B-type Landau-Ginzburg models on Stein manifolds

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Abstract. We summarize the description of the open-closed TFT datum for B-type Landau-Ginzburg models with Stein manifold targets and discuss various constructions which lead to large classes of examples of such models.

1. Introduction

It is well-known [1, 2] that B-type classical two-dimensional topological Landau-Ginzburg (LG) models with D-branes can be defined for any non-compact smooth Calabi-Yau manifold target \( X \) and any holomorphic superpotential \( W : X \to \mathbb{C} \). The quantization of such models is expected to produce non-anomalous 2d open-closed quantum topological field theories (TFTs), which must in turn obey the axioms first introduced in [3]. The analysis of [3] shows that such quantum field theories are equivalent with a **TFT datum**, an algebraic structure subject to certain axioms. It follows that the quantum B-type LG model with D-branes associated to the pair \((X, W)\) is entirely encoded by its TFT datum, which therefore should admit a description in terms of objects naturally associated to such a pair. This description was extracted in [2] through path integral arguments. It was formulated rigorously and further analyzed in [4, 5, 6].

As shown in [5], the Landau-Ginzburg TFT datum simplifies dramatically when the target Calabi-Yau manifold is Stein, leading to a large class of B-type LG models which were not considered previously. These need not be algebraic in the sense that \( X \) need not be the analyticization of an affine complex algebraic variety and \( W \) need not be the analyticization of a regular function. As a consequence, the corresponding TFT datum is described by constructions carried out in the analytic category. In this contribution, we give a brief review of B-type LG models with Stein Calabi-Yau targets, focusing on certain constructions which lead to large classes of examples.

2. Landau-Ginzburg pairs and B-type LG models

The axiomatic treatment of [3] shows that any 2d open-closed TFT is determined by a TFT datum of parity \( \mu \in \mathbb{Z}_2 \), which consists of:
The on-shell bulk algebra $\mathcal{H}$, which is a finite-dimensional and $\mathbb{Z}_2$-graded associative and commutative $\mathbb{C}$-algebra

- The topological D-brane category $\mathcal{T}$, which is a $\mathbb{Z}_2$-graded and Hom-finite $\mathbb{C}$-linear category whose objects are the topological D-branes of the model.

- The bulk-boundary maps $e_a : \mathcal{H} \to \text{Hom}_\mathcal{T}(a, a)$, which are even $\mathbb{C}$-linear maps defined for any $\text{D-brane } a \in \text{Ob}\mathcal{T}$.

- The boundary-bulk maps $f_a : \text{Hom}_\mathcal{T}(a, a) \to \mathcal{H}$, which are $\mathbb{C}$-linear maps of $\mathbb{Z}_2$-degree $\mu$, defined for any $a \in \text{Ob}\mathcal{T}$.

- The bulk trace $\text{Tr} : \mathcal{H} \to \mathbb{C}$, an even and $\mathbb{C}$-linear non-degenerate map having the graded trace property.

- The boundary traces $\text{tr}_a : \text{Hom}_\mathcal{T}(a, a) \to \mathbb{C}$, which are $\mathbb{C}$-linear and non-degenerate maps of $\mathbb{Z}_2$-degree $\mu$ defined for each $a \in \text{Ob}\mathcal{T}$, such that the map $\sum_a \text{tr}_a : \oplus_a \text{Hom}_\mathcal{T}(a, a) \to \mathbb{C}$ has the graded trace property with respect to the associative multiplication induced on $\oplus_a \text{Hom}_\mathcal{T}(a, a)$ by the composition of $\mathcal{T}$.

This data is subject to certain further conditions discussed in [3, 4].

**Definition 2.1.** A Landau-Ginzburg (LG) pair is a pair $(X, W)$ such that:

1. $X$ is a non-compact complex and Kählerian manifold of dimension $d > 0$, which is holomorphically Calabi-Yau in the sense that its holomorphic canonical line bundle $K_X$ is holomorphically trivial.

2. $W \in O(X)$ is a non-constant holomorphic complex-valued function defined on $X$.

Given an LG pair $(X, W)$, let $Z_W^{\text{def}} = \{ x \in X | (\partial W)(x) = 0 \}$ denote the critical set of $W$; this is an analytic subset of $X$, which can be highly singular in general.

**Definition 2.2.** The modified contraction operator of an LG pair $(X, W)$ is the sheaf morphism $\iota_W = i \partial W : TX \to O_X$. We define$^1$:

- The critical sheaf $\mathcal{J}_W^{\text{def}} = \text{im}(\iota_W : TX \to O_X)$.

- The Jacobi sheaf $\text{Jac}_W^{\text{def}} = O_X / \mathcal{J}_W$.

- The Jacobi algebra $\text{Jac}(X, W)^{\text{def}} = \Gamma(X, \text{Jac}_W)$.

- The critical ideal $J(X, W)^{\text{def}} = \mathcal{J}_W(X) = \iota_W(\Gamma(X, TX)) \subseteq O(X)$.

To any LG pair $(X, W)$, one can associate a classical B-type open-closed topological LG model which was constructed in [1, 2]. When the critical set $Z_W$ is compact, the path integral arguments of [2] lead to a proposal for the TFT datum of the corresponding quantum theory, which was refined and analyzed rigorously in [4, 5, 6]; see [7] for a brief review of those results. It was shown in [4, 6] that this proposal satisfies all TFT datum axioms except for the topological Cardy constraint, which is also conjectured to hold. As shown in [5] and summarized below, the Landau-Ginzburg TFT datum admits a simpler equivalent description when $X$ is a Stein manifold.

**Remark 2.3.** The bulk algebra $\mathcal{H}$ and topological D-brane category $\mathcal{T}$ of the B-type LG model parameterized by $(X, W)$ are not only $\mathbb{C}$-linear but also $O(X)$-linear (see [1]).

$^1$ We denote by $O(X)$ the $\mathbb{C}$-algebra of holomorphic complex-valued functions defined on $X$ and by $O_X$ the sheaf of holomorphic complex-valued functions defined on open subsets of $X$, while $\Gamma(X, E)$ denotes the $O(X)$-module of globally-defined holomorphic sections of a holomorphic vector bundle $E$. 

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**1194 (2019) 012010**

doi:10.1088/1742-6596/1194/1/012010
3. B-type LG models with Calabi-Yau Stein manifold target

3.1. Stein manifolds

**Definition 3.1.** Let $X$ be a complex manifold of dimension $d > 0$. We say that $X$ is Stein if it admits a holomorphic embedding as a closed complex submanifold of $\mathbb{C}^N$ for some $N > 1$.

**Remarks**
- There exist numerous equivalent definitions of Stein manifolds.
- Any Stein manifold is Kählerian.
- The analyticization of any non-singular complex affine variety is Stein, but most Stein manifolds are not of this type.

Since all Stein manifolds are Kählerian, one can consider B-type Landau-Ginzburg models with Stein Calabi-Yau target.

**Examples**
- $\mathbb{C}^d$ is a Stein manifold.
- Every domain of holomorphy in $\mathbb{C}^d$ is a Stein manifold.
- Every closed complex submanifold of a Stein manifold is a Stein manifold.
- Any non-singular analytic complete intersection in $\mathbb{C}^N$ is a Stein manifold.
- Any (non-singular) connected open Riemann surface without border is a Stein manifold.

The following result is crucial in Stein geometry:

**Theorem 3.2** (Cartan’s theorem B). For every coherent analytic sheaf $\mathcal{F}$ on a Stein manifold $X$, the sheaf cohomology $H^i(X, \mathcal{F})$ vanishes for all $i > 0$.

For what follows, let $(X, W)$ to be a Landau-Ginzburg pair such that $X$ is a Stein Calabi-Yau manifold of complex dimension $d$.

3.2. The bulk algebra

**Theorem 3.3.** [2] Suppose that the critical locus $Z_W$ is compact. Then:

1. $Z_W$ is necessarily finite.
2. The bulk algebra is concentrated in even degree and can be identified with the Jacobi algebra:

$$\mathcal{H} \cong_{\mathcal{O}(X)} \text{Jac}(X, W).$$

Moreover, we have an isomorphism of $\mathcal{O}(X)$-algebras:

$$\text{Jac}(X, W) \cong_{\mathcal{O}(X)} \mathcal{O}(X)/\text{J}(X, W).$$

3. Suppose that $X$ is holomorphically parallelizable (i.e. $TX$ is holomorphically trivial). Then:

$$\text{J}(X, W) = \langle u_1(W), \ldots, u_d(W) \rangle,$$

where $u_1, \ldots, u_d$ is any global holomorphic frame of $TX$ and $\langle f_1, \ldots, f_n \rangle$ denotes the ideal of $\mathcal{O}(X)$ generated by the holomorphic functions $f_j \in \mathcal{O}(X)$. 


3.3. The topological D-brane category

**Definition 3.4.** A holomorphic factorization of $W$ is a pair $(E,D)$, where $E$ is a $\mathbb{Z}_2$-graded holomorphic vector bundle defined on $X$ and $D \in \Gamma(X, \text{End}_1(E))$ is an odd holomorphic section of $E$ such that $D^2 = \text{Wid}_E$.

**Definition 3.5.** The holomorphic dg category of holomorphic factorizations of $(X,W)$ is the $\mathbb{Z}_2$-graded $O(X)$-linear dg category $\text{F}(X,W)$ defined as follows:

- The objects of $\text{F}(X,W)$ are the holomorphic factorizations of $W$.
- The hom-sets are the $O(X)$-modules $\text{Hom}_{\text{F}(X,W)}(a_1,a_2) \overset{\text{def}}{=} \Gamma(X,\text{Hom}(E_1,E_2))$, endowed with the $\mathbb{Z}_2$-grading:
  $$\text{Hom}^\kappa_{\text{F}(X,W)}(a_1,a_2) = \Gamma(X,\text{Hom}^\kappa(E_1,E_2)), \ \forall \kappa \in \mathbb{Z}_2$$
- and with the defect differentials $\partial_{a_1,a_2}$ determined uniquely by the condition:
  $$\partial_{a_1,a_2}(f) = D_2 \circ f - (-1)^\kappa f \circ D_1, \ \forall f \in \Gamma(X,\text{Hom}^\kappa(E_1,E_2)), \ \forall \kappa \in \mathbb{Z}_2.$$

Here $a_1 := (E_1,D_1)$ and $a_2 := (E_2,D_2)$ are any two holomorphic factorizations of $W$.
- The composition of morphisms is the obvious one.

Let $HF(X,W) \overset{\text{def}}{=} H(\text{F}(X,W))$ be the total cohomology category of the dg category $F(X,W)$.

**Definition 3.6.** A projective analytic factorization of $W$ is a pair $(P,D)$, where $P$ is a finitely generated projective supermodule and $D \in \text{End}_1^{O(X)}(P)$ is an odd endomorphism of $P$ such that $D^2 = \text{Wid}_P$.

**Definition 3.7.** The dg category $\text{PF}(X,W)$ of projective analytic factorizations of $W$ is the $\mathbb{Z}_2$-graded $O(X)$-linear dg category defined as follows:

- The objects of $\text{PF}(X,W)$ are the projective analytic factorizations of $W$.
- The hom-sets $\text{Hom}_{\text{PF}(X,W)}((P_1,D_1),(P_2,D_2)) \overset{\text{def}}{=} \text{Hom}_{O(X)}(P_1,P_2)$ are endowed with $\mathbb{Z}_2$-grading and with the odd differential $\partial := \partial_{(P_1,D_1),(P_2,D_2)}$
  $$\partial(f) = D_2 \circ f - (-1)^{\deg f} f \circ D_1 f, \ \forall f \in \text{Hom}_{O(X)}(P_1,P_2).$$

- The composition of morphisms is the obvious one.

Let $\text{HPF}(X,W) \overset{\text{def}}{=} H(\text{PF}(X,W))$ be the total cohomology category of the dg category $PF(X,W)$.

For any unital commutative ring $R$ and any element $r \in R$, let $\text{MF}(R,r)$ denote category of finite rank matrix factorizations of $W$ over $R$ and $\text{HMF}(R,r)$ denote its total cohomology category (which is $\mathbb{Z}_2$-graded and $R$-linear).

**Theorem 3.8.** [5] Let $(X,W)$ be an LG pair such that $X$ is a Stein manifold. Then:

(i) There exists a natural equivalence of $O(X)$-linear and $\mathbb{Z}_2$-graded dg categories:
  $$\text{F}(X,W) \simeq_{O(X)} \text{PF}(X,W).$$

(ii) If the critical locus $Z_W$ is finite, then the topological D-brane category $T$ is given by:
  $$T \equiv \text{HF}(X,W) \simeq_{O(X)} \text{HPF}(X,W).$$

(iii) Multiplication with elements of the critical ideal $J(X,W)$ acts trivially on $HF(X,W)$, so $T$ can be viewed as a $\mathbb{Z}_2$-graded $\text{Jac}(X,W)$-linear category.

(iv) The even subcategory $T^0$ has a natural triangulated structure.

(v) There exists an $O(X)$-linear dg functor $\Xi : F(X,W) \to \oplus_{p \in Z_W} \text{MF}(O_{X,p},\tilde{W}_p)$ which induces a full and faithful $\text{Jac}(X,W)$-linear functor $\Xi_* : HF(X,W) \to \oplus_{p \in Z_W} \text{HMF}(O_{X,p},\tilde{W}_p)$. 

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Let $\text{Definition 3.9.}$ holomorphically trivial iff it is topologically trivial, in which case we simply say that it is trivial.

By the Oka-Grauert principle, a holomorphic vector bundle $V$.

### 3.4. The remaining objects

When $X$ is a Stein manifold with complex dimension $d$, the remaining objects of the TFT datum are as follows (see [5]):

- The bulk trace is given by:
  \[
  \text{Tr}(f) = \sum_{p \in \mathbb{Z}_W} A_p \text{Res}_p \left[ \frac{\hat{f}_p \hat{\Omega}_p}{\det\hat{\Omega}_p(\partial W)} \right].
  \]

- The boundary trace of the D-brane (holomorphic factorization) $a = (E, D)$ is given by the sum of generalized Kapustin-Li traces:
  \[
  \text{tr}_a(s) = (-1)^{\frac{d(d-1)}{2}} \sum_{p \in \mathbb{Z}_W} A_p \text{Res}_p \left[ \frac{\text{str}(\det\hat{\Omega}_p(\partial D_p) \hat{s}_p) \hat{\Omega}_p}{\det\hat{\Omega}_p(\partial W_p)} \right].
  \]

Here $\Omega$ is a holomorphic volume form on $X$, $A_p$ are normalization constants and $\text{Res}_p$ denotes the Grothendieck residue on $\mathcal{O}_{X,p}$.

- The bulk-boundary and boundary-bulk maps of $a = (E, D)$ are given by:
  \[
  e_a(f) \equiv i^d(-1)^{\frac{d(d-1)}{2}} \sum_{p \in \mathbb{Z}_W} \hat{f}_p \text{id}_{E_p}, \quad \forall f \in \mathcal{H} \equiv \text{Jac}(X, W)
  \]
  \[
  f_a(s) \equiv i^d \sum_{p \in \mathbb{Z}_W} \text{str}(\det\hat{\Omega}_p(\partial D_p) \hat{s}_p), \quad \forall s \in \text{End}_\tau(a) \equiv \Gamma(X, \text{End}(E)).
  \]

### 3.5. Topologically non-trivial elementary holomorphic factorizations

By the Oka-Grauert principle, a holomorphic vector bundle $V$ on a Stein manifold $X$ is holomorphically trivial iff it is topologically trivial, in which case we simply say that it is trivial.

**Definition 3.9.** Let $(X, W)$ be an LG pair where $X$ is a Stein manifold. A holomorphic factorization $(E, D)$ of $W$ is called elementary if $\text{rk} E^0 = \text{rk} E^1 = 1$ and topologically trivial if $E$ is trivial as a $\mathbb{Z}_2$-graded holomorphic vector bundle, i.e. if both $E^0$ and $E^1$ are trivial.

**Construction.** Let $X$ be a Calabi-Yau Stein manifold with $H^2(X, \mathbb{Z}) \neq 0$. Given a non-trivial holomorphic line bundle $L$ on $X$ and non-trivial holomorphic sections $v \in \Gamma(X, L)$ and $u \in \Gamma(X, L^{-1})$, the $\mathbb{Z}_2$-graded holomorphic vector bundle $E \overset{\text{def}}{=} \mathcal{O}_X \oplus L$ admits the odd global holomorphic section $D \overset{\text{def}}{=} \left[ \begin{array}{cc} 0 & v \\ u & 0 \end{array} \right]$. The tensor product $u \otimes v \in H^0(L \otimes L^{-1})$ identifies with a non-trivial holomorphic function $W$ defined on $X$ through any isomorphism $L \otimes L^{-1} \simeq \mathcal{O}_X$ and $(E, D)$ is a non-trivial elementary holomorphic factorization of $W$.

**Remark 3.10.** Let $\text{Pic}^\text{an}(X)$ denote the analytic Picard group of holomorphic line bundles on any Stein manifold $X$. Then the assignment $L \rightarrow c_1(L)$ induces an isomorphism $\text{Pic}^\text{an}(X) \simeq H^2(X, \mathbb{Z})$. When $X$ is the analytic space associated to an algebraic variety, the natural map $\text{Pic}^\text{alg}(X) \rightarrow H^2(X, \mathbb{Z})$ from the algebraic Picard group need not be isomorphism.

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2 Given a holomorphic vector bundle $V$ on $X$, $\hat{\beta}_p$ denotes the germ at $p \in X$ of a holomorphic section $\beta \in \Gamma(X, V)$.

3 Here $\text{str}$ denotes the supertrace on the finite-dimensional $\mathbb{Z}_2$-graded vector space $\text{End}_c(\text{End}_\tau(a))$. 

4. Examples

4.1. Domains of holomorphy in $\mathbb{C}^d$

Let $X = U \subseteq \mathbb{C}^d$ be a domain of holomorphy. Then $U$ is Stein and holomorphically parallelizable (hence Calabi-Yau) with global holomorphic frame $u_i = \partial_i$, where $\partial_i := \frac{\partial}{\partial z_i}$ and $\{z^1, \ldots, z^d\}$ are global holomorphic coordinates on $U$. Let $W \in \mathcal{O}(U)$ have finite critical set. We have [5]:

$$HPV(U, W) = HPV^0(U, W) \cong \mathcal{O}(U)/\langle \partial_1 W, \ldots, \partial_d W \rangle .$$

**Example 4.1.** [5] Suppose that $U$ is contractible. Then $HDF(U, W)$ is equivalent with the $\mathbb{Z}_2$-graded cohomological category $HMF(\mathcal{O}(U), W)$ of analytic matrix factorizations of $W$.

4.2. Open Riemann surfaces

Let $\Sigma$ be any open Riemann surface, i.e. a smooth, connected and non-compact complex Riemann surface without boundary. Such a surface need not be affine algebraic; in particular, it can have infinite genus and an infinite number of Freudenthal ends. Any open Riemann surface is Stein and any holomorphic vector bundle defined on it is trivial. In particular, $\Sigma$ is holomorphically parallelizable and hence Calabi-Yau. Any finitely-generated projective $\mathcal{O}(\Sigma)$-module is free, thus $HPF(\Sigma, W)$ coincides with the $\mathbb{Z}_2$-graded category $HMF(\mathcal{O}(\Sigma), W)$ of finite rank matrix factorizations of $W$ over the ring of holomorphic functions defined on $\Sigma$.

**Proposition 4.2.** [5] Let $W \in \mathcal{O}(\Sigma)$ be a non-constant holomorphic function. Then the bulk algebra of the corresponding B-type LG model is given by:

$$H \simeq \text{Jac}(\Sigma, W) = \mathcal{O}(\Sigma)/\langle v(W) \rangle ,$$

where $v$ is any non-trivial globally-defined holomorphic vector field on $\Sigma$. Moreover, there exists an equivalence of $\mathcal{O}(\Sigma)$-linear $\mathbb{Z}_2$-graded categories:

$$HF(\Sigma, W) \simeq HPF(\Sigma, W) = HMF(\mathcal{O}(\Sigma), W) .$$

More detail on B-type LG models with open Riemann surface targets can be found in [9].

4.3. Analytic complete intersections

Let $X \subset \mathbb{C}^N$ be an analytic complete intersection of complex dimension $d$, defined by the regular sequence of holomorphic functions $f_1, \ldots, f_{N-d} \in \mathcal{O}(\mathbb{C}^N)$. Let $u_1, \ldots, u_d$ be a global holomorphic frame of $TX$. Then $X$ is Stein and holomorphically parallelizable (hence also Calabi-Yau). We have $\mathcal{O}(X) \simeq \mathcal{O}(\mathbb{C}^N)/\langle f_1, \ldots, f_{N-d} \rangle$. Any $W \in \mathcal{O}(X)$ is the restriction to $X$ of a holomorphic function $W \in \mathcal{O}(\mathbb{C}^N)$. Let $Z_W \subset \mathbb{C}^N$ be the critical locus of $W$. Then $Z_W \cap X \subseteq Z_W$, but the inclusion may be strict. This construction provides a large class of B-type LG models with Stein Calabi-Yau targets. The simplest case arises when $X$ is an analytic hypersurface in $\mathbb{C}^N$.

**Proposition 4.3.** [2] Let $X$ be a non-singular analytic hypersurface in $\mathbb{C}^N$ defined by the equation $f = 0$. Then the holomorphic tangent vector fields $(v_{ij})_{1 \leq i < j \leq N}$ defined on $X$ through:

$$v_{ik}^j = (\partial_j f) \delta_{ik} - (\partial_i f) \delta_{jk} \quad (k = 1, \ldots, N)$$

generate each fiber of $TX$. If $W \in \mathcal{O}(\mathbb{C}^N)$ is a holomorphic function, then the critical locus $Z_W$ of the restriction $W \overset{\text{def.}}{=} W|_X$ is defined by the system:

$$f = 0 , \quad \partial_i W \partial_j f - \partial_j W \partial_i f = 0 \quad (1 \leq i < j \leq N) .$$
If $W$ has isolated critical points on $X$, then we have:

$$\text{Jac}(X,W) = O(\mathbb{C}^N)/I,$$

where $I \subset O(\mathbb{C}^N)$ is the ideal generated by $f$ and by the holomorphic functions $\partial_i W \partial_j f - \partial_j W \partial_i f$ with $1 \leq i < j \leq N$.

**Example 4.4.** Let $X$ be the non-singular analytic hypersurface defined in $\mathbb{C}^3$ by the equation $f(x_1, x_2, x_3) = x_1e^{x_2} + x_2e^{x_3} + x_3e^{x_1} = 0$ and $W \in O(\mathbb{C}^3)$ be the holomorphic function given by $W(x_1, x_2, x_3) = x_1^{n+1} + x_2x_3$, where $n \geq 1$. Let $W = W|_X \in O(X)$. The critical locus of $W$ coincides with the origin of $\mathbb{C}^3$, which lies on $X$. Thus $Z_W$ contains the point $(0, 0, 0) \in X$. In this example, we have:

$$\partial_1 f = e^{x_2} + x_3e^{x_1}, \quad \partial_2 f = e^{x_3} + x_1e^{x_2}, \quad \partial_3 f = e^{x_1} + x_2e^{x_3}.$$  

The vector fields of Proposition 2. are given by:

$$v_{23} = (0, \partial_3 f, -\partial_2 f) = (0, e^{x_1} + x_2e^{x_3}, -e^{x_3} - x_1e^{x_2})$$
$$v_{13} = (\partial_3 f, 0, -\partial_1 f) = (e^{x_1} + x_2e^{x_3}, 0, -e^{x_2} - x_3e^{x_1})$$
$$v_{12} = (\partial_2 f, -\partial_1 f, 0) = (e^{x_3} + x_1e^{x_2}, -e^{x_2} - x_3e^{x_1}, 0)$$

and the defining equations (4.3) of $Z_W$ take the form:

$$x_1e^{x_2} + x_2e^{x_3} + x_3e^{x_1} = 0$$
$$\begin{align*}
(n + 1)x_1^n(e^{x_1} + x_2e^{x_3}) - x_2(e^{x_2} + x_3e^{x_1}) &= 0 \\
x_3(e^{x_1} + x_2e^{x_3}) - x_2(e^{x_3} + x_1e^{x_2}) &= 0 \\
(n + 1)x_1^n(e^{x_3} + x_1e^{x_2}) - x_3(e^{x_2} + x_3e^{x_1}) &= 0.
\end{align*}$$

The bulk space $H$ is the Jacobi algebra $\text{Jac}(X,W) = O(\mathbb{C}^3)/I$, where $I \subset O(\mathbb{C}^3)$ is the ideal generated by $f$ and the four holomorphic functions appearing in the left hand side of the previous system. Numerical study shows that, for generic $n \geq 1$, the above transcendental system admits solutions different from $x_1 = x_2 = x_3 = 0$, so $W$ has critical points on $X$ which differ from the origin. For any $k \in \{0, \ldots, n+1\}$, an example of holomorphic (here even algebraic) factorization $(E, D_k)$ of $W$ on $\mathbb{C}^3$ is obtained by taking $E^0 = E^1 = O^{\mathbb{C}^2}_{\mathbb{C}^3}$ with:

$$D_k = \begin{bmatrix} 0 & b_k \\
 a_k & 0 \end{bmatrix}, \quad \text{where } a_k = \begin{bmatrix} x_2^k & x_1^{n+1-k} \\
x_1^k & -x_3 \end{bmatrix} \quad \text{and } b_k = \begin{bmatrix} x_3^k & x_1^{n+1-k} \\
x_1^k & -x_2 \end{bmatrix}$$

This induces a holomorphic factorization $(E|_X, D_k|_X)$ of $W$.

### 4.4. Complements of affine hyperplane arrangements

Let $\mathcal{A}$ be a $d$-dimensional central complex affine hyperplane arrangement, i.e. a finite set of distinct linear hyperplanes $H = \ker \alpha_H \subset \mathbb{C}^d$, where $\alpha_H : \mathbb{C}^d \to \mathbb{C}$ are non-trivial linear functionals. The complex manifold $X = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}} H$ is Stein and parallelizable (hence Calabi-Yau) and coincides with the analyticization of the affine hypersurface in $\mathbb{C}^{d+1}$ defined by:

$$x_{d+1} \prod_{H \in \mathcal{A}} \alpha_H(x_1, \ldots, x_d) = 1.$$
The cohomology ring $H(X, \mathbb{Z})$ is isomorphic as a graded $\mathbb{Z}$-algebra with the *Orlik-Solomon algebra* of $\mathcal{A}$, whose homogeneous components $A^k$ are free $\mathbb{Z}$-modules of finite rank. In particular, the cohomology groups $H^k(X, \mathbb{Z})$ are free Abelian groups of finite rank. The *intersection poset* $L$ of $\mathcal{A}$ is the set of all those subspaces $F \subset \mathbb{C}^d$ (called flats) which arise as finite intersections of hyperplanes from $\mathcal{A}$, ordered by reverse inclusion. Since $\mathcal{A}$ is central, its intersection poset $L$ is a bounded geometric lattice with greatest element given by $\cap_{H \in \mathcal{A}} H$. Let $P_X(t) \overset{\text{def}}{=} \sum_{j=0}^d \text{rk} H^j(X, \mathbb{Z}) t^j$ be the Poincaré polynomial of $X$.

**Proposition 4.5.** [8, Theorem 2.2 & Corollary 3.6] Let $\mu_L$ be the Möbius function of the locally-finite poset $L$. Then $\mu_L(\mathbb{C}^d, F) \neq 0$ for all flats $F \subset L$ and the sign of $\mu_L(\mathbb{C}^d, F)$ equals $(-1)^{\text{codim} F}$. Moreover, we have:

$$P_X(t) = \sum_{F \in L} |\mu_L(\mathbb{C}^d, F)| t^{\text{codim} F} = \sum_{F \in L} \mu_L(\mathbb{C}^d, F) (-t)^{\text{codim} F}.$$ 

One has $\text{rk} H^j(X, \mathbb{Z}) > 0$ for all $j = 0, \ldots, \text{rk} \mathcal{A}$ and $\text{rk} H^j(X, \mathbb{Z}) = 0$ for $j > \text{rk} \mathcal{A}$ (see [8, Chap. 2.5, Exercise 2.5]), where $\text{rk} \mathcal{A} \overset{\text{def}}{=} \text{codim}_C [\cap_{H \in \mathcal{A}} H]$ is the rank of $\mathcal{A}$. This implies [5] that the complement of any $d$-dimensional central complex affine hyperplane arrangement $\mathcal{A}$ with $d \geq 2$ and $\text{rk} \mathcal{A} \geq 2$ admits topologically non-trivial elementary holomorphic factorizations.

**Example 4.6.** Let $\mathcal{A} \subset \mathbb{C}^3$ be the 6-hyperplane arrangement defined by the linear functionals $x, y, z, x-y, x-z$ and $y-z$. Then $P_X(t) = 1 + 6t + 11t^2 + 6t^3$ and hence $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{11}$.

**Example 4.7.** For any $d \geq 2$, let $\mathcal{A} \subset \mathbb{C}^d$ be the Boolean arrangement, defined by the equation $x_1 \cdots x_d = 0$. Then $X \overset{\text{def}}{=} \mathbb{C}^d \setminus (\cup_{H \in \mathcal{A}} H) = (\mathbb{C}^*)^d$ is a complex algebraic torus which can be identified with the analytic space associated to the complex affine variety:

$$X_{\text{alg}} \overset{\text{def}}{=} \text{Spec}(\mathbb{C}[x_1, \ldots, x_d, x_1^{-1}, \ldots, x_d^{-1}]) = \text{Spec} \left( \mathbb{C}[x_1, \ldots, x_{d+1}]/(x_1 \cdots x_{d+1} - 1) \right).$$

In this case, the Orlik-Solomon algebra coincides with the Grassmann $\mathbb{Z}$-algebra on $d$ generators and the Poincaré polynomial of $X$ is $P_X(t) = (t + 1)^d$ (see [8, Example 2.12 & Example 3.3]), hence $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{d(d-1)/2}$. One can show [5] that $\text{Pic}_{\text{alg}}(X) = 0$ while $\text{Pic}_{\text{an}}(X) \simeq \mathbb{Z}^{d(d-1)/2}$. Thus $X$ supports topologically non-trivial elementary holomorphic factorizations, even though all algebraic elementary factorizations on $X$ are topologically trivial.

**Example 4.8.** Let $X \simeq (\mathbb{C}^*)^2$ be the complement of the 2-dimensional Boolean hyperplane arrangement, which embeds in $\mathbb{C}^3$ as the affine hypersurface with equation $x_1 x_2 x_3 = 1$. We have $\text{Pic}_{\text{alg}}(X) = 0$ but $\text{Pic}_{\text{an}}(X) \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$. To describe $\text{Pic}_{\text{an}}(X)$ explicitly, write $X = \mathbb{C}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts on $\mathbb{C}^2$ by:

$$(n_1, n_2) \cdot (z_1, z_2) \overset{\text{def}}{=} (z_1 + n_1, z_2 + n_2), \quad \forall (z_1, z_2) \in \mathbb{C}^2, \forall (n_1, n_2) \in \mathbb{Z}^2.$$
Consider the lattice $\Lambda := \mathbb{Z} \oplus i\mathbb{Z} \subset \mathbb{C}$ and the elliptic curve $X_0 \overset{\text{def}}{=} \mathbb{C}/\Lambda$ of modulus $\tau = i$. The maps $\varphi^\pm : \mathbb{C}^2 \to \mathbb{C}$ given by $\varphi^\pm(z_1, z_2) = z_1 \pm iz_2$ induce surjections $\varphi^\pm : X \to X_0$ which are homotopy retractions of $X$ onto $X_0$. Pullback by $\varphi^\pm$ induces isomorphisms:

$$\varphi^\pm : \text{NS}(X_0) \to H^2(X, \mathbb{Z}) ,$$

which differ only by sign, where $\text{NS}(X_0) \overset{\text{def}}{=} \text{Pic}_{\text{an}}(X_0)/\text{Pic}_{\text{an}}^0(X_0) = \text{Pic}_{\text{alg}}(X_0)/\text{Pic}_{\text{alg}}^0(X_0) \simeq H^2(X_0, \mathbb{Z})$ is the Neron-Severi group of $X$. Let $p_0 \in X_0$ be the mod $\Lambda$ image of the point $z_0 = \frac{1+i}{2} \in \mathbb{C}$ and consider the holomorphic line bundle $L = \mathcal{O}(X_0(p_0))$ on $X_0$. Then the class of $L$ modulo $\text{Pic}_{\text{an}}^0(X_0)$ generates $\text{NS}(X_0)$ and it was shown in [5] that the pullbacks $L_\pm := (\varphi^\pm)^*(L)$ are holomorphic line bundles defined on $X$, each of which generates $\text{Pic}_{\text{an}}(X)$ and which satisfy $L_- = L_+^{-1}$. Up to multiplication by a non-zero complex number, there exists a unique holomorphic section of $\mathcal{O}(X_0(p_0))$ which vanishes at $p_0$. A convenient choice $s_0 \in \Gamma(\mathcal{O}(X_0(p_0)))$ is described by the Riemann-Jacobi theta function (traditionally denoted by $\vartheta_{00}$ or $\vartheta_3$) at modulus $\tau = i$:

$$\vartheta(z) = \vartheta_{00}(z)\big|_{\tau = i} = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 + 2\pi niz} ,$$

which satisfies:

$$\vartheta(z + 1) = \vartheta(z) , \quad \vartheta(z \pm i) = e^{\pi \mp 2\pi iz}\vartheta(z)$$

and vanishes on the lattice $\frac{1+i}{2} + \Lambda$. The $\varphi^\pm$-pullbacks of $s_0$ give global holomorphic sections $s_{\pm} \in H^0(L_{\pm})$, which are described by the $\mathbb{Z}^2$-quasiperiodic holomorphic functions $f_{\pm} \in O(\mathbb{C}^2)$ defined through:

$$f_{\pm}(z_1, z_2) \overset{\text{def}}{=} \vartheta(z_1 \pm iz_2) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 + 2\pi in(z_1 \pm iz_2)} .$$

The tensor product $s_+ \otimes s_- \in H^0(L_+ \otimes L_-) \simeq O(X)$ corresponds to the holomorphic function $f \overset{\text{def}}{=} f_+ f_- \in O(\mathbb{C}^2)$, which satisfies:

$$f(z_1 + 1, z_2) = f(z_1, z_2) , \quad f(z_1, z_2 + 1) = e^{2\pi i z_2} f(z_1, z_2) .$$

The isomorphism $L_+ \otimes L_- \simeq \mathcal{O}_X$ is realized on $\mathbb{Z}^2$-factors of automorphy by the holomorphic function $S : \mathbb{C}^2 \to \mathbb{C}^*$ given by:

$$S(z_1, z_2) \overset{\text{def}}{=} e^{-2\pi z_2^2} ,$$

which satisfies $S(z_1, z_2) = e^{2\pi i z_2}$. The section $s_+ \otimes s_- \in H^0(L_+ \otimes L_-)$ corresponds through this isomorphism to a holomorphic function $W \in H^0(\mathcal{O}_X) = O(X)$ whose lift to $\mathbb{C}^2$ is the $\mathbb{Z}^2$-periodic function:

$$\tilde{W}(z_1, z_2) \overset{\text{def}}{=} S(z_1, z_2) f(z_1, z_2) = e^{-2\pi z_2^2} \vartheta(z_1 + iz_2) \vartheta(z_1 - iz_2) .$$

Applying the construction of Subsection 3.5 gives a topologically non-trivial elementary holomorphic factorization $(E, D)$ of $W$, where $E = \mathcal{O}_X \oplus L_+$ and $D = \begin{bmatrix} 0 & s_- \\ s_+ & 0 \end{bmatrix}$. Notice that $\text{Pic}_{\text{an}}(X_0) = \text{Pic}_{\text{alg}}(X_0)$ by the GAGA correspondence since $X_0$ is a projective variety.
4.5. Complements of anticanonical divisors in Fano manifolds

The following result produces a large class of Calabi-Yau Stein manifolds which are analytifications of non-singular complex affine varieties:

**Proposition 4.9.** [5] Let $Y$ be a non-singular complex projective Fano variety and $D$ be a smooth anticanonical divisor on $Y$. Then the analytification of the complement $X := Y \setminus D$ is a non-compact Stein Calabi-Yau manifold.

**Example 4.10.** Let $\mathbb{P}^d$ be the $d$-dimensional projective space with $d \geq 2$. In this case, we have $K_{\mathbb{P}^d} = O_{\mathbb{P}^d}(-d-1)$. Any irreducible smooth hypersurface $Z$ of degree $d+1$ in $\mathbb{P}^d$ defines an anticanonical divisor, whose complement $X = \mathbb{P}^d \setminus Z$ is a Stein Calabi-Yau manifold.

In this example, we have $H_1(X,\mathbb{Z}) = \mathbb{Z}^{d+1}$ and the torsion part of $H^2(X,\mathbb{Z})$ is isomorphic with $\text{Ext}^1(H_1(X,\mathbb{Z}),\mathbb{Z}) = \text{Ext}^1(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}$. Thus $X$ admits non-trivial holomorphic line bundles and hence also topologically non-trivial elementary holomorphic factorizations.

4.6. The total space of a holomorphic vector bundle

Let $Y$ be a Stein manifold and $\pi : E \to Y$ be a holomorphic vector bundle such that $c_1(E) = c_1(K_Y)$. Then the total space $X$ of $E$ is Stein and Calabi-Yau. A particular case of this construction is obtained by taking $E = K_Y$ of $Y$.

**Example 4.11.** [5] Consider the hypersurface $Z = \{[x,y,z] \in \mathbb{P}^2 \mid x^2 + y^2 + z^2 = 0\}$ in $\mathbb{P}^2$. Then $Y' := \mathbb{P}^2 \setminus Z$ is Stein with $H^2(Y',\mathbb{Z}) \simeq \mathbb{Z}_2$ and $c_1(K_Y) = -\gamma$, where $\gamma$ is the generator of $H^2(Y,\mathbb{Z})$. Let $X$ be the total space of $K_Y$. The pullback $L$ of $K_Y$ to $X$ has non-trivial first Chern class, hence the $\mathbb{Z}_2$-graded holomorphic vector bundle $E = O_X \oplus L$ supports holomorphic factorizations of some non-zero function $W \in O(X)$.

Acknowledgments

This work was supported by the research grants IBS-R003-S1 and IBS-R003-D1. The work of E.M.B. was also supported by the joint Romanian-LIT, JINR, Dubna Research Project, theme no. 05-6-1119-2014/2019 and by the Romanian government grant PN 18090101/2019.

References

[1] Lazaroiu C I 2005 *J. High Energy Phys.* JHEP 505(2005)037
[2] Herbst M and Lazaroiu C I 2005 *J. High Energy Phys.* JHEP 0505(2005)044
[3] Lazaroiu C I 2001 *Nucl. Phys. B* 603 497–530
[4] Babalic E M, Doryn D, Lazaroiu C I and Tavakol M 2018 *Commun. Math. Phys.* 361 1169–1234 (arXiv:1610.09103v3 [math.DG])
[5] Babalic E M, Doryn D, Lazaroiu C I and Tavakol M 2018 *Commun. Math. Phys.* 362 129–165 (arXiv:1610.09813v3 [math.DG])
[6] Doryn D and Lazaroiu C I 2018 *Preprint* arXiv:1802.06261 [math.AG]
[7] Babalic E M, Doryn D, Lazaroiu C I and Tavakol M 2017 *Preprint* arXiv:1709.00684 [math.DG]
[8] Dimca A 2017 *Universitext*, Springer
[9] Lazaroiu C I and Tavakol M 2018 *Preprint*, conference proceedings for “Group 32” Prague