Spin Networks, Wilson Loops and 3nj Wigner Identities

Manu Mathur 1, Atul Rathor 2

S. N. Bose National Centre for Basic Sciences
JD Block, Sector III, Salt Lake City, Kolkata 700106, India

Abstract

We exploit the spin network properties of the magnetic eigenstates of SU(2) Hamiltonian lattice gauge theory and use the Wilson loop operators to obtain a wide class of new identities amongst 3nj Wigner coefficients. We also show that the topological ground states of the SU(2) toric code Hamiltonian lead to Wigner 3nj identities with non-trivial phases. The method is very general and involves only the eigenvalue equations of any gauge invariant operator and their solutions. Therefore, it can be extended to any higher dimensional spin networks as well as larger SU(N) groups.

1 Introduction

The idea of spin networks was first introduced by Roger Penrose as a simple model of discrete quantum space time geometry [1]. They are now ubiquitous in quantum field theories of elementary particles and gravity providing the most economical as well as complete description of their physical Hilbert spaces. In Hamiltonian lattice gauge theories, the spin networks solve the Mandelstam constraints present in the loop basis [2, 3]. In loop quantum gravity, spin networks represent the quantum states of space geometry [1, 2, 4]. They also play important roles in conformal field theories and topological quantum field theories (see references in Lee Smolin in [4]). The spin networks are also useful to describe the degenerate topological ground states of SU(2) toric code model [5] and might be significant in topological quantum computing in the future. Infact, in the past there are proposals to use spin networks as storage devices for quantum information in quantum computing [6]. In this work, we exploit the group theoretical properties of the spin networks in SU(2) lattice gauge theory to obtain a large class of new identities involving various Wigner 3nj coefficients. These identities are obtained by applying various Wilson loop operators on the gauge invariant magnetic field eigenstates and then computing their action on the spin network basis with the help of generalized Wigner Eckart theorem (see section 3). Similar relations have been obtained in the context of quantum gravity in the past [7]. We also use SU(2) toric code Hamiltonian [5] on a two dimensional

---

1 manu@boson.bose.res.in, manumathur14@gmail.com
2 atulrathor@bose.res.in, atulrathor999@gmail.com
torus as it is exactly solvable and has 4 topological ground states. We show that the non-contractible Wilson or Polyakov loops encircling the torus in either direction lead to richer Wigner coefficient identities with topological phases of the ground states (see section 3.3). We will work in \( d = 2, 3 \) space dimensions and consider only simple 1, 2 plaquette contractible or a non-contractible (torus) Wilson loop operators on small lattices. This is to keep the discussion simple. More general cases in higher dimensions and with larger loop operators can be similarly handled. They will lead to higher Wigner 3nj coefficients identities. Further, the method can be generalized to higher SU(N) groups.

The plan of the paper is as follows. In section 2 we discuss the kinematical aspects of lattice gauge theory Hamiltonian and summarize the properties and construction of SU(2) spin networks. This section is for the sake of completeness and setting up the notations to be used in the following sections. More details about the Hamiltonian formulation of lattice gauge theory can be found in [8]. The SU(2) spin networks are discussed in detail in [2, 3, 4]. In section 3 we start with introducing the basic idea of using the magnetic field eigenvalue equation as generating function for the various identities for 3nj Wigner coefficients. In section 3.1 and 3.2 we consider SU(2) gauge theory on a single tetrahedron and a single cube respectively to illustrate the ideas in the simplest possible settings. Even on these simple and small lattices we get identities involving 6j & 9j (tetrahedron case), 12j & 12j (cube) Wigner coefficients respectively. In section 3.3 we consider the topological ground states of the SU(2) toric code Hamiltonian on a small 4 plaquette torus. We show that the identities derived from the contractible Wilson loop operators are independent of topological phases but non-contractible Wilson loop operators lead to identities decorated with the topological phases of the ground state considered (equation (48)). This is expected result as only non-local operators on the entire torus can detect topological charges. The final results are summarized at the end in the summary and discussion section.

### 2 SU(2) Lattice Gauge Theory and Spin Networks

In this section we briefly introduce the electric, magnetic field operators, their algebras and the constraints in the Hamiltonian formulation of lattice gauge theory in \((d + 1)\) dimensions. The construction of spin network states and their properties are briefly discussed in section 2.1. We consider the SU(2) link holonomies \( U_{\alpha\beta}(\vec{n}; \hat{i}) \) and their conjugate electric fields \( E^a_+ (\vec{n}; \hat{i}) \) \( (E^-_a (\vec{n} + \hat{i}; \hat{i}) \) on every link \((\vec{n}; \hat{i})\) (see Figure 1) with \( \alpha, \beta = 1, 2 \) and \( a = 1, 2, 3 \) [8]. The electric fields \( E^a_+ (\vec{n}; \hat{i}) \) \( (E^-_a (\vec{n} + \hat{i}; \hat{i}) \) rotate (anti-rotate) the link holonomies \( U(\vec{n}; \hat{i}) \) from the left (right) and therefore satisfy the following canonical commutation relations:

\[
[E^a_+ (\vec{n}; \hat{i}), U_{\alpha\beta}(\vec{n}; \hat{i})] = \left( \frac{\sigma^a}{2} U(\vec{n}; \hat{i}) \right)_{\alpha\beta}, \quad [E^-_a (\vec{n} + \hat{i}; \hat{i}), U_{\alpha\beta}(\vec{n}; \hat{i})] = - \left( U(\vec{n}; \hat{i}) \frac{\sigma^a}{2} \right)_{\alpha\beta},
\]
where $\sigma^a$ are the Pauli matrices. The above commutation relations and Jacobi identity imply

\[
[E_+^a(n; \hat{i}), E_+^b(n; \hat{i})] = i \epsilon^{abc} E_+^c(n; \hat{i}), \quad [E_-^a(n + \hat{i}; \hat{i}), E_-^b(n + \hat{i}; \hat{i})] = i \epsilon^{abc} E_-^c(n + \hat{i}; \hat{i}).
\] (2)

Here $\epsilon^{abc}$ are the completely antisymmetric structure constants. The left and the right electric fields or the angular momentum operators in (1) are related by a parallel transport as follows:

\[
E_-(n + \hat{i}; \hat{i}) = -U^\dagger(n; \hat{i}) E_+(n; \hat{i}) U(n; \hat{i}),
\] (3)

where we have defined $E_\pm \equiv \sum_a E_\pm^a \sigma^a$. It is easy to check that the left and right electric fields commute with each other and their magnitudes of are equal

\[
[E_+^a(n, \hat{i}), E_-^b(n + \hat{i}; \hat{i})] = 0, \quad \text{Tr}(E_+(n; \hat{i}))^2 = \text{Tr}(E_-(n + \hat{i}; \hat{i}))^2.
\] (4)

The identities (4) imply that the complete set of commuting operators on a link $l = (\vec{n}, \hat{i})$ in the dual electric fields representation are

\[
\left[ \tilde{E}^2(l), E_+^{a=3}(l), E_-^{a=3}(l) \right].
\]

In the above set $\tilde{E}^2(l) \equiv \tilde{E}_+^2(n, \hat{i}) = \tilde{E}_-^2(n + \hat{i}, \hat{i})$. The corresponding eigenvectors, denoted by $|j, m_+, m_-\rangle_l \equiv |j, m_+\rangle_l \otimes |j, m_-\rangle_l$

\[
\tilde{E}^2(l) |j, m_+, m_-\rangle_l = j(l) (j(l) + 1) |j, m_+, m_-\rangle_l, \quad E_+^{a=3}(l) |j, m_+, m_-\rangle_l = m_\pm(l) |j, m_+, m_-\rangle_l.
\] (5)
The SU(2) gauge transformations at a lattice site \( \vec{n} \) are:

\[
U(\vec{n}; \hat{i}) \rightarrow \Lambda(\vec{n}) U(\vec{n}; \hat{i}) \Lambda(\vec{n} + \hat{i}); \quad i = 1, 2.
\] (6)

In (6) \( \Lambda(\vec{n}) \) are the SU(N) gauge degrees of freedom at lattice site \( \vec{n} \). The canonical commutation relations (11) imply that the generators of the above gauge transformations at site \( \vec{n} \) are:

\[
G^a(\vec{n}) \equiv \sum_{i=1}^{d} \left( E^a_+(\vec{n}; \hat{i}) + E^a_-(\vec{n}; \hat{i}) \right).
\] (7)

Physically, \( G^a(\vec{n}) \) represent the sum of all electric field or equivalently the angular momentum operators meeting at the lattice site \( \vec{n} \). We also define the simplest gauge invariant plaquette loop operators

\[
U_p = U(\vec{n}; \hat{i}) U(\vec{n} + \hat{i}; \hat{j}) U^\dagger(\vec{n} + \hat{j}; \hat{i}) U^\dagger(\vec{n}; \hat{j}).
\] (8)

The Kogut Susskind \( H_{KS} \) is

\[
H_{KS} = A \sum_l A_l + B \sum_p B_p,
\] (9)

In (9), \( l \) and \( p \) denote the links and plaquettes, \( A \) and \( B \) are positive constants and

\[
A_l \equiv \sum_{a=1}^{3} E^a(l) E^a(l), \quad B_p \equiv 1 - \frac{1}{2} \text{Tr} \ U_p.
\] (10)

Note that for SU(2) gauge group, \( W_p \equiv \frac{1}{2} \text{Tr}(U_p) \) is Hermitian and its eigenvalues are \( \cos \omega_p \).

The magnetic ordered states with \( B_p = 0 \) implies \( \omega_p = 0, \forall p \).

### 2.1 Wilson Loop and Spin Network States

We consider the Gauss law constraints at every lattice site \( \vec{n} \)

\[
G^a(\vec{n}) |\psi\rangle = 0, \quad a = 1, 2, 3.
\] (11)

The gauge invariant states satisfying (11) are the physical states and belong to the physical Hilbert space \( \mathcal{H}^p \). On the other hand, the most general gauge invariant operators are the Wilson loop operators \( \mathcal{W}_C \) around oriented loops \( C \) which are defined using SU(2) holonomies

\[
\mathcal{W}_C = \frac{1}{2} \text{Tr} \left( \prod_{\vec{l} \in C} U(\vec{l}) \right).
\] (12)
Therefore, the most general gauge invariant (loop) states can be written as
\[
|\mathcal{C}_1, \mathcal{C}_2, \cdots \rangle \equiv \mathcal{W}_\mathcal{C}_1 \mathcal{W}_\mathcal{C}_2 \cdots |0\rangle.
\tag{13}
\]
Here $|0\rangle$ is the gauge invariant strong coupling vacuum state defined by $\vec{E}_a^0(l)|0\rangle = 0, \quad \forall \; l$. The loop states in (13) form a highly over-complete basis and satisfy the Mandelstam constraints $[2, 3]$. A complete and orthonormal basis set in $\mathcal{H}^p$ in SU(2) lattice gauge theory is the spin network basis. This basis is best characterized in the dual formulation in terms of electric fluxes $[12, 3]$ (see (5) or angular momenta). We note that if there are $L$ SU(2) electric flux lines meeting at a vertex $\vec{n}$ then the Gauss law condition (11) states
\[
|j_1, j_2, j_{12}, \cdots, j_T = 0, m_T = 0\rangle_{\vec{n}} = \sum_{all \; m} C_{j_1^T, m_1; j_2^T, m_2}^{j_{12}, m_{12}} C_{j_{12}, m_{12}; j_{13}, m_{13}}^{j_{13}, m_{13}} \cdots C_{j_T, m_T; j_{L-1}; m_{L-1}; j_L, m_L}^{j_{L-1}, m_{L-1}; j_L, m_L} |j_1 m_1 \rangle \otimes |j_2 m_2 \rangle \cdots \otimes |j_L m_L \rangle
\tag{14}
\]
In (14), $j_T \equiv j_{12} \cdots L = 0, \; m_T \equiv m_{12} \cdots L = 0$ and $C_{j, m; \bar{j}, \bar{m}}^{J,M}$ are the standard Clebsch-Gordan coefficients. The Clebsch-Gordan couplings (14) ensure that the spin network basis is gauge invariant ($j_T = 0$)
\[
\mathcal{G}^a(\vec{n})|\vec{J}_{\vec{n}}\rangle = \sum_{i=1}^d \left( E_{+}^a(\vec{n}; \hat{i}) + E_{-}^a(\vec{n}; \hat{i}) \right) |\vec{J}_{\vec{n}}\rangle = 0,
\tag{15}
\]
and they form an orthonormal as well as complete basis in $\mathcal{H}^p$,
\[
\langle \vec{K}_{\vec{n}} | \vec{J}_{\vec{n}} \rangle = \delta_{\vec{K}_{\vec{n}}, \vec{J}_{\vec{n}}}, \quad \sum_{\vec{J}_{\vec{n}}} |\vec{J}_{\vec{n}}\rangle \langle \vec{J}_{\vec{n}}| = \mathcal{I}_{\vec{n}}.
\tag{16}
\]
In (16), $\mathcal{I}_{\vec{n}}$ is an identity operator at lattice site $\vec{n}$. Therefore, on a finite dimensional lattice with $S$ lattice sites, if we label all the vertices by $\vec{n} \equiv v_1, v_2, \cdots, v_S$, then the most general state $|\psi\rangle$ satisfying (11) can be expanded in the spin network basis
\[
|\psi\rangle = \sum_{\left\{\vec{J}_{n_1}, \vec{J}_{n_2}, \cdots, \vec{J}_{n_S}\right\}} \Phi\left(\vec{J}_{n_1}, \vec{J}_{n_2}, \cdots, \vec{J}_{n_S}\right) \left\{\prod_{i=1}^S \otimes |\vec{J}_{n_i}\rangle\right\} \equiv \sum_{\left\{\vec{J}\right\}} \Phi(\vec{J}) \left|\left\{\vec{J}\right\}\right\rangle.
\tag{17}
\]
In (17), the summation over $\left\{\vec{J}_{n}\right\}$ are constrained as $j$ on each link is shared by its two vertices. As an example in Figure 1b, $j(\vec{n}; \hat{1}) \equiv j(\vec{n} + \hat{1}; \vec{3})$ and $j(\vec{n}; \hat{2}) \equiv j(\vec{n} + \hat{2}; \vec{4})$. 


3 Wilson Loops and Wigner Identities

In this section we study the gauge invariant eigenvalue equations of $\mathcal{W}_C$ in $d$ dimensions. These equations will be used later to derive various identities involving Wigner coefficients in $d = 2, 3$. The state $|\psi_\omega\rangle$ satisfies

$$\mathcal{W}_C |\psi_\omega\rangle = \cos \omega |\psi_\omega\rangle.$$  \hfill (18)

In the spin network basis we get

$$\mathcal{W}_C |\psi_\omega\rangle = \sum_{\{\vec K\}} \sum_{\{\vec J\}} \Phi_\omega(\vec J) \mathcal{M}_{[\{\vec J\} \{\vec K\}]} \{\vec K\} = \cos \omega \sum_{\{\vec K\}} \Phi_\omega(\vec K) \{\vec K\}.$$  \hfill (19)

In (19) the $\mathcal{M}_{[\{\vec J\} \{\vec K\}]}$ are the matrix elements of the magnetic flux operator $\mathcal{W}_C$ between the spin network states $|\{\vec J\}\rangle$ and $|\{\vec K\}\rangle$. Using the orthogonality properties of the spin network states we get

$$\sum_{\{\vec K\}} \mathcal{M}_{[\{\vec J\} \{\vec K\}]} \Phi_\omega(\vec K) = \cos \omega \Phi_\omega(\vec J).$$  \hfill (20)

As we will see, the matrix elements $\mathcal{M}_{[\{\vec J\} \{\vec K\}]}$ and the amplitudes $\Phi_\omega(\vec K)$ are the Wigner 3nj coefficients. Therefore, the above identities represent the relationships between different Wigner coefficients. Now we can iterate the above equation multiple times to obtain a family of identities among Wigner coefficients for a given loop $C$ in a $d$ dimensional spin networks. From equation (20) we get the most general identity

$$\sum_{\{\vec K_1, \{\vec K_2, \ldots, \{\vec K_q\}\}} \mathcal{M}_{[\{\vec J\} \{\vec K_1\}]} \mathcal{M}_{[\{\vec K_1\} \{\vec K_2\}]} \cdots \mathcal{M}_{[\{\vec K_{q-1}\} \{\vec K_q\}]} \Phi_\omega(\vec K_q) = (\cos \omega)^q \Phi_\omega(\vec J).$$  \hfill (21)

In this work, we only consider the simple magnetic vacuum cases ($\omega = 0$) for small Wilson loops $C$. In Appendix A, we have computed the amplitudes $\Phi(\vec J)$ for the magnetic vacuum states with $\omega = 0$ using pure gauge conditions on lattices of different shapes, sizes. The matrix elements $\mathcal{M}_{[\{\vec J\}, \{\vec K\}]}$ are computed in Appendix B. The magnetic flux eigenvalue equation (20) then leads to non-trivial identities amongst Wigner coefficients. The more general identities can be similarly derived. In the following sections, we construct simple models on finite lattices to implement the above ideas.
3.1 A Toy Model on a Tetrahedron

We know that the simplest 3nj Wigner coefficients are the 6\_j coefficients. The six values of \( j \) can represent SU(2) flux values on the six edges of a tetrahedron. Therefore, we consider a tetrahedron \( T \) (see Figure 2a) with vertices \( v = a, b, c, d \) and oriented triangular plaquettes or the smallest Wilson loops \( \mathcal{C} \equiv p = \Delta_{abc}, \Delta_{abd}, \Delta_{adc}, \Delta_{bcd} \). We now analyze the Kogut Susskind Hamiltonian on \( T \)

\[
H = A \sum_l A_l + B \sum_p B_p. \tag{22}
\]

Above A and B are +ve constants and

\[
A_l = \sum_{a=1}^6 E^a(l) E^a(l), \quad l = ab, ac, ad, bc, bd, cd; \tag{23}
\]

\[
B_p = \left(1 - \frac{1}{2} \text{Tr} \, \Delta_p\right), \quad \Delta_p = \Delta_{abc}, \Delta_{bad}, \Delta_{bdc}, \Delta_{dac}. \tag{24}
\]

The magnetic field terms are

\[
\Delta_{abc} = U_{ab} U_{bc} U_{ca}, \quad \Delta_{abd} = U_{ad} U_{db} U_{ba}, \quad \Delta_{acd} = U_{ac} U_{cd} U_{da}, \quad \Delta_{bcd} = U_{bd} U_{dc} U_{ca} \tag{25}
\]

All the link operators or the link holonomies along the 6 edges of \( T \) are chosen in the fundamental \( j = \frac{1}{2} \) representation. The SU(2) transformations at the at the 4 vertices are are generated by the 4 Gauss law generators

\[
G^a(a) \equiv E^a(a, 1) + E^a(a, 2) + E^a(a, 3), \quad G^a(b) \equiv E^a(b, 3) + E^a(b, 4) + E^a(b, 5), \quad G^a(c) \equiv E^a(c, 2) + E^a(c, 4) + E^a(c, 6), \quad G^a(d) \equiv E^a(d, 1) + E^a(d, 5) + E^a(d, 6). \tag{25}
\]

We will work with magnetically ordered gauge invariant states \(|\psi_0\rangle\) with 0 magnetic fields. They satisfy

\[
G^a(v) \, |\psi_{\omega=0}\rangle = 0, \quad \forall \, v, \quad B_p \, |\psi_{\omega=0}\rangle = 0, \quad \forall \, p. \tag{26}
\]

\(^3\)By oriented triangle \( \Delta_{abc} \) we mean that it is traversed in the direction \( a \to b \to c \to a \)
Figure 2: Gauge theory on a tetrahedron: (a) The tetrahedron with the j labels represents the Wigner 6j coefficients which are the magnetic ground state amplitudes \( \Phi(\vec{J}) \) in \((29)\) in the spin network basis, (b) Matrix elements \( \mathcal{M}_{\{\vec{J}\} \{\vec{K}\}}^{(bcd)} \) in \((30)\).

The state \( \psi_0 \) can be expanded in the spin network\(^4\) basis \((14)\)

\[
|\psi_0\rangle = \sum_{j_1, j_2, \ldots, j_6} \frac{\Phi(j_1, j_2, \ldots, j_6)}{\text{amplitude on } \mathcal{T}} |j_1 j_2 j_3\rangle_a \otimes |j_3 j_4 j_5\rangle_b \otimes |j_2 j_4 j_6\rangle_c \otimes |j_1 j_5 j_6\rangle_d \text{ spin network on } \mathcal{T}
\]

\[= \sum_{\{\vec{J}\}} \Phi(\vec{J}) |\{\vec{J}\}\rangle \quad (28)\]

The amplitudes \( \Phi(\vec{J}) \), which ensure the magnetic fields are zero, can be constructed by choosing pure gauge conditions on every link. The detail of this calculations are given in Appendix A. The final result is

\[
\Phi(\vec{J}) \equiv \Phi(j_1, j_2, \ldots, j_6) = \Pi(j_1, j_2, \ldots, j_6) \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}. \quad (29)
\]

In \((29)\), \( \Pi(a, b, \ldots) \equiv \sqrt{(2a+1)(2b+1)\cdots} \). The 6j coefficients or the amplitude in \((29)\) is shown in Figure 2a. Using the generalized Wigner Eckart theorem, the matrix elements

\[|j_1, j_2, j_3\rangle_a = (-1)^{j_1-j_2+j_3} \sum_{m_1, m_2, m_3} \binom{j_1 m_1}{j_2 m_2 j_3 m_3} |j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes |j_3, m_3\rangle. \quad (27)\]

\(^4\)In this simple case, the spin network states at the vertex \( a \) are

Similarly we can construct these states at the other 3 vertices.
Figure 3: Graphical representation of the Wilson loop eigenvalue equation (20) on a tetrahedron: $M(\{\vec{J}, \vec{K}\}) \Phi(\vec{K}) = \Phi(\vec{J})$ leading to Wigner coefficient identities (40): $9j \times 6j = 6j$ in (32).

$M(\{\vec{J}, \vec{K}\})$ can be computed (see Appendix B).

$$M^{(bed)}_{\{\vec{J}, \vec{K}\}} = \frac{1}{2} \langle \{\vec{J}\} | \text{Tr} \bigtriangleup_{bed} | \{\vec{K}\} \rangle$$

$$= M^{(bed)}(\vec{J}, \vec{K}) \left[ \begin{array}{ccc} k_4 & k_5 & k_6 \\ j_3 & j_1 & j_2 \\ j_4 & j_5 & j_6 \end{array} \right] \prod_{l=4,5,6} \left\{ j_l, k_l, \frac{1}{2} \right\}. \tag{30}$$

In (30),

$$M^{(bed)}(\vec{J}, \vec{K}) = \delta_{j_1, k_1} \delta_{j_2, k_2} \delta_{j_3, k_3} \frac{\Pi (j_4, j_5, j_6, k_4, k_5, k_6)}{\Pi^4 (j = \frac{1}{2})}.$$

and

$$\{a, b, c\} = \begin{cases} 1 & \text{if } (a, b, c) \text{ form a triangle}, \\ 0 & \text{otherwise}. \end{cases} \tag{31}$$

Thus the matrix elements of the triangular plaquette operator are $9j$ Wigner coefficients of the second kind [9, 10].
3.1.1 6j-9j and 6j-12j Wigner Identities

We now substitute the amplitude \(29\) and the matrix elements \(30\) in the equation \(19\) to get the first identity derived using the simplest Wilson loop operator \(W_{bcd} = \frac{1}{2} \text{Tr} \triangle_{bcd}\) in the fundamental \(j = \frac{1}{2}\) representation

\[
\sum_{\{k\}'} \Pi^2(k_1, k_5, k_6) \begin{bmatrix} k_4 & k_5 & k_6 \\ j_3 & j_1 & j_2 \\ j_4 & j_5 & j_6 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_3 \\ k_4 & k_5 & k_6 \end{bmatrix} = \Pi^4(\frac{1}{2}) \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix}
\]

(32)

The summation over \(\{k\}'\) means that the values of \((k_4, k_5, k_6)\) are restricted by the triangular constraints \(\{j_4, k_4, \frac{1}{2}\}, \{j_5, k_5, \frac{1}{2}\}, \{j_6, k_6, \frac{1}{2}\}\) given in (31). In other words, \(k_4 = j_4 \pm \frac{1}{2}, k_5 = j_5 \pm \frac{1}{2}, k_6 = j_6 \pm \frac{1}{2}\). The reason for the appearance of the factor \(\Pi^4(\frac{1}{2})\) in (32) is that the SU(2) flux raising and lowering Wilson loop operator \(W_{bcd}\) was chosen to be in the fundamental \(j = \frac{1}{2}\) spin representation. If we use the plaquette operator in an arbitrary spin \(s\) representation, \(W_{bcd}^{(s)} = \frac{1}{(2s+1)} \text{Tr} \left( \triangle_{bcd}^{(s)} \right)\), then we get

\[
\sum_{\{k\}'} \Pi^2(k_1, k_5, k_6) \begin{bmatrix} k_4 & k_5 & k_6 \\ j_3 & j_1 & j_2 \\ j_4 & j_5 & j_6 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_3 \\ k_4 & k_5 & k_6 \end{bmatrix} = \Pi^4(s) \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix}
\]

(33)

In (33) the sums are restricted by the triangular constraints: \(\{j_4, k_4, s\}, \{j_5, k_5, s\}, \{j_6, k_6, s\}\).

Theses identities can also be represented geometrically as a fusion of tetrahedron and tetrahedron frustum into a tetrahedron as shown in Figure-3. In Appendix C, we prove (33) explicitly with the help of known results given in the book by Varshalovich et. al. \[9\].

We can also obtain identities involving higher Wigner coefficients by applying larger Wilson loops. As an example, we consider 2 plaquette Wilson loop \(W_{\triangle} \equiv W_{bcda} = \frac{1}{2} \text{Tr} \left( \triangle_{bcd} \triangle_{bda} \right)\) on \(|\psi_0\rangle\). As this loop operator changes 4 edges, we get 12\(j\) Wigner coefficients of second kind \[9\] \[10\] as the matrix elements

\[
M_{\{J\}\{K\}}^{(bcda)} = \frac{1}{2} \langle \{J\} | \text{Tr} \left( \triangle_{bcda} \right) | \{K\} \rangle
\]

\[
= M^{(bcda)}(\vec{J}, \vec{K}) \begin{bmatrix} j_1 & j_3 & j_4 & j_6 \\ k_1 & k_3 & k_4 & k_6 \end{bmatrix} \prod_{l=1,3,4,6} \left\{ j_l, k_l, \frac{1}{2} \right\}
\]

(34)

In (34),

\[
M^{(bcda)}(\vec{J}, \vec{K}) = \delta_{j_2, k_2} \delta_{j_5, k_5} \frac{\prod (j_1, j_3, j_4, j_6, k_1, k_3, k_4, k_6)}{\Pi^4(\frac{1}{2})}
\]
and the matrix elements of the triangular plaquette operator is 12\(j\) coefficient of second kind. We thus get the identity involving Wigner 12\(j\) and Wigner 6\(j\) coefficients:

\[
\sum_{\{k\}'} \Pi^2(k_1, k_3, k_4, k_6) \begin{bmatrix}
  j_1 & j_3 & j_4 & j_6 \\
  j_2 & j_5 & j_2 & j_5 \\
  k_1 & k_3 & k_4 & k_6
\end{bmatrix} \begin{bmatrix}
  k_1 & j_2 & k_3 \\
  k_4 & j_5 & k_6
\end{bmatrix} = \Pi^4 \left(\frac{1}{2}\right) \begin{bmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
\end{bmatrix}
\]

(35)

Here the 4 summations in \(\{k\}'\) are restricted over the triangular constraints defined in (31): 
\(\{j_1, k_1, \frac{1}{2}\}, \{j_3, k_3, \frac{1}{2}\}, \{j_4, k_4, \frac{1}{2}\}, \{j_6, k_6, \frac{1}{2}\}\). We can further generalize (35) for a general plaquette in an arbitrary spin \(s\) representation \(W^{(s)}_{bcd} \equiv W^{(s)}_{boda} = \frac{1}{2} \text{Tr} \Delta^{(s)}_{bcda} \Delta^{(s)}_{bda}\) by simply replacing \(\frac{1}{2}\) by \((s)\) in (35) as well as in the above triangular constraints as was done in the previous \((6j-9j)\) case.

### 3.2 A model on a cube

We now consider a cube with twelve angular momentum \(j_1, j_2, \ldots, j_{12}\) assigned to its 12 edges as shown in Figure 4. The state \(|\psi_0\rangle\) can be expanded in the spin network basis

\[
|\psi_0\rangle = \sum_{j_1,j_2,\ldots,j_6} \Phi(j_1, j_2, \ldots, j_6) |j_1 j_2 j_5 a \otimes j_2 j_3 j_6 b \otimes j_3 j_4 j_7 c \otimes j_4 j_1 j_8 d \rangle
\]

(36)

\[
\equiv \sum_{\{J\}} \Phi(\vec{J}) |\{\vec{J}\}\rangle
\]

(37)

The amplitude \(\Phi(j_1, j_2, \ldots, j_6)\) are now given in terms of the Wigner 12\(j\) coefficients

\[
\Phi(\vec{J}) \equiv \Phi(j_1, j_2, \ldots, j_{12}) = \Pi(j_1, j_2, \ldots, j_{12}) \begin{bmatrix}
  j_1 & j_2 & j_3 & j_4 \\
  j_5 & j_6 & j_7 & j_8 \\
  j_9 & j_{10} & j_{11} & j_{12}
\end{bmatrix}
\]

(38)

### 3.2.1 12j-12j and 12j-18j Wigner Identities

We consider the simplest Wilson plaquette loop operator in \(j = \frac{1}{2}\) representation: \(W_{abcd} = \frac{1}{2} \text{Tr} \square_{abcd}\). The matrix elements \(M_{\{\vec{J}\} \{\vec{K}\}}^{(abcd)}\) are

\[
M_{\{\vec{J}\} \{\vec{K}\}}^{(abcd)} = \frac{1}{2} \langle \{\vec{J}\} | \text{Tr} \square_{abcd} | \{\vec{K}\} \rangle
\]
Figure 4: Gauge theory on a box. The box with the j labels represents the Wigner 12j coefficients which are the magnetic ground state amplitudes (38) in the spin network basis.

Figure 5

\[ M^{(abcd)}(\vec{J}, \vec{K}) = \left( \prod_{l=5, 6, \ldots, 12} \delta_{j_l, k_l} \right) \frac{\Pi (j_1, j_2, j_3, j_4, k_1, k_2, k_3, k_4)}{\Pi^4 \left( \frac{1}{2} \right)} \]

Now from (20) we get

\[ \sum_{\{k\}'} \Pi^2(k_1, k_2, k_3, k_4) \left[ \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \\ k_1 & k_2 & k_3 & k_4 \end{array} \right] \left[ \begin{array}{cccc} j_5 & j_6 & j_7 & j_8 \\ j_9 & j_{10} & j_{11} & j_{12} \end{array} \right] = \left( \prod_{l=5, 6, \ldots, 12} \delta_{j_l, k_l} \right) \frac{\Pi (j_1, j_2, j_3, j_4, k_1, k_2, k_3, k_4)}{\Pi^4 \left( \frac{1}{2} \right)} \]

As before \{k\}' means 4 triangular constraints over summations on the final momenta, i.e. \( k_1, k_2, k_3, k_4 \) and the corresponding initial momenta \( j_1, j_2, j_3, j_4 \) differ by \( \pm \frac{1}{2} \) respectively. We
can also consider a larger Wilson loop $W_{abfgcda}$. We now get

$$M^{(abfgcda)}_{\{\vec{J}\},\{\vec{K}\}} = \langle \{\vec{J}\} | W_{abfgcda} | \{\vec{K}\} \rangle = M^{(abfgcda)}(\vec{J}, \vec{K})$$

$$= \begin{pmatrix} j_1 & j_2 & j_6 & j_{11} & j_7 & j_4 \\
  j_5 & j_3 & j_{10} & j_{12} & j_3 & j_8 \\
  k_1 & k_2 & k_6 & k_{11} & k_7 & k_4 \end{pmatrix}^{18j \text{ coefficient of } 2^{\text{nd}} \text{ kind}}$$

Like in the previous cases

$$M^{(abfgcda)}(\vec{J}, \vec{K}) = \delta_{j_3,k_3} \delta_{j_5,k_5} \delta_{j_8,k_8} \delta_{j_9,k_9} \delta_{j_{10},k_{10}} \delta_{j_{12},k_{12}} \frac{\Pi(j_1,j_2,j_4,j_6,j_7,j_{11},k_1,k_2,k_4,k_6,k_7,k_{11})}{\Pi^4(\frac{1}{2})}.$$
Now from \cite{20} we get following identity for $12j$ coefficients.

$$\sum_{\{k\}'} \prod(k_1, k_2, k_4, k_6, k_7, k_{11}) \left[ \begin{array}{cccccc}
  j_1 & j_2 & j_6 & j_{11} & j_7 & j_4 \\
  j_5 & j_3 & j_{10} & j_{12} & j_3 & j_8 \\
  k_1 & k_2 & k_6 & k_{11} & k_4 & k_7 \\
  j_9 & j_{10} & k_{11} & j_{12} & j_9 & j_{10} \\
  k_1 & k_2 & k_6 & k_{11} & k_4 & k_7 \\
  j_9 & j_{10} & k_{11} & j_{12} & j_9 & j_{10} \\
\end{array} \right] = \prod^4 \left( \frac{1}{2} \right) \left[ \begin{array}{cccc}
  j_1 & j_2 & j_3 & j_4 \\
  j_5 & j_6 & j_7 & j_8 \\
  j_9 & j_{10} & j_{11} & j_{12} \\
\end{array} \right]$$

(41)

As in the cases before, the identity (41) can also be generalized to arbitrary spin $s$ representation by replacing $\frac{1}{2}$ by $s$ and modifying the triangular constraints appropriately.

### 3.3 SU(2) Toric code model & topological ground states

In the previous sections we considered the magnetic ground states to get various Wigner coefficient identities. We now extend this method further and use magnetic ground states with topological charges leading to more general class of identities parametrized by non-trivial ‘topological phase factors’. We consider exactly solvable SU(2) toric code model defined on a two dimensional torus with periodic boundary conditions in both directions\cite{5}. The model has 4 fold degenerate ground states with topological charges $(p, q)$; $p, q = 0, 1$. In this paper we work with a simple 4 plaquette torus $T_2$ to illustrate the ideas. We obtain Wigner coefficient identities with non trivial $(p, q)$ dependent ‘topological phase factors’ only for gauge invariant Wilson lines or Polyakov lines encircling the entire torus $T_2$ in any of the two directions. We further show that these $(p, q)$ dependent phase factors cancel out from the identities for any local gauge invariant Wilson loops. This is as expected because one needs to traverse the entire torus to detect the topological charge of the state \cite{11}.

We consider a small 4 plaquette lattice on torus as shown in Figure 7a. The 4 plaquettes are denoted by $U_p \equiv \square_{abcd}, \square_{badc}, \square_{cdba}, \square_{dcb}$. The SU(2) Kitaev toric code Hamiltonian is

$$H = A \sum_n A_n + B \sum_p B_p,$$

(42)

where $A$ and $B$ are positive constants, $n$ and $p$ denote the sites and plaquettes and

$$A_n \equiv \sum_{a=1}^3 \mathcal{G}^a(n) \mathcal{G}^a(n); \quad B_p \equiv \left( 1 - \frac{1}{2} \text{Tr} U_p \right).$$

(43)

The Gauss law generators $\mathcal{G}^a(n)$ and the plaquette terms $U_p$ have been defined in \cite{7} and \cite{8} respectively. To construct the spin network on a torus, it is convenient to label the 4 angular momenta around a vertex $v(=a, b, c, d)$ in a counter clockwise direction as $j_1^v, j_2^v, j_3^v, j_4^v$ as shown
Figure 7: (a) The 4 plaquette torus $\mathcal{T}_2$ with plaquettes $\square_{abcd}, \square_{badc}, \square_{cdab}, \square_{dcba}$. The 12 Angular momenta are shown on the edges, (b) The topological phase factor $(\eta_p, \eta_q)$: $\eta_p$ denoted by ■ on the vertical links (ad), (bc), $\eta_q$ denoted by ● on the horizontal links (ab), (dc). Note that these phase factors ±1 do not change the plaquette magnetic fields.

As each edge is shared by two vertices there is double counting and we identify:

\[
\begin{align*}
    j_1^a &\equiv j_3^b, & j_2^a &\equiv j_4^b, & j_1^b &\equiv j_3^d, & j_2^b &\equiv j_4^d, \\
    j_1^c &\equiv j_3^d, & j_2^c &\equiv j_4^d, & j_1^d &\equiv j_3^c, & j_2^d &\equiv j_4^c.
\end{align*}
\]

Thus there are 8 angular momenta on 8 edges or links on the 4 plaquette torus $\mathcal{T}$. We combine two set of angular momentum $j_1^v, j_2^v$ to get $j_1^{12}$ and $j_3^v, j_4^v$ to get $j_3^{34}$ and then $j_1^{12}, j_3^{34}$ are add to get the gauge invariant 0 angular momentum states. We set $j_1^{12} = j_3^{34} = j^v$, now the spin network states\footnote{Spin network states at site $v = 1, 2, \ldots, L^2$ is given by}

\[
|\vec{J}_v\rangle \equiv |j_1^v, j_2^v, j_3^v, j_4^v, j^v\rangle = \sum_{m_1, m_2, m_3, m_4} A_{j^v}^{e^{j^v}} C_{j_1^v,m_1,j_2^v,m_2}^{j^v} C_{j_3^v,m_3,j_4^v,m_4}^{j^v} |j_1^v, m_1\rangle \otimes |j_2^v, m_2\rangle \otimes |j_3^v, m_3\rangle \otimes |j_4^v, m_4\rangle \equiv |j_1^v, j_2^v, j^v\rangle.
\]

Here $A_j^p = \frac{(-1)^{(j-m)}}{\sqrt{2j+1}}$. 

Thus the gauge invariant states on $\mathcal{T}_2$ are characterized by 12 angular momenta (8 on the 8 edges and 4 on the 4 vertices). We expand the ground state $|\psi_0\rangle$, satisfying

|j_1^v, j_2^v, j_3^v, j_4^v, j^v\rangle \equiv |j_1^v, j_2^v, j^v\rangle$. Therefore, the spin network states at a vertex $v$ can be represented by $|\vec{J}_v\rangle \equiv |j_1^v, j_2^v, j^v\rangle$. Thus the gauge invariant states on $\mathcal{T}_2$ are characterized by 12 angular momenta (8 on the 8 edges and 4 on the 4 vertices). We expand the ground state $|\psi_0\rangle$, satisfying
(a) The 12j amplitude $\Phi_{p,q}(\vec{J})$

(b) The 18j matrix elements $M_{abcd}^{abcd}(\{\vec{J}\} \{\vec{K}\})$

Figure 8: The amplitude $\Phi_{p,q}(\vec{J})$ and the matrix elements $M_{abcd}^{abcd}(\{\vec{J}\} \{\vec{K}\})$ on the torus $T_2$ (a)

The diagram implies: $\Phi_{p,q}(\vec{J}) = (-1)^{2p(j_a^2 + j_b^2)}(-1)^{2q(j_c^4 + j_d^4)}\Phi_{(0,0)}(\vec{J})$, (b) The matrix elements $M_{abcd}^{abcd}(\{\vec{J}\} \{\vec{K}\})$ do not depend on the topological phases.

$G^a \langle \psi_0 | 0 \rangle = 0$, in the spin network basis

$$|\psi_0\rangle = \sum_{\{J_a, J_b, J_c, J_d\}} \Phi(\vec{J}_a, \vec{J}_b, \vec{J}_c, \vec{J}_d) |\vec{J}_a\rangle \otimes |\vec{J}_b\rangle \otimes |\vec{J}_c\rangle \otimes |\vec{J}_d\rangle .$$ (45)

In Appendix A, we compute the amplitudes $\Phi$ by choosing the standard pure gauge conditions on all the 8 edges to get $B_p = 0$ and then integrating over the SU(2) gauge parameters on the 4 vertices to make $|\psi_0\rangle$ gauge invariant. As expected from the diagram in Figure 8(a), the amplitudes are 12j Wigner coefficients of 2nd kind:

$$\Phi\left(\vec{J}_a, \vec{J}_b, \vec{J}_c, \vec{J}_d\right) = \Pi(\{\vec{J}\}) \begin{bmatrix} j_a^2 = j_d^4 & j_b = j^b & j_c = j_d^4 & j_d^2 = j_a^4 \\ j_1^a = j_3^b & j_1^b = j_3^c & j_1^c = j_3^d & j_1^d = j_3^a \\ j_2^d = j_4^a & j_2^a = j_4^b & j_2^b = j_4^c & j_2^c = j_4^d \end{bmatrix}$$ (46)

We can now construct the topological ground states (see Appendix A) by modifying the pure gauge configurations $|\psi_0\rangle$ with the following $Z_2$ factors $(-1)^p$, $(-1)^q$, $p = 0, 1$ on the links $(ad)$ and $(bc)$, (b) $(-1)^q$, $q = 0, 1$ on the links $(ab)$ and $(cd)$. The magnetic field $B_p$ on each plaquette remains unchanged but the amplitude $\Phi$ gets replaced: $\Phi\left(\vec{J}_a, \vec{J}_b, \vec{J}_c, \vec{J}_d\right) \rightarrow$ 

---

6Equivalently, we can modify the pure gauge configurations on the links $(da), (cb)$ by $(-1)^p$ and $(ba), (cd)$ by $(-1)^q$. They all leave plaquette magnetic fields unchanged.
\( \Phi_{(p, q)} (\vec{J}_a, \vec{J}_b, \vec{J}_c, \vec{J}_d) \) where,

\[
\Phi_{(p, q)} (\vec{J}_a, \vec{J}_b, \vec{J}_c, \vec{J}_d) = (-1)^{2p(j_a^2 + j_b^2)} (-1)^{2q(j_c^2 + j_d^2)} \Phi (\vec{J}_a, \vec{J}_b, \vec{J}_c, \vec{J}_d), \quad p, q = 0, 1. \tag{47}
\]

Now we can generalize identities (20) for topological charge and write down the following identities:

\[
\sum_{\{\vec{k}\}'} \mathcal{M}^{(c)}_{\{\vec{J}\} \{\vec{K}\}} \Phi_{(p, q)} (\vec{K}) = \Phi_{(p, q)} (\vec{J}) \tag{48}
\]

For \( p = q = 0 \) these identities coincide with the standard identities (20) for \( \omega = 0 \). As before, \( \{\vec{k}\}' \) means that the summations over \( \{\vec{k}\} \) are restricted by the representation of the loop operator involved in computing \( \mathcal{M} \). We now consider the Wilson loop to be the plaquette operator \( \square_{abcd} \) in \( s = \frac{1}{2} \) representation:

\[
\mathcal{M}^{abcd}_{\{\vec{J}\} \{\vec{K}\}} = \frac{1}{2} \langle \vec{K}_d | \otimes \langle \vec{K}_c | \otimes \langle \vec{K}_b | \otimes \langle \vec{K}_a | \text{Tr} \square_{abcd} | \vec{J}_a \rangle \otimes | \vec{J}_b \rangle \otimes | \vec{J}_c \rangle \otimes | \vec{J}_d \rangle \tag{49}
\]

are 18\( j \) Wigner coefficients of second kind and can be represented by a ribbon diagram shown in Figure-8b.

\[
\mathcal{M}^{abcd}_{\{\vec{J}\} \{\vec{K}\}} = \mathcal{M}^{abcd} (\vec{J}, \vec{K}) \left[ \begin{array}{cccccc} j_1^a & j_2^a & j_1^d & j_2^d & j_1^b & j_2^b \\ j_1^c & j_2^c & j_1^d & j_2^d & j_1^b & j_2^b \\ k_1^a & k_2^a & k_1^d & k_2^d & k_1^b & k_2^b \\ k_1^c & k_2^c & k_1^d & k_2^d & k_1^b & k_2^b \end{array} \right] \{\vec{j}, \vec{k}, \frac{1}{2}\}. \tag{50}
\]

\[
\mathcal{M}^{abcd} (\vec{J}, \vec{K}) = \delta_{j^a, k_a} \delta_{j^c, k_c} \delta_{j_1^d, k_1^d} \delta_{j_2^d, k_2^d} \delta_{j^b, k_b} \delta_{j, k, \frac{1}{2}} \Pi \left( j_1^a, j_2^a, j_1^d, j_2^d, j_1^b, j_2^b, k_1^a, k_2^a, k_1^d, k_2^d, k_1^b, k_2^b \right). \tag{51}
\]

The constraints in \( \{\vec{j}, \vec{k}, \frac{1}{2}\} \) imply that the initial and the final angular momenta in (50) differ by \( \pm \frac{1}{2} \). Similarly, we can write these matrix elements for other plaquette operators. The action of \( \text{Tr} \square_{abcd} \) in an arbitrary spin \( s \) representation can be obtained by the replacements: \( \{\vec{j}, \vec{k}, \frac{1}{2}\} \to \{\vec{j}, \vec{k}, s\} \) and \( \Pi(\frac{1}{2}) \to \Pi(s) \).
3.3.1 Wigner Identities for 12j-18j and 18j-18j Coefficients

Using above matrix elements obtained above we get;

\[
\sum \limits_{\{k\}'} \prod^2(\vec{k}) \begin{bmatrix} j_1^a & j_2^a & j_1^d & j_2^d & j_1^b & j_2^b \ \\
 k_1^a & k_2^a & k_1^d & k_2^d & k_1^b & k_2^b \end{bmatrix} \begin{bmatrix} k_2^a & k_1^a & j_1^c & j_2^c & j_1^d & j_2^d \\
 j_2^a & j_1^a & j_1^b & j_2^b & j_1^c & j_2^c \end{bmatrix} = \prod^4(\frac{1}{2}) \begin{bmatrix} j_1^a & j_2^a & j_1^c & j_2^c \\
 j_1^b & j_2^b & j_1^d & j_2^d \end{bmatrix}
\]

(51)

Note that the topological phase factors cancel out on the two sides of (51). This can be seen as follows. First note that the matrix elements are computed in the spin network basis and therefore do not depend on the topological charges \((p, q)\). The topological phase appear in only through the amplitudes on both sides. The phase factor on the left hand side of (51) is \((-1)^{2p(k_1^a+k_2^a)}(-1)^{2q(k_1^d+k_2^d)}\). We first focus on the \(p\) dependent phase. The 4 triangular constraints present in the matrix element \(M_{[\vec{J}, \vec{K}]}\) in (50) are \(\{k_1^a, j_1^a, j_2^a\}, \{k_2^a, j_1^a, j_2^a\}\) and \(\{k_1^d, j_1^d, j_2^d\}, \{k_2^d, j_1^d, j_2^d\}\). This implies

\[(-1)^{k_1^a+k_2^a} = (-1)^{j_1^a+j_2^a}\]

and \[(-1)^{k_1^d+k_2^d} = (-1)^{j_1^d+j_2^d}\]. Infact, this cancellation remains valid even if we had chosen the plaquette operator in an arbitrary spin \(s\) representation. We now consider Wigner coefficients identities which get modified by the topological phases.

3.3.2 Non-contractible Wilson loops and Topological sectors

In order to obtain non-trivial identities from the topological ground states, we need to consider non-contractible operators and their eigenvalue equations. These operators, known as Wilson lines or Polyakov lines, are defined as path ordered product of the flux operator along a non-contractible curve \([5, 11]\). On the small torus \(T^2\) with 4 plaquettes, we have two horizontal Wilson lines \(W_{aba}, W_{dcd}\) and two vertical Wilson lines \(W_{ada}, W_{bcb}\):

\[
W_{ada} = U_{ad} U_{da}, \quad \quad W_{bcb} = U_{bc} U_{ca} \\
W_{aba} = U_{ab} U_{ba}, \quad \quad W_{dcd} = U_{dc} U_{cd}
\]

(52)

(53)

Topologically non-trivial ground state are given by (see Appendix A)

\[
|\psi_0\rangle_{(p,q)} = \sum \limits_{\{j\}} \Phi_{(p,q)}(\vec{j}_a, \vec{j}_b, \vec{j}_c, \vec{j}_d) \prod \limits_{v=a,b,c,d} \langle j_v^v^{v'} | j_1^v, j_2^v, j_3^v, j_4^v). \]

(54)
Here $\Phi_{(p,q)}$ are given in (16) and (17). The matrix elements of Wilson line $W_{ada}$ in $s = \frac{1}{2}$ representation are

$$\mathcal{M}_{(J, K)}^{(ada)} = \frac{1}{2} \langle J | W_{ada} | K \rangle$$

$$= M^{(ada)}(J, K) \begin{bmatrix} j^a_1 & j^a_2 & j^d_2 & j^d_2 \\ j^1_1 & j^1_1 & j^d_2 & j^d_2 \\ k^a_2 & k^a_2 & k^d_2 & k^d_2 \\ k^d_2 & k^d_2 & k^d_2 & k^d_2 \end{bmatrix} \{ j_l, k_l, \frac{1}{2} \}$$

where

$$M^{(ada)}(J, K) = \delta_{j^a_1, k^a_1} \delta_{j^a_2, k^a_2} \delta_{j^d_2, k^d_2} \delta_{j^d_2, k^d_2} \delta_{j^a_1, k^a_1} \delta_{j^a_2, k^a_2} \delta_{j^d_2, k^d_2} \delta_{j^d_2, k^d_2} \prod \left( j^a_1, j^a_2, j^d_2, j^d_2, k^a_2, k^d_2, k^d_2, k^d_2 \right) \Pi^4 \left( \frac{1}{2} \right).$$

Now from (48) we get following identities for $12j$ coefficients

$$\sum_{(k)} \left( -1 \right)^{2p k^a_2} \Pi^2 (k^a_2, k^a_2, k^d_2, k^d_2) \begin{bmatrix} j^a_1 & j^a_2 & j^d_2 & j^d_2 \\ j^a_1 & j^a_2 & j^d_2 & j^d_2 \\ k^a_2 & k^a_2 & k^d_2 & k^d_2 \\ k^a_2 & k^a_2 & k^d_2 & k^d_2 \end{bmatrix} \begin{bmatrix} k^a_2 & j^1_1 & j^d_2 & j^d_2 \\ k^a_2 & j^1_1 & j^d_2 & j^d_2 \\ k^d_2 & j^b_2 & j^c_2 & j^d_2 \\ k^d_2 & j^b_2 & j^c_2 & j^d_2 \end{bmatrix}$$

$$= \left( -1 \right)^{2p j^a_2} \Pi^4 \left( \frac{1}{2} \right) \begin{bmatrix} j^2_1 & j^a_2 & j^b_2 & j^c_2 \\ j^a_2 & j^b_2 & j^c_2 & j^d_2 \\ j^b_2 & j^b_2 & j^c_2 & j^d_2 \\ j^d_2 & j^d_2 & j^d_2 & j^d_2 \end{bmatrix}$$

Note that the phase factors present on the links $(ab), (dc)$ and $(bc)$ cancel out as the corresponding $j$s do not change. The only contribution comes from the topological phase present on the link $(ad)$ carrying $j^a_2$ flux. Further, the trivial $p = 0$ identities in (55) are exactly same as the ones obtained from the single cube lattice in section 3.2. It is interesting that the same identities are obtained by analysing gauge theory on two very different underlying lattices with different structure of spin networks, the first on a cube and the second on a torus.

4 Summary & Discussion

We considered SU(2) lattice gauge theory in 2 and 3 space dimensions on finite lattices and the associated physical spin network Hilbert spaces. We obtained various 3nj Wigner coefficient identities by analysing Wilson loop and Wilson line (toric code) operators and their eigenvalue
equations in the SU(2) spin network Hilbert space. All identities are of the form:
\[
\sum_{\{k\}'} M_{\{\vec{k}\} \{\vec{k}'\}}^{\{\vec{j}\} \{\vec{j}'\}} \Phi_\omega(\vec{j}') = \cos \omega \Phi_\omega(\vec{j}),
\]
- In this work we only considered \( \omega = 0 \) cases on finite small lattices. The value of \( n \) is the length of the Wilson loops or Wilson lines, the value of \( m \) depends on the dimensions and size of the lattice.
- The number of summations involved is the number of links in the Wilson loop or line (toric code). The range in the summation can be increased or decreased by considering Wilson loops or lines in the higher or lower spin representations respectively. Generally, identities involving more than 3 summations are not found in the literature \([9, 10]\). In this work we have considered cases up to 6 summations. We can also iterate the above eigenvalue equations multiple times (see equation (21)) to obtain more general results. Similarly, \( \omega \neq 0 \) and higher SU(N) cases will be interesting to analyze.
- The SU(2) toric code Hamiltonian \([42]\) is exactly solvable as all the terms present in it mutually commute. Therefore, its eigenvalue equations can also be used to get general identities with additional topological phase factors.

5 Appendix

A Spin Network Amplitude on Tetrahedron

In this section, we will calculate the amplitudes for the magnetic ground state expanded in the spin network Hilbert space. The final results are used in all the cases discussed in this work. We will work out the details for the tetrahedron \([29]\) and the torus where the amplitude gets modified by topological phases \([47]\). These amplitudes can be fixed using properties of ground states namely \( \mathcal{B}_p |\psi_0\rangle = 0 \) and \( \mathcal{G}_n^a |\psi_0\rangle = 0 \). We begin with states that are magnetically ordered, \( \mathcal{B}_p |\psi_0\rangle = 0, \forall p \) and then ensure \( \mathcal{G}_n^a |\psi_0\rangle = 0, \forall n \) by demanding invariance under SU(2) gauge transformations \([6]\). First, we observe that eigenstates of magnetic fields are necessarily eigenstates of individual links or flux operators \( U_{\alpha\beta} \). As eigen-values of the flux operators are SU(2) matrices, we define SU(2) group manifold \( S^3 \) on every link \( l = 1, 2, \ldots, 6 \):

\[
Z(l) = \begin{bmatrix}
  z_1(l) & z_2(l) \\
  -z_2^*(l) & z_1^*(l)
\end{bmatrix} \quad |z_1(l)|^2 + |z_2(l)|^2 = 1; \quad (z_1, z_2) \in S^3.
\]

The these holonomy operators commute with each other \( [U_{\alpha\beta}(l), U_{\gamma\delta}(l')] = 0, \forall l, l' \). Therefore, we can diagonalize all of them simultaneously. We define the magnetic eigenstates to be
eigenstates of each link operator:

\[ U_{\alpha\beta}(l) \, |Z(l)\rangle = Z_{\alpha\beta}(l) \, |Z(l)\rangle. \] (57)

The eigenvectors can be expanded in terms of angular momentum states with Wigner D functions as Fourier coefficients.

\[ |Z(l)\rangle \equiv |z_1(l), z_2(l)\rangle = \frac{1}{4\pi} \sum_{j=0}^{\infty} \sqrt{(2j+1)} \sum_{m_{\pm}} D_{m_{\pm} m_{\pm}}^j (Z(l)) |j \, m_+\rangle_l \otimes |j \, m_-\rangle_l. \] (58)

To obtain completely ordered states, \( B_p = 0 \) or equivalently \( W(p) = 1 \), we choose pure gauge conditions on every link and write

\[ Z(l) \equiv Z(n, \hat{i}) = \xi(s) \xi(s')^\dagger. \] (59)

Here \( \xi_s \) are the SU(2) matrices defined on the site \( s \). The magnetic eigenvalue equations now decouple into left and right parts:

\[ |Z(l) = Z(l)\rangle = \frac{1}{4\pi} \sqrt{(2j+1)} \sum_{j=0}^{\infty} \sum_{m_{\pm}} |\xi(n)\rangle^j_{m_+} \otimes |\xi(s')\rangle^j_{m_-}. \] (60)

The states at the left and the right vertices of the link \( l \) in (60) are called vertex states and defined as [5]:

\[ |\xi(s)\rangle^j_{m_+} \equiv \sum_{m_{\pm}} D_{m_{\pm} \hat{m}}^j (\xi(s)) |j, m_+\rangle, \quad |\xi(s')\rangle^j_{m_-} \equiv \sum_{m_{\pm}} D_{\hat{m} m_{\pm}}^j (\xi(s')) |j, m_-\rangle. \] (61)

Using pure gauge conditions the completely ordered states on tetrahedron can be written as;

\[ |\psi\rangle = \sum_{all \, j} \sum_{all \, m, n} \prod_{s \in T} \left( |\xi(s)\rangle^j_{m_1} \otimes |\xi(s)\rangle^j_{m_2} \otimes |\xi(s')\rangle^j_{m_3} \right) \] (62)

Now we can integrate over \( \xi(s), s = a, b, c, d \) to get the ground states which satisfies conditions [24].

\[ |\psi_0\rangle = \sum_{all \, j} \sum_{all \, m, s \in T} \left\{ \int_{S^3} d^2 \mu(\xi(s)) \, |\xi(s)\rangle^j_{m_1} \otimes |\xi(s)\rangle^j_{m_2} \otimes |\xi(s')\rangle^j_{m_3} \right\}, \]

\[ = \sum_{\{\hat{J}\}} \Phi[\hat{J}] \prod_s \otimes |j_1, j_2, j_3\rangle_s \text{ loop state at sites} \]
\[
= \sum_{\text{all } j} \Pi(j_1, j_2, j_3, j_4, j_5, j_6) \left\{ \begin{array}{ccc}
j_1 & j_2 & j_3 \\
j_4 & j_5 & j_6
\end{array} \right\} |j_1j_2j_3\rangle \otimes |j_3j_4j_5\rangle \otimes |j_2j_4j_6\rangle \otimes |j_1j_5j_6\rangle \tag{63}
\]

In the above calculation we have used

1. Properties of D-functions which implies
\[
\int_{S^3} d\mu(\xi(s)) \langle \xi(s) | \langle j_1, j_2, j_3 | \rangle \otimes \langle j_4, j_5, j_6 | \rangle \rangle = \sqrt{\Pi(j_1, j_2, j_3)} \left\{ \begin{array}{ccc}j_1 & j_2 & j_3 \\
m_1 & m_2 & m_3 \end{array} \right\} |j_1, j_2, j_3\rangle \tag{64}
\]

2. Loop states \(|j_1, j_2, j_3\rangle\) which are given by
\[
|j_1, j_2, j_3\rangle = (-1)^{j_1-j_2+j_3} \sum_{m_1, m_2, m_3} \left( \begin{array}{ccc}j_1 & j_2 & j_3 \\
m_1 & m_2 & m_3 \end{array} \right) |j_1, m_1\rangle |j_2, m_2\rangle |j_3, m_3\rangle \tag{65}
\]

3. Expression for 6j coefficients
\[
\{ j_1 \ j_2 \ j_3 \\
|j_4 \ j_5 \ j_6 \}\ = \sum_{\text{all } p} p \left( \begin{array}{ccc}j_1 & j_2 & j_3 \\
p_1 & p_2 & p_3 \end{array} \right) \left( \begin{array}{ccc}j_4 & j_5 & j_6 \\
p_1 & p_5 & -p_6 \end{array} \right) \left( \begin{array}{ccc}-p_4 & p_2 & p_6 \\
p_4 & -p_5 & p_3 \end{array} \right)
\]
Thus the amplitudes for the gauge invariant magnetic field ground state in the spin network basis on a tetrahedron are
\[
\Phi(\vec{J}) = \Pi(j_1, j_2, j_3, j_4, j_5, j_6) \left\{ \begin{array}{ccc}j_1 & j_2 & j_3 \\
j_4 & j_5 & j_6 \end{array} \right\} \tag{66}
\]

Thus using the pure gauge conditions and then integrating over all gauge degrees of freedom we can construct the magnetic ground states with \(B_p = 0\). For lattice with non trivial geometry as torus we have more general solutions of \(B_p = 0\) which generate degenerate topological ground states. On a 2 dimensional lattice on torus we can draw two distinct non-contractible loops which can not be deformed into each other. We modify pure gauge conditions \((59)\) by \(Z_2\) factors without changing the magnetic fields as follows \([5]\):

\[
Z_{ab} = \xi_a \eta_q \xi_b^\dagger, \quad Z_{dc} = \xi_d \eta_q \xi_c^\dagger, \quad Z_{bc} = \xi_b \eta_p \xi_c^\dagger, \quad Z_{ad} = \xi_a \eta_p \xi_d^\dagger, \\
Z_{ba} = \xi_b \xi_a^\dagger, \quad Z_{cd} = \xi_c \xi_d^\dagger, \quad Z_{cb} = \xi_c \xi_b^\dagger, \quad Z_{da} = \xi_d \xi_a^\dagger.
\tag{67}
\]

Where \(\eta_p = e^{i2\pi p} = (-1)^{2p}, \ \eta_q = e^{i2\pi q} = (-1)^{2q}\) with \(p, q = 0\). With this solution of ordered states we can integrates all the gauge degrees of freedom to get amplitudes for topological
ground states:

\[ \Phi(p, q) \left( \vec{J}_a, \vec{J}_b, \vec{J}_c, \vec{J}_d \right) = (-1)^{2p(j_a^2 + j_b^2)} (-1)^{2q(j_i^2 + j_i^2)} \]

\[ \times \Pi(\{\vec{J}\}) \]

\[ \begin{aligned}
& j_2^a = j_4^d, 
& j_3^a = j_3^b, 
& j_2^c = j_4^b, 
& j_1^i = j_3^d, 
& j_2^d = j_4^a, 
& j_1^a = j_3^c, 
& j_2^b = j_4^c, 
& j_1^d = j_3^c 
\end{aligned} \]

\[ (68) \]

**B Matrix Elements**

In this section we will calculate matrix elements of various Wilson loop operators in spin network basis. The calculations are done for a triangular loop \( W_C = \triangle_{bcd} \) in the spin \( s = \frac{1}{2} \) representation. We then generalise the results to other cases discussed in this work. The various triangular constraints and the matrix elements are displayed in the dual tetrahedron diagram in the angular momentum space in Figure 9. All angular momenta meeting at a vertex in real space in Figure 2a now lie on a triangle in the dual angular momentum space in Figure 9. The reverse is also true. Therefore, the action of the \( \text{Tr} \ \triangle_{bcd} \) is to move the single dual site \( \tilde{a} \rightarrow \tilde{a}' \) this is shown in Figure 9 by a dotted line with an arrow. As Wilson loops create or destroy half unit of flux on the links so they act like a ladder or creation and annihilation operators for the SU(2) fluxes in the spin network states. The matrix element \( \langle \{\vec{J}\} | \triangle_{bcd} | \{\vec{K}\} \rangle \) will be non-zero only if \( \{ k = j \pm \frac{1}{2} \} \) along the Wilson loop, i.e \( l \in C \). The flux values on all other links, \( l \notin C \), remain unaltered i.e., \( k = j \). Moreover there are new triangular constraints between \( \vec{J} \) and \( \vec{K} \) and \( s = \frac{1}{2} \) which are clearly shown in Figure 9 which represents a 9j Wigner coefficients.

\[ \mathcal{M}^{(bcd)}_{\{\vec{J}\} \{\vec{K}\}} = \frac{1}{2} \langle \{\vec{J}\} | \triangle_{bcd} | \{\vec{K}\} \rangle \]

\[ = \frac{1}{2} \delta_{j_1, k_4} \delta_{j_2, k_5} \delta_{j_3, k_6} \Pi(j_4, k_4, j_5, k_5, j_6, k_6) \left\{ j_4 \ k_4 \ \frac{1}{2} \right\} \left\{ j_5 \ k_5 \ \frac{1}{2} \right\} \left\{ j_6 \ k_6 \ \frac{1}{2} \right\} \left\{ j_1 \ k_1 \ \frac{1}{2} \right\} \left(68\right) \]

\[ = \frac{M^{bcd}}{4} \sum_x (-1)^{R+2x}(2x + 1) \left\{ j_4 \ k_4 \ x \right\} \left\{ j_5 \ k_5 \ x \right\} \left\{ j_6 \ k_6 \ x \right\} \prod_{l=4,5,6} \left\{ j_l \ k_l \ \frac{1}{2} \right\} \]

\[ = \frac{M^{bcd}}{\Pi^4(\frac{1}{2})} \left[ k_4 \ j_3 \ k_6 \ j_1 \ j_6 \ j_2 \right] \prod_{l=4,5,6} \left\{ j_l \ k_l \ \frac{1}{2} \right\} \left(69\right) \]

Here

\[ M^{bcd} = \delta_{j_1, k_4} \delta_{j_2, k_5} \delta_{j_3, k_6} \Pi(j_4, k_4, j_5, k_5, j_6, k_6). \]
Figure 9: The simple action of the operator $\text{Tr} \, \triangle_{bcd}$ on the dual lattice. It moves the single dual site: $\tilde{a} \rightarrow \tilde{a}'$. The matrix elements $M^{(bcd)}$ are 9j coefficients in (69).

It is easy to verify the validity of (69). The 9j matrix structure is uniquely fixed by the triangular constraints of the spin networks. The numerical factor $M^{bcd}$ can be verified by replacing $\triangle_{bcd}$ by an identity operators and getting $M^{bcd} = \langle \{ \vec{J} \} \mid \{ \vec{K} \} \rangle = \delta_{\vec{J}, \vec{K}}$. The matrix elements (69) are geometrical in nature and therefore can be easily generalize to other cases discussed in this work. The associated numerical factors can also be checked by replacing the Wilson loop operator by an identity operator as done in the present case.

C Proof of the 6j-9j identity

In this section, using the known results in the literature, we prove the identities derived in section 3.1. We use the following two identities from Varshalovich \[9\]:

1. We use the standard expansion of 9j in terms of 6j given in \[9\] on page 361, equation (2) for $n=3$.

\[
\begin{bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{bmatrix} = \sum_x \Pi^2(x)(-1)^{R+3x} \begin{bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & x \\ k_3 & j_3 & x \end{bmatrix} .
\]

Here $R = j_1 + j_2 + j_3 + l_1 + l_2 + l_3 + k_1 + k_2 + k_3$. 

24
2. The second identity involves triple summations and it is given on page 472, equation (38):

\[
\sum_{k_1, k_2, k_3} \Pi^2(k_1, k_2, k_3)(-1)^{R+x} \left\{ \begin{array}{ccc} j_1 & k_1 & x \\ k_2 & j_2 & l_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_2 & k_2 & x \\ k_3 & j_3 & l_1 \end{array} \right\} \left\{ \begin{array}{ccc} j_3 & k_3 & x \\ k_1 & j_1 & l_2 \end{array} \right\} = \Pi^2(x) \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ j_1 & j_2 & j_3 \end{array} \right\}. \quad (V-2)
\]

Now we consider left hand side of the identity (32):

\[
\text{LHS} = \sum_{\{k\}'} \Pi^2(k_4, k_5, k_6) \left[ \begin{array}{ccc} k_4 & k_5 & k_6 \\ j_3 & j_1 & j_2 \\ j_4 & j_5 & j_6 \end{array} \right] \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ k_1 & k_5 & k_6 \end{array} \right\} \prod_{l=4,5,6} \{j_l, k_l, s\}
\]

We expand 9j coefficients using (V-1):

\[
\text{LHS} = \sum_{k_4, k_5, k_6} \Pi^2(k_4, k_5, k_6) \sum_x (-1)^{R+x} \Pi^2(x) \left\{ \begin{array}{ccc} k_4 & j_4 & x \\ j_5 & k_5 & j_3 \\ j_4 & k_6 & j_1 \end{array} \right\} \left\{ \begin{array}{ccc} k_5 & j_5 & x \\ j_6 & k_6 & j_1 \end{array} \right\} \prod_{l=4,5,6} \{j_l, k_l, s\}
\]

In above \(R = j_1 + j_2 + j_3 + j_4 + j_6 + k_4 + k_5 + k_6\). Due to triangular constraints \(\{j_l, k_l, s\}\) only non-zero term in the series is \(x = s\); i.e.

\[
\text{LHS} = \Pi^2(s) \sum_{k_4, k_5, k_6} (-1)^{R+s} \Pi^2(k_4, k_5, k_6) \left\{ \begin{array}{ccc} k_4 & j_4 & s \\ j_5 & k_5 & j_3 \end{array} \right\} \left\{ \begin{array}{ccc} k_5 & j_5 & s \\ j_6 & k_6 & j_1 \end{array} \right\} \left\{ \begin{array}{ccc} k_6 & j_6 & s \\ j_4 & k_4 & j_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ k_4 & k_5 & k_6 \end{array} \right\} = \Pi^4(s) \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \quad (\text{using identity (V-2)})
\]

\[
= \text{RHS}
\]

**References**

[1] Penrose R. *Angular momentum: An approach to combinatorial spacetime in Quantum Theory and Beyond*, T. Bastin, ed. 1971 Combinatorial Quantum Theory and Quantized Directions in Advances in Twistor Theory (Cambridge University Press, Cambridge).
[2] Gambini R. and Pullin J. 2000 *Loops, Knots, Gauge Theories and Quantum Gravity* England Cambridge University Press, Loll R 1992 *Nucl. Phys. B* 368 121, 1993 *Nucl. Phys. B* 400 126, Watson N. J. 1994 *Phys. Lett. B* 323 385

[3] Mathur M. 2007 Loop Approach to Lattice Gauge Theories *Nucl.Phys.B* 779 32-62

[4] Rovelli C. and Smolin L. 1995 Spin networks and quantum gravity *Phys. Rev. D* 52 5743, *gr-qc/9505006*. Smolin L. 1997 The Future of spin networks *CGPG-97/2-2*. Seth A. Major 1999 A spin network primer *Am. J Phy.* 67, 972. Hari Dass N. D. and Mathur M. 2007 On loop states in loop-quantum gravity *Class. Quantum Grav.* 24 2179.

[5] Mathur M. and Rathor A. 2021 SU(N) Toric Code and Nonabelian Anyons *arXiv:2110.13841*

[6] Marzuoli A. and Rasettib M. 2002 Spin network quantum simulator *Physics Letters A* 306 79–87, 2005 Computing spin networks *Ann. of Phys* 318 345-407. Mielczarek J. 2018 Spin networks on adiabatic quantum computer *arXiv:1801.06017*

[7] Delfino G. 2011 Deriving identities for Wigner nj-symbols *arXiv:1110.5776v3*. Carbone G., Carfora M and Marzuoli A. 2000 *Commun. Math. Phys.* 212, 571 – 590. Valentin B., Livine Etera R. and Speziale S. 2010 Recurrence relations for spin foam vertices *Class. Quant. Grav.* 27 125002.

[8] Kogut and Susskind L. 1975 *Phys. Rev. D* 11 395.

[9] Varshalovich D. A., Moskalev A. N. and Khersonskii V. K. 1988 *Quantum Theory of Angular Momentum* Singapore World Scientific.

[10] Yutsis A. P., Levinson I. B. and Vanagas V. V. 1962 *Mathematical Apparatus of the Theory of Angular Momentum* Israel program for scientific translations.

[11] Kitaev A. Yu. 2003 Fault-tolerant quantum computation by anyons *Annals Phys.* 303 2-30, 1997 *arXiv: quant-ph/9707021*. A. J. Leggett, The Kitaev models, Lecture 26 (2013).

[12] Robson D. and Webber D. M. 1982 *Z. Phys. C* 15 199. Furmanski W. and Kolawa A. 1987 *Nucl. Phys.B* 291 594, Anisheetty R. and Sharatchandra H. S. 1990 *Phys. Rev. Lett.* 65 813. Burgio G., De Pietri R., Morales-Tecotl H. A., Urrutia L. F. and Vergara J. D. 2000 *Nucl. Phys. B* 566, 547.