Dual Algebraic Pairs and Polynomial Lie Algebras in Quantum Physics:
Foundations and Geometric Aspects

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Abstract

We discuss some aspects and examples of applications of dual algebraic pairs 
\( (\mathcal{G}_1, \mathcal{G}_2) \) in quantum many-body physics. They arise in models whose Hamiltonians \( H \) have invariance groups \( G_i \). Then one can take \( \mathcal{G}_1 = G_i \) whereas another dual partner \( \mathcal{G}_2 = g^D \) is generated by \( G_i \) invariants, possesses a Lie-algebraic structure and describes dynamic symmetry of models; herewith polynomial Lie algebras \( g = g^D \) appear in models with essentially nonlinear Hamiltonians. Such an approach leads to a geometrization of model kinematics and dynamics.

1 Introduction

As is known, group-theoretical and Lie-algebraic methods yield powerful tools for both qualitative (adequate formulations of model kinematics and dynamics) and quantitative (dimension reduction of calculations) analysis of many physical problems\(^1-3\). In quantum many-body physics, where Hilbert spaces \( \mathcal{L} \) of states and all physical observables \( O \) are given in terms of boson \( (a_i, a_i^+) \) and fermion \( (b_j, b_j^+) \) operators with standard commutation relations (CR), Lie-algebraic structures arise in a natural way via using different boson-fermion mappings: \( (a_i, a_i^+, b_j, b_j^+) \) which introduce generators \( F_\alpha \) of finite-dimensional Lie (super)algebras \( g = \text{Span}\{F_\alpha\} \) as (super)symmetry operators and simultaneously as basic dynamic variables (i.e. \( O = O(\{F_\alpha\}) \)) yielding a most adequate formulation of problems under study\(^3\). Such algebras \( g \) generate Lie groups \( G = \exp g = \{\exp F : F \in g\} \) with the key for applications group property of their elements: \( \exp F_1 \exp F_2 = \exp F_3, F_i \in g^{1,2} \).

Depending on the behaviour of model Hamiltonians \( H \) with respect to symmetry transformations one discerns two (used, as a rule, separately) symmetry types\(^1 : \( a) \) invariance groups \( G_i \) of Hamiltonians \( H : [G_i, H] = G_i H - H G_i = 0 \); b) dynamic symmetry algebras \( g^D : [g^D, H] \leq g^D \neq 0 \) (\( \iff \) \( H \in g^D \)). In the first case Hamiltonians are considered to be functions in only \( G_i \)-invariant (Casimir) operators \( \Lambda_j(G_i) \) whose eigenvalues \( \lambda_j \) label energy levels \( E_{\lambda_j} \), and dimensions \( d^{G_i}(\lambda) \) of \( G_i \)-irreducible representations (IR) \( D^\lambda(G_i) \) are equal to the \( E_{\lambda} \)-degeneracy multiplicities \( \mu(\lambda) \). At the same time algebras \( g^D \) already generate total spectra \( \{E_\nu\} \) of "elementary" quantum system within fixed IRs \( D^\lambda(g^D) \) and yield spectral decompositions

\[
L(H)_{g^D} = \sum_\lambda \mu(\lambda) L(\lambda), \quad L(\lambda) = \text{Span}\{|\lambda; \nu\} = D^\lambda_\nu(g^D)|\lambda\rangle
\]

(1)
of Hilbert spaces \( L(H) \) of many-body systems in \( \mu \)-multiple \( g^D \)-invariant subspaces \( L(\lambda) \) generated by actions of the \( g^D \)-operators \( D^\lambda_\nu(g^D) \) on eigenvectors \( |\lambda\rangle \in L(H) \) of
$g^D$-invariant operators $\Lambda_i$. Subspaces $L(\lambda)$ describe formation of ”macroscopic coherent structures" ($g^D$-domains) in $L(H)$ which are stable under the temporal evolution: $|\Psi(0)\rangle \in L(\lambda) \implies |\Psi(t)\rangle = U_H(t)|\Psi(0)\rangle \in L(\lambda)$, $U_H(t) = \exp(-itH)$, $H \in g^D$, but a physical sense of c-numbers $\mu, \lambda, j$ in Eq. (1) still remains unclear. At the same time within many-body models with $G_i$ -invariant Hamiltonians one can reveal deep interrelations between $G_i$ and $g^D$ symmetries which enable not only to elucidate this sense but also to formulate an unified invariant-algebraic approach for an efficient analysis of physical problems in such models. A natural formal description of the latter is given in terms of novel mathematical concepts of dual algebraic pairs (DAP) incorporating actions of both groups $G_i$ and algebras $g^D$ and polynomial Lie algebras (PLA) arising as $g^D$ in models with essentially nonlinear Hamiltonians.

The DAP techniques enabled us to elucidate a few non-trivial questions of quantum physics; however, a number of problems concerning applications of PLA is still unsolved. In this work we briefly discuss these problems and ways of their solution focusing the main attention on geometric aspects. At first we recapitulate fundamentals of the DAP and PLA formalism in the context of quantum many-body physics, restricting ourselves for the sake of simplicity by the boson case and referring to for a general discussion. Then we discuss some aspects of our applications of the DAP techniques in quantum optics and outline prospects of further studies.

## 2 Dual algebraic pairs and polynomial Lie algebras in multiboson physics: a general analysis

The notion of DAP extracted from the vector invariant theory of classical groups by Howe is defined in the context of many-boson systems by

**Definition 1.** Let $a_i = (a_i^\alpha)_{\alpha=1}^m$, $a_i^\dagger = (a_i^\dagger)^\alpha$, $i = 1, \ldots, n$ be $n$ pairs of boson vector operators transforming according to two mutually contragredient fundamental IRs $D^1(G)$ and $\bar{D}^1(G)$ of a certain group $G$:

$$a_i^\dagger \xrightarrow{D^1(G)} a_{i\alpha}^\dagger \xrightarrow{\bar{D}^1(G)} \tilde{a}_{i\alpha} = \sum_{\beta=1}^m u_{\alpha\beta} a_{i\beta}^\dagger.$$  

Consider the associative algebra $A_G^\dagger$ of vector invariants of the group $G$ generated by finite (according to the vector invariant theory) basis $\mathcal{B}_{G1} = \{I_j : [I_j, G] = 0\}_{j=1}^d$ of homogeneous polynomials $I_j = I_j(a_{i}^{\alpha}, a_{i}^{\dagger\alpha})$. Endowing it by the commuting operation $[I_j, I_j] \equiv [I_i, I_j]$ one gets a Lie algebra $g(A_G^\dagger)$ with the basis $\mathcal{B}_{G1}$ and defining CR

$$[I_i, I_j] = f_{ij}([I_i]) \quad ([I_i, f_{ij} + [I_j, f_{ji} + [I_k, f_{jk}]] = 0)$$

where $f_{ij}([I_i])$ are (consistent with the Jacobi identities) polynomials in $I_i$ stemming from CR for $a_i, a_i^\dagger$ and the invariant theory. By the construction two algebraic structures $G_1 = G$ and $G_2 = g(A_G^\dagger)$ commute: $[G_1, G_2] = 0$ and have a common center
Corollary 1 be the Fock space generated by actions of creation operators \( a_{i\beta}^+ \) on the vacuum vector \( |0\rangle \) and carrying (due to Eqs. (2) and the \( \mathcal{G}_2 \) definition) reducible representations of both structures \( \mathcal{G}_1, \mathcal{G}_2 \). Then there holds the decomposition

\[
L(v^\otimes n) \downarrow_{\mathcal{G}_1 \otimes \mathcal{G}_2} = \sum_{[c_i]} L([c_i]), \quad L([c_i]) = \text{Span}\{D^{[c_i]}(\mathcal{G}_1) \otimes D^{[c_i]}(\mathcal{G}_2)|[c_i]\}\}
\]

where \( L([c_i]) \) are \( \mathcal{G}_1 \otimes \mathcal{G}_2 \)-invariant subspaces labeled by eigenvalues \( c_i \) of elements \( C_i = C_i(a_i, a_i^+) = \tilde{C}_i(t_j) \) of the center \( \mathcal{C} = \{C_i\} \) and generated by joint actions \( D^{[c_i]}(\mathcal{G}_1) \otimes D^{[c_i]}(\mathcal{G}_2) \) of both DAP components on some reference vectors \([|c_i\rangle] \in L_F(nm)\).

The Definition 1 entails a very important for physical applications Corollary 1 (sometimes inserted in the DAP definition). Let

\[
L(v^\otimes n) = \text{Span}\{|\{n_{i\beta}\}\rangle = \prod_{i,\beta} (a_{i\beta}^+)^{n_{i\beta}} |0\rangle : a_{i\beta}|0\rangle = 0\} \equiv L_F(nm)
\]

be the Fock space that holds, e.g., for groups \( G = \text{O}(n), \text{U}(n), \text{Sp}(2n) \). But in the general case bases \( \mathcal{B}_{GI} \) contain polynomials \( \tilde{I}_j = I_j(a_{i\alpha}, a_{i\alpha}) = T_j \) of higher orders which form tensor operators \( t = \text{Span}\{T_j\} \) with respect to \( h : [h, t] = t \). Then CR in (3) do not close to linear combinations of invariants \( I_j \in \mathcal{B}_{GI} \), and repeated commutators lead to infinite-dimensional Lie algebras \( g(A_G^I) \), generally, not belonging to well-examined classes of the Kac-Moody algebras. Therefore, for physical applications it is useful to consider (retaining Eq. (5)) DAP with \( \mathcal{G}_2 = \mathcal{E}(\mathcal{B}_{GI}) \) where \( \mathcal{E}(\mathcal{B}_{GI}) \) are defined as enveloping algebras generated by the bases \( \mathcal{B}_{GI} = h \cup t \) and appropriate specifications of CR (3). Such objects, also appeared in other contexts, are called as polynomial deformations of Lie algebras or simply PLA (in view of the absence in the general case one-to-one correspondences between root systems of PLA and usual Lie algebras).

PLA \( \mathcal{E}(\mathcal{B}_{GI}) \) being, by the definition above, specific \( (t\text{-tensor}) \) extensions of usual Lie algebras \( h \) are also \( G \)-invariant subalgebras of the universal enveloping algebra \( \mathcal{U}(w(nm)) \) of the Weyl-Heisenberg algebra \( w(nm) = \text{Span}\{a_{i\alpha}, a_{i\alpha}^+\} \). It enables one to specify completely CR (3) for them and to develop their representation theory (unlike the case of arbitrary PLA). These constructions are especially simple when \( h \)-tensors \( t \) consist of two Hermitian conjugated irreducible tensors \( t^\lambda = \{T_i^\lambda : [T_i^\lambda, T_j^\lambda] = 0\}, \lambda = (t^\lambda)^\dagger \) : \( t = t^\lambda + t^\lambda \). Then CR (3) are specified as follows

\[
a) [h, h] = h, \quad b) [h, t^\lambda] = t^\lambda, \quad [h, t^\lambda] = t^\lambda, \quad c) [T_i^\lambda, T_j^\lambda] = \mathcal{P}_{ij}(h; r), \quad r \subset \mathcal{C}
\]

where \( \mathcal{P}_{ij}(h; r) \) are polynomials of a fixed degree \( s \geq 2 \) in \( F_j \in h, R_i \in r \) which are found with the help of the Jacobi identities and (6b) from the only polynomial
\( \mathcal{P}_{\lambda\lambda}(\ldots) \equiv \mathcal{P}(\ldots) \) (corresponding to “extremal” components \( T^\lambda_\lambda, T^{\lambda}_\lambda \) of tensors \( t^\lambda, t^\lambda \); the latters, in turn, are determined by explicit expressions \( T^\lambda_\lambda, T^{\lambda}_\lambda \in \mathcal{U}(w(nm)) \).

So, bases \( \mathcal{B}_{\text{dif}} = h \cup \{ t^\lambda \cup t^{\lambda}_\lambda \} \), centers \( r \) and \( \mathcal{C} \) (6) define a special (very vast) class of PLA \( \mathcal{E}(\mathcal{B}_{\text{dif}}) = \mathcal{E}_r(h; t^\lambda) \) as the second component of the DAP \( (\mathcal{G}_1 = G, \mathcal{G}_2) \) connected with \( \mathcal{G} \) via the appearance of \( \mathcal{P}, r \subseteq \mathcal{C} \) in CR (6). In fact, PLA \( \mathcal{E}_r(h; t^\lambda) \) can be also examined as abstract PLA beyond the DAP context that is of interest for finding their representations not containing in (5) (as it is the case for usual Lie algebras \( ^1 \)). As an illustration we consider two examples taken from physics \( ^3, ^6 \).

A simplest Example 1 is given by PLA \( \mathcal{E}_{R_1}^P(h = u(1) = \{ V_0 \}; t^\lambda = v^{(1)}_+ = \{ V_+ \}) \) defined by the bases \( \mathcal{B} = \{ V_0, V_+, V_- = V_+^\dagger \}, r = \{ R_1 : [R_1, V_0] = 0 \} \) and CR

\[
[V_0, V_\pm] = \pm V_\pm, \quad [V_-, V_+] = \mathcal{P}(V_0; R_1) = Q(V_0 + 1; R_1) - Q(V_0; R_1) \quad (7)
\]

where (extracted from concrete physical models) polynomials \( Q(V_0; R_1) \) (of the degree \( s + 1 \)) determine the Casimir operators \( C^P, [C^P, V_0] = 0 \), \( C^P|_{L_\mathcal{F}(nm)} \equiv 0 \). (Eq.(5)). (8)

The PLA \( \mathcal{E}_{R_1}^P(u(1); v^{(1)}_+) \) can be also viewed as polynomial deformations \( s_{\mathcal{F}}(2) \) of the Lie algebra \( s(2) = \text{Span}[Y_0, Y_\pm : [Y_0, Y_\pm] = \pm Y_\pm, [Y_-, Y_+] = \pm 2Y_0] \) due to their connection via the generalized Holstein-Primakoff transformation \( ^3, ^6 \)

\[
Y_0 = V_0 - R_0 - J, \quad Y_+ = V_+^{\phi(0_0)} - 1/2, \quad Y_- = (Y_+)\dagger, \quad [Y_\alpha, R_0] = 0 = [Y_\alpha, J] \quad (9)
\]

where \( R_0, -J \) are invariant ”lowest weight” operators and functions \( \phi(V_0) \) are determined via polynomials \( Q(V_0; R_1) \). Furthermore, PLA \( \mathcal{E}_{R_1}^P(u(1); v^{(1)}_+) \) admit two conjugate realizations by (pseudo) differential operators of one complex variable \( z \in \mathbb{C} \)

\[
V_+ = z, \quad V_0 = zd/dz + R_0, \quad V_- = z^{-1}[C^P + Q(zd/dz + R_0; R_1)],
\]

\[
V_- = d/dz, \quad V_0 = zd/dz + R_0, \quad V_+ = [C^P + Q(zd/dz + R_0; R_1)](d/dz)^{-1} \quad (10)
\]

with \( Q(zd/dz + R_0; R_1) = \sum_{k=1}^{k+1} \gamma_k z^k \) being determined from (7) - (8). \( ^3, ^6 \)

Example 2 extends the first one and is given by the PLA \( \mathcal{E}_{R_2}^P(u(2); v^{(2)}_+) \) where \( u(2) = \{ E_{ij} : [E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} \} \) is the two-dimensional unitary Lie algebra, and \( v^{(2)}_+ = \{ V^+_0 \} \) is its 2-nd rank symmetric tensor. All components \( V^+_0 \) and \( V_{ij} = (V^+_0)_{ij} \in v^{(2)}_+ \) are determined (via the specifications: \( [E_{ij}, V^+_0] = ad_{E_{ij}}, V^+_0 = \delta_{jk}V^+_i + \delta_{jl}V^+_k, [E_{ij}, V^+_0] = -[E_{ij}, V^+_0] \) of CR (6b)) by \( u(2) \) adjoint actions

\[
2V^+_{12} = ad_{E_{12}}V^+_0, \quad 2V^+_{22} = ad_{E_{22}}^2V^+_0, \quad 2V^+_{12} = -ad_{E_{12}}V^+_1, \quad 2V^+_{22} = -ad_{E_{12}}^2V^+_1 \quad (11)
\]

on the ”extremal” components \( T^{(2)}_2 = V^+_0 (ad_{E_{12}}V^+_1 = 0 = ad_{E_{22}}^2V^+_1) \) and \( T^{(2)}_2 = V^+_1 (ad_{E_{22}}V^+_1 = 0 = ad_{E_{22}}^2V^+_1) \) which together with \( V_0 = \frac{1}{2}E_{11} \) generate PLA \( s_{\mathcal{F}}(2) \sim \)
\[ \mathcal{E}_{R_1}^T(u(1); v_1^{(1)}) \subset \mathcal{E}_{R_2}^T(u(2); v_2^{(2)}) \] with CR (7). Then, using Eqs. (11) and the Jacobi identities we can calculate all polynomials \( \mathcal{P}_{ij;kl}(\{E_{ij}; R_1\}) = [V_{ij}, V_{kl}^+] \) in specifications of CR (6c) by the \( u(2) \) adjoint actions on \( \mathcal{P} = \mathcal{P}_{11;11}(\ldots), e.g., \mathcal{P}_{11;12}(\ldots) = \frac{1}{2}ad_{E_2}\mathcal{P}(\ldots) \) etc. Evidently, this procedure of “lifting” PLA \( s_{pd}^T(2) \) to PLA \( \mathcal{E}_r(h; t^\lambda) \) is easily extended on the case of any \( h = u(N) + u(M) \) and their irreducible tensors \( t^\lambda \); however, generalizations of Eqs. (9), (10) are still open problems 3.

And now we outline general features of DAP applications in examining multiboson models with the Hilbert spaces \( L(H) = L_F(nm) \) and \( G_{\lambda^-} \)-invariant Hamiltonians

\[ H_{GI}^{n,m} = \hbar \left\{ \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^m \left[ \omega_{ij} a_{i\alpha}^a a_{j\beta} + g_{ij} a_{i\alpha}^a i_{j\beta} + g_{ij} \right] + H_{GI}^{hd}(\{a_{i\alpha}^a, a_{j\beta}^+\}) \right\} \tag{12} \]

where \( H_{GI}^{hd}(\ldots) = H_{GI}^{hd\dagger}(\ldots) \) are polynomials of higher (\( \geq 3 \)) degrees describing essentially nonlinear interactions 3. Then \( H_{GI} \in \mathcal{E}(\mathcal{B}_{GI} = h \cup t) \) where quadratic terms in (12) belong to \( h, H_{GI}^{hd} \in t, \) and the DAP \( (\mathcal{G}_1 = G_{\lambda^-}, \mathcal{G}_2 = \mathcal{E}(\mathcal{B}_{GI}) = g^D) \) naturally arise in such models. Their use reveals a ”synergetic” role of \( G_{\lambda^-} \)-invariance and leads via the introduction of three types of collective variables related to \( r \subset C \) (integrals of motions ), \( g^D \) (”cluster” dynamic variables ) and \( G_{\lambda^-} \) (”hidden” intrinsic parameters ) to a geometrization of model kinematics and dynamics that opens possibilities to apply geometrical methods 8–11 for their analysis.

Indeed, the Hamiltonians (12) can be reformulated in the \( G_{\lambda^-} \)-invariant form:

\[ H_{GI}^{n,m} = H_{GI}^{n,m}(\{I_j\}) = \hbar \left\{ \sum_j \Omega_j F_j + \sum_k v_k T_k + \delta(C_i) \right\}, \quad F_j \in h, \ T_k \in t, \ C_i \in r \tag{13} \]

(with some of coefficients \( \Omega_j, v_k \) being equal to zero), and the decompositions (5) for \( L(H) = L_F(nm) \) can be viewed as specifications of Eq. (1) because subspaces \( L([c_i]) \) have a fibre bundle structure with fibres \( L_\mathcal{E}(\mathcal{B}_{GI})([c_i]; \nu]) \sim (L(\lambda) \in (1)) \) generated by actions \( D_{[c_i]}(\mathcal{E}(\mathcal{B}_{GI})) \) on (labelling the fibre bundle bases) vectors \( [c_i]; \nu] = D_{[c_i]}(G_{\lambda^-}) \). Herewith dimensions \( d_{G_{\lambda^-}}([c_i]) \) of the \( D_{[c_i]}(G_{\lambda^-}) \) IRs are equal to multiplicities \( \mu(\lambda) \) in Eq. (1) and describe degeneracies of all energy levels within a given subspace \( L([c_i]) \). At the (quasi)classical level of analysis, implemented via generalized coherent states (CS) 2,11, the decomposition (5) induces the fibre bundle representation

\[ \mathcal{M}(H) = \bigcup_{[c_i]} \mathcal{M}_{[c_i]}(\{\xi_{a}^I, \zeta_{b}^D\}), \quad \mathcal{M}_{[c_i]}(\{\xi_{a}^I, \zeta_{b}^D\}) = \mathcal{M}_{G_{\lambda^-}}^{[c_i]}(\{\xi_{a}^I\}) \times \mathcal{M}_{g^D}^{[c_i]}(\{\zeta_{b}^D\}) \tag{14} \]

of the model phase spaces \( \mathcal{M}(H) \subseteq C^{nm} \) where fibres \( \mathcal{M}_{[c_i]}(\{\xi_{a}^I, \zeta_{b}^D\}) \) are \( G_{\lambda^-} \otimes g^D \)-invariant algebraic manifolds (or cell complexes) determined via dequantizing subspaces \( L([c_i]) \) and introducing curvilinear coordinates \( \xi_{a}^I \) and \( \zeta_{b}^D \) related to \( G_{\lambda^-} \) and \( g^D \)-generators respectively; herewith the numbers \( c_i \) play the role of topological charges (cf. 10). In general cases coordinates \( \xi_{a}^I, \zeta_{b}^D \) are introduced via using so-called ”mean-field approximations” as standard (”averaging” ) procedures of dequantizing quantum.
problems \(^3\). If \(G_i = \exp(g_i)\) and \(G^D = \exp(g^D = h)\) are Lie groups coordinates \(\xi_i^0, \zeta_i^0\) are associated in a natural way with parameters of special displacement operators \(S_{g_i}(\{\xi_i^0\}) = \exp[\sum \phi_i(\{\xi_i^0\})g_i^0], g_i^0 \in g_i; S_h(\{\zeta_i^0\}) = \exp[\sum \varphi_i(\{\zeta_i^0\})F_i]\) of groups \(G_i, G^D\) which define \(G_i \otimes G^D\)-orbit-type generalized CS \(^2\)

\[
|\{\xi_i^0; \zeta_i^0\}; \psi_0 \rangle \equiv S_{g_i}(\{\xi_i^0\}) S_h(\{\zeta_i^0\}) |\psi_0 \rangle, \quad |\psi_0 \rangle \in L([c_i]) = \text{Span}\{[c_i]; \nu; \kappa\} \quad (15)
\]
on \(L([c_i])\) and implement a re-parametrization \(|\{\alpha_i \beta\} = |\{\alpha_i \beta(\{\xi_i^0; \zeta_i^0\})\}\rangle\) of the Glauber CS \(|\{\alpha_i \beta\} = D_{nm}(\{\alpha_i \beta\})|0\rangle = \exp(\sum [\alpha_i \beta a_i^* \beta - \alpha_i^* \beta a_i \beta])|0\rangle\) via the factorization \(D_{nm}(\{\alpha_i \beta\}) = S_{g_i}(\{\xi_i^0\}) S_h(\{\zeta_i^0\}) D_{11}(\alpha) S_h^0(\{\zeta_i^0\}) S_{g_i}^0(\{\xi_i^0\})\) of \(D_{nm}(\{\alpha_i \beta\})\)^12. However, direct generalizations of Eqs. (15) are less efficient for \(g^D = \mathcal{E}(\mathcal{B}_{GI})\) because explicit expressions for matrix elements \(|[c_i]; \nu; \kappa]\exp[\sum \gamma_i B_i]|c_i; \nu; \kappa\rangle = \text{absent}.

On the other hand, the introduction of three classes of collective variables \((C_i \in r, I_j \in \mathcal{E}(\mathcal{B}_{GI}), G^r_i \subset G_i)\) leads to a dimension reduction of dynamical problems governed by Hamiltonians (13) in both Schroedinger and Heisenberg (for dynamic variables \(I_j = F_j, T_j\)) pictures. Indeed, the Schroedinger and cluster Heisenberg (for \(I_j\)) equations can be written in terms of only variables \(C_i, I_j:\)

\[
a) \ i\hbar \frac{dU_H(t)}{dt}|\Psi_0 \rangle = H U_H(t)|\Psi_0 \rangle, \quad b) \ i\hbar \frac{dI_j(t)}{dt} = [I_j(t), H] = \mathcal{L}(\{I_j(t)\}) \quad (16)
\]
where \(U_H(t)\) is the time-evolution operator induced by \(H = H_{GI}\) from Eq. (13) and Eqs. (16b), in a sense, determine a generalized dynamics on noncommutative algebraic manifolds \(\mathcal{M}_{NC}(\{L_i\}) = \{I_j : \tilde{C}_a(\{I_i\}) = C^a_i\}\) (see (8)). If Hamiltonians (13) do not contain operators \(T_k \in t\) both Eqs. (16) are solved by group-theoretical methods even for time-dependent \(H_{GI} \quad ^{2,3} : U_H(t) = \exp(\sum \nu_a(t)F_a) = \prod_a \exp(\eta_a(t)F_a), I_j(t) = U_H(t)I_j U_H^\dagger(t) = \sum_a B_a(t)I_j, I_j \in \mathcal{B}_{GI}\) where the second (factorized) form of \(U_H(t)\) is more adequate for physical calculations in comparison with the first one. However, such simple expressions are not valid for general (even time-independent) Hamiltonians (13) due to the absence of the group property for elements of \(\exp[\mathcal{E}(\mathcal{B}_{GI})]\) and nonlinearity of \(\mathcal{L}(\{I_j(t)\})\) in Eq. (16b) \(^3\). In this case for \(U_H(t), I_j(t)\) one can get only “\(I_j\)-power series” representations

\[
U_H(t) = \sum_{[k_j]} A^H_{[k_j]}(t) \prod_a I_a^{k_a} \equiv U_H(\{I_j\}; t), \quad I_j(t) = \sum_{[k_j]} B^j_{[k_j]}(t) \prod_a I_a^{k_a} \equiv I_j(\{I_j\}; t) \quad (17)
\]
where the coefficients \(A^H_{[k_j]}(t), B^j_{[k_j]}(t)\) are determined from differential-difference equations obtained via the substitution of Eqs. (17) in (16) and the use of CR (6) \(^3\). These equations define (non-classical) special functions related also with solutions of differential equations stemming from realizations of the type (10) for PLA \(\mathcal{E}(\mathcal{B}_{GI})\).

However, at present, simple analytical expressions for these functions are absent even in the case of simplest PLA \(sl^P_{2d}(2)\) \(^6\) that necessitates to separate “principal parts” (or asymptotics) \(U^P_0(\{I_j\}; t), I^0_j(\{I_j\}; t)\) in \(U_H(\{I_j\}; t) = U^P_0(\{I_j\}; t)\{1 + \ldots\} \quad (18)\).
\[ \epsilon([C_a]) F(t) + \ldots, T_j([I_j]; t) \approx T_j^D([I_j]; t) \] which possess special (simplifying physical calculations) properties and determine quasiclassical factors in model dynamics.6 So, e.g., one can take solutions of classical dynamic equations, obtained via averaging Eqs. (16b), as suitable approximations for \( T_j^D([I_j]; t) \). At the same time asymptotics \( U_H^0([I_j]; t) \) can be obtained from (determined by \( g^D \) CS \( [c_i]; \nu; \xi] = \mathcal{S}_E(B_{GL})([c_i]; [\nu] \in \mathcal{L}_S(B_{GL})([c_i]; [\nu]) \)) quasiclassical representations of \( U_H(t) \):

\[ U_H(t) = \sum_{[c_i]} \int d\mu^{[c_i]}(\xi_0) \int d\mu^{[c_i]}(\xi_1) K_{[c_i]}(\xi_1|\xi_0) \sum_{\nu} \langle [c_i]; [\nu; \xi_1] \langle [c_i]; [\nu; \xi_0] \rangle \]

where \( d\mu^{[c_i]}(\xi_0) \) is a \( E(B_{GL}) \)-invariant measure on \( \mathcal{M}^{[c_i]; \nu}(\xi) \subset \mathcal{M}(H) \) and the \( \nu \)-independent (in view of Eq. (13)) kernel \( K_{[c_i]}(\xi_1|\xi_0) = \langle [c_i]; [\nu; \xi_0] | U_H(t) | [c_i]; [\nu; \xi_1] \rangle = \int \exp[ih^{-1}S^{[c_i]}(z(t))] \Pi d\mu^{[c_i]}(z(t)) \) has the \( E(B_{GL}) \)-path integral form.10,11 Its calculation in the stationary phase approximation determines \( U_H^0([I_j]; t) \). However, the problem of finding adequate \( \mathcal{S}_E(B_{GL})(\xi) \) is not still solved completely.

So, within the DAP framework \( G_i \)-invariance of \( H_{GI} \) classifies states \( |\Psi\rangle \in L(H) \) yielding potential kinematic forms for, generally, degenerate (with \( d^{G_i}([c_i]) \neq 1 \) \( g^D \)-domains \( L([c_i]) \)). Non-degenerate \( g^D \)-domains with the identical IR \( D^{[c_i]=0}(G_i) \equiv \{ I \} \) \( (I \) is the operator identity) describe completely \( G_i \)-invariant \( (G_i \)-scalar) subsystems having unusual (extremal) physical features while degenerate \( g^D \)-domains have "rest" \( G_i \) characteristics stipulating an appearance of critical phenomena in \( L([c_i]) \). At the same time CS techniques and associated path integral schemes provide efficient tools to solve dynamical problems enabling to reveal new cooperative phenomena in \( G_i \)-invariant models.6 Furthermore, \( G_i \)-invariance of \( L([c_i]) \) allows to examine on \( L([c_i]) \) \( G_i \)-dynamics determined by \( g^D \)-invariant "intrinsic" Hamiltonians \( H(g_i^a \in g_i = lnG_i) \) with considering \( g^D \)-variables as "dummy" ones.3,12

3 Dual algebraic pairs in action: applications in polarization and nonlinear quantum optics

In this Section we demonstrate an efficiency of the DAP concept and techniques on recent examples of their applications in quantum optics.

The first example3,12, manifesting the kinematic significance of DAP, is due to the gauge \( SU(2) \) invariance of free light fields described by Hamiltonians \( H_{FI} \) of the form (12) with \( m = 2, \omega^{\alpha\beta}_{ij} = \omega_i \delta_{ij} \delta_{\alpha\beta}, g^{\alpha\beta}_{ij} \equiv 0, H_{G^D}^G \equiv 0 \) and the Hilbert space \( L_F(2m) = \text{Span}\{\{|n_{\pm}\}\} \) where \( i = 1, \ldots, n, \beta = \pm \) label, respectively, spatiotemporal (frequency) and polarization (in the helicity basis) modes of light. Then, taking \( G_i = SU(2) \) \( \equiv \{ \exp[\sum_{n=0,\pm n_u \varphi_n^\pm] : P_0 = \frac{1}{2} \sum a_i^\dagger \varphi_n^\pm a_i - \varphi_n^\pm a_i^\dagger \varphi_n^\pm, P_\pm = \sum a_i^\dagger \varphi_n^\pm a_i \} \), we get DAP \( (G_1 = SU(2) = G_i, g_2 = so^*(2m) \equiv \text{Span}\{E_{ij}, X_{ij}, X_{ij} = (X_{ij})^\dagger : E_{ij} = \sum_{\beta=\pm} a_i^\dagger \beta a_i^\dagger \beta, X_{ij} = a_i a_j - a_i a_j \} = g^D = h) \) acting on \( L_F(2n) \). The decomposition (5) for \( L_F(2n) \) is specified by determining the "polarization domains"

\[ L(c_1 = p) = \text{Span}\{|p; \nu; \kappa\} \propto (P_+)^{\nu+\nu} \mathcal{D}^{\kappa}(\{E_{ij}\})(X_{12})^{\kappa_1}|p\rangle, \ |p\rangle = (a_i^\dagger)^{2p}|0\rangle \] (19)
in \( L_F(2n) = \sum L(c_i) \) as eigenspaces of the \( SU(2) \) Casimir operator \( P^2 = P_0^2 + \frac{1}{2}(P_+P_- + P_-P_+) = C_1 \in C(G_1 = SU(2), G_2 = so^*(2m)) : P^2|p;\nu;\kappa\rangle = c_1(p)|p;\nu;\kappa\rangle \) whose eigenvalues \( c_1(p) = p(p+1), p = 0, \frac{1}{2}, 1, \ldots \) determine values \( p \) of the polarization \( (P)\)-quaispin replacing the non-gauge-invariant usual spin for light fields.

This decomposition of \( L_F(2n) \) provides a new (symmetry) treatment of polarization structure of light that enabled us to reveal an unusual (coherent) sort of unpolarized light \( (P\) - scalar light) given by states \( |0_p\rangle \in L(p = 0) = \text{Span}\{|p = 0;\nu = 0;\kappa \neq 0\rangle \times \prod\langle X_{ij}^+|\kappa_{ij}|0\rangle\} \) (existing for \( L_F(2n), n \geq 2 \)) with characteristic property

\[
P_{\alpha=0,\pm}|0_p\rangle = 0 \iff \langle 0_p|P_+^{a_1}P_+^{a_2}P_0^{a_0}|0_p\rangle = 0 \quad \forall a_1 + a_2 + a_0 \geq 1 \tag{20}\]

of the "polarization vacuum". For \( n = 2 \) (when \( P_\alpha = P_{\alpha a} + P_{2a} \)) in view of Eq. (20) states of \( P\)-scalar light generalize so-called Bell states widely used in quantum physics for examining both fundamental (EPR-paradox, teleportation etc.) and applied (design of quantum computers, optical communication) problems. Furthermore, they give positive solutions of the problem of existence of non-stochastic waves of unpolarized light \( (A. \text{ Fresnel, 1821}) \) having the negative solution in classical optics.

According to general remarks of Section 2 polarization domains \( L(p) \) are dynamically stable under Hamiltonians \( H_{G_1 \otimes g^p} = H_f + H_{so^*(2m)} + H_{SU(2)} \) with

\[
H_f = \sum_i \omega_i E_{ii}, \quad H_{so^*(2m)} = \sum_{ij \neq j}[\omega_{ij} E_{ij} + g_{ij} X_{ij} + g_{ij}^* X_{ij}^+], \quad H_{SU(2)} = \sum_\alpha \Omega_\alpha P_\alpha \tag{21}\]

where \( H_{so^*(2m)} \) and \( H_{SU(2)} \) determine, respectively, dynamics of biphoton clusters \( X_{ij}^+ \) (including their production) and a purely polarization dynamics. These dynamics are adequately described in terms of the \( SU(2)_p \otimes so^*(2m)\)-orbit-type CS of the form (15) with \( S_{so^*(2)}(\xi) = \exp(\xi P_+ - \xi^* P_-), S_{so^*(2m)}(\xi_{ij}) = \exp(\sum\xi^a E_{ii+1} - \xi_{ij}^* E_{i+1j} + \xi_{ij}^* X_{i+1j}^+ - \xi_{ij}^* X_{ij}^+) \) which, in particular, yield elegant solutions of many quantum problems (such, e.g., as calculations of geometric phases, developments of quantum tomography schemes and analysis of quantum interference patterns).

The second example leads to applications of PLA formalism, is given by models with Hamiltonians \( H^{mps}(n; s) = \omega_0 a_0^+ a_0 + H_{G_1}^{so^1} \) from (12), where \( g_{ij}^{0\beta} \equiv 0 \) and

\[
H_{G_1}^{so^1} = H_f(n; s) \equiv \sum_{1 \leq i_1 \ldots i_s \leq n} [g_{i_1 \ldots i_s} a_{i_1}^+ \ldots a_{i_s}^+ a_0 + g_{i_1 \ldots i_s}^* a_{i_1} \ldots a_{i_s}^+ a_0], \quad s \geq 2, \tag{22}\]

acting on the Hilbert space \( L_F(n + 1) = \text{Span}\{|\{n_i\}\rangle\}_{i=0,1}^n \) (the "dummy" label \( \beta = 1 \) is omitted) and describing processes of multiphoton scattering. In the case of arbitrary \( g_{i_1 \ldots i_s} \) Hamiltonians \( H^{mps}(n; s) \) have the invariance groups \( G_i = C_n \otimes U_R(1) \) with both discrete \( (C_n = \{e^{2\pi kN/s}\}_{k=0,1}^{s-1}, N = \sum_{i=1}^n E_{ii}, E_{ii} = a_i^+ a_i \) ) and continuous \( (U_R(1) = \{exp(i\phi R_1)\}, R_1 = |N + s E_0|/[s + 1] \) ) factors. Then \( r = \{R_1\}, B_{G_1} \equiv \{E_{ij} = a_i^+ a_j, V_{i_1 \ldots i_s} = a_{i_1}^+ \ldots a_{i_s}^+ a_0 \in v^s, V_{i_1 \ldots i_s} a_{i_1} \ldots a_{i_s}^+ a_0 \in v^s\}, \) where \( v^s \) is the -r- rank symmetric \( u(n)\) - tensor, and the DAP \( (G_i = C_n \otimes U_R(1), G_2 = g^D = \ldots \)
\[ \mathcal{E}_{R}^{P}(u(n); v^{(s)}) \] acts on \( L_{F}(n+1) \). In view of the \( G_{i} \), Abelian nature the decomposition (5) for \( L_{F}(n+1) \) contains only non-degenerate \( 2j + 1 \)-dimensional \( g^{D} \)-domains

\[ L([c_{1}, c_{2}]) = \text{Span} \{ [c_{1}; \kappa] \propto \mathcal{D}_{R}^{P}(\{E_{ij}\})(V_{1...n}^{+})^{s}[c_{i}], [c_{1}] = (a_{i}^{+})^{k}(a_{0}^{+})^{2j}[0] \} \quad (23) \]

where \( c_{1} = k = 0, 1, \ldots, s - 1, c_{2} = 2j = 0, 1, \ldots \) are determined by eigenvalues of \( G_{i} \)-invariant operators. At the same time, in view of CR (6), \( G_{i} \)-invariant form (13) of the Hamiltonians \( H^{\text{mps}}(n; s) \) can be given by the expressions

\[ H^{\text{mps}}(n; s) = \hbar S_{E}(\xi) \left[ \sum_{i,j=1}^{n} \Omega_{ij}E_{ij} + \tilde{g}V_{1...n}^{+} + \tilde{g}^{*}V_{1...n} + \frac{\omega_{0}}{s}(R_{1} - N) \right] S_{E}^{\dagger}(\xi) \quad (24) \]

\((S_{E}(\xi) = \exp\{\sum_{i>j}[\bar{\xi}_{ij}E_{ij} - \xi_{ij}E_{ji}]\})\) which are most suitable for analyzing Eqs. (16).

However, nowadays we can get only (quasi)classical solutions of these equations, and besides, solely in the case \( n = 1 \) when PLA \( \mathcal{E}_{R}^{P}(u(n); v^{(s)}) \) is reduced to \( su_{ps}^{2}(2) \) defined by Eqs. (7)\textsuperscript{3,6}. For example, in this case Eqs. (16b) are nonlinear analogs

\[ i\hbar \frac{dV_{0}}{dt} = \tilde{g}V_{+} - g^{*}V_{-}, \quad i\hbar \frac{dV_{+}}{dt} = -aV_{+} - \tilde{g}^{*}\mathcal{P}(V_{0}), \quad i\hbar \frac{dV_{-}}{dt} = aV_{-} + \tilde{g}\mathcal{P}(V_{0}) \quad (25) \]

\((V_{0} = \frac{1}{s+1}[N - sE_{00}], V_{+} = V_{1...n}^{+}, V_{-} = V_{1...n})\) of the well-known linear Bloch equations for \( su(2) \). In turn, solutions of Eqs. (25) are equivalent to those of the only equation

\[ d^{2}V_{0}(t)/dt^{2} = a(H - C) - a^{2}V_{0}(t) + 2 | \tilde{g} |^{2} \mathcal{P}(V_{0}(t)) \quad (26) \]

which have in the cluster mean-field approximation \((\langle f(\{V_{a}\}) \rangle = f(\langle V_{a} \rangle))\) quasi-classical solutions in terms of (hyper)elliptic functions \textsuperscript{6} naturally arising in soliton theories \textsuperscript{9,10}. On other hand, using Eqs. (9) in this case one can transform linear Hamiltonians (24) to an essentially nonlinear form

\[ H^{\text{mps}}(1; s) = \hbar[\Delta Y_{0} + Y_{+}g(Y_{0}) + g^{*}(Y_{0})Y_{+} + \delta(R_{1})], \quad g(Y_{0}) = \tilde{g}[\phi(V_{0})]^{1/2} \quad (27) \]

depending on variables \( Y_{a} \in su(2) \) that enabled us to obtain (via path integral representations (18) with using \( SU(2) \) CS of the form (15)) quasi-classical \( SU(2) \)-asymptotics

\[ U_{H}^{0}(\{Y_{a}\}; t) = \exp[\sum_{i} a_{i}(t)Y_{i}], \quad H = H^{\text{mps}}(1; s) \quad (28) \]

of the evolution operators \( U_{H}(t) \) where time-dependent coefficients \( a_{i}(t) \) are determined through solutions of classical versions of Eqs. (26)\textsuperscript{3,6}.

### 4 Conclusion

So, we demonstrated natural appearances and an efficiency of DAP and PLA formalism in examining multiboson models with \( G_{i} \)-invariant Hamiltonians. In conclusion we outline some directions of further studies concerning physical applications.
They include: 1) specifications of quasiclassical representations (18) for $U_H(t)$ based on determining adequate form CS related to exponentials $\exp(\hat{g}^P(\mathcal{B}_{GL}))$ and on generalizations of the transformations (9); 2) extractions of their "group -like" asymptotics (extending (28)) and examinations (in view of Eqs. (10)) of connections of latters with the Maslov quasiclassical asymptotics for partial equations in quantum mechanics$^{14}$; 3) applications of geometric methods $^8,9$ in analysis (cf. $^9,10$) of nonlinear operator evolution equations of the type (26) stemming from the "cluster" Heisenberg equations (16b), (25) and their quasiclassical approximations (taking into account that Eqs. (27) together with transformations $\mathcal{M}_{q_1}^{[\xi]}(\{\xi_1^D\}) \rightarrow S_{2,[j]}^2(\{\zeta_j\})$ of fibers in (14) into the Bloch spheres describe a geometrization of model dynamics).

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References

1. A.O. Barut and R. Racka, *Theory of Group Representations and Applications* (PWN - Polish Sci. Publishers, Warszawa, 1977).
2. A.M. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin e.a., 1986).
3. V.P. Karassiov, *J. Phys.*, A25, 393 (1992); A27, 153 (1994); *Rep. Math. Phys.*, 40:2, 235 (1997); *Yad. Fiz.*, 63, 714 (2000); *J. Rus. Laser Res.*, 21, 370 (2000).
4. R. Howe, *Remarks on Classical Invariant Theory*. (Yale University Preprint, 1976); S. Sternberg, *Lect. Notes Phys.*, 79, 117 (1978).
5. K. Schoutens, A. Sevrin, P. Nieuwenhuizen, *Commun. Math. Phys.*, 124, 87 (1989); M. Rocek, *Phys. Lett.*, A255, 554 (1991).
6. V.P. Karassiov, *Phys. Lett.*, A 238, 19 (1998); *J. Rus. Laser Res.*, 20, 239 (1999); V.P. Karassiov, A.B. Klimov, *Phys. Lett.*, A 189, 43 (1994).
7. H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, 1939).
8. A.T. Fomenko, *Differential Geometry and Topology. Additional Chapters* [Russian] (Moscow University Press, Moscow, 1983); A.S. Mishchenko, *Vector Bundles and Their Applications* [Russian] (Nauka, Moscow, 1984).
9. I. A. Taimanov, *Usp. Math. Nauk*, 52:1, 150 (1997).
10. R. Rajaraman, *Solitons and Instantons*, (North-Holland, Amsterdam, 1982).
11. A. Odzijewicz, *Commun. Math. Phys.*, 114, 577 (1988); 150, 385 (1990).
12. V.P. Karassiov, *Bull. Lebedev Phys. Inst.* (Allerton Press), N9, 34 (1999); V.P. Karassiov, A.V. Masalov, *J. Opt.*, B 4, S366 (2002).
13. D. Bowmester, A.Ekert, A. Zeilinger, *The Physics of Quantum Information* (Springer-Verlag, Berlin e.a., 2000).
14. V.P. Maslov and M.V. Fedorink, *Semi-classical Approximation in Quantum Mechanics* (D. Reidel, Dordrecht, 1981).