1. Introduction

1.1. The history of the Fermi-to-Bose crossover

The change of quantum statistics, from bosonic to fermionic and back, may look trivial in some respects. Indeed, nearly all known bosons are composed of fermions, such as the hydrogen atom which consists of a proton and electron, and, seemingly, the ionization resolves the problem of creating two Fermi systems from a Bose condensate of hydrogen. On the other hand, the resulting Fermi systems would hardly be degenerate or stable.

For converting a degenerate bosonic system into a fermionic one and vice versa, smooth control of interactions is necessary. Such a problem was considered for the first time by Keldysh and Kozlov in 1968 [1] for a gas of excitons which substantially overlap at small electron–hole coupling and transform into a gas of point-like bosons as the coupling increases.

In the 1980s, it became clear that a smooth transform of a Fermi gas into a Bose condensate was already contained in the Bogolubov superconductivity theory [2, 3]. Even at infinitely small attraction, fermions with opposite spins form Cooper...
pairs [4–6], while an adiabatic growth of attraction makes the pairs contract and smoothly cross over into point-like bosons [7–9]. Such transformation of a fermionic system into a bosonic one is a fundamentally many-body effect. Initially the binding of the pairs appears due to the presence of the Fermi surface. Only after the interaction strength overcomes a threshold does the bound state becomes stable in the vacuum. The bound-state emergence is not accompanied by any jumps in system properties. Such change in the statistics has been considered not only for the excitons but also for electrons in superconductors and for quark matter [10]. Physical implementation of this phenomenon took place in experiment with an ultracold gas of Fermi atoms [11].

1.2. Two-body and many-body physics in 2D

In a two-dimensional (2D) system, in the case of attractive interaction of two particles, a bound state always exists, even in a vacuum: an infinitely small interaction may bind two fermions into a boson for a symmetric attractive potential. Nevertheless the problem of crossover between Bose and Fermi statistics naturally appears in a many-body 2D system. Schematically such crossover is shown in figure 1. When the size of the vacuum bound state is larger than the interparticle distance, the system looks like a gas of fermions. Upon contraction of the bound state to a size much smaller than the interparticle distance, the system becomes bosonic. Between the fermionic and bosonic limits, there is the region of strong many-body interactions where the size of fermion pairs is comparable to the interparticle distance.

In uniform 2D systems the long-range order does not appear at finite temperature due to thermal fluctuations of the order parameter. In particular, this prohibits Bose condensation. Nevertheless the reduction of the spatial dimensionality from three to two does not impoverish many-body physics, but makes it more diverse. New phase transitions appear [12–16]. For example, at the Berezinskii–Kosterlitz–Thouless (BKT) transition [12, 13], the superfluid phase is destroyed via proliferation of vortices. The BKT transition became the first known phase transition with nontrivial topology. Currently, topological phase transitions in 2D systems are actively being studied [14–16]. Other examples of interesting physics in 2D include the integer [17] and fractional [18] quantum Hall effects. Explanation of high-temperature superconductivity observed in materials with 2D kinematics is still an open question [19]. A unified theory of the Fermi-to-Bose crossover has to describe weakly interacting Fermi and Bose gases as well as the gas in the intermediate strongly interacting regime. For 3D systems, generalization of the Bardeen–Cooper–Schrieffer (BCS) variational wavefunction (describing the paired state) has been successfully applied [7–9]. In part for this reason, the crossover problem is frequently referred to as the BCS–BEC crossover, where BEC stands for Bose–Einstein condensation. For 2D, construction of such theory was more complicated than for 3D. While in 3D a mean-field theory of Cooper pairs [7–9] qualitatively correctly describes the ground-state properties, for a 2D system it is important to include the fluctuations in the model [20, 21]. The more important role of fluctuations in lower dimensions is in agreement with the Ginzburg–Levanyuk criterion [22, 23]. Note also that in recent years a considerable numerical effort has been applied to the problem of the BCS–BEC crossover in 2D in the framework of the advanced mean-field [21, 24], Monte Carlo simulations [25–28], and Luttinger–Ward (thermodynamic) identities [29]. Also, a generalization of the Nozières–Schmitt-Rink approach [9] with Gaussian fluctuations to the 2D case was performed [20, 30].

The interaction description is a general question for kinematically 2D systems. In the theory of high-temperature superconductivity, for example, purely 2D models are frequently used [31]. These models assume that all motion is in the xy-plane and also the interactions are independent of z. Such a pure 2D model is not always sufficient, which may be seen in the case of 3He on a substrate: the increase of zero-point oscillations relative to the interaction radius brings about a formation of a self-bound liquid state on the substrate of alkali metals (except for lithium) [32]. A unified and correct description of interactions in all regimes is required during construction of a model for a system with tunable statistics. Experiments with such a system may test the applicability of purely 2D models to real systems.

1.3. Human-made and natural quantum 2D systems

A general understanding of 2D many-body phenomena is important for at least four physical systems:

(i) High-temperature layered cuprate superconductors, where the Cooper pair size is comparable to the interparticle distance, as in the strongly interacting regime of the Bose-to-Fermi crossover problem. Earlier cited models

Figure 1. Sizes of the pairs of fermions at different stages of the crossover between the Fermi (right) and Bose (left) statistics.
Ultracold gases in the 2D geometry.

condensed matter, high-energy, and nuclear physics [45, 46].

(iii) Helium thin films and $^3$He submonolayers [40].

(iv) Ultracold atoms in the 2D geometry.

1.4. Scope of the review

This review is devoted to the physics of the Fermi-to-Bose crossover in quantum gases. Preparation of the ultracold Fermi gases with 2D kinematics is described in section 2. Two-body interactions are the subject of section 3. This section also discusses the difference between interactions via 2D and 3D potentials as well as mapping one interaction type onto the other. Experimental methods for degeneracy detection and thermometry are reviewed in section 4. Mean-field approaches to the description of the gases in the 3D and 2D Bose-to-Fermi crossover are discussed in section 5. Observation of the Fermi-to-Bose crossover is presented in section 6. Critical phenomena, including Bose condensation, the BKT transition, and mean-field description of the latter, are discussed in section 7. The equation of state (EoS) is the subject of section 8, which also includes simple and advanced models, possible scale invariance specific to 2D gases, and experimental tests of the EoS. Criteria of two-dimensionality are revisited on the basis of recent experimental progress in section 9. Spin-imbalanced Fermi gases are the subject of section 10. We conclude in section 11.

2. Ultracold atomic Fermi gases with 2D kinematics

2.1. Cooling and trapping ultracold atoms

Experiments with ultracold quantum gases allowed the observation of states of matter and effects that previously were just a matter of theoretical discussion: Bogolubov weakly interacting Bose gas [41]; Bertsch matter [42], which is a Fermi gas with infinite s-wave scattering length and nearly zero interaction range; crossover between the fermionic BCS superfluidity and Bose condensation [11]; Tonks–Girardeau fermionization of a 1D Bose gas [43]; and Efimov trimers [44]. The ultracold gases are used for quantitative tests of theories that are applicable to condensed matter, high-energy, and nuclear physics [45, 46]. The success of experiments is facilitated by a set of conditions:

- both the interactions and the spin composition may be tuned smoothly and reversibly;
- kinematic dimensionality is controlled;
- measurements are performed directly: it is possible to instantaneously image the density distribution owing to absorption of light by the atomic gas, to measure thermodynamic properties, and to observe the momentum space distribution and the difference between the order-parameter phases of subsystems.

Lithium-6 is currently the most popular atom in the experimental studies of the 2D Fermi-to-Bose crossover [47–52] while potassium-40 is another option [53]. Achievement of quantum degeneracy requires cooling to temperature $T$ in the micro- to nanokelvin range [54]. In the simplest case, the atomic gas is prepared in two stages, taking a few seconds each. At the first stage, trapping and cooling of atoms is performed using laser radiation whose frequency is nearly resonant to an atomic electric-dipole transition [55]. The gas is collected in a magneto-optical trap from either an atomic beam or ambient vapor. The trap is schematically shown in figure 2(a). By means of the viscous light pressure force, whose value is adjusted by the Zeeman shift, the atomic gas is gathered near the point where the magnetic field is zero.

The magneto-optical trap collects from millions to billions of atoms. Their state is far from quantum degeneracy. For example, for lithium, the phase space density is just $\sim 10^{-6}$. Further cooling is impossible due to the resonant light used at this stage. Due to re-emission of photons there is a temperature minimum which is few tens of microkelvins for lithium. In addition, further increase of the density is prevented by the light pressure of the atoms upon each other, appearing due to the re-emission. Therefore, the next stage of cooling is needed. At the start of this stage, the resonant light fields are instantaneously extinguished and the atomic gas finds itself in a conservative potential which may exist [36–38], one of which is the nuclear lasagna phase, where the kinematics of particles is predominantly 2D. This phase may be responsible for limiting the rotation period of pulsars [39].

The cooled gas is cooled further by means of evaporation [57]. Particles with the highest energies leave the trap, while the remaining particles collide and try to form the equilibrium distribution at a lower temperature. This repopulates the higher momentum states, which again bring about losses of the energetic particles. To speed up the evaporation, the trap depth is slowly decreased. As a result, the cooling brings the gas into a degenerate state with phase space density $\sim 1$. The ultracold quantum gases are extremely dilute. The interparticle distance ranges from hundreds of nanometers to microns, which is much larger than the intermolecular distance of 3 nm in air, and the typical scale of the interatomic
2.2. Spin states of Fermi atoms

The spin degrees of freedom are fundamental to the properties of Fermi systems. The role of spin in atomic gas is played by the internal states of the atom, which are discussed here using the example of lithium-6. The states of lithium-6 corresponding to the ground-state orbital of the single valence electron $2s^1$ are shown in figure 3. These states differ by the mutual orientation of the valence electron spin $S = 1/2$ and the nuclear spin $I = 1$. In the absence of the magnetic field, the states may be described in the basis of the total angular momentum operator $\hat{F} = \hat{S} + \hat{I}$. The balanced mixture of states $|1\rangle$ and $|2\rangle$ is an analog of the gas of spin-up and spin-down electrons in a solid-state system. Problems requiring more spin diversity may be implemented using a gas of lithium-6, as one may see from figure 3.

In the experiment, an equal mixture of states $|1\rangle$ and $|2\rangle$ is used. In the external magnetic field $B$, which is turned on for controlling the interactions, the states are expressed in basis $|S_z, I_z\rangle$ [61]:

\[
|1\rangle = \cos \theta_+ \left| \frac{1}{2}, 1 \right\rangle - \sin \theta_+ \left| \frac{1}{2}, 0 \right\rangle,
\]

\[
|2\rangle = \cos \theta_- \left| -\frac{1}{2}, 0 \right\rangle - \sin \theta_- \left| \frac{1}{2}, -1 \right\rangle,
\]

where

\[
\sin \theta_\pm = \frac{1}{\sqrt{1 + (Z^\pm + R^\pm)^2/2}},
\]

\[
Z^\pm = \frac{2\mu_B B}{\alpha} \pm 1/2, \quad R^\pm = \sqrt{(Z^\pm)^2 + 2},
\]

$\mu_B$ is the Bohr magneton, and $\alpha/(2\pi\hbar) = 152.1$ MHz is the hyperfine interaction constant. In the field $B = 600–1000$ G typical for the experiments, the first terms of superposition states (1) and (2) are dominant. These terms correspond to projection $S_z = -1/2$. At $B = 800$ G, for example, $\sin^2 \theta_\pm = 0.002$. Despite this, the part of the state corresponding to projection $S_z = 1/2$ is principal in the tunability of the interaction.

2.3. 2D kinematics for noninteracting fermions in anisotropic traps

2D kinematics may be achieved in a strongly anisotropic parabolic potential

\[
V(x, y, z) = \frac{m_0^2 \omega_z^2}{2} + \frac{m^2 \omega_y^2}{2} + \frac{m^2 \omega_x^2}{2}
\]

with $\omega_z \gg \omega_y \equiv \sqrt{\omega_x \omega_y}$, $\omega_x \sim \omega_y$, and $m$ being the atom mass. Due to the anisotropy it is possible to place the absolute majority of the atoms into the ground state of motion along $z$, while according to the Pauli exclusion principle, the fermions...
populate many states of motion in the xy-plane, as shown in figure 4(a). As a result, the gas is kinematically 2D.

Experimentally a series of such traps may be formed by anti-nodes of an electromagnetic standing wave as schematically shown in figure 4(b). The radiation frequency has to be far below that of the electric-dipole transitions. An image of the clouds taken along the plane of motion xy is shown in figure 4(c), where each light strip is a separate 2D system. Here the trapping wavelength is $\lambda = 10.6 \mu m$, which gives 5.3 $\mu m$ separation between the antinodes and, as a result, each cloud is resolved. For imaging, the clouds are illuminated with monochromatic radiation of wavelength 671 nm, which is resonant to the electric-dipole transition $2S_{1/2} \rightarrow 2P_{3/2}$ of the lithium-6 atom. As a result of the resonant absorption, a shadow appears and is further projected onto a charge-coupled device. From the absorption image, the gas density distribution integrated along y is reconstructed [50, 63]. Such photographing is selective to the internal atomic state. The imaging destroys the state of the system due to the energy input, which is much larger than the kinetic energy. Preparation of few tens of clouds in anti-nodes of a standing electromagnetic wave. The gas is shown in dark red, while the intensity of the radiation forming the trap is shown in light purple. (c) Image of the clouds along the y-direction. From [62] © Uspekhi Fizicheskikh Nauk 2016.

3. Tunable two-body interactions

3.1. T-matrix theory of the two-body scattering in 3D and 2D for short-range interactions

3.1.1. The 3D vacuum T-matrix. In the 3D case for symmetric attractive potentials there is a threshold for the formation of the bound state when the depth of the potential well $|U|$ exceeds the effective width of the well $W \sim h^2/(mr_{0}^2)$, where $m$ is the particle mass and $r_{0}$ is the range of the interaction. As an example one may take the exponentially decaying potential $U(r') = U \exp(-r'/r_{0})$ (here and further primed values are used for coordinates in the center-of-mass reference frame of two interacting particles). The attractive Hubbard model with contact interaction on the lattice is another example, where $W$ is the bandwidth, $U$ is the Hubbard onsite attraction, and the role of the interaction range $(r_{0})$ is played by the intersite distance $d$. For $|U| \ll W$ we are in the weak-coupling case (weak-coupling Born approximation) and there is no bound state of the two particles in real space. The s-wave scattering length is negative and is given by

$$a_{3D} = \frac{mU_{0}}{4\pi\hbar^2} < 0$$

where $U_{0} \sim U r_{0}^2$ is a zeroth Fourier component of the interaction. In this case only extended s-wave Cooper pairing is possible in the BCS domain [66].

If the interaction is strong enough (and the Born parameter $|\gamma| = |U| r_{0}^2/(4\pi\hbar^2)$ becomes close to unity) the s-wave scattering length and the corresponding 3D vacuum T-matrix for zero total energy $E$ and zero total momentum $K \equiv p_1 + p_2$ reads
signaling the appearance of a bound state for $|\gamma| = 1$. Note that if the Born parameter is close to but smaller than 1 \((1 - |\gamma|)/|\gamma| \ll 1\), we are in the resonance situation \(a_{3D} \gg r_0\) and obtain the shallow virtual bound state \(E_{b}^3 = \hbar^2/(2m r_0^2)\). For \(|\gamma| = 1\) (at the threshold) an expression for the scattering length \(a_{3D}\) has a pole. For \(|\gamma| > 1\) we are in the BEC domain of local (compact) pairs or molecules (dimers) and the expression for the \(s\)-wave scattering length should be modified. For the bound-state problem (for the negative energies \(E < 0\)) we should use the \(s\)-wave scattering amplitude \(f_0(E)\) which reads

\[
|f_0(E)| = \frac{|\gamma|r_0}{1 - |\gamma| + \sqrt{m|E|r_0^2}}. \tag{6}
\]

Above the threshold (but not very far away from it) we have a two-particle bound state with an energy:

\[
|E_b| \simeq \frac{\hbar^2}{mr_0^2} \left( \frac{|\gamma| - 1}{|\gamma|} \right)^2. \tag{7}
\]

It is convenient to represent the binding energy as \(|E_b^3| = \hbar^2/(ma_{3D}^2)\) and thus to introduce repulsive scattering length \(a_{3D} = |\gamma|r_0/(|\gamma| - 1) > 0\) above the threshold. For \(|\gamma| = 1\), as we discussed, the scattering length diverges. Correspondingly \(1/a_{3D} = 0\) and we are in the unitary limit. Slightly above the threshold for \((|\gamma| - 1)/|\gamma| \ll 1\) we have a shallow (real) bound state \(E_{b}^3 = \hbar^2/(ma_{3D}^2) \ll \hbar^2/(mr_0^2)\) and thus \(a_{3D} \gg r_0\). In the strong-coupling case for \(|U| \gg W\) the binding energy \(|E_b| \sim |U|\) and we have deep levels with \(a_{3D} \ll r_0\).

### 3.1.2. The 2D vacuum T-matrix

In the 2D case (in contrast to the 3D situation) the bound state of two particles in vacuum appears already for infinitely small symmetric attractive potential and the threshold for its appearance is absent. Thus even on the BCS-side of the crossover we always have the coexistence of the two phenomena, the Cooper pairing in momentum space and pairing of two particles in a vacuum (in real space). The BCS–BEC crossover is governed by the ratio between the binding energy of the molecule (it is \(|E_b|/2\) for one particle in the molecule) and the Fermi energy \(E_F\). If we again use the convenient notation for the binding energy \(E_b = \hbar^2/(ma^2)\), then \(2E_b/E_b = k_F^2/a^2\) and \(\sqrt{2E_b/E_b} = k_F a\). Here \(E_F = \hbar^2k_F^2/(2m)\) is the local Fermi energy and \(k_F b\) is the Fermi momentum; in 2D \(E_F = 2m\hbar^2 n/m\), where \(n\) is the local numerical density per spin component. The corresponding vacuum T-matrix for the negative total energy \(E < 0\) reads:

\[
f_0(E < 0) = \frac{mT(E = 0, K = 0)}{4\pi\hbar^2} = \frac{-|\gamma|}{1 - \gamma \ln \left( \frac{W}{|\gamma|} + 1 \right)}, \tag{8}
\]

where \(\gamma = m|U_0|/(4\pi\hbar^2)\) is the effective Born parameter and \(W = \hbar^2/(mr_0^2)\) is again the effective width of the potential well. In (8), \(U_0 \sim Ur_0^2\) is the zeroth Fourier component of the potential in 2D. As a result the pole of the T-matrix which corresponds to the energy of the bound state in (8) reads

\[
|E_b| = \frac{\hbar^2}{ma^2} \frac{W}{\exp \left( \frac{\pi}{\gamma} \right) - 1}. \tag{9}
\]

Correspondingly it is useful to express the scattering amplitude via the binding energy \(|E_b|\) which yields the following elegant expression:

\[
f_0(E < 0) = \frac{1}{\ln \left( \frac{W + |E_b|}{|E_b|} \right)}. \tag{10}
\]

Equation (10) correctly describes both the situation with shallow levels for which \(|E| \sim |E_b| \ll W\) and with deep levels where \(|E| \sim |E_b| \gg W\).

For deep levels

\[
f_0(E < 0) \simeq \frac{|E| |E_b|}{W(|E_b| - |E|)}. \tag{11}
\]

Since \(\gamma \gg 1\) for deep levels, we can expand \(\exp(1/\gamma) \simeq 1 + 1/\gamma\). Thus

\[
|E_b| \simeq \gamma W = \frac{|U_0|}{4\pi r_0^2} \sim U. \tag{12}
\]

For shallow levels we obtain

\[
f_0(E < 0) \simeq \frac{1}{\ln(E_b/E)} \tag{13}
\]

In the shallow-level case \(\exp(1/\gamma) \gg 1\) (or equivalently \(\gamma \ll 1\)) and hence the binding energy reads:

\[
|E_b| = W \exp \left( -\frac{1}{\gamma} \right). \tag{14}
\]

A more conventional scattering amplitude \(f_{2D}\) may be obtained from (13) by analytic continuation to positive energies \(E = \hbar^2q^2/m\). In (13), the binding energy may be expressed via the widely used 2D scattering length \(a_{2D} (a_2 e^{\gamma\pi}/2 \equiv a); E_b = -\hbar^2/(ma_{2D}^2)\), where \(\gamma \pi \approx 0.577\) is Euler’s constant. As a result one obtains:

\[
f_{2D}(q, a_2) = 4\pi f_0(E > 0) = -\frac{2\pi}{\ln[a_2 e^{\gamma\pi}/(2i)]}, \tag{15}
\]

where \(hq \equiv |p_1 - p_2|/2\) is the relative momentum of the two colliding atoms, while \(p_1\) and \(p_2\) are their momenta in the laboratory reference frame. Note that \(f_{2D}\) is different from the standard 2D scattering amplitude [67], which equals \(-f_{2D}/\sqrt{4\pi q}\). Correspondingly, the scattering wavefunction is [67]

\[
\psi_1(\rho') \simeq \exp(iq \cdot \rho') - f_{2D} \exp(iq\rho' - i\pi/4)/\sqrt{8\pi q\rho'}, \tag{16}
\]

where \(\rho'\) is the vector between the two atoms in the \(xy\)-plane. The \(s\)-wave part of the wavefunction of two unbound atoms at large distances asymptotically behaves as \(\psi_1(\rho') \propto \ln(\rho'/a_2)\), where \(\rho' \equiv |\rho'|\).
3.2. Two-body scattering for the 2D kinematics and 3D contact interaction

The situation with atomic gases in the tight parabolic trap bears similarities and differences with the case where both kinematics and the interaction potential are purely 2D. In the experiments, the range $r_0$ of the 3D potential is much smaller than the quantization size $l_z \equiv \sqrt{\hbar/2m_\omega}$. The interaction potential may be regarded as the 3D $\delta$-function on the scale of the problem. As a result, the interactions are quasi-2D rather than 2D because at distances $l_z \ll l_z$ the wavefunction of colliding atoms is determined by 3D scattering length $a_{3D}$, which is the major difference to the purely 2D scattering problem.

The similarity appears in the form of the scattering wavefunction, which at large distances is the same as (16), except that the scattering amplitude $f_{2D}$ is to be replaced by the expression which is specific to the quasi-2D scattering [68]

$$f = f_{2D}(q, a_{3D}, l_z) \equiv \frac{2\pi}{\sqrt{\pi l_z/a_{3D} + w(q^2 l_z^2)/2}},$$

where the function $w(\xi)$ is defined via the limit

$$w(\xi) \equiv \lim_{\xi \to \infty} \left[ \sqrt{\frac{4J}{\pi}} \ln \frac{J}{\xi^2} - \sum_{j=0}^{\lambda} \frac{(2J - 1)!!}{(2j)!!} \ln(j - \xi - i0) \right]$$

(18)

3.3. Parametrization of the quasi-2D scattering via the 2D scattering length

The theory of the 2D many-body systems traditionally uses the 2D $s$-wave scattering length $a_2$ [69, 70]. Parametrization of quasi-2D scattering in terms of $a_2$ is important for relating ultracold-atom experiments to the body of theoretical models formulated for purely 2D systems.

Quasi-2D interaction is parametrized via $a_{3D}$, $l_z$, and $q$ since $f_{2D}$ is a function of those variables. The wavefunction structure (16), which is common for both pure 2D and quasi-2D collisions, opens the way to parametrize quasi-2D collisions using $a_2$. Since the only difference between 2D and quasi-2D is which scattering amplitude ($f_{2D}$ or $f_{Q2D}$) to substitute into (16), the solution of the equation

$$f_{2D}(q, a_{3D}, l_z) = f_{Q2D}(q, a_2)$$

(19)

gives the $a_2$ value for parameterizing the quasi-2D collision [50].

In the limit $q \to 0$ this approach yields the known expression [68]

$$a_2 \simeq 2.96 l_z e^{-\sqrt{\pi/u_{3D}}},$$

(20)

where, as one may note, no discontinuity appears at resonance ($a_2 = 2.96 l_z$ as $a_{3D}$ jumps from $-\infty$ to $+\infty$). In a many-body problem, the momentum $\hbar q$ differs from 0 and may be estimated as $\hbar q = \sqrt{2u_{3D}}$ from the chemical potential $\mu$ that does not include the two-body binding energy [50]. This estimate is exact for deeply degenerate weakly interacting Fermi gases where the colliding particles are on the Fermi surface ($q = k_F$).

In this section and further, $a_2$ is related to the 3D scattering parameters in the limit $r_0 \to 0$. The role of a finite interaction range is discussed in [71].

There is an alternative way of finding the corresponding value of the 2D scattering length and binding energy, which are respectively denoted as $\tilde{a}_2$ and $\tilde{E}_b$ for this case. The calculation is based on [45]:

$$\frac{l_z}{\tilde{a}} = \int_0^\infty \frac{du}{\sqrt{8\pi w^2}} \left(1 - \frac{e^{-w|\tilde{E}_b|/\hbar \omega}}{\sqrt{(1 - e^{-\omega l_z})/2u}}\right),$$

(21)

from where $\tilde{E}_b$ is found. Subsequently $\tilde{a}_2$ is obtained from $\tilde{E}_b = -4\hbar^2/(me^2\tilde{a}_2^2)$.

For low binding energies $|\tilde{E}_b| \ll \hbar \omega$, (20) shows that the effective 2D scattering length $a_2$ may be controlled by changing either $a_{3D}$ or $l_z$. The 3D scattering length $a_{3D}$ may be tuned using magnetically controlled Fano–Feshbach resonances [74].

Here the fundamentals of the Fano–Feshbach resonance are reviewed using $^6$Li atoms as an example. Since the focus is on controlling $a_{3D}$, the 3D kinematics is discussed. In the $s$-wave approximation, the two colliding atoms have to be in orthogonal internal states, e.g. $|1\rangle$ and $|2\rangle$ for $^6$Li. The appearance of the resonance and tunability of the scattering length is illustrated in figure 5. At the interaction of two univalent atoms, the valence electrons are in a superposition of the triplet state of the pair, shown by the gray dashed line, becomes

$$\tilde{V}_{3D}(r) = \tilde{V}_{3D}(r)$$

which is on controlling the bound and free state from each other, any desired scattering length, the singlet-channel bound-state energy (black dashed line) than a scattering resonance appears: in the case of zero binding, an approximate formula may be used to the triplet and singlet state of the electrons. These potentials are shown in figure 5. The external magnetic field $B$ shifts the zero level of the triplet-state kinetic energy since the state has a large magnetic moment, about $2\mu_B$. If the energy of the unbound state of the pair, shown by the gray dashed line, becomes equal to the singlet-channel bound-state energy (black dashed line) than a scattering resonance appears: in the case of zero kinetic energy, the scattering length diverges to $\infty$. By detuning the bound and free state from each other, any desired scattering length between $-\infty$ and $\infty$ may be obtained. Near the resonance, an approximate formula may be used to the $s$-wave scattering length of two unbound particles:

$$a_{3D}(B) = a_{bg} \left(1 + \frac{\Delta}{B - B_0}\right),$$

(22)

where $B_0$ and $\Delta$ are the center location and the width of the resonance respectively, while $a_{bg}$ is the background scattering
length stemming from the triplet channel alone. For the majority of experiments discussed below, the Feshbach resonance with parameters \( B_0 = 832 \) G, \( \Delta = 262 \) G, and \( a_{\text{Bo}} = -1580 \) \( a_0 \) (\( a_0 \) is the Bohr radius) [75] is employed. Note that at such a large magnetic field, the triplet-state part of the electronic spin dominates in the state of the pair of atoms [1] and [2] (see formulas (1) and (2)) since the coefficients \( \sin \theta \) are small. The singlet part of the state of electronic spins, despite being small, is principal for the coupling of the channels \( V_{\text{singlet}}(r') \) and \( V_{\text{triplet}}(r') \), tunability of the interactions, and appearance of the resonance.

In a many-body system, the Fermi atoms are joined into diatomic molecular bosons by tuning the interactions. For this purpose, the magnetic field smoothly changes from larger values, where the bounds state is above the energy of the free state in the triplet channel (as in figure 5), to smaller values. On the bosonic side of the resonance \( (B < B_0) \), at sufficiently large detuning from the resonance, a Bose condensate of molecules appears.

4. Degeneracy detection and thermometry

The form of the trapped gas density profile serves as a source of information about the temperature. In figure 6(a) one may see an example of the linear-density profile \( n_1(x) = \int n(x, y) dy \). Absorption images similar to that of figure 4(c) effectively provide the integration along the line of sight \( y \). The \( n_1(x) \) data are obtained by integrating such an image along \( z \) which, for this particular case, resulted in averaging over 30 nearly identical clouds [50]. At high temperature, \( T > E_F \), the gas is nearly classical, and the density profile is close to the Gaussian distribution. At \( T = 0 \) the edges of the distribution \( n_1(x) \) are sharper and, in case of a noninteracting Fermi gas, the numerical density distribution has the form

\[
n_1(x) = \begin{cases} \frac{n_0}{\pi \rho} \left( 1 - \frac{x^2}{\rho^2} \right)^{3/2} & \text{for } x < R_c, \\ 0 & \text{for } x > R_c, \end{cases}
\]

(23)

where \( R_c \equiv \sqrt{2E_F/m\omega_m^2} \) is the Thomas–Fermi radius. At an arbitrary temperature \( T \) the numerical density profile of a nearly ideal Fermi gas is expressed as

\[
n_1(x) = -\frac{m \omega_m}{2 \pi \hbar} \left( \frac{T}{\hbar \omega_m} \right)^{3/2} \text{Li}_{3/2} \left( -\frac{\omega_m^2}{4T^2} x^2 \right),
\]

(24)

where \( \text{Li}_{3/2} \) is the polylogarithm function of the order \( 3/2 \) and the chemical potential is to be found self-consistently, from the constraint \( N = \int n_1(x) dx \). By fitting this profile to the data of figure 6(a) one may estimate the temperature \( T \). In the experiment the temperatures \( T \lesssim 0.1E_F \) are accessible, which is \( \approx 10 \) nK in absolute units.

For comparison, a trial fit by a Gaussian curve is shown in figure 6(a). One may see that the Gaussian deviates both at the edges and center. For interacting Fermi and Bose gases, the numerical density profiles differ from profile (24). Despite that, since at \( T = 0 \) the dependence of the chemical potential on the density is nearly linear \( (\mu \propto n) \) [76], the closeness of the density profile to (23) firmly indicates a deep degeneracy

and a smallness of the temperature with respect to \( E_F \) and chemical potential \( \tilde{\mu} = \mu + |E_0|/2 \).

Quantitative thermometry of interacting gases is provided by fitting the density distribution at the cloud edge where the gas is nearly classical [77, 78] as shown in figure 7. The interactions are taken into account in various orders by means of the virial expansion of the phase space density \( n_{\lambda_{\text{BB}}} \), where \( \lambda_{AB} = \sqrt{2\pi \hbar^2/(mT)} \) is the thermal de Broglie wavelength:

\[
n_{\lambda_{AB}} = \ln(1 + e^{\mu/T}) + 2b_2 e^{2\mu/T} + 3b_3 e^{3\mu/T} + \ldots
\]

(25)

Here, \( n(\rho) \) is the 2D numerical density profile, \( \mu(\rho) = \mu_0 - V(\rho) \) is the local chemical potential including the binding energy, while \( \mu_0 \) is the trap-center value. The first term of the right-hand side corresponds to the noninteracting gas, while the second and third term, respectively, account for the two- and three-body interactions between the fermions. The expansion parameter is the fermionic fugacity \( e^{\mu/T} \), which has to be \( \ll 1 \) in order to keep the expansion valid. The second virial coefficient may be found within the Beth–Uhlenbeck approach [77–80]:

\[
b_2 = e^{E_0/T} - \int_{-\infty}^{\infty} \frac{\exp[-e^2/(2\pi)] ds}{\pi^2 + [s - \ln(2\pi E_0/T)]^2},
\]

(26)

while the third virial coefficient \( b_3 \) has been calculated in [81, 82]. The fit parameters are \( T \) and \( \mu_0 \).

The virial expansion up to the third order has been used for thermometry in [77]. The gas in this experiment was in the fermionic and strongly interacting regimes; in all cases the magnetic fields corresponded to the values on the Fermi side of the respective 3D Feshbach resonance. Figure 7 shows the fit of the data by the third-order virial expansion.

In [78], the virial expansion is used up to the second order. Importantly, it is shown that the expansion (26) up to the second order is applicable in the whole crossover. In the Bose regime expression (26) converges to the ideal-gas Boltzmann distribution since the dimer–dimer interactions are not accounted for. For fugaicities that are less than but comparable to 1, the second-order virial thermometry estimates the temperature from below [78].

In the deeply degenerate regime, the virial-expansion methods may contain errors related to the shrinking of the cloud edge that is used for the fit, as well as due to the breakdown
of the local density approximation needed for calculating the virial coefficients.

5. Mean-field approaches to the Fermi-to-Bose crossover

5.1. Fermi-to-Bose crossover in 3D

In a dilute 3D Fermi gas (with substantially weak attraction between particles $|a_{3D}|k_F \ll 1$) the BCS critical temperature is given by the famous Gor’kov–Melik-Barkhudarov formula [83]:

$$T_{c}^{\text{BCS}} = 0.28 \varepsilon_F \exp \left( \frac{-1}{|f_0|} \right),$$  \hspace{1cm} (27)

where $a_{3D}$ is the 3D gas parameter $|f_0| = 2|a_{3D}|k_F/\pi$. Correspondingly the BEC critical temperature yields:

$$T_{c}^{\text{BEC}} \simeq 0.2\varepsilon_F \left( 1 + 1.3a_{3D}^2 - n_{3D}^{1/3} \right),$$  \hspace{1cm} (28)

where $n_{3D}$ is the numerical density of the dimers and the non-trivial corrections to the Einstein formula [84], according to Proko’ev, Svistunov, and colleagues, are governed by the dimer–dimer scattering amplitude [85, 86]. In exact calculations of Petrov et al [87] and Brodsky et al [88, 89] the dimer–dimer 3D scattering length is $a_{3D}^2 = 0.6a_{3D}$.

5.2. Fermi-to-Bose crossover in 2D

In the 2D case on the mean-field level the BCS–BEC crossover was first addressed by Miyake [91] (see also important papers [92, 93]) on the basis of the self-consistent Leggett theory [7, 8]. It is important to mention here also the Fisher–Hohenberg theory [94] for weakly interacting repulsive 2D Bose gas, which used the Hartree–Fock ansatz for the chemical potential. According to Miyake the mean-field BCS critical temperature for the BCS domain of extended Cooper pairs is given by

$$T_{c}^{\text{BCS}} = \sqrt{2\varepsilon_F |E_b|}.$$  \hspace{1cm} (29)

Correspondingly, for low temperatures $T \ll |E_b| \ll 2\varepsilon_F$ the chemical potential

$$\mu \simeq \varepsilon_F - |E_b|/2.$$  \hspace{1cm} (30)

Figure 6. (a) Linear numerical density profile $n_1(x)$. Dots: data for $a_{1D}\sqrt{n} = 55$, $B = 1400$ G, $\omega_x/(2\pi) = 94$ Hz, $\omega_y/(2\pi) = 141$ Hz, $\omega_z/(2\pi) = 6020$ Hz, and $N = 660$. Fits of the Thomas–Fermi (24) and Gaussian profiles to the data are, respectively, shown by the solid and dashed curves. (b) 2D numerical density distribution in the $xy$-plane, $n(\rho)$, found from $n_1(x)$. The dots are the data. The curve is the fit of the parabola $n(\rho) = n - \rho^2 n''/2$ for finding the central numerical density $n \equiv n(\rho = 0)$.

Figure 7. Fit of the in-trap density with the third-order virial expansion. Blue points are the average of 20 experimental images. The red solid line is the virial expansion fit. Reprinted figure with permission from [77], Copyright (2016) by the American Physical Society.

Figure 8. Phase diagram of the BCS–BEC crossover in the resonance Fermi gas in 3D (numerical calculations for $T_c$ and $\mu$ versus $1/(k_F a)$) for the first iteration to the self-consistent $T$-matrix approximation) according to [66, 90].

The typical phase diagram of the BCS–BEC crossover in 3D resonance Fermi gas is presented in figure 8. The BCS domain corresponds to the $s$-wave scattering length $a_{3D} < 0$ and the chemical potential $\mu > 0$. In the BEC domain the situation is vice versa: $a_{3D} > 0$, $\mu < 0$. In figure 8 we plot dimensionless critical temperature as a function of the inverse gas parameter $1/(k_F a_{3D})$. 
The mean-field BEC critical temperature according to Fisher and Hohenberg [94] is given by
\[ T_{c}^{\text{BEC}} = \frac{\varepsilon_{F}}{4 \ln(1/f_{2-2})}, \] (31)
where according to Petrov, Baranov, and Shlyapnikov [95] the dimer–dimer scattering amplitude in 2D case reads
\[ f_{2-2} \sim \frac{1}{\ln(1.6(E_{b}/\varepsilon_{F})}. \] (32)

In the Fermi-liquid expansions for normal state and for the BCS–BEC crossover in the superfluid state we should use the 2D gas parameter \( f_{0} \), which is usually defined as the s-wave scattering amplitude for the absolute value of the total energy \( |E| \) equal to 2\( \varepsilon_{F} \). Thus we obtain \( f_{0}(|E| = 2\varepsilon_{F}) \), or correspondingly
\[ f_{0}(k_{F}a) \sim -\frac{1}{2 \ln(k_{F}a)}. \] (33)

This is a standard definition of the 2D gas parameter both in Fermi gas (with strong repulsion) and Fermi gas with (weak and strong) attraction. If we introduce a convenient variable
\[ g \equiv \frac{1}{2f_{0}(k_{F}a)} = \frac{1}{2} \ln\left(\frac{E_{0}}{2\varepsilon_{F}}\right) = -\ln(k_{F}a) \] (34)

and plot the dimensionless critical temperature \( T_{c}/\varepsilon_{F} \) as a function of \( g \), then \( T_{c}^{\text{BEC}}/\varepsilon_{F} \sim 1/[4 \ln(6.4g)] \), \( T_{c}^{\text{BCS}}/\varepsilon_{F} \sim 2 \exp(g) \), and we reproduce the same qualitative picture of the BCS–BEC crossover in figure 9 as the one presented in figure 8. Note that at low temperatures \( T \ll \varepsilon_{F} \leq |E_{b}|/2 \) the one-particle chemical potential in the BEC domain at first (in the intermediate situation \( |E_{b}| \geq 2\varepsilon_{F} \) and for shallow bound states) has the same form as in (30) and contains the factor \( \varepsilon_{F} \). However for deep levels in 2D (when \( \varepsilon_{F} \ll \hbar^{2}/(mr_{0}^{2}) \ll |E_{b}| \sim |U| \) the chemical potential (neglecting corrections of the order of \( \varepsilon_{F}/W \), \( W/|E_{b}| \)) is just \( \mu \sim -|E_{b}|/2 \). This result does not take into account weak dimer–dimer scattering in 2D Fermi gas. When we take this repulsive dimer-dimer interaction into account we finally obtain
\[ \mu \sim -\frac{|E_{b}|}{2} + \frac{\pi n}{2m}f_{2-2}. \] (35)

where \( n \) is the 2D numerical density of the bosonic dimers. Correspondingly, the bosonic chemical potential is
\[ \mu_{\text{Bose}} = 2\mu + |E_{b}| \simeq \frac{\pi n}{m}f_{2-2}. \] (36)
as it should be.

Note also that in the BEC regime \( |E_{b}| > 2\varepsilon_{F} \) there is one more characteristic temperature governed by the Saha formula for the thermodynamic equilibrium [96] in the process \( A + B \leftrightarrow AB \). This crossover temperature is given by \( T_{s} = |E_{b}|/\ln(E_{b}/2\varepsilon_{F}) \). For the temperatures \( T_{s}^{\text{BEC}} < T < T_{s} \) we are in the strong pseudogap regime corresponding to an interesting phase of a normal (nonsuperfluid) dilute gas of composed bosons [90, 97–99]. The pseudogap phenomenon is also pronounced in the intermediate coupling case \( |E_{b}| \sim 2\varepsilon_{F} \) [99].

The pressure value directly indicates the transition between the Bose and Fermi regimes. Whenever the total pressure of the two spin components \( P \) is close to that of an ideal Fermi gas, \( P_{\text{ideal}} \), the system is fermionic, while \( P/P_{\text{ideal}} < 1 \) indicates the bosonic regime. The ratio \( P/P_{\text{ideal}} \) has been measured in [50] at different interaction values at deep degeneracy. The preparation of the ultracold gas of \(^{6}\text{Li} \) in two equally populated spin states and tuning the interactions is discussed in sections 2 and 3, respectively. The confinement of a series of 2D clouds is shown in figure 4(b), and the images of the trapped clouds are in figure 4(c).

For observing the Fermi-to-Bose crossover, the system properties at the center are particularly interesting since the gas is mostly degenerate there. Besides that, all known models are constructed for uniform systems. Therefore, for quantitative comparison, one needs a measurement at the mostly uniform part of the cloud, which the cloud center is. While the image of the trapped gas (figure 4(c)) is taken along the planes of motion and the centers are not seen directly, it is still possible to obtain the pressure and the ratio \( P/P_{\text{ideal}} \) for the cloud centers.

The measurement is based on analyzing the linear density profiles \( n_{1}(x) \), such as the one in figure 6(a), and the use of force balance equation
\[ \nabla_{\perp}P(x,y) = -2n(x,y)\nabla_{\perp}V(x,y,0), \] (37)
where \( V(x,y,0) \) is the trapping potential and \( n(x,y) \) is the numerical density per spin state or density of the bosons. Integrating (37), one finds that for a harmonic potential the central pressure is independent of the interaction: \( P = m\omega_{\perp}^{2}N/\pi \), where \( N \) is the atom number per spin state. Further, the pressure is normalized to the local Fermi pressure, i.e. to the ideal Fermi gas pressure at \( T = 0 \) and at the same numerical density as in the cloud center, \( n \simeq n(0,0) \), \( P_{\text{ideal}} = 2\pi n^{2}\hbar^{2}/m \).

The particle number \( N \) is found by integrating \( n_{1}(x) \). The numerical density profile \( n(x,y) \) is fully reconstructed from the integral \( n_{1}(x) \) owing to the cylindrical symmetry of the potential (3) in stretched coordinates \( (x,y) \equiv y\omega_{\perp}/\omega_{\parallel} \). The inverse Abel transform yields
where $\tilde{\rho} \equiv \sqrt{x^2 + y^2}$. Profile $n(\tilde{\rho})$ is displayed in figure 6(b). It is known that the inverse Abel transform emphasizes noise, especially on the small scale. To avoid the noise in the $n(\tilde{\rho})$ distribution, the small-scale noise in profile $n_1(x)$ is filtered out prior to the substitution into the formula (38). The fit of the parabola $n(\tilde{\rho}) = n - \tilde{\rho}^2 n''/2$ to the data near the origin yields the sought quantity $n$.

The measured normalized pressure at the cloud center, $P_2/P_{\text{ideal}}$, versus the interaction parameter is shown in figure 10. The measurement destroys the quantum-mechanical state of the system. Therefore, for each measurement the system is prepared from the beginning.

The qualitative form of the normalized pressure dependence on the coupling parameter tells us that the system crosses over from fermionic statistics (in the right-hand side of figure 10) to bosonic statistics (in the left-hand side of figure 10). At $a_2\sqrt{n} \gg 1$ the pressure value closely approaches the Fermi pressure. Such high pressure may not be explained by thermal effects since the temperature is $\ll \varepsilon_F$. The pressure measured in the Bose regime is due to a weak repulsion between diatomic molecular bosons. The center of the strong-interaction region $a_2\sqrt{n} = 1$ approximately corresponds to a pressure drop by a factor of 2.

In the Fermi region $a_2\sqrt{n} \gg 5$, the temperature is in the interval $T = (0.02\ldots0.15)\varepsilon_F$. Meanwhile, at $a_2\sqrt{n} = 5$ the temperature of pair breaking is expected at 0.01$\varepsilon_F$ [95], and even lower at higher values of $a_2\sqrt{n}$. Therefore, in the Fermi regime, the superfluid phase is most probably absent and the system is in the Fermi-liquid state.

7. Critical phenomena in the 2D Fermi-to-Bose crossover

7.1. BKT corrections to the mean-field results in dilute gases

In the BEC domain of local pairs or molecules for the dilute 2D Fermi gas ($k_{\text{F}2}a \ll 1$) the mean-field $T_\text{c}^\text{BEC}$ according to Fisher–Hohenberg theory is only slightly different from the exact BKT critical temperature:

$$\frac{T_\text{c}^\text{BEC} - T_\text{c}^\text{BKT}}{T_\text{c}^\text{BEC}} \sim f_{2-2} \ll 1.$$  \hspace{1cm} (39)

In the BCS domain of extended pairs in the weak-coupling case $|E_0| \ll \varepsilon_F$ the mean-field $T_\text{c}^\text{BCS}$ [100] is close again to exact critical temperature $T_\text{c}^\text{BKT}$:

$$\frac{T_\text{c}^\text{BCS} - T_\text{c}^\text{BKT}}{T_\text{c}^\text{BCS}} \sim T_\text{BCS} / \varepsilon_F \sim \sqrt{|E_0| / \varepsilon_F}.$$  \hspace{1cm} (40)

For the intermediate coupling case $|E_0| \sim \varepsilon_F$ the difference between exact and mean-field critical temperatures becomes substantial.

Note that several layers or slabs of dilute gases may be created, such as shown in figure 4(b). Therefore, the possible role of the Josephson (tunneling) coupling [101, 102] between the layers also has to be considered. This coupling can modify the one-particle spectrum from $\varepsilon(p_\perp) = p_\perp^2/(2m)$ to $\varepsilon(p_\perp) = p_\perp^2/(2m) + J[1 - \cos(p_d)d]$, where $d$ is an interlayer distance. The inclusion of $J$ rapidly makes the spectrum quasi-3D and suppresses fluctuations [103]. An experimental way of suppressing Josephson coupling is to make $J$ smaller than the temperature or the inverted duration of the experiment.

Note also that typical for BKT transition creation and destruction of vortex–antivortex pairs as well as specific form of pair correlation function takes place close to exact critical temperature $T_\text{c}^\text{BKT}$, while for low temperature $T \ll T_\text{c}^\text{BKT}$ these features are much less pronounced and mean-field approaches are more applicable.

7.2. Cooper-pair condensation in the strongly interacting regime

A signature of Cooper-pair condensation has been observed in [51]. An equally populated mixture of $^6\text{Li}$ atoms in two spin states has been studied near the $B = 832$ G $3\text{D}$ Feshbach resonance, with $N = 4\ldots5 \times 10^4$ atoms per spin state. The kinematics is restricted by confining in a disc-shaped trap with the anisotropy ratio $\omega_z/\omega_\perp \approx 310$, which gives $E_F/(\hbar\omega_z) = 0.91\ldots1.02$ and, for the considered couplings, $\mu < \hbar\omega_z$. The authors have measured the 2D density distribution $n(\tilde{\rho})$ in the trap (figure 11(a)) and also after abrupt extinction of the potential along the $z$-direction and subsequent evolution in a radial potential (figure 11(b)). All the measurements are performed at the lowest attainable temperatures. The in-trap images (figure 11(a)) show smooth shrinking of the gas as the atom–atom attraction is increased, while a more dramatic effect is observed in the images taken after the evolution (figure 11(b)).

The images preceded by the evolution (figure 11(b)) may be interpreted as Bose condensation of atomic pairs. Upon
preparation of the gas at the reported interaction strength, the magnetic field is quickly shifted to the smallest value on the Bose side of the resonance \( B = 692 \text{ G}, \sqrt{2}l / a_{3D} = 7.11 \). The shift is performed on the time scale \( \ll 1 / \omega_L \). During this shift, the radial density profile does not change, while the atoms bind into small pairs, at least if they were bound prior to the shift. Immediately after the change in the interaction strength, the optical confining potential is instantaneously extinguished, which removes the axial and weakens the radial confinement. As a result the cloud freely expands in the \( z \)-direction and evolves in the radial potential with the frequency \( \omega_{rad} \). The cloud is photographed after the evolution over the quarter of the radial period, \( \tau / 4 \equiv \pi / (2\omega_{exp}) \). The images are shown in figure 11(b). The authors suggest that the density distribution imaged after the evolution corresponds to the momentum distribution right before the release. In this case, the sharp peaks signify Bose condensation of the pairs of atoms.

The interpretation of figure 11(b) depends on the hypothesis that the expansion is collisionless. In this case the radial harmonic potential initially acts as a lens and, as a result, the density distribution after the evolution for \( t = \tau / 4, n(\rho, t = \tau / 4) \), coincides with the initial momentum distribution, \( \tilde{n}(k, t = 0) \), up to a constant factor. Weakening the interaction by ramping the magnetic field right before the expansion helps in reduction of the collision rate.

Superfluid hydrodynamics \[42, 104\], however, is an alternative scenario of the expansion. Such scenario requires nonzero superfluid order parameter \( \psi \) but does not require condensation. In such expansion, nearly all momentum is released along the tightly confined direction, with almost no momentum released radially. Hydrodynamic expansion, therefore, may also produce sharp peaks similar to those of figure 11(b). Therefore, the experiment \[51\] may be observing onset of superfluidity or quasicondensation rather than condensation. The same group also points out that the temperature, at which the peaks appear, coincides with the temperature of the BKT transition, which they measure \[105\].

Superfluid hydrodynamics is also discussed in section 9.

7.3. BKT transition

Proliferation of vortices and the change in the correlation decay law are the two hallmarks of the BKT transition \[12, 13\] from a superfluid to a normal state. While in a weakly interacting Bose gas the appearance \[106\] and pairing \[107\] of vortices has been detected, such effects have not been observed in the Bose-to-Fermi crossover, where the interactions between Bose molecules are typically stronger than in the atomic Bose gases.

Switching from a power-law to exponential decay of the phase correlation in response to the rising temperature has been recently observed \[105\] in the Bose-to-Fermi crossover. The experimental setup and techniques in many respects are the same as in \[51\], which is discussed in the previous section. The first order correlation function \( g_1(\rho) \) is derived from the measured momentum distribution \( \tilde{n}(k) \):

\[
 g_1(\rho) = \int \tilde{n}(k) e^{i k \cdot \rho} d^3 k.
\]  

(41)

The momentum distribution \( \tilde{n}(k) \) is obtained by the focusing technique as explained in the previous section. The result of such measurement of the correlation function is shown in figure 12. All the measurements reflected in this figure are performed at the same interaction strength, \( \ln(k_F a) \approx -0.5 \), belonging to the strongly interacting regime between the Bose and Fermi asymptotes. The only parameter which is varied in the experiment, is the dimensionless temperature \( T = T / T_{BEC}^0 \) where \( T_{BEC}^0 = E_F \sqrt{3} / \pi \) is the ideal-gas Bose-condensation temperature. One may see that at lower temperatures the decay is algebraic \[g_1(r) \propto r^{-\eta(T)}\] with a temperature-dependent exponent \( \eta(T) \), while in a hotter gas the correlation function decays exponentially. The transition from exponential to power low decay signals the onset of superfluidity. Interestingly, the observed value of the exponent \( \eta = 0.6-1.4 \) is well above the values \( \eta < 0.25 \) expected for a homogeneous gas. Inhomogeneity may indeed cause an increase of \( \eta \) compared to uniform gases as found within quantum Monte Carlo simulation \[105\].

The measurement of the correlation decay rests upon the assumption of ballistic cloud expansion after abrupt extinction of
the potential along $z$. In the superfluid phase, the expansion may be going along an alternative scenario of superfluid hydrodynamics. This would complicate the relation between $n(\rho, \tau/4)$ and the sought $\tilde{n}(k, 0)$ as well as the derivation of $g_1(\rho)$ from the data. In the normal phase, however, the expansion should be nearly ballistic due to a low collision number. Therefore, the power-law decay of the measured $g_1(\rho)$ cannot be obtained in a normal gas and does indeed indicate the onset of superfluidity.

Recent calculations of the algebraic-decay exponent have accounted for the trapped-gas inhomogeneity [108, 109] and contribution of the normal fraction [109]. These calculations have shown $\eta$ values close to the measured ones [109]. This agreement may be regarded as evidence against hydrodynamic expansion, because such expansion would have changed the apparent $\eta$ value, making it different from the actual in situ quantity.

8. Equation of state in the crossover

8.1. The pressure in 2D Fermi-to-Bose crossover

At low temperatures the pressure $P = 2n\mu - E$, where we used thermodynamic identity $\mu = (1/2)\partial E/\partial n$ (note that the total density is $2n$). Let us consider first the BCS case and the intermediate case. Taking into account that in a 2D gas with a parabolic spectrum $\varepsilon_F = 2\pi n/m$ and total energy density (energy per unit surface) $E = n\varepsilon_F - nE_b$, we obtain for the total pressure at the mean-field level

$$P \simeq n\varepsilon_F.$$  \hfill (42)

It is a remarkable result which contains only the Pauli pressure.

Note that, the Fermi-gas expansion in the normal (non-superfluid) state both for the total energy and pressure gives the corrections proportional to $f_0(k_Fa)$ and $f_0^2(k_Fa)$, where in terms of density per spin state $n$: $k_F = \sqrt{4\pi n}$.

Therefore deep in the BCS domain instead of a simple expression (42) for the pressure we should have

$$P = n\varepsilon_F \left[ 1 + c_1 f_0(k_Fa) + c_2 f_0^2(k_Fa) \right],$$  \hfill (44)

where $c_1$ and $c_2$ are numerical constants. Note that first-order corrections in $f_0$ to the expression (42) were obtained in [72].

At the same time deep in the BEC domain the total energy is given by:

$$E \simeq -n|E_b| + \frac{\pi \hbar^2 n^2}{m} f_{2-2}.$$  \hfill (45)

where we again neglect the small corrections of the order of $\varepsilon_F/W$ and $W/|E_b|$. Hence for the pressure we obtain

$$P \simeq \frac{\pi \hbar^2 n^2}{m} f_{2-2}.$$  \hfill (46)

A dilute 2D Fermi gas with $s$-wave interaction is one of the few cases where the Fermi-liquid parameters were calculated from first principles [69, 70]. Using the results of [70] one may find the pressure of the interacting normal (non-superfluid) Fermi gas up to the $f_0^2$ term:

$$P = 1 + 2f_0 + (7 - 4\ln 2)f_0^2,$$  \hfill (47)

where $f_0$ is given by (33), while $P_{\text{ideal}} = 2\pi \hbar^2 n^2/m$ and coincides with (42).

8.2. Compressibility and sound velocity in the 2D BCS–BEC crossover

Evaluation of the pressure in the 2D BCS–BEC crossover allows us to calculate the speed of sound $c$ (or equivalently the inverse compressibility $k^{-1} \sim c^2$). Using the well-known thermodynamic identities $c^2 \sim \partial P/\partial \rho$, where $\rho = 2n m$ is the mass density, and $dP = 2n d\mu$, we obtain at low temperatures

$$c^2 = \frac{1}{2m} \frac{\partial P}{\partial n} = \frac{n}{m} \frac{\partial \mu}{\partial n},$$  \hfill (48)

where $\mu$ is the chemical potential. Thus from the expression (44) for the pressure in the BCS domain we obtain for the sound velocity

$$c^2 \simeq \frac{\varepsilon_F^2}{2} \left[ 1 + c_1 f_0 + c_2 f_0^2 \right],$$  \hfill (49)

where $\varepsilon_F = \hbar k_F/m$ is the Fermi velocity. Analogously deep in the BEC domain from (46) we have

$$c^2 \simeq \frac{\varepsilon_F^2}{4} f_{2-2} \left[ 1 + O(f_{2-2}) \right].$$  \hfill (50)

It is important to emphasize that the negative values of the sound velocity squared (or of the inverse compressibility) signal the instability of the ground state towards some kind of phase separation (see, for example, [110–112]).

8.3. Scale invariance and scale anomaly

The Hamiltonian of a 2D uniform gas with pair interactions may generally be taken in the form

$$\hat{H} = \sum_i -\frac{\hbar^2}{2m} \Delta_i + \sum_{i<j} V(\rho_i - \rho_j).$$  \hfill (51)
In the case of the contact potential $V(\rho') \propto \delta^2(\rho')$ the Hamiltonian is invariant under the dilatation transformation $\rho_i \to \lambda \rho_i$ [113]. Such a transformation only changes the Hamiltonian as $H \to H/\lambda^2$, which implies the scale invariance of the solutions.

An external trapping potential breaks the scale invariance. For a symmetric trap $m_2c^2\rho^2/2$, however, the system remains symmetric with the SO(2,1) symmetry [113]. Such symmetry uniquely defines the frequency of the breathing mode in the harmonic trap. Regardless of the interaction strength or particle statistics, the mode frequency is $2\omega$. For the zero-temperature gas the local EoS $\mu \propto n$ and $P \propto n^2\bar{\epsilon}$ is expected [114, 115].

The scale invariance, however, may be broken by regularization of the $\delta^2(\rho')$ potential [113]. The regularization also breaks the SO(2,1) symmetry and the invariance of the breathing mode frequency. On the basis of Monte Carlo simulation of the EoS [25], the frequency of the breathing mode has been calculated [114]. The amount of the scale-anomalous shift of the breathing mode frequency is shown in figure 13. The value of the shift has also been calculated in [116, 117]. Since the anomaly is related to regularization of the interactions, the shift is largest where the interactions are the strongest. The predicted shift is large enough to be detectable in ultracold-atom experiments.

The measurement of the breathing mode frequency [76] did not show any obvious shift with respect to the scale-invariant prediction $2\omega$. The conditions for the calculations and the experiment differ in two respects. First, the EoS has been calculated for zero temperature ($T = 0$) [25], while the temperature $T = 0.42\bar{c}_p$ is reported in the experiment. The frequency shift at $T = 0$ is the upper bound for the finite-$T$ shift [115]; the lower bound, however, is unknown. Second, the shift is calculated for a purely 2D atom–atom interaction potential, while the atoms interact via a 3D potential. While 2D and 3D scattering parameters are expressed via each other as discussed in section 3.3, the regularization procedure and related anomalous shift may be different from the purely 2D case.

### 8.4. Testing zero-temperature models

Available zero-temperature EoSs may be tested using the measurements of the pressure [50] shown in figure 10 and the measurements of the chemical potential [78] displayed in figure 14.

The analytic models for the zero-temperature gas at the Fermi-to-Bose crossover are presented by (47) and (46), for the fermionic and bosonic asymptotes, respectively. Formula (47) describes the gas within the Fermi-liquid model [70]. Strictly speaking, the pairing (superfluid) gap is present at $T = 0$. The gap, however, does not alter the pressure value significantly since the gap is small and the pressure is continuous at the transition. In figure 10, the Fermi-liquid prediction is shown by the dashed curve. This analytical result agrees with the fixed-node diffusion Monte Carlo numerical calculation [25] and auxiliary-field Monte Carlo [27], but differs from the measurements. The discrepancy with the measurement is significant because within the Fermi-liquid theory, the pressure is counted from the Fermi pressure rather than from zero. The difference between the experimental data and the Fermi-liquid theory [70] may be due to the mesoscopic character of the experimental system [50] since the calculation is done for infinitely extended Fermi liquid.

In the Fermi regime, the data of figure 14 are in contradiction with the data of figure 10 discussed above. In 14, the data are consistently below the auxiliary-field Monte Carlo [27] (orange curve), while the data of figure 10 are consistently above.

For a 3D gas, the analytic mean-field approach of Leggett’s type [45, 90] qualitatively correctly predicts the pressure and chemical potential at any interaction level. Upon inclusion of the fluctuations, this model comes into reasonable agreement with the experiments, including the Bose regime, where the original mean-field model has the largest discrepancy.

For a 2D system, the use of a Leggett-type approach is limited. While a two-body quantity (the two-fermion binding energy) shows qualitatively correct behavior increasing with the interaction growth [92], many-body quantities such as the pressure, in contrast, are independent of the interaction. In particular, such a model predicts that the Fermi pressure must be present in the Bose limit, at least in the range $|E_b| \sim 2\bar{c}_p$. The addition of fluctuations into the mean-field model yields qualitatively correct dependence of the pressure on the interaction [21], shown by the green curve in figure 10. Indeed, according to the Ginzburg–Levanyuk criterion [22, 23], the reduction of the dimensionality is accompanied by the increasing role of fluctuations (see also [31]).

Models with qualitatively correct description of the Fermi-to-Bose crossover appeared only in the recent few years, such as the model [21]. These models include quantum diffusion Monte Carlo simulations [25, 28], self-consistent $T$-matrix [29] (see also earlier papers [91, 97, 98]), finite-temperature lattice Monte Carlo [26], and auxiliary-field Monte Carlo [27].
Predictions of some of these models are shown in figure 10, while the latter model is also displayed in figure 14. Results of all the models are given for $T = 0$.

For the Bose regime, the analytic prediction is given by (46). The leading term in the pressure of Bose molecules should be the same as for the point-like bosons: $P_{\text{Bose}} = -P_{\text{ideal}}/[4\ln(k_0a_{1,2}^{-2})]$ [72]. Here $a_{1,2}^{-2}$ is the 2D scattering length for molecule–molecule collisions, which may be related to the respective 3D scattering length $a_{3D}^{-2} = 0.6a_{3D}^{2D}$ [87–89] by equating the scattering amplitudes $f_{2D}(2q \to 0, a_{2D}^{-2}) = f_{2D}(2q \to 0, a_{3D}^{2D}, l_c/\sqrt{2})$. The low-energy limit $2\mu \ll \hbar\omega_c$ yields

$$a_{1,2}^{-2} \simeq 2.09 l_c \exp \left( -\frac{\pi}{2} \frac{l_c}{a_{3D}^{2D}} \right).$$  

(52)

As a result, in the limit of unmodified 3D interactions $a_{3D}^{2D} \ll l_c$, one obtains

$$P_{\text{Bose}} \simeq P_{\text{ideal}} \frac{a_{3D}^{2D}}{l_c \sqrt{8\pi}}.$$  

(53)

which is shown by the dashed line in figure 15. We see that in the Bose regime $P/P_{\text{ideal}}$ scales as $a/l_c$. This scaling is applicable in a wide range of $\ln(k_0a_2)$.

8.5. Finite-temperature equations of state

Recent progress in virial thermometry [77, 78], discussed in section 4, made it possible to take data for testing finite temperature EoSs. The thermometry has been accompanied by the measurement of the chemical potential $\mu$. As a result, a measured local density can be expressed as a function of $\mu$ and $T$. A related dimensionless value, namely phase space density $n\lambda_{3D}^2$, may be expressed as a function of dimensionless variables $\mu/T$ and $|E_0|/T$ [77]. Another convenient dimensionless value is $n/n_0$, where $n_0$ is the local density of an ideal Fermi gas at the same values of $\mu$ and $T$:

$$n_0 = \frac{1}{\lambda_{3D}^2} \ln(1 + e^{|E_0|/T}).$$  

(54)

Combination $n/n_0$ also includes the phase space density times a function of fugacity.

The normalized density, $n/n_0$, is represented in figure 16. The logarithm of the local fugacity, is taken as the horizontal axis. Interestingly, the graph is non-monotonic, which is qualitatively different from the similar measurements in 3D [77].
The authors mention that data are taken above the superfluid transition. Therefore, no superfluidity-related features are expected to be seen in figure 16. This measurement is reported for the fermionic and strongly interacting regime [77].

The measurements [78] of the same quantities are reported in figure 17. Here the measurements cover all the interaction regimes, including where the system is mostly bosonic. In both papers [77, 78], the data are compared to the Luttinger–Ward [29] and lattice quantum Monte Carlo [26] calculations. These two models are known to make the predictions that differ in the strongly interacting regime but agree at the other couplings.

9. Criteria of two-dimensionality revisited

For comparison with 2D models, it is important that the experimental system remains 2D. In this section, the effect of the interactions on the kinematic dimensionality is discussed.

Even a small two-body interaction mixes states of the z-motion of an atom. From the standpoint of a single atom, therefore, even small two-body interaction brings about some deviation from the 2D kinematics. In the absence of many-body interactions, however, the center-of-mass kinematics of the pair is 2D since the confining potential is nearly harmonic. The whole system, therefore, remains kinematically 2D despite any level of two-body interactions. For example, the bosonic asymptote of a Fermi-to-Bose crossover is a 2D monatomic. The whole system, therefore, remains kinematically 2D right before the release, of pairs with the wave function $\psi$. From the standpoint of a single atom, the gas is assumed to be kinematically 2D right before the release, and the boson number per spin state. Within the model, the size along the $z$-direction ($\Delta z$) is calculated for fixed expansion time $t = 600 \mu s$ ($\omega_z/(2\pi) = 5.15$ kHz and $\omega_z/(2\pi) = 26.6$ Hz) for different $N$, number of atoms per spin state. The result is shown in figure 18 together with the data of [64]. For the initial condition, the gas is assumed to be kinematically 2D right before the release, and the predictions are expected to be seen in figure 16.
systems, the experiment may have physics similar to layered electronic systems with in-plane magnetic field, where the Landau diamagnetism is absent. A strong enhancement of the critical temperature has been predicted [103] in such materials. Prospectively, 2D spin-imbalanced systems are home for yet unobserved new superfluid/superconductive phases. One such phase is the Fulde–Ferrell–Larkin–Ovchinnikov superfluid [123, 124] where the fermions pair and condense into states with finite momenta of the pair. In 3D the parameter range for such phase is vanishingly small [125], while in 2D the effect may be less fragile [126]. Phases with the Kohn–Luttinger pairing [127–129] is another interesting example. In this case, the reduction of dimensionality down to 2D together with the spin imbalance strongly increases the critical temperature [130]. Ultimately, the ultracold atoms may be the best system for detecting such effects due to high tunability, purity, and various techniques of direct observations.

Nonuniform systems, such as quantum gases in parabolic traps, are prone to separation into a paired phase and a fully polarized phase. In 3D and 1D, this phenomenon is qualitatively different: in the 3D gases, the paired phase is in the trap center [131], while in a 1D gas the center is occupied by a polarized fraction and the paired atoms are pushed to the edges [132].

The phase separation in a quasi-2D gas has been observed by J E Thomas and co-workers [122]. The images of atoms in the two spin-states are shown in figure 19. The images are taken along the plane of the motion, therefore, the density profiles are the result of the integration along one dimension. The flat center in the column density difference (blue dashed lines) is a signature of a fully balanced core. Qualitatively, therefore, the balanced phase appears in the center similar to that in 3D.

Further studies of the phase separation [52] have shown that there are no discontinuities in the polarization or density.
profiles at the transition. This is qualitatively different from the 3D phase separation [133] where the transition is of the first order. It is suggested [52] that the absence of discontinuities in 2D is due to the enhanced role of fluctuations.

The third dimension may be important for the imbalanced 2D gases. A minority-spin atom moving in the field of the majority atoms may have higher lying axial harmonic-oscillator states admixed to its state of motion.

Expansion of the imbalanced gas into the free 3D space upon the instantaneous trap extinction shows a clear bimodal structure in the plane of the initial 2D motion [52]. Such bimodal profiles are shown in figure 20. The bimodal structure is clearly visible on the Bose side of the 3D Feshbach resonance, where two atoms may remain bound upon removal of the tight confinement. For dimer molecules, the density profile in the 2D plane after long expansion time may be closely related to the velocity profile before the release [52]. This bimodal structure is interpreted as Bose condensation of dimer molecules [52]. Interestingly, the bimodal structure of the expanded gas is visible both in the phase-separation regime as well as outside of this regime. In a weakly repulsive atomic Bose gas, bimodal structure has been observed without condensation [134].

While the weak trapping potential in the xy-plane is a driver of the phase separation, the tendency of forming the population-balanced core hides other interesting phenomena, such as the Fulde–Ferrell–Larkin–Ovchinnikov superfluidity and the $p$-wave pairing driven by the Kohn–Luttinger mechanism [127–129]. Such effects may be seen in a system with a uniform trapping of the atoms in the xy-plane.

10.2. Pressure and polaron effects in the 2D imbalanced case

If we consider the imbalanced Fermi gas with $n_{\uparrow} > n_{\downarrow}$, then the total energy density (energy per unit surface) reads in the BCS domain

$$E = \frac{n_{\uparrow} \varepsilon_{F\uparrow}}{2} + \frac{n_{\downarrow} \varepsilon_{F\downarrow}}{2} - n_{\uparrow}|E_b|.$$  (56)

Correspondingly the pressure in this case at low temperatures is given by

$$P = n_{\uparrow} \varepsilon_{F\uparrow} + n_{\downarrow} \varepsilon_{F\downarrow} - f(E),$$  (57)

where $\varepsilon_{F\uparrow} \equiv \hbar^2 k_{F\uparrow}^2/(2m_\uparrow)$, $\varepsilon_{F\downarrow} \equiv \hbar^2 k_{F\downarrow}^2/(2m_\downarrow)$ are the Fermi energies of the ‘up’ and ‘down’ spins and the function $f(E)$ depends upon the total energy. It is reasonable to assume that the bare ‘up’ and ‘down’ masses $m_\uparrow = m_\downarrow$ are equal. However, the effective ‘up’ and ‘down’ masses can be different due to many-body fermionic polaron effects [111, 112]. In the BCS domain at low temperatures:

$$f(E) \simeq \frac{1}{2} (n_{\uparrow} \varepsilon_{F\uparrow} + n_{\downarrow} \varepsilon_{F\downarrow}),$$  (58)

and hence the pressure is approximately

$$P \simeq f(E).$$  (59)

An interpolating model has been developed in a form applicable both in the Fermi and Bose domain, as well as in the intermediate region of strong interactions [112, 135]. In these papers, the energy density is taken in the form

$$E = \frac{n_{\uparrow} \varepsilon_{F\uparrow}}{2} + \frac{n_{\downarrow} \varepsilon_{F\downarrow}}{2} + n_{\downarrow} E_p(\downarrow),$$  (60)

where $E_p(\downarrow)$ is a negative interaction energy per polaron:

$$E_p(\downarrow) \equiv y_m \left( \frac{\varepsilon_{F\downarrow}}{|E_b|} \right)^2.$$  (61)

In (61), the following analytic approximation for the function $y_m$ is used [136]:

$$y_m \left( \frac{\varepsilon_{F\downarrow}}{|E_b|} \right) = \frac{2}{\ln \left( 1 + \frac{2 \varepsilon_{F\downarrow}}{|E_b|} \right)}.$$  (62)

Deep in the BEC domain $E_p(\downarrow) \simeq -|E_b|$ in agreement with (56).

The total pressure of the two spin components may be expressed according to [122, 135] as

$$P = n_{\uparrow} \mu_{\uparrow} + n_{\downarrow} \mu_{\downarrow} - E,$$  (63)

where $\mu_{\uparrow} = \partial E/\partial n_{\uparrow}$ and $\mu_{\downarrow} = \partial E/\partial n_{\downarrow}$ are the partial chemical potentials of the respective spin components.

It is important to select the correct geometry of the quasi-2D trap to avoid phase separation which is always present in 3D. Note that in the imbalanced case the phase separation is connected with the negative partial compressibilities (negative sound velocities squared for ’up’ and ’down’ spins or, respectively, with the negative values of the derivatives $\partial \mu_{\uparrow}/\partial n_{\uparrow}$, $\partial \mu_{\downarrow}/\partial n_{\downarrow}$) [110–112].
11. Summary and unresolved problems

Fermi-to-Bose crossover in 2D ultracold gases has been discussed. While the interactions at close distances are governed by 3D potentials, the many-body systems can be parametrized in terms of parameters conventional for pure 2D description of many-body systems. A number of quantitative measurements are in agreement with 2D models, nevertheless, criteria of two-dimensionality are still under discussion.

There are a lot of open questions (both theoretical and experimental) in the subjects ranging from the experimental observation of the Kohn–Luttinger superfluidity [127–129] in strongly imbalanced 2D Fermi gases to the Fulde–Ferrell–Larkin–Ovchinnikov phases [123, 124] in case of a small imbalance and so on. Note that a recent experimental discovery of the p-wave superconductivity induced by proximity effect in single-layer graphene (on electron-doped oxide superconductor) [137] is building a bridge between an interesting physics of Dirac semimetals and hexagonal 2D optical lattices with the emerging Dirac points [138].

Acknowledgments

The authors are thankful to Igor Boettcher, Selim Jochim, and Puneet Murthy for valuable discussions. A T acknowledges the financial support by the program of the Presidium of Russian Academy of Sciences ‘Fundamental problems of nonlinear dynamics’ and Russian Foundation for Basic Research (grants No. 14-22-02080-06-m, 15-02-08464, 15-42-02638). M Yu K gratefully acknowledges support from the Basic Research Program of the National Research University Higher School of Economics and RFBR grant 17-02-00135-a.

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