In this paper we address the following question: do the recent advances in the orbit determination of the major natural satellites of Saturn obtained with the analysis of the Cassini mission allow to detect the general relativistic gravitoelectric orbital precessions of the mean longitudes of such moons? The answer is still negative. The present-day down-track accuracy is adequate for Mimas, Enceladus, Tethys, Dione, Rhea and Titan and inadequate for Hyperion, Iapetus and Phoebe. Instead, the size of the systematic errors induced by the mission modelling in the key parameters of the Saturnian gravitational field like the even zonal harmonics \( J^2 \) are larger than the relativistic down-track shifts by about one order of magnitude, mainly for the inner satellites like Mimas, Enceladus, Tethys, Dione, Rhea, Titan and Hyperion. Iapetus and Phoebe are not sensibly affected by such kind of perturbations. Moreover, the bias due to the uncertainty in Saturn’s GM is larger than the relativistic down-track effects for all such moons up to two orders of magnitude (Phoebe). Thus, it would be in principle possible to separately analyze the mean longitudes of each satellite. Proposed linear combinations of the satellites’ mean longitudes would allow to cancel out the impact of the mission modelling in the low-degree even zonal harmonics and GM, but the combined down-track errors would be larger than the combined relativistic signatures by a factor \( 10^3 \).

Keywords: Cassini spacecraft; general relativistic orbit precessions; Saturnian system of natural satellites

1. Introduction
A satisfactorily empirical corroboration of a fundamental theory requires that as many independent experiments as possible are conducted by different scientists in different laboratories. Now, general relativity is difficult to test, especially in the weak-field and slow-motion approximation, valid, e.g., in our Solar System, both because the relativistic effects are very small and the competing classical signals are often quite larger. Until now, Solar System tests of general relativity accurate to better than 1% have been performed by many independent groups only in the gravitational field of the Sun by checking the effects induced by the gravitoelectric
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Schwarzschild part of the space-time metric on the propagation of electromagnetic waves and planetary orbital motions. It is as if many independent experiments aimed to measure fundamental physical effects were conducted always in the same laboratory. Thus, it is worthwhile to try to use different laboratories, i.e., other gravitational fields, to perform such tests, even if their outcomes should be less accurate than those conducted in the Sun's field. In principle, the best candidates other than the Sun are the giant planets of the Solar System like Jupiter and Saturn. To date, the only investigations of this kind are due to Hiscock and Lindblom, who preliminarily analyzed the possibility of measuring the Einstein pericentre precessions of some of the natural satellites of Jupiter and Saturn, and to Iorio and Lainey who investigated the measurability of the Lense-Thirring precessions in the system of the galilean satellites of Jupiter with modern data sets.

The aim of this paper is to investigate if the recent improvements in the ephemerides of the major Saturnian satellites (see Table 1 for their orbital parameters) obtained from the Cassini data allow to detect at least the general relativistic gravitoelectric precessions of their orbits.

Table 1. Orbital parameters of the major Saturnian satellites: $a$ is the semi-major axis, $e$ is the eccentricity and $i$ is the inclination, referred to the local Laplace planes (http://ssd.jpl.nasa.gov/?sat_elem).

| Satellite | $a$ (km) | $e$  | $i$ (deg) |
|-----------|---------|-----|---------|
| Mimas     | 185540  | 0.0196 | 1.572   |
| Enceladus | 238040  | 0.0047 | 0.009   |
| Tethys    | 294670  | 0.0001 | 1.091   |
| Dione     | 377420  | 0.0022 | 0.028   |
| Rhea      | 527070  | 0.0010 | 0.331   |
| Titan     | 1221870 | 0.0288 | 0.280   |
| Hyperion  | 1500880 | 0.0274 | 0.630   |
| Iapetus   | 3560840 | 0.0283 | 7.489   |
| Phoebe    | 12947780| 0.1635 | 175.986 |

2. The relativistic effects investigated

The Einstein pericentre precession is undoubtedly the most famous relativistic orbitale effect, but it is not the only one. Indeed, also the mean anomaly $M$ undergoes

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$a$ In regard to themuch smaller gravitoelectric Lense-Thirring precessions of the planetary orbits, they lie just at the edge of the present-day accuracy in planetary ephemerides and have recently been found in agreement with the latest measurements, although the errors are still large. A 6% test performed in the gravitational field of Mimas with the MGS spacecraft has recently been reported in Ref. 3.

$b$ In their original papers Lense and Thirring proposed to use some of the satellites of Jupiter and Saturn.
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A secular advance which is even larger than that experienced by the pericentre. For small eccentricities it is

\[ M_{-G_E} = \frac{3nGM}{c^2a} \frac{1}{1-e^2} \]

where \( G \) is the Newtonian constant of gravitation, \( M \) is the mass of the central body which acts as source of the gravitational field, \( a, e \) and \( n = \frac{GM}{c^2} = a^2 \) are the semi-major axis, the eccentricity and the Keplerian mean motion, respectively, of the satellite's orbit. Eq. (1), which yields a rate of -130 arcseconds per century for Mercury, has been obtained by using the standard isotropic radial coordinate \( r \) related to the Schwarzschild coordinate \( r^* \) by \( r^* = r(1 + \frac{GM}{c^2}r^2) \). The validity of Eq. (1) has also been numerically checked by integrating over 200 years the Jet Propulsion Laboratory (JPL) equations of motion of all the planets of the Solar System, which are written in terms of just the standard isotropic radial coordinate, with and without the gravitoelectric \( 1=c^2 \) terms in the dynamical force models in order to single out just the post-Newtonian gravitoelectric effects (Einstein, private communication, 2004). The obtained precessions fully agree with those obtained from Eq. (1).

As a consequence, the mean longitude \( \lambda \) = \( \lambda_0 + n \cdot M \), which is one of the non-singular orbital elements used for orbits with small eccentricities and inclinations like those of most of the Solar System main bodies as just the Saturnian satellites (see Table 1), experiences a secular advance

\[ \lambda_{-G_E} = \frac{6nGM}{c^2a} \]  

It is twice the pericentre advance. On the other hand, because of \( M \) in the definition of the systematic uncertainty in the Keplerian mean motion \( n \) must also be accounted for if a secular rate must be extracted from the analysis of such an orbital element.

3. The possibilities offered by the Saturnian system

Is it possible to measure the gravitoelectric orbital precessions of the natural satellites of Saturn, in view of the recent re-exports of their ephemerides obtained by analyzing the first data from the Cassini spacecraft? In this Section we will address this problem in detail. To this aim, we must confront the magnitude of the relativistic effects of interest with the major systematic errors induced by classical forces having the same signatures (Section 3.1), and with the currently available measurement accuracies (Section 3.2).

3.1. The reduction of the impact of the even zonal harmonics

In regard to the first issue, a major source of systematic bias is represented by the even zonal harmonic coefficients \( J_{2n} \); \( n = 2;4;6;\ldots \) of the multipolar expansion of the

\[ J_{2n} \]  

Here \( \lambda \) and \( \mu \) are the argument of pericentre and the longitude of the ascending node, respectively.
Saturn's gravitational potential. Indeed, they induce secular precessions on
\[ \text{even zonals} = -\frac{1}{2}J_i; \]  
(3)
where the coefficients \( -\frac{1}{2} \) are expressed in terms of the satellite's orbital elements \( a; e; i \) and the mass and the radius of the central body. Such classical advances must be accurately modelled because they are much larger than the relativistic ones. Even the latest determinations of the Saturnian gravity field from some Cassini data sets\(^6\) show that the currently available models of the even zonal harmonics do not reach the required accuracy to allow for a measurement of the Einstein precessions. This fact is clearly shown by Table 2. Indeed, we can note that for

\begin{table}[h]
\begin{tabular}{lcccc}
Satellite & \( -\epsilon_E \) & \( -\epsilon_J \) & \( -\epsilon_J \) & \( n_G M \) \\
Mim as & -684.670 & 5926.552 & 5465.353 & 793.461 \\
Enceladus & -367.207 & 2478.985 & 1390.913 & 546.018 \\
Thetys & 215.374 & 1173.870 & 429.231 & 396.440 \\
Dione & -116.003 & 493.910 & 110.229 & 273.492 \\
Rhea & -50.334 & 153.451 & 17.558 & 165.721 \\
TITAN & 6.152 & 8.101 & 0.172 & 46.950 \\
Hyperion & -3.679 & 3.942 & 0.055 & 34.487 \\
Iapetus & -0.424 & 0.186 & 0 (10 \(^{-4}\)) & 9.437 \\
Phoebe & 0.016 & 0.002 & 0 (10 \(^{-7}\)) & 1.361 \\
\end{tabular}
\end{table}

\( M\) in as, Enceladus and Thetys the im modelled precessions due to \( J_2 \) and \( J_4 \) are larger than the relativistic rates by about one order of magnitude. For Dione, Rhea, Titan and Hyperion the bias due to \( J_4 \) is smaller than the relativistic signal. For Iapetus and Phoebe the even zonal harmonics of Saturn do not represent a problem. In addition to the bias due to the \( f|_{\epsilon} \) for the mean longitude there is also the systematic error \( n_G M = (G M) = 4GM a^3 \) induced on the Keplerian mean motion \( n \) by the uncertainty in the Saturn's GM which, as can be seen, is larger than the relativistic shifts for all the satellites.

A way to overcome the problem of the impact of the even zonals is represented by the linear combination approach\(^9\) summarized in the following. Let us write down the expressions of the measured residuals\(^6\) of the mean longitudes \( m_{\text{meas}} \) of

\( \text{The Keplerian orbital elements are not directly observable, so here we are using the words 'measured residuals' in an improper sense. We mean, instead, a set of solve-for parameters of a suitable} \)
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N chosen satellites in terms of the mismodelled Keplerian mean motions, of the classical precessions due to the mismodelled parts of the even zonal harmonics, and of the relativistic precessions, assumed as totally unmismodelled features of motion

\[ n_{\text{m e a s}} = \frac{1}{4G M a^2} \]

where \( n \) are the coefficients of the classical precessions of degree \( j \) of the mean longitude and

\[ n_{\text{m e a s}} = n_{j;1;N} \quad (GM) + \frac{2}{2} \quad n_{j;1;N} \quad J_j + n_{j;1;N} \quad G_E \quad j = 1;2;N; \]

(4)

where \( n \rceil \) are the coefficients of the classical precessions of degree \( j \) of the mean longitude and

\[ n_{\text{m e a s}} = \frac{1}{4G M a^2} \]

(5)

If we look at Eq. (4) as a homogeneous algebraic system of \( N \) linear equations in the \( N \) unknowns \( f \); \( J_j \); \( g \) and solve it for \( f \), we get a linear combination of the satellites' mean longitude residuals which, by construction, is independent of \( (GM) \) and the first \( 2 \) even zonal harmonics, and is sensitive just to the relativistic signatures and to the perturbations of the remaining, uncanceled even zonal harmonics. The cancellation of the low-degree even zonals is important also because in this way one avoids any possible a priori 'imprint' of the relativistic effects they themselves via such spherical harmonics. Indeed, the Saturnian gravity field models obtained as least-square solutions using all the available data from the spacecraft encounters and the satellites' motions around Saturn, so that the relativistic effects they themselves are included in the solved-for parameters like \( fJg \).

A possible combination which cancels out \( J_2 \), \( J_4 \) and \( (GM) \) is

\[ \frac{P_{\text{m e a s}}}{-T_{\text{m e a s}}} + k_1 \frac{P_E}{-T_{\text{m e a s}}} + k_2 \frac{R_E}{-T_{\text{m e a s}}} + k_3 \frac{T_{\text{m e a s}}}{-T_{\text{m e a s}}}; \]

with

\[ k_1 = 0.1296; \]

\[ k_2 = 0.0224; \]

\[ k_3 = 0.0067; \]

\[ \frac{P_{\text{m e a s}}}{-T_{\text{m e a s}}} + k_1 \frac{P_E}{-T_{\text{m e a s}}} + k_2 \frac{R_E}{-T_{\text{m e a s}}} + k_3 \frac{T_{\text{m e a s}}}{-T_{\text{m e a s}}} = 0.0237 \quad 0^2 \]

(7)

As can be argued from Table 2, the impact of the remaining uncanceled even zonal harmonics \( J_6;J_8 \) is negligible.
3.2. The present-day orbital accuracy

In regard to the feasibility of such a measurement, it is useful to consider the down-track shifts and the present-day accuracy according to the latest Saturnian ephemerides SAT240\(^1\)\(^2\) over 81 years. The results are in Table 3. It can be noted that the situation is presently not favorable just for the satellites used for the combination of Eq. (6). This fact reflects on Eq. (6) itself: indeed the total uncertainty calculated by summing in quadrature the errors of Table 3 with the coefficients (7)

\[
\begin{align*}
\sum_{\text{meas}} \left[ (P_{\text{PH}})^2 + (L_{\text{PH}})^2 + (L_{\text{GE}})^2 + (L_{\text{GE}})^2 + (L_{\text{GE}})^2 + (L_{\text{GE}})^2 + (L_{\text{GE}})^2 + (L_{\text{GE}})^2 ) \right]^{1/2} 
\end{align*}
\]

is \(10^3\) times larger than the gravitoelectric shift over 81 years.

About the use of the inner satellites, for which the relativistic shifts are larger than the errors, Table 2 shows that it is not possible to use their mean longitudes without combining them because both the even zonal mismodeled shifts and the bias of \(n\) are quite larger. On the other hand, it turns out that also the linear combination approach does not yield good results. Indeed, the root-sum-square of the errors of a combination with Enceladus, Dione, Rhea and Titan is \(10^3\) times larger than the relativistic combined shifts.

4. Discussion and conclusions

In regard to the measurability of the general relativistic gravitoelectric orbit advances in the Saturn’s system of natural satellites, it turns out that the present-day in proven ephemerides by the Cassini spacecraft are not yet su cient to detect such post-Newtonian effects.

As a general rule, in regard to the single satellites, an about one order of magnitude in proven error in the knowledge of the parameters of the Saturnian gravity field...
Does Cassini allow to measure relativistic orbit elements in the Saturnian system of satellites? Like the even zonal harmonics J and GM would be required to make the competing classical biases smaller than the relativistic precessions. This would make possible to successfully analyze separately the mean longitudes of each of Mimas, Enceladus, Tethys, Dione, Rhea and Titan because the present-day down-track errors are already smaller than the relativistic shifts. The use of the mean longitudes of Hyperion, Iapetus and Phoebe, for which the Saturnian even zonals represent a relatively less important problem, is strictly related to a one-two orders of magnitude improvement in the down-track parts of their orbits.

The linear combination approach yields, in principle, good results in reducing the impact of the mismodelling in the even zonals and GM. The main problem with such strategy relies in the down-track orbital accuracy. Indeed, the down-track errors weighted by their combination’s coefficients must be summed in quadrature, while the weighted relativistic shifts are combined with their proper own signs, so that with the present-day precision the overall error amounts to $10^3$.

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