INDEX OF Varieties over Henselian fields AND EULER CHARACTERISTIC OF COHERENT SHEAVES

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Abstract. Let $X$ be a smooth proper variety over the quotient field of a Henselian discrete valuation ring with algebraically closed residue field of characteristic $p$. We show that for any coherent sheaf $E$ on $X$, the index of $X$ divides the Euler–Poincaré characteristic $\chi(X,E)$ if $p = 0$ or $p > \dim(X) + 1$.

If $0 < p \leq \dim(X) + 1$, the prime-to-$p$ part of the index of $X$ divides $\chi(X,E)$. Combining this with the Hattori–Stong theorem yields an analogous result concerning the divisibility of the cobordism class of $X$ by the index of $X$.

As a corollary, rationally connected varieties over the maximal unramified extension of a $p$-adic field possess a 0-cycle of $p$-power degree (a 0-cycle of degree 1 if $p > \dim(X) + 1$). When $p = 0$, such statements also have implications for the possible multiplicities of singular fibers in degenerations of complex projective varieties.

Introduction

The index of a variety $X$ over a field $K$ is the smallest positive degree of a zero-cycle on $X$, or equivalently, the greatest common divisor of the degrees $[L:K]$ of all finite extensions $L/K$ such that $X(L) \neq \emptyset$. The present paper is devoted to investigating this invariant when $K$ is the quotient field of an excellent Henselian discrete valuation ring with algebraically closed residue field, for instance $\mathbb{Q}_{nr}^p$, the maximal unramified extension of $\mathbb{Q}_p$, or the field of formal Laurent series $\mathbb{C}(((t)))$.

Such fields are $(C_1)$ fields in the sense of Lang [30]. Recall that a field $K$ is said to be $(C_1)$ if every hypersurface of degree $d \leq n$ in $\mathbb{P}_K^n$ possesses a rational point. Smooth hypersurfaces of degree $d \leq n$ in $\mathbb{P}_K^n$ are examples of smooth Fano varieties, in particular they are rationally chain connected (and thus separably rationally connected if $K$ has characteristic 0). It is an old question raised by Lang, Manin, and Kollár, whether smooth proper varieties over $K$ which are either Fano or separably rationally connected always have a rational point if $K$ is $(C_1)$ (see [29], [34] p. 48, Remark 2.6 (ii)]). A positive answer is known to hold when $K$ is a finite field (see [12]). In the separably rationally connected case, a positive answer also holds when $K$ is the function field of a curve over an algebraically closed field (see [10], [9]), from which it follows, by a global-to-local approximation argument, that a positive answer holds when $K$ is the quotient field of an equal characteristic Henselian discrete valuation ring with algebraically closed residue field (see [7] Théorème 7.5]). In the local situation, no direct proof is known. On the other hand, over function fields of curves, the arguments of [10] and [9] rely on

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the study of moduli spaces of rational curves and are thus anchored in geometry; in particular, they shed no light on unequal characteristic local fields with algebraically closed residue fields, such as $\mathbb{Q}_p^{nr}$. It is still unknown whether any smooth proper rationally connected variety over $\mathbb{Q}_p^{nr}$ possesses a rational point.

We can ask for less by considering the index. As the index of a proper variety $X$ over $K$ is an invariant of cohomological nature (being determined by the cokernel of the degree map $\text{CH}_0(X) \to \mathbb{Z}$ from the Chow group of 0-cycles), one would expect that cohomological conditions on $X$, rather than the actual geometry of $X$, should suffice to control the index.

Indeed, various authors have shown that smooth proper varieties over $K = \mathbb{C}((t))$ which satisfy $H^i(X, \mathcal{O}) = 0$ for all $i > 0$ have index 1 (see [37, p. 162], [26, p. 194], [39], [8], Proposition 7.3; the last three references rely on Hodge theory). This statement applies in particular to rationally connected varieties. On the other hand, it is a well-known consequence of the adjunction formula for surfaces that a family of curves $f : X \to S$, where $X$ is a smooth surface and $S$ is a smooth curve, has no multiple fiber if the generic fiber of $f$ is a smooth, geometrically irreducible curve of genus 2. Equivalently, every smooth proper curve of genus 2 over $\mathbb{C}((t))$ has index 1, even though the groups $H^i(X, \mathcal{O})$ do not vanish in this case.

In this paper, we show that these two statements are in fact instances of a single general phenomenon relating the index of $X$ over $K$ and the Euler–Poincaré characteristic of coherent sheaves on $X$. Note that $|\chi(X, \mathcal{O}_X)| = 1$ when $X$ is either a rationally connected variety or a curve of genus 2.

**Theorem 1** (see Theorem [21], Theorem [31]). Let $R$ be a Henselian discrete valuation ring with quotient field $K$ and algebraically closed residue field $k$. Let $X$ be a proper scheme over $K$. Assume $k$ has characteristic 0. Then, for any coherent sheaf $E$ on $X$, the index of $X$ over $K$ divides $\chi(X, E)$.

The assumption that $k$ have characteristic 0 is required to ensure that resolution of singularities holds for integral schemes of finite type over $R$. For the arguments of Theorem [1] to go through in general, it would suffice to know that for any integral proper $K$-scheme $X$, there is a normal proper flat $R$-scheme $\mathcal{X}$ and a birational $K$-morphism $\mathcal{X} \otimes_R K \to X$, such that the special fiber $\mathcal{X} \otimes_R k$ is divisible, as a Cartier divisor on $\mathcal{X}$, by its multiplicity as a Weil divisor. We cannot prove the existence of such models when $k$ has positive characteristic. Nonetheless, using Gabber’s refinement of de Jong’s theorem on alterations, together with a $K$-theoretic dévissage and the Hirzebruch–Riemann–Roch theorem, we show:

**Theorem 2** (see Theorem [32]). Let $R$ be a Henselian discrete valuation ring with quotient field $K$ and algebraically closed residue field $k$. Let $X$ be a smooth proper scheme over $K$. Assume $k$ has characteristic $p > 0$. Then, for any coherent sheaf $E$ on $X$, the prime-to-$p$ part of the index of $X$ over $K$ divides $\chi(X, E)$. If in addition $p > \dim(X) + 1$, the index of $X$ over $K$ divides $\chi(X, E)$.

(By the prime-to-$p$ part of $N$, we mean the largest divisor of $N$ which is prime to $p$.) Coming back to the motivation for our work, we deduce:

**Corollary 3** (see Corollary [36]). The index of a rationally connected variety $X$ over $\mathbb{Q}_p^{nr}$, or more generally over the quotient field of a Henselian discrete valuation ring of characteristic 0 with algebraically closed residue field of characteristic $p > 0$, is a power of $p$. It is 1 if in addition $p > \dim(X) + 1$. 
Theorems 1 and 2 also have a number of unexpected consequences. For instance, we prove that if \( X \) is a variety of general type over \( \mathbb{C}((t)) \), or over the maximal unramified extension of a \( p \)-adic field with \( p > \dim(X) + 1 \), then the index of \( X \) divides the plurigenera \( P_n(X) \) for \( n \geq 2 \) (Example 2.7, Corollary 3.7). In another direction, if \( K = \mathbb{C}((t)) \) or \( K \) is the maximal unramified extension of a \( p \)-adic field with \( p \geq 5 \), then hypersurfaces of degree 6 in \( \mathbb{P}^4_K \) have index 1. More generally, in Section 4, we produce, for any \( d \) and \( N \), an optimal bound on the index of a hypersurface of degree \( d \) in \( \mathbb{P}^N \) over such a field (Theorem 4.1, Proposition 4.3). In particular, if \( d = \prod p_i^{n_i} \) is the prime factorization of \( d \), the property “every hypersurface of degree \( d \) in \( \mathbb{P}^N \) over \( \mathbb{C}((t)) \) contains a zero-cycle of degree 1” holds if and only if \( \max(p_1, \ldots , p_n) \leq N \) (Example 4.6). This should be compared with Lang’s theorem according to which \( \mathbb{C}((t)) \) is a \((C)\) field.

Let us stress the geometric content of such statements: they imply in particular that if \( f : X \to S \) is a dominant morphism between smooth projective complex varieties and if the geometric generic fiber of \( f \) is an irreducible variety of general type (resp., a sextic surface), then the multiplicities of the codimension 1 fibers of \( f \) divide the higher plurigenera of the generic fiber (resp., the morphism \( f \) has no multiple fiber).

Although we use a slightly different method here, Theorem 1 may be proved using only properties of \( K \)-theory as an oriented cohomology theory (in the sense of [32, Definition 1.1.2]), which, assuming \( K \) is a subfield of \( \mathbb{C} \), suggests that the class of \( X(\mathbb{C}) \) in the complex cobordism ring \( \pi_*(MU) \) should be divisible by the index of \( X \) over \( K \). The goal of Section 5 is to show that this is indeed the case, and to prove a similar statement without any assumption on the characteristic of \( K \).

As it turns out, when \( K \) is a subfield of \( \mathbb{C} \), one can give a purely algebraic description of the complex cobordism class of \( X(\mathbb{C}) \). It has long been known (see, e.g., [44, Chapter I]) that the complex cobordism ring \( \pi_*(MU) \) is canonically isomorphic to the graded subring \( \mathbb{A} \) of the infinite polynomial ring \( \mathbb{Z}[b_1, b_2, \ldots] \) (with \( \deg(b_i) = i \)) spanned by the polynomials \( b(X) = \sum_{|I|=\dim(X)} \deg(c_I(-T_X))b_I \) as \( X \) runs over smooth proper varieties over \( \mathbb{C} \), and that \( b(X) \) is equal to the cobordism class of \( X(\mathbb{C}) \) via this isomorphism. Here \( T_X \) denotes the tangent bundle of \( X \) and \( c_I \) is the \( I \)-th Conner–Floyd Chern class. When \( K \) is an arbitrary field, we take this description as our point of departure and define the cobordism ring of \( \text{Spec}(K) \) to be the subring \( \mathbb{L}_K \) of \( \mathbb{Z}[b_1, b_2, \ldots] \) spanned by the polynomials \( b_K(X) = \sum_{|I|=\dim(X)} \deg(c_I(-T_X))b_I \) when \( X \) runs over smooth proper varieties over \( K \). This subring is in fact equal to \( \mathbb{L} \), according to Merkurjev [35, Theorem 8.2]. Under the hypotheses of Theorem 2, we may ask whether the cobordism class \( b_K(X) \) is divisible, in \( \mathbb{L}_K \), by the index of \( X \) over \( K \). In this direction, we show:

**Theorem 4** (see Theorem 5.1). Let \( R \) be a Henselian discrete valuation ring with quotient field \( K \), and algebraically closed residue field \( k \) of characteristic \( p \geq 0 \). Let \( X \) be a smooth proper irreducible \( K \)-scheme. If \( p = 0 \) or \( p > \dim(X) + 1 \), then \( b_K(X) \) is divisible, in the ring \( \mathbb{L}_K \), by the index of \( X \) over \( K \). If \( p > 0 \), then \( b_K(X) \) is divisible, in \( \mathbb{L}_K \), by the prime-to-\( p \) part of the index of \( X \) over \( K \).

Since a full theory of algebraic cobordism is only available in characteristic zero, we use another method to prove Theorem 4, namely the Hattori–Stong theorem [43, Theorem 1], [38, Theorem 1], which allows one to use \( K \)-theory to compute cobordism; thus Theorem 4 becomes a consequence of Theorem 2.
From Theorem 4 and from the well-known fact that $\pi_*(MU)$ is generated by the classes of projective spaces $P^n$ ($n \geq 1$) and Milnor hypersurfaces $H_{m,n}$ (hypersurfaces of bidegree $(1,1)$ in $P^m \times P^n$, with $2 \leq m \leq n$), we deduce the following concrete consequence regarding integral-valued rational characteristic classes:

**Corollary 5** (see Corollary 4.9). Let $R$ be a Henselian discrete valuation ring with quotient field $K$, and algebraically closed residue field $k$ of characteristic $p \geq 0$. Let $d \geq 1$ and let $P \in \mathbb{Q}[c_1, \ldots, c_d]$ be homogeneous of degree $d$ with respect to the grading $\deg(c_i) = i$. Assume that $\deg(P(c_1(T_X), \ldots, c_d(T_X))) \in \mathbb{Z}$ for any $d$-dimensional product $X$ of complex projective spaces and Milnor hypersurfaces. Then, for any smooth proper irreducible $K$-scheme $X$ of dimension $d$, the rational number $\deg(P(c_1(T_X), \ldots, c_d(T_X)))$ is an integer. If $p = 0$ or $p > \dim(X) + 1$, this integer is divisible by the index of $X$ over $K$. If $p > 0$, it is divisible by the prime-to-$p$ part of the index of $X$ over $K$.

At the end of Section 5 we list examples of such polynomials, such as $\frac{1}{2}c_1 - c_1$ if $d$ is odd. As pointed out by Merkurjev in [36, p. 8], properties of Brosnan’s Steenrod operations on the mod $q$ Chow groups show that, for a given prime number $q$, the characteristic class $X \mapsto \frac{1}{q}\deg(c_1(-T_X))$ is integral-valued as long as $I = (\alpha_j)_{j \geq 1}$ with $\alpha_j = 0$ whenever $q^n - 1$ for some natural number $n$. This gives a large supply of integral-valued rational characteristic classes that are not expressible as $\mathbb{Z}$-linear combinations of monomials in the Chern classes of the tangent bundle.

Although the Hattori–Stong theorem tells us that each integral-valued characteristic class $X \mapsto \deg(P(c_1(T_X), \ldots, c_d(T_X)))$ on $d$-dimensional smooth proper varieties is given as the Euler–Poincaré characteristic of $\rho(T_X)$ for some virtual representation $\rho$ of $GL_n$, we are not aware of any general and explicit formulas for $\rho$ in terms of $P$. While one can find a $\rho$ for the characteristic class $\frac{1}{2}c_d$ (for odd $d$), one cannot do this so easily for other classes, for example for $\frac{1}{2}c_1$. The same seems to be the case for the series of characteristic classes $X \mapsto \frac{1}{q}\deg(c_1(-T_X))$ mentioned above. Thus Theorem 4 yields nontrivial divisibility by the index for integral-valued rational characteristic classes which, at least as a practical matter, goes beyond Theorem 2.

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**Notation.** If $N$ and $n$ are integers, the prime-to-$N$ part of $n$ is the largest integer which divides $n$ and is prime to $N$ (or 0 if $n = 0$). Let $X$ be a scheme of finite type over a field $K$. If $X$ is smooth over $K$, we denote by $T_X$ the tangent bundle of $X$, i.e., the locally free sheaf dual to $\Omega^1_{X/K}$. If $X$ is proper over $K$ and $E$ is a coherent sheaf on $X$, we denote the Euler–Poincaré characteristic of $E$ by $\chi(X, E) = \sum_{i \geq 0}(-1)^i \dim_K H^i(X, E)$. Finally, if $R$ is a discrete valuation ring with quotient field $K$, a model of $X$ over $R$ is a flat $R$-scheme of finite type with generic fiber $X$. 
1. INDEX OF SMOOTH PROPER SCHEMES OVER ARBITRARY FIELDS

**Definition 1.1.** Let $X$ be a scheme of finite type over a field $K$. The *index of $X$ over $K$*, denoted $\text{ind}(X)$, is the greatest common divisor of the degrees $[K(x) : K]$ of the closed points $x$ of $X$.

The index of $X$ over $K$ is also characterized by the fact that it generates the subgroup $\text{deg}(Z_0(X)) \subseteq \mathbb{Z}$. By the covariant functoriality of $Z_0(X)$, it follows that $\text{ind}(X)$ divides $\text{ind}(Y)$ for any morphism $Y \to X$ of schemes of finite type over $K$.

**Proposition 1.2.** Let $X$ be a smooth proper $K$-scheme of dimension $d$ and $E$ be a coherent sheaf on $X$. Then the prime-to-$(d+1)!$ part of $\text{ind}(X)$ divides $\chi(X, E)$.

**Proof.** As $X$ is regular and separated, the Grothendieck group of coherent sheaves on $X$ is generated by the classes of locally free sheaves. Thus, we may assume that $E$ is locally free. According to the Hirzebruch–Riemann–Roch theorem, we then have an equality of rational numbers

\[
\chi(X, E) = \varepsilon_*(\text{ch}(E) \cdot \text{td}(T_X)) \in \text{CH}^*(\text{Spec}(K)) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q},
\]

where $\varepsilon : X \to \text{Spec}(K)$ is the structure morphism of $X$, and $\text{ch}(E), \text{td}(T_X) \in \text{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ respectively denote the Chern character of $E$ and the Todd class of the tangent bundle $T_X$ of $X$ (see [13 Corollary 15.2.1]).

**Lemma 1.3.** Let $X$ be a smooth proper irreducible $K$-scheme of dimension $d$ and let $E$ be a vector bundle on $X$. The Chern character $\text{ch}(E) \in \text{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ belongs to the image of $\text{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/d!]$. The Todd class $\text{td}(E) \in \text{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ belongs to the image of $\text{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/(d+1)!].$

**Proof.** Let $r$ denote the rank of $E$. Let $\text{ch}_{r,d} \in \mathbb{Q}[\xi_1, \ldots, \xi_r]$ (resp., $\text{td}_{r,d} \in \mathbb{Q}[\xi_1, \ldots, \xi_r]$) denote the degree $d$ polynomial obtained by truncating the formal power series $\sum_{j=1}^r \exp(\xi_j)$ (resp., $\prod_{j=1}^r 1 - \exp(-\xi_j)$). This polynomial has coefficients in $\mathbb{Z}[1/d!]$ (resp., in $\mathbb{Z}[1/(d+1)!]$), has degree $d$, and is invariant under all permutations of the $\xi_j$’s. Therefore it belongs to the subring $\mathbb{Z}[1/d!][c_1, \ldots, c_d]$ (resp., $\mathbb{Z}[1/(d+1)!][c_1, \ldots, c_d]$) of $\mathbb{Q}[\xi_1, \ldots, \xi_r]$, where $c_1, \ldots, c_r$ denote the elementary symmetric polynomials in the $\xi_j$’s (with the convention that $c_i = 0$ if $i > r$). Now the Chern character of $E$ (resp., the Todd class of $E$) is, by definition, the element of $\text{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ obtained by applying $\text{ch}_{r,d}$ (resp., $\text{td}_{r,d}$) to the Chern classes $c_i(E) \in \text{CH}^*(X)$ of $E$; hence the lemma. \hfill \Box

Thanks to Lemma 1.3 it follows from (1.1) that $\chi(X, E)$ belongs to the image of $\varepsilon_* : \text{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/(d+1)!] \to \text{CH}^*(\text{Spec}(K)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/(d+1)!] = \mathbb{Z}[1/(d+1)!].$

In other words, there exist an integer $m \geq 0$ and a zero-cycle $\zeta$ on $X$ such that

\[
((d+1)!)^m \chi(X, E) = \text{deg}(\zeta),
\]

which proves the proposition. \hfill \Box

**Remark 1.4.** Haudron [13] Theorem 5.1 (ii)] has proved that Proposition 1.2 remains valid without any smoothness assumption on $X$. 
2. Index of the generic fiber of a regular proper scheme over a Henselian discrete valuation ring

The goal of this section is to prove the following theorem:

**Theorem 2.1.** Let $R$ be a Henselian discrete valuation ring with quotient field $K$ and algebraically closed residue field $k$. Let $X$ be a regular proper $K$-scheme. Assume $X$ admits a regular proper model over $R$. Then $\text{ind}(X)$ divides $\chi(X, \mathcal{O}_X)$.

We first recall two elementary facts relating the index of the generic fiber and the multiplicity of the special fiber of a scheme over a discrete valuation ring.

**Definition 2.2.** The *multiplicity* of a noetherian scheme $S$, denoted $\text{mult}(S)$, is the greatest common divisor, over all points $\eta \in S$ of codimension 0, of the length of the artinian local ring $\mathcal{O}_{S, \eta}$.

**Lemma 2.3.** Let $R$ be a discrete valuation ring with quotient field $K$ and residue field $k$. Let $X$ be a flat $R$-scheme of finite type, with generic fiber $X = X \otimes_R K$ and special fiber $Y = X \otimes_R k$.

(i) If $R$ is Henselian and $k$ is algebraically closed, then $\text{ind}(X)$ divides $\text{mult}(Y)$.

(ii) If $X$ is locally factorial (e.g., regular) and is proper over $R$, then $\text{mult}(Y)$ divides $\text{ind}(X)$.

**Proof.** The first assertion follows from [6, §9.1, Corollary 9]. The second assertion was included for the sake of completeness; we shall not use it. It can be proved with a simple computation in intersection theory, see [15, §8.1] for details.

**Proof of Theorem 2.1.** Let $X$ be a regular proper model of $X$ over $R$. It is clear, from the definition of $\text{mult}(Y)$, that $Y$ is divisible by $\text{mult}(Y)$ as a Weil divisor on $X$. As $X$ is regular, it follows that $Y$ is divisible by $\text{mult}(Y)$ as a Cartier divisor on $X$. Thus, Theorem 2.1 results from the combination of Lemma 2.3 (i) and Proposition 2.4 below.

**Proposition 2.4.** Let $R$ be a discrete valuation ring with quotient field $K$. Let $X$ be a normal proper flat $R$-scheme, with generic fiber $X$ and special fiber $Y$. Let $m \geq 1$ be an integer such that $Y$ is divisible by $m$ as a Cartier divisor on $X$. Then $m$ divides $\chi(X, \mathcal{O}_X)$.

**Proof.** By assumption, there is a Cartier divisor $D$ on $X$ satisfying the equality of Cartier divisors $Y = mD$. As $Y$ is effective and $X$ is normal, $D$ is effective, so that $\mathcal{O}_X(-D)$ is a sheaf of ideals of $\mathcal{O}_X$. For $j \in \{0, \ldots, m\}$, the closed subscheme $Y_j$ of $X$ defined by the ideal sheaf $\mathcal{O}_X(-jD)$ is contained in $Y_m = Y$ and may thus be regarded as a closed subscheme of $Y$. The corresponding ideal sheaf $\mathcal{I}_j \subseteq \mathcal{O}_Y$ fits into an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X(-mD) \longrightarrow \mathcal{O}_X(-jD) \longrightarrow i_Y^* \mathcal{I}_j \longrightarrow 0,$$

where $i_Y$ denotes the inclusion of $Y$ in $X$.

Let $i : D \rightarrow Y$ and $i_D : D \rightarrow X$ denote the canonical closed immersions, where in an abuse of notation $D$ stands for the scheme $Y_1$. For every $j \in \{0, \ldots, m-1\}$, there is an exact sequence of $\mathcal{O}_Y$-modules

$$0 \longrightarrow \mathcal{I}_{j+1} \longrightarrow \mathcal{I}_j \longrightarrow i_*(\mathcal{I}_j^{(j)}) \longrightarrow 0,$$
with \( \mathcal{L} = i_\ast D (\mathcal{O}_X (-D)) \). As \( \mathcal{I}_0 = \mathcal{O}_Y \) and \( \mathcal{I}_m = 0 \), we deduce that

\[
\chi(Y, \mathcal{O}_Y) = \sum_{j=0}^{m-1} \chi(D, \mathcal{O}_X^\otimes j).
\]

According to Kleiman’s version of Snapper’s theorem [24, Chapter I, §1], there exists a polynomial \( P \in \mathbb{Q}[X] \) such that \( P(n) = \chi(D, \mathcal{O}_X^\otimes n) \) for all \( n \in \mathbb{Z} \). The \( \mathcal{O}_X \)-module \( \mathcal{O}_X (-mD) \) is free since \( mD \) is a principal divisor on \( X \). Hence \( \mathcal{O}_X^\otimes m \) is a free invertible sheaf on \( D \). In particular, the polynomial \( P \) takes the value \( \chi(D, \mathcal{O}_D) \) on all integer multiples of \( m \). It is therefore a constant polynomial, equal to \( \chi(D, \mathcal{O}_D) \); we conclude, thanks to (2.1), that \( \chi(Y, \mathcal{O}_Y) = m \chi(D, \mathcal{O}_D) \).

Corollary 2.5. Let \( R \) be a Henselian discrete valuation ring with algebraically closed residue field of characteristic 0. Let \( X \) be a smooth proper variety over the quotient field \( K \) of \( R \). Then \( \text{ind}(X) \) divides \( \chi(X, \mathcal{O}_X) \).

Proof. Resolution of singularities for integral \( R \)-schemes of finite type holds, by [45, Theorem 1.1], so that Theorem 2.1 may be applied.

Example 2.6. A smooth proper curve of genus 2 over \( \mathbb{C}((t)) \) has index 1. More generally, a hyperelliptic curve of even genus over \( \mathbb{C}((t)) \) has index 1. These two assertions were already known (see Remark 2.8 below, or [17, Theorem 2]). By Theorem 2.1 they also hold for curves defined over the maximal unramified extension of a \( p \)-adic field, as regular proper models exist in this case (see [1]).

Example 2.7. If \( X \) is a smooth proper surface of general type over \( \mathbb{C}((t)) \), then \( \text{ind}(X) \) divides the plurigenera \( P_n = \dim H^0(X, \mathcal{O}_X(nK_X)) \) for all \( n \geq 2 \).

Indeed, if the canonical class \( K_X \) is nef, then

\[
P_n = \frac{n(n-1)}{2} (K_X)^2 + \chi(X, \mathcal{O}_X)
\]

for \( n \geq 2 \) (see [25, §4], [13, Ch. 10, Proposition 10]). In general, one can always find a smooth proper surface \( X' \) over \( \mathbb{C}((t)) \), birationally equivalent to \( X \), with nef canonical class (see [27, Ch. III, Theorem 2.2]). This implies the claim as both the plurigenera and the index are birational invariants among smooth proper varieties (see [10, §7.1], [15, Proposition 6.8]).

Let us note that \( \text{ind}(X) \) does not necessarily divide the first plurigenus \( P_1 = p_g \). Indeed, smooth quintic surfaces in \( \mathbb{P}^3 \) satisfy \( p_g = 4 \), and their index over \( \mathbb{C}((t)) \) may be equal to 5, as will result from Proposition 4.4 below.

Remark 2.8. Let \( C \) be a smooth proper curve over \( \mathbb{C}((t)) \). By Corollary 2.5, the index of \( C \) divides \( \text{deg}(K_C)/2 \), where \( K_C \) denotes a canonical divisor. A stronger statement is known to hold: even the class of \( K_C \) in \( \text{Pic}(C) \) is divisible by 2 (see Atiyah [3, p. 61], which rests on results of Serre and Mumford). One might wonder whether the same phenomenon occurs in higher dimension. Namely, in the situation of Corollary 2.5 does the 0-dimensional component of the Todd class \( \text{td}(T_X) \in \text{CH}^*(X) \otimes \mathbb{Q} \) belong to the image of \( \text{CH}_0(X) \to \text{CH}_0(X) \otimes \mathbb{Z} \mathbb{Q} \)?
3. INDEX OF PROPER SCHEMES OVER THE QUOTIENT FIELD OF AN EXCELLENT
HENSELIAN DISCRETE VALUATION RING

The following theorem, which builds upon Theorem 2.1, extends Corollary 2.5 in three directions: the residue field of \( R \) may have positive characteristic, the scheme \( X \) may be singular, and any coherent sheaf may be used instead of \( \mathcal{O}_X \).

Recall that a discrete valuation ring \( R \) is excellent if the field extension \( \hat{K}/K \) is separable, where \( K \) denotes the quotient field of \( R \) and \( \hat{K} \) is its completion. This condition is trivially satisfied if \( K \) has characteristic 0 or \( R \) is complete.

**Theorem 3.1.** Let \( R \) be an excellent Henselian discrete valuation ring with algebraically closed residue field \( k \) and quotient field \( K \). Let \( X \) be a proper \( K \)-scheme. Let \( E \) be a coherent sheaf on \( X \).

(i) If \( k \) has characteristic 0, then \( \text{ind}(X) \) divides \( \chi(X, E) \).

(ii) If \( k \) has characteristic \( p > 0 \), the prime-to-\( p \) part of \( \text{ind}(X) \) divides \( \chi(X, E) \).

**Proof.** The Grothendieck group of coherent sheaves on \( X \) is generated by the classes of \( \mathcal{O}_Z \) for all integral closed subschemes \( Z \) of \( X \) (see [5, §8, Lemme 17] and [4, Proposition 1.1]). Moreover, the index of \( X \) divides the index of any subscheme of \( X \). Therefore it suffices to show that \( \text{ind}(Z) \), or the prime-to-\( p \) part of \( \text{ind}(Z) \), divides \( \chi(Z, \mathcal{O}_Z) \), for all integral closed subschemes \( Z \) of \( X \).

This remark proves the theorem in case \( \dim(X) = 0 \). Indeed, for any integral closed subscheme \( Z \) of dimension 0, we have \( \text{ind}(Z) = \deg(Z) = \chi(Z, \mathcal{O}_Z) \). To establish the theorem in general, we argue by induction on \( d = \dim(X) \). Assume that the conclusion of the theorem holds for all proper \( K \)-schemes of dimension \( < d \). In order to prove that it also holds for \( X \), we may assume, thanks to the preceding paragraph, that \( X \) is integral and that \( E = \mathcal{O}_X \).

Let \( \ell \) be a prime number invertible in \( k \). We shall prove that the largest power of \( \ell \) which divides \( \text{ind}(X) \) also divides \( \chi(X, \mathcal{O}_X) \). By Nagata’s compactification theorem [11], there exists a proper model \( \mathcal{X} \) of \( X \) over \( R \). According to a theorem of Gabber and de Jong [22, Theorem 1.4], as \( R \) is excellent, there exist a regular scheme \( \mathcal{Y} \), and a proper, surjective, generically finite morphism \( h : \mathcal{Y} \to \mathcal{X} \) of degree \( N \) prime to \( \ell \). Let \( Y = \mathcal{Y} \otimes_R K \) and let \( f : Y \to X \) denote the morphism induced by \( h \).

Let \( U \subseteq X \) be a dense open subset above which \( f \) is finite and flat. Recall the exact sequence of Grothendieck groups of coherent sheaves

\[
\begin{align*}
G_0(C) \longrightarrow G_0(X) \longrightarrow G_0(U) \longrightarrow 0,
\end{align*}
\]

where \( C = X \setminus U \) (see [5, §8, Proposition 7]). The restriction to \( U \) of the coherent sheaf \( R^qf_*\mathcal{O}_Y \) is 0 for \( q > 0 \), and is a locally free \( \mathcal{O}_U \)-module of rank \( N \) for \( q = 0 \), which we may assume to be a free \( \mathcal{O}_U \)-module of rank \( N \) after further shrinking \( U \).

It then follows, thanks to [21], that the class in \( G_0(X) \) of the bounded complex of coherent sheaves \( Rf_*\mathcal{O}_Y \) is the sum of the class of \( \mathcal{O}_X^N \) and a class coming from \( G_0(C) \). Therefore there exists a virtual coherent sheaf \( F \) on \( C \) such that

\[
\chi(Y, \mathcal{O}_Y) = N\chi(X, \mathcal{O}_X) + \chi(C, F).
\]

As \( C \) is a subscheme of \( X \), the index of \( X \) divides the index of \( C \). Therefore, thanks to the induction hypothesis, if \( p = 0 \) then \( \text{ind}(X) \) divides \( \chi(C, F) \), and if \( p > 0 \), then the prime-to-\( p \) part of \( \text{ind}(X) \) divides \( \chi(C, F) \). Moreover, the index of \( X \) divides the index of \( Y \) because there exists a morphism from \( Y \) to \( X \). The index of \( Y \) in
turn divides $\chi(Y, \mathcal{O}_Y)$ according to Theorem 2.1. All in all, we deduce from (3.2) that the prime-to-$Np$ part of $\text{ind}(X)$ divides $\chi(X, \mathcal{O}_X)$. As $\ell$ is prime to $N$, this concludes the proof. □

Combining Theorem 3.1 and Proposition 1.2 yields:

**Theorem 3.2.** Let $R$ be a Henselian discrete valuation ring with algebraically closed residue field $k$ and quotient field $K$. Let $E$ be a coherent sheaf on $X$. Let $p$ denote the characteristic of $k$.

(i) If $p = 0$, then $\text{ind}(X)$ divides $\chi(X, E)$.

(ii) If $p > 0$, then the prime-to-$p$ part of $\text{ind}(X)$ divides $\chi(X, E)$.

(iii) If $p > \dim (X) + 1$, then $\text{ind}(X)$ divides $\chi(X, E)$.

**Proof.** In view of Lemma 3.3 below, we may assume that $R$ is complete and hence excellent. In this case, we may apply Theorem 3.1 and Proposition 1.2. □

**Lemma 3.3.** Let $R$ be a Henselian discrete valuation ring, with completion $\hat{R}$. Let $K$ and $\hat{K}$ denote the quotient fields of $R$ and $\hat{R}$ respectively. If $X$ is a smooth $K$-scheme, the index of $X$ over $K$ and the index of $X \otimes_K \hat{K}$ over $\hat{K}$ are equal.

**Proof.** Let $P$ be a closed point of $X \otimes_K \hat{K}$. Let $d$ be its degree. We must show that $\text{ind}(X)$ divides $d$. For this we may assume that the residue field $\hat{K}(P)$ of $P$ is separable over $\hat{K}$, by [15, Theorem 9.2]. Under this assumption, there exists a finite extension $L/K$ such that $L \otimes_K \hat{K} = \hat{K}(P)$, by Krasner’s lemma. After replacing $X$ with a neighborhood of $P$, we may assume in addition that $X$ is quasi-projective over $K$. The Weil restriction of scalars $R_{L/K}(X \otimes_K L)$ is then a smooth $K$-scheme (see [6, §7.6, Theorem 4 and Proposition 5]). Applying [6, §3.6, Corollary 10] to it now shows that $X(L) \neq \emptyset$. □

**Example 3.4.** According to Theorem 3.2, Example 2.7 is also valid over the maximal unramified extension of a $p$-adic field with $p \geq 5$.

As immediate consequence of Theorem 3.2 we have:

**Corollary 3.5.** We keep the assumptions of Theorem 3.2. Assume $|\chi(X, \mathcal{O}_X)| = 1$. Then $\text{ind}(X)$ is a power of $p$. If moreover $p = 0$ or $\dim (X) < p-1$, then $\text{ind}(X) = 1$.

Corollary 3.5 applies to geometrically irreducible smooth varieties over $K$ which satisfy $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, for instance Fano varieties or more generally rationally connected varieties when $K$ has characteristic 0 (see [27, Ch. IV]; we say that a variety $X$ over $K$ is rationally connected if $X \otimes_K \hat{K}$ is rationally connected in the sense of [27], where $\hat{K}$ denotes an algebraic closure of $K$). We thus obtain:

**Corollary 3.6.** Let $X$ be a smooth proper rationally connected variety over the maximal unramified extension of a $p$-adic field. The index of $X$ is a power of $p$. If moreover $\dim (X) < p-1$, the index of $X$ is equal to 1.

Corollary 3.6 gives some evidence for the conjecture according to which any rationally connected variety over the maximal unramified extension of a $p$-adic field possesses a rational point.

Theorem 3.2 also has interesting consequences when applied to other coherent sheaves than $\mathcal{O}_X$. As an illustration, we extend Examples 2.7 and 3.4 to higher-dimensional general type varieties. (The argument of Example 2.7 fails in higher
Corollary 3.7. Let $X$ be a smooth proper variety of general type over $\mathbb{C}((t))$ or over the maximal unramified extension of a $p$-adic field with $p > \dim(X) + 1$. Then $\text{ind}(X)$ divides the plurigenera $P_n = \dim H^n(X, \mathcal{O}_X(nK_X))$ for all $n \geq 2$.

Proof. According to Kollár and Lazarsfeld, for $n \geq 2$, the $n$th plurigenus of a smooth proper variety $X$ of general type over a field of characteristic $0$ may be expressed as the Euler–Poincaré characteristic of the coherent sheaf

$$\mathcal{O}_X(nK_X) \otimes \mathcal{I}(\| (n-1)K_X \|),$$

where $\mathcal{I}(\| (n-1)K_X \|)$ denotes the asymptotic multiplier ideal sheaf associated to the complete linear system $\| (n-1)K_X \|$ (see [31, §11.2.C], [28, Example 2.5]). □

4. Application: hypersurfaces in projective space

We illustrate the results of the previous sections by examining the case of hypersurfaces in projective space. It turns out that a simple application of Theorem 3.1 yields the best possible bound on the index of a degree $d$ hypersurface in $\mathbb{P}^N$ over the quotient field of an excellent Henselian discrete valuation ring with algebraically closed residue field, for any $d$ and $N$ (Theorem 4.1 and Proposition 4.4). For certain values of $d$ and $N$, this bound is equal to $1$, thus yielding unexpected existence results for zero-cycles of degree $1$ (Examples 4.2 (i) and 4.6).

Given two integers $d$, $N$, let

$$I_{d,N} = \gcd \left\{ \frac{d}{\delta} \mid \delta \in \{1, \ldots, N\} \text{ and } \delta \text{ divides } d \right\}.$$ 

Theorem 4.1. Let $R$ be an excellent Henselian discrete valuation ring with algebraically closed residue field $k$ and quotient field $K$. For any hypersurface $X \subset \mathbb{P}^N_K$ of degree $d$, one has:

(i) If $k$ has characteristic $0$, then $\text{ind}(X)$ divides $I_{d,N}$.
(ii) If $k$ has characteristic $p > 0$, the prime-to-$p$ part of $\text{ind}(X)$ divides $I_{d,N}$.
(iii) If $p > N$, then $\text{ind}(X)$ divides $I_{d,N}$.

Example 4.2. (i) Let $K$ denote either the field $\mathbb{C}((t))$, or the maximal unramified extension of a $p$-adic field with $p \geq 5$. Then any sextic hypersurface in $\mathbb{P}^3$ over $K$ has a zero-cycle of degree $1$, and so does any hypersurface of degree $12$ in $\mathbb{P}^4$.
(ii) If $d \leq N$, then $I_{d,N} = 1$. In this case, it is even true that $X(K) \neq \emptyset$, according to a theorem of Lang [30].

Proof of Theorem 4.1. For $n \in \{1, \ldots, N\}$, let $X_n$ denote the intersection of $X$ with a linear subspace of $\mathbb{P}^N$ of dimension $n$. Let $p \geq 0$ be the characteristic of $k$. According to Theorem 3.1, the index of $X$, if $p = 0$, or its prime-to-$p$ part if $p > 0$, divides $\chi(X, \mathcal{O}_{X_n})$ for every $n$. On the other hand, as $X_n$ is a degree $d$ hypersurface in $\mathbb{P}^n_K$, we have $\chi(X, \mathcal{O}_{X_n}) = \chi_{d,n}$, where

$$\chi_{d,n} = 1 - (-1)^n \binom{d-1}{n}.$$ 

The following lemma thus concludes the proof of (i) and (ii).

Lemma 4.3. For any $d \geq 1$ and $N \geq 1$, we have $I_{d,N} = \gcd \{ \chi_{d,n} \mid 1 \leq n \leq N \}$. 


Proof. We may rewrite $\chi_{d,n}$ as

$$\chi_{d,n} = 1 - \prod_{1 \leq i \leq n} \left( 1 - \frac{d}{\gcd(i, d)} \cdot \frac{1}{\gcd(i, n)} \right).$$

The integers $i/\gcd(i, d)$ and $d/\gcd(i, d)$ are coprime, and $I_{d,N}$ divides $d/\gcd(i, d)$ for $i \leq N$, therefore $i/\gcd(i, d)$ and $I_{d,N}$ are coprime for $i \leq N$. Thus, if $n \leq N$, each factor appearing the above expression makes sense in $\mathbb{Z}/I_{d,N}\mathbb{Z}$. As $I_{d,N}$ divides $d/\gcd(i, d)$ for $i \leq N$, we conclude that $I_{d,N}$ divides $\chi_{d,n}$ for all $n \leq N$.

It remains to be shown that $\gcd\{\chi_{d,n} : 1 \leq n \leq N\}$ divides $I_{d,N}$. For $N = 1$ this is clear. Assume $N \geq 2$ and $\gcd\{\chi_{d,n} : 1 \leq n \leq N - 1\}$ divides $I_{d,N-1}$. If $I_{d,N} = I_{d,N-1}$, there is nothing to prove. Otherwise $N$ must divide $d$, and the desired result follows from the equality

$$1 - \chi_{d,N} = (1 - \chi_{d,N-1}) \left( 1 - \frac{d}{N} \right)$$

and from the induction hypothesis. \qed

Let us turn to (iii). Denoting by $v_p(n)$ the $p$-adic valuation of an integer $n$, we remark that if $p > N$, then $v_p(I_{d,N}) = v_p(d)$. As $X$ is a degree $d$ hypersurface in projective space, the index of $X$ divides $d$. Therefore $v_p(\text{ind}(X)) \leq v_p(I_{d,N})$ if $p > N$. In view of (ii), this completes the proof of Theorem \[11\] \[11\] \[11\].

Proposition \[4.4\] below shows that the bound given in Theorem \[4.1\] is optimal, as it is attained by “maximally twisted” Fermat hypersurfaces.

**Proposition 4.4.** Let $K$ be the quotient field of a discrete valuation ring $R$, and let $X_{d,N} \subset \mathbb{P}_K^N$ be the smooth hypersurface defined by

\[ (4.1) \quad x_0^d = \sum_{i=1}^{N} \pi^i x_i^d, \]

where $\pi$ is a uniformiser of $R$. Then $\text{ind}(X_{d,N}) = I_{d,N}$ for any $d, N \geq 1$.

**Proof.** For any $\delta \in \{1, \ldots, N\}$ which divides $d$, the $K(\pi^{\delta/d})$-point with coordinates $x_0 = 1$, $x_\delta = \pi^{-\delta/d}$, $x_i = 0$ for $i \notin \{0, \delta\}$ lies on $X_{d,N}$. Therefore $\text{ind}(X_{d,N})$ divides $I_{d,N}$.

To prove that $I_{d,N}$ divides $\text{ind}(X_{d,N})$, we may assume, by extending scalars, that $K$ is complete. Let us fix a closed point $x \in X$ and show that $I_{d,N}$ divides $e = \deg(x)$. As $K$ is complete, the integral closure $R'$ of $R$ in the residue field $K'$ of $x$ is a discrete valuation ring, and the corresponding valuation $v : K'^* \to \mathbb{Z}$ satisfies $v(\pi) = e$ (see \[11\] Ch. II, § 2, Prop. 3 and Cor. 1).

Write the coordinates of $x$ as $x = [x_0 : \cdots : x_N]$ where all $x_i$’s belong to $R'$ and one of them is a unit. For $a = (a_0, \ldots, a_N) \in \mathbb{Z}^{N+1}$, let us denote by $m(a)$ the minimum of $ie + da_i$ over all $i \in \{0, \ldots, N\}$.

**Lemma 4.5.** Assume $I_{d,N}$ does not divide $e$. Then for any $a \in \mathbb{Z}^{N+1}$ such that $v(x_i) \geq a_i$ for all $i$, there exists $b \in \mathbb{Z}^{N+1}$ such that $v(x_i) \geq b_i$ for all $i$ and such that $m(b) > m(a)$.

**Proof.** We first remark that the map $i \mapsto ie + da_i$ is injective on $\{0, \ldots, N\}$. Indeed, assume there exist $i, j$ with $0 \leq i < j \leq N$ such that $ie + da_i = je + da_j$. Let
\[ \delta = \gcd(j - i, d). \] Thus \( \delta \in \{1, \ldots, N\} \) and \( d/\delta \) divides \( e \), since \( (j - i)e = d(a_i - a_j) \).

Thus \( I_{d_N} \) divides \( e \), a contradiction.

In particular, there is a unique \( i(a) \in \{0, \ldots, N\} \) such that \( i(a)e + da_i(a) = m(a) \).

Let \( b_i = a_i \) for \( i \neq i(a) \) and \( b_{i(a)} = a_{i(a)} + 1 \). It is clear that \( m(b) > m(a) \). Let us note, moreover, that \( v(\pi^i x_i^d) > m(a) \) for \( i \neq i(a) \). Thanks to (1.1), it follows that \( v(\pi^i x_i^d) > m(a) \) for \( i = i(a) \) as well. In other words \( i(a)e + dv(x_{i(a)}) > i(a)e + da_{i(a)} \), hence \( v(x_{i(a)}) > a_{i(a)}. \)

As \( v(x_i) \geq v(a_i) \) for all \( i \), we conclude that \( v(x_i) \geq v(b_i) \) for all \( i \).

Assume \( I_{d_N} \) does not divide \( e \). A repeated application of Lemma 4.5 starting with \( 0 \in \mathbb{Z}^{N+1} \), yields an \( a \in \mathbb{Z}^{N+1} \) such that \( v(x_i) \geq a_i \) for all \( i \) and \( m(a) > Ne \). The condition \( m(a) > Ne \) implies \( a_i > 0 \) for all \( i \), which in turn contradicts the hypothesis that one of the \( x_i \)'s is a unit.

**Example 4.6.** Let \( d \geq 1 \). Write \( d = \prod_{i=1}^{n} p_i^{\alpha_i} \) with pairwise distinct prime numbers \( p_i \). Let \( N_0 = \max(p_1^{\alpha_1}, \ldots, p_n^{\alpha_n}) \). Theorem 4.1 and Proposition 4.3 show that the property “any degree \( d \) hypersurface in \( \mathbb{P}^N \) over \( \mathbb{C}((t)) \) possesses a zero-cycle of degree 1” holds if and only if \( N \geq N_0 \).

## 5. AN EXTENSION TO COBORDISM

The goal of the present section is to prove that in the situation of Theorem 3.2 not only does the index of \( X \) over \( K \) (or its prime-to-\( p \) part) divide the Euler–Poincaré characteristic of any vector bundle on \( X \), but it also divides the class of \( X \) in the cobordism ring of \( \text{Spec}(K) \) (Theorem 5.1). Since \( K \) may have positive characteristic, we rely on an ad hoc definition for the cobordism ring of \( \text{Spec}(K) \). The definition we use is motivated by the fact that any element of the complex cobordism ring is determined by the set of its Chern classes (see [41, p. 117, Theorem]). As a consequence of Theorem 5.1 integral-valued rational characteristic classes yield bounds on the index of smooth proper schemes over the quotient field of a Henselian discrete valuation ring with algebraically closed residue field (Corollary 5.6). Examples 5.8 and 5.13.

We start by defining the cobordism ring of \( \text{Spec}(K) \). Let \( \mathbb{Z}[b] = \mathbb{Z}[b_1, b_2, \ldots] \) denote a polynomial ring in countably many variables. Let \( \mathcal{A} \) be the set of all sequences \((\alpha_j)_{j \geq 1}\) of nonnegative integers all but finitely many of which are zero. For \( I \in \mathcal{A} \), let \( |I| = \sum j \alpha_j \) and \( b^I = \prod b_i^{\alpha_j} \). Given a smooth proper connected scheme \( X \) of dimension \( d \) over a field \( K \), we define, following Merkurjev [35], the fundamental polynomial \( b_K(X) \in \mathbb{Z}[b] \) of \( X \) by the formula

\[
\begin{align*}
b_K(X) &= \sum_{|I|=d} \deg(c_I(-T_X))b^I,
\end{align*}
\]

where \( c_I(-T_X) \in CH_0(X) \) denotes the Conner–Floyd Chern class of the virtual vector bundle \(-T_X \in K_0(X)\) associated to \( I \). (The Conner–Floyd Chern classes are polynomials, with integer coefficients, in the usual Chern classes; their definition is recalled below.) For an arbitrary smooth proper \( K \)-scheme \( X \), let \( b_K(X) \) be the sum of the fundamental polynomials of the connected components of \( X \). The cobordism ring of \( \text{Spec}(K) \) is by definition the subring \( \mathbb{L}_K \subset \mathbb{Z}[b] \) generated by the fundamental polynomials of all irreducible smooth proper schemes \( X \) over \( K \).
Even though we shall not use this property, let us remark that all elements of $L_K$ are in fact fundamental polynomials of smooth proper schemes over $K$. Indeed, the map $X \mapsto b_K(X)$ takes disjoint unions to sums, products to products (a consequence of the Whitney sum formula for Conner–Floyd Chern classes, see [2, Theorem 4.1]), and the set of fundamental polynomials of smooth proper $K$-schemes is stable under $b \mapsto -b$ according to Thom (see [40] §5).

We may now state the main result of this section.

**Theorem 5.1.** Let $R$ be a Henselian discrete valuation ring with algebraically closed residue field $k$ and quotient field $K$. Let $X$ be a smooth proper irreducible $K$-scheme. Let $p$ denote the characteristic of $k$.

(i) If $p = 0$, then $b_K(X)$ is divisible by $\text{ind}(X)$ in the ring $L_K$.

(ii) If $p > 0$, then $b_K(X)$ is divisible, in $L_K$, by the prime-to-$p$ part of $\text{ind}(X)$.

(iii) If $p > \dim(X) + 1$, then $b_K(X)$ is divisible by $\text{ind}(X)$ in $L_K$.

**Remark 5.2.** According to Quillen [40] Theorem 6.5, the complex cobordism ring is canonically isomorphic to the Lazard ring, which by definition is the coefficient ring of the universal rank one commutative formal group law. Merkurjev [35, Theorem 8.2] has shown that $L_K$, for any field $K$, is also canonically isomorphic to the Lazard ring. See our brief discussion of complex cobordism below for a more detailed description of the relation between the Lazard ring, the complex cobordism ring and the polynomial ring $Z[b]$.

Before proving Theorem 5.1 we set up some notation and state a few lemmas. For the time being $K$ denotes an arbitrary field.

Let $Z[c] = Z[c_1, c_2, \ldots]$ denote another polynomial ring in countably many variables. Let $I = (\alpha_j)_{j \geq 1} \in \mathcal{I}$. For $n \geq |I|$, consider the monomial symmetric polynomial in $n$ indeterminates

$$\sigma_I = \sum \xi_1^{m_1} \cdots \xi_n^{m_n} \in Z[\xi_1, \ldots, \xi_n],$$

where the sum ranges over all $n$-tuples of nonnegative integers $(m_1, \ldots, m_n)$ such that for each $j \geq 1$, exactly $\alpha_j$ of the $m_i$'s are equal to $j$. As $\sigma_I$ is invariant under permutations of the $\xi_1$'s, there is a unique $c_I \in Z[c]$ such that $\sigma_I = c_I(\sigma_1, \ldots, \sigma_n)$, where $\sigma_i$ denotes the $i$th elementary symmetric polynomial in the $\xi_j$'s. The polynomial $c_I \in Z[c]$ thus defined does not depend on the choice of $n \geq |I|$ (see [33] Ch. I, §2).

We recall that the family $(c_I)_{I \in \mathcal{I}}$ forms a basis of the $Z$-module $Z[c]$, and that for $I \in \mathcal{I}$, the Conner–Floyd Chern class $c_I(E)$ of a virtual vector bundle $E$ on a smooth $K$-scheme $X$ is by definition the image of $c_I$ by the ring homomorphism $Z[c] \to \text{CH}^*(X)$, $c_i \mapsto c_i(E)$.

Let us consider $Q[b]$ and $Q[c]$ as graded algebras by letting $\deg(b_i) = i$ and $\deg(c_i) = i$. The fundamental polynomial of an irreducible smooth proper scheme of dimension $d$ over $K$ is thus homogeneous of degree $d$, and for any $I \in \mathcal{I}$, the polynomial $c_I$ is homogeneous of degree $|I|$. For $d \geq 0$, let $Q[b]_d$ and $Q[c]_d$ denote the homogeneous components of degree $d$ of $Q[b]$ and $Q[c]$, and let $\mathcal{I}_d = \{I \in \mathcal{I}; |I| = d\}$. We define a perfect pairing of finite-dimensional $Q$-vector spaces

$$\langle \ , \ \rangle : Q[c]_d \times Q[b]_d \to Q$$

by declaring that the bases $(c_I)_{I \in \mathcal{I}_d}$ and $(b^I)_{I \in \mathcal{I}_d}$ are dual to each other.
We need to briefly recall a few facts about complex cobordism; we refer the reader to [2, Chapter 1, §§1–4], [14] Chapters I–IV], and [40] §1 for details. Let \( \pi_\ast(MU) \) denote the complex cobordism ring, i.e., the generalized homology of the point with values in the Thom spectrum. To any compact almost complex manifold \( M \) is associated an element \([M]\) of the complex cobordism ring \( \pi_\ast(MU) \). These classes generate \( \pi_\ast(MU) \) as an abelian group. The Hurewicz map \( \pi_\ast(MU) \to H_\ast(MU, \mathbb{Z}) \) and the Thom isomorphism \( H_\ast(MU, \mathbb{Z}) \cong H_\ast(BU, \mathbb{Z}) \) yield a map \( b : \pi_\ast(MU) \to H_\ast(BU, \mathbb{Z}) \), which is known to be injective. If we endow \( H_\ast(BU, \mathbb{Z}) \) with the ring structure induced by the direct sum map \( \oplus : BU \times BU \to BU \), then \( b \) becomes a ring homomorphism. Thus \( b \) identifies the complex cobordism ring \( \pi_\ast(MU) \) with a subring of \( \mathbb{Z}[b] \); we shall denote this subring by \( \mathcal{L} \). For any compact almost complex manifold \( M \), the image in \( \mathbb{Z}[b] \) of \([M]\) is a lattice in \( \pi_\ast(MU) \) as described by the defining formula of the fundamental polynomial \( \xi_1 \) (see [40] (6.2), p. 49).

Let \( \text{Laz} \) denote the Lazard ring, that is, the coefficient ring of the universal one commutative formal group law. Setting \( \lambda(t) \in \mathbb{Z}[b][[t]] \) be the power series \( t + \sum_{n \geq 1} b_n t^{n+1} \) and \( \lambda^{-1}(t) \) the inverse of \( \lambda(t) \) with respect to composition of power series, \( \text{Laz} \) embeds into \( \mathbb{Z}[b] \) via the classifying homomorphism associated to the formal group law \( F(u, v) = \lambda(\lambda^{-1}(u) + \lambda^{-1}(v)) \) with coefficients in \( \mathbb{Z}[b] \) (see for example [2, II, §7]). According to Quillen [40] Theorem 6.5], the image of \( \text{Laz} \) in \( \mathbb{Z}[b] \) is exactly the subring \( \mathcal{L} \). Finally, Merkurjev’s result [35, Theorem 8.2] states that \( \mathcal{L}_K = \mathcal{L} \) for all fields \( K \), a fact that we shall reprove below using the Hattori–Stong theorem (Proposition 5.3).

According to Milnor (see, e.g., [2, p. 86, Corollary 10.8]), the ring \( \mathcal{L} \) is generated by the classes of projective spaces \( \mathbb{P}^n_C \) for \( n \geq 1 \) and of the so-called Milnor hypersurfaces \( H_{m,n} \) with \( 2 \leq m \leq n \), where \( H_{m,n} \) is the smooth hypersurface of bidegree \((1,1)\) in \( \mathbb{P}^n_C \times \mathbb{P}^m_C \) defined by the equation \( \sum_{i=0}^n x_i y_i = 0 \).

**Lemma 5.3.** We have the inclusion \( \mathcal{L} \subseteq \mathcal{L}_K \) of subrings of \( \mathbb{Z}[b] \).

**Proof.** If \( \mathfrak{X} \) is a smooth proper scheme over \( \mathbb{Z} \), the polynomial \( b_K(\mathfrak{X} \otimes \mathbb{Z} K) \) does not depend on the field \( K \). As complex projective spaces and Milnor hypersurfaces possess smooth proper models over \( \text{Spec} \mathbb{Z} \), the lemma follows. \( \square \)

For any \( d \geq 0 \), let \( \mathcal{L}_d = \mathcal{L} \cap \mathbb{Q}[b]_d \) and \( \mathcal{L}_{K,d} = \mathcal{L}_K \cap \mathbb{Q}[b]_d \). We recall that \( \mathcal{L}_d \) is a lattice in \( \mathbb{Q}[b]_d \) (i.e., a finitely generated subgroup containing a basis), see [14] p. 117, Theorem]. As \( \mathcal{L} \subseteq \mathcal{L}_K \subseteq \mathbb{Z}[b] \), the group \( \mathcal{L}_{K,d} \) is also a lattice in \( \mathbb{Q}[b]_d \). Let \( I_{K,d} = \{ c \in \mathbb{Q}[c]_d : \langle c, \mathcal{L}_{K,d} \rangle \subseteq \mathbb{Z} \} \) denote the lattice dual to \( \mathcal{L}_{K,d} \) with respect to \( \langle \cdot, \cdot \rangle \) in \( \mathbb{Q}[c]_d \). Similarly, let \( I_d \subseteq \mathbb{Q}[c]_d \) denote the lattice dual to \( \mathcal{L}_d \).

We need to introduce one more subgroup of \( \mathbb{Q}[c]_d \). For any polynomial \( f \in \mathbb{Z}[t_1, \ldots, t_d] \) invariant under permutations of the \( t_i \)'s, the homogeneous part of degree \( d \) of the power series

\[
(5.3) \quad f(e^{\xi_1}, \ldots, e^{\xi_d}) \prod_{j=1}^d \frac{\xi_j}{1 - e^{-\xi_j}} \in \mathbb{Q}[[\xi_1, \ldots, \xi_d]]
\]

is symmetric in the \( \xi_i \)'s. Thus it can be written \( R_f(\sigma_1, \ldots, \sigma_d) \) for a unique \( R_f \in \mathbb{Q}[c]_d \), where \( \sigma_i \) denotes the \( i \)th elementary symmetric polynomial in the \( \xi_i \)'s. Let \( s_1, s_2, \ldots, s_d \in \mathbb{Z}[c] \) be the Segre polynomials, defined by the formula

\[
(5.4) \quad 1 + \sum_{i \geq 1} s_i t^i = \left( 1 + \sum_{i \geq 1} c_i t^i \right)^{-1} \in \mathbb{Z}[c][[t]].
\]
Let $S_f = R_f(s_1, \ldots, s_d) \in \mathbb{Q}[c_d]$. Let $I'_d \subset \mathbb{Q}[c_d]$ be the subgroup consisting of the polynomials $S_f$ when $f$ ranges over all symmetric polynomials in $\mathbb{Z}[t_1, \ldots, t_d]$. Finally, let $I_K = \bigoplus_{d \geq 0} I_{K,d}$, $I = \bigoplus_{d \geq 0} I_d$, and $I' = \bigoplus_{d \geq 0} I'_d$. The definition of $I'$ is motivated by the following lemma.

**Lemma 5.4.** Let $X$ be a smooth proper irreducible variety over a field $K$, of dimension $d$. Let $m_1, \ldots, m_d$ be nonnegative integers, and let $f = \prod_{i} \tau_i^{m_i}$, where $\tau_i$ denotes the $i$th elementary symmetric polynomial in $t_1, \ldots, t_d$. Then

$$\langle S_f, b_K(X) \rangle = \chi \left( X, \bigotimes_{i=1}^d \left( \bigwedge T_X \right)^{\otimes m_i} \right).$$

**Proof.** The number $\langle S_f, b_K(X) \rangle$ is the degree of the element of $\text{CH}_0(X) \otimes \mathbb{Q}$ obtained by evaluating the polynomial $R_f \in \mathbb{Q}[c]$ on the Chern classes of $T_X$. In addition, if $\xi_1, \ldots, \xi_d$ denote the Chern roots of $T_X$, we have

$$\text{ch} \left( \bigotimes_{i=1}^d \left( \bigwedge T_X \right)^{\otimes m_i} \right) = \prod_{i} \text{ch} \left( \bigwedge T_X \right)^{m_i} = f(\xi^1, \ldots, \xi^d).$$

Hence the lemma results from the Hirzebruch–Riemann–Roch theorem. \qed

As $S_f$ depends linearly on $f$ and as any symmetric polynomial $f \in \mathbb{Z}[t_1, \ldots, t_d]$ is a $\mathbb{Z}$-linear combination of monomials $\prod_{i} \tau_i^{m_i}$, Lemma 5.4 gives the value of $\langle S_f, b_K(X) \rangle$ for any $S_f \in I'_d$. In particular, it implies that $I' \subseteq I_K$; thus, according to Lemma 5.2, we have $I' \subseteq I_K \subseteq I$. On the other hand, a theorem due to Hattori and Stong asserts that $I' = I$ (see [13, Theorem 1], [18, Theorem 1]). As a result, letting $L'_d = \{ b \in \mathbb{Q}[b_d] : \langle I'_d, b \rangle \subseteq \mathbb{Z} \}$ and $L' = \bigoplus_{d \geq 0} L'_d$, we have established the following proposition:

**Proposition 5.5.** For any field $K$, we have $I' = I_K = I$ and $L = L_K = L'$.

We are now in a position to complete the proof of Theorem 5.1.

**Proof of Theorem 5.1.** Let $X$ and $K$ be as in the statement of the theorem. Let $n = \text{ind}(X)$ if $p = 0$ or $p > \text{dim}(X) + 1$, otherwise let $n$ denote the prime-to-$p$ part of $\text{ind}(X)$. According to Theorem 3.2 and Lemma 5.4, we have $\langle I', b_K(X) \rangle \subseteq n \mathbb{Z}$. In other words, the class $b_K(X)$ is divisible by $n$ in $L'$. Thanks to Proposition 5.5, we deduce that it is divisible by $n$ in $L_K$. \qed

Corollary 5.6 is an essentially equivalent reformulation of Theorem 5.1 in terms of characteristic numbers. To ease notation we write $P(T_X)$ for $P(c_1(T_X), \ldots, c_d(T_X))$.

**Corollary 5.6.** Let $R$ be a Henselian discrete valuation ring with algebraically closed residue field $k$ and quotient field $K$. Let $p$ denote the characteristic of $k$. Let $d \geq 1$ and let $P \in \mathbb{Q}[c_1, \ldots, c_d]$ be homogeneous of degree $d$ with respect to the grading $\text{deg}(c_i) = i$. Assume that $\text{deg}(P(T_X)) \in \mathbb{Z}$ for any $d$-dimensional product $X$ of complex projective spaces and Milnor hypersurfaces. Then, for any smooth proper irreducible $K$-scheme $X$ of dimension $d$, the rational number $\text{deg}(P(T_X))$ is an integer, and we have:

(i) If $p = 0$, then $\text{ind}(X)$ divides $\text{deg}(P(T_X))$.
(ii) If $p > 0$, then the prime-to-$p$ part of $\text{ind}(X)$ divides $\text{deg}(P(T_X))$.
(iii) If $p > \text{dim}(X) + 1$ or if the denominators of the coefficients of $P$ are prime to $p$, then $\text{ind}(X)$ divides $\text{deg}(P(T_X))$. 


Proof. Let $Q = P(s_1, \ldots, s_d) \in \mathbb{Q}[c]_d$, where $s_1, \ldots, s_d$ are the Segre polynomials (see (5.4)). The linear form $Q[b, d] \rightarrow \mathbb{Q}$, $b \mapsto \langle Q, b \rangle$ maps $b_K(X)$ to $\deg(P(T_X))$, and is integral-valued on $\mathbf{L}_d$ since $\mathbf{L}_d$ is spanned by the fundamental polynomials of $d$-dimensional products of complex projective spaces and Milnor hypersurfaces. Thus, by Proposition 5.5 it restricts to a homomorphism $\mathbf{L}_{K, d} \rightarrow \mathbb{Z}$. Applying Theorem 5.1 now yields the desired result, noting, for the third assertion, that if $n$ is the lowest common multiple of the denominators of the coefficients of $P$, then the index of $X$ divides $n \deg(P(T_X))$ since $nP(T_X) \in \text{CH}_0(X)$. \hfill \Box

Remark 5.7. If $P \in \mathbb{Q}[c]$ satisfies the hypothesis of Corollary 5.6 Proposition 5.5 implies that for any field $K$ and any smooth proper irreducible $K$-scheme $X$ of dimension $d$, the rational number $\deg(P(T_X))$ may be written, in a way which does not depend on $X$, as a $\mathbb{Z}$-linear combination of Euler–Poincaré characteristics of tensor products of exterior powers of $T_X$. In other words, there exists a virtual representation $\rho$ of $\text{GL}_{d, K}$ such that $\deg(P(T_X)) = \chi(X, \rho(T_X))$.

More precisely, by a theorem of Serre [12, §3.6, Théorème 4], the Grothendieck group of the category of representations of $\text{GL}_{d, K}$ over $K$ does not depend on the field $K$: sending a representation $\rho$ to its character (viewed as a symmetric function in the characters $t_1, \ldots, t_d$ of a maximal torus $\mathbf{G}^d_{d, K} \subset \text{GL}_d$) induces an isomorphism

$$K_0(\text{Rep}_K(\text{GL}_{d, K})) \simeq \mathbb{Z}[t_1, \ldots, t_d]^{\otimes d}[\frac{1}{t_1 \cdots t_d}].$$

Thus, writing $P$ as $S_f$ for an $f \in \mathbb{Z}[t_1, \ldots, t_d]^{\otimes d}$ and letting $\rho$ be the unique virtual representation of $\text{GL}_{d, K}$ with character $f$, we have $\deg(P(T_X)) = \chi(X, \rho(T_X))$ for any field $K$ and any smooth proper irreducible $K$-scheme $X$ of dimension $d$.

The remainder of this section is devoted to examples. From now on, the letter $K$ will always denote the quotient field of a Henselian discrete valuation ring with algebraically closed residue field of characteristic $p \geq 0$.

Example 5.8. The polynomial $P = \frac{1}{2}c_d$ satisfies the hypothesis of Corollary 5.6 if $d$ is odd. Indeed, by Poincaré duality, the topological Euler–Poincaré characteristic of any odd-dimensional compact complex manifold is even.

As a consequence, if $p \neq 2$, the index of any odd-dimensional smooth proper irreducible $K$-scheme $X$ divides $\frac{1}{2}e(X)$. Here $e(X)$ denotes the $\ell$-adic Euler–Poincaré characteristic of $X \otimes_K \hat{K}$ for any $\ell$ invertible in $K$, where $\hat{K}$ is a separable closure of $K$ (see [23 Corollaire 4.9]).

In this example, it is easy to exhibit the virtual representation $\rho$ of $\text{GL}_d$ whose existence is predicted by Remark 5.7. Namely, if $V$ denotes the standard representation of $\text{GL}_d$ and $V^*$ is its dual, the virtual representation

$$\rho(V) = \sum_{i=0}^{d-1} (-1)^i \bigwedge^i V^*$$

satisfies $\frac{1}{2}e(X) = \chi(X, \rho(T_X))$. Indeed, we have $e(X) = \sum_{i=0}^{d} (-1)^i \chi(X, \Omega^i_{X/K})$ according to [23 Proposition 4.11], and $(-1)^i \chi(X, \Omega^i_{X/K}) = (-1)^{d-i} \chi(X, \Omega^{d-i}_{X/K})$ for all $i$ by Serre duality.

Example 5.9. The polynomial $P = \frac{1}{2}c_d^2$ satisfies the hypothesis of Corollary 5.6 if $d$ is odd. To see this, first note that if $X = A \times B$ with $\text{dim}(A) = a$, $\text{dim}(B) = b$, then
then
\[ \text{deg}(c^{a+b}_1(T_\mathcal{X})) = \left(\frac{a+b}{a}\right)\text{deg}(c^a_1(T_\mathcal{M}))\text{deg}(c^b_1(T_\mathcal{N})); \]
thus it suffices to check that \( P \) is integral-valued on odd-dimensional projective spaces and Milnor hypersurfaces. We have \( \frac{1}{2}\deg(c^d_1(T_\mathcal{M})) = \frac{1}{2}(-1)^d(d+1)^d \), which is an integer when \( d \) is odd, and if \( d = m + n - 1 \) with \( 2 \leq m \leq n \), the adjunction formula shows that
\[ \frac{1}{2}\deg(c^d_1(H_{m,n})) = (-1)^d \binom{d-1}{m-1} m^{m-1} n^{n-1} d, \]
which is an integer as well.

Hence, if \( p \neq 2 \), the index of any smooth proper irreducible \( K \)-scheme \( \mathcal{X} \) of odd dimension \( d \) divides \( \frac{1}{2}(K_X)^d \). In contrast with the previous example, we were unable in this case to find a closed-form formula (valid for all odd \( d \)) for a virtual representation \( \rho_d \) of \( \text{GL}_d \) such that \( \frac{1}{2}(K_X)^d = \chi(\rho_d(T_\mathcal{X})) \) for all \( d \)-dimensional \( \mathcal{X} \). Such virtual representations do exist as a consequence of the Hattori–Stong theorem (see Remark 5.7).

**Example 5.10.** Let \( d \) be an even integer. For any compact complex manifold \( \mathcal{X} \) of dimension \( d \), the symmetric bilinear form \( H^d(X, \mathbb{R}) \times H^d(X, \mathbb{R}) \to \mathbb{R} \) given by cup-product is symmetric, and hence has a well-defined signature \( \sigma(X) \). Hirzebruch’s signature formula [20, Theorem 8.2.2] furnishes a homogeneous degree \( d \) polynomial \( P_d \in \mathbb{Q}[c] \) such that \( \sigma(X) = \deg(P_d(T_\mathcal{X})) \) for all such \( \mathcal{X} \). Specifically, to obtain \( P_d \), evaluate the polynomial denoted \( L_d/2 \) in [20, §1.5] at \( p_i = \sum_{j=0}^{2i}(-1)^{i+j}c_{2i-j}, \) with the convention that \( c_0 = 1 \). The first values of \( P_d \) are \( P_2 = \frac{1}{3}c_2^3 - \frac{4}{9}c_2, \) \( P_4 = \frac{1}{90}(14c_4 - 14c_1c_3 + 3c_2^2 + 4c_2c_1^2 - c_1^3) \), etc.

For any field \( k \) and any smooth proper irreducible \( k \)-scheme \( \mathcal{X} \) of even dimension, we may define the “signature” of \( \mathcal{X} \) as \( \text{sig}(\mathcal{X}) = \deg(P_{\dim(\mathcal{X})}(T_\mathcal{X})) \). In case \( \mathcal{X} \) admits an embedding \( \tau : k \hookrightarrow \mathbb{C} \), we have \( \text{sig}(\mathcal{X}) = \sigma(X(\mathbb{C})) \); in particular, \( \text{sig}(\mathcal{X}) \) is an integer and \( \sigma(X(\mathbb{C})) \) is independent of the choice of \( \tau \). Thanks to Proposition 5.6, we conclude that \( \text{sig}(\mathcal{X}) \) is always an integer, even when \( k \) has positive characteristic and \( \sigma(X) \) has no interpretation as the signature of a real quadratic form.

By Corollary 5.6, if \( \mathcal{X} \) is a smooth proper irreducible \( K \)-scheme of even dimension, the index of \( \mathcal{X} \) divides \( \text{sig}(\mathcal{X}) \) if \( p = 0 \) or \( p > \dim(\mathcal{X}) + 1 \), and the prime-to-\( p \) part of \( \text{ind}(\mathcal{X}) \) divides \( \text{sig}(\mathcal{X}) \) otherwise.

A representation \( \rho \) of \( \text{GL}_d \) such that \( \text{sig}(\mathcal{X}) = \chi(\rho(T_\mathcal{X})) \) for any field \( k \) and any smooth proper irreducible \( k \)-scheme \( X \) of even dimension \( d \) is given by

\[ \rho(V) = \bigoplus_{i=0}^{d} \bigwedge_{\mathcal{I}_i} V^*, \]
where \( V \) denotes the standard representation. Indeed, when \( k = \mathbb{C} \), Hodge has proved that \( \sigma(X(\mathbb{C})) = \sum_{i=0}^{d} \chi(X(\mathbb{C})^i) \) (see [21] and [20, Introduction 0.6]), from which it follows that \( \text{sig}(\mathcal{X}) = \sum_{i=0}^{d} \chi(X(\mathbb{C})^i) \) for any field \( k \).

**Example 5.11.** Let \( q \) be a prime number, let \( I = (\alpha_j)_{j \geq 1} \in \mathcal{S} \), and let \( d = |I| \). Assume that \( \alpha_j = 0 \) for every \( j \) which is not of the form \( q^n - 1 \) for some \( n \geq 1 \), and denote by \( s_1, \ldots, s_d \in \mathbb{Q}[c] \) the Segre polynomials (see [5.3]). Then
\[ P = \frac{1}{q}c_I(s_1, \ldots, s_d) \in \mathbb{Q}[c]_d \]
satisfies the hypothesis of Corollary 5.6. Indeed, for any irreducible smooth projective scheme $X$ of dimension $d$ over a field of characteristic $\neq q$, the integer $\deg(c_j(-T_X))$ is divisible by $q$, as follows from [26, Proposition 5.3] applied to the structure morphism of $X$.

Thanks to Corollary 5.6 we conclude that if $p \neq q$, the index of any irreducible smooth proper $K$-scheme of dimension $d$ divides $\frac{1}{q} \deg(c_j(-T_X))$.

**Example 5.12.** Taking $q = 2$ and $I = (d, 0, \ldots)$ in Example 5.11, we see that for any integer $d$, the polynomial $P = \frac{1}{2}s_d$ satisfies the hypothesis of Corollary 5.6.

**Example 5.13.** Let $I = (\alpha_j)_{j \geq 1}$ with $\alpha_d = 1$ and $\alpha_j = 0$ for $j \neq d$. In this case, the polynomial $c_j$ is the $d$th Newton polynomial $Q_d$ (see [2, Part II, §12]). According to Example 5.11 for any irreducible smooth proper $K$-scheme $X$ whose dimension is of the form $d = q^n - 1$ for any integer $q \neq p$ and some $n \geq 1$, the index of $X$ divides $\frac{1}{q} \deg(Q_d(-T_X))$.

This example and Example 5.12 may be combined as follows: fix integers $n, m \geq 0$ and a prime $q \neq p$, and let $d = m(q^n - 1)$. Take $I = (\alpha_j)_{j \geq 1}$ with $\alpha_{q^n-1} = m$ and $\alpha_j = 0$ for $j \neq q^n - 1$. Then $c_j(E)$, for any vector bundle $E$, is the $m$th elementary symmetric function of the $(q^n - 1)$st powers of the Chern roots of $E$. For any irreducible smooth proper $K$-scheme $X$ of dimension $d$, the index of $X$ divides $\frac{1}{q} \deg(c_j(-T_X))$.

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