Variational principle for the relativistic hydrodynamic flows with discontinuities, and local invariants of motion.

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March 22, 2021

Abstract
A rigorous method for introducing the variational principle describing relativistic ideal hydrodynamic flows with all possible types of discontinuities (including shocks) is presented in the framework of an exact Clebsch type representation of the four-velocity field as a bilinear combination of the scalar fields. The boundary conditions for these fields on the discontinuities are found. We also discuss the local invariants caused by the relabeling symmetry of the problem and derive recursion relations linking invariants of different types. These invariants are of specific interest for stability problems. In particular, we present a set of invariants based on the relativistic generalization of the Ertel invariant.

Introduction. In this paper we discuss some problems related to ideal relativistic hydrodynamic (RHD) flows in the framework of the special relativity. They are pertinent to the description of flows with discontinuities, including shocks, in terms of canonical (Hamiltonian) variables based upon the corresponding variational principle and introducing local invariants along with recursion relations. These subjects are of interest from a general point of view and are very useful in solving nonlinear problems, specifically, nonlinear stability investigation, description of the turbulent flows, etc. In particular, the use of the Hamiltonian approach along with additional local invariants of the motion and the corresponding Casimirs allows to improve the nonlinear stability criteria. The necessity to consider the relativistic flows is motivated by a wide area of applications, including the astrophysical and cosmological problems.
Variational principles for the ideal relativistic hydrodynamic (RHD) flows, without discontinuities, have been widely discussed in the literature, see, for instance, [1–3] and citations therein. As for the nonrelativistic flows, the least action principle is conveniently formulated in terms of the subsidiary fields and the corresponding velocity representation known as the Clebsch representation, see [4–9]. These subsidiary fields can be introduced explicitly by means of the Weber transformation, [10], see also [3, 4]. Alternatively, they naturally arise from the least action principle as Lagrange multipliers for necessary constraints. Using these variables allows one to describe the dynamics in terms of canonical (Hamiltonian) variables. The nontrivial character of the Hamiltonian approach is due to the fact that the fluid dynamics corresponds to the degenerated case, see [11, 12].

Recently it was shown [13, 14] that the hydrodynamic flows with discontinuities (including shocks) can be described in terms of a least action principle, which includes (as well as natural boundary conditions) the boundary conditions for the subsidiary fields. In the present paper we show that all type of discontinuities can be described by means of the least action principle in terms of the canonical variables for the relativistic flows.

**Variational principle.** The relativistic least action principle can be formulated in close analogy to the nonrelativistic one. We introduce the action $A$,  

$$A = \int d^4 x \mathcal{L},$$

with the Lagrangian density

$$\mathcal{L} = -\epsilon(n, S) + G J^\alpha Q_\alpha,$$

$$G = (1, \nu_B, \Theta), \quad Q = (\varphi, \mu^B, S), \quad B = 1, 2, 3,$$

where $\nu_B$, $\Theta$, $\varphi$, $\mu^B$ represent subsidiary fields; $n$, $S$ and $\epsilon(n, S)$ denote the particle’s number, entropy and energy proper densities, $J^\alpha = nu^\alpha$ is the particle current, and $u^\alpha$ is the four-velocity, $u^\alpha = u^0(1, v/c)$, $u^0 = 1/\sqrt{1 - v^2/c^2}$; comma denotes partial derivatives. Small Greek indexes run from 0 to 3, and the Latin indexes run from 1 to 3; $x^0 = ct$, $r = (x^1, x^2, x^3)$. The metric tensor, $g^{\alpha\beta}$, corresponds to the flat space-time in Cartesian coordinates, $g^{\alpha\beta} = \text{diag}\{-1, 1, 1, 1\}$. The four-velocity obeys the normalization condition

$$u^\alpha u_\alpha = g_{\alpha\beta} u^\alpha u^\beta = -1.$$ 

Below we consider the four-velocity and the particle density $n$ as dependent variables expressed in terms of the particles current $J^\alpha$,

$$u^\alpha = J^\alpha/|J|, \quad n = |J| = \sqrt{-J^\alpha J_\alpha}.$$ 

The fluid energy obeys the second thermodynamic law

$$d\epsilon = nT dS + n^{-1} w dn \equiv nT dS + W dn,$$

where $T$ is the temperature and $w \equiv \epsilon + p$ is the proper enthalpy density, $p$ is the fluid pressure, $W = w/n$.  

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Variation of the action given by Eq. (1) with respect to the variables $J^\alpha$, $\Theta$, and $Q = (\varphi, \mu^B, S)$, which are supposed to be independent, $A = A[J^\alpha, \varphi, \mu^B, S, \nu_B, \Theta]$, results in the following set of equations

\[ \delta J^\alpha :\implies Wu_\alpha \equiv V_\alpha = -GQ_\alpha , \tag{6} \]

\[ \delta \varphi :\implies J^\alpha,\alpha = 0 , \tag{7} \]

\[ \delta \mu^B :\implies \partial_\alpha(J^\alpha \nu_B) = 0 , \quad \text{or} \quad D\nu_A = 0 , \tag{8} \]

\[ \delta \nu_B :\implies D\mu^B = 0 , \tag{9} \]

\[ \delta S :\implies \partial_\alpha(J^\alpha \Theta) , \quad \text{or} \quad D\Theta = -T , \tag{10} \]

\[ \delta \Theta :\implies DS = 0 , \tag{11} \]

where $D \equiv u^\alpha \partial_\alpha$. Eq. (6) gives us the Clebsch type velocity representation, cf. Ref. [2]. Contracting it with $u^\alpha$ results in the dynamic equation for the scalar potential $\varphi$,

\[ D\varphi = W . \tag{12} \]

Both triplets $\mu^B$ and $\nu_B$ represent the advected subsidiary fields and do not enter the internal energy. Therefore, it is natural to treat one of them, say, $\mu^B$ as the flow line label.

Taking into account that the entropy and particle conservation are incorporated into the set of variational equations, it is easy to make sure that the equations of motion for the subsidiary variables along with the velocity representation reproduces the relativistic Euler equation. The latter corresponds to the orthogonal to the flow lines projection of the fluid stress-energy-momentum $T^{\alpha\beta}$ conservation, cf. Ref. [16, 17],

\[ T^{\alpha\beta}_{\ ,\beta} = 0 , \quad T^{\alpha\beta} \equiv wu^\alpha u^\beta + pg^{\alpha\beta} . \tag{13} \]

We may then write the relativistic Euler equation as

\[ (V_{\alpha,\beta} - V_{\beta,\alpha})u^\beta = TS_\alpha , \tag{14} \]

where the thermodynamic relation

\[ dp = ndW - nTdS \tag{15} \]

is taken into account. The vector $V_\alpha$, sometimes called the Taub current, [15], plays an important role in relativistic fluid dynamics, especially in the description of circulation and vorticity. Note that $W$ can be interpreted as an injection energy (or chemical potential), cf., for instance [16], i.e., the energy per particle required to inject a small amount of fluid into a fluid sample, keeping the sample volume and the entropy per particle $S$ constant. Therefore, $V_\alpha$ is identified with the four-momentum per particle of a small amount of fluid to be injected in a larger sample of fluid without changing the total fluid volume and the entropy per particle.
Boundary conditions. In order to complete the variational approach for the flows with discontinuities, it is necessary to formulate the boundary conditions for the subsidiary variables, which do not imply any restrictions on the physically possible discontinuities (the shocks, tangential and contact discontinuities), are consistent with the corresponding dynamic equations, and thus are equivalent to the conventional boundary conditions, i.e., to continuity of the particle and energy-momentum fluxes intersecting the discontinuity surface $R(x^\alpha) = 0$, cf. Ref. [17],

$$\{J\} = 0, \quad J \equiv J^\alpha n_\alpha,$$  \hfill (16)

$$\{T^{\alpha\beta} n_\beta\} = 0,$$ \hfill (17)

where $n_\alpha$ denotes the unit normal vector to the discontinuity surface,

$$n_\alpha = N_\alpha/N, \quad N_\alpha = R_\alpha, \quad N = \sqrt{N_\alpha N_\alpha},$$ \hfill (18)

and braces denote jump, $\{X\} \equiv X|_{R=+0} - X|_{R=-0}$.

Our aim is to obtain boundary conditions as natural boundary conditions for the variational principle. In the process of deriving the volume equations we have applied integration by parts to the term $J^\alpha G\delta Q_{,\alpha}$. Vanishing of the corresponding surface term along with that resulting from the variation of the surface itself will lead to the appropriate boundary conditions after the variational principle has been specified.

Rewriting the (volume) action with the discontinuity surface being taken into account in explicit form as

$$A = \int d^4x \sum_{\varsigma=\pm} L^\varsigma \theta(\varsigma R),$$ \hfill (19)

where $\theta$ stands for the step-function, we obtain the residual part of the (volume) action in the form

$$\delta A|_{res} = \int d^4x \sum_{\varsigma=\pm} [\varsigma L \delta_D(R) \delta R + \theta(\varsigma R) \partial_\alpha(J^\alpha G\tilde{Q})].$$ \hfill (20)

Here $\delta_D$ denotes Dirac’s delta-function and we omit the index $\varsigma$ labeling the quantities that correspond to the fluid regions divided by the interface at $R = 0$; the superscript $\varsigma \geq 0$ corresponds to the quantities in the regions $R \geq 0$, respectively. Integrating the second term by parts and supposing that the surface integral $\int d^4x \sum_{\varsigma=\pm} \partial_\alpha(\theta(\varsigma R)(u^\alpha G\delta Q))$ vanishes due to vanishing of the variations $\delta Q$ at infinity, we arrive at the residual action expressed by the surface integral

$$\delta A|_{res} = \int d^4x \sum_{\varsigma=\pm} \varsigma \delta_D(R) \left[ L \delta R - R_\alpha J^\alpha G\tilde{Q} \right].$$ \hfill (21)

Here $\tilde{Q}$ designates the limit values of the volume variations, $\tilde{Q}^\pm \equiv (\delta Q)|_{R=\pm 0}$. It is convenient to express these variations in terms of variations of the boundary restrictions of the volume variables, $\delta(X_{R=\pm 0}) \equiv \delta \tilde{X}^\pm$, and variation of the discontinuity surface. It is easy to show that

$$\tilde{X} = \delta \tilde{X} + |N|^{-1} n^\alpha X_{,\alpha} \delta R - X_{,\alpha} P^\alpha_\beta \delta f^\beta,$$ \hfill (22)

where $P^\alpha_\beta = \delta^\alpha_\beta - n^\alpha n_\beta$, and $\delta f^\beta$ is an arbitrary infinitesimal four-vector related to the one-to-one mapping of the surfaces $R = 0$ and $R + \delta R = 0$. 

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Vanishing of the action variation with respect to variations of the surface variables \( \delta R \) and \( \delta f^\beta \) (which are supposed to be independent) results in the following boundary conditions

\[
\delta R := \left\{ p + (u^\alpha n_\alpha)^2 w \right\} = 0,
\]

\[
\delta f^\beta := P^\gamma_\beta \{ W J^\alpha N_\alpha u_\gamma \} = 0, \quad \text{or} \quad P^\gamma_\beta \{ \tilde{J} W u_\gamma \} = 0,
\]

which are equivalent to continuity of the momentum and energy fluxes, cf. Eq. (17). Here we consider that the ‘on shell’ value of the volume Lagrangian density, \( \mathcal{L}_{eq} \), is equal to the pressure,

\[
\mathcal{L}_{eq} = -\epsilon + nG D Q = -\epsilon + w = p.
\]

Now we can complete formulation of the variational principle appropriate both for continuous and discontinuous flows. The independent volume variables are indicated above, and independent variations of the surface variables are \( \delta R \), \( \delta f^\beta \), the variations of the surface restrictions of the generalized coordinates \( \delta \varphi \), \( \delta \mu \), supposed to be equal from both sides of the discontinuity, \( \{ \delta \varphi \} = \{ \delta \mu \} = 0 \), and \( \delta S \) with \( \{ \delta S \} \neq 0 \). Under these assumptions we arrive at the following subset of the boundary conditions

\[
\delta \tilde{\varphi} := \{ J^\alpha n_\alpha \} \equiv \{ \tilde{J} \} = 0 \quad \text{for} \quad \{ \delta \tilde{\varphi} \} = 0,
\]

\[
\delta \tilde{\mu}^B := \{ \nu_B J^\alpha n_\alpha \} \equiv \tilde{J} \{ \nu_B \} = 0 \quad \text{for} \quad \{ \delta \tilde{\mu}^B \} = 0,
\]

\[
\delta \tilde{S}^\pm := J^\alpha n_\alpha \tilde{\Theta}^\pm \equiv \tilde{J} \tilde{\Theta}^\pm = 0.
\]

Eqs. (23)–(25) reproduce the usual boundary conditions, and Eqs. (26), (27) are the boundary conditions for the subsidiary variables. Other boundary conditions for the latter variables do not strictly follow from the variational principle under discussion. But we can find them from the corresponding volume equations of motion, providing, for instance, that they are as continuous as possible.\(^1\) The natural choice corresponds to continuity of their fluxes,

\[
\{ n_\alpha u^\alpha n \mu^B \} \equiv \tilde{J} \{ \mu^B \} = 0,
\]

\[
\{ n_\alpha u^\alpha n \varphi \} \equiv \tilde{J} \{ \varphi \} = 0.
\]

The set of the boundary conditions given by Eqs. (23)–(29) is complete and allows one to describe any type of discontinuities, including shocks. For the latter case \( \tilde{J} \neq 0 \) and we arrive at continuity of the variables \( \nu_B, \mu^B, \varphi \) and zero boundary value of \( \Theta \). For \( \tilde{J} = 0 \) the flow lines do not intersect the discontinuity surface and we obtain very weak restrictions on the boundary conditions.

\(^1\)Note that the choice of the boundary conditions for the fields \( \varphi, \mu^B, \nu_B \) and \( \Theta \) is not unique due to the fact that they play the roles as generalized potentials and therefore possess the corresponding gauge freedom relating to the transformations \( \varphi, \mu^B, \nu_B, \Theta \to \varphi', \mu'^B, \nu'_B, \Theta' \) such that \( u'_\alpha = u_\alpha \) (given by the representation (6)). For instance, it seems possible to use entropy \( S \) as one of the flow line markers. But if we are dealing with discontinuous flows then it is necessary to distinguish the Lagrange markers of the fluid lines, \( \mu^B \), and the entropy, \( S \). Namely, the label of the particle intersecting a shock surface evidently does not change, but the entropy does change. Thus, entropy can be chosen as one of the flow line markers only for the flows without entropy discontinuities.
values of the subsidiary variables, cf. the nonrelativistic case discussed in Refs. [13, 14]. Note that for the specific case \( \mathbf{J} = 0 \) (slide and contact discontinuities) we can simplify the variational principle assuming all both-side variations of the subsidiary variables to be independent.

The above variational principle allows modifications. First, it is possible to exclude constraints, expressing the four-velocity by means of the representation (5). In this case the volume Lagrangian density can be chosen to coincide with the fluid pressure, cf. Ref. [2], where the continuous flows are discussed in detail. Second, we can include into the action the surface term respective for the surface constraints, cf. Refs. [13, 14, 18, 19], where such surface terms are discussed for ideal hydrodynamics and magnetohydrodynamics in the nonrelativistic limit. This can be done for the cases both with excluded and non excluded volume constraints.

**Canonical variables.** Starting from the action in Eq. (1) and Lagrangian density given by Eq. (2) we can introduce the canonical (Hamiltonian) variables according to the general receipt. Let \( Q \) represents the canonical coordinates then

\[
P \equiv \frac{\delta A}{\delta Q,0} = J^0 G \equiv (\pi_\phi, \pi_\mu, \pi_S)
\]

provides the conjugate momenta. Relations (30) cannot be solved for the generalized velocities \( Q,0 \) suggesting that we are dealing with the degenerated (constraint) system, cf. Refs. [3,5,11,12]. But the constraints are of the first type. Thus, performing the Legendre transform with respect to \( Q \) we arrive at the Hamiltonian density

\[
\mathcal{H} = PQ,0 - p(W,S),
\]

where we suppose that the four-velocity is given by representation (3). Making use of the definition (30) and of the time component of the velocity representation, Eq. (6), we can transform the first term in Eq. (31) as

\[
PQ,0 = J^0 GQ,0 = -\pi_\phi V_0 = \pi_\phi V^0.
\]

Taking into account the normalization condition for the Taub current, \( V_\alpha V^\alpha = -W^2 \), we obtain

\[
V^0 = \sqrt{W^2 + V_\alpha V^\alpha}.
\]

Consequently, we arrive at the following Hamiltonian density

\[
\mathcal{H} \equiv \mathcal{H}(P, Q, Q,\alpha; W) = \sqrt{W^2 + V_\alpha V^\alpha} \pi_\phi - p(W,S).
\]

In terms of the canonical coordinates and momenta the space components of the velocity are

\[
\pi_\phi V_a = -PQ,\alpha.
\]

The canonical equations following from this Hamiltonian reproduce in a 3+1 form the dynamical equations above for the variables entering the Taub current representation. Variation of the action with respect to the chemical potential \( W \) results in the identity

\[
n = \frac{\pi_\phi}{\sqrt{1 + V_\alpha V^\alpha/W^2}}.
\]
Obviously, this relation is equivalent to Eq. (33), expressing the particle density $n$ in terms of the variables entering the Hamiltonian.

We emphasize that the Hamiltonian given by Eq. (34) depends not only on the generalized coordinates $\varphi, \mu^B, S$, their spatial derivatives and conjugate momenta, but also on the chemical potential $W$ as well. Evidently, we can consider $W$ as the additional generalized coordinate with zero conjugate momentum, $\pi_W = 0$. This condition is consistent with the dynamic equations due to the fact that $\partial_0 \pi_W = \partial H / \partial W = 0$, cf. Eq. (36).

Bearing in mind that we are dealing with flows having discontinuities, it is seen that in the discussed variant of the least action principle we do not arrive at the additional surface variables except for the variable defining the discontinuity surface, $R$. But this enters the action functional without derivatives. Therefore, the corresponding conjugate momentum is zero-valued. Introducing the Hamiltonian variables for the flows with discontinuities we have to treat $R$ as the surface function, defining some (surface) constraint. The latter is nothing else than continuity of the normal component of the fluid momentum flux, Eq. (23).

**Local invariants and recursion relations.** In addition to energy, momentum, and angular momentum conservation, for the ideal hydrodynamic flows there exist specific local conservation laws related to the advected and frozen-in fields, and corresponding topological invariants (vorticity, helicity, Ertel invariant, etc.), cf. Refs. [3–5, 9, 20] and citations therein for the non-relativistic case. They are caused by the relabeling symmetry, cf. Ref. [9]. Discussion of these problems along with the recursion relations linking the four different types of invariants for the relativistic flows seems insufficient or absent in the literature, see Refs. [1–3, 15, 21] and citations therein. Exploitation of the above description permit us considering these invariants in a simplified form. Here we shall briefly discuss the invariants and recursion relations.

The local Lagrangian invariants, say $I$, correspond to advected (dragged) quantities,

$$DI = 0.$$  \hspace{1cm} (37)

The partial derivative of each scalar Lagrange invariant gives us the simplest example of the Lamb type momentum, $L_\alpha$, which satisfy the following relations

$$DL_\alpha + u^\beta_\alpha L_\beta = 0,$$  \hspace{1cm} (38)

The next type of invariants are vector conserved quantities, $X^\alpha$, being proportional to the four-velocity, i.e.,

$$X^\alpha_{,\alpha} = 0, \quad X^\alpha = |X|u^\alpha.$$  \hspace{1cm} (39)

The trivial example of such quantities is the particle current $J^\alpha = nu^\alpha$. The last type corresponds to the frozen-in fields, $M_{\alpha\beta}$, defined as the antisymmetric tensors obeying the following equation

$$DM_{\alpha\beta} + u^\gamma_{,\alpha} M_{\gamma\beta} + u^\gamma_{,\beta} M_{\alpha\gamma} = 0, \quad M_{\beta\alpha} = -M_{\alpha\beta}.$$  \hspace{1cm} (40)

Now we can derive some recursion relations. First, an arbitrary function of the Lagrangian invariants is also Lagrangian invariant,

$$I'' = F(I, I', \ldots).$$  \hspace{1cm} (41)
Second, multiplication of any invariant by the Lagrange invariant results in the invariant of the same type. Symbolically,

\[ L'_\alpha = IL_\alpha, \quad X'^\alpha = IX^\alpha, \quad M'_{\alpha\beta} = IM_{\alpha\beta}. \] (42)

Third,

\[ L_\alpha = I_{,\alpha}, \] (43)

\[ M_{\alpha\beta} = L_{\alpha\beta} - L_{\beta\alpha}, \] (44)

\[ M_{\alpha\beta} = L_\alpha L'_\beta - L_\beta L'_\alpha, \] (45)

\[ L_\alpha = M_{\alpha\beta} u^\beta, \] (46)

\[ X^\alpha = \epsilon^{\alpha\beta\mu\nu} I_{,\beta} I'_\mu I''_\nu, \] (47)

\[ I''' = n^{-1} \epsilon^{\alpha\beta\mu\nu} u_{\alpha} I_{,\beta} I'_\mu I''_\nu, \quad DI''' = 0. \] (48)

Here \( \epsilon^{\alpha\beta\mu\nu} \) is Levi-Civita tensor. Note that the latter relation follows from the fact that the conserved current defined by Eq. (47) is collinear to the four-velocity, \( X^\alpha = |X|u^\alpha \equiv -|X|J^\alpha/n. \) Consequently, \( |X| = -X^\alpha u_\alpha, \) and \( |X|/n = I \) is an advected scalar. Note also that we can easily arrive at the conserved four-currents, say \( Z^\alpha, \) which are not necessarily collinear to the velocity field. In analogy with relation (47) we find

\[ Z^\alpha = \epsilon^{\alpha\beta\mu\nu} L_{,\beta} I'_\mu I''_\nu. \] (49)

Other examples of such conserved currents are as follows

\[ Z^\alpha = \epsilon^{\alpha\beta\mu\nu} (IL_{\beta\mu})_{,\nu}, \] (50)

\[ Z^\alpha = \epsilon^{\alpha\beta\mu\nu} (IL_{\mu} L'_\beta)_{,\nu}, \] (51)

\[ Z^\alpha = (\star M_{\alpha\beta} I)_{,\beta}, \] (52)

where \( \star M_{\alpha\beta} \) is dual to \( M_{\alpha\beta}. \)

These recursion relations represent a strict analog of the nonrelativistic ones, cf. Refs. [3,5,20] and can be proved by direct calculations. Existence of these four types of invariants is related to the dimensionality of the space-time as becomes evident if we remember that they correspond to the differential forms of 0th, 1st, 2nd and 3rd order, cf., for instance, Ref. [16]. The recursion procedure allows one to study the structure of the invariants.

It is noteworthy that not all invariants could be obtained by means of the recursion procedure. For flows of general type there also exists the Ertel invariant (or the potential vorticity), cf. Ref. [21]. The corresponding conserved four-current is of the form

\[ \mathcal{E}^\alpha = -\frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \omega_\beta \nu S^\mu_{,\nu} = -\star \omega_{\alpha\nu} S^\nu_{,\nu}, \] (53)
where $\omega_{\beta\mu}$ is the (Khalatnikov) vorticity tensor, $\omega_{\beta\mu} = V_{\mu,\beta} - V_{\beta,\mu}$, and $\omega^{\alpha\nu}$ is its dual. The vorticity tensor obeys the following equation

$$D\omega_{\alpha\nu} + \omega_{\alpha\beta} u^\beta_{,\nu} + \omega_{\beta\nu} u^\beta_{,\alpha} = T_{\alpha\nu} - T_{\nu\alpha} \equiv n^{-2}(n_{\alpha}p_{,\nu} - n_{\nu}p_{,\alpha}).$$  \hfill (54)

It can be proved that $E^\alpha$ is divergence-free (the easiest way is to use the above velocity representation) and

$$E^\alpha = \Xi^\alpha, \quad DE = 0,$$  \hfill (55)

where $\Xi$ is a direct generalization of the well-known nonrelativistic potential vorticity. If the vorticity tensor would be the frozen-in quantity then conservation of the vector given by Eq. (53) would follow from the above recursion relations. But for the non-barotropic flows it is not so. Here $\omega_{\alpha\nu}$ obeys Eq. (54) and becomes a frozen-in field only for the barotropic (isentropic, in particular) flows. Nevertheless, the Ertel current is conserved. Therefore, the basic invariants may be obtained by direct calculations.

It is interesting to note that for the non-barotropic flows there exists a conserved current generalizing the helicity current. Consider the reduced Taub vector,

$$\tilde{V}_\alpha \equiv V_\alpha + \Theta S_{,\alpha},$$  \hfill (56)

where $\Theta$ obeys Eq. (10), and the corresponding reduced vorticity tensor

$$\tilde{\omega}_{\alpha\beta} \equiv \tilde{V}_{\beta,\alpha} - \tilde{V}_{\alpha,\beta} = \nu_{A,\beta} \mu^A_{,\alpha} - \nu_{A,\alpha} \mu^A_{,\beta}.$$  \hfill (57)

This tensor is orthogonal to the flow lines,

$$\tilde{\omega}_{\alpha\beta} u^\beta = 0,$$  \hfill (58)

therefore, the reduced helicity current

$$\tilde{Z}^\alpha = \tilde{\omega}^{\alpha\nu} \tilde{V}_\nu$$  \hfill (59)

is conserved for arbitrary flows,

$$\tilde{Z}^\alpha_{,\alpha} = \tilde{\omega}^{\alpha\nu} \tilde{V}_{\nu,\alpha} = \frac{1}{2} \tilde{\omega}^{\alpha\nu} \tilde{\omega}_{\alpha\nu} \equiv \frac{1}{4} \epsilon^{\alpha\nu\beta\gamma} \tilde{\omega}_{\beta\gamma} \tilde{\omega}_{\alpha\nu} = 0.$$  \hfill (60)

Here the “thermassy” field $\Theta$ can be chosen in such a way that its initial value is zero and thus the initial value of the generalized helicity coincides with the conventional one.

To derive a set of local invariants we have to start with those following directly from the hydrodynamic equations and then apply the above recursion relations. As the simplest example consider the non-barotropic flow. Then one can start with the specific entropy $S$ and the Ertel invariant $\Xi$. It is easy to show that the general form of the gauge-independent scalar invariants is

$$I = F(S, \Xi),$$  \hfill (61)

where $F$ is arbitrary function, cf. the nonrelativistic case, [3]. The structure of the complete set of invariants differs for the different type of flows and will be discussed in forthcoming publications. For instance, for the barotropic flows we have the independent frozen in field $M_{\alpha\beta} = \omega_{\alpha\beta}$ in addition to the scalar invariants of the first generation, $S$ and $\Xi$. This fact allows one to obtain a more complicated set of the invariants.
Conclusion. We have shown that it is possible to describe the relativistic ideal fluids with all physically allowable discontinuities in terms of the least action principle both in the Lagrangian and Hamiltonian description. The boundary conditions for the subsidiary variables, entering the Clebsch type velocity representation, are obtained in two different ways: one way follows from the variational principle as natural boundary conditions while the other one was obtained from the dynamical equations under the assumption relating to the absence of the corresponding sources and the maximal continuity compatible with the volume equations. It is possible to change the variational principle in such a way that all boundary conditions will result from it, i.e., they become natural boundary conditions. For this purpose it is necessary to modify the variational principle by adding a surface term with corresponding constraints, similarly to the nonrelativistic case (compare with the papers [13, 14] for the hydrodynamics and [18, 19] for the magnetohydrodynamics). These variants will be discussed in future works.

The approach discussed in this paper allowed us to give a simple treatment of the additional invariants of motion and present a set of recursion relations linking different types of invariants. In particular, we presented a generalization of the helicity invariant for the non-barotropic relativistic flows. This approach is suitable for the general relativity and for the relativistic magnetohydrodynamics as well. The discontinuous flows for the general relativity can be described in analogy with the above discussion and the results will be published elsewhere.

Acknowledgment

This work was supported by INTAS (Grant No. 00-00292).

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