Automated Calculation Scheme for $\alpha^n$ Contributions of QED to Lepton $g - 2$

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This article reports an automated approach to the evaluation of higher-order terms of QED perturbation to anomalous magnetic moments of charged leptons by numerical means. We apply this approach to tenth-order correction due to a particular subcollection of Feynman diagrams, which have no virtual lepton loops. This set of diagrams is distinctive in that it grows factorially in number as the order increases, and also each of the diagrams holds quite a large number of subtraction terms to be treated along renormalization procedure. Thus some automated scheme has long been required to evaluate correctly this class of diagrams. We developed a fast algorithm and an implementation which automates necessary steps to generate from the representation of each Feynman diagram the FORTRAN codes for numerical integration. Currently those diagrams of tenth order are being evaluated.

1. Introduction

The anomalous magnetic moment of electron, also called the electron $g - 2$, is one of the fundamental quantities of particle physics. Since its discovery in 1947\(^1\) it has been measured with steadily increasing precision\(^2,3\). The best values of $g - 2$ of the electron and the positron available in the literature\(^3\) were obtained by the Penning trap experiment.

The electron $g - 2$ (denoted by $a_e = \frac{1}{2}(g - 2)$) is explained almost entirely by the electromagnetic interaction between electron and photon alone, and thus it has provided the most stringent test of QED. An important by-product of the study of the electron $g - 2$ is that currently the most precise estimation of the fine structure constant $\alpha$ can be obtained by combining the measurement and the theory of $a_e$, which yields\(^4\)

$$\alpha^{-1}(a_e) = 137.035 \, 998 \, 834 \,(12)(31)(502),$$

where the uncertainties 12 and 31 are due to $\alpha^4$ and $\alpha^5$ terms, and 502 comes from the experiment\(^1\).

At present a new experiment on the electron $g - 2$ is being carried out by a Harvard group, which will reduce the measurement uncertainty substantially\(^5\). It will enable us to test the validity of QED to a very high degree and to determine $\alpha$ to an unprecedented precision of $7 \times 10^{-10}$ or better. Of course such a feat requires availability of the theoretical calculation of matching precision. The present issue is that the uncertainty due to $\alpha^5$ term may become major source of systematic errors. Therefore a reliable theoretical estimation of $\alpha^5$ corrections is urgently required.

From the viewpoint of obtaining the $\alpha^5$ corrections the numerical integration approach is the only practical choice at present. We employ the numerical evaluation scheme developed by one of the authors (T. K.) and Cvitanović\(^6,7\).

The contribution to the $\alpha^5$ term of the electron $g - 2$ comes from 12672 vertex-type Feynman diagrams, which can be categorized into 6 sets according to their structures and classified further into 32 gauge-invariant sets\(^8,9\). None of 32 sets is dominant so that all must be evaluated. A particularly difficult one is Set V, a huge set con-

\(^1\)Talk presented by T. Aoyama at the 7th International Symposium on Radiative Corrections (RADCOR 2005), Shonan Village, Japan, October 2-7, 2005.
consisting of 6354 vertex diagrams, all of which have radiative corrections only due to virtual photons. We begin by exploiting the equation derived from the Ward-Takahashi identity,

$$\Lambda^\nu(p, q) \simeq -q^\nu \left[ \frac{\partial \Lambda^\nu(p, q)}{\partial q^\nu} \right]_{q=0} - \frac{\partial \Sigma(p)}{\partial p^\nu} ,$$

which relates the sum of 9 vertex diagrams to one self-energy-like diagram. This relation together with the time-reversal symmetry of QED enables us to reduce the number of independent integrals of Set V diagrams drastically from 6354 to 389.

The difficulty of Set V also stems from the fact that many of them have very large number of UV and IR divergences, e.g. maximally 47 UV counterterms are required for some diagrams and many of the others require more than 20 UV counterterms. This makes the previous approach highly impractical since it runs into an extremely severe logistic problem. Toward this difficult problem we present our solution by an automated scheme for code generation, which enables us to obtain renormalized integrals for all diagrams of Set V that are ready to be integrated by numerical means [10].

2. Diagrams without lepton loops

Set V diagrams summed by W-T identity are one-particle-irreducible (1PI) self-energy-like diagrams of 2nd order which have no closed lepton loops. A typical diagram is shown in Figure 1.

They are given by a path consisting of all lepton lines emanating from the incoming lepton and terminating at the outgoing lepton. 2n interaction vertices lie on the path, which are labeled by $v_0, v_1, \ldots, v_{2n-1}$ from the left to the right in our conventions. A photon line labeled by $k$ connects the vertices $v_i_k$ and $v_j_k$ at its ends. Therefore a diagram is uniquely specified by a single-line expression $\{(v_{i_1}, v_{j_1}), \ldots\}$ which represents the pattern of pairings how photon lines connect the vertices if we adopt a proper ordering of pairs.

All Set V diagrams are generated systematically by finding every possible pattern of pairings that satisfies 1PI condition, which reduces to a matter of simple combinatorics. Due to their simple structure the graph-theoretical notions, especially the independent set of closed paths on the diagram, are easily identified. It is crucial in constructing the integrand. Another significant feature is that UV-divergent subdiagrams are easily found as single segments on the lepton path. The UV-divergent structure of the diagram will be identified exactly by forests constructed by the combinations of the subdiagrams. Those properties are quite efficiently handled in an automated manner.

3. Numerical integration formalism

The anomalous magnetic moment $a_e$ is given by the static limit of the magnetic form factor $F_2(q^2)|_{q^2 \to 0}$. The contribution from a 2nd order Feynman diagram $G$ is expressed by an integral over $n$ loop momenta $\{k_r\}$:

$$\frac{1}{i} \Sigma_G = (ie)^{2n} \left[ \prod_{r=1}^{n} \frac{d^4 k_r}{(2\pi)^4} \right] \gamma_{\mu_1}^i \gamma_{\mu_2}^j \prod_{r=1}^{n} \frac{-ig_{\mu_1 \mu_2}}{k_r^2} \prod_{r=1}^{n} \frac{1}{k_r^2}$$

where $\prod_{r=1}^{n} \frac{-ig_{\mu_1 \mu_2}}{k_r^2}$ is a diagram-specific product specified by the pairing pattern in the case of Set V diagrams. The momentum integration is exactly carried out and the amplitude is converted into an integral over 13-dimensional Feynman parameter space:

$$\frac{1}{i} \Sigma_G = \left( \frac{\alpha}{\pi} \right)^n \frac{1}{4^n} \Gamma(n-1) \int (dz)_G \left[ F_0(B_{ij}, A_j, C_{ij}) \right]_U^2 V^{n-1} + F_1(B_{ij}, A_j, C_{ij}) \left[ U^3 V^{n-2} + \cdots \right] .$$

The result is expressed symbolically as a function of quantities $U, B_{ij}, A_j, C_{ij}$ and $V$ (called as
Automated Calculation Scheme for $\alpha^n$ Contributions of QED to Lepton $g - 2$

building blocks), which are homogeneous polynomials of Feynman parameters. The explicit forms of $U$ and $B_{ij}$ are determined by the underlying topological structure of the Feynman diagram $G$ [11]. $A_{ij}, C_{ij}$ and $V$ also reflect the structure of the diagram and are obtained from $U$ and $B_{ij}$.

The amplitude constructed above is divergent in general, and the divergences must be removed before carrying out the integration numerically. We adopt the subtractive on-shell renormalization. In order to perform renormalization numerically we employ a strategy in which we prepare the subtraction terms as integrals over the same domain of the original unrenormalized amplitude, and carry out point-wise subtraction so that the singularities of the original integrand are canceled point-by-point on the Feynman parameter space.

To achieve this we adopt the following intermediate renormalization scheme. We first subtract the UV-divergent part of the unrenormalized integrand, which is identified by simple power-counting rules. This procedure is formulated as $K$-operation. By construction the subtraction term factorizes exactly into UV-divergent part of the renormalization constant $L_g$ associated to the UV-divergent subdiagram $S$ and the lower-order $g - 2$ term $M_{g/S}$. The factorization property is significant in the present formulation in that it allows successive application of $K$-operations.

The whole subtraction terms are given by Zimmermann’s forest formula [12]. Each source of UV-divergence is related to a forest, a set of non-overlapping UV-divergent subdiagrams. The subtraction term associated to a forest is constructed by applying $K$-operations to the unrenormalized integrand successively for the subdiagrams in a proper order. The advantage of the forest approach is that it is readily translatable into code generation. The forests are given by the combination of the subdiagrams, so the complete identification of UV-divergent parts are obtained by purely combinatorial procedure. Thus it is quite efficiently implemented in terms of forests, which enables us to obtain fully UV-renormalized amplitude of the Feynman diagram $G$.

The diagram may have IR-divergences when it contains self-energy subdiagrams. The subtraction of IR divergences is achieved by $I$-operation in a similar manner as that of UV divergences.

Counterterms thus constructed can be identified with only the divergent part of the renormalization constants so that the result of the above step is not fully equivalent to the standard on-shell renormalization. To complete the calculation the difference between the full renormalization and the intermediate renormalization must therefore be evaluated by summing up all subtraction terms. This step is called the residual renormalization, which is also much complicated for the higher-order calculations and is to be automated in the future work.

4. Automated flow of calculation

The flow of the process to generate the numerical integration code for evaluating an individual diagram is schematically depicted as Figure 2

The information of a diagram is provided in a single-line representation indicated by the rectangular box at the upper-left corner. It enables us to construct the unrenormalized amplitude determined directly by the pairing pattern of the diagram, which is converted into the form of Feynman parametric integrals in terms of building blocks by analytically performing the integration over the loop momenta with the help of algebraic manipulation program, FORM [13].

The explicit form of the build blocks are determined from the underlying topological structure. They are obtained by referring to the fundamental set of closed paths on the diagram, which can be extracted from the diagram representation. This step also incorporates algebraic manipulations performed by MAPLE.

The UV divergences of the diagram are identified as Zimmermann’s forests, which are obtained by the non-overlapping combinations of UV-divergent subdiagrams once the complete set of those subdiagrams are found from the diagram representation. The subtraction operation is achieved by $K$-operation, in which simple power-counting rules are applied to the original unrenormalized integrand and the building blocks to result in the proper form of subtraction term. A subtraction term corresponding to a particular forest is constructed by the successive application
of $K$-operations, each of which is associated to the subdiagram in the forest.

The IR divergences remaining in the individual diagram should also be subtracted away by $I$-operations in a similar manner as those of UV divergences, though this subject is not covered in the present article.

Finally, the (intermediate-) renormalized amplitude constructed from the original amplitude and the set of subtraction terms is turned into a FORTRAN code. It is readily processed by the numerical integration system such as VEGAS [14], an adaptive Monte-Carlo integration routine.

5. Current status and concluding remarks

In this article we presented an automated scheme of code generation for evaluating higher-order QED corrections of the electron anomalous magnetic moment by numerical means. We constructed an algorithm and concrete procedure to obtain UV-renormalized amplitudes for a particular set of diagrams without lepton loops in an automated manner. It should be noted that the scheme itself is applicable to an arbitrary order of diagrams of this particular type, though our practical concern is to evaluate the tenth-order corrections.

We obtained a program to list up all the topologically distinct diagrams, and to identify the UV divergent structure of each diagram in terms of forest structures. We implemented our automated procedure as a set of Perl programs with the help of symbolic manipulation systems, FORM and MAPLE. From a single-line representation of a diagram it generates numerical in-
Automated Calculation Scheme for $\alpha^n$ Contributions of QED to Lepton $g - 2$

integration codes in FORTRAN, which are ready to be processed by VEGAS, an adaptive Monte-Carlo integration routine.

The programs have been tested for lower-order diagrams and confirmed that they reproduce the codes for the sixth-order and eighth-order diagrams previously constructed. They are now being applied to tenth-order diagrams. At present, the diagrams which have only vertex renormalization were processed and test runs were performed. Those diagrams corresponds to 2232 vertex diagrams among 6354 Set V diagrams of tenth-order. Crude evaluation showed no sign of divergent behavior, which confirms that our scheme is working well as expected. They are currently put to production runs.

Typical size of the numerical integration code of a diagram amounts to about 80,000 lines of FORTRAN program. It takes 10–20 minutes to generate the code on an ordinary PC. The running time to evaluate $10^6$ samplings times 20 iterations takes about 5–7 hours on 32 CPU cluster. It seems that the estimated computational cost required to accomplish the whole numerical integration in a few percent of uncertainty remain within the manageable scale with up-to-date computational facilities.

The remaining 4122 diagrams have not only UV-divergent self-energy subdiagrams but also IR divergences. The simplest way to deal with the IR problem is to give a small mass $\lambda$ to photons, which requires no further work on the automating code, and is being pursued as a first step. To obtain a result independent of $\lambda$ it is necessary to incorporate IR subtraction terms in a manner similar to that of UV counterterms, which is left as our future issue.

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