Upper Bounds and Extreme Results for Conflict-free Vertex-connection Number

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Abstract. A path of a vertex-colored graph is conflict-free path, if there exists a color used only on one of its vertices; a vertex-colored graph is conflict-free vertex-connected, if there is a conflict-free path between each pair of distinct vertices of the graph. For a connected graph $G$, the minimum number of colors required to make $G$ conflict-free vertex-connected is conflict-free vertex-connection number of $G$, denoted by $vcfc(G)$. In this paper, we first showed an upper bound of $vcfc(G)$ for the general graph by structural method. And then, we gave a partial solution to the conjecture on the conflict-free vertex-connection number by contradiction, posed by Doan and Schiermeyer in [Conflict-free vertex connection number at most 3 and size of graphs, Discuss. Math. Graph Theory].

Keywords: Conflict-free vertex-connection number; Upper bounds; Size of a graph.

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1. Introduction

All graphs mentioned in this paper are simple, undirected and finite. We follow Bondy and Murty’s book [4] for undefined notation and terminology. A block of a graph is a maximal connected subgraph of $G$ that has no cut-vertex. Then a block of a graph is either a cut-edge, called a trivial block, or a maximal 2-connected subgraph, called a nontrivial block.

Even et al. in [14] introduced the concept of hypergraph conflict-free coloring. The coloring was motivated to solve the problem of assigning frequencies to different base stations in cellular networks. There are a number of base stations and clients in the network. Each base station is a vertex in the hypergraph which needs to be allocated to a frequency. For each client, in order to make connection with one of the base station in the range, there must be at least one base station with a unique frequency in the range for fear of interference. Unnecessarily, many different frequencies can be expensive, so this situation may be converted to a conflict-free vertex-coloring problem of a hypergraph seeking for the minimum number of colors which is defined as the conflict-free chromatic number of the hypergraph. Scholars have studied various geometric hypergraphs, more information for the conflict-free coloring can be seen from the papers [3, 9, 10, 23].

As an important deformation and extension of conflict-free coloring of hyper-graphs, conflict-free coloring on neighborhoods [1, 15, 16, 21, 22] and conflict-free (vertex-)connection coloring of simple graphs are introduced and widely developed, which are correspond to the conflict-free coloring of special hypergraphs. Czap et al. [11] introduced the concept of conflict-free connection of the graph...
inspired by the theory of conflict-free coloring of the hypergraph by Even et al. in [14]. A path in an edge-colored graph \( G \) is called conflict-free if there is a color appearing only once on the path. The graph \( G \) is called conflict-free connected if there is a conflict-free path between each pair of distinct vertices of \( G \). For a connected graph \( G \), the minimum number of colors required to make \( G \) conflict-free connected is called the conflict-free connection number of \( G \), denoted by \( \text{cfc}(G) \). Up to now, there have shown the basic characterizations, the bounds of \( \text{cfc}(G) \) for the graph \( G \) in [5, 6, 7, 12], the hardness was also showed that deciding the \( \text{vcfc}(G) \) is NP-hard for the graph \( G \) in [17, 18].

As a natural counterpart of the conflict-free connection, Li et al. in [19] introduced the concept of conflict-free vertex-connection of graphs. A path in a vertex-colored graph is called conflict-free if there is a color appearing on only one vertex of the path. A vertex-colored graph is called conflict-free vertex-connected if there is a conflict-free path between every pair of distinct vertices of \( G \). For a connected graph \( G \), the minimum number of colors required to make \( G \) conflict-free vertex-connected is called the conflict-free vertex-connection number of \( G \), denoted by \( \text{vcfc}(G) \). They characterized some graphs with \( \text{vcfc}(G) = 3 \) and general upper bounds in [19], and Li and Wu showed upper bound of the conflict-free vertex-connection number for a connected graph \( G \) is the conflict-free vertex-connection number of its spanning tree in [20]. Doan and Schiermeyer showed an extreme result for conflict-free vertex-connection number at most 3 and posed a conjecture in [13].

**Conjecture 1.1** [13] Let \( k \geq 3 \) be an integer, and \( G \) be a connected graph of order \( n \). If \( |E(G)| \geq \left( n - \left( \frac{2^k - 2}{2} \right) \right) + 2k - 1 \), then \( \text{vcfc}(G) \leq k \).

Li and Wu [20] show an upper bound of \( G \) that the conflict-free vertex-connection number of the connected graph \( G \) with order \( n \) is no more than the conflict-free vertex-connection number of \( P_n \).

**Lemma 1.2** [19] If \( P_n \) is a path of order \( n \), then \( \text{vcfc}(P_n) = \lceil \log_2(n+1) \rceil \).

**Lemma 1.3** [20] For a connected graph \( G \) of order \( n \), \( \text{vcfc}(G) \leq \text{vcfc}(P_n) \).

Now we give another one upper bound for \( \text{vcfc}(G) \) using the number of cut-vertices and the number of blocks.

**Theorem 1.4** Let \( G \) be a connected graph with \( t \) cut-vertices. Then \( \text{vcfc}(G) \leq \lceil \log_2(t+1) \rceil + 1 \).

From Theorem 1.4 we have the following result.

**Theorem 1.5** Let \( k \geq 3 \) be a positive integer, and \( G \) be a connected graph of order \( n \). If \( |E(G)| \geq \left( n - \left( \frac{2^k - 2}{2} \right) \right) + 2k - 1 \) and \( 1 > 2^k \left( 1 - \left( \frac{1}{2} \right)^{\frac{k}{2}} \right) \) for \( 0 < \varepsilon < 1 \), then \( \text{vcfc}(G) \leq k \).

In addition, we also need another two auxiliary lemmas to show our theorems.

**Lemma 1.6** Let \( G \) be a 2-connected graph and \( x \) be a vertex of \( G \). Then for any two vertices \( u \) and \( v \) in \( G \), there is a \( u-v \) path containing the vertex \( x \).

**Lemma 1.7** [13] Let \( t \) be an integer and \( G \) be a connected graph. If \( G \) contains \( t \) cut-vertices, then \( |E(G)| \leq \left( \frac{n-t}{2} \right) + t \).

### 2. The Proof of Theorems

For a connected graph \( G \) of order \( n \) with \( t \) cut-vertices and \( s \) leaves, if \( t + s = n \), then it is called a cactus-like graph.

Proof of Theorem 1.4. Let \( B_1, B_2, ..., B_k \) be the blocks of \( G \). For each nontrivial block \( B_i \) with \( i \in [k] \), if there is only one cut-vertex in \( B_i \), then we retain the cut-vertex and one neighbor of it in \( B_i \); if there are exactly two cut-vertices \( v_1, v_2 \) in \( B_i \), then we replace \( B_i \) by \( v_1v_2 \); if there are at least \( r \geq 3 \) cut-vertices in \( B_i \), then we construct the structure \( R \): if there is a path, in which each internal vertex is not cut-vertex, between each pair of two cut-vertices, then we replace the path by an edge. Thus we replace \( B_k \) by \( R \); retain the trivial block. Clearly, the resulting graph is a cactus-like graph, denoted by \( H_G \). Let \( H_0 \) contain \( s \) leaves. We call \( H_0 \) a cactus-like graph from \( G \).

We denote by \( H \) the resulting graph by deleting \( s \) leaves of \( H_0 \). It follows that \( \text{vcfc}(H) \leq \lceil \log_2(t+1) \rceil \) by Lemma 1.2 and Lemma 1.3. Let \( TH \) be a spanning tree of \( H \). Clearly, there exists a conflict-free vertex-connection coloring \( c^* : V(TH) \rightarrow \{ \lceil \log_2(t+1) \rceil \} \) for the tree \( T_n \), thus there always exists a unique coloring in the connected subtree of \( T_n \). Now we first define a coloring \( c : V(H_0) \rightarrow \{ \lceil \log_2(t+1) \rceil + 1 \} \) of \( H_0 \) by assigning the vertices of \( V(H) \) with the coloring \( c^* \) to make it conflict-free vertex-connected and assigning the \( s \) leaves with one fresh color \( \lceil \log_2(t+1) \rceil + 1 \). We can check
easily that the coloring is a conflict-free vertex-coloring. Hence, \(\text{vcfc}(H_0) \leq \log_2(t + 1) + 1\). Next, we define a coloring \(c'\) of \(G\): \(c'(v) = c(v)\) for each cut-vertex \(v \in V(G)\); \(c'(v) = \lceil \log_2(t + 1) \rceil + 1\) for the remaining vertices. For each pair of vertices contained in the same block in \(G\), since there is a unique color used on some cut-vertex \(x\) in the block by the coloring, there is a conflict-free path containing \(x\) assigned by a unique color by Lemma 1.6. For each pair of vertices \(u\) and \(v\) contained in two distinct blocks in \(G\), suppose that \(y_1\) is one cut-vertex contained in the block containing \(u\) and \(y_l\) is one cut-vertex contained in the block containing \(v\). Clearly, there is conflict-free path by the coloring of \(H_G\). Assume that there are \(l\) cut-vertices, say \(y_1, y_2, \ldots, y_l\) in order, contained in the path connecting \(y_1\) and \(y_l\). Without loss of generality, let \(y_i\) be the cut-vertex assigned by the unique color in the conflict-free path between \(y_1\) and \(y_l\). Clearly, there is also a conflict-free path containing \(y_i\). Hence, \(\text{vcfc}(G) \leq \lceil \log_2(t + 1) \rceil + 1\).

Proof of Theorem 1.5. Suppose, on the contrary, that there is a connected graph \(G\) such that \(|E(G)| \geq \binom{n-(2^k-2)}{2} + 2^k - 1\), but \(\text{vcfc}(G) \geq k + 1\). Assume that \(G\) has \(t\) cut-vertices. By Theorem 1.4, we have \(\text{vcfc}(G) \leq \lceil \log_2(t + 1) \rceil + 1\). Thus, \(k \leq \lceil \log_2(t + 1) \rceil\).

If \(\lceil \log_2(t + 1) \rceil = \log_2(t + 1)\), then we have \(2^k - 1 \leq t\). Thus, \(\binom{n-(2^k-1)}{2} + 2^k - 1\). On the other hand, we have that \(|E(G)| \leq \binom{n}{2} + t\) by Lemma 1.7. Hence, \(|E(G)| \leq \binom{n-(2^k-1)}{2} + 2^k - 1\) contradicts to \(|E(G)| \geq \binom{n-(2^k-2)}{2} + 2^k - 1\).

If \(\lceil \log_2(t + 1) \rceil = \log_2(t + 1) + \varepsilon\), where \(0 < \varepsilon < 1\), then it is clear to see that \(2^{k-\varepsilon} - 1 < t\). Thus, \(\binom{n}{2} + t \leq \binom{n-(2^k-\varepsilon-1)}{2} + 2^k - 1\). However, \(|E(G)| \leq \binom{n}{2} + t\) by Lemma 1.7. Hence, \(|E(G)| \leq \binom{n-(2^k-\varepsilon-1)}{2} + 2^k - 1\) \(1 > 2^k \left(1 - \left(\frac{1}{2}\right)^k\right)\), which again contradicts to \(|E(G)| \geq \binom{n-(2^k-2)}{2} + 2^k - 1\).

3. Conclusion
We have showed that Theorem 1.5 using a new upper bound of conflict-free vertex-connection number by a nice structure. It illustrated that this method is worked. In the future work, we will prove the conjecture by more structure.

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