Binomial sums about Bernoulli, Euler and Hermite polynomials

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Abstract

Binomial sums about Bernoulli, Euler and Hermite polynomials are examined by making use of the symmetric summation theorem on polynomial differences, which is due to Chu and Magli [European J. Combin. 28 (2007) 921–930]. Several summation formulae are also obtained, including Barbero’s recent one on Bernoulli polynomials reported in [Comptes Rendus Math. 338 (2020) 41–44].

Keywords: binomial coefficient; Bernoulli polynomial; Euler polynomial; Hermite polynomial; recurrence relation.

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1. Introduction and motivation

In classical analysis and combinatorics, the Bernoulli and Euler numbers play an important role, that are defined respectively by

\[ \frac{\tau}{e^\tau - 1} = \sum_{n \geq 0} B_n \frac{\tau^n}{n!} \quad \text{and} \quad \frac{2e^{\tau\tau + 1}}{e^{\tau} + 1} = \sum_{n \geq 0} E_n \frac{\tau^n}{n!}. \]

The corresponding polynomials have the following generating functions:

\[ \frac{\tau e^{\tau\tau}}{e^\tau - 1} = \sum_{n \geq 0} B_n(x) \frac{\tau^n}{n!} \quad \text{and} \quad \frac{2e^{\tau\tau + 1}}{e^{\tau} + 1} = \sum_{n \geq 0} E_n(x) \frac{\tau^n}{n!}. \]

Both Bernoulli and Euler polynomials can be expressed by the corresponding numbers through the binomial relations

\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad \text{and} \quad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k(0)x^{n-k}. \]

They can be characterized by the following general polynomials associated to an arbitrary sequence \( \{a_n\} \) by the binomial sums

\[ A_n(x) = \sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k} \quad \text{for} \quad n = 0, 1, 2, \ldots. \]

Chu and Magli [5] found that these polynomials satisfy the following general algebraic identity, which has interesting applications to classical combinatorial numbers and polynomials, such as Bernoulli and Euler polynomials (cf. [9]).

Lemma 1.1 (Symmetric Difference). For two variables \( x, y \) and three integer parameters \( m, n, \ell \) with \( m, n \) being nonnegative, the following algebraic identity holds:

\[
\sum_{k=0}^{m} \binom{m}{k} A_{n+k+\ell}(x)(y-x)^{m-k} - \sum_{k=0}^{n} \binom{n}{k} A_{m+k+\ell}(y)(x-y)^{n-k} \\
= m!n!\chi(\ell > 0) \frac{(m+n+\ell)!}{(m+n+\ell)!} \sum_{k=1}^{\ell} \binom{m+n+\ell}{\ell-k} (-1)^{\ell-k} \binom{\ell-k}{m} A_{\ell-k}(y)(x-y)^{m+n+\ell+k}.
\]

Here and forth, \( \chi \) denotes, for brevity, the logical function with \( \chi(\true) = 1 \) and \( \chi(\false) = 0 \), otherwise. For two integers \( i, j \) and a natural number \( m \), the notation “\( i \equiv_m j \)” stands for that “\( i \) is congruent to \( j \) modulo \( m \)”.

There exist numerous summation formulæ and identities about the Bernoulli and Euler numbers and polynomials (cf. [1, 2, 4, 6, 7]). Recently, Barbero [3] discovered a new identity about Bernoulli polynomials. We find that Barbero’s
identity is an implication of Lemma \ref{lemma1.1} when \( \ell = 1, m = n \) and \( A_n(x) \) is specified to Bernoulli polynomial. This suggests us to examine further applications of Lemma \ref{lemma1.1}. In the next section, we shall prove a general theorem about Bernoulli polynomials, which contains Barbero’s identity as the special case \( \ell = 1 \). Then in Section 3, an analogous theorem for Euler polynomials will be shown, where three interesting formulae corresponding to \( \ell < 1, \ell = 1 \) and \( \ell = 2 \) will be highlighted. Finally, we illustrate an application to Hermite polynomials in Section 4, where some unusual identities are deduced.

2. Bernoulli polynomials

In Lemma \ref{lemma1.1}, performing first the replacements \( n \to m, y \to n - x \) and then specifying \( A_n(x) \) to Bernoulli polynomial, we have the equality (cf. \cite{9})

\[
\sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+\ell}(x)}{(m+k+1)_\ell} (n-2x)^{m-k} - \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+\ell}(n-x)}{(m+k+1)_\ell} (2x-n)^{m-k} = \frac{m!^2 \chi(\ell > 0)}{(2m+\ell)!} \sum_{k=1}^{\ell} \binom{2m+\ell}{\ell-k} \binom{-k}{m} B_{\ell-k}(n-x)(2x-n)^{2m+k},
\]

(2)

By iterating the recurrence relation

\[ B_m(1+x) = B_m(x) + mx^{m-1}, \]

we can reformulate the polynomial

\[ B_m(n-x) = B_m(n-1-x) + m(n-1-x)^{m-1} = B_m(n-2-x) + m(n-1-x)^{m-1} + m(n-2-x)^{m-1} = B_m(1-x) + m \sum_{i=1}^{n-1} (n-x-i)^{m-1}. \]

According to the reciprocal relation

\[ B_m(1-x) = (-1)^m B_m(x), \]

we deduce further the expression

\[ B_m(n-x) = (-1)^m B_m(x) + m \sum_{i=1}^{n-1} (i-x)^{m-1}. \]

Substituting this into (2) and then simplifying the resultant equation, we get the identity

\[
\Phi_{\ell}(m, n) = 2\chi(\ell \equiv 2 \mod 4) \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+\ell}(x)}{(m+k+1)_\ell} (n-2x)^{m-k} - \frac{m!^2 \chi(\ell > 0)}{(2m+\ell)!} \sum_{k=1}^{\ell} \binom{2m+\ell}{\ell-k} \binom{-k}{m} B_{\ell-k}(n-x)(2x-n)^{2m+k},
\]

(3)

where \( \Phi_{\ell}(m, n) \) is a double sum defined by

\[
\Phi_{\ell}(m, n) = \sum_{k=0}^{m} \binom{m}{k} (2x-n)^{m-k} \sum_{i=1}^{n-1} (i-x)^{m+k+\ell-1} (m+k+1)_{\ell-1}.
\]

(4)

The rightmost fraction can be expressed as a multiple integral with the integration domain

\[
\left\{ x \leq y_{\ell-1} \leq y_{\ell-2} \leq \cdots \leq y_2 \leq y_1 \leq i \right\}
\]

and then reformulated by reversing the integral order as

\[
\frac{(i-x)^{m+k+\ell-1}}{(m+k+1)_{\ell-1}} = \int_x^i dy_{\ell-1} \int_{y_{\ell-2}}^i dy_{\ell-2} \cdots \int_{y_2}^i dy_{3} \int_{y_2}^i dy_{2} \int_{y_1}^{y_{\ell-2}} dy_{\ell-2} \int_x^{y_{\ell-3}} dy_{\ell-3} \int_x^{y_{\ell-2}} dy_{\ell-2} \int_x^{y_{\ell-2}} dy_{\ell-1} = \int_x^i (i-y_1)^{m+k} \frac{(y_1-x)^{\ell-2}}{\ell!} dy_1.
\]
According to the binomial theorem, we get the expression

\[
\Phi_\ell(m, n) = \sum_{k=0}^{\ell} \binom{m}{k} (2x - n)^{m-k} \int_x^y (i - y_1)^{\ell-2} \frac{(y_1 - x)^{\ell-2}}{(\ell-2)!} dy_1
\]

\[
= \sum_{k=0}^{\ell} \int_x^y (y_1 - x)^{\ell-2} \frac{(i - y_1)^m (2x - n + i - y_1)^m}{(\ell-2)!} dy_1.
\]

(5)

Under the change of variable by \(y_1 = i - T(i - x)\), or equivalently \(T = \frac{y_1 - x}{i - x}\), the last integral becomes

\[
\int_x^y (y_1 - x)^{\ell-2} (i - y_1)^m (2x - n + i - y_1)^m dy_1
\]

\[
= (i - x)^{m+\ell-1} \int_0^1 T^m (1 - T)^{\ell-2} \{2x - n + T(i - x)\}^m dT.
\]

Expanding the binomial in the braces “\(\{\cdots\}\)”

\[
\{2x - n + T(i - x)\}^m = \{x - n + i - (1 - T)(i - x)\}^m
\]

\[
= \sum_{j=0}^{m} (-1)^j \binom{m}{j} (1 - T)^j (i - x)^j (x - n + i)^{m-j}
\]

and then evaluating the beta integral by

\[
\int_0^1 T^m (1 - T)^{\ell+j-2} dT = \frac{m!(\ell + j - 2)!}{(m + \ell + j - 1)!}
\]

we find the following expression for the afore-displayed integral:

\[
\int_x^y (y_1 - x)^{\ell-2} (i - y_1)^m (2x - n + i - y_1)^m dy_1
\]

\[
= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(\ell + j - 2)!}{(m + 1)_{\ell+j-1}} (i - x)^{m+j+\ell-1} (x - n + i)^{m-j}.
\]

By substituting this into (5), we get another double sum expression

\[
\Phi_\ell(m, n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(\ell - 1)!}{(m + 1)_{\ell+j-1}} \Omega_n(m + j + \ell - 1, m - j),
\]

(6)

where \(\Omega_n(\lambda, \mu)\) denotes the convolution of arithmetic progressions:

\[
\Omega_n(\lambda, \mu) = \sum_{i=1}^{n-1} (i - x)^\lambda (i + x - n)^\mu.
\]

Summing up, we have established the following theorem.

**Theorem 2.1.** For any variable \(x\) and three integer parameters \(m, n, \ell\) with \(m, n\) being nonnegative, the following algebraic identity holds:

\[
\Phi_\ell(m, n) = 2\chi(\ell \equiv 2 \mod 2) \sum_{k=0}^{m} \binom{m}{k} B_{m+k+\ell}(x) \frac{(n - 2x)^{m-k}}{(m + k + 1)\ell}
\]

\[
- \frac{m^2 \chi(\ell > 0)}{2m + \ell} \sum_{k=1}^{\ell} \binom{2m + \ell}{\ell - k} \binom{-k}{m} B_{\ell-k}(n - x)(2x - n)^{2m+k}.
\]

When \(\ell < 1\), Theorem 2.1 gives a simpler identity.

**Corollary 2.1** (\(\ell < 1; m \geq 0\) and \(n > 0\)).

\[
\Phi_\ell(m, n) = 2\chi(\ell \equiv 2 \mod 2) \sum_{k=0}^{m} \binom{m}{k} B_{m+k+\ell}(x) \frac{(n - 2x)^{m-k}}{(m + k + 1)\ell}.
\]
When $\ell = 1$, the double sum $\Psi_1(m,n)$ reduces to a single term in view of (6). In this case, we recover from Theorem 2.1 the following identity.

**Corollary 2.2** ($\ell = 1$: $m \geq 0$ and $n > 0$).

$$\Omega_n(m,m) = \frac{(-1)^m(n-2x)^{2m+1}}{(2m+1)^{2m}} + 2 \sum_{k=0}^{m} \binom{m}{k} \frac{B_{m+k+1}(x)}{m+k+1} (n-2x)^{m-k}.$$

It is obvious that the formula due to Barbero [3, Theorem 1] is equivalent to Corollary 2.2 under the replacement $x \to \frac{n-y}{2}$. However, our formula looks more elegant.

When $\ell = 2$, we find from Theorem 2.1, by taking into account that

$$B_0(x) = 1 \quad \text{and} \quad B_1(x) = x - \frac{1}{2},$$

the following unusual double sum evaluation.

**Corollary 2.3** ($\ell = 2$: $m \geq 0$ and $n > 0$).

$$\Phi_2(m,n) = \sum_{j=0}^{m} \frac{(-1)^j(m)_j}{(m+1)_j+1} \Omega_n(m+j+1, m-j) = (-1)^m \frac{(n-1)(n-2x)^{2m+1}}{2(2m+1)^{(2m)_m}}.$$

### 3. Euler polynomials

Analogously, making first the replacements $n \to m, y \to n-x$ and then specifying $A_n(x)$ to Euler polynomial in Lemma 1.1, we have another equality (cf. [9])

$$\sum_{k=0}^{m} \binom{m}{k} \frac{E_{m+k+\ell}(x)}{(m+k+1)_\ell} (n-2x)^{m-k} - \sum_{k=0}^{m} \binom{m}{k} \frac{E_{m+k+\ell}(n-x)}{(m+k+1)_\ell} (2x-n)^{m-k}$$

$$= \frac{m!^2 \chi(\ell > 0)}{(2m+\ell)!} \sum_{k=1}^{\ell} \binom{2m+\ell}{\ell-k} (-k)_m E_{\ell-k}(n-x)(2x-n)^{2m+k}.$$

(7)

By iterating the recurrence relation

$$E_m(1+x) = 2x^m - E_m(x),$$

we can reformulate the polynomial

$$E_m(n-x) = 2(n-x-1)^m - E_m(n-1-x)$$

$$= 2(n-x-1)^m - 2(n-x-2)^m + E_m(n-2-x)$$

$$= 2 \sum_{i=1}^{n-1} (-1)^{i-1}(n-x-i)^m - (-1)^n E_m(1-x).$$

According to the reciprocal relation

$$E_m(1-x) = (-1)^m E_m(x),$$

we deduce further the expression

$$E_m(n-x) = 2 \sum_{i=1}^{n-1} (-1)^{1+n-i(i-x)}^m - (-1)^{m+n} E_m(x).$$

Substituting this into (7) and then simplifying the resultant equation, we get the following counterpart identity of that in Theorem 2.1 for Euler polynomials.

**Theorem 3.1.** For any variable $x$ and three integer parameters $m, n, \ell$ with $m, n$ being nonnegative, the following algebraic identity holds:

$$\Psi_\ell(m,n) = 2 \chi(n + \ell \equiv 2 \mod 2) \sum_{k=0}^{m} \binom{m}{k} \frac{E_{m+k+\ell}(x)}{(m+k+1)_\ell} (n-2x)^{m-k}$$

$$- \frac{m!^2 \chi(\ell > 0)}{(2m+\ell)!} \sum_{k=1}^{\ell} \binom{2m+\ell}{\ell-k} (-k)_m E_{\ell-k}(n-x)(2x-n)^{2m+k},$$

(8)

where $\Psi_\ell(m,n)$ is a double sum defined by

$$\Psi_\ell(m,n) = 2 \sum_{k=0}^{m} \binom{m}{k} (2x-n)^{m-k} \sum_{i=1}^{n-1} (-1)^{n-i+1} \frac{(i-x)^{m+k+\ell}}{(m+k+1)_\ell}.$$  

(9)
By carrying out exactly the same procedure as that from (4) to (6), we can write \( \Psi(\ell, m, n) \) in terms of a multiple integral

\[
\Psi(\ell, m, n) = 2 \sum_{k=0}^{m} \left( \sum_{i=1}^{n-1} (-1)^{n-i+1} \binom{m}{k} (2x - n)^{m-k} \right) \int_{x}^{i} dy \int_{y}^{i-1} dy \cdots \int_{y_{1}}^{i} dy_{1} (i - y_{1})^{m+k} dy_{1}
\]

and then derive the following alternative expression

\[
\Psi(\ell, m, n) = 2 \sum_{j=0}^{m} (-1)^{n+j+1} \binom{m}{j} \frac{(\ell)_{j}}{(m+1)_{j+1}} \Omega_{\ell}(m + j + \ell, m - j),
\]

(10)

where \( \Omega_{\ell}(\lambda, \mu) \) stands for the alternating convolution of arithmetic progressions:

\[
\Omega_{\ell}(\lambda, \mu) = \sum_{i=1}^{n-1} (-1)^{i} (i-x)^{\lambda} (i+x-n)^{\mu}.
\]

Theorem 3.1 contains the following three interesting special cases.

**Corollary 3.1** \( (\ell < 1: m \geq 0 \text{ and } n > 0) \).

\[
\Psi(\ell, m, n) = 2 \chi(n + \ell \equiv 2 0) \sum_{k=0}^{m} \binom{m}{k} \frac{E_{m+k+\ell}(x)}{(m+k+1)_{\ell}} (n-2x)^{m-k}.
\]

**Corollary 3.2** \( (\ell = 1: m \geq 0 \text{ and } n > 0) \).

\[
\Psi_{1}(m, n) = \frac{(-1)^{m}(n-2x)^{2m+1}}{(2m+1)(2^{m})} \left\{ \begin{array}{ll}
0, & n \equiv 2 0; \\
+ \sum_{k=0}^{m} \binom{m}{k} \frac{E_{m+k+1}(x)}{m+k+1} (n-2x)^{m-k}, & n \equiv 2 1.
\end{array} \right.
\]

**Corollary 3.3** \( (\ell = 2: m \geq 0 \text{ and } n > 0) \).

\[
\Psi_{2}(m, n) = \frac{(-1)^{m}(n-1)(n-2x)^{2m+1}}{2(2m+1)(2^{m})} \left\{ \begin{array}{ll}
0, & n \equiv 2 1; \\
+ \sum_{k=0}^{m} \binom{m}{k} \frac{E_{m+k+2}(x)}{m+k+1} (n-2x)^{m-k}, & n \equiv 2 0.
\end{array} \right.
\]

### 4. Hermite polynomials

The Hermite polynomials are an important class of orthogonal polynomials (cf. Rainville [8, Chapter 11]). They are defined by the exponential generating function

\[
e^{2x\tau - \tau^{2}} = \sum_{n=0}^{\infty} H_{n}(x) \frac{\tau^{n}}{n!}.
\]

- **Explicit expression**

\[
H_{n}(x) = \sum_{k=0}^{n/2} (-1)^{k} \binom{n}{2k} (2k)! (2x)^{n-2k}.
\]

- **Reciprocal relation**

\[
H_{n}(-x) = (-1)^{n} H_{n}(x).
\]

- **Expansion formula**

\[
H_{n}(x + y) = \sum_{k=0}^{n} (2y)^{k} \binom{n}{k} H_{n-k}(x).
\]
Comparing (1) with the explicit formula of $H_n(x)$, we can see that there exists a reciprocal relation corresponding to Lemma 1.1, where $A_n(x)$ is specified by $H_n(x/2)$. Under the replacements $x \to 2x$ and $y \to 2y$, this reciprocity is stated in the following theorem.

**Theorem 4.1.** For two variables $x, y$ and three integer parameters $m, n, \ell$ with $m, n$ being nonnegative, the following algebraic identity holds:

$$\sum_{k=0}^{m} \binom{m}{k} \frac{H_{m+k+\ell}(x)}{(m+k+1)\ell} (2y-2x)^{m-k} - \sum_{k=0}^{n} \binom{n}{k} \frac{H_{m+k+\ell}(y)}{(m+k+1)\ell} (2x-2y)^{n-k}$$

$$= \frac{m!n!(\ell \geq 0)}{(n+m+\ell)!} \sum_{k=1}^{\ell} \binom{m+n+\ell}{\ell-k} (-1)^{\ell-k} H_{\ell-k}(y)(2x-2y)^{m+n+k}.$$ (11)

When $m = n$ and $x = -y$, this theorem gives the simpler expression below:

$$0 = 2\chi(\ell \equiv 2 \, 1) \sum_{k=0}^{n} (-4y)^{n-k} \binom{n}{k} \frac{H_{n+k+\ell}(y)}{(n+k+1)\ell} + \frac{n!^2 \chi(\ell \geq 0)}{(2n+\ell)!} \sum_{k=1}^{\ell} (-4y)^{2n+k} \binom{2n+\ell}{\ell-k} (-1)^{\ell-k} H_{\ell-k}(y).$$ (12)

In particular, the following summation formulae are believed to be new.

- $\ell \equiv 0$ with $\ell > 0$:
  $$\sum_{k=0}^{\ell} (-4y)^{2n+k} \binom{2n+\ell}{\ell-k} (-1)^{\ell-k} H_{\ell-k}(y) = 0.$$

- $\ell \equiv 2$ with $\ell \leq 0$:
  $$\sum_{k=0}^{n} (-4y)^{n-k} \binom{n}{k} \frac{H_{n+k+\ell}(y)}{(n+k+1)\ell} = 0.$$

- $\ell \equiv 2$ with $\ell > 0$:
  $$0 = \sum_{k=1}^{\ell} (-4y)^{2n+k} \binom{2n+\ell}{\ell-k} (-1)^{\ell-k} H_{\ell-k}(y) + \frac{2(2n+\ell)!}{n!^2} \sum_{k=0}^{n} (-4y)^{n-k} \binom{n}{k} \frac{H_{n+k+\ell}(y)}{(n+k+1)\ell}.$$

Alternatively, for $n = m$ and $y = -x$, the corresponding relation in Theorem 4.1 becomes

$$\sum_{k=0}^{m} \binom{m}{k} \frac{H_{m+k+\ell}(x)}{(m+k+1)\ell} (2n-4x)^{m-k} - \sum_{k=0}^{n} \binom{n}{k} \frac{H_{m+k+\ell}(n-x)}{(m+k+1)\ell} (4x-2n)^{m-k}$$

$$= \frac{m!^2 \chi(\ell \geq 0)}{(2m+\ell)!} \sum_{k=1}^{\ell} \binom{2m+\ell}{\ell-k} (-1)^{\ell-k} H_{\ell-k}(n-x)(4x-2n)^{2m+k}.$$ (13)

Unlike Bernoulli and Euler polynomials, the last expression cannot further be reduced unfortunately. Even though by making use of the expansion

$$H_m(n-x) = \sum_{j=0}^{m} \binom{m}{j} H_j(n-x) = \sum_{j=0}^{m} (-1)^j (2n)^{m-j} \binom{m}{j} H_j(x),$$

we can reformulate the second sum with respect to $k$ in (13) as

$$\sum_{k=0}^{m} \binom{m}{k} \frac{H_{m+k+\ell}(n-x)}{(m+k+1)\ell} (4x-2n)^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} \frac{H_{m+k+\ell}(n-x)}{(m+k+1)\ell} \sum_{j=0}^{m+k+\ell} (-1)^j (2n)^{m+k+\ell-j} \binom{m+k+\ell}{j} H_j(x)$$

$$= \sum_{k=0}^{m} \binom{m}{k} \frac{H_{m+k+\ell}(n-x)}{(m+k+1)\ell} \sum_{i=-\ell}^{m+\ell} (-1)^{m+i+\ell} (2n)^{m+i+\ell} \binom{m+k+\ell}{m+i+\ell} H_{m+i+\ell}(x)$$

$$= \sum_{i=-\ell}^{m+\ell} (-1)^{m+i+\ell} H_{m+i+\ell}(x) \sum_{k=\max(0,i)}^{m} \binom{m}{k} \frac{(m+k+\ell)(2n)^{k-i}(4x-2n)^{m-k}}{(m+k+1)\ell}.$$

However, it is not plausible to simplify this last double sum further.
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