Biased Halfspaces, Noise Sensitivity, and Relative Chernoff Inequalities (Extended Version)

Nathan Keller* and Ohad Klein†

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Abstract

In analysis of Boolean functions, a halfspace is a function \( f : \{-1, 1\}^n \rightarrow \{0, 1\} \) of the form \( f(x) = 1 \{a \cdot x > t\} \), where \( \sum_i a_i^2 = 1 \). We show that if \( f \) is a halfspace with \( E[f] = \epsilon \), then the degree-1 Fourier weight of \( f \) is \( W_1(f) = \Theta(\epsilon^2 \log(1/\epsilon)) \), and the maximal influence of \( f \) is \( I_{\text{max}}(f) = \Theta(\epsilon \min(1, a' \sqrt{\log(1/\epsilon)})) \), where \( a' = \max_i |a_i| \). These results, which determine the exact asymptotic order of \( W_1(f) \) and \( I_{\text{max}}(f) \), provide sharp generalizations of theorems proved by Matulef, O’Donnell, Rubinfeld, and Servedio, and settle a conjecture posed by Kalai, Keller and Mossel.

In addition, we present a refinement of the definition of noise sensitivity which takes into consideration the bias of the function, and show that (like in the unbiased case) halfspaces are noise resistant, and on the other hand, any noise resistant function is well-correlated to a halfspace.

Our main tools are ‘relative’ forms of the classical Chernoff inequality, like the following one: Let \( \{x_i\} \) be independent random variables uniformly distributed in \( \{-1, 1\} \), and let \( a_i \in \mathbb{R} \geq 0 \) be such that \( \sum_i a_i^2 = 1 \). If for some \( t \geq 0 \) we have \( \Pr[\sum_i a_i x_i > t] = \epsilon \), then \( \Pr[\sum_i a_i x_i > t + \delta] \leq \frac{\epsilon}{c} \) holds for \( \delta \leq \frac{c}{\sqrt{\log(1/\epsilon)}} \), where \( c \) is a universal constant.

1 Introduction

Analysis of Boolean functions (that is, functions of the form \( f : \{-1, 1\}^n \rightarrow \{0, 1\} \)), was initiated about 30 years ago, and has grown into a prolific research field, with numerous applications and connections to other fields of mathematics, computer science, physics, and economics (see [38]). Halfspaces (i.e., Boolean functions of the form \( f(x) = 1 (\sum_i a_i x_i > t) \)) have ever been a central object of study in the field; noise sensitivity (which studies the effect of small perturbations of the input on the function output) joined in 1999, bringing thrilling applications to percolation theory. As was shown by Benjamini, Kalai, and Schramm [1], these two notions are closely related, and we further explore the relation in this paper.

Usually, the Boolean functions of interest are unbiased, i.e., satisfy \( E[f] = 1/2 \). As a result, many of the central notions and results in analysis of Boolean functions assume that the function is (roughly) unbiased. Noise sensitivity is a notable example. However, in various applications the effect of the bias is central (e.g., threshold phenomena [23], correlation inequalities [17].
isoperimetry [13], and social choice theory [28]), and thus, getting rid of the assumption on \( \mathbb{E}[f] \) is desirable.

In this paper, we study biased Boolean functions, concentrating on halfspaces, noise sensitivity, and the relation between noise resistance and strong correlation with a halfspace. In particular, we determine the exact asymptotic order of the 1-st degree Fourier weight and the maximal influence of halfspaces, and show that the relation between being resistant to noise and being well-correlated to a halfspace carries from the unbiased case to biased functions, under appropriate definitions. Our techniques are somewhat non-standard to the type of questions we study: while most previous results on these problems were obtained using discrete Fourier analysis and hypercontractivity, our main tool is a relative variant of the Chernoff inequality, which allows comparing the rates of decay of the probability \( \Pr[\sum a_i x_i > t] \) (where \( \{x_i\} \) are independent and uniformly distributed in \( \{-1, 1\} \)), as function of \( t \).

### 1.1 First-degree Fourier weight and maximal influence of halfspaces

A halfspace, or a Linear Threshold Function (LTF), is a Boolean function \( f(x) = 1(\sum_i a_i x_i > t) \), where \( a \in \mathbb{R}^n \) and \( t \in \mathbb{R} \). (The vector \( a \) is usually normalized such that \( \sum_i a_i^2 = 1 \).) In the last half-century, halfspaces have been a central object of study in various areas, such as complexity theory, optimization, machine learning, and social choice theory (see, e.g., [18, 20, 33, 38, 44, 49, 50]).

#### 1.1.1 First-degree Fourier weight of halfspaces

A major tool frequently used in the study of halfspaces is their Fourier expansion – namely, their unique representation as a multilinear polynomial: \( f = \sum_{S \subseteq \{1, 2, \ldots, n\}} \hat{f}(S) x^S \), where \( x^S = \prod_{i \in S} x_i \). Of special importance here are the first-degree (or first-level) Fourier coefficients – the coefficients \( \hat{f}(S) \) which correspond to singletons \( S = \{i\} \), as Chow [7] proved in 1961 that a halfspace is uniquely determined by the set of its first-degree coefficients (together with \( \hat{f}(\emptyset) \)).

By Parseval’s identity, the total Fourier weight of a Boolean function, \( \sum_S \hat{f}(S)^2 \), is equal to \( \mathbb{E}[f] \). It is well-known that most of the Fourier weight of unbiased halfspaces (i.e., halfspaces of the form \( f = 1(\sum_i a_i x_i > 0) \), whose expectation is 1/2) is concentrated on the first degree. Namely, Gotsman and Linial [16] proved that for any unbiased halfspace \( f \), we have \( W^1(f) = \sum_{|S|=1} \hat{f}(S)^2 \geq 1/8 \). (The best currently known bound is \( 1/8 + c \) for some explicit \( c > 0 \) [8], and it is conjectured that the ‘correct’ bound is \( 1/2 \pi \), which is asymptotically attained by the majority function \( f(x) = 1(\sum_i \sqrt{n} x_i > 0) \).) Hence, the first-degree Fourier weight of unbiased halfspaces is within a constant multiplicative factor of the maximal possible weight.

A question that arises naturally is whether a similar phenomenon holds for biased halfspaces. Here, the bounds must depend on the bias of the function, as the Level-1 inequality [6, 21, 47] asserts that for any Boolean function \( f \), we have \( W^1(f) \leq 2\mathbb{E}[f]^2 \log(1/\mathbb{E}[f]) \). (Note that this improves significantly over the bound \( O(\mathbb{E}[f](1 - \mathbb{E}[f])) \) that follows from merely applying Parseval’s identity.) In view of the results for unbiased halfspaces, it makes sense to conjecture that any halfspace \( f \) satisfies

\[
W^1(f) \geq c \mathbb{E}[f]^2 \log(1/\mathbb{E}[f]),
\]

(1)

where \( c \) is a universal constant. Matulef, O’Donnell, Rubinfeld, and Servedio [32] Theorem 48 showed that (1), and actually a more precise bound, holds for halfspaces all of whose coefficients \( a_i \) are sufficiently small, called low-influence halfspaces. This result plays a crucial role in the algorithm of [32] for testing halfspaces, and in the algorithm of O’Donnell and Servedio [39] for
learning halfspaces. Kalai, Keller, and Mossel [26, Open Problem 6.2] asked to determine all functions for which (1) holds (i.e., all functions for which the Level-1 inequality is tight up to a constant factor), and conjectured that (1) holds for all halfspaces. We prove this conjecture, using a relative Chernoff inequality (to be presented in the sequel).

**Theorem 1.1.** There exists a universal constant \( c \) such that for any halfspace \( f = 1(\sum_i a_i x_i > t) \), we have

\[
cE[f]^2 \log \frac{1}{E[f]} \leq W^1(f) \leq 2E[f]^2 \log \frac{1}{E[f]}.
\]

Using the technique of [26, Proposition 5.3], Theorem 1.1 provides a large class of tightness examples for a well-known correlation inequality of Talagrand [47], which asserts that for any two monotone Boolean functions \( f, g \), we have \( \text{Cov}(f, g) = c \varphi(\sum_i f(i)g(i)) \), where \( \varphi(x) = x/\log(e/x) \), and \( c \) is a universal constant.

**Corollary 1.2.** Let \( f = 1(\sum_i a_i x_i > t) \) be a halfspace, and let \( g = 1(\sum_i a_i x_i > -t) \) be the dual halfspace. Then the pair \((f, g)\) is a tightness example for Talagrand’s inequality, meaning that \( \text{Cov}(f, g) = \Theta(\varphi(\sum_i f(i)g(i))) \), where \( \varphi(x) = x/\log(e/x) \).

We generalize Theorem 1.1 to the \( k \)th-degree Fourier weight of halfspaces, showing that the Level-\( k \) inequality (see [38, Chapter 9]) is tight (up to a factor that depends only on \( k \)) for all low-influence halfspaces. In particular, we prove:

**Theorem 1.3.** For any \( k \), there exist constants \( c_1, c_2, c_3, c_4 \) depending only on \( k \) such that for any halfspace \( f = 1(\sum_i a_i x_i > t) \) with \( E[f] \leq c_1 \) and \( I_{\max}(f) \leq c_4 E[f] \), we have

\[
c_2E[f]^2 \left( \log \frac{1}{E[f]} \right)^k \leq W^k(f) \leq c_3E[f]^2 \left( \log \frac{1}{E[f]} \right)^k.
\]

### 1.1.2 The maximal influence of halfspaces

The influence of the \( k \)th coordinate on a Boolean function \( f \) is defined as

\[
I_k(f) = \Pr[f(x) \neq f(x \oplus e_k)],
\]

where \( x \oplus e_k \) is obtained from \( x \) by flipping the \( k \)th coordinate. The total influence of \( f \) is \( I(f) = \sum_k I_k(f) \).

Influences have been studied very extensively in the last decades, and their applications span a wide variety of fields, including percolation theory [1], social choice theory [24, 36], hardness of approximation [10, 19], correlation inequalities [26, 47], etc. (see the survey [27]).

At a first sight, it may seem that the \( k \)th influence of a halfspace \( 1(\sum_i a_i x_i > t) \) is ‘proportional’ to the weight \( a_k \). However, this is not the case; for example, the halfspace \( f = 1(\sqrt{3} x_1 + \frac{1}{2} x_2 > 0) \) is equal to the dictator function \( f(x) = 1(x_1 > 0) \), and the influence of the second coordinate on it is zero. Hence, it is desirable to find a relation between the influences and the weights, to the extent such a relation exists.

In [32, Theorem 36], Matulef et al. proved a lower bound on the maximal influence of halfspaces:

\[
\max_k \{I_k(f)\} \geq \max_i \{a_i\} \cdot \mathbb{E}[f]^6 \log(1/\mathbb{E}[f]),
\]

and used it as another central component in their algorithm for testing halfspaces. The authors of [32] conjectured that the lower bound can be improved to \( \Omega(\max_i \{a_i\} \mathbb{E}[f]) \). This conjecture was later proved by Dzindzalieta and Götze [11].

We determine the exact asymptotic order of the largest influence of a halfspace:
Theorem 1.4. There exist universal constants $c_1, c_2$ such that for any halfspace $f = 1(\sum_i a_i x_i > t)$ with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$, we have

$$c_1 \mathbb{E}[f] \min\{1, a_1 \sqrt{\log(1/\mathbb{E}[f])}\} \leq \max_i I_i(f) \leq c_2 \mathbb{E}[f] \min\{1, a_1 \sqrt{\log(1/\mathbb{E}[f])}\}.$$ 

In view of the aforementioned example, even the fact that there at all exists a fixed relation between the maximal influence of a halfspace and its largest weight is perhaps somewhat surprising.

1.1.3 The vertex boundary of halfspaces

A halfspace $f$ naturally corresponds to the set $1_f = \{x : f(x) = 1\}$ which may be viewed as a subset of the discrete cube graph. A natural isoperimetric question one may ask is: what is the relation between the size of this set (which is, of course, $2^n \cdot \mathbb{E}[f]$, and the size of its boundary. In finite graphs, there are two classical types of boundary of a set $S$: the edge boundary, which consists of the edges that connect a vertex in $S$ with a vertex in the complement of $S$, and the vertex boundary, which consists of the vertices in $S$ that have a neighbor outside $S$ (or, vice versa, of the vertices outside $S$ that have a neighbor in $S$).

It is easy to see that the edge boundary of the set $1_f$ is equal (up to normalization) to the total influence $I(f) = \sum_k I_k(f)$, and thus, is usually easier to deal with. We show that for halfspaces, the asymptotic size of the vertex boundary $\partial(1_f)$ admits a neat expression in terms of $\mathbb{E}[f]$ and the maximal weight $a_1$.

Theorem 1.5. There exist universal constants $c'_1, c'_2$ such that for any halfspace $f = 1(\sum_i a_i x_i > t)$ with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$, we have

$$c'_1 \mathbb{E}[f] \min\{1, a_1 \sqrt{\log(1/\mathbb{E}[f])}\} \leq |\partial(1_f)|/2^n \leq c'_2 \mathbb{E}[f] \min\{1, a_1 \sqrt{\log(1/\mathbb{E}[f])}\}.$$ 

The theorem is proved by showing that for halfspaces, the vertex boundary is approximately equal to the largest influence, and then applying Theorem 1.4. We note that other relations between the measure of the vertex boundary and influences were obtained by Talagrand [18].

1.2 Noise sensitivity of biased functions and correlation with halfspaces

A Boolean function is called noise sensitive if flipping each of its input bits with a small probability affects its output ‘significantly’. Otherwise, it is called noise resistant. Formally, the noise stability of a function $f$ at noise rate $1 - \rho$ is defined as

$$S_\rho(f) = \text{Cov}(f(x), f(y)),$$

(2)

where $y$ is obtained from $x$ by independently keeping each coordinate of $x$ unchanged with probability $\rho$, and replacing it by a random value with probability $1 - \rho$. A sequence of functions $\{f_m : \{-1, 1\}^{nm} \to \{0, 1\}\}$ is called asymptotically noise sensitive if for any constant $\rho \in (0, 1)$, we have $\lim_{m \to \infty} S_\rho(f_m) = 0$. For sake of simplicity, we consider a single function $f$ and say that it is noise sensitive if $S_\rho(f) = o_n(1)$, and is noise resistant otherwise.

Noise sensitivity is a fundamental property of Boolean functions that has been studied extensively over the last two decades. Its applications span several areas, including machine learning (e.g., [9, 31]), hardness of approximation (e.g., [30, 36]), percolation theory (e.g., [17, 42]), and social choice theory (e.g., [24, 36]).

A main result of the seminal work of Benjamini, Kalai and Schramm [1] which initiated the study of noise sensitivity, is that noise resistance is closely related to strong correlation with a halfspace, via the Fourier expansion. Specifically, they showed that:
Theorem 1.6. (a) A monotone Boolean function $f$ is noise resistant if and only if $W^1(f) = \Omega(1)$.

(b) Any unbiased halfspace is noise resistant (and actually, satisfies a stronger property called ‘noise stability’).

(c) For any noise resistant monotone Boolean function $f$, there exists an unbiased halfspace $g$ such that $\text{Cov}(f, g) = \Omega(1)$.

We note that in the non-monotone case the situation is more complex. Indeed, as was shown recently by Mossel and Neeman [35], even the stronger assumption that $f$ is noise stable is not sufficient for assuring existence of a halfspace $g$ such that $\text{Cov}(f, g) = \Omega(1)$.

The definition of noise sensitivity is ‘not interesting’ for highly biased functions (i.e., when $\mathbb{E}[f]$ is close to 0 or to 1), as any such function is clearly noise sensitive. Hence, it is natural to ask what should be the ‘right’ definition of noise sensitivity for highly biased functions. Inspired by Theorem 1.6(a), we propose a ‘Fourier-theoretic’ definition.

Note that Theorem 1.6(a) asserts that an unbiased monotone function is noise resistant if and only if its first-degree Fourier weight is, up to a constant factor, the maximum possible. For general functions, the aforementioned ‘Level-1 inequality’ asserts that $W^1(f) = O(\mathbb{E}[f]^2 \log(1/\mathbb{E}[f]))$. Based on this, we say that $f$ is noise resistant if $W^1(f)$ is within a constant factor of the maximum possible. Formally:

**Definition 1.7.** A function $f : \{-1, 1\}^n \to \{0, 1\}$ is called Fourier noise resistant if $W^1(f) \geq c\mathbb{E}[f]^2 \log(1/\mathbb{E}[f])$, for some universal constant $c$.

Theorem 1.1 allows us to claim that with respect to this definition, the close relation between noise resistance and strong correlation with a halfspace holds also for biased functions. Indeed, one direction (i.e., that any halfspace is Fourier noise resistant) is exactly the assertion of Theorem 1.1. In the converse direction, Mossel and Neeman [35] Proposition 3.2 showed that for any Boolean function $f$, there exists a halfspace $g$ such that $\text{Cov}(f, g) \geq \Omega(W^1(f)/\sqrt{\mathbb{E}[f](1-\mathbb{E}[f])})$. We show the following sharp bound, which is always stronger than the bound of [35] by the Level-1 inequality.

**Theorem 1.8.** For any Boolean function $f$, there exists a halfspace $g$ such that

$$\text{Cov}(f, g) \geq c\sqrt{W^1(f)/\log(c/W^1(f))},$$

where $c$ is an absolute constant. In particular, if $f$ is Fourier noise resistant and $\mathbb{E}[f] \leq 1/2$ then there exists a halfspace $g$ such that $\text{Cov}(f, g) = \Omega(\mathbb{E}[f])$.

Note that the correlation asserted in the theorem is clearly within a constant factor of the maximum possible, as $\text{Cov}(f, g) \leq \mathbb{E}[f]$ for any $f, g$. An interesting feature of Theorem 1.8 is that unlike the classical result of [1], the strong correlation with a halfspace is assured even if the function $f$ is not monotone. This is somewhat surprising, as most known correlation bounds (such as FKG-type inequalities [15]) hold only for monotone functions.

Finally, we show that for monotone functions, strong correlation with a halfspace is implied also by a ‘probabilistic’ notion of noise resistance. Here, the rate of noise we consider is $1 - c/\log(1/\mathbb{E}[f])$, for a fixed ‘small’ constant $c$ (i.e., $\rho = c/\log(1/\mathbb{E}[f])$). It is easy to show (see Section 9) that for this noise rate, any function $f$ satisfies $S_{\rho}(f) = O(\mathbb{E}[f]^2)$. Recalling that the classical definition of noise resistance is $S_{\rho}(f) = \Omega(1)$, which is within a constant factor of the maximal possible value, a natural definition of noise resistance in our setting is the requirement $S_{\rho}(f) = \Omega(\mathbb{E}[f]^2)$.
Proposition 1.9. Let \( f : \{-1, 1\}^n \to \{0, 1\} \) be a monotone function such that \( S_{\log(1/|E|)}(f) = \Omega(\mathbb{E}[f]^2) \). Then \( W^1(f) = \Omega(\mathbb{E}[f]^2 \log(1/\mathbb{E}[f])) \), and hence, there exists a halfspace \( g \) such that \( \text{Cov}(f, g) = \Omega(\mathbb{E}[f]) \).

1.3 Relative Chernoff Inequalities

Tail estimates for weighted sums of independent random variables are among the most frequently used probabilistic tools in combinatorics and theoretical computer science. A standard example is Hoeffding’s inequality which asserts that if \( \{x_i\}_{i=1}^n \) are independent mean-zero random variables with \( \forall i : |x_i| \leq 1 \) and \( \{a_i\}_{i=1}^n \) are real numbers that satisfy \( \sum_i a_i^2 \leq 1 \), then for any \( t > 0 \),

\[
\Pr \left[ \sum_i a_i x_i > t \right] \leq \exp(-t^2/2).
\]

In the commonly-studied case where each \( x_i \) is uniformly distributed in \( \{-1, 1\} \) (also called Rademacher random variables), stronger bounds can be obtained, which essentially state that \( \sum a_i x_i \) is distributed ‘like’ a Gaussian random variable. In particular, there exists a constant \( c \) such that for any \( t > 0 \),

\[
\Pr \left[ \sum_i a_i x_i > t \right] \leq c \Pr[Z > t],
\]  

(4)

where \( Z \sim N(0, 1) \). (This is a result of Eaton [12]; the ‘correct’ value of \( c \) was recently determined by Bentkus and Dzindzalieta [4] to be \( \approx 3.178 \).) This phenomenon is also demonstrated by the Central Limit Theorem, or its more quantitative form, the Berry-Esseen Theorem (see, e.g., [14]), which implies that for any interval \( I \),

\[
\left| \Pr \left[ \sum_i a_i x_i \in I \right] - \Pr[Z \in I] \right| \leq c' \sum_i |a_i|^3 \leq c' \max_i |a_i|,
\]  

(5)

where \( c' \) is an absolute constant. (The claim holds, e.g., for \( c' = 1 \); the best currently known bound on \( c' \) was obtained by Shevtsova [15].) The ‘relative Chernoff inequalities’ we consider in this paper assert that the rate of decay of \( \Pr[\sum a_i x_i > t] \) as function of \( t \) is also essentially equal to that of a Gaussian random variable \( Z \sim N(0, 1) \).

1.3.1 A relative Chernoff inequality, via a general method of Benjamini, Kalai, and Schramm

In a remark ending their seminal paper on the variance of first passage percolation [2], Benjamini et al. suggested a general method for deriving ‘relative’ tail estimates for random variables from hypercontractive inequalities. Essentially, in order to obtain a relative tail estimate for \( f \), one considers the function \( g_t(x) = \max(f(x), t) \), where \( t \) is chosen such that \( \Pr[f > t] = \epsilon \). Then one uses a theorem of Talagrand [40] (Theorem 1.1 below, whose proof relies on hypercontractivity) to show that \( \text{Var}(g_t) \) is ‘small’ (as function of \( \epsilon \)), and deduces an upper bound on the minimal \( \delta \) such that \( \Pr(f > t + \delta) \leq \epsilon/2 \) using Chebychev’s inequality.

In a MathOverflow question [25], Kalai suggested to study the function \( t \mapsto \text{Var}(g_t) \) (where \( g_t \) is as defined above) further, and to derive from it properties of \( f \) and its tail behavior. We are not aware of other development or applications of the method of [2].

As we show in Section 4 one may apply the method of Benjamini et al. to the function \( f = \sum a_i x_i \), to prove the following tail estimate:
Theorem 1.10. Let \( \{x_i\} \) be independent random variables uniformly distributed in \( \{-1, 1\} \), and let \( a_i \in \mathbb{R}_{\geq 0} \) be such that \( \sum_i a_i^2 = 1 \). There exists a universal constant \( c > 0 \) such that if \( t \geq 0 \) and \( \epsilon = \Pr[\sum_i a_i x_i > t] \), then \( \Pr[\sum_i a_i x_i > t + \delta] \leq \frac{\epsilon}{2} \), for \( \delta \leq \frac{c}{\sqrt{\log(1/\epsilon)}} \).

This shows that the ‘relative’ decay of the tail probability \( \Pr[\sum_i a_i x_i > t] \) is essentially equal to that of a Normal random variable \( Z \). Indeed, an easy computation yields that if for some \( t \geq 0 \) we have \( \Pr[Z > t] = \epsilon \), then the minimal \( \delta \) such that \( \Pr[Z > t + \delta] \leq \epsilon/2 \) is of order \( \Theta\left(1/\sqrt{\log(1/\epsilon)}\right) \).

Theorem 1.10 implies that if for some \( t > 0 \), the probability \( \Pr[\sum_i a_i x_i > t] \) is much smaller than the Gaussian-like bound provided by (4), then for any \( t' > t \), the probability \( \Pr[\sum_i a_i x_i > t'] \) will ‘remain’ much smaller than that of a Gaussian random variable. The theorem is clearly tight, e.g., for \( X = \sum_{i=1}^n \frac{1}{\sqrt{n}} x_i \) where \( n \) is sufficiently large; this follows immediately from (5), using the exact rate of decay of the Gaussian distribution.

Since Theorem 1.10 and its variants presented below compare the probabilities \( \Pr[\sum_i a_i x_i > t] \) at different values of \( t \), we call them relative Chernoff inequalities \(^1\).

1.3.2 Refined variants, via log-concavity

For our applications, we will need a refined inequality, which takes into consideration the weights \( a_i \):

Theorem 1.11. Let \( \{x_i\} \) be independent random variables uniformly distributed in \( \{-1, 1\} \), let \( a_i \in \mathbb{R}_{\geq 0} \) be such that \( \sum_i a_i^2 = 1 \), and let \( t \geq 0 \). Denote \( \epsilon = \Pr[\sum_i a_i x_i > t] \), and let \( \delta \) be minimal such that \( \Pr[\sum_i a_i x_i > t + \delta] \leq \frac{\epsilon}{2} \).

If \( B, S \) is any partition of \( \{1, 2, \ldots, n\} \) (which corresponds to ‘big’ and ‘small’ values of the \( a_i \)’s), then one of the following holds:

- \( |B| \geq \frac{1}{2} \log(1/\epsilon) \), or
- \( \delta \leq c \sqrt{\frac{\sum_{i \in S} a_i^2}{\log(1/\epsilon)}} \),

where \( c \) is a universal constant.

We also prove the following inequality, which applies in the slightly more general case of bounded symmetric random variables:

Theorem 1.12. Let \( X = \sum x_i \), where \( \{x_i\} \) are independent symmetric (around 0) random variables with \( |x_i| \leq a_i \) almost surely, and let \( F(t) = \Pr[X > t] \). Set \( m = 2 \max_i \{a_i\} \), and let \( c \in (0, 1) \). If \( \epsilon = F(t) \) for some \( t \geq 0 \), and \( \delta \geq 0 \) is minimal such that \( F(t + \delta) \leq c \cdot F(t) \), then we have

\[
\delta \leq O\left(m + \log (2/c) \sqrt{\frac{\sum_i a_i^2}{\log(1/\epsilon)}}\right).
\]

The main tool in the proof of Theorem 1.12 which we also use to present an alternative proof of Theorem 1.10 is a ‘relaxed log-concavity’ lemma:
Lemma 1.13. Let \( X = \sum_i x_i \) be a sum of independent real random variables, and denote \( F(t) = \Pr[X > t] \). Set \( m = \max_{i \in [n]} \{ \sup x_i - \inf x_i \} \). For any \( b \leq c \leq d \), we have

\[
F(d)F(b) \leq F(c)F(b + d - c - m).
\]

We prove the lemma by constructing an explicit measure-preserving injection from the set \( \{ X_1 > d \} \times \{ X_2 > b \} \) to the set \( \{ X_1 > c \} \times \{ X_2 > b + d - c - m \} \), where \( X_1, X_2 \) are two identical, independent, copies of \( X \). The idea is to swap an appropriate fraction of \( X_1 \) and \( X_2 \), in a way that increases \( X_2 \), at the expense of decreasing \( X_1 \). Theorem 1.11 follows from Lemma 1.13 and Hoeffding’s inequality via some technical computations.

We believe that these ‘relative’ tail estimates and their variants, as well as the log-concavity lemma, will be useful in other contexts as well.

1.4 Organization of the paper

This paper is organized as follows. In Section 2 we present notations and conventions to be used throughout the paper. In Section 3 we prove two concentration lemmas, and in Section 4 we use them to prove the relative Chernoff inequalities. The evaluations of the first-degree Fourier weight, the maximal influence, the vertex boundary size, and the \( k \)'th-degree Fourier weight of halfspaces are presented in Sections 5, 6, 7, and 8 respectively. Finally, we study noise sensitivity of biased functions and its relation to correlation with a halfspace in Section 9.

2 Conventions

In this section we present notations and conventions that we will use throughout the paper.

For a Boolean function \( f \), we denote \( \mu(f) = \Pr[f = 1] = \mathbb{E}_{x \sim \{-1,1\}^n}[f] \). We say that \( f \) is almost unbiased if \( c \leq \mathbb{E}[f] \leq 1 - c \) for a universal constant \( c \) (whose exact value does not matter). On the other hand, \( f \) is said to be strongly biased if \( \mathbb{E}[f] = o(1) \) or \( \mathbb{E}[f] = 1 - o(1) \).

A halfspace is a Boolean function of the form \( f = \mathbb{1}\{a \cdot x > t\} \) with \( a \in \mathbb{R}^n \) and \( t \in \mathbb{R} \). We are always going to assume that \( a_i \in \mathbb{R}_{\geq 0} \) and that \( a_1 \geq \ldots \geq a_n > 0 \). Furthermore, we frequently assume \( \mu(f) \leq \frac{1}{2} \); this mostly does not affect generality since we can alternatively investigate the dual function \( g(x) = 1 - f(-x) \) which shares many properties with \( f \) (note that \( g \) is also a halfspace).

We sometimes identify the halfspace \( f \) with \( 2f - 1 = \text{sgn}(a \cdot x - t) \), where \( \text{sgn} \) is the sign function, and we choose \( \text{sgn}(0) = -1 \). Furthermore, since there are only finitely many \( (2^n) \) values for \( a \cdot x \), we may increase \( t \) a little without changing \( f \). Moreover, notice that as long as we are interested in a particular halfspace \( f = \mathbb{1}\{a \cdot x > t\} \), we may assume that \( a \cdot x \) does not assume any finite set of values since slightly altering \( a \) does not change \( f \).

We now introduce some notation that will be used throughout the paper. For an \( a \in \mathbb{R}_{\geq 0}^n \) as above and an \( s \in \mathbb{R} \) we write \( f_s(x) = \mathbb{1}\{a \cdot x > s\} \). (Notice that the notation \( f_s(x) \) is used only for halfspaces, where \( s \) always denotes the threshold). Additionally, we use the notation \( F(t) = \Pr[s \cdot x > t] \), and so, \( F(t) = \mu(f_t) \). Furthermore, we regularly write \( \epsilon = \mu(f_t) = F(t) \).

The letters \( \beta, \gamma, \delta \) are usually used to describe a significant decay of \( F \); e.g., in several places \( \beta \) is chosen to be minimal \( \in \mathbb{R}_{\geq 0} \) satisfying \( F(t + \beta) \leq \frac{1}{2}F(t) \).

When we write \( \log(x) \) we always mean \( \log_2(x) \); \( \log(x) \) is always assumed to be \( \ln(x) = \log_e(x) \).
3 Two Concentration Lemmas

In this section we prove two concentration results concerning sums of independent random variables. The first is Lemma 1.13 which asserts that if $X = \sum_{i=1}^{n} x_i$, where $\{x_i\}_{i=1}^{n}$ are independent real random variables, and we denote $F(t) = \Pr[X > t]$ and $m = \max_{i \in [n]} \{\sup x_i - \inf x_i\}$, then for any $b \leq c \leq d$, we have

$$F(d)F(b) \leq F(c)F(b + d - c - m). \quad (6)$$

Notice the $(-m)$ in Equation (6) can not in general be omitted. This is because $b, c, d$ might not be ‘aligned’ to values achievable by $X$. To compare, in the context of the usual notion of discrete log-concave distribution, the underlying random variable assumes only integer values. This does not capture the behavior of the variables $X$ discussed in Lemma 1.13.

The second result is a concentration lemma which assumes (in addition) that the random variables are symmetric.

**Lemma 3.1.** Let $x_i$ be independent symmetric (around 0) random variables with $|x_i| \leq a_i$, almost surely. For any $m \geq \max \{a_i\}$ and any $s, t$ such that $0 \leq s \leq t$, we have

$$\Pr \left[ \sum_{i=1}^{n} x_i \in (t - m, t + m) \right] \leq 5 \Pr \left[ \sum_{i=1}^{n} x_i \in (s - m, s + m) \right]. \quad (7)$$

Note that while it may seem that the lemma ‘should’ hold with the constant in the right hand side of (7) equal to 1, we show below by an explicit example that this constant must be at least 2.

We prove both lemmas constructively using *injective measure preserving maps* from the set of events that represent the l.h.s. into the set of events that represent the r.h.s. We introduce several injective transformations which we will use to construct the maps in Section 3.1 and present the proof of the lemmas in Section 3.2.

3.1 Auxiliary injective transformations

**Definition 3.2** (Prefix/suffix flip). Let $r$ be a real number, and let $u, v \in \mathbb{R}^{n}$ be two vectors whose ‘partial sums of differences’ $S_k(u, v) = \sum_{i=1}^{k} (u_i - v_i)$ satisfy $\max_{k \in [n]} \{S_k(u, v)\} \geq r$. We define $SF_r(u, v) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$ as follows. Take $t = \min \{i \in \{0, \ldots, n\} \mid S_i \geq r\}$, and set

$$\forall i \in [n]: SF_r(u, v)_i = \begin{cases} (u_i, v_i), & i \leq t \\ (v_i, u_i), & i > t \end{cases}.$$  

That is, we choose the first index $t$ in which the partial sum $S_k(u, v)$ exceeds $r$, and interchange the coordinates of the vector $(u, v)$ in all indices later than $t$. (This is called a ‘suffix flip’).

For $r \in \mathbb{R}$ and a single vector $u \in \mathbb{R}^{n}$ such that $\max_{k \in [n]} \{S_k(u, -u)\} \geq r$, we define $PF_r(u)$ to be the unique $v \in \mathbb{R}^{n}$ that satisfies $(-v, v) = SF_r(u, -u)$.

That is, we choose the first index $t$ in which the partial sum $\sum_{i \leq t} u_i$ exceeds $r/2$ and flip all coordinates of $u$ with indices no later than $t$. (This is called a ‘prefix flip’).

Notice that the suffix flip is defined (in particular) for any $(u, v)$ such that $\sum u_i \geq \sum v_i + r$, and the prefix flip is defined (in particular) for any $u$ such that $\sum u_i \geq r/2$. Also, notice that $SF_r(u, v)$ is an involution (meaning that $SF_r(SF_r(u, v)) = (u, v)$), and hence, is injective. Moreover, as $PF_r$ is a composition of $SF_r$ (restricted to inputs of the form $(u, -u)$) with the map $x \mapsto (-x)$, it is injective as well.
Definition 3.3 (Single coordinate flip). Let \( x \in \{-1,1\}^n \) be a vector whose ‘partial sums’ 
\( S_k(x) = \sum_{i=1}^k x_i \) satisfy \( \max_k S_k(x) > 0 \). We define \( \text{SCF}(x) \in \{-1,1\}^n \) as follows. Take 
\[ t = \min\{i \in [n] \mid \forall j : S_i \geq S_j \} \] 
and set 
\[ \text{SCF}(x)_i = \begin{cases} -x_i, & i = t \\ x_i, & i \neq t. \end{cases} \]

That is, we flip only a single coordinate of \( x \) – the first among the indices \( k \) for which the partial sum \( S_k(x) \) is maximal.

The map \( \text{SCF} \) is invertible, and thus, injective. To see this, note that if in the map 
\( x \mapsto \text{SCF}(x) \), the \( t \)'th coordinate was flipped, then the latest index in which the maximum 
of \( \{S_k(\text{SCF}(x))\} \) is attained is \( t - 1 \). Indeed, by the definition of \( \text{SCF} \), we have 
\( S_{t-1}(\text{SCF}(x)) = S_t(x) - 1 \), while for any \( t' < t \) we have 
\( S_{t'}(\text{SCF}(x)) \leq S_t(x) - 1 \) and for any \( t'' \geq t \) we have 
\( S_{t''}(\text{SCF}(x)) \leq S_t(x) - 2 \).

In particular, we can define an inverse mapping \( \text{ISCF} \) as follows. Take 
\[ t = \max\{i \in \{0, \ldots, n-1\} \mid \forall j : S_i \geq S_j \} \] 
and set 
\[ \text{ISCF}(y)_i = \begin{cases} -y_i, & i = t + 1 \\ y_i, & i \neq t + 1. \end{cases} \]

We would like to use the map \( \text{SCF} \) not only for \( x \)'s such that \( \max_k S_k(\text{SCF}(x)) > 0 \), but also 
for \( x \)'s for which we only know that \( \sum a_i x_i > 0 \) for some non-negative weights \( a_1, \ldots, a_n \). For 
this, we define the following variant which reorders the coordinates of \( x \) according to the sizes 
of the \( a_i \)'s, applies \( \text{SCF} \), and then reorders the coordinates back.

Definition 3.4. For any \( a \in \mathbb{R}_{\geq 0}^n \) we define the partial map 
\( \text{SCF}_a : \{-1,1\}^n \rightarrow \{-1,1\}^n \) as follows. Canonically choose a permutation 
\( P_a \in S_n \) which satisfies \( a_{P_a(1)} \geq \cdots \geq a_{P_a(n)} \) (e.g., 
using the lexicographic ordering on \( S_n \)). Then, define \( \text{SCF}_a(x) = P_a^{-1}(\text{SCF}(P_a(x))) \).

Analogously, we define \( \text{ISCF}_a : \{-1,1\}^n \rightarrow \{-1,1\}^n \) as 
\( \text{ISCF}_a(y) = P_a^{-1}(\text{ISCF}(P_a(y))) \).

Note that for any \( a \in \mathbb{R}_{\neq 0}^n \), we have \( \text{ISCF}_a \circ \text{SCF}_a = \text{id}_{\text{dom} \text{SCF}_a} \), and hence, \( \text{SCF}_a \) is 
injective. We claim that the function \( \text{SCF}_a(x) \) is defined (in particular) for all \( x \)'s such that 
\( \sum a_i x_i > 0 \). To see this, note that after the re-ordering of the coordinates of \( x \), the function 
\( \text{SCF} \) is applied on a vector \( x' \) that satisfies \( \sum a'_i x'_i > 0 \), where \( a'_1 \geq \cdots \geq a'_n \) (of course, 
the \( a'_i \)'s are a reordering of the \( a_i \)'s). The only reason \( \text{SCF}(x) \) might not be defined, is if 
\( \forall k : S_k(x') = \sum_{i=1}^k x'_i \leq 0 \). But this cannot happen, from the following Abel’s-summation 
argument:

\[
0 \overset{\text{assumption}}{\leq} \sum_{i=1}^n a'_i x'_i = a'_n S_n(x') + \sum_{k=1}^{n-1} ((a'_{k} - a'_{k+1})S_k(x')) \leq 0,
\]

where the last inequality follows from the, apparently wrong, assumption \( \forall k : S_k(x') \leq 0 \).

3.2 Proof of Lemmas 1.13 and 3.1

Now we are ready to present the proofs of the lemmas.
Proof of Lemma 1.13. After subtracting $F(c)F(d)$ from both sides of Inequality (8), it is left to prove that

$$\Pr[X > d] \Pr[X \in (b, c)] \leq \Pr[X > c] \Pr[X \in (b + d - c - m, d)].$$

(8)

Let $\Omega$ be the underlying probability space over which $\{x_i\}_i$ and $X$ are defined. Without loss of generality, $\omega \mapsto (x_i(\omega))_{i=1}^n$ is an injective map. We define an injective measure-preserving map which takes as input a pair $(\omega_1, \omega_2) \in \Omega^2$ for which $X(\omega_1) > d$ and $X(\omega_2) \in (b, c)$ and outputs $(\delta_1, \delta_2) \in \Omega^2$ for which $X(\delta_1) > c$ and $X(\delta_2) \in (b + d - c - m, d)$. This will clearly conclude the proof.

Set $r = (d - c) - m$ (note that if $r \leq 0$ then the assertion holds trivially), and consider

$$(v, u) = SF_r ((x_i(\omega_1))_{i=1}^n, (x_i(\omega_2))_{i=1}^n).$$

Define $\delta_1, \delta_2$ by $\forall i : x_i(\delta_1) = u_i$ and $\forall i : x_i(\delta_2) = v_i$. We claim that $\psi : (\omega_1, \omega_2) \mapsto (\delta_1, \delta_2)$ is the desired map.

First, $\psi$ is well-defined: Since $X(\omega_1) > d$ and $X(\omega_2) \leq c$ we have $X(\omega_1) \geq X(\omega_2) + r$, so according to Definition 3.2 the suffix flip $SF_r$ is defined on the vectors $(x_i(\omega_1))_{i=1}^n, (x_i(\omega_2))_{i=1}^n$. Hence, $v, u$ are well-defined. As we assumed that $\omega \mapsto (x_i(\omega))_{i=1}^n$ is injective, and since $v, u$ are obtained from $(x_i(\omega_1))_{i=1}^n, (x_i(\omega_2))_{i=1}^n$ by swapping some pairs of elements, $\delta_1, \delta_2$ are well-defined as well.

Second, $\psi$ is measure-preserving, as the variables $x_i$ are independent, and $SF_r$ just swaps pairs of identically distributed variables in its input. Furthermore, $\psi$ is injective, since $SF_r$ is invertible from the left (as noted after Definition 3.2).

Hence, it remains to show that the range of $\psi$ is included in the space represented by the r.h.s. of (3), i.e., that $X(\delta_1) > c$ and $X(\delta_2) \in (b + d - c - m, d)$. Observe that (unless $r < 0$, in which case the assertion of the lemma is trivial), we have

$$\sum_{i=1}^n x_i(\omega_1) - \sum_{i=1}^n u_i \in [r, r + m),$$

according to how $SF_r$ is defined – swapping the suffix just after the first index $t_0$ for which $S_{t_0}((x_i(\omega_1))_{i=1}^n, (x_i(\omega_2))_{i=1}^n) \geq r$. (Here we also use the fact that each difference $x_i(\omega_1) - x_i(\omega_2)$ is bounded by $m$, and thus we have $S_{t_0}((x_i(\omega_1))_{i=1}^n, (x_i(\omega_2))_{i=1}^n) \in [r, r + m)$.) Hence, $X(\delta_1) > X(\omega_1) - (r + m) > c$. Moreover,

$$X(\delta_2) = X(\omega_1) + X(\omega_2) - X(\delta_1) \in [r + X(\omega_2), r + m + X(\omega_2)) \subseteq (b + d - c - m, d),$$

as required. □

Proof of Lemma 3.7. Let $0 \leq s \leq t$. We first show

$$\Pr \left[ \sum_{i=1}^n x_i \in (t - m, t + m) \right] \leq \Pr \left[ \sum_{i=1}^n x_i \in (s - m, s + 3m) \right].$$

(9)

We will do this with an injective measure-preserving map. Let $r = t - s - 2m$ (note that the assertion holds trivially if $r \leq 0$). Denote $u_i = x_i$, and define $v = PF_r(u)$. By the definition of $PF_r$, for any $u$ we have

$$\sum_{i=1}^n u_i - \sum_{i=1}^n PF_r(u)_i \in [r, r + 2 \max_i \{a_i\}] \subseteq [r, r + 2m).$$

(10)
Hence, in our case, if $\sum_{i=1}^{n} u_i \in (t - m, t + m]$ then $\sum_{i=1}^{n} v_i \in (s - m, s + 3m]$. The map $(x_i)_{i=1}^{n} \rightarrow (v_i)_{i=1}^{n}$ is injective by Definition 3.2 and is measure-preserving as it only negates some $x_i$’s, which we assumed are symmetric random variables. Therefore, (9) holds, and thus, it is sufficient to show that

$$\Pr_{\mu} \left[ \sum_{i=1}^{n} x_i \in (s + m, s + 3m] \right] \leq 4 \Pr_{\mu} \left[ \sum_{i=1}^{n} x_i \in (s - m, s + m] \right].$$

(11)

For this, we construct four injective, measure-preserving maps from sub-events of $\{\sum_{i=1}^{n} x_i \in (s + m, s + 3m]\}$ to the event

$$\left\{ \sum_{i=1}^{n} x_i \in (s - m, s + m] \right\},$$

whose union of domains covers $\{\omega \mid \sum_{i=1}^{n} x_i(\omega) \in (s + m, s + 3m]\}$.

Let $S = \{i \in [n] \mid |x_i| \leq \frac{m}{2}\}$ and $B = \{i \in [n] \mid |x_i| > \frac{m}{2}\}$. The maps we are going to construct will only negate some of the $x_i$’s, so $B, S$ are reconstructible from the output of any of the maps. Thus, it is sufficient to show that the maps are injective given the partition $B, S$. Moreover, these maps will be measure-preserving as they only negate input variables, in a measurable fashion.

1. The first map $\psi_1$ is defined on inputs $(x_i)_{i=1}^{n}$ with $\sum_{i \in S} x_i \geq s + m$ and $\sum_{i \in [n]} x_i \in (s + m, s + 2m]$. We set $r = m$, apply $\text{PF}_r((x_i)_{i \in S})$ on the $S$-coordinates, and leave the $B$-coordinates unchanged. The map is injective as $\text{PF}_r$ is. Also, the output $y = \psi_1(x)$ satisfies $\sum_{i \in [n]} y_i \in (s - m, s + m]$ by (10) and the definition of $S$.

2. The second map $\psi_2$ is defined on inputs $(x_i)_{i=1}^{n}$ with $\sum_{i \in S} x_i \geq s + m$ and $\sum_{i \in [n]} x_i \in (s + 2m, s + 3m]$. We set $r = 2m$, apply $\text{PF}_r((x_i)_{i \in S})$ on the $S$-coordinates, and leave the $B$-coordinates unchanged. The map is injective and the output $y = \psi_2(x)$ satisfies $\sum_{i \in [n]} y_i \in (s - m, s + m]$ exactly like in the previous case.

3. The third map $\psi_3$ is defined on inputs $(x_i)_{i=1}^{n}$ with $\sum_{i \in S} x_i < s + m$. Notice that for such inputs, we have $\sum_{i \in B} x_i > 0$. First, for every $i \in B$, we extract $s_i = \text{sgn}(x_i)$ and $b_i = |x_i|$. Then, we treat $\sigma = \{s_i\}_{i \in B}$ as a $\{-1, 1\}$-valued vector and $\{b_i\}_{i \in B}$ as a vector of weights. Since

$$\sum_{i \in B} b_i s_i = \sum_{i \in B} |x_i| \text{sgn}(x_i) = \sum_{i \in B} x_i > 0,$$

we can apply the map $\text{SCF}_b$ to the vector $\sigma$ (as noted right after Definition 3.4), to obtain $\sigma' = \text{SCF}_b(\sigma)$. Finally, we define the output $y = \psi_3(x)$ by $\forall i \in B : y_i = \sigma'_ib_i$ and $\forall i \in S: y_i = x_i$. That is, we apply a single coordinate flip to the vector $(x_i)_{i \in B}$, and leave the $S$-coordinates unchanged. The map is injective since $\text{SCF}_b$ is. By the definitions of $\text{SCF}$ and of $B$, we have

$$\sum_{i} x_i - \sum_{i} y_i \in (m, 2m]$$

(12)

(as $\text{SCF}$ flips a single coordinate whose value is between $m/2$ and $m$, being taken from $B$). Therefore, the output might not satisfy $\sum_{i \in [n]} y_i \in (s - m, s + m]$. If it does not, we apply the next map $\psi_4$ on our intermediate “output” $y$. 

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Hence, it is sufficient to prove that

\[ s - m < \sum_{i \in [n]} x_i - 2m \leq \sum_{i \in [n]} y_i, \]

and we assumed that \( \sum_{i \in [n]} y_i \not\in (s - m, s + m] \). Finally, we claim that \( z = \psi_4(y) \) satisfies \( \sum_{i \in [n]} z_i \in (s - m, s + m] \). Indeed, applying again (12) we see that on the one hand,

\[ \sum_{i \in [n]} z_i \geq \sum_{i \in [n]} y_i - 2m > s - m, \]

and on the other hand,

\[ \sum_{i \in [n]} z_i < \sum_{i \in [n]} y_i - m < \sum_{i \in [n]} x_i - 2m \leq s + m. \]

This completes the proof of the lemma. \( \square \)

Two remarks are due.

1. One may wonder whether the constant 5 in Inequality (7) can generally be improved. We believe the correct value is 2; it surely cannot be less. Indeed, consider \( X = \sum_{i \in [n]} x_i \) where \( x_i \sim \{-1, 1\} \) independently from each other. Trivially, \( X \) assumes only values equal to \( n \pmod{2} \). Hence, taking \( n \) to be a large odd integer and \( (m = 1.5, s = 1, t = 2) \), we have

\[ \Pr[X \in (t - m, t + m)] = \Pr[X \in (0.5, 3.5)] \approx 2 \Pr[X \in (-0.5, 2.5)] = 2 \Pr[X \in (s - m, s + m)]. \]

2. One may also wonder whether Inequality (7) can be strengthened in the case where \( m \) is large, so that the constant 5 is replaced by \( 1 + O(\max a_i/m) \). This can indeed be done, by slightly modifying the proof of Lemma 3.1. Specifically, in the beginning of the proof we may define \( r = t - s - 2 \max_i \{a_i\} \), and then due to (13) instead of (9) we get

\[ \Pr_x \left[ \sum_{i=1}^{n} x_i \in (t - m, t + m) \right] \leq \Pr_x \left[ \sum_{i=1}^{n} x_i \in (s - m, s + m + 2 \max_i \{a_i\}) \right]. \]

Hence, it is sufficient to prove that

\[ \Pr_x \left[ \sum_{i=1}^{n} x_i \in (s + m, s + m + 2 \max_i \{a_i\}) \right] \leq O \left( \frac{\max_i \{a_i\}}{m} \right) \Pr_x \left[ \sum_{i=1}^{n} x_i \in (s - m, s + m) \right], \]

and this indeed follows immediately by invoking Lemma 3.1 itself as a black-box.

The following corollary of Lemma 3.1 will be used several times in the sequel, so for sake of convenience we state it explicitly.

**Corollary 3.5.** Let \( f_s = 1_{a \cdot x > s} \) be a family of halfspaces and suppose \( a_1 \geq a_2 \geq \ldots a_n \geq 0 \). Then for any \( s, t \in \mathbb{R} \) such that \( |s| \leq t \), we have \( 5I_1(f_s) \geq I_1(f_t) \).
Proof. Notice that \( I_1(f_s) = \Pr_{x \sim \{-1,1\}^n}[a \cdot x - a_1x_1 \in (r - a_1, r + a_1)] \). Hence, for \( 0 \leq s < t \), the corollary follows immediately from Lemma \ref{lem:1}. To prove the assertion for \( s < 0 < t \), note that if \( s' = -s \), we have

\[
I_1(f_s) = I_1(\mathbb{1}\{a \cdot x \geq s'\}) = \Pr_{x \sim \{-1,1\}^n}[a \cdot x - a_1x_1 \in [s' - a_1, s' + a_1]].
\]

Hence, it is sufficient to prove a variant of Lemma \ref{lem:1} in which the assertion is replaced by

\[ \Pr \left[ \sum_{i=1}^n x_i \in (t - m, t + m) \right] \leq 5 \Pr \left[ \sum_{i=1}^n x_i \in [s' - m, s' + m] \right] \]

(i.e., only the types of open-closed intervals are changed). This variant can be proved by essentially repeating the proof of Lemma \ref{lem:1}; the only change required is slightly modifying the definition of the map \( SF_r \), such that instead of using \( t = \min \{i \in \{0, \ldots, n\} \mid S_i \geq r \} \), one lets \( t = \min \{i \in \{0, \ldots, n\} \mid S_i > r \} \).

The rest of the argument (including the definition of the map \( SCF_r \)) works without change. \( \square \)

\section{Relative Chernoff Inequalities}

In this section we prove our relative Chernoff inequalities, namely, Theorems \ref{thm:10}, \ref{thm:11} and \ref{thm:12}.

First we present the proof of Theorem \ref{thm:10} using the general method of Benjamini et al. \cite{2}, and then we present a proof of all three theorems, via Lemma \ref{lem:13}.

\subsection{Proof of Theorem \ref{thm:10} using the Benjamini-Kalai-Schramm method}

Recall that Theorem \ref{thm:10} asserts that if \( \{x_i\} \) are independent random variables uniformly distributed in \( \{-1,1\} \), and \( a_i \in \mathbb{R}_{\geq 0} \) satisfy \( \sum a_i^2 = 1 \), then there exists a universal constant \( c > 0 \) such that if \( t \geq 0 \) and \( \epsilon = \Pr[\sum a_i x_i > t] \), then \( \Pr[\sum a_i x_i > t + \delta] \leq \frac{c}{\delta} \), for \( \delta \leq \sqrt{\frac{c}{\log(1/\epsilon)}} \).

To prove the theorem, we need the following result of Talagrand \cite{46}, whose proof relies on the hypercontractive inequality \cite{5}.

\begin{theorem}[\cite{46} Theorem 1.5]
Let \( g \) be a real-valued function on the discrete cube, i.e., \( g : \{-1,1\}^n \rightarrow \mathbb{R} \), and let \( L_i f = (f(x) - f(x \oplus e_i))/2 \) for \( i = 1, 2, \ldots, n \). Then

\[
\Var(g) \leq O \left( \sum_i \frac{\|L_i g\|_2^2}{1 + \log(\|L_i g\|_2 / \|L_i g\|_1)} \right). \tag{14}
\]

\end{theorem}

\begin{proof}[Proof of Theorem \ref{thm:10}] We let \( g = \max(t, a \cdot x) \) and \( \epsilon = \Pr[g = t] \), and apply Theorem \ref{thm:11} to \( g \). Now, we bound the terms that appear in (14).

Firstly, as \( \forall i : L_i(g) \leq a_i \), we clearly have

\[
\|L_i g\|_2^2 \leq \Pr[L_i g \neq 0] \|L_i g\|_\infty^2 \leq 2\epsilon a_i^2.
\]

Secondly, by the Cauchy-Schwarz inequality,

\[
\|L_i g\|_1 \leq \sqrt{\Pr[L_i g > 0]} \cdot \sqrt{\mathbb{E}[L_i g^2]} \leq \sqrt{2\epsilon} \|L_i g\|_2,
\]

and thus,

\[
1 + \log(\|L_i g\|_2 / \|L_i g\|_1) \geq \log(1/\sqrt{2\epsilon}) = \Omega(\log(1/\epsilon)).
\]

Substituting into (14) and using the assumption \( \sum a_i^2 = 1 \), we obtain

\[
\Var(g) \leq O \left( \sum_i \frac{\|L_i g\|_2^2}{1 + \log(\|L_i g\|_2 / \|L_i g\|_1)} \right) \leq O \left( \sum_i \frac{2\epsilon a_i^2}{\log(1/\epsilon)} \right) \leq O(\epsilon / \log(1/\epsilon)). \tag{15}
\]

\end{proof}
On the other hand, by Chebyshev’s inequality, we have
\[ \Pr[|g - \mathbb{E}[g]| > \sqrt{2 \text{Var}(g)}] < \frac{\epsilon}{2}. \]
Since for \( t \geq 0 \) we have \( \Pr[g \leq t] \geq 1/2 \geq \epsilon/2 \), we must have \( \mathbb{E}[g] \leq t + \sqrt{2 \text{Var}(g)}/\epsilon \), and so by Markov’s inequality,
\[ \Pr[g > t + \sqrt{8 \text{Var}(f)/\epsilon}] \leq \frac{\epsilon}{2}. \]
Substituting the bound of (15) on \( \text{Var}(g) \), we obtain \( \Pr[g > t + \delta] \leq \epsilon/2 \) for \( \delta = O(1/\sqrt{\log(1/\epsilon)}) \), as asserted. This completes the proof.

4.2 Proof of Theorems 1.10, 1.11, and 1.12, using Lemma 1.13

In the proof, we shall use Lemma 1.13 via the following auxiliary lemma.

**Lemma 4.2.** Let \( X = \sum x_i \) where \( \{x_i\} \) are independent symmetric (around 0) random variables with \( |x_i| \leq a_i \) almost surely, and let \( F(t) = \Pr[X > t] \). Set \( m = 2 \max \{a_i\} \). Then, for every \( t \geq 0 \) and \( \delta > 0 \), we have \( F(t + \delta + m)^l \leq 2F(t)^{l+1} \), where \( l = 1 + \lfloor t/\delta \rfloor \).

**Proof.** Set \( r = t - l\delta < 0 \). Since \( x_i \) are symmetric, we have \( F(r) \geq 1/2 \). Applying repeatedly Lemma 1.13 with \( d = t + \delta + m, c = t, \) and \( b \) taken from the sequence of values \( b = r, r + \delta, r + 2\delta, \ldots, r + (l - 1)\delta \), we get a series of inequalities:
\[
\frac{1}{2} F(t + \delta + m)^l \leq F(r) F(t + \delta + m)^l \leq F(r + \delta) F(t) F(t + \delta + m)^{l-1} \leq F(r + 2\delta) F(t)^2 F(t + \delta + m)^{l-2} \leq \cdots \leq F(r + l\delta) F(t)^l = F(t)^{l+1}.
\]
This completes the proof.

Now we are ready to present the proofs of the theorems. We note that although we already proved Theorem 1.10 above, we present an alternative proof as well, since it is more constructive and may be applicable in settings where Talagrand’s result does not apply.

We begin with a proof of Theorem 1.12, which asserts the following:

**Theorem.** Let \( X = \sum x_i \) where \( \{x_i\} \) are independent symmetric (around 0) random variables with \( |x_i| \leq a_i \) almost surely, and let \( F(t) = \Pr[X > t] \). Set \( m = 2 \max \{a_i\}, \) and let \( c \in (0, 1) \). If \( \epsilon = F(t) \) for some \( t \geq 0 \), and \( \delta \geq 0 \) is minimal such that \( F(t + \delta) \leq c \cdot F(t) \), then we have
\[
\delta \leq O \left( m + \log \left( \frac{2}{c} \right) \sqrt{\frac{\sum a_i^2}{\log(1/\epsilon)}} \right).
\]

**Proof of Theorem 1.12.** Without loss of generality, assume \( \sum a_i^2 = 1 \). Denote \( \epsilon = F(t) \). We split into two cases.
Case 1: \( t < \delta - m \). Let \( \delta' \in (t + m, \delta) \). As \( \delta'' = \delta' - m > t \), we can apply Lemma 4.2 with \( \delta'' \) and \( l = 1 + \lfloor t/\delta'' \rfloor = 1 \) to obtain \( F(t+\delta')/2 = F(t+\delta'+m) \leq F(t)^2 \). Hence, \( F(t+\delta') \leq 2F(t)^2 \).

By minimality of \( \delta \) with respect to \( F(t + \delta) \leq cF(t) \), we must have

\[
2F(t) > c, \tag{16}
\]

as otherwise \( F(t + \delta') \leq (2F(t)) \cdot F(t) \leq cF(t) \) with \( \delta' < \delta \). Using again the minimality of \( \delta \), we deduce that every \( d < t + \delta \) satisfies \( F(d) > cF(t) > c^2/2 \). On the other hand, Hoeffding’s inequality implies \( F(d) \leq \exp(-d^2/2) \). Hence, \( d < \sqrt{2 \log (2/c^2)} \). Since this holds for all \( d < t + \delta \), we have \( \delta < t + \delta < \sqrt{2 \log (2/c^2)} \).

Recall that by (16), \( \epsilon = F(t) > c/2 \), and so, \( \log(1/\epsilon) \leq \log(2/c) \). All in all, we get the required inequality,

\[
\delta \leq \sqrt{2 \log (2/c^2)} < 2 \frac{\log (2/c)}{\sqrt{\log (2/c)}} \leq 2 \frac{\log (2/c)}{\sqrt{\log (1/\epsilon)}}.
\]

Case 2: \( t \geq \delta - m \). If \( \delta \leq m \), we are done. Otherwise, let \( \delta' \in (0, \delta - m) \). Applying Lemma 4.2 with \( \delta' \) and \( l = 1 + \lfloor t/\delta' \rfloor \), we get

\[
F(t + \delta' + m)^t \leq 2F(t)^{t+1}. \tag{17}
\]

As \( t \geq \delta' \), we have \( l = 1 + \lfloor t/\delta' \rfloor \leq 2t/\delta' \). Thus, (17) implies:

\[
F(t + \delta' + m) \leq 2^{1/t} F(t)^{t+1} \leq 2F(t) \cdot F(t)^{\delta'/2t}. \tag{18}
\]

Since \( F(t) = \epsilon \) by assumption, Hoeffding’s inequality yields \( t \leq \sqrt{2 \log(1/\epsilon)} \). Substituting into (18), we get

\[
F(t + \delta' + m) \leq 2F(t) \cdot F(t)^{\delta'/2t} \leq 2 \epsilon \cdot \epsilon^{\delta'/\sqrt{8 \log(1/\epsilon)}}.
\]

On the other hand, by minimality of \( \delta \) and since \( \delta' + m < \delta \), we have \( F(t + \delta' + m) > c \cdot \epsilon \), and so

\[
c/2 < \epsilon^{\delta'/\sqrt{8 \log(1/\epsilon)}} = \exp \left( \log(\epsilon) \delta'/\sqrt{8 \log(1/\epsilon)} \right).
\]

Therefore, \( \log (2/c) > \delta' \sqrt{\log(1/\epsilon)/8} \), or more nicely, \( \delta' < \sqrt{8 \log(2/c) \log(1/\epsilon)} \). Since this inequality holds for every \( \delta' \in (0, \delta - m) \), we have

\[
\delta \leq m + \sqrt{8 \log(2/c) \log(1/\epsilon)}.
\]

This completes the proof.

We now prove Theorems 1.10 and 1.11 together. Namely, we prove:

**Theorem.** Let \( \{x_i\} \) be independent random variables uniformly distributed in \( \{-1, 1\} \), and let \( a_i \in \mathbb{R}_{\geq 0} \) be such that \( \sum_i a_i^2 = 1 \). There exists a universal constant \( c > 0 \) such that if \( t \geq 0 \) and \( \epsilon = \Pr [\sum_i a_i x_i > t] \), then \( \Pr [\sum_i a_i x_i > t + \delta] \leq \frac{c}{2} \), for \( \delta \leq \frac{c}{\sqrt{\log(1/\epsilon)}} \).

Furthermore, if \( B, S \) is any partition of \( \{n\} \) (which corresponds to ‘big’ and ‘small’ values of the \( a_i \)'s), then either \( |B| \geq \frac{1}{2} \log(1/\epsilon) \), or \( \delta \leq c \sqrt{\frac{\sum_{i \in S} a_i^2}{\log(1/\epsilon)}} \).
Proof of Theorems 1.10 and 1.11. Let \( X = \sum_{i \in [n]} a_i x_i \) and \( \epsilon = \Pr[X > t] \) for \( t \geq 0 \). Take \( \delta \geq 0 \) to be minimal such that \( \Pr[X > t + \delta] \leq \frac{\epsilon}{2} \).

First, we prove \( \delta \leq O \left( \frac{1}{\sqrt{\log(1/\epsilon)}} \right) \). Let

\[
\beta = \frac{2}{\sqrt{\log(1/\epsilon)}}, \quad S = \{i \in [n] \mid a_i \leq \beta\}, \quad \text{and} \quad B = [n] \setminus S.
\]

We have \( |B| \beta^2 \leq \sum_{i \in [n]} a_i^2 \leq 1 \), and so \( |B| \leq \frac{1}{4} \log(1/\epsilon) \). By the law of total probability,

\[
\Pr_x[a \cdot x > t] = 2^{-|B|} \sum_{y \in \{-1,1\}^B} \Pr_z \left( \sum_{i \in S} a_i z_i > t - \sum_{i \in B} a_i y_i \right).
\]

Notice that \( 2^{-|B|} \geq \epsilon^{1/4} \), and so each of the probabilities on the right hand side is \( \leq \epsilon^{3/4} \). For \( y \in \{-1,1\}^B \), let \( \delta_y \) be minimal such that

\[
\Pr_z \left( \sum_{i \in S} a_i z_i > t + \delta_y - \sum_{i \in B} a_i y_i \right) \leq \frac{1}{2} \Pr_z \left( \sum_{i \in S} a_i z_i > t - \sum_{i \in B} a_i y_i \right).
\]

Clearly, \( \delta \leq \max \{ \delta_y \mid y \in \{-1,1\}^B \} \). Applying Theorem 1.12 to the random variable \( Z = \sum_{i \in S} a_i z_i \), with \( t - \sum_{i \in B} a_i y_i \) in place of \( t \) and \( \Pr_{z \sim \{-1,1\}^S} \left[ \sum_{i \in S} a_i z_i > t - \sum_{i \in B} a_i y_i \right] \leq \epsilon^{3/4} \) in place of \( \epsilon \), we deduce

\[
\forall y : \delta_y \leq O \left( 2\beta + \sqrt{\frac{\sum_{i \in S} a_i^2}{\log(1/\epsilon^{3/4})}} \right) = O \left( \frac{1}{\sqrt{\log(1/\epsilon)}} \right),
\]

as asserted.

Now, given a partition \( B, S \) of \([n]\), we show that either \( |B| \geq \frac{1}{2} \log(1/\epsilon) \), or \( \delta \leq \sqrt{\frac{\sum_{i \in S} a_i^2}{\log(1/\epsilon)}} \).

Assume that \( |B| < \frac{1}{2} \log(1/\epsilon) \), and note that by (19), we have

\[
\forall y : \Pr_{z \sim \{-1,1\}^S} \left[ \sum_{i \in S} a_i z_i > t - \sum_{i \in B} a_i y_i \right] \leq \frac{\epsilon}{2^{-|B|}} \leq \epsilon^{1/2}.
\]

Let \( \delta_y \) be as in the above proof, and consider the random variable \( Z' = \alpha \sum_{i \in S} a_i z_i \), for \( \alpha = (\sum_{i \in S} a_i^2)^{-1/2} \) (which is needed for rescaling the weights to have sum-of-squares equal 1).

Applying (20) to \( Z' \), with \( t' = \alpha(t - \sum_{i \in B} a_i y_i) \) in place of \( t \) and using \( \Pr[Z' < t'] \leq \epsilon^{1/2} \) which follows from (21), we get

\[
\forall y : \delta_y \leq O \left( \sqrt{\frac{\sum_{i \in S} a_i^2}{\log(1/\epsilon^{1/2})}} \right),
\]

and we conclude with \( \delta \leq O \left( \sqrt{\frac{\sum_{i \in S} a_i^2}{\log(1/\epsilon)}} \right) \) since \( \delta \leq \max_y \delta_y \), as above. \( \square \)
5 First-Degree Fourier Weight of Halfspaces

In this section we prove Theorem 1.1 stating that any halfspace \( f_t = \mathbb{1}\{a \cdot x > t\} \) satisfies \( W^1(f_t) = \Omega(\mu(f_t)^2 \log(1/\mu(f_t))) \) (which is the maximal possible value up to a constant factor, by the aforementioned Level-1 inequality).

We start with an easy lemma describing how ‘large coordinates’ (i.e., coordinates \( i \) for which \( a_i \) is ‘large’) influence a halfspace.

**Lemma 5.1.** Let \( f_t(x) = \mathbb{1}\{a \cdot x > t\} \), where \( a \in \mathbb{R}_0^n \) and \( t \geq 0 \). Denote \( \epsilon = \mu(f_t) = \Pr_{x \sim \{-1, 1\}^n}[a \cdot x > t] \), and let \( \beta \geq 0 \) be minimal such that \( \Pr[a \cdot x > t + \beta] \leq \epsilon/3 \). If \( a_i > \beta/2 \) for some \( i \in [n] \), then \( I_i(f_t) \geq \frac{2}{3} \epsilon \).

**Proof.** Define \( F(s) = \Pr[a \cdot x > s] \) and \( G(s) = \Pr[a \cdot x - a_i x_i > s] \). (Note that \( G \) does not depend on the value of the coordinate \( x_i \).) We have

\[
I_i(f_t) = \Pr[a \cdot x - a_i x_i \in (t - a_i, t + a_i)] = G(t - a_i) - G(t + a_i).
\]

By the definition of \( G \),

\[
F(s) = \Pr[a \cdot x > s] = \frac{1}{2} (\Pr[a \cdot x > s|x_i = 1] + \Pr[a \cdot x > s|x_i = -1])
\]

\[
= \frac{1}{2} (G(s - a_i) + G(s + a_i)),
\]

and therefore, \( G(s) = 2F(s + a_i) - G(s + 2a_i) \). Using this repeatedly, we get

\[
G(s) = 2F(s + a_i) - 2F(s + 3a_i) + 2F(s + 5a_i) - \ldots
\]

Hence,

\[
I_i(f_t) = G(t - a_i) - G(t + a_i)
\]

\[
= 2F(t) - 4F(t + 2a_i) + 4F(t + 4a_i) - 4F(t + 6a_i) + \ldots
\]

\[
\geq 2F(t) - 4F(t + 2a_i) \geq \frac{2}{3} \epsilon,
\]

where the last inequality holds since \( a_i > \beta/2 \). \( \Box \)

We proceed with a lemma which states that in some sense, the influence of a coordinate on a halfspace is ‘proportional’ to its weight. (Recall that there exist halfspaces for which the weight of some coordinate is positive and nevertheless, it has zero influence; the lemma shows that this ‘anomaly’ can be fixed by slightly modifying the function.)

**Lemma 5.2.** For \( \delta > 0 \) define \( e_i^\delta := \mathbb{E}_{x \sim U(0, \delta)} [I_i(f_{t+s})] \). We have

\[
e_i^\delta \geq \frac{a_i}{\delta} \Pr[a \cdot x \in [t + a_i, t + \delta - a_i]].
\]

(22)

**Proof.** By the definition of \( f_{t+s} \), we have \( I_i(f_{t+s}) = \Pr_x[a \cdot x - a_i x_i \in (t + s - a_i, t + s + a_i)] \). Changing the order of integration, we can express \( e_i^\delta \) in a different way:

\[
e_i^\delta = \mathbb{E}_{s \sim U(0, \delta)} \mathbb{E}_x [\mathbb{1}\{a \cdot x - a_i x_i \in (t + s - a_i, t + s + a_i)\}]
\]

\[
= \mathbb{E}_x \mathbb{E}_{s \sim U(0, \delta)} [\mathbb{1}\{a \cdot x - a_i x_i \in (t + s - a_i, t + s + a_i)\}]
\]

\[
= \mathbb{E}_x \left[ \frac{1}{\delta} \int_0^\delta \mathbb{1}\{a \cdot x - a_i x_i \in (t + s - a_i, t + s + a_i)\} \, ds \right].
\]
(Note that there is no difference here between closed-open segments and open segments, as this does not change the integral involved.) We may assume $\delta \geq a_i$, for otherwise the statement of the lemma is trivial. An easy computation confirms that for any $t \in \mathbb{R}$ we have

$$\int_0^\delta 1 \{ r \in (t + s - a_i, t + s + a_i) \} \, ds \geq a_i \cdot 1 \{ r \in [t, t + \delta] \}.$$ 

So we overall obtain

$$e_i^\delta \geq \frac{a_i}{\delta} \mathbb{E}_x \left[ 1 \{ a \cdot x - a_i x_i \in [t, t + \delta] \} \right].$$

(23)

However, $1 \{ a \cdot x - a_i x_i \in [t, t + \delta] \} \geq 1 \{ a \cdot x \in [t + a_i, t + \delta - a_i] \}$, and so we deduce

$$e_i^\delta \geq \frac{a_i}{\delta} \mathbb{E}_x \left[ 1 \{ a \cdot x \in [t + a_i, t + \delta - a_i] \} \right] \geq \frac{a_i}{\delta} \Pr_x \left[ a \cdot x \in [t + a_i, t + \delta - a_i] \right].$$

This completes the proof. \hfill \Box

Now we are ready to prove that any halfspace has a ‘large’ Fourier weight on the first degree.

**Proof of Theorem 1.1.** Let $a \in \mathbb{R}^n_{\geq 0}$ with $\sum a_i^2 = 1$, let $f_i(x) = 1 \{ a \cdot x > t \}$, and let $\epsilon = \mathbb{E}[f_i]$. We prove $W^1(f_i) \geq \Omega(\epsilon^2 \log(1/\epsilon))$.

The main idea of the proof is as follows. We divide the coordinates into a set $B$ of coordinates $i$ whose weight $a_i$ is ‘large’, and a set $S$ of coordinates whose weight is ‘small’ (the exact definition is given below). We show that either $|B|$ is ‘large’, and then by Lemma 5.1 the contribution of the coordinates in $B$ is already sufficient to assure $W^1(f_i) \geq \Omega(\epsilon^2 \log(1/\epsilon))$, or else, the contribution of the coordinates in $S$ will assure $W^1(f_i) \geq \Omega(\epsilon^2 \log(1/\epsilon))$. To show the latter (which is the more complex case), we note that by the Cauchy-Schwarz inequality, we have

$$\sqrt{\sum_{i \in S} a_i^2} \cdot \sqrt{W^1(f_i)} \geq \sqrt{\sum_{i \in S} a_i^2} \cdot \sqrt{\sum_{i \in S} \hat{f}_i(\{i\})^2} \geq \sum_{i \in S} a_i I_i(f_i).$$

(24)

Hence, it is sufficient to show that

$$\sum_{i \in S} a_i I_i(f) \geq \Omega \left( \epsilon \sqrt{\log(1/\epsilon)} \cdot \sqrt{\sum_{i \in S} a_i^2} \right).$$

(25)

We will do so, using the relative Chernoff inequality presented in Section 4.

Let $\beta$ be minimal such that $F(t + \beta) \leq \epsilon/3$, and let $\gamma$ be minimal such that $F(t + \gamma) \leq \epsilon/6$. Denote $\delta = \beta + \gamma$, let $S = \{ i \in [n] \mid a_i \leq \beta \}$ and $B = [n] \setminus S$.

As in Lemma 5.2 we denote $e_i^\delta := \mathbb{E}_{x \sim U(0, \delta)} [I_i(f_{t+\delta})]$. For every $i \in S$, we apply Lemma 5.2 to obtain

$$e_i^\delta \geq \frac{a_i}{\delta} \Pr \left[ a \cdot x \in [t + a_i, t + \delta - a_i] \right].$$

Since $a_i \leq \beta$, the definition of $\beta$ and $\gamma$ implies

$$e_i^\delta \geq \frac{a_i}{\delta} \Pr \left[ a \cdot x \in [t + \beta, t + \gamma] \right] \geq \frac{a_i}{\delta} \cdot \frac{\epsilon}{6}.$$ 

(26)

Applying the ‘strong relative Chernoff inequality’ (Theorem 1.11) to the function $f_i$, with $S, B$ as defined above, we obtain that either $|B| \geq \frac{1}{2} \log(1/\epsilon)$ or $\gamma \leq O \left( \sqrt{\sum_{i \in S} a_i^2 / \log(1/\epsilon)} \right)$. 

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(Formally, the theorem is applied three times, where \( \Pr[a \cdot x > s] \) drops from \( \epsilon \) to \( \epsilon/2, \epsilon/4 \), and then \( \epsilon/6 \) as \( s \) increases.) In the latter case, we have \( \delta \leq 2\gamma \leq O\left(\sqrt{\sum_{i \in S} a_i^2 / \log(1/\epsilon)}\right) \).

We consider three cases:

Case 1: \(|B|\) is large – specifically, \(|B| \geq \frac{1}{2} \log(1/\epsilon)\). Note that by Lemma 5.1, every \( i \in B \) has \( I_i(f_t) \geq 2\epsilon/3 \). Hence, in this case we have \( W^1(f_t) = \sum_i I_i(f_t)^2 \geq \frac{2}{9}\epsilon^2 \log(1/\epsilon) \), as asserted.

Case 2: \( \epsilon \geq \frac{1}{4} \).

In this case, we use the aforementioned theorem of [16] which asserts that any halfspace \( g_r : \{-1,1\}^n \to \{-1,1\} \) satisfies \( W^{\leq 1}(g_r) \geq 1/2 \). This yields \( W^1(2f_t - 1) + W^0(2f_t - 1) \geq \frac{1}{2} \), which in turn implies

\[
W^1(f_t) \geq (1/2 - (1 - 2\epsilon)^2)/4 \geq \frac{1}{16} \geq \frac{1}{3} \epsilon^2 \log(1/\epsilon),
\]
as asserted.

Case 3: \(|B| < \frac{1}{2} \log(1/\epsilon)\), \( \delta \leq O\left(\sqrt{\sum_{i \in S} a_i^2 / \log(1/\epsilon)}\right) \), and \( \epsilon < \frac{1}{4} \). Using (26), we have

\[
\sum_{i \in S} a_i e_i^\delta \geq \frac{\epsilon}{6\delta} \sum_{i \in S} a_i^2.
\]

Since \( \delta \leq O\left(\sqrt{\sum_{i \in S} a_i^2 / \log(1/\epsilon)}\right) \), we deduce

\[
\sum_{i \in S} a_i e_i^\delta \geq \frac{\epsilon}{6\delta} \sum_{i \in S} a_i^2 \geq \Omega\left(\epsilon \sqrt{\log(1/\epsilon)} \cdot \sqrt{\sum_{i \in S} a_i^2}\right).
\]

Recall that by definition, \( e_i^\delta := \mathbb{E}_{s \sim U(0,\delta)}[I_i(f_t+s)] \), and thus, by linearity of expectation, we have

\[
\mathbb{E}_{s \sim U(0,\delta)} \left[ \sum_{i \in S} a_i I_i(f_t+s) \right] = \sum_{i \in S} a_i e_i^\delta.
\]

Hence, (28) implies that there exists \( s \in (0,\delta) \) with

\[
\sum_{i \in S} a_i I_i(f_t+s) \geq \Omega\left(\epsilon \sqrt{\log(1/\epsilon)} \cdot \sqrt{\sum_{i \in S} a_i^2}\right).
\]

To show that (25) holds, and thus complete the proof of the theorem, it is sufficient to show that the inequality (29) holds also for \( s = 0 \). This is achieved in the following proposition.

**Proposition.** In the former settings, where \(|B| \leq \frac{1}{2} \log(1/\epsilon)\) and \( \epsilon < \frac{1}{4} \), for any \( s > 0 \) we have \( \sum_{i \in S} a_i I_i(f_t) \geq \sum_{i \in S} a_i I_i(f_{t+s}) \).

**Proof.** We start by showing that \( \sum_{i \in B} a_i \leq t \). Assume the contrary. We then have

\[
\epsilon = \Pr[a \cdot x > t] \geq \Pr\left[\sum_{i \in B} a_i x_i > t \land \sum_{i \in S} a_i x_i \geq 0\right] \geq \frac{1}{2} \cdot 2^{-|B|} \geq \frac{1}{2} \sqrt{\epsilon}.
\]

Hence, \( \epsilon \geq 1/4 \), which contradicts the assumption.
From the monotonicity of halfspaces, we have
\[ \forall s : \sum_{i \in S} a_i I_i(f_{t+s}) = E_x \left[ \left( \sum_{i \in S} a_i x_i \right) \cdot f_{t+s}(x) \right]. \]
Thus, as \( \forall x : 0 \leq f_{t+s}(x) \leq f_t(x) \), it is sufficient to show that \( \sum_{i \in S} a_i x_i \geq 0 \) whenever \( f_t > 0 \). This indeed holds, as
\[ (f_t(x) > 0) \Rightarrow (a \cdot x > t) \Rightarrow \left( \sum_{i \in S} a_i x_i > t - \sum_{i \in B} a_i \geq 0 \right), \]
where the ultimate inequality holds since \( \sum_{i \in B} a_i \leq t \). This completes the proof of the proposition, and thus also the proof of the theorem. \( \square \)

6 The Maximal Influence of Halfspaces

In this section we prove Theorem 1.4 which asserts that if \( f_t = \mathbb{1}(a \cdot x > t) \) is a halfspace, where \( \|a\|_2 = 1 \) and \( a_1 \geq a_2 \geq \ldots \geq a_n \), then its largest influence \( I_1(f_t) \) is of order \( \Theta \left( \mu(f_t) \min \left( 1, a_1 \sqrt{\log(1/\mu(f_t))} \right) \right) \).

We start with the lower bound, which follows directly from the tools developed in the previous sections.

**Proposition 6.1.** Let \( f_t = \mathbb{1}(a \cdot x > t) \) be a halfspace, where \( \|a\|_2 = 1 \) and \( a_1 \geq a_2 \geq \ldots \geq a_n \), and assume \( \mu(f_t) \leq 1/2 \). We have
\[ I_1(f_t) \geq \Omega \left( \mu(f_t) \min \left( 1, a_1 \sqrt{\log(1/\mu(f_t))} \right) \right). \]

**Proof.** Let \( f_t \) satisfy the assumptions of the proposition. As in the proof of Theorem 1.1, we let \( \beta \) be minimal such that \( F(t + \beta) \leq F(t)/3 \) and \( \gamma \) be minimal such that \( F(t + \gamma) \leq F(t)/6 \), and denote \( \delta = \beta + \gamma \). We also denote \( e_1^\delta := \mathbb{E}_{x \sim U(0,\delta)} \left[ I_1(f_{t+s}) \right] \).

If \( a_1 > \beta \), then by Lemma 5.1 we have \( I_1(f_t) \geq \frac{2}{\delta} \mu(f_t) \), as required in (30). Hence, we may assume \( a_1 \leq \beta \). By Lemma 5.2 we have
\[ e_1^\delta \geq \frac{a_1}{\delta} \Pr \left[ a \cdot x \in [t + a_1, t + \delta - a_1] \right] \geq \frac{a_1}{\delta} \Pr \left[ a \cdot x \in [t + \beta, t + \gamma] \right] \geq \frac{a_1 \mu(f_t)}{\delta}. \]

From Theorem 1.10 we get \( \delta \leq O(1/\sqrt{\log(1/\mu(f_t))}) \), and thus,
\[ e_1^\delta \geq \frac{a_1 \mu(f_t)}{\delta} \geq \Omega \left( a_1 \mu(f_t) \sqrt{\log(1/\mu(f_t))} \right). \]

Since \( e_1^\delta \) is defined as \( \mathbb{E}_{x \sim U(0,\delta)} \left[ I_1(f_{t+s}) \right] \), this implies that there exists \( s \geq 0 \) with
\[ I_1(f_{t+s}) \geq \Omega \left( a_1 \mu(f_t) \sqrt{\log(1/\mu(f_t))} \right). \]

Finally, by Corollary 3.3, for any \( s > 0 \) we have \( 5I_1(f_t) \geq I_1(f_{t+s}) \). (Note that since \( \mu(f_t) \leq \frac{1}{2} \), we may assume \( t \geq 0 \) and so, Corollary 3.3 can indeed be applied.) Hence,
\[ I_1(f_t) \geq \frac{1}{5} I_1(f_{t+s}) \geq \Omega \left( a_1 \mu(f_t) \sqrt{\log(1/\mu(f_t))} \right), \]
as asserted. \( \square \)
To prove the upper bound, we use the following ‘reverse’ version of Corollary \[ \text{[3.5]} \] which asserts that while \( I_1(f_s) \) is a decreasing function of \( s \) up to a constant factor, the ‘normalized’ influence \( I_1(f_s)/E[f_s] \) is increasing up to a constant factor.

**Lemma 6.2.** Let \( a \in \mathbb{R}^n \geq 0 \) satisfy \( a_1 \geq \ldots \geq a_n \). For every \( t \geq s \geq 0 \) with \( E[f_t] > 0 \), we have

\[
\frac{I_1(f_s)}{E[f_s]} \leq O \left( \frac{I_1(f_t)}{E[f_t]} \right).
\]

**Proof.** For any \( r_1, r_2 \in \mathbb{R} \), let \( G(r_1) = \Pr_{x \sim \{-1,1\}^n} [a \cdot x - a_1 x_1 > r_1] \) and \( G(r_1, r_2) = G(r_1) - G(r_2) \). Notice that for any \( r \), we have

\[
E[f_r] = \frac{G(r - a_1) + G(r + a_1)}{2} \in G(r - a_1) \cdot \left[ \frac{1}{2}, 1 \right],
\]

\[
I_1(f_r) = G(r - a_1) - G(r + a_1) = G(r - a_1, r + a_1).
\]

Hence, in order to prove (31), it is sufficient to show that for every \( t \geq s \geq 0 \) with \( E[f_t] > 0 \),

\[
\frac{G(s - a_1, s + a_1)}{G(s - a_1)} \leq O \left( \frac{G(t - a_1, t + a_1)}{G(t - a_1)} \right).
\]

Let \( t \geq s \geq 0 \). Lemma \[ \text{[1.13]} \] (with \( m = 2a_1 \)) asserts that \( G(b)G(d) \leq G(c)G(b + d - c - 2a_1) \), for any \( b \leq c \leq d \). Substitute \((b, c, d) = (s - a_1, s + a_1, t + 3a_1)\) and subtract \( G(c)G(d) \) from both sides to deduce

\[
G(t + 3a_1)G(s - a_1, s + a_1) \leq G(s + a_1)G(t - a_1, t + 3a_1).
\]

It follows that

\[
\frac{G(t + 3a_1)}{G(t - a_1, t + 3a_1)} \leq \frac{G(s + a_1)}{G(s - a_1, s + a_1)}.
\]

Adding 1 to both sides we obtain

\[
\frac{G(t - a_1)}{G(t - a_1, t + 3a_1)} \leq \frac{G(s - a_1)}{G(s - a_1, s + a_1)}.
\]

Taking reciprocal and using Corollary \[ \text{[3.5]} \] we get

\[
\frac{G(s - a_1, s + a_1)}{G(s - a_1)} \leq \frac{G(t - a_1, t + 3a_1)}{G(t - a_1)} \leq \frac{6G(t - a_1, t + a_1)}{G(t - a_1)} \tag{Cor. \[ \text{[3.5]} \]},
\]

and thus, (32) holds, as asserted.

\[ \square \]

We are now ready to prove the upper bound of Theorem \[ \text{[1.4]} \].

**Proposition 6.3.** Let \( f_t = 1\{a \cdot x > t\} \) be a halfspace, where \( \|a\|_2 = 1 \) and \( a_1 \geq a_2 \geq \ldots \geq a_n \), and assume \( \mu(f_t) \leq 1/2 \). We have

\[
I_1(f_t) \leq O \left( \mu(f_t) \min \left( 1, a_1 \sqrt{\log(1/\mu(f_t))} \right) \right).
\]

(33)
Proof. Let $f_i$ satisfy the assumptions. Define $\beta, \gamma,$ and $\delta$ like above, and denote $\epsilon = \mu(f_i)$. Let $S = \{i \in [n] \mid a_i \leq \beta\}$ and $B = [n] \setminus S$. Note that we have

$$I_1(f_i) = 2\mathbb{E}[x_i f_i(x)] \leq 2\mathbb{E}[f_i(x)] \max_x |x_1| = 2\epsilon.$$  

Therefore, if $a_1 \sqrt{\log(1/\epsilon)} = \Omega(1)$ then we are done, since $I_1(f_i) \leq O(\epsilon \cdot \min\left(1, a_1 \sqrt{\log(1/\epsilon)}\right))$, as claimed. Hence, for any required small universal constant $c_1 > 0,$ we may assume

$$a_1 \leq \frac{c_1}{\sqrt{\log(1/\epsilon)}}. \quad (34)$$

By Lemma 5.1, we have $\forall i \in B : I_i(f_i) \geq \Omega(\epsilon),$ and in particular, $W^1(f_i) \geq \Omega(|B|\epsilon^2).$ Since we have $W^1(f_i) \leq O(\epsilon^2 \log(1/\epsilon))$ by the aforementioned Level-1 Inequality, this implies $|B| \leq O(\log(1/\epsilon)).$ Thus,

$$\sum_{i \in B} a_i^2 \leq |B| a_1^2 \leq O(\log(1/\epsilon) a_1^2) \leq \frac{1}{2},$$

and hence,

$$\sum_{i \in S} a_i^2 \geq 1/2. \quad (35)$$

Following the proof of Theorem 1.1 and specifically Inequalities (24) and (27) (notice these inequalities hold regardless of the cases we had there), there exists an $s \geq 0$ with

$$W^1(f_{t+s}) \geq \Omega \left( (\epsilon/\delta)^2 \sum_{i \in S} a_i^2 \right) \geq \Omega \left( (\epsilon/\delta)^2 \right).$$

Using again the Level-1 Inequality, we get $\delta = \Omega(1/\sqrt{\log(1/\epsilon)}),$ as otherwise $W^1(f_{t+s}) = \omega(\epsilon^2 \log(1/\epsilon)),$ while $\mathbb{E}[f_{t+s}] \leq \epsilon.$ In particular,

$$\gamma \geq \delta/2 \geq \Omega(1/\sqrt{\log(1/\epsilon)}). \quad (36)$$

Hence, by (34), we may assume

$$a_1 < \gamma/4. \quad (37)$$

Consider $e_1 = \mathbb{E}_{s \sim U(0, \gamma)} [I_1(f_{t+s})].$ Using Lemma 6.2, we deduce

$$e_1^\gamma = \mathbb{E}_{s \sim U(0, \gamma)} \left[ I_1(f_{t+s}) \mathbb{E}_{f_t \sim s \sim U(0, \gamma)} [f_t] \right] \geq \Omega \left( \frac{I_1(f_t)}{\epsilon} \right) \mathbb{E}_{s \sim U(0, \gamma)} [\mathbb{E}_{f_t \sim s \sim U(0, \gamma)} [f_t]] \frac{\gamma}{\gamma} \text{ def.}$$

On the other hand,

$$e_1^\gamma = \mathbb{E}_{s \sim U(0, \gamma)} \mathbb{E}_{x \sim U(0, \gamma)} [\mathbb{1} \{a \cdot x - a_1 x_1 \in (t + s - a_1, t + s + a_1)\}].$$

By Fubini

$$\leq \frac{2a_1}{\gamma} \mathbb{E}_{x} [\mathbb{1} \{a \cdot x - a_1 x_1 > t - a_1\}].$$

And

$$\leq \frac{2a_1}{\gamma} \Pr_{x} [a \cdot x > t - 2a_1] \leq 12 \frac{\epsilon}{\gamma} a_1,$$
where the last inequality will be justified below. Hence, \( I_1(f_t) \leq O \left( \frac{\epsilon}{\gamma a_1} \right) \), and thus, by (36), we have \( I_1(f_t) \leq O(a_1 \epsilon \sqrt{\log(1/\epsilon)}) \), as asserted.

It only remains to justify why \( \text{Pr}_x [a \cdot x > t - 2a_1] \leq 6 \varepsilon \). This follows from Lemma 1.13. Indeed, applying the lemma with \((b, c, d) = (t - 2a_1, t, t + 4a_1)\), we obtain

\[
F(t - 2a_1)F(t + 4a_1) \leq F(t)^2 = \varepsilon^2.
\]

Since \( 4a_1 < \gamma \) by (37), the definition of \( \gamma \) implies \( F(t + 4a_1) \geq \varepsilon / 6 \). Hence, \( \text{Pr}_x [a \cdot x > t - 2a_1] = F(t - 2a_1) \leq 6 \varepsilon \). This completes the proof.

Consider a linear form \( l(x) = \sum a_i x_i \) where \( x_i \sim \{-1, 1\} \) uniformly and independently. The following corollary is an obvious application of Theorem 1.4, which essentially describes the probability that \( l(x) \) lies in some interval \((a, b)\), by means of the tail probability \( \text{Pr}[l(x) > a] \) and the interval length \( |I| = b - a \). This, for example, generalizes [43, Theorem 4], up to the multiplicative constants. We note that one could also prove this corollary directly, by an argument similar to that of the proof of Theorem 1.4.

**Theorem 6.4.** Let \( l(x) = \sum a_i x_i \) be a linear form with \( \|a\|_2 = 1 \). If \( m \geq \max_i |a_i| \), \( t \geq -m \) and \( \epsilon = \min(1/2, \text{Pr}_x, (-1, 1)^n \mid l(x) > t) \), then

\[
\text{Pr}_{x \sim (-1, 1)^n} [l(x) \in (t, t + 2m)] = \Theta \left( \epsilon \min \left( 1, m \sqrt{\log(1/\epsilon)} \right) \right).
\] (38)

**Proof.** Assume first \( t \geq 0 \). Consider the linear form \( l'(x) = \frac{1}{\sqrt{1 + m^2}} (mx_0 + l(x)) \) where \( x_0 \sim \{-1, 1\} \) is independent of the other variables. (Note that the normalization is intended to keep the sum-of-squares of the coefficients equal 1). Consider

\[
g(x_0, x_1, \ldots, x_n) = 1 \left\{ l'(x) > \frac{1}{\sqrt{1 + m^2}} (t + m) \right\}.
\]

We have \( I_0(g) = \text{Pr} [l(x) \in (t, t + 2m)] \), while

\[
\mu(g) = (\text{Pr}[l(x) > t] + \text{Pr}[l(x) > t + 2m]) / 2 \in [\epsilon/2, \epsilon].
\]

By Theorem 1.4 we have

\[
I_0(g) = \Theta \left( \mu(g) \min \left( 1, \frac{m}{\sqrt{1 + m^2}} \sqrt{\log(1/\mu(g))} \right) \right).
\]

Combining all these implies

\[
\text{Pr} [l(x) \in (t, t + 2m)] = I_0(g) = \Theta \left( \epsilon \min \left( 1, \frac{m}{\sqrt{1 + m^2}} \sqrt{\log(1/\epsilon)} \right) \right).
\] (39)

This concludes the proof, because \( m = \Theta \left( m / \sqrt{1 + m^2} \right) \) unless \( m \geq 1 \), in which case the value of the minimum in Equation 39 is anyways \( \Theta(1) \).

It remains to consider the case \( -m \leq t < 0 \). We claim that the assertion in this case follows from the assertion for \( t = 0 \). On the one hand, let \( l''(x) \) be a small enough perturbation of \( l(x) \), as explained in Section 2, having \( \text{Pr}[l''(x) = 0] = 0 \). Then,

\[
\text{Pr}[l(x) \in (t, t + 2m)] \leq \text{Pr}[|l''(x)| < 2m] \leq 2 \text{Pr}[l''(x) \in (0, 2m)] \leq O \left( \frac{1}{2} \min \left( 1, m \sqrt{\log 2} \right) \right),
\]

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Proposition 7.2. Let $VB$ where the ultimate inequality follows (again) from the case $t = 0$. This completes the proof. □

7 The Vertex Boundary of Halfspaces

Definition 7.1. Let $f : \{-1, 1\}^n \to \{0, 1\}$ be a monotone Boolean function. For $\delta \in \{0, 1\}$, define the $\delta$-vertex-boundary by

$$VB_\delta (f) = \Pr_{x \{\{-1, 1\}^n}[ (f(x) = \delta) \land (\exists i \in [n] : f(x \oplus e_i) \neq \delta) ] .$$

In this section we prove Theorem 1.5 which asserts that for any halfspace $f_t = \mathbb{1}\{a \cdot x > t\}$, where $\|a\|_2 = 1$, $a_1 \geq a_2 \geq \ldots \geq a_n > 0$, and $\mu(f_t) \leq \frac{1}{2}$, we have $VB_1(f_t) = \Theta\left(\mu(f_t)\min\left(1, a_1\sqrt{\log(1/\mu(f_t))}\right)\right)$. In addition, we show that for halfspaces, $VB_1(f_t)$ and $VB_0(f_t)$ cannot be too far from each other, while for general Boolean functions they can be 'very' far.

Remark. All the results in this section apply also to halfspaces $f_t$ having $\mu(f_t) \geq \frac{1}{2}$. The difference is that $\mu(f_t)$ should be replaced by $1 - \mu(f_t)$ and $VB_0(f_t)$ exchanges roles with $VB_1(f_t)$.

We start with a proposition, which, together with Theorem 1.4 implies Theorem 1.5.

Proposition 7.2. Let $f_t = \mathbb{1}\{a \cdot x > t\}$ be a halfspace with $a_1 \geq a_2 \geq \ldots \geq a_n > 0$. Assume that $\mathbb{E}[f_t] \leq \frac{1}{2}$. Then,

$$\frac{1}{2}I_1(f_t) \leq VB_1(f_t) \leq \frac{7}{4}I_1(f_t). \tag{40}$$

Proof. Since $\mu(f_t) \leq \frac{1}{2}$, we may assume w.l.o.g. $t > 0$ as noted in Section 2. We observe that, since $a_i \geq a_k$ for all $k > i$, if for some $x$ we have $f_t(x) \neq f_t(x \oplus e_k)$ and $i < k$ satisfies $x_i = x_k$, then $f_t(x) \neq f_t(x \oplus e_i)$. Hence, setting

$$c_k = \Pr_x \left[ (x_k = 1) \land (f_t(x) \neq f_t(x \oplus e_k)) \land \bigwedge_{i<k} (x_i = -1) \right] ,$$

we have $VB_1(f_t) = \sum_{k=1}^n c_k$. Define auxiliary variables

$$b_k(\delta_1, \ldots, \delta_{k-1}) = \Pr_x \left[ (f_t(x) \neq f_t(x \oplus e_k)) \land \bigwedge_{i<k} (x_i = \delta_i) \right] ,$$

so that $c_k = \frac{1}{2}b_k(-1, \ldots, -1)$. Note that by the law of total probability,

$$I_k(f_t) = \sum_{\delta \sim \{-1, 1\}^{k-1}} b_k(\delta) .$$

We claim that for any $\delta \in \{-1, 1\}^{k-1}$, we have $b_k(\delta) \geq \Omega(c_k)$. Indeed, we have

$$b_k(\delta) = 2^{-(k-1)} \cdot \Pr_{x \sim \{-1, 1\}^{[n]\setminus[k]}} \left[ \sum_{i>k} a_i x_i \in \left( t - \sum_{i<k} a_i \delta_i \right) + (-a_k, a_k) \right] .$$
Hence, Corollary 3.5 (applied to the family of halfspaces \( \{ \{ \sum_{i=k+1}^n a_i x_i > s \} \} \)), using the assumption \( t > 0 \) implies:
\[
\forall \delta : b_k(\delta) \geq \frac{1}{5} b_k(-1, \ldots, -1) = \frac{2}{5} c_k.
\]
Since there are \( 2^{k-1} \) different \( \delta \)'s, we have \( I_k(f_t) = \sum_{\delta \sim (-1,1)^{k-1}} b_k(\delta) \geq \frac{2^k}{5} c_k \), or more tightly, \( I_k(f_t) \geq \frac{2^k+8}{5} c_k \) (as the summand \( b_k(-1, -1, \ldots, -1) \) contributes \( 2c_k \) instead of \( \frac{2}{5} c_k \)). Thus, \( c_k \leq \frac{5}{2^{k+8}} I_k(f_t) \leq \frac{5}{2^{k+8}} I_1(f_t) \). Overall, we have
\[
\frac{1}{2} f_1(f_t) = c_1 \leq V B_1(f_t) = \sum_{k=1}^n c_k \leq \sum_{k=1}^\infty \frac{5}{2^k + 8} I_1(f_t) \leq 1.71 I_1(f_t),
\]
which completes the proof. \( \square \)

**Remark.** The lower bound of (40) is tight, e.g., for the dictatorship \( \mathcal{I}(x_1 > 0) \). As is apparent from (41), the constant 7/4 in the upper bound of (40) is not tight. The majority function achieves \( V B_1(\text{Majority}) \approx 1 \cdot I_1(\text{Majority}) \). Interestingly, there are examples of halfspaces \( f_t \) with \( V B_1(f_t) > I_1(f_t) \). For instance, \( f_t = \mathcal{I}(\{ \sum_{i=1}^n x_i + 4 \cdot \sum_{i=5}^n x_i > 1 \} \) satisfies \( V B_1(f_t) / I_1(f_t) \to 10/9 \) where \( n \to \infty \) is odd. We have no conjecture for what the correct upper bound is.

Theorem 1.5 follows immediately by combining Proposition 7.2 with Theorem 1.4.

We present now a similar argument which establishes a sharp relation between \( V B_0(f_t) \) and \( V B_1(f_t) \) for halfspaces.

**Proposition 7.3.** For any halfspace \( f_t \) with \( \mu(f_t) \leq \frac{1}{2} \), we have
\[
\Omega(V B_1(f_t)) \leq V B_0(f_t) \leq O(\log(1/\mu(f_t))) V B_1(f_t).
\]

**Proof.** Let \( f_t = \mathcal{I}(a \cdot x > t) \) be a halfspace with \( \Pr[f_t = 1] \leq \frac{1}{2} \), and assume without loss of generality \( a_1 \geq a_2 \geq \ldots \geq a_n > 0 \) and \( t > 0 \).

We clearly have \( \Omega(V B_1(f_t)) \leq V B_0(f_t) \), since \( V B_0(f_t) \geq \frac{1}{2} I_1(f_t) \) and \( I_1(f_t) = \Theta(V B_1(f_t)) \) by Proposition 7.2.

For the proof of \( V B_0(f_t) \leq O(\log(1/\mu(f_t))) V B_1(f_t) \), we let \( b_k(\delta) : \{-1, 1\}^{k-1} \to [0, 1] \) be the auxiliary variables from the proof of Proposition 7.2 and set
\[
b_k(\delta_1, \ldots, \delta_{k-1}) = \Pr_x \left[ (f_t(x) = 1) \land \bigwedge_{i<k} (x_i = \delta_i) \right].
\]
Recall that by the law of total probability we have \( I_k(f_t) = \sum_{\delta \sim \{-1,1\}^{k-1}} b_k(\delta) \), so in particular, \( \forall \delta : b_k(\delta) \leq I_k(f_t) \leq I_1(f_t) \). In addition, like in the proof of Proposition 7.2, we observe that
\[
2 V B_0(f_t) = \sum_{k=1}^n b_k(1, \ldots, 1) \leq \sum_{k=1}^{\log(1/\mu(f_t))} \min \left\{ I_1(f_t), O(2^{-k} I_1(f_t)/\mu(f_t)) \right\}
\]
Combining (42) with the two former observations, we obtain
\[
2 V B_0(f_t) = \sum_{k=1}^n b_k(1, \ldots, 1) \leq \sum_{k=1}^{\log(1/\mu(f_t))} \min \left\{ I_1(f_t), O(2^{-k} I_1(f_t)/\mu(f_t)) \right\}
\]
Combining (42) with the two former observations, we obtain
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2 V B_0(f_t) = \sum_{k=1}^n b_k(1, \ldots, 1) \leq \sum_{k=1}^{\log(1/\mu(f_t))} \min \left\{ I_1(f_t), O(2^{-k} I_1(f_t)/\mu(f_t)) \right\}
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Combining (42) with the two former observations, we obtain
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\]
Combining (42) with the two former observations, we obtain
\[
2 V B_0(f_t) = \sum_{k=1}^n b_k(1, \ldots, 1) \leq \sum_{k=1}^{\log(1/\mu(f_t))} \min \left\{ I_1(f_t), O(2^{-k} I_1(f_t)/\mu(f_t)) \right\}
\]
completing the proof. Hence, it is only left to prove (42). For this, consider the family of halfspaces
\[ \{ f^\delta : \{0,1\}^n \rightarrow \{0,1\} \}_{\delta \in \{-1,1\}^{k-1}}, \]
defined as \( f^\delta(x) = f_t(\delta, x) \). Note that for each \( \delta \in \{-1,1\}^{k-1} \), we have \( b'_k(\delta) = 2^{1-k} \mathbb{E}[f^\delta] \) and \( b_k(\delta) = 2^{1-k} I_{\text{max}}(f^\delta) \). Now, observe that
\[ \forall \delta \in \{-1,1\}^{k-1}, \quad \frac{b_k(1, \ldots, 1)}{b'_k(1, \ldots, 1)} = \frac{\mathbb{E}[f_t]}{b'_k(1, \ldots, 1)} \leq O \left( \frac{b_k(\delta)}{b'_k(\delta)} \right). \tag{43} \]
Indeed, an application of Lemma 6.2 to the family of halfspaces \( \{ f^\delta \} \) implies that \( I_{\text{max}}(f^\delta) / \mathbb{E}[f^\delta] \) is (up to a constant) an increasing function of \( t_\delta = t - \sum_{i<k} a_i \delta_i \) in the range \( t_\delta \geq 0 \); In the range \( t_\delta \leq 0 \), \( \mathbb{E}[f^\delta] \) is clearly a decreasing function of \( t_\delta \), while from Lemma 3.5 \( I_{\text{max}}(f^\delta) \) is (up to a constant) an increasing function of \( t_\delta \). This confirms Inequality (43). Hence,
\[ \frac{b_k(1, \ldots, 1)}{b'_k(1, \ldots, 1)} \mu(f_t) = \frac{b_k(1, \ldots, 1)}{b'_k(1, \ldots, 1)} \sum_{\delta \in \{-1,1\}^{k-1}} b'_k(\delta) \leq \sum_{\delta \in \{-1,1\}^{k-1}} O (b_k(\delta)) \leq O(I_k(f_t)). \]
Since \( b'_k(1, \ldots, 1) \leq 2^{1-k} \), we conclude \( b_k(1, \ldots, 1) \leq O(2^{-k} I_k(f_t)/\mu(f_t)) \), as required by (42). This completes the proof. \( \square \)

We conclude this section with an example, suggested by Rani Hod, showing that for general Boolean functions, the difference between \( \text{VB}_1(f) \) and \( \text{VB}_0(f) \) can be very large (in contrast to Proposition 7.3 which should be viewed as a property of halfspaces). The example is based on a random construction of Talagrand [47], originally proposed as an example of a monotone Boolean function \( g \) with ‘maximal’ vertex boundary \( \text{VB}(g) = \Omega(1) \) and ‘maximal’ total influence \( I(g) = \Omega(\sqrt{n}) \).

**Example 7.4.** Let \( b = \sqrt{n} \), \( a = 2^b/b \), and define \( h : \{-1,1\}^n \rightarrow \{0,1\} \) by
\[ h(x) = \bigvee_{i \in [a]} \bigwedge_{j \in S_i} x_j, \]
where, for every \( i \), \( S_i \) is a random subset of \([n]\) of size \( b \). Also, let \( f(x) = h(x) \lor \text{Maj}_n(x) \). We claim the following (proofs will be given below).

1. Always, \( 1/2 \leq \mu(f) \leq 1/2 + 1/\sqrt{n} \).

2. Always, \( \text{VB}_1(f) = O(1/\sqrt{n}) \).

3. With probability \( \geq \Omega(1), \text{VB}_0(f) = \Omega(1) \).

Hence, there exists an almost unbiased Boolean function \( f \) having a multiplicative gap of \( \sqrt{n} \) between \( \text{VB}_0(f) \) and \( \text{VB}_1(f) \). (For comparison, Proposition 7.3 implies that for almost unbiased halfspaces we have \( \text{VB}_0(f) = \Theta(\text{VB}_1(f)) \).) Furthermore, the example implies that apparently, there is no analog to Proposition 7.2 for general functions, as \( f \) and its dual function \( 1-f(-x) \) are similar in terms of Fourier expansion (and in particular, have the same influences), but are very different with respect to the \( \text{VB}_1(\cdot) \) measure.

Let us verify the above claims.
1. It is clear that \( \mu(f) \geq \mu(\text{Maj}_n) = 1/2 \). On the other hand, since \( \mu(\bigwedge_{j \in S_t} x_j) = 2^{-t} \), from a union bound we have \( \mu(h) \leq \frac{1}{n} \) and consequently,
\[
\mu(f) \leq \mu(\text{Maj}_n) + \mu(h) \leq \frac{1}{2} + \frac{1}{b} = \frac{1}{2} + \frac{1}{\sqrt{n}}.
\]

2. For an \( x \in \{-1, 1\}^n \) to be in the upper-boundary of \( f \), either \( f \) is in the upper-boundary of \( h \) or in that of \( \text{Maj}_n \). We have \( \text{VB}_1(\text{Maj}_n) = \Theta(1/\sqrt{n}) \), and \( \text{VB}_1(h) \leq \mu(h) \leq 1/b \leq 1/\sqrt{n} \) as above. Hence, \( \text{VB}_1(f) = O(1/\sqrt{n}) \).

3. For an \( x \in \{-1, 1\}^n \) to be in the lower-boundary of \( f \), it is sufficient that \( x \) is in the lower-boundary of \( h \) and \( \text{Maj}_n(x) = 0 \). Let \( x \in \{-1, 1\}^n \) be chosen uniformly at random among the vectors that satisfy \( \sum x_i = -2c\sqrt{n} \), for a fixed \( c \in (0, 10) \). We want to show that with some positive probability (‘continuously’) depending on \( c \), \( x \) lies in the lower-boundary of \( h \). As \( \mu(\{x : -20\sqrt{n} \leq \sum x_i < 0\}) = \Omega(1) \), this will imply \( \text{VB}_0(f) = \Omega(1) \).

Since \( \mu(h) = o(1) \) as we showed above, it is sufficient to show that
\[
\Pr_x [\exists i \in [a] : |S_i \setminus \text{supp}(x)| \leq 1] = \Omega(1). \tag{44}
\]

Consider a specific \( i \in [a] \). The probability \( P_i(x) \) that \( |S_i \setminus \text{supp}(x)| \leq 1 \) given \( X = |\text{supp}(x)| \), is at least \( (n - X)(X-1)/(n-1) \). Recall \( X = n/2 - c\sqrt{n} \), so a computation gives \( P_i(x) \geq h2^{-b-1} \exp(-cb/\sqrt{n}) \) for \( c = O(1) \). Since there are \( a = 2^b/b \) independent relevant \( i \)'s, the number of those \( i \) with \( |S_i \setminus \text{supp}(x)| \leq 1 \) is approximately distributed \( \text{Poi}(\exp(-O(c))) \), and is nonzero with a constant probability. Thus, \( \text{ VB}_0(f) \) holds, as asserted.

8 \( k \)'th Degree Fourier Weight of Halfspaces

In this section we prove Theorem 1.3 which asserts that the Level-\( k \) inequality is tight for strongly biased halfspaces, up to a multiplicative factor depending only on \( k \). Specifically, we prove the following result which clearly includes Theorem 1.3.

**Theorem 8.1.** There exist universal constants \( c_1,c_2 \) such that for any \( k \in \mathbb{N} \) and for any halfspace \( f_t = 1\{ a \cdot x > t \} \) with \( \epsilon = \mu(f_t) < 2^{-0.999k} \) and \( I_1(f_t) \leq c_1(\mu(f_t)/k) \), we have
\[
W^k(f_t) \geq \frac{(c_2 \log(2k))^{-k}}{k!} 2^{\log(1/\epsilon)^k}. \tag{45}
\]

Let us compare Theorems 1.1 and 8.1. While Theorem 1.1 expresses the tightness of the Level-1 Inequality for halfspaces (up to a multiplicative constant factor), Theorem 8.1 states that even the Level-\( k \) inequalities are tight for halfspaces. However, Theorem 8.1 has two disadvantages. The first is that the requirement that all the influences \( I_1(f_t) \) are somewhat small. (Note that the maximal possible value of an influence is \( 2\mu(f_t) \), and so, we are ‘missing’ a factor of \( O(k) \).) The second is that Equation (15) is not the exact converse of the Level-\( k \) inequality, as there is a factor \( (2k)^{-k} \) off; though, for a constant \( k \) (which is the case highlighted in Theorem 1.3, 15) is indeed tight up to a constant multiplicative factor.

We believe these two deficiencies are actually not inherent, and are side effects of our proof. Specifically, it is plausible one can omit the assumption that \( f_t \)'s influences are small and replace the multiplicative \( (c_2 \log(2k))^{-k} \) in Equation (15) by \( c^k \) for some universal constant \( c > 0 \).
Before we present the proof, some preparations are needed. Let \( f_t = 1\{a \cdot x > t\} \) be a halfspace. Recall that in the proof of Theorem 1, a central role is played by the quantity \( \sum a_i \delta_i \), where

\[
e_i^\delta := \mathbb{E}_{s \sim U(0,\delta)} \left[I_i(f_{t+s})\right] = \mathbb{E}_{s \sim \delta U(0,1)} \left[\widehat{f}_{t+s}(\{i\})\right].
\]

In this section we consider the following degree-\( k \) generalization of this notion.

For a set \( S \), we denote \( a^S = \prod_{i \in S} a_i \). We let \( T = T_k \) be a random variable, distributed as the sum of \( k \) independent \( U(0,1) \)-distributed variables (also called the Irwin-Hall distribution). Then, for any \( S \subset [n] \) with \( |S| = k \) we set

\[
e_S^\delta := \mathbb{E}_{s \sim \delta T} \left[I_{S}(f_{t+s})\right],
\]

and consider the quantity

\[
M = \sum_{|S| = k} a^S e_S^\delta.
\]

The following easy propositions will help us to study this quantity.

**Claim 8.2.** Let \( X = \sum_{i=1}^m X_i \) be the sum of \( m \) independent random variables distributed \( X_i \sim U(0,1) \). Let \( G_m = G_X \) be the cumulative distribution function of \( X \). Then the \( m \)th derivative of \( G_m \) satisfies \( G_m^{(m)}(x) = (-1)^{[x]} \frac{1}{[x]} \), except for \( x \in \{0,\ldots,m\} \).

**Proof.** We have \( G_m(x) = \int_{x-1}^x G_m(t)dt \), and so \( G_m^{(m)}(x) = G_m^{(m-1)}(x) - G_m^{(m-1)}(x-1) \). The assertion follows by induction. \( \square \)

**Claim 8.3.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a function that is differentiable \( m \) times. Let \( a \in \mathbb{R}_{\geq 0} \) and denote \( S = \sum_{i[m]} a_i \). For any \( s \in \mathbb{R} \), we have

\[
\prod_{i=1}^m a_i \inf_{t \in (s-S,s+S)} g^{(m)}(t) \leq \mathbb{E}_{x \sim \{0,\ldots,m\}} \left[x^{[m]} g(s + a \cdot x)\right] \leq \prod_{i=1}^m a_i \sup_{t \in (s-S,s+S)} g^{(m)}(t),
\]

where \( x^{[m]} = \prod_{i=1}^m x_i \).

**Proof.** Integrating each time the inner-most integral, one can find by induction that

\[
\sum_{x \in \{0,\ldots,m\}} x^{[m]} g(s + a \cdot x) = \frac{1}{\prod_{i=1}^m a_i} \int_{-1}^1 \cdots \int_{-1}^1 g^{(m)} \left(s + \sum_{i=1}^m a_i t_i\right) dt_1 \cdots dt_m
\]

and so, Equation (47) follows. \( \square \)

**Proposition 8.4.** Let \( f_t(x) = 1\{a \cdot x > t\} \) be a halfspace with \( \forall i : a_i \geq 0 \), let \( S \subset [n] \) be a set of size \( k \), let \( \delta > 0 \) and let \( e_S^\delta \) be as defined in (10). If \( A = \sum_{i \in S} a_i \leq \delta/2 \), then

\[
e_S^\delta \geq \frac{1}{\delta^k} \prod_{i \in S} a_i \Pr_{z \in \{0,\ldots,n\}} \left[a \cdot z > t + 2A\right] - \sum_{l=1}^k \binom{k}{l} \Pr_{z \in \{0,\ldots,n\}} \left[a \cdot z > t + l\delta - 2A\right].
\]

**Proof.** Notice we have, by definition,

\[
\widehat{f}_{t+s}(S) = \mathbb{E}_{y \in \{0,\ldots,n\}\setminus S} \mathbb{E}_{x \in \{0,\ldots,n\}^S} \left[\sum_{y \in \{0,\ldots,n\}\setminus S} \mathbb{1}\{a \cdot y + a \cdot x - t > s\}\right],
\]

(49)
with the somewhat abusive notation \( a \cdot x = \sum_{i \in S} a_i x_i \) and \( a \cdot y = \sum_{i \notin S} a_i y_i \). Recall that \( T \) is defined as the sum of \( k \) independent \( U(0, 1) \)-distributed variables, and let \( G_k \) be the cumulative distribution function of \( T \). We have

\[
\mathbb{E}_T[\mathbb{1} \{ a \cdot y + a \cdot x - t > \delta T \}] = \Pr_T \left[ T < \frac{a \cdot y + a \cdot x - t}{\delta} \right] = G_k \left( \frac{a \cdot y + a \cdot x - t}{\delta} \right).
\]

Hence, substituting into (49) and using Fubini, we obtain

\[
e^\delta = \mathbb{E}_T \left[ f_{t+\delta} T(S) \right] = \mathbb{E}_{y \in \{-1,1\}^{|n|}} \mathbb{E}_{x \in \{-1,1\}^S} \left[ x^S G_k \left( \frac{a \cdot x + a \cdot y - t}{\delta} \right) \right]. \tag{50}
\]

In the right hand side, for each fixed \( y \), the expectation over \( x \) is of the form \( \sum_{x \in \{-1,1\}^m} x^{[m]} g(s + a \cdot x) \) discussed in Claim 8.3 with \( (k, S, G_k, (a \cdot y - t)/\delta, a/\delta) \) in place of \( (m, [m], g, s, a) \), respectively, and with expectation in place of sum. Thus, by Claim 8.3, each such expectation can be bounded from below by

\[
\prod_{i \in S} \left( \frac{a_i}{\delta} \right) \cdot \inf_{t \in ((a \cdot y - t - A)/\delta, (a \cdot y - t + A)/\delta)} G_k^{(k)}(t).
\]

By Claim 8.2, the \( k \)'th derivative \( G_k^{(k)}(x) \) is piece-wise constant, and specifically, satisfies \( G_k^{(k)}(x) = (-1)^{|x|} \binom{k-1}{|x|} \). Hence, we can partition the \( y \)'s into subsets, such that inside each subset, \( G_k^{(k)}(x) \) is ‘almost’ constant in the range \( ((a \cdot y - t - A)/\delta, (a \cdot y - t + A)/\delta) \).

The ‘main’ subset we consider is \( \{ y : a \cdot y - t \in [A, \delta - A] \} \), for which we have \( ((a \cdot y - t - A)/\delta, (a \cdot y - t + A)/\delta) \subset (0, 1) \), and thus, \( G_k^{(k)}(r) = \binom{k-1}{0} = 1 \) for all \( r \in ((a \cdot y - t - A)/\delta, (a \cdot y - t + A)/\delta) \). Therefore, for all \( y \)'s of this subset we have \( \mathbb{E}_{x \in \{-1,1\}^S} \left[ x^S G_k \left( \frac{a \cdot x + a \cdot y - t}{\delta} \right) \right] \geq \prod_{i \in S} a_i \).

The other subsets correspond to \( y \)'s for which \( (a \cdot y - t + A)/\delta \in [j, j + 1] \); as we are interested only in a lower bound, we may take the contributions of all these subsets with a ‘‘-t’’ sign, and enlarge each such set of \( y \)'s for sake of simplicity. Doing so and substituting into (50), we get

\[
e^\delta \geq \prod_{i \in S} a_i \cdot \left( \Pr_{y \in \{-1,1\}^{|n| \setminus S}} [a \cdot y - t \in [A, \delta - A]] - \sum_{l=1}^{k-1} \binom{k-1}{l} \Pr_{y \in \{-1,1\}^{|n| \setminus S}} [a \cdot y - t > l\delta - A] \right). \tag{51}
\]

One can easily obtain the following two (crude) inequalities:

\[
\Pr_{y \in \{-1,1\}^{|n| \setminus S}} [a \cdot y \in [t + A, t + \delta - A]] \geq \Pr_{z \in \{-1,1\}^{|n|}} [a \cdot z \in [t + 2A, t + \delta - 2A]];
\]

\[
\Pr_{y \in \{-1,1\}^{|n| \setminus S}} [a \cdot y > t + l\delta - A] \leq \Pr_{z \in \{-1,1\}^{|n|}} [a \cdot z > t + l\delta - 2A].
\]

Substituting into (51), we obtain:

\[
e^\delta \geq \prod_{i \in S} a_i \cdot \left( \Pr_{z \in \{-1,1\}^{|n|}} [a \cdot z \in [t + 2A, t + \delta - 2A]] - \sum_{l=1}^{k-1} \binom{k-1}{l} \Pr_{z \in \{-1,1\}^{|n|}} [a \cdot z > t + l\delta - 2A] \right),
\]

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which implies the assertion of the proposition (notice that for \( l = 1 \), we have an extra additive term of 1; this is handled by replacing \( \binom{k-1}{l-1} \) we obtained here with \( \binom{k}{l} \) in the assertion of the proposition, as \( \binom{k-1}{l-1} + 1 = \binom{k}{l} \).

Now we are ready to present the proof of Theorem 8.1.

**Proof of Theorem 8.1.** Let \( f_t(x) \) be a halfspace that satisfies the assumptions of the theorem and denote \( \epsilon = \mu(f) \). Let \( \beta \) be minimal such that \( F(t + \beta) \leq \epsilon / 3 \) and let \( \gamma \) be minimal such that \( \forall l \in \mathbb{N} : F(t + l\gamma) \leq \epsilon / (6k)^l \). Denote \( \delta = \beta + \gamma \). Note that by Theorem 1.10 we have

\[
\delta \leq O \left( \frac{\log(2k)}{\sqrt{\log(1/\epsilon)}} \right).
\]  
(52)

Let \( A \) be the sum of the \( k \) largest weights \( a_i \), and let \( M = \sum_{|S| = k} a^S e^S_S \), where \( a^S = \prod_{i \in S} a_i \) and \( e^S_S \) is as in Proposition 8.4. Similarly to the proof of Theorem 1.1 we are going to prove \( 45 \) by combining upper and lower bounds on \( M \). We shall need two technical claims whose proof will be presented later on.

**Claim 8.5.** For any \( \eta > 0 \), there exists a constant \( c = c(\eta) \) such that for any \( k \) and for any halfspace \( f_t \) with \( \mu(f_t) < 2^{-999k} \) and \( I_1(f_t) \leq c \mu(f_t) / k \), we have:

- \( a_1 \leq \eta / \sqrt{k} \), and
- \( 2ka_1 < \beta \),

where \( a_1 = \max_i a_i \) and \( \beta \) is as defined above.

**Claim 8.6.** Let \( f_t \) be a halfspace that satisfies the assumptions of Theorem 8.1. For any \( x \in \{-1, 1\}^n \) with \( f_t(x) = 1 \), we have \( \sum_{|S| = k} a^S x^S \geq 0 \).

As for all \( s \geq 0 \) and for any \( x \in \{-1, 1\}^n \) we have \( f_t(x) \geq f_{t+s}(x) \geq 0 \), Claim 8.6 implies that

\[
\forall s \geq 0, x \in \{-1, 1\}^n : \sum_{|S| = k} a^S x^S f_t(x) \geq \sum_{|S| = k} a^S x^S f_{t+s}(x).
\]

Taking expectation over \( x \) and using Fubini’s theorem, we get that for any \( s \geq 0 \),

\[
\sum_{|S| = k} a^S \hat{f}_t(S) = \sum_{|S| = k} a^S \mathbb{E}_x [x^S f_t(x)] \geq \sum_{|S| = k} a^S \mathbb{E}_x [x^S f_{t+s}(x)] = \sum_{|S| = k} a^S \hat{f}_{t+s}(S).
\]

In particular, as each \( e^S_S \) is a convex combination of expressions of the form \( \hat{f}_{t+s}(S) \), it follows that

\[
M = \sum_{|S| = k} a^S e^S_S \leq \sum_{|S| = k} a^S \hat{f}_t(S).
\]

By the Cauchy-Schwarz inequality, this implies

\[
M \leq \sqrt{W^k(f_t)} \sqrt{\sum_{|S| = k} (a^S)^2}.
\]  
(53)

On the other hand, by Proposition 8.4 we have

\[
M \geq \sum_{|S| = k} \frac{(a^S)^2}{\delta^k} \left( \Pr_z [a \cdot z > t + 2A] - \sum_{l=1}^{\infty} k^l \Pr_z [a \cdot z > t + l\delta - 2A] \right). \tag{54}
\]
As $2A < 2ka_1 < \beta$ by Claim [8.5], it follows from the definition of $\beta$ that $\Pr_z [a \cdot z > t + 2A] \geq \epsilon/3$. Using again the inequality $2A < \beta$, we have $t + l\delta - 2A > t + l\delta - \beta \geq t + l\gamma$. Thus, by the definition of $\gamma$ we have $\Pr_z [a \cdot z > t + l\delta - 2A] \leq \epsilon/(6k)^l$. Substituting into (54), this yields

$$M \geq \frac{\sum_{|S|=k} (a^S)^2}{\delta^k} \left( \frac{\epsilon}{3} - \sum_{l=1}^\infty k^l \frac{\epsilon}{(6k)^l} \right) \geq \frac{\epsilon}{9\delta^k} \sum_{|S|=k} (a^S)^2. \quad (55)$$

Combining Equations (52), (53) and (55), we obtain

$$W^k(f_t) \geq \frac{\epsilon^2}{81\delta^2k} \sum_{|S|=k} (a^S)^2 \geq \epsilon^2 \frac{\log(1/\epsilon)^k}{O(\log(2k))^k} \sum_{|S|=k} (a^S)^2. \quad (56)$$

Therefore, the assertion of the theorem will follow once we prove the following bound.

**Claim 8.7.** There exists $c > 0$ such that for any $1 \leq m \leq k \in \mathbb{N}$ and for any halfspace $f_t(x) = \mathbb{1}\{a \cdot x > t\}$ with $\mu(f_t) \leq 2^{-999k}$ and $I_{\max}(f_t) \leq c\mu(f_t)$, we have $\sum_{|S|=m} (a^S)^2 \geq 2^{1-m}/m!$.

**Proof of Claim 8.7.** Let $f_t$ be a function that satisfies the assumptions of the claim. Denote $b_i = a_i^2$, and notice $\sum b_i = 1$. For each $m$, let $e_m = \sum_{|S|=m} b^S$, and $s_m = \sum_i b_i^m$. By the classical Newton-Girard formulas which relate elementary symmetric functions to sums-of-powers, we have

$$me_m = \sum_{i=1}^m (-1)^{i-1} s_i e_{m-i}. \quad (57)$$

Notice $e_1 = s_1 = 1$. Furthermore, $s_i$ is a decreasing sequence satisfying

$$s_{i+1} \leq s_i \max_j b_j \leq \frac{s_i}{4k}. \quad \text{Claim 8.5}$$

Now, we prove that $e_{m-1} \leq 2me_m$ for any $m \leq k$. Indeed,

$$me_m \geq \sum_{i=1}^{\lfloor m/2 \rfloor} (s_{2i-1}e_{m+1-2i} - s_{2i}e_{m-2i}) \underbrace{\geq}_{\text{Induction}} \sum_{i=1}^{\lfloor m/2 \rfloor} \left( s_{2i-1}e_{m+1-2i} - s_{2i-1}e_{m-2i} \frac{1}{4k} \right) \geq \sum_{i=1}^{\lfloor m/2 \rfloor} \left( s_{2i-1}e_{m+1-2i} - \frac{2(m+1-2i)}{4k} \right) \geq \frac{s_1}{2} = e_{m-1}/2.$$ 

As $e_1 = 1$, the assertion follows by induction.

**Equation 56** together with Claim 8.7 implies the desired inequality,

$$W^k(f_t) \geq \frac{O(\log(2k))^{-k}}{k!} \epsilon^2 \log(1/\epsilon)^k.$$

It thus remains to prove Claims [8.5] and [8.6].
Proof of Claim 8.5. Let \( f_t \) be a halfspace that satisfies \( \mu(f_t) < 2^{-999k} \) and \( I_1(f_t) \leq c\mu(f_t) \), with a sufficiently small \( c \) to be determined below. (Note that the assumption on the influences of \( f_t \) is weaker than the assumption of Claim 8.5.) Denote \( \epsilon = \mu(f_t) \). By Theorem 1.4, we have \( c\epsilon \geq I_1(f_t) \geq c/d_1\epsilon\sqrt{\log(1/\epsilon)} \), and thus, \( a_1 \leq c/d_1\epsilon\sqrt{\log(1/\epsilon)} \). As by assumption, \( \epsilon \leq 2^{-999k} \), it follows that for a sufficiently small \( c = c(\eta) \), we have \( a_1 \leq \eta/\sqrt{k} \), as desired.

Now, we wish to show \( 2ka_1 \leq \beta \), and for this we use the assumption: \( I_1(f_t) \leq c_1\epsilon/k \), for a sufficiently small \( c_1 \). Consider \( \epsilon_1^\beta \) defined in Lemma 5.2, explicitly, \( \epsilon_1^\beta = \mathbb{E}_{s\sim U(0,\beta)} [I_1(f_{t+s})] \). Equation (23) of that lemma states that \( \epsilon_1^\beta \geq \frac{a_1}{\beta} \Pr[a\cdot x - a_1x_1 \in [t,t+\beta]] \). From Corollary 3.5 we deduce \( \epsilon_1^\beta \leq 5I_1(f_t) \). Hence, either

1. \( \Pr[a\cdot x - a_1x_1 \in [t,t+\beta]] \leq \epsilon/3 \), or
2. \( \frac{a_1}{\beta} \leq \epsilon_1^\beta \leq 5I_1(f_t) \).

If \( \frac{a_1}{\beta} \leq \epsilon_1^\beta \leq 5I_1(f_t) \), then the assumption \( I_1(f_t) \leq c_1\epsilon/k \) implies \( \frac{a_1}{\beta} \leq \frac{\epsilon_1^\beta}{5I_1(f_t)} \leq \frac{\epsilon}{3} \), and thus, \( 2ka_1 \leq \beta \), provided \( c_1 \) is sufficiently small. Thus, we may assume

\[
\Pr[a\cdot x - a_1x_1 \in [t,t+\beta]] \leq \epsilon/3.
\] (58)

Since for any \( r,s \) we have

\[
\Pr[a\cdot x \in (r,s)] \leq \Pr[a\cdot x - a_1x_1 \in (r-a_1,s+a_1)],
\]

and as \( I_1(f_t) = \Pr[a\cdot x - a_1x_1 \in (t-a_1,t+a_1)] \) and similarly for \( f_{t+\beta} \), it follows that

\[
\Pr[a\cdot x - a_1x_1 \in [t,t+\beta]] + I_1(f_t) + I_1(f_{t+\beta}) \geq \Pr[a\cdot x \in (t,t+\beta)] \geq \frac{2\epsilon}{3}.
\]

By (58), this implies

\[
I_1(f_t) + I_1(f_{t+\beta}) \geq \frac{2\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3}.
\] (59)

However, as by Corollary 3.5 \( I_1(f_{t+\beta}) \leq 5I_1(f_t) \), (59) implies \( 6I_1(f_t) \geq I_1(f_t) + I_1(f_{t+\beta}) \geq \epsilon/3 \), which contradicts the assumption \( I_1(f_t) \leq c_1\epsilon/k \) for a sufficiently small \( c_1 \). This completes the proof. \( \square \)

In order to prove Claim 8.6, we need another auxiliary claim.

Claim 8.8. For any halfspace \( f_t \) that satisfies the assumptions of Claim 8.5 with \( \eta = 1/16 \), we have \( t \geq 4\sqrt{k} \).

The following proof method appears in [34]. For completeness we repeat it here.

Proof of Claim 8.8. We will show \( \Pr[\sum a_ix_i > 4\sqrt{k}] > 2^{-999k} \). Since by assumption, \( 2^{-999k} \geq \mu(f_t) = \Pr[\sum a_ix_i > t] \), this will imply \( t > 4\sqrt{k} \).

Partition the \( a_i \)'s into sets \( \{G_s\}_s \), each having sum-of-squares in \([1/256k; 1/128k]\); this is possible since \( a_i \leq 1/(16\sqrt{k}) \). (To be precise, at most one of the sets \( G_s \) may have sum-of-squares less than \( 1/256k \). As will be apparent below, this does not affect the proof, so we neglect that
set.) For each $s$, consider the random variable $X_s = (\sum_{i \in G_s} a_i x_i)^2$. It is easy to see that
\[
\mathbb{E}[X_s] = \sum_{i \in G_s} a_i^2,
\]
and that
\[
\mathbb{E}[X_s^2] = \mathbb{E} \left[ \left( \sum_{i \in G_s} a_i x_i \right)^4 \right] = \sum_{i \in G_s} a_i^4 + 3 \sum_{i \neq j} a_i^2 a_j^2 \leq 3 \left( \sum_{i \in G_s} a_i^2 \right)^2.
\]

Recall that the classical Paley-Zygmund inequality asserts that for any nonnegative random variable $Z$ with a finite second moment and for any $\alpha \in [0,1]$, we have
\[
\Pr [Z > \alpha \mathbb{E}[Z]] \geq \frac{(1 - \alpha)^2 \mathbb{E}[Z]^2}{\text{Var}[Z] + (1 - \alpha)^2 \mathbb{E}[Z]^2}.
\]

Applying this inequality to the random variable $X_s = (\sum_{i \in G_s} a_i x_i)^2$, we get
\[
\Pr \left[ \sum_{i \in G_s} a_i x_i > \lambda \sqrt{\sum_{i \in G_s} a_i^2} \right] = \frac{1}{2} \Pr \left[ X_s > \lambda^2 \mathbb{E}[X_s] \right] \geq \frac{(1 - \lambda^2)^2}{4 + 2(1 - \lambda^2)^2}.
\]

As $1/(16\sqrt{k}) \leq \sqrt{\sum_{i \in G_s} a_i^2}$ by the construction of $G_s$, we infer $\Pr \left[ \sum_{i \in G_s} a_i x_i > 1/(32\sqrt{k}) \right] > 1/10$. Since the number of sets $\{G_s\}_s$ is between $128k$ and $256k$, we obtain
\[
\Pr \left[ \sum_{i \in G_s} a_i x_i > 4\sqrt{k} \right] \geq \Pr \left[ \forall s : \sum_{i \in G_s} a_i x_i > 1/(32\sqrt{k}) \right] > 10^{-256k} > 2^{-999k},
\]
as required. \(
\square
\)

Now we are ready to prove Claim 8.6.

**Proof of Claim 8.6.** Let $f_t$ satisfy the assumptions of the claim, and let $x \in \{-1,1\}^n$ be such that $f_t(x) = 1$. Denote $b_i = b_i(x) = a_i x_i / \sum_{j \in |n_x|} a_j x_j$, so that $|b_i| \leq a_i / t$ for each $i$ and $\sum_i b_i = 1$. Define as before $c_m = \sum_{\|S\|=m} b_S$ and $s_m = \sum_i b_i^m$. It is clear that for proving the claim, it is sufficient to prove $e_k \geq 0$.

It is clear that $\forall r \in \mathbb{N} : |s_{2+r}| \leq s_2 (\max b_i)^r$. As $\forall i : |b_i| \leq a_i / t$, this implies
\[
\forall r \in \mathbb{N} : |s_{2+r}| \leq s_2 \max \{(a_i / t)^r\}. \tag{60}
\]

By Claims 8.5 and 8.8, we have $\forall i : a_i \leq 1/16\sqrt{k}$ and $t \geq 4\sqrt{k}$ provided $c_1$ is sufficiently small, and hence, (60) implies
\[
\forall r \in \mathbb{N} : |s_{2+r}| \leq s_2 \max \{(a_i / t)^r\} \leq s_2/(64k)^r.
\]

Similarly to Claim 8.7, we shall prove by induction that $e_{m-1} \leq 2me_m$ for each $m \leq k$, and so in particular, $e_k \geq 2^k / k! > 0$, as required.
From the Newton-Girard formulas, we have

\[
\forall m \leq k : \quad m e_m = s_1 e_{m-1} + \sum_{i=2}^{m} (-1)^{i-1} s_i e_{m-i} \\
\geq s_1 e_{m-1} - \sum_{i=2}^{m} \frac{s_2}{(64k)^{i-2}} e_{m-i} \\
\geq s_1 e_{m-1} - \sum_{i=2}^{m} \frac{2^{i-1}(m-1)^{i-1}}{(64k)^{i-2}} s_2 e_{m-1} \\
\geq (s_1 - 3m s_2) e_{m-1} \\
\geq (1 - 3k/t^2) e_{m-1} \\
\geq e_{m-1}/2.
\]

This completes the proof. \(\square\)

9 Noise Resistance and Correlation with a Halfspace

Recall that a Boolean function \(f\) is called Fourier noise resistant if its first degree Fourier weight is within a constant factor of the maximal possible value, i.e., if \(W^1(f) \geq c_0 \mu(f)^2 \log(1/\mu(f))\) for a fixed constant \(c_0\). In this section we prove Theorem 1.8 which asserts that for any Boolean function \(f\), there exists a halfspace \(g\) such that \(\text{Cov}(f, g) \geq \Omega(\sqrt{W^1(f) \log(e/W^1(f))})\). This implies that if \(f\) is Fourier noise resistant then it is strongly correlated with some halfspace \(g\).

In addition, we show that in the special case where \(f\) is Fourier noise resistant, one can take the correlating halfspace \(g\) to be unbiased, and also there exists a strongly biased halfspace \(g''\) whose correlation with \(f\) is ‘surprisingly large’. Finally, we prove Proposition 1.9 which provides a ‘probabilistic’ notion of noise sensitivity for biased functions that implies strong correlation with a halfspace.

**Proof of Theorem 1.8** Let \(f\) be a Boolean function, let \(l(x) = \sum f(i)x_i\) be the first Fourier degree of \(f\), and denote \(a = \|l\|_2 = \sqrt{W^1(f)}\). Consider the family of biased halfspaces \(\{g_t(x) = 1\{l(x) > t\}\}_{t \in \mathbb{R}}\). The proof goes as follows: First, we show that the average correlation of \(f\) with \(g_t\) is ‘not very small’. Then we use the Hoeffding inequality to assert that \(\text{Cov}(f, g_t)\) is very small for a large \(|t|\), and deduce that there exists \(t\) such that \(\text{Cov}(f, g_t)\) is ‘large’, as asserted.
Define \( h(t) = \text{Cov}(f, g_t) \). We have

\[
a^2 = \mathbb{E}_x [f(x)l(x)]
\]

\[
= \int_0^\infty \mathbb{E}_x [f(x) \cdot 1_{l(x)>t}] \, dt - \int_{-\infty}^0 \mathbb{E}_x [f(x) \cdot 1_{l(x)<t}] \, dt
\]

\[
= \int_0^\infty \left( \mathbb{E}_x [f(x) \cdot 1_{l(x)>t}] + \mathbb{E}_x [f(x) \cdot 1_{l(x)\geq-t}] - \mathbb{E}_x [f] \right) \, dt
\]

\[
+ \left( \mathbb{E}_x [f(x) \cdot 1_{l(x)\geq-t}] - \mathbb{E}_x [f(x)] \mathbb{E}_x [1_{l(x)\geq-t}] \right) \, dt
\]

\[
= \int_0^\infty h(t) \, dt + \int_{-\infty}^0 h(t) \, dt = \int_{-\infty}^\infty h(t) \, dt.
\]

By Hoeffding’s inequality, for any \( t > 0 \) we have \( \mu(g_t) \leq \Pr[l(x) > |t|] \leq \exp(-t^2/2a^2) \), and therefore, \( |h(t)| \leq \exp(-t^2/2a^2) \) as well. Notice this also justifies the convergence of the above integrals. Let \( r = \sqrt{6\log(2/a)} \). Then

\[
r \int_{-ra}^{ra} h(t) \, dt \leq \int_{-ra}^{ra} t \, h(t) \, dt \leq \int_{-ra}^{ra} \exp \left( -\frac{t^2}{2a^2} \right) \, dt = a \cdot \exp \left( -\frac{r^2}{2} \right) \leq \frac{a^4}{8}.
\]

A symmetric argument implies \( r \int_{-ra}^{ra} h(t) \, dt \leq \frac{a^4}{8} \), and hence, \( a^2 - 2a^2 \leq \int_{-ra}^{ra} h(t) \, dt \). Thus, there exists \( t \in (-ra, ra) \) with \( h(t) \geq \frac{a^2-a^4/4r}{2ra} = \Omega \left( \frac{a}{\sqrt{\log(2/a)}} \right) \), as desired. \( \square \)

Theorem 1.8 is clearly tight (up to a constant factor) for any Fourier noise resistant function, as \( \text{Cov}(f, g) \) cannot exceed \( \mu(f) \). The following tightness example is of a different nature, being unbiased, monotone, and noise sensitive.

**Example 9.1.** Let \( t(x) \) be the classical tribes function defined by Ben-Or and Linial [3]. That is, we divide \( [n] \) into tribes \( T_1, T_2, \ldots, T_n/f \), each of size \( r \), and let \( t(x) = 1 \iff \exists j : (x_i = 1, \forall i \in T_j) \). The tribe size \( r \) is chosen such that \( \mathbb{E}[t(x)] \approx 1/2 \). (The size is \( r \approx \lg n - \lg \log n \).) One can easily show that for any halfspace \( g(x) \), we have \( \text{Cov}(t, g) = o_n(1) \).

Denote by \( \eta \) the maximal correlation of \( t(x) \) with a halfspace, so that \( \eta = o(1) \). Let \( h_r(x) = 1 \{a \cdot x > r\} \) be a halfspace of measure \( \eta \), and let \( f(x) = t(x) \lor h_r(x) \). The function \( f \) is monotone, we clearly have \( \mu(f) \approx \frac{1}{2} \), and by Theorem 1.1 we have

\[
W^1(f) \geq \Omega(W^1(h_r)) = \Omega(\eta^2 \log(1/\eta)).
\]

On the other hand, as \( f = t + h_r - t \cdot h_r \), for any halfspace \( g \) we have

\[
\text{Cov}(f, g) = \text{Cov}(t, g) + \text{Cov}(h, g) - \text{Cov}(t \cdot h, g) \leq \text{Cov}(t, g) + \mu(h) + \mu(t \cdot h) \leq 3\eta,
\]

where the last inequality holds since \( \mu(t \cdot h_r) \leq \mu(h_r) = \eta \), and \( \text{Cov}(t, g) \leq \eta \) by the definition of \( \eta \). Therefore, the correlation of \( f \) with any halfspace is at most \( 3\eta = O(\sqrt{W^1(f)} / \log(e/W^1(f))) \), which means that Theorem 1.8 is sharp for \( f \).

**Remark.** A central feature of Theorem 1.8 is that it holds also for non-monotone functions. In the monotone case, Theorem 1.8 (together with the classical KKL theorem [22]) implies
that for any unbiased monotone function \( f \), there exists a halfspace \( g \) such that \( \text{Cov}(f, g) = \Omega(\sqrt{\log n/n}) \). A stronger (and optimal) result of \( \Omega(\log n/\sqrt{n}) \) was obtained by O’Donnell and Wimmer [40] who used their result to obtain a provably optimal weak learning algorithm for the class of monotone functions.

The result of O’Donnell and Wimmer also shows that Theorem 1.8 is not tight for monotone unbiased functions with a ‘very small’ \( W^1 \). Indeed, while the minimal possible value of \( W^1 \) is \( \nu \sim (\log n)^2/n \) (attained by the tribes function), the result of [40] shows that maximal correlation with a halfspace for a monotone biased function is always at least \( \sqrt{\nu} = \log(n)/\sqrt{n} \) (and not \( \sqrt{\nu}/\sqrt{\log(e/\nu)} \), as we would have obtained if Theorem 1.8 was tight in that range).

Unlike the classical result of Benjamini et al. [1] which states that any noise resistant function has a strong correlation with an unbiased halfspace, Theorem 1.8 does not assure that the correlating halfspace is unbiased. In the following proposition we show that in the special case where \( f \) is noise resistant, one may require the correlating halfspace to be unbiased, like in [1].

**Proposition 9.2.** For any Fourier noise resistant Boolean function \( f : \{−1,1\}^n \to \{0,1\} \), there exists an unbiased halfspace \( g_0 \) with \( \text{Cov}(f, g_0) \geq \Omega(\mu(f)) \).

In the proof of the proposition we use the classical noise operator \( T_\rho \), which lies behind the notion of noise sensitivity. The noise operator is defined as \( T_\rho f(x) = \mathbb{E}[f(y)] \), where \( y \) is obtained from \( x \) by independently keeping each coordinate of \( x \) unchanged with probability \( \rho \), and replacing it by a random value with probability \( 1 - \rho \). It has a convenient representation in terms of the Fourier expansion of \( f \): we have \( T_\rho(f) = \sum S \rho^{|S|} \hat{f}(S) \), and thus, by the Parseval identity, \( \mathbb{E}[f \cdot T_\rho g] = \sum S \rho^{|S|} \hat{f}(S) \hat{g}(S) \), for any \( f, g \). (Note that the noise stability \( S_\rho(f) \) is simply \( \mathbb{E}[f \cdot T_\rho f] \).) The method we use in the proof was introduced in [26].

**Proof.** Let \( f : \{−1,1\}^n \to \{0,1\} \) be a Fourier noise resistant function. Denote \( \mu(f) = \epsilon \), so that \( W^1(f) \geq c_0 \epsilon^2 \log(1/\epsilon) \). Let \( l(x) = f^{-1}(x) \), and denote \( g_0(x) = \text{sgn}(l(x)) \). We show that for an appropriate choice of \( \rho \), the function \( f \) has a strong correlation with the ‘noisy version’ \( T_\rho g_0 \). As \( T_\rho g_0 \) is a convex combination of unbiased halfspaces, this will imply that there exists an unbiased halfspace \( g_0^{'} \) that strongly correlates with \( f \).

Let \( \rho \) be a parameter to be chosen below. Since \( \mathbb{E}[T_\rho g_0] = \mathbb{E}[g_0] = 0 \), we have

\[
\text{Cov}(f, T_\rho g_0) = \mathbb{E}[f \cdot T_\rho g_0] = \sum S \rho^{|S|} \hat{f}(S) \hat{g}_0(S) \geq \rho \sum_{i=1}^n \hat{f}({\{i\}}) \hat{g}_0({\{i\}}) - \rho \sum_{k \geq 2} \rho^k \sqrt{W^k(f)},
\]

(61) where the last inequality uses Cauchy-Schwarz. As \( l(x) = f^{-1}(x) \), we have

\[
\sum_{i=1}^n \hat{f}({\{i\}}) \hat{g}_0({\{i\}}) = \mathbb{E}[l(x)g_0(x)] = \mathbb{E}[l(x)\text{sgn}(l(x))] = \mathbb{E}[|l(x)|] = ||l||_1 \geq ||l||_2 / \sqrt{2},
\]

(62) where the first equality uses Parseval’s identity and the last inequality employs the Khintchine-Kahane inequality. As \( f \) is Fourier noise resistant, we have \( ||l||_2 = \sqrt{W^1(f)} \geq \sqrt{c_0 \epsilon^2 \log(1/\epsilon)} \), and so combining (61) with (62) we get

\[
\text{Cov}(f, T_\rho g_0) \geq \rho \cdot \frac{\sqrt{c_0 \epsilon^2 \log(1/\epsilon)}}{\sqrt{2}} - \sum_{k \geq 2} \rho^k \sqrt{W^k(f)}.
\]

(63)

Using the Level-k Inequalities (see, e.g., [38, Section 9.5]) which assert that

\[
\forall k \leq 2 \log(1/\mu(f)) : \quad W^k(f) \leq \left( \frac{2e}{k} \log(1/\epsilon) \right)^k \epsilon^2
\]

(64)
we obtain
\[ \mathbb{E}[f \cdot T_{\rho}g] \geq \rho \epsilon \cdot \frac{\sqrt{c_0 \log(1/\epsilon)}}{\sqrt{2}} - \sum_{k=2}^{2\log(1/\epsilon)} \rho^k \epsilon \cdot \left(\frac{2e^k}{k} \log(1/\epsilon)\right)^{k/2} - \sum_{k>2\log(1/\epsilon)} \rho^k. \]

Taking \( \rho = (1/2e)^2 \sqrt{c_0 / \log(1/\epsilon)} \), and noting that w.l.o.g. we may assume \( c_0 \leq 1 \), results in
\[ \mathbb{E}[f \cdot T_{\rho}g_0] \geq \frac{c_0 \epsilon}{\sqrt{32e^2}} - c_0 \epsilon \sum_{k>2} (2e)^{-3k/2} - c_0 \epsilon \sum_{k>2\log(1/\epsilon)} (2e)^{-3k/2} \geq \Omega(c_0 \epsilon). \]

Finally, note that by the definition of the noise operator, the function \( T_{\rho}g_0 \) is a convex combination of unbiased halfspaces of the form \( \frac{\sqrt{k}}{BD} \), where \( \forall i : \alpha_i \in \{0,1\} \). Hence, there exists an unbiased halfspace \( g'_0 \) such that \( \text{Cov}(f, g'_0) = \Omega(c_0 \epsilon) \), as asserted. \( \Box \)

Interestingly, one cannot assure that the linear form associated with the correlating halfspace is simply \( l = f^{-1} \) like in the unbiased case, as can be seen in the following example.

**Example 9.3.** Let \( f : \{-1,1\}^5 \to \{0,1\} \) be defined as \( f(x) = 1 \iff \sum_{i=1}^5 x_i \in \{-1,3,5\} \). We have \( \hat{f} = 1/16 \) for all \( 1 \leq i \leq 5 \), and so, the linear form \( f^{-1} \) defines the majority function \( g_0(x) = \mathbb{1}\{\sum_{i=1}^5 x_i > 0\} \). However, a direct computation shows that \( \text{Cov}(f, g_0) = -1/16 \) is not even positive! On the other hand, if we reverse the sign of one variable, i.e., consider \( g'_0 = \mathbb{1}\{x_1 + x_2 + x_3 + x_4 - x_5 > 0\} \), we obtain \( \text{Cov}(f, g'_0) = 1/8 \).

Similar examples can be constructed for large values of \( n \) as well.

We now present another proposition which shows that any Fourier noise resistant function correlates well with a strongly biased halfspace. This result is somewhat surprising, as biased functions correlate badly in general.

**Proposition 9.4.** For any function \( f : \{-1,1\}^n \to \{0,1\} \) that satisfies \( \epsilon = \mu(f) \) and \( W^1(f) \geq \alpha^2 \log(1/\epsilon) \), for some \( \alpha = \omega\left(\epsilon^2\right) \), there exists a halfspace \( g_s : \{-1,1\}^n \to \{0,1\} \) such that \( \mathbb{E}[fg] \geq \Omega(\sqrt{\alpha \epsilon}) \) and \( \mathbb{E}[g_s] \leq \epsilon^{\alpha/8} \). Consequently, if \( \alpha = \Omega(1) \), \( \text{Cov}(f, g_s) \geq \sqrt{\alpha \epsilon} - \epsilon^{1+\alpha/8} = \Omega(\epsilon) \).

**Proof.** Let \( s = \frac{1}{2} \sqrt{\alpha \log(1/\epsilon)} \), and consider the halfspace \( g_s(x) = \mathbb{1}\{l(x) > s\} \) (with \( \|l\|_2 = 1 \)). By Hoeffding’s inequality, we have \( \mathbb{E}[g_s] \leq \epsilon^{\alpha/8} \). Hence, in order to prove the proposition we have to show that
\[ \mathbb{E}[fg_s] \geq \Omega(\sqrt{\alpha \epsilon}). \] (65)
Denote \( l = \frac{1}{\|f\|_2} f^{-1} \). We clearly have
\[ \sqrt{W^1(f)} = \langle f, l \rangle = \mathbb{E}[\mathbb{1}\{l(x) \leq s\} f(x) l(x)] + \mathbb{E}[\mathbb{1}\{l(x) > s\} f(x) l(x)]. \] (66)
Since
\[ \mathbb{E}[\mathbb{1}\{l(x) \leq s\} f(x) l(x)] \leq s \epsilon \leq \frac{1}{2} \sqrt{W^1(f)}, \]
it follows from (66) that
\[ \mathbb{E}[\mathbb{1}\{l(x) > s\} f(x) l(x)] \geq \frac{1}{2} \sqrt{W^1(f)}. \]
Note that for any $u \geq s$,
\[
\mathbb{E}[g_s(x)l(x)f(x)] \leq u \mathbb{E}[g_s(x)f(x)] + \int_u^\infty \Pr[l(x) > t] \, dt = u \mathbb{E}[fg_s] + \int_u^\infty \Pr[l(x) > t] \, dt.
\]
Since $\|l\|_2 = 1$, Hoeffding’s inequality yields $\Pr[l(x) > t] \leq \exp(-t^2/2)$. Thus,
\[
\int_u^\infty \Pr[l(x) > t] \, dt \leq \int_u^\infty \frac{t}{u} \Pr[l(x) > t] \, dt \leq \int_u^\infty \frac{t}{u} \exp(-t^2/2) \, dt = \frac{1}{u} \exp(-u^2/2).
\]
Taking $u = 2\sqrt{\log(1/\epsilon)}$ and combining the four previous inequalities, we obtain
\[
s \epsilon \leq \frac{1}{2} \sqrt{W^1(f)} \leq u \mathbb{E}[fg_s] + \frac{1}{u} \epsilon^2.
\]  
Finally, note that we have $\frac{1}{u} \epsilon^2 \leq \frac{1}{2} s \epsilon$, as otherwise we have $\sqrt{\alpha} \log(1/\epsilon) = us \leq 2\epsilon$, and hence, $\alpha \leq 4\epsilon^2$, which contradicts the assumption. Therefore, (67) gives
\[
\frac{\sqrt{\alpha}}{8} \epsilon \leq \frac{1}{2} s \epsilon \leq \mathbb{E}[fg_s].
\]
This proves (65), thus completing the proof of the proposition.

We conclude this section with a ‘probabilistic’ notion of noise resistance that implies strong correlation with a halfspace. Recall that by the definition of [1], a function $f$ is called noise resistant if $S_\rho(f) = \Omega(1)$ for any constant $\rho$. If we want to generalize this definition to biased functions, the rate of the noise we consider must depend on the expectation of the function (e.g., it is clear that if the rate of noise is larger than $\min \{\mu(f), 1 - \mu(f)\}$, then the noise destroys the function almost completely). In order to find a natural rate of noise for biased functions, we use (once again) the relation of noise sensitivity to the Fourier expansion of the function.

Using the expansion $S_\rho(f) = \mathbb{E}[f \cdot T_\rho f] = \sum S^2 \rho^k |\hat{f}(S)|^2$ and the aforementioned Level-$k$ inequality (Equation (64) above), we get
\[
S_\rho(f) = \sum_k \rho^k W^k(f) \leq \sum_k \left( \frac{2\epsilon \rho}{k} \log(1/\epsilon) \right)^k \epsilon^2 + \sum_{k > 2 \log(1/\epsilon)} \rho^k,
\]
where $\epsilon = \mu(f)$. By Stirling’s approximation, this implies
\[
S_\rho(f) - \frac{1 - \rho^{2\log(1/\epsilon)}}{1 - \rho} \leq \epsilon^2 \sum_k \frac{c_1}{\sqrt{k}} \left( \frac{c_2 \rho \log(1/\epsilon)}{k!} \right)^k \leq \epsilon^2 \exp(c_3 \rho \log(1/\epsilon)),
\]
where $c_i$ are universal constants. It follows that if $\rho = o(1/\log(1/\epsilon))$ then $S_\rho(f)$ is very small for any function $f$. Hence, we consider noise rate of $\rho = \Theta(1/\log(1/\epsilon))$, for which $S_\rho(f)$ can be as large as $\mu(f)^2$, and say that $f$ is noise resistant if $S_\rho(f) = \Omega(\mu(f)^2)$.

We show that this notion of noise resistance implies strong correlation with a halfspace. Specifically, we prove Proposition [1.9] which asserts that if a monotone Boolean function $f$ satisfies $S_\rho(f) = \Omega(\mu(f)^2)$, where $\rho = c/\log(1/\mu(f))$ for a sufficiently small universal constant $c$, then $W^1(f) = \Omega(\mu(f)^2 \log(1/\mu(f)))$, and hence, there exists an unbiased halfspace $g_0$ such that $\text{Cov}(f, g_0) = \Omega(\mu(f))$. 

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Proof of Proposition 1.9. Recall that the quantitative version of the BKS noise sensitivity theorem [29] asserts that for any monotone $f$ and for any $k$ which satisfies $W^1(f) \leq \exp(-2(k-1))$, we have

$$W^k(f) = \sum_{|S|=k} \hat{f}(S)^2 \leq \frac{5e}{k} W^1(f) \left( \frac{2e \log(k/W^1(f))}{k-1} \right)^{k-1}. \quad (68)$$

Let $f$ be a function that satisfies the assumptions of the proposition. Denote $\epsilon = \mu(f)$ and let $\alpha$ satisfy $W^1(f) = \epsilon^\alpha$. We shall compute an upper bound for $\mathbb{S}_\rho(f)$ when $\rho = c/\log(1/\epsilon)$, for a constant $c$ to be specified below. Set $T = \log(1/\epsilon)/2$, and note that since $W^1(f) \leq \epsilon$ by the Poincaré inequality, we may apply (68) for all $k \leq T$. Hence, we have

$$\mathbb{S}_\rho(f) = \sum_{k=1}^n \rho^k W^k(f) \leq \sum_{k=1}^T W^1(f) \left( 6 \log(k/W^1(f)) \right)^{k-1} \rho^k + \sum_{k=T}^\infty \rho^k$$

$$\leq W^1(f) \rho \sum_{k=1}^T \left( \frac{6c(\alpha + 1) \log(1/\epsilon)}{\log(1/\epsilon)} \right)^{k-1} + \sum_{k=T}^\infty \rho^k$$

$$\leq O \left( \frac{W^1(f)}{\log(1/\epsilon)} \left( 1 + \left( \frac{\alpha + 1}{100} \right)^{\log(1/\epsilon)} \right) \right) + O(\epsilon^3), \quad (69)$$

where the last inequality follows by taking $c$ to be a sufficiently small constant and the ‘100’ in the denominator can be taken to be any constant (determined by $c$).

For $\alpha \leq 99$, (69) implies $\mathbb{S}_\rho(f) = O \left( \frac{W^1(f)}{\log(1/\epsilon)} + \epsilon^3 \right)$. As we have $\mathbb{S}_\rho(f) = \Omega(\epsilon^2)$ by assumption, this implies $W^1(f) \geq \Omega(\epsilon^2 \log(1/\epsilon))$, as asserted.

For $\alpha > 99$, (69) yields $\epsilon^{2-\alpha} \leq O \left( \left( \frac{\alpha + 1}{100} \right)^{\log(1/\epsilon)} \right) = O \left( \epsilon^{-\log((\alpha+1)/100)} \right)$, which can not happen for $\epsilon$ small enough (note that we may assume $\epsilon$ is small, as we control $c$). This completes the proof. $\Box$

Remark. In [37, Theorem 3.10.4], O’Donnell presented another notion of noise resistance that is satisfied by biased halfspaces. He showed that for any halfspace $g_\ell$ with $\epsilon = \mu(g_\ell)$,

$$\text{NS}_{\delta/\log(1/\epsilon)}(g_\ell) = O(\epsilon \sqrt{\delta}), \quad (70)$$

where $\text{NS}_\rho(g_\ell) = \frac{1}{2} - \frac{1}{2} S_{1-2\rho}(g_\ell)$. (This provides a biased version of Peres’s noise stability theorem for halfspaces [41].) In light of the previous results, one might wonder whether every monotone function $f$ that satisfies (70) is well-correlated with some halfspace. This indeed holds for unbiased functions, as in this case, (70) implies that $f$ is noise stable (according to the notation of [41]), and consequently, satisfies $W^1(f) = \Omega(1)$, which in turn implies that $f$ correlates well with a halfspace.

However, this does not generalize to the biased setting, as can be seen in the following example. Let

$$f(x) = \bigvee_{i \in [a]} \bigwedge_{j \in [b]} x_{i,j}$$

be a variant of the tribes function, with $a = 1/\epsilon$ and $b = 2 \log(1/\epsilon)$. Clearly, $\mu(f) \approx \epsilon$. It can be shown that on the one hand, $f$ satisfies (70), and on the other hand, $f$ does not correlate well with any halfspace (i.e., $\text{Cov}(f, g_\ell) = o(\epsilon)$ for any halfspace $g_\ell$).
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