On the potential energy in a gravitationally bound two-body system with arbitrary mass distribution

Klaus Wilhelm • Bhola N. Dwivedi

Abstract The potential energy problem in a gravitationally bound two-body system has recently been studied in the framework of a proposed impact model of gravitation [Wilhelm and Dwivedi 2015]. The result was applied to the free fall of the so-called Mintrop–Ball in Göttingen with the implicit assumption that the mass distribution of the system is extremely unbalanced. An attempt to generalize the study to arbitrary mass distributions indicated a conflict with the energy conservation law in a closed system. This necessitated us to reconsider an earlier assumption made in selecting a specific process out of two options [Wilhelm et al. 2013]. With the result obtained here we can now make an educated selection and reverse our choice. The consequences are presented and discussed in detail for several processes. Energy and momentum conservation could now be demonstrated in all cases.

Keywords gravity; potential energy; closed systems; impact model

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1 Introduction

An earlier study on the potential energy problem [Wilhelm and Dwivedi 2015] had been motivated by the remark that the “potential energy is a rather mysterious quantity” [Carlip 1998]. It led to the identification of the “source region” of the potential energy for the special case of a system with two masses $M_E$ and $M_M$ subject to the condition $M_E \gg M_M$. An attempt to generalize the study without this condition required either violations of the energy conservation principle as formulated by von Laue [1920] for a closed system, or a reconsideration of an assumption we made concerning the gravitational interaction process [Wilhelm et al. 2013]. The changes necessary to comply with the energy conservation principle will be the theme of this article, in addition to a generalization of the potential energy concept for a system of two spherically symmetric bodies A and B with masses $m_A$ and $m_B$ without the above condition.

We will again exclude any further energy contributions, such as rotational or thermal energies, and make use of the fact that the external gravitational potential of a spherically symmetric body of mass $m$ and radius $r_m$ is that of a corresponding point mass at the centre, i.e.

$$\phi_m(r) = -G \frac{m}{r} \quad (r > r_m) ,$$

where $r = |r|$ is the distance from the centre of the sphere and $G_N = 6.67384(80) \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$, the constant of gravity. The direction of $r$ will be reckoned positive in later calculations.

The energy $E_m$ and momentum $p$ of a free particle with mass $m$ moving with a velocity $v$ relative to an inertial reference system are related by

$$E_m^2 - p^2 c_0^2 = m^2 c_0^4 ,$$

1In this context it is of interest that Brillouin [1998] discussed this problem in relation to electrostatic potential energy.

2This and other constants are taken from 2010 CODATA at [http://physics.nist.gov/cuu/Constants]
where \( c_0 = 299792458 \text{ m s}^{-1} \) (exact\(^3\)) is the speed of light in vacuum and

\[
p = v \frac{E_m}{c_0}
\]

(Einstein 1905a,b). For an entity in vacuum with no rest mass \((m = 0)\), such as a photon (cf. Einstein 1905c; Lewis 1926; Okun 2009), the energy-momentum relation in Eq. (2) reduces to

\[
E = p_X c_0
\]

\( E \) is the energy and \( p_X \) the momentum.

2 Gravitational impact model and quadrupoles

In analogy to Eq. (4), we have postulated hypothetical massless entities (named “quadrupoles”) that obey the energy-momentum equation

\[
E_G = p_G c_0
\]

and constructed a gravitational impact model with a background flux of these entities modified by gravitational centres (for details see Wilhelm et al. 2013). Equal absorption and emission number rates proportional to the mass of a body had been assumed. Nevertheless, the energy–and momentum–change rates must be different in order to emulate the gravitational attraction in line with Newton’s law of gravity. We, therefore, had to introduce a reduction parameter \( Y \ll 1 \) and defined the energy absorption rate by

\[
\frac{dE_{G,ab}}{dt} = p_G c_0 \frac{dN_m}{dt}
\]

and the energy emission rate by

\[
\frac{dE_{G,m}}{dt} = p_G c_0 (1 - Y) \frac{dN_m}{dt},
\]

where \( dN_m/\text{dt} = m c_0^2/(2h) \) is half the intrinsic de Broglie frequency of a body with mass \( m \).

According to Newton’s third law, the interaction rate of quadrupoles with bodies \( A \) and \( B \) must be the same for both bodies even if \( m_A \neq m_B \):

\[
\frac{dN_{m_A,m_B}}{dt} = \frac{dN_{m_B,m_A}}{dt}
\]

The rate required to emulate Newton’s law of gravitation critically depends on the details of the process. A spherically symmetric emission of a liberated quadrupole had been assumed by Wilhelm et al. (2013). Further studies summarized below have, however, indicated that an anti-parallel emission with respect to the incoming quadrupole is more appropriate, because conflicts with the energy and momentum conservation principles in closed systems can be avoided by the second choice. It leads to a momentum transfer rate from \( m_A \) to \( m_B \) of \((2 - 3Y)\) \( p_G \) \( dN_{m_A,m_B}/\text{dt} \) between \( m_A \) and \( m_B \) by interacting quadrupoles. This implies that \((2 - Y)\) \( p_G \) \( dN_{m_A,m_B}/\text{dt} \) (that would have been balanced by interactions from the opposite direction) will not be absorbed by \( m_B \) from the background.

Consequently, Eq. (17) of Wilhelm et al. (2013) has to be modified to a momentum transfer rate of

\[
\frac{dP_{m_A,m_B}(r)}{dt} = -2 p_G Y \frac{dN_{m_A,m_B}(r)}{dt} = -2 p_G Y \frac{\kappa_G c_0}{4 \pi \hbar} \frac{m_A m_B}{r^2} = K_G(r) = -G_Y \frac{m_A m_B}{r^2}
\]

where \( h = 6.626068 \times 10^{-34} \text{ J s} \) is the Planck constant and the last two terms are Newton’s law of gravitation.

The directionally emission assumption reduces the absorption coefficient by a factor of two, i.e. \( \kappa_G^2 = \kappa_G/2 \), and requires, with constant \( \eta_G \), a corresponding increase of the spatial density of quadrupoles to \( \rho_G^2 = 2 \rho_G \), as well as twice the quadrupole energy density \( \epsilon_G \) in Eq. (20) of Wilhelm et al. (2013). All other quantities are not affected, in particular, the relative mass accretion rate \( \lambda \) will not change.

In Fig. \( 1 \) we show the new relationships between quadrupole energy, their spatial density and the corresponding energy density. The extreme logarithmic scale makes it difficult to see the factor of two, however, we indicate in the modified figure not only the range of the most likely electron mass radius \( r_{\text{ce}} \) from our earlier studies (Wilhelm and Dwivedi 2011, 2013), but also the parameter ranges of \( Y \) and the spatial energy density in the upper panels that do not yield a realistic impact model (cf. Wilhelm et al. 2013). The influence of a potential shielding effect by a third body placed between two gravitational centres (cf. Drude 1897) is also affected and will be treated in Sect. 4.

3 The potential energy

3.1 Classical mechanics

We assume that two spherically symmetric bodies \( A \) and \( B \) with masses \( m_A \) and \( m_B \), respectively, are placed

\(^3\text{Follows from the definition of the SI base unit “metre” (Bureau International des Poids et Mesures, BIPM, 2006).}\)

\(^4\text{For weak gravitational interactions, the spatial density of quadrupoles with reduced momentum is very small compared to that of the background.}\)
in space remote from other gravitational centres at a distance of \( r + \Delta r \) reckoned from the position of A. Initially both bodies are at rest with respect to an inertial reference frame represented by the centre of gravity of both bodies. The total energy of the system then is with Eq. (2) for the rest energies and Eq. (1) for the potential energy

\[
E_S = (m_A + m_B) E_0^2 - G N \frac{m_A m_B}{r + \Delta r}.
\]  

(10)

The evolution of the system during the approach of A and B from \( r + \Delta r \) to r can be described in classical mechanics. According to Eq. (11), the attractive force between the bodies during the approach is approximately constant for \( r \gg \Delta r \), resulting in accelerations of \( b_A = |K(r)|/m_A \) and \( b_B = -|K(r)|/m_B \), respectively. Since the duration \( t \) of the free fall of both bodies is the same, the approach of A and B can be formulated as

\[
\Delta r = s_A - s_B = \frac{1}{2} \left( b_A - b_B \right) t^2 = \frac{1}{2} \left( \frac{1}{m_A} + \frac{1}{m_B} \right) |K(r)| t^2,
\]  

(11)

showing that \( s_A m_A = -s_B m_B \), i.e. the centre of gravity stays at rest. Multiplication of Eq. (11) by \( |K(r)| \) gives the corresponding kinetic energy equation

\[
|K(r)| \Delta r = \frac{1}{2} \left( \frac{K^2(r) t^2}{m_A} + \frac{K^2(r) t^2}{m_B} \right) = \frac{1}{2} \left( m_A v_A^2 + m_B v_B^2 \right) = T_A + T_B,
\]  

(12)

The kinetic energies \( T_A \) and \( T_B \) should, of course, be the difference of the potential energy in Eq. (10) at distances of \( r + \Delta r \) and r. We find indeed

\[
-G N \frac{m_A m_B}{r + \Delta r} + G N \frac{m_A m_B}{r} \approx G N \frac{m_A m_B}{r^2} \Delta r = |K(r)| \Delta r.
\]  

(13)

3.2 Quadrupole energy deficiency

We may now ask the question, whether the impact model can provide an answer to the "mysterious" potential energy problem in a closed system. Since the model implies a secular increase of mass of all bodies, it obviously violates a closed-system assumption. The

\[
\text{Eqs. (2) and (3) together with } E_0 = m c^2 \text{ (Einstein 1905b) and } \gamma = 1/\sqrt{1 - v^2/c^2} \text{ yield the relativistic kinetic energy of a massive body: } T = E - E_0 = E_0 (\gamma - 1). \text{ The evaluations for } T_A \text{ and } T_B \text{ agree in very good approximation with Eq. (12) for small } v_A \text{ and } v_B.
\]
increase is, however, only significant over cosmological time scales, and we can neglect its consequences in this context. A free single body will, therefore, still be considered as a closed system with constant mass. In a two-body system both masses $m_A$ and $m_B$ will be constant in such an approximation, but now there are quadrupoles interacting with both masses.

The number of quadrupoles travelling at any instant of time from one mass to the other can be calculated from the interaction rate in Eq. (9) multiplied by the travel time $r/c_0$

$$\Delta N_{m_A,m_B}(r) = \frac{k_G^*}{8 \pi \hbar} \frac{m_A m_B}{r}.$$

(14)

The same number is moving in the opposite direction. The energy deficiency of the interacting quadrupoles with respect to the corresponding background then is together with Eq. (2) for each body

$$\Delta E_Q(r) = -p_G Y \kappa_G \frac{c_0}{8 \pi \hbar} \frac{m_A m_B}{r} = -G_G^* \frac{c_0}{8 \pi \hbar} \frac{m_A m_B}{r} = -G_N \frac{m_A m_B}{2 r}.$$

(15)

The last term shows – with reference to Eq. (1) – that the energy deficiency $\Delta E_Q$ equals half the potential energy of body A at a distance $r$ from body B and vice versa.

We now apply Eq. (15) and calculate the difference of the energy deficiencies for separations of $r + \Delta r$ and $r$ for interacting quadrupoles travelling in both directions and get

$$2 [\Delta E_Q(r + \Delta r) - \Delta E_Q(r)] = -G_N m_A m_B \left( \frac{1}{r + \Delta r} - \frac{1}{r} \right) = |K(r)| \Delta r.$$

(16)

Consequently, the difference of the potential energies between $r + \Delta r$ and $r$ in Eq. (13) is balanced by this difference of the total energy deficiencies.

The physical processes involved can be described as follows:

1. The number of quadrupoles on their way for a separation of $r + \Delta r$ is smaller than that for $r$, because the interaction rate depends on $r^{-2}$ according to Eq. (9), whereas the travel time is proportional to $r$.
2. A decrease of $r + \Delta r$ to $r$ during the approach of A and B increases the number of quadrupoles with reduced energy.
3. The energies liberated by energy reductions are available as potential energy and are converted into kinetic energies of the bodies A and B.
4. With Eqs. (2) and (3) and the approximations in Footnote 4 it follows that the sum of the kinetic energies $T_A$ and $T_B$, the masses A and B plus the total energy deficiencies of the interacting quadrupoles can indeed be considered to be a closed system as defined by von Laue (1920).

Further details of the interaction of quadrupoles with massive particles have been presented in Wilhelm et al. (2013) explaining the actual conversion of the liberated energy into kinetic energy.

4 Multiple quadrupole interactions and the perihelion advances

In large gravitational centres, such as the Sun, multiple interactions have to be expected before the quadrupoles are emitted with reduced energy and momentum. The process assumed in Wilhelm et al. (2013) led to secondary emission centres in the direction of the orbiting body. Using published data on the secular perihelion advances of the inner planets of the solar system and the asteroid Icarus Wilhelm and Dwivedi (2014) found that the effective gravitational centre is displaced from the centre of the Sun by approximately 4400 m. Since an analytical derivation of this value from the mass distribution of the Sun was beyond the scope of the study, the modified process just has to account, at least in principle, for such a displacement.

The proportionality of the linear term in the binomial theorem with the exponent in

$$(1 - Y)^n \approx 1 - n Y \quad \text{for} \quad Y \ll 1$$

(17)

suggests that a linear superposition of the effects of multiple intercations will be a good approximation, if $n$ is not too large. Energy reductions according to Eq. (7) are therefore not lost as claimed by Drude (1897), but they are redistributed to other emission locations within the Sun. This has two consequences: (1) The total energy reduction is still dependent on the solar mass, and (2) since emissions from matter closer to the surface of the Sun in the direction of an orbiting object is more likely to escape into space than quadrupoles from other locations, the effective gravitational centre is displaced from the centre of the Sun towards that object.

5 Bodies in motion and photon-quadrupole interactions

5.1 Moving massive bodies

Based on the impact model developed for massive bodies at rest (Wilhelm et al. 2013), we applied the same concept to bodies in motion and to photons
The modified interaction process between quadrupoles and massive bodies presented above leads to the same consequences in Sect. 2 of that paper, whenever the gravitational absorption coefficient, the spatial density of the quadrupoles, or the constant $Y \kappa$ are concerned. The relations are:

$$\kappa^*_G = \kappa_G / 2, \quad \rho^*_G = 2 \rho_G \quad \text{and} \quad G^*_G = G_G / 2.$$  

The other quantities and the results are not affected, because the changes of the gravitational absorption coefficient and the spatial density of the quadrupoles cancel each other.

5.2 Photons

The deflection of light by gravitational centres according to the General Theory of Relativity (Einstein 1916) and its observational detection by Dyson et al. (1920) leave no doubt that a photon is deflected by a factor of two more than expected relative to a corresponding massive particle. Since in our concept the interaction rate between photons and quadrupoles is twice as high as for massive particles of the same total energy, the reflection of a quadrupole from a photon with a momentum of $(1 - Y)p_G$ must also be anti-parallel to the incoming one, i.e. a momentum of $-2Yp_G$ will be transferred. Otherwise the correct deflection angle for photons cannot be obtained. This modified interaction process has one further important advantage: the reflected quadrupole can interact with the deflecting gravitational centre and – through the process outlined in the paragraph just before Eq. (9) – transfers $2Yp_G$, in compliance with the momentum conservation principle. In the old scheme, the violation of this principle had no observational consequences, because of the extremely large masses of relevant gravitational centres, but the adherence to both the momentum and energy conservation principles is very encouraging and clearly favours the new concept.

Basically the same arguments are relevant for the longitudinal interaction between photons and quadrupoles. The momentum transfer per interaction will be doubled, but the gravitational absorption coefficient will be reduced by a factor of two. Together with an increased quadrupole density, all quantities and results are the same as before. However, a detailed analysis shows that the momentum conservation principle is now also adhered to.

6 Conclusions

In the framework of a recently proposed gravitational impact model (Wilhelm et al. 2013) – with a modification discussed in this work – the physical processes during the conversion of gravitational potential energy into kinetic energy have been described for two bodies with masses $m_A$ and $m_b$ and the source of the potential energy could be identified. Multiple interactions of quadrupoles leading to shifts of the effective gravitational centre of a massive body from the “centre of gravity” are significantly affected by the modified concept, however, without, changing the results presented in Wilhelm and Dwivedi (2014). The interaction of quadrupoles with photons had to be modified as well, but the modification did not change the results, with the exception that now both the energy and momentum conservation principles are fulfilled.
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