CAT(0) SPACES WITH POLYNOMIAL DIVERGENCE OF GEODESICS

Nataša Macura
Department of Mathematics
Trinity University
nmacura@trinity.edu
One Trinity Place
San Antonio TX 78212

Abstract. We construct a family of finite 2-complexes whose universal covers are CAT(0) and have polynomial divergence of desired degree. This answers a question of Gersten, namely whether such CAT(0) complexes exist.

1. Introduction

In [5] Gersten defined divergence of a CAT(0) space, generalizing the classical idea of the divergence of geodesics in manifolds, and showed it to be a quasi-isometry invariant. He constructed a CAT(0) 2-complex with quadratic divergence, therefore showing that the aphorism of Riemannian geometry that geodesics diverge either linearly or exponentially fails for CAT(0) spaces. In later work, Gersten [5], and M. Kapovich and Leeb,[7] showed that the aphorism also fails for 3-manifolds since there exist graph manifolds with quadratic divergence of geodesics. In this paper we exhibit a family of CAT(0) groups $G_d, d \in \mathbb{N}$, such that the divergence of $G_d$ is polynomial of degree $d$. We construct $G_d$ inductively as an HNN extension of $G_{d-1}$, starting with $G_1 = \mathbb{Z} \oplus \mathbb{Z}$. Each $G_d$ has a 2-dimensional presentation complex $X_d$ whose universal cover $\tilde{X}_d$ is a CAT(0) cube complex. We prove that the divergence of $\tilde{X}_d$ is polynomial of degree $d$. The groups described here turn out to be the family of examples W. Dison and T. Riley introduced and named hydra groups in [4]. W. Dison and T. Riley show that hydra groups have finite-rank free subgroups with huge distortion and use this class

Keywords and phrases: divergence, CAT(0) spaces
Mathematics Subject Classification 2010: 20F65, 20F67, 57M20.
of groups to construct elementary examples of groups whose Dehn functions are equally large.

Divergence of geodesics, as well as in its higher dimensional generalizations received renewed interest in recent work of a number of authors. In [1] A. Abrams, N. Brady, P. Dani, M. Duchin and R. Young define higher divergence functions, which measure isoperimetric properties "at infinity", and give a characterization of the divergence of geodesics in RAAGs as well as upper bound for filling loops at infinity in the mapping class group. J. Behrstock and R. Charney ([2]) give a group theoretic characterization of geodesics with super-linear divergence in the Cayley graph of a right-angled Artin group $A_{\Gamma}$ with connected defining graph $\Gamma$ and use this to determine when two points in an asymptotic cone of $A_{\Gamma}$ are separated by a cut-point.

We propose a modified version of Gersten’s question: are there CAT(0) spaces with isolated flats ([6]) and super-linear and sub-exponential divergence of geodesics. Our examples, like those of Gersten and M. Kapovich do not have isolated flats. So the aphorism may yet hold for CAT(0) spaces with isolated flats.

The organization of the paper is as follows. In Section 2 we recall the definitions and results concerning divergence and CAT(0) spaces, that are pertinent to our proofs. When studying the divergence, we use the language and techniques of detour functions developed in [8], since they facilitate simple and intuitive arguments. The equivalence class of detour functions of a proper metric space $X$ is the divergence in Gersten’s sense, and it is a quasi-isometry invariant if $X$ has a weak form of geodesic extension property. In Section 3 we define the complexes $X_d$, and analyze geometric properties of the complexes $X_d$ and $\tilde{X}_d$ pertinent to proof of the polynomial divergence in Sections 4 and 5.

In Section 4 we show that the detour function of $\tilde{X}_d$ is bounded above by a polynomial of degree $d$ and in Section 5 we show that there are geodesics $\gamma_0$ and $\gamma_d$ in $\tilde{X}_d$ which actually do diverge polynomially with degree $d$, therefore establishing that the divergence of $\tilde{X}_d$ is polynomial of degree $d$.

I am grateful to Daniel Allcock for his help with the first version of this paper, and to the anonymous referee for careful reading and helpful suggestions on the exposition of the paper.
2. Detour functions and CAT(0) spaces

2.1. Detour functions and divergence. Detour functions were introduced in [8] in order to classify mapping tori of polynomially growing automorphisms of free groups; they provide a language and techniques to study divergence of geodesics in proper metric spaces, and are invariant under quasi-isometries. We recall the definition and the main results used in this paper, and refer the reader to [8] and [5] for detailed expositions on detour functions and divergence.

Let $X$ be a proper metric space, $O$ a point in $X$, and $r \geq 0$ a real number. Let $S(O, r)$ and $B(O, r)$, be the sphere and the open ball, of radius $r$ centered at $O$. We say that a path $\alpha$ in $X$ is an $r$-detour path if $\alpha$ does not intersect $B(O, r)$. The $r$-detour distance $\delta_r(P, Q)$, between two points $P, Q \in X \setminus B(O, r)$ is the infimum of the lengths of all $r$-detour paths $\alpha$ that connect $P$ and $Q$. In the case $P$ and $Q$ are in different components of $X \setminus B(O, r)$, we define their detour distance to be infinite. Since $X$ is a proper metric space, if the detour distance $\delta_r(P, Q)$ is finite, Arzela-Ascoli Theorem implies the existence of a detour path $\alpha$ such that $|\alpha| = \delta_r(P, Q)$.

We call such $\alpha$ a minimal length or shortest detour path. As indicated above, we suppress the point $O$ from notation if it is understood from the context and, when necessary, we will talk about $(O, r)$-detour path and $(O, r)$-detour distance. A detour function roughly speaking, assigns to each positive real number $r$ the maximum of all $r'$-detour distances between points on the sphere of radius $r$, where $r' = ar - b$, for $0 \leq a \leq 1$ and $b > 0$.

**Definition 2.1.** Let $(X, d)$ be a proper metric space. Given a point $O \in X$ let $B_r = B(O; r)$ be the open ball of radius $r$ centered at $O$ and $S_r = S(O; r)$ the sphere of the radius $r$. A detour function of $(X, O)$ is a pair $(\phi, \mu)$ such that $\phi$ is a linear function, $\phi(x) = ax - b$, $0 < a, b; a \leq 1$, and $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{\infty\}$ is defined in the following way:

$$\mu_O(r) = \max_{P, Q \in S_r} \{\delta_{\phi(r)}(P, Q) : P, Q \in S_r\}.$$ 

In [8] we introduced a (weak) version of geodesic extension property for a metric space $X$ that implies the existence of a detour function $(\phi, \mu)$, $\phi(x) = ax - b$, such that, if $(\psi, \mu')$, $\psi(x) = cx - d$, is a detour function and $a \leq c, d \geq b$, then $(\phi, \mu)$ and $(\psi, \mu')$, are equivalent in the following sense. We say that $f \preceq g$ if there are constants $A, B, C, D, E > 0$ such that

$$f(x) \leq Ag(Bx + C) + Dx + E$$

for every $x > 0$. 


We define two functions \( f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{\infty\} \), to be equivalent, \( f \sim g \), if \( f \preceq g \) and \( g \preceq f \). This gives equivalence relation capturing the qualitative agreement of growth rates. The square complexes constructed in Section 3 satisfy a strong version of the geodesic extension property, that is that every geodesic can be extended to an infinite geodesic ray, which implies that any two detour functions are equivalent. In particular, the equivalence class of the detour function does not depend on the choice of the base point. In the remainder of the paper, we select \( a = 1 \) and \( b = 0 \), and a point \( O \in X \), and take the detour function of a proper metric space \( X \) to be the function \( \mu_X : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{\infty\} \) defined by

\[
\mu_X(r) = \max \{ \delta_r(P, Q) : P, Q \in S(O, r) \}.
\]

The equivalence class of the detour function of \( X \) is the divergence in Gersten’s sense (see e.g.\[5\]), and bounding a detour function \( \mu_X \) from above and below by polynomials of degree \( d \) shows that the divergence of geodesics is polynomial of degree \( d \).

2.2. **CAT(0) spaces.** We recall the definition of a CAT(0) space and several properties that such a space enjoys, and refer the reader to [3] for a detailed treatment of the topic.

Let \((X, d)\) be a metric space and let \((E, d_E)\) be the Euclidean plane, where \( d \) and \( d_E \) are the respective metrics. A geodesic triangle \( \Delta = \Delta(P, Q, R) \) in \( X \) consists of three points \( P, Q, R \in X \), its vertices, and a choice of three geodesic segments \( \gamma_{PQ}, \gamma_{QR} \) and \( \gamma_{PR} \) joining the vertices, its sides. If the point \( T \) lies in the union of \( \gamma_{PQ}, \gamma_{QR} \) and \( \gamma_{PR} \), then we write \( T \in \Delta \).

A geodesic triangle \( \Delta_E = \Delta(P_E, Q_E, R_E) \) in \( E \) is called a comparison triangle for the triangle \( \Delta(P, Q, R) \) if \( d(P, Q) = d_E(P_E, Q_E), \ d(P, R) = d_E(P_E, R_E) \) and \( d(Q, R) = d_E(Q_E, R_E) \). A point \( T_E \) on \( \gamma_{P EQ_E} \) is called a comparison point for \( T \) in \( \gamma_{PQ} \) if \( d(P, T) = d_E(P_E, T_E) \). Comparison points for points on \( \gamma_{PR} \) and \( \gamma_{QR} \) are defined in the same way.

**Definition 2.2.** A metric space \( X \) is a CAT(0) space if it is a geodesic metric space all of whose triangles satisfy the CAT(0) inequality:

Let \( \Delta \) be a geodesic triangle in \( X \) and let \( \Delta_E \) be a comparison triangle in the Euclidean plane \( E \). Then, \( \Delta \) is said to satisfy the CAT(0) inequality if for all \( S, T \in \Delta \) and all comparison points \( S_E, T_E \in \Delta_E \), \( d(S, T) \leq d_E(S_E, T_E) \).
A metric space $X$ is said to be of non-positive curvature if it is locally a CAT(0) space, i.e. for every $x \in X$ there exists $r_x > 0$ such that the ball $B(x, r_x)$ with the induced metric, is a CAT(0) space.

We will use the orthogonal projections onto complete, convex, subsets of CAT(0) spaces, called projections in [3, II.2]. We review selected parts of a proposition [3, II.2, Proposition 2.4], which gives the construction of such a projection $\pi_C : X \rightarrow C$.

**Proposition 2.3.** Let $X$ be a CAT(0) space, and let $C$ be a convex subset which is complete in the induced metric. Then,

1. for every $x \in X$, there exists a unique point $\pi(x) \in C$ such that $d(x, \pi(x)) = d(x, C) = \inf_{y \in C} d(x, y)$;
2. if $x'$ belongs to the geodesic segment connecting $x$ and $\pi(x)$, then $\pi(x) = \pi(x')$.
3. the map $x \mapsto \pi(x)$ is a retraction from $X$ onto $C$ which does not increase distances.

Throughout the rest of the paper $\pi_C : X \rightarrow C$ will denote the projection onto complete, convex, subset $C$ of $X$, as described in Proposition 2.3. We will also make use of the property of a CAT(0) space that a local geodesic is a global geodesic. We will use the following property of projections.

**Remark 2.4.** Let $O$ be a point in a CAT(0) space $\tilde{X}$, and let $O'$ be the closest point projection of $O$ to geodesic $\omega$. If $P$ and $Q$ are points on $\omega$ such that $Q$ is contained in the segment of $\omega$ connecting $O'$ and $P$, then $d(O, Q) \leq d(O, P)$.

We note that the above remark is a consequence of CAT(0) inequality applied to the triangle $OOP$.

As a matter of general terminology and notation, a “path $\alpha$” refers to both the path as a continuous function $\alpha : [0, t] \rightarrow \mathbb{R}$ and the image of $\alpha$ in the metric space $X$, and $|\alpha|$ stands for the length of a path $\alpha$. We will use $\alpha \ast \beta$ to denote the path which is the concatenation of paths $\alpha$ and $\beta$, or $\gamma_1 \gamma_2$ for a geodesic which is a concatenation of geodesics paths $\gamma_1$ and $\gamma_2$.

3. **Square complexes $\tilde{X}_d$ and their geometric properties**

Let $G_1$ be the group $\mathbb{Z} \oplus \mathbb{Z}$, generated by $a_0$ and $a_1$, and let $X_1$ be the flat torus obtained by isometric identification of the edges of the Euclidean unit square $C_1 = I \times I$. Orient the horizontal edges of the unit square from
left to right, and the vertical ones with upward positive direction. We call this orientation torus orientation. Denote two opposite directed edges of the square $C_1$ by $a_0$ and the other two by $a_1$. We will use the same notation $(a_0, a_1)$ for the corresponding (directed) loops in $X_1$. The group $G_1$ acts properly and cocompactly by isometries on the Euclidean plane $\widetilde{X}_1$ with the quotient space $X_1$.

We define the CAT(0) groups $G_d, d \geq 2$, inductively, taking $G_d$ to be the HNN extension of $G_{d-1}$ that amalgamates the infinite cyclic subgroups of $G_{d-1}$ generated by $a_{d-1}$ and $a_0$ respectively. If we denote the stable letter in this extension by $a_d$ the resulting group $G_d$ has a presentation

\[ \{a_0, \ldots, a_d \mid a_0 a_1 = a_1 a_0, a_i^{-1} a_0 a_i = a_{i-1}, \text{for } 2 \leq i \leq d\}. \]

We construct a presentation complex $X_d$ of $G_d$ by a standard topological construction of gluing with a tube, see [3, II.11] on $X_{d-1}$. Let $C_d = I \times I$ be the Euclidean unit square with the torus orientation. Label two opposite directed edges by $a_d$ and identify them to obtain a cylinder (tube) $U_d$. The remaining two edges of $C_d$ map to loops in $U_d$, and we label them $a_{d-1}$ and $a_0$ respectively. The complex $X_d$ obtained by gluing the cylinder $U_d$ to $X_{d-1}$, with the identification map the orientation preserving isometry prescribed by the labeling of the edges, is a graph of spaces with one vertex and one edge. The vertex space is $X_{d-1}$ and the edge space $S^1$. We will call the universal cover $\widetilde{X}_d$ of $X_d$ the $d$-th square complex. We will refer to $d$ as the height of $\widetilde{X}_d$.

The resulting complex $X_d$ is a non-positively curved cube complex (see [3, II.11]) and therefore $\widetilde{X}_d$ is a CAT(0) cube complex. We note that it is not difficult to see that the “link condition” ([3, II.5]), is satisfied: for each vertex $P \in X_d$ every injective loop in $\text{Lk}(P, X_d)$ has length at least $2\pi$. The preimage of $X_{d-1}$ in $\widetilde{X}_d$ consists of infinitely many disjoint convex (hence isometrically embedded) copies of $\widetilde{X}_{d-1}$. We call such a copy of $\widetilde{X}_{d-1}$ a vertex complex in $\widetilde{X}_d$.

Let $s_d, d \geq 1$ be the line segment in $C_d$ that connects the midpoints of the opposite edges labeled $a_d$. We also denote by $s_d$ the image of $s_d$ in $U_d$, as well as its image in $X_d$ after the gluing. Every component $H$ of the preimage of $s_d \subset X_d$ under the covering map is isometric to the real line and separates $\widetilde{X}_d$. We call $H$ a hyperplane in $\widetilde{X}_d$ (even though it is just a line). Every hyperplane $H$ is geodesic contained in a single component ostar($H$) of the preimage of $\text{Int}(U_d)$ under the covering map. Since $U_d$ is a cylinder, the closure star($H$) of ostar($H$) is isometric to a flat strip. We call ostar($H$)
the open star of $H$. For each edge $e$ labeled $\tilde{a}_d$ in $\tilde{X}_d$, there is a unique hyperplane $H$ that intersects $e$, and we say that $H$ corresponds to the edge $e$.

Let $\phi_0, \phi_{d-1} : S^1 \rightarrow U_d \rightarrow X_d$ be the inclusions of $S^1$ into $X_d$ that wrap $S^1$ isometrically once around $a_0$ and $a_{d-1}$, respectively. For each hyperplane $H$ there are lifts $\tilde{\phi}_0, \tilde{\phi}_{d-1}$ of $\phi_0, \phi_{d-1}$, such that $\omega_0 = \tilde{\phi}_0(\mathbb{R}) \subset \text{star}(H)$ and $\omega_{d-1} = \tilde{\phi}_{d-1}(\mathbb{R}) \subset \text{star}(H)$. We call the bi-infinite geodesic path $\omega_0$ consisting of copies of $\tilde{a}_0$ the smooth trace of the hyperplane $H$. The rugged trace $\omega_{d-1}$ is the bi-infinite geodesic path consisting of copies of $\tilde{a}_{d-1}$.

Let $\mathcal{H}$ be the collection of all hyperplanes in $\tilde{X}_d$, and let $U = \cup\{\text{ostar}(H_i) : H_i \in \mathcal{H}\}$. Each connected component $V$ of $\tilde{X}_d \setminus U$ is a copy of the universal cover of the square complex $X_{d-1}$, and we called such $V$ a vertex complex in $\tilde{X}_d$. Since a vertex complex $V_{d-1}$ in $\tilde{X}_d$ is a copy of a square complex of height $d - 1$, we can talk about hyperplanes and vertex complexes in $V_{d-1}$. The vertex complexes in this case are isometric copies of $\tilde{X}_{d-2}$. In the same fashion, for every vertex $P$ in $\tilde{X}_d$, there is a sequence of sub-complexes $V_i \subset \tilde{X}_d$, $1 \leq i \leq d$, such that $P \in V_1 \subset V_2 \subset \ldots V_i \ldots \subset V_{d-1} \subset \tilde{X}_d$. Each $V_i$ is a copy of an $i$-th square complex, and we call each such $V_i$ an $i$-vertex complex, or a vertex complex of height $i$. A hyperplane $H_i$ in $V_i$ separates $V_i$, but does not separate $V_{i+1}$. An edge labeled $\tilde{a}_i$ has the height, denoted by $\text{height}(\tilde{a}_i)$, equal to $i$.

If $\text{star}(H) \cap V \neq \emptyset$ for a hyperplane $H \subset \tilde{X}_d$ and a $(d-1)$-vertex complex $V$ then, $\text{star}(H) \cap V$ is a geodesic path, the smooth or rugged trace of $H$. We will call this geodesic edge path the trace of $H$ in $V$ and will say that the hyperplane $H$ and the vertex complex $V$ are adjacent.

A $d$-segment (or a $d$-line) is a geodesic segment (line) that is a concatenation of edges labeled $\tilde{a}_d$. When the orientation is of importance, we will call a finite or infinite oriented segment a ray. We use suffixes to indicate the endpoints of a segment or the initial endpoint of a ray: $\omega_{PQ}$ is a segment with initial endpoint $P$ and the terminal endpoint $Q$, $\gamma_P$ stands for a geodesic ray issuing at $P$. We call the orientation of edges in $\tilde{X}_d$ induced by the orientation on $X_d$ the standard edge orientation. If $\gamma_P : [0, r] \rightarrow \tilde{X}_d$ is a geodesic ray in 1-skeleton of $\tilde{X}_d$, then $\gamma_P$ induces a $\gamma_P$-orientation on each edge it traces, by choosing the positive direction to be the one of increasing values of the parameter $t \in [0, r]$. We say that a geodesic $\gamma_P$ traces an edge $e$ in a positive direction if the $\gamma_P$-direction on $e$ coincides with the standard direction.
Definition 3.1. A geodesic segment $u$ is a positive geodesic segment if it is contained in 1-skeleton of $\widetilde{X}_m$, and if $u(t)$ traces all the edges in the positive direction.

We conclude our study of basic geometric properties of square complexes with the geodesic extension property. Since it is easily observed that every 1-cell in $\widetilde{X}_d$ is contained in a boundary of at least two 2-cells, $\widetilde{X}_d$ has no free faces, and Proposition 5.10, Chapter II, in [3], implies that $\widetilde{X}_d$ has the geodesic extension property. We formalize the above result in the following proposition.

Proposition 3.2. Every non-constant geodesic $\gamma$ in $\widetilde{X}_d$ can be extended to an infinite geodesic ray.

4. POLYNOMIAL DIVERGENCE OF GEODESICS

The following theorem is our main result.

Theorem 4.1. The $d$-th square complex $\widetilde{X}_d$ has degree $d$ polynomial divergence of geodesics.

In this section we show that there is a degree $d$ polynomial $q_d$ such that the detour function of $\widetilde{X}_d$ is bounded above by $q_d$, and we start by stating and proving a lemma.

Lemma 4.2. Let $H$ be a hyperplane in $\widetilde{X}_d$, and let $\omega$ be a trace of $H$. Let $O, Q$ be points in $\widetilde{X}_d$ such that $Q \in \text{star}(H)$ and $d(O, Q) = r$. If $\xi \subset \text{star}(H)$ is any bi-infinite geodesic parallel to $\omega$ such that $d(O, \xi) \leq r$, then there is a point $P \in \xi \cap S(O, r)$, and an $(O, r)$-detour path $\beta$ connecting $Q$ and $P$ such that $|\beta| \leq 2r + 1$.

Proof. Let $\zeta$ be the bi-infinite geodesic parallel to $\xi$ through $Q$ and let $O'$ be the projection of the point $O$ to the infinite flat strip $U$ bounded by $\xi$ and $\zeta$. Since the lemma trivially holds when $Q \in \xi$, we can assume that $Q \notin \xi$. This implies that $O' \neq Q$, since otherwise we would have $d(\xi, O) > r$. Let $\beta'_\perp$ be the geodesic segment in $U$ perpendicular to $\xi$ through the point $O'$, and let $U_1$ be the component of $U \setminus \beta'_\perp$ that contains $Q$. We claim that the endpoints of $\beta'_\perp$ are the projections of $O$ to $\zeta$ and $\xi$. If $O \in U$, then $O' = O$ and the claim follows directly from the definition of $\beta'_\perp$. If $O \notin U$, then either $O' \in \zeta$, or $O' \in \xi$. If, say, $O' = O_\zeta \in \zeta$, then $O_\xi = \beta'_\perp \cap \xi$ is the closest point to $O$ on $\xi$: $O_\xi$ is the point closest to $O$ on $\zeta$, and the distance
between $\zeta$ and $\xi$ is equal to the length of $\beta_1'$. A similar argument holds if $O' = O_\xi \in \xi$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{starH.png}
\caption{Illustration of the proof of Lemma 4.2}
\end{figure}

Let $\beta_1'$ be the geodesic segment in $U_1$ parallel to $\beta_1'$ and at the distance $r$ from $\beta_1'$, and denote by $Q_1$ and $P_1$ the intersections of $\zeta$ and $\xi$ with $\beta_1'$ respectively, see Figure 4. Since $d(O, P_1) \geq d(O', P_1) \geq r$, and $d(O, O_\xi) = d(O, \xi) \leq r$, there is a point $P$ in $\xi \cap U_1$ such that $d(O, P) = r$.

To construct the desired detour path $\beta$, take $\beta_1$ to be the segment of $\zeta$ connecting $Q$ and $Q_1$. The length of $\beta_1$ is equal to the distance between points $Q$ and $Q_1$, which, by construction, is not larger than $r$. Let $\beta_2$ be the segment of $\xi$ connecting $P$ and $P_1$, and note that the length of $\beta_2$ is also less than or equal to $r$. Remark 2.4 implies that $\beta_1$ and $\beta_2$ do not intersect $B(O, r)$, and $\beta = \beta_1 \beta_1 \beta_2$ is then a detour path connecting $Q$ and $P$ of length not larger than $2r + 1$. $\square$

**Lemma 4.3.** Let $O \in \tilde{X}_d$ and let $P, Q \in S(O, r) \cap E$ be points contained in subcomplex $E$ of $\tilde{X}_d$ which is a copy of $\tilde{X}_1$. Then there is an $(O, r)$-detour path $\alpha$ connecting $P$ and $Q$ of length at most $\pi r + 2r$.

**Proof.** Let $O'$ be the projection of $O$ to $E$. By the properties of projections, $d(O', P) \leq r$ and $d(O'Q) \leq r$. Let $\gamma_P$ and $\gamma_Q$ be geodesics connecting $O'$ and $P, Q$ respectively. By the geodesic extension property we can extend $\gamma_P$ and $\gamma_Q$ to infinite geodesic rays $\gamma_P'$ and $\gamma_Q'$. Let $P'$ and $Q'$ be the points of intersection of the sphere $S(O', r)$ and the geodesic rays $\gamma_P', \gamma_Q'$ respectively. Let $\beta_P$ be the segment of $\gamma_P'$, that connect $P$ and $P'$, and let $\beta_Q$ be the segments of $\gamma_Q'$ that connect $Q$ and $Q'$. Remark 2.4 implies that $d(O, S) \geq r$ for any point $S$ in $\beta_P$. Similarly, $d(O, S) \geq r$ for a point $S \in \beta_Q$. 

Since $E$ is copy of $\tilde{X}_1$, it is the Euclidean plane, and $P'$ and $Q'$ lie on the sphere $S(O', r)$, there is an $(O', r)$-detour path $\beta$ connecting $P'$ and $Q'$ of length at most $\pi r$. We note that, since the properties of projections imply that $d(O', T) \leq d(O,T)$ for any point $T \in \beta$, $\beta$ is also an $(O, r)$ detour path. The desired $(O, r)$-detour path that connects $P$ and $Q$ is $\beta_P \beta_Q$. \hfill \Box \\

**Proposition 4.4.** There is a polynomial $q_d$, of degree $d$, such that for any point $O$ in $\tilde{X}_d$, and any two points $P,Q$ on the sphere $S(O, r) \subset \tilde{X}_d$, there is a path $\alpha$ in $\tilde{X}_d \setminus B(O, r)$ connecting $P$ and $Q$ such that the length of $\alpha$ is at most $q_d(r)$.

*Proof.* We prove the statement of the proposition by induction. Lemma 4.3 provides the base of induction, with $q_1(r) = (2 + \pi)r$.

Let $q_{d-1}$ be a polynomial of degree $d - 1$ such that for any point $O \in \tilde{X}_{d-1}$, and any two points $P',Q' \in S(O', r) \subset \tilde{X}_{d-1}$ there is a path $\alpha'$ in $\tilde{X}_{d-1} \setminus B(O, r)$ connecting $P'$ and $Q'$ such that the length $|\alpha'| \leq q_{d-1}(r)$. Let $O, P, Q \in \tilde{X}_d$ be as in the statement of the proposition. If $P$ and $Q$ are contained in the same vertex complex $V_{d-1}$, the claim of the Proposition follows directly from the induction hypothesis, otherwise, let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be the collection of all hyperplanes in $\tilde{X}_d$ such that each $H_i$ either separates $P$ and $Q$, or $\{P, Q\} \cap star(H_i) \neq \emptyset$.

Without loss of generality we can assume that either $P \in star(H_1)$, or $H_1$ is the hyperplane in $\mathcal{H}$ closest to $P$, and that every hyperplane $H_i$ (for $i = 2, \ldots, m - 1$) separates $H_{i-1}$ and $\{H_{i+1}, H_{i+2}, \ldots, H_m\}$. Since $d(P, Q) \leq 2r$, there are no more than $2r$ hyperplanes separating $P$ and $Q$ and therefore $m \leq 2r + 2$. Let $Y_i$ be the component of $\tilde{X}_d \setminus H_i$ that contains $H_{i+1}$ and let $V_i$, $i = 1, \ldots, m - 1$, be the (unique) vertex complex contained in $Y_i$ that intersects $star(H_i)$. Note that then $V_i$ also intersects $star(H_{i+1})$: $H_{i+1}$ is contained in $Y_i$ and no hyperplane separates $H_i$ and $H_{i+1}$. If $P \notin star(H_1)$, let $V_0$ be the vertex complex containing $P$. Similarly, if $Q \notin star(H_m)$ let $V_m$ be the vertex complex containing $Q$. If $P \in star(H_1)$ let $Q_0 = P$. If $P \in V_0$, then $d(O, star(H_1) \cap V_0) \leq r$ and we take $Q_0$ to be a point in $star(H_1) \cap V_0 \cap S(0, r)$.

If $Q \in star(H_m)$ let $P_m = Q$, otherwise let $P_m$ be a point in $star(H_m) \cap V_m \cap S(0, r)$. By Lemma 4.2 there is a point $Q_{m-1} \in V_{m-1} \cap star(H_m) \cap S(O, r)$ and an $(O, r)$-detour path $\beta_m$ connecting $Q_{m-1}$ and $P_m$ such that $|\beta_m| \leq 2r + 1$.

For $1 < i < m - 2$, we let $Q_i$ be a point in $star(H_{i+1}) \cap V_i \cap S(0, r)$ such a point exists since $d(O, star(H_{i+1}) \cap V_i) \leq r$. By Lemma 4.2 for every
1 ≤ i < m, there is a point \( P_i \in V_i \cap \text{star}(H_i) \cap S(O_i, r) \), and an \((O, r)\)-detour path \( \beta_i \) connecting \( Q_{i-1} \) and \( P_i \) such that \( |\beta_i| \leq 2r + 1 \).

The point \( P_i, Q_i \), chosen as above for \( i = 1 \ldots, m - 1 \), are both contained in \( V_i \), and, by the induction hypothesis, for each \( i = 1 \ldots, m - 1 \) there is a detour path \( \alpha_i \) of length at most \( q_{d-1}(r) \) connecting \( P_i \) and \( Q_i \) in the vertex space \( V_i \), and outside the ball \( B(O, r) \).

If \( P \notin \text{star}(H_1) \), let \( \alpha_0 \) be the detour path of length at most \( q_{d-1}(r) \) connecting \( P \) and \( Q_0 \) in the vertex space \( V_0 \). Similarly, if \( Q \notin \text{star}(H_m) \), let \( \alpha_m \) be a detour path of length at most \( q_{d-1}(r) \) connecting \( P_m \) and \( Q \) in the vertex space \( V_m \). In the case \( P \in \text{star}(H_1) \) \((Q \in \text{star}(H_m) \) we will take \( \alpha_0 \) \((\alpha_m)\), to be the empty paths.

Then the path

\[
\alpha = \beta_P \ast \alpha_0 \ast \beta_1 \ast \alpha_1 \ast \beta_2 \ast \ldots \ast \alpha_{m-1} \ast \beta_m \ast \alpha_m \ast \bar{\beta}_Q
\]

is a detour path connecting \( P \) and \( Q \) and \( |\alpha| \leq (2r + 3)q_{d-1}(r) + (2r + 2)(2r + 1) \).

5. **Lower bound on the detour function**

We complete our proof of degree \( d \) polynomial divergence in complexes \( \tilde{X}_d \) by showing that there are two geodesic rays in \( \tilde{X}_d \), emanating from the same point \( O \), that diverge at least polynomially with degree \( d \). The two such infinite rays are \( \gamma_0 \) and \( \gamma_d \) which are the infinite concatenations of edges \( \tilde{a}_0 \) and \( \tilde{a}_d \) respectively. As a matter of convention, we use \( \gamma_d \) and \( \omega_d \) to denote either a segment, a ray, or a line which is a concatenation of edges \( \tilde{a}_d \), and we call them \( d \)-segment, \( d \)-ray and \( d \)-line respectively. We will also

---

**Figure 2.** Illustration of the proof of Proposition 4.4
consider a finite oriented segment to be a ray, issuing from the its initial endpoint.

**Definition 5.1.** We call the pair of geodesic rays $\gamma_0$ and $\gamma_d$ both issuing from a vertex $T \in \tilde{X}_d$ a basic $d$-corner at $T$ and denote it by $(\gamma_d, \gamma_0)_T$.

**Definition 5.2.** Let $\gamma$ and $\gamma'$ be geodesic rays in $\tilde{X}_d$. An $(r, O)$-detour path between geodesic rays $\gamma$ and $\gamma'$ is any $(r, O)$-detour path connecting $P \in \gamma$ and $Q \in \gamma'$, $P, Q$ outside $B(O, r)$.

**Theorem 5.3.** There is a polynomial $p_d$ of degree $d$ and with a positive leading coefficient, such that the length of any $(r, O)$-detour path in $\tilde{X}_d$ over a basic $d$-corner $(\gamma_d, \gamma_0)_O$ is bounded below by $p_d(r)$.

5.1. Intuitive approach. Our general approach is to prove Proposition 5.3 by induction on $d$. We first discuss the motivation for this approach and explain a technical difficulty that it encounters. We start with the observation that any detour path $\alpha \subseteq \tilde{X}_d$ over a basic $d$-corner $(\gamma_d, \gamma_0)_O$ has to intersect every hyperplane that $\gamma_d$ intersects. Let $n$ be the greatest integer less than or equal to $r$, let $j$ be an integer $j \in \{1, \ldots, n\}$, and let $H_j$ be the hyperplane that intersects $\gamma_d$ at distance $j - 1/2$ from $O$. Note that the $j$-th vertex of $\gamma_d$ (the vertex at the distance $j$ from $O$) is contained in both $\text{star}(H_j)$ and $\text{star}(H_{j+1})$. We denote this vertex by $T_j$. For every $1 \leq j \leq n - 1$, let $\alpha_j$ be a component of $\alpha \setminus (\text{os}_{\text{star}}(H_j) \cup \text{os}_{\text{star}}(H_{j+1}))$ that connects the rugged trace $\omega_j$ of $H_j$ and the smooth trace $\gamma_{0,j+1}^1$ of $H_{j+1}$. The geodesics $\omega_j$, which is a $(d - 1)$-ray, and $\gamma_{0,j+1}^1$ intersect at $T_j$ and form a basic $(d - 1)$-corner at $T_j$. We note that, since $d(\alpha_j, O) \geq r$ and $d(O, T_j) = j$, the path $\alpha_j$ does not intersect the ball of radius $r - j$ centered at $T_j$, making $\alpha_j$ into an $(r - j)$-detour over a basic $(d - 1)$-corner. We would like to use the hypothesis of induction and claim that $|\alpha_j| \geq p_{d-1}(r - j)$, but $\alpha_j$ might not be contained in the vertex complex $V_{d-1}$, (a copy $\tilde{X}_{d-1}$) that contains $T_j$. To continue the proof by induction, we would need the hypothesis of the induction to be that the length of an $r$-detour path over a $(d - 1)$-corner in $\tilde{X}_d$ is bounded below by $p_{d-1}(r)$. This assumption is more general than our original statement, which brings the following additional technical difficulty to the proof. If $\alpha_T \subset \tilde{X}_d$ is a detour path over a $(d - 1)$-corner $(\gamma_{d-1, \gamma_0})_T$, where $T$ is contained in a vertex complex $V_{d-1}$, and if we do not require that $\alpha_T \subset V_{d-1}$, then $\alpha_T$ does not necessarily intersect the hyperplanes in $V_{d-1}$ that separate its endpoints, making such detour paths unsuitable for
induction process. We tackle this difficulty by reformulating our statement in terms of almost detour paths (to be defined).

The motivation for our approach is to describe a canonical way to modify an $r$-detour path $\alpha_T$, as above, to obtain a path $\alpha'_T$, of length not more than the length of $\alpha_T$, and such that $\alpha'_T$ intersects all the hyperplanes in $V_{d-1}$ that separate its endpoints. If we can then show that there is a polynomial $p'_{d-1}$ of degree $d-1$ such that the length of $\alpha'_T$ is bounded below by $p'_{d-1}(r)$, then $p'_{d-1}(r)$ would also give a lower bound for the length of $\alpha_T$. The first natural question to consider is if there is a polynomial $p'_{d-1}$ of degree $d-1$ such the length of the closest point projection of $\alpha_T$ to the vertex complex $V_{d-1}$ is bounded below by $p'_{d-1}(r)$. We note that $\pi_{d-1}(\alpha_T)$ might not be a detour path, and could intersect the ball $B(O, r)$ in a collection of segments that are rugged or smooth sides of hyperplanes in $\tilde{X}_d$. It turns out that the projections of detour paths are not, in general, long enough, but if we only allow projections in the cases when they are contained in the rugged sides of hyperplanes, we get the desired lower bound. We proceed with this approach since, as we will show, this is sufficient to obtain paths that behave well under induction.

In the following two subsections we introduce the terminology necessary to define almost detour paths, which will be paths that connect two points on the sphere $S(O, r)$, and intersect the ball $B(O, r)$ only in geodesics of a very particular form, we will call such geodesics legal shortcuts. Every detour path is an almost detour path, but we will show that, given a detour path $\alpha$, we can obtain an almost detour path $\alpha'$, with the same endpoints as $\alpha$, and of length no longer than the length of $\alpha$, and which has the following property: if the closest point projection $\pi_k(\alpha'')$ of an arc $\alpha'' \subset \alpha'$ to a vertex complex $V_k$, $k \geq 1$ and $O \in V_k$, is a $k$-segment $\sigma_k$, then $\pi_k(\alpha'') = \alpha''$. This property is stated and proved in Lemma 5.16, which is the most technical part of this section. Our modified approach will also require us to consider more general corners in addition to the basic ones, and we introduce raised corners in the next subsection.

5.2. Raising rays and lines. We recall that a positive ray is geodesic ray in 1-skeleton of $\tilde{X}_m$ that traces all its edges in a positive direction (3.1).

**Definition 5.4.** A raising $d$-ray $\zeta_d$ in $\tilde{X}_m$, $m \geq d$, is a concatenation $\sigma_d u$ of a $d$-segment $\sigma_d$ and a positive geodesic segment $u$, such that, if $0 \leq t_1 \leq t_2$, and if $u(t_1)$, $u(t_2)$ are contained in the interiors of the edges $e_1$ and $e_2$ respectively, then $d + 1 \leq \text{height}(e_1) \leq \text{height}(e_2)$. 

We call $\sigma_d$ the $d$-segment of $\zeta_d$. We allow for $\sigma_d$ to be a single point, or for $u$ to be an empty path, but not both at the same time. In the case that $u$ is an empty path, $\sigma_d$ cannot be a single point, and is considered to be a raising $d$-ray.

Note that a raising $d$-ray can be either an infinite ray or, a finite segment, in which case we use the term ray to emphasize the importance of the orientation. It follows directly from the definition, and the group presentation, that every raising ray is a local geodesic, and therefore a geodesic.

**Definition 5.5.** Let $\zeta_d$ be a raising $d$-ray and let $\gamma_0$ be a 0-segment, both issuing from a vertex $T \in \tilde{X}_m$, $m \geq d$. We call the pair $(\zeta_d, \gamma_0)_T$ a raised $d$-corner at $T$.

**Definition 5.6.** Let $\sigma_d$ be a $d$-segment in $\tilde{X}_m$, $m \geq d \geq 1$. A geodesic $\bar{u}_1\sigma_du_2$, where $u_1$ and $u_2$ are positive (possibly empty) raising $(d + 1)$-rays, (Figure 5.2) is called a raising $\sigma_d$-line.

![Figure 3](https://via.placeholder.com/150)

*Figure 3. A raising $\sigma_d$-line*

**Remark 5.7.** The following observation is a direct consequence of the definition of a raising $\sigma_d$-line: if $\bar{u}_1\sigma_du_2$ is a raising $\sigma_d$-line and $\sigma'$ a ray contained in $\bar{u}_1\sigma_du_2$ such that the $\sigma'$ traces its first edge $e$ in the positive direction, then all the edges in $\sigma'$ have positive direction and height bigger than or equal to the height of $e$. 
We also note that any subray of a raising $\sigma_d$-line issuing from a point $P$ in $\sigma_d$ is a raising $d$-ray.

**Remark 5.8.** Let $V_i \subseteq \tilde{X}_m$ be a vertex complex in $\tilde{X}_m$, $i \leq m$, let $H_i$ be a hyperplane in $V_i$, and let $\omega$ be a geodesic in 1-skeleton of $\tilde{X}_m$. If $\omega$ intersects $\text{o star}(H_i)$, then the intersection of $\text{star}(H_i)$ and $\omega$ is contained in a single edge labeled $\tilde{a}_i$.

**Proof.** The only edges contained in the o star($H_i$) are edges labeled $\tilde{a}_i$, and, since $\text{o star}(H_i) \cap \omega \neq \emptyset$, we conclude that there is an edge $e$ labeled $\tilde{a}_i$, and a point $Q \in \text{Int}(e) \cap \omega$. If there is a point $Q' \in \text{star}(H_i) \cap \omega$ not contained in $e$, then convexity of $\text{star}(H_i)$ implies that the geodesic connecting $Q$ and $Q'$ is contained in $\text{star}(H_i)$. Moreover, since $\text{star}(H_i)$ embeds isometrically into $\tilde{X}_m$, such a geodesic would intersect the interior of one of the two cubes adjacent to $e$, which contradicts the fact that the segment of $\omega$ connecting $Q$ and $Q'$ is the unique geodesic that connects these two points, and is contained in the 1-skeleton of $\tilde{X}_m$.

The above remark implies that, if $\omega$ also intersects a component $C$ of $V_i \text{\text{o star}}(H_i)$, then it intersects the trace of $H_i$ adjacent to $C$ in a single point.

**Lemma 5.9.** Let $V_i$ be an $i$-vertex complex in $\tilde{X}_m$, $1 \leq i \leq m$, and let $H_i$ be a hyperplane in $V_i$ with the rugged side $\sigma_{i-1}$. If $\omega$ is a raising $\sigma_d$-line, $1 \leq d \leq i$, such that $\omega$ intersects $\sigma_{i-1}$ at a point $P$, and such that $\text{ostar}(H_i) \cap \omega \neq \emptyset$, then $\omega \setminus \text{ostar}(H_i)$ has exactly one component $\omega_P$ containing $P$, and $\omega_P$ is a positive raising $i$-ray issuing at $P$.

If $S$ is any point on $\sigma_{i-1}$, and $\sigma$ the segment of $\sigma_{i-1}$ connecting $S$ and $P$ then the concatenation $\sigma \omega_P$ is a raising $(i-1)$-ray.

**Proof.** Since $\omega$ is a geodesic contained in 1-skeleton, $\omega \cap \text{star}(H_i)$ is contained in a single edge labeled $\tilde{a}_i$ (Remark 5.8). Let $Q$ be a point in $\text{ostar}(H_i) \cap \omega$, and let $\omega_Q$ be the subray of $\omega$, issuing at $Q$, and that contains $P$. Since $P$ is contained in the rugged trace $\sigma_{i-1}$ of $H_i$, $\omega_Q$ traces $\tilde{a}_i$ in the positive direction, and Remark 5.7 implies that it is a positive raising ray. Then $\omega_P = \omega_Q \setminus \text{ostar}(H_i)$ is the component of $\omega \setminus \text{ostar}(H_i)$ containing $P$, and, since it is a subray of $\omega_Q$, is also a positive raising ray.

The last statement of the lemma follows directly from the definition of a raising ray.
5.3. Legal shortcuts and almost detour paths.

**Definition 5.10.** A shortcut is a geodesic contained in the open ball \( B(O, r) \subseteq \tilde{X}_m \). If \( \omega \) is a geodesic such that \( S(O, r) \cap \omega \neq \emptyset \) we call a point \( P \in S(O, r) \cap \omega \) an endpoint of the shortcut \( \omega \cap B(O, r) \).

**Definition 5.11.** Let \( O \) be a point in \( \tilde{X}_m \). A shortcut \( \omega \subseteq \tilde{X}_m \) is \( O \)-legal if it is raising \( \sigma_d \)-line for a \( d \)-segment \( \sigma_d \subseteq V_d \), where \( V_d \subseteq \tilde{X}_m \) is a \( d \)-vertex complex, \( 1 \leq d \leq m \), such that \( O \in V_d \).

The following properties of legal shortcuts are direct consequences of the above definition, and we list them to provide the reader with different aspects of legal shortcuts that we use in our proofs.

1. If \( \omega \subseteq \tilde{X}_m \) is an \( O \)-legal shortcut, and if \( V_i \cap \omega \neq \emptyset \) for a vertex complex \( V_i \subseteq \tilde{X}_m \), containing the point \( O \), then \( V_i \cap \omega \) is also an \( O \)-legal shortcut.
2. If \( V_i \cap \omega = \emptyset \) for a vertex complex \( V_i \subseteq \tilde{X}_m \), containing the point \( O \), then all the edges in \( \omega \) have the height greater or equal to \( i + 1 \).
3. If a raising \( \sigma_d \)-line \( \omega \) for a \( d \)-segment \( \sigma_d \subseteq V_d \) is an \( O \)-legal shortcut, and \( O \in V_d \subseteq V_i \) are vertex complexes in \( \tilde{X}_m \) containing \( O \), then \( \omega \cap V_i \setminus V_d \), is either empty, or consist of one or two positive raising \( i \)-rays.

**Definition 5.12.** A path \( \alpha \subseteq \tilde{X}_m \) is an almost \((r, O)\)-detour path if the intersection \( \alpha \cap B(O, r) \) is a collection of \( O \)-legal shortcuts.

**Definition 5.13.** An almost \((r, O)\)-detour path \( \alpha \subseteq \tilde{X}_m \) over a raised corner \((\zeta_d, \gamma_0)_T \) is an almost \((r, O)\)-detour path with initial endpoint \( P \in \zeta_d \), \( P \notin B(O, r) \), and such that \( \alpha \) intersects \( \gamma_0 \) at a point \( Q \neq O \).

We also require that, if there is a shortcut \( \omega \subseteq \alpha \) that contains both points \( O \) and \( P \), then \( \omega \) is a raising \( \sigma_d \)-line for a \( d \)-segment \( \sigma_d \) (that is, has no edges of height less than \( d \)).

We call the arc \( \alpha' \) of \( \alpha \) connecting the points \( P \) and \( Q \) a truncated almost detour path.

Note that we allow for \( \alpha \) to contain \( O \), and therefore intersect \( \gamma_0 \) multiple times.

5.4. The main result.

**Proposition 5.14.** For every \( d \in \mathbb{N} \) there is a polynomial \( p_d \) of degree \( d \), and with a positive leading coefficient, such that for any \( m, i \) such that
m \geq i \geq d$, and any almost $(r,O)$-detour path $\alpha$ in $\tilde{X}_m$ over a raised $i$-corner $(\zeta_i,\gamma_0)_O$, the length of the corresponding truncated almost detour path $\alpha'$ is bounded below by $p_d(r)$.

**Remark 5.15.** Our definition of an almost detour path over a raised corner implies that $p_d(r) \leq p_d(r')$ for $r \leq r'$.

Since a minimal length $r$-detour path $\alpha$ over a basic $d$-corner $(\gamma_d,\gamma_0)_O$ is also an almost $r$-detour path over $(\gamma_d,\gamma_0)_O$, the statement of Theorem 5.3 follows directly from Proposition 5.14. It remains to prove Proposition 5.14.

### 5.5. Proof of Proposition 5.14

We will prove the proposition by induction on the degree of the polynomial $p_d$ that gives the lower bound on the divergence. We first introduce the necessary terminology, applied throughout the statements and proofs of four lemmas that follow, and conclude the section with the proof of Proposition 5.14.

Let $\alpha$ be a minimal length almost $(r,O)$-detour path in $\tilde{X}_m$ over a raised $i$-corner $(\zeta_i,\gamma_0)_O$. For each $k \in \mathbb{N}$, $i \leq k \leq m$, let $V_k$ be the $k$-vertex complex in $\tilde{X}_m$ containing $O$. Let $\pi_k : \tilde{X}_m \rightarrow V_k$ be the projection onto $V_k$. Let $E = \{e_1,\ldots,e_n\}$ be the set of all edges in $\tilde{X}_m$ such that $\zeta_i \cap \text{Int}(e) \neq \emptyset$, and note that $n \geq r$. Since $\zeta_i$ is a geodesic, we can assume that the distance $d(O,e_j) = j - 1$ for an edge $e_j$ in $E$.

Since $\zeta_i$ is a raising ray, the height $\text{height}(e_p) \leq \text{height}(e_j)$ for edges $e_p,e_j$ in $E$ such that $p \leq j$, and any edge $e_j$ labeled $\tilde{a}_k$ is contained in $V_k$. Let $H_j$ be the hyperplane in $V_k$ that corresponds to such an $e_j$.

Then $e_j = \pi_k(e_j)$ is contained in $\pi_k(\zeta_i)$ and the hyperplane $H_j$ separates $O$ and $\pi_k(P)$, and therefore also $Q$ and $\pi_k(P)$. Since $Q$ and $\pi_k(P)$ are the endpoints of $\pi_k(\alpha)$, $H_j$ intersects $\pi_k(\alpha)$. The following lemma establishes that, if $\alpha$ is a minimal length almost detour path, as assumed, then $\alpha$ intersects each $H_j$.

We remark that the lemma describes the procedure in which, by replacing arcs of $\alpha$ that do not intersect the hyperplanes by shortcuts, any detour path $\alpha$, over a raised $i$-corner, can be modified without increasing length to an almost detour path over a raised $i$-corner that intersects the appropriate hyperplanes. In particular, all the edges in the newly introduced shortcuts have height bigger than or equal to $i$, ensuring that no shortcut in the resulting almost detour path that contains both points $O$ and $P$ will contain edges of height strictly less than $i$. By the same argument, if $\alpha$ is an an
almost detour path over a raised \( i \)-corner, the resulting path is also an almost detour path over a raised \( i \)-corner.

**Lemma 5.16.** Let \( \alpha \) be a minimal length \((r, O)\)-detour path in \( \overline{X}_m \) over a raised \( i \)-corner \((\gamma_i, \gamma_0)\).

1. If \( \pi_k(\alpha) \cap \text{ostar}(H) \neq \emptyset \) for a hyperplane \( H \) in \( V_k \), then \( \pi_k(\alpha) \cap \text{ostar}(H) \subseteq \alpha \).
2. If \( H \) is a hyperplane in \( V_k \), where \( k \geq i + 1 \), whose rugged side is contained in the vertex space \( V_{k-1} \), and \( Y_O \) the component of \( V_k \setminus \text{ostar}(H) \) such that \( O \in V_{k-1} \subseteq Y_O \), then \( \pi_k(\alpha) \subseteq Y_O \).

**Proof.** Let \( V_{k-1} \subseteq V_k \) be the \((k-1)\)- and \( k \)-vertex complexes, respectively, that contain the point \( O \), and let \( \pi_k : \overline{X}_m \rightarrow V_k, \pi_{k-1} : \overline{X}_m \rightarrow V_{k-1} \) be the corresponding projections. We note that, since \( \pi_m(\alpha) = \alpha \), Statement 1 is true for \( k = m \), and will first prove that, for any \( k \), Statement 1 implies Statement 2.

If \( H \) is a hyperplane in \( V_k \) whose rugged side \( \sigma_{k-1} \) is contained in the vertex space \( V_{k-1} \), then \( H \) does not separate the endpoints \( \pi_k(P) \) and \( Q \) of \( \pi_k(\alpha) \): by the definition of a raised corner, every hyperplane in \( V_k \), for \( k \geq i + 1 \), that separates \( \pi_k(P) \) and \( Q \) intersects the vertex complex \( V_{k-1} \) in its smooth side. Assume that \( \pi_k(\alpha) \) intersects \( Y_C = V_k \setminus Y_O \), and note that this implies that \( \text{ostar}(H) \cap \pi_k(\alpha) \neq \emptyset \), and Statement 1 of the lemma further implies that \( \text{ostar}(H) \cap \alpha = \text{ostar}(H) \cap \pi_k(\alpha) \neq \emptyset \). Let \( \alpha_A \) be the connected component of \( \alpha \setminus \text{ostar}(H) \) that contains \( P \), and let \( \alpha_B \) be the connected component of \( \alpha \setminus \text{ostar}(H) \) that contains \( Q \). We denote the endpoints of \( \alpha_A \) and \( \alpha_B \) contained in \( \sigma_{k-1} \) by \( A \) and \( B \) respectively. Let \( \alpha' \) be the arc of \( \alpha \) connecting \( A \) and \( B \).

Let \( S \) be either of the points \( A, B \). If \( S \in B(O, r) \), then \( S \) is contained in a legal shortcut \( \omega \) which intersects \( \text{ostar}(H) \cap B(O, r) \). Lemma 5.9 implies that there is a positive raising \( k \)-ray \( u_S \) issuing from \( S \), contained in \( \omega \), and such that \( u_S \) does not intersect \( \text{ostar}(H) \). Since \( u_S \) contains \( S \) and does not intersect \( \text{ostar}(H) \), \( u_S \subseteq \alpha_A \). If \( S \) is not in the open ball \( B(O, r) \), take \( u_S \) to be the trivial path. Let \( u_A \) and \( u_B \) be the two positive raising rays corresponding to the points \( A \) and \( B \) and let \( \sigma \) be the segment of \( \sigma_{k-1} \) connecting \( A \) and \( B \), as in Figure 5.5. The path \( \theta = \overline{\alpha_A} \sigma \overline{u_B} \) is a legal shortcut with initial endpoint in \( \alpha_A \) and terminal endpoint in \( \alpha_B \). The segment \( \sigma \) is the unique geodesic segment connecting \( A \) and \( B \) and if \( \alpha' \neq \sigma \), or, equivalently \( \alpha \cap \text{ostar}(H) \neq \emptyset \), then \( |\alpha'| > |\sigma| \). This would further imply that \( \alpha_A \sigma \alpha_B \) is an almost detour path connecting \( P \) and \( Q \) of length
strictly shorter than the length of $\alpha$. Since $|\alpha_A \sigma \alpha_B| < |\alpha|$ contradicts our assumption that $\alpha$ is an almost detour path of minimal length, we conclude that $\alpha' = \sigma$ and therefore $\pi_k(\alpha) \subseteq Y_O$ as claimed.

We proceed to prove that, if Statement 2 of the lemma holds for $k, i+1 \leq k \leq m$, then Statement 1 holds for $k - 1$. First note that the projection $\pi_{k-1}(S)$ to $V_{k-1}$ of a point $S \notin V_{k-1}$ is contained in an edge labeled $\tilde{a}_{k-1}$ or $\tilde{a}_0$. If such $\pi_{k-1}(S)$ is contained in $\text{ostar}(h)$ for a hyperplane $h$ in $V_{k-1}$, then it is contained in the interior of an edge $e$ labeled $\tilde{a}_{k-1}$, since edges labeled $\tilde{a}_0$ do not intersect $\text{ostar}(h)$.

Let $S$ be a point in $\alpha$ such that $\pi_{k-1}(S)$ is contained in the interior of an edge $e \in V_{k-1}$ labeled $\tilde{a}_{k-1}$. Denote by $\sigma_{k-1}$ the $(k - 1)$-line that contains $e$, let $H$ be the hyperplane in $V_k$ such that $\sigma_{k-1} \subset V_{k-1}$ is the rugged side of $H$, and let $Y_C = V_k \setminus Y_O$, where $Y_O$ is as in Statement 2 of the lemma. We assume $S \neq \pi_{k-1}(S)$, and proceed to show that implies $\pi_k(S) \in Y_C$, creating a contradiction to Statement 2 of the lemma.

Let $\gamma_S$ be the geodesic that connects $S$ and $\pi_{k-1}(S)$, and let $\tau \subset Y_C$ be the smooth side of $H$. Since $H$ is the only hyperplane in $V_k$ such that $e \subset \text{star}(H)$, $\gamma_S$ intersects $\text{ostar}(H)$.

Figure 4. Shortening an almost detour path.
If $S \in \text{ostar}(H)$, then $\pi_k(S) = S \in Y_C$. If $S \notin \text{ostar}(H)$, then $\gamma_S$ intersects $\tau$. If $S \in V_k$, then $\gamma_S \subset V_k$, and, since $\gamma_S$ cannot intersect $\tau$ twice, $\pi_k(S) = S \in Y_C$.

If $S \notin V_k$, let $L \in \{k, \ldots, m - 1\}$ be such that $S \in V_L$ and $S \notin V_{L-1}$. For $k \leq l < L$, let $H_l$ be the hyperplane in $V_{l+1}$ adjacent to $V_l$, and let $\gamma^l_D$ be a geodesic in $V_L$ connecting $S$ and a point $D$ in $V_l$. Denote by $Y'O$ the component of $V_{l+1}\setminus\text{ostar}(H_l)$ such that $O \in V_l \subset Y'O$, and let $Y'C = V_{l+1}\setminus Y'O$.

Since $S = \pi_L(S) \subset Y'L^{-1}$, Statement 2 implies that $V_{L-1}$ contains the smooth trace $\tau_{L-1}$ of $H^{L-1}$, and we also note that $\pi_{L-1}(S) \in \tau_{L-1}$. Separation properties of hyperplanes in $V_L$ imply that any geodesic $\gamma^L_D$ intersect $\tau_{L-1}$.

If we assume that $\pi_l(S) \in \tau_l \subset Y'_C$, and that any geodesic $\gamma^l_D$ intersect $\tau_l \subset \text{star}(H_l)$, Statement 2 implies that $V_{l-1}$ contains the smooth trace $\tau_{l-1}$ of $H^{l-1}$, and separation properties of hyperplanes in $V_l$ imply that any geodesic $\gamma'^{l-1}_D$ intersect $\tau_{l-1}$. We, deduce, by induction that the $V_l$ contains the smooth trace $\tau_l$ of $H^l$ for every $k \leq l < L$, that $\pi_l(S) \in \tau_l(S)$, and also that, for every such $l$, any $\gamma^l_D$ intersects $\tau_l$, see Figure 5.5. We turn

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{Projections onto vertex complexes}
\end{figure}

our attention back to $\gamma_S$, concluding, based on the above discussion, that it intersect $\tau_k$, and that $\tau_k$ also contains $\pi_k(S)$. Since $\gamma_S$ cannot intersect
\[ \tau \text{ twice } \tau \text{ is contained in the component of } V_k \setminus \text{ostar}(H) \text{ that contains } \tau_k, \]

which is \( Y_C \), and therefore \( \pi_k(S) \in Y_C \).

Since the rugged side of \( H \) is contained in \( V_{k-1} \), this provides the contradiction, which we aimed for, to Statement 2 of the lemma that \( \pi_k(\alpha) \cap Y_C = \emptyset \). We conclude that \( S = \pi_{k-1}(S) \), establishing our claim that \( \pi_{k-1}(\alpha) \cap \text{ostar}(h) \) is contained in \( \alpha \). \( \square \)

We highlight the following straightforward corollary of the second statement of Lemma 5.16.

**Corollary 5.17.** If \( H \) is a hyperplane in \( V_k \), where \( k \geq i + 1 \), whose rugged side is contained in the vertex space \( V_{k-1} \) then \( \alpha \) does not intersect \( H \).

**Lemma 5.18.** We may assume that no proper arc of \( \alpha \) is an almost detour path over a raised \( i \)-corner, \( i \geq d \), based at \( O \).

*Proof. If there is such a proper arc \( \alpha' \), we can replace \( \alpha \) by \( \alpha' \). Since \( \alpha \) has finite length and \( |\alpha'| < |\alpha| \), this process terminates. \( \square \)*

In particular, Lemma 5.18 implies that there are no raising \( i \)-rays, \( i \geq d \), issuing at \( O \) and intersecting \( \alpha \setminus B(O,r) \) at any point other than \( P \). We next show that such an almost detour path does not intersect \( \zeta_i \) at any other points except \( P \) and, possibly, \( O \).

**Lemma 5.19.** \( \alpha \cap \zeta_i \subset \{O, P\} \).

*Proof. If \( \alpha \) intersects \( \zeta_i \) in a point other then \( \{O, P\} \), then \( \alpha \) contains a shortcut \( \omega \) that intersects \( \zeta_i \) in a point other then \( \{O, P\} \).

We first consider the case that such a shortcut \( \omega \) contains \( O \). Then \( O \) separates \( \omega \) into two components \( \omega_1 \) and \( \omega_2 \). If one of the components, say \( \omega_1 \), contains \( P \) then \( \omega \) is an raising \( i \)-line, and \( \omega_2 \) is a raising \( i \)-ray that intersects \( \alpha \setminus B(O,r) \) at a point other than \( P \). If neither of the rays \( \omega_1, \omega_2 \) contain \( P \), but if one of them, say \( \omega_1 \), intersects \( \zeta_i \) at a point other than \( O \), then \( \omega_1 \) is a raising \( i \)-ray that intersects \( \alpha \setminus B(O,r) \) at a point other than \( P \).

In either case, we have a contradiction with Lemma 5.18.

If \( \omega \) does not contain \( O \), and intersect \( \zeta_i \) at point(s) different then \( P \), let \( Z_0 \neq P \) be the point in \( \omega \cap \zeta_i \) closest to \( O \). \( Z_0 \) separates \( \alpha \) into arcs \( \alpha_P \) and \( \alpha_Q \) containing \( P \) and \( Q \) respectively, and also separates \( \omega \) into two components. We denote the component of \( \omega \setminus \{Z_0\} \) contained in \( \alpha_Q \) by \( \omega_0 \). We consider \( \omega_0 \) as a ray issuing from \( Z_0 \).

Let \( \delta \) be the segment of \( \zeta_i \) connecting \( O \) and \( Z_0 \), and let \( e_j \) be the edge in \( \delta \) such that \( Z_0 \) is an endpoint of \( e_j \). We want to show that the height
of every edge in $\omega$ is greater than or equal to the height of $e_j$. We start by proving that $\omega$ does not intersect $H_j$, the hyperplane that corresponds to the edge $e_j$. By the definition of the point $Z_0$ and the segment $\delta$, $\omega$ does not intersect $e_j$ at any point other than $Z_0$. If $\omega$ intersects $H_j$ at a point $Z_1 \notin e_j$, then the segment of $\omega$ connecting $Z_0$ and $Z_1$ would be the geodesic between $Z_0$ and $Z_1$ in $\text{star}(H_j)$. Since such a geodesic is not contained in 1-skeleton, it cannot be a part of a shortcut.

Let $h_j$ be the height of the edge $e_j$, and, consistent with our notation, let $V_{h_j}$ be the $h_j$-vertex complex containing $O$. Since $\omega$ does not intersect $H_j$, the closest point projection $\pi_{h_j}(\omega)$ does not intersect $H_j$ either (Lemma 5.16, statement (1)). Note that $H_j$ separates $V_{h_j}$ into two components, one of which contains $O$ and $V_{h_j-1}$, and the other one containing $Z_0$. Since $\pi_{h_j}(\omega)$ contains $Z_0$ and does not intersect $H_j$, it does not intersect $V_{h_j-1}$ either. This further implies that, since $\pi_{h_j}(\omega \cap V_{h_j}) = \omega \cap V_{h_j}$, $\omega$ does not intersect $V_{h_j-1}$.

Since $\omega$ is an $O$-legal shortcut, the observation (item 2) after definition 5.11 implies that the height of every edge in $\omega$ is greater or equal to the height of $e_j$, which is greater or equal to $i$.

If the height of the first edge $e$ that $\omega_0$ traces is equal to the height of $e_j$, then $\delta \omega_0$ traces $e_j$ and $e$ in the same direction. If the height of $e$ is strictly larger than the height of $e_j$, then Corollary 5.17 implies that $\omega_0$ traces all its edges in positive direction. This, together with properties of legal shortcuts, makes $\delta \omega_0$ a raising $i$-ray that intersects $\alpha \setminus B(O,r)$ at a point different than $P$, contradicting Lemma 5.18 again.

The following lemma establishes the base of induction for our inductive proof of Proposition 5.14.

**Lemma 5.20.** Let $\alpha'$ be a minimal length almost $(r,O)$-detour path over a raised $d$-corner $(\zeta_d, \gamma_0)$ based at $O$, with the initial endpoint $P \in \zeta_d$. Let $Q$ be the point of intersection of $\alpha'$ and $\gamma_0$, and let $\alpha$ be the arc of $\alpha'$ connecting $P$ and $Q$. Then the length of $\alpha$ is at least $r - 1$.

**Proof.** The claim follows directly from the above observation that $\alpha$ intersects both traces of $\text{star}(H)$ for every hyperplane $H_j$, $1 \leq j \leq n - 1$: the distance between such two intersections is at least 1, which implies that the length of $\alpha$ is at least $r - 1$. 

We can now complete the proof of Proposition 5.14.
By Lemma 5.20, for $i \geq 1$, the length of a truncated almost $r$-detour path over a raised $i$-corner is at least $p_1(r) = r - 1$, which establishes the base of induction. We proceed to prove that the existence of a polynomial $p_{d-1}$ of degree $d-1$, where $d \geq 2$, as in the statement of the proposition, implies the existence of a polynomial $p_d$.

Let $\zeta_i = \sigma_i u_{i+1}$, where $\sigma_i$ is an $i$-segment and $u_{i+1}$ is a raising $(i+1)$-ray. For each edge $e_j$ in $\sigma_i u_{i+1}$, $j \leq n-1$, we denote by $T_j$ the terminal endpoint of $e_j$. Since $\zeta_i$ traces each edge $e_j$ in $u_{i+1}$ in positive direction, the point $T_j$ lies in the rugged trace of the hyperplane corresponding to $e_j$, and on the smooth trace of the hyperplane corresponding to the edge $e_{j+1}$. If $\zeta_i$ also traces the edges in $\sigma_i$ in positive orientation, the same conclusion holds for all $T_j$ in $\{T_1, \ldots, T_{n-1}\}$: $T_j$ is contained in the rugged trace of the hyperplane $H_j$ and the smooth side of the hyperplane $H_{j+1}$, as illustrated in Figure 5.5.

![Figure 6](image_url)

**Figure 6.** Intersections of hyperplanes and an almost detour path $\alpha$ when $\zeta_i$ traces the edges in $\sigma_i$ in positive direction.

If $\zeta_i$ traces the edges in $\sigma_i$ in negative direction, as illustrated in Figure 5.5, then, for $j \geq 2$, the vertex $T_j$ which is the terminal endpoint of an edge $e_j \in \sigma_i$ lies in the rugged trace of the hyperplane corresponding to the edge $e_j$, and on the smooth trace of the hyperplane corresponding to the edge...
\(e_{j-1}\). The vertex \(T_1\) lies on the rugged trace of the hyperplane corresponding to the edge \(e_1\) and on the smooth path \(\gamma_0\).

We will show that there is a collection \(\{\alpha_j|j \in \{1,\ldots,n-1\}\}\) of disjoint arcs \(\alpha_j\) of \(\alpha\), such that \(\alpha_j\) is a truncated almost detour path over a raised \((i_j-1)\)-corner, \(i_j \geq d\), based at \(T_j \in V_k\).

Let \(q\) to be the element of \(\{0,1,\ldots,n-1\}\) defined in the following way. If \(\zeta_i\) traces all the edges \(e_j, j \in \{1,\ldots,n-1\}\), in the positive direction, we let \(q = 0\). If \(\zeta_i = \sigma_i\) and if it traces all the edges \(e_j, j \in \{1,\ldots,n-1\}\), in the negative direction, we let \(q = n - 1\). Otherwise, let \(q\) be such that \(\zeta_i\) traces each edge \(e_j\) in negative direction for \(j \leq q\), and in positive direction the for \(j > q\). Let \(S_n = P\), and \(S_0 = Q\). For each \(j \in \{1,\ldots,n-1\}\) we inductively define a point \(S_j\) in the rugged side of the hyperplane \(H_j\) in the following way. If \(j\) is such that \(0 \leq q < j \leq n - 1\), let \(S_j\) to be the point of intersection of \(\alpha\) and the rugged side of the hyperplane \(H_j\) such that the arc \(\beta_j\) of \(\alpha\) connecting \(S_j\) and \(P\) does not intersect \(H_j\). If \(1 \leq j \leq q\) choose \(S_j\) be the point in the intersection of \(\alpha\) and the rugged side of the hyperplane \(H_j\) such that the arc \(\beta_j\) of \(\alpha\) connecting \(S_j\) and \(Q\) does not intersect \(H_j\).

Lemma 5.19 implies that \(S_j \neq T_j\)
If $S_j$ is outside the ball $B(O, r)$ we let $P_j = S_j$. If $S_j \in B(O, r)$, then it is contained in a shortcut $\omega$ such that $\omega \cap \text{ostar}(H_j) \neq \emptyset$, and, since $S_j$ is contained in the rugged side of $H_j$, Lemma 5.16 implies that there is an endpoint $P_j$ of $\omega$ such that the oriented segment of $\omega$ connecting $S_j$ and $P_j$ is an positive raising ray which does not intersect $H_j$, and such that the geodesic connecting $T_j$ and $P_j$ is a raising $(i_j - 1)$-ray. We note that $P_j$ is contained in $\beta_j$.

Let $\zeta_j^j$ be the raising $(i_j - 1)$-ray connecting $T_j$ and $P_j$, and let $\gamma_0^j$ the 0-ray issuing at $T_j$. We note that, for $j > q$, $\zeta_j^j$ does not contain $S_{j+1}$: if it did, the segment of $\zeta_j^j$ between $T_j$ and $S_{j+1}$ would be the unique geodesic segment connecting $T_j$ and $S_{j+1}$, and, the definition of $S_{j+1}$ together with our observation that $S_{j+1} \neq T_{j+1}$, imply that such geodesic segment is not contained in 1-skeleton. By the same argument, $S_{j-1}$ is not contained in $\zeta_j^j$ for $j \leq q$.

We let $\alpha_j$ be the arc of $\beta_j$ connecting $P_j$ and $P_{j+1}$ in the case $j \geq q$, and the arc of $\beta_j$ connecting $P_j$ and $P_{j-1}$ for $j < q$. The above discussion implies that the point $P_j$ is contained in the arc of $\beta_j$ connecting $S_j$ and $S_{j+1}$ for $j > q$, and in the arc of $\beta_j$ connecting $S_j$ and $S_{j-1}$ for $j \leq q$, and therefore the arcs $\alpha_j$ are disjoint. Each $\alpha_j$ is an almost detour path and Lemma 5.19 implies that $\alpha_j$ intersects $\gamma_0^j$ at a point different than $T_j$. Therefore, for every $j \in \{1, \ldots, n-1\}, \alpha_j$ is a subarc of $\alpha$, which is a truncated almost detour path over a $(i_j - 1)$-corner $(\zeta_j^j, \gamma_0^j)$ and Since $d(O,T_j) \leq j$, and $\alpha$ is an an almost $(r, O)$-detour path, $d(T_j, A) \geq r - j$ for every point $A \in \alpha_j$ which is not contained in a legal shortcut. Therefore $\alpha_j$ is an $(r - j)$-almost detour path over a raised $i_j - 1$-corner, for $i_j \geq d$. By the hypothesis of the induction $|\alpha_j| \geq p_{d-1}(r - j)$. Then the length of

$$|\alpha| \geq \sum_{j=1}^{n-1} |\alpha_j| \geq \sum_{j=1}^{n-1} p_{d-1}(r - j),$$

and $p_d(r) = \sum_{j=1}^{n-1} p_{d-1}(r - j)$ is a polynomial of degree $d$ which is a lower bound for the detour function of $\tilde{X}_m$.

References

[1] A. Abrams, N. Brady, P. Dani, and M. Duchin. Pushing fillings in right-angled artin groups. ArXiv:1004.4253, 2010.
[2] J. Behrstock and R. Charney. Divergence and quasimorphisms of right-angled artin groups. Mathematische Annalen, (352):339–356, 2012.
[3] M. Bridson and A. Haefliger. *Metric spaces of nonpositive curvature*. Grundlehren der Mathematischen Wissenschaften, Vol. 319, Springer-Verlag, Heidelberg, 1999.

[4] W. Dison and T. Riley. Hydra groups. arXiv:1002.1945v2, 2010.

[5] S. M. Gersten. Quadratic divergence of geodesics in $CAT(0)$–spaces. *Geometric and Functional analysis*, 4(1):37–51, 1994.

[6] G.C. Hruska. Nonpositively curved 2-complexes with isolated flats. *Geom. Topol.*, 8(205-275), 2004.

[7] M. Kapovich and B. Leeb. 3-manifold groups and nonpositive curvature. *Geometric and Functional Analysis*, 8(5):841–852, 1998.

[8] N. Macura. Detour functions and quasi-isometries. *Quarterly Journal of Mathematics*, 53(2):207–239, 2002.