Motzkin numbers and flag codes

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Abstract

Motzkin numbers have been widely studied since they count many different combinatorial objects. In this paper we present a new appearance of this remarkable sequence in the network coding setting through a particular case of multishot codes called flag codes. A flag code is a set of sequences of nested subspaces (flags) of a vector space over the finite field $F_q$. If the list of dimensions is $(1,\ldots,n-1)$, we speak about a full flag code. The flag distance is defined as the sum of the respective subspace distances and can be represented by means of the so-called distance vectors. We show that the number of distance vectors corresponding to the full flag variety on $F_q^n$ is exactly the $n$-th Motzkin number. Moreover, we can identify the integer sequence that counts the number of possible distance vectors associated to a full flag code with prescribed minimum distance.

Keywords: Motzkin numbers, Motzkin paths, flag codes, flag distance, distance vectors.

1 Introduction

Network coding is the most efficient way to send data across a network modelled as a directed acyclic multigraph with multiple senders and receivers. The key is that the intermediate nodes can perform random linear combinations of the incoming inputs. This improves considerably the information flow although it can also lead to error propagation and erasures (see [1]). To solve this problem, in [10] the authors consider subspaces of $F_q^n$ as codewords and subspace codes as collections of subspaces. In particular, if all the subspaces have the same dimension, we have constant dimension codes. Consult [19] to have an overview of the most representative works in this subject.

On the other side, subspace codes can be considered as one-shot subspace codes since sending a codeword (a subspace) requires just a channel use. This gives rise to the idea, first suggested in [15], of using the channel several times to get multishot subspace codes. Under this approach codewords are sequences of subspaces. In particular, it is possible to consider constant type flags, that is, sequences of nested subspaces with fixed dimensions as it was proposed in

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[12]. In this seminal paper, collections of flags are called flag codes and they are presented as a generalization of constant dimension codes. The recent works [4, 5, 6, 11] deal with different questions related to the parameters and construction of flag codes.

In this paper we focus on the distance parameter associated to full flag codes, that is, those whose sequence of dimensions is $(1, \ldots, n-1)$. As usual in the multishot context, the distance between flags is computed as the sum of the subspace distances corresponding to each shot, which provokes that many different combinations of them can give the same flag distance value. This fact was carefully pointed out in [4], where the authors introduce the notion of distance vector (associated to a given distance value) in order to complete the information concerning the distance of pairs of flags. In this framework, it naturally arises the question of knowing in how many different ways the distance between a couple of flags can be distributed. We provide the answer to this question by showing that the cardinality of the set of distance vectors corresponding to the full flag variety on $\mathbb{F}_q^n$ is given by the $n$-th Motzkin number.

Motzkin numbers have been widely investigated as they appear in a great variety of combinatorial objects (see [17, 18] for more details). Here we look at them in terms of certain lattice paths from $(0, 0)$ to $(n, 0)$ that consist of horizontal steps $(1, 0)$, up steps $(1, 1)$ and down steps $(1, -1)$ and never goes below the x-axis (Motzkin paths). Thus, we exhibit a correspondence between distance vectors and Motzkin paths that results to be crucial for our purposes. This idea has been inspired by the combinatorial approach to flag codes developed in [3] and some interesting talks with Paulo Almeida and Alessandro Neri.

The remain of the paper is organized as follows: in Section 2 we remember some basic definitions and results on Motzkin numbers, subspace codes and flag codes. In Section 3 we recall the concept of distance vector associated to a couple of flags and we pose the problem of counting such objects. In Section 4 we establish a bijection between the set of distance vectors associated to the full flag variety on $\mathbb{F}_q^n$ and the set of Motzkin paths with $n$ steps which permits to compute the cardinality of the former set as well as the one of some important subsets of it. Moreover, we show that such a bijection takes the flag distance associated to a distance vector to the area under the corresponding Motzkin path. Finally, in Section 5 we propose some open questions that could be addressed taking into account our work.

2 Preliminaries

In this section we briefly recall some basic background on Motzkin numbers, subspace codes and flag codes.

2.1 Motzkin numbers

Let us consider the set of lattice paths in $\mathbb{Z}^2$ whose permitted steps are the up diagonal step $(1, 1)$ denoted by $U$, the down diagonal step $(1, -1)$, denoted by $D$, and the horizontal step $(1, 0)$, denoted by $H$. Let $U(x, y)$ represent the set
of all unrestricted lattice paths running from \((0, 0)\) to \((x, y)\) and using the steps \(U, D\) and \(H\).

**Definition 2.1.** A *Motzkin path* of length \(n\) is a lattice path in \(U(n, 0)\) that 
ever runs below the \(x\)-axis. We denote by \(\mathcal{M}_n\) the set of Motzkin paths of 
length \(n\). A *Dyck path* is a Motzkin path that does not contain horizontal steps. The 
set of Dyck paths of length \(2n\) will be denoted by \(\mathcal{D}_n\).

It is clear that, if horizontal steps are not allowed, the length of a Motzkin 
path (a Dyck path, in this case) must be even. On the other hand, it is well 
known that the cardinality of the set \(\mathcal{M}_n\) is the \(n\)-th Motzkin number \(M_n\) 
(sequence A001006 in [17]) and the cardinality of \(\mathcal{D}_n\) is the \(n\)-th Catalan number 
\(C_n\) (sequence A000108 in [17]). More precisely, we can compute 
\[M_0 = C_0 = 1\] and 
\[M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-k-2}, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} \quad (n \geq 1).\]

The Motzkin numbers sequence, whose first ten terms are 
\[1, 1, 2, 4, 9, 21, 51, 127, 323, 835,\]

was introduced by Theodor Motzkin [14], while counting possible sets of non-intersecting 
chords joining some of \(n\) points on a circle. Since this seminal paper 
it has been the subject of numerous studies over the last forty years. See [2] 
and [7] for a couple of the earliest surveys where the authors exhibit different 
combinatorial objects counted by this sequence, including Motzkin paths.

A commonly used and concise way of representing a Motzkin path \(p \in \mathcal{M}_n\) 
is by its corresponding Motzkin word (see [8], for instance).

**Definition 2.2.** A *Motzkin word* of length \(n\) is a sequence of \(n\) letters in the 
apphabet \(\{U, D, H\}\), namely \(p = p_1 p_2 \ldots p_n\), with the constraint that the number 
of occurrences of the letter \(U\) is equal to the number of occurrences of the letter 
\(D\) and, for every \(i \in \{1, \ldots, n\}\), the number of occurrences of \(U\) in the subword 
\(p_1 p_2 \ldots p_i\) is not smaller than the one of \(D\).

In the following, we will not distinguish between a Motzkin path and the 
corresponding Motzkin word.

![Figure 1: The Motzkin path \(p = UUHUDDHDH\) in \(\mathcal{M}_9\).](image)
Remark 2.3. Observe that if we see the letters $U$ and $D$ as an open and closed parenthesis respectively, a Motzkin path is a word in $\{U,H,D\}$ such that the subword on the letters $U$ and $D$ forms a balanced parenthesization, that is, the pairs of parantheses on them are correctly matched. This viewpoint will be useful for us in Section 4.

Example 2.4. To the Motzkin path $p = UUHUDDHDH$ in Figure 1 it corresponds the parenthesization $((()))$.

Definition 2.5. The area of a Motzkin path $p$, denoted by $A(p)$, is the area of the region above the $x$-axis and below the path $p$.

Remark 2.6. For the Motzkin path in Figure 1 we have $A(p) = 12$. It is clear that for any $p \in M_n$, it holds $0 \leq A(p) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. In fact, if we denote by $M_n(k)$ the set of Motzkin paths with length $n$ and area $k$ with $0 \leq k \leq \left\lfloor \frac{n^2}{4} \right\rfloor$, the cardinality of that set, denoted $T(n,k)$, is given by the sequence A129181 in [17]. As far as we know, at the moment there is not a recurrence formula to calculate this sequence. We present here a table that covers it for values $n \in \{0,1,\ldots,8\}$.

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0   | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 1   | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 2   | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 3   | 1  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 4   | 1  | 3  | 3  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |
| 5   | 1  | 4  | 6  | 4  | 3  | 2  | 1  |    |    |    |    |    |    |    |    |    |
| 6   | 1  | 5  | 10 | 10 | 8  | 7  | 5  | 3  | 1  | 1  |    |    |    |    |    |    |
| 7   | 1  | 6  | 15 | 20 | 19 | 18 | 16 | 12 | 8  | 6  | 3  | 2  | 1  |    |    |    |
| 8   | 1  | 7  | 21 | 35 | 40 | 41 | 41 | 36 | 29 | 23 | 18 | 12 | 9  | 5  | 3  | 1  |

Table 1: First terms of the sequence A129181.

Of special interest is the set of Motzkin paths that never touch the $x$-axis in intermediate steps or the set of Motzkin paths that do no have horizontal steps on the $x$-axis.

Definition 2.7. A return of a Motzkin path $p \in M_n$ is a point of $p$ different from $(0,0)$ and $(n,0)$ and belonging to the $x$-axis. We say that $p$ is an irreducible or elevated Motzkin path if it does not have returns. We denote by $E_n$ the set of elevated Motzkin paths in $M_n$. Its cardinality is the integer $E_n$ given by

$$E_0 = E_1 = 0, \quad E_n = M_{n-2}, \quad n \geq 2.$$ (1)

Example 2.8. The lattice path $p = UUHUDDHD$ is an elevated Motzkin path in $E_8$.

In terms of its Motzkin word, a Motzkin path is elevated if the number of occurrences of $U$ is greater than the number of occurrences of $D$ in every subword $p_1 \ldots p_i$, for $1 \leq i \leq n - 1$. On the other hand, a Motzkin path with returns contains at least a subword of the form either $DU$ or $DH$. 

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Definition 2.9. A Motzkin path of length \( n \) without horizontal steps on the \( x \)-axis will be called a Riordan path of length \( n \). The set of Riordan paths of length \( n \) is denoted by \( \mathcal{R}_n \). The cardinality of \( \mathcal{R}_n \) is given by the Riordan numbers \( R_n \). In [17] these numbers correspond to the sequence A005043.

Remark 2.10. In the literature related to Motzkin paths there are many papers that deal with the problem of counting variations of them in terms of the appearance of different elements such as returns, horizontal steps, peaks (subwords of the form \( UD \)) and valleys (subwords of the form \( DU \)), for instance. In Section 4 we will consider some of them (elevated Motzkin paths and Riordan paths) as translation of particular situations coming from the context of flag codes.

2.2 Subspace codes and flag codes

Consider a prime power \( q \) and a positive integer \( n \geq 2 \). Let \( \mathbb{F}_q \) be the finite field with \( q \) elements. The set of subspaces of the vector space \( \mathbb{F}_q^n \) can be seen as a metric space by using different metrics. Among them, here we will use the so-called injection distance.

Definition 2.11. The injection distance between two subspaces \( U, V \subseteq \mathbb{F}_q^n \) is defined as

\[
d_I(U, V) = \max\{\dim(U), \dim(V)\} - \dim(U \cap V). \tag{2}\]

In particular, if \( U, V \) have the same dimension, say \( 1 \leq k < n \), then we have

\[
d_I(U, V) = k - \dim(U \cap V) = \dim(U + V) - k. \tag{3}\]

Hence, if for any \( 1 \leq k < n \) we denote by \( \mathcal{G}_q(k, n) \) the Grassmannian, that is, the set of \( k \)-dimensional subspaces of \( \mathbb{F}_q^n \), we can consider error-correcting codes in \( \mathcal{G}_q(k, n) \) as follows.

Definition 2.12. A constant dimension code \( C \) of length \( n \) and dimension \( k \) is a nonempty subset of \( \mathcal{G}_q(k, n) \). The minimum distance of \( C \) is defined as

\[
d_I(C) = \min\{d_I(U, V) \mid U, V \in C, U \neq V\}
\]

whenever \( |C| \geq 2 \). In case \( |C| = 1 \), we put \( d_I(C) = 0 \).

Note that the minimum distance of a code \( C \) is an integer such that

\[
0 \leq d_I(C) \leq \begin{cases} k & \text{if } 2k \leq n, \\ n - k & \text{if } 2k \geq n. \end{cases} \tag{4}\]

Remark 2.13. Another frequent metric used when working with subspace codes of \( \mathbb{F}_q^n \) is the subspace distance. It is given by

\[
d_S(U, V) = \dim(U + V) - \dim(U \cap V). \tag{5}\]

Observe that, if \( U, V \in \mathcal{G}_q(k, n) \), then \( d_S(U, V) = 2(k - \dim(U \cap V)) = 2d_I(U, V) \). Hence, in the context of constant dimension codes, the injection distance and the subspace distance are equivalent metrics. Consult [19] and the references therein for more information on this class of codes.
Let us now consider flags on $\mathbb{F}_q^n$ with the same type, that is, sequences of nested subspaces of $\mathbb{F}_q^n$ whose list of dimensions is fixed. In this way we get flag codes (of constant type). This idea was first proposed in [12]. Let us recall the basics on this family of codes.

**Definition 2.14.** A flag $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ on $\mathbb{F}_q^n$ is a sequence of $\mathbb{F}_q$-vector subspaces of $\mathbb{F}_q^n$ satisfying

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_r \subsetneq \mathbb{F}_q^n.$$  

The type of $\mathcal{F}$ is the vector $(\dim(\mathcal{F}_1), \ldots, \dim(\mathcal{F}_r))$. If it equals $(1, 2, \ldots, n-1)$, we say that $\mathcal{F}$ is a full flag. We say that $\mathcal{F}_i$ is the $i$-th subspace of $\mathcal{F}$.

The set of all the flags on $\mathbb{F}_q^n$ of a fixed type vector $(t_1, \ldots, t_r)$ is said to be the flag variety $\mathcal{F}_q((t_1, \ldots, t_r), n)$ or the full flag variety $\mathcal{F}_q(n)$, if the type vector is $(1, \ldots, n-1)$. The distance defined in (2) can be extended to the flag variety as follows: given two flags $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}_1', \ldots, \mathcal{F}_r')$ in $\mathcal{F}_q((t_1, \ldots, t_r), n)$, the (injection) flag distance between them is the value

$$d_\mathcal{F}(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^r d_1(\mathcal{F}_i, \mathcal{F}'_i). \quad (6)$$

**Remark 2.15.** The subspace distance $d_S$ defined in (5) can also be extended to the flag variety. Given $\mathcal{F}$ and $\mathcal{F}'$ as above, the sum of subspace distances

$$\sum_{i=1}^r d_S(\mathcal{F}_i, \mathcal{F}'_i) = 2d_\mathcal{F}(\mathcal{F}, \mathcal{F}') \quad (7)$$

is an equivalent distance to $d_\mathcal{F}$. In this paper we will always work with the injection flag distance and we will simply write flag distance.

**Definition 2.16.** A flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is a nonempty subset $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$. Its minimum distance is given by

$$d_\mathcal{F}(\mathcal{C}) = \min \{d_\mathcal{F}(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C}, \mathcal{F} \neq \mathcal{F}'\}$$

when $|\mathcal{C}| \geq 2$. If $|\mathcal{C}| = 1$, we put $d_\mathcal{F}(\mathcal{C}) = 0$.

For each dimension $t_i$ in the type vector of a flag code $\mathcal{C}$, we can consider the constant dimension code in the Grassmannian $\mathcal{G}_q(t_i, n)$ consisting of the set of the $i$-th subspaces of flags in $\mathcal{C}$. This set is called the $i$-projected code of $\mathcal{C}$ and we denote it by $\mathcal{C}_i$. It is clear that $|\mathcal{C}_i| \leq |\mathcal{C}|$ for every $i = 1, \ldots, r$. Moreover, if given an index $1 \leq i \leq r$ we have that $|\mathcal{C}_i| = |\mathcal{C}|$, we can deduce that for any couple of flags $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ we have that $\mathcal{F}_i \neq \mathcal{F}_i'$. This fact leads to the following definition.

**Definition 2.17.** We say that two flags $\mathcal{F}$ and $\mathcal{F}'$ of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ collapse at their $i$-th subspace if $\mathcal{F}_i = \mathcal{F}_i'$. On the other hand, two different flags $\mathcal{F}$ and $\mathcal{F}'$ are said to be disjoint if they do not collapse at any subspace, i.e., if $\mathcal{F}_i \neq \mathcal{F}_i'$, for every $1 \leq i \leq r$. Similarly, a flag code $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$ is disjoint if it consists of disjoint flags. In terms of the projected codes, a flag code $\mathcal{C}$ is disjoint if, and only if, $|\mathcal{C}_1| = \cdots = |\mathcal{C}_r| = |\mathcal{C}|$. 


3 Distance vectors: how to spread the flag distance

In sight of the flag distance definition given in (6), one realizes that a fixed distance value could possibly be obtained by adding different subspace distances combinations. This fact is reflected in the following example.

Example 3.1. Let \{e_1, e_2, e_3, e_4, e_5, e_6\} be the standard basis of the \(\mathbb{F}_q\)-vector space \(\mathbb{F}_q^6\). Consider the flag code \(\mathcal{C}\) of type \((1, 3, 5)\) on \(\mathbb{F}_q^6\) given by the set of flags:

\[
\begin{align*}
\mathcal{F}^1 &= \langle \{e_1\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_3, e_4, e_5\} \rangle, \\
\mathcal{F}^2 &= \langle \{e_5\}, \{e_4, e_5\}, \{e_1, e_2, e_4, e_5\} \rangle, \\
\mathcal{F}^3 &= \langle \{e_6\}, \{e_4, e_5, e_6\}, \{e_2, e_3, e_4, e_5, e_6\} \rangle, \\
\mathcal{F}^4 &= \langle \{e_2\}, \{e_2, e_5\}, \{e_2, e_3, e_4, e_5, e_6\} \rangle.
\end{align*}
\]

Observe that it holds:

\[
d_f(\mathcal{C}) = d_f(\mathcal{F}^2, \mathcal{F}^3) = 1 + 0 + 1 = 2 = 1 + 1 + 0 = d_f(\mathcal{F}^3, \mathcal{F}^4).
\]

We can follow that, even if the pairs \(\mathcal{F}^2, \mathcal{F}^3\) and \(\mathcal{F}^3, \mathcal{F}^4\) are both at distance 2, the first one presents a collapse in the second subspace, that is, \(\mathcal{F}^2 = \mathcal{F}^3\), whereas the second pair collapses at the third one. As a consequence, to totally capture the relative position of two flags, it is necessary to provide more precise information beyond the distance in absolute terms. In [4] the authors deal with this question by defining distance vectors. Let us recall this definition.

Definition 3.2. Given two different flags \(\mathcal{F}, \mathcal{F}'\) of type \(t = (t_1, \ldots, t_r)\) on \(\mathbb{F}_q^n\), their associated distance vector is

\[
d(\mathcal{F}, \mathcal{F}') = (d_f(\mathcal{F}_1, \mathcal{F}'_1), \ldots, d_f(\mathcal{F}_r, \mathcal{F}'_r)) \in \mathbb{Z}^r.
\]

Notice that the sum of the components of \(d(\mathcal{F}, \mathcal{F}')\) is the flag distance \(d_f(\mathcal{F}, \mathcal{F}')\) defined in (6). Given a positive integer \(n \geq 2\) and a type vector \(t = (t_1, \ldots, t_r)\), we denote by \(D(t,n)\) the maximum possible value of the flag distance in \(\mathcal{F}_q(t, n)\) that, as a consequence of (4), is

\[
D(t,n) = \sum_{t_i \leq \left\lfloor \frac{n}{2} \right\rfloor} t_i + \sum_{t_i > \left\lfloor \frac{n}{2} \right\rfloor} (n - t_i). \tag{8}
\]

In particular, the set \(\{0, 1, \ldots, D(t,n)\}\) contains all the possible values for the flag distance in \(\mathcal{F}_q(t, n)\). When working with the full type vector, we simply write

\[
D^n = \left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even}, \\ \frac{n^2 - 1}{4} & \text{if } n \text{ is odd} \end{cases} \tag{9}
\]

to denote the maximum possible distance between full flags on \(\mathbb{F}_q^n\).

Definition 3.3. Let \(d\) be an integer such that \(0 \leq d \leq D(t,n)\). We define the set of distance vectors associated to \(d\) for the flag variety \(\mathcal{F}_q(t, n)\) as

\[
\mathcal{D}(d, t, n) = \{d(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(t, n), \ d_f(\mathcal{F}, \mathcal{F}') = d\} \subseteq \mathbb{Z}^r.
\]

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On the other hand, the set of distance vectors for the flag variety $\mathcal{F}_q(t, n)$ is
\[ \mathcal{D}(t, n) = \{ d(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(t, n) \} \subseteq \mathbb{Z}^r. \]
and it holds
\[ \mathcal{D}(t, n) = \bigcup_d \mathcal{D}(d, t, n), \]
where $d$ takes all the integers between $0 \leq d \leq D(t,n)$. When working with the full flag variety, we drop the type vector and simply write $\mathcal{D}(d, n)$ and $\mathcal{D}(n)$, respectively.

At this point we propose several important questions: how many possible distance vectors could correspond to a given couple of arbitrary flags $\mathcal{F}$, $\mathcal{F}'$ on $\mathbb{F}^n_q$, that is, what is the cardinality of $\mathcal{D}(n)$? In particular, what happens if these flags do not collapse? What if they never share two consecutive subspaces? What if $\mathcal{F}$, $\mathcal{F}'$ belong to a flag code with prescribed minimum distance? We address these questions in the following section.

### 4 The bijection

In the remain of the paper we will always work with full flag codes. More precisely, we will show that the number of distance vectors associated to the full flag variety $\mathcal{F}_q(n)$ is given by the Motzkin number $M_n$.

Let us first recall those properties that characterize distance vectors. The following result is based on [3, Th. 3.8] and [4, Th. 3.9]. For completeness here we include a shorter proof adapted to the full type case.

**Theorem 4.1.** Consider integers $\delta_1, \ldots, \delta_{n-1} \geq 0$ and put $\delta_0 = \delta_n = 0$. Then $(\delta_1, \ldots, \delta_{n-1})$ is a distance vector in $\mathcal{D}(n)$ if, and only if,
\[ \delta_i \in \{ \delta_{i-1} - 1, \delta_{i-1}, \delta_{i-1} + 1 \}, \tag{10} \]
for all $1 \leq i \leq n$.

**Proof.** Suppose that $(\delta_1, \ldots, \delta_{n-1})$ is a distance vector. Then, there exists a pair of full flags $\mathcal{F}, \mathcal{F}' \in \mathcal{F}_q(n)$ satisfying $\delta_j = d_I(\mathcal{F}_j, \mathcal{F}_j')$, for every $1 \leq j \leq n - 1$. Let us see that $\delta_i \in \{ \delta_{i-1} - 1, \delta_{i-1}, \delta_{i-1} + 1 \}$. Note that $\delta_1, \delta_{n-1} \in \{0, 1\}$ given that $\delta_1, \delta_{n-1}$ are, respectively, distances between lines and hyperplanes of $\mathbb{F}^n_q$. Moreover, since $\delta_0 = \delta_n = 0$, the stated condition holds for both $i = 1, n$. Consider now any index $2 \leq i \leq n - 1$. Note that
\[ \dim(\mathcal{F}_{i-1} + \mathcal{F}'_{i-1}) \leq \dim(\mathcal{F}_i + \mathcal{F}_i') \leq \dim(\mathcal{F}_{i-1} + \mathcal{F}'_{i-1}) + 2. \]
Hence, $\dim(\mathcal{F}_i + \mathcal{F}_i') = \dim(\mathcal{F}_{i-1} + \mathcal{F}'_{i-1}) + k$, with $k \in \{0, 1, 2\}$, and we have
\[ \delta_i = d_I(\mathcal{F}_i, \mathcal{F}_i') = \dim(\mathcal{F}_i + \mathcal{F}_i) - i = \dim(\mathcal{F}_{i-1} + \mathcal{F}'_{i-1}) + k - i = \dim(\mathcal{F}_{i-1} + \mathcal{F}'_{i-1}) - (i - 1) + (k - 1) = d_I(\mathcal{F}_{i-1}, \mathcal{F}'_{i-1}) + (k - 1) = \delta_{i-1} + (k - 1). \]
where \( k - 1 \in \{-1, 0, 1\} \).

For the converse, assume that \( (\delta_1, \ldots, \delta_{n-1}) \) is a vector satisfying (10). This condition, along with the fact that \( \delta_0 = \delta_n = 0 \), implies \( \delta_i \leq \min\{i, n - i\} \), i.e., every \( \delta_i \) is an admissible value for the injection distance between \( i \)-dimensional subspaces of \( \mathbb{F}_q^n \). Using induction, we build flags \( F, F' \) in \( F_q(n) \) with \( d(F, F') = (\delta_1, \ldots, \delta_{n-1}) \). First of all, by means of (10), it is clear that \( \delta_1 \in \{0, 1\} \). If \( \delta_1 = 0 \), just take \( F_1 = F'_1 \). Otherwise, consider any two different lines \( F_1, F'_1 \).

Assume now that for some \( 1 \leq i < n - 1 \), we have found nested subspaces \( F_1 \subseteq \cdots \subseteq F_i \) and \( F'_1 \subseteq \cdots \subseteq F'_i \) such that \( \delta_j = d_I(F_j, F'_j) = \dim(F_j + F'_j) - j \), for every \( 1 \leq j \leq i \). Let us give convenient \( u, u' \in \mathbb{F}_q^n \) and subspaces \( F_{i+1} = F_i + \langle u \rangle, F'_{i+1} = F'_i + \langle u' \rangle \) such that \( \delta_{i+1} = d_I(F_{i+1}, F'_{i+1}) \) by distinguishing three situations.

- \( \delta_{i+1} = \delta_i \). In this case, notice that \( \dim(F_i + F'_i) - i = \delta_i = \delta_{i+1} \leq n - (i+1) \). Thus, \( \dim(F_i + F'_i) \leq n - 1 \) and we can choose \( u \in \mathbb{F}_q^n \setminus (F_i + F'_i) \) to form \( F_{i+1} = F_i + \langle u \rangle \). Now, if \( \delta_i \neq 0 \), we can take \( u' \in (F_i + F'_i) \setminus F'_i \) and \( F'_{i+1} = F'_i + \langle u' \rangle \). In case \( \delta_i = 0 \), then \( \delta_{i+1} = 0 \) and we just put \( F'_{i+1} = F_{i+1} = F_i + \langle u \rangle \). Note that, in any case, \( F_{i+1}, F'_{i+1} \) are subspaces of dimension \( i + 1 \) with

\[
\dim(F_{i+1} + F'_{i+1}) = \dim(F_i + F'_i + \langle u \rangle) = \dim(F_i + F'_i) + 1 = \delta_i + i + 1.
\]

As a consequence, \( \delta_{i+1} = \delta_i = \dim(F_{i+1} + F'_{i+1}) - (i + 1) \) as desired.

- \( \delta_{i+1} = \delta_i + 1 \). Hence, \( \dim(F_i + F'_i) - i = \delta_i = \delta_{i+1} - 1 \leq n - (i+1) - 1 = n - i - 2 \) and \( \dim(F_i + F'_i) \leq n - 2 \). This allows us to choose two linearly independent vectors \( u, u' \in \mathbb{F}_q^n \) such that

\[
\dim(F_i + F'_i + \langle u \rangle + \langle u' \rangle) = \dim(F_i + F'_i) + 2.
\]

The subspaces \( F_{i+1} = F_i + \langle u \rangle \) and \( F'_{i+1} = F'_i + \langle u' \rangle \) satisfy the required condition.

- \( \delta_{i+1} = \delta_i - 1 \). In this situation, notice that \( \delta_i \geq 1 \) and then \( F_i \neq F'_i \).

Therefore, we can find nonzero vectors \( u \in F'_i \setminus F_i \) and \( u' \in F_i \setminus F'_i \) to form two \( (i + 1) \)-dimensional subspaces \( F_{i+1} = F_i + \langle u \rangle \) and \( F'_{i+1} = F'_i + \langle u' \rangle \) with \( d_I(F_{i+1}, F'_{i+1}) = \delta_{i+1} \).

The previous theorem turns out to be the key to establish our bijection between the set of distance vectors corresponding to the full flag variety \( F_q(n) \) and the set of Motzkin paths of length \( n \).

**Theorem 4.2.** Given a positive integer \( n \geq 2 \), there is a bijection between the set of distance vectors \( D(n) \) and the set of Motzkin paths \( M_n \).

**Proof.** Recall that in this paper we do not distinguish between a Motzkin path and the corresponding Motzkin word. Consider \( (\delta_1, \ldots, \delta_{n-1}) \in D(n) \) and put \( \delta_0 = \delta_n = 0 \). In light of Theorem 4.1, for any \( i \in \{1, \ldots, n\} \), it holds

\[
\delta_i - \delta_{i-1} \in \{-1, 0, 1\}.
\]


Consider the following map:

\[
\Psi : \mathcal{D}(n) \to \mathcal{M}_n
\]

\[
(\delta_1, \ldots, \delta_{n-1}) \mapsto p_1 \ldots p_n,
\]

where, for every \(1 \leq i \leq n\), we take

\[
p_i = \begin{cases} 
U & \text{if } \delta_i - \delta_{i-1} = 1, \\
H & \text{if } \delta_i - \delta_{i-1} = 0, \\
D & \text{if } \delta_i - \delta_{i-1} = -1.
\end{cases}
\]

Let us see that the word \(\Psi((\delta_1, \ldots, \delta_{n-1}))\) is a Motzkin word. To do so, assume that, for some \(1 \leq i \leq n\), in the subword \(p_1 \ldots p_i\), the letter \(D\) appears more times than \(U\). This means that, up to the \(i\)-th step, the difference \(\delta_j - \delta_{j-1} = -1\) occurs more times than \(\delta_j - \delta_{j-1} = 1\). As a result, we have that \(\delta_i = \sum_{j=1}^i \delta_j - \delta_{j-1} < 0\), which is a contradiction. Similarly, since \(\sum_{j=1}^n \delta_j - \delta_{j-1} = \delta_n - \delta_0 = 0\), the number of occurrences of both \(U\) and \(D\) coincides.

On the other hand, given \(p \in \mathcal{M}_n\), we have that \(p = p_1 \ldots p_n\) is a sequence of \(n\) letters \(p_i \in \{U, H, D\}\), for \(i = 1, \ldots, n\) such that, in particular, \(p_1 \in \{U, H\}\) and \(p_n \in \{H, D\}\). Now, for \(i = 1, \ldots, n\), we consider the following associated values: \(u_i\) (resp. \(h_i, d_i\)) is the number of occurrences of an \(U\) (resp. \(H, D\)) step at or before \(p_i\). Note that \(d_1 = 0\) and \(u_n = d_n\). Moreover, for every \(1 \leq i \leq n\), it holds \(u_i + h_i + d_i = i\) and \(u_i \geq d_i\). From these values, we define a map \(\Phi : \mathcal{M}_n \to \mathcal{D}(n)\) as follows:

\[
\Phi(p_1 \ldots p_n) = (u_1 - d_1, u_2 - d_2, \ldots, u_{n-1} - d_{n-1}).
\]

As a consequence of the definition of \(u_i, h_i, d_i\), we have that

\[
(u_i - d_i) - (u_{i-1} - d_{i-1}) = \begin{cases} 
1 & \text{if } p_i = U, \\
0 & \text{if } p_i = H, \\
-1 & \text{if } p_i = D,
\end{cases}
\]

with \(2 \leq i \leq n\). As \(u_1 - d_1 \in \{0, 1\}\), by means of Theorem 4.1, \(\Phi(p_1 \ldots p_n)\) is a distance vector.

Finally, we prove that \(\Phi\) and \(\Psi\) are mutually inverse. To do so, consider \((\delta_1, \ldots, \delta_{n-1}) \in \mathcal{D}(n)\). Let us see that, if \(\Phi \circ \Psi((\delta_1, \ldots, \delta_{n-1})) = \Phi(p_1 p_2 \ldots p_n) = (u_1 - d_1, u_2 - d_2, \ldots, u_{n-1} - d_{n-1})\) (as defined in (13) and (12)), then \(\delta_i = u_i - d_i\), for all \(i \in \{1, \ldots, n-1\}\). We work by induction on \(n\). Clearly, if \(\delta_1 = 1\) (resp. \(\delta_1 = 0\)), then \(p_1 = U\) (resp. \(p_1 = H\)) and \(u_1 - d_1 = 1 = \delta_1\) (resp. \(u_1 - d_1 = 0 = \delta_1\)). Assume that the result is true, for \(k < n - 1\), that is, \(\delta_k = u_k - d_k\). We now that \(\delta_{k+1} - \delta_k \in \{-1, 0, 1\}\). Suppose, for instance, that \(\delta_{k+1} = -1\). Then \(p_{k+1} = U\) and, as \(\delta_k = u_k - d_k\) by induction hypothesis, we have \(u_{k+1} - d_{k+1} = (u_k - d_k) + 1 = \delta_k + 1\). Thus, \(u_{k+1} - d_{k+1} = \delta_{k+1}\) as desired. The cases where \(\delta_{k+1} - \delta_k = 0\) or \(\delta_{k+1} - \delta_k = -1\) are analogous.

Take now \(p_1 p_2 \ldots p_n\) a Motzkin word of length \(n\) such that \(\Psi \circ \Phi(p_1 p_2 \ldots p_n) = p'_1 p'_2 \ldots p'_n\). Let us see that \(p_i = p'_i\), for any \(i \in \{1, \ldots, n\}\). Let us work again by induction on \(n\). It is clear that, if \(p_1 = U\) (resp. \(p_1 = H\)) then \(u_1 - d_1 = 1\) (resp. \(u_1 - d_1 = 0\)) and \(p'_1 = U = p_1\) (resp. \(p'_1 = H = p_1\)). Assume that \(p_k = p'_k\)}
for $k < n$. If $p'_{k+1} = U$, then $u_{k+1} - d_{k+1} - (u_k - d_k) = 1$ which means that $p_{k+1} = U = p'_{k+1}$. The cases $p'_{k+1} = H$ or $p'_{k+1} = D$ go analogously. Hence $\Psi$ is a bijection as we wanted to show.

**Example 4.3.** To the distance vector $v = (1, 2, 2, 3, 2, 1, 1, 0)$ it corresponds the Motzkin path $\Psi(v) = UUHUDDHDH$ represented in Figure 1. For it, we have $u_6 = 3, h_6 = 1, d_6 = 2$, for instance. In fact, $\Phi \circ \Psi(v) = (1 - 0, 2 - 0, 2 - 0, 3 - 0, 3 - 1, 3 - 2, 3 - 2, 3 - 3) = v$.

We can now provide the answer to the first question proposed in Section 3.

**Corollary 4.4.** The number of distance vectors in $D(n)$ is the $n$-th Motzkin number $M_n$.

**Remark 4.5.** Note that in our work [3], given a pair of full flags $F$ and $F'$ on $F_n$, we defined their distance path $\Gamma(F, F')$ as the polygonal path in $\mathbb{Z}^2$ from $(0, 0)$ to $(n, 0)$ whose intermediate vertices are the points $(i, d_i(F, F'))$ for every $0 < i < n$. Therefore, by virtue of Theorem 4.1, our distance paths are Motzkin paths and conversely.

### 4.1 Distance vectors corresponding to disjoint flags

Recall from Section 2 that a couple of flags $F, F'$ on $F_n$ are disjoint if they do not share any subspace, that is, if they do not present collapses at any dimension. On the other hand, if $F, F'$ collapse at dimension $i$, that is $F_i = F'_i$, then the $i$-th component in $d(F, F')$ is equal to zero.

By means of (13), for a distance vector $v = (\delta_1, \ldots, \delta_{n-1}) \in D(n)$ such that $\delta_i = 0$, we have that $u_i = d_i$. Graphically, the Motzkin path $\Psi(d)$ intersects the $x$-axis at the point $(i, 0)$. In other words, two full flags collapsing at dimension $i$ induce a Motzkin path in which the point $(i, 0)$ is a return. This fact enables us to state the following result. Its proof can be easily obtained by using the maps $\Psi$ and $\Phi$ defined in the proof of Theorem 4.2, which are still bijections when restricted to $E_n$ and the set of distance vectors in $D(n)$ without null components.

**Proposition 4.6.** Given a positive integer $n \geq 2$, there is a bijection between the set of distance vectors in $D(n)$ corresponding to pairs of disjoint flags and the set of elevated Motzkin paths $E_n$.

**Corollary 4.7.** The number of distance vectors in $D(n)$ corresponding to pairs disjoint flags is the $n$-th number $E_n$ defined in (1).

### 4.2 Flag distance equals area

Let us show that the map $\Psi$ takes flag distance to area. In other words, if $v = (\delta_1, \ldots, \delta_{n-1}) \in D(d, n)$, that is, $d = \delta_1 + \cdots + \delta_{n-1}$, then $A(\Psi(v)) = d$.

To do this, we work first with elevated Motzkin paths, which means that our corresponding distance vectors have no null component, by Proposition 4.6. Let us describe a decomposition of any such a distance vector that may result very convenient for our purposes.
Proposition 4.8. Consider \( v = (\delta_1, \ldots, \delta_{n-1}) \in \mathcal{D}(n) \) with \( \delta_i \neq 0 \) for every \( i \in \{1, \ldots, n-1\} \). Then, there is \( r \geq 1 \) a positive integer and vectors \( v^k = (v^k_1, \ldots, v^k_{n-1}) \in \mathbb{F}_2^{n-1} \) for \( k \in \{1, \ldots, r\} \) such that
\[
\delta_i = \sum_{k=1}^r v^k_i.
\] (14)

Proof. Take \( v = (\delta_1, \ldots, \delta_{n-1}) \in \mathcal{D}(n) \) with \( \delta_i \neq 0 \) for \( i \in \{1, \ldots, n-1\} \). Note that, in this case, \( \delta_1 = \delta_{n-1} = 1 \). Take \( r = \max\{\delta_1, \ldots, \delta_{n-1}\} \) and, for any \( k \in \{1, \ldots, r\} \) define a vector \( v^k = (v^k_1, \ldots, v^k_{n-1}) \) such that \( v^k_i = 0 \) if \( k > \delta_i \) and \( v^k_i = 1 \) otherwise. With this choice the desired equality (14) holds.

Remark 4.9. Note that, if we denote \( \rho_k = \sum_{i=1}^{n-1} v^k_i \), from the previous decomposition, we have that
\[
d = \sum_{i=1}^{n-1} \delta_i = \sum_{i=1}^{n-1} \sum_{k=1}^r v^k_i = \sum_{k=1}^r \sum_{i=1}^{n-1} v^k_i = \sum_{k=1}^r \rho_k.
\] (15)

Example 4.10. Take the distance vector \( v = (1, 2, 3, 2, 1, 2, 2, 1) \in \mathcal{D}(9) \). Thus, \( r = 3 \), \( v_1 = (1, 1, 1, 1, 1, 1, 1, 1) \), \( v_2 = (0, 1, 1, 1, 0, 1, 1, 0) \), \( v_3 = (0, 0, 1, 0, 0, 0, 0, 0) \). In fact, we can put
\[
\begin{align*}
0 & 0 1 0 0 0 0 0 \rightarrow \rho_3 = 1 \\
0 & 1 1 1 0 1 1 0 \rightarrow \rho_2 = 3 + 2 \\
1 & 1 1 1 1 1 1 1 \rightarrow \rho_1 = 9 \\
\downarrow & \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
1 & 2 3 2 1 2 2 1 & 14 \\
\| & \| \| \| \| \| \| \| \\
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \delta_6 & \delta_7 & \delta_8
\end{align*}
\]
and observe how the flag distance, \( d = 14 \) in this case, can be decomposed “vertically” and “horizontally”.

Following the ideas in [9], let us see that the area under a Motzkin path \( p \in \mathcal{M}_p \) can be decomposed into horizontal strips determined by the up and down steps.

Proposition 4.11. Let \( p \in \mathcal{E}_n \) be an elevated Motzkin path. Then
\[
A(p) = \sum_{p_j=D} j - \sum_{p_i=U} i.
\] (16)

Proof. Given \( p = p_1p_2 \ldots p_n \in \mathcal{M}_n \), as it was pointed out in Remark 2.3, the letters \( U \) and \( D \) in \( p \) form a balanced parenthesization, so they can be naturally paired up. Moreover, every couple of matched steps \( p_i = U \) and \( p_j = D \) determines a horizontal strip of area \( j - i \) below the Motzkin path \( p \). The sum of areas of all the corresponding horizontal strips gives us expression (16) for \( A(p) \).

\]

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Example 4.12. Let us take \( p \in \mathcal{M}_9 \) with \( p = UUDDUHDD \). In this case, we have the paired steps \((p_1, p_3), (p_2, p_5), (p_6, p_8)\) and \((p_3, p_4)\).

![Figure 2: The Motzkin path \( p = UUDDUHDD \) in \( \mathcal{M}_9 \).](image)

Then we have that
\[
A(p) = (9 - 1) + (5 - 2) + (8 - 6) + (4 - 3) = 8 + 3 + 2 + 1 = 14.
\]

Observe that our path \( p \) satisfies \( p = \Psi(v) \) where \( v \) is the vector considered in Example 4.10. Moreover, the areas corresponding to horizontal strips at height 1, 2 or 3, as showed in the picture above, correspond, respectively, to the sum of the non-zero components of \( v_1, v_2 \) or \( v_3 \).

Remark 4.13. Note that the null distance vector in \( \mathcal{D}(n) \) goes by \( \Psi \) to the “flat” Motzkin path. Besides, a null component in a distance vector \( v \) does not contribute to the computation of the corresponding value of the flag distance and gives a return on \( \Psi(v) \). On the other hand, a Motzkin path \( p \) with a finite number of returns can be decomposed into a finite number of elevated Motzkin paths whose sum of areas gives \( A(p) \). In Figure 3 we can see a Motzking path \( p \in \mathcal{M}_9 \) with a return in \((4,0)\). This path corresponds to the distance vector \( v = (1,2,1,0,1,2,2,1) \in \mathcal{D}(10,9) \). Then \( p \) can be decomposed into two elevated Motzkin paths \( p' = UUDD \in \mathcal{M}_4 \) and \( p'' = UUHDD \in \mathcal{M}_5 \) such that \( A(p) = A(p') + A(p'') = 10 \).

![Figure 3: The Motzkin path \( p = UUDDUHDD \) in \( \mathcal{M}_9 \).](image)

Proposition 4.8 along with Proposition 4.11 and Remark 4.13, give straightforwardly the following result.
Theorem 4.14. Consider the map $\Psi : \mathcal{D}(n) \to \mathcal{M}_n$ defined in Theorem 4.2. Given a distance vector $v = (\delta_1, \ldots, \delta_{n-1}) \in \mathcal{D}(n)$, we have that $A(\Psi(v)) = \delta_1 + \cdots + \delta_{n-1}$. In other words, the map $\Psi$ takes flag distance to area and the restriction $\Psi |_{\mathcal{D}(d,n)} : \mathcal{D}(d,n) \to \mathcal{M}_n(d)$ is a bijection for any integer $d \in [0, \lfloor \frac{n^2}{4} \rfloor]$.

Corollary 4.15. The number of distance vectors in $\mathcal{D}(n)$ corresponding to a fixed flag distance value $d$ equals the number of Motzkin paths $p \in \mathcal{M}_n$ such that $A(p) = d$. This number is the term $T(n,d)$ of the sequence A129181 in [17].

Observe that, as a consequence of Theorem 4.14, we could directly deduce the bound given in (9) and, by Theorem 4.1, conclude that it is sharp. It is enough to see that $\lfloor \frac{n^2}{4} \rfloor$ is the largest possible area under a Motzkin path with $n$ steps. This area is attained by the path $p = p_1 \cdots p_n$ with $p_1 = \cdots = p_k = U$ and $p_{k+1} = \cdots = p_n = D$, if $n$ is even ($n = 2k, k \geq 1$), or with $p_1 = \cdots = p_k = U$, $p_{k+1} = H$ and $p_{k+2} = \cdots = p_n = D$, if $n$ is odd ($n = 2k + 1, k \geq 1$).

Remark 4.16. An analogous result to Theorem 4.14 was proved also in [3]. In that paper, the authors define distance paths in a distance support and show that the area under a distance path corresponds to its associated value of the flag distance by counting lattice points in the path, or below it, and applying Pick’s Theorem.

4.3 The number of possible distance vectors of a flag code

As a consequence of Theorem 4.14, we can easily compute the number of possible distance vectors that can be potentially associated to a full flag code with prescribed minimum distance.

Definition 4.17. Given a full flag code $\mathcal{C} \subseteq \mathcal{F}_q(n)$, its set of distance vectors is defined as

$$\mathcal{D}(\mathcal{C}) = \{ d(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C}, \ d_f(\mathcal{F}, \mathcal{F}') = d_f(\mathcal{C}) \}.$$  

Example 4.18. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{F}_q^4$ as $\mathbb{F}_q$-vector space and consider the full flag code $\mathcal{C}$ on $\mathbb{F}_q^4$ given by

$$\mathcal{F}^1 = \langle (e_1), \ (e_1, e_2), \ (e_1, e_2, e_4) \rangle,$$
$$\mathcal{F}^2 = \langle (e_1), \ (e_1, e_3), \ (e_1, e_2, e_3) \rangle,$$
$$\mathcal{F}^3 = \langle (e_2), \ (e_2, e_3), \ (e_1, e_2, e_3) \rangle.$$  

Notice that

$$d_f(\mathcal{F}^1, \mathcal{F}^2) = 0 + 1 + 1 = 2,$$
$$d_f(\mathcal{F}^1, \mathcal{F}^3) = 1 + 1 + 1 = 3,$$
$$d_f(\mathcal{F}^2, \mathcal{F}^3) = 1 + 1 + 0 = 2.$$  

Hence, we have $d_f(\mathcal{C}) = 2$ and $\mathcal{D}(\mathcal{C}) = \{(0,1,1),(1,1,0)\} \subseteq \mathcal{D}(2,4)$. On the other hand, there are more distance vectors in $\mathcal{D}(2,4)$. It suffices to apply Theorem 4.1 and see that $(1,0,1)$ is an element in $\mathcal{D}(2,4)$. 

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Motzkin numbers and flag codes

Directly from Corollary 4.15 we can state the following results concerning full flag codes.

**Corollary 4.19.** Given a full flag code $C$ in $F_q^n$, the number of distance vectors in $\mathcal{D}(d_f(C), n)$ equals the number of Motzkin paths $p \in \mathcal{M}_n$ such that $A(p) = d_f(C)$, i.e., the value $T(n, d_f(C))$.

**Remark 4.20.** Notice that the previous result counts the number of possible distance vectors associated to the minimum distance of a flag code. Nevertheless, this number might not coincide with the exact number of distance vectors of the code. For instance, if we take $F^1$ and $C$ as in Example 4.18, it holds $d_f(C) = 2$ and $|\mathcal{D}(C)| = 2$. Moreover, if we consider $C' = \{F^1, F^4\}$, with

$$\mathcal{F}^4 = (\langle e_2 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_2, e_3 \rangle),$$

then we also have $d_f(C') = 2$ but $|\mathcal{D}(C')| = 1$. Even more, the sets $\mathcal{D}(C)$ and $\mathcal{D}(C')$ do not share any distance vector and form a partition of $\mathcal{D}(2, 4)$, which contains $T(4, 2) = 3$ elements.

On the other hand, in the special case of a flag code on $F_q^n$ with minimum distance $D^n$, we can give the exact number of its distance vectors (observe that $T(n, D^n) = 1$), as stated in the next result.

**Corollary 4.21.** Let $C$ be a full flag code in $F_q^n$ such that $d_f(C) = D^n$. Then $C$ has a unique distance vector.

We finish this section by computing the number of possible distance vectors that a disjoint full flag code can have.

**Proposition 4.22.** Let $C$ be a disjoint full flag code in $F_q^n$. Then $d_f(C) \geq n - 1$ and the number of possible distance vectors for $C$ is $T(n - 2, d_f(C) - n + 1)$.

**Proof.** First of all, notice that, if $C$ is disjoint, then its minimum distance is attained by distance vectors $(\delta_1, \ldots, \delta_{n-1})$ with no zero components, i.e., $\delta_i \geq 1$, for every $1 \leq i \leq n - 1$. Hence, such vectors can be decomposed into

$$(\delta_1, \ldots, \delta_{n-1}) = (1, \ldots, 1) + (\gamma_0, \gamma_1, \ldots, \gamma_{n-3}, \gamma_{n-2})$$

with $\gamma_i = \delta_{i+1} - 1$, for every $0 \leq i \leq n - 2$. Notice that $\gamma_0 = \gamma_{n-2} = 0$ and, for every $1 \leq i \leq n - 2$, it holds

$$\gamma_i - \gamma_{i-1} = \delta_{i+1} - \delta_i \in \{-1, 0, 1\}.$$

As a consequence, and by means of Theorem 4.1, $(\delta_1, \ldots, \delta_{n-1})$ can be naturally identified with the distance vector $(\gamma_1, \ldots, \gamma_{n-3}) \in \mathcal{D}(n - 2)$, associated to the distance value

$$\sum_{i=1}^{n-3} \gamma_i = \sum_{i=1}^{n-3} (\delta_{i+1} - 1) = \sum_{i=0}^{n-2} (\delta_{i+1} - 1) = d_f(C) - n + 1.$$

Hence, the number of distance vectors in $\mathcal{D}(d_f(C), n)$ with no zero components, that is, the number of potential distance vectors for $C$, coincides with the number $T(n - 2, d_f(C) - n + 1)$. 

\(\blacksquare\)
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In the next table we can see the number of possible distance vectors associated to a disjoint flag code $C$ on $\mathbb{F}_q^n$ with prescribed minimum distance $d$ for small values of $n$.

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 2   | -  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 3   | -  | -  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 4   | -  | -  | -  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |
| 5   | -  | -  | -  | -  | 1  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |
| 6   | -  | -  | -  | -  | -  | 1  | 3  | 3  | 1  | 1  |    |    |    |    |    |    |    |
| 7   | -  | -  | -  | -  | -  | -  | 1  | 4  | 6  | 4  | 3  | 2  | 1  |    |    |    |    |
| 8   | -  | -  | -  | -  | -  | -  | -  | 1  | 5  | 10 | 10 | 8  | 7  | 5  | 3  | 1  | 1  |

Table 2: Numbers $T(n-2,d-n+1)$ for small values of $n$.

4.4 Other cases, other sequences

As it was done in Subsection 4.1 with the case of disjoint flag codes, it can be very useful to interpret the properties of a given family of flags in terms of a concrete family of Motzkin paths in order to localize a relevant sequence of integers to count the number of associated distance vectors. To this end we contemplate a last family of flags.

**Proposition 4.23.** Let $\mathcal{H}_n$ be the set of full flags on $\mathbb{F}_q^n$ such that given $\mathcal{F}, \mathcal{F}' \in \mathcal{H}_n$, they never share consecutive subspaces. Hence, the number of possible distance vectors corresponding to couples of flags in $\mathcal{H}_n$ is given by the sequence of Riordan numbers, that is, sequence A005043 in [17].

**Proof.** It is enough to observe that a couple of flags in $\mathcal{F}, \mathcal{F}' \in \mathcal{H}_n$ never presents consecutive collapses, that is, their associated distance vector $d(\mathcal{F}, \mathcal{F}')$ never has two consecutive null components. Thus, the Motzkin path $\Psi(d(\mathcal{F}, \mathcal{F}'))$ does not have horizontal steps in the $x$-axis and, as consequence, it is a Riordan path.

5 Conclusions and open questions

In this paper we have addressed the problem of counting the number of distance vectors associated with the full flag variety $\mathcal{F}_q(n)$. Moreover, we have provided the number of possible distance vectors for a full flag code with prescribed minimum distance either in the general case or in the disjoint one. The key to compute these cardinalities is to associate biunivocally a Motzkin word of length $n$ with a distance vector in $D(n)$.

The problem of calculating the possible number of distance vectors when we consider flags of general type $t = (t_1, \ldots, t_r)$ is still open. We believe that, to deal with this question, it could be useful the use of Motzkin paths of higher rank where other kind of steps are allowed (see [13]).

On the other hand, taking into account the definition of the distance in the context of multishot codes ([15, 16]), not necessarily flag codes, we think that our work can be a starting point in the study of the possible distributions of the total distance by using lattice paths.
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