Robust entropy expansiveness implies generic domination

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Abstract

Let $f : M \rightarrow M$ be a $C^r$-diffeomorphism, $r \geq 1$, defined on a compact boundaryless $d$-dimensional manifold $M$, $d \geq 2$, and let $H(p)$ be the homoclinic class associated with the hyperbolic periodic point $p$. We prove that if there exists a $C^1$ neighbourhood $U$ of $f$ such that, for every $g \in U$, the continuation $H(pg)$ of $H(p)$ is entropy expansive, then for $g$ in an open and dense subset of $U$, there is a $Dg$-invariant dominated splitting for $H(pg)$ of the form $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$ where all $F_j$ are one dimensional and not hyperbolic. Moreover, if $H(p)$ is isolated then $E$ is contracting and $G$ is expanding and if the indices of the periodic points in $H(pg)$ are equal to index $(p)$ then $H(pg)$ is hyperbolic for $g$ in an open and dense subset of $U$.

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1. Introduction

In this paper we study the consequences of robust entropy expansiveness for a diffeomorphism $f$ to the dynamical behaviour of the tangent map $Df$. In this direction we obtain that the tangent bundle has a $Df$-invariant dominated splitting of the form $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$ where $E$ is contracting, $G$ is expanding and all the $F_j$ are one dimensional and not hyperbolic.

Let $M$ be a compact connected boundaryless Riemannian $d$-dimensional manifold, $d \geq 2$, and $f : M \rightarrow M$ be a homeomorphism. Let $K$ be a compact invariant subset of $M$ and $\text{dist} : M \times M \rightarrow \mathbb{R}^+$ a distance in $M$ compatible with its Riemannian structure. For $G, J \subset K$, $n \in \mathbb{N}$ and $\delta > 0$ we say that $G$ ($n, \delta$)-spans $J$ with respect to $f$ if for each $y \in J$ there is $x \in G$ such that $\text{dist}(f^j(x), f^j(y)) \leq \delta$ for all $j = 0, \ldots, n - 1$. Let $r_n(\delta, J)$ denote the minimum
cardinality of a set that \((n, \delta)\)-spans \(J\). Since \(K\) is compact we know that \(r_n(\delta, J) < \infty\). We define
\[
h(f, J, \delta) \equiv \lim_{n \to \infty} \frac{1}{n} \log(r_n(\delta, J))
\]
and the topological entropy of \(f\) restricted to \(J\) as
\[
h(f, J) \equiv \lim_{\delta \to 0} h(f, J, \delta).
\]
The last limit exists since \(h(f, J, \delta)\) increases as \(\delta\) decreases to zero.

**Definition 1.1.** Let \(K\) be a compact invariant subset of \(M\). For \(x \in K\) let us denote
\[
\Gamma_\epsilon(x, f) \equiv \{y \in M : d(f^n(x), f^n(y)) \leq \epsilon, n \in \mathbb{Z}\}.
\]
We will simply write \(\Gamma_\epsilon(x)\) instead of \(\Gamma_\epsilon(x, f)\) when it is understood which \(f\) we refer to.

Following Bowen (see [Bo]) we say that \(f/K\) is entropy expansive or \(h\)-expansive for short, if and only if there exists \(\epsilon > 0\) such that
\[
h^*_f(\epsilon) \equiv \sup_{x \in K} h(f, \Gamma_\epsilon(x)) = 0.
\]

**Theorem 1.1 ([Bo, theorem 2.4]).** For all homeomorphisms \(f\) defined on a compact invariant set \(K\),
\[
h(f, K) \leq h(f, K, \epsilon) + h^*_f(\epsilon)
\]
in particular \(h(f, K) = h(f, K, \epsilon)\) if \(h^*_f(\epsilon) = 0\) holds.

A similar notion to \(h\)-expansiveness, albeit weaker, is the notion of asymptotically \(h\)-expansiveness introduced by Misiurewicz [Mi]: let \(K\) be a compact metric space and \(f : K \to K\) be a homeomorphism. We say that \(f\) is asymptotically \(h\)-expansive if and only if
\[
\lim_{\epsilon \to 0} h^*_f(\epsilon) = 0.
\]
Thus, we do not require that for a certain \(\epsilon > 0\), \(h^*_f(\epsilon) = 0\) but that \(h^*_f(\epsilon) \to 0\) when \(\epsilon \to 0\). It has been proved by Buzzi [Bu] that any \(C^\infty\) diffeomorphism defined on a compact manifold is asymptotically \(h\)-expansive. Examples of diffeomorphisms that are neither entropy expansive nor asymptotically entropy expansive can be found in [Mi, PaVi].

Next we recall the notions of a dominated and finest dominated splitting.

**Definition 1.2.** We say that a compact \(f\)-invariant set \(\Lambda \subset M\) admits a dominated splitting if the tangent bundle \(T_{\Lambda}M\) has a continuous \(Df\)-invariant splitting \(E \oplus F\) and there exist \(C > 0, 0 < \lambda < 1, \) such that
\[
\|Df^n|E(x)| \cdot \|Df^{-n}|F(f^n(x))\| \leq C\lambda^n \quad \forall x \in \Lambda, \quad n \geq 0.
\]
If \(T_{\Lambda}M = E_1 \oplus E_2 \oplus \cdots \oplus E_k\) with \(E_j Df\)-invariant, \(j = 1, \ldots, k\) then we say that \(E_1 \oplus E_2 \oplus \cdots \oplus E_k\) is dominated if for all \(1 \leq j \leq k - 1\)
\[
(E_1 \oplus \cdots \oplus E_j) \oplus (E_{j+1} \oplus \cdots \oplus E_k)
\]
is a dominated splitting.

We say that \(E_1 \oplus E_2 \oplus \cdots \oplus E_k\) is the finest dominated splitting if for all \(1 \leq j \leq k\) there is no possible decomposition of \(E_j\) as two invariant sub-bundles satisfying inequality (1).

We warn the reader that the notion of \(h\)-expansiveness is not a straightforward generalization of the notion of expansiveness. If a diffeomorphism \(f\) is expansive then for a certain \(\epsilon > 0\) the subset \(\Gamma_\epsilon(x)\) defined above is equal to \([x]\) and hence \(f\) is \(h\)-expansive.

The reciprocal of this statement is not true: if the topological entropy \(h(f)\) of a map \(f : M \to M\) vanishes then \(f\) is \(h\)-expansive but \(f\) may not be expansive. This is the case when \(f\) is the identity map or a Morse–Smale diffeomorphism.
Recall that the non-wandering set of \( f : M \to M \) is defined as \( \Omega(f) = \{ x \in M : \forall U \text{ neighbourhood of } x, \exists n > 0 \text{ such that } f^n(U) \cap U \neq \emptyset \} \). This set is \( f \)-invariant and hence if \( f/\Omega(f) \) is \( h \)-expansive then \( f \) is \( h \)-expansive. In particular axiom A diffeomorphisms with the no cycle condition (see [Gu]) provide a class of robustly \( h \)-expansive diffeomorphisms containing the Morse–Smale diffeomorphisms. To see that axiom A diffeomorphisms are \( h \)-expansive it suffices to note that if \( f \) is axiom A then \( f/\Omega(f) : \Omega(f) \to \Omega(f) \) is expansive. If \( f \) is axiom A and \( \Omega(f) \) is an infinite set then \( f \) has positive topological entropy. Hence such diffeomorphisms provide a class of robustly entropy-expansive maps with positive topological entropy.

We are interested in enlarging the class of such examples and in giving properties that the tangent map of \( h \)-expansive diffeomorphisms should have. We restrict our study to homoclinic classes \( H(p) \) associated with saddle-type hyperbolic periodic points. Recall that the homoclinic class \( H(p) \) of a saddle-type hyperbolic periodic point \( p \) of \( f \in \Diff^1(M) \) is the closure of the transverse intersections between the unstable manifold and the stable manifold of the orbit \( \mathcal{O}(p) \) of \( p \). These homoclinic classes are invariant by \( f \) and persist under perturbations. We wish to establish the properties of these classes under the assumption that they are robustly \( h \)-expansive.

**Definition 1.3.** Let \( M \) be a compact boundaryless \( C^\infty \) manifold and \( f : M \to M \) be a \( C^1 \) diffeomorphism. Let \( H(p) \) be a \( f \)-homoclinic class associated with the \( f \)-hyperbolic periodic point \( p \). We say that \( f/H(p) \) is \( C^1 \) robustly \( h \)-expansive, or robustly \( h \)-expansive for short, if there is a \( C^1 \) neighbourhood \( \mathcal{U} \) of \( f \) such that for any \( g \in \mathcal{U} \), the continuation \( H(pg) \) of \( H(p) \) is \( h \)-expansive.

In [PaVi, theorem B] we obtain that if \( H(p) \) is isolated (see definition 1.5) and there is a dominated splitting on \( H(p) \) of the form

\[
T_{H(p)} M = E \oplus F_1 \oplus \cdots \oplus F_k \oplus G
\]

with \( E \) contracting, \( G \) expanding and all \( F_j \), \( j = 1, \ldots, k \), one dimensional and not hyperbolic, then \( f/H(p) \) is \( h \)-expansive. Moreover, since the dominated splitting is preserved under \( C^1 \) perturbations this result holds for a \( C^1 \) neighbourhood \( \mathcal{U}(f) \subset \Diff^1(M) \), i.e. \( h \)-expansiveness is \( C^1 \) robust.

In fact in [PaVi, theorem B] it is assumed that we are far from tangencies without explicitly mentioning it in the hypothesis of this theorem. There, the existence of a dominated splitting in a residual subset of a \( C^1 \) neighbourhood of \( f \), with the form given above, is a consequence of [ABCDW, corollary 3] and [Go, corollary 6.6.2, theorem 6.6.8] under the assumptions that \( H(p) \) is isolated, has a dominated splitting and \( H(p) \) is far from tangencies.

This paper intends to continue [PaVi] in the reverse direction: we analyse the consequences of \( h \)-expansiveness holding in a \( C^1 \) neighbourhood \( \mathcal{U}(f) \subset \Diff^1(M) \) of \( f \). Our main results are the following:

**Theorem A.** Let \( M, f : M \to M \) and \( H(p) \) be as in definition 1.3. Then \( H(p) \) has a dominated splitting \( E \oplus F \), with \( \dim(E) = \dim(W^s(p)) \).

In fact [PaVi, example 2] shows that in dimension greater than or equal to three the existence of a dominated splitting for \( H(p) \) is not enough to guarantee \( h \)-expansiveness, so it is natural to search for stronger properties to characterize \( h \)-expansiveness.

**Corollary 1.2.** Let \( M, f : M \to M, \mathcal{U} \) and \( H(p) \) be as in definition 1.3. Then for \( g \) in an open and dense subset of \( \mathcal{U} \), \( H(pg) \) has a dominated splitting of the form

\[
T_{H(pg)} M = E \oplus F_1 \oplus \cdots \oplus F_k \oplus G,
\]

where all \( F_j \) are one dimensional and not hyperbolic.
Is it true that if \( f/H(p) \) is robustly \( h \)-expansive then \( T_{H(p)}M \) decomposes as in corollary 1.2?

If \( H(p) \) is isolated then we may refine the previous result. Before we announce precisely this result, let us recall the definitions of chain recurrent set, recurrence class, isolated homoclinic class and heterodimensional cycles.

**Definition 1.4.** The chain recurrent set of a diffeomorphism \( f \), denoted by \( R(f) \), is the set of points \( x \) such that, for every \( \epsilon > 0 \), there is a closed \( \epsilon \)-pseudo-orbit joining \( x \) to itself: there is a finite sequence \( x = x_0, x_1, \ldots, x_n = x, n \geq 1 \), such that \( \text{dist}(f(x_i), x_{i+1}) < \epsilon \).

The recurrence class of \( x \) is the set of points \( y \in M \) such that for any \( \epsilon > 0 \) there are \( \epsilon \)-pseudo-orbits connecting \( x \) with \( y \) and \( y \) with \( x \).

**Remark 1.3.** Generically a recurrence class which contains a periodic point \( p_g \) coincides with \( H(p_g) \) \([BC]\).

**Definition 1.5.** We say that \( H(p) \) is (robustly) isolated if there are neighbourhoods \( U \) of \( f \) in \( \text{Diff}^1(M) \) and \( U \) of the homoclinic class \( H(p) \) in \( M \) such that, for every \( g \in U \), the continuation \( H(p_g) \) of \( H(p) \) coincides with the intersection of the chain recurrent set \( R(g) \) of \( g \) with the neighbourhood \( U \).

**Remark 1.4.** The classical definition of isolated homoclinic class requires that it is isolated in the non-wandering set, i.e. there is a neighbourhood \( U \) of \( H(p) \) such that \( \Omega(f) \cap U = H(p) \).

**Definition 1.6.** We say that \( \Gamma \) is a cycle if \( \Gamma = \{ p_0, 0 \leq i \leq n, p_0 = p_n \} \), where \( p_i \) are hyperbolic periodic points of \( f \) and \( W^s(p_i) \cap W^u(p_{i+1}) \neq \emptyset \), for all \( 0 \leq i \leq n - 1 \). \( \Gamma \) is called a heterodimensional cycle if, for some \( i \neq j \), \( \dim(W^u(p_i)) \neq \dim(W^u(p_j)) \).

Recall that the index of a hyperbolic periodic point \( p \) is the dimension of its stable manifold \( W^s(p) \). Also, a diffeomorphism \( f \) has a heterodimensional cycle if there are transitive hyperbolic sets \( \Lambda \) and \( \Sigma \) having different indices (dimension of the stable bundle) such that the unstable manifold of \( \Lambda \) meets the stable one of \( \Sigma \) and vice versa.

**Theorem B.** Let \( f : M \to M \), \( \mathcal{U} \) and \( H(p) \) be as in definition 1.3. If \( f/H(p) \) is isolated then for \( g \) in an open and dense subset of \( \mathcal{U} \), \( H(p_g) \) has a dominated splitting of form (2) where \( E \) is contracting and \( G \) is expanding. Moreover, we have one of the following:

(a) for an open dense subset \( V \subset \mathcal{U} \), \( H(p_g) \) is hyperbolic if the indices of periodic points in \( H(p_g) \) are in a \( C^1 \) robust way equal to index \( (p) \),

(b) there is \( g \) arbitrarily near \( f \) having a robust heterodimensional cycle in \( H(p_g) \).

**Corollary 1.5.** For \( g \) in an open dense subset of \( \mathcal{U} \) we have that for every \( k > 0 \), \( g^k \) is robustly \( h \)-expansive.

**Remark 1.6.** If there is \( g \) arbitrarily \( C^1 \) near \( f \) such that \( H(p_g) \) has periodic points of different indices then, by \([BDi, \text{theorem 3.}]\), \( f/H(p) \) is not hyperbolic.

If we do not assume that \( H(p) \) is isolated but we know that \( f \) cannot be approximated by \( g \) exhibiting a heterodimensional cycle, then by \([Cr, \text{main theorem}]\) we have that for \( g \) in an open and dense subset \( \mathcal{R} \subset \mathcal{U}(f) \), \( H(p_g) \) has a dominated splitting of the form \( E^{s} \oplus E^{c} \oplus E^{u} \) where \( E^{s} \) is not hyperbolic and \( \dim(E^{s}) \leq 2 \), \( E^{c} \) is contracting and \( E^{u} \) is expanding. Moreover, if \( \dim(E^{c}) = 2 \) then \( E^{c} = E^{c}_{1} \oplus E^{c}_{2} \) dominated.

After writing this paper we were warned that Asaoka has obtained related results in \([As]\). In particular, for any manifold of dimension at least three, he constructs a hyperbolic invariant
set that exhibits $C^1$-persistent homoclinic tangencies. This provides an open subset of $C^1$-diffeomorphisms in which generic diffeomorphisms admit no symbolic extensions, and so $f$ cannot be $h$-expansive [BoDo]. The tools to get this result are analogous to those in [DN], as it occurs in the present paper and in [PaVi] too.

We point out that in [DF] Díaz and Fisher use similar techniques to those in [PaVi] to obtain that every partially hyperbolic diffeomorphism with a one-dimensional centre bundle has a principal symbolic extension. They also show that there are no symbolic extensions $C^1$-generically among diffeomorphisms containing non-hyperbolic robustly transitive sets with a centre indecomposable bundle of dimension at least 2.

For surface diffeomorphisms, the results achieved by Díaz and Fisher follow under weaker assumptions, as can be seen in [PaVi].

In a forthcoming work the authors, together with Díaz and Fisher, will address the case where the centre bundle splits into one-dimensional sub-bundles. This will generalize results obtained in [PaVi, DF].

By [BoDo, theorem 8.6] the existence of principal symbolic extensions is equivalent to $f$ being asymptotically $h$-expansive. In the present paper, since we are looking for a counterpart of robust $h$-expansiveness in terms of properties of the tangent map, we do not emphasize the notion of symbolic extensions.

1.1. Idea of the proofs

The proof of theorem A goes by contradiction: under the hypothesis that there is not a dominated splitting in $T_H(p)M$, we profit from some ideas of [PV, Ro] to create a flat tangency between $W^s(p)$ and $W^u(p)$. We remark that in [PV, Ro] for the case that dim$(M) > 2$ it was proved that if $r \geq 2$ and $g$ has a homoclinic tangency then there are diffeomorphisms arbitrarily $C^1$-close to $g$ exhibiting persistent homoclinic tangencies (thus generalizing the results of [Nh1, Nh2]). Here, since we are dealing with the $C^1$ topology, our arguments are simpler than the ones presented in [Nh1, Nh2]. We first produce a flat tangency for a $C^2$ diffeomorphism $g$ nearby $f$, and afterwards we create an arc of tangencies between $W^{s}(p_g, g)$ and $W^{u}(p_g, g)$.

Next we follow [DN] to perform another $C^1$ perturbation with support in a small neighbourhood of the arc of tangencies leading to the appearance of arbitrarily small horseshoes with positive entropy contradicting $h$-expansiveness for the perturbed diffeomorphism $G$. Therefore $Df/T_H(p)M$ admits a dominated splitting.

Remark 1.7. We produce a flat tangency between $W^s(p)$ and $W^u(p)$ at a homoclinic point $x$ associated with $p$. This avoids proving that $x \in H(p_G)$ since $x$ is still a homoclinic point of the final perturbation $G$. This differs from what is done in other papers in which the tangency is created at a periodic orbit $q$ homoclinically related to $p$.

Moreover, either there is a dominated splitting (see definition 1.2) which has the form $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$ where all $F_j$ are one dimensional and not hyperbolic or again we contradict robustness of $h$-expansiveness using [Go, theorem 6.6.8].

For the proof of theorem B we assume some specific generic properties described in section 3 and that $H(p)$ is isolated. These allow us to prove that the extremal sub-bundles $E$ and $G$ are, respectively, contracting and expanding. Moreover, if the index of periodic points of $H(p_g)$ is robustly the index of $p$ then for an open dense subset of $\mathcal{U}(f)$ the dominated splitting defined on $T_H(p)M$ is hyperbolic. This proof is done in two steps: first we show that the extremal sub-bundles are hyperbolic using the fact that $H(p)$ is isolated [BDPR]. Second we prove in lemma 3.2 that if in a $C^1$ robust way the indices of periodic points in $H(p_g)$ are
the same for \( g \in \mathcal{U}(f) \) then for an open and dense subset \( \mathcal{U}_1 \) of \( \mathcal{U}(f) \) we have that \( H(p_g) \) is hyperbolic.

2. Robust entropy expansiveness implies domination

In this section we prove theorem A assuming that \( f/H(p) \) is robustly entropy expansive.

Let \( H(p) \) be a \( f \)-homoclinic class associated with the hyperbolic periodic point \( p \). Assume that there is a \( C^1 \) neighbourhood \( \mathcal{U} \) of \( f \) such that for any \( g \in \mathcal{U} \) it holds that there is a continuation \( H(p_g) \) of \( H(p) \) such that \( H(p_g) \) is \( h \)-expansive.

In order to simplify notation and arguments we assume from now on that \( p \) is a hyperbolic fixed point. Let \( x \in W^s(p) \cap W^u(p) \) be a transverse homoclinic point associated with the point \( p \).

We define \( E(x) \) as \( T_xW^s(p) \) and \( F(x) \) as \( T_xW^u(p) \). Since \( p \) is hyperbolic we have that \( E(x) \oplus F(x) = T_xM \). Moreover, \( E(x) \) and \( F(x) \) are \( Df \)-invariant, i.e. \( Df(E(x)) = E(f(x)) \) and \( Df(F(x)) = F(f(x)) \). Denote by \( H_t(p) \) the set of the transverse homoclinic points associated with \( p \). Then, it can be proved that \( H(p) \equiv H_t(p) \). Here \( \bar{A} \) stands for the closure in \( M \) of the subset \( A \subset M \). So if we prove that there is a dominated splitting for \( H_t(p) \) we are done since we can extend, by continuity, the splitting to the closure \( H(p) \) of \( H_t(p) \) (see [SV]).

We will use the following result proved in [Fr]:

**Lemma 2.1 ([Fr, lemma 1.1]).** Let \( M \) be a closed \( n \)-manifold, \( f : M \to M \) a \( C^1 \) diffeomorphism, and \( \mathcal{U}(f) \) a given neighbourhood of \( f \). Then, there exist \( U_0(f) \subset \mathcal{U}(f) \) and \( \delta > 0 \) such that if \( g \in U_0(f) \), \( S = \{ p_1, p_2, \ldots, p_m \} \subset M \) is a finite set, and \( L_1, \ldots, L_m \) are linear maps, \( L_i : T_{p_i}M \to T_{p_i}M \), satisfying \( \| L_i - D_{p_i}g \| \leq \delta \), \( i = 1, \ldots, m \) then there is \( \tilde{g} \in U(f) \) satisfying \( \tilde{g}(p_i) = g(p_i) \) and \( D_{p_i}\tilde{g} = L_i \). Moreover, if \( U \) is any neighbourhood of \( S \) then we may choose \( \tilde{g} \) so that \( \tilde{g}(x) = g(x) \) for all \( x \in \{ p_1, p_2, \ldots, p_m \} \cup (M \setminus U) \).

**Remark 2.2.** The statement given in [Fr] is slightly different from that above, but the proof of our statement is contained there.

2.1. Existence of dominated splitting: proof of theorem A

Under the hypothesis of theorem A, we shall prove that there is a dominated splitting for \( H_t(p) \), the set of the transverse homoclinic points associated with \( p \).

The proof is done in several steps. (1) Assuming that there is no dominated splitting for \( H_t(p) \) in lemma 2.3 we perform a perturbation \( g \) of \( f \) exhibiting a homoclinic point \( x \in H(p_g) \) with a small angle between \( W^s_{loc}(x, g) \) and \( W^u_{loc}(x, g) \). (2) In proposition 2.5, we perform another perturbation (that we still denote by \( g \) of \( f \) to create a tangency between \( E^s(x, g) = T_xW^s(p_g) \) and \( E^u(x, g) = T_xW^u(p_g) \), \( x \in W^s(p_g) \cap W^u(p_g) \)). (3) In proposition 2.6 through another perturbation we create an arc of flat tangencies \( \beta \subset H(p_g) \). (4) Finally, in section 2.2 we perform a sequence of perturbations of \( g \) leading to \( G \), still near \( f \), presenting a sequence of two by two disjoint small horseshoes \( H_{\epsilon_n} \subset H(p_G) \), \( \epsilon_n \to 0 \) as \( n \to \infty \). Moreover, we can select the sequence \( \epsilon_n \) in such a way that none of them are a constant of \( h \)-expansiveness of \( G \). Since the entropy of each of these small horseshoes is positive, we arrive at a contradiction to \( h \)-expansiveness of \( f \).

To start let us assume, by contradiction, that \( H_t(p) \) has no dominated splitting. Then, by [MPP, section 3.6, proof of theorem F] the following property holds: for all \( m \in \mathbb{Z}^* \) there exists \( x_m \) such that for all \( 0 \leq n \leq m \),

\[
\| Df^n|E(x_m) \| \cdot \| Df^{-n}|F(f^n(x_m)) \| > 1/2.
\]
Lemma 2.3. Assume that inequality (∗) holds. Then, given γ > 0 and ε > 0 there is \( m > 0 \) and \( g \) an \( \epsilon\)-C\(^{-1}\) perturbation of \( f \) with a homoclinic point \( x_g \) associated with \( p_g \) such that the angle at \( x_g \) between \( W^s_{\text{loc}}(x_g, g) \) and \( W^u_{\text{loc}}(x_g, g) \) is less than \( \gamma \).

Proof. Let \( U_0 \) be a C\(^1\) neighbourhood of \( f \) such that all \( g \in U_0 \) are h-expansive and assume that there is \( \gamma_0 > 0 \) such that for all \( g \) in \( U_0 \) the angle at \( x_g \) between \( W^s_{\text{loc}}(x_g, g) \) and \( W^u_{\text{loc}}(x_g, g) \) is greater than or equal to \( \gamma_0 \).

Next assuming that the angle is greater than \( \gamma_0 \), we perform a sequence of arbitrarily small perturbations along a finite segment of orbit of a homoclinic point \( x_m \) pushing \( T_{f/(x_m)}W^u(p) \) into \( T_{f/(x_m)}W^s(p) \) leading to an arbitrarily small angle between \( T_{f/(x_m)}W^u(p) \) and \( T_{f/(x_m)}W^s(p) \). This will contradict the angle is greater than \( \gamma_0 \).

To do so we proceed as follows: let \( \epsilon > 0 \) be small enough such that any \( C^1\)-\( \epsilon\)-perturbation of \( f \) gives a diffeomorphism \( g \in U_0 \). Let \( \epsilon' > 0 \) be such that any \( \epsilon'\)-perturbation of the derivatives along a finite orbit of \( f \) can be realized via lemma 2.1 by a \( C^1\)-\( \epsilon'\)-perturbation of \( f \).

By inequality (∗) there are vectors \( v_m \in F(x_m) \) and \( w_m \in E(x_m) \) with \( \|v_m\| = \|w_m\| = 1 \) such that

\[
\frac{\|Df^j(w_m)\|}{\|Df^j(v_m)\|} > \frac{1}{2}, \quad \forall j, \quad 1 \leq j \leq m.
\]

Let us define \( T_j : T_{f/(x_m)}M \to T_{f/(x_m)}M \) a linear map such that \( T_1|_{E(f/(x_m))} = (1 + \epsilon')\text{id} \) and \( T_j|_{F(f/(x_m))} = \text{id}, \quad j = 0, \ldots, m \). Note that \( T_j \) stretches \( E = T_{x_m}W^s(x_m, f) \) and leaves \( F = T_{x_m}W^u(x_m, f) \) unchanged. Let \( w_m \in E_{x_m} \) and \( v_m \in F_{x_m} \) be unit vectors. Let \( P : T_{x_m}M \to T_{x_m}M \) be a linear map satisfying \( P = \text{id} \) in \( E(x_m) \) and \( P = \text{id} + L \) in \( F(x_m) \) where \( L : F(x_m) \to E(x_m) \) is a linear map such that \( L(w_m) = \epsilon'w_m \) and \( \|L\| = \epsilon' \). Finally define \( G_0 = T_1 \cdot Df_{x_m} \cdot P \), and \( G_j = T_{j+1} \cdot Df^j_{/(x_m)} \) for \( j = 1, \ldots, m - 1 \). By lemma 2.1 there exists a diffeomorphism \( g : M \to M \) such that \( g \) is \( \epsilon\)-near \( f \), keeps the orbit of \( x_m \) unchanged for \( j = 0, 1, \ldots, m \) and \( Dg^j_{/(x_m)} = G_j \). We assume that the support of the perturbation does not intersect a small neighbourhood of \( p \). It follows that \( x_m \) continues to be a homoclinic point of \( g \). Moreover, we do not change \( E(f^j(x_m)), \quad j \in \mathbb{Z}, \quad \) and \( F(f^j(x_m)) \) is changed only for \( j \geq 0 \). Thus such bundles are the stable and unstable directions of a homoclinic point of a diffeomorphism \( g \in U_0 \). We obtain that \( v_m \mapsto v_m + \epsilon'w_m = u \) and after \( m \) iterates we have \( u_m = Dg^m(u) = Dg^m(v_m + \epsilon'w_m) = Df^m(v_m) + (1 + \epsilon')^mDf^m(\epsilon'w_m) \).

Given \( \epsilon' > 0 \) we may find \( m > 0 \) such that \( (1 + \epsilon')^m \geq 4 + 2/\gamma_0 \) where \( \gamma_0 > 0 \) for all \( x \in H_1(p_e), \quad g \in U_0 \), and \( \Omega(E(x), F(x)) \) stands for the angle between \( E(x) \) and \( F(x) \). With this choice of \( m \), by [Ma2, lemma II.10] we have

\[
\frac{\|Df^m(v_m)\|}{\|Df^m(u_m)\|} = \frac{\|u_m - (1 + \epsilon')^mDf^m(\epsilon'w_m)\|}{\|u_m\|} \geq \frac{\gamma_0}{1 + \gamma_0} \frac{\gamma_0}{1 + \gamma_0} \left| \epsilon'(1 + \epsilon')^mDf^m(\epsilon'w_m) \right|.
\]

Dividing the inequality \( \|Df^m(v_m)\| \geq \frac{\gamma_0^2}{1 + \gamma_0} \|\epsilon'(1 + \epsilon')^mDf^m(\epsilon'w_m)\| - \|Df^m(\epsilon'w_m)\| \) by \( \frac{\gamma_0^2}{1 + \gamma_0} \|Df^m(v_m)\| \) and taking into account that by hypothesis

\[\frac{\|Df^m(v_m)\|}{\|Df^m(u_m)\|} > \frac{1}{2} \quad \text{and} \quad \epsilon'(1 + \epsilon')^m \geq 4 + 2/\gamma_0,\]

we find

\[
\frac{\gamma_0}{1 + \gamma_0} > \frac{\epsilon'(1 + \epsilon')^m}{2} - 1 > 1 + 1/\gamma_0 = \frac{1 + \gamma_0}{\gamma_0},
\]

arriving at a contradiction. Hence \( \Omega(Dg^m(u), w_m) < \gamma \), proving lemma 2.3. □

Let us recall the following result which may be found in [BDP, Lemma 4.16]; see also [BDPR, lemma 3.8].
**Theorem 2.4.** Let $p$ be a hyperbolic periodic point and $H(p)$ its homoclinic class. Assume that $H(p)$ is not trivial. Then there exists an arbitrarily small $C^1$ perturbation $g$ of $f$ and a hyperbolic periodic point $q$ of $H(p_g)$ with period $\pi(q)$ and homoclinically related to $p_g$ such that $Dg^{\pi(q)}$ has only positive real eigenvalues of multiplicity one.

Observe that in the previous result, since $q_g$ is homoclinically related to $p_g$, we have $H(p_g) = H(q_g)$. So, to simplify notation, we may assume directly that $p = q$ and moreover that $g = f$, and that $p$ is a fixed point. We order the eigenvalues of $Df_p$ labelling them as $0 < \lambda_2 < \cdots < \lambda_1 < 1 < \mu_1 < \cdots < \mu_{d-k}$ so that the less contracting and the less expanding ones are, respectively, $\lambda_1$ and $\mu_1$.

By a small $C^1$ perturbation we may also assume that locally, in a neighbourhood $V$ of $p$, we have linearizing coordinates so that

$$
 f(x) = \sum_{j=1}^{k} \lambda_j a_j u_j + \sum_{j=1}^{d-k} \mu_j a_{k+j} u_{k+j},
$$

where $\{u_1, \ldots, u_k, u_{k+1}, \ldots, u_d\}$ is a base of $\mathbb{R}^d$ formed by the eigenvectors of $Df_p$, $\|u_j\| = 1$, $j = 1, \ldots, d$, and we have written $x = \sum_{j=1}^{d} a_j u_j$ for $x \in V$. In particular, $p$ has been identified with 0.

The lines in $W^{ue}(p)/V$ corresponding to the eigenvalues $\lambda_j$ may be extended to all of $W^s(p)$ by backward iteration by $f$, giving us a foliation by lines of $W^s(p)$. Similarly for $W^u(p)$ we have a foliation by lines obtained by forward iteration by $f$.

Now, let us assume that $g$ is near $f$, $f = g$ in a small neighbourhood of $p$, and that there is a small angle between $T_i W^s(p_g)$ and $T_i W^u(p_g)$ where $x$ is a $g$-homoclinic point associated with $p$. That is, there is $\gamma$ small such that

$$
 \angle(T_i W^s(p_g), T_i W^u(p_g)) < \gamma.
$$

By theorem 2.4, we may assume that all the eigenvalues of $Df_p^{\pi(q)}$ are positive with multiplicity one and that we have linearizing coordinates in a small neighbourhood of $p$.

The next proposition establishes that if the angle between $T_i W^s(p_g)$ and $T_i W^u(p_g)$ is small, then we can create a tangency between $T_i W^s(p_g)$ and $T_i W^u(p_g)$, for some $g$ near $g$.

Let $U_0(f)$ and $U(f)$ be $C^1$ neighbourhoods of $f$ where lemma 2.1 holds.

**Proposition 2.5.** There are $\gamma > 0$ and $U_0(g) \subset U(f)$ such that if $\angle(E^s(x, g), E^u(x, g)) < \gamma$, where $E^s(x, g) = T_i W^s(p_g)$ and $E^u(x, g) = T_i W^u(p_g)$, then there is $g \in U_0(g)$ exhibiting a tangency between $W^s(x, \tilde{g})$ and $W^u(x, \tilde{g})$. Moreover $E^s(x, \tilde{g}) + E^u(x, \tilde{g})$ has dimension $d - 1$ and there is $N > 0$ so that if $\{u\}$ is the subspace common to $E^s(x, \tilde{g})$ and $E^u(x, \tilde{g})$, then $(D\tilde{g})^N(\{u\})$ is tangent to $u_1$, the eigenvector corresponding to the less contracting eigenvalue and $(D\tilde{g})^{-N}(\{u\})$ is tangent to $u_{k+1}$, the eigenvector corresponding to the less expanding eigenvalue of $D\tilde{g}$.

**Proof.** Let $U(f), U_0(f)$ and $\delta$ be as in lemma 2.1. Shrinking $U_0$, if necessary, we may assume that $\text{clos}(U_0(f)) \subset U(f)$. Hence there is some $C > 0$ such that $\sup\|Dg\| : g \in U_0(f) \leq C$.

By lemma 2.3 for $0 < \gamma < \delta/C$ there is $g \in U_0(f)$, $x \in W^s(p_g, g) \cap W^u(p_g, g)$ so that

$$
 \angle(E^s(x, g), E^u(x, g)) < \gamma.
$$

Hence, there exist $v \in E^s_{\perp}$ and $w \in E^s$ such that $v + w \in E^s$, $\|w\| = 1$, $\|v\| < \gamma$. Let $T : T_i M \rightarrow T_i M$ be such that $T|_{E^s_{\perp}} = 0$, $T(w) = -v$ and $\|T\| < \delta/C$. Let $L : T_{g^{-1}(x)} M \rightarrow T_x M$ be defined by $L = (Id + T) \circ Dg^{-1}(x)g$. Then we have

$$
 \|L - D_{g^{-1}(x)}g\| < \delta \quad \text{and} \quad w \in L(E^u(g^{-1}(x))$. 

Take a neighbourhood $U$ of $g^{-1}(x)$ such that $O_p(x) \cap U = \{g^{-1}(x)\}$. Using lemma 2.1 we find $\tilde{g} \in U(f)$ such that $g^j(x) = \tilde{g}^j(x)$ for all $j \in \mathbb{Z}$. $\tilde{g} = g$ outside $U$, and $D_x g \cdot \tilde{g} = L$. Hence $x \in W^s(p_{\tilde{g}}, \tilde{g}) \cap W^u(p_{\tilde{g}}, \tilde{g})$ since its forward and backward orbits continue to converge to $p_{\tilde{g}}$. Moreover, $w \in E^s(x, \tilde{g}) \cap E^u(x, \tilde{g})$ and so the intersection of $W^s(p_{\tilde{g}})$ and $W^u(p_{\tilde{g}})$ is not transverse at the point $x$.

Since the eigenvalues of $Df_p$ are all real positive and of multiplicity one and $f = g$ in a small neighbourhood of $p$, iterating $N$ times by $\tilde{g}$ we have a vector $D^N \tilde{g}(w)$ almost tangent to $(u_1)$ corresponding to the less contracting eigenvalue at $p$. Again by lemma 2.1 we can perturb $\tilde{g}$ outside a small neighbourhood of $p$ to let the direction of $(D \tilde{g})^N(w)$ coincide with $(u_1)$. Similarly we obtain $(D \tilde{g})^{-N}(w)$ tangent to the line corresponding to the less expanding eigenvector of $D \tilde{g}$.

From proposition 2.5 we may assume for $f$ itself that there is a homoclinic point of tangency $x \in W^s(p) \cap W^u(p)$ with properties analogous to those of $\tilde{g}$. The next lemma asserts that under these hypotheses, we can obtain an arc $\beta$ of non-traversal homoclinic points in $W^s(p) \cap W^u(p)$.

Recall that in linearizing coordinates in a neighbourhood $V$ of $p$ the expression of $f$ is given by equation (3). In particular, we can speak about an arc being tangent to a given direction determined by the eigenvectors of $Df_p$.

**Proposition 2.6.** Let $p$ be a hyperbolic fixed point for $f$ of index $k$ and $x \in W^s(p) \cap W^u(p)$ such that the intersection at $x$ is not transverse. Then by an arbitrarily small $C^1$ perturbation we may obtain a diffeomorphism $g$ with $x \in W^s(p_g, g) \cap W^u(p_g, g)$ such that the intersection at $x$ is flat, and there exists a small arc $\beta$ contained in the intersection of the stable and unstable manifolds of $p$. Moreover, there is $N > 0$ such that $g^N(\beta) \subset W^s_{loc}(p_g)$ is tangent to the eigenspace corresponding to the less contracting eigenvalue and similarly $g^{-N}(\beta) \subset W^u_{loc}(p_g)$ is tangent to the eigenspace corresponding to the less expanding eigenvalue.

**Proof.** Since $p$ is a hyperbolic saddle, $W^s(p)$ is a Euclidean $k$-dimensional hyperplane and $W^u(p)$ a Euclidean $(d-k)$-dimensional hyperplane both immersed in $M$. If the intersection at $x$ of $W^s(p)$ and $W^u(p)$ is not transverse we have a vector $w \neq 0$ in $T_xW^s(p) \cap T_xW^u(p)$, i.e. we have a tangency between $W^s(p_f)$ and $W^u(p_f)$ at the homoclinic point $x$. Using lemma 2.1 we may assume that the subspace generated by $w$ is the intersection between $T_xW^s(p)$ and $T_xW^u(p)$, i.e. $T_xW^s(p) + T_xW^u(p)$ has dimension $d - 1$ and by proposition 2.5 we also have that for some $N > 0$, $(Df)^N(w)$ is tangent to $u_1$ the eigenvector corresponding to the less contracting eigenvalue and $(Df)^{-N}(w)$ is tangent to $u_{k+1}$ the eigenvector corresponding to the less expanding eigenvalue of $Df_p$.

Moreover, we may also assume that $k \geq d - k$ (otherwise we may take $f^{-1}$ instead of $f$) and, again by lemma 2.1, that the tangent space $T_xW^s(x)$ intersects trivially $(T_xW^u(x))^\perp$ the orthogonal complement of $T_xW^u(x)$. Under these assumptions the orthogonal projection of $W^u(x)$ into $W^u(x)$ is locally a diffeomorphism in a suitable neighbourhood of $x$. Let us choose a small disc $D_x \subset W^u(x)$ and $N > 0$ such that $f^N(D_x) \subset W^u(x)$, and let $L_x$ be a small disc in $W^u(x)$ such that $f^{-N}(L_x) \subset W^u(x)$. $L_x$ projects onto $L'_x \subset D_x$ diffeomorphically. Via a local chart we may identify $D_x$ with

$$\{y \in \mathbb{R}^d / y_{k+1} = \cdots = y_d = 0; y_1^2 + \cdots + y_k^2 = 1\},$$

with $x$ identified with the origin 0 and $w$ having the direction of 0 for $y_1$, the space spanned by $(1, 0, \ldots, 0)$, tangent at 0 to $L'_x$. $L_x$ may be viewed as the graph of a map $\Gamma: L'_x \to (T_xW^u(x))^\perp$ with $\frac{\partial}{\partial \theta} |_{\theta=0} = 0$. To simplify notation we write $(y_1, \ldots, y_k) = Y_1$ and $(y_{k+1}, \ldots, y_d) = Y_2$. Hence if $(Y_1, Y_2) \in L_x$, then $Y_2 = \Gamma(Y_1(Z))$, where, given $L'_x$, $Y_1(Z)$ is a local coordinate map from a neighbourhood of 0 in $\mathbb{R}^{d-k}$ to $D_x$. 


**Claim 2.1.** There is a $C^1$ perturbation $g$ of $f$, $g \in \mathcal{U}(f)$, with a flat intersection at $x \in D_x \cap L_x$, $D_x \subset W^u_x(x)$ and $L_x \subset W^s_x(x)$. This flat intersection contains a small arc $\beta$.

**Proof.** Define $h : M \to M$ by

$$h(Y_1, Y_2) = (Y_1, Y_2 - G(Y_1, Y_2)\Gamma(y_1, 0, \ldots, 0)).$$

Here $G$ is a $C^\infty$-bump function, $0 \leq G(Y_1, Y_2) \leq 1$, which vanishes in the boundary of the ball $B(0, \epsilon')$, and is equal to 1 in $B(0, \epsilon'/4)$, such that $\|\nabla G\| < \frac{1}{2}$, where $\nabla G$ is the gradient of $G$. With this choice it is not difficult to see that $h$ is a diffeomorphism as $C^1$ close to the identity map as we wish and $h = \text{id}$ off a small ball $B(x, \epsilon')$ (see [SV, section 2]).

Now consider the diffeomorphism $g = h \circ f$. Then, by the choice of $h$, $g$ is a small $C^1$ perturbation of $f$ such that $x$ is a flat $g$-point contained in $W'(p_{\epsilon}) \cap W^u(p_{\epsilon})$ and moreover there is an arc $\beta \subset W'(p_{\epsilon}) \cap W^u(p_{\epsilon})$ with $x \in \beta$ such that $T_x \beta$ is parallel to $u$ by construction.

Indeed, since $x \in W'(p_{\epsilon}) \cap W^u(p_{\epsilon})$ we have that $\lim_{n \to -\infty} f^n(x) = \lim_{n \to +\infty} f^n(x) = p$ and so $x$ is neither forward recurrent nor backward recurrent by $f$. This implies that we may choose the support $B(x, \epsilon')$ of the perturbation in such a way that for $n \neq 0$, $g^n(B(x, \epsilon')) \cap B(x, \epsilon') = \emptyset$. Hence if $y \in W^u_x(x, g)$ then for $\epsilon > 0$ small enough we obtain that $y \in W^u_x(x, g)$ too. But $h$ sends the arc $\beta$ passing through $x$ in $W^u_{\epsilon}(x, g)$ onto an arc $\gamma$ included in $W^u_{\epsilon}(x, f) = W^u_{\epsilon}(x, g)$ and passing through $x$ too. Therefore $g^{-1} = f^{-1} \circ h^{-1}$ sends the arc $\gamma$ into $\beta$ which iterated successively by $f^{-1}$ converges to $p$. Hence $\beta$ is contained in both the local stable and unstable manifold of $x$ which in turn is contained in $W'(p_{\epsilon}) \cap W^u(p_{\epsilon})$. Thus $\beta$ is an arc of flat intersection between $W'(p_{\epsilon})$ and $W^u(p_{\epsilon})$. This completes the proofs of claim 2.1.

**Claim 2.2.** There is a $C^1$ perturbation $\tilde{g} \in \mathcal{U}(f)$ so that if $\beta$ is the common arc of $W'(x, g)$ and $W^u(x, g)$ given by claim 2.1 then, in linearizing coordinates around $p$, $\tilde{g}^\beta(\beta)$ is a straight segment tangent to the vector $u_1$ corresponding to the less contracting eigenvalue $\lambda_1$.

**Proof.** Note that for every $n \geq N$, with $N$ given by proposition 2.5, we have $g^n(\beta) \subset V \cap W^u_{\text{loc}}(p_{\epsilon})$ and $g^{-n}(\beta) \subset V \cap W^u_{\text{loc}}(p_{\epsilon})$, where $V$ is the neighbourhood of $p$ in which equation (3) holds. Let $D$ be a fundamental domain contained in $W^u_{\text{loc}}(p_{\epsilon})$ identified with $R^k$ where $k = \dim (W^u_{\text{loc}}(p_{\epsilon}))$.

We may assume that (a) $g^N(\beta) \subset D$, (b) $\text{dist}(g^N(\beta), \partial D)$ is bounded away from zero, (c) by proposition 2.5, $g^N(x)$ is an interior point of $g^N(\beta)$ in which $T g^N(\beta) = \langle u_1 \rangle$, (d) $g^N(\beta)$ does not intersect the strong stable manifold $W^s_{\text{loc}}(p_{\epsilon})$, given by $p_{\epsilon}$ and the subspace $\langle u_2, \ldots, u_k \rangle$.

Note that condition (c) says that the arc $g^N(x)$ is almost parallel to $u_1$ in a neighbourhood of $x$ in $\beta$. Hence, since $\beta$ can be chosen arbitrarily small, by a $C^1$ perturbation $h$ as small as we wish, performed in a neighbourhood $W$ of $\beta$ with $W \cap W^u_{\text{loc}}(p_{\epsilon}) \subset \text{int}(D)$, $W \cap W^s_{\text{loc}}(p_{\epsilon}) = \emptyset$ and such that $h$ is equal to the identity outside $W$, we may assume that $\beta$ itself is a straight segment parallel to $u_1$. Defining $\tilde{g} = h \circ g$ the thesis of Claim 2.2 follows.

Similarly we can achieve that for $\beta$ sufficiently small $\tilde{g}^{-n}(\beta) \subset W^u_{\text{loc}}(p_{\epsilon})$ is a straight segment tangent to the eigenvector $u_{k+1}$ corresponding to the less expanding eigenvalue $\mu_1$.

Summarizing, we have that $\beta \subset W^s_{\epsilon}(x, \tilde{g}) \cap W^u_{\epsilon}(x, \tilde{g})$, $\tilde{g}^\beta(\beta)$ is tangent to the eigenspace corresponding to the less contracting eigenvalue and $\tilde{g}^{-n}(\beta)$ is tangent to the eigenspace corresponding to the less expanding eigenvalue of $D_{\epsilon} \tilde{g}$. By construction $\tilde{g}$ continues to be linear in a neighbourhood of $p$ which guarantees that for $n > N$ $\tilde{g}^n(\beta)$ is also tangent to $u_1$ and similarly $\tilde{g}^{-n}(\beta)$ is tangent to $u_{k+1}$. All together complete the proof of proposition 2.6.
2.2. Creating small horseshoes

The previous result gives a diffeomorphism $\tilde{g} \in C^1$ near $f$, that we return to call $g$, such that the intersection between $W^u(p_\sigma)$ and $W^s(p_\sigma)$, in a local chart around $x$ such that $T_xW^u(x) \cap T_xW^s(x) = \{u\}$, contains a segment $\beta = \{su : -\delta \leq s \leq \delta\}$. Moreover, in linearizing coordinates given by equation (3) in a neighbourhood $V_\epsilon$ of $p$, we have for $n \geq N$ that $g^n(\beta)$ is tangent to $u_1$ the less contracting eigenvector of $Dg_p$ and $Dg^{-n}(\beta)$ is tangent to $u_{k+1}$ the less expanding eigenvector of $Dg_p$. Although $V_\epsilon$ may not coincide with $V$, in which $f$ is linear, we will write $V$ instead of $V_\epsilon$ to simplify notation.

Next, as in [DN, lemmas 5.1 and 6.3], we perturb $g$ obtaining a diffeomorphism $G$, such that $G$ coincides with $g$ outside a small neighbourhood of $\beta$, in order to create a sequence of small horseshoes $H_n \subset H(pG)$ associated with $W^u_{loc}(x, G)$ and $W^s_{loc}(x, G)$. These horseshoes will have positive topological entropy and will be built in such a way that neither $\epsilon/2, \epsilon/4, \ldots \epsilon/2^n, \ldots$ will be constants of $h$-expansiveness for $H(pG)$. Therefore the diffeomorphism $G$ will not be $h$-expansive, contradicting our hypothesis.

To do so we proceed as follows: first, since we are working in a $C^1$ neighbourhood of $f$ and $C^r, r \geq 2$, diffeomorphisms are dense in $\text{Diff}^1(M)$ we may assume that $g$, the diffeomorphism obtained in proposition 2.6, is of class $C^r, r \geq 2$. We split the proof into two cases, according to the index of $p$.

2.2.1. index $(p) = d - 1$. Let us assume first that $p$ is of index $d - 1$, i.e. $\dim(W^u(p_\sigma)) = 1$. This will simplify the techniques involved. We may assume that $N = 0$ and so that $\beta$ itself, the segment of tangency, is contained in the local stable manifold of $p$ in a local chart which is a linearizing neighbourhood $V$ of $p$.

Let $\psi : [0, \delta] \to \mathbb{R}$ be a $C^\infty$ bump function satisfying the following:

1. $\psi(s) = 1/5$, for $s \in [0, \delta/16]$; this implies that $\psi^{(k)}(0) = \psi^{(k)}(\delta/16) = 0$ for all $k \geq 1$.
2. $\psi'(s) < 0$ for $s \in (\delta/16, \delta/8)$.
3. $\psi(s) = 0$ for all $s \in [\delta/8, \delta/4]$; this implies that $\psi^{(k)}(\delta/8) = \psi^{(k)}(\delta/4) = 0$ for all $k \geq 1$.
4. $\psi'(s) > 0$ for $s \in (\delta/4, 3\delta/8)$.
5. $\psi(s) = 1$ for all $s \in [3\delta/8, \delta]$; this implies that $\psi^{(k)}(3\delta/8) = \psi^{(k)}(\delta) = 0$ for all $k \geq 1$.

Next, consider $b : (-\delta, 5\delta/4) \to \mathbb{R}$ such that

$$b(s) = \psi(s) \quad \text{for all} \; s \in [0, \delta],$$

$$b(s) = \frac{1}{5} \psi(2(s + \delta/2)) \quad \text{for all} \; s \in [-\delta/2, 0],$$

and in general

$$b(s) = \frac{1}{5^n} \psi(2^n(s + \delta(1 - 1/2^n))) \quad \text{for all} \; s \in [-\delta(1 - 1/2^n), -\delta(1 - 1/2^{n-1})].$$

Put also

$$b(s) = 5\psi\left(\frac{s - \delta}{2}\right) \quad \text{for} \; s \in [\delta, 5\delta/4].$$

It is easy to see that $b(s)$ is $C^\infty$ at $(-\delta, 5\delta/4]$. Moreover, we may assume that for $s \in [0, \delta]$, $|b'(s)| \leq 24/\delta$ and $|b''(s)| \leq K/\delta^2$, for some $K > 0$.

Hence for $s \in [-\delta(1 - 1/2^n), -\delta(1 - 1/2^{n-1})]$ we have

$$|b'(s)| = \frac{1}{5^n} 2^n \left|\psi'\left(2^n \left(s + \frac{2^n - 1}{2^n} \delta\right)\right)\right| \leq \frac{24 \cdot 2^n}{5^n \delta^2}.$$
and
\[
|b''(s)| = \frac{4^n}{5^n} \psi'' \left( 2^n \left( s + \frac{2^n - 1}{2^n} \delta \right) \right) \leq \frac{4^n K}{5^n \delta^2}.
\]

Therefore \(|b'(s)| \to 0\) and \(|b''(s)| \to 0\) when \(s \to -\delta\). Setting \(b(\delta) = 0\) we have that \(b'(\delta) = b''(\delta) = 0\) and \(b\) is of class \(C^2\) on \([-\delta, 5\delta/4]\).

Let \(w\) be the unit vector in \(T_x M\) tangent to the expanding eigenvector of \(D g_p\) (recall we are assuming that \(\text{dim}(W^u(p)) = 1\)). Then \(w\) is not contained in \(T_x W^u(x, g) + T_x W^s(x, g)\) since \(T_x W^u(x, g)\) is tangent to \(T_x W^s(x, g)\). Let \((S, s, t)\) be coordinates in a local chart around \(x\), where \((S, s, 0)\) are the coordinates of \(W^u_{loc}(p_G)\), \((0, s, 0)\) are the coordinates of \(\beta\) and \((0, -1, t)\) are those of \(W^s_{loc}(p_G)\); the coordinates of \(x\) are \((0, 0, 0)\). Moreover let us assume that the interval \((0, -\delta, 5\delta/4), 0)\) is totally contained in \(\beta\). In the plane given by the origin \(0\) (identified with \(x\)) and the vectors \(u\) and \(w\) we consider the graph of the function \(\hat{l} : [b/4, 5\delta/4] \to \mathbb{R}\) given by

\[
\hat{l}(s) = \epsilon_1 \cdot (s - \delta/2)(\delta - s), \quad s \in [b/4, 5\delta/4].
\]

Observe that for \(s \in [b/4, 5\delta/4]\), \(\hat{l}(s)\) vanishes at \(s = \delta/2\) and \(s = \delta\) and it has a maximum value equals to \(\delta \epsilon_1/16\) at \(s = 3\delta/4\). Now we extend \(\hat{l}\) to \([-\delta, 5\delta/4]\) in the following way:

\[
\hat{l}(s) = \epsilon_2 \cdot (s + \delta/4)(-s), \quad s \in [-3\delta/8, \delta/8],
\]

\[
\hat{l}(s) = \epsilon_3 \cdot (s + 5\delta/8)(-\delta/2 - s), \quad s \in [-11\delta/16, -7\delta/16],
\]

and in general for \(n \geq 1\):

\[
\hat{l}(s) = \epsilon_{n+1} \cdot (s + (1 - 3/2^n+1)(-\delta(1 - 1/2^n+1) - s), \quad s \in [-\delta(1 - 5/2^n+2), -\delta(1 - 9/2^n+2)].
\]

For \(s \in [-\delta(1 - 5/2^n+2), -\delta(1 - 9/2^n+2)]\), \(\hat{l}\) vanishes only at \(s_n = -\delta(1 - 3/2^n+1) + s_n\) and \(s_n = -\delta(1 - 1/2^n-1)\) and it has a maximum value \(\delta \epsilon_n/5^n \cdot 2^n+2\) at \((s_n + s_n)/2\).

We complete the definition of \(\hat{l}\) in \([-\delta, 5\delta/4]\) setting \(\hat{l}(s) = 0\) elsewhere. Finally, let \(l(s) = \hat{l}(s)\b s\) for all \(s \in [-\delta, 5\delta/4]\). Then \(l(s)\) is \(C^\infty\) in \([-\delta, 5\delta/4]\) and \(C^2\) in \([-\delta, 5\delta/4]\).

Denote by \(B_s\) a small \((d - 1)\)-dimensional disc around \(x\) contained in a fundamental domain of \(W^u_{loc}(p_G)\) whose coordinates in the local chart are \((S, s, 0)\). Analogously denote by \(B_s\) a small one-dimensional disc contained in \(W^u(p_G)\) around \(x\) whose coordinates in the local chart are \((0, s, 0)\). Note that \(B_s\) is characterized by \(t = 0\), and \(B_s\) is in the arc \(\beta\) contained in \(B_s\) parametrized by \(s \in [-\delta, 5\delta/4]\). As we have noticed before, the point \(x\) is identified with \((0, 0, 0)\).

Now, pick another \(C^\infty\) bump function \(\psi\) such that \(\psi\) vanishes outside an \(\epsilon\)-neighbourhood of \(\beta, \epsilon \geq 2\epsilon_1\), and is equal to \(1\) in the \(\epsilon/2\) neighbourhood of \(\beta\).

Let \(h : M \to M\) be given by

\[
(S, s, t) \mapsto (S, s, t + l(s) \psi(\|T\|))
\]

and \(h = \text{id} \) outside \(B(\beta, \epsilon)\) where \(\epsilon\) is such that the \(\epsilon\)-neighbourhood of \(\beta\) does not intersect \(\Gamma \cap g(V) \cap g^{-1}(V)\) where we recall that \(\Gamma\) is the linearizing neighbourhood of \(p\).

Now, letting \(G = h \circ g\), we get, by construction, that \(G\) is a small perturbation of \(g\), and, as in proposition 2.6, it is not difficult to see that \(B_s \subset W^u_{loc}(x, G) \subset W^u(p_G)\) and \((0, s, l(s)) \subset W^u_{loc}(x, G) \subset W^u(p_G)\). Furthermore, it is straightforward to show that \(W^s(p_G)\) and \(W^u(p_G)\) intersect transversely at the points

\((0, \delta/2, 0), (0, \delta, 0), (0, -\delta/4, 0), (0, 0, 0), \ldots, (0, -\delta(1 - 3/2^{n+1}), 0), (0, -\delta(1 - 1/2^n-1), 0), \ldots\)
and the absolute value of the tangent of the angles at the points
\((0, -\delta(1 - 3/2^{n+1}), 0), (0, -\delta(1 - 1/2^{n-1}), 0)\)
is\(\frac{\epsilon_{n+1}\delta}{2^n/2^{n+1}}, n \in \mathbb{N}\).

We denote by \(\beta'\) the graph of \(l(s)\) in the plane \(\partial W_s\). If we choose \(\epsilon, \epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_n \geq 0\) and \(\delta \geq 0\), we may obtain that the perturbation \(G = h \circ g\) is \(C^1\) small (see [Nh1]). Moreover, we can also assume the following:

1. \(G = g\) on \(V \cap g^{-1}(V)\).
2. \(W^u_{loc}(p) = W^u_{loc}(p_G)\) and \(W^u_{loc}(p_G) = W^u_{loc}(p_G)\). Here \(loc > 0\) stands for a suitable small positive number,
3. \(W^u(\partial G) \cup W^u_G(\partial G) \subset V \setminus G(V)\). In particular \(\beta \cup \beta' \subset V \setminus G(V)\),
4. \(G^t(W^u_{loc}(x, G)) \subset V\) for all \(t \geq 0\) and there is \(T > 0\) such that \(G^{−k}(W^u_{loc}(x, G)) \subset V\) for all \(k \geq T\),
5. \(G^{−t}(\beta \cup \beta') \subset V \setminus G^{-1}(V)\).

We point out that item (5) above follows from the fact that we may reduce the value of \(\delta\), if it were necessary, in order to ensure it.

**Lemma 2.7.** There exists a sequence \(\epsilon_n \downarrow 0\) such that \(G = h \circ g\) is not \(h\)-expansive.

**Proof.** Recall that we are working in a linearizing neighbourhood \(V\) of \(p\) with respect to \(g\). Set

\[V^n = V \cap g(V) \cap \cdots \cap g^n(V)\]and \([V^n = V \cap g^{-1}(V) \cap \cdots \cap g^{-k}(V)\].

Let \(\gamma' = G^{−T}(\beta') \subset V \setminus G^{-1}(V)\) and denote by \((0, 0, d_0), (0, 0, d_\infty)\) the coordinates of the end points of \(\gamma'\) corresponding, respectively, to \(s = 5\delta/4\) and \(s = −\delta\). In the same way we label all points in \(\gamma'\) corresponding to the transverse intersections of \(\beta\) with \(\beta'\): \((0, 0, d_1)\) corresponds to \((0, \delta/2, 0)\) and \((0, 0, d'_1)\) corresponds to \((0, \delta, 0)\), \((0, 0, d_2)\) corresponds to \((0, −\delta/4, 0)\) and \((0, 0, d'_2)\) corresponds to \((0, 0, d_3)\) corresponds to \((0, −\delta/2, 0)\), and so on, labelling the image by \(G^{−T}\) of all the points of transverse intersection between \(\beta\) and \(\beta'\).

Take small arcs \(a^1_t\) and \(a^{−1}_t\) contained in \(V \setminus G^{-1}(V)\) tangent to the direction of the eigenvector corresponding to the weakest contracting eigenvalue of \((DG)_p\) at the points \((0, 0, d_1)\) and \((0, 0, d'_1)\). Multiply them by a \((d − 2)\)-dimensional disc \(C\) of diameter \(c\). Analogously take small arcs \(a^n_t\) and \(a^{−n}_t\) tangent to the direction corresponding to the eigenvector of the expanding eigenvalue of \((DG)_p\) at the points \((0, \delta/2, 0)\) and \((0, 0, d'_1)\) and contained in \(V \setminus G(V)\). By the \(\lambda\)-lemma, [PdeM, lemma 7.1], the forward orbits of \(a^n_t\) and \(a^{−n}_t\) contain arcs arbitrarily \(C\) near \(W^s(p_G)\) and the backward orbits of \(a^n_t \times C\) and \(a^{−n}_t \times C\) contain \((d − 1)\)-dimensional discs arbitrarily \(C\) near \(W^s(p_G)\). By the way we have chosen \(a^n_t\) and \(a^{−n}_t\) and the assumption about the eigenvalues of \((DG)_p\) (all positive real), we have that there is \(k_1 = k_1(\epsilon, \delta)\) such that for \(k \geq k_1\) in \(U\) we have \(\text{dist}(G^{−k}(a^1_t), \beta') < \epsilon_1\delta^2/32\) and \(\text{dist}(G^{−k}(a^{−1}_t), \beta') < \epsilon_1\delta^2/32\). Moreover, we may choose \(c > 0\) small such that \(G^{−k}(a^n_t \times C)\) and \(G^{−k}(a^{−n}_t \times C)\) cut \(\beta'\) but is contained in the \(\epsilon/4\) neighbourhood of \(\beta\) and therefore \(\psi = 1\) there.

In the local coordinates we have chosen, we pick a thin rectangle \(R_1\) with top and bottom given by \(G^{−k_1}(a^1_t \times C)\) and \(G^{−k_1}(a^{−1}_t \times C)\) and bounded in its sides by segments parallel to the \(w\)-axis which is transverse to \(D_s\). Increasing \(k_1\) and reducing \(c, a^n_t\) and \(a^{−n}_t\), if it were necessary, we may assume that \(G^{k_1}(R)\) is contained in the \(c\)-neighbourhood of the graph of \(\beta'\) restricted to \([3\delta/8, 9\delta/8]\).
conclude that there exists $H(p_G)$ both with periodic points $a$ applying again the lemma. Increasing $\lambda$-lemma we have that positive iterates by $(g_2 \circ g_1)^{-1}$ give thin subrectangles crossing all of $R_1$ and hence the stable manifold of $p_1$ cuts $\mathcal{W}^s_{loc}(x) \subset \mathcal{W}^s(p_G)$ and analogously positive iterates by $g_2 \circ g_1$ give subrectangles close to $\beta'$ in the Hausdorff metric and therefore the unstable manifold of $p_1$ cuts $\mathcal{W}^u_{loc}(x) \subset \mathcal{W}^u(p_G)$.

Claim 2.3. There is $\{\varepsilon_n\}_{n=1}^{\infty}$ such that with every $\varepsilon_n$ a horseshoe $H_{\varepsilon_n}$ is associated, with $H_{\varepsilon_n} \subset H(p_G)$ and $\lim_{n \to \infty} \text{diam}(H_{\varepsilon_n}) = 0$.

Proof. Let us choose $\varepsilon_2 > 0$ and construct $H_{\varepsilon_2}$. For this, pick $\varepsilon_2 \leq \varepsilon_1$ such that $G^{-1}(a^i \times C)$ and $G^{-1}(a^i \times C)$ are at a distance greater than $\varepsilon_2$ from $(S, s, 0)$. Since $\delta \leq \varepsilon_2$ for all $n \geq 3$ we have that no part of the graph of $l(s)$ for $s \in [-\delta, \delta/4]$ cuts $R_1$.

We found a new rectangle $R_2$ disjoint from $R_1$ contained in $V_{k+2} \setminus V_{k+1}$ with $k_2 > k_1$ applying again the $\lambda$-lemma. Increasing $k_2$ and reducing the corresponding values of $c_2$, $a_2^2$, and $a_2^3$, if it were necessary, we may assume that $G^{\pm 1}(R_2)$ is contained in the $c_2$-neighbourhood of the graph of $\beta'$ restricted to $[-5\delta/16, \delta/16]$. By construction when we iterate by $G$ the images of $R_1$ and $R_2$ cannot intersect since in $V \setminus G(V)$ there are only one iterate of $R_1$ and one iterate of $R_2$ (namely $R_1$ and $R_2$). We then have for $G$ two disjoint small horseshoes, $H_1$, $H_2$ both with periodic points $p_1$, $p_2$ homoclinically related to $p$ (the proof that $p_2$ is homoclinically related to $p$ is the same as that to $p_1$). Hence both $H_1$ and $H_2$ are included in $H(p_G)$.

Next we choose $\varepsilon_3 \leq \varepsilon_2 \leq \varepsilon_1$ so that $G^{-1}(a_2^3 \times C_2)$ and $G^{-1}(a_2^3 \times C_2)$ are at a distance greater than $\varepsilon_3$ from $(S, s, 0)$. For such $\varepsilon_3$, there is a horseshoe $H_{\varepsilon_3}$ disjoint from $H_{\varepsilon_2}$ and $H_{\varepsilon_1}$ but still contained in $H(p_G)$. This construction follows the same steps as before: first find a thin rectangle $R_3$ cutting the graph of $l(s)$ only for $s \in [-21\delta/32, -15\delta/32]$, $R_3 \cap R_1 = \emptyset$, $R_3 \cap R_2 = \emptyset$. Then find an appropriate positive real number $k_3 > k_2$ such that $G^{k_3}(R_3)$ is contained in the $c_3$-neighbourhood of the graph of $\beta'$ restricted to $[-21\delta/32, -15\delta/32]$.

In this way we may pick the sequence $\varepsilon_n$ such that with every $n$ a horseshoe $H_{\varepsilon_n}$ is associated satisfying $(1) \lim_{n \to \infty} \text{diam}(H_{\varepsilon_n}) \to 0$, $(2) H_{\varepsilon_1} \cap H_{\varepsilon_n} = \emptyset$ and $(3) H_{\varepsilon_n} \subset H(p_G)$ for all $n \in \mathbb{Z}^+$. This proves claim 2.3. □

Since the topological entropy of $H_n$ is positive for all $n$, and $H_n \subset H(p_G)$, we conclude that $G/H(p_G)$ is not $h$-expansive, violating robustness of $h$-expansiveness. The proof of lemma 2.7 is complete. □

Then, the final conclusion is that we cannot have that for all $m \in \mathbb{Z}^+$ there exists $x_m$ such that for all $0 \leq n \leq m$, $\|Df^n[E(x_m)]\| \cdot \|Df^{-n}[F(f^n(x_m))]\| > 1/2$. In other words, we conclude that there exists $m > 0$ such that for all homoclinic point $x \in \mathcal{W}^s(p_f) \cap \mathcal{W}^u(p_f)$ there is $1 \leq k \leq m$ such that

$$\|Df^k/E(x)\| \cdot \|Df^{-k}/F(f^k(x))\| \leq \frac{1}{2}.$$

From this point (see [SV, theorem A]), we can construct a dominated splitting $E \oplus F$ for the set of points in $\mathcal{W}^s(p_f) \cap \mathcal{W}^u(p_f)$ such that $\dim(E(x)) = \dim(\mathcal{W}^s(p_f))$. Afterwards, taking into account that $\text{clos}(\mathcal{W}^s(p_f) \cap \mathcal{W}^u(p_f)) = H(p_f)$ we extend this splitting by continuity to the whole homoclinic class.
Thus in the case of $p$, a periodic point of index $d - 1$, the proof of theorem A follows.

2.2.2. index $(p) = k < d - 1$. For the general case of index $(p) = k < d = \dim(M)$ the proof is similar, the perturbation $h$ of $g$ given $G = h \circ g$ has to be adapted as we sketch below.

Let $w$ be the unit vector in $T_x M$ tangent to the less expanding eigenvector of $Dg_p$. Then $w$ is not contained in $T_x W^s(x, g) + T_x W^u(x, g)$, see propositions 2.5 and 2.6. In a local chart around $x$, $(0, s, 0)$ represent the coordinates of the arc $\beta$ but the coordinates $(S, s, T)$ are such that $S$ is a $(k - 1)$-dimensional vector, and $T$ a $(d - k)$-dimensional vector that we split as $(t, T') = T$ with $t$ one dimensional. As in the codimension one case we have that $(0, [-\delta, 5\delta/4], 0, 0)$ is totally contained in $\beta$. In the plane given by the origin 0 (identified with $x$) and the vectors $u$ corresponding to $(0, 1, 0, 0)$ and $w$ corresponding to $(0, 0, 1, 0)$ we, as above, consider the graph of the function $l : [\delta/4, 5\delta/4] \to \mathbb{R}$ given by

$$l(s) = \epsilon_1 \cdot (s - \delta/2)(\delta - s), \quad s \in [\delta/4, 5\delta/4].$$

Now we extend $l$ to $[-\delta, 5\delta/4]$ and define the $C^2$-function $I(s)$ as in the codimension one case.

Put coordinates in the local chart $Y = (S, s, t, T)$ and denote by $B_t$ a small $k$-dimensional disc around $x$ contained in a fundamental domain of $W^s_{\text{loc}}(p_x)$ whose coordinates in the local chart are $(S, s, 0, 0)$. Analogously denote by $B_s$ a small $(d - k)$-dimensional disc contained in $W^u(p_x)$ around $x$ whose coordinates in the local chart are $(0, s, 0, T)$. Note that $B_t$ is characterized by $t = 0$, $T = 0$, and $B_s$ contains the arc $\beta$ contained in $B_t$, parametrized by $s \in [-\delta, 5\delta/4]$. The point $x$ is identified with $(0, 0, 0, 0)$.

Now, pick a $C^\infty$ bump function $\varphi$ such that $\varphi$ vanishes outside a $\epsilon$ neighbourhood of $\beta$, $\epsilon \geq 2\epsilon_1$, and is equal to 1 in the $\epsilon/2$ neighbourhood of $\beta$.

Let $h : M \to M$ be given by

$$(S, s, t, T) \mapsto (S, s, t + I(s)\varphi(\|Y\|), T)$$

and $h = \text{id}$ outside $B(\beta, \epsilon)$ where $\epsilon$ is such that the $\epsilon$-neighbourhood of $\beta$ does not intersect $U \cap g(U) \cap g^{-1}(U)$.

Now, letting $G = h \circ g$, we get, by construction, that $G$ is a small perturbation of $g$, and, as in proposition 2.6, it is not difficult to see that $B_t \subset W^s_{\text{loc}}(x, G) \subset W^s(p_G)$ and $(0, s, l(s), T) \subset W^u_{\text{loc}}(x, G) \subset W^u(p_G)$.

The rest of the proof of theorem A follows in a similar way to that of the codimension one case.

3. Proofs of corollary 1.2 and theorem B

Let $\mathcal{G} \subset \text{Diff}^1(M)$ be the subset of $f \in \text{Diff}^1(M)$ satisfying the following properties:

G1 $f$ is Kupka–Smale, i.e. all periodic points are hyperbolic and their stable and unstable manifolds intersect transversely.

G2 the periodic points of $f$ are dense in $\Omega(f)$.

G3 for any pair of saddles $p, q$, either $H(p, f) = H(q, f)$ or $H(p, f) \cap H(q, f) = \emptyset$.

G4 the chain recurrence classes of $f$ form a partition of the chain recurrent set of $f$.

G5 every chain recurrence class containing a periodic point $p$ is the homoclinic class associated with that point.

G6 for any saddle $p$ of $f$, $H(p, f)$ depends continuously on $g \in \mathcal{G}$.

We have that $\mathcal{G}$ is a residual subset of $\text{Diff}^1(M)$; see [ABCDW, section 2.1].
After [ABCDW, corollary 3] and [Go, corollary, 6.6.2], we have the following result:

**Theorem 3.1** ([Go, theorem 6.6.8]). There is a residual subset $I \subset G$ of $\text{Diff}^1(M)$ such that if $f \in I$ has a homoclinic class $H(p_f)$ which contains hyperbolic saddles of indices $i \leq j$ then either

1. for any neighbourhood $U$ of $H(p_f)$ and any $C^1$ neighbourhood $\mathcal{U}$ of $f$ there is a diffeomorphism $g \in \mathcal{U}$ with a homoclinic tangency associated with a saddle of the homoclinic class $H(p_g, g)$, where $p_g \in U$ is the continuation of $p$;

or

2. there is a dominated splitting $T_{H(p_f)}M = E \oplus F_1 \oplus \cdots \oplus F_{j-i} \oplus G$

with $\dim(E) = i$ and $\dim(F_h) = 1$ for all $h$ and $\dim(G) = \dim(M) - j$. Moreover, the sub-bundles $F_h$ are not hyperbolic.

**Proof of corollary 1.2.** Let $H(p_f) \subset M$ be a homoclinic class robustly entropy expansive. By theorem A we have a dominated splitting defined on $T_{H(p_f)}M$, and so does $T_{H(g)}M$ for every $g \in V$, $V$ a small neighbourhood of $f$. Taking $g \in V \cap G$, $G$ the residual subset defined above, we can apply theorem 3.1. Under the hypotheses of robustly entropy expansive, the first alternative in theorem 3.1 leads to a contradiction. Thus the second alternative holds, and we conclude the proof of corollary 1.2.

**Proof of theorem B.** The proof that the splitting has form (2) follows from corollary 1.2. Since $H(p)$ is isolated it is a robustly transitive set maximal invariant in a neighbourhood $U \subset M$ and hence, according to [BDPR, theorem D], the extremal sub-bundles $E$ and $G$ are contracting and expanding, respectively.

Alternative (a) in theorem B is a consequence of the following lemma.

**Lemma 3.2.** Let $H(p)$ be isolated, $T_{H(p)}M$ has a dominated splitting of form (2), such that in a $C^1$ robust way the indices of periodic points in $H(p_g)$, $g$ near $f$, are the same and equal to index$(p)$. Then there is an open and dense subset $U_1$ of $U(f)$ in the $C^1$ topology such that $H(p_g)$ is hyperbolic for all $g \in U_1$.

**Proof.** We follow the lines of the proof in [BDi, section 6]. Since $H(p)$ is isolated by [BC, corollary 1.13] or [Ab, theorem A] it is robustly isolated. Let $E$ and $F$ be sub-bundles such that $T_{H(p)}M = E \oplus F$ is $m$-dominated, for all $g \in U(f)$, with $\dim(E) = \text{index}(p)$. We need to prove that $\|Df^n_{E(x)}\| \to 0$ as $n \to +\infty$ and $\|Df^{-n}_{F(x)}\| \to 0$ as $n \to +\infty$ for any $x \in H(p_g)$ in order to prove that $H(p_g)$ is hyperbolic. Let us show only that $\|Df^n_{E(x)}\| \to 0$ as $n \to +\infty$, the other one being similar. For this, it is enough to show that for any $x \in H(p_g)$ there exists $k = k(x)$ such that $\prod_{i=0}^k \|Dg^m_{E(x)}\| \leq \frac{1}{2}$.

Arguing by contradiction, assume this does not hold. Then, there exist $z \in H(p_g)$ such that $\prod_{k=0}^m \|Dg_{E(x)}\| \geq \frac{1}{2} \forall k \geq 0$.

As in the proof of [Ma2, theorem B] we may find $y \in H(p_g) \cap \Sigma(g)$, where $\Sigma(g)$ is a set of total probability measure, such that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Dg^m_{E(x)}g^m / E(g^m_{E(x)})\| \geq 0.$$
Thus there is a perturbation $h$ of $g$ such that $h$ has a non-hyperbolic periodic point in $H(p_k)$. After a new perturbation we obtain periodic points $P$ and $Q$ contained in a small neighbourhood $U$ of $H(p_k)$ and with different indices. Since $H(p)$ is $C^1$ robustly isolated $P, Q \in H(p_k)$ contradicting our assumption that in a $C^1$ robust way the indices of periodic points in $H(p)$ are the same and equal to index $(p)$. □

Now assuming that there are $g$ arbitrarily close to $f$ such that in $H(p_k)$ there are periodic points of different indices let us show that alternative (b) in theorem B holds.

First let us prove that in this case the eigenvalues of periodic points are robust in $\mathbb{R}$. This will be a consequence of the next lemma and corollary 3.4

**Lemma 3.3.** Let us assume that there is a periodic point $q \in H(p)$ with expanding complex eigenvalues such that index $(q) <$ index $(p)$. Then there is an arbitrarily small perturbation of $f$ creating a tangency inside the perturbed homoclinic class $H(p_k)$.

**Proof.** By [BDi] we have that $C^1$ generically there is a robust heterodimensional cycle between $p$ and $q$ and that $W^s(p) \cap W^u(q)$ contains a compact arc $\alpha$ homeomorphic to $[0, 1]$. Consider a disc $D \subset W^s(p)$ with dimension $s = \dim(W^s(p))$, $D$ homeomorphic to $[0, 1] \times [-1, 1]^{s-1}$ by a homeomorphism $h$ such that $h([0, 1] \times [0]^{s-1}) = \alpha$. Iterating by $f^{-n(q)}$ this arc $\alpha$ spirals around $q$ while $D$ stretches approaching $W^s(q)$. Since $W^s(q) \cap W^u(p) \neq \emptyset$ there is a $C^1$ small perturbation of $f$ creating a tangency between $W^s(p_k)$ and $W^u(p_k)$. □

**Corollary 3.4.** If there is a periodic point $q \in H(p)$ with expanding complex eigenvalues such that index $(q) <$ index $(p)$ then $H(p)$ is not $C^1$ robustly h-expansive.

Corollary 3.4 implies that the eigenvalues of periodic points in $H(p)$ are real numbers in a robust way. By [ABCDW] for $C^1$ generic diffeomorphisms the set of indices of the hyperbolic periodic points in a homoclinic class form an interval in $\mathbb{N}$. Thus by [BDi, theorem 2.1] there are diffeomorphisms arbitrarily close to $f$ with $C^1$ robust heterodimensional cycles proving alternative (b) in theorem B.

4. *Proof of corollary 1.5*

In this section we prove corollary 1.5. Let $\mathcal{G}$ be the $C^1$-residual subset of $\text{Diff}^1(M)$ verifying properties G1–G6, as described at the beginning of section 3. We start establishing the following result:

**Lemma 4.1.** If $f \in \mathcal{G}$ then $f^k \in \mathcal{G}$ for all $k > 0$.

**Proof.** Let $f \in \mathcal{G}$ and assume that $f$ also satisfies the following property:

($\ast$) if two periodic points of $f$ are in a homoclinic class and have the same index then the stable manifold of one of them intersects the unstable manifold of the other one and reciprocally.

Note that property ($\ast$) is generic; see [Ar, proposition 20]. Here recall that the index of a hyperbolic periodic point is the dimension of its stable manifold.

Next we verify, step by step, that $f^k$, for $k > 0$, also satisfies properties G1–G6 that define $\mathcal{G}$.

$f^k, k > 0$, satisfies G1. Clearly if $f$ is Kupka–Smale the same holds for $f^k, k > 1$, since both have the same periodic points and if all periodic points of $f$ are hyperbolic then the same is true for $f^k$. Moreover, if $x \in W^s(p, f^k) \cap W^u(q, f^k)$ then $x \in W^s(p, f) \cap W^u(q, f)$ and since the last intersection is transverse the same is true for $f^k$.\]

\]
\]
f^k, k > 0, satisfies G2. If p is a periodic point for f then it is a periodic point for f^k and Ω( f) = Ω(f^k) so that density of periodic points of f^k in Ω(f^k) holds.

f^k, k > 0, satisfies G3. Assume that for p, q, f-periodic points, we have H(p, f^k) ∩ H(q, f^k) ≠ ∅, k > 1. Then it holds that H(p, f) ∩ H(q, f) ≠ ∅ and by assumption we have H(p, f) = H(q, f). By [Ar, proposition 20], if two periodic points are in the same homoclinic class and have the same index then the stable manifold of one of them intersects the unstable manifold of the other one. In particular, this implies that W^s(f_i(p), f) ∩ W^u(f_j(p), f) ≠ ∅ for all 0 ≤ i, j ≤ π(p), where π(p) is the period of p with respect to f. Now, if x ∈ H(p, f) then it is accumulated by points y_n of W^s(x, f) ∩ W^u(x, f) for some index i_n, where j_n(x) = 0, 1, ..., π(p). Since this set is finite there is a pair (j, i) which corresponds to infinitely many values of n. We may assume that the same (j, i) corresponds to all n ∈ N so that y_n ∈ W^s(j_m(p), f^k) ∩ W^u(j_m(p), f^k) such that y_n → x when l → ∞, which implies that y_n → x and hence x ∈ H(p, f^k). Thus H(p, f) = H(p, f^k) and analogously H(q, f) = H(q, f^k). Thus H(p, f^k) = H(q, f^k).

f^k, k > 0, satisfies G4. That the chain recurrence classes form a partition of the chain recurrent set holds for all C^1 diffeomorphisms so that it holds for both f and f^k; see [Co].

f^k, k > 0, satisfies G5. Let p be a f-periodic point contained in a f^k-chain recurrence class CR(p, f^k). Clearly H(p, f^k) ⊂ CR(p, f^k). Since H(p, f) = H(p, f^k) we have H(p, f) ⊂ CR(p, f^k). By assumption H(p, f) = CR(p, f), where CR(p, f) denotes the f-chain recurrence class of p. Thus CR(p, f) ⊂ CR(p, f^k). But for any point x, its f^k-chain recurrence class CR(f^k) is always contained in its f-chain recurrence class CR(f). Thus CR(p, f^k) = CR(p, f).

f^k, k > 0, satisfies G6. Assume now that H(p, f) is a point of continuity of the map f ↦ H(p, f), where p is a saddle of f. It is known that G ↦ H(p_G, G) is lower semicontinuous. Let [G_n, G_n] ∈ Diff^1(M) such that G_n → f^k. Since f^k has p as a hyperbolic periodic point we may assume that there is p_{G_n}, a hyperbolic periodic point for G_n. Let H(p_{G_n}, G_n) be its homoclinic class. Taking a convergent subsequence of [H(p_{G_n}, G_n)], if it were necessary, there is K ⊂ H(p, f^k), such that H(p_{G_n}, G_n^k) → K. If we prove that K = H(p, f^k) then H(p, f^k) is a point of continuity of the map G ↦ H(p_G, G). We claim that K ⊂ CR(p, f^k). It is not difficult to prove that p ∈ K. By definition of K, given y ∈ K and ε > 0 there is N ∈ N such that for all n ≥ N, there is y_n ∈ H(p_{G_n}, G_n) such that dist(y_n, y) < ε, dist(f^k(y_n), G_n(y_n)) < ε and dist(G_n(y_n), f^k(y_n)) < ε. Indeed, for all n ≥ N, there is a neighbourhood U(y_n) ⊂ H(p_{G_n}, G_n) such that all z ∈ U(y_n) has these properties. Hence we may assume that the orbit of y_n is dense in H(p_{G_n}, G_n), and so there is m_n such that dist(G_m(y_n), p_{G_n}) < ε. For x ∈ K we consider the corresponding x_n ∈ H(p_{G_n}, G_n) and the ε-pseudo-orbit

{x_n, f^k(x_n), G_n(x_n), f^k(G_n(x_n)), G_n^2(x_n), f^k(G_n^2(x_n)), G_n^3(x_n), ..., G_n^{m_n}(x_n), p_{G_n}, p}

and analogously we may find an ε-pseudo-orbit from p to x. Hence K ⊂ CR(p, f^k). But we have seen in item 5 that CR(p, f^k) = H(p, f^k) proving that H(p, f^k) is a point of continuity of G ↦ H(p_G, G).

Thus, f^k, k > 0, satisfies properties G1–G6, completing the proof of the lemma. □
Back to the proof of corollary 1.5, lemma 4.1 together with [PaVi, theorem B], implies that $C^1$-generically robust $h$-expansiveness of $f / H(p, f)$ implies the same property for $f^k / H(p, f^k)$.

We complete the proof taking into account that, by theorem B and [PaVi, theorem B], the existence of a dominated splitting as that in (2) is equivalent to robust $h$-expansiveness at $H(p)$. The existence of such a splitting for $g / H(p, g)$ implies the same for $g^k / H(p, g^k)$, $k \geq 1$, and the fact that $H(p, f)$ is isolated implies that $H(p, f^k)$ is also isolated. Thus, for $g$ in an open and dense subset of $U$, we have not only that $g / H(p, g)$ is robustly $h$-expansive but also that, for every $k > 0$, $g^k / H(p, g^k)$ is robustly $h$-expansive. This completes the proof.

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