The covariant chiral kinetic equation (CCKE) is derived from the 4-dimensional Wigner function by an improved perturbative method under the static equilibrium conditions. The chiral kinetic equation in 3-dimensions can be obtained by integration over the time component of the 4-momentum. There is freedom to add more terms to the CCKE allowed by conservation laws. In the derivation of the 3-dimensional equation, there is also freedom to choose coefficients of some terms in \( dx_0/d\tau \) and \( dx/d\tau \) (\( \tau \) is a parameter along the worldline, and \((x_0, x)\) denotes the time-space position of a particle) whose 3-momentum integrals are vanishing. So the 3-dimensional chiral kinetic equation derived from the CCKE is not uniquely determined in the current approach. To go beyond the current approach, one needs a new way of building up the 3-dimensional chiral kinetic equation from the CCKE or directly from covariant Wigner equations.

I. INTRODUCTION

The chiral or axial vector anomaly is the anomalous nonconservation of a chiral or axial vector current of fermions arising from quantum effects, it is also called Adler-Bell-Jackiw (ABJ) anomaly [1, 2]. The chiral anomaly manifests itself in the pion’s decay into two photons, which involves an AVV (axial-vector-vector) coupling with the chiral current and the photon field being the axial vector and vector field respectively. In magnetic fields, the chiral anomaly will lead to an electric current along the magnetic field resulting from an imbalance of chirality, which is called the chiral magnetic effect (CME) [3–6], for reviews, see, e.g., Ref. [6–9]. This effect gives another example of AVV coupling among the vector current, the chiral chemical potential of fermions and the magnetic field. The CME is also associated with the chiral vortical effect in which an electric current is induced by the vorticity in a system of charged particles [10–12]. In anomalous hydrodynamics the CME and CVE must coexist in order to guarantee the second law of thermodynamics [13–15]. The CME has recently been confirmed in materials such as Dirac and Weyl semi-metals associated with the chiral vortical effect in which an electric current is induced by the vorticity in a system of charged fermions and the magnetic field. The CME is also called Adler-Bell-Jackiw anomaly [1, 2]. The chiral anomaly manifests itself in the pion’s decay into two photons, which involves an AVV (axial-vector-vector) coupling with the chiral current and the photon field being the axial vector and vector field respectively. In magnetic fields, the chiral anomaly will lead to an electric current along the magnetic field resulting from an imbalance of chirality, which is called the chiral magnetic effect (CME) [3–6], for reviews, see, e.g., Ref. [6–9]. This effect gives another example of AVV coupling among the vector current, the chiral chemical potential of fermions and the magnetic field. The CME is also associated with the chiral vortical effect in which an electric current is induced by the vorticity in a system of charged particles [10–12]. In anomalous hydrodynamics the CME and CVE must coexist in order to guarantee the second law of thermodynamics [13–15]. The CME has recently been confirmed in materials such as Dirac and Weyl semi-metals associated with the chiral vortical effect in which an electric current is induced by the vorticity in a system of charged fermions and the magnetic field. The CME is also called Adler-Bell-Jackiw anomaly [1, 2].
over the time component of the 4-momentum. In this paper we will give a systematic derivation of the CCKE in an improved method in comparison with Ref. [56]. We will show how the 3D chiral kinetic equation can be derived from the CCKE in details. The 3D chiral kinetic equation cannot be uniquely determined in our current approach due to some free coefficients.

The paper is organized as follows. In Section II we introduce the Wigner function and its solutions in static-equilibrium conditions. In Section III we give a systematic and improved derivation of the CCKE in 4D. In Section IV we show that there is freedom of adding more terms to the CCKE allowed by conservation laws. In Section VI we derive the chiral kinetic equation in 3D from the CCKE. In the final section, we give a summary of the main results.

We use the same sign convention for $Q$ and $\gamma_5$ as in Ref. [55, 54]. The energy of a massless fermion with a three-momentum $p$ is denoted as $E_p = |p|$.

II. WIGNER FUNCTION AND ITS SOLUTIONS IN STATIC-EQUILIBRIUM CONDITIONS

In a background electromagnetic field, the quantum mechanical analogue of a classical phase-space distribution for fermions is the gauge invariant Wigner function $W_{\alpha\beta}(x, p)$ which satisfies the equation of motion [52, 53],

$$ (\gamma^\mu K^\mu - m) W(x, p) = 0 $$

(1)

where $x = (x_0, \mathbf{x})$ and $p = (p_0, \mathbf{p})$ are space-time and energy-momentum 4-vectors. For the constant field strength $F_{\mu\nu}$, the operator $K^\mu$ is given by $K^\mu = p^\mu + i\mathbf{\hat{F}} \mathbf{\nabla}^\mu$ with $\nabla^\mu = \partial^\mu - Q F^{\mu\nu} \partial_\nu$. The Wigner function can be decomposed in 16 independent generators of Clifford algebra,

$$ W = \frac{1}{4} \left[ \mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{I}_{\mu\nu} \right], $$

(2)

whose coefficients $\mathcal{F}$, $\mathcal{P}$, $\mathcal{V}_\mu$, $\mathcal{A}_\mu$, and $\mathcal{I}_{\mu\nu}$ are the scalar, pseudo-scalar, vector, axial-vector and tensor components of the Wigner function respectively.

For massless or chiral fermions, the equations for $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ are decoupled from other components of the Wigner function, from which one can obtain independent equations for vector components $\mathcal{F}_\mu^s(x, p)$ of the Wigner function for right-handed ($s = +$) and left-handed ($s = -$) fermions,

$$ p^\mu \mathcal{F}_\mu^s(x, p) = 0, $$

$$ \mathbf{\nabla}^\mu \mathcal{F}_\mu^s(x, p) = 0, $$

$$ 2s(p^\lambda \mathcal{F}_\lambda^s - p^\rho \mathcal{F}_\rho^s) = -\epsilon^{\mu\nu\lambda\rho} \mathbf{\nabla}_\mu \mathcal{F}_\nu^s, $$

(3)

where $\mathcal{F}_\mu^s(x, p)$ are defined as

$$ \mathcal{F}_\mu^s(x, p) = \frac{1}{2} [\mathcal{V}_\mu(x, p) + s\mathcal{A}_\mu(x, p)]. $$

(4)

One can derive a formal solution of $\mathcal{F}_\mu^s$ satisfying Eq. (3) by a perturbation in powers of space-time derivative $\partial_\mu^s$ and field strength $F_{\mu\nu}$. The solution at the zeroth and first order reads

$$ \mathcal{F}_\mu^p(x, p) = p^\rho f_s \delta(p^2), $$

$$ \mathcal{F}_\mu^{(1)}(x, p) = -\frac{s}{2} Q^\rho p_\beta \frac{df_s}{dp_0} \delta(p^2) - \frac{s}{p^2} Q F^{\rho\lambda} \epsilon_{p\lambda}, $$

(5)

Here $p_0 \equiv u \cdot p$ is the particle energy in the co-moving frame of the fluid, $u^\mu$ is the fluid velocity, $\tilde{F}^{\rho\lambda} = \frac{1}{2} \rho^{\rho\lambda} F_{\mu\nu}$, $\tilde{\Omega}^{\eta} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \Omega_{\nu\sigma}$ with $\Omega_{\nu\sigma} = \frac{1}{2} (\partial_\nu u_\sigma - \partial_\sigma u_\nu)$, where $\epsilon^{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ are anti-symmetric tensors with $\epsilon^{\mu\nu\rho\sigma} = 1(-1)$ and $\epsilon_{\mu\nu\rho\sigma} = -1(1)$ for even (odd) permutations of indices 0123, so we have $\epsilon^{0123} = -\epsilon_{0123} = 1$. Instead of $\Omega_{\nu\sigma}$, $\tilde{\Omega}^{\eta}$, $F_{\mu\nu}$ and $\tilde{F}^{\rho\lambda}$, we will also use the vorticity vector $\omega^\rho = \frac{1}{2} \epsilon^{\rho\sigma\beta} u_\sigma \partial_\beta u_\rho$, the electric field $E^\mu = F_{\mu\nu} u^\nu$, and the magnetic field $B^\mu = \frac{1}{2} \epsilon^{\mu\rho\sigma} u_\sigma \tilde{F}_{\rho\lambda}$. In Eq. (5) $f_s$ is the distribution function for chiral fermions at the zeroth order,

$$ f_s(x, p) = \frac{2}{(2\pi)^3} [\Theta(p_0) f_{FD}(p_0 - \mu_s) + \Theta(-p_0) f_{FD}(-p_0 + \mu_s)]. $$

(6)

Here $f_{FD}(y) \equiv 1/\{\exp(\beta y) + 1\}$ is the Fermi-Dirac distribution function, with $\beta = 1/T$ and $\mu_s$ being the temperature inverse and the chemical potential for chiral fermions with chirality $s = \pm 1$ respectively. We can express $\mu_s$ in terms of the scalar and pseudo-scalar (or chiral) chemical potentials, $\mu_s = \mu + s \mu_5$. 
The solution to the Wigner function in Eq. (9) is the result of following static-equilibrium conditions \[55\],
\[
\Delta^\sigma \Delta^\rho \left( \partial_\sigma u_\beta + \partial_\beta u_\sigma - \frac{2}{3} \Delta_{\alpha\beta} \Delta^{\rho\sigma} \partial_\rho u_\sigma \right) = 0,
\]
\[
T \Delta^\sigma \partial_\rho \frac{\mu}{T} + QE^\sigma = 0,
\]
\[
u^\mu \partial_\rho u^\sigma - \Delta^{\rho\sigma} \partial_\rho \ln T = 0,
\]
\[
\partial_\sigma \mu = 0, \quad u^\sigma \partial_\sigma \mu = 0,
\]
\[
u^\sigma \partial_\sigma T + \frac{1}{3} T \Delta^{\rho\sigma} \partial_\rho u_\sigma = 0.
\]
For simplicity we assume in this paper the equilibrium conditions with constant temperature, then the above constraints are reduced to the following conditions \[55\]
\[
\partial_\sigma \mu = 0, \quad \partial_\sigma T = 0,
\]
\[
\partial^\rho u^\sigma + \partial^\sigma u^\rho = 0,
\]
\[
\partial_\sigma \mu = -Q E^\sigma.
\]
From above conditions, we can derive following identities \( u_\rho \partial^\rho u^\sigma = 0, \partial_\sigma u^\sigma = 0, \partial_\mu \omega^\mu = 0, \epsilon^{\mu\nu\alpha\beta} \omega_\nu u_\alpha B_\beta = 0, \Omega_{\mu\nu} = \partial_\mu u_\nu, \) etc., which are used in deriving the covariant chiral kinetic equation in this paper.

**III. DERIVATION OF CCKE**

Now we start to derive the covariant chiral kinetic equation from the second line of Eq. (1). To this end we insert the solution \[1\] into it,
\[
\nabla_{\mu} [f_0 + f_{1s}] = \delta(p^2) \left[ p^\mu \nabla_\mu f_s + s Q \frac{1}{p^2} \Omega^{\mu\lambda} p_\lambda \tilde{F}_{\mu\nu} p^\nu f_s 
\]
\[
- \frac{s}{2} \tilde{\Omega}^{\mu\lambda} p_\lambda (\nabla_{\mu} f_s) - s Q \frac{1}{p^2} \Omega^{\mu\lambda} p_\lambda (\nabla_{\mu} f_s) \right] = 0,
\]
whose derivation is given in Appendix A. We can simplify \( \nabla_{\mu} f_s \) as
\[
\nabla_{\mu} f_s = \nabla_{\mu} (u_\nu \partial_\nu f_s) = \Omega_{\mu\nu} \partial_\nu f_s + u_\nu \partial_\nu \nabla_{\mu} f_s,
\]
using \( f_s = df_s/dp_0 = u^\nu \partial_\nu f_s, \partial_\mu u_\nu = \Omega_{\mu\nu} \) and \( [\partial_\nu, \nabla_{\mu}] = 0 \). Then Eq. (10) can be rewritten as
\[
\left[ p^\mu \nabla_\mu f_s - s Q \frac{1}{p^2} \tilde{F}^{\mu\nu\lambda} p_\lambda (\nabla_{\mu} f_s) + s Q \frac{1}{p^2} \Omega^{\mu\lambda} p_\lambda \tilde{F}_{\mu\nu} p^\nu u^\rho \partial_\rho f_s 
\]
\[
- \frac{s}{2} \tilde{\Omega}^{\mu\lambda} p_\lambda \Omega_{\mu\nu} \partial_\nu f_s - \frac{s}{2} \tilde{\Omega}^{\mu\lambda} p_\lambda u_\nu \partial_\nu \nabla_{\mu} f_s \right] \delta(p^2) = 0.
\]
Now we try to rewrite the last term \( \sim \partial_\nu \nabla_{\mu} f_s \). We assume that the CCKE holds at the momentum integral level, we then look at the momentum integral of the last term in Eq. (11)
\[
\int dp_0 \delta (p^2) \tilde{\Omega}^{\mu\lambda} p_\lambda u_\nu \partial_\nu \nabla_{\mu} f_s
\]
\[
= \int dp_0 \frac{d}{dp_0} \left[ \delta (p^2) \tilde{\Omega}^{\mu\lambda} p_\lambda \nabla_{\mu} f_s \right] - \int dp_0 \frac{d}{dp_0} \left[ \delta (p^2) \tilde{\Omega}^{\mu\lambda} p_\lambda \nabla_{\mu} f_s \right] - \int dp_0 \delta (p^2) \tilde{\Omega}^{\mu\lambda} u_\lambda \nabla_{\mu} f_s
\]
\[
= \int dp_0 \delta (p^2) \left[ \frac{2p_0}{p^2} \tilde{\Omega}^{\mu\lambda} p_\lambda - \tilde{\Omega}^{\mu\lambda} u_\lambda \right] \nabla_{\mu} f_s
\]
\[
(12)
\]
where we have used in the first equality \( d/dp_0 = u^\nu \partial_\nu \) and dropped the surface term. So Eq. (11) becomes
\[
\left[ \left( p^\mu - s Q \frac{1}{p^2} \tilde{F}^{\mu\lambda} p_\lambda - s \frac{p_0}{p^2} \tilde{\Omega}^{\mu\lambda} p_\lambda + \frac{s}{2} \tilde{\Omega}^{\mu\lambda} u_\lambda \right) \nabla_{\mu} f_s 
\]
\[
+ \left( - \frac{s}{2} \tilde{\Omega}^{\mu\lambda} p_\lambda \Omega_{\mu\nu} + s Q \frac{1}{p^2} \Omega^{\mu\lambda} p_\lambda \tilde{F}_{\mu\nu} p^\nu u_\nu \right) \partial_\nu f_s \right] \delta(p^2) = 0,
\]
(13)
from which we can extract the equations of motion

\[
\frac{dx^\mu}{d\tau} = \rho - sQ \frac{1}{p^2} \tilde{F}_{\nu\lambda}^\mu p_\lambda - s \frac{p_0}{p^2} \tilde{\Omega}_{\nu\lambda}^\mu p_\lambda + \frac{s}{2} \tilde{\Omega}_{\nu\lambda}^\mu u_\lambda,
\]

\[
\frac{dp^\mu}{d\tau} = Q F^{\nu\mu} p_\nu + sQ^2 \frac{1}{4p^2} F^{\nu\lambda} \tilde{F}_{\nu\lambda}^\mu + sQ \frac{1}{p^2} \Omega_{\nu\lambda}^\mu \tilde{F}_{\nu\lambda}^\mu p_\lambda u^\mu
\]

\[-\frac{s}{8} \tilde{\Omega}_{\nu\lambda}^\mu \tilde{\Omega}_{\nu\lambda}^\mu + \frac{1}{2} sQ F^{\nu\mu} \tilde{\Omega}_{\nu\lambda}^\mu - sQ \frac{p_0}{p^2} F^{\nu\mu} \tilde{\Omega}_{\nu\lambda}^\mu,
\]

where \(m_0\) is an arbitrary mass scale which is irrelevant to the physics we discuss in this paper and \(\tau\) denotes a parameter along the worldline. In deriving Eq. (13) we have used two identities \(F^{\nu\mu} \tilde{F}_{\nu\lambda}^\mu = -\frac{1}{4} F^{\nu\sigma} \tilde{F}_{\nu\sigma}^\mu\) and \(\Omega^{\nu\mu} \tilde{\Omega}_{\nu\lambda}^\mu = \frac{1}{4} \Omega^{\nu\sigma} \tilde{\Omega}_{\nu\sigma}^\mu\) to rewrite two terms of \(dp^\mu/d\tau\), whose proof is given in Eq. (A8). To compare with the previous result in Ref. [55], we rewrite the vorticity part of Eq. (14) in terms of time-like and space-like components of vorticity and field strength tensor under the conditions in (8).

\[
\frac{dx^\mu}{d\tau} = \rho - sQ \frac{1}{p^2} \tilde{F}_{\nu\lambda}^\mu p_\lambda + s \left( \frac{1}{2} - \frac{p_0^2}{p^2} \right) \omega^\mu + s \frac{p_0}{p^2} (p \cdot \omega) u^\mu,
\]

\[
\frac{dp^\mu}{d\tau} = Q F^{\nu\mu} p_\nu + sQ^2 \frac{1}{4p^2} F^{\nu\lambda} \tilde{F}_{\nu\lambda}^\mu + \frac{1}{2} sQ (E \cdot \omega) u^\mu - sQ \frac{1}{p^2} (p \cdot \omega) (p \cdot E) u^\mu + s \frac{1}{p^2} p_0 (p \cdot \omega) E^\mu,
\]

where \(p_0 \equiv p \cdot u\). The detailed derivation of the vorticity part of Eq. (16) is given in Appendix B. So the covariant chiral kinetic equation reads

\[
\delta(p^2) \left( \frac{dx^\mu}{d\tau} \partial^\sigma f_s + \frac{dp^\mu}{d\tau} \partial^\sigma f_s \right) = 0,
\]

with \(dx^\mu/d\tau\) and \(dp^\mu/d\tau\) being given by Eq. (13) or (15).

Note that the covariant chiral kinetic equation (15) is different from Eq. (12) of Ref. [56] in two places: (a) there is an additional term \(sQ p^{-2} (p \cdot \omega) e^{\mu\nu\rho\sigma} p_\nu p_\rho B_\sigma\) to the last term of \(dp^\mu/d\tau\) in Eq. (12) of Ref. [56]; (b) there is a factor 2 in the term \(s(p \cdot \omega)(p \cdot u) u^\mu/p^2\) (the last term of \(dx^\mu/d\tau\)) in Eq. (12) of Ref. [56].

We note that the term in (a) is vanishing when combined with \(\partial^\sigma f_s\) if we use \(\partial^\mu f_s = u^\mu d f_s/dp_0\). The term in (b) is also vanishing when combined with \(\partial^\mu f_s\) if we use \(\partial^\mu f_s = \partial^\mu (u \cdot p) - \partial^\mu \mu d f_s/dp_0\) and Eq. (8), so it seems that the factor of this term would be irrelevant. However the coefficient of this term is essential to obtain the correct energy-momentum tensor from \(dx^\mu/d\tau\). We will demonstrate it in great details in the rest part of the paper.

IV. FREEDOM OF ADDING MORE TERMS

Our starting point is the covariant chiral kinetic equation (15). But the problem with this equation is that one cannot get the correct energy-momentum tensor from \(dx^\mu/d\tau\). One has to add terms to \(dx^\mu/d\tau\) and \(dp^\mu/d\tau\) to achieve this goal. Actually there is a freedom to do so provided the correct vector and axial vector currents and energy-momentum tensor are obtained from \(dx^\mu/d\tau\) after integration over momentum. In this section, we will give the concrete form of additional terms allowed by the covariant chiral kinetic equation and conservation laws.

We know from Eq. (15) that we need to modify the vorticity terms. Suppose we add a new term \(X^\mu\) in linear order of \(\omega\) to \(dx^\mu/d\tau\), which will bring a new term \(Y^\mu\) to \(dp^\mu/d\tau\) accordingly in order to keep Eq. (16) hold. So we obtain the constraint

\[
X^\sigma \partial^\sigma f_s + Y^\sigma \partial^\sigma f_s = 0.
\]

We assume \(X^\mu\) has the following form

\[
X^\mu = sC_1 (p, u) \omega^\mu + sC_2 (p, u) (p \cdot \omega) u^\mu + sC_3 (p, u) (p \cdot \omega) \tilde{p}^\mu,
\]

where \(\tilde{p}^\mu \equiv p^\mu - p_0 u^\mu\) and \(C_{1,2,3}(p, u)\) are functions of \(u^\mu\) and \(p^\mu\) as follows,

\[
C_1 (p, u) = C_{10} + C_{11} \frac{p_0^2}{p^2},
\]

\[
C_2 (p, u) = C_{20} + C_{21} \frac{p_0^2}{p^2},
\]

\[
C_3 (p, u) = C_{30} + C_{31} \frac{p_0^2}{p^2}.
\]
\[ C_2(p, u) = C_{20} \frac{p_0}{p^2} + C_{21} \frac{1}{p_0}, \]
\[ C_3(p, u) = C_{30} \frac{1}{p^2}, \]

(19)

with \( \{C_{10}, C_{11}, C_{20}, C_{21}, C_{30}\} \) being dimensionless coefficients to be determined. With \( X^\sigma \) in Eq. (13), as shown in Appendix [C] we can solve \( Y^\sigma \) from Eq. (17) as,

\[ Y^\sigma = -sQ[C_1(p, u)(\omega \cdot E) + C_3(p, u)(p \cdot \omega)(p \cdot E)]u^\sigma + sQ\tilde{p}^\sigma C_4(p, \omega), \]

(20)

where the function \( C_4(p, \omega) \) is defined by

\[ C_4(p, \omega) = C_{40}(\omega \cdot E) \frac{1}{p_0} + C_{41} \frac{1}{p^2}(p \cdot \omega)(p \cdot E), \]

(21)

with \( C_{40} \) and \( C_{41} \) being two dimensionless coefficients to be determined.

With these terms in (18,20), Eq. (15) is modified to

\[ m_0 \frac{dx^\mu}{d\tau} = p^\mu - sQ \frac{1}{p^2} \tilde{F}^{\mu\lambda} p_\lambda \]
\[ + s \left[ \frac{1}{2} + C_{10} + (C_{11} - 1) \frac{p_0^2}{p^2} \right] \omega^\mu \]
\[ + s \left[ (C_{20} + 1) \frac{p_0}{p^2} + C_{21} \frac{1}{p_0} \right] (p \cdot \omega) u^\mu \]
\[ + sC_{30} \frac{1}{p^2}(p \cdot \omega)\tilde{p}^\mu, \]

\[ m_0 \frac{dp^\mu}{d\tau} = QF^{\mu\nu} p_\nu + sQ \frac{p^\mu}{4p^2} F^{\mu\nu} \tilde{F}_{\nu\lambda} \]
\[ + sQ \left( \frac{1}{2} - C_{10} - C_{11} \frac{p_0^2}{p^2} \right) (\omega \cdot E) u^\mu \]
\[ - sQ(C_{30} + 1) \frac{1}{p^2}(p \cdot \omega)(p \cdot E) u^\mu \]
\[ + sQ\tilde{p}^\mu \left[ C_{40}(\omega \cdot E) \frac{1}{p_0} + C_{41} \frac{1}{p^2p_0}(p \cdot \omega)(p \cdot E) \right] \]
\[ + sQ \frac{1}{p^2} p_0(p \cdot \omega)E^\mu. \]

(22)

The next task is to obtain the constraints for coefficients \( \{C_{10}, C_{11}, C_{20}, C_{21}, C_{30}\} \) by conservation laws and those for coefficients \( \{C_{40}, C_{41}\} \) by matching the power (energy rate) to the force for quasi-particles. Note that all these coefficients in the vorticity terms.

V. CONSTRAINTS FOR COEFFICIENTS

In this section, we give the constraints for \( \{C_{10}, C_{11}, C_{20}, C_{21}, C_{30}\} \) by computing the currents and energy-momentum tensor. The detailed derivation is given in Appendix [D].

A. Constraint from currents

The currents for chiral fermions with chirality \( s = \pm 1 \) are given by,

\[ j^\mu_s = m_0 \int d^4p \delta(p^2) \frac{dx^\mu}{d\tau} f_s. \]

(23)

For the electromagnetic term of \( dx^\mu/d\tau \) in Eq. (22), we obtain the currents whose derivation is given in Eq. (D1),

\[ j^\mu_s(\text{EM}) = -sQ \int d^4p \delta(p^2) \frac{1}{p^2} \tilde{F}^{\mu\lambda} p_\lambda f_s = \xi^\mu_s B^\mu, \]

(24)
where we have defined the coefficient (no summation over \( s \) is implied)

\[
\xi^s_B = \frac{sQ}{4\pi^2} \int_0^\infty dE_p \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right].
\]  

(25)

Then one can reproduce the chiral magnetic effect for the vector and axial vector currents as

\[
j^\mu(EM) = \sum_s j^\mu_s(EM) = (\xi^+_B + \xi^-_B) B^\mu = \xi_B B^\mu,
\]

\[
j^\mu(EM) = \sum_s s j^\mu_s(EM) = (\xi^+_B - \xi^-_B) B^\mu = \xi_B B^\mu,
\]

(26)

where \( \xi_B = Q\mu_5/(2\pi^2) \) and \( \xi_{B5} = Q\mu/(2\pi^2) \) are coefficients in Eqs. (22-23) of Ref. [55].

For the vorticity part of \( dx^\mu/d\tau \) in Eq. (22), we obtain the currents induced by the vorticity whose derivation is shown in Eq. (D2)

\[
j^\mu(\omega) = \left( C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} + 1 \right) \xi_s \omega^\mu,
\]

(27)

where \( \xi_s \) is defined by (no summation over \( s \) is implied)

\[
\xi_s = \int d^4 p \delta(p^2)s f_s
\]

\[
= \frac{1}{2\pi^2} s \int_0^\infty dE_p E_p \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right].
\]

(28)

The vector and axial vector currents in the chiral vortical effect are given by,

\[
\begin{align*}
\xi^+_v(\omega) &= \frac{1}{2} C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} + 1, \\
\xi^-_v(\omega) &= \frac{1}{2} C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} + 1.
\end{align*}
\]

(29)

where \( \xi = \mu\mu_5/\pi^2 \) and \( \xi_5 = T^2/6 + (\mu^2 + \mu_5^2)/(2\pi^2) \) are coefficients in Eqs. (22-23) of Ref. [55]. To match Eq. (27) with the currents of the chiral vortical effect in (29), we obtain the first constraint for \( \{C_{10}, C_{11}, C_{30}\} \),

\[
C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} = 0.
\]

(30)

### B. Constraint from stress tensor

The energy momentum tensor in the relativistic chiral kinetic theory can be obtained by

\[
T^{\rho\sigma} = \frac{1}{\xi} m_0 \int d^4 p \delta(p^2) \sum_s \left[ p^\rho \frac{dx^\sigma}{d\tau} f_s + p^\sigma \frac{dx^\rho}{d\tau} f_s \right].
\]

(31)

First we look at the electromagnetic field part \( T^{\rho\sigma}(EM) \), which we obtain from Eq. (D5),

\[
T^{\rho\sigma}(EM) = \frac{1}{\xi} Q u^{(\rho} B^\sigma),
\]

(32)

where \( \xi \) is the same as the coefficient of \( j^\mu(\omega) \) in Eq. (29). The energy-momentum tensor (32) is just the result in Ref. [55].

Now we work on the vorticity part of \( T^{\mu\nu} \). Inserting the last three terms of \( dx^\mu/d\tau \) in Eq. (22) into Eq. (31), we obtain

\[
T^{\rho\sigma}(\omega) = \left( \frac{1}{2} C_{10} - \frac{1}{4} C_{11} + \frac{1}{4} C_{30} + 1 \right) \phi u^{(\rho} \omega^{s\sigma)},
\]

(33)

whose derivation is given in Eq. (D8). To match Eq. (33) with the stress tensor of the vorticity part in Ref. [55], \( T^{\mu\nu}(\omega) = n_5 u^{(\rho} \omega^{s\sigma)} \) [see Eq. (D10) for a derivation], we arrive at the second constraint for coefficients,

\[
\frac{1}{2} C_{10} - \frac{1}{4} C_{11} + \frac{1}{4} C_{30} + 1 = 1.
\]

(34)
Combining Eqs. (30,31) we obtain two independent constraints

\[
C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} = 0, \\
C_{20} - \frac{2}{3} C_{21} = 1.
\] (35)

We see that \(C_{20}\) and \(C_{21}\) cannot all be zero.

VI. CHIRAL KINETIC EQUATION: FROM 4D TO 3D

From the covariant chiral kinetic equation (16) with \(dx^\mu/d\tau\) and \(dp^\mu/d\tau\) given by Eq. (22), we can obtain its 3D version by integrating over \(p_0\),

\[
I = \int dp_0 \delta(p^2) \left\{ \frac{dx^\sigma}{d\tau} \partial_\sigma \hat{f}_s + \frac{dp^\sigma}{d\tau} \partial_\sigma f_s \right\}
\]

\[
= I_{x0} + I_{x0}^{(EM)} + I_x + I_p,
\] (36)

where we work in the co-moving frame with \(u^\mu = (1, 0, 0)\), and \(I_{x0}\), \(I_{x0}^{(EM)}\), \(I_x\) and \(I_p\) are from \((dx^\mu/d\tau)\partial_\sigma \hat{f}_s\), \((dp^\mu/d\tau)\partial_\sigma f_s\), \((dx^i/d\tau)\partial_i \hat{f}_s\), \((dp^i/d\tau)\partial_i f_s\) respectively. We evaluate each term in Appendix E. In evaluation of these terms we imply an additional integration over \(p\), i.e. \(\int d^3p\), so we can drop complete derivative terms in \(p\) (or \(E_p\)) which are vanishing at the boundary.

The results for \(I_{x0}\) are derived in Eqs. (111,13).

\[
I_{x0}(0) = \frac{1}{(2\pi)^3} \partial_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right],
\]

\[
I_{x0}^{(EM)} = \frac{1}{(2\pi)^3} \left\{ sQ(p \cdot B) \frac{1}{2 E_p^3} \partial_0 \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right] - sQ(p \cdot B) \frac{d}{dE_p} \frac{1}{2 E_p^3} \partial_0 \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right] \right\},
\]

\[
= I_{x0}^{(EM)} + I_{x0}^{2(EM)}
\]

\[
I_{x0}(\omega) = \frac{1}{(2\pi)^3} \left[ (C_{20} - C_{21} + 1)(p \cdot \omega) \frac{1}{E_p^3} \partial_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \right],
\] (37)

where we have labeled two terms of \(I_{x0}^{(EM)}\) as \(I_{x0}^{1(EM)}\) and \(I_{x0}^{2(EM)}\) for later use. The results for \(I_x\) are derived in Eqs. (110,12),

\[
I_x(0) = \int dp_0 \delta(p^2) p^i \partial_i \hat{f}_s
\]

\[
= \frac{p_i}{E_p^2} \frac{1}{(2\pi)^3} \partial_i \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right],
\]

\[
I_x^{(EM)} = \frac{1}{(2\pi)^3} sQ_B \frac{1}{2 E_p^3} \partial_i \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right]
\]

\[
- \frac{1}{(2\pi)^3} sQ(p \times E) \frac{1}{2 E_p^3} \partial_i \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right]
\]

\[
+ \frac{1}{(2\pi)^3} sQ(p \times E) \frac{d}{dE_p} \frac{1}{2 E_p^3} \partial_i \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right],
\]

\[
= I_{x}^{1(EM)} + I_{x}^{2(EM)} + I_{x}^{3(EM)}
\]

\[
I_x(\omega) = \frac{1}{(2\pi)^3} \left[ \left( 1 + C_{10} - \frac{1}{2} C_{11} \right) \frac{1}{E_p} \omega^i \partial_i \left[ f_F(E_p - \mu_s) + f_F(E_p + \mu_s) \right] + \frac{1}{(2\pi)^3} sC_{30} (p \cdot \omega) p_i \frac{3}{2 E_p^3} \partial_i \left[ f_F(E_p - \mu_s) + f_F(E_p + \mu_s) \right] \right],
\] (38)

where we have labeled three terms of \(I_x^{(EM)}\) as \(I_{x}^{1(EM)}\), \(I_{x}^{2(EM)}\) and \(I_{x}^{3(EM)}\) for later use.
The results for $I_{p0}$ and $I_p$ are derived in Eqs. (41, 42, 43, 44).

$$I_{p0}(\text{EM}) = \frac{1}{2\pi^3} Q(p \cdot E) \frac{1}{E_p} \frac{d}{dE_p} [f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s)]$$

$$+ \frac{1}{2\pi^3} s Q^2(E \cdot B) \frac{1}{2E^2_p} \frac{d}{dE_p} [f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s)],$$

$$I_{p0}(\omega) = \frac{1}{2\pi^3} \frac{d}{dE_p} [f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s)],$$

$$I_p(\text{EM}) = \frac{1}{2\pi^3} \left\{ Q(E \cdot \partial p \cdot [f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s)]$$

$$+ \left[ Q \frac{p \times B + s Q^2(E \cdot B)}{E_p^3} \right] \partial_i \frac{d}{dE_p} [f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s)] \right\},$$

$$I_p(\omega) = \frac{1}{2\pi^3} s Q \left[ \frac{1}{E^2_p} (p \cdot \omega) E_i - C_{40} \frac{1}{E^2_p} (\omega \cdot E) p_i - C_{41} \frac{2}{E^2_p} (p \cdot \omega)(p \cdot E) p_i \right]$$

$$\partial_i \frac{d}{dE_p} [f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s)],$$

(39)

We now extract $d\mathbf{p}/d\tau$ from Eqs. (39, 40) for particles (it is similar for anti-particles). For an on-shell particle the energy is not an independent phase space variable, its rate $dE_p/d\tau$ from $I_{p0}$ can be determined by (see, e.g., Ref. [69])

$$\frac{dE_p}{d\tau} = \frac{1}{E_p} \mathbf{p} \cdot \frac{d\mathbf{p}}{d\tau}.$$  

(41)

So in derivation of the 3D chiral kinetic equation from the 4D one, the $p_0$ degree of freedom is fixed and is not a kinematic variable in the 3D equation.

First we look at $I_p(\text{EM})$ in Eq. (39) from which we can obtain the energy rate from the electromagnetic field

$$\frac{dE_p}{d\tau}(\text{EM}) = Q(P \cdot E) \frac{1}{E_p} + s Q^2(E \cdot B) \frac{1}{2E^2_p}.$$  

(42)

Let us compare it with $d\mathbf{p}/d\tau$ extracted from $I_p(\text{EM})$ in Eq. (40),

$$\frac{d\mathbf{p}}{d\tau}(\text{EM}) = Q(\mathbf{E} + \frac{\mathbf{P}}{E_p} \times \mathbf{B}) + s Q^2(E \cdot B) \mathbf{p} \frac{1}{E^3_p}.$$  

(43)

But Eqs. (42, 43) do not satisfy Eq. (41) due to a factor 2 difference in the $E \cdot B$ term. The solution to this problem is to define $d\mathbf{p}/d\tau$ in such a way that Eq. (41) is satisfied

$$\frac{d\mathbf{p}}{d\tau}(\text{EM}) = Q(\mathbf{E} + \frac{\mathbf{P}}{E_p} \times \mathbf{B}) + s Q^2(E \cdot B) \mathbf{p} \frac{1}{2E^3_p},$$  

(44)

where the last term is only one-half of that in Eq. (13). Note that our current solution of $d\mathbf{p}/d\tau$ is based on the first order solution of the Wigner function which is linear in the electromagnetic field except the anomaly term. The other half of the anomaly term can be grouped with other second order terms of the electromagnetic fields which belong to the higher order solutions. This is beyond the scope of our current paper and for a future study.

In the same way, the energy rate from the vorticity can be obtained from $I_{p0}(\omega)$ in Eq. (39),

$$\frac{dE_p}{d\tau}(\omega) = -s Q(C_{30} + 1) \frac{1}{2E_p} (E \cdot \omega)$$

$$+ s Q(C_{30} + 1) \frac{3}{2E^2_p} (p \cdot \omega)(p \cdot E).$$  

(45)

We can compare it with $d\mathbf{p}/d\tau$ extracted from $I_p(\omega)$ in Eq. (40),

$$\frac{d\mathbf{p}}{d\tau}(\omega) = s Q \frac{1}{E^2_p} [(p \cdot \omega)E - C_{40}(\omega \cdot E)p$$

$$- 2C_{41} \frac{1}{E^2_p} (p \cdot \omega)(p \cdot E)p].$$  

(46)
Applying Eq. (41) for Eqs. (45-46), we obtain the following constraints for coefficients

\[ C_{40} = \frac{1}{2}(C_{30} + 1), \]
\[ C_{41} = \frac{1}{2} - \frac{3}{4}(C_{30} + 1). \]  

(47)

If we choose \( C_{30} = 0 \), we obtain \( C_{40} = 1/2 \) and \( C_{41} = -1/4 \). Then from Eqs. (43-44) we have

\[ \frac{dp}{d\tau} = Q \left( E + \frac{p}{|p|} \times B \right) + sQ^2(E \cdot B)\Omega \\
+ sQ \frac{1}{|p|^2} \left[ (p \cdot \omega)E - \frac{1}{2}(\omega \cdot E)p + \frac{1}{2|p|^2}(p \cdot \omega)(p \cdot E)p \right], \]  

(48)

where \( \Omega = p/(2|p|^3) \) is the Berry curvature in 3-momentum space. Equation (48) differs from Eq. (22) of Ref. [56] only by an additional term perpendicular to \( p \): \( sQ|p|^{-1}(p \cdot \omega)E_T \) with \( E_T \equiv E - (p \cdot E)p \). This additional term is allowed by the energy and momentum constraint (41).

Now we try to extract \( dx_0/d\tau \) and \( dx/d\tau \) from \( I_{z0} \) and \( I_z \) in Eqs. (47-48). We can combine Eqs. (47-48) to obtain \( dx_0/d\tau \) and \( dx/d\tau \) for fermions (one can also obtain results for anti-fermions similarly)

\[ \frac{dx_0}{d\tau} = 1 + \mathcal{C}_B sQ(\Omega \cdot B) + \left( 4 - \frac{2}{3}C_{21} \right) s|p|(\Omega \cdot \omega), \]
\[ \frac{dx}{d\tau} = \dot{p} + sQB(\dot{p} \cdot \Omega) + \mathcal{C}_E sQ(E \times \Omega) \\
+ s \left( 1 - \frac{1}{2}C_{30} \right) \frac{\omega}{|p|} + 3sC_{30}(\Omega \cdot \omega)p, \]  

(49)

where we have used the constraint (35). Here \( \mathcal{C}_B \) and \( \mathcal{C}_E \) are prefactors of the \( sQ(\Omega \cdot B) \), \( sQ(E \times \Omega) \) terms, respectively, that need to be determined. The freedom to choose \( \mathcal{C}_B \) and \( \mathcal{C}_E \) is because the integration over \( p \) of \( I_{z0}^0(EM) \), \( I_{z0}^2(EM) \), \( I_z^2(EM) \) in Eqs. (47-48) are all vanishing, we can make choices as to keep or drop them following some physical reasons. So \( \mathcal{C}_B \) and \( \mathcal{C}_E \) can be either 1 or 2. The value 2 is due to the fact that the \( p \) integrals of \( I_{z0}^0(EM) \) and \( I_{z0}^2(EM) \) are equal and so are those of \( I_z^2(EM) \) and \( I_z^2(EM) \). We can set \( \mathcal{C}_E \) to 1 to match the previous result.

We can now combine Eq. (43-44) and (45-46) to obtain the 3D chiral kinetic equation,

\[ \frac{dx_0}{d\tau} = 1 + \mathcal{C}_B sQ(\Omega \cdot B) + \left( 4 - \frac{2}{3}C_{21} \right) s|p|(\Omega \cdot \omega), \]
\[ \frac{dx}{d\tau} = \dot{p} + sQB(\dot{p} \cdot \Omega) + \mathcal{C}_E sQ(E \times \Omega) \\
+ s \left( 1 - \frac{1}{2}C_{30} \right) \frac{\omega}{|p|} + 3sC_{30}(\Omega \cdot \omega)p, \]
\[ \frac{dp}{d\tau} = Q \left( E + \frac{p}{|p|} \times B \right) + sQ^2(E \cdot B)\frac{p}{2|p|^3} \\
+ sQ \frac{1}{|p|^2} \left[ (p \cdot \omega)E - \frac{1}{2}(C_{30} + 1)(\omega \cdot E)p \right] \\
+ \frac{s}{2}(1 + 3C_{30}) \frac{1}{|p|^3}(p \cdot \omega)(p \cdot E)p \]  

(50)

We see that the chiral kinetic equation (50) is not uniquely determined due to a freedom to choose the coefficients \( \{C_{21}, C_{30}, \mathcal{C}_B, \mathcal{C}_E\} \) where \( \mathcal{C}_B \) and \( \mathcal{C}_E \) can be set to 1 or 2. If we choose \( \{C_{21}, C_{30}, \mathcal{C}_B, \mathcal{C}_E\} = \{0, 0, 1, 1\} \), Eq. (50) reproduces the result of Ref. [56] except an additional term perpendicular to \( p \) in \( dp/d\tau \) which is allowed by the energy and momentum constraint (41). Another possible choice of \( C_{30} \) is \( C_{30} = 2/3 \). In this case the vorticity terms in \( dx/d\tau \) in Eq. (50) read

\[ \frac{dx}{d\tau}(\omega) = \frac{s}{|p|}(\dot{p} \cdot \omega)p + \frac{s}{2|p|^3}\omega. \]  

(51)
When calculating the vorticity contribution to the current from $dx/d\tau$ by integration over $p$, one can verify that the first term of $\Delta E^m_p$ contributes to 1/3 of the chiral vortical effect while the second term contributes to the rest 2/3. In comparison to the result of Ref. [61]60, the 1/3 contribution corresponds to that from the spin-vorticity coupling energy, while the rest 2/3 contribution corresponds to that from the magnetization current.

We can show that the $sQ(\Omega \cdot B)$ term in $dx_0/d\tau$ in Eq. (50) can be absorbed into the distribution function as the magnetic moment energy $\Delta E^m_p = sQ \frac{p}{2T} (p \cdot B)$ [57]. Similarly the term $2s|p|(\Omega \cdot \omega)$ in $dx_0/d\tau$ in Eq. (50) can also be absorbed into the distribution function as the spin-vorticity coupling energy $\Delta E^\omega_p = s\frac{1}{2} (p \cdot \omega)$ [57]. But once including the energy corrections into the distribution functions, $dx/d\tau$ has to be modified in order to reproduce the CME and CVE, which has been done in Ref. [61]60.

We should point out that our current approach is based on perturbation in which the zero-th order Wigner functions involve the Fermi-Dirac distribution functions for free and massless fermions. If the zero-th order distributions are modified by including e.g. energy corrections from the magnetic moment and spin-vorticity coupling of fermions, our current perturbation method breaks down. Another key assumption of our approach is the static equilibrium conditions under which we obtain the vector and axial vector components of Wigner functions up to the first order. Due to these two assumptions, we cannot fix the coefficients $\{C_{21}, C_{30}, \mathcal{E}_B, \mathcal{E}_E\}$ in the 3D chiral kinetic equation. To go beyond our current approach, one needs a new and consistent way of building up the 3D chiral kinetic equation from the CCKE or directly from covariant Wigner equations.

VII. SUMMARY

The chiral kinetic equation is an important aspect of chiral fermions, which is closely related to the Berry phase and monopole in momentum space. We derive the covariant chiral kinetic equation (CCKE) in the 4-dimensional Wigner function approach using an improved perturbative method under the static equilibrium conditions. The chiral kinetic equation in 3-dimensions can be obtained by integration over the time component of the 4-momentum from the CCKE. In the 3-dimensional equation, due to the on-shell condition, $p_0$ degree of freedom is removed and is not a phase space variable. There is a freedom to add more terms into the CCKE whose coefficients can be constrained by conservation laws. Moreover, in the 3-dimensional equation, the coefficients of the $sQ(\Omega \cdot B)$ term of $dx_0/d\tau$ and the $sQ(\mathbf{E} \times \Omega)$ term of $dx/d\tau$ cannot be fixed in the current formalism where $\Omega$ is the Berry curvature in 3-momentum space. Therefore the 3-dimensional chiral kinetic equation is not uniquely determined up to some coefficients in our current approach. By one set of coefficients we can recover the the 3-dimensional chiral kinetic equation derived in Ref. [56] except a transverse electric field term allowed by the energy and momentum constraint. To go beyond our current approach, one needs a new and consistent way of building up the 3D chiral kinetic equation from the CCKE or directly from covariant Wigner equations.

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Appendix A: Derivation of Eq. (9)

In this appendix, we evaluate $\nabla_\mu J^\mu_{(1)s}$ with $J^\mu_{(1)s}$ given in Eq. (5). The final result is Eq. (9). First we list some useful formula for the field strengths and vorticity tensors,

\begin{align}
E^{\mu\nu} &= E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} u_\rho B_\sigma, \\
F^{\mu\nu} &= B^\mu u^\nu - B^\nu u^\mu + \epsilon^{\mu\nu\rho\sigma} u_\rho u_\sigma, \\
\Omega^{\mu\nu} &= \epsilon^{\mu\nu\rho\sigma} \omega_\rho - \epsilon^{\nu\mu\rho\sigma} \omega_\rho, \\
\tilde{\Omega}^{\mu\nu} &= \epsilon^{\mu\nu\rho\sigma} \omega_\rho - \epsilon^{\nu\mu\rho\sigma} \omega_\rho, \\
\Omega^{\mu\nu} &= \frac{1}{2} (\partial^\mu u^\nu - \partial^\nu u^\mu) = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \tilde{\Omega}_{\rho\sigma}, \\
\tilde{\Omega}^{\mu\nu} &= \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \Omega_{\rho\sigma}. 
\end{align}

(A1)
We evaluate the vorticity and electromagnetic field terms as follows,

\[
\nabla_\mu [\tilde{\Omega}^{\mu \lambda} p_\lambda f_s(p^2)] = \tilde{\Omega}^{\mu \lambda} \nabla_\mu [p_\lambda f'_s(p^2)] = -Q \tilde{\Omega}^{\mu \lambda} F_{\mu \kappa} f'_s(p^2) + \tilde{\Omega}^{\mu \lambda} p_\lambda (\nabla_\mu f'_s)\delta(p^2) + 2Q \frac{1}{p^2} \tilde{\Omega}^{\mu \lambda} p_\lambda F_{\mu \kappa} p^\kappa f'_s(p^2)
\]

\[
= Q \frac{1}{p^2} \left[ 2\tilde{\Omega}^{\mu \lambda} p_\lambda F_{\mu \kappa} p^\kappa - p^2 \tilde{\Omega}^{\mu \lambda} F_{\mu \lambda} \right] f'_s(p^2) + \tilde{\Omega}^{\mu \lambda} p_\lambda (\nabla_\mu f'_s)\delta(p^2)
\]

\[
= -2Q \frac{1}{p^2} \tilde{\Omega}^{\mu \lambda} p_\lambda F_{\mu \kappa} p^\kappa f'_s(p^2) + \tilde{\Omega}^{\mu \lambda} p_\lambda (\nabla_\mu f'_s)\delta(p^2),
\]

(A2)

\[
\nabla_\mu \left[ \frac{1}{p^2} \tilde{\Phi}^{\mu \lambda} p_\lambda f_s(p^2) \right] = 2Q \frac{1}{p^2} \tilde{\Phi}^{\mu \lambda} p_\lambda F_{\mu \nu} p^\nu f_s(p^2) - Q \frac{1}{p^2} \tilde{\Phi}^{\mu \lambda} F_{\mu \lambda} f_s(p^2)
\]

\[
+ \frac{1}{p^2} \tilde{\Phi}^{\mu \lambda} p_\lambda (\nabla_\mu f_s)\delta(p^2) + 2Q \frac{1}{p^2} \tilde{\Phi}^{\mu \lambda} p_\lambda F_{\mu \nu} p^\nu f_s(p^2)
\]

\[
= \frac{1}{p^2} \tilde{\Phi}^{\mu \lambda} p_\lambda (\nabla_\mu f_s)\delta(p^2) + \frac{1}{p^2} \left[ 4\tilde{\Phi}^{\mu \lambda} p_\lambda F_{\mu \nu} p^\nu - p^2 \tilde{\Phi}^{\mu \lambda} F_{\mu \lambda} \right] f_s(p^2)
\]

\[
= \frac{1}{p^2} \tilde{\Phi}^{\mu \lambda} p_\lambda (\nabla_\mu f_s)\delta(p^2),
\]

(A3)

where we have used \(\nabla_\mu \tilde{\Omega}^{\mu \lambda} = \partial_\mu \tilde{\Omega}^{\mu \lambda} = 0\) under static-equilibrium conditions and \(\nabla_\mu \tilde{\Phi}^{\mu \lambda} = 0\) for constant electromagnetic field. We also used following formula,

\[
\nabla_\mu p_\lambda = -Q F_{\mu \lambda},
\]

\[
\nabla_\mu \delta(p^2) = 2Q F_{\mu \nu} p^\nu \frac{1}{p^2} \delta(p^2),
\]

\[
\nabla_\mu \frac{1}{p^2} = 2Q F_{\mu \nu} p^\nu \frac{1}{p^2},
\]

(A4)

and

\[
\tilde{\Omega}^{\mu \lambda} p_\lambda F_{\mu \kappa} p^\kappa + \tilde{\Omega}^{\mu \lambda} p_\lambda \tilde{\Phi}_{\mu \kappa} p^\kappa = \frac{1}{2} p^2 \tilde{\Omega}^{\rho \sigma} F_{\rho \sigma},
\]

\[
4\tilde{\Phi}^{\mu \lambda} p_\lambda F_{\mu \nu} p^\nu - p^2 F_{\rho \sigma} \tilde{\Phi}^{\rho \sigma} = 0,
\]

\[
4\tilde{\Phi}^{\mu \lambda} p_\lambda F_{\mu \nu} p^\nu - p^2 \tilde{\Omega}^{\rho \sigma} \tilde{\Phi}_{\rho \sigma} = 0.
\]

(A5)

We can prove the first identity of (A5) by observing

\[
\tilde{\Omega}^{\mu \nu} F_{\nu \lambda} p^\lambda = -\omega^\mu (p \cdot E) - (u \cdot p)(\omega \cdot E) u^\mu,
\]

\[
\Omega^{\mu \nu} \tilde{F}_{\nu \lambda} p^\lambda = (p \cdot E) u^\mu + (u \cdot p)(\omega \cdot E) u^\mu - (\omega \cdot E) p^\mu,
\]

\[
\tilde{\Omega}^{\mu \nu} F_{\nu \mu} = 2(\omega \cdot E),
\]

where we have used \(\varepsilon^\mu = \Omega^{\mu \nu} u_\nu = u_\nu \partial^\mu u^\nu = 0\) and \(\epsilon_{\nu \lambda \rho \sigma} \omega^\nu u^\rho B^\sigma = 0\) and

\[
-\epsilon^{\mu \rho \lambda \sigma} \omega_{\nu \lambda \rho \sigma} u_\mu B_\beta E^\rho u^\sigma p^\lambda = (p \cdot \omega) E^\mu + (u \cdot p)(\omega \cdot E) u^\mu - (\omega \cdot E) p^\mu.
\]

(A6)

The last two identities in Eq. (A5) are results of following identities

\[
F^{\mu \nu} \tilde{F}_{\nu \lambda} p^\lambda = -\frac{1}{4} F^{\rho \sigma} \tilde{F}_{\rho \sigma} p^\mu,
\]

\[
\tilde{\Omega}^{\mu \nu} \tilde{F}_{\nu \lambda} p^\lambda = -\frac{1}{4} F^{\rho \sigma} \tilde{\Omega}_{\rho \sigma} p^\mu.
\]

(A7)

The proof of the first identity of Eq. (A7) is as follows,

\[
F^{\mu \nu} \tilde{F}_{\nu \lambda} p^\lambda = -(p \cdot B) E^\mu - (E \cdot B)(u \cdot p)u^\mu - \epsilon^{\mu \rho \lambda \sigma} \omega_{\nu \lambda \rho \sigma} u_\alpha B_\beta E^\rho u^\sigma p^\lambda
\]

\[
= -\frac{1}{4} F^{\rho \sigma} \tilde{F}_{\rho \sigma} p^\mu,
\]

(A8)

where we have used \(u \cdot E = u \cdot B = 0, 4(E \cdot B) = F^{\rho \sigma} \tilde{F}_{\rho \sigma}, \) and

\[
-\epsilon^{\mu \rho \lambda \sigma} \omega_{\nu \lambda \rho \sigma} u_\mu B_\beta E^\rho u^\sigma p^\lambda = (p \cdot B) E^\mu + (E \cdot B)(u \cdot p)u^\mu - (E \cdot B) p^\mu.
\]

(A9)

The proof of the second identity of (A7) is similar.
Appendix B: Derivation of the vorticity part of Eq. (15)

We rewrite the vorticity part in Eq. (14). The vorticity part of \( \frac{dx^\mu}{d\tau} \) is treated as

\[
I_1(\omega) = -s \frac{p_0}{p^2} \tilde{\Omega}^{\mu\lambda} p_\lambda \\
= -s \frac{p_0}{p^2} (\omega^\mu u^\nu - \omega^\nu u^\mu) p_\nu \\
= -s \frac{p_0}{p^2} \omega^\mu + s \frac{p_0}{p^2} (p \cdot \omega) u^\mu,
\]

\[
I_2(\omega) = \frac{s}{2} \omega^\mu,
\]

where we have used \( \epsilon^\alpha = \Omega^{\alpha\beta} u_\beta = 0 \). So we combine the two terms,

\[
I_1(\omega) + I_2(\omega) = s \left( \frac{1}{2} - \frac{p_0^2}{p^2} \right) \omega^\mu + s \frac{p_0}{p^2} (p \cdot \omega) u^\mu.
\] (B1)

We deal with each term in the vorticity part of \( \frac{dp^\mu}{d\tau} \) as follows. The first term is vanishing,

\[
I_3(\omega) = -s \frac{p^\mu}{8} \Omega_{\nu\lambda} \tilde{\Omega}^{\rho\lambda} \\
= -s \frac{1}{16} p^\mu \epsilon^{\rho\lambda\alpha\beta} (\partial_\rho u_\lambda) (\partial_\alpha u_\beta) \\
= -s \frac{1}{16} p^\mu \epsilon^{\rho\lambda\alpha\beta} \partial_\rho (u_\lambda \partial_\alpha u_\beta) \\
= -s \frac{8}{8} p^\mu \partial_\rho \omega^\rho = 0.
\] (B2)

The second term is rewritten as

\[
I_4(\omega) = s Q \frac{1}{p^2} u^\mu \Omega_{\nu\lambda} p_\lambda \tilde{F}^{\nu\kappa} p_\kappa \\
= s Q u^\mu (E \cdot \omega) - s Q \frac{p_0^2}{p^2} (E \cdot \omega) u^\mu - s Q \frac{1}{p^2} (p \cdot \omega) (p \cdot E) u^\mu.
\] (B3)

The third term is rewritten as

\[
I_5(\omega) = \frac{1}{2} s Q F^{\mu\kappa} \tilde{\Omega}_{\nu\lambda} u^\lambda = -\frac{1}{2} s Q (E \cdot \omega) u^\mu.
\] (B4)

The fourth term is rewritten as

\[
I_6(\omega) = -s Q \frac{p_0}{p^2} F^{\mu\nu} \tilde{\Omega}_{\nu\lambda} p_\lambda \\
= s Q \frac{p_0^2}{p^2} (E \cdot \omega) u^\mu + s Q \frac{1}{p^2} p_0 (p \cdot \omega) E^\mu,
\] (B5)

where we have used \( \epsilon^\alpha = \Omega^{\alpha\beta} u_\beta = 0 \). We combine these four terms as

\[
\sum_{i=3}^{6} I_i(\omega) = \frac{1}{2} s Q (E \cdot \omega) u^\mu - s Q \frac{1}{p^2} (p \cdot \omega) (p \cdot E) u^\mu + s Q \frac{1}{p^2} p_0 (p \cdot \omega) E^\mu.
\] (B6)

Appendix C: Derivation of Eq. (20)

In this appendix, we give a derivation of Eq. (20). Inserting Eq. (18) into (17) we obtain
\[ 0 = [sC_1(p, u)\omega^\sigma + sC_2(p, u)(p \cdot \omega)u^\sigma \\
+ sC_3(p, u)(p \cdot \omega)p^\rho \partial_\rho f_s + Y^\sigma \partial_\sigma f_s] \\
= sC_1(p, u)\omega^\sigma[\partial_\sigma^\rho(u^\rho) - \partial_\sigma^\rho]\frac{df_s}{dp_0} \\
+ sC_2(p, u)(p \cdot \omega)u^\sigma[\partial_\sigma^\rho(u^\rho) - \partial_\sigma^\rho]\frac{df_s}{dp_0} \\
+ sC_3(p, u)(p \cdot \omega)p^\rho[\partial_\rho(u^\rho) - \partial_\rho]\frac{df_s}{dp_0} + (Y \cdot u)\frac{df_s}{dp_0} \\
= [sQC_1(p, u)(\omega \cdot E) + sQC_3(p, u)(p \cdot \omega)(p \cdot E) + (Y \cdot u)]\frac{df_s}{dp_0} \quad (C1) \]

which leads to the following form of \( Y^\sigma \)

\[ Y^\sigma = -sQ[C_1(p, u)(\omega \cdot E) + C_3(p, u)(p \cdot \omega)(p \cdot E)]u^\sigma + Y^\sigma, \]

\[ Y^\sigma \partial_\sigma f_s = -sQ[C_1(p, u)(\omega \cdot E) + C_3(p, u)(p \cdot \omega)(p \cdot E)]u^\sigma \partial_\sigma f_s, \quad (C2) \]

where we have used \( \bar{Y}^\sigma \equiv Y^\sigma - (Y \cdot u)u^\sigma \), and we assume \( \bar{Y}^\sigma \) has the form

\[ \bar{Y}^\sigma = sQ\bar{p}^\rho C_4(p, \omega), \quad (C3) \]

with the function \( C_4(p, \omega) \) being defined by

\[ C_4(p, \omega) = C_{40}(\omega \cdot E) \frac{1}{p_0} + C_{41} \frac{1}{p^2 p_0} (p \cdot \omega)(p \cdot E), \quad (C4) \]

where \( C_{40} \) and \( C_{41} \) are two dimensionless coefficients to be determined. We have used following identities derived from the static and equilibrium conditions in \( \Sigma \).

\[ \partial_\mu f_s = u^\mu \frac{df_s}{dp_0}, \]

\[ \omega^\sigma \partial_\sigma u_\rho = \omega^\sigma \Omega_{\sigma \rho} = \epsilon_{\sigma\rho\tau\mu} u^\tau \omega^\mu = 0, \]

\[ u^\sigma \partial_\sigma f_s = u^\sigma[\partial_\sigma^\rho(u^\rho) - \partial_\sigma^\rho]\frac{df_s}{dp_0} = 0, \]

\[ \bar{p}^\rho \partial_\rho u_\rho = \bar{p}^\rho \partial_\rho u_\rho = \frac{1}{2} \bar{p}^\rho \partial_\rho \Omega_{\sigma \rho} = 0. \quad (C5) \]

Appendix D: Derivation of formula in Section (V)

In this Appendix, we give detailed derivations of currents and energy momentum tensor in Section (V).

1. Currents

For the electromagnetic field part of \( dx^\mu / d\tau \) in Eq. (22), we evaluate the current as

\[ j^\mu_{(EM)} = -sQ \int d^4p \delta(p^2) \frac{1}{p^2} \bar{F}^\mu_\lambda p_\lambda f_s \]

\[ = \frac{1}{2} sQ \int d^4p \delta(p^2) \frac{1}{p_0} (B^\mu p_0 - (p \cdot B)u^\mu + \epsilon^{\mu\lambda\rho\sigma} p_\lambda E_\rho u_\sigma) f_s \]

\[ = \frac{1}{2} sQB^\mu \int d^4p \delta(p^2) \frac{1}{p_0} \frac{df_s}{dp_0} \]

\[ = -\frac{1}{2} sQB^\mu \int \frac{d^3p}{(2\pi)^3} \frac{1}{p_0} \frac{d}{dE_p} [f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s)] \]

\[ = B^\mu \frac{sQ}{4\pi^2} \int_0^\infty dE_p [f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s)] \]

\[ \equiv \xi_B B^\mu, \quad (D1) \]
We have used the assumption that \( f \) into Eq. (23) and obtain

\[
\frac{d}{dp_0} \frac{d\delta(p^2)}{dp_0} f_s
\]

Here we have carried out the momentum integrals in the co-moving frame of the fluid with

\[
\frac{d}{dp_0} \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} f_s
\]

\[
(\omega_3) = \frac{1}{2} C_{30} \omega_3 \int d^4 p \delta(p^2) f_s + \frac{1}{2} s C_{30} \omega_3 \int d^4 p \delta(p^2) f_s
\]

where we have dropped the complete derivative in \( p_0 \) and those integrals whose integrands are odd in \( \bar{p} \). This gives the derivation of Eq. (24).

Now we derive Eq. (27) for the currents induced by the vorticity. We put the vorticity part of \( dx^\mu/d\tau \) in Eq. (22) into Eq. (23) and obtain

\[
j_s^\mu(\omega) = s \left( \frac{1}{2} + C_{10} \right) \omega^\mu \int d^4 p \delta(p^2) f_s - \frac{1}{2} s(C_{11} - 1) \omega^\mu \int d^4 p_0 \frac{d\delta(p^2)}{dp_0} f_s
\]

where we have defined (no summation over \( s \) is implied)

\[
\xi_s = \int d^4 p \delta(p^2) s f_s
\]

\[
\frac{1}{2 \pi^2} \int_0^\infty dE_p E_p \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right].
\]

(D2)

In the first equality of Eq. (D2) we have used the fact that \( f_s \) is given by Eq. (6) and depends on momentum through \( p_0 = p \cdot u \), so the integral proportional to \( u^\mu \) is vanishing since the integrand is odd in spatial momentum due to \( p \cdot \omega = \bar{p} \cdot \omega \). We have dropped complete integrals in \( p_0 \) in Eq. (D2). We have also used following integrals,

\[
\int d^4 p \delta(p^2) f_s = \frac{d^3 p}{(2\pi)^3} \int_{0}^{\infty} \frac{1}{E_p} \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right],
\]

\[
\int d^4 p \frac{d\delta(p^2)}{dp_0} f_s = - \int d^4 p \delta(p^2) f_s - \int d^4 p \delta(p^2) p_0 \frac{df_s}{dp_0} - \int d^4 p \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} f_s
\]

\[
= \int d^4 p \delta(p^2) f_s,
\]

\[
\int d^4 p \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} f_s = \int d^4 p \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} f_s - \int d^4 p \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} f_s
\]

\[
= - \Delta^\lambda \mu \int d^4 p \delta(p^2) f_s.
\]

(D4)

Here we have carried out the momentum integrals in the co-moving frame of the fluid with \( u^\mu = (1, 0) \) and \( \bar{p}^\mu = (0, p) \). We have used the assumption that \( f_s \) depends on \( p \) through \( u \cdot p \) so it is isotropic in spatial momentum, \( \bar{p} \cdot \bar{p} \rightarrow -\frac{1}{2}|p|^2 \Delta \bar{p} \). We have also dropped complete integrals in \( p_0 \) and \( E_p \).

2. Energy-momentum tensor

We can derive the energy-momentum tensor in Eq. (2) in a similar way to how we derive the currents in the last subsection.

We evaluate the electromagnetic field part \( T^{\rho \sigma}(EM) \) as

\[
T^{\rho \sigma}(EM) = -\frac{1}{2} s Q \int d^4 p \delta(p^2) \left( \frac{1}{p^2} \right) \left( p^\rho p_\lambda \tilde{F}^{\sigma \lambda} f_s + p^\sigma p_\lambda \tilde{F}^{\rho \lambda} f_s \right)
\]

\[
= \frac{1}{4} s Q \int d^4 p \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} f_s \left( p^\rho p_\lambda \tilde{F}^{\sigma \lambda} + p^\sigma p_\lambda \tilde{F}^{\rho \lambda} \right)
\]

\[
= -\frac{1}{3} s Q u^{\rho} B^{\sigma} \left[ \frac{1}{2} \int d^4 p \delta(p^2) f_s + \int d^4 p_0 \frac{d\delta(p^2)}{dp_0} \frac{|p|^2 df_s}{p_0 dp_0} \right]
\]
\[ \rho_\sigma T \left( \frac{d^4}{d^3} \delta(p^2) \right) \cdot \left( \frac{d^4}{d^3} \delta[p^0(p \cdot \omega)] \right) = 0, \]

where we have replaced \( \rho_\sigma \rightarrow -|p|^2 \Delta_\rho \) which is true in 3D momentum integrals for isotropic momentum distributions. Note that we have not shown terms linear in \( \bar{p} \) in Eq. (D6) which give vanishing integrals. In Eq. (D5) we also used the integral

\[
\int d^4p \delta(p^2)f_s = \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right],
\]
\[
\int d^4p \delta(p^2) \frac{|p|^2 df_s}{p_0 dp_0} = \int \frac{d^3p}{(2\pi)^3} \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right] = -2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right]. \tag{D7}
\]

Here we have dropped the complete integral in \( E_p \) which is vanishing.

For the vorticity part, we insert the last three terms of \( d\mu^\mu/d\tau_7 \) in Eq. (22) into Eq. (31) and obtain

\[
T^{\mu\sigma}(\omega) = \frac{1}{2} \left( \frac{1}{2} + C_{10} \right) u^{(\rho \omega)} \int d^4pp_0 \delta(p^2)sf_s - \frac{1}{4}(C_{11} - 1) \int d^4p \frac{d\delta(p^2)}{dp_0} p_0 p^{(\rho \omega)} sf_s - \frac{1}{4}(C_{20} + 1) \int d^4p \frac{d\delta(p^2)}{dp_0} p^{(\rho \omega)} \cdot (p \cdot \omega) sf_s + \frac{1}{2} C_{21} \int d^4p \delta(p^2) \frac{1}{p_0} (p \cdot \omega) p^{(\rho \omega)} sf_s - \frac{1}{4} C_{50} \int d^4p \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} (\bar{p} \cdot \omega) sf_s = \frac{1}{2} \left( \frac{1}{2} + C_{10} \right) u^{(\rho \omega)} n_5 - \frac{1}{4}(C_{11} - 1) u^{(\rho \omega)} n_5 + \frac{1}{4}(C_{20} + 1) u^{(\rho \omega)} n_5 - \frac{1}{6} C_{21} u^{(\rho \omega)} n_5 + \frac{1}{4} C_{50} u^{(\rho \omega)} n_5 = n_5 u^{(\rho \omega)} \left( \frac{1}{2} C_{10} - \frac{1}{4} C_{11} + \frac{1}{4} C_{30} + \frac{1}{4} C_{20} - \frac{1}{6} C_{21} + \frac{3}{4} \right), \tag{D8}
\]

where the summation over \( s \) was implied and we have used following integrals

\[
\int d^4pp_0 \delta(p^2)sf_s = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} s \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] = n_5,
\]
\[
\int d^4p \frac{d\delta(p^2)}{dp_0} p_0^2 sf_s = -2n_5 - \int d^4p \delta(p^2) \frac{d}{dp_0} sf_s = n_5,
\]
\[
\int d^4p \frac{d\delta(p^2)}{dp_0} p^{(\rho \omega)} \cdot (p \cdot \omega) sf_s = -\int d^4p p^{(\rho \omega)} (\bar{p} \cdot \omega) \delta(p^2) d(sf_s) = -\frac{1}{3} \omega_\rho u^{(\rho \omega \rho)} n_5,
\]
\[
\int d^4p \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} (\bar{p} \cdot \omega) p^{(\rho \omega)} sf_s = -\frac{1}{3} \omega_\lambda \Delta_\rho u^{(\rho \omega \rho)} \int d^4p \delta(p^2) E_p^2 \frac{1}{p_0} sf_s = -\frac{1}{3} n_5 u^{(\rho \omega \rho)}.
\]

\[
\int d^4p \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} (\bar{p} \cdot \omega) p^{(\rho \omega)} sf_s = -\frac{1}{3} \omega_\rho u^{(\rho \omega \rho)} \int d^4p \delta(p^2) E_p^2 \frac{d(sf_s)}{dp_0} = -n_5 u^{(\rho \omega \rho)}. \tag{D9}
\]
For comparison, we can obtain $T^{\rho\sigma}(\omega)$ from $\mathcal{J}^{\rho}_{(1)s}(x,p)$ in Eq. (36) by the definition in Ref. [55],

$$
T^{\rho\sigma}(\omega) = \frac{1}{2} \int d^4p \rho^\sigma \frac{\partial}{\partial p_0} \mathcal{J}^{\rho}_{(1)s} + p^\rho \mathcal{J}^{\rho}_{(1)s} \\
\quad \to -\frac{s}{4} \int d^4p \left[ p^\rho p_\beta \Omega_{\sigma\beta} + p^\sigma p_\beta \Omega_{\rho\beta} \right] \frac{df_s}{dp_0}(p^2) \\
\quad = -\frac{s}{4} \int d^4p \left[ (p_0^2 u^\sigma u_\beta - \frac{1}{3} p_0^2 E_f^0) \Omega_{\rho\beta} + (p_0^2 u^\rho u_\beta - \frac{1}{3} p_0^2 E_f^0) \Omega_{\sigma\beta} \right] \frac{df_s}{dp_0}(p^2) \\
\quad = -\frac{s}{3} \int d^4p E_f^2 \left[ u^\rho u_\beta \Omega_{\sigma\beta} + u^\sigma u_\beta \Omega_{\rho\beta} \right] \frac{df_s}{dp_0}(p^2) \\
\quad = u^{(\sigma,\omega,\rho)} \int \frac{d^3p}{(2\pi)^3} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \\
\quad = n_s u^{(\sigma,\omega,\rho)},
$$
(D10)

where we also used Eq. (D9) and the summation over $s = \pm 1$ is implied.

Appendix E: Evaluation of each term in Eq. (36)

In this appendix we evaluate each term in Eq. (36) with $dx^\mu/d\tau$ and $dp^\mu/d\tau$ given by Eq. (22). We work in the co-moving frame with $u^\mu = (1, 0)$.

1. Evaluation of $I_{x0}$

The term $I_{x0}$ consists of three parts, $I_{x0}(0)$ of the zeroth order contribution, $I_{x0}(EM)$ of the first order in electromagnetic field, and $I_{x0}(\omega)$ of the first order in vorticity. The detailed derivation is as follows,

$$
I_{x0}(0) = \int dp_0 \delta(p^2) p_0 \partial_0^\sigma f_s \\
\quad = \frac{1}{(2\pi)^3} \partial_0^\sigma \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right],
$$
(E1)

$$
I_{x0}(EM) = sQ \int dp_0 \frac{1}{p^2} \delta(p^2)(p \cdot B) \partial_0^\sigma f_s \\
\quad = \frac{1}{2} sQ \hat{p} \cdot \hat{B} \int dp_0 \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} \partial_0^\sigma f_s \\
\quad = \frac{1}{2} sQ \left( \frac{1}{2} \frac{1}{E_f^2} \partial_0^\sigma \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right] \\
\quad - \frac{1}{2} \frac{1}{E_f^2} \partial_0^\sigma \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right] \right),
$$
(E2)

$$
I_{x0}(\omega) = s \int dp_0 \delta(p^2) \left[ (C_{20} + 1) \frac{p_0}{p^2} + C_{21} \frac{1}{p_0} \right] (p \cdot \omega) \partial_0^\sigma f_s \\
\quad = \frac{1}{2} s(C_{20} + 1)(p \cdot \omega) \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial_0^\sigma f_s - sC_{21}(p \cdot \omega) \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_0^\sigma f_s \\
\quad \to \frac{1}{(2\pi)^3} s(C_{20} - C_{21} + 1)(p \cdot \omega) \frac{1}{E_f^2} \partial_0^\sigma \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right],
$$
(E3)

where the involved integrals are evaluated as

$$
\int dp_0 \delta(p^2) \frac{1}{p_0} \partial_0^\sigma f_s = \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_0^\sigma f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} \partial_0^\sigma f_s \\
\quad = \frac{1}{(2\pi)^3} \left\{ \frac{1}{E_f^2} \partial_0^\sigma \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right] \right\}
$$
\[ -\frac{1}{E_p^2} \frac{d}{dE_p} \partial^p_0 \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right] \], \quad (E4) \]

\[ \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial^p_0 f_s = -\int dp_0 \delta(p^2) \frac{d}{dp_0} \partial^p_0 f_s \]

\[ = -\frac{1}{(2\pi)^3 E_p} \frac{d}{dE_p} \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \quad (E_p \text{ in integrand}) \]

\[ \rightarrow \frac{1}{(2\pi)^3 E_p^2} \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \quad (E5) \]

\[ \int dp_0 \delta(p^2) \frac{1}{p_0} \partial^p_0 f_s = \frac{1}{(2\pi)^3 E_p^2} \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right]. \quad (E6) \]

In the last line of the integral \[(E5),\] we have implied that there will be a \( \int d^3 p \) integral so we have

\[ \int d^3 p (p \cdot \omega) \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial^p_0 f_s \]

\[ \rightarrow -\int d\Omega_p (p \cdot \omega) \int_0^\infty dp E_p^2 E_p^\mu \frac{d}{dE_p} \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \]

\[ = -\int d\Omega_p (p \cdot \omega) \int_0^\infty dp E_p \frac{d}{dE_p} \left\{ E_p^2 \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \right\} \]

\[ + 2 \int d\Omega_p (p \cdot \omega) \int_0^\infty dp E_p \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \]

\[ \sim \int d^3 p (p \cdot \omega) \frac{2}{E_p^2} \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \quad (E7) \]

where the complete integral term in \( E_p \) is vanishing at two limits \( E_p = 0, \infty \). The above integral can also be treated in a slightly different way,

\[ \int d^3 p (p \cdot \omega) \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial^p_0 f_s \]

\[ \rightarrow -\int d^3 p (p \cdot \omega) \frac{1}{E_p} \frac{d}{dE_p} \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \]

\[ = -\int d^3 p \frac{1}{E_p^2} (p \cdot \omega) p \cdot \nabla \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \]

\[ = -\int d^3 \nabla \cdot \left\{ \frac{1}{E_p^2} (p \cdot \omega) p \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \right\} \]

\[ + \int d^3 \nabla \cdot \left[ \frac{1}{E_p^2} (p \cdot \omega) p \right] \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \]

\[ = \int d^3 p (p \cdot \omega) \frac{2}{E_p^2} \partial^p_0 \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \quad (E8) \]

where we have used

\[ \frac{d}{dE_p} \rightarrow \frac{1}{E_p} p \cdot \nabla, \]

\[ \nabla \cdot \left[ \frac{1}{E_p^2} (p \cdot \omega) p \right] = (p \cdot \omega) \nabla \cdot \left( \frac{p}{E_p} \right) + \frac{1}{E_p^2} p \cdot \nabla (p \cdot \omega) \]

\[ = \frac{2}{E_p^2} (p \cdot \omega), \quad (E9) \]

and the total divergence term is vanishing.
2. Evaluation of \( I_x \)

Now we work on \( I_x = I_x(0) + I_x(\text{EM}) + I_x(\omega) \), where \( I_x(0) \) is the zeroth order contribution, and \( I_x(\text{EM}) \) and \( I_x(\omega) \) are the first order contribution from electromagnetic field and vorticity respectively. We evaluate these terms as

\[
I_x(0) = \int dp_0 \delta(p^2) p^i \partial^x_i f_s
= \frac{p_i}{E_p (2\pi)^3} \partial^x_i \left[ f_{\text{FD}}(E_p - \mu_s) + f_{\text{FD}}(E_p + \mu_s) \right],
\]

\[
I_x(\text{EM}) = -sQB^i \int dp_0 p_0 \frac{1}{p^2} \delta(p^2) \partial_i f_s + sQ \epsilon^{0ijk} \int dp_0 \frac{1}{p^2} \delta(p^2) p_j E_k \partial_i f_s
= \frac{1}{2} sQB_i \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial_i f_s - \frac{1}{2} sQ(p \times E)_i \int dp_0 \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} \partial_i f_s
= \frac{1}{(2\pi)^3} \frac{1}{2} sQB_i \frac{E_p^2}{E_p} \partial^x_i \left[ f_{\text{FD}}(E_p - \mu_s) - f_{\text{FD}}(E_p + \mu_s) \right]
- \frac{1}{(2\pi)^3} \frac{1}{2} sQ(p \times E)_i \frac{E_p^2}{E_p} \partial^x_i \left[ f_{\text{FD}}(E_p - \mu_s) + f_{\text{FD}}(E_p + \mu_s) \right]
+ \frac{1}{(2\pi)^3} \frac{1}{2} sQ(p \times E)_i \frac{1}{E_p} \frac{d}{dE_p} \partial^x_i \left[ f_{\text{FD}}(E_p - \mu_s) + f_{\text{FD}}(E_p + \mu_s) \right],
\]

\[
I_x(\omega) = s \int dp_0 \delta(p^2) \left[ \frac{1}{2} + C_{10} + (C_{11} - 1) \frac{p_0^2}{p^2} \right] \omega^i \partial_i f_s
+ sC_{30} \int dp_0 \delta(p^2) \frac{1}{p^2} (p \cdot \omega) \bar{p}^i \partial_i f_s
\]

\[
= \frac{1}{(2\pi)^3} s \left( \frac{1}{2} + C_{10} \right) \frac{1}{E_p} \omega^i \partial^x_i \left[ f_{\text{FD}}(E_p - \mu_s) + f_{\text{FD}}(E_p + \mu_s) \right]
- \frac{1}{(2\pi)^3} s(C_{11} - 1) \omega^i \partial^x_i \left[ f_{\text{FD}}(E_p - \mu_s) + f_{\text{FD}}(E_p + \mu_s) \right]
- \frac{1}{(2\pi)^3} sC_{30} \bar{p}^i \omega_j \frac{3}{2E_p} \partial^x_j \left[ f_{\text{FD}}(E_p - \mu_s) + f_{\text{FD}}(E_p + \mu_s) \right]
+ \frac{1}{(2\pi)^3} sC_{30} (p \cdot \omega) \bar{p} \frac{3}{2E_p} \partial^x \left[ f_{\text{FD}}(E_p - \mu_s) + f_{\text{FD}}(E_p + \mu_s) \right],
\]

where we have used following integrals

\[
\int dp_0 \frac{d\delta(p^2)}{dp_0} \partial_i f_s = - \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_i f_s
= - \frac{1}{E_p} \frac{d}{dE_p} \partial^x_i \left[ f_{\text{FD}}(E_p - \mu_s) - f_{\text{FD}}(E_p + \mu_s) \right], \quad (E_p^0 \text{ in integrand})
\]

\[
\int dp_0 \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} \partial_i f_s = \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_i f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} \partial_i f_s
\]
In Eqs. (E13,E15,E16) we have implied integrals over 3-momentum besides those over derivative terms in \( E \). We now work on

\[
\hat{p}_0 \frac{d\delta(p^2)}{dp_0} \partial_i f_s = -\int dp_0 \delta(p^2) \partial_i f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} \partial_i f_s \]

\[
= -\int dp_0 \delta(p^2) \partial_i f_s - \frac{1}{(2\pi)^3} \frac{1}{E_p^3} \frac{d}{dE_p} \partial^2 \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right], \tag{E14}
\]

\[
\int dp_0 \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} \partial_i f_s = \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_i f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} \partial_i f_s \]

\[
= \frac{1}{(2\pi)^3} \frac{1}{E_p^3} \frac{d}{dE_p} \partial^2 \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right]
- \frac{1}{(2\pi)^3} \frac{1}{E_p^3} \frac{d}{dE_p} \partial^2 \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right], \tag{E15}
\]

\[
\int dp_0 \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} \partial_i f_s = \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_i f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} \partial_i f_s \]

\[
= \frac{1}{(2\pi)^3} \frac{1}{E_p^3} \frac{d}{dE_p} \partial^2 \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right]
- \frac{1}{(2\pi)^3} \frac{1}{E_p^3} \frac{d}{dE_p} \partial^2 \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right], \tag{E16}
\]

In Eqs. (E13,E15,E16) we have implied integrals over 3-momentum besides those over \( p_0 \), so we can drop the complete derivative terms in \( E_p \) which are vanishing at two limits \( E_p = 0, \infty \).

3. Evaluation of \( I_{p_0} \)

We now work on \( I_{p_0} = I_{p_0}(EM) + I_{p_0}(\omega) \), where \( I_{p_0}(EM) \) and \( I_{p_0}(\omega) \) denote the first order contribution from electromagnetic field and vorticity, respectively. The results are

\[
I_{p_0}(EM) = Q(p \cdot E) \int dp_0 \delta(p^2) \partial_0^p f_s + \frac{1}{2} sQ^2(E \cdot B) \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial_0^p f_s
= \frac{1}{(2\pi)^3} Q(p \cdot E) \frac{1}{E_p} \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right]
+ \frac{1}{(2\pi)^3} sQ^2(E \cdot B) \frac{1}{2E_p^2} \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right], \tag{E17}
\]

\[
I_{p_0}(\omega) = -sQ \int dp_0 \delta(p^2) \left[ \left( \frac{1}{2} - C_{10} - C_{11} \frac{p^2}{p^2} \right) (E \cdot \omega) \right.
+ \left( C_{30} + 1 \right) \frac{1}{p^2} (p \cdot \omega) (p \cdot E) \] \partial_0^p f_s
= -sQ \left( \frac{1}{2} - C_{10} \right) (E \cdot \omega) \int dp_0 \delta(p^2) \partial_0^p f_s
- \frac{1}{2} sQ C_{11} (E \cdot \omega) \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial_0^p f_s
+ \frac{1}{2} sQ (C_{30} + 1) (p \cdot \omega) (p \cdot E) \int dp_0 \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} \partial_0^p f_s
\[ I_p \omega = \frac{1}{(2\pi)^3} sQ(C_{30} + 1) \left[ -\frac{1}{2E_p^2} (E \cdot \omega) + \frac{3}{2E_p^3} (p \cdot \omega)(p \cdot E) \right] \]

\[ \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \]

(E18)

where we have used Eq. 345 in \( I_{s0}(\omega) \). We list following integrals involved in \( I_{p0}(\omega) \) and \( I_{p0}(\omega) \),

\[ \int dp_0 \frac{d(p^2)}{dp_0} \frac{d}{dp_0} f_s = -\int dp_0 \delta(p^2) \frac{d}{dp_0} f_s \]

\[ = -\frac{1}{(2\pi)^3} \frac{1}{E_p} \frac{d^2}{dE_p^2} \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right], \quad (E_p^0 \text{ in integrand}) \]

\[ = \frac{1}{(2\pi)^3} \frac{1}{E_p^2} \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right], \quad (E_p^0 \text{ in integrand}) \]

\[ \int dp_0 \frac{d(p^2)}{dp_0} p_0 \frac{d}{dp_0} f_s = -\int dp_0 \delta(p^2) \frac{d}{dp_0} f_s - \int dp_0 p_0 \delta(p^2) \frac{d}{dp_0} f_s \]

\[ = -\frac{1}{(2\pi)^3} \frac{1}{E_p} \frac{d^2}{dE_p^2} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \]

\[ -\frac{1}{(2\pi)^3} \frac{1}{E_p^2} \frac{d}{dE_p^2} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \quad (E_p^0 \text{ in integrand}) \]

\[ \quad \rightarrow \frac{1}{(2\pi)^3} \frac{1}{E_p} \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \quad (E_p \text{ in integrand}) \]

\[ \int dp_0 \frac{d(p^2)}{dp_0} \frac{1}{p_0} \frac{d}{dp_0} f_s \]

\[ \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} f_s \]

\[ = \frac{1}{(2\pi)^3} \frac{1}{E_p^3} \frac{d}{dE_p^2} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \]

\[ -\frac{1}{(2\pi)^3} \frac{1}{E_p^2} \frac{d}{dE_p^2} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \quad (E_p^2 \text{ in integrand}) \]

\[ \rightarrow \frac{3}{E_p^2} \frac{1}{(2\pi)^3} \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right], \quad (E_p^0 \text{ in integrand}) \]

(E21)

where we have implied that there will be integrals over \( p \) besides those over \( p_0 \) so we have dropped the complete derivatives in the integral \( \int_0^\infty dE_p \) (with \( E_p = |p| \)) since they are vanishing at two limits \( E_p = 0, \infty \).

4. Evaluation of \( I_p \)

Finally we work on \( I_p = I_p(\omega) + I_p(\omega)_0 \), where \( I_p(\omega) \) and \( I_p(\omega)_0 \) denote the zeroth order contribution, the first order contribution from electromagnetic field and vorticity, respectively. Now we evaluate \( I_p(\omega) \) and \( I_p(\omega)_0 \),

\[ I_p(\omega) = Q \int dp_0 \delta(p^2) \left[ p_0 E_i + (p \times B)_i \right] \frac{d}{dp_0} f_s \]

\[ + \frac{1}{2} sQ^2 (E \cdot B) p_i \int dp_0 \frac{d(p^2)}{dp_0} \frac{1}{p_0} \frac{d}{dp_0} f_s \]

\[ = \frac{1}{(2\pi)^3} Q E_i \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s) \right] \]

\[ + \frac{1}{(2\pi)^3} \left[ Q E + Q \frac{p}{E_p^2} \right] \frac{d}{dE_p} \left[ f_{FD}(E_p - \mu_s) + f_{FD}(E_p + \mu_s) \right], \quad (E22) \]

\[ I_p(\omega)_0 = -sQ \int dp_0 \delta(p^2) \frac{1}{p_0} p_0 (p \cdot \omega) E_i \frac{d}{dp_0} f_s \]
Here we have implied that there will be integrals over $p$ besides those over $p_0$ so we have dropped the complete derivatives in the integral $\int_0^\infty dE_p$ (with $E_p = |p|$) since they are vanishing at two limits $E_p = 0, \infty$.

\[ + sQ \int dp_0 \delta(p^2) p_i \left[ -C_{40}(\omega \cdot E) \frac{1}{p_0} + C_{41} \frac{1}{p^2 p_0} (p \cdot \omega)(p \cdot E) \right] \partial_i^p f_s \]
\[ = \frac{1}{2} sQ(p \cdot \omega) E_i \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial_i^p f_s \]
\[ - sQC_{40}(\omega \cdot E) p_i \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_i^p f_s \]
\[ - \frac{1}{2} sQC_{41}(p \cdot \omega)(p \cdot E) p_i \int dp_0 \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 sQ} \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 sQ} \int dp_0 \frac{d\delta(p^2)}{dp_0} \partial_i^p f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_i^p f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{1}{E_p} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{1}{E_p} \partial_i^p f_s - \frac{1}{(2\pi)^3 E_p^2} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{2}{E_p} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{2}{E_p} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p^2} \partial_i^p f_s \]
\[ \times \partial_i^p [f_{FD}(E_p - \mu_s) - f_{FD}(E_p + \mu_s)] \]

where we have used

\[ \int dp_0 \frac{d\delta(p^2)}{dp_0} \frac{1}{p_0} \partial_i^p f_s = \frac{1}{(2\pi)^3 E_p} \frac{1}{E_p} \partial_i^p f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \partial_i^p f_s - \int dp_0 \delta(p^2) \frac{1}{p_0} \frac{d}{dp_0} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{1}{E_p} \partial_i^p f_s - \frac{1}{(2\pi)^3 E_p^2} \partial_i^p f_s - \frac{1}{(2\pi)^3 E_p^3} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p^2} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{2}{E_p} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p^2} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{2}{E_p} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p^2} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{2}{E_p} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p^2} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{2}{E_p} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p^2} \partial_i^p f_s \]
\[ = \frac{1}{(2\pi)^3 E_p} \frac{2}{E_p} \partial_i^p f_s \]

Here we have implied that there will be integrals over $p$ besides those over $p_0$ so we have dropped the complete derivatives in the integral $\int_0^\infty dE_p$ (with $E_p = |p|$) since they are vanishing at two limits $E_p = 0, \infty$.

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