Maximum likelihood estimation for disk image parameters

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Abstract

We present a novel technique for estimating disc parameters from its 2D image. It is based on the maximal likelihood approach utilising both edge coordinates and the image intensity gradients. We emphasise the following advantages of our likelihood model. It has closed-form formulae for parameter estimating, therefore requiring less computational resources than iterative algorithms. The likelihood model naturally distinguishes the outer and inner annulus edges.

The proposed technique was evaluated on both synthetic and real data.

1 Introduction

Circles and disks are among the most basic geometric primitives. Therefore, the detection and fitting problems are widespread over different fields: microwave engineering [1], particle physics [2,3], pattern recognition [4], quality control [5], robotic systems [6,7,8], and others. Since a disk edge is a circle, detecting a circle and detecting a disk are closely related problems. In this paper, we apply disk fitting in the experimental astronomy field.

The techniques may be coarsely divided into the two following classes. First, convolutional image-based techniques such as the circle Hough transform [9,10], or the phase-coded annulus [11]. They work with a 2D image represented as an array of pixels. Second, point-based statistical techniques, for instance, maximal likelihood estimation. Here, input data is a list of points presumable residing on a circle. A review of different approaches is given, for instance, in [12].

Since we limit to fit only a single one disk or annulus per an image frame in this work, the 2D image array can be straightforwardly transformed to the list

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of points by a gradient filtering followed by thresholding such as Otsu thresholding [13]. As soon as the maximal likelihood fitting follows the gradient filtering, we also use normalised gradient vectors during the likelihood fitting.

The paper is organised as the following. A gradient-based maximal likelihood estimation with an analytic solution is derived in Section 2. In Section 3 we describe full detection pipeline and evaluate it on synthetic data. In Section 4 we show how the expectation minimisation technique applied on top of maximal likelihood can recognise an astronomical telescope entrance pupil. The major results are discussed in the Conclusion.

2 Gradient-based likelihood model

When a table of noisy \((x, y)\)-pairs is known, under a specific natural assumption we may derive the following log-likelihood loss function to estimate the centre and the radius for a circle:

\[
\ln p(x_i, y_i | \theta) = -\frac{1}{2} \ln 2\pi \sigma - \frac{1}{2\sigma^2} \sum_i \left( \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} - R \right)^2,
\]

(1)

where \(\theta\) is the set of distribution parameters: the centre \(x_0, y_0\), the radius \(R\). The measured points are denoted by \(x_i, y_i\). For instance, this approach is used in [14, 15, 2].

However, since there is no analytic solution for minimisation of (1), one needs either to use iterative numerical methods or to change the loss function:

\[
\ln p(x_i, y_i | \theta) = -\frac{1}{2} \ln 2\pi \sigma - \frac{1}{2\sigma^2} \sum_i \left( (x_i - x_0)^2 + (y_i - y_0)^2 - R^2 \right)^2,
\]

(2)

as it is done in [1, 16].
We propose the following likelihood model:

\[
\ln p(x_i, y_i, n_{xi}, n_{yi} | \theta) = - \ln 2\pi \sigma - \frac{1}{2\sigma^2} \sum_i \left[ (n_{xi} (x_0 - x_i) + n_{yi} (y_0 - y_i) - R)^2 \\
+ (n_{yi} (x_0 - x_i) - n_{xi} (y_0 - y_i))^2 \right],
\]

(3)

where \( n_{xi}, n_{yi} \) is the measured normalised gradient at the point \( x_i, y_i \). The sign before \( R \) is negative for outer disk edges and positive for inner annulus edges. In essence, the gradient \( n_{xi}, n_{yi} \) is the normal vector at the circle point \( x_i, y_i \).

Let us show how equation (3) is derived. We have the following four quantities for each point: \( x_i, y_i, n_{xi}, n_{yi} \). Since the points are supposed to be located at the circle with some precision, the quantities are not independent random variables. It is known that the vectors normal to the circle are crossed at its centre, so we assume that the following relations are held:

\[
n_{xi} = \frac{x_0 - x_i}{\sqrt{(x_0 - x_i)^2 + (y_0 - y_i)^2}} + \epsilon_{xi},
\]

(4)

\[
n_{yi} = \frac{y_0 - y_i}{\sqrt{(x_0 - x_i)^2 + (y_0 - y_i)^2}} + \epsilon_{yi},
\]

(5)

where \( \epsilon_{xi} \) and \( \epsilon_{yi} \) are random variables with zero mean. Consider the following two new random variables which geometrical meaning is explained in Fig. 1:

\[
d_l = n_{xi} (x_0 - x_i) + n_{yi} (y_0 - y_i),
\]

(6)

\[
d_r = n_{yi} (x_0 - x_i) - n_{xi} (y_0 - y_i).
\]

(7)

Using equations (4) and (5), one may find mean values for \( d_l \) and \( d_r \):

\[
E[d_l] = E\left[ \sqrt{(x_0 - x_i)^2 + (y_0 - y_i)^2} \right] = R,
\]

(8)

\[
E[d_r] = 0,
\]

(9)

where \( E[\cdot] \) denotes a mean operator. Here we assume that \( \epsilon_{xi} \) and \( \epsilon_{yi} \) are independent from \( x_i \) and \( y_i \).

There can also be an alternative geometric interpretation for (3). It is easy to show that every circle is located at some two-dimensional plane in the four-dimensional space of \( x, y, n_x, n_y \). Equation (3) suggests minimal distances between the plane and the measurement points in this four-dimensional space.

The model has two important properties. First, it can be reduced to the exact model (1). Indeed, expanding and collecting the terms in (3) we have the following:

\[
\ln p(x_i, y_i, n_{xi}, n_{yi} | \theta) = - \ln 2\pi \sigma - \frac{1}{2\sigma^2} \sum_i \left[ (x_0 - x_i)^2 + (y_0 - y_i)^2 + R^2 \\
-2Rn_{xi} (x_0 - x_i) - 2Rn_{yi} (y_0 - y_i) \right],
\]

(10)
which is the same as (1), when equations (4) and (5) are substituted in (10), and $\epsilon_{xi} = \epsilon_{yi} = 0$.

Second, it has the following simple analytic solution for the maximal likelihood estimators:

$$R = \frac{\sum_{i} n_{xi} \frac{x_i}{N} + \sum_{i} n_{yi} \frac{y_i}{N} - \sum_{i} x_i n_{xi}}{1 - \left( \frac{\sum_{i} n_{xi}}{N} \right)^2 - \left( \frac{\sum_{i} n_{yi}}{N} \right)^2},$$

(11)

$$x_0 = \frac{\sum_{i} x_i}{N} + R \frac{\sum_{i} n_{xi}}{N},$$

(12)

$$y_0 = \frac{\sum_{i} y_i}{N} + R \frac{\sum_{i} n_{yi}}{N},$$

(13)

$$\sigma^2 = \frac{1}{2N} \sum_{i} \left[ (n_{xi} (x_0 - x_i) + n_{yi} (y_0 - y_i) - R)^2 + (n_{yi} (x_0 - x_i) - n_{xi} (y_0 - y_i))^2 \right].$$

(14)

They use a similar approach in [17] where authors implicitly constructed normal vectors from the model parameters and the measured point positions. However, since they didn’t measure the normal vectors, their estimates are obtained through the iterative procedure.

### 3 Maximal likelihood disk fitting

Let us recall that we are interested in locating a single disk on a 2D image. We suppose that the image is represented as a 2D grey scale array. The following
two steps are performed on the data to convert the array to the list of circle points coordinates for applying equation (3).

First, the gradient filters are applied to produce the horizontal gradient $g_{x,nm}$ and vertical gradient $g_{y,nm}$ component maps. The following convolution kernels are used to evaluate the gradients:

$$K_{x,nm} = d_n s_m,$$
$$K_{y,nm} = s_n d_m,$$

$$s \equiv (t^4, t, 1, t, t^4),$$
$$d \equiv (-2t^4, -t, 0, t, 2t^4),$$
$$t \equiv \exp\left(-\frac{1}{2s^2}\right)$$

where $s^2$ is selected to be 2 to fit the kernel into five points. We also ignore the Gaussian norm in the kernel, since we are interested in normalised gradient vectors rather than absolute gradient values. Kernels (15) and (16) are separable, so the filtering can be efficiently applied as a sequence of two 1D

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Table 1: The algorithm evaluation results for the disk centre point $x_0,y_0$. 25%, 50%, and 75% percentiles are given for the absolute difference between the true and estimated parameters.

| Points | 30 | 50 | 75 | 25 | 50 | 75 | 25 | 50 | 75 | 25 | 50 | 75 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|
| Percentile | 25% | 50% | 75% | 25% | 50% | 75% | 25% | 50% | 75% | 25% | 50% | 75% |
| Technique | Noise level | | | | | | | | | | | |
| This paper | 1 | 0.49 | 1.21 | 2.69 | 0.35 | 0.87 | 1.97 | 0.24 | 0.64 | 1.48 | 0.17 | 0.46 | 1.13 | 0.15 | 0.40 | 1.02 |
| 256 | 0.55 | 1.32 | 2.99 | 0.38 | 0.96 | 2.18 | 0.27 | 0.71 | 1.67 | 0.20 | 0.52 | 1.33 | 0.17 | 0.45 | 1.22 |
| 1024 | 0.75 | 1.87 | 4.25 | 0.51 | 1.35 | 3.23 | 0.37 | 1.00 | 2.61 | 0.28 | 0.76 | 2.29 | 0.24 | 0.68 | 2.21 |
| Li, et al. | 1 | 0.11 | 0.25 | 0.46 | 0.08 | 0.18 | 0.35 | 0.06 | 0.13 | 0.27 | 0.05 | 0.10 | 0.22 | 0.04 | 0.09 | 0.21 |
| 256 | 0.11 | 0.24 | 0.46 | 0.08 | 0.18 | 0.34 | 0.06 | 0.13 | 0.26 | 0.04 | 0.10 | 0.21 | 0.04 | 0.09 | 0.21 |
| 1024 | 0.11 | 0.25 | 0.48 | 0.08 | 0.18 | 0.36 | 0.06 | 0.14 | 0.29 | 0.05 | 0.11 | 0.25 | 0.04 | 0.10 | 0.24 |

Table 2: The algorithm evaluation results for the disk radius $R$. The table layout is the same as for Table 1.

| Points | 30 | 50 | 75 | 25 | 50 | 75 | 25 | 50 | 75 | 25 | 50 | 75 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|
| Percentile | 25% | 50% | 75% | 25% | 50% | 75% | 25% | 50% | 75% | 25% | 50% | 75% |
| Technique | Noise level | | | | | | | | | | | |
| This paper | 1 | 0.33 | 0.80 | 2.19 | 0.32 | 0.71 | 1.86 | 0.33 | 0.70 | 1.71 | 0.36 | 0.72 | 1.63 | 0.37 | 0.73 | 1.62 |
| 256 | 0.38 | 0.93 | 2.54 | 0.38 | 0.85 | 2.16 | 0.42 | 0.88 | 2.11 | 0.45 | 0.90 | 2.04 | 0.47 | 0.94 | 2.07 |
| 1024 | 0.62 | 1.50 | 4.00 | 0.64 | 1.45 | 3.76 | 0.72 | 1.50 | 3.54 | 0.80 | 1.59 | 3.70 | 0.81 | 1.63 | 3.65 |
| Li, et al. | 1 | 0.09 | 0.21 | 0.42 | 0.07 | 0.15 | 0.31 | 0.05 | 0.11 | 0.25 | 0.04 | 0.08 | 0.21 | 0.03 | 0.07 | 0.20 |
| 256 | 0.09 | 0.20 | 0.41 | 0.07 | 0.14 | 0.30 | 0.05 | 0.11 | 0.23 | 0.04 | 0.08 | 0.20 | 0.03 | 0.07 | 0.21 |
| 1024 | 0.09 | 0.21 | 0.43 | 0.07 | 0.15 | 0.34 | 0.05 | 0.11 | 0.29 | 0.04 | 0.09 | 0.25 | 0.03 | 0.08 | 0.25 |
convolutions. When both gradient coordinates are known, the gradient norm may be evaluated.

Second, Otsu threshold selection method [13] applies to the gradient norm array to detect pixels belonging to the circle. The thresholding technique relies on the data distribution itself, so the Gaussian norm in kernels (15)–(16) may be safely ignored here too. All the image pixels are divided into the circle and the background using the threshold. There may be too many first class pixels, so the smaller fixed-size subset is chosen at random. Adjusting subset size helps to find a balance between computational expenses and accuracy. Finally, for every pixel from the subset we know its integer coordinates $x_i = n, y_i = m$ and normalised gradient vector $n_{xi} = g_{x,nm}, n_{yi} = g_{y,nm}$. Then, the gradient-based maximum likelihood algorithm is applied to the list to estimate circle parameters.

To evaluate the algorithm performance, we generate 10000 sample disk images $640 \times 480$ pixels each. Each disk has a random size and a random position. The radius is distributed uniformly between 30 and 270 pixels. The position is distributed uniformly between zero and the maximal dimension to allow existing of partial disk segments. The disk pixels have a value of 255, the background is 0, and then we apply Poisson noise of different scales $\lambda$ to this input dataset. A sample is shown at Fig. 2. For each combination of the noise level $\lambda$ and the points number each input image is processed as described above, the circle parameters are estimated using equations (11)–(14), and an a-posteriori estimation error is evaluated. The maximal absolute parameter difference is used as an accuracy measure.

The proposed algorithm is also compared to the algorithm of Li et al. (referred as Li’s algorithm below) presented in [17]. Since that algorithm is iterative, the termination criterium is chosen to be so that relative error is less than $10^{-4}$ to ensure that the accuracy is limited by the data properties rather than an insufficient number of iterations.

The evaluation results are given in Table 1 and Table 2. We may see that the proposed algorithm has an important drawback. It has a considerably lower accuracy than the algorithm from [17] has. The origin of this additional error is clear. When we add normalised vectors $(n_{x2}, n_{y2})$ into consideration, we also introduce additional measurements error. On the other hand, it takes only about $60 \times 10^{-6}$s for the proposed algorithm to process one circle, while it takes $2 \times 10^{-3}$s for Li’s algorithm to reach the obtained accuracy. The execution times were measured at the Intel i5-7200U CPU for Python based implementation. They are consistent within this paper but should be thought of as relative values.

The execution times can be made comparable by limiting the maximal iteration number for Li’s algorithm. The median error for Li’s algorithm is about 20 pixels in that case. It is natural to try using the results of the proposed algorithm as initial estimates for iterative Li’s algorithm. In that case, Li’s algorithm execution time is reduced to $0.9 \times 10^{-3}$s keeping the same good accuracy as in Tables 1 and 2.

The evaluation results make us optimistic about the algorithm applications. Since the proposed algorithm is not an iterative one, its major advance is the
An astronomical telescope focuses coming from stars parallel beams of light at a focal plane. Usually, two-mirror optical schemes are common nowadays. A typical optical scheme for a modern optical astronomical telescope is given at Fig. 3. Note, that the part of the light energy is dissipated by the secondary mirror back surface. It is usually called a central screening. The outer size of the beam is limited by the primary mirror size. An optical scheme part limiting the amount of light passing through the whole system is called an actual aperture stop. We may imagine that the aperture stop splits the whole optical system into two parts. An image of the actual aperture stop produced by the back of the optical system is called an exit pupil. An entrance pupil is the image of the
actual aperture stop produced by the front optical system part in the reverse
direction. The entrance pupil is identical to the actual aperture stop in case of
typical astronomical telescope optical scheme given at Fig. 3. The pupil image
can be obtained by putting a detector at the exit pupil. The sample for the real
telescope is given at Fig. 4.

Installing additional equipment in the exit pupil is used while examining the
optics quality \cite{18, 19} or optical turbulence profiling \cite{20}. Since the telescope
optical system geometry is always a subject of thermal expansions and bending
by gravity, the exact position for the exit pupil is known with limited accuracy.
In real conditions, an additional adjustment would be required. The sharpness
of the outer pupil edge may be controlled to achieve the goal. Note, that the
inner pupil edge can be unsharp at the same moment since it is produced by
the secondary mirror edge that is located at some distance from the primary
mirror. This is why the outer edge locating, followed by spatial filtering, is
required before sharpness detection.

The proposed algorithm naturally distinguishes between the disk or annulus
outer edges and the inner ones. Outer edges have gradients directed towards
the circle centre where they are crossed. At the same time, inner edges have
gradients directed in the reverse direction. We could craft an algorithm detecting
inner edges by changing the sign for $R$ in equation (3), assuming that $R$ is
positive.

Two classes of edge points are obtained by the edge detection algorithm in
the case of images similar to Fig. 4. The first class is consisted from the outer
edge points. The second class consists of the inner edge points and the spiders
edge points. Since the class is a hidden variable, it seems to be natural to apply
the expectation maximisation algorithm on the following two-component model.
The probability density function are defined as follows:

$$p_1(x_i, y_i, n_{xi}, n_{yi}|\theta) = \frac{\tau}{2\pi\sigma_1^2} \exp\left(-\frac{1}{2\sigma_1^2} \left((n_{xi}(x_{0,1} - x_i) + n_{yi}(y_{0,1} - y_i) - R)^2 + (n_{yi}(x_{0,1} - x_i) - n_{xi}(y_{0,1} - y_i))^2\right)\right),$$

$$p_2(x_i, y_i|\theta) = \frac{1 - \tau}{2\pi\sigma_2^2} \exp\left(-\frac{1}{2\sigma_2^2} \left((x_{0,2} - x_i)^2 + (y_{0,2} - y_i)^2\right)\right).$$

(20)

(21)

Here, $p_1$ is following from equation (3) and represents the outer edge point
distribution. $p_2$ is obtained from equation (3) by setting $R = 0$ and collecting
the terms. The vector $\theta$ stands for the probability density function parameters
to be determined: $\tau, x_{0,1}, y_{0,1}, R, \sigma_1, x_{0,2}, y_{0,2}, \sigma_2$.

The expectation maximisation algorithm iteratively minimises the goal function:

$$Q(\theta'|\theta) = \sum_i \left(T_{i,1}' \ln p_1(x_i, y_i, n_{xi}, n_{yi}|\theta) + T_{i,2}' \ln p_2(x_i, y_i|\theta)\right),$$

(22)

where weights $T_{i,1}'$ and $T_{i,2}'$ are evaluated based on the previously estimated
distribution parameters $\theta'$ as the following:

$$T'_{i,1} = \frac{p_1(x_i, y_i, n_{xi}, n_{yi}|\theta')}{p_1(x_i, y_i, n_{xi}, n_{yi}|\theta') + p_2(x_i, y_i|\theta')} ,$$

$$T'_{i,2} = \frac{p_2(x_i, y_i|\theta')}{p_1(x_i, y_i, n_{xi}, n_{yi}|\theta') + p_2(x_i, y_i|\theta')} .$$

The solution for the following optimisation problem:

$$\theta = \arg\min_\theta Q(\theta|\theta')$$

can be found analytically. Parameters $x_{0,1}, y_{0,1}, R, \sigma_1$ are evaluated using modified equations (11)-(14) where each $\sum a_i$ is substituted for $\sum T'_{i,1} a_i$, and $N$ is substituted for $\sum T'_{i,1}$. Other parameters are evaluated using ordinarily equations for the Gaussian mixture EM model.

The initial parameter estimation is required to start the iteration process. The following has been found to be working satisfactory. The pupil centre is estimated as the solution for the following linear equation set:

$$x_0 \sum_i n_{yi}^2 - y_0 \sum_i n_{xi} n_{yi} = \sum_i n_{yi} (n_{yi} x_i - n_{xi} y_i) ,$$

$$x_0 \sum_i n_{xi} n_{yi} - y_0 \sum_i n_{xi}^2 = \sum_i n_{xi} (n_{yi} x_i - n_{xi} y_i) ,$$

while other parameters are evaluated as follows:

$$x_{0,1} = x_{0,2} = x_0 ,$$

$$y_{0,1} = y_{0,2} = y_0 ,$$

$$\sigma_2 = R = \sqrt{\frac{\sum_i ((x_i - x_0)^2 + (y_i - y_0)^2)}{N}} ,$$

$$\sigma_1 = 100 .$$

We examined two data sets of real data: one for the focused pupil image and another for the defocused pupil. Each data set consists of 1000 individual images obtained in a series under the same conditions. The pupil parameters estimator accuracy can be evaluated by running the algorithm for every individual image frame.

In the focused pupil case, we have found that the standard deviation for the pupil centre $\sigma_c = 0.85$ pix. and the standard deviation for the outer pupil radius $\sigma_R = 0.42$ pix. Given the average outer radius $R = 113.5$ pix, we have a relative accuracy of 0.4% here. For the defocused pupil case, $\sigma_c = 1.2$ pix. and $\sigma_R = 0.82$ pix, while the average outer radius remains almost the same $R = 113.4$ pix.

It takes approximately $120 \times 10^{-3}$ s to process a single $659 \times 493$ frame using Python programming language in both cases. Most of the time is spent at the gradient filtering stage. However, our production C++ based algorithm
implementation requires only $6 \times 10^{-3}s$ that allows us to process approximately 160 frames per second.

Hence, we have found that the proposed algorithm is able to cope with real data.

5 Conclusion

In this paper, the new likelihood disk fitting model was presented. We showed how using additional information about image gradients can change the model properties. We obtained likelihood model \([3]\) that can be reduced to the exact likelihood model \([1]\) and has simple equations \([11]-[14]\) for its solution at the same time. The likelihood model parameters evaluating doesn’t require iterative techniques because of the equations’ simplicity. So, compared to iterative techniques, less computational resources are generally required, and the computational time is more predictable, which may be important in real-time image processing.

At the same time, the proposed model also has the following drawbacks. First, additional input data are required. However, when gradient-based edge detection algorithms are applied during the image processing pipeline, the required data is usually already available. Second, it can only apply to disks or annuluses. Fitting real circles may require further algorithm complication, such as considering a circle as a thin annulus and using expectation maximisation technique to fit the annulus parameters using both its outer and inner edges. Third, it has a lower accuracy than can be reached by other algorithms under the same conditions.

We believe that in Sections 3 and 4, we demonstrated that the proposed likelihood model may be efficiently used both solely and for estimating initial parameters for iterative circle fitting algorithms.

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