The 4-girth-thickness of the complete multipartite graph

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Abstract

The $g$-girth-thickness $\theta(g,G)$ of a graph $G$ is the smallest number of planar subgraphs of girth at least $g$ whose union is $G$. In this paper, we calculate the 4-girth-thickness $\theta(4,G)$ of the complete $m$-partite graph $G$ when each part has an even number of vertices.

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1 Introduction

The thickness $\theta(G)$ of a graph $G$ is the smallest number of planar subgraphs whose union is $G$. Equivalently, it is the smallest number of parts used in any edge partition of $E(G)$ such that each set of edges in the same part induces a planar subgraph.

This parameter was introduced by Tutte [20] in the 60s. The problem to calculate the thickness of a graph $G$ is an NP-hard problem [16] and a few of exact results can be found in the literature, for example, if $G$ is a complete graph [2, 5, 6], a hypercube [15], or a complete multipartite graph for some particular values [21, 22]. Even for the complete bipartite graph there are only partial results [7, 13].

Some generalizations of the thickness for complete graphs have been studied, for instance, the outerthickness $\theta_o$, defined similarly but with outerplanar instead of planar [12], the $S$-thickness $\theta_S$, considering the thickness on a surface $S$ instead of the plane [4], and the $k$-degree-thickness $\theta_k$ taking a restriction on the planar subgraphs: each planar subgraph has maximum degree at most $k$ [9].

The thickness has applications in the design of circuits [1], in the Ringel’s earth-moon problem [14], and to bound the achromatic numbers of planar graphs [3], etc. See the survey [17].
In [19], the author introduced the $g$-girth-thickness $\theta(g, G)$ of a graph $G$ as the minimum number of planar subgraphs of girth at least $g$ whose union is $G$, a generalization of the thickness owing to the fact that the $g$-girth-thickness is the usual thickness when $g = 3$ and also the arboricity number when $g = \infty$ because the girth of a graph is the size of its shortest cycle or $\infty$ if it is acyclic. See also [11].

In this paper, we obtain the 4-girth-thickness $\theta(4, K_{n_1,n_2,\ldots,n_m})$ of the complete $m$-partite graph $K_{n_1,n_2,\ldots,n_m}$ when $n_i$ is even for all $i \in \{1, 2, \ldots, m\}$.

2 Calculating $\theta(4, K_{n_1,n_2,\ldots,n_m})$

Given a simple graph $G$, we define a new graph $G \bowtie G$ in the following way: If $G$ has vertex set $V = \{w_1, w_2, \ldots, w_n\}$, the graph $G \bowtie G$ has as vertex set two copies of $V$, namely, $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and two vertices $x, y$ are adjacent if $w_i w_j$ is an edge of $G$, for the symbols $x, y \in \{u, v\}$. For instance, if $w_1w_2$ is an edge of a graph $G$, the graph $G \bowtie G$ has the edges $u_1u_2, v_1v_2, u_1v_2$ and $v_1u_2$. See Figure 1.

![Figure 1: An edge of $G$ produces four edges in $G \bowtie G$.](image)

On the other hand, an acyclic graph of $n$ vertices has at most $n - 1$ edges and a planar graph of $n$ vertices and girth $g < \infty$ has at most $\frac{g-2}{g-2}(n-2)$ edges, see [8]. Therefore, a planar graph of $n$ vertices and girth at least 4 has at most $2(n-2)$ edges for $n \geq 4$ and at most $n-1$, otherwise. In consequence, the 4-girth-thickness $\theta(4, G)$ of a graph $G$ is at least $\left\lceil \frac{|E(G)|}{2(n-2)} \right\rceil$ for $n \geq 4$ and at least $\left\lceil \frac{|E(G)|}{n-1} \right\rceil$, otherwise.

Lemma 2.1. If $G$ is a tree of order $n$ then $G \bowtie G$ is a bipartite planar graph of size $2(2n-2)$.

Proof. By induction over $n$. The basis is given in Figure 1 for $n = 2$. Now, take a tree $G$ with $n + 1$ vertices. Since it has at least a leaf, we say, the vertex $w_1$ incident to $w_2$, then we delete $w_1$ from $G$ and by induction hypothesis, $H \bowtie H$ is a bipartite planar of size $2(2n-2)$ edges for $H = G \setminus \{w_1\}$. Since $H$ is connected, the vertex labeled $w_2$ has at least a neighbour, we say, the vertex labeled $w_3$, then $u_2v_3v_2$ is a path in $H \bowtie H$ and the edge $u_2v_2 \notin E(H \bowtie H)$. Add the paths $u_2v_1v_2$ and $u_2u_1v_2$ to $H \bowtie H$ such that both of them
are “parallel” to $u_2v_3v_2$ and identify the vertices $u_2$ as a single vertex as well as the vertices $v_2$. This proves that $G \bowtie G$ is planar. To verify that is bipartite, given a proper coloring of $H \bowtie H$ with two colors, we extend the coloring putting the same color of $v_3$ to $v_1$ and $u_1$. Then the resulting coloring is proper. Due to the fact that we add four edges, $H \bowtie H$ has $2(2n - 2) + 4 = 2(2(n + 1) - 2)$ edges and the lemma follows.

Now, we recall that the arboricity number or $\infty$-girth-thickness $\theta(\infty, G)$ of a graph $G$ equals (see [18])

$$\max \left\{ \left\lfloor \frac{|E(H)|}{|V(H)|-1} \right\rfloor : H \text{ is an induced subgraph of } G \right\}.$$ 

We have the following theorem.

**Theorem 2.2.** If $G$ is a simple graph of $n \geq 2$ vertices and $e$ edges, then

$$\left\lceil \frac{e}{n-1} \right\rceil \leq \theta(4, G \bowtie G) \leq \theta(\infty, G).$$

**Proof.** Since $G \bowtie G$ has $2n \geq 4$ vertices, $4e$ edges and

$$\frac{|E(G \bowtie G)|}{2(|V(G \bowtie G)| - 2)} = \frac{4e}{2(2n - 2)} = \frac{e}{n-1}$$

it follows the lower bound

$$\left\lceil \frac{e}{n-1} \right\rceil \leq \theta(4, G \bowtie G).$$

To verify the upper bound, take an acyclic edge partition $\{F_1, F_2, \ldots, F_{\theta(\infty, G)}\}$ of $E(G)$. Therefore, $\{F_1 \bowtie F_1, F_2 \bowtie F_2, \ldots, F_{\theta(\infty, G)} \bowtie F_{\theta(\infty, G)}\}$ is an edge partition of $E(G \bowtie G)$ (where $F_i \bowtie F_i := E((F_i) \bowtie (F_i))$ and $(F_i)$ is the induced subgraph of the edge set $F_i$ for all $i \in \{1, 2, \ldots, \theta(\infty, G)\}$). Indeed, an edge $x_jy'_j \in E(G \bowtie G)$ is in $F_i \bowtie F_i$ if and only if $w_jw'_j \in E(G)$ is in $F_i$. By Lemma 2.1 the result follows.

**Corollary 2.3.** If $G$ is a simple graph of $n \geq 2$ vertices and $e$ edges with $\theta(\infty, G) = \left\lceil \frac{e}{n-1} \right\rceil$, then

$$\theta(4, G \bowtie G) = \left\lfloor \frac{e}{n-1} \right\rfloor.$$ 

Next, we estimate the arboricity number of the complete $m$-partite graph.

**Lemma 2.4.** If $K_{n_1, n_2, \ldots, n_m}$ is the complete $m$-partite graph then $\theta(\infty, G) = \left\lceil \frac{e}{n-1} \right\rceil$ where $n = n_1 + n_2 + \ldots + n_m$ and $e = n_1n_2 + n_1n_3 + \ldots + n_{m-1}n_m$.

**Proof.** By induction over $n$. The basis is trivial for $K_{1,1}$. Let $G = K_{n_1, n_2, \ldots, n_m}$ with $n > 2$ and $H = G \setminus \{u\}$ a proper induced subgraph of $G$ for any vertex $u$. By the induction hypothesis,

$$\theta(\infty, H) = \max \left\{ \left\lceil \frac{|E(F)|}{|V(F)|-1} \right\rceil : F \leq H \right\} = \left\lceil \frac{|E(H)|}{(n-1)-1} \right\rceil,$$ 

where $F \leq H$ indicates that $F$ is an
induced subgraph of $H$. Since $u$ is an arbitrary vertex and by the hereditary property of the induced subgraphs, we only need to show that

\[
\frac{|E(H)|}{n-2} \leq \frac{e}{n-1}
\]

because

\[
\max \left\{ \left[ \frac{|E(F)|}{|V(F)| - 1} \right] : F \leq G \right\} = \max \left\{ \left[ \frac{e}{n-1} \right], \left[ \frac{|E(H)|}{n-2} \right] : H = G \setminus \{u\}, u \in V(G) \right\}.
\]

We prove it in the following way. Without loss of generality, $u$ is a vertex in a part of size $n_m$.

Since

\[
\begin{align*}
n_1 &+ n_1n_2 + \ldots + n_1n_m + n_2^2 + n_1n_2 + \ldots + n_1n_m + \\
n_2 &+ \ldots + n_2n_m + n_2n_1 + n_2^2 + \ldots + n_2n_m \\
&\vdots \\
n_{m-1} &+ n_{m-1}n_1 + n_{m-1}n_2 + \ldots + n_{m-1}n_m
\end{align*}
\]

then $e + n_1 + n_2 + \ldots + n_{m-1} \leq n(n_1 + n_2 + \ldots + n_{m-1})$ and

\[
\begin{align*}
en - e - n(n_1 + n_2 + \ldots + n_{m-1}) + (n_1 + n_2 + \ldots + n_{m-1}) &\leq en - 2e \\
(n-1)(e - (n_1 + n_2 + \ldots + n_{m-1})) &\leq e(n - 2)
\end{align*}
\]

and the result follows. \qed

Now, we can prove our main theorem.

**Theorem 2.5.** If $G = K_{2n_1,2n_2,\ldots,2n_m}$ is the complete $m$-partite graph then $\theta(4, G) = \left\lceil \frac{n}{n-1} \right\rceil$ where $n = n_1 + n_2 + \ldots + n_m$ and $e = n_1n_2 + n_1n_3 + \ldots + n_{m-1}n_m$.

**Proof.** We need to show that $G = K_{n_1,n_2,\ldots,n_m} \bowtie K_{n_1,n_2,\ldots,n_m}$. Let $(W_1, W_2, \ldots, W_m)$ be an $m$-partition of $K_{n_1,n_2,\ldots,n_m}$. The graph $K_{n_1,n_2,\ldots,n_m} \bowtie K_{n_1,n_2,\ldots,n_m}$ has the partition $(U_1 \cup V_1, U_2 \cup V_2, \ldots, U_m \cup V_m)$ where $U_i$ and $V_i$ are copies of $W_i$ for $i \in \{1, 2, \ldots, m\}$. Take two vertices $x_i$ and $y_j$ in different parts, without loss of generality, $U_1 \cup V_1$ and $U_2 \cup V_2$. If the vertex $x_i$ is in $U_1$ and $y_j$ is in $U_2$ then they are adjacent because $w_iw_j$ is an edge of $K_{n_1,n_2,\ldots,n_m}$ is $m$-complete. Similarly for $x_i \in V_1$ and $y_j \in V_2$. If $x_i$ is in $U_1$ and $y_j$ is in $V_2$, then also they are adjacent because $w_iw_j$ is an edge of $K_{n_1,n_2,\ldots,n_m}$. By Corollary 2.3 and Lemma 2.4 the theorem follows. \qed

Due to the fact that $\theta(4, G) = \theta(3, G) = \theta(G)$ for any triangle-free graph $G$, we obtain an alternative proof for the thickness of the complete bipartite graph $K_{2n_1,2n_2}$ that is given in [7].

**Corollary 2.6.** If $G = K_{2n_1,2n_2}$ is the complete bipartite graph then $\theta(G) = \left\lceil \frac{e}{n-1} \right\rceil$ where $n = n_1 + n_2$ and $e = n_1n_2$.  

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