Control of state and state entanglement with a single auxiliary subsystem

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Abstract

We present a strategy to control the evolution of a quantum system. The novel aspect of this protocol is the use of a single auxiliary subsystem. Two applications are given, one which allows for state preservation and another which controls the degree of entanglement of a given initial state.

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I. INTRODUCTION

Protocols for control and manipulation of quantum systems are essential for the development of quantum information theory [1]. Advances of this area can provide proper tools to avoid decoherence and to conduct quantum evolution to desirable results. Some examples of those strategies are: Quantum Zeno Effect (QZE) [2–4], Super Zeno Effect [5], strong continuous coupling [6, 7], Bang-Bang control [8, 9], etc.

In the present contribution we show a new possibility to control the evolution of a quantum system through successive interactions with a single auxiliary subsystem. These interactions are described as unitary evolutions (in a finite time period), that can stop or inhibit the evolution of the system of interest.

To make our results concrete, we present the protocol based on two examples. In the first one, we study a system of interest composed by two coupled qubits ($S_b$ and $S_c$) sharing one excitation. A third two level system ($S_a$) is the auxiliary subsystem responsible for the control. Interactions between $S_b$ and $S_a$ are inserted in the evolution of the system $S_b - S_c$ and control its dynamics. The quantity of such interactions, as well as their duration are parameters that allows for several forms of control. We performed an analytical calculation for the state vector of the global system after $N$ interactions with $S_a$. The Hamiltonian form of this system permits the mapping of the global evolution ($S_a - S_b - S_c$) in the real euclidian subspace, suggesting a geometrical interpretation for this dynamics. We also pointed out the differences between the present dynamics and the QZE.

In the second example we show how to control the entanglement dynamics presented in Ref. [10], through interactions with a single auxiliary subsystem. In Ref. [10], two initially entangled atoms undergo different time evolutions. One of them interacts with an electromagnetic mode in a cavity and the other one evolves freely. The dynamics is nondissipative and the entanglement oscillates. The introduction of an auxiliary subsystem, that interacts with the atom in the cavity, allows for the control of entanglement dynamics. An empirical implementation for this process may be realized with the experimental setup used in Ref. [11], where a two level atom interacts with two electromagnetic modes preserved in the same microwave cavity. In such empirical implementation, the two level atom and one of the modes in the cavity ($M_a$) compose the system of interest, and the second mode ($M_b$) acts as an auxiliary subsystem.
II. TWO QUBITS DYNAMICS

Let us consider the system of interest composed by two coupled qubits ($S_b - S_c$) and another qubit ($S_a$) as an auxiliary system. The hamiltonian that governs the interaction between $S_b$ and $S_c$ is given by:

\[
H_{bc} = \epsilon_a \langle 1_a | 1_a \rangle + \epsilon_b \langle 1_b | 1_b \rangle + \epsilon_c \langle 1_c | 1_c \rangle + I_a \otimes \hbar G_{bc} (\sigma^b_+ \sigma^c_+ + \sigma^b_- \sigma^c_-),
\]

where $\sigma_+ = |1\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$, $G_{bc}$ is the coupling coefficient, $I_a$ is the identity matrix on the subsystem $S_a$. The coefficients $\epsilon_a$, $\epsilon_b$ and $\epsilon_c$ are the eigenvalues of the free hamiltonian.

The goal is to control the dynamics in subsystem $S_b - S_c$ through interactions between the auxiliary qubit $S_a$ and $S_b$. The hamiltonian for these auxiliary interactions is

\[
H_{ab} = \epsilon_a \langle 1_a | 1_a \rangle + \epsilon_b \langle 1_b | 1_b \rangle + \epsilon_c \langle 1_c | 1_c \rangle + \hbar G_{ab} (\sigma^a_+ \sigma^b_+ + \sigma^a_- \sigma^b_-) \otimes I_c,
\]

where $I_c$ is the identity matrix on $S_c$ and $G_{ab}$ is the coupling coefficient between $S_a$ and $S_b$.

Suppose that $S_a - S_b - S_c$ share one excitation and $\epsilon_a = \epsilon_b = \epsilon_c$. Since the operators $H_{bc}$ and $H_{ab}$ preserve the excitation number, we can write in one excitation subspace the time evolution operators as

\[
\hat{U}_{bc}(\theta) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -i \sin \theta \\
0 & -i \sin \theta & \cos \theta
\end{bmatrix},
\]

and

\[
\hat{U}_{ab}(\phi) = \begin{bmatrix}
\cos \phi & -i \sin \phi & 0 \\
-i \sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

where $\theta = G_{bc} t_{bc}$, $\phi = G_{ab} t_{ab}$, $t_{bc}$ ($t_{ab}$) is the interaction time between $S_b$ and $S_c$ ($S_a$ and $S_b$).

The control of $S_b - S_c$ dynamics is induced by unitary operators ($\hat{U}_{ab}$) inserted $N$ times in the free evolution of $S_b - S_c$. The number of interventions and the durations of each one are the parameters that specify the control. The general expression for the state vector of the global system submitted to this control is given by:

\[
| \psi_N \rangle = (\hat{U}_{ab}(\phi) \hat{U}_{bc}(\theta))^N | \psi(0) \rangle,
\]

where $\theta = G_{bc} t_{bc}$, $\phi = G_{ab} t_{ab}$, $t_{bc}$ ($t_{ab}$) is the interaction time between $S_b$ and $S_c$ ($S_a$ and $S_b$).
where the time evolution of $S_b - S_c$ was divided by $N$ interactions with the auxiliary subsystem.

In the appendix we calculate the vector state $|\psi_N\rangle$ for a general initial state. Now, let us consider the initial state $|\psi(0)\rangle = |0_a\rangle |0_b, 1_c\rangle$ which goes through a quantum transition when submitted to the time evolution $\hat{U}_{bc}(\frac{\pi}{2}) |\psi(0)\rangle = |0_a\rangle |1_b, 0_c\rangle$. We show the inhibition of this transition through a sequence of unitary interactions with the single auxiliary subsystem.

As it is shown in the appendix, the global evolution of this system can be mapped on $R^3$, therefore we may represent a sequence of $N$ interactions with the auxiliary subsystem as:

$$\vec{r}_N = \left[ R_3 (\phi) R_1 \left( -\frac{\pi}{2N} \right) \right]^N \vec{r}(0) = \begin{bmatrix} ac (1 - \cos N\phi) + b \sin N\phi \\ bc (1 - \cos N\phi) - a \sin N\phi \\ (1 - c^2) \cos N\phi + c^2 \end{bmatrix}, \quad (6)$$

where $\phi \neq 2\pi$ and $\vec{r}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Taking the limit $N \to \infty$ we have

$$\lim_{N \to \infty} \vec{r}_N = \vec{r}(0),$$

as $a \to 0$, $b \to 0$ and $c \to 1$.

To give a geometrical interpretation of this effect consider a vector $\vec{r}$ in the euclidian subspace. In the present dynamics, the rotations $R_1 (-\theta)$ around the axis $Ox$ are clockwise. Therefore, when $\vec{r}$ has a positive $y$ component, the rotations $R_1 (-\theta)$ will reduce the $z$ projection of the vector, but when $\vec{r}$ has a negative $y$ component the rotation $R_1 (-\theta)$ will do the opposite, tending to compensate the previous decrease. The rotations $R_3 (\phi)$ move the vector through the subspaces where the $y$ component is positive and where it is negative. Therefore, when we study the dynamics of $\vec{r}$ inserted by rotations $R_3 (\phi)$ we notice that the decreasing of the $z$ projection, induced by $R_1 (-\theta)$ when $y > 0$, is compensated by the increasing of the same projection, also induced by $R_1 (-\theta)$, but when $y < 0$. Choosing the angle of the rotations $R_3 (\phi)$ and the number of interventions $N$ it is possible to preserve the projection $z$ of $\vec{r}$ or even freeze the dynamics of $\vec{r}$. In Fig.1 we explicitate another geometric point of view for the effect: the net effect of $R_1 (-\theta)$ and $R_3 (\phi)$ concerns a rotation around the vector $\hat{n}$, when the relation $\theta/\phi$ decreases, $\hat{n}$ gets closer to $\hat{e}_3$ and the state will be always closer to the initial one.
FIG. 1: The curves at the unitary sphere give the evolution of the terminal point of the vector $\vec{r}$ (represented with its initial point at the origin) starting at $\vec{r} = \hat{e}_3$ and subjected to $N$ rotations concerning $\mathbf{R}_3(\phi)\mathbf{R}_1(-\pi/2N) = \mathbf{R}(\hat{n},\varphi)$. The line segments explicitate the direction of each rotation axis $\hat{n}$. (a) $\phi = \pi/16$ and three values for $N$: $N = 10$, $N = 20$ and $N = 40$. Higher $N$ corresponds to $\hat{n}$ closer to $\hat{e}_3$ and to curves closer to the initial point. (b) $N = 20$ and three values for $\phi$: $\phi = \pi/32$, $\phi = \pi/16$ and $\phi = \pi/8$. Higher $\phi$ corresponds to $\hat{n}$ closer to $\hat{e}_3$ and to curves closer to the initial point.

For the initial vector $|\psi(0)\rangle = |0_a\rangle|0_b,1_c\rangle$, which corresponds to $(\vec{r}(0))^T = (0,0,1)$, the survival probability is

$$P_{001} = |\vec{r}_N \cdot \vec{r}(0)|^2. \quad (8)$$

The structure of hamiltonians $H_{bc}$ and $H_{ab}$ conserves the state vector in the subspace $\{|1_a,0_b,0_c\rangle, -i|0_a,1_b,0_c\rangle, |0_a,0_b,1_c\rangle\}$, allowing us to assume the presented geometrical interpretations. However, the control of quantum state through interactions with a single auxiliary system is not restricted to systems with such symmetry.

From a broader point of view, this effect occurs when the accumulation of interactions
with the same auxiliary subsystem changes the signal and reduces the absolute value of the quantum transition rate. In Fig. 2 we show the function $\frac{dP_{001}}{dt}$ of the system $S_a - S_b - S_c$. The oscillations of $\frac{dP_{001}}{dt}$ induce oscillations on the behavior of $P_{001}(t)$ (increase-decrease). Therefore, for appropriate values of the parameters ($N$ and interaction time with the auxiliary subsystem) we may have the increase of the function $P_{001}(t)$, at some time intervals, compensating the decrease in other time intervals, inducing (in average) preservation of the initial state, as it is shown in Fig. 2.

![Fig. 2: Probability $P_{001}(t)$ and quantum transition rate $\frac{dP_{001}}{dt}(t)$ with $\phi = \frac{\pi}{10}$, $\theta = \frac{\pi}{2N}$](image)

This inhibition of quantum transition induced by the increase on $N$ is similar to the discrete QZE, but is structured differently: the QZE as presented in Ref. [12], has a system of interest interacting with $N$ auxiliary (probe) subsystems. After each interaction the complete information about the occurrence of the quantum transition is available on the probe. This fact implies the cancelation of the transition rate after each interaction [13]. Therefore, we may characterize the interactions between system and probe as a measurement process (pre-measurement). The net effect of $N$ interactions between a single auxiliary system and the system of interest can not be characterize as a measurement process at all. Consequently for the later dynamics, the quantum transition rate is not necessarily null
Another difference between these two effects is that in the dynamics presented here a transition like $|1_a⟩|0_b,0_c⟩ → |0_a⟩|0_b,1_c⟩$ is intermediated by $S_b$, i.e., if only one auxiliary subsystem interacts $N$ times with the system of interest, the excitation present in $S_a$ may return to the subsystem $S_c$ inducing on $P_{001}$ larger values than the survival probability observed in the QZE. A comparison between these two probabilities is shown in Fig.3.

III. CONTROL OF THE ENTANGLEMENT DYNAMICS

Let us consider a system composed by two space-separated atoms, one of them is isolated (atom $B$) and the other (atom $A$) is coupled with an electromagnetic mode ($M_1$), as studied in Ref. [10]. If there is an initial entanglement between atoms $B$ and $A$ (or between atom $B$ and mode $M_1$) it changes over time, even if the initially entangled systems are not coupled.
This entanglement dynamics takes place because of the coupling between atom \( A \) and mode \( M_1 \). In this section we proposed a control of this entanglement dynamics through successive interactions with a single auxiliary subsystem.

Suppose the initial state is

\[ |\psi(0)\rangle = |g_a\rangle (\alpha|1_1, g_b\rangle + \beta|0_1, e_b\rangle), \]  

where the mode \( M_1 \) and the atom \( B \) are entangled. The coefficients \( \alpha \) and \( \beta \) give the intensity of the initial entanglement.

The coupling between atom \( A \) and \( M_1 \) can be described by the Jaynes-Cummings model. After a time evolution the vector state of the system has the form

\[ |\psi(t)\rangle = \alpha (\cos(gt)|g_a\rangle|1_1, g_b\rangle - i \sin(gt)|e_a\rangle|0_1, g_b\rangle) + \beta|0_1, e_b\rangle, \]  

where \( g \) is the coupling coefficient between atom \( A \) and \( M_1 \). We consider the atomic transition frequency of atoms \( A \) and \( B \) resonant with the frequency of \( M_1 \). Notice that when \( t = \frac{\pi}{2} \) we have the entanglement swap, the entanglement initially present in the subsystem \( M_1 \)-atom \( B \) is completely transferred to atom \( A \)-atom \( B \) subsystem.

To quantify and study the entanglement dynamics we write the concurrence [14] between \( M_1 \) and atom \( B \) (details of this calculation are in Ref. [10]).

\[ C_{M_1,B}(t) = 2|\alpha \beta \cos(gt)|. \]  

The concurrence oscillates assuming null values when \( gt = \frac{k\pi}{2} \), where \( k \) is an odd number.

It is possible to control the entanglement dynamics through \( N \) interactions with a single auxiliary subsystem. In the presentation of such control we consider that \( M_1 \) is in a cavity that supports two orthogonal modes \( (M_1 \text{ and } M_2) \). An empirical realization of this system (in a microwave cavity) is reported in Ref. [11].

The modes have different frequencies \( (\omega_1 - \omega_2 = \delta) \). The difference between atomic energy levels may be controlled by Stark effect. The detuning between the modes allows for the atom \( A \), when coupled to \( M_1 \) (\( M_2 \)), not to interact with \( M_2 \) (\( M_1 \)). Therefore it is possible to control the coupling time between atom \( A \) and the modes \( M_1 \) and \( M_2 \).

A sequence of interactions between the atom \( A \) and \( M_2 \), inserted into the time evolution shown in (9) and (10), is responsible for the control on the dynamics of entanglement between
atom $B$ and $M_1$. The evolution of the global system is composed by $N$ steps, each one in two stages. At the first stage the atom $A$ interacts with $M_1$ and at the second stage the atom $A$ interacts with $M_2$ (the second stage is responsible for the control).

In the first stage the time evolution is governed by the hamiltonian

$$H_1 = \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2 + \hbar \omega_1 |e_a\rangle\langle e_a| + \hbar \omega_1 |e_b\rangle\langle e_b| + \hbar g (\sigma^+_a a_1 + \sigma^-_a a_1^\dagger),$$

$$= H'_1 + \hbar \omega_1 |e_b\rangle\langle e_b|,$$  \hspace{1cm} (12)

and in the second stage by the hamiltonian

$$H_2 = \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2 + \hbar \omega_2 |e_a\rangle\langle e_a| + \hbar \omega_1 |e_b\rangle\langle e_b| + \hbar g (i \sigma^+_a a_2 - i \sigma^-_a a_2^\dagger),$$

$$= H'_2 + \hbar \omega_1 |e_b\rangle\langle e_b|,$$ \hspace{1cm} (13)

where $\sigma^+_a = |e_a\rangle\langle g_k|$, $\sigma^-_a = |g_k\rangle\langle e_a| \ (k = a, b)$. $H'_1$ and $H'_2$ act only on the subsystem composed by atom $A$, $M_1$ and $M_2$. Notice that the coupling coefficient in the second stage (in $H_2$) is imaginary, this is due to the orthogonal mode’s polarization.

The unitary time evolution operators for the first and second stages are:

$$e^{-iH_1 t/\hbar} = e^{-iH'_1 t/\hbar} e^{-i\omega_1 |e_b\rangle\langle e_b| t},$$ \hspace{1cm} (14)

Written on the basis $\{|0_1, g_a, 1_1\}, |0_1, e_a, 0_1\}, |1_1, g_a, 0_1\}$ the operators $e^{-iH'_1 t/\hbar}$ and $e^{-iH'_2 t/\hbar}$ assume the form:

$$e^{-iH'_1 t_1/\hbar} = e^{-i\omega_1 t_1} \begin{bmatrix} e^{i\delta t_1} & 0 & 0 \\ 0 & \cos(gt_1) & -i \sin(gt_1) \\ 0 & -i \sin(gt_1) & \cos(gt_1) \end{bmatrix},$$ \hspace{1cm} (15)

$$e^{-iH'_2 t_2/\hbar} = e^{-i\omega_2 t_2} \begin{bmatrix} \cos(gt_2) & -\sin(gt_2) & 0 \\ \sin(gt_2) & \cos(gt_2) & 0 \\ 0 & 0 & e^{-i\delta t_2} \end{bmatrix},$$ \hspace{1cm} (16)

The time evolution is given by

$$|\psi_N\rangle = \left(e^{-iH'_2 t_2/\hbar} e^{-iH'_1 t_1/\hbar}\right)^N |\psi(0)\rangle,$$ \hspace{1cm} (17)

and the initial vector state of the global system is
\[ |\psi(0)\rangle = |g_a\rangle (\alpha|1_1, g_b\rangle + \beta|0_1, e_b\rangle) |0_2\rangle, \] (18)

where \( M_2 \) (auxiliary subsystem) is prepared in the vacuum state.

After \( N \) steps the interaction time between the atom \( A \) and \( M_1 \) is \( T = N t_1 \). Let us consider \( T = N t_1 = \frac{\pi}{2g} \) (as it is in Fig.4). When there is no participation of the auxiliary subsystem in the dynamics the concurrence between the mode \( M_1 \) and the atom \( B \) is null, because the entanglement is completely transferred to the subsystem \( \text{atom}_A-\text{atom}_B \).

Therefore, with the intervention of the auxiliary subsystem the entanglement dynamics is inhibited, \( i.e. \) the enhancement on the number of interactions with the auxiliary subsystem allows for the preservation of the concurrence initial value, even when the total time for the interaction of the system (atom \( A \), atom \( B \) and \( M_1 \)) is \( T = N t_1 = \frac{\pi}{2g} \) (time of the entanglement swap), as it is shown in Fig.4.

To summarize, we have presented a new strategy to control the evolution of a quantum system. This strategy requires only unitary interactions between the system of interest and a single auxiliary subsystem. We discuss two examples for the strategy application. In the first one the auxiliary subsystem controls excitation exchange between two qubits. In the second example, a single auxiliary subsystem is used to control the entanglement of a system composed by an isolated atom and a Jaynes-Cummings atom.

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**APPENDIX: CALCULATION OF \( |\psi_N\rangle \)**

In this section we calculate explicitly the vector state \( |\psi_N\rangle \). Let us focus on the action of the unitary matrix \( \tilde{U}_{ab}(\phi)\tilde{U}_{bc}(\theta) \) on the vector \( |\xi\rangle \) written in the basis \( \{ |1_a, 0_b, 0_c\rangle , -i |0_a, 1_b, 0_c\rangle , |0_a, 0_b, 1_c\rangle \} \):

\[
\tilde{U}_{ab}(\phi)\tilde{U}_{bc}(\theta) |\xi\rangle = \tilde{U}_{ab}(\phi)\tilde{U}_{bc}(\theta) \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 \cos \phi - (\xi_2 \cos \theta + \xi_3 \sin \theta) \cos \phi \\ \xi_1 \sin \phi + (\xi_2 \cos \theta + \xi_3 \sin \theta) \cos \phi \\ \xi_3 \cos \theta - \xi_2 \sin \theta \end{bmatrix}. \quad (A.1)
\]
FIG. 4: Concurrence between mode $M_1$ and atom $B$ with $\delta = 8 \times 10^5 s^{-1}$, $g = 1.5 \times 10^4 s^{-1}$ and $gt_2 = \frac{\pi}{2}$.

This action may be mapped as a rotation on the real euclidian subspace, by choosing conveniently the rotation matrix and assuming that $\xi_1$, $\xi_2$ and $\xi_3$ (components of $|\xi\rangle$) are real.

Defining

$$ R_1 (\varphi) \equiv R (\hat{e}_1, \varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{bmatrix}, \tag{A.2} $$

$$ R_3 (\varphi) \equiv R (\hat{e}_3, \varphi) = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{A.3} $$

and the vector

$$ \vec{r} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \tag{A.4} $$

we notice that the action of the matrix $R_1 (-\theta)$ and $R_3 (\phi)$, over $\vec{r}$ produces the same effect.
on its components as the action of \( \hat{U}_{bc}(\theta) \) and \( \hat{U}_{ab}(\phi) \) on the components of \( |\xi\rangle \), i.e.,

\[
R_3(\phi)R_1(-\theta) \hat{r} = \begin{bmatrix}
\xi_1 \cos \phi - (\xi_2 \cos \theta + \xi_3 \sin \theta) \sin \phi \\
\xi_1 \sin \phi + (\xi_2 \cos \theta + \xi_3 \sin \theta) \cos \phi \\
\xi_3 \cos \theta - \xi_2 \sin \theta
\end{bmatrix}.
\]

(A.5)

The orthogonal matrix \( R_3(\phi)R_1(-\theta) \) may be written as

\[
R_3(\phi)R_1(-\theta) = R(\hat{n}, \varphi) = \exp(\varphi \hat{n} \cdot \hat{J}),
\]

(A.6)

where \( \hat{n} \cdot \hat{J} \) is the generator of rotations around the axis defined by the unitary vector \( \hat{n} = a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3 \), \( \varphi \) is the angle of rotation around \( \hat{n} \), \( \hat{J} = \hat{e}_1 J_1 + \hat{e}_2 J_2 + \hat{e}_3 J_3 \) where \( J_1, J_2 \) and \( J_3 \) are the generators of rotation around the axis \( Ox, Oy \) and \( Oz \), respectively.

\[
J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

(A.7)

In order to calculate the action of \( \left( \hat{U}_{ab}(\phi)\hat{U}_{bc}(\theta) \right)^N \) over the state vector \( |\psi_0\rangle \), we use the mapping of the operator \( \hat{U}_{ab}(\phi)\hat{U}_{bc}(\theta) \) on the orthogonal matrix \( R_3(\phi)R_1(-\theta) \) and the identity:

\[
[R_3(\phi)R_1(-\theta)]^N = \left[ \exp(\varphi \hat{n} \cdot \hat{J}) \right]^N = \exp(N \varphi \hat{n} \cdot \hat{J}).
\]

(A.8)

The calculation of \( [R_3(\phi)R_1(-\theta)]^N \) is reduced now to finding the axis \( \hat{n} \) and the angle \( \varphi \).

a. Finding \( \hat{n} \) and \( \varphi \)

Explicitly,

\[
\varphi \hat{n} \cdot \hat{J} = \varphi \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}.
\]

(A.9)

After some algebra we have

\[
\left( \varphi \hat{n} \cdot \hat{J} \right)^{2n} = \varphi^{2n} (-1)^{n+1} \left( \hat{n} \cdot \hat{J} \right)^2, \quad \left( \varphi \hat{n} \cdot \hat{J} \right)^{2n+1} = \varphi^{2n+1} (-1)^n \left( \varphi \hat{n} \cdot \hat{J} \right),
\]

(A.10)
and we may write

\[
\exp \left( \varphi \hat{n} \cdot \vec{J} \right) = 1 + (1 - \cos \varphi) \left( \hat{n} \cdot \vec{J} \right)^2 + \sin \varphi \left( \hat{n} \cdot \vec{J} \right).
\]  

(A.11)

The calculation of \( \left( \hat{n} \cdot \vec{J} \right)^2 \) is given by:

\[
\left( \hat{n} \cdot \vec{J} \right)^2 = \begin{bmatrix}
a^2 - 1 & ab & ac \\
ab & b^2 - 1 & bc \\
ac & bc & c^2 - 1
\end{bmatrix}.
\]  

(A.12)

Substituting equations (A.9) and (A.12) in (A.11) we get

\[
\exp \left( \varphi \hat{n} \cdot \vec{J} \right) = \begin{bmatrix}
(1 - a^2) \cos \varphi + a^2 & ab (1 - \cos \varphi) - c \sin \varphi & ac (1 - \cos \varphi) + b \sin \varphi \\
ab (1 - \cos \varphi) + c \sin \varphi & (1 - b^2) \cos \varphi + b^2 & bc (1 - \cos \varphi) - a \sin \varphi \\
ac (1 - \cos \varphi) - b \sin \varphi & bc (1 - \cos \varphi) + a \sin \varphi & (1 - c^2) \cos \varphi + c^2
\end{bmatrix}.
\]  

(A.13)

Comparing (A.13) with the product

\[
\mathbf{R}_3 (\varphi) \mathbf{R}_1 (-\theta) = \begin{bmatrix}
\cos \phi - \sin \phi \cos \theta & \text{sin} \phi \cos \phi \cos \theta & \text{cos} \phi \cos \theta \\
\text{cos} \phi \sin \phi \cos \theta & \text{cos} \phi \cos \phi \cos \theta & \text{sin} \phi \cos \theta \\
0 & -\sin \theta & \text{cos} \theta
\end{bmatrix},
\]  

(A.14)

we get the following expressions for \( \sin \varphi \), \( \cos \varphi \) and components of \( \hat{n} \):

\[
\sin \varphi = 2 \cos \phi \frac{\theta}{2} \sqrt{\sin^2 \phi + \cos^2 \phi + \sin^2 \phi},
\]  

(A.15a)

\[
\cos \varphi = \frac{\cos \phi + \cos \theta + \cos \phi \cos \theta - 1}{2},
\]  

(A.15b)

\[
a = -\frac{\sin \theta (\cos \phi + 1)}{4 \cos \theta \cos \phi \sqrt{\sin^2 \phi + \sin^2 \phi}},
\]  

(A.16a)

\[
b = -\frac{\sin \phi \sin \theta}{4 \cos \theta \cos \phi \sqrt{\sin^2 \phi + \sin^2 \phi}},
\]  

(A.16b)

\[
c = \frac{\sin \phi (\cos \theta + 1)}{4 \cos \theta \cos \phi \sqrt{\sin^2 \phi + \sin^2 \phi}}.
\]  

(A.16c)
This result allows us to calculate \([R_3(\phi)R_1(-\theta)]^N\).

\[
[R_3(\phi)R_1(-\theta)]^N = \left[\exp\left(\varphi \hat{n} \cdot \vec{J}\right)\right]^N = \exp\left(N \varphi \hat{n} \cdot \vec{J}\right).
\] (A.17)

The exponential \(\exp\left(N \varphi \hat{n} \cdot \vec{J}\right)\) will have the form identical to the matrix (A.13), but with the substitution of \(\varphi\) by \(N \varphi\). The components \(a, b\) and \(c\), as well as \(\sin \varphi\) and \(\cos \varphi\) are shown in (A.15) and (A.16).

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