Estimation of minimum energy pathways and transition states from geometric properties of the initial and final states

Mark C. Palenik¹,²

U.S. Naval Research Laboratory, Chemistry Division, Washington, DC 20375, United States

Reaction pathways and transition states are sometimes estimated by interpolating internal coordinates based on bond lengths between reactant and product structures. Typically, it is impossible to directly translate these interpolated geometries into atomic positions, because bond lengths and Cartesian coordinates are not isomorphic. Linear synchronous transit uses least-squares minimization to avoid this problem but fails when a best fit for the atomic positions cannot be found. As an alternative, we transform a velocity vector from internal coordinates to Cartesian coordinates and obtain a first-order differential equation for the reaction path which is numerically integrable even when linear synchronous transit fails. As a demonstration, we calculate transition states and reaction pathways for HONO elimination from dimethylnitramine.

The minimum energy pathway (MEP) between reactant and product geometries represents the “path of least resistance”¹ over which a reaction can occur and therefore defines a reaction coordinate. It can be defined mathematically from eigenvectors of the Hessian of potential energy with respect to nuclear coordinates or from a minimal action principle². In transition state theory, the highest point on the reaction pathway is used to calculate reaction rates.

Often, it is useful to have a quick estimate of the MEP and the transition state, which can be refined with electronic structure theory. Linear synchronous transit (LST), for example, uses coordinates based on bond lengths as a heuristic for the true reaction coordinate. It is usually impossible to directly interpolate structures along such coordinates, because an arbitrary set of internal coordinates will not necessarily correspond to a set of Cartesian atomic positions. As a simple example, consider three atomic nuclei, A, B, and C, with internal coordinates specified by their bond lengths. According to the triangle inequality, there is no set of Cartesian coordinates where the bond length $A - B$ is greater than the sum of $A - C$ and $B - C$, even though this is a perfectly valid point in the space of internal coordinates. Additionally, even if there is a set of Cartesian positions corresponding to the desired internal coordinates, it can be difficult to retrieve these positions when the internal coordinates are complicated functions of bond lengths. LST attempts to solve both of these problems by minimizing the mean-squared difference between the desired internal coordinates and the actual coordinates computed from atomic positions. However, LST tends to produce discontinuous paths and is prone to outright failure at times, when no suitable minimum can be found.

Measuring the mean-squared difference in internal coordinates is equivalent to measuring the distance between the two structures using a euclidean metric. Linear interpolation through internal coordinates is also what one would expect from the dynamics of a euclidean space.

Zhu et al.⁴ recognize that using a euclidean metric in internal coordinates (which they also define as a function of bond lengths) effectively induces a non-euclidean metric in Cartesian coordinates. Rather than using interpolation, they estimate the MEP as a geodesic between reactant and product geometries using this induced metric. The downside to their method is that the geodesic equation is a second-order differential equation involving derivatives of the metric tensor, which form a complicated three-index tensor. To solve the geodesic equation, they must seed their algorithm with a guess for the geodesic, slice it into a number of pieces, numerically approximate the path length integral, and explicitly minimize it by adjusting points along the curve.

In this work, we similarly use differential geometry to estimate the MEP and transition state. Although we use bond length-based internal coordinates, this method is not prone to failure like LST. We begin by defining a velocity vector field in internal coordinates that points from every location toward the desired product geometry. We then, through a series of steps, transform this vector field into a vector field in Cartesian coordinates. Because velocity is the first-derivative of position, this defines a first-order differential equation, which only involves two-index tensors (matrices), and which can be numerically integrated without starting from a seed path. By arc-length parameterizing the path, we can also obtain an estimate for the transition state at the halfway point.

Geometrically speaking, the transformation from the set of Cartesian nuclear coordinates $\{x^i\}$ to the set of internal bond length derived coordinates $\{q^j\}$ can be thought of as a mapping between from a manifold $M$ to another manifold $P$. Although the mapping from Cartesian to internal coordinates, $M \rightarrow P$, is defined, its inverse, from $P \rightarrow M$, usually is not. An invertible mapping between manifolds is called a bijection. Bijective maps are defined by two properties: they are one-to-one (each unique point on $M$ corresponds to a unique point on $P$) and onto (every point on $P$ has some corresponding point on $M$).

$M \rightarrow P$ can easily be made one-to-one simply by removing translational and rotational degrees of freedom.

¹Electronic mail: mark.palenik@nrl.navy.mil
from the Cartesian basis. The second property of a bijection, however, is much less trivial. As we demonstrated earlier, internal coordinates can easily arise that have no corresponding Cartesian atomic positions. Therefore, the mapping \( M \rightarrow P \) cannot be made bijective.

An unfortunate consequence of this is that vectors cannot be directly transformed from \( P \) to \( M \), because the transformation law for vectors in this direction requires the Jacobian \( \partial x^i / \partial q^j \). Although we can define the Jacobian \( \partial q^i / \partial x^j \), it can only be inverted if \( M \rightarrow P \) is bijective. Fortunately, there is a related type of geometric object called a 1-form, which transforms from \( P \) to \( M \) using \( \partial q^i / \partial x^j \). In euclidean spaces, 1-forms are identical to their corresponding vectors, but in non-euclidean spaces, the two objects are related by the metric tensor. We can therefore take a velocity 1-form on \( M \) and then convert the 1-form on \( M \) back into a vector, which we integrate to create a reaction pathway.

We start with generalized coordinates labeled \( q^i \) that we wish to interpolate between an initial value \( q^{i0} \) and a final value \( q^{i1} \) over the time interval \( 0 \leq t \leq 1 \), while finding the corresponding Cartesian positions with translational and rotational degrees of freedom removed. If we initialize \( q^i = q^{i0} \) at \( t = 0 \), the time derivative, or velocity, of \( q^i \) is then given by

\[
\frac{dq^i}{dt} = \frac{q^{i1} - q^{i0}}{1 - t}.
\]

The right-hand side defines a vector field on \( P \) at every position \( q^i \) and time \( t \) that sends an object from \( q^i \) to \( q^{i1} \) at \( t = 1 \).

Because the space of generalized coordinates is Euclidean, the vector \( dq^i / dt \) is identical to the 1-form \( dq_i / dt \). We can pullback the 1-form \( dq_i / dt \) from \( P \) to \( M \), and employing Einstein notation, with an implicit sum over repeated indices, we have

\[
\frac{dx_j}{dt} = \frac{dq^i}{dx^j} \frac{dq_i}{dt}.
\]

This object is a 1-form in the cotangent bundle on \( M \). However, to generate a path, we need to integrate a velocity vector through the tangent bundle on \( M \). Using the metric tensor, we can transform \( dx_j / dt \) from a 1-form to a vector by

\[
\frac{dx^k}{dt} = g^{jk} \frac{dq^i}{dx^j} \frac{q^{i1} - q_i}{1 - t},
\]

where

\[
g^{jk} = (g_{jk})^{-1} = \left[ \frac{\partial q^i}{\partial x^j} \frac{\partial q^j}{\partial x^k} \right]^{-1},
\]

and the inverses are a matrix inverses.

Integrating Eq. (3) from \( t = 0 \) to \( t = 1 \) takes \( x^i \) along a continuous path from the initial structure to the final structure. However, \( x^i \) will not necessarily change at a uniform rate, and therefore, a good estimation for the transition state may not lie at \( t = 0.5 \). To remedy this, we can reparametrize \( x \), using the arc-length parameter \( \tau \), where

\[
\frac{d\tau}{dt} = \frac{dx^i}{dt} \frac{dx^j}{dt} g_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} = \frac{\partial q^a (q^{1a} - q_a) g^{ij} (q^{1i} - q_i) \partial q^c}{\partial x^j} (1 - t)^2 \partial x^i.
\]

leaving us with

\[
\frac{dx^k}{d\tau} = g^{jk} \frac{dq^i}{dx^j} \frac{q^{i1} - q_i}{\sqrt{\frac{\partial q^a}{\partial x^j} (q^{1a} - q_a) g^{ij} (q^{1i} - q_i)} \partial x^m} \frac{\partial q^c}{\partial x^m}.
\]

The structure halfway between the initial and final points in this parametrization can then serve as an estimate for the transition state.

The downside to an arc-length parametrization is that the total length of the path is not known ahead of time, as the path through Cartesian coordinates is not the shortest possible path through generalized coordinates. However, in practice, the integration can be performed until \( q_i \) and \( q_i^1 \) differ by some small amount, at which point, the end of the path has been reached.

We might foresee a problem arising in situations of high symmetry, where the generalized coordinates do not uniquely specify Cartesian positions even after translations and rotations are removed. As an example, consider an initial structure composed of several atoms in a plane and a final structure where one atom is displaced perpendicular to the plane. If the generalized coordinates are bond lengths, structures where the atom is displaced in front of or behind the plane will have an equivalent representation in this basis.

In such a situation, the vector \( q^{i1} - q^{i0} \) is orthogonal to the Cartesian manifold and will result in the zero vector.

FIG. 1. The path through generalized coordinates, \( \Delta q \) is orthogonal to the Cartesian manifold, represented by the solid black line embedded in a higher dimensional space. Nudging \( x \) toward the final Cartesian positions by a small amount \( \Delta x \) fixes the problem here.
FIG. 2. Initial structure (left) and final structure (right) for HONO elimination from dimethylnitramine. The middle image has the true transition state overlaid with the guess from our method.

FIG. 3. Energy along the estimated MEP (blue) and IRC pathway (green), computed with M06 6-311++G(3df,3pd). The horizontal axis is total distance traversed through generalized coordinates as a fraction of reaction path length.

More generally, one could imagine a situation where the generalized coordinates may or may not uniquely specify the Cartesian geometry, but the velocity vector through generalized coordinates is orthogonal to the Cartesian manifold. We have depicted an example of such a situation in Fig. 1 where a manifold consisting of a single Cartesian coordinate is embedded in a two-dimensional manifold of generalized coordinates.

In many cases, it will suffice to give the system a small “nudge” in the correct direction, toward the Cartesian coordinates of the product. That is, if \( \frac{dx^k}{d\tau} \) goes to zero, but we are still not at the correct final geometry, we can set \( \frac{dx^k}{d\tau} = x^{k1} - x^k \). Such situations should, however, be rare and did not arise in our tests.

We found several reactions where LST failed to produced a guess for the transition state, and in all cases our algorithm was able to produce a continuous path.

A simple example of such a reaction is HONO elimination from dimethylnitramine. Nitramine compounds are essential ingredients in many propellants and explosives, and dimethylnitramine is the simplest member of the nitramine series. It can decompose via N-N and C-H bond scission according to

\[
C_2H_6N_2O_2 \rightarrow CH_3NCH_2 + HONO. \tag{1}
\]

The initial and final states of this reaction, optimized with the M06 functional and 6-311++G(3df,3pd) basis set in Gaussian09, are depicted on the left and right of Figure 2 respectively. The middle image is our estimate of the transition state overlaid in 3D with the true, optimized transition state. The estimated transition state is taken from the point halfway along the pathway calculated by our algorithm and is depicted with translucent atoms for clarity. The two structures are aligned so that their centers of mass are at the origin and the RMS distance between their nuclei is minimized.

Although we achieved reasonable results using interatomic distances divided by the sum of their covalent radii as our generalized coordinates, we chose to use the basis of coordinates defined in the paper by Zhu et al.

\[
q_{kl} = \exp \left[ -\alpha \frac{r_{kl} - r^e_{kl}}{r^e_{kl}} \right] + \beta \frac{r^e_{kl}}{r_{kl}}, \tag{7}
\]

where \( r_{kl} \) is the distance between atoms \( k \) and \( l \) and \( r^e_{kl} \) is the sum of their covalent radii. The parameters \( \alpha \) and \( \beta \) are taken from the same paper and set to 1.7 and 0.01 respectively. In some of our tests, these coordinates generated drastically different reaction paths from those generated with \( q_{kl} = r_{kl}/r^e_{kl} \). Single point calculations typically revealed the paths generated with the coordinates of Zhu et al. to be lower in energy and therefore closer to the true minimum energy pathway.

The estimated MEP was generated by integrating Equation (6) using fifth-order Runge Kutta with an adaptive step size. We tested various accuracies of the stepper
routine between $10^{-8}$ and $10^{-4}$ without noticeable degradation of the pathway. The calculation was terminated when the RMS difference between initial and final geometries was less than $10^{-4}$ Å. For loosest integration accuracy, the processor time to compute the reaction pathway was approximately 0.02 s, and for the tightest, it was approximately 0.08 s.

In Figure 2, we plotted the energy at 110 points along the path computed by our algorithm and at 610 points along the true MEP as determined by an intrinsic reaction coordinate (IRC) calculation. Energies were computed with the 6-311++G(3df,3pd) basis set and M06 functional and are given in kcal/mol relative to the reactants. The horizontal axis measures total distance traversed at a given point along the reaction path, $\sum \alpha |q^{n+1} - q^n|$. A vertical dashed line is placed halfway along the path, at the location where we obtained a transition state estimate on the estimated path. Both the true transition state and the maximum of our estimated path lie relatively near this point, although on different sides.

We might also ask whether it is actually important to convert the velocity 1-form in Cartesian coordinates into a vector, or if we could get equally good results by producing a path from integrating the 1-form, even though it is not formally correct. We attempted this for comparison, and found that without using $q^k$ to create a vector, the algorithm made strange leaps in geometry, produced transition states that were further from the optimized transition state, and required a significantly higher number of Runge-Kutta steps than the original version.

As a second comparison, we also attempted to generate a continuous LST path from a first-order differential equation using the coordinates in Eq. (7). We began by linearly interpolating between $q^0$ and $q^3$ on the interval $0 \leq t \leq 1$ and writing the LST least-squares minimization as a function of time. We then differentiated this equation with respect to time to get

$$\frac{d}{dt} \left( q^j - (1-t)q^0 - tq^1 \right)^2 = \frac{d}{dt} \lambda (t) \left( x^j - x^1 \right),$$

where the right-hand side enforces an optional constraint to ensure that the computed geometry at $t = 1$ matches the correct product geometry. To turn the constraint off, we set $\lambda(t) = t/(t - 1)$, which enforces $\lim_{t \to 1} dx^j/dt = (x^j - x^1)/(1 - t)$. To turn the constraint on, we set $\lambda(t) \equiv 0$. We attempted to integrate this equation with both Runge-Kutta and a stiff Rosenbrock method.

With or without constraint, the integration step size often tended to go to zero, prematurely terminating the calculation not only for Reaction 1 but in most systems we tested. Without the constraint, even when integration succeeded, the final geometries typically did not closely resemble the desired products. Both of these issues likely have to do with the inherently discontinuous nature of the LST reaction pathway. Successful calculations with the constraint turned on produced poor reaction pathways and took significantly longer to run than our method.

The method we developed in Eq. (8) provides a good estimate of MEPs and transition states even in systems where LST fails. It is based on the premise that a straight line through some set of generalized coordinates results in the minimum energy pathway between two states. Although this path is usually not realizable in terms of actual atomic positions, a velocity vector can still be defined in generalized coordinates that takes the reactant geometry to the product geometry over the interval $0 \leq t \leq 1$. This velocity vector can then be transformed from the high-dimensional manifold of generalized coordinates to a corresponding vector in $\mathbb{R}^{3N}/\mathbb{Z}^{3N}$ by pulling back a 1-form and acting on it with the inverse metric tensor. For HONO elimination from dimethylnitramine, we were able to generate a continuous reaction pathway in a fraction of a second with an estimated transition state that was near the true energy maximum of the path and qualitatively similar to the true transition state.

**Acknowledgments**

This work was supported by the Office of Naval Research (ONR) directly and through the Naval Research Laboratory. We thank Igor V. Schweigert for providing transition state and IRC calculations for dimethylnitramine.

1. J. D. Dunitz, Philos. Trans. R. Soc. Lond., B, Biol. Sci. 272, 99 (1975)
2. A. Tachibana and K. Fukui, Theor. Chim. Acta. 51, 275 (1979)
3. K. Fukui, Accounts of Chemical Research 14, 363 (1981)
4. H. Eyring, J. Chem. Phys. 3, 107 (1935)
5. E. Wigner, The Journal of Chemical Physics 5, 720 (1937)
6. D. G. Truhlar, B. C. Garrett, and S. J. Klippenstein, J. Phys. Chem. 100, 12771 (1996)
7. K. A. Halgren and W. N. Lipscomb, Chem. Phys. Lett. 49, 225 (1977)
8. S. Smidstrup, A. Pedersen, K. Stokbro, and H. Jønsson, J. Chem. Phys. 140, 214106 (2014)
9. X. Zhu, K. C. Thompson, and T. J. Martinez, J. Chem. Phys. 150, 164103 (2019)
10. Y. Zhao and D. G. Truhlar, Theor. Chem. Acc. 120, 215 (2008)
11. M. J. Frisch, G. W. Trucks, H. B. Schlegel, G. E. Scuseria, M. A. Robb, J. R. Cheeseman, G. Scalmani, V. Barone, G. A. Petersson, H. Nakatsuji, X. Li, M. Caricato, J. B. Foresman, D. J. Fox, and others, Gaussian 09, Wallingford CT.