Resolution of Chern–Simons–Higgs Vortex Equations

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Abstract

It is well known that the presence of multiple constraints of non-Abelian relativistic Chern–Simons–Higgs vortex equations makes it difficult to develop an existence theory when the underlying Cartan matrix $K$ of the equations is that of a general simple Lie algebra and the strongest result in the literature so far is when the Cartan subalgebra is of dimension 2. In this paper we overcome this difficulty by implicitly resolving the multiple constraints using a degree-theorem argument, utilizing a key positivity property of the inverse of the Cartan matrix deduced in an earlier work of Lusztig and Tits, which enables a process that converts the equality constraints to inequality constraints in the variational formalism. Thus this work establishes a general existence
theorem which settles a long-standing open problem in the field regarding the general solvability of the equations.

**Keywords:** Chern–Simons–Higgs vortex equations, non-Abelian gauge theory, Lie algebras, Cartan matrices, constraints, calculus of variations, minimization.

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1 Introduction

The main result of this paper is a proof of a general existence theorem for the doubly-periodic solutions of the relativistic non-Abelian Chern–Simons–Higgs vortex equations whose Cartan matrix $K$ is that of an arbitrary simple Lie algebra. The motivation of our work originates from theoretical physics and development of new methods of mathematical analysis to tackle systems of nonlinear equations of physical interests. Below we begin with a description of some of the field-theory aspects of our study.

**Field-theoretical origins.** In 1957, Abrikosov [1] predicted in the context of the Ginzburg–Landau theory [18,30,55,64] that vortices of a planar lattice structure may appear in superconductors of the second type. Due to the complexity of the problem, a complete mathematical understanding of such Abrikosov vortices has not been obtained yet, although in an over simplified situation, known as the Bogomol’nyi–Prasad–Sommerfield (BPS) limit [9, 53], an existence and uniqueness theorem is proved under a necessary and sufficient condition [67] where the Abrikosov lattice structure is realized through gauge-periodic boundary conditions conceptualized and formulated by ’t Hooft [63] so that the governing elliptic equation is defined over a doubly periodic planar domain. Since early 1980, there has been a growing interest in accommodating electrically charged vortices in condensed-matter physics [27–29,69] such that the presence of the Chern–Simons dynamics becomes imperative due to the Julia–Zee theorem which says that vortices in the classical Yang–Mills–Higgs models can only carry magnetism. For a recent review on these subjects and related literature, see [72]. It is relevant to describe such a non-go theorem in the context of the simplest Yang–Mills–Higgs model here.

**The Julia–Zee theorem.** Consider the classical Yang–Mills–Higgs (YMH) model in the adjoint representation of $SU(2)$ over the Minkowski spacetime $\mathbb{R}^{2,1}$ of signature $(+−−)$ and use $\mu, \nu = 0, 1, 2$ to denote the spacetime coordinate indices. Let $A_\mu$ and $\phi$ be the gauge and Higgs fields written as isovectors, respectively. Then the action density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} + \frac{1}{2} D^\mu \phi \cdot D_\mu \phi - \frac{\lambda}{4} (|\phi|^2 - 1)^2,$$

where the field strength tensor $F_{\mu \nu}$ is defined by $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e A_\mu \times A_\nu$ and the gauge-covariant derivative $D_\mu$ is defined by $D_\mu \phi = \partial_\mu \phi + e A_\mu \times \phi$. Here $\lambda, e$ are positive
coupling constants. The Euler–Lagrange equations of (1.1) in the static limit are

\[ D_i D_i \phi = -e^2(|A_0|^2 \phi - [\phi \cdot A_0] A_0) + \lambda \phi (|\phi|^2 - 1), \]

(1.2)

\[ D_j F_{ij} = e (A_0 \times F_{i0} + \phi \times D_i \phi), \]

(1.3)

\[ \Delta A_0 = -e (\partial_i [A_i \times A_0] + A_i \times [\partial_i A_0 + e A_i \times A_0]) + e^2 \phi \times (A_0 \times \phi), \]

(1.4)

over the space \( \mathbb{R}^2 \), among which (1.4) is also the Gauss law constraint. The two-dimensional nature of these equations implies that their solutions may be interpreted as the YMH vortices. On the other hand, the associated energy density, or Hamiltonian, of (1.1) may be calculated to be

\[ H = \frac{1}{4} |F_{ij}|^2 + \frac{1}{2} |\partial_i A_0 + e A_i \times A_0|^2 + \frac{1}{2} |D_i \phi|^2 + \frac{1}{2} e^2 |A_0 \times \phi|^2 + \frac{\lambda}{4} (|\phi|^2 - 1)^2, \]

(1.5)

which is positive definite. In [59] it is proved that under the finite-energy condition \( E = \int_{\mathbb{R}^2} H \, dx < \infty \), any solution of (1.2)–(1.4) must have a trivial temporal component for its gauge field:

\[ A_0 = 0. \]

(1.6)

This result, first observed in [40] for the Abelian Higgs model, later referred to as the Julia–Zee theorem [59], has an important physical implication. Recall that, based on consideration on the manner of interactions, 't Hooft [62] proposed that the electromagnetic field \( F_{\mu\nu} \) in the YMH theory is to be defined by the formula

\[ F_{\mu\nu} = \frac{1}{|\phi|} \phi \cdot F_{\mu\nu} - \frac{1}{e|\phi|^3} \phi \cdot (D_\mu \phi \times D_\nu \phi). \]

(1.7)

Hence, inserting (1.6) into (1.7), we obtain \( F_{01} = F_{02} = 0 \), which indicates that there is no induced electric field in the model. In other words, the YHM vortices, Abelian or non-Abelian, are purely magnetic and electrically neutral.

**Emergence of the Chern–Simons type models.** The development of theoretical physics requires the presence of both electrically and magnetically charged vortices, also called dyons, since such dually charged vortices have found applications in a wide range of areas including high-temperature superconductivity [43, 47], optics [8], the Bose–Einstein condensates [37, 42], the quantum Hall effect [58], and superfluids [17, 54, 57]. Thanks to the studies of Jackiw–Templeton [38], Schonfeld [50], Deser–Jackiw–Templeton [19, 20], Paul–Khare [52], de Vega–Schaposnik [65, 66], and Kumar–Khare [45], starting from the early 1980’s, it has become accepted that, in order to accommodate electrically charged vortices, one needs to introduce into the action Lagrangian a Chern–Simons topological term [15, 16], which has also become a central structure in anyon physics [29, 68, 69]. On the other hand, despite of the importance of electrically charged vortices with the added Chern–Simons dynamics, it has been a difficult issue until rather recently [12] to construct finite-energy solutions of the field equations because of the indefiniteness of the action functional as a consequence of the Minkowski signature of spacetime and the presence of electricity. In 1990, it came as a fortune
that Hong, Kim, and Pac [35] and Jackiw and Weinberg [39] showed that when one uses only the Chern–Simons term and switches off the usual Maxwell term in the Abelian Higgs model one can achieve a BPS structure and thus arrive at a dramatic simplification of the governing equations. Subsequently, the ideas of [35,39] were extended to non-Abelian gauge field theory models and a wealth of highly interesting systems of nonlinear elliptic equations of rich structures governing non-Abelian Chern–Simons–Higgs vortices was unearthed [22–24]. More recently, these ideas have also been further developed in supersymmetric gauge field theory in the context of the Bagger–Lambert–Gustavsson (BLG) model [4–7,11,25,31] and the Aharony–Bergman–Jafferis–Maldacena (ABJM) model [2,14,33,44] which have been the focus of numerous activities in contemporary field-theoretical physics.

The present work is a complete resolution of the most general relativistic Chern–Simons–Higgs vortex equations defined over a doubly periodic planar domain with the Cartan matrix of an arbitrary simple Lie algebra. In the next section, we describe the vortex equations and some of the key technical issues, including methodology. In the section that follows, we state our main existence theorem. In the subsequent sections, we prove this theorem.

2 Vortex equations, technical issues, and methodology

Let $K = (K_{ij})$ be the Cartan matrix of a finite-dimensional semisimple Lie algebra $L$. We are interested in the relativistic Chern–Simons–Higgs vortex equations [22,24,70,71] of the form

$$\Delta u_i = \lambda \left( \sum_{j=1}^{n} \sum_{k=1}^{n} K_{kj} e^{u_j} e^{u_k} - \sum_{j=1}^{n} K_{ji} e^{u_j} \right) + 4\pi N_i \sum_{j=1}^{N_i} \delta_{p_{ij}}(x), \quad i = 1, \ldots, n, \quad (2.1)$$

where $n \geq 1$ is the rank of $L$ which is the dimension of the Cartan subalgebra of $L$, $\delta_p$ denotes the Dirac measure concentrated at the point $p$, $\lambda > 0$ is a coupling constant, and the equations are considered over a doubly periodic domain $\Omega$ resembling a lattice cell housing a distribution of point vortices located at $p_{ij}, j = 1, \ldots, N_i, i = 1, \ldots, n$. For an existence theory the ultimate goal is to obtain conditions under which (2.1) allows or fails to allow a solution. In order to see the technical difficulties of the problem, we take the beginning situation $n = 1$ as an illustration for which the underlying gauge group may be either $U(1)$ or $SU(2)$ which is of fundamental importance in applications, such that (2.1) takes the scalar form

$$\Delta u = \lambda e^u (e^u - 1) + 4\pi N \sum_{j=1}^{N} \delta_{p_{ij}}(x), \quad x \in \Omega. \quad (2.2)$$

Let $u_0$ be doubly periodic modulo $\Omega$ and satisfy $\Delta u_0 = -\frac{4\pi N}{|\Omega|} + 4\pi \sum_{j=1}^{N} \delta_{p_{ij}}(x)$. Then $u = u_0 + v$ recasts (2.2) into

$$\Delta v = \lambda e^{u_0 + v} (e^{u_0 + v} - 1) + \frac{4\pi N}{|\Omega|}, \quad x \in \Omega, \quad (2.3)$$
which is the Euler–Lagrange equation of the action functional

\[
S(v) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla v|^2 + \frac{\lambda}{2} e^{2u_0+2v} - \lambda e^{u_0+v} + \frac{4\pi N}{|\Omega|} v \right\} \, dx.
\]

(2.4)

It is easily seeing by taking constants as test functions that (2.4) is not bounded from below over the space of doubly periodic functions. In order to tackle such a difficulty, it is to take into account of a natural constraint arising from integrating (2.3). That is,

\[
\int_{\Omega} \left( e^{u_0+v} - \frac{1}{2} \right)^2 \, dx = \frac{|\Omega|}{4} - \frac{4\pi N}{\lambda},
\]

(2.5)

which indicates that there fails to permit a solution when \( \lambda \leq \frac{16\pi N}{|\Omega|} \). At a first glance, it may be attempting to believe that a solution may be obtained by minimizing (2.4) subject to (2.5). Unfortunately, there arises a Lagrange multiplier issue which prohibits a minimization process under the equality constraint (2.5). To overcome this issue, we may take the decomposition \( v = c + w \) where \( c \in \mathbb{R} \) and \( \int_{\Omega} w \, dx = 0 \) and rewrite the constraint (2.5) as

\[
e^{2c} \int_{\Omega} e^{2u_0+2w} \, dx - e^c \int_{\Omega} e^{u_0+w} \, dx + \frac{4\pi N}{\lambda} = 0,
\]

(2.6)

which becomes a solvable quadratic equation in \( \xi = e^c \) if and only if the discriminant of (2.6) stays nonnegative,

\[
\left( \int_{\Omega} e^{u_0+w} \, dx \right)^2 - \frac{16\pi N}{\lambda} \int_{\Omega} e^{2u_0+2w} \, dx \geq 0,
\]

(2.7)

so that \( c \) may be represented as (say)

\[
c = \ln \left( \int_{\Omega} e^{u_0+w} \, dx + \sqrt{\left( \int_{\Omega} e^{u_0+w} \, dx \right)^2 - \frac{16\pi N}{\lambda} \int_{\Omega} e^{2u_0+2w} \, dx} \right) - \ln \left( 2 \int_{\Omega} e^{2u_0+2w} \, dx \right).
\]

(2.8)

Then it may be shown that a solution of (2.3) can be obtained by minimizing the action functional

\[
I(w) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla w|^2 + \frac{\lambda}{2} e^{2u_0+2c+2w} - \lambda e^{u_0+c+w} \right\} \, dx + 4\pi Nc,
\]

(2.9)

descending from (2.4), subject to the inequality constraint (2.7), with \( c \) given by (2.8), when \( \lambda \) is sufficiently large so that a minimizer occurs in the exterior of the constraint class which rules out the Lagrange multiplier problem mentioned earlier. Finally, using the maximum principle and a continuity argument, it may be shown that there is a critical value of \( \lambda \), say \( \lambda_c > 0 \), such that there is no solution when \( \lambda < \lambda_c \) and solution exists when \( \lambda > \lambda_c \). See [10,71] for details. Later, it was shown in [60] that there is solution at \( \lambda = \lambda_c \) as well. More results on existence and asymptotic behavior of doubly periodic solutions for (2.3) can be found in [21,50]. Thus our understanding about the scalar case (2.2) or (2.3) is fairly satisfactory.
When \( G = SU(3) \) so that \( n = 2 \), things already become rather complicated because now one needs to resolve two coupled quadratic constraints. This more complicated problem was studied by Nolasco and Tarantello [51] who refined and improved the inequality-constrained minimization method developed in [10] for the \( n = 1 \) situation and showed that solutions in this \( n = 2 \) situation exist as well when \( \lambda \) is sufficiently large. Note that, since in this case we are treating a system of equations, for which the maximum principle cannot be used and thus a continuity argument as that in [10] is not available, an existence result under the condition that \( \lambda \) is sufficient large may be the best one can hope for. See [13] for some new development regarding more generalized \( 2 \times 2 \) systems arising in the Chern–Simons theory.

The contribution of the present article is a successful settlement of the situation when the gauge group \( G \) is any compact group and in particular the Cartan matrix \( K \) of the equations is that of an arbitrary simple Lie algebra, of rank \( n \). The system now consists of \( n \) nonlinear equations and results in \( n \) quadratic constraints, which cannot be resolved explicitly as in [10, 51]. In order to unveil the constraints difficulties we encounter and the varied levels of efficiency of the implicit constraint-resolution methods we use, we shall present a special method that works for the \( SU(4) \) situation, which has been extended to tackle the \( SU(N) \) situation in [34], and then a general method that works for all situations. The special method may be described as a “squeeze-to-the-middle” implicit-iteration strategy whose validity depends on the structure of the Cartan matrix of \( SU(N) \). The general method, on the other hand, uses a degree-theorem argument, which does not depend on the detailed specific numeric structures of the Cartan matrix. Rather, we shall see that, for a simple Lie algebra (say), things may be worked out miraculously to ensure an acquirement of all the needed \textit{apriori} estimates so that the multiple quadratic constraints allow an implicit resolution in any situation under consideration.

### 3 Chern–Simons–Higgs equations and existence theorem

Our purpose is to carry out a complete resolution of the existence of doubly periodic solutions to (2.1) with very general Cartan matrix \( K \). In order to treat the system in a unified framework, we need some suitable assumption on the matrix \( K \). For a semi-simple Lie algebra, we know that the associated Cartan matrix has the property: the diagonal entries \( K_{ii} \) assume the same positive integer 2, all off-diagonal entries \( K_{ij} \) \((i \neq j)\) can only assume the non-positive integers \(-3, -2, -1, 0, \) and \( K_{ji} = 0 \) if \( K_{ij} = 0 \). This motivates us to consider (2.1) with a general matrix \( K \), which satisfies

\[
K^\tau = PS,
\]

where \( P \) is a diagonal matrix with

\[
P \equiv \text{diag}\{P_1, \ldots, P_n\}, \quad P_i > 0, \quad i = 1, \ldots, n,
\]
$S$ is a positive definite matrix of the form

$$S \equiv \begin{pmatrix}
\alpha_{11} & -\alpha_{12} & \cdots & \cdots & \cdots & -\alpha_{1n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha_{i1} & -\alpha_{i2} & \cdots & \alpha_{ii} & \cdots & -\alpha_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha_{n1} & -\alpha_{n2} & \cdots & \cdots & -\alpha_{nn-1} & \alpha_{nn}
\end{pmatrix},$$ (3.3)

$$\alpha_{ii} > 0, \quad i = 1, \ldots, n, \quad \alpha_{ij} = \alpha_{ji} \geq 0, \quad i \neq j = 1, \ldots, n,$$ (3.4)

and all the entries of $S^{-1}$ are positive. (3.5)

In fact the assumptions on the matrix on $K$ are broad enough to cover all simple Lie algebras realized as $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \ [36, 41]$, thanks to the work of Lusztig and Tits [46].

As a result of (3.5), the entries of $(K^\tau)^{-1} = S^{-1} P^{-1}$ are all positive. In particular, we have

$$R_i \equiv \sum_{j=1}^{n} ((K^\tau)^{-1})_{ij} > 0, \quad i = 1, \ldots, n.$$ (3.6)

Here is our main existence theorem for (2.1).

**Theorem 3.1** Consider the non-Abelian Chern–Simons system (2.1) over a doubly periodic domain $\Omega$ with the matrix $K$ satisfying (3.1)–(3.5). Let $p_1, \ldots, p_{n_i} (i = 1, \ldots, n)$ be any given points on $\Omega$, which need not to be distinct. Then there hold the following conclusions.

(i) (Necessary condition) If the system (2.1) admits a solution, then

$$\lambda > \lambda_0 \equiv \frac{16\pi \sum_{i=1}^{n} \sum_{j=1}^{n} P_{i}^{-1}(K^{-1})_{ji} N_{j}}{|\Omega| \sum_{i=1}^{n} \sum_{j=1}^{n} P_{i}^{-1}(K^{-1})_{ji}}.$$ (3.7)

(ii) (Sufficient condition) There exists a large constant $\lambda_1 > \lambda_0$ such that when $\lambda > \lambda_1$ the system (2.1) admits a solution $(u_1^\lambda, \ldots, u_n^\lambda)$.

(iii) (Asymptotic behavior) The solution $(u_1^\lambda, \ldots, u_n^\lambda)$ of (2.1) obtained in (ii) satisfies

$$\int_{\Omega} (e^{u_i^\lambda} - R_i)^2 dx \to 0 \quad \text{as} \quad \lambda \to \infty, \quad i = 1, \ldots, n,$$ (3.8)

where $R_i (i = 1, \ldots, n)$ are defined by (3.6).

(iv) (Quantized integrals) The solution $(u_1^\lambda, \ldots, u_n^\lambda)$ of (2.1) obtained in (ii) possesses the following quantized integrals

$$\int_{\Omega} \left( \sum_{j=1}^{n} K_{ji} e^{u_i^\lambda} - \sum_{j=1}^{n} \sum_{k=1}^{n} K_{kj} K_{ji} e^{u_i^\lambda} e^{u_k^\lambda} \right) dx = \frac{4\pi N_i}{\lambda}, \quad i = 1, \ldots, n.$$ (3.9)

In the subsequent sections we prove the theorem.
4 Necessary condition and variational formulation

In this section we shall find a necessary condition for the existence of solutions of (2.1) and present a variational formulation.

For convenience, we first use the translation

\[ u_i \rightarrow u_i + \ln R_i, \quad i = 1, \ldots, n, \]  

(4.1)

to recast the system (2.1) into a normalized form:

\[ \Delta u_i = \lambda \left( \sum_{j=1}^{n} \sum_{k=1}^{n} \tilde{K}_{jk} \tilde{K}_{ij} e^{u_j} e^{u_k} - \sum_{j=1}^{n} \tilde{K}_{ij} e^{u_j} \right) + 4\pi \sum_{j=1}^{n} \delta_{p_j}(x), \quad i = 1, \ldots, n \]  

(4.2)

whose vector version reads

\[ \Delta \mathbf{u} = \lambda \tilde{K} \mathbf{U} \tilde{K} (\mathbf{U} - \mathbf{1}) + 4\pi \mathbf{s}, \]  

(4.3)

where the notation

\[ \tilde{K} \equiv K^T R = P S R, \quad R \equiv \text{diag}\{R_1, \ldots, R_n\}, \]  

(4.4)

\[ \mathbf{u} \equiv (u_1, \ldots, u_n)^T, \quad \mathbf{U} \equiv \text{diag}\{e^{u_1}, \ldots, e^{u_n}\}, \quad \mathbf{U} \equiv (e^{u_1}, \ldots, e^{u_n})^T, \]  

(4.5)

\[ 1 \equiv (1, \ldots, 1)^T, \quad \mathbf{s} \equiv \left( \sum_{s=1}^{N_1} \delta_{p_1s}, \ldots, \sum_{s=1}^{N_n} \delta_{p_ns} \right)^T, \]  

(4.6)

will be observed throughout this work. Note that, since the matrix \( S \) is positive definite, so are the matrices

\[ A \equiv P^{-1} S^{-1} P^{-1} \quad \text{and} \quad Q \equiv R S R. \]  

(4.7)

To solve (4.2) or (4.3) over a doubly periodic domain, we need to introduce some background functions to remove the Dirac source terms. Let \( u_i^0 \) be the solution of the following problem [3]

\[ \Delta u_i^0 = 4\pi \sum_{s=1}^{N_i} \delta_{p_is} - \frac{4\pi N_i}{|\Omega|}, \quad \int_{\Omega} u_i^0 \, dx = 0, \]

and \( u_i = u_i^0 + v_i, \quad i = 1, \ldots, n. \) In the sequel we will use the \( n \)-vector notation

\[ \mathbf{v} \equiv (v_1, \ldots, v_n)^T, \quad \mathbf{N} \equiv (N_1, \ldots, N_n)^T, \quad \mathbf{0} \equiv (0, \ldots, 0)^T. \]  

(4.8)

Thus the system (4.2) or (4.3) becomes

\[ \Delta v_i = \lambda \left( \sum_{j=1}^{n} \sum_{k=1}^{n} \tilde{K}_{jk} \tilde{K}_{ij} e^{u_j^0+v_j} e^{u_k^0+v_k} - \sum_{j=1}^{n} \tilde{K}_{ij} e^{u_j^0+v_j} \right) + \frac{4\pi N_i}{|\Omega|}, \quad i = 1, \ldots, n, \]  

(4.9)
or

\[ \Delta \mathbf{v} = \lambda \mathbf{K} \mathbf{U} \mathbf{K}^{-1} (\mathbf{U} - \mathbf{1}) + \frac{4\pi N}{|\Omega|}, \quad (4.10) \]

where

\[ \mathbf{U} = \text{diag} \{ e^{u_1^{0}+v_1}, \ldots, e^{u_n^{0}+v_n} \}, \quad \mathbf{U} = (e^{u_1^{0}+v_1}, \ldots, e^{u_n^{0}+v_n})^\tau. \quad (4.11) \]

We now unveil a necessary condition for the existence of solutions of \((2.1)\). To this end, we rewrite \((4.10)\), after multiplying both sides of \((4.10)\) by \(\mathbf{A}\), equivalently as

\[ \Delta \mathbf{A} \mathbf{v} = \lambda \mathbf{U} \mathbf{Q} (\mathbf{U} - \mathbf{1}) + \mathbf{b}, \quad (4.12) \]

where \(\mathbf{A}\) and \(\mathbf{Q}\) are defined in \((4.7)\), and

\[ \mathbf{b} \equiv (b_1, \ldots, b_n)^\tau \equiv 4\pi A N = 4\pi P^{-1} S^{-1} P^{-1} N. \quad (4.13) \]

Noting \((3.5)\), we obtain

\[ b_i > 0, \quad i = 1, \ldots, n, \quad (4.14) \]

which will be used in the sequel.

Let \(\mathbf{v}\) be a solution of \((4.10)\), which is of course a solution of \((4.12)\). Then integrating \((4.12)\) over \(\Omega\), we obtain the natural constraint

\[ \int_{\Omega} \mathbf{U} \mathbf{Q} (\mathbf{U} - \mathbf{1}) \, dx + \frac{\mathbf{b}}{\lambda} = 0, \quad (4.15) \]

which implies

\[ \int_{\Omega} \mathbf{U}^\tau \mathbf{Q} (\mathbf{U} - \mathbf{1}) \, dx + \frac{1^\tau \mathbf{b}}{\lambda} = 0. \quad (4.16) \]

We may rewrite \((4.16)\) as

\[ \int_{\Omega} \left( \mathbf{U} - \frac{1}{2} \right)^\tau \mathbf{Q} \left( \mathbf{U} - \frac{1}{2} \right) \, dx = \frac{|\Omega|}{4} 1^\tau \mathbf{Q} 1 - \frac{1^\tau \mathbf{b}}{\lambda} \]

\[ = \frac{|\Omega|}{4} 1^\tau P^{-1} (K^\tau)^{-1} 1 - \frac{4\pi 1^\tau P^{-1} (K^\tau)^{-1} N}{\lambda}, \quad (4.17) \]

where we have used the fact

\[ (K^\tau)^{-1} 1 = S^{-1} P^{-1} 1 = R 1 \quad (4.18) \]

and \((4.13)\).

Since the matrix \(\mathbf{Q}\) is positive definite, \((4.17)\) gives a necessary condition for the existence of solutions of \((4.10)\):

\[ \frac{|\Omega|}{4} 1^\tau P^{-1} (K^\tau)^{-1} 1 - \frac{4\pi 1^\tau P^{-1} (K^\tau)^{-1} N}{\lambda} > 0, \]
which establishes (i) in Theorem 3.1.

Now we show that the system (4.10) admits a variational formulation. To see this we consider the system (4.10) in its equivalent formulation, (4.12).

Now since the matrices $A$ and $Q$ defined in (4.7) are symmetric, we see that the equations (4.12) are the Euler–Lagrange equations of the functional

$$I(v) = \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega} \partial_i v^\tau A \partial_i v dx + \frac{\lambda}{2} \int_{\Omega} (U - 1)^\tau Q (U - 1) dx + \int_{\Omega} \frac{b^\tau v}{|\Omega|} dx.$$  

(4.19)

Here and in what follows we use the notation (4.7), (4.8), (4.11) and (4.13) without explicit reference.

We observe that the functional (4.19) is not bounded from below. So we cannot conduct a direct minimization. To deal with this problem, we will find a critical point of the functional $I$ by using a constrained minimization approach developed in [10], later refined by [51]. Recently such a treatment was extended by [32] to solve the system associated with some general $2 \times 2$ Cartan matrices. To carry out this constrained minimization, the main difficulty is how to resolve the constraints, which will be the focus of the next three sections.

5 The constraints

In this section we identify a family of integral constraints under which our variational functional will be minimized.

We start by decomposing the Sobolev space $W^{1,2}(\Omega)$ into $\hat{W}^{1,2}(\Omega) = \mathbb{R} \oplus \hat{W}^{1,2}(\Omega)$, where

$$\hat{W}^{1,2}(\Omega) \equiv \left\{ w \in W^{1,2}(\Omega) \left| \int_{\Omega} w dx = 0 \right. \right\}$$

is a closed subspace of $W^{1,2}(\Omega)$. Then, for any $v_i \in W^{1,2}(\Omega)$, we have $v_i = c_i + w_i, c_i \in \mathbb{R}, w_i \in \hat{W}^{1,2}(\Omega), i = 1, \ldots, n$. To save notation, in the sequel, we also interchangeably use $W^{1,2}(\Omega), \hat{W}^{1,2}(\Omega)$ to denote the spaces of both scalar and vector-valued functions. Hence, if $v = w + c \in W^{1,2}(\Omega)$, with $w \equiv (w_1, \ldots, w_n)^\tau \in \hat{W}^{1,2}(\Omega)$ and $c \equiv (c_1, \ldots, c_n)^\tau \in \mathbb{R}^n$, satisfies the constraint (4.16), we obtain

$$\text{diag}\{e^{c_1}, \ldots, e^{c_n}\} \tilde{Q} \begin{pmatrix} e^{c_1} \\ \vdots \\ e^{c_n} \end{pmatrix} - P^{-1} R \text{diag}\{a_1, \ldots, a_n\} \begin{pmatrix} e^{c_1} \\ \vdots \\ e^{c_n} \end{pmatrix} + \frac{b}{\lambda} = 0,

(5.1)$$
where
\[ \tilde{Q} \equiv \tilde{Q}(w) \equiv R\tilde{S}R, \]
\[ \tilde{S} \equiv \begin{pmatrix}
\alpha_{11}a_{11} & -\alpha_{12}a_{12} & \cdots & \cdots & \cdots & -\alpha_{1n}a_{1n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha_{i1}a_{i1} & -\alpha_{i2}a_{i2} & \cdots & \alpha_{ii}a_{ii} & \cdots & -\alpha_{in}a_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha_{n1}a_{n1} & -\alpha_{n2}a_{n2} & \cdots & \cdots & \cdots & \alpha_{nn}a_{nn}
\end{pmatrix} \quad (5.3) \]
and we adapt the notation
\[ a_i \equiv a_i(w_i) \equiv \int_{\Omega} e^{u_0^i + w_i} dx, \quad i = 1, \ldots, n. \]
\[ a_{ij} \equiv a_{ij}(w_i, w_j) \equiv \int_{\Omega} e^{u_0^i + u_0^j + w_i + w_j} dx, \quad i, j = 1, \ldots, n. \]

Since the matrix \( S \) is positive definite, from (5.2) and (5.3) we see that \( \tilde{Q} \) is positive definite. (5.6)

Now the system (5.1) can be rewritten in its component form:
\[ e^{2c_i} R_i^2 a_{ii} - e^{c_i} \left( \frac{R_i a_i}{P_i} + \sum_{j \neq i} e^{c_j} R_j R_{ij} a_{ij} \right) + \frac{b_i}{\lambda} = 0, \quad i = 1, \ldots, n. \]
(5.7)
For any \( w \in \dot{W}^{1,2}(\Omega) \), we see that the equations (5.7) with respect to \( c \) are solvable only if
\[ \left( \frac{R_i a_i}{P_i} + \sum_{j \neq i} e^{c_j} R_j R_{ij} a_{ij} \right)^2 \geq \frac{4 R_i^2 b_i a_{ii}}{\lambda}, \quad i = 1, \ldots, n. \]
(5.8)
In order to ensure (5.8), it is sufficient to take the following inequality-type constraints
\[ \frac{a_i^2}{a_{ii}} \geq \frac{4 R_i a_{ii}^2 b_i}{\lambda}, \quad i = 1, \ldots, n. \]
(5.9)
Define the admissible set
\[ \mathcal{A} \equiv \left\{ w \big| w \in \dot{W}^{1,2}(\Omega) \text{ such that (5.9) is satisfied} \right\}. \]
(5.10)
Therefore, for any \( w \in \mathcal{A} \), we can obtain a solution of (5.7) by solving the system
\[ e^{c_i} = \frac{1}{2 R_i^2 a_{ii}} \left\{ \left( \frac{R_i a_i}{P_i} + \sum_{j \neq i} e^{c_j} R_j R_{ij} a_{ij} \right)^2 + \left( \frac{R_i a_i}{P_i} + \sum_{j \neq i} e^{c_j} R_j R_{ij} a_{ij} \right)^2 \right\} - \frac{4 b_i R_i^2 a_{ii}}{\lambda} \equiv f_i(e^{c_1}, \ldots, e^{c_n}), \quad i = 1, \ldots, n. \]
(5.11)
In the next two sections we aim at resolving the constraints (5.7) by solving (5.11).
6 Resolving the $SU(4)$ constraints

In this section we present a direct/concrete method for resolving the constraints (5.7) when $K$ is the Cartan matrix of $SU(4)$. Since in this case the coupling between the equations enjoys some special properties, we will see that the constraints allow a “squeeze-to-the-middle” solution process to be effectively carried out, which is of independent interest.

For $SU(4)$, the associated Cartan matrix $K = (K_{ij})$ is given by

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \quad (6.1)$$

Obviously, now $K$ satisfies all the requirement in Theorem 3.1 with $P = I$. Note that, in this case,

$$K^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad (6.2)$$

$$R = \text{diag} \left \{ \frac{3}{2}, \frac{2}{3} \right \}, \quad (6.3)$$

$$b = (b_1, b_2, b_3)^\tau = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} (N_1, N_2, N_3)^\tau. \quad (6.4)$$

Then the constraints (5.7) are

$$e^{2c_1} a_{11} - e^{c_1} \left( \frac{a_1}{3} + \frac{2a_{12}}{3} e^{c_2} \right) + \frac{2b_1}{9\lambda} = 0, \quad (6.5)$$

$$e^{2c_2} a_{22} - e^{c_2} \left( \frac{a_2}{4} + \frac{3a_{12}}{8} e^{c_1} + \frac{3a_{23}}{8} e^{c_3} \right) + \frac{b_2}{8\lambda} = 0, \quad (6.6)$$

$$e^{2c_3} a_{33} - e^{c_3} \left( \frac{a_3}{3} + \frac{2a_{23}}{3} e^{c_2} \right) + \frac{2b_3}{9\lambda} = 0. \quad (6.7)$$

Furthermore, for any $w \in \tilde{W}^{1,2}(\Omega)$, to ensure the solvability of the equations (6.5)–(6.7) with respect to $c$, the required inequality-type constraints (5.9) take the form

$$\frac{a_i^2}{a_{ii}} \geq \frac{8b_i}{\lambda}, \quad i = 1, 2, 3. \quad (6.8)$$

The admissible set $\mathcal{A}$ reads

$$\mathcal{A} \equiv \left \{ w \mid w \in \tilde{W}^{1,2}(\Omega) \text{ such that } (6.8) \text{ is satisfied} \right \}. \quad (6.9)$$
Hence, for any \( w \in \mathcal{A} \), to get a solution of (6.5)–(6.7), it suffices to solve

\[
e^{c_1} = \frac{\frac{a_1}{3} + \frac{2a_{12}}{3}e^{c_2} + \sqrt{\left(\frac{a_1}{3} + \frac{2a_{12}}{3}e^{c_2}\right)^2 - \frac{8b_{12}a_{22}}{9\lambda}}}{2a_{11}} \equiv f_1(e^{c_2}),
\]

\[
e^{c_2} = \frac{\frac{a_2}{3} + \frac{3a_{12}}{8}e^{c_1} + \frac{3a_{23}}{8}e^{c_3} + \sqrt{\left(\frac{a_2}{3} + \frac{3a_{12}}{8}e^{c_1} + \frac{3a_{23}}{8}e^{c_3}\right)^2 - \frac{b_{22}a_{22}}{2\lambda}}}{2a_{22}} \equiv f_2(e^{c_1}, e^{c_3}),
\]

\[
e^{c_3} = \frac{\frac{a_3}{3} + \frac{2a_{23}}{3}e^{c_2} + \sqrt{\left(\frac{a_3}{3} + \frac{2a_{23}}{3}e^{c_2}\right)^2 - \frac{8b_{33}a_{33}}{9\lambda}}}{2a_{33}} \equiv f_3(e^{c_2}).
\]

To solve these equations, we simply need to find a positive zero of the function

\[
F(t) \equiv t - f_2(f_1(t), f_3(t)), \quad t \in [0, \infty).
\]

We have

**Proposition 6.1** For any \( w \in \mathcal{A} \), there exists a unique positive solution \( t_0 \) for the equation

\[
F(t) \equiv t - f_2(f_1(t), f_3(t)) = 0.
\]

In view of this proposition we see that, for any \( w \in \mathcal{A} \), the system (6.10)–(6.12) with respect to \( \mathbf{c} \) admits a unique solution. However, as shown in the next section for the general case, there is no guarantee for the uniqueness of a solution to the general system (5.11).

**Proof.** For any \( w \in \mathcal{A} \), we see from (6.10)–(6.12) that

\[
f_i(t) > 0, \quad \forall t \geq 0, \ i = 1, 3, \quad f_2(t, s) > 0, \quad \forall t, s \geq 0,
\]

which imply

\[
F(0) = -f_2(f_1(0), f_3(0)) < 0.
\]

A direct computation gives

\[
\lim_{t \to \infty} \frac{f_1(t)}{t} = \frac{2a_{12}}{3a_{11}},
\]

\[
\lim_{t \to \infty} \frac{f_2(t, t)}{t} = \frac{3(a_{12} + a_{23})}{8a_{22}},
\]

\[
\lim_{t \to \infty} \frac{f_3(t)}{t} = \frac{2a_{23}}{3a_{33}}.
\]

Then, using the above limits and the Hölder inequality, we have

\[
\lim_{t \to \infty} \frac{F(t)}{t} = 1 - \lim_{t \to \infty} \frac{f_2(f_1(t), f_3(t))}{t}
\]

\[
= 1 - \frac{a_{12}^2}{4a_{11}a_{22}} - \frac{a_{23}^2}{4a_{22}a_{33}}
\]

\[
\geq 1 - \frac{1}{4} - \frac{1}{4} > 0.
\]
Then we see that

\[
\lim_{t \to \infty} F(t) = \infty. \tag{6.19}
\]

Since we have \( F(0) < 0 \), then the function \( F(\cdot) \) admits at least one zero \( t_0 \in (0, \infty) \).

Now we prove that the zero of \( F(\cdot) \) is also unique. In fact we easily check that

\[
\frac{df_1(t)}{dt} = \frac{2a_1 f_1(t)}{\sqrt{(a_1/3 + 2a_1/3 t)^2 - 8b_1 a_1/9\lambda}}, \tag{6.20}
\]

\[
\frac{\partial f_2(t, s)}{\partial t} = \frac{3a_1 f_2(t, s)}{\sqrt{(a_1/4 + 3a_1/8 t + 3a_2/8 s)^2 - b_2/2\lambda}}, \tag{6.21}
\]

\[
\frac{\partial f_2(t, s)}{\partial s} = \frac{3a_1 f_2(t, s)}{\sqrt{(a_1/4 + 3a_1/8 t + 3a_2/8 s)^2 - b_2/2\lambda}}, \tag{6.22}
\]

\[
\frac{df_3(t)}{dt} = \frac{2a_2 f_3(t)}{\sqrt{(a_1/3 + 2a_2/3 t)^2 - 8b_3 a_1/9\lambda}}, \tag{6.23}
\]

which are all positive. Thus, the functions \( f_i(t), (i = 1, 3) \) are strictly increasing for all \( t > 0 \).

Then from (6.20)–(6.23), (6.11)–(6.12) and the constraints (6.8) we obtain

\[
\frac{dF(t)}{dt} = 1 - \frac{1}{t} f_2(f_1(t), f_3(t)) \times
\frac{a_1^2 f_1(t)}{\sqrt{(a_1/3 + 2a_1/3 t)^2 - 8b_1 a_1/9\lambda}} + \frac{a_2^2 f_3(t)}{\sqrt{(a_1/3 + 2a_2/3 t)^2 - 8b_2 a_1/9\lambda}}
\times
\left[
\frac{a_1^2 f_2(f_1(t), f_3(t))}{\sqrt{(a_1/3 + 2a_1/3 t)^2 - 8b_1 a_1/9\lambda}} + \frac{a_2^2 f_3(t)}{\sqrt{(a_1/3 + 2a_2/3 t)^2 - 8b_3 a_1/9\lambda}}
\right]
\]

\[
> 1 - \frac{3}{8} f_2(f_1(t), f_3(t)) \frac{a_1 f_1(t) + a_2 f_3(t)}{t} \times
\frac{a_1^2 f_1(t)}{\sqrt{(a_1/3 + 2a_1/3 t)^2 - 8b_1 a_1/9\lambda}} + \frac{a_2^2 f_3(t)}{\sqrt{(a_1/3 + 2a_2/3 t)^2 - 8b_2 a_1/9\lambda}}
\times
\left[
\frac{a_1^2 f_2(f_1(t), f_3(t))}{\sqrt{(a_1/3 + 2a_1/3 t)^2 - 8b_1 a_1/9\lambda}} + \frac{a_2^2 f_3(t)}{\sqrt{(a_1/3 + 2a_2/3 t)^2 - 8b_3 a_1/9\lambda}}
\right]
\]

\[
> 1 - \frac{3}{8} f_2(f_1(t), f_3(t)) \frac{a_1 f_1(t) + a_2 f_3(t)}{t}
\]

\[
= 1 - \frac{f_2(f_1(t), f_3(t))}{t} = \frac{F(t)}{t}, \quad t > 0,
\]

so the uniqueness of the zero of \( F(t) \) over \([0, \infty)\) follows from the monotonicity of \( F(t)/t \) for \( t > 0 \) and the proof of the proposition is complete.

Using Proposition 6.1 for any \( w \in \mathcal{A} \), we see that the equations (6.5)–(6.7) with respect to \( c \) admit a unique solution \( c(w) = (c_1(w), c_2(w), c_3(w))^T \) determined by (6.10)–(6.12), such that \( v = (v_1, v_2, v_3)^T \) defined by

\[
v_i = w_i + c_i(w), \quad i = 1, 2, 3, \tag{6.24}
\]

satisfies the constraints (4.15) for the \( SU(4) \) case.
7 Solving the constraints in general

In this section we carry out a new way to resolve the constraints (5.7) by solving (5.11) for the general case via a topological-degree-theory argument. To this end, we consider the system

\[ \mathbf{F}(t) \equiv t - \mathbf{f}(t) = 0, \quad t \equiv (t_1, \ldots, t_n)^\tau \in \mathbb{R}_+^n, \tag{7.1} \]

where \( \mathbb{R}_+^n \equiv (\mathbb{R}_+)^n, \mathbf{f}(t) \equiv (f_1(t), \ldots, f_n(t))^\tau \). We have

**Proposition 7.1** For any \( w \in \mathcal{A} \), the system \( (7.1) \) admits a solution \( t \in (0, \infty)^n \).

**Proof.** To conduct a degree-theory argument, we deform the system (7.1) as

\[ \mathbf{F}(\epsilon, t) \equiv t - \mathbf{f}(\epsilon, t) = 0, \quad t \in \mathbb{R}_+^n, \quad \epsilon \in [0, 1], \tag{7.2} \]

where

\[
\begin{align*}
\mathbf{f}(\epsilon, t) &\equiv (f_1(\epsilon, t), \ldots, f_n(\epsilon, t))^\tau, \\
f_i(\epsilon, t) &\equiv \frac{1}{2 R_i^2 \alpha_{ii} a_{ii}} \left\{ \left( \frac{R_i a_i}{P_i} + \sum_{j \neq i} t_j R_j \alpha_{ij} a_{ij} \right) \\
&\quad + \sqrt{\left( \frac{R_i a_i}{P_i} + \sum_{j \neq i} t_j R_j \alpha_{ij} a_{ij} \right)^2 - \frac{4 \epsilon b_i R_i^2 \alpha_{ii} a_{ii}}{\lambda}} \right\},
\end{align*}
\]

\( i = 1, \ldots, n. \tag{7.4} \)

Then, we see that, to solve the system \( (7.1) \), we need to find a solution of

\[ \mathbf{F}(1, t) = 0, \quad t \in \mathbb{R}_+^n. \tag{7.5} \]

To facilitate our statement, we use the convention that we write

\[(\alpha_1, \ldots, \alpha_n)^\tau < (\leq) (\beta_1, \ldots, \beta_n)^\tau \text{ if } \alpha_i < (\leq) \beta_i, \quad i = 1, \ldots, n, \]

and we use the same convention for matrices.

To proceed, we establish the following key *a priori* estimates.

**Lemma 7.1** For any \( w \in \mathcal{A} \) and \( \epsilon \in [0, 1] \), every solution \( t \) of \( (7.2) \) satisfies

\[
\begin{align*}
0 < a_i t_i &\leq |\Omega|, \quad i = 1, \ldots, n, \tag{7.6} \\
0 < t_i &\leq 1, \quad i = 1, \ldots, n. \tag{7.7}
\end{align*}
\]

By virtue of Lemma \( 7.1 \) we immediately obtain the following.
Corollary 7.1 For any $w \in \mathcal{A}$, every solution $(c_1, \ldots, c_n)$ of (5.11) satisfies

\begin{align}
& a_i c_i \leq |\Omega|, \quad i = 1, \ldots, n, \\
& c_i \leq 1, \quad i = 1, \ldots, n.
\end{align}

(7.8) (7.9)

Proof of Lemma 7.1 For any $w \in \mathcal{A}$ and $\epsilon \in [0, 1]$, let $t$ be a solution of (7.2). By (7.2)–(7.4) we readily get the left-hand sides of (7.6)–(7.7).

From (7.2) it is straightforward to see that

\[ t_i = f_1(\epsilon, t) \leq \frac{R_i a_i}{P_i} + \sum_{j \neq i} t_j R_i R_j \alpha_{ij} a_{ij}, \quad i = 1, \ldots, n, \]

(7.10) which can be rewritten as

\[ R_i^2 \alpha_{ii} a_{ii} t_i - \sum_{j \neq i} t_j R_i R_j \alpha_{ij} a_{ij} \leq \frac{R_i a_i}{P_i}, \quad i = 1, \ldots, n. \]

(7.11)

Noting the expression of $Q$, we rewrite (7.11) in a vector form

\[ \tilde{Q} t \leq P^{-1} R a, \]

(7.12) where $a \equiv (a_1, \ldots, a_n)^\tau$. By Hölder’s inequality we obtain

\[ a_{ij}^2 \leq a_{ii} a_{jj}, \quad a_i \leq |\Omega|^{\frac{1}{2}} a_i^{\frac{1}{2}}, \quad i, j = 1, \ldots, n. \]

(7.13)

Therefore, noting that $t_i > 0$, $i = 1, \ldots, n$, the expression of $Q$, and (7.13), we arrive at

\[ \text{diag}\left\{ a_{11}^{\frac{1}{2}}, \ldots, a_{mm}^{\frac{1}{2}} \right\} Q \text{diag}\left\{ a_{11}^{\frac{1}{2}}, \ldots, a_{mm}^{\frac{1}{2}} \right\} t \leq \tilde{Q} t. \]

(7.14)

On the other hand, by (3.5) and the expression of $Q$ in (4.7) we see that all the entries of $Q^{-1}$ are positive.

(7.15)

Therefore, by (7.12), (7.14), and (7.15), we get

\[ \begin{array}{c}
t \leq \text{diag}\left\{ a_{11}^{-\frac{1}{2}}, \ldots, a_{mm}^{-\frac{1}{2}} \right\} Q^{-1} \text{diag}\left\{ a_{11}^{-\frac{1}{2}}, \ldots, a_{mm}^{-\frac{1}{2}} \right\} P^{-1} R a \\
= \text{diag}\left\{ a_{11}^{-\frac{1}{2}}, \ldots, a_{mm}^{-\frac{1}{2}} \right\} Q^{-1} \text{diag}\left\{ a_{11}^{-\frac{1}{2}}, \ldots, a_{mm}^{-\frac{1}{2}} \right\} P^{-1} R 1.
\end{array} \]

(7.16)

From (7.13), (7.15), and (7.16) we infer that

\[ \text{diag}\{ a_1, \ldots, a_n \} t \]
\[ \leq \text{diag}\left\{ a_1 a_{11}^{-\frac{1}{2}}, \ldots, a_n a_{mm}^{-\frac{1}{2}} \right\} Q^{-1} \text{diag}\left\{ a_1 a_{11}^{-\frac{1}{2}}, \ldots, a_n a_{mm}^{-\frac{1}{2}} \right\} P^{-1} R 1 \]
\[ \leq |\Omega| Q^{-1} P^{-1} R 1 \]
\[ = |\Omega| 1, \]

(7.17)
where (4.18) is also used. Then the right-hand sides of (7.6) follow from (7.17). By Jensen’s inequality, we have $a_i \geq |\Omega|, i = 1, \ldots, n$, which together with (7.17) imply the right-hand sides of (7.7). The proof of Lemma 7.1 is complete.

By the definition of $F(\epsilon, t)$, for any $\epsilon \in [0, 1]$, it is easy to see that $F(\epsilon, t)$ is a smooth function from $\mathbb{R}^n_+$ into $\mathbb{R}^n$.

Let us define

$$\tilde{\Omega} \equiv (0, r_0)^n, \quad (7.18)$$

where $r_0 > 1$ is a constant.

Then by (7.7) we see that, for every $w \in \mathcal{A}$ and $\epsilon \in [0, 1]$, $F(\epsilon, t)$ has no zero on the boundary of $\tilde{\Omega}$. Consequently, the Brouwer degree

$$\deg(F(\epsilon, t), \tilde{\Omega}, 0) \quad (7.19)$$

is well defined.

To prove Proposition 7.1 it is sufficient to show that

$$\deg(F(1, t), \tilde{\Omega}, 0) \neq 0. \quad (7.20)$$

Since $F(\epsilon, t)$ is smooth, by homotopy invariance [48], we have

$$\deg(F(1, t), \tilde{\Omega}, 0) = \deg(F(0, t), \tilde{\Omega}, 0). \quad (7.21)$$

Now we only need to calculate the degree $\deg(F(0, t), \tilde{\Omega}, 0)$.

Note for any $w \in \mathcal{A}$ the system

$$F(0, t) = 0 \quad (7.22)$$

is reduced into

$$t_i - \frac{R_i a_i}{P_i} + \sum_{j \neq i} t_j R_j a_{ij} a_{ij} = 0, \quad i = 1, \ldots, n. \quad (7.23)$$

By the definition of $\tilde{Q}$ we rewrite the system (7.23) equivalently in a vector form

$$\tilde{Q} t = P^{-1} R a. \quad (7.24)$$

In view of (5.6) the matrix $\tilde{Q}$ is of course invertible. Then we see that the system (7.24), i.e., (7.22), has a unique solution

$$t = \tilde{Q}^{-1} P^{-1} R a,$$

which, belonging to $\tilde{\Omega}$, is not a boundary point of $\tilde{\Omega}$ by (7.7).

Noting (5.6), the determinant of $\tilde{Q}$ is positive, which implies that the Jacobian of $F(0, t)$ is positive everywhere. Therefore, by the definition of the Brouwer degree, we have $\deg(F(0, t), \tilde{\Omega}, 0) = 1$, which implies $\deg(F(1, t), \tilde{\Omega}, 0) = 1$. Then the proof of Proposition 7.1 is complete.

By Proposition 7.1 we see that for any $w \in \mathcal{A}$, the constraint equations (5.7) admit a solution $c(w) = (c_1(w), \ldots, c_n(w))^\tau$ determined by (5.11), such that $v = w + c(w) = (w_1 + c_1(w), \ldots, w_n + c_n(w))^\tau$ satisfies the constraints (4.15).
8 Constrained minimization

In this section we solve the equation (4.9) by finding a critical point of the functional $I$ via a constrained minimization procedure. To do this, we consider the constrained functional

$$J(w) \equiv I(w + c(w)), \quad w \in \mathcal{A},$$

where $c(w)$ is the solution of the constraint equations (5.7) determined by Proposition 7.1.

Since, for every $w \in \mathcal{A}$, $v = w + c(w)$ satisfies the constraints (4.15), we have

$$\int_{\Omega} (U - 1)^{\tau} Q (U - 1) dx = \int_{\Omega} 1^{\tau} Q (1 - U) dx - \frac{1^{\tau} b}{\lambda}$$

Then the functional $J$ can be expressed as

$$J(w) = \frac{1}{2} \sum_{i=1}^{n} \int_{\Omega} \partial_i w^{\tau} A_{ij} \partial_j w dx + \frac{\lambda}{2} 1^{\tau} P^{-1} R (1 - U) dx + b^{\tau} c - \frac{1^{\tau} b}{2}$$

where and in the following we use the notation (4.7), (4.13).

We easily see that the functional $J$ is Fréchet differentiable in $\mathcal{A}$. We aim to show that the functional $J$ admits a minimizer, say $w$, in the interior of $\mathcal{A}$. Then $v = w + c(w)$ is a critical point of the functional $I$.

To proceed further, we need the following inequalities.

Lemma 8.1 For any $w \in \mathcal{A}$ and $s \in (0, 1)$, there hold the inequalities

$$\int_{\Omega} e^{u_0 + w} dx \leq \left( \frac{\lambda}{4P_i b_i a_{ii}} \right)^{\frac{1-s}{s}} \left( \int_{\Omega} e^{su_0 + w} dx \right)^{\frac{1}{s}}, \quad i = 1, \ldots, n. \quad (8.4)$$

Such type of inequalities were first established in [49].

For our purposes, we will also need the Moser–Trudinger inequality [26]

$$\int_{\Omega} e^{w} dx \leq C \exp \left( \frac{1}{16\pi} \| \nabla w \|_2^2 \right), \quad \forall w \in \dot{W}^{1,2}(\Omega), \quad (8.5)$$

where $C > 0$ is a constant.

Let $\alpha_0$ and $\beta_0$ be the smallest eigenvalues of $A$ and $Q$ defined by (4.7), respectively.
Lemma 8.2 For every \( w \in \mathcal{A} \), the functional \( J \) satisfies
\[
J(w) \geq \frac{\alpha_0}{4} \sum_{i=1}^{n} \| \nabla w_i \|_2^2 - C(\ln \lambda + 1),
\] (8.6)
where \( C > 0 \) is a constant independent of \( \lambda \).

**Proof.** By (4.7) and the definition of \( J \) we see that
\[
J(w) \geq \frac{\alpha_0}{2} \sum_{i=1}^{n} \| \nabla w_i \|_2^2 + \sum_{i=1}^{n} b_i c_i.
\] (8.7)

Noting (5.11) we obtain
\[
e^{c_i} \geq \frac{a_i}{2P_i R_i \alpha_{ii} a_{ii}}, \quad i = 1, \ldots, n,
\] (8.8)
which together with (5.9) yield
\[
e^{c_i} \geq \frac{2P_i b_i}{\lambda R_i a_i} = \frac{2P_i b_i}{\lambda R_i \int_{\Omega} e^{u_0^i + w_i} dx}, \quad i = 1, \ldots, n.
\] (8.9)

Then we have
\[
c_i \geq \ln \frac{2P_i b_i}{R_i} - \ln \lambda - \ln \int_{\Omega} e^{u_0^i + w_i} dx, \quad i = 1, \ldots, n.
\] (8.10)

Using Lemma 8.1 and the Moser–Trudinger inequality (8.5) we arrive at
\[
\ln \int_{\Omega} e^{u_0^i + w_i} dx \leq \frac{1}{s} \left\{ \ln \lambda - \ln \left( 4P_i^2 b_i \alpha_{ii} \right) \right\} + \frac{1}{s} \ln \int_{\Omega} e^{s u_0^i + s w_i} dx
\]
\[
\leq \frac{1}{s} \left\{ \ln \lambda - \ln \left( 4P_i^2 b_i \alpha_{ii} \right) \right\} + \frac{C}{s} + \max_{\Omega} u_0^i, \quad i = 1, \ldots, n.
\] (8.11)

Combining (8.7), (8.10), and (8.11), we have
\[
J(w) \geq \left( \frac{\alpha_0}{2} - \frac{s \max_{1 \leq i \leq n} \{ b_i \}}{16\pi} \right) \sum_{i=1}^{n} \| \nabla w_i \|_2^2 - \frac{1}{s} \sum_{i=1}^{n} b_i \left\{ \ln \lambda - \ln \left( 4P_i^2 b_i \alpha_{ii} \right) + \ln C \right\}
\]
\[
- \sum_{i=1}^{n} b_i \left\{ \ln (2R_i P_i b_i \alpha_{ii}) + \max_{\Omega} u_0^i \right\}.
\] (8.12)

Therefore, taking \( s \) suitably small in (8.12) we obtain the lemma.

Using Lemma 8.2 we see that the functional \( J \) is bounded from below and coercive in \( \mathcal{A} \). Then noting that \( J \) is weakly lower semicontinuous in \( \mathcal{A} \), we conclude that \( J \) admits a minimizer in \( \mathcal{A} \). In the following we establish some estimates as in [10][51] to show that this minimizer belongs to the interior of \( \mathcal{A} \) when \( \lambda \) is sufficiently large.
Lemma 8.3 There exists a constant $C > 0$ independent of $\lambda$ such that

$$\inf_{w \in \partial \mathcal{A}} J(w) \geq \frac{|\Omega| \lambda}{2} \min_{1 \leq i \leq n} \left\{ \frac{R_i}{P_i} \right\} - C(1 + \ln \lambda + \sqrt{\lambda}). \quad (8.13)$$

Proof. By the definition of $\mathcal{A}$, we see that at least one of the following equalities

$$\frac{a_i^2}{a_{ii}} = \frac{4a_{ii} P_i^2 b_i}{\lambda}, \quad i = 1, \ldots, n \quad (8.14)$$

must hold on the boundary $\partial \mathcal{A}$.

If the case $i = 1$ in (8.14) holds, by (7.17) we obtain

$$a_1 e^{c_1} \leq \frac{R_1}{P_1} (Q^{-1})_{11} a_1^2 a_{11} - \sum_{j=2}^{n} \frac{R_j}{P_j} (Q^{-1})_{1j} a_1 a_j a_{11}^{-1} a_{jj}^{-1}$$

$$\leq \frac{R_1}{P_1} (Q^{-1})_{11} a_1^2 a_{11} + \frac{1}{2} |\Omega| \sum_{j=2}^{n} \frac{R_j}{P_j} (Q^{-1})_{1j} a_1 a_{11}^{-1}$$

$$= (Q^{-1})_{11} \frac{4P_1 R_1 b_1 a_{11}}{\lambda} + \frac{2P_1 \sum_{j=2}^{n} \frac{R_j}{P_j} (Q^{-1})_{1j}}{\sqrt{\lambda}} \sqrt{b_1 |\Omega| a_{11}}. \quad (8.15)$$

If other cases happen, similar estimates as (8.15) can be established.

Using (7.8) and (8.15) leads to

$$\frac{\lambda}{2} \sum_{i=1}^{n} \frac{R_i}{P_i} \int_{\Omega} (1 - e^{c_i e^{v_i^0 + w_i}})dx$$

$$\geq \frac{|\Omega| \lambda R_1}{2P_1} - 2(Q^{-1})_{11} P_1 R_1 b_1 a_{11} - P_1 \sum_{j=2}^{n} \frac{R_j}{P_j} (Q^{-1})_{1j} \sqrt{b_1 |\Omega| a_{11}}. \quad (8.16)$$

By (8.16) and estimating $c_i$ ($i = 1, \ldots, n$) as that in Lemma 8.2, we get the lemma.

Now we need to choose some suitable functions in the interior of $\mathcal{A}$ to estimate the value of the functional $J$ in $\mathcal{A}$.

Recall that in [60] Tarantello proved that, for $\mu$ sufficiently large, the problems

$$\Delta v = \mu e^{v^0_i + v} (e^{v^0_i + v} - 1) + \frac{4\pi N_i}{|\Omega|}, \quad i = 1, \ldots, n \quad (8.17)$$

admit solutions $v_i^\mu$ ($i = 1, \ldots, n$), such that $u_i^0 + v_i^\mu < 0$ in $\Omega$, $e_i^\mu = \frac{1}{|\Omega|} \int_{\Omega} v_i^\mu dx \to 0$ and $u_i^\mu = v_i^\mu - c_i^\mu \to -u_i^0$ pointwise as $\mu \to \infty$, $i = 1, \ldots, n$. Then it follows that

$$\lim_{\mu \to \infty} \int_{\Omega} e^{v^0_i + u_i^\mu} dx = |\Omega|, \quad \lim_{\mu \to \infty} \int_{\Omega} e^{v^0_i + v_j^0 + u_i^\mu + u_j^\mu} dx = |\Omega|, \quad i, j = 1, \ldots, n. \quad (8.18)$$

Consequently, in view of the expression of $Q$, we have

$$\lim_{\mu \to \infty} \tilde{Q}(w^\mu) = |\Omega| Q. \quad (8.19)$$
Noting (5.6), we know that $\tilde{Q}(w^\mu)$ is invertible, which together with (8.19) yield

$$\lim_{\mu \to \infty} \tilde{Q}^{-1}(w^\mu) = \frac{1}{|\Omega|} Q^{-1}. \quad (8.20)$$

Therefore, by the definition of $A$, (8.18) and (8.20), we conclude that, for a fixed $\lambda' > 0$ sufficiently large and any $\varepsilon \in (0, 1)$, there exists a $\mu_\varepsilon \gg 1$, such that

$$w^{\mu_\varepsilon} = (u_1^{\mu_\varepsilon}, \ldots, u_n^{\mu_\varepsilon})^T \in \text{int} A, \quad (8.21)$$

for every $\lambda > \lambda'_0$, and

$$a_{ij}(w_i^{\mu_\varepsilon}, w_j^{\mu_\varepsilon}) < (1 + \varepsilon)|\Omega| < 2|\Omega|, \quad i, j = 1, \ldots, n, \quad (8.22)$$

$$\frac{(1 - \varepsilon)}{|\Omega|} Q^{-1} < \tilde{Q}^{-1}(w^{\mu_\varepsilon}) < \frac{(1 + \varepsilon)}{|\Omega|} Q^{-1} < \frac{2}{|\Omega|} Q^{-1}. \quad (8.23)$$

At this point we can establish the following comparison result.

**Lemma 8.4** For $w^{\mu_\varepsilon}$ given by (8.21), we have

$$J(w^{\mu_\varepsilon}) - \inf_{w \in \partial A} J(w) < -1, \quad (8.24)$$

when $\lambda$ is large enough.

**Proof.** Since $w^{\mu_\varepsilon} \in \text{int} A$, by (5.11) and the Jensen inequality, we arrive at the estimates

$$e_{c_i}(w^{\mu_\varepsilon}) = \frac{R_i a_i + \sum_{j \neq i} e_{c_j}(w^{\mu_\varepsilon}) R_i R_j a_{ij}}{2 R_i^2 a_{ii}} \times \left( 1 + \frac{1 - \frac{4 b_i R_i^2 a_{ii}}{\lambda (P_i R_i + \sum_{j \neq i} e_{c_j} R_i R_j a_{ij})^2}}{1 - \frac{4 b_i R_i^2 a_{ii}}{\lambda (P_i R_i + \sum_{j \neq i} e_{c_j} R_i R_j a_{ij})}} \right)$$

$$\geq \frac{R_i a_i + \sum_{j \neq i} e_{c_j}(w^{\mu_\varepsilon}) R_i R_j a_{ij}}{R_i^2 a_{ii}} - \frac{2 b_i}{\lambda (P_i R_i + \sum_{j \neq i} e_{c_j} R_i R_j a_{ij})}$$

$$\geq \frac{R_i |\Omega|}{P_i} + \sum_{j \neq i} e_{c_j}(w^{\mu_\varepsilon}) R_i R_j a_{ij} - \frac{2 P_i b_i}{\lambda |\Omega| R_i}, \quad i = 1, \ldots, n. \quad (8.25)$$

Here and to the end of this section we understand that

$$a_i = a_i(w_i^{\mu_\varepsilon}), \quad a_{ij} = a_{ij}(w_i^{\mu_\varepsilon}, w_j^{\mu_\varepsilon}), \quad i, j = 1, \ldots, n. \quad (8.26)$$
Then it follows from (8.25) and (8.22) that

\[
R^2_i \alpha_{ii} a_{ii} e^{c_i(w^{i \varepsilon})} - \sum_{j \neq i} e^{c_j(w^{i \varepsilon})} R_i R_j \alpha_{ij} a_{ij} \geq \frac{R_i |\Omega|}{P_i} - \frac{2 P_i R_i b_i}{\lambda |\Omega|} \alpha_{ii} a_{ii} \\
\geq \frac{R_i |\Omega|}{P_i} - \frac{4 \alpha_{ii} P_i R_i b_i}{\lambda}, \quad i = 1, \ldots, n, (8.27)
\]

which can be expressed in a vector form:

\[
\tilde{Q}(w^{i \varepsilon})(e^{c_1(w^{i \varepsilon})}, \ldots, e^{c_n(w^{i \varepsilon})})^\tau \geq |\Omega| P^{-1} R 1 - \frac{4 P R}{\lambda} \text{diag} \{\alpha_{11}, \ldots, \alpha_{nn}\} b. (8.28)
\]

Noting that all the entries of \(Q^{-1}\) are positive, using (8.28) and (8.23), we have

\[
(e^{c_1(w^{i \varepsilon})}, \ldots, e^{c_n(w^{i \varepsilon})})^\tau \geq (1 - \varepsilon) Q^{-1} P^{-1} R 1 - \frac{8 Q^{-1} P R}{\lambda |\Omega|} \text{diag} \{\alpha_{11}, \ldots, \alpha_{nn}\} b \\
= (1 - \varepsilon) 1 - \frac{8 Q^{-1} P R}{\lambda |\Omega|} \text{diag} \{\alpha_{11}, \ldots, \alpha_{nn}\} b. (8.29)
\]

Hence from (8.29) we get

\[
\int_\Omega \left(1 - e^{c_i(w^{i \varepsilon})} e^{u_0 + w^{i \varepsilon}}\right) dx \leq |\Omega| \varepsilon + \frac{8}{\lambda} \sum_{j=1}^n (Q^{-1})_{jj} P_j R_j b_j \alpha_{jj}, \quad i = 1, \ldots, n. (8.30)
\]

Using (7.9) and (8.30), we see that there exists a constant \(C_\varepsilon\) depending only on \(\varepsilon\) such that

\[
J(w^{i \varepsilon}) \leq \frac{|\Omega| \lambda \varepsilon}{2} \sum_{j=1}^n \frac{R_i}{P_i} + C_\varepsilon. (8.31)
\]

Then it follows from Lemma 8.3 and (8.31) that

\[
J(w^{i \varepsilon}) - \inf_{w \in \partial \mathcal{A}} J(w) \leq \frac{|\Omega| \lambda \varepsilon}{2} \left( \sum_{j=1}^n R_i P_j \varepsilon - \min_{1 \leq i \leq n} \left\{ \frac{R_i}{P_i} \right\} \right) + C \left( \sqrt{\lambda} + \ln \lambda + 1 \right), (8.32)
\]

where \(C > 0\) is a constant independent of \(\lambda\). Now taking \(\varepsilon\) suitably small and \(\lambda\) sufficiently large in (8.32), we get the lemma.

Therefore by Lemma 8.2 and Lemma 8.4 we conclude that there exists a large \(\lambda_1 > \max\{\lambda_0, \lambda'_0\}\) such that, for all \(\lambda > \lambda_1\), the functional \(J\) has a minimizer in the interior of \(\mathcal{A}\), say

\[
w^\lambda \in \text{int} \mathcal{A}. (8.33)
\]

It is straightforward to check that

\[
v^\lambda = w^\lambda + c(w^\lambda) (8.34)
\]

is a critical point of \(I\) and accordingly a solution of the system (4.10). Hence the second conclusion of Theorem 3.1 follows.
9 Asymptotic behavior and quantized integrals

In this section we prove the last two conclusions of Theorem 3.1. We first establish the asymptotic behavior of the solution obtained above as \( \lambda \to \infty \).

Lemma 9.1 Let \( v^{\lambda} \) be given by (8.34). Then,

\[
\lim_{\lambda \to \infty} \int_{\Omega} (e^{u_0^{\lambda} + v_i^{\lambda}} - 1)^2 \, dx = 0, \quad i = 1, \ldots, n.
\]  

(9.1)

Proof. For any \( \varepsilon \in (0, 1) \), we conclude from (8.31) that there exist constants \( \lambda_\varepsilon > 0 \) and \( C_\varepsilon > 0 \) such that

\[
J(w^{\lambda}) = \inf_{w \in A} J(w) \leq \frac{|\Omega| \lambda \varepsilon}{2} \sum_{i=1}^{n} \frac{R_i}{P_i} + C_\varepsilon,
\]

(9.2)

for all \( \lambda > \lambda_\varepsilon \).

Since \( Q \) is positive definite and \( \beta_0 \) being the smallest eigenvalue of \( Q \), we get

\[
\int_{\Omega} (U - 1)^T Q (U - 1) \, dx \geq \beta_0 \sum_{i=1}^{n} \int_{\Omega} (e^{u_0^{i} + v_i} - 1)^2 \, dx.
\]

(9.3)

Estimating \( c_i \) as that in Lemma 8.2 and using (9.3), we see that

\[
J(w^{\lambda}) \geq \frac{\beta_0 \lambda}{2} \sum_{i=1}^{n} \int_{\Omega} (e^{u_0^{i} + v_i} - 1)^2 \, dx - C(\ln \lambda + 1)
\]

(9.4)

where \( C > 0 \) is a constant independent of \( \lambda \).

Therefore, we infer from (9.2) and (9.4) that

\[
\lim_{\lambda \to \infty} \sup_{\lambda} \sum_{i=1}^{n} \int_{\Omega} (e^{u_0^{i} + v_i^{\lambda}} - 1)^2 \, dx \leq \frac{\varepsilon |\Omega|}{\beta_0} \sum_{i=1}^{n} \frac{R_i}{P_i}, \quad \forall \varepsilon \in (0, 1).
\]

(9.5)

Since \( \varepsilon \in (0, 1) \) is arbitrary, the lemma follows immediately.

Then using the translation (4.1) and Lemma 9.1 we get the third conclusion of Theorem 3.1

Finally, we can establish the quantized integrals (3.9). In fact, for the obtained solution, integrating the equations (4.9), we see that desired quantized integrals follow.

The proof of Theorem 3.1 is now complete.

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