Boundary structure of convex sets in the hyperbolic space

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Received: 8 November 2017 / Accepted: 22 May 2018 / Published online: 11 June 2018
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Abstract We prove some results concerning the boundary of a convex set in $\mathbb{H}^n$. This includes the convergence of curvature measures under Hausdorff convergence of the sets, the study of normal points, and, for convex surfaces, a generalized Gauss equation and some natural characterizations of the regular part of the Gaussian curvature measure.

Keywords Convex sets · Hyperbolic space · Curvature measures · Gauss equation

Mathematics Subject Classification 52A55 · 53C45

1 Introduction

In this short note, we discuss some properties of the boundary of convex sets with non-empty interior in the real $n$-dimensional hyperbolic space $\mathbb{H}^n$. Most of the results we will present are well-known for convex sets in the Euclidean space $\mathbb{R}^n$ (see [23] for a general reference on the subject) and hence they are naturally expected to be true in $\mathbb{H}^n$ too. However, we have not found detailed proofs in the literature.

First, we prove that the curvature measures introduced by Kohlmann via a Steiner polynomial type formula, [18], are stable under Hausdorff convergence of convex sets (see Sect. 2). In dimension 2, this can be applied to get the validity of a generalized
Gauss–Codazzi equation for the boundary of a convex set in $\mathbb{H}^3$, giving as in the smooth case the equivalence of the 2nd curvature measure and of the intrinsic curvature measure (in the sense of BIC surfaces) up to a constant factor (see Theorem 11). This latter in turn guarantees that Kohlmann’s 2nd curvature measure we use here is the same as the extrinsic measure of convex sets of $\mathbb{H}^3$ introduced by Alexandrov, whose construction exploits the local euclidean character of $\mathbb{H}^n$, [3, p 397].

In a second part we will deal with the regularity of boundary points, proving that a.e. point is normal. This roughly means that the neighborhood of a.e. point in the boundary can be suitably approximated by a smooth surface. This permits to define a concept of local curvature $\text{LocCurv}(q)$ at every normal point $q$ (see Corollary 9). Using Gauss equation and the smooth approximation around normal points we will finally proof that $\text{LocCurv}(q)$ is the regular part, given by Lebesgue decomposition theorem, of the curvature measure (see Sect. 6). Since every surface with lower bounded curvature is locally isometric to the boundary of a convex set in a space-form, the local curvature $\text{LocCurv}(q)$ can be used to characterize almost everywhere the Gaussian curvature with an approach which differs from the one followed by Machigashira, [19], who exploited in his definition the upper excess of geodesic triangles.

Throughout this paper, we will consider either local properties of (arbitrary) convex sets or global properties of compact convex sets. In a different direction, it should be recalled that a rich theory has been developed to study the boundary structure of (a special class of) non-compact convex sets in $\mathbb{H}^n$, notably when $n = 3$, [6,13,24,25] and [7, sections 3.6 and 6.8.4]. The central object are sets which arise as convex hulls $\mathcal{C}(\Lambda)$ of a given subset $\Lambda \subset S^{n-1}$, where $S^{n-1}$ is seen as the boundary sphere at infinity of $\mathbb{H}^n$. This theory is based on the observation that hyperbolic convex hulls can be considered as Euclidean convex hulls in the Klein projective model of the hyperbolic space. The boundaries of 3-dimensional convex hulls are locally trivial by an intrinsic metric viewpoint, since the boundary metric is hyperbolic and the intrinsic curvature measure is thus constant. However they can present rich extrinsic and global structures.

2 Convergence of curvature measures

Let $\mathcal{K}(\mathbb{H}^n)$ be the set of compact convex sets in $\mathbb{H}^n$ with nonempty interior. For any $K \in \mathcal{K}(\mathbb{H}^n)$ and $\rho > 0$ define the set

$$K_\rho := \{ x \in \mathbb{H}^n : d_{\mathbb{H}^n}(x, K) \leq \rho \}.$$

The maps $f_K : \mathbb{H}^n \setminus K \to \partial K$ and $F_K : \mathbb{H}^n \setminus K \to T_{\partial K} \mathbb{H}^n$ are defined by the relations

$$d_{\mathbb{H}^n}(f_K(x), x) = d_{\mathbb{H}^n}(x, K) \quad \text{and} \quad x = \exp_{f_K(x)}(d(K, x)F_K(x)),$$

and are well-defined since $K$ is convex. For $\beta \subset \mathbb{H}^n$, define also

$$M_\rho(K, \beta) = f_K^{-1}(\beta \cap \partial K) \cap (K_\rho \setminus K).$$
Following [18], given a convex set $K$ and $\rho > 0$, let us define a Radon measure $\mu_{\rho}$ on the Borel $\sigma$-algebra of the hyperbolic space $B(\mathbb{H}^n)$ by

$$
\mu_{\rho}(K, \beta) = \operatorname{Vol}_{\mathbb{H}^n}(M_{\rho}(K, \beta)).
$$

Set

$$
\ell_{n+1-r}(t) := \int_0^t \sinh^{n-r}(x) \cosh^r(x) dx, \quad r = 0, \ldots, n.
$$

Then P. Kohlmann proved the following Steiner-type formula.

**Theorem 1** (Theorem 2.7 in [18]) There exists a family $\{\Phi_{r}(K, \cdot)\}_{r=0}^{n}$ of measures on $B(\mathbb{H}^n)$ such that

$$
\mu_{\rho}(K, \beta) = \sum_{r=0}^{n} \ell_{n+1-r}(\rho) \Phi_{r}(K, \beta), \quad \forall \beta \in B(\mathbb{H}^n). \tag{1}
$$

To our purpose, it is worth that, whenever $\eta = \partial K \cap \beta$ is a $C^3$ surface, the Borel measures $\Phi_{r}(K, \cdot)$ have the nice expression

$$
\Phi_{r}(K, \beta) = \binom{n}{r} \int_{\eta} H_{n-r}^{K}(q) d\sigma_{\partial K}(q), \tag{2}
$$

where $\sigma_{\partial K}$ is the surface measure on $\partial K$ induced by $\operatorname{Vol}_{\mathbb{H}^n}$. Hence Theorem 1 recovers the classical regular Steiner formula in $\mathbb{H}^n$, [4]. Here, $H_{k}^{K}(q)$ is the $k$-th symmetric function of the principal curvatures of $\partial K$ at $q$. In particular $H_{0}^{K} = 1$, $H_{1}^{K}$ is the mean curvature of $\partial K \subset \mathbb{H}^n$ and $H_{n}^{K}$ its Gaussian curvature.

**Remark 2** For completeness, we recall that Kohlmann proved also for non-regular convex sets a more general integral representation for the Borel measures $\Phi_{r}(K, \cdot)$. This is given by

$$
\Phi_{r}(K, \beta) = \binom{n}{r} \int_{\Pi^{-1}(\beta) \cap \mathcal{N}_{K}} g(v) \tilde{H}_{n-r}(v) d\mathcal{H}^n(v), \quad r = 0, \ldots, n,
$$

where

- $\mathcal{N}_{K} \subset T_{\mathbb{H}^n}$ is the unitary normal bundle along $\partial K$,
- $\Pi : T\mathbb{H}^n \rightarrow \mathbb{H}^n$ is the standard projection on the base point,
- the product $g(v) \tilde{H}_{r}(v)$ is defined for $\mathcal{H}^n$-a.e. $v \in \mathcal{N}_{K}$ by the algebraic expressions

$$
\tilde{H}_{r}(v) := \binom{n}{r}^{-1} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \prod_{j=1}^{r} \tilde{k}_{i_j}(v),
$$

where $\tilde{k}_{i}(v)$ is the $i$-th symmetric function of the principal curvatures of $\partial K$ at $v$.\[\square\]
and
\[ g(v) := \infty^{-s} \prod_{i=s+1}^{n} \sqrt{\tilde{k}_{i}^{2}(v) + 1}. \]

- the (almost everywhere defined) generalized principal curvatures \( \tilde{k}_i(v) \) can be roughly seen as limits as \( \epsilon \to 0 \) of the principal curvatures of the parallel sets \( \partial K_{\epsilon} \) at point \( \exp \Pi(v)(\epsilon v) \), and \( s = s(v) \) is such that \( \tilde{k}_i(v) = \infty \) if and only if \( i \leq s \).

The pretty involved rigorous definitions and further details can be found in [18, Sections 1 and 2].

As in the classical setting of convex bodies in Euclidean space, [23], the curvature measures introduced by Kohlmann are solid enough to be weak continuous with respect to the topology induced on \( K(H^n) \) by the Hausdorff distance
\[ \text{dist}_{H}(K, L) := \inf\{\lambda > 0 : K \subset B_\lambda(L) \text{ and } L \subset B_\lambda(K)\}, \]
with \( B_\lambda(L) := \{q \in H^n : d_{H^n}(q, L) < \lambda\} \).

**Theorem 3** Let \( \{K_j\}_{j=1}^{\infty} \subset K(H^n) \) be a sequence of convex sets such that \( K_j \to K \) as \( j \to \infty \) in the Hausdorff topology. Then for every \( r = 0, \ldots, n \) we have
\[ \Phi_r(K_j, \cdot) \to \Phi_r(K, \cdot) \]
as \( j \to \infty \), weakly in the sense of measure.

According to the Steiner formula (1), this latter theorem is a direct consequence of the following

**Proposition 4** Let \( \{K_j\}_{j=1}^{\infty} \subset K(H^n) \) be a sequence of convex sets such that \( K_j \to K \) as \( j \to \infty \) in the Hausdorff topology. Then, for every \( \rho > 0 \),
\[ \mu_\rho(K_j, \cdot) \to \mu_\rho(K, \cdot) \]
as \( j \to \infty \), weakly in the sense of measure.

**Proof** We start proving the following lemma

**Lemma 5** Let \( K, L \subset H^n \) be closed convex sets and \( x \in H^n \).
Let \( d = \min\{d_{H^n}(x, f_K(x)); d_{H^n}(x, f_L(x))\} \) and \( \delta = \text{dist}_H(K, L) \). Then
\[ d_{H^n}(f_K(x), f_L(x)) \leq \arccosh \left( \frac{\cosh(d + \delta)}{\cosh(d - \delta)} \right), \]
whenever \( \delta < d \). In particular
\[ d_{H^n}(f_K(x), f_L(x)) \leq 2(tanh(d))^{1/2} \delta^{1/2} + O(\delta), \quad \text{as } \delta \to 0. \]
lemma 1.8.9. the inequality is trivial if \( x \in K \cap L \). If \( x \in K \setminus L \), then
\[
d_{\mathbb{H}^n}(f_K(x), f_L(x)) = d_{\mathbb{H}^n}(x, f_L(x)) = d_{\mathbb{H}^n}(x, L) \leq \delta,
\]
so that (3) is satisfied. So let us suppose \( x \notin (K \cup L) \) and \( d = d_{\mathbb{H}^n}(x, f_K(x)) \). By definition of \( \delta \), we have \( L \cap B_{\delta}(f_K(x)) \neq \emptyset \), which implies \( d_{\mathbb{H}^n}(L, x) \leq d + \delta \) and \( f_L(x) \in B_d + \delta \). Let \( \gamma_1 \) be the minimizing geodesics connecting \( f_L(x) \) and \( f_K(f_L(x)) \). It holds \( L(\gamma_1) = d_{\mathbb{H}^n}(f_L(x), K) \leq \delta \). Let \( P \) be the totally geodesic hyperbolic hyperplane which is a support plane to \( K \) at \( f_K(x) \) and is orthogonal to \( F_K(x) \).

Suppose first that \( f_L(x) \) and \( f_K(x) \) are in the same half-space with respect to \( P \). Then the angle \( \alpha := x f_K(x) f_L(x) \geq \pi / 2 \) so that by the hyperbolic cosine law
\[
\cosh(d_{\mathbb{H}^n}(f_K(x), f_L(x))) \cosh(d_{\mathbb{H}^n}(f_K(x), x)) \leq \cosh(d_{\mathbb{H}^n}(x, f_L(x))) \leq \cosh(d + \delta).
\]
Hence
\[
\cosh(d_{\mathbb{H}^n}(f_K(x), f_L(x))) \leq \frac{\cosh(d + \delta)}{\cosh(d)} \leq \frac{\cosh(\delta)}{\cosh(d - \delta)}.
\]

Suppose now that \( f_L(x) \) and \( f_K(x) \) are in opposite half-spaces with respect to \( P \). Let \( b \) be the intersection point of \( \gamma_1 \) with \( P \), so that \( d_{\mathbb{H}^n}(b, f_L(x)) \leq \delta \). Let \( E \) be the point obtained by projecting \( f_L(x) \) onto the (convex) geodesic \( \gamma_2 \) of \( \mathbb{H}^n \) which pass through \( x \) and \( f_K(x) \). Note that \( f_K(x) \) is the projected of \( B \) onto \( \gamma_2 \). By the Hyperbolic Busemann-Feller Lemma, [10, Proposition II.2.4], projection onto a complete convex set in \( \mathbb{H}^n \) is distance decreasing. In particular \( d_{\mathbb{H}^n}(E, f_K(x)) \leq d_{\mathbb{H}^n}(b, f_L(x)) \leq \delta < d \), and \( E \) is between \( x \) and \( f_K(x) \) on \( \gamma_2 \). Then \( d_{\mathbb{H}^n}(x, E) = d_{\mathbb{H}^n}(x, f_K(x)) - d_{\mathbb{H}^n}(E, f_K(x)) \geq d - \delta \). Pythagoras’ theorem for hyperbolic triangles gives
\[
\cosh(d_{\mathbb{H}^n}(x, f_L(x))) = \cosh(d_{\mathbb{H}^n}(E, f_L(x))) \cosh(d_{\mathbb{H}^n}(x, E))
\]
and
\[
\cosh(d_{\mathbb{H}^n}(f_K(x), f_L(x))) = \cosh(d_{\mathbb{H}^n}(E, f_L(x))) \cosh(d_{\mathbb{H}^n}(f_K(x), E)).
\]
All together implies
\[
\cosh(d_{\mathbb{H}^n}(f_K(x), f_L(x))) \leq \cosh(\delta) \frac{\cosh(d + \delta)}{\cosh(d - \delta)}
\]
as desired. \( \square \)

We come back to the proof of Proposition 4, inspired by [23, Theorem 4.1.1]. Let \( \{K_j\}_{j=1}^{\infty} \subset K(\mathbb{H}^n) \) be a sequence of convex sets such that \( K_j \rightarrow K \) as \( j \rightarrow \infty \) in the Hausdorff topology. Let \( U \subset \mathbb{H}^n \) be an open set and let \( q \in M_B(K, U) \setminus \partial K_p \), so that \( f_K(q) \in U \). According to Lemma 5, for \( j \) large enough \( f_K_j(q) \in U \) and
\[ d_{\mathbb{H}^n}(q, K_j) \leq d_{\mathbb{H}^n}(q, K) + O((\text{dist}_H(K, K_j))^{1/2}) < \rho. \] Then for \( j \) large enough \( q \in M_\rho(K_j, U) \), so that \( M_\rho(K, U) \setminus \partial K_\rho \subset \liminf_{j \to \infty} M_\rho(K_j, U). \) By Fatou’s lemma

\[ \mu_\rho(K, U) = \text{Vol}_{\mathbb{H}^n}(M_\rho(K, U) \setminus \partial K_\rho) \leq \liminf_{j \to \infty} \mu_\rho(K_j, U). \tag{4} \]

On the other hand, let \( \delta_j = \text{dist}_H(K, K_j) \). Then, reasoning as above, \( M_{\rho+\delta_j}(K, \mathbb{H}^n) \subset M_\rho(K, \mathbb{H}^n) = M_{\rho+\delta_j}(K, \mathbb{H}^n), \) where \( \delta_j \to 0 \) as \( j \to \infty. \) Since \( M_\rho(K, \mathbb{H}^n) \in \mathcal{K}(\mathbb{H}^n) \), by Theorem 1

\[ \mu_{\rho_j}(M_\rho(K, \mathbb{H}^n), \mathbb{H}^n) = \mu_\rho(K, \mathbb{H}^n) + O(\delta_j), \]

so that

\[ \limsup_{j \to \infty} \mu_\rho(K_j, \mathbb{H}^n) \leq \mu_\rho(K, \mathbb{H}^n). \]

This latter, together with (4) and standard measure theory, [8, p. 196], concludes the proof. \( \square \)

### 3 Boundary structure

In this section we study the regularity of the boundary of a convex set \( K \in \mathcal{K}(\mathbb{H}^n). \) As in the Euclidean setting, the main tool is the first and second differentiability property of real convex functions at almost every point.

Up to an isometry of the ambient space, the boundary of \( K \) is locally the graph of a (horo)convex function. Namely, let \( q_0 \in \partial K. \) Consider the Poincaré half-space model \( \mathbb{H}^n \) with its global coordinates system \((x, z) \in \mathbb{R}^{n-1} \times (0, +\infty) \) and metric \( g_{\mathbb{H}^n} = z^{-2}(dx^2 + dz^2) \) of constant curvature \(-1\), where \( dx^2 = \sum_{i=1}^{n-1} dx_i^2 \) is the Euclidean metric of \( \mathbb{R}^{n-1}. \) Up to an isometry of \( \mathbb{H}^n \) we can suppose that \( q_0 = (0, 1)_E \) and, since \( K \) has nonempty interior, that \((0, 1 + t) \subset \text{int}(K) \) for \( t \) small enough. Consider the horosphere \( h = \{(x, 1)_E : x \in \mathbb{R}^{n-1}\} \) and note that \( x \mapsto (x, 1)_E \) is an isometry from \( \mathbb{R}^{n-1} \) to \( h. \) As in [15], let us introduce a coordinate system \((\xi, \zeta)_h \) on \( \mathbb{H}^n, \) called horospherical coordinates, defined by \((\xi, \zeta)_h = (\xi, e^{-\zeta})_E \) so that \( h = \{(\xi, 0)_h : \xi \in \mathbb{R}^{n-1} \} \cong \mathbb{R}^{n-1} \) and \( \zeta \) is the signed hyperbolic distance from \((\xi, e^{-\zeta})_E \) to the horosphere \( h, \) which is realized by the “vertical” unit speed geodesic \( t \in [0, \zeta] \mapsto (x, t)_h. \) We have chosen the sign of \( \zeta \) in such a way that \((0, \zeta)_h \notin K(u) \) for \( \zeta > 0 \) small enough. Note also that the hyperbolic metric in horospherical coordinates at \((\xi, \zeta)_h \) is given by

\[ e^{2\zeta} d\xi^2 + d\zeta^2. \tag{5} \]
Thanks to the relative position of $K$ and $h$, there exists $R > 0$ and $u : \mathbb{B}_R \subset \mathbb{R}^{n-1} \rightarrow (0, +\infty)$ such that a neighborhood of $x_0$ in $\partial K$ coincides with the graph

$$g_h(u) := \{(x, u(x))_h \in \mathbb{H}^n : x \in \mathbb{B}_R\}$$

(6)

of $u$ on $\mathbb{B}_R$. Here and on, $\mathbb{B}_R(x) \subset \mathbb{R}^{n-1}$ denotes the Euclidan ball of radius $R$ centered at $x$, and we just write $\mathbb{B}_R$ whenever $x = 0$. Since $K$ is convex, $u$ is horoconvex,1 which means that the function $h_u : \mathbb{B}_R \rightarrow \mathbb{R}$ defined by

$$h_u(x) = e^{-2u(x)} + |x|^2$$

is convex. By the way, this guarantees that both $u$ and $h_u$ are Lipschitz. Conversely, whenever $v : \mathbb{B}_R \subset \mathbb{R}^{n-1} \rightarrow (0, +\infty)$ is horoconvex, then the graph $g_h(v)$ of $v$, defined as in (6), is an open set of the boundary of a convex set $K = K(v) \in \mathcal{K}(\mathbb{H}^3)$. Note that, in order to define the whole convex set $K(v)$ containing $g_h(v)$ (and thus rigorously justify some subsequent arguments), one can for instance choose

$$K(v) = \{(x, z) \in \mathbb{H}^n : x \in \mathbb{B}_R \text{ and } v(x) \leq z \leq \zeta\},$$

(7)

with $\zeta > \max_{\mathbb{B}_R} v$ large enough to be fixed later. The equivalence between convexity of sets and horoconvexity of functions is proved for instance in [17] for smooth convex sets and in [15, Proposition 2.6] for non-necessarily smooth convex sets in $\mathbb{H}^3$. The generalization to any dimension is straightforward.

Since $h_u$ is convex, according to a celebrated theorem by Busemann-Feller and Alexandrov, [2,12], the set of points

$$D_u'' = \{x \in \mathbb{B}_R : h_u \text{ is 2 times differentiable at } x\}$$

has full Lebesgue measure in $\mathbb{B}_R$. Namely, for $x \in D_u''$, the subgradient $\partial_x h_u$ of $h_u$ at $x$, defined by

$$\partial_x h_u := \{v \in \mathbb{R}^{n-1} : \forall y \in \mathbb{B}_R, \ h_u(y) \geq h_u(x) + \langle v, y - x \rangle\},$$

contains a unique element $\nabla_x h_u$ and there exists a symmetric matrix $\nabla^2_x h_u$ such that

$$h_u(y) = h_u(x) + \langle \nabla_x h_u, y - x \rangle + \frac{1}{2} \left\langle \nabla^2_x h_u(y - x), y - x \right\rangle + o(|y - x|^2).$$

An easy computation shows then that also $u$ is two times differentiable at $x$, i.e.

$$u(y) = \bar{u}(y) + o(|y - x|^2),$$

(8)

1 The definition of horoconvex function could be misleading, since there is in the literature a well studied concept of horoconvex sets (also called h-convex or horospherically convex) which denote sets in $\mathcal{K}(\mathbb{H}^n)$ having at each point a supporting horosphere. This is not directly related to horoconvex functions.
where
\[ \tilde{u}(y) = u(x) + \langle \nabla_x u, y - x \rangle + \frac{1}{2} \left( \nabla_x^2 u(y - x), y - x \right), \] (9)
with \( \nabla_x u = -\frac{1}{2} e^{2u(x)} (\nabla_x h_u - 2x) \) and
\[ \nabla_x^2 u = -\frac{1}{2} e^{2u(x)} \nabla_x^2 h_u + e^{2u(x)} I_{\mathbb{R}^n-1} + 2 \nabla_x u \otimes \nabla_x u. \]

In what follows we want to prove that almost every point of the boundary is normal, that is that the second differentiability property at a.e. point of the horoconvex representation is preserved under changes of the reference horosphere.

**Smooth** points are those boundary points admitting a unique support hyperbolic (hyper)plane. The smoothness of a point \( q \in \partial K \) clearly do not depend on the chosen horosphere \( \mathfrak{h} \), since it is equivalent to the fact that the smooth convex surface \( g_{\mathfrak{h}}(\tilde{u}) \) is tangent to \( \partial K \) at \( q \). In particular, by Rademacher’s theorem a.e. boundary point is smooth. So let \( q \in \partial K \) be a smooth point. There is a unique exterior unit normal \( v_q \) to \( \partial K \) at \( q \), and hence a unique interior tangent horosphere \( \mathfrak{h}_q \), that is the horosphere orthogonal to \( v_q \) and such that \( \exp_q(-t v_q) \in \text{int}(K) \) for \( t > 0 \) small enough. Since \( \partial K \) is tangent to \( \mathfrak{h}_q \) at \( q \), it admits a (unique) local parametrization via a horoconvex function \( u_q : \mathcal{B}_\epsilon \subset \mathfrak{h}_q \to \mathbb{R} \), for some small \( \epsilon > 0 \), with
\[ u_q(0) = 0, \quad \text{and} \quad \nabla_0 u_q = 0. \] (10)

We say that \( q \) is a **normal** boundary point if \( u_q \) is two times differentiable at \( q \). By the way, note that at normal points, \( \nabla u_q \) is differentiable, in the sense that
\[ \sup_{x \in \mathcal{B}_\epsilon} \sup_{V \in \partial_x u_q} |V - \nabla_x \tilde{u}_q| = \sup_{x \in \mathcal{B}_\epsilon} \sup_{V \in \partial_x u_q} |V - \nabla_0^2 u_q(x)| = o(\epsilon), \] (11)
see [22, Corollary 2.4]. By properties (10), here the expression for \( \tilde{u}_q \) simplifies as
\[ \tilde{u}_q(x) = \frac{1}{2} \nabla_0^2 u_q(x, x). \]

**Proposition 6** Let \( K \in \mathcal{K}(\mathbb{H}^n) \). Then \( \mathcal{H}^{n-1} \text{-a.e. point in } \partial K \text{ is normal.} \)

For the corresponding result in the Euclidean setting, see [2,12] or [23, Theorem 2.5.5].

**Proof (of Proposition 6)** Let \( \mathfrak{h} \) be a horosphere of \( \mathbb{H}^n \) and \( u : \mathbb{B}_R \subset \mathfrak{h} \to \mathbb{R} \) a horoconvex function locally parametrizing an open set of \( \partial K \). According to what said above, the set \( \mathcal{D}_u^0 \) of two times differentiable points has full Lebesgue measure in \( \mathbb{B}_R \). Since \( u \) is Lipschitz we deduce that \( x \in \mathbb{B}_R \mapsto (x, u(x))_{\mathfrak{h}} \in \partial K \subset \mathbb{H}^n \) is biLipschitz onto its image, and thus \( \{(x, u(x))_{\mathfrak{h}} : x \in \mathcal{D}_u^0\} \) has full Hausdorff measure in \( g_{\mathfrak{h}}(u|_{\mathbb{B}_R}) \subset \partial K \). We are going to show that every such a point is normal.
Fix $x_0 \in D''$. Let $h_1 \cong \mathbb{R}^{n-1}$ be the horosphere $\{(y, \zeta)_{h} \in \mathbb{H}^3 : \zeta \equiv u(x_0)\}$, and let $h_2 \cong \mathbb{R}^{n-1}$ be a horosphere of $\mathbb{H}^n$ tangent to the graph of $u$ at $q := (x_0, u(x_0))_{h}$. As above, we introduce two horospherical coordinates systems $(y, z)_{h_i}$, $i = 1, 2$, on $\mathbb{H}^n$ relative to the horospheres $h_1$ and $h_2$, and fix the Euclidean coordinates on $h_i$ in such a way that $q = (0, 0)_{h_i}$ for $i = 1, 2$. Since both graphs $g_{h_1}(u|_{\mathbb{B}_R})$ and $g_{h_2}(\bar{u}|_{\mathbb{B}_R})$ are tangent to $h_2$ at $0$, by the implicit function theorem they have respective local representations as graphs $g_{h_2}(u')$ and $g_{h_2}(\bar{u}')$ of functions $u', \bar{u}' : \Omega \rightarrow \mathbb{R}$ defined on an open neighborhood $\Omega$ of $0 \in h_2$. Both $u'$ and $\bar{u}'$ are differentiable at $0$ and $\nabla_{0}u' = \nabla_{0}\bar{u}' = 0$. Finally, since $\bar{u} \in C^\infty$, also $\bar{u}' \in C^\infty$.

We will use repeatedly the following

**Lemma 7** Let $\mathfrak{h} \subset \mathbb{H}^n$ be a fixed horosphere, then

$$|a - b| \leq e^{\max\{|c_\mathfrak{a}|, |c_\mathfrak{b}|\}}d_{\mathbb{H}^n}((a, \zeta_\mathfrak{a})_{\mathfrak{h}}, (b, \zeta_\mathfrak{b})_{\mathfrak{h}}).$$  \hfill (12)

**Proof** Since the projection on a convex set is a contraction, [10, Proposition II.2.4], by definition of the induced metric also the projection on the boundary of the convex set is a contraction. The inequality (12) follows from projecting on the convex horosphere $\{(y, e^{-\max\{\zeta_\mathfrak{a}, \zeta_\mathfrak{b}\}}) : y \in \mathbb{R}^{n-1}\}$, together with a simple explicit calculation. \hfill $\square$

Let $(\xi, 0)_{h_2} \in h_2$ such that $|\xi| < \epsilon$. We need to prove that $|u'(\xi) - \bar{u}'(\xi)| = o(\epsilon^2)$. Define points $A, \tilde{A} \in \mathbb{H}^n$ and $y_A, \tilde{y}_A \in h_1$ by $A := (\xi, u'(\xi))_{h_2} =: (y_A, u(y_A))_{h_1}$ and $\tilde{A} := (\xi, \bar{u}'(\xi))_{h_2} =: (\tilde{y}_A, \bar{u}'(\tilde{y}_A))_{h_1}$. This is clearly possible for $\epsilon$ small enough. According to (12), for $\epsilon \ll 1$,

$$|y_A| \leq 2d_{\mathbb{H}^n}(A, q) \leq 2d_{\mathbb{H}^n}(A, (\xi, 0)_{h_2}) + 2d_{\mathbb{H}^n}((\xi, 0)_{h_2}, q) = 2u'(|\xi|) + 2\arcsinh(|\xi|/2) = o(\epsilon).$$

The second differentiability of $u$ at $0$ gives then $|u(y_A) - \bar{u}(y_A)| = o(\epsilon^2)$, i.e.

$$d_{\mathbb{H}^n}(A, B) = o(\epsilon^2),$$ \hfill (13)

where we have posed $B := (y_A, \bar{u}(y_A))_{h_1} =: (\xi_B, \bar{u}'(\xi_B))_{h_2}$, for some (unique) $\xi_B \in h_2$. Applying again Lemma 7 with $\mathfrak{h} = h_2$ we get

$$|\xi - \xi_B| \leq 2d_{\mathbb{H}^n}(A, B) = o(\epsilon^2).$$

Since $\bar{u}'$ is smooth,

$$|\bar{u}'(\xi) - \bar{u}'(\xi_B)| \leq L|\xi - \xi_B| = o(\epsilon^2),$$

\hfill $\square$
Finally, this latter and (13) give
\[ |u'(\xi) - \tilde{u}'(\xi)| = d_{\mathbb{H}^n}(A, \tilde{A}) \leq d_{\mathbb{H}^n}(A, B) + d_{\mathbb{H}^n}(B, \tilde{A}) \leq o(\epsilon^2). \]

\[ \square \]

4 BiLipschitz approximation around normal points

In this section we will show that around a.e. (normal) point of \( \partial K \), \( \epsilon \)-balls are (1 + o(\epsilon^2))-biLipschitz equivalent to \( \epsilon \)-balls on a smooth manifold. In the case of 2 dimensional surfaces, one can choose the smooth manifold to be of constant curvature. This property will be subsequently exploited to give a characterization of the regular part of the curvature measure for convex surfaces; see Sect. 6.

**Proposition 8** Let \( K \in \mathcal{K}(\mathbb{H}^n) \) and let \( q \in \partial K \) be a normal point. For \( \epsilon \ll 1 \), the metric ball \( B_{\epsilon}(q) \subset \partial K \) is biLipschitz equivalent to an open set \( U_{\epsilon} \subset \partial \tilde{K} \) contained in the boundary of a smooth convex set \( \tilde{K} \in \mathcal{K}(\mathbb{H}^n) \), with Lipschitz constants smaller than \( (1 + o(\epsilon^2)) \) as \( \epsilon \to 0 \).

**Corollary 9** Let \( K \in \mathcal{K}(\mathbb{H}^3) \) and let \( q \in \partial K \) be a normal point. For \( \epsilon \ll 1 \), the metric ball \( B_{\epsilon}(q) \subset \partial K \) is biLipschitz equivalent to an open set \( U_{\epsilon} \subset S_k \) contained in a smooth surface \( S_k \) of constant sectional curvature \( k = k(q) \), with Lipschitz constants smaller than \( (1 + o(\epsilon^2)) \) as \( \epsilon \to 0 \).

We call the (unique) real number \( k \) given by Corollary 9 the local curvature of \( \partial K \) at \( q \), and we denote it by \( \text{LocCurv}_{\partial K}(q) \).

**Proof (of Proposition 8)** Let \( q \in \partial K \) be a normal point and let \( u = u_q : \mathbb{B}_R \subset \mathbb{H} \to \mathbb{R} \) be the corresponding local parametrization as the graph of a horoconvex function over a horosphere \( \mathbb{H} = \mathbb{H}_q \). In particular, for \( \epsilon \) close to 0, the asymptotic relation (11) holds. We let \( \tilde{K}_q \) be the smooth hypersurface of \( \mathbb{H}^n \) given by the graph \( g_n(q, \tilde{u}) \). For any horoconvex function \( v : \mathbb{B}_R \subset \mathbb{H} \to (0, +\infty) \) we define \( \tilde{d}_v \) the metric induced on \( \partial K(v) \) by \( \mathbb{H}^3 \) and \( d_v(x, y) = \tilde{d}_v((x, v(x))_n, (y, v(y))_n) \) for any \( x, y \in \mathbb{B}_R \).

Fix \( \epsilon > 0 \) and let \( p, s \in \mathbb{B}_R \) such that \( d_u(p, 0) < \epsilon \) and \( d_u(s, 0) < \epsilon \). Let \( \gamma : [0, L] \to \mathbb{H} \) be a Lipschitz curve with \( \gamma(0) = p \) and \( \gamma(L) = s \) and with a.e. unit
speed with respect to the Euclidean metric of \( h \). Denote by \( \hat{\gamma}(t) := (\gamma(t), u(\gamma(t)))_h \) and \( \tilde{\gamma}(t) := (\gamma(t), \tilde{u}(\gamma(t)))_h \) the corresponding lifted Lipschitz curves in \( \partial K \) and \( \partial \tilde{K} \) respectively. We can choose \( \gamma \) in such a way that \( \nabla_{\gamma(t)}u \) exists for a.e. \( t \in [0, L] \) and the length \( \hat{L} \) of \( \hat{\gamma} : [0, L] \to \partial K \) is less then \( (1 + \varepsilon^2) d_u(p, s) \). Since \( d_u(p, s) < 2\varepsilon \) and the identity map \((\mathbb{B}_R, d_{\mathbb{B}_R-1}) \to (\mathbb{B}_R, d_u)\) is biLipschitz, there exists a constant \( \alpha \) independent of \( p, s \) such that \( \gamma([0, L]) \subset \mathbb{B}_{\alpha \varepsilon} \) and \( L \leq \alpha d_u(p, s) \). We can compute

\[
\int_0^L |\hat{\gamma}(t)|_{\partial K} dt = \int_0^L \left( e^{2u(\gamma(t))} + |\nabla_{\gamma(t)}\tilde{u}, \dot{\gamma}(t)|^2 \right)^{1/2} dt
\]

and

\[
\int_0^L |\tilde{\gamma}(t)|_{\partial K} dt = \int_0^L \left( e^{2u(\gamma(t))} + |\nabla_{\gamma(t)}u, \dot{\gamma}(t)|^2 \right)^{1/2} dt.
\]

Note that

\[
e^{2u(\gamma(t))} = e^{2u(\gamma(t)) + o(\varepsilon^2)} = e^{2u(\gamma(t))} + o(\varepsilon^2),
\]

and, using also (11),

\[
\langle \nabla_{\gamma(t)}u, \dot{\gamma}(t) \rangle^2 - \langle \nabla_{\gamma(t)}\tilde{u}, \dot{\gamma}(t) \rangle^2 = |\nabla_{\gamma(t)}u - \nabla_{\gamma(t)}\tilde{u}, \dot{\gamma}(t)| |\nabla_{\gamma(t)}u + \nabla_{\gamma(t)}\tilde{u}, \dot{\gamma}(t)| \\
\leq |\nabla_{\gamma(t)}u - \nabla_{\gamma(t)}\tilde{u}| |\nabla_{\gamma(t)}u + \nabla_{\gamma(t)}\tilde{u}| \\
= o(\varepsilon)O(\varepsilon) = o(\varepsilon^2).
\]

Putting this all together gives

\[
\int_0^L |\hat{\gamma}(t)|_{\partial K} dt = \int_0^L \left( e^{2u(\gamma(t))} + |\nabla_{\gamma(t)}\tilde{u}, \dot{\gamma}(t)|^2 \right)^{1/2} dt \\
= \int_0^L \left( e^{2u(\gamma(t))} + |\nabla_{\gamma(t)}u, \dot{\gamma}(t)|^2 + o(\varepsilon^2) \right)^{1/2} dt \\
= \int_0^L \left\{ \left( e^{2u(\gamma(t))} + |\nabla_{\gamma(t)}u, \dot{\gamma}(t)|^2 \right)^{1/2} + o(\varepsilon^2) \right\} dt \\
= \hat{L} + L o(\varepsilon^2) \\
\leq \hat{L} + \alpha d_u(p, s) o(\varepsilon^2) \\
\leq d_u(p, s)(1 + o(\varepsilon^2)),
\]

from which

\[
d_{\hat{u}}(p, s) \leq d_u(p, s)(1 + o(\varepsilon^2)).
\]

Interchanging the role of \( u \) and \( \tilde{u} \) we get the converse relation

\[
d_u(p, s) \leq d_{\hat{u}}(p, s)(1 + o(\varepsilon^2)).
\]
The proposition is thus proved with \( U_\varepsilon = g_\varepsilon(\bar{u}|V_\varepsilon) \), where \( V_\varepsilon := \{ x \in h : d_{\alpha}(x, 0) < \varepsilon \} \).

**Proof (of Corollary 9)** Let \( \partial \bar{K} \) be the smooth surface containing \( q \) given by Proposition 8. Let \( S_k \) be a smooth surface of constant sectional curvature \( k \) equal to the sectional curvature of \( \partial \bar{K} \) at the point \( q \), and let \( p \) be any reference point in \( S_k \).

According to the asymptotic development of the distance function in geodesic normal coordinates, (see [9,16] or [27, Lemma 9]), distances in \( B_\varepsilon \subset \partial \bar{K} \) are equivalent to distances in the geodesic ball \( B_\varepsilon^{S_k}(p) \) of \( S_k \) up to the third order as \( \varepsilon \to 0 \). Hence the corollary follows.

\[ \Box \]

5 Generalized Gauss theorem

In this section we focus on convex sets of \( \mathbb{H}^3 \). In particular, we want to prove that the well-known Gauss theorem connecting the scalar curvature of a surface, its extrinsic Gaussian curvature and the curvature of the ambient space generalizes to non-smooth convex surfaces in \( \mathbb{H}^3 \). This will be done considering a smooth approximation of the convex surface (locally parametrized as a horoconvex function) and showing that Gauss equation passes to the limit. As a corollary, we obtain also that Alexandrov's approach to extrinsic curvature of convex surfaces in \( \mathbb{H}^3 \) is in fact equivalent to the one by Kohlmann we use here.

First, recall that the boundary of a convex set \( K \in \mathcal{K}(\mathbb{H}^3) \) has an intrinsic curvature measure \( \omega_{\partial K} \) which generalizes to the boundary of non-smooth convex sets the integral of the usual sectional curvature in case of smooth surfaces.

In fact, given a convex set \( K \in \mathcal{K}(\mathbb{H}^3) \), since its boundary has curvature in the sense of Alexandrov lower bounded by \(-1\), the surface \( \partial K \) endowed with the metric induced by \( \mathbb{H}^3 \) turns out to be a special case of surface of Bounded Integral Curvature (often shortened in BIC surface in the literature); see for instance [11, p. 359] and [19] or [5, Theorem 3.2]. There exist several equivalent definitions of a BIC surface, [1,21] and [26, Section 2]. For instance, one can define it as a topological surface \( S \) endowed with a metric \( d \) obtained as a uniform limit of a sequence of distances \( \{d_j\} \) induced by a sequence of smooth Riemannian metrics \( \{g_j\} \) on \( S \), with the property that the \( \int_S \text{Sect}_{g_j} \text{dVol}_{g_j} \) is uniformly bounded along the sequence. The curvature measure \( \omega_S \) is then the weak limit in the sense of measures of the sequence of measures \( \alpha \mapsto \int_{\alpha} \text{Sect}_{g_j} \text{dVol}_{g_j} \).

Let \( v : B_R \subset h \to \mathbb{R} \) be any horoconvex function. For \( k = 0, 1, 2 \), let us introduce Borel measures \( \hat{H}_k^v \) defined for every Borel set \( \beta \subset B_R \) by

\[
\left( \begin{array}{c} n \\ k \end{array} \right) \hat{H}_k^v(\beta) = \Phi_{3-k}(K(v), g_\varepsilon(v|\beta)).
\] (14)

Since \( v \) is Lipschitz, its graph \( g_\varepsilon(v|\beta) \) restricted to \( \beta \in B(\mathbb{R}^2) \) is a Borel set of \( \mathbb{H}^3 \), so that the measures \( \hat{H}_k^v \) are well defined. According to (2), whenever \( v \) is smooth,
is the $k$-th symmetric function of the principal curvatures of $\partial K$ integrated with respect to the area measure of $\partial K$.

Moreover, we introduce also the Borel measure $\hat{\omega}_v$ on $\mathbb{B}_R$, where for $\beta \in \mathcal{B}(\mathbb{B}_R)$,

$$\hat{\omega}_v(\beta) = \omega_{\partial K}(g_h(v|\beta))$$

is the intrinsic curvature measure of the horograph of $v$ over $\beta$. As above, by the definition of the intrinsic curvature, whenever $v$ is smooth,

$$\hat{\omega}_v(\beta) = \int_\beta \text{Sect}_{\partial K(v)}((x, v(x))) dH_0^v(x).$$

Now, fix a convex set $K = K(u) \in \mathcal{K}(\mathbb{H}^n)$. As in the previous sections, let $u : \mathbb{B}_R \subset h \to \mathbb{R}$ be a horoconvex function, defined on a horosphere $h$, whose graph gives a local parametrization of $\partial K$. There exists a sequence of $C^\infty$ horoconvex functions $u_j : \mathbb{B}_R \to (0, +\infty)$ such that $u_j$ converges to $u$ uniformly and $W^{1,2}_{\text{loc}}$ on $\mathbb{B}_R$ as $j \to \infty$. To prove this assertion, one can approximate $h(u)$ uniformly and $W^{1,2}_{\text{loc}}$ by a sequence $\{h(u_j)\}$ of convex functions $\mathbb{B}_R \to \mathbb{R}$ explicitly constructed via mollification, [14, p. 123 and 238].

Let $K(u)$ and $K(u_j)$ be defined as in (7) with $\zeta = \sup_j \max_{\mathbb{B}_R} u_j + 1 < +\infty$. By [15, Lemma 3.7] we deduce

**Lemma 10** Let $u : B_R \to (0, +\infty)$ be a horoconvex function. There exist constants $C, C' > 0$ (depending on $u$) such that for any $0 < r < R$ and any other horoconvex function $v : B_R \to (0, +\infty)$, if $\sup_{\mathbb{B}_r} |u - v| < \epsilon$, then

$$\sup_{x, y \in \mathbb{B}_r} |d_u(x, y) - d_v(x, y)| \leq C \epsilon + C' \sqrt{\epsilon r}.$$

In particular $d_{u_j}$ converges uniformly to $d_u$ on $\mathbb{B}_R$ as $j \to \infty$.

Since the $u_j$’s are smooth, by (2) the measures $\mathring{H}^{u_j}_k$ are absolutely continuous with respect to the Lebesgue measures of $\mathbb{R}^2$, and the classical Gauss–Codazzi equation

$$\forall q \in g_h(u_j), \quad H_2^K(u_j)(q) = \text{Sect}_{\partial K(u_j)}(q) - 1$$

implies

$$\frac{1}{3} \Phi_1(K(u_j), g_h(u_j|\beta)) = \omega_{\partial K(u_j)}(g_h(u_j|\beta)) - \Phi_3(K(u_j), g_h(u_j|\beta)).$$

Projecting measures onto $h$, this latter gives

$$\mathring{H}^{u_j}_2(\beta) = \tilde{\omega}_{u_j}(\beta) - \mathring{H}^{u_j}_0(\beta).$$

(16)
Note that \( \int_{\beta} \text{Sect}_{K(u_j)}(x) d\hat{H}_0^{u_j}(x) = \hat{\omega}_{u_j}(\beta) \). Since \( u_j \to u \) uniformly, by Lemma 10 and by definition of the curvature measure of a BIC surface, we get that

\[
\omega_{\partial K(u_j)} \to \omega_{\partial K(u)}
\]

and equivalently

\[
\hat{\omega}_{u_j} \to \hat{\omega}_u
\]

weakly in the sense of measure as \( j \to \infty \). Lemma 10 implies also that \( \text{dist}_{\mathbb{H}_3}(K(u_j), K(u)) \to 0 \) as \( j \to \infty \). Thanks to Theorem 3, we have then that (16) passes to the limit as \( j \to \infty \) giving the following results, which can be seen as a version of Gauss–Codazzi equation (15) for non-smooth convex surfaces.

**Theorem 11 (Generalized Gauss Theorem)** Let \( K \in \mathcal{K}(\mathbb{H}^3) \) and \( u : \mathbb{H} \to \mathbb{R} \) a horoconvex local parametrization of \( \partial K \), \( \mathbb{H} \) being a horosphere of \( \mathbb{H}^n \). Then

\[
\hat{H}_2^u + \hat{H}_0^u = \hat{\omega}_u
\]

for every Borel set \( \alpha \subset \partial K \).

**Remark 12** The measure \( \hat{H}_0^u \) is absolutely continuous with respect to the Lebesgue measure \( \lambda \) of \( \mathbb{H} = \mathbb{R}^2 \). Namely, one has for every Borel set \( \beta \subset \mathbb{R}^2 \),

\[
\hat{H}_0^u(\beta) = \int_{\beta} \sqrt{e^{4u(x)} + e^{2u(x)}|\nabla_x u|^2} d\lambda(x),
\]

where \( \nabla_x u \) is the gradient of \( u \) at \( x \) defined for \( \lambda \)-a.e. point \( x \). When \( u \) is smooth, formula (19) can be deduced from (5) via a direct calculation. For general horoconvex function \( u \), one can apply (19) to the approximating sequence of smooth horoconvex functions \( u_j \) introduced above, and use the fact that \( u_j \to u \) uniformly and \( W^{1,2}_{loc} \) in \( \mathbb{R} \), that \( \nabla u \) is in \( L^2_{loc} \) since \( u \) is Lipschitz, and that the weak distributional gradient of \( u \) coincides with the a.e. defined pointwise gradient.

By the way, note that

\[
\hat{H}_0^u(\beta) = \Phi_3(K(v), g(v|\beta)) = \mathcal{H}^2(g(v|\beta))
\]

is the 2-dimensional Hausdorff measure of \( g(v|\beta) \). This follows from the co-area formula applied to the Lipschitz map \( \mathbb{R}^2 \to \mathbb{H}^3 \) defined by \( x \mapsto (x, u(x))_\mathbb{H} \).

**Remark 13** The generalized Gauss theorem for convex sets in \( \mathbb{H}^3 \) was already stated in [3, Chapter XII, p. 397] and [20, Section V.2]. Actually, to define the extrinsic curvature

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of the boundary of a convex set, Alexandrov used a different approach based on the local Euclidean character of $\mathbb{H}^3$. The validity of (18) proves also that Alexandrov’s definition of extrinsic curvature is equivalent to Kohlmann’s definition of the curvature measure $\Phi_1(K, \cdot)$.

6 The regular part of the curvature measure

In this section we will prove that the local curvature at normal points $\text{LocCurv}$ defined in Sect. 4 is in fact the regular part of the curvature measure. Thanks to Gauss equation (18) here one can consider indifferently (up to a constant factor) the intrinsic curvature measure $\omega_{\partial K}(\cdot)$ or the extrinsic curvature measure $\Phi_1(K, \cdot)$. Since both measures $\Phi_1(K, \cdot)$ and $\mathcal{H}^2$ (restricted to Borel sets) are Radon, according to Lebesgue decomposition theorem one can write

$$\Phi_1(K, \cdot) = \Phi_1(K, \cdot)_{\text{reg}} + \Phi_1(K, \cdot)_{\text{sing}}$$

where the singular part $\Phi_1(K, \cdot)_{\text{sing}}$ is supported by a set of nul 2-dimensional Hausdorff measure, while $\Phi_1(K, \cdot)_{\text{reg}}$ is absolutely continuous with respect to $\mathcal{H}^2$.

**Theorem 14** Let $K \in K(\mathbb{H}^3)$. For every Borel set $\alpha \subset \partial K$ it holds

$$(\omega_{\partial K})_{\text{reg}}(\alpha) = \frac{1}{3} \Phi_1(K, \alpha)_{\text{reg}} + \mathcal{H}^2(\alpha) = \int_{\alpha} \text{LocCurv}_{\partial K}(q) d\mathcal{H}^2(q). \quad (21)$$

Note that $\text{LocCurv}_{\partial K}$ is a.e. defined. Its local integrability will be a byproduct of the proof.

**Proof** The first equality is a direct consequence of (20) and Gauss equation (18). Clearly we can prove the second equality locally.

As in Sect. 3, let $u : \mathbb{B}_R \subset \mathfrak{h} \rightarrow \mathbb{R}$ be a horoconvex function defined on a horosphere $\mathfrak{h}$ such that its graph $g_{\mathfrak{h}}(u)$ is an open set of $\partial K$. Projecting onto $\mathfrak{h}$, the identity (21) corresponds to

$$(\hat{H}_2^u)_{\text{reg}}(\beta) + \hat{H}_0^u(\beta) = \int_{\beta} \text{LocCurv}_{\partial K}(((x, u(x))_{\mathfrak{h}}) d\hat{H}_0^u(x)$$

for any Borel set $\beta \subset \mathbb{B}_R \subset \mathfrak{h}$. According to a version of the Lebesgue decomposition theorem, [14, p. 42], this latter is equivalent to

$$\lim_{\epsilon \rightarrow 0} \frac{\hat{H}_2^u(\mathbb{B}_\epsilon(x)) + \hat{H}_0^u(\mathbb{B}_\epsilon(x))}{\hat{H}_0^u(\mathbb{B}_\epsilon(x))} = \text{LocCurv}_{\partial K}(((x, u(x))_{\mathfrak{h}}) \quad (22)$$

for $\hat{H}_0^u$-a.e. $x \in \mathbb{B}_R$, and in view of Remark 12 it is enough to check the latter equality for $\lambda$-a.e. $x \in \mathbb{B}_R$. Indeed, we are going to prove that (22) holds at any $x$ at which $u$ is twice differentiable.
Fix such an $x$. Define $\bar{u} : \mathfrak{h} \to \mathbb{R}$ as in (9) so that (8) holds. Set $q := (x, u(x))_{\mathfrak{h}}$ and $K(\bar{u})$ as in (7) so that $g_{\mathfrak{h}}(\bar{u}|_{B_\varepsilon(x)}) \subset \partial K(\bar{u})$. We have

$$\text{dist}_{H}(g_{\mathfrak{h}}(u|_{B_\varepsilon(x)}), g_{\mathfrak{h}}(\bar{u}|_{B_\varepsilon(x)})) \leq \sup_{B_\varepsilon(x)} |u - \bar{u}| = o(\varepsilon^2). \quad (23)$$

Fix $\rho > 0$ and let $p \in M_\rho(K(u), g(u|_{B_\varepsilon(x)}))$. Either $p \in K(\bar{u})$ or not. In the first case, since

$$d_{\mathbb{H}^3}(p, K(u)) = d_{\mathbb{H}^3}(p, f_K(u)(p)) = o(\varepsilon^2)$$

and since $f_K(u)(p) \in g(u|_{B_\varepsilon(x)})$, we get for $\varepsilon$ small enough

$$p \in A^u_\varepsilon := \{(y, z)_{\mathfrak{h}} \in \mathbb{H}^3 : y \in B_{2\varepsilon}(x) \text{ and } \bar{u}(y) \leq z \leq u(y)\}. \quad (24)$$

A direct calculation gives $\text{Vol}_{\mathbb{H}^3}(A^u_\varepsilon) = o(\varepsilon^3)$. On the other hand, if $p \notin K(\bar{u})$, (23) and Lemma 5 imply

$$d_{\mathbb{H}^3}(f_K(u)(p), f_K(\bar{u})(p)) \leq 2(\tanh \rho)^{1/2} o(\varepsilon) + o(\varepsilon^2) = o(\varepsilon),$$

while

$$d_{\mathbb{H}^3}(p, f_K(\bar{u})(p)) = d_{\mathbb{H}^3}(p, K(\bar{u})) \leq d_{\mathbb{H}^3}(p, f_K(u)(p)) + d_{\mathbb{H}^3}(f_K(u)(p), K(\bar{u})) \leq \rho + o(\varepsilon^2).$$

Combining this latter with (24) we deduce that

$$M_\rho(K(u), g(u|_{B_\varepsilon(x)})) \subset M_{\rho + o(\varepsilon^2)}(K(\bar{u}), g(\bar{u}|_{B_{\varepsilon + o(\varepsilon)}(x)})) \cup A^u_\varepsilon,$$

which in turn implies

$$\mu_\rho(K(u), g(u|_{B_\varepsilon(x)})) \leq \mu_{\rho + o(\varepsilon^2)}(K(\bar{u}), g(\bar{u}|_{B_{\varepsilon + o(\varepsilon)}(x)})) + o(\varepsilon^3).$$

Note that for $1 \leq k \leq 3$ it holds $\ell_k(\rho + o(\varepsilon^2)) = \ell_k(\rho) + o(\varepsilon^2)$. Then Theorem 1 gives

$$\sum_{r=0}^2 \ell_{3-r}(\rho) \Phi_r(K(u), g(u|_{B_\varepsilon(x)})) \leq \sum_{r=0}^2 (\ell_{3-r}(\rho) + o(\varepsilon^2)) \Phi_r(K(\bar{u}), g(\bar{u}|_{B_{\varepsilon + o(\varepsilon)}(x)})).$$

that is

$$\sum_{k=0}^2 \ell_{k+1}(\rho) \binom{3}{k} H^u_k(B_\varepsilon(x)) \leq \sum_{k=0}^2 (\ell_{k+1}(\rho) + o(\varepsilon^2)) \binom{3}{k} \hat{H}^u_k(B_{\varepsilon + o(\varepsilon)}(x)). \quad (25)$$
Moreover, interchanging the roles of \( u \) and \( \bar{u} \) we also get the converse relation

\[
\sum_{k=0}^{2} \ell_{k+1}(\rho) \binom{3}{k} \hat{H}_k^u(B_{\varepsilon}(x)) \geq \sum_{k=0}^{2} (\ell_{k+1}(\rho) - o(\varepsilon^2)) \binom{3}{k} \hat{H}_k^u(B_{\varepsilon-o(\varepsilon)}(x)).
\]

(26)

Since \( \bar{u} \) is smooth, from (2), (14) and Lebesgue differentiation theorem we have that the limits

\[
h_k^\bar{u}(x) := \lim_{\varepsilon \to 0} \frac{\hat{H}_k^\bar{u}(B_{\varepsilon-o(\varepsilon)}(x))}{\lambda(B_{\varepsilon}(x))} = \lim_{\varepsilon \to 0} \frac{\hat{H}_k^\bar{u}(B_{\varepsilon+o(\varepsilon)}(x))}{\lambda(B_{\varepsilon}(x))}, \quad 0 \leq k \leq 2,
\]

(27)

exist and are finite. Letting \( \varepsilon \to 0 \), (25) and (26) thus give

\[
\sum_{k=0}^{2} \ell_{k+1}(\rho) \binom{3}{k} \hat{H}_k^u(B_{\varepsilon}(x)) = \sum_{k=0}^{2} \ell_{k+1}(\rho) \binom{3}{k} \hat{H}_k^u(B_{\varepsilon}(x))
\]

\[
\lim_{\varepsilon \to 0} \sum_{k=0}^{2} \ell_{k+1}(\rho) \binom{3}{k} \hat{H}_k^u(B_{\varepsilon}(x)) = \sum_{k=0}^{2} \ell_{k+1}(\rho) \binom{3}{k} h_k^\bar{u}(x).
\]

Since this latter is true for arbitrary small enough positive \( \rho \), it yields

\[
\lim_{\varepsilon \to 0} \frac{\hat{H}_k^u(B_{\varepsilon}(x))}{\lambda(B_{\varepsilon}(x))} = h_k^\bar{u}(x), \quad 0 \leq k \leq 2.
\]

Hence

\[
\lim_{\varepsilon \to 0} \frac{\hat{H}_2^u(B_{\varepsilon}(x)) + \hat{H}_0^u(B_{\varepsilon}(x))}{\lambda(B_{\varepsilon}(x))} = h_2^\bar{u}(x) + h_0^\bar{u}(x),
\]

which gives

\[
\lim_{\varepsilon \to 0} \frac{\hat{H}_2^u(B_{\varepsilon}(x)) + \hat{H}_0^u(B_{\varepsilon}(x))}{\hat{H}_0^u(B_{\varepsilon}(x))} = \lim_{\varepsilon \to 0} \frac{\hat{H}_2^u(B_{\varepsilon}(x)) + \hat{H}_0^u(B_{\varepsilon}(x))}{\lambda(B_{\varepsilon}(x))} \lim_{\varepsilon \to 0} \frac{\lambda(B_{\varepsilon}(x))}{\hat{H}_0^u(B_{\varepsilon}(x))}
\]

\[
= \frac{h_2^\bar{u}(x)}{h_0^\bar{u}(x)} + 1.
\]

(28)
On the other hand, by (16), (27) and Lebesgue differentiation,

\[
\frac{h^2}{h_0^2}(x) + 1 = \lim_{\epsilon \to 0} \frac{\hat{H}^2_{\epsilon}(B_{\epsilon}(x)) + \hat{H}^2_0(B_{\epsilon}(x))}{\hat{H}^2_0(B_{\epsilon}(x))}
\]

\[
= \lim_{\epsilon \to 0} \frac{\omega_{\partial K}(\bar{u})(g^h(\bar{u} | B_{\epsilon}(x)))}{\mathcal{H}^2(g^h(\bar{u} | B_{\epsilon}(x)))}
\]

\[
= \lim_{\epsilon \to 0} \frac{\int_{g^h(\bar{u} | B_{\epsilon}(x))} \text{Sect}_{\partial \bar{K}}(p) d\mathcal{H}^2(p)}{\mathcal{H}^2(g^h(\bar{u} | B_{\epsilon}(x)))}
\]

\[
= \text{Sect}_{\partial \bar{K}}(q)
\]

\[
= \text{LocCurv}_{\partial \bar{K}}(q),
\]

which together with (28) prove (22), and hence (21).

In [19], Machigashira studied the (intrinsic) curvature measures of a surface \((S, d)\) which has curvature bounded below by \(k \in \mathbb{R}\) in the sense of Alexandrov. Namely, he introduced the so called curvature regular points, which form a set of full \(\mathcal{H}^2\) measure in \(S\), and characterized the regular part of the curvature measure in terms of the asymptotics of the upper excess at curvature regular points. To this end, he defined the Gaussian curvature function \(G\) by

\[
G(x) = \inf_{d > 0} G_d(x); \quad G_d(x) := \liminf_{\Delta \to [x], x \in \Delta, \Delta \in U_d} \frac{e(\Delta)}{\text{Area}(\Delta)}.
\]

and proved that

\[
(\omega_{(S, d)})_{\text{reg}}(\alpha) = \int_{\alpha} G(q) d\mathcal{H}^2(q).
\]

The liminf in (29) is taken for geodesic triangles converging to the point \(x\), containing \(x\) in their interior, and with interior angles greater than \(d > 0\). Moreover the excess \(e(\Delta)\) of the geodesic triangle \(\Delta\) is defined so that \(e(\Delta) + \pi\) is the sum of the interior angles of \(\Delta\).

As discussed above, according to a well-known result by Alexandrov any \(CBB(k)\) surface is locally isometric to an open set of the boundary \(\partial K\) of a convex set \(K\) in the simply connected space form of constant curvature \(k\). Clearly, up to a rescaling of the metric, we can always suppose that one has \(k \geq -1\) locally, and hence take \(K \subset \mathbb{H}^3\) and \(\omega_{(S, d)} = \omega_{\partial K}\). Either \(K \in \mathcal{K}(\mathbb{H}^3)\), i.e. it has non-empty interior, or \(K\) is isometric to the double of a convex set \(\mathcal{K}\) of \(\mathbb{H}^2\). In the first case, Theorem 14 applies to \((S, d)\) and a further characterization of the regular part of the curvature measure of a \(CBB(k)\) surface is given by the relation

\[
G(q) = \text{LocCurv}_{\partial \bar{K}}(q).
\]
In the second case, all points but the ones in $\partial K$, hence up to a set of null $\mathcal{H}^2$-measure, have a hyperbolic neighborhood, so that for those points trivially

$$G(q) = -1 = \text{LocCurv}_{\partial K}(q).$$

Remark 15 The relation (30) together with the biLipschitz approximation around normal points given by Corollary 9, will be used in [27] to get a further characterization of the regular part of $CBB(K)$ surfaces via the asymptotic expansion of the extent around normal points.

Acknowledgements I would like to thank F. Fillastre for many useful discussions, especially concerning Euclidean convex sets, as well as for reading a first version of this article. I’m indebted to J. Bertrand for suggesting the application to $CBB(k)$ surfaces and for pointing me out Alexandrov’s theorem on the a.e. second differentiability of convex functions. This research has been conducted as part of the project Labex MME-DII (ANR11-LBX-0023-01). The author is member of the “Gruppo Nazionale per l’Analisi Matematica, le Probabilità e le loro Applicazioni” (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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