GRAVITATIONAL INSTABILITY OF SOLIDS ASSISTED BY GAS DRAG: SLOWING BY TURBULENT MASS DIFFUSIVITY

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Received 2010 November 15; accepted 2011 June 14; published 2011 August 12

ABSTRACT

The Goldreich & Ward (axisymmetric) gravitational instability of a razor thin particle layer occurs when the Toomre parameter \( Q_T = c_p \Omega_0 / \pi G \Sigma_p < 1 \) (\( c_p \) being the particle dispersion velocity). Ward extended this analysis by adding the effect of gas drag upon particles and found that even when \( Q_T > 1 \), sufficiently long waves were always unstable. Youdin carried out a detailed analysis and showed that the instability allows chondrule-sized (~1 mm) particles to undergo radial clumping with reasonable growth times even in the presence of a moderate amount of turbulent stirring. The analysis of Youdin includes the role of turbulence in setting the thickness of the dust layer and in creating a turbulent particle pressure in the momentum equation. However, he ignores the effect of turbulent mass diffusivity on the disturbance wave. Here, we show that including this effect reduces the growth rate significantly, by an amount that depends on the level of turbulence, and reduces the maximum intensity of turbulence the instability can withstand by 1–3 orders of magnitude. The instability is viable only when turbulence is extremely weak and the solid to gas surface density of the particle layer is considerably enhanced over minimum-mass-nebula values. A simple mechanistic explanation of the instability shows how the azimuthal component of drag promotes instability while the radial component hinders it. A gravito-diffusive overstability is also possible but never realized in the nebula models.

Key words: planetary systems – protoplanetary disks

1. INTRODUCTION

1.1. Preliminary Remarks

It is believed that clumping of solid material to form the terrestrial planets and the putative cores of gas giants involved three stages. The first and third stages are relatively well understood. First, grains that survived shocked entry into the solar nebula and those that condensed from the cooling gas, collided and stuck by van der Waals attraction; such a process is considerably speeded up by turbulence but is still effective. Dominik (2005) and is able to form cm-or-larger sized particles. Thereafter, growth by binary accretion becomes problematic in a turbulent dust layer due to shattering. In the third stage, planetesimals, bodies 1 km and larger which are akin to the present day asteroids, merged in binary fashion by physical and/or gravitational capture as they collided (Kokubo & Ida 2000; Chambers 2001; Kenyon & Bromley 1993).

Least understood is the middle stage, namely, the growth from cm to km sized bodies. Early on Safronov (1972) and Goldreich & Ward (1973, henceforth collectively referred to as SGW) independently suggested that particles settled to the midplane and underwent a gravitational instability (GI). This caused clumps of some characteristic size to contract until centrifugal force became strong enough to balance self-gravity. Goldreich & Ward (1973) suggested that bodies of size \( \lesssim 0.5 \) km having the density of solid material could form in this way on a dynamical timescale.

The SGW scenario ignored the presence of global turbulence in the solar nebula which stabilizes the dust layer against GI by making the basic state particle layer more diffuse and introducing a turbulent pressure into the particle dynamics. A possible source of disk turbulence is magneto-rotational instability (MRI). It has been argued (Gammie 1996) that MRI may be confined to only the upper layers of a disk which are sufficiently ionized by cosmic rays. Nevertheless, this does not imply a completely quiescent mid-plane since vortical eddies in the unstable layers can induce fluctuations at the midplane. Dullemond & Dominik (2004) found that laminar settling of dust grains could not account for the observed flaring of disks suggesting that turbulence is keeping small particles aloft throughout the disk height.

Even if one discounts global turbulence, an obstacle to the SGW picture of midplane GI remains: the dust disk itself can generate turbulence. The gas component feels a small outward directed pressure force and therefore rotates at a slightly slower speed than the Keplerian speed of solid particles. This creates an Ekman-like layer whose density stratification is not strong enough to stabilize the layer. The turbulent state is such that the critical particle density required for the mid-plane GI is not realized (Weidenschilling 1980, 1984; Cuzzi et al. 1993).

A variant of midplane GI involves suppression of turbulence in the mixed particle–gas layer by the vertical density gradient. This variant is valid only for very small particles, which are so well trapped to the gas that the particle–gas mixture behaves as a single fluid. In this limit, Sekiya (1998) calculated the (mean) vertical structure of the turbulent dust layer by assuming that the mean profiles of density and velocity lie at the instability/stability boundary, i.e., characterized by a Richardson number equal to its critical value of \( 1/4 \). He found that if the ratio of solid to gas surface density in the disk is much larger than the cosmic value, the layer achieves the critical density needed for GI. This type of analysis produces a cusp at the midplane in the mean density profile as the surface density of solids in the disk is increased. Youdin & Shu (2002) and later Youdin & Chiang (2004) studied this situation further, suggesting that the cusp was not a mathematical artifact but indicative of the inability of the turbulence to stir up particles near the midplane. They proposed that the required enrichment of solids could occur as...
particles drift inward by gas drag. Garaud & Lin (2004) studied the Richardson number instability as the particle layer slowly settles. They also find (their Figure 11) that, at the levels of the pressure force appropriate to a minimum mass nebula, GI sets in before the Richardson number instability only if the column density of particles is ∼1/5 the column density of the gas; this represents a factor of ∼30 increase over the minimum-mass-nebula model, in agreement with previous works. Because of the small particle limitation, this variant of midplane GI is precluded by even the faintest breath of global turbulence (Cuzzi & Weidenschilling 2006).

Alternate scenarios of growth in dense midplane layers, which do not rely on GIs, have been developed by Weidenschilling in a number of papers (see Weidenschilling 2000, for a review). A review of this stage is given by Cuzzi & Weidenschilling (2006) and recent work (Cuzzi et al. 2001; Johansen et al. 2006) suggests that the problem is more complex than envisioned in the early works.

1.2. The Gas-drag Mediated Instability

The subject of the present work is a gravitationally driven instability made possible by gas drag. What makes the instability interesting is that, although it is relatively slow, it is unconditional. The instability was proposed and studied by Ward (1976, 2000) in the absence of turbulence and associated particle dispersion. Safronov (1987, 1991) also briefly considers this instability under the same conditions and writes down the dispersion relation in the limit where particle stopping time is short compared to the timescale of the perturbation. A detailed analysis (Youdin 2005a, 2005b, henceforth papers I and II) suggested that the gas-drag assisted instability remains viable in the presence of a moderate amount of nebula turbulence. This analysis included the effect of turbulence in the particle momentum equation (via an effective pressure) and in setting the height of the particle layer. However, a turbulent diffusion term must also be present in the equation for particle mass conservation. Here, we correct this omission and find significant reductions in growth rate. In particular, we find that the instability is not viable for nominal values of nebula turbulence unless favorable assumptions are made regarding local conditions. These include some combination of elevated solid/gas ratio, and particle size neither too small nor too large.

During review of the revised version of this paper, we were sent a new paper, Youdin (2011), that also corrects for the lack of turbulent mass diffusivity in the 2005 papers. In addition, it uses updated models for the radial turbulent diffusivity of particles, the radial particle dispersion, and the height of the particle sub-disk. These models are based on the analysis of Youdin & Lithwick (2007) and the corrections they imply become important for particles whose stopping times \( \tau_s \) non-dimensionalized by the orbital frequency \( \Omega_0 \) are such that \( \tau_s \sim \Omega/\Omega_0 \gtrsim 1 \). The present calculations were therefore redone to incorporate the newer models, however, the basic conclusions remained the same.

2. ANALYSIS

2.1. Dispersion Relation

When a mass diffusion term with coefficient \( D \) is included, the linearized and vertically integrated mass and momentum equations for the particles, Equations (6) and (7) in paper I, become

\[
\frac{\partial \sigma}{\partial t} + \frac{\partial u}{\partial x} = D \frac{\partial^2 \sigma}{\partial x^2},
\]

\[
\frac{\partial u}{\partial t} - 2v\Omega_0 = -\frac{\partial \sigma}{\partial x} - \frac{u}{\tau_s},
\]

\[
\frac{\partial v}{\partial t} + (2 - q)v\Omega_0 = -\frac{v}{\tau_s},
\]

where \( q \equiv 3/2 \). The equations are written relative to a box at radius \( R \) revolving at the local angular velocity \( \Omega(R) \) of particles; we have defined \( \Omega_0 \equiv \Omega(R) \). Affixed to the box are local Cartesian coordinates \((x, y)\), where \( x \) is radial and \( y \) is azimuthal. The corresponding velocity components are \((u, v)\); they represent turbulent mean quantities that have then been vertically averaged. The quantity \( \sigma \) is the relative perturbation in particle surface density defined so that

\[
\Sigma = \Sigma_0 (1 + \sigma),
\]

where \( \Sigma_0 \) and \( \Sigma \) are the basic state and total (basic state + perturbation) surface densities, respectively. Throughout the analysis used to obtain Equations (1)–(3), one assumes that \( \Sigma_0 \) is locally uniform and can therefore be freely moved into and out of \( x \) and \( y \) derivatives. The basic state velocity of particles has been taken to be Keplerian in deriving Equations (1)–(3). The gas flow has also been taken to be Keplerian and is left unperturbed. In this model therefore, there is only one-way coupling from the gas to the particles. (One consequence of two-way coupling are streaming instabilities, e.g., Youdin & Goodman 2005). Figure 4 (lower) in Cuzzi et al. (1993) shows that the turbulent mean flow departs from being Keplerian by only about 0.2%. The basic state flow of the particles should also have an inward radial drift arising from gas drag (Adachi et al. 1976); this has been left out in Equations (1)–(3). However, since the (vertically averaged) drift velocity is locally uniform with respect to \( x \) and \( y \), it would cause the instability wave to merely drift inward without affecting its local growth rate. Effects that involve the fact that the drift velocity has a vertical dependence are not included in the analysis. We found (except for the gravito-diffusive mode that was never realized in the nebula calculations) that the phase and group velocity of the instability wave is zero in the drifting frame. This comes about simply because the wave oscillation frequency is zero in some neighborhood of the most amplified wavenumber. Thus, the wave drift speed is the same as the particle drift speed and this will limit the total amount of wave growth. This is accounted for in the study by comparing the \( e \)-folding growth time to a local drift timescale

\[
t_{\text{drift}} = R/u_{\text{drift}}.
\]

Complete loss of solids from the entire nebula is a weaker constraint.

Axisymmetric disturbances are being considered. Thus, the only spatial derivatives that appear are those with respect to the radial coordinate \( x \).

The quantity \( \chi \equiv \Phi + \Pi' \) consists of the gravitational potential \( \Phi \) and \( \Pi' \equiv \sigma c_s^2 \) which arises from modeling the effect of turbulence on particle momentum by an effective pressure, \( c_s \) being the radial particle dispersion velocity (Youdin & Lithwick 2007). The effective pressure term arises from the \( rr \) component of the particle Reynolds stress tensor in the
Reynolds-averaged particle momentum equation. It should be noted that neither the present treatment nor that of Youdin (2011) has incorporated all the effects of turbulence in the particle momentum equation. A Reynolds average of the original equation reveals that, in addition to an effective pressure, several other turbulent correlations arise that require closure models. Compared to previous treatments (Ward 2000; Youdin 2005a, 2005b) the only new term is the right-hand side of Equation (1) and represents turbulent diffusion of surface density. This term arises from the correlation $\langle \rho' u' \rangle$ in the Reynolds averaged particle mass conservation equation (the primes denote turbulent fluctuations and the angle brackets denote a suitable Reynolds average). This has been modeled using gradient diffusion as $-D\delta \rho/\delta x_j$.

Substituting into Equations (1)–(3) perturbations of the form,

$$\langle \sigma, u, v \rangle = (\tilde{\sigma}, \tilde{u}, \tilde{v}) e^{i(kx - \omega t)},$$

where $(\tilde{\sigma}, \tilde{u}, \tilde{v})$ are complex constants, gives a linear homogeneous system

$$A(\tilde{\sigma}, \tilde{u}, \tilde{v})^T = 0,$$

where $A$ is a matrix. The solvability condition for this system, namely, that the determinant of $A$ vanish, gives the dispersion relation:

$$(\omega + i/t_s)(\omega + iDK^2)(\omega + i/t_s) + i\Omega^2Dk^2 - t_s^{-1} = 0,$$

$$C(k) \equiv \Omega_0^2 + k^2\gamma^2 - 2\pi G\Sigma_p kT(kh_p),$$

$$D(k) \equiv \nu + iDK^2 + i\Omega^2Dk^2 - t_s^{-1} = 0,$$

where $h_p$ is the height of the particle sub-disk and $T(kh_p) = 1/(1 + kh_p)$ is a factor that approximates finite thickness effects on the potential of self-gravity in the context of a vertically integrated model. In the limit $t_s \to \infty$ (vanishing drag) and $D \to 0$, Equation (8) reduces to the SGW form $\omega^2 = C(k)$. Since turbulent mass diffusivity occurs only in the product $DK^2$, its effect is arbitrarily small for sufficiently long waves. However, the most amplified wave has a finite $k$ and therefore turbulent mass diffusivity has a non-negligible effect on growth rate.

Following Youdin (2005a) let us introduce the non-dimensional variables

$$\gamma \equiv -i\omega/\Omega_0, \quad \kappa \equiv \pi G\Sigma_p k/\Omega_0^2, \quad t_s \equiv \Omega_0 t_s,$$

and the Toomre and Roche parameters

$$Q_T \equiv \frac{c_p\Omega_0}{\pi G\Sigma_p}, \quad Q_R \equiv \frac{h_p\Omega_0^3}{\pi G\Sigma_p}.$$

The additional parameter resulting in the present case is the non-dimensional diffusivity:

$$\hat{D} \equiv \frac{D\Omega_0^3}{(\pi G\Sigma_p)}.$$

As can be checked, the condition $F < 0$ gives instability in the limit of zero gas drag ($t_s \to \infty$) and zero mass diffusivity ($\hat{D} \to 0$). Equation (13) expands to the following cubic equation for $\gamma$:

$$\gamma^3 + (2\tau_s^{-1} + \hat{D}k^2)\gamma^2 + (\tau_s^{-2} + F + 2\hat{D}k^2\tau_s^{-1})\gamma + \tau_s^{-1}(F - 1) + \hat{D}k^2(1 + \tau_s^{-2}) = 0.$$

Using Descartes’ rule of signs and clever reasoning, Youdin (2005a) concluded that a necessary and sufficient condition for instability (for the case of zero mass diffusivity) is that $F(\kappa, Q_T, Q_R) < 0$. A similar analysis does not appear to be possible in the present case. However, Youdin (2011) has shown that, even with mass diffusivity, one can always find a wavenumber $k$ such that the system is unstable.

The slow instabilities considered are such that the growth rates can be much smaller than the particle stopping rate, i.e., $\gamma \ll 1/t_s$. In this limit, an explicit expression is obtained for the growth rate from Equation (13):

$$\gamma_{\text{approx}} = \tau_s(1 - F) - \hat{D}k^2(1 + \tau_s^{-2} + 1),$$

which clearly shows the damping effect of $\hat{D}$. The validity of Equation (16) can be checked a posteriori.

### 2.2. Implementation

To implement the analysis for various disk conditions we follow exactly the treatment of papers I and II updated with the models in Youdin (2011) for the effect of turbulence on particles. Defining $\varpi \equiv R/AU$, the nebula model employed is

$$\Sigma_g = 1700f_g\varpi^{-3/2} \text{ gm cm}^{-2},$$

$$\Sigma_p = 10f_p\varpi^{-3/2} \text{ gm cm}^{-2},$$

$$c_g = 10^5\varpi^{-1/4} \text{ cm s}^{-1},$$

for the gas-disk surface density, particle-layer surface density, and sound speed, respectively. Apart from the factors $f_g$ and $f_p$, this is just the minimum mass model. The calculation of the non-dimensional stopping time, $\tau_s$, uses the Epstein and Stokes formulae as appropriate:

$$\tau_s = \begin{cases} 4 \times 10^{-4}(a/\text{mm})3/5 f_g^{-1}, & a/\lambda_{\text{mfp}} \lesssim 2.002; \\ 9 \times 10^{-5}(a/\text{mm})^2(\varpi/0.3)^{-5/4}, & \text{otherwise,} \end{cases}$$

where

$$\lambda_{\text{mfp}} = f_g^{-1}a^{2.75}$$

is the mean free path. The resulting $\tau_s$ depends on both particle size and radial location in the disk. Equation (20) is valid when the particle Reynolds number $Re_p = 2a\Delta u/\nu \lesssim 1$. This condition is satisfied in all the plots we present by adjusting the range of the abscissa if necessary. Here, $\Delta u$ is the magnitude of the particle velocity relative to the gas which depends on $\tau_s$ according to Equation (A1) in Youdin (2011), and $\nu = 2.45 \times 10^8 \sigma_m^{3/5} f_g^{-1} \text{ cm}^2 \text{ s}^{-1}$ is the kinematic viscosity.

The radial component of the particle dispersion velocity, $c_p$, is calculated using Equation (27) in Youdin (2011):

$$c_p = \frac{1 + 2\tau_s^2 + (5/4)\tau_s^2}{1 + \tau_s} \sqrt{\sigma_g c_g}.$$
The radial mass diffusivity due to turbulence is written as
\[ D = \frac{v_g}{Sc_T}, \]  
(23)
where \( v_g \) is the turbulent momentum diffusivity of the gas and \( Sc_T \) is the turbulent Schmidt number. Following Youdin (2011)
\[ Sc_T = \frac{(1 + \tau_s^2)}{1 + \tau_s + 4\tau_s^2}. \]  
(24)
The turbulent viscosity of the gas, \( \nu_g \), is defined via an \( \alpha \) parameter:
\[ \nu_g = \alpha_g \epsilon_g^2/\Omega_0, \]  
(25)
where the turbulence parameter \( \alpha_g \) is a measure of the local turbulence intensity of the gas and \( \epsilon_g \) is the sound speed of the gas. After some substitutions one obtains
\[ \frac{\alpha_g}{\nu_g} = \frac{1 + \tau_s + 4\tau_s^2}{(1 + \tau_s^2)} \left( \frac{Q_T \Sigma_g}{\Sigma_p} \right)^2, \]  
(26)
where \( Q_T \) is the Toomre parameter of the disk gas:
\[ Q_T = \frac{c_s \Omega_0}{\pi G \Sigma_g}. \]  
(27)
For the present nebula model we obtain
\[ \left( \frac{Q_T \Sigma_g}{\Sigma_p} \right)^2 = 10^8 f_p^{-2} \sigma^{-1/2}, \]  
(28)
whose large leading coefficient accounts for the sensitivity to \( \alpha_g \) observed in the results.

The model of Youdin (2011) used to determine the height \( h_p \) of the particle disk is
\[ h_p = h_g \sqrt{\frac{\alpha_g}{\tau_s \psi + \alpha_g}}, \]  
(29)
where
\[ \psi = 1 + \frac{2 \pi G \Sigma_p}{\Omega_0^2 h_p} = 1 + 2/Q_R. \]  
(30)
Equations (29) and (30) lead to a quadratic equation for \( h_p/h_g \) which is explicitly solved.

Finally, we discuss the constraint due to inward radial drift of the instability wave as motivated earlier. We have
\[ u_{\text{drift}} = R/|u_{\text{drift}}|. \]  
(31)
The analysis of Nakagawa et al. (1986), extended to include terms quadratic in \( \tau_s \) gives for the equilibrium (i.e., neglecting acceleration terms) radial drift speed of the particle layer:
\[ u_{\text{drift}} = -2\eta u_K \left[ \frac{\tau_s}{(1 + \phi_p(z))^2 + \tau_s^2} \right], \]  
(32)
where \( u_K = \Omega_0 R \) is the Keplerian speed,
\[ \eta = -\frac{R}{2\rho_g u_K^2} \frac{\partial p}{\partial R} = 1.3 \times 10^{-3} \sigma^{1/2} \]  
(33)
is the pressure gradient parameter, and \( \phi_p(z) \equiv \rho(z)/\rho_g(z) \) is the ratio of particle to gas density (the so-called particle loading) which in general depends on the vertical position \( z \) in the disk. Using vertical averages for both numerator and denominator we estimate
\[ \phi_p \equiv \frac{\rho(z)}{\rho_g(z)} \approx \frac{\Sigma h_g}{\Sigma h}, \]  
(34)

2.3. Mechanistic Interpretation of the Instability

Before we begin let us note that gas drag (identified by the parameter \( \tau_\nu \)) enters not only as an explicit term in the momentum equations but also through setting the turbulent diffusivity, effective pressure, and height of the particle layer. That is, \( \tau_\nu \) also affects the parameters \( D, Q_T \), and \( Q_p \). In this subsection we focus only on the role of the explicit gas drag terms in the momentum equations.

GI occurs when the increase of mutual gravity between approaching particles outweighs restoring forces. As is well known, an important restoring force in Keplerian systems and for axisymmetric perturbations comes from the outwardly increasing angular momentum. Consider a perturbation mode (in radial velocity and density) such that particles accumulate in a toroidal region and deplete in the neighboring region. Self-gravity is not needed at this stage. A circle of particles at the inner edge of this region is displaced outward. If gas drag is absent, the circle will conserve its specific angular momentum \( \mu_\nu \) and it will slow down more steeply than the local Keplerian speed slows with radius. It will therefore have a deficit of centripetal acceleration relative to the pull of the central gravitator and will experience an inward restoring force. Similarly, a circle at the outer edge of the toroidal region will be displaced inward and will speed up faster than the local Keplerian value and experience an outward restoring force. This restoring force is what makes Keplerian disks stable via Rayleigh’s criterion and is characterized by a \( +\Omega_0^2 \) term on the right-hand side of the dispersion relation \( \omega^2 = \omega^2 \) in the drag-free (SGW) case.

Let us now consider the same picture but in the presence of the azimuthal gas drag term in the momentum equation. In this case, the azimuthal speed of an outward/inward displacing particle circle will remain closer to the local Keplerian value and there will not be as much of a deficit/excess of centripetal acceleration compared to the pull of the central gravitator. The restoring force is thus diminished. This is the mechanism by which the azimuthal gas drag term promotes instability. Ultimately, what has to overcome whatever restoring force remains is the increase in mutual gravity among the accumulating particles.

Note that for an outwardly displacing circle, the gas flow is a tailwind relative to the particles and gas drag supplies them with more angular momentum.

In actual fact, the gas revolves slightly slower than Keplerian due to pressure support. Hence an outwardly/inwardly displaced circle will actually experience a slightly smaller tailwind/larger headwind than envisioned above. The net result is that everything that was described in the preceding paragraphs should be imagined to be taking place in a reference frame drifting inward (in the vertically averaged model).

Consider the limit of large drag; specifically, assume that the stopping time \( \tau_s \) is very short compared to the timescale \( 1/|\omega| \) of the perturbation. In this case the azimuthal speed of the displaced circle will always remain very close to that of the local gas, namely, Keplerian. In this limit, therefore, the deficit/excess of angular momentum of displaced circles and hence the restoring tendency is completely annulled. One can see this explicitly by writing the dispersion relation (8) for this case:
\[ t_s^{-1} \left[ C(k) - (\omega + iDk^2)i t_s^{-1} \right] + \Omega_0^2 Dk^2 - \Omega_0^2 t_s^{-1} = 0. \]  
(35)
One observes that the restoring force term (\( \Omega_0^2 \) in \( C(k) \)) is canceled by the last term in Equation (35).
To begin with let us set the mass diffusivity \( D = 0 \) and plot the non-dimensional growth-rate \( \text{Re}(\gamma) \) (maximized with respect to wavenumber and the three roots of the cubic) versus \( \tau_s \); see Figure 1. The first case, \( QT = QR = 0.2 \) (Figure 1(a)), is unstable in the limit of zero explicit gas drag \( (\tau_s \to \infty) \) in the classical SGW way. Observe that in this case, increasing overall gas drag (lowering \( \tau_s \)) reduces the growth rate (solid line). If one artificially retains only the azimuthal gas drag (dashed line; radial drag turned off) then the instability is enhanced in accordance with the discussion of the preceding paragraphs. If
only radial drag is retained (chain dotted; azimuthal drag turned off) we see that the instability is further weakened compared to the case with both drag terms active. We thus conclude that azimuthal/radial drag promotes/hinders instability. This is easy to appreciate: radial gas drag slows radial compressive and rarefactive particle motions.

The second case, \( Q_T = Q_R = 2 \) (Figure 1(b)), is neutrally stable without gas drag \( (\tau_s \to \infty) \) and with \( \hat{D} = 0 \), i.e., neutrally stable in the SGW limit. It is only for this case that one may say “gas drag assists instability.” Optimal enhancement occurs for \( \tau_s = 1 \). If only azimuthal drag is present (dashed line, radial drag suppressed) then the more drag the better, i.e., the optimum disappears, again in agreement with the discussion in earlier paragraphs. As \( \tau_s \) decreases, radial drag diminishes the instability, while azimuthal drag enhances it, leading to an optimum \( \tau_s \). When only radial drag is present, the maximum growth rate is zero and thus is not shown. An inspection of all three roots of the cubic shows that this makes physical sense. In the absence of gas drag there is a zero root and two purely oscillating roots. In the presence of radial drag alone the two oscillating roots are damped, while the zero root remains zero.

Next, let us introduce a little diffusivity, \( \hat{D} = 1.0 \). The case \( Q_T = Q_R = 0.2 \) (SGW unstable) is shown in Figure 1(c). Diffusivity does reduce growth rates (compare with Figure 1(a)) as expected, however, above a certain \( \tau_s \) \( (\approx 2) \) for this case azimuthal drag is stabilizing. This is contrary to our physical explanation and it will be suggested below that a gravito-diffusive instability mechanism is at play. Inspection of the individual roots showed that this behavior always corresponded to a pair of complex conjugate \( \gamma \) roots with non-zero oscillation frequency (overstability, \( \text{Im}(\gamma) \neq 0 \)). This can also be described as a pair of growing waves propagating radially inward and outward. The bump or local maximum in the solid line near \( \tau_s \approx 1 \) shows that gas drag assisted behavior is present to the left of the plot. This was not the case for \( \hat{D} = 0 \). That is, mass diffusivity allows the gas drag assisted instability to occur at lower \( Q_T \). If we add mass diffusivity to the case \( (Q_T = Q_R = 2) \) that is gas drag assisted without diffusivity, we observe (Figure 1(d)) the same qualitative behavior to the left of the plot as in \( \hat{D} = 0 \) case (apart from an expected decrease in peak growth rate). However, to the right of the plot, the gravito-diffusive mode appears for \( \tau_s \) above \( \approx 200 \), and closer inspection reveals that for \( \tau_s \gtrsim 400 \) the diffusive case has a larger growth rate that the non-diffusive case. Note that the gravito-diffusive mode survives in the gas-free limit \( \tau_s \to \infty \) when there is some mechanism of mass diffusivity still active to keep \( \hat{D} \) finite. An overstability with momentum diffusivity (viscosity) is known to exist in the context of planetary rings (for a review see Schmidt et al. 2009). In the present case, we have encountered overstability with mass diffusivity. We shall not speculate on whether it is relevant in other astrophysical contexts. Varying \( Q_T \) in the \( \tau_s \to \infty \) limit (for the razor thin case, \( Q_R = 0 \)) shows (see Figure 2) that diffusivity allows the Toomre instability to occur at all \( Q_T \), albeit with decreasing growth rate as \( Q_T \) increases. It is for these reasons that we believe this mode (which is overstable in nature) to be gravito-diffusive in nature and invite the reader to explain it mechanistically.

Since \( \hat{D} \) decreases with increasing \( \tau_s \) in the case of stirring by a gas, the question arises whether the gravito-diffusive mode can be realized for particles in a solar nebula. To investigate this, a search was conducted in the range \( 10^{-3} \leq \alpha_g \leq 10^{-3} \), \( 0.1 \leq a \leq 100 \) cm, and \( 0.1 \leq R \leq 100 \) AU. The most amplified mode never had \( \text{Im}(\gamma) \neq 0 \).

Figures 1(e) and (f) show that when \( \hat{D} = 100 \) the distinction between high \( Q_T \) and low \( Q_T \) disappears. Both now display gas drag assisted behavior except at sufficiently large \( \tau_s \) where the gravito-diffusive mode appears.

A different mechanical interpretation of the instability has been given by Goodman & Pindor (2000) as follows. Consider a localized clump of positive density perturbation in the particle layer in the form of an axisymmetric ring. At the inner edge of the ring, particles will feel an extra outward gravitational force and, to maintain radial equilibrium in absence of the gas, will need to travel at less than Keplerian speed. In the presence of gas the particles will therefore be energized by gas drag and drift toward the center of the ring. At the outer edge of the ring, particles will feel an extra inward gravitational force and will need to travel at faster than Keplerian speed to maintain radial equilibrium. If gas drag is now turned on, they will be slowed down and drift inward.

Compared to our explanation that of Goodman & Pindor (2000) assumes equilibrium between gravitational forces and...
Table 1 summarizes the five models considered in paper II and repeated here with the mass diffusivity term added. The reference model has the following parameter values: particle size, \( a = 1 \) cm, enhancement/depletion factors (relative to the minimum mass solar nebula) of gas, \( f_g = 1 \), and of particles, \( f_p = 1 \). Figure 3 is analogous to Figure 4 (top) in paper II and shows \( e \)-fold growth times for the reference case and various weak to moderate values of \( \alpha_g \). (Here and henceforth, growth times are plotted for the fastest mode among all three roots and all wavenumbers.) The lower set of curves (of lighter weight) are for zero mass diffusivity. The upper curves (heavier weight) are for the case with diffusivity. One concludes that including mass diffusivity increases growth times by about two orders of magnitude in the asteroid belt region at 3 AU. It appears that in this region turbulence levels must be such that \( \alpha_g < 10^{-7} \) to get growth times smaller than \( 10^6 \) years, a typical disk lifetime, and the situation becomes worse in the terrestrial planet region. However, the instability could perhaps play a role in the outermost regions of the disk for weak turbulence levels.

For the same turbulence levels as in Figure 3, we now consider the effect of particle size \( a \) at \( R = 3 \) AU; see Figure 4. Growth time (heavy lines) decreases for larger particles. Also shown is the characteristic drift time of the instability wave which also decreases with particle size but more slowly than the growth time giving rise to a possibility that \( t_{\text{grow}} < t_{\text{drift}} \). One observes that such a cross over occurs (for particle radii in the range of the plot) when the turbulence levels is sufficiently weak. Furthermore, the cross over occurs for smaller particles as the turbulence diminishes.

Paper II also considers the following four alternative models each of which makes the instability stronger: (1) an increase in the surface density of particles by a factor of \( f_p = 4 \), (2) a depletion in gas by a factor of \( f_g = 0.1 \), (3) an increase in the particle radius to \( a = 1 \) cm, and, finally, (4) a model designated “all” that incorporates all of these changes.

The “all” model, in which solids are enhanced relative to gas by a factor of 40 compared to the minimum mass solar nebula, produces the fastest growth times of all the models: these are

\[
\begin{array}{|c|c|c|}
\hline
\text{Name} & \text{Particle Size} & \text{Gas Ratio} & \text{Particle Ratio} \\
\hline
\text{Reference} & 1 \text{ mm} & 1.0 & 1.0 \\
f_g = 0.1 & 1 \text{ mm} & 0.1 & 1.0 \\
f_g = 4 & 1 \text{ mm} & 1.0 & 4.0 \\
a = 1 \text{ cm} & 1 \text{ cm} & 1.0 & 1.0 \\
\text{All} & 1 \text{ cm} & 0.1 & 4.0 \\
\hline
\end{array}
\]
Figure 7. Maximum turbulence level $\alpha_g$ the instability can tolerate. There are two curves for each model. The upper curve (light weight) is for zero mass diffusivity while the lower one (heavy weight) is for non-zero diffusivity. The line-type denotes the condition that sets the maximum $\alpha_g$: solid, nebula lifetime; --- radial drift; ......, wavelength. The abscissa starts at $R = 0.3$ AU in (d) and $R = 0.2$ AU in (e) to ensure a particle Reynolds number $\geq 1$.

plotted in Figure 5 for larger turbulence intensities $\alpha_g$ than in the previous figure up to the (global) value $\alpha_g \sim 0.01$ required to account for observed disk accretion rates. The increase in growth time at 3 AU from the non-diffusive to the diffusive treatment is now just less than an order of magnitude. Thus, a solids-enhanced nebula is much less affected by turbulent diffusivity. This is not surprising because in the limit of vanishing gas density, none of the bothersome gas-related obstacles to GI will occur. For the standard turbulence intensity of $\alpha_g = 0.01$ the instability is not strong enough ($e$-fold time barely less than a million years). It should be noted that unlike the reference model, the “all” model produces a peak in growth rate at about 7 AU.

For the same turbulence levels as in Figure 5 we now consider the effect of particle size. Figure 6 shows growth times (heavy lines) versus particle size for the particle enriched and gas depleted case ($f_g = 0.1, f_p = 4$). Again, these should be compared with the thin lines which show the corresponding drift time. One concludes that there exists a range of particles sizes for which $t_{\text{grow}} < t_{\text{drift}}$ only if the turbulence is weaker than $\alpha_g \lesssim 10^{-3}$.

Paper II defines a “viable” instability as the satisfaction of three conditions. (1) $e$-folding times reasonably smaller than the disk lifetime, in particular $t_{\text{grow}} < 10^5$ yr. (2) Growth times less than the drift time: $t_{\text{grow}} < t_{\text{drift}}$ where $t_{\text{drift}}$ is given in Equation (31) of paper II. (3) Wavelength of the fastest mode reasonably shorter than the radius: $\lambda_f < R/2$. Figure 7 shows the maximum value of turbulence intensity $\alpha_g$ that is still able to produce a viable GI of solids. The condition that is violated when $\alpha_g$ exceeds the maximum value is depicted using different line types: solid, dashed, and dotted for the three conditions, respectively. One can see that under the favorable conditions represented by the “all” model, the instability becomes non-viable in the terrestrial planet region when the turbulence level exceeds $\alpha_g = 10^{-4}$, and everywhere when $\alpha_g > 10^{-3}$.

Finally, Figure 8 shows the relative error in the maximum growth rate given by the approximate formula (16) valid for
small growth rates compared to stopping times. The value plotted is the relative error in the maximum growth rate with respect to wavenumber. For the two cases shown, the approximate formula gives answers correct to within 10% if $\gamma_{\text{approx}} \lesssim 10^{-3}$.

4. CONCLUSION

Under conditions of the minimum mass solar nebula at the distance of the terrestrial planets and the asteroid belt, the gas-drag mediated instability remains viable for chondrule sized particles only for very weak turbulence levels $\alpha_g < 10^{-8}$. Even under the optimistic conditions of enhancement of the ratio of particle column density to gas column density by a factor of 40 and 1 cm particles (both of which promote the instability) the instability remains viable only if $\alpha_g < 10^{-4}$. Things look a little more optimistic further out. At 10 AU, under standard conditions the instability becomes too slow compared to nebula lifetime when $\alpha_g > 10^{-7}$ while under the most optimistic conditions, viability of the instability is destroyed by exceedingly large wavelength for $\alpha_g > 2 \times 10^{-4}$. Clearly, research should be undertaken to pin down turbulence levels at the disk mid-plane.

We thank the referee for useful suggestions and pointing out several typographical errors and one logical error in the original manuscript. We also thank Dr. A. Youdin for suggestions that improved the paper.

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Figure 8. Locus of the relative error in the growth rate (maximized over wavelength) predicted by the approximate formula (16) as compared with the actual value (separately maximized over wavelength). Each locus was obtained by varying the radius $R$ and plotting the relative error vs. $\gamma_{\text{approx}} \tau_s$. $D \neq 0$ for both cases: \( \cdots \cdots \) “all” model and $\alpha_g = 10^{-5}$; \( \cdots \cdots \) $f_p = 4$ model and $\alpha_g = 10^{-7}$. 

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