Spin chain from marginally deformed $AdS_3 \times S^3$

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We derive a spin chain Hamiltonian from a fast spinning string in the marginally deformed $AdS_3 \times S^3$. This corresponds to a closed trajectory swept out by the $SU(2)$ or $SL(2)$ spin vector on the surface of one-parameter deformed two-sphere or hyperboloid in the background of anisotropic magnetic field interaction. In the limit of small deformation, a class of general Landau-Lifshitz equation with a nontrivial anisotropic matrix can be derived.

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I. INTRODUCTION

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence has revealed deep relation between string theory and gauge theory, in particular the correspondence between IIB strings on $AdS_5 \times S^5$ and $N = 4$ super Yang-Mills theory (SYM) at the 't Hooft’s large $N$ limit\cite{1, 2, 3}. While the correspondence was mostly tested for those Bogomol’nyi-Prasad-Sommerfield (BPS) states where partial supersymmetry protects it against quantum correction\cite{4}, semiclassical analysis on the near-BPS sector is also considered in Ref.\cite{5}; the integrability of this Berenstein-Mandalena-Nastase (BMN) limit allows for quantitative tests of correspondence beyond BPS states where the energy of classical string solutions is compared to the anomalous dimension of SYM operators with large $R$-charge. On the other hand, intimate relation between SYM dynamics and integrable spin chain was realized in Ref.\cite{6}, because the planar limit of the $\beta$-function was realized in Ref.\cite{6}, because the planar limit of the $\beta$-function was identified with the Hamiltonian of integrable spin chains.

The string/spin chain correspondence was pushed further in Ref.\cite{7}, where a classical string spinning on $S^3 \in S^5$ were identified with semiclassical coherent states in the $SU(2)$ spin chain system. Later this identification was explored in the full $SU(3)$ sector\cite{8} and $SL(2)$ sector\cite{9}. A few examples of generalization were discussed in the past: the fast spinning string in the marginally $\beta$-deformed $N = 4$\cite{10} corresponds to an anisotropic XXZ spin chain\cite{11}. The Melvin’s magnetic-deformed background was also studied in Ref.\cite{12}.

Here we are interested in another integrable system, $AdS_3 \times S^3$ background\cite{13, 14}. As discussed in Ref.\cite{15}, asymmetric marginal deformations of $G = SU(2)_k$ and $SL(2, R)_k$ Wess-Zumino-Witten (WZW) models create new families of exact string vacua. The corresponding geometry is a deformed $S^3$ or $AdS_3$ which includes, in particular, geometric coset $S^3$ or $AdS_2$. In general, the effect of deformation is to turn on an electromagnetic field along some Cartan direction inside $G$, which in turn induces a gravitational back-reaction on the metric and the three-form antisymmetric tensor.

The plan of this report is as follows. We derive spin chain models in the fast spinning limit from the deformed $SU(2)_k$ in section II, and deformed $SL(2)_k$ in section III. They correspond to the deformed configuration space of spin vector, a squashed two-sphere and hyperboloid, respectively. In section IV, we take the limit of small deformation and find the system can be exactly derived from a class of general Landau-Lifshtiz equation with a nontrivial anisotropic matrix. In section V, we have a discussion and comments.

II. DEFORMED $SU(2)_k$ SPIN CHAIN MODEL

The $SU(2)$ manifold is a three-sphere of unit radius. The $SU(2)_k$ WZW model then corresponds to a three-sphere of radius $k$ at classical level. In Ref.\cite{16}, the authors discussed asymmetric marginal deformation in the background of three-sphere which breaks the bosonic affine algebra $SU(2)_L \times SU(2)_R$ into $U(1)_L \times SU(2)_R$. The deformed metric is given by

$$ds^2 = \frac{k}{4}[-dt^2 + d\beta^2 + \sin^2 \beta d\alpha^2 + (1 - 2H^2)(d\gamma + \cos \beta d\kappa)^2]$$

(1)

We request $0 \leq H^2 \leq 1/2$ to avoid non-unitary gauge field ($H^2 < 0$) and closed time-like geodesics ($H^2 > 1/2$). For trivial $H = 0$ and at level $k = 1$, one recovers the unit round $S^3$ as Hopf fibration. For maximal $H^2 = 1/2$, the $S^1$ fiber degenerates and we are left with $S^2$.

Now we consider a spinning string by sending $\alpha \rightarrow t + \alpha$ in Eq.\cite{17}, with gauge choice $t = \kappa \tau$, the Polyakov action reads,

$$S = \sqrt{\frac{k}{16\pi}} \int \int d\tau d\sigma - 2H^2 \cos^2 \beta \kappa^2$$

$$-(1 - 2H^2 \cos^2 \beta)\alpha'^2 - \beta'^2 - (1 - 2H^2)\gamma'^2$$

$$-2(1 - 2H^2) \cos \beta \alpha' \gamma' + 2(1 - 2H^2 \cos^2 \beta)\kappa'$$

$$+ 2(1 - 2H^2) \cos \beta \kappa' \gamma' + (1 - 2H^2 \cos^2 \beta)\kappa'^2$$

$$+ \beta'^2 + (1 - 2H^2) \gamma'^2 + 2(1 - 2H^2) \cos \beta \kappa' \gamma', \quad (2)$$

where the dot and prime are derivative with respect to worldsheet time and space coordinates. Vanishing of the
agonal components of the stress-energy tensor puts on a Virasora constraint:

\[
(1 - 2H^2 \cos^2 \beta) \dot{\alpha} \alpha' + \beta \beta' \\
+ (1 - 2H^2) \dot{\gamma} \gamma' + 2(1 - 2H^2 \cos^2 \beta) \kappa \alpha' \\
+ 2(1 - 2H^2) \cos \beta (\kappa \gamma' + \dot{\alpha} \gamma' + \dot{\alpha}' \gamma) = 0.
\]  

(3)

Now we take fast spinning limit \( \kappa \to \infty, \dot{X}^\mu \to 0 \) but keep \( \kappa \dot{X}^\mu \) finite for \( \mu \neq t \). At this limit, the contribution of kinetic term, i.e. \( \dot{X}^2 \), is ignorable thus can be dropped. In the other words, the string is frozen and energy is contributed mostly from curving the string. The the relevant part of Eq. (2) becomes

\[
S = \frac{\sqrt{\lambda k}}{16\pi} \int d\tau d\sigma - 2H^2 \cos^2 \beta \kappa^2 + \pi_\alpha \dot{\alpha} \\
+ \pi_\gamma \dot{\gamma} - \beta^2 - \Delta \gamma'^2,
\]

(4)

where the Virasora constraint has been applied to eliminate \( \alpha' \). Two comments are in order: At first, we notice that there is no explicit time derivative in the Hamiltonian. Therefore, one may find a mapping \( \Gamma : \sigma \to S^2 \) from string worldsheet coordinate \( \sigma \) to the deformed two-sphere. The periodicity of closed string ensures this \( \Gamma(\sigma) \) a closed trajectory on the sphere surface. To see this explicitly, one may introduce the spin vector pointing to an arbitrary point on the surface,

\[
\vec{n} = (\sqrt{\Delta} \cos \gamma, \sqrt{\Delta} \sin \gamma, n_z(\beta, H))
\]

(5)
such that the following constraint is satisfied

\[
((\sqrt{\Delta}))^2 + (n_z')^2 = 1.
\]

(6)

In generic, there may not be an analytic solution for \( n_z(\beta, H) \) but it can be solved numerically. It is not surprising that the norm \( |\vec{n}| \) is not a constant as in [7] but a function of \( \beta \) and \( H \) due to the deformation. However, the constraint (6) implies \( \Gamma \) is an uniform map. This reflects the very fact that string has no infrastructure for stretching. Figure 1 shows that the configuration space of the spin vector is deformed from a unit two-sphere.

Second, the conjugate variables \( \Pi_\alpha \)'s are angular momenta associated with \( \alpha \) and \( \gamma \). Those terms can be further removed by the other Virasora constraint due to tracelessness of stress-energy tensor. However, to keep them in the above action is useful in comparison with spin chain model. The angular momentum \( \Pi_\alpha \) gives us information of length of spin chain,

\[
J = J_0 - \int d\sigma 2H^2 \cos^2 \beta,
\]

(7)

where the total length of chain for vanishing \( H \) is defined as

\[
J_0 = \frac{\sqrt{\lambda k}}{16\pi} \int d\sigma \equiv \int d\tilde{\sigma}.
\]

(8)

The term \( \Pi_\gamma \) is called the Wess-Zumino term and it is related to the surface area bounded by \( \Gamma \) in the undeformed case. Without knowing the explicit form of \( n_z \), one can still write the spin vector with small \( H \) expansion,

\[
n_z \simeq \cos \beta + \cos^3 \beta H^2 + \mathcal{O}(H^4).
\]

(9)

The Hamiltonian from (10) becomes

\[
\mathcal{H} = \int \int dt d\tilde{\sigma} H^2 \cos^2 \beta + \frac{\lambda k^2}{128\pi^2} \int \int \tilde{t} \tilde{d}\tilde{\sigma} \beta^2 + \Delta \gamma'^2,
\]

(10)

where the derivative of \( \beta \) and \( \gamma \) is with respective to \( \tilde{\sigma} \). Up to a rescale of coordinates, it is easy to see that the above agrees with Ref.[7] for vanishing \( H \). The first term in Eq. (10) shows a quadrupole type interaction

\[
\mathcal{H}_{\text{int}} \simeq J_0 \int \int (\vec{n} \cdot \vec{B})^2.
\]

(11)

Here we have claimed that the deformation of \( SU(2)_k \) manifold can be described by an external magnetic field
$B_z = H$ with the quadrupole type interaction\textsuperscript{20}. At last, we would like to comment on the NS-NS B-field which has been ignored before. The deformed B-field can be obtained locally\textsuperscript{15} with

$$B_{[2]} = \frac{k}{4}(1 - 2H^2) \cos \beta da \wedge d\gamma,$$

which couples to string worldsheet in the way

$$S_B = \frac{\sqrt{X}}{4\pi} \int \int d\tau d\sigma \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}. \quad (13)$$

Since we have taken the limit where $\dot{X}^\mu \to 0$, it is obvious $S_B$ becomes irrelevant in this limit and action $H$ is still valid.

### III. DEFORMED $SL(2)_k$ SPIN CHAIN MODEL

In this section, we consider the hyperbolic deformation of $AdS_3$, or correspondingly $SL(2)_k$ WZW model\textsuperscript{15}. The isometry after deformation reduces to $SO(1,1)_L \times SL(2)_R$. The resulting metric reads,

$$ds^2 = \frac{k}{4} \left[ d\rho^2 - \cosh^2 \rho dt^2 + (1 - 2H^2)(d\phi + \sinh \rho dt)^2 + \gamma^2 \right], \quad (14)$$

where the string also sits at fixed $a$ and $\beta$ inside $S^3$. Now we consider fast spinning string, say, let $\gamma \to t + \gamma$, and at the same time rescale $d\phi \to \sinh \rho dt$. With gauge choice $t = \kappa \tau$ and taking the limit $\kappa \to \infty$, $X^\mu \to 0$ but keeping $\kappa X^\mu$ finite, one reaches

$$S = \frac{\sqrt{\lambda k}}{16\pi} \int \int d\tau d\sigma - 2H^2 \sinh^2 \rho \kappa^2 \phi $$

$+ 2(1 - 2H^2) \sinh^2 \rho \kappa \dot{\phi} + 2\kappa \gamma - \rho'' - \Delta_h \phi'^2$, 

$\Delta_h \equiv (1 - 2H^2) \sinh^2 \rho [1 + (1 - 2H^2) \sinh^2 \rho], \quad (15)$

where the term associated with $\dot{\gamma}$ gives us the same angular momentum as the undeformed one. One may introduce the spin vector moving over a deformed hyperboloid with signature $(-, +, +, +)$. Let $n_z(\rho, H)$ be the spin vector, then

$$\vec{n} = (\sqrt{\Delta_h} \cos \phi, \sqrt{\Delta_h} \sin \phi, n_z(\rho, H)) \quad (16)$$

such that $-((\sqrt{\Delta_h})^2 + (n_z')^2 = 1$ is satisfied. We again do not know the analytic form for $n_z(\rho, H)$ but it can be solved numerically. Figure 2 shows that the configuration space of the spin vector is deformed from a hyperboloid. For small $H$ expansion, we obtain

$$n_z \simeq \frac{1}{2} \cosh 2\rho + \left[ \frac{1}{2} \left( \log \coth \rho - 2 \cosh 2\rho \right) - \frac{1}{4} \cosh \rho \right] H^2 + O(H^4). \quad (17)$$

One can also read the Hamiltonian from Eq.\textsuperscript{15},

$$H = \int \int dt d\bar{\sigma} H^2 \sinh^2 \rho + \frac{\lambda k^2}{128\pi^2} \int \int dt d\bar{\sigma} \rho'^2 + \Delta_h \phi'^2, \quad (18)$$

where the first term can be rewritten as

$$H_{int} \simeq J_0 \int \int H^2 \rho \left( n_z - \frac{1}{2} \right). \quad (19)$$

### IV. LANDAU-LIFSHITZ EQUATION AS $H \to 0$ LIMIT

In the previous discussion, we have absorbed the infinite $\kappa$ by redefinition of $\sigma$ on the string worldsheet such that the Hamiltonian density appears finite. String sees the deformation via both interaction term $H_{int}$ and $\Delta_h$ in front of $\gamma''$ or $\phi''$. In order to go around the complication due to deformation, one may take a new double scaling limit where both $H \to 0$ and $\kappa \to \infty$ limits are taken, keep $\kappa H$ and $\kappa X \to H \to 0$ and $\kappa X \to X$ and we will only focus on the deformed $SU(2)_k$ in the following discussion. The equations of motion for $\beta$ and $\gamma$ after rescaling are given by

$$\beta'' - \sin \beta \gamma - \sin \beta \cos (\beta')^2 - H^2 \sin 2\beta = 0$$

$$\sin \beta \beta' + (\sin^2 \beta' \gamma')' = 0 \quad (20)$$

It has been long established that the classical Heisenberg spin chain is completely integrable and its equation of motion is a particular case of a more general Landau-Lifshitz equation\textsuperscript{16}

$$\partial_t \vec{S} = \vec{\partial}_\phi \vec{S} + \vec{S} \times \vec{J} \vec{S}, \quad (21)$$
where
\[ \mathcal{J} = \begin{pmatrix} \sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \\ -\sin \beta \cos \gamma & -\sin \beta \sin \gamma & -\cos \beta \\ \cos \beta & \sin \beta & 0 \end{pmatrix}, \]
(22)
\[ J \] is the unit matrix for the isotropic case \(^6\), but expected to be nontrivial for the anisotropic one. Indeed, a straight calculation shows that \(^{21}\) can be derived from (21), provided \(^{22}\) and the anisotropic matrix \( J = \text{diag}[j_1, j_2, j_3] \) where
\[ j_1 = j_2 = 1, \quad j_3 = 1 - 2H^2. \]
(23)
This general Landau-Lifshitz equation can be seen as continuous limit of the inhomogeneous Heisenberg spin chain model
\[ \mathcal{H} = -J_0 \sum_i S_i \cdot S_{i+1} + \sum_i aS_i^3 \cdot S_{i+1}, \]
(24)
where the anisotropy parameters \( a \propto j_3 - j_1 = j_3 - j_2 = -2H^2 \). Integrability of (21) provides the evidence that this marginally deformed \( SU(2)_k \) background is integrable, at least in the double scaling limit of fast spinning and small deformation.

V. DISCUSSION

In this report, we have investigated the fast spinning limit of classical string in the deformed background \( AdS_5 \times S^3 \). In particular, we have considered the asymmetric marginally deformed \( SU(2)_k \) and \( SL(2)_k \) WZW model. Four comments are in order. Since only the component \( j_3 \) is deviated from identity in the case of deformed \( SU(2)_k \), it is curious if there exists a more general correspondence between nontrivial \( j_i \)'s and other kinds of deformation. However, if the deformed \( SU(2)_k \) discussed in Ref.\(^{13}\) is unique, those other deformation, if exist, must be relevant or irrelevant. In that sense, integrability makes the system still under control even outside the reach of marginal deformation. Second, the \( H \rightarrow 0 \) limit implies that we may use the same coherent states as constructed in Refs.\(^{6,7}\) to build up a discrete sigma model, whose continuous limit will recover the action of fast spinning string. Thirdly, if one could start with a deformed WZW sigma model \(^{15}\) whose target space is given by (1) and calculate the dilatation operator, the same spinning string can be constructed whose thermodynamic limit may give rise to the general Landau-Lifshitz equation \(^{21}\). At last, solutions to Eq.(21) has been intensely studied in recent decades. In particular, it has been found that the (multi-)soliton solutions may correspond to (multi-)magnon-like excitation first discussed in Ref.\(^{18}\). We plan to come back to these issues in future publications.\(^{19}\)

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