Abstract. In this paper, we show that for every $N \in \mathbb{N}$, isometric $N$-Jordan operators on Hilbert spaces are power regular. Moreover, the only normaloid or 2-isometric, $N$-Jordan operators are isometries.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ stands for the space of all bounded linear operators on $\mathcal{H}$. An operator $T$ in $B(\mathcal{H})$ is called an isometric $N$-Jordan operator if $T = A + Q$ where $A$ is an isometry, $Q$ is a nilpotent operator of order $N$, that is, $Q^N = 0$ but $Q^{N-1} \neq 0$, and $AQ = QA$. Note that the notions of isometric 1-Jordan and isometry coincide. The dynamic and spectral properties of $N$-Jordan operators have been studied in [7]. It follows from Proposition 1.1 of [7] that these operators are injective. We note that for an $N$-Jordan operator $T$, $T^*T$ is invertible. Indeed, Corollary 1.2 of [7] states that the operator $T$ is bounded below, and so for every $h \in \mathcal{H}$,

$$\|T^*Th\| \geq |\langle T^*Th, h \rangle| = \|Th\|^2 \geq c\|h\|^2$$

for some $c > 0$, which implies that $T^*T$ is also bounded below. Thus, $T^*T$ is surjective, and so is invertible. It is easy to see that if $A$ is a unitary operator then

$$(T^*T)^{-1} = 3I - 3TT^* + T^2T^*.$$
For a positive integer \( m \) an operator \( S \in B(\mathcal{H}) \) is an \( m \)-isometry if

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} S^* S^k = 0.
\]

The operator \( S \) is called a strict \( m \)-isometry if it is \( m \)-isometry but not an \( m - 1 \)-isometry. It is proved in [3] that every isometric \( N \)-Jordan operator is a strict \( (2N - 1) \)-isometry. Moreover, it is shown in [2] that if \( A \) is an \( m \)-isometry then \( A + Q \) is a \( (2N - m - 2) \)-isometry. Recently, such operators have been considered by several authors; for example, see [4, 6].

2. Main results

Recall that an operator \( S \) in \( B(\mathcal{H}) \) is power regular if \( \lim_{n \to \infty} \| S^n h \|^{\frac{1}{n}} \) exists for every \( h \in \mathcal{H} \). It is known that compact operators, normal operators and decomposable operators are power regular (see [1] and references therein). It is important in operator theory that which operators are power regular. Indeed, in this case for each \( h \in \mathcal{H} \), the spectral radius of the restriction of the operator \( S \) to the subspace \( \bigvee_{n=0}^\infty \{ S^n h \} \) is in the closed interval \([a, r(S)]\), where \( a = \lim_{n \to \infty} \| S^n h \|^{\frac{1}{n}} \) and \( r(S) \) is the spectral radius of \( S \). In this section, we will show that if \( T = A + Q \) is an isometric \( N \)-Jordan operator then

\[
\lim_{n \to \infty} \frac{\| T^n \|}{n^{N-1}} = \frac{\| Q^{N-1} \|}{(N-1)!};
\]

in particular, \( T \) is power regular. Also, we see that if \( T \) is normaloid or 2-isometry then it is an isometry. It is proved in [7] that the eigenvectors of an isometric \( N \)-Jordan operator corresponding to distinct eigenvalues are orthogonal. In the next proposition, we generalize this result to distinct approximate eigenvalues. As usual, \( \sigma_{ap}(T) \) denotes the approximate point spectrum of an operator \( T \). Besides, in what follows \( \langle \cdot, \cdot \rangle \) denotes the inner product of the Hilbert space \( \mathcal{H} \).

**Theorem 2.1.** If \( \lambda \) and \( \mu \) are distinct approximate eigenvalues of an isometric \( N \)-Jordan operator \( T = A + Q \), and \( (x_n)_n, (y_n)_n \) are sequences of unit vectors such that \( (T - \lambda)x_n \to 0 \) and \( (T - \mu)y_n \to 0 \), then \( \langle x_n, y_n \rangle \to 0 \).

**Proof.** First note that by induction, for each positive integer \( k \),

\[
\lim_{n \to \infty} (T^k - \lambda^k)x_n = 0;
\]
which in turn implies that \( \lim_{n \to \infty} \| T^k x_n \| = |\lambda|^k \). By Proposition 1 of [7], \( \sigma_{ap}(T) = \sigma_{ap}(A) \) and so \( |\lambda| = 1 \). Therefore, since \( A \) is isometric, for every integer \( k \) we have

\[
\lim_{n \to \infty} \left\| \sum_{i=0}^{N-1} \binom{k}{i} Q^i A^{N-1-i} x_n \right\| = 1. \tag{1}
\]

Consequently,

\[
\lim_{n \to \infty} \left\| \sum_{i=0}^{N-2} \binom{k}{i} Q^{i+1} A^{N-2-i} x_n \right\| = \lim_{n \to \infty} \left\| \sum_{i=0}^{N-1} \binom{k+1}{i} Q^i A^{N-1-i} x_n - \sum_{i=0}^{N-1} \binom{k}{i} Q^i A^{N-1-i} x_n \right\| \leq 2.
\]

In the next step we get

\[
\lim_{n \to \infty} \left\| \sum_{i=0}^{N-3} \binom{k+2}{i} Q^{i+2} A^{N-3-i} x_n \right\| \leq 2^2.
\]

Continuing this process, in the last step we obtain

\[
\lim_{n \to \infty} \| Q^{N-2} A x_n + kQ^{N-1} x_n \| \leq 2^{N-2}.
\]

Therefore,

\[
\limsup_{n \to \infty} \| Q^{N-1} x_n \| \leq \frac{2^{N-2} + \| Q^{N-2} \|}{k},
\]

for every positive integer \( k \). Thus, \( \lim_{n \to \infty} \| Q^{N-1} x_n \| = 0 \). On the other hand, by (1)

\[
\limsup_{n \to \infty} \left\| \sum_{i=0}^{N-2} \binom{k}{i} Q^i A^{N-2-i} x_n \right\| = \limsup_{n \to \infty} \left\| \sum_{i=0}^{N-1} \binom{k}{i} Q^i A^{N-1-i} x_n - \binom{k}{N-1} Q^{N-1} x_n \right\| \leq \limsup_{n \to \infty} \left\| \sum_{i=0}^{N-1} \binom{k}{i} Q^i A^{N-1-i} x_n \right\| + \limsup_{n \to \infty} \left( \binom{k}{N-1} \right) \left\| Q^{N-1} x_n \right\| = 1;
\]

moreover,

\[
\liminf_{n \to \infty} \left\| \sum_{i=0}^{N-2} \binom{k}{i} Q^i A^{N-2-i} x_n \right\| = \liminf_{n \to \infty} \left\| \sum_{i=0}^{N-1} \binom{k}{i} Q^i A^{N-1-i} x_n - \binom{k}{N-1} Q^{N-1} x_n \right\|
\]
\[ \geq \lim_{n \to \infty} \left\| \sum_{i=0}^{N-1} \binom{k}{i} Q^i A^{N-1-i} x_n \right\| - \limsup_{n \to \infty} \left( \frac{k}{N-1} \right) \left\| Q^{N-1} x_n \right\| = 1. \]

Hence
\[ \lim_{n \to \infty} \left\| \sum_{i=0}^{N-2} \binom{k}{i} Q^i A^{N-2-i} x_n \right\| = 1. \]

The same argument shows that \( \lim_{n \to \infty} \| Q^{N-2} x_n \| = 0 \), and by continuing this process we get
\[ \lim_{n \to \infty} \| Q^{N-2} x_n \| = 0 \]

which implies that \( \lim_{n \to \infty} (A - \lambda)x_n = 0 \). Similarly, \( \lim_{n \to \infty} (A - \mu)y_n = 0 \). Thus Cauchy-Schwarz inequality shows that
\[ 0 = \lim_{n \to \infty} \langle (A^* A - I) x_n, y_n \rangle = (\lambda \mu - 1) \lim_{n \to \infty} \langle x_n, y_n \rangle, \]

hence \( \lim_{n \to \infty} \langle x_n, y_n \rangle = 0 \).

**Theorem 2.2.** If \( T = A + Q \) is an isometric \( N \)-Jordan operator, then
\[ \lim_{n \to \infty} \frac{\| T^n \|}{n^{N-1}} = \frac{\| Q^{N-1} \|}{(N-1)!}. \]

**Proof.** An application of the polar identity shows that if \( h_1, h_2, ..., h_{N-1} \) are in \( \mathcal{H} \), then \( \left\| \sum_{i=0}^{N-1} \binom{n}{i} h_i \right\|^2 \) is a polynomial in \( n \) of degree at most \((N-1)^2\). So for every \( h \in \mathcal{H} \) and \( n \geq 1 \),
\[ \| T^n h \|^2 = \left\| \sum_{i=0}^{N-1} \binom{n}{i} A^{N-1-i} Q^i h \right\|^2 = \sum_{i=0}^{(N-1)^2-1} a_i n^i + \left( \frac{n}{N-1} \right)^2 \| Q^{N-1} h \|^2 \]

for some coefficients \( a_i \), \( 1 \leq i \leq (N-1)^2 - 1 \). Thus,
\[ \left| \frac{\| T^n h \|^2}{\left( \frac{n}{N-1} \right)^2} - \| Q^{N-1} h \|^2 \right| = \left| \frac{\sum_{i=0}^{(N-1)^2-1} a_i n^i}{\left( \frac{n}{N-1} \right)^2} \right| \]

which converges uniformly to zero on \( S = \{ h : \| h \| \leq 1 \} \) because there is a positive constant \( c \) such that \( |a_i| \leq c \) for all \( h \in S \) and \( i = 0, 1, ..., (N-1)^2 - 1 \).
1)² − 1. It follows that
\[ \lim_{n \to \infty} \frac{\|T^n\|^2}{(nN-1)^2} = \lim_{n \to \infty} \sup_{\|h\| \leq 1} \frac{\|T^n h\|^2}{(nN-1)^2} = \sup_{\|h\| \leq 1} \lim_{n \to \infty} \frac{\|T^n h\|^2}{(nN-1)^2} = \sup_{\|h\| \leq 1} \|Q^{N-1} h\|^2 = \|Q^{N-1}\|^2. \]

Consequently,
\[ \lim_{n \to \infty} \frac{\|T^n\|}{n^{N-1}} = \frac{\|Q^{N-1}\|}{(N-1)!}. \]

\[ \square \]

**Corollary 2.3.** Every isometric $N$-Jordan operator $T = A + Q$ is power regular. Moreover, for every nonzero $h \in \mathcal{H}$, the spectral radius of the restriction of the operator $T$ to the subspace $\bigvee_{n=0}^\infty \{T^n h\}$ which is denoted by $r_h(T)$ is 1.

**Proof.** As we have observed in the proof of the above theorem, for every $h \in \mathcal{H}$
\[ \lim_{n \to \infty} \frac{\|T^n h\|}{(nN-1)} = \|Q^{N-1} h\|. \]

Hence, $\lim_{n \to \infty} \|T^n h\|^{\frac{1}{n}}$ is 0 or 1. Moreover, by the spectral radius formula, for every operator $S$ in $B(\mathcal{H})$, $r(S) = \lim_{n \to \infty} \|S^n\|^\frac{1}{n}$ hence for every nonzero $h \in \mathcal{H}$
\[ \lim_{n \to \infty} \|T^n h\|^{\frac{1}{n}} \leq r_h(T) \leq r(T). \]

On the other hand it follows from Proposition 1.1 of [7] that $r(T) = 1$ thus $r_h(T) = 1$.

\[ \square \]

**Corollary 2.4.** Suppose that $T = A + Q$ is an isometric $N$-Jordan operator such that $A$ is invertible. Then $T(\ker Q^{N-1}) = \ker Q^{N-1}$.

**Proof.** First note that since $TQ = QT$ we have $T(\ker Q^{N-1}) \subseteq \ker Q^{N-1}$. Moreover, by Proposition 1.1 of [7] the operator $T$ is invertible; so for every $h \in \ker Q^{N-1}$ there exists $g \in \mathcal{H}$ such that $h = Tg$. Thus,
\[ \frac{\|T^n g\|}{n^{N-1}} = \frac{\|T^{n-1} h\|}{n^{N-1}} \leq \|T^{-1}\| \frac{\|T^n h\|}{n^{N-1}}. \]
By the proof of the previous theorem,
\[ \lim_{n \to \infty} \frac{\|T^n h\|}{n^{N-1}} = \frac{\|Q^{N-1} h\|}{(N-1)!} = 0 \]
which implies that
\[ \frac{\|Q^{N-1} g\|}{(N-1)!} = \lim_{n \to \infty} \frac{\|T^n g\|}{n^{N-1}} = 0. \]
Hence, \( \ker Q^{N-1} \subseteq T(\ker Q^{N-1}) \).

In the sequel, we will show that the only isometric \( N \)-Jordan normaloid or 2-isometric operators are isometries. Note that an operator \( S \) is normaloid if \( r(S) = \|S\| \).

**Proposition 2.5.** Suppose that the operator \( T = A + Q \) is an isometric \( N \)-Jordan operator. If \( T \) is normaloid or 2-isometry then \( N = 1 \).

**Proof.** If \( T \) is normaloid by Proposition 1.1 of [7], \( r(T) = r(A) = 1 \) which implies that \( \|T\| = 1 \). Hence \( T \) is a power bounded operator. Now, Proposition 1.6 of [7] says that \( N = 1 \). Now suppose that \( T \) is a 2-isometry and \( N > 1 \). We observe that

\[
0 = \sum_{k=0}^{2} (-1)^k \binom{2}{k} (A^* + Q^*)^k (A + Q)^k
\]
\[
= A^2Q^2 + 2Q^*Q + 2A^*Q^*Q^2 + Q'^2A^2 + 2Q^*AQ + Q'^2Q^2.
\]

For the simplicity call the last statement \( \Delta \). Thus,
\[
0 = Q^{N-2} \Delta Q^{N-2} = 2Q^{N-1}Q^{N-1}
\]
which implies that \( Q^{N-1} = 0 \). Thus, we get a contradiction.

Recall that an operator \( S \) in \( B(H) \) is \( k \)-paranormal, if \( \|S^{k+1} h\| \geq \|Sh\|^{k+1} \) and is \( * \)-paranormal if \( \|S^2 h\| \geq \|S' h\|^2 \) for all unit vectors \( h \). These operators are normaloid (see [5]). Thus, as a corollary of the preceding proposition the only isometric \( N \)-Jordan, \( k \)-paranormal or \( * \)-paranormal operators are isometries.

The commutativity of \( A \) and \( Q \) is essential in the preceding proposition as the following example shows.

**Example 2.6.** Define the operators \( A \) and \( Q \) on \( l^2(\mathbb{C}) \) by \( A(\alpha_1, \alpha_2, \cdots) = (0, \alpha_1, \alpha_2, \cdots) \) and \( Q(\alpha_1, \alpha_2, \cdots) = (0, -2\alpha_1, 0, 0, \cdots) \). Observe that \( Q^2 = 0, Q \neq 0 \) and \( A \) is normaloid but \( AQ \neq QA \).
We note that if $T = A + Q$ is an isometric $N$-Jordan $m$-isometry with $m \geq 3$, then $N$ is not necessarily 1. Indeed, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then it is easy to see that $T = A + Q$ is a 3-isometry on the space $\mathbb{C} \oplus \mathbb{C}$, $Q^2 = 0$ and $AQ = QA$.

References

[1] A. Atzmon, Power regular operators, Trans. Amer. Math. Soc., 347(1995), 3101-3109.
[2] T. Bermúdez, A. Martionón, V. Müller, J. A. Noda, Perturbation of $m$-isometries by nilpotent operators, Abstract and Applied Analysis, 2014; Article ID 745479, 6 pages.
[3] T. Bermúdez, A. Martionón, J. A. Noda, An isometry plus a nilpotent is an $m$-isometry. Applications, J. Math. Anal. Appl., 407(2013), 505-512.
[4] G. Gu and M. Stankus, Some results on higher order isometries and symmetries: products and sums with a nilpotent operator, Linear Algebra Appl. 469(2015), 500-509.
[5] C. S. Kubrusly, B. P. Duggal, A note on $k$-paranormal operators, Operators and Matrices 4(2010), 213-223.
[6] T. Le, Algebraic properties of operators roots of polynomials, J. Math. Anal. Appl. 421(2015) 1238-1246.
[7] S. Yarmahmoodi, K. Hedayatian, B. Yousefi, Supercyclicity and hypercyclicity of an isometry plus a nilpotent, Abstract and Applied Analysis, 2011; Article ID 686832, 11 pages.

Karim Hedayatian
Department of Mathematics, College of Sciences, Shiraz University, Shiraz 7146713565, Iran
hedayati@shirazu.ac.ir(khedayatian@gmail.com)

Saeed Yarmahmoodi
Department of Mathematics, Marvdasht University, Islamic Azad University, Fars, Iran
saedyarmahmoodi@gmail.com