Abstract

Finite order invariants (Vassiliev invariants) of knots are expressed in terms of weight systems, that is, functions on chord diagrams satisfying the four-term relations. Weight systems have graph analogues, so-called 4-invariants of graphs, i.e. functions on graphs that satisfy the four-term relations for graphs. Each 4-invariant determines a weight system.

The notion of weight system is naturally generalized for the case of embedded graphs with an arbitrary number of vertices. Such embedded graphs correspond to links; to each component of a link there corresponds a vertex of an embedded graph. Recently, two approaches have been suggested to extend the notion of 4-invariants of graphs to the case of combinatorial structures corresponding to embedded graphs with an arbitrary number of vertices. The first approach is due to V. Kleptsyn and E. Smirnov, who considered functions on Lagrangian subspaces in a $2^n$-dimensional space over $\mathbb{F}_2$ endowed with a standard symplectic form and introduced four-term relations for them. On the other hand, the second approach, the one due to Zhukov and Lando, suggests four-term relations for functions on binary delta-matroids. In this paper, we prove that the two approaches are equivalent.

Finite order invariants (Vassiliev invariants) of knots are expressed in terms of weight systems, that is, functions on chord diagrams satisfying four-term relations. The vector space over $\mathbb{C}$ spanned by chord diagrams considered modulo four-term relations is supplied with a Hopf algebra structure. The notion of weight system is naturally extended from functions on...
chord diagrams (which can be interpreted as embedded graphs with a single vertex) to functions on arbitrary embedded graphs.

In [1], to each embedded graph a Lagrangian subspace in a symplectic space over the field $\mathbb{F}_2$ is associated. V. Kleptsyn, E. Smirnov in [7] rediscovered this construction. They introduced four-term relations in the vector space spanned by Lagrangian subspaces, and showed that linear functionals satisfying these four-term relations produce weight systems. They constructed a Hopf algebra of Lagrangian subspaces and a quotient Hopf algebra of Lagrangian subspaces modulo the four-term relations.

Meanwhile, Lando and Zhukov in [10] constructed a Hopf algebra of binary delta-matroids, introduced four-term relations for them and constructed a quotient Hopf algebra modulo the four-term relations. The correspondence between delta-matroids and embedded graphs allows one to associate a weight system to a linear functional on the latter Hopf algebra. The main result of the present paper is the proof of equivalence of these two approaches; in particular, we establish an isomorphism between the Hopf algebra of Lagrangian subspaces and the Hopf algebra of binary delta-matroids. This isomorphism is given by the mapping $\nu_E$, which establishes (according to Theorem 2.1) a one-to-one correspondence between the set of Lagrangian subspaces in $V_E$, the vector space spanned by the elements of a finite set $E$ as well as their duals, and binary delta-matroids on the set $E$.

1 Necessary information about delta-matroids

A set system $(E; \Phi)$ is a pair consisting of a finite set $E$ and a set $\Phi \subset 2^E$ of subsets of $E$. The set $E$ is called the ground set and the elements of the set $\Phi$ are called the feasible subsets of this system.

Two set systems $(E_1; \Phi_1), (E_2; \Phi_2)$ are said to be isomorphic if there exists a one-to-one correspondence $E_1 \rightarrow E_2$, which identifies the subsets $\Phi_1 \subset 2^{E_1}$ with $\Phi_2 \subset 2^{E_2}$. Below, we will not distinguish between isomorphic set systems.

A set system $(E; \Phi)$ is said to be proper if the set $\Phi$ is nonempty. In our paper, we consider only proper set systems if otherwise is not stated explicitly. We denote by $\Delta$ the set symmetric difference operation, that is, $A \Delta B = (A \setminus B) \sqcup (B \setminus A)$. A delta-matroid is a set system $(E; \Phi)$ that satisfies the following symmetric exchange axiom (SEA): for any two feasible subsets $\phi_1$ and $\phi_2 \in \Phi$ and for any element $e \in \phi_1 \Delta \phi_2$ there exists an element $e' \in \phi_1 \Delta \phi_2$ such that $\phi_1 \Delta \{e, e'\} \in \Phi$.

Let $G$ be an (abstract) simple graph. We will consider more general ob-
jects, namely, framed graphs, that is, graphs all whose vertices are endowed with an element 0 or 1 of the field $\mathbb{F}_2$. To each framed graph $G$, with the set of vertices $V(G)$, one can assign its adjacency matrix $A(G)$ (of dimension $|V(G)| \times |V(G)|$) on the intersection of the row $v$ and the column $v'$ of which $(v \neq v')$, there is the element 1 of the field $\mathbb{F}_2$ if the vertices $v$ and $v'$ are neighbors (that is, are connected by an edge), and the element 0, otherwise. In turn, the diagonal elements are equal to the frames of the corresponding vertices.

A framed graph $G$ is said to be non-degenerate if its adjacency matrix $A(G)$, considered as a matrix over the field $\mathbb{F}_2$, is non-degenerate, i.e. if its determinant equals 1. Let us define the set system $(V(G); \Phi(G))$, $\Phi(G) \subset 2^{V(G)}$ in the following way:

$$V(G) \quad \text{is the set of the vertices of } G$$

$$\Phi(G) = \{ U \subset V(G) \mid G_U \text{ is non-degenerate} \},$$

where $G_U$ denotes the subgraph in $G$ induced by the vertex set $U$.

**Theorem 1.1** ([2]) The set system $(V(G); \Phi(G))$ is a delta-matroid.

We call this delta-matroid the non-degeneracy delta-matroid of the graph $G$.

Non-degeneracy delta-matroids of framed graphs are examples of binary delta-matroids. To introduce the notion of binary delta-matroid, we need the operation of twisting. For a set system $D = (E; \Phi)$ and a subset $E' \subset E$, let us define the twist $D \ast E'$ of the set system $D$ by the subset $E'$ by the equation

$$D \ast E' = (E; \Phi \Delta E') = (E; \{ \phi \Delta E' \mid \phi \in \Phi \}).$$

Obviously, twisting of set systems by a subset is an involution, $D \ast E' \ast E' = D$.

**Theorem 1.2** ([4]) The twist of a non-degeneracy delta-matroid of a framed graph by any subset is a delta-matroid.

**Definition 1.1** ([4]) A binary delta-matroid is the result of twisting the non-degeneracy delta-matroid of a framed graph by (maybe an empty) subset.

Denote by $\mathcal{B}_E$ the set of binary delta-matroids with the ground set $E$. 

3
2 Binary delta-matroids and Lagrangian subspaces (set-theoretic bijection)

In this section we establish a one-to-one correspondence between the set of binary delta-matroids (on a finite set $E$) and the set of Lagrangian subspaces in the symplectic space $V_E$ over the field $\mathbb{F}_2$ associated with the set $E$.

Let $E$ be a finite set and $E^\vee$ be its copy. Denote by $e^\vee$ the element of $E^\vee$ corresponding to the element $e$ in $E$. We denote by $\vee: E \cup E^\vee \to E \cup E^\vee$ the bijection of $E \cup E^\vee$, which exchanges the elements $e$ and $e^\vee$ for all $e \in E$. For $Y \subset E \cup E^\vee$, denote by $Y^\vee$ the image of $Y$ under the map $\vee$.

A \textit{symplectic structure} on a vector space is a nondegenerate skew symmetric form on it. Symplectic structures exist only on even-dimensional spaces. Denote by $V_E$ the $2|E|$-dimensional space over the field $\mathbb{F}_2$ spanned by the elements of the set $E \cup E^\vee$. Let us introduce a symplectic structure $\langle \cdot, \cdot \rangle$ on $V_E$ by the rule $(e, e^\vee) = (e^\vee, e) = 1$, and $(u, v) = 0$ otherwise.

A subspace $L$ of a symplectic space is said to be \textit{isotropic} if the restriction of the symplectic form to $L$ is zero, i.e. $(u, v) = 0$ for all $u$ and $v$ in $L$. The dimension of an isotropic subspace of a symplectic space cannot exceed half of the dimension of the symplectic space itself. An isotropic subspace whose dimension is half the dimension of the symplectic space is called a \textit{Lagrangian subspace}. Denote by $\mathcal{L}_E$ the set of Lagrangian subspaces in $V_E$.

\textbf{Definition 2.1} (mapping $\nu_E$) Let $L$ be an arbitrary Lagrangian subspace in $V_E$. Denote by $\nu_E(L)$ the set system $\nu_E(L) = (E; \Psi_L)$, where a subset $Y \subset E$ belongs to $\Psi_L$ if and only if $L \cap (Y^\vee \cup (E \setminus Y)) = 0$; Here the angle brackets denote the vector subspace in $V_E$ spanned by the elements inside, and 0 is the zero vector of the space $V_E$.

\textbf{Example 2.2} Let $E$ be a 2-element set, $E = \{1, 2\}$, then $L = \langle 1^\vee + 2 + 2^\vee, 1+2 \rangle$ is a Lagrangian subspace in $V_E$. It consists of four elements, namely, $0, 1^\vee + 2 + 2^\vee, 1 + 2, 1 + 1^\vee + 2^\vee$. Then $\nu_E(L) = (E; \{\{1\}, \{2\}, \{1, 2\}\})$. (In [10], this set system is denoted by $s_{25}$.) Indeed, we have

for $Y = \emptyset$, $\langle Y^\vee \cup (E \setminus Y) \rangle = \langle 1, 2 \rangle$, $L \cap \langle 1, 2 \rangle \ni 1 + 2$;
for $Y = \{1\}$, $\langle Y^\vee \cup (E \setminus Y) \rangle = \langle 1^\vee, 2 \rangle$, $L \cap \langle 1^\vee, 2 \rangle = 0$;
for $Y = \{2\}$, $\langle Y^\vee \cup (E \setminus Y) \rangle = \langle 1, 2^\vee \rangle$, $L \cap \langle 1, 2^\vee \rangle = 0$;
for $Y = \{1, 2\}$, $\langle Y^\vee \cup (E \setminus Y) \rangle = \langle 1^\vee, 2^\vee \rangle$, $L \cap \langle 1^\vee, 2^\vee \rangle = 0$.

\textbf{Theorem 2.1} The mapping $\nu_E$ is a bijection between the set of Lagrangian subspaces $\mathcal{L}_E$ and the set $\mathcal{B}_E$ of binary delta-matroids on the set $E$.

\footnote{A similar mapping is considered in [12].}
We split the proof of this theorem into several lemmas.

**Definition 2.3** We say that a Lagrangian subspace $L$ in $V_E$ is **graphic** if for each $e \in E$ there exists an element $v_e \in L$ such that $(v_e, e) = 1$ and $(v_e, e') = 0$ for all $e' \in E, e' \neq e$.

By dimension consideration, the collection of such elements \{v_e\}, $e \in E$ forms a basis in the space $L$.

**Example 2.4** The Lagrangian subspace $L$ from Example 2.2 is not a graphic one. Indeed, for the element $e = 1 \in E$, there are two elements $v_e$ such that $(e, v_e) = 1$. (namely, $1^\vee + 2 + 2^\vee$ and $1 + 1^\vee + 2^\vee$), but for any such element $v_e$ the equality $(2, v_e) = 1$ holds as well.

The subspace $\langle 1^\vee, 2^\vee \rangle$ is an example of a graphic Lagrangian subspace in $V_{\langle 1^\vee, 2^\vee \rangle}$. (For $e = 1$, we can take $v_e = 1^\vee$, for $e = 2$ we take $v_e = 2^\vee$).

**Lemma 2.5** The mapping $\nu_E$ determines a bijection between graphic Lagrangian subspaces in $V_E$ and non-degeneracy delta-matroids of framed graphs on the set of vertex $E$.

**Proof.** Let $L \subset V_E$ be a graphic Lagrangian subspace; assign a symmetric $|E| \times |E\vee|$-matrix $A(L)$ over $\mathbb{F}_2$ to this subspace as follows: put $(v_e, e^\vee)$ on the intersection of the row $e$ and (The symmetry of the matrix follows from the fact that $L$ is Lagrangian: indeed, the equations $(v_e, e) = (v_{e'}, e') = 1$ (for $e \neq e'^\vee$), $(v_e, e') = (v_{e'}, e) = 0$ and $(v_e, v_{e'}) = 0$ imply that $(v_e, e'^\vee) = (v_{e'}, e^\vee)$ for all $e$ and $e'$. One can obtain an arbitrary symmetric matrix in this way. Conversely, from a symmetric matrix one can reconstruct the Lagrangian subspace. Indeed, $L$ is the Lagrangian subspace in $V_E$ spanned by the vectors $v_e = e^\vee + \sum_{e' \in E} A(L)_{e,e'^\vee} e'$.

On the other hand, to each framed graph $G$ with the vertex set $E$ its adjacency matrix $A(G)$ over $\mathbb{F}_2$ is associated. By putting $A(L) = A(G)$, we get a one-to-one correspondence between the two sets. Let us prove that under this correspondence the set system $\nu_E(L)$ assigned to the Lagrangian subspace $L$, is taken to the non-degeneracy delta-matroid of the graph $G$. Indeed, the subset $Y \subset E$ is feasible, $Y \in \Phi_L$, if and only if the sub-matrix $A|Y$ is non-degenerate over $\mathbb{F}_2$. The last statement is equivalent to the assertion that the subspace $L \cap \langle Y^\vee \cup (E \setminus Y) \rangle$ contains only a zero vector.

Let us prove the last statement. The subspace $L \cap \langle Y^\vee \cup (E \setminus Y) \rangle$ contains a non-zero vector if and only if there exists a non-zero linear combination $\sum_{e \in E} \lambda_e v_e$ (here $v_e = e^\vee + \sum_{e' \in E} A(L)_{e,e'^\vee} e'$) in $L$ belonging to
This means that there exist \( \lambda_e \in \mathbb{F}_2, e \in E \), not all equal to 0 and such that \( \sum_{e \in E} \lambda_e v_e^* = 0 \), where

\[
v_e^* = \begin{cases} 
  e^\vee + \sum_{e' \in Y} A(L)_{e,e' \vee e'}, & \text{if } e \in E \setminus Y \\
  \sum_{e' \in Y} A(L)_{e,e' \vee e'}, & \text{if } e \in Y 
\end{cases}
\]

(here \( v_e^* \) is the restriction of \( v_e \) to \( Y \cup (E \setminus Y) \)). This statement is equivalent to degeneracy of the matrix

\[
\begin{pmatrix} 0 & A \vert_Y \\
E & * \end{pmatrix},
\]

(here 0 is the zero matrix of the appropriate size), and hence of the matrix \( A \vert_Y \). We arrive at a contradiction. \( \square \)

For an arbitrary \( L \in \mathcal{L}_E \) and for an arbitrary \( e \in E \) denote by \( L \ast e \) the Lagrangian subspace obtained from \( L \) by the linear transformation of the space \( V_E \) of the form \( e \mapsto e^\vee, \ e^\vee \mapsto e \), acting trivially on the other vectors of the basis.

**Lemma 2.6** For an arbitrary \( L \in \mathcal{L}_E \) and an arbitrary \( e \in E \) the following statement is true: \( \nu_E(L) \ast e = \nu_E(L \ast e) \). In other words, local duality of Lagrangian subspaces descends to twisting of delta-matroids under the map \( \nu_E \).

**Proof.** Let \( Y \subset E \) be an arbitrary subset. Note that

\[
(L \ast e) \cap (Y^\vee \cup (E \setminus Y)) = L \cap ((Y^\vee \Delta \{e^\vee\}) \cup (E \setminus (Y \Delta \{e\}))).
\]

It follows that \( Y \) is a feasible subset for \( \nu_E(L \ast e) \) if and only if \( L \cap ((Y^\vee \Delta \{e^\vee\}) \cup (E \setminus (Y \Delta \{e\}))) = 0 \). Thus \( Y \Delta e \) is feasible for \( \nu_E(L) \) or, equivalently, \( Y \) is feasible for \( \nu_E(L) \ast e \). \( \square \)

Clearly, the operations \( \ast e \) and \( \ast e' \) specified by (not necessarily distinct) elements \( e, e' \in E \) commute with each other; therefore, the operation \( \ast E' \) is well defined for an arbitrary subset \( E' \subset E \).

**Lemma 2.7** For any Lagrangian subspace \( L \in \mathcal{L}_E \), there exists a subset \( E' \subset E \) such that the Lagrangian subspace \( L \ast E' \) is graphic.

**Proof.** We start with the choice of a “good” basis of \( L \). We proceed as follows.

Choose a vector \( e_1 \) from the standard basis \( E \cup E^\vee \) of \( V_E \) such that there exists a vector \( v_1 \in L \) such that \( (e_1, v_1) = 1 \). (Pick \( v_1 \) for the first element
Then pick a vector $e_2$ from the standard basis in $V_E$ such that there exists a vector $v_2 \in L$, with $(e_2, v_2) = 1$. Add the vector $v'_2 = v_2 - (e_1, v_2)v_1$ to the “good basis”. Repeat the procedure to obtain a basis in $L$ (similarly to the Gram–Schmidt process). Then apply to $L$ the local duality through the set of those $e_1, e_2, \ldots, e_{|E|}$ that belong to $E^\circ$. We obtain the subspace $L_1$. It corresponds to the matrix $A(L_1)$ (which is symmetric as long as $L_1$ is a Lagrangian space).

**Corollary 2.8** (follows from Lemmas 2.5, 2.6 and 2.7) The mapping $\nu_E$ takes every Lagrangian subspace in $V_E$ to a binary delta-matroid over the set $E$.

Now we can complete the proof of Theorem 2.1.

Let us prove that $\nu_E : \mathcal{L}_E \to \mathcal{B}_E$ is an injection. Suppose the converse. Then there exist distinct Lagrangian subspaces $L_1, L_2 \in \mathcal{L}_E$, such that $\nu_E(L_1) = \nu_E(L_2)$. Let $E' \subset E$ be the set corresponding to $L_1$ in Lemma 2.7. Then

$$\nu_E(L_1 * E') = \nu_E(L_1) * E' = \nu_E(L_2) * E' = \nu_E(L_2 * E'),$$

by Lemma 2.6. But it is shown in Lemma 2.5 that the equation $\nu_E(L_1 * E') = \nu_E(L_2 * E')$ implies that $L_1 * E' = L_2 * E'$. Therefore, $L_1 * E' * E' = L_2 * E' * E'$, i.e. $L_1 = L_2$.

Now let us prove that $\nu_E : \mathcal{L}_E \to \mathcal{B}_E$ is a surjection. Indeed, for every binary delta-matroid $B \in \mathcal{B}_E$ there exists a subset $E' \subset E$ such that $B * E'$ is a graphic delta-matroid. There exists a Lagrangian subspace $L \in \mathcal{L}_E$ such that $\nu_E(L) = B * E'$. Now $\nu_E(L) * E' = B$ and, by Lemma 2.6, $\nu_E(L) * E' = \nu_E(L * E')$, i.e. $\nu_E(L * E') = B$.

Theorem 2.1 is proven.

### 3 Lagrangian subspaces and binary delta-matroids of embedded graphs

Denote by $\mathcal{G}_E$ the set of connected ribbon graphs with the set of ribbons labeled by the elements of $E$.

In [1], a mapping from $\mathcal{G}_E$ to $\mathcal{L}_E$ is constructed. It has the following form. Let $\Gamma$ be a connected ribbon graph with the set of ribbons $E$ interpreted as the union of two sets of closed topological disks called vertices $V(G)$ and edges $E(G)$ satisfying the following conditions:

- edges and vertices intersect by disjoint line segments;
Figure 1: A ribbon graph without discs removed around the centers of
the vertices, with elements $h_e, h_{e\vee}$ of the first relative homology group
$H_1(F_\Gamma, \partial F_\Gamma)$ assigned to the edge $e$

- each such segment lies in the closure of precisely one edge and one
  vertex;
- each edge contains two such segments.

Given a ribbon graph $\Gamma$, remove small open discs from the centers of
the vertices, which are discs. Let $F_\Gamma$ denote the resulting two-dimensional
surface with a boundary.

To each $e \in E$, we associate $h_e$, an element of the relative homology
group $H_1(F_\Gamma, \partial F_\Gamma)$. This element is represented by a segment going along
the edge $e$ and connecting the boundaries of the discs that are removed from
the vertices incident to the edge $e$.

On the other side, to each element $e\vee \in E\vee$ we may associate an element
$h_{e\vee}$ in the relative homology group $H_1(F_\Gamma, \partial F_\Gamma)$ that is represented by a
segment that goes across the edge $e$ and connects the opposite sides of this
edge (see Fig. 1).

To each continuous cycle $\gamma : S^1 \to F_\Gamma$, we associate the vector
$\sum_{e\in E}((\gamma, h_e)h_e + (\gamma, h_{e\vee})h_{e\vee})$ in $V_E$. (The brackets $(\cdot, \cdot)$ in this formula
denote the intersection form between the first absolute and relative homol-
ogy for the given surface with boundary $F_\Gamma$). As shown in [1, 7], the subspace
of $V_E$ formed by the vectors that correspond to all cycles $\gamma$, is Lagrangian.
Denote this subspace by $\pi_E(\Gamma)$.

On the other hand, Bouchet [4] assigned to each ribbon graph a set
system whose ground set is the set of edges of the graph: a subset of edges
is feasible if the restriction of the given graph to this subset is a quasi-tree, that is, a ribbon graph with a connected boundary. Bouchet showed that the set system assigned to a ribbon graph in such a way is a delta-matroid. We denote this delta-matroid by $\rho E(\Gamma)$.

**Theorem 3.1** The mapping $\nu E$ is compatible with the mappings $\pi E$ and $\rho E$. Namely, for an arbitrary $\Gamma \in G_E$ the following identity holds:

$$\rho E(\Gamma) = \nu E(\pi E(\Gamma)).$$

**Proof.** Let first $\Gamma$ be a ribbon graph with a single vertex, i.e. a (framed) chord diagram. Then the statement is true, since both mappings are compatible with the mapping which assigns to a chord diagram $\Gamma$ the adjacency matrix of its intersection graph. Conversely, each of the mappings is compatible with the twist operation on the corresponding ribbon graphs $\rho E(\Gamma \ast e) = (\pi E(\Gamma)) \ast e$. For an arbitrary ribbon graph $\Gamma$ find a set $E' \subset E$ such that $\Gamma \ast E'$ has a single vertex; then $(\rho E(\Gamma)) \ast E' = \nu E(\pi E(\Gamma) \ast E') = \nu E(\pi E(\Gamma)) \ast E'$, i.e. $(\rho E(\Gamma)) \ast E' = \nu E(\pi E(\Gamma)) \ast E'$, hence $\rho E(\Gamma) = \nu E(\pi E(\Gamma))$ as required. ■

### 4 Hopf Algebras Isomorphism

Let $n = |E|$. Denote by $L_n$ the set of isomorphism classes of Lagrangian subspaces $L_E \subset V_E$ with respect to bijections of $n$-element sets.

Let $B_n$ denote the set of isomorphism classes of binary delta-matroids on $n$ elements.

Klepsyn and Smirnov in [7] introduce the structure of a graded commutative and cocommutative Hopf algebra on the infinitely dimensional vector space

$$\mathbb{C}L = \mathbb{C}L_0 \oplus \mathbb{C}L_1 \oplus \cdots,$$

where $\mathbb{C}L_n$ is the vector space over $\mathbb{C}$ freely spanned by the set $L_n$. Multiplication in this Hopf algebra is given by the operation of direct sum of Lagrangian subspaces in the direct sum of symplectic spaces, which is extended to $\mathbb{C}L$ by linearity. The comultiplication $\mathbb{C}L \to \mathbb{C}L \otimes \mathbb{C}L$ assigns to a Lagrangian subspace $L \subset V_E$ the sum of the tensor products of the Lagrangian subspaces

$$L \mapsto \sum_{I \subset E} L_I \otimes L_{E \setminus I},$$

where, for a subset $I$ of the set $E$, $L_I \subset V_I$ denotes the subspace, which is the symplectic reduction of the Lagrangian subspace $L$ (see [7]). This multiplication can be naturally transferred to the vector space $\mathbb{C}L$, spanned
by the Lagrangian subspaces, considered up to renumbering finite element sets.

Meanwhile, in [10], a graded Hopf algebra of binary delta-matroids is constructed

\[ \mathbb{C}B = \mathbb{C}B_0 \oplus \mathbb{C}B_1 \oplus \cdots, \]

where the subspace \( \mathbb{C}B_n \) is freely spanned over \( \mathbb{C} \) by the set \( B_n \). The multiplication in this Hopf algebra is given by the direct sum of set systems extended to \( \mathbb{C}B \) by linearity. The coproduct of a given set system \((E; \Psi)\) is the sum

\[ \mu(E; \Psi) = \sum_{E' \subset E} \Psi|_{E'} \otimes \Psi|_{E \setminus E'}, \]

where the set \( \Psi|_{E'} \) consists of those elements of the set \( \Psi \) that are contained in \( E' \).

The mapping \( \nu_E \) (see Def. 2.1) is equivariant with respect to bijections of finite sets both on the set of Lagrangian subspaces and on the set of binary delta-matroids. Hence the set of such mappings defines a graded linear mapping

\[ \nu : \mathbb{C}L \to \mathbb{C}B, \quad \nu_n : \mathbb{C}L_n \to \mathbb{C}B_n, \quad n = 0, 1, 2, \ldots. \]

This linear mapping appears to be an isomorphism:

**Theorem 4.1** The mapping \( \nu : \mathbb{C}L \to \mathbb{C}B \) is a graded isomorphism of Hopf algebras.

**Proof.** The mapping \( \nu \) transfers the multiplication and the comultiplication in the Hopf algebra of Lagrangian subspaces to the multiplication and the comultiplication, respectively, in the algebra of binary delta-matroids. This can be seen from the definitions above. \(\blacksquare\)

## 5 Four-term relations and weight systems

In [14] V. A. Vassiliev introduced the four-term relations for functions on chord diagrams. He proved that any invariant of order at most \( n \) determines a function on chord diagrams that satisfies these relations. Such a function is called a weight system. Every four-term relation corresponds to a chord diagram and to a pair of chords with neighboring ends in it. The remaining three diagrams that participate in this relation can be built from the initial one by application of one of the two (mutually commuting) Vassiliev moves, and their compositions. In [9] Vassiliev moves were extended
to framed diagrams, which are chord diagrams associated to ribbon graphs with possibly twisted ribbons, and the corresponding four-term relations were described.

Kleptsyn and Smirnov in [7] extended Vassiliev moves to Lagrangian subspaces. Let, as above, $E$ be a finite set, $V_E$ be the vector space over $\mathbb{F}_2$ spanned by the elements of the set $E \sqcup E'$, and let $e, e' \in E$ be two distinct elements in $E$. Then the first Vassiliev move, assigned to a pair $e, e'$, is a linear mapping $V_E \to V_E$ preserving all the basis vectors except for the vectors $e^\vee, e'^\vee$. The action on these vectors is defined as follows:

$$
e^\vee \mapsto e^\vee + e'; \quad e'^\vee \mapsto e'^\vee + e.$$

Notice that the first Vassiliev move is symmetric with respect to the transposition of the elements $e$ and $e'$.

The second Vassiliev move for the pair $e, e'$ is a linear mapping $V_E \to V_E$ obtained from the first move by conjugation with respect to the twist along the element $e' \in E$, see Sec. 2. In contrast to the first move, the description of the second one depends on the order of elements in the pair $e, e'$. The action of each Vassiliev move on the set of Lagrangian subspaces is induced by its action on $V_E$.

In [10], the authors define the first and the second Vassiliev moves for binary delta-matroids $\mathcal{B}_E$. To define the second Vassiliev move, they use the recently introduced (see [13]) concept of handle sliding for delta-matroids. In [10], it is shown (see Proposition 4.10) that the action of the first and second Vassiliev moves on the space $V_E$ as defined by Kleptsyn–Smirnov coincides with the one defined by Zhukov and Lando for binary delta-matroids. Taking into account Theorem 2.1, we obtain the following statement.

**Theorem 5.1** The graded Hopf algebras isomorphism $\nu : \mathbb{C}\mathcal{L} \to \mathbb{C}\mathcal{B}$ descends to a graded quotient Hopf algebras isomorphism $\nu : \mathbb{F}\mathbb{C}\mathcal{L} \to \mathbb{F}\mathbb{C}\mathcal{B}$, that of the Hopf algebras $\mathbb{C}\mathcal{L}$ and $\mathbb{C}\mathcal{B}$ modulo the corresponding four-term relations.

**References**

[1] Booth, Richard F.; Borovik, Alexandre V.; Gelfand, Israel M.; Stone, David A. *Lagrangian matroids and cohomology*. Ann. Comb. 4 (2000), no. 2, 171–182.
[2] A. Bouchet, *Greedy algorithms and symmetric matroids*, Math. Programm. 38 (1987), 147–159

[3] Bouchet, A.(F-LMNS-IC) *Representability of \( \Delta \)-matroids*. Combinatorics (Eger, 1987), 167–182, Colloq. Math. Soc. Janos Bolyai, 52, North-Holland, Amsterdam, 1988.

[4] A. Bouchet, *Maps and delta-matroids*, Discrete Math. 78 (1989), 59–71

[5] S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás–Riordan polynomial*, J. of Combin. Theory Ser. B 99 (2009) 617–638

[6] C. Chun, I. Moffatt, S. D. Noble, R. Rueckriemen, *Matroids, delta-matroids and embedded graphs*, arXiv: 1403.0920v1, 45 pp.

[7] V. Kleptsyn, E. Smirnov *Ribbon graphs and bialgebra of Lagrangian subspaces*, Journal of Knot Theory and Its Ramifications Vol. 25, No. 12 (2016) 1642006

[8] S. K. Lando, *On a Hopf algebra in graph theory*, J. Comb. Theory, Ser. B, vol. 80 (2000), 104–121.

[9] S. K. Lando, *J-invariants of ornaments and framed chord diagrams*, Funct. Anal. Appl., 40(1) (2006), 1–13.

[10] S. Lando, V. Zhukov, *Delta-matroids and Vassiliev invariants*, arXiv:1602.00027

[11] S. Lando, A. Zvonkin, *Graphs on surfaces and their applications*, Springer, 2004.

[12] Malic G. *An action of the Coxeter group BCn on maps on surfaces, Lagrangian matroids and their representations* arXiv:1507.01957v3

[13] Iain Moffatt, Eunice Mphako-Banda, *Handle slides for delta-matroids*, arXiv:1510.07224, 12 pp.

[14] V. A. Vassiliev, *Cohomology of knot spaces*, in: Theory of singularities and its applications, 23-69, Adv. Soviet Math., 1, Amer. Math. Soc., Providence, RI, 1990.