THE REPRESENTATION TYPE OF HECKE ALGEBRAS OF TYPE B

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Abstract. This paper determines the representation type of the Iwahori-Hecke algebras of type B when \( q \neq \pm 1 \). In particular, we show that a single parameter non-semisimple Iwahori-Hecke algebra of type B has finite representation type if and only if \( q \) is a simple root of the Poincaré polynomial, confirming a conjecture of Uno’s [18].

1. Introduction

In this paper we determine the representation type of the Hecke algebras of type B. Previously, Uno [18] determined the representation type of the (one parameter) Iwahori-Hecke algebras for the rank 2 Coxeter groups and the Coxeter groups of type A. We build upon Uno’s work to study the Hecke algebras of type B; in particular, we settle Uno’s conjecture in this case.

Let \( R \) be an integral domain and suppose that \( q \) and \( Q \) are invertible elements of \( R \). The Iwahori-Hecke algebra \( H_{q,Q}(B_n) \) of type \( B_n \) is the unital associative \( R \)-algebra with generators \( T_0, T_1, \ldots, T_{n-1} \) and relations

\[
(T_0 + 1)(T_0 - Q) = 0, \quad (T_i + 1)(T_i - q) = 0 \quad \text{for} \quad 1 \leq i \leq n - 1, \\
T_0T_10_1T_1 = T_1T_0T_1T_0, \quad T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i \quad \text{for} \quad 1 \leq i \leq n - 2, \\
T_iT_j = T_jT_i \quad \text{for} \quad 0 \leq i < j - 1 \leq n - 2.
\]

We will determine the representation type of \( \mathcal{H} \).

Let \( \mathcal{H}(A_{n-1}) \) be the subalgebra of \( \mathcal{H} \) generated by \( T_1, \ldots, T_{n-1} \); then \( \mathcal{H}(A_{n-1}) \) is isomorphic to the Iwahori-Hecke algebra of the symmetric group of degree \( n \).

Let \( e \in \{1, 2, 3, \ldots, \infty\} \) be the multiplicative order of \( q \) in \( R \).

1 (Uno) [18, Proposition 3.7, Theorem 3.8] Suppose that \( R \) is a field and that \( q \neq 1 \). Then \( \mathcal{H}(A_{n-1}) \) is of finite representation type if and only if \( n < 2e \).

Note that although Uno stated the theorem only in the case where \( R \) is the field of complex numbers; this is not essential in his proof. We also remark that K. Erdmann and D. K. Nakano [10, Theorem 1.2] have determined the representation type of all of the blocks of \( \mathcal{H}(A_{n-1}) \); so (1) also follows from their result.

The following reduction theorem, together with (1), will allow us to assume that \( -Q \) is a power of \( q \).

2 (Dipper–James) [7, Theorem 4.17] Suppose that \( Q \neq -q^f \) for any \( f \in \mathbb{Z} \). Then \( \mathcal{H}_{q,Q}(B_n) \) is Morita equivalent to

\[
\bigoplus_{m=0}^{n} \mathcal{H}_q(A_{m-1}) \otimes \mathcal{H}_q(A_{n-m-1}).
\]

Key words and phrases. Hecke algebras, representation type.
Combining the last two results we obtain.

**Corollary 3.** Suppose that $R$ is a field and that $Q \neq -q^f$ for any $f \in \mathbb{Z}$. Then $\mathcal{H}_{q,Q}(B_n)$ is of finite representation type if and only if $n < 2e$.

It remains to determine the representation type of $\mathcal{H}$ when $Q = -q^f$ for some $f \in \mathbb{Z}$. When $Q = -q^f$ the relation for $T_0$ becomes $(T_0 + 1)(T_0 + q^f) = 0$. If $e$ is finite we may assume that $0 \leq f < e$. It is convenient to renormalize $T_0$ as $-T_0$, when $0 \leq f \leq \frac{e}{2}$, and as $-q^{-f}T_0$, when $\frac{e}{2} < f < e$; in this way, the relation for $T_0$ becomes $(T_0 - 1)(T_0 - q^f) = 0$ where $0 \leq f \leq \frac{e}{2}$ whenever $e$ is finite.

Henceforth we assume that $q$ is a primitive $e^{th}$ root of unity in $R$ and that $T_0$ satisfies the relation $(T_0 - 1)(T_0 - q^f) = 0$ where $0 \leq f \leq \frac{e}{2}$. As the $R$–algebra $\mathcal{H}$ now only depends on $q$ we now write $\mathcal{H} = \mathcal{H}_q(B_n)$, or $\mathcal{H}_{R,q}(B_n)$ when we wish to emphasize the choice of $R$.

The main result of this paper is the following. We will consider the cases $q = \pm 1$ (that is, $e = 1$ and $e = 2$) separately in [4].

**Theorem 4.** Suppose that $R$ is a field and that $e \geq 3$ and $0 \leq f \leq \frac{e}{2}$. Then $\mathcal{H}_q(B_n)$ is of finite representation type if and only if

\[ n < \min\{e, 2f + 4\}. \]

Uno [15] asked whether the representation type of a non–semisimple single parameter Iwahori–Hecke algebra is finite if and only if $q$ is a simple root of the Poincaré polynomial of the corresponding finite Coxeter group. With the assumptions currently in place, the one parameter Hecke algebra of type $B$ corresponds to $e$ being even and $f = \frac{e}{2} - 1$; so our result gives an affirmative answer to Uno’s question in type $B$. In fact, if $e$ is even and $\mathcal{H}$ is not semisimple then Theorem [4] says that $\mathcal{H}$ is of finite representation type if and only if $\frac{e}{2} = f + 1 \leq n < e$; this is if and only if $q$ is a simple root of the Poincaré polynomial ($q$ is a root of the factor $x^e - 1 = 0$). If $\mathcal{H}$ is a non–semisimple one parameter Hecke algebra of type $B$ with $e$ odd then, by Corollary [15], $\mathcal{H}$ is of finite representation type if and only if $e \leq n < 2e$; again, this is if and only if $q$ is a simple root of the Poincaré polynomial (this time $q$ is a root of $x^{2e} - 1 = 0$).

The proof of Theorem [4] will occupy all of this paper. In sections 2 and 3 we recall the results that we need from the representation theory of algebras and from the representation theory of $\mathcal{H}$; section 4 shows that $\mathcal{H}$ has infinite representation type when $n \geq e$; section 5 shows that $\mathcal{H}$ has infinite representation type when $n \geq 2f + 4$; finally, section 6 shows that $\mathcal{H}$ has finite representation type in the remaining cases.

### 2. Preliminaries on representation type

An algebra $A$ has finite representation type if there are only a finite number of isomorphism classes of indecomposable $A$–modules; otherwise, $A$ has infinite representation type. This section summarizes the results that we need on the representation type of algebras. More details can be found in the books of Auslander, Reiten and Smalø [3] and Benson [4].

Suppose that $K$ is a field. We always assume that $K$ is a splitting field for $A$. The following two results are well–known. Throughout the paper, all modules are right modules.
Lemma 5. Let $A$ be a finite dimensional $K$–algebra.

(i) Suppose that $I$ is a two–sided ideal of $A$ such that $A/I$ has infinite representation type. Then $A$ is of infinite representation type.

(ii) Suppose that $B$ is a direct summand of $A$ as a $(B,B)$–bimodule. Then

(a) If $B$ is of infinite representation type then so is $A$.

(b) If $A$ is of finite representation type then so is $B$.

Lemma 6. Let $A$ be a finite dimensional $K$–algebra and let $P_1, \ldots, P_l$ be the complete set of projective indecomposable $A$–modules, up to isomorphism. Then

(i) $A$ is Morita equivalent to $\text{End}_A(P_1 \oplus \cdots \oplus P_l)$.

(ii) for each $i$ the algebra $\text{End}_A(P_i)$ has infinite representation type if and only if $\text{End}_A(P_i) \cong K[x]/(x^m)$ for some integer $m \geq 0$ (which depends on $i$).

For any $A$–module $M$ let $\text{Rad} M$ be the Jacobson radical of $M$. Let $D_1, \ldots, D_l$ be a complete set of isomorphism classes of simple $A$–modules and let $P_1, \ldots, P_l$ be the corresponding projective indecomposables.

In the theory of algebras Dynkin diagrams are valued graphs, with the underlying graph being the usual Dynkin diagram; see, for example, [5, VII.3, p241]. If $A$ is a symmetric algebra, then the separation diagram of $A$ is the valued graph with vertices $\{1, \ldots, l, 1', \ldots, l'\}$ and edges $i \rightarrow j'$ where $a = [\text{Rad} P_i / \text{Rad}^2 P_i : S_j]$ and $b = [\text{Rad} P_i / \text{Rad}^2 P_i : S_j]$.

The following result is fundamental, and may be derived from the theory of hereditary algebras.

Theorem 7 (Gabriel). Suppose that $A$ is an indecomposable algebra such that $\text{Rad}^2 A = 0$. Then $A$ is of finite representation type if and only if the separation diagram of $A$ is a disjoint union of Dynkin diagrams of finite type as a valued graph.

The Auslander–Reiten quiver of $A$ is the directed graph with vertices the indecomposable $A$–modules and edges the irreducible morphisms between the indecomposables (a map $\varphi : M \rightarrow N$ is irreducible if $\varphi$ has no left or right inverse and whenever $\varphi$ factorizes as $\varphi = \theta \psi$ then either $\theta$ has a right inverse or $\psi$ has a left inverse).

Theorem 8 (Auslander). Let $A$ be an indecomposable algebra and suppose that the Auslander–Reiten quiver of $A$ has a connected component which has a finite number of vertices. Then $A$ is of finite representation type.

Uno used Auslander–Reiten sequences and Theorem 8 to prove the following.

Theorem 9 (Uno [5, Theorem 3.6]). Suppose that $A$ is a symmetric indecomposable algebra and that the decomposition matrix of $A$ can be written in the form

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 1 \\
0 & \cdots & 0 & 1
\end{pmatrix}
$$

Then $A$ is of finite representation type.
It turns out that in the cases where $\mathcal{H}$ has finite representation type all of the non–semisimple blocks of $\mathcal{H}$ satisfy the hypotheses of Theorem 3, hence they are of finite type. Hence, in principle, we can compute all of the indecomposable modules using Auslander–Reiten sequences when $\mathcal{H}$ has finite type.

We remark that Uno’s paper does not actually contain the statement above. However, the result can be extracted from his paper because the assumptions of Theorem 3 appear as [15, Theorem 3.4] and these are all that are used in the proof of his Theorem 3.6.

3. Results from the Representation Theory of $\mathcal{H}_q(B_n)$

We now turn to the representation theory of $\mathcal{H}_q(B_n)$.

Let $\ast$ be the anti–involution of $\mathcal{H}_q(B_n)$ determined by $T_i^\ast = T_i$ for $0 \leq i < n$. If $M$ is a right $\mathcal{H}_q(B_n)$-module then $\text{Hom}_K(M, K)$ is naturally a left $\mathcal{H}_q(B_n)$-module and it becomes a right module by twisting the $\mathcal{H}_q(B_n)$-action by the anti–involution $\ast$. We call this the dual of $M$; $M$ is self–dual if it is isomorphic to its dual.

It is well–known that $\mathcal{H}_q(B_n)$ is a symmetric algebra; hence we have the following.

**Lemma 10.** The algebra $\mathcal{H}_q(B_n)$ is a symmetric algebra. In particular, if $D$ is a simple $\mathcal{H}_q(B_n)$–module and $P$ is its projective cover then $P/\text{Rad} P \cong D$, $\text{Soc} P \cong D$ and $P$ is self-dual.

As $P$ is self-dual, the dual of the radical series of $P$ is the socle series of $P$. We remark that this does not mean that the radical series must be symmetric with respect to its middle layer.

Applying Lemma 3(ii)(a) to the inclusion $\mathcal{H}_q(B_m) \hookrightarrow \mathcal{H}_q(B_n)$, for $m \leq n$, yields the following.

**Corollary 11.** Suppose that $m \leq n$ and that $\mathcal{H}_q(B_m)$ is of infinite representation type. Then $\mathcal{H}_q(B_n)$ is of infinite representation type.

Recall that a partition of $n$ is an non–increasing sequence $\sigma = (\sigma_1 \geq \sigma_2 \geq \ldots)$ of non–negative integers such that $|\sigma| = n$ where $|\sigma| = \sum \sigma_i$. A bipartition of $n$ is an ordered pair $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of partitions $\lambda^{(1)}$ and $\lambda^{(2)}$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| = n$; we write $\lambda \vdash n$ and $|\lambda| = n$. The set of bipartitions is naturally a poset with partial order $\bowtie$ where $\lambda \bowtie \mu$ if for all $k \geq 1$ and $l \geq 1$

$$\sum_{i=1}^k \lambda_i^{(1)} \geq \sum_{i=1}^k \mu_i^{(1)} \text{ and } |\lambda^{(1)}| + \sum_{j=1}^l \lambda_j^{(2)} \geq |\mu^{(1)}| + \sum_{j=1}^l \mu_j^{(2)}.$$  

If $\lambda \bowtie \mu$ we say that $\lambda$ dominates $\mu$. If $\lambda \bowtie \mu$ and $\lambda \neq \mu$ we write $\lambda \triangleright \mu$.

Let $\mathcal{A} = \mathbb{Z}[t, t^{-1}]$ where $t$ is an indeterminate. Then Dipper, James and Murphy have shown that there exist a family $\{ S_{\mathcal{A}}^\lambda | \lambda \vdash n \}$ of free $\mathcal{A}$–modules which are equipped with operators $T_0, T_1, \ldots, T_{n-1} \in \text{End}_{\mathcal{A}}(S^{\lambda})$ which satisfy the defining relations of $\mathcal{H}(B_n)$. Consequently, $S^{\lambda} = S_{\mathcal{A}}^\lambda \otimes_{\mathcal{A}} K$ is a $\mathcal{H}_q(B_n)$–module, where $\mathcal{H}_q(B_n)$ is the Hecke algebra defined over the field $K$ with $q \in K$ and we consider $K$ as an $\mathcal{A}$–module by letting $t$ act on $K$ as multiplication by $q$.

The module $S^{\lambda}$ is a Specht module of $\mathcal{H}_q(B_n)$. It comes equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle uT_i, v \rangle = \langle u, vT_i \rangle$ for $0 \leq i < n$ and all $u, v \in S^{\lambda}$. 


Consequently, the module
\[ \text{rad} S^\lambda = \{ u \in S^\lambda \mid \langle u, v \rangle = 0 \text{ for all } v \in S^\lambda \} \]
is an \( \mathcal{H} \)–submodule of \( S^\lambda \). Set \( D^\lambda = S^\lambda / \text{rad} S^\lambda \).

The modules \( S^\lambda \) enjoy the following properties. Theorem 12(i) is proved by modifying the proof of [9] Theorem 6.1. The others are stated in [3, Theorem 6.5] and [3, Theorem 6.6].

**Theorem 12** (Dipper–James–Murphy [9]). Suppose that \( K \) is a field.

(i) Any \( \mathcal{H}_q(B_n) \)–submodule of \( S^\lambda \) contains \( \text{rad} S^\lambda \) or is contained in \( \text{rad} S^\lambda \).

In particular, the module \( D^\lambda \) is either 0 or an absolutely irreducible self–dual \( \mathcal{H}_q(B_n) \)–module.

(ii) \( \{ D^\mu : D^\mu \neq 0 \} \) is a complete set of pairwise non–isomorphic irreducible \( \mathcal{H}_q(B_n) \)–modules.

(iii) If \( D^\mu \neq 0 \) then the decomposition multiplicity \( [S^\lambda : D^\mu] \neq 0 \) only if \( \lambda \geq \mu \); further, \( [S^\mu : D^\mu] = 1 \).

In particular, if \( D^\mu \neq 0 \) then \( D^\mu \) is the unique head of \( S^\mu \) and \( \text{Rad} S^\mu = \text{rad} S^\mu \); consequently, \( S^\mu \) is indecomposable and if \( S^\mu \neq D^\mu \) then \( S^\mu \) has Loewy length at least 2. If \( D^\mu \neq 0 \) let \( P^\mu \) be the corresponding principal indecomposable \( \mathcal{H} \)–module; in other words, \( P^\mu \) is the projective cover of \( D^\mu \). Let \( d_{\lambda \mu} = [S^\lambda : D^\mu] \) be the multiplicity of \( D^\mu \) as a composition factor of \( S^\lambda \).

It is implicit in the work of Dipper, James and Murphy that \( \mathcal{H}_q(B_n) \) is a cellular algebra in the sense of Graham and Lehrer [12] (compare [3]). Consequently, the theory of cellular algebras gives us the following result.

**Corollary 13.** Let \( P \) be a projective \( \mathcal{H} \)–module.

(i) Then \( P \) has a Specht filtration; thus, there exist bipartitions \( \nu_1, \ldots, \nu_k \) and a filtration \( P = P^k \supset P^{k-1} \supset \cdots \supset P^1 > 0 \) such that \( P^i/P^{i-1} \cong S^\nu_i \), for \( 1 < i \leq k \), and \( i < j \) whenever \( \nu_i \triangleright \nu_j \).

(ii) Suppose that \( P = P^\mu \) for some bipartition \( \mu \) with \( D^\mu \neq 0 \). Then the Specht filtration can be chosen so that \( d_{\lambda \mu} = \# \{ 1 \leq i \leq k \mid \nu_i = \lambda \} \). In particular, if \( \lambda \) is maximal in the dominance ordering such that \( d_{\lambda \mu} \neq 0 \) then \( P^\mu \) has a submodule isomorphic to \( S^\lambda \).

**Proof.** These results are implicit in the work of Graham–Lehrer [12] (and slightly more explicit in [16, Lemma 2.19]). The existence of the filtration is exactly [12, Lemma 2.9(ii)]; that we can order the bipartitions \( \nu_i \) by dominance follows from the choice of \( \Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_d \) as a maximal chain of ideals in the proof of this result. Part (ii) follows by combining Lemma 2.10(i) and Theorem 3.7(ii) of [12]. \( \square \)

We remark that some care must be taken when working with Specht filtrations because Specht modules indexed by different bipartitions can be isomorphic when \( \mathcal{H} \) is not semisimple. This technicality can be avoided by working with a modular system and lifting the projective module to the discrete valuation ring where the Specht filtration is unambiguously defined.

In principle, Theorem 12(ii) produces all of the irreducible \( \mathcal{H} \)–modules; however, determining when \( D^\lambda \) is non–zero is still a difficult problem. The non–zero \( D^\lambda \) have now been classified by the first author; to describe this result we need some more nomenclature.
The diagram $[\lambda]$ of a bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is the set of nodes

$$[\lambda] = \{ (i; j, k) \mid 1 \leq j \leq \lambda^{(k)}_i \text{ for } i \geq 1 \text{ and } k = 1, 2 \},$$

which we will think of as being an ordered pair of arrays of boxes in the plane.

Given two nodes $x = (i; j, k)$ and $y = (i', j', k')$ we say that $y$ is below $x$ if either $k = k'$ and $i < i'$, or $k < k'$. Further, $x \in [\lambda]$ is removable if $[\lambda] \setminus \{ x \}$ is the diagram of a bipartition; similarly, $y \notin [\lambda]$ is addable if $[\lambda] \cup \{ y \}$ is the diagram of a bipartition. The content of $x$ is $c(x) = j - i + (k - 1)f$ and the residue of $x$ is $\text{res}(x) = c(x) \mod e$ — recall that $q$ is an $e^{th}$ root of unity and $Q = -q^f$. If $r = \text{res}(x)$ we call $x$ an $r$–node.

An $r$–node $x$ is normal if whenever $y$ is a removable $r$–node below $x$ then there are more removable $r$–nodes between $x$ and $y$ than there are addable $r$–nodes, and there are at least as many removable $r$–nodes below $x$ as addable $r$–nodes below $x$. In addition, $x$ is good if there are no normal $r$–nodes above $x$. Here, $0 \leq r < e$.

Finally, a bipartition $\mu$ is Kleshchev if either $\mu = (0, (0))$ or $\mu$ contains a good node $x$ such that $[\mu] \setminus \{ x \}$ is the diagram of a Kleshchev bipartition.

**Theorem 14** (Ariki [2]). Suppose that $\mu$ is a bipartition of $n$. Then $D^\mu \neq 0$ if and only if $\mu$ is a Kleshchev bipartition.

The proof of this result builds on the next theorem which reveals the deep connections between the representation theory of $\mathcal{H}_q(B_n)$ and the representation theory of the Kac–Moody algebra $U(\mathfrak{sl}_e)$ of type $A^{(1)}_{e-1}$ in characteristic zero. Let $\Lambda_0, \ldots, \Lambda_{e-1}$ be the fundamental weights of $U(\mathfrak{sl}_e)$ and for each dominant weight $\Lambda$ let $L(\Lambda)$ be the corresponding integral highest weight module.

Let $\mathcal{H}_q(B_n)\text{-mod}$ be the category of finite dimensional right $\mathcal{H}_q(B_n)$–modules and $\mathcal{H}_q(B_n)\text{-proj}$ be the category of finite dimensional projective $\mathcal{H}_q(B_n)$–modules. Finally, let $\mathcal{K}_0(\mathcal{C})$ be the Grothendieck group of the category $\mathcal{C}$.

**Theorem 15** (Ariki [3]). For $i = 0, 1, \ldots, e-1$ there exist exact functors

$$e_i, f_i : \mathcal{H}_q(B_n)\text{-mod} \to \mathcal{H}_q(B_{n+1})\text{-mod}$$

such that the operators induced by these and suitably defined operators $h_i$ for $i = 0, 1, \ldots, e-1$ and $d$ give $\mathcal{K}_0 = \bigoplus_{n \geq 0} \mathcal{K}_0(\mathcal{H}_q(B_n)\text{-proj}) \otimes \mathbb{Q}$ the structure of a $U(\mathfrak{sl}_e)$–module. Furthermore, $L(\Lambda_0 + \Lambda_f)$ is an $U(\mathfrak{sl}_e)$–module and if $K$ is a field of characteristic zero then the principal indecomposable $\mathcal{H}_q(B_n)$–modules correspond to elements of the Lusztig–Kashiwara canonical basis of $L(\Lambda_0 + \Lambda_f)$ under this isomorphism.

A modular system with parameters is a modular system $(K, \mathcal{O}, k)$ such that $K$ is a field of characteristic zero and $\mathcal{O}$ is a discrete valuation ring with residue field $k$, together with parameters $t \in \mathcal{O} \to K$ and $q \in k$ such that $t$ and $q$ have the same multiplicative order in $K$ and $k$ respectively and $t$ maps to $q$ under the canonical map $\mathcal{O} \to k$. Let $\mathcal{H}_k = \mathcal{H}_{k,q}(B_n)$ and $\mathcal{H}_R = \mathcal{H}_{R,t}(B_n)$, for $R \in \{ K, \mathcal{O} \}$, be the corresponding Hecke algebras. Then $\mathcal{H}_R \cong \mathcal{H}_\mathcal{O} \otimes \mathcal{O} R$ for $R \in \{ K, \mathcal{O} \}$.

Let $\Lambda = \Lambda_0 + \Lambda_f$ and fix a highest weight vector $v_\Lambda$ in $L(\Lambda)$.

If $\mu$ is a Kleshchev bipartition then we write $P^K_\mu$ for the principal indecomposable $\mathcal{H}_R$–module for $R \in \{ K, \mathcal{O}, k \}$. So if $\mathcal{O}$ is a complete discrete valuation ring then $P^K_\mu = P_0 \otimes \mathcal{O} k$ and $P^K_\mu$ is a direct summand of $P_0 \otimes \mathcal{O} K$. By abuse of notation, we also let $[P^K_\mu]$ denote both the equivalence class of $P^K_\mu$ in $\mathcal{K}_0$ and the corresponding canonical basis element of $L(\Lambda)$.
Corollary 16. Let \((K, O, k)\) be a modular system with parameters. Suppose that \(\mu\) is a Kleshchev bipartition such that \([P^\mu_K]\) is isomorphic to \(L\) below. Then the decomposition map sends \([P^\mu_K]\) to \([P^\mu]\).

Proof. Let \(O \subset \hat{O}\) be an embedding into a complete discrete valuation ring \(\hat{O}\) where \(\hat{O}\) has residue field \(k\) and \(K \subset \hat{K}\) is a field extension such that \(O \subset \hat{K}\). Since \(D^\mu\) is absolutely irreducible, \(P^\mu_K = P^\mu_{\hat{K}} \otimes \hat{K}\). Hence it is enough to prove the statement under the assumption that \(O\) is a complete discrete valuation ring.

Set \(N = m_1! \ldots m_l!\) and let \(M_O = f^{(m_1)} \ldots f^{(m_l)} 1_{B_0}\) where \(1_{B_0}\) is the trivial \(H_{O,1}(B_0)\)-module. Then \(M_O\) is a projective \(H_{O}\)-module by the definition of the \(f_i\). Therefore, by Corollary 13 applied to \(H_{O}\), the module \(M_O\) has a Specht filtration. Let \(M_K = M_O \otimes K\); then, by assumption, \([M_K] = N[P^\mu_K]\); therefore, \([M_K] = N[S^\mu_K] + \sum_{\lambda, \mu} N\delta_{\lambda\mu}[S^\lambda_K]\), where \(\delta_{\lambda\mu} = [S^\lambda_K : D^\mu_K]\). Consequently, there is a surjective homomorphism \(M_O \to S^\mu_O \otimes N\); tensoring with \(\bar{k}\) gives a surjective homomorphism \(M_k \to D_k \otimes N\) (as \(D^\mu\) is the head of \(S^\mu\)). Since \(M_k\) is a projective \(\mathcal{H}_k\)-module there exists a map \(\psi\) which makes the following diagram commute.

\[
\begin{array}{ccc}
M_k & \xrightarrow{\mu} & \Psi \\
\downarrow \varphi & & \downarrow \\
P^\mu \otimes N & \rightarrow & D^\mu \otimes N \rightarrow 0
\end{array}
\]

Now, \(\varphi\) is surjective so \(\psi\) must also be surjective. On the other hand, \(P^\mu_k\) is a projective \(\mathcal{H}_k\)-module so we also have a commutative diagram

\[
\begin{array}{ccc}
P^\mu \otimes N & \xrightarrow{\mu} & M_k \\
\downarrow \psi & & \downarrow \\
P^\mu_k & \rightarrow & P^\mu \otimes N \rightarrow 0
\end{array}
\]

Hence, \(P^\mu_k \otimes N\) is a direct summand of \(M_k\). Note that
\[
\dim_k P^\mu_k \otimes N = \dim_K (P^\mu_O \otimes K)^\otimes N \geq \dim_K P^\mu K \otimes N = \dim_K M_K = \dim_k M_k \geq \dim_k P^\mu_k \otimes N;
\]

thus, \(M_k = P^\mu_k \otimes N\). However, \(M_K = P^\mu_K \otimes N\); so we have shown that the modular reduction of \(P^\mu_k \otimes N\) is \(P^\mu_k \otimes N\), as required. \(\square\)

In order to apply the last two results we need to set up the machinery for computing the canonical basis elements \([P^\lambda_k]\). Let \(v\) be an indeterminant over \(\mathbb{Z}\) and let \(\mathcal{A} = \mathbb{Z}[v, v^{-1}]\). The Fock space is the infinite dimensional \(\mathcal{A}\)-module

\[\mathcal{F}_{\mathcal{A}} = \bigoplus_{\lambda \in \Lambda} \mathcal{A} \lambda.\]

Let \(U_{\mathcal{A}}(\hat{sl}_e)\) be Lusztig’s \(\mathcal{A}\)-form of the quantum group of \(U(\hat{sl}_e)\). Then there are \(v\)-analogues \(E_i\) and \(F_i\) of the operators \(e_i\) and \(f_i\) which act on \(\mathcal{F}_{\mathcal{A}}\) and give it the structure of a \(U_{\mathcal{A}}(\hat{sl}_e)\)-module; an explicit description of \(E_i\) and \(F_i\) is given below. The \(U_{\mathcal{A}}(\hat{sl}_e)\)-submodule of \(\mathcal{F}_{\mathcal{A}}\) generated by the bipartition \(((0),(0))\) is isomorphic to \(L_{\mathcal{A}}(\Lambda)\), the \(\mathcal{A}\)-form of \(L(\Lambda)\). (Recall that \(\Lambda = \Lambda_0 + \Lambda_f\)).
Identifying \( L_{sf}(\Lambda) \) with \( U_{sf}(\widehat{\mathfrak{sl}_e}) \cdot ((0), (0)) \), Theorem \[11\] can be reinterpreted as saying that if \( \mu \) is a Kleshchev bipartition then there exist polynomials \( d_{\lambda\mu}(v) \) such that

\[(17) \quad [P_{\ell\ell}^\mu] = \mu + \sum_{\lambda \vdash n} d_{\lambda\mu}(v)\lambda, \quad \text{where} \ d_{\lambda\mu}(v) \in v\mathbb{Z}[v],\]

and \( d_{\lambda\mu} = d_{\mu\lambda}(1) \). Uglov \[17\] has given an explicit algorithm for computing the canonical basis elements \([P_{\ell\ell}^\lambda]\). Uglov actually works with a different Fock space; however, we can compute the canonical basis of inside our Fock space by modifying his algorithm. For the applications we have in mind it is enough to know that if \( F_{i_1}^{(m_1)} \ldots F_{i_r}^{(m_r)} \cdot ((0), (0)) \) can be written in the form of the right hand side of \((17)\) then it is an element of the canonical basis of \( L_{sf}(\Lambda) \); hence, in such situations we may apply Corollary \[10\]. We now recall from \[3\] how \( U_{sf}(\widehat{\mathfrak{sl}_e}) \) acts on \( F_{\ell\ell} \).

Suppose that \( \lambda \) is a bipartition of \( n - 1 \) and that \( \mu \) is a bipartition of \( n \). We write \( \lambda \xrightarrow{i} \mu \) if \([\mu] \setminus [\lambda] = \{x\}\) and \( \text{res}(x) = q^i \). For \( 0 \leq i < e \) let

\[
N_i^\ell(\nu, \lambda) = \# \left\{ \nu \xrightarrow{i} \alpha | \alpha \triangleright \lambda \right\} - \# \left\{ \beta \xrightarrow{i} \lambda | \nu \triangleright \beta \right\},
\]

\[
N_i^\ell(\lambda, \mu) = \# \left\{ \lambda \xrightarrow{i} \alpha | \mu \triangleright \alpha \right\} - \# \left\{ \beta \xrightarrow{i} \mu | \beta \triangleright \lambda \right\}.
\]

By \[3\] Prop. 2.6 the action of \( E_i \) and \( F_i \) on \( F_{\ell\ell} \) is determined by

\[
E_i \lambda = \sum_{\nu \xrightarrow{-i} \lambda} v^{-N_i^\ell(\nu, \lambda)} \nu, \quad \text{and} \quad F_i \lambda = \sum_{\lambda \xrightarrow{+i} \mu} v^{N_i^\ell(\lambda, \mu)} \mu.
\]

We will only need the formula giving the action of \( F_i \). There are similar formulæ for the action of the remaining generators of \( U_{sf}(\widehat{\mathfrak{sl}_e}) \).

The final tool that we shall need is the analogue of the Jantzen sum formula for Hecke algebras of type \( B \) over an arbitrary field \( K \). The setup is a little technical; we include it for completeness. Let \( p \) be the maximal ideal of \( K[t] \) generated by \( t - q \) and let \( \mathcal{O} = K[t]/p \), where \( t \) is an indeterminate over \( K \); then \( K \cong \mathcal{O}/p \) (so \( \mathcal{O} \) is a localized ring and, in particular, a discrete valuation ring). Let \( \mathcal{H}_\ell \) be the Hecke algebra over \( \mathcal{O} \) with parameters \( t \) and \( -q^i(t-q+1)^n \); then \( \mathcal{H}_\ell \otimes K(t) \) is semisimple and \( \mathcal{H}_K \) is the reduction of \( \mathcal{H}_\ell \) modulo \( p \). Let \( \nu_p \) be the \( p \)-adic valuation on \( \mathcal{O} \).

Previously we defined the residue \( \text{res}(x) \) of a node \( x = (i, j, k) \) to be \( \text{res}_\mathcal{O}(x) = v^{q^{-i}(t-q+1)^n} \). The relationship between these two definitions is that \( \text{res}_\mathcal{O}(x) \otimes 1_K = q^{\text{res}(x)} \).

Let \( \lambda \) be a bipartition and for each node \( x = (i, j, k) \in [\lambda] \) let \( r_x \) be the corresponding rim hook (so \( r_x \) is a rim hook in \([\lambda(k)]\)); the point is that \([\lambda] \setminus r_x \) is the diagram of a bipartition. Let \( \ell_\ell(r_x) \) be the leg length of \( r_x \) and define \( \text{res}_\mathcal{O}(r_x) = \text{res}_\mathcal{O}(f_x) \) where \( f_x \) is the foot node of \( r_x \). The definitions of these terms can be found, for example, in \[14\].

Suppose that \( \lambda \) and \( \mu \) are bipartitions of \( n \). If \( \lambda \not\triangleright \mu \) let \( g_{\lambda\mu} = 1 \); otherwise set

\[
g_{\lambda\mu} = \prod_{x \in [\lambda]} \prod_{y \in [\mu]} (\text{res}_\mathcal{O}(r_x) - \text{res}_\mathcal{O}(r_y))^{\varepsilon_{xy}},
\]

where \( \varepsilon_{xy} = (-1)^{\ell_\ell(r_x) + \ell_\ell(r_y)} \). The \( g_{\lambda\mu} \) are not as complicated as their definition suggests; they have a nice combinatorial interpretation, see \[14\] Example 3.39.

Finally, let \( S_\mathcal{O}^\lambda \) and \( S_K^\lambda \) be the Specht modules for \( \mathcal{H}_\ell \) and \( \mathcal{H}_K \) respectively. For each \( i \geq 0 \) define \( S_\mathcal{O}^\lambda(i) = \{ u \in S_\mathcal{O}^\lambda | \langle u, v \rangle \in p^i \text{ for all } v \in S_\mathcal{O}^\lambda \} \). Then the Jantzen
filtration of $S^\lambda_K$ is the filtration

$$S^\lambda_K = S^\lambda_K(0) \geq S^\lambda_K(1) \geq S^\lambda_K(2) \geq \ldots$$

where $S^\lambda_K(i) = (S^\lambda_K(i) + pS^\lambda_O)/pS^\lambda_O$. In particular, note that $\text{rad } S^\lambda_K = S^\lambda_K(1)$.

We can now state the analogue of Jantzen’s sum formula for $\mathcal{H}_K$.

**Theorem 18** (James–Mathas [14], Theorem 4.6). Let $\lambda$ be a bipartition of $n$. Then

$$\sum_{i > 0} |S^\lambda_K(i)| = \sum_{\mu \triangleright \lambda} \nu_p(g_{\lambda\mu})|S^\mu_K|.$$  

in the Grothendieck group $K_0(\mathcal{H}_K/\text{mod})$ of $\mathcal{H}_K$.

In general, if $\lambda \triangleright \mu$ then $\nu_p(g_{\lambda\mu})$ is non-zero only if it is possible to remove a rim hook $r_x$ from $\lambda$ and reattach it to $\mu$ without changing the residue $\text{res}(r_x)$ of the foot node. In fact, we will only apply this result when $n < e$; in this situation we have $\nu_p(g_{\lambda\mu}) \in \{0, 1\}$ so the technicalities above can be ignored.

All of the composition factors of a Specht module belong to the same block (for example, because $\mathcal{H}$ is cellular); we abuse notation and say that $\lambda$ is contained in the block $B$ if $S^\lambda$ is contained in $B$. Say that two bipartitions $\lambda$ and $\mu$ are linked by hooks if there is a sequence of bipartitions $\lambda = \nu_1, \ldots, \nu_l = \mu$ such that, for each $i$, $[\nu_{i+1}] \setminus r_{y_i} = [\nu_i] \setminus r_{x_i}$ and $\text{res}(r_{x_i}) = \text{res}(r_{y_i})$ for some nodes $x_i \in [\nu_{i+1}]$ and $y_i \in [\nu_i]$. 

**Proposition 19.** Suppose that $S^\lambda$ and $S^\mu$ are in the same block. Then $\lambda$ and $\mu$ are linked by hooks.

**Proof.** By definition, $S^\lambda$ and $S^\mu$ are in the same block if and only if there exists a sequence of bipartitions $\mu = \nu_1, \ldots, \nu_l = \lambda$ such that $S^\nu_i$ and $S^\nu_{i+1}$ have a common composition factor. Thus, it is enough to prove that if $D^\mu \neq 0$ appears in $S^\lambda$ then $\lambda$ and $\mu$ are linked by hooks. If $\lambda \neq \mu$ and $\lambda \triangleright \mu$ by Theorem [13](iii). The sum formula implies that $D^\mu$ appears in $S^\nu$ for some $\nu$ such that $\lambda \triangleright \nu$ and $\lambda$ and $\nu$ are linked by hooks. By induction on dominance $\nu$ and $\mu$ are linked by hooks so we are done. \qed

If $\lambda$ is a bipartition let $\text{res}(\lambda)$ be the **multiset** $\{\text{res}(x) \mid x \in [\lambda]\}$. Then as a corollary of the Proposition we have the following (there is an easier proof).

**Corollary 20** (Dipper–James–Murphy [8], Corollary 8.7). Suppose that $S^\lambda$ and $S^\mu$ are in the same block. Then $\text{res}(\lambda) = \text{res}(\mu)$ (as multisets).

By the Corollary we can define the **residue** of a block $B$ to be the multiset $\text{res}(B) = \text{res}(\lambda)$ where $\lambda$ is any bipartition contained in $B$.

In fact, Grojnowski [13] has recently shown that the converse of Corollary 20 is true; so, two Specht modules $S^\lambda$ and $S^\mu$ belong to the same block if and only if $\text{res}(\lambda) = \text{res}(\mu)$. We will not need this stronger result.

4. The representation type when $n = e$

In this section we will prove the following result. Recall that we are assuming that $e \geq 3$.

**Theorem 21.** Suppose $n \geq e$. Then $\mathcal{H}$ has infinite representation type.
Proof. By Corollary $[11]$ we may assume that $n = e$. Further, by Lemma $[9]$ it is enough to show that one block of $\mathcal{H}$ has infinite representation type; we will show that the block $B$ with residues $\{0,1,\ldots,e-1\}$ has infinite representation type.

There will be several cases to consider. To begin suppose that $f \neq 0$. Because all of the residues in $B$ are distinct a bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ appears in $B$ only if $\lambda^{(1)}$ and $\lambda^{(2)}$ are both hook partitions; that is, $\lambda = ((a,1^b),(c,1^d))$ for some $a,b,c,d$. Define bipartitions

$$
\lambda_k = ((0), (k, 1^{e-k})), \quad \mu_k = ((k, 1^{e-k}), (0)), \quad \lambda_{k,l} = ((f-l, 1^{e-f-k}), (k, 1^l)),
$$

for $1 \leq k \leq e$.

It is easily checked that all of these bipartitions belong to $B$. Certainly, the two sets of bipartitions $\{\lambda_k\}$ and $\{\mu_k\}$ are disjoint; the restrictions on $k$ and $l$ ensure that $\lambda_{k,l} \neq \lambda_m$ and $\lambda_{k,l} \neq \mu_m$ for any $m$. Consequently, this is a complete list of the bipartitions which appear in $B$, with no repeats.

**Proposition 22.** Suppose that $n = e$ and that $1 \leq f \leq \frac{e}{2}$.

(i) The complete set of Kleshchev bipartitions in $B$ is

$$\{ \lambda_k \mid 1 \leq k < e \} \cup \{ \lambda_{k,l} \mid 1 \leq k \leq e - f \text{ and } 0 \leq l < f \}.$$

(ii) For $1 \leq k < e$ we have

$$[P^{\lambda_k}] = [S^{\lambda_k}] + [S^{\lambda_{k+1}}] + [S^{\lambda_{k+1}}], \quad \text{if } k < e - f,$$

$$[S^{\lambda_{e-f-1}}], \quad \text{if } k = e - f,$$

$$[S^{\lambda_{e-f-k}}] + [S^{\lambda_{e-f-k-1}}], \quad \text{if } k > e - f.$$

(iii) For $1 \leq k \leq e - f$ and $0 \leq l < f$ we have

$$[P^{\lambda_{k,l}}] = [S^{\lambda_{k,l}}] + \begin{cases} [S^{\lambda_{k-l}}] + [S^{\mu_{f+k}}] + [S^{\mu_{f+k}}], & \text{if } k \neq 1 \text{ and } l = 0, \\ [S^{\mu_{f+l}}] + [S^{\mu_{f+l}}], & \text{if } k = 1 \text{ and } l = 0, \\ [S^{\lambda_{k-l}}] + [S^{\lambda_{k-l}}] + [S^{\lambda_{k-l}}], & \text{if } k \neq 1 \text{ and } l \neq 0, \\ [S^{\lambda_{k-l}}] + [S^{\mu_{f+1}}] + [S^{\mu_{f+1}}], & \text{if } k = 1 \text{ and } l \neq 0. \end{cases}$$

Proof. It is easy to see that the bipartitions $\lambda_k$ and $\mu_k$, for $1 \leq k \leq e$, are not Kleshchev. Next suppose that $1 \leq k \leq e - f$. Then a straightforward computation shows that

$$F_{f+k} \cdots F_{e-1} F_0 F_{f+k-1} \cdots F_{f+1} F_1 \cdots F_f ((0), (0)) = F_{f+k} \cdots F_{e-1} F_0 ((0), (k, 1^{f-1})) \begin{cases} \lambda_k + v \lambda_{k+1} + v \lambda_{k+1} + v^2 \lambda_{k,f-1}, & \text{if } 1 \leq k < e - f, \\ \lambda_{e-f} + v \lambda_{e-f+1} + v^2 \lambda_{e-f+1}, & \text{if } k = e - f. \end{cases}$$

Note that $((0), (k, 1^{f-1}))$ has two addable 0–nodes, an addable $f$–node and an addable $f + k$–node which is a 0–node if $k = e - f$, and when we add an addable $r$–node, there is no removable $r$–node.

This shows that $\lambda_k$ is Kleshchev for $1 \leq k \leq e - f$; the formula for $[P^{\lambda_k}]$ now follows from Corollary $[13]$ (and the remarks after $[17]$).
The remaining cases are similar: for \( \lambda_k \) with \( e - f < k < e \) we have \( f \geq 2 \) and we compute
\[
F_{k-e+f} \cdots F_{k-e+f-j} \cdots F_1 F_0 \cdots F_{k} ((0), (0)) = F_{k-e+f} \cdots F_{k-e+f-j} \cdots F_1 F_0 ((0), (e - f)) = F_{k-e+f} \cdots F_{j-1} \left[ ((0), (k)) + v((k - e + f), (e - f)) \right] = F_{k-e+f} \left[ ((0), (k, 1^e-k-1)) + v((k - e + f), (e - f, 1^e-k-1)) \right] = \lambda_k + v \lambda_{k+1} + v \lambda_{k-e-k} + v^2 \lambda_{k-e-k};
\]
for \( \lambda_{k,l} \) with \( 1 \leq k \leq e - f \) and \( l = 0 \) compute
\[
F_{j+k-1} \cdots F_j F_{j+k} \cdots F_{e-k} F_{j-1} \cdots F_0 ((0), (0)) = F_{j+k-1} \cdots F_j F_{j+k} \cdots F_{e-k} F_{j-1} \cdots F_0 ((f), (0)) = F_{j+k-1} \cdots F_j ((f, 1^e-k-f), (0)) = \left\{ \begin{array}{ll}
\lambda_{k,0} + v \lambda_{k-1,0} + v \mu_{k+1} + v^2 \mu_{f+k+1}, & \text{if } k \neq 1, \\
\lambda_{1,0} + v \mu_1 + v^2 \mu_{f+1}, & \text{if } k = 1.
\end{array} \right.
\]
for \( \lambda_{k,l} \) with \( 0 < l < f \) and \( 2 \leq k \leq e - f \) compute
\[
F_{j-l} \cdots F_{j-1} F_j F_{j+k-l} \cdots F_{j+k} \cdots F_{e-k} F_{j-1-l} \cdots F_0 ((0), (0)) = F_{j-l} \cdots F_{j-1} F_j F_{j+k-l} \cdots F_{j+k} ((f-l, 1^e-l-k), (0)) = F_{j-l} \left[ ((f - l, 1^e-l-k), (k, 1^{l-1})) + v((f - l, 1^e-l-k), (k - 1, 1^{l-1})) \right];
\]
and, finally, for \( \lambda_{1,l} \) with \( 0 < l < f \) compute
\[
F_{j-l} \cdots F_{j-1} F_j F_{j+l} \cdots F_{e-l} F_{j-1-l} \cdots F_0 ((0), (0)) = F_{j-l} \cdots F_{j-1} F_j ((f - l, 1^e-l), (0)) = F_{j-l} \left[ ((f - l, 1^e-l), (1^l)) + v((f - l, 1^e-l), (0)) \right].
\]
In each case, an application of \( b \) now completes the proof. \( \square \)

By Lemma \( 3(i) \) in order to prove Theorem \( 2 \) it is enough to show that \( \text{End}_P(P^{\lambda}) \) is not isomorphic to \( k[x]/(x^m) \) for any \( m \). We need to consider several cases. First we observe that
\[
[P^{\lambda_1}] = [S^{\lambda_1}] + [S^{\lambda_2}] + [S^{\lambda_{2,f-1}}] + [S^{\lambda_{1,f-1}}]
\]
by Proposition \( 2(i) \). We will use this to determine the structure of \( P^{\lambda_1} \).

\textit{Case 1: } \( e \geq 5 \) and \( f \geq 2 \). Then \( e - f \geq 3 \) since \( e - f \geq \frac{e}{2} = 2.5 \). By (17) we can use Proposition \( 2 \) to compute the decomposition numbers \( d_{\lambda \mu} \) for the four Specht modules appearing in (23); this gives the following table (omitted entries are zero).

| \( D^{\lambda_1} \) | \( D^{\lambda_2} \) | \( D^{\lambda_{2,f-1}} \) | \( D^{\lambda_{1,f-1}} \) |
|-------------------|-------------------|-------------------|-------------------|
| \( S^{\lambda_1} \) | 1 | . | . | . |
| \( S^{\lambda_2} \) | 1 | 1 | . | . |
| \( S^{\lambda_{2,f-1}} \) | 1 | 1 | 1 | 1 |
| \( S^{\lambda_{1,f-1}} \) | 1 | 0 | 0 | 1 |

Consequently, \( [P^{\lambda_1}] = 2[D^{\lambda_1}]+2[D^{\lambda_2}]+[D^{\lambda_{1,f-1}}]+2[D^{\lambda_{2,f-1}}]+[D^{\lambda_{3,f-1}}]. \) By Corollary \( 3(ii) \), \( P^{\lambda_1} \) has submodule isomorphic to \( S^{\lambda_{1,f-1}} \). Therefore, \( S^{\lambda_{1,f-1}} \) has both
a simple head and simple socle; so, looking at the submatrix of the decomposition matrix above, the Loewy structure of $S^{\lambda_1,f-1}$ is

$$S^{\lambda_1,f-1} = \frac{D^{\lambda_1,f-1}}{D^{\lambda_2,f-1}}.$$  

We also have

$$S^{\lambda_2} = \frac{D^{\lambda_2}}{D^{\lambda_1}}.$$  

Considering the dual of $\text{Rad} S^{\lambda_1,f-1}$ and $S^{\lambda_2}$, we conclude that $D^{\lambda_2}$ and $D^{\lambda_2,f-1}$ appear in $\text{Rad} P^{\lambda_1}/\text{Rad}^2 P^{\lambda_1}$. If $0 \to D^{\lambda_1} \to X \to D^{\lambda_1} \to 0$ is an exact sequence then, for $1 \leq i < n$, $T_i - q$ acts invertibly on $X$ so that $T_i$ acts as $-1$ on $X$. Similarly, $T_0$ also acts as multiplication by a scalar on $X$ since $f \neq 0$. Therefore, every such exact sequence splits and so $\text{Ext}^1(D^{\lambda_1}, D^{\lambda_1}) = 0$; consequently, $D^{\lambda_1}$ is not a composition factor of $\text{Rad} P^{\lambda_1}/\text{Rad}^2 P^{\lambda_1}$.

Again, by Corollary 3, $P^{\lambda_1}$ has a Specht filtration so $\text{Rad} P^{\lambda_1}/S^{\lambda_1,f-1}$ has a Specht filtration whose successive quotients are $S^{\lambda_2}$ and $S^{\lambda_2,f-1}$. This means that $\text{Rad} P^{\lambda_1}/\text{Rad}^2 P^{\lambda_1} = D^{\lambda_2} \oplus D^{\lambda_2,f-1}$.

We now use the fact that $\text{Rad} S^{\lambda_1,f-1} = \frac{D^{\lambda_2,f-1}}{D^{\lambda_1}}$ to prove that $D^{\lambda_1}$ is contained in $\text{Rad} S^{\lambda_2,f-1}/\text{Rad}^2 S^{\lambda_2,f-1}$. As $\text{Rad} S^{\lambda_1,f-1}$ has a unique head there is a surjection $P^{\lambda_2,f-1} \to \text{Rad} S^{\lambda_1,f-1}$. Note also that $P^{\lambda_2,f-1}$ has a Specht filtration with successive quotients $S^{\lambda_2,f-1}$ and $S^{\lambda_2,f-2}$, $S^{\lambda_1,f-1}, S^{\lambda_1,f-2}$. Since $S^{\lambda_2,f-2}, S^{\lambda_1,f-1}, S^{\lambda_1,f-2}$ must map to $D^{\lambda_1}$, and each of these has unique head which is not isomorphic to $D^{\lambda_1}$, this surjection induces a map $S^{\lambda_2,f-1} \to \text{Rad} S^{\lambda_1,f-1}$. Therefore, $\text{Rad} S^{\lambda_2,f-1}/\text{Rad}^2 S^{\lambda_2,f-1}$ contains $D^{\lambda_1}$ as a summand. Let $U$ be a module such that $\text{Rad}^2 S^{\lambda_2,f-1} \subseteq U \subseteq \text{Rad} S^{\lambda_2,f-1}$ and $\text{Rad} S^{\lambda_2,f-1}/U \cong D^{\lambda_1}$ and set

$$V = (\text{Rad} P^{\lambda_1})/(U + S^{\lambda_1,f-1}).$$

Then there is a short exact sequence

$$0 \to \frac{D^{\lambda_2,f-1}}{D^{\lambda_1}} \to V \to \frac{D^{\lambda_2}}{D^{\lambda_1}} \to 0$$

and $V/\text{Rad} V \cong D^{\lambda_2} \oplus D^{\lambda_2,f-1}$. Hence, we also have $0 \to D^{\lambda_1} \to \text{Rad} V \to D^{\lambda_1} \to 0$ and consequently $\text{Rad} V = D^{\lambda_2} \oplus D^{\lambda_1}$. Therefore, $\text{Rad}^2 P^{\lambda_1}/\text{Rad}^3 P^{\lambda_1}$ contains $D^{\lambda_2} \oplus D^{\lambda_1}$ and it follows that $P^{\lambda_1}/\text{Rad} P^{\lambda_1}$ contains $D^{\lambda_2} \oplus D^{\lambda_1}$ as an $\mathcal{H}$-submodule, that another $D^{\lambda_1}$ appears as the head of $P^{\lambda_1}/\text{Rad} P^{\lambda_1}$, and that these are the only ways in which $D^{\lambda_1}$ appear as a composition factor of $P^{\lambda_1}/\text{Rad}^3 P^{\lambda_1}$. Therefore, we have that $\text{End}_{\mathcal{H}}(P^{\lambda_1}/\text{Rad}^3 P^{\lambda_1}) \cong \text{End}_{\mathcal{H}}(P^{\lambda_1})$ is not of finite representation type by Lemma 5(ii). Hence, $\text{End}_{\mathcal{H}}(P^{\lambda_1})$ has infinite representation type by Lemma 5(ii).

Case 2: $e = 4$ and $f = 2$. By Proposition 22 and (23), we have the same table as above if we replace $\lambda_3,f-1$ by $\lambda_3$. Thus

$$[P^{\lambda_1}] = 4[D^{\lambda_1}] + 2[D^{\lambda_2}] + [D^{\lambda_3}] + 2[D^{\lambda_2,f-1}] + [D^{\lambda_1,1}].$$

The argument used in case 1 shows that $P^{\lambda_1}/\text{Rad}^3 P^{\lambda_1}$ contains $D^{\lambda_2} \oplus D^{\lambda_1}$ as an $\mathcal{H}$-submodule. Hence, $\text{End}_{\mathcal{H}}(P^{\lambda_1})$ is again of infinite type.

Case 3: $f = 1$. Again using Proposition 22 and (23) we find that

$$[P^{\lambda_1}] = 4[D^{\lambda_1}] + 2[D^{\lambda_2}] + [D^{\lambda_1,0}] + 2[D^{\lambda_2,0}] + \delta_{e \geq 4}[D^{\lambda_3,0}],$$

where $\delta_{e \geq 4}$ is the Kronecker delta function.
where $\delta_{c \geq 4} = 1$ if $e \geq 4$ and $\delta_{c \geq 4} = 0$ otherwise. Almost the same argument as before again shows that $P^{k_1}/\text{Rad}^2 P^{k_1}$ contains $D^{k_1} \oplus D^{k_1}$ as an $\mathcal{H}$-submodule; however, to show that $\text{Rad} S^{\lambda_{2,0}}/\text{Rad}^2 S^{\lambda_{2,0}}$ contains $D^{k_1}$ we need to argue in a different way: since $[P^{\lambda_{2,0}}] = [S^{\lambda_{2,0}}] + [S^{e_2}] + [S^{\mu_3}]$ and $[S^{\mu_2}] = [D^{\lambda_{2,0}}] + [D^{\lambda_{1,0}}]$, $[S^{\mu_1}] = [D^{\lambda_{3,0}}] + [D^{\lambda_{2,0}}]$ do not contain $D^{k_1}$, the surjection $P^{\lambda_{2,0}} \rightarrow \text{Rad} S^{\lambda_{2,0}}$ induces a surjection $S^{\lambda_{2,0}} \rightarrow \text{Rad} S^{\lambda_{1,0}}$. Thus, once again $\mathcal{H}$ is of infinite representation type by Lemma 5 and Lemma 6.

**Case 4:** $f = 0$. It remains to consider the case $f = 0$. This case is somewhat degenerate as 1 is the only possible eigenvalue for the action of $T_0$ upon a representation. Because $f = 0$ there are no bipartitions of the form $\lambda_{k,1}$ in $B$ so Proposition 22 cannot hold in this case. A similar argument shows that we have the following simpler statement when $f = 0$.

**Proposition 24.** Suppose that $f = 0$.

(i) $\{ \lambda_k \mid 1 \leq k < e \}$ is the complete set of Kleshchev bipartitions in $B$.

(ii) For $1 \leq k < e$ we have $[P^{k_1}] = [S^{k_1}] + [S^{k_{+1}}] + [S^{k_2}] + [S^{k_{+3}}]$.

To apply this result we do not have to work as hard as in the previous cases. First observe that for $1 \leq k < e$ the module $D^{k_1}$ can be constructed by letting $T_0$ act as 1 on the simple $\mathcal{H}(A_e)$-module $D^{k_{1,-k}}$ (by induction both modules have the same dimension; namely, $(q^{-1}, 0)$). Let $M^{k_1}$ be the $\mathcal{H}$-module obtained by letting $T_0$ act as $(1, 0)$ on the $\mathcal{H}(A_e)$-module $D^{k_{1,-k}} \oplus D^{k_{1,-k}}$. Then $M^{k_1}$ cannot be semisimple as an $\mathcal{H}$-module because $T_0$ does not act as a scalar. On the other hand, the socle of $M^{k_1}$ is simple (being isomorphic to $D^{k_1}$); hence, $M^{k_1}$ is indecomposable. This implies that $[\text{Rad} P^{k_1}/\text{Rad}^2 P^{k_1} : D^{k_1}] \neq 0$.

Now $[S^{k_1}] = [D^{k_1}] + [D^{k_{1,-1}}]$ for $1 \leq k < e$ by Proposition 24. Hence, $S^{k_1}$ is indecomposable and $[\text{Rad} P^{k_1}/\text{Rad}^2 P^{k_1} : D^{k_{1,-2}}] \neq 0$ for $2 \leq k < e$. As $D^{k_1}$ is self-dual, by taking duals we also have

$$[\text{Rad} P^{k_1}/\text{Rad}^2 P^{k_1} : D^{k_{1,+2}}] \neq 0,$$

for $1 \leq k < e - 1$. Combining these facts we conclude that

$$\text{Rad} P^{k_1}/\text{Rad}^2 P^{k_1} \supset D^{k_1} \oplus D^{k_2},$$

$$\text{Rad} P^{k_1}/\text{Rad}^2 P^{k_1} \supset D^{k_{1,-1}} \oplus D^{k_1} \oplus D^{k_{1,+1}}, \text{ for } 2 \leq k < e - 1,$$

and that $\text{Rad} P^{k_{e-1}}/\text{Rad}^2 P^{k_{e-1}} \supset D^{k_{e-2}} \oplus D^{k_{e-1}}$. Therefore, the separation diagram of $B$ contains

$$\xymatrix{\ldots & \cdot & 

\text{Consequently, } B \text{ has infinite representation type by Theorem 5.}$$

This completes the proof of Theorem 24.

5. **The representation type when $n \geq 2f + 4$**

Recall that we are assuming that $e \geq 3$ and $0 \leq f \leq \frac{e}{2}$. This section is devoted to the proof of the following result.

**Theorem 25.** Suppose $n \geq 2f + 4$. Then $\mathcal{H}$ has infinite representation type.

**Proof.** By Corollary 1 it is enough to show that $\mathcal{H}_n$ has infinite representation type when $n = 2f + 4$. Also, by the last section we may assume that $e > n$. Let $B$ be the block $B$ with residues $\{-1, 0, 0, 1, 1, \ldots, f, f, f + 1\}$; so $B$ contains the
bipartition \(((2, 2), (2'f))\). We will show that \(B\) has infinite representation type. To do this we need to consider two cases separately.

**Case 1:** Suppose that \(f = 0\) (and \(e > n = 4\)). We consider the block \(B\) with residues \(\{-1, 0, 0, 1\}\). It is easy to see that there are precisely six bipartitions in this block; namely, \(\lambda_1 = ((0), (2'))\), \(\lambda_2 = ((1), (2, 1))\), \(\lambda_3 = ((1'), (2))\), \(\lambda_4 = ((2), (1'))\), \(\lambda_5 = ((2, 1), (1))\) and \(\lambda_6 = ((2'), (0))\). Furthermore, of these bipartitions only \(\lambda_1\) and \(\lambda_2\) are Kleshchev. Since

\[
F_0F_1F_{e-1}F_0((0), (0)) = F_0\left[((0), (2,1)) + v((2,1), (0))\right] = \lambda_1 + v\lambda_2 + v\lambda_5 + v^2\lambda_6
\]

and

\[
F_1F_{e-1}F_0^{(2)}((0), (0)) = F_1F_{e-1}((1), (1)) = F_1\left[((1), (1')) + v((1'), (1))\right] = \lambda_2 + v\lambda_4 + v\lambda_3 + v^2\lambda_5,
\]

by Corollary 16 the transpose of the decomposition matrix of \(B\) is

| \(D^{\lambda_1}\) | \(S^{\lambda_2}\) | \(S^{\lambda_3}\) | \(S^{\lambda_4}\) | \(S^{\lambda_5}\) | \(S^{\lambda_6}\) |
|-----------------|---------|---------|---------|---------|---------|
| \(D^{\lambda_2}\) | 1       | 1       | 0       | 0       | 1       |
| \(D^{\lambda_1}\) | 0       | 1       | 1       | 1       | 0       |

Let \(\mu_1 = ((0), (2,1))\) and \(\mu_2 = ((1), (1'))\). We first observe that \(P^{\lambda_2}\) is the induced module \(P^{\mu_2} \uparrow^B\); this follows from a calculation in the Grothendieck group. Similarly, we also have that \(P^{\lambda_1} = P^{\mu_1} \uparrow^B\). Further, \(P^{\mu_i}\) is uniserial of length 2 and \([P^{\mu_i}] = 2[D^{\lambda_i}], \) for \(i = 1, 2\). Next note that \(D^{\lambda_i} = S^{\mu_i}, \) for \(i = 1, 2\); hence, the branching rules for Specht modules imply that \([D^{\lambda_1} \uparrow^B] = 2[D^{\lambda_1}] + [D^{\lambda_2}]\) and \([D^{\mu_2} \uparrow^B] = [D^{\lambda_1}] + 2[D^{\lambda_2}].\)

Next note that there are surjective homomorphisms \(P^{\lambda_i} \twoheadrightarrow D^{\mu_i} \uparrow^B\) for \(i = 1, 2\). As \(D^{\mu_i} \uparrow^B\) is self-dual, \(D^{\mu_i} \uparrow^B\) is a uniserial module whose top and bottom is isomorphic to \(D^{\lambda_i}.\) To conclude, each \(P^{\lambda_i}\) has a submodule \(M_i\) such that both \(P^{\lambda_i}/M_i\) and \(M_i\) are isomorphic to the uniserial module \(D^{\mu_i} \uparrow^B.\)

Now we prove that \(\text{Ext}^1(D^{\lambda_2}, D^{\lambda_1}) \neq 0.\) As \(\text{Ext}^1(D^{\lambda_1}, D^{\lambda_1}) \neq 0\), we know that the radical series of \(P^{\lambda_1}\) has the form

\[
D^{\lambda_1} \supset D^{\lambda_2} \supset D^{\lambda_1} \uparrow^B
\]

Assume that \(\text{Ext}^1(D^{\lambda_2}, D^{\lambda_2}) = 0.\) Then \(P^{\lambda_2}\) must have the form

\[
\begin{align*}
D^{\lambda_2} \\
D^{\lambda_1} \uparrow^B
\end{align*}
\]

Consider a homomorphism \(P^{\lambda_2} \twoheadrightarrow D^{\lambda_2} \subset \text{Rad} P^{\lambda_1}/\text{Rad}^2 P^{\lambda_1}.\) The assumption that \(\text{Rad} P^{\lambda_2}/\text{Rad}^2 P^{\lambda_2} = D^{\lambda_1}\) together with the radical structure of \(P^{\lambda_1}\) imply
that this map lifts to a homomorphism \( P^{\lambda_2} \rightarrow P^{\lambda_1} \), whose image is of the form
\[
\begin{align*}
D^{\lambda_2} \\
D^{\lambda_1} \\
D^{\lambda_0}
\end{align*}
\]
This contradicts our assumption that \( \text{Rad}^2 P^{\lambda_2} / \text{Rad}^3 P^{\lambda_2} = D^{\lambda_2} \oplus D^{\lambda_0} \).

Hence, \( \text{Ext}^1(D^{\lambda_2}, D^{\lambda_0}) \) and \( \text{Ext}^1(D^{\lambda_1}, D^{\lambda_0}) \) are both non-zero. Therefore, \( \text{Rad} P^{\lambda_1} / \text{Rad}^2 P^{\lambda_1} \) and \( \text{Rad} P^{\lambda_2} / \text{Rad}^2 P^{\lambda_2} \) both contain \( D^{\lambda_1} \oplus D^{\lambda_2} \) as an \( \mathcal{H} \)–submodule. Therefore, the separation diagram of \( B \) is \( \square \). Consequently, \( B \) is of infinite type, by Gabriel’s theorem, and \( \mathcal{H} \) has infinite representation type by Lemma 3(ii).

Case 2: Suppose that \( 0 < f \leq \frac{e}{2} \). Now we are considering the block \( B \) of \( \mathcal{H} \) with residues \( \{-1, 0, 1, 1, \ldots, f, f, f + 1\} \). Notice that \( f + 2 \) is a residue in \( B \) only if \( f + 1 \geq e - 1 \) and \( -2 \) is a residue in \( B \) only if \( e - 2 \leq f + 1 \); however, \( e > 6 \) because \( e > n = 2f + 4 \) and \( f \geq 1 \); so \( f + 2 \) and \( -2 \) cannot be residues in \( B \). Consequently, all of the bipartitions in \( B \) have the form \( (a, b, (2f+1)) \).

Define bipartitions \( \lambda_1 = ((0), (2f+2)), \lambda_2 = ((1), (2f+1, 1)), \lambda_3 = ((1^2), (2f+1)), \lambda_4 = ((2), (2f, 1^2)) \) and \( \lambda_5 = ((2, 1), (2f, 1)) \). We will show that \( B \) has infinite representation type by computing \( \text{End}_\mathcal{H}(P^{\lambda_2}) \).

**Lemma 26.** The bipartitions \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) and \( \lambda_5 \) are all Kleshchev. Furthermore, \( [P^{\lambda_2}] = [S^{\lambda_1}] + [S^{\lambda_2}] + [S^{\lambda_3}] + [S^{\lambda_4}] + [S^{\lambda_5}] \) and the first five rows of the decomposition matrix of \( B \) are as follows (all omitted entries are zero):

| \( S^{\lambda_1} \) | \( S^{\lambda_2} \) | \( S^{\lambda_3} \) | \( S^{\lambda_4} \) | \( S^{\lambda_5} \) |
|-----------------|-------------|-------------|-------------|-------------|
| \( D^{\lambda_1} \) | 1           | .           | .           | .           |
| \( D^{\lambda_2} \) | .           | 1           | .           | .           |
| \( D^{\lambda_3} \) | 0           | 1           | 1           | .           |
| \( D^{\lambda_4} \) | 0           | 1           | 0           | 1           |
| \( D^{\lambda_5} \) | 1           | 1           | 1           | 1           |

**Proof.** We use Corollary 14 to compute \( [P^{\lambda_i}] \), for \( i = 1, \ldots, 5 \). We find that
\[
F_1 \ldots F_{f+1}F_{e-1}F_0(2)F_1 \ldots F_f((0), (0)) = F_1 \ldots F_{f+1}F_{e-1}F_0(2)((0), (1^f))
\]
\[
= F_1 \ldots F_{f+1}\left( (1), (1^{f+2}) \right) + v((1^2), (1^{f+1})) \right) 
\]
\[
= F_1 \left( (1), (2f, 1^2) \right) + v((1^2), (2f, 1)) 
\]
\[
= ((1), (2f+1, 1)) + v((2), (2f, 1^2)) + v((1^2), (2f+1)) + v^2((2, 1), (2f, 1)).
\]

As in Case 1, by Corollary 16 this shows that \( \lambda_2 \) is Kleshchev, proves the formula for \( [P^{\lambda_2}] \) and thus gives the second column of the decomposition matrix of \( B \). The remaining claims follow from the following calculations, which we leave to the reader.

\[
F_0 \ldots F_{f+1}F_{e-1}F_0 \ldots F_f((0), (0)) = \lambda_1 + v\lambda_2 + v\lambda_5 + \ldots
\]
\[
F_1 \ldots F_{f+1}F_0 \ldots F_fF_0((0), (0)) = \lambda_3 + v\lambda_5 + \ldots
\]
\[
F_{e-1}F_2 \ldots F_{f+1}F_0F_1(2)F_2 \ldots F_fF_0((0), (0)) = \lambda_4 + v\lambda_5 + \ldots
\]

Here, “+ ...” indicates a linear combination of more dominant terms and if \( e = \infty \) then we replace \( F_{e-1} \) by \( F_{-1} \). \( \square \)
Consequently, \([P^{\lambda_2}] = 4[D^{\lambda_2}] + 2[D^{\lambda_1}] + 2[D^{\lambda_3}] + 2[D^{\lambda_4}] + [D^{\lambda_5}]\). Now, by Corollary 13(ii), \(P^{\lambda_2}\) has a submodule isomorphic to \(S^{\lambda_5}\). The composition factors \(D^{\lambda_3}\) and \(D^{\lambda_4}\) of \(S^{\lambda_5}\) cannot appear in \(\text{Rad} P^{\lambda_2}/\text{Rad}^2 P^{\lambda_2}\) because then \(D^{\lambda_5}\) would appear in the head of \(P^{\lambda_2}\). On the other hand, because \(S^{\lambda_2}, S^{\lambda_3}\) and \(S^{\lambda_4}\) are indecomposable, \(D^{\lambda_3} \oplus D^{\lambda_3} \oplus D^{\lambda_4}\) appears in \(\text{Rad} P^{\lambda_2}/\text{Rad}^2 P^{\lambda_2}\). Recall, again, that \(P^{\lambda_2}/S^{\lambda_5}\) has a Specht filtration with successive quotients \(S^{\lambda_2}, S^{\lambda_3}\) and \(S^{\lambda_4}\). Therefore, the \(\mathcal{H}\)-modules \(D^{\lambda_3}\) and \(D^{\lambda_4}\) which appear in \(\text{Rad} P^{\lambda_2}/\text{Rad}^2 P^{\lambda_2}\) are composition factors of \(S^{\lambda_3}\) and \(S^{\lambda_4}\). Since \(\text{Rad} S^{\lambda_3} = \text{Rad} S^{\lambda_4} = D^{\lambda_2}\) and other \(D^{\lambda_2}\) are the head and the socle of \(P^{\lambda_2}\), \(D^{\lambda_2}\) does not appear in \(\text{Rad} P^{\lambda_2}/\text{Rad}^2 P^{\lambda_2}\). Thus \(\text{Ext}^1(D^{\lambda_2}, D^{\lambda_2}) = 0\) and we see that \(P^{\lambda_2}/\text{Rad}^3 P^{\lambda_2}\) contains \(D^{\lambda_2} \oplus D^{\lambda_2}\) as an \(\mathcal{H}\)-submodule by the same argument as before. Consequently, \(\text{End}_{\mathcal{H}}(P^{\lambda_2}/\text{Rad}^3 P^{\lambda_2}) \cong k[x, y]/(x^2, xy, y^2)\) and \(B\) is of infinite representation type by Lemma 3. Therefore, \(\mathcal{H}\) has infinite representation type by Lemma 3(iii).

This completes the proof of Theorem 23. \( \Box \)

6. Finite representation type

In this final section we show that \(\mathcal{H}_q(B_n)\) is of finite representation type when \(n < \min\{e, 2f + 4\}\). To do this we use a different combinatorial description of bipartitions which was suggested to the first author by Fomin. Recall that we are assuming that \(q\) is a primitive \(e^{th}\) root of unity in \(R\) and that \(T_0\) satisfies the relation \((T_0 - 1)(T_0 - q^f) = 0\) where \(0 \leq f \leq \frac{e}{2}\). We note that if \(K\) is a field of characteristic zero then there is a different argument by Geck [11, Corollary 9.7].

First, consider a partition \(\lambda\). The diagram of \(\lambda\) is the set
\[
\{(i, j) \in \mathbb{N}^2 \mid 1 \leq j \leq \lambda_i \text{ and } i \geq 1\},
\]
which we think of as an array of boxes in the plane. Just as we can identify \(\lambda\) with its diagram we can also identify \(\lambda\) with its border
\[
\{(i, j) \in \mathbb{N}^2 \mid \lambda_i < j \leq \lambda_{i-1} + 1\}
\]
(we set \(\lambda_0 = \infty\)). We can think of the border of \(\lambda\) as a (doubly infinite) path from \((\infty, 1)\) to \((1, \infty)\). Writing 0 for each vertical edge and 1 for each horizontal edge in the border we identify \(\lambda\) with a doubly infinite sequence of 0’s and 1’s. We call this the path sequence of \(\lambda\). A path sequence is also called a Maya diagram.

For example, if \(\lambda\) is the partition \((4, 2, 1)\) then by looking at the diagram of \(\lambda\)

\[
\begin{array}{cccccccc}
\times & \times & \times & \times & 0 & 1 & 1 & \ldots \\
\times & \times & 0 & 1 & 1 \\
\times & 0 & 1 \\
0 & 1 \\
\vdots
\end{array}
\]

the path sequence of \(\lambda\) is \(\ldots 00101011011 \ldots \). Here the crosses mark the nodes in the diagram of \(\lambda\) and the 0’s and 1’s are the nodes in the border of \(\lambda\).

In order to keep track of the contents of the nodes we insert a horizontal bar into the path sequence after the node which appears on the diagonal \(\{(i, i) \mid i \in \mathbb{N}\}\). The example above becomes \(\ldots 00101011011 \ldots \). The bar divides the path sequence into two regions which we refer to as the left and right regions of the path.
Note that the number of 1’s in the left region is always the same as the number of 0’s in the right region. Conversely, any sequence
\[ \ldots p_{-2}p_{-1}p_0|p_1p_2p_3 \ldots \]
of zeros and ones with \( P_- = \sum_{i \leq 0} p_i < \infty, \ P_+ = \sum_{i > 0} (1 - p_i) < \infty \) and \( P_- = P_+ \) always corresponds to a partition.

A rim hook in (the diagram of) \( \lambda \) corresponds to a subsequence \( 1 \ldots 0 \) in the path sequence of \( \lambda \); explicitly, if \( x = (i, j) \in [\lambda] \) then the rim hook \( r_x \) corresponds to the border path from \( (\lambda'_1 + 1, j + 1) \) to \( (i, \lambda_1 + 1) \). As there is a bijection between the nodes and rim hooks of \( \lambda \), the number of subsequences of a path sequence of the form \( 1 \ldots 0 \) is equal to \( n \). Hence, it follows that removing a rim hook from \( \lambda \) is the same as swapping a 0 and a 1 in the path sequence for \( \lambda \). Further, the leg length of the hook is equal to the number of zeros in the corresponding \( 1 \ldots 0 \) subsequence minus 1. Conversely, wrapping a rim hook onto \( \lambda \) is the same as changing a 0…1 subsequence to \( 1 \ldots 0 \). These observations will allow us to rephrase the Jantzen sum formula \( [3] \) in terms of path sequences.

Given \( \mathbf{p} = \ldots p_{-2}p_{-1}p_0|p_1p_2p_3 \ldots \) let \( \mathbf{p}_i = \ldots p_{-2}p_{-1}p_0|p_1p_2p_3 \ldots \) be the sequence of left partial sums; that is, \( p_i = \sum_{j \leq i} p_j \). For example, if \( \lambda = (0) \) then the path sequence is \( 0 = \ldots 000|111 \ldots \) and \( \mathbf{p} = \ldots 000|123 \ldots \).

The content of a node \( x = (i, j) \in [\lambda] \) is \( c(x) = j - i \). For a partition \( \lambda \) let \( c_k(\lambda) = \# \{ x \in [\lambda] \mid c(x) = k \} \). Using the notation of the last paragraph, if \( \mathbf{p} = \ldots p_{-2}p_{-1}p_0|p_1p_2p_3 \ldots \) is the path sequence of \( \lambda \) then the content multiplicities \( c_k(\lambda) \) are given by the sequence \( \ldots c_{2-1}c_0c_1c_2 \ldots = \mathbf{p} - \mathbf{0} \).

We now turn to path sequences for bipartitions. If \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \) is a bipartition then we have a path sequence \( \ldots p_{-2}p_{-1}p_0|p_1p_2p_3 \ldots \) for \( \lambda^{(1)} \) and a path sequence \( \ldots s_{-2}s_{-1}s_0|s_1s_2 \ldots \) for \( \lambda^{(2)} \). Recall that the content of node \( x = (i, j, k) \in [\lambda] \) is defined to be \( c(x) = j - i + (k - 1)f \) (and \( \text{res}(x) = c(x) \pmod e \)). In particular, \( p_0 \) corresponds to a node of content 0 and \( s_0 \) corresponds to a node of content \( f \). Accordingly, we shift the nodes in the path sequence for \( \lambda^{(2)} \) by \( f \) positions to the right and define the path sequence of the bipartition \( \lambda \) to be the sequence \( \{(p_i, s_i - f)\} \) of ordered pairs which we write with two separation bars as follows
\[
\ldots p_{-1} p_0 | p_1 \ldots p_f | p_{f+1}p_{f+2} \ldots \\
\ldots s_{-1-f} s_{-f} | s_1 \ldots s_0 | s_1 \ldots s_2 \ldots \\
\]

We refer the three regions in the path sequence of a bipartition as the left, middle and right regions of the sequence.

For example, if \( \lambda = ( (4, 2, 1), (2^2, 1) ) \) and \( f = 2 \) then the contents in \( \lambda \) and its path sequence are as follows.
\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & -1 & 0 & 0 \\
-2 & 2 & 1 & 2 \\
\end{pmatrix}
\]
and
\[
\ldots 000101|011011 \ldots \\
\ldots 000001|010100 \ldots \\
\]

As before, it is easy to see that we can recover the contents of a bipartition from the partial sums \( \sum_{j \leq i} (p_j + s_{j-f}) \) from the path sequence.

We now develop a calculus with which to analyze path sequences of bipartitions. Let \( A = 0 \); \( B = 1 \); \( C = 0 \) and \( D = 1 \) be the four possible ordered pairs which can appear in the path sequence of a bipartition. Define \( a_l, a_m \) and \( a_r \) to be the number of \( A \)'s in the left, middle and right regions, respectively, of the path sequence;
similarly, we define $b_l, b_m, b_r, c_l, c_m, c_r, d_l, d_m$ and $d_r$. Notice that $c_l$ and $d_r$ are both infinite; all of the other quantities are non-negative and finite.

27 Suppose that $n \leq 2f + 3$. Then

(i) $b_l + d_l = a_m + c_m + a_r + c_r$.

(ii) $b_r + c_r = a_l + d_l + a_m + d_m$.

(iii) $f = a_m + b_m + c_m + d_m$.

(iv) $2f + 3 \geq (b_l + d_l)(a_m + c_m + a_r + c_r) + (a_l + d_l + a_m + d_m)(b_r + c_r)$

$+ (b_m + d_m)(a_r + c_r) + (a_l + d_l)(b_m + c_m) + a_l b_l + a_m b_m + a_r b_r$.

Proof. Parts (i) and (ii) both follow from the fact mentioned earlier that the number of ones in a region to the left of a bar is equal to the number of zeros in the region to the right of the same bar. Part (iii) is true by definition. Finally, (iv) follows by counting the rim hooks in $\lambda$ (recall that the rim hooks in the path sequence for a partition correspond to subsequences of the form $1 \ldots 0$); this is the only place where we use the restriction on $n$. 

We want to understand this system of inequalities when $n \leq 2f + 3$. As a first step, adding parts (i) and (ii) and subtracting (iii) we see that

\begin{equation}
(28)
    b_l + b_m + b_r = f + a_l + a_m + a_r.
\end{equation}

Careful inspection shows that by combining parts (i), (ii) and (iv) of (27), and omitting some terms for the second inequality, we have

\[
2f + 3 \geq (a_m + c_m + a_r + c_r)^2 + (a_l + d_l + a_m + d_m)^2 + (b_m + d_m)a_r
\]
\[
           + (b_m + d_m)c_r + (a_l + d_l)(b_m + c_m) + a_l b_l + a_m b_m + a_r b_r
\]
\[
\geq (a_l + a_m + a_r)(a_m + b_m + c_m + d_m) + a_l a_m + a_m a_r + a_l^2 + a_r^2
\]
\[
+ a_r^2 + a_l(b_l + d_l) + a_r(b_r + c_r)
\]
\[
= (a_l + a_m + a_r)f + a_l a_m + a_m a_r + a_l^2 + a_r^2
\]
\[
+ a_l(a_m + c_m + a_r + c_r) + a_r(a_l + d_l + a_m + d_m);
\]
the last line again uses (i)–(iii) of (27). Finally, throwing away a few more terms and rearranging gives

\[
2f + 3 \geq (a_l + a_m + a_r)(f + a_l + a_m + a_r).
\]

Now, if $a_l + a_m + a_r \geq 2$ then the right hand side is greater than or equal to $2f + 4$; as this is impossible we must therefore have

\begin{equation}
(29)
    a_l + a_m + a_r \leq 1.
\end{equation}

In particular, note that at most one of $a_l, a_m$ and $a_r$ can be non-zero. Notice that this is the first time that the assumption $n \leq 2f + 3$ has really been needed; it is exactly what is required to ensure that $a_l + a_m + a_r \leq 1$.

Using similar arguments it is possible to classify the possible path sequences when $n \leq 2f + 3$; however, we won’t need this.

Theorem 30. Suppose that $n < \min\{e, 2f + 4\}$. Then $\mathcal{H}$ is of finite representation type.

Proof. Fix a block $B$ of $\mathcal{H}$ and recall from Corollary 29 that two simple modules belong to the same block only if they have a common multiset of residues. Note that because $n < e$ the number of distinct residues contained in the diagram of $\lambda$
is strictly less than \( n \); consequently, we can find a \( k \), with \( 0 \leq k < e \), such that 
\(-k - 1 \pmod{e}\) is not a residue in \( B \).

Suppose that \( D^\lambda \) appears in \( B \). Then the contents of all of the nodes in \( \lambda^{(1)} \) are contained in the interval \([−k,e−k−1]\).

\textit{Case 1: Suppose that } \( e−k \geq f \). As \(-k−1 \pmod{e}\) is not a residue for the block \( B \) the contents of the nodes in \( \lambda^{(2)} \) are all contained in the interval \([−k,e−k]\) — note that \( f \in [−k,e−k] \). Therefore, the multiset of residues for the block \( B \) is the same as the multiset of contents for \( B \). Consequently, we can unwrap a rim hook from \( \lambda \) and wrap it back on again without changing the residue of the foot node only if the resulting bipartition \( \mu \) has the same multiset of contents as \( \lambda \).

Recall that unwrapping a rim hook from a partition is the same as swapping the ends of a \( 1\ldots0 \) subsequence to give \( 0\ldots1 \) and that wrapping a hook back on changes some \( 0\ldots1 \) into \( 1\ldots0 \). Now, the contents of a bipartition \( \lambda \) are determined by the partial sums in the path sequence of \( \lambda \); because of this, the only way to unwrap a rim hook from \( \lambda \) and then wrap it back on to give a bipartition \( \mu \) with the same multiset of contents is by interchanging some \( A \) and \( B \) in the path sequence:

\[
\lambda = \ldots B \ldots A \ldots \quad \longrightarrow \quad \mu = \ldots A \ldots B \ldots.
\]

Moreover, \( \lambda \triangleright \mu \) if and only if \( A \) moves to the left. (Note that \( |\mu| = |\lambda| \) in this case as the number of \( 1\ldots0 \) subsequences in the two path sequences is the same.)

If \( a_l + a_m + a_r = 0 \) then the path sequence for \( \lambda \) does not contain any \( A \)'s so by the sum formula, Theorem \( \ref{prop19} \), and by Proposition \( \ref{prop19} \), the Specht module \( S^\lambda = D^\lambda = P^\lambda \) is the only simple module in the block \( B \). In particular, \( B \) is semisimple and so of finite type in this case.

If \( a_l + a_m + a_r \neq 0 \) then \( a_l + a_m + a_r = 1 \) by \( \ref{prop19} \). In this case by Proposition \( \ref{prop19} \) and \( \ref{prop19} \) the block \( B \) contains at most \( f + 2 \) bipartitions; namely, the bipartitions \( \lambda_0, \ldots, \lambda_{f+1} \) whose path sequences contain exactly one \( A \) and \( (f+1) B \)'s and which agree with the path sequence for \( \lambda \) on all of the \( C \)'s and \( D \)'s. By ordering these bipartitions according to the location of the (unique) \( A \) in their path sequence we may assume that \( \lambda_{f+1} \triangleright \cdots \triangleright \lambda_0 \) (for example, \( A \) occupies the leftmost position in the path sequence of \( \lambda_0 \) and the rightmost position in \( \lambda_{f+1} \)).

For our purposes, it is enough to prove that \( S^{\lambda_0} = D^{\lambda_0} \) and \( [S^{\lambda_i}] = [D^{\lambda_i}] + [D^{\lambda_{i−1}}] \) for \( 0 < i \leq f \); in particular, we do not need to know that \( D^{\lambda_i} \neq 0 \), for \( 0 \leq i \leq f \). In fact these modules are always non–zero; we include the proof below because it yields the remarkable fact that when \( n < \min\{e,2f+4\} \) the number of Specht modules belonging to a block is either 1, \( f + 2 \) or \( e − f + 2 \).

The removable nodes in a partition correspond to the 10 subsequences in the path sequence. Suppose that \( 0 < i < f \). Then \( \lambda_i \) contains a removable node \( x \in [\lambda_i^{(2)}] \); furthermore, this node is automatically good because if \( x \) is a \( r \)-node then there is no addable \( r \)-node below \( x \) because \( |\lambda_i^{(2)}| < e \). Let \( \mu_i \) be the bipartition with \( [\mu_i] = [\lambda_i] \setminus \{x\} \). Then \( \mu_i \) is Kleshchev because either the path sequence for \( \mu_i \) contains an \( A \) (so \( \mu_i \) is Kleshchev by induction on \( n \)), or \( D^{\mu_i} = S^{\mu_i} \neq 0 \) (by the second last paragraph); hence, \( \lambda_i \) is Kleshchev. On the other hand, \( \lambda_{f+1} \) is not Kleshchev because either we can apply induction after removing a node from \( \lambda_{f+1}^{(1)} \), or the path sequence for \( \lambda_{f+1} \) is \( \ldots BBBA \ldots \) in which case it is easy to see that \( \lambda_{f+1} \) is not Kleshchev.

Now \( S^{\lambda_0} = D^{\lambda_0} \) by Theorem \( \ref{prop19}(iii) \) since \( \lambda_i \triangleright \lambda_0 \) for \( i > 0 \). To complete the proof we claim that \( [S^{\lambda_i}] = [D^{\lambda_i}] + [D^{\lambda_{i−1}}] \), for \( i = 1, \ldots, f \), and \( S^{\lambda_{f+1}} = D^{\lambda_f} \). To
see this we apply the sum formula. As discussed earlier, the leg length of a hook 1...0 in a path sequence is given by the number of 0's strictly contained in the subsequence. Consequently, when we unwrap the hook $B...A$ from $\lambda_i$ and wrap it back on to give some $\lambda_l$ then, modulo 2, the difference in the leg lengths of the two rim hooks is equal to the number of $B$'s which are strictly contained in the subsequence for the rim hook. Therefore, by Theorem 18, for $i = 1, \ldots, f + 1$
\[
\sum_{j>0} [S^{\lambda_i}(j)] = [S^{\lambda_i-1}] - [S^{\lambda_i-2}] + \cdots + (-1)^{i-1}[S^{\lambda_0}].
\]
As we already know that $\lambda_0, \ldots, \lambda_f$ are Kleshchev, and that $\lambda_{f+1}$ is not, our claim now follows by induction on $i$. Consequently, the decomposition matrix of the block $B$ is
\[
\begin{bmatrix}
S^{\lambda_0} & D^{\lambda_0} & \ldots & D^{\lambda_f} \\
S^{\lambda_1} & 1 & 1 & \\
\vdots & \ddots & \ddots & \\
S^{\lambda_f} & 1 & 1 & \\
S^{\lambda_{f+1}} & 1 & & \\
\end{bmatrix}
\]
and $B$ has finite representation type by Theorem 3.

Case 2: Suppose that $0 < e - k < f$. In this case the contents of the nodes in $\lambda^{(2)}$ are contained in the interval $[e - k, 2e - k - 1]$. Renormalizing $T_0$ as $q^e T_0$ the relation for $T_0$ becomes $(T_0 - 1)(T_0 - q^{e-f}) = 0$. The Specht module $S^{\lambda}$ is relabelled as $S^{(\lambda^{(2)}, \lambda^{(1)})}$ and the residues in $[\lambda]$ are all changed by adding $e - f \equiv -f \pmod{e}$. Consequently, the residues for $\lambda$ are all contained in the interval $[e - f - k, 2e - f - k - 1]$. Therefore, with this renormalization, the multiset of residues for $B$ is the same as the multiset of contents for $B$. Consequently, we can repeat the argument of Case 1 to deduce that decomposition matrix for $B$ has the form above; so, again, $B$ has finite representation type by Theorem 3. \hfill \Box

References

[1] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(m,1,n)$, J. Math. Kyoto Univ., 36 (1996), 789–808.
[2] ———, On the classification of simple modules for cyclotomic Hecke algebras of type $G(m,1,n)$ and Kleshchev multipartitions, Osaka J. Math., 38 (2001), 827–837.
[3] S. Ariki and A. Mathas, The number of simple modules of the Hecke algebras of type $G(r,1,n)$, Math. Z., 233 (2000), 601-623.
[4] ———, The Hecke algebras with a finite number of indecomposable modules, in “Representation Theory of Algebraic Groups and Quantum Groups”, T. Shoji et al. (eds.), Advanced Studies in Pure Math., In press.
[5] M. Auslander, I. Reiten, and S. O. Smalø, Representation theory of Artin algebras, Cambridge studies in advanced mathematics, 36, CUP, 1997.
[6] D. J. Benson, Representations and cohomology, Cambridge studies in advanced mathematics, 30, CUP, 1991.
[7] R. Dipper and G. James, Representations of Hecke algebras of type $B_n$, J. Algebra, 146 (1992), 454–481.
[8] R. Dipper, G. James, and A. Mathas, Cyclotomic $q$–Schur algebras, Math. Z., 229 (1999), 385–416.
[9] R. Dipper, G. James, and E. Murphy, Hecke algebras of type $B_n$ at roots of unity, Proc. L.M.S. (3), 70 (1995), 505–528.
[10] K. Erdmann and D.K. Nakano, Representation type of Hecke algebras of type A, Trans. A.M.S., 354 (2002), 275–285.
[11] M. Geck, Brauer trees of Hecke algebras, Comm. Alg., 20 (1992), 2937–2973.
[12] J. J. Graham and G. I. Lehrer, Cellular algebras, Invent. Math., 123 (1996), 1–34.
[13] I. Grojnowski, Blocks of the cyclotomic Hecke algebra, preprint 1999.
[14] G. D. James and A. Mathas, The Jantzen sum formula for cyclotomic q–Schur algebras, Trans. A.M.S., (2000), 5381–5404.
[15] M. Jimbo, K. C. Misra, T. Miwa, and M. Okado, Combinatorics of U_q(\hat{sl}(n)) at q = 0, Comm. Math. Phys., 136 (1991), 543–566.
[16] A. Mathas, Hecke algebras and Schur algebras of the symmetric group, Univ. Lecture Notes, 15, A.M.S., 1999.
[17] D. Uglov, Canonical bases of higher-level q–deformed Fock spaces and Kazhdan-Lusztig polynomials, in Physical combinatorics (Kyoto, 1999), Boston, MA, 2000, Birkhäuser Boston, 249–299.
[18] K. Uno, On representations of non–semisimple specialized Hecke algebras, J. Algebra, 149 (1992), 287–312.

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