ON LIPSCHITZ AND D.C. SURFACES OF FINITE CODIMENSION IN A BANACH SPACE

LUDĚK ZAJÍČEK, Praha

Abstract. Properties of Lipschitz and d.c. surfaces of finite codimension in a Banach space, and properties of generated \( \sigma \)-ideals are studied. These \( \sigma \)-ideals naturally appear in the differentiation theory and in the abstract approximation theory. Using these properties, we improve an unpublished result of M. Heisler which gives an alternative proof of a result of D. Preiss on singular points of convex functions.

Keywords: Banach space, Lipschitz surface, d.c. surface, multiplicity points of monotone operators, singular points of convex functions, Aronszajn null sets.

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1. Introduction

Let \( X \) be a real separable Banach space. A number of \( \sigma \)-ideals of subsets of \( X \) are considered in the literature. Besides the most classical system of first category sets mention the \( \sigma \)-ideals of Haar null sets, Aronszajn (equivalently Gaussian) null sets (see [2]), \( \Gamma \)-null sets (see [12], [11]) and \( \sigma \)-lower or upper) porous sets (see e.g., [21]). In some questions of the differentiability theory and of the abstract approximation theory, the \( \sigma \)-ideals \( \mathcal{L}^1(X) \) and \( \mathcal{DC}^1(X) \) generated by Lipschitz and d.c. Lipschitz hypersurfaces (i.e., “graphs” of Lipschitz and of d.c. Lipschitz functions), respectively, naturally appear. These \( \sigma \)-ideals are proper subsystems of all \( \sigma \)-ideals mentioned above. The sets from \( \mathcal{L}^1(X) \) were used in \( \mathbb{R}^2 \) (under a different but equivalent definition) by W.H. Young (under the name “ensemble ridée”) and by H. Blumberg (under the name “sparse set”); cf. [20, p. 294]. These sets were used in \( \mathbb{R}^n \) e.g., (implicitly) by P. Erdős [4], and in infinite-dimensional spaces (possibly for the first time) in [18] and [17]. The sets from \( \mathcal{DC}^1(X) \) were probably first applied in [19] (cf. [2, p. 93]). In some articles (e.g., [18], [19], [20], [13]) also sets from smaller \( \sigma \)-ideals \( \mathcal{L}^n(X) \) and \( \mathcal{DC}^n(X) \) generated by Lipschitz and d.c. Lipschitz surfaces of codimension \( n > 1 \) were used.

In the present article we prove some properties of Lipschitz and Lipschitz locally d.c. surfaces of finite codimension (Section 3; Proposition 3.6 and Proposition 3.7).

Using these properties, we study in Section 4 sets which are projections of sets from \( \mathcal{L}^n(X) \) on a closed space \( Y \subset X \) of codimension \( d < n \). The study of such projections was suggested by D. Preiss in connection with a result of [13] (see Remark 4.7(i)). M. Heisler [7] proved that any such projection is a first category set in \( Y \), which provides (together with a result of [19]) an alternative proof of a result of [13]. We prove that each such projection is also a subset of an Aronszajn null set in \( Y \) (and even a subset of a set from a smaller class \( C^*_1 \)). As a consequence, we obtain a result on projections of sets of multiplicity of monotone operators (Theorem 4.9 which improves both [13, Theorem 1.3.] and the corresponding result of [7].
Our proof is more transparent than that of [7] and gives stronger results, since it uses “perturbation” Proposition\textsuperscript{5.7}. To prove (and apply it), we need some results on perturbations of finite-dimensional subspaces. These results are collected in Preliminaries, where also needful results on d.c. mappings are recalled.

2. Preliminaries

We consider only real Banach spaces. By sp\{M\} we denote the linear span of the set M. A mapping is called K-Lipschitz if it is Lipschitz with a (not necessarily minimal) constant K. A bijection f is called bilipschitz (K-bilipschitz) if both f and f\(^{-1}\) are Lipschitz (K-Lipschitz).

A real function on an open convex subset of a Banach space is called d.c. (delta-convex) if it is a difference of two continuous convex functions. Hartman’s notion of d.c. mappings between Euclidean spaces [6] was generalized and studied in [16].

**Definition 2.1.** Let X, Y be Banach spaces, C ⊂ X an open convex set, and let F : C → Y be a continuous mapping. We say that F is d.c. if there exists a continuous convex function f : C → R such that y^* \circ F + f is convex whenever y^* ∈ Y^*; \|y^*\| ≤ 1.

It is easy to see (cf. [16 Corollary 1.8.]) that, if Y is finite dimensional, then F is d.c. if and only if y^* \circ F is d.c. for each y^* ∈ Y^* (or for each y^* from a fixed basis of Y^*). Note also that each d.c. mapping is locally Lipschitz ([16 p. 10]). If X is finite-dimensional, then each locally d.c. mapping is d.c. (see. [16 p. 14]) but it is not true (see [9]) if X is infinite-dimensional. We will need also the following well-known facts on d.c. mappings.

**Lemma 2.2.** Let X, X_1, Y, Y_1, Y_2, Z be Banach spaces.

1. Let f : X → Y be d.c. and let g : X_1 → X, h : Y → Y_1 be linear and continuous. Then both f \circ g and h \circ f are d.c.
2. A mapping f = (f_1, f_2) : X → Y_1 × Y_2 is d.c. if and only if both f_1 and f_2 are d.c.
3. If g : X → Y, h : X → Y are d.c. and a, b ∈ R, then ag + bh is d.c.
4. If f : X → Y is locally d.c. and g : Y → Z is locally d.c., then g \circ f is locally d.c.
5. Suppose that G : X → Y is a linear isomorphism, g : X → Y is a locally d.c. bilipschitz bijection, and the range of g − G is contained in a finite dimensional space. Then g\(^{-1}\) is locally d.c.

**Proof.** The statements (i) and (ii) are very easy (cf. [16 Lemma 1.5. and Lemma 1.7.]) and (iii) follows from (i) and (ii). The statement (iv) is a special case of [16 Theorem 4.2.] and (v) is a special case of [5 Theorem 2.1.]

We will need some notions and results concerning distances of two subspaces of a Banach space, which are well-known from the perturbation theory of linear operators ([5, 8, 11]). Let X be a Banach space and S(X) be the unit sphere of X. Let Y and Z be closed non-trivial subspaces of X. Then the gap between Y and Z (called also the opening or the deviation of Y and Z) is defined by

\[\gamma(Y, Z) = \max\{\sup_{y \in Y \cap S(X)} \text{dist}(y, Z), \sup_{z \in Z \cap S(X)} \text{dist}(z, Y)\}\]

We set \(\gamma(\{0\}, \{0\}) := 0\) and \(\gamma(Y, Z) = 1\) if one and only one of Y, Z is \{0\}. The gap need not be a metric on the set of all non-trivial subspaces of X; this property has
the distance $\rho(Y, Z)$ between $Y$ and $Z$ defined as the Hausdorff distance between $Y \cap S(X)$ and $Z \cap S(X)$.

We will work with the gap $\gamma(Y, Z)$. However, since it is easy to prove (see e.g., [8]) that (for nontrivial $Y$, $Z$) always

$$\rho(Y, Z)/2 \leq \gamma(Y, Z) \leq \rho(Y, Z),$$

we could work also with $\rho(Y, Z)$. We will need the following well-known facts.

**Lemma 2.3.** Let $X$ be a Banach space and $F, \tilde{F}, K$ be finite dimensional subspaces of $X$. Then:

(i) If $\gamma(F, \tilde{F}) < 1$, then $\dim F = \dim \tilde{F}$.

(ii) If $F \cap K = \{0\}$, then there exists $\omega > 0$ such that $\gamma(F, \tilde{F}) < \omega$ implies $\tilde{F} \cap K = \{0\}$.

(iii) If $E \oplus F = X$, then there exists $\omega > 0$ such that $\gamma(F, \tilde{F}) < \omega$ implies $E \oplus \tilde{F} = X$.

**Proof.** Statement (i) is proved in [5] (see [1] Theorem 2.1) and (ii) is an easy consequence of (2.1). (We can also apply [1, Theorem 5.2] with $Y := F$, $Z := K$ and $X := F \oplus K$.) The statement (iii) immediately follows from [1, Theorem 5.2].

The following simple lemma is also essentially well-known. Although it is not stated explicitly in [10], it follows from [10] Theorem 2.2.] which works with complex Banach spaces. Since the formulation of [10, Theorem 2.2.] is rather complicated and we work with real spaces, for the sake of completeness we give a proof.

**Lemma 2.4.** Let $X$ be a Banach space and $(v_1, \ldots, v_n)$ be a basis of a space $V \subset X$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that the inequalities $\|w_1 - v_1\| < \delta, \ldots, \|w_n - v_n\| < \delta$ imply that $W := \text{sp}\{w_1, \ldots, w_n\}$ is $n$-dimensional and $\gamma(V, W) < \varepsilon$.

**Proof.** First we will show that there exists $\eta > 0$ and $\delta^* > 0$ such that the inequality

$$\left(2.2\right) \quad \left\| \sum_{i=1}^{n} c_i w_i \right\| \geq \eta \|c\|_{\infty}$$

holds whenever $\|w_1 - v_1\| < \delta^*, \ldots, \|w_n - v_n\| < \delta^*$ and $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ is arbitrary. To this end observe that there exists $\eta^* > 0$ such that (2.2) holds for $\eta = \eta^*, w_i = v_i$, and arbitrary $c$. Put $\eta := \eta^*/2$ and $\delta^* := \eta^*/2n$. Then the inequalities $\|w_1 - v_1\| < \delta^*, \ldots, \|w_n - v_n\| < \delta^*$ imply that, for each $0 \neq c \in \mathbb{R}^n$,

$$\left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_{\infty}} v_i \right\| - \left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_{\infty}} w_i \right\| \leq \left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_{\infty}} (v_i - w_i) \right\| < n\delta^* = \eta^*/2.$$

Consequently, using the definition of $\eta^*$, we obtain

$$\left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_{\infty}} w_i \right\| \geq \left\| \sum_{i=1}^{n} \frac{c_i}{\|c\|_{\infty}} v_i \right\| - \eta^*/2 \geq \eta^* - \eta^*/2 = \eta,$$

which implies (2.2).
Now set $\delta := \min\{\delta^* + \varepsilon \eta / 2n\}$ and suppose that the inequalities $\|w_1 - v_1\| < \delta, \ldots, \|w_n - v_n\| < \delta$ hold. Let $w = \sum_{i=1}^n c_i w_i$ with $\|w\| = 1$ be given. Set $v = \sum_{i=1}^n c_i v_i$. Since $\|c\|_{\infty} \leq 1/\eta$ by (2.2), we obtain $\|v - w\| \leq \sum_{i=1}^n |c_i| \delta \leq n(1/\eta) \delta \leq \varepsilon / 2$.

Consequently, $\sup_{v \in W \cap S(X)} \text{dist}(w, V) < \varepsilon$. By a quite symmetrical way we obtain $\sup_{v \in W \cap S(X)} \text{dist}(v, W) < \varepsilon$, so $\gamma(V, W) < \varepsilon$. Since we can suppose $\varepsilon < 1$, we know that $W$ is $n$-dimensional by Lemma 2.3(i).

**Lemma 2.5.** Let $X, Y$ be Banach spaces and $F : X \to Y$ be a linear isomorphism. Then there exists $C > 0$ such that

$$C^{-1} \gamma(F(V), F(W)) \leq \gamma(V, W) \leq C \gamma(F(V), F(W)),$$

whenever $V$ and $W$ are subspaces of $X$.

**Proof.** We can clearly suppose that $V$ and $W$ are non-trivial. Since $F^{-1}$ is also a linear isomorphism, it is clearly sufficient to find $D > 0$ such that $\gamma(V, W) \leq D \gamma(F(V), F(W))$ always holds. Choose $K > 0$ such that $F$ is $K$-bilipschitz and consider $v \in V$ with $\|v\| = 1$. We can clearly find $\tilde{w} \in F(W)$ for which $\|\tilde{w} - F(v)\| \leq K \gamma(F(V), F(W))$. Since $\|F(v)\| \leq K$, we have $\|F(v) - F(v)\| \cdot \tilde{w} \leq 2K \gamma(F(V), F(W))$, and therefore $\|v - F(v)\| - F^{-1}(\tilde{w}) \leq 2K^2 \gamma(F(V), F(W))$. Since the roles of $V$ and $W$ are symmetric, we can clearly set $D := 2K^2$.

**Lemma 2.6.** Let $X$ be an infinite dimensional Banach space, $V, W \subset X$ non-trivial finite dimensional spaces, and $\delta > 0$. Then there exists a space $\tilde{V} \subset X$ with $\gamma(V, \tilde{V}, V) < \delta$ and $\tilde{V} \cap W = \{0\}$.

**Proof.** Denote $n := \dim V$, choose an $n$-dimensional space $Y \subset X$ with $Y \cap (V + W) = \{0\}$ and a linear bijection $L : V \to Y$. For $t > 0$, set $\tilde{V}_t := \{v + tL(v) : v \in V\}$. It is easy to check that each $\tilde{V}_t$ is an $n$-dimensional space with $\tilde{V}_t \cap W = \{0\}$. Applying Lemma 2.4 to a basis $v_1, \ldots, v_k$ of $V$ and $w_i := v_i + tL(v_i)$, it is easy to see that $\gamma(V, \tilde{V}_t) \to 0$ ($t \to 0^+$), which implies our assertion.

**Lemma 2.7.** Let $X$ be a Banach space, $1 \leq n < \dim X$, and $K \geq 1$. Let $X = E \oplus F$, where $F$ is an $n$-dimensional space. Suppose that the canonical mapping $\mu : E \oplus F \to E \times F$ (where $E \times F$ is equipped with the maximum norm) is $K$-bilipschitz. Then there exists $\omega > 0$ such that if $\tilde{F} \subset X$ is a closed space with $\gamma(F, \tilde{F}) < \omega$, then $X = E \oplus \tilde{F}$ and the canonical mapping $\tilde{\mu} : E \oplus \tilde{F} \to E \times \tilde{F}$ is $2K$-bilipschitz.

**Proof.** Distinguishing the cases $\lambda < 1$, $\lambda = 1$ and $\lambda > 1$, it is easy to check that there exists $1 > \omega_0 > 0$ such the inequalities

$$K \max(1 + \omega, \lambda) + \omega \leq 2K \max(1, \lambda),$$

$$K^{-1} \max(1 - \omega, \lambda) - \omega \geq (2K)^{-1} \max(1, \lambda)$$

hold for each $\lambda \geq 0$ and $0 < \omega < \omega_0$. By Lemma 2.3(iii), we can choose $0 < \omega < \omega_0$ such that $X = E \oplus \tilde{F}$ whenever $\gamma(F, \tilde{F}) < \omega$. Let $\tilde{F}$ with $\gamma(F, \tilde{F}) < \omega$ be given, and consider arbitrary $\tilde{f} \in \tilde{F}$ and $e \in E$. We will prove

$$(2K)^{-1} \max(\|\tilde{f}\|, \|e\|) \leq \|\tilde{f} + e\| \leq 2K \max(\|\tilde{f}\|, \|e\|).$$
Since the case \( \tilde{f} = 0 \) is trivial, by homogeneity of the norm we can suppose \( \|\tilde{f}\| = 1 \) and find \( f \in F \) with \( \|f - \tilde{f}\| < \omega \). Applying (2.3) to \( \lambda := \|e\| \), we obtain

\[
\|\tilde{f} + e\| \leq \|f + e\| + \omega \leq K \max(\|f\|, \|e\|) + \omega \\
\leq K \max(1 + \omega, \|e\|) + \omega \leq 2K \max(1, \|e\|)
\]

and

\[
\|\tilde{f} + e\| \geq \|f + e\| - \omega \geq K^{-1} \max(1 - \omega, \|e\|) - \omega \geq (2K)^{-1} \max(1, \|e\|).
\]

Thus, (2.4) holds, and \( \tilde{\mu} \) is \( (2K) \)-bilipschitz. \( \square \)

3. Properties of Lipschitz surfaces of finite codimension

If \( X \) is a Banach space and \( X = E \oplus F \), then we denote by \( \pi_{E,F} \) the projection of \( X \) on \( E \) along the space \( F \).

**Definition 3.1.** Let \( X \) be a Banach space and \( A \subset X \).

(i) Let \( F \) be a closed subspace of \( X \). We say that \( A \) is an \( F \)-Lipschitz surface if there exists a topological complement \( E \) of \( F \) and a Lipschitz mapping \( \varphi : E \to F \) such that \( A = \{x + \varphi(x) : x \in E\} \).

(ii) Let \( 1 \leq n < \dim X \) be a natural number. We say that \( A \) is a Lipschitz surface of codimension \( n \) if \( A \) is an \( F \)-Lipschitz surface for some \( n \)-dimensional space \( F \subset X \).

(iii) If we consider in (i) mappings \( \varphi : E \to F \) which are d.c. (resp. Lipschitz d.c., locally d.c., Lipschitz locally d.c.), we obtain the notions of an \( F \)-d.c. surface, d.c. surface of codimension \( n \) (resp. \( F \)-d.c. surface, d.c. surface, etc.).

A Lipschitz surface (resp. d.c. surface, etc.) of codimension 1 is said to be a Lipschitz hypersurface (resp. \( d.c. \) hypersurface, etc.).

(iv) The \( \sigma \)-ideals of sets which can be covered by countably many Lipschitz surfaces (d.c. surfaces) of codimension \( n \) will be denoted by \( \mathcal{L}^n(X) \) (\( \mathcal{DC}^n(X) \)), respectively.

**Lemma 3.2.** Let \( X \) be a Banach space, \( F \subset X \) a space of dimension \( n \) (\( 1 \leq n < \dim X \)), and \( A \subset X \). Then the following properties are equivalent.

(i) \( A \) is an \( F \)-Lipschitz surface (resp. an \( F \)-d.c. surface, an \( F \)-Lipschitz d.c. surface, an \( F \)-Lipschitz locally d.c. surface).

(ii) There exists a topological complement \( \tilde{E} \) of \( F \) such that \( \pi|_A : A \to \tilde{E} \) is a bijection and \( (\pi|_A)^{-1} \) is Lipschitz (resp. d.c., etc.), where \( \tilde{\pi} := \pi_{E,F} \).

(iii) If \( X = F \oplus E \) and \( \pi := \pi_{E,F} \), then \( \pi|_A : A \to E \) is a bijection and \( (\pi|_A)^{-1} \) is Lipschitz (resp. d.c., etc.).

(iv) If \( X = F \oplus E \), then there exists a Lipschitz mapping (resp. a d.c. mapping, etc.) \( \varphi : E \to F \) such that \( A = \{x + \varphi(x) : x \in E\} \).

**Proof.** In the proof we use Lemma 2.2(i)-(iii).

If (i) holds, then there exists a topological complement \( \tilde{E} \) of \( F \) and a Lipschitz (d.c., etc.) mapping \( \tilde{\varphi} : \tilde{E} \to F \) such that \( A = \{x + \tilde{\varphi}(x) : x \in \tilde{E}\} \). Set \( \tilde{\pi} := \pi_{E,F} \).

Then clearly \( \tilde{\pi}|_A : A \to \tilde{E} \) is a bijection and \( (\tilde{\pi}|_A)^{-1} \) is Lipschitz (d.c., etc.), since \( (\tilde{\pi}|_A)^{-1}(\tilde{e}) = \tilde{e} + \tilde{\varphi}(\tilde{e}) \).
Now let $\tilde{E}$ be as in (ii), and let $E$ and $\pi$ be as in (iii). Since $\pi|_E : \tilde{E} \to E$ is clearly a linear isomorphism, $(\pi|_E)^{-1} = \tilde{\pi}|_E$, $\pi|_A = (\pi|_E) \circ (\tilde{\pi}|_A)$ and $(\pi|_A)^{-1} = (\tilde{\pi}|_A)^{-1} \circ (\tilde{\pi}|_E)$, we easily obtain (iii).

Letting $\varphi(x) := (\pi|_A)^{-1}(x) - x$ for $x \in E$, we easily see that (iii) implies (iv). The implication (iv) ⇒ (i) is trivial. \qed

Remark 3.3. (i) Every Lipschitz surface of codimension $n$ in $X$ is clearly a closed subset of $X$.

(ii) If $S \subset X$ is a Lipschitz (resp. d.c., etc.) surface of codimension $n \geq 2$, then $S$ is a subset of a Lipschitz (resp. d.c., etc.) surface of codimension $n - 1$. Indeed, suppose that $S = \{x + \varphi(x) : x \in E\}$, where $\varphi : E \to F$, $X = E \oplus F$, and $F$ is $n$-dimensional. Choose $0 \neq v \in F$, and write $F = \text{sp}\{v\} \oplus \tilde{F}$. Set $\tilde{E} := E + \text{sp}\{v\}$ and, for $x \in \tilde{E}$, define $\tilde{\varphi}(x) := \pi_{E,F}(\varphi(x))$. Set $\tilde{S} := \{y + \tilde{\varphi}(y) : y \in \tilde{E}\}$. It is easy to see that $S \subset \tilde{S}$ and $\tilde{\varphi} : \tilde{E} \to F$ is Lipschitz (resp. d.c., etc) if $\varphi$ is Lipschitz (resp. d.c., etc.). Consequently, if $\dim X > n \geq 2$, then $\mathcal{L}^n(X) \subset \mathcal{L}^{n-1}(X)$. If $X$ is separable, then this inclusion is proper, see Remark 3.8 which shows that no Lipschitz surface of codimension $n - 1$ belongs to $\mathcal{L}^n(X)$ (if $\dim X < \infty$, it is sufficient to use by the obvious way basic properties of Hausdorff dimension).

(iii) If $X$ is separable, then the $\sigma$-ideal $\mathcal{DC}^n(X)$ coincides with the $\sigma$-ideal generated by Lipschitz d.c. surfaces (or Lipschitz locally d.c. surfaces, or locally d.c. surfaces). It easily follows from local Lipschitzness of d.c. functions, from the well-known fact that each Lipschitz convex function defined an open ball in a space $E$ can be extended to a Lipschitz convex function on $E$, and from separability of $X$.

(iv) It is not difficult to show that $\mathcal{DC}^n(X)$ is a proper subset of $\mathcal{L}^n(X)$ (if $\dim X > n \geq 1$); see [17, p. 295] for $n = 1$.

Remark 3.4. Suppose that $X = E \oplus F$ and $F$ is finite dimensional. An easy argument using local compactness of $F$ shows that $\pi_{E,F}(A)$ is closed in $E$ whenever $A$ is closed and bounded in $X$. Consequently, $\pi_{E,F}(A)$ is an $F_\sigma$ subset of $E$ whenever $A$ is closed in $X$.

We will need the following well-known easy consequence of the Brouwer’s Invariance of Domain Theorem. Because of the lack of a suitable reference, we present a short proof.

Lemma 3.5. Let $C$, $\tilde{C}$ be Banach spaces with $0 < \dim C = \dim \tilde{C} < \infty$ and let $f : \tilde{C} \to C$ be an injective continuous mapping such that $f^{-1} : f(\tilde{C}) \to \tilde{C}$ is Lipschitz. Then $f(\tilde{C}) = C$.

Proof. We can clearly suppose that $C = \tilde{C}$ and $X := C = \tilde{C}$ is an Euclidean space. The Brouwer’s Invariance of Domain Theorem implies that $f(X)$ is open in $X$. Let $y_n \to y$, where $y_n \in f(X)$. Then $(y_n)$ is bounded and, since $f^{-1}$ is Lipschitz, $(x_n) := (f^{-1}(y_n))$ is bounded as well. Choose a subsequence $x_{n_k} \to x \in X$. Then $f(x_{n_k}) = y_{n_k} \to f(x) = y$. Thus, we have proved that $f(X)$ is closed; the connectivity of $X$ implies $f(X) = X$. \qed

Proposition 3.6. Let $X$ be a Banach space, $S \subset X$ a Lipschitz surface of codimension $n$, and let $X = D \oplus F$ with $\dim F = n$. Let $\psi = \pi_{D,F}|_S : S \to D$ be injective
and \( \psi^{-1} : \psi(S) \to S \) be Lipschitz. Then \( S \) is an \( F \)-Lipschitz surface. Moreover, if \( S \) is a Lipschitz locally d.c. surface of codimension \( n \), then \( S \) is an \( F \)-Lipschitz locally d.c. surface.

**Proof.** Choose an \( n \)-dimensional space \( \tilde{F} \) such that \( S \) is an \( \tilde{F} \)-Lipschitz surface.

Since the case \( F = \tilde{F} \) is obvious by Lemma 3.2, we suppose \( F \neq \tilde{F} \). Put \( K := F \cap \tilde{F} \) and choose spaces \( C, \tilde{C} \) such that \( F = K \oplus C \) and \( \tilde{F} = K \oplus \tilde{C} \). Then clearly \( 1 \leq \dim C = \dim \tilde{C} < \infty \). Choose a topological complement \( Z \) of the (finite dimensional) space \( F + \tilde{F} = K \oplus C \oplus \tilde{C} \) and denote \( E := Z \oplus C \), \( \tilde{E} := Z \oplus \tilde{C} \).

Clearly \( X = F \oplus E = \tilde{F} \oplus \tilde{E} \).

By Lemma 3.2 \( \varphi := \pi_{E,F} \mid_S : S \to \tilde{E} \) is a bilipschitz bijection. It is easy to see (proceeding similarly as in the proof of Lemma 3.2) that \( \varphi := \pi_{E,F} \mid_S : S \to E \) is injective and \( \varphi^{-1} : \varphi(S) \to S \) is Lipschitz. So Lemma 3.2 implies that, to prove the first part of the assertion, it is sufficient to verify \( \varphi(S) = E \).

To this end choose an arbitrary \( e \in E \) and write \( e = z + c \), where \( z \in Z \) and \( c \in C \). For each \( x \in \tilde{C} \), put \( f(x) := \varphi \circ (\varphi^{-1})(x + z) - z \). Clearly \( f(x) \in (F + \tilde{F}) \cap E = C \); so \( f : \tilde{C} \to C \). It is easy to see that \( f \) is continuous injective and \( f^{-1}(y) = \varphi \circ \varphi^{-1}(y + z) - z \) for each \( y \in f(\tilde{C}) \). Consequently, \( f^{-1} \) is Lipschitz, and so \( f(\tilde{C}) = C \) by Lemma 3.2. For \( c := f^{-1}(e) \) we have \( \varphi((\varphi^{-1})(e + z)) = e + z = e \); so \( \varphi(S) = E \).

To prove the second part of the assertion, we suppose that \( (\varphi^{-1}) : \tilde{E} \to X \) is moreover locally d.c. Then \( g := \varphi \circ (\varphi^{-1}) = \pi_{E,F} \circ (\varphi^{-1}) \) is clearly Lipschitz and it is locally d.c. by Lemma 2.2 (i). Since \( \varphi(S) = E \), we have that \( g : \tilde{E} \to E \) is a bijection and \( \varphi^{-1} = \varphi \circ \varphi^{-1} \) is Lipschitz. Choose a linear bijection \( L : \tilde{C} \to C \), and let \( G : \tilde{E} \to E \) be the mapping which assigns to a point \( \tilde{c} = \bar{c} + z \) \((\bar{c} \in \tilde{C}, z \in Z)\) the point \( G(\tilde{c}) := L(\bar{c}) + z \). Then clearly \( G \) is a linear isomorphism. Since \( G(\bar{c}) - \tilde{c} \in C + \tilde{C} \) and \( g(\bar{c}) - \tilde{c} \in F + \tilde{F} \), we obtain that \( g - G \) has a finite dimensional range. Consequently, Lemma 2.2 (v) implies that \( g^{-1} \) is locally d.c. Thus, Lemma 2.2 (iv) implies that \( \varphi^{-1} = (\varphi^{-1}) \circ g^{-1} \) is locally d.c. So, Lemma 3.2 implies that \( S \) is an \( F \)-Lipschitz locally d.c. surface.

\[ \square \]

**Proposition 3.7.** Let \( X \) be a Banach space, \( F \subset X \) an \( n \)-dimensional space, and \( A \subset X \) an \( F \)-Lipschitz (resp. \( F \)-Lipschitz locally d.c.) surface. Then there exists \( \varepsilon > 0 \) such that if \( \tilde{F} \subset X \) is an \( n \)-dimensional space with \( \gamma(F, \tilde{F}) < \varepsilon \), then \( A \) is an \( \tilde{F} \)-Lipschitz (resp. \( \tilde{F} \)-Lipschitz locally d.c.) surface.

**Proof.** Choose \( E \) such that \( X = E \oplus F \), and choose \( K \geq 1 \) such that the canonical mapping \( \gamma : E \oplus F \to E \times F \) is \( K \)-bilipschitz. Choose a corresponding \( \omega > 0 \) by Lemma 2.7. Denote \( \pi := \pi_{E,F} \), and choose \( L \geq 1 \) such that \( (\pi|_A)^{-1} \) is Lipschitz with constant \( L \). Choose \( \varepsilon > 0 \) such that \( \varepsilon < \omega \) and

\[ 2K^2L\varepsilon < 1/2. \]

Now suppose that an \( n \)-dimensional space \( \tilde{F} \) with \( \gamma(F, \tilde{F}) < \varepsilon \) be given. Since \( \varepsilon < \omega \), we have that \( X = E \oplus \tilde{F} \) and the canonical mapping \( \tilde{\gamma} : E \oplus \tilde{F} \to E \times \tilde{F} \) is \( 2K \)-bilipschitz. By Proposition 3.6 it is sufficient to prove that, putting \( \tilde{\pi} := \pi_{E,\tilde{F}} \), the mapping \( (\tilde{\pi}|_A)^{-1} \) is Lipschitz with constant \( 2L \); i.e., that

\[ \|x - y\| \leq 2L\|\tilde{\pi}(x) - \tilde{\pi}(y)\| = 2L\|\tilde{\pi}(x - y)\|, \quad x, y \in A. \]
Thus, consider \(x, y \in A\), \(x \neq y\), and write \(x - y = e_1 + f = e_2 + \tilde{f}\), where \(e_1 = \pi(x - y) \in E\), \(e_2 = \pi(x - y) \in E\), \(f \in F\), and \(\tilde{f} \in F\). We know that \(\|x - y\| \leq L\|e_1\|\) and so \(\|\tilde{f}\| \leq 2K\|x - y\| \leq 2KL\|e_1\|\).

If \(\tilde{f} = 0\), then (3.2) is obvious. If \(\tilde{f} \neq 0\), put \(\tilde{z} := \|\tilde{f}\|^{-1}\tilde{f}\), and find \(z \in F\) such that \(\|\tilde{z} - z\| \leq \varepsilon\). Then \(f_2 := \|\tilde{f}\|z\) satisfies \(\|f - f_2\| \leq \varepsilon\|\tilde{f}\|\), and so
\[
K^{-1}\|e_1 - e_2\| \leq \|e_1 - e_2 - f + f_2\| = \|f - f_2\| \leq \varepsilon\|\tilde{f}\| \leq 2KL\|e_1\|.
\]
Thus, by (3.1), we obtain \(\|e_1 - e_2\| \leq \|e_1\|/2\), and so \(\|e_2\| \geq \|e_1\|/2\). Therefore \(\|x - y\| \leq L\|e_1\| \leq 2L\|e_2\|\), which proves (3.2) and finishes the proof.

\[\square\]

**Remark 3.8.** I do not know whether the analogies of Proposition 3.6 and Proposition 3.7 hold for Lipschitz d.c. surfaces.

### 4. Projections of Lipschitz Surfaces of Finite Codimension

**Definition 4.1.** Let \(X\) be a separable Banach space, and let a finite-dimensional space \(V \subset X\) be given. We define the following classes of sets:

- \(A(V)\) is the system of all Borel sets \(B \subset X\) such that \(V \cap (B + a)\) is Lebesgue null (in \(V\)) for each \(a \in X\). For \(0 \neq v \in X\), we put \(A(v) := A(sp\{v\})\).
- \(A^*(V, \varepsilon)\) (where \(0 < \varepsilon < 1\)) is the system of all Borel sets \(B \subset X\) such that \(B \in A(W)\) for each space \(W\) with \(\gamma(V, W) < \varepsilon\), and \(A^*(V)\) is the system of all sets \(B\) such that \(B = \bigcup_{k=1}^{\infty} B_k\), where \(B_k \in A^*(V, \varepsilon_k)\) for some \(0 < \varepsilon_k < 1\).
- \(C^*_d\) (where \(d \in \mathbb{N}\)) is the system of those \(B \subset X\) that can be written as \(B = \bigcup_{k=1}^{\infty} B_k\), where each \(B_k\) belongs to \(A^*(V_k)\) for some \(V_k\) with \(\dim V_k = d\).
- \(A\) is the system of those \(B \subset X\) that can be, for every complete sequence \((v_k)\) in \(X\), written as \(B = \bigcup_{k=1}^{\infty} B_k\), where each \(B_k\) belongs to \(A(v_k)\).

Note that \(C^*_1\) coincide with \(C^*\) from [14] and \(A\) is the system of all Aronszajn null sets. For basic properties of sets from \(A\) see [2]. Lemma 2.3 easily implies the following fact.

**Lemma 4.2.** Let \(X, Y\) be Banach spaces and \(F : X \to Y\) a linear isomorphism. Let \(S \subset X\) belong to \(A^*(V, \delta)\). Then there exists \(\varepsilon > 0\) such that \(F(S) \in A^*(F(V, \varepsilon))\) (in the space \(Y\)).

**Proposition 4.3.** Let \(X\) be a separable infinite dimensional Banach space and \(k \in \mathbb{N}\). Then \(C^*_1 \subset C^*_2 \subset \cdots \subset A\) and all inclusions are proper.

**Proof.** To prove the inclusions \(C^*_d \subset A\), it is sufficient to show that \(A^*(V, \varepsilon) \subset A\) whenever \(V \subset X\) is a \(d\)-dimensional space and \(\varepsilon > 0\). Let \(V, \varepsilon\) and \(B \in A^*(V, \varepsilon)\) be given. Choose a basis \((v_1, \ldots, v_d)\) of \(V\) and consider an arbitrary complete sequence \((u_i)\) in \(X\). Choose a \(\delta > 0\) that corresponds to \((v_1, \ldots, v_d)\) and \(\varepsilon\) by Lemma 2.3. We can clearly choose \(n \in \mathbb{N}\) and vectors \(w_1, \ldots, w_d\) in \(U := sp\{u_1, \ldots, u_n\}\) such that \(\|u_i - v_i\| < \delta\), \(i = 1, \ldots, d\). Then, denoting \(W := sp\{w_1, \ldots, w_d\}\), we have \(\gamma(V, W) < \varepsilon\), and so \(B \in A(W)\). Consequently, by the Fubini theorem, \(B \in A(U)\). Using [2] Proposition 6.29, we easily obtain that \(B\) can be decomposed as \(B = \bigcup_{i=1}^{n} B_i\), where \(B_i \in A(u_i)\). So, \(B \in A\), and \(C^*_d \subset A\) is proved.

To prove \(C^*_d \subset C^*_{d+1}\), consider a \(B \in A^*(V, \varepsilon)\), where \(\dim V = d\) and \(1 > \varepsilon > 0\). Choose a basis \(v_1, \ldots, v_d\) of \(V\) with \(\|v_i\| = 1\) and find a corresponding \(\delta > 0\) by
Lemma [2.4] Now choose an arbitrary \( Z \supset V \) with \( \dim Z = d + 1 \). To prove \( B \in A^*(Z, \delta) \), consider an arbitrary \((d + 1)\)-dimensional \( W \) with \( \gamma(W, Z) < \delta \). By the definition of \( \gamma \), find \( w_1, \ldots, w_d \in W \) with \( \|w_1 - v_1\| < \delta, \ldots, \|w_d - v_d\| < \delta \), and set \( \tilde{W} := \text{sp}\{w_1, \ldots, w_d\} \). The choice of \( \delta \) implies that \( \gamma(\tilde{W}, V) < \varepsilon \), and so \( \tilde{W} \cap (B + a) \) is Lebesgue null in \( \tilde{W} \) for each \( a \in X \). Consequently, by Fubini theorem, \( W \cap (B + a) \) is Lebesgue null in \( W \) for each \( a \in X \). So \( B \in A^*(Z, \delta) \), and \( C_1^* \subset C_{d+1} \) follows.

A construction of a set in \( A \setminus C_1^* \) is presented in the proof of [14, Proposition 13]. Moreover, it is shown in [14] that this set \( (F_2(I)) \) meets any 2-dimensional affine space in a 2-dimensional Lebesgue null set, which shows that even \( C_1^* \setminus C_1^* \neq \emptyset \). It is not difficult to modify that construction and obtain a set in \( C_{d+1}^* \setminus C_d^* \) for each \( d \) (see Remark [14]). However, since the notation is somewhat complicated in the general case, we will give a detailed proof for \( d = 2 \) only.

Our construction starts quite similarly as the construction of a set from \( A \setminus C^* \) on p. 20 of [14]. Namely, by the same procedure as in [14] we can define positive numbers \( c_0, c_1, c_2, \ldots \) and nonzero vectors \( u_0, u_1, u_2, \ldots \) in \( X \) such that both \( \{u_n - 3 : n \in \mathbb{N}\} \) and \( \{u_n : n \in \mathbb{N}\} \) are dense in \( X \), and the formula

\[
F(x) = \sum_{j=0}^{\infty} c_j x_{j+1} u_j \quad \text{(where } x = (x_1, x_2, \ldots)\text{)}
\]

defines a linear injective mapping of \( \ell_2 \) to \( X \).

As in [14], we set \( I := \{x \in \ell_\infty : 1 \leq x_k \leq 2\} \), and equip \( I \) with the topology of pointwise convergence (so it is a compact metrizable space) and with the measure \( \mu \) defined as the product of countably many copies of the Lebesgue measure on \([1, 2]\).

Choose two sequences \( \xi_1, \xi_2, \ldots \) and \( \zeta_1, \zeta_2, \ldots \) such that \( 0 < \xi_j < 1/(j + 1)! \), \( 0 < \zeta_j < 1/(j + 1)! \), and

\[
\lim_{k \to \infty} \sum_{j=k}^{\infty} c_{3j-2} \xi_j^2 u_{3j-3} = 0, \quad \lim_{k \to \infty} \sum_{j=k}^{\infty} c_{3j-1} \zeta_j^2 u_{3j-1} = 0.
\]

Now, for \( x \in I \), set

\[
G(x) = \sum_{k=1}^{\infty} c_{3k-2} \xi_j^3 x_1 x_3 \ldots x_{2k-1} u_{3k-2} + \sum_{k=1}^{\infty} c_{3k-1} \zeta_j^2 x_2 x_4 \ldots x_{2k} u_{3k-1} + \sum_{k=1}^{\infty} x_k c_{3k} u_{3k}.
\]

Then \( G : I \to X \) is a continuous mapping. Indeed, we have \( G = F \circ H \), where

\[
H(x_1, x_2, \ldots) := (0, \xi_1 x_1, \xi_2 x_2, x_1, \xi_3 x_1 x_3, \xi_4 x_2 x_4, x_2, \xi_5 x_1 x_3 x_5, \xi_6 x_2 x_4 x_6, x_3, \ldots),
\]

and \( H : I \to \ell_\infty \) is clearly continuous. So, \( G(I) \) is compact.

Let \( e_j \) be the \( j \)-th member of the canonical basis of \( \ell_\infty \). Observe that if \( x \in I \), \( k_1, k_2 \in \mathbb{N}, t, \tau \in \mathbb{R} \) and \( x + t e_{2k_1-1} + \tau e_{2k_2} \in I \), then \( G(x + t e_{2k_1-1} + \tau e_{2k_2}) = G(x) + t v_{k_1}(x) + \tau w_{k_2}(x) \), where

\[
v_k(x) := \sum_{j=k}^{\infty} c_{3j-2} \xi_j^3 (x_1 x_3 \ldots x_{2j-1}) u_{3j-2} + c_{6k-3} u_{6k-3},
\]

\[
w_k(x) := \sum_{j=k}^{\infty} c_{3j-1} \zeta_j^2 (x_2 x_4 \ldots x_{2j}) u_{3j-1} + c_{6k} u_{6k}.
\]

Now consider \( x, y \in I \) such that \( x \neq y \) and \( tG(x) + (1-t)G(y) \in G(I) \) for infinitely many real \( t \). Since \( F \) is a linear injection of \( \ell_\infty \) to \( X \), for any such \( t \) we have \( tH(x) + (1-t)H(y) = H(z) \) for some \( z \in I \). Considering \((3k+1)\)-th coordinates
of \( H(z) \) we obtain \( z = tx + (1 - t)y \). Consequently, considering \((3k - 1)\)-th and
\(3k\)-th coordinates of \( H(z) \), we obtain that, for each \( k \in \mathbb{N} \),
\[
tx_1x_3 \ldots x_{2k-1} + (1 - t)y_1y_3 \ldots y_{2k-1} = (tx_1 + (1 - t)y_1) \cdots (tx_{2k-1} + (1 - t)y_{2k-1}),
\]
\[
tx_2x_4 \ldots x_{2k} + (1 - t)y_2y_4 \ldots y_{2k} = (tx_2 + (1 - t)y_2) \cdots (tx_{2k} + (1 - t)y_{2k}).
\]
Since the above equalities hold for infinitely many \( t \), we infer that \( x \) and \( y \) differ
at most in one odd coordinate and at most in one even coordinate (otherwise
one of right sides, for sufficiently large \( k \), is a polynomial in \( t \) of degree grater
than one, which is impossible). Consequently, there exist \( k_1, k_2 \in \mathbb{N} \) such that
\( y \in x + sp\{e_{2k_1-1}, e_{2k_2}\} \); so \( G(y) \in G(x) + sp\{v_{k_1}(x), w_{k_2}(x)\} \).

The above analysis shows that the set of lines, which contain any fixed point
\( G(x) \), \( x \in I \), and meet the set \( G(I) \) in an infinite set, can be covered by countably
many planes containing \( G(x) \). Therefore \( G(I) \) meets any 3-dimensional affine
subspace of \( X \) in a set of three dimensional Lebesgue measure zero. Consequently,
\( G(I) \in C^*_2 \).

Now suppose that \( G(I) \in C^*_2 \), hence \( G(I) = \bigcup_{n=1}^{\infty} B_n \), where \( B_n \in \mathcal{A}^*(V_n, \varepsilon_n) \)
and \( V_n \) are two-dimensional subspaces of \( X \). Write \( V_n = sp\{p_n, q_n\} \) and choose
\( \delta_n > 0 \) (by Lemma 2.4) such that \( \gamma(V_n, sp\{v, w\}) < \varepsilon_n \) whenever \( \|v - p_n\| < \delta_n \)
and \( \|w - q_n\| < \delta_n \). For any given \( n \) find \( k_1, k_2 \) such that
\[
\sum_{j=k_1}^{\infty} 2^j c_{3j-2} \xi_1^j \|u_{3j-2}\| < c_{6k_1-3} \delta_n/2, \quad \|u_{6k_1-3} - p_n\| < \delta_n/2,
\]
\[
\sum_{j=k_2}^{\infty} 2^j c_{3j-1} \xi_2^j \|u_{3j-1}\| < c_{6k_2} \delta_n/2, \quad \|u_{6k_2} - q_n\| < \delta_n/2.
\]
For any \( x \in I \) we have
\[
\|v_{k_1}(x) - c_{6k_1-3}p_n\| \leq \sum_{j=k_1}^{\infty} 2^j c_{3j-2} \xi_1^j \|u_{3j-2}\| + c_{6k_1-3} \|u_{6k_1-3} - p_n\| < c_{6k_1-3} \delta_n,
\]
\[
\|w_{k_2}(x) - c_{6k_2}q_n\| \leq \sum_{j=k_2}^{\infty} 2^j c_{3j-1} \xi_2^j \|u_{3j-1}\| + c_{6k_2} \|u_{6k_2} - q_n\| < c_{6k_2} \delta_n.
\]
So \( \|v_{k_1}(x)/c_{6k_1-3} - p_n\| < \delta_n \) and \( \|w_{k_2}(x)/c_{6k_2} - q_n\| < \delta_n \), which shows that the
plane \( G(x) + sp\{v_{k_1}(x), w_{k_2}(x)\} \) meets \( B_n \) in a two-dimensional Lebesgue null set.

Hence the set
\[
\{ (t, \tau) : x + te_{2k_1-1} + \tau e_{2k_2} \in G^{-1}(B_n) \} = \{ (t, \tau) : G(x) + tv_{k_1}(x) + \tau w_{k_2}(x) \in B_n \}
\]
is Lebesgue null, and Fubini theorem gives \( \mu(G^{-1}(B_n)) = 0 \). (Note that \( G^{-1}(B_n) \)
is Borel, since \( G \) is continuous.) But this contradicts \( I = \bigcup_{n=1}^{\infty} G^{-1}(B_n) \), and we
infer that \( G(I) \notin C^*_2 \).

\section*{Remark 4.4.} For an arbitrary \( d \in \mathbb{N} \), we obtain as above that \( G_d(I) \in C^*_d+1 \setminus C^*_d \),
where \( G_d = F \circ H_d \),
\[
H_d(x) := (0, \xi_1^1 x_1, \ldots, \xi_1^d x_d, x_1, \xi_2^1 x_1 x_{d+1}, \ldots, \xi_2^d x_d x_{2d}, x_2, \xi_3^1 x_1 x_{d+1} x_{2d+1}, \ldots),
\]
and \( (\xi_1^1), \ldots, (\xi_d^i) \) are suitably chosen sequences.
Proposition 4.5. Let $X$ be a separable infinite dimensional space, $S$ a Lipschitz surface of codimension $n \geq 2$, and $P : X \to Y$ a continuous linear mapping onto a Banach space $Y$ such that $\dim(\ker(P)) < n$. Then there exists an $n$-dimensional space $D \subset Y$ and $0 < \varepsilon < 1$ such that $P(S) \in C^*(D, \varepsilon)$ in $Y$. Consequently, $P(S)$ is a first category subset of $Y$ which is Aronszajn null in $Y$.

Proof. Denote $K := \ker P$. Choose a space $F \subset X$ such that $\dim F = n$ and $S$ is an $F$-Lipschitz surface. Using Lemma 3.7, Lemma 2.3, and Lemma 2.6, we can choose a space $V$ with $\dim V = n$ such that $S$ is an $V$-Lipschitz surface and $V \cap K = \{0\}$. Choose a closed space $H \subset X$ such that $X = H \oplus (K \oplus V)$. Denoting $Z := H \oplus V$, we have $X = Z \oplus K$. Set $\pi := \pi_{Z,K}$. Using Lemma 3.7 and Lemma 2.3, we find $0 < \delta < 1$ such that $\gamma(V, W) < \delta$ implies that $S$ is a $W$-Lipschitz surface and $W \oplus (H \oplus K) = X$. Now consider an arbitrary $W \subset Z$ such that $\gamma(V, W) < \delta$. We can choose a Lipschitz mapping $\varphi : H \oplus K \to W$ such that $S = \{h + k + \varphi(h + k) : h \in H, k \in K\}$. Consequently, $\pi(S) = \{h + \varphi(h + k) : h \in H, k \in K\}$. Now consider an arbitrary $a = h_0 + w_0 \in Z$. Then $(\pi(S) + a) \cap W = \{w_0 + \varphi(-h_0 + k) : k \in K\}$. Since the mapping $\psi : K \to W$ defined by $\psi(k) := w_0 + \varphi(-h_0 + k)$ is Lipschitz and $\dim K < \dim W$, we obtain that $(\pi(S) + a) \cap W$ is Lebesgue null in $W$. Thus, $\pi(S)$ is an $F_\sigma$ set by Remark 3.3(i) and Remark 3.4, we obtain that $\pi(S) \in C^*(V, \delta)$ in $Z$. Since $F := P|Z$ is a linear isomorphism with $F(\pi(S)) = P(S)$, Lemma 4.2 implies that $P(S) \in C^*(D, \varepsilon)$ for $D := F(V)$ and some $\varepsilon > 0$. Consequently, $P(S)$ is Aronszajn null in $Y$ by Lemma 4.3. Thus, $\text{int} P(S) = \emptyset$. Since $P(S)$ is an $F_\sigma$ set, we obtain that $P(S)$ is a first category set.

As an immediate consequence, we obtain the following result.

Proposition 4.6. Let $X$ be a separable infinite dimensional Banach space, $n \geq 2$, $A \in \mathcal{L}^n(X)$, and let $P : X \to Y$ be a continuous linear mapping onto a Banach space $Y$ such that $\dim(\ker(P)) < n$. Then $P(S)$ is a subset of a set from $C^*_n$ in $Y$. Consequently, $P(S)$ is a first category subset of $Y$ which is a subset of an Aronszajn null set in $Y$.

Remark 4.7. Let $X$, $Y$, $P$ and $n$ be as in Proposition 4.6.

(i) Let $f$ be a continuous convex function on $X$ and $B_n := \{x \in X : \dim(\partial f(x)) \geq n\}$. Then [13] Theorem 1.3] states that $P(A)$ is a first category set. Using results of [19], it is easy to see that [13] Theorem 1.3] is equivalent to the statement that $P(A)$ is a first category for each $A \in \mathcal{DC}^n(X)$, but the proof of [13] is direct, it does not use [19].

(ii) The result that $P(A)$ is a first category for each $A \in \mathcal{L}^n(X)$ is due to Heisler [7].

(iii) An example from [7] shows that there exists $A \in \mathcal{DC}^n(X)$ such that $P(A) \notin \mathcal{L}^1(Y)$.

(iv) It is not known whether $P(A)$ is $\sigma$-porous or $\Gamma$-null for each $A \in \mathcal{L}^n(X)$ (resp. $A \in \mathcal{DC}^n(X)$). The negative answer seems to be probable.

Remark 4.8. Let $X$ be a separable infinite dimensional space. Proposition 1.6 easily implies that the inclusions $\mathcal{L}^n(X) \subset \mathcal{L}^{n-1}(X)$ ($n > 1$) are proper. Indeed, no Lipschitz surface $S$ of codimension $n - 1$ can belong to $\mathcal{L}^n(X)$, since there is a surjective continuous linear projection of $S$ on a space $E$ of codimension $n - 1$.

Proposition 1.6 implies the following result which improves both [13] Theorem 1.3] and [7] Theorem 5.6].
Theorem 4.9. Let $X$ be a separable infinite dimensional space, $n \geq 2$, and let $T : X \to X^\ast$ be a monotone (multivalued) operator. Denote by $B_n$ the set of all $x \in X$ for which the convex cover of $T(x)$ is at least $n$-dimensional. Let $P : X \to Y$ be a continuous linear mapping onto a Banach space $Y$ such that $\dim(\ker(P)) < n$. Then $P(B_n)$ is a subset of a set from $C_n^\ast$ in $Y$. Consequently, $P(B_n)$ is a first category subset of $Y$ which is a subset of an Aronszajn null set in $Y$.

Proof. Since $B_n \in \mathcal{L}^n(X)$ by \cite{18}, the assertion follows from Proposition 4.6.

\[\square\]

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*Authors address: Luděk Zajíček*, Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: zajicek@karlin.mff.cuni.cz.