ASSOCIATIVE TRIPLES AND YANG-BAXTER EQUATION

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ABSTRACT. We introduce triples of associative algebras as a tool for building solutions to the Yang-Baxter equation. It turns out that the class of R-matrices thus obtained is related to a Hecke-like condition, which is formulated for associative algebras with symmetric cyclic inner product. R-matrices for a subclass of the $A_n$-type Belavin-Drinfel’d triples are derived in this way.

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1. Introduction.

The canonical Faddeev-Reshetikhin-Takhtajan recipe for constructing quantum matrix groups [RTF] is based on a solution to the Yang-Baxter equation (YBE), which is assumed to be a priori known. On the other hand, the theory of quantum groups was designed as an environment for constructing such solutions [D] being of interest for mathematics and physics. The quasi-classical limit of YBE, the classical Yang-Baxter equation (cYBE), has clear algebraic interpretation in terms of Manin triples, which have been classified for semisimple Lie algebras in [BD]. Transition from Manin triples to quantum universal enveloping algebras brings about the notions of quantum double and universal solution of YBE [D]. The possibility to quantize an arbitrary Manin triple has been proven in [EK], and for universal R-matrices there were derived rather complicated although explicit formulas, in the semisimple case, [ESS]. At the same time, one can raise the question what structure particulars of an associative ring itself may be responsible for YBE, without appealing to the

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intricate technique of the Hopf algebra theory. An explicit formula for \( sl_n(\mathbb{C}) \) R-matrices, known as the GGS conjecture, was proposed in [GGS]. It has been confirmed [GH, Sch1] in many cases and a combinatorial proof was recently found in [Sch2]. In the present paper, we pursue another approach making use of cyclic inner product on associative algebras and keeping an analogy with Manin triples. Such a point of view has led us to definition of associative triples, which allow to explain many examples and quantize a wide class of Belavin-Drinfel’d triples associated to the special linear Lie algebras. In associative triples, R-matrices naturally into the sum of two solutions to YBE, one of them being a part of the canonical element of the cyclic inner product and the other belonging to a smaller subalgebra. That ”smaller” solution should satisfy a Hecke-like condition. The problem of building R-matrices in a given algebra is thus reduced to finding an associative triple (if that possible) with its total algebra as the homomorphic pre-image. For \( \text{Mat}_n(\mathbb{C}) \), we take a proper extension of \( \text{Mat}_n(\mathbb{C}) \oplus \text{Mat}_n(\mathbb{C}) \) for the role of such a pre-image.

2. Cyclic inner product and YBE. Throughout the paper, we assume \( \mathcal{M} \) to be a finite dimensional associative algebra with unit over \( \mathbb{C} \). Nevertheless, there are interesting infinite-dimensional examples that can be understood in the framework of our construction. Arbitrary fields are also admissible, and presence of unit is not necessary in certain cases.

The Yang-Baxter equation in \( \mathcal{M} \otimes^3 \) is

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},
\]

where \( R \in \mathcal{M} \otimes^2 \) and the subscripts specify the way of embedding \( \mathcal{M} \otimes^2 \) into \( \mathcal{M} \otimes^3 \). Supposing \( R \) being a deformation of the unit, \( R = 1 \otimes 1 + \lambda r + o(\lambda) \), \( \lambda \in \mathbb{C} \), the element \( r \) satisfies the classical Yang-Baxter equation

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.
\]

The square brackets mean the commutator \( [\alpha_1, \alpha_2] = \alpha_1\alpha_2 - \alpha_2\alpha_1 \), \( \alpha_i \in \mathcal{M} \).

Suppose \( \mathcal{M} \) is endowed with a non-degenerate symmetric cyclic inner product \( (\cdot, \cdot) \), that is \( (\alpha_1 \alpha_2, \alpha_3) = (\alpha_2, \alpha_3 \alpha_1) \) for all \( \alpha_i \in \mathcal{M} \). Choose a basis \( \{\alpha_i\} \in \mathcal{M} \) and let \( \{\alpha^j\} \) be its dual: \( (\alpha_i, \alpha^j) = \delta^j_i \) (the Kronecker symbols).

**Definition 1.** The element \( \sigma_{\mathcal{M}} = \alpha_i \otimes \alpha^i \in \mathcal{M} \otimes^2 \) (summation over repeating upper and lower indices understood throughout the paper) is called *permutation* in \( \mathcal{M} \).

For unital algebras, cyclic inner products are in one-to-one correspondence with linear functionals \( t_{\mathcal{M}} \) obeying \( t_{\mathcal{M}}(\alpha \beta) = t_{\mathcal{M}}(\beta \alpha) = (\alpha, \beta) \). In the endomorphism algebra of the vector space \( \mathbb{C}^n \) with the trace functional, the permutation is the flip operator: \( x \otimes y \rightarrow y \otimes x \in \mathbb{C}^n \otimes \mathbb{C}^n \).

**Proposition 1 ([BFS]).** *Permutation satisfies YBE.* For all \( \alpha \in \mathcal{M} \)

\[
(\alpha \otimes 1)\sigma_{\mathcal{M}} = \sigma_{\mathcal{M}}(1 \otimes \alpha), \quad (1 \otimes \alpha)\sigma_{\mathcal{M}} = \sigma_{\mathcal{M}}(\alpha \otimes 1).
\]
Remark that associative algebras with a functional $t$ such that the bilinear form $\mathcal{M} \otimes \mathcal{M} \to k$, $\alpha_1 \otimes \alpha_2 \to t(\alpha_1 \alpha_2)$, is non-degenerate are called Frobenius. For those algebras, the canonical element fulfills the braid relation too, in analogy with the conventional matrix permutation [BFS].

Recall that a Manin triple $(\mathfrak{g}, \mathfrak{a}, \mathfrak{a}^*)$ comprises a Lie algebra $\mathfrak{g}$ with an ad-invariant inner product and its two lagrangian (maximal isotropic) Lie subalgebras $\mathfrak{a}$ and $\mathfrak{a}^*$ with zero intersection. The canonical element $\alpha_i \otimes \beta^i \in \mathfrak{a} \otimes \mathfrak{a}^*$ solves cYBE. To find a possible "quantification" of this construction, consider first the situation when a Manin triple $(\mathcal{M}, \mathfrak{A}, \mathfrak{A}^*)$ is formed by the commutator Lie algebras of associative algebras, and the ad-invariant 2-form is induced by a non-degenerate cyclic inner product. Consider $\mathfrak{A}$ and $\mathfrak{A}^*$ as bimodules for each other, the left ($\triangleright$) and right ($\triangleleft$) actions being induced via duality from multiplication. The product in $\mathcal{M}$ is expressed by the formula

$$\alpha \beta = \alpha \triangleright \beta + \alpha \triangleleft \beta, \quad \beta \alpha = \beta \triangleright \alpha + \beta \triangleleft \alpha, \quad \alpha \in \mathfrak{A}, \quad \beta \in \mathfrak{A}^*. \tag{3}$$

Associativity is encoded in the following two equations:

$$(\alpha_1 \triangleright \beta_1, \alpha_2 \triangleleft \beta_2) + (\alpha_1 \triangleleft \beta_1, \alpha_2 \triangleright \beta_2) = (\beta_1 \triangleright \alpha_2, \beta_2 \triangleleft \alpha_1) + (\beta_1 \triangleleft \alpha_2, \beta_2 \triangleright \alpha_1),$$

$$(\alpha_2 \alpha_1, \beta_1 \beta_2) = (\beta_2 \alpha_2, \alpha_1 \triangleright \beta_1) + (\alpha_1 \triangleleft \beta_1, \beta_2 \triangleleft \alpha_2), \quad \alpha_i \in \mathfrak{A}, \quad \beta_i \in \mathfrak{A}^*. \tag{4}$$

Conversely, the linear sum $\mathfrak{A} \oplus \mathfrak{A}^*$ is equipped with the structure of an associative algebra (3), provided the actions $\triangleleft$ and $\triangleright$ satisfy these two conditions. Then, the following statement holds true.

**Proposition 2.** Natural pairing between $\mathfrak{A}$ and $\mathfrak{A}^*$ induces a non-degenerate cyclic inner product on $\mathcal{M} = \mathfrak{A} \oplus \mathfrak{A}^*$ such that $\mathfrak{A}$ and $\mathfrak{A}^*$ are isotropic. The canonical element $\alpha_i \otimes \alpha^i \in \mathfrak{A} \otimes \mathfrak{A}^* \subset \mathcal{M} \otimes^2$ of the pairing satisfies YBE.

**Proof.** The first statement is immediate. Verification of the second is conducted in the next section in a more general setting $\blacksquare$

For the above construction, unit is not necessary. But if it is present, the sum

$$1 \otimes 1 + \lambda \alpha_i \otimes \alpha^i \tag{4}$$

is also a solution to YBE for an arbitrary value of the scalar parameter $\lambda$ (cf. concluding remarks). This is true because $(\mathcal{M}, \mathfrak{A}, \mathfrak{A}^*)$ is a Manin triple of the corresponding commutator Lie algebras. Associativity imposes strong restrictions on the algebras $\mathfrak{A}$ and $\mathfrak{A}^*$; for instance, they cannot be simultaneously unital or they would consist of the zero elements only. Nevertheless, there is a non-trivial example, rather infinite dimensional, describing the XXX-spin chains in the theory of integrable models [F,KS]. That is the Yang matrix

$$R(z, u) = 1 \otimes 1 + \lambda \frac{P}{z - u},$$

where $P$ is the identity matrix.
where $\mathcal{P}$ is the conventional permutation operator acting on $\mathbb{C}^n \otimes \mathbb{C}^n$. The element $\frac{1}{z-u}$ is a brief way of writing the formal power series $\sum_{k \geq 0} \frac{u^k}{z^{k+1}}$. It represents the canonical element of pairing $\text{Res}_0$ between $\mathbb{C}[z]$ and $\mathbb{C}[\frac{1}{z}]$. In this example, the algebra $\mathcal{M}$ is formed by the Laurent polynomials with matrix coefficients, but $R(z, u)$ requires extension by the Laurent series.

Propositions 1 and 2 can be understood with the help of the following construction. Let $\mathcal{N}_\pm$ be two linear subspaces in $\mathcal{M}$ paired via the inner product. There exist two bijections $\mathcal{N}_\pm \rightarrow \mathcal{N}_+^\pm$ inducing projectors $\pi_\pm: \mathcal{M} \rightarrow \mathcal{N}_\pm$. If $\{\alpha_i\}$ is a basis in $\mathcal{N}_+$ and $\{\beta_i\}$ its dual in $\mathcal{N}_-$, then $\pi_+$ is given by $\mu \rightarrow \alpha_i(\mu, \beta^i)$. The projector $\pi_-$, the conjugate to $\pi_+$ with respect to the inner product, acts as $\mu \rightarrow (\alpha_i, \mu)\beta^i$. Denote $\mathcal{M}_\pm = \{\mu|\pi_\pm(\alpha_i\mu)\alpha_2 = \alpha_1\pi_\pm(\mu\alpha_2), \alpha_1, \alpha_2 \in \mathcal{N}_\pm\}$. The subspaces $\mathcal{M}_\pm$ contain, in particular, the normalizers for $\mathcal{N}_\pm$, i.e. the maximal subalgebras in $\mathcal{M}$ for which $\mathcal{N}_\pm$ are bimodules. In particular, $\mathcal{N}_\pm \subset \mathcal{M}_\pm$ if $\mathcal{N}_\pm$ are subalgebras.

**Proposition 3.** Suppose $\mathcal{M} = \mathcal{M}_+ + \mathcal{M}_-$ as a linear space. The canonical element $Q = \alpha_i \otimes \beta^i \in \mathcal{N}_+ \otimes \mathcal{N}_- \subset \mathcal{M}^{\otimes 2}$ satisfies YBE.

**Proof.** It will be sufficient to evaluate the middle tensor components of YBE on elements of $\mathcal{M}_+$ and $\mathcal{M}_-$ separately, since they span entire $\mathcal{M}$. For arbitrary $\mu \in \mathcal{M}_+$, the Yang-Baxter equation

$$\alpha_i\alpha_j \otimes (\beta^i\alpha_k, \mu) \otimes \beta^j\beta^k = \alpha_j\alpha_i \otimes (\alpha_k\beta^i, \mu) \otimes \beta^k\beta^j$$

is rewritten as

$$\alpha_i\alpha_j \otimes (\beta^i, \alpha_k\mu) \otimes \beta^j\beta^k = \alpha_j\alpha_i \otimes (\beta^i, \mu\alpha_k) \otimes \beta^k\beta^j$$

and therefore

$$\pi_+(\alpha_k\mu)\alpha_j \otimes \beta^j\beta^k = \alpha_j\pi_+(\mu\alpha_k) \otimes \beta^k\beta^j.$$

Similarly one verifies YBE by pairing with elements of $\mathcal{M}_-$. ■

The above statement is an apparent generalization of Propositions 1 and 2. In the first case, one has $\mathcal{N}_\pm = \mathcal{M}_\pm = \mathcal{M}$ and, in the second, already $\mathcal{M}_+ + \mathcal{M}_-$ gives entire $\mathcal{M}$. Let us present another example, quite exotic, where the normalizers for $\mathcal{N}_\pm$ are very small whereas both $\mathcal{M}_\pm$ coincide with the whole $\mathcal{M}$. Take $\mathcal{M}$ to be the algebra of $n \times n$ matrices and denote $e^i_k$ the matrix units. Let $\sigma$ be a permutation of the set of indices $I = \{1, \ldots, n\}$. Put $\mathcal{N}_+ = \text{Span}\{e^i_i\}_{i \in I}$ and $\mathcal{N}_- = \text{Span}\{e^i_i\}_{i \in I}$; then the canonical element with respect to the trace pairing reads $Q = \sum_{i \in I} e^i_i \otimes e^i_i$. Now observe that $\mathcal{M}_+ = \mathcal{M}$. Indeed, for any matrix $u = u^i_k e^i_k \in \mathcal{M}$ one has $\pi_+(ue^i_i) = u^i_i e^i_i$ and $\pi_+(e^i_i u) = u^i_i e^i_i$. So one gets the identity

$$\pi_+(e^i_i u) e^k_k = u^i_i e^k_k e^2_k = u^2_k e^k_k e^2_k = e^i_i \pi_+(ue^k_k).$$

Note that the normalizers for $\mathcal{N}_\pm$ consist of diagonal matrices only.

Proposition 3 supplies one with solutions that seem to be quite distant from those related to quantum groups. We view it as a tool for constructing parts of $R$-matrices of real interest.
3. Associative triples.

From now on we assume the subspaces $\mathfrak{N}_\pm$ to be subalgebras. Suppose $\mathfrak{M}$ can be represented as the linear sum of three subalgebras $\mathfrak{N}_- + \mathfrak{D} + \mathfrak{N}_+$ such that

i) $\mathfrak{N}_-$ is dual to $\mathfrak{N}_+$ with respect to the inner product,

ii) $\mathfrak{N}_\pm$ are $\mathfrak{D}$-bimodules,

iii) $\mathfrak{D}$ is orthogonal to $\mathfrak{N}_- + \mathfrak{N}_+$.

In the sequel, $\mathfrak{M}_\pm$ will stand for the subalgebras $\mathfrak{D} + \mathfrak{N}_\pm$.

**Definition 2.** The set $(\mathfrak{M}, \mathfrak{M}_+ \mathfrak{M}_-)$ is called **associative triple** with diagonal $\mathfrak{D} = \mathfrak{M}_+ \cap \mathfrak{M}_-$. A triple is called **trivial** if it coincides with its diagonal and **disjoint** if $\mathfrak{D} = \{0\}$.

It follows from condition iii) that restriction of the cyclic form to $\mathfrak{D}$ is nondegenerate. Product of two elements from $\mathfrak{N}_\pm \mathfrak{N}_\mp$ may contribute to every part of $\mathfrak{M}$, in analogy with (3), including the diagonal. Thus associative triples generalize decomposition of $\text{Mat}_n(\mathbb{C})$ into the diagonal, strictly upper and lower triangular matrix subalgebras. Let us give one more definition before formulating the basic statement of the paper.

**Definition 3.** An element $S \in \mathfrak{M} \otimes^2$, where $\mathfrak{M}$ is an algebra with a non-degenerate inner product, is said to satisfy **Hecke condition** in $\mathfrak{M} \otimes^2$ if

$$S_{21}S - \sigma_{\mathfrak{M}}S = \lambda^2$$

with some scalar $\lambda$.

It is convenient to represent $\frac{1}{\lambda} = \omega = q - q^{-1}$ assuming $q^2 \neq 1$ and $\lambda \neq 0$. For a matrix algebra $\mathfrak{M}$, this is the conventional Hecke condition. Moreover, that is the only case when the permutation $\sigma_{\mathfrak{M}}$ is invertible [BFS]. Then, $\sigma_{\mathfrak{M}}^2 = 1$ and one can combine $S$ with $\sigma_{\mathfrak{M}}$ getting a close quadratic equation on $\sigma_{\mathfrak{M}}S$.

**Theorem 1.** The canonical element $Q \in \mathfrak{N}_+ \otimes \mathfrak{N}_-$ with respect to the cyclic inner product satisfies YBE. The element $R = S + Q \in \mathfrak{M} \otimes^2$, where $S \in \mathfrak{D} \otimes^2$, is a solution to YBE if $S$ is a solution to YBE fulfilling the Hecke condition in $\mathfrak{D} \otimes^2$.

**Proof.** The first assertion follows from Proposition 3 because $\mathfrak{N}_\pm$ are bimodules for $\mathfrak{M}_\pm$ and $\mathfrak{M}_+ + \mathfrak{M}_- = \mathfrak{M}$. Verification of YBE is reduced to checking

$$S_{12}Q_{13}Q_{23} + Q_{12}S_{13}Q_{23} + Q_{12}Q_{13}S_{23} + S_{12}Q_{13}S_{23}$$

$$= Q_{23}Q_{13}S_{12} + Q_{23}S_{13}Q_{12} + S_{23}Q_{13}Q_{12} + S_{23}Q_{13}S_{12}.$$
\[ \alpha_j \delta_i \otimes \alpha_k \tilde{\beta}^i \otimes \beta^k \beta^j + \delta_j \alpha_i \otimes \alpha_k \beta_i \otimes \beta^k \tilde{\beta}^j + \alpha_j \delta_i \otimes \delta_k \beta_i \otimes \tilde{\beta}^k \beta^j + \alpha_j \delta_i \otimes \delta_k \tilde{\beta}^i \otimes \tilde{\beta}^k \beta^j \]

Since \( \mathcal{M} \) is spanned by \( \mathcal{N}_\pm \) and \( \mathcal{D} \), it suffices to check this identity separately pairing the middle tensor component with their elements.

Step 1. Pairing with \( \alpha \in \mathcal{N}_+ \).

\[ 0 + \alpha_k \alpha_j \tilde{\beta}^i \beta^k + \delta_k \alpha_j \beta^i \tilde{\beta}^j + 0 = 0 + \delta_j \alpha \alpha_k \beta^i \tilde{\beta}^j + \alpha_j \alpha \delta_k \beta^i \tilde{\beta}^j + 0. \]

Step 2. Pairing with \( \beta \in \mathcal{N}_- \).

\[ \delta_i \alpha_j \beta^i \tilde{\beta}^j + \alpha_i \delta_j \beta^i \tilde{\beta}^j + 0 + 0 = \alpha_j \delta_i \beta^i \tilde{\beta}^j + \delta_j \alpha_i \beta^i \tilde{\beta}^j + 0 + 0. \]

Step 3. Pairing with \( \delta \in \mathcal{D} \). The first and third terms on each sides turn zero. In the fourth terms, we perform the substitution \( S \rightarrow \sigma \mathcal{D} \) of the last factors employing the Hecke condition. For example,

\[ S_{12} Q_{13} S_{23} = S_{12} S_{21} Q_{13} = (S_{12} (\sigma \mathcal{D})_{12} + \lambda^2) Q_{13} = S_{12} Q_{13} (\sigma \mathcal{D})_{23} + \lambda^2 Q_{13}. \]

The last term will appear on both sides of the equation and thus vanish. The resulting equation is

\[ 0 + \alpha_i \delta_j \otimes \tilde{\beta}^i \beta^j + 0 + \alpha_i \delta_j \otimes \beta^i \tilde{\beta}^j + 0 + \alpha_j \delta_i \otimes \tilde{\beta}^i \beta^j, \]

and it holds identically ■

Note that representation of the Cremmer-Gervais R-matrix [CG] by the sum of two solutions to YBE was used in [H1]. Introduced in this paper, associative triples is an algebraic scheme adopted to solving the specific system of equations arising from such a representation.

Associative triples form a category \( \mathcal{AT} \), with the subcategory \( \mathcal{AT}_0 \) of trivial triples. Morphisms in \( \mathcal{AT} \) are algebraic maps preserving elements of triples. The category \( \mathcal{AT} \) admits the following operations with its objects:

1. Transposition \( \mathcal{M}' \).

\[ (\mathcal{M}, \mathcal{M}_+, \mathcal{M}_-)' = (\mathcal{M}, \mathcal{M}_-, \mathcal{M}_+), \quad t_{\mathcal{M}'} = t_{\mathcal{M}}. \]

2. Sum \( \mathcal{M}^1 \oplus \mathcal{M}^2 \).

\[ (\mathcal{M}^1 \oplus \mathcal{M}^2, \mathcal{M}^1_+ \oplus \mathcal{M}^2_+, \mathcal{M}^1_- \oplus \mathcal{M}^2_-), \quad t_{\mathcal{M}^1 \oplus \mathcal{M}^2} = t_{\mathcal{M}^1} \oplus t_{\mathcal{M}^2}. \]

3. Product by objects of \( \mathcal{AT}_0 \).

\[ (\mathcal{A} \otimes \mathcal{M}, \mathcal{A} \otimes \mathcal{M}_+, \mathcal{A} \otimes \mathcal{M}_-), \quad t_{\mathcal{A} \otimes \mathcal{M}} = t_{\mathcal{A}} \otimes t_{\mathcal{M}}. \]
4. Double \((D(\mathcal{M}), D(\mathcal{M})_+, D(\mathcal{M})_-)\).

\[
D(\mathcal{M}) = \mathcal{D} \oplus \mathcal{M} \oplus \mathcal{M}, \\
D(\mathcal{M})_+ = \mathcal{D} \oplus \mathcal{M} \oplus \text{id}(\mathcal{M}), \\
D(\mathcal{M})_- = \mathcal{D} \oplus \mathcal{D} \oplus \text{id}(\mathcal{D}) + \{0\} \oplus \mathcal{N}_+ \oplus \mathcal{N}_-.
\]

In the definition of \(D(\mathcal{M})_-\) we identify the first and last copies of \(\mathcal{D}\). The diagonal here is \(\mathcal{D} \oplus \text{id}(\mathcal{D}) \oplus \text{id}(\mathcal{D})\). The functional \(t_D\) is induced by \(t_{\mathcal{M}}\), and the restriction of \(t_{D(\mathcal{M})}\) to the last addend coincide with \(-t_{\mathcal{M}}\) (that is reflected by the notation).

5. Skew double \(S(\mathcal{M})\). This is a disjoint triple \(\mathcal{M} + \mathcal{M}^*\) of an algebra and its dual equipped with nil multiplication. Formula (3) degenerates into the dual regular actions of \(\mathcal{M}\) on \(\mathcal{M}^*\).

Introduction of associative triples is motivated by the idea of solving YBE in a smaller algebra and to create a tool for the induction process. Propositions 1 and 2 describe two extreme cases of trivial and disjoint triples providing quite exotic examples. To find more interesting solutions, one has to admit non-trivial \(\mathcal{D}\) and \(\mathcal{N}_\pm\) simultaneously. Such applications are considered in the rest of the paper, and this section is finished with the following statement.

**Proposition 4.** Let \(\mathcal{M}\) be an associative triple with the diagonal \(\mathcal{D}\) and \(S\) satisfies the Hecke condition in \(\mathcal{D} \otimes 2\). Then, the element \(R = S + Q\) satisfies the Hecke condition in \(\mathcal{M}\) if and only if \(\sigma_D Q + Q^2 = 0\).

**Proof.** Taking into account \(S_{12}Q = QS\) and \(\sigma_{\mathcal{D}} = \sigma_D + Q + Q_{21}\), one has

\[
R_{21}R = S_{21}S + S_{21}Q + QS + Q_{21}Q = \lambda(1 \otimes 1) + \sigma_{\mathcal{D}} S + Q_{21}Q = \lambda(1 \otimes 1) + \sigma_{\mathcal{D}} R - \sigma_D Q - Q^2
\]

as required. \(
\)

4. Examples.

1. Drinfeld-Jimbo R-matrix for \(\mathfrak{sl}_n(\mathbb{C})\) [J].

One can build, by recursion,

\[
R_n = q \sum_{i=1}^{n} e_i^i \otimes e_i^i + \sum_{i,j=1}^{n} e_i^i \otimes e_j^j + \omega \sum_{i<j=1}^{n} e_i^i \otimes e_j^j, \quad \omega = q - q^{-1},
\]

if assumes \(\mathcal{M} = \text{Mat}_n(\mathbb{C})\) with the ordinary matrix trace, \(\mathcal{M}_+ = \sum_{i,j=1}^{n-1} \mathcal{C} e_j^i + \sum_{i=1}^{n} \mathcal{C} e_n^i\), and \(\mathcal{M}_- = \sum_{i,j=1}^{n-1} \mathcal{C} e_j^i + \sum_{i=1}^{n} \mathcal{C} e_i^n\). So the subalgebra \(\mathcal{M}_+\) is formed by the last matrix column without the bottom entry. Its dual \(\mathcal{M}_-\) is spanned by the bottom matrix line except the right-most diagonal element. The diagonal subalgebra \(\mathcal{D}\) is \(\text{Mat}_{n-1}(\mathbb{C}) \oplus \mathcal{C} e_n^n\). The one-dimensional R-matrix \(R_1 = q e_n^n \otimes e_n^n\) fulfills the Hecke condition in \(\text{Mat}_1(\mathbb{C})\). By induction
assumption, that holds for the matrix $R_{n-1}$. The direct sum $R_{n-1} + R_1$ satisfies the Yang-
Baxter equation but not the Hecke condition since the unit matrix $1_n \otimes 1_n$ is not equal to the 
sum $1_{n-1} \otimes 1_{n-1} + 1_1 \otimes 1_1$. To fix the situation, we put $\omega S = R_{n-1} + R_1 + P_{n-1} \otimes P_1 + P_1 \otimes P_{n-1}$, 
where $P_{n-1}$ and $P_1$ stand for the projectors to the corresponding matrix subalgebras. Thus 
defined, $S$ solves the Yang-Baxter equation too. The matrix $Q = \sum_{i=1}^{n-1} e_n^i \otimes e^i_n$ fulfills the 
condition of Proposition 4, so $S + Q = \frac{1}{\omega} R_n$ is the Hecke matrix by induction.

Let us pursue another representation of $\text{Mat}_n(\mathbb{C})$ as an associative triple, taking $\mathcal{M}_\pm$ to be 
the subalgebras of upper and lower triangular matrices. Then $\mathcal{D}$ is a commutative algebra 
formed by the diagonal matrices. The Hecke condition on $S = a^{ik} e_i^k \otimes e^k_i \in \mathcal{D} \otimes^2$, which is 
an apparent solution to YBE, boils down to the system

$$(6) \quad a^{ii} + a^{ik} a^{ki} = a^{ii} a^{ii}, \quad k \neq i, \quad a^{ik} a^{ki} = a^{im} a^{mi}, \quad k \neq i, \quad m \neq i.$$ 

It is satisfied by the numbers $a^{ii} = \pm q^{\pm 1}/\omega$ and $a^{ik} = b^{ik}/\omega$, $b^{ik} b^{ki} = 1$ for $i \neq k$ (note that 
the classical limit $R \to 1$ exists, after the proper rescaling by $\omega$, only in case of $a^{ii} = q/\omega$).

This is the principal solution of the Hecke condition in the commutative algebra $\mathbb{C}^n \otimes \mathbb{C}^n$. It 
deviates from the standard solution $b^{ik} = 1$ exactly by the Reshetikhin diagonal twist $[R]$. In 
general, let $\tilde{S} = F S F^{-1} z_{21}$, where $F$ is an invertible element from $\mathcal{D} \otimes^2$, be a solution to YBE 
too. Then, the Hecke condition is fulfilled and $F Q F^{-1} = Q$. So $\tilde{R} = F (S + Q) F^{-1} = \tilde{S} + Q$ 
is again an $R$-matrix for $\mathcal{M}$.

2. Baxterization procedure.

The baxterization operation converts a constant matrix solution $R$ to YBE to that with 
spectral parameter

$$(7) \quad R(z, u) = z R - u R^{-1}_{21},$$ 

provided $R$ satisfies the conventional Hecke condition

$$(R P)^2 = \omega(R P) + I$$

with the matrix permutation $\mathcal{P}$. Parameters $z$ and $u$ are usually represented in the exponential form; then (7) is the trigonometric solution to YBE. Set $\mathfrak{N}_+ = z \text{Mat}_n(\mathbb{C})[z]$, $\mathfrak{N}_- = \frac{1}{z} \text{Mat}_n(\mathbb{C})[\frac{1}{z}]$, and $\mathcal{D} = \text{Mat}_n(\mathbb{C})$. The cyclic inner product on $\mathcal{M}$ is given by the formula 
$(A z^k, B z^m) = \text{Tr}(A B) \delta^{k,-m}$. Thus we built an associative triple on $\text{Mat}_n(\mathbb{C})[z, \frac{1}{z}]$. Now, 
put $S = \frac{R}{\omega}$ and $Q = \frac{-u}{z} \mathcal{P}$. The result will be proportional to (7) because $\omega \mathcal{P} = \tilde{R} - R_{21}^{-1}$.

Again, as with the Yang matrix, one has to extend the Laurent polynomial algebra to that of the Laurent series.

5. On quantization of Belavin-Drinfel’d triples for $sl_{n-1}(\mathbb{C})$.

Consider a semisimple Lie algebra $\mathfrak{g}$ with the Cartan subalgebra $\mathfrak{h}$ and the polarization 
$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$. Let $\Delta$ and $\Delta^\pm$ be, respectively, the systems of its all, positive, and 
negative roots. Recall [BD] that non-skew-symmetric classical $r$-matrices associated to $\mathfrak{g}$
are in one-to-one correspondence with Belavin-Drinfel’d (BD) triples \((\Gamma_1, \Gamma_2, \tau)\) consisting of two subsets of positive simple roots \(\Gamma_i, i = 1, 2\), and a bijection \(\tau: \Gamma_1 \to \Gamma_2\) preserving lengths of the roots with respect to the Killing form. Besides, for every \(\alpha \in \Gamma_1\) there is a positive integer \(k\) such that \(\tau^{k-1}(\alpha) \in \Gamma_1 \) and \(\tau^k(\alpha) \notin \Gamma_1\). It follows from [RS] (see also [BZ]) that for any BD triple the double Lie algebra \(D(g)\) is isomorphic to the direct sum \(g \oplus g\) with the invariant scalar product being the difference of Killing forms on the two addends. There is a geometric description of \(g^*\) as a Lie subalgebra in \(D(g)\) [S]. Let \(\Delta_i\) be the subsystems of roots generated by \(\Gamma_i, i = 1, 2\). Denote \(g_i\) the semisimple Lie algebras with the root systems \(\Delta_i\), and the Cartan subalgebras \(h_i\). The bijection \(\tau\) is extended to a Lie algebra isomorphism such that \(g_1 \oplus \tau(g_1)\) has trivial intersection with the diagonal embedding \(g \oplus \text{id}(g)\). Choose subspaces \(h^0_i \subset (h_i)^\perp \subset h\) containing their orthogonal complements in \((h_i)^\perp\) such that the equation \(\eta_1 + \eta_2^0 = \tau(\eta_1) + \eta_2^0, \eta_1 \in h_1, \eta_2^0 \in h_2^0\), has only zero solution. Then, \(g^*\) is the subalgebra in \((n_+ + g_1 + h^0_1) \oplus (n_- + g_2 + h^0_2)\) obtained by identification of \(g_1\) and \(g_2\) via \(\tau\). The classical r-matrix for \(g\) is recovered from the canonical element of the pairing by projecting \(g \oplus g\) to the first addend.

We restrict further considerations to the case \(g = sl_{n-1}(\mathbb{C})\); then \(D(g)\) is represented in the direct sum of two matrix algebras \(\mathfrak{M} = \text{Mat}_n(\mathbb{C})\). The cyclic inner product is induced by the functional \(t_{\mathfrak{M}^2} = \text{Tr} \circ \text{Tr}\), the difference of the corresponding traces. Denote \(\mathcal{A}(l)\) the associative envelope of a Lie algebra \(l\) in \(\mathfrak{M}^2\). Algebra \(\mathcal{A}(g_1)\) is isomorphic to a direct sum of matrix algebras \(\text{Mat}_{m_i}(\mathbb{C})\). Suppose that \(\tau\) may be extended to the isomorphism \(\hat{\tau}: \mathcal{A}(g_1) \to \mathcal{A}(g_2)\). That means that the restriction of \(\tau\) to every connected component of \(\Gamma_1\) preserves orientation induced by that of the Dynkin diagram. Let \(\hat{\Gamma}\) be the set of diagonal matrix idempotents \(\eta_i, i = 1, ..., n\), and \(\hat{\Gamma}_i = \hat{\Gamma} \cap \mathcal{A}(g_i)\); then \(\hat{\tau}\) is reduced to a bijection \(\hat{\Gamma}_1 \to \hat{\Gamma}_2\). We impose one more condition on the BD triple assuming that for every \(\eta \in \hat{\Gamma}_1\) there is the smallest positive integer \(m(\eta)\) such that \(\hat{\tau}^{m(\eta)}(\eta) \notin \hat{\Gamma}_1\). Again, this condition means that the subspace spanned by \(\hat{\Gamma}_1 \oplus \hat{\tau}(\hat{\Gamma}_1)\) has trivial intersection with the subalgebra \(\mathfrak{M} \oplus \text{id}(\mathfrak{M})\). This requirement holds, for example, if \(\hat{\Gamma}_1 \cap \hat{\Gamma}_2 = \emptyset\) or \(\hat{\tau}(\eta_i) = h_k \Rightarrow k > i\).

Denote \(\mathfrak{B}\) the associative subalgebra in \((\mathcal{A}(g_1) + n_+) \oplus (\mathcal{A}(g_2) + n_-)\) obtained by identification of \(\mathcal{A}(g_1)\) and \(\mathcal{A}(g_2)\) via \(\hat{\tau}\); define \(\mathfrak{d}_1\) and \(\mathfrak{d}_2\) as commutative subalgebras spanned by \(\hat{\Gamma}_1 \oplus \{0\}\) and \(\{0\} \oplus \hat{\Gamma}_2\) correspondingly. Both \(\mathfrak{d}_i\) have dimension \(n - m\), where \(m = \text{card}(\hat{\Gamma}_1)\); they are orthogonal to \(\mathfrak{B}\), which is also \(\mathfrak{d}_i\)-invariant. So the sum \(\mathfrak{B} + \mathfrak{d}_1 + \mathfrak{d}_2\) is an associative algebra, and its intersection \(\mathfrak{d}\) with \(\mathcal{A}(g)\) is a subalgebra in \(\mathfrak{C}\hat{\Gamma} \oplus \mathfrak{C}\hat{\Gamma}\).

**Lemma 1.** Projections \(\gamma_i: \mathfrak{d} \to \mathfrak{d}_i, i = 1, 2\), have the full rank \(n - m\).

**Proof.** We will check the statement only for \(\gamma_2\) in view of the symmetry \(\hat{\tau} \to \hat{\tau}^{-1}\). Consider the sets \(\hat{\Gamma}_\eta = \{\eta, \hat{\tau}(\eta), ..., \hat{\tau}^{m(\eta)}(\eta)\}\) if \(\eta \in \hat{\Gamma}_1 \setminus \hat{\Gamma}_2\) and \(\hat{\Gamma}_\eta = \{\eta\}\) in case of \(\eta \in \hat{\Gamma}_1 \setminus (\hat{\Gamma}_1 \cup \hat{\Gamma}_2)\). They do not intersect for different \(\eta\) and clearly \(\hat{\Gamma}_\eta \cap (\hat{\Gamma}_1 \cup \hat{\Gamma}_2) = \emptyset\). The one-dimensional subspace spanned by \(\sum_{\xi \in \hat{\Gamma}_\eta} \xi + \sum_{\xi \in \hat{\Gamma}_\eta} \xi\) is evidently contained in \(\mathcal{A}(g)\) which is embedded in \(\mathfrak{M} \oplus \mathfrak{M}\) diagonally. It is also contained in \(\mathfrak{B} + \mathfrak{d}_1 + \mathfrak{d}_2\) because its projection to \(\mathfrak{C}\hat{\Gamma}_1 \oplus \mathfrak{C}\hat{\Gamma}_2\) lies in the subalgebra of \(\mathfrak{B}\) spanned by \(\hat{\Gamma}_1 \oplus \hat{\tau}(\hat{\Gamma}_1)\). So it is a subspace of \(\mathfrak{d}\) and its projection \(\text{ro} \mathfrak{d}_2\) is \(\mathbb{C}\eta\). □

9
Corollary 1. Algebras $\mathcal{B}$, and $\mathfrak{d}_i$, $i = 1, 2$, are bimodules for $\mathfrak{d}$.

Proof. Concerning the algebras $\mathfrak{d}_i$, this immediately follows from Lemma 1. By the very definition, $\mathfrak{d}$ is a subalgebra in $\mathcal{B} + \mathfrak{d}_1 + \mathfrak{d}_2$, and the latter is a direct sum of algebras. Therefore, this is also true for the algebra $\mathcal{B}$. ■

Introduce algebras

$$
\mathcal{M} = \mathfrak{d}_2 \oplus \mathcal{R} \oplus \mathcal{R}, \quad \mathcal{D} = \gamma_2(\mathfrak{d}) \oplus \mathfrak{d},
$$

$$
\mathfrak{N}_- = \{0\} \oplus \mathcal{R} \oplus \text{id}(\mathcal{R}), \quad \mathfrak{N}_+ = \gamma_2(\mathfrak{d}) \oplus \{0\} \oplus \gamma_2(\mathfrak{d}) \oplus \{0\} \oplus \mathcal{B}.
$$

The non-degenerate cyclic inner product on $\mathcal{M}$ is determined by the functional $t_{\mathcal{M}} = t_{\mathcal{d}_2} \oplus t_{\mathcal{R}^2} = t_{\mathcal{d}_2} \oplus t_{\mathcal{R}} \oplus t_{\mathcal{R}}$, where $t_{\mathcal{d}_2}$ is $t_{\mathcal{R}}$ restricted to $\mathcal{d}_2$.

Theorem 2. Suppose $\tau$ can be extended to the automomorphism $\hat{\tau}: \mathcal{A}(\mathfrak{g}_1) \to \mathcal{A}(\mathfrak{g}_2)$ of associative algebras with no stable points. Then, the algebras $\mathcal{M}$ and $\mathfrak{N}_+ = \mathcal{D} + \mathfrak{N}_+$ form an associative triple.

Proof. In fact, algebras $\mathfrak{N}_+$ are ideals in $\mathfrak{N}_\pm$. That is evident for $\mathfrak{N}_+$ and follows from Corollary 1 for $\mathfrak{N}_-$. Algebras $\mathfrak{N}_\pm$ are isotropic and orthogonal to $\mathcal{D}$. It remains to show that they are mutually dual with respect to the inner product. That will imply $\mathfrak{N} \cap (\mathfrak{N}_- + \mathfrak{N}_+) = \{0\}$ and $\mathcal{M} = \mathfrak{N}_- + \mathcal{D} + \mathfrak{N}_+$, taking into account $\dim(\mathcal{D}) \geq n - m$ (Lemma 1). We will compute the dual bases explicitly. The canonical element $Q \in \mathfrak{N}_+ \otimes \mathfrak{N}_-$ includes two addends:

$$(0 \oplus e_\gamma \oplus \theta_1(\gamma)e_{\hat{\tau}(\gamma)}) \otimes \sum_{k=0}^{m_2(\gamma)} (0 \oplus f_{\hat{\tau}^{-k}(\gamma)} \oplus f_{\hat{\tau}^{-k}(\gamma)}),$$

$$-(0 \oplus \theta_2(\gamma)f_{\hat{\tau}^{-1}(\gamma)} \oplus f_\gamma) \otimes \sum_{k=0}^{m_1(\gamma)} (0 \oplus e_{\hat{\tau}^k(\gamma)} \oplus e_{\hat{\tau}^k(\gamma)}),$$

Here $\gamma \in \Delta^+$ and $e_\gamma, f_\gamma$ are the corresponding positive and negative root vectors normalized to $(e_\gamma, f_\gamma) = 1$ with respect to the trace pairing. Functions $\theta_i$ take two values: $\theta_i(\gamma) = 1$ if $\gamma \in \Delta_i$ and $\theta_i(\gamma) = 0$ otherwise. The integer number $m_2(\gamma)$ means the same for $\tau^{-1}$ as $m_1(\gamma) = m(\gamma)$ for $\tau$. If $\gamma \not\in \Delta_i$, we assume $m_i(\gamma) = 0$. There is nothing new so far in comparison with the quasi-classical situation, and this part of the canonical element is the same as that of the classical r-matrix (isomorphism $\tau$ coincides with $\hat{\tau}$ on these elements). The distinction appears in the sector of diagonal matrices, so we exhibit this part of the canonical element $Q$ in more detail:

$$-(\eta \oplus 0 \oplus \eta) \otimes (0 \oplus \eta \oplus \eta), \quad \eta \not\in \hat{\Gamma}_2 \cup \hat{\Gamma}_1,$$

$$-\sum_{k=0}^{m(\eta)} (\eta \oplus 0 \oplus \eta) \otimes (0 \oplus \hat{\tau}^k(\eta) \oplus \hat{\tau}^k(\eta)), \quad \eta \in \hat{\Gamma}_1 \setminus \hat{\Gamma}_2,$$
\[- \sum_{k=0}^{m(\eta)} (0 \oplus \hat{\tau}^{-1}(\eta) \oplus \eta) \otimes (0 \oplus \hat{\tau}^k(\eta) \oplus \hat{\tau}^k(\eta)), \quad \eta \in \hat{\Gamma}_2 \cap \hat{\Gamma}_1, \]
\[-(0 \oplus \hat{\tau}^{-1}(\eta) \oplus \eta) \otimes (0 \oplus \eta \oplus \eta), \quad \eta \in \hat{\Gamma}_2 \setminus \hat{\Gamma}_1. \]

This shows that \( N_+ \) and \( \mathcal{N}_- \) are in duality. It follows from the proof of Lemma 1 that the vectors
\[ m(\eta) \sum_{k,i=0}^{m(\eta)} (\eta \oplus \hat{\tau}^k(\eta) \oplus \hat{\tau}^i(\eta)), \quad \eta \in \hat{\Gamma} \setminus \hat{\Gamma}_2, \]
belong to \( \mathcal{D} \). They form an orthonormal set and, by dimensional arguments, span entire \( \mathcal{D} \).

To accomplish construction of the R-matrices, we should satisfy the Hecke condition for some \( S \in \mathcal{D} \otimes \mathcal{D} \). As an algebra with cyclic inner product, \( \mathcal{D} \) is isomorphic to \( \mathbb{C}^{n-m} \). This problem has been solved during the study of the standard R-matrix for \( gl_n(\mathbb{C}) \). Finally, to get R-matrices lying in \( \text{Mat}_{n \otimes n}(\mathbb{C}) \), let us take the projection \( \pi: \mathfrak{g} \rightarrow \{0\} \oplus \mathbb{R} \oplus \{0\} \). The result is the sum of two terms
\[ (\pi \otimes \pi)(S) = \sum_{\eta_i, \eta_j \in \hat{\Gamma} \setminus \hat{\Gamma}_2} a^{ij} \sum_{l=0}^{m(\eta_i)} \hat{\tau}^l(\eta_i) \otimes \sum_{k=0}^{m(\eta_j)} \hat{\tau}^k(\eta_j), \]
\[ (\pi \otimes \pi)(Q) = - \sum_{\eta \in \hat{\Gamma}_1}^{m(\eta)} \sum_{k=1}^{m(\eta)} \eta \otimes \hat{\tau}^k(\eta) + \sum_{\gamma \in \Delta^+} e_\gamma \otimes f_\gamma + \sum_{\sigma, \gamma \in \Delta^+, \sigma \prec \gamma} e_\gamma \wedge f_\sigma. \]

Symbol \( \prec \) means the partial ordering in \( \Delta^+ \) defined by \( \tau: \sigma \prec \gamma \) if \( \tau^k(\sigma) = \gamma \) for some \( k > 0 \). Numbers \( a^{ik} \) satisfy equation (6) and provide the principal solution of the Hecke condition in the commutative algebra \( \mathbb{C}\hat{\Gamma} \setminus \hat{\Gamma}_2 \otimes \mathbb{C}\hat{\Gamma} \setminus \hat{\Gamma}_2 \). Let us stress that we obtain, in this way, R-matrices which do not tend to unit in the classical limit \( q \to 1 \). To do so, we take \( a^{ii} = -q^{-1} \) instead of \( q \) for some \( i \).

The deduced formula goes over into that for the standard \( gl_n(\mathbb{C}) \) corresponding to empty \( \hat{\Gamma}_i \). Indeed, the first and the third terms in the expression for \( (\pi \otimes \pi)(Q) \) vanish, and summation over \( l \) and \( k \) in \( (\pi \otimes \pi)(S) \) is cancelled. Another extreme possibility is the BD triple with \( \hat{\Gamma}_1 = \{\eta_1, \eta_2, ..., \eta_{n-1}\} \), \( \hat{\Gamma}_2 = \{\eta_2, \eta_3, ..., \eta_n\} \), and the isomorphism \( \hat{\tau}: \eta_i \rightarrow \eta_{i+1} \). It leads to the solution to YBE called the Cremmer-Gervais R-matrix. In this case, the algebra \( \mathcal{D} \) is one-dimensional, so the Hecke condition is evidently fulfilled for any scalar \( S = \lambda \). Thus we come to the one parameter Cremmer-Gervais solution in the form \( \lambda + (\pi \otimes \pi)(Q) \). This is in agreement with [H1]. Putting \( \lambda = q/\omega \), we get the R-matrix of [H2] for the special value of parameter \( p = 1 \).
6. Remarks. Analysis shows that the developed technique does not apply, as it is, to the orthogonal series of simple Lie algebras. The intuitive explanation may be the fact that rings coincide as linear spaces with their commutator Lie algebras. Another indication is that R-matrices for orthogonal Lie algebras do not satisfy the Hecke condition. So a modification of this approach for the other classical series is an open problem. It has to be emphasized that associative triples are just an algebraic construction, probably the simplest, naturally adopted for solving the system of equations arising from decomposition of an R-matrix into the sum of two solutions to YBE. Let us demonstrate how an extension of this scheme explains deformation of the Yang R-matrix with a constant unitary R-matrix, [BFS].

Consider a disjoint triple $\mathcal{M} = \mathcal{M}_- + \mathcal{M}_+$ with $\mathcal{M}_{p,m}$ isotropic and select the subspaces $\mathcal{M}_c^\pm$ of “constants” in $\mathcal{M}_\pm$ annihilated by the actions $\triangleright$ and $\triangleleft$: $\mathcal{M}_\pm \triangleright \mathcal{M}_c^- = 0 = \mathcal{M}_c^- \triangleleft \mathcal{M}_\pm^+$. The sum $\mathcal{M}_c^\pm = \mathcal{M}_c^- + \mathcal{M}_c^+$ is a direct sum of algebras. In the case $\mathcal{M} = \text{Mat}_n(C)[z, \frac{1}{z}]$ considered in Section 2, $\mathcal{M}_c^\pm$ is formed by constant matrices. It can be shown that for every solution $S \in \mathcal{M}_c^\pm \otimes \mathcal{M}_c^\pm$ to YBE satisfying the unitarity condition $S_{21} = 1 \otimes 1$ the sum $S + Q$ is a solution, too.

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