Hyper-Mahler measures via Goncharov–Deligne cyclotomy

Yajun Zhou

To Prof. Weinan E’s 60th birthday

Abstract. The hyper-Mahler measures \( m_k(1 + x_1 + x_2), k \in \mathbb{Z}_{>1} \) and \( m_k(1 + x_1 + x_2 + x_3), k \in \mathbb{Z}_{>1} \) are evaluated in closed form, via multiple polylogarithms at roots of unity, which came under scrutiny by Goncharov and Deligne. Some infinite series related to these hyper-Mahler measures are also explicitly representable as Goncharov–Deligne periods of levels 1, 2, 3, 4, 6, 8, 10 and 12.

1. Introduction

For an \( n \)-variate Laurent polynomial \( P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) that does not vanish identically, its \( k \)-Mahler measure [48, Definition 1] is given by the following integral:

\[
m_k(P) := \int_0^1 dt_1 \cdots \int_0^1 dt_n \log^k |P(e^{2\pi it_1}, \ldots, e^{2\pi it_n})|,
\]

where \( k \) is a positive integer. When \( k = 1 \), this becomes the logarithmic Mahler measure \( m_1(P) \) of \( P \). When the integer \( k \) is greater than 1, the quantity \( m_k(P) \) is a hyper-Mahler measure.

Smyth [18, Appendix 1] has evaluated a logarithmic Mahler measure

\[
m_1(1 + x_1 + x_2 + x_3) = \frac{7\zeta_3}{2\pi^2}
\]

in terms of Apéry’s constant \( \zeta_3 := \sum_{n=1}^{\infty} \frac{1}{n^3} \). Borwein et al. (see [15, (6.10)] and [13, (4.17)]) have computed a hyper-Mahler measure

\[
m_2(1 + x_1 + x_2 + x_3) = \frac{24 \text{Li}_4\left(\frac{1}{2}\right) - \frac{\pi^4}{32} + 21\zeta_3 \log 2 - \pi^2 \log^2 2 + \log^4 2}{\pi^2}
\]

involving a polylogarithmic constant \( \text{Li}_4\left(\frac{1}{2}\right) := \sum_{n=1}^{\infty} \frac{1}{2^n n^4} \). If we introduce an alternating double sum \( \zeta_{-3,1} := \sum_{m>n>0} (-1)^m \frac{(-1)^n}{m^3 n} \), then we

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Table 1. Selected closed forms for $k$-Mahler measures $m_k(1 + x_1 + x_2)$, where $\omega := e^{2\pi i/3}$

| $k$ | $m_k(1 + x_1 + x_2)$ |
|-----|-----------------------|
| 1   | $\frac{3}{2\pi} \text{Im} \, \zeta(2)$ |
| 2   | $-\frac{2}{\pi} \text{Im} \, \zeta(2,1,1) + \frac{\pi^2}{2}$ |
| 3   | $-\frac{2}{23\pi} \text{Im} \, \zeta(2,3) + \frac{2\zeta_3}{2} + \frac{\pi}{2\pi} \text{Im} \, \zeta(2)$ |
| 4   | $-\frac{2 \cdot 3^2 \cdot 5 - 11}{13\pi} \text{Im} \, \zeta(2,1,1) - \frac{3 \cdot 7^2}{13\pi} \text{Im} \, \zeta(2,3) - \frac{3 \zeta_3}{\pi} \text{Im} \, \zeta(2) + \frac{3^2}{2^2} \text{Im} \, \zeta(2)^2 - \frac{5923\pi^4}{2^4 \cdot 3^2 \cdot 5 - 11}$ |

The table entries can condense (1.3) into

$$m_2(1 + x_1 + x_2 + x_3) = \frac{12\zeta_{3,1}}{\pi^2} + \frac{\pi^2}{20},$$

in the light of a proven relation between $\zeta_{3,1}$ and $\zeta_4$ \cite[(5.5)]{9}.

In this work, we study the hyper-Mahler measures $m_k(1 + x_1 + x_2)$ and $m_k(1 + x_1 + x_2 + x_3)$ in detail, for all $k \in \mathbb{Z}_{>1}$. For every positive integer $k$, we express the quantity $\pi^2 m_k(1 + x_1 + x_2)$ as a $\mathbb{Q}$-linear combination of special values for Goncharov’s \cite{39} multiple polylogarithm

$$\text{Li}_{a_1, \ldots, a_n}(z_1, \ldots, z_n) := \sum_{\ell_1, \ldots, \ell_n > 0, \ell_j, j = 1}^n \prod_{j=1}^n \frac{\ell_j}{\ell_j !^j},$$

where $a_1, \ldots, a_n$ are positive integers and $z_1, \ldots, z_n$ are third roots of unity (see Table 1 for the first few), with $|a_1 - 1| + |z_1 - 1| > 0$ to ensure convergence \cite[(0.1)]{32}. Meanwhile, $\mathbb{Q}$-linear combinations of Hoffman’s (alternating) multiple zeta values \cite{44} (also known as Euler–Zagier numbers \cite{68, 61})

$$\zeta_{A_1, \ldots, A_n} := \text{Li}_{|A_1|, \ldots, |A_n|} \left( \frac{A_1}{|A_1|}, \ldots, \frac{A_n}{|A_n|} \right),$$

enter the representations of $\pi^2 m_k(1 + x_1 + x_2 + x_3)$ (see Table 2 for examples).

In §2, we verify all the statements in the last paragraph. On the qualitative side, we do so by linking Broadhurst’s formula \cite[(9)]{21} for moments of random walks (§§2.1–2.2) to Brown’s homotopy-invariant iterated integrals \cite[§1.1]{25} on the moduli spaces $\mathcal{M}_{0,n}$ of genus-zero curves with $n$ marked points (§§2.3–2.4). On the quantitative side, with Panzer’s \texttt{HyperInt} package \cite{57} that implements Brown’s algorithm \cite{23, 24, 25} as well as Au’s \texttt{MultipleZetaValues} package \cite{4} that reduces certain special values of multiple polylogarithms, we generate Tables 1 and 2 as stated.

\footnote{While defining $\text{Li}_{a_1, \ldots, a_n}(z_1, \ldots, z_n)$ and $\zeta_{a_1, \ldots, a_n}$, we sort the positive integer parameters $a_1, \ldots, a_n$ as in Maple 2022’s built-in functions \texttt{MultiPolylog} and \texttt{MultiZeta} (which is Hoffman’s \cite{44} order—see also \cite{61, 32, 6, 22, 37, 36, 4, 38} for the same practice). In the \texttt{HyperInt} package \cite{57} for Maple, our $\text{Li}_{a_1, \ldots, a_n}(z_1, \ldots, z_n)$ is represented by $\text{Mpl}([a_n, \ldots, a_1], [z_n, \ldots, z_1])$ (the latter of which follows Zagier’s \cite{68} order—see also \cite{39, 42, 23, 24, 25, 50, 34} for the same convention).}
Table 2. Selected closed forms for $k$-Mahler measures $m_k(1 + x_1 + x_2 + x_3)$

| $k$ | $m_k(1 + x_1 + x_2 + x_3)$ |
|-----|-----------------------------|
| 1   | $\frac{7\zeta_3}{2\pi^2}$ |
| 2   | $\frac{2\zeta^2}{2^2 \cdot 3\zeta_{-3,1}} + \frac{\pi^2}{2\pi^2} + \frac{\pi^2}{2\pi^2}$ |
| 3   | $\frac{3 \cdot 5 \cdot 23\zeta_3}{2^2 \cdot 3^2 \cdot \zeta_{-3,1,1}} + \frac{61\zeta_3}{23}$ |
| 4   | $\frac{2^2 \cdot 3^2 \cdot 13\zeta_{-5,1}}{\pi^2} + \frac{2^2 \cdot 3^4 \cdot \zeta_{-3,1,1,1}}{\pi^2} - 2 \cdot 3 \zeta_{-3,1} + \frac{2^2 \cdot 3 \cdot 5 \cdot \zeta_{-3,1,1,1}}{\pi^2} + \frac{2^1 \cdot 3^4 \cdot 5^2 \cdot 11\zeta_{-5,1,1,1}}{\pi^2} - \frac{1867\pi^4}{2^4 \cdot 3 \cdot 5 \cdot 7}$ |
| 5   | $\frac{3 \cdot 5 \cdot 2297\zeta_3}{2^2 \cdot 3^2 \cdot 5 \cdot 19\zeta_{-3,1,1}} + \frac{2^2 \cdot 3 \cdot 5 \cdot \zeta_{-3,1,1,1}}{\pi^2} + \frac{2^2 \cdot 3 \cdot 5 \cdot \zeta_{-3,1,1,1}}{\pi^2} + \frac{2^2 \cdot 3^2 \cdot \zeta_{-3,1,1,1,1}}{\pi^2} - \frac{2626\pi^5}{\pi^2}$ |
| 6   | $\frac{2\pi^2}{2^2 \cdot 3 \cdot 5 \cdot 19\zeta_{-7,1}} + \frac{2 \cdot 3^2 \cdot 11 \cdot 29 \cdot 43\zeta_{-3,1,1,1}}{\pi^2} + \frac{2^4 \cdot 3^5 \cdot 5 \cdot 13\zeta_{-3,1,1,1,1}}{\pi^2} - \frac{2^2 \cdot 3^4 \cdot 5 \cdot 2^2 \cdot 11\zeta_{-5,1,1,1,1}}{\pi^2}$ |
|     | $\frac{2^2 \cdot 3^4 \cdot 5 \cdot \zeta_{-3,1,1,1,1}}{\pi^2} + \frac{3 \cdot 11^2 \cdot \zeta_{-3,1,1,1,1}}{2^2} - \frac{3^2 \cdot 5^2 \cdot 1213\zeta_3}{2^2 \cdot 3 \cdot 5 \cdot 13\zeta_{-5,1,1,1,1}} - \frac{26459029\pi^6}{\pi^2}$ |

In §3, we extend our methods to several families of infinite series occurring in recent studies of number theory and high energy physics. While these infinite sums may or may not be directly related to hyper-Mahler measures, they can all be expressed algorithmically in terms of multiple polylogarithms at roots of unity. Our automated evaluations provide either new patterns for infinite series or succinct alternatives to existent treatments of the same objects.

2. Integral representations of $k$-Mahler measures

2.1. Broadhurst’s formula for zeta Mahler measures. Consider the zeta Mahler measure (see [1, (1.1)] and [15, (2.3)])

\begin{equation}
W_n(s) := \int_0^1 dt_1 \cdots \int_0^1 dt_n \left| \sum_{j=1}^n e^{2\pi i t_j} \right|^s = \int_0^1 dt_1 \cdots \int_0^1 dt_{n-1} \left| 1 + \sum_{j=1}^{n-1} e^{2\pi i t_j} \right|^s,
\end{equation}

whose $k$-th order derivatives at $s = 0$ evaluate the $k$-Mahler measures:

\begin{equation}
\frac{d^k W_n(s)}{ds^k} \bigg|_{s=0} = \int_0^1 dt_1 \cdots \int_0^1 dt_{n-1} \log^k \left| 1 + \sum_{j=1}^{n-1} e^{2\pi i t_j} \right| =: m_k \left( 1 + \sum_{j=1}^{n-1} x_j \right).
\end{equation}

By definition, we have

\begin{equation}
W_1(s) = 1.
\end{equation}

The evaluation

\begin{equation}
W_2(s) = \frac{2s \Gamma \left( \frac{1+s}{2} \right)}{\sqrt{\pi} \Gamma \left( 1 + \frac{s}{2} \right)}
\end{equation}
can be found in [1, Theorem 1(1)]. In [48, Theorem 14] and [11, (25)], it was written as an equivalent binomial coefficient \( \frac{s}{2} \).

In the next lemma, we recapitulate Broadhurst’s integral representation [21, (9)] of \( W_n(s) \), which will be instrumental in the rest of this section.

**Lemma 2.1 (Broadhurst’s formula for \( W_n(s) \)).** For \( n \in \mathbb{Z}_{>0} \), the integral representation

\[
W_n(s) = 2^{s+1} \frac{\Gamma \left( 1 + \frac{s}{2} \right)}{\Gamma \left( -\frac{s}{2} \right)} \int_0^\infty J_0^n(x) \frac{dx}{x^{s+1}}
\]

is valid for \( s \in \left( \max\{-2, -\frac{3}{2}\}, 0 \right) \), where \( \Gamma(\sigma) := \int_0^\infty t^{\sigma-1}e^{-t} \, dt, \sigma > 0 \) defines Euler’s gamma function and \( J_0(x) := \sum_{\ell=0}^\infty \frac{(-1)^\ell}{(\ell!)^2} \left( \frac{x}{2} \right)^{2\ell} \) is the zeroth-order Bessel function of the first kind. \( \blacksquare \)

**Remark 2.2.** For \( n \in \{1, 2\} \), Broadhurst’s integral representations are compatible with (2.3) and (2.4), as noted by Borwein–Straub–Wan [14, Example 2.2]. For \( n = 2 \), one may also check this compatibility against a critical case of the Weber–Schaafheitlin integral [63, §14.31(2)]. \( \square \)

### 2.2. Some integral formulae involving Bessel functions

In this subsection, we will reformulate Broadhurst’s integral representations of \( W_3(s) \) and \( W_4(s) \) with the Parseval–Plancherel identity for Hankel transforms [63, §14.3(3)]

\[
\frac{1}{2} \int_0^\infty \left[ \int_0^\infty J_\nu(\sqrt{ux}) f(x) x \, dx \right] \left[ \int_0^\infty J_\nu(\sqrt{ux}) g(x) x \, dx \right] \, du.
\]

(2.6)

\[
= \int_0^\infty f(x) g(x) x \, dx,
\]

where \( J_\nu(X) := \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!(\nu+\ell+1)} \left( \frac{X}{2} \right)^{2\ell+\nu} \) is the Bessel function of order \( \nu \in \mathbb{R} \). This motivates the study of the integrals in the next proposition.

**Proposition 2.3 (Weber–Schaafheitlin integral and its generalizations).** For \( s \in (-2, 2) \), we have

\[
\int_0^\infty J_{1+\frac{s}{2}}(\sqrt{ux}) \frac{J_0(x)}{x^{s/2}} \, dx = \left\{ \begin{array}{ll}
0, & u \in (0, 1), \\
\frac{1}{u^{2+\nu}} \Gamma \left( \frac{1}{2} + \frac{s}{2} \right) \left( \frac{u-1}{u} \right)^{\frac{s}{2}}, & u \in (1, \infty),
\end{array} \right.
\]

(2.7)

\[
\int_0^\infty J_{1+\frac{s}{2}}(\sqrt{ux}) \frac{J_0^2(x)}{x^{s/2}} \, dx = \left\{ \begin{array}{ll}
\int_0^{2\arcsin \frac{u}{2}} \frac{u-4\sin^2 \frac{\phi}{2}}{2^\nu \pi u^{2+\nu} \Gamma \left( \frac{1}{2} + \frac{s}{2} \right)} \, d\phi, & u \in (0, 4), \\
\int_0^\pi \frac{u-4\sin^2 \frac{\phi}{2}}{2^\nu \pi u^{2+\nu} \Gamma \left( \frac{1}{2} + \frac{s}{2} \right)} \, d\phi, & u \in (4, \infty).
\end{array} \right.
\]

(2.8)

**Proof.** See [63, §13.46(1)]. \( \blacksquare \)
Combining the formulae in the proposition above with the observation that

\[ \int_0^\infty J_{1+\frac{\pi}{2}} \left( \sqrt{ux} \right) \frac{J_0(x)}{x^{s/2}} \, dx \]

\[ = \int_0^\infty J_{1+\frac{\pi}{2}} \left( \sqrt{ux} \right) \frac{J_0(x)}{x^{s/2}} \, dx + \int_0^\infty \frac{\partial}{\partial x} \frac{J_0 \left( \sqrt{ux} \right)}{ \sqrt{ux} x^{s/2}} \, dx \]

\[ = \int_0^\infty J_{1+\frac{\pi}{2}} \left( \sqrt{ux} \right) \frac{J_0(x)}{x^{s/2}} \, dx - \frac{1}{2^s u} \frac{\pi}{\Gamma \left( 1 + \frac{s}{2} \right)} , \]

we will apply the Parseval–Plancherel identity (2.6) to \( W_3(s) - W_1(s) \) and \( W_4(s) - W_2(s) \), in the corollary below.

**Corollary 2.4** (Hankel–Broadhurst representations for \( W_3(s) \) and \( W_4(s) \)).

Set \( \varphi = e^{\pi i/3} \) and \( Q(x) := \left( 1 - \frac{\varphi}{x} \right) (1 - x \varphi) \). Define five rational functions

(2.10)

\[ A_3(t, \tau) := \frac{\left( 1 - \frac{\tau}{t} \right) \left( 1 - t \tau \right)}{Q(t)} , \quad B_3(\tau) := \frac{\tau^2}{(1 + \tau)^2 Q(\tau)} , \quad C_3(\tau) := \frac{(1 + \tau)^2}{3\tau} \]

(2.11)

\[ A_4(t, \tau) := \frac{\left( 1 - \frac{\tau}{t} \right) \left( 1 - t \tau \right)}{t} , \quad B_4(\tau) := -\frac{\tau}{(1 - \tau)^2} \]

When \( s \in (-\frac{1}{2}, 0) \), we have

(2.12)

\[ W_3(s) - 1 = \frac{3^{s+\frac{\pi}{2}} \sin \frac{\pi s}{2}}{\pi^2} \int_0^1 \left[ \int_{-1}^1 A_3^2(t, \tau) \frac{dt}{Q(t)} - C_3^2(\tau) \int_{-1}^1 Q(t) \right] \frac{\tau - 1}{\tau + 1} B_3^2(\tau) \, d\tau \]

\[ + \frac{3^{s+\frac{\pi}{2}} \sin \frac{\pi s}{2}}{\pi^2} \int_1^\infty \left\{ \int_{-1}^1 \left[ A_3^2(t, \tau) - C_3^2(\tau) \right] \frac{dt}{Q(t)} \right\} \frac{\tau - 1}{\tau + 1} B_3^2(\tau) \, d\tau , \]

where all the paths of integration are straight line segments; when \( s \in (-1, 0) \), we have

(2.13)

\[ W_4(s) - \frac{2\Gamma \left( \frac{1+s}{2} \right)}{\sqrt{\pi} \Gamma \left( 1 + \frac{s}{2} \right)} \]

\[ = \frac{\sin \frac{\pi s}{2}}{\pi^3} \int_1^0 \left[ \int_{-1}^1 A_4^2(t, \tau) \frac{dt}{t} \int_{-1}^1 A_4^2(T, \tau) \frac{dT}{T} - \pi i B_4^2(\tau) \right] \frac{\tau + 1}{\tau - 1} B_4^2(\tau) \, d\tau \]

\[ + \frac{\sin \frac{\pi s}{2}}{\pi^3} \int_{-1}^0 \left[ \int_{-1}^1 A_4^2(t, \tau) \frac{dt}{t} \left\{ \int_{-1}^1 \left[ A_4^2(T, \tau) - B_4^2(\tau) \right] \frac{dT}{T} \right\} \right] \frac{\tau + 1}{\tau - 1} B_4^2(\tau) \, d\tau , \]

where the path of integration for each \( f \) runs counterclockwise along the unit circle, and the path of integration for \( \tau \) from \(-1\) to \(0\) is a straight line segment.

**Proof.** For \( W_3(s) \), make the variable substitutions \( \phi = 2 \arctan \frac{1+t}{\sqrt{3(1-t)}} \) [which satisfies \( 4 \sin^2 \frac{\phi}{2} = \frac{1+(1+t)^2}{1+t} \)] and \( u = \frac{1+(1+t)^2}{1-\tau + \tau^2} \) after invoking (2.6). For \( W_4(s) \), use \( \phi = \arg t \) [which satisfies \( 4 \sin^2 \frac{\phi}{2} = -\frac{(1-t)^2}{1} \) for \( |t| = 1 \)] and \( u = -\frac{(1-\tau^2)}{\tau} \) instead. □
2.3. Period structure of \( m_k(1 + x_1 + x_2) \). As a variation on Goncharov’s notation [40, §1.2], we define \( \mathbb{Q}_k \)-vector spaces

\[
\mathfrak{z}_k(N) := \text{span}_\mathbb{Q} \left\{ \text{Li}_{a_1, \ldots, a_n}(z_1, \ldots, z_n) \in \mathbb{C} \mid z_1^{a_1} \cdots z_n^{a_n} = 1 \right\}
\]

for positive integers \( k \) and \( N \). With the understanding that empty sums are zero (so \( \text{span}_\mathbb{Q} \emptyset = \{0\} \) ), we have \( \mathfrak{z}_1(1) = \{0\} \). Retroactively, we set \( \mathfrak{z}_0(N) := \mathbb{Q} \).

A number \( \text{Li}_{a_1, \ldots, a_n}(z_1, \ldots, z_n) \in \mathfrak{z}_k(N) \) is referred to as a cyclotomic multiple zeta value (CMZV) of weight \( k \) and level \( N \) [60, (1)]. Some authors [60, 4] write \( \text{CMZV}_{k,N} \) or \( \text{CMZV}_N^k \) for \( \mathfrak{z}_k(N) \). At level 1, a CMZV \( \zeta_{a_1, \ldots, a_n}(z_1, \ldots, z_n) = \text{Li}_{a_1, \ldots, a_n}(1, \ldots, 1) \in \mathfrak{z}_{a_1, \ldots, a_n}(1) \) is simply a multiple zeta value (MZV); at level 2, a CMZV becomes an alternating multiple zeta value (AMZV).

It is clear from the definition (2.14) that we have a set inclusion \( \mathfrak{z}_k(N) \subseteq \mathfrak{z}_k(M) \) if \( N \) divides \( M \). Furthermore, we have an elementary fact that (see [60, Remark 1.3] or [4, Lemma 4.1])

\[
\pi_i = \frac{N}{N-2} \left[ \text{Li}_1(e^{2\pi i/N}) - \text{Li}_1(e^{-2\pi i/N}) \right] \in \mathfrak{z}_1(N)
\]

when \( N \in \mathbb{Z}_{\geq 3} \).

For each fixed \( N \), the \( \mathbb{Q} \)-algebra \( \mathfrak{z}(N) := \sum_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{z}_k(N) \) is filtered by weight [40, §1.2]:

\[
\mathfrak{z}_j(N) \mathfrak{z}_k(N) \subseteq \mathfrak{z}_{j+k}(N), \quad \text{for all } j, k \in \mathbb{Z}_{\geq 0}.
\]

In what follows, we will refer to members of \( \mathfrak{z}_k(N) \) as Goncharov–Deligne periods, in honor of the pioneers [40, 32] who worked on them.

**Theorem 2.5** (\( \pi i m_k(1 + x_1 + x_2) \) as Goncharov–Deligne periods). For \( n \in \mathbb{Z}_{\geq 0} \), define

\[
X_3^{(n)}(\tau) := \int_{-1}^{\tau} \frac{\log^n A_3(t, \tau) \, dt}{Q(t)} - \frac{\pi \log^n C_3(\tau)}{\sqrt{3}}, \quad Y_3^{(n)}(\tau) := \frac{\tau - 1}{\tau + 1} \frac{\log^n B_3(\tau)}{Q(\tau)},
\]

\[
\Xi_3^{(n)}(\tau) := \int_{-1}^{\tau} [\log^n A_3(t, \tau) - \log^n C_3(\tau)] \frac{dt}{Q(t)},
\]

using the rational functions in (2.10). For each positive integer \( k \), we have

\[
\pi i m_k(1 + x_1 + x_2) \in \text{span}_\mathbb{Q} \left\{ \left[ \frac{\sqrt{3} \log^n 3}{\pi - 2n_2} \int_0^1 X_3^{(n_3)}(\tau) Y_3^{(n_1)}(\tau) \, d\tau \right] \right|_{n_1 + 2n_2 + n_3 + n_4 = k} \}
\]

\[
+ \text{span}_\mathbb{Q} \left\{ \left[ \frac{\sqrt{3} \log^n 3}{\pi - 2n_2} \int_1^\tau \Xi_3^{(n_3)}(\tau) Y_3^{(n_1)}(\tau) \, d\tau \right] \right|_{n_1 + 2n_2 + n_3 + n_4 = k} \}
\]

\[
\subseteq i \mathfrak{z}_{k+1}(6).
\]

**Proof.** The “\( \in \)” part is a direct consequence of (2.2) and (2.12).

To prove the “\( \subseteq \)” part, we follow Brown’s algorithmic studies [23, 24, 25] of \( L(\mathfrak{M}_{0,n}) \), the totality of homotopy-invariant iterated integrals [25, §1.1] on the
moduli spaces \( \mathcal{M}_{0,n} \) of Riemann spheres with \( n \) marked points. Thanks to the filtration property (2.16) and the facts that

\[
\log 3 = -\text{Li}_1(-\varrho) - \text{Li}_1\left(-\frac{1}{\varrho}\right) \in \mathfrak{F}_1(6), \quad \pi^2 = 6 \text{Li}_2(1) \in \mathfrak{F}_2(6),
\]

we may reduce our task to the demonstration of

\[
\mathfrak{F}_{1+\ell}(6) \subseteq \mathfrak{F}_{1+\ell+1}(6)
\]

for \( n_3, n_4 \in \mathbb{Z}_{\geq 0} \) and \( n_3 + n_4 = k - 1 \).

Using the notation of Frellesvig–Tommasini–Wever [37, §2], we define the generalized polylogarithm\(^2\) (GPL) by the recursion

\[
G(\alpha_1, \ldots, \alpha_n; z) := \int_0^z \frac{d x}{x - \alpha_1} G(\alpha_2, \ldots, \alpha_n; x)
\]

for any complex-valued vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \) with non-vanishing modulus \( |\alpha| := \sqrt{|\alpha_1|^2 + \cdots + |\alpha_n|^2} \neq 0 \), along with the additional settings that

\[
G(0, \ldots, 0; z) := \frac{\log^n z}{n!}, \quad G(0; z) := 1.
\]

Here, it is always understood that the integration path in (2.22) is a straight line segment joining 0 to \( z \). For convenience, we will use a short-hand \( G(\alpha; z) := G(\alpha_1, \ldots, \alpha_{\omega(\alpha)}; z) \) and refer to \( \omega(\alpha) \) (the number of components in the vector \( \alpha \in \mathbb{C}^{\omega(\alpha)} \)) as the weight of the GPL \( G(\alpha; z) \).

Set \( \mathfrak{G}_0^{(z)}(N) = \mathbb{Q} \) and \( \mathfrak{G}_0^{(z)}(N) = \mathbb{Q} \). For every \( k \in \mathbb{Z}_{\geq 0} \), we construct two \( \mathbb{Q} \)-vector spaces

\[
\mathfrak{G}_k^{(z)}(N) := \text{span}_{\mathbb{Q}} \left\{ G(\alpha_1, \ldots, \alpha_k; z) \in \mathbb{C} \left| \alpha_1^N, \ldots, \alpha_{k-1}^N \in \{0,1\}; \alpha_k^N = 1 \right. \right\},
\]

\[
\mathfrak{H}_k^{(z)}(N) := \text{span}_{\mathbb{Q}} \left\{ (\pi)^l G(\alpha_1, \ldots, \alpha_{k-l}; z) \in \mathbb{C} \left| \alpha_1^N, \ldots, \alpha_{k-l}^N \in \{0,1\}; l \in \mathbb{Z}_{\geq 0} \right. \right\}.
\]

The \( \mathbb{Q} \)-algebras \( \mathfrak{G}^{(z)}(N) := \sum_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{G}_k^{(z)}(N) \) and \( \mathfrak{H}^{(z)}(N) := \sum_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{H}_k^{(z)}(N) \) are filtered by weight:

\[
\mathfrak{G}_j^{(z)}(N) \subseteq \mathfrak{G}_{j+k}^{(z)}(N), \quad \mathfrak{H}_j^{(z)}(N) \subseteq \mathfrak{H}_{j+k}^{(z)}(N),
\]

thanks to a shuffle identity (see [10, §5.4] or [37, (2.4)])

\[
G(\alpha_1, \ldots, \alpha_{\omega_{w_1+w_2}}; z) G(\alpha_{w_1+1}, \ldots, \alpha_{w_1+w_2-1}; z) = \sum_{\sigma \in \Pi_{w_1+w_2}} G(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(w_1+w_2)}; z),
\]

where \( \Pi_{w_1+w_2} \) consists of \( \frac{(w_1+w_2)!}{w_1!w_2!} \) permutations \( \sigma: \mathbb{Z} \cap [1, w_1+w_2] \to \mathbb{Z} \cap [1, w_1+w_2] \) that satisfy \( \sigma^{-1}(m) < \sigma^{-1}(n) \) whenever \( 1 \leq m < n \leq w_1 \) or \( w_1+1 \leq m < w_1+w_2+1 \).

\(^2\) The GPL \( G(\alpha_1, \ldots, \alpha_n; z) \) is implemented in Maple as \text{GeneralizedPolylog}([\alpha_1, \ldots, \alpha_n], z) \) by Frellesvig [36]. The same GPL corresponds to \( \text{Hlog}(z, [\alpha_1, \ldots, \alpha_n]) \) in Panzer’s \text{HyperInt} package [57] for Maple, where \( \text{Hlog} \) stands for “hyperlogarithm”, another name for GPL. We also note that Panzer used “GPL” as an abbreviation for “Goncharov polylogarithm” [57, §1], referring to the same object as our GPL.
where the equivalence between two definitions stems from the GPL shuffle algebra

(2.30b) \[ \text{Li}_{a_1, \ldots, a_n}(z_1, \ldots, z_n) = (-1)^n G \left( \frac{1}{z_1}, \frac{1}{z_1 z_2}, \ldots, \frac{1}{z_1 z_2 \cdots z_n}; 1 \right) \]

when \( \prod_{j=1}^n z_j \neq 0 \), and vice versa (see [57, (1.3)] or [37, (2.6)])

(2.29) \[ G(0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, \alpha_n; z) = (-1)^n \text{Li}_{a_1, \ldots, a_n} \left( \frac{z}{\alpha_1}, \frac{\alpha_1}{\alpha_2}, \ldots, \frac{\alpha_{n-1}}{\alpha_n} \right) \]

when \( \prod_{j=1}^n \alpha_j \neq 0 \). Especially, we have \( \mathfrak{g}_k^{(1)}(N) = \mathfrak{z}_k(N) \).

To facilitate further analysis, we define

(2.30a) \[ \mathfrak{g}_k^{(z)}(N) := \text{span}_{\mathbb{Q}} \left\{ G(\alpha; 1) G(\beta; z)(\pi i)^\ell (\log z)^k \frac{1}{\ell - w(\alpha) - w(\beta)} \in \mathbb{C} \mid G(\alpha; 1) \in \mathfrak{g}(\alpha; \ell - w(\alpha) - w(\beta)) \right\} \]

(2.30b) \[ \equiv \text{span}_{\mathbb{Q}} \left\{ G(\alpha; 1) G(\beta; z)(\pi i)^\ell \in \mathbb{C} \mid G(\alpha; 1) \in \mathfrak{g}(\alpha; N) \right\} \]

where the equivalence between two definitions stems from the GPL shuffle algebra [34, p. 24]. By the shuffle identity (2.27), we have \( \mathfrak{g}_j^{(z)}(N) \mathfrak{g}_k^{(z)}(N) \subseteq \mathfrak{g}_{j+k}^{(z)}(N) \) for \( j, k \in \mathbb{Z}_{\geq 0} \). Combining (2.30b) with the recursive definition of GPL in (2.22), we see that if \( \alpha^N \in \{0, 1\} \), then

(2.31) \[ \int_0^z \frac{g(x)}{x - \alpha} \, dx \in \mathfrak{g}_{k+1}^{(z)}(N) \]

so long as the integral in question converges, and \( g(x) \in \mathfrak{g}_k^{(z)}(N) \) on the path of integration.

In view of the facts that

(2.32) \[ \log A_3(t, \tau) := \log \left( 1 - \frac{\tau}{t} \right) \frac{1}{Q(t)} = G(\tau; t) + G \left( \frac{1}{t}; \frac{1}{\tau} \right) - G(\tau; \frac{1}{t}) - G \left( \frac{1}{\tau}; t \right), \]

(2.33) \[ \log C_3(\tau) := \log \left( 1 + \tau \right) = 2G(-1; \tau) - G(0; \tau) - G(-\frac{1}{\tau}; 1) \]

\[ \in \mathfrak{g}_1^{(\tau)}(6), \]

and

(2.34) \[ \frac{1}{Q(t)} = \frac{1}{i \sqrt{3}} \left( \frac{1}{t - \frac{1}{\tau}} - \frac{1}{t - \frac{1}{\tau}} \right), \quad \frac{\pi i}{3} = G \left( \frac{1}{\tau}; 1 \right) - G(-\frac{1}{\tau}; 1) \in \mathfrak{g}_1^{(\tau)}(6), \]

\(^3\text{The equations (2.28) and (2.29) can also be used as working definitions for an MPL when its series form (1.4) does not converge.}\)
we have the following arguments for \( k \in \mathbb{Z}_{>0} \):

\[
X_3^{(k-1)}(\tau) := \int_{-1}^{\tau} \frac{dt}{Q(t)} \log^{k-1} \left( 1 - \frac{1}{t} \right) \left( 1 - t \tau \right) - \frac{\pi \log^{k-1} (1+\tau)^2}{\sqrt{3}}
\]

\[
\in \text{span}_{\mathbb{Q}} \left\{ \frac{G(\alpha_1, \ldots, \alpha_k; z)}{i\sqrt{3}} \mid \alpha_1 \in \{0, \frac{1}{\tau}, \ldots, \frac{\tau}{\sqrt{3}} \}, \alpha_2, \ldots, \alpha_k \in \{ 0, \frac{1}{\tau}, \ldots, \frac{\tau}{\sqrt{3}} \} \right\}
\]

Here, the “\( \in \)” step issues from the GPL recursion (2.22) and the shuffle identity (2.27) (along with Panzer’s logarithmic regularizations at \( t = \tau \) [57, §2.3] that preserve the shuffle structure), while the “\( \subseteq \)” step is a result of the fibration basis in Brown’s algorithm [57, Lemma 2.14 and Corollary 3.2]. Concretely speaking, through the fibration procedure, one can rewrite a GPL \( G(\alpha; \tau) \) (in which the components of \( \alpha \) belong to \( \{ 0, \frac{1}{\tau}, \tau \} \)) as a member in the following \( \mathbb{Z} \)-module:

\[
\text{span}_{\mathbb{Z}} \left\{ G(\beta_1, \ldots, \beta_w(\alpha); \tau) \mid \beta_1, \ldots, \beta_w(\alpha) \in \{ 0, \frac{1}{\tau}, -1, 0, 1 \} \right\},
\]

by induction on the weight \( w(\alpha) \) and applications of the GPL differential form\(^4\) [64, (8.8)]

\[
\frac{d}{d \log(\alpha_j - 1)} \left( \frac{d \log(\alpha_j - 1) - d \log(\alpha_{j+1} - \alpha_j)}{\sqrt{3}} \right)
\]

\[
= \sum_{j=1}^{n} G(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_n; z) \left[ d \log(\alpha_{j-1} - \alpha_j) - d \log(\alpha_{j+1} - \alpha_j) \right],
\]

where an element under the caret is removed, and \( \alpha_0 := z, \alpha_{n+1} := 0 \). Thus, it is clear that \( G(\alpha; \tau) \in \frac{g_3^{(\tau)}(6)}{i\sqrt{3}} \). Likewise, one can show inductively that \( G(\alpha; -1) \) (in which the components of \( \alpha \) belong to \( \{ 0, \frac{1}{\tau}, \tau \} \)) is a member of

\[
\text{span}_{\mathbb{Z}} \left\{ G(\beta_1, \ldots, \beta_j; \tau) G(\beta_{j+1}, \ldots, \beta_w(\alpha); -1) \mid \beta_1, \ldots, \beta_w(\alpha) \in \{ 0, \frac{1}{\tau}, -1, 0, 1 \} \right\}.
\]

This justifies the relation \( X_3^{(k-1)}(\tau) \in \frac{g_3^{(\tau)}(6)}{i\sqrt{3}} \) for every \( k \in \mathbb{Z}_{>0} \).

Reasoning in a similar vein as the last paragraph, one can verify that \( X_3^{(k-1)}(\tau) \in \frac{g_3^{(\tau)}(6)}{i\sqrt{3}} \) for every \( k \in \mathbb{Z}_{>0} \).

\(^4\)Such fibration procedures are automated by Panzer’s HyperInt package [57, Lemma 2.14 and Corollary 3.2], where \( \text{fibrationBasis}(HLog(\tau; [\alpha_1, \ldots, \alpha_{w(\alpha)}]), [\tau]) \) produces a unique representation of \( G(\alpha; \tau) \) as a member of (2.36).

\(^5\)As in [64, §8.1, p. 297], we adopt the convention that the 1-form \( d \log(\alpha_j - \alpha_j) \) is zero when \( \alpha_j = \alpha_j \). Accordingly, logarithmic regularizations [57, §2.3] are implicit when we encounter \( G(z, \ldots; z) \) and \( G(\ldots, 0, 0) \).
With the additional observation that
\[
\Upsilon_3^{(n)}(\tau) = \frac{1}{3} \left( \frac{1}{\tau - \varrho} + \frac{1}{\tau - \frac{1}{\varrho}} - \frac{2}{\tau + 1} \right) \log^n \frac{\tau^2}{(1 + \tau)^2 Q(\tau)}
\]
together with the scaling property \([37, (2.3)]\) of GPLs and the fibration of GPLs with respect to regularizations at \(\tau\), we may complete all the integrations over \(\tau\) (with Panzer’s logarithmic regularizations at \(\varrho\)) and arrive at
\[
\pi m_k(1 + x_1 + x_2) \in i\mathcal{G}_{k+1}^{(1)}(6) + i\mathcal{G}_{k+1}^{(2)}(6) = i\mathcal{G}_{k+1}^{(1)}(6) = i\mathcal{G}_{k+1}^{(2)}(6)
\]
for all \(k \in \mathbb{Z}_{>0}\). Here, the set inclusion \(\mathcal{G}_{k+1}^{(2)}(6) \subseteq \mathcal{G}_{k+1}^{(1)}(6)\) descends from the relation \(G(\frac{1}{2}; \tau) = \log(1 - \tau) = -\pi i \in \mathcal{G}_{1+1}^{(1)}(6)\) and parameter rescaling by a factor of \(\varrho\) for members of \(\mathcal{G}_{(6)}(6)\).

While the last paragraph finishes the proof of (2.19), we still have a slightly stronger result:
\[
\pi i m_k(1 + x_1 + x_2) \in \text{span}_Q \left\{ (\pi i)^f G(\alpha_1, \ldots, \alpha_j; \tau)G(\alpha_{j+1}, \ldots, \alpha_n; z) \log^3 z \in \mathbb{C} \mid \alpha_1, \ldots, \alpha_n, z \in \left\{ \frac{1}{2}, -1, 0, 1 \right\}, \tau \in \left\{ 1, \frac{1}{\varrho} \right\} \right\}
\]

Here, the “\(\in\)” step traces back to (2.35)–(2.39), along with all the foregoing fibrations, while the “\(\subseteq\)” step comes from the observation that
\[
\log 3 = G(1; \varrho) + 2G(-1; \varrho) \in \mathcal{G}_{(6)}(2),
\]
and the fibration of GPLs with respect to \(\varrho\).

**Remark 2.6.** If one defines a multiple zeta value (MZV) \(\zeta_{\alpha_1, \ldots, \alpha_n} = \text{Li}_{\alpha_1, \ldots, \alpha_n}(1, \ldots, 1)\) via the iterative constructions of GPL on the right-hand side of (2.28), then one gets the Drinfel’d integral representation of MZV (see [33, §2] or [68, §9]). In general, the representation of an MPL via the iterative integrations for the corresponding GPL is called the Leibniz–Kontsevich formula (see [41, Theorem 2.2] or [42, §1.2]) by Goncharov, who ascribed it to a private communication with Kontsevich that generalized the Leibniz representation for \(\zeta_n\) as an \(n\)-dimensional integral.

By considering mixed Tate motives [42, (2) and (3)], Goncharov–Manin asked [42, Conjecture 4.5] if certain integral periods on \(\mathcal{M}_{0,n}\) (generalizations of the Drinfel’d integral representation of MZV) are still representable as \(\mathbb{Q}\)-linear combinations of MZVs. This deep question has been answered in the positive by Brown [25, Theorem 1.1], whose algorithmic approach [23, 24, 25] to the Goncharov–Manin periods on \(\mathcal{M}_{0,n}\) [along with generalizations to a larger set \(L(\mathcal{M}_{0,n})\)] has been implemented in Panzer’s HyperInt package [57].
In the following, the absolute fibration operation \( \text{Fib}(f) \) (where \( f \) is an expression involving a formal variable \( g \)) consists of four tasks in a sequel:

1. Evaluate \( \text{fibrationBasis}(f) \) in \text{HyperInt}, which (among other things) reduces the alternating multiple zeta values (AMZVs) \( \zeta_{A_1, \ldots, A_n} \in \mathcal{S}_{|A_1|+\cdots+|A_n|}(2) \) [see (1.5)] and a special class of multiple polylogarithms [16, (2–3)]

\[
\text{Li}_{1, \ldots, 1}(z, 1, \ldots, 1) = \frac{(-1)^n}{n!} \log^n(1 - z);
\]

2. Set \( \varrho = \frac{1}{2} + \frac{i\sqrt{3}}{2} \) and \( \delta_{\varrho} := \frac{\text{Im} \varrho}{\text{Im} \varrho} = 1 \), before evaluating logarithms and simplifying all the complex arguments of (multiple) polylogarithms;

3. Set \( \varrho = \frac{1}{2} + \frac{i\sqrt{3}}{2} = \frac{1}{\varrho} \) and \( \delta_{\varrho} = -\varrho \), and \( \varrho = \frac{1}{2} - \frac{i\sqrt{3}}{2} \).

4. Convert to HyperLog form in \text{HyperInt}.

Define the \( k \)-th Hankel–Broadhurst integral

\[
\mathcal{S}^\text{HB}_k := \text{fib} \text{Fib} \left. \frac{d^k W_3(s)}{ds^k} \right|_{s=0}
\]

by computing the \( k \)-th order derivative of (2.12) at \( s = 0 \) with \text{HyperInt} [where \( Q(x) := (1 - \varrho)(1 - x \varrho) \) and \( \varrho \) is treated as a formal variable], before evaluating its absolute fibration (“Fib”) and relative fibration basis (“\( \text{fib}_f \)”)

The last relative fibration sends the \( k \)-th Hankel–Broadhurst integral to \( \phi_k(2) = \sum_{\ell=0}^{k+1} \delta_{\ell}^k(2) \mathcal{S}_{k+1-\ell}(2) \), where \( \phi_k(2) \) is a \( \mathbb{Q} \)-vector space generated by the Remiddi–Vermaseren [59] harmonic polylogarithms of weight \( k + 1 \). Members in the latter \( \mathbb{Q} \)-vector space can be decomposed into Lyndon words [58] by Maitre’s HPL package [54] for \text{Mathematica}. We use “Lyn” to denote this Lyndon word decomposition together with subsequent reduction by “\( \text{fib}_f \)”. The processed Lyndon words Lyn \( \mathcal{S}^\text{HB}_k \), \( k \in \{1, 2, 3, 4\} \) can now be fed into Au’s \text{Mathematica} package \text{MultipleZetaValues} (v1.1.0), which performs automated (and provable [4, §5]) reductions for members of \( \mathcal{S}_k(6) \), \( k \in \mathbb{Z} \cap [1, 5] \).

This confirms all the entries in Table 1. It is worth noting that all those tabulated formulae suggest that

\[
\pi \text{Im} \varrho(1 + x_1 + x_2) \in \mathcal{S}_{k+1}(3),
\]

which is stronger than the relation (2.19) established in Theorem 2.5.

To get ready for the proof of a sharper statement (2.44) in the next theorem, we recall the multi-dimensional polylogarithm

\[
L_{\alpha_1, \ldots, \alpha_n}(x) \equiv \text{Li}_{\alpha_1, \ldots, \alpha_n}(x, 1, \ldots, 1) := \sum_{\ell_1 > \cdots > \ell_n > 0}^{x^{\ell_1}} \prod_{j=1}^{n} \ell_{j}^{\alpha_j}
\]

1. If all the components of \( \alpha = (\alpha_1, \ldots, \alpha_k) \) belong to \( \{-1, 0, 1\} \), then HPL\{\( \alpha_1, \ldots, \alpha_k, \ z\)\} (in Maitre’s HPL package [54]) is equal to \((-1)^{n_1(\alpha)} G(\alpha; z)\), where \( n_1(\alpha) := \# \{ j \in \mathbb{Z} \cap [1, k] \mid \alpha_j = 1 \} \) counts the number of 1’s in the components of \( \alpha \).

2. Prior to Au’s analytic work on MPLs, the Henn–Smirnov–Smirnov database [43] provided empirical reduction formulae for CMZVs in \( \mathcal{S}_k(6) \), \( k \in \mathbb{Z} \cap [1, 6] \).

3. The filtration property of \( \mathcal{S}(3) \) helps us to quickly identify certain terms as Goncharov–Deligne periods at third roots of unity. For example, the relation \( \pi i \zeta_3 \in \mathcal{S}_3(3) \) builds upon \( \pi i = 3[\text{Li}_1(\omega) - \text{Li}_1(\omega^2)] \in \mathcal{S}_3(3) \) [cf. (2.15)], \( \zeta_3 = \text{Li}_3(1) \in 3(3), \) and \( 3(3) \mathcal{S}_3(3) \subseteq 3(3) \).
from the work of Borwein–Broadhurst–Kanalitzer [16, §2]. For brevity, we write

\[ L_A(x) = L_{a_1,\ldots,a_n}(x) \]

for a vector \( A = (a_1, \ldots, a_{d(A)}) \in \mathbb{Z}_{>0}^{d(A)} \), while calling \( d(A) \) the depth of \( L_A \), and \( w(A) := \sum_{j=1}^{d(A)} a_j \) the weight of \( L_A \). In a trigonometric analysis of the hyper-Mahler measures for \( 1 + x_1 + x_2 \), Borwein–Borwein–Straub–Wan [8, Theorem 26] introduced a \( \mathbb{Q} \)-linear combination over certain multi-dimensional polylogarithms of weight \( n \in \mathbb{Z}_{>1} \) and all possible depths

\[ \rho_n(X) := \frac{(-1)^n n!}{4^n} \sum_{\substack{w(A) \equiv n, a_1 = 2 \\ a_2, \ldots, a_{d(A)} \in \{1,2\}}} 4^{d(A)} L_A(X^2), \quad X \in (0,1), \]

and defined additionally that \( \rho_0(X) = 1, \rho_1(X) = 0 \).

**Theorem 2.7** \((\pi i m_k(1 + x_1 + x_2) \) as Goncharov–Deligne periods, reprise). For each positive integer \( k \), we have

\[ \pi m_k(1 + x_1 + x_2) \]

\[ = 2 \int_0^{\pi/6} \rho_k(2 \sin \theta) \, d\theta + 2 \sum_{j=0}^k \binom{k}{j} \int_0^{\pi/6} \rho_j \left( \frac{1}{2 \sin \theta} \right) \log^{k-j}(2 \sin \theta) \, d\theta \in i \mathfrak{S}_{k+1}(3). \]

**Proof.** The integral representation in (2.47) is due to Borwein–Borwein–Straub–Wan [8, Theorem 17].

We reparametrize the integral over \( \theta \in (0, \pi/6) \) with \( \theta = \arctan \frac{1+t}{\sqrt{1-t^2}}, t \in (-1,0) \), so that

\[ \int_0^{\pi/6} \rho_k(2 \sin \theta) \, d\theta = - \frac{i}{2} \int_{-1}^0 \left( \frac{1}{t - \theta} - \frac{1}{t - \theta^*} \right) \rho_k \left( \frac{1+t}{\sqrt{1-t^2}} \right) \, dt. \]

By induction on the weight \( w(A) \) and applications of the GPL recursion (2.22) to \( z = \frac{(1+t)^2}{1-t+t^2} \), we obtain

\[ L_A \left( \frac{(1+t)^2}{1-t+t^2} \right) \]

\[ \in \text{span}_\mathbb{Z} \left\{ (\pi i)^t G(\alpha_1, \ldots, \alpha_j; t) G(\alpha_{j+1}, \ldots, \alpha_{w(A) - \ell}; -1) : \alpha_1, \ldots, \alpha_{w(A) - \ell} \in \mathbb{Z}, \ell \in \mathbb{Z} \cap \{0, w(A); \} \right\} \]

for \( q = e^{\pi i/3} \) and

\[ i \int_0^{\pi/6} \rho_k(2 \sin \theta) \, d\theta \]

\[ \in \text{span}_\mathbb{Q} \left\{ (\pi i)^t G(\alpha_1, \ldots, \alpha_{k+1}; -1) : \alpha_1, \ldots, \alpha_{k+1} \in \mathbb{Z}, \ell \in \mathbb{Z} \cap \{0, w(A)+j; \} \right\} \]

\[ \subseteq \text{span}_\mathbb{Q} \left\{ (\pi i)^{k+1-n} G(\alpha_1, \ldots, \alpha_n; 1) : \alpha_1, \ldots, \alpha_n \in \mathbb{Z}, n \in \mathbb{Z} \cap \{0, k+1; \} \right\} \subseteq \mathfrak{S}_{k+1}(3) \]

for \( \omega = e^{2\pi i/3} \).

Arguing as in the last paragraph, we can check that

\[ i \int_0^{\pi/6} \log^k(2 \sin \theta) \, d\theta \in \mathfrak{S}_{k+1}(3). \]
Meanwhile, a constructive proof for the relation [13, §2.1]

\[
\int_0^{\pi/2} \log^k(2 \sin \theta) \, d \theta \in \pi \mathfrak{Z}_k(1) \subseteq \pi \mathfrak{Z}_k(3)
\]

is known. Combining the last two displayed equations with the fact that \( \pi i \in \mathfrak{Z}_k(3) \) [see (2.15)], we get

\[
i \int_0^{\pi/2} \log^k(2 \sin \theta) \, d \theta \in \mathfrak{Z}_{k+1}(3).
\]

Each one of the remaining integrals \( \int_{\pi/6}^{\pi/2} \rho_j \left( \frac{1}{2 \sin \theta} \right) \log^{k-j}(2 \sin \theta) \, d \theta \) (where \( j \in \mathbb{Z} \cap [2, k] \)) contributes a summand to the right-hand side of a vanishing identity

\[
0 = \lim_{\epsilon \to 0^+} \left( \int_{1-\epsilon}^{1+i \epsilon} + \int_{1-\epsilon}^{-i \epsilon} + \int_{1-\epsilon}^{1/\epsilon} + \int_{1-\epsilon}^{-1/\epsilon} \right) \rho_j \left( \sqrt{\frac{-z}{(1-z)^2}} \right) \left[ \log \frac{(1-z)^2}{-z} \right]^{k-j} \frac{dz}{iz},
\]

where the path of integration for each \( \int \) runs counterclockwise along a circular arc centered at the origin. Concretely speaking, we have

\[
2^{k-j+2} \int_{\pi/6}^{\pi/2} \rho_j \left( \frac{1}{2 \sin \theta} \right) \log^{k-j}(2 \sin \theta) \, d \theta,
\]

which is equal to the sum of

\[
- \lim_{\epsilon \to 0^+} \left( \int_{1-\epsilon}^{1+i \epsilon} + \int_{1-\epsilon}^{-i \epsilon} \right) \rho_j \left( \sqrt{\frac{-z}{(1-z)^2}} \right) \left[ \log \frac{(1-z)^2}{-z} \right]^{k-j} \frac{dz}{iz} \in \text{span}_Q \left\{ \int_0^1 \rho_j \left( \sqrt{\frac{-z}{(1-z)^2}} \right) \left[ \log \frac{(1-z)^2}{-z} \right]^{k-j-n} \frac{dz}{iz} \mid n \in \mathbb{Z} \cap [0, k-j] \right\}
\]

and

\[
- \lim_{\epsilon \to 0^+} \left( \int_{1-\epsilon}^{1-i \epsilon} + \int_{1-\epsilon}^{-i \epsilon/\epsilon} \right) \rho_j \left( \sqrt{\frac{-z}{(1-z)^2}} \right) \left[ \log \frac{(1-z)^2}{-z} \right]^{k-j} \frac{dz}{iz} \in i \text{span}_Q \left\{ \int_1^{i \epsilon} (\pi i)^n Z_i G \left( \alpha_1, \ldots, \alpha_{j-l-n}; \frac{1-\zeta}{-z} \right) \times \left[ \log \frac{(1-z)^2}{-z} \right]^{k-j} \frac{dz}{iz} \right\} \]

\[
\times \left\{ \begin{array}{c}
Z_i \in \mathfrak{Z}_1(1) \\
\alpha_1, \ldots, \alpha_{j-l-n} \in \{0, 1\} \\
n \in \mathbb{Z} \cap [0, j]; \ell \in \mathbb{Z} \cap [0, j-n]
\end{array} \right\} + i \text{span}_Q \left\{ \int_1^{1/\epsilon} (\pi i)^n Z_i G \left( \alpha_1, \ldots, \alpha_{j-l-n}; \frac{1-\zeta}{-z} \right) \times \left[ \log \frac{(1-z)^2}{-z} \right]^{k-j} \frac{dz}{iz} \right\} \]

\[
\times \left\{ \begin{array}{c}
Z_i \in \mathfrak{Z}_1(1) \\
\alpha_1, \ldots, \alpha_{j-l-n} \in \{0, 1\} \\
n \in \mathbb{Z} \cap [0, j]; \ell \in \mathbb{Z} \cap [0, j-n]
\end{array} \right\}.
\]
Here, the multi-dimensional polylogarithm behaves like \( L_{A}(\Xi \pm i0^+) \in \varphi_{w(A)}^{(1/\Xi)}(1) + \pi i \varphi_{w(A)}^{(1/\Xi)}(1) \), with a jump discontinuity \([55, 57]\) \( L_{A}(\Xi + i0^+) - L_{A}(\Xi - i0^+) \in \pi i \varphi_{w(A)}^{(1/\Xi)}(1) \) \([\text{see } (2.30b) \text{ for the definition of } \varphi_{k}^{(z)}(N)]\), across its branch cut where \( \Xi > 1 \). To handle the right-hand side of (2.56), we may switch back to the parameter \( t = \frac{\pi z - 1}{z - e} \), as follows:

\[
(2.58) \quad \int_{0}^{1} \rho_{j} \left( \sqrt{-\frac{z}{(1 - z)^2}} \right) \left[ \frac{(1 - z)^2}{z} \right]^{k - j - n} \frac{(\pi i)^n}{iz} dz = i \int_{-1}^{1} \rho_{j} \left( \frac{\sqrt{1 - t + t^2}}{1 + t} \right) \left[ \frac{(1 - t)2^2}{1 - t + t^2} \right]^{k - j - n} \frac{(\pi i)^n}{t - \tilde{\vartheta}} \left( 1 - \frac{1}{t - \tilde{\vartheta}} \right) dt \\
\quad \in i \text{span}_{Q} \left\{ (\pi i)^{k+1-\ell} G(\alpha_{1}, \ldots, \alpha_{j} ; u) \right\} \times \\
\quad \times G(\alpha_{j+1}, \ldots, \alpha_{i} ; \tilde{\vartheta}) \in C \left\{ \alpha_{1}, \ldots, \alpha_{i} \in \{ 0, -1, \frac{1}{2} \} ; u \in \{-1, \frac{1}{2} \} \right\} \\
\quad \subseteq i \text{span}_{Q} \left\{ (\pi i)^{k+1-\ell} G(\alpha_{1}, \ldots, \alpha_{i} ; 1) \right\} \left\{ \alpha_{1}, \ldots, \alpha_{i} \in \{ 0, 1, \omega = -\frac{1}{2} \} ; \tilde{\vartheta} \in \text{C} \right\} \\
\quad \subseteq i \mathcal{I}_{3k+1}(3).
\]

Likewise, by integrating GPLs over \( t = \frac{\pi z - 1}{z - e} \in [-1, 0] \), one can identify the right-hand side of (2.57) with a subset of \( i \mathcal{I}_{3k+1}(3) \), as we did in (2.50).

Therefore, we have \( \pi m_{k}(1 + x_{1} + x_{2}) \in i \mathcal{I}_{3k+1}(3) \) as claimed. \( \square \)

**Remark 2.8.** If we reparametrize all the integrals in (2.47) with \( \theta = \arctan \frac{1 + t}{\sqrt{3(1 - t)}} \), \( t \in (-1, 1) \), then we have

\[
(2.47') \quad \mathcal{I}_{BWS}^{k} := \int_{-1}^{0} \left( \frac{1}{t - \tilde{\vartheta}} - \frac{1}{t - \frac{1}{\tilde{\vartheta}}} \right) \frac{\rho_{j} \left( \sqrt{Q(t)} \right)}{\pi i} dt \\
\quad + \frac{k}{j} \int_{0}^{1} \left( \frac{1}{t - \tilde{\vartheta}} - \frac{1}{t - \frac{1}{\tilde{\vartheta}}} \right) \frac{\rho_{j} \left( \sqrt{Q(t)} \right)}{2\pi j} \log^{k-j} \frac{(1 + t)^{2}}{Q(t)} dt,
\]

where \( Q(t) := (1 - \tilde{\vartheta})(1 - t \tilde{\vartheta}) \). Evaluating in HyperInt and HPL, one can check that \( \text{Lyn} \rho_{k}^{BWS} = \text{Lyn} \rho_{k}^{HB}, k \in \{1, 2, 3, 4\} \).

**Remark 2.9.** Let

\[
(2.59) \quad \text{Ls}_{k}(\varphi) := - \int_{0}^{\varphi} \log^{k-1} \left| 2 \sin \frac{\vartheta}{2} \right| d \vartheta
\]

be Lewin’s log-sine integral \([52, 53]\) for \( k \in \mathbb{Z}_{> 0} \). From Borwein–Straub \([12, \text{Theorem 3}]\), one sees that \( \text{Ls}_{k}(2\pi/3) \in i \mathcal{I}_{3k}(3) \). According to (2.51), we also have \( \text{Ls}_{k}(\pi/3) \in i \mathcal{I}_{3k}(3) \), which is not immediate from the Borwein–Straub reduction \([12, \text{Theorem 3}]\) of log-sine integrals. Borwein–Borwein–Straub–Wan \([8, (108) \text{ and } (109)]\) proposed conjectural expressions for \( m_{k}(1 + x_{1} + x_{2}) \) and \( m_{k}(1 + x_{1} + x_{2}) \), using log-sine integrals at \( \pi/3 \) and \( 2\pi/3 \). These conjectures are equivalent to the corresponding formulae in Table 1, as one can check with Au’s algorithm \([4, \S 5]\). \( \square \)
adapted to the explicit construction of Borwein–Straub [12, Theorem 3], we have Lewin’s generalized log-sine integral [53, Appendix A.1(20)] for \( (2.60) \)

Here, the "step builds upon a more general result before specializing to \( \omega \). To prove this, use the GPL recursion (2.22) (with an integration path connecting \( (2.62) \)) to show inductively that \( (2.63) \)

To prove this, use the GPL recursion (2.22) (with an integration path connecting \( \omega^2 \) and \( z \)) to show inductively that

\[
(2.60) \quad L_{\varphi}^{(m)}(\varphi) := -\int_0^{\varphi} \varphi^m \log^{k-1-m} \left| 2 \sin \frac{\varphi}{2} \right| \, d\varphi
\]

be Lewin’s generalized log-sine integral [53, Appendix A.1(20)] for \( k, m \in \mathbb{Z}_{\geq 0} \). By the explicit construction of Borwein–Straub [12, Theorem 3], we have \( L_{\varphi}^{(m)}(2\pi/3) \in i^{m+1} \mathcal{S}_k(3) \). Meanwhile, we note that the proof of the theorem above can be readily adapted to

\[
L_{\varphi}^{(m)} \left( \frac{\pi}{3} \right)
\]

\[
= -\frac{1}{2^{k-1-m}} \int_1^{\varphi} \left( \frac{\log z}{t} \right)^m \left[ \log \left( \frac{1-z}{z} \right) \right]^{k-1-m} \frac{dz}{iz}
\]

\[
\in i^{m+1} \text{span}_{\mathbb{Q}} \left\{ (\pi i)^{k-\ell} G(\alpha_1, \ldots, \alpha_{\ell}; u) \in \mathbb{C} \mid \alpha_1, \ldots, \alpha_{\ell} \in \{0, 1\} : w \in \{1, \varphi\} \right\}
\]

\[
\subseteq i^{m+1} \mathcal{S}_k(3).
\]

Here, the "\( \subset \)" step builds upon a more general result

\[
(2.62) \quad \text{span}_{\mathbb{Q}} \left\{ G(\alpha_1, \ldots, \alpha_k; z) \mid \alpha_1, \ldots, \alpha_k, z \in \{0, 1, \varphi, \frac{1}{\varphi}\} \right\} \subseteq \mathcal{S}_k(3).
\]

To prove this, use the GPL recursion (2.22) (with an integration path connecting \( \omega^2 \) and \( z \)) to show inductively that

\[
\text{span}_{\mathbb{Q}} \left\{ G(\alpha_1, \ldots, \alpha_k; u) \mid \alpha_1, \ldots, \alpha_k, w \in \{0, 1, \varphi, \frac{1}{\varphi}\} : \alpha_1 \neq 1, \alpha_k \neq 0 \right\}
\]

\[
\subseteq \text{span}_{\mathbb{Q}} \left\{ (\pi i)^{k-\ell} G(\alpha_1, \ldots, \alpha_j; z) G(\alpha_{j+1}, \ldots, \alpha_{\ell}; \omega^2) \in \mathbb{C} \mid \alpha_1, \ldots, \alpha_{\ell} \in \{0, 1, \omega^2\} \right\}
\]

\[
\text{before specializing to } z = \omega := e^{2\pi i/3}.
\]

**2.4. Period structure of \( m_k(1 + x_1 + x_2 + x_3) \).** With the \( \mathbb{Q} \)-vector space \( g_k^{(2)}(2) \) introduced in (2.30b), we can extend our proof of Theorem 2.5 to the \( k \)-Mahler measures of \( 1 + x_1 + x_2 + x_3 \).

**Theorem 2.11** \( \pi^2 m_k(1 + x_1 + x_2 + x_3) \) as Goncharov–Deligne periods. For \( n \in \mathbb{Z}_{\geq 0} \), define\(^9\)

\[
(2.64) \quad X^{(n)}_k(\tau) := \int_1^{\tau} \left[ \log A_k(t, \tau) + \log B_k(\tau) \right]^{n} \, dt, \quad \bar{X}^{(n)}_k(\tau) := \int_1^{\tau} \frac{\log^n A_k(t, \tau)}{t} \, dt.
\]

\[
(2.65) \quad \Xi^{(n)}_k(\tau) := \int_1^{-1} \left[ \log A_k(t, \tau) + \log B_k(\tau) \right]^{n} - \delta_{n, 0} \, dt, \quad \bar{\Xi}^{(n)}_k(\tau) := \int_1^{-1} \frac{\log^n A_k(t, \tau)}{t} \, dt
\]

\(^9\)We set the Kronecker delta as \( \delta_{a,b} = 1 \) when \( a = b \), and \( \delta_{a,b} = 0 \) otherwise.
using the rational functions in (2.11). For each positive integer \( k \), we have

\[
\pi^2 m_k(1 + x_1 + x_2 + x_3) - \pi^2 m_k(1 + x_1)
\in \text{span}_Q \left\{ \frac{1}{\pi - 2n_2} \int_{1}^{-1} X_4^{(n_2)}(\tau) X_4^{(n_3)}(\tau) \frac{\tau + 1}{\tau - 1} \, d\tau \mid (\tau,1) \in \mathbb{Z}^{n_1}\right. \\
\left. \left. (\tau) \in \mathbb{Z}^{n_2}\right) \in \mathbb{Z}^{n_3} = k \right\}
\]

(2.66)

\[
+ \text{span}_Q \left\{ \frac{1}{\pi - 2n_2} \int_{-1}^{0} \Xi_4^{(n_2)}(\tau) \Xi_4^{(n_3)}(\tau) \frac{\tau + 1}{\tau - 1} \, d\tau \mid (\tau) \in \mathbb{Z}^{n_1}\right. \\
\left. \left. (\tau) \in \mathbb{Z}^{n_2}\right) \in \mathbb{Z}^{n_3} = k \right\}
\]

(2.67)

which simplifies the earlier proof of the same statement by Kurokawa–Lalín–Ochiai [48, Proposition 4]. Thus, it indeed suffices to work out (2.66) for our goal in (2.67).

**Proof.** As we may recall, Borwein–Straub [13, (2.4), (2.5), (4.3)] devised a recursion for \( m_k(1 + x_1), k \in \mathbb{Z}^{n_2} \) and showed that

\[
m_k(1 + x_1) \in \text{span}_Q \left\{ \prod_{j=1}^{n} \zeta(a_j) \mid a_j \in \mathbb{Z}^{n_2}, j \in [1, n]; \sum_{j=1}^{n} a_j = k \right\}
\]

(2.68)

which reduces our task to the demonstration of

\[
\left\{ \int_{1}^{-1} X_4^{(n_2)}(\tau) X_4^{(n_3)}(\tau) \frac{\tau + 1}{\tau - 1} \, d\tau \right. \\
\left. \left. + \int_{-1}^{0} \Xi_4^{(n_2)}(\tau) \Xi_4^{(n_3)}(\tau) \frac{\tau + 1}{\tau - 1} \, d\tau \right) \in \mathbb{Z}^{n_1}\right. \\
\left. \left. (\tau) \in \mathbb{Z}^{n_2}\right) \in \mathbb{Z}^{n_3} = k \right\} \subseteq \mathfrak{g}_k(2)
\]

(2.69)

we may reduce our task to the demonstration of

\[
\left\{ \int_{1}^{-1} X_4^{(n_2)}(\tau) X_4^{(n_3)}(\tau) \frac{\tau + 1}{\tau - 1} \, d\tau \right. \\
\left. \left. + \int_{-1}^{0} \Xi_4^{(n_2)}(\tau) \Xi_4^{(n_3)}(\tau) \frac{\tau + 1}{\tau - 1} \, d\tau \right) \in \mathbb{Z}^{n_1}\right. \\
\left. \left. (\tau) \in \mathbb{Z}^{n_2}\right) \in \mathbb{Z}^{n_3} = k \right\} \subseteq \mathfrak{g}_k(2)
\]

(2.70)

for \( n_2, n_3 \in \mathbb{Z}^{n_2} \) and \( n_2 + n_3 = k - 1 \).

With

\[
\log A_4(t, \tau) := \log \frac{(1 - t)(1 - t\tau)}{t} = (1 - t\tau) - G(0; t)
\]

(2.71)

in hand, we have the following result for \( k \in \mathbb{Z}^{n_2} \):

\[
\Xi_4^{(k)}(\tau) := \int_{1}^{\tau} \frac{dt}{t} \log \frac{(1 - \frac{t}{\tau})(1 - t\tau)}{t} 
\]

(2.72)

\[
\in \text{span}_Q \left\{ G(\alpha_1, \ldots, \alpha_k; z) \in \mathbb{C} \mid \alpha_1, \ldots, \alpha_k \in \{0, 1\} \right\}
\]

\[
\subseteq \mathfrak{g}_{k+1}(2)
\]

(2.73)

Here, the rationales behind both the “\( \in \)” step and the “\( \subseteq \)” step are identical to what we provided for (2.35). For the “\( \subseteq \)” step, the fibration output resides in the following \( \mathbb{Z} \)-module:

(2.74)
as we inductively apply the GPL recursion (2.22) (with \( z = \tau \) and \( z = 1 \)) to the GPL differential form (2.37) [where \( \log(\alpha_{j-1} - \alpha_j), \log(\alpha_{j+1} - \alpha_j) \in \{ \log(\tau - 1), \log(\tau - 1), \log(\tau - 1) \} \) ]. Taking into account

\[
\log \mathcal{B}_4(\tau) := \log \frac{-\tau}{(1 - \tau)^2} \equiv \pi i + G(0; \tau) - 2G(1; \tau) \quad (\text{mod } 2\pi i \mathbb{Z}),
\]

we may check that \( X_4^{(k)}(\tau) \in \pi i \log^k \mathcal{B}_4(\tau) + \text{span}_Q \{ \tilde{X}_4^{(j)}(\tau) \log^{k-j} \mathcal{B}_4(\tau) | j \in \mathbb{Z} \cap [0, k] \} \subseteq \mathfrak{g}_{k+1}(2) \).

Likewise, one can verify that \( \{ \Xi_4^{(k)}(\tau), \Xi_4^{(k)}(\tau) \} \subseteq \mathfrak{g}_{k+1}(2) \) holds for all \( k \in \mathbb{Z}_{\geq 0} \).

Thus far, we see that

\[
\begin{align*}
\tilde{X}_4^{(n_2)}(\tau)X_4^{(n_3)}(\tau) &\in \mathfrak{g}_{n_2+n_3+2}(2), \\
\Xi_4^{(n_2)}(\tau)\Xi_4^{(n_3)}(\tau) &\in \mathfrak{g}_{n_2+n_3+2}(2).
\end{align*}
\]

As we expand

\[
\frac{\tau + 1}{\tau - 1} = \frac{1}{\tau} - \frac{2}{\tau - 1},
\]

and invoke the closure property (2.31) for integrations over \( \tau \), we may deduce

\[
\pi^2 m_k(1 + x_1 + x_2 + x_3) \in \mathfrak{g}_{k+2}(2) \cap \mathbb{R} = \mathfrak{b}_{k+2}(2). \quad \square
\]

Thanks to the data mine of Blümlein–Broadhurst–Vermaseren [6], we can reduce every AMZV (up to weight 8) to a polynomial in \( \mathbb{Q}[\log 2, \pi^2, \zeta_3, \zeta_{-3}, \zeta_5, \zeta_{-3,1,1,\ldots}] \), using a look-up table that ships with HyperInt. This leads us to the evaluations in Table 2.

All the entries in the aforementioned look-up table for \( \mathfrak{g}(2), k \in \mathbb{Z} \cap [1, 8] \) are analytically provable by Deligne’s standard relations [69, Definition 1.1] for \( \mathbb{Q} \)-linearly dependent multiple polylogarithms.\(^1\)

Before closing this section, we compare our results to some related works on hyper-Mahler measures.

In their foundational paper [48, Theorem 3], Kurokawa, Lalín, and Ochiai established a remarkable connection between the \( \mathbb{Q} \)-vector space \( \mathfrak{g}(1) \) incorporating MZVs of weight \( k \) and the hyper-Mahler measure \( m_k(1 + x_1) \) for \( k \in \mathbb{Z}_{>1} \):

\[
m_k(1 + x_1) = (-1)^k k! \sum_{a_1 + \cdots + a_n = k \atop a_j \geq 2, j \in \mathbb{Z} \cap (1,n]} \frac{\zeta_{a_1,\ldots,a_n}}{2^n} \in \mathfrak{g}(1).
\]

The same extends to the \( k = 1 \) case, where \( m_1(1 + x_1) = 0 \) and an empty sum over \( \mathfrak{g}(1) = \{ 0 \} \) must vanish.

Using Lalín’s method [49], Kurokawa–Lalín–Ochiai originally demonstrated (2.78) by writing the hyper-Mahler measures \( m_k(1 + x_1) \) as analogs of the Drinfel’d integral representation of MZVs (see [33, §2] or [68, §9]), the latter of which are special cases of Goncharov–Manin periods [42] on \( \mathfrak{g}_{0,n} \).

\(^1\)Some provable non-standard \( \mathbb{Q} \)-linear relations in \( \mathfrak{g}(N) \) are covered by Au’s method [4, §5.1], where \( N \in \{ 6, 8, 10, 12 \} \).
Our Theorems 2.7–2.11 extend the Kurokawa–Lalín–Ochiai sum rule (2.78) to Goncharov–Deligne periods in \(3_{k+1} (3)\) and \(3_{k+2} (2)\) for \(k \in \mathbb{Z}_{>0}\):

\[
(2.79) \quad \pi i m_k(1 + x_1 + x_2) \in 3_{k+1} (3),
\]

\[
(2.80) \quad (\pi i)^2 m_k(1 + x_1 + x_2 + x_3) \in 3_{k+2} (2).
\]

If we set \(m_0(1 + x_1 + x_2) = m_0(1 + x_1 + x_2 + x_3) = 1\), then the same is true for \(k = 0\).

The hyper-Mahler measures for the multivariate family \(R_m(x_1, \ldots, x_m, z) = z + \prod_{j=1}^m \frac{1-z}{1-x_j}\) are expressible as \(\mathbb{Q}\)-linear combinations of multiple polylogarithms whose arguments are fourth roots of unity, as established by Lalín–Lechasseur [50, Theorem 1.1]. The results of Lalín–Lechasseur hinged on a representation of \(m_k(a + x_1)\) via multi-dimensional polylogarithms of \([-2 \text{sgn} \log|a|]\) [50, Theorem 4.5] (extending a result of Akatsuka [1, Theorem 7], in a similar spirit as Borwein–Brown–Straub–Wan [8, Theorem 26]) and subsequent computations of certain motivic periods on \(\mathfrak{M}_{0,n}\) [50, §§7–8].

We are not sure whether there are systematic classifications of all the hyper-Mahler measures \(m_k(1 + \sum_{j=1}^\ell x_j)\), where \(k\) and \(\ell\) are positive integers. The conjectural characterizations of \(m_1(1 + \sum_{j=1}^4 x_j)\) and \(m_1(1 + \sum_{j=1}^5 x_j)\) by Rodríguez-Villegas [17, §8] suggest that the moduli spaces \(\mathfrak{M}_{g,n}\) for \(g > 0\) are needed to quantify these logarithmic Mahler measures for polynomials in 4 and 5 variables.

### 3. Some infinite series behind hyper-Mahler measures

#### 3.1. Non-linear Euler sums revisited

We open this section with a non-exhaustive overview of some infinite sums that revolve around the harmonic numbers of order \(r\)

\[
(3.1) \quad \mathcal{H}_n^{(r)} := \sum_{k=1}^\infty \left[ \frac{1}{k^r} - \frac{1}{(k + n)^r} \right].
\]

Aside from their connections to the hyper-Mahler measures (see §§3.2–3.3 for details) investigated in the current work, these infinite series play significant roles in the quantitative understanding of Feynman diagrams in quantum electrodynamics and quantum chromodynamics [19, 46, 29, 30, 6].

Studies of “linear Euler sums” like

\[
(3.2) \quad \sum_{n=1}^\infty \frac{\mathcal{H}_n^{(r)}}{n^s} = \zeta_{s,r} + \zeta_{r+s}, \quad r \in \mathbb{Z}_{>0}, s \in \mathbb{Z}_{>1}
\]

were initiated by Euler [35], in response to Goldbach’s challenge in 1742 [9, §1]. (Here, we note that \(\mathcal{H}_n^{(r)} := \sum_{k=1}^n \frac{1}{k^r}\) for \(n \in \mathbb{Z}_{>0}\).) Borwein–Zucker–Boersma [9] generalized the Euler–Goldbach double sums to \(\{p, q\}(s, t) := \sum_{m>n>0} \frac{\chi_m p_m \chi_q(n)}{m^n n^t}\) for Dirichlet characters \(p\) and \(q\). This allows one to convert infinite sums like [9, Table 1]

\[
(3.3) \quad \sum_{n=1}^\infty \frac{\mathcal{H}_n^{(r)}}{n^s}, \quad \sum_{n=1}^\infty \frac{(-1)^n \mathcal{H}_n^{(r)}}{n^s}, \quad \sum_{n=1}^\infty \frac{\mathcal{H}_n^{(r)}}{(2n-1)^s}, \quad \sum_{n=1}^\infty \frac{(-1)^n \mathcal{H}_n^{(r)}}{(2n-1)^s}
\]

into a single integral over polylogarithmic expressions [9, Table 2]. Although predating Brown’s work [23, 24, 25], the integral representations of Borwein–Zucker–Boersma [9, Table 2] belong to Brown’s \(L(\mathfrak{M}_{0,n})\) class. A linear Euler sum \(\zeta_{-3,1}\)
in its polylogarithmic avatar \( \zeta_{-3.1} = 2 \text{Li}_4 \left( \frac{1}{2} \right) - \frac{\pi^4}{48} + \frac{7 \zeta(2) \log 2}{12} - \frac{\pi^2 \log^2 2}{12} + \frac{\log^4 2}{48} \) [9, (5.5)] featured prominently in the Laporta–Remiddi formula [51, (5)] for the 3-loop contribution to electron’s magnetic moment.

The following “non-linear Euler sums”

\[
\sum_{n=1}^{\infty} \frac{\left[ H_n^{(1)} \right]^2}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n \left[ H_n^{(1)} \right]^2}{n^2}, \quad \sum_{n=1}^{\infty} \frac{\left[ H_n^{(2)} \right]^2}{n^2}
\]

had been considered by Borwein–Borwein [7], De Doelder [31], and Mező [56] before Panzer’s HyperInt package [57] became available. In retrospect, these authors have effectively converted their series of interest into members of \( L(M_0,n) \) and have referred to the classical treatise on polylogarithms [53] for integral evaluations, without regard to the algebraic geometry of the moduli spaces \( M_{0,n} \). Like their linear brethren, the non-linear Euler sums found their ways into perturbative expansions in high energy physics. While considering on-shell charge renormalization, Broadhurst [19, (42)] differentiated a certain hypergeometric expression and encountered non-linear Euler sums in the form of

\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^a} \prod_{i=1}^{k} \frac{1}{(2\ell_i - 1)^{b_i}}.
\]

More sophisticated analogs of such infinite series showed up in the \( \varepsilon \)-expansions of Kalmykov–Veretin [46, (4)], Davydychev–Kalmykov [29, (1.11)] and Kalmykov–Ward–Yost [47, Theorems A and B].

For the prototypic non-linear Euler sums like those in (3.4), we have the following characterization of their period structure, which in turn, generalizes the Xu–Wang algorithm [66].

**Theorem 3.1 (Non-linear Euler sums, HPLs, and AMZVs).** (a) For \( |z| < 1 \) and \( r_1, \ldots, r_M \in \mathbb{Z}_{>0} \), we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} \text{H}_{n}^{(r_j)} \right] \frac{1}{1-z} \quad \text{span}_Q \left\{ L_A(z) \mid w(A) = \sum_{j=1}^{M} r_j \right\} = \frac{\Theta^{(z)}_{\sum_{j=1}^{M} r_j}}{1-z},
\]

where the multi-dimensional polylogarithms \( L_A(z) \) are defined by (2.45), and the \( \mathbb{Q} \)-vector space \( \Theta_{k}(1) \) is given in (2.24). By extension, we have (alternating) MZV representations (cf. [66, Theorems 2.2 and 4.2])

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} \text{H}_{n}^{(r_j)} \right] \frac{1}{n^{s+1}} \in \mathfrak{F}_{k+1}(1),
\]

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} \text{H}_{n}^{(r_j)} \right] \frac{(-1)^n}{n^{s}} \in \mathfrak{F}_{k}(2),
\]

where \( s \in \mathbb{Z}_{>0} \) and \( k = s + \sum_{j=1}^{M} r_j \).

(b) Define

\[
\mathfrak{F}_{n}^{(r)} := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^r}, \quad n \in \mathbb{Z}_{>0}.
\]
For \( |z| < 1 \), \( r_1, \ldots, r_M \in \mathbb{Z}_{>0} \), and \( r_{\tau_1}, \ldots, r_{\tau_T} \in \mathbb{Z}_{>0} \), we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_n^{(r_j)} \right] \left[ \prod_{j=1}^{T} H_{n_j}^{(r_j)} \right] z^n \in \frac{\Phi_k^{(z)}_{(2)}(z)}{1 - z},
\]

where \( \Phi_k^{(z)}_{(2)} \) [defined in (2.24)] is a subspace of \( \mathcal{H}_k^{(2)} \) [defined in (2.25)], the latter being the \( \mathbb{Q} \)-vector space spanned by all the Remiddi–Vermaseren harmonic polylogarithms (HPLs) [59] of weight \( k \). By extension, we have AMZV representations (cf. [66, Theorem 4.2])

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_n^{(r_j)} \right] \left[ \prod_{j=1}^{T} H_{n_j}^{(r_j)} \right] \frac{1}{n^{s+1}} \in \mathcal{G}_{k+1}(2),
\]

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_n^{(r_j)} \right] \left[ \prod_{j=1}^{T} H_{n_j}^{(r_j)} \right] \frac{(-1)^n}{n^s} \in \mathcal{G}_k(2),
\]

where \( s \in \mathbb{Z}_{>0} \) and \( k = s + \sum_{j=1}^{M} r_j + \sum_{j=1}^{T} \tau_j \).

**PROOF.** (a) First, we note that [56, §5.1]

\[
\sum_{n=1}^{\infty} H_n^{(r)} z^n = \frac{\text{Li}_r(z)}{1 - z}
\]

fits into (3.6) when \( M = 1 \). Suppose that (3.6) holds for positive integers \( M \) up to a certain \( M_0 \in \mathbb{Z}_{>1} \). In the next paragraph, we show that the same will apply to the \( M_0 + 1 \) case.

According to our induction hypothesis, we have the following relation for \( M \in \mathbb{Z} \cap [1, M_0] \):

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_n^{(r_j)} \right] z^n \in \text{span}_\mathbb{Q}\left\{ \int_0^z \frac{L_A(x) \, dx}{x(1 - x)} \bigg| w(A) = \sum_{j=1}^{M} r_j \right\}
\]

\[
\subseteq \text{span}_\mathbb{Q}\left\{ L_A(z) \bigg| w(A) = 1 + \sum_{j=1}^{M} r_j \right\},
\]

where the “\( \subseteq \)” step involves the standard recursion for GPL [see (2.22)] and the conversions between MPL and GPL [see (2.28)–(2.29)]. Since

\[
\int_0^z \frac{L_{a_1,a_2,\ldots,a_n}(x)}{x} \, dx = L_{a_1+1,a_2,\ldots,a_n}(z),
\]

it also follows that

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_n^{(r_j)} \right] \frac{z^n}{n^s} \in \text{span}_\mathbb{Q}\left\{ L_A(z) \bigg| w(A) = s + \sum_{j=1}^{M} r_j \right\}
\]
for $s \in \mathbb{Z}_{>0}$ and $M \in \mathbb{Z} \cap [1, M_0]$. As we note that $H_0^{(r)} = 0$ and $H_n^{(r)} - H_{n-1}^{(r)} = \frac{1}{n^r}$, we may deduce the following result for $k = \sum_{j=1}^{M_0+1} r_j$:

$$\sum_{n=1}^{\infty} \prod_{j=1}^{M_0+1} H_n^{(r_j)} z^n = \sum_{n=1}^{\infty} \prod_{j=1}^{M_0+1} H_n^{(r_j)} - \prod_{j=1}^{M_0+1} H_{n-1}^{(r_j)} z^n$$

\(\subseteq\) \[\mathbb{Q} \text{Li}_{r}(z) + \text{span}_{\mathbb{Q}} \left\{ \sum_{n=1}^{\infty} \prod_{j=1}^{M'} H_n^{(r_j)} z^n \bigg| k = s + \sum_{j=1}^{M'} r_j', s \in \mathbb{Z}_{>0}, M' \in \mathbb{Z} \cap [1, M_0] \right\} \]

where the “$$\subseteq$$” step follows from induction.

Therefore, the relation in (3.6) holds for all positive integers $M$. As a consequence, we may also extend (3.16) to all $M \in \mathbb{Z}_{>0}$.

As we closely examine the derivation of (3.16), we see that on its right-hand side, the first component of the vector $A$ is at least $s$. This implies that the left-hand side of (3.16) converges at the point $z = 1$ (resp. $z = -1$) when $s \in \mathbb{Z}_{>1}$ (resp. $s \in \mathbb{Z}_{>0}$). Specializing (3.16) to the $z = 1$ (resp. $z = -1$) scenario, we arrive at (3.7) [resp. (3.8)].

(b) Direct computations reveal that

$$(1 - z) \sum_{n=1}^{\infty} \prod_{j=1}^{M_0+1} H_n^{(r_j)} \frac{z^n}{n^s} = -\frac{\text{Li}_{r}(-z)}{1-z}$$

holds for $|z| < 1$, where

$$(3.19) \quad -\text{Li}_{r}(-z) = G_{\mathbb{Q}}(0, \ldots, 0, 1; -z) = G_{\mathbb{Q}}(0, \ldots, 0, -1; z) \in \mathfrak{G}_{r}^{(z)}(2).$$

Here, in the penultimate step, we have resorted to the scaling property [37, (2.3)] for GPLs: $G(a_1, a_2, \ldots, a_k; -z) = G(-a_1, -a_2, \ldots, -a_k; z)$ if $a_k \neq 0$. This scaling property also entails the relation $\mathfrak{G}_{r}^{(z)}(2) = \mathfrak{G}_{r}^{(-z)}(2)$ for all $k \in \mathbb{Z}_{>0}$.

After this, repeated invocations of the GPL recursion in (2.22) take us to

$$(3.20) \quad \sum_{n=1}^{\infty} \prod_{j=1}^{M_0+1} H_n^{(r_j)} \frac{z^n}{n^s} \in \mathfrak{G}_{r+s}^{(z)}(2),$$

so it follows that

$$(3.21) \quad (1 - z) \sum_{n=1}^{\infty} \prod_{j=1}^{M_0+1} H_n^{(r_j)} \frac{z^n}{n^s} \in \mathfrak{G}_{r+s}^{(z)}(2),$$

which confirms (3.10) for $M = M = 1$.

Subsequently, we can prove (3.10) by inductions on $M$ and $M$, as we did for part (a).

Like what we have encountered before, the relation in (3.10) lifts to

$$(3.22) \quad \sum_{n=1}^{M} \prod_{j=1}^{M_0+1} H_n^{(r_j)} \prod_{j=1}^{M} H_n^{(r_j)} \frac{z^n}{n^s} \in \mathfrak{G}_{w}^{(z)}(2)$$
for \( s \in \mathbb{Z}_{>0} \) and \( w = s + \sum_{j=1}^{M} r_j + \sum_{j=1}^{M} r_j \). Here, if \( s = 1 \), then the left-hand side of (3.22) is a \( \mathbb{Q} \)-linear combination of functions in the form of

\[
\int_{0}^{z} \frac{G(\alpha_1, \ldots, \alpha_k; x)}{x(1-x)} \, dx = G(0, \alpha_1, \ldots, \alpha_k; z) - G(1, \alpha_1, \ldots, \alpha_k; z)
\]

where \( \alpha_1, \ldots, \alpha_k \in \{-1, 0, 1\}, k = \sum_{j=1}^{M} r_j + \sum_{j=1}^{M} r_j \), so that it converges when \( z = -1 \); if \( s \in \mathbb{Z}_{>1} \), then the left-hand side of (3.22) is a \( \mathbb{Q} \)-linear combination of functions in the form of

\[
\int_{0}^{z} \frac{G(\alpha_1, \ldots, \alpha_k; x)}{x} \, dx = G(0, \alpha_1, \ldots, \alpha_k; z)
\]

where \( \alpha_1, \ldots, \alpha_k \in \{-1, 0, 1\}, k = s-1+\sum_{j=1}^{M} r_j + \sum_{j=1}^{M} r_j \), so that it converges when \( z = \pm 1 \). Therefore, special cases of (3.22) lead us to (3.11) and (3.12). 

**Remark 3.2.** In the statements of the theorem above and its corollaries in two subsections to follow, we only pinpoint the \( \mathbb{Q} \)-vector spaces in which the infinite sums reside. The accompanying proofs contain effective algorithms to evaluate each individual series explicitly.

For example, in Table 3, we compute a few generating functions by the procedures in the proof of the theorem above, and convert some of them to classical polylogarithms via the Frellesvig–Tommasini–Wever method [37]. Here, the generating functions \( G^{(1,2)}(x) \) and \( G^{(1,3)}(x) \) have been reported by Mező [56, Theorem 9] in the form of classical polylogarithms. Our induction step (3.17) is inspired by Mező’s derivation of \( G^{(1,2)}(x) \) [56, (24)].

For \( r, \ell \in \mathbb{Z}_{>0} \) and \( |z| < 1 \), the formula \( \sum_{n=1}^{\infty} H_n^{(r)} \frac{z^n}{n} = \text{Li}_{\ell+r}(z) + \text{Li}_{\ell+r}(z, 1) \in \mathcal{G}_{\ell+r}(1) \) in Table 3 builds inductively on GPL recursions. Likewise, for \( r, \ell, \) and \( z \) in the same ranges, one can prove \( \sum_{n=1}^{\infty} H_n^{(r)} \frac{z^n}{n} = -\text{Li}_{\ell+r}(-z) - \text{Li}_{\ell+r}(z, -1) \in \mathcal{G}_{\ell+r}(2) \) recursively.

**Remark 3.3.** From the generating functions given by the theorem above, one can construct moment sequences that express harmonic numbers. For instance, if we pick a generating function

\[
G^{(r_1, \ldots, r_M)}(z) := \sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n}^{(r_j)} \right] \frac{z^n}{n^s} \in \mathcal{G}_{s+\sum_{j=1}^{M} r_j}(1)
\]

for certain positive integers \( r_1, \ldots, r_M \) and \( s \), then we can extract its Taylor coefficients by residue calculus:

\[
\prod_{j=1}^{M} H_{n}^{(r_j)} \left[ \frac{1}{n^s} \right] = \oint_{|z|=1} \frac{G^{(r_1, \ldots, r_M)}(z)}{z^n \, 2\pi i z} \, dz = \oint_{|w|=1} \frac{G^{(r_1, \ldots, r_M)}(1/w)}{(1/w)^n} \, dw \frac{1}{2\pi i w},
\]

where \( n \in \mathbb{Z}_{>0} \). In view of the jump discontinuities [55, 57] \( G^{(r_1, \ldots, r_M)}(1/w + i0^+) - G^{(r_1, \ldots, r_M)}(1/w - i0^+) \in \pi i \delta^{(w)}(1) \) [cf. (2.30b)] for \( 0 < w < 1 \), we can reformulate the last contour integral in the complex \( w \)-plane as a moment
Table 3. Some generating functions of Mező type

| r  | $G^{(r),m}(x) := \sum_{n=1}^{\infty} \left[ H_n^{(r)} \right]^m x^n, x \in (0, 1) $ |
|----|------------------------------------------------------------------|
| 1  | $G^{(1),1}(x) = \frac{\text{Li}_s(x)}{1-x}$                        |
|    | $G^{(1),2}(x) = \frac{\text{Li}_s(x) + 2\text{Li}_1(x)}{1-x}$     |
|    | $G^{(1),3}(x) = \frac{\text{Li}_s(x) + 3\text{Li}_2(x) + 6\text{Li}_{1,1}(x)}{1-x}$ |
|    | $= \frac{x^2}{2} \left[ \text{Li}_1(x) \right]^3 + \frac{2}{3} \left[ \text{Li}_1(x) \right]^2 \log x + 3 \text{Li}_1(1-x) + \text{Li}_4(x) - 3\zeta_3$ |
|    | $G^{(1),4}(x) = \frac{\text{Li}_s(x) + 4\text{Li}_{1,1}(x) + 6\text{Li}_2(x) + 4\text{Li}_3(x)}{1-x}$ |
|    | $+ 12\text{Li}_{1,2}(x) + 12\text{Li}_{1,1,2}(x) + 24\text{Li}_{1,1,1,1}(x)$ |
|    | $= -\frac{4\zeta_3}{3} \text{Li}_1(x) + \frac{x^2}{3} \left[ \text{Li}_1(x) \right]^3 - \frac{2}{3} \left[ \text{Li}_1(x) \right]^3 (2\pi i - \log x) + \frac{2}{3} \left[ \text{Li}_1(x) \right]^3 + \frac{2x^4}{1-x}$ |
|    | $\quad + \left[ \text{Li}_2(x) \right]^2 - 4 \text{Li}_1(x) \text{Li}_3(x) - 8 \text{Li}_4 \left( \frac{1}{1-x} \right) - 8 \text{Li}_4 \left( \frac{3}{1-x} \right) - 12 \text{Li}_4(1-x) - 7 \text{Li}_4(x)$ |
| 2  | $G^{(2),1}(x) = \frac{\text{Li}_s(x) + 2\text{Li}_2(x)}{1-x}$       |
|    | $G^{(2),2}(x) = \frac{\text{Li}_s(x) + 4\text{Li}_2(x)}{1-x}$     |
|    | $= 4\zeta_3 \text{Li}_1(x) + \frac{x^2}{3} \left[ \text{Li}_1(x) \right]^3 - \frac{2}{3} \left[ \text{Li}_1(x) \right]^3 (\pi i + \log x) - \frac{2}{3} \left[ \text{Li}_1(x) \right]^3 + \frac{2x^4}{1-x}$ |
|    | $\quad + \left[ \text{Li}_2(x) \right]^2 - 4 \text{Li}_1(x) \text{Li}_3(x) - 4 \text{Li}_4 \left( \frac{1}{1-x} \right) - 4 \text{Li}_4 \left( \frac{3}{1-x} \right) - 3 \text{Li}_4(x)$ |

relation

\begin{equation}
\left( 3.27 \right) \quad \prod_{j=1}^{M} H_{n_j}^{(r_j)} \quad \frac{1}{n^s} = \int_0^1 x^{n-1} L_s^{(r_1, \ldots, r_M)}(x) \, dx,
\end{equation}

where $n \in \mathbb{Z}_{>0}$ and $L_s^{(r_1, \ldots, r_M)}(x) \in g_s^{(x)}$ for $\text{Re} \, n > 0$ that vanishes at all the positive integers, it is in fact identically vanishing for all $\text{Re} \, n > 0 \{2, \text{Theorem 2.8.1} \}$. In particular, for the same $L_s^{(r_1, \ldots, r_M)}(x) \in g_s^{(x)}$, as given above, we have

\begin{equation}
\left( 3.28 \right) \quad \prod_{j=1}^{M} H_{n_j}^{(r_j)} \quad \frac{1}{(2n - 1)^s} = \frac{1}{2s} \int_0^1 x^{n-\frac{s}{2}} L_s^{(r_1, \ldots, r_M)}(x) \, dx
\end{equation}

$\quad = \frac{1}{2s-1} \int_0^1 X^{2n-\frac{s}{2}} L_s^{(r_1, \ldots, r_M)}(X^2) \, dX,$
a moment relation that will become useful later in Corollary 3.12. □

3.2. Infinite series related to \( m_k(1 + x_1 + x_2) \). Following the footsteps of Broadhurst [20, §2 and 6], Davydychev–Kalmykov [30, §2], and Borwein–Borwein–Straub–Wan [8, §5] at an early stage of the present research, we converted \( m_k(1 + x_1 + x_2) \) into infinite series by differentiating a formula of Borwein–Borwein–Straub–Wan–Zudilin [15, Theorem 7.2] (see also [13, Theorem 4.2]):

\[
W_3(s) = \frac{3^{s+\frac{3}{2}} \left( \frac{1}{2} \right)^{s+\frac{1}{2}}}{\Gamma(2 + s)} \mathcal{F}_2 \left( 1 + \frac{3}{2}, 1 + \frac{3}{2} ; \frac{1}{4} \right), \quad |s| < 2.
\]

With \( c_n := \frac{(n!)^2}{(2n+1)!} = \frac{1}{(n+1)_{n+1}} \),

\[
\frac{\pi}{\sqrt{3}} \frac{d^k W_3(s)}{ds^k} \bigg|_{s=0} \in \text{span}\left\{ \sum_{n=0}^{\infty} c_n \prod_{j=1}^{M} b_n^{(r_j)} : r_1, \ldots, r_M \in \mathbb{Z}_{>0}; \sum_{j=1}^{M} r_j = k \right\},
\]

by slightly modifying the proof of [8, Theorem 10].

In the terminology of Kalmykov–Ward–Yost [47, §2.1], the infinite series appearing in (3.30) are special cases of “inverse binomial sums”. In the next corollary, we explore such infinite sums in broader context.

Corollary 3.4 (Inverse binomial sums via Goncharov–Deligne periods). In the following, we set \( r_1, \ldots, r_M \in \mathbb{Z}_{>0} \) and \( r_1', \ldots, r_M' \in \mathbb{Z}_{>0} \). We will also need new \( \mathbb{Q} \)-vector spaces

\[
g_k(z_1, \ldots, z_n; z)(N) := \text{span}\left\{ (\pi i)^{k-\ell} G(\alpha_1, \ldots, \alpha_j; z) Z_{\ell-j} \in \mathbb{C}^{\{ \alpha_1, \ldots, \alpha_j \in \{0, 1, \ldots, n\} \}} \right\},
\]

\[
g_{k,N}^{z_1, \ldots, z_n} := \text{span}\left\{ \prod_{j=1}^{j} G_j \right\},
\]

which extend \( g_k^{(\xi)}(N) = g_k[1, e^{2\pi i/N}, \ldots, e^{2\pi i(N-1)/N}; z](N) \) (defined in (2.30b)).

(a) If

\[
\left\{ \begin{array}{ll}
\left| \frac{1 - \xi}{\xi} \right|^2 \leq 4 |\xi|, & \text{or} \quad \left| \frac{1 - \xi}{\xi} \right|^2 < 4 |\xi|,

s \in \mathbb{Z}_{>0},

\end{array} \right.
\]

then (cf. [47, (1.1), (5.1a), and (5.1b)])

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} \mathcal{H}_{n-1}^{(r_j)} \right] \left[ \frac{(1 + \frac{3}{2})^{\delta_{n, s}}}{n^{s+1}(2n)} \right] \left[ \frac{(1 - \xi)^2}{\xi} \right] \in \mathbb{g}_{k+1}^{(\xi)} \cap \mathbb{g}_{k+1,1}^{\{1, \frac{1}{\xi}, e^{\frac{2\pi i}{\xi}}\}}
\]
where \( k = s + \sum_{j=1}^{M} r_j \). More generally, for any moment sequence\(^{11}\) \( a_{n,M'} = \int_{0}^{1} x^{n-1} (1-x)^{n-1} f_{M'}(x) \, dx \), \( n \in \mathbb{Z}_{>0} \) satisfying \( f_{M'}(x) \in \mathcal{F}_{M'}(1) \), we have

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} \mathcal{H}^{(r_j)}_{n-1} \left[ \frac{1+\xi}{\xi} \right]^{\delta_{0,s}} n \left[ 1 - \left( 1 - \frac{\xi}{2} \right)^{2n} \right] a_{n,M'} \in \mathcal{H}_{k+1}(2) \cap \mathcal{H}_{k+1}(1),
\]

for \( k = s + M' + \sum_{j=1}^{M} r_j \), so long as the infinite series are convergent.

In particular, for \( \xi = \phi := e^{\pi/3} \), we have

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} \mathcal{H}^{(r_j)}_{n-1} \left[ \frac{-1}{n^{s+1/2n}} \right] \in \mathcal{H}_{k+1}(2) \cap \mathcal{H}_{k+1}(3),
\]

where \( s \in \mathbb{Z}_{>0} \), \( k = s + \sum_{j=1}^{M} r_j \); for \( \xi = 1 - \rho := \frac{3\sqrt{3}}{2} \), \( \rho \), we have (cf. [16, §6])

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} \mathcal{H}^{(r_j)}_{n-1} \left[ \frac{\sqrt{3} \delta_{0,s}}{n^{s+1/2n}} \right] \in \mathcal{H}_{k+1}(2) \cap \mathcal{H}_{k+1}(10),
\]

where \( s \in \mathbb{Z}_{>0} \), \( k = s + \sum_{j=1}^{M} r_j \); for \( \xi = i \), we have

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} \mathcal{H}^{(r_j)}_{n-1} \left[ \frac{i \delta_{0,s} \cdot 3^n}{n^{s+1/2n}} \right] \in \mathcal{H}_{k+1}(2) \cap \mathcal{H}_{k+1}(4),
\]

where \( s \in \mathbb{Z}_{>0} \), \( k = s + \sum_{j=1}^{M} r_j \); for \( \xi = \omega := e^{2\pi i/3} \), we have

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} \mathcal{H}^{(r_j)}_{n-1} \left[ \frac{-1}{n^{s+1/2n}} \right] \in \mathcal{H}_{k+1}(2) \cap \mathcal{H}_{k+1}(6),
\]

where \( s \in \mathbb{Z}_{>0} \), \( k = s + \sum_{j=1}^{M} r_j \); for \( \xi = -1 \), we have (cf. [62, (1.1)])

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} \mathcal{H}^{(r_j)}_{n-1} \left[ \frac{2^{2n}}{n^{s+1/2n}} \right] \in \mathcal{H}_{k+1}(2),
\]

where \( s \in \mathbb{Z}_{>0} \), \( k = s + \sum_{j=1}^{M} r_j \).

We also have the following quadratic analog of inverse binomial sums:

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} \mathcal{H}^{(r_j)}_{n-1} \left[ \frac{2^{4n}}{n^{s+2/2n}} \right] \in \mathcal{H}_{k+2}(4),
\]

where \( s \in \mathbb{Z}_{>0} \), \( k = s + \sum_{j=1}^{M} r_j \).

\(^{11}\text{For example, the function } f_1(x) = \log x \in \delta_{1}^{(2)}(1) \text{ produces a sequence } a_{n,1} = \frac{2}{n^{(2)}(n)} [H^{(1)}_{n-1} - H^{(1)}_{2n-1}], \text{ while } f_2(x) = \log^2 x \in \delta_{2}^{(2)}(1) \text{ brings us another sequence } a_{n,2} = \frac{2}{n^{(2)}(n)} \left( [H^{(1)}_{n-1} - H^{(1)}_{2n-1}]^2 + H^{(2)}_{n-1} - H^{(2)}_{2n-1} \right).
(b) If

\[
\begin{cases}
|1 + \tau^2| \leq 2|\tau|, \\
|1 + \tau^2| < 2|\tau|,
\end{cases}
\quad s \in \mathbb{Z}_{>0},
\]

then we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \left\{ \prod_{j'=1}^{M'} \left[ H_{2n-1}^{(r_{j''})} + \overline{H}_{2n-1}^{(r_{j''})} \right] \right\} \frac{(i\sqrt{3})^{\delta_{0,s}}}{n^{s+1}(2\pi n)} \frac{1 + \tau^2}{\tau} 2^n
\in \mathcal{G}_{k+1,4}^{(r)} \cap \mathcal{G}_{k+1,2}^{(i,-i,\tau, -1, -\tau, \tau)}
\]

where \( k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j''} \). More generally, for any moment sequence \( b_{n,M''} = \int_0^1 x^{n-1} (1-x)^{n-1} g_{M''}(x) \, dx, n \in \mathbb{Z}_{>0} \) satisfying \( g_{M''}(x) \in \mathcal{G}_{M''}^{(r)}(1) \), we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \left\{ \prod_{j'=1}^{M'} \left[ H_{2n-1}^{(r_{j''})} + \overline{H}_{2n-1}^{(r_{j''})} \right] \right\} \frac{(i\sqrt{3})^{\delta_{0,s}}}{n^{s+1}(2\pi n)} \frac{1 + \tau^2}{\tau} 2^n b_{n,M''}
\in \mathcal{G}_{k+1,4}^{(r)} \cap \mathcal{G}_{k+1,2}^{(i,-i,\tau, -1, -\tau, \tau)}
\]

for \( k = s + M'' + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j''} \), so long as the infinite series are convergent.

In particular, for \( \tau = \theta \), we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \left\{ \prod_{j'=1}^{M'} \left[ H_{2n-1}^{(r_{j''})} + \overline{H}_{2n-1}^{(r_{j''})} \right] \right\} \frac{(i\sqrt{3})^{\delta_{0,s}}}{n^{s+1}(2\pi n)} \frac{1 + \tau^2}{\tau} 2^n
\in \mathcal{G}_{k+1,4}^{(i,\omega)} \cap \mathcal{G}_{k+1,2}^{(i,\omega)}
\]

where \( s \in \mathbb{Z}_{>0}, k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j''} \); for \( \tau = \theta := i\sqrt{3} \), we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \left\{ \prod_{j'=1}^{M'} \left[ H_{2n-1}^{(r_{j''})} + \overline{H}_{2n-1}^{(r_{j''})} \right] \right\} \frac{(i\sqrt{3})^{\delta_{0,s}}}{n^{s+1}(2\pi n)} \frac{1 + \tau^2}{\tau} 2^n
\in \mathcal{G}_{k+1,4}^{(i,\omega)} \cap \mathcal{G}_{k+1,2}^{(i,\omega)}
\]

where \( s \in \mathbb{Z}_{>0}, k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j''} \); for \( \tau = \theta := e^{\pi i/4} \), we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \left\{ \prod_{j'=1}^{M'} \left[ H_{2n-1}^{(r_{j''})} + \overline{H}_{2n-1}^{(r_{j''})} \right] \right\} \frac{(i\sqrt{3})^{\delta_{0,s}}}{n^{s+1}(2\pi n)} \frac{1 + \tau^2}{\tau} 2^n
\in \mathcal{G}_{k+1,4}^{(i,\omega)} \cap \mathcal{G}_{k+1,2}^{(i,\omega)}
\]

where \( s \in \mathbb{Z}_{>0}, k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j''} \); for \( \tau = i/\theta = e^{\pi i/6} \), we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \left\{ \prod_{j'=1}^{M'} \left[ H_{2n-1}^{(r_{j''})} + \overline{H}_{2n-1}^{(r_{j''})} \right] \right\} \frac{(i\sqrt{3})^{\delta_{0,s}}}{n^{s+1}(2\pi n)} \frac{1 + \tau^2}{\tau} 2^n
\in \mathcal{G}_{k+1,4}^{(i,\omega)} \cap \mathcal{G}_{k+1,2}^{(i,\omega)}
\]
where $s \in \mathbb{Z}_{>0}$, $k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j'}$; for $\tau = 1$, we have

$$
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \frac{\sum_{j=1}^{M'} H_{n-1}^{(r_{j'})}}{\sum_{j'=1}^{M'} \Pi_{n-1}^{(r_{j'})}} \left( \frac{22n}{n+1} \right) \in \mathbb{Z}_{>0},
$$

where $s \in \mathbb{Z}_{>0}$, $k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j'}$.

We also have the following quadratic analog of inverse binomial sums:

$$
\sum_{n=1}^{\infty} \frac{1}{n(\frac{4n}{2n})} \left[ \frac{(1-\xi)^2}{\xi} \right] \left( \frac{1-\xi}{\xi} \right)^{n-1} \int_{0}^{1} x^{n-1}(1-x)^{n-1} \, dx
$$

$$
\in -\frac{(1-\xi)^2}{2\xi} \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{M} H_{n-1}^{(r_j)}}{\prod_{j'=1}^{M'} \Pi_{n-1}^{(r_{j'})}} \left( \frac{1-\xi}{\xi} \right)^{n-1} \int_{0}^{1} x^{n-1}(1-x)^{n-1} \, dx
$$

PROOF. From the asymptotic behavior $\frac{1}{n(\frac{4n}{2n})} = \frac{1}{4} \sqrt{\frac{\pi}{n}} \left[ 1 + O \left( \frac{1}{n} \right) \right]$ as $n \to \infty$, we see that all the infinite sums of interest indeed converge for the prescribed ranges of $s$, $\xi$, and $\tau$.

(a) First, we consider the $s = 0$ case in (3.34). By (3.6), we have

$$
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \frac{1}{n(\frac{2n}{2n})} \left[ \frac{(1-\xi)^2}{\xi} \right]
$$

$$
= -\frac{(1-\xi)^2}{2\xi} \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{M} H_{n-1}^{(r_j)}}{\prod_{j'=1}^{M'} \Pi_{n-1}^{(r_{j'})}} \left( \frac{1-\xi}{\xi} \right)^{n-1} \int_{0}^{1} x^{n-1}(1-x)^{n-1} \, dx
$$

$$
\in -\frac{(1-\xi)^2}{2\xi} \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{M} H_{n-1}^{(r_j)}}{\prod_{j'=1}^{M'} \Pi_{n-1}^{(r_{j'})}} \left( \frac{1-\xi}{\xi} \right)^{n-1} \int_{0}^{1} x^{n-1}(1-x)^{n-1} \, dx
$$

Like the proof of Theorem 2.5, we will compute the integrals over multidimensional polylogarithms using Brown’s method. By direct applications of the GPL recursion (2.22) [with $z = -\frac{(1-\xi)^2}{\xi} (1-x)$], we can show inductively that

$$
L_{A} \left( \frac{(1-\xi)^2}{\xi} x(1-x) \right)
$$

$$
\in \text{span}_{\mathbb{Z}} \left\{ G(\beta_1, \ldots, \beta_{w(A)}; x) \mid \beta_1, \ldots, \beta_{w(A)} \in \{0, 1, \frac{1}{\xi}, \frac{\xi}{1} \} \right\}.
$$

Furthermore, the partial fraction expansion

$$
\frac{(1-\xi)^2}{1 + (1-\xi)^2 x(1-x)} = \frac{1}{1-x} - \frac{1}{1+\xi (x-1)}
$$

$$
\text{tells us that}
$$

$$
-\frac{1 + \xi (1-\xi)^2}{2\xi} \int_{0}^{1} L_{A} \left( \frac{(1-\xi)^2}{\xi} x(1-x) \right) \frac{dx}{1 + (1-\xi)^2 x(1-x)}
$$

$$
\in \text{span}_{\mathbb{Q}} \left\{ G(\beta_1, \ldots, \beta_{w(A)}+1; 1) \in \mathbb{C} \mid \beta_1, \ldots, \beta_{w(A)+1} \in \{0, 1, \frac{1}{\xi}, \frac{\xi}{1} \} \right\}
$$

$$
\subseteq \mathcal{Z}_{w(A)+1}(2) \cap \mathcal{Z}_{w(A)+1,1},
$$
where the “⊆” step comes from the GPL recursion in (2.22), and the “⊆” step draws on another round of fibration with respect to $\xi$ which is deducible from repeated applications of the $z = 1$ and $z = \xi$ cases of the GPL recursion (2.22) to the GPL differential form (2.37) for $d\log(\alpha_j - \alpha_j), d\log(\alpha_j + 1 - \alpha_j) \in \{ d\log \frac{1}{\alpha_j}, d\log \frac{\xi}{\alpha_j}, d\log \frac{1+\xi}{\alpha_j} \}$.

Then, we work with positive integers $s$. By the GPL recursion in (2.22), we have [cf. (3.14)]

$$
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \frac{z^n}{n} 
= \sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \int_0^z t^{n-1} dt \in \text{span}_Q \left\{ \int_0^z \frac{L_A(x)}{1-x} dx \mid w(A) = \sum_{j=1}^{M} r_j \right\}
\subseteq \text{span}_Q \left\{ L_A(z) \mid w(A) = 1 + \sum_{j=1}^{M} r_j \right\}.
$$

Subsequent inductions in the spirit of (3.15) bring us

$$
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \frac{z^n}{n^s} \in \text{span}_Q \left\{ L_A(z) \mid w(A) = s + \sum_{j=1}^{M} r_j \right\}
$$
for all $s \in \mathbb{Z}_{>0}$. Accordingly, we arrive at

$$
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \frac{1}{n^{s+1}} \left[ \frac{(1-\xi)^2}{\xi} \right]^n 
= \sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \frac{1}{n^s} \left[ \frac{(1-\xi)^2}{\xi} \right] \int_0^1 x^n(1-x)^n \frac{dx}{x(1-x)} 
\in \text{span}_Q \left\{ \int_0^1 L_A \left( -\frac{(1-\xi)^2}{\xi}x(1-x) \right) \frac{dx}{x(1-x)} \mid w(A) = s + \sum_{j=1}^{M} r_j \right\},
$$

where

$$
\int_0^1 L_A \left( -\frac{(1-\xi)^2}{\xi}x(1-x) \right) \frac{dx}{x(1-x)}
\in \text{span}_Q \left\{ G(\beta_1, \ldots, \beta_{w(A)}+1; 1) \mid \beta_1, \ldots, \beta_{w(A)}+1 \in \left\{ 0,1,\frac{1}{2},\frac{1}{3} \right\}, \beta_j \in \{0,1\}, \beta_{w(A)+1} \neq 0, \ell \in \mathbb{Z} \cap [0, w(A)+1]; j \in \mathbb{Z} \cap [0, w(A)+1-\ell] \right\}
\subseteq g_{w(A)+1}^{(e)} \cap g_{w(A)+1,1},
$$
follows from recursions and fibrations of GPLs.
Table 4. Selected closed forms for inverse binomial sums, where \( \omega := e^{2\pi i/3} \), \( \varrho := e^{\pi i/3} \), \( \varsigma := e^{2\pi i/5} \), \( G := \text{Im} \text{Li}_2(i) \), \( \lambda := \log 2 \), and \( A := \log 3 \)

\[
(\tau, m, \xi) \sum_{n=1}^{2m} \left[ \binom{r}{n} \right] m \frac{1}{n} \text{Li}_2^{(r)} \left( \frac{1 - \xi}{\xi} \right)^{\frac{1}{2} - 1} \frac{1}{2} \int_{0}^{1} G^{(r)}(z) \left( 1 - \xi \right)^{z(1 - z)} \text{d} x
\]

(1, 1, \varrho)

(1, 2, \varrho)

(1, 3, \varrho)

(1, 4, \varrho)

(2, 1, \varrho)

(2, 2, \varrho)

(1, 1, \rho^2)

(1, 2, \rho^2)

(1, 3, \rho^2)

(1, 4, \rho^2)

(2, 1, \rho^2)

(2, 2, \rho^2)

This completes the proof of (3.34). To generalize it to (3.34∗), simply note that

\[
L_A \left( -\frac{(1 - \xi)^2}{\xi} x(1 - x) \right) f_M(x)
\]

\[
\in \text{span}_Q \left\{ G(\beta_1, \ldots, \beta_w(A); x) \mid \beta_1, \ldots, \beta_w(A) \in \{0, 1, \frac{1}{2}, \frac{\xi}{2}, \frac{\xi^2}{2}, \frac{\xi^3}{2}, \frac{\xi^4}{2}, \frac{\xi^5}{2} \} \right\}
\]

follows from the condition \( f_M(x) \in S_M^{(3)}(1) \) and GPL shuffle algebra.

With \( \xi = \varrho \) in (3.34), we obtain \( \left\{ 1, \frac{1}{2}, \frac{\xi}{2}, \frac{\xi^2}{2} \right\} = \{1, \varrho, \frac{1}{2} \} \). This hearkens back to the scenario in (2.62), so the left-hand side of (3.35) is in \( Z_{k+1}(3) \).

Setting \( \xi = 1 - \rho = 2\sqrt{-2} \) in (3.34), we get \( \left\{ 1, \frac{1}{2}, \frac{\xi}{2}, \frac{\xi^2}{2}, \frac{\xi^3}{2}, \frac{\xi^4}{2}, \frac{\xi^5}{2} \right\} = \{1, \frac{1}{2}, \frac{\xi}{2}, \frac{\xi^2}{2}, \frac{\xi^3}{2}, \frac{\xi^4}{2}, \frac{\xi^5}{2} \} \).

By a variation on the arguments for (2.62), we may use GPL fibration in the
variable \( z \) to prove\(^{12} \)

\[
\{ 1, \frac{1 + \xi}{\xi^2 + z}, \frac{1 + \xi^3}{\xi^6 + z} \} \subseteq \mathfrak{g}_{k+1} \left[ 1, -\xi, -\xi^2, -\xi^3, -\xi^4, -\xi^5 \right]
\]

(3.58)

for \( \xi = e^{2\pi i/5} \), while checking that

\[
-\frac{(1 + \xi)\xi(\xi + z)}{\xi^4 + z} = \frac{1 + \sqrt{5}}{2}, \quad -\frac{(1 + \xi)\xi(\xi^3 + z)}{\xi^6 + z} = \frac{1 - \sqrt{5}}{2}
\]

(3.59)

for \( z = 1 \). This identifies the left-hand side of (3.36) as a CMZV of level 10.

Specializing (3.34) to \( \xi \in \{ i, \omega, -1 \} \), while noting that \( \pi i \in \mathbb{Z}_1(N), N \in \mathbb{Z}_{\geq 3} \) [cf. (2.15)] and \( \mathfrak{g}_{k+1}^{(1)}(2) \cap \mathbb{R} = \mathcal{B}_{k+1}(2) \), one can verify (3.37)–(3.39).

Combining the following integral along a straight line contour (on which \( |1 - z|^2 \leq 2|z| \) holds)

\[
\int_1^{10} \left( \frac{1 - z^2}{iz} \right)^{2n-1} \frac{dz}{iz} = \int_1^{10} \left[ \frac{(1 - z^2)^2}{z^2} \right]^n \frac{dz}{1 - z^2} = -\frac{2^{4n-2}}{n(n^{2n})}
\]

(3.60)

with the relation \( \mathfrak{g}_{k+1}^{(z)}(2) \subseteq \mathfrak{g}_{k+1}^{(z)}(4) \) (provable by fibration), we can confirm (3.40) by integrating (3.34).

It is clear from the analysis so far that (3.34)–(3.40) extend retroactively to the cases where \( M = 0 \), and that the designated \( \mathbb{Q} \)-vector spaces are not affected if we replace any single factor of \( H_{n-1}^{(r)} = H_{n}^{(r)} - \frac{1}{\tau} \) by \( H_{n}^{(r)} \).

(b) Combining (3.10) with the observation that

\[
H_{n-1}^{(r)} = 2^{r-1} \left[ H_{2n-1}^{(r)} - \Pi_{n}^{(r)} \right], \quad r \in \mathbb{Z}_{>0},
\]

we may put down

\[
\sum_{n=1}^{\infty} \left\{ \prod_{j=1}^{M} H_{n}^{(r_j)} \left\{ \prod_{j=1}^{M'} \left[ H_{2n-1}^{(r'_j)} + \Pi_{n}^{(r'_j)} \right] \right\} z^{2n-1} \right\}
\]

(3.62)

\[
\in \text{span}_{\mathbb{Q}} \left\{ \frac{g(z)}{1 - z} - \frac{g(-z)}{1 + z} \mid g(z) \in \mathfrak{g}^{(z)}(\sum_{j=1}^{M} r_j + \sum_{j=1}^{M'} r'_j, 2) \right\}
\]

When \( s = 0 \), to evaluate integral representation for the left-hand side of (3.42), we need

\[
\int_0^1 G \left( \alpha_1, \ldots, \alpha_w; \frac{1 + \tau^2}{\tau}\sqrt{X(1 - X)} \right) \frac{dX}{\sqrt{X(1 - X)} \left[ 1 + \frac{1 + \tau^2}{\tau}\sqrt{X(1 - X)} \right]}
\]

(3.63)

\[
= \frac{2(1 - \tau^2)}{\tau} \int_{1/2}^{1} G \left( \alpha_1, \ldots, \alpha_w; \frac{1 + \tau^2}{\tau}\sqrt{x(1 - x)} \right) \frac{dx}{\sqrt{x(1 - x)} \left[ 1 + \frac{1 + \tau^2}{\tau}\sqrt{x(1 - x)} \right]}
\]

\[
= 4 \int_0^{1/2} G \left( \alpha_1, \ldots, \alpha_w; \frac{(1 + \tau^2)t}{(1 + t^2)\tau} \right) \left( \frac{1}{\tau + t} - \frac{1}{t + \tau} \right) dt,
\]

---

\(^{12}\)The rational functions in (3.58) paraphrase [4, Example 4.7]. We also note that the \( M = 1 \) case of (3.36) has already been covered by [4, Theorem 7.2]. For the fibration here, the GPL recursion integrals run from \( -\xi \) (or \( -\xi^4 \)) to \( z \), and the GPL differential forms involve terms like \( d \log \left( 1 + \frac{(1 + \xi)(\xi + z)}{\xi^4 + z} \right) = d \log \frac{\xi^2 + z}{\xi^4 + z} \) and \( d \log \left( 1 + \frac{1 + \xi(\xi^3 + z)}{\xi^6 + z} \right) = d \log \frac{\xi^6 + z}{\xi^6 + z} \).
where $x = \frac{1}{1+\tau}$, $\alpha_1, \ldots, \alpha_w \in \{-1, 0, 1\}$, $\alpha_w \neq 0$ and $w = \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j'}$.

Again, by recursive applications of (2.22) for $z = \frac{\tau + \tau^2 w}{1 + \tau^2}$, we can show that

$$G \left( \alpha_1, \ldots, \alpha_w; \pm \frac{1 + \tau^2 w}{1 + \tau^2} \right) \in \text{span}_Q \left\{ G(\beta_1, \ldots, \beta_w; t) \mid \beta_1, \ldots, \beta_w \in \{0, \tau, i, -\frac{1}{\tau}, -\tau, -i, -\frac{1}{\tau} \} \right\},$$

Therefore, the left-hand side of (3.42) belongs to the Q-vector space

$$\text{span}_Q \left\{ G(\beta_1, \ldots, \beta_{k+1}; t) \mid \beta_1, \ldots, \beta_{k+1} \in \{0, \tau, i, -\frac{1}{\tau}, -\tau, -i, -\frac{1}{\tau} \} \right\},$$
when $s = 0$. Fibrating the GPLs in (3.65) with respect to $\tau$, we realize that (3.65) is a subspace of $\mathfrak{g}_{k+1}^{(r)}(4)$.

Then we handle (3.42) for $s \in \mathbb{Z}_{>0}$. By the GPL recursion in (2.22), we may promote (3.62) to

$$
\sum_{n=1}^{\infty} \left\{ \prod_{j=1}^{M} \mathcal{H}_{n-1}^{(r_j)} \right\} \left\{ \prod_{j'=1}^{M'} \left[ \mathcal{H}_{2n-1}^{(r'_{j'})} + \overline{\mathcal{H}}_{2n-1}^{(r'_{j'})} \right] \right\} \frac{z^{2n}}{n^s} 
\in \text{span}_\mathbb{Q} \left\{ g(z) + g(-z) \mid g(z) \in \mathfrak{g}^{(z)}_{s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r'_{j'}}(2) \right\}
$$

for all $s \in \mathbb{Z}_{>0}$. Accordingly, the left-hand side of (3.42) is still in the $\mathbb{Q}$-vector space $\mathfrak{g}_{k+1}^{(r)}(4)$.

This completes the proof for a weaker version of (3.42). The same arguments extend to the $\mathfrak{g}_{k+1}^{(r)}(4)$ part of (3.42), since $g_{M''}(x) \in \mathfrak{g}_{M''}(1)$ entails $g_{M''}(\frac{1}{1 + \tau^2}) + g_{M''}(1 - \frac{1}{1 + \tau^2}) = 0$.

To verify (3.42) and (3.42*) in their entirety, we need contour deformations. Concretely speaking, if we have $f_k(z) \in \mathfrak{g}_{k}^{(z)}(2)$ and $g_{M''}(z) \in \mathfrak{g}_{M''}(1)$. To do this, we may deform the contour of

$$
\int_{0}^{1} G \left( \frac{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{w+s;:}; \pm \frac{1 + \tau^2}{\tau} \sqrt{X(1 - X)} }{X(1 - X)} \right) dX
= 4 \int_{0}^{1} G \left( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{w+s;:}; \pm \frac{(1 + \tau^2) t}{1 + t^2} \right) \frac{d t}{t},
$$

where $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{w+s} \in \{ -1, 0, 1 \}$, $\alpha_w \neq 0$ and $w = \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r'_{j'}$. Accordingly, the left-hand side of (3.42) is still in the $\mathbb{Q}$-vector space $\mathfrak{g}_{k+1}^{(r)}(4)$.

(3.69) = \left\{ \int_{-i0+}^{\infty - i0+} f_k \left( \pm \frac{(1 + \tau^2) t}{1 + t^2} \right) g_{M''} \left( \frac{1}{1 + t^2} \right) h(t) \frac{d t}{2\pi i} \right\}

where $h(t) \in \left\{ \frac{1}{\tau + t} - \frac{1}{\tau - t}; -\frac{1}{t} \right\}$

to tight loops enveloping the branch cuts, so that the net contribution to the integral results from the following jump discontinuities [55, 57]

$$
(3.70) \quad \left\{ \begin{array}{l}
    f_k(\pm(\Xi + i0^+)) - f_k(\pm(\Xi - i0^+)) \in \pi i \mathfrak{g}_{k+1}^{(\pm1/\Xi)}(2), \\
    g_{M''}(\Xi + i0^+) - g_{M''}(\Xi - i0^+) \in \pi i \mathfrak{g}_{M''-1}^{(1/\Xi)}(1),
\end{array} \right.
$$

for $\Xi > 1$ (with the understanding that $\mathfrak{g}_{1}^{(z)}(1) = \{ 0 \}$). Here, in the complex $t$-plane, the end points of these branch cuts belong to the set $\{ i, -i, \tau, -\frac{1}{\tau}, -\tau, \frac{1}{\tau} \}$. 
Table 6. Continuation of Table 4, with \( \omega := e^{2\pi i/3}, \varphi := e^{\pi i/3}, \theta = e^{\pi i/4}, \lambda := \log 2, \text{ and } A := \log 3 \)

| \( (r, m, \tau) \) | \[ \sum_{n=1}^{(m)} \left[ \left( \begin{array}{c} 1 \end{array} \right)^{m} \left( \frac{1 + \tau^{2}}{(1 + \tau^{2})^{t}} \right)^{n-1} \right] \] | \( \int_{0}^{1} \left( \frac{1 + \tau^{2}}{(1 + \tau^{2})^{t}} \right) d\tau \) |
| \hline | \( (1, 1, \varphi) \) | \( -\frac{2\varphi}{\sqrt{3}} \text{ Im } L_{2,1}(\omega) - \frac{2\varphi}{3\sqrt{3}} \text{ Im } L_{2,1}(i) + \frac{3\varphi}{3\sqrt{3}} \text{ Im } L_{1,1,1,1} \left( \frac{3}{4}, 1, -1 \right) - \frac{7\varphi}{3\sqrt{3}} \text{ Im } L_{1,2}(\omega) \) |
| | \( (1, 2, \varphi) \) | \( -\frac{2\varphi}{\sqrt{3}} \text{ Im } L_{2,1}(\omega) - \frac{2\varphi}{3\sqrt{3}} \text{ Im } L_{2,1}(i) + \frac{3\varphi}{3\sqrt{3}} \text{ Im } L_{1,1,1,1} \left( \frac{3}{4}, 1, -1 \right) - \frac{7\varphi}{3\sqrt{3}} \text{ Im } L_{1,2}(\omega) \) |
| | \( (2, 1, \varphi) \) | \( \frac{\varphi}{\sqrt{3}} \text{ Im } L_{2,1}(\omega) - \frac{\varphi}{3\sqrt{3}} \text{ Im } L_{2,1}(i) + \frac{\varphi}{3\sqrt{3}} \text{ Im } L_{1,1,1,1} \left( \frac{3}{4}, 1, -1 \right) + \frac{\varphi}{3\sqrt{3}} \text{ Im } L_{1,2}(\omega) \) |

Recursive applications of (2.22) with \( z = i \) and \( z = t \) lead us to

\[
(3.71) \quad g_{k-1}^{(1+\varphi)}(2) \subseteq g_{k-1}^{(1)}(4) \quad \text{and} \quad g_{k'-1}^{(1+\varphi)}(1) \subseteq g_{k'-1}^{(1)}(4)
\]

for \( \Xi = \left( \frac{1 + \varphi^{2}}{(1 + \varphi^{2})^{t}} \right) \), hence the full characterization stated in (3.42) and (3.42').

With \( \pi i \in 3_{1}(N), N \in \mathbb{Z}_{\geq 3} \) [cf. (2.15)] in mind, we can deduce (3.43)–(3.47) from (3.42), except that we need the next two paragraphs to account for the CMZV portrayal in (3.44) and (3.46).

Setting \( R_{n}(z) := i(z e^{\pi i/5} + \frac{1}{2z e^{\pi i/5}}) \), we consider

\[
(3.72) \quad \{ i, -i, \tau, -\frac{1}{\tau}, -\frac{1}{\tau}, \frac{1}{\tau} \} \bigg|_{\tau = i/\varphi} = \{ i, -i, R_{0}(z), R_{1}(z), R_{2}(z), R_{3}(z) \} \bigg|_{z = e^{\pi i/5}}.
\]

The right-hand side of the equation above is a \( \mathbb{Q} \)-linear subspace of \( \mathcal{F}_{k+1}(60) \), as one can check by the fibration basis with respect to \( z \).\textsuperscript{13}

A naïve set inclusion \( g_{k+1}^{(1/\varphi)}(4) \subseteq \mathcal{F}_{k+1}(12) \) [based on a weaker form of (3.42)] is sharpened by the following invocation of the scaling property [37, (2.3)] for GPLs:

\[
(3.73) \quad \{ i, -i, \tau, -\frac{1}{\tau}, -\frac{1}{\tau}, \frac{1}{\tau} \} \bigg|_{\tau = i/\varphi} \subseteq \{ 1, \varphi, \varphi^{2}, -1, -\varphi, -\varphi^{2} \} \subseteq \mathcal{F}_{k+1}(6),
\]

so the statement in (3.46) is true.

\textsuperscript{13}To show that \( g_{k+1}^{(i, -i, z R_{0}(z), z R_{1}(z), z R_{2}(z), z R_{3}(z))} \bigg|_{z = e^{\pi i/5}} \subseteq \mathcal{F}_{k+1}(60) \), one may apply the GPL recursion (2.22) to the GPL differential form (2.37), while noting that \( d \log(z + e^{\pi i/5}) = d \log(z - e^{7\pi i/15}) + d \log(z + e^{2\pi i/15}) \) and so forth.
With
\[
\int_{1}^{i} \left( \frac{1 + \tau^2}{\tau} \right)^{2n} \frac{i \, d\tau}{1 + \tau^2} = -\frac{2^{4n-2}}{n!} \tag{3.74}
\]
in hand, we can prove (3.48) by integrating (3.42) along a straight line segment (on which \(|1 + \tau^2| \leq 2|\tau|\)) and applying Panzer’s logarithmic regularization [57, §2.3] at \(\tau = i\).

It is clear that the \(\mathbb{Q}\)-vector spaces appearing on the right-hand sides of (3.43)–(3.48) incorporate their counterparts in (3.35)–(3.40) as subsets. Therefore, in view of (3.61), we can replace any occurrence of \(H_{2n-1}^{(r)} + \overline{H}_{2n-1}^{(r)}\) in (3.43)–(3.48) by \(H_{2n-1}^{(r)}\), without affecting the corresponding CMZV characterizations.

\[\square\]

**Remark 3.5.** With Panzer’s HyperInt [57] and Au’s MultipleZetaValues [4], one can verify all the entries of Tables 4–6. To keep our analytic expressions within manageable size, we have suppressed the explicit fibrations of
\[
\begin{align*}
\left\{ \text{Li} \left( -\frac{1}{\sqrt{3}} \right) = -G(0, 0, r_3(z); 1) \right|_{z=e^{i\pi/6}} \right.
\text{Li} \left( -\frac{1}{\sqrt{3}} \right) &= -G(0, 0, r_3(z); 1) \right|_{z=e^{i\pi/6}}
\text{where } r_n(z) := \sum_{r=0}^{n} \frac{e^{i\pi r/6} + 1}{2e^{i\pi r/6} + 1},
\end{align*}
\]
into CMZVs of level 10.

\[\square\]

**Remark 3.6.** In [47, (2.1)], Kalmykov–Ward–Yost considered
\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \left[ \prod_{j=1}^{M'} H_{2n-1}^{(r_j') \frac{1}{2}} \right] \left( \frac{1 + \tau^2}{\tau} \right)^{2n} \tag{3.76}
\]
among other things. In their capacities as \(\mathbb{Q}\)-linear combinations of the left-hand side in (3.43), these Kalmykov–Ward–Yost series are also explicitly representable as members of \(g_{k+1}^{(r)}(4)\) for \(s \in \mathbb{Z}_{\geq 0}\) and \(k = s + \sum_{j=1}^{M} r_j + \sum_{j=1}^{M'} r_j'\). For \(\tau = \varrho\) and small values of \(k\), Kalmykov–Veretin [46, Appendix B] proposed closed-form formulae for some special cases of (3.76), based on numerical evidence. Diligent readers may check these experimental findings analytically, using Au’s package [4]. For generic \(\tau\) and small values of \(k\), Davydychev–Kalmykov [30] evaluated (3.76) in terms of classical polylogarithms and their associates.

\[\square\]

**Remark 3.7.** We note that
\[
H_{n-1}^{(r)} + 2\delta_{1,r} \log 2 - 2^{r-1} H_{2n-1}^{(r)} + \overline{H}_{2n-1}^{(r)}
\]
is always a member of \(g_{k+1}^{(\varrho)}(2)\). Therefore, the relation in (3.42) can be rewritten as
\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right] \left( \prod_{j=1}^{M'} H_{n-1}^{(r_j') \frac{1}{2}} + 2\delta_{1,r_j'} \log 2 \right) \frac{(iv)^{\delta_{0,s}}}{n^{r+1} \binom{2n}{n}} \in g_{k+1}^{(\varrho)} \cap \mathfrak{z}_{k+1}(12),
\]
\[\square\]
whose $s = 0$ case combines with (2.41) into

\[
\sum_{n=0}^{\infty} c_n \prod_{j=1}^{M} H_{n-1}^{(r_j)} \in g_k^{(q)}(4) \cap \mathcal{F}_{k+1}(12).
\]

Unfortunately, this result is not sharp enough to recover the MPLs at sixth roots exhibited in Theorem 2.5, let alone the third roots sharpened by Theorem 2.7.

We initially confirmed the sufficiency of third roots in the MPL characterizations of $m_k(1 + x_1 + x_2)$ for small positive integers $k$ by checking manually that these series representations [cf. (3.30)] of hyper-Mahler measures fit into the framework of (3.34*) and (3.35). Later afterwards, we performed variable substitutions and contour deformations on the Borwein–Borwein–Straub–Wan integrals (Theorem 2.7) that led us to $\pi m_k(1 + x_1 + x_2) \in \mathcal{F}_{k+1}(3)$ as a whole, for all positive integers $k$.

In parallel to the inverse binomial sums treated in the last corollary, we will soon present period characterizations for the binomial sums, the latter of which played their part in some hyper-Mahler measures studied by Kurokawa–Lalin–Ochiai [48, Theorem 19].

**Corollary 3.8 (Binomial sums via GPLs).** As before, we set $r_1, \ldots, r_M \in \mathbb{Z}_{>0}$ and $r'_1, \ldots, r'_M \in \mathbb{Z}_{>0}$.

(a) If $4|\chi| \leq |1 + \chi|^2$ and $s \in \mathbb{Z}_{>0}$, then we have (cf. [47, (1.1) and (5.2b)])

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} H_{n-1}^{(r_j)} \left( \frac{2n}{n^s} \left[ \frac{\chi}{(1 + \chi)^2} \right] \right)^n \in g_k^{(q)}(2) \cap \mathcal{G}_{k,1}^{\{-1, \frac{1}{2}\}},
\]

where $k = s + \sum_{j=1}^{M} r_j$.

(b) If $2|\nu| \leq |1 + \nu|^2$ and $s \in \mathbb{Z}_{>0}$, then we have

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} H_{n-1}^{(r_j)} \left( \frac{2n}{n^s} \left[ \frac{\nu}{1 + \nu^2} \right]^2 \right) \in g_k^{(q)}(4) \cap \mathcal{G}_{k,2}^{\{-1, \frac{1}{2}, \frac{3}{2}, -\nu, \frac{1}{2}\}},
\]

where $k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r'_{j'}$.

**Proof.** All the infinite series of our concern are convergent, thanks to the asymptotic behavior $(\frac{2n}{n^s}) = \frac{4^n}{\sqrt{n\pi}} [1 + O(\frac{1}{n})]$ as $n \to \infty$.

(a) When $s \in \mathbb{Z}_{>0}$, the left-hand side of (3.79) is

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} H_{n-1}^{(r_j)} \left[ \frac{16\chi}{(1 + \chi)^2} \right]^n \int_{0}^{1} x^{n}(1 - x)^{n} \frac{dx}{\sqrt{x(1 - x)}}
\]

\[
\in \text{span}_{\mathbb{Q}} \left\{ \int_{0}^{1} L_{A} \left( \frac{16\chi x(1 - x)}{(1 + \chi)^2} \right) \frac{dx}{\sqrt{x(1 - x)}} \left| w(A) = s + \sum_{j=1}^{M} r_j \right\} \right. .
\]
at the beginning of the last subsection, we once evaluated $m_3 \in \text{span}_A (\Omega(x,t))$ to the first summand on the right-hand side of (3.83), then GPLs with respect to (3.79).

To verify the claim in (3.80), use GPL fibrations and contour deformations as in our proof of (3.79).

\[ L_A(\Xi(x,t)) \]

where $\Xi(x,t) := \frac{(1+t)^2}{1+x^2}$. The following fibration of the multi-dimensional polylogarithm

\[ L_A(\Xi(x,t)) \]

\begin{align*}
\in \text{span}_Q \left\{ \left( \frac{\pi}{\text{Im} \Xi(x,t)} \right)^\ell \mathbb{Z} \log^r \frac{1}{\Xi(x,t)} \bigg| \mathbb{Z} \in \mathbb{Z}_{\geq 0} \bigg( \mathbb{Z} \in \mathbb{Z}(1) \bigg) \bigg( \ell + j + r = u(\mathbb{A}) \bigg) \right\} \\
+ \text{span}_Q \left\{ \left( \frac{\pi}{\text{Im} \Xi(x,t)} \right)^\ell \mathbb{Z} L_{B} \left( \frac{1}{\Xi(x,t)} \right) \log^r \frac{1}{\Xi(x,t)} \bigg| \mathbb{Z} \in \mathbb{Z}_{\geq 0} \bigg( \mathbb{Z} \in \mathbb{Z}(1) \bigg) \bigg( \ell + j + u(B) + r = u(\mathbb{A}) \bigg) \right\}
\end{align*}

highlights its branch cuts [55, 57], on which $\Xi(x,t) > 1$. If $f(\Xi(x,t))$ belongs to the first summand on the right-hand side of (3.83), then (3.84) remains valid, for two reasons: (1) One can shrink the contour to tight loops wrapping around the branch cuts inside the unit circle, whose boundary points belong to $\{0, -1, \frac{1}{2}, \frac{1}{3} \}$; (2) Across the branch cuts, the integrand $f(\Xi(x,t))$ has jumps in the Q-vector space $\mathbb{Q} 1/(\Xi(x,t)) (1)$, so (3.84) results from fibrations of GPLs with respect to $t$ and integrations of GPLs along the branch cuts.

The arguments in the last paragraph take us to the the right-hand side of (3.79).

(b) When $s \in \mathbb{Z}_{>0}$, the left-hand side of (3.80) is

\[ \sum_{n=1}^{\infty} \prod_{j=1}^{M} \left[ \frac{1}{H_{n-1}^{(r_j)}} \right] \prod_{j=1}^{M'} \left[ \frac{1}{H_{2n-1}^{(r_j')}} + \frac{1}{H_{2n-1}^{(r_j')}} \right] \frac{1}{n \tau} \int_{|\tau|=1} \left[ \frac{(1 + \tau^2 + \eta^2)\tau^2}{(1 + \eta^2)\tau} \right]^{2n} d\tau \]

\[ \in \text{span}_Q \left\{ \int_{|\tau|=1} \frac{G(\alpha_{1}, \ldots, \alpha_{w} ; \frac{(1 + \tau^2)^2}{(1 + \eta^2)\tau})}{2\pi i \tau} d\tau \bigg| \alpha_{1}, \ldots, \alpha_{w-1} \in \{-1, 0, 1\} \bigg( \alpha_{w} \in \{-1, 1\} \bigg) \right\},
\]

in the light of (3.66).

To verify the claim in (3.80), use GPL fibrations and contour deformations as in our proof of (3.79).

\[ 3.3. \text{ Infinite series related to } m_k(1+x_1+x_2+x_3). \] Akin to the descriptions at the beginning of the last subsection, we once evaluated $m_k(1+x_1+x_2+x_3), k \in \mathbb{N}$.
$Z \cap [1, 6]$ by taking $k$-th order derivatives of Crandall’s hypergeometric representation [15, (6.8)]:

$$W_4(s) = \frac{2^s}{\Gamma(s)} \left[ \frac{1}{\sqrt{\pi}} \right] \left[ \frac{1 + \frac{s}{2}}{s + 1 + \frac{s}{2}} \right] \frac{3}{2} \log \left( \frac{1 + \frac{s}{2}}{s + 1 + \frac{s}{2}} \right) \left( \frac{1}{2} \right)$$

(3.86)

$$+ \frac{2^s \Gamma \left( \frac{1 + \frac{s}{2}}{s + 1 + \frac{s}{2}} \right)}{\sqrt{\pi} \Gamma \left( \frac{1 + \frac{s}{2}}{s + 1 + \frac{s}{2}} \right)} F_3 \left( \frac{1 + \frac{s}{2}}{s + 1 + \frac{s}{2}} \right).$$

This produced the following types of series involving $h^{(r)}_n := H^{(r)}_n - 3H^{(r)}_{n+1} + 2\delta_{1,r} \log 2$ and $\tilde{h}^{(r)}_n := 3H^{(r)}_n - H^{(r)}_{n+1} - 2\delta_{1,r} \log 2$:

$$\pi \sum_{n=0}^{\infty} \prod_{j=1}^{\tilde{r}} h^{(r)}_{n+1} \prod_{j=1}^{r'} \prod_{j=1}^{r''} \prod_{j=1}^{s} \sum_{n=0}^{\infty} \frac{\tilde{h}^{(r)}_{n+1}}{(2n+1)^3},$$

(3.87)

where $r_0 + \sum_{j=1}^{\tilde{r}} r_j = k - 3$, $s_0 + \sigma + \sum_{j=1}^{\tilde{r}} s_j = k$, $r_0 \in 2Z_{\geq -1}$, $s_0 \in 2Z_{\geq 0}$, $\sigma \in Z_{\geq 3}$, and $\varepsilon \in \{0, 1\}$. (Here, both $\ell$ and $\tilde{\ell}$ may be zero, whereupon the empty products are equal to 1.) All such infinite series have closed forms, which add up to the values in Table 2.

According to the next corollary, the first type of series in (3.87) are always representable as $\frac{1}{\pi} \times$ times $Q$-linear combinations of AMZVs of the same weight, and so are Broadhurst’s series in (3.5).

**Corollary 3.9 (Euler sums, HPLs, and Goncharov–Deligne periods).** For $s \in Z_{\geq 0}$, $r_1, \ldots, r_M \in Z_{\geq 0}$, and $r'_1, \ldots, r'_{M'} \in Z_{\geq 0}$, we have

$$\sum_{n=1}^{\infty} \prod_{j=1}^{M} \left[ H^{(r_j)}_{n+1} \right] \prod_{j'=1}^{M'} \left[ H^{(r'_{j'})}_{n+1} \right] \frac{\varepsilon^{2n-1}}{(2n-1)s+1} \in \mathcal{G}_{k+1}(2),$$

(3.88)

where $|z| < 1, k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r'_{j'}$. For $s \in Z_{\geq 0}$, $r_1, \ldots, r_M \in Z_{\geq 0}$, and $r'_1, \ldots, r'_{M'} \in Z_{\geq 0}$, the result above extends to

$$\sum_{n=1}^{\infty} \prod_{j=1}^{M} \left[ H^{(r_j)}_{n+1} \right] \prod_{j'=1}^{M'} \left[ H^{(r'_{j'})}_{n+1} \right] \frac{1}{(2n-1)s+2} \in \mathcal{G}_{k+2}(2),$$

(3.89)

$$\sum_{n=1}^{\infty} \prod_{j=1}^{M} \left[ H^{(r_j)}_{n+1} \right] \prod_{j'=1}^{M'} \left[ H^{(r'_{j'})}_{n+1} \right] \frac{(-1)^n}{(2n-1)s+1} \in \mathcal{G}_{k+1}(4),$$

(3.90)

where $k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r'_{j'}$.

**Proof.** To prove (3.88) for $s = 0$, divide both sides of (3.62) by $z$ and integrate. For progress onto all $s \in Z_{\geq 0}$, apply the operations $\int f(z) \rightarrow \int_{0}^{z} f(z) \frac{dz}{z}$ iteratively. In fact, this method allows us to deduce a slightly stronger statement:

$$\sum_{n=1}^{\infty} \prod_{j=1}^{M} \left[ H^{(r_j)}_{n+1} \right] \prod_{j'=1}^{M'} \left[ H^{(r'_{j'})}_{n+1} \right] \frac{\varepsilon^{2n-1}}{(2n-1)s+1} \in \mathcal{G}_{s+1}(2) \in \mathcal{G}_{s+2}(2).$$

(3.91)
When \( s \in \mathbb{Z}_{>0} \) (resp. \( s \in \mathbb{Z}_{=1} \)), the integral representation for the left-hand side of (3.88) converges at \( z = i \) (resp. \( z = 1 \)), according to variations on the proof of Theorem 3.1. Thus, the period characterizations in (3.89) and (3.90) immediately appear in the following series representation:

\[
\sum_{n=1}^{\infty} \prod_{j=1}^{M} H_n^{(r_j)} \left( \prod_{j'=1}^{M'} H_n^{(r_{j'})} + 2\delta_{1,r_{j'}} \right) \frac{1}{(2n-1)^{s+2}} \in 3(N)
\]

and (3.90) to \( Z \) on Hoffman’s treatment of the hyper-Mahler measures in Theorem 2.11. It is difficult to gain an arithmetic or combinatorial understanding for the magic for \( m \) cancelations of AMZVs bearing “undesirable weights” in the series representations (3.93) of (3.88) converges at \( z = 3 + \infty \). We found it difficult to gain an arithmetic or combinatorial understanding for the magic cancelations of AMZVs bearing “undesirable weights” in the series representations for \( m_{5}(1 + x_1 + x_2 + x_3) \). We circumvented this obstacle by turning to a holistic treatment of the hyper-Mahler measures in Theorem 2.11.
We conclude this work by pointing out that the techniques in the last three corollaries can be combined to take some recently proven series evaluations [26, 28] a fair distance.

**Corollary 3.12 (GPL and CMZV representations of generalized Sun series).** If \( r_1, \ldots, r_M \in \mathbb{Z}_{>0}, r'_1, \ldots, r'_M \in \mathbb{Z}_{>0}, |1 + \tau^2| \leq 2|\tau|, \) and \( s \in \mathbb{Z}_{>0}, \) then we have

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}(r_j) \right] \left[ \prod_{j=1}^{M'} H_{2n-1}(r'_j) \sum_{n=1}^{\infty} \right]
\]

\[
\in i \left( g_{k+1}(4) \cap \mathcal{S}_{k+1}(12) \right),
\]

where \( k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r'_j. \)

**Proof.** Noting that

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}(r_j) \right] \left[ \prod_{j=1}^{M'} H_{2n-1}(r'_j) \sum_{n=1}^{\infty} \right]
\]

\[
\in i \left( g_{k+1}(4) \cap \mathcal{S}_{k+1}(12) \right),
\]

where \( k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r'_j. \) It is possible to use CMZVs at a lower level to represent a special subclass of (3.98), namely

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}(r_j) \right] \left[ \prod_{j=1}^{M'} H_{2n-1}(r'_j) \sum_{n=1}^{\infty} \right]
\]

\[
\in i \mathcal{S}_{k+1}(3),
\]

where \( k = s + \sum_{j=1}^{M} r_j. \)

**Proof.** Noting that

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}(r_j) \right] \left[ \prod_{j=1}^{M'} H_{2n-1}(r'_j) \sum_{n=1}^{\infty} \right]
\]

\[
\in i \mathcal{S}_{k+1}(10),
\]

where \( k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r'_j. \) If \( k + 1 \leq 3, \) then the left-hand side of (3.99) belongs to \( \mathcal{S}_{k+1}(10). \)

**Proof.** Noting that

\[
\sum_{n=1}^{\infty} \left[ \prod_{j=1}^{M} H_{n-1}(r_j) \right] \left[ \prod_{j=1}^{M'} H_{2n-1}(r'_j) \sum_{n=1}^{\infty} \right]
\]

\[
\in i \mathcal{S}_{k+1}(10),
\]

we may identify the left-hand side of (3.97) with a member of

\[
\text{span}_{\mathbb{Q}} \left\{ \int_{0}^{\infty} g \left( \frac{1 + \tau^2 t}{(1 + t^2)^2} \right) \frac{dt}{\pi t} \right\},
\]

by virtue of the symmetry displayed in (3.91) and a variable substitution \( X = \frac{1}{1 + \tau^2}. \)

This effectively reduces our target to an integral

\[
\int_{\infty}^{\infty} G \left( \alpha_1, \ldots, \alpha_{k+1}; \frac{(1 + \tau^2 t)(1 + t^2)}{(1 + t^2)^2} \right) \frac{dt}{\pi t}, \quad \alpha_1, \ldots, \alpha_{k+1} \in \{-1, 0, 1\},
\]

whose contour can be deformed to loops wrapping around the branch cuts in the upper half-plane, on which \( \frac{(1 + \tau^2 t)(1 + t^2)}{(1 + t^2)^2} \in (-\infty, -1) \cup (1, \infty) \subset \mathbb{R}. \) The end points of
Table 7. Selected closed forms for binomial sums of Sun’s type, where
\( \omega := e^{2\pi i/3}, \theta := e^{\pi i/3}, \rho := \sqrt{3}i, \zeta := e^{2\pi i/5}, G := \text{Im Li}_2(i), \) and
\( \lambda := \log 2 \)

| \( s \) | \( \sum_{n=1}^{\infty} \frac{(2n-2)!}{n(2n-1)!} \frac{1}{2n-1} \) | \( \sum_{n=1}^{\infty} \frac{u^{(1)}_{n-1} - 2^{4(n-1)/2}}{u^{(1)}_{n-1} + 2^{4(n-1)/2}} \) |
|---|---|---|
| 1 | \( \frac{3\pi^2}{24} - 2 \text{Im Li}_2(\omega) \) | \( 3 \text{Im Li}_{2,1}(\omega) + \frac{\pi^2}{24} \) |
| 2 | \( 3 \text{Im Li}_{2,1}(\omega) + \frac{\pi^2}{24} \) | \( 3 \text{Im Li}_{2,2}(\omega) + \frac{\pi^2}{24} \) |
| 3 | \( -\frac{\pi^2}{24} \) | \( -\frac{\pi^2}{24} \) |
| 4 | \( \frac{\pi^2}{24} \) | \( -\frac{\pi^2}{24} \) |

these branch cuts are in \( \{ i, \tau, \frac{\pi}{2}, -\tau, -\frac{\pi}{2} \} \cap \{ w \in \mathbb{C} | \text{Im } w > 0 \} \). As we integrate with respect to \( t \) (resp. \( \tau \)) before (resp. after) integration over branch cuts, we arrive at the claim in the right-hand side of (3.97).

The rationales behind the right-hand sides of (3.98) and (3.99) are similar to those for (3.43) and (3.44).

Enlisting the help from (3.28), we may equate the left-hand side of (3.98) with\(^{14}\)

\[
\frac{1}{2s} \sum_{n=1}^{\infty} \frac{(2n-2)!}{n(2n-1)!} \int_0^{r+1} \frac{1}{x^{n+1}} \frac{r^{1, \ldots, r_M}}{2n-1} \, dx^{2n-1} = \frac{1}{2s} \int_{r=i}^{r=\pi} \frac{1}{x^{s+1}} \frac{r^{1, \ldots, r_M}}{2n-1} \, dx^{2n-1} = \frac{1}{2s} \int_{r=i}^{r=\pi} \frac{1}{x^{s+1}} \left( \frac{\log x}{r^2} \right) \arcsin \frac{1 + \tau^2}{\tau^2} \, dx^{2n-1}
\]

\[(3.103)\]

\(^{14}\)To construct Tables 7 and 8, we need the following special cases of \( L^{(1)}_s(x) = \frac{1}{2s} \int_{r=i}^{r=\pi} \frac{1}{x^{s+1}} \frac{r^{1, \ldots, r_M}}{2n-1} \, dx^{2n-1} \):

- \( L_1^{(1)}(x) = \frac{1}{2} \left( G_{1,1}^{(1,1)} \left( \frac{x}{1+x} \right) - G_{1,1}^{(1,1)} \left( \frac{x+1}{x} \right) \right) \) for \( 0 < x < 1 \): \( L_1^{(1)}(x) = L_1(x), L_2^{(1)}(x) = -L_1(x) + \frac{x^2}{6}, \)
- \( L_{3,1}^{(1)}(x) = L_3(x) + \frac{2}{3} \log^2 x - \zeta_3, \) and \( L_{4,1}^{(1)}(x) = -L_4(x) + \zeta_4 + \log x + \frac{2}{3} \log^2 x + \frac{x^3}{12} + \frac{x^4}{234}.\)
where \( L_{k+1}^{(r_1, \ldots, r_M)}(X) \in \mathcal{g}_k(X) \) (1) and \( k = s + \sum_{j=1}^{M} r_j \). Introducing a new variable \( u = -\tau^2 \), one can reveal the expression above as member of

\[
\begin{aligned}
&\text{span}_Q \left\{ (\pi i)^{k-\ell} Z_{\ell-j} \int_1^{1/\rho} G\left( \alpha_1, \ldots, \alpha_j; -\left(1 - u\right)^2 \right) \frac{du}{\ell^u} : \alpha_1, \ldots, \alpha_j \in \{0, 1\} \right\} \\
&\subseteq \text{span}_Q \left\{ (\pi i)^{k-\ell} Z_{\ell-j} \int_1^{1/\rho} G\left( \alpha_1, \ldots, \alpha_j; u\right) \frac{du}{\ell^u} : \alpha_1, \ldots, \alpha_j \in \{0, 1\} \right\} \\
&\subseteq \frac{g_{k+1}[1, \omega, \omega^2; \tau](3)}{i} \subseteq i3_{k+1}(3).
\end{aligned}
\]

If the weight \( k+1 \) does not exceed 3 in the case of (3.99), then we are effectively working with \( g_{k+1}^S \), where the subset \( S \) of \( \{i, -i, \tau, -\frac{1}{\rho}, -\frac{1}{\rho}, \frac{1}{\rho}, \frac{1}{\rho}\} = \{i, -i, i\rho, -\frac{1}{i\rho}, -i\rho, \frac{1}{i\rho}\} \) contains no more than 3 members. By virtue of GPL rescaling and the fact that \( \log \rho = \text{Re} \text{Li}_1(e^{\pi i/\rho}) - \text{Li}_1(e^{2\pi i/\rho}) \in 3(5) \), we only need to check that \( G(\alpha; 1) \in 3_{\omega(\alpha)}(10) \) when the components of \( \alpha \) involve no more than 2 different numbers from one of the following sets:

\[
\{1, -1, \rho, \frac{1}{\rho}, -\rho, \frac{1}{\rho}\}, \quad \{\frac{1}{\rho}, -\frac{1}{\rho}, 1, \frac{1}{\rho}, -1, -\frac{1}{\rho}\}, \quad \{\rho, -\rho, \rho^2, 1, -\rho^2, -1\}.
\]

This can be done automatically by Au’s MZIteratedIntegralDoableQ function.

**Remark 3.13.** As revealed by Table 7, two \( 3_{k+1}(10) \) cases of (3.99) cover an infinite series evaluated in [28, §4]:

\[
\begin{aligned}
\sum_{n=0}^{\infty} \frac{\binom{2n}{n} \left[ 9H_{2n+1}^{(1)} + \frac{32}{2n+1} \right]}{(2n+1)^2(-16)^n} &= 4\pi \zeta_3, \\
&= 14\zeta_3.
\end{aligned}
\]

Charlton–Gangl–Lai–Xu–Zhao [28, §§2–3] verified another infinite series due to Zhi-Wei Sun, namely

\[
\begin{aligned}
\sum_{n=0}^{\infty} \frac{\binom{2n}{n} \left[ 9H_{2n+1}^{(1)} + \frac{32}{2n+1} \right]}{(2n+1)^2(-16)^n} &= 40 \text{Im} \text{Li}_4(i) + \frac{5\pi \zeta_3}{12}.
\end{aligned}
\]

by maneuvering identities of polylogarithms. Au [5, Corollary 3.4] achieved the same by an application of the Wilf–Zeilberger method. At the time of this writing, we do not have an automated proof of (3.107), since Au’s MultipleZetaValues package currently does not simplify CMZVs in \( 3_{12}(1) \).

**Corollary 3.14** (Generalized Wang–Chu series as Goncharov–Deligne periods). For \( s \in \mathbb{Z}_{\geq 0} \), \( r_1, \ldots, r_M \in \mathbb{Z}_{\geq 0} \), and \( r'_1, \ldots, r'_M' \in \mathbb{Z}_{>0} \), we have \( \text{cf. [26, (17), (18), (27)–(29)]} \)

\[
\begin{aligned}
\sum_{n=1}^{\infty} &\left\{ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right\} \left\{ \prod_{j=1}^{M'} H_{2n-1}^{(r'_j)} + \Pi_{2n-1}^{(r'_j)} \right\} \frac{(2n-2)^2}{2^{4n}(2n-1)^2+1} \in 3_{k+2}(4) \frac{\pi i}{3}, \\
\sum_{n=1}^{\infty} &\left\{ \prod_{j=1}^{M} H_{n-1}^{(r_j)} \right\} \left\{ \prod_{j=1}^{M'} H_{2n-1}^{(r'_j)} + \Pi_{2n-1}^{(r'_j)} \right\} \frac{(2n)^2}{2^{4n+1}} \in 3_{k+2}(4) \frac{\pi i}{3}.
\end{aligned}
\]
Table 8. Selected closed forms for quadratic analogs of binomial sums, where $G := \text{Im} \, \text{Li}_2(i)$ and $\lambda := \log 2$

\[
\sum_{n=0}^{\infty} \frac{(2n-2)^2}{2^{6n}} = \frac{1}{2} \int_0^\infty \left[ \int \log^2 \left( \frac{1 + r^2 + \pi \log \frac{s}{1 + r^2}}{1 + r^2} \frac{1 + r^2 + \pi \log \frac{s}{1 + r^2}}{1 + r^2} \right) \, dr \right] \, dv
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-2)^2}{2^{5n}(2n-1)^2} = \frac{1}{2} \int_0^\infty \int \left[ \int \left( \frac{1 + r^2 + \pi \log \frac{s}{1 + r^2}}{1 + r^2} \frac{1 + r^2 + \pi \log \frac{s}{1 + r^2}}{1 + r^2} \right) \, dr \right] \, dv
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-2)^2}{2^{5n}(2n-1)^2} = \frac{1}{2} \int_0^\infty \int \left[ \int \left( \frac{1 + r^2 + \pi \log \frac{s}{1 + r^2}}{1 + r^2} \frac{1 + r^2 + \pi \log \frac{s}{1 + r^2}}{1 + r^2} \right) \, dr \right] \, dv
\]

\[
\sum_{n=1}^{\infty} \frac{(2n-2)^2}{2^{5n}(2n-1)^2} = \frac{1}{2} \int_0^\infty \int \left[ \int \left( \frac{1 + r^2 + \pi \log \frac{s}{1 + r^2}}{1 + r^2} \frac{1 + r^2 + \pi \log \frac{s}{1 + r^2}}{1 + r^2} \right) \, dr \right] \, dv
\]
where \( k = s + \sum_{j=1}^{M} r_j + \sum_{j'=1}^{M'} r_{j}' \).

**Proof.** Observe that

\[
\frac{\pi}{2} \binom{2n - 2}{n - 1} = \int_0^{\pi/2} (2 \cos \phi)^{2n - 2} \, d\phi = \int_1^{i} \left( \frac{1 + \tau^2}{\tau} \right)^{2n-1} \frac{d\tau}{i(1 + \tau^2)},
\]

where the integral contour in the complex \( \tau \)-plane is a straight line segment, on which \(|1 + \tau^2| \leq 2|\tau|\). Hence, the last corollary allows us to equate the left-hand side of (3.108) with a member of the following \( \mathbb{Q} \)-vector space:

\[
\frac{1}{\pi} \text{span}_{\mathbb{Q}} \left\{ \int_1^{i} \frac{(\pi i)^\ell G(\alpha_1, \ldots, \alpha_{k+1-\ell}; \tau) \, d\tau}{1 + \tau^2} \mid \alpha_1, \ldots, \alpha_{k+1-\ell} \in \{0, 1, i, -1, -i\}, \ell \in \mathbb{Z}_{\geq 0} \right\}.
\]

Appealing to the GPL recursion in (2.22) and applying Panzer’s logarithmic regularization [57, §2.3] at \( \tau = i \), we may recognize the expression above as a subspace of \( \frac{1}{\pi i} \mathbb{Z}^{k+2}(4) \).

To prove (3.109), first verify

\[
\int_0^{1} \frac{1}{1 + v^2} \left( \frac{v}{1 + v^2} \right)^{2n} \frac{dv}{1 + v^2} = \frac{1}{2} \int_0^{1/2} [x(1 - x)]^{n-\frac{1}{2}} \, dx = \frac{\pi(2n)}{2^{2n+1}}
\]

by a substitution \( v = \sqrt{x}/\sqrt{1-x} \), then integrate the relation (3.80) in Corollary 3.8(b).

\[\blacksquare\]

**Remark 3.15.** Some special cases of (3.108) and (3.109) have been investigated by Cantarini–D’Aurizio [27, §§2–3] in the Fourier–Legendre framework. See Table 8 for more examples.

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Program in Applied and Computational Mathematics (PACM), Princeton University, Princeton, NJ 08544

Email address: yajunz@math.princeton.edu

Current address: Academy of Advanced Interdisciplinary Studies (AAIS), Peking University, Beijing 100871, P. R. China

Email address: yajun.zhou.1982@pku.edu.cn