BOHR RADIUS FOR CERTAIN CLOSE-TO-CONVEX HARMONIC MAPPINGS

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Abstract. Let \( H \) be the class of harmonic functions \( f = h + \bar{g} \) in the unit disk \( D := \{ z \in \mathbb{C} : |z| < 1 \} \), where \( h \) and \( g \) are analytic in \( D \). Let

\[ P_0^H(\alpha) = \{ f = h + \bar{g} \in H : \text{Re}(h'(z) - \alpha) > |g'(z)| \text{ with } 0 \leq \alpha < 1, \ g'(0) = 0, \ z \in \mathbb{D} \} \]

be the class of close-to-convex mappings defined by Li and Ponnusamy [34]. In this paper, we obtain the sharp Bohr-Rogosinski radius, improved Bohr radius and refined Bohr radius for the class \( P_0^H(\alpha) \).

1. Introduction

The classical inequality of Bohr says that if \( f \) is an analytic function in the unit disk \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) with the following Taylor series expansion

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]

such that \( |f(z)| < 1 \) in \( \mathbb{D} \), then the majorant series \( M_f(r) \) associated with \( f \) satisfies the following inequality

\[ M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \leq 1 \text{ for } |z| = r \leq 1/3, \]

and the constant 1/3, known as Bohr radius, cannot be improved. In 1914, H. Bohr [17] obtained the inequality (1.2) for \( r \leq 1/6 \) and subsequently improved by 1/3 later, Weiner, Riesz and Schur independently obtained the constant 1/3. An observation shows that the quantity \( 1 - |a_0| \) is equal to \( d(f(0), \partial f(\mathbb{D})) \). Therefore, the inequality (1.2) is called Bohr inequality, can be written in the following form

\[ \sum_{n=1}^{\infty} |a_n z^n| \leq 1 - |a_0| = d(f(0), \partial f(\mathbb{D})) \]

for \( |z| = r \leq 1/3 \), where \( d \) is the Euclidean distance. It is important to note that the constant 1/3 is independent of the coefficients of the Taylor series (1.1). This fact can be elucidated by saying that Bohr inequality occurs in the class \( B \) of analytic self maps of the unit disk \( \mathbb{D} \). Analytic functions \( f \in B \) of the form (1.1) satisfying the inequality (1.2) for \( |z| = r \leq 1/3 \), are sometimes said to satisfy the classical...
Bohr phenomenon. The notion of the Bohr phenomenon can be generalized to the class $F$ consisting of analytic functions $f$ from $D$ to a given domain $\Omega \subseteq \mathbb{C}$ such that $f(D) \subseteq \Omega$ and the class $F$ is said to satisfy the Bohr phenomenon if there exists largest radius $r_\Omega \in (0, 1)$ such that the inequality (1.3) holds for $|z| = r \leq r_\Omega$ and for all functions $f \in F$. We say the largest radius $r_\Omega$ is the Bohr radius for the class $F$. The Bohr radius has been obtained for the class $F$ when $\Omega$ is convex domain [8], simply connected domain [1], the exterior of the closed unit disk, the punctured unit disk, and concave wedge domain (see [9]). In 1997, Boas and Khavinson [15] generalized the Bohr inequality in several complex variables by finding multidimensional Bohr radius. In 2020, Liu and Ponnusamy [36] obtained multidimensional analogues of refined Bohr inequality.

There are many improved versions of Bohr’s inequality (1.2) in various forms obtained by several authors. In 2020, Kayumov and Ponnusamy [33] obtained several interesting improved versions of Bohr inequality. For more results on this, we refer the reader to glance through the articles (see [22, 23, 27, 32, 33, 35, 38, 40]). In 2017, Kayumov and Ponnusamy [27] introduced Bohr-Rogosinski radius motivated by Rogosinski radius for bounded analytic functions in $D$. Rogosinski radius is defined as follows: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $D$ and its corresponding partial sum of $f$ is defined by $S_N(z) := \sum_{n=0}^{N-1} a_n z^n$. Then, for every $N \geq 1$, we have $|\sum_{n=0}^{N-1} a_n z^n| < 1$ in the disk $|z| < 1/2$ and the radius $1/2$ is sharp. Motivated by Rogosinski radius, Kayumov and Ponnusamy have considered the Bohr-Rogosinski sum $R_N^f(z)$ which is defined by

$$R_N^f(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n.$$  

It is worth to point out that $|S_N(z)| = |f(z) - \sum_{n=N}^{\infty} a_n z^n| \leq |R_N^f(z)|$. Therefore, it is easy to see that the validity of Bohr-type radius for $R_N^f(z)$, which is related to the classical Bohr sum (Majorant series) in which $f(0)$ is replaced by $f(z)$, gives Rogosinski radius in the case of bounded analytic functions in $D$. We have the following interesting results by Kayumov and Ponnusamy [27].

**Theorem 1.5.** [27] Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $D$ and $|f(z)| \leq 1$. Then

$$|f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq 1$$

for $|z| = r \leq R_N$, where $R_N$ is the positive root of the equation $\psi_N(r) = 0$, $\psi_N(r) = 2(1 + r)r^N - (1 - r)^2$. The radius $R_N$ is the best possible. Moreover,

$$|f(z)|^2 + \sum_{n=N}^{\infty} (|a_n| + |b_n|)|z|^n \leq 1$$

for $R'_N$, where $R'_N$ is the positive root of the equation $(1 + r)r^N - (1 - r)^2$. The radius $R_N$ is the best possible.
Recently, Kayumov and Ponnusamy [27] have proved the following improved version of Bohr’s inequality.

**Theorem 1.8.** [27] Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \), \(|f(z)| \leq 1 \) and \( S_r \) denote the image of the subdisk \(|z| < r \) under mapping \( f \). Then

\[
B_1(r) := \sum_{n=0}^{\infty} |a_n| r^n + \frac{16}{9} \left( \frac{S_r}{\pi} \right) \leq 1 \quad \text{for} \quad r \leq \frac{1}{3}
\]

and the numbers \( 1/3, 16/9 \) cannot be improved. Moreover,

\[
B_2(r) := |a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \frac{9}{8} \left( \frac{S_r}{\pi} \right) \leq 1 \quad \text{for} \quad r \leq \frac{1}{2}
\]

and the numbers \( 1/2, 9/8 \) cannot be improved.

Bohr’s phenomenon for the complex-valued harmonic mappings have been studied extensively by many authors (see [1, 8, 13, 12]). Improved Bohr inequality for locally univalent harmonic mappings have been discussed by Evdoridis et al. [21].

A complex-valued function \( f = u + iv \) is harmonic if \( u \) and \( v \) are real-harmonic in \( \mathbb{D} \). Every harmonic function \( f \) has the canonical representation \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \) known respectively as the analytic and co-analytic parts of \( f \). A locally univalent harmonic function \( f \) is said to be sense-preserving if the Jacobian of \( f \), defined by \( J_f(z) := |h'(z)|^2 - |g'(z)|^2 \), is positive in \( \mathbb{D} \) and sense-reversing if \( J_f(z) \) is negative in \( \mathbb{D} \). Let \( \mathcal{H} \) be the class of all complex-valued harmonic functions \( f = h + \overline{g} \) defined in \( \mathbb{D} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \) such that \( h(0) = h'(0) - 1 = 0 \) and \( g(0) = 0 \). A function \( f \in \mathcal{H} \) is said to be in \( \mathcal{H}_0 \) if \( g'(0) = 0 \). Thus, every \( f = h + \overline{g} \in \mathcal{H}_0 \) has the following form

\[
f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n.
\]

In 2013, Ponnusamy et al. [39] considered the following classes

\[
\mathcal{P}_H^0 = \{ f = h + \overline{g} \in \mathcal{H} : \Re h'(z) > |g'(z)| \quad \text{with} \quad g'(0) = 0 \quad \text{for} \quad z \in \mathbb{D} \}.
\]

Motivated by the class \( \mathcal{P}_H^0 \), Li and Ponnusamy [34] have studied the following class \( \mathcal{P}_H^0(\alpha) \) defined by

\[
\mathcal{P}_H^0(\alpha) = \{ f = h + \overline{g} \in \mathcal{H} : \Re (h'(z) - \alpha) > |g'(z)| \quad \text{with} \quad 0 \leq \alpha < 1, \quad g'(0) = 0 \quad \text{for} \quad z \in \mathbb{D} \}.
\]

We have the following coefficient bounds and growth estimates for the class \( \mathcal{P}_H^0(\alpha) \).

**Lemma 1.12.** [34] Let \( f \in \mathcal{P}_H^0(\alpha) \) and be given by (1.11). Then for any \( n \geq 2 \),

(i) \(|a_n| + |b_n| \leq \frac{2(1-\alpha)}{n} \); 

(ii) \(||a_n| - |b_n|| \leq \frac{2(1-\alpha)}{n} \);
(iii) $|a_n| \leq \frac{2(1-\alpha)}{n}$.

All the inequalities are sharp, with extremal function $f(z) = (1-\alpha)(-z - 2 \log(1-z)) + \alpha z$.

**Lemma 1.13.** [13] Let $f = h + g \in \mathcal{P}_h^0(\alpha)$ with $0 \leq \alpha < 1$. Then

$$|z| + \sum_{n=2}^{\infty} \frac{2(1-\alpha)(-1)^{n-1}}{n} |z|^n \leq |f(z)| \leq |z| + \sum_{n=2}^{\infty} \frac{2(1-\alpha)}{n} |z|^n.$$  

Both inequalities are sharp.

The organization of this paper is follows: In section 2 we obtain sharp Bohr-Rogosinski radius for the class $\mathcal{P}_h^0(\alpha)$ of close-to-convex harmonic mappings. In section 3, we establish interesting sharp improved-Bohr radius $\mathcal{P}_h^0(\alpha)$. In section 4, we prove sharp refined-Bohr radius as well as Bohr-type inequality for the class $\mathcal{P}_h^0(\alpha)$. Section 6 is devoted for the proofs of main the results.

2. **Bohr-Rogosinski Radius for the class $\mathcal{P}_h^0(\alpha)$**

We first prove the following Bohr-Rogosinski radius for the class $\mathcal{P}_h^0(\alpha)$.

**Theorem 2.1.** Let $f \in \mathcal{P}_h^0(\alpha)$ be given by (1.11). Then, for $N \geq 2$,

$$|f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(D))$$

for $|z| = r \leq r_N(\alpha)$, where $r_N(\alpha)$ is the smallest root of the equation

$$r - 1 - 2(1-\alpha) \left( r - 1 + \ln(2(1-r)^2) + \sum_{n=1}^{N-1} \frac{r^n}{n} \right) = 0 \text{ in } (0, 1).$$

The constant $r_N(\alpha)$ is the best possible.

![Figure 1. The graph of $r_3(\alpha)$ and $r_{10}(\alpha)$ in (0, 1).](image-url)
Bohr radius for certain close-to-convex harmonic mappings

| α   | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  | 0.6  | 0.7  | 0.8  | 0.9  |
|-----|------|------|------|------|------|------|------|------|------|
| \(r_2(\alpha)\) | 0.2771 | 0.3115 | 0.3477 | 0.3866 | 0.4296 | 0.4785 | 0.5367 | 0.6109 | 0.7187 |
| \(r_3(\alpha)\) | 0.3121 | 0.3493 | 0.3877 | 0.4281 | 0.4717 | 0.5201 | 0.5764 | 0.6463 | 0.7453 |
| \(r_6(\alpha)\) | 0.3248 | 0.3653 | 0.4070 | 0.4508 | 0.4978 | 0.5493 | 0.6080 | 0.6786 | 0.7736 |
| \(r_{10}(\alpha)\) | 0.3251 | 0.3657 | 0.4078 | 0.4522 | 0.4999 | 0.5527 | 0.6130 | 0.6859 | 0.7832 |

**Table 1.** This table shows the value of the roots \(r_N(\alpha)\) for different values of \(\alpha\) in \([0,1)\) and \(N = 2,3,6,10\).

**Theorem 2.4.** Let \(f \in \mathcal{P}^0_H(\alpha)\) be given by (1.11). Then, \(N \geq 2\),

\[
|f(z)|^2 + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(D))
\]

for \(|z| = r \leq r_N(\alpha)\), where \(r_N(\alpha) \in (0,1)\) is the smallest root of the equation

\[
\left( r - 2(1-\alpha)(r+\ln (1-r)) \right)^2 - 2(1-\alpha) \left( \ln (2-2r) - 1 + \sum_{n=2}^{N-1} \frac{r^n}{n} \right) - 1 = 0.
\]

The constant \(r_N(\alpha)\) is the best possible.

**Figure 2.** The graph of \(r_3(\alpha)\) and \(r_8(\alpha)\) of the equation (2.6).

| α   | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  | 0.6  | 0.7  | 0.8  | 0.9  |
|-----|------|------|------|------|------|------|------|------|------|
| \(r_3(\alpha)\) | 0.4102 | 0.4399 | 0.4708 | 0.5038 | 0.5399 | 0.5807 | 0.6291 | 0.6903 | 0.7783 |
| \(r_8(\alpha)\) | 0.4304 | 0.4613 | 0.4933 | 0.5273 | 0.5644 | 0.6060 | 0.6547 | 0.7152 | 0.7994 |

**Table 2.** Values of \(r_N(\alpha)\) for \(N = 3\) and \(8\) when \(\alpha \in [0,1)\).

**Theorem 2.7.** Let \(f \in \mathcal{P}^0_H(\alpha)\) be given by (1.11). Then for a positive integer \(N \geq 2\),

\[
|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq d(f(0), \partial f(D))
\]
for $|z| = r \leq r_{m,N}(\alpha)$, where $r_{m,N}(\alpha) \in (0, 1)$ is the smallest root of the equation

$$r^m - 1 - 2(1 - \alpha) \left( r^m - 1 + \ln \left( (1 - r^m)(2 - 2r) \right) + \sum_{n=1}^{N-1} \frac{r^n}{n} \right) = 0.$$  

The constant $r_{m,N}(\alpha)$ is the best possible.

**Figure 3.** The graphs $r_{2,2}(\alpha)$, $r_{3,2}(\alpha)$ and $r_{7,2}(\alpha)$ of the equation (2.9).

**Figure 4.** The graphs $r_{25,2}(\alpha)$, $r_{150,2}(\alpha)$ and $r_{5,3}(\alpha)$ of the equation (2.9).

**Figure 5.** The graphs $r_{15,3}(\alpha)$, $r_{25,3}(\alpha)$ and $r_{180,3}(\alpha)$ of the equation (2.9).

**Figure 6.** The graphs $r_{5,5}(\alpha)$, $r_{15,5}(\alpha)$ and $r_{35,5}(\alpha)$ of the equation (2.9).

For different values of $\alpha$, $m$ and $N$, in above the corresponding radii are represented on $x$-axis cut by the increasing curves all have asymptotes at $x = 1$ have been shown in Figures 3-6.
Theorem 2.10. Let $f \in \mathcal{P}_N^h(\alpha)$ be given by (1.11). Then
\begin{equation}
 r + |h(r)|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \leq d \left( f(0), \partial f(\mathbb{D}) \right), \text{ for } r \leq r_p(\alpha),
\end{equation}

where $r_p(\alpha)$ is the smallest root of the equation
\begin{equation}
 r^p + r - 1 - 2(1 - \alpha) (r - 1 + \ln (2 - 2r)) = 0 \text{ in } (0, 1).
\end{equation}

The radius $r_p(\alpha)$ is the best possible.
Figure 7. The roots $r_7(\alpha)$ and $r_{35}(\alpha)$ of the equation $r^p + r - 1 - 2(1 - \alpha) (r - 1 + \ln (2 - 2r)) = 0$ in $(0, 1)$.

| $\alpha$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $r_7(\alpha)$ | 0.3249 | 0.3653 | 0.4069 | 0.4503 | 0.4963 | 0.5456 | 0.5992 | 0.6579 | 0.7231 |
| $r_{35}(\alpha)$ | 0.3251 | 0.3657 | 0.4078 | 0.4522 | 0.5000 | 0.5529 | 0.6136 | 0.6872 | 0.7867 |

Table 5. The roots $r_7(\alpha)$ and $r_{35}(\alpha)$ of the equation $r^p + r - 1 - 2(1 - \alpha) (r - 1 + \ln (2 - 2r)) = 0$ for $\alpha \in [0, 1)$.

3. Improved Bohr Radius for the class $P_0^\alpha_H$

In 2020, Kayumov and Ponnusamy have obtained several improved versions of Bohr inequality for analytic functions.

**Theorem 3.1.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$, $|f(z)| \leq 1$ and $S_r$ denotes the image of the subdisk $|z| < r$ under mapping $f$. Then

$$B_1(r) := \sum_{n=0}^{\infty} |a_n| r^n + \frac{16}{9} \left( \frac{S_r}{\pi} \right) \leq 1 \quad \text{for} \quad r \leq \frac{1}{3}$$

and the numbers $1/3$, $16/9$ cannot be improved. Moreover,

$$B_2(r) := |a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \frac{9}{8} \left( \frac{S_r}{\pi} \right) \leq 1 \quad \text{for} \quad r \leq \frac{1}{2}$$

and the numbers $1/2$ and $9/8$ cannot be improved.

**Theorem 3.4.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and $|f(z)| \leq 1$. Then

$$|a_0| + \sum_{n=1}^{\infty} \left( |a_n| + \frac{1}{2} |a_n|^2 \right) r^n \leq 1 \quad \text{for} \quad r \leq \frac{1}{3}$$

and the numbers $1/3$ and $1/2$ cannot be improved.

The primary object of this section is to generalize the harmonic versions of Theorem 3.1 and Theorem 3.4 for the class $P_0^\alpha_H(\alpha)$. It is interesting to investigate Theorem 3.1 when $S_r/\pi$ has certain power. Therefore in order to generalize Theorem 3.1, we consider a $N^{th}$ degree polynomial in $S_r/\pi$ of the form

$$P \left( \frac{S_r}{\pi} \right) = \left( \frac{S_r}{\pi} \right)^N + \left( \frac{S_r}{\pi} \right)^{N-1} + \cdots + \frac{S_r}{\pi}.$$
**Theorem 3.6.** Let $f \in \mathcal{P}_H^0(\alpha)$ be given by (1.11). Then

$$r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n + P \left( \frac{S_r}{\pi} \right) \leq d(f(0), \partial f(D)) \tag{3.7}$$

for $r \leq r_N(\alpha)$, where $P(w) = w^N + w^{N-1} + \cdots + w$, a polynomial in $w$ of degree $N - 1$, and $r_N(\alpha) \in (0, 1)$ is the smallest root of the equation

$$r - 1 - 2(1-\alpha) (r - 1 + \ln (2 - 2r)) + P \left( r^2 - 4(1-\alpha)^2(r^2 + \ln (1 - r^2)) \right) = 0. \tag{3.8}$$

The constant $r_N(\alpha)$ is the best possible.

| $\alpha$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $r_2(\alpha)$ | 0.2734 | 0.3027 | 0.3320 | 0.3618 | 0.3923 | 0.4241 | 0.4574 | 0.4927 | 0.5303 |
| $r_3(\alpha)$ | 0.2732 | 0.3023 | 0.3314 | 0.3607 | 0.3907 | 0.4217 | 0.4540 | 0.4878 | 0.5230 |
| $r_4(\alpha)$ | 0.2732 | 0.3023 | 0.3313 | 0.3606 | 0.3905 | 0.4213 | 0.4533 | 0.4867 | 0.5212 |
| $r_5(\alpha)$ | 0.2732 | 0.3023 | 0.3313 | 0.3606 | 0.3904 | 0.4213 | 0.4532 | 0.4864 | 0.5208 |

**Table 6.** The roots $r_N(\alpha)$ of equation (3.8) when $N = 2, 3, 4, 5$ and $\alpha \in [0, 1)$.

As a consequence of Theorem 3.6, we obtain the following interesting corollary.

**Corollary 3.9.** Let $f \in \mathcal{P}_H^0(\alpha)$ be given by (1.11). Then

$$r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n + \left( \frac{S_r}{\pi} \right) \leq d(f(0), \partial f(D)) \tag{3.10}$$

for $r \leq r_\alpha$, where $r_\alpha \in (0, 1)$ is the smallest root of the equation

$$r^2 + r - 1 - 2(1-\alpha)(3 - 2\alpha)(r + \ln (1 - r)) - 2(1-\alpha)(\ln 2 - 1) = 0. \tag{3.11}$$

The radius $r_\alpha$ is the best possible.

![Figure 8](image-url)
The polylogarithm function is defined by a power series in \( z \), which is also a Dirichlet series in \( s \). That is
\[
Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots,
\]
is valid for arbitrary complex order \( s \) and for all complex arguments \( z \) with \( |z| < 1 \). Therefore, the dilogarithm function is denoted by \( Li_2(z) \), is a particular case of the polylogarithm. The following theorem is the generalization of the harmonic version of Theorem 3.4 by considering the right hand side \( d(f(0), \partial f(\mathbb{D})) \) instead of 1.

**Theorem 3.12.** Let \( f \in \mathcal{P}_H^0(\alpha) \) be given by (1.11). Then
\[
(3.13) \quad r + \sum_{n=2}^{\infty} \left( |a_n| + |b_n| + (|a_n| + |b_n|)^2 \right) r^n \leq d(f(0), \partial f(\mathbb{D})) \quad \text{for} \quad r \leq r_\alpha,
\]
where \( r_\alpha \in (0, 1) \) is the smallest root of the equation
\[
(3.14) \quad r - 1 - 2(1-\alpha)(r - 1 + \ln(2-2r)) + 4(1-\alpha^2)(Li_2(r) - r) = 0,
\]
where \( Li_2(z) \) is a dilogarithm. The constant \( r_\alpha \) is best possible.

4. Refined Bohr Radius for the Class \( \mathcal{P}_H^0(\alpha) \)

In 2020, Ponnusamy et al. [38] established the following refined Bohr inequality by applying a refined version of the coefficient inequalities.

**Theorem 4.1.** [38] Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \) and \( |f(z)| \leq 1 \). Then
\[
\sum_{n=0}^{\infty} |a_n| r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq 1
\]
for \( r \leq 1/(2 + |a_0|) \) and the numbers \( 1/(1 + |a_0|) \) and \( 1/(2 + |a_0|) \) cannot be improved. Moreover,
\[
|a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq 1
\]
for \( r \leq 1/2 \) and the numbers \( 1/(1 + |a_0|) \) and \( 1/2 \) cannot be improved.

Next we prove the harmonic analogue of Theorem 4.1.

**Theorem 4.2.** Let \( f \in \mathcal{P}_H^0(\alpha) \) be given by (1.11). Then
\[
(4.3) \quad r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + \frac{1}{1 - r^N} \sum_{n=2}^{\infty} n(|a_n| + |b_n|)^2 r^{2n} \leq d(f(0), \partial f(\mathbb{D})) \quad \text{for} \quad r \leq r_N(\alpha),
\]
where \( r_N(\alpha) \in (0, 1) \) is the smallest root of the equation

\[
(4.4) \quad r - 1 - 2(1 - \alpha)\left(r - 1 + \ln (2 - 2r) + \frac{2(1 - \alpha)}{1 - r^N}(r^2 + \ln (1 - r^2))\right) = 0.
\]

Here \( r_N(\alpha) \) is the best possible.

Here is a table showing the roots of \( r_2(\alpha) \) and \( r_{25}(\alpha) \) when \( \alpha \in [0, 1) \):

| \( \alpha \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|--------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( r_2(\alpha) \) | 0.3148 | 0.3527 | 0.3920 | 0.4338 | 0.4793 | 0.5304 | 0.5904 | 0.6651 | 0.7693 |
| \( r_{25}(\alpha) \) | 0.3158 | 0.3542 | 0.3942 | 0.4368 | 0.4835 | 0.5361 | 0.5977 | 0.6741 | 0.7792 |

Table 8. In this table, we obtained the roots of \( r_N(\alpha) \) for \( N = 2 \) and 25 when \( \alpha \in [0, 1) \).

In the following, we prove two interesting results which are harmonic analogue of refined Bohr inequality.

**Theorem 4.5.** Let \( f \in \mathcal{P}_{H}^0(\alpha) \) be given by (1.11). Then

\[
(4.6) \quad r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + \left(\frac{1}{1 + |a_2| + |b_2|} + \frac{r_m}{1 - r_m}\right) \sum_{n=3}^{\infty} n^{m-1}(|a_n| + |b_n|)^m r^{mn} \leq d(f(0), \partial f(\mathbb{D})), \text{ for } r \leq r_m(\alpha),
\]

where \( r_m(\alpha) \in (0, 1) \) is the smallest root of the equation

\[
(4.7) \quad r - 1 - 2(1 - \alpha)(r - 1 + \ln (2 - 2r)) - 2^m(1 - \alpha)^m \left(\frac{1}{1 + |a_2| + |b_2|} + \frac{r_m}{1 - r_m}\right) \left(r^m + \frac{r^{2m}}{2} + \ln (1 - r^m)\right) = 0.
\]

The constant \( r_m(\alpha) \) is the best possible.
Theorem 4.8. Let $f \in \mathcal{P}_H^0(\alpha)$ be given by (1.11). Then
\[ \begin{align*}
    (4.9) & \quad r + \frac{(1 - (1 + |a_2| + |b_2| - (|a_2| + |b_2|)^2)) r}{1 - (|a_2| + |b_2|)r} + \sum_{n=3}^{\infty} (|a_n| + |b_n|) r^n \\
    & \quad \leq d(f(0), \partial f(D)) \text{ for } r \leq r_{\alpha},
\end{align*} \]
where $r_{\alpha} \in (0, 1)$ is the smallest root of the equation
\[ (4.10) \quad r - \frac{(1 - (|a_2| + |b_2|)^2) r}{1 - (|a_2| + |b_2|)r} - 2(1 - \alpha) \left( r + \frac{r^2}{2} - 1 + \ln (2 - 2r) \right) = 0. \]
The constant $r_{\alpha}$ is the best possible.

5. Bohr-Type Inequality for the class $\mathcal{P}_H^0(\alpha)$

We now prove the following Bohr-type inequality for the class of functions $\mathcal{P}_H^0(\alpha)$.

Theorem 5.1. Let $f \in \mathcal{P}_H^0(\alpha)$ be given by (1.11). Then
\[ \begin{align*}
    (5.2) & \quad |f(z)| + \sqrt{|J_f(z)|} r + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r^n \leq d(f(0), \partial f(D)) \text{ for } r \leq r_N(\alpha),
\end{align*} \]
where $r_N(\alpha) \in (0, 1)$ is the smallest root of the equation
\[ (5.3) \quad r - 1 - 2(1 - \alpha) \left( 2r - 1 + \frac{r^2}{2} + \cdots + \frac{r^{N-1}}{N-1} + \ln 2 + 2 \ln(1 - r) \right) + \left( \alpha + (1 - \alpha) \left( \frac{1 + r}{1 - r} \right) \right) r = 0. \]
The radius $r_N(\alpha)$ is the best possible.

6. Proof of the main results

Proof of Theorem 2.1. Let $f \in \mathcal{P}_H^0(\alpha)$ be given by (1.11). Then from Theorem 2.1 we have
\[ (6.1) \quad |f(z)| \geq |z| + (1 - \alpha) \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{n} |z|^n \text{ for } |z| < 1. \]
The Euclidean distance between $f(0)$ and the boundary of $f(D)$ is given by
\[ (6.2) \quad d(f(0), \partial f(D)) = \liminf_{|z| \to 1} |f(z) - f(0)|. \]
Since $f(0) = 0$, from (1.14) and (6.2) we obtain
\[ (6.3) \quad d(f(0), \partial f(D)) \geq 1 + \sum_{n=2}^{\infty} 2(1 - \alpha) \frac{(-1)^{n-1}}{n}. \]
Using Lemmas 1.12 and 1.13, for $|z| = r_N(\alpha)$, we obtain

\begin{equation}
|f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n 
\leq r + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)r^n}{n} + \sum_{n=N}^{\infty} \frac{2(1 - \alpha)r^n}{n} 
= r - 2(1 - \alpha)(r + \ln(1 - r)) - 2(1 - \alpha) \left( \ln(1 - r) + \sum_{n=1}^{N-1} \frac{r^n}{n} \right) 
= r - 2(1 - \alpha) \left( r + \ln((1 - r)^2) + \sum_{n=1}^{N-1} \frac{r^n}{n} \right).
\end{equation}

It is easy to see that

\begin{equation}
r - 2(1 - \alpha) \left( r + \ln((1 - r)^2) + \sum_{n=1}^{N-1} \frac{r^n}{n} \right) \leq 1 + 2(1 - \alpha)(\ln 2 - 1)
\end{equation}

for $r \leq r_N(\alpha)$, where $r_N(\alpha)$ is the smallest root of

\[ r - 1 - 2(1 - \alpha) \left( r - 1 + \ln(2(1 - r)^2) + \sum_{n=1}^{N-1} \frac{r^n}{n} \right) = 0 \]

in $(0,1)$. Let $H_1 : [0,1) \rightarrow \mathbb{R}$ be defined by

\[ H_1(r) := r - 1 - 2(1 - \alpha) \left( r - 1 + \ln(2(1 - r)^2) + \sum_{n=1}^{N-1} \frac{r^n}{n} \right). \]

The existence of a root $r_N(\alpha)$ is ensured by the following fact that $H_1$ is a continuous function with the properties $H_1(0) = -1 - 2(1 - \alpha)(\ln 2 - 1) < 0$ and $\lim_{r \rightarrow 1} H_1(r) = +\infty$. Let $r_N(\alpha)$ to be the smallest root of $H_1(r) = 0$ in $(0,1)$. Therefore, we have $H_1(r_N(\alpha)) = 0$. That is

\begin{equation}
r_N(\alpha) - 1 - 2(1 - \alpha) \left( r_N(\alpha) - 1 + \ln(2(1 - r_N(\alpha))^2) + \sum_{n=1}^{N-1} \frac{r_n^{r_N(\alpha)}}{n} \right) = 0.
\end{equation}

In view of (6.3), (6.4) and (6.5) for $|z| = r \leq r_N(\alpha)$, it follows that

\[ |f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n \leq d(f(0), \partial f(\mathbb{D})). \]

In order to show that the constant $r_N(\alpha)$ is the best possible constant, we consider the following function $f = f_\alpha$ by

\begin{equation}
f_\alpha(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)z^n}{n}.
\end{equation}

It is easy to show that $f_\alpha \in P^0_{r_N(\alpha)}$. For $f = f_\alpha$, it can be seen that

\begin{equation}
d(f(0), \partial f(\mathbb{D})) = 1 + 2(1 - \alpha)(\ln 2 - 1).
\end{equation}
For $f = f_\alpha$ and $|z| = r_N(\alpha)$, a simple computation using (6.6) and (6.8) shows that

$$
|f(z)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)r^n = r_N(\alpha) + \sum_{n=2}^{\infty} \frac{2(1-\alpha)r_N(\alpha)}{n} + \sum_{n=N}^{\infty} \frac{2(1-\alpha)r_N(\alpha)}{n}
$$

$$
= r_N(\alpha) - 2(1-\alpha) \left( r_N(\alpha) + \ln((1-r_N(\alpha))^2) + \sum_{n=1}^{N-1} \frac{r_n(\alpha)}{n} \right)
$$

$$
= 1 + 2(1-\alpha)(\ln 2 - 1)
$$

$$
= d(f(0), \partial f(\mathbb{D})).
$$

Therefore, the radius $r_N(\alpha)$ is the best possible. This completes the proof. □

**Proof of Theorem 2.4.** Let $f \in P_0^0(\alpha)$ be given by (1.11). Then in view of Lemmas 1.12 and 1.13 for $|z| = r$, we obtain

$$
|f(z)|^2 + \sum_{n=N}^{\infty} |a_n|r^n \leq \left( r + \sum_{n=2}^{\infty} \frac{2(1-\alpha)r^n}{n} \right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\alpha)r^n}{n}.
$$

A simple computation shows that

$$
\left( r + \sum_{n=2}^{\infty} \frac{2(1-\alpha)r^n}{n} \right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\alpha)r^n}{n} \leq 1 + 2(1-\alpha)(\ln 2 - 1)
$$

for $r \leq r_N(\alpha)$, where $r_N(\alpha)$ is the smallest root of $H_2(r) = 0$ in $(0, 1)$, where $H_2 : [0, 1) \rightarrow \mathbb{R}$ is defined by

$$
H_2(r) = \left( r - 2(1-\alpha)(r + \ln(1-r)) \right)^2 - 2(1-\alpha) \left( \ln(2-2r) - 1 + \sum_{n=1}^{N-1} \frac{r^n}{n} \right) - 1.
$$

Then $H_2$ is a continuous function with $H_2(0) = -1 - 2(1-\alpha)(\ln 2 - 1) < 0$ and $\lim_{r \to 1} H_2(r) = +\infty$. Therefore, $H_2(r) = 0$ has a root in $(0, 1)$ and we choose the smallest root to be $r_N(\alpha)$. Therefore, we have $H_2(r_N(\alpha)) = 0$. That is

$$
\left( r_N(\alpha) - 2(1-\alpha)(r_N(\alpha) + \ln(1-r_N(\alpha))) \right)^2 - 2(1-\alpha) \left( \ln(2-2r_N(\alpha)) - 1 + \sum_{n=1}^{N-1} \frac{r_n(\alpha)}{n} \right) - 1 = 0.
$$

Using (6.3), (6.9) and (6.10) for $|z| = r \leq r_N(\alpha)$, we obtain

$$
|f(z)|^2 + \sum_{n=N}^{\infty} |a_n|r^n \leq d(f(0), \partial f(\mathbb{D})).
$$
In order to show that \( r_N(\alpha) \) is the best possible, we consider the function \( f = f_\alpha \) defined by (6.7). For \( f = f_\alpha \) and \( |z| = r_N(\alpha) \), a simple computation using (6.8) and (6.11) shows that
\[
|f(z)|^2 + \sum_{n=N}^{\infty} |a_n| r^n = \left( r_N(\alpha) + \sum_{n=2}^{\infty} \frac{2(1-\alpha)(r_N(\alpha))^n}{n} \right)^2 + \sum_{n=N}^{\infty} \frac{2(1-\alpha)(r_N(\alpha))^n}{n}
\]
\[
= 1 + 2(1-\alpha)(\ln 2 - 1)
\]
\[
= d(f(0), \partial f(D)).
\]
Therefore, the radius \( r_N(\alpha) \) is the best possible. \( \square \)

**Proof of Theorem 2.7.** Let \( f \in \mathcal{P}_H(\alpha) \) be given by (1.11). Using Lemmas 1.12 and 1.13 for \( |z| = r \), we obtain
\[
|f(z^m)| + \sum_{n=N}^{\infty} |a_n||z|^n \leq r^m + \sum_{n=2}^{\infty} \frac{2(1-\alpha)r^m n}{n} + \sum_{n=N}^{\infty} \frac{2(1-\alpha)r^n}{n}. \tag{6.12}
\]
A simple computation shows that
\[
r^m + \sum_{n=2}^{\infty} \frac{2(1-\alpha)r^m n}{n} + \sum_{n=N}^{\infty} \frac{2(1-\alpha)r^n}{n}
\]
\[
= r^m - 2(1-\alpha)(r^m + \ln(1-r^m)) - 2(1-\alpha) \left( r + \ln(1-r) + \sum_{n=2}^{N-1} \frac{r^n}{n} \right)
\]
\[
= r^m - 2(1-\alpha) \left( r^m + r + \ln(1-r)(1-r^m) + \sum_{n=2}^{N-1} \frac{r^n}{n} \right)
\]
\[
\leq 1 + 2(1-\alpha)(\ln 2 - 1)
\]
for \( r \leq r_{m,N}(\alpha) \), where \( r_{m,N}(\alpha) \) is the smallest root of \( H_3(r) = 0 \) in \((0,1)\), where \( H_3 : [0,1) \to \mathbb{R} \) is defined by
\[
H_3(r) := r^m - 1 - 2(1-\alpha) \left( r^m + r - 1 + \ln(2-2r)(1-r^m) + \sum_{n=2}^{N-1} \frac{r^n}{n} \right).
\]
In view of the same line of argument as in the proof of Theorem 2.1, we can show that \( H_3(r) = 0 \) has a root in \((0,1)\) and we choose \( r_{m,N}(\alpha) \) to be the smallest root of \( H_3(r) \). Therefore, we have \( H_3(r_{m,N}(\alpha)) = 0 \). That is
\[
r_{m,N}(\alpha) - 1 - 2(1-\alpha)G_{m,N}(r) = 0, \tag{6.14}
\]
where
\[
G_{m,N} = \left( r_{m,N}(\alpha) + r_N(\alpha) - 1 + \ln \left( (2-2r_N(\alpha))(1-r_{m,N}(\alpha)) \right) + \sum_{n=2}^{N-1} \frac{r^n}{n} \right).
\]
In view of (6.3), (6.12) and (6.13) for \(|z| = r_{m,N}(\alpha)\), we obtain

\[ |f(z^m)| + \sum_{n=N}^{\infty} |a_n|r^n \leq d(f(0), \partial f(D)). \]

To show that the radius \(r_{m,N}(\alpha)\) is the best possible, we consider the function \(f = f(\alpha)\) defined by (6.7). For \(f = f_\alpha\) and \(|z| = r_{m,N}(\alpha)\), a simple calculation using (6.8) and (6.14) shows that

\[ |f(z^m)| + \sum_{n=N}^{\infty} |a_n|r^n \]
\[ = r^m_{m,N}(\alpha) - 2(1 - \alpha) \left( r^m_{m,N}(\alpha) + r + \ln \left( (1 - r_{m,N}(\alpha))(1 - r^m_{m,N}(\alpha)) \right) + \sum_{n=2}^{N-1} \frac{r^m_{m,N}(\alpha)}{n} \right) \]
\[ = 1 + 2(1 - \alpha)(\ln 2 - 1) = d(f(0), \partial f(D)). \]

Hence, the radius \(r_{m,N}(\alpha)\) is the best possible. This completes the proof.

Proof of Theorem 2.10. Let \(f \in \mathcal{P}_H^0(\alpha)\) be given by (1.11). Applying Lemmas 1.12 and 1.13 for \(|z| = r\), we obtain

\[ (6.15) \quad r + |h(r)|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \leq r + |h(r)|^p + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)r^n}{n}. \]

It is not difficult to show that

\[ (6.16) \quad r + |h(r)|^p + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)r^n}{n} = r^p + r - 2(1 - \alpha)(r + \ln(1 - r)) \]
\[ \leq 1 + 2(1 - \alpha)(\ln 2 - 1) \]

for \(r \leq r_p(\alpha)\), where \(r_p(\alpha)\) is the smallest root of \(H_4(r) = 0\) in \((0, 1)\) and \(H_4 : [0, 1] \to \mathbb{R}\) is defined by

\[ H_4(r) := r^p + r - 1 - 2(1 - \alpha)(r - 1 + \ln(2 - 2r)). \]

By the same argument used in the proof of Theorem 2.1, we can show that \(H_4(r)\) has a root in \((0, 1)\) and we choose \(r_p(\alpha)\) to be the smallest root of \(H_4(r)\). Therefore, \(H_4(r_p(\alpha)) = 0\). That is

\[ (6.17) \quad r^p_p(\alpha) + r_p(\alpha) - 1 - 2(1 - \alpha)(r_p(\alpha) - 1 + \ln(2 - 2r_p(\alpha))) = 0, \]

Using (6.3), (6.15) and (6.16) for \(|z| = r \leq r_p(\alpha)\), we obtain

\[ r + |h(r)|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \leq d(f(0), \partial f(D)). \]

In order to show that \(r_p(\alpha)\) is the best possible, we consider the function \(f = f_\alpha\) defined by (6.7). For \(f = f_\alpha\) and \(|z| = r_p(\alpha)\), a simple calculation using (6.8) and (6.17) shows that
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\[ r_p(\alpha) + |h(r_p(\alpha))|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \]

\[ = r_p^p(\alpha) + r_p(\alpha) - 2(1 - \alpha)(r_p(\alpha) + \ln(1 - r_p(\alpha))) \]

\[ = 1 + 2(1 - \alpha)(\ln 2 - 1) \]

Therefore, the radius \( r_p(\alpha) \) is the best possible. This completes the proof. \( \square \)

**Proof of Theorem 3.6.** Let \( f \in P_0^H(\alpha) \) be given by (1.11). For the analytic functions \( h \) and \( g \), the area of the disk \( |z| < r \) under the harmonic map \( f \) is \( S_r \) is given by

\[ S_r = \frac{1}{\pi} \int_{D_r} (|h'(z)|^2 - |g'(z)|^2) \, dx \, dy, \]

\[ \frac{1}{\pi} \int_{D_r} |h'(z)|^2 \, dx \, dy = \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \]

\[ \frac{1}{\pi} \int_{D_r} |g'(z)|^2 \, dx \, dy = \sum_{n=2}^{\infty} n|b_n|^2 r^{2n}. \]

Therefore, in view of (6.18), (6.19) and (6.20) and Lemma 1.12 we obtain

\[ \frac{S_r}{\pi} = \frac{1}{\pi} \int_{D_r} (|h'(z)|^2 - |g'(z)|^2) \, dx \, dy \]

\[ = r^2 + \sum_{n=2}^{\infty} |a_n|^2 r^{2n} - \sum_{n=2}^{\infty} n|b_n|^2 r^{2n} \]

\[ = r^2 + \sum_{n=2}^{\infty} n (|a_n| + |b_n|) (|a_n| - |b_n|) r^{2n} \]

\[ \leq r^2 + \sum_{n=2}^{\infty} \frac{4(1 - \alpha)^2 r^{2n}}{n^2} \]

\[ = r^2 - 4(1 - \alpha)^2 (r^2 + \ln(1 - r^2)). \]

Using Lemmas 1.12 and 1.13 for \( |z| = r \), we obtain

\[ r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + P \left( \frac{S_r}{\pi} \right) \]

\[ \leq r + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)r^n}{n} + P(r^2 - 4(1 - \alpha)^2 (r^2 + \ln(1 - r^2))) \]

\[ = r - 2(1 - \alpha)(r + \ln(1 - r)) + P(r^2 - 4(1 - \alpha)^2 (r^2 + \ln(1 - r^2))) \]

\[ \leq 1 + 2(1 - \alpha)(\ln 2 - 1) \]
for \( r \leq r(\alpha) \), where \( r(\alpha) \) is the smallest root of \( H_5(r) = 0 \) in \((0, 1)\) where \( H_5 : [0, 1) \to \mathbb{R} \) be defined by
\[
H_5(r) := r - 1 - 2(1 - \alpha)(r - 1 + \ln(2 - 2r)) + P(r^2 - 4(1 - \alpha)^2 (r^2 + \ln(1 - r^2)))
\]
Clearly, \( H_5(r(\alpha)) = 0 \). That is
\[
(6.22) \quad r(\alpha) - 1 - 2(1 - \alpha)(r(\alpha) - 1 + \ln(2 - 2r(\alpha))) + P(G(r, \alpha)) = 0,
\]
where
\[
G(r, \alpha) = r(\alpha)^2 - 4(1 - \alpha)^2 (r(\alpha)^2 + \ln(1 - r(\alpha)^2)).
\]
From (6.3), (6.21) and (6.22), \(|z| = r(\alpha)\), we obtain
\[
r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + P \left( \frac{S_r}{\pi} \right) \leq d(f(0), \partial f(\mathbb{D})).
\]
To show that the radius \( r(\alpha) \) is the best possible, we consider the function defined by (6.7). For \( f = f_\alpha \) and \(|z| = r(\alpha)\), a simple computation using (6.8) and (6.22) shows that
\[
r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + P \left( \frac{S_r}{\pi} \right) = r - 2(1 - \alpha)(r + \ln(1 - r)) + P(G(r, \alpha))
\]
\[
= 1 + 2(1 - \alpha)(\ln 2 - 1) = d(f(0), \partial f(\mathbb{D})).
\]
Thus, the radius \( r(\alpha) \) is the best possible. This completes the proof. \(\square\)

**Proof of Theorem 3.12.** Let \( f \in P^0_{H_6}(\alpha) \) be given by (1.11). Using Lemmas 1.12 and 1.13, for \(|z| = r\), we obtain
\[
(6.23) \quad r + \sum_{n=2}^{\infty} (|a_n| + |b_n| + (|a_n| + |b_n|)^2) r^n \leq r + \sum_{n=2}^{\infty} \left( \frac{2(1 - \alpha)}{n} + \frac{4(1 - \alpha)^2}{n^2} \right) r^n.
\]
An easy computation shows that
\[
(6.24) \quad r + \sum_{n=2}^{\infty} \left( \frac{2(1 - \alpha)}{n} + \frac{4(1 - \alpha)^2}{n^2} \right) r^n
\]
\[
= r - 2(1 - \alpha)(r + \ln(1 - r)) + 4(1 - \alpha)^2 (Li_2(r) - r)
\]
\[
\leq 1 + 2(1 - \alpha)(\ln 2 - 1)
\]
for \( r \leq r(\alpha) \), where \( r(\alpha) \) is the smallest root of \( H_6(r) = 0 \) in \((0, 1)\) and \( H_6 : [0, 1) \to \mathbb{R} \) be defined by
\[
H_6(r) := r - 2(1 - \alpha)(r - 1 + \ln(2 - 2r)) + 4(1 - \alpha)^2 (Li_2(r) - r) - 1.
\]
Thus, we have \( H_6(r(\alpha)) = 0 \). That is
\[
(6.25) \quad r(\alpha) - 2(1 - \alpha)(r(\alpha) - 1 + \ln(2 - 2r(\alpha))) + 4(1 - \alpha)^2 (Li_2(r(\alpha)) - r(\alpha)) - 1 = 0.
\]
Using (6.3), (6.23) and (6.24) for $|z| = r \leq r(\alpha)$, we obtain

$$r + \sum_{n=2}^{\infty} \left( |a_n| + |b_n| + (|a_n| + |b_n|)^2 \right) r^n \leq d(f(0), \partial f(\mathbb{D})).$$

In order to show that $r(\alpha)$ is the best possible, we consider the function defined by (6.7). For $f = f_\alpha$ and $|z| = r(\alpha)$, a simple computation using (6.8) and (6.25) shows that

$$r(\alpha) + \sum_{n=2}^{\infty} \left( |a_n| + |b_n| + (|a_n| + |b_n|)^2 \right) r^n(\alpha)$$

$$= r(\alpha) - 2(1 - \alpha)(r(\alpha) + \ln(1 - r(\alpha))) + 4(1 - \alpha)^2(Li_2(r(\alpha)) - r(\alpha))$$

$$= 1 + 2(1 - \alpha)(\ln 2 - 1)$$

$$= d(f(0), \partial f(\mathbb{D})).$$

Therefore, the radius $r(\alpha)$ is the best possible. This completes the proof.

**Proof of Theorem 4.2.** Let $f \in \mathcal{P}_\mathcal{H}(\alpha)$ be given by (1.11). In view of Lemmas 1.12 and 1.13 for $|z| = r$, we obtain

$$r + \sum_{n=2}^{\infty} \left| |a_n| + |b_n| + (|a_n| + |b_n|)^2 \right| r^n + \frac{1}{1 - r^N} \sum_{n=2}^{\infty} n(|a_n| + |b_n|)^2 r^{2n}$$

$$\leq r + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)r^n}{n} + \frac{1}{1 - r^N} \sum_{n=2}^{\infty} \frac{4(1 - \alpha)^2 r^{2n}}{n}.$$

A simple computation shows that

$$r + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)r^n}{n} + \frac{1}{1 - r^N} \sum_{n=2}^{\infty} \frac{4(1 - \alpha)^2 r^{2n}}{n}$$

$$= r - 2(1 - \alpha)(r + \ln(1 - r)) - \frac{4(1 - \alpha)^2}{1 - r^N}(r^2 + \ln(1 - r^2))$$

$$\leq 1 + 2(1 - \alpha)(\ln 2 - 1)$$

for $r \leq r(\alpha)$, where $r(\alpha)$ is the smallest root of $H_7(r) = 0$ in $(0, 1)$ and $H_7 : [0, 1) \rightarrow \mathbb{R}$ be defined by

$$H_7(r) := r - 2(1 - \alpha)(r - 1 + \ln(2 - 2r)) - \frac{4(1 - \alpha)^2}{1 - r^N}(r^2 + \ln(1 - r^2)) - 1.$$

Thus, we have $H_7(r(\alpha)) = 0$. That is

$$r(\alpha) - 2(1 - \alpha)(r - 1 + \ln(2 - 2r(\alpha)))$$

$$- \frac{4(1 - \alpha)^2}{1 - r^N(\alpha)} \left( r^2(\alpha) + \ln(1 - r^2(\alpha)) \right) - 1 = 0.$$
Using (6.3), (6.26) and (6.27) for $|z| = r \leq r(\alpha)$, we obtain
\[
r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n + \frac{1}{1 - rN} \sum_{n=2}^{\infty} n(|a_n| + |b_n|)^2 r^{2n} \leq d(f(0), \partial f(D)).
\]
In order to show that $r(\alpha)$ is the best possible, we consider the function $f = f_\alpha$ defined by (6.7). For $f = f_\alpha$ and $|z| = r(\alpha)$, a simple calculation using (6.8) and (6.28) shows that
\[
r(\alpha) + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n(\alpha) + \frac{1}{1 - rN(\alpha)} \sum_{n=2}^{\infty} n(|a_n| + |b_n|)^2 r^{2n}(\alpha)
\]
\[
= r(\alpha) - 2(1 - \alpha)(r(\alpha) + \ln(1 - r(\alpha))) - \frac{4(1 - \alpha)^2}{1 - rN(\alpha)}(\alpha^2 + \ln(1 - \alpha^2))
\]
\[
= 1 + 2(1 - \alpha)(\ln 2 - 1)
\]
\[
= d(f(0), \partial f(D)).
\]
Hence the radius $r(\alpha)$ is the best possible. This completes the proof. \qed

**Proof of Theorem 4.5.** Let $f \in \mathcal{P}_h^\theta(\alpha)$ be given by (1.11). Using Lemmas 1.12 and 1.13 for $|z| = r$, we obtain
\[
(6.29)
\]
\[
r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n + \left( \frac{1}{1 + |a_2| + |b_2|} + \frac{r^m}{1 - r^m} \right) \sum_{n=3}^{\infty} n^{m-1}(|a_n| + |b_n|)^m r^{mn}
\]
\[
\leq r + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)r^n}{n} + \left( \frac{1}{1 + |a_2| + |b_2|} + \frac{r^m}{1 - r^m} \right) \sum_{n=3}^{\infty} \frac{2^{m-1}(1 - \alpha)^m n^{m-1}r^{mn}}{n}.
\]
Let
\[
Q_m(r) := \frac{1}{1 + |a_2| + |b_2|} + \frac{r^m}{1 - r^m}.
\]
A computation using $Q_m(r)$ shows that
\[
(6.30)
\]
\[
r + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)r^n}{n} + Q_m(r) \sum_{n=3}^{\infty} \frac{2^{m}(1 - \alpha)^m n^{m-1}r^{mn}}{n}
\]
\[
= r - 2(1 - \alpha)(r + \ln(1 - r)) - 2^{m}(1 - \alpha)^m Q_m(r) \left( r^m + \frac{r^{2m}}{2} + \ln(1 - r^m) \right)
\]
\[
\leq 1 + 2(1 - \alpha)(\ln 2 - 1)
\]
for $r \leq r_m(\alpha)$, where $r_m(\alpha)$ is the smallest root of of $H_s(r)$ in $(0, 1)$ and $H_s(r) :=
\[
r - 1 - 2(1 - \alpha)(r - 1 + \ln(2 - 2r)) - 2^{m}(1 - \alpha)^m Q_m(r) \left( r^m + \frac{r^{2m}}{2} + \ln(1 - r^m) \right).
\]
Then we have $H_8(r_m(\alpha)) = 0$. That is
\begin{equation}
(6.31) \quad r_m(\alpha) - 1 - 2(1 - \alpha) \left( r_m(\alpha) - 1 + \ln(2 - 2r_m(\alpha)) \right) - 2^m(1 - \alpha)^m Q_m(r) \left( r_m(\alpha) + \frac{r^{2m}(\alpha)}{2} + \ln(1 - r_m^m(\alpha)) \right) = 0.
\end{equation}

With the help of (6.3), (6.29) and (6.30) for $|z| = r_m(\alpha)$, we obtain
\[
r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n + \left( \frac{1}{1 + |a_2| + |b_2|} + \frac{r^m}{1 - r_m} \right) \sum_{n=3}^{\infty} n^{m-1} (|a_n| + |b_n|)^n r^{mn} \leq d(f(0), \partial f(D)).
\]

To show that $r_m(\alpha)$ is the best possible, we consider the function $f = f_\alpha$ defined by (6.7). For the function $f = f_\alpha$ and $|z| = r_m(\alpha)$, an simple computation using (6.8) and (6.31) shows that
\[
\begin{align*}
    r_m(\alpha) + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n(\alpha) + \left( \frac{1}{1 + |a_2| + |b_2|} + \frac{r^m}{1 - r_m} \right) \\
    \times \sum_{n=3}^{\infty} n^{m-1} (|a_n| + |b_n|)^n r_m(\alpha) \\
    = r_m(\alpha) - 2(1 - \alpha) \left( r_m(\alpha) + \ln(1 - r_m(\alpha)) \right) \\
    - 2^m(1 - \alpha)^m Q_m(r_m(\alpha)) \left( r_m(\alpha) + \frac{r^{2m}(\alpha)}{2} + \ln(1 - r_m(\alpha)) \right) \\
    = 1 + 2(1 - \alpha)(\ln 2 - 1) \\
    = d(f(0), \partial f(D)).
\end{align*}
\]

Therefore, the radius $r_m(\alpha)$ is the best possible. This completes the proof. \qed

**Proof of Theorem 4.8** Let $f \in P^0_H(\alpha)$ be given by (1.11). By using Lemmas 1.12 and 1.13 for $|z| = r$, we obtain
\begin{equation}
(6.32) \quad r + \frac{[1 - (1 + |a_2| + |b_2|) - (|a_2| + |b_2|)^2] r}{1 - (|a_2| + |b_2|) r} + \sum_{n=3}^{\infty} (|a_n| + |b_n|) r^n \leq r + \frac{[1 - (1 + |a_2| + |b_2|) - (|a_2| + |b_2|)^2] r}{1 - (|a_2| + |b_2|) r} + \sum_{n=3}^{\infty} 2(1 - \alpha)r^n.
\end{equation}

Let $R(r)$ be defined
\[
R(r) = \frac{[1 - (1 + |a_2| + |b_2|) - (|a_2| + |b_2|)^2] r}{1 - (|a_2| + |b_2|) r}.
\]
A simple computation shows that
\[
(6.33) \quad r + R(r) + \sum_{n=3}^{\infty} \frac{2(1-\alpha)r^n}{n} = r + R(r) - 2(1-\alpha) \left( r + \frac{r^2}{2} + \ln(1-r) \right) \\
\leq 1 + 2(1-\alpha)(\ln 2 - 1)
\]
for \( r \leq r(\alpha) \), where \( r(\alpha) \) is the smallest root of \( H_\alpha(r) \) in \((0,1)\) and \( H_\alpha : [0,1) \to \mathbb{R} \) be defined by
\[
H_\alpha(r) := r + R(r) - 2(1-\alpha) \left( r + \frac{r^2}{2} - 1 + \ln(2-2r) \right) - 1.
\]
Then we have \( H_\alpha(r(\alpha)) = 0 \). That is
\[
(6.34) \quad r(\alpha) + R(r(\alpha)) - 2(1-\alpha) \left( r(\alpha) + \frac{r^2(\alpha)}{2} - 1 + \ln(2-2r(\alpha)) \right) - 1 = 0.
\]
From \((6.3), (6.32)\) and \((6.33)\) for \( |z| = r \leq r(\alpha) \), we obtain
\[
r + \frac{1 - (1 + |a_2| + |b_2| - (|a_2| + |b_2|)^2)r}{1 - (|a_2| + |b_2|)r} + \sum_{n=3}^{\infty} (|a_n| + |b_n|)r^n \leq d(f(0), \partial f(\mathbb{D})).
\]
To show that the radius \( r(\alpha) \) is the best possible, we consider the function \( f = f_\alpha \) defined in \((6.7)\). For \( f = f_\alpha \) and \( |z| = r(\alpha) \), a simple calculation using \((6.8)\) and \((6.34)\) shows that
\[
r(\alpha) + \frac{1 - (1 + |a_2| + |b_2| - (|a_2| + |b_2|)^2)r(\alpha)}{1 - (|a_2| + |b_2|)r(\alpha)} + \sum_{n=3}^{\infty} (|a_n| + |b_n|)r^n(\alpha) \\
= r(\alpha) + R(r(\alpha)) - 2(1-\alpha) \left( r(\alpha) + \frac{r^2(\alpha)}{2} + \ln(1-r(\alpha)) \right) \\
= 1 + 2(1-\alpha)(\ln 2 - 1) \\
= d(f(0), \partial f(\mathbb{D})).
\]
Hence the radius \( r(\alpha) \) is the best possible. This comples the proof. \( \square \)

**Proof of Theorem 5.1.** Let \( f \in \mathcal{P}_\mathcal{H}(\alpha) \) be given by \((1.11)\). It is not difficult to show that
\[
|h'(z)| \leq \alpha + (1-\alpha) \left( \frac{1 + r}{1 - r} \right).
\]
Therefore, we have
\[
|J_f(z)| = |h'(z)|^2 - |g'(z)|^2 \leq |h'(z)|^2.
\]
Using Lemmas 1.12 and 1.13 for \(|z| = r_N(\alpha)|\), we obtain

\begin{align}
|f(z)| & + \sqrt{J_f(z)}r + \sum_{n=N}^\infty (|a_n| + |b_n|)r^n \\
& \leq r + 2(1 - \alpha)(-r - \ln(1 - r)) + \left(\alpha + (1 - \alpha)\left(\frac{1 + r}{1 - r}\right)\right) r \\
& + 2(1 - \alpha)\left(-r - r^2/2 - \cdots - \frac{r^{N-1}}{N-1} - \ln(1 - r)\right) \\
& = r - 2(1 - \alpha)\left(2r + \frac{r^2}{2} + \cdots + \frac{r^{N-1}}{N-1} + 2\ln(1 - r)\right) + \left(\alpha + (1 - \alpha)\left(\frac{1 + r}{1 - r}\right)\right) r.
\end{align}

A simple calculations shows that

\begin{align}
r - 2(1 - \alpha)\left(2r + \frac{r^2}{2} + \cdots + \frac{r^{N-1}}{N-1} + 2\ln(1 - r)\right) + \left(\alpha + (1 - \alpha)\left(\frac{1 + r}{1 - r}\right)\right) r \\
& \leq 1 + 2(1 - \alpha)(\ln 2 - 1)
\end{align}

for \(r \leq r(\alpha)\), where \(r(\alpha)\) is the smallest root of \(H_{10}(r)\) in \((0, 1)\). Here \(H_{10} : [0, 1) \to \mathbb{R}\) defined by

\begin{align}
H_{10}(r) : = r - 1 - 2(1 - \alpha)\left(2r - 1 + \frac{r^2}{2} + \cdots + \frac{r^{N-1}}{N-1} + \ln 2 + 2\ln(1 - r)\right) \\
& + \left(\alpha + (1 - \alpha)\left(\frac{1 + r}{1 - r}\right)\right) r.
\end{align}

Therefore, we have \(H_{10}(r_N(\alpha)) = 0\), which shows that

\begin{align}
r_N(\alpha) - 1 - 2(1 - \alpha)\left(2r_N(\alpha) - 1 + \frac{r_N^2(\alpha)}{2} + \cdots + \frac{r_N^{N-1}(\alpha)}{N-1} + \ln 2 + 2\ln(1 - r_N(\alpha))\right) \\
& + \left(\alpha + (1 - \alpha)\left(\frac{1 + r_N(\alpha)}{1 - r_N(\alpha)}\right)\right)^2 = 0,
\end{align}

Using (6.3), (6.35) and (6.36) for \(|z| = r \leq r_N(\alpha)|\), we obtain

\begin{align}
|f(z)| + |J_f(z)| + \sum_{n=N}^\infty (|a_n| + |b_n|)r^n & \leq d(f(0), \partial f(\mathbb{D})).
\end{align}

In order to show that \(r_N(\alpha)|\) is sharp, we consider the function \(f = f_\alpha\) defined by (6.7). For \(f = f_\alpha\) and \(|z| = r_N(\alpha)|\), a simple calculation using (6.8) and (6.37) shows
that

\[
|f(z)| + \sqrt{|Jf(z)|}r + \sum_{n=N}^{\infty} (|a_n| + |b_n|) r_n^\alpha
\]

\[
= r_N^\alpha - 2(1 - \alpha) \left( 2r_N^\alpha + \frac{r_N^2\alpha}{2} + \cdots + \frac{r_N^{N-1}\alpha}{N-1} + 2\ln(1 - r_N^\alpha) \right)
\]

\[
+ \left( \alpha + (1 - \alpha) \left( \frac{1 + r_N^\alpha}{1 - r_N^\alpha} \right) \right) r
\]

\[
= 1 + 2(1 - \alpha)(\ln 2 - 1)
\]

\[
= d(f(0), \partial f(D)).
\]

Therefore, the radius \( r_N^\alpha \) is the best possible. This completes the proof. \( \square \)

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