Neutrino-Neutrino Interactions and Flavor Mixing in Dense Matter

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An algebraic approach to the neutrino propagation in dense media is presented. The Hamiltonian describing a gas of neutrinos interacting with each other and with background fermions is written in terms of the appropriate SU(N) operators, where N is the number of neutrino flavors. The evolution of the resulting many-body problem is formulated as a coherent-state path integral. Some commonly used approximations are shown to represent the saddle-point solution of the path integral for the full many-body system.

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I. INTRODUCTION

Neutrino propagation in dense matter is encountered near the core of a core-collapse supernovae, in the Early Universe, and possibly in the gamma-ray bursts. In particular, neutrino interactions play a crucial role in core-collapse supernovae. Neutrino oscillations in a core-collapse supernova differ from the matter-enhanced neutrino oscillations in the Sun as in the former there are additional effects coming from both neutrino-neutrino scattering and antineutrino flavor transformations. Exact integration of the neutrino evolution equations with such terms in the supernova environment turns out to be a very difficult problem. A “mean-field” type approximation was proposed in Ref. which was adopted in exploratory calculations of the conditions for r-process nucleosynthesis in supernovae. Analytical solutions were investigated in the limiting cases where off-diagonal terms dominate. In addition, various collective effects were explored.

The purpose of this article is to formulate the problem of neutrino propagation in dense media algebraically. An algebraic formulation of the problem should make hidden symmetries evident and also provide a framework to look for exact solutions or systematic approximations. In the next section we introduce the second-quantized formalism and the appropriate SU(2) algebra for two flavors. In this section we also show that the evolution operator for the standard MSW problem of neutrinos, mixing with each other and interacting with background electrons, can be written down exactly using an algebraic ansatz. In this and subsequent sections we utilize a path-integral approach to the underlying many-body problem, the details of which are sketched in the Appendix. In Section III, we introduce the neutrino-neutrino interaction and elucidate its algebraic nature. These arguments are expanded into the situations where antineutrinos are present in Section IV. The full problem with three flavors of both neutrinos and antineutrinos is discussed in Section V. Finally, a discussion of our results in Section VI concludes the paper.

II. SECOND-QUANTIZED FORMALISM

In our presentation we find it convenient to use the second-quantized form. For simplicity in this section we assume that only two flavors of neutrinos mix, which we take to be the electron neutrino, $\nu_e$, and a combination of muon and tau neutrinos, which we denote by $\nu_x$. (In the limit $\theta_{13} = 0$, one particular combination of the mu and tau flavors decouple and the results in this section become exact with $\nu_x$ being the combination orthogonal to the decoupled one). We first consider a situation where there are no antineutrinos. (We relax both of these assumptions in the subsequent sections). The Hamiltonian describing mixing and interaction of the neutrinos with
where \( a^+_\tau(p) \) and \( a^-_\tau(p) \) are the creation operators for the left-handed \( \nu_\tau \) and \( \nu_\tau \) with momentum \( p \), respectively, and \( a_\tau(p) \) and \( a^\dagger_\tau(p) \) are the corresponding annihilation operators. In Eq. (1), \( \sqrt{2} G_F N_e(x) = V_e \) is the Wolfenstein potential describing the interaction of neutrinos with electrons in neutral, unpolarized matter [23]; \( N_e = n_{e^-} - n_{e^+} \) is the net electron density; \( \theta \) is the vacuum mixing angle; and \( \Delta m^2 = m^2_3 - m^2_2 \). In writing Eq. (1) we omitted a term proportional to the identity (this includes the other Wolfenstein potential \( V_\nu(x) = -(1/\sqrt{2}) G_F N_n(x) \) describing the neutral-current interaction of neutrinos with neutrons). The presence of the Mikheev, Smirnov, Wolfenstein (MSW) resonance [22, 24, 25] is manifest in the first term.

The inherent SU(2) symmetry of the problem can be implemented by the operators

\[
J_+(p) = a^+_\tau(p)a_\tau(p), \quad J_-(p) = a^-_\tau(p)a_\tau(p), \quad J_0(p) = \frac{1}{2} (a^+_\tau(p)a_\tau(p) - a^-_\tau(p)a_\tau(p)),
\]

which satisfy the commutation relations

\[
[J_+(p), J_-(q)] = 2 \delta^3(p - q) J_0(p), \quad [J_0(p), J_\pm(q)] = \pm \delta^3(p - q) J_\pm(p).
\]

These equations describe as many commuting SU(2) algebras as the number of distinct values of the neutrino momenta \( p \) permitted by the appropriate physical situation. Each \( J(p) \) is realized in the \( j = 1/2 \) representation due to the fermionic nature of neutrinos. The global SU(2) operators,

\[
\mathcal{J}_\pm = \int d^3p J_\pm(p), \quad \mathcal{J}_0 = \int d^3p J_0(p),
\]

also satisfy the SU(2) commutation relations and play an important role in the problem. Representation of the algebra spanned by the operators \( \mathcal{J} \) of Eq. (1) is obtained by adding \( N \) different copies of SU(2) each with \( j = 1/2 \) where \( N \) is the number of allowed values of neutrino momenta.

In terms of the SU(2) generators of Eq. (2), we can write the neutrino Hamiltonian in Eq. (1) as follows:

\[
H_\nu = \int d^3p \frac{\delta m^2}{2p} \left[ \cos 2\theta J_0(p) + \frac{1}{2} \sin 2\theta (J_+(p) + J_-(p)) \right] - \sqrt{2} G_F \int d^3p N_e J_0(p).
\]

Note that in the last term of Eq. (5) we kept the electron density inside the momentum integral since it depends on the direction of the neutrino momentum (neutrino traveling in different directions will, in principle, see different electron densities), but of course not on the absolute value.

In general we are interested in finding the operator \( U \), describing the evolution of the system:

\[
i \frac{\partial U}{\partial t} = H_\nu U.
\]

In most cases, however, the physical interest is in finding the evolution of a particular state of the system. One may start, for example, with a state in which all permitted electron neutrino states are occupied and all \( \nu_\tau \) states are empty i.e.,

\[
|\phi\rangle = \prod_{p \in \mathcal{P}} a^\dagger_\tau(p)|0\rangle,
\]

where \( |0\rangle \) is the particle vacuum. \( \mathcal{P} \) denotes the set of all allowed neutrino momenta. The state \( |\phi\rangle \) is annihilated by the operators \( J_-(p) \) for all \( p \) and also by \( \mathcal{J}_- \). It is the lowest-weight eigenstate of the representation with \( j = N/2 \) of the global SU(2) operators \( \mathcal{J}_0 \) and \( \mathcal{J}_\pm \).

The evolution operator can be found by employing the unitary ansatz

\[
U = \exp \left( \int d^3p (p,t) J_+(p) \right) \exp \left( \int d^3p \log(1 + |\tau(p,t)|^2) J_0(p) \right) \exp \left( - \int d^3p \tau^\dagger(p,t) J_- (p) \right).
\]
Here $\tau(p, t)$ is a function to be determined by substituting this ansatz into Eq. (3). One can differentiate $U$ of Eq. (8) using the operator chain rule. Differentiation introduces the operator $J_0(p')$ between the first and second exponentials, and the operator $J_-(p')$ between the second and third exponentials. Since, for example, $J_0(p')$ does not commute with the first exponential, to write this operator before the first exponential, one needs to introduce the identity operator in the form $\exp \left( \int d^3p \tau(p, t) J_+(p) \right)$ after the operator $J_0(p')$ and use the identity

$$\exp(\mathcal{O}_1) \mathcal{O}_2 \exp(-\mathcal{O}_1) = \mathcal{O}_2 + [\mathcal{O}_1, \mathcal{O}_2] + \frac{1}{2!} [\mathcal{O}_1, [\mathcal{O}_1, \mathcal{O}_2]] + \cdots,$$

valid for any two arbitrary operators $\mathcal{O}_1$ and $\mathcal{O}_2$. Moving all such terms to the left of the exponentials we find

$$i \frac{\partial U}{\partial t} = \left[ \int d^3p \left( i \frac{\hat{\tau}(p, t) \hat{\tau}^*(p, t) - \tau(p, t) \hat{\tau}^*(p, t)}{1 + |\tau(p, t)|^2} J_0(p) \right) + \int d^3p \left( i \frac{\hat{\tau}(p, t) \hat{\tau}^*(p, t)}{1 + |\tau(p, t)|^2} J_+(p) - i \frac{\hat{\tau}^*(p, t)}{1 + |\tau(p, t)|^2} J_-(p) \right) \right] U,$$

where the dot denotes derivative with respect to time. Eq. (6) is satisfied if $\tau(p, t)$ obeys the equations:

$$i \frac{\dot{\tau}(p, t) \tau^*(p, t) - \tau(p, t) \dot{\tau}^*(p, t)}{1 + |\tau(p, t)|^2} = \frac{\delta m^2}{2p} \cos 2\theta - \sqrt{2}G_F N_e,$$

and

$$i \frac{\dot{\tau}(p, t)}{1 + |\tau(p, t)|^2} = -i \frac{\dot{\tau}^*(p, t)}{1 + |\tau(p, t)|^2} = \frac{1}{2} \frac{\delta m^2}{2p} \sin 2\theta,$$

i.e. equations for different neutrino momenta decouple as expected. In order to satisfy the condition $U(t = 0) = 1$, $\tau(p, t = 0)$ should be zero for all $p$. Using Eqs. (10) and (11), one can show that $\tau(p, t)$ is the solution of the following nonlinear (Riccati-type) first-order differential equation:

$$i \tau(p, t) = \frac{1}{2} \frac{\delta m^2}{2p} \sin 2\theta (1 - \tau(p, t)^2) + \left( \frac{\delta m^2}{2p} \cos 2\theta - \sqrt{2}G_F N_e \right) \tau(p, t).$$

$\tau(p, t)$ can be interpreted as the ratio of the one-body neutrino wave functions

$$\tau(p, t) = \frac{\psi_x(p, t)}{\psi_e(p, t)}.$$

To see this, one only needs to substitute Eq. (13) into Eq. (12) together with the normalization condition

$$|\psi_e|^2 + |\psi_x|^2 = 1.$$

This way, it becomes clear that Eq. (12) is equivalent to the Schrödinger equation

$$i \frac{\partial}{\partial t} \begin{pmatrix} \psi_e \\ \psi_x \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A - \Delta \cos 2\theta & \Delta \sin 2\theta \\ \Delta \sin 2\theta & -A + \Delta \cos 2\theta \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_x \end{pmatrix},$$

where we defined

$$\Delta = \frac{\delta m^2}{2p}, \quad A = \sqrt{2}G_F N_e.$$

For $\tau(t = 0)$ to be zero, we need the initial conditions $\psi_x(t = 0) = 0$ and $\psi_e(t = 0) = 1$. Eq. (15) is nothing but the standard MSW evolution equation for two flavors. Note that our ansatz for the evolution operator provides the "coset-space" formulation of this problem (cf. Eq. (12) with the formulation in the Appendix of Ref. [26]). The initial state $|\phi\rangle$, given in Eq. (7), then evolves into the state

$$\exp \left( -\frac{1}{2} \int_p d^3p \log(1 + |\tau(p, t)|^2) \right) \exp \left( \int_p d^3p \tau(p, t) J_+(p) \right) |\phi\rangle,$$

where $\tau(p, t)$ is obtained by solving Eq. (12). Both the initial state in Eq. (7) and evolved state in Eq. (17) are normalized to unity. The evolved state in Eq. (17) is the standard normalized SU(2) coherent state.
III. NEUTRINO-NEUTRINO INTERACTIONS

In some astrophysical environments, such as supernovae and the Early Universe, where neutrino density can become very large [27], the Hamiltonian in Eq. 11 is no longer sufficient. Neutrino self interactions must also be taken into account. In most cases one is mainly interested in the forward scattering of neutrinos from other neutrinos. These are described by the Hamiltonian

\[ H_{\nu\nu} = \frac{G_F}{\sqrt{2}V} \int d^3p \, d^3q \, R_{pq} \left[ a_x^\dagger(p)a_x(p)a_x^\dagger(q)a_x(q) + a_x^\dagger(p)a_x(p)a_y^\dagger(q)a_x(q) + a_x^\dagger(p)a_x(p)a_z^\dagger(q)a_x(q) \right] \]

where we defined

\[ R_{pq} = (1 - \cos \vartheta_{pq}). \]

Here, \( \vartheta_{pq} \) is the angle between the momentum directions of the neutrinos with momenta \( \mathbf{p} \) and \( \mathbf{q} \) and we used box quantization conditions for a box with volume \( V \). In forward scattering, neutrinos exchange their momenta by exchanging a \( Z \) boson with momentum \( \mathbf{p} - \mathbf{q} \) or a \( Z \) boson with zero momentum. The coefficient \( 1 - \cos \vartheta_{pq} \) ensures that the neutrinos which move parallel to each other do not undergo forward scattering.

Numerical studies indicate that the addition of a neutrino background results in very interesting physical effects such as coherent flavor transformation. For details see, for example, Refs. 3, 4, 5, 6, 7 and 16, 17, 18, 19, 20. The major difficulty in studying the effects of a neutrino background on flavor evolution is the inherent nonlinearity of the problem. On the other hand, the contribution from \( H_{\nu\nu} \) given in Eq. (18) in terms of the SU(2) generators of Eq. (2) presents a difficulty in numerical studies, it is relatively easier to work with in this algebraic approach. The noncommutativity of the first integral in Eq. (5) with generators of the total angular momentum operators \( J_z \) ensures

\[ [H_{\nu\nu}, J_z] = 0 \]

although the Hamiltonian \( H_{\nu} \) does not:

\[ [H_{\nu}, J_z] \neq 0. \]

On the other hand, the last term of \( H_{\nu} \) in Eq. (5) is proportional to \( J_0 \) for a constant electron density. Thus if \( \delta m^2 \) were zero the evolution problem for \( H_{\nu} + H_{\nu\nu} \) could have been solved in the \( J \)-basis for a constant electron density. We see that, although \( H_{\nu\nu} \) presents a difficulty in numerical studies, it is relatively easier to work with in this algebraic approach. The noncommutativity of the first integral in Eq. (5) with generators of the total angular momentum operators \( J \), however, is the salient difficulty of the algebraic approach to the problem.

One possible approach to the problem of finding the evolution operator with \( H_{\nu} + H_{\nu\nu} \) is to seek a path integral representation. One can appropriately use the SU(2) coherent states

\[ |z(t)\rangle = N \exp \left( \int_\mathcal{P} d^3p \, z(p,t)J_z(p) \right) |\phi\rangle. \]

Here \( |\phi\rangle \) is the state defined in Eq. 7 and the normalization constant \( N \) is given by

\[ N = \exp \left( -\frac{1}{2} \int_\mathcal{P} d^3p \log(1 + |z(p,t)|^2) \right). \]
Clearly when $z(p, t) = \tau(p, t)$ the state in Eq. (24) becomes the exact solution of the evolution with $H_\nu$ alone. Path integral representation of the matrix element of the evolution operator calculated with $H_\nu + H_{\nu\nu}$ between two states $|z(t_i)\rangle$ and $|z'(t_f)\rangle$ is given by

$$
\langle z'(t_f)|U|z(t_i)\rangle = \int D[z, z^*] e^{iS[z, z^*]} \tag{26}
$$

where the path integral measure is

$$
D[z, z^*] = \lim_{N \to \infty} e^{-2\sum_{\alpha=1}^{N} \int_{p} dp \log(1 + |z(p, t_o)|^2) \prod_{\alpha=1}^{N} \prod_{p \in P} 2! \int \frac{dz(p, t_o)}{2\pi i}}. \tag{27}
$$

Here, the exponential factor arises because the $SU(2)$ coherent states are overcomplete and therefore require a weight function in the resolution of identity. A detailed derivation can be found in the Appendix. In the above formula $dz^2$ refers to $dz \, dz^*$. The action functional is given by

$$
S[z, z^*] = \int_{t_i}^{t_f} dt \left( i \frac{\partial}{\partial t} - H_\nu - H_{\nu\nu} \right) - i \log \langle z'(t_f)|z(t_f)\rangle. \tag{28}
$$

The leading contribution to this path integral comes from the stationary path $|z(t)\rangle$ which minimizes the action functional. As in any variational problem, stationary path can be found by solving the Euler-Lagrange equations

$$
\left( \frac{d}{dt} \frac{\partial}{\partial \dot{z}} - \frac{\partial}{\partial z} \right) L(z, z^*) = 0, \quad \left( \frac{d}{dt} \frac{\partial}{\partial \dot{z}^*} - \frac{\partial}{\partial z^*} \right) L(z, z^*) = 0, \tag{29}
$$

where

$$
L[z, z^*] = \langle i \frac{\partial}{\partial t} - H_\nu - H_{\nu\nu} \rangle \tag{30}
$$

which plays the role of the Lagrangian. To solve the Euler-Lagrange equations, however, we first need to calculate the Lagrangian as a function of $z$ and $z^*$. This can be done by substituting $H_\nu$ from Eq. (5) and $H_{\nu\nu}$ from Eq. (20) into the Lagrangian of Eq. (30) and using the expectation values

$$
\langle J_+(p) \rangle = \langle J_-(p) \rangle^* = \frac{z^*(p, t)}{1 + |z(p, t)|^2}, \quad \langle J_0(p) \rangle = -\frac{1}{2} \frac{1 - |z(p, t)|^2}{1 + |z(p, t)|^2}, \tag{31}
$$

which are valid for $p \in \mathcal{P}$. When $p \notin \mathcal{P}$, however, these expectation values are equal to zero. When $p \neq q$ coherent states also obey

$$
\langle J_a(p)|J^\dagger_a(q) \rangle = \langle J_a(p) \rangle \langle J^\dagger_a(q) \rangle \tag{32}
$$

for $a = 0, \pm$. Using above formulas and Eqs. (5), (30) and (20) together with

$$
\langle i \frac{d}{dt} \rangle = i \int dp \frac{\dot{z}(p) z^*(p)}{1 + |z(p)|^2}, \tag{33}
$$

we obtain the following expression for the Lagrangian:

$$
L[z, z^*] = L_\nu - \frac{1}{4} \sqrt{2} G_F \int \frac{d^3 p}{2} \frac{B(p)(1 - |z(p)|^2) + 2 B_{ex}(p) z(p) + 2 B_{ex}(p) z^*(p)}{1 + |z(p)|^2}. \tag{34}
$$

Here we defined

$$
L_\nu = \frac{1}{i} \int \frac{d^3 p}{2} \frac{\dot{z}(p, t) z^*(p, t)}{1 + |z(p, t)|^2} + \frac{1}{2} \int \frac{d^3 p}{2} \left( \frac{\delta m^2}{2p} \cos 2\theta - \sqrt{2} G_F N_c \right) \frac{1}{2} \frac{1 - |z(p, t)|^2}{1 + |z(p, t)|^2}
$$

$$
- \frac{1}{2} \int \frac{d^3 p}{2p} \frac{\delta m^2}{2p} \sin 2\theta \frac{z(p, t) + z^*(p, t)}{1 + |z(p, t)|^2} \tag{35}
$$

We use the shorthand notation $\langle \mathcal{O} \rangle = \langle z(t)|\mathcal{O}|z(t)\rangle$.
and
\[ B(p) = \frac{\sqrt{2} G_F}{V} \int_P d^3q R_{pq} \frac{1 - |z(q,t)|^2}{1 + |z(q,t)|^2}, \quad B_{ex}(p) = B_{ex}^* = 2 \frac{\sqrt{2} G_F}{V} \int_P d^3q R_{pq} \frac{z(q,t)}{1 + |z(q,t)|^2}. \] (36)

Euler-Lagrange equations which follow from this Lagrangian are
\[ i \dot{\psi}(p,t) = \beta(p,t) + \alpha(p,t) z(p,t) \beta^*(p,t) z(p,t)^2 \] (37)
and its complex conjugate. The coefficients \( \alpha \) and \( \beta \) are given by:
\[ \alpha(p,t) = -\frac{\delta m^2}{2p} \cos 2\theta + \sqrt{2} G_F N_e + B(p), \quad \beta(p,t) = \frac{1}{2} \frac{\delta m^2}{2p} \sin 2\theta + \frac{1}{2} B_{ex}(p). \] (38)
As in the previous section, \( z(p,t) \) can be interpreted as the ratio of the one-body neutrino wave functions \( \psi_e, \psi_x \):
\[ z(p,t) = \frac{\psi_x(p,t)}{\psi_e(p,t)}. \] (39)
with the normalization condition
\[ |\psi_e|^2 + |\psi_x|^2 = 1. \] (40)
If we substitute Eq. (39) into Eq. (37), we see that Eq. (37) is equivalent to the Schrodinger equation
\[ i \frac{\partial}{\partial t} \begin{pmatrix} \psi_e \\ \psi_x \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A + B - \Delta \cos 2\theta & B_{ex} + \Delta \sin 2\theta \\ B_{ex} - \Delta \sin 2\theta & A - B - \Delta \cos 2\theta \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_x \end{pmatrix}. \] (41)
where \( \Delta \) and \( A \) are defined in Eq. (16). \( B \) and \( B_{ex} \) are obtained by substituting Eq. (39) into Eq. (36):
\[ B = \frac{\sqrt{2} G_F}{V} \int_P d^3q R_{pq} \left[ |\psi_e(q,t)|^2 - |\psi_x(q,t)|^2 \right], \] (42)
\[ B_{ex} = 2 \frac{\sqrt{2} G_F}{V} \int_P d^3q R_{pq} \psi_e(q,t) \psi_x^*(q,t). \] (43)
Eq. (41) is commonly used to study neutrino propagation with neutrino-neutrino interactions (see Refs. [3] through [18]). Here we illustrated that it is not an exact result, but represents the saddle point solution to the many-body problem.
There is an alternative method to obtain the approximate results outlined above. This method, typically employed in the random-phase approximation approach to many-body problems, approximates the product of two commuting arbitrary operators \( \hat{O}_1 \) and \( \hat{O}_2 \) as
\[ \hat{O}_1 \hat{O}_2 \sim \hat{O}_1 \langle \xi | \hat{O}_2 | \xi \rangle + \langle \xi | \hat{O}_1 | \xi \rangle \hat{O}_2 - \langle \xi | \hat{O}_1 | \xi \rangle \langle \xi | \hat{O}_2 | \xi \rangle, \] (44)
provided that the condition
\[ \langle \xi | \hat{O}_1 | \xi \rangle = \langle \xi | \hat{O}_1 | \xi \rangle \langle \xi | \hat{O}_2 | \xi \rangle \] (45)
is satisfied. In Eq. (44) the state \( |\xi\rangle \) must be appropriately chosen so that Eq. (45) is satisfied. Since the quantity \( R_{pq} \) is zero when the operators do not commute (i.e., when \( p = q \)), one can apply this approximation technique to the quadratic element \( J(p) \cdot J(q) \) which appear in the neutrino-neutrino forward scattering Hamiltonian \( H_{\nu\nu} \) given in Eq. (20). It follows from Eq. (32) that the SU(2) coherent states satisfy the condition stated in Eq. (45). Using the symmetry \( R_{pq} = R_{qp} \), one can then replace \( H_{\nu\nu} \) by
\[ H_{\nu\nu} \sim 2 \frac{\sqrt{2} G_F}{V} \int d^3p d^3q R_{pq} \left( J_0(p) \langle J_0(q) \rangle + \frac{1}{2} J_+(p) \langle J_-(q) \rangle + \frac{1}{2} J_-(p) \langle J_+(q) \rangle \right), \] (46)
where the expectation values are calculated using the SU(2) coherent states (the variable \( z(t) \) of these coherent states must be obtained in a self-consistent way). In writing Eq. (46) we omitted a term proportional to identity. The
total Hamiltonian for the two neutrino flavors propagating in the presence of electrons and other neutrinos in the background becomes approximately

\[
H_\nu + H_{\nu\nu} \sim \int d^3p \left( \frac{\delta m^2}{2p} \cos 2\theta - \sqrt{2}G_F N_e - B(p) \right) J_0(p) + \int d^3p \left( \frac{1}{2} \sin^2 2\theta + \frac{1}{2} B_{\nu\nu}(p) \right) J_+(p). \tag{47}
\]

Here \(B(p), B_{\nu\nu}(p)\) and \(B_{\nu\nu}(p)\) are given in Eq. 50 and we used the expectation values given in Eq. 31. Since the above approximate Hamiltonian is linear in the SU(2) generators we can use the same ansatz for the evolution operator of the system as we did in section II, i.e.

\[
U = \exp \left( \int d^3p z(p,t) J_+(p) \right) \exp \left( \int d^3p \log(1+|z(p,t)|^2) J_0(p) \right) \exp \left( - \int d^3p z^*(p,t) J_-(p) \right). \tag{48}
\]

It is straightforward to show that, substituting this ansatz into the equation

\[
i\frac{d}{dt} U = (H_\nu + H_{\nu\nu}) U \tag{49}
\]
yields nothing but Eq. 37 for \(z(p,t)\). It is also straightforward to generalize this linearization scheme to the situations where antineutrinos and more flavors are present.

Corrections to the path integral will naturally arise as a result of the deviations from the classical (i.e., stationary) path given by Eq. 37 or Eq. 41. To calculate the effect of small deviations one can carry out a Taylor expansion in order to eliminate the exponential factor in the path integral measure in Eq. 27:

\[
\beta(p,t) = \frac{z(p,t)}{\sqrt{1 + |z(p,t)|^2}} \quad \beta^*(p,t) = \frac{z^*(p,t)}{\sqrt{1 + |z(p,t)|^2}} \tag{50}
\]

The Jacobian resulting from this change of variables conveniently cancels out the exponential factor. It is straightforward to show that

\[
e^{-\frac{1}{2} \sum_{\alpha=1}^N \int d\beta \log(1+|\beta_{t\alpha}|^2)} \prod_{\alpha=1}^N \int \frac{d^2z(p,t_{t\alpha})}{2\pi} = \prod_{\alpha=1}^N \int \frac{d^2\beta(p,t_{t\alpha})}{2\pi} \tag{51}
\]

The path integral then takes the form

\[
\langle z'(t_f)|U|z(t_i)\rangle = \int \lim_{N \to \infty} \prod_{\alpha=1}^N \frac{d\beta(p,t_{t\alpha})d\beta^*(p,t_{t\alpha})}{i\pi} e^{iS[\beta,\beta^*]} \tag{52}
\]

where the action functional, written in terms of the new variables is as follows:

\[
S[\beta,\beta^*] = \int_{t_i}^{t_f} dt \left[ \int dp \left( \frac{\beta(p,t)\beta^*(p,t) - \beta(p,t)\beta^*(p,t)}{2i} + \frac{1}{2} \left( \frac{\delta m^2}{2p} \cos 2\theta - \sqrt{2}G_F N_e \right) (1 - 2|\beta(p,t)|^2) \right) 
- \frac{1}{2} \frac{\delta m^2}{2p} \sin 2\theta \left( \beta(p,t) + \beta^*(p,t) \sqrt{1 - |\beta(p,t)|^2} \right) \right]
+ \frac{1}{2} \frac{\sqrt{2}G_F}{V} \int d^3p d^3q R_{pq} \left[ \frac{(|\beta(p,t)|^2 + |\beta(q,t)|^2) - 2|\beta(p,t)|^2|\beta(q,t)|^2}{(1 - |\beta(p,t)|^2)(1 - |\beta(q,t)|^2)(\beta(p,t)\beta^*(q,t) + \beta^*(p,t)\beta(q,t))} \right] \right] \right]
+ \frac{1}{2} \frac{\delta m^2}{2p} \sin 2\theta \left( \beta(p,t) + \beta^*(p,t) \sqrt{1 - |\beta(p,t)|^2} \right) \right]
+ \frac{1}{2} \frac{\sqrt{2}G_F}{V} \int d^3q R_{pq} \frac{3|\beta(p,t)|^2 - 2}{2\sqrt{1 - |\beta(p,t)|^2}} \frac{\beta(p,t)^2}{\sqrt{1 - |\beta(p,t)|^2}} \tag{53}
\]

One obtains the following equation of motion (and its complex conjugate) from the variation of this action:

\[
i\dot{\beta}(p,t) = \left( \frac{\delta m^2}{2p} \cos 2\theta - \sqrt{2}G_F N_e \right) - \frac{\sqrt{2}G_F}{V} \int d^3q R_{pq} (1 - 2|\beta(q,t)|^2) \right) \right) \beta(p,t) \tag{54}
\]

\[
+ \left( \frac{\delta m^2}{2p} \sin 2\theta + \frac{\sqrt{2}G_F}{V} \int d^3q R_{pq} \left( 1 - |\beta(q,t)|^2 \right) \beta(q,t) \right) \right) + \left( \frac{\delta m^2}{2p} \sin 2\theta + \frac{\sqrt{2}G_F}{V} \int d^3q R_{pq} \left( 1 - |\beta(q,t)|^2 \right) \beta(q,t) \right) \right) - \frac{\beta(p,t)^2}{\sqrt{1 - |\beta(p,t)|^2}} \tag{55}
\]
This classical path is the same as the one given in Eq. (57). This can be shown directly by substituting the transformation in Eq. (50) into Eq. (37). We will denote the classical path as $\beta_{cl}$. Since the first order variations are zero on the classical path by definition, the Taylor expansion around $\beta_{cl}$ yields

$$S[\beta, \beta^*] = S[\beta_{cl}, \beta_{cl}^*] + \frac{1}{2} (\beta - \beta_{cl})^T \left( \frac{\delta^2 S}{\delta \beta \delta \beta} \right)_{cl} (\beta - \beta_{cl})$$

$$+ (\beta - \beta_{cl})^T \left( \frac{\delta^2 S}{\delta \beta^* \delta \beta'} \right)_{cl} (\beta^* - \beta_{cl}^*) + \ldots$$

Here $(\ldots)_{cl}$ indicates that the derivatives are to be calculated on the classical path $\beta_{cl}$ and $(\beta - \beta_{cl})^T \left( \frac{\delta^2 S}{\delta \beta^* \delta \beta'} \right)_{cl} (\beta^* - \beta_{cl}^*)$ is a short hand notation for the matrix product

$$\sum_{p,k} \sum_{q,m} (\beta(p,t_k) - \beta_{cl}(p,t_k))^T \left( \frac{\delta^2 S}{\delta \beta(p,t_k) \delta \beta(p,t_m)} \right)_{cl} (\beta(p,t_m) - \beta_{cl}(p,t_m)),$$

and similarly for the other terms. The sums over $p$ and $q$ run through the allowed momentum modes and the sums over $k$ and $m$ run from 1 to $N$ which is the number of time intervals we introduced in path integral. $N \to \infty$ limit should be taken as explained in the Appendix. For small deviations from the classical path, one can ignore the higher order terms in the expansion (55) and substitute it in Eq. (52):

$$\langle z'(t_f)|U|z(t_i) \rangle = e^{iS[\beta_{cl}, \beta_{cl}^*]} \int \lim_{N \to \infty} \prod_{k=1}^{N} \prod_{p \in \mathcal{P}} d \tilde{\beta}(p,t_k) d \tilde{\beta}^*(p,t_k) e^{i \left( \frac{1}{2} \tilde{\beta}^T \left( \frac{\delta^2 S}{\delta \beta \delta \beta} \right)_{cl} \tilde{\beta} + \frac{1}{2} \tilde{\beta}^* \left( \frac{\delta^2 S}{\delta \beta^* \delta \beta'} \right)_{cl} \tilde{\beta}^* + \frac{1}{2} \tilde{\beta}^* \left( \frac{\delta^2 S}{\delta \beta \delta \beta'} \right)_{cl} \tilde{\beta} + \frac{1}{2} \tilde{\beta} \left( \frac{\delta^2 S}{\delta \beta^* \delta \beta} \right)_{cl} \tilde{\beta} \right)}$$

where we defined $\tilde{\beta} = \beta - \beta_{cl}$. The classical action $S[\beta_{cl}, \beta_{cl}^*]$ is taken out of the integration since it does not depend on $\tilde{\beta}$. The lowest order quantum corrections are captured by the Gaussian integral in Eq. (57). The result of the integration is

$$\langle z'(t_f)|U|z(t_i) \rangle = \lim_{N \to \infty} (i\pi)^{N+P} \frac{e^{iS[\beta_{cl}, \beta_{cl}^*]}}{\sqrt{\text{Det} (KM - L^T K^-L)}}.$$  

Here $P$ denote the number of allowed momentum modes. The matrices $K, M$ and $L$ are given as follows:

$$K(p,k,q,m) = \frac{1}{2} \left( \frac{\delta^2 S}{\delta x(p,t_k) \delta x(q,t_m)} \right)_{cl}$$

$$M(p,k,q,m) = \frac{1}{2} \left( \frac{\delta^2 S}{\delta y(p,t_k) \delta y(q,t_m)} \right)_{cl}$$

$$L(p,k,q,m) = \frac{1}{2} \left( \frac{\delta^2 S}{\delta x(p,t_k) \delta y(q,t_m)} \right)_{cl}$$

where $x = (\tilde{\beta} + \tilde{\beta}^*)/2$ and $y = (\tilde{\beta} - \tilde{\beta}^*)/2i$. The fundamental difficulty involved in the calculation of the determinant in Eq. (58) is the existence of non-diagonal terms in the matrices $K, M$ and $L$. These terms are generated by the $\int d^3p \, d^3q \, R_{pq} \ldots$ integral in the action given in Eq. (53). A complete analysis of these determinants is beyond the scope of the present paper and will be given elsewhere.

IV. NEUTRINO-ANTINEUTRINO INTERACTIONS

Environments such as core-collapse supernovae and the Early Universe contain copious amounts of antineutrinos as well as neutrinos. When antineutrinos are added to the picture the effective flavor evolution Hamiltonian becomes

$$H = H_\nu + H_\bar{\nu} + H_{\nu\nu} + H_{\nu\bar{\nu}} + H_{\bar{\nu}\bar{\nu}}.$$  

Here, $H_\nu$ and $H_{\nu\nu}$ are given in Eqs. (1) and (18). $H_\nu$ and $H_{\bar{\nu}\bar{\nu}}$ are the same as $H_\nu$ and $H_{\nu\nu}$, respectively except that the neutrino operators $a$ and $a^\dagger$ are replaced by the antineutrino operators $b$ and $b^\dagger$ and the sign of the $N_e$ term is reversed. The neutrino-antineutrino forward scattering Hamiltonian $H_{\nu\bar{\nu}}$ is given as follows:

$$H_{\nu\bar{\nu}} = - \frac{\sqrt{2}G_F}{V} \int d^3p \, d^3q \, R_{pq} \left[ a_\nu^\dagger(p) a_\nu(p) b_\bar{\nu}^\dagger(q) b_\bar{\nu}(q) + a_\bar{\nu}^\dagger(p) a_\bar{\nu}(p) b_\nu^\dagger(q) b_\nu(q) + a_\nu^\dagger(p) a_\nu(p) b_\bar{\nu}(q) b_\bar{\nu}(q) \right].$$

$$+ a_\bar{\nu}^\dagger(p) a_\bar{\nu}(p) b_\nu(q) b_\nu(q) + a_\nu^\dagger(p) a_\nu(p) b_\bar{\nu}(q) b_\bar{\nu}(q) \right].$$
In addition to the neutrino operators defined in Eq. (2), we now define the antineutrino SU(2) operators\(^2\)

\[
\bar{J}_+ (p) = b_+^\dagger (p) b_+ (p), \quad \bar{J}_- (p) = b_-^\dagger (p) b_- (p), \quad \bar{J}_0 (p) = \frac{1}{2} \left( b_+^\dagger (p) b_- (p) - b_-^\dagger (p) b_+ (p) \right).
\]  

(63)

These operators also obey SU(2) commutation relations

\[
[\bar{J}_+ (p), \bar{J}_- (q)] = 2\delta (p - q) \bar{J}_0 (p), \quad [\bar{J}_0 (p), \bar{J}_\pm (q)] = \pm \delta (p - q) \bar{J}_\pm (p)
\]

(64)

and they commute with the neutrino operators

\[
[J_i (p), \bar{J}_j (q)] = 0.
\]

(65)

Written in terms of \(J_i (p)\) and \(\bar{J}_i (p)\), the Hamiltonian in Eq. (61) takes the following form:

\[
\begin{align*}
H_\nu + H_\nu & + H_{\nu\nu} + H_{\nu\bar{\nu}} + H_{\bar{\nu}\bar{\nu}} \\
& = \int d^3 p \left( \frac{\delta m^2}{2\rho} \cos 2\theta - \sqrt{2} G_F N_e \right) \bar{J}_0 (p) + \frac{1}{2} \int d^3 p \frac{\delta m^2}{2\rho} \sin 2\theta (J_+ (p) + J_- (p)) \\
& + \int d^3 p \left( - \frac{\delta m^2}{2\rho} \cos 2\theta - \sqrt{2} G_F N_e \right) \bar{J}_0 (p) + \frac{1}{2} \int d^3 p \frac{\delta m^2}{2\rho} \sin 2\theta (J_+ (p) + J_- (p)) \\
& + \frac{\sqrt{2} G_F}{V} \int d^3 p d^3 q R_{pq} (J (p) \cdot J (q) + \bar{J} (p) \cdot \bar{J} (q)) \\
& - \frac{\sqrt{2} G_F}{V} \int d^3 p d^3 q R_{pq} (-2J_0 (p) \bar{J}_0 (q) + J_+ (p) \bar{J}_- (q) + J_- (p) \bar{J}_+ (q))
\end{align*}
\]

(66)

where a term proportional to identity is omitted.

As we did in the previous section, we introduce the coherent states\(^3\)

\[
|z (t), \bar{z} (t)\rangle = N \bar{N} e^{\int d^3 p z (p, t) J_+ (p) e^{\int d^3 \bar{p} \bar{z} (\bar{p}, t) \bar{J}_- (\bar{p})} |\phi\rangle.
\]

(67)

Here \(|\phi\rangle\) is analogous to the state defined in Eq. (4), i.e., all permitted \(\nu_e\) and \(\bar{\nu}_e\) states are occupied and all other neutrino flavor states are empty:

\[
|\phi\rangle = \prod_{p \in \mathcal{P}} a_+^\dagger (p) \prod_{\bar{p} \in \bar{\mathcal{P}}} b_-^\dagger (\bar{p}) |0\rangle.
\]

(68)

In the above formulas, \(\mathcal{P}\) and \(\bar{\mathcal{P}}\) denote the set of all allowed momentum modes for neutrinos and antineutrinos, respectively. In what follows, we will drop the symbols \(\mathcal{P}\) and \(\bar{\mathcal{P}}\) from the formulas by adopting the convention that the non-overlined quantities such as \(p, q\) will take values in \(\mathcal{P}\) whereas the overlined quantities such as \(\bar{p}, \bar{q}\) take values in \(\bar{\mathcal{P}}\).

The constants

\[
N = \exp \left( -\frac{1}{2} \int d^3 p \log (1 + |z(p, t)|^2) \right), \quad \bar{N} = \exp \left( -\frac{1}{2} \int d^3 \bar{p} \log (1 + |\bar{z}(\bar{p}, t)|^2) \right)
\]

(69)

in Eq. (67) normalize the coherent states:

\[
\langle z, \bar{z} | z, \bar{z} \rangle = 1.
\]

(70)

A path integral representation of the evolution operator can be given in terms of these coherent states as

\[
\langle z' (t_f), \bar{z}' (t_f) | U | z (t_i), \bar{z} (t_i) \rangle = \int D[z, z^*, \bar{z}, \bar{z}^*] e^{iS[z, z^*, \bar{z}, \bar{z}^*]}
\]

(71)

\(^2\)Time-reversal invariance requires that the order of the flavors is flipped in the definition of the antineutrino algebra as compared with the definition of the neutrino algebra in Eq. (2).

\(^3\)In these formulas \(z\) is an independent complex number, not the complex conjugate of \(z\). Complex conjugates will be denoted by a star such as \(z^*\) and \(\bar{z}^*\).
where the action functional $S[z, z^*, \bar{z}, \bar{z}^*]$ is

$$S[z, z^*, \bar{z}, \bar{z}^*] = \int_{t_i}^{t_f} dt (i \partial_\xi - (H_\nu + H_\bar{\nu} + H_\nu^\alpha + H_\bar{\nu}^\alpha)) - i \ln\langle z' (t_f), \bar{z}' (t_f) | z(t_f), \bar{z}(t_f) \rangle. \quad (72)$$

Once more we can find the stationary path by solving the Euler-Lagrange equations for

$$L[z, z^*, \bar{z}, \bar{z}^*] = \langle i \partial_\xi - (H_\nu + H_\bar{\nu} + H_\nu^\alpha + H_\bar{\nu}^\alpha) \rangle.$$

The linear terms in the Lagrangian can be calculated using

$$\langle J_+(p) \rangle = \langle J_-(p) \rangle^* = \frac{z^* (p, t)}{1 + |z(p, t)|^2}, \quad \langle J_0 (p) \rangle = -\frac{1}{2} \frac{1 - |z(p, t)|^2}{1 + |z(p, t)|^2}, \quad (74)$$

$$\langle \bar{J}_+ (\bar{p}) \rangle = \langle \bar{J}_- (\bar{p}) \rangle^* = \frac{\bar{z}(\bar{p}, t)}{1 + |\bar{z}(\bar{p}, t)|^2}, \quad \langle \bar{J}_0 (\bar{p}) \rangle = \frac{1}{2} \frac{1 - |\bar{z}(\bar{p}, t)|^2}{1 + |\bar{z}(\bar{p}, t)|^2}, \quad (75)$$

which are valid for $p \in \mathcal{P}$ and $\bar{p} \in \bar{\mathcal{P}}$. When $p \notin \mathcal{P}$ or $\bar{p} \notin \bar{\mathcal{P}}$ the expectation values are zero. To calculate the quadratic terms we use the identities

$$\langle J_a (p) J_a^\dagger (q) \rangle = \langle J_a (p) \rangle \langle J_a^\dagger (q) \rangle, \quad \langle \bar{J}_a (\bar{p}) \bar{J}_a^\dagger (\bar{q}) \rangle = \langle \bar{J}_a (\bar{p}) \rangle \langle \bar{J}_a^\dagger (\bar{q}) \rangle, \quad \langle J_a (p) \bar{J}_a^\dagger (\bar{q}) \rangle = \langle J_a (p) \rangle \langle \bar{J}_a^\dagger (\bar{q}) \rangle. \quad (76)$$

Here $a = 0, \pm$ and we assumed $p \neq q$ and $\bar{p} \neq \bar{q}$. Also note that

$$\langle i \frac{d}{dt} \rangle = i \left( \int d^3 p \frac{\dot{z}(p) z^*(p)}{1 + |z(p)|^2} + \int d^3 \bar{p} \frac{\dot{\bar{z}}(\bar{p}) \bar{z}^*(\bar{p})}{1 + |\bar{z}(\bar{p})|^2} \right). \quad (77)$$

Using Eqs. (74) - (77) Lagrangian can be found as follows:

$$L = L_\nu + L_\bar{\nu} - \frac{1}{4} \int d^3 p B(p) \frac{(1 - |z(p)|^2)}{1 + |z(p)|^2} + \frac{2 B_{xx}(p) z(p) + 2 B_{xx}(p) z^*(p)}{1 + |z(p)|^2} \quad (78)$$

Here $L_\nu$ is the Lagrangian given in Eq. (35) and $L_\bar{\nu}$ is the same as $L_\nu$ except that we substitute $\bar{z}(\bar{p})$ in place of $z(p)$ and change the sign of $N_\alpha$. In the above equation we also defined

$$B(p) = \sqrt{2} G_f \frac{1 - |z(q, t)|^2}{1 + |z(q, t)|^2} \quad (79)$$

and

$$B_{xx}(p) = B_{xx}^*(p) = 2 \sqrt{2} G_f \frac{z(q, t)}{1 + |z(q, t)|^2} \quad (80)$$

Equations of motion which result from this Lagrangian are as follows:

$$i \dot{z}(p, t) = \beta (p, t) - \alpha (p, t) z(p, t) - \beta^* (p, t) z^2 (p, t), \quad (81)$$

$$i \dot{\bar{z}}(\bar{p}, t) = -\beta^* (\bar{p}, t) + \alpha (\bar{p}, t) \bar{z}(\bar{p}, t) + \beta (\bar{p}, t) \bar{z}^2 (\bar{p}, t) \quad (82)$$

---

4 Here the expectation values are calculated using the states in Eq. (77):

$$\langle O \rangle = \langle z(t), \bar{z}(t) | O | z(t), \bar{z}(t) \rangle.$$
and the complex conjugates of these equations. The coefficients \( \alpha \) and \( \beta \) are given by:

\[
\alpha(p, t) = -\frac{\delta m^2}{2p} \cos 2\theta + \sqrt{2}G_F N_e + B(p) \quad \beta(p, t) = \frac{1}{2} \frac{\delta m^2}{2p} \sin 2\theta + B_{ex}(p).
\]

(83)

The coefficients \( \alpha \) and \( \beta \) are the same except that the sign of the \( \delta m^2 \) is changed.

We can again write the parameters \( z \) and \( \bar{z} \) in terms of the one-body neutrino and antineutrino wave functions \( \psi_e, \psi_x, \psi_e \) and \( \psi_x \) as follows:

\[
z(p, t) = \frac{\psi_x(p, t)}{\psi_e(p, t)} \quad \text{and} \quad \bar{z}(\bar{p}, t) = \frac{\bar{\psi}_x(\bar{p}, t)}{\bar{\psi}_e(\bar{p}, t)}.
\]

(84)

We also have the normalization conditions

\[
|\psi_e|^2 + |\psi_x|^2 = 1 \quad \text{and} \quad |\bar{\psi}_e|^2 + |\bar{\psi}_x|^2 = 1.
\]

(85)

Substituting Eq. (84) into Eqs. (81) and (82) we see that they are equivalent to the Schrödinger equations

\[
i \frac{\partial}{\partial t} \begin{pmatrix} \psi_e(p, t) \\ \psi_x(p, t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A + B - \Delta \cos 2\theta & B_{ex} + \Delta \sin 2\theta \\ B_{ex} - \Delta \sin 2\theta & A - B - \Delta \cos 2\theta \end{pmatrix} \begin{pmatrix} \psi_e(p, t) \\ \psi_x(p, t) \end{pmatrix},
\]

(86)

and

\[
i \frac{\partial}{\partial t} \begin{pmatrix} \bar{\psi}_e(\bar{p}, t) \\ \bar{\psi}_x(\bar{p}, t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A + B + \Delta \cos 2\theta & B_{ex} - \Delta \sin 2\theta \\ B_{ex} + \Delta \sin 2\theta & A - B + \Delta \cos 2\theta \end{pmatrix} \begin{pmatrix} \bar{\psi}_e(\bar{p}, t) \\ \bar{\psi}_x(\bar{p}, t) \end{pmatrix}.
\]

(87)

In the above formula \( B \) and \( B_{ex} \) are obtained by substituting Eq. (85) into Eqs. (59) and (60), i.e.,

\[
B = \frac{\sqrt{2}G_F}{V} \int d^3q R_{pq} (|\psi_e(q, t)|^2 - |\psi_x(q, t)|^2) - \sqrt{2}G_F \int d^3q R_{pq} (|\bar{\psi}_e(\bar{q}, t)|^2 - |\bar{\psi}_x(\bar{q}, t)|^2),
\]

(88)

\[
B_{ex} = 2\frac{\sqrt{2}G_F}{V} \int d^3q R_{pq} \psi_e(q, t) \psi^*_x(q, t) - 2\sqrt{2}G_F \int d^3q R_{pq} \bar{\psi}_e(\bar{q}, t) \bar{\psi}_x^*(\bar{q}, t).
\]

(89)

V. THREE NEUTRINO FLAVORS

In this section we generalize our formalism to three neutrino flavors, i.e., when \( \theta_{13} \neq 0 \). SU(3) symmetry of the neutrinos can be represented by the operators

\[
T_{ij}(p) = \psi_i(p) a_j(p) \quad \text{and} \quad \bar{T}_{ij}(p) = \bar{\psi}_j(p) b_i(p)
\]

(90)

where \( i \) and \( j \) run over the flavor indices \( e, \mu \) and \( \tau \). These operators generate orthogonal SU(3) algebras:

\[
[T_{ij}(p), T_{kl}(q)] = \delta(p - q) \delta_{kj} T_{il}(p) - \delta_{li} T_{kj}(p),
\]

(91)

\[
[T_{ij}(p), \bar{T}_{kl}(q)] = -\delta(p - q) \delta_{kj} \bar{T}_{il}(p) - \delta_{li} \bar{T}_{kj}(p),
\]

\[
[T_{ij}(p), \bar{T}_{kl}(q)] = 0.
\]

The effective Hamiltonian describing the propagation of neutrinos in a background of electrons is given by

\[
H = \sum_{i,j} \int d^3p \left[(\gamma_{ij}(p) + \omega_{ij}(p)) T_{ij}(p) + (\gamma_{ij}(p) - \omega_{ij}(p)) \bar{T}_{ij}(p) \right] + \frac{G_F}{\sqrt{2}V} \int d^3pd^3q R_{pq} \sum_{i,j} (T_{ij}(p) - \bar{T}_{ij}(p))(T_{ji}(q) - \bar{T}_{ji}(q)).
\]

(92)

In this Hamiltonian, the coefficients \( \gamma_{ij}(p) \) are the elements of the symmetric, traceless matrix \( \Gamma \) which is given below:

\[
\Gamma = (\gamma_{ij}) = \frac{1}{3} Q_{23} Q_{13} Q_{12} \left( \begin{array}{ccc}
-\frac{\delta m^2_{21}}{2p} & 0 & 0 \\
0 & \frac{\delta m^2_{13}}{2p} & \frac{\delta m^2_{13}}{2p} \\
0 & 0 & \frac{\delta m^2_{13} + \delta m^2_{23}}{2p}
\end{array} \right) \left( \begin{array}{c}
Q_{12}^T Q_{13}^T Q_{23}^T
\end{array} \right).
\]

(93)
Here \( Q_{12}, Q_{13} \) and \( Q_{23} \) are neutrino mixing matrices in vacuum:

\[
Q_{23}Q_{13}Q_{12} = \begin{pmatrix}
1 & 0 & 0 \\
0 & C_{23} & S_{23} \\
0 & -S_{23} & C_{23}
\end{pmatrix}\begin{pmatrix}
C_{13} & 0 & S_{13}^* \\
0 & 1 & 0 \\
-S_{13} & 0 & C_{13}
\end{pmatrix}\begin{pmatrix}
C_{12} & S_{12} & 0 \\
-S_{12} & C_{12} & 0 \\
0 & 0 & 1
\end{pmatrix},
\tag{94}
\]

where \( C_{13}, \) etc. is the short-hand notation for \( \cos \theta_{13}, \) etc. Since \( S_{13} \) may be multiplied by a phase, we explicitly indicated its complex conjugate. Note that individual matrices, not their matrix elements, are called \( Q_{23}, Q_{13}, \) and \( Q_{12}, \) respectively in Eq. \( (127) \).

The terms in the Hamiltonian which are proportional to \( \gamma_{ij} \) represent vacuum oscillations of the neutrinos. The coefficients \( \omega_{ij} \) are real and symmetric. They are the elements of the diagonal, traceless matrix \( \Omega \) given below:

\[
\Omega = (\omega_{ij}) = \frac{1}{3} \begin{pmatrix}
2V_c & 0 & 0 \\
0 & -V_c & 0 \\
0 & 0 & -V_c
\end{pmatrix}.
\tag{95}
\]

Here \( V_c \) is the Wolfenstein potential described earlier.

A path integral formula for the evolution operator of the system can be constructed using SU(3) coherent states which are given by

\[
|\vec{z}, \vec{z}'\rangle = N \bar{N} e^{\int P dp \phi(z_p, p) \bar{z}^* \bar{z}'(p) \tau e^{\int P dp \phi(z_p, p) \bar{z}^* \bar{z}'(p)}} |\phi\rangle.
\tag{96}
\]

These coherent states are defined with respect to the reference state \( |\phi\rangle \) which is defined as in Eq. \( (98) \). The normalization constants \( N \) and \( \bar{N} \) are given by

\[
N = \exp \left( -\frac{1}{2} \int d^3 p \ln (1 + |z_\mu(p, t)|^2 + |z_\tau(p, t)|^2) \right), \quad \bar{N} = \exp \left( -\frac{1}{2} \int d^3 \bar{p} \ln (1 + |\bar{z}_\mu(\bar{p}, t)|^2 + |\bar{z}_\tau(\bar{p}, t)|^2) \right).
\tag{97}
\]

Neutrino SU(3) coherent states are characterized by two complex numbers that we denoted by \( z_\mu \) and \( z_\tau \) in Eq. \( (96) \). We used \( \bar{z}_\mu \) and \( \bar{z}_\tau \) for the SU(3) symmetry of the antineutrinos. As a practical convention, we also define \( z_\mu(p) = \bar{z}_\mu(\bar{p}) = 1 \) in what follows.

Evolution operator can be given by the following path integral formula in terms of the SU(3) coherent states:

\[
\langle \vec{z}'(t_f), \vec{z}'(t_f) | U | \vec{z}(t_i), \vec{z}(t_i) \rangle = \int D[\vec{z}, \vec{z}] e^{iS[\vec{z}, \vec{z}]},
\tag{98}
\]

where the measure is given by Eq. \( (127) \) of the Appendix. In this formula

\[
S[\vec{z}, \vec{z}] = \int_{t_i}^{t_f} dt \langle \vec{z}(t), \vec{z}(t) \rangle \frac{d}{dt} - H(t) | \vec{z}(t), \vec{z}(t) \rangle - i \ln \langle \vec{z}'(t_f), \vec{z}'(t_f) | \vec{z}(t_f), \vec{z}(t_f) \rangle
\tag{99}
\]

plays the role of a classical action. The derivation of these formulas and the exact expression for the integral measure can be found in the Appendix.

As before, we write down the Lagrangian

\[
L[\vec{z}, \vec{z}] = \langle i \frac{d}{dt} - H \rangle
\tag{100}
\]

and solve the Euler-Lagrange equations

\[
\left( \frac{d}{dt} \frac{\partial}{\partial \vec{z}_n} - \frac{\partial}{\partial \vec{z}_n} \right) L[\vec{z}, \vec{z}] = 0 \quad \text{and} \quad \left( \frac{d}{dt} \frac{\partial}{\partial \vec{z}_n} - \frac{\partial}{\partial \vec{z}_n} \right) L[\vec{z}, \vec{z}] = 0
\tag{101}
\]

5 For example

\[
\sum_k |z_k(p)|^2 = 1 + |z_\mu(p)|^2 + |z_\tau(p)|^2.
\]

6 Here we use the short hand notation \( \langle \mathcal{O} \rangle = \langle \vec{z}(t), \vec{z}(t) | \mathcal{O} | \vec{z}(t), \vec{z}(t) \rangle, \) etc.
for \( n = \mu, \tau \) and those for \( z_n^+ \) and \( \bar{z}_n^+ \). We use the expectation values of the SU(3) generators which are given below:

\[
\langle T_{ij}(p) \rangle = \frac{z_n^+(p) z_j(p)}{\sum_k |z_k(p)|^2}, \quad \langle \bar{T}_{ij}(\bar{p}) \rangle = \frac{\bar{z}_n(\bar{p}) \bar{z}_j(\bar{p})}{\sum_k |\bar{z}_k(\bar{p})|^2}
\]  

(102)

for \( i, j = e, \mu, \tau \) (with the convention \( z_n(p) = \bar{z}_n(p) = 1 \)). Here we assumed \( p \in \mathcal{P} \) and \( \bar{p} \in \bar{\mathcal{P}} \). If \( p \notin \mathcal{P} \) or \( \bar{p} \notin \bar{\mathcal{P}} \) then the expectation values are zero. The quadratic terms in the Lagrangian are calculated using the identities

\[
\langle T_{ij}(p) T_{ji}(q) \rangle = \langle T_{ij}(p) \rangle \langle T_{ji}(q) \rangle, \quad \langle T_{ij}(p) \bar{T}_{ji}(\bar{q}) \rangle = \langle T_{ij}(p) \rangle \langle \bar{T}_{ji}(\bar{q}) \rangle,
\]

which are valid for \( p \neq p \) and \( \bar{p} \neq \bar{q} \). The expectation value of the time derivative term is

\[
\langle i \frac{d}{dt} \rangle = i \sum_{n(\neq e)} \left( \int d^3p \frac{\dot{z}_n(p) z_n^+(p)}{\sum_k |z_k(p)|^2} + \int d^3\bar{p} \frac{\dot{\bar{z}}_n(\bar{p}) \bar{z}_n^+(\bar{p})}{\sum_k |\bar{z}_k(\bar{p})|^2} \right).
\]

(105)

Using formulas (102)–(105) we find the Lagrangian as

\[
L[z, \bar{z}] = i \sum_{n(\neq e)} \left( \int d^3p \frac{\dot{z}_n(p) z_n^+(p)}{\sum_k |z_k(p)|^2} + \int d^3\bar{p} \frac{\dot{\bar{z}}_n(\bar{p}) \bar{z}_n^+(\bar{p})}{\sum_k |\bar{z}_k(\bar{p})|^2} \right)
\]

(106)

\[
- \sum_{i,j} \int d^3p \left( \gamma_{ij}(p) + \omega_{ij}(p) \right) \frac{\dot{z}_j(p) z_i(p)}{\sum_k |z_k(p)|^2} - \sum_{i,j} \int d^3\bar{p} \left( \gamma_{ij}(\bar{p}) - \omega_{ij}(\bar{p}) \right) \frac{\dot{\bar{z}}_j(\bar{p}) \bar{z}_i(\bar{p})}{\sum_k |\bar{z}_k(\bar{p})|^2}.
\]

Here \( Y_{ij}(p) \) is given by

\[
Y_{ij}(p) = \frac{\sqrt{2} G_F}{V} \left( \int d^3q R_{pq} \frac{z_i(q) z_j^+(q)}{\sum_k |z_k(q)|^2} - \int d^3q R_{pq} \frac{\bar{z}_{i}(\bar{q}) \bar{z}_{j}^+(\bar{q})}{\sum_k |\bar{z}_k(\bar{q})|^2} \right).
\]

(107)

The equations of motion derived from this Lagrangian are given below:

\[
i \dot{z}_n(p) = \sum_i \left[ (\gamma_{ni}(p) + \omega_{ni}(p) + Y_{ni}(p)) z_i(p) - (\gamma_{ei}(p) + \omega_{ei}(p) + Y_{ei}(p)) z_n(p) z_i(p) \right],
\]

(108)

\[
i \dot{\bar{z}}_n(\bar{p}) = \sum_i \left[ (\gamma_{ni}(\bar{p}) - \omega_{ni}(\bar{p}) - Y_{ni}(\bar{p})) \bar{z}_i(\bar{p}) - (\gamma_{ei}(\bar{p}) - \omega_{ei}(\bar{p}) - Y_{ei}(\bar{p})) \bar{z}_n(\bar{p}) \bar{z}_i(\bar{p}) \right],
\]

(109)

where \( n = \mu, \tau \). We write the parameters \( z_i \) and \( \bar{z}_i \) in terms of the neutrino wavefunctions as \( z_i(p) = \psi_i(p)/\psi_e(p) \) and \( \bar{z}_i(\bar{p}) = \bar{\psi}_i^+(\bar{p})/\bar{\psi}_e^+(\bar{p}) \), i.e.,

\[
z_e(p) = \frac{\psi_e(p)}{\psi_e(p)} = 1, \quad z_\mu(p) = \frac{\psi_\mu(p)}{\psi_e(p)}, \quad z_\tau(p) = \frac{\psi_\tau(p)}{\psi_e(p)}
\]

(110)

and

\[
\bar{z}_e(\bar{p}) = \frac{\bar{\psi}_e^+(\bar{p})}{\bar{\psi}_e^+(\bar{p})} = 1, \quad \bar{z}_\mu(\bar{p}) = \frac{\bar{\psi}_\mu^+(\bar{p})}{\bar{\psi}_e^+(\bar{p})}, \quad \bar{z}_\tau(\bar{p}) = \frac{\bar{\psi}_\tau^+(\bar{p})}{\bar{\psi}_e^+(\bar{p})}
\]

(111)

where the one body wavefunctions are normalized as follows:

\[
|\psi_e(p)|^2 + |\psi_\mu(p)|^2 + |\psi_\tau(p)|^2 = 1, \quad |\bar{\psi}_e(\bar{p})|^2 + |\bar{\psi}_\mu(\bar{p})|^2 + |\bar{\psi}_\tau(\bar{p})|^2 = 1.
\]

(112)

If the Eqs. (110) and (111) are substituted in Eqs. (108) and (109), we find that \( \psi_i(p) \) and \( \bar{\psi}_i(\bar{p}) \) obey the following Schrodinger equations:

\[
i \frac{d}{dt} \begin{pmatrix} \psi_e(p) \\ \psi_\mu(p) \\ \psi_\tau(p) \end{pmatrix} = (\Gamma(p) + \Omega(p) + Y(p)) \begin{pmatrix} \psi_e(p) \\ \psi_\mu(p) \\ \psi_\tau(p) \end{pmatrix},
\]

(113)
Here, the matrix $Y$ is the matrix formed by the elements $Y_{ij}$ given in Eq. (107). Note that as one goes from the neutrino to antineutrino equations only the signs of the $\delta m^2$ terms change. If we substitute Eq. (110) and (111) in Eq. (107) we see that $Y_{ij}$ can be written in terms of the one-body wavefunctions as

$$Y_{ij}(p) = \frac{\sqrt{2G_F}}{V} \left( \int d^3q R_{pq} \psi_i(q) \psi_j^*(q) - \int d^3\bar{q} R_{pq} \bar{\psi}_i(\bar{q}) \bar{\psi}_j^*(\bar{q}) \right).$$ (115)

VI. CONCLUSIONS

In this article an algebraic approach to the neutrino propagation in dense media is presented. The Hamiltonian describing a gas of neutrinos interacting with each other and background fermions is written in terms of the appropriate SU(2) (for two flavors) or SU(3) (for three flavors) operators. Neutrinos as well as antineutrinos are considered. The evolution of the resulting many-body problem is formulated as either an SU(2) or an SU(3) coherent-state path integral.

The evolution operator for the entire system is calculated using two different approximations, namely the saddle-point approximation and the operator product linearization approximation. In our notation the one-body polarization vector is defined as

$$\bar{\psi}_i(\bar{p}) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \psi_e(p) \\
\psi_\mu(p) \\
\psi_\tau(p) \end{array} \right).$$ (114)

For two flavors. For three neutrino flavors one simply calculates the trace above with the SU(3) generators, resulting in an eight-dimensional vector. The polarization vector of Eq. (117) satisfies the equation

$$i\dot{P}(q) = \text{Tr}([J(q), H]|\rho).$$ (118)

It is straightforward to show that linearizing the Hamiltonian of Eq. (111) using Eq. (114) and substituting this linearized Hamiltonian in Eq. (118) yields the evolution equations of the polarization vectors as stated, e.g., in Refs. 3 and 6.

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Appendix: Path Integral Representation of the Evolution Operator

In this appendix, we drive the SU(3) path integral formula in Eq. (98). The path integral formula for SU(2) coherent states follows very similar lines.

The coherent states defined in Eq. (96) admit the following resolution of identity:

$$I = \int \prod_p \frac{3!}{(2\pi i)^2} \frac{d^2 z_e(p)}{(\sum_k |z_k(p)|^2)^{3/2}} \prod_p \frac{3!}{(2\pi i)^2} \frac{d^2 z_\mu(p)}{(\sum_k |z_k(p)|^2)^{3/2}} \frac{d^2 z_\tau(p)}{(\sum_k |z_k(p)|^2)^{3/2}} |z, \bar{z} \rangle \langle z, \bar{z}|,$$ (119)
One can use this resolution of identity to write down the path integral formula for the matrix element of the evolution operator $U$. The procedure is well established \[3\] and will only be outlined in what follows: We start by dividing the time interval $[0, T]$ into $N$ infinitesimally small pieces:

$$t_0 = 0, \quad t_1 = \varepsilon, \quad t_2 = 2\varepsilon, \ldots, \quad T = t_N = N\varepsilon. \quad (120)$$

Ignoring the time dependence of the Hamiltonian in each infinitesimal time interval we can write

$$\langle z'(T), z'(T)|U(T)|z(0), z(0)\rangle = \langle z'(T), z'(T)e^{-i\varepsilon H(t_N)}e^{-i\varepsilon H(t_{N-1})}\ldots e^{-i\varepsilon H(t_1)}|z(0), z(0)\rangle. \quad (121)$$

We then insert a resolution of identity to the left of each exponential. We will put an additional label to the variables $z_i(p)$ and $\bar{z}_i(p)$ in the resolution of identity which is inserted just next to $e^{-i\varepsilon H(t_\alpha)}$. This way, we obtain

$$\langle z'(T), z'(T)|U(T)|z(0), z(0)\rangle = \int \prod_{\alpha=1}^{N} \prod_{p} \frac{3!}{(2\pi i)^2} d^2z_\mu(p, t_\alpha) d^2z_\tau(p, t_\alpha) \prod_{\alpha=1}^{N} \prod_{\bar{p}} \frac{3!}{(2\pi i)^2} d^2\bar{z}_\mu(\bar{p}, t_\alpha) d^2\bar{z}_\tau(\bar{p}, t_\alpha) \times \langle z'(T), z'(T)|z(t_N), \bar{z}(t_N)\rangle \prod_{\alpha=1}^{N} \langle z(t_\alpha), \bar{z}(t_\alpha)\rangle e^{-i\varepsilon H(t_\alpha)}|z(t_{\alpha-1}), \bar{z}(t_{\alpha-1})\rangle. \quad (122)$$

Assuming that only the continuous paths will contribute to the integral when we take $N \to \infty$ limit, we can write

$$|z(t_{\alpha-1}), \bar{z}(t_{\alpha-1})\rangle = \left(1 - \varepsilon \frac{d}{dt}\right) |z(t_{\alpha}), \bar{z}(t_{\alpha})\rangle. \quad (123)$$

In this case the infinitesimal propagator $\langle z(t_\alpha), \bar{z}(t_\alpha)\rangle|z(t_{\alpha-1}), \bar{z}(t_{\alpha-1})\rangle$ can be written as

$$\langle z(t_{\alpha}), \bar{z}(t_{\alpha})|1 - i\varepsilon H(t_\alpha)(1 - \varepsilon \frac{d}{dt})|z(t_\alpha), \bar{z}(t_\alpha)\rangle = e^{i\varepsilon S(z(t_{\alpha}), \bar{z}(t_{\alpha})(i\frac{d}{dt} - H(t_\alpha))|z(t_\alpha), \bar{z}(t_\alpha)\rangle. \quad (124)$$

Substituting this into Eq. \[122\] and taking the limits $N \to \infty$ and $\varepsilon \to 0$ such that $N\varepsilon = T$ we arrive at the following path integral formula for the propagator:

$$\langle z'(T), z'(T)|U(T)|z(0), \bar{z}(0)\rangle = \int D[z, \bar{z}]e^{iS[z, \bar{z}]}.$$ 

(125)

In this formula $S[z, \bar{z}]$ is given by

$$S[z, \bar{z}] = \int_{0}^{T} dt \langle z(t), \bar{z}(t)\rangle i\frac{d}{dt} - H(t)|z(t), \bar{z}(t)\rangle - i\log\langle z'(T), \bar{z}'(T)|z(T), \bar{z}(T)\rangle \quad (126)$$

and the path integral measure is

$$D[z, \bar{z}] = \lim_{N \to \infty} e^{-3\sum_{\alpha=1}^{N} \int dp \log(\sum_k |z_k(p, t_\alpha)|^2)} e^{-3\sum_{\alpha=1}^{N} \int \bar{p} \log(\sum_k |\bar{z}_k(\bar{p}, t_\alpha)|^2)} \prod_{\alpha=1}^{N} \prod_{p} \frac{3!}{(2\pi i)^2} d^2z_\mu(p, t_\alpha) d^2z_\tau(p, t_\alpha) \prod_{\alpha=1}^{N} \prod_{\bar{p}} \frac{3!}{(2\pi i)^2} d^2\bar{z}_\mu(\bar{p}, t_\alpha) d^2\bar{z}_\tau(\bar{p}, t_\alpha). \quad (127)$$

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