Hardy’s type inequality for the over critical exponent
associated with the Dunkl transform

Rahmouni Atef
University of Carthage, Faculty of Sciences of Bizerte
Department of Mathematics Bizerte 7021 Tunisia.
Atef.Rahmouni@fsb.rnu.tn

Abstract

For the Hardy space $H^p(\mathbb{R}^d)$, $0 < p \leq 1$, we shall improve a Hardy’s type inequality associated with Dunkl transform respect to the measures $d\mu_k$ homogeneous of degree $\gamma$, on the strip $(2\gamma + d)(2 - p) \leq \sigma < 2\gamma + d + p(N + 1)$, where $N = \lfloor(2\gamma + d)(1/p - 1)\rfloor$ is the greatest integer not exceeding $(2\gamma + d)(1/p - 1)$.

2010 Mathematics Subject Classification: 42B10, 42B30, 33C45.

keywords: Hardy space, Hardy’s type inequality, Dunkl transform.

1 Introduction

In recent years the topic of Hardy type inequalities and their applications seem to have grown more and more popular. Although Hardy’s original result dates back to the 1920’s, some new versions are stated and old ones are still being improved almost a century later. One of the reasons for the popularity of Hardy type inequalities is their usefulness in various applications.

The first definition of Hardy spaces was in terms of analytic functions in the unit disc and their boundary values. In the last two decades, the theory was developed in $\mathbb{R}^d$ by real variable methods like Poisson integrals, Riesz transforms, and maximal functions. The subsequent discovery of the atomic decomposition theory of $H^p(\mathbb{R}^d)$ spaces marks an important step of further developments on its real variable theory. Using the grand maximal function, R. Coifman [1] first shows that an element in $H^p(\mathbb{R}^d)$ can decomposed into a sum of a series of basic elements. Then the study on some analytic problems on $H^p(\mathbb{R}^d)$ is summed up to investigate some properties of these basic elements, and therefore the problems because quite simple. Taibleson and Weiss [15] gave the definition of molecules belonging to $H^p$, and showed that every molecule is in $H^p$ with continuous embedding map. By the atomic decomposition and the molecule characterization, the proof of $H^p$ boundedness of the operators on Hardy space becomes easier. The theory of $H^p$ have been extensively studied in [6, 7, 8]

In the setting of the euclidian case, Hardy’s inequality for Fourier transform asserts that for all $f \in H^p(\mathbb{R}^d)$, $0 < p \leq 1$

$$\int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^p}{|\xi|^{d(2-p)}} d\xi \leq \|f\|_{H^p(\mathbb{R}^d)}^p, \quad 0 < p \leq 1 \quad (1)$$
where $H^p(\mathbb{R}^d)$ indicates the real Hardy space. Recently, an extension has been given by [10], the latter establish a Hardy’s type inequality associated with the euclidean Fourier transform for over critical exponent $\sigma > d(2 - p)$.

In the same context we prove a Hardy’s type inequality associated with Dunkl transform. We consider the differential-difference operators $T_j, j = 1, \ldots, d$, on $\mathbb{R}^d$ introduced by C.F. Dunkl in [3] and called Dunkl operators in the literature, associated with the finite reflection group $G$ and the multiplicity function $k$, are given for a function $f$ of class $C^1$ on $\mathbb{R}^d$ by

$$T_j f(y) = \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \alpha_j \frac{f(y) - f(\sigma \alpha y)}{< \alpha, y>}. $$

For $y \in \mathbb{R}^d$, the initial problem $T_j u(\cdot, y)(x) = y_j u(x, y); j = 1, \ldots, d$ with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel [4]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$. The Dunkl kernel has the Laplace-type representation [11]

$$E_k(x, y) = \int_{\mathbb{R}^d} e^{-y \cdot z} d\Gamma_x(z); \ x \in \mathbb{R}^d, y \in \mathbb{C}^d,$$

where $< y, z > := \sum_{j=1}^d y_j z_j$ and $\Gamma_x$ is a probability measure on $\mathbb{R}^d$, such that $supp(\Gamma_x) \subset \{z \in \mathbb{R}^d : |z| \leq |x|\}$.

In particular cases, we have

$$|E_k(x, y)| \leq 1, \ x, y \in \mathbb{R}^d. \quad (2)$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and it was introduced by Dunkl [5], where already many basic properties were established. Dunkl’s results were completed and extended later on by de Jeu [2]. The Dunkl transform of a function $f \in L^1(\mathbb{R}^d)$ is

$$\mathcal{F}_\phi(f)(x) := c_k \int_{\mathbb{R}^d} E_k(-ix, y)f(y)d\mu_k(y), \ x \in \mathbb{R}^d,$$

where $c_k$ is the Mehta-type constant given by $c_k = \left( \int_{\mathbb{R}^d} e^{-|y|^2/2} d\mu_k(y) \right)^{-1}$. We denote by $d\mu_k$ the measure on $\mathbb{R}^d$ given by $d\mu_k(y) := w_k(y) dy$ and by $L^p(\mathbb{R}^d), 0 < p \leq \infty$, the space of measurable functions $f$ on $\mathbb{R}^d$, such that

$$\|f\|_{L^p_k} = \left( \int_{\mathbb{R}^d} |f(y)|^p d\mu_k(y) \right)^{1/p}, \text{ if } p > 0 \quad \text{and} \quad \|f\|_{L^\infty_k} = \text{ess sup}_{y \in \mathbb{R}^d} |f(y)|.$$

In this paper, we obtain an improved Hardy’s type inequality associated with the Dunkl transform. So, for the Hardy space $H^p(\mathbb{R}^d), 0 < p \leq 1$ we establish a Hardy’s type inequality for the strip $(2\gamma + d)(2 - p) \leq \sigma < 2\gamma + d + p(N + 1)$.

Throughout this paper, $C$ stands for a positive constant that can be changed from line to line.
2 Preliminaries: (Reflection groups, Root systems and Multiplicity functions)

Let us begin to recall some results concerning the root systems. A useful reference for this topic is the book by Humphreys [9].

We consider \( \mathbb{R}^d \) with the euclidean inner product \( \langle \cdot , \cdot \rangle \) and norm \( |y| := \sqrt{\langle y, y \rangle} \). For \( \alpha \in \mathbb{R}^d \setminus \{0\} \), let \( \sigma_\alpha \) be the reflection in the hyperplane \( H_\alpha \subset \mathbb{R}^d \) orthogonal to \( \alpha \), i.e.

\[
\sigma_\alpha y := y - \frac{2 \langle \alpha, y \rangle}{|\alpha|^2} \alpha.
\]

A finite set \( \mathcal{R} \subset \mathbb{R}^d \setminus \{0\} \) is called a root system, if \( \mathcal{R} \cap \mathcal{R} = \{-\alpha, \alpha\} \) and \( \sigma_\alpha \mathcal{R} = \mathcal{R} \) for all \( \alpha \in \mathcal{R} \). We assume that it is normalized by \( |\alpha|^2 = 2 \) for all \( \alpha \in \mathcal{R} \). For a root system \( \mathcal{R} \), the reflections \( \sigma_\alpha, \alpha \in \mathcal{R} \), generate a finite group \( G \subset O(d) \), the reflection group associated with \( \mathcal{R} \). All reflections in \( G \) correspond to suitable pairs of roots. For a given \( \beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in \mathcal{R}} H_\alpha \), we fix the positive subsystem \( \mathcal{R}_+ := \{ \alpha \in \mathcal{R} : \langle \alpha, \beta \rangle > 0 \} \). Then, for each \( \alpha \in \mathcal{R} \) either \( \alpha \in \mathcal{R}_+ \) or \( \alpha \in \mathcal{R}_- \). Let \( k : \mathcal{R} \to \mathbb{C} \) be a multiplicity function on \( \mathcal{R} \) (i.e. a function which is constant on the orbits under the action of \( G \)). For abbreviation, we introduce the index

\[
\gamma = \gamma_k := \sum_{\alpha \in \mathcal{R}_+} k(\alpha).
\]

Throughout the paper, we assume that the multiplicity is non-negative, that is, \( k(\alpha) \geq 0 \) for all \( \alpha \in \mathcal{R} \). Moreover, let \( w_k \) denote the weight function

\[
w_k(y) := \prod_{\alpha \in \mathcal{R}_+} |\langle \alpha, y \rangle|^{2k(\alpha)}, \quad y \in \mathbb{R}^d,
\]

which is \( G \)-invariant and homogeneous of degree \( 2\gamma \).

3 Hardy-type inequality

The atom decomposition theory of \( H^p(\mathbb{R}^d) \) spaces marks an important step of further developments on its real variable theory. Using the grand maximal function, R. R. Coifman [11] first shows that an element in \( H^p(\mathbb{R}) \) can decomposed into a sum of a series of basic elements. Then the study on some analytic problems on \( H^p(\mathbb{R}^d) \) is summed up to investigate some properties of these basic elements, and therefore the problems because quite simple. These basic elements are called atoms. Let us now make the definition of an atom.

Definition 1. Let \( 0 < p \leq 1 \leq q \leq \infty \) with \( p \neq q \). A function \( a(x) \in L^q(\mathbb{R}^d) \) is called a \((p,q,s)\)-atom with the center at \( x_0 \), if it satisfies the following conditions

(i) \( \text{Supp} \ a \subset B(x_0, r) \)

(ii) \( \|a\|_{L^q(\mathbb{R}^d)} \leq |B(x_0, r)|^{\frac{1}{q} - \frac{1}{p}} \),
(iii) $\int_{\mathbb{R}^d} a(y) y^\ell d\mu_k(y) = 0$, for all monomials $y^\ell$ with $|\ell| \leq s$ with $s \geq N = \left[ (2\gamma + d)(\frac{1}{p} - 1) \right]$, where $\lfloor \cdot \rfloor$ denotes, as usual, the “greatest integer not exceeding” function.

Here, (i) means that an atom must be a function with compact support, (ii) is the size condition of atoms, and (iii) is called the cancelation moment condition. Moreover, $B(x_0, r)$ is the ball centered at $x_0$ with radius $r$. Clearly, $a(p, \infty, s)$ atom must be $a(p, q, s)$ atom, if $p < q < \infty$.

Using the atomic decomposition, we define the Hardy space $H^p(\mathbb{R}^d)$ to be the collection of functions $f$ satisfying $f = \sum_{j=0}^\infty \beta_j a_j$, where $a_j$ are $H^p(\mathbb{R}^d)$-atoms and $\beta_j$ is a sequence of complex numbers with $\sum_{j=0}^\infty |\beta_j|^p < \infty$. $H^p(\mathbb{R}^d)$ is equipped with a norm as follows

$$\|f\|_{H^p(\mathbb{R}^d)} = \inf \left\{ \sum_{j=0}^\infty |\beta_j|^p \right\},$$

where the infimum is taken over all atoms decompositions of $f$.

**Theorem 1.** Let $0 < p \leq 1$, and $N = \left\lfloor (2\gamma + d)(1/p - 1) \right\rfloor$, the greatest integer not exceeding $(2\gamma + d)(1/p - 1)$. Then for any $f \in H^p(\mathbb{R}^d)$ the Dunkl transform of $f$ satisfies the following Hardy’s type inequality

$$\int_{\mathbb{R}^d} \frac{|\mathcal{F}_\sigma(f)(y)|^p}{|y|^{\sigma}} d\mu_k(y) \leq C \|f\|_{H^p(\mathbb{R}^d)}^p,$$  \hspace{1cm} (3)

provide that

$$(2\gamma + d)(2 - p) \leq \sigma < 2\gamma + d + p(N + 1)$$  \hspace{1cm} (4)

where $C$ is a constant does not depends on $f$.

**Remarks.**

1. Note that the collection of all real $\sigma$ satisfying the condition (4) is a nonempty set since $2\gamma + d + p(N + 1) - (2\gamma + d)(2 - p) > 0$.

2. For the critical case $\sigma_0 = (2\gamma + d)(2 - p)$ has been extensively studied in [14].

3. It would be interesting to know if this is the best possible improved.

**Proof.** Let $f = \sum_{j=0}^\infty \beta_j a_j \in H^p(\mathbb{R}^d)$, being element of $H^p(\mathbb{R}^d)$ where $a_j$ are atoms. Since $0 < p \leq 1$ it follows

$$\int_{\mathbb{R}^d} \frac{|\mathcal{F}_\sigma(f)(y)|^p}{|y|^{\sigma}} d\mu_k(y) \leq C \sum_{j=0}^\infty |\beta_j|^p \int_{\mathbb{R}^d} \frac{|\mathcal{F}_\sigma(a_j)(y)|^p}{|y|^{\sigma}} d\mu_k(y).$$

In order to prove the theorem, it is enough to prove,

$$\int_{\mathbb{R}^d} \frac{|\mathcal{F}_\sigma(f)(y)|^p}{|y|^{\sigma}} d\mu_k(y) \leq C.$$  \hspace{1cm} (5)
Let us now take $\rho$ an arbitrary nonnegative real number, and decomposing the left hand side of (5) as

$$\int_{\mathbb{R}^d} \frac{|\mathcal{F}_\gamma(a_j)(y)|^p}{|y|^\sigma} d\mu_k(y) = \int_{|y|<\rho} \frac{|\mathcal{F}_\gamma(a_j)(y)|^p}{|y|^\sigma} d\mu_k(y) + \int_{|y|\geq\rho} \frac{|\mathcal{F}_\gamma(a_j)(y)|^p}{|y|^\sigma} d\mu_k(y)$$

$$:= S_1 + S_2.$$ 

To estimate $S_1$, we may use Taylor’s theorem in several variables with integral’s remainder for the function $y \mapsto E_k(ix,y)$, we obtain

$$E_k(ix,y) = \sum_{n=0}^N V_k(\prec iy,\succ^n(x)) + R_{N+1}(x,y),$$

where

$$R_{N+1}(x,y) = \frac{1}{(N+1)!} \int_0^1 (1-t)^{N+1} \left[ \int_{\mathbb{R}^d} \prec iy,z \succ^{N+1} e^{t \prec iy,z \succ} d\Gamma(z) \right] dt,$$

and $V_k$ is the intertwining operator (see, [4] [12]), defined on $\mathbb{C}[\mathbb{R}^d]$ (the algebra of polynomial functions on $\mathbb{R}^d$) by

$$V_k(p) = \int_{\mathbb{R}^d} f(z)d\Gamma(z), \quad x \in \mathbb{R}^d.$$ 

Since $\int_{\mathbb{R}^d} a_j(y) y^\ell d\mu_k(y) = 0$, for every $|\ell| \leq N$ where $N = \left\lceil (2\gamma + d) \left(\frac{1}{p} - 1\right) \right\rceil$, we can write

$$\mathcal{F}_\gamma(a_j)(x) = \int_{B(0,r)} \left[ E_k(-ix,y) - \sum_{n=0}^N \frac{V_k(\prec iy,\succ^n(x))}{n!} a_j(y) d\mu_k(y) \right], \quad x \in \mathbb{R}^d.$$ 

Hence, from (2), it follows that

$$|\mathcal{F}_\gamma(a_j)(x)| \leq c_k \int_{B(0,r)} |R_{N+1}(x,y)||a_j(y)| d\mu_k(y).$$

But it is clear that

$$|R_{N+1}(x,y)| \leq \frac{1}{(N+1)!} [|x|,|y|]^{N+1}.$$ 

Now with the help of properties $(i)$, $(ii)$ and $(iii)$ for $a(p,\infty,s)$-atoms of $H^p(\mathbb{R}^d)$, we get the following results

$$|\mathcal{F}_\gamma(a_j)(x)| \leq C \int_{B(0,r)} |x|^{N+1} |y|^{N+1} \mu_k(B(0,r))^{-\frac{1}{p}} d\mu_k(y)$$

$$\leq C r^{N+1+(2\gamma+d)(1-\frac{1}{p})} |x|^{N+1},$$
where we have used (see, \[16\]),
\[\mu_k(B(0, r)) = \frac{r^{2\gamma+d}}{c_k 2^\gamma d/\Gamma(\gamma+d/2+1)}.\]

Integrating with respect to the measure \(d\mu_k\) over the domain \(0 < |y| < \rho\), we obtain
\[S_1 := \int_{|y|<\rho} \frac{|\mathcal{F}_\varphi(a_j)(y)|^p}{|y|^\sigma} d\mu_k(y) \leq Cr^{p(N+2\gamma+d+1)-(2\gamma+d)} \int_{|y|<\rho} |y|^{p(N+1)-\sigma} d\mu_k(y) \leq Cr^{-(2\gamma+d)+p(N+2\gamma+d+1)} \rho^{2\gamma+d+p(N+1)-\sigma}

that is
\[S_1 \leq Cr^{-(2\gamma+d)+p(N+2\gamma+d+1)} \rho^{2\gamma+d+p(N+1)-\sigma} \tag{6}\]
provide that \(\sigma < 2\gamma+d+p(N+1)\) which follows from the inequality \([4]\).

Now to estimate \(S_2\), we may apply Hölder’s inequality for \(q = \frac{2}{p}\) and Plancherel formula to get
\[S_2 \leq \left( \int_{\mathbb{R}^d} (|a_j(y)|^p)^{\frac{q}{2}} d\mu_k(y) \right)^{\frac{2}{q}} \left( \int_{|y|\geq\rho} |y|^{\frac{2\sigma}{p-2}} d\mu_k(y) \right)^{\frac{2-p}{2}} \leq C \|a_j\|_{L^2(\mathbb{R}^d)}^p \rho^{\frac{(2\gamma+d)(2-p)}{2}-\sigma},
\]
provide that \(\frac{(2\gamma+d)(2-p)}{2} < \sigma\), which is a consequence of the left hand side of \((4)\). Taking into account that
\[\|a_j\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |a_j(y)|^2 d\mu_k(y) \leq \int_{B(0,r)} \mu_k(B(0,r))^\frac{1}{2} d\mu_k(y) \leq Cr^{-\frac{(2\gamma+d)(2-p)}{p}},\]
We obtain \(\|a_j\|_{L^p(\mathbb{R}^d)} \leq C r\frac{(2\gamma+d)(2-p)}{2}\) and hence,
\[S_2 \leq Cr^{-\frac{(2\gamma+d)(2-p)}{p}} \rho\frac{(2\gamma+d)(2-p)}{2}-\sigma. \tag{7}\]

**Case 1.** If \(\sigma_0 = (2\gamma+d)(2-p)\). We put \(\rho = \frac{1}{r}, \forall r > 0\), then we have \(S_1 \leq C\) and \(S_2 \leq C\).

**Case 2.** If \((2\gamma+d)(2-p) < \sigma < (2\gamma+d) + p(N+1)\). We shall discuss the cases \(0 < r < 1\) and \(r \geq 1\).
Hence, in order to deal with the case $0 < r < 1$, we need more precise estimates, so we consider the set $\Upsilon_\rho$; the collection of all numbers $\rho$ satisfying

$$
\frac{(2\gamma + d)(2 - p)}{(2\gamma + d)(2 - p) - 2\sigma} \log(r) \leq \log(\rho) \leq \frac{(2\gamma + d) - p(N + 2\gamma + d + 1)}{(2\gamma + d) + p(N + 1) - \sigma} \log(r). \quad (8)
$$

To prove that the collection $\Upsilon_\rho$ above is a nonempty set it is enough to prove that

$$
\frac{(2\gamma + d)(2 - p)}{(2\gamma + d)(2 - p) - 2\sigma} \times \frac{(2\gamma + d) + p(N + 1) - \sigma}{(2\gamma + d) - p(N + 1 + (2\gamma + d))} \leq 1 \quad (9)
$$

which is a different formulation of the left hand side of (4), that is $(2\gamma + d)(2 - p) \leq \sigma$.

Using the fact that $(2\gamma + d) + p(N + 1) - \sigma > 0$ and the right hand side of (8) it follows that

$$
S_1 \leq Cr^{-(2\gamma + d) + p(N + 2\gamma + d + 1)}\rho^{2\gamma + d + p(N + 1) - \sigma} \leq C. \quad (10)
$$

Using the left hand side of (8) and the fact that $(2\gamma + d)(2 - p) < 0$, we obtain

$$
S_2 \leq C. \quad (11)
$$

Combining (10) and (11) the result follows for the case $0 < r < 1$.

Now, to deal with the case $r \geq 1$, we may take

$$
\rho = r^{\frac{2\gamma + d - p(N + 1 + 2\gamma + d)}{2\gamma + d + p(N + 1) - \sigma}} \quad (12)
$$

so, using the fact that $r \geq 1$, we obtain

$$
\rho = r^{\frac{(2\gamma + d)(2 - p)}{(2\gamma + d)(2 - p) - 2\sigma}}. \quad (13)
$$

Combining (6), (12) and (7) together with (13), the proof of the main theorem is completed.

\[\Box\]

References

[1] R. R. Coifman, A real-variable characterization of $H^p$, Studia Math. 51, (1974), 269-274.

[2] M. F. E. De Jeu, The Dunkl transform, Invent. Math. 113 (1993), 147-162.

[3] C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311 (1989), 167–183.

[4] C. F. Dunkl, Integral kernels with reflection group invariance, Canad. J. Math. 43 (1991), 1213–1227.

[5] C. F. Dunkl, Hankel transforms associated to finite reflection groups, Contemp. Math. 138 (1992), 123-138.
[6] C. Fefferman and E. M. Stein, $H^p$ spaces of several variables, Acta Math. 129, (1972), 137-193.

[7] G. B. Folland and E. M. Stein, Hardy Spaces on Homogeneous Groups, Princeton University Press, Princeton, NJ, 1982.

[8] J. Garcia-Cuerva and J. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North Holland, 1985.

[9] J.E. Humphreys, Reflection groups and Coxeter groups Cambridge: Cambridge Univ. Press 1990, 1–204.

[10] A. Rahmouni and M. Assal, An improved Hardy’s inequality associated with the Euclidean Fourier transform, Acta Mathematica Scientia, (2013), 1-6.

[11] M. Rössler, Positivity of Dunkl’s intertwining operator, Duke Math. J. 98 (1999), 445-463.

[12] M. Rössler, A positive radial product formula for the Dunkl kernel, Trans. Amer. Math. Soc. 355 (2003), 2413-2438.

[13] E. M. Stein, Harmonic Analysis, real variable Methods, orthogonality and oscillatory integrals, Princeton Univ. Press, Princeton, NJ, 1993.

[14] F. Soltani, Maximal BochnerRiesz operators on Hardy-type spaces in the Dunkl setting, Integr. Transf. Spec. F., (2012), 1–15.

[15] M. H. Taibleson and G. Weiss, The molecular characterization of certain Hardy spaces, Astérisque 77, (1980), Société Math. de France, Paris, 67-149.

[16] S. Thangavelu and Y. Xu, Convolution operator and maximal function for the Dunkl transform, J. Anal. Math. 97 (2005), 25-56.