Application of the unified control and detection framework to detecting stealthy integrity cyber-attacks on feedback control systems

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Abstract: This draft addresses issues of detecting stealthy integrity cyber-attacks on automatic control systems in the unified control and detection framework. A general form of integrity cyber-attacks that cannot be detected using the well-established observer-based technique is first introduced as kernel attacks. The well-known replay, zero dynamics and covert attacks are special forms of the kernel attacks. Existence conditions for the kernel attacks are presented. It is demonstrated, in the unified framework of control and detection, that all kernel attacks can be structurally detected when not only the observer-based residual, but also the control signal based residual signals are generated and used for the detection purpose. Based on the analytical results, two schemes for detecting the kernel attacks are then proposed, which allow reliable attack detection without loss of control performance. While the first scheme is similar to the well-established moving target method and auxiliary system aided detection scheme, the second detector is realised with encrypted transmissions of control and monitoring signals in the feedback control system that prevents adversary to gain system knowledge by means of eavesdropping attacks. Both schemes are illustrated by examples of detecting replay, zero dynamics and covert attacks and an experimental study on a three-tank control system.

Keywords: Cyber-security of control systems, observer-based detection of integrity cyber-attacks, unified framework of control and detection, kernel attacks, residual generation, observer-based detectors.

1 Introduction

Automatic control systems are essential system parts of many industrial cyber-physical systems (CPSs) and their flawless operations are of elemental importance for optimal system operation and high product quality. It is therefore not surprising that automatic control systems are often immediate targets of cyber-attacks on industrial CPSs. Driven by the
rapidly increasing industrial demands for higher cyber-security, detection of cyber-attacks on automatic control systems has drawn incredible research attention in the current decade. Excellent reviews of state of the art of research in this thematic area can be found in the recent surveys published in (Ding et al., 2018; Giraldo et al., 2018; Dibaji et al., 2019; Yan et al., 2019; Tan et al., 2020; Zhang et al., 2021; Zhou et al., 2021).

Among various types of cyber-attacks, integrity attacks are specially directed to automatic control systems (Dibaji et al., 2019; Griffioen et al., 2019). By injecting attack signals into system input and output channels, e.g. via I/O and network interfaces, integrity attacks can lead to remarkable system performance degradations and even catastrophic damages. An early and reliable detection of integrity attacks is becoming a vital requirement on cyber-security of industrial CPSs, for instance, for power control systems (Mohan et al., 2020). Thanks to its well-established theoretical framework in the past three decades, observer-based fault detection technique (Ding, 2013) is widely accepted as an efficient method, among numerous ones, to deal with detection of integrity attacks on control systems (Dibaji et al., 2019; Griffioen et al., 2019; Tan et al., 2020). Unfortunately, different from technical faults, cyber-attacks are artificially created and can be designed and generated by an adversary. It is particularly insidious, when cyber-attacks are generated in such a way that they cannot be detected using the known detection techniques. Such cyber-attacks are called stealthy. This observation and some real examples with stealthy cyber-attacks have strongly motivated researchers to improve the existing detection schemes and develop alternative solutions. In this regard, a great number of results have been reported about detecting the so-called replay, zero dynamics and covert attacks, which are stealthy integrity attacks as the standard observer-based fault detection technique cannot detect them without modifications on the applied algorithms (Dibaji et al., 2019; Griffioen et al., 2019; Tan et al., 2020). Representative solutions are the watermark detection scheme (Mo et al., 2015), the moving target method (Weerakkody and Sinopoli, 2015) and the auxiliary system aided detection scheme (Schellenberger and Zhang, 2017), just citing the initial works on these methods. Our work is motivated by the above observation and in particular driven by the questions like: what is the general form of stealthy integrity attacks? what are the existence conditions for such stealthy integrity attacks? is it possible to develop a general observer-based scheme applied to detecting integrity attacks in automatic control systems? Satisfactory answers to these questions could help us (i) to reveal possible weakness of observer-based detection technique by dealing with integrity cyber-attacks, and thus (ii) to prevent new variations of stealthy integrity attacks, and (iii) to develop new detection schemes, in particular such ones that are able to detect major types of integrity attacks. The main objective of our work is to investigate possible answers to the above questions. Different from the reported studies, our work will study the issues of stealthy integrity attacks in the unified framework of control and detection.

Inspired by the work in (Zhou and Ren, 2001), and based on the parameterisations of observers and observer-based residual generators (Ding, 2013), Ding et al., 2010 proposed an observer-based realisation and implementation of all stabilising (dynamic output) controllers whose core is an observer-based residual generator, and demonstrated its successful applications. In the recent decade, on the basis of this work, a new unified framework of control and detection has been established, which generalises the integrated design schemes for control and detection initiated by Nett et al. (1988) and further developed in the past decades (Kisgaard et al., 1996; Stoustrup et al., 1997; Khosrowjerdi et al., 2004; Henry
and Zolghadri, 2005; Wang and Yang, 2009). It has been applied to fault diagnosis in automatic control systems with uncertainties, fault-tolerant control and, more recently, to control performance degradation monitoring, detection and recovery (Ding, 2020). The basic idea behind the control and detection unified framework is that any controller is indeed residual-driven and can be implemented in form of an observer and an observer-based residual generator. This allows to extend the residual-based detection space to the overall measurement space spanned by the system inputs and outputs. As a result, it can be expected that the system capability for detecting cyber-attacks is (considerably) enhanced.

The intended contributions of our work are summarised as

- revealing that any attacks lying in the system kernel space cannot be detected by an observer-based detection system. In this context, the concept of kernel attacks is introduced, which provides us with a general expression of all stealthy integrity attacks (with respect to the observer-based detection technique);
- presenting existence conditions that integrity attacks are stealthy in the unified framework of control and detection, and based on them,
- proposing two schemes for detecting the kernel attacks (thus including detecting replay, zero dynamics and covert attacks). The first one is a natural extension of the observer-based detection schemes to a unified control and detection system, while the second one is dedicated to a detection scheme with encrypted transmissions of control and monitoring signals in the feedback control system under consideration. This is helpful to prevent adversary to gain system knowledge by means of eavesdropping attacks.

The paper is organised as follows. In Section 2, the unified framework of control and detection is first presented together with the necessary control theoretical and mathematical preliminaries. It is followed by a short review of replay, zero dynamics and covert attacks. Section 3 is dedicated to the study on stealthy integrity attacks and introduction of the concept of kernel attacks as a general form of stealthy integrity attacks. In Section 4, existence conditions for stealthy integrity attacks are first investigated and presented. They build the basis for the development of two schemes for detecting kernel attacks. These two schemes are presented in Sections 4 and 5, respectively. Their capability of detecting the kernel attacks are illustrated and demonstrated by examples and experimental results in Section 6.

Throughout this paper, standard notations known in linear algebra and advanced control theory are adopted. In addition, \( \mathcal{RH}_\infty \) is used to denote the set of all stable systems. In the context of cyber-attacks, when signal \( \xi \) is attacked, it is denoted by \( \xi^a \), and the corresponding (injected) attack signal by \( a_\xi \), i.e. \( \xi^a = \xi + a_\xi \).

2 Preliminaries of system models, the unified framework of control and detection, and stealthy integrity attacks

As the methodological basis of our work, we first introduce the unified framework of control and detection. It is followed by a short review of system descriptions of stealthy integrity cyber-attacks on feedback control systems.
2.1 System representations and controller parameterisation

2.1.1 System factorisations, observer-based residual generation and kernel space

Consider a nominal plant model

\[ y(z) = G_u(z)u(z), \quad y(z) \in \mathbb{C}^m, \quad u(z) \in \mathbb{C}^p \]  \hspace{1cm} (1)

with \( u \) and \( y \) as the plant input and output vectors. It is assumed that \( G_u(z) \) is a proper real-rational matrix and its minimal state space realisation is given by the following discrete-time linear time invariant (LTI) system

\[ x(k + 1) = Ax(k) + Bu(k), \quad x(0) = x_0, \]  \hspace{1cm} (2)

\[ y(k) = Cx(k) + Du(k), \]  \hspace{1cm} (3)

where \( x \in \mathbb{R}^n \) is the state vector and \( x_0 \) is the initial condition of the system. Matrices \( A, B, C, D \) are appropriately dimensioned real constant matrices. A coprime factorisation of a transfer function matrix over \( \mathcal{RH}_\infty \) gives a further system representation form and factorises the transfer matrix into two stable and coprime transfer matrices. The left and right coprime factorisations (LCF and RCF) of \( G_u(z) \) are given by

\[ G_u(z) = \hat{M}^{-1}(z) \hat{N}(z) = N(z)M^{-1}(z), \]  \hspace{1cm} (4)

where the state space realisations of the left and right coprime pairs (LCP and RCP) \( (\hat{M}(z), \hat{N}(z)) \) and \( (M(z), N(z)) \) are

\[ \hat{M}(z) = (A - LC, -L, C, I), \quad \hat{N}(z) = (A - LC, B - LD, C, D), \]  \hspace{1cm} (5)

\[ M(z) = (A + BF, B, F, I), \quad N(z) = (A + BF, B, C + DF, D). \]  \hspace{1cm} (6)

Correspondingly, there exist RCP and LCP \( (\hat{X}(z), \hat{Y}(z)) \) and \( (X(z), Y(z)) \) so that the so-called Bezout identity holds

\[ \begin{bmatrix} X(z) & Y(z) \\ -\hat{N}(z) & \hat{M}(z) \end{bmatrix} \begin{bmatrix} M(z) & -\hat{Y}(z) \\ N(z) & \hat{X}(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \]  \hspace{1cm} (7)

The state space computation formulas for \( (\hat{X}(z), \hat{Y}(z)) \) and \( (X(z), Y(z)) \) are

\[ \hat{X}(z) = (A + BF, L, C + DF, I), \quad \hat{Y}(z) = (A + BF, -L, F, 0), \]  \hspace{1cm} (8)

\[ X(z) = (A - LC, -(B - LD), F, I), \quad Y(z) = (A - LC, -L, F, 0). \]  \hspace{1cm} (9)

In (5)–(9), (real) matrices \( F \) and \( L \) are selected such that \( A + BF \) and \( A - LC \) are Schur matrices (Zhou, 1998; Ding, 2014).

We now consider an observer-based residual generator

\[ \hat{x}(k + 1) = A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)), \]  \hspace{1cm} (10)

\[ r_0(k) = y(k) - \hat{y}(k), \quad \hat{y}(k) = C\hat{x}(k) + Du(k) \]  \hspace{1cm} (11)
with \( r_0(k) \) being the primary form of a residual vector. It can be equivalently written as

\[
\dot{x}(k + 1) = (A - LC) \dot{x}(k) + (B - LD) u(k) + Ly(k),
\]

\[\implies r_0(z) = y(z) - \hat{y}(z) = \hat{M}(z)y(z) - \hat{N}(z)u(z).\] (12)

Note that if there exists no uncertainty in the plant and \( x(0) = \hat{x}(0) \), it holds

\[r_0(z) = 0 \implies y(z) = \hat{M}^{-1}(z)\hat{N}(z)u(z),\]

which illustrates the interpretation of LCF as an observer-based residual generator. It is well-known that given plant model (13), all LTI residual generators can be parameterised by

\[r(z) = R(z)r_0(z) = R(z)(y(z) - \hat{y}(z)), R(z) \in \mathcal{RH}_\infty,\] (13)

where \( R(z) \) is the parameterisation transfer function matrix (Ding, 2013).

Remark 1 Hereafter, we may drop out the domain variable \( z \) or \( k \) when there is no risk of confusion.

2.1.2 Parameterisation of stabilising controllers and basics of the unified control and detection framework

Consider the feedback control loop

\[y(z) = G_u(z)u(z), u(z) = K(z)y(z)\]

with the plant model \( G_u(z) \) and controller \( K(z) \). It is a well-known result that all stabilising controllers can be parameterised by

\[K(z) = -\left(\hat{X}(z) - Q(z)\hat{N}(z)\right)^{-1}\left(\hat{Y}(z) + Q(z)\hat{M}(z)\right)\]

\[= -\left(\hat{Y}(z) + M(z)Q(z)\hat{X}(z) - N(z)Q(z)\right)^{-1}\] (14)

(15)

with the parameter system \( Q(z) \in \mathcal{RH}_\infty \), where the four coprime pairs \( (\hat{M}, \hat{N}) \), \( (M, N) \), \( (\hat{X}, \hat{Y}) \) and \( (X, Y) \) are given in (5)-(9) and satisfy Bezout identity (7). The parameterisation expression (14)-(15) is called Youla parameterisation (Zhou, 1998). It follows from (5)-(9) and Bezout identity (Ding et al., 2010; Ding, 2020) that any (stabilising) output feedback controller

\[u(z) = K(z)y(z) + v(z)\] (16)

with \( v(z) \) being the reference signal can be equivalently written as

\[
\dot{x}(k + 1) = A\dot{x}(k) + Bu(k) + Lr_0(k),
\]

\[u(z) = F\dot{x}(z) - Q(z)r_0(z) + \bar{v}(z),\] (18)

\[\bar{v}(z) = \left(X(z) - Q(z)\hat{N}(z)\right)v(z).\] (19)

In other words, any output feedback controller is an observer-based controller and driven by the residual signal \( r_0 \).
Recall that the basis of an observer-based fault diagnosis is residual generation and evaluation (Ding, 2013). Thus, (17)-(18) reveal that both diagnosis and control are driven by the residual signal and can be integratedly realised by sharing a common observer-based residual generator as the information provider. By means of the observer parameterisation (Ding, 2013), we gain a deeper insight into the information aspect of a feedback controller that the control signal $u(k)$ in (18) is an estimate for $Fx(k) + \bar{v}(k)$ and satisfies

$$\forall x(0), u(k), \lim_{k \to \infty} (u(k) - Fx(k) - \bar{v}(k)) = 0, \quad (20)$$

when there exists no uncertainty in the plant. The observer-based realisation of stabilising feedback controllers (17)-(18) and the estimator interpretation (20) of (any) output feedback controllers are the basics of the unified control and detection framework and build the basis for our study on attack detection schemes presented in the subsequent work.

### 2.2 Integrity attacks under consideration

The system configuration under consideration in the first part of our study is sketched in Figure 1, in which the controller and attack detection system are networked with the plant (equipped with sensors, actuators and a computation system like micro-controllers). Via the communication network, the plant receives the control signal $u(k)$ and sends the sensor signal $y(k)$ to the control and monitoring system.

![Figure 1: System configuration under consideration](image)

Recall that our major attention is paid to integrity cyber-attacks that are injected into the system I/O interface via the network, cause (considerable) changes in the system dynamics, but cannot be detected by a standard observer-based detector. As reviewed in (Dibaji et al., 2019), such cyber-attacks include zero dynamics, covert and replay attacks. Below is a short description of these attack types.

#### 2.2.1 Zero dynamics attacks

Roughly speaking, a zero dynamics attack is referred to an attack $a_u(k)$ on the actuators, which causes no response at the system output over the detection time interval $[k_0, k_0 + N]$
and thus cannot be detected (Teixeira et al., 2015). The corresponding attack model is

\[ x(k+1) = Ax(k) + B (u(k) + a_u(k)), \]
\[ y^a(k) = Cx(k) + D (u(k) + a_u(k)) \]

with \( y^a(k) \) satisfying the condition

\[ \forall k \in [k_0, k_0 + N], y^a(k) = y(k). \]

It is obvious that the existence condition of zero dynamics attacks can be expressed by means of the LCF of the plant as

\[ \hat{N}(z) a_u(z) = 0. \]  

### 2.2.2 Covert attacks

Introduced by (Smith, 2015), covert attacks are modelled by

\[ x(k+1) = Ax(k) + B (u(k) + a_u(k)), \]
\[ y^a(k) = Cx(k) + D (u(k) + a_u(k)) + a_y(k) \]

with \( a_u(k) \) and \( a_y(k) \) denoting attacks on the actuators and sensors, respectively, and satisfying

\[ a_y(z) + G_u(z) a_u(z) = 0 \implies \forall k \in [k_0, k_0 + N], y^a(k) = y(k). \]

It is straightforward that the existence condition for covert attacks is

\[ \hat{M}(z) a_y(z) + \hat{N}(z) a_u(z) = 0. \]

### 2.2.3 Replay attacks

As described in (Mo et al., 2015), replay attacks are performed on the assumption that the plant under attacks is operating in the steady state, which yields

\[ y(k) \approx y(k-i), i = 1, \ldots. \]

Consequently, the attacker can “replay”, e.g. over the time interval \([k, k + M]\), the sensor data collected in the past (for instance, by means of an eavesdropping attack), and simultaneously inject signals in the actuators. Denote by \( y(k_0 + i), k_0 + M < k, i = 0, 1, \ldots, M \), the data collected and saved by the attacker. Replay attacks can be modelled by

\[
\begin{align*}
x(j+1) &= Ax(j) + B (u(j) + a_u(j)), j \in [k, k + M], \\
y^a(j) &= Cx(j) + D (u(j) + a_u(j)) + a_y(j), \\
an_j &= y(k_0 + j - k) - (Cj + D (u(j) + a_u(j))).
\end{align*}
\]

As a result,

\[ \forall j \in [k, k + M], y^a(j) = y(j-k + k_0) \approx y(j) = Cx(j) + Du(j). \]

Notice that the attack signal \( a_y(j) \) depends on the plant state vector \( x(j) \) and is, therefore, not a pure additive attack signal.

In summary, it can be seen that the above three types of attacks have one thing in common that they do not cause changes in the measurement output and hence cannot be traced by the output variables. As a result, these attacks cannot be detected using an observer-based detection scheme. In this context, they are called stealthy attacks (Dibaji et al., 2019).
2.3 Problem formulation

The goal of our work is to investigate cyber-attacking issues in the unified control and detection framework. We will first deal with the following three problems:

- study on general system structural conditions, under which the above-mentioned three types of attacks cannot be detected using an observer-based detector. Based on the achieved results, a general class of stealthy integrity cyber-attacks, the so-called kernel cyber-attacks, are then defined;

- derivation of system structural conditions, under which any integrity cyber-attacks, as sketched in Figure 1, can be (structurally) uniquely detected, and based on them,

- development of an alternative attack detection scheme that ensures a reliable detection of the integrity cyber-attacks shown in Figure 1.

A major reason why an integrity cyber-attack could be performed stealthily is that the attacker has knowledge of system dynamics. One potential tool to gain such knowledge is to collect sufficient plant input and output data by means of eavesdropping attacks, which enable, for instance, the identification of the plant model and even controller parameters. Under this consideration, we will, in the further part of our work, propose an alternative system configuration that leads to an encrypted data transmission aiming at preventing attackers to gain system knowledge.

3 Kernel attacks: a general form of stealthy integrity attacks

In this section, we investigate the existence conditions for stealthy attacks and generalise the different types of stealthy integrity attacks, including the three types of integrity attacks introduced in the previous section, as the so-called kernel attacks. To this end, we consider, in the sequel, the system configuration sketched in Figure 1.

3.1 Observer-based attack detection Strategy

For our purpose, we extend the nominal model (1)-(3) to the following attack model,

\[
x(k+1) = Ax(k) + B(u(k) + a_u(k)) + \omega(k), \\
y^a(k) = Cx(k) + D(u(k) + a_u(k)) + a_y(k) + \nu(k),
\]

where \(\omega(k), \nu(k)\) represent the process and measurement noise vectors, and \(a_y(k), a_u(k)\) denote the attack signals on the actuators and sensors, respectively. With respect to the system configuration shown in Figure 1, an observer-based attack detector consists of (i) a residual generator as given in (10)-(11) with the generated residual vector \(r_0(k)\),

\[
r_0(k) = y^a(k) - \hat{y}^a(k), \quad \hat{y}^a(k) = C\hat{x}(k) + Du(k),
\]

(ii) a residual evaluation function

\[
J(k) = J(||r_0(k)||)
\]
with $\|r_0(k)\|$ denoting a certain norm of $r_0(k)$, and (iii) detection logic described by

$$\begin{cases} J(k) \leq J_{th} \implies \text{attack-free}, \\ J(k) > J_{th} \implies \text{attack is detected}, \end{cases}$$

where $J_{th}$ is the threshold. In order to achieve an optimal attack detection, the observer gain matrix $L$, the evaluation function $J(k)$ and the threshold $J_{th}$ are designed taking into account of the statistic properties of $\omega(k), \nu(k)$. Suppose that $\omega(k), \nu(k)$ are uncorrelated with the state and input vectors and satisfy

$$\begin{align*}
\omega(k) &\sim \mathcal{N}(0, \Sigma_\omega), \nu(k) \sim \mathcal{N}(0, \Sigma_\nu), x(0) \sim \mathcal{N}(0, \Pi_0), \\
\mathcal{E} \left( \begin{bmatrix} \omega(i) \\ \nu(i) \\ x(0) \end{bmatrix} \begin{bmatrix} \omega(j) \\ \nu(j) \\ x(0) \end{bmatrix}^T \right) &= \begin{bmatrix} \Sigma_\omega & S \\ S^T & \Sigma_\nu \end{bmatrix} \delta_{ij} 0 \\ 0 & \Pi_0 \end{bmatrix}, \delta_{ij} = \begin{cases} 1, i = j, \\ 0, i \neq j \end{cases}
\end{align*}$$

with known matrices $\Sigma_\omega, \Sigma_\nu, S$. In this case, the observer gain matrix can be determined using the (steady) Kalman filter algorithm,

$$
L_K := L = (APC^T + S) \Sigma_{r}^{-1}, P = APA^T + \Sigma_\omega - L_K \Sigma_r L_K^T, \quad \Sigma_r = CPC^T + \Sigma_\nu = \mathcal{E} \left( r_0(k) r_0^T(k) \right), \mathcal{E} \left( r_0(i) r_0^T(j) \right) = \Sigma_r \delta_{ij},
$$

the $\chi^2$ test statistic is used as the evaluation function,

$$J(k) = r_0^T(k) \Sigma_{r}^{-1} r_0(k) \sim \chi^2(m),$$

and finally the threshold $J_{th}$ is determined by means of $\chi^2_\alpha(m)$ for a given upper-bound of false alarm rate $\alpha$ (Ding, 2014).

### 3.2 Kernel attacks

We now study the generalisation of stealthy attacks and their existence conditions. Corresponding to the above described observer-based attack detection strategy, we introduce the following definition.

**Definition 1** Given system model (30)-(31) with $\omega(k) = 0, \nu(k) = 0$, and observer-based attack detector (10)-(11), an integrity attack is stealthy if

$$\forall u, r_0(z) = y^a(z) - \hat{y}^a(z) = 0.$$

For our purpose, the following definition of the so-called kernel space is introduced.

**Definition 2** Given the plant model (1) and a corresponding LCP $\left( \hat{M}(z), \hat{N}(z) \right)$, we call the $\mathcal{H}_2 \times \mathcal{H}_2$ subspace $\mathcal{K}_P$ defined by

$$\mathcal{K}_P = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} : \begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = 0, \begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{H}_2 \right\}$$

kernel space of the plant.
It is evident that the kernel space $\mathcal{K}_P$ consists of all (bounded) input and output pairs $(u, y)$ satisfying
\[
\begin{bmatrix}
-\hat{N}(z) & \hat{M}(z)
\end{bmatrix}
\begin{bmatrix}
u(z)
\end{bmatrix} = 0.
\]

$\mathcal{K}_P$ is a closed subspace in $\mathcal{H}_2$ (Vinnicombe, 2000).

We are now in the position to present the existence condition of stealthy attacks defined in Definition 1.

**Theorem 1** Given plant model (30)-(31) with $\omega(k) = 0, \nu(k) = 0$, and an observer-based attack detector (10)-(11), an integrity attack is stealthy if and only if
\[
\begin{bmatrix}
u(z)
y^a(z)
\end{bmatrix} \in \mathcal{K}_P.
\]  

(37)

**Proof.** Without loss of generality, assume that the LCP $\left(\hat{M}, \hat{N}\right)$ is given as described in (3). Then, it follows from the well-known parameterisation of observer-based residual generators (Ding, 2013) that all observer-based residual generators (attack detectors) of the form (10)-(11) can be written as
\[
y^a(z) - \hat{y}^a(z) = R(z) \begin{bmatrix} -\hat{N}(z) & \hat{M}(z) \end{bmatrix} \begin{bmatrix}
u(z)
y^a(z)
\end{bmatrix},
\]
where $R(z)$ is a stable and invertible dynamic post-filter. Consequently, $y^a(z) = \hat{y}^a(z)$ if and only if
\[
\begin{bmatrix}
-\hat{N}(z) & \hat{M}(z)
\end{bmatrix}
\begin{bmatrix}
u(z)
y^a(z)
\end{bmatrix} = 0 \iff \begin{bmatrix}
u(z)
y^a(z)
\end{bmatrix} \in \mathcal{K}_P.
\]

The theorem is proved. 

In this context, we introduce the definition of kernel attacks, which gives a general form of integrity attacks that cannot be detected using an observer detector. As will be demonstrated in the example given below, the zero dynamics, covert and replay attacks are special forms of the kernel attacks.

**Definition 3** Given system model (30)-(31), an integrity attack is called kernel attack when condition (37) holds.

**Example 1** We first check a zero dynamics attack. It is evident that for $\omega(k) = 0, \nu(k) = 0$,
\[
r_0(z) = y^a(z) - \hat{y}^a(z) = \begin{bmatrix} -\hat{N}(z) & \hat{M}(z) \end{bmatrix} \begin{bmatrix}
u(z)
y^a(z)
\end{bmatrix} = -\hat{N}(z)a_u(z).
\]

It follows from (24) that
\[
\hat{N}(z)a_u(z) = 0 \iff \begin{bmatrix} -\hat{N}(z) & \hat{M}(z) \end{bmatrix} \begin{bmatrix}
u(z)
y^a(z)
\end{bmatrix} = 0,
\]

i.e. the zero dynamics attack is a kernel attack.
Now, consider the residual dynamics under a covert attack, which is given by
\[ r_0(z) = [-\hat{N}(z) \quad \hat{M}(z)] \begin{bmatrix} u(z) \\ y^a(z) \end{bmatrix} = \hat{N}(z)a_u(z) + \hat{M}(z)a_y(z). \]

According to (28), it holds
\[ \hat{N}(z)a_u(z) + \hat{M}(z)a_y(z) = 0 \Rightarrow \begin{bmatrix} -\hat{N}(z) & \hat{M}(z) \end{bmatrix} \begin{bmatrix} u(z) \\ y^a(z) \end{bmatrix} = 0. \]

Therefore, the covert attack is obviously a kernel attack.

Concerning the residual dynamics under a replay attack, recall the relation (29). It turns out, for \( j \in [k, k + M] \),
\[ r_0(j) = y^a(j) - \hat{y}^a(j) \approx Cx(j) + Du(j) - (C\hat{x}(j) + Du(j)) = 0 \]
\[ \Rightarrow \begin{bmatrix} u \\ y^a \end{bmatrix} \in \mathcal{K}_P. \]

Thus, the replay attack is a kernel attack.

Given the plant dynamics described by (30)-(31) with \( \omega(k) = 0, \nu(k) = 0 \), the dynamics of the observer-based attack detector (10)-(11) is described by
\[ r_0(z) = y^a(z) - \hat{y}^a(z) = \hat{M}(z)a_y(z) + \hat{N}(z)a_u(z). \]

Consequently, if the attack pair \((a_u, a_y)\) is constructed satisfying
\[ \hat{M}(z)a_y(z) + \hat{N}(z)a_u(z) = 0, \]
it cannot be detected. Hence, we have the following theorem.

**Theorem 2** Given the plant model (30)-(31) and an observer-based attack detector (10)-(11), the pair \((a_u, a_y)\) builds a kernel attack if it satisfies (40).

**Remark 2** We would like to point out that a replay attack does not satisfy (40), since it is, in fact, not an additive type of attacks, as remarked in the previous section.

Recall that the kernel space \( \mathcal{K}_P \) is a structural property of the plant and determined by the dynamics of the nominal plant. As a result, if an attacker is in possession of knowledge of the plant dynamics, kernel attacks could be constructed according to (40) and injected in the plant without being detected by the observer-based attack detector (10)-(11). In fact, recall that any LTI observer-based residual generators, including the parity relation based one and diagnosis observer, can be parameterised by (Ding, 2013)
\[ r(z) = R(z)r_0(z) = R(z)(y(z) - \hat{y}(z)), R(z) \neq 0, R(z) \in \mathcal{RH}_\infty. \]

The following corollary is obvious.

**Corollary 1** Given plant model (30)-(31), any attack cannot be detected by an LTI attack detector of the form (41), if and only if the signal pair \((u, y^a)\) satisfies (37), or if the attack signal pair \((a_u, a_y)\) is constructed satisfying (40).

At the end of this section, we would like to emphasise that the concept of the kernel attacks and the associated existence conditions given in Theorems 1-2 and Corollary 1 are described in terms of the LCF or kernel space of the system under consideration. They are the system structural properties and independent of the observer design and the evaluation schemes adopted by the attack-detector. Our subsequent investigation on detecting stealthy integrity attacks will be carried out in this context.
4 Analysis and detection of kernel attacks

This section deals with detecting kernel attacks on the feedback control systems shown in Figure 2. Consider that the observer-based attack detector (10)-(11) performs the online detection by means of the (online) data $(u(k), y^a(k))$, whose dimension is $p + m$. On the other hand, the parameterisation of observer-based residual generators and Theorem 1 as well as Corollary 1 reveal that the residual signals that belong to the $m$-dimensional kernel space $K_P$ are only effective in detecting attacks which do not belong to $K_P$. In other words, in order to detect kernel attacks successfully, generating additional signals to cover the overall $(p + m)$-dimensional data space is an effective alternative solution. In fact, the so-called moving target or auxiliary system schemes reported, for instance, in (Weerakkody and Sinopoli, 2015; Griffioen et al., 2021; Schellenberger and Zhang, 2017; Dibaji et al., 2019) for detecting zero dynamics and covert attacks, are special realisations of this idea.

![Figure 2: Schematic description of the control loop under attack](image)

4.1 Analysis of closed-loop dynamics under kernel attacks

Consider the feedback control loop with the plant model (30)-(31) and controller described by (16), where $K(z)$ satisfies (14)-(15). For the sake of the structural analysis, $\omega(k)$ and $\nu(k)$ are assumed to be zero. It yields, under attacks,

$$y^a(z) = G_u(z)u^a(z) + a_y(z),$$
$$u^a(z) = K(z)y^a(z) + v(z) + a_u(z),$$

and the control loop configuration can be equivalently sketched by Figure 2. It turns out

$$\begin{bmatrix} u^a(z) \\ y^a(z) \end{bmatrix} = \begin{bmatrix} I & -K(z) \\ -G_u(z) & I \end{bmatrix}^{-1} \begin{bmatrix} a_u(z) + v(z) \\ a_y(z) \end{bmatrix},$$

$$\begin{bmatrix} \hat{V}(z) & -\hat{U}(z) \\ -\hat{N}(z) & \hat{M}(z) \end{bmatrix}^{-1} \begin{bmatrix} \hat{V}(z)(a_u(z) + v(z)) \\ \hat{M}(z)a_y(z) \end{bmatrix},$$

where

$$\hat{V} = X - Q\hat{N} \in \mathcal{RH}_\infty, \hat{U} = -Y - Q\hat{M} \in \mathcal{RH}_\infty,$$

and $\hat{V}, \hat{M}$ are invertible. Recall the Bezout identity (7) and extend it to

$$\begin{bmatrix} X - Q\hat{N} & Y + Q\hat{M} \\ -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} M & -\hat{Y} - MQ \\ N & \hat{X} - NQ \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \iff (44)$$

$$\begin{bmatrix} X - Q\hat{N} & Y + Q\hat{M} \\ -\hat{N} & \hat{M} \end{bmatrix}^{-1} = \begin{bmatrix} M & -\hat{Y} - MQ \\ N & \hat{X} - NQ \end{bmatrix} \in \mathcal{RH}_\infty.$$
As a result, we have

**Theorem 3** Given the plant model (30)-(31) and controller (16) with \( K(z) \) satisfying (14)-(15), it holds

\[
\begin{bmatrix}
\bar{a}_u \\
\bar{a}_y
\end{bmatrix} = \begin{bmatrix}
X - Q\hat{N} & Y + Q\hat{M} \\
-\hat{N} & \hat{M}
\end{bmatrix} \begin{bmatrix}
u^a \\
y^a
\end{bmatrix} - \begin{bmatrix}
\bar{v} \\
0
\end{bmatrix},
\tag{46}
\]

\( \bar{v} = \hat{V}v, \bar{a}_u = \hat{V}a_u, \bar{a}_y = \hat{M}a_y. \)

It follows immediately from Theorem 3 that, using signals \( y^a(k), u^a(k) \) and \( v(k) \),

- the attack signals \( a_y(k), a_u(k) \) could be structurally detected in the sense that

\[
\begin{bmatrix}
a_u(z) \\
a_y(z)
\end{bmatrix} \neq 0 \iff \begin{bmatrix}
\bar{a}_u(z) \\
\bar{a}_y(z)
\end{bmatrix} \neq 0 \iff \begin{bmatrix}
X - Q\hat{N} & Y + Q\hat{M} \\
-\hat{N} & \hat{M}
\end{bmatrix} \begin{bmatrix}
u^a \\
y^a
\end{bmatrix} - \begin{bmatrix}
\bar{v} \\
0
\end{bmatrix} \neq 0, \text{ and}
\]

- if \( \hat{V}^{-1} \in \mathcal{RH}_\infty, \hat{M}^{-1} \in \mathcal{RH}_\infty \), i.e. both the plant and controller are stable, the attack pair \( (a_y, a_u) \) could also be (structurally) uniquely identified according to

\[
\begin{bmatrix}
a_u \\
a_y
\end{bmatrix} = \begin{bmatrix}
I & -K \\
-G_u & I
\end{bmatrix} \begin{bmatrix}
u^a \\
y^a
\end{bmatrix} + \begin{bmatrix}
-v \\
0
\end{bmatrix},
\]

except that the transfer matrix

\[
\begin{bmatrix}
M & -\hat{Y} - MQ \\
N & \hat{X} - NQ
\end{bmatrix}
\tag{47}
\]

has a transmission zero at \( z = z_0 \), i.e.

\[
\text{rank} \begin{bmatrix}
M(z_0) & -\hat{Y}(z_0) - M(z_0)Q(z_0) \\
N(z_0) & \hat{X}(z_0) - N(z_0)Q(z_0)
\end{bmatrix} < m + p, \tag{48}
\]

and

\[
\begin{bmatrix}
a_u(z) \\
a_y(z)
\end{bmatrix} = \begin{bmatrix}
a_u(z_0) \\
a_y(z_0)
\end{bmatrix}.
\tag{49}
\]

Considering that transmission zeros of transfer matrix (47) are structural properties of the plant and the controller, and whose number is limited, we will not address this class of possible attacks whose realisation requires not only full knowledge of the plant and controller, but also very special forms of attack signals.

It is worth noting that, according to the relations given in (44)-(45), attacks \( a_y, a_u \) can also be (structurally) detected using the relation

\[
\begin{bmatrix}
u^a \\
y^a
\end{bmatrix} - \begin{bmatrix}
M \\
N
\end{bmatrix} \bar{v} = \begin{bmatrix}
M & -\hat{Y} - MQ \\
N & \hat{X} - NQ
\end{bmatrix} \begin{bmatrix}
\bar{a}_u \\
\bar{a}_y
\end{bmatrix}
\]

\[
= \begin{bmatrix}
I & -\hat{Y} - MQ \\
0 & \hat{X} - NQ
\end{bmatrix} \begin{bmatrix}
a_u \\
\hat{N}a_u + \hat{M}a_y
\end{bmatrix}.
\tag{50}
\]
Before we continue our study on applying signals \( v(k), y^a(k) \) and \( u^a(k) \) for attack detection, we would like to discuss about relations (46) and (50), which is helpful to gain a deep insight into our solutions and two different implementation forms of attack detectors. To this end, we first check the transfer function matrix

\[
\begin{bmatrix}
X - Q\hat{N} & Y + Q\hat{M} \\
-\hat{N} & \hat{M}
\end{bmatrix} = \begin{bmatrix}
\hat{V} & -\hat{U} \\
-\hat{N} & \hat{M}
\end{bmatrix}
\]

on the right-hand side of (46). While the LCP \((\hat{M}, \hat{N})\) builds the kernel space of the plant, the pair \((\hat{V}, \hat{U})\) is left coprime and spans the kernel space of the controller (16). In other words, the signal \( r_u(z) \) defined by

\[
u(z) - v(z) = K(z)y(z) \implies \hat{V}(z)(u(z) - v(z)) - \hat{U}(z)y(z) =: r_u(z)
\]
can be viewed as a residual vector generated based on the controller configuration. Since

\[
\dim \begin{bmatrix} r_0 \\ r_u \end{bmatrix} = m + p,
\]

any changes (caused, for instance, by attacks) in the space spanned by the plant input and output vectors, \((u, y)\), can be (structurally) uniquely detected. It is of interest to notice that the residual vector \( r_u(k) \) can be generated as well using an observer of the form

\[
\begin{bmatrix}
r_{u,1}(k) \\
r_{u,2}(k)
\end{bmatrix} = \begin{bmatrix}
u(k) - v(k) - F\hat{x}_u(k) \\
y(k) - D(u(k) - v(k)) - C\hat{x}_u(k)
\end{bmatrix},
\]

\[r_u(z) = r_{u,1}(z) + Q(z)r_{u,2}(z).
\]

We now consider the left-hand side of (50). In the attack-free case, it is indeed a residual generator based on the closed-loop dynamics,

\[
\begin{bmatrix}
r_{u,c}(z) \\
r_{y,c}(z)
\end{bmatrix} := \begin{bmatrix} u^a(z) \\ y^a(z) \end{bmatrix} - \begin{bmatrix} M(z) \\ N(z) \end{bmatrix} \bar{v}(z),
\]

whose state space representation is given by

\[
\hat{x}_v(k + 1) = (A - LC)\hat{x}_v(k) + (B - LD)u(k) - v(k) + F\hat{x}_u(k) - q(k), q(z) = Q(z)(C\hat{x}_v(z) + Dv(z)),
\]

\[
\hat{x}_c(k + 1) = (A + BF)\hat{x}_c(k) + B\bar{v}(k),
\]

\[
\begin{bmatrix}
r_{u,c}(k) \\
r_{y,c}(k)
\end{bmatrix} = \begin{bmatrix} u^a(k) \\ y^a(k) \end{bmatrix} - \begin{bmatrix} F \\ C + DF \end{bmatrix}\hat{x}_c(k) - \begin{bmatrix} I \\ D \end{bmatrix} \bar{v}(k).
\]

Hence, using the closed-loop dynamics based residual vectors, \( r_{u,c} \) and \( r_{y,c} \), with

\[
\dim \begin{bmatrix} r_{u,c} \\ r_{y,c} \end{bmatrix} = m + p,
\]

it is possible to detect attacks uniquely as well.

In summary, in order to detect all kernel attacks uniquely, we can use the (online) data \( v(k), y^a(k) \) and \( u^a(k) \) to generate either the observer-based residuals \( r_0 \) and \( r_u \) or the closed-loop dynamics based residuals \( r_{u,c} \) and \( r_{y,c} \). Unfortunately, \( u^a(k) \) is only available on the side of the plant, as shown in Figures 1 and 2. This motivates us to propose a detection scheme described in the next sub-section.
4.2 A conceptual scheme for detecting kernel attacks

For the realisation of the detection solution based on (46) given in Theorem 3, we propose the following conceptual detection scheme.

It follows from the relation

\[
\left( X(z) - Q(z)\hat{N}(z) \right) (u(z) - v(z)) = - \left( Y(z) + Q(z)\hat{M}(z) \right) y(z)
\]

in attack-free case that signal

\[
r_{u,0}(z) := X(z)u(z) + Y(z)y(z) - \left( X(z) - Q(z)\hat{N}(z) \right) v(z)
\]

builds a residual signal satisfying

\[
r_{u,0}(z) = r_u(z) + Q(z)r_0(z).
\]

Recall that \( v \) is available at the monitoring side, while the signals \( y, u \) exist on the plant side with \( u \) being corrupted by \( a_u \). For our purpose, we propose to generate the residual signal \( r_{u,0} \) using the following algorithm:

- compute

\[
r_{en}(z) := X(z)u^a(z) + Y(z)y(z);
\]

- transmit \( r_{en}(k) \) to the monitoring and control side;

- compute, on the monitoring and control side,

\[
r_{u,0}(z) = r_{en}^a(z) - \left( X(z) - Q(z)\hat{N}(z) \right) v(z).
\]

Here, it is supposed that \( r_{en}(z) \) is attacked by the attack signal \( a_{r_{en}}(k) \), i.e.

\[
r_{en}^a(k) = r_{en}(k) + a_{r_{en}}(k).
\]

Figure 3 shows the corresponding system configuration.

Next, we check the dynamics of \( r_{u,0}, r_0 \) without considering noises. Recall that

\[
(\left( X(z) - Q(z)\hat{N}(z) \right) (u(z) - v(z)) = - \left( Y(z) + Q(z)\hat{M}(z) \right) y^a(z).
\]

It yields

\[
\begin{bmatrix}
    r_{u,0} \\
    r_0
\end{bmatrix} = \begin{bmatrix}
    0 & 0 & I \\
    -\hat{N} & -\hat{M} & 0
\end{bmatrix} \begin{bmatrix}
    u \\
    y^a \\
    r_{en}^a
\end{bmatrix} - \begin{bmatrix}
    \bar{v} \\
    0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    X - Q\hat{N} & -Y - Q\hat{M} & I \\
    \hat{N} & -\hat{M} & 0
\end{bmatrix} \begin{bmatrix}
    a_u \\
    a_y \\
    a_{r_{en}}
\end{bmatrix}.
\]
Consequently,
\[
\begin{bmatrix}
    r_u(z) \\
    r_0(z)
\end{bmatrix} = 0
\]
if and only if \(a_{ren}, a_u, a_y\) solve
\[
\begin{align*}
    \hat{N}(z)a_u(z) + \hat{M}(z)a_y(z) &= 0, \\
    a_{ren}(z) &= Y(z)a_y(z) - X(z)a_u(z).
\end{align*}
\]
\[(59)\]
\[(60)\]
We summary the above results in the following theorem.

**Theorem 4** Given the plant model (30)-(31), the controller
\[
u(z) = K(z)y^a(z) + v(z),
\]
with \(K(z)\) satisfying (14)-(15) and the system configuration shown in Figure 3, where \(a_{ren}, a_u, a_y\) are the attack signals, \(r^a_{ren}, y^a, u, v\) are the system signals being available at the monitoring side and used for the attack detection purpose, then attacks \((a_{ren}, a_u, a_y)\) are stealthy, i.e. they cannot be detected using the available system signals, if and only if the conditions (59)-(60) are satisfied.

### 4.3 Design and construction of residual generator \(r_{u,0}\)

It follows from Theorem 4 that an attacker could design attack signals \(a_{ren}, a_u, a_y\) so that conditions (59)-(60) are satisfied when the attacker is in possession of knowledge of the plant dynamics (regarding to (59)) and the construction of system (56) (regarding to (60)). This demands that knowledge of the residual generator (56) and (55) should be protected from the attacker so that the attacker could not be able to construct the attack signal \(a_{ren}\) according to (60). To this end, encryption of the concerning system dynamics is the major task of designing and constructing the residual generator \(r_{u,0}\) described by (56) and (55). This will be realised in two steps.

At first, the residual generator (55) is constructed at two different sides of the networked control system, as shown in (56) and (57). In a certain sense, the dynamic system
\[
r_{en}(z) = \begin{bmatrix} X(z) & Y(z) \end{bmatrix} \begin{bmatrix} u^a(z) \\ y(z) \end{bmatrix}
\]
can be interpreted as an encoding algorithm and thus is called encoder. The residual signal \( r_{u,0} \) is then generated by a decoding algorithm in the form

\[
  r_{u,0}(z) = r_{en}(z) - \left( X(z) - Q(z)\hat{N}(z) \right) v(z).
\]

Note that even if knowledge of the (encoding) system (56) is protected, the attacker could identify the system dynamics using possibly eavesdropped signals \( r_{en}, u^a \) and \( y \). In order to protect the system dynamics from being identified, the involved system \((X, Y)\) is further encrypted in the next step.

Recall that the state space form of the encoder (56) is given by

\[
  \dot{\xi}(k+1) = (A - LC)\xi(k) + Ly(k) + (B - LD)u^a(k),
\]

\[
  r_{en}(k) = u^a(k) - F\xi(k).
\]

Moreover, the following lemma can be proved.

**Lemma 1** Given \( \left( \hat{M}_i, \hat{N}_i \right), (X_i, Y_i), i = 1, 2, \) subject to

\[
  \hat{M}_i = (A - L_i C, -L_i, C, I), \hat{N}_i = (A - L_i C, B - L_i D, C, D),
\]

\[
  X_i = (A - L_i C, -(B - L_i D), F_i, I), Y_i = (A - L_i C, -L_i, F_i, 0),
\]

it holds

\[
  \begin{bmatrix}
    X_1(z) & Y_1(z) \\
    X_2(z) & Y_2(z)
  \end{bmatrix} = R_{12}(z) \begin{bmatrix}
    X_2(z) & Y_2(z) \\
    X_1(z) & Y_1(z)
  \end{bmatrix} + Q_{11}(z) \begin{bmatrix}
    -\hat{N}_1(z) & \hat{M}_1(z) \\
    -\hat{N}_2(z) & \hat{M}_2(z)
  \end{bmatrix}
\]

(63)

\[
  R_{12}(z) = R_{21}^{-1}(z) = I + (F_1 - F_2) (zI - A_{F_2})^{-1} B \in \mathcal{RH}_\infty, A_{F_1} = A + BF_i,
\]

\[
  \dot{Q}_{11}(z) = F_1 (zI - A_{L_2})^{-1} (L_2 - L_1) - \dot{R}_{12}(z) Q_{21}(z) \in \mathcal{RH}_\infty, A_{L_i} = A - L_i C,
\]

\[
  \dot{Q}_{12}(z) = F_1 (zI - A_{L_1})^{-1} (L_2 - L_1) - (F_1 - F_2) (zI - A_{F_2})^{-1} L_2 \in \mathcal{RH}_\infty,
\]

\[
  \dot{R}_{12}(z) = (F_1 - F_2) (zI - A_{F_2})^{-1} L_2 \in \mathcal{RH}_\infty,
\]

\[
  Q_{21}(z) = Q_{12}^{-1}(z) = I + C (zI - A_{L_2})^{-1} (L_1 - L_2) \in \mathcal{RH}_\infty.
\]

(64)

(65)

(66)

The proof is given in Appendix.

Lemma 1 reveals that varying the gain matrices \( F_2 \) and \( L_2 \) in \((X, Y)\) to \( F_1 \) and \( L_1 \) is equivalent to adding (i) a (stable and invertible) post-filter to \( r_{en}(z) \) and (ii) additional residual signal \( r_0 \). On the basis of this result, we propose to switch \( F \) and \( L \), denoted by \( F_\sigma \) and \( L_\sigma \), among a set of values as follows:

\[
  F_\sigma \in \mathcal{F} := \left\{ F_i \in \mathcal{R}^{p \times n}, A + BF_i \text{ is Schur}, i \in \mathcal{I} \right\}, \mathcal{I} = \left\{ 1, \ldots, \kappa \right\},
\]

\[
  L_\sigma \in \mathcal{L} := \left\{ L_i \in \mathcal{R}^{n \times m}, A - L_i C \text{ is Schur}, i \in \mathcal{I} \right\},
\]

where \( \sigma \in \mathcal{I} \) is the switching law that is to be protected so that it is unknown for the attacker. Let \( F_0 \) and \( L_0 \) denote the gain matrices \( F \) and \( L \) adopted in the control law (14)-(15), and

\[
  r_{0,\sigma}(z) = \hat{M}_\sigma(z)y(z) - \hat{N}_\sigma(z)u^a(z), \sigma \in \mathcal{I},
\]

\[
  r_{en,\sigma}(z) = X_\sigma(z)u^a(z) + Y_\sigma(z)y(z),
\]

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Theorem 5 Given the plant model [30]-[37], the control law \( K(z) \) satisfying [14]-[15] and the system configuration shown in Figure 3, it holds

\[
\begin{align*}
\hat{M}_0 &= (A - L_\sigma C, -L_\sigma, C, I), \quad \hat{N}_0 = (A - L_\sigma C, B - L_\sigma D, C, D), \\
X_\sigma &= (A - L_\sigma C, -(B - L_\sigma D), F_\sigma, I), \quad Y_\sigma = (A - L_\sigma C, -L_\sigma, F_\sigma, 0).
\end{align*}
\]

Then, we have

\[
\begin{align*}
\hat{M}_0(z) &= P_0(z) r_{0,p}(z), \quad r_{0,p}(z) = \hat{M}_0(z) y(z) - \hat{N}_0(z) u(z), \\
P_0(z) &= I + C (zI - A_{L_\sigma})^{-1} (L_0 - L_\sigma) \in \mathcal{RH}_\infty, A_{L_\sigma} = A - L_\sigma C, \\
r_{en,\sigma}(z) &= P_u(z) r_{en,0}(z) + Q(z) r_{0,p}(z), \quad r_{en,0}(z) = X_0(z) u(z) + Y_0(z) y(z), \\
P_u(z) &= I + (F_0 - F_i) (zI - A_{F_0})^{-1} B \in \mathcal{RH}_\infty, A_{F_0} = A + BF_0, \\
Q(z) &= F_i (zI - A_{L_i})^{-1} (L_0 - L_i) - (F_i - F_0) (zI - A_{F_0})^{-1} L_0 \in \mathcal{RH}_\infty,
\end{align*}
\]

where

\[
\begin{align*}
\hat{M}_0 &= (A - L_0 C, -L_0, C, I), \quad \hat{N}_0 = (A - L_0 C, B - L_0 D, C, D), \\
X_0 &= (A - L_0 C, -(B - L_0 D), F_0, I), \quad Y_0 = (A - L_0 C, -L_0, F_0, 0).
\end{align*}
\]

The proof of this theorem follows immediately from Lemma [1].

Remark 3 Although in the attack-free case

\[
r_0(z) = r_{0,p}(z) = \hat{M}_0(z) y(z) - \hat{N}_0(z) u(z) = r_{0,n}(z),
\]

we would like to call the reader’s attention to the differences between the residual signals \( r_{0,p} \) and \( r_0 \). While \( r_0 \) is realised on the monitoring and control side, \( r_{0,p} \) is generated on the plant side. Here, \( r_{0,n} \) denotes the influence of the noises on the residual vector. Moreover, in case of attacks,

\[
\begin{align*}
\hat{M}_0(z) &= P_0(z) r_{0,p}(z), \quad r_{0,p}(z) = \hat{M}_0(z) y(z) - \hat{N}_0(z) u(z) + r_{0,n}(z), \\
r_0(z) &= \hat{M}_0(z) y(z) - \hat{N}_0(z) u(z) = \hat{N}_0(z) a_u(z) + \hat{M}_0(z) a_y(z) + r_{0,n}(z),
\end{align*}
\]

Theorem [5] demonstrates that switching the gain matrices \( (F_\sigma, L_\sigma) \) can be equivalently interpreted as (i) switching post-filters \( P_{u,\sigma}(z) \) and \( P_{0,\sigma}(z) \) to \( r_{en,0}(z) \) and \( r_{0,p}(z) \),

\[
\begin{align*}
P_{u,\sigma}(z) &\in \{ P_{u,i}(z) \in \mathcal{RH}_\infty, P_{u,i} = I + (F_0 - F_i) (zI - A_{F_0})^{-1} B, i \in \mathcal{I} \}, \\
P_{0,\sigma}(z) &\in \{ P_{0,i}(z) \in \mathcal{RH}_\infty, P_{0,i} = I + C (zI - A_{L_i})^{-1} (L_0 - L_i), i \in \mathcal{I} \},
\end{align*}
\]

and (ii) adding additional residual signal \( Q_\sigma(z) r_{0,p}(z) \) with a switching post-filter \( Q_\sigma(z) \),

\[
Q_\sigma(z) \in \{ Q_i(z) \in \mathcal{RH}_\infty, i \in \mathcal{I}, \\
Q_i = F_0 (zI - A_{L_i})^{-1} (L_0 - L_i) P_{0,i}(z) - (F_i - F_0) (zI - A_{F_0})^{-1} L_0 \}
\]

Note that \( Q_\sigma(z) r_{0,p}(z) \) is noise. In the remaining part of this work, \( F_0 \) and \( L_0 \) are used to denote the gain matrices \( F \) and \( L \) adopted in the control law [14]-[15], and correspondingly the LCP \( (\hat{M}, \hat{N}), (X, Y) \) are denoted by \( (\hat{M}_0, \hat{N}_0), (X_0, Y_0) \), respectively.
Remark 4 The encrypting effect of adding switched post filters and noises by switching the gain matrices among different values is analogous to the existing approaches, for instance, reported in (Weerakkody and Sinopoli, 2015; Griffioen et al., 2021; Schellenberger and Zhang, 2017; Dibaji et al., 2019), although in our proposed method both the design and (online) computations are considerably less demanding. In addition, the signal $r_{en}$ is encoded on the plant side before the transmission and the real residual signal $r_{u,0}$ is recovered by a decoding algorithm on the monitoring and control side.

The switched encoder system plays a central role for detecting the kernel attacks successfully. Their use is to prevent an attacker from identifying the dynamics of encoding system (56) so that the attack signal $a_{r,\sigma}(k)$ is set to be

$$a_{r,\sigma}(z) = Y_\sigma(z)a_y(z) - X_\sigma(z)a_u(z).$$

On the other hand, the needed online computations for the implementation of

$$r_{en,\sigma}(z) = X_\sigma(z)u\sigma(a)(z) + Y_\sigma(z)y(z)$$

that is to be performed on the plant side should be considered and kept as less as possible.

Let $\sigma(k_s)$ denote the switching law with $k_s$ as switching time instant, $F_{\sigma(k_s)}$ and $L_{\sigma(k_s)}$ be the operating mode of the gain matrices between two successive switching time instants $k_s = k_0, k_1$. On the assumption that

- the attacker could access $y(k), u\sigma(k)$, even
- have knowledge of $F_i$ and $L_i$ and so that $(X_i, Y_i), i = 1, \cdots, \kappa$, are known,
- the switching law $\sigma(k_s)$ is shared only by the monitoring system and the plant system but kept hidden from the attacker,

the LCP $(X_{\sigma(k_0)}, Y_{\sigma(k_0)})$ (i.e. the encoder running over the time interval $[k_0, k_1]$) should not be detected or identified by the attacker using the data collected over $[k_0, k_1]$. This can be formulated as an inverse problem of fault isolation or identification. It is well-known that if the time interval $[k_0, k_1]$ is sufficiently short with respect to the complexity (e.g. the order) of $(X_{\sigma(k_0)}, Y_{\sigma(k_0)})$ and the mode number $\kappa$, with high confidential $(X_{\sigma(k_0)}, Y_{\sigma(k_0)})$ cannot be detected or identified. On the other hand, in order to guarantee the stability of the switched system, the switching law $\sigma(k_s)$ is to be designed to satisfy the so-called average dwell time (ADT) condition (Hespanha and Morse, 1999; Zhao et al., 2012). Recall that $r_{en,\sigma}$ is only used for the detection purpose and $(X_{\sigma(k_0)}, Y_{\sigma(k_0)})$ has, different from the existing approaches, no influence on the system control performance. As a result, $(X_{\sigma(k_0)}, Y_{\sigma(k_0)})$ together with the switching law $\sigma(k_s)$ can be designed so that (i) $(X_{\sigma(k_0)}, Y_{\sigma(k_0)})$ is not identifiable over the time interval, (ii) the ADT condition is satisfied. Since the major focus of this work is on detecting kernel attacks, we will not discuss about the design of the switching issues for $F_\sigma$ and $L_\sigma$ in more details. The reader can refer to, for instance, the approach of cryptographically secure pseudo random number generator (PRNG) described in (Griffioen et al., 2021) or the approach proposed by (Schellenberger and Zhang, 2017).
4.4 Realisation of the detection scheme

In this sub-section, we describe the realisation of the detection scheme proposed in the previous sub-section. To this end, two issues are to be addressed: (i) real-time implementation of the residual generators, and (ii) design of test statistic and threshold setting. Concerning the first issue, the major tasks consist of

- computation on the plant side:
  \[ r_{en,\sigma}(z) = X_{\sigma}(z)u^a(z) + Y_{\sigma}(z)y(z), \]  
  \[ (74) \]

- signal transmissions from the plant side to the monitoring side:
  \[ r_{en,\sigma}^a(k) = r_{en,\sigma}(k) + a_{re,\sigma}(k), y^a = y(k) + a_y(k), \]

- computation on the monitoring and control side:
  \[ r_{u,0}(z) = r_{en,\sigma}^a(z) - P_{u,\sigma}(z)\bar{v}_0(z), \bar{v}_0(z) = \left(X_0(z) - Q(z)\bar{N}_0(z)\right)v(z), \]
  \[ (75) \]
  \[ r_0(z) = \bar{M}_0(z)y^a(z) - \bar{N}_0(z)u(z), \]  
  \[ (76) \]

and under consideration of the plant model \([30]-[31]\) with the process and sensor noises satisfying \([32]-[33]\). It follows from \([58]-[59]\) that the state space realisations of \([74]-[76]\) are described respectively by

\[ \zeta(k+1) = (A - L_{\sigma}C)\zeta(k) + L_{\sigma}y(k) + (B - L_{\sigma}D)u^a(k), \]
\[ r_{en,\sigma}(k) = u^a(k) - F_{\sigma}\zeta(k) \]  
\[ (77) \]
\[ (78) \]

as well as

\[ \dot{x}(k+1) = (A - L_0C)\dot{x}(k) + (B - L_0D)u(k) + L_0y^a(k), \]
\[ x_v(k+1) = (A - L_0C)x_v(k) + (B - L_0D)v(k), \]
\[ r_0(k) = y^a(k) - (C\dot{x}(k) + Du(k)), \]  
\[ \bar{v}_0(z) = v(z) - Fx_v(z) - Q(z)(C\bar{x}_v(z) + Dv(z)), \]  
\[ r_{u,0}(z) = r_{en,\sigma}^a(z) - P_{u,\sigma}(z)\bar{v}_0(z). \]  
\[ (79) \]
\[ (80) \]
\[ (81) \]
\[ (82) \]
\[ (83) \]

Next, the influences of \(\omega(k), \nu(k)\) on \(r_{u,0}(k)\) and \(r_0(k)\) during attack-free operations are analysed aiming at setting an optimal threshold. It turns out

\[ e(k+1) = (A - L_0C)e(k) + \omega(k) - L_0\nu(k), e(k) = x(k) - \dot{x}(k), \]
\[ r_0(k) = Ce(k) + \nu(k), \]  
\[ (84) \]
\[ (85) \]
\[ r_{u,0}(z) = r_{en,\sigma}^a(z) - P_{u,\sigma}(z)\bar{v}_0(z) = P_{u,\sigma}(z)(r_{en,0}(z) - \bar{v}_0(z)) + Q_{\sigma}(z)r_0(z), \]  
\[ (86) \]
\[ r_{en,0}(z) - \bar{v}_0(z) = -Q(z)r_0(z) \implies r_{u,0}(z) = (Q_{\sigma}(z) - P_{u,\sigma}(z)Q(z))r_0(z), \]  
\[ (87) \]

which implies that the residual vector

\[
\begin{bmatrix} r_{u,0}(z) \\ r_0(z) \end{bmatrix} = \begin{bmatrix} \hat{Q}_{\sigma}(z) \\ I \end{bmatrix} r_0(z), \quad \hat{Q}_{\sigma} = Q_{\sigma} - P_{u,\sigma}Q
\]

20
is a normally distributed color noise vector. In order to achieve an optimal attack detection, a post-filter \( P(z) \) is added as follows

\[
r(z) = \begin{bmatrix} r_u(z) \\ r_{0,K}(z) \end{bmatrix} = P(z) \begin{bmatrix} r_{u,0}(z) \\ r_0(z) \end{bmatrix},
\]

\[
P(z) = \begin{bmatrix} I & -\bar{Q}_\sigma(z) \\ 0 & Q_{K0}(z) \end{bmatrix}, Q_{K0}(z) = I + C(zI - A_{L_K})^{-1}(L_0 - L_K)
\]

where \( L_K \) is the Kalman filter gain matrix satisfying (34). Correspondingly, \( r_K(k) \sim \mathcal{N}(0, \Sigma_r) \) and is white with \( \Sigma_r \) given in (35). It is remarkable that the residual vector \( r_u \) is fully decoupled from the noises \( \omega(k), \nu(k) \). In order to define a practical and easily computing (scale) test statistic, \( r_u(z) \) is treated as a (quasi-) random vector with a covariance matrix whose inverse is approximated by \( \lambda I \), where \( \lambda > 0 \) is a sufficiently large number. As a result, we set the test statistic equal to

\[
J(k) = \lambda r_u^T(k)r_u(k) + r_{0,K}^T(k) \Sigma_r^{-1} r_{0,K}(k) \sim \chi^2(m),
\]

which is subject to \( \chi^2 \) distribution with \( m \) degrees of freedom in the attack-free operation, and the threshold

\[
J_{th} = \chi^2_\alpha(m)
\]

for a given upper-bound of false alarm rate \( \alpha \).

**Remark 5** It is noteworthy that detecting kernel attacks is in the foreground of our study. In order to highlight the basic ideas and major results in this regard clearly, only process and measurement noises are taken into account. The above simplified handling of \( r_u \) follows from the geometric interpretation of the \( \chi^2 \) text statistic (Ding, 2020). If unknown inputs and model parameter variations are to be considered, advanced fault detection methods could be applied (Ding, 2020).

When the control loop is attacked, the dynamics of the observer-based attack detector (75)-(76) is governed by

\[
r_{u,0}(z) = r_{en,\sigma}(z) - P_{u,\sigma}(z)\tilde{v}_0(z)
\]

\[
= X_\sigma(z)u^a(z) + Y_\sigma(z)y(z) + a_{re_n}(z) - P_{u,\sigma}(z)\tilde{v}_0(z)
\]

\[
= r_{u,\sigma}(z)(X_0(z)u^a(z) + Y_0(z)y(z)) + Q_\sigma(z)r_{0,p}(z) + a_{re_n}(z) - P_{u,\sigma}(z)\tilde{v}_0(z)
\]

\[
= a_1(z) + \dot{Q}_\sigma(z)r_{0,n}(z),
\]

\[
a_1(z) = P_{u,\sigma}(z)(X_0(z)a_u(z) - Y_0(z)a_y(z)) + a_{re_n}(z),
\]

\[
r_0(z) = \dot{M}_0(z)y^a(z) - \dot{N}_0(z)u(z) = a_2(z) + r_{0,n}(z),
\]

\[
a_2(z) = \dot{M}_0(z)a_y(z) + \dot{N}_0(z)a_u(z),
\]

where \( r_{0,n}(z) \) describes the influence of the noises on the residuals \( r_{0,p}(z) \) and \( r_0(z) \) and is given by

\[
e(k + 1) = (A - L_0C)e(k) + \omega(k) - L_0\nu(k), r_{0,n}(k) = Ce(k) + \nu(k).
\]
Hence,

\[
\begin{align*}
    r(z) &= P(z) \begin{bmatrix} r_{u,0}(z) \\ r_0(z) \end{bmatrix} = \begin{bmatrix} r_u(z) \\ r_{0,K}(z) \end{bmatrix} = \begin{bmatrix} a_1(z) - Q_\sigma(z)a_2(z) \\ Q_{K0}(z)(a_2(z) + r_{0,n}(z)) \end{bmatrix} \\
    J(k) &= \lambda r_u^T(k)r_u(k) + r_{0,K}^T(k)\Sigma_r^{-1}r_{0,K}(k) \\
    &= \lambda \tilde{a}_1^T(k)\tilde{a}_1(k) + (\tilde{a}_2(k) + r_K(k))^T\Sigma_r^{-1}(\tilde{a}_2(k) + r_K(k)) \sim \chi^2(\delta, m), \\
    \tilde{a}_1(z) &= a_1(z) - Q_\sigma(z)a_2(z), \tilde{a}_2(z) = Q_{K0}(z)a_2(z), r_K(z) = Q_{K0}(z)r_{0,n}(z),
\end{align*}
\]

where \( \chi^2(\delta, m) \) denotes a noncentral \( \chi^2 \) distribution with

\[
\delta = \lambda \tilde{a}_1^T(k)\tilde{a}_1(k) + \tilde{a}_2^T(k)\Sigma_r^{-1}\tilde{a}_2(k)
\]

as the noncentrality parameter and \( m \) the degree of freedom. As well-known (Ding, 2014), the test statistic (91) and the threshold (92) lead to the maximal fault detectability and guarantee the FAR bounded by \( \alpha \). Moreover, from (94), (95) and (97) it can be evidently seen that all attacks, \( a_r, a_u, a_y \), can be well detected as far as the dynamics of the encoded signal \( r_{en,\sigma} \) or equivalently \( (X_\sigma, Y_\sigma) \) is not identified.

As summary of the proposed detection scheme, the configuration of the detection system including data transmissions is sketched in Figure 4.

![Figure 4: Schematic description of the proposed attack detection system](image)

At the end of this section, we would like to underline the following points:

- the detection scheme proposed in this section and based on the residual signals \( r_0, r_{0,u} \) can be analogously realised as well using the alternative residual signals \( r_{u,c}, r_{y,c} \) defined in (54);
- the test statistic (91) and the threshold (92) deliver the optimal attack detection only on the assumptions of (i) the statistic features of the noises being specified by (32)-(33), and (ii) the additive character of the kernel attacks being under consideration (Ding, 2020), and
in case that the noises cannot be described by (32)-(33) or/and the cyber-attacks are presented e.g. in multiplicative form like false data injection attacks (Liang et al., 2017; Griffioen et al., 2019), sophisticated detection schemes are needed. Some of these methods are reported in (Li and Ding, 2020; Ding, 2020).

5 An encrypted configuration of feedback control and detection systems

The basis for the execution of kernel attacks is that attackers have knowledge of plant dynamics. Among numerous possibilities to gain such information, eavesdropping attacks enable collecting sufficient amount of process data which can then be used for identifying the plant dynamics. It is state of the art that in real industrial applications plant input and output data, \( u(k) \) and \( y(k) \), are often transmitted between the control and monitoring station and the plant via networks. Such system configurations make an identification of the plant dynamics considerably easy. In this section, we propose an encrypted configuration scheme of feedback control systems. The core of the alternatively configured control systems consists in the transmission of encoded system signals, instead of \( u(k) \) and \( y(k) \), from which a direct identification of the plant dynamics without \textit{a priori} knowledge becomes almost impossible. The basis for this encrypted configuration is the so-called functionalisation of dynamic controllers introduced in the unified framework of control and detection.

5.1 Functionalisation of all stabilising feedback controllers

Recall the observer-based realisation of all stabilising controllers given in (17)-(19). It can be divided into several functional modules:

- an observer and an observer-based residual generator,

\[
\dot{x}(k + 1) = A\hat{x}(k) + Bu(k) + L_0r_0(k), \quad r_0(k) = y(k) - \hat{y}(k), \quad \hat{y}(k) = C\hat{x}(k) + Du(k),
\]

which serve as an information provider for the controller and diagnostic system, and deliver a state estimation, \( \hat{x} \), and the primary residual, \( r_0 = y - \hat{y} \),

- control law

\[
u(z) = F_0\hat{x}(z) - Q(z)r_0(z) + \hat{V}(z)v(z),
\]

including

- a feedback controller: \( F_0\hat{x}(z) - Q(z)r_0(z) \) and
- a feed-forward controller: \( \hat{V}(z)v(z), \hat{V} = X_0 - Q\hat{N}_0 \), and in addition, for the detection purpose,

- detector \( R(z)r_0(z) \) with \( R(z) \) as a stable post-filter.

This modular structure provides us with a clear parameterisation of the functional modules:

- the state observer is parameterised by \( L_0 \),
the feedback controller by $F_0, Q$,  
the feed-forward controller by $\hat{V}$, and  
the detector by $R$.

Although all five parameters listed above are available for the design and online optimisation objectives, they have evidently different functionalities, as summarised below:

- $F_0, L_0$ determine the stability and eigen-dynamics of the closed-loop,
- $R, \hat{V}$ have no influence on the system stability, and $R$ serves for the optimisation of the detectability, while $\hat{V}$ for the tracking behavior, and
- $Q$ is used to enhance the system robustness and control performance. The design and update of $Q$ will have influence on the system dynamics and stability, when parameter uncertainties or degradations are present in the system.

It is evident that the above five parameters have to be, due to their different functionalities, treated with different priorities. Recall that system stability and eigen-dynamics are the fundamental requirement on an automatic control system. This requires that the system stability should be guaranteed, also in case of cyber-attacks. Differently, $Q, R, \hat{V}$ are used to optimise control or detection performance. In case that a temporary system performance degradation is tolerable, the real-time demand and the priority for an online optimisation of $Q, R, \hat{V}$ are relatively lower. Under these considerations, we propose in the next sub-section an encrypted control system configuration based on the above controller functionalisation.

### 5.2 An encrypted system configuration scheme

To begin with, we would like to emphasise that the objective of the system configuration proposed in the sequel is to prevent system knowledge from attackers in the manner that the plant model cannot be identified using the data possibly collected by attackers by means of eavesdropping attacks. Moreover, the basic requirements on the system control performance like the stability are to be met.

The proposed encrypted system configuration mainly consists of

- on the plant side, an observer-based state feedback controller and residual generator,

\[
\dot{x}(k + 1) = A\dot{x}(k) + Bu(k) + L_0 r_{0,p}(k),
\]

\[
u(k) = F_0 \dot{x}(k) + \gamma(k) \implies \dot{x}(k + 1) = (A + BF_0) \dot{x}(k) + B\gamma(k) + L_0 r_{0,p}(k) \tag{98}
\]

\[
= (A - L_0 C) \dot{x}(k) + (B - L_0 D) u(k) + L_0 y(k), \tag{99}
\]

where $\gamma$ is the signal (vector) received from the monitoring and control side,

- on the monitoring and control side,

\[
\gamma(z) = \hat{V}(z)v(z) - Q(z)r_{0,p}(z),
\]

where $r_{0,p}$ is received from the plant side and $v$ is the reference vector,
transmission from the plant side to the monitoring and control side, $r_{0,p}(k)$,

- transmission from the monitoring and control side to the plant side, $\gamma(k)$.

Depending on applications, the following functional modules can be further realised and integrated on the monitoring and control side, for instance,

- reconstructing $y(k)$,

\[
\dot{x}(k+1) = (A + BF_0) \dot{x}(k) + B\gamma(k) + L_0r_{0,p}(k),
\]

\[
y(k) = r_{0,p}(k) + \hat{y}(k) = (C + DF_0) \dot{x}(k) + D\gamma(k) + r_{0,p}(k)
\]  \hspace{1cm} (100) \hspace{1cm} (101)

with $r_{0,p}(k)$ received from the plant side,

- tuning $Q$ using $r_{0,p}(k)$ and $v(k)$ to enhance the stability margin, as reported in (Li et al., 2019), or

- recovering control performance degradation using $y(k)$ and $u(k)$, as described in (Ding, 2020).

It is evident that, according to the observer-based realisation of all stabilising controllers, the control input $u(k)$ acted on the actuators (located on the plant side) is given by

\[
u(k) = F_0\dot{x}(k) + \gamma(k) \iff u(z) = K(z)y(z) + v(z)
\]

with $K$ satisfying (14)-(15). Different from the standard system configuration, for instance the one shown in Figure 1, the observer-based state feedback controller and residual generator (98)-(99) running on the plant side serve as

- an encoder for an encrypted transmission of the plant measurement $y(k)$, i.e. $r_{0,p}(k)$ instead of $y(k)$,

- a decoder for control input $u(k) = F_0\dot{x}(k) + \gamma(k)$, and

- a local controller guaranteeing the basic control performance like the stability even if the communication between the both sides of the control system is considerably attacked.

Simultaneously, the recovering algorithm (100)-(101) running on the monitoring and control side acts (i) as a decoder for $y$ and (ii) $\gamma = Vv - Qr_{0,p}$ as an encoder for an encrypted transmission of the control signal from the monitoring and control side to the plant.

Considering that, during the attack-free operation, $r_{0,p}$ is noise (and even white noise when $L_0$ is set to be the Kalman filter gain matrix), it is obviously impossible to identify the plant model $G_u$ by means of $r_{0,p}$ and $\gamma$ that could be eavesdropped during their transmission. As a result, it can be claimed that the encrypted control system configuration proposed in this sub-section fully fulfills the design requirements.
5.3 The associated attack detection scheme

Figure 5 sketches schematically the proposed encrypted system configuration. On the assumptions that

- the control loop under consideration is configured as sketched in Figure 5,
- the attacker has no knowledge about the plant model \( G_u \), and
- both \( \gamma \) and \( r_{0,p} \) are corrupted by the attack signals \( a_\gamma \) and \( a_{r_0} \) respectively, i.e.
  \[
  \begin{align*}
  \gamma^a(k) &= \gamma(k) + a_\gamma(k) \implies u^a(k) = \gamma(k) + a_\gamma(k) + F_0\hat{x}(k), \\
  r_{0}^a(k) &= r_{0,p}(k) + a_{r_0}(k),
  \end{align*}
  \]

we propose the following attack detection scheme performed on the monitoring and control side. Similar to the controller, the attack detector is also distributedly realised on the both sides of the control system.

![Figure 5: Schematic description of the encrypted control and detection system configuration](image)

Remember that in the attack-free case

\[
\begin{align*}
X_0(z)u(z) + Y_0(z)y(z) - \gamma(z) &= X_0(z)u(z) + Y_0(z)y(z) - (\bar{v}(z) - Q(z)r_{0,p}(z)) \\
&= u(z) - (F_0\hat{x}(z) + \gamma(z)) = 0.
\end{align*}
\]

(102)

It motivates us to encrypt the detector as follows. At first, the encoded signal \( \beta(k) \) is generated on the plant side,

\[
\beta(k) = F_0\hat{x}(k) - F_\sigma\hat{x}(k),
\]

(103)
where $F_\sigma$ is a switched feedback gain introduced in the previous section. It follows from Lemma 1 and Theorem 5 that

$$
\beta(z) = (F_0 - F_\sigma) \hat{x}(z) = u^a(z) - F_\sigma \hat{x}(z) - (u^a(z) - F_0 \hat{x}(z))
= R_{0\sigma}(z) (X_0(z)u^a(z) + Y_0(z)y(z)) + Q_{0\sigma}(z)r_{0,p}(z),
$$

(104)

$$
R_{0\sigma}(z) = P_{u,\sigma}(z) - I = (F_0 - F_\sigma)(zI - A_{F_0})^{-1} B,
$$

(105)

$$
Q_{0\sigma}(z) = (F_0 - F_\sigma)(zI - A_{F_0})^{-1} L_0.
$$

Here, $P_{u,\sigma}(z)$ is given in Theorem 5. The encoded signal $\beta$ is then sent to the monitoring and control side, at which a residual signal is generated by decoding $\beta$ as follows

$$
r_\beta(z) = \beta^a(z) - R_{0\sigma}(z) \gamma(z),
$$

(106)

where

$$
\beta^a(k) = \beta(k) + a_\beta(k)
$$

denotes the corrupted signal $\beta$ due to the cyber-attack $a_\beta$. It turns out, remembering (102),

$$
r_\beta(z) = a_\beta(z) + R_{0\sigma}(z) (X_0(z)u^a(z) + Y_0(z)y(z) - \gamma(z)) + Q_{0\sigma}(z)r_{0,n}(z)
= a_\beta(z) + R_{0\sigma}(z)X_0(z)a_\gamma(z) + Q_{0\sigma}(z)r_{0,n}(z).
$$

(107)

with $r_{0,n}$ denoting the influence of the noises on the residual vector. Therefore, it holds, on the monitoring and control side,

$$
\begin{bmatrix}
  r_\beta(z) \\
  r_0^a(z)
\end{bmatrix} =
\begin{bmatrix}
  I & R_{0\sigma}(z)X_0(z) & 0 \\
  0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
  a_\beta(z) \\
  a_\gamma(z) \\
  a_{r_0}(z)
\end{bmatrix}
+ \begin{bmatrix}
  Q_{0\sigma}(z) \\
  I
\end{bmatrix} r_{0,n}(z).
$$

(108)

As a result, we have

**Theorem 6** Given the plant model (30)-(31), the control law $K$ satisfying (14)-(15) and residuals $r_\beta$ and $r_0^a$ that are realised in the encrypted system configuration shown in Figure 5, the attacks $a_\beta, a_\gamma$ and $a_{r_0}$ are stealthy, if and only if the conditions,

$$
a_{r_0}(k) = 0, a_\beta(z) + R_{0\sigma}(z)X_0(z)a_\gamma(z) = 0,
$$

(109)

are satisfied.

Theorem 6 reveals that

- an (additive) attack on the residual signal $r_{0,p}$ can be (structurally) directly detected, and

- keeping $a_\beta, a_\gamma$ stealthy is almost impossible, since condition (106) can hardly be satisfied, (i) without system knowledge, (ii) without knowing the purpose of using and transmitting $\beta$ and $\gamma$, and (iii) in particular when $R_{0\sigma}(z)$ is a switched system.
5.4 Implementation of the control and detection systems

Now, we summarise the implementation issues of the proposed control and detection systems.

On the plant side, the state observer (98) (equivalently (99)) builds the core of the system implementation. Based on the state estimate \( \hat{x}(k) \), the control input \( u^a(k) \), the residual signal \( r_{0,p}(k) \) as well as the encoded signal \( \beta(k) \) are formed,

\[
\begin{align*}
    u^a(k) &= F_0 \hat{x}(k) + \gamma^a(k), \\
    r_{0,p}(k) &= y(k) - C \hat{x}(k) - Du^a(k), \\
    \beta(k) &= (F_0 - F_\sigma) \hat{x}(k).
\end{align*}
\]

(110) (111) (112)

For running the realisation algorithms, the system on the plant side receives the signal \( \gamma^a \) from the monitoring and control side. It sends the residual signal \( r_{0,p} \) and encoded signal \( \beta \) to the system running on the monitoring and control side. It is of considerable interest to remark that the state observer (98) serves both as a decoder for the control signal, as given in (110), and as an encoder for the controller and for the generation of residual signal \( r_{\beta} \) (that are implemented on the monitoring and control side), as described by (111) and (112).

On the monitoring and control side, \( \gamma(k) \) is first computed as follows

\[
\begin{align*}
    x_v(k+1) &= (A - L_0C) x_v(k) + (B - L_0D)v(k), \\
    \gamma(z) &= v(z) - F_0 x_v(z) - Q(z) (C x_v(z) + D v(z) - r^a_0(z)).
\end{align*}
\]

(113) (114)

Then, \( r_{\beta} \) is generated as

\[
\begin{align*}
    x_{\beta}(k+1) &= (A + BF_0) x_{\beta}(k) + B \gamma(k), \\
    r_{\beta}(k) &= \beta^a(k) - (F_\sigma - F_0) x_{\beta}(k).
\end{align*}
\]

(115) (116)

It is worth emphasising that computation (113)-(114) serves as an encoder for the control signal, while the system (115)-(116) acts as a decoder.

Next, for detecting attacks \( a_{\beta}, a_{\gamma} \) and \( a_{r_0} \) optimally, the residual vector

\[
\begin{bmatrix}
    r_{\beta}(z) \\
    r_{0}(z)
\end{bmatrix} = \begin{bmatrix}
    I \\
    0
\end{bmatrix} \begin{bmatrix}
    x_v(z) \\
    Q_K0(z)
\end{bmatrix} =: \begin{bmatrix}
    r_u(z) \\
    r_0(z)
\end{bmatrix}
\]

(117)

and the test statistic

\[
J(k) = \lambda r_u^T(k) r_u(k) + r_{0,K}^T(k) \Sigma^{-1}_{r} r_{0,K}(k)
\]

are built with \( Q_{K0} \) as given in (89), which is analogue to the result described in Sub-section 4.4. We have

\[
J(k) = \lambda r_u^T(k) r_u(k) + r_{0,K}^T(k) \Sigma^{-1}_{r} r_{0,K}(k) \sim \chi^2(m),
\]

(118)

and thus the threshold is set to be

\[
J_{th} = \chi^2_\alpha(m)
\]

(119)

for a given upper-bound of false alarm rate \( \alpha \). In case of attacks,

\[
\begin{align*}
    J(k) &= \lambda r_u^T(k) r_u(k) + r_{0,K}^T(k) \Sigma^{-1}_{r} r_{0,K}(k) \\
         &= \lambda a^T(k) a_1(k) + (a_2(k) + r_{K}(k))^T \Sigma^{-1}_{r} (a_2(k) + r_{K}(k)) \sim \chi^2(\delta, m) \\
    a_1(z) &= a_\beta(z) + R_{0\sigma}(z) X_0(z) a_\gamma(z) - Q_{0\sigma}(z) a_{r_0}(z), \\
    a_2(z) &= Q_{K0}(z) a_{r_0}(z), r_{K}(k) \sim \mathcal{N}(0, \Sigma_r),
\end{align*}
\]

(120) (121)

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where $\chi^2(\delta, m)$ denotes a noncentral $\chi^2$ distribution with

$$\delta = \lambda a_1^T(k)a_1(k) + a_2^T(k)\Sigma^{-1}_r a_2(k)$$

as the noncentrality parameter and $m$ the degree of freedom.

6 Examples and experimental study

6.1 Examples of detecting typical kernel attacks

As examples, we will demonstrate that the zero dynamics, covert and replay attacks as kernel attacks can be well detected using the detection schemes proposed in Sections 4 and 5.

Example 2 Consider a zero dynamics attack satisfying (24). For our purpose of detecting $a_u$, applying both detection schemes presented in Sub-sections 4.3.4.4 and Section 5 results in

- by detector (88) whose dynamics with respect to the (possible) attack signals is described by (97):

$$r(z) = \begin{bmatrix} r_u(z) \\ r_{0,K}(z) \end{bmatrix} = \begin{bmatrix} P_{u,\sigma}(z)X_0(z)a_u(z) + a_{ren}(z) \\ r_K(z) \end{bmatrix},$$

where $a_{ren}$ denotes the (possible) attack signal on the transmitted signal $r_{en}$ that builds an (encoded) part of $r_u$.

- by detector (117) whose dynamics with respect to the (possible) attack signals is described by (108):

$$r(z) = \begin{bmatrix} r_u(z) \\ r_{0,K}(z) \end{bmatrix} = \begin{bmatrix} a_\beta(z) + R_{0,\sigma}(z)X_0(z)a_\gamma(z) \\ r_K(z) \end{bmatrix}, a_u(z) = a_\gamma(z)$$

with the (additional) attack signal $a_\beta$ on the encoded signal $\beta$.

It is evident that in the former case, $a_u$ can be detected using $r$ as far as the attacker could not identify $P_{u,\sigma}$ or equivalently $X_\sigma$ and thus set $a_{ren}$ equal to $-P_{u,\sigma}X_0a_u$. For the latter case, as long as the switched system $R_{0,\sigma}X_0$ could not be identified, it is impossible for the attacker to construct $a_\beta$ equal to $-R_{0,\sigma}X_0a_u$. Consequently, both $a_\beta$ and $a_u$ can be detected. We would like to emphasise that in this case it is impossible to identify the plant dynamics $(\hat{M}_0, \hat{N}_0)$ using eavesdropped data $\gamma(k), r_{0,p}(k)$.

Example 3 Now, consider the both detection systems under a covert attack satisfying (40). It holds,

- by detector (88):

$$r(z) = \begin{bmatrix} r_u(z) \\ r_{0,K}(z) \end{bmatrix} = \begin{bmatrix} P_{u,\sigma}(z)(X_0(z)a_u(z) - Y_0(z)a_y(z)) + a_{ren}(z) \\ r_K(z) \end{bmatrix},$$

with the (possible) additional attack $a_{ren}$ on the transmitted signal $r_{en}$.

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• by detector (117):

\[
\begin{bmatrix}
    r_u(z) \\
    r_{0,K}(z)
\end{bmatrix} =
\begin{bmatrix}
    a_\beta(z) + R_{0\sigma}(z)X_0(z)a_\gamma(z) - Q_{0\sigma}(z)a_\gamma(z) \\
    Q_{K0}(z)a_\gamma(z)
\end{bmatrix},
\]

(125)

\[a_u(z) = a_\gamma(z), a_y(z) = a_\gamma(z)\]

where it is assumed that the attack signal \(a_y\) is added to the transmitted signal \(r_{0,p}\), since \(r_{0,p}\) instead of \(y\) is transmitted from the plant side to the monitoring and control side.

It is clear that in the first case, \(a_u\) and \(a_y\) can be detected as far as the attacker could not identify \((X_\sigma, Y_\sigma)\). It is of considerable interest to notice the results in the second case. Using the detector (117), we can identify the attack \(a_y\) (\(a_{\gamma}\)),

\[a_{\gamma}(z) = Q_{K0}^{-1}(z)r_{0,K}(z), Q_{K0}^{-1}(z) = I + C(zI - A + L_0C)^{-1}(L_K - L_0),\]

and moreover estimate \(a_u\) based on

\[\hat{M}_0(z)a_y(z) + \hat{N}_0(z)a_u(z) = 0 \iff \hat{N}_0(z)a_u(z) = -\hat{M}_0(z)Q_{K0}^{-1}(z)r_{0,K}(z),\]

when \((a_y, a_u)\) is a covert attack. In this case, \(a_\beta\) can also be estimated in terms of

\[a_\beta(z) = -R_{0\sigma}(z)a_u(z) + Q_{0\sigma}(z)a_y(z).\]

Finally, as far as \(R_{0\sigma}(z), Q_{0\sigma}(z)\) are not identified, any attacks of \(a_\beta, a_u, a_u\) can be detected. This example clearly demonstrates the advantage of the detector (117) over the detector (88) and other reported attack detectors.

Example 4 We now address the detection issue of replay attacks under the assumption of steady operation, i.e.

\[y(k) \approx y(k - i), u(k) = u(k - i), i = 1, \ldots.\]

(126)

Since in our detection schemes proposed in the last two sections additional signals, \(r_{en}\) and \(\beta\), are transmitted from the plant side to the monitoring and control side, it is assumed that the attacker has collected and saved the (attack-free) data \(r_{en}(j), \beta(j), j \in [k_0, k_0 + M]\). When the data are replayed over the time interval \([k, k + M]\), \(k > k_0 + M\), it holds

\[r_{en}^a(i) = r_{en}(i - (k - k_0)), \beta^a(i) = \beta(i - (k - k_0)), i \in [k, k + M].\]

(127)

Moreover, an attack signal on the actuators is injected, for instance,

\[a_u(i) = a_\gamma(i), i \in [k, k + M].\]

It turns out

• by detector (88):

\[
\begin{bmatrix}
    r_u(z) \\
    r_{0,K}(z)
\end{bmatrix} \approx\begin{bmatrix}
    \Delta r_{en}(z) - \Delta r_K(z) \\
    r_K(z)
\end{bmatrix},
\]

(128)

\[
\Delta r_{en}(i) = r_{en,\sigma(i - (k - k_0))}(i - (k - k_0)) - r_{en,\sigma(i)}(i),
\]

\[
r_{en,\sigma(i - (k - k_0))}(z) = X_{\sigma(i - (k - k_0))}(z)u(z^{-(k-k_0)}) + Y_{\sigma(i - (k - k_0))}(z)y(z^{-(k-k_0)}),
\]

\[
r_{en,\sigma(i)}(z) = X_{\sigma(i)}(z)u(z) + Y_{\sigma(i)}(z)y(z),
\]

\[
\Delta r_K(i) = r_{K,\sigma(i - (k - k_0))}(i - (k - k_0)) - r_{K,\sigma(i)}(i),
\]

(129)

\[
r_{K,\sigma(i - (k - k_0))}(z) = Q_{\sigma(i - (k - k_0))}(z)r_K(z^{-(k-k_0)}), r_{K,\sigma(i)}(z) = Q_{\sigma(i)}(z)r_K(z),
\]

(130)

due to assumptions (126) and (127).
In comparison with the existing detection methods, it is clear that the mean and co-variance matrix of \( r_u(z) \) are generally different, which leads to

\[
\Delta r_K \approx (Q_{\sigma(i-(k-k_0))} - Q_{\sigma(i)}) r_K \neq 0.
\]

Consequently, both the mean and co-variance matrix of \( r_u(z) \) will change, which can be well detected using the generalised likelihood ratio (GLR) method (Ding, 2020). The second case is similar to the first one so that the replay attack can be detected in general, thanks to the fact that

\[
R_{0\sigma(i-(k-k_0))} [ \begin{array}{cc} X_{\sigma(i-(k-k_0))} & Y_{\sigma(i-(k-k_0))} \end{array} ] \neq R_{0\sigma(i)} [ \begin{array}{cc} X_{\sigma(i)} & Y_{\sigma(i)} \end{array} ],
\]

\[
Q_{0\sigma(i-(k-k_0))} \neq Q_{0\sigma(i)}.
\]

In comparison with the existing detection methods, it is clear that

- the two detection schemes proposed in this work guarantee structural detection of any kernel attacks, while the most existing methods can be generally applied to detecting a special type of kernel attacks;

- in particular, both methods deliver reliable detection of replay attacks without adding (additional) signals like a watermark in \( u \) (Mo et al., 2015). In fact, \( \Delta r_{en} - \Delta r_K \) and \( \Delta \beta - \Delta r_{0,K} \) delivered by the detectors (88) and (117), respectively, act like a watermark but without any influence on the control performance;

- the design of both detectors are straightforward without complicated computations, and

- the required online computations are less demanding.

### 6.2 Experimental study

Experimental study on detecting cyber-attacks on a real three-tank control system is running and the achieved results will be reported.
7 Conclusions

In this work, we have studied issues of detecting stealthy integrity cyber-attacks in the unified control and detection framework. The first effort has been dedicated to the general form of integrity cyber-attacks that cannot be detected using the well-established observer-based detection technique. It has been demonstrated that any attacks lying in the system kernel space cannot be detected by an observer-based detection system. Correspondingly, the concept of kernel attacks has been introduced. The replay, zero dynamics and covert attacks which are widely investigated in the literature are the examples of kernel attacks. Our further effort has been focused on the existence conditions of stealthy integrity attacks. To this end, the unified framework of control and detection has applied. It has been revealed that all kernel attacks can be structurally detected when residual generation is extended to the space spanned by the control signal. In other words, not only the observer-based residual, but also the control signal based residual signals are needed for a reliable detection of kernel attacks. As a result of this work, the necessary and sufficient conditions for detecting kernel attacks are given.

Based on the analytical results in the first part of our study, we have proposed two schemes for detecting kernel attacks. Using the known results and methods of the unified control and detection framework, both schemes result in reliable detection of kernel attacks without any loss of control performance. While the first detector is configured similar to the existing methods like the moving target method and auxiliary system aided detection scheme (Weerakkody and Sinopoli, 2015; Schellenberger and Zhang, 2017; Dibaji et al., 2019; Griffioen et al., 2019), the second detector is realised with the encrypted transmissions of control and monitoring signals in the feedback control system that prevent adversary to gain system knowledge by means of eavesdropping attacks. The theoretical basis for such detector configurations is the observer-based, residual-driven realisation of all stabilising feedback controllers. In particular, the functionalisation of controllers in the unified control and detection framework plays an essential role in developing the second detection scheme.

It should be remarked that our study in this work has been performed on the assumptions that (i) the LTI system models are not corrupted with model uncertainties, and (ii) the kernel attacks are presented in the additive form (although the replay attack is a multiplicative signal, it is handled as an additive one). In this context, the concept of kernel attacks and the derived existence conditions are in fact the expressions of structural properties of the feedback control system under consideration. So far, the proposed detection schemes would work well in laboratory conditions, but cannot be directly applied in real industrial applications without modifications. This fact motivates our future work to deal with cyber-attacks in the multiplicative form, for instance, false data injection attacks (Liang et al., 2017; Griffioen et al., 2019), and on automatic control systems with uncertainties. The unified control and detection framework and the associated detection methods developed recently (Li and Ding, 2020; Ding, 2020) could serve as efficient tools.

Appendix Proof of Lemma 1
Since

\[ R_{12}(z)F_2 = F_2 + (F_2 - F_1) (zI - A_{F_2})^{-1} BF_2 \]
\[ = F_1 + (F_2 - F_1) (zI - A_{F_2})^{-1} BF_2 \]
\[ = F_1 + (F_2 - F_1) (zI - A_{F_2})^{-1} (zI - A), \]

it turns out

\[ R_{12}(z)Y_2(z) = -(F_1 + (F_2 - F_1) (zI - A_{F_2})^{-1} (zI - A)) (zI - A_{L_2})^{-1} L_2, \]
\[ R_{12}(z)X_2(z) = R_{12}(z) - (F_1 + (F_2 - F_1) (zI - A_{F_2})^{-1} (zI - A)) (zI - A_{L_2})^{-1} (B - L_2 D). \]

Moreover, the relation

\[ (zI - A) (zI - A_L)^{-1} L = (I + LC (zI - A)^{-1})^{-1} L \]
\[ = L (I - C (zI - A + LC)^{-1} L) = L\hat{M}(z) \]

leads to

\[ R_{12}(z)Y_2(z) = -F_1 (zI - A_{L_2})^{-1} L_2 + \hat{R}_{12}(z)\hat{M}_2(z) \tag{134} \]
\[ R_{12}(z)X_2(z) = R_{12}(z) - F_1 (zI - A_{L_2})^{-1} (B - L_2 D) - \]
\[ (F_2 - F_1) (zI - A_{F_2})^{-1} (I - L_2 C (zI - A + L_2 C)^{-1} (B - L_2 D) \]
\[ = I - F_1 (zI - A_{L_2})^{-1} (B - L_2 D) + (F_2 - F_1) (zI - A_{F_2})^{-1} L_2 D \]
\[ + (F_2 - F_1) (zI - A_{F_2})^{-1} L_2 C (zI - A + L_2 C)^{-1} (B - L_2 D) \]
\[ = I - F_1 (zI - A_{L_2})^{-1} (B - L_2 D) - \hat{R}_{12}(z)\hat{N}_2(z). \tag{135} \]

Next, we consider \((zI - A_{L_2})^{-1} L_2\) and \((zI - A_{L_2})^{-1} (B - L_2 D)\). It is straightforward that

\[ (zI - A_{L_2})^{-1} L_2 - (zI - A_{L_1})^{-1} L_1 \]
\[ = (zI - A_{L_2})^{-1} (L_2 - (zI - A + L_2 C) (zI - A_{L_1})^{-1} L_1) \]
\[ = (zI - A_{L_2})^{-1} L_2 (I - C (zI - A_{L_1})^{-1} L_1) - (zI - A) (zI - A_{L_1})^{-1} L_1 \]
\[ = (zI - A_{L_2})^{-1} L_2 \hat{M}_1(z) - (zI - A_{L_2})^{-1} L_1 \hat{M}_1(z) \implies \]
\[ (zI - A_{L_2})^{-1} L_2 = (zI - A_{L_1})^{-1} L_1 + (zI - A_{L_2})^{-1} (L_2 - L_1) \hat{M}_1(z) \tag{136} \]

as well as

\[ (zI - A_{L_2})^{-1} (B - L_2 D) - (zI - A_{L_1})^{-1} (B - L_1 D) = \]
\[ (zI - A_{L_2})^{-1} (I - (zI - A + L_2 C) (zI - A_{L_1})^{-1} B - ((zI - A_{L_2})^{-1} L_2 - (zI - A_{L_1})^{-1} L_1) D \]
\[ = (zI - A_{L_2})^{-1} (L_1 - L_2) C (zI - A_{L_1})^{-1} B - (zI - A_{L_2})^{-1} (L_2 - L_1) \hat{M}_1(z)D = \]
\[ (zI - A_{L_2})^{-1} (L_1 - L_2) \left( \hat{M}_1(z)D + C (zI - A_{L_1})^{-1} B \right) = - (zI - A_{L_2})^{-1} (L_2 - L_1) \hat{N}_1(z) \]
\[ \implies (zI - A_{L_2})^{-1} (B - L_2 D) = (zI - A_{L_1})^{-1} (B - L_1 D) - (zI - A_{L_2})^{-1} (L_2 - L_1) \hat{N}_1(z). \tag{137} \]

Furthermore, it is well-known that

\[ \begin{bmatrix} -\hat{N}_2(z) & \hat{M}_2(z) \\ -\hat{N}_1(z) & \hat{M}_1(z) \end{bmatrix} = Q_{21}(z) \begin{bmatrix} -\hat{N}_1(z) & \hat{M}_1(z) \\ -\hat{N}_2(z) & \hat{M}_2(z) \end{bmatrix} \iff \tag{138} \]
\[ \begin{bmatrix} -\hat{N}_1(z) & \hat{M}_1(z) \end{bmatrix} = Q_{12}(z) \begin{bmatrix} -\hat{N}_2(z) & \hat{M}_2(z) \end{bmatrix}. \tag{139} \]
Summarising (134)-(139) leads to
\[
R_{12}(z)Y_2(z) = Y_1(z) - (F_1 (zI - A_{L_2})^{-1} (L_2 - L_1) - \hat{R}_{12}(z)Q_{21}(z)) \hat{M}_1(z),
\]
\[
R_{12}(z)X_2(z) = X_1(z) + (F_1 (zI - A_{L_2})^{-1} (L_2 - L_1) - \hat{R}_{12}(z)Q_{21}(z)) \hat{N}_1(z) \implies
\]
\[
[ X_1(z) \ Y_1(z) ] = R_{12}(z)\left[ \begin{array}{cc} X_2(z) & Y_2(z) \end{array} \right] + Q_{11}(z) \left[ \begin{array}{cc} -\hat{N}_1(z) & \hat{M}_1(z) \end{array} \right]
\]
as well as
\[
R_{12}(z)Y_2(z) = Y_1(z) - F_1 (zI - A_{L_2})^{-1} (L_2 - L_1) \hat{M}_1(z) + \hat{R}_{12}(z)\hat{M}_2(z),
\]
\[
R_{12}(z)X_2(z) = X_1(z) + F_1 (zI - A_{L_2})^{-1} (L_2 - L_1) \hat{N}_1(z) - \hat{R}_{12}(z)\hat{N}_2(z) \implies
\]
\[
[ X_1(z) \ Y_1(z) ] = R_{12}(z)\left[ \begin{array}{cc} X_2(z) & Y_2(z) \end{array} \right] + \hat{Q}_{12}(z) \left[ \begin{array}{cc} -\hat{N}_2(z) & \hat{M}_2(z) \end{array} \right],
\]
\[
Q_{12}(z) = F_1 (zI - A_{L_2})^{-1} (L_2 - L_1)Q_{12}(z) - \hat{R}_{12}(z).
\]
Since
\[
(zI - A_{L_2})^{-1} (L_2 - L_1)Q_{12}(z) = (zI - A_{L_2})^{-1} (L_2 - L_1) (I + C (zI - A_{L_1})^{-1} (L_2 - L_1))
\]
\[
= (zI - A_{L_2})^{-1} (I + (L_2 - L_1) C (zI - A_{L_1})^{-1}) (L_2 - L_1) = (zI - A_{L_1})^{-1} (L_2 - L_1),
\]
we finally have
\[
\hat{Q}_{12}(z) = F_1 (zI - A_{L_1})^{-1} (L_2 - L_1) - (F_1 - F_2) (zI - A_{F_2})^{-1} L_2.
\]
The lemma is proved.

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