Cosmological Parameter Estimation from the Two-dimensional Genus Topology: Measuring the Shape of the Matter Power Spectrum

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Abstract

We present measurements of the two-dimensional genus of the SDSS-III Baryon Oscillation Spectroscopic Survey (BOSS) catalogs to constrain cosmological parameters governing the shape of the matter power spectrum. The BOSS data are divided into 12 concentric shells over the redshift range 0.2 < z < 0.6, and we extract the genus from the projected two-dimensional galaxy density fields. We compare the genus amplitudes to their Gaussian expectation values, exploiting the fact that this quantity is relatively insensitive to nonlinear gravitational collapse. We extract the genus amplitude from a combination of low redshift shells and high redshift shells, and we find that the genus amplitude is only weakly sensitive to the equation of state of dark energy. We use the fact that on quasi-linear scales the genus amplitude is only weakly sensitive to nonlinear gravitational collapse, and we extract the genus amplitude from a combination of low redshift shells and high redshift shells.}

1. Introduction

The Minkowski functionals (MFs) are a class of statistics that describe the morphology and topology of excursion sets of a field. They have a long history of application within cosmology (Melott et al. 1989; Gott et al. 1990; Park & Gott 1991; Park et al. 1992, 2001; Mecke et al. 1994; Schmalzing & Buchert 1997; Schmalzing & Gorski 1998). Early adopters measured the MFs of the cosmic microwave background (Gott et al. 1990; Park & Gott 1991; Schmalzing & Gorski 1998; Hikage et al. 2006) and nascent large-scale structure catalogs (Melott et al. 1989; Gott et al. 1992; Park et al. 1992). More recent applications have involved the measurement of the MFs from modern cosmological data sets (see, e.g., Hikage et al. 2001, 2002, 2003; Park et al. 2005; James et al. 2009; Gott et al. 2009; Choi et al. 2010; Zhang et al. 2010; Blake et al. 2014; Wiegand et al. 2014; Wang et al. 2015; Buchert et al. 2017). In this work we study one particular MF: the genus. This statistic is a topological quantity that has a simple and intuitive geometric interpretation and is relatively insensitive to nonlinear physics. This makes it a valuable probe for cosmology.

Our focus is on the genus of the matter density field at redshifts z < 0.6, as traced by galaxies. By directly comparing the measured genus amplitude to its Gaussian expectation value, one can measure the cosmological parameters that dictate the shape of the matter power spectrum (Tomita 1986; Doroshkevich 1970; Adler 1981; Gott et al. 1986; Hamilton et al. 1986; Melott et al. 1989). By comparing the genus amplitude at high and low redshift, one can also infer the equation of state of dark energy wDE by exploiting the fact that this quantity is a standard ruler (Park & Kim 2010). Further information regarding the N-point functions (N > 2) can be extracted from the shape of the genus curve (Matsubara 1994; Pogosyan et al. 2009; Gay et al. 2012; Codis et al. 2013).

Extracting information from large-scale structure using the genus, and eliminating systematic effects such as shot noise, nonlinear gravitational evolution, and redshift-space distortion, has been the subject of a series of recent works by the authors. In Appleby et al. (2017) we used mock galaxy catalogs to study various systematic effects that could bias our reconstruction of cosmological parameters. In Appleby et al. (2018b) we applied these lessons to two-dimensional shells of mock galaxy lightcone data, to test the constraining power of the statistic. In this work we further refine our analysis, extract the two-dimensional genus from a galaxy catalog, and use this information for cosmological parameter estimation.

In this work, we measure the genus of two-dimensional shells of the SDSS-III Baryon Oscillation Spectroscopic Survey (BOSS) LOWZ and CMASS galaxy catalogs (Alam et al. 2015). After extracting the genus curves from the data, we compare the measurements to their theoretical expectation value. We use the fact that on quasi-linear scales the genus amplitude is only weakly sensitive to nonlinear gravitational collapse (Melott et al. 1988; Matsubara & Yokoyama 1996; Park & Kim 2010). This allows us to use simple Gaussian statistics at relatively small scales, as we quantify in what follows. In a companion paper (S. Appleby et al. 2020, in preparation) we extract the genus amplitudes from a combination of BOSS data and the Sloan Digital Sky Survey (SDSS) main galaxy sample and use this quantity as a standard ruler to place constraints on the expansion history of the universe.

The paper will proceed as follows. In Section 2 we review the theory underpinning the genus of random fields. The data, mask, mock catalogs, and construction of covariance matrices are described in Section 3. In Section 4 we discuss the results of our analysis, and in Section 5 we present conclusions.
are described in Section 3. Finally, in Section 4 we place constraints on cosmological parameters, and then we close with a discussion in Section 5.

2. Genus—Theory

The theory underlying the MFs of random fields was derived in Doroshkevich (1970), Adler (1981), Tomita (1986), Hamilton et al. (1986), Ryden et al. (1989), Gott et al. (1987), and Weinberg et al. (1987) for Gaussian fields and expanded in Matsubara (1994, 2003), Matsubara & Suto (1996), Melott et al. (1988), Matsubara & Yokoyama (1996), Hikage et al. (2008), Pogosyan et al. (2009), Gao et al. (2012), and Codis et al. (2013) for non-Gaussian generalizations. In this paper we will be concerned with the three-dimensional matter density field as traced by galaxies, $\delta_{3D}$, and specifically two-dimensional slices\(^6\) of this field in planes perpendicular to the line of sight, $\delta_{2D}$. The two-dimensional power spectrum $P_{2D}$ of $\delta_{2D}$ is related to its three-dimensional counterpart $P_{3D}(k)$ as

$$P_{2D}(k_x, z) = \frac{2}{\pi} \int dk_x P_{3D}(k_x, z) \frac{\sin^2(k_x \Delta)}{k_x^2 \Delta^2},$$

where $k = \sqrt{k_x^2 + k_y^2}$, and we have performed real-space top hat smoothing along $x_i$, where $\Delta$ is the thickness of the slice. Here $k_x$ and $k_y$ are the Fourier modes perpendicular and parallel to $x_i$, respectively.

For the low-redshift matter density that we will probe via galaxy catalogs, the underlying three-dimensional power spectrum is given by

$$P_{3D}(k_x, k_y, z) = b^2 \left(1 + \frac{k_x^2}{k_y^2}\right) P_\text{m}(z, k) + P_{\text{SN}},$$

where $P_\text{m}(z, k)$ is the matter power spectrum at redshift $z$, and $P_{\text{SN}}$ is the shot-noise power spectrum, which we estimate as $P_{\text{SN}} = 1/n$, where $n$ is the number density of the galaxy catalog. $b = \Omega_\text{m}/b$ is the redshift-space distortion parameter, $b$ is the linear galaxy bias, and $\gamma \approx 3(1 - w_{de})/(5 - 6w_{de})$. Expression (2) accounts for linear redshift-space distortion and shot noise.

For two-dimensional slices of a three-dimensional Gaussian field, the genus per unit area is given by (Hamilton et al. 1986)

$$g_{2D}(\nu) = \frac{1}{2(2\pi)^{3/2}} \sigma_0^2 \nu e^{-\nu^2/2},$$

$$\sigma_0^2 = \langle \delta_{2D}^2 \rangle, \quad \sigma_0^2 = \langle \nabla \delta_{2D}^2 \rangle,$$

where $\sigma_{0,1}$ are cumulants of the two-dimensional field and $\nu$ is a constant-density threshold. For a Gaussian field the shape of the genus curve is fixed as a function of threshold $g_{2D} \sim \nu e^{-\nu^2/2}$, and only the amplitude,

$$A_{2D} \equiv \frac{1}{2(2\pi)^{3/2}} \sigma_0^2,$$

contains information. We can relate the cumulants to the power spectrum as

$$\sigma_0^2 = \frac{1}{2(2\pi)^{3/2}} \int d^2k_x e^{-|k_x|^2} P_{2D}(k_x, z),$$

$$\sigma_1^2 = \frac{1}{2(2\pi)^{3/2}} \int d^2k_x k_x^2 e^{-|k_x|^2} P_{2D}(k_x, z),$$

where we have smoothed the two-dimensional field using a Gaussian kernel of width $R_G$.

The genus amplitude (4) is proportional to the ratio of $\sigma_1$, $\sigma_0$ cumulants and as such will be insensitive to the total amplitude of the power spectrum. This is a generic property of the MFs.

For a weakly non-Gaussian field, one can perform the so-called Edgeworth expansion of the genus in $\sigma_0$ as follows (Matsubara 1994, 2003):

$$g_{2D}(\nu_A) = A_{2D}^{(2D)} e^{-\nu_A^2/2} \left[H_1(\nu_A) + \frac{2}{3} (S(1) - S(0)) \right] \times \tilde{H}_2(\nu_A) + \frac{1}{3} (S(2) - S(0)) H_0(\nu_A) \sigma_0 + O(\sigma_0^2),$$

where $A_{2D}^{(2D)}$ is the Gaussian amplitude (4) and the skewness parameters $S(0), S(1), S(2)$ are related to the three-point cumulants

$$S^{(0)} = \frac{\langle \delta_{2D}^3 \rangle}{\sigma_0^3},$$

$$S^{(1)} = \frac{3}{4} \frac{\langle \delta_{2D}^2 (\nabla^2 \delta_{2D}) \rangle}{\sigma_0^2 \sigma_1^2},$$

$$S^{(2)} = -3 \frac{\langle (\nabla \delta_{2D} \cdot \nabla \delta_{2D})(\nabla^2 \delta_{2D}) \rangle}{\sigma_1^4},$$

$H_0(x)$ are Hermite polynomials, the first few of which are given by $H_0(x) = 1, H_1(x) = x, H_2(x) = 1 - x^2$, and $H_3(x) = x^3 - 3x$. We have defined $\nu_A$ as the density threshold such that the excursion set has the same area fraction as a corresponding Gaussian field

$$f_A = \frac{1}{\sqrt{2\pi}} \int_{\nu_A}^{\infty} e^{-t^2/2} dt,$$

where $f_A$ is the fractional area of the field above $\nu_A$. This choice of $\nu_A$ parameterization eliminates the non-Gaussianity in the one-point function (Gott et al. 1987; Weinberg et al. 1987; Melott et al. 1988). The expansion given by Equation (7) has been continued to arbitrary order in Pogosyan et al. (2009).

The amplitude of the genus, which is the coefficient of $H_1(\nu_A)$ in Equation (7), is not modified by the non-Gaussian effect of gravitational collapse to leading order in the $\sigma_0$ expansion. We can therefore directly compare the measured genus amplitude to the expectation value (4), after smoothing the field over suitably large scales. The quantity $A_{2D}^{(2D)}$ is defined as the ratio of $\sigma_1, \sigma_0$ cumulants and so will be a measure of the shape of the underlying linear matter power spectrum. It follows that the cosmological parameters to which this statistic will be sensitive are the dark matter fraction $\Omega_\text{dm} h^2$, primordial power spectral index $n_s$, and also weakly the baryon fraction $\Omega_\text{b} h^2$. Conversely, it will be practically

\(^6\) More precisely, we generate shells centered on the observer. The BOSS data are sufficiently distant that we can use the distant observer approximation to predict the amplitude of the genus curve.
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insensitive to the amplitude of the power spectrum and any linear bias factors. The genus is closely related to the Euler characteristic, a central quantity in topological analysis. The Euler characteristic can be expressed as the difference between number of connective components and holes in an excursion set. The genus can be formally defined as an integral over the intrinsic curvature of an excursion set boundary, and for a field without a boundary they are linearly related. For a detailed discussion we direct the reader to Adler & Taylor (2010) and Pranav et al. (2019).

One important caveat associated with our approach is that we are dealing with a masked field and hence a bounded domain. The genus of a field with a boundary will not be represented by expression (7), which was derived assuming an unbounded domain. Calculating topological statistics for fields with a boundary requires a detailed understanding of the interaction between the excursion set and mask (Adler & Taylor 2010; Pranav et al. 2019). However, in this work we are not extracting the genus of the bounded field, but rather estimating the genus of the full sky using an observed subset of data. In what follows, we measure the integrated Gaussian curvature per unit area of the masked sky using the method outlined in Schmalzing & Gorski (1998) and applied to large-scale structure in Appleby et al. (2018b). This approach provides an unbiased estimate of the full-sky genus, which can be compared to analytic result (7) (Tomita 1986; Matsubara 2003). The boundary does not significantly enter into our analysis, as we estimate the full-sky genus by sampling from available pixels. We confirm that our analysis is unbiased in Appendix C; by applying our method to full-sky and masked mock galaxy data. We also confirm that the flat-sky approximation implicit within our analysis is valid. If one wanted to measure the genus of a bounded field, a different expansion from Equation (7) should be used—the Gaussian Kinematic Formula (Adler & Taylor 2010)—as the boundary can generate a highly asymmetric curve (Pranav et al. 2019). We stress that there is a fundamental distinction between calculating the genus of a field with a boundary and estimating the genus per unit area of the full sky by sampling from an observed patch. Our approach is the latter, so we must test that our estimator is unbiased.

We will measure genus curves from shells of galaxy data and extract cosmological information by comparing the amplitude of the curves to their Gaussian expectation value. In the following sections we elucidate the galaxy catalogs used in this analysis and the mock catalogs used to construct the covariance matrices required for statistical inference.

3. Observational Data

To measure the genus over the redshift range $0.2 < z < 0.6$, we use the SDSS-III BOSS (York et al. 2000). The 12th release of the SDSS (Alam et al. 2015) imaged 9, 376 deg$^2$ of the sky in the ugriz bands (Fukugita et al. 1996). The survey was performed with the 2.5 m Sloan telescope (Gunn et al. 2006) at the Apache Point Observatory in New Mexico. The resulting extragalactic catalog contains 1,372,737 unique galaxies, with redshifts measured using an automated pipeline (Bolton et al. 2012).

The BOSS data are decomposed into two distinct catalogs. The LOWZ sample consists of galaxies at redshift $z < 0.4$ that are selected using various color–magnitude cuts that are intended to match the evolution of a passively evolving stellar population. In this way, a bright and red “low-redshift” galaxy population is selected with the intention of extending the SDSS-I and SDSS-II luminous red galaxies to higher redshift and increased number density.

The CMASS “high-redshift” $0.4 < z < 0.7$ galaxies are selected using a set of color–magnitude cuts. The $(g−r)$ and $(r−i)$ cuts are specified to segregate “high-redshift” galaxies. However, the sample is not biased toward red galaxies, as some of the color limits imposed on the SDSS-I/II sample have been removed. The color–magnitude cut is varied to ensure that massive objects are sampled as uniformly as possible with redshift. We direct the reader to Reid et al. (2016) for further details of the galaxy samples, including details of targeting algorithms.

The CMASS and LOWZ samples are provided with the galaxy weights $w_{\text{cp}}$, $w_{\text{noz}}$, and $w_{\text{ystot}}$ to account for observational systematics. $w_{\text{cp}}$ represents the “close pairs” weight, which accounts for the subsample of galaxies that are not assigned a spectroscopic fiber owing to fiber collisions. This sample is not random, as these missed galaxies must be within a fiber collision radius (62") of another target. This systematic is corrected by upweighting the nearest galaxy by $w = (1 + n)$, where $n$ is the number of neighbors without a redshift. The spectroscopic pipeline failed to obtain a redshift for 1.8% (0.5%) of CMASS (LOWZ) targets. In the case of such a failure, a similar upweighting scheme is adopted to that for $w_{\text{cp}}$—the nearest neighbor of any failed redshift galaxy is upweighted by $w_{\text{noz}}$. Note that failed redshift galaxies could be first upweighted by a factor $w_{\text{cp}}$—in this case $w_{\text{cp}}$ is added as a weight to its nearest neighbor (for $w_{\text{noz}}$). The upweighted object must be classified as a “good” galaxy. The $w_{\text{ystot}}$ weight applies to the CMASS sample only and is used to remove noncosmological fluctuations in the CMASS target density due to stellar density and seeing. The LOWZ galaxies are generally bright compared to the CMASS galaxies and do not show significant density variations because of noncosmological fluctuations from stellar density and seeing. Therefore, the LOWZ targets do not require the $w_{\text{ystot}}$ weight (Reid et al. 2016). The total galaxy weight adopted in this analysis is

$$w_{\text{tot}} = w_{\text{ystot}}(w_{\text{cp}} + w_{\text{noz}} - 1).$$

We measure the genus of two-dimensional shells of the BOSS LOWZ and CMASS data. To construct a set of two-dimensional galaxy number density fields, we first bin the galaxies into redshift shells. The redshift shell boundaries are fixed such that each shell has constant comoving thickness $Δ = 80$ Mpc, assuming a flat $\Lambda$CDM input cosmology with parameters presented in Table 1. As the slice thickness is chosen in terms of comoving distance, we have introduced a cosmological parameter dependence—if we select an incorrect cosmology, our bins will not have uniform thickness. However, in Appendix B we vary the cosmology used to generate the slices and verify that our results are robust to this choice, for reasonable variations of the parameters $h$ and $Ω_m$. The redshift shells are concentric and nonoverlapping over the range $0.25 < z < 0.6$. To generate a constant number density sample in each redshift shell, we apply a lower stellar mass cut in each shell to fix $n_{\text{cut}} = 6.5 \times 10^{-5}$(Mpc$^{-3}$). The galaxy stellar mass estimates are derived using the Wisconsin PCA BC03 model (Tinker et al. 2017). With this choice, the shot-noise
contribution to power spectrum (2) is approximately given by \( R_{\text{SN}} = 1/\tilde{n}_{\text{cut}} \) and is constant in each redshift shell.

In Table 2 we present the redshift limits of the shells that we choose for the CMASS and LOWZ samples. We select a total of \( N_z = 12 \) slices (6/6 from the LOWZ/CMASS catalogs, respectively). The full LOWZ and CMASS catalogs span the ranges 0.15 < \( z < 0.45 \) and 0.45 < \( z < 0.7 \), respectively. However, we only select LOWZ data for 0.25 < \( z < 0.38 \) and CMASS data for 0.45 < \( z < 0.6 \). For LOWZ data below \( z < 0.25 \), the density field that we generate does not have sufficient volume to accurately reconstruct the genus curve, which will affect our genus amplitude measurements. For LOWZ data at \( z > 0.38 \) and CMASS data at \( z > 0.6 \), the catalogs do not possess sufficient number density, and the genus measurements would be significantly affected by shot noise. In Figure 1 we present the full redshift distribution of the LOWZ and CMASS samples. The number density of our sample \( \tilde{n}_{\text{cut}} \) is exhibited as a dashed black horizontal line, and the redshift bin limits are presented as vertical yellow dashed lines.

To generate two-dimensional density fields at each redshift, the galaxies are binned into a regular HEALPix\(^8\) (Gorski et al. 2005) lattice on the sphere according to their galactic latitude and longitude (\( b, \ell \)). We use nearest pixel binning, applying the weight \( w_{\text{cut,m}} \) to the nearest pixel center to which the \( m \)th galaxy belongs. This generates a set of density fields \( \delta_{ij} \equiv (n_{ij} - \tilde{n}_{ij})/\tilde{n}_{ij} \), where \( 1 \leq j \leq N_z \) denotes the redshift bin (of which there are \( N_z = 12 \) in total) and \( 1 \leq i \leq N_{\text{pix}} \) is the pixel identifier on the unit sphere. \( \tilde{n}_{ij} \) is the mean number of galaxies contained within an unmasked pixel at each redshift shell, and \( n_{ij} \) is the number of galaxies contained within pixel \( i \) in redshift slice \( j \). We use \( N_{\text{pix}} = 12 \times 512^2 \) pixels.

3.1. Two-dimensional Masks

The mask is an equal-area pixel map of the same number of pixels as the galaxy maps. Each pixel is defined by a weight \( \Theta_i \), \( 1 \leq i \leq N_{\text{pix}} \), obtained by binning the survey angular selection function into the HEALPix basis. The resulting weights lie in the range \( 0 \leq \Theta_i \leq 1 \).

In what follows, we apply a mask weight cut and only use pixels with \( \Theta_i > \Theta_{\text{cut}} \), with \( \Theta_{\text{cut}} = 0.8 \). We account for the angular selection function by directly weighting each galaxy with \( w_{\text{cut}} \) as described in Section 3, so the mask \( \Theta_i \) is converted to a binary map—\( \Theta_i = 1 \) if \( \Theta_i > \Theta_{\text{cut}} \) or \( \Theta_i = 0 \) otherwise.

We apply the binary mask \( \Theta_i \) to the galaxy fields \( \delta_{ij} \rightarrow \Theta_i \times \delta_{ij} \) and then smooth \( \delta_{ij} \) in harmonic space with a Gaussian kernel of width \( \theta_j = R_G/d_c(z_j) \), where \( R_G = 20 \) Mpc is a constant comoving scale and \( d_c(z_j) \) is the comoving distance to the center of the \( j \)th redshift slice. We denote the smoothed density fields as \( \tilde{\delta}_{ij} \). We also smooth the mask \( \Theta_i \) with the same angular scale \( \theta_j \) at each redshift, generating a set of smoothed masks \( \tilde{\Theta}_{ij} \). We then apply a second cut to the density fields: \( \tilde{\delta}_{ij} = 0 \) if \( \tilde{\Theta}_{ij} < \Theta_{\text{cut}} \) and \( \delta_{ij} = \tilde{\delta}_{ij}/\tilde{\Theta}_{ij} \) if \( \tilde{\Theta}_{ij} > \Theta_{\text{cut}} \). Finally, we reapply the original, unsmoothed binary mask \( \Theta_i \), as \( \delta_{ij} \rightarrow \Theta_i \delta_{ij} \). This method removes regions of the density field in the vicinity of the mask boundary, where the true field is not accurately reproduced.

We extract the genus from each redshift shell using the method described in Appleby et al. (2017) and divide by the total comoving area \( A_c = 4\pi f_{\text{sky},j} d_c^2(z_j) \), where \( f_{\text{sky},j} \) is the fraction of sky that is unmasked in the \( j \)th redshift slice. We measure the genus at 201 values of the threshold \( \nu_A \), equispaced over the range \(-2.5 < \nu_A < 2.5\), and then take the average over every four values to obtain \( N_{\nu_A} = 50 \) measurements. We label the measured values \( g_{ij}^j \), where \( j \) runs over the redshift shells and \( 1 \leq n \leq N_{\nu_A} \), over the \( N_{\nu_A} = 50 \), \( \nu_A \) thresholds. The two-dimensional density fields are presented in Figures D4 and D5 of Appendix D.

In Figure 2 we exhibit the genus curves measured from the \( N_z = 6 \) shells of LOWZ (top panel) and CMASS (bottom panel) data. The curves extracted from the six LOWZ shells exhibit large scatter compared to the CMASS data, due to cosmic variance and the smaller volume available at low redshift.

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8 http://healpix.sourceforge.net

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Table 1

| Parameter | Fiducial Value |
|-----------|----------------|
| \( \Omega_m \) | 0.307 |
| \( \theta_b \) | 0.677 |
| \( \Delta \) | 80 Mpc |
| \( R_G \) | 20 Mpc |

Note. \( \Delta \) is the thickness of the two-dimensional slices of the density field, and \( R_G \) is the Gaussian smoothing scale used in the two-dimensional planes perpendicular to the line of sight.

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Table 2

| LOWZ | CMASS |
|------|-------|
| \( 0.25 < z < 0.271 \) | \( 0.453 < z < 0.476 \) |
| \( 0.271 < z < 0.292 \) | \( 0.476 < z < 0.500 \) |
| \( 0.292 < z < 0.313 \) | \( 0.500 < z < 0.524 \) |
| \( 0.313 < z < 0.334 \) | \( 0.524 < z < 0.548 \) |
| \( 0.334 < z < 0.356 \) | \( 0.548 < z < 0.573 \) |
| \( 0.356 < z < 0.378 \) | \( 0.573 < z < 0.598 \) |

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Figure 1. Number densities of the LOWZ and CMASS galaxy catalogs as a function of redshift. The black dashed line is the density cut that we apply in this work, and the yellow dashed lines represent the redshift bin limits used to generate the shells.
These genus curves will be used to extract cosmological information. However, before we can do so, we must estimate the statistical uncertainty of the measurements. In the following section we describe the mock catalogs used to generate the relevant covariance matrices.

### 3.2. Mock Catalogs

To estimate the covariance between the binned $g^n_j$ genus measurements, we use $N_r = 500$ Multidark patchy mocks (Kitaura et al. 2016; Rodríguez-Torres et al. 2016). Full details of their creation can be found in Kitaura et al. (2016). Briefly, the mocks were generated using an iterative procedure to reproduce a reference galaxy catalog using approximate gravity solvers and a statistical biasing model (Kitaura et al. 2014). The reference catalog arises from the Big-MultiDark N-body simulation, which used Gadget-2 (Springel 2005) to evolve $3840^3$ particles in a $(2.5h^{-1}\text{Mpc})^3$ volume. Halo abundance matching was used to reproduce the clustering of the observational data. The Patchy code (Kitaura et al. 2014, 2015) is used to match the two- and three-point clustering statistics with the full reference simulation in different redshift bins. Stellar masses are assigned and mock light cones are generated, accounting for the survey mask and selection effects. The resulting mock catalogs accurately reproduce the number density, two-point correlation function, selection function, and survey geometry of the DR12 observational data. The simulated data were generated using a Planck cosmology with $\Omega_m = 0.307$, $\Omega_b = 0.048$, $n_s = 0.961$, $H_0 = 67.77 \text{km s}^{-1}\text{Mpc}^{-1}$.

For each mock catalog, we repeat our analysis: sort the galaxies into redshift shells, apply a mass cut, and then bin the surviving galaxies into pixels on the sphere. We then apply our masking and smoothing procedure and construct the genus curves $g^n_{p,j,n}$, where $p$, $j$, $n$ represent the $p$th mock realization, $j$th redshift shell, and $n$th $\nu_A$ threshold. As for the actual data, we measure the genus at 201 values of $\nu_A$ over the range $-2.5 < \nu_A < 2.5$ and then average every four points to obtain $N_A = 50$ values. From these measurements the 12 covariance matrices $\Sigma_{m,n}(\nu_A)$—one for each redshift shell—can be constructed as

$$\Sigma_{m,n}(\nu_A) = \frac{1}{N_A} - 1 \sum_{p=1}^{N_r} (g^n_{p,j} - \langle g^n_{j} \rangle) (g^m_{p,j} - \langle g^m_{j} \rangle),$$

where $\langle g^n_{j} \rangle$ is the average value of the genus curve at the $n$th $\nu_{A,n}$ threshold and $j$th redshift bin. We assume that the genus curves obtained at different redshift slices are uncorrelated. In Figure 3 we present one covariance matrix obtained from the mock catalogs, which is representative of all covariance matrices generated. One can observe strong positive correlation between $\nu_A$ thresholds separated by $\Delta \nu_A < 0.5$ (red) and a weaker, negative correlation between thresholds of larger separation (blue).

### 4. Results—Cosmological Parameter Estimation

Finally, we use the measured genus curves $g^n_j$ and reconstructed covariance matrices $\Sigma_{m,n}(\nu_A)$ to constrain cosmological parameters. In each redshift shell, we minimize the
following $\chi_j^2$ function:

$$\chi_j^2 = \sum_{n=1}^{N_a} \sum_{m=1}^{N_h} \Delta g_j^m \Sigma_{n,m}(z_j) \Delta g_j^m,$$

where

$$\Delta g_j^m = g_j^m - A_{G,j}^{(2D)} e^{-\nu A_s^2/2} [a_0 H_0(\nu A_s) + a_2 H_2(\nu A_s) + a_3 H_3(\nu A_s)].$$

This functional form matches the theoretical expansion (7). The parameters varied are $a_{0,j}$, $a_{2,j}$, $a_{3,j}$, which are assumed to be arbitrary constants in each shell, and $\Omega_c h^2$, $\Omega_b h^2$, $n_s$, the cosmological parameters that dictate the genus amplitude $A_{G,j}^{(2D)}$, which is defined in Equation (4) and related to the three-dimensional matter power spectrum via Equations (1) and (2).

$A_{G,j}^{(2D)}$ is sensitive to the shape of the linear matter power spectrum and hence to the parameters $\Omega_c h^2$, $\Omega_b h^2$, $n_s$. Conversely, it is practically insensitive to the amplitude of the power spectrum, so we fix $\ln[10^{10} A_s] = 3.089$ according to its Planck best fit (Aghanim et al. 2018) and linear galaxy bias $b = 2$, inferred from the mock catalogs. In Appendix A we discuss the sensitivity of the genus to the amplitude of the power spectrum and argue that for sparse galaxy data some residual sensitivity exists owing to the presence of a shot-noise contribution. For the smoothing scales used in this work, the sensitivity can be safely neglected.

In principle, the parameters $a_{0,2}$ are sensitive to cosmology, as they are related to the three-point cumulants and hence the shape of the bispectrum. However, in this work we do not utilize this information and treat these parameters as free over the range $-1 < a_0, a_2 < 1$. Here $a_0$, $a_2$ are the leading-order corrections to the genus shape due to non-Gaussianity generated by gravitational collapse. The parameter $a_3$ should be of order $\sigma_3^2$ in the perturbative non-Gaussian expansion of the field, and we include it as a check that higher-order terms remain negligible. We assign uniform priors of $0.5 < n_s < 1.2$ and $0.05 < \Omega_c h^2 < 0.2$, $0.018 < \Omega_b h^2 < 0.026$ on the cosmological parameters.

In Figures D1 and D2 in Appendix D we exhibit the two-dimensional marginalized contours obtained from each individual LOWZ and CMASS slice for the parameters $\Omega_c h^2$, $\Omega_b h^2$, $n_s$, $a_0$, $a_2$. Each colored contour corresponds to the result from a particular redshift slice. Although the parameter uncertainties are large when only using individual shells, we find some general trends. The sensitivity of $A_{G,j}^{(2D)}$ to $\Omega_b h^2$ is extremely weak—we obtain no significant constraint within the prior range selected. The parameters $a_{0,2}$ are effectively independent of $n_s$ and $\Omega_c h^2$; this is due to the fact that $a_{0,2}$ are coefficients of even Hermite polynomials, whereas the genus amplitude is the coefficient of the odd polynomial $H_1(\nu A_s)$. Conversely, there exists a strong correlation between $a_0$, $a_2$ for the same reason (both are even polynomial coefficients). We do not plot $a_3$, as it is included simply as a consistency check. This parameter is present at the $\sim 1\%$ level but does not significantly impact our results. It is one of multiple terms that would be induced at order $O(\sigma_3^2)$.

For the cosmological parameters, there is a strong negative degeneracy between $n_s$ and $\Omega_c h^2$. Both parameters can increase the amount of small-scale power, by tilting the power spectrum and shifting the peak, respectively.

To improve the constraining power of the statistic, in Figure 4 we present the combined constraint on $\Omega_c h^2$, $n_s$ (left panel) and $a_2$, $a_0$ (right panel), obtained by combining all six CMASS shells (blue contour), the combined LOWZ data (brown contour), and all 12 shells combined (purple contour). Specifically, we fit a single set of parameters $\Omega_c h^2$, $n_s$, $a_2$, $a_0$, $a_3$ separately to all six LOWZ and CMASS genus curves, and also to the entire set of 12 curves. We assume that the genus

---

**Figure 4.** Left panel: marginalized 68%/95% contours for the parameters $\Omega_c h^2$ and $n_s$, obtained from the combined LOWZ (brown), CMASS (blue), and all 12 shells of BOSS data (purple). The black star is the Planck best-fit value assuming a flat ΛCDM model. Right panel: two-dimensional 68%/95% contours for the parameters $a_0$, $a_2$, with color scheme identical to the left panel. We observe a degeneracy between the two parameters.
measurements at each redshift are independent and so sum the $\chi^2_j$ contributions from each redshift shell and minimize the following $\chi^2$ functions:

$$\chi^2_{\text{lowz}} = \sum_{j=1}^{6} \chi^2_j,$$  \hspace{1cm} (16)

$$\chi^2_{\text{cmass}} = \sum_{j=7}^{12} \chi^2_j,$$ \hspace{1cm} (17)

$$\chi^2_{\text{all}} = \sum_{j=1}^{12} \chi^2_j.$$ \hspace{1cm} (18)

We include the parameter $\Omega_b h^2$ and marginalize over it, despite this quantity being effectively unconstrained within our prior range. We only present two pairs of contours in the main body of the text because all other combinations are not informative. For completeness we provide the full corner plot in Figure D3 in Appendix D. The CMASS data provide a tighter constraint compared to the LOWZ data, as expected owing to the larger volume being probed at high redshift. All data are self-consistent and also in agreement with Planck measurements of these parameters (black star). The parameters $a_0$, $a_2$ are correlated and represent $\sim 5\%$ corrections to the shape of the genus curve, relative to its Gaussian form. This indicates that the non-Gaussian perturbative expansion in $\sigma_0$ is valid at the scales being probed in this work.

Due to the strong degeneracy between $n_s$ and $\Omega_b h^2$, we cannot simultaneously constrain these parameters using the genus amplitude alone. However, a tight constraint on the particular combination $n_s^{3/2} \Omega_b h^2$ can be derived from the data. If we rotate the contours into the $n_s^{3/2} \Omega_b h^2 - n_s$ plane, we effectively obtain a one-dimensional constraint on $n_s^{3/2} \Omega_b h^2$. In Figure 5 we present the marginalized one-dimensional posterior likelihood for this parameter combination, for the LOWZ (brown), CMASS (blue), combined LOWZ and CMASS (purple), and Planck (black) data. For the Planck constraint we use the publicly available baseline LCDM Markov Chain Monte Carlo chains. The BOSS data are fully consistent with the Planck result, indicating that the shape of the linear matter power spectrum is consistent between $z \lesssim 1$ and $z \sim 1000$ over the scales probed in this analysis.

Finally, in Figure 5 (bottom panel) we present the best fit and $1\sigma$ uncertainties of the combination $n_s^{3/2} \Omega_b h^2$ as a function of redshift, obtained from the 12 LOWZ and CMASS redshift shells individually (colored points/error bars). The Planck best fit is shown as a solid black line, and the combined result from all LOWZ and CMASS slices is presented as blue/brown lines and solid bands (the bands indicate $1\sigma$ limits). The results are self-consistent, and fully consistent with the Planck result. In Table 3 we present our results in tabulated form.

Our results provide a tight constraint on the combination $n_s^{3/2} \Omega_b h^2$, which dictates the shape of the linear matter power spectrum. The genus amplitude is consistent with early universe measurement of the power spectrum, indicating conservation of $P_{\text{DD}}(z, k)$ with redshift. This is expected for the LCDM model, for which the linear power spectrum shape is conserved from the last scattering surface to the present time, and only the amplitude varies. We emphasize that the amplitude cannot be measured efficiently using topological statistics.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Top panel: one-dimensional marginalized posterior likelihood for the parameter combination $n_s^{3/2} \Omega_b h^2$. The blue and brown distributions correspond to the CMASS and LOWZ data, respectively, and the purple distribution corresponds to the combined result of all CMASS and LOWZ shells. The black distribution is the Planck posterior for this parameter combination, assuming a flat LCDM model. Bottom panel: one-dimensional marginalized best fit and $1\sigma$ uncertainty on $n_s^{3/2} \Omega_b h^2$ from each LOWZ and CMASS shell (points and error bars). The solid brown/blue lines and shaded areas are the best fit and $1\sigma$ uncertainty from the combined LOWZ/CMASS shells, and the black solid line is the Planck best fit for this parameter combination.}
\end{figure}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{Data} & $n_s^{3/2} \Omega_b h^2$ & $a_0$ & $a_2$ & $a_3$ \\
\hline
\textbf{LOWZ} & 0.108$^{+0.006}_{-0.006}$ & 0.048$^{+0.006}_{-0.006}$ & 0.023$^{+0.008}_{-0.008}$ & $-0.003^{+0.004}_{-0.005}$ \\
\hline
\textbf{CMASS} & 0.112$^{+0.005}_{-0.005}$ & 0.033$^{+0.004}_{-0.004}$ & 0.041$^{+0.005}_{-0.005}$ & 0.005$^{+0.003}_{-0.003}$ \\
\hline
\textbf{ALL} & 0.112$^{+0.004}_{-0.004}$ & 0.037$^{+0.003}_{-0.003}$ & 0.036$^{+0.004}_{-0.004}$ & 0.003$^{+0.003}_{-0.003}$ \\
\hline
\end{tabular}
\caption{Marginalized Best Fit and $1\sigma$ Uncertainty on the Parameter Combination $n_s^{3/2} \Omega_b h^2$, and the Hermite Polynomial Coefficients $a_{0,2,3}$ from the Combined LOWZ and CMASS and Combination of All 12 Shells Used (“ALL”).}
\end{table}

\section{5. Discussion}

In this work we have measured the two-dimensional genus of shells of BOSS LOWZ and CMASS data. After extracting the genus curves, we used them to place cosmological parameter constraints using the amplitude of the curve. The genus amplitude provides a measure of the shape of the underlying linear matter power spectrum; hence, we were able...
to constrain $\Omega_c h^2$, $n_s$. The parameters $\Omega_c h^2$ and $n_s$ present negative correlation, as both can act to affect the slope of the power spectrum on the smoothing scales adopted in this study. We found that the genus amplitude is effectively insensitive to the baryon fraction. We were able to place a tight constraint on $n_s^{3/2} \Omega_c h^2 = 0.108 \pm 0.010$ and $n_s^{3/2} \Omega_c h^2 = 0.112 \pm 0.005$ for the LOWZ and CMASS data sets, respectively, with a total constraint $n_s^{3/2} \Omega_c h^2 = 0.112 \pm 0.004$ after combining all data. Our constraints are completely consistent with the Planck best-fit values for a flat $\Lambda\text{CDM}$ model.

Our results are practically insensitive to reasonable variations of $A_s$ and linear galaxy bias $b$. However, the MFs can be sensitive to these quantities for sparse galaxy samples, as the relative amplitude between the matter power spectrum and the shot-noise contribution modifies the genus amplitude. We discuss this caveat further in Appendix A. Shot noise is the dominant issue in the reconstruction of MFs from a continuous field inferred from a point distribution. It modifies the genus amplitude, but also the shape of the MFs, as the noise can be non-Gaussian. When shot noise is a significant contributor to the field, we lose the interpretation that the genus amplitude is a measure of the slope of the matter power spectrum.

Given that our analysis is applied to a spectroscopic galaxy catalog, one might question the logic of using two-dimensional slices of data when we have access to accurate redshift information. The reasoning is twofold. First, the galaxy catalog is sparse, and we mitigate this issue by taking thick slices along the line of sight. Binning galaxies in this way is simply a smoothing choice, so we can interpret our decision as anisotropic smoothing perpendicular and parallel to the line of sight. Smoothing on larger scales parallel to the line of sight has advantages, such as allowing us to use linear redshift-space distortion physics. Second, in future work we wish to compare our results with higher-redshift photometric redshift catalogs, which will require us to bin galaxies into thick shells. An understanding of how photometric redshift uncertainty modifies our analysis must be further explored before this comparison can be made.

Our analysis can be refined in a number of ways. We have only used information extracted from the amplitude of the genus curve. For a non-Gaussian field, the shape of the genus contains information on the three-point function of the density field—by relating the Hermite polynomial coefficients $a_{0,2}$ to the three-point cumulants, one can extract information on the shape of the galaxy bispectrum. We intend to perform this comparison in future work.

In addition, one could use combinations of the MFs to improve cosmological parameter constraints. For a two-dimensional field, there are three MFs: the area fraction and perimeter length of the excursion set, and the genus. The area fraction is used in this work in an implicit way, to rescale the $\nu$ constant-density threshold to $\nu_A$, effectively Gaussianizing the one-point function and mitigating the non-Gaussianity of the field. Our focus has been on the genus (Appleby et al. 2017, 2018b), but a generalization to include the perimeter length will be considered in the future.

The redshift-space distortion correction to the genus has been calculated at linear order in Matsubara (1996). It would be of interest to study the nonlinear effects of the velocity field (Codis et al. 2013), all the way down to the small-scale finger-of-God effect. Better understanding of this systematic can be used to reduce the uncertainty of our measurements, as it would allow us to smooth the density field on smaller scales parallel to the line of sight.

Finally, it would be of interest to consider methods by which we can break the parameter degeneracy between $\Omega_c h^2$ and $n_s$. One method would be to combine measurements of the genus at different smoothing scales. On small scales we can expect to be predominantly sensitive to $n_s$, whereas by using a large smoothing scale one will be increasingly sensitive to the peak position of the power spectrum. This dependence will rotate the two-dimensional contour in the $n_s$-$\Omega_c h^2$ plane. To perform this test, we must understand the covariance between genus measurements at different scales. We can also use overlapping redshift bins and measure the two-dimensional genus as a continuous function of $z$. Finally, the two- and three-dimensional genus amplitudes will also be sensitive to the power spectrum slope at different scales. In future work we will combine these measurements to simultaneously constrain the parameters governing the shape of the matter power spectrum.

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Some of the results in this paper have been derived using the healpy and HEALPix package.

Appendix A
Sensitivity of Genus to Galaxy Bias and Power Spectrum Amplitude

In Appendix A we discuss the extent to which the genus amplitude is sensitive to the linear galaxy bias and primordial power spectrum amplitude $A_s$. We begin by repeating the definition of the two-dimensional genus amplitude as the ratio of cumulants,

$$A_{G}^{(2D)} = \frac{1}{2(2\pi)^{3/2}} \int \frac{dk}{k} \frac{k^3 e^{-k^2 R^2} P_{2D}(k, z)}{\int \frac{dk}{k} e^{-2k^2 R^2} P_{2D}(k, z)},$$

where the two-dimensional power spectrum $P_{2D}$ is related to the three-dimensional power spectrum according to

$$P_{2D}(k, z) = \frac{2}{\pi} \int dk_3 P_{3D}(k, k_3, z) \frac{\sin^2[\Delta k_3]}{(\Delta k_3)^2},$$

where $k = \sqrt{k^2 + k_3^2}$. When extracting the genus from a galaxy catalog, we are probing the matter field measured in redshift space, reconstructed from a discrete point distribution. For such a field the underlying three-dimensional power spectrum in Equation (A2) can be approximated on large scales as

$$P_{3D}(k, k_3, z) = b^2 \left(1 + \beta \frac{k^2}{k_3^2}\right) P_{\text{lin}}(z, k) + P_{\text{SN}},$$

where $P_{\text{lin}}(z, k)$ is the linear matter power spectrum at redshift $z$, $P_{\text{SN}}$ is the shot-noise power spectrum that we take as $P_{\text{SN}} = 1/\bar{n}_{\text{cut}}$, $\beta = \Omega_m^2/b$, $b$ is the linear galaxy bias, and $\gamma \simeq 3(1 - w_{de})/(5 - 6w_{de})$.

The conventional wisdom in topological analysis is that the MFs are insensitive to the amplitude of the power spectrum and hence also any linear bias factors. However, galaxy bias and power spectrum amplitude enter into the genus amplitude in two ways. First, the presence of the redshift-space distortion parameter $\beta \sim b^{-1}$ in Equation (A3) introduces weak dependence on the galaxy bias. In Figure A1 we present the dimensionless ratio $a_{\text{cut}} = A_{G,\text{rs}}^{(2D)}/A_{G,\text{lin}}^{(2D)}$, where $A_{G,\text{rs}}$ and $A_{G,\text{lin}}$ are the genus amplitudes in redshift and real space, respectively. We have fixed all cosmological parameters to their Planck best-fit values and used $\beta = 0$ in Equation (A3) to calculate $A_{G,\text{lin}}^{(2D)}$. We have calculated $a_{\text{cut}}$ for three different values of the linear galaxy bias $b = 1.8$, 2, and 2.2, which are shown as green, black, and blue lines in the figure. The net effect of redshift-space distortion is to decrease the genus amplitude by roughly 9%. The red solid area indicates the sensitivity of the redshift-space distortion effect due to galaxy bias—varying the galaxy bias over the range $1.8 < b < 2.2$ introduces a weak $\sim O(1\%)$ variation in the genus amplitude.

The second effect, which generates sensitivity to the amplitude of the power spectrum $b^2 A_s$, is shot noise. The fact that we are attempting to infer the properties of a continuous fluid from a discrete point distribution introduces a noise contribution to the total measured power spectrum—we have approximated this effect via the term $P_{\text{SN}} = 1/\bar{n}_{\text{cut}}$ in Equation (A3), where $\bar{n}_{\text{cut}}$ is the galaxy number density, selected to be constant in each redshift shell.

We see that the genus amplitude is actually a measurement of the sum of two power spectra—the underlying matter power spectrum and $P_{\text{SN}}$. Generically, the fact that $P_{\text{SN}} \neq 0$ implies that the relative amplitudes of $b^2 P_{\text{lin}}$ and $P_{\text{SN}}$ will impact the genus amplitude. This was observed in Kim et al. (2014) and Appleby et al. (2017), where the genus amplitude was found to be a function of galaxy number density and sampling selection (or bias). This effect is minimized by Gaussian smoothing in the plane, which suppresses the contribution of $P_{\text{SN}}$, and also selecting as high a number density sample as possible. If the number density of the galaxy catalog is sufficiently large, the effect of shot noise can be safely neglected and the genus amplitude becomes insensitive to scale-independent bias factors $b(z)$ and also the primordial amplitude $A_s$. However, for a sparse galaxy sample we can expect some sensitivity to both.

In Figure A2 we exhibit $A_{G}^{(2D)}$—Equation (A1)—using the power spectrum (A3) assuming cosmological parameters and smoothing scales $\Delta$, $R_G$ in Table 1. To avoid conflating different systematic effects, we focus on the real-space genus amplitude and set $\beta = 0$. We keep the cosmological parameters fixed and vary the number density $\bar{n}_{\text{cut}}$ and bias $b$. The red and gray filled regions cover the area between two limiting bias values $1.8 < b < 2.2$. The red region corresponds to our fiducial number density choice $\bar{n}_{\text{cut}} = 6.5 \times 10^{-5}$ Mpc$^{-3}$ for the BOSS galaxy sample, and the gray region corresponds to a denser sample $\bar{n}_{\text{cut}} = 6.5 \times 10^{-4}$ Mpc$^{-3}$. The green, black, and blue lines correspond to bias factors $b = 1.8$, 2.0, and 2.2, respectively.

Two conclusions can be drawn from Figure A2. First, shot noise modifies the genus amplitude, and a sparser galaxy sample (red region) will generate a larger genus amplitude than a denser sample (gray region). Second, if the galaxy sample is sufficiently sparse, then the genus amplitude becomes sensitive to galaxy bias (or more precisely the combination $b^2 A_s$)—the width of the filled regions indicates the sensitivity of $A_{G}^{(2D)}$ to bias, for given number density and fixing all other cosmological parameters. For our fiducial number density $\bar{n} = 6.5 \times 10^{-5}$ Mpc$^{-3}$ a $\sim 10\%$ variation in the galaxy bias generates a

![Figure A1](image-url)
\(~0.5\%\) uncertainty in the genus amplitude (see red shaded area). One can also observe a faint redshift evolution in the red shaded region—this is due to the amplitude of the matter power spectrum decreasing with redshift relative to the shot-noise term, which we have fixed to be constant at each redshift. However, if we increase the number density of the galaxy sample by an order of magnitude (gray shaded region), the sensitivity of the genus amplitude to the bias becomes negligible—as the shot noise decreases, the genus amplitude becomes insensitive to the amplitude of the power spectrum. The redshift evolution is also suppressed by selecting a denser sample.

A more general word of caution is required: the shot-noise contribution is not perfectly represented by a white-noise power spectrum with \(P_{\text{SN}} = 1/\bar{n}\). In fact, as we stray into scales at which shot noise affects our results, the expansion of the genus in Hermite polynomials will not provide a good representation of the Poissonian signal, as the Poisson distribution has a different moment-generating function. \(R_G > \bar{r}\), where \(\bar{r}\) is the mean galaxy separation of the catalog, is an important condition on our analysis. Shot noise has been discussed in Appleby et al. (2017) and further in S. Appleby et al. (2020, in preparation).

We note that both the redshift-space distortion effect and shot noise introduce a redshift dependence to the genus amplitude. For the smoothing scales and redshifts used in this work, the redshift-space distortion effect systematically decreases the genus amplitude with increasing redshift. Conversely, the shot-noise contribution increases the genus amplitude with redshift. Both effects are \(~\mathcal{O}(1\%)\) and will act to practically cancel one another.

To completely suppress the shot noise and bias effects, the condition \(\bar{r} \ll R_G\) is required. For the smoothing scales and number densities considered in this work, the genus amplitude is only weakly sensitive to the combination of \(b\), \(A_m\), and \(P_{\text{SN}}\). Nevertheless, we must include the \(P_{\text{SN}}\) contribution to the power spectrum to avoid systematic bias in our cosmological parameter reconstruction.

\section*{Appendix B}

\textbf{Effect of Variation of Cosmological Parameters on Measured Genus Amplitudes}

At numerous points in our analysis when measuring the genus curves from the BOSS galaxy catalog, we have been forced to fix the distance–redshift relation. Specifically, we smoothed the field with constant comoving scale \(R_G = 20\) Mpc, which corresponds to \(\theta_G = R_G/d_L(z)\) angular smoothing on the unit sphere. We measured the genus per unit area, and so divided the genus by the total area \(A_j = 4\pi d^2_L(z_j)\) of each data shell. We also selected redshift shells of thickness 80 Mpc and fixed a constant number density of \(\bar{n}_{\text{av}} = 6.5 \times 10^{-5}\) Mpc\(^{-3}\). Each of these dimension-full operations has forced us to select a distance–redshift relation, which we have taken throughout to be the Planck best-fit, flat \(\Lambda\)CDM cosmology presented in Table 1. In Appendix B we consider how robust our measurements of the genus curve are to such a choice.

To test this, we take an all-sky mock galaxy catalog generated from a known cosmological parameter set and repeat our analysis according to the main body of the paper. We then repeat our analysis again for three different, incorrect cosmological models. For each model, we completely repeat our analysis and compare the resulting genus amplitude measurements.

The data that we use for this test are the Horizon Run 4 all-sky mock galaxy light cones (Kim et al. 2015). Horizon Run 4 is a dense, cosmological scale dark matter simulation in which \(N = 6300^3\) particles in a volume of \(V = (3150\text{ Mpc}/h)^3\) are gravitationally evolved. The simulation uses a modified GOTPM code,\(^9\) and the initial conditions are estimated using second-order Lagrangian perturbation theory (L’Huilier et al. 2014). The cosmological parameters used are \(h = 0.72\), \(n_s = 0.96\), \(\Omega_m = 0.26\), \(\Omega_b = 0.048\). A single all-sky mock galaxy light cone out to \(z = 0.7\) is used in this work. Details of the numerical implementation and the method by which mock galaxies are constructed can be found in Hong et al. (2016). The mock galaxies are defined using the most bound halo particle galaxy correspondence scheme, and the survival time of satellite galaxies post merger is estimated via the merger timescale model described in Jiang et al. (2008).

We begin by repeating the analysis of the paper. Using the correct cosmological parameters \(h = 0.72\), \(\Omega_m = 0.26\) to infer the distance–redshift relation, we bin the galaxies into redshift shells of width \(\Delta = 80\) Mpc and pixels on the sphere. We apply a mass cut to fix the galaxy number density \(\bar{n} = 6.5 \times 10^{-5}\) Mpc\(^{-3}\) in each shell. As for the BOSS data, we take six redshift shells over the range \(0.25 < z < 0.4\) and six over the range \(0.45 < z < 0.6\) and measure the genus curves from these shells. Finally, we extract the genus amplitude \(A^{(2D)}\) from these curves, using the method described in Appleby et al. (2018b).

In Figure B1 we present the genus amplitudes extracted from the mock galaxy light cone in real (blue squares) and redshift (yellow squares) space, for the 12 shells. The blue and yellow dashed lines correspond to the Gaussian theoretical prediction (A1), with \(\beta = 0\) (blue dashed) and \(\beta = \Omega_m^{0.11}/b\) (yellow dashed), taking \(b = 2\).

\footnote{For a description of the original GOTPM code, please see Dubinski et al. (2004). A description of the modifications introduced in the Horizon Run project can be found at https://astro.kias.re.kr/~kjhan/GOTPM/index.html.}
In this appendix we test the impact of the mask on our estimator of the genus. Throughout this work we have estimated the genus of a two-dimensional field on the unit sphere as

$$g_{2D} = \frac{1}{2\pi} \int_{\partial Q} \kappa d\ell,$$

where $\partial Q$ is the excursion set boundary, $d\ell$ is an infinitesimal line element on that boundary, and $\kappa$ is the curvature of $\partial Q$. We have extracted this quantity from masked fields on the two-sphere and equated them to the results of Matsubara (2003), where the genus was calculated for a perturbatively non-Gaussian field in flat space assuming periodic boundary conditions. Hence, two implicit assumptions have been made in our analysis: first, that the Euclidean space expectation value calculated in Matsubara (2003) can be compared to our numerical extraction of the genus from the two-sphere, and second, the survey boundary and mask do not significantly contaminate our estimation of $g_{2D}$.

It is important to stress that definition (C1) of the genus and the Edgeworth expansion (7) are only applicable for a field with no boundary, for example, flat slices with periodic boundary conditions or a field defined on the complete two-sphere. The topology of any field with a boundary will significantly differ from one without (Adler & Taylor 2010; Pranav et al. 2019). However, we are not attempting to extract the genus of the masked field. Rather, one can consider our approach as taking a patch of a full field on the sphere and estimating Equation (C1) from that patch by sampling the excursion set boundary. We must therefore check that our estimator of Equation (C1) is unbiased. We note that this was already done in Schmalzing & Gorski (1998) for the cosmic microwave background and Gaussian random fields. It was found that if one restricts the numerical algorithm to pixels sufficiently distant from the boundary, one can obtain an unbiased estimate of the full-sky genus curve. We now confirm this result for our analysis.

To perform this check, we use the Horizon Run 4 $z = 0$ snapshot box. This is a $3150 Mpc/h^3$ cosmological scale dark matter box with periodic boundary conditions. We confine our analysis to real space and use mock galaxy data with a galaxy mass cut to generate a number density $\bar{n} = 7.5 \times 10^{-5} (\text{Mpc})^{-3}$. The details of the data are not important for this discussion; one could instead use dark matter particle data or even Gaussian random fields.

### Appendix C

**Effect of the Mask**

In this appendix we test the impact of the mask on our estimator of the genus. Throughout this work we have estimated the genus of a two-dimensional field on the unit sphere as

$$g_{2D} = \frac{1}{2\pi} \int_{\partial Q} \kappa d\ell,$$

where $\partial Q$ is the excursion set boundary, $d\ell$ is an infinitesimal line element on that boundary, and $\kappa$ is the curvature of $\partial Q$. We have extracted this quantity from masked fields on the two-sphere and equated them to the results of Matsubara (2003), where the genus was calculated for a perturbatively non-Gaussian field in flat space assuming periodic boundary conditions. Hence, two implicit assumptions have been made in our analysis: first, that the Euclidean space expectation value calculated in Matsubara (2003) can be compared to our numerical extraction of the genus from the two-sphere, and second, the survey boundary and mask do not significantly contaminate our estimation of $g_{2D}$.

It is important to stress that definition (C1) of the genus and the Edgeworth expansion (7) are only applicable for a field with no boundary, for example, flat slices with periodic boundary conditions or a field defined on the complete two-sphere. The topology of any field with a boundary will significantly differ from one without (Adler & Taylor 2010; Pranav et al. 2019). However, we are not attempting to extract the genus of the masked field. Rather, one can consider our approach as taking a patch of a full field on the sphere and estimating Equation (C1) from that patch by sampling the excursion set boundary. We must therefore check that our estimator of Equation (C1) is unbiased. We note that this was already done in Schmalzing & Gorski (1998) for the cosmic microwave background and Gaussian random fields. It was found that if one restricts the numerical algorithm to pixels sufficiently distant from the boundary, one can obtain an unbiased estimate of the full-sky genus curve. We now confirm this result for our analysis.

To perform this check, we use the Horizon Run 4 $z = 0$ snapshot box. This is a $3150 Mpc/h^3$ cosmological scale dark matter box with periodic boundary conditions. We confine our analysis to real space and use mock galaxy data with a galaxy mass cut to generate a number density $\bar{n} = 7.5 \times 10^{-5} (\text{Mpc})^{-3}$. The details of the data are not important for this discussion; one could instead use dark matter particle data or even Gaussian random fields.

### Table B1

| Model | $\Omega_m$ | $h$ |
|-------|------------|-----|
| Fid   | 0.26       | 0.72|
| II    | 0.35       | 0.72|
| III   | 0.35       | 0.677|
| IV    | 0.26       | 0.677|

Note. “Fid” denotes the correct cosmological model used in the simulation.

Next, we repeat our analysis, using three different incorrect cosmological models to infer the distance–redshift relation. The specific models used are labeled II, III, IV in Table B1. For each cosmology the entire process of redshift binning, mass cut, smoothing, and extracting the genus curve and amplitude is repeated. We arrive at a set of 12 genus amplitudes for each cosmological model, which we compare to those obtained using the correct cosmology.

In Figure B2 we present the fractional difference between the genus amplitudes from the wrong cosmology and the “correct” values inferred from the fiducial cosmological parameter set, $\Delta A_{2D}^{(2D)} / A_{\text{fid}}^{(2D)} = (A_{2D}^{(2D)} - A_{\text{fid}}^{(2D)}) / A_{\text{fid}}^{(2D)}$, where $A_{\text{fid}}$ are the correct values. Although one can observe a mild systematic change with cosmology, the effect is at the $\sim 1\%$ level. Specifically, fixing $h = 0.72$ and varying $\Omega_m$ generates a marginally lower genus amplitude (see blue curve). Varying $h$ but selecting the correct value of $\Omega_m = 0.26$ does not produce a definite systematic bias (red curve). In all cases, the statistical scatter of the measurements dominates.
The Astrophysical Journal, 896:145 (18pp), 2020 June 20

Appleby et al.

We first generate flat, two-dimensional fields by taking slices of the cube of thickness $\Delta = 80$ Mpc along an arbitrary $x_3$-direction within the box. The resulting two-dimensional slices are smoothed in the $(x_1, x_2)$ plane using a Gaussian kernel of width $R_G = 20$ Mpc. These fields are periodic, possess no boundary, and so should agree with established theoretical results (Matsubara 2003). The genus is estimated from these slices using a marching squares algorithm outlined in Appleby et al. (2018a). We repeat our analysis for $N = 30$ slices and estimate the mean and standard deviation of the genus curves.

Next, we take a point at the center of the snapshot box and generate concentric shells of data of thickness $\Delta = 80$ Mpc. Ten shells are generated at a comoving distance $1000$ Mpc $< r < 2000$ Mpc from the central point, approximately matching the typical distance of the shells in the LOWZ and CMASS samples. These shells are smoothed with angular scale $\theta_{G,i} = 20/r_i$, where $r_i$ is the comoving radial distance from the center of the box to the center of the $i$th shell. These shells are $4\pi r_i^2$ in extent—that is, they are “all-sky.” We extract the genus per unit area from the shells using the numerical algorithm outlined in Schmalzing & Gorski (1998) and used throughout this work.

Finally, we apply the LOWZ and CMASS masks to the all-sky shells and faithfully repeat our analysis within the main body of the paper. Specifically, we apply the binary masks to the all-sky data, then smooth the data and mask separately, and reapply the smooth mask and then the original mask. We then extract the genus from the masked data.

Our results are presented in Figure C1. In the top panel we exhibit the genus curves extracted from the flat two-dimensional slices (gray shaded region), the $4\pi$, “all-sky” shells (green points/error bars), and masked shells (blue/red points and error bars). The gray shaded region and all error bars represent the $1\sigma$ standard deviation of the slices and shells, respectively. In the bottom panel we extract the amplitudes $A^{(2D)}$ of these curves—the points and error bars are again the mean and standard deviation of the slices and shells. The dashed gray horizontal line in the bottom panel is the Gaussian, analytic prediction (4) from the Edgeworth expansion (7) assuming Euclidean space and periodic boundary conditions.

In both the top and bottom panels, one can observe no significant difference in the genus curves or amplitudes as measured from the flat slices, all-sky shells, and masked shells. This indicates that at the distances probed in this work the flat-sky approximation can be used, and the mask does not bias our reconstruction of the genus curve. We stress that we are not estimating the genus of the masked field, which would be severely affected by the boundary. We are estimating the full-sky genus from a subset of the field. The theoretical, Gaussian prediction (gray horizontal dashed line in the bottom panel) is slightly high relative to the numerical reconstructions, but the difference is less than 1%. This is likely a detection of gravitational smoothing (Melott et al. 1988) and will not significantly impact our estimation of cosmological parameters.

**Appendix D**

**Ancillary Results**

In Appendix D we provide supporting results for completeness. The two-dimensional marginalized 68%/95% contours for the parameter set $\Omega_b h^2$, $n_s$, $\Omega_m h^2$, $a_0$, $a_2$ are presented in Figures D1 (the six LOWZ shells) and D2 (the six CMASS shells). $\Omega_b h^2$ is effectively unconstrained over the prior range taken, and we observe no significant correlation between $a_0$, $a_2$ and $n_s$, $\Omega_m h^2$. The full corner plot for the combined LOWZ and CMASS data is presented in Figure D3.

The 12 density fields used in our analysis are presented in Figures D4 and D5. These maps have been smoothed with angular scale $\theta_G = R_G/d_c(z, \Omega_m, h)$, where $R_G = 20$ Mpc and we have used the Planck cosmological parameters to infer the comoving distance $d_c$ to the center of each slice and projected onto a Cartesian background. The left/right columns correspond to the north/south Galactic data, respectively, and the maps are arrayed in ascending order of redshift. The genus curves extracted from these maps are exhibited in Figure 2 for the LOWZ (top panel) and CMASS (middle panel) shells.
Figure D1. Corner plot of the parameters varied in this work. $\Omega_b h^2$ is effectively unconstrained over its prior range, and $a_0, a_2$ are uncorrelated with the cosmological parameters $\Omega_c h^2, n_s$. The colors of the contours match Figure 2 in the main body of the paper—-we plot six contours that represent the results of each LOWZ shell.
Figure D2. Same as Figure D1, but for the six CMASS data shells.
Figure D3. After combining all six LOWZ (brown contours) and CMASS (blue contours) shells, two-dimensional marginalized contours, fitting a single set of parameters $\Omega_c h^2$, $\Omega_b h^2$, $n_s$, $a_{0,2}$ to all LOWZ (brown contours) and CMASS (blue contours) shells.
Figure D4. Density fields of the LOWZ shells in the north Galactic plane (left panels) and south Galactic plane (right panels), in ascending redshift order. All maps have been smoothed with a Gaussian kernel of width $R_G = 20$ Mpc. The spherical shells have undergone Cartesian projection, and the gray pixels have been masked.
Figure D5. Same as Figure D4, but for the CMASS shells.
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