NOTES ON A THEOREM OF KATZNELSON AND ORNSTEIN

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Abstract. Let \( \log f' \) be an absolutely continuous and \( f''/f' \in L_p(S^1, d\ell) \) for some \( p > 1 \), where \( \ell \) is Lebesgue measure. We show that there exists a subset of irrational numbers of unbounded type, such that for any element \( \tilde{\rho} \) of this subset, the linear rotation \( R_{\tilde{\rho}} \) and the shift \( f_t = f + t \mod 1 \), \( t \in [0,1) \) with rotation number \( \tilde{\rho} \), are absolutely continuously conjugate. We also introduce a certain Zygmund-type condition depending on a parameter \( \gamma \), and prove that in the case \( \gamma > \frac{1}{2} \) there exists a subset of irrational numbers of unbounded type, such that every circle diffeomorphism satisfying the corresponding Zygmund condition is absolutely continuously conjugate to the linear rotation provided its rotation number belongs to the above set. Moreover, in the case of \( \gamma > 1 \), we show that the conjugacy is \( C^1 \)-smooth.

1. Introduction. We study orientation-preserving aperiodic diffeomorphisms \( f \) of the circle \( S^1 = \mathbb{R}/\mathbb{Z} \). Poincaré (1885) noticed that the orbit structure of such \( f \) is determined by some irrational mod 1, called the rotation number of \( f \) and denoted \( \rho = \rho(f) \), in the following sense: for any \( \xi \in S^1 \), the mapping \( f^j(\xi) \to j\rho \mod 1 \), \( j \in \mathbb{Z} \), is orientation-preserving. Some fifty years later, Denjoy proved that if \( f \) is an orientation-preserving \( C^1 \)-diffeomorphism of the circle with irrational rotation number \( \rho \) and \( \log f' \) has bounded variation then the orbit \( \{f^j(\xi)\}_{j \in \mathbb{Z}} \) is dense and the mapping \( f^j(\xi) \to j\rho \mod 1 \) can therefore be extended by continuity to a homeomorphism \( h \) of \( S^1 \), which conjugates \( f \) to the linear rotation \( R_{\rho} : \xi \to \xi + \rho \mod 1 \), that is, such that \( f = h^{-1} \circ R_{\rho} \circ h \).

In this context, a natural question to ask is under what condition one can obtain higher smoothness for the conjugation \( h \)? The first local result asserting regularity of the conjugation of the circle diffeomorphism to the linear rotation was obtained by Arnold [1]. He proved that, for typical irrational rotation number \( \rho \) and for analytic diffeomorphism \( f \) sufficiently close to the linear rotation \( R_{\rho} \), the conjugation \( h \)
is analytic. Later on, Moser extended this result for sufficiently smooth but not analytic diffeomorphisms [14].

In the end of the 70’s the first global result, that is, a result which is not requiring the closeness of diffeomorphism to the linear rotation, was proved by Herman [5]. It was shown that if \( f \in C^k (k \geq 3) \), its rotation number is irrational and satisfies a certain Diophantine condition i.e., \( \rho \in \mathcal{D} = \{ \alpha : |e^{2\pi in\alpha} - 1| \geq C_\delta |n|^{-1-\delta}, \text{ for any } \delta > 0 \text{ and } n \in \mathbb{Z} \} \), then \( h \) is in fact \( k - 1 - \epsilon \) times differentiable for any \( \epsilon > 0 \), and is analytic if \( f \) is analytic. We notice that \( \ell(\mathcal{D}) = 1 \), that is Herman’s theorem holds for a set of rotation numbers of full measure. The proof of Herman’s theorem is based on an application of the Schwarzian derivative and therefore the condition \( k \geq 3 \) is crucial. Later, Yoccoz [20] extended Herman’s theorem for all Diophantine numbers. At the same time Hawkins and Schmidt [4] showed that, for every irrational number \( \alpha \in [0,1) \) of unbounded type, there exists a \( C^2 \) circle diffeomorphism \( f \) with \( \rho(f) = \alpha \), for which the conjugating map \( h \) between \( f \) and \( R_\rho \) is singular.

In the end of the 80’s two different approaches to the Herman’s theory were developed by Katznelson and Ornstein [9, 10] and Khanin and Sinai [11, 12]. These approaches gave sharp results on the smoothness of the conjugacy in the case of diffeomorphisms with low smoothness. Recently, Khanin and Teplinsky [13] developed a conceptually new approach which is entirely based on the idea of cross-ratio distortion estimates. Quite surprisingly this simple and elementary approach allows to prove stronger results. Let us briefly recall some details of above results. It is well known that the smoothness of a conjugacy \( h \) is strongly related to sharp estimates of \( K_n = \max \{ |\log(f^{q_n}(\xi))| \} \), where \( q_n \) is a first return time of \( f \) and it is defined as: \( q_n = \min \{ k \in \mathbb{N} : \|k\rho\| < \|q_{n-1}\rho\| \} \), \( q_0 = 1 \). More precisely, in the works [5, 9, 11, 12, 13] and [20] it was shown that if \( f \in C^k \), \( k \geq 2 + \epsilon \) and the rotation number \( \rho \) is irrational, then the sequence \( (K_n) \) tends to zero exponentially fast. This fact together with Diophantine-type conditions on rotation numbers ensure that the conjugating map \( h \) is at least \( C^1 \)-smooth. Following Katznelson and Ornstein [10] we define a class of low smoothness circle diffeomorphisms as follows.

**Definition 1.1.** We say that a circle diffeomorphism \( f \) belongs to KO class if \( \log f^{'} \) is absolutely continuous, \( f^{''}/f^{'} \in L_p(S^1, d\ell) \) for some \( p > 1 \), and the rotation number \( \rho \) of \( f \) is irrational.

Katznelson and Ornstein [10] proved that within KO class the sequence \( (K_n) \) belongs to \( \ell_2 \). Moreover, if the rotation number is of bounded type, then \( h \) is absolutely continuous. In view of the above results the following question arises naturally.

*Is it possible to extend the results on absolute continuous linearization for a larger class of rotation numbers which will include some rotation number of unbounded type?* Seemingly the results of Hawkins and Schmidt forbid such an extension. However, here we are interested not in all but in typical diffeomorphisms satisfying the KO conditions. A natural way to introduce typical diffeomorphisms is to consider one-parameter families such that the rotation number is changed monotonically with the parameter. The simplest example of such family is given by a shift corresponding to an additive constant, \( f_t = f + t \), \( t \in [0,1] \). In other words, we are interested whether the result of Katznelson and Ornstein [10] can be extended to a larger class of rotation numbers within a family \( f_t \).

In this paper, we give an affirmative answer to the above question. We show that for any KO diffeomorphism \( f \), that is, the diffeomorphism satisfying KO conditions, there exists a subset \( \mathcal{I} \) of irrational numbers of unbounded type, such that for any
\( \hat{\rho} \in \mathcal{I} \), the linear rotation \( R_{\hat{\rho}} \) and the shift \( f_t = f + t \) with rotation number \( \hat{\rho} \), are absolutely continuously conjugate. To formulate this statement we need the following notions. Let \( \alpha \in (0, 1) \) be an irrational number. We use the continued fraction representation

\[
\alpha = 1/(a_1 + 1/(a_2 + ...)) : = [a_1, a_2, ..., a_s, ...]
\]
of a given number \( \alpha \). The sequence of positive integers \( (a_s) \) with \( s \geq 1 \), called partial quotients, and is uniquely determined for each \( \alpha \). Now we define a subset of irrational numbers by using two given sequences of natural numbers. Let \( (i_n) \) be a strictly increasing sequence of natural numbers, \( (v_n) \) be an unbounded sequence of natural numbers and \( M \) be a natural number. Denoting the set of all irrational numbers \( \alpha = [a_1, a_2, ..., a_s, ...] \) such that \( a_{i_n} \leq v_n \) and \( a_s \leq M \) for any \( s \in \mathbb{N} \setminus \{i_n, n = 1, 2, ...\} \) by \( \mathcal{I}(i_n, v_n, M) \), we set

\[
\mathcal{I}(i_n, v_n) = \bigcup_{M=1}^\infty \mathcal{I}(i_n, v_n, M).
\]

Our first main result is given by the following theorem.

**Theorem 1.2.** Let \( f \) be a KO diffeomorphism of the circle. Then for any unbounded sequence of natural numbers \( (v_n) \), there exists a strictly increasing sequence \( i_n = i_n(f, v_n) \) of natural numbers, such that for any \( \hat{\rho} \in \mathcal{I}(i_n, v_n) \), the conjugating map \( h \) between \( f_{t_0} \) and \( R_{\hat{\rho}} \) and its inverse \( h^{-1} \) are absolutely continuous and \( h', (h^{-1})' \in L^2 \). Here \( t_0 = t_0(f, \hat{\rho}) \) is the unique value of a parameter \( t \) such that \( \rho(f_{t_0}) = \hat{\rho} \).

This result extends the result of Katznelson and Ornstein \[10\]. It is clear that the union of the sets \( \mathcal{I}(i_n, v_n) \) contains the set of all irrational numbers of bounded type. Since the set \( \mathcal{I}(i_n, v_n) \) depends on \( v_n \), generally we cannot say much about its Lebesgue measure. We may say that the main feature of this theorem is the arbitrariness of KO diffeomorphism \( f \), and the fact that \( (v_n) \) can tend to infinity arbitrarily fast. Further, we show that the set of rotation numbers can be extended to an unbounded set for a certain subclass of KO. For this we impose a certain Zygmund condition on \( f' \) which defines a subclass within the KO class. Let us consider the following one-parameter family of functions: \( \Phi_\gamma : [0, 1) \to [0, +\infty) \), \( \Phi_\gamma(0) = 0 \) and

\[
\Phi_\gamma(x) = \frac{x}{(\log \frac{x}{2})^\gamma}, \text{ where } 0 < x < 1 \text{ and } \gamma > 0.
\]

Denote by \( \Delta^2 f'(\xi, \tau) \) the second symmetric difference of \( f' \) i.e.,

\[
\Delta^2 f'(\xi, \tau) = f'(\xi + \tau) + f'(\xi - \tau) - 2f'(\xi)
\]

where \( \xi \in S^1 \) and \( \tau \in [0, \frac{1}{2}] \). Suppose that there exists a constant \( C > 0 \) such that the following inequality holds:

\[
\|\Delta^2 f'(\cdot, \tau)\|_{L^\infty(S^1)} \leq C\Phi_\gamma(\tau).
\]

Denote by \( \mathcal{Z}_{\Phi_\gamma} \) the class of circle diffeomorphisms \( f \), whose derivatives \( f' \) satisfy \( \Phi_\gamma \). Below we work with this class.

We note that the class of continuous functions satisfying \( \Phi_\gamma \) is a subclass of Zygmund class \( \Lambda^* \) (see \[21\] for the definition). The Zygmund class \( \Lambda^* \) plays a key role in analysis of the trigonometric series. The class \( \Lambda^* \) was applied to the circle homeomorphisms for the first time by Hu and Sullivan (see \[6\], \[10\]) who extended the classical Denjoy’s theorem to this class. The main motivation comes
from the idea that for many rigidity type problems, the Zygumnd condition is more natural than the usual smoothness spaces when one tries to obtain sharp results.

For example, as we have seen above in the $C^2$-class in general one can guarantee absolute continuity of a linearizing conjugacy only for rotation numbers of bounded type. Below we study related questions in the Zygumnd classes which allow us to extend the class of rotation numbers beyond the bounded type.

Note that for the case of $\gamma \in (1/2, 1]$ the class of continuous functions satisfying the condition $\| \_ \|_1$ was studied by Weiss and Zygumnd in [19]. They proved that in this case the function satisfies KO conditions (see Theorem 6.1 below). Our next main result is the following theorem.

**Theorem 1.3.** Let $f \in Z_\Phi_\gamma$ be a circle diffeomorphism with irrational rotation number $\rho$ and $\gamma \in (1/2, 1]$. Suppose that for some $\alpha \in (0, \gamma - 1/2)$ the partial quotients of $\rho$ satisfies $a_n \leq Cn^\alpha$, $C > 0$. Then the conjugating map $h$ between $f$ and $R_\rho$ and its inverse $h^{-1}$ are absolutely continuous and $h'(nh^{-1}) \in L_2$.

This theorem extends the result of Katzenelson and Ornstein [10]. The theorem is applicable to a set of rotation numbers which includes some irrational numbers of unbounded type. However, the Lebesgue measure of this set is equal to zero.

We next consider the case of $C^1$-smooth linearization. We again consider the Zygumnd class $Z_\Phi_\gamma$ but now assume that $\gamma > 1$. Note that in this case, the class $Z_\Phi_\gamma$ is a subclass of $C^2$ (see Theorem 6.2) and it is wider than $C^{2+\epsilon}$. Our next main result is as follows.

**Theorem 1.4.** Let $f \in Z_\Phi_\gamma$ be a circle diffeomorphism with irrational rotation number $\rho$ and $\gamma > 1$. Suppose that for some $\alpha \in (0, \gamma - 1)$ the partial quotients of $\rho$ satisfies $a_n \leq Cn^\alpha$, $C > 0$. Then the conjugating map $h$ between $f$ and $R_\rho$ and its inverse $h^{-1}$ are $C^1$ diffeomorphisms.

In this theorem, if $1 < \gamma \leq 2$ then the Lebesgue measure of the set of rotation numbers is equal to zero, but if $\gamma > 2$ and $1 < \alpha < \gamma - 1$ then the set of rotation numbers has full Lebesgue measure.

The paper is organized as follows. In Section 2, we present the basic notions and classical inequalities. We also estimate the ratio of lengths of intervals of dynamical partition. In Section 3, we derive an estimate for $K_n(t)$ corresponding to $f_t$, (Theorem 3.2) which plays an important role in the proof of the first main theorem. In Section 4, we obtain a uniform (in parameter) estimate for $K_n(t)$. In Section 5, we prove Theorem 1.2. The next sections are devoted to the proofs of the second and third main theorems. In Section 6, we briefly discuss the properties of the class $Z_\Phi_\gamma$. Using these properties, in Section 7 we get a sharp estimate for $K_n$ (Theorem 7.1). In Section 8 we prove Theorem 1.3 and Theorem 1.4. In the last section we discuss on some extensions of our main theorems.

2. Dynamical partition and universal estimates. Dynamical partition.

Let $f$ be a circle homeomorphism with irrational rotation number $\rho$. Taking a point $\xi_0 \in S^1$ we define the $n$-th fundamental segment $I_0^n := I_0^n(\xi_0)$ as the circle arc $[\xi_0, f^{n}(\xi_0)]$ if $n$ is even and $[f^{n}(\xi_0), \xi_0]$ if $n$ is odd. We denote two sets of closed intervals of order $n$: $q_n$ “long” intervals $\{ I_0^{n-1} := f(I_0^{n-1}), 0 \leq i < q_n \}$ and $q_{n-1}$ “short” intervals $\{ I_j^n := f^j(I_0^n), 0 \leq j < q_{n-1} \}$. The long and short intervals are mutually disjoint except for the endpoints and cover the whole circle. The partition obtained by the above construction will be denoted by $P_n := P_n(\xi_0, f)$ and it is
called the \( n \)-th dynamical partition of \( S^1 \). Obviously, partition \( P_{n+1} \) is a refinement of partition \( P_n \). Indeed, the short intervals are members of \( P_{n+1} \) and each long interval \( I_i^{n-1} \in P_n \), \( 0 \leq i < q_n \), is partitioned into \( a_{n+1} + 1 \) intervals belonging to \( P_{n+1} \) such that
\[
I_i^{n-1} = I_i^{n+1} \cup \bigcup_{s=0}^{a_{n+1}-1} I_{i+q_{n-1}+sq_n}^n.
\] (2)

Denjoy’s theory. We first state the following definition which was introduced in [10].

**Definition 2.1.** An interval \( I = (a, b) \) is \( q_n \)-small and its endpoints \( a, b \) are \( q_n \)-close if \( \{f^j(I)\}_{0}^{q_n-1} \) are disjoint.

Next we introduce two quantities which were also defined in [10]. Then we provide estimates for these quantities which are valid for any circle diffeomorphisms \( f \in C^{1+BV} \) (\( f' \) has bounded variation) with irrational rotation number \( \rho \). These estimates have very important applications in the theory of circle homeomorphisms. Their elementary proofs can be found in [9], [10] and [12].

- \( K_n := K_n(f) = \max_{\xi} |\log(f^{q_n}(\xi))'| = |\log(f^{q_n})'|_0. \)
- \( \hat{K}_n := \hat{K}_n(f) = \sup_{\xi, \eta} \left|\log(f^{k}(\xi))' - \log(f^{k}(\eta))'\right| \) the supremum being taken for all \( k, 0 \leq k < q_n \) and intervals \( (\xi, \eta) \) which are \( q_n \)-small.

The following inequalities hold for any circle diffeomorphisms \( f \in C^{1+BV} \)

(a) Denjoy’s inequality: \( K_n \leq v; \)
(b) Finzi’s inequality: \( \hat{K}_n \leq v; \)

where \( v = \text{Var}_{S^1} \log f' \).

**Family of circle diffeomorphisms and universal estimates.** Let \( f \) be a \( C^{1+BV} \) diffeomorphism of the circle. Consider a family of circle diffeomorphisms \( f_t = f + t \), where \( t \in \mathcal{I} = \{t \in [0, 1]: \rho_t = \rho(f_t) - \text{is irrational}\} \). Similarly as above we can define \( K_n(t) := K_n(f_t) \) and \( \hat{K}_n(t) := \hat{K}_n(f_t) \) for every \( f_t, t \in \mathcal{I} \). An important note is that both Denjoy’s and Finzi’s inequalities hold uniformly for every \( f_t \) with the same constant \( v \), that is

(a’) Uniform Denjoy’s inequality: \( \sup_{t \in \mathcal{I}} K_n(t) \leq v; \)
(b’) Uniform Finzi’s inequality: \( \sup_{t \in \mathcal{I}} \hat{K}_n(t) \leq v; \)

where \( v = \text{Var}_{S^1} \log f'_t \).

The proofs of these inequalities follow from a simple observation: \( v = \text{Var}_{S^1} \log f'_t \) does not depend on \( t \).

Our next discussion is related to the study of some properties of dynamical partition of \( S^1 \) generated by \( f_t \). Denote \( d_n(t) := d_n(f, t) = ||f^{q_n(t)}_t - \text{Id}||_0 = \max_{\xi \in S^1} |f^{q_n(t)}_t(\xi)|. \) It is easy to see that the sequence \( d_n(t) \) is monotone decreasing: \( d_{n+1}(t) \leq d_n(t) \).

Note that we equip \( S^1 \) with the usual metric \( |x - y| = \inf \{|\tilde{x} - \tilde{y}|, \text{ where } \tilde{x}, \tilde{y} \text{ range over the lifts of } x, y \in S^1 \text{ respectively}\}. \) We will need the following elementary but important theorem.

**Theorem 2.2.** Let \( f \) be a \( C^{1+BV} \) diffeomorphism of the circle. Assume that its rotation number is irrational. Then the following statements hold:
(a) for any intervals \( I^{n+m,t} = I_0^{n+m,t}(\eta) \) and \( I^{n,t} = I_0^{n,t}(\xi) \) such that \( I^{n+m,t} \subset I^{n,t} \), we have
\[
\frac{|I^{n+m,t}|}{|I^{n,t}|} \leq e^v(1 + e^v) \frac{d_{n+m}(t)}{d_n(t)};
\]

(b) for any \( n \geq 1 \) and \( m \geq 0 \) we have
\[
\frac{|I_0^{n+m,t}(\xi)|}{|I_0^{n,t}(\xi)|} \leq e^{2v}(1 + e^v) \frac{d_{n+m}(t)}{d_n(t)};
\]

(c) Let \( \lambda = (1 + e^{-v})^{-1/2} \). Then
\[
\frac{d_{n+m}(t)}{d_n(t)} \leq \lambda^m, \text{ for } m \text{ even, and } \frac{d_{n+m}(t)}{d_n(t)} \leq \lambda^{m-1}, \text{ for } m \text{ odd.}
\]

**Proof.** Select the point \( \xi^* \in S^1 \) such that \( d_n(t) = |I_0^{n,t}(\xi^*)| \). Due to the combinatorics of trajectories, there exists \( 0 \leq i < q_{n+1}(t) \) such that either \( I_0^{n,i}(\xi^*) \subset f_t(I_0^{n,t}(\xi^*)) \cup f_t^{-q_{n+1}(t)}(I_0^{n,t}(\xi^*)) \) or \( I_0^{n,t}(\xi^*) \subset f_t(I_0^{n,t}(\xi^*)) \cup f_t^{-q_{n+1}(t)}(I_0^{n,t}(\xi^*)) \). By applying uniform Denjoy’s inequality to the last two relations, we get \( |I_0^{n,t}(\xi^*, t)| \leq (1 + e^v)|f_t(I_0^{n,t}(\xi^*))| \). Then uniform Finsi’s inequality implies
\[
\frac{|I^{n+m,t}|}{|I^{n,t}|} \leq e^v|f_t(I_0^{n+m,t}(\eta))| \leq e^v(1 + e^v) \frac{d_{n+m}(t)}{d_n(t)}.
\]

To prove (b), first notice that for even \( m \) a stronger statement follows immediately from (3) that is
\[
\frac{|I_0^{n+m,t}(\xi)|}{|I_0^{n,t}(\xi)|} \leq e^v(1 + e^v) \frac{d_{n+m}(t)}{d_n(t)}.
\]

If \( m \) is odd the proof is similar but requires a little modification. In this case the interval \( I_0^{n+m,t}(\xi) \) is not inside \( I_0^{n,t}(\xi) \). Therefore,
\[
\frac{|I_0^{n+m,t}(\xi)|}{|I_0^{n,t}(\xi)|} = \frac{|I_0^{n+m,t}(\xi_{-q_{n+1}(t)})|}{|I_0^{n,t}(\xi)|} \frac{|I_0^{n+m,t}(\xi)|}{|I_0^{n+m,t}(\xi_{-q_{n+1}(t)})|} \leq e^{2v}(1 + e^v) \frac{d_{n+m}(t)}{d_n(t)}.
\]

Next we prove (c). By the property of dynamical partition, it is easy to see that for any \( \zeta \in S^1 \) and \( t \in \mathcal{I} \), we have \( |I_0^{n-1,t}(\zeta)| \geq |I_0^{n+1,t}(\zeta)| + |I_0^{n,t}(\zeta_{q_{n+1}(t)} - q_{n}(t))| \) and \( I_0^{n+1,t}(\zeta) \subset I_0^{n,t}(\zeta_{q_{n+1}(t)}) \). The last two relations and uniform Denjoy’s inequality imply
\[
\frac{|I_0^{n-1,t}(\zeta)|}{|I_0^{n+1,t}(\zeta)|} \geq 1 + \frac{|I_0^{n,t}(\zeta_{q_{n+1}(t)} - q_{n}(t))|}{|I_0^{n,t}(\zeta_{q_{n+1}(t)})|} \geq 1 + e^{-v}.
\]

By induction, we get \( |I_0^{n+m,t}(\zeta)| \leq (1 + e^{-v})^{-m} |I_0^{n,t}(\zeta)| \) for even \( m \). If we pick out the point \( \zeta = \xi^* \in S^1 \) such that \( d_{n+m}(t) = |I_0^{n+m,t}(\xi^*)| \), then we get
\[
\frac{d_{n+m}(t)}{d_n(t)} \leq \frac{|I_0^{n+m,t}(\xi^*)|}{|I_0^{n,t}(\xi^*)|} \leq \lambda^m.
\]

In case of odd \( m \) we have
\[
\frac{d_{n+m}(t)}{d_n(t)} = \frac{d_{n+m}(t)}{d_{n+1}(t)} \frac{d_{n+1}(t)}{d_n(t)} \leq \lambda^{m-1}
\]

since \( d_{n+1}(t) \leq d_n(t) \).
Denote $\tilde{d}_n = \sup_{t \in \mathcal{T}} d_n(t)$.

**Remark.** The inequality (4) implies $\tilde{d}_n \leq \lambda^2 \tilde{d}_{n-2}$, $n \geq 2$. Furthermore, inequalities (5) and (7) imply $d_{n+m} \leq \lambda^m \tilde{d}_n$ for $m$ even and $d_{n+m} \leq \lambda^{m-1} \tilde{d}_n$ for $m$ odd. The useful convention $q_{-1} = 0$, $q_0 = 1$ imply that $I_0^{-1}(\xi) = [\xi - 1, \xi]$, $I_0^0(\xi) = [\xi, f(\xi)]$.

It follows that $\tilde{d}_n \leq \lambda^n$ for $n$ even, and $\tilde{d}_n \leq \lambda^{n+1}$ for $n$ odd.

**The ratio distortion.** Various types of ratio and cross-ratio distortion estimates are used in dynamical systems. The cross-ratio distortions were used for the first time by Yoccoz in [20] and later by de Melo and van Strien in [2] and by Świątek in [17]. The asymptotic estimates for a cross-ratio distortion with respect to smooth monotone function were studied in [18]. The ratio of three points $a, b, c$ is

$$Q(a, b, c) = \left\| \frac{[a, b]}{[b, c]} \right\|.$$

Their ratio distortion with respect to the function $f$ is

$$R(a, b, c; f) = \frac{Q(f(a), f(b), f(c))}{Q(a, b, c)}. \quad (8)$$

Let $f \in C^1$ where the derivative of $f$ does not have zeros on $S^1$. Taking the limit in (8) when $b \to c$ we obtain

$$R(c, I; f) = \frac{|f(I)|}{|I|} \frac{1}{f'(c)} \text{ where } I = [a, c].$$

This ratio distortion is one of main tools of the proofs in this paper. Notice that this distortion is multiplicative with respect to composition that is, for any two functions $f$ and $g$ we have

$$R(c, I; f \circ g) = R(c, I; f) \cdot R(g(c), g(I); f). \quad (9)$$

Martingale convergence in $L_p$. Suppose $f$ satisfies KO conditions. Using dynamical partitions $P_n^t$ we define a sequence of step functions on the circle as follows: $M^t_n(x) \equiv 0$, $x \in S^1$ and for any $n \geq 1$ we set

$$M^t_n(x) = \frac{1}{|I^{n,t}|} \int_{I^{n,t}} f''(s) \, ds, \text{ if } x \in I^{n,t}, \quad I^{n,t} \in P_n^t. \quad (10)$$

Denoted by $(P_n^t)$ the sequence of algebras generated by dynamical partitions. A simple calculation shows that the sequence of $(M^t_n)$ is a martingale with respect to $(P_n^t)$ for any $t \in \mathcal{T}$. Moreover, using Hölder’s inequality we obtain

$$||M^t_n||_p^p = \int_{S^1} |M^t_n(x)|^p \, dx = \sum_{I^{n,t} \in P_n^t} \int_{I^{n,t}} |M^t_n(x)|^p \, dx$$

$$= \sum_{I^{n,t} \in P_n^t} \frac{1}{|I^{n,t}|^{p-1}} \int_{I^{n,t}} \left| \frac{f''(x)}{f'(x)} \right|^p \, dx \leq \sum_{I^{n,t} \in P_n^t} \int_{I^{n,t}} \left| \frac{f''(x)}{f'(x)} \right|^p \, dx = \left\| \frac{f''}{f'} \right\|_p^p.$$

Hence, $(M^t_n)$ is a $L_p$-bounded martingale for any $t \in \mathcal{T}$. According to Doob’s theorem we have the following.

**Theorem 2.3.** Suppose $f$ satisfies the KO conditions. Then for any $t \in \mathcal{T}$ as $n \to \infty$

$$M^t_n \to \frac{f''}{f'} \text{ almost surely and in } L_p.$$
Following Katznelson and Ornstein, one can define the difference of martingales, that is
\[
\Theta^t_n(x) = M^t_n(x) - M^t_{n-1}(x), \quad n \geq 1.
\] (11)
The martingale property implies the following.

**Statement 1.** Let a diffeomorphism \( f \) satisfy the KO conditions. Then for any \( t \in I \) the following equality holds:
\[
\int_I \Theta^t_n(x) dx = 0 \quad \text{for any } I \in P^t_{n-1}.
\]

The following proposition was proven by Katznelson and Ornstein in [10].

**Proposition 1.** Let \((G_n)\) be a \( L^p \)-bounded martingale, \( 1 < p \leq 2 \). Define \( g_n = G_n - G_{n-1} \). Then
\[
\sum_{n=1}^\infty \|g_n\|_p^2 < \infty.
\]

Since \((M_n^t)\) is a \( L^p \)-bounded martingale for any \( t \in I \), Proposition 1 has the following immediate consequence.

**Corollary 1.** Let the diffeomorphism \( f \) satisfies the KO conditions and \( 1 < p \leq 2 \). Then for any \( t \in I \) we have
\[
\sum_{n=1}^\infty \|\Theta^t_n\|_p^2 < \infty.
\]

3. \( \ell^2 \)-convergence of \( K_n(t) \). In this section we prove the \( \ell^2 \) convergence of \( K_n(t) \).

We use the following sequences:
\[
\varepsilon_n(t) = d_{n-1}^t(t) + \sum_{k=n+1}^\infty \frac{d_{k-2}(t)}{d_{n-1}(t)} \|\Theta^t_k\|_p, \quad n \geq 0, \tag{12}
\]
\[
\eta_n(t) = \sum_{k=1}^n \frac{d_{n-1}(t)}{d_{k-1}(t)} \frac{d_n(t)}{d_k(t)} \varepsilon_{k-1}(t), \quad n \geq 1,
\]
\[
\tau_n(t) = \sum_{k=1}^n \frac{d_n(t)}{d_k(t)} (\eta_k(t) + \varepsilon_k(t)), \quad n \geq 1.
\]

**Lemma 3.1.** Let a diffeomorphism \( f \) satisfy the KO conditions. Then the sequences \((\varepsilon_n(t))\), \((\eta_n(t))\) and \((\tau_n(t))\) belong to \( \ell^2 \) for any \( t \in I \).

**Proof.** By Theorem 2.2 we get
\[
\varepsilon_n(t) \leq \frac{1}{\lambda} \left( \lambda^{\frac{p}{2}} + \sum_{k=n+1}^\infty \lambda^{k-(n+1)} \|\Theta^t_k\|_p \right).
\]
This implies
\[
\varepsilon^2_n(t) \leq \frac{2}{\lambda^2} \left( \lambda^{\frac{2p}{2}} + \left( \sum_{k=n+1}^\infty \lambda^{k-(n+1)} \|\Theta^t_k\|_p \right)^2 \right).
\]

\[\dagger\] Note that \( d_{-1}(t) = \max_{\xi_0 \in S^1} |\varphi^{-1}(t_0, \xi_0)| = 1.\]
On the other hand applying Cauchy–Schwarz inequality we obtain
\[
\left( \sum_{k=n+1}^{\infty} \lambda^{k-(n+1)} \| \Theta_k^n \|_p \right)^2 \leq \frac{1}{1 - \lambda} \sum_{k=n+1}^{\infty} \lambda^{k-(n+1)} \| \Theta_k^n \|_p^2.
\]

Therefore
\[
\sum_{n=0}^{\infty} C_n^2(t) \leq \frac{2}{\lambda^2(1 - \lambda^2)} + \frac{2}{\lambda^2(1 - \lambda^2)} \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} \lambda^{k-(n+1)} \| \Theta_k^n \|_p^2 = \frac{2}{\lambda^2(1 - \lambda^2)} + \frac{2}{\lambda^2(1 - \lambda^2)} \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} \| \Theta_k^n \|_p^2 \leq \infty.
\]

Similarly one can show that the sequences \((\eta_n(t))\) and \((\tau_n(t))\) belong to \(\ell_2\).

\[\square\]

**Theorem 3.2.** Let a diffeomorphism \(f\) satisfy the KO conditions. Then for any \(t \in I\) there exists a constant \(C_3 = C_3(f) > 0\) such that

\[K_n(t) \leq C_3 \cdot \tau_n(t).\]  

(12)

It is easy to see that, by Lemma 3.1, \(K_n(t)\) belongs to \(\ell_2\). The proof of Theorem 3.2 will be provided in the next section after we prove the following two lemmas.

**Lemma 3.3.** Suppose a diffeomorphism \(f\) satisfies the KO conditions. Then for any \(t \in I\) there exists \(C_4 = C_4(f) > 0\) such that

\[| \log \mathcal{R}(\xi_0, I_0^{n-1}; f_t^{\eta_n(t)}) - \frac{(-1)^{n-1} \eta_n(t)}{2} \sum_{s=0}^{\eta_n(t)-1} \int_{I_s^{n-1}} f''(x) \, dx | \leq C_4 \varepsilon_n(t),\]  

(13)

\[| \log \mathcal{R}(\xi_0, I_0^{n}; f_t^{\eta_n(t)}) - \frac{(-1)^{n} \eta_n(t)-1}{2} \sum_{s=0}^{\eta_n(t)-1} \int_{I_s} f''(x) \, dx | \leq C_4 \varepsilon_{n+1}(t).\]  

(14)

**Proof.** We prove the first inequality. Take any \(t \in I\) and fix it. To simplify the notations, below we write the formulae without \(t\). By multiplicativity of \(\mathcal{R}(\xi_0, I_0^{n-1}; f^{\eta_n})\) with respect to composition, we have

\[\log \mathcal{R}(\xi_0, I_0^{n-1}; f^{\eta_n}) = \sum_{s=0}^{\eta_n-1} \log \left[ \frac{f(\eta_s) - f(\xi_s)}{\eta_s - \xi_s} \cdot \frac{1}{f''(\xi_s)} \right],\]  

(15)

where \(\eta_s = f^s(\xi_{n-s})\) and \(\xi_s = f^s(\xi_0)\) are end-points of the interval \(I_s^{n-1}\). Since the diffeomorphism \(f\) satisfies the KO conditions we have

\[f(\eta_s) = f(\xi_s) + f'(\xi_s)(\eta_s - \xi_s) + \int_{\xi_s}^{\eta_s} f''(x)(\eta_s - x) \, dx\]  

(16)

for any \(0 \leq s < \eta_n\). Using (15) and (16) together with Hölder’s inequality one can show that

\[\log \mathcal{R}(\xi_0, I_0^{n-1}; f^{\eta_n}) = \sum_{s=0}^{\eta_n-1} \log \left[ 1 + \int_{\xi_s}^{\eta_s} \frac{f''(x)}{f'(\xi_s)} \cdot \frac{\eta_s - x}{\eta_s - \xi_s} \, dx \right] = \sum_{s=0}^{\eta_n-1} \left[ \int_{\xi_s}^{\eta_s} \frac{f''(x)}{f'(x)} \cdot \frac{\eta_s - x}{\eta_s - \xi_s} \, dx \right] + O\left(\varepsilon_{n-1}^{1/2}\right).\]  

(17)
We note that the interval \([\xi_s, \eta_s]\) is a \((n - 1)\)-th fundamental segment. It follows that the integral \(\int_{\xi_s}^{\eta_s} \) changes sign depending on the parity of \(n\). More precisely,
\[
\int_{\xi_s}^{\eta_s} \frac{f''(x)}{f'(x)} \cdot \frac{\eta_s - x}{\eta_s - \xi_s} \, dx = (-1)^{n-1} \int_{l_{s}^{-1}}^{N} \frac{f''(x)}{f'(x)} \cdot \frac{\eta_s - x}{\eta_s - \xi_s} \, dx.
\]

Using this we rewrite the right hand side of (17) in the following form
\[
\log R(\xi_0, I_0^{-1}; f^n) = (-1)^{n-1} \sum_{s=0}^{q_n-1} \int_{l_{s}^{-1}}^{N} \frac{f''(x)}{2f'(x)} \, dx + \frac{1}{2} \sum_{s=0}^{q_n-1} \left( \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{\eta_s - \xi_s} \right) \frac{\eta_s - x}{\eta_s - \xi_s} \, dx + O\left(d_{n-1}^{\frac{1}{2}}\right).
\]

Next we estimate the second sum of (18). It is obvious that for any natural number \(N\) one has
\[
\sum_{s=0}^{q_n-1} \int_{l_{s}^{-1}}^{N} \frac{f''(x)}{f'(x)} \cdot \left( \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{2} \right) \, dx = \sum_{k=1}^{N} \sum_{s=0}^{q_n-1} \int_{l_{s}^{-1}}^{N} \left( \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{2} \right) \Theta_k(x) \, dx + (19)
\]

\[
+ \frac{q_n-1}{2} \int_{l_{s}^{-1}}^{N} \left( \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{2} \right) \cdot \left( \frac{f''(x)}{f'(x)} - \mathcal{M}_N(x) \right) \, dx := A_n + B_n.
\]

First we estimate \(|B_n|\). We set
\[
U_n = \bigcup_{s=0}^{q_n-1} I_{s}^{-1}.
\]

It is clear that
\[
|B_n| \leq \left( t(U_n) \right)^{\frac{1}{q}} \left\| \mathcal{M}_N - \frac{f''}{f'} \right\|_p.
\]

According to Theorem 2.3 we have
\[
\lim_{n \to \infty} \left\| \mathcal{M}_N - \frac{f''}{f'} \right\|_p = 0.
\]

So, one can choose a sufficiently large number \(N\) such that
\[
\left\| \mathcal{M}_N - \frac{f''}{f'} \right\|_p \leq d_{n-1}^{\frac{1}{2}}.
\]

Thus, \(|B_n| \leq d_{n-1}^{\frac{1}{2}}.\) Now we estimate \(|A_n|\). For this, we divide the first sum of \(A_n\) into three terms corresponding to summation over: \(1 \leq k \leq n, \ k = n + 1\) and \(n + 2 \leq k \leq N\) and we estimate each term separately.

Let \(1 \leq k \leq n\). In this case, since \(l_{s}^{-1} \subseteq I_{s}^{-1} \in \mathbb{P}_k\) the piecewise constant function \(\Theta_k\) takes a constant value on \(l_{s}^{-1}\). Therefore
\[
\int_{l_{s}^{-1}} \left( \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{2} \right) \Theta_k(x) \, dx = 0.
\]
Hence
\[
\left| \sum_{k=1}^{n} \sum_{s=0}^{q-1} \int_{I_{k-1}^{n}} \left( \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{2} \right) \Theta_k(x) \, dx \right| = 0. \tag{20}
\]

Next, let \( k = n + 1 \). It is obvious that
\[
\left| \sum_{s=0}^{q-1} \int_{I_{n+1}^{n}} \left( \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{2} \right) \Theta_{n+1}(x) \, dx \right| \leq \left( \ell(U_n) \right)^{1/q} \| \Theta_{n+1} \|_p. \tag{21}
\]

Finally, let \( n + 2 \leq k \leq N \). We define a new piecewise constant function \( L_{k,s} \) on \( I_{n+1}^{n} \) as follows
\[
L_{k,s}(x) = \eta_s - \partial_r(I) \eta_s - \xi_s - \frac{1}{2}, \quad x \in I \subset [\xi_s, \eta_s], \quad 0 \leq s < q_n \quad \text{and} \quad I \in P_{k-1},
\]
where \( \partial_r(I) \) is the right end-point of interval \( I \). By construction, the function \( L_{k,s} \) takes a constant value on every interval of \( P_{k-1} \). Therefore, from Statement 1 it follows that
\[
\int_{I_{n+1}^{n}} \Theta_k(x)L_{k,s}(x) \, dx = 0. \tag{22}
\]

Moreover, by Theorem 2.2 we get
\[
\left| \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{2} - L_{k,s}(x) \right| \leq e^\nu (1 + e^\nu) \frac{d_{k-2}}{d_{n-1}}.
\]

Using this inequality and (22) we have
\[
\left| \sum_{k=n+2}^{N} \sum_{s=0}^{q-1} \left[ \int_{I_{k-1}^{n}} \left( \frac{\eta_s - x}{\eta_s - \xi_s} - \frac{1}{2} - L_{k,s}(x) \right) \Theta_k(x) \, dx + \int_{I_{n+1}^{n}} \Theta_k(x)L_{k,s}(x) \, dx \right] \tag{23}
\]
\[
\leq e^\nu (1 + e^\nu) \sum_{k=n+2}^{N} \frac{d_{k-2}}{d_{n-1}} \| \Theta_k \|_p.
\]

Using (20), (21) and (23) we obtain
\[
|A_n| \leq e^\nu (1 + e^\nu) \sum_{k=n+1}^{N} \frac{d_{k-2}}{d_{n-1}} \| \Theta_k \|_p.
\]

Hence
\[
|A_n| + |B_n| \leq e^\nu (1 + e^\nu) \varepsilon_n.
\]

Then, this inequality together with (18) imply the inequality (13).

The inequality (14) can be proved in the same manner as above, but there will be some changes in the estimate of \( A_n \). Since each short interval of \( P_n \) is preserved when it is being passed from partition \( P_n \) to \( P_{n+1} \), the first sum of (19) is divided into three parts: \( 1 \leq k \leq n + 1, k = n + 2 \) and \( n + 3 \leq k \leq N \). These three sums will be estimated similarly to the above and the estimate of \( |A_n| + |B_n| \) will be \( \varepsilon_{n+1} \).

We also need the following lemma for the proof of Theorem 3.2.
Lemma 3.4. Suppose a diffeomorphism $f$ satisfies the KO conditions. Then for any $t \in \mathcal{I}$ there exists $C_5 = C_5(f) > 0$ such that

$$
\left| \log \left[ \frac{|I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \right] \right| - \left| \frac{|I_0^{-1}| - |I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \right| \leq \frac{C_5}{2} \sum_{s=0}^{n-1} \int_{I_s}^{|f'|} \frac{f''(x)}{f'(x)} \, dx \leq C_5 \varepsilon_n(t),
$$

Equality (24) comes from the multiplicativity of ratio distortion with respect to $|I_0|^{-1}$. Using (24)-(26) and similar arguments as in the proof of Lemma 3.3 we get

$$
\left| \log \left[ \frac{|I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \right] \right| - \left| \frac{|I_0^{-1}| - |I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \right| \leq \frac{C_5}{2} \sum_{s=0}^{n-1} \int_{I_s}^{|f'|} \frac{f''(x)}{f'(x)} \, dx \leq C_5 \varepsilon_n(t).
$$

Proof. We prove only the first inequality, the second inequality can be handled similarly. For simplicity we again omit $t$ from the notations. The following three exact relations are crucial for our proof:

$$
\log \left[ \frac{|I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \right] = \sum_{s=0}^{n-1} \log \left[ \frac{|I_{s}^{-1}|}{|I_0^{-1}|} \right] \cdot \frac{|I_{s}^{-1}| - |I_{q_n(t)}^{-1}|}{|I_0^{-1}|}.
$$

Equality (24) comes from the multiplicativity of ratio distortion with respect to composition. The equalities (25) and (26) follows from (16). Since $|I_s^{-1}| \sim |I_{s+1}^{-1}| - |I_{s}^{-1}|$, using (24)-(26) and similar arguments as in the proof of Lemma 3.3 we get

$$
\log \left[ \frac{|I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \right] - \left| \frac{|I_0^{-1}| - |I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \right| = \sum_{s=0}^{n-1} \int_{|f'|} \frac{f''(x)}{f'(x)} \, dx + \sum_{s=0}^{n-1} \int_{|f'|} \frac{f''(x)}{f'(x)} \, dx - \frac{C_5}{2} \sum_{s=0}^{n-1} \int_{|f'|} \frac{f''(x)}{f'(x)} \, dx + O(\varepsilon_n) = \frac{(-1)^{n} \varepsilon_n}{2} \sum_{s=0}^{n-1} \int_{I_{s+1}^{|f'|}} \frac{f''(x)}{f'(x)} \, dx + O(\varepsilon_n)
$$

as required.

3.1. Proof of Theorem 3.2

Proof. In fact the proof of Theorem 3.2 follows closely to the proof in Khanin and Teplinsky [13]. We need the following two relations:

$$
\frac{|I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \cdot \frac{|I_0^{-1}| - |I_{q_n(t)}^{-1}|}{|I_0^{-1}|} - 1 = \frac{|I_0^{-1}|}{|I_{q_n(t)}^{-1}|} \cdot \frac{|I_{q_n(t)}^{-1}| - |I_0^{-1}|}{|I_{q_n(t)}^{-1}|} - 1,
$$

(27)

$$
\left( f_{t_0}^{I_0^{-1}}(\xi_0) \right)^{R(\xi_0, I_0^{-1}; f_{t_0}^{I_0^{-1}}(\xi_0))} \left( f_{t_0}^{I_0^{-1}}(\xi_0) \right)^{R(\xi_0, I_0^{-1}; f_{t_0}^{I_0^{-1}}(\xi_0))} = \frac{|I_{q_n(t)}^{-1}|}{|I_0^{-1}|} \cdot \left( 1 - (f_{t_0}^{I_0^{-1}}(\xi_0))^{R(\xi_0, I_0^{-1}; f_{t_0}^{I_0^{-1}}(\xi_0))} \right).
$$

(28)
The equality (27) is easily verified. The equality (28) comes from the definitions of \( \mathcal{R}(\xi_0, I_0^{n-1,t} : f_t^{q_n(t)}) \) and \( \mathcal{R}(\xi_0, I_0^{n,t} : f_t^{q_n(t)}) \). Define

\[
m_n(t) = \exp \left( \frac{(-1)^n}{2} \sum_{s=0}^{q_n(t)-1} \int_{I_n^{s,t}} f''(x) dx \right).
\]

It is clear that

\[
\sum_{s=0}^{q_n(t)-1} \int_{I_n^{s,t}} f''(x) dx + \sum_{s=0}^{q_n(t)-1} \int_{I_n^{s+1,t}} f''(x) dx = \int_{S^1} f''(x) dx = 0.
\]

Therefore, we have

\[
\exp \left( \frac{(-1)^n+1}{2} \sum_{s=0}^{q_n(t)-1} \int_{I_n^{s+1,t}} f''(x) dx \right) = m_n(t).
\]

Due to (27) and Lemma 3.4 we have

\[
m_{n+1}(t) - 1 = \left| \frac{I_0^{n+1,t}}{I_0^{n,t}} \right| (m_n(t) - 1) + O(\varepsilon_n(t)),
\]

which is iterated into

\[
m_{n+1}(t) - 1 = O(\kappa_{n+1}(t)),
\]

where

\[
\kappa_{n+1}(t) = |I_0^{n+1,t}||I_0^{n,t}| \sum_{k=1}^{n+1} \varepsilon_{k-1}(t) |I_0^{n,t}| |I_0^{n,t}|
\]

It is easy to see that \( \kappa_{n+1}(t) \in \ell_2 \) and by Theorem 2.2 \( \kappa_n = O(\eta_n) \). Relations (28), (30) and Lemma 3.3 imply

\[
(f_t^{q_n(t)}(\xi_0))' - 1 = \left| \frac{I_0^{n,t}}{I_0^{n,t}} \right| (1 - (f_t^{q_n(t)}(\xi_0))') + O(\zeta_n(t)),
\]

which is iterated into

\[
(f_t^{q_n(t)}(\xi_0))' - 1 = O \left( \left| \frac{I_0^{n,t}}{I_0^{n,t}} \right| \sum_{k=1}^{n} \frac{\zeta_k(t)}{|I_0^{k,t}|} \right),
\]

where \( \zeta_n(t) = \eta_n(t) + \varepsilon_n(t) \). The proof of Theorem 3.2 now follows from Theorem 2.2.

4. Uniform estimates for \( \varepsilon_n(t) \), \( \eta_n(t) \) and \( \tau_n(t) \). In the following theorem we will show that the sequences \( \varepsilon_n(t) \), \( \eta_n(t) \) and \( \tau_n(t) \) tend to zero uniformly in \( t \in \mathcal{I} \) as \( n \) tends to infinity.

**Theorem 4.1.** Let \( \tilde{\varepsilon}_n = \sup_{t \in \mathcal{I}} \varepsilon_n(t) \), \( \tilde{\eta}_n = \sup_{t \in \mathcal{I}} \eta_n(t) \) and \( \tilde{\tau}_n = \sup_{t \in \mathcal{I}} \tau_n(t) \). Then

\[
\lim_{n \to \infty} \tilde{\varepsilon}_n = 0, \quad \lim_{n \to \infty} \tilde{\eta}_n = 0 \quad \text{and} \quad \lim_{n \to \infty} \tilde{\tau}_n = 0.
\]

\[\text{In the inequality (14) the estimate is } \varepsilon_{n+1}(t), \text{ however here we use the inequality } \frac{d_n(t)}{d_{n-1}(t)} \varepsilon_{n+1}(t) \leq \varepsilon_n(t).\]
Proof. To prove this theorem, we first show that \( \hat{\Theta}_n = \sup_{t \in \mathbb{I}} \|\Theta'_n\|_p \to 0 \) as \( n \to \infty \). It is clear that \( \mathcal{M}^i_n(x) = \mathcal{M}^i_{n-1}(x) \) on the short intervals of \( \mathbb{P}_{n-1} \). Therefore

\[
\|\Theta'_n\|_p^p = \int_{\mathcal{S}^1} \left| \mathcal{M}^i_n(x) - \mathcal{M}^i_{n-1}(x) \right|^p \, dx =
\]

\[
\sum_{i=0}^{q_{n-1}(t)-1} \int_{I_{n-2,i}} \left| \mathcal{M}^i_n(x) - \frac{1}{|I_{n-2,i}|} \int_{I_{n-2,i}} f''(y) \, dy \right|^p \, dx.
\]

Utilizing Hölder’s inequality and Fubini’s theorem we obtain

\[
\int_{I_{n-2,i}} \left| \mathcal{M}^i_n(x) - \frac{1}{|I_{n-2,i}|} \int_{I_{n-2,i}} f''(y) \, dy \right|^p \, dx \leq
\]

\[
\frac{1}{|I_{n-2,i}|} \int_{I_{n-2,i}} \left( \int_{I_{n-2,i}} \left| \mathcal{M}^i_n(x) - \frac{f''(y)}{f'(y)} \right|^p \, dx \right) dy.
\]

From (2) and Hölder’s inequality it follows

\[
\frac{1}{|I_{n-2,i}|} \int_{I_{n-2,i}} \left( \sum_{i=0}^{q_{n-1}(t)-1} \left| \int_{I_{n-2,i}} f''(s) - \frac{f''(y)}{f'(y)} \, ds \right|^p \right) dy \leq
\]

\[
\frac{1}{|I_{n-2,i}|} \int_{I_{n-2,i}} \left( \sum_{i=0}^{q_{n-1}(t)-1} \int_{I_{n-2,i}} \left| f''(s) - \frac{f''(y)}{f'(y)} \right| \, ds \right) dy =
\]

\[
\frac{1}{|I_{n-2,i}|} \int_{I_{n-2,i}} \left( \int_{I_{n-2,i}} \left| f''(s) - f''(y) \right| \, ds \right) dy.
\]

It is well known that continuous functions with compact support are dense in \( L_p \). Therefore for any \( \epsilon > 0 \) there exists a continuous function \( W_\epsilon \) which has compact support and an \( L_p \) integrable function \( V_\epsilon \) such that

\[
\frac{f''}{f'} = W_\epsilon + V_\epsilon \quad \text{and} \quad \|V_\epsilon\|_p \leq \epsilon.
\]

Taking sufficiently small \( \epsilon > 0 \) and using the above expansion of \( f''/f' \) we get

\[
\frac{1}{|I_{n-2,i}|} \int_{I_{n-2,i}} \left( \int_{I_{n-2,i}} \left| \frac{f''(s)}{f'(s)} - \frac{f''(y)}{f'(y)} \right|^p \, ds \right) dy \leq
\]

\[
\frac{2^{p-1}}{|I_{n-2,i}|} \int_{I_{n-2,i}} \left( \int_{I_{n-2,i}} \left| W_\epsilon(s) - W_\epsilon(y) \right|^p \, ds \right) dy + 2 \cdot 4^{p-1} \int_{I_{n-2,i}} |W_\epsilon(y)|^p \, dy.
\]

It is easy to see that

\[
\frac{2^{p-1}}{|I_{n-2,i}|} \int_{I_{n-2,i}} \left( \int_{I_{n-2,i}} \left| W_\epsilon(s) - W_\epsilon(y) \right|^p \, ds \right) dy \leq 2^{p-1}|I_{n-2,i}| \cdot \omega^p(d_{n-2}, W_\epsilon)
\]

(36)

where \( \omega(\cdot, W_\epsilon) \) is the modulus of continuity of \( W_\epsilon \). Finally, summing relations (32)-(36) we obtain

\[
\|\Theta'_n\|_p^p \leq 4^p (\omega^p(d_{n-2}, W_\epsilon) + \epsilon^p).
\]

This implies

\[
\hat{\Theta}_n^p \leq 4^p (\omega^p(d_{n-2}, W_\epsilon) + \epsilon^p).
\]
Taking the limit as $n \to \infty$ we get
\[
\lim_{n \to \infty} \sup \tilde{\Theta}_n \leq 4\epsilon.
\]
Since $\epsilon > 0$ is arbitrarily small, we have
\[
\lim_{n \to \infty} \tilde{\Theta}_n = 0.
\]
Next, we estimate $\tilde{\varepsilon}_n$. According to Theorem 2.2 there exists a constant $C_1 > 0$ such that
\[
\tilde{\varepsilon}_n \leq C_1 \left( \lambda^{n} + \sum_{k=n+1}^{\infty} \lambda^{-(n+1)} \tilde{\Theta}_k \right).
\]
Let $\hat{\Theta}_n = \sup \{ \Theta_m : m \geq n \}$. It is easy to see that $\Theta_n \geq \hat{\Theta}_{n+1}$ for all $n \geq 1$ and $\lim_{n \to \infty} \hat{\Theta}_n = 0$. By monotonicity of $\hat{\Theta}_n$ and the above inequality, we have
\[
\tilde{\varepsilon}_n \leq C_1 \left( \lambda^{n} + \sum_{k=n+1}^{\infty} \lambda^{-(n+1)} \hat{\Theta}_k \right) \leq \frac{C_1}{1-\lambda} \left( \lambda^{n} + \hat{\Theta}_{n+1} \right).
\]
(37)
Hence $\lim_{n \to \infty} \tilde{\varepsilon}_n = 0$. Next, we estimate $\tilde{\eta}_n$. Due to Theorem 2.2 the monotonicity of $\tilde{\Theta}_n$ and inequality (37) we get
\[
\tilde{\eta}_n \leq \frac{C_1^3}{\lambda^2(1-\lambda)} \sum_{k=1}^{n} \lambda^{2(n-k)}(\lambda^{k-1} + \tilde{\Theta}_k) \leq \frac{C_1^3}{\lambda^2(1-\lambda)} \left( n \lambda^{n-1} + \sum_{k=1}^{[n/2]} \lambda^{2(n-k)} \tilde{\Theta}_k + \sum_{k=[n/2]+1}^{n} \lambda^{2(n-k)} \tilde{\Theta}_k \right)
\]
\[
\leq \frac{C_1^3}{\lambda^2(1-\lambda)^2} \left( n \lambda^{n-1} + \lambda^n \tilde{\Theta}_1 + \tilde{\Theta}_{[n/2]} \right)
\]
where $[\cdot]$ is the integer part of a given number. Therefore $\lim_{n \to \infty} \tilde{\eta}_n = 0$. Now we are going to estimate $\tilde{\tau}_n$. From Theorem 2.2 it follows that
\[
\tilde{\tau}_n \leq C_1 \left( \sum_{k=1}^{n} \lambda^{n-k} \tilde{\varepsilon}_k + \sum_{k=1}^{n} \lambda^{n-k} \tilde{\eta}_k \right).
\]
Inequalities (37), (38) and $\hat{\Theta}_k \leq \hat{\Theta}_{[n/2]}$ imply
\[
\sum_{k=1}^{n} \lambda^{n-k} \tilde{\varepsilon}_k + \sum_{k=1}^{n} \lambda^{n-k} \tilde{\eta}_k \leq \frac{C_1^3}{\lambda^2(1-\lambda)^2} \left( n^2 \lambda^{n-1} (2 + \hat{\Theta}_1) + \sum_{k=1}^{n} \lambda^{n-k} \hat{\Theta}_{[n/2]} \right).
\]
Similarly to the proof of inequality (38) it can be shown that
\[
\sum_{k=1}^{n} \lambda^{n-k} \hat{\Theta}_{[n/2]} \leq \frac{2}{1-\lambda} \left( \lambda^{[n/2]} \hat{\Theta}_0 + \hat{\Theta}_{[n/2]} \right).
\]
Hence, the last three inequalities imply
\[
\tilde{\tau}_n \leq \frac{2C_1^4}{\lambda^2(1-\lambda)^3} \left( n^2 \lambda^{n-1} (2 + \hat{\Theta}_1) + \lambda^{[n/2]} \hat{\Theta}_0 + \hat{\Theta}_{[n/2]} \right).
\]
Thus $\lim_{n \to \infty} \tilde{\tau}_n = 0$, which concludes the proof of Theorem 4.1.
5. Proof of Theorem 1.2. To prove our first main theorem we use a theory which was developed by Katznelson and Ornstein in \[10\]. The following sufficient condition for absolute continuity of the conjugacy was proved there.

**Theorem 5.1.** Let the diffeomorphism \( f \) satisfies the Denjoy’s conditions that is, \( \log f' \) has bounded variation and the rotation number \( \rho \) is irrational. Assume

\[
\sum_{n=1}^{\infty} (a_n K_n)^2 < \infty.
\]

Then the conjugating map \( h \) between \( f \) and \( R_\rho \) and its inverse \( h^{-1} \) are absolutely continuous and \( h', (h^{-1})' \in L_2 \).

Proof of Theorem 1.2. Let the diffeomorphism \( f \) satisfy the KO conditions and \( (v_n) \) be an unbounded sequence of natural numbers. According to Theorem 4.1 we can find a subsequence \((\tilde{t}_n)n\) of \((t_n)n\) such that \( n v_n \tilde{t}_n \leq 1 \) for all \( n = 1, 2, \ldots \). Without loss of generality we may assume that the sequence \((i_n)n\) is a strictly increasing sequence. Let \( \mathcal{I}(i_n, v_n) \) be the set of irrational numbers which was defined in Section 1. It is clear that for any \( \tilde{\rho} \in \mathcal{I}(i_n, v_n) \) there exists a natural number \( M \), such that \( \tilde{\rho} \in \mathcal{I}(i_n, v_n, M) \). Now we consider the family of diffeomorphisms \( f_t = f + t, t \in \mathcal{I} \). Note that \( \rho(f_t) \) is a continuous and nondecreasing function of \( t \). Moreover, it is strictly increasing at irrational values (see \[8\]). Due to this note for any \( \tilde{\rho} \in \mathcal{I}(i_n, v_n, M) \) there exits a unique \( t_0 \in \mathcal{I} \) such that \( \rho(f_{t_0}) = \tilde{\rho} \). By Theorem 3.2 and Lemma 3.1 we get

\[
K_n(t_0) \leq C_3 \cdot \tau_n(t_0) \quad \text{and} \quad \sum_{n=1}^{\infty} K_n^2(t_0) < \infty.
\]

On the other hand we have

\[
\tau_n(t_0) \leq \tilde{\tau}_n.
\]

Inequalities (39), (40) and \( n v_n \tilde{\tau}_n \leq 1 \) imply

\[
\sum_{s=1}^{\infty} (a_s K_s(t_0))^2 = \sum_{s=1}^{\infty} (a_s K_s(t_0))^2 + \sum_{n=1}^{\infty} (a_{i_n} K_{i_n}(t_0))^2 \\
\leq M^2 \sum_{s=1}^{\infty} K_s^2(t_0) + C_3^2 \sum_{n=1}^{\infty} (v_n \tilde{\tau}_n)^2 < \infty.
\]

Thus, the claim of the theorem follows from Theorem 5.1.

6. **Theorem of Weiss-Zygmund.** In this section we provide brief facts about continuous functions \( \mathcal{K} : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) satisfying inequality (1) for different values of parameter \( \gamma > 0 \). These facts will be used in the proofs of Theorems 1.3 and 1.4. First we consider the case \( \gamma \in \left( \frac{1}{2}, 1 \right) \). The following theorem was proved by Weiss and Zygmund in \[19\].

**Theorem 6.1.** Let \( \mathcal{K} : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) be 1-periodic and continuous on \( \mathbb{R}^1 \). Assume that for some \( \gamma \in \left( \frac{1}{2}, 1 \right) \) the function \( \mathcal{K} \) satisfies the inequality

\[
\|\Delta^2 \mathcal{K}(\cdot, \nu)\|_{L^\infty([0,1])} \leq C \Phi_\gamma(\nu).
\]

Then \( \mathcal{K} \) is absolutely continuous and \( \mathcal{K}' \in L_p([0,1]) \) for every \( p > 1 \).
Theorem 6.2. Let $K : \mathbb{R}^1 \to \mathbb{R}^1$ be 1-periodic and continuous on $\mathbb{R}^1$. Assume that the function $K$ satisfies (41) for some $\gamma > 1$. Then $K \in C^1(\mathbb{R}^1)$.

Although this result is probably not new, we were not able to find a proper reference for it. Therefore, we provide a complete proof here.

Proof. According to Weiss-Zygmund theorem $K$ is absolutely continuous, hence $K'$ exists almost everywhere and $K$ is the antiderivative of $K'$. To prove the theorem we take any two points $\xi$ and $\eta$ that are Lebesgue points for $K'$ such that $\xi - \eta \in (0, 1)$ and obtain a uniform estimate for $|K'(\xi) - K'(\eta)|$. Hence we show that $K'$ is uniformly continuous on its set of Lebesgue points. Thus, it can be continuously extended to the whole of $\mathbb{R}^1$. Let us consider the function $D_\tau K(x) = K(x + \tau) - K(x)$ where $x \in \mathbb{R}^1$ and $\tau \in (0, 1)$. By inequality (41) we have

$$D_\tau K(x) = D_\tau K(x - \tau) + O(\tau Q_\gamma(\tau)) \quad \text{for all } x \in \mathbb{R}^1 \text{ and } \tau \in (0, 1),$$

where $Q_\gamma(\tau) = \Phi_\gamma(\tau)/\tau$. We set $\tau := \xi - \eta$. Replacing $x$ by $x_n = \eta + \tau 2^{-n}$ and $\tau$ by $\tau 2^{-n}$, $n = 1, 2, \ldots$ in (42), we obtain

$$D_{\tau 2^{-n}} K(x_n) = D_{\tau 2^{-n}} K(\eta) + O(\tau 2^{-n} Q_\gamma(\tau 2^{-n})).$$

It is easy to see

$$D_{\tau 2^{-n}} K(x_n) - D_{\tau 2^{-n}} K(\eta) = D_{\tau 2^{-n+1}} K(\eta) - 2D_{\tau 2^{-n}} K(\eta).$$

Thus

$$D_{\tau 2^{-n+1}} K(\eta) = 2D_{\tau 2^{-n}} K(\eta) + O(\tau 2^{-n} Q_\gamma(\tau 2^{-n})).$$

By iterating from $n = 1$ to $N$ we obtain

$$\frac{D_\tau K(\eta)}{\tau} = \frac{2^N}{\tau} D_{\tau 2^{-N}} K(\eta) + O\left(\sum_{n=1}^{N} Q_\gamma(\tau 2^{-n})\right).$$

(44)

Since the point $\eta$ is the Lebesgue point for $K'$ and $\gamma > 1$

$$\lim_{N \to \infty} \frac{2^N}{\tau} D_{\tau 2^{-N}} K(\eta) = K'(\eta).$$

Taking the limit as $N \to \infty$ in (44) we get

$$\frac{D_\tau K(\eta)}{\tau} = K'(\eta) + O(P_\gamma(\tau)), \quad \text{(45)}$$

where

$$P_\gamma(\tau) = \sum_{n=1}^{\infty} Q_\gamma(\tau 2^{-n}).$$

Similarly, replacing $x$ by $x_n = \xi - \tau 2^{-n}$ and $\tau$ by $\tau 2^{-n}$, $n = 1, 2, \ldots$ in (42) we obtain

$$\frac{D_\tau K(\xi - \tau)}{\tau} = K'(\xi) + O(P_\gamma(\tau)). \quad \text{(46)}$$
Taking \( \tau = \xi - \eta \) we get
\[
|K'(\xi) - K'(\eta)| = O\left(\mathcal{P}_\gamma(|\xi - \eta|)\right).
\]
This proves uniform continuity of \( K' \) on the set of Lebesgue points, thus \( K' \) coincides almost everywhere with a continuous function. It is obvious that this continuous function is a derivative of \( K \).

It turns out that the functions satisfying relation (41) have “a considerable degree of continuity”.

**Theorem 6.3.** Let \( K : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) be 1-periodic and continuous on \( \mathbb{R}^1 \), and satisfies the inequality (41) for some \( \gamma \in (0, 1) \). If \( \gamma \in (0, 1) \) then
\[
\omega(\delta, K) = O\left(\delta(\log \frac{1}{\delta})^{1-\gamma}\right).
\]
If \( \gamma = 1 \) then
\[
\omega(\delta, K) = O\left(\delta(\log \log \frac{1}{\delta})\right),
\]
where \( \omega(\cdot, K) \) is the modulus of continuity of \( K \).

**Proof.** The proof follows closely to that of Theorem 6.2. Let us take any \( \eta \in \mathbb{R}^1 \) and fix it. Taking any \( \xi \in (\eta, \eta + 1) \) we set \( \tau := \xi - \eta \). In the same way as in the proof of Theorem 6.2 we obtain
\[
D_\tau K(\eta) = 2^N D_{2^{-N}} K(\eta) + O\left(\sum_{n=1}^{N} Q_\gamma(2^{-n})\right)
\]
for any \( N \in \mathbb{N} \) and \( \tau \in (0, 1) \). Suppose \( \delta > 0 \) is small. Choose a number \( n_0 \in \mathbb{N} \) such that \( \frac{1}{2} < 2^{n_0} \delta < 1 \). Taking \( N := n_0 \), \( \tau := 2^{n_0} \delta \) i.e., \( \xi = 2^{n_0} \delta + \eta \) one has
\[
D_{\delta} K(\eta) = \frac{D_{2^{n_0} \delta} K(\eta) \delta}{2^N \delta} + O\left(\sum_{n=1}^{N} Q_\gamma(2^{N-n} \delta)\right).
\]
It is clear that
\[
\left|\frac{D_{2^{n_0} \delta} K(\eta) \delta}{2^N \delta}\right| \leq 4 \max_{x \in [0,1]} |K(x)|
\]
and
\[
\sum_{n=1}^{N} Q_\gamma(2^{N-n} \delta) = 2 \sum_{n=1}^{N} \frac{1}{2^{N-n+1}} Q_\gamma(2^{N-n} \delta) \leq 2 \int_{2^{N}}^{1} \frac{Q_\gamma\left(\frac{\delta}{x}\right)}{x} dx.
\]
Hence
\[
|D_{\delta} K(\eta)| = O\left(\int_{2^{-N}}^{1} \frac{\delta Q_\gamma\left(\frac{\delta}{x}\right)}{x} dx\right).
\]
It is easy to see that the last integral is estimated by \( \delta(\log \frac{1}{\delta})^{1-\gamma} \) if \( \gamma \in (0, 1) \) and \( \delta(\log \log \frac{1}{\delta}) \) if \( \gamma = 1 \). That concludes the proof of Theorem 6.3. \( \square \)
7. Sharp estimate for Denjoy’s inequality. In this section we obtain a sharp estimate for $K_n$ when $f \in Z_{K_n}$ and $\gamma > \frac{1}{2}$. For this we use the same strategy as above. Define a function $\Omega : (0, 1) \times (0, +\infty) \to \mathbb{R}$,

$$
\Omega(\delta, \gamma) = \begin{cases} 
\delta (\log \frac{1}{\delta})^{1-\gamma} & \text{if } (\delta, \gamma) \in (0, 1) \times (0, 1); \\
\delta (\log \log \frac{1}{\delta}) & \text{if } (\delta, \gamma) \in (0, 1) \times \{1\}; \\
\delta & \text{if } (\delta, \gamma) \in (0, 1) \times (1, +\infty).
\end{cases}
$$

In fact, $\Omega(\delta, \gamma)$ is the modulus of continuity of functions satisfying inequality (41). Using this function we define the following sequences:

$$
\Lambda_n = \mathcal{Q}_\gamma(d_{n-1}) + \Omega(d_{n-1}, \gamma), \quad \gamma > \frac{1}{2}, \quad n \geq 0,
$$

$$
\Upsilon_n = \sum_{k=1}^{n} d_{n-1} d_k \Lambda_k, \quad n \geq 1,
$$

$$
\Psi_n = \sum_{k=1}^{n} d_k \left( \Upsilon_k + \Lambda_k \right), \quad n \geq 1,
$$

where $d_n = \|f^q_n - \text{Id}\|_0$. Now we provide a sharp estimate for Denjoy’s inequality.

**Theorem 7.1.** Let $f \in Z_{\mathcal{K}_\gamma}$, $\gamma > \frac{1}{2}$ be a diffeomorphism of the circle with irrational rotation number. There exists a constant $C_6 = C_6(f) > 0$ such that

$$
K_n \leq C_6 \Psi_n.
$$

To prove this theorem we use the same strategy as in Theorem 3.2, that is using Zygmund condition we strengthen the estimates of Lemmas 3.3 and 3.4 so that they imply the proof of Theorem 7.1.

**Lemma 7.2.** Let $f$ satisfy the conditions of Theorem 7.1 Then for any $\xi_0 \in S^1$ there exists a constant $C_7 = C_7(f) > 0$ such that

$$
|\log \mathcal{R}(\xi_0, I_0^{n-1}; f^{q_n})| = \frac{1}{2} \sum_{s=0}^{n-1} \int_{I_s^{n-1}} \frac{f''(x)}{f'(x)} dx \leq C_7 \cdot \Lambda_n, \quad (51)
$$

$$
|\log \mathcal{R}(\xi_0, I_0^n; f^{q_n})| = \frac{1}{2} \sum_{s=0}^{n-1} \int_{I_s^n} \frac{f''(x)}{f'(x)} dx \leq C_7 \cdot \Lambda_{n+1}. \quad (52)
$$

**Proof.** We prove only the first inequality. The second one can be proved similarly. Since $\mathcal{R}(\xi_0, I_0^{n-1}; f^{q_n})$ is multiplicative with respect to composition, we have

$$
\log \mathcal{R}(\xi_0, I_0^{n-1}; f^{q_n}) = \sum_{s=0}^{q_n-1} \log \left[ \frac{f(\eta_s) - f(\xi_s)}{\eta_s - \xi_s} \frac{1}{f'(\xi_s)} \right], \quad (53)
$$

where $\eta_s = f^s(\xi_{q_n-s-1})$ and $\xi_s = f^s(\xi_0)$ are the end-points of interval $I_s^{n-1}$. Setting $\nu_s = (\eta_s + \xi_s)/2$, $\vartheta_s = (\eta_s - \xi_s)/2$ for every $0 \leq s < q_n$ and using inequality (1), one has

$$
\frac{f(\eta_s) - f(\xi_s)}{\eta_s - \xi_s} \frac{1}{f'(\xi_s)} = \frac{1}{2\vartheta_s f'(\xi_s)} \int_0^{\vartheta_s} \left( f'(\nu_s + x) + f'(\nu_s - x) \right) dx = \quad (54)
$$
there exists a constant \( \gamma > \) differentiable in the case continuous and

**Proof of Theorems 1.3 and 1.4.**

**Lemma 7.3.** Let \( \Omega = 1 \) and it is absolute

According to Weiss-Zygmund theorem and Theorem 6.2 the function \( f' \) is absolute continuous and \( f'' \in L_p(S^1) \) for every \( p > 1 \) in the case \( \gamma \in (\frac{3}{2}, 1) \) and it is differentiable in the case \( \gamma > 1 \). Therefore

\[
\frac{f'(\eta_s) - f'(\xi_s)}{2 f'(\xi_s)} = \int_{\xi_s}^{\eta_s} \frac{f''(x) dx}{2 f'(x)} + \int_{\xi_s}^{\eta_s} \frac{f''(x)}{2 f'(x)} \left( \int_{\xi_s}^{x} \frac{f''(z)}{f'(z)} dz \right) dx.
\]  

By Theorems 6.2 and 6.3 we have

\[
\sum_{s=0}^{q_n-1} \left[ \left( f'(\eta_s) - f'(\xi_s) \right)^2 + \int_{\xi_s}^{\eta_s} \frac{f''(x)}{2 f'(x)} \left( \int_{\xi_s}^{x} \frac{f''(z)}{f'(z)} dz \right) dx \right] = O\left( \Omega(d_{n-1}, \gamma) \right).
\]  

It is clear \( \Phi_\gamma(|I_s^{-1}|) \leq |I_s^{-1}| \cdot Q_\gamma(d_{n-1}) \) for all \( 0 \leq s < q_n \).

Hence

\[
\sum_{s=0}^{q_n-1} \Phi_\gamma(|I_s^{-1}|) \leq Q_\gamma(d_{n-1}).
\]  

Summing \( 53 \) to \( 58 \) we obtain

\[
\log R(\xi_0, I_0^{-1}; f^{n+1}) = \sum_{s=0}^{q_n-1} (-1)^{n-1} \int_{I_s^{n+1}} \frac{f''(x)}{2 f'(x)} \, dx + O\left( \Omega(d_{n-1}, \gamma) + Q_\gamma(d_{n-1}) \right).
\]  

This proves the inequality \( 51 \).

**Lemma 7.3.** Let \( f \) satisfy the conditions of Theorem 7.1 Then for any \( \xi_0 \in S^1 \) there exists a constant \( C_S = C_S(f) > 0 \) such that

\[
\left| \log \frac{|I_{q_0-1}^{-1}|}{|I_0^{-1}||I_0^{-1}| - |I_{q_0-1}^{-1}||I_{q_0-1}^{-1}||I_{q_0-1}^{-1}||I_{q_0-1}^{-1}|} \right| \leq \frac{(\gamma)^n}{2} \sum_{s=0}^{q_n-1} \int_{I_s^{n+1}} f''(x) \, dx \leq C_S \cdot \Lambda_n,
\]

\[
\left| \log \frac{|I_{q_0+1}^{-1}|}{|I_0^{-1}||I_0^{-1}| - |I_{q_0+1}^{-1}||I_{q_0+1}^{-1}||I_{q_0+1}^{-1}||I_{q_0+1}^{-1}|} \right| \leq \frac{(\gamma)^n}{2} \sum_{s=0}^{q_n-1} \int_{I_s^{n+1}} f''(x) \, dx \leq C_S \cdot \Lambda_n.
\]

**Proof.** The proof of this lemma is similar to the proof of Lemma 3.4.

8. **Proof of Theorems 1.3 and 1.4.** To prove Theorem 1.3 we use the sufficient condition provided in Theorem 5.1 and to prove Theorem 1.4 we use the following sufficient condition for \( C^1 \)-smoothness of the conjugacy which was developed by Khanin and Sinai in [12].
**Theorem 8.1.** Let the diffeomorphism $f$ satisfies the Denjoy’s conditions. Assume that
\[ \sum_{n=1}^{\infty} a_n K_n < \infty. \]  
(60) Then the conjugating map $h$ between $f$ and $R_\rho$ and its inverse $h^{-1}$ are $C^1$ diffeomorphisms.

We use the following elementary lemma in the proof of Theorems 1.3 and 1.4

**Lemma 8.2.** The following estimate holds
\[ \Psi_n = O \left( \frac{1}{n^{\gamma}} \right). \]

Proof. The proof of this lemma follows from Theorem 2.2 and the definitions of $Q_\gamma(\cdot)$ and $\Omega(\cdot, \gamma)$.

**Proof of Theorems 1.3 and 1.4.** Let $f \in Z_{\Phi_\gamma}$ be a circle diffeomorphism with irrational rotation number $\rho$ and $\gamma \in (1/2, 1]$. Suppose that for some $\alpha \in (0, \gamma - 1/2)$ the partial quotients of $\rho$ satisfies $a_n \leq C n^\alpha$. According to Theorem 7.1 and Lemma 8.2 we have
\[ \sum_{n=1}^{\infty} \left( a_n K_n \right)^2 \leq C \sum_{n=1}^{\infty} \frac{1}{n^{2(\gamma - \alpha)}}. \]
Since $\gamma - \alpha > 1/2$ the last sum converges. Thus the claim of Theorem 1.3 implies from Theorem 5.1. Similarly, if $\gamma > 1$ and for some $\alpha \in (0, \gamma - 1)$ the partial quotients of $\rho$ satisfies $a_n \leq C n^\alpha$ then
\[ \sum_{n=1}^{\infty} a_n K_n \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\gamma - \alpha}}. \]
Since $\gamma - \alpha > 1$ the last sum converges. By Theorem 8.1 $h$ and $h^{-1}$ are $C^1$ diffeomorphisms. Theorems 1.3 and 1.4 are proved.

**9. Extensions of main theorems.** One of the main results of this paper is an extension the main result of Katznelson and Ornstein [10] to the larger class of rotation numbers within the specific family $f_t = f + t$, $t \in [0, 1]$. In fact, this result remains valid for any family of circle diffeomorphisms $f : S^1 \times T \to S^1$, where $T$ is a parameter space, which we assume to be a compact. Denote by $f_t(x) = f(x, t)$. Thus, the following holds.

**Theorem 9.1.** Let $f : S^1 \times T \to S^1$ be a family of circle diffeomorphisms. Assume
\[(i) \quad \frac{f''(x)}{f'(x)} \in L_p, \quad p > 1 \quad \text{for any } t \in T;\]
\[ (ii) \quad t \to \frac{f''(x)}{f'(x)} \quad \text{is a continuous mapping from } T \text{ into } L_p, \quad p > 1. \]
Then there exists a subset $\mathcal{U}$ of irrational numbers of unbounded type such that if $\rho(f_t) \in \mathcal{U}$ for some $t \in T$ then the conjugation map $h$ between $f_t$ and $R_{\rho(f_t)}$ and its inverse $h^{-1}$ are absolute continuous and $h', (h^{-1})' \in L_2$.

Since $S^1 \times T$ is compact, the condition (ii) implies $\bar{v} = \sup_{t \in T} Var_{S^1} \log f_t' < \infty$. Hence $\lambda = \sup_{t \in T} (1 + e^{-v(t)})^{-1/2} < 1$. These uniform estimates ensure uniformity of constants in Theorem 2.2. Further, again using compactness of $T$ and the condition (ii) one can show that $\frac{f''(x)}{f'(x)}$ can be approximated by continuous functions such that the modulus of continuities of those functions tend to zero uniformly in $t$. This
implies uniform estimates for $\varepsilon_n(t)$, $\eta_n(t)$ and $\tau_n(t)$. The rest of the proof of this theorem follows exactly the proof of Theorem 2.2.

Theorem 1.3 extends Katznelson and Ornstein’s theorem [10] to a larger class of rotation numbers for the circle diffeomorphisms satisfying inequality (1). The function $\Phi_2$ in the inequality (1) has been chosen in specific form in order to describe the set of rotation numbers in precise form. Theorem 1.3 can be extended in the following way. Let $\Phi : [0, \varsigma] \to [0, +\infty)$ be a non-decreasing function satisfying

$$\int_0^\varsigma \frac{\Phi^2(s)}{s} ds < \infty. \quad (61)$$

Consider a set of circle diffeomorphisms $f$, such that those derivatives $f'$ satisfy

$$||\Delta^2(f'(\cdot, \tau))||_{L^{\infty}(S^1)} \leq C\tau \Phi(\tau), \quad (62)$$

for some constant $C > 0$. It is obvious that this class is wider than the class considered in Theorem 1.3 and therefore the following theorem extends it.

**Theorem 9.2.** Let $f$ be a circle diffeomorphism such that its derivative $f'$ satisfies the inequality (62) with some $\Phi$ satisfying (61). Then there exists a subset $U$ of irrational numbers of unbounded type such that if $\rho(f) \in U$ then the conjugating map $h$ between $f$ and $R_{\rho(f)}$ and its inverse $h^{-1}$ are absolute continuous and $h'$, $(h^{-1})'$ $\in L_2$.

Note that Weiss - Zygmund’s theorem holds for functions satisfying inequality (62) (see [15]). Using this fact and inequalities (61) and (62) one can show that there exists a sequence $(\beta_n) \in \ell_2$ such that

$$K_n \leq C\beta_n. \quad (63)$$

Hence the claim of this theorem follows from Theorems 1.2 and 5.1. If we assume that the function $\Phi$ in (62) satisfies

$$\int_0^\varsigma \frac{\Phi(s)}{s} ds < \infty, \quad (64)$$

then it can be shown that the sequence $(\beta_n)$ in (63) belongs to $\ell_1$. Therefore using Theorems 1.2 and 5.1 one can prove the following theorem.

**Theorem 9.3.** Let $f$ be a circle diffeomorphism such that its derivative $f'$ satisfies the inequality (62) with some $\Phi$ satisfying (64). Then there exists a subset $U$ of irrational numbers of unbounded type such that if $\rho(f) \in U$ then the conjugating map $h$ between $f$ and $R_{\rho(f)}$ and its inverse $h^{-1}$ are $C^1$ diffeomorphisms.

This theorem extends Theorem 1.4.

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