Online Learning for Unknown Partially Observable MDPs

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Abstract

Solving Partially Observable Markov Decision Processes (POMDPs) is hard. Learning optimal controllers for POMDPs when the model is unknown is harder. Online learning of optimal controllers for unknown POMDPs, which requires efficient learning using regret-minimizing algorithms that effectively tradeoff exploration and exploitation, is even harder, and no solution exists currently. In this paper, we consider infinite-horizon average-cost POMDPs with unknown transition model, though known observation model. We propose a natural posterior sampling-based reinforcement learning algorithm (POMDP-PSRL) and show that it achieves $O(T^{2/3})$ regret where $T$ is the time horizon. To the best of our knowledge, this is the first online RL algorithm for POMDPs and has sub-linear regret.

1. Introduction

Reinforcement learning (RL) considers the sequential decision making problem of an agent in an unknown environment with the goal of minimizing the total cost. The agent faces a fundamental exploration-exploitation trade-off: should it exploit the available information to minimize the cost or should it explore the environment to gather more information for future decisions? Maintaining a proper balance between exploration and exploitation is a fundamental challenge in RL and is measured with the notion of cumulative regret: the difference between the cumulative cost of the learning algorithm and that of the best policy.

The problem of balancing exploration and exploitation in RL has been successfully addressed for MDPs and algorithms with near optimal regret bounds are known (Bartlett & Tewari, 2009; Jaksch et al., 2010; Ouyang et al., 2017b; Azar et al., 2017; Fruit et al., 2018; Jin et al., 2018; Abbasi-Yadkori et al., 2019b; Zhang & Ji, 2019; Zanette & Brunskill, 2019; Hao et al., 2020; Wei et al., 2020; 2021). MDPs assume that the state is perfectly observable by the agent and the only uncertainty is about the underlying dynamics of the environment. However, in many real-world scenarios such as robotics, healthcare and finance, the state is not fully observed by the agent and only a partial observation is available. These scenarios are modeled by Partially Observable Markov Decision Processes (POMDPs). In addition to the uncertainty in the environment dynamics, the agent has to deal with the uncertainty about the underlying state. It is well known (Kumar & Varaiya, 2015) that introducing an information or belief state (a posterior distribution over the states given the history of observations and actions) allows the POMDP to be recast as an MDP over the belief state space. The resulting algorithm requires a posterior update of the belief state which needs the transition and observation model to be fully known. This presents a significant difficulty when the model parameters are unknown. Thus, managing the exploration-exploitation trade-off for POMDPs is a significant challenge and to the best of our knowledge, no online RL algorithm is known that has sub-linear regret.

In this paper, we consider infinite-horizon average-cost POMDPs with finite states, actions and observations. The underlying state transition dynamics is unknown, though we assume the observation kernel to be known. We propose a Posterior Sampling Reinforcement Learning algorithm (POMDP-PSRL) and prove that it achieves a Bayesian regret bound of $O(T^{2/3})$ under some technical assumptions, where $T$ is the time horizon of agent-environment interactions. The POMDP-PSRL algorithm is a natural extension of the TSDE algorithm for MDPs (Ouyang et al., 2017b) with two main differences. First, in addition to the posterior distribution on the environment dynamics, the algorithm maintains a posterior distribution on the underlying state. Second, since the state is not fully observable, the agent cannot keep track of the number of visits to state-action pairs, a quantity that is crucial in the design of algorithms for tabular MDPs. Instead, we introduce a notion of pseudo count and carefully handle its relation with the true counts to obtain sub-linear regret. To the best of our knowledge, POMDP-PSRL is the first online RL algorithm for POMDPs with sub-linear regret.
1.1. Related Literature

We review the related literature in two main domains: efficient exploration for MDPs, and learning in POMDPs.

Efficient exploration in MDPs. To balance the exploration and exploitation, two general techniques are used in the basic tabular MDPs: optimism in the face of uncertainty (OFU), and posterior sampling. Under the OFU technique, the agent constructs a confidence set around the system parameters, selects an optimistic parameter associated with the minimum cost from the confidence set, and takes actions with respect to the optimistic parameter. This principle is widely used in the literature to achieve optimal regret bounds (Bartlett & Tewari, 2009; Jaksch et al., 2010; Azar et al., 2017; Fruit et al., 2018; Jin et al., 2018; Zhang & Ji, 2019; Zanette & Brunskill, 2019; Wei et al., 2020).

An alternative technique to encourage exploration is posterior sampling (Thompson, 1933). In this approach, the agent maintains a posterior distribution over the system parameters, samples a parameter from the posterior distribution, and takes action with respect to the sampled parameter (Strens, 2000; Osband et al., 2013; Fonteneau et al., 2013; Gopalan & Mannor, 2015; Ouyang et al., 2017b). In particular, (Ouyang et al., 2017b) proposes TSDE, a posterior sampling-based algorithm for the infinite-horizon average-cost MDPs.

Extending these results to the continuous state MDPs has been recently addressed with general function approximation (Osband & Van Roy, 2014; Dong et al., 2020; Ayoub et al., 2020; Wang et al., 2020), or in the special cases of linear function approximation (Abbasi-Yadkori et al., 2019a,b; Jin et al., 2020; Hao et al., 2020; Wei et al., 2021; Wang et al., 2021), and Linear Quadratic Regulators (Ouyang et al., 2017a; Dean et al., 2018; Cohen et al., 2019; Mania et al., 2019; Simchowitz & Foster, 2020; Lale et al., 2020a). In general, POMDPs can be formulated as continuous state MDPs by considering the belief as the state. However, computing the belief requires the knowledge of the model parameters and thus unobserved in the RL setting. Hence, learning algorithms for continuous state MDPs cannot be directly applied to POMDPs.

Learning in POMDPs. To the best of our knowledge, the only existing work with sub-linear regret in POMDPs is (Azizzadenesheli et al., 2017). However, their definition of regret is not with respect to the optimal policy, but with respect to the best memoryless policy (a policy that maps the current observation to an action). With our natural definition of regret, their algorithm suffers linear regret. Other learning algorithms for POMDPs either consider linear dynamics (Lale et al., 2020b; Tsiamis & Pappas, 2020) or do not consider regret (Shani et al., 2005; Ross et al., 2007; Poupart & Vlassis, 2008; Cai et al., 2009; Liu et al., 2011; 2013; Doshi-Velez et al., 2013; Katt et al., 2018; Azizzadenesheli et al., 2018) and are not directly comparable to our setting.

2. Preliminaries

An infinite-horizon average-cost Partially Observable Markov Decision Process (POMDP) can be specified by \((S, A, \theta, C, O, \eta)\) where \(S\) is the state space, \(A\) is the action space, \(C : S \times A \to [0, 1]\) is the cost function, and \(O\) is the set of observations. Here \(\eta : S \to \Delta_O\) is the observation kernel, and \(\theta : S \times A \to \Delta_S\) is the transition kernel such that \(\eta(\theta(\cdot|s)) = \mathbb{P}(a_t = a|s_t = s)\) and \(\theta(s'|s, a) = \mathbb{P}(s_{t+1} = s'|s_t = s, a_t = a)\) where \(a_t \in O, s_t \in S\) and \(a_t \in A\) are the observation, state and action at time \(t = 1, 2, 3, \ldots\). Here, for a finite set \(X, \Delta_X\) is the set of all probability distributions on \(X\). We assume that the state space, the action space and the observations are finite with size \(|S|, |A|, |O|\), respectively.

Let \(\mathcal{F}_t\) be the information available at time \(t\) (prior to action \(a_t\)), i.e., the sigma algebra generated by the history of actions and observations \(a_1, a_1, \ldots, a_{t-1}, a_t, o_t\) and let \(\mathcal{F}_{t+}\) be the information after choosing action \(a_t\). Unlike MDPs, the state is not observable by the agent and the optimal policy cannot be a function of the state. Instead, the agent maintains a belief \(h_t(\cdot;\theta) \in \Delta_S\) given by \(h_t(s;\theta) := \mathbb{P}(s_t = s|\mathcal{F}_t;\theta)\), as a sufficient statistic for the history of observations and actions. Here we use the notation \(h_t(\cdot;\theta)\) to explicitly show the dependency of the belief on \(\theta\). After taking action \(a_t\) and observing \(a_{t+1}\), the belief \(h_t\) can be updated as

\[
h_{t+1}(s';\theta) = \frac{\sum \eta(\theta(\cdot|s'))(s'|s, a)h_t(s;\theta)}{\sum \sum \eta(\theta(\cdot|s'))(s'|s, a)h_t(s;\theta)}.
\]

This update rule is compactly denoted by \(h_{t+1}(\cdot;\theta) = \tau(h_t(\cdot;\theta), \theta)\), with the initial condition

\[
h_1(s;\theta) = \frac{\eta(\theta(\cdot|s))(s;\theta)}{\sum \eta(\theta(\cdot|s))(s;\theta)},
\]

where \(h(\cdot;\theta)\) is the distribution of the initial state \(s_1\) (denoted by \(s_1 \sim \theta\)). A deterministic stationary policy \(\pi : \Delta_S \to A\) maps a belief to an action. The long-term average cost of a policy \(\pi\) can be defined as

\[
J_\pi(h;\theta) := \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ C(s_t, \pi(h_t(\cdot;\theta))) \right].
\]

Let \(J(h;\theta) := \inf_\pi J_\pi(h, \theta)\) be the optimal long-term average cost that in general may depend on the initial state distribution \(h\), though we will assume it is independent of the initial distribution \(h\) (and thus denoted by \(J(\theta)\)), and the following Bellman equation holds:

Assumption 1 (Bellman optimality equation). There exist \(J(\theta) \in \mathbb{R}\) and a bounded function \(V(\cdot;\theta) : \Delta_S \to \mathbb{R}\) such
that for all \( b \in \Delta_s \),
\[
J(\theta) + v(b; \theta) = \\
\min_{a \in A} \left\{ c(b, a) + \sum_{o \in O} P(o | b, a; \theta) v(b'; \theta) \right\},
\]
(3)

where \( v \) is called the relative value function, \( b' = \tau(b, a, o; \theta) \) is the updated belief, \( c(b, a) := \sum_s C(s, a | b) \) is the expected cost, and \( P(o | b, a; \theta) \) is the probability of observing \( o \) in the next step, conditioned on the current belief \( b \) and action \( a \), i.e.,
\[
P(o | b, a; \theta) = \sum_{s' \in S} \sum_{s \in S} \eta(o | s') \theta(s' | s, a) b(s).
\]
(4)

Various conditions are known under which Assumption 1 holds, e.g., when the MDP is weakly communicating (Bertsekas, 2017). Note that if Assumption 1 holds, the policy \( \pi^* \) that minimizes the right hand side of (3) is the optimal policy. More precisely,

**Proposition 1.** Suppose Assumption 1 holds. Then, the policy \( \pi^*(h; \theta) : \Delta_s \to A \) given by
\[
\arg\min_{a \in A} \left\{ c(b, a) + \sum_{o \in O} P(o | b, a; \theta) v(b'; \theta) \right\}
\]
is the optimal policy with \( J_* (h; \theta) = J(\theta) \) for all \( h \in \Delta_s \).

The proof can be found in Appendix B.

Note that if \( v \) satisfies the Bellman equation, so does \( v + \text{const} \). Therefore, without loss of generality, and since \( v \) is bounded, we can assume that \( \inf_{b \in \Delta_s} v(b; \theta) = 0 \) and define the span of a POMDP as \( \text{sp} (\theta) := \sup_{b \in \Delta_s} v(b; \theta) \). Throughout the paper, we consider the class \( \Theta_H \) of POMDPs that satisfy Assumption 1 and have \( \text{sp} (\theta) \leq H \) for all \( \theta \in \Theta_H \).

**The learning protocol.** We consider the problem of an agent interacting with an unknown randomly generated POMDP \( \theta_* \), where \( \theta_* \in \Theta_H \) is randomly generated according to the probability distribution function \( f (\cdot) \). After the initial generation of \( \theta_* \), it remains fixed, but unknown to the agent. The agent interacts with the POMDP \( \theta_* \) in \( T \) steps. Initially, the agent starts from state \( s_1 \) that is randomly generated according to the conditional probability mass function \( h_* (\cdot; \theta_*) \). At time \( t = 1, 2, 3, \cdots, T \), the agent observes \( o_t \sim \eta (\cdot | s_t) \), takes action \( a_t \), and suffers cost of \( C(s_t, a_t) \). The environment, then determines the next state \( s_{t+1} \) which is chosen from the probability distribution \( \theta_* (\cdot; s_t, a_t) \). Note that although the cost function \( C \) is assumed to be known, the agent cannot observe the value of \( C(s_t, a_t) \) since the state \( s_t \) is unknown to the agent. The goal of the agent is to minimize the cumulative Bayesian regret defined as
\[
R_T := \mathbb{E} \left[ \sum_{i=1}^T \left[ C(s_t, a_t) - J(\theta_*) \right] \right],
\]
(6)

where the expectation is with respect to the prior distribution \( f (\cdot) \) for \( \theta_* \), the prior distribution \( h_* (\cdot; \theta_*) \) for \( s_1 \), the randomness in the state transitions, and the randomness in the algorithm. The Bayesian regret is widely considered in the MDP literature (Osband et al., 2013; Gopalan & Mannor, 2015; Ouyang et al., 2017b,a).

### 3. The POMDP-PSRL Algorithm

We propose the POMDP-Posterior Sampling Reinforcement Learning (POMDP-PSRL) algorithm (Algorithm 1) that maintains a joint distribution on the unknown parameter \( \theta_* \), as well as the state \( s_t \). The algorithm takes the prior distributions \( h \) and \( f \) as input. At time \( t \), the agent maintains the posterior probability distribution function (pdf) \( f_t (\cdot) \) on the unknown parameter \( \theta_* \), as well as the posterior conditional probability mass function (pmf) \( h_t (\cdot; \theta) \) on the state \( s_t \) for \( \theta \in \Theta_H \). Upon taking action \( a_t \) and observing \( o_{t+1} \), the posterior distribution at time \( t + 1 \) can be updated by applying the Bayes’ rule as
\[
f_{t+1} (\theta) = \frac{\sum_{s, a} \eta(o_{t+1} | s') h(s_t | s, a) h_t (s_t; \theta) f_t (\theta)}{\int \sum_{s, a} \eta(o_{t+1} | s') h(s_t | s, a) h_t (s_t; \theta) f_t (\theta) d\theta},
\]
(7)

with the initial condition
\[
f_1 (\theta) = \frac{\sum_{s, a} \eta(o_1 | s) h(s_0 | s, a) f (\theta)}{\int \sum_{s, a} \eta(o_1 | s) h(s_0 | s, a) f (\theta) d\theta},
\]
(8)

Recall that \( h_t (\cdot; \theta) \) has a compact notation for (1). In the special case of perfect observation at time \( t \), \( h_t (s; \theta) = 1(s_t = s) \) for all \( \theta \in \Theta_H \) and \( s \in S \). Moreover, the update rule of \( f_{t+1} \) reduces to that of fully observable MDPs (see Eq. (4) of (Ouyang et al., 2017b)) in the special case of perfect observation at time \( t \) and \( t + 1 \).

Let \( n_t (s, a) = \sum_{\tau=1}^{t-1} 1(s_\tau = s, a_\tau = a) \) be the number of visits to state-action \( (s, a) \) by time \( t \). The number of visits \( n_t \) plays an important role in learning for MDPs (Jaksch et al., 2010; Ouyang et al., 2017b) and is one of the two criteria to determine the length of the episodes in the TSDE algorithm for MDPs (Ouyang et al., 2017b). However, in POMDPs, \( n_t \) is not \( F_{t-1} \)-measurable since the states are not observable. Instead, we define the pseudo-count \( \tilde{n}_t \) as
\[
\tilde{n}_t (s, a) := \max \{ \tilde{n}_{t-1} (s, a), \text{ceil} (n_{t} (s, a)) \},
\]
(9)

\[
\tilde{n}_t (s, a) := \mathbb{E} [ n_t (s, a) | F_{t-1} ] + 1, \]
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Algorithm 1 POMDP–PSRL

Input: \( f(\cdot), h(\cdot) \)

Initialization: \( t \leftarrow 1, t_k \leftarrow 0 \)

Observe \( o_t \) and compute \( f_t, h_t \) according to (8)

for episodes \( k = 1, 2, \cdots \) do

\[ T_{k-1} \leftarrow t - t_k \]

\( t_k \leftarrow t \)

Generate \( \theta_k \sim f_{t_k}(\cdot) \) and compute \( \pi_k(\cdot) = \pi^*(\cdot; \theta_k) \)

from (5)

while \( t \leq t_k + T_{k-1} \) and \( \tilde{m}_t(s, a) \leq 2\tilde{m}_{t_k}(s, a) \) for all \((s, a) \in S \times A\) do

Choose action \( a_t = \pi_k(h_t(\cdot; \theta_k)) \) and observe \( o_{t+1} \)

Update \( f_{t+1}, h_{t+1} \) according to (7)

\( t \leftarrow t + 1 \)

end while

end for

Similar to the TSDE algorithm for fully observable MDPs, POMDP–PSRL algorithm proceeds in episodes. In the beginning of episode \( k \), POMDP \( \theta_k \) is sampled from the posterior distribution \( f_{t_k} \) where \( t_k \) denotes the start time of episode \( k \). The optimal policy \( \pi^*(\cdot; \theta_k) \) is then computed and used during the episode. Note that the input of the policy is \( h_t(\cdot; \theta_k) \). The intuition behind such a choice (as opposed to the belief \( b_t(\cdot) := \int h_t(\cdot; \theta)f_t(\theta)d\theta \)) is that during episode \( k \), the agent treats \( \pi^*(\cdot; \theta_k) \) and adopts the optimal policy with respect to it. Consequently, the input to the policy should also be the conditional belief with respect to the sampled \( \theta_k \).

The length of episode \( k \) is denoted by \( T_k \) and is determined by two criteria. The first criterion triggers when \( T_k = T_{k-1} + 1 \). The second criterion triggers when the pseudo count \( \tilde{m}_t(s, a) \) for any state-action pair is doubled during episode \( k \). These criteria are previously introduced in the TSDE algorithm (Ouyang et al., 2017b) except that TSDE uses the true count \( n_t \) rather than \( \tilde{m}_t \).

The main result of the paper relies on the following technical assumptions on the parameter estimation.

Assumption 2. The true conditional belief \( h_t(\cdot; \theta_\ast) \) and the approximate conditional belief \( h_t(\cdot; \theta_k) \) satisfy

\[
\sum_s \mathbb{E}[h_t(s; \theta_\ast) - h_t(s; \theta_k) | F_t, \theta_k] \leq \frac{K_1(|S|, |A|, |O|, t)}{\sqrt{T_k}},
\]

with probability at least \( 1 - \delta \), for any \( \delta \in (0, 1) \). Here \( K_1(|S|, |A|, |O|, t) \) is a constant that is polynomial in its input parameters and \( t \) hides the logarithmic dependency on \(|S|, |A|, |O|, T, \delta\).

Note that at time \( t = t_k \), the left hand side of (10) is indeed zero due to the property of posterior sampling. Assumption 2 states that the gap between conditional posterior function for the sampled POMDP \( \theta_k \) and the true POMDP \( \theta_\ast \) decreases with episodes as better approximation of the true POMDP is available. Moreover, in the case of perfect observation, \( h_t(s; \theta) = \mathbb{1}(s_t = s) \), this assumption is clearly satisfied. There has been recent work on computation of approximate information states as required in Assumption 2 (Subramanian et al., 2020).

Assumption 3. There exists an \( \mathcal{F}_t \)-measurable estimator \( \hat{\theta}_t : S \times A \rightarrow \Delta_S \) such that

\[
\sum_s [\hat{\theta}_t(s'|s, a) - \hat{\theta}_t(s|s, a)] \leq \frac{K_2(|S|, |A|, |O|, t)}{\sqrt{\max\{1, \tilde{m}_t(s, a)\}}}
\]

with probability at least \( 1 - \delta \), for any \( \delta \in (0, 1) \), uniformly for all \( t = 1, 2, 3, \cdots, T \), where \( K_2(|S|, |A|, |O|, t) \) is a constant that is polynomial in its input parameters and \( t \) hides the logarithmic dependency on \(|S|, |A|, |O|, T, \delta\).

In the case of perfect observation, \( \tilde{m}_t(s, a) = n_t(s, a) \) and Assumption 3 is satisfied for \( \hat{\theta}_t(s'|s, a) = \frac{n_t(s, a, s')}{\tilde{m}_t(s, a)} \) where \( n_t(s, a, s') \) denotes the number of visits to \( s, a \) such that the next state is \( s' \) by time \( t \) (Jaksch et al., 2010; Ouyang et al., 2017b). There has been extensive work on estimation of transition dynamics of MDPs, e.g., (Grunewalder et al., 2012). Now, we state the main result of the paper.

Theorem 1. Under Assumptions 1, 2 and 3, the regret of POMDP–PSRL algorithm is upper bounded as

\[
R_T \leq \tilde{O}(HK_2(|S|, |A|)|T|^{2/3}),
\]

where \( K_2 := K_2(|S|, |A|, |O|, t) \) in Assumption 3.

The exact constants are known (see proof and Appendix A) though we have hidden the dependence above.

4. Analysis

We now prove Theorem 1. The following key lemma states that the pseudo count \( \tilde{m}_t \) cannot be too smaller than the true count \( n_t \).

Lemma 2. For any \( \alpha \in [0, 1] \) and \((s, a) \in S \times A\),

\[
\mathbb{P}(\tilde{m}_t(s, a) < \alpha n_t(s, a)) \leq \alpha.
\]

Proof. For any \( \alpha \in [0, 1] \),

\[
\tilde{m}_t(s, a) \mathbb{1}(\alpha n_t(s, a) > \tilde{m}_t(s, a)) \leq \alpha n_t(s, a).
\]

By taking conditional expectation with respect to \( F_{(t-1)}^+ \) from both sides and the fact that \( \mathbb{E}[n_t(s, a)|F_{(t-1)}^+] = \tilde{m}_t(s, a) \), we have

\[
\tilde{m}_t(s, a) \mathbb{E}[\mathbb{1}(\alpha n_t(s, a) > \tilde{m}_t(s, a))|F_{(t-1)}^+] \leq \alpha \tilde{m}_t(s, a).
\]
We claim that
\[
E\left[\mathbb{1}(\alpha n_t(s, a) > \tilde{m}_t(s, a)) | \mathcal{F}_{(t-1)_+}\right] \leq \alpha, \text{ a.s. (14)}
\]
If this claim is true, taking another expectation from both sides completes the proof.

To prove the claim, let \( \Omega_0, \Omega_+ \) be the subsets of the sample space where \( \tilde{m}_t(s, a) = 0 \) and \( \tilde{m}_t(s, a) > 0 \), respectively. We consider these two cases separately: (a) On \( \Omega_0 \) one can divide both sides of (13) by \( \tilde{m}_t(s, a) \) and reach (14) by the fact that \( \tilde{n}_t(s, a)/\tilde{m}_t(s, a) \leq 1 \). (b) Note that by definition \( \tilde{n}_t(s, a) \leq \tilde{m}_t(s, a) \), which implies that \( \tilde{n}_t(s, a) = 0 \) on \( \Omega_0 \). Thus, \( n_t(s, a) \mathbb{1}(\Omega_0) = 0 \) almost surely (this is because \( E[n_t(s, a) \mathbb{1}(\Omega_0)] = E[n_t(s, a) \mathbb{1}(\Omega_0)|\mathcal{F}_{(t-1)_+}] = E[\tilde{n}_t(s, a) \mathbb{1}(\Omega_0)] = 0 \). Therefore,
\[
\mathbb{1}(\Omega_0)\mathbb{1}(\alpha n_t(s, a) > \tilde{m}_t(s, a)) = 0, \text{ a.s.},
\]
which implies
\[
\mathbb{1}(\Omega_0)E\left[\mathbb{1}(\alpha n_t(s, a) > \tilde{m}_t(s, a)) | \mathcal{F}_{(t-1)_+}\right] = 0, \text{ a.s.},
\]
which means on \( \Omega_0 \), the left hand side of (14) is indeed zero, almost surely. Hence, the claim is true. \( \square \)

The parameter \( \alpha \) will be tuned later to balance two terms and achieve \( \tilde{O}(T^{2/3}) \) regret bound (see Lemma 5).

A key property of posterior sampling is that conditioned on the information at time \( t \), the sampled \( \theta_t \) and the true \( \theta_s \) have the same distribution (Osband et al., 2013; Russo & Van Roy, 2014). Since the episode start time \( t_k \) is a stopping time with respect to the filtration \( (\mathcal{F}_t)_{t \geq 1} \), we use a stopping time version of this property:

**Lemma 3** (Lemma 2 in (Ouyang et al., 2017b)). For any measurable function \( g \) and any \( \mathcal{F}_{t_k} \)-measurable random variable \( X \), we have
\[
E[f(\theta_t, X)] = E[f(\theta_s, X)].
\]

We now proceed with the formal proof of Theorem 1.

**Proof.** First, note that \( E[C(s_t, a_t)|\mathcal{F}_t, \theta_s] = c(h_t(\cdot; \theta_s), a_t) \) for any \( t \geq 1 \). Thus, we can write:
\[
R_T = E\left[\sum_{t=1}^{T} C(s_t, a_t) - J(\theta_s)\right]
\]
\[
= E\left[\sum_{t=1}^{T} \left[ c(h_t(\cdot; \theta_s), a_t) - J(\theta_s) \right] \right].
\]
During episode \( k \), by the Bellman equation for the sampled POMDP \( \theta_k \) and that \( \theta_t = \pi(\cdot|t|; \theta_k) \), we can write:
\[
c(h_t(\cdot; \theta_k), a_t) - J(\theta_k) = v(h_t(\cdot; \theta_k); \theta_k) - \sum_o P(o|h_t(\cdot; \theta_k), a_t; \theta_k)v(b'; \theta_k),
\]
where \( b' = \tau(h_t(\cdot; \theta_k), a_t, \alpha, \theta_k) \). Using this equation, we proceed by decomposing the regret as
\[
R_T = E\left[\sum_{t=1}^{T} \left[ c(h_t(\cdot; \theta_s), a_t) - J(\theta_s) \right] \right]
\]
\[
= E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_k+1-1} \left[ c(h_t(\cdot; \theta_s), a_t) - J(\theta_s) \right] \right]
\]
\[
= R_0 + R_1 + R_2 + R_3,
\]
where \( K_T \) is the number of episodes up to time \( T \), \( t_k \) is the start time of episode \( k \), and
\[
R_0 := E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_k+1-1} \left[ J(\theta_k) - J(\theta_s) \right] \right],
\]
\[
R_1 := E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_k+1-1} \left[ v(h_t(\cdot; \theta_k); \theta_k) - v(h_{t+1}(\cdot; \theta_k); \theta_k) \right] \right],
\]
\[
R_2 := E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_k+1-1} \left[ c(h_t(\cdot; \theta_s), a_t) - c(h_{t+1}(\cdot; \theta_k), a_t) \right] \right],
\]
\[
R_3 := E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_k+1-1} \left[ c(h_t(\cdot; \theta_s), a_t) - c(h_{t+1}(\cdot; \theta_k), a_t) \right] \right].
\]
\( R_0 \) is bounded by \( E[K_T] \) (see Lemma 6) by using the property of posterior sampling. \( R_1 \) is bounded by \( HE[K_T] \) with a telescoping argument (see Lemma 7). \( R_3 \) is bounded by \( K_1K_2(\max\{1, \tilde{m}u(s_t, a_t)\}) \) where \( K_1 := K_1(|S|, |A|, |O|, \epsilon) \) is the constant in Assumption 3 (see Lemma 8). To bound \( R_2 \), we show that (see Lemma 4),
\[
R_2 \leq \tilde{R}_2 + H K_1 E\left[\sum_{k=1}^{K_T} \frac{T_k}{\sqrt{t_k}}\right] + 1,
\]
where
\[
\tilde{R}_2 := HE\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_k+1-1} \left[ \theta_s(s'|s_t, a_t) - \theta_k(s'|s_t, a_t) \right] \right].
\]
\( \tilde{R}_2 \) is the dominating term in the final \( \tilde{O}(T^{2/3}) \) regret bound and can be bounded by \( H + 12HK_2(\max\{1, \tilde{m}u(s_t, a_t)\}) \) where \( K_2 := K_2(|S|, |A|, |O|, \epsilon) \) is the constant in Assumption 3. The detailed proof can be found in Lemma 5. However, we sketch the main steps of the proof here. By Assumption 3, one can show that
\[
R_2 \leq \tilde{O}\left(E\left[\sum_{t=1}^{T} \frac{H K_2}{\max\{1, \tilde{m}u(s_t, a_t)\}}\right]\right).
\]
Now, let $E_2$ be the event that $\tilde{m}_t(s, a) \geq \alpha t_t(s, a)$ for all $s, a$. Note that by Lemma 2 and union bound, $P(E_2^c) \leq |S| |A| \alpha$. Thus,

$$R_2 \leq \tilde{O}\left(\sum_{t=1}^{T} \frac{HK_2}{\sqrt{\max\{1, m_t(s_t, a_t)\}}}(\mathbb{I}(E_2) + \mathbb{I}(E_2^c))\right)$$

$$\leq \tilde{O}\left(HK_2 \sum_{t=1}^{T} \frac{K_2}{\sqrt{\max\{1, m_t(s_t, a_t)\}}} \right) + K_2 |S| |A| \alpha \tau$$

Algebraic manipulation of the inner summation yields

$$\tilde{R}_2 \leq \tilde{O}\left(HK_2 \sqrt{|S| |A| T} + HK_2 |S| |A| T \alpha\right).$$

Optimizing over $\alpha$ implies $\tilde{R}_2 = \tilde{O}(HK_2 (|S| |A| T)^{2/3})$. Substituting upper bounds for $R_0, R_1, R_2$ and $R_3$, we get

$$R_T = R_0 + R_1 + R_2 + R_3 \leq (1 + H)E[K_T] + 12HK_2 (|S| |A| T)^{2/3} + (H + 1)K_1 \sum_{k=1}^{K_T} \frac{T_k}{\sqrt{t_k}} + 2 + H.$$ 

From Lemma 9, we know that $E[K_T] = \tilde{O}(\sqrt{|S| |A| T})$ and $\sum_{k=1}^{K_T} \frac{T_k}{\sqrt{t_k}} = \tilde{O}(\sqrt{|S| |A| \sqrt{T}}).$ Therefore,

$$R_T \leq \tilde{O}(HK_2 (|S| |A| T)^{2/3}).$$

4.1. Auxiliary Lemmas

**Lemma 4.** The term $R_2$ can be bounded as

$$R_2 \leq H E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \sum_{s, a_t} \left| \theta_k(s'|s_t, a_t) - \theta_k(s'|s_t, a_t) \right| \right] + HK_1 E\left[\sum_{k=1}^{K_T} \frac{T_k}{\sqrt{t_k}} \right] + 1,$$

where $K_1 := K_1(|S|, |A|, |O|, \varepsilon)$ in Assumption 2.

**Proof.** Recall that

$$R_2 = H \tilde{O}\left(\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \left[ v(h_{t+1}(\cdot; \theta_k); \theta_k) - v(h_{t+1}(\cdot; \theta_k); \theta_k) \mathbb{P}(t_{t+1} = o|F_t, \theta_s, \theta_k) \right] \right),$$

where $h' = \tau(h_t(\cdot; \theta_k), \alpha_t; \theta_k)$ and $h_{t+1}(\cdot; \theta_k) = \tau(h_t(\cdot; \theta_k), \alpha_t; \alpha_{t+1} = \theta_k)$. Conditioned on $F_t, \theta_s, \theta_k$, the only random variable in $h_{t+1}(\cdot; \theta_k)$ is $o_{t+1}$ ($a_t = \pi^*(h_t(\cdot; \theta_k); \theta_k)$ is measurable with respect to the sigma algebra generated by $F_t, \theta_s$). Therefore,

$$\mathbb{E}[v(h_{t+1}(\cdot; \theta_k); \theta_k)|F_t, \theta_s, \theta_k] = \sum_{o \in O} v(h'; \theta_k) \mathbb{P}(o_{t+1} = o|F_t, \theta_s, \theta_k).$$

We proceed by showing that $\mathbb{P}(o_{t+1} = o|F_t, \theta_s, \theta_k) = \mathbb{P}(o|h_t(\cdot; \theta_s), a_t; \theta_k)$ by total law of probability and that $\mathbb{P}(o_{t+1} = o|s_{t+1} = s', F_t, \theta_s, \theta_k) = \eta(o|s')$, we can write

$$\mathbb{P}(o_{t+1} = o|F_t, \theta_s, \theta_k) = \sum_{s'} \eta(o|s') \mathbb{P}(s_{t+1} = s'|F_t, \theta_s, \theta_k).$$

Note that

$$\mathbb{E}[v(h_{t+1}(\cdot; \theta_k); \theta_k)|s_{t+1} = s'|F_t, \theta_s, \theta_k] = \sum_{s} \mathbb{P}(s_{t+1} = s'|s_t = s, F_t, \theta_s, \theta_k) \mathbb{P}(s_t = s|F_t, \theta_s) \mathbb{E}[v(h_{t+1}(\cdot; \theta_k); \theta_k)|F_t, \theta_s, \theta_k].$$

Thus, $\mathbb{P}(o_{t+1} = o|F_t, \theta_s, \theta_k) = \sum_{s, s'} \eta(o|s') \theta_s(s'|s_t, a_t) h_t(s; \theta_s) = \mathbb{P}(o|h_t(\cdot; \theta_s), a_t; \theta_s).$}

Combining (16) with (15) and substituting into $R_2$, we get

$$R_2 = \tilde{O}(HK_2 (|S| |A| T)^{2/3}).$$

Recall that for any $\theta \in \Theta_H$, $\mathbb{P}(o|h_t(\cdot; \theta), a_t; \theta) = \sum_{s'} \eta(o|s') \sum_{s} \theta(s'|s_t, a_t) h_t(s; \theta_s)$. Thus, $R_2 = \tilde{O}(HK_2 (|S| |A| T)^{2/3}).$
have
\[
\mathbb{E}\left[ v(h'; \theta_k) \sum_{s} \theta_s(s'|s, a_t) h_t(s; \theta_s) \mid \mathcal{F}_t, \theta_s, \theta_k \right] = v(h'; \theta_k) \mathbb{E}\left[ \theta_s(s'|s_t, a_t) \mid \mathcal{F}_t, \theta_s, \theta_k \right].
\] (18)

Similarly, for the second term on the right hand side of (17), we have
\[
\mathbb{E}\left[ v(h'; \theta_k) \sum_{s} \theta_k(s'|s, a_t) h_t(s; \theta_s) \mid \mathcal{F}_t, \theta_s, \theta_k \right] = v(h'; \theta_k) \mathbb{E}\left[ \theta_k(s'|s_t, a_t) \mid \mathcal{F}_t, \theta_s, \theta_k \right].
\] (19)

Replacing (18), (19) into (17) and using the tower property of conditional expectation, we get
\[
R_2 = \mathbb{E}\left[ \sum_{k=1}^{K_T} \sum_{t=1}^{t_k-1} \left[ \sum_{s} \sum_{a} v(h'; \theta_k) \eta(o|s') \left( \theta_s(s'|s_t, a_t) - \theta_k(s'|s_t, a_t) \right) \right] \right]
+ \mathbb{E}\left[ \sum_{k=1}^{K_T} \sum_{t=1}^{t_k-1} \left[ \sum_{s} \sum_{a} v(h'; \theta_k) \eta(o|s') \sum_{s} \theta_k(s'|s, a_t) \left( h_t(s; \theta_s) - h_t(s; \theta_k) \right) \right] \right].
\] (20)

Since \( \sup_{b \in \Delta_S} v(b, \theta_k) \leq H \) and \( \sum_{a} \eta(o|s') = 1 \), the inner summation for the first term on the right hand side of (20) can be bounded as
\[
\sum_{a \in O} v(h'; \theta_k) \eta(o|s') \left( \theta_s(s'|s_t, a_t) - \theta_k(s'|s_t, a_t) \right)
\leq H \left| \theta_s(s'|s_t, a_t) - \theta_k(s'|s_t, a_t) \right|.
\] (21)

Let \( K_1 := K_1(|S|, |A|, |O|, \iota) \) be the constant in Assumption 2 and define event \( E_1 \) as the successful event of Assumption 2 where \( \sum_{s} \mathbb{E}\left[ h_t(s; \theta_s) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right] \leq \frac{\sqrt{C}}{\delta} \) happens. To bound the second term of (20), we can write
\[
\sum_{s} \sum_{a \in O} v(h'; \theta_k) \eta(o|s') \sum_{s} \theta_k(s'|s, a_t) \left| \mathbb{E}\left[ h_t(s; \theta_s) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right] \right|
\leq H \sum_{s} \mathbb{E}\left[ h_t(s; \theta_s) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right]
= H \mathbb{E}\left[ h_t(s; \theta_s) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right] \left| (1(E_1) + 1(E_1^c)) \right|
\leq H \frac{K_1}{\sqrt{t_k}} + 2H \mathbb{E}(E_1^c),
\] (22)
where the first inequality is by the fact that \( \sup_{b \in \Delta_S} v(b, \theta_k) \leq H \), \( \sum_{a} \eta(o|s') = 1 \) and \( \sum_{s} \theta_k(s'|s, a_t) = 1 \). Note that \( \mathbb{P}(E_1^c) \leq \delta \) by Assumption 2. Thus, substituting (21) and (22) into (20) and using the tower property of conditional expectation, we obtain
\[
R_2 \leq H \mathbb{E}\left[ \sum_{k=1}^{K_T} \sum_{t=1}^{t_k-1} \left[ \sum_{s} \theta_s(s'|s_t, a_t) - \theta_k(s'|s_t, a_t) \right] \right]
+ HK_1 \mathbb{E}\left[ \sum_{k=1}^{K_T} \frac{T_k}{\sqrt{t_k}} \right] + 2HT\delta.
\]

Choosing \( \delta = 1/(2HT) \) completes the proof.

**Lemma 5.** The term \( R_2 \) can be bounded as
\[
R_2 \leq H + 12HK_2(|S||A|T)^{2/3},
\] where \( K_2 := K_2(|S|, |A|, |O|, \iota) \) in Assumption 3.

**Proof.** Recall that \( R_2 = \mathbb{E}\left[ \sum_{k=1}^{K_T} \sum_{t=1}^{t_k-1} \left[ \sum_{s} \theta_s(s'|s_t, a_t) - \theta_k(s'|s_t, a_t) \right] \right]. \) (23)

We proceed by bounding the inner term of the above equation. For notational simplicity, define \( z := (s, a) \) and \( z_t := (s_t, a_t) \). Let \( \theta_{t_k} \) be the estimator in Assumption 3 and define the confidence set \( B_k \) as
\[
B_k := \left\{ \theta \in \Theta_H : \sum_{s \in S} \left| \theta_s(z) - \hat{\theta}_k(s'|z) \right| \leq \frac{K_2}{\sqrt{\max\{1, \bar{m}_{t_k}(z)\}}} \right\} \forall z \in S \times A \}
\]
where \( K_2 := K_2(|S|, |A|, |O|, \iota) \) is the constant in Assumption 3. Note that \( B_k \) reduces to the confidence set used in (Jaksch et al., 2010; Ouyang et al., 2017b) in the case of perfect observation. By triangle inequality, the inner term in (23) can be bounded by
\[
\sum_{s} \left| \theta_s(z_t) - \hat{\theta}_{t_k}(s'|z_t) \right| \leq \sum_{s} \left| \theta_s(z_t) - \hat{\theta}_{t_k}(s'|z_t) \right| + \sum_{s} \left| \hat{\theta}_{t_k}(s'|z_t) - \hat{\theta}_{t_k}(s'|z_t) \right| \leq 2(1(\theta_s \notin B_k) + 1(\theta_k \notin B_k)) + \frac{2K_2}{\sqrt{\max\{1, \bar{m}_{t_k}(z_t)\}}}.
\]

Substituting this into (23) implies
\[
R_2 \leq 2H \mathbb{E}\left[ \sum_{k=1}^{K_T} \sum_{t=1}^{t_k-1} \left( 1(\theta_s \notin B_k) + 1(\theta_k \notin B_k) \right) \right]
+ 2HK_1 \mathbb{E}\left[ \sum_{k=1}^{K_T} \sum_{t=1}^{t_k-1} \frac{K_2}{\sqrt{\max\{1, \bar{m}_{t_k}(z_t)\}}} \right]. \quad (24)
\]

We need to bound these two terms separately.
Bounding the first term. For the first term we can write:

$$
E \left[ \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} (\mathbb{1}(\theta_s \notin B_k) + \mathbb{1}(\theta_k \notin B_k)) \right] = E \left[ \sum_{k=1}^{K_T} T_k (\mathbb{1}(\theta_s \notin B_k) + \mathbb{1}(\theta_k \notin B_k)) \right] \\
\leq T \varepsilon \left[ \sum_{k=1}^{K_T} \mathbb{1}(\theta_s \notin B_k) + \mathbb{1}(\theta_k \notin B_k) \right] \\
\leq T \sum_{k=1}^{K_T} \mathbb{1}(\theta_s \notin B_k) + \mathbb{1}(\theta_k \notin B_k) \\
\leq T \sum_{k=1}^{K_T} \mathbb{1}(\theta_s \notin B_k) + \mathbb{1}(\theta_k \notin B_k) \\
\leq \frac{1}{2}.
$$

(T5)

Bounding the second term. To bound the second term of (24), observe that by the second criterion of the algorithm in choosing the episode length, we have $2\bar{m}_{t_k}(z_t) \geq \bar{m}_t(z_t)$. Thus,

$$
E \left[ \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \frac{K_2}{\max(1, \bar{m}_{t_k}(z_t))} \right] \\
\leq E \left[ \sum_{t=1}^{T} \frac{\sqrt{2K_2}}{\max(1, \bar{m}_t(z_t))} \right] \\
= \sum_{t=1}^{T} \sum_{z} \mathbb{1}(z_t = z) \frac{\sqrt{2K_2}}{\max(1, \bar{m}_t(z_t))} \\
\leq \sum_{t=1}^{T} \sum_{z} \mathbb{1}(z_t = z) \frac{\sqrt{2K_2}}{\max(1, \bar{m}_t(z_t))} \\
\leq 3 \sum_{z} \sqrt{n_{T+1}(z)}.
$$

Since $\sum_z n_{T+1}(z) = T$, Cauchy Schwartz inequality implies

$$
3 \sum_z \sqrt{n_{T+1}(z)} \leq 3 \sqrt{|S||A| \sum_z n_{T+1}(z)} = 3 \sqrt{|S||A|T}.
$$

Therefore, the first term of (26) can be bounded by

$$
\sum_{t=1}^{T} \sum_{z} \mathbb{1}(\bar{m}_t(z) < \alpha n_t(z)) \leq \sqrt{2K_2 |S||A| T}.
$$

Substituting this bound in (26) along with the bound on the second term of (26), we obtain

$$
E\left[ \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \frac{K_2}{\max(1, \bar{m}_{t_k}(z_t))} \right] \\
\leq 3 \sqrt{2K_2 |S||A| T}.
$$

(27)

By substituting (25) and (27) into (24), we get

$$
\hat{R}_2 \leq H + 12K_2 |S||A|T^{2/3}.
$$

□
5. Conclusions

In this paper, we have presented one of the first online reinforcement learning algorithms for POMDPs. Solving POMDPs is a hard problem. Designing an efficient learning algorithm that achieves sublinear regret is even harder. We show that the proposed POMDP-PSRL algorithm achieves a Bayesian regret upper bound of \( O(T^{2/3}) \) under two technical assumptions related to belief state approximation and transition kernel estimation. There has been recent work that does approximate belief state computation, as well as estimates transition dynamics of continuous MDPs, and in future work, we will try to incorporate such estimators. We also assume that the observation kernel is known. Note that without it, it is very challenging to design online learning algorithms for POMDPs. Posterior sampling-based algorithms in general are known to have superior numerical performance as compared to OFU-based algorithms for bandits and MDPs. In future work, we will also do an experimental investigation of the proposed algorithm. An impediment is that available POMDP solvers mostly provide approximate solutions which would lead to linear regret. In the future, we will also try to improve the regret to \( O(\sqrt{T}) \).

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References

Abbasi-Yadkori, Y., Bartlett, P., Bhatia, K., Lazic, N., Szepesvari, C., and Weisz, G. Politex: Regret bounds for policy iteration using expert prediction. In International Conference on Machine Learning, pp. 3692–3702, 2019a.

Abbasi-Yadkori, Y., Lazic, N., Szepesvari, C., and Weisz, G. Exploration-enhanced politex. arXiv preprint arXiv:1908.10479, 2019b.

Ayoub, A., Jia, Z., Szepesvari, C., Wang, M., and Yang, L. Model-based reinforcement learning with value-targeted regression. In International Conference on Machine Learning, pp. 463–474. PMLR, 2020.

Azar, M. G., Osband, I., and Munos, R. Minimax regret bounds for reinforcement learning. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pp. 263–272. JMLR. org, 2017.

Azizzadenesheli, K., Lazaric, A., and Anandkumar, A. Experimental results: Reinforcement learning of pomdps using spectral methods. arXiv preprint arXiv:1705.02553, 2017.

Azizzadenesheli, K., Yue, Y., and Anandkumar, A. Policy gradient in partially observable environments: Approximation and convergence. arXiv e-prints, pp. arXiv–1810, 2018.

Bartlett, P. L. and Tewari, A. Regal: A regularization based algorithm for reinforcement learning in weakly communicating mdps. In Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence, pp. 35–42. AUAI Press, 2009.

Bertsekas, D. P. Dynamic programming and optimal control, vol i and ii, 4th edition. Belmont, MA: Athena Scientific, 2017.

Cai, C., Liao, X., and Carin, L. Learning to explore and exploit in pomdps. Advances in Neural Information Processing Systems, 22:198–206, 2009.

Cohen, A., Koren, T., and Mansour, Y. Learning linear-quadratic regulators efficiently with only \( \sqrt{T} \) regret. In International Conference on Machine Learning, pp. 1300–1309. PMLR, 2019.

Dean, S., Mania, H., Matni, N., Recht, B., and Tu, S. Regret bounds for robust adaptive control of the linear quadratic regulator. arXiv preprint arXiv:1805.09388, 2018.

Dong, K., Peng, J., Wang, Y., and Zhou, Y. Root-n-regret for learning in markov decision processes with function approximation and low bellman rank. In Conference on Learning Theory, pp. 1554–1557. PMLR, 2020.

Doshi-Velez, F., Pfau, D., Wood, F., and Roy, N. Bayesian nonparametric methods for partially-observable reinforcement learning. IEEE transactions on pattern analysis and machine intelligence, 37(2):394–407, 2013.

Fonteneau, R., Korda, N., and Munos, R. An optimistic posterior sampling strategy for bayesian reinforcement learning. In NIPS 2013 Workshop on Bayesian Optimization (BayesOpt2013), 2013.

Fruit, R., Pirotta, M., Lazaric, A., and Ortner, R. Efficient bias-span-constrained exploration-exploitation in reinforcement learning. In International Conference on Machine Learning, pp. 1573–1581, 2018.

Gopalan, A. and Mannor, S. Thompson sampling for learning parameterized markov decision processes. In Conference on Learning Theory, pp. 861–898. PMLR, 2015.
with rkhs embeddings. *arXiv preprint arXiv:1206.4655*, 2012.

Hao, B., Lazic, N., Abbasi-Yadkori, Y., Joulani, P., and Szepesvari, C. Provably efficient adaptive approximate policy iteration. *arXiv preprint arXiv:2002.03069*, 2020.

Jaksch, T., Ortner, R., and Auer, P. Near-optimal regret bounds for reinforcement learning. *Journal of Machine Learning Research*, 11(Apr):1563–1600, 2010.

Jin, C., Allen-Zhu, Z., Bubeck, S., and Jordan, M. I. Is Q-learning provably efficient? In *Advances in Neural Information Processing Systems*, pp. 4863–4873, 2018.

Jin, C., Yang, Z., Wang, Z., and Jordan, M. I. Provable efficient reinforcement learning with linear function approximation. In *Conference on Learning Theory*, pp. 2137–2143. PMLR, 2020.

Katt, S., Oliehoek, F., and Amato, C. Bayesian reinforcement learning in factored pomdps. *arXiv preprint arXiv:1811.05612*, 2018.

Kumar, P. R. and Varaiya, P. *Stochastic systems: Estimation, identification, and adaptive control*. SIAM Classic, 2015.

Lale, S., Azizzadenesheli, K., Hassibi, B., and Anandkumar, A. Explore more and improve regret in linear quadratic regulators. *arXiv preprint arXiv:2007.12291*, 2020a.

Lale, S., Azizzadenesheli, K., Hassibi, B., and Anandkumar, A. Logarithmic regret bound in partially observable linear dynamical systems. *arXiv preprint arXiv:2003.11227*, 2020b.

Liu, M., Liao, X., and Carin, L. The infinite regionalized policy representation. In *ICML*, 2011.

Liu, M., Liao, X., and Carin, L. Online expectation maximization for reinforcement learning in pomdps. In *IJCAI*, pp. 1501–1507, 2013.

Mania, H., Tu, S., and Recht, B. Certainty equivalence is efficient for linear quadratic control. *arXiv preprint arXiv:1902.07826*, 2019.

Osband, I. and Van Roy, B. Model-based reinforcement learning and the eluder dimension. *arXiv preprint arXiv:1406.1853*, 2014.

Osband, I., Russo, D., and Van Roy, B. (more) efficient reinforcement learning via posterior sampling. In *Advances in Neural Information Processing Systems*, pp. 3003–3011, 2013.

Ouyang, Y., Gagrani, M., and Jain, R. Learning-based control of unknown linear systems with thompson sampling. *arXiv preprint arXiv:1709.04047*, 2017a.

Ouyang, Y., Gagrani, M., Nayyar, A., and Jain, R. Learning unknown markov decision processes: A thompson sampling approach. In *Advances in Neural Information Processing Systems*, pp. 1333–1342, 2017b.

Poupart, P. and Vlassis, N. Model-based bayesian reinforcement learning in partially observable domains. In *Proc Int. Symp. on Artificial Intelligence and Mathematics.*, pp. 1–2, 2008.

Ross, S., Chaib-draa, B., and Pineau, J. Bayes-adaptive pomdps. In *NIPS*, pp. 1225–1232, 2007.

Russo, D. and Van Roy, B. Learning to optimize via posterior sampling. *Mathematics of Operations Research*, 39(4):1221–1243, 2014.

Shani, G., Brafman, R. I., and Shimony, S. E. Model-based online learning of pomdps. In *European Conference on Machine Learning*, pp. 353–364. Springer, 2005.

Simchowitz, M. and Foster, D. Naive exploration is optimal for online lqr. In *International Conference on Machine Learning*, pp. 8937–8948. PMLR, 2020.

Strens, M. A bayesian framework for reinforcement learning. In *ICML*, volume 2000, pp. 943–950, 2000.

Subramanian, J., Sinha, A., Seraj, R., and Mahajan, A. Approximate information state for approximate planning and reinforcement learning in partially observed systems. *arXiv preprint arXiv:2010.08843*, 2020.

Thompson, W. R. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25(3/4):285–294, 1933.

Tsiamis, A. and Pappas, G. Online learning of the kalman filter with logarithmic regret. *arXiv preprint arXiv:2002.05141*, 2020.

Wang, R., Salakhutdinov, R. R., and Yang, L. Reinforcement learning with general value function approximation: Provably efficient approach via bounded eluder dimension. *Advances in Neural Information Processing Systems*, 33, 2020.

Wang, T., Zhou, D., and Gu, Q. Provably efficient reinforcement learning with linear function approximation under adaptivity constraints. *arXiv preprint arXiv:2101.02195*, 2021.

Wei, C.-Y., Jafarnia-Jahromi, M., Luo, H., Sharma, H., and Jain, R. Model-free reinforcement learning in
infinite-horizon average-reward markov decision processes. In *International Conference on Machine Learning*, pp. 10170–10180. PMLR, 2020.

Wei, C.-Y., Jafarnia-Jahromi, M., Luo, H., and Jain, R. Learning infinite-horizon average-reward mdps with linear function approximation. *International Conference on Artificial Intelligence and Statistics*, 2021.

Zanette, A. and Brunskill, E. Tighter problem-dependent regret bounds in reinforcement learning without domain knowledge using value function bounds. In *International Conference on Machine Learning*, 2019.

Zhang, Z. and Ji, X. Regret minimization for reinforcement learning by evaluating the optimal bias function. In *Advances in Neural Information Processing Systems*, 2019.
A. Full Upper Bound on the Expected Regret

The exact expression for the upper bound of the expected regret in Theorem 1 is

\[ R_T = R_0 + R_1 + R_2 + R_3 \leq (1 + H)E[K_T] + 12H K_2(\lvert S \rvert \lvert A \rvert T)^{2/3} + (H + 1)K_1 \sum_{k=1}^{K_T} \frac{T_k}{\sqrt{T_k}} + 2 + H \]

\[ \leq (1 + H)\sqrt{2T(1 + \lvert S \rvert \lvert A \rvert \log T)} + 12H K_2(\lvert S \rvert \lvert A \rvert T)^{2/3} + 7(H + 1)K_1 \sqrt{2T(1 + \lvert S \rvert \lvert A \rvert \log T)} \log \sqrt{2T} + 2 + H. \]

B. Omitted Proofs

Proof of Proposition 1. We prove that for any policy \( \pi \), \( J_\pi(h, \theta) \geq J_{\pi^*}(h, \theta) = J(\theta) \) for all \( h \in \Delta_S \). Let \( \pi : \Delta_S \rightarrow A \) be an arbitrary policy. We can write

\[ J_\pi(h, \theta) = \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} E[C(s_t, \pi(h_t))|s_1 \sim h] \]

\[ = \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \mathbb{E}[C(s_t, \pi(h_t))|\mathcal{F}_t, s_1 \sim h]|s_1 \sim h \right] \]

\[ = \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[c(h_t, \pi(h_t))|s_1 \sim h] \]

\[ \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[J(\theta) + v(h_t, \theta) - v(h_{t+1}, \theta)|s_1 \sim h] = J(\theta), \]

with equality attained by \( \pi^* \) completing the proof. \( \square \)

Lemma 6. [Lemma 3 in (Ouyang et al., 2017b)] The term \( R_0 \) is bounded as \( R_0 \leq E[K_T] \).

Proof. \( R_0 = E \left[ \sum_{k=1}^{K_T} \sum_{t=k}^{T_k-1} \left[ J(\theta_k) - J(\theta_s) \right] \right] = E \left[ \sum_{k=1}^{\infty} \mathbb{1}(t_k \leq T)T_kJ(\theta_k) \right] - TE[J(\theta_s)]. \)

By monotone convergence theorem and the fact that \( J(\theta_k) \geq 0 \) and \( T_k \leq T_{k-1} + 1 \) (the first criterion in determining the episode length in Algorithm 1), the first term can be bounded as

\[ E \left[ \sum_{k=1}^{\infty} \mathbb{1}(t_k \leq T)T_kJ(\theta_k) \right] = E \left[ \sum_{k=1}^{\infty} \mathbb{1}(t_k \leq T)T_kJ(\theta_k) \right] \leq E \left[ \sum_{k=1}^{\infty} \mathbb{1}(t_k \leq T)(T_{k-1} + 1)J(\theta_k) \right]. \]

Note that \( \mathbb{1}(t_k \leq T)(T_{k-1} + 1) \) is \( \mathcal{F}_k \)-measurable. Thus, by the property of posterior sampling (Lemma 3), \( E[\mathbb{1}(t_k \leq T)(T_{k-1} + 1)J(\theta_k)] = E[\mathbb{1}(t_k \leq T)(T_{k-1} + 1)J(\theta_s)] \). Therefore,

\[ R_0 \leq E \left[ \sum_{k=1}^{\infty} \mathbb{1}(t_k \leq T)(T_{k-1} + 1)J(\theta_s) \right] - TE[J(\theta_s)] \]

\[ = E \left[ J(\theta_s)(K_T + \sum_{k=1}^{K_T} T_{k-1}) \right] - TE[J(\theta_s)] \]

\[ = E[J(\theta_s)K_T] + E \left[ J(\theta_s)(\sum_{k=1}^{K_T} T_{k-1} - T) \right] \leq E[K_T], \]

where the last inequality is by the fact that \( \sum_{k=1}^{K_T} T_{k-1} - T \leq 0 \) and \( 0 \leq J(\theta_s) \leq 1 \). \( \square \)
Lemma 7. The term $R_1$ is bounded as

$$R_1 \leq HE[K_T].$$

Proof. Recall that

$$R_1 = E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \left[ v(h_t(\cdot; \theta_k); \theta_k) - v(h_{t+1}(\cdot; \theta_k); \theta_k) \right] \right].$$

Since the terms inside the inner summation telescope

$$\sum_{t=t_k}^{t_{k+1} - 1} \left[ v(h_t(\cdot; \theta_k); \theta_k) - v(h_{t+1}(\cdot; \theta_k); \theta_k) \right] = v(h_{t_k}(\cdot; \theta_k); \theta_k) - v(h_{t_{k+1}}(\cdot; \theta_k); \theta_k) \leq H,$$

where the inequality is by the fact that $0 \leq v(b; \theta_k) \leq H$ for all $b \in \Delta_S$. Substituting the bound into the definition of $R_1$, we obtain

$$R_1 \leq E\left[\sum_{k=1}^{K_T} H\right] = HE[K_T].$$

Lemma 8. The term $R_3$ can be bounded as

$$R_3 \leq K_1 E\left[\sum_{k=1}^{K_T} \frac{T_k}{\sqrt{t_k}}\right] + 1,$$

where $K_1 := K_1(|S|, |A|, |O|, \iota)$ is the constant in Assumption 2.

Proof. Recall that

$$R_3 := E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \left[ c(h_t(\cdot; \theta_*), a_t) - c(h_t(\cdot; \theta_k), a_t) \right] \right].$$

By the definition of $c$, we can write

$$R_3 = E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \sum_s C(s, a_t) \left[ h_t(s; \theta_*) - h_t(s; \theta_k) \right] \right]$$

$$= E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \sum_s C(s, a_t) E\left[ h_t(s; \theta_*) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right] \right]$$

$$\leq E\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \sum_s \left| E\left[ h_t(s; \theta_*) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right] \right| \right],$$

where the inequality is by the fact that $0 \leq C(s, a_t) \leq 1$. Let $K_1 := K_1(|S|, |A|, |O|, \iota)$ be the constant in Assumption 2 and define event $E_1$ as the successful event of Assumption 2 where $\sum_s E\left[ h_t(s; \theta_*) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right] \leq \frac{K_1}{\sqrt{t_k}}$ happens. We can write

$$\sum_s \left| E\left[ h_t(s; \theta_*) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right] \right| = \sum_s \left| E\left[ h_t(s; \theta_*) - h_t(s; \theta_k) \mid \mathcal{F}_t, \theta_k \right] \right| (1(E_1) + 1(E_1^c))$$

$$\leq \frac{K_1}{\sqrt{t_k}} + 2 \mathbb{1}(E_1^c).$$
Recall that by Assumption 2, $P(E_1^T) \leq \delta$. Therefore,

$$R_3 \leq K_1 E \left[ \sum_{k=1}^{K_T} \frac{T_k}{\sqrt{T_k}} \right] + 2T\delta.$$  

Choosing $\delta = 1/(2T)$ completes the proof.

**Lemma 9.** The following inequalities hold:

1. The number of episodes $K_T$ can be bounded as $K_T \leq \sqrt{2T(1 + |S||A| \log T)} = \tilde{O}(\sqrt{|S||A|T}).$

2. The following inequality holds: $\sum_{k=1}^{K_T} \frac{T_k}{\sqrt{T_k}} \leq \sqrt{2T}(1 + |S||A| \log T) \log \sqrt{2T} = \tilde{O}(|S||A|\sqrt{T}).$

**Proof.** We first provide an intuition why these results should be true. Note that the length of the episodes is determined by two criteria. The first criterion triggers when $T_k = T_{k-1} + 1$ and the second criterion triggers when the pseudo counts doubles for a state-action pair compared to the beginning of the episode. Intuitively speaking, the second criterion should only happen logarithmically, while the first criterion occurs more frequently. This means that one could just consider the first criterion for an intuitive argument. Thus, if we ignore the second criterion, we get $T_k = O(k)$, $K_T = O(\sqrt{T})$, and $t_k = O(k^2)$ which implies $\sum_{k=1}^{K_T} \frac{T_k}{\sqrt{T_k}} = O(K_T) = O(\sqrt{T})$. The rigorous proof is stated in the following.

1. Define macro episodes with start times $t_{m_i}$ given by $t_{m_i} = t_1$ and

   $$t_{m_i} := \min\{t_k > t_{m_i-1} : \tilde{m}_{t_k}(s, a) > 2\tilde{m}_{t_{k-1}}(s, a) \text{ for some } (s, a)\}.$$

   Note that a new macro episode starts when the second criterion of episode length in Algorithm 1 triggers. Let $M_T$ be the random variable denoting the number of macro episodes by time $T$ and define $m_{M_T+1} = K_T + 1$.

Let $\tilde{T}_i$ denote the length of macro episode $i$. Note that $\tilde{T}_i = \sum_{k=m_i}^{m_i+1} T_k$. Moreover, from the definition of macro episodes, we know that all the episodes in a macro episode except the last one are triggered by the first criterion, i.e., $T_k = T_{k-1} + 1$ for all $m \leq k \leq m_i + 1 - 2$. This implies that

$$\tilde{T}_i = \sum_{k=m_i}^{m_i+1-1} T_k = T_{m_i+1} - 1 + \sum_{j=1}^{m_i+1 - m_i - 1} (T_{m_i-1} + j) \geq 1 + \sum_{j=1}^{m_i+1 - m_i - 1} (1 + j) = \frac{(m_i+1 - m_i)(m_i+1 - m_i + 1)}{2}.$$

This implies that $m_{i+1} - m_i \geq \sqrt{2\tilde{T}_i}$. Now, we can write:

$$K_T = m_{M_T+1} - 1 = \sum_{i=1}^{M_T} (m_{i+1} - m_i) \leq \sum_{i=1}^{M_T} \sqrt{2\tilde{T}_i} \leq \sqrt{2M_T \sum_i \tilde{T}_i} = \sqrt{2M_T T},$$

where the last inequality is by Cauchy-Schwartz.

Now suffices to show that $M_T \leq 1 + |S||A| \log T$. Let $T_{s,a}$ be the start times at which the second criterion is triggered at state-action pair $(s, a)$, i.e.,

$$T_{s,a} := \{t_k \leq T : \tilde{m}_{t_k}(s, a) > 2\tilde{m}_{t_{k-1}}(s, a)\}.$$

We claim that $|T_{s,a}| \leq \log(\tilde{m}_{T+1}(s, a))$. To prove this claim, assume by contradiction that $|T_{s,a}| \geq \log(\tilde{m}_{T+1}(s, a)) + 1$, then

$$\tilde{m}_{T_{K_T}}(s, a) \geq \prod_{t_k \leq T, \tilde{m}_{t_k}(s, a) \geq 1} \frac{\tilde{m}_{t_k}(s, a)}{\tilde{m}_{t_k-1}(s, a)} \geq \prod_{t_k \in T_{s,a}, \tilde{m}_{t_k}(s, a) \geq 1} \frac{\tilde{m}_{t_k}(s, a)}{\tilde{m}_{t_k-1}(s, a)} = \prod_{t_k \in T_{s,a}, \tilde{m}_{t_k}(s, a) \geq 1} 2 = 2^{|T_{s,a}|+1} \geq \tilde{m}_{T+1}(s, a),$$
which is a contradiction. The second inequality is by the fact that \( \tilde{m}_t(s,a) \) is non-decreasing, and the third inequality is by the definition of \( T_{s,a} \). Therefore,

\[
M_T \leq 1 + \sum_{s,a} |T_{s,a}| \leq 1 + \sum_{s,a} \log(\tilde{m}_{T+1}(s,a)) \leq 1 + |S||A| \log(\sum_{s,a} \tilde{m}_{T+1}(s,a)/|S||A|) = 1 + |S||A| \log T, \tag{29}
\]

where the third inequality is due to the concavity of \( \log \) and the last inequality is by the fact that \( \tilde{m}_{T+1}(s,a) \leq T \).

2. First, we claim that \( T_k \leq \sqrt{2T} \) for all \( k \leq K_T \). To see this, assume by contradiction that \( T_{k^*} > \sqrt{2T} \) for some \( k^* \leq K_T \).

By the first stopping criterion, we can conclude that \( T_{k^*-1} > \sqrt{2T} - 1, T_{k^*-2} > \sqrt{2T} - 2, \ldots, T_1 > \max\{ \sqrt{2T} - k^* + 1, 0 \} \) since the episode length can increase at most by one compared to the previous one. Note that \( k^* \geq \sqrt{2T} - 1 \), because otherwise \( T_1 > 2 \) which is not feasible since \( T_1 \leq T_0 + 1 = 2 \). Thus, \( \sum_{k=1}^{K_T} T_k > 0.5\sqrt{2T}(\sqrt{2T} + 1) > T \) which is a contradiction.

We now proceed to lower bound \( t_k \). By the definition of macro episodes in part (1), during a macro episode length of the episodes except the last one are determined by the first criterion, i.e., for macro episode \( i \), one can write \( T_k = T_{k-1} + 1 \) for \( m_i \leq k \leq m_{i+1} - 2 \). Hence, for \( m_i \leq k \leq m_{i+1} - 2 \)

\[
t_{k+1} = t_k + T_k = t_k + T_{m_i} + 1 - (m_i - 1) \geq t_k + k - m_i + 1.
\]

Recursive substitution of \( t_k \) implies that \( t_k \geq t_{m_i} + 0.5(k - m_i)(k - m_i + 1) \) for \( m_i \leq k \leq m_{i+1} - 1 \). Thus,

\[
\sum_{k=1}^{K_T} \frac{T_k}{\sqrt{T_k}} \leq \frac{\sum_{i=1}^{M_T} \frac{m_i + 1}{\sqrt{T_{m_i} + 0.5(k - m_i)(k - m_i + 1)}} - \sum_{i=1}^{M_T} \frac{1}{\sqrt{T_{m_i}}}}{\sum_{k=m_i}^{m_{i+1} - 1} \frac{1}{\sqrt{k - m_i}}} \leq \frac{\sum_{i=1}^{M_T} \frac{1}{\sqrt{T_{m_i}}}}{\sum_{k=m_i}^{m_{i+1} - 1} \frac{1}{\sqrt{k - m_i}}} \leq M_T \sum_{i=1}^{M_T} \frac{1}{\sqrt{T_{m_i}}}. \tag{30}
\]

The denominator of the summats at \( k = m_i \) is equal to \( \sqrt{T_{m_i}} \). For other values of \( k \) it can be lower bounded by \( 0.5(k - m_i)^2 \). Thus,

\[
\sum_{k=m_i}^{m_{i+1} - 1} \frac{1}{\sqrt{T_{m_i} + 0.5(k - m_i)(k - m_i + 1)}} \leq \sum_{i=1}^{M_T} \frac{1}{\sqrt{T_{m_i}}} + \sum_{i=1}^{M_T} \frac{M_T}{\sqrt{T_{m_i} - 1}} \leq M_T + \sum_{j=1}^{M_T} \frac{1}{\sqrt{T_{m_i} - 1}} \leq M_T + \sqrt{2} \sum_{j=1}^{M_T} \log(m_{i+1} - m_i)
\]

\[
\leq M_T(1 + \sqrt{2}) + \sqrt{2}M_T \log \frac{1}{M_T} \sum_{i=1}^{M_T} (m_{i+1} - m_i)
\]

\[
\leq M_T(1 + \sqrt{2}) + \sqrt{2}M_T \log \sqrt{2T} \leq 7M_T \log \sqrt{2T},
\]

where the second inequality is by \( t_{m_i} \geq 1 \), the third inequality is by the fact that \( \sum_{j=1}^{K_T} 1/j \leq 1 + \int_1^{K_T} \frac{dx}{x} = 1 + \log K \), the forth inequality is by concavity of \( \log \) and the fifth inequality is by the fact that \( \sum_{i=1}^{M_T} (m_{i+1} - m_i) = m_{M_T+1} - 1 = K_T \) and \( K_T/M_T \leq \sqrt{2T/M_T} \leq \sqrt{2T} \) (see (28)). Substituting this bound into (30) and using the upper bound on \( M_T \) (29), we can write

\[
\sum_{k=1}^{K_T} \frac{T_k}{\sqrt{T_k}} \leq \sqrt{2T} \left( 7M_T \log \sqrt{2T} \right)
\]

\[
\leq 7\sqrt{2T}(1 + |S||A| \log T) \log \sqrt{2T}.
\]

\qed