Abstract. A method is presented for computing minimal answers of the form $\bigvee A$ in disjunctive deductive databases under the disjunctive stable model semantics. Such answers are constructed by repeatedly extending partial answers. Our method is complete (in that every minimal answer can be computed) and does not admit redundancy (in the sense that every partial answer generated can be extended to a minimal answer), whence no non-minimal answer is generated. For stratified databases, the method does not (necessarily) require the computation of models of the database in their entirety. Compilation is proposed as a tool by which problems relating to computational efficiency and the non-existence of disjunctive stable models can be overcome. The extension of our method to other semantics is also considered.

Keywords: Disjunctive deductive databases, minimal answers, perfect models, disjunctive stable models, cyclic sets, strong covers, compilation.

Introduction

A propositional disjunctive deductive database [Gr86, He88, Lb92] consists of logical rules of the form

$$A_1 \land A_2 \land \ldots \land A_h \land \neg A_{h+1} \land \ldots \land \neg A_{h+r} \rightarrow B_1 \lor B_2 \lor \ldots \lor B_k$$

where each $A_i, B_j$ is a predicate. In such databases, answers (to queries) can themselves be disjunctive, thus an answer is a disjunction of predicates $\bigvee A$ which is logically implied (under the chosen semantics) by the database. Of course if $A$ is non-minimal (i.e., there is some set $B \subset A$ such that $\bigvee B$ is logically implied), then $A$ is redundant, and we are therefore only interested in the computation of minimal answers.
Minimal answer computation is also of wider interest as a result of the fact that many questions relating to minimal answers (e.g., determining whether a given predicate belongs to some minimal answer) are $\Sigma_2^p$-complete ([Ei93, Jo96, Mi82]), and naturally occurring problems residing at $\Sigma_n^p$ ($n \geq 2$) in the polynomial-time hierarchy are rare [St77, Wa86, Wr77].

Of course one obvious way to compute minimal answers is to compute all answers, and then eliminate non-minimal answers by subsumption. Methods for achieving this are well known. For example, in a positive database we can compute answers using the hyper-resolution operator:

\[
\frac{\bigwedge_{i \leq r} A_i \rightarrow \bigvee B; A_i \lor \bigvee P_i (i \leq r)}{\bigvee B \cup \bigcup_{i \leq r} P_i}
\]

and this may be achieved by simple forward application, backward application [Ra89, Lb92], or by a combination of backward and forward [Jo98]. In each case however there is the possibility that non-minimal answers will be generated. For example if our database consists of the rules \{A \lor B, A \rightarrow C, B \rightarrow D, C \rightarrow D\}, then in order to generate the minimal answer D, forward application of the hyper-resolution operator above must generate a non-minimal answer (i.e., A \lor D, B \lor D or C \lor D). Indeed it is fairly evident that a query answering method will always risk generating non-minimal answer if it is based upon the construction of (a structure which in effect encodes) a proof of the answer being generated: clearly the existence of such a proof in no way precludes the possibility of a proof of some smaller disjunction using other rules within the database.

The approach presented in [Bra95] to query processing is to apply unfolding (similar to a middle-out application of the above hyper-resolution operator), which transforms a non-positive database into an equivalent set of conditional facts (i.e., rules whose bodies contain only negative atoms), and from which the set of answers can be generated using hyper-resolution. Subsumption is again required to generate minimal answers.

An alternative approach is to generate answers by computing models of the database. For example in [Ya94, Fe95, Fe95a], bottom-up methods for computing minimal, perfect and disjunctive stable models are presented using the notion of a model tree. Answers can then be derived by picking a predicate from each of the branches (i.e., models) in the tree [Fe95a], but again subsumption would be required to compute minimal answers. A further disadvantage is that such an approach requires the computation of all models (of the appropriate type) in their entirety.

An alternative approach to query processing, again using model trees, is presented in [Ya96, Ya02], where it is observed that top-down query processing for positive databases can be achieved by the application of a model generating procedure to the dual
database. It is not clear that this approach can be extended to non-positive databases, and as noted in [Ya96], answer minimality is not addressed.

The purpose of this paper is to present a query answering method which, in contrast to existing approaches, generates only minimal answers. For the most part we work under the disjunctive stable model semantics [Pr91], although the extension of our results to other semantics is considered.

Answers will be generated iteratively $A_1$, $A_1 \lor A_2$, $A_1 \lor A_2 \lor A_3$, ..., $A_1 \lor A_2 \lor \ldots \lor A_n$, where, for each disjunction $A_1 \lor A_2 \lor \ldots \lor A_r$ generated, we guarantee that $A_1 \lor A_2 \lor \ldots \lor A_r$ can indeed be extended to a minimal answer by the construction of a suitable set of cyclic strong covers [Jo99a], each of which can be viewed as a partial (disjunctive stable) model of the database. Our method is complete (in the sense that every minimal answer can be generated), and does not admit redundancy (in the sense that every partial answer generated can be extended to a minimal answer), whence no non-minimal answer is generated.

In Section 1 we review some background and terminology, and in particular we present the conditions (in terms of disjunctive stable model membership), under which a disjunction of predicates can be extended to a minimal answer. The computation of disjunctive stable models in their entirety is clearly undesirable, and, under certain circumstances, this can be avoided using the concept of a cyclic strong cover, which we re-introduce in Section 2. Section 2 also presents results which demonstrate that we may view cyclic strong covers as partial disjunctive stable models.

In Section 3 we focus on stratified databases. For such databases, disjunctive stable models coincide with perfect models, whence cyclic strong covers may be viewed as partial perfect models. More importantly, cyclic strong covers can always be extended to full perfect models, and hence provide a characterisation of reasoning under the perfect model semantics. This then allows a characterisation of partial minimal answers in terms of cyclic strong covers. Using this, a method for constructing minimal answers is derived in terms of repeated extension of cyclic strong covers, and moreover this can be achieved without necessarily constructing perfect models in their entirety.

In Section 4 we consider unstratified databases. For such databases, total cyclic strong covers coincide with disjunctive stable models, but the possible non-existence of disjunctive stable models means that (non-total) cyclic strong covers cannot always be extended to full disjunctive stable models. This suggests that partial minimal answers can only be witnessed as such using total cyclic strong covers, and (hence) that a direct extension of the methods of Section 3 to unstratified databases requires the (undesirable) computation of disjunctive stable models in their entirety.

In Section 5 we show that this problem can be addressed by partitioning the database into extensional and intensional components, and pre-processing (compiling) the
construction of cyclic strong covers within the intension. This greatly simplifies, and hence reduces the cost of, the run-time computation (which then takes place within the extensional database), this saving being of particular importance given the computational complexity of the problem. An important property of our compilation is that it does not need to be repeated following updates to the extensional database.

In Section 6 we question whether the techniques of the present paper might, from the viewpoint of computational efficiency, be usefully combined with other techniques that are themselves known to generate non-minimal answers. We briefly consider the adaptation of our methods to other semantics in Section 7, and in Section 8 consider the issues that arise when lifting our methods to the first order level. Our conclusions and suggestions for further research are presented in Section 9. An extended worked example is presented in Appendix A, and an alternative form of pre-processing is presented briefly in Appendix B.

§1. Minimal answers

In this section we review some background and terminology, and in particular we state the conditions (in terms of disjunctive stable model membership), under which a disjunction of predicates can be extended to a minimal answer.

1.1 Notation. Throughout we assume that \( \mathcal{L} \) is a finite propositional language (i.e., a finite set of predicates). A literal is a predicate (a positive literal) or its negation (a negative literal), and we will use \( A, B, C, \ldots, P, Q, \ldots \) to denote arbitrary sets of literals. \( Q^- = \{ P \in \mathcal{L} | \neg P \in Q \} \), \( Q^+ = \{ P \in \mathcal{L} | P \in Q \} \) and \( \overline{Q} = \{ \neg K | K \in Q \} \). \( Q \) is total iff \( Q^- \cup Q^+ = \mathcal{L} \), and consistent iff \( Q^- \cap Q^+ = \emptyset \).

Throughout \( T \) will denote a disjunctive deductive database in \( \mathcal{L} \) consisting of rules \( C \) of the form \( A_1 \land A_2 \land \ldots \land A_h \land \neg A_{h+1} \land \ldots \land \neg A_{h+r} \rightarrow B_1 \lor B_2 \lor \ldots \lor B_k \), where each \( A_i, B_j \) is a predicate and \( k > 0 \). We may assume without loss of generality that if \( r > 0 \), then \( h > 0 \).

\[ \text{antec}(C) = \{ A_1, A_2, \ldots, A_h \} \] denotes the set of antecedents of \( C \), \( \mathcal{N}(C) = \{ A_{h+1}, A_{h+2}, \ldots, A_{h+r} \} \), and \( \text{conseq}(C) = \{ B_1, B_2, \ldots, B_k \} \) denotes the consequent of \( C \). \( T \) is said to be positive iff \( \mathcal{N}(C) = \emptyset \) for each \( C \in T \).

1.2 Definition. A set \( M \subseteq \mathcal{L} \) is a model of \( C \) (written \( M \models C \)) iff \( \text{antec}(C) \subseteq M \) and \( M \cap \mathcal{N}(C) = \emptyset \) implies that \( \text{conseq}(C) \cap M \neq \emptyset \). \( M \) is a model of \( T \) (written \( M \models T \)) iff \( M \models C \) for each \( C \in T \).
A total consistent set of literals identifies a truth value for each predicate, which in turn allows us to determine truth values for rules. Given a consistent set of literals \( \mathcal{C} \), we let \( T/\mathcal{C} \) denote those rules in \( T \) whose predicates are given a truth value by \( \mathcal{C} \).

1.3 Definition. If \( \mathcal{C} \) is a consistent set of literals, let \( T/\mathcal{C} = \{ C \in T \mid \text{conseq}(C) \cup \text{antec}(C) \cup \mathcal{N}(C) \subseteq \mathcal{C}^+ \cup \mathcal{C}^- \} \).

1.4 Definitions.

(a) If \( C \) is a rule, let \( \text{pos}(C) = \bigwedge \text{antec}(C) \rightarrow \bigvee \text{conseq}(C) \), i.e., \( \text{pos}(C) \) is formed from \( C \) by removing the negative literals from the body of \( C \).

(b) If \( N \subseteq \mathcal{L} \), then the Gelfond-Lifschitz transformation [Ge88] is given by \( T|_gN = \{ \text{pos}(C) \mid C \in T, \mathcal{N}(C) \cap N = \emptyset \} \). Notice that \( T|_gN \) is positive, and can therefore be interpreted straightforwardly using the minimal model semantics.

(c) If \( M \subseteq \mathcal{L} \), then \( M \) is a disjunctive stable model of \( T \) [Pr91] iff \( M \) is a minimal model of \( T|_gM \).

(d) If \( \Phi \) is a formula in \( \mathcal{L} \), then we write \( T \models \Phi \) iff \( \Phi \) is true in every disjunctive stable model of \( T \).

(e) If \( \mathcal{A} \subseteq \mathcal{L} \), then \( \bigvee \mathcal{A} \) is a minimal answer in \( T \) iff \( T \models \bigvee \mathcal{A} \) and there is no proper subset \( \mathcal{B} \subset \mathcal{A} \) such that \( T \models \bigvee \mathcal{B} \).

Disjunctive stable models are a straightforward generalisation of stable models for non-disjunctive databases [Ge88]. Note that every disjunctive stable model of \( T \) is a minimal model of \( T \), and that for positive databases the converse holds.

Given a minimal answer \( \bigvee \mathcal{A} \), we will for the sake of brevity also refer to the set \( \mathcal{A} \) as a minimal answer. If \( \bigvee \mathcal{A} \) is a minimal answer and \( A \in \mathcal{A} \), then there must be some disjunctive stable model \( M \) of \( T \) such that \( M \cap \mathcal{A} = \{ A \} \) (for otherwise \( T \models (\mathcal{A} - \{ A \}) \)). Conversely if \( A \) belongs to a disjunctive stable model \( M \), then by the minimality of \( M \) we must have that \( T \models A \lor \bigwedge (\mathcal{L} - M) \), whence if \( \mathcal{A} \subseteq \{ A \} \cup (\mathcal{L} - M) \) is a minimal answer, then \( \emptyset \neq M \cap \mathcal{A} \subseteq \{ A \} \), and hence \( A \in \mathcal{A} \). This then yields the following which is a direct analogue of a result originally presented (for the minimal model semantics) by Minker.

1.5 Theorem [Mi82]. A predicate \( A \) belongs to some minimal answer (in \( T \)) iff \( A \) belongs to some disjunctive stable model of \( T \).

If we are to try to construct minimal answers as a sequence \( A_1, A_1 \lor A_2, ..., \), then Theorem 1.5 dictates that each such \( A_i \) needs to belong to some disjunctive stable model \( M_i \). The following theorem indicates that we also need to consider the
relationship between the models $M_i$.

1.6 Theorem. A set of predicates $\{A_i|i \leq r\}$ is contained in a minimal answer iff for each $i \leq r$ we may find a disjunctive stable model $M_i$ of $T$ such that

(a) $M_i \cap \{A_j|j \leq r\} = \{A_i\}$, and

(b) $T \models \bigvee_{i \leq r} A_i \vee \bigvee_{i \leq r} (\mathcal{L} - M_i)$.

Proof ($\rightarrow$). Suppose that $A_1 \lor A_2 \lor \ldots \lor A_{r+s}$ is a minimal answer, then for each $i \leq r$ we may find a disjunctive stable model $M_i$ such that $M_i \cap \{A_j|j \leq r+s\} = \{A_i\}$. If $M$ is a disjunctive stable model of $T$ with $M \nvdash A_1 \lor A_2 \lor \ldots \lor A_r$, then there is some $j > r$ such that $A_j \in M$, where $A_j \in \bigcap_{i \leq r} (\mathcal{L} - M_i)$.

($\leftarrow$). Let $A \subseteq \{A_i|i \leq r\} \cup \bigcap_{i \leq r} (\mathcal{L} - M_i)$ be a minimal answer, then for each $i \leq r$, $\emptyset \neq M_i \cap A \subseteq \{A_i\}$, whence $A_i \in A$. ■

§2. Cyclic strong covers

Theorem 1.6 characterises partial minimal answers in terms of disjunctive stable models. Ideally however we would prefer not to have to construct models of the database in their entirety. With this in mind, this section re-introduces cyclic strong covers [Jo99a] which we will show function (to some extent) as partial disjunctive stable models, and which allow an appropriate re-formulation of Theorem 1.6. Strong covers provide the notion of model-hood, and cyclicness captures minimality.

2.1 Definition [Jo99a]. Let $Q$ be a consistent set of literals in $\mathcal{L}$. A **strong cover** of $Q$ (in $T$) is a consistent set of literals $C \supseteq Q$ such that for each $C \in T$

$$\text{conseq}(C) \subseteq C \implies \text{antec}(C) \cup \text{N}(C) \cap C \neq \emptyset.$$  

Notice that the above definition suggests a top-down construction of strong covers. Clearly if $C$ is a strong cover, then $\mathcal{L} - C^+ \models T$. Conversely if $M \subseteq \mathcal{L}$ then $M$ is a model of $T$ iff $\overline{M} \cup (\mathcal{L} - M)$ is a strong cover in $T$.

In [Jo96] we introduced the notion of a cyclic tree\(^\dagger\) which facilitates reasoning about minimal, perfect and disjunctive stable models. We first present an example to motivate the following definition of such trees, and then detail their essential properties.

\(^\dagger\)Throughout this paper we will use what is referred to in [Jo96] as unfactored cyclic trees.
Further examples, motivation, and details of the top-down construction of such trees are discussed at length in [Jo96, Jo98, Jo98a, Jo99, Jo99a].

2.2 Example. Suppose that $T$ consists of the following rules:

1. $Q_2 \land Q_3 \land \neg R_1 \rightarrow Q_1 \lor Q_5$
2. $Q_1 \land \neg R_2 \rightarrow Q_2$
3. $S_2 \land \neg R_3 \rightarrow Q_3$
4. $S_3 \rightarrow Q_1 \lor Q_2 \lor Q_6$
5. $S_2 \lor R_5$
6. $S_1 \rightarrow Q_3 \lor Q_2$
7. $S_3 \lor R_7$
8. $Q_5 \land R_1 \rightarrow Q_2$

and we wish to determine whether $Q_1$ lies in some disjunctive stable model of $T$.

Suppose that $Q_1$ lies in the disjunctive stable model $M$, then by the minimality of $M$ we may find a rule $C \in T$ such that $\mathcal{N}(C) \cap M = \emptyset$ (i.e., $\text{pos}(C) \in T_{|_g} M$) and $M - \{Q_1\} \not\models \text{pos}(C)$. There are only two possibilities for $C$, namely rules 1 and 4. Suppose we guess that $C$ is rule 1, then $\{Q_1, Q_2, Q_3\} \subseteq M \subseteq \mathcal{L} - \{Q_5, R_1\}$. We will represent this application of rule 1 using the “rule node” $rn_1$ in the tree $T_1$ (Figure 2.2(i)). (Only $Q_2$ and $Q_3$ (i.e. the antecedants) are depicted, since we wish to examine these predicates further.)

Suppose now that we apply the same argument to $Q_2$. If $C$ is a rule in $T$ such that $\mathcal{N}(C) \cap M = \emptyset$ and $M - \{Q_2\} \not\models \text{pos}(C)$, then (since $\{Q_1, Q_2, Q_3\} \subseteq M \subseteq \mathcal{L} - \{Q_5, R_1\}$) $C$ must be rule 2, thus yielding the tree $T_2$ (Figure 2.2(i)), and the further constraint that $\{Q_1, Q_2, Q_3\} \subseteq M \subseteq \mathcal{L} - \{Q_5, Q_1, R_2\}$.

The left hand branch of $T_2$ forms a “cycle”. There is no point working with $Q_1$ or $Q_2$ alone (since we have already done so), thus we look for a rule $C \in T$ such that $\mathcal{N}(C) \cap M = \emptyset$ and $M - \{Q_1, Q_2\} \not\models \text{pos}(C)$, the only candidate being rule 4. Rule 7 then terminates the branch (since rule 7 has no antecedants), thus yielding $T_3$ (Figure 2.2(ii)), and the new constraint $\{Q_1, Q_2, Q_3, S_3\} \subseteq M \subseteq \mathcal{L} - \{Q_5, R_1, R_2, Q_6, R_7\}$.

![Figure 2.2(i).](image-url)
We thus move on to examine $Q_3$: if $M - \{Q_3\} \not\models pos(C)$, then (given the existing constraints on $M$) $C$ must be rule 3. Rule 5 can then be used to handle $S_2$, and again this terminates the branch, yielding $T_4$ (Figure 2.2(ii)), and the constraint $\{Q_1, Q_2, Q_3, S_3, S_2\} \subseteq M \subseteq L - \{Q_5, R_1, R_2, Q_6, R_7, R_3, R_5\}$.

Thus we are then left with the problem of finding a disjunctive stable model $M$ satisfying the above constraint. This appears harder than our initial problem (i.e., of finding such a model containing $Q_1$). Note however that if $M$ is any (disjunctive stable) model of $T$ that is disjoint from $\{Q_5, R_1, R_2, Q_6, R_7, R_3, R_5\}$, then (using the rules in the tree) we may infer that $\{Q_1, Q_2, Q_3, S_3, S_2\} \subseteq M$. In particular, if we can show that $T \not\models \bigvee\{Q_5, R_1, R_2, Q_6, R_7, R_5\}$, then we may infer the existence of a disjunctive stable model containing $Q_1$.

Notice in the tree construction above, that it is the intention that each predicate lies in the intended model $M$. Each “rule node” is labelled with a rule $C \in T$ such that $\mathcal{N}(C) \cap M = \emptyset$, and if $rn_C$ has parent $n$, then there is a subset $\mathcal{P}$ of the predicates above $n$ such that $M - \mathcal{P} \not\models pos(C)$. Thus, $\text{antec}(C) \subseteq M - \mathcal{P}$, and $\emptyset = \text{conseq}(C) \cap (M - \mathcal{P}) = M \cap (\text{conseq}(C) - \mathcal{P})$. Since $M \models pos(C)$ we must have that $\text{conseq}(C) \cap \mathcal{P} \neq \emptyset$.

The following definition captures these features, and also provides a precise definition of the subset $\mathcal{P} = \text{CYC}(n)$ (the “cycle above $n$”) that we wish to work with, this choice being motivated by the properties (Theorem 2.4) of the resulting trees. [Jo96] compares this choice of $\mathcal{P}$ with a number of alternatives.
2.3 Definition [Jo96]. If $P$ is a predicate, then a cyclic tree for $P$ in $T$ contains rule nodes and predicate nodes, satisfying conditions (i) - (v) below.

(i) Each rule node $rn$ is labelled with a rule $C \in T$ (written $rn_C$). Each predicate node $n$ is labelled with a predicate $R \in \mathcal{L}$ (and we write $lab(n) = R$). We set $Pred(T) = \{lab(n) \mid n \text{ is a predicate node in } T\}$.

(ii) The root node (at the top of the tree) is a predicate node labelled with $P$.

(iii) If $rn_C$ is a rule node, then for each predicate $R \in \text{antec}(C)$, $rn_C$ has a child node labelled with $R$. $rn_C$ has no other child nodes. If $rn_C$ has parent $n$, then $\text{conseq}(C) \cap \text{CYC}(n) \neq \emptyset$ and $\text{antec}(C) \cap \text{CYC}(n) = \emptyset$, where $\text{CYC}(n) = \{lab(m) \mid m \geq n, \exists n'(n' \geq m, lab(n') = lab(n))\}$.

Let $\mathcal{O}(rn_C) = \text{conseq}(C) - \text{CYC}(n)$.

(iv) If $n$ is a predicate node, then $n$ is not a leaf node, and its child is a rule node satisfying (iii) above. If $n$ is not the root, then its parent node is a rule node $rn_D$ (with $lab(n) \in \text{antec}(D)$).

(v) $\mathcal{O}(T) = \bigcup \{\mathcal{O}(rn_C) \mid rn_C \text{ occurs in } T\}$ and $\mathcal{N}(T) = \bigcup \{\mathcal{N}(C) \mid rn_C \text{ occurs in } T\}$ are both disjoint from $Pred(T)$.

Let $\mathcal{S}(T) = \overline{Pred(T)} \cup \mathcal{O}(T) \cup \mathcal{N}(T)$.

Notice that the definition of cyclic trees is inherently top-down. It is easy to show that the size of cyclic trees is limited [Jo96, Section 5], in that every branch through a cyclic tree has length at most $|\mathcal{L}| \ast (|\mathcal{L}| + 1)/2$. Clearly if $T$ is a cyclic tree in $T$, then $T$ is also a cyclic tree in any superset of $T$. Notice also that if $C$ is a rule labelling some rule node in $T$, then each predicate appearing in $C$ appears in $Pred(T) \cup \mathcal{O}(T) \cup \mathcal{N}(T)$. (In the terminology of Definition 1.3, $C \in T/\mathcal{S}(T)$.)

The following theorem captures the essential properties of cyclic trees.

2.4 Theorem [Jo96, Jo99a].

(a) If the predicate $P$ belongs to some disjunctive stable model $M$, then we may find a cyclic tree $T$ for $P$ in $T$ such that $Pred(T) \subseteq M \subseteq \mathcal{L} - (\mathcal{O}(T) \cup \mathcal{N}(T))$.

(b) If $T$ is a cyclic tree in $T$ and $M \models T |_g (\mathcal{L} - \mathcal{N}(T)) \land \neg \bigvee \mathcal{O}(T)$, then $Pred(T) \subseteq M$.

(c) If $T$ is a cyclic tree in $T$ and $M \models T \land \neg \bigvee (\mathcal{O}(T) \cup \mathcal{N}(T))$, then $Pred(T) \subseteq M$.

Thus if $M$ is a disjunctive stable model of $T$, then we can find a set $\{T_i \mid i \leq m\}$ of cyclic trees in $T$ such that $\bigcup_{i \leq m} Pred(T_i) = M$ and $M \subseteq \mathcal{L} - \bigcup_{i \leq m} (\mathcal{O}(T_i) \cup \mathcal{N}(T_i))$, i.e., $\mathcal{L} - M \supseteq \bigcup_{i \leq m} (\mathcal{O}(T_i) \cup \mathcal{N}(T_i))$. $\overline{\mathcal{M}} \cup (\mathcal{L} - M)$ thus has the property identified in the following definition.

2.5 Definition [Jo99]. Let $C$ be a consistent set of literals, then $C$ is said to be cyclic
(in $T$) iff there is a set of cyclic trees $\{T_i \mid i \leq m\}$ in $T$ such that
(i) $C^- = \bigcup_{i \leq m} Pred(T_i)$, and
(ii) $C^+ \supseteq \bigcup_{i \leq m} (O(T_i) \cup N(T_i))$.

It is in fact possible to allow some pruning of the cyclic trees within the definition/
construction of a cyclic set (see [Jo99a, Theorem 4.1]) although we will not consider
this issue in this paper. Again note that if $C$ is cyclic in $T$, then $C$ is also cyclic in any
superset of $T$.

Where there is no ambiguity (or required emphasis), we will, for the sake of brevity,
refer to a cyclic set/strong cover in $T$ as simply cyclic/a strong cover.

Following on from Theorem 2.4 we see for example that if $C$ is cyclic, then any
model of $T|_g (\mathcal{L} - C^+) \land \neg \bigvee C^+$ will contain $C^-$. In particular, if $C$ is a total cyclic strong
cover with $M = C^- = \mathcal{L} - C^+$, then $M \models T$ (whence $M \models T|_g M$) and if $M^* \subseteq M$
with $M^* \models T|_g M$, then $M^* \models T|_g (\mathcal{L} - C^+) \land \neg \bigvee C^+$, whence $C^- = M \subseteq M^*$. Thus
$M = M^*$ and $M$ is a disjunctive stable model of $T$. Conversely, as indicated above,
if $M$ is a disjunctive stable model, then $\overline{M} \cup (\mathcal{L} - M)$ is a total cyclic strong cover,
whence we have part (a) of the following theorem.

2.6 Theorem [Jo99a].
(a) If $M \subseteq \mathcal{L}$, then $M$ is a disjunctive stable model of $T$ iff $\overline{M} \cup (\mathcal{L} - M)$ is a cyclic
strong cover in $T$.
(b) If $Q$ is a set of literals, then $T \models \bigvee Q$ iff there is no total cyclic strong cover of $Q$
in $T$.
(c) A consistent set of literals $C$ is cyclic in $T$ iff for each $P \in C^-$ there is a cyclic tree
$T$ for $P$ in $T$ such that $S(T) \subseteq C$.

Note that if $C$ is cyclic in $T$ and $T$ is a cyclic tree with $S(T) \subseteq C$, then for each
rule $C$ labelling a rule node in $T$ we have that $C \in T/S(T) \subseteq T/C$ and $C^- \models C$ (since
$\text{conseq}(C) \cap \text{Pred}(T) \neq \emptyset$). Thus $C$ is also cyclic in $\{C \in T/C \mid C^- \models C\}$. We can thus
characterise cyclicness in terms of disjunctive stable models.

2.7 Theorem.
(a) A consistent set of literals $C$ is cyclic in $T$ iff $C^-$ is a disjunctive stable model of
$\{C \in T/C \mid C^- \models C\}$.
(b) If $C$ is a strong cover in $T$, then $C$ is cyclic in $T$ iff $C^-$ is a disjunctive stable model
of $T/C$.

Proof (a). ($\rightarrow$). By the above remark, $C$ is a total cyclic strong cover in $\{C \in$
T/C | C− ⊨ C}. 

(→). Every predicate in \( \{ C ∈ T/C | C− ⊨ C \} \) appears in \( C^+ ∪ C− \). Thus by Theorem 2.6(a), \( C = C− ∪ ((C^+ ∪ C−) − C−) \) is cyclic in \( \{ C ∈ T/C | C− ⊨ C \} \), whence cyclic in \( T \).

(b). If \( C \) is a strong cover in \( T \), then \( C− ⊨ T/C \), whence the result follows trivially from part (a).

Theorems 2.6(a) and 2.7 indicate that cyclic strong covers can in some sense be regarded as partial disjunctive stable models. Because of the possible non-existence of disjunctive stable models, it is of course not the case that every cyclic strong cover can be extended to a disjunctive stable model. By Theorem 2.6(b), if \( C \) is a cyclic strong cover, then \( C \) may be extended to a total cyclic strong cover (i.e., a disjunctive stable model) iff \( T ∉ \bigvee C \), and by Theorem 2.4(c) this is the case iff \( T ∉ \bigvee C^+ \). We can also show (Theorem 2.9 below) that the disjunctive stable models extending a cyclic strong cover are precisely the disjunctive stable models of some reduced database, and thus that an inability to extend cyclic strong covers is simply the phenomena of the non-existence of such models in a different guise. Definition 2.8 below defines our reduced database.

2.8 Definition. If \( D \) is a strong cover in \( T \) and \( C \) is rule in \( T \), then let \( C_D \) be formed from \( C \) by replacing each predicate in \( D− \) by \( \text{TRUE} \), and each predicate in \( D^+ \) by \( \text{FALSE} \).

Note that if \( \text{conseq}(C) ∩ D− ≠ \emptyset \), then \( C_D \) reduces to \( \text{TRUE} \). Also, if \( \text{conseq}(C) ⊆ D^+ \) then \( \text{conseq}(C_D) \) reduces to \( \text{FALSE} \), but since \( D \) is a strong cover there must be some literal in the body of \( C \) which evaluates to \( \text{FALSE} \), whence \( C_D \) as a whole reduces to \( \text{TRUE} \).

In general \( C_D \) reduces to \( \text{TRUE} \) \( (C_D ≡ \text{TRUE}) \) iff \( \text{conseq}(C) ∩ D− ≠ \emptyset \) or \( \text{antec}(C) ∩ D^+ ≠ \emptyset \) or \( \text{N}(C) ∩ D− ≠ \emptyset \).

Let \( T_D = \{ C_D : C ∈ T, C_D ≠ \text{TRUE} \} \). Clearly a rule \( C^* \) is in \( T_D \) iff there is a rule \( C ∈ T \) such that

(i) \( \text{conseq}(C) ∩ D− = \emptyset \) and \( \text{conseq}(C^*) = \text{conseq}(C) − D^+ \),
(ii) \( \text{antec}(C) ∩ D^+ = \emptyset \) and \( \text{antec}(C^*) = \text{antec}(C) − D− \), and
(iii) \( \text{N}(C) ∩ D− = \emptyset \) and \( \text{N}(C^*) = \text{N}(C) − D^+ \)

and that under these conditions we have that \( C^* = C_D \). Notice that only predicates from \( L − (D^+ ∪ D−) \) appear in \( T_D \).

2.9 Theorem. Suppose that \( D \) is a cyclic strong cover in \( T \) and \( D− ⊆ M ⊆ L − D^+ \).

Then \( M \) is a disjunctive stable model of \( T \) iff \( M − D− \) is a disjunctive stable model of \( T_D \).
The proof is not especially instructive, and we thus delegate it to an accompanying technical report [Jo03a].

Since cyclic strong covers are themselves disjunctive stable models of some sub-database, the theorem above also allows us to characterise when one cyclic strong cover extends another.

2.10 Corollary. Suppose that $\mathcal{D}$ is a cyclic strong cover in $T$ and $\mathcal{G}$ is a consistent set of literals with $\mathcal{G} \supseteq \mathcal{D}$.

(a) $\mathcal{G}$ is a strong cover in $T$ iff $\mathcal{G} - \mathcal{D}$ is strong cover in $T_\mathcal{D}$.

(b) $\mathcal{G}$ is a cyclic strong cover in $T$ iff $\mathcal{G} - \mathcal{D}$ is cyclic strong cover in $T_\mathcal{D}$.

The proof is again delegated to the accompanying technical report [Jo03a].

As mentioned earlier, the (top-down) construction of cyclic trees is detailed in [Jo96]. The (top-down) construction of cyclic strong covers is detailed in [Jo98, Jo99, Jo99a], using the operators detailed below in Definition 2.11, and representing the cyclic strong covers generated as the branches of an extended deduction tree. A simple algorithm for the construction of cyclic strong covers can then be presented, based upon the use of two stacks, and operating in space that is quadratic in $|\mathcal{L}|$ (the size of the language) [Jo98]. Appendix A contains an example of a similar such tree construction.

2.11 Definition. Suppose that $\mathcal{Q}$ is cyclic, then a strong cover $\mathcal{C}$ of $\mathcal{Q}$ is said to be a constructible extension of $\mathcal{Q}$ iff we can find a sequence $\mathcal{Q} = \mathcal{Q}_0 \subseteq \mathcal{Q}_1 \subseteq \ldots \subseteq \mathcal{Q}_r = \mathcal{C}$ ($r \geq 0$) such that for each $i \leq r$ there is a rule $C_i \in T$ such that $\text{conseq}(C_i) \subseteq \mathcal{Q}_{i-1}$, $\mathcal{Q}_{i-1} \cap (\text{antec}(C_i) \cup \mathcal{N}(C_i)) = \emptyset$ and either

(i) $\mathcal{Q}_i = \mathcal{Q}_{i-1} \cup \{A_i\}$ for some $A_i \in \text{antec}(C_i)$, or

(ii) $\mathcal{Q}_i = \mathcal{Q}_{i-1} \cup S(T)$, where $T$ is a cyclic tree for some $A_i \in \mathcal{N}(C_i)$ in $T$ (such that $\mathcal{Q}_{i-1} \cup S(T)$ is consistent).

If $\mathcal{Q} = \mathcal{R} \cup \{\neg A\}$, where $\mathcal{R}$ is cyclic and $A \not\in \mathcal{R}^- \cup \mathcal{R}^+$, then a constructible extension of $\mathcal{Q}$ is formed by taking a constructible extension of $\mathcal{Q}' = \mathcal{R} \cup S(T)$, where $T$ is a cyclic tree for $A$ in $T$ such that $\mathcal{Q}'$ is consistent.

It will not prove necessary to define constructible extensions for sets more complex than those above. Theorem 2.6(c) implies that constructible extensions are themselves cyclic strong covers, and also yields the following corollary.

2.12 Corollary. Suppose that $\mathcal{D}$ is a cyclic strong cover of $\mathcal{Q}$, then we may find a constructible extension $\mathcal{C}$ of $\mathcal{Q}$ such that $\mathcal{C} \subseteq \mathcal{D}$.
The construction of cyclic strong covers can (by Theorem 2.6(a)) be amended to compute disjunctive stable models by implicitly adding the denial rules \( \{ P \land \neg P \rightarrow \text{FALSE} \mid P \in L \} \) to \( T \) (thus forcing all strong covers of \( \{ \text{FALSE} \} \) to be total). This approach to the computation of disjunctive stable models differs from that presented in [Fe95], in that it is top-down, and moreover does not require the database to be transformed within an extended language. Such computation of disjunctive stable models can be enhanced by the application of the results presented in Theorem 2.9 and Corollary 2.10.

§3. Stratified databases

For stratified databases, cyclic strong covers provide a characterisation of reasoning under the perfect model semantics [Jo99]. We will show that they can be viewed as partial perfect models, and we then use cyclic strong covers to define the notion of a cyclic state which provides a characterisation of partial minimal answers. Our method for constructing minimal answers is then presented in terms of an extension mechanism for cyclic states. As a result of the fact that cyclic strong covers can always be extended to full perfect models, this can be achieved without necessarily computing perfect models in their entirety.

3.1 Definitions. Throughout Section 3 we assume that \( T \) is stratified [Pr88], by which we mean that there is a level function \( \ell : L \rightarrow \mathbb{N} \) such that for each rule \( A_1 \land A_2 \land \ldots \land A_h \land \neg A_{h+1} \land \ldots \land \neg A_{h+r} \rightarrow B_1 \lor B_2 \lor \ldots \lor B_k \) in \( T \): (i) \( \ell(B_j) = \ell(B_1) \) for each \( j \leq k \), (ii) \( \ell(A_i) \leq \ell(B_1) \) for each \( i \leq h \), and (iii) \( \ell(A_{h+i}) < \ell(B_1) \) for each \( i \leq r \). We define \( \ell(C) = \ell(B_1) \).

If \( T \) is stratified, then a model \( M \) of \( T \) is perfect [Pr88] iff for each \( \alpha \), there is no set \( M' \subset \{ P \in M \mid \ell(P) = \alpha \} \) such that \( \{ P \in M \mid \ell(M) < \alpha \} \cup M' \models \{ C \in T \mid \ell(C) = \alpha \} \). It is well known [Pr91] that for stratified databases, perfect and disjunctive stable models coincide.

Theorem 1.6 dictates the requirements on a disjunction if it is to be extended to a minimal answer. As mentioned earlier however, we certainly do not wish to (have to) compute perfect models in their entirety, and in this respect the following result is key.

3.2 Theorem [Jo99]. Let \( Q \) be a set of literals, then \( T \models \bigvee Q \) iff \( Q \) has no cyclic strong cover in \( T \).
Theorem 2.6(a) dictated that perfect models coincide with total cyclic strong covers. If \( \mathcal{C} \) is a cyclic strong cover then (applying Theorem 3.2 with \( \mathcal{Q} = \mathcal{C} \)) we have that \( T \not\models \bigvee \mathcal{C} \), whence there is a perfect model \( M \) of \( T \) such that \( M \not\models \bigvee \mathcal{C} \), i.e., \( \mathcal{C} \subseteq \overline{M} \cup (\mathcal{L} - M) \). Thus (as in Section 1) cyclic strong covers may be regarded as partial perfect models, but crucially in this case they may always being extended to a full perfect model.

We now seek to show that we can use such partial models to compute minimal answers. Firstly we reformulate Theorem 1.6 to give a characterisation of partial minimal answers in terms of cyclic strong covers.

**3.3 Theorem.** A set of predicates \( \{ A_i | i \leq r \} \) is contained in a minimal answer iff for each \( i \leq r \) we may find a cyclic strong cover \( \mathcal{C}_i \) of \( \{ \neg A_i \} \cup \{ A_j | j \leq r, j \neq i \} \) such that \( T \models \bigvee_{i \leq r} A_i \lor \bigvee_{i \leq r} \mathcal{C}_i^+ \).

**Proof** (\( \rightarrow \)). By Theorem 1.6, let \( \{ M_i | i \leq r \} \) be a set of perfect models of \( T \) such that for each \( i \leq r \), \( M_i \cap \{ A_j | j \leq r \} = \{ A_i \} \), and \( T \models \bigvee_{i \leq r} A_i \lor \bigvee_{i \leq r} (\mathcal{L} - M_i) \). If \( \mathcal{C}_i = \overline{M_i} \cup (\mathcal{L} - M_i) \), then it is easy to see that \( \mathcal{C}_i \) satisfies the required conditions. (\( \leftarrow \)). By Theorem 3.2, we may, for each \( i \), pick a perfect model \( M_i \) of \( T \) such that \( M_i \not\models \bigvee \mathcal{C}_i \). It is then easy to check that the models \( M_i \) satisfy the conditions of Theorem 1.6.

We will now use the above result to construct minimal answers. At each stage in our construction we will generate a sequence of predicates \( (A_1, A_2, \ldots, A_r) \) which is witnessed (as representing a partial minimal answer) by a corresponding sequence of cyclic strong covers \( (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r) \) satisfying the conditions of Theorem 3.3.

Given such a partial answer (and witnessing cyclic strong covers) \( ((A_i, \mathcal{C}_i) | i \leq r) \), we then seek to extend this to a sequence of the form \( ((A_i, \mathcal{D}_i) | i \leq r + 1) \) satisfying the conditions of Theorem 3.3, and for which \( \mathcal{D}_i \supseteq \mathcal{C}_i \) for each \( i \leq r \). This extension will take place via two distinct phases.

Firstly we pick \( A_{r+1} \) such that \( \{ \neg A_{r+1} \} \cup \{ A_i | i \leq r \} \) has a cyclic strong cover \( \mathcal{F}_{r+1} \) and for each \( i \leq r \), \( \{ A_{r+1} \} \cup \mathcal{C}_i \) has a cyclic strong cover \( \mathcal{F}_i \). Note that \( ((A_i, \mathcal{F}_i) | i \leq r) \) continues to satisfy the conditions of Theorem 3.3 (since each \( \mathcal{F}_i \supseteq \mathcal{C}_i \)), but that \( ((A_i, \mathcal{F}_i) | i \leq r + 1) \) may not, since there is no guarantee that \( T \models \bigvee_{i \leq r+1} A_i \lor \bigvee_{i \leq r+1} \mathcal{F}_i^+ \bigvee_{i \leq r+1} \).

The second phase thus attempts to find (for each \( i \leq r + 1 \)) a cyclic strong cover \( \mathcal{D}_i \) of \( \mathcal{F}_i \) such that \( T \models \bigvee_{i \leq r+1} A_i \lor \bigvee_{i \leq r+1} \mathcal{D}_i^+ \), and it is the existence of the sets \( \{ \mathcal{D}_i | i \leq r + 1 \} \) which verifies the choice of \( A_{r+1} \) and \( \{ \mathcal{F}_i | i \leq r + 1 \} \) made in the first phase. If such verification is not possible, then we truncate \( ((A_i, \mathcal{F}_i) | i \leq r + 1) \) in order
to seek alternative extensions of \( ((A_i,F_i)|i \leq r) \).

The following definition captures these ideas.

### 3.4 Definitions.

(a) A **cyclic state** of length \( r \) \((r \geq 1)\) is a sequence \( S = ((A_i,C_i)|i \leq r) \) such that
   
   (i) each \( A_i \) is a predicate,

   (ii) each \( C_i \) is a cyclic strong cover of \( \{\neg A_i\} \cup \{A_j | j \leq r, j \neq i\} \), and

   (iii) for \( 1 \leq j < r \), \( T \models \bigvee_{i \leq j} A_i \lor \bigwedge_{i \leq j} C_i^+ \).

(b) A cyclic state \( S^* = ((A_i,D_i)|i \leq r+k) \) \((k \geq 0)\) is an extension of \( S = ((A_i,C_i)|i \leq r) \) iff \( C_i \subseteq D_i \) for each \( i \leq r \).

(c) A cyclic state \( S = ((A_i,C_i)|i \leq r) \) is said to be:
   
   (i) **verified** iff \( T \models \bigvee_{i \leq r} A_i \lor \bigwedge_{i \leq r} C_i^+ \) (cf., Theorem 3.3),

   (ii) **verifiable** iff \( S \) has a verified extension,

   (iii) **total** iff each \( C_i \) is total, and

   (iv) **complete** iff \( T \models \bigvee_{i \leq r} A_i \).

If \( ((A_i,C_i)|i \leq r) \) is a cyclic state, note that \( A_i \neq A_j \) for \( i \neq j \) (since \( C_i \) is consistent). Property (a)(iii) is a mere technical property capturing our intention that a cyclic state will only be extended in length if it is verified. This property also ensures that if \( ((A_i,C_i)|i \leq r) \) is a cyclic state of length \( r > 1 \), then truncation yields a verified cyclic state \( ((A_i,C_i)|i \leq r-1) \).

Note the obvious fact that if, for each \( i \leq r \), \( D_i \) is a cyclic strong cover of \( C_i \), then \( S^* = ((A_i,D_i)|i \leq r) \) is a cyclic state. Moreover if \( ((A_i,C_i)|i \leq r) \) is verified, then so is \( S^* \). The following proposition captures some of the basic properties of cyclic states.

### 3.5 Proposition.

(a) If \( T \models \bigvee \{A_i|i \leq r\} \), then \( \{A_i|i \leq r\} \) is a minimal answer iff there is a cyclic state of the form \( ((A_i,C_i)|i \leq r) \).

(b) If \( T \models \bigvee A \), and for each \( A \in A \), \( C_A \) is a cyclic strong cover of \( \{\neg A\} \cup (A \setminus \{A\}) \), then for any non-empty subset \( \{A_1,A_2,\ldots,A_r\} \subseteq A \), we have that \( ((A_i,C_{A_i})|i \leq r) \) is a verified cyclic state.

(c) Every cyclic state of length 1 is verified.

(d) If \( S = ((A_i,C_i)|i \leq r) \) is verified, then \( S \) has a complete total extension \( ((A_i,D_i)|i \leq r+k) \) such that \( \{A_j | r < j \leq r+k\} \subseteq \bigcap_{i \leq r} C_i^+ \).

(e) A set of predicates \( \{A_i|i \leq r\} \) is contained in a minimal answer iff there is a verified cyclic state of the form \( ((A_i,C_i)|i \leq r) \).

(f) \( S \) is verifiable iff \( S \) has a verified extension of the same length iff \( S \) has a complete extension iff \( S \) has a total complete extension.
Proof (a) This follows immediately from Theorem 3.3.

(b) This is trivial from the fact that if \( j \leq r \), then \( A \subseteq \{A_i|i \leq j\} \cup \bigcap_{i \leq j} C_{A_i}^+ \).

(c) A cyclic state of length 1 contains a single pair of the form \((A, C)\), where \( C \) is a cyclic strong cover and \( A \in C^- \), whence the result follows immediately from Theorem 2.4(c).

(d) Let \( A \) be a minimal answer with \( A \subseteq \{A_i|i \leq r\} \cup \bigcap_{i \leq r} C_i^+ \). For each \( i \leq r \), \( C_i \) is a cyclic strong cover of \( \{\neg A_i\} \cup (A - \{A_i\}) \), whence \( A_i \in A \). For \( i \leq r \), let \( D_i \) be a total cyclic strong cover of \( C_i \).

We now present the notion an immediate extension, this being our mechanism that will be used to extend cyclic states. We require that our extension mechanism is correct, in the sense that repeated application should generate only partial answers, and this will be guaranteed by Theorem 3.3 and the use of verification.

We also require completeness, meaning that every minimal answer can be generated via repeated application of our extension mechanism. Now every minimal answer is witnessed by a total complete cyclic state, thus in order to achieve completeness we will insist that whenever \( S^* \) is a total complete extension of \( S \), there is a (proper) immediate extension \( S' \) of \( S \) such that \( S^* \) is an extension of \( S' \). The minimal answer represented by \( S^* \) can then be constructed from \( S \) via a sequence of immediate extensions.

Of course initially we start our computation of minimal answers with the empty sequence, whose immediate extensions we take to be of the form \((A, C)\), where \( C \) is a constructible extension of \( \{\neg A\} \) (Definition 2.11).

3.6 Extending unverified cyclic states.

Suppose now that \( S = ((A_i, C_i)|i \leq r) \) is an unverified cyclic state. Our aim is
to find cyclic strong covers $D_i$ of $C_i$ such that $T \models \bigvee_{i \leq r} A_i \lor \bigvee_{i \leq r} D_i^+$. There are broadly two ways in which we can extend the sets $C_i$: either by adding some negative literal to some $C_{i_0}$ (which in turn will probably require the addition of further literals (both positive and negative) in order to re-form a cyclic strong cover); or the addition of some positive literal (i.e., predicate) to some $C_{i_0}$ (which again would probably require the addition of further literals). However, in order to achieve our desired goal, we specifically need to extend $\bigcap_{i \leq r} C_i^+$, and this suggests that searching for positive literals which can be used to simultaneously extend all $C_i$ would be the more fruitful option. This option is also more appealing since it is the more constrained (i.e., has a narrower search space), and also because it allows within it an integral test to check whether or not the current cyclic state is indeed unverified. The following result encapsulates these ideas, and also shows that our approach to extending unverified cyclic states satisfies the completeness criteria above.

3.6.1 Lemma. Suppose that $S = ((A_i, C_i) | i \leq r)$ is unverified, and $C$ is a cyclic strong cover of $\{A_i | i \leq r\} \cup \bigcap_{i \leq r} C_i^+$. Suppose also that $S$ is verifiable, and that $((A_i, D_i) | i \leq r)$ is a verified cyclic state such that for each $i \leq r$, $D_i \supseteq C_i$.

Then $F = \bigcap_{i \leq r} D_i^+ - (C^+ \cup \bigcup_{i \leq r} C_i^-) \neq \emptyset$, and for each $A \in F$ and each $i \leq r$ we may find a constructible extension $C_i^*$ of $\{A\} \cup C_i$ such that $C_i^* \subseteq D_i$.

Proof. First note that $C^+ \not\supseteq \bigcap_{i \leq r} D_i^+$, for otherwise $C$ would be a cyclic strong cover of $\{A_i | i \leq r\} \cup \bigcap_{i \leq r} C_i^+$, thus contradicting the fact that $T \models \bigvee_{i \leq r} A_i \lor \bigvee_{i \leq r} D_i^+$.

If $A \in \bigcap_{i \leq r} D_i^+$, then $A \not\in \bigcup_{i \leq r} D_i^- \supseteq \bigcup_{i \leq r} C_i^-$, and hence $F \neq \emptyset$.

Finally, if $i \leq r$ then $\{A\} \cup C_i \subseteq D_i$, and the existence of $C_i^*$ is then given by Corollary 2.12.

This then allows us to define immediate extensions of unverified cyclic states.

3.6.2 Definition. Suppose that we are given a cyclic state $S = ((A_i, C_i) | i \leq r)$, then an immediate extension $S^*$ of $S$ is formed as follows. Pick a constructible extension $C$ of $\{A_i | i \leq r\} \cup \bigcap_{i \leq r} C_i^+$. (If no such $C$ exists, i.e., $S$ is verified, then immediate extensions of $S$ are as given in Definition 3.7.3 below.)

Pick a predicate $A \in L - (C^+ \cup \bigcup_{i \leq r} C_i^-)$ such that for each $i \leq r$, $\{A\} \cup C_i$ has a constructible extension $C_i^*$, and let $S^* = ((A_i, C_i^*) | i \leq r)$.

If no such predicate $A$ exists, i.e., $S$ is not verifiable, then $S$ has no immediate extension and the truncation of $S$ is given by $S^* = ((A_i, C_i) | i \leq r - 1)$.
Note that if the predicate \( A \) exists, then the extension formed is a proper extension, since \( A \in \mathcal{L} - \mathcal{C}^+ \subseteq \mathcal{L} - \bigcap_{i \leq r} \mathcal{C}^+_i \).

The constraint that \( A \in \mathcal{L} - (\mathcal{C}^+ \cup \bigcup_{i \leq r} \mathcal{C}^-_i) \) clearly allows us to limit the search space, but nevertheless, it is still the case that we are *blindly* picking an element of \( \mathcal{L} - (\mathcal{C}^+ \cup \bigcup_{i \leq r} \mathcal{C}^-_i) \), and then determining whether the sets \( \mathcal{C}^*_i \) exist. Note however that if we make an incorrect choice for \( A_{r+1} \) then the cost of doing so is the cost of computing some of the sets \( \mathcal{C}^*_i \). In fact this computation is not totally wasted, and we return to this point in the notes following Definition 3.7.3. In Section 5 we will see that compilation allows us to partially overcome this need to make a blind choice.

Constructible extensions \( \mathcal{C} \) of \( \{A_i|i \leq r\} \cup \bigcap_{i \leq r} \mathcal{C}^+_i \) can be computed with no additional effort by computing constructible extensions \( \mathcal{C}' \) of \( \{A_i|i \leq r\} \), and then computing constructible extensions \( \mathcal{C} \) of \( \mathcal{C}' \cup \bigcap_{i \leq r} \mathcal{C}^+_i \). This gives us a free useful test, since (by Theorem 3.2) if no such \( \mathcal{C}' \) exists, then \( T \models \bigvee_{i \leq r} A_i \), whence \( \{A_i|i \leq r\} \) is a minimal answer and no further extension steps are required.

This partitioning of the computation of constructible extensions of \( \{A_i|i \leq r\} \cup \bigcap_{i \leq r} \mathcal{C}^+_i \) is also beneficial due to the fact that if we are successful in extending \( ((A_i, \mathcal{C}_i)|i \leq r) \) to a verified cyclic state \( S' = ((A_i, \mathcal{D}_i)|i \leq r) \), then the subsequent extension of \( S' \) will again require the use (or computation) of the cyclic strong covers of \( \{A_i|i \leq r\} \). This is illustrated in the following section.

### 3.7 Extending verified cyclic states.

Suppose that \( S = ((A_i, \mathcal{C}_i)|i \leq r) \) is verified (but \( T \not\models \bigvee_{i \leq r} A_i \), i.e., \( \{A_i|i \leq r\} \) has a cyclic strong cover). An extension of \( S \) will have the form \( S^* = ((A_i, \mathcal{D}_i)|i \leq r + 1) \), where (i) for \( i \leq r \), \( \mathcal{D}_i \) is a cyclic strong cover of \( \mathcal{C}_i \cup \{A_{r+1}\} \), and (ii) \( \mathcal{D}_{r+1} \) is a cyclic strong cover of \( \{-A_{r+1}\} \cup \{A_i|i \leq r\} \).

Cyclic strong covers of \( \{-A_{r+1}\} \cup \{A_i|i \leq r\} \) can be computed by first computing a constructible extension \( \mathcal{C} \) of \( \{A_i|i \leq r\} \), and then extending to a constructible extension of \( \{-A_{r+1}\} \cup \mathcal{C} \). We thus have the choice of either guessing \( A_{r+1} \) first, and then attempting to find the sets \( \mathcal{D}_i \) (if such exist), or choosing \( \mathcal{C} \) first, and then attempting to find \( A_{r+1} \). The latter option is the more fruitful for three reasons: Firstly, we have already computed the constructible extensions of \( \{A_i|i \leq r\} \) above. Secondly, given \( \mathcal{C} \) we can (by Lemma 3.7.1 below) prune the search space by insisting that \( A_{r+1} \in \mathcal{L} - (\mathcal{C}^+ \cup \bigcup_{i \leq r} \mathcal{C}^-_i) \), and finally given \( \mathcal{C} \), a predicate \( A_{r+1} \) together with sets \( \mathcal{D}_i \ (i \leq r+1) \) are guaranteed to exist (Lemma 3.7.2 below). Lemma 3.7.1 shows that this strategy for extending \( S \) satisfies our completeness condition identified in the remarks following Proposition 3.5.

#### 3.7.1 Lemma.

Suppose that \( S = ((A_i, \mathcal{C}_i)|i \leq r) \) is a verified cyclic state, and that
\((A_i, F_i)|i \leq r + k\) is a complete extension of \(S\) with \(k > 0\).

Then we may find a constructible extension \(\mathcal{C}\) of \(\{A_i|i \leq r\}\) such that

(i) \(A_{r+1} \not\in \mathcal{C}^+ \cup \bigcup_{i \leq r} C_i^−\),
(ii) \(\{\neg A_{r+1}\} \cup \mathcal{C}\) has a constructible extension \(\mathcal{D}_{r+1}\) such that \(\mathcal{D}_{r+1} \subseteq \mathcal{F}_{r+1}\), and
(iii) for each \(i \leq r\), \(\{A_{r+1}\} \cup C_i\) has a constructible extension \(\mathcal{D}_i\) such that \(\mathcal{D}_i \subseteq \mathcal{F}_i\).

**Proof.** Trivially \(A_{r+1} \in \bigcap_{i \leq r} \mathcal{F}_i^+ \subseteq \mathcal{L} - \bigcup_{i \leq r} \mathcal{F}_i^− \subseteq \mathcal{L} - \bigcup_{i \leq r} C_i^−\). Now \(\mathcal{F}_{r+1}\) is a cyclic strong cover of \(\{\neg A_{r+1}\} \cup \{A_i|i \leq r\}\), whence by Corollary 2.12 we may find a constructible extension \(\mathcal{C}\) of \(\{A_i|i \leq r\}\) such that \(\mathcal{C} \subseteq \mathcal{F}_{r+1}\). Since \(\neg A_{r+1} \in \mathcal{F}_{r+1}\), we must have that \(A_{r+1} \not\in \mathcal{C}^+\).

Again using Corollary 2.12, parts (ii) and (iii) follow trivially from the facts that \(\mathcal{F}_{r+1}\) is a cyclic strong cover of \(\{\neg A_{r+1}\} \cup \mathcal{C}\), and (for each \(i \leq r\)) \(\mathcal{F}_i\) is a cyclic strong cover of \(\{A_{r+1}\} \cup C_i\). \(\blacksquare\)

**3.7.2 Lemma.** Suppose that \(S = ((A_i, C_i)|i \leq r)\) is a verified cyclic state, and that \(\mathcal{C}\) is a cyclic strong cover of \(\{A_i|i \leq r\}\). Then we may find a predicate \(A_{r+1} \in \mathcal{L} - (\mathcal{C}^+ \cup \bigcup_{i \leq r} C_i^−)\) such that

(i) \(\{\neg A_{r+1}\} \cup \mathcal{C}\) has a constructible extension \(\mathcal{D}_{r+1}\), and
(ii) for each \(i \leq r\), \(\{A_{r+1}\} \cup C_i\) has a constructible extension \(\mathcal{D}_i\).

**Proof.** For each \(i \leq r\), let \(\mathcal{F}_i\) be a total cyclic strong cover of \(C_i\), and let \(\mathcal{F}\) be a total cyclic strong cover of \(\mathcal{C}\).

Since \(T \models \bigvee\{A_i|i \leq r\} \lor \bigwedge_{i \leq r} \mathcal{F}_i^+\) we cannot have that \(\mathcal{F} \supseteq \{A_i|i \leq r\} \cup \bigcap_{i \leq r} \mathcal{F}_i^+\), whence pick \(A_{r+1} \in \{A_i|i \leq r\} \cup \bigcap_{i \leq r} \mathcal{F}_i^+ - \mathcal{F}\). But then \(A_{r+1} \not\in \mathcal{C}^+ \supseteq \{A_i|i \leq r\}\), whence for each \(i \leq r\), \(A_{r+1} \in \mathcal{F}_i^+ \subseteq \mathcal{L} - \mathcal{F}_i^− \subseteq \mathcal{L} - C_i^−\). In addition, since \(\mathcal{F}\) is total, \(\neg A_{r+1} \not\in \mathcal{F}\).

Parts (i) and (ii) then follow from Corollary 2.12 and the facts that \(\{\neg A_{r+1}\} \cup \mathcal{C} \subseteq \mathcal{F}\) and \(\{A_{r+1}\} \cup C_i \subseteq \mathcal{F}_i\). \(\blacksquare\)

This then gives us our method of extending \(((A_i, C_i)|i \leq r)\).

**3.7.3 Definition.**

Let \(S = ((A_i, C_i)|i \leq r)\) be a verified cyclic state, then an immediate extension \(S^*\) of \(S\) is formed as follows. Let \(\mathcal{C}\) be a constructible extension of \(\{A_i|i \leq r\}\). (If no such \(\mathcal{C}\) exists then \(T \models \bigwedge_{i \leq r} A_i\) and \(\bigwedge_{i \leq r} A_i\) is a minimal answer.)

Pick \(A_{r+1} \in \mathcal{L} - (\mathcal{C}^+ \cup \bigcup_{i \leq r} C_i^−)\) such that

(i) \(\{\neg A_{r+1}\} \cup \mathcal{C}\) has a constructible extension \(\mathcal{D}_{r+1}\), and
(ii) for each \(i \leq r\), \(\{A_{r+1}\} \cup C_i\) has a constructible extension \(\mathcal{D}_i\).
Let \( S^* = (\langle A_i, D_i \rangle | i \leq r + 1) \).

Notes.
1. Note in both Definitions 3.6.2 and 3.7.3, that cyclic strong covers of \( \{ K \} \cup D \) could be computed from constructible extensions of \( \{ K \} \) in the reduced database \( T_D \) (by Theorem 2.9). In the case when \( K = \neg A_{r+1} \) (Definition 3.7.3(i)) it may be preferable to adopt a different strategy (see 3 below).

2. As mentioned in Section 3.6, cyclic strong covers may be computed and then found to be obsolete as far as extending the current partial answer is concerned. Note however that the computation of any cyclic strong cover \( C \) is not wasted (provided \( C^- \neq \emptyset \)) since it can still be employed in the derivation of other minimal answers: for each \( A \in C^- \), \( \langle (A, C) \rangle \) is a (verified) cyclic state of length 1.

3. Again we see that the constraint \( A_{r+1} \in L - (C^+ \cup \bigcup_{i \leq r} C_i^-) \) allows us to prune the search space, but that beyond this we are still making a blind choice of \( A_{r+1} \), and then testing whether it satisfies conditions (i) and (ii) above. This blind search is again not so detrimental, since it can provide useful information: A constructible extension \( D_{r+1} \) of \( \{ \neg A_{r+1} \} \cup C \) may be computed by first computing a constructible extension \( C' \) of \( \{ \neg A_{r+1} \} \), and then computing \( D_{r+1} \) as a constructible extension of \( C' \cup C \). In the case when no such \( C' \) exists, we can immediately discount \( A_{r+1} \) (from belonging to any minimal answer). In the case when \( C' \) exists, but cannot be extended to \( D_{r+1} \), then as above, for each \( A \in C'^- \), the pair \( (A, C') \) can still be employed in the derivation of other minimal answers, and the computation of \( C' \) is not wasted.

4. If \( S = (\langle A_i, C_i \rangle | i \leq r) \) is verified, then by Proposition 3.5(d) we may find a minimal answer \( A \) such that \( \{ A_i | i \leq r \} \subseteq A \subseteq \{ A_i | i \leq r \} \cup \bigcap_{i \leq r} C_i^+ \). Moreover, it is easy to amend the proof of Lemma 3.7.2 to show that given a constructible extension \( C \) of \( \{ A_i | i \leq r \} \) we can find some \( A_{r+1} \in A - \{ A_i | i \leq r \} \) (and hence in \( \bigcap_{i \leq r} C_i^+ \)) satisfying conditions (i) and (ii) of Definition 3.7.3.

Note however that in Definition 3.7.3 we are not able to insist that \( A_{r+1} \) is chosen from \( \bigcap_{i \leq r} C_i^+ \), since this would compromise completeness. For example if \( T = \{ A \lor B, A \rightarrow C, B \rightarrow D \} \), then \( C \lor D \) is a minimal answer, but the only constructible extension of \( \neg C \) (resp. \( \neg D \)) is \( \{ \neg C, \neg A, B \} \) (resp. \( \{ \neg D, \neg B, A \} \)), whence the only cyclic states of length 1 (generated using constructible extensions) representing a sub-answer of \( C \lor D \) are \( \langle (C, \{ \neg C, \neg A, B \}) \rangle \) and \( \langle (D, \{ \neg D, \neg B, A \}) \rangle \). This unfortunate inability to further limit the search space could be overcome by allowing verified states to be extended in the manner similar to that suggested in Section 3.6.2, but then such a requirement effectively insists upon the computation of perfect models in their entirety. We will see in Section 5 that compilation goes quite some way to overcoming this inability.
5. We have already mentioned that our method is both complete and correct. Notice that a truncation step is in effect an undo operation, and clearly we wish to prevent circularity by insisting that extension steps following truncation do not redo what has previously been undone. With this proviso, it is then clear that any sequence of cyclic states generated via immediate extension and truncation will eventually generate a complete cyclic state (i.e., a minimal answer).

§4. Unstratified databases

For unstratified databases, cyclic strong covers are not necessarily extendible to a disjunctive stable model, and we are thus (apparently) unable to characterise partial minimal answers without computing total cyclic strong covers (i.e., disjunctive stable models). In addition, testing verification, i.e., whether \( T \models \bigvee_{i \leq r} A_i \lor \bigvee \bigcap_{i \leq r} C_i^+ \) (cf., Theorem 3.3) requires the computation of total cyclic strong covers (Theorem 2.6(b)), at least in the case when the cyclic state in question is not verified.

As discussed in [Jo99a] (and mentioned in Section 2 above), disjunctive stable models can be generated by (implicitly) adding to our database, for each predicate \( P \in \mathcal{L} \), the denial rule \( P \land \neg P \rightarrow \text{FALSE} \): The disjunctive stable models of the database are unaffected, and every strong cover of \{FALSE\} is then total. Cyclic states then encode sequences of disjunctive stable models satisfying the conditions of Theorem 1.6, and a (top-down) construction of cyclic strong covers [Jo98, Jo99, Jo99a] provides us with a means of testing verification. The application of these denial rules within such a top-down construction amounts to the application of an unrestricted splitting rule (e.g., [Ya96]).

As mentioned earlier, the computation of models in their entirety is undesirable, and in the following section we show that a partitioning of the database can be used to (partially) alleviate this need.

§5. Compilation

It is natural to ask how our method compares in terms of computational efficiency with methods that also generate non-minimal answers. Is the checking required at each stage cost effective in relation to the saving (i.e., of not generating non-minimal answers)?

In this section we show that compiling (pre-processing) the computation of cyclic strong covers can be employed to greatly simplify and reduce the cost of the run-
time computation. In addition, as a by-product, compilation is shown to resolve the problems raised in Section 4 above.

Throughout this section we assume that $L$ is the disjoint union of $\text{EXT}(L)$ and $\text{INT}(L)$. If $Q$ is a set of literals, then $Q_{\text{ext}} = \{K \in Q | K \in \text{EXT}(L) \text{ or } \neg K \in \text{EXT}(L)\}$, and $Q_{\text{int}} = \{K \in Q | K \in \text{INT}(L) \text{ or } \neg K \in \text{INT}(L)\} = Q - Q_{\text{ext}}$. Notice that $(Q^-_{\text{ext}}) = (Q_{\text{ext}})^-$, etc. We also make the (usual) assumption that for each rule $C$, either

(i) $\text{conseq}(C) \subseteq \text{EXT}(L)$ and $\text{antec}(C) \cup \mathcal{N}(C) = \emptyset$, or

(ii) $\text{conseq}(C) \subseteq \text{INT}(L)$ and $\text{antec}(C) \cup \mathcal{N}(C) \neq \emptyset$ (whence $\text{antec}(C) \neq \emptyset$, cf., Section 1.1).

In case (ii) the assumption that the body of $C$ is non-empty if of course a technical requirement that can be achieved artificially without loss of generality.

We let $\text{EXT}(T) = \{C \in T | \text{conseq}(C) \subseteq \text{EXT}(L)\}$, and $\text{INT}(T) = T - \text{EXT}(T) = \{C \in T | \text{conseq}(C) \subseteq \text{INT}(L)\}$.

Note that a rule in $\text{EXT}(T)$ has the form $\bigvee E$, where $E \subseteq \text{EXT}(L)$. In particular, minimal and disjunctive stable models of $\text{EXT}(T)$ coincide, thus if $\Phi$ is a formula in $\text{EXT}(L)$, then $\text{EXT}(T) \models \Phi$ iff $\Phi$ is true in every minimal model of $\Phi$. In particular if $\mathcal{F} \subseteq \text{EXT}(L)$, then $\text{EXT}(T) \models \bigvee \mathcal{F}$ iff there is a rule $\bigvee E \in \text{EXT}(T)$ such that $E \subseteq \mathcal{F}$.

This partitioning of $L$ can be viewed as a very weak form of stratification, and indeed yields a weakened form of Theorem 3.2 as follows: Let us say that a consistent set of literals $Q$ is int-total iff $Q^- \cup Q^+ \supseteq \text{INT}(L)$. If $\mathcal{C}$ is an int-total cyclic strong cover, then $\text{EXT}(T) \not\models \bigvee \mathcal{C}_{\text{ext}}^+$, whence we may find a minimal model $M_0 \subseteq \text{EXT}(L)$ of $\text{EXT}(T)$ such that $M_0 \cap \mathcal{C}_{\text{ext}}^+ = \emptyset$. But then it is easy to show that $\mathcal{C}_{\text{ext}}^- \subseteq M_0$, and hence that $\mathcal{C} \cup \overline{M_0} \cup (\text{EXT}(L) - M_0)$ is a total cyclic strong cover extending $\mathcal{C}$. This then yields the following result.

5.1 Theorem [Jo99a].

(a) Every int-total cyclic strong cover can be extended to a total cyclic strong cover.
(b) If $Q$ is a set of literals, then $T \models \bigvee Q$ iff $Q$ has no int-total cyclic strong cover.

Compilation is based upon the assumption that $\text{EXT}(T)$ is relatively transient, in contrast to $\text{INT}(T)$ which is assumed to be relatively static. Compilation is then the pre-processing of $\text{INT}(T)$ so that subsequent run-time query processing requires a manipulation of $\text{EXT}(T)$ only. Whilst (the less frequent) modifications to $\text{INT}(T)$ necessitate recompilation, the more frequent modifications to $\text{EXT}(T)$ do not.

Since query processing requires the construction of int-total cyclic strong covers, we need to partition this construction into a computation step against $\text{INT}(T)$, and then a further step against $\text{EXT}(T)$.
For strong covers this partitioning is trivial, since a strong cover in \( T \) is simply a set that is both a strong cover in \( \text{INT}(T) \) and a strong cover in \( \text{EXT}(T) \). Moreover note that a consistent set of literals \( C \) is a strong cover in \( \text{EXT}(T) \) iff \( \text{EXT}(T) \neq \bigvee C^+_\text{ext} \).

For cyclic trees, every leaf node is a rule node. Moreover by assumptions (i) and (ii) above, a rule node \( rn_C \) is a leaf node iff \( C \in \text{EXT}(T) \). If \( rn_C \) is such a leaf node, with parent \( n \), then \( \text{conseq}(C) \cap \text{CYC}(n) \neq \emptyset \), whence the branch to \( n \) must contain a predicate node labelled with an extensional predicate. If \( m \) is the top-most such predicate node, then \( \text{CYC}(m) = \{\text{lab}(m)\} \), whence the child of \( m \) must be a rule node of the form \( rn_D \), where \( \text{conseq}(D) \cap \text{CYC}(m) \neq \emptyset \), whence \( D \in \text{EXT}(T) \), i.e., \( m = n \). Thus predicate nodes are labelled with extensional predicates iff their child is a leaf node.

Let us therefore say that a partial cyclic tree satisfies the conditions of Definition 2.3, with the exception that every leaf node is a predicate node labelled with an extensional predicate. Such trees are defined (constructed) entirely within \( \text{INT}(T) \). To complete a partial cyclic tree in order to form a cyclic tree, we need to extend each such leaf node with a rule node \( rn_C \), where \( C \in \text{EXT}(T) \). In order to ensure that the extended tree continues to satisfy the conditions of Definition 2.3, we must have that \( \text{lab}(n) \in \text{conseq}(C) \) (since \( \text{CYC}(n) = \{\text{lab}(n)\} \)) and \( O(rn_C) = \text{conseq}(C) - \{\text{lab}(n)\} \) is disjoint from \( \text{Pred}(T) \). The extended tree \( T' \) then has the properties that \( \text{Pred}(T') = \text{Pred}(T), \text{N}(T') = \text{N}(T) \) and \( O(T') = O(T) \cup \bigcup \{O(rn_C)\} \) if \( rn_C \) is a leaf in \( T' \).

5.2 Definition [Jo99, Jo99a]. A consistent set of literals \( C \) is a weakly cyclic cover (in \( \text{INT}(T) \)) iff

(i) \( C \) is a strong cover in \( \text{INT}(T) \), and

(ii) there is a set \( \{T_i \mid i \leq m\} \) of partial cyclic trees such that \( C^- = \bigcup_{i \leq m} \text{Pred}(T_i) \), and \( C^+ \supseteq \bigcup_{i \leq m} (O(T_i) \cup N(T_i)) \).

Let \( C \) be a weakly cyclic cover in \( \text{INT}(T) \) and \( f \) be a function \( f : C^-_{\text{ext}} \rightarrow \text{EXT}(T) \) such that for each \( P \in C^-_{\text{ext}} \), \( f(P) = \bigvee E_P \), where \( P \in E_P \) and \( (E_P - \{P\}) \cap C^-_{\text{ext}} = \emptyset \). The set \( C \cup \bigcup \{E_P - \{P\} \mid P \in C^-_{\text{ext}}\} \) is said to be a completion of \( C \) in \( T \).

Thus a completion of a weakly cyclic cover is formed by completing each of the cyclic trees which form the weakly cyclic cover. Note that if \( D \) is a completion of \( C \), then \( D - C \subseteq \text{EXT}(\mathcal{L}) \). Note also that the computation of weakly cyclic covers takes place entirely in \( \text{INT}(T) \), and the computation of completions takes place entirely in \( \text{EXT}(T) \).

In fact it is easy to observe that a partial tree \( T \) can always be extended to a cyclic tree in \( \text{INT}(T) \cup \{E\mid E \in \text{Pred}(T)_{\text{ext}}\} \) by appending to each predicate leaf node \( n \) the rule node labelled with the unit rule \( \text{lab}(n) \). We can thus easily show that a consistent
set of literals $C$ is a weakly cyclic cover (in $\text{INT}(T)$) iff $C$ is a cyclic strong cover in $\text{INT}(T) \cup \{ E \mid E \in C^- \}$. This then allows us to apply the previous results concerning cyclic strong covers to weakly cyclic covers.

Proposition 5.3 summarises the properties of weakly cyclic covers presented in [Jo99, Jo99a].

5.3 Proposition [Jo99, Jo99a].

(a) Let $D$ be a cyclic strong cover. Then $D$ is a weakly cyclic cover, and moreover if $C$ is a weakly cyclic cover with $C \subseteq D$, there is a completion $C'$ of $C$ such that $C' \subseteq D$.

(b) If $C$ is a weakly cyclic cover and $D$ is a completion of $C$, then $D$ is a cyclic strong cover iff $\text{EXT}(T) \not= \bigvee D^+_{ext}$.

(c) If $Q$ is a set of literals, then $T \models \bigvee Q$ iff whenever $C$ is an int-total weakly cyclic cover of $Q$ and $D$ is a completion of $C$, then $\text{EXT}(T) \models \bigvee D^+_{ext}$.

Since weakly cyclic covers are characterised by $\text{INT}(T)$, we can easily show that if $Q$ is a set of literals, then $C$ is a weakly cyclic cover of $Q$ iff $C$ has the form $C = C' \cup Q_{ext}$ where $C'$ is a weakly cyclic cover of $Q_{int}$ and $C' \cup Q_{ext}$ is consistent. This then yields the following corollary.

5.4 Corollary.

(a) $D$ is a cyclic strong cover of $Q$ iff there is a weakly cyclic cover $C'$ of $Q_{int}$ such that $C' \cup Q_{ext}$ is consistent, and $D$ is a completion of $C' \cup Q_{ext}$ with $\text{EXT}(T) \not= \bigvee D^+_{ext}$.

(b) If $Q$ is a set of literals, then $T \models \bigvee Q$ iff whenever $C'$ is an int-total weakly cyclic cover of $Q_{int}$ such that $C' \cup Q_{ext}$ is consistent, and $D$ is a completion of $C' \cup Q_{ext}$, then $\text{EXT}(T) \models \bigvee D^+_{ext}$.

5.5 The compilation process.

Clearly the compilation process requires the computation of int-total weakly cyclic covers. As suggested in Sections 2 and 4, forcing weakly cyclic covers to be int-total can be simply achieved by implicitly adding $\{ P \land \neg P \rightarrow \text{FALSE} \mid P \in \text{INT}(L) \}$ to $\text{INT}(T)$. Note that during compilation we can (if desired) apply subsumption to remove redundancy, since if $C$ and $D$ are int-total weakly cyclic covers with $C \subseteq D$, then any completion of $D$ contains a completion of $C$. An appropriate set of int-total weakly cyclic covers can thus be generated using a “constructible” approach (cf., Definition 2.11 and Corollary 2.12), the details of which are discussed further in [Jo98a, Jo99, Jo99a].

We assume therefore that the compilation process generates a set $\text{COMP}$ of int-total
weakly cyclic covers, such that every int-total weakly cyclic cover is a superset of some element of COMP.

We now turn our attention to the computation of minimal answers following compilation. Firstly note that cyclic strong covers can be computed using COMP as follows. If \( Q \) is a consistent set of literals, and \( S \) is an int-total cyclic strong cover of \( Q \), then we may find some \( C \in \text{COMP} \) such that \( C \supseteq Q \) int, \( C \cup Q \) ext is consistent, and a completion \( D \) of \( C \cup Q \) ext such that \( D \subseteq S \) (whence \( \text{EXT}(T) \neq \bigvee D^+ \) ext and \( D \) is a cyclic strong cover of \( Q \)). Let us denote by \( \text{COMP}(Q) \) the set of cyclic strong covers of \( Q \) so obtained, i.e.,

\[
\text{COMP}(Q) = \{ D \mid \exists C \in \text{COMP}, C \supseteq Q \text{ int, } C \cup Q \text{ ext is consistent, } D \text{ is a completion of } C \cup Q \text{ ext, and } \text{EXT}(T) \neq \bigvee D^+ \}.
\]

As in Section 3, our run-time computation of minimal answers is again based upon two distinct processes, depending on whether the cyclic state to be extended is verified or not.

5.6 Extending unverified cyclic states.

As in Section 3, our aim is to extend an unverified cyclic state \( S = ((A_i, C_i)|i \leq r) \) to a verified cyclic state \( S^* = ((A_i, D_i)|i \leq r) \), thus again we need to extend \( \bigcap_{i \leq r} C_i^+ \) by the simultaneous addition of some predicate(s) to each \( C_i^+ \). Since each \( C_i \) will already be int-total, such predicates must be taken from \( \text{EXT}(L) \). With this in mind, note that if \( C \) is an int-total cyclic strong cover, and \( B \subseteq \text{EXT}(L) \), then \( B \cup C \) is a cyclic strong cover iff \( B \cap C_{ext} = \emptyset \) and \( \text{EXT}(T) \neq \bigvee B \lor \bigvee C_{ext}^+ \).

As in Section 3.6 we first perform a test to ensure that \( S \) is not verified, and then use the results of this test to constrain the search space.

5.6.1 Theorem. Let \( S = ((A_i, C_i)|i \leq r) \) be a cyclic state, where each \( C_i \) is int-total, and \( V = \{ A_i| i \leq r \} \cup \bigcap_{i \leq r} C_i^+ \). Suppose that \( S \) is not verified and that \( D \in \text{COMP}(V) \). Suppose also that \( S \) is verifiable, and that \( ((A_i, D_i)|i \leq r) \) is a verified cyclic state, with each \( D_i \supseteq C_i \).

Then we may find a rule \( \bigvee \mathcal{E} \in \text{EXT}(T) \) such that for each \( i \leq r \), \( \mathcal{E} - D \subseteq D_i^+ - C_i^- \), \( \mathcal{E} \cap (D - C_i) \neq \emptyset \) and \( \text{EXT}(T) \neq \bigvee (\mathcal{E} - D) \lor \bigvee (C_i^+)_{ext} \).

Proof. Let \( \mathcal{W} = \{ A_i| i \leq r \} \cup \bigcap_{i \leq r} D_i^+ \), then \( V \subseteq \mathcal{W} \), \( \mathcal{W}_{int} = \mathcal{V}_{int} \) and \( \mathcal{W} - V \subseteq \text{EXT}(L) \).

Now \( D \) is a completion of some \( C \cup \mathcal{V}_{ext} \), where \( C \) is a weakly cyclic cover of \( \mathcal{V}_{int} = \mathcal{W}_{int} \) and \( C \cup \mathcal{V}_{ext} \) is consistent.

If \( C \cup \mathcal{W}_{ext} \) is inconsistent, then there is some \( P \in C^- \) such that \( P \in \mathcal{W}_{ext} - \mathcal{V}_{ext} \),
and (by the definition of a completion) there is some rule \( \forall E \in \text{EXT}(T) \) such that \( P \in \mathcal{E} \) and \( \mathcal{E} - \{P\} \subseteq D \), whence \( \mathcal{E} \subseteq D \cup (W_{\text{ext}} - V_{\text{ext}}) \). If \( C \cup W_{\text{ext}} \) is consistent, then \( D \cup (W_{\text{ext}} - V_{\text{ext}}) \) is a completion of \( C \cup V_{\text{ext}} \cup (W_{\text{ext}} - V_{\text{ext}}) = C \cup W_{\text{ext}} \), whence by Corollary 5.4(b), there is some \( \forall E \in \text{EXT}(T) \) such that \( \mathcal{E} \subseteq D \cup (W_{\text{ext}} - V_{\text{ext}}) \).

But then \( \mathcal{E} - D \subseteq (W_{\text{ext}} - V_{\text{ext}})^+ \subseteq \bigcap_{i \leq r} D_i^+ \subseteq \mathcal{L} - \bigcup_{i \leq r} D_i^- \subseteq \mathcal{L} - \bigcup_{i \leq r} C_i^- \). Suppose that \( \mathcal{E} \cap (D - C_i) = \emptyset \), then \( \mathcal{E} \cap D \subseteq C_i \) and therefore \( \mathcal{E} = (\mathcal{E} - D) \cup (\mathcal{E} \cap D) \subseteq D_i \cup C_i \subseteq D_i, \) thus contradicting the fact that \( D_i \) is a strong cover. Similarly \( \text{EXT}(T) \not\models (\mathcal{E} - D) \lor (C_i^+) \), since \( (\mathcal{E} - D) \lor C_i^+ \subseteq D_i \).

**5.6.2 Definition.**

Suppose that \( S = ((A_i, C_i) | i \leq r) \) is a cyclic state, where each \( C_i \) is int-total. An **immediate extension** \( S^* = ((A_i, D_i) | i \leq r) \) of \( S \) is computed as follows. Pick \( D \in \text{COMP}((A_i | i \leq r) \cup \bigcap_{i \leq r} C_i^+) \). By Corollary 5.4, if no such \( D \) exists, then \( S \) is verified, in which case immediate extensions of \( S \) are defined in Definition 5.7.3 below.

Pick \( \forall E \in \text{EXT}(T) \) such that for each \( i \leq r, (\mathcal{E} - D) \cap C_i^- = \emptyset, \mathcal{E} \cap (D - C_i) \neq \emptyset \) and \( \text{EXT}(T) \not\models (\mathcal{E} - D) \lor (C_i^+) \). We then set \( D_i = (\mathcal{E} - D) \cup C_i \).

Notice that there must be some \( i_0 \) for which \( C_{i_0} \subseteq D_{i_0} \), for otherwise \( \mathcal{E} - D \subseteq \bigcap_{i \leq r} C_i^+ \), thus \( E \subseteq D \lor (\mathcal{E} - D) \subseteq D \lor \bigcap_{i \leq r} C_i^+ \subseteq D \); contradicting the fact that \( \text{EXT}(T) \not\models D_{i_0}^+ \). Thus \( S^* \) is a proper extension of \( S \).

If no such rule \( \forall E \) exists in \( \text{EXT}(T) \) then \( S \) is not verifiable (by Theorem 5.6.1), \( S \) has no immediate extension, and the truncation of \( S \) is given by \( ((A_i, C_i) | i \leq r - 1) \).

Theorem 5.6.1 demonstrates that our extension step satisfies the required completeness property (given after Proposition 3.5).

**5.7 Extending verified cyclic states.**

We first present two results (whose proofs are virtually identical to those of Lemmas 3.7.1 and 3.7.2) detailing the extension mechanism for verified cyclic states and showing that it satisfies the required completeness property.

**5.7.1 Lemma.** Suppose that \( S = ((A_i, C_i) | i \leq r) \) is a verified cyclic state such that each \( C_i \) is int-total, and let \( ((A_i, F_i) | i \leq r + k) \) be a complete total extension of \( S \) with \( k > 0 \). Then we may find a \( C \in \text{COMP}((A_i | i \leq r)) \) such that

(i) \( A_{r+1} \notin C^+ \cup \bigcup_{i \leq r} C_i^- \),

(ii) there is a \( D_{r+1} \in \text{COMP}((-A_{r+1}) \cup C) \) such that \( D_{r+1} \subseteq F_{r+1} \), and

(iii) for each \( i \leq r \), there is a \( D_i \in \text{COMP}((A_{r+1}) \cup C_i) \) such that \( D_i \subseteq F_i \).

**5.7.2 Lemma.** Suppose that \( S = ((A_i, C_i) | i \leq r) \) is a verified cyclic state where
each \( C_i \) is int-total, and that \( C \in \text{COMP}(\{A_i|i \leq r\}) \). Then we may find a predicate \( A_{r+1} \in \mathcal{L} - (\mathcal{C}^+ \cup \bigcup_{i \leq r} C_i^-) \) such that

(i) \( \{\neg A_{r+1}\} \cup C \) has an int-total cyclic strong cover \( \mathcal{D}_{r+1} \), and

(ii) for each \( i \leq r \), \( \{A_{r+1}\} \cup C_i \) has an int-total cyclic strong cover \( \mathcal{D}_i \).

In Lemma 5.7.2, note that when \( A_{r+1} \in \text{INT}(\mathcal{L}) \), we must have that \( A_{r+1} \in \mathcal{C}^- \cap \bigcap_i C_i^+ \) (since \( C \) and each \( C_i \) is int-total), in which case \( C \) itself is an int-total cyclic strong cover of \( \{\neg A_{r+1}\} \cup C \), and for each \( i \leq r \), \( C_i \) is itself an int-total cyclic strong cover of \( \{A_{r+1}\} \cup C_i \).

In the case when \( A_{r+1} \in \text{EXT}(\mathcal{L}) \), condition (i) can be characterised by the existence of a rule \( \bigvee \mathcal{E} \in \text{EXT}(\mathcal{L}) \) such that \( A_{r+1} \in \mathcal{E} \), \( (\mathcal{E} - \{A_{r+1}\}) \cap C_{\text{ext}}^- = \emptyset \) and \( \text{EXT}(T) \models \bigvee (\mathcal{E} - \{A_{r+1}\}) \bigvee C_{\text{ext}}^+ \), whence we may take \( D_{r+1} = \{\neg A_{r+1}\} \cup (\mathcal{E} - \{A_{r+1}\}) \cup C \) (i.e., \( D_{r+1} \) is a completion of \( \{\neg A_{r+1}\} \cup C \)). Condition (ii) can be characterised by the condition \( \text{EXT}(T) \models A_{r+1} \bigvee \bigvee (C_i^+ \cap C_{\text{ext}}) \), in which case we may take \( D_i = \{A_{r+1}\} \cup C_i \). As mentioned earlier, in both cases, the test against \( \text{EXT}(T) \) is of course a simple subsumption check.

Note that in both cases, \( A_{r+1} \not\in \{A_i|i \leq r\} \) since \( A_{r+1} \not\in C^+ \supseteq \{A_i|i \leq r\} \).

### 5.7.3 Definition.

Suppose that \( \mathcal{S} = ((A_i, C_i)|i \leq r) \) is a verified cyclic state, where each \( C_i \) is int-total. An immediate extension \( \mathcal{S}^* \) of \( \mathcal{S} \) is computed as follows. Let \( C \in \text{COMP}(\{A_i|i \leq r\}) \).

(As in Section 3.7.3, if no such \( C \) exists, then \( T \models \bigvee_{i \leq r} A_i \) and \( \bigvee_{i \leq r} A_i \) is a minimal answer.) There are now two cases (at least one of which must apply), depending on whether we (try to) extend with a predicate in \( \text{INT}(\mathcal{L}) \) or \( \text{EXT}(\mathcal{L}) \).

(a) Pick a predicate \( A_{r+1} \in \text{INT}(\mathcal{L}) \cap C^- \cap \bigcap_{i \leq r} C_i^+ \), and let \( \mathcal{S}^* \) be formed by extending \( \mathcal{S} \) with the pair \( (A_{r+1}, C) \).

(b) Pick a predicate \( A_{r+1} \in \text{EXT}(\mathcal{L}) - (\mathcal{C}^+ \cup \bigcup_{i \leq r} C_i^-) \) such that

(i) for each \( i \leq r \), \( \text{EXT}(T) \not\models A_{r+1} \bigvee \bigvee (C_i^+ \cap C_{\text{ext}}) \), and

(ii) there is a rule \( \bigvee \mathcal{E} \in \text{EXT}(T) \) such that \( A_{r+1} \in \mathcal{E} \), \( (\mathcal{E} - \{A_{r+1}\}) \cap C_{\text{ext}}^- = \emptyset \) and \( \text{EXT}(T) \not\models \bigvee (\mathcal{E} - \{A_{r+1}\}) \bigvee C_{\text{ext}}^+ \).

\( \mathcal{S}^* \) is then formed from \( \mathcal{S} \) by replacing each \( C_i \) by \( \{A_{r+1}\} \cup C_i \), and then extending with the pair \( (A_{r+1}, \{\neg A_{r+1}\} \cup (\mathcal{E} - \{A_{r+1}\}) \cup C) \).

### 5.8 Compilation for query processing.

Compilation has been previously studied for query processing under the minimal model [Ya02], perfect model [Jo99], disjunctive stable model [Jo99a], possible model [Jo02], and disjunctive well-founded [Jo03] semantics. Each of these approaches compiles a specific query (as opposed to using compilation to generate all (minimal) an-
swers). For example in order to compile a query $\bigvee Q$ under the disjunctive stable model semantics, the approach presented in [Jo99a] computes int-total weakly cyclic covers $C$ of $Q$, and the run-time processing then attempts to find a completion $D$ of some such $C$ for which $\text{EXT}(T) \not\models \bigvee D^+_{ext}$. (In the terminology of the current section, $T \models \bigvee Q$ iff $\text{COMP}(Q) = \emptyset$.)

Compilation of the GCWA in first order non-recursive positive databases is discussed in [He88]. In such databases, evaluation under the GCWA reduces to testing minimal answer membership, whose compilation is achieved by a combination of resolution and subsumption.

5.9 Other forms of pre-processing.

Compilation as presented above is not the only form of pre-processing that can be applied. For example, in our approach we have partitioned the database into two components, the first of which, $\text{EXT}(T)$, consists of simple disjunctions. Our crucial result, Theorem 5.1, was dependent on the resulting properties of $\text{EXT}(T)$, specifically that testing whether $\text{EXT}(T) \models \bigvee Q_{ext}$ can be achieved without recourse to cyclic strong covers that are total in $\text{EXT}(L)$. But this property is (by Theorem 3.2) shared by all stratified databases, and we therefore note that the techniques of the current section can be extended to the case where the database is suitably partitioned into two components, one of which is stratified. This approach was discussed briefly in the context of query processing in [Jo99a, Section 6].

An alternative, which is explored in [Jo99] for stratified databases, is to employ cyclic trees in order to transform a database into an equivalent database which has specific properties. For example we show in Appendix B that we can transform a database into a positive database which is equivalent under the disjunctive stable model semantics, and in which we can compute minimal answers of the original program.

As noted in the introduction, a form of pre-processing referred to as unfolding has been studied in [Bra94, Bra95] in relation to a wide class of semantics. This pre-processing transforms the database into a set of conditional facts (i.e., rules for which $\text{antec}(C) = \emptyset$). Such facts are of course closely related to answers, and in this respect one might argue that unfolding is closer to full scale answer generation than it is to pre-processing. More importantly (from the database perspective), such a transformation could not be immune to changes in $\text{EXT}(T)$. In [Jo99, Section 4], cyclic trees are employed to transform a stratified database into an equivalent database whose rules have the property that $\text{antec}(C) \subseteq \text{EXT}(L)$. Such a transformation is immune to changes in $\text{EXT}(T)$, although whether this result can be extended to the disjunctive stable model semantics is unclear.
§6. Query answers

Let us say that a query to a database is an expression of the form \(?\bigvee \mathcal{H}\)\text{, where } \mathcal{H} \subseteq \mathcal{L}\text{, and an answer (to } \mathcal{H}\text{) is a set } \mathcal{A} \subseteq \mathcal{H}\text{ such that } T \models \bigvee \mathcal{A}\text{. Our approach can be modified to compute minimal answers to such queries using the following result.}

6.1 Theorem. A set \(\{A_i|i \leq r\} \subseteq \mathcal{H}\) is contained in a minimal answer to the query \(?\bigvee \mathcal{H}\) iff for each \(i \leq r\) we may find a disjunctive stable model \(M_i\) of \(T\) such that

\[M_i \cap \{A_j|j \leq r\} = \{A_i\} \text{ and } T \models \bigvee_{i \leq r} A_i \lor \bigvee_{i \leq r}(\mathcal{H} - M_i).\]

As in Section 3, for stratified databases, \(\{A_i|i \leq r\} \subseteq \mathcal{H}\) is contained in a minimal answer to the query \(?\bigvee \mathcal{H}\) iff for each \(i \leq r\) we may find a cyclic strong cover \(C_i\) of \(\neg A_i \cup \{A_j|j \leq r, j \neq i\}\) such that

\[T \models \bigvee_{i \leq r} A_i \lor \bigvee(\mathcal{H} \cap \bigcap_{i \leq r} C_i^+).\]

As discussed in [Jo99, Section 5], when \(r = 1\) the above theorem indicates that Theorem 1.5 does not carry over to queries of the form \(?\bigvee \mathcal{H}\). Specifically it enables us to show that a predicate belongs to some minimal answer to the query \(?\bigvee \mathcal{H}\) iff it is contained in some disjunctive stable model \(M\) of \(T\) for which there is no disjunctive stable model \(M'\) such that \(M' \cap \mathcal{H} \subset M \cap \mathcal{H}\).

Computing minimal answers to a query \(?\bigvee \mathcal{H}\) is equivalent to computing minimal answers that are contained in \(\mathcal{H}\). This process is clearly going to be more efficient than computing all minimal answers, since our search is constrained in \(\mathcal{H}\). This then raises the following question. When attempting to compute the set of all minimal answers, might it be useful from the viewpoint of computational efficiency, to first apply a technique which generates answers \(\bigvee \mathcal{H}\), and then apply the results of the current paper to find minimal answers within \(\mathcal{H}\)?

§7. Other semantics

The perfect/disjunctive stable model semantics are of course not the only available semantics. For both stratified and unstratified databases, we could employ the possible model semantics [Sa94] based upon inclusive (rather than exclusive) disjunction. In this case possible models can be characterised by total supported strong covers [Jo99b], and using this characterisation, the methods of the current paper can be adapted to the possible model semantics. For unstratified databases, possible models are ultimately defined in terms of stable models, whose possible non-existence gives rise to the same issues as found in the current paper. It is also worthy of note that for a certain class of
stratified databases, the answers (and hence minimal answers) under the perfect model semantics coincide with those under the possible model semantics [Jo02, Corollary 7.3.7].

For unstratified databases we could alternatively employ the disjunctive well-founded semantics (DWFS) [Bra94]. DWFS can be constructed as the union of an increasing sequence \( \emptyset = D_0 \subseteq D_1 \subseteq \ldots \) [Bra98], where each \( D_{\alpha+1} \) consists of a set \( D_{\alpha+1}^+ \) of disjunctions of predicates, and a set \( D_{\alpha+1}^- \) of negative literals. \( D_{\alpha+1}^+ \) is constructed from \( D_{\alpha}^- \) which in turn is constructed from \( D_{\alpha}^+ \). In [Jo01, Theorem 2.6] it is shown that \( D_{\alpha+1}^+ = \{ \bigvee P : T|_g(\mathcal{L} - D^-_{\alpha}) \models \bigvee P \} \), and thus the computation of minimal answers in each \( D_{\alpha+1}^+ \) can be achieved by applying the methods of Section 3 to the positive database \( T|_g(\mathcal{L} - D^-_{\alpha}) \) (using of course minimal rather than perfect models). However, the minimality of some answer in \( D_{\alpha+1}^+ \) does not necessarily imply its minimality in \( D_{\alpha+2}^+ \) or (hence) in DWFS as a whole.

We can overcome this problem by computing the sequence \( D_0^- \subseteq D_1^- \subseteq \ldots \) independently of the sets \( D_{\alpha}^+ \), since ([Jo01, Corollary 4.10]) \( \neg Q \in D_{\alpha+1}^- \) iff for each quasicyclic tree \( T \) for \( Q \) in \( T \), either \( T|_g(\mathcal{L} - D^-_{\alpha}) \models \bigvee \mathcal{N}(T) \) or \( T|_g(\mathcal{L} - \mathcal{N}(T) \cup D^-_{\alpha}) \models \bigvee \mathcal{O}(T) \). Having computed \( \text{DWFS}^- = \bigcup_{\alpha} D^-_{\alpha} \), we can then compute \( \text{DWFS}^+ = \bigcup_{\alpha} D^+_{\alpha} = \{ \bigvee P : T|_g(\mathcal{L} - \text{DWFS}^-) \models \bigvee P \} \) [Jo01, Theorem 2.8], and in particular we can apply the techniques of Section 3 to the positive database \( T|_g(\mathcal{L} - \text{DWFS}^-) \) (again using minimal rather than perfect models) in order to compute minimal answers in \( \text{DWFS}^+ \). In this respect it is shown in [Jo01, Lemma 1.5] that only predicates in \( \mathcal{L} - \text{DWFS}^- \) can appear in a minimal answer in \( \text{DWFS}^+ \).

§8. The first order level

We have presented a new, and somewhat novel solution to the problem of computing minimal answers for propositional databases. In view of the relationship with the wider issues of computational complexity we consider these results to be of interest in themselves.

From the viewpoint of deductive databases however, “real” databases are of course first order (but function free). The rules in such databases represent the set of their ground instances (this defining issues such as logical inference), but for the sake of computational efficiency we require methods that apply directly to the first order level. In particular, grounding of the entire database, and then subsequently applying a propositional query answering method would be considered inappropriate, since the resulting ground database would often be large.

In spite of these comments, first order methods are typically developed in the first
instance for the propositional level, and then lifted to the first order level. In this sense, our results presented to-date constitute the first step. In this section we briefly consider the issues, difficulties and constraints that a first order lifting encounters. We also mention briefly an additional advantage that such a lifting would have over the first order query processing method presented in [Jo98].

Suppose then that $T$ is a first order database, and the set of ground instances of rules in $T$ is denoted by $gr(T)$. A minimal answer in $T$ is simply a set of ground positive atoms that is a minimal answer in $gr(T)$. Our basic method (i.e., without compilation) requires the computation of cyclic strong covers in $gr(T)$, so let us consider how this can be achieved (without computing $gr(T)$).

8.1 Constructing cyclic trees.

The definition and (top-down) construction of cyclic trees for first order (function free) databases is discussed at length in [Jo96, Jo98, Jo98a, Jo00]. Each rule within the tree is linked to the cycle above (cf., Definition 2.3(iii)) by a unifying instantiation, and a significant characteristic of cyclic trees (for first order databases) is that these unifying instantiations have the effect of making $Pred(T) \cup O(T) \cup N(T)$ (naturally) ground.

Cyclic trees thus provide no additional complications at the first order level. Indeed we note as an aside that a conceptually simple approach to the generation of minimal answers is to pre-compute all cyclic trees in $T$, and (using such) transform $T$ into an equivalent positive database (see Appendix B). Minimal answers in $T$ can be computed using strong covers (as opposed to (total) cyclic strong covers) in the transformed database (Theorem B.4), the computation of which can be achieved using the methods of [Jo97] as is discussed below.

8.2 Constructing cyclic strong covers.

Our approach to constructing cyclic strong covers at the propositional level is to employ the operator presented in Section 2.11. Specifically given $Q$ and a rule $C \in T$ such that $\text{conseq}(C) \subseteq Q^+$ and $Q \cap (\text{antec}(C) \cup N(C)) = \emptyset$, we may extend $Q$ to either

- $Q' = Q \cup \{P\}$ for some $P \in \text{antec}(C)$, or
- $Q' = Q \cup S(T)$, where $T$ is a cyclic tree for some $R \in N(C)$ in $T$ such that $Q \cup S(T)$ is consistent.

As mentioned in Section 2, we can then represent the generation of cyclic strong covers as an (extended deduction) tree construction [Jo98, Jo99, Jo99a]. For example if $\text{antec}(C) = \{P\}$, $N(C) = \{R\}$ and $R$ has cyclic trees $T_1, T_2, ..., T_r$ in $T$, then the application of $C$ yields the following subtree:
Of course if $R$ has no cyclic tree in $T$, then we may, in effect, discard $\neg R$ as a child of $rn_C$.

Consider now the first order case. As is normal in research into deductive databases we will assume that for each rule $C$, if $x$ is a variable which appears in $\text{conseq}(C)$, then $x$ also appears in $\text{antec}(C) \cup N(C)$.

Suppose for the sake of simplicity that $Q$ is ground, and that $C$ has the form of $P(y, z) \land \neg R(y, z) \rightarrow V(y) \lor W(y)$. We pick a substitution $\theta$ which unifies $\text{conseq}(C)$ with some subset of $Q$, say $y\theta = a$. We then compute a cyclic tree (say $T_1$) for (an instance of) $R(a, z)$, this resulting in a further instantiation $\eta$ of $z$, say $z\eta = b$. Assuming that $Q \cap (\text{antec}(C\theta\eta) \cup N(C\theta\eta)) = \emptyset$, we then compute the remaining cyclic trees $T_2, ..., T_r$ for $R(a, b)$, thus yielding the tree

$$Q \rightarrow \begin{array}{c} \neg R \rightarrow \begin{array}{c} S(T_1) \quad S(T_2) \quad ... \quad S(T_r) \end{array} \end{array}$$

Notice that the tree is still ground, but importantly we have not taken an arbitrary instance of $C$: the instantiation of variables in $\text{conseq}(C)$ results from the unifier $\theta$, and the instantiation of any other variables in $N(C)$ results from the unifiers employed in the construction of the first cyclic tree $T_1$. The issue of how to handle instances of $R(a, z)$ which have no cyclic trees is discussed below in Section 8.3.

Suppose instead that $C$ had the form of $P(x, y, z) \land \neg R(y, z) \rightarrow V(y) \lor W(y)$, then the resulting tree would have the form

$$Q \rightarrow \begin{array}{c} \neg R \rightarrow \begin{array}{c} S(T_1) \quad S(T_2) \quad ... \quad S(T_r) \end{array} \end{array}$$
In this case the child nodes of $\neg R(a, b)$ are again ground. The left hand child of $rn_{C\theta_\eta}$ contains a variable however, indicating that this extension of $Q$ resulting from the application of $C\theta_\eta$ represents $Q \cup \{P(c, a, b) | c \text{ is a constant in } L\}$.

But now if $C$ had the form $S(x, z) \land P(x, y, z) \land \neg R(y, z) \rightarrow V(y) \lor W(y)$, then the application of $C\theta_\eta$ results in the tree

and again $Q \cup \{P(c, a, b) | c \text{ is a constant in } L\}$ and $Q \cup \{S(c, b) | c \text{ is a constant in } L\}$ are valid extension of $Q$. Unfortunately, these are not the only extensions that we need to consider: for example if $L$ contains just the two constants $a$ and $b$, then $Q \cup \{S(a, b), P(b, a, b)\}$ and $Q \cup \{S(b, b), P(a, a, b)\}$ also need to be considered, and thus the branches individually do not represent the totality of all covers. Worse still, if $T$ contained a rule whose head had the form $S(w, b) \lor P(w, a, b)$, then such a rule would somehow have to be applied to both branches simultaneously, thus effectively destroying our view of cyclic cover construction as a tree traversal.

A solution to this problem, presented in [Jo97, Section 4], is to insist that variables which appear in $\text{antec}(C)$ but not in $\text{conseq}(C) \cup N(C)$ occur only in semi-definite predicates [Jo97]. Such predicates can in particular only occur in the heads of rules which are definite (i.e., contain a single atom in their head), whence this eliminates the possibility of applying the rule to two branches simultaneously.

The problem of how to handle body variables that are not instantiated by the unifier also occurs in the study of hyper-tableaux [Ba97], where the problem of making a blind instantiation of such variables is well known. In [Ba97], Baumgartner presents
a solution to this problem by employing an additional \textbf{Link} inference rule which effectively creates separate instantiated copies of the problematic atoms in a demand driven way, as such instantiations are required by available extension steps. This approach is similar to that employed in [Jo97], where such variables are instantiated by so called \textit{eq-rules}, again in a demand driven way. The technical differences between the methods of [Ba97] and [Jo97] make a direct comparison difficult, but the approach presented in [Ba97] does appear to be more general, and we conjecture that it could be adapted to allow the above-mentioned restriction in terms of semi-definite predicates to be lifted.

A similar argument can be made in the case when the initial set $Q$ is non-ground (although in fact the methods described in Section 3 could be employed without this requirement).

8.3 Negative subgoals with no cyclic trees.

In the example above, the atom $\neg R(a, z)$ becomes ground as a result of the construction of $T_1$. In addition, we have to also allow for instances of $R(a, z)$ that have no cyclic tree. There are two obvious solutions to this problem:

(i) Impose a further constraint, namely that every variable appearing in $N(C)$ also appears in $\text{conseq}(C)$, whence the unifier $\theta$ causes $N(C)$ to become ground.

(ii) Implicitly add to the database a rule of the form $K \lor \text{TRUE}$, for each $K \in \mathcal{H}$. This forces every atom to have at least one cyclic tree, but the downside of this strategy is that instances of $R(a, z)$ (which otherwise have no cyclic tree) are then chosen blindly.

In the top-down query processing method presented in [Jo98] we insisted that every variable appearing in $N(C)$ also appears in $\text{antec}(C)$ (this being a common assumption made of deductive databases). This had the effect of ensuring that negative subgoals became grounded before expansion, as a result of the prior expansion of the positive subgoals arising from $\text{antec}(C)$. It is unclear, although certainly worthy of investigation, whether such an assumption, together with application of the demand driven instantiation of atoms in $\text{antec}(C)$ (and hence in $N(C)$) as described in [Ba97], would resolve this issue.

8.4 Answer size.

In [Jo98, Jo98a], we presented a top-down query processing method applicable to first order (stratified) databases. Suppose for example that we have a unary predicate $Q$, then the query $?Q(x)$ represents the query $\bigvee\{Q(a) | a \text{ is a constant in } \mathcal{L}\}$. An answer of the form $\bigvee_{1 \leq r} Q(a_i)$ is witnessed as being an answer by the absence of a cyclic strong cover, which in turn is represented by an extended deduction tree whose root is
labelled with \( \{ Q(a_i) \mid i \leq r \} \). The difficulty in this approach is deciding when to fix the size of the answer that we are seeking.

Conceptually the simplest approach is to start with the goal \(? \bigvee_{i \leq m} Q(x_i)\), where each \( x_i \) is a variable, and \( m \) is the number of constants in \( L \). As an extended deduction tree is generated, the variables \( x_i \) become instantiated, and eventually yield an answer. Of course \( m \) will typically be very large, and this approach is therefore extremely inefficient (allowing massive redundancy) in the case when the answers actually being generated are much smaller than \( m \). For non-unary predicates the potential redundancy is even worse.

The converse approach would be to first compute answers of size 1 (with initial goal \(? Q(x_1)\)), then answers of size 2 (with initial goal \(? Q(x_1) \lor Q(x_2)\)), and so on. The difficulty with this approach is knowing when the process can halt, since the computation of no new answers of size \( k \) implies nothing about the existence of minimal answers of size \( k + 1 \). (Indeed it is shown in [Jo96] that for propositional databases, the problem of determining whether a minimal answer of size \( k \) exists is \( \Sigma_2^P \)-complete.)

The approach presented in the present paper encounters neither of these problems.

### 8.5 Disjunctive logic programs.

The insistence that deductive databases are function free is of course motivated by the desire that queries should have a finite number of answers (and in the disjunctive case, that answers should themselves be finite). Disjunctive logic programs on the other hand allow rules with function symbols (i.e., they are full first order), whence query processing encounters the problems of first order undecidability. This is also true of minimal answer generation and testing [Ba97a, In02].

Limitations would therefore need to be imposed on the use of function symbols to allow an extension of our methods. Restrictions on the use of function symbols which yield a fragment of clausal logic decidable by hyper-resolution are presented in [Ge02], and we conjecture that similar results could be obtained in the present context. In addition, some preliminary results on the computation of cyclic trees in infinite propositional languages are to be found in [Jo01, Section 4].

It is also worth noting that at the first order level, subsumption can be employed to provide alternative definitions of the notion of a minimal answer ([Ba97, In02]). In [In02] it is shown that under such definitions, the problem of infinite answer sets can occur even in the function free case.

We intend to pursue these issues related to the first order level in a sequel.
§9. Conclusions and open questions

We have presented a method of computing (only) minimal answers of the form $\bigvee A$ in disjunctive deductive databases (and to the best of our knowledge this is the first such method). The method achieves this by generating (and extending) partial minimal answers, with verification being employed at each stage to ensure that a new predicate used to extend a partial answer has the required properties. We have also discussed the problems inherent in extending the method to unstratified databases under the disjunctive stable model semantics. The possible absence of disjunctive stable models would seem to imply the need to force the generation of disjunctive stable models in their entirety by the addition of denial rules of the form $P \land \neg P \rightarrow \text{FALSE}$. Compilation has been proposed as a solution to this problem, and also as a means of simplifying and improving the efficiency of the run-time computation.

The following observations and open questions are suggested by our results.

1. How does the method presented in the current paper compare in terms of computational efficiency with methods that also generate non-minimal answers? Is it useful, again from the viewpoint of computational efficiency, to combine the two, using the latter to generate answers $\bigvee H$, and then the current method to find minimal answers within $H$? This also raises the following question.

2. Let us say that a set $\{H_\alpha | \alpha \leq \beta\}$ of subsets of $L$ is a minimal answer mask iff for each minimal answer $A$ in $T$ there is some $\alpha \leq \beta$ such that $A \subseteq H_\alpha$.

How do we determine whether a given set is a minimal answer mask? Of course the set of all answers, and the set of all minimal answers are both minimal answer masks, and from which the computation of minimal answers is simple/trivial respectively. Is there a method of computing minimal answer masks so that, when combined with the methods of the current paper (as suggested in 1 above), we achieve some computational gain?

3. What is the computational complexity of determining whether:
   (i) there exists a non-empty cyclic strong cover;
   (ii) a given literal belongs to some maximal cyclic strong cover;
   (iii) every cyclic strong cover can be extended to a disjunctive stable model;
   (iv) a given literal belongs to some cyclic strong cover $C$ for which there is no cyclic strong cover $D$ such that $D^+ \cup D^- \supset C^+ \cup C^-$;
   (v) a given set is a minimal answer mask;
   (vi) given a formula $\Phi$, there is a subset $T^* \subseteq T$ such that $T^*$ has a disjunctive stable
(iv) The question given in (iv) is perhaps somewhat artificial, but it is of interest since a similar problem is shown in [Ei98] to be $\Sigma^P_3$-complete. Questions (vi)/(vii) ask whether there is a “consistent” subset of $T$ with which $\Phi$ can be inferred/is consistent. The property given in (vi) has application in argumentation-based dialogues in multi-agent systems [Pa02].

4. In [Pr91, Pr91a], Przymusinski introduced the notion of a disjunctive stationary model, these being consistent sets of literals $I$ such that $I^+$ is a minimal model of $T|_g(\mathcal{L} - I^-)$ and $\mathcal{L} - I^-$ is a minimal model of $T|_gI^+$. Total disjunctive stationary models clearly coincide with disjunctive stable models, and thus disjunctive stationary models can (also) be regarded as partial disjunctive stable models.

For non-disjunctive databases, stationary models are related to (supported) strong covers [Jo02, Section 4.2], but in the disjunctive case it is doubtful that any such relationship exists with cyclic strong covers. For example if $T = \{C \lor E, B \rightarrow D, B \rightarrow A, C \land \neg E \rightarrow A \lor B, A \land C \land \neg D \rightarrow B \lor E\}$, then $\{A, C, \neg E\}$ is a disjunctive stationary model, yet there is no cyclic strong cover containing $\neg A$.

Similarly the analogue of Theorem 2.9 does not hold for disjunctive stationary models. For example if $T = \{C \lor E, D \rightarrow A, \neg D \rightarrow B, \neg B \rightarrow D, C \land \neg E \rightarrow A \lor B\}$, then again $I = \{A, C, \neg E\}$ is a disjunctive stationary model, and indeed $\overline{T} = \{\neg A, \neg C, E\}$ is a strong cover. $T|_\overline{T} = \{\neg D \rightarrow B, \neg B \rightarrow D\}$ (Definition 2.8) has disjunctive stable models $\{B\}$ and $\{D\}$, but $\{A, B, C\}$ is not a disjunctive stable model of $T$.

Ultimately, the difference between disjunctive stationary models and cyclic strong covers lies in the motivation for their definition. Cyclicness insists that $\mathcal{C}^-$ is contained with any model of $T|_g(\mathcal{L} - \mathcal{C}^+) \land \neg \lor \mathcal{C}^+$, whereas stationarity insists that $I^+$ is one (of possibly many) minimal models of $T|_g(\mathcal{L} - I^-)$.

5. As indicated in Section 7, DWFS can be characterised using the variant notion of quasi-cyclic trees. We note however that again there appears to be no direct relationship between DWFS and cyclic strong covers. For example if $T = \{A \lor D, B \lor E, C \lor F, \neg A \rightarrow B, \neg B \rightarrow C, \neg C \rightarrow A, \neg D \rightarrow E, \neg E \rightarrow F, \neg F \rightarrow D\}$, then there is no non-empty cyclic strong cover, yet $\text{DWFS} = \{A \lor D, B \lor E, C \lor F\}$.

As an aside we recall that in [Jo01, Section 2] we showed that if $I$ is a disjunctive stationary model, then $I^+ \models \text{DWFS}^+$ and $I^- \supseteq \overline{\text{DWFS}}^-$, which in turn suggested the following question: “Given a minimal model $M$ of $\text{DWFS}^+$, is it the case that $M \cup \overline{\text{DWFS}}^-$ can always be extended to a disjunctive stationary model?” The above
database, having no disjunctive stationary models, shows the answer to this question to be negative.

6. If $T$ is stratified, and $S = ((A_i, C_i) | i \leq r)$ is a verified cyclic state, is it the case that we can construct a reduced database $T[S]$ such that the complete cyclic states in $T[S]$ characterise the complete extensions of $S$ in $T$?

7. Suppose that $T$ is unstratified, and that $S = ((A_i, C_i) | i \leq r)$ is a verified cyclic state. Suppose that some $C_i$ cannot be extended to a total cyclic strong cover, then (by the remarks following Theorem 2.7) $T \models \bigvee C_i^+$, where $C_i^+ \supseteq \{A_j | j \leq r, j \neq i\}$. What can we then infer (if anything) about the minimal answers related to $S$?

The only obvious (and weak) statement we can make, is that each $A_i$ is contained in some minimal answer in $T/C_i$ (by Theorem 2.7).

8. What is the relationship (if any) between the methods of the current paper, and other $\Sigma^P_2$ - complete problems, for example those relating to argumentation-based dialogues in multi-agent systems [Pa02].

9. Is there a weaker property than stratification which guarantees that every cyclic strong cover may be extended to a disjunctive stable model?

10. Our characterisations (e.g., Theorem 3.3) of minimal answers can easily be extended to the case when we consider answers to be disjunctions of literals (as opposed to disjunctions of predicates). We have not however detailed further the extension of our results to this case.

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Appendix A: Worked examples

Let $\text{INT}(\mathcal{L}) = \{Q_1, Q_2, Q_3, Q_4\}$ and $\text{INT}(T)$ consist of the following rules

1. $Q_2 \land Q_3 \land \neg R_1 \rightarrow Q_1 \lor Q_4$
2. $Q_1 \land \neg R_2 \rightarrow Q_2$
3. $S_2 \land \neg R_3 \rightarrow Q_3$
4. $S_3 \rightarrow Q_1 \lor Q_2 \lor Q_4$
5. $S_1 \rightarrow Q_2 \lor Q_3$
6. $R_1 \rightarrow Q_2$

then the partial cyclic trees (cf., Section 5) in $\text{INT}(T)$ are as follows.

\begin{itemize}
  \item $T_1$
  \item $T_2$
  \item $T_3$
  \item $T_4$
  \item $T_5$
  \item $T_6$
  \item $T_7$
  \item $T_8$
\end{itemize}

There is one further tree (for $Q_2$) which yields the same $S(T)$ set as $T_1$, whence we omit it. Below we present, for each tree, the set $S(T)$.

1. $T_1 : \\{\neg Q_1, \neg Q_2, \neg Q_3, Q_4, R_1, R_2, R_3, \neg S_2, \neg S_3\}$
By Theorem 2.6(c) (and the characterisation of weakly cyclic covers in terms of cyclic strong covers), if $C$ is a weakly cyclic cover containing $\neg Q_i$, then there must be some partial tree $T$ for $P$ such that $S(T) \subseteq C$. (By the above remark, $T_1$ must also count as a tree for $Q_2$.)

As indicated in Sections 2, we can depict the generation of weakly cyclic covers using (the analogues) of the operations presented in Section 2.11: in particular, negative atoms are expanded with the relevant $S(T)$ sets. For the sake of brevity in this example, we first compute the minimal strong covers of the above $S(T)$ sets in INT($T$):

1. $C_1 = \{\neg Q_1, \neg Q_2, \neg Q_3, Q_4, R_1, R_2, R_3, \neg S_2, \neg S_3\}$
2. $C_2 = \{\neg Q_1, \neg S_3, Q_2, Q_4, \neg R_2, R_1\}$
3. $C_3 = \{\neg Q_2, \neg R_1\}$
4. $C_{4a} = \{\neg Q_2, \neg S_1, Q_3, S_2\}$ and $C_{4b} = \{\neg Q_2, \neg S_1, Q_3, \neg R_3\}$
5. $C_{5a} = \{\neg Q_2, \neg S_3, Q_1, Q_4, Q_3, S_2\}$, $C_{5b} = \{\neg Q_2, \neg S_3, Q_1, Q_4, Q_3, \neg R_3\}$, $C_{5c} = \{\neg Q_2, \neg S_3, Q_1, Q_4, \neg R_1\}$
6. $C_6 = \{\neg Q_3, \neg S_2, R_3\}$
7. $C_{7a} = \{\neg Q_3, \neg S_1, Q_2, Q_1, R_1\}$, $C_{7b} = \{\neg Q_3, \neg S_1, Q_2, \neg R_2, R_1\}$
8. $C_8 = \{\neg Q_4, \neg S_3, Q_1, Q_2, R_1\}$.

Thus if $C$ is a weakly cyclic cover containing $\neg Q_i$, then it must contain one of (i) $C_1, C_2$, when $i = 1$; (ii) $C_1, C_3, C_{4a}, C_{4b}, C_{5a}, C_{5b}, C_{5c}$, when $i = 2$; (iii) $C_6, C_{7a}, C_{7b}$, when $i = 3$; and (iv) $C_8$ when $i = 4$.

We represent the int-total weakly cyclic covers as the branches through the following tree.
Note that the path to $\mathcal{C}_{5c}$ is not int-total, but has not been expanded since the set of literals along the path contains the set of literals along the path to the sibling $\mathcal{C}_3$, the extensions of which have already been generated. Notice also that in expanding $\neg Q_2$ we have not employed $\mathcal{C}_1$, since it would generate an inconsistent branch.

Although such trees allow us to represent weakly cyclic covers, it is unclear whether they provide a means of choosing cyclic covers during the computation of minimal answers.

Below we enumerate the int-total weakly cyclic covers identified. The reader will note that in some of these sets we have (for the sake of readability) left duplicates in. The notes on the right indicate the branch of the above tree.

**Case A.** $\neg Q_1 \in \mathcal{C}$
1. $\{\neg Q_1, \neg Q_2, \neg Q_3, Q_4, R_1, R_2, R_3, \neg S_2, \neg S_3\}$ $\mathcal{C}_1$
2. $\{\neg Q_1, \neg S_3, Q_2, Q_4, \neg R_2, R_1, \neg Q_3, \neg S_2, R_3\}$ $\mathcal{C}_2, \mathcal{C}_6$
3. $\{\neg Q_1, \neg S_3, Q_2, Q_4, \neg R_2, R_1, \neg Q_3, \neg S_1, Q_2, \neg R_2, R_1\}$ $\mathcal{C}_2, \mathcal{C}_{7b}$
4. $\{\neg Q_1, \neg S_3, Q_2, Q_4, \neg R_2, R_1, Q_3, S_1, S_2\} \quad \mathcal{C}_2, Q_3 \in \mathcal{C}$
5. $\{\neg Q_1, \neg S_3, Q_2, Q_4, \neg R_2, R_1, Q_3, S_1, \neg R_3\} \quad \mathcal{C}_2, Q_3 \in \mathcal{C}$

**Case B.** $Q_1 \in \mathcal{C}$ and $\neg Q_4 \in \mathcal{C}$
6. $\{Q_1, \neg Q_4, \neg S_3, Q_1, Q_2, R_1, \neg Q_3, \neg S_2, R_3\} \quad \mathcal{C}_8, \mathcal{C}_6$
7. $\{Q_1, \neg Q_4, \neg S_3, Q_1, Q_2, R_1, \neg Q_3, \neg S_1, Q_2, Q_1, R_1\} \quad \mathcal{C}_8, \mathcal{C}_{7a}$
8. $\{Q_1, \neg Q_4, \neg S_3, Q_1, Q_2, R_1, \neg Q_3, \neg S_1, Q_2, \neg R_2, R_1\} \quad \mathcal{C}_8, \mathcal{C}_{7b}$
9. $\{Q_1, \neg Q_4, \neg S_3, Q_1, Q_2, R_1, Q_3, S_1, S_2\} \quad \mathcal{C}_8, Q_3 \in \mathcal{C}$
10. $\{Q_1, \neg Q_4, \neg S_3, Q_1, Q_2, R_1, Q_3, S_1, \neg R_3\} \quad \mathcal{C}_8, Q_3 \in \mathcal{C}$
Case C. \( Q_1 \in \mathcal{C}, Q_4 \in \mathcal{C} \) and \( \neg Q_2 \in \mathcal{C} \)

11. \( \{Q_1, Q_4, \neg Q_2, \neg R_1, Q_3, S_2\} \)
12. \( \{Q_1, Q_4, \neg Q_2, \neg R_1, Q_3, \neg R_3\} \)
13. \( \{Q_1, Q_4, \neg Q_2, \neg R_1, \neg Q_3, \neg S_2, R_3\} \)
14. \( \{Q_1, Q_4, \neg Q_2, \neg S_1, Q_3, S_2\} \)
15. \( \{Q_1, Q_4, \neg Q_2, \neg S_1, Q_3, \neg R_3\} \)
16. \( \{Q_1, Q_4, \neg Q_2, \neg S_3, Q_1, Q_4, Q_3, S_2\} \)
17. \( \{Q_1, Q_4, \neg Q_2, \neg S_3, Q_1, Q_4, Q_3, \neg R_3\} \)
18. \( \{Q_1, Q_4, \neg Q_2, \neg S_3, Q_1, Q_4, \neg R_1\} \)

Case D. \( Q_1 \in \mathcal{C}, Q_4 \in \mathcal{C} \) and \( Q_2 \in \mathcal{C} \)

19. \( \{Q_1, Q_4, Q_2, \neg Q_3, \neg S_2, R_3, S_3, R_1\} \)
20. \( \{Q_1, Q_4, Q_2, \neg Q_3, \neg S_1, Q_2, Q_1, R_1, S_3\} \)
21. \( \{Q_1, Q_4, Q_2, \neg Q_3, \neg S_1, Q_2, \neg R_2, R_1, S_3\} \)
22. \( \{Q_1, Q_4, Q_2, Q_3, S_1, R_1, S_2\} \)
23. \( \{Q_1, Q_4, Q_2, Q_3, S_1, R_1, \neg R_3\} \)

A.1 Example. Suppose now that \( \text{EXT}(T) \) contains a single rule \( R_1 \lor S_3 \). There are then precisely 3 minimal int-total cyclic strong covers:

\[ \mathcal{D}_1 = \{Q_1, \neg Q_4, \neg S_3, R_1, Q_2, Q_3, S_1, S_2\} \]
\[ \mathcal{D}_2 = \{Q_1, Q_4, \neg Q_2, \neg R_1, S_3, Q_3, S_2\} \]
\[ \mathcal{D}_3 = \{Q_1, Q_4, \neg Q_2, \neg S_3, R_1, Q_3, S_2\} \]

from 9
from 11
from 16

[Note that \( \mathcal{D} = \{Q_1, Q_4, Q_2, Q_3, S_1, S_2\} \) is a completion of (22), but it is not a strong cover since \( \text{EXT}(T) \models \bigvee \mathcal{D}^+_{\text{ext}} \).]

Note that \( Q_1 \) and \( Q_3 \) do not therefore belong to any minimal answer. \( ((Q_2, \mathcal{D}_2)) \) is a (verified) cyclic state of length 1, thus let us consider its extension using the approach detailed in Section 5.7.3. We pick an int-total cyclic strong cover \( \mathcal{C} \) of \( \{Q_2\} \), the only choice being \( \mathcal{C} = \mathcal{D}_1 \).

Suppose first that we intend to (try to) extend using a predicate \( Q \) in \( \text{INT}(\mathcal{L}) \), then we require that \( Q \in \mathcal{D}_1^- \cap \mathcal{D}_2^+ \), and (not surprisingly) the only option is \( Q_4 \). We may then extend the cyclic state to \( ((Q_2, \mathcal{D}_2), (Q_4, \mathcal{D}_1)) \). There are no int-total cyclic strong covers of \( \{Q_2, Q_4\} \), whence \( Q_2 \lor Q_4 \) is a minimal answer (and clearly is the only minimal answer contained in \( \text{INT}(\mathcal{L}) \)).

Suppose that we now try to extend \( ((Q_2, \mathcal{D}_2)) \) with a predicate \( E \) in \( \text{EXT}(\mathcal{L}) \). Section 5.7.3 demands that \( E \notin \mathcal{D}_1^+ \cup \mathcal{D}_2^- \), and also that there exists a rule \( \bigvee \mathcal{E} \in \text{EXT}(T) \) such that \( E \in \mathcal{E} \) and \( \text{EXT}(T) \nvdash \bigvee (\mathcal{E} - \{E\}) \lor \bigvee (\mathcal{D}_1^+_{\text{ext}} \setminus \mathcal{E}) \). \( S_3 \) satisfies the conditions of Definition 5.7.3, with \( (\mathcal{E} - \{E\}) \cup \mathcal{D}_1 = \mathcal{D}_1 \), and \( ((Q_2, \mathcal{D}_2), (S_3, \mathcal{D}_1)) \) is an immediate extension of \( ((Q_2, \mathcal{D}_2)) \). \( \{Q_2, S_3\} \) has no int-total cyclic strong cover, whence \( Q_2 \lor S_3 \) is a minimal answer.
A.2 Example. Let $\text{EXT}(T) = \{S_3 \lor R_2, R_1 \lor R_2, S_1 \lor S_2 \lor R_3\}$.

Suppose that we start with the cyclic strong cover $F_1 = \{Q_1, Q_4, Q_2, \neg Q_3, \neg S_2, S_1, R_3, S_3, R_1\}$ of $\{\neg Q_3\}$ (by completing 19). We then look for an int-total cyclic strong cover of $\{Q_3\}$, say $F_2 = \{Q_1, Q_4, \neg Q_2, \neg R_1, R_2, Q_3, S_2\}$ (by completing 11).

If we wish to extend using a predicate in $\text{INT}(L)$, then we pick such a predicate in $F_2^- \cap F_1^+$, the only option being $Q_2$. $S_1 = ((Q_3, F_1), (Q_2, F_2))$ is not verified, since (for example) $G_1 = \{Q_1, Q_4, Q_2, Q_3, S_3, S_1, R_1, \neg R_3, S_2\}$ (from 23) is an int-total cyclic strong cover of $\{Q_2, Q_3\} \cup (F_1^+ \cap F_2^+) = \{Q_2, Q_3, Q_1, Q_4\}$. If $S_1$ is verifiable, then (by Theorem 5.6.1) there is a rule $\forall \epsilon \in \text{EXT}(T)$ such that $(\epsilon - G_1) \cap (F_1^- \cup F_2^-) = \emptyset$, $\epsilon \cap (G_1 - F_1) \neq \emptyset$ and $\text{EXT}(T) \not\models (F_1^+ \cup (\epsilon - G_1))_{\text{ext}}$. $S_1 \lor S_2 \lor R_3$ is such a rule, and therefore $((Q_3, F_1 \cup \{R_3\}), (Q_2, F_2 \cup \{R_3\}))$ is an immediate (and verified) extension of $S_1$, which can then be extended using the methods of Section 5.7.

A.3 Example. Suppose again that $\text{EXT}(T) = \{S_3 \lor R_2, R_1 \lor R_2, S_1 \lor S_2 \lor R_3\}$ and we start with the cyclic strong cover $F_1 = \{Q_1, Q_4, Q_2, \neg Q_3, \neg S_2, S_1, R_3, S_3, R_1\}$ of $\{\neg Q_3\}$ (by completing 19). We then look for a cyclic strong cover of $\{Q_3\}$, say $G_1 = \{Q_1, Q_4, Q_2, Q_3, S_3, S_1, R_1, \neg R_3, S_2\}$ (from 23).

If we attempt to extend $((Q_3, F_1))$ using a predicate in $\text{EXT}(L)$, then $R_3$ satisfies the conditions of Section 5.7.3, and $S_2 = ((Q_3, F_1), (R_3, G_1))$ is an immediate extension of $((Q_3, F_1))$.

It is easy to check that $\{Q_3, R_3\} \cup (F_1^+ \cap G_1^+) = \{Q_3, R_3, Q_1, Q_2, Q_4, S_1, S_3, R_1\}$ has no int-total cyclic strong cover, and thus $S_2$ is verified.

Again, $\{Q_3, R_3\}$ has an int-total cyclic strong cover, for example $G_2 = \{Q_1, Q_4, \neg Q_2, \neg S_1, S_2, R_3, Q_3\}$ (from 14). We look for a predicate in $\text{EXT}(L) - (G_2^+ \cup F_1^- \cup G_1^-) = \{S_1, S_3, R_1, R_2\}$ satisfying the conditions of Section 5.7.3. $S_1$ satisfies these conditions, and $((Q_3, F_1), (R_3, G_1), (S_1, G_2))$ is an immediate extension of $S_2$. It is easy to check that $\{Q_3, R_3, S_1\}$ has no int-total cyclic strong cover, whence is a minimal answer.

Appendix B: Transformation to a positive database.

In [Fe95] it is shown that the disjunctive stable models of $T$ can be computed from the perfect models of the stratified evidential transformation of $T$. In [Jo99] we employed cyclic trees (and variants there-of) to present a number of forms of pre-processing for stratified databases. In this section we extend one of these forms to the disjunctive stable model semantics. Specifically we show that the disjunctive stable models (and minimal answers) of $T$ can be computed (in a one-to-one fashion) from...
the minimal models (and minimal answers) of a transformed positive database in some extended language. Moreover the computation of minimal answers in the transformed database does not require the computation of cyclic trees.

For each cyclic tree $T$ in $T$ introduce a new predicate $Q_T$, and let $\mathcal{L}^* = \mathcal{L} \cup \{Q_T | T$ is a cyclic tree in $T\} \cup \{\text{FALSE}\}$. For each $P \in \mathcal{L}$, let $\phi(P) = \\land \{Q_T | T$ is a cyclic tree for $P$ in $T\}$.

For each rule $C$: $A_1 \land A_2 \land \ldots \land A_h \land \neg A_{h+1} \land \neg A_{h+2} \land \ldots \land \neg A_{h+r} \rightarrow B_1 \lor B_2 \lor \ldots \lor B_k$, let $C^*$ be the rule $A_1 \land A_2 \land \ldots \land A_h \land \phi(A_{h+1}) \land \phi(A_{h+2}) \land \ldots \land \phi(A_{h+r}) \rightarrow B_1 \lor B_2 \lor \ldots \lor B_k \lor \text{FALSE}$. Let $T^* = \{C^* | C \in T\} \cup \{P \lor Q_T | P \in \mathcal{L}, T$ is a cyclic tree for $P$ in $T\} \cup T'$, where $T' = \\{P \land \phi(P) \rightarrow \text{FALSE} | P \in \mathcal{L}\} \cup \{R \rightarrow Q_T | T$ is a cyclic tree in $T, R \in \mathcal{O}(T) \cup \mathcal{N}(T)\}$.

If $T$ contains no rule $C$ for which $\text{antec}(C) = \emptyset$ then $\emptyset$ is the unique disjunctive stable model of $T$ (in which case there are then no answers in $T$). We may therefore assume that $T$ contains at least one rule $C_0$ for which $\text{antec}(C_0) = \emptyset$ (whence $\text{antec}(C_0) \cup \mathcal{N}(C_0) = \emptyset$, cf., Section 1.1) and under these assumptions we note that $\mathcal{L}^* - \mathcal{L}$ is a minimal model of $T^*$.

Notice that $T^*$ is a positive database. We show that (minimal) models of $T^*$ characterise disjunctive stable models of $T$. Note also that $T'$ is definite (i.e., each rule’s head contains a single predicate), thus given $N \subseteq \mathcal{L}^*$, let $\text{cl}(N)$ be the (unique) smallest model of $T'$ containing $N$. Notice that if $M \subseteq \mathcal{L}$, then $\text{cl}(M) \cap \mathcal{L} = M$.

**B.1 Theorem.**

(a) If $M$ is a disjunctive stable model of $T$, then $\text{cl}(M) \models T^* \land \lnot \text{FALSE}$.

(b) Suppose that $N \subseteq \mathcal{L}^*$ and $N \models T^* \land \lnot \text{FALSE}$. Let $M = N \cap \mathcal{L}$, then $M$ is a disjunctive stable model of $T$ with $\text{cl}(M) \subseteq N$.

**Proof** (a) If $P \in M$, then there is some cyclic tree $T$ for $P$ in $T$ such that $[\mathcal{O}(T) \cup \mathcal{N}(T)] \cap M = \emptyset$, whence $Q_T \notin \text{cl}(M)$ (and in particular $\phi(P)$ is false in $\text{cl}(M)$).

Similarly if $P \in \mathcal{L} - M$, and $T$ is a cyclic tree for $P$ in $T$, then by Theorem 2.4(c) we must have that $[\mathcal{O}(T) \cup \mathcal{N}(T)] \cap M \neq \emptyset$ (else $\text{Pred}(T) \subseteq M$), whence $Q_T \in \text{cl}(M)$.

It thus follows trivially that $\text{cl}(M) \models T^* \land \lnot \text{FALSE}$.

(b) Suppose that $C \in T$ with $\text{antec}(C) \subseteq M$ and $M \cap \mathcal{N}(C) = \emptyset$. If $P \in \mathcal{N}(C)$, then $P \notin N$, whence $Q_T \in N$ for each cyclic tree $T$ for $P$ in $T$. Since $N \models C^*$ it then follows that $N \cap \text{conseq}(C^*) = M \cap \text{conseq}(C) \neq \emptyset$.

Suppose that $P \in M$. Since $\text{FALSE} \notin N$ there is a cyclic tree $T$ for $P$ in $T$ such that $Q_T \notin N$, whence $[\mathcal{O}(T) \cup \mathcal{N}(T)] \cap M = \emptyset$. By Theorem 2.4(c), $\text{Pred}(T) \subseteq M$ and by Theorem 2.6, $\overline{M} \cup (\mathcal{L} - M)$ is cyclic and $M$ is a disjunctive stable model. □
We may then easily obtain a one-to-one mapping between disjunctive stable models of $T$ and minimal models of $T^* \land \neg \text{FALSE}$.

**B.2 Corollary.**

(a) If $M \subseteq L$, then $M$ is a disjunctive stable model of $T$ iff $cl(M)$ is a minimal model of $T^* \land \neg \text{FALSE}$.

(b) Every minimal model $N$ of $T^* \land \neg \text{FALSE}$ has the form $N = cl(N \cap L)$, where $N \cap L$ is a disjunctive stable model of $T$.

**Proof (a).** If $M$ is a disjunctive stable model of $T$, then by the above theorem, $cl(M) \models T^* \land \neg \text{FALSE}$. Suppose that $N \subseteq cl(M)$ is a model of $T^*$, then by Theorem B.1(b), $N \cap L$ is a disjunctive stable model of $T$ with $N \cap L \subseteq cl(M) \cap L = M$. By the minimality of $M$ we must have that $M = N \cap L$, and hence that $cl(M) = cl(N \cap L) \subseteq N$.

For the converse, suppose that $cl(M)$ is a minimal model of $T^* \land \neg \text{FALSE}$. By part (b) of the above theorem, $M = cl(M) \cap L$ is a disjunctive stable model of $T$.

(b). If $N$ is a minimal model of $T^* \land \neg \text{FALSE}$, then by the above theorem, $N \cap L$ is a disjunctive stable model of $T$, with $cl(N \cap L) \subseteq N$ and $cl(N \cap L) \models T^* \land \neg \text{FALSE}$. By the minimality of $N$ we must have that $N = cl(N \cap L)$.

**B.3 Corollary.**

(a) If $A \subseteq L$, then $T \models \bigvee A$ iff $T^* \models \text{FALSE} \lor \bigvee A$.

(b) If $A \subseteq L$, then $A$ is a minimal answer in $T$ iff $\{\text{FALSE}\} \cup A$ is a minimal answer in $T^*$.

(c) $T$ has a disjunctive stable model iff $T^* \not\models \text{FALSE}$.

(d) If $P \in L$, then $P$ belongs to some disjunctive stable model of $T$ iff $T^* \not\models \text{FALSE} \lor \phi(P)$.

**Proof (a).** Suppose that $T \models \bigvee A$, and that $N \models T^* \land \neg \text{FALSE}$. By Theorem B.1(b), $N \cap L \cap A \neq \emptyset$. Conversely suppose that $T^* \models \text{FALSE} \lor \bigvee A$, and $M$ is a disjunctive stable model of $T$, then by Theorem B.1(a), $\emptyset \neq A \cap cl(M) = A \cap L \cap cl(M) = A \cap M$.

(b). Suppose that $A$ is a minimal answer in $T$. By the remarks preceding Theorem B.1, $L^* \setminus L$ is a model of $T^*$, whence $T^* \not\models \bigvee A$. Thus if $B \subseteq \{\text{FALSE}\} \cup A$ is a minimal answer in $T^*$, then $B = \{\text{FALSE}\} \cup B'$, where $B' \subseteq A$. But then by part (a), $T \models \bigvee B'$, thus contradicting the minimality of $A$.

For the converse, suppose that $\{\text{FALSE}\} \cup A$ is a minimal answer in $T^*$. Part (a) dictates that $T \models \bigvee A$. If $B \subseteq A$ with $T \models \bigvee B$, then $T^* \models \text{FALSE} \lor B$, thus contradicting the minimality of $\{\text{FALSE}\} \cup A$.
The proofs of (c) and (d) are trivial.  

We can thus compute minimal answers in $T$ via minimal answers in $T^*$. The following theorem (analogous to Theorem 3.3) provides the basis for such a computation, and also indicates that having computed cyclic trees in $T$, we do not need to repeat the process in $T^*$.

**B.4 Theorem.** If $A = \{A_i | i \leq r\} \subseteq \mathcal{L}$, then $A$ can be extended to a minimal answer in $T$ iff for each $i \leq r$ we may find a cyclic tree $T_i$ for $A_i$ in $T$ and a strong cover $C_i \subseteq \mathcal{L}^*$ of $\{A_j | j \leq r, j \neq i\} \cup \{Q_{T_i}, \text{FALSE}\}$ in $T^*$ such that $T^* \models \text{FALSE} \lor \bigvee_{i \leq r} A_i \lor \bigvee (\mathcal{L} \cap \bigcap_{i \leq r} C_i)$.

**Proof** ($\to$). Suppose that $A'$ is a minimal answer in $T$ containing $A$, and that for each $i \leq r$, $M_i$ is a disjunctive stable model of $T$ such that $M_i \cap A' = \{A_i\}$. Let $T_i$ be a cyclic tree for $A_i$ in $T$ for which $\text{Pred}(T_i) \subseteq M_i \subseteq \mathcal{L} - (O(T_i) \cup N(T_i))$.

By Corollary B.2(a), $cl(M_i)$ is a minimal model of $T^* \land \neg \text{FALSE}$ (whence $C_i = \mathcal{L}^* - cl(M_i)$ is a strong cover in $T^*$). It is easy to check that $C_i \supseteq \{A_j | j \leq r, j \neq i\} \cup \{Q_{T_i}, \text{FALSE}\}$.

Finally if $A \in A' - A$, then for each $i \leq r$ $A \notin M_i$, whence $A \notin cl(M_i)$, thus $\{\text{FALSE}\} \cup A' \subseteq \{\text{FALSE}\} \cup \{A_i | i \leq r\} \cup \bigcup_i (\mathcal{L} \cap \bigcap_{i \leq r} C_i)$, and the result then follows from Corollary B.3(a).

($\leftarrow$). Let $B \subseteq \{A_i | i \leq r\} \cup \bigcup_i (\mathcal{L} \cap \bigcap_{i \leq r} C_i)$ be such that $\{\text{FALSE}\} \cup B$ is a minimal answer in $T^*$ (whence $B$ is a minimal answer in $T$). But now $C_i$ is a strong cover in $T^*$, thus $\mathcal{L}^* - C_i \models T^*$, and $\emptyset \neq (\mathcal{L} - C_i) \cap (\{\text{FALSE}\} \cup B) \subseteq \{A_i\}$. Thus $\{A_i | i \leq r\} \subseteq B$.

Note that if we were to assume a partitioning of $\mathcal{L}$ into $\text{EXT}(\mathcal{L}) \cup \text{INT}(\mathcal{L})$ (as in Section 5) then the pre-processing described above would not be immune to changes in $\text{INT}(T)$. To accomplish such immunity we would (as in Section 5) need to employ cyclic trees in $\text{INT}(T) \cup \{E | E \in \text{EXT}(\mathcal{L})\}$, and this would allow us to transform the database into an equivalent database in which rules contain only *extensional* negative subgoals.