On the structure of $C^*$—algebras generated by the components of polar decompositions

A. Lebedev, A. Odzijewicz

Belarus State University, Department of Mathematics and Mechanics,
Skariny av. 4, Minsk, 220050, Belarus;
Institute of Theoretical Physics, University in Bialystok, ul. Lipowa 41,
PL-15-424 Bialystok, Poland

Abstract

In the present paper we study the structure of $C^*$—algebras generated by the components of the polar decompositions of operators in Hilbert space satisfying certain commutation relations.

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Contents

1 Introduction 2

2 Preliminaries. Partial isometries and endomorphisms 5

3 Extensions of commutative $C^*$—algebras by endomorphisms 10

4 The structure of the algebra $\{A_0, U\}$ 20

5 The structure of the algebra $B = \{1, |a|, U\}$ 27

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1 Introduction

Let $a$ be a linear bounded operator acting in a Hilbert space $H$. Suppose that the operator $a$ and its adjoint $a^*$ satisfy the following relation

$$aa^* \in \{1, a^*a\}, \quad (1.1)$$

where we denote by $\{\alpha_\lambda, \lambda \in \Lambda\}$ the $C^*$-algebra generated by the family of operators $\alpha_\lambda, \lambda \in \Lambda$. The problems we shall deal with in this paper concern the description of the structure of the $C^*$-algebra $\{1, a\}$.

1.1 $C^*$-algebras defined by condition (1.1) are the natural generalizations of algebras satisfying the relation

$$aa^* = \gamma(a^*a) \quad (1.2)$$

where $\gamma$ is a continuous function on the spectrum of the operator $a^*a$. They appear in different problems of quantum physics (see, for example, [1, 2]).

In the simplest case when $\gamma$ is a linear map

$$aa^* = qa^*a + h \quad (1.3)$$

they include:

(i) the commutative algebras generated by normal operators:

$$a^*a = aa^*$$

($q = 1, \ h = 0$) the theory of which forms a cornerstone of the spectral theory of operators in Hilbert spaces.

(ii) Toeplitz algebra:

$$aa^* = 1$$

($q = 0, \ h = 1$)
(iii) Heisenberg algebra:

\[ aa^* - a^*a = h \]  \hspace{2cm} (1.4)

(q = 1, \ h > 0).

This is in fact not a C*-algebra situation: the operators satisfying (1.4) are unbounded and play the principal role (associated with the position and momentum operators) in the quantum mechanics theory.

In the general case (q \neq 0, 1, h \neq 0) the quantum algebra generated by relation (1.3) could be interpreted as a q-deformation of the Heisenberg algebra. The models of quantum mechanics based on the q-deformation of the Heisenberg algebra has been studied, for example, in [3].

In the physics of many bosons system that appear in a natural way in quantum optics and nuclear physics one arrives at the Shrodinger operator \( H \) given by

\[ H = a + a^* + D(a^*a) \]  \hspace{2cm} (1.5)

where the operator \( a \) satisfies (1.1) (see [2]).

For the corresponding quantum system the algebra arising from (1.1) plays the role of 'symmetry algebra'. It enables us to establish the interrelation between the spectral problem for the operators considered and the theory of orthogonal polynomials and to solve the problem in many concrete cases.

Let us mention also that the algebra given by (1.1) finds applications in the theory of basic hypergeometric functions and integrable systems (see [4]).

Along with the algebra \( \{1, a\} \) it is reasonable to consider the following objects and arising problems.

Let

\[ a = U|a| \]  \hspace{2cm} (1.6)

be the standard polar decomposition of \( a \). Here \( |a| = \sqrt{a^*a} \) and \( U \) is a partial isometry defined by

\[ U(|a|\xi) = a\xi, \quad \xi \in H. \]  \hspace{2cm} (1.7)
Then (1.1) means that

\[ U|a|^2 U^* \in \{1, |a|^2\} = \{1, |a|\}. \quad (1.8) \]

Note also that since \( U^*U \) is the orthogonal projection onto \( \text{Im} \, |a| \) it follows that \( U^*U|a| = |a| \) and therefore

\[ U|a|^2 U^* = U|a|U^*U|a|U^* = (U|a|U^*)^2. \quad (1.9) \]

Thus we conclude that (1.1) is equivalent to

\[ U|a|U^* \in \{1, |a|\}. \quad (1.10) \]

In addition (1.9) means that the mapping

\[ U(\cdot)U^* : \{|a|\} \rightarrow \{1, |a|\} \]

is a morphism.

This leads to the tight interrelation of the algebras induced by (1.1) with the crossed products of \( C^* \)-algebras by semigroups of endomorphisms. This interrelation is clarified in section 4 (see in particular Theorem 4.7 and Remark 4.8).

In connection with the matter considered in the paper it is reasonable to mention that a great number of various irreducible representations of the relation (1.1) along with the physical applications of the \( C^* \)-algebras arising are presented in the book [5].

1.2 The main aim of the article is to reveal the structure of the \( C^* \)-algebras \( \{1, |a|, U\} \) and \( \{1, a\} \subset \{1, |a|, U\} \) where \( a \) satisfies relation (1.1) and \( U \) is the partial isometry taken from the polar decomposition (1.6).

The paper is organized as follows.

We start with an auxiliary section 2 where we present a number of (mostly known) facts on partial isometries and the associated mappings of the algebra of linear operators in a Hilbert space that we shall use in the subsequent sections.
Sections 3 and 4 present the main framework (the general operator algebraic picture that one can attend ‘from on high’) of the structures and objects that should be involved in the investigation of the algebras in question. As a result we obtain the necessary ‘coefficient’ algebra (given in Theorem 3.8) and the crucial property 4.2 that enables us to reveal the structure of the arising algebras up to $^\ast$-algebraic isomorphism (Theorem 4.7).

Finally in sections 5 and 6 we apply the results obtained to the initial algebras $\{1, |a|, U\}$ and $\{1, a\}$ under investigation and give the description of their structure.

2 Preliminaries. Partial isometries and endomorphisms

This is a starting auxiliary section. Here we list some facts about partial isometries and the associated mappings of the algebra of linear operators in a Hilbert space that we shall use in the subsequent sections. Most of the facts presented here are known and we give the proofs for the sake of completeness.

Recall that a linear bounded operator $U$ in a Hilbert space $H$ is called a partial isometry if there exists a closed subspace $H_1 \subset H$ such that

$$\|U\xi\| = \|\xi\|, \quad \xi \in H_1$$

and

$$U\xi = 0, \quad \xi \in H \ominus H_1.$$ 

The space $H_1$ is called the initial space of $U$ and $U(H_1)$ is called the final space of $U$.

Hereafter we list the well known characteristic properties of a partial isometry (see for example [6], problem 98):

**Theorem 2.1** The following statements are equivalent:

1) $U$ is a partial isometry,
2) $U^\ast$ is a partial isometry,
3) $U^*U$ is a projection (onto the initial space of $U$),
4) $UU^*$ is a projection (onto the final space of $U$),
5) $UU^*U = U$ and $U^*UU^* = U^*$.

One can associate with any linear bounded operator $V$ acting in a Hilbert space $H$ the following mapping $\delta_V : L(H) \to L(H)$ of the algebra $L(H)$ of all linear bounded operators in $H$:

$$\delta_V(b) = V b V^*, \quad b \in L(H) \quad (2.1)$$

It is clear that $\delta_V$ is a linear positive mapping and $\delta_V(b^*) = [\delta_V(b)]^*$ for any $b \in L(H)$.

The next proposition shows that $\delta_V$ generates an endomorphism of some subalgebra $A \subset L(H)$ only under rather restrictive assumptions on $V$ (namely $V$ should be a partial isometry possessing special additional properties).

**Proposition 2.2** For a given operator $V \in L(H)$ the following statements are equivalent:

(i) There exists a subalgebra $A \subset L(H)$ containing the identity such that the mapping

$$V(\cdot)V^* : L(H) \to L(H)$$

generates an endomorphism of $A$.

(ii) All the operators $V^k$, $k = 1, 2, ...$ are partial isometries.

(iii) The family $V^kV^*V^k$, $k = 1, 2, ...$ is a commutative decreasing family of projections.

**Proof:** In view of statement 3) of Theorem 2.1 (iii) implies (ii).

On the other hand if (ii) is true then $V^kV^*V^k = V^k$, $k = 1, 2, ...$ by statement 5) of Theorem 2.1. Therefore for any $k \geq l$ we have

$$V^*V^kV^*V^lV^* = V^*V^kV^{k-l}V^*V^lV^* = V^*V^{k-l}V^*V^l = V^*V^{k-l}V^*V^l = V^*V^{k-l}V^*V^l = V^*V^k$$

and in the same way one can verify that

$$V^*V^lV^*V^kV^* = V^*V^k.$$
Thus (ii) implies (iii).

If (i) is true then all the mappings $V^k(\cdot)V^k$, $k = 1, 2, \ldots$ are also endomorphisms of the algebra $A$ (mentioned in (i)) and therefore (since $1 \in A$)

$$V^k V^k^* = V^k 1 V^k^* = V^k (1 \cdot 1)V^k^* = (V^k 1 V^k^*)(V^k 1 V^k^*) = (V^k V^k^*)^2, \quad k = 1, 2, \ldots$$

Thus (ii) is true.

Finally if (iii) is true then $V(\cdot)V^*$ generates an endomorphism of the algebra $A = \{1, VV^*, V^2V^2^*, \ldots, V^kV^k^*, \ldots\}$.

In view of this result it is reasonable to consider the mapping $\delta_V$ in the situation when $V$ is a partial isometry in more detail.

In the next lemma we list some simple properties of the mapping $\delta_V$ in this case.

**Lemma 2.3** Let $V$ be a partial isometry then

1) $\delta_V \delta_V^* \delta_V = \delta_V$ and $\delta_V^* \delta_V \delta_V^* = \delta_V$.

2) For any $b \in L(H)$ we have

$$\delta_V(b) = VV^*\delta_V(b) = \delta_V(b)V V^* \quad \text{and} \quad \delta_V^*(b) = V^*V \delta_V^*(b) = \delta_V^*(b)V^* V.$$

3) $\text{Ker} \delta_V = \{b \in L(H) : V^*VbV^*V = 0\}$.

In particular if $V^*V = 1$ (that is $V$ is an isometry) then $\text{Ker} \delta_V = 0$

4) $\text{Ker} \delta_V^* = \{b \in L(H) : VV^*bVV^* = 0\}$.

In particular if $VV^* = 1$ (that is $V^*$ is an isometry) then $\text{Ker} \delta_V^* = 0$

**Proof:** The first and the second statements follow from the equalities $VV^*V = V$ and $V^*VV^* = V^*$.

Let us verify the third statement.

If $V^*VbV^*V = 0$ then

$$\delta_V(b) = VbV^* = VV^*VbV^*V = V(V^*VbV^*V)V^* = 0$$

that is $b \in \text{Ker} \delta_V$.

On the other hand if $b \in \text{Ker} \delta_V$ then

$$0 = VbV^* = V^*(VbV^*)V = V^*VbV^*V$$
Thus 3) is proved.

Statement 4) can be proved in the same way. □

In fact if we do not look for endomorphisms of some algebra \( A \subset L(H) \) (as in Proposition 2.2) but only for morphisms \( \delta_V : A \to \delta_V(A) \) nevertheless (as the next lemma shows) this situation is closely related to the property of \( V \) being a partial isometry.

**Lemma 2.4** Let \( A \) be a \( C^* \)-subalgebra of \( L(H) \).

I. If \( V \) is a partial isometry and \( V^*V \in A' \) (where \( A' \) is the commutant of \( A \)) then

(i) \( \delta_V : A \to \delta_V(A) \) is a morphism,

(ii) \( V\alpha = \delta_V(\alpha)V \) and \( \alpha V^* = V^*\delta_V(\alpha) \) for every \( \alpha \in A \).

II. If \( A \) contains the identity and \( VV^*, V^*V \in A' \) then the following statements are equivalent:

(i) \( V^*V \) is a projection (that is \( V \) is a partial isometry),

(ii) \( VV^* \) is a projection,

(iii) \( V(\cdot)V^* : A \to VAV^* \) is a morphism,

(iv) \( V^*(\cdot)V : A \to V^*AV \) is a morphism.

**Proof:** I. The first statement here follows from the observation that for any \( \alpha, \beta \in A \) we have

\[
V(\alpha \cdot \beta)V^* = VV^*V\alpha \cdot \beta V^* = V\alpha V^*V\beta V^* = (V\alpha V^*)(V\beta V^*)
\]

and the second follows from the relations

\[
V\alpha = VV^*V\alpha = V\alpha V^*V = (V\alpha V^*)V
\]
along with the passage to the adjoint operators.

II. Here (i) and (ii) are equivalent in view of Theorem 2.1. If (i) is true then $V$ is a partial isometry and (iii) follows from statement (i) in I. Similarly (ii) implies (iv). On the other hand if (iii) is true then

$$VV^* = V1V^* = V(1 \cdot 1)V^* = (V1V^*)(V1V^*) = (VV^*)^2$$

thus (ii) is true.

Similarly (iv) implies (i).

The next result pushes the matter a bit further and clarifies that if in addition to what is presumed in part I of Lemma 2.4 $\delta V$ generates an endomorphism of $A$ then the interrelation between $A$ and $V$ is much closer.

**Lemma 2.5** Let $V$ be a partial isometry and $A$ be a $C^*$—subalgebra of $L(H)$. If $V^*V \in A'$ and $VAV^* \subset A$ then $\delta V : A \to A$ is an endomorphism and $V^*V^k \in A'$, $k = 1, 2, ...$

**Proof:** By part I of Lemma 2.4 in the situation under consideration we have that $\delta V : A \to A$ is an endomorphism.

Applying statement (ii) of part I of Lemma 2.4 $k$ times we have for any $\alpha \in A$

$$V^*V^k\alpha = V^*(V^k\alpha V^*)V^k$$

and

$$\alpha V^*V^k = V^*(V^k\alpha V^*)V^k$$

Thus $V^*V^k \in A'$, $k = 1, 2, ....$

In connection with the foregoing lemma it is reasonable to mention the following properties of partial isometries

**Lemma 2.6** Let $V$ be a partial isometry such that

$$[V^*V, V^kV^*k] = 0, \quad k = 1, 2, ... \quad (2.2)$$

where $[\alpha, \beta]$ is the commutator of the sets $\alpha$ and $\beta$. Then
1) \( V^l V^l \in \{ VV^*, V^2V^*, ..., V^k V^{*k}, ... \} \), \( l = 1, 2, ... 

2) \( V^* V^k V^l = V^{k-1} V^* V^l \)

3) All the operators \( V^k \) are partial isometries and the family \( V^k V^{*k}, k = 1, 2, ... \) is a commutative decreasing family of projections.

**Proof:** 1) This follows from Lemma 2.5 since the mapping \( V(\cdot)V^* \) is an endomorphism of the algebra \( \{ VV^*, V^2V^*, ..., V^k V^{*k}, ... \} \).

2). Let \( 1 \leq k \leq l \) then applying (2.2) we obtain

\[
V^* V^k V^l = (V^* V)(V^k V^{*k-1})V^l = (V^k V^{*k-1})(V^* V)V^l = (V^k V^{*k-1})(V^* V)\]

\[
V^l - k = V^{k-1} V^* V^l
\]

3). Applying 2) \( k \) times we have

\[
(V^k V^{*k})^2 = V^k V^{*k} V^k V^* V^k = V^k V^{*k-1} V^* V^k V^k = \]

\[
V^k V^{*k-2} (V^* V^{k-1} V^* V^k) = V^k V^{*k-2} V^{k-2} V^k = ... = V^k V^k
\]

this means that \( V^k \), \( k = 1, 2, ... \) are isometries and therefore 3) is true. \( \square \)

### 3 Extensions of commutative \( \text{C}^* \)-algebras by endomorphisms

Throughout this section we fix a partial isometry \( U \) and a commutative \( \text{C}^* \)-algebra \( A_0 \subset L(H) \) containing the identity and proceed to the description of the structures and relations associated with the mappings \( \delta_U \) and \( \delta_{U^*} \) (2.1).

To shorten the notation we denote \( \delta_U \) simply by \( \delta \) and \( \delta_{U^*} \) by \( \delta_* \) thus

\[
\delta(b) = UbU^*, \quad b \in L(H)
\]

(3.1)

and

\[
\delta_*(b) = U^*bU, \quad b \in L(H)
\]

(3.2)
Our goal in this section is the following:

starting from the algebra $A_0$ and the mappings $\delta$ and $\delta_*$ we wish to construct a commutative extension $A \supset A_0$ such that both the mappings $\delta$ and $\delta_*$ are endomorphisms of $A$.

We shall construct the algebra $A$ in two steps. At the first step we construct the extensions $A_\infty$ ($\infty A$) (see (3.7) and (3.10)) such that $\delta$ (or $\delta_*$ respectively) generate their endomorphisms. This is done in Theorems 3.4 and 3.3. Then as the second step we extend the algebra $A_\infty$ ($\infty A$) further to obtain the algebra $A = \infty(A_\infty) = (\infty A)_\infty$ (see (3.13) and (3.16)) that is stable as with respect to $\delta$ so also with respect to $\delta_*$. This is done in Theorems 3.7 and 3.8 which also describe the subtle internal hierarchical structure of the algebras arising in the process of extension.

It is reasonable to note that since we are looking for a commutative extension $A$ of $A_0$ and in view of Lemmas 2.4 and 2.5 among the natural assumptions on $\delta$ (or $\delta_*$ respectively) we have $\delta^k(1) \in A_0'$ ($\delta_*^k(1) \in A_0'$).

Now we start to produce the desired extensions of $A_0$.

**Theorem 3.1** Let $\delta_*^k(1)$, $k = 0, n$ be the projections and $\delta_*^k(1) \in A_0'$ and $\delta^k(A_0) \subset A_0'$, $k = 0, n$ (where $\delta_*^0(x) = \delta^0(x) = x$, $x \in L(H)$, in particular $\delta_*^0(1) = \delta^0(1) = 1$). Then

$$[\delta^k(A_0), \delta_*^l(A_0)] = 0, \quad 0 \leq k, l \leq n \quad (3.3)$$

and

$$[\delta^k(A_0), \delta^l(A_0)] = 0, \quad 0 \leq k, l \leq n \quad (3.4)$$
Proof: Take some \( k \geq l \) then for any \( \alpha, \beta \in A_0 \) we have
\[
\delta^k(\alpha) \cdot \delta^l(\beta) = U^k \alpha U^k U^s l \beta U^l = U^k \alpha U^{k-l} (U^l U^s l) \beta U^l =
\]
\[
U^k \alpha U^{k-l} \beta (U^l U^s l) U^l = U^k \alpha U^{k-l} \beta U^l = U^k \alpha U^{k-l} (U^k U^{k-l}) \beta U^l =
\]
\[
U^k \alpha U^{k-l} \beta U^s k-l U^l = U^k \alpha (U^{k-l} \beta U^{k-l}) U^k =
\]
\[
U^k \alpha \delta^{k-l}(\beta) U^k = \delta^k(\alpha \cdot \delta^{k-l}(\beta))
\]
On the other hand
\[
\delta^l(\beta) \cdot \delta^k(\alpha) = U^s l \beta U^s l U^k \alpha U^k = U^s l \beta (U^s l U^k) U^s l \alpha U^k =
\]
\[
U^s l U^l U^s l \beta U^{s k-l} \alpha U^k = U^l \beta U^{s k-l} \alpha U^k = U^k (\delta^{k-l}(\beta) \cdot \alpha) U^k =
\]
\[
\delta^k(\delta^{k-l}(\beta) \cdot \alpha)
\]
Comparing these two relations and bearing in mind that by assumption of the theorem \( \delta^{k-l}(\beta) \in A_0' \) we conclude that
\[
\delta^k(\alpha \cdot \delta^{k-l}(\beta)) = \delta^k(\delta^{k-l}(\beta) \cdot \alpha)
\]
and therefore (3.3) is proved.

To prove (3.4) we take \( k, l, \alpha, \beta \) as above and observe that
\[
\delta^k(\alpha) \cdot \delta^l(\beta) = U^k \alpha U^s k U^l \beta U^s l = U^k \alpha U^{k-l} (U^s l U^l) \beta U^s l =
\]
\[
U^k \alpha U^{k-l} \beta U^s l U^l = U^k \alpha U^{k-l} \beta U^s l = U^l (U^{k-l} \alpha U^{k-l}) \beta U^s l =
\]
\[
U^l (\delta^{k-l}(\alpha) \cdot \beta) U^s l = \delta^l(\delta^{k-l}(\alpha) \cdot \beta)
\]
On the other hand
\[
\delta^l(\beta) \cdot \delta^k(\alpha) = U^l \beta U^s l U^k \alpha U^k = U^l \beta (U^s l U^l) U^{k-l} \alpha U^k =
\]
\[
U^l U^s l U^l \beta U^{k-l} \alpha U^k = U^l \beta U^{k-l} \alpha U^k = U^l (\beta \cdot \delta^{k-l}(\alpha)) U^s l =
\]
\[
\delta^l(\beta \cdot \delta^{k-l}(\alpha))
\]
which (as in the previous situation) implies (3.4). \( \Box \)
3.2 For every $n = 0, 1, \ldots$ set

$$A_n = \{A_0, \ldots, \delta^n(A_0)\}. \quad (3.5)$$

and

$$nA = \{A_0, \ldots, \delta^n(A_0)\}. \quad (3.6)$$

**Theorem 3.3** Let $\delta_k(1), \; k = 1, \ldots, n$ be the projections and

$$\delta(A_0) \subset A_0 \quad \text{and} \quad \delta_*(1) \in A_0'$$

then $nA$ is a commutative algebra and

(i) $\delta_k(1) \in A_0', \; k = 1, 2, \ldots$

(ii) For any $0 \leq l \leq k \leq n$ we have

$$\delta^k_*(A_0) \cdot \delta^l_*(A_0) \subset \delta^k_*(A_0)$$

in particular $\delta^n_*(A_0)$ is an ideal in $nA$.

(iii) $nA$ is the set of operators of the form

$$\alpha_0 + \delta_*(\alpha_1) + \ldots + \delta^n_*(\alpha_n), \; \alpha_i \in A_0.$$

(iv) $\delta : nA \to n-1A$ is a morphism.

**Proof:** Since $\delta(A_0) \subset A_0$ and $\delta_*(1) \in A_0'$ Lemma 2.3 implies (i).

Therefore (as $\delta^k(A_0) \subset A_0 \subset A_0'$, $k = 1, 2, \ldots$) all the conditions of Theorem 3.1 are satisfied it follows that $nA$ is commutative.

To verify (ii) observe that in the proof of Theorem 3.1 it was established that for any $k \geq l$ and $\alpha, \beta \in A_0$ we have

$$\delta^k_*(\alpha) \cdot \delta^l_*(\beta) = \delta^k_*(\alpha \cdot \delta^{k-l}(\beta))$$

Since by the assumption we have $\delta^{k-l}(\beta) \in A_0$ this equality implies (ii).

Clearly (iii) follows from (ii).

Finally by the assumption we have $UU^* = \delta(1) \in A_0$. Therefore we conclude
(using (ii)) that for any $\alpha \in A_0$ and any $1 \leq k \leq n$ the following relations hold
\[
\delta(\delta^k(\alpha)) = UU^*\alpha U^kU^* = UU^*\delta^{k-1}(\alpha)UU^* = \delta^{k-1}(\alpha)UU^* \in \delta^{k-1}(A_0).
\]
Thus $\delta : nA \hookrightarrow n-1A$.
Note also that
\[
U^*U = \delta_*(1) \in nA \subset (nA)'
\]
and therefore $\delta : nA \hookrightarrow n-1A$ is a morphism by Lemma 2.4. So (iv) is proved. \(\square\)

The next two theorems describe the natural extensions of the algebra $A_0$ that are stable under the morphisms $\delta$ and $\delta_*$ respectively.

**Theorem 3.4** Let $\delta^k(1), \, k = 0, 1, ...$ be the projections and
\[
\delta^k(1) \in A_0', \quad \delta^k(A_0) \subset A_0', \quad \delta_*(1) \in [\delta^k(A_0)]', \quad k = 0, 1, ...
\]
then
\[
A_\infty = \{A_0, \delta(A_0), ..., \delta^k(A_0), ...\}
\] (3.7)
is the minimal commutative $C^*$-algebra containing $A_0$ and such that
\[
\delta : A_\infty \rightarrow A_\infty
\]
is an endomorphism.
In addition we have
\[
U\alpha = \delta(\alpha)U \quad \text{for any} \quad \alpha \in A_\infty
\] (3.8)
and
\[
\delta^k(1) \in (A_\infty)', \quad k = 1, 2, ...
\] (3.9)

**Proof:** The commutativity of $A_\infty$ follows from Theorem 3.1.
The minimality of $A_\infty$ is clear from the construction.
So let us verify that $\delta$ is an endomorphism of $A_\infty$.
Take any $k, l \geq 0$ and $\alpha, \beta \in A_0$. We have
\[
\delta(\delta^k(\alpha) \cdot \delta^l(\beta)) = U\delta^k(\alpha)\delta^l(\beta)U^* = U(U^*U)\delta^k(\alpha)\delta^l(\beta)U^* = 
\]
\[
U\delta^k(\alpha)U^*U\delta^l(\beta)U^* = \delta(\delta^k(\alpha)) \cdot \delta(\delta^l(\beta))
\]
which implies that $\delta$ is an endomorphism of $A_\infty$.
In addition the assumption

$$\delta_* (1) \in [\delta^k (A_0)]', \quad k = 0, 1, ...$$

implies $\delta_* (1) \in (A_\infty)'$ and therefore (3.8) follows from Lemma 2.4.
Finally since $\delta_* (1) \in (A_\infty)'$ and $\delta$ is an endomorphism of $A_\infty$ (3.9) follows from Lemma 2.5. $\square$

**Theorem 3.5** Let $\delta_*^k (1), \ k = 0, 1, ...$ be the projections and

$$\delta_*^k (1) \in A_0', \quad \delta^k (A_0) \subset A_0', \quad \delta (1) \in [\delta_*^k (A_0)]', \quad k = 0, 1, ...$$

then

$$\infty A = \{ A_0, \delta_* (A_0), ..., \delta_*^k (A_0), ... \}$$

(3.10)

is the minimal commutative $C^*$-algebra containing $A_0$ and such that

$$\delta_* : \infty A \to \infty A$$

is an endomorphism.
In addition we have

$$U^* \alpha = \delta_* (\alpha) U^* \quad \text{for any} \quad \alpha \in \infty A \quad (3.11)$$

and

$$\delta_*^k (1) \in (\infty A)', \quad k = 1, 2, ... \quad (3.12)$$

**Proof:** The commutativity of $\infty A$ follows from Theorem 3.1.
The minimality of $\infty A$ is clear from the construction.
The property that $\delta_*$ is an endomorphism of $\infty A$ follows from the observation that for any $k,l \geq 0$ and $\alpha, \beta \in A_0$ we have

$$\delta_* (\delta_*^k (\alpha) \cdot \delta_*^l (\beta)) = U^* \delta_*^k (\alpha) \delta_*^l (\beta) U = U^* (U U^*) \delta_*^k (\alpha) \delta_*^l (\beta) U =$$

$$U^* \delta_*^k (\alpha) U U^* \delta_*^l (\beta) U = \delta_* (\delta_*^k (\alpha) \cdot \delta_*^l (\beta))$$

In addition the assumption

$$\delta (1) \in [\delta_*^k (A_0)]', \quad k = 0, 1, ...$$

15
implies \( \delta(1) \in (\infty A)' \) and thus (3.11) follows from Lemma 2.4. Finally (3.12) can be established just in the same way as it was done for (3.9) in the proof of the previous theorem.

\[ \square \]

Remark 3.6 One can notice certain asymmetry (between \( \delta \) and \( \delta_\ast \)) as in the statement of Theorem 3.1 so also in the statements of Theorems 3.4 and 3.5. This means that we can also formulate and prove the analogous statements by exchanging the conditions on \( \delta \) and \( \delta_\ast \).

Now we present the description of the extensions of the algebra \( A_0 \) that are stable as with respect to \( \delta \) so also with respect to \( \delta_\ast \).

**Theorem 3.7** Let \( \delta(A_0) \subset A_0 \).

I. The following statements are equivalent:

(i) There exists a commutative \( C^\ast \)-algebra \( A \supset A_0 \) such that both the mappings \( \delta \) and \( \delta_\ast \) are endomorphisms of \( A \).

(ii) All the operators \( \delta^k(1), k = 0, 1, \ldots \) are projections and \( \delta_\ast(1) \in A_0' \).

II. If condition (ii) of part I is satisfied then the algebra \( \infty A \) (3.10) is the minimal commutative \( C^\ast \)-algebra such that \( \delta \) and \( \delta_\ast \) are endomorphisms of \( \infty A \).

Moreover we have:

(i) For any \( 0 \leq l \leq k \)

\[ \delta^k(A_0) \cdot \delta^l(A_0) \subset \delta^k(A_0). \]

(ii) For every \( n \) the subalgebra \( nA \subset \infty A \) defined by (3.9) is the set of the operators of the form

\[ \alpha_0 + \delta_\ast(\alpha_1) + \ldots + \delta_\ast^n(\alpha_n), \quad \alpha_i \in A_0. \]

(iii) \( \delta : nA \rightarrow n-1A, \quad n = 1, 2, \ldots \quad \delta_\ast : nA \rightarrow n+1A, \quad n = 0, 1, \ldots, \)

(iv) \( U\alpha = \delta(\alpha)U \) and \( U^*\alpha = \delta_\ast(\alpha)U^* \) for any \( \alpha \in \infty A. \)
Proof: If (i) is true then (as $1 \in A_0 \subset A$) we have that $\delta_s(1) \in A \subset A'_0$. In addition since $\delta$ is an endomorphism of $A$ Proposition 2.2 tells us that all the operators $\delta^k_s(1), \ k = 0, 1, \ldots$ are projections. Thus (i) implies (ii). The implication (ii) $\Rightarrow$ (i) follows as a byproduct from Part II of the theorem (one can take $A = \infty A$). So let us prove this part.

II. Note that in the situation under consideration all the conditions of Theorem 3.3 are satisfied. This implies the commutativity of $\infty A$ along with (i) and (ii). The statement (iv) of Theorem 3.3 also implies that $\delta$ is an endomorphism of $\infty A$ and $\delta : nA \to n-1 A, \ n = 1, 2, \ldots$ Clearly $\delta_s : nA \to n+1 A, \ n = 0, 1, \ldots$ which proves (iii) along with the property $\delta_s : \infty A \hookrightarrow \infty A$.

Since $UU^* = \delta(1) \in A_0 \subset \infty A \subset (\infty A)'$

Lemma 2.4 implies that $\delta_s$ is a morphism (and by the foregoing notes it is an endomorphism) of $\infty A$.

Finally as $UU^*, U^*U \in (\infty A)'$ Lemma 2.4 implies (iv) as well. \hfill \Box

The next result is the natural generalization of the theorem just proved and it gives the solution to the problem we are considering in this section.

Theorem 3.8 I. The following statements are equivalent:

(i) There exists a commutative $C^*$-algebra $A \supset A_0$ such that both the mappings $\delta$ and $\delta_s$ are endomorphisms of $A$.

(ii) All the operators $\delta^k_s(1), \ k = 0, 1, \ldots$ are projections and

$\delta^k_s(1) \in A'_0, \ \delta^k(A_0) \subset A'_0, \ \delta_s(1) \in [\delta^k(A_0)]', \ k = 0, 1, \ldots$

II. If condition (ii) of part I is satisfied then

1) $\infty(A_\infty) = \{A_\infty, \delta_s(A_\infty), \ldots, \delta^k_s(A_\infty), \ldots\}$ \hfill (3.13)

is the minimal commutative $C^*$-algebra containing $A_0$ and such that

$\delta : \infty(A_\infty) \to \infty(A_\infty) \quad \text{and} \quad \delta_s : \infty(A_\infty) \to \infty(A_\infty)$ \hfill (3.14)
are endomorphisms.
Moreover we have:

(i) For any $0 \leq l \leq k$
\[
\delta^k_*(\infty) \cdot \delta^l_*(\infty) \subset \delta^k_*(\infty)
\]

(ii) For every $n$ the subalgebra $\n_*(\infty) \subset \infty_*(\infty)$ given by (3.6) (with $A_0$ substituted by $\infty A$) is the set of operators of the form
\[
\alpha_0 + \delta_*(\alpha_1) + \cdots + \delta^n_*(\alpha_n), \quad \alpha_i \in \infty.
\]

(iii) $\delta : \n_*(\infty) \rightarrow \n-1_*(\infty), \quad n = 1, 2, \ldots \quad \delta : \n_*(\infty) \rightarrow \n+1_*(\infty), \quad n = 0, 1, \ldots$

(iv)
\[
U\alpha = \delta(\alpha)U \quad \text{and} \quad U^*\alpha = \delta_*(\alpha)U^* \quad \text{for any} \quad \alpha \in \infty_*(\infty).
\] (3.15)

2)
\[
\infty_*(\infty) = (\infty A)_\infty
\] (3.16)
where $(\infty A)_\infty = \{\infty A, \delta(\infty A), \ldots, \delta^k(\infty A), \ldots\}$
Moreover we have:

(i) For any $0 \leq l \leq k$
\[
\delta^k(\infty) \cdot \delta^l(\infty) \subset \delta^k(\infty)
\]

(ii) For every $n$ the subalgebra $(\infty A)_n \subset (\infty A)_\infty$ given by (3.6) (with $A_0$ substituted by $\infty A$) is the set of operators of the form
\[
\alpha_0 + \delta(\alpha_1) + \cdots + \delta^n(\alpha_n), \quad \alpha_i \in \infty.
\]

(iii) $\delta_* : (\infty A)_n \rightarrow (\infty A)_{n-1}, \quad n = 1, 2, \ldots \quad \delta : (\infty A)_n \rightarrow (\infty A)_{n+1}, \quad n = 0, 1, \ldots$
**Proof:** I. If (i) is true then Proposition 2.2 tells us that all the operators 
\( \delta^k(1), \ k = 0, 1, ... \) are projections. In addition we have

\[
\delta^k(1) \in A \subset A', \quad \delta^k(A_0) \subset A \subset A'
\]

and

\[
\delta^*(1) \in A \subset [\delta^k(A_0)]', \ k = 0, 1, ...
\]

Thus (i) implies (ii). As in the previous theorem the implication (ii) \( \Rightarrow \) (i) follows as a byproduct from Part II of the theorem (one can take \( A = \infty(A_\infty) \)). So let us prove this part.

II. Since in the situation under consideration all the assumptions of Theorem 3.4 are satisfied it follows that \( A_\infty \) is a commutative algebra and

\[
\delta : A_\infty \hookrightarrow A_\infty
\]

is an endomorphism and

\[
\delta^*(1) \in (A_\infty)' \quad (3.17)
\]

In view of (3.17) and (3.18) we can apply Theorems 3.7 and 3.3 (with \( A_0 \) substituted by \( A_\infty \)) and thus all the statements in 1) are proved. Let us verify 2).

Observe that in the situation considered all the assumptions of Theorem 3.5 are satisfied (in particular \( \delta^*(1) \in [\delta^k(A_0)]', \ k = 0, 1, ... \) since \( \delta(1) \in \infty(A_\infty) \) and \( \delta^k(A_0) \subset \infty(A_\infty) \)). Thus

\[
\delta^* : \infty A \hookrightarrow \infty A
\]

is an endomorphism.

Since \( \delta(1) \in \infty(A_\infty) \) and \( \infty A \subset \infty(A_\infty) \) it follows that

\[
\delta(1) \in (\infty A)' \quad (3.20)
\]

In view of (3.19) and (3.20) we can apply Theorem 3.7 (where we exchange \( \delta' \) for \( \delta^* \)) to the algebra \( \infty A \) (instead of \( A_0 \)) and it follows that both the mappings

\[
\delta : (\infty A)_\infty \to (\infty A)_\infty \quad \text{and} \quad \delta^* : (\infty A)_\infty \to (\infty A)_\infty
\]

19
are endomorphisms and the properties (i), (ii) and (iii) of 2) are true as well. Moreover (3.21) along with the construction of \( (\infty A_\infty) \) and \((\infty A)_\infty \) imply (3.16). \( \Box \)

4 The structure of the algebra \( \{A_0, U\} \)

Throughout this section we fix a partial isometry \( U \) and a commutative algebra \( A_0 \subset L(H) \) containing the identity and satisfying the assumptions (ii) of Part I of Theorem 3.8.

The purpose of the section is to describe the structure of the \( C^* \)-algebra \( \{A_0, U\} \) generated by \( A_0 \) and \( U \).

To simplify the notation we denote by \( A \) the algebra

\[
A = \infty(A_\infty) = (\infty A)_\infty
\]

where \( \infty(A_\infty) \) and \((\infty A)_\infty \) are the algebras defined by (3.13) and (3.16) respectively.

As it will be clarified the algebra \( A \) plays here the crucial role of the 'coefficient' algebra.

**Lemma 4.1** The algebra \( \{A_0, U\} \) is the uniform closure of finite sums of the form

\[
U^n \beta_{-n} + ... + \beta_0 + ... + \beta_n U^n
\]

where \( \beta_k \in A \) and \( \beta_{\pm k} = U^k U^{k*} \beta_{\pm k} = \beta_{\pm k} U^k U^{k*} \), \( k = 0, ..., n \).

This algebra is also the uniform closure of finite sums of the form

\[
\alpha_{-n} U^{n*} + ... + \alpha_0 + ... + U^n \alpha_n
\]

where \( \alpha_k \in A \) and \( \alpha_{\pm k} = U^{k*} U^k \alpha_{\pm k} = \alpha_{\pm k} U^{k*} U^k \), \( k = 0, ..., n \).

**Proof:** Follows in a routine way from the definition of \( \{A_0, U\} \) along with (3.14), (3.15) and the observation that

\[
U^k U^{k*}, U^{k*} U^k \in \infty(A_\infty) = A \quad k = 1, 2, ...
\]
and the equalities
\[ U^k U^k U^k = U^k, \quad U^k U^k U^k = U^k, \quad k = 1, 2, ... \]

\[ \square \]

4.2 We shall say that the algebra \( \{A_0, U\} \) possesses the property \( (*) \) if for every element \( b \in \{A_0, U\} \) having the form (4.2) the following inequality holds:
\[ \|b\| \geq \|\beta_0\| \tag{4.4} \]

Clearly the algebra \( \{A_0, U\} \) possesses the property \( (*) \) iff for every element \( b \in \{A_0, U\} \) having the form (4.3) the following inequality holds:
\[ \|b\| \geq \|\alpha_0\| \tag{4.5} \]

Indeed. Due to (3.15) one can always pass from (4.2) to (4.3) (and back) leaving \( \beta_0 = \alpha_0 \) invariant under the passage.

**Theorem 4.3**  
If \( \{A_0, U\} \) possesses the property \( (*) \) then for any element \( b \) of the form (4.2) we have
\[ \|b\| \geq \max_{i=0, n} \|\beta_{\pm i}\| \tag{4.6} \]
and for any element \( b \) of the form (4.3) we have
\[ \|b\| \geq \max_{i=0, n} \|\alpha_{\pm i}\| \tag{4.7} \]

**Proof:** Let us prove (4.6) first.

Fix some \( k \in \overline{1, n} \). Since \( \|U^k\| \leq 1 \) it follows that
\[ \|b\| \geq \|U^k b\| \tag{4.8} \]

We have
\[ U^k b = U^k U^n \beta_{-n} + ... + U^k U^k \beta_{-k} + ... + \]
\[ U^k U^s \beta_{-s} + ... + U^k \beta_0 + ... + U^k \beta_n U^n \tag{4.9} \]
Note that

1) For \( k > l \) we have (using the equality \( U^{s*}U^{s*} = U^s \), (3.15) and bearing in mind that \( U^{s*}U^{s*} \in \infty(A_\infty) = A \):

\[
U^kU^{l*} \beta_{-l} = U^kU^{k*}U^{(l-k)*} \beta_{-l} = \delta^k(1)U^{(l-k)*} \beta_{-l} = \\
U^{(l-k)*}\delta^l(1) \beta_{-l} = U^{(l-k)*}(U^{l-k}U^{(l-k)*}\delta^l(1) \beta_{-l}) = U^{(l-k)*}\beta'_{-l+k} \tag{4.10}
\]

where \( \beta'_{-l+k} = U^{l-k}U^{(l-k)*}\beta'_{-l+k} \) and \( \beta'_{-l+k} \in A \).

2)

\[
U^kU^{k*} \beta_{-k} = \beta'_0 \in A \tag{4.11}
\]

3) For \( k \geq l > 0 \) we have

\[
U^kU^{l*} \beta_{-l} = U^{k-l}U^{l*}U^{l*} \beta_{-l} = \delta^{k-l}(U^{l*}U^{l*} \beta_{-l})U^{k-l} = \\
\delta^{k-l}(U^{l*}U^{l*} \beta_{-l})U^{k-l}U^{(k-l)*}U^{k-l} = \beta'_{k-l}U^{k-l}
\]

where \( \beta'_{k-l} = \beta'_{k-l}U^{k-l}U^{(k-l)*} \) and \( \beta'_{k-l} \in A \).

4)

\[
U^k \beta_0 = \delta^k(\beta_0)U^kU^{k*}U^k = \beta'_k U^k \tag{4.12}
\]

where \( \beta'_k = \beta'_k U^kU^{k*} \) and \( \beta'_k \in A \).

5) For \( l > 0 \) we have

\[
U^k \beta_{l}U^l = \delta^k(\beta_{l})U^{l+k} = \delta^k(\beta_{l})U^{l+k}U^{(l+k)*}U^{l+k} = \beta'_{l+k}U^{l+k} \tag{4.13}
\]

where \( \beta'_{l+k} = \beta'_{l+k}U^{l+k}U^{(l+k)*} \) and \( \beta'_{l+k} \in A \).

Now (4.8), (4.9) and (4.10)-(4.13) along with the property (\( * \)) imply

\[
\|b\| \geq \|U^k b\| = \|U^{(n-k)*} \beta_{n-k} + \ldots + \beta'_0 + \ldots + \beta'_{n+k}U^{n+k}\| \geq \\
\|\beta'_0\| = \|U^kU^{k*} \beta_{-k}\| = \|\beta_{-k}\| \tag{4.14}
\]

22
Since $\|b\| = \|b^*\|$ inequality (4.14) being applied to $b^*$ implies

$$\|b\| = \|b^*\| \geq \|\beta_k^*\| = \|\beta_k\|$$

thus finishing the proof of (4.6).

(4.7) follows from (4.6) by exchanging $U$ for $U^*$.

4.4 Thus if $\{A_0, U\}$ possesses the property $(\ast)$ then all the coefficients in (4.2) and (4.3) are uniquely determined and $\alpha_0 = \beta_0$ and for every $n = 0, \pm 1, \ldots$ the mapping

$$N_n : \{A_0, U\} \rightarrow A$$

given by

$$N_n(b) = \beta_n \quad (4.15)$$

given by

$$N_n(b) = \beta_n \quad (4.16)$$

is correctly defined (as well as the corresponding mapping exploiting the coefficients $\alpha_n$ is correctly defined).

The next Lemma 4.5 gives a number of norm estimates of sums of elements in $C^*$-algebras. This lemma being useful in its own right also plays an important role in the proof of Theorem 4.6 that presents a certain formula for the norm calculation of the elements of $\{A_0, U\}$.

The estimates presented in the lemma are probably known (in particular the components of the statement of the lemma are given in [8], Lemma 7.3 and [9], Lemma 22.3). The proof of the lemma can be obtained as a simple modification of the reasoning given in the proof of [8], Lemma 22.3.

**Lemma 4.5** For any $C^*$-algebra $B$ and any elements $d_1, \ldots, d_m \in B$ we have

$$\left(\sum_{i=1}^{m} d_i\right)^2 \leq m \left(\sum_{i=1}^{m} d_i d_i^*\right) \quad (4.17)$$

and

$$\left(\sum_{i=1}^{m} d_i\right)^2 \leq m \left(\sum_{i=1}^{m} d_i^* d_i\right) \quad (4.18)$$
On the other hand

$$\left\| \sum_{i=1}^{m} |d_i| \right\|^2 \geq \frac{1}{m} \left\| \sum_{i=1}^{m} d_i^* d_i \right\|$$  \hspace{1cm} (4.19)$$

and

$$\left\| \sum_{i=1}^{m} \sqrt{d_i d_i^*} \right\|^2 \geq \frac{1}{m} \left\| \sum_{i=1}^{m} d_i^* d_i \right\|$$  \hspace{1cm} (4.20)$$

**Theorem 4.6** Let \( \{A_0, U\} \) be an algebra such that the pair \( A_0, U \) satisfies the assumptions (ii) of Part I of Theorem 3.8. If the algebra \( \{A_0, U\} \) possesses the property \((\ast)\) then for any element \( b \) of the form \( (4.2) \) we have

$$\|b\| = \lim_{k \to \infty} 4^{k} \sqrt{\|N_0[(bb^*)^{2k}]\|}$$  \hspace{1cm} (4.21)$$

where \( N_0 \) is the mapping defined by \((4.16)\).

**Proof:** Applying \((4.17)\) to the operator

$$b = U^n \beta_{-n} + \ldots + \beta_0 + \ldots + \beta_n U^n = d_{-n} + \ldots + d_0 + \ldots + d_n$$

we obtain

$$\|b\|^2 \leq (2n + 1) \left\| \sum_{i=-n}^{n} d_i d_i^* \right\| = (2n + 1) \|N_0(bb^*)\|$$

where

$$d_i d_i^* = U^{-i} \beta_i \beta_i^* U^{-i}, \quad i < 0;$$

$$d_i d_i^* = \beta_i U U^* \beta_i^*, \quad i \geq 0,$$

in either case \( d_i d_i^* \in \infty(A_\infty) = A \).

On the other hand as \( \{A_0, U\} \) possesses the property \((\ast)\) we have

$$\|b\|^2 = \|bb^*\| \geq \|N_0(bb^*)\|$$

thus

$$\|N_0(bb^*)\| \leq \|bb^*\| = \|b\|^2 \leq (2n + 1) \|N_0(bb^*)\|$$  \hspace{1cm} (4.22)$$

24
Applying (4.22) to \((bb^\ast)^k\) and having in mind that \((bb^\ast)^k = (bb^\ast)^{k\ast}\) and 

\[
\|N_0 [(bb^\ast)^{2k}]\| \leq \|((bb^\ast)^k \cdot (bb^\ast)^{k\ast})\| = \|b\|^{4k} \leq (4kn + 1)\|N_0 [(bb^\ast)^{2k}]\| \tag{4.23}
\]

since being written in the form (1.2) \((bb^\ast)^k\) has not more than \((4kn + 1)\) summands.

So

\[
4\sqrt{\|N_0 [(bb^\ast)^{2k}]\|} \leq \|b\| \leq 4\sqrt[4k]{4kn + 1} \cdot 4\sqrt{\|N_0 [(bb^\ast)^{2k}]\|} \tag{4.24}
\]

Observing the equality

\[
\lim_{k \to \infty} 4\sqrt[4k]{4kn + 1} = 1
\]

we conclude that

\[
\|b\| = \lim_{k \to \infty} 4\sqrt{\|N_0 [(bb^\ast)^{2k}]\|}
\]

The proof is complete.

The next result shows that the property \((\ast)\) plays the crucial role in the determination of the structure of the algebra \(\{A_0, U\}\) once the structure of \(A = I(A_\infty)\) is elucidated.

**Theorem 4.7** Let \(\{A_0^1, U_1\}\) and \(\{A_0^2, U_2\}\) be two algebras such that both the pairs \(A_i^i, U_i, \ i = 1, 2\) satisfy the assumptions (ii) of Part I of Theorem \(3.8\). Suppose that there exists an isomorphism

\[
\varphi : A_0^1 \to A_0^2
\]

such that under the mapping \(U_1 \to U_2\) the isomorphism \(\varphi\) give rise to the isomorphism

\[
\varphi : A^1 \to A^2 \tag{4.25}
\]

(where \(A^i = I(A^i_\infty), \ i = 1, 2\) are the algebras given by (3.13) defined by the pairs \(A_i^i, U_i\) respectively).

If both the algebras \(\{A_0^i, U_i\}, \ i = 1, 2\) possess the property \((\ast)\) then the mappings

\[
\varphi : A_0^1 \to A_0^2, \ U_1 \to U_2
\]

give rise to the isomorphism

\[
\{A_0^1, U_1\} \cong \{A_0^2, U_2\}
\]

25
**Proof:** Consider an operator \( b \in \{A_0^1, U_1\} \) having the form

\[
b = U_1^* \beta_{-n} + ... + \beta_0 + ... + \beta_n U_1^n
\]

where \( \beta_k \in A^1 \).

Let \( \varphi(b) \in \{A_0^2, U_2\} \) be the operator given by

\[
\varphi(b) = U_2^* \varphi(\beta_{-n}) + ... + \varphi(\beta_0) + ... + \varphi(\beta_n) U_2^n
\]

To prove the theorem it is enough to verify the equality \( \|b\| = \|\varphi(b)\| \).

By Theorem 4.6 we have

\[
\|b\| = \lim_{k \to \infty} 4^k \sqrt{\|N_0 [(bb^*)^{2k}]\|}
\]

where

\[
N_0 : \{A_0^1, U_1\} \to A^1
\]

is described in 4.4.

Similarly

\[
\|\varphi(b)\| = \lim_{k \to \infty} 4^k \sqrt{\|N_0 [(\varphi(b)\varphi(b^*))^{2k}]\|}
\]

where

\[
N_0 : \{A_0^2, U_2\} \to A^2
\]

Observe that (4.26) and (4.27) along with the assumptions of the theorem imply

\[
N_0 [(\varphi(b)\varphi(b^*))^{2k}] = N_0 [(\varphi(bb^*))^{2k}] = \varphi(N_0 [(bb^*)^{2k}])
\]

and therefore (in view of (4.23))

\[
\|N_0 [(\varphi(b)\varphi(b^*))^{2k}]\| = \|N_0 [(bb^*)^{2k}]\|
\]

This along with (4.29) and (4.28) implies the equality

\[
\|b\| = \|\varphi(b)\|
\]

and finishes the proof. \(\square\)
Remark 4.8 It is worth mentioning that the property (•) and the results of Theorem 4.7 type play a fundamental role in the theory of crossed products of $C^*$-algebras by discrete groups (semigroups) of automorphisms (endomorphisms). Namely this property is a characteristic property of the crossed product and it enables one to construct its faithful representations. The importance of the property (•) for the first time (probably) was clarified by O’Donovan [10] in connection with the description of $C^*$-algebras generated by weighted shifts. The most general result establishing the crucial role of this property in the theory of crossed products of $C^*$-algebras by discrete groups of automorphisms was obtained in [11] (see also [9], Chapters 2, 3 for complete proofs and various applications) for an arbitrary $C^*$-algebra and amenable discrete group. The relation of the corresponding property to the faithful representations of crossed products by endomorphisms generated by isometries was investigated in [12, 13]. The properties of this sort proved to be of great value not only in pure $C^*$-theory but also in various applications such as, for example, the construction of symbolic calculus and developing the solvability theory of functional differential equations (see [14, 15]).

5 The structure of the algebra $B = \{1, |a|, U\}$.

Now we return to the consideration of the initial object of our investigation. Throughout this section we fix an operator $a \in L(H)$ satisfying relation (1.1), take the partial isometry $U$ from the polar decomposition (1.6) and set $A_0 = \{1, |a|\}$.

The aim of the section is to describe the structure of the $C^*$-algebra $B = \{1, |a|, U\}$.

Following the path already exploited in the preceding sections we start with the examination of the properties of the partial isometry $U$ along with the properties of the induced mappings and the arising algebras (extensions of $A_0$).

**Theorem 5.1** Let $U$ be the partial isometry defined by polar decomposition (1.7) of an operator $a$ satisfying relation (1.1) then
1). $U^*U \in \{1, |a|\}''$, where we denote by $\mathcal{A}''$ the bicommutant of an algebra $\mathcal{A}$ (that is the Von Neumann algebra generated by $\mathcal{A}$).

2). $U^kU^*k \in \{1, |a|\}''$, $k = 1, 2, ...$

3). $[U^*lU^l, U^kU^*k] = 0$, $k = 1, 2, ...$; $l = 1, 2, ...$

4). $U^*U^kU^*l = U^{k-1}U^*l$, $1 \leq k \leq l$

5). $(U^kU^*k)^2 = U^kU^*k$, that is $U^k$ and $U^*k$ are partial isometries.

6). If $k \geq l$ then

\[ U^*kU^kU^*lU^l = U^*lU^lU^*kU^*k = U^*kU^*k \]

and

\[ U^kU^*kU^lU^*l = U^lU^*lU^kU^*k = U^kU^*k \]

7). 

\[ \delta : \{|a|\} \to \delta(\{|a|\}) \subset \{1, |a|\} \] (5.1)

is a morphism and

\[ Ub = \delta(b)U \quad \text{and} \quad U^*\delta(b) = bU^*, \quad b \in \{|a|\} \] (5.2)

8). $\delta^k\{|a|\} \subset \{1, |a|, UU^*, ..., U^{k-1}U^*\}$, $k = 1, 2, ...$

9). $\delta^k(\{|a|\}) = U^kU^*k\delta^k(\{|a|\}) = \delta^k(\{|a|\})U^kU^*k$

that is $U^kU^*k$ is the projection onto the invariant subspace where the whole of the algebra $\delta^k(\{|a|\})$ acts.

10). For every $\alpha \in \{|a|\}$ we have $\delta_\alpha\delta(\alpha) = \alpha = U^*U\alpha = \alpha U^*U$.

**Proof:**

1). We have that $U^*U$ is the orthogonal projection onto $\text{Im} |a| = (\text{Ker} |a|)^\perp$. Therefore $U^*U$ is the spectral projection of $|a|$ corresponding to the set
σ(|a|) \ {0} (where we denote by σ(α) the spectrum of an operator α). Thus by the spectral theorem (see, for example, [4], §17) \( U^*U \in \{1, |a|\}'' \).

7). The statement that \( \delta : \{|a|\} \to \overline{\delta(\{|a|\})} \) is a morphism and \( (5.2) \) follow from 1) in view of part I of Lemma 2.4. In addition \( (1.10) \) means that \( \delta(|a|) \in \{1, |a|\} \) and therefore \( \delta\{|a|\} \subset \{1, |a|\} \).

8). By \( (5.1) \) we have that

\[ \delta\{|a|\} \subset \{1, |a|\} \]

Thus

\[ \delta^2\{|a|\} \subset \delta(\{1, |a|\}) = \{\delta(1), \delta(|a|)\} \subset \{UU^*, 1, |a|\}. \]

Continuing this reasoning we obtain 8).

2). Since \( U^*U \) is the spectral projection corresponding to the interval \((0, |a|)\) (see the proof of 1)) it follows that there exists a sequence \( \alpha_n \) of elements of \( \{|a|\} \) such that

\[ \alpha_n \xrightarrow{\text{strongly}} U^*U \quad (5.3) \]

Therefore we have

\[ U\alpha_nU^* \xrightarrow{\text{strongly}} U(U^*U)U^* = UU^* \]

Since (by 8)) \( U\alpha_nU^* \subset \{1, |a|\}'' \) it follows that \( UU^* \subset \{1, |a|\}'' \).

The further proof goes by induction.

Suppose that \( U^kU^* \in \{1, |a|\}'' \), \( k = 1, n-1 \). Taking the sequence \( \alpha_n \) mentioned above we have

\[ U^n\alpha_nU^{n*} \xrightarrow{\text{strongly}} U^n(U^*U)U^{n*} = U^{n-1}(UU^*U)U^{n*} = U^nU^{n*} \quad (5.4) \]

But due to 8) and the assumption of the induction we have

\[ U^n\alpha_nU^{n*} \subset \{1, |a|, UU^*, ..., U^{n-1}U^{n-1*}\} \subset \{1, |a|\}'' \]

and therefore \( (5.4) \) implies \( U^nU^{n*} \in \{1, |a|\}'' \). So 2) is proved.

3), 4), 5) and 6). These follow from 1) and 2) along with Lemma 2.6.
9). In view of 5) we have the relations \( U^k U^* U^k = U^k \) and \( U^* U^k U^* U^k = U^* U^k \) that imply 9).

10). Recall that since \( U^* U \) is the orthogonal projection onto \( \text{Im} \, |a| = (\text{Ker} \, |a|)^{\perp} \) it follows that \( U^* U |a| = |a| = |a| U^* U \). This along with 1) implies 10). □

**Remark 5.2** Most of the properties listed in Theorem 5.1 are known (we have presented them for the sake of completeness). In particular, one can find some generalizations of the properties 3) and 7) in Propositions 28 and 29 in [5], Section 2.1 that also contains a lot of important information on the subject.

5.3 Now as in section 2 we introduce the necessary for our future goals extensions of the algebra \( A_0 = \{1, |a|\} \).

Having in mind 3.2 and statements 2) and 6) of the theorem just proved for every \( n = 0, 1, \ldots \) we set

\[
A_n = \{1, |a|, UU^*, \ldots, U^nU^{n*}\} \subset \{1, |a|\}''
\]  

(5.5)

Clearly

\[
\delta(A_n) \subset A_{n+1}
\]

is a morphism (the latter follows from Lemma 2.4).

Now bearing in mind (3.7) we set

\[
A_\infty = \{1, |a|, UU^*, \ldots, U^nU^{n*}, \ldots\} \subset \{1, |a|\}''
\]  

(5.6)

Evidently

\[
\delta : A_\infty \to A_\infty \text{ is an endomorphism.}
\]  

(5.7)

and in particular

\[
\delta^k(A_0) \subset A_\infty \subset A_0'
\]  

(5.8)

Note also that by statement 1) of Theorem 5.1 and (5.6) we have

\[
\delta^*_k(1) = U^* U \in (A_\infty)' \subset [\delta^k(A_0)]', \quad k = 0, 1, \ldots
\]  

(5.9)
which along with (5.7) implies (by Lemma 2.5)
\[ \delta^k_*(1) = U^* k U^k \in (A_\infty)' \subset A'_0, \quad k = 1, 2, \ldots \quad (5.10) \]
Moreover (5.7) along with (5.9) also imply (by Lemma 2.4)
\[ Ub = \delta(b)U \quad \text{and} \quad U^* \delta(b) = b U^*, \quad b \in A_\infty \quad (5.11) \]

Observe now that (5.8), (5.9) and (5.10) mean that for the algebra \( A_0 = \{1, |a|\} \) and the operator \( U \) under consideration all the assumptions of Theorem 3.8 are satisfied.

Thus we arrived at the next statement.

**Theorem 5.4** Let \( a \) be an operator satisfying relation (1.1) and \( U \) be the partial isometry defined by polar decomposition (1.6). Let \( A_\infty \) be the algebra given by (5.6) then

1) \[ A = \infty(A_\infty) = \{A_\infty, \delta_*(A_\infty), \ldots, \delta^k_*(A_\infty), \ldots\} \quad (5.12) \]
is the minimal commutative C*-algebra containing \( A_0 = \{1, |a|\} \) and such that
\[ \delta : A \to A \quad \text{and} \quad \delta_* : A \to A \quad (5.13) \]
are endomorphisms.
Moreover we have:

(i) For any \( 0 \leq l \leq k \)
\[ \delta^k_*(A_\infty) \cdot \delta^l_*(A_\infty) \subset \delta^k_*(A_\infty) \]

(ii) For every \( n \) the subalgebra
\[ n(A_\infty) = \{A_\infty, \ldots, \delta^n_*(A_\infty)\} \subset A \quad (5.14) \]
is the set of operators of the form

\[ \alpha_0 + \delta_*(\alpha_1) + ... + \delta^n(\alpha_n), \quad \alpha_i \in A_{\infty}. \]

(iii) \( \delta : n(A_{\infty}) \to n-1(A_{\infty}), \ n = 1, 2, ... \)
\( \delta_* : n(A_{\infty}) \to n+1(A_{\infty}), \ n = 0, 1, ... \)

(iv)

\[ U\alpha = \delta(\alpha)U \quad \text{and} \quad U^*\alpha = \delta_*(\alpha)U^* \quad \text{for any} \ \alpha \in A. \] (5.15)

2) Let

\[ \infty A = \{ A_0, \delta_*(A_0), ..., \delta^n(A_0), ... \} \] (5.16)

Then

\[ A = \infty(A_{\infty}) = (\infty A)_{\infty} \] (5.17)

where \( (\infty A)_{\infty} = \{ \infty A, \delta(\infty A), ..., \delta^k(\infty A), ... \} \)
Moreover we have:

(i) For any \( 0 \leq l \leq k \)

\[ \delta^k(\infty A) \cdot \delta^l(\infty A) \subset \delta^k(\infty A) \]

(ii) For every \( n \) the subalgebra

\[ (\infty A)_n = \{ \infty A, ..., \delta^n(\infty A) \} \subset A \] (5.18)

is the set of operators of the form

\[ \alpha_0 + \delta(\alpha_1) + ... + \delta^n(\alpha_n), \quad \alpha_i \in \infty A. \]

(iii) \( \delta_* : (\infty A)_n \to (\infty A)_{n-1}, \ n = 1, 2, ... \)
\( \delta : (\infty A)_n \to (\infty A)_{n+1}, \ n = 0, 1, ... \)

Now we proceed to the description of the algebra \( B = \{ 1, |a|, U \} \).

In this situation Lemma 4.3 turns into

32
Lemma 5.5  The algebra $\mathcal{B} = \{1, |a|, U\}$ is the uniform closure of finite sums of the form

$$U^{n*} \beta_{-n} + \ldots + \beta_0 + \ldots + \beta_n U^n$$

(5.19)

where $\beta_k \in A$ ($A$ is given by (5.12)) and $\beta_{\pm k} = U^k U^{k*} \beta_{\pm k} = \beta_{\pm k} U^k U^{k*}$, $k = 0, \ldots, n$.

This algebra is also the uniform closure of finite sums of the form

$$\alpha_{-n} U^{n*} + \ldots + \alpha_0 + \ldots + U^n \alpha_n$$

(5.20)

where $\alpha_k \in A$ and $\alpha_{\pm k} = U^{k*} U^k \alpha_{\pm k} = \alpha_{\pm k} U^{k*} U^k$, $k = 0, \ldots, n$.

5.6 Using the notation of 4.2 we say that the algebra $\mathcal{B}$ possesses the property $(\ast)$ if for every element $b \in \mathcal{B}$ having the form (5.19) the following inequality holds:

$$\|b\| \geq \|\beta_0\|$$

(5.21)

In view of Theorem 4.3 we conclude that

if $\mathcal{B}$ possesses the property $(\ast)$ then for any element $b$ of the form (5.19) we have

$$\|b\| \geq \max_{i=0,n} \|\beta_{\pm i}\|$$

(5.22)

and for any element $b$ of the form (5.20) we have

$$\|b\| \geq \max_{i=0,n} \|\alpha_{\pm i}\|$$

(5.23)

In the situation considered Theorem 4.7 turns into

Theorem 5.7  Let $a_1 = U_1 |a_1|$ and $a_2 = U_2 |a_2|$ be the polar decompositions of operators $a_1$ and $a_2$ and both the operators $a_1$ and $a_2$ satisfy the condition (1.4). Suppose that the mapping

$$|a_1| \rightarrow |a_2|, \quad U_1 \rightarrow U_2$$

(5.24)
gives rise to the isomorphism

\[ A^1 \cong A^2 \]

(\text{where } A^i, \ i = 1, 2 \text{ are defined by (5.11) for the operators } a_i \text{ and } U_i \text{ respectively}).

If both the algebras \( B_i = \{1, |a_i|, U_i\}, \ i = 1, 2 \) possess the property (\text{\star}) then the mapping (5.24) gives rise to the isomorphism

\[ B_1 \cong B_2 \]

6 The structure of the algebra \( B = \{1, a\} \).

Now we return to the consideration of the structure of the algebra \( B = \{1, a\} \). In this situation along with the mapping \( \delta \) (3.14) we shall introduce the mapping

\[ \varphi : \{1, |a|\} \to \{1, \delta(|a|)\} \subset \{1, |a|\} \]

given by

\[ \varphi [(a^*a)^k] = (aa^*)^k = \delta [(aa^*)^k] = \delta^k(a^*a), \quad \varphi(1) = 1. \quad (6.1) \]

One can easily verify that for every \( b \in \{1, |a|\} \) we have

\[ ab = \varphi(b)a \quad \text{and} \quad ba^* = a^*\varphi(b) \quad (6.2) \]

Indeed. The first equality follows from the equalities

\[ a \cdot (a^*a)^k = (aa^*)^k \cdot a = \varphi[(a^*a)^k] \cdot a \]

and

\[ a \cdot 1 = 1 \cdot a = \varphi(1) \cdot a. \]

And the second follows from the first one by passage to the adjoint operator. Note that for any \( k, l \geq 0 \) we also have the following relations:

if \( k > l \) then

\[ a^k a^{*l} = b a^{k-l} \quad \text{where} \quad b \in \{1, |a|\}; \quad (6.3) \]
if $k < l$ then
\[ a^k a^* = a^{*(k-l)}b \text{ where } b \in \{1, |a|\}; \tag{6.4} \]

and
\[ a^k a^*k \in \{1, |a|\}. \tag{6.5} \]

where we take $a^0 = a^{*0} = 1$.

Indeed. If $k \geq l$ then using (6.2) several times we have
\[ a^k a^* = a^{k-1}(aa^*)a^{*-1} = \varphi^{k-1}(aa^*) \cdot a^{k-1} \cdot a^{*-1} = \]
\[ \varphi^{k-1}(aa^*) \cdot \varphi^{k-2}(aa^*) \cdots \cdot \varphi^{k-l}(aa^*) \cdot a^{k-l} = ba^{k-l} \]

where $\varphi^{k-1}(aa^*) \cdot \varphi^{k-2}(aa^*) \cdots \cdot \varphi^{k-l}(aa^*) = b \in \{1, |a|\}$. Thus (6.3) and (6.5) are proved.

(6.4) follows from (6.3) since for $k < l$ we have
\[ a^k a^* = (a^k a^*)^{**} = (a^l a^{*-k})^* = (ba^{*-k})^* = a^{*(l-k)}b^* \]

The next useful observation is given by

**Lemma 6.1** For any two operators of the form
\[ a^{*l_i}b_{l_i,m_i}a^{m_i}, \quad i = 1, 2 \]

where $l_i, m_i \geq 0$, $b_{l_i,m_i} \in \{1, |a|\}$ we have
\[ a^{*l_1}b_{l_1,m_1}a^{m_1} \cdot a^{*l_2}b_{l_2,m_2}a^{m_2} = a^{*l_3}b_{l_3,m_3}a^{m_3} \]

where $l_3, m_3 \geq 0$, $b_{l_3,m_3} \in \{1, |a|\}$ and
\[ m_1 - l_1 + m_2 - l_2 = m_3 - l_3 \]

**Proof:** Follows in a routine way from (6.2) - (6.5). \qed

Applying this lemma and using again (6.2) - (6.5) we conclude that the
algebra $B = \{1, a\}$ is the uniform closure of finite sums of elements of the form

$$b = b_{-n} + \ldots + b_0 + \ldots + b_n, \quad n \geq 0$$  \hspace{1cm} (6.6)

where $b_k, \; -n \leq k \leq n$ is a finite sum of the form

$$b_k = \sum_{m-l=k} a^l b_{l,m} a^m, \quad b_{l,m} \in \{1, |a|\}, \; l, m \geq 0$$

6.2 Let $B_0$ be the subalgebra of $B$ generated by the elements of the form

$$a^l b a^l, \; l \geq 0, \; b \in \{1, |a|\}$$

Lemma 6.1 shows that $B_0$ is indeed a $C^*$-subalgebra of $B$. Note also that $B_0$ is a commutative algebra since it is a subalgebra of $\mathcal{A}_\infty (\mathcal{A}_\infty)$ (see (5.12)). Using the elements of $B_0$ one can rewrite (6.6) in the form

$$b = a^s \beta_{-s} + \ldots + \beta_0 + \ldots + \beta_a a^n$$  \hspace{1cm} (6.7)

where $\beta_k \in B_0, \; -n \leq k \leq n$ (cf (5.19)).

6.3 We shall say that the algebra $B$ possesses the property $(\ast \ast)$ if for any element $b \in B$ of the form (6.6) the following inequality holds

$$\|b\| \geq \|b_0\|$$ \hspace{1cm} (6.8)

Remark. One can easily see that if the algebra $B = \{1, |a|, U\}$ considered in the previous section possesses the property $(\ast)$ (see (5.22)) then the algebra $B$ being a subalgebra of $B$ possesses the property $(\ast \ast)$.

If $B$ possesses the property $(\ast \ast)$ then the mapping

$$N_0 : B \rightarrow B_0$$  \hspace{1cm} (6.9)

given by

$$N_0(b) = b_0$$

for any $b$ having the form (6.6) is correctly defined.
Now the analogue to Theorem 4.6 for the situation considered is

**Theorem 6.4** Let $a$ be an operator satisfying relation (1.1) and $B = \{1, a\}$. If $B$ possesses the property $\ast\ast$ (6.3) then for any element $b$ of the form (6.6) we have

$$\|b\| = \lim_{k \to \infty} 4^k \sqrt[4k]{\|N_0[\{(bb^*)^{2k}\}]}$$  \hspace{1cm} (6.10)

where $N_0$ is the mapping defined by (6.7).

**Proof:** The same as for Theorem 4.6. One should apply the estimates from Lemma 4.5 to operator (6.6)

$$b = b_{-n} + ... + b_0 + ... + b_n = d_{-n} + ... + d_0 + ... + d_n$$

and observe that in view of Lemma 6.1 $d_i d_i^* \in B_0$ $-n \leq i \leq n$. 

Following the same path one can obtain the next result which is the analogue to Theorem 5.7 for the algebra under consideration.

**Theorem 6.5** Let $B_i = \{1, a_i\}$, $i = 1, 2$ where both the operators $a_i$, $i = 1, 2$ satisfy the condition (1.1). Suppose that the mapping $a_1 \mapsto a_2$ generates the isomorphism

$$B_{01} \cong B_{02}$$

where $B_{0i}$, $i = 1, 2$ is the algebra described in 6.2 (corresponding to the algebra $B_i$).

If both the algebras $B_i$, $i = 1, 2$ possess the property $\ast\ast$ then the mapping $a_1 \mapsto a_2$ gives rise to the isomorphism

$$B_1 \cong B_2.$$  

One can consider the results presented above from a bit different point of view.

**6.6** Let us introduce a certain notion of ’degree’ for finite products of elements $a$ and $a^*$.  
For any $l \geq 0$ we set

$$\deg a^l = l \text{ and } \deg a^{*l} = -l.$$  \hspace{1cm} (6.11)
where we take $a^0 = a^{*0} = 1$ and thus $\deg 1 = 0$.

For a finite product $b$ of these elements we set its degree $\deg b$ to be equal to the sum of the degrees of its factors.

Remark. Of course in general the number $\deg b$ depends not on the element $b$ as an element of $L(H)$ but on the representation of $b$ in the form of a finite product of elements $a$ and $a^*$ and therefore it is reasonable to consider the number $\deg$ as being defined on the free semigroup generated by $a$ and $a^*$.

Clearly $\deg b^* = -\deg b$ and if $\deg b_1 = k_1$, $\deg b_2 = k_2$ then $\deg (b_1 \cdot b_2) = k_1 + k_2$.

Evidently the algebra $B = \{1, a\}$ is the uniform closure of finite sums of the form

$$b = b_{-n} + \ldots + b_0 + \ldots + b_n$$

(6.12)

where $n \geq 0$ and $b_i, -n \leq i \leq n$ is a finite sum of the form

$$b_i = \sum \beta_{ki}, \quad \deg \beta_{ki} = i$$

One can verify that

if an element $b$ is a finite product of elements $a$ and $a^*$ and $\deg b = k$ then $b$ can be represented in the form

$$b = a^{*t} b_{l,m} a^m$$

where $l, m \geq 0$, $b_{l,m} \in \{1, |a|\}$ and $m - l = k$.

So we can reformulate the notions from 6.3 in the following way.

6.7 Let $\overline{B}_0$ be the subalgebra of $B$ generated by finite products $b$ such that $\deg b = 0$.

Observe that $\overline{B}_0 = B_0$ where the latter algebra is that defined in 6.2.

Thus the algebra $B$ possesses the property (**) iff for any element $b \in B$ of the form (6.12) the following inequality holds

$$\|b\| \geq \|b_0\|$$

(6.13)

Therefore Theorem 6.5 is equivalent to the next statement.
**Theorem 6.8** Let $B_i = \{1, a_i\}, \; i = 1, 2$ where both the operators $a_i, \; i = 1, 2$ satisfy the condition (1.1). Suppose that the mapping $a_1 \mapsto a_2$ generates the isomorphism $B_{01} \cong B_{02}$ where $B_{0i}, \; i = 1, 2$ is the algebra described in 6.7 (corresponding to the algebra $B_i$).

If for any element $b_i \in B_i$ having the form (6.12) the inequality (6.13) holds then the mapping $a_1 \mapsto a_2$ gives rise to the isomorphism $B_1 \cong B_2$.

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