The aim of this paper is to specify the full spectrum of the classical Rabi system and to describe it in a uniform framework. Although the Rabi model has been investigated for over seventy years, its complete spectrum has not been determined, and some of its important properties are unknown.

The Rabi model was introduced by Rabi [25] to describe the action of a rapidly-varying, weak magnetic field on an oriented atom possessing nuclear spin. Now it is one of the ubiquitous models describing various quantum systems, including cavity and circuit quantum electrodynamics (QED), quantum dots, polaronic physics and trapped ions [3, 4, 7, 10, 12, 24]. Coupling between atoms and radiation is usually very weak. In such cases, the rotating wave approximation (RWA) is well-justified in the Jaynes–Cummings model. However, recent achievements in circuit QED have enabled the exploration of regimes—e.g. the ultrastrong and the deep strong coupling regimes of light-atom interaction—in which the Jaynes–Cummings model begins to fail and the observed phenomena can be explained only by the full Rabi system. More importantly, there is now evidence that even for weak couplings the RWA predicts behavior that differs significantly from that of the full system [19, 31]. For this reason, the Rabi model has now been revisited by many authors with the intent of a detailed analysis of its spectrum. To be more specific, the Hamiltonian in question is

\[ H = a^\dagger a + \mu \sigma_z + \lambda (\sigma_+ + \sigma_-)(a^\dagger + a), \] (1)

where \( a, a^\dagger \) are the photon annihilation and creation operators satisfying the canonical commutation relation \([a, a^\dagger] = 1\) which arise due to the harmonic oscillator representation of quantized electromagnetic field; \( \mu, \lambda \) are the level separation and photon-atom coupling constants, and the \( \sigma \) spin operators are here taken to be

\[
\begin{align*}
\sigma_+ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \sigma_- &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

The RWA is obtained by neglecting \( \sigma_- a \) and \( \sigma_+ a^\dagger \).

The Bargmann–Fock Hilbert space \( \mathcal{H} \), which we will be using, is a space of entire functions of one complex variable \( z \in \mathbb{C} \) with a scalar product given by

\[
(f, g) := \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(z)} g(z) e^{-|z|^2} d(\Re(z)) d(\Im(z)).
\]

In this Hilbert space, the operators \( a\dagger \) and \( a \) become multiplication and differentiation with respect to a complex variable \( z \). This stems from the commutation relation \([\partial_z, z] = 1\). Thus, the Rabi model in this representation is described by the following system of two differential equations

\[
\begin{align*}
(z + \lambda) \frac{d\psi_1}{dz} &= (E - \lambda z) \psi_1 - \mu \psi_2, \\
(z - \lambda) \frac{d\psi_2}{dz} &= (E + \lambda z) \psi_2 - \mu \psi_1,
\end{align*}
\]

where \( E \) is the energy, for details see [18]. In this representation, the two component wave function \( \psi = (\psi_1, \psi_2) \) is an element of a Hilbert space \( \mathcal{H}^2 = \mathcal{H} \times \mathcal{H} \). An entire function \( f(z) \) belongs to \( \mathcal{H} \) if it has proper growth at infinity see, e.g., [1, 36]. Namely their growth order must be less or equal 2, and if it is equal 2, then it must be of the type less than 1/2. Additionally, an entire function of growth order 2 and type 1/2 can be normalizable but in
this case an separate analysis is necessary. For definition of the growth order and type see, e.g., \[24\]. One can estimate from the above the growth order of entire solutions of a differential equation (or a system of differential equations) with rational coefficients by the degree at infinity of these coefficients, see \[20\]. These estimations for the equations describing the Rabi model show that all entire solutions of these equations have the growth order less or equal to one, so they are normalizable.

Let us remark here that in most cases when the Bargmann-Fock representation is used, it is not checked if obtained eigenfunctions are normalizable. We call attention to the fact that the Bargmann-Fock representation has several peculiar properties. For example, from the fact that an entire function \(f(z)\) is an element of \(\mathcal{H}\), it does not follow that entire function \(g(z) := zf(z)\) belongs to \(\mathcal{H}\).

The energy spectrum of the Rabi system has been investigated by many authors and various approaches are used. The most popular ones are numerical techniques equivalent to a large-scale diagonalization of a suitably defined finite subspace of the full Hilbert space \[5\, 9\, 8\, 11\, 35\]. Thus, the eigenvalues and eigenstates determined in this way are inaccurate by construction. Moreover, from such results it is difficult to extract the complete physical description of the system.

Schweber was the first to apply the Bargmann space to the Rabi model \[32\], and the first known analytical results are due to Swain \[33\, 34\], who employs a continued fraction technique to calculate transition amplitudes for spontaneous absorption and emission, and from this he tries to obtain the spectrum and the eigenfunctions of the Rabi Hamiltonian. However, this approach is ineffective. Reik and others \[26\] adapt Judd’s method \[15\], used originally for the Jahn-Teller system. Whereas Judd uses a power series substitution, Reik and others apply a Neumann series. They show that if \(x := E + \lambda^2\) is an integer and parameters of the system satisfy some equalities, then the Neumann series terminates, giving isolated, exact solutions known as the Judd solutions. These conditions were also obtained by Kuś \[18\] using another substitution. As described in \[18\], if, for a given \(n \in \mathbb{N}\), parameters of the problem \((\lambda, \mu)\) belong to the following algebraic set

\[J_n := \{ (\lambda, \mu) \in \mathbb{R}^2 \mid P_n(\lambda^2, \mu^2) = 0 \}, \]

where \(P_n(\lambda^2, \mu^2)\) is a certain polynomial of degree \(n\), then \(x = n\) is a doubly degenerate eigenvalue of the problem. Geometrically speaking, for each \(n \in \mathbb{N}\), Eq. \(4\) defines \(n\) algebraic ovals in the \((\lambda, \mu)\)-plane. The two corresponding eigenstates are known and given in terms of elementary functions \[17\]. These eigenstates have also been expressed explicitly by means of the Neumann series expansion \[20\]. Judd’s solutions can be recognized as quasi-exact solutions \[16\] in that only finitely many states are expressible in terms of elementary functions when appropriate conditions are met. In later work, it is shown \[2\] that for \(x \notin \mathbb{N}\), the spectrum of the Rabi problem can be determined as zeros of certain transcendental functions \(G_\pm(x, p) = 0\). For any pair \(p = (\lambda, \mu)\) of parameters there exist discrete values of admissible energy. Moroz \[23\] proposes yet another method of determining the spectrum.

Following Braak’s work \[2\], the Rabi model was deemed completely solved \[30\]. In other words, consensus held that all the spectral values of \(x\) were known, as well as the corresponding eigenvalues. However, it is demonstrated in this Letter that this conclusion is not completely true. To advance the argument, it is pertinent to establish what complete solution means. For arbitrary fixed values of parameters \(p = (\lambda, \mu) \in \mathbb{R}^2\), all values of energy \(E\) (or parameter \(x\)) for which system \(3\) has a non-zero entire solution \((\psi_1(z; x, p), \psi_2(z; x, p))\) must be specified. Thus, in three dimensional parameters space \(\mathcal{P} \subset \mathbb{R}^3\) those points \((x, \lambda, \mu) \in \mathcal{P}\) for which system \(3\) has an entire solution must be distinguished. All such points form a complicated set \(\mathcal{S}\), which is called the spectral set. To represent it graphically it is cut with a plane \(\mu = \mu_0\) or a plane \(x = x_0\). For a full description of \(\mathcal{S}\) both types of intersections are required. Subsets of \(\mathcal{S}\) with fixed values \(x = x_0\) are denoted by \(\mathcal{S}_{x_0}\). For a generic value of \(x\), the set \(\mathcal{S}_x\) contains infinitely many curves. An example for \(x = 2 + \pi\) is shown in Fig. \[1a\].

Maciejewski et al. \[22\] show that if \(x\) is not an integer, then the spectrum can be defined by one closed-form condition expressed as the Wronskian of the confluent Heun functions \(W(x, \lambda, \mu) = 0\). As an alternative description, a large part of spectral set \(\mathcal{S}\) is given by

\[W := \{(x, \lambda, \mu) \in \mathcal{P} \mid W(x, \lambda, \mu) = 0 \} \subset \mathcal{S}. \]

The Heun function generalizes the Gauss hypergeometric, wave spheroidal, Lamé and Mathieu functions, and finds numerous applications in quantum physics \[13\, 29\]. Since the general solution is expressible by this function, so are also the Juddian ones, which correspond to its degenerate cases. This fact was used in \[35\] to obtain all parts of the spectrum, using uniform formulae for the solutions regardless of the value of \(x\), and the authors use both a corrected version of Braak’s method (checking more than one \(z\) value for the \(G_+\) functions) and Wronskians, both of which elements can already be found in \[22\].

It is now shown that \(\mathcal{W}\) contains all points \((x, \lambda, \mu)\) of the spectral set \(\mathcal{S}\) with non-integer \(x\). This follows from considerations in earlier work \[22\]. The crux of the problem is the determination of points of the spectral set with \(x \in \mathbb{Z}\). Such values of \(x\) are not considered in other recent work \[2\]. Clearly, Judd states give such points, i.e., \(3_n \subset S_{n}\). Hence, the question is, if, apart from Judd states, there exist states with integer values of spectral parameter \(x\). This Letter argues that they do exist. For a fixed integer value of spectral parameter \(x = n\), the set...
$\mathcal{S}_n$ has two components. The first one, $\mathcal{S}_n$, consists of a finite number of curves corresponding to classical Judd states. The second one, $\mathcal{F}_n$, consists of infinitely many curves, corresponding to states that have not received attention in prior works. We can provisionally define it as $\mathcal{F}_n = \mathcal{S}_n \setminus \mathcal{J}_n$ (with a more specific definition to follow), and one goal of the remainder of the Letter will be to understand its properties. A graphical example is given in Fig. 16, where ovals $\mathcal{J}_n$ on the $(\lambda, \mu)$-plane are drawn with dashed lines, and curves $\mathcal{F}_n$ are drawn with continuous lines. That there are infinitely many branches of $\mathcal{F}_n$ can be intuitively seen from the fact, that the curves go continuously through the $\lambda = 0$ line, and in that case the solution of the system (3) is $\psi_1 = c_1 z^{E-\mu} + c_2 z^{E+\mu}$, $\psi_2 = c_1 z^{E-\mu} - c_2 z^{E+\mu}$. These are entire functions only for non-negative integer values of $n$ where $\mathcal{F}_n$ crosses the $\lambda = 0$ axis.

The spectrum of the model for $\mu = 1$ is shown in Fig. 14. The new elements of spectrum, denoted by gray squares, lie on the energy baselines $E + \lambda^2 = n$, where $n$ is an integer, much like the Judd eigenstates. However, these new elements are not degenerated. Another phenomenon is apparent in Fig. 2, which shows the spectrum for $\mu = 3/2$. Also plotted, with dotted lines, are the energy baselines corresponding to half-integer values of $n$. Manifestly, for each line of the spectrum with $E > 3$, for half-integer energy baseline between pairs of Judd states, the levels of the same parity seem to intersect. However, a magnification of the bottom-left corner of Fig. 2 shows that at these points the curves do not cross.

As aforementioned, for determination of the spectrum, the problem is reduced to the purely mathematical question of assessing if, for a given $(x, \lambda, \mu)$, system (3) admits an entire solution. The approach taken to this problem is based on the classical analytic theory of complex differential equations (see Inc. [13]). Although presented for the Rabi model alone, it is applicable to a wide class of systems.

It is convenient to transform system (3) to one second-order equation, putting $z = \lambda(2y - 1)$ and $\psi_1(y) = \exp(2\lambda^2 y)v(y)$. Eliminating $\psi_2$ from system (3), the confluent Heun equation is obtained [28],

$$v'' + \left(\alpha + \frac{\beta + 1}{y} + \frac{\gamma + 1}{y - 1}\right)v' + \left(\frac{\theta}{y} + \frac{\xi}{y - 1}\right)v = 0, \quad (6)$$

with parameters $\alpha = 4\lambda^2$, $\beta = -x$, $\gamma = -1 - x$, and

$$\theta = 4\lambda^2 + \mu^2 - x^2, \quad \xi = x(x - 4\lambda^2) - \mu^2. \quad (7)$$

Instead of $\theta$ and $\xi$, the parameter used are $\delta := 2\lambda^2$, and

$$\eta := \frac{1}{2} (1 + x + x^2) - \mu^2 - 2\lambda^2 (x + 1). \quad (8)$$

Eq. (6) has two regular singular points at $y = 0$, with exponents $(\rho_1^{(0)}, \rho_2^{(0)}) = (x, 0)$, and at $y = 1$, with exponents $(\rho_1^{(1)}, \rho_2^{(1)}) = (x + 1, 0)$. Infinity is an irregular singular point.

From the general theory (see Whittaker [Ch. X in [37]), it is known that at each regular singular point $y_* \in (0, 1)$ there are two independent local solutions. If the difference of exponents $(\rho_1^{(1)} - \rho_2^{(1)})$ is not an integer, then these solutions are of the form

$$v_i^*(y) = (y - y_*)^{\rho_i^*} h_i^*(y), \quad i = 1, 2, \quad (9)$$

where $h_i^*(y)$ are holomorphic at $y_*$. Initially it is assumed that $x$ is not a non-negative integer and that it belongs to the spectrum. Then the corresponding eigenvector is given by an entire solution $v(y)$ of (6). An expansion of $v(y)$ at a singular point $y_*$ coincides, up to a multiplicative constant, with a local solution holomorphic at $y_*$. In the considered case, the only locally holomorphic solutions are those corresponding to exponents $\rho_2^{(0)} = \rho_2^{(1)} = 0$. Thus, solutions $v_2^{(0)}(y)$ and $v_2^{(1)}(y)$ must coincide in the common part of their domains of definitions as they are expansions of one entire solution $v(y)$. Hence the Wronskian of these solutions must vanish. This gives a restriction on the parameters of the problem, and, in effect, it allows for the determination of the spectrum of the problem. However, expansions $v_2^{(0)}(y)$ and $v_2^{(1)}(y)$ are taken around different points. This can be dealt with by exploiting the fact that both these solutions can be expressed in terms of confluent Heun functions [28]. Denoting

$$H_0(y) := \text{HeunC}(a_0; y), \quad H_1(y) := \text{HeunC}(a_1; 1 - y),$$

the confluent Heun functions [28], with parameters $a_0 := (\alpha, \beta, \gamma, \delta, \eta)$ and $a_1 := (\alpha - \gamma, \beta, -\delta, \delta, \eta)$, it is found that $v_2^{(0)}(y) = H_0(y)$ and $v_2^{(1)}(y) = H_1(y)$. Thus the Wronskian of solutions is given by

$$w(x, p; y) := H_1(y)H_2^2(y) - H_1^2(y)H_2(y). \quad (10)$$

The Wronskian shown in (10) must vanish in the intersection of domains of solutions. But if it vanishes at one point of this intersection, then it vanishes identically. Taking $y = 1/2$ and the set $W(x, p) := w(x, p; 1/2)$, the zeros of $W(x, p)$ then determine the spectrum of the problem with the assumption that $x$ is not a non-negative integer. This explains why $\mathcal{W}$, given by (5), is a subset of the spectral set $\mathcal{S}.

If $x$ is a non-negative integer, then the problem is more complex. In such a case, generally, only local solutions corresponding to exponents $\rho_1^{(i)}$ with $i = 0, 1$ have the form (9), i.e., they are locally holomorphic. Solutions corresponding to exponents $\rho_2^{(0)} = \rho_2^{(1)} = 0$ may have logarithmic terms, and such solutions have the following
form

\[ v_2^{(i)}(y) = r_i v_1^{(i)}(y) \ln(y - y_i) + g_i(y), \quad i = 0, 1, \quad (11) \]

where \( r_i \) are constants depending on parameters, \( y_0 = 0 \), \( y_1 = 1 \), and \( g_i(y) \) are holomorphic at \( y_i \), for \( i = 0, 1 \).

If \( x \) is a non-negative integer that belongs to the spectrum, the corresponding eigenvector is given by the entire solution \( v(y) \) of (\ref{heuns}). If local expansions of \( v(y) \) at \( y_i \) do not vanish at \( y_i \), then they have to coincide, up to a multiplicative constant, with local solutions of \( v_2^{(i)}(y) \) with \( r_1 = r_2 = 0 \). By the Frobenius method \cite{14}, condition \( r_1 = 0 \) implies \( r_2 = 0 \) because \( v_2^{(i)} \) must be proportional. Thus, either logarithmic terms are present in both local solutions, or are not present at all.

Moreover, condition \( x = n \in \mathbb{N} \) is a necessary condition for the confluent Heun function to be a polynomial of degree \((n - 1)\). It coincides with the condition

\[ \delta_{n-1} =: \frac{\varphi}{\alpha} + \frac{1}{2}(\beta + \gamma) + n = 0, \quad (12) \]

(see condition (1.5a) in Fiziev \cite{9}); additionally, the condition \( r_1 = 0 \) coincides with the condition \( \Delta_{n-1} \) in Fiziev \cite{9}, and it guarantees that the confluent Heun function \( v(y) = Q_{n-1}(y) \) is a polynomial of degree \((n - 1)\).

It is now demonstrated that, assuming \( x = n \in \mathbb{N} \) and \( r_i = 0 \), for \( i = 0 \) or \( i = 1 \), Eq. (\ref{heuns}) has another independent entire solution. To wit, the dependent variable in (\ref{heuns}) is changed, setting \( v(y) := \exp[-4\lambda^2 y] w(y) \). Again, this yields the confluent Heun equation of the form (\ref{heuns}), but with new parameters \((\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\eta}) = (-\alpha, \beta, \gamma, \delta, \eta)\). Consequently, condition (1.5a) from Fiziev \cite{9} can be alternatively expressed as

\[ \frac{\varphi}{\alpha} + \frac{1}{2}(\tilde{\beta} + \tilde{\gamma}) + n + 1 = 0 \quad (13) \]

and it is automatically fulfilled. Moreover, it can be shown that condition \( \Delta_{n+1} \) for the transformed equation coincides with \( r_i = 0 \). Thus, the transformed equation admits a polynomial solution \( w(y) = R_n(y) \) of degree \( n \). To rephrase, with the given assumption, eigenvalues \( x = n \in \mathbb{N} \) are doubly degenerate. These are the classical Judd eigenstates \cite{13, 15}. Both eigenstates are expressed in terms of polynomial and exponential functions. In \cite{27} these solution are expressed as truncated Neumann series, but using transcendent functions, i.e., Bessel functions, is unnecessary here.

The above analysis of integer values of \( x \) is incomplete. Assuming that \( x = n \in \mathbb{N} \) belongs to the spectrum, and \( v(y) \) is the corresponding eigenvalue given by an entire solution of Eq. (\ref{heuns}), an expansion of \( v(y) \) at singular point \( y_i \) may have the form proportional to local solutions \( v_1^{(i)}(y) \) for \( i = 0, 1 \), which correspond to the non-zero exponents. It seems that this possibility was overlooked in earlier investigations. The only place where it is mentioned that for \( x = n \) there are other more complicated solutions is paper \cite{27}. However, only two numerical values of the parameters are given without any further discussion.

Let us note that, for \( x = n \), solutions \( v_1^{(i)}(y) \) are locally holomorphic without additional conditions (such as vanishing of the logarithmic term). Thus, in some sense, it is more natural to assume that local expansions of \( v(y) \) at singular points are proportional to \( v_1^{(i)}(y) \) than that they are proportional to \( v_2^{(i)}(y) \) which corresponds to already mentioned Judd states. In this case, local solutions \( v_1^{(0)}(y) \) and \( v_1^{(1)}(y) \), as two expansions of the same entire function \( v(y) \), must be proportional to each other in their common domain of definition. So,

\[ v_1^{(0)}(y) = \alpha v_1^{(1)}(y), \quad (14) \]

for a certain constant \( \alpha \), and for all \( y \) belonging to the common part of the domains of definition of considered solutions. Denoting \( v^{(0)}(y) := v_1^{(0)}(y) \) and \( v^{(1)}(y) := v_1^{(1)}(y) \) the condition shown in \cite{14} is expressed in a Wronskian form,

\[ w(y) := \det \begin{bmatrix} v^{(0)}(y) & v^{(1)}(y) \\ v^{(0)}(y) & v^{(1)}(y) \end{bmatrix} = 0. \quad (15) \]

As previously mentioned, if the Wronskian vanishes at one point, then it vanishes identically. Thus, in this case, the condition that \( x = n \) belongs to the spectrum can be written in the form \( w(1/2) = 0 \). This provides a condition on the parameters \( n \), and \( p = (\mu, \lambda) \) of the problem. Unlike the classical Judd solutions, the series does not terminate, and solutions \( v^{(0)}(y) \) and \( v^{(1)}(y) \) can only be found recursively. Moreover, these solutions are given as expansions at different points. This problem is now solved explicitly using Heun confluent functions. For the properties and more information on those functions see Ronveaux \cite{28} Section B]. It is found that

\[ v^{(1)}(y) := (y - 1)^{-\gamma} \text{HeunC}(c_1, 1 - y), \]

\[ v^{(0)}(y) := y^{-\beta}(y - 1)^{-\gamma} \text{HeunC}(c_0, y), \]

\[ c_1 = (-\alpha, -\gamma, \beta, -\delta, \delta + \eta), \quad c_0 = (\alpha, -\beta, -\gamma, \delta, \eta). \]

Denoting \( H_i(z) = \text{HeunC}(c_i, z) \), and \( h_i = H_i(1/2), h'_i = H'_i(1/2) \) for \( i = 0, 1 \), it is found that the Wronskian \( w(1/2) \) is proportional to the following function

\[ F_n(\lambda, \mu) := h_0 h'_1 + h_1 (2n h_0 + h'_0). \quad (16) \]

In this case, parameters \( c_0 \) and \( c_1 \) are expressed in terms of \( n \) and \( p = (\lambda, \mu) \).

For each \( n \in \mathbb{N} \), the following set is defined

\[ \mathcal{F}_n := \{ (\lambda, \mu) \in \mathbb{R}^2 \mid F_n(\lambda, \mu) = 0 \}. \quad (17) \]

This is the second component of spectral set \( \mathbb{S}_n \).
FIG. 1: a) Curves $S_x$ for $x = 2 + \pi$ on $(\lambda, \mu)$-plane. b) Curves $\mathcal{F}_5$ (continuous lines), $\mathcal{G}_5$ (dashed lines) in the $(\lambda, \mu)$ plane. c) Energy spectrum for resonant case $\mu = 1$. Gray circles are Juddian points and gray squares represent new elements of the spectrum. Energy baselines $E + \lambda^2 = p$ with $p \in \mathbb{N}_0$ are plotted with dashed lines.

FIG. 2: Energy spectrum for $\mu = 3.4$. An apparent level crossing is marked with a diamond, shown on magnification of the corner. The range of variables in this inset is $\lambda \in [0.806, 0.817]$ and $E \in [3.835, 3.850]$.

To summarize, the most significant result of this Letter is that the Judd solutions are only a special, finite subset of all eigenstates with integer $x = E + \lambda^2$. Here presented are hitherto unknown closed form conditions that also allow the system to have such eigenstates. In the Jud- dian case, the conditions are polynomial in $\mu$ and $\lambda$, and hence provide a finite number of solutions (e.g. $\mu$ in terms of $\lambda$). In contrast, the newly-discovered conditions are seemingly transcendental and yield, as numerical investigations show, infinitely many curves in the parameter plane $(\lambda, \mu)$, so there are infinitely many choices of $\mu$ for a given $\lambda$. Together with considerations for non-integer $x$, this analysis fully describes in a uniform framework the spectrum of the Rabi model, i.e. provides eigenenergies, eigenstates and the corresponding restrictions on all the other parameters.

Finally, we wish to comment on the questions of solvability and integrability of the Rabi model touched upon in [2]. It should first be made clear that the frequently used word “analytic(al)” has two meanings in this context. First, in connection with entire functions, meaning complex differentiable, as used by Bargmann himself in the title of his article [1]. Second and less rigorous, in context of solvability, where it usually means “expressible by elementary functions” or “explicit formulae”. We avoid the latter meaning and speak of solvability in that case, for example in quadratures or in terms of Liouvillian functions [39]. It should thus be understood that although the eigenfunctions here are analytic in the first sense, they do not, in general amount to what one would call explicit solutions. They can be written as Heun functions, which was discovered in [22], but one has to understand the jump in complexity from the $2F_1$ hypergeometric function to HeunC$(z)$, which is given by a series $\sum_n c_n z^n$, whose coefficients have to be determined recursively, i.e., $c_n$ are not given as explicit functions of the index $n$ (except for degenerate cases, see the next paragraph). Let us recall here, that for any linear system, a solution around a regular singular point can be constructed by the Frobenius method, as such series so one cannot consider providing
such a series a proof of solvability, for then all such systems would be solvable. With symbolic packages such as Maple now offering the Heun functions, the approach of [22] or [35] can be said to at least provide explicit formulae but the $G_{\pm}$ functions of [2] are just Frobenius series which do not indicate the solvability let alone integrability claimed therein.

Since there is the distinguished subset $\mathcal{J}_n$ of the spectrum, the Rabi model can be called quasi-solvable, as the Heun functions reduce to polynomials in the Juddian solutions – for a part of the spectrum we know both the exact eigenenergies and the eigenstates explicitly in terms of elementary functions. However, since the $Z_2$ symmetry seems to have no direct impact on the solvability, we feel that the parity operator is not enough to speak of quantum integrability for this system, for which another commuting operator would be required, providing more than a finite discrete symmetry.

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[1] V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform. Comm. Pure Appl. Math., 14:187–214, 1961.
[2] D. Braak. Integrability of the Rabi model. Phys. Rev. Lett., 107:100401, 2011.
[3] J. Casanova, G. Romero, I. Lizuain, J. J. García-Ripoll, and E. Solano. Deep Strong Coupling Regime of the Jaynes-Cummings Model. Phys. Rev. Lett., 105(26):263603, 2010.
[4] A. Crespi, S. Longhi, and R. Osellame. Photonic Realization of the Quantum Rabi Model. Phys. Rev. Lett., 108(16):163601, 2012.
[5] C. Durst, E. Sigmund, P. Reinkeker, and A. Scheuing. Treatment of non-adiabatic Hamiltonians by matrix continued fractions. I. Electronic two-level system coupled to a single vibrational mode. J. Phys. C: Solid State Phys., 19:2701–2720, 1986.
[6] C. Emary and R. F. Bishop. Bogoliubov transformations and exact isolated solutions for simple nonadiabatic Hamiltonians. J. Math. Phys., 43:3916–3926, 2002.
[7] D. Englund, A. Farao, I. Fushman, N. Stoltz, P. Petroff, and J. Vučković. Controlling cavity reflectivity with a single quantum dot. Nature, 450(7171):857–861, 2007.
[8] I. D. Feranchuk, L. I. Komarov, and A. P. Ulyanenkov. Two-level system in a one-mode quantum field: numerical solution on the basis of the operator method. J. Phys. A: Math. Gen., 29:4035–4047, 1996.
[9] P. P. Fiziev. Novel relations and new properties of confluent Heun’s functions and their derivatives of arbitrary order. J. Phys. A, 43(3):035203, 9 pages, 2010.
[10] P. Forn-Díaz, J. Lisenfeld, D. Marcos, J. J. García-Ripoll, E. Solano, C. J. P. M. Harman, and J. E. Mooij. Observation of the Bloch-Siegert Shift in a Qubit-Oscillator System in the Ultrastrong Coupling Regime. Phys. Rev. Lett., 105(23):237001, 2010.
[11] R. Graham and M. Höhnerbach. Quantum chaos of the two-level atom. Phys. Lett. A, 101(2):61–65, 1984.
[12] G. Günter, A. A. Anappara, J. Hees, A. Sell, G. Biasiol, L. Sorba, S. De Liberato, C. Ciuti, A. Tredicucci, A. Leitenstorfer, and R. Huber. Sub-cycle switch-on of ultrastrong light-matter interaction. Nature, 458(7235):178–181, 2009.
[13] M. Hortacsu. Heun Functions and their uses in Physics. Proceedings of the 13th Regional Conference on Mathematical Physics, Antalya, Turkey, October 27-31, 2010. U. Camci and I. Semiz eds., pp. 23-39. World Scientific, 2013. ArXiv e-prints:1101.0471 [math-ph], 2011.
[14] E. L. Ince. Ordinary Differential Equations. Dover Publications, New York, 1944.
[15] B. R. Judd. Exact solutions to a class of Jahn-Teller systems. J. Phys. C: Solid State Phys., 12(9):1685, 1979.
[16] R. Koç, M. Koca, and H Tüüncüiler. Quasi exact solution of the Rabi Hamiltonian. J. Phys. A: Math. Gen., 35:9425–9430, 2002.
[17] M. Kuś. On the spectrum of a two-level system. J. Math. Phys., 26(11):2792–2795, 1985.
[18] M. Kuś and M. Lewenstein. Exact isolated solutions for the class of quantum optical systems. J. Phys. A, 19(2):305–318, 1986.
[19] J. Larson. Absence of Vacuum Induced Berry Phases without the Rotating Wave Approximation in Cavity QED. Phys. Rev. Lett. 108:033601, 2012.
[20] I. Laine. Nevanlinna Theory and Complex Differential Equations. Walter de Gruyter, Berlin, New York, 1993.
[21] B. Ya. Levine. Lectures on Entire Functions. American Mathematical Society, Providence, Rhode Island, 1996.
[22] A. J. Maciejewski, M. Przybylska, and T. Stachowiak. How to calculate spectra of Rabi and related models. ArXiv e-prints:1210.1130 [math-ph], 2012.
[23] A. Moroz. On the spectrum of a class of quantum models. EPL, 100(6), 2012.
[24] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. Garcia-Ripoll, D. Zueco, T. Hümmer, E. Solano, A. Marx, and R. Gross. Circuit quantum electrodynamics in the ultrastrong-coupling regime. Nature Physics, 6:772–776, 2010.
[25] I. I. Rabi. On the process of space quantization. Phys. Rev., 49:324–328, 1936.
[26] H. G. Reik and M. Doucha. Exact solution of the Rabi hamiltonian by known functions? Phys. Rev. Lett., 57:787–790, 1986.
[27] H. G. Reik, H. Nusser, and L. A. A. Ribeiro. Exact solution of non-adiabatic model Hamiltonians in solid state physics and optics. J. Phys. A: Math. Gen., 15(11):3491, 1982.
[28] A. Ronveaux. Heun’s Differential Equations. Oxford University Press, Oxford, 1995.
[29] S. Yu. Slavyanov and W. Lay. Special functions. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.
[30] E. Solano. The dialogue between quantum light and matter. Physics, 4:68, 2011.
[31] M. Schiavo, M. Bordyuh, B. Oztom, H. E. Tureci. Phys. Rev. Lett. 109:053601, 2012.
[32] S. Schweber. On the application of Bargmann Hilbert spaces to dynamical problems. Ann. Phys., 41(2):205–226, 1967.
[33] S. Swain. A continued fraction solution to the problem...
of a single atom interacting with a single radiation mode in the electric dipole approximation. J. Phys. A : Math., Nucl. Gen., 6(2):192–204, 1973.

[34] S. Swain. Continued fraction expressions for the eigen-solutions of the hamiltonian describing the interaction between a single atom and a single field mode: Comparisons with the rotating wave solutions J. Phys. A : Math., Nucl. Gen., 6(12):1919–1934, 1973.

[35] É. A. Tur. Energy Spectrum of the Hamiltonian of the Jaynes-Cummings Model without Rotating-Wave Approximation. Opt. Spectrosc., 91:899–902, 2001.

[36] A. Vourdas. Analytic representations in quantum mechanics. J. Phys. A, Math. Gen., 39:R65, 2006.

[37] E. T. Whittaker and G. N. Watson. A Course of Modern Analysis. Cambridge University Press, London, 1935.

[38] H. Zhong, Q. Xie, M. T. Batchelor and C. Lee. Analytical eigenstates for the quantum Rabi model. J. Phys. A: Math. Theor., 46:415302, 2013.

[39] H. Zoladek. The Extended Monodromy Group and Liouvillian First Integrals. J. Dynam. Control Systems, 4, 1:1–28, 1998.