ON A COMPACTIFICATION OF THE MODULI SPACE OF THE RATIONAL NORMAL CURVES

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ABSTRACT. For any odd \( n \), we construct a smooth minimal (i.e. obtained by adding an irreducible hypersurface) compactification \( \mathcal{M}_n \) of the quasi-projective homogeneous variety \( S_n = \mathbb{P}GL(n+1)/SL(2) \) that parameterizes the rational normal curves in \( \mathbb{P}^n \). \( \mathcal{M}_n \) is isomorphic to a component of the Maruyama scheme of the semi-stable sheaves on \( \mathbb{P}^n \) of rank \( n \) and Chern polynomial \( (1 + t)^{n+2} \). This will allow us to explicitly compute the Betti numbers of \( \mathcal{M}_n \).

In particular \( \mathcal{M}_3 \) is isomorphic to the variety of nets of quadrics defining twisted cubics, studied by G. Ellingsrud, R. Piene and S. Strømme [EPS].

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1. INTRODUCTION

A rational normal curve \( C_n \), or equivalently a Veronese curve, is a smooth, rational, projective curve of degree \( n \), in the complex projective space \( \mathbb{P}^n \): in particular the Hilbert polynomial of \( C_n \) is \( P_{C_n}(d) = nd + 1 \). For a description of some interesting properties of this curve, see [H].

The set \( S_n \) of the rational normal curves is an homogeneous quasi-projective variety isomorphic to \( \mathbb{P}GL(n+1)/SL(2) \). The purpose of the paper is to describe a nice compactification of such variety, by considering some vector bundles on \( \mathbb{P}^n \), called Schwarzenberger bundles [Schw]. In particular we compute the Euler characteristic of such compactification and its Betti numbers.

There are several ways to define a compactification of the variety \( S_n \): probably the most natural way is to consider the closure \( \mathcal{H}_n \) of the open sub-scheme of the Hilbert scheme \( \text{Hilb}^{P_{C_n}}(\mathbb{P}^n) \), parameterizing the rational normal curves in \( \mathbb{P}^n \). In [PS], the authors describe such compactification in the case \( n = 3 \). In particular, they show that \( \mathcal{H}_3 \subseteq \text{Hilb}^{3d+1}(\mathbb{P}^3) \) is a smooth irreducible variety of dimension 12. Only recently, it was proven by M. Martin-Deschamps and R. Piene [MP] that \( \mathcal{H}_4 \) is singular. Moreover it is not difficult to verify, with the help of the algorithm described in [NS], that \( \mathcal{H}_5 \) and \( \mathcal{H}_6 \) are singular in the points represented by the

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5-fold and 6-fold lines respectively. Therefore we can suspect that \( \mathcal{H}_n \) is singular for any \( n \geq 4 \) (see also [Kap], remark 2.6).

Another natural compactification is given by the closure \( \mathcal{C}_n \) of the quasi projective variety \( S_n \) considered as an open subset of the Chow variety \( \mathcal{C}_{1,n}(\mathbb{P}^n) \) that parameterizes the effective cycles of dimension 1 and degree \( n \) in \( \mathbb{P}^n \).

In [ES], a third natural compactification \( \mathcal{M}_n \) of \( S_n \) is described: this is made by considering the space of all the \( 2 \times n \) matrices with linear forms as entries. In fact all the rational normal curves in \( \mathbb{P}^n \) is the zero locus of the 2-minors of such a matrix. In particular, when \( n = 3 \), \( \mathcal{M}_3 \) can be seen as the variety parametrizing the nets of quadrics in \( \mathbb{P}^3 \) and \( \mathcal{H}_3 \) is the blow-up of \( \mathcal{M}_3 \).

In [C], it is shown that for any odd \( n \), the projective variety \( \mathcal{M}_n \) is isomorphic to a smooth irreducible component of the Maruyama scheme \( \mathcal{M}_{\mathbb{P}^n}(n; c_1, \ldots, c_n) \) parameterizing the semistable sheaves on \( \mathbb{P}^n \) of rank \( n \) and with Chern polynomial \( c_t = \sum c_t t^i = (1 + t)^{n+2} \). \( \mathcal{M}_n \) can be seen as the quotient of a projective space \( \mathbb{P}^N \), by the action of a reductive algebraic group \( G \). This description will allow us to apply a technique of Bialynicki-Birula [B], to compute the Betti numbers of \( \mathcal{M}_n \) (see also [ES]).

Throughout the paper we will use the following notations:

- \( V, W, I \) are complex vector spaces of dimension \( n + 1 \), \( m + k \) and \( k \) respectively, where \( m \geq n \).
- For any \( A \in \mathbb{P}(\text{Hom}(W, V \otimes I)) \), the cokernel \( \mathcal{F}_A \) of the associated map
  \[
  A^*: I \otimes \mathcal{O}_V(1) \longrightarrow W \otimes \mathcal{O}_V(1)
  \]
  is a coherent sheaf of rank \( m \). If \( A^* \) is a vector bundle, then it is said Steiner bundle of rank \( m \), and it is contained in the exact sequence:
  \[
  0 \longrightarrow I \otimes \mathcal{O}_V \longrightarrow W \otimes \mathcal{O}_V(1) \longrightarrow \mathcal{F}_A \longrightarrow 0. 
  \]
  Moreover if \( k = 2 \) and \( n = m \), then all the Steiner bundles are Schwarzenberger bundles (see also [ES]).
- \( \mathbb{G}(k, n+1) (\cong \mathbb{G}(k-1, \mathbb{P}^n)) \) is the Grassmanian of the \( k \)-subspaces of \( V \) or equivalently of the \( k-1 \) subspaces of the projective space \( \mathbb{P}^n \).
- Let \( G = \text{SL}(I) \times \text{SL}(W) \) and \( X = \mathbb{P}(\text{Hom}(W, V \otimes I)) \): we will study the natural action of \( G \) on \( X \) and we will denote by \( X^s \) (resp. \( X^{ss} \)) the open subset of the stable (resp. semi-stable) points of \( X \).
- For any \( A \in X \), \( \text{Stab}_G(A) = \{(P, Q) \in G | PAQ^{-1} = kA \text{ for some } k \in \mathbb{C}^* \} \) is the stabilizer of \( A \) by the group \( G \).
- \( \mathcal{M}_{n,m,k} = X^{ss}/G \) (resp. \( X^s/G \)) is the categorical (resp. geometric) quotient of \( X \) by \( G \). In particular, if \( n = m \), we will denote \( \mathcal{M}_{n,k} = \mathcal{M}_{n,n,k} \).
- \( V^* = \mathbb{C}[x_0, \ldots, x_n]_1 \) is the dual space of \( V \).
- For any \( A \in X \), \( D(A) \) is the degeneracy locus of \( A \) and \( D_0(A) \) is the variety of all the points \( x \in \mathbb{P}^n \) such that \( \text{rank } A_x = 0 \).
- \( S = \{A \in X | D(A) = \emptyset\} = \{A \in X | S(\mathcal{F}_A) = \emptyset\} \subseteq X^s \).
- \( S_{n,m,k} = S/G \) is the moduli space of the rank \( m \) Steiner bundles on \( \mathbb{P}^n = \mathbb{P}(V) \): in particular \( S_{n,k} = S_{n,n,k} \) is the moduli space of the “classical” Steiner bundles or rank \( n \) on \( \mathbb{P}(V) \).
- For any matrix \( A \in \mathcal{M}(k \times (m+k), V^*) \), if \( A = (a_{i,j}) \) we define \( i_s(A) = \min\{j = 0, \ldots, n+k-1 | a_{i,j} \neq 0 \} \) (we will often write \( i_s \) instead of \( i_s(A) \)).
- \( j(n) = \left\lfloor \frac{n+k}{2} \right\rfloor \) where \( \lfloor m \rfloor \) denotes the integer part of \( m \).
- For any coherent sheaf \( \mathcal{E} \) of rank \( r \) on \( \mathbb{P}^n \) and for any \( t \in \mathbb{Z} \), we write \( \mathcal{E}(t) \) instead of \( \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(t) \). \( \mathcal{E}_N \) will denote the normalized of \( \mathcal{E} \), i.e. \( \mathcal{E}_N = \mathcal{E}(t_0) \) where \( t_0 \in \mathbb{Z} \) is such that \( -r < c_1(\mathcal{E}(t_0)) \leq 0 \).
Moreover, we define the slope of $E$ as the number $\mu(E) = \frac{c_1(E)}{r}$ and $E$ is said to be $\mu$-stable if it is Mumford-Takemoto stable.

In the first part of the paper we describe the (semi-)stable points of the projective space $\mathbb{P}(\text{Hom}(W, V \otimes I))$ under the action of $\text{SL}(I) \times \text{SL}(W)$ (see [MFK] for an introduction to the geometric invariant theory) and in particular we will prove that, if $m < \frac{n}{2}$, then all the Steiner bundles are defined by stable matrices, i.e. $S_{n,m,k} \subseteq \mathbb{P}(\text{Hom}(W, V \otimes I))^k/(\text{SL}(I) \times \text{SL}(W))$.

In the second part of the paper, we investigate some properties of $S_{n,m,2}$ and in particular of $S_{n,2}$, the moduli space of the Schwarzenberger bundles. By the previous correspondence of bundles and curves, $\mathcal{M}_{n,n,2}$ gives us a compactification of the set of the rational normal curves in $\mathbb{P}^n$.

We define a filtration of $\mathcal{M}_{n,m,2}$ and we show that the compactification is obtained by adding an irreducible hypersurface.

Moreover in [C] it is shown that, if $k = 2$ and $m$ is odd, then $A \in \mathbb{P}(\text{Hom}(W, V \otimes I))$ is stable if and only if the correspondent coherent sheaf $F_A$ is $\mu$-stable. This yields the theorem:

**Theorem 1.1.** $M_{n,m,2}$ is isomorphic to the connected component of the Maruyama moduli space $\mathcal{M}_{\mathbb{P}^n}(m, c_1, \ldots, c_n)$ containing the Steiner bundles. Such component is smooth and irreducible.

In the last two sections we compute the Betti and Hodge numbers of the smooth projective variety $\mathcal{M}_{n,m,2}$. This formula will be obtained by studying a natural action of $\mathbb{C}^*$ on $\mathcal{M}_{n,m,2}$: in particular we will describe its fixed points and we will compute the weights of the action of $\mathbb{C}^*$ defined in the tangent spaces of the variety at the fixed points.

2. **The categorical quotient of $\mathbb{P}(\text{Hom}(W, V \otimes I))$ by $\text{SL}(I) \times \text{SL}(W)$**

We are interested in the study of the action of $G = \text{SL}(I) \times \text{SL}(W)$ on the projective space $X = \mathbb{P}(\text{Hom}(W, I \otimes V))$. In fact, as shown in the introduction, each $A \in X^{ss}$, such that $A^* : I \otimes \mathcal{O}_{\mathbb{P}^n} \to W \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ is injective, corresponds to a coherent sheaf $F_A$ contained in the exact sequence $[\mathbb{I}]$.

Furthermore $F_A \simeq F_B$ if and only if $PA = BQ$ for some $P \in \text{SL}(I)$ and $Q \in \text{SL}(W)$ (see for instance [AC] or [MT]).

**Lemma 2.1.** Let $A \in X^{ss}$. Then both $A : W \to I \times V$ and $A^* : I \to W \times V$ are injective.

**Proof.** Let $A : W \to I \times V$ be non-injective. Then we can suppose that the first column of $A$ is zero. Let us consider the 1-dimensional parameter subgroup $\gamma : \mathbb{C}^* \to G$ defined by $t \mapsto (\text{Id}, \text{diag}(t^{-(m+k-1)}, t, \ldots, t)) \in \text{SL}(I) \times \text{SL}(W)$: then $\lim_{t \to 0} \gamma(t)A = 0$ and, by the Hilbert-Mumford criterion, the matrix $A$ cannot be semi-stable.

Let us suppose now that $A^* : I \to W \times V$ is not injective: i.e. the first row of $A$ is zero. In this case it suffices to consider the 1-dimensional parameter subgroup $\mu : t \mapsto (\text{diag}(t^{-k-1}, t, \ldots, t), \text{Id}) \in \text{SL}(I) \times \text{SL}(V)$ in order to have $\lim_{t \to 0} \mu(t)A = 0$. \hfill \Box

As a direct consequence of the lemma, it follows that for any $A \in X^{ss}$, the sheaf $F_A$ is well-defined as the cokernel of $A^*$ and is contained in the sequence $[\mathbb{I}]$. Moreover it results
\[ T_A := A(W) \in G(m + k, I \otimes V). \] Thus, in order to study the (semi-)stable point of \( X \) by the action of \( G \), it suffices to study the action of \( \text{SL}(I) \) on the variety \( G(m + k, I \otimes V) \): in particular we have that the categorical quotient \( M_{n,m,k} := X^s//G \) is isomorphic to the quotient \( G(m + k, I \otimes V)^s//\text{SL}(I) \).

Let us recall first the following known result:

**Proposition 2.2.** Let \( T \in G(m + k, I \otimes V) \). The following are equivalent:

1. \( T \) is semi-stable (resp. stable) under the action of \( \text{SL}(I) \);
2. for any non-empty subspace \( I' \subseteq I \)
   \[ \frac{\dim T'}{\dim T} \leq \frac{\dim T}{\dim I} \quad \text{(resp. <)} \]
   where \( T' = (I' \otimes V) \cap T \).

**Proof.** See for instance [NT] (prop. 5.1.1) \( \square \)

As a corollary we get a description of the (semi-)stable points of \( X \) by the action of \( G \):

**Theorem 2.3.** \( A \in X \) is not stable under the action of \( G \) if and only if with respect to suitable bases of \( W \) and \( I \), it results \( i_0(A) \geq i_1(A) \geq \cdots \geq i_{k-1}(A) \) and there exists \( s \in \{0, \ldots, k-1\} \) such that:

\[ \text{either } i_s(A) \geq \frac{m + k}{k}(k - 1 - s) \text{ if } s \neq k - 1 \quad \text{or } \quad i_{k-1}(A) > 0 \quad (2) \]

**Theorem 2.4.** \( A \in X \) is not semi-stable under the action of \( G \) if and only if with respect to suitable bases of \( W \) and \( I \), it results \( i_0(A) \geq i_1(A) \geq \cdots i_{k-1}(A) \) and there exists \( s \in \{0, \ldots, k-1\} \) such that:

\[ i_s(A) \geq \frac{m + k}{k}(k - 1 - s) \quad (3) \]

**Corollary 2.5.** \( X^s = X^{ss} \) if and only if \( (m, k) = 1 \)

**Proof.** If there exists \( A \in X \) properly semi-stable, then there exists \( s \in \{0, \ldots, k-2\} \) such that

\[ i_s(A) = \frac{m + k}{k}(k - 1 - s). \]

Since \( 1 \leq k - 1 - s \leq k - 1 \), such \( s \) exists if and only if \( (m, k) \neq 1 \). \( \square \)

Now we are interested to study the stability of the matrices defining the Steiner bundles and thus we will consider all the matrices \( A \) such that \( \text{rank } A_x = k \) for any \( x \in \mathbb{P}^n \): in [AO] it is shown that if \( n = m \) (boundary format) then all such matrices are stable. We generalize such result with the following:

**Theorem 2.6.** If \( m < \frac{nk}{n} \) then every indecomposable vector bundle \( \mathcal{F}_A \) is defined by a G.I.T. stable matrix \( A \).

Before proving the theorem, we remind the following known lemma:

**Lemma 2.7.** Let \( F \) be a vector bundle of rank \( f \) on a smooth projective variety \( X \) such that \( c_{f-k+1}(F) \neq 0 \) and let \( \phi : \mathcal{O}_X^k \to F \) be a morphism with \( k \leq f \). Then the degeneracy locus \( D(\phi) = \{ x \in X | \text{rank}(\phi_x) \leq k - 1 \} \) is nonempty and \( \text{codim } D(\phi) \leq f - k + 1 \).
proof of theorem 2.4. Let \( \mathcal{F}_A \) be an indecomposable vector bundle. Then for any base of \( W \) and \( I, i_{k-1}(A) = 0 \), otherwise \( \mathcal{F}_A = \mathcal{F}' \oplus \mathcal{O}_{\mathbb{P}^n}(1) \) for some vector bundle \( \mathcal{F}' \).

Let \( I' \subseteq I \) of dimension \( r \): if \( s = \dim(I' \otimes V) \cap T_A \) and \( I'' \subseteq I \) is such that \( I' \oplus I'' = I \), then the restriction of \( A^* \) in \( I'' \) defines a morphism of vector bundles \( A': \mathcal{O}_{\mathbb{P}^n}^{k-r} \to \mathcal{O}_{\mathbb{P}^n}(1)^{m+k-s} \).

Let us suppose \( s > m - n + r \), then \( c_{(m+k-s)-(k-r)+1}(\mathcal{O}_{\mathbb{P}^n}(1)^{m+k-s}) \neq 0 \): lemma 2.7 implies that the degeneracy locus of \( A' \) is not empty, which leads to a contradiction.

Thus:
\[
\dim(I' \otimes V) \cap T_A \leq m + k - n - k + r = m - n + r;
\]
and in particular, if \( m < \frac{n k}{m} \), it results \( \dim(I' \otimes V) \cap T_A < \frac{r(m+k)}{k} \), i.e. \( A \) is G.I.T. stable. \( \square \)

Remark 2.8. By lemma 2.4 we have that if \( A \in X^{ss} \), then \( A: W \to I \otimes V \) is injective, thus it results \( X^{ss} = \emptyset \) if \( m > kn \). Furthermore it is easy to see that if \( m = nk \) the only point of \( M_{n, kn} \) is represented by the vector bundle \( I \otimes T_{\mathbb{P}^n} \).

3. Compactification of \( S_{n,m,2} \)

So far we have studied the G.I.T. compactification of \( S_{n,m,k} \) for any value of \( n, m \) and \( k \).

From now, we restrict our study to the case \( k = 2 \): in particular we know that the moduli space \( S_{n,2} \) is uniquely composed by Schwarzenberger bundles and thus it is isomorphic to \( \mathbb{P} GL(n+1)/SL(2) \).

Hence \( M_{n,2} \) is a compactification of the set of rational normal curves in \( \mathbb{P}^n \).

After a short review of the previous section, we define a \( G \)–invariant filtration of the space \( M_{n,m,2} \) and we study some properties of it.

Theorems 2.3 and 2.4 become:

Theorem 3.1. Let \( j(m) = \lfloor \frac{m+3}{2} \rfloor \). \( A \in X \) is not stable if and only if

either \( A \sim \begin{pmatrix} 0 & \cdots & 0 & f_{j(m)+1} & \cdots & f_{m+2} \\ g_1 & \cdots & g_{j(m)} & g_{j(m)+1} & \cdots & g_{m+2} \end{pmatrix} \) or \( A \sim \begin{pmatrix} 0 & \cdots & * \\ 0 & \cdots & * \end{pmatrix} \)

Theorem 3.2. If \( n \) is odd then \( X^{ss} = X^s \), i.e. there are not properly semi-stable points in \( X \). If \( n \) is even then \( A \in X \) is not semi-stable if and only if

either \( A \sim \begin{pmatrix} 0 & \cdots & 0 & f_{j(m)+2} & \cdots & f_{m+2} \\ g_1 & \cdots & g_{j(m)+1} & g_{j(m)+2} & \cdots & g_{m+2} \end{pmatrix} \) or \( A \sim \begin{pmatrix} 0 & \cdots & * \\ 0 & \cdots & * \end{pmatrix} \)

Lemma 3.3. Let \( m \) be even and for any \( i = 1, 2 \) let us define the subspaces \( I_1^i = \langle f_0^i \ldots f_{l_1^i} \rangle \) and \( I_0 = \langle g_0^i \ldots g_{l_2^i} \rangle \) of \( \mathbb{C}[x_0, \ldots, x_n] \) of dimension \( \frac{m}{2} + 1 \). Moreover let

\[
A^i = \begin{pmatrix} 0 & \cdots & 0 & f_0^i & \cdots & f_{l_1^i} \\ g_0^i & \cdots & g_{l_2^i} & 0 & \cdots & 0 \end{pmatrix} \quad i = 1, 2
\]

Then
\[
A^1 \sim A^2 \quad (4)
\]
if and only if

\[ \text{either } I_f^1 = I_f^2 \text{ and } I_g^1 = I_g^2 \text{ or } I_f^1 = I_g^2 \text{ and } I_g^1 = I_f^2 \]  
(5)

**Proof.** Let us suppose that (4) holds, then \( A^1 \) and \( A^2 \) have the same degeneracy locus and this implies:

\[ V(I_f^1) \cup V(I_g^1) = V(I_f^2) \cup V(I_g^2). \]

Since \( V(I_f^1) \), \( V(I_g^1) \), \( V(I_f^2) \) and \( V(I_g^2) \) are irreducible, it results \( V(I_f^1) = V(I_f^2) \) and \( V(I_g^1) = V(I_g^2) \) or \( V(I_f^1) = V(I_g^2) \) and \( V(I_g^1) = V(I_f^2) \), thus (4) holds.

Vice-versa let us suppose \( I_f^1 = I_f^2 \) and \( I_g^1 = I_g^2 \) and let \( B_1, B_2 \in \text{SL}(\frac{m}{2} + 1) \) be the respective base change matrices. Then

\[ A_1 \begin{pmatrix} B_2 \\ 0 \\ B_1 \end{pmatrix} = A_2. \]

Otherwise if \( I_f = I_g^1 \) and \( I_g = I_f^1 \) then if \( C_1, C_2 \in \text{SL}(\frac{m}{2} + 1) \) are the respective base change matrices, then

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 \begin{pmatrix} 0 & C_2 \\ C_1 & 0 \end{pmatrix} = A_2. \]

Thus (4) holds. \( \square \)

**Theorem 3.4.** Let \( m \) be even. Then

\[ (X^{ss} \setminus X^s) / G \simeq S^2 G \left( \frac{m}{2}, \mathbb{P}(V) \right). \]

(6)

**Proof.** Let \( A \in X^{ss} \setminus X^s \). Then

\[ A \sim \begin{pmatrix} 0 & \cdots & 0 & f_{j(m)+1} & \cdots & f_{m+2} \\ g_1 & \cdots & g_{j(m)} & g_{j(m)+1} & \cdots & g_{m+2} \end{pmatrix} \]

and thus if we consider the 1-dimensional parameter subgroup defined by the weights \( \beta = (-1,1) \) and \( \gamma = (-1, \ldots, -1, 1, \ldots, 1) \), it results:

\[ \lim_{t \to 0} tA = \begin{pmatrix} 0 & \cdots & 0 & f_{j(m)+1} & \cdots & f_{m+2} \\ g_1 & \cdots & g_{j(m)} & 0 & \cdots & 0 \end{pmatrix}. \]

Thus the points of \( (X^{ss} \setminus X^s) / G \) are in one-one correspondence with the orbits of the matrices \( \begin{pmatrix} 0 & \cdots & 0 & * & \cdots & * \\ * & \cdots & 0 & \cdots & 0 \end{pmatrix} \in X^{ss} \) by the action of \( G \). The previous lemma implies the isomorphism in (4). \( \square \)

**Remark 3.5.** Since \( G(m+2,2(n+1)) \simeq G(2n-m,2(n+1)) \), it follows that \( \mathcal{M}_{n,m,2} \simeq \mathcal{M}_{n,2n-m-2,2} \). In particular \( \mathcal{M}_{n,2} \) parameterizes the \( n \times 2 \) matrices with entries in \( V^* \): in fact a rational normal curve is the zero locus of the minors of such a matrix.

In the case \( n = 3 \), we have that \( \mathcal{M}_{3,2} \) is isomorphic to the variety of the nets of quadrics that define the twisted cubics in \( \mathbb{P}^3 \). In [EPS], the authors describe this variety and they show that there exists a natural morphism from the Hilbert scheme compactification \( \mathcal{H}_3 \) to \( \mathcal{M}_{3,3,2} \). It would be interesting to know if there exist a canonical morphism, \( \mathcal{H}_n \to \mathcal{M}_{n,n,2} \), for any odd \( n \).
For any \( \omega \in I \) we define \( R_\omega = \omega \otimes V \subseteq I \otimes V \): by theorems 3.1 and 3.2 we have that an injective matrix \( A : W \rightarrow I \otimes V \) is semi-stable (resp. stable) if and only if

\[
\dim R_\omega \cap T_A \leq \frac{m + 2}{2} \quad (\text{resp.} <)
\]

for any \( \omega \in I \).

For any \( j = 0, 1, \ldots \) we construct the subsets:

\[
S^j = \{ A \in X^{ss} | \exists \omega \in I \text{ such that } \dim R_\omega \cap T_A \geq j + m - n \} \subseteq X^{ss}
\]

and

\[
\tilde{S}^j = \{ A \in X^{ss} | \dim D(A) \geq j - 2 \} \subseteq X^{ss}.
\]

Such subsets of \( X \) define two filtrations:

\[
\emptyset = S^{j_0 + 1} \subseteq S^{j_0} \subseteq \cdots \subseteq S^1 = X^{ss}
\]

\[
\emptyset \subseteq \cdots \subseteq \tilde{S}^{j_0 + 1} \subseteq \tilde{S}^{j_0} \subseteq \cdots \subseteq \tilde{S}^1 = X^{ss}
\]

where \( j_0 = j(m) + n - m \). It results \( S^{j_0} = X^{ss} \setminus X^s \) and in particular it is empty if \( m \) is odd.

Furthermore we have:

**Theorem 3.6.**

1. \( S^j \subseteq \tilde{S}^j \subseteq S^{j-1} \) for any \( j \geq 2 \);
2. \( S^2 = \tilde{S}^2 \);
3. \( S^1 = \tilde{S}^1 = X^{ss} \).

In particular such subsets define a unique filtration \( G \)-invariant:

\[
\emptyset = S^{j_0 + 1} \subseteq \tilde{S}^{j_0 + 1} \subseteq S^{j_0} \subseteq \tilde{S}^{j_0} \subseteq \cdots \subseteq S^3 \subseteq \tilde{S}^3 \subseteq S^2 \subseteq \tilde{S}^2 \subseteq S^1 = \tilde{S}^1 = X^{ss}
\]

**Proof.** See [C] (thm 2.1). \( \square \)

**Remark 3.7.** In general \( S^i \neq \tilde{S}^i \): let us consider, for instance, \( n = m = 3 \) and

\[
A = \begin{pmatrix}
0 & 0 & x_0 & x_1 & x_2 \\
0 & x_1 & 0 & 0 & x_3 \\
\end{pmatrix}.
\]

Since \( D(A) = \{(0 : 0 : t_1 : t_2)\} \simeq P^1 \), \( A \in \tilde{S}^3 \); but \( S^3 = \emptyset \) (see also prop. 3.9).

**Corollary 3.8.** If \( m \) is odd and \( A \in X^s = X^{ss} \) then \( \text{codim } D(A) \geq \frac{m + 1}{2} \).

If \( m \) is even and \( A \in X^{ss} \) (resp. \( X^s \)) then \( \text{codim } D(A) \geq \frac{m}{2} \) (resp. \( > \)).

**Proof.** It suffices to notice that the previous theorem implies that \( \tilde{S}^{j_0 + 1} = \emptyset \) and that \( S^{j_0} \) is the set of the properly semi-stable points of \( X \). \( \square \)

**Proposition 3.9.** If \( m \) is odd, \( A \in X \) is stable and \( \text{codim } D(A) = \frac{m + 1}{2} \), then, up to the action of \( \text{SL}(I) \times \text{SL}(W) \times \text{SL}(V) \), we have

\[
A \simeq \begin{pmatrix}
x_0 & \ldots & x_{t-1} & 0 & \ldots & 0 & x_t \\
0 & \ldots & 0 & x_0 & \ldots & x_{t-1} & x_{t+1} \\
\end{pmatrix},
\]

where \( t = \frac{m+1}{2} \).
Thus, if \( j < j \) compactificate the moduli space of the rational normal curves in \( M \) all the sheaves in \( S \) irreducible. In particular we show that \( F \) if the stability of the cokernels.

A is the image of Corollary 3.11.

case, the two requirements are equivalent.

Let Theorem 3.13. D and since \( \text{codim} D = t \), it must be \( \text{codim} F(x_0, \ldots, x_{t-1}, y_0, \ldots, y_t) = t \) this implies that \( < x_0, \ldots, x_{t-1} > = < y_0, \ldots, y_t > \).

Moreover \( x_t \neq ay_t \) for any \( a \in \mathbb{C} \) otherwise \( A \) cannot be stable.

Remark 3.10. The matrix above can exist if \( n + 1 \geq t + 1 = \frac{m+2}{2}, \) i.e. if \( m \leq 2n - 1 \).

Since \( A : W \hookrightarrow I \otimes V \) injective, it must be \( m + 2 \leq 2(n + 1), \) i.e. \( m \leq 2n \) thus in the odd case, the two requirements are equivalent.

Corollary 3.11. Let \( V_i = X^{ss} \setminus S^i \) e \( \tilde{V}_i = X^{ss} \setminus \tilde{S}^i \).

Then such subsets define a \( G \)-invariant increasing filtration:

\[
\emptyset = V_1 \subseteq V_2 \subseteq \tilde{V}_2 \subseteq \tilde{V}_3 \subseteq \ldots \\
\ldots \subseteq \tilde{V}_j_0 \subseteq V_{j_0} \subseteq \tilde{V}_{j_0} \subseteq V_{j_0+1} = X^{ss}.
\]

In particular \( V_2 \) is the set of matrices that define vector bundles and \( V_{j_0} \) is the open set of the stable points in \( X \).

Remark 3.12. If \( n \) is odd then \( V_{j(m)} = X^s = \tilde{V}_{j(m)+1} = V_{j(m)+1} = X^{ss} \).

Otherwise if \( m \) is even then \( S^{j(m)}/G \simeq S^{2}\mathbb{G}(n, \mathbb{P}^n) \) (theorem 3.4).

All these results are needed to prove the following theorem:

Theorem 3.13. Let \( k = 2 \) and \( m \in \mathbb{N} \) odd. \( A \in \mathbb{G}(m + 2, I \otimes V) \) is G.I.T. stable if and only if \( \mathcal{F}_A \) is \( \mu \)-stable.

Proof. See (thm. 3.1). □

Theorem 3.13 is a direct consequence of this equivalence within the stability of the maps and the stability of the cokernels.

4. Dimension of \( S^j/G \)

For any \( j < j(m) \) we calculate the dimension of \( S^j/G \subseteq \mathcal{M}_{n,m,2} \) and we show that it is irreducible. In particular we show that \( S^j/G \) is the irreducible hypersurface that parameterizes all the sheaves in \( \mathcal{M}_{n,2} \) that are not bundles or, on the other hand, all the points added to compactificate the moduli space of the rational normal curves in \( \mathbb{P}^n \).

We remind that:

\[
S^j = \{ A \in X^{ss} | \exists 0 \neq \omega \in I \text{ such that } \dim(T_A \cap R_\omega) \geq j + m - n \}.
\]

Thus, if \( j < j(m) \),

\[
\frac{S^j}{SL(W)} \simeq \{ T \in \mathbb{G}(m + 1, \mathbb{P}(I \otimes V))^{ss} | \exists \omega \in I^* : \dim(T \cap \mathbb{P}(R_\omega)) \geq j + m - n - 1 \}.
\]
Let us define the incidence correspondence $\mathcal{I}_j \subseteq G(m + 1, \mathbb{P}(I \otimes V)) \times \mathbb{P}(I)$ as:

$$
\mathcal{I}_j = \{(T, [\omega])|T \in G(m + 1, \mathbb{P}(I \otimes V))^{**}, [\omega] \in \mathbb{P}(I), \dim(T \cap \mathbb{P}(R_\omega)) \geq j + m - n - 1\}
$$

and let $p_1$ and $p_2$ be the respective projections. Since $S^1 = X^{**}$, we can suppose $2 \leq j < j(m)$. Let us fix $[\omega] \in \mathbb{P}(I)$: then

$$p_2^{-1}(\omega) \simeq \{T \in G(m + 1, \mathbb{P}(I \otimes V))^{**} | \dim(T \cap \mathbb{P}(\omega \otimes V)) \geq j + m - n - 1\}
$$

and:

$$
\dim p_2^{-1}(\omega) = (n + 1 - (j + m - n))(j + m - n) + \\
+ (2(n + 1) - (m + 2))(m + 2 - (j + m - n)) = \\
= 2mn - m^2 + 3n - m + (n - m)j + j^2.
$$

Hence $\mathcal{I}_j$ is irreducible (see [4], theorem 11.14) of dimension $2mn - m^2 + 3n - m + 1 + (n - m)j + j - j^2$.

Now, if $T \in p_1(\mathcal{I}_j)$ is a generic point, $p_1^{-1}(T)$ is discrete, i.e. $\dim p_1^{-1}(T) = 0$ that implies:

$$
\dim S_j / SL(W) = \dim p_1(\mathcal{I}_j) = 2mn - m^2 + 3n - m + 1 + (n - m)j + j - j^2.
$$

Furthermore $S^j / SL(W)$ is irreducible.

Since all the points of $S^j$ are stable under the action of $G$ (we are supposing $j < j(n)$), theorem [1,4] implies

$$
\dim(S^j/G) = \dim S^j - \dim G = \dim(S^j/SL(W)) - \dim SL(I).
$$

Hence we have:

**Theorem 4.1.** $S^j/G$ is irreducible of codimension $(j + m - n)(j - 1) - 1$ for any $2 \leq j < j(m)$.

In particular:

**Corollary 4.2.** If $n = m$ (boundary format) $S^2/G$ is an irreducible hypersurface of $\mathcal{M}_{n,2}$ such that

$$
\mathcal{M}_{n,2} \setminus (S^2/G) \simeq S_{n,2}.
$$

By theorem [3,4], we know that, if $m$ is even, the variety $\mathcal{M}_{n,m,2} \setminus (S^{j(m)}/G)$ is isomorphic to $S^2G(\frac{m}{2}, \mathbb{P}(V))$ and thus it is irreducible of dimension $(n - \frac{m}{2})(\frac{m}{2} + 1)$, i.e. the G.I.T. quotient $S^{j(m)}/G$ is of codimension $(n - \frac{m}{2})(\frac{m}{2} + 1)$.

If $m$ is odd, then $S^{j(m)} = \emptyset$.

5. A torus action on $\mathcal{M}_{n,m,2}$

In the following two sections we compute the Euler characteristic of $\mathcal{M}_{n,m,2}$ and an implicit formula for its Hodge numbers. For this purpose, we will use the technique of Bialynichi-Birula [3], that is based on the study of the action of a torus on a smooth projective variety: such method was extensively used in the last decade to compute the Betti numbers of smooth moduli spaces (see for instance [K]).

In fact let an algebraic torus $T$ act on a smooth projective variety $Z$ and let $Z^T$ be its fixed points set. Then the Euler characteristics of $Z$ and $Z^T$ are equal. Furthermore if $T = \mathbb{C}^*$ is 1-dimensional, then all the cohomology groups of $Z$ and their Hodge decomposition may be reconstructed from the Hodge structure of the connected components $Z_i^T$ of $Z^T$. In order to do that, we fix a point $z_i \in Z_i^T$ for any component and we consider the action of $T$ on the tangent space $T_{z_i}Z$: let $n_i$ be the number of positive weights of $T$ acting on $T_{z_i}Z$, then we have:
Theorem 5.1 (Bialynichi-Birula). There is a natural isomorphism:
\[ H^{p,q}(Z) = \bigoplus_i H^{p-n_i,q-n_i}(Z_i). \]

Proof. See [3] and [4].

Thus let us consider now the action of \( T = \mathbb{C}^* \) on \( M_{n,m,2} \) defined by the morphism \( \rho : \mathbb{C}^* \to \text{GL}(V) \) with weights \( c = (1, 2, 2^2, \ldots, 2^n) \): this choice is motivated by the fact that
\[ c_i - c_j = c_{i'} - c_{j'} \quad \text{if and only if} \quad i = i' \quad \text{and} \quad j = j'. \]

(7)

that will be useful later on.

For any \( t \in \mathbb{C}^* \), we will write \( t(\cdot) \) to denote the image of \( \cdot \) by the map \( \rho(t) \).

Let \( A = \begin{pmatrix} f_0 & \cdots & f_{m+1} \\ g_0 & \cdots & g_{m+1} \end{pmatrix} \in M_{n,m,2} \) be a fixed point then
\[ t(A) = \begin{pmatrix} t(f_0) & \cdots & t(f_{m+1}) \\ t(g_0) & \cdots & t(g_{m+1}) \end{pmatrix} \sim A \]

for any \( t \in \mathbb{C}^* \). Thus it is defined a morphism \( \tilde{\rho} : \mathbb{C}^* \to \text{Aut}(I) \times \text{Aut}(W) \), such that \( \rho(t)(A) = \tilde{\rho}(t)(A) \) for any \( t \in \mathbb{C}^* \).

Thus for any fixed point \( A \), \( \rho \) induces an action of \( \mathbb{C}^* \) on \( I \) and \( W \); let \( P(t) \) and \( Q(t) \) be the components of \( \tilde{\rho} \) in \( \text{Aut}(I) \) and \( \text{Aut}(W) \) respectively, then \( t(A) = P(t) A Q(t)^{-1} \) for any \( t \) in \( \mathbb{C}^* \).

We can suppose that such action is diagonal and that it is defined by the weights \((a_0, a_1)\) and \((b_0, \ldots, b_{m+1})\) respectively (at the moment we do not fix any order for such weights, we will do it later on).

If \( f_k = \sum r_i x_i \) then \( \sum r_i t^{c_i} x_i = t(f_k) = \sum r_i t^{a_0 - b_k} x_i \), and since \( c_i \neq c_j \) if \( i \neq j \), it must be \( f_k = r_{i_k} x_{i_k} \) for a suitable \( i_k \in \{0, \ldots, n\} \) and with \( r_{i_k} \in \mathbb{C} \); moreover it results \( a_0 - b_k = c_{i_k} \) for any \( k \) such that \( r_{i_k} \neq 0 \).

Similarly we have \( g_k = s_{j_k} x_{j_k} \) with \( s_{j_k} \in \mathbb{C} \), \( j_k \in \{0, \ldots, n\} \) and \( a_1 - b_k = c_{j_k} \) for any \( k \) such that \( r_{j_k} \neq 0 \).

Thus the matrix \( A \) is monomial with respect to the bases of \( I \) and \( W \) chosen. Moreover the weights \((a_0, a_1)\) and \((b_0, \ldots, b_{m+1})\) are the solution of a system:
\[ \begin{cases} a_0 - b_k = c_{i_k} & \forall \ k \text{ s.t. } f_k \neq 0 \\ a_1 - b_k = c_{j_k} & \forall \ k \text{ s.t. } g_k \neq 0 \end{cases} \]

(8)

Since \( A \) is stable, there exists \( \tilde{k} \) such that \( f_{\tilde{k}} g_{\tilde{k}} \neq 0 \), thus, by (8), it follows that \( a_0 - a_1 = c_{i_{\tilde{k}}} - c_{j_{\tilde{k}}} \) : it is easy to check that if (8) admits a solution, then such solution is unique up to an additive constant; for this reason we can suppose \( a_0 = 0 \).

Now we can fix an order on the base of \( W \) chosen (we did not do it before): in fact we can suppose \( f_k = 0 \) if and only if \( k > k_0 \) where \( k_0 \in \{1, \ldots, m + 1\} \); moreover we can take \( b_0 \geq b_1 \geq \cdots \geq b_{k_0} \) and, if \( k_0 \leq m \), we can also take \( b_{k_0+1} \geq b_{k_0+2} \geq \cdots \geq b_{m+1} \). In particular we have \( c_{i_0} \leq c_{i_1} \leq \cdots \leq c_{i_{k_0}} \) and \( c_{j_{k_0+1}} \leq \cdots \leq c_{j_{m+1}} \), that implies \( i_0 \leq i_1 \leq \cdots \leq i_{k_0} \) and \( j_{k_0+1} \leq \cdots \leq j_{m+1} \).

Let \( k_1, \ldots, k_z \leq k_0 \) be such that \( f_{k_j} g_{k_j} \neq 0 \) for any \( j = 1, \ldots, z \): it must be \( z \geq 1 \) and \( a_1 = c_{j_{k_z}} - c_{i_{k_z}} \). Thus (8) becomes:
\[ \begin{cases} b_k = -c_{i_k} & \forall \ k \leq k_0 \\ b_k = a_1 - c_{j_k} & \forall \ k > k_0 \\ a_1 = c_{j_{k_s}} - c_{i_{k_s}} & \forall \ s = 1, \ldots, z \end{cases} \]

(9)

By (8) and since \( (f_{k_s}, g_{k_s}) \neq (f_{k_j}, g_{k_j}) \) if \( s = 2, \ldots, z \), we can either suppose \( z = 1 \) or \( a_1 = 0 \) that implies \( i_{k_s} = j_{k_s} \) for any \( s = 1, \ldots, z \). Thus we have to distinguish two cases:
1. \(a_1 \neq 0, z = 1\)
2. \(a_1 = 0, z \geq 1\)

Under each of these hypothesis, it is easy to show that the system \(\Omega\) admits a unique solution that defines a fixed point \(A \in \mathcal{M}_{n,m,2}\) by the action of \(\rho\).

In order to have a total description of the fixed points, we will consider each case separately:

1. Let us define
   \[
   A_{I,J} = \begin{pmatrix}
   x_{i_0} & x_{i_1} & \cdots & x_{i_t} & 0 & \cdots & 0 \\
   0 & \cdots & 0 & x_{j_1} & \cdots & x_{j_l}
   \end{pmatrix}
   \]
   where \(I = (i_0, \ldots, i_t)\) and \(J = (j_0, \ldots, j_l)\), with \(i_1 < \cdots < i_t, j_1 < \cdots < j_l, i_0 < j_0\) and \(i_0 \neq i_s, j_0 \neq j_s\) for any \(s = 1, \ldots, t\).

2. The matrices fixed by \(\rho\) with \(a_1 = 0\) are given by
   \[
   A^i_\omega = (\omega_1 x_{i_1}, \ldots, \omega_{m+2} x_{i_{m+2}})
   \]
   with \(\omega = (\omega_1, \ldots, \omega_{m+2}) \in \mathbb{R}^{m+2}\) and \(i = (i_1, \ldots, i_{m+2})\) where \(0 \leq i_1 \leq \cdots \leq i_{m+2} \leq n\).

Proposition 5.2. Let \(l \in \mathbb{N}\) be odd and let the group \(\text{SL}(I)\) act on \(Y = \mathbb{P}(I)^{l} = \mathbb{P}(I) \times \cdots \times \mathbb{P}(I)\); for any \(\omega \in Y\), let \(I_k(\omega) = \{ j \in (1, \ldots, n) | \omega_i = \omega_k \}\); then
   \[
   Y^s = Y^{ss} = \{ \omega \in Y | \#I_k(\omega) < \frac{l}{2} \} \text{ for any } k = 1, \ldots, l\}.
   \]

Proof. It is a direct consequence of the Hilbert-Mumford criterion for stability. (see also [MFK]).

The Hodge numbers of \(M_l = \mathbb{P}(I)^l / \text{SL}(I)\) are given by the following:

Theorem 5.3. Let \(l\) be odd. Then:
   \[
   h^{p,q}(M_l) = \begin{cases} 
   0 & \text{if } p \neq q \\
   1 + (l - 1) + \cdots + \left( \min(p, l - 3 - p) \right) & \text{if } p = q
   \end{cases}
   \]

In particular, the Poincaré polynomial is:
   \[
   P_l(M_l) = 1 + h^{1,1} t^2 + \cdots + h^{1,2} t^3 + \cdots + t^{2l-6}.
   \]

and the Euler characteristic is given by:
   \[
   \chi(M_l) = \sum_{p=0, \ldots, l-3} h^{p,p}
   \]
Proof. See [K] pag. 193.

By the classification of the fixed points of $\mathcal{M}_{n,m,2}$, we thus have:

**Corollary 5.4.** $h^{p,q}(\mathcal{M}_{n,m,2}) = 0$ for any $p \neq q$.

We are now ready to compute the Euler characteristic of $\mathcal{M}_{n,m,2}$:

**Theorem 5.5.** Let $m$ be odd and let $t = \frac{m+1}{2}$. Then the Euler characteristic of $\mathcal{M}_{n,m,2}$ is given by:

$$
\chi(\mathcal{M}_{n,m,2}) = \left(\frac{n+1}{2}\right) \left(\frac{n}{t}\right)^2 + \sum_{d=1}^{n-t} \left(\frac{n+1}{t-d}\right) \left(\frac{n+1-t+d}{2d+1}\right) \chi(\mathbb{P}(I)^{2d+1}/\text{SL}(I)).
$$

(13)

**Proof.** By theorem [5.4], it results $\chi(\mathcal{M}_{n,m,2}) = \sum_i \chi(M^T_{\mathcal{M}_{n,m,2}})_i$, where $(M^T_{\mathcal{M}_{n,m,2}})_i$ are the connected components of the fixed point of $\mathcal{M}_{n,m,2}$ under the action of the torus $T$ considered.

The points $A_{I,J}$, defined in (10), represent discrete components of such space and since they are uniquely determined by $I = (i_0, \ldots, i_t)$ and $J = (j_0, \ldots, j_t)$, with $i_1 < \cdots < i_t$, $j_1 < \cdots < j_t$, and $i_0 < j_0$, it is easy to compute that they are exactly

$$
\left(\frac{n+1}{2}\right) \left(\frac{n}{t}\right)^2.
$$

(14)

On the other hand, the matrices $A^i_i$, defined in (11), form connected components determined by $i = (i_1, \ldots, i_{m+2})$ and isomorphic to $M_{l(i)}$, where $l(i)$ is defined in (12).

Let $d(i) = m + 2 - l(i)$ the number of the couples of equal terms in $i$: for any $d \geq 1$ the number of the admissible vectors $i = (i_1, \ldots, i_{m+2})$ with $d(i) = d$ is given by

$$
\binom{n+1}{d} \binom{n+1-d}{m+2-2d}.
$$

thus the Euler characteristic of the set of such matrices is given by

$$
\sum_{d=m-n+1}^{t-1} \binom{n+1}{d} \binom{n+1-d}{m+2-2d} \chi(M_{m+2-2d}) = \sum_{d=1}^{n-t} \binom{n+1}{t-d} \binom{n+1-t+d}{2d+1} \chi(M_{2d+1}).
$$

(15)

(13) is obtained by summing (14) with (15).

6. Betti numbers

We compute the numbers $n_i$ for any fixed point in a connected component $(M^T_{\mathcal{M}_{n,m,2}})_i$. We remind that $n_i$ represents the number of positive weights of $T = \mathbb{C}^*$ acting on the tangent space of $\mathcal{M}_{n,m,2}$ at the fixed points. These numbers will yield to the computation of the Betti numbers of $\mathcal{M}_{n,m,2}$ for any odd $m$.

In particular we get a topological description of the moduli space of the rational normal curves on $\mathbb{P}^n$ for any odd $n$.

Let $A \in \mathcal{M}_{n,m,2}$ be a fixed point for $\rho$. Then $\rho$ induces an action on the tangent space $T_A\mathcal{M}_{n,m,2}$. By theorem [1.1] such vector space is isomorphic to the tangent space of the...
Maruyama scheme $\mathcal{M}_{P^n}(m; c_1, \ldots, c_n)$ at the point corresponding to the sheaf $\mathcal{F}_A$ and thus it is isomorphic to $\text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A)$ (see [Mar] and [Mar2]).

By the sequence (1) that defines the sheaf $\mathcal{F}_A$, it is easily checked that $\text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A)$ is contained in the exact sequence:

$$0 \to \text{Hom}(\mathcal{F}_A, \mathcal{F}_A) \to \text{Hom}(W \otimes \mathcal{O}_{P^n}(1), \mathcal{F}_A) \to \text{Hom}(I \otimes \mathcal{O}_{P^n}(1), \mathcal{F}_A) \to \text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A) \to 0.$$  

Moreover $H^0(\mathcal{F}_A) = (W \otimes V)/a(I)$ where $a : I \to W \otimes V$ is the map induced by $A^* : I \otimes \mathcal{O}_{P^n} \to W \otimes \mathcal{O}_{P^n}(1)$ and $H^0(\mathcal{F}(-1)) = W$.

Thus, it results $\text{Hom}(W \otimes \mathcal{O}_{P^n}(1), \mathcal{F}_A) = W^* \otimes H^0(\mathcal{F}_A(-1)) = W^* \otimes W$ and $\text{Hom}(I \otimes \mathcal{O}_{P^n}, \mathcal{F}_A) = I^* \otimes (W \otimes V)/a(I)$.

In particular the weights of the action $\rho$ on $\text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A)$ are easily computed using the sequence:

$$0 \to C \to W^* \otimes W \to I^* \otimes \frac{W \otimes V}{a(I)} \to \text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A) \to 0. \tag{16}$$

In the previous section we have seen that for any fixed matrix $A \in X$, $\rho$ induces an action on $I$ and $W$ defined by the weights $(a_0, a_1)$ and $(b_0, \ldots, b_{m+1})$ described by $[9]$, where, we remind, $c = (1, 2, \ldots, 2^n)$.

For any $A \in (\mathcal{M}_{n, m, 2})$ we write $n(A)$ in place of $n_1$ and moreover we define $n_1(A)$ as the number of the positive weights of $\rho$ on $W^* \otimes W$ and similarly $n_2(A)$ as the number of the positive weights on $I^* \otimes (W \otimes V)/a(I)$. Thus by the sequence (14), it results $n(A) = n_2(A) - n_1(A)$.

In order to calculate $n_1(A)$ and $n_2(A)$ for all the fixed matrices by the action of $\rho$, we need to distinguish the cases described above:

**Proposition 6.1.**

1. Let $A_{I, J}$ be defined as in (14); then:

$$n_1(A_{I, J}) = 4tn + 2t + 2n - 1 - \sum_{s=0}^t i_s - \sum_{s=0}^t j_s - \sum_{i_s > i_0} i_s - \sum_{j_s > j_0, s \geq 1} j_s$$

$$- \#\{s = 1, \ldots, t | j_s > j_0\} - i_0 \cdot \#\{s = 1, \ldots, t | j_s \leq i_0\}$$

and

$$n_2(A_{I, J}) = \binom{m + 2}{2}.$$

2. Let $A^i_{\omega}$ be defined as in (14); then:

$$n_1(A^i_{\omega}) = 2(m + 2)n - 2 \sum_{s=0}^{m+1} i_s$$

and

$$n_2(A^i_{\omega}) = \binom{m + 2}{2} + \frac{m + 2 - l(i)}{2}.$$

**Proof.** It is just a direct computation. \(\square\)
Proposition 6.1 and theorem 5.3 give us the right ingredients to apply theorem 5.1 of Bialynichi-Birula. Thus we have an algorithm to compute the Betti numbers of $M_{n,m,2}$ for any $m \geq n$, and in particular of $M_{n,n,2}$ the compactification of the variety $S_n$ of the rational normal curves.

In fact, let $b_i(n) = \dim H^i(M_{n,n,2}, \mathbb{Q})$: the following table provides the values of $b_i(n)$, for $n = 2, 3, 5, 7$ and for all the even $i = 0, \ldots, 36$.

| $n$ | $b_0$ | $b_2$ | $b_4$ | $b_6$ | $b_8$ | $b_{10}$ | $b_{12}$ | $b_{14}$ | $b_{16}$ | $b_{18}$ | $b_{20}$ | $b_{22}$ | $b_{24}$ | $b_{26}$ | $b_{28}$ | $b_{30}$ | $b_{32}$ | $b_{34}$ | $b_{36}$ |
|-----|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2   | 1     | 1     | 1     | 1     | 1     | 1       |         |         |         |         |         |         |         |         |         |         |         |         |         |
| 3   | 1     | 1     | 3     | 4     | 7     | 8       | 10      | 8       | 7       | 4       | 3       | 1       | 1       |         |         |         |         |         |         |
| 5   | 1     | 1     | 3     | 4     | 8     | 11      | 18      | 24      | 35      | 45      | 61      | 74      | 93      | 106     | 122     | 128     | 134     | 128     | 122     |
| 7   | 1     | 1     | 3     | 4     | 8     | 11      | 18      | 24      | 35      | 45      | 61      | 77      | 100     | 134     | 165     | 205     | 242     | 289     | 334     | 400     |
| 9   | 1     | 1     | 3     | 4     | 8     | 11      | 18      | 24      | 35      | 45      | 61      | 77      | 100     | 134     | 165     | 205     | 242     | 289     | 334     | 400     |

See also [EPS], for the computation of the Betti numbers of $M_{3,3,2}$.

By this table, it seems that, for any $i \geq 0$ and $n >> 0$, the value of $b_i(n)$ is constant. In particular we have:

**Proposition 6.2.** $b_2(n) = h^{1,1}(M_{n,n,2}) = 1$ for any odd $n$.

**Proof.** By prop. 6.1, it follows that $n_i(A^I_{\omega}) \geq 3$ for any $A^I_{\omega}$ defined as in (11). Thus, by theorem 5.3, the points represented by the matrices $A^I_{\omega}$, do not give any contribute to $b_2(n)$.

Moreover it is not difficult to see that, if $n > 3$, the only matrix $A_{I,J}$, as in (10), such that $n_i(A_{I,J}) = 1$ is given by $I = (n-t, n-t+1, \ldots, n)$ and $J = (n-t-2, n-t+1, n-t+2, \ldots, n)$, where $t = \frac{n+1}{2}$.

**Remark 6.3.** It would be interesting to have a description of the Chow rings of $M_{n,m,2}$: in [Sch], the author studies the Chow ring of the Hilbert compactification $H_3$ of the moduli space of the twisted cubics in $\mathbb{P}^3$.

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