HARNACK ESTIMATES FOR THE POROUS MEDIUM EQUATION WITH POTENTIAL UNDER GEOMETRIC FLOW

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Abstract. Let \((M,g(t)), t \in [0,T]\) be a closed Riemannian \(n\)-manifold whose Riemannian metric \(g(t)\) evolves by the geometric flow \(\frac{\partial}{\partial t} g_{ij} = -2S_{ij}\), where \(S_{ij}(t)\) is a symmetric two-tensor on \((M,g(t))\). We discuss differential Harnack estimates for positive solution to the porous medium equation with potential, \(\frac{\partial u}{\partial t} = \Delta u^p + S \cdot u\), where \(S = g^{ij}S_{ij}\) is the trace of \(S_{ij}\), on time-dependent Riemannian metric evolving by the above geometric flow.

1. Introduction

There are many results about the Harnack estimates for parabolic equations. The study of differential Harnack estimates and applications for parabolic equation originated in the famous paper [11] of Li and Yau, in which they discovered the celebrated differential Harnack estimate for any positive solution to the heat equation with potential on Riemannian manifolds with a fixed Riemannian metric. After then, this method plays an important role in the study of geometric flows, for instance, Hamilton proved Harnack inequalities for the Ricci flow on Riemannian manifolds with weakly positive curvature operator [7] and mean curvature flow [8], also see [3, 5]. Also, recently many authors obtained a differential Harnack estimate for solutions of the parabolic equation on Riemannian manifold along the geometric flow, for instance, Fang in [6], proved differential Harnack estimates for backward heat equation with potentials under an extended Ricci flow and Ishida in [10] studied differential Harnack estimates for heat equation with potentials along the geometric flow.

Let \(M\) be a closed Riemannian manifold with a one parameter family of Riemannian metric \(g(t)\) evolving by the geometric flow

\[
\frac{\partial}{\partial t} g_{ij}(x,t) = -2S_{ij}(x,t)
\]

where \(S_{ij}\) is a general time-dependent symmetric two-tensor on \((M,g(t))\). For example, (1.1) becomes Ricci flow whenever \(S_{ij} = R_{ij}\) is the Ricci tensor, where it introduced by Hamilton [9].

In [4], Cao and Zhu obtained Aronson-Benilan estimates for the porous medium equation (PME) along the geometric flow, for instance, Fang in [6], proved differential Harnack estimates for backward heat equation with potentials under an extended Ricci flow and Ishida in [10] studied differential Harnack estimates for heat equation with potentials along the geometric flow.

Differential equations (1.2) is a nonlinear parabolic equation and has applications in mathematics and

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physics. For \( p > 1 \) differential equations PME describes physical processes of gas through porous medium, heat radiation in plasmas \((15)\). Motivated by the above works, in this paper, we consider equation of type \((1.2)\) with a linear forcing term \((1.3)\)
\[
\frac{\partial u}{\partial t} = \Delta u^p + Su
\]
under the geometric flow \((1.1)\), where \( S = g^{ij}S_{ij} \), \( \Delta \) is Laplace operator with \( n \) respect to the evolving metric \( g(t) \) of the geometric flow \((1.1)\) and prove differential Harnack estimates for positive solutions to \((1.3)\). Notice also that for any smooth solution \( u \) of \((1.3)\) we have
\[
\frac{\partial}{\partial t} \left( \int_M u \, d\mu \right) = \int_M \frac{\partial u}{\partial t} \, d\mu + u \frac{\partial u}{\partial t} = \int_M \left( \frac{\partial u}{\partial t} - Su \right) d\mu = \int_M \Delta u^p d\mu = 0.
\]
For \( p = 1 \), \((1.3)\) is simply the equation
\[
\frac{\partial u}{\partial t} = \Delta u + Su,
\]
where differential Harnack estimates for positive solution to \((1.3)\) have been studied in \((10)\). Suppose that \( u \) is positive solution of \((1.3)\) and \( v = \frac{p}{p-1} u^{p-1} \). Then we can rewrite \((1.3)\) as follows
\[
\frac{\partial v}{\partial t} = (p-1)v\Delta v + |\nabla v|^2 + (p-1)Sv.
\]
To state the main results of the current article, analogous to definition from Müller \((13)\) we introduce evolving tensor quantises associated with the tensor \( S_{ij} \).

**Definition 1.1.** Let \( g(t) \) be a solution of the geometric flow \((1.1)\) and let \( X = X^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(M) \) be a vector field on \((M, g(t))\). We define
\[
\mathcal{I}(S, X) = (R^{ij} - S^{ij})X_iX_j,
\]
\[
\mathcal{H}(S, X) = \frac{\partial S}{\partial t} + S - 2\nabla_iSX^i + 2S^{ij}X_iX_j,
\]
\[
\mathcal{D}(S) = \frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2,
\]
\[
\mathcal{E}(S, X) = \mathcal{D}(S) + 2\mathcal{I}(S, X) + 2(2\nabla^iS_{ij} - \nabla_jS)X^j.
\]

2. MAIN RESULTS

The main results of this paper are the following.

**Theorem 2.1.** Let \( g(t), t \in [0, T) \) be a solution to the geometric flow \((1.1)\) on a closed Riemannian \( n \)-manifold \( M \) satisfying
\[
\mathcal{E}(S, X) \geq 0, \quad \mathcal{H}(S, X) \geq 0, \quad \text{Ric} \geq -(n-1)k_1, \quad -k_2g \leq S_{ij} \leq k_3g, \quad S \geq 0
\]
for all vector fields \( X \) and all time \( t \in [0, T) \). Suppose \( u \) is a smooth positive solution to equation \((1.3)\) with \( p > 1 \) and \( v = \frac{p}{p-1} u^{p-1} \). Then for any \( d \in [2, \infty) \), on the geodesic ball \( Q_{\rho, T} \), we have
\[
\frac{|\nabla v|^2}{v} - 2\frac{v^t}{v} - \frac{S}{v} - \frac{d}{t} \leq \frac{2n(p-1)}{1+n(p-1)} \left( \frac{E_1v_{\text{max}}}{\rho^2} + E_2 \right)
\]
where \( E_1 = (p^2n + \frac{1}{2}\sqrt{k_1} + \frac{p}{2})c_1(p-1) \), \( E_2 = \sqrt{c_2(k_2 + k_3)^2} + 1 \) and \( c_1, c_2 \) are absolute positive constants.
Let \( \rho \to \infty \), we can get the gradient estimates for the nonlinear parabolic equation \([1, 3]\).

**Corollary 2.2.** Let \( g(t), t \in [0, T) \) be a solution to the geometric flow \((1.1)\) on a closed Riemannian \( n \)-manifold \( M \) satisfying

\[
(2.3) \quad \mathcal{E}(S, X) \geq 0, \quad \mathcal{H}(S, X) \geq 0, \quad \text{Ric} \geq -(n-1)k_1, \quad -k_2g \leq S_{ij} \leq k_3g, \quad S \geq 0
\]

for all vector fields \( X \) and all time \( t \in [0, T) \). Suppose \( u \) is a bounded smooth positive solution to equation \((2.3)\) with \( p > 1 \) and \( v = \frac{p}{p-1}u^{p-1} \). Then for any \( d \in [2, \infty) \), on the geodesic ball \( Q_{\rho,T} \), we have

\[
(2.4) \quad \frac{|\nabla v|^2}{v} - 2\frac{v_t}{v} - \frac{S}{v} - \frac{d}{t} \leq \frac{2n(p-1)}{1+n(p-1)}E_2
\]

where \( E_2 = \sqrt{c_2}(k_2 + k_3)^2 + 1 \) and \( c_2 \) is absolute positive constant.

As an application, we get the following Harnack inequality for \( v \).

**Theorem 2.3.** With the same assumption as in Corollary 2.2 if \( d \geq 2 \), then for any points \( (x_1, t_1) \) and \( (x_2, t_2) \) on \( M \times [0, T) \) with \( 0 < t_1 < t_2 \) we have the following estimate

\[
(2.5) \quad v(x_1, t_1) \leq v(x_2, t_2)\left(\frac{t_2}{t_1}\right)^\frac{\Gamma}{2\nu_{\min}} + \left(\frac{n(p-1)}{1+n(p-1)}E_2(t_2-t_1)\right)
\]

where \( E_2 \) is the constants in Corollary 2.2 and \( \Gamma = \inf_\gamma \int_{t_1}^{t_2} (S + |\frac{\partial \gamma}{\partial t}|^2) \) with the infimum taking over all smooth curves \( \gamma(t) \) in \( M \), \( t \in [t_1, t_2] \), so that \( \gamma(t_1) = x_1 \) and \( \gamma(t_2) = x_2 \).

Our results in this article are similar to those of Cao and Zhu [4] in the case \( S_{ij} = R_{ij} \).

3. Examples

3.1. **Static Riemannian manifold.** In this case we have \( S_{ij} = 0 \) and \( S = 0 \). Then \( \mathcal{D} = 0, \mathcal{H}(S, X) = 0 \) and \( \mathcal{I}(S, X) = R^{ij}X_iX_j \). Thus the assumption in Theorems 2.1, 2.3 and Corollary 2.2 can be replace by \( R_{ij} \geq 0 \).

3.2. **The Ricci flow.** The Ricci flow defined for the first time by Haimlton as follow

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.
\]

In this case we get \( S_{ij} = R_{ij} \) and \( S = R \) the scalar curvature. Along the Ricci flow we have

\[
\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2, \quad 2|\nabla R| - \nabla R = 0.
\]

Therefore we obtain

\[
\mathcal{I}(S, X) = 0, \quad \mathcal{D}(S) = 0, \quad \mathcal{E}(S, X) = 0, \quad \mathcal{H}(S, X) = \frac{\partial R}{\partial t} + \frac{R}{t} - 2\nabla_iRX^i + 2R_{ij}X_iX_j.
\]

Notice that for any vector field \( X = X^i \frac{\partial}{\partial x^i} \) on \( M \), if \( g(t) \) be complete solution to the Ricci flow with bounded curvature and nonnegative curvature operator then from [7] we have \( \mathcal{H}(S, X) \geq 0 \), that is \( g(t) \) has weakly positive curvature operator. Hence, the assumption in Theorems 2.1, 2.3 and Corollary 2.2 hold.
3.3. List’s extended Ricci flow. Extended Ricci flow defined by List in [12] as follows
\[
\begin{align*}
\frac{\partial}{\partial t}g_{ij} &= -2R_{ij} + 4\nabla_i f \nabla_j f, \\
\frac{\partial}{\partial t}\alpha &= \Delta \alpha,
\end{align*}
\]
where \( f : M \to \mathbb{R} \) is a smooth function. In this case, \( S_{ij} = R_{ij} - 2\nabla_i f \nabla_j f \) and \( S = R - 2|\nabla f|^2 \).
Along the extended Ricci flow we have
\[
\frac{\partial S}{\partial t} = \Delta S + 2|Ric|^2 + 4|\Delta f|^2, \quad 2\nabla^i S_d - \nabla_i S + 4\Delta f \nabla_i f = 0.
\]

Therefore we obtain
\[
\mathcal{I}(S, X) = 2|\nabla_X f|^2 \geq 0, \quad \mathcal{D}(S) = 4|\Delta f|^2, \quad \mathcal{E}(S, X) = 4|\Delta f - \nabla_X f|^2 \geq 0.
\]

3.4. Müller coupled with harmonic map flow. Let \((N, h)\) be a fixed Riemannian manifold. The harmonic-Ricci flow on \( M \) introduced by Müller in [14] as follows
\[
\begin{align*}
\frac{\partial}{\partial t}g_{ij} &= -2R_{ij} + 2\alpha(t)\nabla_i f \nabla_j f, \\
\frac{\partial}{\partial t}\tau_f &= \tau_f f,
\end{align*}
\]
where \( \tau_f \) is the tension field of the map \( f : M \to N \) with respect to the metric \( g(t) \) and \( \alpha(t) \) is positive non-increasing real function respect to \( t \). In this case, \( S_{ij} = R_{ij} - \alpha(t)\nabla_i f \nabla_j f \) and \( S = R - \alpha(t)|\nabla f|^2 \).
Along this flow we have
\[
\frac{\partial S}{\partial t} = \Delta S + 2|Ric|^2 + 2\alpha(t)|\tau_f f|^2 - \left( \frac{\partial \alpha(t)}{\partial t} \right)|\nabla f|^2, \quad 2\nabla^i S_d - \nabla_i S + 2\alpha(t)\tau_f f \nabla_i f = 0.
\]

Therefore we obtain
\[
\mathcal{I}(S, X) = \alpha(t)|\nabla_f \nabla^i f X_i X_j| = \alpha(t)|\nabla_X f|^2 \geq 0, \quad \mathcal{D}(S) = 2\alpha(t)|\tau_f f|^2 - \left( \frac{\partial \alpha(t)}{\partial t} \right)|\nabla f|^2.
\]

and
\[
\mathcal{E}(S, X) = 2\alpha(t)|\tau_f f - \nabla_X f|^2 - \left( \frac{\partial \alpha(t)}{\partial t} \right)|\nabla f|^2.
\]

Thus \( \mathcal{E}(S, X) \geq 0 \) holds if \( \alpha(t) \geq 0 \) and \( \alpha(t) \geq 0 \) be an non-increasing function. Notice, to the best our knowledge, it is still unknown wether \( \mathcal{H}(S, X) \geq 0 \) is preserved by the under harmonic-Ricci flow in particular case extended Ricci flow under suitable assumptions.

4. PROOFS OF THE RESULTS

In this section, we suppose that \( u \) is smooth positive solution to equation (1.3) and \( v = \frac{u}{p-1}u^{p-1} \). In the order to prove the main results, we need the following lemmas and proposition.

**Lemma 4.1.** Let \((M, g(t))\) be a complete solution to the geometric flow (1.1) in some time interval \([0, T]\). Suppose that \( v \) is a positive solution of (1.2),
\[
\mathcal{L} = \frac{\partial}{\partial t} - (p - 1)v \Delta
\]
and
\[
F = \frac{[\nabla v]^2}{v} - b \frac{v_t}{v} + (1 - b) \frac{S}{v} - \frac{d}{t}.
\]

\[
F = -b(p - 1)\Delta v + (1 - b) \frac{[\nabla v]^2}{v} - b(p - 1)S + (1 - b) \frac{S}{v} - \frac{d}{t}.
\]
then for any constants $b, d$ we have

\[
\mathcal{L}(F) = 2p\nabla_i F \nabla_j v - \left[\frac{1 - b}{v} + p - 1\right] \left(\frac{\partial S}{\partial t} - 2\nabla_i S \nabla^j v + 2S^{ij} \nabla_i v \nabla_j v\right)
- 2(p - 1)(R^{ij} - S^{ij}) \nabla_i v \nabla_j v - \frac{b}{2} S_{ij}\]
\]

\[
+ \frac{(b - 2)^2}{2} p (p - 1)|S_{ij}|^2 + (p - 1)(1 - b)D(S)
\]
\[
- \frac{1}{b} F^2 - [(p - 1)S - \frac{2(1 - b) S}{v} + \frac{2d}{b} t] F - \frac{(1 - b)^2 S^2}{v^2} + (1 - b)\frac{1}{v^2} S
\]
\[
+ \frac{1 - b}{v^2} - \frac{d^2}{b} t^2 + \frac{2d}{b} (1 - b)\frac{1}{v^2} - \frac{d}{t} (p - 1)\frac{1}{S} t + \frac{2}{b} \frac{1}{d} \frac{1}{S} t v
\]
\[
- \frac{2 - b}{b} + \frac{d}{t^2} - (p - 1)(2\nabla^i S_{it} - \nabla_i S)\nabla_t v.
\]

**Proof.** First of all, we have the following evolution equations, under the flow (1.4),

\[
\frac{\partial}{\partial t}(\Delta v) = 2S^{ij} \nabla_i \nabla_j v + \Delta(v_t) - g^{ij} \frac{\partial}{\partial t}(\Gamma^k_{ij}) \nabla_k v
\]

\[
\frac{\partial}{\partial t}|\nabla v|^2 = 2S^{ij} \nabla_i v \nabla_j v + 2\nabla^i v_t \nabla_i v
\]

\[
g^{ij} \frac{\partial}{\partial t} \Gamma^k_{ij} = -g^{kl}(2\nabla^i S_{it} - \nabla_i S).
\]

Then from (1.2), (4.3) and (4.6) we get

\[
\frac{\partial}{\partial t}(\Delta v) = 2S^{ij} \nabla_i \nabla_j v + \frac{1}{2} (p - 1)v \Delta v^2 + (p - 1)(\nabla v)^2 + 2(p - 1)\nabla_i (\Delta v) \nabla_i v
\]
\[
+ \frac{1}{2} \Delta |\nabla v|^2 + \frac{1}{2} (p - 1) \Delta(S v) + (2\nabla^i S_{it} - \nabla_i S) \nabla_t v.
\]

Using the Bochner-Weitzenböck formula

\[
\frac{1}{2} \Delta |\nabla v|^2 = \nabla_i (\Delta v) \nabla_i v + |\nabla v|^2 + \nabla^i \nabla_i \nabla_j v,
\]

we obtain

\[
\mathcal{L}(\Delta v) = 2p\nabla_i (\Delta v) \nabla^i v + 2S^{ij} \nabla_i \nabla_j v + (p - 1)(\nabla v)^2 + 2|\nabla v|^2 + 2R^{ij} \nabla_i v \nabla_j v
\]
\[
+ (p - 1)v \Delta S + 2(p - 1)\nabla_i S \nabla^i v + (p - 1) \Delta v
\]
\[
+ (2\nabla^i S_{it} - \nabla_i S) \nabla_t v.
\]

On the other hand, again (1.2) results that

\[
\mathcal{L}(|\nabla v|^2) = 2S^{ij} \nabla_i v \nabla_j v + 2(p - 1)|\nabla v|^2 \Delta v + 2\nabla_i |\nabla v|^2 \nabla_i v + 2(p - 1)v \nabla_i S \nabla^i v
\]
\[
+ 2(p - 1)S|\nabla v|^2 - 2(p - 1)v|\nabla v|^2 - 2(p - 1)v R^{ij} \nabla_i v \nabla_j v,
\]

it follows that

\[
\mathcal{L}(\frac{|\nabla v|^2}{v}) = \frac{1}{v} \mathcal{L}(|\nabla v|^2) - \frac{|\nabla v|^2}{v^2} \mathcal{L}(v) + 2(p - 1)\nabla_i \left(\frac{|\nabla v|^2}{v}\right) \nabla_t v
\]
\[
= 2p\nabla_i \left(\frac{|\nabla v|^2}{v}\right) \nabla^i v + 2S^{ij} \nabla_i v \nabla_j v + 2(p - 1)\frac{|\nabla v|^2}{v^2} \Delta v + \frac{|\nabla v|^4}{v^2}
\]
\[
+ 2(p - 1)\nabla_i S \nabla^i v + (p - 1)\frac{|\nabla v|^2}{v^2} - 2(p - 1)|\nabla v|^2
\]
\[
- 2(p - 1)R^{ij} \nabla_i v \nabla_j v.
\]
Also, we obtain

\[ (4.11) \quad \mathcal{L}(\frac{S}{v}) = 2p\nabla_i\frac{S}{v}\nabla^i v + \frac{[\nabla v]^2}{v^2} S - \frac{2}{v} \nabla_i S \nabla^i v + \frac{1}{v} \frac{\partial S}{\partial t} - (p - 1) \frac{S^2}{v} - (p - 1) \Delta S. \]

From (4.2), (4.8), (4.10) and (4.11) we get

\[
\begin{align*}
\mathcal{L}(F) & = (1 - b) \mathcal{L}(\frac{[\nabla v]^2}{v^2}) - b(p - 1) \mathcal{L}(\Delta v) - b(p - 1) \mathcal{L}(S) + (1 - b) \mathcal{L}(\frac{S}{v}) - \mathcal{L}(\frac{d}{t}) \\
& = 2p\nabla_i F \nabla^i v + \frac{1 - b}{v} \frac{\partial S}{\partial t} - 2\nabla_i S \nabla^i v + 2S \nabla^i v \nabla_i v - 2(p - 1) [\nabla v]^2 \\
& \quad - 2b(p - 1) S \nabla_i \nabla_j v - b(p - 1)^2 (\Delta v)^2 + 2(1 - b) (p - 1) \frac{[\nabla v]^2}{v^2} \\
& \quad - b(p - 1)^2 S \Delta v + (1 - b)(p - 1) \frac{[\nabla v]^2}{v^2} S + (1 - b) \frac{\nabla v | d_i v}{v^2} + (1 - b) \frac{\nabla v | v}{v^2} S \\
& \quad - (1 - b)(p - 1) \frac{S^2}{v} - b(p - 1)(2\nabla^i S \nabla_i v + \frac{d}{t}) \\
& = -b(p - 1)^2 (\Delta v)^2 + 2(1 - b)(p - 1) \frac{[\nabla v]^2}{v^2} \Delta v - b(p - 1)^2 \Delta v v \\
& \quad + (1 - b)(p - 1) \frac{[\nabla v]^2}{v^2} S + (1 - b) \frac{\nabla v | d_i v}{v^2} S - (1 - b)(p - 1) \frac{S^2}{v} \\
& = -\frac{1}{b} \left( -F + (1 - b) \frac{[\nabla v]^2}{v^2} v - b(p - 1) S + (1 - b) \frac{S}{v} \right)^2 \\
& \quad - 2\frac{1}{b} \left( (p - 1) (F - (1 - b) \frac{\nabla v | d_i v}{v^2} S + b(p - 1) S - (1 - b) \frac{S}{v} + \frac{d}{t}) \\
& \quad + (p - 1) S (F - (1 - b) \frac{[\nabla v]^2}{v^2} S + b(p - 1) S - (1 - b) \frac{S}{v} + \frac{d}{t}) \\
& \quad + (1 - b)(p - 1) \frac{[\nabla v]^2}{v^2} S + (1 - b) \frac{\nabla v | d_i v}{v^2} S - (1 - b)(p - 1) \frac{S^2}{v} \\
& \quad + \frac{1}{b} \frac{[\nabla v]^2}{v^2} - \frac{d}{bt} \right) - 2\frac{d}{bt} (1 - b) \frac{\nabla v | d_i v}{v^2} v - d(p - 1) \frac{S}{t} + 2\frac{1 - b d S}{t} \frac{S}{v} - \frac{1 - b d}{t},
\end{align*}
\]

we have

\[ (4.12) \quad \mathcal{L}(F) = 2p\nabla_i F \nabla^i v + \frac{1 - b}{v} \frac{\partial S}{\partial t} - 2\nabla_i S \nabla^i v + 2S \nabla^i v \nabla_i v - b(p - 1)(1 - b) |S_{ij}|^2 \\
\quad + (p - 1)(1 - b) D(S) - 2(p - 1) |\nabla v|^2 - 2b(p - 1) S \nabla_i \nabla_j v \\
\quad - \frac{1}{b} F^2 - (p - 1) S - \frac{2(1 - b) S}{v} + \frac{2d}{bt} F - \frac{(1 - b)^2 S^2}{v^2} + (1 - b) \frac{[\nabla v]^2}{v^2} S \\
\quad + \frac{1}{b} \frac{[\nabla v]^2}{v^2} - \frac{d^2}{bt^2} (1 - b) \frac{\nabla v | d_i v}{v^2} v - d(p - 1) \frac{S}{t} + 2\frac{1 - b d S}{t} \frac{S}{v} - \frac{1 - b d}{t},
\]

\[ \mathcal{L}(F) = 2p\nabla_i F \nabla^i v + \frac{1 - b}{v} \frac{\partial S}{\partial t} - 2\nabla_i S \nabla^i v + 2S \nabla^i v \nabla_i v - b(p - 1)(1 - b) |S_{ij}|^2 \\
\quad + (p - 1)(1 - b) D(S) - 2(p - 1) |\nabla v|^2 - 2b(p - 1) S \nabla_i \nabla_j v \\
\quad - \frac{1}{b} F^2 - (p - 1) S - \frac{2(1 - b) S}{v} + \frac{2d}{bt} F - \frac{(1 - b)^2 S^2}{v^2} + (1 - b) \frac{[\nabla v]^2}{v^2} S \\
\quad + \frac{1}{b} \frac{[\nabla v]^2}{v^2} - \frac{d^2}{bt^2} (1 - b) \frac{\nabla v | d_i v}{v^2} v - d(p - 1) \frac{S}{t} + 2\frac{1 - b d S}{t} \frac{S}{v} - \frac{1 - b d}{t},
\]

\[ -2\frac{1}{b} \frac{d}{t} + \frac{d}{t^2} - b(p - 1)(2\nabla^i S \nabla_i v).
\]
Evolution equation (4.13) results that (4.13).

**Definition 4.2.** Suppose that \( g(t) \) evolves by (1.1). Let \( S \) be the trace of \( S_{ij} \) and \( X = X^i \frac{\partial}{\partial x^i} \) be a vector field on \( M \). We define

\[
\mathcal{E}_6(S, X) = (b - 1)\mathcal{D}(S) + 2\mathcal{I}(S, X) + b(2\nabla^i S_{ij} - \nabla_j S)X^j
\]

where \( b \) is a constant.

**Proposition 4.3.** Let \( g(t), t \in [0, T] \) be a solution to the geometric flow (1.1) on a closed Riemannian \( n \)-manifold \( M \) satisfying

\[
(4.14) \quad \mathcal{E}_6(S, X) \geq 0, \quad \mathcal{H}(S, X) \geq 0, \quad \text{Ric} \geq -(n-1)k_1, \quad -k_2 \leq S_{ij} \leq k_3, \quad S \geq 0
\]

for all vector fields \( X \) and all time \( t \in [0, T) \). Suppose \( u \) is a smooth positive solution to equation (1.3) with \( p > 1 \) and \( v = \frac{\max}{\rho} u^{p-1} \). Then for any \( b \in [2, \infty) \) and \( d \geq b \), on the geodesic ball \( Q_{\rho,T} \), we have

\[
(4.15) \quad \frac{[\nabla v]^2}{v} - b \frac{v_1}{v} - (b - 1)\frac{S}{v} - \frac{d}{t} \leq b\alpha \frac{E_4 v_{\text{max}}}{\rho^2} + E_5 + E_6
\]

where \( \alpha = \frac{bn(p-1)}{2 + bn(p-1)} \), \( E_4 = \frac{(k_2^2 + \frac{\sqrt{2}}{2} + \frac{\sqrt{7}}{2})c_3(p-1)}{4(n-1)} \), \( E_5 = \sqrt{c_1(k_2 + k_3)^2 + \frac{2}{b}(k_2 + k_3) + 1} \), and \( E_6 = n(k_2 + k_3)(b - 2) \sqrt{\frac{b(p-1)}{2}} \).

**Proof.** Let \( x, x_0 \) and \( d(x, x_0, t) \) be the geodesic distance \( x \) from \( x_0 \) with respect to the metric \( g(t) \). Choose a smooth cut-off function \( \psi(s) \) defined on \( [0, \infty) \) with \( \psi(s) = 1 \) for \( 0 \leq s \leq \frac{1}{2}, \psi(s) = 0 \) for \( 1 \leq s \) and \( \psi(s) > 0 \) for \( \frac{1}{2} < s < 1 \) such that \(-c_1 \psi \leq \psi'(s) \leq 0, -c_2 \leq \psi''(s) \leq c_2 \) and \(-c_2 \psi \leq |\psi'|^2 \leq c_2 \psi \) for some absolute constants \( c_1, c_2 > 0 \). For any fixed point \( x_0 \in M \) and any positive number \( \rho > 0 \), we define \( \phi(x, t) = \psi(\frac{r(x, t)}{2\rho}) \) on

\[
(4.16) \quad Q_{\rho,T} = B(x_0, 2\rho) \times [0, T) \subset M \times [0, \infty)
\]

where \( B(x_0, 2\rho) \) is a ball of radius \( 2\rho \) centered at \( x_0 \) and \( r(x, t) = d(x, x_0, t) \).

Using an argument of Calabi [2], we can assume every where smooth ness of \( \phi(x, t) \) with support in \( Q_{\rho,T} \). By the Laplacian comparison theorem in [1], the Laplacian of the distance function satisfies

\[
(4.17) \quad \Delta r(x, t) \leq (n-1)\sqrt{|k_1|}{\coth(2\sqrt{|k_1|}|\rho|)}, \quad \forall x \in M, \quad d(x, x_0) \geq 2\rho.
\]

From the definition of \( \phi \) and direct calculation shows that

\[
\frac{|\nabla \phi|^2}{\phi} = \frac{|\psi'|^2|\nabla r|^2}{4\rho^2} \leq \frac{c_1}{\rho^2},
\]

and

\[
\Delta \phi = \frac{\psi' \Delta r}{2\rho} + \frac{\psi''|\nabla r|^2}{4\rho^2} \geq -\frac{c_1}{2\rho} (n-1)\sqrt{|k_1|}{\coth(2\sqrt{|k_1|}|\rho|)} - \frac{c_1}{4\rho^2} \geq -\frac{c_1\sqrt{|k_1|}}{2\rho} - \frac{c_1}{4\rho^2}.
\]

On the other hand, since along the geometric flow (1.1), for a fixed smooth path \( \gamma : [a, b] \to M \) whose length at time \( t \) is given by \( d(\gamma) = \int_{a}^{b} |\gamma'(s)|_{g(t)} ds \), where \( s \) is the arc length along the path, we have

\[
\frac{\partial d(\gamma)}{\partial t} = -\frac{1}{2} \int_{a}^{b} |\gamma'(s)|_{g(t)}^{-1} S_{ij}(X, X) ds.
\]
where $X$ is the unit tangent vector to the path $\gamma$. $-k_2g \geq S_{ij} \leq k_3g$ results that $-(k_2 + k_3)g \leq S_{ij} \leq (k_2 + k_3)g$, then

$$\sup_M |S_{ij}|^2 \leq n(k_2 + k_3)^2.$$ 

Now, we get

$$ \frac{\partial \phi}{\partial t} = \frac{\psi'}{2\rho} \frac{\partial r}{\partial t} = \frac{\psi'}{2\rho} \int S_{ij}(X, X) ds \leq \sqrt{c_2(k_2 + k_3)^2}. $$

Suppose that $t\phi F$ achieves its positive maximum value at $(v_0, t_0)$. Then at $(x_0, t_0)$, we have

$$\nabla(t\phi F)(x_0, t_0) = 0, \quad \frac{\partial}{\partial t}(t\phi F)(x_0, t_0) \geq 0, \quad L(t\phi F)(x_0, t_0) \geq 0.$$ 

Suppose that

$$y = \frac{[\nabla v]^2}{v} + \frac{S}{v}, \quad \tilde{y} = t\phi y, \quad z = \frac{v_t}{v} + \frac{d}{b}\phi^t, \quad \tilde{z} = t\phi z$$

then $F = y - b\tilde{z}$, $t\phi F = \tilde{y} - b\tilde{z}$ and

$$\mathcal{L}(F) = 2p \nabla_i F \nabla_i v - \frac{1 - b}{v} p - 1|\frac{\partial S}{\partial t} - 2\nabla_i S \nabla_i v + 2S^i_j \nabla_i v \nabla_j v$$

$$- 2(p - 1)(R^i_j - S^i_j) \nabla_i v \nabla_j v - 2(p - 1)|\nabla^2 v + \frac{b}{2} S_{ij}|^2$$

$$+ \frac{(b - 2)^2}{2} (p - 1)|S_{ij}|^2 + (p - 1)(1 - b)D(S) - \frac{1}{b} F^2$$

$$- |(p - 1)S - \frac{2(1 - b)}{v} S + \frac{2d}{b}\phi^t F - \frac{b - 1}{b} y^2 - \frac{(b - 1)(b - 2)}{b} y v$$

$$- \frac{d^2}{b t^2} + \frac{2d}{b t} (1 - b)|\nabla v|^2 - d(p - 1)\frac{S}{t} + 2 - b d S - \frac{2 - 1}{b} b d t$$

$$+ \frac{d}{t^2} - b(p - 1)(2\nabla^i S_{ij} - \nabla_i S)\nabla^j v.$$ 

Therefore

$$t\phi \mathcal{L}(t\phi F) = t\phi^2 + t^2 \phi \phi^t \phi + t^2 \phi^2 \mathcal{L}(F) - (p - 1)t^2 \phi^2 \Delta \phi$$

$$- 2t^2 (p - 1) \phi \nabla_i \phi \nabla^i F$$

$$= \phi(\tilde{y} - b\tilde{z}) + t\phi (\tilde{y} - b\tilde{z}) - (p - 1)tv\phi^t (\tilde{y} - b\tilde{z})$$

$$- 2t^2 (p - 1) \phi \nabla_i \phi \nabla^i F + 2pt^2 \phi^2 \nabla_i F \nabla_i v$$

$$- t^2 \phi^2 \left[ \frac{1 - b}{v} + p - 1 \right] H(S, \nabla v) - 2(p - 1)t^2 \phi^2 |\nabla^2 v + \frac{b}{2} S_{ij}|^2$$

$$+ \frac{(b - 2)^2}{2} (p - 1)t^2 \phi^2 |S_{ij}|^2 - \frac{1}{b} (\tilde{y} - b\tilde{z})^2$$

$$- |(p - 1)S - \frac{2(1 - b)}{v} S + \frac{2d}{b t^2} |\nabla v|^2 + \frac{b - 1}{b} y^2$$

$$- \frac{(b - 1)(b - 2)}{b} t\phi \frac{S}{v} + \frac{2d}{b (1 - b)t\phi^2 |\nabla v|^2}{v}$$

$$- d(p - 1)t\phi^2 S + 2 - b d S - \frac{2 - 1}{b} b d t\phi^2$$

$$- t^2 \phi^2 (p - 1)E_0(S, \nabla v) - \frac{d}{b} (d - b)\phi^2.$$
On the other hand, we have
\[
\int_0^2 \phi^2 \nabla_i v \nabla^i v = - \int_0^2 \phi F \nabla_i \phi \nabla^i v \leq \int_0^2 \phi \nabla_i \phi \nabla^i v \leq \frac{\sqrt{c_1}}{\rho} \tilde{y}^2 (t_0 \phi)^2 (\tilde{y} - \tilde{z}),
\]
\[-(p-1)t_0 v \Delta \phi (\tilde{y} - \tilde{z}) \leq (p-1)t_0 v \left( \frac{c_1 \sqrt{k_1}}{2 \rho} + \frac{c_1}{4 \rho^2} \right) (\tilde{y} - \tilde{z}),
\]
\[-2(p-1)t_0^2 v \nabla_i \phi \nabla^i v = 2(p-1)t_0^2 v |\nabla \phi|^2 F \leq 2(p-1)t_0 v \frac{c_1}{\rho^2} (\tilde{y} - \tilde{z}),
\]
and
\[
-2(p-1)t_0^2 \phi^2 |\nabla^2 v + \frac{b}{2} S_{ij}|^2 \leq - \frac{2(p-1)t_0^2 \phi^2}{n} (\Delta v + \frac{b}{2} S)^2
\]
\[
(4.20) \quad - \frac{2}{b^2 n(p-1)} \left( \tilde{y} - \tilde{z} + (b-1)\tilde{y} - \frac{b(b-2)}{2}(p-1)t_0 \phi S \right)^2.
\]
Thus
\[
0 \leq t_0 \phi L(t_0 \phi F)
\]
\[
\leq (\tilde{y} - \tilde{z}) + t_0 \sqrt{c_2(k_2 + k_3)}(\tilde{y} - \tilde{z}) + (p-1)t_0 v \left( \frac{c_1 \sqrt{k_1}}{2 \rho} + \frac{c_1}{4 \rho^2} \right) (\tilde{y} - \tilde{z})
\]
\[
(4.21) \quad + 2(p-1)t_0 v \frac{c_1}{\rho^2} (\tilde{y} - \tilde{z}) + 2p \frac{\sqrt{c_1}}{\rho} \tilde{y}^2 (t_0 v)^2 (\tilde{y} - \tilde{z})
\]
\[- \frac{2}{b^2 n(p-1)} \left( \tilde{y} - \tilde{z} + (b-1)\tilde{y} - \frac{b(b-2)}{2}(p-1)t_0 \phi S \right)^2
\]
\[
+ \frac{(b-2)^2}{2}(p-1)t_0^2 \phi^2 S^2 - \frac{1}{b}(\tilde{y} - \tilde{z})^2.
\]
Notice that \((r + s)^2 \geq r^2 + 2rs\) results that
\[
- \frac{2}{b^2 n(p-1)} (\tilde{y} - \tilde{z})^2 - \frac{4(b-1)}{b^2 n(p-1)} \tilde{y}(\tilde{y} - \tilde{z}) + \frac{2(b-2)}{bn} t_0 \phi S(\tilde{y} - \tilde{z})
\]
hence
\[
0 \leq (\tilde{y} - \tilde{z}) + t_0 \sqrt{c_2(k_2 + k_3)}(\tilde{y} - \tilde{z}) + (p-1)t_0 v \left( \frac{c_1 \sqrt{k_1}}{2 \rho} + \frac{c_1}{4 \rho^2} \right)(\tilde{y} - \tilde{z})
\]
\[
+ 2(p-1)t_0 v \frac{c_1}{\rho^2} (\tilde{y} - \tilde{z}) + 2p \frac{\sqrt{c_1}}{\rho} \tilde{y}^2 (t_0 v)^2 (\tilde{y} - \tilde{z}) - \frac{1}{b_0}(\tilde{y} - \tilde{z})^2
\]
\[- \frac{4(b-1)}{b^2 n(p-1)} \tilde{y}(\tilde{y} - \tilde{z}) + \frac{2(b-2)}{bn} t_0 \phi S(\tilde{y} - \tilde{z}) + \frac{(b-2)^2}{2}(p-1)t_0^2 \phi^2 S^2
\]
\[
(\tilde{y} - \tilde{z}) \left[ - \frac{4(b-1)}{b^2 n(p-1)} \tilde{y} + 2p \frac{\sqrt{c_1}}{\rho} \tilde{y}^2 (t_0 v)^2 + \frac{c_1 \sqrt{k_1}}{2 \rho} (p-1) + \frac{c_1}{4 \rho^2} + 2 \frac{c_1}{\rho^2} \right]
\]
\[
+ \frac{2(b-2)}{b} t_0 \phi S(k_2 + k_3) + 1
\]
\[
+ \frac{(b-2)^2}{2}(p-1)t_0^2 n^2(k_2 + k_3)^2 - \frac{1}{b \alpha}(\tilde{y} - \tilde{z})^2.
\]
where \( \alpha = \frac{bn(p-1)}{2\gamma^m(p-1)} \). For \( a > 0 \) inequality \(-ax^2 + bx \leq \frac{b^2}{4a} \) implies that

\[
(4.22) \quad - \frac{4(b-1)}{b^2n(p-1)} \hat{y} + 2p \sqrt{c_1} \hat{y}^2 (t_0 v) \leq \frac{b^2 p^2 n c_1}{4(b-1)p^2} (p-1) t_0 v.
\]

Therefore

\[
0 \leq (\hat{y} - b\hat{z}) \left[ \left( \frac{b^2 p^2 n c_1}{4(b-1)p^2} + \frac{c_1 \sqrt{k}}{2p} + \frac{c_1}{4p^2} + 2 \frac{c_1}{\rho^2} \right) (p-1) t_0 v + t_0 \sqrt{c_2} (k_2 + k_3)^2 + \frac{2(b-2)}{b} t_0 (k_2 + k_3) + 1 \right] + t_0 n(k_2 + k_3)(b-2) \sqrt{\frac{b(p-1)\alpha}{2}}.
\]

If \( 0 \leq -ax^2 + bx + c \) for \( a, b, c > 0 \) then \( x \leq \frac{b}{a} + \sqrt{\frac{c}{a}} \). Hence

\[
(4.24) \quad \hat{y} - b\hat{z} \leq b\alpha \left[ \left( \frac{b^2 p^2 n c_1}{4(b-1)p^2} + \frac{c_1 \sqrt{k}}{2p} + \frac{c_1}{4p^2} + 2 \frac{c_1}{\rho^2} \right) (p-1) t_0 v + t_0 \sqrt{c_2} (k_2 + k_3)^2 + \frac{2(b-2)}{b} t_0 (k_2 + k_3) + 1 \right] + t_0 n(k_2 + k_3)(b-2) \sqrt{\frac{b(p-1)\alpha}{2}}.
\]

This completes the proof. \( \square \)

**Proof of Theorem 2.1** In Proposition 4.3 suppose that \( b = 2 \). Then inequality (4.15) results that (2.2).

**Proof of Corollary 2.2** If \( u \) is bounded on \( M \times [0, T] \), then assume that \( \rho \to \infty \), therefore inequality Theorem 2.1 results that (2.2).

**Proof of Theorem 2.3** For any curve \( \gamma(t), t \in [t_1, t_2] \), from \( \gamma(t_1) = x_1 \) to \( \gamma(t_2) = x_2 \), we have

\[
\log \frac{v(x_1, t_1)}{v(x_2, t_2)} = \int_{t_1}^{t_2} \frac{d}{dt} \log v(\gamma(t), t) dt = \int_{t_1}^{t_2} \frac{v_t}{v} + \frac{v d\gamma}{v dt} dt.
\]

Since for any \( x, y \), inequality \( xy \geq -\frac{x^2}{2} - \frac{y^2}{2} \) results that

\[
\nabla v. \frac{d\gamma}{dt} \geq -\frac{\nabla v}{2} \frac{d\gamma}{dt} - \frac{1}{2} \frac{d\gamma}{dt}^2.
\]

Hence

\[
\log \frac{v(x_1, t_1)}{v(x_2, t_2)} \geq \int_{t_1}^{t_2} \left( \frac{v_t}{v} - \frac{\nabla v}{2v} \frac{d\gamma}{dt} - \frac{1}{2} \frac{d\gamma}{dt}^2 \right) dt.
\]
Corollary 2.2 implies that
\[
\log \frac{v(x_1, t_1)}{v(x_2, t_2)} \geq \int_{t_1}^{t_2} \left( -\frac{n(p-1)}{1+n(p-1)} E_2 - \frac{S}{2v_{\min}} - \frac{d}{2t} - \frac{1}{2v_{\min}} \left| \frac{d\gamma}{dt} \right|^2 \right) dt \\
= -\frac{n(p-1)}{1+n(p-1)} E_2(t_2 - t_1) - \left( \frac{t_2}{t_1} \right)^d - \frac{1}{2v_{\min}} \int_{t_1}^{t_2} \left( S + \left| \frac{d\gamma}{dt} \right|^2 \right) dt.
\]
By exponentiating we arrive at (2.5).

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