Quantum Cyclic Code

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Abstract

In this paper, we define and study quantum cyclic codes, a generalisation of cyclic codes to the quantum setting. Previously studied examples of quantum cyclic codes were all quantum codes obtained from classical cyclic codes via the CSS construction. However, the codes that we study are much more general. In particular, we construct cyclic stabiliser codes with parameters $[[5, 1, 3]]$, $[[17, 1, 7]]$ and $[[17, 9, 3]]$, all of which are not CSS. The $[[5, 1, 3]]$ code is the well known Laflamme code and to the best of our knowledge the other two are new examples. Our definition of cyclicity applies to non-stabiliser codes as well; in fact we show that the $((5, 6, 2))$ nonstabiliser first constructed by Rains et al [10] and latter by Arvind et al [2] is cyclic.

We also study stabiliser codes of length $4^n + 1$ over $\mathbb{F}_2$ for which we define a notation of BCH distance. Much like the Berlekamp decoding algorithm for classical BCH codes, we give efficient quantum algorithms to correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors when the BCH distance is $d$.

1 Introduction

One of the biggest challenge in implementation of quantum computation is to deal with quantum errors efficiently. The subtle nature of quantum phenomenon like entanglement and superposition needs to be preserved from both the environment as well as from faulty circuits, for any of the speedups to be realised. Quantum error correcting codes provide a way to make this this possible. Despite strange phenomenons like the no-cloning theorem, a sufficiently detailed theory of quantum error correcting exists[8, 7, 4]. It has already provided a foundation for fault-tolerant quantum computing via the implementation of error resistant quantum circuits and quantum storage elements. Besides it plays an important role in various areas like quantum cryptography and quantum key distribution protocols.

In this paper, we study a certain class of quantum codes which we believe is a natural generalisation of the class of cyclic code in the classical setting. We give complete characterisation such codes and study the stabiliser case in depth (Section 3). Previously, Calderbank et al[4, Section 5]
had constructed quantum stabiliser codes whose underlying totally isotropic set (see Section 2 for a definition) is cyclic. In the literature, there has been work [9, 1] in trying to use classical cyclic codes to build efficient quantum codes via the CSS construction[6, 12]. As concrete examples we construct a [[5, 1, 3]], a [[17, 1, 7]] and a [[17, 9, 3]] quantum cyclic code none of which are CSS. The 5 qubit code is the well known Laflamme code and to the best of our knowledge the other are new. Besides we give some examples of nonstabiliser cyclic codes as well (Section 5).

We also study a restricted family of cyclic stabiliser codes for which we can define a notion of BCH distance. Much like the classical case these family of code have efficient (polynomial time) quantum decoding algorithm within the BCH limit, i.e. if the BCH distance is \(d = 2t + 1\) then we can correct up to \(t\) errors.

2 Preliminaries

We now give a brief overview of the notation used in this paper. For a prime power \(q = p^k\), \(\mathbb{F}_q\) denotes the unique finite field of cardinality \(q\). In this paper, we study quantum codes over the alphabet \(\mathbb{F}_p\). Most of what we say carry over any extension \(\mathbb{F}_{p^k}\) as well.

We consider the \(p\)-dimensional Hilbert space \(\mathcal{H} = \mathbb{C}^2(\mathbb{F}_p)\) of all functions from \(\mathbb{F}_p\) to the set of complex numbers \(\mathbb{C}\). This Hilbert space plays the role of the alphabet set in the quantum setting. The set \(\{|a\} | a \in \mathbb{F}_p\}\) where \(|a\) stands for the function that takes value 1 at \(a\) and 0 otherwise, forms an orthonormal basis for the Hilbert space \(\mathcal{H}\). For a positive integer \(n\), an element \(a = (a_1, \ldots, a_n)^T \in \mathbb{F}_p^n\) will be considered as column vectors. As is standard in quantum computing, by \(|a\) we mean the vector \(|a_1\rangle \otimes \ldots \otimes |a_n\rangle\). Thus, the set \(\{|a\} | a \in \mathbb{F}_p^n\}\) forms a basis for the \(n\)-fold tensor product \(\mathcal{H}^\otimes n\).

Quantum errors are captured by what are known as the Weyl operators. For \(a\) and \(b\) in \(\mathbb{F}_p\) define the unitary operators \(U_a\) and \(V_b\) as \(U_a|x\rangle = |x + a\rangle\) and \(V_b|x\rangle = \zeta^{bx}|x\rangle\), where \(\zeta\) is a primitive \(p\)-th root of unity. The operator \(U_a\) is thought of as a flip in the alphabet \(\mathbb{F}_p\) and \(V_b\) is thought of as a flip in the phase. The operator \(U_aV_b\) constitutes a flip in both the alphabet and phase. It is sufficient to consider only the Weyl operators when designing quantum codes as they form a basis of the Hilbert space \(\mathcal{B}(\mathcal{H})\) of operators from \(\mathcal{H}\) onto itself. To extend these operators onto \(\mathcal{H}^\otimes n\), for a positive integer \(n\), define for \(a\) and \(b\) in \(\mathbb{F}_p^n\) the Weyl operators \(U_a\) and \(V_b\) on \(\mathcal{H}^\otimes n\) as \(U_a|x\rangle = |x + a\rangle\) and \(V_b|x\rangle = \zeta^{bx}|x\rangle\) respectively. To capture errors at \(t\) locations, we define the joint weight \(w(a, b)\) for a pair \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) in \(\mathbb{F}_p^n\) as the number of positions \(i\) such that either \(a_i\) or \(b_i\) is not zero. We extend this definition to Weyl operators, the weight \(w(U_aV_b)\) is the joint weight \(w(a, b)\). Consider the transmission of any pure state \(|\psi\rangle\) in \(\mathcal{H}^\otimes n\). Occurrence of a quantum error at \(t\) positions is modelled as the channel applying an unknown Weyl operator \(U_aV_b\) of weight \(t\) on the transmitted message \(|\psi\rangle\). An quantum code over \(\mathbb{F}_p\) of length \(n\) is a subspace of the \(n\)-fold tensor product product \(\mathcal{H}^\otimes n\). There is by now a significant literature on quantum codes[8, 7, 4]. A quantum code, being a subspace, is completely captured by the projection into it. Therefore we often express a quantum code by giving its projection operator. A quantum code of length \(n\), dimension \(K\) and distance \(d\) over \(\mathbb{C}^2(\mathbb{F}_p)\) will be called an \(((n, K, d))\) quantum code.

We now discuss special quantum codes called stabiliser codes or additive codes. To this end fix a finite field \(\mathbb{F}_p\) and a positive integer \(n\). Let \(\mathcal{W}_{n,p}\) denote the group of unitary operators generated by the Weyl operators. The error group \(\mathcal{E}_{n,p}\) is just the group \(\mathcal{W}_{n,p}\), if the characteristic \(p\) is odd, and is the group generated by \(\mathcal{W}_{n,p} \cup i\mathcal{W}_{n,p}\), \(i\) being the complex number \(\sqrt{-1}\), when \(p\) is 2. We will drop the subscripts \(n\) and \(p\) when the quantities are clear from the context. For a subset \(S\) of the error group \(\mathcal{E}\), the stabiliser code \(C_S\) associated with \(S\) is the subspace of vectors \(|\varphi\rangle \in \mathcal{H}^\otimes n\) such that \(U|\varphi\rangle = |\varphi\rangle\) for all \(U\) in \(S\). Without loss of generality we can assume that \(S\) is actually a
subgroup of \( \mathcal{E} \). Furthermore, for the code \( \mathcal{C}_S \) to be be nontrivial, \( \mathcal{S} \) should be Abelian and should not contain \( \omega I \) for any nontrivial root of unity \( \omega \). We call such a subgroup \( \mathcal{S} \) a Gottesman subgroups of the error group \( \mathcal{E} \).

Consider a pair \( (a, b) \) of elements of \( \mathbb{F}_p^n \) as elements of the vector space \( \mathbb{F}_p^{2n} \) over the scalars \( \mathbb{F}_p \). The symplectic inner product between two elements \( u = (a, b) \) and \( v = (c, d) \) of \( \mathbb{F}_p^{2n} \) is defined as \( \langle u, v \rangle = a^{T} d - b^{T} c \). A subset \( S \) of \( \mathbb{F}_p^{2n} \) is called totally isotropic [4] with respect to the symplectic inner product, if for any two elements \( u \) and \( v \) of \( S \), \( \langle u, v \rangle = 0 \). In the rest of the paper, unless otherwise mentioned, by totally isotropic set we mean totally isotropic with respect to the symplectic inner product define above.

Stabiliser codes, or equivalently their corresponding Gottesmann subgroups, are intimately connected to totally isotropic subspace of \( \mathbb{F}_p^{2n} \). Depending on whether the characteristic \( p \) is odd or 2, the exact nature of the correspondence is slightly different. Calderbank et al [5, 4] were the first to study this connection when characteristic of the underlying field \( \mathbb{F}_p \) is 2. Later Arvind and Parthasarathy [3] studied the case when \( p \) is an odd prime. We summaries these results in a form convenient for our purposes.

**Theorem 2.1** ([5, 3]). Let \( p \) be any prime and \( n \) a positive integer. If \( S \) is a totally isotropic subspace of \( \mathbb{F}_p^{2n} \) there exists \( n \times 2n \) matrices \( L \) and \( M \) such that

1. \( L^T M \) is symmetric,

2. \( S \) is the image of the map \( \phi_{L,M} \) from \( \mathbb{F}_p^n \) to \( \mathbb{F}_p^{2n} \) defined as \( \phi_{L,M}(a) = (L a, M a) \) and

3. The set of operators \( \mathcal{S} = \{ \alpha_a U_L V_M | a \in \mathbb{F}_p^n \} \) forms a Gottesman subgroup where \( \alpha_a \) is defined as

\[
\alpha_a = \left\{ \begin{array}{ll}
\zeta^{a^{T} L^T M a} & \text{when } p \neq 2, \\
i^{a^{T} L^T M a} & \text{when } p = 2
\end{array} \right.
\]

where \( \zeta \) is a primitive \( p \)-th root of unity and \( i = \sqrt{-1} \).

4. The projection operator to the associated code \( \mathcal{C}_S \) is given by

\[
P = \sum_{U \in \mathcal{S}} U = \sum_{a \in \mathbb{F}_p^n} \alpha_a U_L V_M a.
\]

When studying stabiliser codes, we will concentrate only on the underlying totally isotropic subspace of \( \mathbb{F}_p^{2n} \).

One possible way of constructing quantum codes, or totally isotropic subspaces of \( \mathbb{F}_p^n \times \mathbb{F}_p^n \), is by taking classical codes \( C_1 \) and \( C_2 \) of length \( n \) such that \( C_1 \) is orthogonal to \( C_2 \) (here the orthogonality is with respect to the usual inner product \( a^{T} b \)). It is easy to verify then that \( C_1 \times C_2 \) is isotropic. This construction is called the CSS construction [6] and the resultant quantum stabiliser codes are called CSS codes.

Let \( S \) be a subspace of \( \mathbb{F}_p^{2n} \). By the *centraliser* of \( S \), denoted by \( \overline{S} \), we mean the subspace of all \( u \) in \( \mathbb{F}_p^{2n} \), such that \( \langle u, v \rangle = 0 \), for all \( v \) in \( S \). If \( S \) is totally isotropic, \( \overline{S} \) contains \( S \). We have the following theorem on the properties of the stabiliser code \( \mathcal{C}_S \) associated to the set \( S \).

**Theorem 2.2** ([5, 3]). Let \( S \) be a totally isotropic subspace of \( \mathbb{F}_p^{2n} \) and let \( \mathcal{C} \) be the associated stabiliser code. Then, the dimension the subspace \( S \) is always less than \( n \). If \( S \) has dimension \( n - k \) for some positive integer \( k \) then dimension of its centraliser \( \overline{S} \), as a subspace of \( \mathbb{F}_p^{2n} \), and the dimension of the code \( \mathcal{C} \), as a Hilbert space, are \( n + k \) and \( p^k \) respectively. Furthermore, if the minimum weight \( \min \{ w(u) | u \in \overline{S} \setminus S \} \) is \( d \) then \( \mathcal{C} \) can detect up to \( d - 1 \) errors and correct up to \( \left\lfloor \frac{d - 1}{2} \right\rfloor \) errors.
Let \( \mathcal{C} \) be a stabiliser code associated with an \( n - k \) dimensional totally isotropic subspace \( S \) of \( \mathbb{F}_p^{2n} \). By the stabiliser dimension of \( \mathcal{C} \) we mean the integer \( k \). Similarly, we call the weight \( \min \{ w(u) | u \in S \setminus \mathcal{S} \} \) the distance of \( \mathcal{C} \). A stabiliser code of length \( n \), stabiliser dimension \( k \) and distance \( d \) will be called an \( [[n, k, d]]_p \) stabiliser code. By theorem 2.2, an \( [[n, k, d]]_p \) stabiliser code is an \( ((n, p^k, d))_p \) quantum code. As usual we will drop the subscript \( p \) when it is clear from the context.

### 3 Quantum Cyclic codes

In this section we define quantum cyclic codes and study some of its properties. Recall that a classical code over \( \mathbb{F}_p \) is cyclic if and only if for all code words \( u = (u_1, \ldots, u_n) \), its right shift \( (u_n, u_1, \ldots, u_{n-1}) \) is also a code word. Let \( N \) denote the right shift operator over \( \mathbb{F}_p^n \), i.e. the operator that maps \( u = (u_1, \ldots, u_n) \) to \( (u_n, u_1, \ldots, u_{n-1}) \). Consider the unitary operator \( \mathcal{N} \) defined on the tensor product \( \mathcal{H}^\otimes n \) as follows \( \mathcal{N}[u] = |Nu\rangle \).

**Definition 3.1.** A quantum code \( \mathcal{C} \) is defined to be cyclic if the shift operator \( \mathcal{N} \) maps \( \mathcal{C} \) to itself, i.e. \( \mathcal{N}\mathcal{C} = \mathcal{C} \).

We have the following result on the projection operator associated to a quantum cyclic code.

**Proposition 3.2.** A quantum code \( \mathcal{C} \) is cyclic if and only if its projection operator commutes with \( \mathcal{N} \).

**Proof.** Let \( \mathcal{H} \) be any Hilbert space and let \( \mathcal{C} \) be any subspace with the associated projection operator being \( P \). Let \( U \) be any unitary operator on \( \mathcal{H} \). If \( UP = PU \) then \( UC = UPH = PUH \). However, since \( \mathcal{H} \) is the underlying Hilbert space we have \( UH = \mathcal{H} \). Thus \( UC = \mathcal{C} \).

Conversely, suppose that \( UC = \mathcal{C} \). We prove \( UP = PU \) by showing that for all \( \langle \psi | \psi \rangle \) in \( \mathcal{H} \), \( UP|\psi\rangle = PU|\psi\rangle \). Let \( C^\perp \) be the orthogonal complement of \( \mathcal{C} \). Since any unitary map preserves inner product, we have \( UC^\perp = C^\perp \).

Any vector \( |\psi\rangle \) can be expressed uniquely as \( |\psi_1\rangle + |\psi_2\rangle \) where \( |\psi_1\rangle \in \mathcal{C} \) and \( |\psi_2\rangle \in C^\perp \). Therefore,

\[
UP|\psi\rangle = U|\psi_1\rangle = P(U|\psi_1\rangle + U|\psi_2\rangle) = PU|\psi\rangle.
\]

Thus if \( UC = \mathcal{C} \) then \( UP = PU \). The result then follows by taking \( U = \mathcal{N} \).

\[ \square \]

Let \( S \) be a subspace of \( \mathbb{F}_p^n \times \mathbb{F}_p^n \). We say that \( S \) is separately cyclic if for all \( (a, b) \) in \( S \) \( (Na, Nb) \) is also in \( S \). We have the following property on centralisers of separately cyclic sets.

**Proposition 3.3.** Let \( S \) be any separately cyclic set then its centraliser is also separately cyclic.

In the context of cyclic stabiliser codes, separately cyclic sets are interesting because of the following property.

**Proposition 3.4.** A stabiliser code \( \mathcal{C} \) is cyclic if and only if its associated totally isotropic subspace \( S \) and its centraliser \( \overline{S} \) are separately cyclic.

**Proof.** Let \( S \) be the totally isotropic set associated with \( \mathcal{C} \). Let \( P \) denote the projector to \( \mathcal{C} \). Then \( \mathcal{C} \) is cyclic if and only if \( N^\dagger P N = P \).

We make use of Theorem 2.1 for the proof. The projector \( P \) is given by the expression

\[
P = \sum_a \alpha_a U_L a V_M a
\]

and \( S \) is \( \{(L a, M a) | a \in \mathbb{F}_p^n \} \) for \( L, M \) and \( \alpha \) as in Theorem 2.1. Since \( N^\dagger U_L a V_M a N = \)
$U_N P V_N$, it is necessary that $S$ is separately cyclic. Otherwise the support of $N^TPN$ will not match with that of $P$.

Conversely, if $S$ is separately cyclic then we have $(NLa, NMa) \in S$ for all $a \in \mathbb{F}_p^n$ where $L$ and $M$ are as in Theorem 2.1. Also note that the inverse of the shift operation $N$ is just $N^T$. Therefore $L^TN^TNM = L^TM$. Hence the scalars $\alpha_a$ are also preserved and hence $N^TPN = P$.

The cyclicity of the centraliser $S$ follows from Proposition 3.3.

\[\square\]

Classical cyclic codes over $\mathbb{F}_p$ of length $n$, $n$ coprime to $p$, are ideals of the polynomial ring $\mathbb{F}_p[X]/(X^n - 1)$. The goal of the rest of the section is to develop an algebraic characterisation of cyclic stabiliser codes along similar lines. We fix some conventions for the rest of the paper. Fix a prime $p$ and a positive integer $n$ coprime to $p$. Let $R$ denote the cyclotomic ring $\mathbb{F}_p[X]/X^n - 1$ of polynomials modulo $X^n - 1$. For the vector $a = (a_0, \ldots, a_{n-1})$ in $\mathbb{F}_p^n$, associate the polynomials $a(X) = a_0 + \ldots + a_{n-1}X^{n-1}$ in the ring $R$. Often, we need to interchange between these two perspectives of an element in $\mathbb{F}_p^n$. When we think of them as a vector, we use the bold face Latin letter. On the other hand, when thinking of them as polynomials we use the corresponding plain face letter. For example the polynomial associated with the vector $a$ is either written as $a(X)$ or often just $a$. In the ring $R$, the polynomial $X$ has a multiplicative inverse namely $X^{n-1}$. Often, we just write $X^{-1}$ or just $\frac{1}{X}$ to denote this inverse.

First we have the following characterisation of separately cyclic subspaces of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ in terms of polynomials in $R$.

**Lemma 3.5.** A subspace $S$ of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ is separately cyclic if and only if there exists degree $n - 1$ polynomials $g(X), f(X)$ and $h(X)$ in $\mathbb{F}_p[X]$ such that $g(X)$ and $h(X)$ are factors of $X^n - 1$ as polynomials in $\mathbb{F}_p[X]$ and elements of $S$ as a pair of polynomials in $R \times R$ are precisely $(a(X)g(X), a(X)f(X) + b(X)h(X))$ where $a(X)$ and $b(X)$ vary over all degree $n - 1$ polynomials in $\mathbb{F}_p[X]$.

**Proof.** Consider any subspace $S$ of $\mathbb{F}_p^n \times \mathbb{F}_p^n$. Clearly if there exists polynomials satisfying the above mentioned conditions, then $S$ is separately cyclic.

To prove the converse, assume that $S$ is separately cyclic. Define $A$ and $B$ to be the projections of $S$ onto the first and last $n$ coordinates respectively, i.e. $A = \{a|(a, b) \in S\}$ and $B = \{b|(a, b) \in S\}$. Since $S$ is separately cyclic, $A$ and $B$ are cyclic subspaces of $\mathbb{F}_p^n$ and hence are ideals of the ring $R$. Let $g(X)$ be the factor of $X^n - 1$ that generates $A$. Since $g(X)$ is an element of $A$ there exists a polynomial $f(X)$ in $R$ such that $(g, f) \in S$. Fix any such polynomial $f$. To construct $h(X)$, consider the set $B_0 = \{b(x)|(0, b) \in S\}$. Clearly $B_0$ is also cyclic and therefore ideal of the ring $R$. Let $h(X)$ denote the factor of $X^n - 1$ that generates $B_0$. Our claim is that these are the required polynomials.

Since $S$ is separately cyclic we have $(X^ig(X), X^if(X))$ are all elements of $S$. As $S$ is a subspace, by taking appropriate linear combinations, we have that for any two degree $n - 1$ polynomials $a(X)$ and $b(X)$, $(ag, af + bh)$ is an element of $S$. On the other hand, consider any arbitrary $(u, v) \in S$. Clearly $u$ is an element of $A$ and hence $u(X) = a(X)g(X)$. Subtract from the $(u, v)$ the element $(ag, af) \in S$. We have $(0, v - af)$ is in $S$ and hence $v - af$ is in $B_0$. Therefore $v - af = bh$ for some polynomial $b$.

\[\square\]

The triple of polynomials $(g, f, h)$ play a crucial role in unravelling the structure of a separately cyclic subspace $S$. We have the following definition.
Definition 3.6 (Generating triple). Let $S$ be a separately cyclic subspace of $F_p^n \times F_p^m$. A generating triple for $S$ is a triple of polynomials $(g, f, h)$ in $F_p[X]$ the polynomials pairs $(g, f)$ and $(0, h)$ are in $S$ and every element of $S$ as a polynomial pair in $R \times R$ is of the form $(ag, af + bh)$ for some polynomials $a(X)$ and $b(X)$ in $F_p[X]$.

We want to express the isotropic condition $a^Td = b^Tc$ in terms of polynomials. The following definition on pairs of polynomials over $F_p$ that will play the role of the isotropic condition in the setting of separately cyclic subspace.

Definition 3.7 (Isotropic pairs of polynomial). Let $a(X)$, $b(X)$, $c(X)$ and $d(X)$ are polynomials in $F_p[X]$. We say that the pairs $(a, b)$ and $(c, d)$ are isotropic pairs of polynomial modulo $X^n - 1$ for some $n$ coprime to $p$ if and only if

$$a(X)d(X^{-1}) = b(X)c(X^{-1}) \mod X^n - 1.\]$$

Notice that for any two vectors $u$ and $v$ in $F_p^n$, if $u(X)$ and $v(X)$ denote the corresponding polynomials in $R$, then the coefficient of $X^k$ in the product $u(X)v(X^{-1}) \mod X^n - 1$ is the inner product $u^TN^kv$, where $N$ is the right shift operator. An immediate consequence of this observation is the following.

Proposition 3.8. Let $S$ be a separately cyclic subspace of $F_p^n \times F_p^m$. An element $(u, v)$ is isotropic to all elements of $S$ with respect to the symplectic inner product if and only if the corresponding pair of polynomials $(u, v)$ is isotropic modulo $X^n - 1$ with all polynomial pairs $(a, b)$ in $S$.

As a corollary we have the following proposition

Corollary 3.9. A separately cyclic subset $S$ of $F_p^n \times F_p^m$ is totally isotropic if and only if for every pair of elements $(a, b)$ and $(c, d)$ of $S$, the corresponding polynomials $(a, b)$ and $(c, d)$ are isotropic modulo $X^n - 1$. Furthermore, the centraliser $S$ is the pair of all polynomials $(c, d)$ that are isotropic to all pairs of polynomials in $S$.

We are now ready to characterise all cyclic stabiliser codes. It is sufficient to characterise separately cyclic, totally isotropic subsets of $F_p^n \times F_p^m$.

Theorem 3.10. A subspace $S$ of $F_p^n \times F_p^m$ is totally isotropic and separately cyclic if and only if there exists polynomials $g$, $f$ and $h$ in $F_p[X]$ such that

1. The polynomials $g$ and $h$ divide $X^n - 1$ as polynomials over $F_p$.

2. $g(X^{-1})h(X) = g(X)h(X^{-1}) = 0 \mod X^n - 1$,

3. The pair $(g, f)$ is isotropic to itself modulo $X^n - 1$, i.e. $g(X)f(X^{-1}) = f(X)g(X^{-1})$.

4. $S$ is precisely the set of polynomial pairs of the form $(ag, af + bh)$ where $a$ and $b$ varies over polynomials over $F_p$.

Proof. Assume that $S$ is both separately cyclic and totally isotropic. Let $(g, f, h)$ be the triple generating $S$. Since $S$ is totally isotropic and since $(g, f)$ is an element of $S$, it should be isotropic modulo $X^n - 1$ with every polynomial pair in $S$. In particular it should be so with itself and the element $(0, h)$. The shows that $g$, $f$ and $h$ satisfies the conditions in the theorem.

To prove the converse, assume that polynomials $g$, $f$ and $h$ exists with the above mentioned properties. Then clearly $S$ is separately cyclic. It is straightforward to then check that any two elements $(a_i g, a_i f + b_i h)$, for $i = 1, 2$ form a isotropic pair of polynomials modulo $X^n - 1$. Hence $S$ is isotropic.
A similar proof together with corollary 3.9 gives the following characterisation of the centraliser of a separately cyclic totally isotropic subspace.

**Theorem 3.11.** Let $S$ be a totally isotropic, separately cyclic subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ generated by the triple $(g, f, h)$ then the centraliser $\mathcal{S}$ is the set of all polynomial pairs $(c, d)$ such that

1. $c(X)h(X^{-1}) = c(X^{-1})h(X) \equiv 0 \mod X^n - 1$,

2. $(g, f)$ and $(c, d)$ forms a isotropic pairs of polynomials, i.e $g(X)d(X^{-1}) = f(X)c(X^{-1}) \mod X^n - 1$.

4 **Explicit constructions and decoding algorithm**

In this section we give an explicit constructions of cyclic stabiliser codes over the binary alphabet $\mathbb{F}_2$. We define the notion of a BCH distance for such codes and give an efficient algorithm to decode such codes within the BCH limit.

Consider the unique quadratic extension $\mathbb{F}_4 = \mathbb{F}_2(\eta)$ of $\mathbb{F}_2$ obtain by adjoining a root $\eta$ of the irreducible polynomial $X^2 + X + 1$ in $\mathbb{F}_2[X]$. The conjugate root $\eta + 1$ of $X^2 + X + 1$ in $\mathbb{F}_4$ will we denoted by $\eta'$. For this section, we identify the set $\mathbb{F}_2^2 \times \mathbb{F}_2^2$ with the vector space $\mathbb{F}_4^n$ over $\mathbb{F}_4$ as follows: every pair of element $(a, b)$ where $a$ and $b$ are vectors in $\mathbb{F}_2^n$ will be identified with vector $a + \eta b$ in $\mathbb{F}_4$. It is convenient to extend this identification to polynomials as well. As before, let $\mathcal{R}$ denote the cyclotomic ring $\mathbb{F}_2[X]/X^n - 1$. We identify the set $\mathcal{R} \times \mathcal{R}$ with the cyclotomic ring $\mathcal{R}(\eta) = \mathbb{F}_4[X]/X^n - 1$ over the field extension $\mathbb{F}_4$ by identifying the pair $(a, b)$ of polynomials in $\mathcal{R} \times \mathcal{R}$ with the polynomial $a(X) + \eta b(X) \in \mathcal{R}(\eta)$. The codes that we give in this section will in fact be linear stabiliser codes [4] defined as follows.

**Definition 4.1 (Linear stabiliser codes).** A stabiliser code is said to be linear if the underlying totally isotropic subspace $S$ of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ as a subset of $\mathbb{F}_4^n$ is a $\mathbb{F}_4$-linear subspace of $\mathbb{F}_4^n$.

We first show the following result on separately cyclic subspaces of that are $\mathbb{F}_4$ linear.

**Proposition 4.2.** Let $n$ be any positive odd integer. Let $S$ be any separately cyclic subspace of $\mathbb{F}_2^n \times \mathbb{F}_2^n$. Then $S$ is $\mathbb{F}_4$ linear if and only if $S$ is an ideal of $\mathbb{F}_4[X]/X^n - 1$.

**Proof.** Clearly $S$ is separately cyclic if and only if $S$ is cyclic as in a classical cyclic code over $\mathbb{F}_4$. The result then follows from the theory of classical cyclic codes. \hfill $\Box$

As a corollary we have the following result.

**Corollary 4.3.** Let $C$ be any linear stabiliser code and let $S$ be the underlying totally isotropic set. Then $C$ is cyclic if and only if $S$ and its centraliser $\mathcal{S}$ are ideals of $\mathbb{F}_4[X]/X^n - 1$.

Thus constructing linear stabiliser codes that are also cyclic involves computing factors of $X^n - 1$ over $\mathbb{F}_4$ such that the underlying ideal as a set of $\mathbb{F}_2^n \times \mathbb{F}_2^n$ is totally isotropic. To this end we fix some more notation.

Consider the Forbenius automorphism $\sigma$ on $\mathbb{F}_4$ that maps the root $\eta$ its conjugate $\eta' = \eta + 1$. We extend this to the ring $\mathbb{F}_4[X]/X^n - 1$ defined by $\sigma(a(X) + \eta b(X)) = a(X) + \eta' b(X)$. We first state the following result on the irreducible factors of $X^{4n+1} - 1$ over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_4$ respectively.

**Lemma 4.4.** Let $f(X)$ be any irreducible factor of $X^{4n+1} - 1$ over $\mathbb{F}_2$. 

7
1. A root $\beta$ of $X^n - 1$ is a root of $f(X)$ if and only if $\beta^{-1}$ is also a root.

2. Furthermore if $f(X) \neq X - 1$, the degree of $f(X)$ is an even number $2t$ and it splits into two irreducible factors $f_1(X, \eta)$ and $f_2(X, \eta)$ of degree $t$ each such that $f_2 = \sigma(f_1)$.

Proof. Let $f(X)$ be any irreducible factor of $X^{4m+1} - 1$ over $\mathbb{F}_2$. Consider any root $\beta$ of $f(X)$.

To prove Property 1, note that $\sigma^{2m}$ is a field automorphism and $\sigma^{2m}(\beta) = \beta^{4m} = \beta^{-1}$. Hence $f(\beta^{-1}) = \sigma^{2m}(f(\beta)) = 0$.

Now consider Property 2. Consider any root $\beta$ of $f(X)$. Since $f(X) \neq X - 1$, we have $\beta \neq 1$ and hence $\beta \neq \beta^{-1}$. As a result, that roots of $f(X)$ comes in pairs; for every root $\beta$ its inverse $\beta^{-1}$ is also a root. Hence, $f(X)$ has to be of even degree.

Let the degree of $f$ be the even number $2t$ then the splitting field of $f(X)$ over $\mathbb{F}_2$ is $\mathbb{F}_{4t}$ and therefore contains $\mathbb{F}_4$. Furthermore any irreducible factor $f_1(X)$ of $f(X)$ over $\mathbb{F}_4$ has degree exactly the degree of the extension $\mathbb{F}_{4t}/\mathbb{F}_4$ which is $t$. Since $\sigma$ is a field automorphism of $\mathbb{F}_4$, we have $f(X) = f_1(X)\sigma(f_1(X))$ over $\mathbb{F}_4$.

The following theorem gives us a generic way of constructing linear cyclic codes over $\mathbb{F}_2$ of length $4m + 1$.

**Theorem 4.5.** Let $n = 4m + 1$ and let $g(X)$ be any factor of $X^n - 1$ over $\mathbb{F}_2$. Let $h(X, \eta)$ be any product of irreducible factors of $X^n - 1/g$ over $\mathbb{F}_4$, such that for any irreducible factor $r(X, \eta)$ of $X^n - 1$ over $\mathbb{F}_4$, $r(X, \eta)$ divides $h(X, \eta)$ if and only if $\sigma(r) = r(X, \eta')$ does not. Then there is a linear stabiliser code whose underlying totally isotropic set $S$ is the ideal generated by $g(X)h(X, \eta)$ in $\mathcal{R}(\eta)$ and hence is cyclic. Furthermore, the centeriser $\mathcal{F}$ of $S$ is the ideal generated by $h(X, \eta)$.

Proof. Let the polynomials be as in the theorem. Clearly the ideal $S$ generated by $g(X)h(X, \eta)$ is separately cyclic. All that remains is to show that $S$ is isotropic. Consider the ring $\mathbb{F}_4[X]/(X^n - 1)$. Define via Chinese remaindering an element $a(X)$ such that $a(X) = 0 \mod g$ and for all irreducible factor $r(X, \eta)$

$$a(X) = \begin{cases} \eta \mod r(X, \eta) & \text{if } r(X, \eta) \nmid h(X, \eta), \\ \eta' \mod r(X, \eta) & \text{if } r(X, \eta) \mid h(X, \eta), \end{cases}$$

We first show that $a(X)$ is a polynomial over $\mathbb{F}_2[X]$ instead of $\mathbb{F}_4[X]$. To prove this consider the action of the Frobenius $\sigma$ on $\mathcal{R}(\eta)$ defined by $\sigma(u(X, \eta)) = u(X, \eta')$. It is sufficient to show that $\sigma(a) = a$. Consider any irreducible factor $r$ of $X^n - 1$ over $\mathbb{F}_4$. Since $h(X, \eta)$ contains in it exactly one of the factors $r$ or $\sigma(r)$, let us assume, without loss of generality, that $r \nmid h(X, \eta)$ and $a$ modulo $r$ and $\sigma(r)$ are $\eta$ and $\eta'$ respectively. Therefore $\sigma(a)$ modulo $\sigma(r)$ and $\sigma^2(r) = r$ are $\eta'$ and $\eta$ modulo respectively. Therefore $a = \sigma(a)$ mod $r\sigma(r)$ for every irreducible factor of $X^n - 1/g$.

Furthermore since $a = 0 \mod g$, we have $\sigma(a) = a$ over $\mathcal{R}(\eta)$.

It is easy to verify that the ideal $S$ as a separately cyclic set is generated by the triple $(g, ag, 0)$. By Theorem 3.10, $S$ is isotropic if and only if the equation

$$g(X)g(X^{-1})a(X) = g(X)g(X^{-1})a(X^{-1}). \quad (1)$$

holds modulo $X^n - 1$. By Chinese remaindering it is sufficient to verify Equation 1 modulo $g$ and $X^n - 1/g$ separately.

Clearly Equation 1 holds modulo $g$. Since $g(X)$ is a product of irreducible factors of $X^n - 1$ over $\mathbb{F}_2$, for any root $\beta$ of $X^n - 1$, we have $g(\beta^{-1}) = 0$ if and only if $g(\beta) = 0$ (Lemma 4.4). By both $g(X)$ and $g(X^{-1})$ are invertible modulo $X^n - 1/g$. Therefore we need to show that $a(X^{-1}) = a(X)$ modulo $X^n - 1/g$. It follows from the construction of $a(X)$ that it satisfies the
equation \( a^2 = a + 1 \mod X^n - 1/g \). Therefore \( a(X^{-1}) \approx a(X)^{4m} = a(X) + 2m \mod X^n - 1/g \) and hence \( a(X^{-1}) = a(X) \mod X^n - 1/g \) as \( 2m = 0 \) in characteristic 2.

We call the codes that are characterised by the above theorem as \( 4^m + 1 \)-codes and the pair of polynomials \((g(X), h(X, \eta))\) the generating pair of the code. We have the following theorem on the dimension of the code.

**Theorem 4.6.** Let \( C \) be \( 4^m + 1 \)-code generated by the pair \((g, h)\) then the stabiliser dimension is given by \( \text{deg}(g) \).

**Proof.** The set \( S \) is an ideal of \( \mathbb{F}_4[X]/X^n - 1 \). Let \( d_1 \) and \( d_2 \) denote the degree of \( g \) and \( h \) respectively then clearly \# \( S \) is \( 4^{n-(d_1+d_2)} \). Also note that \( gh \sigma(h) = X^n - 1 \). Therefore \( d_1 + 2d_2 = n \).

If \( k \) is the \( \mathbb{F}_2 \) dimension of \( S \) then \# \( S \) = \( 2^k \). Comparing we have \( k = 2n - 2d_2 - 2d_2 \). Thus we have \( k = n - d_1 \). Therefore by 2.2 we have the required result.

We now recall some concepts from the theory of classical cyclic codes.

**Definition 4.7 (BCH distance).** Let \( g(X) \) be a factor of the polynomial \( X^n - 1 \) over the finite field \( \mathbb{F}_q \) for some prime power \( q \) and let \( n \) coprime to \( q \). The BCH distance is the largest integer \( d \) such that the consecutive distinct powers \( \beta^1, \beta^{d+1}, \ldots, \beta^{d+c-2} \) are roots of \( g \), for some primitive \( n \)-th root \( \beta \).

For a \( 4^m + 1 \)-code generated by the pair \((g, h)\), by the BCH distance we mean the BCH distance of \( h(X, \eta) \) as a factor of \( X^n - 1 \) over \( \mathbb{F}_4 \). We have the following theorem about the distance.

**Theorem 4.8.** The distance of a \( 4^m + 1 \) code is at least its BCH distance.

**Proof.** The centraliser \( \overline{S} \) of the underlying totally isotropic subspace \( S \) is an ideal generated by \( h(X, \eta) \) and hence can be thought of as classical cyclic code over \( \mathbb{F}_4 \). Hence the \( \mathbb{F}_4 \)-weight of any non-zero element in \( \overline{S} \) is at least the BCH distance. As the distance of the code is the the minimum weight of \( \overline{S} \setminus \overline{S} \) (Theorem 2.2), we have the desired result.

To demonstrate the construction for specific cases take \( m = 1 \). The polynomial \( X^5 - 1 = (X - 1)(X^4 + X^3 + X^2 + X + 1) \) over \( \mathbb{F}_2 \). Further, over \( \mathbb{F}_4 \), the degree 4 irreducible factor factorises into \((X^2 + \eta X + 1)\) and \((X^2 + \eta/X + 1)\). If we pick \( g = X - 1 \) and \( h(X, \eta) \) to be any one of the factors, we get a \([5, 1, 3]\) code which turns out to be the Laflamme code.

The case \( m = 2 \) is more interesting. The polynomial \( X^{17} - 1 \) factorises into three factors.

\[
x^{17} - 1 = (x + 1)(x^8 + x^7 + x^6 + x^4 + x^2 + x + 1)(x^8 + x^5 + x^4 + x^3 + 1).
\]

We have two possibilities for \( g(X) \) here. In one case \( g(X) = X - 1 \) and in the other is \( X - 1 \) times one of the degree 8 factors. By choosing \( h(X, \eta) \) appropriately in these cases we get a \([17, 1, 7]\) and a \([17, 9, 1]\) code respectively. We skip the details for lack of space. To the best of our knowledge, these are new codes.

### 4.1 Decoding \( 4^m + 1 \)-codes within the BCH limit

Let \( C \) be \( 4^m + 1 \)-code with BCH distance \( d = 2t + 1 \). Much like in the classical case, we show that there is an efficient quantum algorithm to correct any quantum error of weight at most \( t \). Here by efficient we mean polynomial in the code length \( n = 4^m + 1 \). There are two key algorithms that we use: (1) The quantum phase finding algorithm and (2) The Berlekamp decoding algorithm for classical BCH codes.

The decoding algorithm for classical BCH code can be seen as follows:
Theorem 4.9 (Berlekamp). Let $g(X)$ be a factor of $X^n - 1$ of BCH distance $d = 2t + 1$ over a finite field $\mathbb{F}_q$, $q$ and $n$ coprime. Let $e(X)$ be any polynomial of weight at most $t$ over $\mathbb{F}_p$. Given a polynomial $r(X) = e(X) \mod g(X)$, there is a polynomial time algorithm to recover $e(X)$.

Proof sketch. Since $r(X) = e(X) \mod g(X)$ we have $r(X) = e(X) + c(X)$ for some $c(X)$ in the ideal generated by $g(X)$. Note that $c(X)$ is a valid, but as of now unknown, code word in the code generated by $g(X)$. We can think of the computational task of recovering $e(X)$ as that task of recovering the sent message $c(X)$ for a received message $r(X)$ for which there is an efficient algorithm due to Berlekamp [13, p-98,6.7].

For the rest of the section fix a $4^m + 1$-code $C$ with BCH distance $d = 2t + 1$. Let $n = 4^m + 1$ denote its length and assume that $(g, h)$ is a generating pair for $C$. Assume that we have transmitted a quantum message $|\varphi\rangle \in C$ over the quantum channel and received the corrupted state $|\psi\rangle = U_a V_b |\varphi\rangle$, where the vectors $a$ and $b$ are unknown to us. If we can design an algorithm that recovers $a$ and $b$ without actually disturbing $|\psi\rangle$, then we have an error correction algorithm as we recover the sent message by applying the inverse map $V_b^T U_a^T$ on $|\psi\rangle$. We show that this is possible provided the joint weight $w(a, b) \leq t$.

Consider the polynomial $e(X, \eta) = a(X^{-1}) + \eta b(X^{-1})$ as a polynomial over $\mathbb{F}_4[X]/X^n - 1$. Clearly, the $F_4$-weight of $e(X, \eta)$ is also at most $t$ and polynomials $a(X), b(X)$ can be recovered once $e(X, \eta)$ is recovered. We prove that $e(X, \eta)$ can be recovered modulo $h(X, \eta)$.

Let $S$ be the underlying totally isotropic set associated with $C$. Since $(g, h)$ is the generating pair for $S$, the factor $g(X) h(X, \eta)$ of $X^n - 1$ generates $S$ as an ideal of $\mathbb{F}_4[X]/X^n - 1$. We have the following proposition

Proposition 4.10. For any $(c, d)$ in $S$, there is an efficient quantum algorithm to compute the polynomial $d(X)a(X^{-1} - c(X)b(X^{-1})$.

Proof. Recall that the code $C$ is the set of vectors stabilised by the corresponding Gottesman subgroup $S$. Let $U = \zeta U_c V_d$ be the element in $S$ corresponding to the pair $(c, d) \in S$. It can be easily show that $|\psi\rangle$ is an eigen vector of $U$ with eigen value $(-1)^{d^T a - c^T b}$ and using phase finding one can recover the inner product $(-1)^{d^T a - c^T b}$. Repeating the algorithm with $(N^k c, N^k d)$, all the inner products $d^T N^k a - c^T N^k b$ can be recovered. Since these are precisely the coefficients the polynomial $d(X) a(X^{-1} - c(X)b(X^{-1})$ modulo $X^n - 1$, this is sufficient to prove the claim.

Since $\tilde{g} = gh$ generate the set $S$ as an ideal, both $\tilde{g}$ and $\eta \tilde{g}$ belong to $S$. Using the algorithm in Proposition 4.10 with $\tilde{g}$ and $\eta \tilde{g}$, it is straight forward but tedious to show that the polynomial $e(X, \eta)$ can be computed modulo $h(X, \eta)$. We are now in the setting of Theorem 4.9 where $h(X, \eta)$ as a polynomial in $\mathbb{F}_4[X]/X^n - 1$ playing the role of the generator polynomial. Since $h(X, \eta)$ has BCH distance $2t + 1$, we can recover $e(X, \eta)$ and hence $(a, b)$ using the Berlekamp algorithm. Thus we have the following theorem

Theorem 4.11. Let $C$ be a $4^{n+1}$ code of length $n = 4^m + 1$ and BCH distance $d = 2t + 1$. There is quantum algorithm that takes time polynomial in $n$ to correct errors of weight at most $t$.

5 Cyclic codes that are not stabiliser

In this section we give examples for certain nonstabiliser codes that are cyclic. There has been some work in the construction of nonstabiliser quantum code. Rains et al [10] used computer search to construct the first example of a ($(5,6,2)$) quantum code which is not stabiliser. Shortly,
Roychowdhury and Vatan [11] gave few more examples of such codes. Arvind et al [2] gave a generic method to construct quantum codes for Gottesman subgroups of the error group, some of which turn out to be nonstabiliser. We summarise their result in the following proposition.

**Proposition 5.1** ([2]). Let $S$ be a Gottesman subgroup of the error group. Then

1. For any character $\chi$ of $S$, the set $S_\chi = \{\chi(s)s | s \in S\}$ is also a Gottesman subgroup of the error group with $P_\chi = \sum_{s \in S} \chi(s)s$ as the corresponding stabiliser code.

2. The codes $P_\chi$ and $P_\varphi$ are orthogonal unless $\chi = \varphi$.

3. An element $A$ in the algebra $\mathbb{C}[S]$ is a projection if and only if there is a subset $B$ of characters of $S$ such that $A = \sum_{\chi \in B} P_\chi$.

We call the codes thus generated from Gottesman subgroups, pseudo-stabiliser codes. In particular, Arvind et al [2] have show that the $((5, 6, 2))$ code of Rains et al [10] is a pseudo-stabiliser code.

Consider any Gottesman subgroup $S$ that is separately cyclic. Then the corresponding stabiliser code $C$ is cyclic. For any character $\chi$ of $S$, $S_\chi$ is also separately cyclic as the underlying totally isotropic subspace $S$ of $F_2^{2n}$ is same for $S$ and $S_\chi$. Hence the corresponding stabiliser code $C_\chi$ is cyclic and therefore $N^\dagger P_\chi N = P_\chi$. By Proposition 5.1, any pseudo-stabiliser code with support in $S$ is given by the sum of projections $P_B = \sum_{\chi \in B} P_\chi$. Thus we have $N^\dagger P_B N = P_B$. As a result we have the following proposition.

**Proposition 5.2.** A pseudo-stabiliser code whose underlying Gottesman subgroup is cyclic is also cyclic.

The above proposition gives us ways to construct cyclic pseudo-stabiliser code by first constructing a cyclic stabiliser code. In particular, the $((5, 6, 2))$ code is a pseudo-stabiliser code whose underlying Gottesman subgroup is separately cyclic. This gives a concrete example for a nonstabiliser cyclic quantum code.

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