SUBLINEAR EXTENSIONS OF POLYGONS

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Abstract. Every convex n-gon is a projection of a polytope with o(n) facets.

1. Introduction

A polytope \( P \subset \mathbb{R}^n \) is a convex hull of a finite subset of \( \mathbb{R}^n \). A linear extended formulation (or extension) of \( P \) is a polytope \( Q \subset \mathbb{R}^d \) such that \( P \) can be obtained from \( Q \) as an image under a linear projection from \( \mathbb{R}^d \) to \( \mathbb{R}^n \). The size of extended formulation is the number of facets of \( Q \). The extension complexity of a polytope \( P \) is the smallest size of any extended formulation of \( P \), that is, the minimal possible number of inequalities in the representation of \( Q \). Extended formulations of small size are interesting in combinatorial optimization because they allow to reduce the complexity of linear programming problems useful for numerous applications [4, 6].

Extended formulations proved to be a powerful tool in combinatorial optimization after the 1991 paper by Yannakakis [15], and they attract a significant amount of attention in present time. However, there are many basic questions on extended formulations that remain unsolved. For instance, consider a convex polygon, the simplest polytope from combinatorial point of view. How small and how large the extension complexity of a convex \( n \)-gon can be? The first part of this question has been answered in [2], it has been proved that the extension complexity of regular \( n \)-gon is \( \Theta(\log n) \). The question of worst-case complexity is still unsolved; the following version of this question has been mentioned in [3] as one of important problems of extended formulations theory.

Problem 1.1. Does a convex polygon need a linear number of inequalities in any linear representation in the worst case?

A thorough investigation of extensions of polygons has been undertaken in [5], where the worst-case extension complexity of \( n \)-gon has been bounded from below by \( \Omega(\sqrt{n}) \). Actually, it has been proven that every generic \( n \)-gon (that is, a polygon whose vertex coordinates are algebraically independent over \( \mathbb{Q} \)) has extension complexity at least \( \sqrt{2n} \). The best upper bound for the worst-case extension complexity known so far was \((6n + 6)/7\), as shown in [13]. These papers left Problem 1.1 open, as well as [8 11], which present geometric techniques of attacking the problem, and [12], which shows that hidden vertices are necessary to construct optimal extensions of convex polygons. Actually, Problem 1.1 has been open even for a more general class of semidefinite formulations, which have more expressive power than linear formulations [7].

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A linear algebraic characterization of extended formulations has been given by Yannakakis in [15]. Recall that the nonnegative rank of a nonnegative matrix $M$ (that is, a matrix whose entries are nonnegative) is the smallest $k$ such that $M$ is a sum of $k$ nonnegative rank one matrices. Assume a polytope $P \subset \mathbb{R}^n$ is defined by conditions $c_f(x) \geq \beta_f$, where $f$ runs over all facets of $P$, and every $c_f$ is a linear functional on $\mathbb{R}^n$. The entries of a slack matrix of $P$ are defined by $S^i_k = c_f(v) - \beta_f$, where $v$ runs over all vertices of $P$. It is not hard to see that the rank of $S$ is the dimension of $P$; the Yannakakis’ result states that the nonnegative rank of $S$ equals the extension complexity of $P$, see also Corollary 5 of [8]. These results provide a reformulation of Problem 1.1 which has been studied in [1, 9, 10, 14] in linear algebraic terms but remained open.

In this paper we give a negative answer for Problem 1.1 by showing that every convex $n$-gon admits an extended formulation with $o(n)$ facets. As a corollary, we prove that the nonnegative rank of a nonnegative rank three $m \times n$ matrix is $o(n)$. Also, we provide an open set of convex $n$-gons which have extension complexity $O(\sqrt{n})$, which shows that an $\Omega(\sqrt{n})$ lower bound for extension complexities of all generic $n$-gons, obtained in [5], cannot be improved.

2. Preliminaries

The following notation is used throughout our paper. We denote by $S^i_j$ the number located in the intersection of $i$th row and $j$th column of matrix $S$. A submatrix of $S$ formed by rows with indexes from a set $R$ and columns with indexes from $C$ will be denoted by $S[R,C]$. We refer to either a row or a column of a matrix as a line of it. We sometimes write $\exp_{a,b}$ instead of $a^b$ to avoid complicated expressions in upper indexes. By $\conv A$ we denote the convex hull of a set $A$.

Let $h$ be a positive real and $r = (r_1, \ldots, r_n)$ a sequence of real numbers. We say that $r$ is $h$-increasing (or $h$-decreasing) if $hr_i \leq r_{i+1}$ (or $hr_i \geq hr_{i+1}$, respectively) holds for any $i$. A sequence is $h$-monotone if it is either $h$-increasing or $h$-decreasing.

For any three points $u_1, u_2, u_3 \in \mathbb{R}^2$, we denote by $\Delta(u_1, u_2, u_3)$ the oriented area of the parallelepiped spanned by vectors $u_2u_1$ and $u_2u_3$. Assume $v_1, \ldots, v_{n+1} \in \mathbb{R}^2$ and $v_{n+1} = v_1$; we say that $(v_1, \ldots, v_n)$ is a proper sequence if $\Delta(v_1, v_2, v_3) > 0$ and, for every $j$, the segment $\conv \{v_j, v_{j+1}\}$ is an edge of $\conv \{v_1, \ldots, v_n\}$. We define the slack matrix $S$ of $(v_1, \ldots, v_n)$ by $S^i_j = \Delta(v_i, v_j, v_{j+1})$. The result of Yannakakis states that the nonnegative rank of $S$ equals the extension complexity of $\conv \{v_1, \ldots, v_n\}$. The following two propositions will be helpful in what follows.

**Lemma 2.1.** If $P_1$ and $P_2$ are convex polytopes, then the extension complexity of $\conv \{P_1 \cup P_2\}$ does not exceed the sum of those of $P_1$ and $P_2$.

**Proof.** This is a special case of Proposition 2.8 of [6].

**Lemma 2.2.** Assume $w \in \mathbb{R}^2$ and $v = (v_1, \ldots, v_n)$ is a proper sequence. If there are distinct indexes $p_1, p_2$ satisfying $\Delta(w, v_{p_1-1}, v_{p_1}) \Delta(w, v_{p_1}, v_{p_1+1}) < 0$, then, for any $q \notin \{p_1, p_2\}$, it holds that $\Delta(w, v_{q-1}, v_q) \Delta(w, v_q, v_{q+1}) > 0$.

**Proof.** The two supporting lines containing $w$ pass through $v_{p_1}$ and $v_{p_2}$, respectively. It suffices to note that $\conv \{v_q, v_{q+1}\}$ and $w$ lie on the same side of $\conv \{v_{p_1}, v_{p_2}\}$ if, and only if, $w$ and $\conv v$ are separated by $\conv \{v_q, v_{q+1}\}$.
The following concept is crucial for our further considerations. We say that a proper sequence \((v_1, \ldots, v_n)\) is admissible if, for any indexes \(p, q, r, t\) satisfying either (i) \(\min\{p, q-1\} > r > t\) or (ii) \(q > p > \max\{r, t\}\), it holds that \(S^p_{\tau}\Delta^q_{\nu} > S^p_{\tau}S^q_{\nu}\).

The rest of our paper has the following structure. We construct an extension of size \(17\sqrt{n}\) for any admissible \(n\)-gon in Section 3, and we give a technical characterization of admissible polygons in Section 4. The existence of admissible polygons will be shown in Section 5. We will prove that, for any \(n\) fixed in advance, there is a sufficiently large \(m\) such that the vertex set of every convex \(m\)-gon has a subset \(A\) of cardinality \(n\) such that \(\text{conv} A\) is admissible. This will allow us to prove the main results in Section 6.

### 3. Admissible polygons and their extensions

In this section we prove that, if the sequence of vertices of a convex \(n\)-gon is admissible, then the extension complexity of it is at most \(17\sqrt{n}\). The following lemma gives a bound on the nonnegative ranks of certain submatrices of the slack matrix.

**Lemma 3.1.** Let \((v_1, \ldots, v_n)\) be a proper sequence; denote \(U = \{p, p+1, \ldots, q\}\). Assume \(a_i^k\) are non-positive numbers and define the entries of a matrix \(S\) by \(S^i_j = S^i + \sum_{t \in U} a_i^k S_t^j\), for any \(i \in \{1, \ldots, n\} \setminus U\) and \(j \in U \setminus \{q\}\). If the entries of \(S\) are nonnegative, then its nonnegative rank is at most eight.

**Proof.** 1. For \(\ast \in \{<, >\}\), we denote by \(W_{\ast}\) the set of all \(w\) satisfying \(\Delta(w, v_p, v_q) + 0, \Delta(w, v_{q-1}, v_q) + 0, \Delta(w, v_p, v_{p+1}) + 0\), and by \(W_{\ast}\) the closure of \(W_{\ast}\). Lemma 3.2 shows that \(\Delta(w', \nu, \nu_0) = 0\) holds for any \(j \in U \setminus \{q\}\) and \(w' \in W_{\ast}\). Also, since \(W_{\ast}\) is an intersection of three half-planes, any finite subset of \(W_{\ast}\) lies in the convex hull of at most four points from \(W_{\ast}\).

2. Since matrices of nonnegative rank at least nine form an open subset within \(n \times n\) nonnegative matrices, we can assume that \(a_i^k \geq 0\) for any \(i \in \{1, \ldots, n\} \setminus U\). Denote the affine combination \(w_i / a_i + \sum_{t \in U} a_i^k v_t / a_i\) by \(w_i\); then we have \(a_i \Delta(v_i, v_a, v_b) = \Delta(v_i, v_a, v_b) + \sum_{t \in U} a_i^k \Delta(v_t, v_a, v_b)\). The lemma’s assumption shows that \(w_i \in W_{\ast}\) whenever \(a_i \neq 0\); in particular, we get \(S^i_j = a_i \Delta(w_i, v_j, v_{j+1})\). From Step 1 it follows that there are \(w_1^\ast, \ldots, w_q^\ast \in W_{\ast}\) such that, for every \(i\) satisfying \(a_i \neq 0\), \(w_i\) is a convex combination \(\sum_{s=1}^n \lambda_i^s w_i^s\), which implies \(S^i_j = \sum_{s=1}^n a_i \lambda_i^s \Delta(w_i^s, v_j, v_{j+1}) = \sum_{s=1}^n |a_i| \lambda_i^s \cdot \Delta(w_i^s, v_j, v_{j+1})\).

Now we prove the main result of the section in a special case when \(n\) is a square.

**Lemma 3.2.** Assume \(m > 3\) and \((v_0, \ldots, v_{m^2-1})\) is an admissible sequence. Then, the extension complexity of \(\text{conv}\{v_0, \ldots, v_{m^2-1}\}\) does not exceed \(15m - 3\).

**Proof.** Denote \(P_{\sigma} := \{\sigma m + 2, \ldots, \sigma m + m - 2\}, P := \bigcup_{\sigma=0}^{m-1} P_{\sigma}\). For any \(r \in P_0\), we define the vectors \(A^r, B^r \in \mathbb{R}^{m^2}\) by \(A^r = 0\) whenever \(i - r \notin m\mathbb{Z}\),

\[
A^r_{\sigma m + r} = \left(\prod_{t=0}^{\sigma - 1} S^t_{\tau m + r}\right)^{-1} \left(\prod_{r=\sigma + 1}^{m - 1} S^m_{\tau m + r}\right)^{-1},
\]

\[
B^r_{\sigma m + r} = \left(\prod_{t=0}^{[j/m] - 1} S^t_{\tau m + r}\right) \left(\prod_{r=[j/m] + 1}^{m - 1} S^m_{\tau m + r}\right) \cdot S^j_{r + [j/m] m}.
\]
Now define $S^j_i := S^j_i - \sum_{r=2}^{m-2} A^j_i B^j_r$, for any $i, j \in \{0, \ldots, m^2 - 1\}$. It is a routine to check that the entries of $S$ are nonnegative, and submatrices $S[P,P]$ are zero. Since the matrix $S[P,P]$ can be obtained from $S$ by subtracting $m - 3$ nonnegative rank one matrices and removing $6m$ lines, the nonnegative rank of $S$ can exceed that of $S[P,P]$ by at most $7m - 3$. Now it suffices to note that, by Lemma 3.1, the nonnegative rank of $S[P \setminus P_a,P_a]$ does not exceed eight.

Now, the main result of this section follows from Lemma 2.1 and Lemma 3.2 because $n - \lceil \sqrt{n} \rceil^2 < 2\sqrt{n}$.

**Theorem 3.3.** Let $(v_1, \ldots, v_n)$ be an admissible sequence. Then, the extension complexity of $\text{conv}\{v_1, \ldots, v_n\}$ is less than $17\sqrt{n}$.

4. Characterization of admissible sequences

In what follows we denote the distance between points $u$ and $v$ by $d(u,v)$, denote the (absolute value of) measure of the angle between $\overrightarrow{u_2u_1}$ and $\overrightarrow{u_2u_3}$ by $\alpha(u_1, u_2, u_3)$, and also $\beta(u_1, u_2, u_3) = \pi - \alpha(u_1, u_2, u_3)$. Let $v = (v_1, \ldots, v_n)$ be a sequence of points. We define the edge sequence of $v$ as $d(v_1, v_2), \ldots, d(v_{n-1}, v_n)$, the angle sequence as $\beta(v_1, v_2, v_3), \ldots, \beta(v_{n-2}, v_{n-1}, v_n)$, and the edge-by-angle sequence as $d(v_2, v_3)\beta(v_1, v_2, v_3), \ldots, d(v_{n-2}, v_{n-1}, v_n)\beta(v_{n-2}, v_{n-1}, v_n)$. We say that $v$ is $\theta$-thin if the sum of elements of angle sequence is at most $\theta$. In the rest of our paper $n$ always denotes an integer, and we assume $n \geq 20$.

The main result of this section is a sufficient condition on $v$ to be admissible, stated in terms of monotonicity of the edge, angle and edge-by-angle sequences. Let us prove two easy lemmas first.

**Lemma 4.1.** Let $v = (v_1, \ldots, v_n)$ be a proper $\theta$-thin sequence. Then, any subsequence of $v$ is $\theta$-thin. If $\theta < 1$, $h > 2$, and the edge sequence of $v$ is $h$-decreasing, then any subsequence of $v$ is has $(h-1)$-decreasing edge sequence.

**Proof.** The first assertion follows from that $\sum_{k=2}^{n-1} \beta(v_{k-1}, v_k, v_{k+1}) = \alpha(v_2, v_1, v_n) + \alpha(v_1, v_n, v_{n-1})$. Further, the assumption on $\theta$ guarantees that the angle $\alpha(v_p, v_{q-1}, v_q)$ is obtuse whenever $p < q - 1$, so that $d(v_p, v_q) < d(v_{q-1}, v_q)$. Thus,

$$\frac{d(v_p, v_q)}{h-1} \geq \sum_{t=1}^{\infty} d(v_{q-1}, v_q) h^{-t} > \sum_{k=q}^{r-1} d(v_k, v_{k+1}) \geq d(v_q, v_r)$$

holds whenever $p < q < r$, and this proves the second assertion. \hfill \Box

**Lemma 4.2.** For any proper $1$-thin sequence $(v_1, \ldots, v_n)$, it holds that $2d(v_1, v_n) > d(v_1, v_2) + \ldots + d(v_{n-1}, v_n)$.

**Proof.** The projection of $\overrightarrow{v_iv_j}$ on $\overrightarrow{v_1v_n}$ has length at least $d(v_i, v_j) \cos 1 > 0.5d(v_i, v_j)$. \hfill \Box

Now we prove the lower and upper bounds on $\beta(v_1, v_{n-1}, v_n)$ and $\beta(v_1, v_2, v_n)$ in terms of measures of angles and lengths of edges of $\text{conv} v$.

**Lemma 4.3.** Assume $\theta < 0.5$ and $h > 4$. If $(v_1, \ldots, v_n)$ is a proper $\theta$-thin sequence with $h$-decreasing edge sequence, then

$$(1 - 2h^{-1})^n \leq \frac{\beta(v_1, v_{n-1}, v_n)}{\sum_{k=2}^{n-1} \beta(v_{k-1}, v_k, v_{k+1})} \leq 1,$$
\[
\left(\frac{\sin \theta}{\theta}\right)^n \leq \frac{\beta(v_1, v_2, v_n) d(v_2, v_n)}{\sum_{k=2}^{n-1} \beta(v_{k-1}, v_k, v_{k+1}) d(v_k, v_{k+1})} \leq \left(\frac{h}{h-1}\right)^n.
\]

**Proof.** Note that \(\alpha(v_1, v_{n-1}, v_{n-2}) = \arcsin \left(\frac{d(v_1, v_{n-2})}{d(v_1, v_{n-1})}\right)\) sin \(\beta(v_1, v_{n-2}, v_{n-1})\). Similarly to the proof of Lemma 4.3 we get \(d(v_1, v_{n-2}) < d(v_1, v_{n-1})\). Also,
\[
d(v_1, v_{n-1}) < d(v_1, v_{n-2}) + d(v_{n-2}, v_{n-1}) \leq (1 + h^{3-n}) d(v_1, v_{n-2}),
\]
so we get from Taylor expansion of \(\arcsin(t \sin x)\) at \(t = 1\) that
\[
(1 - 2h^{3-n}) \beta(v_1, v_{n-2}, v_{n-1}) \leq \alpha(v_1, v_{n-1}, v_{n-2}) \leq \beta(v_1, v_{n-2}, v_{n-1}),
\]
so the first inequalities follow by induction from \(\beta(v_1, v_{n-1}, v_n) = \beta(v_{n-2}, v_{n-1}, v_n) + \alpha(v_1, v_{n-1}, v_{n-2})\).

Similarly, we have \(\alpha(v_n, v_2, v_3) = \arcsin \left(\frac{d(v_3, v_n)}{d(v_2, v_n)}\right)\) sin \(\beta(v_2, v_3, v_n)\), and then
\[
\frac{\sin \beta(v_2, v_3, v_n)}{\beta(v_2, v_3, v_n)} \leq \frac{\alpha(v_n, v_2, v_3)}{d(v_2, v_3)} \leq 1.
\]

Combining this result with \(\beta(v_1, v_2, v_3) = \beta(v_1, v_2, v_3) + \alpha(v_n, v_2, v_3)\), we get
\[
\frac{\sin \beta(v_2, v_3, v_n)}{\beta(v_2, v_3, v_n)} \leq \frac{\beta(v_1, v_2, v_3) d(v_2, v_n)}{(\beta(v_1, v_2, v_3) d(v_2, v_n) + \beta(v_2, v_3, v_n) d(v_3, v_n))} \leq \frac{d(v_2, v_n)}{d(v_2, v_3)}.
\]

By Lemma 4.3 \(\beta(v_2, v_3, v_n) \leq \theta\) and \((h-1)d(v_2, v_n) \leq hd(v_2, v_3)\), so the second inequalities follow by induction. \(\square\)

Lemma 4.5 will allow us to prove the main result of the section. The following three lemmas are special cases of it.

**Lemma 4.4.** Let \(v = (v_1, \ldots, v_m)\) be a proper sequence. If indexes \(p, q, r, t\) satisfy \(q > p > r > t\), then \(S_p^p S_q^q > S_q^p S_t^t\).

**Proof.** Note that \(v' = (v_{t+1}, v_{r+1}, v_{p+1}, v_{q+1})\) is a subsequence of \(v\), so \(v'\) is proper itself. If \(v_{t+1} v_{p+1}\) and \(v_{r+1} v_{q+1}\) are collinear, the result is straightforward. Otherwise, we act on \(v\) with an appropriate affine transformation, and we get \(v_p = (a, 0), v_{p+1} = (1, 0), v_r = (0, 1), v_{r+1} = (0, b)\) with some positive \(a\) and \(b\). The rest is done by straightforward checking. \(\square\)

**Lemma 4.5.** Let \(v = (v_1, \ldots, v_n)\) be a proper \(n^{-1}\)-thin sequence. Assume the edge sequence of \(v\) is \(n^2\)-decreasing and angle sequence of \(v\) is monotone. Then, for any \(p, q, r, t\) satisfying \(p > q > r > t + 1\), it holds that \(S_p^p S_q^q > S_q^p S_t^t\).

**Proof.** Note that \(a > b > c\) implies \(2 \sin \beta(v_a, v_b, v_c) > \beta(v_a, v_b, v_c)\), and
\[
2S^2 = d(v_i, v_{i+1})d(v_{i+1}, v_j) \sin \beta(v_i, v_{i+1}, v_j) = d(v_i, v_{i+1})d(v_{i+1}, v_j) \sin \beta(v_{i+1}, v_i, v_j).
\]

Consequently,
\[
\frac{S_p^p S_q^q}{S_q^p S_t^t} > \frac{d(v_t, v_p)d(v_{r+1}, v_q)}{4d(v_q, v_p)d(v_r, v_t)} \frac{\beta(v_t, v_p, v_{p+1}) \beta(v_r, v_{r+1}, v_q)}{\beta(v_q, v_p, v_{p+1}) \beta(v_t, v_r, v_{r+1})}.
\]

Note that \(\beta(v_r, v_{r+1}, v_q) \geq \beta(v_r, v_{r+1}, v_{r+2})\) and \(n \sin n^{-1} \geq 1 - n^{-2}\). Denoting \(\beta_k := \beta(\nu_{k-1}, v_k, v_{k+1})\) and using Lemma 4.3 we get
\[
\frac{S_p^p S_q^q}{S_q^p S_t^t} > \frac{d(v_t, v_p)d(v_{r+1}, v_q)}{d(v_q, v_p)d(v_t, v_r)} \frac{\left(\sum_{k=t+1}^p \beta_k\right) \beta_{r+1}}{\left(\sum_{k=q+1}^r \beta_k\right) \left(\sum_{k=t+1}^r \beta_k\right) \left(1 - 2n^{-2}\right)^n}.
\]
The first fraction is at least \(n^2 - 1\) by Lemma 4.1 since \((\beta_k)\) is a monotone sequence, we have \(n \beta_{k+1} > \min \left\{ \sum_{k=q+1}^{p} \beta_k, \sum_{k=r+1}^{t} \beta_k \right\}\). This implies that
\[
(S^p_q S'^r_q / S^p_q S'^r_q) > 0.25(n^2 - 1)(1 - 2n^{-2})n^{-1} > 1.
\]

**Lemma 4.6.** Let \(v = (v_1, \ldots, v_n)\) be a proper \(n^{-1}\)-thin sequence. Assume the edge sequence of \(v\) is \(n^2\)-decreasing and edge-by-angle sequence of \(v\) is monotone. Then, for any indexes \(p, q, r, t\) satisfying \(q > p > t > r\), it holds that \((S^p_q S'^r_q / S^p_q S'^r_q) > 1\).

**Proof.** Similarly to the proof of Lemma 4.5 we get
\[
\frac{S^p_q S'^r_q}{S^p_q S'^r_q} > \frac{d(v_t, v_p) d(v_{r+1}, v_q)}{4(d(v_q, v_{p+1}) d(v_t, v_{r+1}) \beta(v_r, v_{r+1}, v_q) \beta(v_p, v_{p+1}, v_q) \beta(v_r, v_{r+1}, v_t)}.
\]
Denoting \(\gamma_k := d(v_k, v_{k+1})\beta(v_{k-1}, v_k, v_{k+1})\), we use Lemma 4.3 and get
\[
\frac{S^p_q S'^r_q}{S^p_q S'^r_q} > \frac{\left(\sum_{k=r+1}^{p-1} \gamma_k\right) \gamma_p}{\left(\sum_{k=p+1}^{q-1} \gamma_k\right) \left(\sum_{k=r+1}^{l-1} \gamma_k\right)} \cdot \frac{d(v_t, v_p)}{d(v_q, v_{p+1})} \cdot \frac{n^3 \left(\sin n^{-1}\right)^n}{4(n^2 - 1)^n}.
\]
Again, the first fraction is at least \(n^{-1}\) because \((\gamma_k)\) is a monotone sequence; the second fraction is at least \(n^{-1}\) because the edges of \(v\) are \(n^2\)-decreasing. Again, we can see that \((S^p_q S'^r_q / S^p_q S'^r_q) > 1\).

Now we put the results of Lemmas 4.4, 4.5, and 4.6 together.

**Corollary 4.7.** Let \(v = (v_1, \ldots, v_n)\) be a proper \(n^{-1}\)-thin sequence. Assume the edge sequence of \(v\) is \(n^2\)-decreasing, the angle sequence of \(v\) is monotone, the edge-by-angle sequence of \(v\) is monotone. Then, \(v\) is admissible.

**Proof.** Consider indexes \(p, q, r, t\); note that \(S^p_q S'^r_q = 0\) when either \(p = q\) or \(r = t\). Then, assume \(p \neq q\) and \(r \neq t\). If \(\min\{p, q-1\} > r > t\), then either \(p > q > r > t\) or \(q > p > r > t\). In the former case, Lemma 4.4 implies \(S^p_q S'^r_q > S^p_q S'^r_q\); in the latter case, this follows from Lemma 4.4. If \(q > p > \max\{r, t\}\), then either \(q > p > r > t\) or \(q > p > t > r\). Again, we deduce \(S^p_q S'^r_q > S^p_q S'^r_q\) from Lemma 4.4 in the former case; in the latter case, this follows from Lemma 4.6.

## 5. Extracting an admissible subsequence

The goal of this section is to show that, for any \(n\) fixed in advance, any sufficiently large proper sequence contains an admissible subsequence of length \(n\).

Let \(r = (r_1, \ldots, r_m)\) be a sequence of reals and \(i = (i_0, \ldots, i_l)\) a strictly increasing sequence of indexes. In the following lemma, we refer to the sequence \((\sum_{k=i_0+1}^{i_1} r_k, \ldots, \sum_{k=i_{l-1}+1}^{i_l} r_k)\) as a sum-subsequence of \(r\) corresponding to \(i\).

**Lemma 5.1.** Let \(r = (r_1, \ldots, r_m)\) be a sequence of positive reals and \(p, q, h, \) positive integers. If \(m = \exp_2 \left(\sum_{k=0}^{p-1} h^k + \sum_{k=0}^{q-1} h^k\right)\), then \(r\) contains either an \(h\)-increasing sum-subsequence of length \(p\) or a \(h\)-decreasing sum-subsequence of length \(q\).

**Proof.** If \(s\) was a non-decreasing sequence of length \(\sum_{k=0}^{p-1} h^k\), then a sum-subsequence of \(s\) defined by \(s_0 = 0\) and \(s_i = \sum_{k=0}^{i-1} h^k\), for \(r \in \{1, \ldots, p\}\), would be \(h\)-increasing. This shows that proving the result for \(h = 1\) is sufficient, and we do this by induction on \((p, q)\). If \(p = q = 1\), then the result is trivial, and we proceed by
Lemma 5.2. Let \( v = (v_1, \ldots, v_n) \) be a proper \( n^{-1} \)-thin sequence. Assume the edge sequence of \( v \) is \( n^2 \)-decreasing. Denote \( \beta_k := \beta(v_{k-1}, v_k, v_{k+1}) \) and \( \gamma_k := d(v_k, v_{k+1})\beta(v_{k-1}, v_k, v_{k+1}) \). Then, for every \( p \), we have

1. \( \beta(v_1, v_p, v_n) \leq \sum_{k=2}^{p-1} \beta_k + 0.1 \sum_{k=p+1}^{n-1} \beta_k \).
2. \( \beta(v_1, v_p, v_n) \geq 0.9 \sum_{k=2}^{p} \beta_k \).
3. \( \beta(v_1, v_p, v_n)d(v_p, v_n) \leq 0.1 \sum_{k=2}^{p-1} \gamma_k + 1.1 \sum_{k=p}^{n-1} \gamma_k \).
4. \( \beta(v_1, v_p, v_n)d(v_p, v_n) \geq 0.9 \sum_{k=p}^{n-1} \gamma_k \).

Proof. The inequalities (2) and (4) follow from Lemma 4.3 because \((1 - 2n^{-2})^n \text{ and } (n \sin n^{-1})^n \text{ are both greater than } 0.9\). To prove (1), note that \( \beta(v_1, v_p, v_n) = \beta(v_1, v_p, v_{p+1}) + \alpha(v_p, v_{p+1}) \text{ and } \beta(v_1, v_{p+1}, v_{p+1}) \leq \sum_{k=2}^{p} \beta_k \text{ by Lemma 4.3} \). Also, \( \alpha(v_p, v_{p+1}) = \arcsin \left( \frac{d(v_{p+1}, v_n)}{d(v_p, v_n)} \sin \beta(v_p, v_{p+1}, v_n) \right) \), which does not exceed \( \frac{d(v_{p+1}, v_n)}{d(v_p, v_n)} \beta(v_p, v_{p+1}, v_n) \). Using Lemma 4.3 again, we get

\[
\alpha(v_p, v_{p+1}) \leq \sum_{k=p+1}^{n-1} \frac{d(v_k, v_{k+1})}{d(v_p, v_n)} \beta_k \leq \frac{d(v_{p+1}, v_{p+2})}{d(v_p, v_n)} \sum_{k=p+1}^{n-1} \beta_k \leq n^{-2} \sum_{k=p+1}^{n-1} \beta_k.
\]

To prove (2), note that \( \beta(v_1, v_p, v_n) = \beta(v_{p-1}, v_p, v_n) + \alpha(v_{p-1}, v_p, v_1) \), and \( \beta(v_{p-1}, v_p, v_n)d(v_p, v_n) \leq 1.1 \sum_{k=p}^{n-1} \gamma_k \text{ by Lemma 4.3} \). Further, \( \alpha(v_{p-1}, v_p, v_1) \) equals \( \arcsin \left( \frac{d(v_{p-1}, v_{p-1})}{d(v_1, v_p)} \sin \beta(v_1, v_{p-1}, v_p) \right) \leq \beta(v_1, v_{p-1}, v_p) \). By Lemma 4.3,

\[
\alpha(v_{p-1}, v_p, v_1)d(v_p, v_n) \leq \sum_{k=2}^{p-1} \beta_k d(v_p, v_n) \leq \sum_{k=2}^{p-1} \frac{d(v_p, v_n) \gamma_k}{d(v_k, v_{k+1})} \leq 0.1 \sum_{k=2}^{p-1} \gamma_k.
\]

\( \Box \)

The following lemma points out a sufficient condition for monotonicity of angles of a subsequence.

Lemma 5.3. Let \( v = (v_1, \ldots, v_n) \) be a proper \( n^{-1} \)-thin sequence. Assume the edge sequence of \( v \) is \( n^2 \)-decreasing. If the angle sequence of \( v \) is \( 3 \)-monotone, then any subsequence of \( v \) has monotone angle sequence.

Proof. Assume \( p, q, r, s \in \{1, \ldots, n\} \) and \( p < q < r < s \). We use Lemma 5.2 to see that, if \( \{\beta_k\} \) is decreasing,

\[
\beta(v_p, v_q, v_r) \geq 0.9 \sum_{k=p+1}^{q} \beta_k > \sum_{t=1}^{+\infty} 3^{-t} \beta_q > \sum_{k=q+1}^{n-1} \beta_k > \beta(v_q, v_r, v_s).
\]
If \((\beta_k)\) is increasing, we get
\[
\beta(v_p, v_q, v_r) \leq \sum_{t=1}^{+\infty} 3^{-t} \beta_{q+1} + 0.1 \sum_{k=q+1}^{r-1} \beta_k < 0.9 \sum_{k=q+1}^{r} \beta_k < \beta(v_q, v_r, v_k).
\]
This shows that any subsequence of \(v\) has a monotone angle sequence. \(\square\)

The following lemmas help us finalize the proof of the main result.

**Lemma 5.4.** If \(N > 2\pi nm\), then any proper sequence \(u = (u_1, \ldots, u_N)\) has an \(n^{-1}\)-thin subsequence of length \(m\).

**Proof.** Follows from that \(\sum_{k=2}^{N-1} \beta(u_{k-1}, u_k, u_{k+1}) < 2\pi\). \(\square\)

**Lemma 5.5.** Assume \(N > 2\pi \exp_2(2n)^{2n}\) and \(u = (u_1, \ldots, u_N)\) is a proper sequence. Then, \(u\) has an \(n^{-1}\)-thin subsequence of length \(n\) whose edge sequence is \(n^{2}\)-monotone.

**Proof.** By Lemma 5.4, \(u\) has an \(n^{-1}\)-thin subsequence \(u'\) of length \(\exp_2(2n)^{2n}\). By Lemma 5.1, the edge sequence of \(u'\) admits a \(2n^{2}\)-monotone sum-subsequence, and the result follows from Lemma 4.2. \(\square\)

**Lemma 5.6.** Assume \(N \geq \exp_2 \exp_9 n\) and \(u = (u_1, \ldots, u_N)\) is a proper \(N^{-1}\)-thin sequence whose edge sequence is \(N^{2}\)-decreasing. Then, \(u\) has a subsequence of length \(n\) whose angle (or edge-by-angle) sequence is \(3\)-monotone.

**Proof.** Denote \(\beta_k := \beta(u_{k-1}, u_k, u_{k+1})\). By Lemma 5.1, there is an increasing sequence \(i_1, \ldots, i_n\) such that the sequence \((s_i) = \left(\sum_{k=i_n}^{i_{i+1}} \beta_k\right)\) is \(6\)-monotone. We will use Lemma 5.2 to show that \((u_{i_1}, \ldots, u_{i_n})\) has \(3\)-monotone angle sequence. Actually, if \(s\) is increasing, then
\[
\beta(u_{i_{t-1}}, u_{i_t}, u_{i_{t+1}}) \leq s_{t-1} + 0.1s_t < 0.3s_t \leq \beta(u_{i_t}, u_{i_{t+1}}, u_{i_{t+2}})/3.
\]
Similarly, if \(s\) is decreasing, we would have
\[
\beta(u_{i_{t-1}}, u_{i_t}, u_{i_{t+1}}) \geq 0.9s_{t} \geq 3s_t + 2.4s_{t+1} \geq 3\beta(u_{i_t}, u_{i_{t+1}}, u_{i_{t+2}}).
\]

To prove the statement for edge-by-angle sequences, denote \(\gamma(a, b, c) := d(u_a, u_c)\beta(u_a, u_b, u_c)\). Again, Lemma 5.1 shows that there is an increasing sequence \(j_1, \ldots, j_n\) such that the sequence \((r_i) = \left(\sum_{k=j_i}^{j_{i+1}} \gamma(k-1, k, k+1)\right)\) is \(6\)-monotone. If \(r\) is decreasing, we get by Lemma 5.2
\[
\gamma(j_{t-1}, j_t, j_{t+1}) \geq 0.9r_t \geq 0.3r_t + 3.6r_{t+1} \geq 3\gamma(j_t, j_{t+1}, j_{t+2}),
\]
and if \(r\) is increasing, then
\[
\gamma(j_{t-1}, j_t, j_{t+1}) \leq 0.1r_{t-1} + 1.1r_t \leq 0.2r_{t+1} \leq \gamma(j_t, j_{t+1}, j_{t+2})/3,
\]
so that \((u_{j_1}, \ldots, u_{j_n})\) has \(3\)-monotone edge-by-angle sequence. \(\square\)

We finalize the section with a proof of desired result.

**Theorem 5.7.** If \(m > \exp \exp_3 \exp_2 \exp_6 \exp_2 \exp_6 n\), then any proper sequence \(v\) of length \(m\) has an admissible subsequence of length \(n\).
Proof. For positive $x$, we have $\exp\exp 3x > 2\pi x \exp_p(2x)^2$, so by Lemma 6.5 there is an $N^{-1}$-thin subsequence of length $N = \exp_p 2 \exp_p 2 \exp_p n$ whose edge sequence is $N^2$-monotone. Up to symmetry, we can assume that the edge sequence is decreasing, and then by Lemma 5.6 there is a subsequence $v'$ of length $\exp_p 2 \exp_p 2 \exp_p n$ whose angle sequence is $3$-monotone. Finally, we find a subsequence $v''$ of $v'$ of length $n$ with monotone edge-by-angle sequence. The angle sequence of $v''$ is monotone by Lemma 5.3, the edge sequence of $v''$ is $n^2$-decreasing by Lemma 4.1, and $v''$ is $n^{-1}$-thin again by Lemma 4.1. Therefore, $v''$ is admissible.

6. The main result

We are ready to prove the main theorem. Recall that the results of Sections 4 and 5 were proven under assumption that $n \geq 20$, which has been mentioned in the beginning of Section 4.

Theorem 6.1. If $m > \exp\exp\exp\exp\exp\exp 40$, then the extension complexity of convex $m$-gon does not exceed $25m (\ln\ln\ln\ln\ln\ln m)^{-0.5}$.

Proof. Let $n = [0.5 \ln\ln\ln\ln\ln\ln m]$, then $n \geq 20$ and Theorem 5.7 is applicable. Therefore, the sequence of vertices of a convex $m$-gon $P$ contains admissible disjoint subsequences $v_1, \ldots, v_k$ of length $n$ such that $m - kn \leq \exp\exp\exp\exp\exp_p 2 \exp_p 2 \exp_p n$. The extension complexity of $\text{conv } v_i$ is at most $17\sqrt{n}$ by Theorem 3.3. Finally, we use Lemma 2.1 to bound the extension complexity of $P$ from above by $17m/\sqrt{n} + \exp\exp\exp\exp_p 2 \exp_p 2 \exp_p n$.

Also, let us follow the argument from [13] to deduce a linear algebraic formulation of Theorem 6.1. Let $A$ be a nonnegative $m$-by-$n$ matrix with classical rank equal to $3$. Consider the simplex $\Delta \subset \mathbb{R}^m$ consisting of points with nonnegative coordinates which sum to 1. Since $\Delta$ has $m$ facets, the intersection of $\Delta$ with the column space of $A$ is a polygon $I$ with $k$ vertices, and $k \leq m$. Form a matrix $S$ of column coordinate vectors of vertices of $I$, then $A = SB$ with $B$ nonnegative. Since $S$ is a slack matrix for $I$, it has nonnegative rank at most $25m (\ln\ln\ln\ln\ln\ln m)^{-0.5}$, and so does $A$.

Finally, let us remark that our paper shows the existence of a generic $n$-gon with extension complexity $O(\sqrt{n})$. To see this, note that the set of admissible $n$-gons is open by its definition and non-empty by Theorem 5.7. Clearly, almost all polygons from this open set are generic, and they have extensions of size at most $17\sqrt{n}$ by Theorem 3.3. This result shows that an $\Omega(\sqrt{n})$ lower bound for extension complexities of all generic $n$-gons, obtained in [5], cannot be improved.

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