Quantum spectrum of black holes

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The black hole as the thermodynamical system in equilibrium possesses the periodicity of motion in imaginary time, that allows us to formulate the quasi-classical rule of quantization. The rule yields the equidistant spectrum for the entropy of non-rotating black holes as well as for the appropriately scaled entropy in the case of rotation. We clarify and discuss a role of quasi-normal modes.

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I. INTRODUCTION

Recently, we have used the technique of quantum thermal geodesics confined behind horizons of black hole in order to derive a quasi-classical spectrum of masses for rotating Kerr and BTZ black holes. For the Kerr black hole, we have found the following relation between the orbital momentum $J$ and mass $M$:

$$J = \frac{2\sqrt{l}}{l + 1} M_{\text{Kerr}}^2,$$  \hspace{1cm} (1)

where we put the gravitational constant equal to unit ($G = 1$). The parameter $l$ takes the values of

$$l = \{1, \frac{3}{2}, 2, 3, \infty\},$$  \hspace{1cm} (2)

representing the ratio of external horizon area to the internal one

$$l = \frac{A_+}{A_-},$$

as follows from the consistency of mapping the analytically continued space for radial geodesics completely confined behind the horizons. Therefore, the Kerr black hole with respect to rotation and radial motion has got two quantum numbers: the first is the integer or, generically, half integer momentum $J$, while the second is the ‘loop number’ $l$. It is important to emphasize that the quantum spectrum of Kerr black hole possesses the loop-duality: the spectrum is invariant under the action of duality transform

$$l \leftrightarrow \frac{1}{l}.$$

The extremal black hole corresponds to $l = 1$. The limit of $l \to \infty$ gives the Schwarzschild black hole, so that $J \to 2M^2/\sqrt{l} \to 0$, which indicates a breaking down such the quantization method in the case of Schwarzschild black hole.

The same note concerns the BTZ black hole, in which case we have found the following spectrum:

$$J = \frac{2k}{k^2 + 1} M_{\text{BTZ}}\ell,$$  \hspace{1cm} (3)

where $\ell$ is the curvature radius of AdS$_3$ space-time, while $k$ is the loop number for the BTZ black hole. The loop-duality

$$k \leftrightarrow \frac{1}{k}$$

remains the spectrum invariant. Again, the non-rotating limit of $k \to \infty$ misses the quantization in the form of (3).

Therefore, for non-rotating black holes we need a consistent quantization procedure supplemental to the case of $J \neq 0$. In section II we use the quasi-classical method for the periodic motion in purely imaginary time, which corresponds to the thermodynamical ensemble, that is the case of geodesics confined behind the horizons. The offered approach is applied to the Schwarzschild black hole in section III and to the BTZ black hole in section IV. We compare the procedure of quantization at $J = 0$ with that of $J \neq 0$ and clarify the difference.

We make notes on the connection of quantization with quasi-normal modes (see reviews in [7]). Several remarks are devoted to the comparison with another quantization procedure developed by [8]. Our results are summarized in Conclusion.

II. QUASI-CLASSICAL METHOD AND THERMODYNAMICAL ENSEMBLE

Let us start with a system possessing the only dynamical degree of freedom, the generalized coordinate $q$, moving periodically. Then, the quasi-classical quantization rule is the following:

$$\int p \ dq = 2\pi \hbar n, \hspace{1cm} n \in \mathbb{N},$$  \hspace{1cm} (4)

where $p$ is the momentum canonically conjugated to $q$, while $n \gg 1$ is the quantum number. In [9] we have
neglected a possible shift of \( n \) due to reflections at turn-
points. The energetic density of levels \( dn/dE \) can be easily
derived by differentiating \( \text{(4)} \) with respect to energy \( E \), so that
\[
\frac{dn}{dE} = \frac{1}{2\pi \hbar} \int \frac{\partial p}{\partial q} dq.
\] (5)
The Hamilton equations give
\[
\frac{\partial E}{\partial p} = \dot{q},
\] (6)
where \( \dot{q} = dq/dt \) is the velocity of motion. Therefore,
\[
\frac{dn}{dE} = \frac{\tau(E)}{2\pi \hbar}, \quad \tau(E) = \int dt,
\] (7)
where \( \tau \) is the period of motion depending on the energy. Introducing the phase frequency
\[
\omega(E) = \frac{2\pi}{\tau(E)},
\]
we get
\[
\frac{dn}{dE} = \frac{1}{\hbar \omega(E)}.
\] (8)
Thus, we have reminded the ordinary result on the spacing between the levels of energy: \( \Delta E = \hbar \omega(E) \Delta n \).
Summing up the number of levels in a given interval of energy we get
\[
n\hbar = \int \frac{dE}{\omega(E)},
\] (9)
or equivalently
\[
2\pi \hbar n = \int \tau(E) dE.
\] (10)
Next, in a thermodynamical equilibrium, a system is moving periodically in purely imaginary time, so that the period is fixed by the inverse temperature \( \beta = 1/T \). Therefore, we have got the substitution
\[
\tau(E) \mapsto -i\hbar \beta(E), \quad E \mapsto i\mathcal{E},
\] (11)
with \( \mathcal{E} \) denoting the Euclidean energy. So, we derive
\[
\int \beta(\mathcal{E}) d\mathcal{E} = 2\pi n, \quad n \in \mathbb{N}.
\] (12)
In the framework of quantum thermal geodesics confined behind the horizon, the black hole is represented by definite microstates, namely, the system of particles on geodesics in analytically continued space defined behind the horizon. Each particle is ascribed to a winding number \( n_W \) determining the number of cycles per period, i.e. \( \beta_{n_W} = \beta/n_W \). Summing up contributions by microstates is equal to summing over the particles and cycles:
\[
\sum_{\text{micro.}} \int \beta(\mathcal{E}) d\mathcal{E} \bigg|_{\text{micro.}} = \sum_{\text{cycle}} \sum_{n_W} \int \beta(\mathcal{E}) d\mathcal{E}.
\] (13)
Since
\[
\sum_{n_W} \frac{\beta}{n_W} = \beta,
\]
we get the same overall period, while the summing over particles gives the total energy of the black hole, i.e. its mass \( M \). Therefore, for the non-rotating black hole with the single external characteristics, the mass \( M \), the quantization rule of \( \text{(12)} \) takes the form
\[
\int \beta(M) dM = 2\pi n, \quad n \in \mathbb{N}.
\] (14)
Using the thermodynamical relation
\[
dM = T dS,
\]
where \( S \) is the entropy, we arrive to
\[
S = 2\pi n.
\] (15)
Thus, the quasi-classical quantization of non-rotating black hole results in the equidistant quantization of its entropy\(^1\), which is in agreement with the argumentation by J.Bekenstein in his pioneering paper on the quantum spectrum of black hole area \( \mathcal{A} \) as well as with further developments in \( \text{[10]} \).

III. KERR BLACK HOLE: \( J = 0 \) AND \( J \neq 0 \)

A. Schwarzschild black hole

Non-rotating black hole satisfies the condition of single dynamical variable, the black hole mass. Therefore, we straightforwardly get the quasi-classical spectrum
\[
S_n = 4\pi M_n^2 = 2\pi n \quad \Rightarrow \quad M_n^2 = \frac{n}{2}.
\] (16)
This result coincides with the quasi-classical limit of spectrum obtained in \( \text{[3]} \) in the framework of quantizing the effective dynamical system of black hole in terms of its global external characteristics. Then, after canonical transformation one gets the quantum system equivalent to harmonic oscillator. This transform uses the canonically conjugated pair of black hole mass \( M \) and periodic angle-like variable \( P \) as conjectured by authors of

\(^1\) We do not consider charged black holes, too, since the derivation has been based on the fact of single dynamical quantity of black hole, its mass.
However, we can point out a problem related with a constructing of Hermitian phase operator conjugated to the occupation number of oscillator in quantum mechanics as reviewed in \[11\]. Nevertheless, the problem is irrelevant in the quasi-classical approximation, but it is important while exact quantization. In addition, the oscillator-like spectrum yields the entropy, not the total energy, that could change the situation. Thus, the result of this section is in agreement with that of obtained by the method of \[8\].

Similar ideas were used by H.Kastrup in \[12\]. However, the obtained spectrum \(M_n^2 = n/4\) includes the additional one half, which reflects a systematic miss caused by the heuristic correspondence of energy multiplied by the period in imaginary time, with the quantized adiabatic invariant. So, we have improved this guess by more strict argumentations. Ref. \[12\] contains a discussion on the black hole entropy and thermodynamics, too.

B. Quasi-normal modes and quantum spectrum

Remarkably, one could explore independent determination of phase frequency \(\omega(E)\) in order to use the quantization rule in the form of \[9\], which is a formal expression of Bohr’s correspondence principle: classical frequencies reproduce increments of energy between the levels at high quantum numbers, i.e. \(\Delta E = \hbar \omega(E) \Delta n\). Such classical frequencies correspond to quasi-normal modes \[8\]. Those modes have got both real and imaginary terms (see original evaluations in \[13\], recent analytical results were obtained in \[14, 15\]). In \[16, 17\] authors use the real parts in order to quantize the black hole spectrum. However, as we have argued in section II, the thermodynamical system is inherently periodic with imaginary time. This fact is exactly reproduced by tower of the imaginary parts in quasi-normal modes. So, we insist that procedure based on the quantization performed above, if only one uses the imaginary part of classical frequencies, which are universal for classical fields with various spins, while the substitution of real parts of quasi-normal modes in quantization rule \[10\] seems to be misleading. Nevertheless, the real parts of quasi-normal frequencies could contain some other physical information.

This fact could be especially important in connection with attempts to use Bohr’s correspondence principle in the framework of Loop Quantum Gravity (see recent reviews in \[18\] and references therein) in order to fix both Immirzi parameter and quantum spacing of black hole horizon area \[18, 20\]. In that case one exploits the real parts of quasi-normal modes to relate it with the area law and minimal value of spin in its network. Unfortunately, to our opinion, again, the substitution of real parts of quasi-normal frequencies in Bohr’s correspondence principle is misleading in the context of black hole thermodynamics.

In addition, authors of \[14, 18\] argue for the real parts of quasi-normal modes cannot be straightforwardly applied to other black holes except the simplest case of Schwarzschild black holes in the context of Loop Quantum Gravity. Such the argumentation invalidates some proposals in \[21\]. Moreover, in \[14\] one finds a discussion, why real parts of asymptotic quasi-normal modes cannot be used in semi-classical considerations of Loop Quantum Gravity, at suggested in \[19, 20\].

C. Kerr black hole

At \(J \neq 0\) we have two dynamical variables, the mass \(M\) and orbital momentum \(J\), so one has to take into account the quantization of angular motion. Moreover, it is important to pay attention to both horizons, the external and internal ones. Indeed, the angular velocities of horizons are equal to

\[ \Omega_\pm = \frac{a}{r_\pm + a^2}, \]  

where \(a = J/M\), and \(r_\pm\) are the radii of horizons. The quantization of horizon-area ratio leads to strict relation between the mass and orbital momentum shown in \[11\]. Then, \(M\) and \(J\) are not independent at fixed loop \(l\) of \[2\], that makes the single-variable quantization of \[9\], \[10\] or \[14, 16\] irrelevant. Under relation \[11\] we get

\[ r_+ \Omega_+ = a, \quad r_- \Omega_- = J, \]

The temperatures at horizons are given by

\[ \beta_+ = 8\pi M \frac{l}{l - 1}, \quad \beta_- = \frac{\beta_+}{l}. \]

(19)

So, the self-dual angle of rotation per thermodynamical period is equal to

\[ \Delta \phi = \beta_+ \Omega_+ = \beta_- \Omega_- = 4\pi \frac{\sqrt{l}}{l - 1}. \]

(20)

The corresponding winding numbers for the ground state at horizons are given by

\[ n_+ = \frac{2l}{l - 1}, \quad n_- = n_+ \frac{2l}{l - 1} = \frac{2}{l - 1}. \]

(21)

Introduce a multiple period consistent for both horizons,

\[ \tau = \sqrt{\beta_+ \beta_-} = \frac{\beta_+}{\sqrt{l}}, \]

(22)

which gives the following rotation angles

\[ \Delta \phi_+ = \tau \Omega_+ = 2\pi \frac{2l}{l - 1} = 2\pi n_-, \]
\[ \Delta \phi_- = \tau \Omega_- = 2\pi \frac{2l}{l - 1} = 2\pi n_+. \]
Therefore, at both horizons the Kerr black hole makes rotations by angles multiple to $2\pi$ per the specified time period. The multiplication factors are identical to winding numbers. Thus, the complete periodicity with account of rotation takes place at $\tau = \beta_+/\sqrt{l}$. Note, that due to

$$T \, dS = dM - \Omega_+ dJ$$

we can deduce

$$\int_0^M \tau(M) dM \left(1 - \Omega_+ \frac{dJ}{dM}\right) = \frac{S}{\sqrt{l}} = 2\pi J,$$  \hspace{1cm} (24)

that provides the correct quantization of entropy $S$ as it was obtained in $[3]$. Thus, we should modify the quantization rule of $[13]$ by

$$\frac{T}{\beta} S = 2\pi n,$$  \hspace{1cm} (25)

valid in the case of rotation, though $n$ could be a subset of integer numbers.

In the quasi-classical approach, the horizon area spectrum is given by

$$\mathcal{A} = 8\pi \sqrt{l} J.$$  \hspace{1cm} (26)

In the same limit, formula (26) reproduces the spectrum obtained in $[3]$, if only one puts the loop $l = 1$. To our opinion, the reason for such the correspondence is transparent: if one ignores the dynamics on inner horizon (as in $[3]$), one gets the consistent quantization supposing a coherent rotation of both horizons, i.e. putting $l = 1$.

Finally, it is interesting to note, that combining the cases of $J = 0$ and $J \neq 0$ at $l = 1$, one could ascribe the spectrum of $M^2 = n/2$ to points of ‘daughter trajectories’ of main trajectory $J = M^2$ in the plane of $\{M^2, J\}$.

IV. BTZ BLACK HOLE: $J = 0$ AND $J \neq 0$

At $J = 0$ we use the quantization rule of $[13]$ to deduce

$$S_n = 2\pi \ell \sqrt{\frac{M_n}{2G}} = 2\pi n, \quad M_n = 2G \frac{n^2}{\ell^2}.$$  \hspace{1cm} (27)

The horizon area spectrum $\mathcal{A}_0 = 4G \, S_n$ is also equidistant.

At $J \neq 0$, after taking into account the spectrum of $[3]$, we find that the horizons rotate with angle velocities

$$\Omega_+ = \frac{1}{\ell k}, \quad \Omega_- = k^2 \Omega_+,$$  \hspace{1cm} (28)

while the corresponding temperatures are given by

$$\beta_+ = \pi \ell \frac{k}{k^2 - 1} \sqrt{\frac{kl}{GJ}}, \quad \beta_- = \frac{\beta_+}{k}.$$  \hspace{1cm} (29)

Two horizons consistently rotate by angles multiple to $2\pi$ at the period of

$$\tau = 4\pi \ell \frac{k}{k - 1},$$  \hspace{1cm} (30)

so that the angles are determined by the winding numbers of ground state at the horizons,

$$\Delta \phi_+ = \tau \Omega_+ = 2\pi \frac{2}{k - 1} = 2\pi n^{-W},$$  \hspace{1cm} (31)

$$\Delta \phi_- = \tau \Omega_- = 2\pi \frac{k^2}{k - 1} = 2\pi k n^{-W}.$$  \hspace{1cm} (31)

The ratio

$$\frac{\tau}{\beta_+} = 4(k + 1) \sqrt{\frac{GJ}{k \ell}}$$  \hspace{1cm} (32)

gives

$$\frac{\tau}{\beta_+} S = 2\pi J (2k + 2),$$  \hspace{1cm} (33)

which is consistent with $[25]$. The obtained result disagrees with consideration in $[17]$.

V. CONCLUSION

In the present paper we have used the periodic motion of thermodynamical ensemble in imaginary time in order to formulate the quasi-classical quantization rule for single-variable dynamical system of black hole, i.e. non-rotating black hole. The rule has given the equidistant quantization of entropy. The application of method to Schwarzschild and BTZ black holes has been considered. We have emphasized the difference with the treatment in terms of quasi-normal modes: to our opinion the use of real parts of frequencies is misleading in the problem under study, while the imaginary parts of quasi-normal modes reproduce our result. This fact makes irrelevant the treatment of quantum spectrum for black holes in terms of Loop Quantum Gravity as suggested in $[19, 20]$ as well as in the quasi-classical framework of $[16]$. Nevertheless, the real parts of quasi-normal frequencies could have another physical sense.

We have clarified the difference of single-variable approach with the case of rotating black holes. So, one has to take into account consistent multiple folding of rotation for both horizons. This consistency has required to scale the full period of motion for the black hole as a whole. This scaling has resulted in the modified quantization rule, which guarantees the appropriate equidistant quantization of scaled entropy.

The mentioned consistency adjusting the rotation of both horizons, was not generically taken into account in approach of $[3]$, which, therefore, is theoretically sound at $l \rightarrow 1$, only, in the quasi-classical limit, since quantization of periodic phase has some principal problems $[11]$. 

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