Excitation of Orbital Eccentricities by Repeated Resonance Crossings: Requirements

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ABSTRACT

Divergent migration of planets within a viscous circumstellar disk can engender resonance crossings and dramatic excitation of orbital eccentricities. We provide quantitative criteria for the viability of this mechanism. For the orbits of two bodies to diverge, a ring of viscous material must be shepherded between them. As the ring diffuses in radius by virtue of its intrinsic viscosity, the two planets are wedged further apart. The ring mass must be smaller than the planetary masses so that the crossing of an individual resonance lasts longer than the resonant libration period. At the same time, the resonance crossing cannot be of such long duration that the disk’s direct influence on the bodies’ eccentricities interferes with the resonant interaction between the two planets. This last criterion is robustly satisfied because resonant widths are typically tiny fractions of the orbital radius. We evaluate our criteria not only for giant planets within gaseous protoplanetary disks, but also for shepherd moons that bracket narrow planetary rings in the solar system. A shepherded ring of gas orbiting at a distance of 1 AU from a solar-type star and having a surface density of less than 500 g/cm$^2$, a dimensionless alpha viscosity of 0.1, and a height-to-radius aspect ratio of 0.05 can drive two Jovian-mass planets through the 2:1 and higher-order resonances so that their eccentricities amplify to values of several tenths. Because of the requirement that the disk mass in the vicinity of the planets be smaller than the planet masses, divergent resonance crossings may figure significantly into the orbital evolution of planets during the later stages of protoplanetary disk evolution, including the debris disk phase.

Subject headings: celestial mechanics — planetary systems — planets and satellites: individual (Uranus, $\epsilon$ ring) — accretion, accretion disks
Orbital eccentricities of extrasolar giant planets can be surprisingly large compared to their counterparts in the solar system (see, e.g., the review by Marcy, Cochran, & Mayor 2000). Figure 1 displays the distribution of eccentricities of 93 extrasolar planets, downloaded from the California and Carnegie Planet Search website (http://exoplanets.org). Aside from those “hot Jupiters” whose eccentricities were likely damped by tidal interactions with their central stars, most giant planets occupying stellocentric distances between $\sim 0.2$ and $\sim 2$ AU have eccentricities near 0.35, and a few have eccentricities ranging up to 0.93.

A variety of mechanisms have been introduced to excite planetary eccentricities. These theories can be divided into three categories: those that rely on interactions between the planet and another point mass, be it a star or planet; those that rely on interactions between the planet and the circumstellar disk from which that planet coalesced; and “hybrid” theories that implicate both another body and the disk. Kozai-resonant forcing by a binary stellar companion (Holman, Touma, & Tremaine 1997) and violent encounters between two or more planets formed within close proximity (Rasio & Ford 1996; Weidenschilling & Marzari 1996; Ford, Havlickova, & Rasio 2001; Marzari & Weidenschilling 2002) belong in the first category. In the second category, one of the most recent and inclusive studies is by Goldreich & Sari (2002), who demonstrate that interactions between a planet and a disk at first-order Lindblad resonances can excite the planet’s eccentricity provided that eccentricity exceeds a threshold value. In the third category belongs the formation scenario proposed for the planetary system GJ 876 (Marcy et al. 2001; Lee & Peale 2002); this scenario involves convergent orbital migration of two planets, followed by capture of these planets into a mean-motion resonance and continued migration within that resonance. This is a hybrid mechanism because the required planetary migration is driven by an underlying disk, while each planet’s eccentricity is directly excited by the other planet. A different hybrid scenario may have played out for planet c in the system Upsilon Andromedae (Butler et al. 1999; Chiang, Tabachnik, & Tremaine 2001). A primordial disk may have directly excited the eccentricity of the outermost planet, d; secular interactions between d and c could then have siphoned off the eccentricity of the former to grow that of the latter (Chiang & Murray 2002). As a bonus, this mechanism can also explain the heretofore puzzling alignment of orbital apsides exhibited by c and d. This process has been explored both in the adiabatic (Chiang & Murray 2002) and impulsive (Malhotra 2002) limits.

Yet another hybrid mechanism has been introduced by Chiang, Fischer, & Thommes (2002, hereafter CFT), who point out that if two planets migrate within a circumstellar disk such that their orbital trajectories diverge, then their eccentricities can increase as the planets cross a series of mean-motion resonances. The closer the initial ratio of orbital
Fig. 1.— Orbital eccentricities of 93 planets, downloaded from the California and Carnegie Planet Search website (http://exoplanets.org). The spike near zero eccentricity mostly represents “hot Jupiters” whose orbital periods are shorter than 20 days and whose eccentricities were likely damped by tidal interactions with their parent stars. The remaining distribution appears to peak near an eccentricity of 0.35.
periods is to unity, the greater the number of resonances that are crossed and the greater
the mutual excitation of eccentricities. Jupiter-mass planets on slowly divergent orbits that
cross the 2:1 and higher-order resonances have their eccentricities grown from initial values
\( \lesssim 0.05 \) to values \( \sim 0.4 \). Chiang & Murray (2002) discuss in which of the known multiple
planet systems divergent resonance crossings might have occurred.

In initially proposing the mechanism of repeated resonance crossings via divergent mi-
gration, CFT ignored the disk’s direct effects on the planets’ eccentricities. This follow-up
paper to their proposal restores such effects. We quantitatively evaluate the conditions under
which divergent migration and the excitation of eccentricities by resonance crossings can oc-
cur. In §2, we provide three such necessary requirements and evaluate them for giant planets
embedded within gaseous circumstellar disks. In §3, we test one of these requirements by
numerical integration of Jovian-mass planets on divergent trajectories. There, we also show
explicitly which mean-motion resonances are the principal amplifiers of orbital eccentricities.
In §4, we evaluate these conditions for the comparatively tiny shepherd moons that confine
narrow planetary rings in our own solar system; the divergent migration of ring shepherds
affords the most accessible laboratory we have for studying repeated resonance crossings.
A summary and critical discussion of our principal findings, including directions for future
research, are given in §5.

2. REQUIREMENTS FOR PLANETS

We consider the planets to be embedded within a circumstellar disk having surface
density \( \Sigma \) and kinematic viscosity \( \nu = \alpha c_s h \) at stellocentric distance \( r \), where \( c_s \) is the gas
sound speed, \( h = c_s/\Omega \) is the vertical gas scale height, \( \Omega \) is the orbital angular frequency,
and \( 0 < \alpha < 1 \) is a dimensionless constant. Planetary masses, orbital semi-major axes, and
orbital eccentricities are denoted by \( M, a, \) and \( e \), respectively, with subscript 1 (2) denoting
the inner (outer) planet. The masses of the disk and of the planets are assumed to be small
compared to the mass of the star, \( M_* \). In numerically evaluating quantities, we will take
\( c_s = 1 (r/\text{AU})^{-9/14} \text{km/s}, h/r = 0.05 (r/\text{AU})^{2/7}, \alpha = 10^{-2}, M/M_* = 10^{-3}, \) and \( M_* = M_\odot \)
(see, e.g., Chiang et al. 2001; Hartmann et al. 1998).

We assume that the planets open gaps about their orbits and migrate only by dint of
disk viscosity. This mode of migration is referred to as “Type II” (Ward 1997). An upper
limit for the gap-opening mass, above which planets do open gaps but below which planets
might not, is given by
\[
\max M_{\text{gap}} = \frac{2c_s^3}{3\Omega G} \approx 8 \left(\frac{r}{\text{AU}}\right)^{6/7} M_\oplus, \tag{1}
\]

where \(G\) is the gravitational constant and \(M_\oplus\) represents an Earth mass (Lin & Papaloizou 1993; Rafikov 2002). A planet exceeding this mass excites strongly non-linear density waves at principal Lindblad resonances situated near the location of the torque cut-off, i.e., having azimuthal wavenumbers \(m \approx \Omega r/c_s\). These waves shock upon launch, releasing the angular momentum which they carry to disk gas. We define the gap width, \(w\), to be the distance between the planet and the disk edge which it shepherds. The gap width is estimated by setting the viscous torque, \(3\pi \nu \Sigma \Omega r^2\), equal to the total tidal torque, \(\Sigma \Omega^2 r^4 (r/w)^3 (M/M_\ast)^2\), exerted over all principal Lindblad resonances up to wavenumber \(m_{\text{max}} \approx 2r/3w\) to one side of the planet:

\[
\frac{w}{r} \approx 0.2 \left(\frac{10^{-2}}{\alpha}\right)^{1/3} \left(\frac{r}{h}\right)^{2/3} \left(\frac{M/M_\ast}{10^{-3}}\right)^{2/3} \tag{2}
\]

(Goldreich & Sari 2002, hereafter GS, and references therein). This estimate employs the standard, linear torque formula for principal Lindblad resonances [see, e.g., equation (10) of Goldreich & Porco 1987].

Based on the estimate of the upper bound on \(M_{\text{gap}}\) given by equation (1), we would expect Jupiter-mass \((M = M_J = 300 M_\oplus)\) planets at \(r \lesssim 30\ \text{AU}\) to open gaps. Uranus-mass \((M = M_U = 14 M_\oplus)\) planets open gaps inside \(r \lesssim 1\ \text{AU}\). Equation (1) notwithstanding, bodies having \(M < M_U\) can open gaps throughout the disk, depending on the local viscosity, temperature, and surface density; more precise estimates of \(M_{\text{gap}}\) are provided by Rafikov (2002; see his Figure 5), who employs a realistic prescription of wave dissipation by non-linear steepening to calculate the conditions for which Type I migration solutions transition to Type II solutions (see also Ward 1997; Ward & Hourigan 1989; Goodman & Rafikov 2001).

We identify the following three requirements for divergent resonance crossings.

### 2.1. Ring Shepherding

For the ratio of the orbital period of the outer planet to that of the inner planet to diverge, a ring of viscous material must be shepherded between the two planets. As the ring diffuses in radius by virtue of its intrinsic viscosity, the two planets are wedged further apart.
The requirement of ring shepherding supersedes the requirement originally proposed by CFT that the viscous diffusion time, \( t_D = \frac{r^2}{\nu} \), increase with \( r \). Even if the latter condition were satisfied, material must exist just outside of the inner planet to drive that planet further inwards; otherwise, disk material just outside of the outer planet would cause the outer planet's orbit to converge upon that of the inner one.

A ring will fail to be shepherded if it does not span enough wave dissipation lengths. For example, if a wave that is excited by the inner planet near the inner ring boundary propagates across the ring to the outer ring boundary and damps there, then disk gas at the latter edge would be forced past the outer planet. Similar conclusions obtain for waves excited in the ring by the outer planet. Such "anti-shepherding" has been observed in numerical simulations by Bryden et al. (2000).

How many dissipation lengths can be accommodated by the rings of interest here? For the giant planetary masses under consideration, the lengthscale over which waves dissipate, \( l_d \), is not likely to greatly exceed \( h \) since the waves are already marginally non-linear at launch. The ring width is \( \Delta r = a_2 - a_1 - 2w \). Just prior to the planets crossing a \( p:p + q \) resonance of order \(|q|\), where \( p \) and \( q \) are integers and \( p > p + q > 0 \), we have \( a_2 = \left[ \frac{p}{(p + q)} \right]^{2/3} a_1 \). Then we require

\[
\frac{\Delta r}{l_d} \approx 20 \left[ \left( \frac{p}{p + q} \right)^{2/3} - 1 - 0.3 \left( \frac{10^{-2}}{\alpha} \right)^{1/3} \left( \frac{r/h}{20} \right)^{2/3} \left( \frac{M/M_*}{10^{-3}} \right)^{2/3} \right] \frac{r/h}{20} \frac{h}{l_d} \gg 1. 
\] (3)

If \( p = 2 \) and \( q = -1 \), then for our choice of normalizations, \( \Delta r/l_d \approx 5 \). The ring would be barely wide enough for waves not to propagate across it, and then only for \( \alpha \gtrsim 10^{-2} \). For \( \alpha < 10^{-2} \), the gaps excavated by both planets would be so wide that no ring could exist. The simulations by Bryden et al. (2000; see their section 4) employ an effective \( \alpha \lesssim 10^{-2} \); it is therefore not surprising that their rings fail to be confined. It would be interesting to repeat their simulations with \( \alpha \gtrsim 10^{-2} \).

Our calculation is sufficiently imprecise—most notably with respect to the dissipation length, which we have taken to be \( l_d = h \)—that ring confinement may be impractical for the low-order (\(|q| = 1\)) resonant configurations that generate the largest eccentricity jumps. Even if \( w \ll r \), then \( \text{max}(\Delta r)/h = 10 \). Our uncertainty is probably best addressed by high-fidelity numerical simulations that can follow the evolution and dissipation of non-linear density waves.
2.2. Slow Crossing

For two planets to mutually excite their eccentricities by resonant interaction, the duration of passage through the resonance must be longer than the resonant libration period (see, e.g., Dermott, Malhotra, & Murray 1988):

\[
\frac{\Delta a_{\text{res}}}{|\dot{a}_2 - \dot{a}_1|} \gg T_l, \tag{4}
\]

where \(\Delta a_{\text{res}}\) is the width of the resonance and \(T_l\) is the resonant libration period. This requirement is tested numerically in §3.

With no important loss of generality, let us consider the migration of the inner planet through a resonance of order \(|q|\) and take the outer planet to be the perturber on a fixed orbit. To order of magnitude,

\[
T_l \sim \frac{2\pi}{\Omega_1} \sqrt{\frac{M_2}{M_1}} e_1^{-|q|/2}, \tag{5}
\]

\[
\Delta a_{\text{res}} \sim a_1 \sqrt{\frac{M_2}{M_*}} e_1^{1/2}. \tag{6}
\]

These expressions are appropriate for \(e \lesssim 0.3\) orbits in \(|q| > 1\) resonances, and \(0.3 \gtrsim e \gtrsim 0.1(M_2/M_*)^{1/3} \approx 0.01(M_2/M_J)^{1/3}\) orbits in the 2:1 resonance [Dermott et al. 1988; Murray & Dermott 1999; see the latter’s equations (8.58) and (8.76)]. Insertion of (5) and (6) into (4) yields

\[
\frac{a_1}{|\dot{a}_1|} \gg \frac{2\pi}{\Omega_1} \frac{M_*}{M_2} e_1^{-|q|/2}. \tag{7}
\]

Now it is usually remarked (see, e.g., Ward 1997 and GS) that for Type II drift, \(|a/\dot{a}| \sim a^2/\nu\), the viscous diffusion time of the disk. But this statement cannot be true in the limit that the planet mass greatly exceeds the ring mass. We generalize the Type II drift velocity by setting the viscous torque equal to the rate of change of angular momentum of either the planet or the ring, whichever is more massive:

\[
|\dot{a}_1| \sim \frac{6\pi \nu \Sigma r}{\max(M_1, 2\pi \Sigma r \Delta r)} \tag{8}
\]

This expression is a rough estimate of the instantaneous drift velocity; it can be used to estimate the timescale for the radial dimension of the ring-planet system to expand by
distances up to but not exceeding $\Delta r$. Ignoring differences between $a_1$ and $r$, we employ (8) to re-write our criterion (7) as

$$\max \left( \frac{M_1}{2\pi \Sigma r \Delta r}, 1 \right) \gg \frac{h/r}{0.05} \frac{h/\Delta r}{0.2} 10^{-2} \frac{\alpha |q|}{e_1} 10^{-3} \frac{M_2}{M_*}. \quad (9)$$

Thus, for our choice of normalizations, the planet mass must exceed the ring mass for the divergent migration to be sufficiently slow. Note in particular the dependence of our criterion (9) on the resonance order, $|q|$.

2.3. Fast Crossing

At the same time, the traversal of the resonance cannot occur so slowly that the disk’s direct influence on the planet’s eccentricity interferes with the resonant interaction between the two planets; in other words,

$$\frac{\Delta a_{\text{res}}}{|\dot{a}_2 - \dot{a}_1|} \ll \left( \frac{e}{\dot{e}} \right)_{\text{disk--induced}}. \quad (10)$$

That this inequality might fail to be true is a concern raised by Goldreich & Sari (2002); among the three requirements we discuss in this paper, it is the easiest one to satisfy.

We again restrict ourselves to analyzing the motion of the inner planet and take the outer planet to reside on a fixed orbit. Following GS, and ignoring differences between $a_1$ and $r$, we write

$$\min \left( \frac{e_1}{\dot{e}_1} \right)_{\text{disk--induced}} \sim \left( \frac{w}{r} \right)^4 \frac{M_*^2}{M_1 \Sigma r^2} \frac{1}{\Omega}. \quad (11)$$

This timescale is derived by relating either the total first-order Lindblad torque or the total first-order co-rotation torque to the conservation of the Jacobi constant of the planet. It is a minimum estimate because it considers either type of torque alone. If both types of torques operate, their effects tend to cancel and the timescale for eccentricity change would increase by a factor of $1/0.046 \sim 22$ (GS, and references therein). Conservatively proceeding with the minimum timescale, we combine equations (2), (6), (8), and (11) to re-write condition (10) as
\[
10^{-2} \left( \frac{\alpha}{10^{-2}} \right)^{1/3} \left( \frac{h/r}{0.05} \right)^{2/3} \left( \frac{M_2/M_\ast}{10^{-3}} \right)^{1/2} \left( \frac{M_1/M_\ast}{10^{-3}} \right)^{-2/3} \frac{|q|/2}{0.1} \ll 1, \quad (12)
\]

where we have taken \( M_1 \gg 2\pi \Sigma r \Delta r \), in accordance with requirement (9). Notice that there is no explicit dependence on the ring surface density; increasing \( \Sigma \) increases the rate of migration, but it also increases the rate of change of eccentricity by the same amount. Because inequality (12) is well satisfied, we conclude that each resonance crossing proceeds undistorted by the direct effects of the disk on the planet’s eccentricity. However, the disk’s direct effects would dominate while the planets migrate between resonances.

3. NUMERICAL EXPERIMENTS

We test relation (4) by numerical orbit integrations of two Jupiter-mass planets on divergent trajectories. Planet 1, of mass \( M_1 = 10^{-3} M_\odot \), is taken to occupy initially an orbit of semi-major axis 1 AU and osculating eccentricity \( e_1 = 0.01 \) about a 1 \( M_\odot \) star. Planet 2, of identical mass to planet 1, initially occupies a co-planar orbit of semi-major axis 1.5 AU and eccentricity \( e_2 = 0.01 \). The periastra of the two orbits are initially 0° apart. Planet 1 is placed at periastron, while planet 2 is located at apastron. Similar initial conditions were employed by CFT; this configuration, integrated forward by \( 2 \times 10^6 \) yr with no differential migration imposed, does not betray any instability; semi-major axes remain within 1% of their initial values, and osculating eccentricities average to 0.025 each.

We execute 31 integrations characterized by 31 differential migration timescales. For a given integration, following CFT, we impose a drag force \( \vec{F}_{\text{drag}} = -M_1 \vec{v}_1/t_{\text{drag}} \) on planet 1, where \( M_1 \) and \( \vec{v}_1 \) are the mass and instantaneous velocity of planet 1, and \( t_{\text{drag}} \) is the timescale over which \( a_1 \) decays. No drag force is applied to planet 2. The drag force is imposed at \( t_{\text{start}} = 500 \) yr and removed at \( t_{\text{stop}} = t_{\text{start}} + 0.2 \times t_{\text{drag}} \). This prescription reduces \( a_1 \) from 1 AU to 0.67 AU nearly linearly with time; thus, \( \dot{a}_1 = -1.65 \) AU/\( t_{\text{drag}} \). In each simulation, the two planets cross the 2:1 and 3:1 resonances, among numerous higher-order resonances. Inserting the above speed into inequality (4), and evaluating \( \Delta a_{\text{res}} \) and \( T_l \) using the analytic expressions given in Murray & Dermott [1999, see their equations (8.47) and (8.58)], we calculate that \( t_{\text{drag}} \gg 1 \times 10^4 \) yr for the 2:1 resonance to excite eccentricities effectively, assuming a pre-crossing eccentricity of 0.025. If the 2:1 resonance is effective, and eccentricities become excited to \( \sim 0.1 \), then we calculate that \( t_{\text{drag}} \gg 3 \times 10^4 \) yr for the 3:1 resonance to be similarly effective. How well do these order-of-magnitude criteria compare with numerical experiment?

Figure 2 displays the final eccentricities of planets 1 and 2 as a function of \( t_{\text{drag}} \). The
Fig. 2.— Final eccentricities of diverging Jovian-mass planets 1 and 2, as a function of differential migration timescale, $t_{\text{drag}}$. Solid circles refer to the inner planet 1, while open circles refer to the outer planet 2. Dotted lines mark the minimum migration timescales required for the 2:1 and 3:1 resonances to excite eccentricities effectively, as estimated using relation (4).
final eccentricities equal the average eccentricities from \( t = t_{\text{stop}} \) to \( t = t_{\text{stop}} + 10^4 \) yr; these final values represent averages over secular timescales. The final eccentricities increase from \( \sim 0.05 \) to \( \sim 0.25 \) as \( t_{\text{drag}} \) increases from \( 10^2 \) yr to \( 10^5 \) yr; for \( t_{\text{drag}} > 10^5 \) yr, the final eccentricities saturate, as we expect. We consider Figure 2 to largely verify relation (4). Requirement (4) is not as stringent as we might have guessed; even if \( t_{\text{drag}} = 3 \times 10^3 \) yr, modest eccentricities of 0.09–0.15 can still be excited.

Figure 3 displays sample time evolutions of \( e_1 \) and \( e_2 \) for various choices of \( t_{\text{drag}} \). Jumps in eccentricity that occur upon crossing mean-motion resonances, most having \( |q| > 1 \), are evident for \( t_{\text{drag}} > 10^4 \) yr. Eccentricity excitation is suppressed for shorter migration timescales. For \( t_{\text{drag}} > 10^5 \) yr, the final eccentricities are insensitive to \( t_{\text{drag}} \). Figure 4 identifies those mean-motion resonances that are responsible for the eccentricity jumps.

4. PROSPECTS FOR PLANETARY RINGS AND SHEPHERD MOONS

Narrow planetary rings shepherded by satellites potentially furnish a miniature stage upon which the processes of divergent migration and repeated resonance crossings unfold. We take the \( \epsilon \) ring of Uranus (Elliot & Nicholson 1984; Goldreich & Porco 1987; Porco & Goldreich 1987; Chiang & Goldreich 2000; Mosqueira & Estrada 2002) as our showcase example. To what extent does the \( \epsilon \) ring satisfy the 3 requirements above?

4.1. Ring Shepherding

Ring confinement is manifestly occurring. The dissipation length, \( l_d \), for the \( \epsilon \) ring whose optical depth is of order unity is on the order of a scale height, \( h \). Near the ring boundary, we evaluate \( l_d \sim h \sim c_s/\Omega \sim (1 \text{ cm/s})/(2 \times 10^{-4} \text{ rad/s}) \sim 50 \) m (see, e.g., Chiang & Goldreich 2000, and references therein). The ring spans a width \( \Delta r \sim 60 \) km; thus \( \Delta r/l_d \sim 10^3 \).

4.2. Slow Crossing

The \( \epsilon \) ring is incredibly thin compared to its protoplanetary disk counterpart; in the ring interior, \( h \sim 5 \) m (Chiang & Goldreich 2000), while \( r \sim 5 \times 10^4 \) km.\(^1\) Moreover, the source of

\(^1\)Velocity dispersions are \( \sim 10 \times \) greater near the ring boundary than in the ring interior due to resonant perturbations by shepherd satellites (Borderies, Goldreich, & Tremaine 1982; Chiang & Goldreich 2000).
Fig. 3.— Eccentricity evolution of diverging Jovian-mass planets. The simulation in each panel is characterized by a differential migration timescale, $t_{\text{drag}}$, displayed in the upper right-hand corner. Solid lines correspond to $e_1$, while dotted lines correspond to $e_2$. Jumps in eccentricity are most pronounced for $t_{\text{drag}} \geq 10^4$ yr. These jumps are due to the crossing of numerous mean-motion resonances, most of high-order; these resonances are identified in Figure 4. Secular oscillations in eccentricity are evident in those panels for which $t_{\text{drag}} \geq 10^5$ yr.
Fig. 4.— Evolution of (a) the eccentricity of the inner migrating planet, and (b) the ratio of orbital periods of the outer planet to that of the inner planet, for $t_{\text{drag}} = 10^6$ yr. Crossings of various mean-motion resonances are indicated. Eccentricity jumps are most marked for the 2:1, 7:3, 5:2, 8:3, 3:1, 4:1, and 5:1 resonances.
viscosity is relatively well understood in rings (but see §4.4); interparticle collisions for a ring of optical depth unity provide $\alpha \sim 1$ (Goldreich & Tremaine 1978). The $\epsilon$ ring shepherds, Cordelia and Ophelia, have estimated masses of $M_1 \sim 5 \times 10^{19}$ g and $M_2 \sim 9 \times 10^{19}$ g, respectively, and orbital eccentricities of $e_1 = 0.47(\pm 0.41) \times 10^{-3}$, and $e_2 = 10.1(\pm 0.4) \times 10^{-3}$, respectively (Porco & Goldreich 1987). We assemble these data to re-normalize condition (9) as

$$\max \left( \frac{M_1}{2\pi \Sigma r \Delta r}, 1 \right) \gg 0.2 \cdot \frac{h/r}{10^{-7}} \cdot \frac{h/\Delta r}{8 \times 10^{-5}} \cdot \frac{10^{-3} \alpha}{e_1^{[\eta]} / 10^{-9}} \cdot \frac{10^{-9} M_2 / M_U}{10^{-9}}. \quad (13)$$

It is a remarkable coincidence that the numerical coefficient on the right-hand side of equation (13) remains the same as that for (9) despite the vast difference in scales between planetary rings and protoplanetary disks. The mass of the $\epsilon$ ring is estimated to be $2 \times 10^{19}$ g (Chiang & Goldreich 2000). Then the left-hand side of (13) equals $\max(3, 1) = 3$. Thus, the requirement of slow crossings is marginally satisfied.

### 4.3. Fast Crossing

We use the properties of the $\epsilon$ ring cited above to re-normalize (12) as

$$2 \times 10^{-5} \left( \frac{\alpha}{1} \right)^{1/3} \left( \frac{h/r}{10^{-7}} \right)^{2/3} \left( \frac{M_2 / M_U}{10^{-9}} \right)^{1/2} \left( \frac{M_1 / M_U}{10^{-9}} \right)^{-2/3} \frac{e_1^{[\eta]} / 2}{0.03} \ll 1. \quad (14)$$

Thus, eccentricity changes directly induced by the ring during resonance passage can be ignored.

### 4.4. Observational Signatures

Have resonance crossings left their imprint on Ophelia and Cordelia? The present ratio of orbital semi-major axes of the two moons is $(a_2/a_1)_0 = 1.08065 \pm 0.00004$ (Porco & Goldreich 1987). The nearest first-order resonance that lies at a smaller ratio of semi-major axes is the 10:9 resonance: $(a_2/a_1)_{10:9} = 1.07277 \pm 10^{-6}$, where the uncertainty measures

\[ \S4.2 \text{ and } \S4.4, \text{ we evaluate } h \text{ in the ring interior because the ring’s intrinsic flux of angular momentum due to interparticle collisions can be assessed there relatively undistorted by torques exerted by satellites, while in } \S4.1, \text{ we take } h \text{ near the ring boundary because satellite-excited waves damp in that vicinity.} \]
the width of the resonance as roughly estimated by equation (6). To have crossed the 10:9 resonance, the satellite orbits must have expanded by \( \sim [(a_2/a_1)_{10:9} - (a_2/a_1)_{10:9}] r \sim 400 \text{ km} \). Since this distance exceeds the radial width of the \( \epsilon \) ring (\( \Delta r \approx 60 \text{ km} \)), which we take to be the maximum length by which the ring-satellite system has expanded during its lifetime (but see also the last paragraph of this section), we conclude that the moons could not have crossed any first-order resonance in the past. This conclusion is consistent with Ophelia, the more massive of the pair, currently possessing the greater orbital eccentricity of the two; repeated resonance crossings would predict the opposite to be true. In our view, Ophelia’s aberrantly large eccentricity is primordial.

What about future crossings? The ratio of \( (a_2/a_1)_0 \) sits closest to, but does not overlap with, the 9:8 resonance: \( (a_2/a_1)_{9:8} = 1.08169 \pm 10^{-6} \). The satellite orbits must diverge by another \( \delta a_{9:8} \sim [(a_2/a_1)_{9:8} - (a_2/a_1)_0] r \sim 39 \text{ km} \) before resonance encounter.\(^2\) The event will be long in the waiting; the time of encounter lies \( \delta t_{9:8} = \delta a_{9:8}/\dot{a}_1 \sim 2 \times 10^4 \text{ yr} \) in the future.\(^3\) Using relations derived by Dermott, Malhotra, & Murray (1988; see their Appendix B), we estimate the perturbation eccentricity after resonance crossing to be on the order of \( 6 \times 10^{-4} \) (= \( e_{\text{crit}} \) as defined by Dermott et al. 1988). This will be a minor perturbation for Ophelia but will be relatively significant for Cordelia.

What about the \( \epsilon \) ring’s direct effects on satellite eccentricities? Both first-order co-rotation torques and first-order Lindblad torques operate simultaneously in the \( \epsilon \) ring system, resulting in eccentricity damping on a timescale that is \( 22 \times \) greater than that given by equation (11) (GS; Goldreich & Tremaine 1980). We estimate an exponential decay time for the satellite eccentricity of \( \sim 3.6 \times 10^7 \text{ yr} \), much longer than either \( \delta t_{9:8} \) or the likely age of the ring.\(^4\)

We note in passing that our expressions imply that the \( \epsilon \) ring is a young creation of the solar system. For the ring width to grow from \( \Delta r/2 \) to its current width of \( \Delta r \) would require of order \( \delta t_{\Delta r} \sim \delta t_{9:8} \Delta r/\delta a_{9:8} \sim 3 \times 10^4 \text{ yr} \). If the viscous flux of angular momentum across the ring midline were higher in the past, we expect \( \delta t_{\Delta r} \) to approximate well the age of the ring. Even if the viscous flux were to remain constant at all stages of ring evolution—as might be expected if \( \Sigma \nu \propto \Sigma/\tau \) were conserved, where we have assumed that the ring’s vertical optical

\(^2\)The 9:8 resonance splits into two sub-resonances, the so-called \( e \) and \( e' \) resonances (see Murray & Dermott 1999). The separation between them is 1 km and is due almost entirely to the oblateness of Uranus. The widths of these sub-resonances do not overlap.

\(^3\)Tides raised by the shepherds on the planet also engender divergent migration, but over timescales of order \( 10^{10} \text{ yr} \).

\(^4\)Tides raised by the planet on the shepherds damp their orbital eccentricities over \( \sim 8 \times 10^7 \text{ yr} \).
depth $\tau \gg 1$ in the ring’s past—then we should multiply $\delta t_{\Delta r}$ by a logarithm that is unlikely to increase our estimate of the ring age by more than an order of magnitude. Goldreich & Porco (1987) estimate an upper limit to the ring age by dividing the present-day viscous torque into the angular momentum required to be transferred from the inner shepherd to the outer shepherd to separate their orbital semi-major axes by their current difference of $\sim 4000$ km. Using our estimate for the viscous torque, we evaluate their upper limit to be $\sim 1 \times 10^6$ yr. We consider this upper limit to be a gross one, since it assumes the satellite separation to be originally much smaller than its current value. In our view, a closer approximation to the ring age is obtained by assuming the ring width to be originally much smaller than its current value and the ring edges to occupy the same first-order resonances with the shepherds as they do now, in which case the initial satellite separation is within $\sim 98\%$ of its current value. Additional uncertainty is introduced by the magnitude of the ring viscosity. Our above expressions employ the viscosity appropriate to a dilute gas (Goldreich & Tremaine 1978) of sound speed $c_s \sim 0.1$ cm/s and turbulence parameter $\alpha \approx \tau/(\tau^2 + 1) \approx 1$. Then $\nu = \alpha c_s^2/\Omega \approx 50$ cm$^2$/s. If ring particles are instead closely packed and behave more like an incompressible liquid, then the viscosity attains its minimum value of $\Omega(\Sigma/\rho)^2 \approx 0.3$ cm$^2$/s (Borderies, Goldreich, & Tremaine 1985), where $\rho \approx 2$ g/cm$^3$ is the internal density of a ring particle. If this minimum viscosity obtains, then our estimate of the ring age, $\sim \delta t_{\Delta r}$, and the upper limit on the ring age obtained by Goldreich & Porco (1987), would need to be revised upwards to $5 \times 10^6$ yr and $2 \times 10^8$ yr, respectively. We close by noting that (1) none of the four timescales we have computed, ranging from $3 \times 10^4$ yr to $2 \times 10^8$ yr, is as long as the age of the solar system, $4 \times 10^9$ yr, and (2) none of our conclusions in §4.2–§4.3 would change if the minimum viscosity were to obtain and actual diffusion timescales were $\sim 10^2$ times longer than what we have assumed in those sections, since requirement (13) would be only better satisfied and requirement (14) would remain satisfied.

5. SUMMARY AND DISCUSSION

We have derived three requirements for the mutual excitation of orbital eccentricities of two secondary bodies by repeated resonance crossings within a circumprimary disk, a process originally proposed by Chiang, Fischer, & Thommes (2002).

1. For the orbits of two bodies to diverge, a ring of viscous material must be shepherded between them. This requires that density waves excited within the ring by each body damp near the edge where they are launched.

2. The divergent migration must be slow enough that the timescale for traversal of the resonant width exceeds the resonant libration period. In the context of Jovian-mass
planets within a gaseous protoplanetary disk, this requires that the ring mass be small compared to the planet mass so that so-called “Type II” migration occurs over a timescale longer than the viscous spreading time of the ring.

3. At the same time, the divergent migration must not be so slow that the disk’s direct effects on the bodies’ eccentricities dominate during resonance passage. This criterion appears well satisfied for a variety of parameters because resonant widths are typically small fractions of the orbital radius and the duration of each passage is consequently short.

These necessary conditions are embodied in equations (3), (9), and (12), respectively, with normalizations appropriate to Jovian-mass planets in protoplanetary disks.

The $\epsilon$ ring of Uranus and its attendant shepherds, Cordelia and Ophelia, satisfy these requirements. Unfortunately, these moons probably have yet to undergo their first resonance crossing, so that we cannot look to them for a signature of the process. In $2 \times 10^4 - 3 \times 10^6$ yr, we expect the radial diffusion of the $\epsilon$ ring to wedge Cordelia and Ophelia through the 9:8 resonance, after which Cordelia’s eccentricity is likely to increase by $\sim 10^{-3}$. Ophelia’s eccentricity will probably remain at its current high and likely primordial value of $10^{-2}$. The range in timescales reflects our uncertainty in the magnitude of the ring viscosity. The Cassini spacecraft will probe analogs of the $\epsilon$ ring around Saturn (e.g., the Maxwell ringlet). We look forward to sifting the abundance of new observations of narrow ringlets around Saturn for signs of divergent resonance crossings.

In the context of giant planets embedded within protoplanetary disks, a shepherded ring of gas in the vicinity of $r \sim 1$ AU and having a surface density of $\Sigma \lesssim 500$ g/cm$^2$, a dimensionless viscosity parameter of $\alpha \sim 0.1$, and an aspect ratio of $h/r \approx 0.05$ can drive two Jovian-mass planets through the 2:1 and higher-order resonances so that their eccentricities magnify dramatically. The above-cited upper limit to the surface density is a factor of $\sim 3$ lower than that of the minimum-mass solar nebula; repeated resonance crossings, in the context of the orbital evolution of giant planets, is a process that is therefore restricted to the later stages of the evolution of the protoplanetary disk (ages $t \gtrsim 10^6$ yr).

Future work on this subject should incorporate numerical simulations to verify that gaseous rings having the parameters that we have outlined can indeed be shepherded by giant planets and drive divergent migration. Perhaps our chief uncertainty lies in the exact manner by which density waves excited by planets having masses $M > \text{max } M_{\text{gap}}$ dissipate. A potential outcome might be that such waves, launched at one ring edge, release a great deal of their angular momentum at the far ring edge, rendering ring confinement impossible for low-order resonant configurations such as the 2:1. It would be worthwhile to measure the
extent of eccentricity excitation engendered by the crossings of only high-order resonances, for which the requirements of ring shepherding are less severe. Finally, divergent migration within particle disks (see, e.g., Fernandez & Ip 1984; Murray et al. 1998) should also be explored, particularly in light of our finding that divergent resonance crossings are most effective when the ring mass is small compared to the planet mass.

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