COMMUTATIVITY OF INTEGRAL QUASI-ARITHMETIC MEANS ON MEASURE SPACES

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Abstract. Let \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) be finite measure spaces for which there exist \(A \in \mathcal{L}\) and \(B \in \mathcal{M}\) with \(0 < \lambda(A) < \lambda(X)\) and \(0 < \mu(B) < \mu(Y)\), and let \(I \subseteq \mathbb{R}\) be a non-empty interval. We prove that, if \(f\) and \(g\) are continuous bijections \(I \to \mathbb{R}^+\), then the equation

\[
    f^{-1}\left( \int_X f \left( g^{-1}\left( \int_Y g \circ h \, d\mu \right) \right) \, d\lambda \right) = g^{-1}\left( \int_Y g \left( f^{-1}\left( \int_X f \circ h \, d\lambda \right) \right) \, d\mu \right)
\]

is satisfied by every \(L \otimes M\)-measurable simple function \(h : X \times Y \to I\) if and only if \(f = cg\) for some \(c \in \mathbb{R}^+\) (it is easy to see that the equation is well posed). An analogous, but essentially different, result, with \(f\) and \(g\) replaced by continuous injections \(I \to \mathbb{R}\) and \(\lambda(X) = \mu(Y) = 1\), was recently obtained in [Indag. Math. 27 (2016), 945–953].

1. Introduction

Let \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) be measure spaces, and \(f\) and \(g\) be real-valued continuous injections defined on a non-empty interval \(I \subseteq \mathbb{R}\) (which may be bounded or unbounded, and need not be open or closed). In this note, we examine conditions under which the equation

\[
    f^{-1}\left( \int_X f \left( g^{-1}\left( \int_Y g \circ h \, d\mu \right) \right) \, d\lambda \right) = g^{-1}\left( \int_Y g \left( f^{-1}\left( \int_X f \circ h \, d\lambda \right) \right) \, d\mu \right)
\]

(1)

is satisfied by every \(h\) in a suitable class of \(L \otimes M\)-measurable functions \(X \times Y \to I\), taking \(f\) and \(g\) as unknowns and assuming the equation is well posed (notations and terminology, if not explained, are standard or should be clear from the context).

When \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) are probability spaces, the left- and right-hand side of (1) can be interpreted as “partially mixed” integral quasi-arithmetic means. The interest in functional equations involving generalized means dates back at least to G. Aumann [1] and has been a subject of extensive research, see, e.g., [4], [5], [9, 10], and references therein.

In particular, (1) is naturally related to the vast literature on permutable mappings [11], and is motivated by the study of certainty equivalences, a notion first introduced by S. H. Chew [3] in connection to the theory of expected utility and decision making under uncertainty, see [8] and [12] for current trends in the area.

The equation was recently addressed in [7], where it was observed, among other things, that (1) is well posed if \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) are probability spaces and \(h(X \times Y) \subseteq I\) for every “test function” \(h\), see [7, Proposition 2] (“\(\subseteq\)” means, as usual, “contained in a compact subset

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of”). It follows that, if \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) are probability spaces, then both the left- and the right-hand side of (1) is well defined provided that \(h : X \times Y \to I\) is an \(\mathcal{L} \otimes \mathcal{M}\)-measurable simple function, namely, \(h = \sum_{n=1}^{n} \alpha_{i} 1_{E_{i}}\), where \(\alpha_{1}, \ldots, \alpha_{n} \in I\) and \(E_{1}, \ldots, E_{n} \in \mathcal{L} \otimes \mathcal{M}\) are disjoint sets such that \(E_{1} \cup \cdots \cup E_{n} = X \times Y\).

With this in mind, we call a measure space \((S, \mathcal{C}, \gamma)\) non-degenerate if there exists \(A \in \mathcal{C}\) with \(0 < \gamma(A) < \gamma(S)\). Here, then, comes the main theorem of [7], which was stated in that paper under the assumption that (1) is satisfied for all \(\mathcal{L} \otimes \mathcal{M}\)-measurable functions \(h : X \times Y \to I\) for which \(h(X \times Y) \subseteq I\), but is actually true, as is transparent from its proof, in the following (more general) form.

**Theorem 1.** Let \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) be non-degenerate probability spaces, and \(f, g : I \to \mathbb{R}\) be continuous injections. Then equation (1) is satisfied by every \(\mathcal{L} \otimes \mathcal{M}\)-measurable simple function \(h : X \times Y \to I\) if and only if \(f = ag + b\) for some \(a, b \in \mathbb{R}\) with \(a \neq 0\).

Now we may ask what happens if \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) are not probability spaces, and in the next section we give a partial answer to this question.

## 2. Main result

It is easy to check (we omit details) that (1) is still well posed if \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) are non-degenerate finite measure spaces, and \(f\) and \(g\) are continuous bijections \(I \to \mathbb{R}^{+}\) (throughout, \(\mathbb{R}^{+}\) denotes the set of positive reals and \(\mathbb{N}^{+}\) the set of positive integers). Accordingly, we have the following analogue of Theorem 1.

**Theorem 2.** Let \((X, \mathcal{L}, \lambda)\) and \((Y, \mathcal{M}, \mu)\) be non-degenerate finite measure spaces, and \(f, g : I \to \mathbb{R}^{+}\) be continuous bijections, where \(I \subseteq \mathbb{R}\) is a (necessarily open) interval. Then equation (1) is satisfied by every \(\mathcal{L} \otimes \mathcal{M}\)-measurable simple function \(h : X \times Y \to I\) if and only if \(f = cg\) for some \(c \in \mathbb{R}^{+}\).

**Proof.** The “if” part follows by Fubini’s theorem (viz., [2, Theorem 3.4.4]) and the fact that, if \((S, \mathcal{C}, \gamma)\) is a measure space, \(w\) a continuous bijection \(I \to \mathbb{R}^{+}\), and \(h : S \to I\) a \(\mathcal{C}\)-measurable function such that \(w \circ h\) is \(\gamma\)-integrable, then

\[
\frac{1}{w} \left( \int_{S} w \circ h \, d\gamma \right) = (aw)^{-1} \left( \int_{S} (aw) \circ h \, d\gamma \right)
\]

for every \(a \in \mathbb{R}^{+}\) (we omit details, cf. [7, Proposition 3] for the case of probability spaces).

As for the “only if” part, set \(D := \mathbb{R}^{+} \times \mathbb{R}^{+}\). By hypothesis, there are determined \(A \in \mathcal{L}\) and \(B \in \mathcal{M}\) such that \(\alpha_{1} := \lambda(A)\), \(\alpha_{2} := \lambda(A^{c})\), \(\beta_{1} := \mu(B)\), and \(\beta_{2} := \mu(B^{c})\) belong to \(\mathbb{R}^{+}\), where \(A^{c} := X \setminus A\) and \(B^{c} := Y \setminus B\). Hence, for all \(x, y, z, w \in I\) the function

\[
h = x1_{A \times B} + y1_{A \times B^{c}} + z1_{A^{c} \times B} + w1_{A^{c} \times B^{c}}
\]

is an \(\mathcal{L} \otimes \mathcal{M}\)-measurable simple function \(X \times Y \to I\), so we can plug (2) into (1) and obtain

\[
f^{-1}(\alpha_{1} f(g^{-1}(\beta_{1} g(x) + \beta_{2} g(y))) + \alpha_{2} f(g^{-1}(\beta_{1} g(z) + \beta_{2} g(w))))
\]

\[
= g^{-1}(\beta_{1} g(f^{-1}(\alpha_{1} f(x) + \alpha_{2} f(z))) + \beta_{2} g(f^{-1}(\alpha_{1} f(y) + \alpha_{2} f(w))))
\]

(3)
Set $\varphi := f \circ g^{-1}$ on $\mathbb{R}^+$. Of course, $\varphi$ is a continuous bijection on $\mathbb{R}^+$, and we derive from (3), through the change of variables $x \mapsto g^{-1}(s)$, $y \mapsto g^{-1}(t)$, $z \mapsto g^{-1}(u)$, and $w \mapsto g^{-1}(v)$, that
\[ \varphi^{-1}((\alpha_1 \varphi(\beta_1 s + \beta_2 t) + \alpha_2 \varphi(\beta_1 u + \beta_2 v)) = \beta_1 \varphi^{-1}(\alpha_1 \varphi(s) + \alpha_2 \varphi(u)) + \beta_2 \varphi^{-1}(\alpha_1 \varphi(t) + \alpha_2 \varphi(v)) \] 
for every $s, t, u, v \in g(I) = \mathbb{R}^+$. Moreover, if we take $\Phi$ to be the function
\[ D \to \mathbb{R}^+ : (x, y) \mapsto \varphi^{-1}(\alpha_1 \varphi(x) + \alpha_2 \varphi(y)), \]
then (4) can be conveniently rewritten as
\[ \Phi(\beta_1 x + \beta_2 y) = \beta_1 \Phi(x) + \beta_2 \Phi(y), \] for all $x, y \in D$. (6)
Let $\preceq$ be the product order on $\mathbb{R} \times \mathbb{R}$ induced by the usual order on $\mathbb{R}$, and note that
\[ \Phi(x) < \Phi(y), \] for all distinct $x, y \in D$ with $x \preceq y$. (7)
Indeed, $\varphi$ being a continuous bijection on $\mathbb{R}^+$ entails that $\varphi$ is strictly monotone. So, assume $\varphi$ is strictly increasing (respectively, strictly decreasing), and let $x, y, z, w \in \mathbb{R}^+$ be such that $x \leq z$, $y \leq w$, and $(x, y) \neq (z, w)$. Then
\[ \alpha_1 \varphi(x) + \alpha_2 \varphi(y) < \alpha_1 \varphi(z) + \alpha_2 \varphi(w) \] (respectively, $\alpha_1 \varphi(x) + \alpha_2 \varphi(y) > \alpha_1 \varphi(z) + \alpha_2 \varphi(w)$), and since $\varphi$ is strictly increasing (respectively, decreasing) if and only if so is $\varphi^{-1}$, we conclude that $\Phi(x, y) < \Phi(z, w)$, which is what we wanted to prove.

On the other hand, it is straightforward to check that $\Phi$ is surjective. Indeed, pick $z \in \mathbb{R}^+$. By the surjectivity of $\varphi$, there exist $x, y \in \mathbb{R}^+$ such that $\alpha_1 \varphi(x) + \alpha_2 \varphi(y) = \frac{1}{2} \varphi(z) > 0$, viz., $\alpha_1 \varphi(x) + \alpha_2 \varphi(y) = \varphi(z)$, which, by (5), is equivalent to $\Phi(x, y) = z$.

With this said, set $\xi_n := \Phi(1/n, 1/n)$ for every $n \in \mathbb{N}^+$. By (7), $(\xi_n)_{n \geq 1}$ is a strictly decreasing sequence of positive reals. Hence, the limit of $\xi_n$ as $n \to \infty$ exists, and is non-negative and equal to $\xi := \inf_{n \geq 1} \xi_n$. Suppose for a contradiction that $\xi > 0$. Then, we infer from the surjectivity of $\Phi$ that $\xi = \Phi(\bar{x}, \bar{y})$ for some $\bar{x}, \bar{y} \in \mathbb{R}^+$, which is, however, impossible, because $\frac{1}{n} < \min(\bar{x}, \bar{y})$, and hence, by (7), $\xi_n < \xi$, for all sufficiently large $n \in \mathbb{N}^+$.

So, using that a local base at $0 := (0, 0)$ (in the usual topology of $\mathbb{R}^2$) is given by the squares of the form $[-1/n, 1/n] \times [-1/n, 1/n]$ with $n \in \mathbb{N}^+$, it follows from the above that
\[ \lim_{x \to 0} \Phi(x) = 0. \]
By letting $x \to 0$ (respectively, $y \to 0$) in (6), we therefore find that
\[ \Phi(\beta_1 x) = \beta_1 \Phi(x) \] and
\[ \Phi(\beta_2 y) = \beta_2 \Phi(y), \] for all $x, y \in D$.
Together with (6), this in turn implies that
\[ \Phi(x + y) = \Phi(x) + \Phi(y), \] for all $x, y \in D$.
(8)
But $D$ is a subsemigroup of the group $(\mathbb{R}^2, +)$ with $\mathbb{R}^2 = D - D := \{ x - y : x, y \in D \}$ and $\Phi$ is continuous, so we get from (8) and [6, Theorems 5.5.2 and 18.2.1] that there exist $a, b \in \mathbb{R}$
such that \( \Phi(x, y) = ax + by \) for all \( x, y \in \mathbb{R}^+ \), and actually, it is immediate that \( a, b \geq 0 \) and \( a + b \neq 0 \), since \( \Phi \) is a positive function. In addition, we derive from (5) that

\[
\alpha_1 \varphi(x) + \alpha_2 \varphi(y) = \varphi(ax + by), \quad \text{for all } x, y \in \mathbb{R}^+.
\]

(9)

Now, we have already observed that \( \varphi \) is strictly monotone. Suppose for a contradiction that \( \varphi \) is strictly decreasing. Then, \( \varphi \) being a bijection of \( \mathbb{R}^+ \) gives that \( \varphi(z) \to 0^+ \) as \( z \to \infty \), and assuming \( a \neq 0 \) (the case when \( b \neq 0 \) is similar), this implies by (9) that

\[
0 < \alpha_2 \varphi(1) = \varphi(ax + b) - \alpha_1 \varphi(x) < \varphi(ax + b) \leq \varphi(ax),
\]

which is, however, impossible in the limit as \( x \) goes to \( \infty \).

Thus, \( \varphi \) is a strictly increasing continuous bijection of \( \mathbb{R}^+ \), and hence \( \varphi(z) \to 0^+ \) as \( z \to 0 \).

Taking \( \varphi(0) := 0 \) and letting \( x \to 0 \) (respectively, \( y \to 0 \)) in (9), we can therefore conclude that

\[
\alpha_2 \varphi(y) = \varphi(by) \quad \text{and} \quad \alpha_1 \varphi(x) = \varphi(ax), \quad \text{for all } x, y \in \mathbb{R}^+.
\]

It follows \( a, b \in \mathbb{R}^+ \), and in combination with (9), this yields

\[
\varphi(x + y) = \varphi(x) + \varphi(y), \quad \text{for all } x, y \in \mathbb{R}^+.
\]

So, considering that \( \varphi \) is continuous and applying [6, Theorems 5.5.2 and 18.2.1] to the function \( \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} : (x, y) \mapsto \varphi(x) \) shows that there is a constant \( c \in \mathbb{R}^+ \) such that \( \varphi(x) = cx \) for all \( x \in \mathbb{R}^+ \), which is equivalent to \( f = cg \).

\[\square\]

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