Half-line compressions and finite sections of discrete Schrödinger operators with integer-valued potentials

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Abstract. We study 1D discrete Schrödinger operators $H$ with integer-valued potential and show that, (i), invertibility (in fact, even just Fredholmness) of $H$ always implies invertibility of its half-line compression $H_+$ (zero Dirichlet boundary condition, i.e. matrix truncation). In particular, the Dirichlet eigenvalues avoid zero – and all other integers. We use this result to conclude that, (ii), the finite section method (approximate inversion via finite and growing matrix truncations) is applicable to $H$ as soon as $H$ is invertible. The same holds for $H_+$.

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1 Introduction

Discrete Schrödinger operators. We look at so-called discrete Schrödinger operators in 1D, each acting via

$$(Hx)_n = x_{n-1} + v(n)x_n + x_{n+1}, \quad n \in \mathbb{Z},$$

as a bounded linear operator $H$ on $\ell^2(\mathbb{Z})$. The (bounded) function $v : \mathbb{Z} \to \mathbb{R}$ is referred to as the potential of $H$. The operator $H$ acts via matrix-vector multiplication by a two-sided infinite matrix $(H_{ij})_{i,j \in \mathbb{Z}}$ with main diagonal $H_{ii} = v(i)$, super and subdiagonal $H_{i,i \pm 1} = 1$ and all other entries equal to zero.

Now let $\mathbb{Z}_+ := \{k \in \mathbb{Z} : k \geq 0\}$ and, given the operator (1) on $\ell^2(\mathbb{Z})$, we refer to the corresponding operator

$$(H_+x)_n = x_{n-1} + v(n)x_n + x_{n+1}, \quad n \in \mathbb{Z}_+, \quad \text{where} \quad x_{-1} := 0,$$

on $\ell^2(\mathbb{Z}_+)$ as the half-line compression $H_+$ of $H$.

From the operator perspective, $H_+$ equals $PHP\mid_{\text{im } P} : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$, where $P$ is the operator of multiplication by the characteristic function of $\mathbb{Z}_+$ (i.e. the orthogonal projection of $\ell^2(\mathbb{Z})$ onto $\ell^2(\mathbb{Z}_+)$). From the matrix perspective, $H_+$ corresponds to the one-sided infinite submatrix $(H_{ij})_{i,j \in \mathbb{Z}_+}$ of the two-sided infinite matrix $(H_{ij})_{i,j \in \mathbb{Z}}$ behind $H$.

In the same style, we denote the compression of $H$ to $\ell^2(\mathbb{Z}_-)$ by $H_-$, where $\mathbb{Z}_- := -\mathbb{Z}_+$, the Dirichlet condition is $x_1 = 0$, and the corresponding matrix is $(H_{ij})_{i,j \in \mathbb{Z}_-}$.

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Finite sections. For \( l, r \in \mathbb{Z} \) with \( l < r \), let \( H_{l..r} \) denote the compression of \( H \) to the (space of functions on the) interval \( \{l, \ldots, r\} \) – again with homogeneous Dirichlet conditions. The matrix behind \( H_{l..r} \) is \((H_{ij})_{i,j=l}^r\), which is a so-called finite section of \( H \). The finite section method (FSM) consists in approximating \( H \) by a sequence \((H_n)\) of finite sections

\[
H_n := H_{l_n..r_n} \quad \text{with} \quad l_n \to -\infty \quad \text{and} \quad r_n \to +\infty
\]

for the asymptotic inversion (i.e. the approximation of \( H^{-1} \)) or the spectral approximation of \( H \). For a fixed choice of cut-off sequences \((l_n)\) and \((r_n)\), we will simply call the sequence \((H_n)\) from (3) itself the FSM of \( H \).

The FSM \((H_n)\) is called applicable to \( H \) if all but finitely many \( H_n \) are invertible and \( H_n^{-1}b \to H^{-1}b \) for all \( b \in \ell^2(\mathbb{Z}_+) \). Here, \( H_n^{-1} \) is extended by zero, and the invertibility of \( H \) is implicitly assumed as well. In particular, the FSM, if applicable, can be used to approximately solve equations \( Hx = b \).

By a standard result of numerical analysis (“consistency+stability=convergence” a.k.a. Lax equivalence theorem [6] a.k.a. Polski’s theorem [12], see e.g. [1, 5]), one has

\[
\text{the FSM } (H_n) \quad \text{is applicable to } H \quad \iff \quad \begin{cases} 
(a) & H \text{ is invertible,} \\
(b) & \text{all but finitely many } H_n \text{ are invertible, and} \\
(c) & \text{all inverses are uniformly bounded.}
\end{cases}
\]

Practically, condition (a) is mandatory anyway as assumption for the unique solvability of the equation \( Hx = b \), and conditions (b) and (c) are typically a bit annoying to check.

The study of banded Toeplitz operators [4, 1] was a first instance where an operator class (on the half-line, finitely many constant diagonals, not limited to the tridiagonal setting (2) with \( v \equiv \text{const} \)) was identified for which condition (a) always implies (b) and (c), so that the applicability of the FSM already follows from the invertibility of the operator. Of course, such a situation simplifies FSM matters a lot: let us call \( H \) FSM-simple if condition (a) implies (b) and (c). For a particular operator \( H \), this means, the FSM is applicable or \( H \) is not even invertible.

Example: The Fibonacci Hamiltonian. The standard 1D model for the study of electrical conduction properties of quasicrystals is the so-called Fibonacci Hamiltonian. It is the discrete Schrödinger operator (1) with potential

\[
v(n) = \chi_{[1-\alpha,1)}(n\alpha \mod 1), \quad n \in \mathbb{Z},
\]

where \( \chi_I \) refers to the characteristic function of the interval \( I \) and \( \alpha = \frac{1}{2}(\sqrt{5} - 1) \) is the golden ratio. Since \( \alpha \) is irrational, \( v \) is not periodic – but it has very interesting combinatorial features instead. In [11], using many of those properties, it is shown that the FSM is applicable to the Fibonacci Hamiltonian, whence it is FSM-simple.

Our results. Already then, Albrecht Böttcher sparked the discussion whether all those arguments using the complicated structure of (5) were really necessary in [11] and what we would say if one day it turned out that every discrete Schrödinger operator (1) with a \{0,1\}-valued (or even just integer) potential is FSM-simple.

We tried to give periodic counter-examples right away, failed to produce any and recently proved in [17, 2] that none exist. In the current paper, as our first result, we even prove that periodicity has nothing to do with this non-existence; in short, Böttcher was right:
Theorem 1.1. Let $H$ be a discrete Schrödinger operator (1) on $\ell^2(\mathbb{Z})$ with an integer-valued potential $v : \mathbb{Z} \to \mathbb{Z}$. Then both $H$ and $H_+$ are FSM-simple.

Of course, this makes some of the twisted Fibonacci arguments in [11] redundant although, for a FSM-simple operator, one is still left with proving invertibility in order to conclude applicability of the FSM.

Conditions (b) and (c) in (4), and hence the applicability of the FSM to $H$, are connected to the invertibility of half-line compressions of so-called limit operators of $H$, see Lemma 2.4 below. That’s why the proof of Theorem 1.1 relies on our second main result which is, moreover, interesting in its own right:

Theorem 1.2. For a discrete Schrödinger operator $H$ on $\ell^2(\mathbb{Z})$, as in (1), with an integer-valued potential $v : \mathbb{Z} \to \mathbb{Z}$, and its half-line compression $H_+$, as in (2), the following implication holds:

$$H \text{ is a Fredholm operator on } \ell^2(\mathbb{Z}) \implies H_+ \text{ is invertible on } \ell^2(\mathbb{Z}_+).$$

(6)

In particular, $H_+$ is invertible if $H$ is invertible.

Essential spectra and Dirichlet eigenvalues. In many situations, including periodic, almost-periodic, Sturmian and pseudo-ergodic potentials, the arrow in implication (6) points in the other direction (even stronger, with $H_+$ Fredholm $\implies H$ invertible). More precisely, this is the case when $H$ is a limit operator of $H_+$. In short, this means that every finite subword of the potential $v$ of $H$ (understood as an infinite string) occurs infinitely often (at least up to arbitrary precision) in the right half of $v$.

Under this condition, by Lemma 2.3 below, the spectrum of $H$ is equal to both the essential spectra of $H$ and $H_+$,

$$\text{spec } H = \text{spec } \text{ess } H = \text{spec } \text{ess } H_+.$$  

(7)

The spectrum of $H_+$ is typically a larger set. It arises from (7) by adding eigenvalues of $H_+$, the so-called Dirichlet-eigenvalues. Their name addresses their cause: the truncation, i.e. introduction of a homogeneous Dirichlet boundary condition.

If one is actually trying to compute $\text{spec } H$ via a truncation technique like the FSM then such Dirichlet eigenvalues typically appear and lead to so-called spectral pollution, being, erroneously, caused by the method rather than by the physical problem behind the operator $H$.

For example, for a $p$-periodic potential, one can show (see [2] and the references there) that (7) consists of at most $p$ closed intervals. Dirichlet eigenvalues of $H_+$ can only occur in the closure of the gaps in (7), and each gap contains at most one Dirichlet eigenvalue. Similar results (see [3]) extend to aperiodic limits like the Fibonacci Hamiltonian, where (7) is a Cantor set, so that the number of gaps is infinite.

Remarkably, by our Theorem 1.2, zero is never among the Dirichlet eigenvalues, as well as all other integers $z \in \mathbb{Z}$, which is seen by applying the same result to $H - zI$.

Outline of the paper. After introducing all the necessary tools and language in Section 2, we prove both our theorems in Section 3 of the paper. The final section, Section 4 shows some possible directions of extension but also some examples showing the limits of what is possible.
2 Notations and tools

Spaces and operators. Let \( \mathbb{I} \in \{ \mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_- \} \) here and in what follows. We demonstrate our results in the simplest case, \( \ell^2(\mathbb{I}) \), the standard space of all complex-valued sequences \( x = (x_k)_{k \in \mathbb{I}} \) for which \( \|x\| = \sum_{k \in \mathbb{I}} |x_k|^2 < \infty \). Extension to \( \ell^p \) with \( p \in [1, \infty] \) is possible, see Section 4.

The statements in this section are not limited to discrete Schrödinger operators (1) but apply to all so-called band operators on \( \ell^2(\mathbb{I}) \): operators \( A \) whose matrix representation \( (A_{ij})_{i,j \in \mathbb{I}} \) has uniformly bounded entries and support on finitely many diagonals only. Such an operator \( A \) is always bounded on \( \ell^2(\mathbb{I}) \). All this is clearly the case for our discrete Schrödinger operators \( H, H_+ \) and \( H_- \).

Fredholmness, essential spectrum and limit operators. Recall that a bounded linear operator \( A \) on a Banach space \( X \) is a Fredholm operator if its coset, \( A + K(X) \), modulo compact operators \( K(X) \), is invertible in the so-called Calkin algebra \( L(X)/K(X) \). This holds if and only if the nullspace of \( A \) has finite dimension and the range of \( A \) has finite codimension in \( X \). In particular, Fredholm operators have a closed range. The essential spectrum of \( A \) is the spectrum of \( A + K(X) \) in the Calkin algebra \( L(X)/K(X) \), i.e. the set of all \( \lambda \in \mathbb{C} \) for which \( A - \lambda I \) is not a Fredholm operator.

Since the coset \( A + K(X) \) cannot be affected by changing finitely many matrix entries, its study takes place “at infinity”. This is where limit operators [13, 14, 7] come in:

Definition 2.1. For a band operator \( A \) on \( \ell^2(\mathbb{I}) \), we look at all its translates \( S^{-k}A^k \) with \( k \in \mathbb{Z} \) and speak of a limit operator, \( A_h \), on \( \ell^2(\mathbb{Z}) \) if, for a particular sequence \( h = (h_n) \) in \( \mathbb{Z} \) with \( |h_n| \to \infty \), the corresponding sequence of translates, \( S^{-h_n}A^h \), converges strongly to \( A_h \).

Hereby, \((Sx)_n = x_{n-1}\) denotes the shift operator on \( \ell^2(\mathbb{Z}) \) and a sequence \( A_n \) is said to converge strongly to \( A \) if \( A_n x \to Ax \) for all \( x \in \ell^2(\mathbb{Z}) \). Moreover, let \( \text{Lim}(A) \) denote the set of all limit operators of \( A \), together with the following local versions: For an integer sequence \( g = (g_n) \) with \( |g_n| \to \infty \), we put

\[
\text{Lim}_g(A) := \{ A_h : h \text{ is a subsequence of } g \}
\]

as well as \( \text{Lim}_+(A) := \text{Lim}_{(1,2,...)}(A) \) and \( \text{Lim}_-(A) := \text{Lim}_{(-1,-2,...)}(A) \).

From the matrix perspective, \( A_h = (A_{ij})_{i,j \in \mathbb{Z}} \) is the limit operator of \( A = (A_{ij})_{i,j \in \mathbb{I}} \) with respect to the sequence \( h = (h_n) \) in \( \mathbb{Z} \), if, for all \( i,j \in \mathbb{Z} \),

\[
A_{i+h_n,j+h_n} \to A_{ij} \quad \text{as} \quad n \to \infty.
\]}

(8)

Here is the announced connection to Fredholm operators.

Lemma 2.2. For a band operator \( A \) on \( \ell^2(\mathbb{I}) \), it holds that the following are equivalent:

(a) \( A \) is a Fredholm operator on \( \ell^2(\mathbb{I}) \),

(b) all limit operators of \( A \) are invertible on \( \ell^2(\mathbb{Z}) \) [13, 10].

As a consequence of Lemma 2.2,

\[
\text{spec}_{\text{ess}} A = \bigcup_{A_h \in \text{Lim}(A)} \text{spec} A_h.
\]

(9)

Let us generalize the definition of a half-line compression \( A_+ := P A P : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+) \), and similarly \( A_- \), from below (2) to band operators \( A \) on \( \ell^2(\mathbb{Z}) \). Then we have the following result leading to formula (7) above.
Lemma 2.3. If a band operator $A$ on $\ell^2(\mathbb{Z})$ is limit operator of its half-line compression $A_+$, in short, $A \in \text{Lim}(A_+)$, then

$$\text{spec } A = \text{spec}_{\text{ess}} A = \text{spec}_{\text{ess}} A_+.$$ 

Proof. The inclusion $\text{spec}_{\text{ess}} A_+ \subset \text{spec}_{\text{ess}} A$ holds by (9) since $\text{Lim}(A_+) = \text{Lim}_+(A) \subset \text{Lim}(A)$. The inclusion $\text{spec}_{\text{ess}} A \subset \text{spec } A$ is standard, of course. Finally, $\text{spec } A \subset \text{spec}_{\text{ess}} A_+$ holds by (9) and $A \in \text{Lim}(A_+)$, see our assumption. ■

Limit operators and the FSM. Fix integer sequences $l = (l_n)_{n \in \mathbb{N}}$ and $r = (r_n)_{n \in \mathbb{N}}$ with $l_n < r_n$ for all $n \in \mathbb{N}$ and $l_n \to -\infty$ and $r_n \to +\infty$ as $n \to \infty$. For a band operator $A$ on $\ell^2(\mathbb{Z})$, resp. $\ell^2(\mathbb{Z}_+)$, look at its sequence $(A_n)_{n \in \mathbb{N}}$ of finite sections

$$A_n := A_{l_n..r_n} = (A_{ij})_{i,j=l_n}^r,$$ 

resp.

$$A_n := A_{0..r_n} = (A_{ij})_{i,j=0}^r,$$ 

$n \in \mathbb{N}$.

Recall from our introduction that we call the sequence $(A_n)$ itself the finite section method (FSM) of $A$ and say that it is applicable if the conditions (a), (b), (c) hold in (4).

Lemma 2.4. [15, 8, 9] a) The two-sided case: The FSM with cut-off sequences $l = (l_n)_{n \in \mathbb{N}}$ and $r = (r_n)_{n \in \mathbb{N}}$ as above is applicable to a band operator $A$ on $\ell^2(\mathbb{Z})$ if and only if the following operators are invertible:

- (a) the operator $A$ itself,
- (b) all operators $L_+$ with $L \in \text{Lim}(A)$,
- (c) all operators $R_-$ with $R \in \text{Lim}_r(A)$.

b) The one-sided case: The FSM with cut-off sequence $r = (r_n)_{n \in \mathbb{N}}$ as above is applicable to a band operator $A_+$ on $\ell^2(\mathbb{Z}_+)$ if and only if the following operators are invertible:

- (d) the operator $A_+$ itself,
- (e) all operators $R_-$ with $R \in \text{Lim}_r(A_+)$. 

Without loss, we can restrict ourselves to the study of half-line compressions in the positive direction. Indeed, by an elementary reflection technique, one can connect the study of $R_-$ with $R \in \text{Lim}_r(A)$ to the study of

$$R_- := (R^-)_+, \quad \text{where} \quad R^- := \Phi R \Phi$$ 

with the flip operator $(\Phi x)_n = x_{-n}$ for all $n \in \mathbb{Z}$. In matrix language, $R^- = (R_{-i,-j})_{i,j \in \mathbb{Z}}$ if $R = (R_{ij})_{i,j \in \mathbb{Z}}$. Note that $R^- \in \text{Lim}_r(A^-)$. It is straightforward to check that $R$ and $R^-$ are simultaneously invertible on $\ell^2(\mathbb{Z})$ and that $R_-$ is invertible on $\ell^2(\mathbb{Z}_-)$ if and only if $R^-_+$ is invertible on $\ell^2(\mathbb{Z}_+)$. 

Discrete Schrödinger operators and transfer matrices. In order to examine the kernel of $H$ or $H_+$, it is useful to reformulate the scalar three-term recurrence

$$0 = (Hx)_n = x_{n-1} + v(n)x_n + x_{n+1}, \quad n \in \mathbb{Z}$$ 

as a vector-valued two-term recursion

$$
\begin{pmatrix}
  x_n \\
  x_{n+1}
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 \\
  -1 & -v(n)
\end{pmatrix}
\begin{pmatrix}
  x_{n-1} \\
  x_n
\end{pmatrix}
.$$ 

(10) The matrix $T(n) := \begin{pmatrix}
  0 & 1 \\
  -1 & -v(n)
\end{pmatrix}$ from (10) is the so-called (and well-known) transfer matrix.
3 The proofs of our results

The proof of Theorem 1.1 uses Theorem 1.2. The latter follows from the following two results:

**Proposition 3.1.** Let $H_+$ be a discrete Schrödinger operator on the half-line, $\ell^2(\mathbb{Z}_+)$, as in (2), with an integer-valued potential $v: \mathbb{Z}_+ \to \mathbb{Z}$. Then $H_+$ is injective.

**Proof.** In order to show that $H_+$ is injective, take a vector $x \in \ell^2(\mathbb{Z}_+)$ with $H_+x = 0$. We further assume that $x \neq 0$, which will lead to a contradiction. Recall from (2) that the Dirichlet boundary condition $x_{-1} = 0$ holds. Since $x_0 = 0$ would imply that $x = 0$, we can, without loss of generality, assume that $x_0 = 1$. By (10) and the definition of the transfer matrices $T(k)$, the other entries of $x$ satisfy the condition

$$
\begin{pmatrix}
  x_n \\
  x_{n+1}
\end{pmatrix} = T(n) \ldots T(0) \begin{pmatrix}
  x_{-1} \\
  x_0
\end{pmatrix} = T(n) \ldots T(0) \begin{pmatrix}
  0 \\
  1
\end{pmatrix}, \quad n \in \mathbb{N}.
$$

(11)

Since, by assumption, $v(k) \in \mathbb{Z}$ for all $k \in \mathbb{Z}_+$, we find that $T(k) = \begin{pmatrix}
  0 & 1 \\
  -1 & -v(k)
\end{pmatrix} \in \mathbb{Z}^{2 \times 2}$. Thus (11) implies that $x_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}_+$. As an integer sequence in $\ell^2(\mathbb{Z}_+)$, $x$ can only have finitely many non-zero entries. Thus there exists an $n \in \mathbb{N}$ such that $x_n = x_{n+1} = 0$. Using the invertibility of the transfer matrices, we get

$$
\begin{pmatrix}
  x_{-1} \\
  x_0
\end{pmatrix} = T(0)^{-1} \ldots T(n)^{-1} \begin{pmatrix}
  x_n \\
  x_{n+1}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0
\end{pmatrix}
$$

(12)

and hence, $x_0 = 0$. This contradicts the assumption $x_0 \neq 0$. Hence we have shown that $H_+$ cannot have any nontrivial vectors $x$ with $H_+x = 0$, i.e. it is injective.

So, after this proposition, all that is missing for the invertibility of the self-adjoint operator $H_+ : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ with integer-valued potential is the closedness of its range.

**Lemma 3.2.** Let $H$ be a discrete Schrödinger operator, (1), on $\ell^2(\mathbb{Z})$ and $H_+$ its half-line compression, (2), on $\ell^2(\mathbb{Z}_+)$. Then the following implications hold,

$$
(i) \iff (ii) \implies (iii) \iff (iv),
$$

where

(i) $H$ is a Fredholm operator,

(ii) $H$ has a closed range,

(iii) $H_+$ is a Fredholm operator,

(iv) $H_+$ has a closed range.

**Proof.** $(i) \implies (ii)$ and $(iii) \implies (iv)$ are standard. Their reverse directions follow, once we show that always $\dim \ker(H) \leq 2$ and $\dim \ker(H_+) \leq 1$. For $H_+$ this is shown in the proof of Proposition 3.1: A vector $x$ in the kernel of $H_+$ is given by $x_{-1} = 0, x_0 = a$ and all other $x_n$ by (11). So we have one degree of freedom, $a$, (or possibly just $\ker(H_+) = \{0\}$). For $x \in \ker(H)$, one has two degrees of freedom, say $x_{-1}$ and $x_0$, and the recurrence (11) is going forward and (appropriately inverted) backward from there.

The implication $(i) \implies (iii)$ holds by Lemma 2.2 and $\text{Lim}(H) \supset \text{Lim}_+(H) = \text{Lim}(H_+)$.  

**Proof of Theorem 1.2.** Let $H$ be a discrete Schrödinger operator on $\ell^2(\mathbb{Z})$ with an integer-valued potential $v: \mathbb{Z} \to \mathbb{Z}$ and let $H$ be a Fredholm operator. Since $H_+$ is self-adjoint, it suffices to show that it is a) injective and has b) a closed range.

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a) follows by Proposition 3.1 and \(v(n) \in \mathbb{Z}\) for all \(n \in \mathbb{Z}\),
b) follows from Lemma 3.2. ■

**Proof of Theorem 1.1.** We start with the two-sided case, \(H\) on \(\ell^2(\mathbb{Z})\). Let \(H\) be invertible. By Lemma 2.2, all \(L \in \text{Lim}_-(H)\) and \(R \in \text{Lim}_+(H)\) are also invertible. Because the convergence in (8) is by being eventually constant, \(L\) and \(R\) are again discrete Schrödinger operators with integer potential.

Clearly, also \(R^-\) is invertible and a discrete Schrödinger operator with integer potential.

Applying Theorem 1.2 to \(H = L\) and \(H = R^-\), we get that \(L_+\) and \(R_+^-\), and consequently also \(R^-\), is invertible on the corresponding half-line space.

Because this is true for all \(L \in \text{Lim}_-(H)\) and \(R \in \text{Lim}_+(H)\), we get, by Lemma 2.4 a), that the FSM with arbitrary cut-off sequences \(l = (l_n)_{n \in \mathbb{N}}\) and \(r = (r_n)_{n \in \mathbb{N}}\) with \(l_n < r_n\) for all \(n \in \mathbb{N}\) and \(l_n \to -\infty\) and \(r_n \to +\infty\) as \(n \to \infty\) is applicable to \(H\).

In the one-sided case, \(H_+\) on \(\ell^2(\mathbb{Z}_+)\), assume that \(H_+\) is invertible. Now argue with Lemma 2.4 b) exactly as above (without the operators \(L_+\)) to see that the FSM is applicable to \(H_+\). ■

**Corollary 3.3.** For a discrete Schrödinger operator \(H\) on \(\ell^2(\mathbb{Z})\), as in (1), with an integer-valued potential \(v : \mathbb{Z} \to \mathbb{Z}\), and its half-line compression \(H_+\), as in (2), the following holds:

If \(H\) has a closed range then the FSM is applicable to \(H_+\).

**Proof.** \(\text{im}(H)\) closed \(\overset{\text{T}3.2}{\longrightarrow}\) \(H\) Fredholm \(\overset{\text{T}1.2}{\longrightarrow}\) \(H_+\) invertible \(\overset{\text{T}1.1}{\longrightarrow}\) FSM applicable to \(H_+\). ■

## 4 Limitations and possible directions of extension

**Limitations.**

- We cannot replace “integer-valued” by “rational-valued” in Theorems 1.1, 1.2 and Proposition 3.1, see the following example:

**Example 4.1** Let \(v\) be the 3-periodic, two-sided extension of the vector \((\frac{1}{2}, 2, \frac{1}{2})\) with \(v(1) = 2\). Then it is shown in [2, Example 4.2] that the corresponding operator \(\tilde{H}\) from (1) is invertible, while \(H_+\) from (2) is not invertible. \(H_+\) is Fredholm, though: note that \(\text{Lim}(H_+) = \{H, S^{-1}HS, S^{-2}HS^2\}\). So \(H_+\) must lack injectivity. In particular, the claims of Proposition 3.1 and Theorem 1.2, do obviously not apply in this situation. Theorem 1.1: also not, see [2]. ■

- Unlike \(H_+\), the full-space operator \(H\) is not automatically injective if it has an integer-valued potential. See the following example:

**Example 4.2** We construct a potential \(v \in \{0, 1\}^\mathbb{Z}\) with \(v_\pm := v|_{\mathbb{Z}_\pm}\) periodic each. \(v_+\) first: In the sense of (11), we get from \((\frac{x}{x_0})\) to \((\frac{x}{x_{kp}})\) by applying \(M^k\) with \(M := T(p-1)\ldots T(0)\), where \(k \in \mathbb{N}\) and \(p\) is the period of \(v_+\). Then \(x_{kp} = \alpha \lambda_1^k + \beta \lambda_2^k\), where \(\lambda_i \in \text{spec } M\) and \(\alpha, \beta\) depend on \((\frac{x}{x_0})\), e.g. [2]. Noting that \(\lambda_1 \lambda_2 = \det M = \prod_{k=0}^{i-1} \det T(k) = 1\), we choose \(p = 5\) and \(v|_{\{0,\ldots,4\}} = 10101\), so that \(|\lambda_1| < 1\) and \((\frac{x}{x_0}) \neq 0\) has no component in the eigenspace of \(\lambda_2\),
whence $\beta = 0$ and $x_{kp}$ (and hence $x_n$) decays exponentially as $k,n \to +\infty$. Now choose $v_-$ with period $q$ so that also

$$\begin{pmatrix} x_{-kq-1} \\ x_{-kq} \end{pmatrix} = \left( T(-q)^{-1} \ldots T(-1)^{-1} \right)^k \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix}, \quad k \in \mathbb{N}$$

decays exponentially. The trick is to take $v_-$ as reflection of $v_+$ and replace every 0 by 000 and every 1 by 11. The result is $x_{-kq} = x_{kp}$ since $T_0 = T_0^{-3}$ and $T_1 = T_1^{-2}$ for $T_\gamma = \left( \begin{smallmatrix} 0 & 1 \\ -1 & -\gamma \end{smallmatrix} \right)$.

Consequently, $q = 12$, $v|_{\{-12,\ldots,-1\}} = 110001100011$ and $Hx = 0$ with $x \in \ell^2(\mathbb{Z}) \setminus \{0\}$.

- The majority of our results (all but Lemma 3.2) have only very restricted counterparts in the non-selfadjoint situation

$$(Hx)_n = ax_{n-1} + v(n)x_n + bx_{n+1}, \quad n \in \mathbb{Z}, \quad (13)$$

with $a, b \in \mathbb{C}$, where $T(n) = \left( \begin{smallmatrix} 0 & -1 \\ a & b \end{smallmatrix} \right)$.

Precisely, all results that require an integer potential do in fact rely on Proposition 3.1 and integer transfer matrices $T(n) \in \mathbb{Z}^{2 \times 2}$, which then translates to $a$ and all $v(n)$ being integer multiples of $b$.

However, for both our theorems, we need, besides injectivity and a closed range of the half-line compression, also injectivity of its adjoint (which is now different). Arguing via Proposition 3.1 again, in order to guarantee integer transfer matrices, also $\frac{b}{a}$ and all $\frac{v(n)}{a}$ have to be integer. So in total, all $v(n)$ have to be integer multiples of $a = \pm b \neq 0$.

Possible extensions.

- The results are clearly not different when $H_-$ instead of $H_+$ is considered.
- Also the starting point of the half-axis, here zero, is clearly irrelevant.
- We have shown that also compressions to finite intervals (i.e. finite sections), with homogeneous Dirichlet conditions on both ends, $l_n \to -\infty$ and $r_n \to +\infty$, inherit invertibility from $H$ for large enough $n$.

Note that, in the general situation of all integer-valued potentials $v$, there is no interval length $L$ for which all finite sections $H_{l_n \ldots r_n}$ are invertible as soon as they have sufficient size, $r_n - l_n + 1 =: L_n \geq L$. Indeed, for every odd $L \in \mathbb{N}$, $v$ can contain $L$ consecutive zeros, which makes a corresponding finite section of size $L_n = L$ singular (simple exercise by row operations), but $H$ can still be invertible.

For more specific classes, the situation is different: For the Fibonacci Hamiltonian $H$, it is shown in [17] that $L = 6$, and for its half-line compression $H_+, L = 1$ (with $l_n \equiv 0$).

- We can clearly pass from $\ell^2$ to $\ell^p$, even with $p \in [1,\infty]$. Instead of the strong convergence of $S^{-hn}AS^{hn}$, one then looks at the so-called $\mathcal{P}$-convergence [16].

- We can generalize from potentials $v$ with values in $\mathbb{Z}$ to values in a set $R \subseteq \mathbb{C}$ with

(i) $-1, 0, 1 \in R$,  

(ii) $v(n) \in R$ for all $n \in \mathbb{Z}$,  

(iii) $(R, +, \cdot)$ is a ring,  

(iv) 0 is an isolated point of $R$.

From (i),(ii),(iii) it follows that $T(n) \in \mathbb{R}^{2 \times 2}$ and hence $x_n \in R$ in (11) for all $n \in \mathbb{Z}_+$.

From (iv) it follows that, just as two lines below (11), $x \in \ell^2(\mathbb{Z}_+)$ implies $x \in c_{00}(\mathbb{Z}_+)$. 

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By (i) and (iii), \( s - t = s + (-1) \cdot t \in R \) for all \( s, t \in R \). By (iii), with every \( r \in R \), also \( r^2, r^3, \ldots \in R \), so that, by (iv), the only \( r \in R \) with \( |r| < 1 \) is \( r = 0 \). As a consequence of these observations, \( R \) is discrete with \( |s - t| \geq 1 \) for all \( s, t \in R \) with \( s \neq t \).

Examples of such a set \( R \) are grids \( R = r^1 \mathbb{Z} + r^2 \mathbb{Z} + \cdots + r^n \mathbb{Z} \) with \( n \in \mathbb{N} \) and a fixed \( r \in \mathbb{C} \) with \( r^n = 1 \). In particular, this leads to \( R = \mathbb{Z} \) for \( n \in \{1, 2\} \), \( R = \mathbb{Z} + i \mathbb{Z} \) for \( n = 4 \), while \( n \in \{3, 6\} \) lead to the same honeycomb grid \( R \) (since 3 is odd and \( \mathbb{Z} = -\mathbb{Z} \)). Note that \( n > 6 \) is impossible as it leads to the existence of different \( s, t \in R \) with \( |s - t| < 1 \) and \( n = 5 \) has the same problem as its grid coincides with the one for \( n = 10 > 6 \).

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References

[1] A. Böttcher and B. Silbermann: *Introduction to Large Truncated Toeplitz Matrices*, Springer, Berlin, Heidelberg, 1999.

[2] F. Gabel, D. Gallaun, J. Großmann, M. Lindner and R. Ukena: Finite sections of periodic Schrödinger operators, https://arxiv.org/abs/2110.09339.

[3] F. Gabel, D. Gallaun, J. Großmann, M. Lindner and R. Ukena: Spectral approximation of aperiodic Schrödinger operators, in preparation.

[4] I. Gohberg and I. A. Feldman: *Convolution equations and projection methods for their solution*, Transl. of Math. Monographs, 41, Amer. Math. Soc., Providence, R.I., 1974 [Russian original: Nauka, Moscow, 1971].

[5] R. Hagen, S. Roch and B. Silbermann: *C*-Algebras and Numerical Analysis, Marcel Dekker, Inc., New York, Basel, 2001.

[6] P. D. Lax and R. D. Richtmyer: Survey of the Stability of Linear Finite Difference Equations, *Comm. Pure Appl. Math.* 9 (1956), 267–293.

[7] M. Lindner: *Infinite Matrices and their Finite Sections: An Introduction to the Limit Operator Method*, Frontiers in Mathematics, Birkhäuser, 2006.

[8] M. Lindner: The finite section method and stable subsequences, *Applied Numerical Mathematics* 60 (2010), 501–512.

[9] M. Lindner and S. Roch: Finite sections of random Jacobi operators, *SIAM J. Numer. Anal.* 50 (2012), 287–306.

[10] M. Lindner and M. Seidel: An affirmative answer to a core issue on limit operators, *J. Funct. Anal.* 267 (2014), 901–917.

[11] M. Lindner and H. Söding: Finite sections of the Fibonacci Hamiltonian, in A. Böttcher, D. Potts, P. Stollmann, and D. Wenzel, eds., *The Diversity and Beauty of Applied Operator Theory*, Operator Theory: Advances and Applications 268, Birkhäuser, 2018, 381–396.

[12] N. I. Polski: Projection methods for solving linear equations, *Uspekhi Mat. Nauk*, 18 (1963), 179–180 (in Russian).
[13] V. S. Rabinovich, S. Roch and B. Silbermann: Fredholm Theory and Finite Section Method for Band-dominated operators, *Integral Equations Operator Theory* **30** (1998), 452–495.

[14] V. S. Rabinovich, S. Roch and B. Silbermann: *Limit Operators and Their Applications in Operator Theory*, Birkhäuser, 2004.

[15] V. S. Rabinovich, S. Roch and B. Silbermann: On finite sections of band-dominated operators, *Operator Theory: Advances and Applications* **181** (2008), 385–391.

[16] M. Seidel: Fredholm theory for band-dominated and related operators: a survey, *Linear Algebra Appl.* **445** (2014), 373-394.

[17] L. Weber: *Eindimensionale Quasikristalle, endliche Abschnitte und Invertierbarkeit*, Bachelor thesis, TU Hamburg, 2018.

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