A time-fractional Lotka-Volterra model with mutualistic predation

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Abstract

In this work, we present a predator-prey type model where the two predator species engage in mutualistic predation. Constant effort harvesting is also incorporated. Linear stability analysis of the resulting models is performed and it is shown that harvesting is sustainable. We next introduce a fractionalised time derivative to the model and investigate its effect. It is revealed that the so-called memory concept of the fractional derivative damps out oscillations in the population numbers so that the system as a whole settles on a equilibrium quicker than it would with integer time derivatives.

Keywords: Population modelling, Fractional calculus, Lotka-Volterra, Stability and harvesting, Mutualistic predation

1. Introduction

Ecological models are of great importance for environmental decision making because they provide stakeholders with a conceptual framework and a

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“laboratory” for studying the consequences of alternative policies and management scenarios [1]. The Lotka-Volterra (or predator-prey) model – an important and popular prototype model appearing in various fields of applied mathematics – has been studied extensively in different forms due to its descriptive power, tractability and diverse applications [2–5]. As examples of its diverse application it has been employed to probe the interactive behaviour amongst species and determine the factors playing a key role in curing some diseases such as HIV [6] and has also been utilised to forecast the number of the Internet users in China [7] with a good degree of accuracy.

Given the biological realities of how various species interact with each other, different models have been proposed to investigate such modes of interaction. Predation and competition for a resource like food, shelter, or nesting sites are common interactions to consider [8]. However, sometimes it is seen that different species interact in such a way that it brings benefits to all parties. A common form of mutualistic interaction has at least one partner species behaving as a forager — this is the role pollinating insects play in their partnership with associated plant life [9]. Other mutualisms have evolved from exploitative interactions such as plants responding to herbivores so that the animals’ consumption becomes an important means of seed dispersal. Paradoxically, sometimes even parasitism can result in a benefit to the host [10].

In this study, we consider a model of three species (two predators and one prey) with mutualistic predation. An example of the type of interaction the model idealises is the mutualistic predation of spotted dolphins and yellowfin tuna upon schools of fish [11]. The mutual benefit afforded to a predator by existence of different predatory groups can be due simply to the increased predator numbers and the conflicting prey responses which diminishes the effectiveness of the prey defences [12]. There need not be any form of coordination on the part of the predators.

Cooperative interspecific hunting, however, does occur — for well over a century there have been reports of individual badgers (taxidea taxus) and coyotes (canis latrans) working together to hunt [13]. More recently, Bshary et al. [14], reported on groupers (plectropomus pessuliferus) in the Red Sea region signalling a giant moray eel (gymnothorax javanicus) to assist them in hunting small reef fish, which had escaped into crevices too small for the groupers to pursue after them. Effectively, the groupers herd the prey into
a hunting ground where the eel’s form is an advantage and signal the eel to the prey’s presence thus saving the eel the effort of finding the prey. In turn, as the eel hunts it flushes out the reef fish back where the groupers have the size advantage over the prey.

The cooperation of predators can also be longer term, as observed in New Zealand by Zaeschmar et al [15], where mixed groups of false killer whales (*pseudorca crassidens*) and common bottlenose dolphins (*tursiops truncatus*) were seen travelling together and hunting schools of fish. However, in this case, and in the case of the yellowfin tuna and spotted dolphins, it is believed that the main reason for the two species travelling together is that by doing so, both species decrease the risk of predation by groups like sharks [11].

When human activity such as commercial harvesting is considered, however, the aforementioned behaviour of cooperation may result in the undesirable death of one species because they are essentially inseparable from the economically valuable species (i.e. a by-catch of dolphins when fishing for tuna). Therefore, understanding harvesting in this context is an important issue for the management of fisheries.

Much research has been conducted to improve the understanding of how harvesting effects population densities and leads commercial fisheries to be managed in such a way that the maximum sustainable yield (MSY) is extracted [16–21]. Following this example, we investigate the properties of harvested multi-species systems. We consider constant effort harvesting on the species, and after discussing the stability condition of the systems, the limitation of MSY measures are derived. In order to find a qualitative understanding of the properties of the harvesting strategy, the output results of the models are simulated as well, and interestingly it is demonstrated that under the obtained stability condition, the fish harvest would enhance the stability of the system.

Long term cooperation between species acting as predators in a particular location and ecosystem that they otherwise would not be engaged in implies a process of learning and memory. Time derivatives of integer order are determined by the properties of the differentiable function only in an infinitesimal neighbourhood of the considered moment. That is, they are local in time. As a result, ordinary differential equations with integer-order derivatives are describing instantaneous reactions and are not capable of incorporating, in and of themselves, processes with memory. To give our models some additional
realism, we investigate the role of memory with the help of fractional calculus (FC) and find a relation between the stability of the system and fractional order. FC, is a generalisation of the familiar calculus and a powerful tool for modelling complicated phenomena with long-term memory effect \[22,24\], causing this tool to be widely applied in different fields including viscoelastic materials, combustion science, biology, economics and engineering \[25\].

The next section is given over to a brief discussion of foraging and predation. Since good mathematical models allow ecologists and scholars to interpret the environment better the processes are expressed, firstly, as integer-derivative models in order to investigate the system in a familiar setting. In effect, we first propose a single predator model with harvesting and discuss its stability. As a next step, mutualistic predation is incorporated by extending the model with two predators and linear stability analysis of the 3 species model is derived.

In the following section, we present some basic definitions of fractional calculus. Then by “fractionalizing” the existing model, we introduce the memory/learning concept of the species. The relation between species memory and the stability of the system is also investigated.

2. Foraging and predation

Sometimes groups of two different species associate for mutual benefit. One such example is the habit of yellowfin tuna and spotted dolphins to swim together in the eastern tropical Pacific. This behaviour is believed to be based on the rationale of mutual hunting (the feeding hypothesis) and mutual protection (the predation hypothesis) \[11\]. Thus, the interaction between these two species could be categorised as mutualism as both groups benefit from the association.

Making up their lack of size by living together, lanternfish are small mesopelagic fish of the large Myctophidae family and found in oceans worldwide. This fish partakes in diel vertical migrations, which means that during daylight hours, it remains within the gloomy bathypelagic zone, but towards sundown, it begins to rise into the epipelagic zone to feed on plankton. But sometimes, lanternfish come to the surface during the day in order to spawn and then begin to descend back into the dark depths \[26\].
It has been observed that the dolphins, by swimming beneath the lanternfish, pen them within the surface zone of the ocean, forcing them to pack more closely together. Dolphins herd small fish into a so-called “bait-balls” and then attack the fish near the surface of the sea disrupting the school. Images of related hunting techniques can be seen in [15]. It is thought that, by attacking the prey together, yellowfin tuna and spotted dolphins catch more prey than they might have acting as single species groups.

3. Integer models

3.1. Single predator

The initial model of predator and prey is as follows:

\[
\begin{align*}
\frac{dX}{dT} &= rX \left(1 - \frac{X}{K}\right) - aXY - H_1(X), \\
\frac{dY}{dT} &= \frac{aXY}{1 + \sigma X} - kY - H_2(Y),
\end{align*}
\]

where \(X\) and \(Y\) are the population densities of prey and predator respectively, \(T\) is time, \(r\) is the prey growth rate, \(K\) is the environmental carrying capacity for the prey, \(a\) is the feeding rate of predators, \(\sigma\) is the predator growth saturation factor and \(k\) is the predator death rate. The functions \(H_1\) and \(H_2\) are terms indicating some form of harvesting. All parameters are positive reals.

A natural question which arises when a species is commercially important is how does the harvesting of the species effect the natural equilibrium of the ecology? Further, how heavily can a species be harvested and still be sustainable? In this work we will consider system harvest functions which are proportional to the density of species population (so-called constant harvest quota) for both prey and predator \(H(X) = h_1X\) and \(H(Y) = h_2Y\) respectively. We assume that either both species have market value or that one species is caught as by-catch.

Substituting the rescalings \(X = Kx, Y = ky/a\) and \(T = t/k\) in to (1) we
arrive at the system

\[
\frac{dx}{dt} = \rho x (1 - x) - xy - \varepsilon_1 x,
\]

\( (2) \)

\[
\frac{dy}{dt} = \frac{\psi xy}{1 + \phi x} - y - \varepsilon_2 y,
\]

where \( \rho = r/k, \psi = aK/k, \phi = K\sigma, \varepsilon_1 = h_1/k \) and \( \varepsilon_2 = h_2/k \).

The system (2) has three stationary points: \( E_1 = (0,0) \) extinction of both species; \( E_2 = (1 - \varepsilon_1/\rho, 0) \) predator only extinction; and \( E_3 = (\omega, \rho(1 - \omega) - \varepsilon_1) \) predator/prey co-existence, where

\[
\omega = \frac{1 + \varepsilon_2}{\psi - \phi(1 + \varepsilon_2)}.
\]

Non-zero harvesting clearly shifts the prey-only equilibrium, \( E_2 \), to lower densities of prey. For the co-existence point, \( E_3 \), the prey population density is unaffected by harvesting of the prey itself but is shifted to higher densities as rate of predator harvesting is increased. The harvesting of prey is effectively competition for the predators reducing their equilibrium population in concert with any direct harvesting.

Linearising the system (2) about the stationary points \((x^*, y^*)\) we can determine each point’s linear stability by considering the eigenvalues of the resulting Jacobian matrix

\[
J(x^*, y^*) = \begin{pmatrix}
\rho (1 - 2x^*) - y^* - \varepsilon_1 & -x^* \\
\frac{\psi y^*}{1 + \phi x^*} & (\psi - \phi)x^* - \frac{1}{1 + \phi x^*} - \varepsilon_2
\end{pmatrix}
\]

\( (3) \)

From (3), it can easily be deduced that \( E_1 \) is stable if and only if \( \rho < \varepsilon_1 \). That is, if the rate of harvesting of the prey outstrips its growth rate, the system is driven to extinction.

Also, it can be concluded that the linearised system has two eigenvalues of

\[
\lambda_1 = -\rho \left( 1 + \frac{\varepsilon_1}{\rho} \right) - \varepsilon_1 \quad \text{and} \quad \lambda_2 = \frac{\psi (\rho - \varepsilon_1)}{\rho + \phi (\rho - \varepsilon_1)} - (1 + \varepsilon_2)
\]
at $E_2$, and thus $E_2$ is a stable node only when

$$\psi < (1 + \varepsilon_2) \left[ \frac{\rho}{(\rho - \varepsilon_1)} + \phi \right] .$$

which corresponds to the coexistence point being biologically irrelevant. For fixed point $E_3$, we have

$$J(E_3) = \left( \begin{array}{cc} -\left((\rho \omega + \varepsilon_1) - \omega \right) \\ \psi(\rho(1-\omega) - \varepsilon_1) \\ (1 + \phi \omega)^2 \\ 0 \end{array} \right). \quad (4)$$

Thus, the trace and determinant of the Jacobian evaluated at Eq. (4) are as follows:

$$\text{Tr}(J(E_3)) = -(\rho \omega + \varepsilon_1), \quad \text{det}(J(E_3)) = \frac{\psi \omega (\rho(1-\omega) - \varepsilon_1)}{(1 + \phi \omega)^2}.$$

Based on the Routh-Hurwitz conditions, if det$(J) < 0$ then $E_3$ will be a saddle point (regardless of the sign of Tr$(J)$), and if det$(J) > 0$ then $E_3$ will be asymptotically stable when Tr$(J) < 0$, a centre when Tr$(J) = 0$, and unstable when Tr$(J) > 0$. Since Tr$(J)$ at $E_3$ is always negative, the only thing left is the sign of det$(J)$, so system (2) is asymptotically stable if and only if $\omega < 1 - \varepsilon_1/\rho$. It is also worth noting that $E_3$ cannot be a saddle point in our system because to this end, det$(J)$ must be negative which is biologically irrelevant.

Hence, the coexistence point is always linearly stable when $\omega < 1 - \varepsilon_1/\rho$, and the $x$ and $y$ trajectories will be under-damped when (Fig. 1)

$$((\rho + \varepsilon_1 \phi)(1 + \varepsilon_2) - \varepsilon_1 \psi)^2 < \frac{4}{\psi} \rho(1 + \varepsilon_2)(\psi - \phi(1 + \varepsilon_2))^2 \times $$

$$[ (1 - \varepsilon_1)(\psi - \phi(1 + \varepsilon_2)) - (1 + \varepsilon_2)] \quad (5)$$

3.2. Two predators model

In this section, system (1) is extended to model interactions amongst three species: one prey and two predators. The predators are not treated as isolated hunters which no direct interaction. Rather, we consider the predator
Figure 1: The evolution of prey (blue line) and predator (red line) densities in system (2) with $\rho = 1$, $\psi = 15$, $\phi = 2$, $\epsilon_1 = 0.4$ and $\epsilon_2 = 1$. Also $\omega = \frac{1 + \epsilon_2}{\psi - \phi(1 + \epsilon_2)} = 2/11 < 0.6$ with initial condition $(0.5, 0.1)$, and the phase plane of the model is plotted on the right side of the figure.

species to be cooperative. The model of the 3 species with type II mutualism [9, 27] functional response for the predators is

\[
\begin{align*}
\frac{dX}{dT} &= rX \left(1 - \frac{X}{K}\right) - X (aY + bZ + \xi YZ) - H_1(X), \\
\frac{dY}{dT} &= \frac{XY (a + \xi_1 Z)}{1 + \sigma_1 X} - k_1 Y - H_2(Y), \\
\frac{dZ}{dT} &= \frac{XZ (b + \xi_2 Y)}{1 + \sigma_2 X} - k_2 Z - H_3(Z).
\end{align*}
\]

(6)

where $X$ is the prey population density (lanternfish), $Y$ and $Z$ are population densities of distinct predators (tuna fish and dolphin respectively), $T$ is time, $r$ is the prey growth rate, $K$ is the environmental carrying capacity for the prey, $a$, $b$, $\xi$, $\xi_1$ and $\xi_2$ are the feeding rate of predators, $\sigma_1$ and $\sigma_2$ are the predator growth saturation factor and $k_1$ and $k_2$ are the predator death rate. As before, all parameters are positive reals and the terms $H_j$ for $j = 1, 2, 3$ are the harvesting functions.

The number of parameters in system (6) can be reduced by considering the following transformations

\[
X = Kx, \quad Z = \left(\frac{a}{\xi}\right) z, \quad Y = by, \quad T = \frac{t}{ab}
\]
thus we arrive the following dimensionless system:

\[
\begin{align*}
\frac{dx}{dt} &= \rho x (1 - x) - x (y + \eta z + y z) - \varepsilon_1 x, \\
\frac{dy}{dt} &= \psi xy (1 + \eta_1 z) - \gamma_1 y - \varepsilon_2 y, \quad (7) \tag{7} \\
\frac{dz}{dt} &= \beta x z (1 + \xi_2 y) - \gamma_2 z - \varepsilon_3 z.
\end{align*}
\]

where \( \rho = r/ab, \ \psi = K/b, \ \beta = K/a, \ \eta = 1/\xi, \ \eta_1 = \xi_1/\xi, \ \phi_1 = \sigma_1 K, \ \phi_2 = \sigma_2 K, \ \gamma_1 = k_1/ab, \ \gamma_2 = k_2/ab \) and \( \varepsilon_j = h_j/ab \) (for \( j = 1, 2, 3 \)).

3.2.1. Linear stability analysis

There are five stationary points for model (7):

- \( E_1 = (0, 0, 0) \), \( E_2 = (1, 0, 0) \);
- \( E_3 = (x^*, \rho [1 - x^*] - \varepsilon_1, 0) \) s.t \( x^* = \frac{\gamma_1 + \varepsilon_2}{\psi - \phi_1 (\gamma_1 + \varepsilon_2)} \);
- \( E_4 = (x^*, 0, \frac{1}{\eta_1} [\rho (1 - x^*) - \varepsilon_1]) \) s.t \( x^* = \frac{\gamma_2 + \varepsilon_3}{\beta - \phi_2 (\gamma_2 + \varepsilon_3)} \); and
- \( E_5 = (x^*, \frac{\rho (1 - x^*) - \eta z^* - \varepsilon_1}{1 + z^*}, \frac{(\gamma_1 - \varepsilon_2)(1 + \phi_1 x^*)}{\psi x^*} - 1) \) s.t \( x^* = \frac{\gamma_2 - \varepsilon_3}{\beta - \phi_2 (\gamma_2 - \varepsilon_3)} \).

where \( E_1 \) is total population extinction, \( E_2 \) is prey only, \( E_3 \) and \( E_4 \) are equilibria of partial co-existence (the prey with one of the predators), and \( E_5 \) is the co-existence of all three species.

Linearising the system (7) about the stationary points \( (x^*, y^*, z^*) \), we can determine each point’s linear stability by considering the eigenvalues of the resulting Jacobian matrix.

\[
J (x^*, y^*, z^*) = \\
\begin{pmatrix}
\rho (1 - 2x^*) - (y^* (1 + z^*) + \eta z^*) - \varepsilon_1 & -x^* (1 + z^*) & -x (\eta + y^*) \\
\psi y^* (1 + \eta_1 z^*) & \psi x^* (1 + \eta z^*) & \psi x^* y^* \\
\beta z^* (1 + \xi_2 y^*) & \beta x^* z^* & \beta x^* (1 + \xi_2 y^*) - \gamma_2 - \varepsilon_3 \\
\frac{1}{1 + \phi_1 x^*} & \frac{1}{1 + \phi x^*} & \frac{1}{1 + \phi_2 x^*}
\end{pmatrix}
\]
According to [8], it can quickly be determined that the origin in the linearised system has eigenvalues of \( \rho - \varepsilon_1, -\gamma_1 - \varepsilon_2 \) and \( -\gamma_2 - \varepsilon_2 \). Which means that \( E_1 \) is a stable point for the system only if \( \rho < \varepsilon_1 \).

Similar analysis as performed in earlier sections allows us to determine that \( E_2 \) is stable when \( \varepsilon_2 > \frac{\psi}{1 + \phi_1} - \gamma_1 \) and \( \varepsilon_3 > \frac{\beta}{1 + \phi_2} - \gamma_2 \). As for \( E_3 \), it can be deduced that under the following condition the system considered is stable:

\[
\varepsilon_1 > \rho(1 - x^*), \quad \varepsilon_2 < \frac{\psi}{\phi_1} - \gamma_1, \quad \varepsilon_3 > \frac{\beta x^*}{1 + \phi_2 x^*} - \gamma_2. \tag{9}
\]

Similarly, the stability of \( E_4 \) requires the following inequalities to hold,

\[
\varepsilon_1 < \rho(1 - x^*), \quad \varepsilon_2 > \frac{\psi x^*(1 + z^*)}{1 + \phi_1 x^*} - \gamma_1, \quad \varepsilon_3 < \frac{\beta}{\phi_2} - \gamma_2 \tag{10}
\]

It can easily be shown that \( E_5 \) is a stable point if and only if (Figs. 2 and 3)

\[
\varepsilon_1 < \rho(1 - x^*) - \eta z^*, \quad \varepsilon_2 < \gamma_1 - \frac{\psi}{\beta}(\gamma_2 - \varepsilon_3), \quad \gamma_2 < \frac{\beta}{\phi_2} < \varepsilon_3 < \gamma_2 \tag{11}
\]

**Figure 2**: The evolution of prey and predators densities in system \( \hat{\mathbf{E}} \) with parameter values \( (\rho, \psi, \gamma_1, \gamma_2, \phi_1, \phi_2, \eta, \beta) = (2.5, 5.5, 1.5, 3.4, 1.4, 0.8, 0.1, 3.5) \), initial densities \( (0.75, 0.5, 0.25) \) and no harvesting \( (\varepsilon_j = 0, \forall j = \{1, 2, 3\}) \). The fixed point of the system \( E_5^* = (0.65, 0.43, 0.81) \) is stable for aforementioned parameter values. The trajectory as a function of the state variables is plotted on the right side of the figure.

**4. Fractional models**

FC is a powerful tool which has been employed in different fields of science to model complex systems with non-local behaviour and long-term memory
Figure 3: The evolution of prey and predators densities in system (7) with parameter values \((\rho, \psi, \gamma_1, \gamma_2, \phi_1, \phi_2, \eta, \beta, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (2.8, 3, 1.2, 1.5, 1.4, 0.8, 0.01, 4.5, 0.766, 0.168, 0.015)\) and initial densities \((1, 0.5, 0.5)\). The fixed point of the system \(E^*_0 = (0.44, 0.62, 0.24)\) is stable for aforementioned parameter values. The trajectory as a function of the state variables is plotted on the right side of the figure.

This powerful approach can lead to models which capture more of the phenomena under scrutiny while still keeping the model parameters to a minimum. In this section, some basic definitions of FC, including the Caputo derivative and related concepts, are presented. From that point of departure, we incorporate fractional time derivatives into our models and investigate the effects. The stability of the systems with fractional order time derivative is analysed and we contrast the dynamics to those of the integer derivative models.

4.1. Basic definition of fractional calculus and its concept

The adjective “fractional” in FC is a historical remnant and this calculus is a generalisation from integer order derivatives to arbitrary real-valued order, not merely rational order. Interestingly, FC is not a uniquely defined generalisation and there exists multiple known approaches and new versions of fractional derivatives is an area of continuing research. However, there are some accepted common definitions in the literature and generally speaking, it can be formulated as follows.

\[
\frac{d^\alpha f}{dt^\alpha} = \begin{cases} 
D^\alpha f(t) & \alpha > 0 \\
 f(t) & \alpha = 1 \\
I^\alpha f(t) & \alpha < 0 
\end{cases}
\]  

(12)
where $D^\alpha$ and $I^\alpha$ are the fractional derivative and fractional integral respectively \[28\]. Various definitions of fractional derivatives have been proposed, including the Caputo derivative \[13\]. It is worth noting that all definitions of the fractional derivative coincide when the order is integer, however, this need not be the case for non-integer order. For example, a non-integer order Caputo derivative of a constant is zero while the Riemann-Liouville derivative of the same constant is not. Also, each definition of the fractional derivative has its own characteristics which lead to different physical interpretations and thus application. From the physical point of view indeed, what the memory of a system is and how it can be defined depends so much on a deep understanding of the phenomena.

We will use the Caputo definition in the remainder of this section. The Caputo derivative is defined as follows for function $f(t)$ (for other formula, see \[28, 29\]).

$$C^\alpha_a D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{D^n f(\tau)}{(t - \tau)^{\alpha-n+1}} d\tau, \quad n - 1 \leqslant \alpha < n \quad (13)$$

In equation \[13\], if $0 < \alpha < 1$, then we can rewrite the equation as follows,

$$C^\alpha_a D_t^\alpha f(t) = \int_a^t w(t - \tau) Df(\tau) d\tau \quad s.t \; w(t) = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \quad (14)$$

where $w(t)$ is a weight function whose task is the storage of the system memory over time \[24\]. Therefore, the bigger $\alpha$, the smaller weight function, and vice versa. That is, the bigger the value that the weight function has, the closer $\alpha$ will be to zero and as a result the information of the system will be saved with a higher rate. In other words, any arbitrary fractional model made on the basis of Caputo derivative (with $0 < \alpha < 1$) will retain close to complete memory of the past when the fractional order is close to zero. It is also worth mentioning that if the Riemann-Liouville definition is utilised the opposite situation is true because the Caputo derivative is the left inverse of the Riemann-Liouville integral \[24\]. It is expected that when the system maintains a near total memory of its past then the system keeps trying to stay in its past condition (position) and resists changing over time. So, based on this interpretation, oscillations in the population numbers of a species will be damped out in a system with fractional time derivative of order less than unity.
Figure 4: Oscillation damping property of system (15) with initial densities (0.2,0.25) and $\psi = 19$; other parameters are chosen from (Fig. 1). (a and b) Numerical values of the lanternfish and tuna v.s time respectively. (c): Phase portrait respect to time and (d): Phase portrait of the system given.

4.2. Fractional Models

Replacing the time derivatives of system (2) with Caputo derivatives, we arrive at the time fractional version model of the system

$$D^\alpha x(t) = \rho x (1 - x) - xy - \varepsilon_1 x,$$

$$D^\alpha y(t) = \frac{\psi xy}{1 + \phi x} - y - \varepsilon_2 y,$$

(15)
The fractionalised model of system (7) is also given in equation (16).

\[
D^\alpha x(t) = \rho x (1 - x) - x (y + \eta z + yz) - \varepsilon_1 x,
\]
\[
D^\alpha y(t) = \frac{\psi xy (1 + \eta z)}{1 + \phi_1 x} - \gamma_1 y - \varepsilon_2 y,
\]
\[
D^\alpha z(t) = \frac{\beta xz (1 + \xi y)}{1 + \phi_2 x} - \gamma_2 z - \varepsilon_3 z.
\]

We have solved both systems (15) and (16) numerically [30] and the results are shown in Figs. 4 and 5.

From the simulations presented in figures 4 and 5, the effect of reducing the order of the time derivative can be seen. As the fractional order \( \alpha \) is decreased, the system (with Caputo derivative) stabilises faster. That is the higher “memory” the system has of past states, the greater the damping of oscillations in system dynamics. The simulations demonstrate that, even with quite moderate reductions in \( \alpha \), the amplitude of population density oscillations is strongly retarded.

5. Conclusion

The mutualistic interactions amongst species is of great importance in the field of conservation ecology, so gaining an understanding of such interactions can make a noteworthy contribution to species maintenance. With this in mind, we introduced a modified Lotka-Volterra model to study interaction amongst three species with mutualistic predation. The motivation is based on the observed feeding behaviour of spotted dolphins and yellowfin tuna upon schools of lanternfish. Along with obtaining the stability conditions for the model, we also investigated the impact of “memory effects” on the species interaction via fractional calculus. Our analysis reveals that the fractionalised system dampens out induced oscillating inherent in predator-prey models and reaches the local stable point sooner than the integer model does. In other words, stability is more robust when the species exhibit “memory”.

Moreover, assuming that either both species have market value or that one species is caught as by-catch, we have investigated the effect of constant harvesting quota with the proposed models. We have also discussed the local stability behaviour of all the equilibrium states of the system. The
Figure 5: Oscillation damping property of system (16); parameters are chosen from (Fig. 3). (a, b and c) Numerical values of the lanternfish, tuna fish and dolphin v.s time respectively. (f): Phase portrait respect to time and (e and d): Phase portrait of the system given.
Figure 6: Phase plan of system (15) with integer order derivative \((6a)\) and fractional order derivative \((6b)\). Also, initial densities are \((x, y) = (0.2, 0.15)\) and parameter values \((\rho, \psi, \phi, \varepsilon_1, \varepsilon_2)\) are considered \((1, 50, 0.02, 0, 0)\) respectively, completely holding criterion \((5)\).

Output of all models shows that stability and extinction of the ecosystem are affected by economic interest/harvesting. In fact, applying constants harvest under the stability condition make the system stabilises faster. In other words, exploitation of species can be regarded as a favour to the ecosystem to be stabilised sooner, provided that harvesting does not exceed the stability range.

An assumption of differential equation based population models is that the quantities under consideration are sufficiently large such that the addition or loss of an individual to that population may be considered an infinitesimal change. That is the population may take on a continuum of values rather than being strictly discrete. Further, it is well known that many such models can exhibit, for the certain parameter values and initial conditions, large fluctuations — particularly those with Hopf bifurcations — whereby the assumption of the “largeness” of the population may very well breakdown. In practical terms the population has either, already gone extinct even though the model predicts a return to healthy levels, or its reduced to a level vulnerable to external perturbations such that it future viability such be discussed in probabilistic terms. These issues imply the model should incorporate stochasticity at low population levels. Alternatively, assuming a memory like behaviour can be attributed to the population, a non-integer order time derivative arguably extends the validity of the model by preventing
the wild swings in population numbers from essentially extinct to thriving.

It has been argued that ecological systems, while being stable for moderate numbers of species interactions or moderate strength of the inter-species connections, will become unstable to small perturbations once some threshold value for interaction number or interaction strength is breached [31, 32]. In 2012, similar findings were presented, claiming that any feasible co-existence of the system species will be unstable when the pair-wise competitive interactions are sufficiently strong [33]. Further, Goh presented findings that the continuum of globally stable Lotka-Volterra models of mutualism among three or more species is smaller than the continuum of globally stable Lotka-Volterra models of competition among the same number of species [34]. Thus, if diversity has an adverse effect on stability in competitive system it has even more so in mutualistic ones.

More recently, however, Butler and O’Dwyer have to some extent overturned this understanding through proposing a consumer-producer-resource model. They demonstrated, for a model of \( N \) bacteria species consuming \( N \) abiotic resources that the stability is guaranteed for all feasible equilibria. For an extension of the model where the bacteria also produce some or all of the resources to mutual benefit of all consumers, stability of all feasible solutions can be guaranteed provided that mutualistic interactions are symmetric [35].

Our system is one with moderate to strong interactions between mutualistic predators and a prey (biotic resource) species. In addition, its possible to consider weak to strong memory effects via the fractional derivative order. For the 1 or 2 predators thus far analysed then is no need for symmetry in the interactions. As future work we will investigate how stability of feasible solutions in the model changes as the number of predators and prey increase.

Acknowledgements

M.A. Matlob expresses his appreciation for many useful and enlightening discussions with Dr. Yousef Jamali of Tarbiat Modares University.
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