The Eigensharp Property for Unit Graphs Associated with Some Finite Rings

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Abstract: Let $R$ be a commutative ring with unity. The unit graph $G(R)$ is defined such that the vertex set of $G(R)$ is the set of all elements of $R$, and two distinct vertices are adjacent if their sum is a unit in $R$. In this paper, we show that for each prime, $p$, $G(Z_p)$ and $G(Z_{2p})$ are eigensharp graphs. Likewise, we show that the unit graph associated with the ring $\mathbb{Z}[x]/(x^2)$ is an eigensharp graph.

Keywords: commutative ring; unit graph; graph join; biclique; biclique partition number; eigensharp graph

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1. Introduction

Studying rings by associating various graphs with the ring via its algebraic structure has attracted the attention of many researchers. Beck [1] introduced the zero-divisor graph; Anderson and Badawi [2] introduced the total graph. Grimaldi [3] defined the unit graph $G(Z_n)$ associated with the finite ring $\mathbb{Z}_n$, where the author studied some properties of a graph, such as the Hamilton cycles, covering number, independence number, and chromatic polynomial. The units of a ring play a crucial role in determining the structure of the ring, and many features of a ring can be known from these units. So, it is natural to make a connection between a ring with a graph whose edges have a strong relationship with the units of the ring. The unit graph of a ring is one of such graphs.

In 2010, Ashrafi et al. [4] generalized the unit graph $G(Z_n)$ to $G(R)$ for an arbitrary (commutative) ring $R$, and considered standard concepts of graph theory such as connectedness, chromatic index, diameter, girth, and planarity of $G(R)$. Akbari et al. [5] studied the unit graph of a noncommutative ring. Maimani et al. [6] showed that the unit graphs is Hamiltonian if and only if the ring $R$ is generated by its units. Heydari and Nikmehr [7] investigated the case when the ring $R$ is a left Artinian ring. Afkhami and Khosh-Ahang [8] studied the unit graphs of rings of polynomials and power series.

A biclique is a complete bipartite subgraph of $G$. The complete bipartite graphs $K_{1,n}$ are called stars, denoted by $S_n$. A collection $\mathcal{H}_G = \{B_1, B_2, \ldots, B_k\}$ of subgraphs of $G$ is called a biclique partition covering of a graph $G$ if $B_i$ is a biclique subgraph for all $i = 1, 2, \ldots, k$, and for every edge $e \in E(G)$, there exists exactly one $B_i \in \mathcal{H}_G$, such that $e \in E(B_i)$. The biclique partition number of a graph $G$, denoted by $bp(G)$, is given by

$$bp(G) = \min \{ |\mathcal{H}_G| : \mathcal{H}_G \text{and is a biclique partition covering of } G \}.$$
for example, [9–13]). When Graham and Pollak [14] first studied this parameter for the complete graph, they were motivated by a network addressing problem. For more details about graph addressing, please see [15]. The adjacency matrix of $G$, denoted by $A(G)$, is a square matrix of order $|V(G)|$, with the $ij$th entry equaling 1 if $v_i, v_j$ is an edge of $G$ and 0 otherwise. Witsenhausen (see, for example, [14]) showed that for a graph $G$

$$\max\{a_+(G), a_-(-G)\} \leq bp(G),$$

where $a_+(G)$ and $a_-(G)$ are the number of positive and negative eigenvalues of the adjacency matrix $A(G)$, respectively. We repeatedly use this fact below. We say that $G$ is an eigensharp graph if $bp(G) = \max\{a_+(G), a_-(G)\}$, and it is almost eigensharp if $bp(G) = \max\{a_+(G), a_-(G)\} + 1$. Certain families of graphs, including complete graphs $K_n$, complete bipartite graphs $K_{m,n}$, trees, cycles $C_n$ with $n = 4$ or $n \neq 4k$, and various graph products, are eigensharp (see, for example, [16–19]).

The unit graph $G(R)$ is defined such that the vertex set of $G(R)$ is the set of all elements of the ring $R$, and two distinct vertices are adjacent if their sum is a unit in $R$. In this paper, we show that for each prime $p$, $G(Z_p)$, $G(Z_{2p})$, and $G(Z_{4p}/\langle x^2 \rangle)$ are eigensharp graphs.

2. Preliminaries

In this paper, $R$ is assumed to be a commutative ring with unity. An element $a$ is said to be a unit in $R$ if $a$ has a multiplicative inverse. The set $U(R)$ is defined to be the set of all units in $R$. Moreover, the polynomial ring over $Z_n$ is denoted by $Z_n[x]$. In particular, $a$ is a unit in $Z_n$ if the greatest common divisor between $n$ and $a$ is equal to 1. For example, $U(Z_5) = \{1, 2, 3, 4\}$ and $U(Z_6) = \{1, 5\}$.

Several properties of the unit graph are provided in [4], from which we cite the following Theorem:

**Theorem 1.** [4] Let $R$ be a finite ring. If $2 \in U(R)$, then for every $x \in U(R)$, degree $(x) = |U(R)| - 1$ and for every $x \in R - U(R)$, degree $(x) = |U(R)|$.

All graphs in this paper are finite undirected simple graphs. For a graph $G = (V(G), E(G))$, the set $V(G)$ denotes the vertex set of $G$, and $E(G)$ denotes the edge set of $G$. The degree of a vertex in $G$ is defined as the number of edges emanating from the vertex. A graph $G$ is said to be $(n, m)$-semiregular if each vertex in $G$ has a degree $n$ or $m$.

For a simple graph $G$, the adjacency matrix $A(G)$ is a symmetric matrix with real eigenvalues such that the algebraic multiplicity is equal to geometric multiplicity for each eigenvalue. We refer to it as multiplicity. It can be proved that $a_+(G) > 0$ and $a_-(G) > 0$ for any non-null graph $G$.

The multiplicity of an eigenvalue $\lambda_i$ is the number of linearly independent eigenvectors associated with it. If $\lambda_i, 1 \leq i \leq j$ are the distinct eigenvalues of the adjacency matrix $A(G)$ with multiplicity $r_i$, then $\sigma(A(G)) = \left( \begin{array}{ccc} \lambda_1 & \lambda_2 & \ldots & \lambda_j \\ r_1 & r_2 & \ldots & r_j \end{array} \right)$ is called the spectrum of $G$. For example,

$$\sigma(A(K_n)) = \left( \begin{array}{cc} n-1 & -1 \\ 1 & n-1 \end{array} \right) \quad \text{and} \quad \sigma(A(K_{n,m})) = \left( \begin{array}{cc} \sqrt{nm} & 0 & -\sqrt{nm} \\ 1 & nm-2 & 1 \end{array} \right).$$

The join of two graphs $G$ and $H$, denoted by $G \vee H$, is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. $G \vee H$ is a complete bipartite graph if both $G$ and $H$ are independent vertices. The following Theorem was proved in [20].

**Theorem 2.** [20] Suppose that $G$ and $H$ are two regular graphs. Then, $a_-(G \vee H) = a_-(G) + a_-(H) + 1$ and $a_+(G \vee H) = a_+(G) + a_+(H) - 1$. Consequently, if each $G$ and $H$ are eigensharp graphs with $bp(G) = a_-(G)$ and $bp(H) = a_-(H)$, then $G \vee H$ is an eigensharp graph.
3. Unit Graph Associated with Rings \( Z_p \) and \( Z_{2p} \)

In this section, we obtain the biclique partition number of \( G(Z_p) \), and we prove that \( G(Z_p) \) is an eigensharp graph.

**Theorem 3.** For each prime \( p \), the graph \( G(Z_p) \) is eigensharp.

**Proof.** If \( p = 2 \) and 3, then \( G(Z_p) \) is isomorphic to \( P_2 \) and \( P_3 \), respectively. Hence, \( bp(G(Z_2)) = bp(G(Z_3)) = 1 \) with

\[
\sigma(A(G(Z_2))) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \sigma(A(G(Z_3))) = \begin{pmatrix} 0 & \sqrt{2} \\ 1 & -\sqrt{2} \end{pmatrix}.
\]

Hence, for \( p = 2 \) or 3, \( G(Z_p) \) is an eigensharp graph. Now, for \( p \geq 5 \), let \( V = \{0, 1, \ldots, p - 1\} \) and \( E = \{e_{r,s} : r + s \in U(Z_p)\} \) be the vertex set and the edge set of \( G(Z_p) \), respectively. Because \( U(Z_p) = \{1, 2, \ldots, p - 1\} \) and 2 \( \in U(Z_p) \), then \( |U(Z_p)| = p - 1 \). From Theorem 1, it follows that for every \( x \in U(Z_p) \), degree \( (x) = p - 2 \); for every \( x \notin U(Z_p) \), degree \( (x) = p - 1 \). We notice that the vector \( i \in \{0, 1, \ldots, (p - 1)\} \) is an eigenvector for \( H \) with multiplicity 1. Hence, for \( p - 3 \)–regular graph. It has been found and from several computations for different \( p \)'s that \( A(H) \) is a \((p - 1) \times (p - 1)\) matrix that has the form

\[
A(H) = \begin{pmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 \\ 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & \cdots & 1 & 0 & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 \end{pmatrix}.
\]

The entries of \( A(H) \) are all 1, except 0 on the main and secondary diagonals. Notably, the first \( \frac{p - 1}{2} \) columns are linearly independent. The \( \frac{p - 1}{2} \)th column is the same as the \( \frac{p - 3}{2} \)th column. The \( \frac{p - 3}{2} \)th column is the same as the \( \frac{p - 5}{2} \)th column, \ldots, the last column is the same as the first column. Thus, the column rank is \( \frac{p - 1}{2} = \left\lfloor \frac{p}{2} \right\rfloor \). We show that \( H \) is eigensharp graph with \( bp(H) = a_-(H) \).

Because nullity \( (A(H)) = \left\lfloor \frac{p}{2} \right\rfloor \), then \( \lambda = 0 \) is an eigenvalue of \( A(H) \) with multiplicity \( \left\lfloor \frac{p}{2} \right\rfloor \). We notice that the vector \( D^0 \), where \( r = 2, 3, \ldots, \left\lfloor \frac{p}{2} \right\rfloor \) is defined as a \((p - 1) \times 1\) vector, and all entries are 0 except the first and last entries, which are 1; the \( r \)th and \((p - r)\)th entries are \(-1\), which is an eigenvector for \( A(H) \) with eigenvalue \( \lambda = 2 \). Moreover, because \( \text{trace}(A(H)) = 0 \), then the value \((p - 3)\) is an eigenvalue of \( A(H) \) of multiplicity 1. Hence,

\[
\sigma(A(H)) = \begin{pmatrix} 0 & -2 & 0 \\ \left\lfloor \frac{p}{2} \right\rfloor & \left\lfloor \frac{p}{2} \right\rfloor - 1 & \frac{p - 3}{2} \end{pmatrix}.
\]

Therefore, \( a_-(H) = \left\lfloor \frac{p}{2} \right\rfloor - 1 \geq \sigma_+ (H) \), and so \( bp(H) \geq \left\lfloor \frac{p}{2} \right\rfloor - 1 \).
Let \( \mathcal{H}_H = \{ B_i(X_i, Y_i) : 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor - 1 \} \) be a collection of subgraphs of \( H \) such that, for each \( i, X_i = \{ i, p - i \} \) and \( Y_i = \{ i + 1, i + 2, \ldots, (p - i) \} \) and

\[
E(B_i) = \{ e_{i+j}, e_{p-i-j} : i + 1 \leq j \leq (p - i) \}.
\]

For each \( j : 1 \leq j \leq (p - 2i) - 1, i + j = 0 \mod p \) only if \( j = p - i \), which is completely impossible. Similarly, \( (p - i) + (i + j) \neq 0 \mod p \). So, \( E(B_i) \) is a nonempty set.

Hence, \( B_i \) is isomorphic to \( K_{2, (p-1) - 2i} \). Note that no pair of edges of \( H \) belongs to a common \( B_i(X_i, Y_i) \), and

\[
\sum_{i=1}^{\left\lfloor \frac{p}{2} \right\rfloor - 1} |E(B_i)| = \sum_{i=1}^{\left\lfloor \frac{p}{2} \right\rfloor - 1} 2((p - 1) - 2i) = \frac{1}{2} (p - 1)(p - 3) = |E(H)|.
\]

Thus, \( \mathcal{H}_H = \{ B_i(X_i, Y_i) : 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor - 1 \} \) is a biclique partition of \( H \) with cardinality \( \left\lfloor \frac{p}{2} \right\rfloor - 1 \), which implies that \( G(Z_p) \) is an eigensharp graph. □

Now, we show that \( G(Z_{2p}) \) is an eigensharp graph.

**Remark 1.** If \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), where \( A, B, C, \) and \( D \) are block matrices, and if \( CD = DC \), then

\[
\det(M) = \det(AD - BC).
\]

See [21], Theorem 3.

**Theorem 4.** The graph \( G(Z_{2p}) \) is eigensharp.

**Proof.** Note that the graph \( G(Z_{2p}) \) is a graph with \( 2p \) vertices. Suppose that the vertex set is \( V(G(Z_{2p})) = \{ 0, 1, 2, \ldots, 2p - 1 \} \). Then, the two distance vertices in \( G(Z_{2p}) \) are adjacent if their sum is an odd number less than \( 2p \) and not equal to \( p \).

Now, the adjacency matrix of \( A(G(Z_{2p})) = \begin{bmatrix} 0 & A(K_p) \\ A(K_p) & 0 \end{bmatrix} \) where \( A(K_p) \) is the adjacency matrix of the complete graph \( K_p \). Using Remark 1, we claim that the spectrum of \( \sigma(A(G(Z_{2p}))) = \left( \begin{array}{ccc} p - 1 & 1 & -p \\ 1 & 1 & p - 1 \\ -p & p - 1 & 1 \end{array} \right) \). To prove this claim, we notice that

\[
\det(\lambda I - A(G(Z_{2p}))) = \det(\lambda^2 I - A^2(K_p)) = \sigma(A(K_p)) \sigma(-A(K_p)) = \left( \begin{array}{ccc} p - 1 & 1 & -p \\ 1 & 1 & p - 1 \\ -p & p - 1 & 1 \end{array} \right).
\]

So, \( bp(G(Z_{2p})) \geq p \). On the other hand, let \( \mathcal{H}_G(Z_{2p}) = \{ S_{2k} : 0 \leq k \leq p - 1 \} \) be the set of \( p \) disjoint stars in \( G(Z_{2p}) \) generated by the vertices \( 2k, 0 \leq k \leq p - 1 \). Then, \( \mathcal{H}_G(Z_{2p}) \) is a biclique partition of cardinality \( p \). Hence, the graph \( G(Z_{2p}) \) is eigensharp. □

**4. Unit Graph Associated with the Ring \( \mathbb{Z}_n[x]/(x^2) \)**

In this section, we consider the ring \( \mathbb{Z}_n[x]/(x^2) = \{ a + bX : a, b \in \mathbb{Z}_p, X = x + \langle x^2 \rangle \} \), where \( \langle x^2 \rangle = \{ x^2P(x) : P(x) \in \mathbb{Z}_n[x] \} \) is the ideal of \( \mathbb{Z}_n[x] \) generated by \( x^2 \). We show that the unit graph \( G(\mathbb{Z}_n[x]/(x^2)) \) is eigensharp. We denote the graph \( G(\mathbb{Z}_p[x]/(x^2)) \) by \( G_p(x^2) \).

Let \( s = p^2 - p \) and \( I_p \) be a \( p \times p \) matrix, where all entries are ones; let \( I_p \) be a \( p \times 1 \) matrix, where all entries are ones, \( N_p \) be the zero matrix of size \( p \times p \), and \( 0_p \) be the zero matrix of size \( p \times 1 \). For \( m = 1, 2, \ldots, \frac{p^2 - 1}{2} \) define the partition matrix \( F(m) \) as the \( s \times 1 \) matrix such that all the submatrices entries are \( 0_p \), except for the \( m \)th row, which is the submatrix \( I_p \), and the \( (p - m) \)th row is the submatrix \( -1_p \). Furthermore, for \( r = 2, 3, \ldots, \frac{p^2 - 1}{2} \), defines the partition matrix \( H(r) \) as the \( s \times 1 \) matrix, where all the submatrices are \( 0_p \), except the
Theorem 5. For each prime \( p \), \( G_p(x^2) \) is an eigensharp graph.

Proof. Let \( a + bX \in Z_p[x]/\langle x^2 \rangle \). Then, \( a + bX \) is a unit if and only if \( a \) is a unit in \( Z_p \). Thus,

\[
U(Z_p[x]/\langle x^2 \rangle) = \{ r + sX : r, s \in Z_p, r \neq 0 \},
\]

hence, \( |U(Z_p[x]/\langle x^2 \rangle)| = p\). Because \( 2 \in U(Z_p[x]/\langle x^2 \rangle) \), then, by Theorem 1, \( G_p(x^2) \) is a \( (p(p-1), p(p-1)-1) \)–semiregular graph.

\( T = \{ 0, X, 2X, \ldots, (p-1)X \} \) is an independent set of \( G_p(x^2) \) with each vertex of \( T \) having a degree \( p(p-1) \). For \( v = a + bX \notin T \) and \( u = t + sX \in V(G_p(x^2)) \), such that \( v \neq u \) and \( t \in Z_p \setminus \{ p-a \} \), we have \( v + u \in U(G_p(x^2)) \). Thus, \( v \) is adjacent with each vertex in \( G_p(x^2) \), except \( \{ a + bX, (p-a), (p-a) + X, \ldots, (p-a) + (p-1)X \} \), i.e., \( v \) has a degree \( p^2 - (p+1) = p(p-1) \).

Now, we consider the subgraph \( W \) of \( G_p(x^2) \) induced by \( V(W) = V(G_p(x^2)) \setminus T \). Let \( m = (p(p-1) - p-1) \). Then, \( W \) is an \( m \)–regular graph with

\[
|E(W)| = \frac{1}{2} (p(p-1) - 1 - p)(p(p-1)) = \frac{1}{2} p^4 - 3p^3 + p^2 + p.
\]

It is clear that \( G_p(x^2) \) is isomorphic to \( T \vee W \). Mainly, we show that \( W \) is an eigensharp graph with \( bp(W) = a_-(W) \) and, by Theorem 2, \( G_p(x^2) \) is an eigensharp.

The adjacency matrix of \( W \) is

\[
A(W) = \begin{bmatrix}
A(K_p) & I_p & I_p & \cdots & I_p & I_p & N_p \\
I_p & A(K_p) & I_p & \cdots & I_p & N_p & I_p \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & I_p & \cdots & \cdots \\
\vdots & \vdots & \vdots & \cdots & I_p & \cdots & \cdots \\
N_p & I_p & I_p & \cdots & I_p & A(K_p) & I_p \\
I_p & N_p & I_p & \cdots & I_p & I_p & A(K_p)
\end{bmatrix}.
\]

Now, we show that

\[
\sigma(A(W)) = \begin{pmatrix}
p^2 - 2p - 1 & p - 1 & -(p+1) & -1 \\
1 & p^3 & (p-1)^2\end{pmatrix}.
\]

First, because each row of \( A(W) \) has \( p^2 - 2p - 1 \) ones entries, then \( A(W)1_s = (p^2 - 2p - 1)1_s \).
Second, because $\lambda = -1$ is an eigenvalue of $A(K_p)$ of multiplicity $p - 1$, then, it is clear that $\lambda = -1$ is an eigenvalue of $A(W)$ of multiplicity $(p - 1)^2$.

Third, if we look to the submatrix in the $(j, 1)$ entry of $A(W)F^{(m)}$, we obtain

\[
\begin{cases}
0_p, & \text{if } j \notin \{m, p-m\}, \\
(p-1)1_p, & \text{if } j = m \\
-(p-1)1_p, & \text{if } j = p-m.
\end{cases}
\]

where $j = 1, 2, \ldots, p-1$ and $m = 1, 2, \ldots, \frac{p-1}{2}$. Thus, $F = \{F^{(m)} : m = 1, 2, \ldots, \frac{p-1}{2}\}$ is a set of linearly independent eigenvectors of $A(W)$ corresponding to the eigenvalue $\lambda = p - 1$.

Fourth, similar to the third case, the $(j, 1)$ entry of $A(W)H^{(r)}$ is

\[
\begin{cases}
0_p, & \text{if } j \notin \{1, m, p-m, p-1\}, \\
(p+1)1_p, & \text{if } j \in \{1, p-1\} \\
(p+1)1_p, & \text{if } j \in \{p-m, m\}.
\end{cases}
\]

where $j = 1, 2, \ldots, p-1$ and $r = 2, 3, \ldots, \frac{p-1}{2}$. Thus, $H = \{H^{(m)} : r = 2, 3, \ldots, \frac{p-1}{2}\}$ is a set of linearly independent eigenvectors of $A(W)$ corresponding to the eigenvalue $\lambda = -(p + 1)$. Therefore, the set

\[
Q = \{1, s, F^{(1)}, F^{(2)}, \ldots, F^{(\frac{p-1}{2})}, H^{(2)}, H^{(3)}, \ldots, H^{(\frac{p-1}{2})}\}
\]

consists of $p - 1$ linearly independent eigenvectors, and because the multiplicity of $\lambda = -1$ is $(p - 1)^2$, then

\[|Q| + (p - 1)^2 = s = p^2 - p.\]

Hence, we obtain $s$ linearly independent eigenvectors of the matrix $A(W)$, which is of size $s \times s$.

Therefore, the characteristic polynomial of $A(W)$ is

\[
P(\lambda) = (\lambda + 1)(p-1)^2(\lambda - p^2 + 2p + 1)(\lambda + p + 1)^{\frac{p-3}{2}}(\lambda - p + 1)^{\frac{p-1}{2}},
\]

which gives $\sigma(A(W)) = \left\{\frac{p^2 - 2p - 1}{1}, \frac{p - 1}{p - 1}, \frac{-2p + 1}{\frac{p-3}{2}}, \frac{p - 1}{(p - 1)^2}\right\}$, thus

\[bp(W) \geq (p - 1)^2 + \frac{p - 3}{2} = \left\lfloor \frac{p}{2} \right\rfloor + (p - 1)^2 - 1.\]

Let $[i] : 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor - 1$ denote the class of vertices

\[\{i, i + X, i + 2X, \ldots, i + (p - 1)X\}.
\]

Let $[p - i] = \{p - i, p - i + X, p - i + 2X, \ldots, p - i + (p - 1)X\}$. Define $\varphi = [i] \cup [p - i]$, $\ell = \bigcup_{j=1}^{\left\lfloor \frac{p}{2} \right\rfloor} [i + j]$. Then, $|\varphi| = 2p$ and $|\ell| = p - 2l - 1$. Now, define $F_i : 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor - 1$ be a biclique subgraph of $W$, such that

\[V(F_i) = \varphi \cup \ell\]

and

\[E(F_i) = \{e_{rs} : r \in \varphi, s \in \ell\}.
\]
Then, $F_i$ is isomorphic to $K_{2p,p(p−2i−1)}$ with no pair of edges of $E(W)$, which belongs to a common $F_i$ and

$$\sum_{i=1}^{\lfloor \frac{p}{2} \rfloor -1} |E(F_i)| = 2p^2 \sum_{i=1}^{\lfloor \frac{p}{2} \rfloor -1} (p−2i−1) = \frac{1}{2} p^2 (p−1)(p−3).$$

Moreover, $B_j = \{ j + tx : 1 ≤ j ≤ p−1, 0 ≤ t ≤ p−2 \}$ is a complete subgraph of $W$. Now, consider the disjoint stars $S_{j+tx}$ in $B_j$ generated by the vertices

$$\{ j + tx : 1 ≤ j ≤ p−1, 0 ≤ t ≤ p−2 \}.$$

Then,

$$\sum_{j=1}^{p−1} \sum_{t=0}^{p−2} |E(S_{j+tx})| = \sum_{j=1}^{p−1} \left( \frac{p}{2} \right) = \frac{1}{2} p(p−1)^2.$$

and

$$\sum_{i=1}^{\lfloor \frac{p}{2} \rfloor -1} |E(F_i)| + \sum_{j=1}^{p−1} \sum_{t=0}^{p−2} |E(S_{j+tx})| = \frac{1}{2} p^2 (p−1)(p−3) + \frac{1}{2} p (p−1)^2 = |E(W)|,$$

which implies that

$$\mathcal{H}_W = \left\{ F_i, S_{j+tx} : 1 ≤ i ≤ \left\lfloor \frac{p}{2} \right\rfloor −1, 1 ≤ j ≤ p−1, 0 ≤ t ≤ p−2 \right\}$$

is a biclique partition of $W$ with cardinality $\left\lfloor \frac{p}{2} \right\rfloor + (p−1)^2 −1$. Therefore, $W$ is an eigensharp graph with $bp(W) = \left\lfloor \frac{p}{2} \right\rfloor + (p−1)^2 −1$, which implies that $G_p(x^2)$ is an eigensharp graph. $\square$

5. Conclusions

In this study, for each prime $p$, we proved that the graphs $G(Z_p), G(Z_{2p})$ and $G\left(\frac{Z_p[x]}{(x^2)}\right)$ are eigensharp. We showed that $G(Z_p)$ is isomorphic to a graph $K_1 \vee H$, where $H$ is a certain subgraph of $G(Z_p)$ and $G\left(\frac{Z_p[x]}{(x^2)}\right)$ is isomorphic to $T \vee W$, where $T$ is a certain independent set of $G\left(\frac{Z_p[x]}{(x^2)}\right)$ and $W$ is a certain subgraph of $G\left(\frac{Z_p[x]}{(x^2)}\right)$. Then, the adjacency matrices for $H$ and $W$ were studied to show that $a_-(H) = bp(H)$ and $a_-(W) = bp(W)$, which yields, by Theorem 2, that both graphs $G(Z_p)$ and $G\left(\frac{Z_p[x]}{(x^2)}\right)$ are eigensharp. The spectrum of the graph $A(G(Z_{2p}))$ was found to demonstrate that $bp(G(Z_{2p})) ≥ p$. We also described a biclique partition for $G(Z_{2p})$ with cardinality $p$; we hence concluded that $G(Z_{2p})$ is eigensharp.

Finally, we raise the following question: Does the eigensharp property hold for $Z_{p^2}, Z_{pq}$ and $Z_{p[x]}(x^2)$? We have attempted several examples to answer this question, but our research is still ongoing.

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