We evaluate some new three-parameter families of finite reciprocal sums involving Horadam numbers. We are also able to state the results for the associated infinite sums. Some Fibonacci and Lucas sums are presented as examples.

1. Introduction and Motivation

The Horadam sequence $(w_n)_{n \geq 0} = (w_n(a, b; p, q))_{n \geq 0}$ is recursively defined as

$$w_0 = a, \quad w_1 = b, \quad w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2,$$

where $a$, $b$, $p$, and $q$ are arbitrary (possibly complex) numbers [13]. The sequences

$$(u_n(p, q)) = (w_n(0, 1; p, q)) \quad \text{and} \quad (v_n(p, q)) = (w_n(2, p; p, q))$$

are the Lucas sequences of the first and second kind, respectively. The most well-known Lucas sequences are the Fibonacci numbers $F_n = u_n(1, -1)$, the Lucas numbers $L_n = v_n(1, -1)$, the Pell numbers $P_n = u_n(2, -1)$, the Pell–Lucas numbers $Q_n = v_n(2, -1)$, and the balancing numbers $B_n = u_n(6, 1)$. All sequences are indexed in the On-Line Encyclopedia of Integer Sequences [28].

Denote by $\alpha$ and $\beta$, with $|\alpha| > |\beta|$, the distinct roots of the characteristic equation $x^2 - px + q = 0$ with the following discriminant: $\Delta = p^2 - 4q \neq 0$.

The Binet formulas for $w_n$, $u_n$, and $v_n$, where $n$ is a nonnegative integer, are given by

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \quad u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad v_n = \alpha^n + \beta^n,$$

where $A = b - a\beta$ and $B = b - a\alpha$.

We also need an expression for negatively subscripted Horadam numbers. Thus, for negative subscripts the sequences are given by

$$w_{-n} = \frac{av_n - w_n}{q^n}, \quad u_{-n} = -u_n q^{-n}, \quad v_{-n} = v_n q^{-n}.$$
We require the following identity \[15, \text{formula (2.12)}\] in the sequel:

\[w_n w_{n+r+s} - w_{n+r} w_{n+s} = e_w q^n u_r u_s,\]  

(3)

where \(e_w = -AB = pab - qa^2 - b^2\) and \(n, r, \text{ and } s\) are integers. For the Fibonacci numbers, \(e_F = -1\), while for the Lucas numbers, \(e_L = 5\).

The aim of the present study is to evaluate a family of finite and infinite reciprocal sums involving the Horadam sequence. The interest in evaluating Fibonacci and Lucas (related) reciprocal sums in the closed form is not new. The topic has challenged the mathematical community for decades. Thus, in 1974, Miller [25] proposed the problem of proving that

\[
\sum_{i=0}^{\infty} \frac{1}{F_{2i}} = \frac{7 - \sqrt{5}}{2},
\]

(4)

Miller’s proposal stimulated a great interest in the series of reciprocal Fibonacci numbers, which led to many proofs and generalizations (see the survey paper [5] for more information and references). Note that, in [25], the author’s name of the problem is indicated incorrectly as Millin (see the editorial note in [29, p. 92]).

In 1974, Good [10] showed that

\[
\sum_{i=0}^{N} \frac{1}{F_{2i}} = 3 - \frac{F_{2N-1}}{F_{2N}}.
\]

Allowing \(N\) to approach infinity, we get (4). Hoggatt and Bicknell [11] gave 11 methods for finding the value of sum (4). Shortly later, in [12], they proved a more general formula

\[
\sum_{i=0}^{\infty} \frac{1}{F_{k2i}} = \frac{1}{F_k} + \frac{\Phi^2 + 1}{\Phi(\Phi^{2k} + 1)},
\]

where \(\Phi = \frac{1 + \sqrt{5}}{2}\) is the golden ratio. In 1990, André-Jeannin [2, Theorem 2] expressed the infinite reciprocal series

\[
\sum_{i=1}^{\infty} \frac{1}{u_{k(i+1)}} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{v_{k(i+1)}},
\]

with an odd positive integer \(k\) in terms of the Lambert series

\[
\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}, \quad |x| < 1.
\]

Melham [24] considered analogs of the sequences \(u_n\) and \(v_n\) for the recurrence \(w_n = pw_{n-1} - w_{n-2}\) and obtained analogs of André-Jeannin’s results for these sequences. In 1997, André-Jeannin [3, Theorem 2′] again studied the reciprocals of second-order recurrences and evaluated the series

\[
\sum_{i=1}^{\infty} \frac{q^m}{w_{mi+n} w_{m(i+k)+n}} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{w_{mi+n} w_{m(i+k)+n}}
\]
for integers \( n \geq 0, \ m \geq 1, \) and \( k \geq 1 \). Some years later, Hu et al. [16, Theorem 1] obtained a general result, which contained the evaluation of the finite (and infinite) series

\[
\sum_{i=1}^{N-1} \frac{q^{mi}}{w_{mi+n}w_{m(i+1)+n}},
\]

where \( n, \ m, \) and \( N \geq 2 \) are integers, as a special case. Laohakosol and Kuhapatanakul [19] extended this result to reciprocal sums of the second-order recurrence sequences with nonconstant coefficients.

In [1], the first author obtained a series of closed-form expressions for finite and infinite Fibonacci–Lucas sums with products of Fibonacci or Lucas numbers in the denominator of the summand. His results generalized numerous identities, such as the identities from [4, 27].

More types of Fibonacci and Lucas (related) reciprocal series, both finite or infinite and alternating or non-alternating, were studied in [6, 8, 9, 14, 22, 23, 26]. For studies focused on reciprocal sums with three and more factors, we refer the reader to [7, 18, 20, 21].

The series studied in the present paper are three-parameter series of the form

\[
\sum_{i=1}^{N} \frac{q^{m(i-k)}}{w_{m(i-k)+n}w_{m(i+k)+n}} \quad \text{and} \quad \sum_{i=1}^{N} \frac{q^{m(2i-k)}}{w_{m(2i-k)}w_{m(2i+k)}},
\]

where \( m, \ k, \) and \( n \) are integers. To the best of our knowledge, these types of Horadam reciprocal series have not been considered in the available literature. For all series, we provide closed forms in the finite and infinite cases by using an elementary approach.

We require the following telescoping summation identities with any integers \( N \) and \( t \):

\[
\sum_{i=1}^{N} (f(i + t) - f(i)) = \sum_{i=1}^{t} (f(i + N) - f(i)) \quad (5)
\]

and

\[
\sum_{i=1}^{2N} (\pm 1)^i(f(i + 2t) - f(i)) = \sum_{i=1}^{2t} (\pm 1)^i(f(i + 2N) - f(i)). \quad (6)
\]

Telescoping identities are often used to find the sums of finite and infinite series of Fibonacci and Lucas numbers in the closed form [1, 6, 17, 23, 30].

2. New Families of Reciprocal Horadam Series

Our first main result is the following statement:

**Theorem 1.** Let \( m, \ k, \) and \( n \) be integers and let \( N \) be a natural number. Then

\[
\sum_{i=1}^{N} \frac{q^{m(i-k)}}{w_{m(i-k)+n}w_{m(i+k)+n}} = \frac{1}{e_{wm}u_{2km}} \sum_{i=1}^{2k} \left( \frac{w_{m(i-k)}}{w_{m(i-k)+n}} - \frac{w_{m(i+N-k)}}{w_{m(i+N-k)+n}} \right) \quad (7)
\]
or, equivalently,

\[ u_{2km} \sum_{i=1}^{N} q_{mi}^{r/m} = u_{mN} \sum_{i=1}^{2k} q_{mi}^{r/m}. \]

**Proof.** Writing \( n - r \) for \( n \) in identity (3), we find

\[ w_{n-r}w_{n+s} - w_{n}w_{n-r+s} = e_{w}q^{n-r}u_{r}u_{s}. \]

Therefore, writing \( mi - km \) for \( n \), \( 2km \) for \( s \), and \(-n\) for \( r \), we get

\[ w_{m(i-k)+n}w_{m(i+k)} - w_{m(i-k)}w_{m(i+k)+n} = e_{w}q^{m(i-k)+n}u_{-n}u_{2km} = -e_{w}q^{m(i-k)}u_{n}u_{2km}, \]

where, in the last step, we have used (2).

We now divide identity (8) by \( w_{m(i-k)+n}w_{m(i+k)+n} \) to obtain

\[ \frac{q^{m(i-k)}}{w_{m(i-k)+n}w_{m(i+k)+n}} = \frac{1}{e_{w}u_{m}u_{2km}} \left( \frac{w_{m(i-k)}}{w_{m(i-k)+n}} - \frac{w_{m(i+k)}}{w_{m(i+k)+n}} \right). \]

We identify

\[ f(i) = \frac{w_{m(i-k)}}{w_{m(i-k)+n}} \quad \text{and} \quad t = 2k \]

and use these equalities in the summation formula (5) with regard for (9).

The theorem is proved.

In particular, evaluating (7) for \( k = 1 \) and \( k = 2 \), we obtain

\[ \sum_{i=1}^{N} q_{mi}^{r/m} \frac{u_{m(i-1)+n}u_{m(i+1)+n}}{w_{m(i-1)+n}w_{m(i+1)+n}} = \frac{q^{m}}{e_{w}u_{m}u_{2m}} \left( \frac{w_{0}}{w_{n}} + \frac{w_{m}}{w_{m+n}} - \frac{w_{m(N+1)}}{w_{m(N+1)+n}} - \frac{w_{mN}}{w_{mN+n}} \right) \]

and

\[ \sum_{i=1}^{N} q_{mi}^{r/m} \frac{u_{m(i-2)+n}u_{m(i+2)+n}}{w_{m(i-2)+n}w_{m(i+2)+n}} = \frac{q^{2m}}{e_{w}u_{m}u_{4m}} \left( \frac{w_{-m}}{w_{-m+n}} + \frac{w_{0}}{w_{n}} + \frac{w_{m}}{w_{m+n}} + \frac{w_{2m}}{w_{2m+n}} \right) \]

\[ \quad \quad \quad \quad \quad - \frac{w_{m(N-1)}}{w_{m(N-1)+n}} - \frac{w_{mN}}{w_{mN+n}} - \frac{w_{m(N+1)}}{w_{m(N+1)+n}} - \frac{w_{m(N+2)}}{w_{m(N+2)+n}}. \]

Setting \( n = mk \) in Theorem 1, we get the following corollary:

**Corollary 1.** For integers \( m \) and \( k \) and a natural number \( N \), the following equality is true:

\[ \sum_{i=1}^{N} q_{mi}^{m(i-k)} = \frac{1}{e_{w}u_{mk}u_{2mk}} \sum_{i=1}^{2k} \left( \frac{w_{m(i-k)}}{w_{mi}} - \frac{w_{m(i+N-k)}}{w_{m(i+N)}} \right) \]
or, equivalently,

\[ u_{2mk} \sum_{i=1}^{N} \frac{q^{mi}}{u_{mi}u_{m(i+2k)}} = u_{mN} \sum_{i=1}^{2k} \frac{q^{mi}}{u_{mi}u_{m(i+N)}}. \]

The associated infinite series are evaluated in the next corollary.

**Corollary 2.** Let \( m, k, \) and \( n \) be integers. Then

\[
\sum_{i=1}^{\infty} \frac{q^{m(i-k)}}{w_{m(i-k)+n}w_{m(i+k)+n}} = \frac{1}{e^{\alpha^m u_{2km}}} \left( \sum_{i=1}^{2k} \frac{w_{m(i-k)+n}}{w_{mi}} - \frac{2k}{\alpha^n} \right)
\]

and, especially with \( n = mk, \)

\[
\sum_{i=1}^{\infty} \frac{q^{m(i-k)}}{w_{m(i+2k)}} = \frac{1}{e^{\alpha^m u_{2km}}} \left( \sum_{i=1}^{2k} \frac{w_{m(i-k)} - 2k}{\alpha^{mn}} \right).
\]

**Proof.** According to (1),

\[
\lim_{N \to \infty} \frac{w_N}{w_{N+r}} = \frac{1}{\alpha^r}.
\]

Taking limits as \( N \to \infty \) on both sides of identity (7) and using (10), we complete the proof.

As special cases of our results obtained so far, we have new Fibonacci and Lucas identities.

**Corollary 3.** For integers \( m \) and \( n, \) and a natural number \( N, \)

\[
\sum_{i=1}^{N} \frac{(-1)^m(i-1)}{F_{m(i-1)+n}F_{m(i+1)+n}} = \frac{1}{5F_nF_{2m}} \left( \frac{F_{m(N+1)}}{F_{m(N+1)+1}} + \frac{F_{mN}}{F_{mN+1}} - \frac{F_m}{F_{m+n}} \right),
\]

\[
\sum_{i=1}^{N} \frac{(-1)^m(i-1)}{L_{m(i-1)+n}L_{m(i+1)+n}} = \frac{1}{5F_nF_{2m}} \left( \frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{L_{mN}}{L_{mN+n}} - \frac{L_{m(N+1)}}{L_{m(N+1)+n}} \right),
\]

\[
\sum_{i=1}^{\infty} \frac{(-1)^m(i-1)}{F_{m(i-1)+n}F_{m(i+1)+n}} = \frac{1}{F_nF_{2m}} \left( \frac{2}{\Phi n} - \frac{F_m}{F_{m+n}} \right),
\]

\[
\sum_{i=1}^{\infty} \frac{(-1)^m(i-1)}{L_{m(i-1)+n}L_{m(i+1)+n}} = \frac{1}{5F_nF_{2m}} \left( \frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{2}{\Phi n} \right),
\]

where \( \Phi = \frac{1 + \sqrt{5}}{2}. \)

**Proof.** We use Theorem 1 and Corollary 2 with \( w_n = F_n \) and \( w_n = L_n, \) respectively, and \( k = 1. \) Recall that \( e^F = -1 \) and \( e^L = 5. \)

The corollary is proved.
We now mention that equations (11)–(14) have been recently discovered by the second author and appeared in [8, Theorem 1.2].

Note that it is clear from equation (7) that it does not hold for \(m, n, k = 0\). The next theorem addresses the case of \(n = 0\).

**Theorem 2.** Let \(m\) and \(k\) be integers and let \(N\) be a natural number. Then

\[
\sum_{i=1}^{N} q^{m(i-k)} w_{m(i-k)} w_{m(i+k)} = \frac{1}{e_w u_{2km}} \sum_{i=1}^{2k} \left( \frac{w_m(i+N-k)+1}{w_m(i-k)} - \frac{w_m(i-k)+1}{w_m(i-k)} \right)
\]

or, equivalently,

\[
u_{2km} \sum_{i=1}^{N} \frac{q^m}{w_{m(i-k)} w_{m(i+k)}} = u_{mN} \sum_{i=1}^{2k} \frac{q^m}{w_{m(i-k)} w_{m(i+N-k)}}.
\]

**Proof.** We divide identity (8) by \(w_{m(i-k)} w_{m(i+k)}\) to obtain

\[
e_w u_{2km} u_n \frac{q^m(i-k)}{w_{m(i-k)} w_{m(i+k)}} = \frac{w_m(i+k)+n}{w_m(i+k)} - \frac{w_m(i-k)+n}{w_m(i-k)},
\]

where \(n\) is now arbitrary and can be set equal to 1. This yields

\[
e_w u_{2km} q^{m(i-k)} w_{m(i-k)} w_{m(i+k)} = \frac{w_m(i+k)+1}{w_m(i+k)} - \frac{w_m(i-k)+1}{w_m(i-k)}.
\]

(16)

Thus, the required result now follows as a result of summation over \(i\) by using (5) with

\[f(i) = \frac{w_{m(i-k)}+1}{w_m(i-k)}.
\]

The theorem is proved.

Letting \(N \to +\infty\) in (15), we arrive at the following corollary:

**Corollary 4.** Let \(m\) and \(k\) be integers. Then

\[
\sum_{i=1}^{\infty} q^{m(i-k)} w_{m(i-k)} w_{m(i+k)} = \frac{1}{e_w u_{2km}} \left( 2k \alpha - \sum_{i=1}^{2k} \frac{w_m(i-k)+1}{w_m(i-k)} \right).
\]

Working with the Lucas numbers and taking \(k = 1\), we immediately get the following results:

\[
\sum_{i=1}^{N} \frac{(-1)^{m(i-1)}}{L_m(i-1)L_m(i+1)} = \frac{1}{5F_{2m}} \left( \frac{L_m(N+1)+1}{L_m(N+1)} + \frac{L_mN+1}{L_mN} - \frac{L_m+1}{L_m} - \frac{1}{2} \right)
\]

and

\[
\sum_{i=1}^{\infty} \frac{(-1)^{m(i-1)}}{L_m(i-1)L_m(i+1)} = \frac{1}{5F_{2m}} \left( 2\Phi - \frac{L_m+1}{L_m} - \frac{1}{2} \right).
\]
The above Lucas sums were also evaluated in [8]. However, these results are formulated in a different form as follows:

\[
\sum_{i=1}^{N} \frac{(-1)^{m(i-1)}}{L_{m(i-1)}L_{m(i+1)}} = \frac{1}{2F_{2m}} \left( \frac{F_{m(N+1)}}{L_{m(N+1)}} + \frac{F_{mN}}{L_{mN}} - \frac{F_{m}}{L_{m}} \right),
\]

\[
\sum_{i=1}^{\infty} \frac{(-1)^{m(i-1)}}{L_{m(i-1)}L_{m(i+1)}} = \frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2L_{m}^2}.
\]

The reason for the differences in expressing these sums is that the special case of Theorem 2 with \( w_n = v_n \) has a different expression.

As the family of series involving the terms of Lucas sequences is of independent interest, we present an expression containing the terms of Lucas sequences of each kind as a separate theorem.

**Theorem 3.** For integers \( m \) and \( k \), the following identities are true:

\[
\sum_{i=1}^{N} \frac{q^{m(i-k)}}{u_{m(i-k)}v_{m(i+k)}} = \frac{1}{2u_{2km}} \sum_{i=1}^{2k} \left( \frac{u_{m(i+N-k)}}{v_{m(i+N-k)}} - \frac{u_{m(i-k)}}{v_{m(i-k)}} \right)
\]

\[
= \frac{u_{mN}}{u_{2km}} \sum_{i=1}^{2k} \frac{q^{m(i-k)}}{v_{m(i-k)}v_{m(i+N-k)}},
\]

and

\[
\sum_{i=1}^{N} \frac{q^{mi}}{u_{mi}u_{m(i+2k)}} = \frac{1}{2u_{2km}} \sum_{i=1}^{2k} \left( \frac{v_{mi}}{u_{mi}} - \frac{v_{m(i+N)}}{u_{m(i+N)}} \right)
\]

\[
= \frac{u_{mN}}{u_{2km}} \sum_{i=1}^{2k} \frac{q^{mi}}{u_{m2}u_{m(i+N)}},
\]

**Proof.** The proof is similar to the proof of Theorem 1. Here, we use the equality

\[
u_{m(i+k)}u_{m(i-k)} - u_{m(i-k)}v_{m(i+k)} = 2q^{m(i-k)}u_{2mk},
\]

which is obtained by setting \( s = m(i + k) \) and \( t = m(i - k) \) in the identity

\[u_{s}v_{t} - u_{s}u_{t} = -2q^{s}u_{t-s}.
\]

Dividing identity (19) by \( v_{m(i-k)}v_{m(i+k)} \), we get

\[
\frac{2u_{2km}q^{m(i-k)}}{v_{m(i-k)}v_{m(i+k)}} = \frac{u_{m(i+k)}}{v_{m(i+k)}} - \frac{u_{m(i-k)}}{v_{m(i-k)}},
\]

(20)
while dividing the same identity (19) by \( u_{m(i-k)}u_{m(i+k)} \) and shifting the index \( i \), we find

\[
\frac{2u_{2km}q^{mi}}{u_{mi}u_{m(i+2k)}} = \frac{v_{mi}}{u_{mi}} - \frac{v_{m(i+2k)}}{u_{m(i+2k)}}. \tag{21}
\]

Identities (17) and (18) now follow as a result of summation, over \( i \), of both sides of each equality (20) and (21) and noting that the sum on the right-hand side, in each case, telescopes according to the telescoping summation formula (6).

As a by-product, from Theorems 2 and 3, we obtain the following relation involving Lucas sequences of the first and second kind:

\[
2^k X_{i=1}^{2k} \left( \frac{u_{m(i+N-k)} - u_{m(i-k)}}{v_{m(i+N-k)} - v_{m(i-k)}} \right) = \frac{q_{mN}}{u_{2km}u_{m(i+N)}}.
\]

Similarly, comparing the second part of Theorem 3 with Corollary 1 \((w_n = u_n)\), we get the following interesting relation:

\[
\sum_{i=1}^{2k} \left( \frac{v_{mi} - v_{m(i+N)}}{u_{mi} - u_{m(i+N)}} \right) = 2^k \Delta \sum_{i=1}^{2k} \left( \frac{u_{m(i+N-k)+1} - u_{m(i-k)}}{u_{m(i+N)} - u_{mi}} \right).
\]

We conclude this section with the observation that the identity in Theorem 2 is, in general, violated for sequences with \( w_0 = a = 0 \), such as the Lucas sequence of the first kind. We now give a nonsingular version of the theorem. The proof is similar to the proof of Theorem 1 and, hence, is omitted.

**Theorem 4.** Let \( m \) and \( k \) be integers and let \( N \) be a natural number. Then

\[
\sum_{i=1}^{N} \frac{q^{mi}}{w_{mi}w_{m(i+2k)}} = \frac{1}{e_w u_{2km}} \sum_{i=1}^{2k} \left( \frac{w_{m(i+N)} + 1}{w_{m(i+N)}} - \frac{w_{mi} + 1}{w_{mi}} \right) = \frac{u_{mN}}{u_{2km}} \sum_{i=1}^{2k} \frac{q^{mi}}{w_{mi}w_{m(i+N)}}
\]

and, in addition,

\[
\sum_{i=1}^{\infty} \frac{q^{mi}}{w_{mi}w_{m(i+2k)}} = \frac{1}{e_w u_{2km}} \left( 2k\alpha - \sum_{i=1}^{2k} \frac{w_{mi} + 1}{w_{mi}} \right).
\]

### 3. Some Other Horadam Series

The next achievement of the present paper is the following theorem:

**Theorem 5.** Let \( m, k, \) and \( n \) be integers and let \( N \) be a natural number. Then

\[
\sum_{i=1}^{2N} \frac{(\pm 1)^i q^{m(i-k)}}{w_{m(i-k)+n}w_{m(i+k)+n}} = \frac{1}{e_w u_{n} u_{2km}} \sum_{i=1}^{2k} (\pm 1)^i \left( \frac{w_{m(i-k)}}{w_{m(i-k)+n}} - \frac{w_{m(i+2N-k)+n}}{w_{m(i+2N-k)+n}} \right) \tag{22}
\]
or, equivalently,
\[
\sum_{i=1}^{2N} \frac{(\pm)^i q^{m(i-k)}}{w_m(i-k)+n w_m(i+k)+n} = \sum_{i=1}^{2k} \frac{(\pm)^i q^{m(i)}}{w_m(i-k)+n w_m(i+2N-k)+n}.
\]

**Proof.** By using
\[
f(i) = \frac{w_m(i-k)}{w_m(i-k)+n} \quad \text{and} \quad t = k
\]
in (6), we obtain, by (9),
\[
f(i + 2k) - f(i) = \frac{w_m(i+k)}{w_m(i+k)+n} - \frac{w_m(i-k)}{w_m(i-k)+n} = - \frac{e_w u_n u_{2km} q^{m(i-k)}}{w_m(i-k)+n w_m(i+k)+n}
\]
and
\[
f(i + 2N) - f(i) = \frac{w_m(i+2N-k)}{w_m(i+2N-k)+n} - \frac{w_m(i-k)}{w_m(i-k)+n}.
\]
Substituting these values in the summation formula (6), we get the required identity (22). The theorem is proved.

Letting \( N \) tend to infinity, we immediately obtain the following corollary from (22):

**Corollary 5.** Let \( m, k, \) and \( n \) be integers. Then
\[
\sum_{i=1}^{\infty} \frac{(-1)^i q^{m(i-k)}}{w_m(i-k)+n w_m(i+k)+n} = \frac{1}{e_w u_n u_{2km}} \sum_{i=1}^{2k} \frac{(-1)^i w_m(i-k)}{w_m(i+k)+n}.
\]

**Theorem 6.** Let \( m, k, \) and \( n \) be integers and let \( N \) be a natural number. Then
\[
\sum_{i=1}^{N} \frac{q^{m(2i-k)}}{w_m(2i-k)+n w_m(2i+k)+n} = \frac{1}{e_w u_n u_{2km}} \sum_{i=1}^{k} \left( \frac{w_m(2i-k)}{w_m(2i-k)+n} - \frac{w_m(2(i+N)-k)}{w_m(2(i+N)-k)+n} \right)
\]
or, equivalently,
\[
\sum_{i=1}^{N} \frac{q^{2mi}}{w_m(2i-k)+n w_m(2i+k)+n} = \sum_{i=1}^{k} \frac{q^{2mi}}{w_m(2i-k)+n w_m(2(i+N)-k)+n}.
\]

**Proof.** We write \( 2i \) for \( i \) in (9) to obtain
\[
\frac{q^{m(2i-k)}}{w_m(2i-k)+n w_m(2i+k)+n} = \frac{1}{e_w u_n u_{2km}} \left( \frac{w_m(2i-k)}{w_m(2i-k)+n} - \frac{w_m(2i+k)}{w_m(2i+k)+n} \right).
\]
Further, we use
\[
f(i) = \frac{w_m(2i-k)}{w_m(2i-k)+n} \quad \text{and} \quad t = k
\]
in (5).

The theorem is proved.
In the limit as \( N \) tends to infinity in Theorem 6, we get the following result:

**Corollary 6.** Let \( m, k, \) and \( n \) be integers. Then

\[
\sum_{i=1}^{\infty} \frac{q^{m(2i-k)}}{u_m(2i-k) + n u_m(2i+k)} = \frac{1}{e_w u_{2km}} \left( \sum_{i=1}^{k} \frac{u_m(2i-k)}{u_m(2i-k) + n} - \frac{k}{\alpha^n} \right).
\]

We now list some Fibonacci and Lucas series, which follow from Corollary 6:

\[
\sum_{i=1}^{\infty} \frac{1}{F_m(2i-1) + n F_m(2i+1)} = \frac{(-1)^m}{F_n F_{2m}} \left( \frac{1}{\Phi^n} - \frac{F_m}{F_{m+n}} \right),
\]

\[
\sum_{i=1}^{\infty} \frac{1}{F_{2m}(i-1) + n F_{2m}(i+1)} = \frac{1}{F_n F_{4m}} \left( \frac{2}{\Phi^n} - \frac{F_{2m}}{F_{2m+n}} \right),
\]

\[
\sum_{i=1}^{\infty} \frac{1}{L_m(2i-1) + n L_m(2i+1)} = \frac{(-1)^m}{5 F_n F_{2m}} \left( \frac{L_m}{L_{m+n} - \frac{1}{\Phi^n}} \right),
\]

\[
\sum_{i=1}^{\infty} \frac{1}{L_{2m}(i-1) + n L_{2m}(i+1)} = \frac{1}{5 F_n F_{4m}} \left( \frac{2}{L_n} + \frac{L_{2m}}{L_{2m+n} - \frac{2}{\Phi^n}} \right).
\]

The next theorem is a nonsingular version of the first identity in Theorem 6 and Corollary 6 for \( n = 0 \).

**Theorem 7.** Let \( m \) and \( k \) be integers and let \( N \) be a natural number. Then

\[
\sum_{i=1}^{N} \frac{q^{m(2i-k)}}{u_m(2i-k) u_m(2i+k)} = \frac{1}{e_w u_{2km}} \sum_{i=1}^{k} \left( \frac{u_m(2(i+N)-k+1)}{u_m(2(i+N)-k)} - \frac{u_m(2i-k+1)}{u_m(2i-k)} \right)
\]

\[
= \frac{u_{2mN}}{u_{2km}} \sum_{i=1}^{k} \frac{q^{m(2i-k)}}{u_m(2i-k) u_m(2i+N-k)}
\]

and

\[
\sum_{i=1}^{\infty} \frac{q^{m(2i-k)}}{u_m(2i-k) u_m(2i+k)} = \frac{1}{e_w u_{2km}} \left( k \alpha - \sum_{i=1}^{k} \frac{u_m(2i-k+1)}{u_m(2i-k)} \right). \tag{23}
\]

**Proof.** We write \( 2i \) for \( i \) in identity (16) to obtain

\[
- \frac{e_w u_{2km} q^{m(2i-k)}}{u_m(2i-k) u_m(2i+k)} = \frac{u_m(2i-k+1)}{u_m(2i-k)} - \frac{u_m(2i+k+1)}{u_m(2i+k)}.
\]

This yields the required result upon summation over \( i \) by using (5) with

\[
f(t) = \frac{u_m(2i-k+1)}{u_m(2i-k)} \quad \text{and} \quad t = k.
\]

Taking the limit as \( N \to \infty \), we obtain (23).
4. Conclusions

We evaluate some new three-parameter families of reciprocal Horadam sums in the closed form. The proposed approach is elementary and based on clever telescoping. It seems possible to extend the results obtained in the present paper to reciprocal sums involving three and four Horadam numbers as factors in the denominator. This problem will be studied elsewhere in our future project.

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