Magnetic dynamo C-flows in Riemannian compact manifold as generalized Arnold’s metric

by

L.C. Garcia de Andrade

Departamento de Física Teórica – IF – Universidade do Estado do Rio de Janeiro-UERJ
Rua São Francisco Xavier, 524
Cep 20550-003, Maracanã, Rio de Janeiro, RJ, Brasil
Electronic mail address: garcia@dft.if.uerj.br

Abstract

It is shown that C-flows in Riemannian three-dimensional compact manifold can be naturally considered as generalized dynamo Arnold’s Riemann metric in compact manifolds, the so-called cat map dynamo. The generalized solution of self-induction equation in the background of this metric shows that one is allowed to consider stretching along both directions of the flow, instead of compressed in one direction and stretched in the other such as in Arnold’s dynamo. Though this solution can be considered as unrealistic, at least for incompressible flows, there is another generalized solution which considers distinct stretch and compression or exponential damping are anisotropic, and the dynamo flow is compressed and stretched non-uniformly along distinct directions. Riemann curvature tensor components are computed by making use of Cartan’s calculus of differential forms. PACS numbers: 02.40.Hw:Riemannian geometries
I Introduction

One of the most important issues in the investigation of magnetic structures in solar and plasma physics is the study of existence of realistic magnetic dynamos [1]. On this track an important role has been played by less realistic dynamo maps in compact Riemannian manifolds such as torus. Among this, so to speak more naive dynamo examples, the cat fast dynamo map investigated by Arnold et al [2] and later further developed by Childress and Gilbert [3]. This dynamo flow makes use of a Riemann metric which is stretched along one direction and compressed in the other. More recently conformal mappings to Arnold's metric [4] have shown also to yield new dynamo solutions. The conformal geodesic Anosov [5] flows, of negative constant Riemannian curvature, were also investigated by Vishik [6] and Friedlander and Vishik [7]. Using other approach using the stretching of fluid particles was recently investigated by Kambes [8] where the sectional curvature does not depend on the flow speed but just on the geometrical quantities of the flow. More recently Chicone and Latushkin [9] have provided us with an elementary proof of the fact that the geodesic flow on a unit tangent bundle of a two dimensional manifold, constant negative curve provides an example of a kinematic fast magnetic dynamo problem. Taking the advantage of the Chicone-Latushkin theorem [10], in this paper it is shown that a generalized class of Arnold’s dynamos can be obtained by the auxiliary $C^{-}\text{flows}$, where $C$ case stands for classical. Though all these in dynamos have the common feature of being inspired by the stretch-twist and fold Vainshtein-Zeldovich [11] method, and generally, physically realistic dynamos possesses stretching in one direction and compression in the other, the generalized $C^{-}\text{flows}$ dynamos considered here may be considered as stretched in both coordinates of the compact Riemannian manifold, or either compressed in both directions, the present dynamo solution also contemplates distinct stretch and compression in distinct direction, which one could call non-uniform stretching [12]. Actually is easy to show that Arnold’s dynamo metric is a particular case of the $C^{-}\text{flow}$ metric.

The paper is organised as follows: Section II presents a review of Arnold dynamo solution. Section III presents the dynamo solution of the self-induction equation in compact Riemannian space. In section IV the computation of the Riemann curvature components is computed by making use of Cartan’s calculus of differential forms and V presents the conclusions.
II Arnold’s dynamos in Riemannian manifolds

The Arnold metric line element can be defined as [2]

\[ ds^2 = e^{-2\lambda z} dp^2 + e^{2\lambda z} dq^2 + dz^2 \] (II.1)

which describes a dissipative dynamo model on a 3D Riemannian manifold. By dissipative here, we mean that contrary to the previous section, the resistivity \( \eta \) is small but finite. The flow build on a toric space in Cartesian coordinates \((p, q, z)\) given by \( T^2 \times [0, 1] \) of the two dimensional torus. The coordinates \( p \) and \( q \) are build as the eigenvector directions of the toric cat map in \( \mathcal{R}^3 \) which possesses eigenvalues as \( \chi_1 = \frac{(3+\sqrt{5})}{2} > 1 \) and \( \chi_2 = \frac{(3-\sqrt{5})}{2} < 1 \) respectively. This represents a simple global translation and is not changed at every point in the manifold. Let us now recall the Arnold et al [4] vector analysis forms, definition of a orthogonal basis in the Riemannian manifold \( \mathcal{M}^3 \)

\[ \vec{e}_p = e^{\lambda z} \frac{\partial}{\partial p} \] (II.2)
\[ \vec{e}_q = e^{-\lambda z} \frac{\partial}{\partial q} \] (II.3)
\[ \vec{e}_z = \frac{\partial}{\partial q} \] (II.4)

Assume a magnetic vector field \( \vec{B} \) on \( \mathcal{M} \)

\[ \vec{B} = B_p \vec{e}_p + B_q \vec{e}_q + B_z \vec{e}_z \] (II.5)

The vector analysis formulas in this frame are

\[ \nabla f = [e^{\lambda z} \partial_p f, e^{-\lambda z} \partial_q f, \partial_z f] \] (II.6)

where \( f \) is the map function \( f : \mathcal{R}^3 \rightarrow \mathcal{R} \). The Laplacian is given by

\[ \Delta f = \nabla^2 f = [e^{2\lambda z} \partial_p^2 f + e^{-2\lambda z} \partial_q^2 f + \partial_z^2 f] \] (II.7)

while the divergence is given by

\[ \nabla \cdot \vec{B} = div \vec{B} = div[B_p \vec{e}_p + B_q \vec{e}_q + B_z \vec{e}_z] = [e^{\lambda z} \partial_p B_p + e^{-\lambda z} \partial_q B_q + \partial_z B_z] \] (II.8)
Thus one may write

\[ \text{div} \vec{e}_p = \text{div} \vec{e}_q = \text{div} \vec{e}_z = 0 \]  

(II.9)

and the curl is written as

\[ \text{curl} \vec{B} = \text{curl}[B_p \vec{e}_p + B_q \vec{e}_q + B_z \vec{e}_z] \]  

(II.10)

where

\[ \text{curl}_p \vec{B} = e^{-\lambda z}(\partial_q B_z - \partial_z (e^{\lambda z} B_q)) \]  

(II.11)

\[ \text{curl}_q \vec{B} = -e^{\lambda z}(\partial_p B_z - \partial_z (e^{-\lambda z} B_p)) \]  

(II.12)

\[ \text{curl}_z \vec{B} = e^{\lambda z} \partial_p B_q - e^{-\lambda z} \partial_q B_p \]  

(II.13)

and

\[ \text{curl} \vec{e}_p = -\lambda \vec{e}_q \]  

(II.14)

\[ \text{curl} \vec{e}_q = -\lambda \vec{e}_p \]  

(II.15)

\[ \text{curl} \vec{e}_z = 0 \]  

(II.16)

The Laplacian operators of the frame basis are

\[ \Delta \vec{e}_p = -\text{curl} \text{curl} \vec{e}_p = -\lambda^2 \vec{e}_p \]  

(II.17)

\[ \Delta \vec{e}_q = -\text{curl} \text{curl} \vec{e}_q = -\lambda^2 \vec{e}_q \]  

(II.18)

\[ \Delta \vec{e}_z = 0 \]  

(II.19)

from these expressions Arnold et al [2] were able to build the self-induced equation in this Riemannian manifold as

\[ \partial_t B_p + v \partial_z B_p = -\lambda v B_p + \eta[\Delta - \lambda^2] B_p - 2\lambda e^{\lambda z} \partial_p B_z \]  

(II.20)

\[ \partial_t B_q + v \partial_z B_q = +\lambda v B_q + \eta[\Delta - \lambda^2] B_p - 2\lambda e^{-\lambda z} \partial_q B_z \]  

(II.21)

\[ \partial_t B_z + v \partial_z B_z = \eta[\Delta - 2\lambda \partial_z] B_z \]  

(II.22)

Decomposing the magnetic field on a Fourier series, Arnold et al were able to yield the following solution

\[ b(p, q, z, t) = e^{\lambda \nu t} b(p, q, z - \nu t, 0) \]  

(II.23)
where \( B(x,y,z,t) = b(p,q,z,t) \) and the fast dynamo limit \( \eta = 0 \) was used. Now with these formulas at hand, we are able to compute the new solution of the self-induced magnetic equation in the background of Riemann Arnold’s line element, which can be given in the next section.

### III Dynamo \( C - \)flows as generalized cat map metric

Earlier Arnold and Avez [13] investigated the auxiliary \( C - \)flow metric given by

\[
\begin{align*}
\text{ds}^2 &= \lambda_1^{2z} dp^2 + \lambda_2^{2z} dq^2 + dz^2 \\
\end{align*}
\]  

where now coordinates \( p \) and \( q \) are given by the global transformations

\[
\begin{align*}
p &= [\lambda_1 x + (1 - \lambda_1)y] \\
q &= [\lambda_2 x + (1 - \lambda_2)y]
\end{align*}
\]  

The inverse transformations are easily obtained as

\[
\begin{align*}
x &= \frac{[(1 - \lambda_2)p - (1 - \lambda_1)q]}{\beta} \\
y &= \frac{[\lambda_1 p - \lambda_2 q]}{\beta}
\end{align*}
\]  

where \( \beta := (\lambda_1 + \lambda_2 - 2\lambda_1 \lambda_2) \). Note that when \( \lambda_1 := \lambda_2^{-1} \) the \( C - \)flow metric under the above coordinate maps reduces to the Arnold’s cat fast dynamo metric. The vector analysis formulas, in the \( C - \)flows metric reads

\[
\begin{align*}
\nabla &= (\lambda_1^{-z} \partial_p, \lambda_2^{-z} \partial_q, \partial_z) \\
\nabla \cdot \vec{B} &= \lambda_1^{-z} \partial_p B_p + \lambda_2^{-z} \partial_q B_q + \partial_z B_z
\end{align*}
\]  

Here the general expression for the flow is

\[
\vec{u} := v_p \lambda_1^{-z} \vec{e}_p + v_q \lambda_2^{-z} \vec{e}_q + \vec{e}_z
\]  

To simplify matters one shall adopt the Childress-Gilbert [3] choice \( \vec{u} := \vec{e}_z \). From this choice it is easy to note that

\[
(\vec{B} \cdot \nabla) \vec{e}_z = 0
\]
and

$$(\vec{u} \nabla) \vec{B} = \partial_z [B_p \lambda_1^{-z} \vec{e}_p + \lambda_2^{-z} B_q + B_z \vec{e}_z] \quad (III.33)$$

Taking the definitions $\mu_A = \log \lambda_A$ where $(A = 1, 2)$, one is able to write the self-induction equation as

$$\partial_t B_p + \partial_z B_p - \mu_1 B_p = \eta(\nabla^2 - \mu_1^2)B_p - \mu_1 \lambda_1^{-z} \partial_z B_p \quad (III.34)$$

$$\partial_t B_q + \partial_z B_q - \mu_2 B_q = \eta(\nabla^2 - \mu_2^2)B_q - \mu_2 \lambda_2^{-z} \partial_z B_q \quad (III.35)$$

$$\partial_t B_z + \partial_z B_z = \eta(\nabla^2 - 2\mu_1^2)(-\lambda_1^{-z} B_p + \lambda_2^{-z} \partial_z B_q) \quad (III.36)$$

Let us consider the case of the highly conductive non-dissipative flow, where the resistivity $\eta$ vanishes and these equations reduce to

$$\partial_t B_p + \partial_z B_p - \mu_1 B_p = 0 \quad (III.37)$$

$$\partial_t B_q + \partial_z B_q - \mu_2 B_q = 0 \quad (III.38)$$

$$\partial_t B_z + \partial_z B_z = 0 \quad (III.39)$$

Thus these equations yield the solution

$$(B_p, B_q, B_z)(p, q, z, t) = (\lambda_1^t B_p^0, \lambda_2^t B_q^0, B_z)(p, q, z - t) \quad (III.40)$$

From this expression is easy to see that if one makes the assumption above about the constants $\lambda_A$ one obtains the Childress-Gilbert solution for fast dynamo metric in the limit of non-dissipative flows. Thus this solution represents the generalized case where the stretch can be given in both directions $\lambda_1 > 0$ and $\lambda_2 > 0$, compressed along both directions, or $\lambda_1 > 0$ and $\lambda_2 < 0$, which are however, unrealistic or even unphysical models, while the last option, $\lambda_1 > 0$ and $\lambda_2 < 0$ or vice-versa represents the stretch and compression along the $(p, q)$ directions is a physical dynamo solution which is more general than Arnold’s solution since now the stretch is non-uniform.
**IV Riemann curvature of $C$–flow dynamos**

Due to the importance of constant negative curvature of geodesic Anosov flows for dynamo maps, in this section one computes the Riemann curvature for C-flows. The C-flow generalization to Arnold metric can be expressed in terms of the Cartan [14] frame basis form $\omega^i$ ($i = 1, 2, 3$), as

$$ds^2 = (\omega^p)^2 + (\omega^q)^2 + (\omega^z)^2 \quad \text{(IV.41)}$$

The basis form are write as

$$\omega^p = \lambda_1^z dp \quad \text{(IV.42)}$$
$$\omega^q = \lambda_2^z dq \quad \text{(IV.43)}$$

and

$$\omega^z = dz \quad \text{(IV.44)}$$

Application of the exterior differentiation of this basis form yields

$$d\omega^p = \omega_z \wedge \omega_p \quad \text{(IV.45)}$$
$$d\omega^z = 0 \quad \text{(IV.46)}$$

by Poincaré lemma, and

$$d\omega^q = \omega_z \wedge \omega_q \quad \text{(IV.47)}$$

Assuming that our manifold is Riemannian, the Cartan torsion 2–forms of non-Riemannian geometry vanishes, and one obtains, from Cartan first structure equations

$$T^p = 0 = d\omega^p + \omega^p_q \wedge \omega^q + \omega^p_z \wedge \omega^z \quad \text{(IV.48)}$$
$$T^q = 0 = d\omega^q + \omega^q_p \wedge \omega^p + \omega^q_z \wedge \omega^z \quad \text{(IV.49)}$$
$$T^z = 0 = \omega^p_q \wedge \omega^q + \omega^p_z \wedge \omega^z \quad \text{(IV.50)}$$

the following constraints

$$\omega^z \wedge \omega^p + \omega^p_q \wedge \omega^q + \omega^p_z \wedge \omega^z = 0 \quad \text{(IV.51)}$$
$$\omega^z \wedge \omega^q + \omega^q_p \wedge \omega^p + \omega^q_z \wedge \omega^z = 0 \quad \text{(IV.52)}$$
$$\omega^z_q \wedge \omega^q + \omega^z_p \wedge \omega^p = 0 \quad \text{(IV.53)}$$
From these relations one obtains the following Cartan connection one forms

\[ \omega^p_z = -\alpha \omega^q \] (IV.54)

\[ \omega^q_z = -\alpha \omega^p \] (IV.55)

and

\[ \omega^p_q = (1 + \alpha)\omega^z \] (IV.56)

where \( \alpha \) is constant. Substitution of these connection form components into the second Cartan equation

\[ R^i_j = R^i_{jk} \omega^k \wedge \omega^j = d\omega^i_j + \omega^i_l \wedge \omega^l_j \] (IV.57)

where \( R^i_j \) is the Riemann curvature 2-form. After straightforward algebra one obtains the following components of Riemann curvature for the C-flows dynamo

\[ R^q_{zzq} = -\alpha + \alpha^2 \] (IV.58)

which is constant and negative if \( 0 < \alpha < 1 \) and

\[ R^q_{zzp} = -\alpha < 0 \] (IV.59)

while other components are easily computed. For Asonov geodesic dynamo flows the important issue is to compute the Gaussian curvature which is given by

\[ d\omega^q_z = \alpha \omega^p \wedge \omega^z \] (IV.60)

\[ d\omega^p_z = -\alpha \omega^q \wedge \omega^z \] (IV.61)

so the first scalar curvature is \( K_1 = \alpha \), which by analogy yields \( K_2 = -\alpha \). Since the Gaussian curvature is the product of \( K_1 \) and \( K_2 \) one obtains \( K_G = -\alpha^2 < 0 \) and one certainly has a negative curvature as necessarily for compact Riemannian Anosov manifolds.

V Conclusions

In conclusion, we obtain a class of C-flow dynamos in three-dimensional Riemannian manifold which generalizes the Arnold’s Riemann metric. This solution of magnetic dynamo presents
no pathologies in the and it is easy to show the magnetic flux and energy definitely grow which indicates that this solution can be considered a more realistic dynamo. More modern approaches to the dynamo problem, namely chaotic dynamos have been recently addressed by Reyel et al [15] in the realm of plasma physics by investigating the quasi-two dimensional fast kinematic dynamo instabilities of chaotic fluid flow. This investigation, however is numerically and not analytically as was performed in this paper. Analytical solutions are still important to even guide us in building more general numerical solutions. Though our solution is more of a mathematical nature in certain sense without fold and reconnection [16], more realistic dynamo maps have considered recently [17] and one could this path to build more general realistic dynamos.

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