Exponential integrability of noncommutative Lipschitz martingales

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Abstract. In this paper, we study the John-Nirenberg theorems for the Lipschitz spaces and the atomic decomposition for Hardy spaces in the noncommutative martingale setting. By adapting Cuculescu’s construction for the increasing adapted sequences, we obtain a distribution function inequality form of John-Nirenberg theorem for noncommutative martingale Lipschitz spaces. Using this result, we obtain the exponential integrability and the sharp moment inequalities, which not only extend but also improve the corresponding results for bmo spaces. As an application, we show Lipschitz space is also the dual space of noncommutative Hardy space defined via symmetric atoms.

1. Introduction

The purpose of this paper is to study the exponential integrability of noncommutative Lipschitz martingales and their atomic decompositions. The paper [23] of Pisier and Xu is the foundation of modern noncommutative martingale theory. The authors introduced the noncommutative setup and established the noncommutative Burkholder-Gundy inequalities and Fefferman-Stein duality theorem between $H^1$ and BMO. Since then, the theory of noncommutative martingales has been rapidly developed. We refer the reader to [11] for noncommutative Doob’s maximal inequality, to [13, 14] for noncommutative Burkholder/Rosenthal inequalities and ergodic theorems, to [20, 22] for the noncommutative Gundy and Davis decompositions, to [24, 25] for noncommutative martingale transform and conditioned square functions, to [8, 9, 10] and the references therein for noncommutative differential subordinates and good-$\lambda$ inequalities. We should also mention another two works directly related with the objectives of this paper, which are the John-Nirenberg theorems for noncommutative martingale BMO spaces by Mei and the first author [7] (see also [3, 15]) and the atomic decomposition for Hardy spaces with small exponents by Chen, Randrianantoanina and Xu [4] (see also [2]).

Before describing our main results, we recall the classical John-Nirenberg inequalities in the martingale theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n \geq 0}$
an increasing sequence of sub-$\sigma$-algebras of $F$ with the associated conditional expectations $(E_n)_{n \geq 0}$. The bmo($\Omega$) space is defined as the set of all $x \in L_1(\Omega)$ with the norm
\[ \|x\|_{\text{bmo}} = \sup_n \| (E_n | x - x_n |^2)^{1/2} \|_\infty < \infty. \]
There are three equivalent forms of the classical John-Nirenberg theorem. The one appearing usually in the application is the exponential integrability: There exist two universal constants $c_1, c_2 > 0$ such that if $\|x\|_{\text{bmo}} < c_2$, then
\[ \sup_n \| E_n (e^{c_1 |x - E_n x|}) \|_\infty < 1. \] (1.1)
The second one is a distribution function inequality: For any $n \geq 1$, $E \in F_n$ and $\lambda > 0$, there exist two universal positive constants $c_1, c_2 > 0$ such that
\[ \frac{1}{P(E)} P(\{ \omega \in E : |x(\omega) - E_n (x(\omega))| > \lambda \}) \leq c_2 e^{-c_1 \lambda \|x\|_{\text{bmo}}}. \] (1.2)
The last one is the $p$-moment inequality with sharp order $c_p \leq O(\sqrt{p})$ as $p \to \infty$: For $2 \leq p < \infty$,
\[ \|x\|_{\text{bmo}} \leq \sup_n \| (E_n | s^2(x) - s_n^2(x)|^{p/2})^{1/p} \|_\infty \leq c_p \|x\|_{\text{bmo}}, \] (1.2)
where $s(x) = (\sum_{k=1}^\infty E_k(x) - E_{k-1}(x))^2)^{1/2}$ and $s_n(x) = (\sum_{k=1}^n E_k(x) - E_{k-1}(x))^2)^{1/2}$.

In the noncommutative setting, after [15], Mei and the first author [7] systematically studied the John-Nirenberg theorem for martingale BMO spaces. They obtained all the equivalent formulations by first establishing the $p$-moment inequalities corresponding to (1.2) via an interpolation argument involving Hardy and BMO spaces. This approach was a surprise since it is quite different from the commutive one, which usually begins with the distribution function inequality (1.1) via a stopping time argument. However, it is well-known that the noncommutative distribution function inequalities are usually very difficult to deal with due to the noncommutativity of operator product and the lack of an efficient analogue of the notion of stopping times (see [10] for an advance on good-$\lambda$ inequalities). Therefore, before the present paper it had been an open problem to have a direct or constructive proof of the noncommutative John-Nirenberg theorem via the distribution function inequality.

Recently, for Hardy space—the predual of BMO space, Chen, Randrianantoanina and Xu [4] provide a constructive proof of the atomic characterization; moreover, their constructive proof can be used to establish the duality between Hardy spaces with small exponents and Lipschitz spaces. These inspire us to further understand the John-Nirenberg type theorems and in particular exponential integrability for martingale Lipschitz spaces. By a moment of thought, the interpolation arguments for BMO spaces in [7] do not work any more for Lipschitz spaces, which prompts us to search for other proofs.

In the present paper, we find a direct proof based on Cuculescu’s construction for the adapted non-decreasing sequences of operators. This approach does not involve anymore the BMO interpolation or the Fefferman-Stein duality, which applies immediately to martingale Lipschitz spaces. Moreover, this seemly easy
variant of Cuculescu’s construction allows us to handle more noncommutative inequalities such as the square function and maximal inequalities, and we will explore it elsewhere.

Let $\mathcal{M}$ be von Neumann algebra and $\tau$ a normal faithful trace with $\tau(1) = 1$. Let $(\mathcal{M}_n)_{n \geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$’s is weak*-dense in $\mathcal{M}$. Let $\mathcal{E}_n$ be the conditional expectation with respect to $\mathcal{M}_n$. The lattice of projections of $\mathcal{M}$ will be denoted by $\mathcal{P}(\mathcal{M})$. Let $\beta \geq 0$. The column Lipschitz space of noncommutative martingales is defined by

$$\Lambda^c_\beta(\mathcal{M}) = \{ x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^c_\beta} = \max\{\|\mathcal{E}_1(x)\|_\infty, \sup_n \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \frac{\|\mathcal{E}_n(x)\|}{\tau(e^{1/2})} \} < \infty\}$$

and more generally, for $0 < p < \infty$ the $p$-moment Lipschitz space is defined by

$$\Lambda^c_{\beta,p}(\mathcal{M}) = \{ x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^c_{\beta,p}} = \max\{\|\mathcal{E}_1(x)\|_\infty, \sup_n \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \frac{\|\mathcal{E}_n(x)\|}{\tau(e^{1/2})} \} < \infty\}.$$

The $l^\infty$-norm of $x$ is defined as $L_\infty$-norm of column square function $s_n(x)$, and is the same as $L_2$-norm of $x$ when $p = 2$. So $\Lambda^c_{\beta,2}(\mathcal{M})$ is just $\Lambda^c_\beta(\mathcal{M})$. On the other hand, when $\beta = 0$, this is just the bmo space. We refer the reader to the preliminary section for more details of bmo spaces, square functions and other notions used here and below.

Let $\beta \geq 0$ and $x \in \Lambda^c_\beta(\mathcal{M})$ with $\mathcal{E}_1(x) = 0$. In this paper, we will firstly show the distribution function inequality: For all $n \geq 1$, $P \in \mathcal{P}(\mathcal{M}_n)$ and $\lambda > 0$, we have

$$\frac{1}{\tau(P)}\tau(1(\lambda, \infty)(P s_n^2(x - x_n) P)) \leq e^{\lambda} \cdot e^{-\frac{\lambda}{\|x\|_{\Lambda^c_\beta}}}.$$

And then we conclude the exponential integrability: We have that for any $0 < a < (\|x\|_{\Lambda^c_\beta})^{-1}$

$$\sup_{n \geq 1} \sup_{P \in \mathcal{P}(\mathcal{M}_n)} \|\mathcal{E}_n(e^{a P s_n^2(x - x_n) P})\|_\infty \leq K_a < \infty,$$

where $K_a = (1 + a e^2 \int_0^\infty e^{(a-1)/\lambda}) d\lambda$. Finally, we obtain the $p$-moment characterization: Let $0 < p < \infty$, there exist two constants $\alpha_p$ and $\beta_p$ such that

$$\alpha_p^{-1}\|x\|_{\Lambda^c_\beta} \leq \|x\|_{\Lambda^c_{\beta,p}} \leq \beta_p\|x\|_{\Lambda^c_\beta}$$

(1.3)

with

$$\alpha_p = 1 \text{ for } 2 \leq p < \infty, \quad \alpha_p \leq C^{\beta+2/p} \text{ for } 0 < p < 2,$$

$$\beta_p \leq c\sqrt{p} \text{ for } 2 \leq p < \infty, \quad \beta_p = 1 \text{ for } 0 < p < 2,$$

where $c$ and $C$ are two universal positive constants.

Note that the characterization (1.3) for $1 \leq p < \infty$ without explicit estimates over $\alpha_p$ and $\beta_p$ has been obtained in [4] as a corollary of their atomic decomposition and duality results. While our direct method gives the sharp order $\beta_p \leq c\sqrt{p}$ for $2 \leq p < \infty$, and moreover by an unusual real interpolation, the scale of $p$ in (1.3) may extend to $0 < p < 1$. Starting with (1.3), we also obtain the moment characterization in terms of symmetric spaces in a more direct way, which is different from the interpolation arguments given in [3].
As an application, we show that the noncommutative Hardy space $h^p_{\mathcal{P},E}(\mathcal{M})$ ($0 < p \leq 1$) defined via symmetric atoms is also a predual space of Lipschitz space $\Lambda^p_{\mathcal{P},-1}(\mathcal{M})$.

2. Preliminaries and notations

2.1. Symmetric Banach function spaces. Let $((0, \infty), \mathcal{F}, \mathbb{P})$ be the Lebesgue measure space and $L_0(0, \infty)$ be the space of all Lebesgue measurable real-valued functions defined on $(0, \infty)$. Let $x \in L_0(0, \infty)$. Recall that the distribution function of $x$ is defined by

$$\lambda_s(x) = \mathbb{P}\{|x| > s\}, \quad s > 0$$

and its non-increasing rearrangement by

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}, \quad t > 0.$$ 

Let $E$ be a quasi-Banach subspace of $L_0(0, \infty)$, simply called a quasi-Banach function space on $(0, \infty)$ in the sequel. $E$ is called symmetric if for any $g \in E$ and any measurable function $f$ with $\mu_t(f) \leq \mu_t(g)$ for all $t \geq 0$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$.

A symmetric quasi-Banach space $E$ on $(0, \infty)$ is said to have the Fatou property if for every net $(x_i)_{i \in I}$ in $E$ satisfying $0 \leq x_i \uparrow$ and $\sup_{i \in I} \|x_i\|_E < \infty$ the supremum $x = \sup_{i \in I} x_i$ exists in $E$ and $\|x_i\|_E \uparrow \|x\|_E$. We say that $E$ has order continuous norm if for every net $(x_i)_{i \in I}$ in $E$ such that $x_i \downarrow 0$ we have $\|x_i\| \downarrow 0$.

The Köthe dual of a symmetric Banach space $E$ on $(0, \infty)$ is given by

$$E^* = \{f \in L_0(0, \infty) : \int_0^\infty |f(t)g(t)|dt < \infty : \forall g \in E\},$$

with the norm $\|f\|_{E^*} := \sup\{\int_0^\infty |f(t)g(t)|dt : \|g\|_E \leq 1\}$. The space $E^*$ is symmetric and has the Fatou property. A symmetric Banach function space $E$ on $(0, \infty)$ has order continuous norm if and only if it is separable, which is also equivalent to the statement $E^* = E^{**}$. We refer to [1, 3, 5] for more details.

For any $s > 0$, we define the dilation operator $D_s$ on $L_0(0, \infty)$ by

$$(D_sf)(t) = f(t/s), \quad t > 0, \quad f \in E.$$ 

For a quasi-Banach function space $E$ on $(0, \infty)$, the lower and upper Boyd indices $p_E$ and $q_E$ of $E$ are respectively defined by

$$p_E := \lim_{s \to \infty} \frac{\log s}{\log \|D_s\|} \quad \text{and} \quad q_E := \lim_{s \to 0^+} \frac{\log s}{\log \|D_s\|}.$$ 

For a symmetric quasi-Banach function space $E$ on $(0, \infty)$, $D_s$ is a bounded linear operator on $E$ for every $s > 0$ and $0 \leq p_E \leq q_E \leq \infty$. If $E$ is a symmetric Banach function space on $(0, \infty)$, then $1 \leq p_E \leq q_E \leq \infty$ and

$$\frac{1}{p_E} + \frac{1}{q_E} = 1, \quad \frac{1}{p_E^*} + \frac{1}{q_E^*} = 1. \quad (2.1)$$

Note that if $E$ is a separable symmetric Banach space on $(0, \infty)$ with $1 < p_E \leq q_E < \infty$, then $E$ automatically have the Fatou property.

Given a quasi-Banach function space $E$ on $(0, \infty)$, for $0 < r < \infty$, $E^{(r)}$ will denote the quasi-Banach function space on $(0, \infty)$ defined by $E^{(r)} = \{x : |x|^r \in E\}$. 


equipped with the quasi-norm $\|x\|_{E(r)} = \|\|x\|^r\|_E^{1/r}$. Note that
\[ p_{E(r)} = rp_E, \quad q_{E(r)} = rq_E. \] (2.2)
and if $0 < p, q < \infty$, then
\[ (E^{(p)})^{(q)} = E^{(pq)}. \] (2.3)
If $E$ is a symmetric Banach function space and $p \geq 1$, then $E^{(p)}$ is a symmetric Banach function space.

Let $E_i$ be a quasi-Banach function space on $(0, \infty)$ for $i = 1, 2$. The pointwise product space $E_1 \odot E_2$ is defined by
\[ E_1 \odot E_2 = \{ f \in L_2(0, \infty) : f = f_1f_2, f_i \in E_i, i = 1, 2 \} \]
with functional $\| \cdot \|_{E_1 \odot E_2}$ being defined by
\[ \|f\|_{E_1 \odot E_2} = \inf\{\|f\|_{E_1}\|f\|_{E_2} : f = f_1f_2, f_i \in E_i, i = 1, 2 \}. \]

We need the following lemmas (see [15, Theorem 1 (iii), Corollary 2] and [16, Theorem 6]).

**Lemma 2.1.** Let $E$ and $F$ be two symmetric Banach function spaces on $(0, \infty)$.
1. If $0 < p < \infty$, then $(E \odot F)^{(p)} = E^{(p)} \odot F^{(p)}$.
2. $L_1(0, \infty) = E \odot E^\times$.
3. If $1 < p < \infty$, then $(E^{(p)})^\times = (E^\times)^{(p')} \odot L_{p'}(0, \infty)$.

**Lemma 2.2.** Let $E$ be a symmetric Banach function space on $(0, \infty)$ which is separable or has the Fatou property and $p_E > p$. Then $E^{(\frac{1}{p})}$ can be renormed as a symmetric Banach function space.

**Proof.** We first consider the case of $0 < p \leq 1$. Since $E$ is symmetric and $\frac{1}{p} \geq 1$, we have that $E^{(\frac{1}{p})}$ is a symmetric Banach function space (see [16, p 53]). For $p > 1$, by [5, Theorem 3.2], we have that $E$ is an interpolation space for the couple $(L_p(0, \infty), L_{\infty}(0, \infty))$. It follows that $E^{(\frac{1}{p})}$ is an interpolation space for the couple $(L_1(0, \infty), L_{\infty}(0, \infty))$ (see [5, Theorem 3.5]). Thus according to Lemma 2.2 in [3], we get that $E^{(\frac{1}{p})}$ can be renormed as a symmetric Banach function space. \hfill \Box

### 2.2. Noncommutative symmetric spaces

Throughout this paper, $\mathcal{M}$ will always denote a von Neumann algebra with a normal faithful normalized trace $\tau$. The unit of $\mathcal{M}$ will be denoted by $1$. For each $0 < p \leq \infty$, let $L_p(\mathcal{M}, \tau)$ or simply $L_p(\mathcal{M})$ be the associated noncommutative $L_p$-spaces. For $x \in L_p(\mathcal{M})$ we denote the right (resp. left) supports of $x$ by $r(x)$ (resp. $l(x)$), which is the least projection $e$ such that $xe = x$ (resp. $ex = x$).

The set of all the $\tau$-measurable operators is denoted by $L_0(\mathcal{M})$. For $x \in L_0(\mathcal{M})$, the distribution function $\lambda(x)$ of $x$ is defined by $\lambda_t(x) = \tau(1_{(t, \infty)}(|x|))$ for $t > 0$, where $1_{(t, \infty)}(|x|)$ is the spectral projection of $|x|$ in the interval $(t, \infty)$, and the rearrangement function $\mu(x)$ of $x$ by $\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}$ for $t > 0$. Given a symmetric quasi-Banach function space $(E, \| \cdot \|_E)$ on $(0, \infty)$, we define the corresponding noncommutative space by setting
\[ E(\mathcal{M}, \tau) = \{ x \in L_0(\mathcal{M}) : \mu_t(x) \in E \} \]
equipped with the quasi-norm
\[ \|x\|_{E(\mathcal{M}, \tau)} := \|\mu_t(x)\|_E. \]
It is well-known that $E(\mathcal{M}, \tau)$ (denoted by $E(\mathcal{M}$ for convenience) is a quasi-Banach space and is referred to as the noncommutative symmetric quasi-Banach space associated with $(\mathcal{M}, \tau)$ corresponding to the function space $(E, \| \cdot \|_E)$. Note that if $0 < p < \infty$ and $E = L_p(0, \infty)$, then $E(\mathcal{M}, \tau) = L_p(\mathcal{M}, \tau)$ is the usual noncommutative $L_p$-space associated with $(\mathcal{M}, \tau)$. We refer to [1, 3] for more details and historical references on these spaces.

For $0 < p < \infty$ and $0 < q \leq \infty$, the noncommutative Lorentz space is defined as

$$L_{p,q}(\mathcal{M}) = \{ x \in L_0(\mathcal{M}) : \| x \|_{p,q} < \infty \}$$

where

$$\| x \|_{p,q} = \begin{cases} \left( \int_0^\infty \left[ \frac{1}{p} \mu_t(x) \right]^q dt \right)^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup_{t \geq 0} t^p \mu_t(x), & q = \infty. \end{cases}$$

Equipped with $\| \cdot \|_{p,q}$, $L_{p,q}(\mathcal{M})$ is a quasi-Banach space which can be renormed to a Banach space when $p > 1$ and $q \geq 1$. We refer to [29] for more details on the noncommutative Lorentz spaces.

In what follows, unless otherwise specified, we always denote by $E$ a symmetric quasi-Banach function space on $(0, \infty)$. For $1 \leq p \leq \infty$, $p'$ denotes the conjugate index of $p$.

### 2.3. Noncommutative martingales

Let $(\mathcal{M}_n)_{n \geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$’s is weak*-dense in $\mathcal{M}$. Then for $n \geq 1$, there exists a unique trace preserving conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$. It is well-known that if $\tau_n$ denotes the restriction of $\tau$ on $\mathcal{M}_n$, then $\mathcal{E}_n$ extends to a contractive projection from $L_p(\mathcal{M}, \tau)$ onto $L_p(\mathcal{M}_n, \tau_n)$ for all $1 \leq p \leq \infty$. More generally, if $E$ is a symmetric Banach function space on $(0, \infty)$ that belongs to $\text{Int}(L_1, L_\infty)$—the interpolation spaces, then for every $n \geq 1$, $\mathcal{E}_n$ is bounded from $E(\mathcal{M}, \tau)$ onto $E(\mathcal{M}_n, \tau_n)$.

A sequence $x = (x_n)_{n \geq 1}$ in $L_1(\mathcal{M}) + \mathcal{M}$ is called a noncommutative martingale with respect to $(\mathcal{M}_n)_{n \geq 1}$ if $\mathcal{E}_n(x_{n+1}) = x_n$ for every $n \geq 1$.

If in addition, all $x_n$’s belong to $E(\mathcal{M})$ then $x$ is called a $E(\mathcal{M})$-martingale. In this case, we set $\| x \|_E = \sup_{n \geq 1} \| x_n \|_E$. If $\| x \|_E < \infty$, then $x$ is called a bounded $E(\mathcal{M})$-martingale.

Let $x = (x_n)_{n \geq 1}$ be a noncommutative martingale with respect to $(\mathcal{M}_n)_{n \geq 1}$. Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the usual convention that $x_0 = 0$. The sequence $dx = (dx_n)$ is called the martingale difference sequence of $x$. In the sequel, for any operator $x \in L_1(\mathcal{M}) + \mathcal{M}$ we denote $x_n = \mathcal{E}_n(x)$ for $n \geq 1$.

The column and row conditioned Hardy spaces $h^c_E(\mathcal{M})$ and $h^r_E(\mathcal{M})$ of noncommutative martingales are respectively defined to be the completions of the space of all finite martingales $x \in E(\mathcal{M}) \cap \mathcal{M}$ under the associated quasi-norms $\| x \|_{h^c_E} = \| s_c(x) \|_E$ and $\| x \|_{h^r_E} = \| s_r(x) \|_E$, where $s_c(x)$ and $s_r(x)$ are the column and row conditioned square functions of $x$, defined by (with the convention that $\mathcal{E}_0 = \mathcal{E}_1$)

$$s_c(x) = \left( \sum_{k \geq 1} \mathcal{E}_{k-1} |dx_k|^2 \right)^{1/2} \quad \text{and} \quad s_r(x) = s_c(x^*).$$

In general, we have no explicit description of elements in $h^c_E(\mathcal{M})$ or $h^r_E(\mathcal{M})$, that is, not all the elements of $h^c_E(\mathcal{M})$ and $h^r_E(\mathcal{M})$ can be represented by a martingale. However, if $E = L_p(0, \infty)$ for $0 < p < \infty$, then $h^c_E(\mathcal{M}) = h^c_p(\mathcal{M})$ and
\[ h_r^\ast(M) = h^\ast_r(M), \] namely the column and row conditioned \( H_p \)-spaces of noncommutative martingales. As remarked in \[ 27, \] if \( E \) is a symmetric Banach function space on \((0, \infty)\) with the Fatou property which is an interpolation space for the couple \((L_p(0, \infty), L_q(0, \infty))\) for some \(1 < p < q < \infty\), then every element of \( h_r^\ast(M) \) and \( h_c^\ast(M) \) can be represented by a martingale.

Define the column bmo space of noncommutative martingales as

\[ \text{bmo}_c^\ast(M) = \{x \in L_1(M) : \|x\|_{\text{bmo}_c^\ast} < \infty\} \]

equipped with the norm

\[ \|x\|_{\text{bmo}_c^\ast} = \max\{\|\mathcal{E}_1(x)\|_\infty, \sup_n \|\mathcal{E}_n|x - x_n|\|^\frac{1}{2}\infty\} \]

and the row bmo space by

\[ \text{bmo}_r^\ast(M) = \{x \in L_1(M) : x^* \in \text{bmo}_c^\ast(M)\} \]
equipped with the norm \( \|x\|_{\text{bmo}_r^\ast} = \|x^*\|_{\text{bmo}_c^\ast} \). We refer to \[ 7, 19 \] for more information on these spaces.

For \( \beta \geq 0 \). Recall that the column Lipschitz space of noncommutative martingales is defined by

\[ \Lambda_c^\beta(M) = \{x \in L_2(M) : \|x\|_{\Lambda_c^\beta} < \infty\} \]
equipped with the norm

\[ \|x\|_{\Lambda_c^\beta} = \max\{\|\mathcal{E}_1(x)\|_\infty, \sup_n \sup_{e \in P(M_n)} \|\frac{(x - x_n)e}{\tau(e)}\|_{\beta + \frac{1}{2}}\} \]
with \( P(M_n) \) denoting the lattice projections of \( M_n \). The row Lipschitz space is defined by

\[ \Lambda_r^\beta(M) = \{x \in L_2(M) : x^* \in \Lambda_r^\beta(M)\} \]
equipped with the norm \( \|x\|_{\Lambda_r^\beta} = \|x^*\|_{\Lambda_r^\beta} \). Note that when \( \beta = 0 \), we recover the bmo spaces \( \text{bmo}_c^\ast(M) \), \( \text{bmo}_r^\ast(M) \).

Let \( 0 < p < \infty \). We define

\[ \Lambda_c^{\beta,p}(M) = \{x \in L_2(M) : \|x\|_{\Lambda_c^{\beta,p}} < \infty\}, \]

where

\[ \|x\|_{\Lambda_c^{\beta,p}} = \max\{\|\mathcal{E}_1(x)\|_\infty, \sup_n \sup_{e \in P(M_n)} \|\frac{(x - x_n)e}{\tau(e)}\|_{\beta + \frac{1}{2}}\} \]

and

\[ \Lambda_r^{\beta,p}(M) = \{x \in L_2(M) : x^* \in \Lambda_r^{\beta,p}(M)\}. \]

Let \( \Lambda_c^{0,p}(M) \) and \( \Lambda_r^{0,p}(M) \) be their subspaces of all \( x \) with \( \mathcal{E}_1(x) = 0 \). Note that \( \Lambda_c^{\beta,2}(M) = \Lambda_c^\beta(M) \) and \( \Lambda_r^{\beta,2}(M) = \Lambda_r^\beta(M) \). When \( \beta = 0 \), we recover the bmo spaces \( \text{bmo}_c^\ast(M) \), \( \text{bmo}_r^\ast(M) \) (see \[ 7 \]).

In the sequel, \( 1_{(\lambda, \infty)}(a) \) denotes the spectral projection of a self-adjoint operator corresponding to the interval \((\lambda, \infty)\), and \( P(M) \) always denotes the lattice projections of \( M \).
3. The John-Nirenberg theorem

3.1. A distribution function inequality and exponential integrability.
In this subsection, we give a constructive proof of the John-Nirenberg type theorem for noncommutative martingale Lipschitz spaces in terms of a distribution function inequality. We begin with a lemma which plays an important role in our construction.

**Lemma 3.1.** Suppose that $x$ and $y$ are two self-adjoint operators in $L_0(\mathcal{M})$ and $x \leq y$. For a fixed number $\lambda \geq 0$, set

$$R_1^\lambda = 1_{(-\infty, \lambda)}(x), \quad R_2^\lambda = R_1^{\lambda} 1_{(-\infty, \lambda)}(R_1^{\lambda} y R_1^{\lambda}).$$

Then we have $R_2^\lambda = 1_{(-\infty, \lambda)}(y)$.

**Proof.** Let $H = L_2(\mathcal{M})$. Since $x \leq y$, we have for any $\xi \in 1_{(-\infty, \lambda)}(y)(H)$,

$$\begin{align*}
\langle x 1_{(-\infty, \lambda)}(y)(\xi), 1_{(-\infty, \lambda)}(y)(\xi) \rangle & \leq \langle y 1_{(-\infty, \lambda)}(y)(\xi), 1_{(-\infty, \lambda)}(y)(\xi) \rangle \\
& \leq \lambda \|\xi\|^2
\end{align*}$$

which implies that $1_{(-\infty, \lambda)}(y) \leq 1_{(-\infty, \lambda)}(x)$. Thus for any $\xi \in 1_{(-\infty, \lambda)}(y)(H)$,

$$\begin{align*}
\langle R_1^{\lambda} y R_1^{\lambda} 1_{(-\infty, \lambda)}(y)(\xi), 1_{(-\infty, \lambda)}(y)(\xi) \rangle & = \langle (y 1_{(-\infty, \lambda)}(y))(\xi), 1_{(-\infty, \lambda)}(y)(\xi) \rangle \\
& \leq \lambda \|\xi\|^2.
\end{align*}$$

It follows that $1_{(-\infty, \lambda)}(y) \leq R_2^\lambda$.

On the other hand, for any $\xi \in R_2^\lambda(H)$, we have

$$\begin{align*}
\langle 1_{(-\infty, \lambda)}(R_1^{\lambda} y R_1^{\lambda}) \cdot y \cdot 1_{(-\infty, \lambda)}(R_1^{\lambda} y R_1^{\lambda})(\xi), \xi \rangle & = \langle 1_{(-\infty, \lambda)}(R_1^{\lambda} y R_1^{\lambda}) \cdot R_1^{\lambda} y R_1^{\lambda} \cdot 1_{(-\infty, \lambda)}(R_1^{\lambda} y R_1^{\lambda})(\xi), \xi \rangle \\
& \leq \lambda \|\xi\|^2
\end{align*}$$

which implies that $R_2^\lambda \leq 1_{(-\infty, \lambda)}(y)$. Thus we have that $R_2^\lambda = 1_{(-\infty, \lambda)}(y)$. \hfill \square

As an application, we obtain the following Cuculescu’s construction for the adapted increasing sequence of operators.

**Proposition 3.2.** Let $s > 0$ and $0 < p < \infty$. Suppose that $(x_n)_{n \geq 1} \subset L_p(\mathcal{M})$ is an adapted increasing sequence of self-adjoint operators, i.e. $x_n \in L_p(\mathcal{M}_p)$. Then there exists a decreasing sequences $(e_n)_{n \geq 0}$ of projections in $\mathcal{M}$ such that for every $n \geq 1$

(i) $e_n = 1_{(-\infty, s)}(x_n)$;
(ii) $e_n$ commutes with $e_{n-1}x_n e_{n-1}$;
(iii) moreover, if $e = \land_{n \geq 1} e_n$, then

$$\forall n \geq 1 \quad e x_n e \leq s e \quad \text{and} \quad \tau(e^{-1}) \leq \sup_n \frac{\|x_n\|^p_{L_p}}{s^p e^p}.$$

**Proof.** We define a sequence of projections $(e_n)_{n \geq 0}$ by putting $e_0 = 1$ and

$$e_n = 1_{(-\infty, s)}(e_{n-1}x_n e_{n-1}), \quad n \geq 1.$$
It follows immediately that \((e_n)_{n \geq 0}\) is decreasing and (ii) holds. On the other hand, by using Lemma 3.1 repeatedly, we have that \(e_n = 1_{(-\infty, x)}(x_n)\), thus we get (i). Finally, by the definition of \(e\), we have that

\[
\tau(1 - e_n) = \tau(1_{[s, +\infty)}(x_n)) \leq \|x_n\|_{p}\e^{-|s|/\beta}.
\]

Letting \(n \to \infty\), we obtain the second inequality of (iii). \(\Box\)

Below is the distribution function inequality form of the John-Nirenberg theorem.

**Theorem 3.3.** Let \(\beta \geq 0\) and \(x \in \Lambda_{\beta}^\infty(\mathcal{M})\) with \(\mathcal{E}_1(x) = 0\). Then for all natural number \(n\), \(P \in \mathcal{P}(\mathcal{M}_n)\) and \(\lambda > 0\), we have

\[
\frac{1}{\tau(\lambda)} \tau(\lambda, \infty)(\frac{P s_n^2(x - x_n)P}{\tau(P)}} \leq e^2 \cdot e^{-\frac{\lambda}{\|x\|_{p}\beta}}. \tag{3.1}
\]

**Proof.** Fix \(n \in \mathbb{N}\), \(P \in \mathcal{P}(\mathcal{M}_n)\) and \(\lambda > 0\). Set \(y_{n-1} = 0\) and

\[y_m = \frac{P s_n^2(x_m - x_n)P}{\tau(P)}, \quad m \geq n.\]

Consider the sequence \(R^\lambda_N = (R^\lambda_m)_{m \geq n-1}\) of projections associated with \((y_m)_{m \geq n-1}\), given by \(R^\lambda_{n-1} = P\) and, inductively,

\[R^\lambda_m = R^\lambda_{m-1}1_{[0, \lambda)}(R^\lambda_{m-1}y_mR^\lambda_{m-1}), \quad m \geq n.\]

Then noting that \((y_m)_{m \geq n-1}\) is increasing, by Proposition 3.2, we have that \(R^\lambda_m = 1_{(0, \lambda)}(y_m)\). Let \(N \geq n\) and \(\mu \geq 0\). By the trivial inequality \(R^\lambda_N \leq R^\lambda_{N+\mu}\) and the properties of the projection \(R^\lambda_{N+\mu}\),

\[
\tau(P - R^\lambda_N) = \tau((P - R^\lambda_{N+\mu})(P - R^\lambda_N)) \leq \tau((P - R^\lambda_N)(P - R^\lambda_{N+\mu})(\frac{y_N - \lambda P}{\mu})(P - R^\lambda_{N+\mu})) \leq \frac{1}{\mu} \sum_{k=n}^{N} \tau((R^\lambda_{k-1} - R^\lambda_k)(y_N - y_{k-1} + y_{k-1} - \lambda P)).
\]

Using the inequality \(R^\lambda_{k-1}y_{k-1}R^\lambda_{k-1} \leq \lambda PR^\lambda_{k-1}\), we further get that

\[
\tau(P - R^\lambda_N) \leq \frac{1}{\mu} \sum_{k=n}^{N} \tau((R^\lambda_{k-1} - R^\lambda_k)(y_N - y_{k-1})) \leq \frac{1}{\mu} \sum_{k=n}^{N} \tau(R^\lambda_{k-1} - R^\lambda_k) \cdot \|\epsilon_{k-1}(y_N - y_{k-1})\|_{\infty}.
\]
By duality, we can write
\[
\|\mathcal{E}_{k-1}(y_N - y_{k-1})\|_\infty = \sup_{\|z\|_1 \leq 1, \ z \in L_1^1(P_M k^{-1}P)} \tau(z(y_N - y_{k-1}))
\]
\[
\leq \sup_{\|z\|_1 \leq 1, \ z \in L_1^1(P_M k^{-1}P)} \frac{\tau(z P s^2(x - x_{k-1})P)}{\tau(P)^{2\beta}}
\]
\[
= \sup_{P' \in P(P_M k^{-1}P)} \frac{1}{\tau(P')} \frac{\tau(P' P s^2(x - x_{k-1})P')}{\tau(P)^{2\beta}},
\]
where the last equality comes from the density of linear combinations of mutually disjoint projections in \(L_1(P_M k^{-1}P)\). Adding the above two simple observations, we get the estimate
\[
\tau(P - R_{N}^{\lambda + \mu}) \leq \frac{1}{\mu} \tau(P - R_{N}^{\lambda}) \sup_{k \geq n} \sup_{P' \in P(P_M k^{-1}P)} \frac{\|x - x_{k-1})PP\|_2^2}{\tau(P')\tau(P)^{2\beta}}.
\]
Noting that \(P' \leq P\), we have that
\[
\tau(P - R_{N}^{\lambda + \mu}) \leq \frac{1}{\mu} \tau(P - R_{N}^{\lambda}) \|x\|_\lambda^2.
\]
Now suppose \(\|x\|_\lambda = 1\). Let \(\mu = e\) and \(\lambda = k_0 e, k_0 \in \mathbb{N}\). Then
\[
\tau(P - R_{N}^{(\lambda + k_0)e}) \leq \frac{1}{e} \tau(P - R_{N}^{k_0 e}) \leq e^{-k_0} \tau(P - R_{N}^{e}) \leq e^{-k_0} \tau(P).
\]
Given an arbitrarily fixed \(\lambda > 0\), choosing a nonnegative integer \(k\) such that \(ke \leq \lambda < (k + 1)e\), we get that
\[
\tau(P - R_{N}^{\lambda}) \leq \tau(P - R_{N}^{ke}) \leq e^{-(k-1)} \cdot \tau(P) \leq e^2 \cdot e^{-k} \tau(P).
\]
For the general case, we set \(\bar{x} = \frac{x}{\|x\|_\lambda} e^{-\frac{\lambda}{\|x\|_\lambda}}\). It follows that
\[
\tau(P - R_{N}^{\lambda}) \leq e^2 \cdot e^{-\frac{\lambda}{\|x\|_\lambda}} \cdot \tau(P).
\]
Thus we obtain that
\[
\lim_{N \to \infty} \tau(P - R_{N}^{\lambda}) \leq e^2 \cdot e^{-\frac{\lambda}{\|x\|_\lambda}} \cdot \tau(P),
\]
giving the desired result. \(\square\)

With the help of Theorem 3.3, we obtain the following John-Nirenberg theorem in terms of exponential integrability.

**Theorem 3.4.** Let \(\beta > 0\) and \(x \in \Lambda_\beta(M)\) with \(E_1(x) = 0\). Then for any \(0 < a < (\|x\|_\lambda^\beta)^{-1}\)
\[
\sup_{n \geq 1} \sup_{P \in P(M_n)} \|E_n(e^{\frac{a P s^2(x - x_n)P}{\tau(P)^{2\beta}}})\|_\infty \leq K_a < \infty,
\]
where \(K_a = (1 + ae^2 \int_0^\infty e^{(a-1/s)}ds)\).
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PROOF. It suffices to consider the case of $\|x\|_{\Lambda_\beta} = 1$ and $0 < a < e^{-1}$. Fix $n$ and $P \in \mathcal{P}(\mathcal{M}_n)$. By duality, we have

$$\|E_n(e^{\frac{ap_s^2(x-x_n)}{\tau(P)^{2\beta}}})\|_\infty = \sup_{P' \in \mathcal{P}(\mathcal{P}_{\mathcal{M}_n}P)} \frac{1}{\tau(P')} \tau(e^{\frac{aP'_s^2(x-x_n)}{\tau(P')^{2\beta}}} P')$$

$$= \sup_{P' \in \mathcal{P}(\mathcal{P}_{\mathcal{M}_n}P)} \frac{1}{\tau(P')} \tau(e^{\frac{aP'_s^2(x-x_n)}{\tau(P')^{2\beta}}} P')$$

$$\leq \sup_{P' \in \mathcal{P}(\mathcal{P}_{\mathcal{M}_n}P)} \frac{1}{\tau(P')} \tau(e^{\frac{aP'_s^2(x-x_n)}{\tau(P')^{2\beta}}} P').$$

Set $\Phi(s) = e^{as} - 1$. Then by the equality $\tau(\Phi(|x|)) = \int_0^\infty \lambda_s(|x|) d\Phi(s)$ and Theorem 3.3, we have that for any $P' \in \mathcal{P}(\mathcal{P}_{\mathcal{M}_n}P)$

$$\tau(e^{\frac{aP'_s^2(x-x_n)}{\tau(P')^{2\beta}}} P') = \int_0^\infty \lambda_s(P'_s(x-x_n)) \frac{P'}{\tau(P')^{2\beta}} \int_0^\infty \lambda_s(x) \frac{P'}{\tau(P')^{2\beta}} ds$$

$$\leq a \int_0^\infty e^{as} \cdot \tau(P') e^{as} ds$$

$$= \tau(P') \int_0^\infty e^{(a-\frac{1}{2})s} ds.$$

Putting the preceding results together, we obtain that

$$\|E_n(e^{\frac{ap_s^2(x-x_n)}{\tau(P)^{2\beta}}})\|_\infty \leq \sup_{P' \in \mathcal{P}(\mathcal{P}_{\mathcal{M}_n}P)} \frac{1}{\tau(P')} \left( \tau(e^{\frac{aP'_s^2(x-x_n)}{\tau(P')^{2\beta}}} P') + \tau(P') \right)$$

$$\leq 1 + ae^2 \int_0^\infty e^{(a-\frac{1}{2})s} ds$$

$$\leq K_a.$$

Thus we obtain the desired inequality. \hfill \square

3.2. The $p$-moment characterization. Next we consider the John-Nirenberg inequality in terms of $p$-moment inequalities.

THEOREM 3.5. Let $\beta \geq 0$, $x \in \Lambda_\beta(M)$ and $0 < p < \infty$. Then there exist two constants $\alpha_p$ and $\beta_p$ such that

$$\alpha_p^{-1} \|x\|_{\Lambda_\beta} \leq \|x\|_{\Lambda_\beta^p} \leq \beta_p \|x\|_{\Lambda_\beta^p}$$

(3.2)

with

(i) $\alpha_p = 1$ for $2 \leq p < \infty$, (ii) $\alpha_p \leq C^{\beta+2/p}$ for $0 < p < 2$,

(iii) $\beta_p \leq c\sqrt{p}$ for $2 \leq p < \infty$, (iv) $\beta_p = 1$ for $0 < p < 2$,

where $c$ and $C$ are two universal positive constants. The same inequalities hold for $\|\cdot\|_{\Lambda_\beta^p}$ and $\|\cdot\|_{\Lambda_\beta^p}$.

REMARK 3.6. The order $\beta_p \leq c\sqrt{p}$ as $p \to \infty$ improves the previous ones $c_p$ in [7] or [4]; and moreover it is sharp, which can be seen from the commutative result (see Theorem 2.50 in [31]).

We need the following lemmas: when $\mathcal{M}$ is commutative, one can find them for instance in [6, 21]; the noncommutative versions can be deduced in a similar way, see e.g. the book manuscript of Xu [30].
Lemma 3.7. Let $0 < p_k, q_k \leq \infty$ ($k = 0, 1$) and $p_0 \neq p_1, 0 < \theta < 1, 0 < q \leq \infty$. Then
\[
(L_{p_0, q_0}(\mathcal{M}), L_{p_1, q_1}(\mathcal{M}))_{\theta, q} = L_{p, q}(\mathcal{M})
\]
with equivalent quasi-norms, where $1/p = (1 - \theta)/p_0 + \theta/p_1$. More precisely,
\[
eq c^{\theta - \min\{1/q, 1/q_0\}} (1 - \theta)^{- \min\{1/q, 1/q_1\}} \|x\|_{L_{p, q}(\mathcal{M})}
\]
where $c$ is a positive numerical constant.

Lemma 3.8. Let $0 < p_k, q_k, r_k, s_k \leq \infty$ ($k = 0, 1$) and $p_0 \neq p_1, q_0 \neq q_1, 0 < \theta < 1$. Suppose that
\[
T : L_{p_k, r_k}(\mathcal{M}) \to L_{q_k, s_k}(\mathcal{M}), \text{ with the norm } M_k, \ k = 0, 1.
\]
Put $1/p = (1 - \theta)/p_0 + \theta/p_1, 1/q = (1 - \theta)/q_0 + \theta/q_1$. Then for $0 < r \leq \infty$,
\[
T : L_{p, r}(\mathcal{M}) \to L_{q, r}(\mathcal{M}) \text{ with the norm } M \leq c^{\theta (1 - \theta)^\alpha} M_0^{\frac{\alpha}{r}} M_1^{\theta}
\]
where $c$ is a positive numerical constant, $\alpha_i = \min\{1/r, 1/s_i\} - \max\{1/r, 1/r_i\}, \ i = 0, 1.$

Proof of Theorem 3.5. We only need to prove the column case, since the row case can be done by replacing $x$ with $x^*$. First we show the inequality on the right hand side. Fix $n$ and $P \in \mathcal{P}(\mathcal{M}_n)$. For $2 \leq p < \infty$, by Theorem 3.3, we have that
\[
(\tau(P))^{-\beta - \frac{1}{p}} \|x - x_n\|_{h_\rho} \leq (\tau(P))^{-\frac{1}{p}} \left( \frac{p}{2} \int_0^\infty s^{\frac{p}{2} - 1} \cdot \lambda_s \left( \frac{P s^2 (x - x_n) P}{(\tau(P))^{2\beta}} \right) ds \right)^\frac{1}{p}
\]
\[
\leq \left( \frac{p}{2} \right)^{\frac{1}{p}} \left( \frac{p}{2} \right)^{1 + \frac{1}{2p}} \Gamma\left( \frac{p}{2} \right)^{\frac{1}{p}} \|x\|_{\Lambda^p_\rho}
\]
which implies that $\|x\|_{\Lambda^p_\theta, \rho} \leq e^{1/2 + 1/p} \left( (p/2) \Gamma(p/2) \right)^{1/p} \|x\|_{\Lambda^p_\rho}$. Noting that
\[
\lim_{p \to \infty} \frac{e^{\frac{1}{2p} + \frac{1}{2p}} \left( \frac{p}{2} \right)^{\frac{1}{p}} \Gamma(p/2)^{1/p}}{\left( \frac{p}{2} \right)^{\frac{1}{p}}} = \frac{\sqrt{2}}{2},
\]
and thus there exists a constant $c$ such that $\beta_p \leq c\sqrt{p}$ for $p \geq 2$. For $0 < p < 2$. Set $2/p = 1/q + 1$. By Hölder’s inequality, one obtains
\[
\|x - x_n\|_{h_\rho} \leq (\tau(P))^{-\frac{1}{q}} \|x - x_n\|_{h_\rho}.
\]
Thus we have
\[
(\tau(P))^{-\beta - \frac{1}{p}} \|x - x_n\|_{h_\rho} \leq (\tau(P))^{-\beta - \frac{1}{q}} \|x - x_n\|_{h_\rho}
\]
which implies that
\[
\|x\|_{\Lambda^p_{\theta, \rho}} \leq \|x\|_{\Lambda^p_{\theta, 2}} = \|x\|_{\Lambda^p_\rho}.
\]
We turn to the inequality on the left hand side. First we consider the case of $2 \leq p < \infty$. Choose $q$ such that $1 = 2/p + 1/q$. Fix $n$. By Hölder’s inequality, one gets

$$\sup_{e \in \mathcal{P}(\mathcal{M}_n)} \frac{\| (x - x_n) e \|_2^2}{(\tau(e))^{1+2\beta}} = \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \frac{\tau(e) s_n^2(x - x_n) e}{(\tau(e))^{1+2\beta}} \leq \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \frac{1}{(\tau(e))^{1+2\beta}} (\| e \|_q \cdot \| e s_n^2(x - x_n) e \|_2) = \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \| (x - x_n) e \|_2^2 \left( \frac{1}{(\tau(e))^{1+2\beta}} \right)^{\frac{1}{p}}.$$ 

Thus we have $\| x \|_{A_{\beta,p}} \leq \| x \|_{A_{\beta,p}}$.

Now we consider the case of $0 < p < 2$. Set

$$q_0 = \begin{cases} \frac{p}{p\beta + 1} & \text{if } \frac{p}{p\beta + 1} \leq 1; \\ 1 & \text{if } \frac{p}{p\beta + 1} > 1. \end{cases}$$

For each $n$, view $x - x_n$ as a left multiplication operator from $L_{p\beta+q_0}(\mathcal{M})$ to $h_p^c(\mathcal{M})$. We claim the following assertion:

$$\sup_n \| x - x_n : L_{p\beta+q_0}(\mathcal{M}) \to h_p^c(\mathcal{M}) \| = \begin{cases} \| x \|_{A_{\beta,p}} & \text{if } \frac{p}{p\beta + 1} \leq 1, \\ (\beta + \frac{1}{p}) \| x \|_{A_{\beta,p}} & \text{if } \frac{p}{p\beta + 1} > 1. \end{cases} \quad (3.3)$$

The subcase $p/(p\beta + 1) > 1$. One has $p > 1$ and $L_{p\beta+q_0} = L_{p\beta+1}$. By approximation, for any $a \in \mathcal{M}_n$ with $\| a \|_{L_{p\beta+1}} \leq 1$, there exists a sequence $(\beta + 1/p) \sum k \geq 1 \lambda_k \frac{e_k}{(\tau(e_k))^{\beta+1/p}}$ with $e_k$’s in $\mathcal{M}_n$ and $\sum k \geq 1 |\lambda_k| \leq 1$ such that $(\beta + 1/p) \sum k \geq 1 \lambda_k \frac{e_k}{(\tau(e_k))^{\beta+1/p}}$ converges to $a$ in $L_{p\beta+1}(\mathcal{M})$. Thus by the property of the norm $\cdot \| h_p^c$, we have that

$$\| (x - x_n) a \|_{h_p^c} = \| (\beta + \frac{1}{p}) \sum k \geq 1 |\lambda_k| \frac{(x - x_n) e_k}{(\tau(e_k))^{\beta+1/p}} \|_{h_p^c} \leq (\beta + \frac{1}{p}) \sum k \geq 1 |\lambda_k| \| (x - x_n) e_k \|_{h_p^c} \leq (\beta + \frac{1}{p}) \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \| (x - x_n) e \|_{h_p^c},$$

which implies that

$$\| x - x_n : L_{p\beta+1}(\mathcal{M}) \to h_p^c(\mathcal{M}) \| \leq (\beta + \frac{1}{p}) \| x \|_{A_{\beta,p}}.$$

Conversely, for any $e \in \mathcal{M}_n$, nothing that

$$\| (\beta + \frac{1}{p}) \frac{e}{(\tau(e))^{\beta+1/p}} \|_{L_{p\beta+1}} = 1,$$

we get that

$$\sup_{e \in \mathcal{P}(\mathcal{M}_n)} \| (\beta + \frac{1}{p}) \frac{(x - x_n) e}{(\tau(e))^{\beta+1/p}} \|_{h_p^c} \leq \| x - x_n : L_{p\beta+1}(\mathcal{M}) \to h_p^c(\mathcal{M}) \|. $$
It follows that

\[
(\beta + \frac{1}{p})\|x\|_{\Lambda_{\beta,p}} \leq \sup_n \|x - x_n : L_{\frac{p}{p+n+\theta}}(\mathcal{M}) \to h^c_p(\mathcal{M})\|.
\]

In the subcase \(p/(p\beta + 1) \leq 1\), one has \(L_{\frac{p}{p+n+\theta},\psi}(\mathcal{M}) = L_{\frac{p}{p+n+\theta}}(\mathcal{M})\). Observe that all \(a \in \mathcal{M}\) with \(\|a\|_{\frac{p}{p+n+\theta}} \leq 1\) can be approximated by the sum \(\sum_{k \geq 1} \lambda_k (\tau(e_k))^{\frac{\beta}{p}+\frac{1}{p+n+\theta}}\) in \(L_{\frac{p}{p+n+\theta}}(\mathcal{M})\), where \(e_k\)'s are in \(\mathcal{M}\) and \(\sum_{k \geq 1} |\lambda_k|^{p/(p\beta + 1)} \leq 1\). Thus if \(p < 1\), by the property of the quasi-norm \(\| \cdot \|_{h^c_p}\) and nothing that \(|\lambda_k| \leq 1\) for all \(k\), \(p/(\beta + 1) \leq 1\), we have that

\[
\|(x - x_n)a\|_{h^c_p} \leq \sum_{k \geq 1} |\lambda_k|^{\frac{p}{p+n+\theta}} \left\| \frac{(x - x_n)e_k}{(\tau(e_k))^{\frac{\beta}{p}+\frac{1}{p+n+\theta}}} \right\|_{h^c_p} \leq \sum_{k \geq 1} |\lambda_k|^{\frac{p}{p+n+\theta}} \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \left\| \frac{(x - x_n)e}{(\tau(e))^{\frac{\beta}{p}+\frac{1}{p}}} \right\|_{h^c_p} \leq \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \left\| \frac{(x - x_n)e}{(\tau(e))^{\frac{\beta}{p}+\frac{1}{p}}} \right\|_{h^c_p}.
\]

Next if \(p \geq 1\), nothing that \(|\lambda_k| \leq 1\) for all \(k\) and \(p/(\beta + 1) \leq 1\),

\[
\|(x - x_n)a\|_{h^c_p} \leq \sum_{k \geq 1} |\lambda_k|^{\frac{p}{p+n+\theta}} \left\| \frac{(x - x_n)e_k}{(\tau(e_k))^{\frac{\beta}{p}+\frac{1}{p+n+\theta}}} \right\|_{h^c_p} \leq \sum_{k \geq 1} |\lambda_k|^{\frac{p}{p+n+\theta}} \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \left\| \frac{(x - x_n)e}{(\tau(e))^{\frac{\beta}{p}+\frac{1}{p}}} \right\|_{h^c_p} \leq \sup_{e \in \mathcal{P}(\mathcal{M}_n)} \left\| \frac{(x - x_n)e}{(\tau(e))^{\frac{\beta}{p}+\frac{1}{p}}} \right\|_{h^c_p}.
\]

Putting the above two inequalities together, we get that

\[
\sup_n \|x - x_n : L_{\frac{p}{p+n+\theta}}(\mathcal{M}) \to h^c_p(\mathcal{M})\| \leq \|x\|_{\Lambda_{\beta,p}}.
\]

The converse inequality is true because of the fact that for \(e \in \mathcal{M}\) the \(L_{p/(\beta + 1)}\)-norm of \(e/(\tau(e))^{\beta + 1/p}\) equals 1. Thus we conclude the claim (3.3).

Now choose \(0 < \theta < 1\) and \(p_1 > 2\) such that \(\theta/p + (1 - \theta)/p_1 = 1/2\). Set

\[
q_1 = \begin{cases} \frac{p_1}{p_1 + 1} & \text{if } \frac{p_1}{p_1 + 1} \leq 1, \\ 1 & \text{if } \frac{p_1}{p_1 + 1} > 1 \end{cases}
\]

and

\[
q = \begin{cases} 2 & \text{if } 2 \leq 2 \leq 1, \\ 1 & \text{if } 2 > 1. \end{cases}
\]

Then by a similar argument of the proof of the claim (3.3), we have that

\[
\sup_n \|x - x_n : L_{\frac{p_1}{p_1 + 1}}(\mathcal{M}) \to h^c_{p_1}(\mathcal{M})\| = \left\{ \begin{array}{ll} \|x\|_{\Lambda_{\beta,p_1}} & \text{if } \frac{p_1}{p_1 + 1} \leq 1, \\ (\beta + \frac{1}{p_1})\|x\|_{\Lambda_{\beta,p_1}} & \text{if } \frac{p_1}{p_1 + 1} > 1. \end{array} \right. \tag{3.4}
\]
and
\[ \sup_n \| x - x_n : L_{2^{\frac{1}{p} + q}}(\mathcal{M}) \rightarrow h^c_\beta(\mathcal{M}) \| = \begin{cases} \| x \|_{A^\beta_\beta} & \text{if } \frac{2}{2^{\beta + 1}} \leq 1, \\ (\beta + \frac{1}{\beta}) \| x \|_{A^\beta_\beta} & \text{if } \frac{2}{2^{\beta + 1}} > 1. \end{cases} \] (3.5)

Since \( h^c_{p,q}(\mathcal{M}) \) embeds into the column \( L_{p,q} \) space with constant 1, \( (h^c_{p}(\mathcal{M}), h^c_{p,1}(\mathcal{M}))_{\theta,q} \) embeds into \( h^c_{2,q}(\mathcal{M}) \) with the same constant as in the inclusion
\[ (L_p(\mathcal{M}), L_{p,1}(\mathcal{M}))_{\theta,q} \subset L_{2,q}(\mathcal{M}) \]
which in turn is provided explicitly in Lemma 3.7. Set
\[ C(p, \beta) = \begin{cases} 1 & \text{if } \frac{p}{2^{\beta + 1}} \leq 1, \\ \beta + \frac{1}{\beta} & \text{if } \frac{p}{2^{\beta + 1}} > 1. \end{cases} \]
Together (3.3),(3.4),(3.5) and by Lemma 3.8, we get that
\[ \| x \|_{A^\beta_\beta} = C^{-1}(2, \beta) \sup_n \| x - x_n : L_{2^{\frac{1}{p} + q}}(\mathcal{M}) \rightarrow h^c_\beta(\mathcal{M}) \| \]
\[ \leq C^{-1}(2, \beta) \sup_n \| x - x_n : L_{p, \beta}^{\frac{1}{p}}(\mathcal{M}) \rightarrow h^c_{2,q}(\mathcal{M}) \| \]
\[ \leq C^{-1}(2, \beta) c_\theta \min\{1/q, 1/p\} - \max\{1/q, 1/p_1\} (1 - \theta) \min\{1/q, 1/p_1\} - \max\{1/q, 1/q_1\} \]
\[ \sup_n \| x - x_n : L_{p, \beta}^{\frac{1}{p}}(\mathcal{M}) \rightarrow h^c_{p,q}(\mathcal{M}) \|^{1-\theta} \sup_n \| x - x_n : L_{p, \beta}^{\frac{1}{p}}(\mathcal{M}) \rightarrow h^c_{p}(\mathcal{M}) \|^{\theta} \]
\[ \leq c(\theta(1 - \theta))^{-\beta + \frac{2}{\theta}} \| x \|_{A^\beta_{\beta, p}}^{1-\theta} \| x \|_{A^\beta_{\beta, p}}^{\theta}. \]

Since \( x \in A^\beta_\beta(\mathcal{M}) \), by the inequality on the right hand side in the case of \( 2 \leq p < \infty \), we deduce that
\[ \| x \|_{A^\beta_\beta} \leq c(\theta(1 - \theta))^{-\beta + \frac{2}{\theta}} \| x \|_{A^\beta_{\beta, p}}^{1-\theta} \| x \|_{A^\beta_{\beta, p}}^{\theta}. \]
It follows that
\[ \| x \|_{A^\beta_\beta} \leq c(\theta(1 - \theta))^{-\beta + \frac{2}{\theta}} \| x \|_{A^\beta_{\beta, p}}^{1-\theta} \| x \|_{A^\beta_{\beta, p}}^{\theta}. \]

We get the desired estimate by taking \( C = c^{1/(\theta(1 - \theta))(\theta(1 - \theta))^{-1/\theta}} \).

Remark 3.9. (i) Theorem 3.5 (or its proof) actually tells us that the Lipschitz space \( L_\beta(\mathcal{M}) \) coincides with \( L_{\beta, p}(\mathcal{M}) \) for any \( 2 \leq p < \infty \). While for \( 0 < p < 2 \), if a priori we assume that \( x \in L_\beta(\mathcal{M}) \), then the norms are equivalent; so it would be interesting to show a distribution inequality as (3.1) starting with \( x \in L_{\beta, p}(\mathcal{M}) \) with \( L_1(x) = 0 \).

(ii) Another closely related question is whether there exists a direct approach to the John-Nirenberg theorem for mixed BMO or Lipschitz spaces via the distribution function inequality; an indirect way has been provided in [7, Theorem 3.20].

4. Symmetric space moment characterization and atomic decomposition

In this section, we show that the noncommutative Hardy space \( h^c_{p, E}(\mathcal{M}) \) \( (0 < p \leq 1) \) defined via symmetric atoms is also a predual space of Lipschitz space \( A^c_{\beta, \infty} \). For this purpose, we first provide the symmetric space moment characterization of the noncommutative Lipschitz spaces.
4.1. Symmetric space moment inequality. We first introduce the symmetric Lipschitz spaces $\Lambda^{\beta}_{\beta,E}(\mathcal{M})$ and $\Lambda^{0}_{\beta,E}(\mathcal{M})$.

**Definition 4.1.** Let $\beta \geq 0$. Let $E$ be a separable symmetric Banach function space on $(0, \infty)$ with $1 < p_E \leq q_E < \infty$. We define

$$\Lambda^{\beta}_{\beta,E}(\mathcal{M}) = \{ x \in L_2(\mathcal{M}) : \|x\|_{\Lambda^{\beta}_{\beta,E}} < \infty \},$$

where

$$\|x\|_{\Lambda^{\beta}_{\beta,E}} = \max\{\|\mathcal{E}_1(x)\|_\infty, \sup_n \sup_{e \in \mathcal{P}(\mathcal{M}_n)} (\tau(e))^{-\beta} \|(x - x_n)\|_E \frac{e}{\|e\|_E} \}$$

and

$$\Lambda^{0}_{\beta,E}(\mathcal{M}) = \{ x \in L_2(\mathcal{M}) : x^* \in \Lambda^{\beta}_{\beta,E}(\mathcal{M}) \}.$$

Let $\Lambda^{0,c}_{\beta,E}(\mathcal{M})$ and $\Lambda^{0,r}_{\beta,E}(\mathcal{M})$ be their subspaces of all $x$ with $\mathcal{E}_1(x) = 0$.

**Theorem 4.2.** Let $\beta \geq 0$ and $x \in \Lambda^{\beta}_{\beta}(\mathcal{M})$. Let $E$ be a separable symmetric Banach function space on $(0, \infty)$ with $1 < p_E \leq q_E < \infty$. Then we have that

$$\alpha_E^{-1} \|x\|_{\Lambda^{\beta}_{\beta}} \leq \|x\|_{\Lambda^{\beta}_{\beta,E}} \leq \beta_E \|x\|_{\Lambda^{\beta}_{\beta}}.$$

The constants $\alpha_E$ and $\beta_E$ depend on $E$. The same inequalities hold for $\| \cdot \|_{\Lambda^{\beta}_{\beta}}$ and $\| \cdot \|_{\Lambda^{0}_{\beta,E}}$.

**Lemma 4.3.** Let $E$ be a separable symmetric Banach function space on $(0, \infty)$ with $1 < p_E \leq q_E < \infty$. Choose $p$ such that $q_E < p$. Then we have that $E = F(p') \odot L_p(0, \infty)$, where $F = (E^\infty(\frac{1}{p'}))^\infty$.

**Proof.** By (2.1), it follows that $1 < p' < p_E' \leq q_E' < \infty$. Since $E^\infty$ is symmetric and has the Fatou property, by Lemma 2.2, $E^\infty(\frac{1}{p'})$ can be renormed as a symmetric Banach function space. Then by (2.3) and (3) of Lemma 2.1, we have that

$$E = (E^\infty)\infty = ([E^\infty(\frac{1}{p'})]^{(p')}\infty = ([E^\infty(\frac{1}{p'})]^{(p')}\odot L_p(0, \infty) = F(p') \odot L_p(0, \infty).$$

□

We will use repeatedly the following fact which follows from Theorem 2 of [18].

**Lemma 4.4.** Let $E$ and $F$ be two symmetric Banach function spaces on $(0, \infty)$. Then $E \odot F$ is a symmetric quasi-Banach function space on $(0, \infty)$ and the following formula holds:

$$\|\chi_{[0,t]}\|_{E \odot F} = \|\chi_{[0,t]}\|_E \|\chi_{[0,t]}\|_F \text{ for } t \in (0, \infty).$$

**Proof of Theorem 4.2.** We may assume $\mathcal{E}_1(x) = 0$. Let $F$, $p$, and $p'$ be as in Lemma 4.3. Fix $n$ and $P \in \mathcal{P}(\mathcal{M}_n)$. By definition $\mu(P) = \chi_{[0,\tau(P)]}(t)$, and thus by Lemma 4.3 and Lemma 4.4, we have

$$\|P\|_E = \|P\|_{F(p')} \|P\|_p.$$
Therefore, by the equality $E^{\frac{1}{p}} = F^{(\frac{1}{p})} \odot L^2_2(0, \infty)$, we have that
\[
(\tau(P))^{-\beta}(x - x_n) = (\tau(P))^{-\beta} \frac{P}{\|P\|_E} \frac{P}{\|P\|_E} s^2_c(x - x_n) \frac{P}{\|P\|_E} \frac{1}{(\tau(P))^{\frac{1}{p}}} \leq (\tau(P))^{-\beta} \frac{P}{\|P\|_E} \frac{P}{\|P\|_E} s^2_c(x - x_n) P \frac{1}{(\tau(P))^{\frac{1}{p}}} = (\tau(P))^{-\beta} \frac{1}{(\tau(P))^{\frac{1}{p}}} \leq \|x\|_{\beta,E} \leq \beta_E \|x\|_{\beta,E}
\]
which implies that $\|x\|_{\beta,E} \leq \|x\|_{\beta,p}$. Then by Theorem 3.5, we have that $\|x\|_{\beta,E} \leq \beta_E \|x\|_{\beta,E}$. Conversely, choose $p$ such that $p < p_E$. Then by Lemma 2.2, $E^{(\frac{1}{p})}$ can be renormed as a symmetric Banach function space. Thus by (2) of Lemma 2.1, we have that $L_2(0, \infty) = E^{(\frac{1}{p})} \odot F^{(\frac{1}{q})}$. It follows that $L_2(0, \infty) = E \odot F$ by (1) of Lemma 2.1, where $F = (E^{(\frac{1}{p})})^{(\frac{1}{q})}$. Hence, for all $e \in P(\mathcal{M}_n)$, by Lemma 4.4, we have
\[
(\tau(e))^{\frac{1}{p}} = \|e\|_E \|e\|_F.
\]
Using the equality $L_2(0, \infty) = E^{(\frac{1}{p})} \odot F^{(\frac{1}{q})}$, one has
\[
\|x - x_n\|_E = \|x - x_n\|_E \leq \|e\|_E \|e\|_F \|e\|_F \|e\|_F \|e\|_F \|e\|_F \|e\|_F \|e\|_F = \|x - x_n\|_E \|e\|_E \|e\|_E \|e\|_E \|e\|_E \|e\|_E \|e\|_E \|e\|_E
\]
which implies $\|x\|_{\beta,E} \leq \|x\|_{\beta,E}$. Therefore, by Theorem 3.5, we get the desired result
\[
\|x\|_{\beta,E} \leq \beta_E \|x\|_{\beta,E}.
\]

**Remark 4.5.** When $0 < p_E \leq q_E \leq 1$, it is easy to obtain the left inequality from the proof of Theorem 4.2.

### 4.2. The atomic Hardy space $h_{p,E}(\mathcal{M})$ for $0 < p \leq 1$

**Definition 4.6.** Let $0 < p \leq 1$. Let $E$ be a separable symmetric Banach function space on $(0, \infty)$ and $1 < p_E \leq q_E < \infty$. An element $a \in L^p(\mathcal{M})$ is called a $(p, E)_c$-atom with respect to $(\mathcal{M}_n)_{n \geq 1}$, if there exist $n \geq 1$ and a projection $e \in \mathcal{M}_n$ such that:
(i) $E_n(a) = 0$;
(ii) $r(a) \leq e$;
(iii) $\|a\|_{E_E} \leq (\tau(e))^{1 - 1/p} \|e\|_{E_E}^{-1}$.
Replacing (ii) by $l(a) \leq e$, we have the notion of a $(p, E)_c$-atom.

Note that if $E = L_q(0, \infty)$ for $1 < q < \infty$, they are $(p, q)$-atoms defined in [4, Definition 4.1].
Definition 4.7. Let $0 < p \leq 1$. Let $E$ be a separable symmetric Banach function space on $(0, \infty)$ and $1 < p_E \leq q_E < \infty$. We define $h^c_{p,E}(\mathcal{M})$ as the space of all operators $x \in L_p(\mathcal{M})$ which admits a decomposition

$$x = \sum_k \lambda_k a_k,$$

where for each $k$, $a_k$ is either a $(p,E)_{c}\text{-atom}$ or an element of the unit ball of $L_p(\mathcal{M}_1)$, and $\lambda_k \in \mathbb{C}$ satisfying $\sum_k |\lambda_k|^p < \infty$. We equip this space with the $p$-norm:

$$\|x\|_{h^c_{p,E}} = \inf \left(\sum_k |\lambda_k|^p\right)^{\frac{1}{p}},$$

where the infimum is taken over all decompositions of $x$ described above. We also define the subspace:

$$h^0_{p,E}(\mathcal{M}) = \{x \in h^c_{p,E}(\mathcal{M}) : \mathcal{E}_1(x) = 0\}.$$

Similarly, we define $h^r_{p,E}(\mathcal{M})$ and $h^{0,r}_{p,E}(\mathcal{M})$.

Lemma 4.8. Let $0 < p \leq 1$. If $a$ is a $(p,E)_{c}\text{-atom}$, then

$$\|a\|_{h^c_{p}} \leq 1.$$

The similar inequality holds for $(p,E)_{r}\text{-atom}$.

Proof. Let $e$ be a projection associated with $a$ satisfying (i)-(iii) of Definition 4.6. Then $s_\varepsilon(a) = e s_\varepsilon(a) = s_\varepsilon(a)e$ (see [2, Proposition 2.2]). Nothing that $p < p_E$, similar with the proof of (4.3),

$$\|a\|_{h^c_{p}} \leq \|s_\varepsilon(a)\|_{E,F} \|e\|_{F}$$

$$= \|a\|_{h^c_{p}} \frac{(\varepsilon(e))^{\frac{1}{p}}}{\|e\|_{E}} \text{ by (4.2))}$$

$$\leq \frac{\tau(e)}{\|e\|_{E}} \|e\|_{E} \text{ by (iii) of Definition 4.6) }$$

$$= 1 \text{ ( by } L_1(0, \infty) = E \otimes E^\times \text{ and Lemma 4.4).}$$

Thus we obtain the desired result.

Lemma 4.9. Let $E$ be a separable symmetric Banach function space on $(0, \infty)$ and $1 < p_E \leq q_E < \infty$. For a given $q_E < q_0 < \infty$ and $q = \max\{2, q_0\}$, $L_q(\mathcal{M})$ embeds densely and continuously into $h^c_{p,E}(\mathcal{M})$.

Proof. Let $x \in L_q(\mathcal{M})$. We decompose it as a linear combination of two atoms:

$$x = C_q C_E \|x - \mathcal{E}_1(x)\|_q \frac{x - \mathcal{E}_1(x)}{C_q C_E \|x - \mathcal{E}_1(x)\|_q} + \|\mathcal{E}_1(x)\|_q \frac{\mathcal{E}_1(x)}{\|\mathcal{E}_1(x)\|_q},$$

where $C_E$ is the constant in the inequality $\|y\|_E \leq C_E \|y\|_q$ for all $y \in L_q(\mathcal{M})$, and $C_q$ is the constant in the noncommutative Burkholder inequality $\|y\|_{h^c_p} \leq C_q \|y\|_q$ for all $y \in L_q(\mathcal{M})$ (see [14]). Then $\mathcal{E}_1(x)/\|\mathcal{E}_1(x)\|_q \in L_q(\mathcal{M}_1) \subset L_1(\mathcal{M}_1)$ and

$$\frac{\|\mathcal{E}_1(x)\|_1}{\|\mathcal{E}_1(x)\|_q} \|_1 \leq \frac{\|\mathcal{E}_1(x)\|_1}{\|\mathcal{E}_1(x)\|_q} \leq 1.$$
Also,
\[
\frac{x - E_1(x)}{C_q C_E \|x - E_1(x)\|_q} = \frac{x - E_1(x)}{C_q C_E \|x - E_1(x)\|_q} \cdot 1 = ae.
\]

Clearly, \( E_1(a) = 0 \) and
\[
\frac{x - E_1(x)}{C_q C_E \|x - E_1(x)\|_q} \leq \frac{C_E \|x - E_1(x)\|_{h_E^q}}{C_q C_E \|x - E_1(x)\|_q} \leq 1.
\]

Thus
\[
\|x\|_{h_E^q} \leq C_q C_E \|x - E_1(x)\|_q + \|E_1(x)\|_q \leq (2C_q C_E + 1)\|x\|_q.
\]

The density is trivial. \(\square\)

**Proposition 4.10.** Let \( 0 < p \leq 1 \) and \( \beta = 1/p - 1 \). Let \( E \) be a separable symmetric Banach function space on \((0,\infty)\). If \( 1 < p_E \leq q_E < \infty \), then \((h_{p,E}^0)\) is a \((\beta, E^x)\)-atom with constant \( 1 \) (see [3, Theorem 2.5]), we have that
\[
\varphi(y) = \tau(y^* x)
\]
for all \( x \in h_{p,E}^0(M) \), and \( \|\varphi\|_{(h_{p,E}^0(M))^*} \leq \|y\|_{\Lambda_{\beta,E^x}} \); \( (i) \) Every \( y \in \Lambda_{\beta,E^x}^0(M) \) defines a continuous linear functional on \( h_{p,E}^0(M) \) by
\[
\varphi(y) = \tau(y^* x)
\]

**Proof.** (i) Let \( y \in \Lambda_{\beta,E^x}^0(M) \). If \( a \) is a \((p,E)\)-atom with \( E_n(a) = 0 \) for some \( n \geq 1 \) and \( a = ae \) for some projection \( e \in M_n \), satisfying \( \|a\|_{h_E^q} \leq (\tau(e))^{1-1/p} \|e\|_{E^x}^{-1} \), then using the duality inclusion \( h_E^q(M) \subset (h_E^q(M))^* \) with constant \( 1 \) (see [3, Theorem 2.5]), we have that
\[
|\tau(y^* a)| = |\tau((y - y_n)^* ae)| \leq \|y - y_n\|_{h_E^q} \|a\|_{h_E^q} \leq \|y - y_n\|_{h_E^q} (\tau(e))^{1-1/p} \|e\|_{E^x}^{-1} \leq \|y\|_{\Lambda_{\beta,E^x}^0}.
\]

(ii) Let \( \varphi \in (h_{p,E}^0(M))^* \). Set \( q = \max\{q_0, 2\} \), where \( q_E < q_0 < \infty \). By Lemma 4.9, we can find \( y \in L_q(M) \) such that
\[
\varphi(y) = \tau(y^* x), \forall x \in L_q(M).
\]

Fix \( n \geq 1 \) and \( e \in P(M_n) \). For a fixed arbitrary and small enough \( \varepsilon > 0 \), by the inclusion \((h_E^q(M))^* \subset h_E^q(M) \) with constant \( C_E \), we may choose \( x \in L_q(M) \) such that \( \|x\|_{h_E^q} \leq 1 \) so that
\[
C_E |\tau(e(y - y_n)^* x)| + \varepsilon \geq \|(y - y_n)e\|_{h_E^q}.
\]

Clearly, we may assume that \( E_n(x) = 0 \) and \( xe = x \). Set
\[
a = \frac{x}{\|x\|_{h_E^q} (\tau(e))^{1/p} \|e\|_{E^x}}.
\]
Then \( a \) is a \((p, E)_c\)-atom and
\[
\| \varphi \| \geq |\tau(y - y_n)^*a)| = \frac{1}{\| x \|_{\Lambda^E_0(\tau(e))^{1/p-1}}\| e \|_{E^\times}} |\tau(e(y - y_n)^*x)| \\
\geq \frac{1}{C_{E}(\tau(e))^2}\| e \|_{E^\times} \|((y - y_n)e\|_{\Lambda^{E^\times}_0} - \varepsilon).
\]

By the arbitrariness of \( \varepsilon \), and taking supremum over \( n \) and \( e \in P(M_n) \), we get
\[
C_{E}\| \varphi \| \geq \| y \|_{\Lambda^{E^\times}_0}.
\]

By Proposition 4.10 and Theorem 4.2, we arrive at the main result of this subsection.

**Theorem 4.11.** Let \( 0 < p \leq 1 \) and \( \beta = 1/p - 1 \). Let \( E \) be a separable symmetric Banach function space on \((0, \infty)\). If \( 1 < p_E \leq q_E < \infty \), then \( (h^{0,c}_{p,E}(M))^* = \Lambda^{0,c}_{0,E^\times} \) with equivalent norms. The similar results hold for the row and mixture spaces.

**Remark 4.12.** By Lemma 4.8, we have the obvious inclusion \( h^{c}_{p,E}(M) \subset h^{c}_{p}(M) \) for \( 0 < p \leq 1 \). When \( p = 1 \), the converse inclusion is also true [3]; it is, however, an open question in the cases \( 0 < p < 1 \) even though they have the same dual from Theorem 4.11 and [4, Theorem 5.4].

**Remark 4.13.** We may consider the crude symmetric atoms. Let \( 0 < p \leq 1 \). Let \( E \) be a separable symmetric Banach function space on \((0, \infty)\) with \( 1 < p_E \leq q_E < \infty \). An element \( a \in L^p(M) \) is called a \((p, E)_c\)-crude atom with respect to \((M_n)_{n \geq 1}\), if there exist \( n \geq 1 \) and a factorization \( a = yb \) such that:

(i) \( E_n(y) = 0 \);
(ii) \( b \in L^{p,E}(M_n), \ b \in E^\times(M_n) \) and \( \| b \|_{E^\times} \leq 1 \) for \( 0 < p < 1 \);
(iii) \( \| y \|_{E^\times} \leq 1 \).

Similarly, we define the notion of a \((p, E)_r\)-crude atom with \( a = yb \) replaced by \( a = by \).

Similar to Definition 4.7, we define \( h^{c}_{p,E;\text{crude}}(M) \) based on the \((p, E)_c\)-crude atoms as building blocks.

Note that if \( a \) is a \((p, E)_c\)-atom with the associated projection \( e \in M_n \), then \( a \) is a \((p, E)_c\)-crude atom. Indeed, for \( 0 < p < 1 \), we write
\[
a = \frac{\| e \|_{E^\times} a}{\| e \|_{E^\times} a} = yb;
\]
for \( p = 1 \), we write \( a = \frac{\| e \|_{E^\times} a}{\| e \|_{E^\times} a} = yb \).

One can check that \( h^{c}_{p,E;\text{crude}}(M) \subset h^{c}_{p,E}(M) \) for \( 0 < p \leq 1 \). So we have that
\[
h^{c}_{p,E}(M) \subset h^{c}_{p,E;\text{crude}}(M) \subset h^{c}_{p}(M)
\]
for \( 0 < p \leq 1 \). As commented the previous remark, it is an interesting question to show the three spaces are the same. We shall take care of it elsewhere.
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