$SL(2)$-solution of the pentagon equation and invariants of three-dimensional manifolds

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Abstract

Building on a classical solution to the pentagon equation, constructed earlier by the author and E. V. Martyushev and related to the flat geometry invariant under the group $SL(2)$, we construct an algebraic complex corresponding to a triangulation of a three-manifold. In case if this complex is acyclic (which is confirmed by examples), we use it for constructing a manifold invariant.

1 Introduction

Recently, acyclic complexes of a new kind were invented which can be put in correspondence to triangulations of three- and four-dimensional manifolds [1, 2, 3]. Euclidean metric values (such as edge lengths, dihedral angles and Euclidean coordinates of vertices) were ascribed to the elements of these triangulations. The linear spaces entering in the complex were constructed out of infinitesimal variations of such values. The result was the construction of invariants of piecewise-linear manifolds which were expressed in terms of the torsion of the complex together with volumes of simplices of different dimensions entering in the triangulation.

The basis for constructing these new acyclic complexes consisted in algebraic relations corresponding in a natural sense to elementary rebuildings of a manifold triangulation — Pachner moves, namely the move $2 \rightarrow 3$ (two tetrahedra having a common face are replaced by three tetrahedra having a common edge; we call such relation pentagon equation) in the three-dimensional case, and the move $3 \rightarrow 3$ in the four-dimensional case. Then, the complex was built in such way as to ensure the construction of the invariant of all Pachner moves.

In the three-dimensional case, the invariant passes successfully a standard test of distinguishing the lens spaces [6]. The more complicated four-dimensional case requires further investigations.
In the present paper, we are dealing again with three-manifolds but, instead of using the solution to pentagon equation related to Euclidean values, we are using the SL(2)-solution found in paper [5]. We continue to use this name for it, although a bigger group arises, too, in our constructions — the group of area-preserving affine motions of a plane.

Two representations of the manifold $M$ fundamental group $\pi_1(M)$ enter in a natural way in the construction of acyclic complex. To explain this, consider once again the Euclidean case. In paper [4], we put the vertices of a triangulation of manifold $M$ in a three-dimensional Euclidean space, while in paper [6] we put there the vertices of the triangulation of universal cover of manifold $M$. One can say that every vertex of $M$’s triangulation was multiplied to $\text{card}(\pi_1(M))$ its copies, and the transition from one copy to another was determined by the image of an element of $\pi_1(M)$ in the group $E_3$ of Euclidean motions of the three-dimensional space with respect to some representation $f$: $\pi_1(M) \rightarrow E_3$. In addition to representation $f$, one can consider other representations of group $\pi_1(M)$, namely in the linear spaces of differentials of which our complex (corresponding to the universal cover) is built. Here the work with nontrivial representations was initiated in short note [7].

Two similar kinds of representations can be considered in the SL(2)-case as well. The experience of studying the Euclidean case shows that it is exactly the use of nontrivial representations that leads to the most interesting manifold invariants. Still, we will confine ourselves in this paper to the simplest case where both representations are trivial. We hope to study the invariants corresponding to nontrivial representations in subsequent papers.

The contents of the remaining sections of this paper is as follows: in section 2 we write out the solution to pentagon equation from [5] together with some new ideas. In section 3 we construct an algebraic complex on this basis. In section 4 we study the behavior of the torsion of the complex (assuming its acyclicity) under the Pachner moves $2 \rightarrow 3$ and $1 \rightarrow 4$, and propose a formula for the manifold invariant based on this study. In section 5 the invariant for sphere $S^3$ and projective space $\mathbb{R}P^3$ is computed (the complexes turning out indeed acyclic). In the final section 6 we discuss the results and plans for future research.

## 2 SL(2)-solution of pentagon equation

The Pachner move $2 \rightarrow 3$ is pictured in Figure 1: adjacent tetrahedra $EABC$ and $ABCD$ which belong to a triangulation of a three-dimensional oriented manifold are replaced with three tetrahedra $ABED$, $BCED$ and $CAED$. We put in correspondence to every oriented edge a real number, for instance, number $\lambda_{AB}$ to edge $AB$, and assume that for all edges

$$\lambda_{BA} = -\lambda_{AB}. \quad (1)$$

If necessary, we can extend the field to which numbers $\lambda$ and related values belong to the field $\mathbb{C}$ of complex numbers. On the other hand, we are not considering at this

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moment an interesting question of what we will get if $\lambda$’s belong to a field of a finite characteristic.

For every oriented two-dimensional face, we construct a value $S$ which is the circulation of values $\lambda$; e.g., for face $ABC$, by definition,

$$S_{ABC} = \lambda_{AB} + \lambda_{BC} + \lambda_{CA}.$$  \hspace{1cm} (2)

Now we ascribe numerical values to dihedral angles at the edges of a tetrahedron. Consider an oriented tetrahedron $BCED$ and its oriented edge $ED$. We say that the “value of the dihedral angle” at $ED$ is, by definition,

$$\alpha = \frac{1}{2} S_{BDE} + S_{BEC} - S_{BDC}.$$  \hspace{1cm} (3)

Thus, $\alpha$ changes its sign both when we change the orientation of the tetrahedron (without changing the orientation of edge $ED$) and when we change the orientation of edge $ED$ (without changing the orientation of the tetrahedron).

Introduce similar values for tetrahedra $CAED$ and $ABED$:

$$\beta = \frac{1}{2} S_{CDA} + S_{CEA} - S_{CDE},$$  \hspace{1cm} (4)

$$\gamma = \frac{1}{2} S_{ADB} + S_{AEB} - S_{ADE}.$$  \hspace{1cm} (5)

We assume from now on that (if the contrary is not stated explicitly) we always choose the positive orientation for all tetrahedra belonging to the manifold triangulation, i.e., the orientation determined by the order of tetrahedron vertices coincides with the fixed orientation of the whole manifold. We set by definition

$$\omega_{ED} = -\omega_{DE} = \alpha + \beta + \gamma.$$  \hspace{1cm} (6)

Under the agreement of the previous paragraph, this value is uniquely determined by the oriented edge $ED$. Moreover, its definition generalizes in the obvious way for
the case where edge $ED$ is common for more than three tetrahedra. Due to the reasons which will soon be clear, we will call $\omega_{ED}$ the *curvature around edge $ED$*.

Return now to Figure 1. The following formula takes place:

$$S_{ABC} = S_{ADE}S_{BDE}S_{CDE} \frac{\partial \omega_{ED}}{\partial \lambda_{ED}},$$

where all values are calculated at such $\lambda_{ED}$ for which $\omega_{ED} = 0$. The relation (7) is exactly the solution of pentagon equation that lies in the foundation of our subsequent constructions. In its left-hand side, there is the value $S_{ABC}$ corresponding exactly to the face which is present in the left diagram of Figure 1 but absent from the right diagram, and a similar statement is true for the right-hand side of (7) as well.

The validity of relation (7) can be verified directly. The ideas that can lead to it can be found in paper [5] (cf. formula (15) of that paper).

The condition $\omega_{ED} = 0$ is equivalent to the bilinear relation

$$S_{ADB}S_{CDE} + S_{BDC}S_{ADE} + S_{CDA}S_{BDE} = 0,$$

and (8) is transformed into itself under any transposition of letters $A, \ldots, E$. On the other hand, the relation (8) holds if we assume that $A, \ldots, E$ are points in the usual *plane* $\mathbb{R}^2$, while $S_{\cdots}$ are the oriented areas of triangles. Besides the bilinear relations, the areas obey, of course, linear relations of type $S_{ABC} + S_{ACD} = S_{ABD} + S_{BCD}$, which is in accordance with such formulas as (2).

This motivates the following constructions. Let $A, B, D$ and $E$ be points in the plane $\mathbb{R}^2$. We remark at once that we don’t need to calculate, e.g., distances between points or angles (in the Euclidean sense) within this plane, but only areas of figures. Let all areas of triangles with vertices in these points be given. Then one can find that

$$\overrightarrow{EB} = \frac{S_{EBA}\overrightarrow{ED} + S_{EDB}\overrightarrow{EA}}{S_{EDA}},$$

where $S_{EBA}$ is the oriented area of triangle $EBA$ and so on.

Let us add one more point, $C$, to our four points. Replacing in (9) the pair $A, B$ first with $B, C$ and then with $C, A$, we get two more relations:

$$\overrightarrow{EC} = \frac{S_{ECB}\overrightarrow{ED} + S_{EDC}\overrightarrow{EB}}{S_{EDB}},$$

$$\overrightarrow{EA} = \frac{S_{EAC}\overrightarrow{ED} + S_{EDA}\overrightarrow{EC}}{S_{EDC}}.$$
So, we consider formulas (9), (10) and (11) simply as linear relations imposed on some nonzero vectors $\overrightarrow{EA}$, $\overrightarrow{EB}$, $\overrightarrow{EC}$ and $\overrightarrow{ED}$ lying in $\mathbb{R}^2$.

If (8) holds, then these relation agree with each other. We will not, however, require the validity of (8), but instead of this we change $\overrightarrow{EA}$ to $\overrightarrow{EA}_{\text{new}}$ in the l.h.s. of (11), and express $\overrightarrow{EA}_{\text{new}}$ in terms of $\overrightarrow{ED}$ and $\overrightarrow{EA}$, substituting expressions (10) and (9) for $\overrightarrow{EC}$ and $\overrightarrow{EB}$. We get, after some calculation:

$$\overrightarrow{EA}_{\text{new}} = \overrightarrow{EA} + \omega_{ED} S_{EDA} \overrightarrow{ED}. \quad (12)$$

From the viewpoint of Figure 1, the relation (9) deals with tetrahedron $ABED$, (10) — with tetrahedron $BCED$ and (11) — with $CAED$. Their successive use corresponds, one can say, to going around the edge $ED$. Namely, we see that the element of $SL(2)$ which corresponds to this going around is determined by the following transformation of (unnormed) bases in the plane $\mathbb{R}^2$:

$$(\overrightarrow{ED}, \overrightarrow{EA}) \rightarrow (\overrightarrow{ED}, \overrightarrow{EA} + \omega_{ED} S_{EDA} \overrightarrow{ED}).$$

If we return to considering $E$, $D$ and $A$ in formula (12) as points in the plane $\mathbb{R}^2$, $\overrightarrow{ED}$ and $\overrightarrow{EA}$ — as vectors with corresponding origins and ends, and $S_{EDA}$ — as the area of triangle $EDA$, then the transformation (12) depends only on the vector $\overrightarrow{ED}$ and number $\omega_{ED}$, and not on the vector $\overrightarrow{EA}$.

### 3 Construction of the algebraic complex

The algebraic complex that we are going to build in this Section has the following form:

$$0 \rightarrow \mathbb{R}^6 \xrightarrow{f_1} \mathbb{R}^{3\#\text{vertices}} \xrightarrow{f_2} \mathbb{R}^{\#\text{edges}} \xrightarrow{f_3} \mathbb{R}^{\#\text{edges}} \xrightarrow{f_4} \mathbb{R}^{3\#\text{vertices}} \xrightarrow{f_5} \mathbb{R}^6 \rightarrow 0. \quad (13)$$

Here $\#\text{vertices}$ is, of course, the number of vertices in the simplicial complex, while $\#\text{edges}$ — the number of edges. The bases are fixed in all spaces, so the mappings are identified with matrices. We now pass on to defining the mappings $f_i$ in the sequence (13).

We put all the vertices of the complex in the two-dimensional coordinate space $\mathbb{R}^2$, i.e. we ascribe to every vertex $A$ real numbers $x_A$ and $y_A$. We assume that values $x_A$ and $y_A$ can take infinitesimal variations $dx_A$ and $dy_A$. Besides, we put in correspondence to every vertex $A$ one more real number $\kappa_A$ which, too, can take a variation $d\kappa_A$. We understand the second left nonzero space $\mathbb{R}^{3\#\text{vertices}}$ in the sequence (13) as consisting of columns of differentials $(dx_A, dy_A, d\kappa_A, dx_B, dy_B, d\kappa_B, \ldots)^T$ (the superscript $T$ means matrix transposing).

The leftmost nonzero space $\mathbb{R}^6$ is the five-dimensional Lie algebra of infinitesimal affine area-preserving transformations of the plane $\mathbb{R}^2$ plus (direct sum) the one-dimensional space of differentials $d\kappa$. A vector in this space is represented by a column

$$(dt_1, dt_2, dt_3, dx, dy, d\kappa)^T, \quad (14)$$
which corresponds to the element \((dt_3 \quad dt_1 \quad dt_2 \quad dt_3)\) of Lie algebra \(\mathfrak{sl}(2)\), the translation by vector \(\left(\begin{array}{c} dx \\ dy \end{array}\right)\) in \(\mathbb{R}^2\) and the translation of parameter \(\kappa\) by \(d\kappa\). The mapping \(f_1\) in sequence (13) is as follows: to a vector (14) there correspond, in every vertex \(A\), the vector \(\left(\begin{array}{c} dx_A \\ dy_A \end{array}\right) = \left(\begin{array}{c} dt_3 \\ dt_1 \\ dt_2 \\ -dt_3 \end{array}\right) \left(\begin{array}{c} x_A \\ y_A \end{array}\right) + \left(\begin{array}{c} dx \\ dy \end{array}\right)\) (15) and
\[d\kappa_A = d\kappa + \frac{x_A dy - y_A dx}{2}.\] (16)

The meaning of formulas (15) and (16) will become clear when we start proving that the sequence (13) is a complex.

Now we ascribe to every edge \(AB\) the value
\[\lambda_{AB} = S_{OAB} + \kappa_B - \kappa_A,\] (17)
where \(S_{OAB}\) is the (oriented) area of triangle \(OAB\), \(O\) is the origin of coordinates. The left one of the two spaces \(\mathbb{R}^{\#\text{edges}}\) in (13) consists of columns of differentials \(d\lambda_{AB}\) for all edges. The mapping \(f_2\), by definition, is obtained by differentiating the equality (17), i.e. the value \(d\lambda_{AB}\) which is obtained from given \(dx, dy\) and \(d\kappa\) by means of \(f_2\) is
\[d\lambda_{AB} = \frac{y_B}{2} dx_A - \frac{x_B}{2} dy_A - \frac{y_A}{2} dx_B + \frac{x_A}{2} dy_B + d\kappa_B - d\kappa_A.\] (18)

As the discussion in Section 2 shows, the curvatures \(\omega_{AB}\) around all edges are zero if all \(\lambda_{AB}\) are obtained from vertex coordinates by formulas (17) (the adding of values \(\kappa\) obviously does not interfere with the vanishing of the curvatures). Now we introduce into consideration the differentials of all curvatures. They will form the right one of the two spaces \(\mathbb{R}^{\#\text{vertices}}\) in sequence (13). The matrix of mapping \(f_3\) is, by definition, \((\partial \omega_a / \partial \lambda_b)\), where \(a\) numbers the edges of the complex and at the same time the rows of this matrix, while \(b\) — the edges and the columns. The partial derivatives are calculated on the basis of relation (10) or its generalization to greater number of tetrahedra, the “angles” \(\alpha, \beta, \gamma, \ldots\) being calculated according to formulas of type (3), (4), (5), where, of course, values \(\lambda\) are substituted according to formulas (2) and (1).

The right one of the two spaces \(\mathbb{R}^{3\#\text{vertices}}\) in sequence (13) will be the direct sum of Lie algebras \(\mathfrak{sl}(2)\), one copy of the algebra for every vertex in the complex. The mapping \(f_4\) is constructed in the following way. Consider, for instance, a vertex \(E\). The element \(d\gamma_E \in \mathfrak{sl}(2)\) in which \(f_4\) transforms a given configuration of values \(d\omega\) on the edges of complex equals, by definition, the sum over all edges \(beginning\) in \(E\) of the elements of this Lie algebra corresponding to infinitesimal basis transformations of type (12).

Assume we are considering an edge \(ED\) whose coordinates are
\[\left(\begin{array}{c} x_{ED} \\ y_{ED} \end{array}\right) = \left(\begin{array}{c} x_D - x_E \\ y_D - y_E \end{array}\right).\] (19)
After some calculation, we find from formula (12), where we replace \( \omega_{ED} \) with \( d\omega_{ED} \) and consider, instead of the transformation from the group \( SL(2) \), its differential (i.e., subtract the identity matrix), the following element of algebra \( \mathfrak{sl}(2) \):

\[
\frac{1}{2} d\omega_{ED} \begin{pmatrix}
-x_{ED} y_{ED} & x_{ED}^2 \\
-y_{ED}^2 & x_{ED} y_{ED}
\end{pmatrix}.
\]  

(20)

To construct the algebraic complex, we need only the structure of vector space, thus, we will identify the expression (20) with the column vector

\[
d\gamma_{ED} = \frac{d\omega_{ED}}{2} \begin{pmatrix}
x_{ED}^2 \\
x_{ED} y_{ED} \\
y_{ED}^2
\end{pmatrix}
\]

(21)

Summing over all vertices \( D \) joined with \( E \) by the edges, we get the three components of vector \( d\gamma \in \mathbb{R}^{3 \# \text{vertices}} \) corresponding to vertex \( E \):

\[
(d\gamma_E)_1 = \sum_D x_{ED}^2 \frac{d\omega_{ED}}{2},
\]  

(22)

\[
(d\gamma_E)_2 = \sum_D x_{ED} y_{ED} \frac{d\omega_{ED}}{2},
\]

(23)

\[
(d\gamma_E)_3 = \sum_D y_{ED}^2 \frac{d\omega_{ED}}{2}.
\]

(24)

It remains to construct the right space \( \mathbb{R}^6 \) and mapping \( f_5 \). The space will consist of column vectors with components \( d\beta_1, \ldots, d\beta_6 \). By definition, values \( d\beta \) obtained by means of \( f_5 \) from given \( d\gamma \) are (the sums are taken over all vertices \( A \) in the complex):

\[
d\beta_1 = \sum_A (d\gamma_A)_1, \quad d\beta_2 = \sum_A (d\gamma_A)_2, \quad d\beta_3 = \sum_A (d\gamma_A)_3,
\]

(25)

\[
d\beta_4 = \sum_A (y_A (d\gamma_A)_1 - x_A (d\gamma_A)_2),
\]

(26)

\[
d\beta_5 = \sum_A (y_A (d\gamma_A)_2 - x_A (d\gamma_A)_3),
\]

(27)

\[
d\beta_6 = \sum_A (y_A^2 (d\gamma_A)_1 - 2x_A y_A (d\gamma_A)_2 + x_A^2 (d\gamma_A)_3).
\]

(28)

The sequence (13) is constructed. We will also use for the linear spaces entering in it somewhat looser but convenient notations in the style of papers [2] and [3], and write it the following way:

\[
0 \rightarrow (\mathfrak{sl}(2) \text{ and translations}) \xrightarrow{f_1} \begin{pmatrix}
\frac{dx_A}{d\gamma_A} \\
\frac{dy_A}{d\gamma_A} \\
\frac{d\lambda_{AB}}{d\gamma_A} \\
\frac{d\omega_{AB}}{d\gamma_A} \\
\frac{d\gamma_A}{d\gamma_A} \\
\frac{d\beta}{d\gamma_A}
\end{pmatrix} \xrightarrow{f_2} (d\lambda_{AB}) \xrightarrow{f_3} (d\omega_{AB}) \xrightarrow{f_4} (d\gamma_A) \xrightarrow{f_5} (d\beta) \rightarrow 0.
\]

(29)
Here the word “translations” in the leftmost nonzero space means “global translations” \( dx, \ dy \) and \( d\kappa \) acting according to formulas (15) and (16); the next space consists of differentials \( dx_A, dy_A, d\kappa_A \) for every vertex \( A \) and so on.

Now we prove that the sequence (29) is indeed a complex, that is the composition of any two successive mappings is zero. We start with the composition \( f_2 \circ f_1 \). Substitute in (18) expressions (15) and (16) for \( dx_A, dy_A \) and \( d\kappa_A \), as well as similar expressions for \( dx_B, dy_B \) and \( d\kappa_B \). The result of a direct calculation is then \( d\lambda_{AB} = 0 \). We see that the terms added to \( d\kappa \) in the right-hand side of (16) are chosen so as to compensate the change of the area of triangle \( OAB \) arising when its vertices \( A \) and \( B \) are shifted by the vector \( \left( \begin{array}{c} dx \\ dy \end{array} \right) \) while the origin of coordinates \( O \) remains in its place.

The fact that \( f_3 \circ f_2 = 0 \) is evident from geometric considerations: if the areas of all triangles are determined by their coordinates (in other words, all the triangles can be placed in \( \mathbb{R}^2 \)), then all the curvatures are zero.

The fact that \( f_4 \circ f_3 = 0 \) is also evident from geometric considerations. Imagine a vertex \( E \) and edges going out of it. A transformation of the basis in the plane \( \mathbb{R}^2 \) can be put in correspondence to any closed path that does not intersect edges. Indeed, as formulas (9) and (10) show, the system of coordinates in \( \mathbb{R}^2 \) is uniquely extended from a given tetrahedron to an adjacent one (having a common face with the first one), but if the curvatures are nonzero we can get a new system of coordinates on returning to the initial tetrahedron (which is demonstrated by formula (12)). If, however, the path can be contracted into a point in such way that it does not intersect edges at any moment, we must, of course, arrive at the system of coordinates identical to the initial one. For the case of infinitesimal curvatures \( d\omega \), this means that the sum of algebra \( \mathfrak{sl}(2) \) elements of the form (20) over all vertices \( D \) joined by an edge with \( E \) will be zero for any \( d\lambda \). Thus, the right-hand sides of expressions (22), (23) and (24) will be zero, as desired.

It remains to check that \( f_5 \circ f_4 = 0 \). We substitute the expressions (22)–(24) in formulas (25)–(28), take into account the relations of type (19) and get zero. This is quite evident for the expressions (25): the curvature at each edge gives, in view of \( d\omega_{AB} = -d\omega_{BA} \), the mutually opposite contributions at its two ends; on summing according to (25), one gets zero. The explanation is a bit harder for (26), (27) and (28); we have to say simply that these expressions were specially invented in such way as to get zeros.

4 The behavior of the torsion under moves 2 \( \rightarrow \) 3 and 1 \( \rightarrow \) 4 and the formula for the invariant

There are reasons to believe (see the next section of the present paper) that at least in many interesting cases the complex (29) is acyclic, i.e. the images of mappings coincide exactly with the kernels of the next mappings. If this is true then one can define the torsion \( \tau \) of the complex (29) as the product of some minors in matrices of mappings.
$f_1, \ldots, f_5$ taken in the alternating powers $+1$ and $-1$. Recall that the bases in all spaces are fixed. Some of the basis vectors in each space correspond to the rows of the minor belonging to the left (from this space) mapping, while the rest of them — to the columns of the minor belonging to the right mapping.

We choose the exponents $+1$ for the mappings with odd numbers and $-1$ for those with even numbers:

$$\tau = \frac{\text{minor } f_1 \cdot \text{minor } f_3 \cdot \text{minor } f_5}{\text{minor } f_2 \cdot \text{minor } f_4}. \quad (30)$$

Now we consider the local rebuilding of type $2 \to 3$ of the simplicial complex, pictured in Figure 1. The new edge $DE$ adds one new basis vector in both spaces $(d\lambda_{AB})$ and $(d\omega_{AB})$. We include the corresponding column and row in minor $f_3$ (thus, the remaining minors in (30) rest intact). Considerations using the triangular form of the matrices and similar to those given in section 3 of paper [4], show that this makes minor $f_3$ to be multiplied by $\partial \omega_{ED}/\partial \lambda_{ED}$. Taking into account formula (7), this shows that the value

$$\tau \cdot \prod_{\text{over all}} S \quad (31)$$

does not change under the move $2 \to 3$.

Note that we define both the torsion and our future invariant *to within a sign*. This allows us not to care about the orientation of the two-dimensional faces in the complex.

Consider now the rebuilding of type $1 \to 4$: we add the new vertex $E$ inside the tetrahedron $ABCD$. In the same way as for the move $2 \to 3$, we add to the minor of mapping $f_3$ a row and a column corresponding to edge $ED$. We then add to the minor of mapping $f_2$ rows corresponding to the three remaining edges, $EA$, $EB$ and $EC$, and columns corresponding to $dx_E$, $dy_E$ and $d\kappa_E$. Similarly, we add to the minor of mapping $f_4$ columns corresponding to $EA$, $EB$ and $EC$ and three rows corresponding to the three components of $d\gamma_E$. Again, considerations using the triangular form of matrices show that minor $f_2$ and minor $f_4$ are simply multiplied by $3 \times 3$ minors corresponding to the new rows and columns. The first of these $3 \times 3$ minors is calculated using formula (18) and turns out to equal $1/2 S_{ABC}$, while the second one is calculated using formulas (22), (23) and (24) and turns out to equal $S_{EAB}S_{EBC}S_{ECA}$ (both — to within their signs), which makes the whole torsion (where we must take into account also the minor of $f_3$ which behaves the same way as under a move $2 \to 3$) to be multiplied by $2 \prod_{\text{over new}} S^{-1}$.

This all shows that the following value is invariant under all Pachner moves and can be thus attributed to the manifold $M$ itself:

$$I_{SL(2)}(M) = \tau \cdot \prod_{\text{over all}} S \cdot 2^{-\# \text{vertices} - 1}. \quad (32)$$
Minus one is added to the exponent of number 2 in order that the invariant be equal to 1 for the sphere $S^3$, see below Subsection 5.1.

5 Examples

5.1 Sphere $S^3$

We take the triangulation of the sphere consisting of two tetrahedra, each of which has vertices $A$, $B$, $C$ and $D$. We must ascribe to these vertices some coordinates in the plane $\mathbb{R}^2$ (on which, of course, our determinant will not depend); for instance, we can take: $A(0, 0)$, $B(x_B, 0)$, $C(0, y_C)$ and $D(x_D, y_D)$.

In the same way as in the Euclidean case (see Section 6 of paper [4]), the mapping $f_3$: $(d\lambda) \to (d\omega)$ is the identical zero, because the curvature $\omega$ around any of the six edges is made up of two mutually opposite summands. Thus, minor $f_3 = 1$ (as the determinant of a $0 \times 0$ matrix). To calculate the minors of $f_1$ and $f_2$, we must choose six basis differentials in the space of values $\begin{pmatrix} dx_A \\ dy_A \\ d\kappa_A \end{pmatrix}$ for the rows of minor $f_1$ (the calculations go very easily if we choose $dx_A$, $dy_A$, $d\kappa_A$, $dx_B$, $dy_B$ and $dx_C$), and use the remaining six basis differentials for columns of minor $f_2$. The result is:

$$\frac{\text{minor } f_1}{\text{minor } f_2} = \pm 8. \quad (33)$$

The calculations in the right side of the sequence are similar although slightly harder. Their result is:

$$\frac{\text{minor } f_5}{\text{minor } f_4} = \pm \frac{4}{S_{ABC}S_{ABD}S_{ACD}S_{BCD}}. \quad (34)$$

Substituting the right-hand sides of relations (33) and (34) in formula (30) and then in formula (32), we get the announced result:

$$I_{SL(2)}(S^3) = 1$$

(here we omit the $\pm$ sign).

5.2 Projective space $\mathbb{R}P^3$

We take the same triangulation as in section 6 of paper [4], where we considered the Euclidean case. Namely, there are again 4 vertices $A$, $B$, $C$ and $D$, but now 12 edges, 16 two-dimensional faces and 8 tetrahedra, see Figure 2. Here we have, for example, two edges $b$ and $b'$ instead of the single edge $AB$ in Subsection 5.1 of the present work, and so on. Note that any two edges bearing identical notations in Figure 2, for instance, the two copies of edge $f$, are identified, but $f$ and $f'$ are different edges.
Figure 2: Triangulation of $\mathbb{R}P^3$

Despite all these differences, formulas (33) and (34) remain valid for $\mathbb{R}P^3$ as well. We must only specify that we employ the six “primed” differentials $d\lambda$ for the rows of minor $f_2$, that is, $d\lambda_b', \ldots, d\lambda_f'$ in place of $d\lambda_{AB}, \ldots, d\lambda_{CD}$ from Subsection 5.1. Similarly, we use the “primed” differentials $d\omega$ for constructing the columns of minor $f_4$.

“Unprimed” $d\lambda$ and $d\omega$ remain for constructing the minor of mapping $f_3$ consisting of the partial derivatives of the six values $\omega_b, \omega_c, \omega_d, \omega_h, \omega_g$ and $\omega_f$ with respect to the variables $\lambda_b, \lambda_c, \lambda_d, \lambda_h, \lambda_g, \lambda_f$. This $6 \times 6$ minor is equal to the product of its six elements, namely,

$$
\frac{\partial \omega_b}{\partial \lambda_f}, \frac{\partial \omega_f}{\partial \lambda_b}, \frac{\partial \omega_c}{\partial \lambda_g}, \frac{\partial \omega_g}{\partial \lambda_c}, \frac{\partial \omega_d}{\partial \lambda_h} \text{ and } \frac{\partial \omega_h}{\partial \lambda_d}, \tag{35}
$$

while the rest of its elements are zero. The reason for this is that these remaining elements are made up of pairs of mutually opposite summands corresponding to tetrahedra having opposite orientations (we are, essentially, repeating the argumentation from section 6 of paper [4]). As for the derivatives (35), each of them is a sum of two identical terms, corresponding to two tetrahedra with the same orientation. A derivative of such sort for one tetrahedron is given in paper [5], formula (10). Changing the notations of that formula to those of the present paper, multiplying by 2 and ignoring the possible minus sign, we get, for example, the formula

$$
\frac{\partial \omega_b}{\partial \lambda_f} = 2 \frac{S_{ABC}}{S_{ABC}S_{ABD}}
$$
(it is also not very difficult to deduce this from formulas of type (3), (4), (5)).

For the whole minor, we get

$$\text{minor } f_3 = \frac{64}{S_{ABC}S_{ABD}S_{ACD}S_{BCD}}.$$

Hence,

$$I_{SL(2)}(\mathbb{R}P^3) = 64. \quad (36)$$

6 Discussion

Our result (36) confirms the hypothesis stated in the end of paper [5]: in the case of trivial representations of the fundamental group, the Euclidean and $SL(2)$ invariants yield the same result (the Euclidean invariant for $\mathbb{R}P^3$, as calculated in [4], was equal to $1/8$, but this was simply because we defined it there in a slightly different way, analogous to our present definition in the power $(-1/2)$).

In the terminology of paper [5], we have globalized the “local” pentagon equation in the present paper, i.e., we have shown how it can be used for studying not only a local rebuilding $2 \to 3$ but the whole manifold. This globalization turned out to be harder than we expected when writing the paper [5]. Recall that, in the Euclidean case, the algebraic complex analogous to (29) was symmetric with respect to its middle in the following sense: the matrices of mappings equidistant from the ends of the complex could be obtained from each other by means of transposing (see the sections of papers [2] and [3] devoted to the three-dimensional case). For the complex (29), not only this property is no longer valid, but even its analogue of any kind could not be found as yet. In general, the feeling is that the geometric and algebraic sense of sequence (29) has not yet been discovered in full.

It seems that the exactness of sequence (29) in some of its terms can be shown for arbitrary manifolds using ideas parallel to those of section 2 in paper [2]. There is also another idea: to consider the simplicial complex corresponding to the universal cover of the manifold, construct for it the algebraic complex of type (29), and then consider the subcomplexes of the algebraic complex. They must correspond to different representations of the fundamental group (which we have discussed in the Introduction). It looks plausible that this may help both to prove the acyclicity and construct new manifold invariants. In the Euclidean case, such activity was initiated in paper [7].

Finally, there is a very intriguing question about possible existence of some quantum relations from which our solution to pentagon equation can be obtained as a semiclassical limit. This question is motivated by the fact that, in the Euclidean case, our solution to pentagon equation [4] can be obtained by a limiting procedure from the quantum $6j$-symbols. Note, however, that even in the Euclidean case the similar question remains open for manifolds of dimensionality more than three [1, 2, 3].

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