Visualizing the quantum interaction picture in phase space

Bahar Mehmani$^1$ and Andrea Aiello$^{1,2}$

$^1$Max Planck Institute for the Science of Light, Günther-Scharowsky-Straße 1/Bau 24, D-91058 Erlangen, Germany
$^2$Institute for Optics, Information and Photonics, University Erlangen-Nürnberg, Staudtstraße 7/B2, D-91058 Erlangen, Germany

E-mail: bahar.mehmani@mpg.mpl.de

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Abstract
We present a graphical example of the interaction picture-time evolution. Our aim is to help students understand in a didactic manner the simplicity that this picture provides. Visualizing the interaction picture unveils its advantages, which are hidden behind the involved mathematics. Specifically, we show that the time evolution of a driven harmonic oscillator in the interaction picture corresponds to a local transformation of a phase space-reference frame into the one that is co-rotating with the Wigner function.

Online supplementary data available from stacks.iop.org/EJP/33/1367/mmedia
(Some figures may appear in colour only in the online journal)

1. Introduction

One of the first lessons students learn in quantum mechanics courses is the procedure of calculating the time evolution or the dynamics of a system. In order to calculate the time evolution of a system in quantum mechanics, one has to choose from among three pictures: the Schrödinger picture (SP), the Heisenberg picture (HP) and the interaction picture (IP) [1]. Since the Schrödinger equation is the first tool students become familiar with, they generally prefer to choose SP and avoid IP for its somewhat more complicated mathematical formalism when facing a time-dependent Hamiltonian. Here we want to change this image and illustrate the elegance of IP with a simple example.

In fact, SP is more suitable for studying closed and conservative systems. In the case where the Hamiltonian describing a system is complicated due to the presence of the environment and/or external force, it is usually impossible to have an exact solution to the Schrödinger equation. This can be tackled by adopting the HP, in which only the observables are time dependent and their evolution is described by the Heisenberg equation of motion. However, this picture is unable to give any information regarding the dynamics of the state itself. IP is the most suitable picture when the Hamiltonian of a system can be written as a sum of two parts:
a time-independent term of which the eigenstates and eigenenergies are known, and a usually
time-dependent term which influences the dynamics of the system. Employing IP enables one
to set the dynamics arising from the time-independent Hamiltonian aside and focus on the
dynamics of the state caused by the time-dependent Hamiltonian. That is why this picture is
frequently employed in quantum optics, for example, where matter interacts with the radiation
field.

Despite the provided advantages, the underlying mathematics of IP is somewhat difficult
for students and therefore it may not seem enlightening in its abstract Hilbert-space
representation. In this paper we present a pictorial example of IP in phase space which is
a familiar concept to an average student from classical and statistical mechanics courses.
Unlike SP and HP which have been mapped into classical active and passive transformations
[1, 2], to the best of our knowledge, there is no such direct comparison for the IP in the physics
literature. Illustrating the IP time evolution of an example system in phase space also has the
advantage of understanding the concept of unitary transformations in the Hilbert space.

We employ some well-established concepts of modern physics, such as Wigner
distribution [4–7] and coherent and squeezed states of light [3]. A self-consistent
summary of these subjects can be found in the supplementary data, available from
stacks.iop.org/EJP/33/1367/mmedia.

The rest of this paper is organized as follows. We review the SP and HP in phase space in
sections 2 and 3. In section 4 we introduce two types of time-dependent potential to the system:
a linear and a quadratic term (in terms of the bosonic creation and annihilation operators).
Then we take the IP to study the dynamics of the initially squeezed state under the driving
potentials. We explicitly show that the IP-time evolution of the state in the Hilbert space
corresponds to a transformation to a local frame that is co-rotating with the Wigner function in
phase space. Furthermore, we show that switching from one picture to another is equivalent to
performing different active and/or passive transformations in phase space. Finally, we conclude
in section 5. The detailed calculations are presented in the supplementary data, available from
stacks.iop.org/EJP/33/1367/mmedia.

2. Basics of the quantized harmonic oscillator

Let us consider single-mode harmonic oscillator with unit mass and frequency $\omega_0$. The
Hamiltonian describing the system is given by

$$\hat{H}_0 = \frac{\omega_0}{2} (\hat{p}^2 + \hat{x}^2).$$

Here $\hat{x}$ and $\hat{p}$ represent quadratures of the quantized harmonic oscillator and are defined in
terms of the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^\dagger$ as

$$\hat{x} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger),$$

$$\hat{p} = \frac{1}{i\sqrt{2}} (\hat{a} - \hat{a}^\dagger),$$

where $\hbar$ is set to 1. Alternatively, this system may be described in terms of $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{H}_0 = \omega_0 (\hat{a}^\dagger \hat{a} + \frac{1}{2}),$$

where the commutator of the creation and the annihilation operators is $[\hat{a}, \hat{a}^\dagger] = 1$. $\hat{H}_0$ has the
energy eigenvalues $E_n$ defined as

$$E_n = \omega_0 (n + \frac{1}{2}) \text{ with } n = 0, 1, 2, \ldots \tag{4}$$
The energy eigenstates (number states) $|n\rangle$ are such that
\[ \hat{H}_0 |n\rangle = E_n |n\rangle. \] (5)
Any superposition of the number states is a solution to the time-dependent Schrödinger equation for a quantized harmonic oscillator described by the Hamiltonian $\hat{H}_0$. There are two classes of such superpositions that are of great interest: coherent and squeezed states.

A coherent state $|\gamma\rangle$ is described as the displaced vacuum state $|0\rangle$ and is generated by a canonical transformation of the vacuum state $|0\rangle$ in phase space
\[ |\gamma\rangle = \hat{D}(\gamma) |0\rangle, \] (6)
where $\hat{D}(\gamma)$ is the unitary displacement operator defined as $\hat{D}(\gamma) = e^{(\gamma \hat{a}^\dagger - \gamma^* \hat{a})}$. (7)
The displacement parameter $\alpha$ is generally a complex number and may be decomposed into
\[ \alpha = \frac{1}{\sqrt{2}} (a + ib). \] (8)
Squeezed states are of interest because they provide reduced fluctuations in one quadrature, when compared with the coherent state. A squeezed state is described by the unitary squeezing operator $\hat{S}(\zeta)$ in a similar fashion. However, the difference lies in the fact that $\hat{S}(\zeta)$ is not linear in $\hat{a}$ and $\hat{a}^\dagger$ and is defined as
\[ \hat{S}(\zeta) = \exp\left(\frac{(\zeta^* \hat{a}^2 - \zeta \hat{a}^\dagger)^2}{2}\right), \quad \zeta = se^{i\theta}. \] (9)
where $s = |\zeta|$ is the strength of the squeezing and $\theta = \arg(\zeta)$ determines the direction along which the squeezing is performed. In phase space the squeezed states correspond to the Gaussian distributions with unequal widths as opposed to the symmetric distribution of the coherent state [9].

Throughout this paper, for the sake of simplicity and without the loss of generality, we take both the displacement parameter $\alpha$ and the squeezing parameter $\zeta$ as real numbers.

The time evolution of both above-mentioned states can be clearly visualized in phase space. This is done with the help of the Wigner function and Weyl transform [4–6]. Here instead of representing a state with a density matrix in the Hilbert space, we work with a phase-space distribution function which is known as the Wigner function and is constructed from the density matrix. Knowing the state of the quantum system at any time, we can build up the symmetric characteristic function [12], $\text{Tr}[\hat{D}(\chi) \hat{\rho}(t)]$, and from there the corresponding SP-Wigner function $W(x, p; t)$ (see the supplementary data, available from stacks.iop.org/EJP/33/1367/mmedia) is given by
\[ W(x, p; t) = \frac{1}{(2\pi)^2} \int d\xi \int d\eta e^{-i(\xi x + \eta p)} \text{Tr}[\hat{D}(\xi) \hat{\rho}(t)], \] (10)
where $\hat{D}(\chi)$ is the displacement operator with the parameter $\chi$ defined as
\[ \chi \equiv \frac{1}{\sqrt{2}} (-\eta + i\xi). \] (11)
Both integrations are from $-\infty$ to $\infty$.

3. SPs and HPs versus passive and active transformations

Let us first explicitly show for a single-mode harmonic oscillator how SP and HP map to active and passive transformations in phase space, respectively. For a harmonic oscillator with the
Hamiltonian given by (3), the unitary time-evolution operator $\hat{U}_0$, which governs the evolution of the state, reads

$$\hat{U}_0(t) = e^{-i\hat{H}t} = e^{-i\omega_0 t \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)}.$$  

(12)

If the system is initially in an ideal squeezed state, its state which is described by the density matrix $\hat{\rho}$ is given by

$$\hat{\rho}(0) = \hat{D} \left( \frac{\mu_x}{\sqrt{2}} \right) \hat{S}(s) \left| 0 \right> \left< 0 \right> \hat{S}(s) \hat{D} \left( -\frac{\mu_x}{\sqrt{2}} \right).$$  

(13)

with $\mu_x$ representing the displacement along the $x$-axis in phase space and $s$ denoting the real squeezing parameter, then the time evolution of the state is given by

$$\hat{\rho}(t) = \hat{U}_0(t) \hat{\rho}(0) \hat{U}_0^\dagger(t).$$  

(14)

Knowing $\hat{\rho}(t)$, the Wigner function at time $t$ can be derived by first calculating the symmetric characteristic function as the expectation value of the displacement operator $\hat{D}(\chi)$ as

$$\text{Tr} \left[ \hat{\rho}(t) \hat{D}(\chi) \right] = \exp \left[ i \xi \mu_x \cos(\omega_0 t) - i \eta \mu_x \sin(\omega_0 t) \right]$$

$$\times \exp \left[ -\frac{\xi^2}{4} \left( \frac{\cos^2(\omega_0 t)}{2\sigma_p^2} + \frac{\sin^2(\omega_0 t)}{2\sigma_r^2} \right) \right]$$

$$\times \exp \left[ -\frac{\eta^2}{4} \left( \frac{\cos^2(\omega_0 t)}{2\sigma_p^2} + \frac{\sin^2(\omega_0 t)}{2\sigma_r^2} \right) \right]$$

$$\times \exp \left[ -\frac{\xi \eta}{8} \sin(2\omega_0 t) \left( \frac{1}{\sigma_r^2} - \frac{1}{\sigma_p^2} \right) \right],$$  

(15)

where we substitute $e^{-2s} = 2\sigma_s^2$ and $e^{2t} = 2\sigma_s^2$ in the coefficients of $\xi^2, \eta^2$ and $\xi \eta$. Substituting (15) into the definition of the Wigner function given by (10) and performing two Gaussian integrals over $\xi$ and $\eta$ provide the SP-Wigner function as (for detailed calculations see the supplementary data, available from stacks.iop.org/EJP/33/1367/mmedia)

$$W_{0_{\text{SP}}}(x, p; t) = \frac{1}{\pi} \exp \left\{ -\frac{1}{2} \left[ r - R(-\omega_0 t) \mu \right]^T \cdot \Gamma^{-1}(t) \cdot \left[ r - R(-\omega_0 t) \mu \right] \right\},$$  

(16)

where the superscripts ‘$T$’ and ‘$-1$’ denote matrix transposition and inversion operations, respectively. The suffix SP denotes the Schrödinger picture and index 0 in $W_0(x, p; t)$ represents the free evolution of the system. $W_{0_{\text{SP}}}(x, p; t)$ is a Gaussian distribution with vectors $\mu$ and $r$ representing the initial displacement of the Wigner function and the coordinates of phase space, respectively, as

$$\mu = \begin{pmatrix} \mu_x \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} x \\ p \end{pmatrix}.$$  

(17)

The covariance matrix $\Gamma^{-1}(t)$ is given by a clockwise rotation of the squeezing matrix through the angle $\omega_0 t$

$$\Gamma^{-1}(t) \equiv R(-\omega_0 t) \cdot \Xi^{-1} \cdot R(\omega_0 t),$$  

(18)

where the diagonal $2 \times 2$ squeezing matrix $\Xi^{-1}$ is described by

$$\Xi^{-1} = \begin{pmatrix} \frac{1}{\sigma_r^2} & 0 \\ 0 & \frac{1}{\sigma_p^2} \end{pmatrix}. $$  

(19)

Note that the matrix $R(\omega_0 t)$ defines a counterclockwise rotation by $\omega_0 t$ as

$$R(\omega_0 t) = \begin{pmatrix} \cos \omega_0 t & -\sin \omega_0 t \\ \sin \omega_0 t & \cos \omega_0 t \end{pmatrix}. $$  

(20)
The free time evolution of the $\frac{1}{4}$-contour of the SP-Wigner function of an ideal squeezed state is represented in figure 1. As illustrated in figures 1(a)–(i), the elliptical distribution performs two types of rotations: (i) a global clockwise rotation (active transformation) of the distribution about the origin of the phase space on a circular path defined by the initial amount of displacement; (ii) a local counterclockwise rotation of the squeezing direction through the same angle $\omega_0 t$ about its centroid. From the mathematical point of view, the global rotation is produced by the clockwise rotation matrix $R(-\omega_0 t)$ acting on the vector $\mu$, whereas the action of the rotation matrix $R(\omega_0 t)$ on the squeezing matrix $\Sigma^{-1}$ amounts to local counterclockwise rotations of the distribution around its centroid.

In (16), if we make a transformation

$$r' = R(\omega_0 t) r,$$

in this local frame, which is attached to the centroid of the Wigner function, the counterclockwise rotation of the Wigner function about its centroid cancels out its global
clockwise rotations about the origin. In terms of operators in the Hilbert space, this is equivalent to
\[
\dot{x}(t) = \dot{x}(0) \cos(\omega_0 t) + \dot{p}(0) \sin(\omega_0 t),
\]
which arises when we employ the HP. Given \( \hat{H}_0 \), the time evolution of operators \( \hat{x} \) and \( \hat{p} \) is given in the above form by the Heisenberg equation of motion. Thus in phase space the passive transformation (21) corresponds to changing the reference frame to the one in which \( x(t) \) and \( p(t) \) are rotated back to their initial values \( x(0) \) and \( p(0) \). The Wigner function in the new local frame described by (21) reads
\[
W_0(x', p') = \frac{1}{\pi} \exp \left[ -\frac{1}{2} (r' - \mu)^T \cdot \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \cdot (r' - \mu) \right].
\]
(23)
\( W_0(x', p') \) represents the HP-Wigner function. Remember in HP the operators \( \hat{x} \) and \( \hat{p} \) evolve according to (22a) and (22b). Hence we see that the passive transformation (21) in phase space corresponds to HP.

4. IP and mixed rotations

In this section we study the dynamics of the harmonic oscillator in the presence of a time-dependent potential \( \tilde{V}(t) \). This situation is described by
\[
\hat{H} = \hat{H}_0 + \tilde{V}(t),
\]
(24)
with \( \hat{H}_0 \) being defined in (3). \( \tilde{V}(t) \) influences the temporal behaviour of the system and is taken to be a linear driving potential
\[
\tilde{V}(t) = g(e^{-i\omega_1 t} a \hat{a} + e^{i\omega_1 t} a^* \hat{a}^i),
\]
(25)
where \( g \) is a real coupling constant and \( \alpha \) is a complex parameter. Physically, this corresponds to a forced harmonic oscillator with the driving frequency \( \omega_1 \).

In the IP both the state and observables are time dependent. The observables evolve in time according to the Heisenberg equation of motion as
\[
\hat{O}_1(t) = \hat{U}_0^\dagger(t) \hat{O} \hat{U}_0(t),
\]
(26)
where \( \hat{U}_0(t) \) is given by (12). Accordingly, the time-dependent potential in the IP, \( \tilde{V}_1 \), reads as
\[
\tilde{V}_1(t) = g(e^{-i\Omega t} a \hat{a} + e^{i\Omega t} a^* \hat{a}^i),
\]
(27)
where \( \Omega \) is defined as the sum of the natural frequency \( \omega_0 \) and the driving frequency \( \omega_1 \):
\[
\Omega \equiv \omega_0 + \omega_1.
\]
(28)
When \( \Omega = 0 \) (\( \omega_1 = -\omega_0 \)), the IP potential becomes time independent. We shall study this case in more detail in section 4.2. For the moment we assume \( \Omega \neq 0 \).

The time evolution of the state is described by the IP unitary transformation \( \hat{U}_1(t) \) as
\[
\hat{\rho}_1(t) = \hat{U}_1(t) \hat{\rho}(0) \hat{U}_1^\dagger(t),
\]
(29)
where \( \hat{U}_1(t) \) is given by the Magnus series [18]
\[
\hat{U}_1(t) = \exp \{-i[\hat{A}_1(t) + \hat{A}_2(t) + \hat{A}_3(t) + \cdots] \}.
\]
(30)
The first three terms in the exponential are given by
\[
\hat{A}_1(t) = \int_0^t dt_1 \hat{V}_1(t_1),
\]
(31a)
\[ \hat{A}_2(t) = \frac{1}{2\pi} \int_0^t dt_2 \int_0^{t_2} dt_1 \left[ \hat{V}_1(t_1), \hat{V}_1(t_2) \right], \]  
\[ \hat{A}_3(t) = \frac{1}{3!} \int_0^t dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \left[ \left[ \hat{V}_1(t_1), \hat{V}_1(t_2) \right], \hat{V}_1(t_3) \right] + \left[ \hat{V}_1(t_3), \left[ \hat{V}_1(t_2), \hat{V}_1(t_1) \right] \right] dt_1, \]  
\[ \text{(31b)} \]
\[ \text{where } \hat{V}_1(t) \text{ is given by (27). Thus the time evolution of the state of the system can be derived by calculating } \hat{A}_1(t), \hat{A}_2(t), \ldots, \text{ and subsequently constructing } \hat{U}_1(t), \text{ which ultimately enables us to calculate } \rho_1(t). \text{ Then it is straightforward to calculate the IP-Wigner function and present a classical correspondence to } \hat{U}_1(t). \text{ For the first two terms } \hat{A}_1(t) \text{ and } \hat{A}_2(t), \text{ we have} \]
\[ \hat{A}_1(t) = i \left[ v(t) \hat{a}^\dagger - v^*(t) \hat{a} + \frac{g}{\Omega} \left( \alpha^* \hat{a}^\dagger - \alpha \hat{a} \right) \right], \]  
\[ \text{(32)} \]
\[ \text{where } v(t) \text{ is defined as} \]
\[ v(t) = -\frac{g}{\Omega} \alpha^* e^{i\Omega t}, \]  
\[ \text{(33)} \]
\[ \text{with } \alpha^* \text{ being the complex conjugate of the displacement parameter defined in the driving potential given by (25). For } \hat{A}_2(t) \text{ we obtain} \]
\[ \hat{A}_2(t) = \frac{g^2|\alpha|^2}{\Omega^2} (\Omega t - \sin \Omega t). \]  
\[ \text{(34)} \]

The commutator of \( \hat{V}_1(t) \) at two different times turns into a purely imaginary function of time. Hence the higher order terms in the Magnus series vanish and we are able to write down the exact solution for the equations of motion as
\[ \hat{U}_1(t) = e^{-i \frac{\Omega g^2|\alpha|^2 (\Omega t - \sin \Omega t)}{\Omega^2}} \hat{D}(v(t)) \hat{D} \left( \frac{g}{\Omega} \alpha^* \right). \]  
\[ \text{(35)} \]

For an initially ideal squeezed state described by (13), the IP state at later time \( t \) reads
\[ \hat{\rho}_1(t) = \hat{D}(v(t)) \hat{D} \left( \frac{g}{\Omega} \alpha^* \right) \hat{\rho}(0) \hat{D} \left( -\frac{g}{\Omega} \alpha^* \right) \hat{D}(-v(t)), \]  
\[ \text{(36)} \]

where we substituted equation (35) for \( \hat{U}_1(t) \) in (29). Knowing the state, we can construct the IP-Wigner function following the same line of calculations as in section 3 and obtain
\[ W^\text{I}_1(x, p; t) = \frac{1}{\pi} \exp \left\{ - \frac{1}{2} \left[ \left[ r - \mu_1(t) \right]^T : \Xi^{-1} : \left[ r - \mu_1(t) \right] \right] \right\}, \]  
\[ \text{(37)} \]

where the suffix I represents the IP, and index 1 represents the evolution of the Wigner function that takes place in the presence of the interaction Hamiltonian \( \hat{H}_1(t) \) given by (25). We note that the IP-Wigner function remains Gaussian with the covariance matrix \( \Xi \) and vector \( \mu_1(t) \) being defined as
\[ \mu_1(t) = \left( \begin{array}{c} \mu_1(t) \\ 0 \end{array} \right) + \frac{g}{\Omega} \left( \begin{array}{c} a \\ -b \end{array} \right) - \frac{g}{\Omega} R(\Omega t) \left( \begin{array}{c} a \\ -b \end{array} \right). \]  
\[ \text{(38)} \]

In order to see how the Wigner function evolves in the IP, we look at the equation of motion of its centroid given by
\[ \left( \langle x_1(t) \rangle - \mu_x \right)^2 + \left( \langle p_1(t) \rangle \right)^2 = 2 \left( \frac{g}{\Omega} \right)^2 (a^2 + b^2)(1 - \cos \Omega t). \]  
\[ \text{(39)} \]
As shown in figure 2, the time evolution of the $\frac{1}{2}$-contour of the IP-Wigner function corresponds to a displacement of its centroid in phase space together with a counterclockwise parallel transformation of the distribution on a circle with a time-dependent radius described by equation (39). There are two important points that can be inferred from figure 2. (i) In the IP the squeezing direction remains invariant during the evolution, which in our example was in the $x$-direction. (ii) The evolution of the IP-Wigner function corresponds to the displacement of its centroid around a circle as was the case for the free Hamiltonian SP time evolution shown in figure 1. The reason is that the IP unitary transformation $\hat{U}_I(t)$ is defined as two successive displacement operators apart from a phase factor. The local rotations which arise in SP are absent in IP and it is exactly because of this property that for the Hamiltonian (24), choosing the IP makes the study of the time evolution of the state much simpler. We will see in the next part that both the above-mentioned properties get spoiled when we transform back to the SP.

4.1. Back to the SP

Transforming back to the SP amounts to performing the unitary transformation

$$\hat{\rho}(t) = \hat{U}_0 \hat{\rho}_I(t) \hat{U}_0^\dagger,$$

(40)
which as we showed in section 3 corresponds to global and local rotations of the SP-Wigner function through angle $\omega_0 t$. The symmetric characteristic function is straightforward to establish. As a result in the presence of the interaction Hamiltonian $\hat{H}_1(t)$, we have

$$W_{\text{SP}}^1(x, p; t) = \frac{1}{\pi} \exp \left\{ -\frac{1}{2} [r - \mu_S(t)]^\top \Gamma^{-1} [r - \mu_S(t)] \right\}. \quad (41)$$

The $2 \times 2$ matrix $\Gamma^{-1}$ is given by (18) and the mean value vector $\mu_S(t)$ is defined as the clockwise rotation of the initial displacement about the origin through angle $\omega_0 t$ plus a clockwise rotation through angle $\omega_1 t$ and a counterclockwise rotation through angle $\omega_0 t$:

$$\mu_S(t) = R(-\omega_0 t) \begin{pmatrix} \mu_x \\ 0 \end{pmatrix} + \frac{g}{\Omega} \left[ R(-\omega_0 t) \begin{pmatrix} a \\ -b \end{pmatrix} - R(\omega_1 t) \begin{pmatrix} a \\ -b \end{pmatrix} \right]. \quad (42)$$

We note that the global rotation of the centroid of the distribution follows a complicated path given by

$$\left( \langle x_S(t) \rangle - \mu_x \cos \omega_0 t \right)^2 + \left( \langle p_S(t) \rangle + \mu_x \sin \omega_0 t \right)^2 = 2 \left( \frac{g}{\Omega} \right)^2 \left( a^2 + b^2 \right)(1 - \cos \Omega t). \quad (43)$$

Equation (43) resembles a glissette [19]. A glissette is defined as the locus of a generator point which is moving along a given curve. The generator point in (43) is $(\mu_x \cos \omega_0 t, -\mu_x \sin \omega_0 t)$,
Figure 4. Time evolution of the \( \frac{1}{\pi} \)-contour of \( W_{SP}^1(x, p; t) \) given by (41) \((\sigma_x = 1, \sigma_p = \frac{1}{2}, \mu_x = -2, a = 1, b = -1, \omega_1 = 2, \omega_0 = 1, g = 5)\).

described as the centre of the circle on the right-hand side of (43). This point is rolling along the time-dependent curve given by the left-hand side of (43). The resulting glissette is shown in figure 3 for different values of \( \Omega \). On top of the rotations induced by \( \hat{U}_0 \), there are two more rotations of the driven displacement given by \( \alpha^* \): (i) a clockwise rotation through angle \( \omega_0 t \); (ii) a counterclockwise rotation through angle \( \omega_1 t \). As a result, the SP-Wigner function in the presence of the interaction ‘dances’ in a complicated manner, on a complicated path in comparison to simple parallel transform on a circular path in IP. This is illustrated in figure 4 for a fixed value of \( \Omega \).

Choosing the counterclockwise rotation of the \( xp \)-plane given by (21) transforms the SP-Wigner function into the IP-Wigner function in the local \( x'p' \) frame. This corresponds to

\[
W_{SP}^1(x, p; t) \rightarrow W_{I}^1(x', p'; t) = \frac{1}{\pi} \exp \left[ -\frac{1}{2} (r' - \mu_1(t))^T \cdot \Sigma^{-1} \cdot (r' - \mu_1(t)) \right].
\]  

(44)

In fact, by choosing the counterclockwise rotation of the coordinates, we let the observables of the system evolve under the free Hamiltonian \( \hat{H}_0 \). Thus the evolution of the state in the new local frame is only governed by the interaction Hamiltonian. In other words, we separate the time evolution of the state from that of the observables which is the essence of IP [20].
4.2. The case $\Omega = 0$

When the driving frequency of the interaction Hamiltonian given in equation (25) is equal to that of $\hat{H}_0$ with an opposite sign, the IP potential given by equation (27) becomes time independent. As a result the Wigner function reads

$$W_{\Omega=0}^I(x, p; t) = \frac{1}{\pi} \exp \left[ -\frac{1}{2} (r - \mu_1(t)|_{\Omega=0})^T \Xi \cdot (r - \mu_1(t)|_{\Omega=0}) \right],$$

where the components of $\mu_1(t)|_{\Omega=0}$ are given by

$$\langle x_1(t) \rangle|_{\Omega=0} = \mu_x - gbt,$$

$$\langle p_1(t) \rangle|_{\Omega=0} = -gat.$$  \hspace{1cm} (46a) \hspace{1cm} (46b)

Thus the centroid of the ellipse follows a line instead of a circle and the Wigner function will only be parallel displaced along this line in time. This is shown in figure 5.

4.3. Quadratic Hamiltonian

So far we have considered an interaction Hamiltonian that is linear in the creation and annihilation operators which preserves the covariance matrix of the Gaussian distribution of the Wigner function. Now we consider a quadratic interaction term in the Hamiltonian to illustrate the deformation of the covariance matrix. We show that in this case the contour of the Wigner function initially squeezed in the $p$-direction expands into a coherent state and then becomes squeezed in the $x$-direction. We choose the quadratic Hamiltonian to be

$$\hat{H}(t) = \omega_0 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + i \kappa [e^{2i\omega t} \hat{a}^2 - e^{-2i\omega t} (\hat{a}^\dagger)^2],$$

where $\kappa$ is a real coupling constant. A physical realization of the above Hamiltonian may be the interaction of a coherent light beam with a nonlinear optical medium. In that case, $\kappa$ contains...
the nonlinear susceptibility and the pump field amplitude. In order to derive the unitary time-evolution operator, we take the group-theoretical approach \cite{21–28}. In what follows we use the Lie algebra of SU(1,1) \cite{21, 22} realized in terms of $\hat{a}$ and $\hat{a}^\dagger$,  
\[
\hat{K}_0 = \frac{1}{2} \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad \hat{K}_+ = \frac{1}{2} (\hat{a}^\dagger)^2, \quad \hat{K}_- = \frac{1}{2} \hat{a}^2, \]

such that  
\[
[\hat{K}_0, \hat{K}_\pm] = \pm \hat{K}_\pm, \quad [\hat{K}_-, \hat{K}_+] = 2\hat{K}_0. \]

Thus in terms of these operators we have a Hamiltonian with a linear interaction term (in $\hat{K}_\pm$) as  
\[
\hat{H}(t) = 2\omega_0 \hat{K}_0 + i\kappa (e^{2i\omega_0 t} \hat{K}_- - e^{-2i\omega_0 t} \hat{K}_+). \]

It is straightforward to calculate the unitary time-evolution operator in the IP using (30). In this case the IP potential $\hat{V}_I(t)$ becomes independent of time and as a result the Magnus series contains only the first term $\hat{A}_1(t)$ given by (31a) as  
\[
\hat{U}_I(t) = e^{\kappa t(\hat{K}_- - \hat{K}_+)}. \]
which resembles a squeezing operator with a time-dependent parameter. We assume the system
starts its evolution from a vacuum squeezed state. Then the symmetric characteristic function
reads
\[
\text{Tr} \left[ \hat{\rho} (t) \hat{D} (\chi) \right] = \exp \left\{ - \frac{1}{8} \left[ \frac{\xi^2}{(\sigma^2 e^\kappa t)^2} + \frac{\eta^2}{(\sigma^2 e^\kappa t)^2} \right] \right\}.
\]
(52)

We realize that the quadratic interaction conformally transforms the covariance matrix given
by (19). Furthermore, at a certain time the two elements of the covariance matrix become
identical; thus a squeezed state transforms into a coherent state as shown in figure 6. For the
IP-Wigner function we obtain
\[
W_{2}^{\text{IP}} (x, p; t) = \frac{1}{\pi} \exp \left\{ - \frac{1}{2} [r^T \cdot \Xi_2^{-1} (t) \cdot r] \right\},
\]
(53)
where index 2 represents the quadratic interaction term. \( \Xi_2^{-1} (t) \) is the transformed covariance
matrix defined as
\[
\Xi_2^{-1} (t) = \begin{pmatrix} \frac{e^{2\kappa t}}{\sigma_x^2} & 0 \\ 0 & \frac{e^{3\kappa t}}{\sigma_p^2} \end{pmatrix}.
\]
(54)
The transformation of the time-dependent density matrix back to the SP amounts to a global clockwise rotation of the SP-Wigner function in phase space. This is straightforward to prove and yields to
\[
W_{SP}^2(x, p; t) = \frac{1}{\pi} \exp \left\{ -\frac{1}{2} |p| \cdot \Gamma_2^{-1}(t) \cdot r \right\},
\]
where \( \Gamma_2^{-1}(t) \) is defined as the clockwise rotated breathing covariance matrix
\[
\Gamma_2^{-1}(t) \equiv R(-\omega_0 t) \cdot \Sigma_2^{-1}(t) \cdot R(\omega_0 t).
\]
This is shown in figure 7. It is worth noting that since the centroid of the Wigner function coincides with the origin of the phase space, the local counterclockwise rotation of the elliptical distribution around the centroid is cancelled out by the global clockwise rotations around the origin as illustrated in figure 7. However, this is not the case for an ideal squeezed state shown in figure 1.

5. Summary

We presented a pictorial representation of time evolution of a driven harmonic oscillator being initially in a squeezed state in IP and SP. This has been done by employing the Wigner–Weyl representation to map the density matrices and observables in the Hilbert space on distribution functions and variables in phase space. We took the initial state of the system squeezed state. We showed that the IP time evolution corresponds to a parallel transformation of the distribution function on a circular path in phase space. In transforming back to SP, we showed that the evolution of the Wigner function becomes more complicated. We also showed that a quadratic interaction term amounts to time-dependent squeezing. A classical picture of such interaction corresponds to conformal deformation of the Wigner function contours.

References

[1] Auletta G and Parisi G 2001 Foundations and Interpretation of Quantum Mechanics (Singapore: World Scientific) p 48
[2] Kim Y S and Noz M E 1991 Phase Space Picture of Quantum Mechanics: Group Theoretical Approach (Singapore: World Scientific) p 19
[3] Mandel L and Wolf E 1995 Optical Coherence and Quantum Optics (Cambridge: Cambridge University Press) pp 1042–45
[4] Case W B 2008 Wigner functions and Weyl transforms for pedestrians Am. J. Phys. 76 937–46
[5] Hillery M, O’Connell R F, Scully M O and Wigner E P 1984 Distribution functions in physics: fundamentals Phys. Rep. 106 121–67
[6] Agarwal G S 1987 Wigner-function description of quantum noise in interferometers J. Mod. Opt. 34 909–21
[7] Schleich W P 2001 Quantum Optics in Phase Space 1st edn (Berlin: Wiley) pp 321–42
[8] Kim Y S and Wigner E P 1990 Canonical transformation in quantum mechanics Am. J. Phys. 58 439–48
[9] Han D, Kim Y S and Noz M E 1988 Linear canonical transformations of coherent and squeezed states in the Wigner phase space Phys. Rev. A 37 807–14
[10] Han D, Kim Y S and Noz M E 1989 Linear canonical transformations of coherent and squeezed states in the Wigner phase space: II. Quantitative analysis Phys. Rev. A 40 902–12
[11] Glauber R J 1963 Coherent and incoherent states of the radiation field Phys. Rev. 131 2766–88
[12] Barnett S M and Radmore P M 1997 Methods in Theoretical Quantum Optics (Oxford: Clarendon) p 106
[13] Scully M O and Zubairy M S 2002 Quantum Optics (Cambridge: Cambridge University Press) p 51
[14] Yuen H P 1976 Two-photon coherent states of the radiation field Phys. Rev. A 13 2226–43
[15] Čaves C M 1981 Quantum-mechanical noise in an interferometer Phys. Rev. D 23 1693–708
[16] Walls D F 1983 Squeezed states of light Nature 306 141–6
[17] Muñoz-Tapia R 1993 Quantum mechanical squeezed state Am. J. Phys. 61 1005–8
[18] Magnus W 1954 On the exponential solution of differential equations for a linear operator Comm. Pure Appl. Math. 7 649–73

For a general review on the Magnus series both from the mathematical and the physical point of view see Blanes S, Casas F, Oteo J A and Ros J 2009 The Magnus expansion and some of its applications Phys. Rep. 470 151–238
[19] Walker G 1939 The theory of roulettes and glissettes Natl Math. Mag. 13 223–9
[20] Barnett S M and Radmore P M 1997 Methods in Theoretical Quantum Optics (Oxford: Clarendon) p 14
[21] Wódkiewicz K and Eberly J H 1985 Coherent states, squeezed fluctuations, and the SU(2) and SU(1,1) groups in quantum-optics applications J. Opt. Soc. Am. B 2 458–66
[22] Gerry C C 1985 Dynamics of SU(1,1) coherent states Phys. Rev. A 31 2721–3
[23] Dattoli G and Torre A 1988 Algebraic view to the quantum anharmonic oscillator Phys. Rev. A 37 1571–5
[24] Dattoli G, Solimeno S and Torre A 1987 Algebraic view of the optical propagation in a nonhomogeneous medium Phys. Rev. A 35 1668–72
[25] Han D, Kim Y S and Noz M E 1989 Linear canonical transformations of coherent and squeezed states in the Wigner phase space: II. Quantitative analysis Phys. Rev. A 40 902–12
[26] Orlowski A and Wódkiewicz K 1990 On the SU(1, 1) phase-space description of reduced and squeezed quantum fluctuations J. Mod. Opt. 37 295–301
[27] Vourdas A 1990 SU(2) and SU(1,1) phase states Phys. Rev. A 41 1653–61
[28] Zoubi H and Ben-Aryeh Y 1998 The evolution of harmonic oscillator Wigner functions described by the use of group representation Quantum Semiclass. Opt. 10 447–58