SYMMETRIES AND STRING FIELD THEORY IN D=2

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Abstract

Prof. Wigner was well-known for his contributions to group theory and symmetries, especially the representations of the Poincaré group. Similarly, symmetry also plays a decisive role in string theory. In fact, because of these symmetries, in two dimensions the theory is, remarkably enough, exactly soluble, giving us the first non-perturbative information from string theory.

There are two ways in which to formulate 2D string theory, either as a matrix model or as a Liouville theory [1-4]. Matrix models are enormously powerful, and the emergence of a new symmetry, called $w(\infty)$, helps to explain why the model is exactly soluble. Its generators obey:

$$[Q_{jm}, Q_{j'm'}] = (jm' - j'm)Q_{j+j'-1,m+m'}$$

where $j = 0, 1/2, 1, 3/2, ...$ and $m = -j, ... + j$.

However, in matrix models the string degrees of freedom have been completely removed, and hence the field theoretic origin of $w(\infty)$ is rather obscure. Furthermore, the generators of $w(\infty)$ involve the so-called “discrete states,” which are unusual physical states defined only at discrete momenta.

Liouville theory, by contrast, has the opposite problem: it manifestly has all the string degrees of freedom intact, but the theory is notoriously difficult to solve, even at the free level. The purpose of this paper is to show how to reformulate Liouville theory as a second quantized field theory, so that $w(\infty)$ and the discrete states emerge naturally.

To re-write Liouville theory (for $\mu = 0$) as a field theory, we will generalize the 26 dimensional non-polynomial closed string field theory, recently proposed by the author [5] and also by the Kyoto and MIT groups [6,7], which solves the long-standing problem of a covariant, closed string field theory.

Using this non-polynomial theory in two dimensions, we will show that the origin of $w(\infty)$ comes from the gauge group of the non-polynomial action. Furthermore, we will show how to reproduce the standard shifted Shapiro-Virasoro four-point amplitude from the non-polynomial Liouville field theory. This gives a cohesive, integrated description of the many seemingly divergent results found in this rapidly growing field.

Further details of this paper can be found in ref. [8].
that: $\delta A^a_\mu = f_{abc} A^b_\mu \Lambda^c + \ldots$. And lastly, we write down an action $\mathcal{L}$ which is an invariant under this variation. We find, for example, that the three-vertex is proportional to the structure constant of the group $f_{abc}$.

However, $w(\infty)$ emerges from matrix models in a highly unorthodox fashion, largely because the string degrees of freedom have been completely eliminated.

The goal of this paper is to reformulate closed string Liouville theory as a second quantized field theory, so that $w(\infty)$ emerges in the usual way. We will show that the second quantized fields transform under a certain representation of the group, that the action is an invariant of the group, and that the three-vertex is proportional to the structure constant of the group.

The action of the 26 dimensional string theory is given by the non-polynomial closed string action, first written down by the author [5] and the Kyoto and MIT groups [6,7]:

$$\mathcal{L} = \langle \Phi | Q | \Phi \rangle + \sum_{n=3}^{\infty} \alpha_n \langle \Phi^n \rangle$$

(2)

where $Q = Q_0 (b_0 - \bar{b}_0)$, $Q_0$ is the usual BRST operator, and where the field $\Phi(X)$ transforms as:

$$\delta | \Phi \rangle = | Q \Lambda \rangle + \sum_{n=1}^{\infty} \beta_n | \Phi^n \Lambda \rangle$$

(3)

In ref. 8, we have generalized this action to describe two dimensional string theory. In this paper, we will show that:

(a) the vertices are BRST invariant
(b) the action reproduces the shifted Shapiro-Virasoro amplitude
(c) the three-vertex is proportional to the structure constant of $w(\infty)$.

## 2 BRST Invariance of Vertices

We first wish to show that the three-vertex is BRST invariant, i.e.

$$\sum_{i=1}^{3} Q_i | V_3 \rangle = 0$$

(4)

Naively, this calculation appears to be trivial, since the vertex function simply represents a delta function across three overlapping strings. Hence, we expect that the three contributions to $Q$ cancel exactly. However, this calculation is actually rather delicate, since there are potentially anomalous contributions at the joining points. To resolve this issue, we must use point-splitting, pioneered in refs. [9,10].

We wish to make a conformal map from the multi-sheeted, three-string, world-sheet configuration in the $\rho$-plane to the flat, complex $z$-plane. Fortunately, the conformal map for the $N$-point function to the complex $z$-plane is known. The map is given by [5]:

$$\frac{d\rho(z)}{dz} = C \prod_{i=1}^{N} \frac{\sqrt{(z - z_i)(z - \bar{z}_i)}}{\prod_{i=1}^{N} (z - \gamma_i)}$$

(5)

where $\gamma_i$ correspond to the points at infinity (the external lines) and the points $z_i$ correspond to the $i$th vertex.
\[
\oint_{C_1+C_2+C_3} \frac{dz}{2\pi i} e(z) \left\{ -\frac{1}{2} \left( \frac{dz'}{dz} \right) \partial_{\mu}(z') \partial^\mu(z) + \left( \frac{dz'}{dz} \right)^2 \frac{dc}{dz} b(z') + \frac{Q}{2} \partial^2 \phi(z) \right\} |V_3\rangle
\]

where \( z' \) is infinitesimally close to \( z \), where \( \mu \) ranges over the \( D \) dimensional string modes as well as the \( \phi \) mode, where \( b \) and \( c \) are the usual reparametrization ghosts, and \( C_i \) are infinitesimally small curves in the \( z \)-plane which encircle the joining point \( z_0 \).

The major complication to this calculation is that the Liouville \( \phi \) field does not transform as a scalar. Instead, it transforms as:

\[
\phi(\rho) \rightarrow \phi(z) + \frac{Q}{2} \log \left| \frac{dz'}{d\rho} \right|^2
\]

This means that the energy-momentum tensor \( T \) transforms as:

\[
T_{\rho\rho} \rightarrow \left( \frac{dz}{d\rho} \right)^2 T_{zz} + \left( \frac{Q}{2} \right)^2 S
\]

where \( S \) is the Schwartzian, given by \((z'''/z')-(3/2)(z''/z')^2\). This complicates the calculation considerably, since it means that there is subtle insertion factor located at delta-function curvature singularities in the vertex function. These add non-trivial \( \phi \) contributions to the calculation.

The final calculation is rather long [8], so we only summarize the result:

\[
\left\{ -pc(z_0) \left[ \frac{D}{24} - \frac{13}{12} + \frac{1}{24} + \frac{1}{8} Q^2 \right] - \frac{dc(z_0)}{dz} \left[ \frac{5D}{96} - \frac{65}{48} + \frac{5}{96} + \frac{5}{32} Q^2 \right] \right\} |V_3\rangle
\]

which cancels if:

\[
D - 26 + 1 + 3Q^2 = 0
\]

which is precisely the consistency equation for Liouville theory in \( D \) dimensions. Thus, the vertex is BRST invariant.

### 3 Shifted Shapiro-Virasoro Amplitude

The next major test of the theory is whether it reproduces the shifted Shapiro-Virasoro amplitude. This calculation is highly non-trivial, since the conformal map between the multi-sheeted string-scattering Riemann sheet to the complex plane is very involved. Unlike the light cone theory, or even Witten’s open string theory, the non-polynomial theory yields very complicated conformal maps.

Fortunately, for the four-point function, all conformal maps are known exactly, in terms of elliptic functions, and the calculation can be performed (see ref. 11).

For the four point function, the map in eq. (5) can be integrated exactly, giving:
where $\Pi$ is a third elliptic function, $z = ia_i + b_i$ and $\bar{z}_i = -ia_1 + b_1$ for complex $a_i$ and $b_i$, and:

$$A_i = \frac{[(\gamma_i - b_1)^2 + a_i^2][(\gamma_i - b_2)^2 + a_i^2]}{\prod_{j=1,j \neq i}^4(\gamma_i - \gamma_j)}$$  \hspace{1cm} (12)

$$\omega_i = \frac{a_1 + b_1 g_1 - \gamma_i g_1}{b_1 - a_1 g_1 - \gamma_i}$$

$$f_i = \frac{1}{2}(1 + \omega^2)^{-1/2}(k^2 + \omega^2)^{-1/2}$$

$$\times \ln \frac{(k^2 + \omega^2)^{1/2} - (1 - \omega^2)^{1/2}nu}{(k^2 + \omega^2)^{1/2} + (1 + \omega^2)^{1/2}nu}$$

$$\phi = \arctan \left( \frac{y - b_2 + a_1 g_1}{a_1 + g_1 b_1 - g_1 y} \right)$$  \hspace{1cm} (13)

where: $A^2 = (b_1 + b_2)^2 + (a_1 + a_2)^2$, $B^2 = (b_1 - b_2)^2 + (a_1 - a_2)^2$, $g_1^2 = [4a_1^2 - (A - B)^2][(A + B)^2 - 4a_1^2]$, $g = 2/(A + B)$, $y_1 = b_1 - a_1 g_1$, $k^2 = 1 - k^2 = 4AB/(A + B)^2$, $u = \sin^{-1}(1 - k^2 \sin^2 \phi)$

This explicit conformal map allows us to calculate the four-point amplitude. We first write the amplitude in the $\rho$ plane, and then make a conformal map to the $z$-plane. Let the modular parameter be $\hat{\tau} = \tau + i\theta$, where $\tau$ is the distance between the splitting strings, and $\theta$ is the relative rotation. Let $\gamma_1 = (0, \hat{x}, 1, \infty)$. Then, with a fair amount of work, one can find the Jacobian from $\hat{\tau}$ to $\hat{x}$:

$$\frac{d\hat{\tau}}{d\hat{x}} = \frac{\pi C}{2K(k)g\hat{x}(1 - \hat{x})(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)}$$  \hspace{1cm} (14)

Then the four point amplitude can be written as:

$$A_4 = \langle V_3 | \frac{b_0 \bar{b}_0}{L_0 + L_0 - 2} | V_3 \rangle$$

$$= \int d\tau \int d\theta A_G \left\langle \prod_{i=1}^4 c(\gamma_i) \bar{c}(\gamma_i) V(\gamma_i) \right\rangle$$  \hspace{1cm} (15)

where:

$$A_G = \int \frac{dz}{2\pi i} \frac{\prod_{i \leq i'}(\gamma_i - \gamma_j)}{dw \prod_{j=1}^4(z - \gamma_j)}$$  \hspace{1cm} (16)

$$= \frac{g}{\pi C} \hat{x}(1 - \hat{x})K(k)(\gamma_1 - \gamma_3)^3(\gamma_2 - \gamma_4)^3$$  \hspace{1cm} (17)

Putting everything together, we finally find:

$$A_4 = \int d^2\hat{x} |\hat{x}^{2p_1} \hat{x}^{2p_2} (1 - \hat{x})^{2p_2} |^2$$  \hspace{1cm} (18)

for shifted momenta $p$. This is precisely the shifted Shapiro-Virasoro amplitude, as expected.
First, notice that the field $|\Phi\rangle$ can be decomposed into a massless tachyon, the discrete states $|j,m\rangle$ (labeled by $SU(2)$ quantum numbers $j,m$), and BRST trivial states:

$$|\Phi\rangle = \varphi|0\rangle + \sum_{j,m} \psi_{j,m}|j,m\rangle + ...$$  \hspace{1cm} (19)$$

where $...$ represent the BRST trivial states.

Now take the matrix element of three strings: $\langle \Phi|\langle \Phi|\langle \Phi|V_3\rangle$. When $\langle \Phi|$ is physical and on-shell, only the tachyon and discrete states survive. To compute the three-vertex function between three discrete states, we will make a conformal transformation from the three-vertex $\rho$ complex plane to the complex $z$-plane. Under a complex transformation, $|j,m\rangle$ remains the same, i.e. $\Omega|j,m\rangle = |j,m\rangle$ since $|j,m\rangle$ satisfies the Virasoro conditions. Let us therefore insert $\Omega\Omega^{-1}$ inside the three-string vertex function. The $|j,m\rangle$ states remain the same, but the vertex function $|V_3\rangle_\rho$ transforms onto $|V_3\rangle_z$, which is defined on the complex $z$-plane.

Thus, it is a simple matter to show (for the holomorphic part), that:

$$\langle j_1, m_1|\langle j_2, m_2|\langle j_3, m_3|V_3\rangle_\rho = \langle j_1, m_1|\langle j_2, m_2|\langle j_3, m_3|V_3\rangle_z$$

$$= \langle \psi_{j_1,m_1}(0)\psi_{j_2,m_2}(1)\psi_{j_3,m_3}(\infty) \rangle$$

$$\sim (j_1m_2 - j_2m_1)\delta_{j_3,j_1+j_2+1,2} \delta_{m_3,m_1+m_2}$$  \hspace{1cm} (20)$$

where $Q_{j,m} = \oint \frac{dz}{2\pi i} \psi_{j,m}(z)$. That $w(\infty)$ emerges from the interaction of discrete states has been stressed in refs. [12-14]. What we have shown is that this result easily generalizes to closed string field theory.

Thus, we have shown that the three-vertex function of string field theory reproduces the structure constants of $w(\infty)$, as expected. Thus, $w(\infty)$ emerges naturally as part of the gauge invariance of the string field, as in ordinary gauge theory. (However, the geometric picture of $w(\infty)$ as area-preserving diffeomorphisms appears to be lost.)

One advantage of this formulation is that the notorious $c = 1$ barrier found in matrix models is easily broken. (However, we do not expect the model to be exactly soluble beyond this barrier.)

In summary, we have successfully formulated 2D string theory as a second quantized field theory for closed strings. Once Liouville theory is formulated as a field theory, the mysteries of 2D string theory (e.g. the field theoretic origin of $w(\infty)$, discrete states, etc.) become transparent, because everything is formulated as a standard gauge theory. See [8] for more details.

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6 References

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