $1_{\mathcal{A}^{c(s)}} \in f_{s,j(n)}(1_{\mathcal{A}^{d(s)}}|0,D)$ for all $n \in \mathbb{N}$. Thus $\varphi^s(1_{d(s)}) = 1_{\mathcal{A}^{c(s)}}$.

**Step 6.** For all $s \in \mathcal{S}$, the map $\varphi^s$ has a unique extension (still denoted by $\varphi^s$) as a continuous linear map from $\mathcal{A}^{d(s)}$ to $\mathcal{A}^{c(s)}$ with:

- $\|\varphi^s\|_{\mathcal{A}^{d(s)}} \leq D(s)$,
- $\|\varphi^s\|_{\mathcal{A}^{c(s)}} \leq \sqrt{2}D(s)$,
- $L^{c(s)} \circ \varphi^s \leq D(s)L^{d(s)}$,
- $\forall a \in \mathcal{A}^{d(s)} \quad \varphi^s(a^*) = \varphi^s(a)^*$.

Fix $s \in \mathcal{S}$. The map $\varphi^s : \text{dom}(L^{d(s)}) \to \text{dom}(L^{c(s)})$ satisfies $L^{c(s)} \circ \varphi^s \leq D(s)L^{d(s)}$ by Step (4).

For all $a \in \text{dom}(L^{d(s)})$, we have $\|\varphi^s(a)\|_{\mathcal{A}^{c(s)}} \leq D(s)||a||_{\mathcal{A}^{d(s)}}$ by construction, so $\varphi^s : \text{dom}(L^{d(s)}) \to \text{sa}(\mathcal{A}^{c(s)})$ is a continuous linear map. Hence it is uniformly continuous on the dense subspace $\text{dom}(L^{d(s)})$ of $\text{sa}(\mathcal{A}^{d(s)})$ and therefore, it has a uniformly continuous extension as a linear map from $\text{sa}(\mathcal{A}^{d(s)})$ to $\text{sa}(\mathcal{A}^{c(s)})$ with norm at most $D(s)$. As a side note, if $L^{d(s)}(a) = \infty$ (equivalently if $a \in \text{sa}(\mathcal{A}^{d(s)}) \setminus \text{dom}(L^{d(s)})$) then the inequality $L^{c(s)} \circ \varphi^s(a) \leq D(s)L^{d(s)}(a) = \infty$ is trivial.

For all $a \in \mathcal{A}^{d(s)}$, we then set:

$$\varphi^s(a) = \varphi^s(Re(a)) + i\varphi^s(Im(a)).$$
where \( \Re a = \frac{a_1 + a_2}{2} \) and \( \Im a = \frac{a_1 - a_2}{2} \) are the respective real and imaginary parts of \( a \).

First, we note that if \( a \in \mathfrak{a}(\mathfrak{A}(d(s))) \) then \( \Re a = 0 \) and thus our extension of \( \varphi^s \) to \( \mathfrak{A}(d(s)) \) agrees with \( \varphi^s \) on \( \mathfrak{a}(\mathfrak{A}(d(s))) \), which justifies that we keep the notation \( \varphi^s \). Moreover, it is straightforward that \( \varphi^s \) is linear — owing to the linearity of \( \varphi^s \) restricted to \( \mathfrak{a}(\mathfrak{A}(d(s))) \).

A quick computation shows that for all \( a \in A \):

\[
||\varphi^s(a)||_{\mathfrak{R}(s)}^2 = ||\varphi^s(a)\ast\varphi^s(a)||_{\mathfrak{R}(s)}^2 \\
\leq ||\varphi^s(\Re a)\ast\varphi^s(\Im a)||_{\mathfrak{R}(s)}^2 \\
\leq 2D(s)^2||a||^2_{\mathfrak{R}(s)}.
\]

Last, by construction, \( \varphi^s(a^*) = \varphi^s(\Re a - i\Im a) = \varphi^s(\Re a) - i\varphi^s(\Im a) = \varphi^s(a)^* \) so \( \varphi^s \) preserves the adjoint operation.

This concludes our step.

**Step 7.** For all \( s, s' \in \mathcal{S} \), if \( d(s) = c(s') \), then \( \varphi^s \circ \varphi^{s'} = \varphi^{s \circ s'} \).

Let \( s, s' \in \mathcal{S} \) such that \( d(s) = c(s') \). By assumption, \( (\mathfrak{A}(d(s)), L(d(s))) = (\mathfrak{A}(c(s'), L(c(s'))) \), so the statement of this step is at least meaningful. We also note that it is obviously sufficient to prove that \( \varphi^s \circ \varphi^{s'}(a) = \varphi^{s \circ s'}(a) \) for all \( a \in \mathfrak{a}(\mathfrak{A}(d(s))) \).

Moreover, by assumption:

\[
\forall n \in \mathbb{N}, \quad (\mathfrak{A}(c(s')), L(c(s'))) = \text{dom}(\varphi_{n}^{s'}) = \text{dom}(\varphi_{n}^{s}) = (\mathfrak{A}(d(s)), L(d(s))).
\]

Let \( a \in \mathfrak{a}(\mathfrak{A}(d(s))) \) and write \( l = L(d(s))(a) \). For all \( n \in \mathbb{N} \), let \( a_n \in \mathfrak{j}_{i(s')} \{ l \} \), and write \( b_n = \varphi_{n}^{s'}(a_n) \). Note that \( b_n \in \mathfrak{A}(c(s)) \). Let \( c_n \in \mathfrak{i}_{j(s')} \{ D(s') \} \) — in particular, \( c_n \in \mathfrak{i}_{j(s')} \{ a(l), D(s') \} \).

In addition, let \( e_n \in \mathfrak{t}_{j(s')} \{ \varphi_{j(s')}^{s)}(a) \} \), so that by Definition (2.2), we have \( e_n \in \mathfrak{f}_{g(s)} \{ a(l), D(s \circ s') \} \). By construction, \( (e_n)_{n \in \mathbb{N}} \) converges to \( \varphi^{s \circ s'}(a) \) in \( \mathfrak{A}(c(s)) \), while \( (c_n)_{n \in \mathbb{N}} \) converges to \( \varphi^s(a) \) in \( \mathfrak{A}(d(s)) \).

Let \( d_n \in \mathfrak{t}_{j(s')} \{ \varphi_{j(s')}^{s)}(b_n) \} \), so that \( d_n \in \mathfrak{A}(c(s)) \). Now, since we chose \( c_n \in \mathfrak{t}_{j(s')} \{ D(s') \} \), and since \( \gamma_{d(s)} = \gamma_{j(s')}^{-1} \), we also have by symmetry \( b_n \in \mathfrak{t}_{j(s')} \{ D(s') \} \). So by Definition (2.2), we conclude that \( d_n \in \mathfrak{i}_{s \circ j(s')} \{ D(s') \}, D(s) \).

Let \( h_n \in \mathfrak{f}_{s \circ j(s')} \{ \varphi^s(a) \} \), By construction, \( (h_n)_{n \in \mathbb{N}} \) converges to \( \varphi^s \circ \varphi^{s'}(a) \) in \( \mathfrak{A}(c(s)) \). On the other hand, by Lemma (2.5):

\[
||d_n - h_n||_{\mathfrak{A}(c(s))} \leq D(s) \left( ||c_n - \varphi^{s'}(a)||_{\mathfrak{A}(c(s))} + 4D(s')\chi (\tau_{j(s')}^s) \right)
\]

so

\[
0 \leq \lim \sup_{n \to \infty} ||d_n - h_n||_{\mathfrak{A}(c(s))} \leq D(s) \left( \lim_{n \to \infty} ||c_n - \varphi^{s'}(a)||_{\mathfrak{A}(c(s))} + 4D(s') \lim_{n \to \infty} \chi (\tau_{j(s')}^s) \right) = 0
\]

so \( (d_n)_{n \in \mathbb{N}} \) converges to \( \varphi^s \circ \varphi^{s'}(a) \) as well.

Let \( \varepsilon > 0 \) and let \( D = \max \{ D(s), D(s \circ s') \} \).

Let \( N_1 \in \mathbb{N} \) such that for all \( n \geq N_1 \), we have \( ||d_n - \varphi^s \circ \varphi^{s'}(a)||_{\mathfrak{A}(c(s))} < \frac{\varepsilon}{4} \).
Let $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $\|e_n - \varphi^{\circ n}(a)\|_{\mathcal{A}(\varepsilon)} < \frac{\varepsilon}{4}$.

Let $N_3 \in \mathbb{N}$ such that for all $n \geq N_3$, we have $\chi \left( \tau_{j(n)}^s \right) < \frac{\varepsilon}{4l+1}$.

Last, by assumption, there exists $N_4 \in \mathbb{N}$ such that for all $n \geq N_4$ and for all $a' \in \text{sa} \left( \mathfrak{A}^{(d(s'))} \right)$ with $L(d(s'))(a') \leq l$, we have:

$$\left\| \varphi_{j(n)}^{s'}(a') - \varphi_{j(n)}(a') \right\|_{\mathcal{A}(\varepsilon)} < \frac{\varepsilon}{4}.$$

We record that by assumption, we have:

$$d_n \in t_{\gamma_{j(n)}} \left( \varphi_{j(n)}^{s}(b_n) \right) \subseteq t_{\gamma_{j(n)}} \left( \varphi_{j(n)}^{s}(b_n) \right) \subseteq t_{\gamma_{j(n)}} \left( \varphi_{j(n)}^{s}(b_n) \right).$$

And:

$$e_n \in t_{\gamma_{j(n)}} \left( \varphi_{j(n)}^{s}(a) \right) \subseteq t_{\gamma_{j(n)}} \left( \varphi_{j(n)}^{s}(a) \right).$$

For all $n \geq \max \{N_1, N_2, N_3, N_4\}$, we thus have:

$$0 \leq \varphi^{\circ n}(a) - \varphi^{\circ n}(a) \leq \left\| \varphi^{\circ n}(a) - e_n \right\|_{\mathcal{A}(\varepsilon)} + \left\| e_n - d_n \right\|_{\mathcal{A}(\varepsilon)} + \left\| d_n - \varphi^{\circ n}(a) \right\|_{\mathcal{A}(\varepsilon)}$$

$$\leq \frac{2\varepsilon}{4} + \left\| e_n - d_n \right\|_{\mathcal{A}(\varepsilon)}$$

$$\leq \frac{2\varepsilon}{4} + \left\| \varphi^{\circ n}(a_n) - \varphi^{\circ n}(b_n) \right\|_{\mathcal{A}(\varepsilon)} + 2lD \left( \tau_{j(n)}^{c(s)} \right) \text{ by Prop (1.26)}$$

$$\leq \frac{3\varepsilon}{4} + \left\| \varphi^{\circ n}(a_n) - \varphi^{\circ n}(b_n) \right\|_{\mathcal{A}(\varepsilon)} \leq \varepsilon.$$

Hence, as $\varepsilon > 0$ is arbitrary, we conclude:

$$\left\| \varphi^{\circ n}(a) - \varphi^{\circ n}(a) \right\|_{\mathcal{A}(\varepsilon)} = 0 \text{ so } \varphi^{\circ n}(a) = \varphi^{\circ n}(a).$$

This concludes this step.

**Summary 2.14.** By setting $\Psi(s) = (\mathfrak{A}^s, L^s)$ for all $s \in S^{(0)}$, and $\Psi(s) = \varphi^n$ for all $s \in \mathcal{S}$, we have proven that $\Psi$ is a functor from $(\mathcal{S}, S^{(0)}, d, c, o)$ to the category of $F$-quasi-Leibniz quantum compact metric spaces whose arrows are Lipschitz linear maps.

We now turn to various additional properties for our maps $\varphi^n(s \in \mathcal{S})$ under more stringent conditions. We begin with positivity.

**Step 8.** If, for some $s \in \mathcal{S}$ and for all $n \in \mathbb{N}$, the map $\varphi^n$ is positive, and $\{a \in \text{dom} \left( L(d(s)) \right) : a \geq 0 \}$ is dense in $\{a \in \text{sa} \left( \mathfrak{A}^{d(s)} \right) : a \geq 0 \}$, then $\varphi^n$ is positive as well.

Let $a \in \text{dom} \left( L(d(s)) \right)$ with $a \geq 0$ and write $l = L(d(s))(a)$; we also write $D = D(s)$. Let $\mu \in \mathcal{F}(\mathfrak{A}^{(d(s)))}$. Let $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N$, we can find $f_n \in \mathfrak{A}^{(d(s))}$ such that $\|f_n - \varphi^n(a)\|_{\mathcal{A}(\varepsilon)} < \varepsilon$, with $f_n \in t_{\gamma_{j(n)}} \left( \varphi^n(b_n) \right) \subseteq t_{\gamma_{j(n)}} \left( \varphi^n(b_n) \right)$, where $b_n \in t_{\gamma_{j(n)}} \left( a(l) \right)$.

There exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have:

$$\chi \left( \tau_{j(n)}^d \right) < \frac{\varepsilon}{3(l+1)} \text{ and } \chi \left( \gamma_{j(n)}^{c(s)} \right) < \frac{\varepsilon}{3(Dl + 1)}.$$
Let $N = \max\{N_1, N_2\}$ and $n \geq N$.

We will need, for this step only, a notation for the tunnels involved in the following computation. We write $\tau^{d(s)}_{j(n)} = (\mathcal{D}^{d(s)}_n, \mathcal{R}^{d(s)}_n, \pi^{d(s)}_n, \rho^{d(s)}_n)$ and $\gamma^{c(s)}_{j(n)} = (\mathcal{D}^{c(s)}_n, \mathcal{R}^{c(s)}_n, \pi^{c(s)}_n, \rho^{c(s)}_n)$.

By Definition (1.17) and [18, Proposition 2.12], there exists $\eta_n \in \mathcal{F}(\mathcal{D}^{c(s)}_{j(n)})$ such that $\mu(\mathcal{F}_{j(n)}^{c(s)}(\mu, \eta_n)) = \frac{\epsilon}{3(Dl + 1)}$. Since $\varphi^{a}_{j(n)}$ is positive and unital, the map $\mu_n = \eta_n \circ \varphi^{a}_{j(n)}$ is a state of $\mathcal{D}^{d(s)}_{j(n)}$. Again by Definition (1.17), there exists $\nu_n \in \mathcal{F}(\mathcal{D}^{d(s)}_{j(n)})$ with $\mu(\mathcal{F}_{j(n)}^{c(s)}(\nu_n, \mu_n)) = \frac{\epsilon}{3(Dl + 1)}$.

Moreover, by Definition (1.24), since $b_n \in \mathcal{T}_{d(s)}(a \{l\})$, there exists $d_n \in \mathcal{D}^{d(s)}_{j(n)}$ with $\mathcal{R}^{d(s)}_{j(n)}(d_n) \leq l$ such that $\pi^{d(s)}_n(d_n) = a$ and $\rho^{d(s)}_n(d_n) = b_n$. There also exists $d'_n \in \mathcal{D}^{c(s)}_{j(n)}$ such that $\mathcal{R}^{c(s)}_{j(n)}(d'_n) \leq Dl$, $\pi^{c(s)}_n(d'_n) = f_n$ and $\rho^{c(s)}_n(d'_n) = \varphi^{a}_{j(n)}(b_n)$.

We then compute:

\[ (2.4) \]
\[ |\mu(\varphi^{a}(a)) - \nu_n(a)| \leq |\mu(\varphi^{a}(a)) - \mu(f_n)| + |\mu(f_n) - \nu_n(a)| \]
\[ \leq \|\varphi^{a}(a) - f_n\|_{\mathfrak{A}} + \mu(f_n) - \nu_n(a) \]
\[ \leq \frac{\epsilon}{3} + |\mu(f_n) - \nu_n(a)| \]
\[ \leq \frac{\epsilon}{3} + |\mu(f_n) - \mu_n(b_n)| + |\mu_n(b_n) - \nu_n(a)| \]
\[ \leq \frac{\epsilon}{3} + |\mu(f_n) - \eta_n(\varphi^{a}_{j(n)}(b_n))| + |\mu_n \circ \rho^{d(s)}_n(d_n) - \nu_n \circ \pi^{d(s)}_n(d_n)| \]
\[ \leq \frac{\epsilon}{3} + |\mu \circ \pi^{c(s)}_n(d'_n) - \eta_n \circ \rho^{c(s)}_n(d'_n)| + \lim_{n \to \infty} \mu(\mathcal{F}_{j(n)}^{c(s)}(\mu_n, \mu_n)) \]
\[ \leq \frac{\epsilon}{3} + Dl \lim_{n \to \infty} \mu(\mathcal{F}_{j(n)}^{c(s)}(\nu_n, \mu_n)) \]
\[ \leq \frac{\epsilon}{3} + \frac{Dl \epsilon}{3Dl} + \frac{\epsilon}{3l} = \epsilon. \]

Now, since $a \geq 0$, for any $n \geq N$, we have $\nu_n(a) \geq 0$. As the limit of the sequence of nonnegative numbers $(\nu_n(a))_{n \in \mathbb{N}}$ by Expression (2.4), we conclude that $\mu(\varphi^{a}(a)) \geq 0$. As $\mu \in \mathcal{F}(\mathcal{A}^{c(s)})$ was an arbitrary state, we conclude that $\varphi^{a}(a) \geq 0$.

By continuity of $\varphi^{a}$, since the space of positive elements in $\mathcal{A}^{c(s)}$ is closed in norm, and by assumption that any positive element of $\mathcal{A}^{d(s)}$ is the limit of positive elements in $\mathcal{D}^{d(s)}$, we conclude that $\varphi^{a}$ is a positive map, as claimed.

**Step 9.** If, for some $s \in \mathcal{S}$ and for all $n \in \mathbb{N}$, the map $\varphi^{a}_{j(n)}$ is a unital *-morphism, then $\varphi$ is a unital *-morphism; in particular $\|\varphi\|_{\mathfrak{A}} = 1$.

Fix $s \in \mathcal{S}$ such that for all $n \in \mathbb{N}$, the map $\varphi^{a}_{j(n)}$ is a unital *-morphism. For this step, set $D = \mathcal{D}(s)$.

We begin by proving that $\varphi^{a}$, restricted to dom $(\mathfrak{A})$, is a Jordan-Lie morphism. Let $a, a' \in \text{dom}(\mathcal{D}(s))$, is a Jordan-Lie morphism. Let $a, a' \in \text{dom}(\mathcal{D}(s))$, is a Jordan-Lie morphism. Let $a, a' \in \text{dom}(\mathcal{D}(s))$, is a Jordan-Lie morphism. For each $n \in \mathbb{N}$, let $b_n \in f_{j(n)}(a \{l\}, D)$ and $b'_n \in f_{j(n)}(a' \{l\}, D)$. By construction, $(b_n)_{n \in \mathbb{N}}$ converges to $\varphi^{a}(a)$ while $(b'_n)_{n \in \mathbb{N}}$ converges to $\varphi^{a}(a')$. On the other hand, by Lemma (2.6) (in particular, using the notations therein), for all $n \in \mathbb{N}$, we have:

\[ b_n \circ b'_n \in f_{j(n)}(a \circ a' \{M(||a||_{\mathfrak{A}}, ||a'||_{\mathfrak{A}}, l, l), D) \]
so \((b_n \circ b'_n)_{n \in \mathbb{N}}\) converges to \(\varphi^s(a \circ a')\). By uniqueness of the limit, we conclude:
\[
\varphi^s(a \circ a') = \varphi^s(a) \circ \varphi^s(a').
\]
The same results holds for the Lie product. Thus \(\varphi^s\) restricted to \(\text{dom}(L^{d(s)})\) is a Jordan-Lie morphism. By continuity, \(\varphi^s\) is a Jordan-Lie morphism on from \(sa(\mathfrak{A}^{d(s)})\) to \(sa(\mathfrak{A}^{c(s)})\).

We note that by construction, \(\varphi^s(a^s) = \varphi^s(a)^s\).

Let now \(a, b \in \mathfrak{A}^{d(s)}\). We first note that:
\[
\varphi^s(a \circ b) = \varphi^s(\Re a \circ \Re b) - \varphi^s(\Im a \circ \Im b) + i(\varphi^s(\Re a \circ \Re b) + \varphi^s(\Im a \circ \Im b))
\]
\[
= \varphi^s(\Re a) \circ \varphi^s(\Re b) - \varphi^s(\Im a) \circ \varphi^s(\Im b) + i(\varphi^s(\Re a) \circ \varphi^s(\Im b) + \varphi^s(\Im a) \circ \varphi^s(\Re b))
\]
\[
= \varphi^s(a) \circ \varphi^s(b).
\]
The same computation holds for the Lie product. Therefore:
\[
\varphi^s(ab) = \varphi^s(\Re(ab)) + i\varphi^s(\Im(ab))
\]
\[
= \varphi^s(a \circ b) + i\varphi^s\{a, b\}
\]
\[
= \varphi^s(a) \circ \varphi^s(b) + i\varphi^s\{a, b\}
\]
\[
= \varphi^s(a)\varphi^s(b).
\]
Thus \(\varphi\) is a unital \(*\)-morphism. This concludes our step.

We now prove a non-triviality result.

**Step 10.** If, for some \(s, s' \in \mathcal{S}\), we have \(c(s) = c(s')\) and \(d(s) = d(s')\), then:
\[
\liminf_{n \to \infty} \text{mkD}^{d(s)}_\mathfrak{A}^{d(s)}(\varphi^s_n, \varphi^{s'}_n) \leq \text{mkD}^{d(s)}_\mathfrak{A}^{d(s)}(\varphi^s, \varphi^{s'}) \leq \limsup_{n \to \infty} \text{mkD}^{d(s)}_\mathfrak{A}^{d(s)}(\varphi^s_n, \varphi^{s'}_n).
\]
Moreover, if for some \(s \in S\), we have \(d(s) = c(s)\), then:
\[
\liminf_{n \to \infty} \text{mkD}^{L_d}_\mathfrak{A}(\varphi^s_n) \leq \text{mkD}^{L_d}_\mathfrak{A}(\varphi^s) \leq \limsup_{n \to \infty} \text{mkD}^{L_d}_\mathfrak{A}(\varphi^s_n).
\]

Fix \(s, s' \in \mathcal{S}\) with the desired properties and write \(D = \max\{1, D(s), D(s')\}\). Let \(\varepsilon > 0\). Let \(E\) be a finite subset of \(\{a \in sa(\mathfrak{A}^{d(s)}): L^{d(s)}(a) \leq 1\}\) such that for all \(a \in sa(\mathfrak{A}^{d(s)})\) with \(L^{d(s)}(a) \leq 1\) there exists \(a' \in E\) and \(t \in \mathbb{R}\) such that:
\[
\|a - (a' + t1\mathfrak{A})\|_{\mathfrak{A}^{d(s)}} < \frac{\varepsilon}{12D}.
\]
Let \(N \in \mathbb{N}\) such that for all \(n \geq N\), we have for all \(a' \in E\):
\[
\text{Haus}_{\|\cdot\|_{\mathfrak{A}^{d(s)}}} \left(\tilde{f}_{s,n}(a'|1, D), \{\varphi^s(a')\}\right) < \frac{\varepsilon}{8}
\]
and:
\[
\text{Haus}_{\|\cdot\|_{\mathfrak{A}^{d(s)}}} \left(\tilde{f}_{s',n}(a'|1, D), \{\varphi^{s'}(a')\}\right) < \frac{\varepsilon}{8}
\]
while:
\[
\chi \left(\tau^s_n\right) < \frac{\varepsilon}{12D}.
\]
Note that \(N\) exists since \(E\) is finite and by construction of \(\varphi^s\) above.

Fix \(n \geq N\).

Let \(G_n \subseteq \mathfrak{A}^{d(s)}\) be given by Lemma (1.27), so that:
• for all \(a \in \mathfrak{A} \left(\mathfrak{G}^{d(s)}_{j(n)}\right)\) with \(L^{d(s)}_{j(n)}(a) \leq 1\), there exists \(t \in \mathbb{R}\) and \(a' \in G_n\) such that \(\|a - (a' + t1_{\mathfrak{A}})\|_{\mathfrak{A}^e(s)} \leq \frac{\varepsilon}{12D}\).

• for all \(a \in G_n\) there exists \(a' \in E\) such that \(a \in t_{j(n)}(a'[1]),\) and conversely, for all \(a \in E\) there exists \(a' \in G_n\) such that \(a \in t_{j(n)}(a'[1]).\)

Let \(a \in \mathfrak{A} \left(\mathfrak{G}^{d(s)}\right)\) with \(L^{d(s)}(a) \leq 1\). There exists \(a' \in E\) and \(t \in \mathbb{R}\) such that \(\|a - (a' + t1_{\mathfrak{A}})\|_{\mathfrak{A}^e(s)} \leq \frac{\varepsilon}{12D} < \frac{\varepsilon}{D}\). We then have:

\[
\begin{align*}
\left\| \varphi^s(a') - \varphi^s(a) \right\|_{\mathfrak{A}^e(s)} & \leq \left\| \varphi^s(a - (a' + t1_{\mathfrak{A}})) \right\|_{\mathfrak{A}^e(s)} \\
& \quad + \left\| \varphi^s(a' + t1_{\mathfrak{A}}) - \varphi^s(a' + t1_{\mathfrak{A}}) \right\|_{\mathfrak{A}^e(s)} \\
& \quad + \left\| \varphi^s(a - (a' + t1_{\mathfrak{A}})) \right\|_{\mathfrak{A}^e(s)} \\
& \leq \frac{D\varepsilon}{4D} + \left\| \varphi^s(a') - \varphi^s(a') \right\|_{\mathfrak{A}^e(s)} + \frac{D\varepsilon}{4D} \\
& \leq \frac{\varepsilon}{2} + \left\| \varphi^s(a') - \varphi^s(a') \right\|_{\mathfrak{A}^e(s)}.
\end{align*}
\]

Let \(a_n \in G_n \cap t_{j(n)}(a'[1]).\) Let \(c_n = \varphi^s_{j(n)}(a_n)\) and \(d_n = \varphi^s_{j(n)}(a_n).\) Let \(c \in t_{j(n)}(c_n|D)\) and \(d \in t_{j(n)}(d_n|D).\) In particular, \(c \in f_{s,j(n)}(a'|1,D)\) and \(d \in f_{s,j(n)}(a'|1,D).\) Thus by construction, \(\left\| \varphi^s(a') - c \right\|_{\mathfrak{A}^e(s)} \leq \frac{\varepsilon}{8}\).

We then compute:

\[
\begin{align*}
\left\| \varphi^s(a') - \varphi^s(a') \right\|_{\mathfrak{A}^e(s)} & \leq \frac{\varepsilon}{4} + \left\| d - c \right\|_{\mathfrak{A}^e(s)} \\
& \leq \frac{\varepsilon}{4} + 2 \frac{\varepsilon}{12D} + \left\| \varphi^s_{j(n)}(a_n) - \varphi^s_{j(n)}(a_n) \right\|_{\mathfrak{A}^e(s)} \text{ by Proposition (1.26)} \\
& \leq \frac{\varepsilon}{2} + \text{mkD}^{d(s)}_{\mathfrak{A}^e(s)} \left( \varphi^s_{j(n)}, \varphi^s_{j(n)} \right).
\end{align*}
\]

Hence for all \(a \in \mathfrak{A} \left(\mathfrak{G}^{d(s)}\right)\) with \(L^{d(s)}(a) \leq 1:\)

\[
\left\| \varphi^s(a) - \varphi^s(a) \right\|_{\mathfrak{A}^e(s)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \text{mkD}^{d(s)}_{\mathfrak{A}^e(s)} \left( \varphi^s_{j(n)}, \varphi^s_{j(n)} \right),
\]

i.e. \(\text{mkD}^{d(s)}_{\mathfrak{A}^e(s)} \left( \varphi^s, \varphi^s \right) \leq \text{mkD}^{d(s)}_{\mathfrak{A}^e(s)} \left( \varphi^s_{j(n)}, \varphi^s_{j(n)} \right) + \varepsilon.\)

We proceed symmetrically to show the converse inequality. Namely, let \(a_n \in \mathfrak{G}^{d(s)}_{j(n)}\) with \(L^{d(s)}_{j(n)}(a_n) \leq 1\). There exists \(t \in \mathbb{R}\) and \(a'_n \in G_n\) such that \(\|a_n - (a'_n + t1_{\mathfrak{A}})\|_{\mathfrak{A}^e(s)} \leq \frac{\varepsilon}{4D}\). As before, we have:

\[
\begin{align*}
\left\| \varphi^s_{j(n)}(a_n) - \varphi^s_{j(n)}(a_n) \right\|_{\mathfrak{A}^e(s)} & \leq \frac{\varepsilon}{2} + \left\| \varphi^s_{j(n)}(a'_n) - \varphi^s_{j(n)}(a'_n) \right\|_{\mathfrak{A}^e(s)}.
\end{align*}
\]

Let \(a \in E\) such that \(a \in t_{j(n)}(a'[1]).\) Let \(c \in t_{j(n)} \left( \varphi^s_{j(n)}(a_n) \right) D\). Note that by symmetry, we have \(a_n \in t_{j(n)}(a|[1,D])\), so that by construction \(c \in f_{s,j(n)}(a|[1,D])\), and thus \(\|\varphi^s(a) - c\|_{\mathfrak{A}^e(s)} < \frac{\varepsilon}{8}\).

Similarly, let \(d \in t_{j(n)} \left( \varphi^s_{j(n)}(a_n) \right) D\), and note \(d \in f_{s,j(n)}(a|[1,D])\) so \(\|\varphi^s(a) = d\| < \frac{\varepsilon}{8}\).
From this, we then obtain as before that \( \| \varphi_{j(n)}^{s'}(a'_n) - \varphi_{j(n)}^{s}(a'_n) \|_{\mathfrak{A}_{j(n)}} \leq \frac{\varepsilon}{2} + mkD_{\mathfrak{A}_{j(n)}}^{d_{j(n)}}(\varphi^s, \varphi^{s'}) \). Thus we get:

\[
mkD_{\mathfrak{A}_{j(n)}}^{d_{j(n)}}(\varphi_{j(n)}^s, \varphi_{j(n)}^{s'}) \leq mkD_{\mathfrak{A}_{j(n)}}^{d_{j(n)}}(\varphi^s, \varphi^{s'}) + \varepsilon.
\]

Hence we have shown that, for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), we have:

\[
mkD_{\mathfrak{A}_{j(n)}}^{d_{j(n)}}(\varphi^s, \varphi^{s'}) \leq \varepsilon \leq mkD_{\mathfrak{A}_{j(n)}}^{d_{j(n)}}(\varphi_{j(n)}^s, \varphi_{j(n)}^{s'}) \leq mkD_{\mathfrak{A}_{j(n)}}^{d_{j(n)}}(\varphi^s, \varphi^{s'}) + \varepsilon.
\]

Thus:

\[
\lim_{n \to \infty} mkD_{\mathfrak{A}_{j(n)}}^{d_{j(n)}}(\varphi_{j(n)}^s, \varphi_{j(n)}^{s'}) = mkD_{\mathfrak{A}_{j(n)}}^{d_{j(n)}}(\varphi^s, \varphi^{s'}).\]

In particular, we have:

\[
\liminf_{n \to \infty} mkD_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^s_n, \varphi^{s'}_n) \leq \limsup_{n \to \infty} mkD_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^s_n, \varphi^{s'}_n).
\]

If \( c(s) = d(s) \) then our argument above can be applied as well when we replace \( \varphi_{n}^{s'} \) and \( \varphi^{s'} \) with the identity of \( \mathfrak{A}_{j(n)}^{d_{j(n)}}(\varphi^s_n, \varphi^{s'}_n) \) and \( \mathfrak{A}_{j(n)}^{d_{j(n)}} \), and adjusting the proof accordingly, in which case we conclude:

\[
\liminf_{n \to \infty} mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^s_n) \leq \limsup_{n \to \infty} mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^s_n).
\]

Now, assume \( u \in \mathcal{S} \) such that \( d(u) = c(u) \) and \( \lim_{n \to \infty} mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^s_n) = 0 \). By this step, we conclude that \( mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^s_n) = 0 \) so \( \varphi^u \) is the identity of \( \mathfrak{A}_{n}^{d_{j(n)}} \).

From this, and by definition, we conclude that if all units \( u \) of \( \mathcal{S} \) satisfy \( \lim_{n \to \infty} mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^s_n) = 0 \) then \( \Psi \) is a functor.

Moreover, if \( s \in \mathcal{S} \) is invertible and we set \( u = s' \circ s \) and \( v = s' \circ s \), where \( s' \) is the inverse of \( s \), and if:

\[
\lim_{n \to \infty} mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^s_n) = \lim_{n \to \infty} mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^{s' \circ s}_n) = 0
\]

and

\[
\lim_{n \to \infty} mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^u_n) = \lim_{n \to \infty} mkf_{\mathfrak{A}_{n}}^{d_{j(n)}}(\varphi^{s' \circ s}_n) = 0
\]

then \( \varphi^u \) and \( \varphi^v \) are the identities of \( \mathfrak{A}_{n}^{d_{j(n)}} \) and \( \mathfrak{A}_{n}^{d_{j(n)}} \). Since moreover \( \varphi^u = \varphi^{s' \circ s} = \varphi^s \circ \varphi^{s'} \) and \( \varphi^v = \varphi^{s' \circ s} = \varphi^{s' \circ s} \circ \varphi^s \), we conclude that \( \varphi^s \) is invertible with inverse \( \varphi^{s'} \).

This concludes our step and our theorem. \( \square \)

**Remark 2.15.** We note that the morphisms obtained by Theorem (2.13) are certainly not unique in general. There are many different choices being made in their construction: the choice of the tunnels and the choices of subsequences of target sets from compactness. The proof shows how all these choices can be made in a coherent fashion to preserve composition, and ensure that the limit of identity automorphisms is the identity — thus preserving inverse relationships.

**Remark 2.16.** We need an additional assumption regarding the density of the positive elements with finite L-seminorm within the positive elements of the underlying C*-algebra for Item (1) of Theorem (2.13) which does not appear for the rest of the theorem, and in particular for Item (3) regarding *-morphisms.
This assumption is needed for the proof of Item (1) as we have no multiplicative property for this step, and not for Item (3) because *-morphisms are automatically positive. We note that for classical metric space, this special assumption is always satisfied because the truncation of a Lipschitz function is Lipschitz. On the other hand, we also note that when L-seminorms are obtained from ergodic group actions as in [24], which is an often used construction, then again this special property is met, using a standard regularization argument akin to the proof of the density of the domain of such L-seminorms in [24]. However, in general, the situation can be more involved and the Leibniz property does not seem to ensure that this special condition will always hold.

We proved our main Theorem (2.13) with the construction of the dual propinquity in [18]. We could just as easily make the same argument, with trivial changes, adapted to the original construction in [12] based on journeys — this would technically allow us to use somewhat more general classes of tunnels. Among the possible choices of tunnels, we could then employ the tunnels obtained directly from bridges [15]. The reason one would wish to adapt the argument in such an easy manner is if it is possible to extract more information about the functor constructed by Theorem (2.13) based on a more stringent choice of tunnels — for instance, when spaces converge for the quantum propinquity [15].

3. Applications

As a first observation, we apply Theorem (2.13) to the construction of a single *-morphism. The result is interesting because it is not necessarily easy to construct morphisms between $C^*$-algebras, yet this first corollary shows a new method based on metric approximations.

**Corollary 3.1.** Let $F$ be an admissible function. Let $(\mathfrak{A}, L^\mathfrak{A})$ and $(\mathfrak{B}, L^\mathfrak{B})$ be two $F$-quasi-Leibniz quantum compact metric spaces, and let $(\mathfrak{A}_n, L^\mathfrak{A}_n)_{n\in\mathbb{N}}$ and $(\mathfrak{B}, L^\mathfrak{B}_n)_{n\in\mathbb{N}}$ be two sequences of $F$-quasi-Leibniz quantum compact metric spaces such that:

$$\lim_{n\to\infty} \Lambda_F^*(\mathfrak{A}_n, L^\mathfrak{A}_n, (\mathfrak{A}, L^\mathfrak{A})) = \lim_{n\to\infty} \Lambda_F^*(\mathfrak{B}_n, L^\mathfrak{B}_n, (\mathfrak{B}, L^\mathfrak{B})) = 0.$$ 

Let $D > 0$. If for each $n \in \mathbb{N}$, there exists a Lipschitz morphism (resp. a Lipschitz linear map) $\varphi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ such that:

1. \(\forall n \in \mathbb{N} \quad L^\mathfrak{B}_n \circ \varphi_n \leq D L^\mathfrak{A}_n\),
2. \(||\varphi_n||_{\mathfrak{B}_n} \leq D\),

then there exists a Lipschitz morphism (resp. a Lipschitz linear map) $\varphi : \mathfrak{A} \to \mathfrak{B}$ such that:

1. \(||\varphi||_{\mathfrak{B}} \leq D\)
2. \(L^\mathfrak{B} \circ \varphi \leq L^\mathfrak{A}\).

If moreover $(\mathfrak{A}_n, L^\mathfrak{A}_n) = (\mathfrak{B}_n, L^\mathfrak{B}_n)$ for all $n \in \mathbb{N}$ then:

$$\liminf_{n\to\infty} \text{mkl}_{L^\mathfrak{A}}(\varphi_n) \leq \text{mkl}_{L^\mathfrak{A}}(\varphi) \leq \limsup_{n\to\infty} \text{mkl}_{L^\mathfrak{A}}(\varphi_n).$$

If furthermore, $\varphi_n$ is a bijection for infinitely many $n \in \mathbb{N}$, then $\varphi$ may be chosen to be a bijection as well.

**Proof.** All but the last statement result from applying Theorem (2.13) to the semigroupoid ($\{s\}, \{a,b\}, c, d, o$) with $d(s) = a$ and $c(s) = b$ (thus $o$ is the empty
map). Setting \((\mathfrak{A}^a, L^a) = (\mathfrak{A}, L^a), (\mathfrak{B}^b, L^b) = (\mathfrak{B}, L^b)\) and for all \(n \in \mathbb{N}\), setting \((\mathfrak{A}_n, L^n) = (\mathfrak{A}, L^a), (\mathfrak{B}_n, L^n) = (\mathfrak{B}, L^b)\), and \(\varphi^a_n = \varphi^b_n\). Theorem \((2.13)\) applies.

If \(\varphi_n\) is invertible for infinitely many \(n \in \mathbb{N}\), then up to extracting a subsequence, we may as well assume that \(\varphi_n\) is invertible for all \(n \in \mathbb{N}\). Then we employ the groupoid \((s, s', a, b), \{a, b\}, d, e, \circ\), with the same definitions as above, with \(a\) and \(b\) serving as the units for the homonymous domains, and \(s'\) the inverse of \(s\), with \(\varphi_n^* = \varphi_n^{-1}\) for all \(n \in \mathbb{N}\). Theorem \((2.13)\) applies again. 

As an obvious remark, we point out that if we are given two convergent sequences \((\mathfrak{A}_n, L^a)_{n \in \mathbb{N}}\) and \((\mathfrak{B}_n, L^b)_{n \in \mathbb{N}}\) of quasi-Leibniz quantum compact metric spaces which are entry-wise fully quantum isometric, then by \([12]\), their respective limits \((\mathfrak{A}, L^a)\) and \((\mathfrak{B}, L^b)\) are also fully quantum isometric:

\[
0 \leq \Lambda((\mathfrak{A}, L^a), (\mathfrak{B}, L^b)) = \lim_{n \to \infty} \Lambda((\mathfrak{A}_n, L^n), (\mathfrak{B}, L^b)) = \lim_{n \to \infty} 0 = 0.
\]

This coincidence property was a strong motivation behind the construction of the propinquity, and is proven using in part the quasi-Leibniz property. Thus our present Theorem \((2.13)\) provides another illustration of the same principle that using the quasi-Leibniz property and the definition of the propinquity enables us to build *-morphisms between C*-algebras, albeit Theorem \((2.13)\) application in the single morphism case is only new and interesting for Lipschitz morphisms which are not full quantum isometries.

We now turn to the main motivation for this paper: the existence of a nontrivial action of a group on the limit of a sequence of quasi-Leibniz quantum compact metric spaces, when the group is the limit of groups acting on each term of the sequence. One way to see the importance of this result is with a sight on mathematical physics applications: a model obtained as a limit for the propinquity of action of a group on the limit of a sequence of quasi-Leibniz quantum compact metric spaces, when the group is the limit of groups acting on each term of the sequence.

There are in fact different manners to formalize this problem. We will study the equivariant propinquity in a subsequent paper \([19]\), to keep the present work of manageable length, and also because we take the perspective that this is a work on mathematical physics applications: a model obtained as a limit for the propinquity of covariant models for some group will bear the same covariance.

We introduce a natural notion of compact, metric monoid convergence. We will use the term monoid to mean an associative magma with an identity element (a unit). A metric monoid \((G, \delta_G)\) is a monoid \(G\) endowed with a distance \(\delta_G\) which induces a topology on \(G\), and for which the multiplication of \(G\) is continuous.

**Definition 3.2.** Let \((G_1, \delta_1)\) and \((G_2, \delta_2)\) be two metric monoids and \(\varepsilon > 0\). An ordered pair \(\varsigma_i : G_1 \to G_2\) and \(\varsigma_2 : G_2 \to G_1\) of functions is called \(\varepsilon\)-near isometric isomorphism from \((G_1, \delta_1)\) to \((G_2, \delta_2)\) when, for all \(\{j, k\} = \{1, 2\}\) and \(g, g' \in G_j\):

1. \(\delta_k(\varsigma_j(g), \varsigma_j(g')) \leq \varepsilon,
2. \(\delta_j(\varsigma_k(\varsigma_j(g)), g) \leq \varepsilon,
3. |\delta_k(\varsigma_j(g), \varsigma_j(g')) - \delta_j(g, g')| \leq \varepsilon,
4. \(\delta_k(\varsigma_j(e_j), e_k) \leq \varepsilon.

**Definition 3.3.** A sequence \((G_n, \delta_n)_{n \in \mathbb{N}}\) of compact monoids converges (in the sense of Gromov-Hausdorff for compact monoids) to a compact monoid \((G, \delta)\) when, for all \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that if \(n \geq N\) then there exists a \(\varepsilon\)-near isometric isomorphism from \(G_n\) to \(G\).
We refer to [19] for a more detailed presentation of the basic properties of the notion of convergence introduced in Definition (3.3); we only record here that convergence of compact monoid in this sense is induced by a metric up to isometric isomorphism — and thus in particular, the limit is unique up to an isometric isomorphism of monoids.

A special case of obvious interest is given by converging sequences of closed subgroups for the Hausdorff metric induced by a continuous function on a compact group: the following proposition shows that such a sequence would also converge in the sense of Definition (3.3) even though we do not a priori assume more than convergence in a metric sense, with no algebraic condition.

**Proposition 3.4.** Let \((G, \delta)\) be a compact metric group, with \(\delta\) invariant by left and right translations. If \(H \subseteq G\) is a closed subgroup of \(G\) and \((H_n)_{n \in \mathbb{N}}\) is a sequence of closed subgroups of \(G\) converging to \(H\) for the Hausdorff distance \(\text{Haus}_\delta\) induced by \(\delta\) on the closed subsets of \(G\), then \((H_n, \delta)_{n \in \mathbb{N}}\) converges to \((H, \delta)\) in the sense of Definition (3.3).

**Remark 3.5.** If \(\ell\) is a length function over a group \(G\) such that \(\ell(g^{-1}hg) = \ell(h)\) for all \(g, h \in G\), then the distance \(\delta : g, h \in G \mapsto \ell(h^{-1}g)\) is both left and right invariant, i.e. translation invariant.

**Proof.** For any closed subset \(F\) of \(G\) and \(g \in G\), we choose an element \(p_F(g) \in F\) such that:

\[
\delta(g, p_F(g)) = \min \{\delta(g, h) : h \in F\}.
\]

The existence of such an element is guaranteed by compactness; our choice is arbitrary: in particular, we have no expectation of any regularity for the function \(p_F\) thus defined.

For all \(n \in \mathbb{N}\), set:

\[
\varphi_n : g \in H \mapsto p_{H_n}(g) \quad \text{and} \quad \psi_n : g \in H_n \mapsto p_H(g).
\]

We immediately note that \(\varphi_n\) and \(\psi_n\) both fix the identity element of \(G\) for all \(n \in \mathbb{N}\).

Let \(N \in \mathbb{N}\) such that, for all \(n \geq N\), we have \(\text{Haus}_\delta(H_n, H) < \frac{\varepsilon}{3}\). Let \(n \geq N\). Note that by definition of \(\text{Haus}_\delta\), we have \(\max\{\delta(\varphi_n(g), g), \delta(\psi_n(h), h)\} < \frac{\varepsilon}{3}\) for all \(g \in H\) and \(h \in H_n\). It is a matter of routine computation that \((\varphi_n, \psi_n)\) is an \(\varepsilon\)-near isometric isometry from \((H_n, \delta)\) to \((H, \delta)\) as desired. \(\square\)

We are now ready to prove our main application of Theorem (2.13).

**Theorem 3.6.** Let \((G, \delta_G)\) be a compact metric monoid, Let \((G_n, \delta_n)_{n \in \mathbb{N}}\) be a sequence of compact metric monoids, converging to \((G, \delta_G)\). Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of positive numbers with \(\lim_{n \to \infty} \varepsilon_n = 0\) and such that for each \(n \in \mathbb{N}\), there exists a \(\varepsilon_n\)-near isometric isomorphism \((\varphi_n, \psi_n)\) from \((G_n, \delta_n)\) to \((G, \delta_G)\).

Let \(D : G \to [0, \infty)\) be a continuous function. Let \(K : [0, \infty) \to [0, \infty)\) be continuous with \(K(0) = 0\).

Let \(C\) be a nonempty class of \(F\)-quasi-Leibniz quantum compact metric spaces for some admissible function \(F\), and let \(T\) be a class of tunnels compatible with \(C\). Let \((\varphi_n, L_n)_{n \in \mathbb{N}}\) be a sequence in \(C\), such that for each \(n \in \mathbb{N}\), there exists an action \(\alpha_n\) by Lipschitz linear endomorphisms of \(G_n\) on \(\varphi_n\), where:

\[
\forall n \in \mathbb{N}, \ g \in G_n \quad \|\alpha_n^g\|_{\varphi_n} \leq D(\varphi_n(g)) \quad \text{and} \quad L_n \circ \alpha_n^g \leq D(\varphi_n(g))L_n,
\]
and
\[ \forall n \in \mathbb{N}, \, g, g' \in G_n, \, a \in \mathfrak{A}(\mathfrak{A}_n) \quad \left\| \alpha^g_n(a) - \alpha^{g'}_n(a) \right\|_{\mathfrak{A}_n} \leq K (\delta_n(g, g')) L_n(a). \]

If there exists \((\mathfrak{A}, L) \in C\) such that:
\[ \lim_{n \to \infty} \Lambda^\mathfrak{A}_n (\mathfrak{A}_n, L_n(a), (\mathfrak{A}, L)) = 0 \]
then there exists a strongly continuous action \(\alpha\) of the monoid \(G\) by Lipschitz linear endomorphisms such that for all \(g \in G\):
\[ \|\alpha^g\|_{\mathfrak{A}} \leq D(g) \quad \text{and} \quad L \circ \alpha^g \leq D(g)L, \]
and for all \(g, h \in G\):
\[ \liminf_{n \to \infty} \text{mkD}_{\mathfrak{A}_n}^L (\alpha^g_n, \alpha^h_n) \leq \text{mkD}_{\mathfrak{A}_n}^L (\alpha^g, \alpha^h) \leq \limsup_{n \to \infty} \text{mkD}_{\mathfrak{A}_n}^L (\alpha^g_n, \alpha^h_n), \]
while for all \(g, h \in G\), and \(a \in \text{dom}(L)\), we have:
\[ \left\| \alpha^g(a) - \alpha^h(a) \right\|_{\mathfrak{A}} \leq L(a)K(\delta_G(g, h)). \]

If moreover for all \(n \in \mathbb{N}\), the actions \(\alpha_n\) of the monoid \(G_n\) is by positive unital linear maps, then \(\alpha\) is a strongly continuous action of \(G\) by positive unital linear maps.

Remark 3.7. With the notations of Theorem (3.6), if \(G\) is a group, and \(G_n\) is a group for all \(n \in \mathbb{N}\), then of course \(\|\alpha^g\|_{\mathfrak{A}} = 1\) as \(\alpha^g\) is a \(*\)-automorphism for all \(g \in G\).

Remark 3.8. We emphasize that while Theorem (3.6) requires convergence of the quantum metric spaces and of the groups or monoids, there is not requirement of convergence involving the actions themselves.

Proof. Since \(G\) is a compact metric space, it is separable. Let \(E\) be a countable dense subset of \(G\). Let \(H\) be the sub-monoid generated by \(E\). Since \(H\) consists of all the finite products of elements of the countable set \(E\), it is itself countable.

As a monoid, \(H\) is also a semigroup in a trivial manner (formally, the semigroupoid is \((H, \{e\}, c, d, \cdot)\) with \(c\) and \(d\) the constant functions equal to \(e\) and \(\cdot\) the multiplication of \(H\), and \(e\) is the unit of \(H\). With these notations, we would set \((\mathfrak{A}^c, L^c) = (\mathfrak{A}, L)\) and \((\mathfrak{A}^c_n, L^c_n) = (\mathfrak{A}_n, L_n)\) for all \(n \in \mathbb{N}\), though we will not use this heavier notation here).

Let \(g \in H\), \(n \in \mathbb{N}\) and \(a \in \text{dom}(L_n)\) with \(L_n(a) \leq 1\). We now compute:
\[ \left\| \alpha^g_n(a) \circ \alpha^{g'}_n(a) - \alpha^g_n(gg')(a) \right\|_{\mathfrak{A}_n} = \left\| \alpha^g_n(g) \alpha^{g'}_n(a) - \alpha^g_n(gg')(a) \right\|_{\mathfrak{A}_n} \]
\[ \leq K(\delta_n(\alpha_n(g), \alpha_n(g')) L_n(a)) \]
\[ \leq K(\delta_n(\alpha_n(g), \alpha_n(g'))). \]

By Definition (3.2), the sequence \((\delta_n(\alpha_n(g), \alpha_n(g'))))_{n \in \mathbb{N}}\) converges to 0. So:
\[ \lim_{n \to \infty} K(\delta_n(\alpha_n(g), \alpha_n(g')))) = 0. \]
We also record that \(\alpha_n^{e_n}\), for \(e_n \in G_n\) the unit of \(G_n\), is the identity map.
Thus we fit the hypothesis of Theorem (2.13). Therefore, there exists an action \( \alpha \) of \( H \) on \( \mathfrak{A} \), with the property that, for all \( a \in \mathfrak{A} \) with \( L(a) \leq 1 \) and for all \( g, h \in H \):

\[
\|\alpha^g(a) - \alpha^h(a)\| \leq \limsup_{n \to \infty} K(\delta_n(\mathfrak{a}_n(g), \mathfrak{a}_n(h))) \leq K(\delta_G(g, h)),
\]

and in addition, Expressions (3.1) and (3.2) hold for \( g \in H \). Note that \( \alpha \) is a monoid action, i.e. in particular the unit of \( G \) acts as the identity.

It thus immediately follows by homogeneity that for all \( a \in \text{dom } (L) \) and \( g \in H \):

\[
\|\alpha^g(a) - \alpha^h(a)\| \leq L(a)K(\delta_G(g, h)).
\]

Therefore, \( g \in H \mapsto \alpha^g(a) \) is uniformly continuous for all \( a \in \text{dom } (L) \) since \( \lim_{n} K = 0 \), and thus it can be extended uniquely to a uniformly continuous function \( g \in G \mapsto \alpha^g(a) \), with the additional property that for all \( g, h \in G \):

\[
\|\alpha^g(a) - \alpha^h(a)\| \leq L(a)\text{diam } (\mathfrak{A}, L)K(\delta_G(g, h))
\]

by continuity of \( K \) (and \( \delta_G \)).

It is straightforward that for all \( a \in \text{dom } (L) \):

1. for all \( g, h \in G \), we have by continuity of the multiplication on \( G \) that \( \alpha^g \circ \alpha^h(a) = \alpha^{gh}(a) \),
2. for all \( g \in G \), the lower semi-continuity of \( L \) and the continuity of \( D \) gives us \( L \circ \alpha^g(a) \leq D(g)\|a\| \),
3. for all \( g \in G \), we also have by continuity that \( \|\alpha^g(a)\| \leq D(g)\|a\|_{\mathfrak{A}} \).

In particular, Expression (3.1) holds for all \( g \in G \).

Therefore, \( \alpha \) is a monoid action of \( G \) via Lipschitz linear maps.

Let \( g \in G \). Let \( \varepsilon > 0 \), and let \( R > 0 \) be some upper bound for the convergent sequence \( \text{diam } (\mathfrak{A}_n, L_n) \) \( n \in \mathbb{N} \). Since \( K \) is continuous and \( K(0) = 0 \), there exists \( \delta > 0 \) such that for \( t \in [0, \delta) \), we have \( K(t) < \frac{\varepsilon}{2} \).

Let \( h \in H \) with \( \delta_G(g, h) < \frac{\delta}{4} \).

Now, there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), we have \( \varepsilon_n < \frac{\varepsilon}{4} \). Let \( n \geq N \). By Definition (3.2), we have:

\[
|\delta_n(\mathfrak{a}_n(g), \mathfrak{a}_n(h)) - \delta_G(g, h)| < \frac{\delta}{3}.
\]

Hence \( \delta_n(\mathfrak{a}_n(g), \mathfrak{a}_n(h)) < \delta \).

We now compute, for any \( a \in \text{dom } L_n \) with \( L_n(a) \leq 1 \):

\[
\left\| a - \alpha_{\mathfrak{a}_n}(g)(a) \right\|_{\mathfrak{A}_n} - \left\| a - \alpha_{\mathfrak{a}_n}(h)(a) \right\|_{\mathfrak{A}_n} \leq \left\| \alpha_{\mathfrak{a}_n}(g)(a) - \alpha_{\mathfrak{a}_n}(h)(a) \right\|_{\mathfrak{A}_n} \leq K(\delta_n(\mathfrak{a}_n(g), \mathfrak{a}_n(h))) \leq \frac{\varepsilon}{2}.
\]

Therefore:

\[
\limsup_{n \to \infty} mK\ell_n(\alpha_{\mathfrak{a}_n}(g)) \leq \limsup_{n \to \infty} mK\ell_n(\alpha_{\mathfrak{a}_n}(h)) + \frac{\varepsilon}{2},
\]

and

\[
\liminf_{n \to \infty} mK\ell_n(\alpha_{\mathfrak{a}_n}(g)) \geq \liminf_{n \to \infty} mK\ell_n(\alpha_{\mathfrak{a}_n}(h)) - \frac{\varepsilon}{2}.
\]

Now:

\[
\left\| a - \alpha^g(a) \right\|_{\mathfrak{A}} - \left\| a - \alpha^h(a) \right\|_{\mathfrak{A}} \leq \left\| \alpha^g(a) - \alpha^h(a) \right\|_{\mathfrak{A}} \leq K(\delta_G(g, h)) < \frac{\varepsilon}{2}.
\]
Thus:
\[ \|a - \alpha^g(a)\|_{\mathfrak{A}} \leq \frac{\varepsilon}{2} + \|a - \alpha^h(a)\|_{\mathfrak{A}} \]
\[ \leq \frac{\varepsilon}{2} + \limsup_{n \to \infty} m_k \ell_n(\alpha^{\varepsilon_n(h)}) \quad \text{since} \quad h \in H, \]
\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \limsup_{n \to \infty} m_k \ell_n(\alpha^{\varepsilon_n(h)}) = \varepsilon + \limsup_{n \to \infty} m_k \ell_n(\alpha^{\varepsilon_n(g)}), \]
and therefore:
\[ m_k \ell_n(\alpha^g) \leq \limsup_{n \to \infty} m_k \ell_n(\alpha^{\varepsilon_n(g)}) + \varepsilon. \]

Similarly:
\[ m_k \ell_n(\alpha^g) \geq \liminf_{n \to \infty} m_k \ell_n(\alpha^{\varepsilon_n(h)}) - \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we conclude that for all \( g \in G \):
\[ \liminf_{n \to \infty} m_k \ell_n(\alpha^{\varepsilon_n(g)}) \leq m_k \ell_n(\alpha^g) \leq \limsup_{n \to \infty} m_k \ell_n(\alpha^{\varepsilon_n(g)}). \]

Thus Expression (3.2) now holds for all \( g \in G \).

Let now \( a \in \mathfrak{sa}(\mathfrak{A}) \), \( \varepsilon > 0 \) and \( g \in G \). Since \( D \) is continuous, let \( \eta > 0 \) such that if \( \delta_G(g, h) < \eta \), then \( |D(g) - D(h)| < \varepsilon \).

By density, there exists \( a' \in \operatorname{dom}(L) \) such that \( \|a - a'\| < \frac{\varepsilon}{\operatorname{max}(L(a'), \eta)} \). Now, there exists \( \eta_2 > 0 \) such that if \( t \in [0, \eta_2] \) then \( K(t) < \frac{\varepsilon}{\operatorname{max}(L(a'), \eta)} \). Let now \( h \in G \) with \( \delta_G(g, h) < \min\{\eta, \eta_2\} \). We compute:
\[ \|\alpha^g(a) - \alpha^h(a)\|_{\mathfrak{A}} \leq \|\alpha^g(a - a')\|_{\mathfrak{A}} + \|\alpha^g(a') - \alpha^h(a')\|_{\mathfrak{A}} + \|\alpha^h(a - a')\|_{\mathfrak{A}} \]
\[ \leq D(g) \|a - a'\|_{\mathfrak{A}} + L(a') K(\delta_G(g, h)) + D(h) \|a - a'\|_{\mathfrak{A}} \]
\[ \leq \varepsilon. \]

Therefore, for all \( a \in \mathfrak{sa}(\mathfrak{A}) \), the function \( g \in G \mapsto \alpha^g(a) \) is continuous, as desired.

We now turn to an application of our work to certain examples. We can begin to address two related difficult questions using our work in this paper, and we will see that quantum metric geometric ideas might continue to prove helpful in their study.

Ergodic actions of compact groups on unital C*-algebras have been of great interests, and it has been challenging to determine, for a given compact group, even compact Lie groups, what are all the C*-algebras on which the group acts. For instance, it remains unclear what the closure is to determine the closure of certain classes of quasi-Leibniz quantum compact metric spaces is at the time of this writing. When restricting our focus to finite dimensional algebras whose quantum metrics come from an ergodic action of compact groups, our present work will enable us to make significant advance is answering the closure question.

As a first simple observation, working with groups enables us to strengthen the conclusions of Theorem (3.6), by obtaining actions by full quantum isometries. The following proposition will be used to this end in our next theorem on group actions.
Proposition 3.9. If \( \alpha \) is an action of a compact group \( G \) on some quasi-Leibniz quantum compact metric space \( (\mathfrak{A}, L) \) by 1-Lipschitz automorphisms (i.e. \( L \circ \alpha^g \leq L \) for all \( g \in G \)), then for all \( g \in G \), the automorphism \( \alpha^g \) is a full quantum isometry of \( (\mathfrak{A}, L) \).

Proof. Let \( g \in G \). By assumption, for all \( a \in \mathfrak{sa}(\mathfrak{A}) \), we have \( L \circ \alpha^g(a) \leq L(a) \).

Thus for all \( a \in \text{dom}(L): \)

\[
L(a) = L(\alpha^g(\alpha^{-1}(a))) \leq L(\alpha^{-1}(a)).
\]

Now, since the above holds for all \( g \in G \), then we also have \( L(a) \leq L \circ \alpha^g(a) \) for all \( a \in \text{dom}(L) \). Hence \( L \circ \alpha^g \leq L \leq L \circ \alpha^g \) and thus \( L \circ \alpha^g = L \).

Consequently, \( \alpha^g \) is a full quantum isometry for any \( g \in G \). \( \square \)

As an important corollary of Theorem (3.6), we obtain:

Theorem 3.10. Let \( (G, \delta_G) \) be a compact metric group. Let \( (G_n, \delta_n)_{n \in \mathbb{N}} \) be a sequence of compact metric groups, converging to \((G, \delta_G)\). Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of positive numbers with \( \lim_{n \to \infty} \varepsilon_n = 0 \) and such that for each \( n \in \mathbb{N} \), there exists a \( \varepsilon_n \)-near isometric isomorphism \((s_n, \tau_n)\) from \((G_n, \delta_n)\) to \((G, \delta_G)\).

Let \( D : G \to [0, \infty) \) be a continuous function. Let \( K : [0, \infty) \to [0, \infty) \) be continuous with \( K(0) = 0 \).

Let \( C \) be a nonempty class of \( F \)-quasi-Leibniz quantum compact metric spaces for some admissible function \( F \), and let \( T \) be a class of tunnels compatible with \( C \). Let \( (A_n, L_n)_{n \in \mathbb{N}} \) be a sequence in \( C \), such that for each \( n \in \mathbb{N} \), there exists an action \( \alpha_n \) by Lipschitz automorphisms of \( G_n \) on \( A_n \), where:

\[
\forall n \in \mathbb{N}, \ g \in G_n \quad L_n \circ \alpha_n^g \leq D(s_n(g))L_n,
\]

and

\[
\forall n \in \mathbb{N}, \ g, g' \in G_n, \ a \in \mathfrak{sa}(A_n) \quad \left\| \alpha_n^g(a) - \alpha_n^{g'}(a) \right\|_{\mathfrak{A}_n} \leq K(\delta_n(g, g'))L_n(a).
\]

If there exists \((A, L) \in C\) such that:

\[
\lim_{n \to \infty} \Lambda^*_T((A_n, L_n), (A, L)) = 0
\]

then there exists a strongly continuous action \( \alpha \) of the group \( G \) by Lipschitz automorphisms such that for all \( g \in G \):

\[
L \circ \alpha^g \leq D(g)L,
\]

and for all \( g \in G \):

\[
\liminf_{n \to \infty} \text{mkf}_L \left( \alpha_n^\delta(g) \right) \leq \text{mkf}_L \left( \alpha^g \right) \leq \limsup_{n \to \infty} \text{mkf}_L \left( \alpha_n^\delta(g) \right),
\]

while for all \( g, h \in G \), and \( a \in \text{dom}(L) \), we have:

\[
\left\| \alpha^g(a) - \alpha^h(a) \right\|_{\mathfrak{A}} \leq L(a)K(\delta_G(g, h)).
\]

Furthermore:

1. If \( D \leq 1 \) then for all \( g \in G \), the \(*\)-automorphism \( \alpha^g \) is a full quantum isometry.

2. If for all \( n \in \mathbb{N} \), the action \( \alpha_n \) of \( G_n \) on \( A_n \) is ergodic, then the action \( \alpha \) of \( G \) on \( \mathfrak{A} \) is also ergodic.
Proof. As a corollary of Theorem (3.6), there exists an action of $G$ on $\mathfrak{A}$ by Lipschitz *-endomorphisms, only noting that we can apply Theorem (2.13) to obtain an action of $G$ by *-automorphisms.

By Proposition (3.9), if $D \leq 1$ then in fact, $\alpha^g$ is a full quantum isometry of $G$. Let $\mathfrak{A}$, $L$.

We now turn to the ergodicity property.

**Step 1.** We assume, for the rest of this proof, that for all $n \in \mathbb{N}$, the action $\alpha^n$ of $G_n$ on $\mathfrak{A}_n$ is ergodic, namely that $\{a \in \mathfrak{A} : \forall g \in G^n \alpha^g(a) = a\} = C\Lambda_n$. Our goal is to prove that $\alpha$ is also ergodic.

For this and the next few steps, we will need to be more specific about the construction of the action $\alpha$ provided by Theorem (2.13).

We start this proof at Step 4 of the proof of our main theorem (2.13). Thus, we have chosen a tunnel $\tau_n$ in $T$ from $\mathfrak{A}$, $L$ to $\mathfrak{A}_n$, $L_n$ with:

$$\chi(\tau_n) \leq \Lambda_T((\mathfrak{A}_n, L_n), (\mathfrak{A}, L)) + \frac{1}{n+1},$$

and we let $\gamma_n = \tau_n^{-1}$ for all $n \in \mathbb{N}$. To simplify notations, we have extracted a subsequence from $(\mathfrak{A}_n, L_n)_{n \in \mathbb{N}}$ which we still denote by $(\mathfrak{A}_n, L_n)_{n \in \mathbb{N}}$ such that for all $g \in G$, $a \in \text{dom}(L)$ and $l \geq L(a)$:

$$\lim_{n \to \infty} \text{Haus}_\mathbb{R}[ \{f_{n,g}(1/l, 1), \{\alpha^g(a)\} \} ] = 0,$$

where $f_{n,g}(1/l, 1)$ is a shorthand for $f_{\tau_n, \tau_n^a} \{\alpha^n(a)\}$. $\gamma_n$. We now turn to the ergodicity property.

**Step 1.** We assume, for the rest of this proof, that for all $n \in \mathbb{N}$, the action $\alpha^n$ of $G_n$ on $\mathfrak{A}_n$ is ergodic, namely that $\{a \in \mathfrak{A} : \forall g \in G^n \alpha^g(a) = a\} = C\Lambda_n$. Our goal is to prove that $\alpha$ is also ergodic.

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$$\chi(\tau_n) \leq \Lambda_T((\mathfrak{A}_n, L_n), (\mathfrak{A}, L)) + \frac{1}{n+1},$$

and we let $\gamma_n = \tau_n^{-1}$ for all $n \in \mathbb{N}$. To simplify notations, we have extracted a subsequence from $(\mathfrak{A}_n, L_n)_{n \in \mathbb{N}}$ which we still denote by $(\mathfrak{A}_n, L_n)_{n \in \mathbb{N}}$ such that for all $g \in G$, $a \in \text{dom}(L)$ and $l \geq L(a)$:

$$\lim_{n \to \infty} \text{Haus}_\mathbb{R}[ \{f_{n,g}(1/l, 1), \{\alpha^g(a)\} \} ] = 0,$$

where $f_{n,g}(1/l, 1)$ is a shorthand for $f_{\tau_n, \tau_n^a} \{\alpha^n(a)\}$. $\gamma_n$. We now turn to the ergodicity property.

**Step 2.** It is sufficient to show that if $a \in \text{dom}(L)$ and $\alpha^g(a) = a$ for all $g \in G$, then $a \in \text{R} \mathfrak{A}$.

Indeed, should this be the case, then, first, for all $g \in G$ we have $L(\alpha^g(a)) \leq L(a)$ so $\alpha^g(a) \in \{b \in \mathfrak{A} : L(b) \leq L(a), \|b\|_{\mathfrak{A}} \leq \|a\|_{\mathfrak{A}}\}$. As $L$ is an L-semi-norm, the latter set is compact for $\|\cdot\|_{\mathfrak{A}}$. Thus, $E(a) = \int_G \alpha^g(a) d\lambda(g)$ lies in the same set, and in particular, it lies in dom $(L)$.

Since $\alpha^g(E(a)) = E(a)$ for all $g \in G$, we conclude that $E(a) \in \text{R} \mathfrak{A}$, by the assumption of our step.

Let $a \in \mathfrak{A}$ such that $\alpha^g(a) = a$ for all $g \in G$, so $\|a - E(a)\|_{\mathfrak{A}} = 0$. By density of dom $(L)$, there exists $(a_n)_{n \in \mathbb{N}}$ in dom $(L)$ converging for $\|\cdot\|_{\mathfrak{A}}$ to $a$. 


By continuity, the function $g \in G \mapsto \|\alpha^g(a_n) - \alpha^h(a)\|_\mathfrak{A}$ is pointwise convergent to 0, and it is bounded. Hence by Lebesgues’ dominated convergence theorem (as $\lambda$ is a bounded measure, so bounded functions are integrable), we conclude:

$$\|E(a_n) - E(a)\|_\mathfrak{A} \leq \int_G \|\alpha^g(a_n) - \alpha^g(a)\| \, d\lambda(g) \quad \text{as} \quad n \to \infty.$$  

Hence $E(a) \in \mathbb{R}1_{\mathfrak{A}}$ since $(E(a_n))_{n \in \mathbb{N}} \in \mathbb{R}1_{\mathfrak{A}}$ and $\mathbb{R}1_{\mathfrak{A}}$ is closed in $\mathfrak{A}$. As $a = E(a)$, we conclude that $a \in \mathbb{R}1_{\mathfrak{A}}$.

Last, if $a \in \mathfrak{A}$ and for all $g \in G$, we have $\alpha^g(a) = a$, then for all $g \in G$ we have $\alpha^g(\frac{1}{n}(a + a^*) = \frac{1}{n}(a + a^*)$ and $\alpha^g(\frac{1}{n}(a - a^*)) = \frac{1}{n}(a - a^*)$, so:

$$\alpha^g(a) = \alpha^g\left(\frac{1}{2}(a + a^*) + \frac{1}{2n}(a - a^*)\right) = \frac{1}{2}(a + a^*) + \frac{1}{2n}(a - a^*) = a.$$  

This concludes our proof of our first step.

**Step 3.** If $a \in \text{dom}(L)$ and for all $g \in G$, we have $\alpha^g(a) = a$, then $a \in \mathbb{R}1_{\mathfrak{A}}$.

Let now $a \in \text{dom}(L)$ such that $\alpha^g(a) = a$ for all $g \in G$. Our goal is to prove $a \in \mathbb{R}1_{\mathfrak{A}}$. Without loss of generality, we shall assume that $L(a) \leq 1$.

Let $\varepsilon > 0$. As $K$ is continuous and $K(0) = 0$, there exists $\delta > 0$ such that if $t \in \mathbb{R}$, $|t| < \delta$ then $K(t) < \frac{\varepsilon}{7}$. Let $F \subseteq G$ be a finite, $\frac{\varepsilon}{7}$-dense subset of $G$.

Let $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, we have $\varepsilon_n < \frac{\varepsilon}{7}$. Note that by assumption, $\kappa_n(F)$ is $\delta$-dense: if $g \in G_n$ then $\varpi(g) \in G$, so there exists $h \in F$ such that $\delta_G(g, h) < \frac{\varepsilon}{7}$. Now:

$$\delta_{G_n}(g, \kappa_n(h)) \leq \delta_{G_n}(g, \kappa_n \circ \varpi_n(g)) + \delta_{G_n}(\kappa_n \circ \varpi_n(g), \kappa_n(h))$$  

$$\leq \varepsilon_n + \delta_{G}(\varpi_n(g), h) + \varepsilon_n$$  

$$\leq 2\varepsilon_n + \frac{\delta}{3}$$  

$$< \delta.$$  

Let $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $\chi(\tau_n) < \frac{\varepsilon}{7}$.

For each $g \in F$, let $N_g \in \mathbb{N}$ such that if $n \geq N_g$ and $c \in f_{n,g}(a|1,1)$ then $\|c - \alpha^g(a)\|_\mathfrak{A} < \frac{\varepsilon}{7}$. Let $M = \max\{N, N_g : g \in F\} \in \mathbb{N}$.

Let $n \geq M$.

Let $b_n \in \tau_{n,g}(a|1)$. For all $g \in G$, let $c_{n,g} \in \epsilon_{\tau_n}(\alpha_n^{\kappa_n(g)}(b)|1)$. Note that $c_{n,g} \in f_{n,g}(a|1,1)$. Thus by Proposition (1.26):

$$\begin{aligned}
\|b_n - \alpha_n^{\kappa_n(g)}(b_n)\|_\mathfrak{A} &\leq 2\chi(\tau_n) \cdot 1 + \|a - c_{n,g}\|_\mathfrak{A} \\
&\leq 2\varepsilon_n + \|a - \alpha^g(a)\|_\mathfrak{A} + \|\alpha^g(a) - c_{n,g}\|_\mathfrak{A} \\
&< \frac{3\varepsilon_n}{7},
\end{aligned}$$  

since $\|a - \alpha^g(a)\|_\mathfrak{A} = 0$.

Let now $g \in G_n$ and $h \in \kappa_n(F) \subseteq G_n$ such that $\delta_{G_n}(g, h) < \delta$. We estimate:

$$\begin{aligned}
\|b_n - \alpha_n^g(b_n)\|_\mathfrak{A} &\leq \|b_n - \alpha_n^g(b_n)\|_\mathfrak{A} + \|\alpha_n^h(b_n) - \alpha_n^g(b_n)\|_\mathfrak{A} \\
&\leq \frac{3\varepsilon_n}{7} + K(\delta_{G_n}(g, h))
\end{aligned}$$  


Hence (as $\lambda_n(G_n) = 1$):

$$
\|b_n - E_n(b_n)\|_{A_n} = \left\| b_n - \int_{G_n} \alpha_n^g(b_n) d\lambda_n(g) \right\|_{A_n} \\
\leq \int_{G_n} \|b_n - \alpha_n^g(b_n)\| d\lambda_n(g) \\
\leq 4\varepsilon / 7.
$$

(3.3)

To ease notations, let $E_n(b_n) = t_n 1_{A_n}$ for all $n \in \mathbb{N}$. The sequence $(t_n)_{n \in \mathbb{N}}$ is bounded since, for all $n \in \mathbb{N}$:

$$
|t_n| = \|E_n(b_n)\|_{A_n} \leq \|b_n\|_{A_n} \leq \chi(\tau_n) + \|a\|_{A_n} \leq 1 + \|a\|_{A_n} \text{ by Proposition (1.26)}.
$$

Let $f : \mathbb{N} \to \mathbb{N}$ be strictly increasing so that $(t_{f(n)})_{n \in \mathbb{N}}$ converges to some $t \in \mathbb{R}$. Let $N' \in \mathbb{N}$ such that, for $n \geq N'$, we have $|t - t_{f(n)}| < \frac{\varepsilon}{7}$.

Observe that by construction, $x1_{A_n} \in \tau_n (x1_{A_n}|l)$ for any $x \in \mathbb{R}$, $l \geq 0$ and $n \in \mathbb{N}$.

Let $n \geq \max\{M, N'\}$. We compute:

$$
\|a - t1_{A_n}\|_A \leq \|a - t_{f(n)}1_{A_n}\|_A + |t_{f(n)} - t| \\
< \|a - t_{f(n)}1_{A_n}\|_A + \frac{\varepsilon}{7} \\
\leq \frac{2\varepsilon}{7} + \|b_{f(n)} - t_{f(n)}1_{A_{f(n)}}\|_A_{f(n)} + \frac{\varepsilon}{7} \text{ by Proposition (1.26)}, \\
= \|b_{f(n)} - E_{f(n)}(b_{f(n)})\|_{A_{f(n)}} + \frac{3\varepsilon}{7} \\
\leq \frac{4\varepsilon}{7} + \frac{3\varepsilon}{7} = \varepsilon \text{ by Eq. (3.3)}.
$$

Hence $a \in R1_{A_n}$ as desired.

This concludes our proof. \qed

We now apply Theorem (3.10) to determining certain closures of quasi-Leibniz quantum compact metric spaces under the propinquity.

**Corollary 3.11.** Let $G$ be a compact group, of unit $e \in G$, endowed with a continuous length function $\ell$ such that:

$$
\forall g, h \in G \quad \ell(hgh^{-1}) = \ell(g).
$$

Let $\mathcal{G}$ be a collection of closed subgroups of $G$, closed for the Hausdorff distance $\text{Haus}_\ell$ induced by $\ell$.

Let $\mathcal{C}$ be the class of all quasi-Leibniz quantum compact metric spaces $(A, L)$ such that:

1. there exists $H \in \mathcal{G}$ and there exists a strongly continuous ergodic action $\alpha$ of $H$ on $A$, 

We have now applied Theorem (3.10) to determining certain closures of quasi-Leibniz quantum compact metric spaces under the propinquity.
(2) for all \(a \in \mathfrak{sa}(\mathfrak{A})\):
\[
L(a) \supset \sup \left\{ \frac{\|a - \alpha^g(a)\|_\mathfrak{A}}{\ell(g)} : g \in G \setminus \{e\} \right\},
\]

(3) for all \(g \in G\), the automorphism \(\alpha^g\) is 1-Lipschitz.

The class \(\mathcal{C}\) is not empty and closed for the dual propinquity.

Proof. Let \((\mathfrak{A}, L) \in \mathcal{C}\). For all \(g \in G\) and \(a \in \text{dom}(L)\), note that \(\|a - \alpha^g(a)\| \leq \ell(g)L(a)\) by construction; thus \(\|\alpha^g(a) - \alpha^h(a)\|_\mathfrak{A} \leq L(a)\ell(gh^{-1})\). Denoting \(\delta(g, h) = \ell(gh^{-1})\) for all \(g, h \in G\), we thus have:
\[
\|\alpha^g(a) - \alpha^h(a)\|_\mathfrak{A} \leq L(a)\delta(g, h).
\]

By Proposition (3.9), the automorphisms \(\alpha^g\) are full quantum isometries for \(L\) for all \(g \in G\).

Moreover, we note in passing that by construction:
\[
L(\alpha^h(a)) = \sup \left\{ \frac{\|\alpha^g(\alpha^h(a)) - \alpha^h(a)\|_\mathfrak{A}}{\ell(g)} : g \in G \setminus \{e\} \right\}
\]
\[
= \sup \left\{ \frac{\|\alpha^h(\alpha^{-1}g(a)) - a\|_\mathfrak{A}}{\ell(g)} : g \in G \setminus \{e\} \right\}
\]
\[
= \sup \left\{ \frac{\|\alpha^{-1}g(a) - a\|_\mathfrak{A}}{\ell(h^{-1}gh)} : g \in G \setminus \{e\} \right\} = L(a),
\]
so the class \(\mathcal{C}\) is not empty. (There exists an irreducible representation of \(G\) on some \(C^\infty\) as \(G\) is compact. Therefore, setting \(\alpha^g : M \in \mathfrak{M}_n \mapsto U^gM(U^g)^*\) for all \(g \in G\) defines an ergodic action of \(G\) on a full matrix algebra; as seen, \((\mathfrak{M}_n, a \mapsto \sup \left\{ \|a^n(a) - a\|_\mathfrak{M} : g \in G \setminus \{e\} \right\})\) lies in \(\mathcal{C}\).

Let now \((\mathfrak{A}_n, L_n)_{n \in \mathbb{N}}\) be any sequence in \(\mathcal{C}\) converging to some \((\mathfrak{B}, L_B)\) for \(\Lambda_T\). For each \(n \in \mathbb{N}\), there exists by assumption on \(\mathcal{C}\), a group \(H_n \in \mathcal{G}\), and an ergodic action \(\alpha_n\) of \(H_n\) on \(\mathfrak{A}_n\). Now, \(\mathcal{G}\) is closed for Haus, so it is compact (as \(G\) is compact). Thus there exists a subsequence \((H_{f(n)})_{n \in \mathbb{N}}\) converging to some closed subgroup \(H \in \mathcal{G}\) of \(G\) for Haus. By Proposition (3.4), the sequence \((H_{f(n)})_{n \in \mathbb{N}}\) converges to \(H\) in the sense of Definition (3.3) as well.

Thus, \((\mathfrak{A}_{f(n)}, L_{f(n)})_{n \in \mathbb{N}}\) meets the hypothesis of Theorem (3.10). Therefore, there exists an ergodic action \(\beta\) of \(G\) on \((\mathfrak{B}, L_B)\) by full quantum isometries, such that for all \(a \in \mathfrak{sa}(\mathfrak{B})\) and \(g \in G\), we have \(\|a - \beta^g(a)\|_\mathfrak{B} \leq \ell(g)L_{\mathfrak{B}}(a)\). Thus \((\mathfrak{B}, L_B) \in \mathcal{C}\), as desired. \(\square\)

In particular, we see that:

**Corollary 3.12.** Let \(\ell\) be a continuous length function on the \(d\)-torus \(T^d\). Let \(\mathcal{C}\) be the class quasi-Leibniz quantum compact metric spaces \((\mathfrak{A}, L)\) such that:

(1) \(\mathfrak{A}\) is finite dimensional,

(2) there exists an ergodic action \(\alpha\) of \(T^d\) on \(\mathfrak{A}\),
(3) for all \(a \in sa(\mathfrak{A})\), we have:
\[
L(a) = \sup \left\{ \frac{\|a - \alpha^g(a)\|_\mathfrak{A}}{\ell(g)} : g \in \mathbb{T}^d \setminus \{(1, \ldots, 1)\} \right\}.
\]

All the infinite dimensional quasi-Leibniz quantum compact metric spaces in the closure of \(\mathcal{C}\) are quantum tori.

Proof. Quantum tori are the only (infinite dimensional) C*-algebras carrying an ergodic action of the torus \(\mathbb{T}^d\). \(\square\)

An interesting direction for future research exploits Theorem (3.10) as follows. Let \(G\) be a compact Lie group. In general, it is a delicate problem to determine all C*-algebras on which \(G\) acts ergodically and classify them. However, we now propose an approach to this question. Indeed, \(G\) acts ergodically on many matrix algebras. Theorem (3.10) suggests that limits of such finite dimensional C*-algebras, endowed with the natural L-seminorm from the ergodic action of \(G\) and some fixed choice of length function on \(G\), will carry ergodic actions of \(G\). We make two observations. First, we conjecture that in fact, all these C*-algebras will lie in the closure of the class of these finite dimensional C*-algebras, for the propinquity. Second, we believe that as finite dimensional actions are of course tightly related to the representations of \(G\) (or projective representations, more generally), the question of finding all C*-algebras on which \(G\) acts ergodically will amount to understanding under what conditions these actions give rise to converging sequences for the propinquity.

A natural group to start exploring the application of our suggested technique is \(SU(2)\). Rieffel proved in [31] that all C*-algebras of continuous functions on co-adjoint orbits of \(SU(2)\) are obtained as limits, for the propinquity, of full matrix algebras carrying the adjoint action of \(SU(2)\) induced by well-chosen irreducible representations of \(SU(2)\). More generally, it would be natural to ask whether one can obtain all possible C*-algebras of continuous sections of the sort of matrix-bundles over \(SU(2)\)-homogeneous spaces, which are all the examples of C*-algebras with an ergodic action of \(SU(2)\) by [33]. As a prelude to this effort, we offer the following result. Note that in general, the closure of finite quasi-Leibniz quantum compact metric spaces contain all nuclear, quasi-diagonal quasi-Leibniz quantum compact metric space, and in particular, it contains all the quantum tori, including the non-type I variety [10]. It is very unclear what conditions in general are needed on a class of quantum metrics spaces built over type I C*-algebras to ensure that the closure of this class for the propinquity consists only of type I C*-algebras. Bringing together our present work with the powerful results in [33], we however obtain:

**Corollary 3.13.** Let \(\ell\) be a continuous length function over \(SU(2)\). If \(\mathcal{C}\) is the class of all finite dimensional quasi-Leibniz quantum compact metric spaces \((\mathfrak{A}, L)\) such that:

1. there exists a strongly ergodic action \(\alpha\) of \(SU(2)\) on \(\mathfrak{A}\) by *-automorphisms,
2. for all \(a \in sa(\mathfrak{A})\):
\[
L(a) = \sup \left\{ \frac{\|a - \alpha^g(a)\|_\mathfrak{A}}{\ell(g)} : g \in SU(2), g \neq I \right\}
\]

then the closure of \(\mathcal{C}\) consists of quasi-Leibniz quantum compact metric spaces of the form \((\mathfrak{A}, L)\) where \(\mathfrak{A}\) is type I.
Proof: This results from Corollary (3.12) and [33], whose corollary is that a C*-algebra admits an ergodic action of SU(2) only if it is type I. \(\square\)

Another application of Theorem (2.13) involves inductive limits of groups. Notably, the following result does not require that the inductive limit be locally compact. This is a step in a possibly new approach to construct actions of non-locally compact groups on C*-algebras.

An inductive sequence of groups, for our purpose, is given by a sequence of groups \((G_n)_{n \in \mathbb{N}}\) together with group monomorphisms \(\theta_n : G_n \to G_{n+1}\). We will assume all the groups \(G_n\) to be topological groups with a separable, first countable topology — hence, sequences will prove sufficient to test continuity.

The inductive limit group \(G = \lim_{\to n \to \infty} (G_n, \theta_n)\) is the quotient of \(\prod_{n \in \mathbb{N}} G_n\) by the equivalence relation:

\[(g_n)_{n \in \mathbb{N}} \equiv (g'_n)_{n \in \mathbb{N}} \iff \exists N \in \mathbb{N} \ \forall n \geq N \ \theta_n(g_n) = \theta_n(g'_n).
\]

Let \(N \in \mathbb{N}\). We define, for \(g \in G_N\), the element \(\eta_N(g)\) as the equivalence class in \(G\) of:

\[
\begin{cases}
\text{identity of } G_n \text{ if } n < N, \\
g \text{ if } n = N, \\
\theta_n(g) \text{ if } n > N.
\end{cases}
\]

Thus defined, the maps \(\eta_n\) are group monomorphisms for all \(n \in \mathbb{N}\). We henceforth identify \(G_n\) with \(\eta_n(G_n)\) in \(G\) for all \(n \in \mathbb{N}\). Thus \(G = \bigcup_{n \in \mathbb{N}} G_n\) with \(G_n \subseteq G_{n+1}\) for all \(n \in \mathbb{N}\).

The inductive topology on \(G\) is by definition, the final topology for the maps \(\theta_n\). With the above identification, it becomes the largest topology which makes, for all \(n \in \mathbb{N}\), the inclusion of \(G_n\) in \(G\) continuous. In particular, \(U \subseteq G\) is open in the inductive topology if and only if for all \(N \in \mathbb{N}\), the set \(U \cap G_N\) is open in \(G_N\).

In our next theorem, we work directly with inductive limits seen as increasing unions of groups with the inductive space topology. As a side note, this topology does not turn \(G\) to a topological group in general.

**Theorem 3.14.** Let \(F\) be an admissible function, \(C\) a nonempty class of \(F\)-quasi-Leibniz quantum compact metric spaces and \(T\) an appropriate class of tunnels for \(C\).

Let \(G\) be a group with unit \(e\) and \((G_n)_{n \in \mathbb{N}}\) an increasing sequence, for inclusion, of subgroups of \(G\) such that \(G = \bigcup_{n \in \mathbb{N}} G_n\). We assume that for all \(n \in \mathbb{N}\), the group \(G_n\) is a separable first-countable topological group and that the inclusion map \(G_n \hookrightarrow G_{n+1}\) is continuous. We endow \(G\) with the inductive limit topology.

Let \((\mathfrak{A}_n, L_n)_{n \in \mathbb{N}}\) be a sequence in \(C\), and for each \(n \in \mathbb{N}\), let \(\alpha_n\) be an action of \(G_n\) by \(\ast\)-automorphisms of \(\mathfrak{A}_n\).

Let \(D : G \to [0, \infty)\) be a continuous function such that for all \(n \in \mathbb{N}\) and for all \(g \in G_n\):

\[L_n \circ \alpha_n^g \leq D(g) L_n.\]

Let \(K : G \times G \to [0, \infty)\) be continuous, such that \(K(g, g) = 0\) for all \(g \in G\) and:

\[\forall n \in \mathbb{N}, g, h \in G_n, a \in \text{dom}(L_n) \quad \|\alpha_n^g(a) - \alpha_n^h(a)\|_{\mathfrak{A}_n} \leq L_n(a) K(h, g).
\]

If \((\mathfrak{A}, L) \in C\) satisfies:

\[
\lim_{n \to \infty} \Lambda^L_T((\mathfrak{A}_n, L_n), (\mathfrak{A}, L)) = 0,
\]

then...
then there exists an strongly continuous action $\alpha$ of $G$ on $\mathfrak{A}$ such that:

1. for all $g \in G$, we have $L \circ \alpha^g \leq D(g)L$,
2. for all $g, h \in G$, and $a \in \text{dom}(L)$, we have:
   $$\|\alpha^g(a) - \alpha^h(a)\|_{\mathfrak{A}} \leq L(a)K(\delta g, h),$$
3. the following inequalities hold for all $g \in G$, and for all $N \in \mathbb{N}$ such that $g \in G_N$:
   $$\liminf_{n \to \infty} mk\ell_{L_n}(\alpha^g_n) \leq mK(\alpha^g) \leq \limsup_{n \to \infty} nk\ell_{L_n}(\alpha^g_n),$$

Proof. For any $n \in \mathbb{N}$, the group $G_n$ is assume separable, so it admits a dense countable subgroup $H_n$. If $U$ is nonempty open in $G$ then for some $N \in \mathbb{N}$, we have $U \cap G_N$ is open and not empty, and thus $H_N \cap U$ is not empty as well. Consequently, $H = \bigcup_{n \in \mathbb{N}} H_n$ is a dense subgroup $\bigcup_{n \in \mathbb{N}} G_n$. Notably, $H$ is countable.

Let $g \in H$. For all $n \in \mathbb{N}$, if $g \in H_n$ then write $\varphi_n^g = \alpha_n^g$; otherwise set $\varphi_n^g$ to be the identity of $\mathfrak{A}_n$.

If $g, g' \in H$, there exists $N \in \mathbb{N}$ such that $g, g' \in G_N$. For $n \geq N$ we then have:

$$\|\varphi_n^g \circ \varphi_n^{g'} - \varphi_n^{g'}\|_{\mathfrak{A}_n} = \|\varphi_n^g \circ \alpha_n^g - \alpha_n^g\|_{\mathfrak{A}_n} = 0$$

and thus, we meet Theorem (2.13) hypothesis (noting that $mk\ell_{L_n}(\alpha^g_n) = 0$ for all $n \in \mathbb{N}$ and $\alpha_n$ are actions by *-automorphisms). We thus conclude that there exists an action $\alpha$ of $H$ on $\mathfrak{A}$ by *-automorphisms such that in particular:

$$\|\alpha^h(a) - \alpha^g(a)\|_{\mathfrak{A}} \leq L(a)K(h, g),$$

reasoning in the same manner as in the proof of Theorem (3.6).

Let now $g \in G$, and $N \in \mathbb{N}$ such that $g \in G_N$. Since $H_N$ is dense in $G_N$, there exists a sequence $(g_k)_{k \in \mathbb{N}}$ in $H_N$ such that $\lim_{k \to \infty} g_k = g$. Thus in particular, if $a \in \text{dom}(L)$ then:

$$\|\alpha^{g_k}(a) - \alpha^g(a)\|_{\mathfrak{A}} \leq L(a)K(g_k, g) \xrightarrow{p,q} L(a)K(g, g) = 0$$

and thus $(\alpha^{g_k}(a))_{k \in \mathbb{N}}$ is Cauchy in $sa(\mathfrak{A})$, which is complete. It thus converges.

Note that moreover, if we had chosen $N' \in \mathbb{N}$ for which $g \in G_{N'}$ and some other sequence $(h_k)_{k \in \mathbb{N}}$ converging in $G_{N'}$ to $g$, then, without loss of generality, we may assume $N' \geq N$; the sequence $(g_k)_{k \in \mathbb{N}}$ obtained by interleaving $(g_k)_{k \in \mathbb{N}}$ and $(h_k)_{k \in \mathbb{N}}$ converges to $g$ in $G_{N'}$ and thus as above, $(\alpha^{g_k}(a))_{k \in \mathbb{N}}$ must converge for all $a \in \text{dom}(L)$; thus both subsequences $(\alpha^{g_k}(a))_{k \in \mathbb{N}}$ and $(\alpha^{h_k}(a))_{k \in \mathbb{N}}$ have the same limit. We denote this limit by $\alpha^g(a)$.

A simple argument shows that $\alpha^g$ is a *-morphism and $\alpha^g \circ \alpha^g(a) = \alpha^{g^2}(a)$ for all $a \in \mathfrak{A}$, so $\alpha$ is an algebraic action of $G$ on $\mathfrak{A}$ by *-automorphisms.

By construction, we then note that for all $a \in \text{dom}(L)$, the following inequality holds:

$$\|\alpha^g(a) - \alpha^h(a)\|_{\mathfrak{A}} = \lim_{k \to \infty} \|\alpha^{g_k}(a) - \alpha^{h_k}(a)\|_{\mathfrak{A}} \leq \limsup_{k \to \infty} L(a)K(g_k, h_k) = L(a)K(g, h).$$

Let $a \in sa(\mathfrak{A})$. Let $\varepsilon > 0$. There exists $a' \in \text{dom}(L)$ such that $\|a - a'\|_{\mathfrak{A}} < \varepsilon$. 

Since $K$ is continuous, there exists an open subset $U$ of $G$ such that $e \in U$ and for all $g \in U$ we have $K(g, e) < \frac{\delta_{\text{diam}(\mathcal{A} \cdot \mathcal{L}(\alpha'))} + 1}{\alpha}$. We then have for all $g \in U$:

$$
\|\alpha^g(a) - a\|_\mathcal{A} = \|\alpha^g(a) - \alpha^e(a)\|_\mathcal{A}
\leq \|\alpha^g(a - a')\|_\mathcal{A} + \|\alpha^g(a') - \alpha^e(a')\|_\mathcal{A} + \|\alpha^e(a - a')\|_\mathcal{A}
\leq 2\|a - a'\|_\mathcal{A} + \varepsilon.
$$

We conclude that we have proven that for all $a \in \mathcal{A}$:

$$
\lim_{g \to e} \|\alpha^g(a) - a\|_\mathcal{A} = 0
$$

and thus it follows immediately that $\alpha$ is strongly continuous.

Now, the same general techniques used in the proof of Theorem (3.10) applies to complete the theorem — with $K$ is now a function over $G \times G$. □

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