A localization of bicategories via homotopies

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Abstract

We give conditions on a pair \((C, \Sigma)\), where \(\Sigma\) is a family of arrows of a bicategory \(C\), such that the bicategorical localization with respect to \(\Sigma\) can be constructed by dealing only with the 2-cells, that is without adding neither objects nor arrows to \(C\). We show that in this case, the 2-cells of the localization can be given by \(\sigma\)-homotopies, a notion defined in this article which is closely related to Quillen’s notion of homotopy for model categories but depends only on a single family of arrows. Considering the pair \((\mathcal{C}_{fc}, W)\) given by the weak equivalences between fibrant-cofibrant objects, the localization result of this article has a natural application to the construction of the homotopy bicategory of a model bicategory, which we develop elsewhere.

1 Introduction

The subject of this article is the localization of a bicategory \(C\), that is the process of making a family \(\Sigma\) of arrows of \(C\) into equivalences in an appropriate universal sense. As far as we know, this situation was first considered in [7], where a bicategorical version of the calculus of fractions of [4] is given and a construction of the localization is performed in this case. A pseudofunctor \(C \xrightarrow{i} \mathcal{E}\) is the localization of \(C\) with respect to \(\Sigma\) if it is universal in the following sense: For any bicategory \(D\), precomposition with \(i \circ Hom(\mathcal{E}, D) \xrightarrow{i} Hom_{\Sigma, \Theta}(C, D)\) is a pseudoequivalence of bicategories, where \(Hom_{\Sigma, \Theta}(C, D)\) stands for the full subbicategory of \(Hom(C, D)\) consisting of those pseudofunctors that map the arrows of \(\Sigma\) to equivalences. As a motivation, let us consider also the example of the homotopy category of a model category [8]. The homotopy category of a given model category is its localization with respect to the weak equivalences, and a construction of it is given in [8] in which the arrows are given by the homotopy classes of arrows of \(C\).

As it is well-known, the localization of a category always exists and can be constructed by adding formal inverses, that is by identifying classes of zigzags; however this construction is unmanageable in practice. This is a motivation for the constructions in [4], where zigzags of length 2 suffice, and in [8], where the candidates for the inverses are already present in the model category and the localization can be constructed as a quotient (see also [4, §3.1] for a detailed explanation of this situation in an abstract context).

This paper deals with the situation analogue to that of [8], [9], that is the construction of the localization as a quotient, but in dimension 2. For an arbitrary bicategory \(C\) and a family \(\Sigma\) of arrows, we consider a notion of \(\sigma\)-homotopy between arrows of \(C\), that is a bicategorical notion of homotopy which depends only on the family \(\Sigma\). The \(\sigma\)-homotopies
can be thought of something that would be an actual 2-cell if the arrows of Σ were equivalences, and when this is the case we can associate to each σ-homotopy \( H \) a 2-cell \( \hat{H} \). We can apply pseudofunctors to σ-homotopies, and thus for any pseudofunctor \( C \xrightarrow{F} \mathcal{D} \) which maps the weak equivalences to equivalences we can construct in this way a 2-cell \( \hat{F}H \) of \( \mathcal{D} \). The σ-homotopies are the basic ingredient for the following construction which we do in this paper.

**The bicategory \( \mathcal{H}o(\mathcal{C}, \Sigma) \) and the 2-functor \( \mathcal{C} \xrightarrow{i} \mathcal{H}o(\mathcal{C}, \Sigma) \).** The objects and the arrows of \( \mathcal{H}o(\mathcal{C}, \Sigma) \) are those of \( \mathcal{C} \). A 2-cell \( f \Rightarrow g \in \mathcal{H}o(\mathcal{C}, \Sigma) \) is given by the class \([H^n, \ldots, H^2, H^1]\) of a finite sequence \( f \xrightarrow{H^n} f_1 \xrightarrow{H^2} \cdots f_{n-1} \xrightarrow{H^1} g \) of σ-homotopies, where \([H^n, \ldots, H^2, H^1] = [K^m, \ldots, K^2, K^1]\) if and only if for every pseudofunctor \( F \) as above, \( \hat{F}H^n \circ \cdots \hat{F}H^2 \circ \hat{F}H^1 = \hat{F}K^m \circ \cdots \hat{F}K^2 \circ \hat{F}K^1 \). There is a projection 2-functor \( \mathcal{C} \xrightarrow{i} \mathcal{H}o(\mathcal{C}, \Sigma) \), which is the identity on objects and arrows and maps a 2-cell \( \mu \) of \( \mathcal{C} \) to the class of a σ-homotopy \( I\mu \) which satisfies that \( \hat{F}I\mu = F\mu \) for any \( F \) as above.

We prove the following fundamental fact regarding the 2-functor \( i \) (Theorem 3.46): for an arbitrary pair \((\mathcal{C}, \Sigma)\), \( \text{Hom}(\mathcal{H}o(\mathcal{C}, \Sigma), \mathcal{D}) \xrightarrow{i^*} \text{Hom}_{\Sigma, \Theta}(\mathcal{C}, \mathcal{D}) \) satisfies the properties which would make it an isomorphism of bicategories if it were well defined, i.e. as soon as we could show that \( i \) maps the arrows of \( \Sigma \) to equivalences.

Going back to the example of the homotopy category of a model category, a reason why in this case the candidates for inverses are present is that, as is well known, any weak equivalence between fibrant-cofibrant objects can be factored as a section followed by a retraction. We say that an arrow is split if it is either a retraction or a section. Recall that \( \Sigma \) is said to satisfy the “3 for 2” condition if, for any three arrows that satisfy \( fg = h \), whenever two of them are in \( \Sigma \), so is the third. We have adequately weakened these notions for them to be considered in bicategories (\( w \) stands for “weak”), and we have showed (Proposition 3.54): if \( \Sigma \) satisfies 3 for 2, then any \( w \)-split arrow in \( \Sigma \) is mapped to an equivalence by \( i \).

Combining the two results above, it follows the main theorem of this article (Theorem 3.56): If \( \Sigma \) satisfies 3 for 2 and each arrow of \( \Sigma \) can be written as a composition of \( w \)-split arrows of \( \Sigma \), then \( \mathcal{C} \xrightarrow{i} \mathcal{H}o(\mathcal{C}, \Sigma) \) is the localization of \( \mathcal{C} \) with respect to \( \Sigma \).

A bicategory with weak equivalences can be defined, with the approach of [3], as a pair \((\mathcal{C}, W)\), where \( W \) is a family of arrows of a bicategory \( \mathcal{C} \) that satisfies 3 for 2. The axioms of model category can be modified in a natural way in order to define the notion of model bicategory, [1], see also [2]. The bicategory with weak equivalences \((\mathcal{C}_{fc}, W)\) given by the weak equivalences between fibrant-cofibrant objects of a model bicategory \( \mathcal{C} \) satisfies the hypothesis of Theorem 3.56 and this allows for an application of this result to the construction of the homotopy bicategory of a model bicategory, which we develop in [1]. We note here that since localizations of bicategories are by definition characterized only up to equivalence, we can already take \( \mathcal{H}o(\mathcal{C}_{fc}, W) \) as the localization of \( \mathcal{C} \) at \( W \), as opposed to the 1-dimensional case in which all the objects of the model category form part of the homotopy category.
2 Preliminaries on bicategories

While the theory of bicategories is nowadays well-established, it is still convenient to explicitly define its basic concepts in order to fix the notation that we will use throughout the paper.

A bicategory \( C \) consists of all the following:

1. A family of objects that we will denote by \( X, Y, Z, \ldots \).
2. For each pair of objects \( X, Y \in C \) a category \( C(X,Y) \) whose objects are the arrows \( X \to Y \) of \( C \) and whose arrows are the 2-cells \( \alpha : f \Rightarrow g \) between those arrows. Thus we have a vertical composition of 2-cells which we denote by “\( \circ \)”, and identity 2-cells “\( \text{id}_f \)”.
   We abuse the notation by denoting indistinctly \( f \circ \text{id}_f \) or \( \text{id}_f \circ f \), thus \( f = \text{id}_f \) as 2-cells.

3. For each \( X, Y, Z \in C \), a functor \( C(Y,Z) \times C(X,Y) \to C(X,Z) \). This is a horizontal composition which we denote by “\( * \)”, for each configuration

\[
\begin{array}{c}
\begin{array}{c}
X \xrightarrow{f_1} Y \\
\alpha \downarrow \\
X \xrightarrow{f_2} Y \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
Y \xrightarrow{g_1} Z \\
\beta \downarrow \\
Y \xrightarrow{g_2} Z \\
\end{array}
\end{array}
\]

we have

\[
g_1 * f_1 \Rightarrow g_2 * f_2.
\]

All these data has to satisfy the following axioms:

**H1.** For each \( X \xrightarrow{f} Y \xrightarrow{g} Z \in C \), \( \text{id}_g * \text{id}_f = \text{id}_{g \circ f} \).

**H2.** For each configuration \( X \xrightarrow{f} Y \xrightarrow{g} Z \), \( (\delta * \beta) \circ (\gamma * \alpha) = (\delta \circ \gamma) * (\beta \circ \alpha) \). This is the “Interchange law”.

In order to avoid parenthesis, we consider “\( * \)” more binding than “\( \circ \)”, thus \( (\delta * \beta) \circ (\gamma * \alpha) \) above could be written as \( \delta * \beta \circ \gamma * \alpha \).

4. Finally, part of the structure of \( C \) is given by the identities, the unitors and the associator as follows:

**I.** For each \( X \in C \), we have a 1-cell \( X \xrightarrow{\text{id}_X} X \).

**U.** For each \( X \xrightarrow{f} Y \in C \), we have invertible 2-cells \( f * \text{id}_X \Rightarrow f, \text{id}_Y * f \Rightarrow f \).

**A.** For each \( W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \in C \), we have an invertible 2-cell \( f * (g * h) \Rightarrow (f * g) * h \).

We will use these same letters \( \theta, \rho, \lambda \) for any bicategory, and we will denote the inverses of these 2-cells also by the same letters. The unitors and the associators are required to satisfy the well-known pentagon and triangle identities ([6, XII,6]) and are required to be natural in each of the variables. These naturalities are expressed by the following equalities of 2-cells which we record here for convenience:

**N\( \lambda \).** For each \( X \xrightarrow{f} Y \in C \), \( \lambda \circ \alpha * \text{id}_X = \alpha \circ \lambda \).
Nρ. For each $X \xrightarrow{f} Y \in \mathcal{C}$, $\rho \circ id_Y \circ \alpha = \alpha \circ \rho$.

Nθ. For each configuration $W \xrightarrow{f_1} X \xrightarrow{g_1} \frac{\beta_1}{\alpha_1} Y \xrightarrow{h_1} \frac{\gamma_1}{\delta_1} Z \in \mathcal{C}$, $\theta \circ \gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha \circ \theta$.

2.1. As it is well-known, in order to have a horizontal composition of general 2-cells it is enough to have horizontal compositions between an arrow and a 2-cell:

Assume that for each $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{C}$, we have 2-cells

$X \xrightarrow{\alpha * f} Z$ and $X \xrightarrow{g * \alpha} Z$, subject to the axioms:

W1. For each $X \xrightarrow{f_1} Y \xrightarrow{g_1} \frac{\beta_1}{\alpha_1} Z \in \mathcal{C}$, $(g_2 \circ \alpha) \circ (\beta \circ f_1) = (\beta \circ f_2) \circ (g_1 \circ \alpha)$.

W2. For each $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{C}$, $id_g \circ f = g \circ id_f = id_{g \circ f}$.

W3. For each $X \xrightarrow{f} Y \xrightarrow{g} \frac{\alpha_1}{\gamma_1} Z \in \mathcal{C}$, $(\beta \circ f) \circ (\alpha \circ f) = (\beta \circ \alpha) \circ f$.

For each $X \xrightarrow{f_1} Y \xrightarrow{g_1} \frac{\beta_1}{\alpha_1} X \xrightarrow{h_1} \frac{\gamma_1}{\delta_1} Z \in \mathcal{C}$, $(\beta \circ g) \circ (g \circ \alpha) = g \circ (\beta \circ \alpha)$.

Then these axioms allow to define, for each configuration as in W1, the horizontal composition $\beta \circ \alpha$ by either one of the two compositions there. The correspondence between the sets of axioms “H” and “W” is thus clear.

We will use this fact in order to define the horizontal composition of a bicategory by defining only the horizontal compositions of 2-cells with arrows. We note also that the axioms Nλ and Nρ above involve only these sorts of compositions, and as for axiom Nθ, it is an easy exercise to show that it is equivalent to the following three axioms, corresponding to putting two identity 2-cells out of the three $\alpha$, $\beta$ and $\gamma$ in Nθ:

Nθ1. For $W \xrightarrow{f_1} X \xrightarrow{g_1} Y \xrightarrow{h_1} \frac{\beta_1}{\alpha_1} Z \in \mathcal{C}$, $\theta \circ h \circ (g \circ \alpha) = (h \circ g) \circ \alpha \circ \theta$.

Nθ2. For $W \xrightarrow{f_1} X \xrightarrow{g_1} Y \xrightarrow{h_1} \frac{\beta_1}{\alpha_1} Z \in \mathcal{C}$, $\theta \circ h \circ (\beta \circ f) = (h \circ \beta) \circ f \circ \theta$.

Nθ3. For $W \xrightarrow{f_1} X \xrightarrow{g_1} Y \xrightarrow{h_1} \frac{\beta_1}{\alpha_1} Z \in \mathcal{C}$, $\theta \circ \gamma \circ (g \circ f) = (\gamma \circ g) \circ f \circ \theta$.

Coherence. There is a well-known coherence theorem (see for example [5]) which generalizes the coherence theorem for tensor categories. Given any sequence of composable
arrows, the parenthesis determine the order in which the compositions are performed. The coherence theorem states that the arrows resulting of any choice of parenthesis (and adding or subtracting identities) are canonically isomorphic by an unique 2-cell built with the associators and the unitors. This justifies the following abuse of notation which greatly simplifies the computations:

2.2. We write any horizontal composition of arrows omitting the parenthesis and the identities. In this way, the associator and the unitors disappear in the diagrams of 2-cells.

Elevators calculus. In addition to the usual pasting diagrams, we will use the Elevators calculus\(^1\) to write equations between 2-cells. In this article, each elevator represents a composition of 2-cells in a bicategory. Objects are omitted, arrows are composed from right to left, and 2-cells from top to bottom. Axiom H2 shows that the correspondence between elevators and 2-cells is a bijection. Axiom W1 is the following basic equality for the elevator calculus.

\[
\begin{align*}
\begin{array}{c}
g_1 \bigg| f_1 \\
g_2 \bigg| f_1
\end{array}
&= \begin{array}{c}
g_1 \bigg| f_2 \\
g_2 \bigg| f_2
\end{array}
\end{align*}
\]

(2.3)

This allows to move cells up and down when there are no obstacles, as if they were elevators.

Using the basic move (2.3) we form configurations of cells that fit valid equations in order to prove new equations.

Definition 2.4. A pseudofunctor \( \mathcal{C} \xrightarrow{F} \mathcal{D} \) between bicategories is given by a family of functors \( \mathcal{C}(X,Y) \xrightarrow{F} \mathcal{D}(FX,FY) \), one for each pair of objects \( X,Y \) of \( \mathcal{C} \), invertible 2-cells \( \text{id}_{FX} \xrightarrow{\xi_X} F(\text{id}_X) \), one for each object \( X \) of \( \mathcal{C} \) and natural isomorphisms \( \ast \circ (F \times F) \xrightarrow{\phi} F \circ \ast : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \rightarrow \mathcal{C}(X,Z) \) with components \( Fg \ast Ff \xrightarrow{\phi_{g,f}} F(g \ast f) \), one for each triplet \( X,Y,Z \) of objects of \( \mathcal{C} \). As with the associators and unitors, we will omit the subindexes of \( \xi \) and \( \phi \), and use the same letters for the inverses. The following equalities are required to hold:

\[
\begin{align*}
\begin{array}{c}
Ff
\end{array}
&= \begin{array}{c}
Ff
\end{array}
\end{align*}
\]

For each \( X \xrightarrow{f} Y \in \mathcal{C} \), P1. \( Ff \xrightarrow{\xi} Ff \) P2. \( Ff \xrightarrow{\phi} Ff \)

\( ^1\)Developed in 1969 by the second author for draft use.
For each \( W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z \in C \), \( \text{P3} \).

\[
\begin{array}{c}
Fh & Fg & Ff \\
\downarrow \phi & & \downarrow \phi \\
F(hgf) & & F(hgf)
\end{array}
\]

\[=\]

\[
\begin{array}{c}
Fh & Fg & Ff \\
\downarrow \phi & & \downarrow \phi \\
F(hgf) & & F(hgf)
\end{array}
\]

We will often use the naturality of \( \phi \), thus we make it explicit:

For each \( X \xrightarrow{f_1} \xrightarrow{g_1} Z \in C \), \( \text{N}\phi \).

\[
\begin{array}{c}
Fg_1 & Ff_1 & Fg_1 \\
\downarrow \phi & & \downarrow \phi \\
F(g_1 f_1) & & F(g_2 f_2)
\end{array}
\]

\[=\]

\[
\begin{array}{c}
Fg_1 & Ff_1 & Fg_1 \\
\downarrow \phi & & \downarrow \phi \\
F(g_1 f_1) & & F(g_2 f_2)
\end{array}
\]

A 2-functor is a pseudofunctor such that all the 2-cells \( \phi \) and \( \xi \) are identities.

**Definition 2.5.** A pseudonatural transformation \( \theta : F \Rightarrow G : C \rightarrow D \) between pseudofunctors consists of a family of arrows \( \theta_X : FX \Rightarrow GX \), one for each \( X \in C \) and a family of invertible 2-cells \( \theta f \), one for each \( X \xrightarrow{f} Y \in C \), satisfying the following axioms:

**PN0.** For each \( X \in C \),

\[
\begin{array}{c}
\theta_X \\
\downarrow \xi \\
\theta_X
\end{array}
\]

\[=\]

\[
\begin{array}{c}
\theta_X \\
\downarrow \xi \\
\theta_X
\end{array}
\]

**PN1.** For each \( X \xrightarrow{f} Y \xrightarrow{g} Z \in C \),

\[
\begin{array}{c}
Gg & Gf & \theta_X \\
\downarrow \theta_f & & \downarrow \phi \\
Gg & Gf & \theta_X
\end{array}
\]

\[=\]

\[
\begin{array}{c}
Gg & Gf & \theta_X \\
\downarrow \theta_f & & \downarrow \phi \\
Gg & Gf & \theta_X
\end{array}
\]
PN2. For each $X \xrightarrow{f} Y \in \mathcal{C}$, \[
\begin{array}{c|c|c}
Gf & \theta_X & Gf \\
\downarrow \theta_f & \downarrow & \downarrow \theta_f \\
Gg & \theta_X & Gf \\
\downarrow \theta_g & \downarrow & \downarrow \theta_g \\
\theta_Y & Ff & \theta_Y \\
\end{array}
\]
As a particular case, we have the notion of pseudonatural transformation between 2-functors. A 2-natural transformation between 2-functors is a pseudonatural transformation such that $\theta_f$ is the equality for every arrow $f$ of $\mathcal{C}$.

**Definition 2.6.** A modification $\rho : \theta \Rightarrow \eta : F \Rightarrow G : \mathcal{C} \rightarrow \mathcal{D}$ between pseudonatural transformations is a family of 2-cells $\theta_X \xRightarrow{\rho_X} \eta_X$ of $\mathcal{D}$, one for each $X \in \mathcal{C}$ such that:

PM. For each $X \xrightarrow{f} Y \in \mathcal{C}$,
\[
\begin{array}{c|c|c}
Gf & \theta_X & Gf \\
\downarrow \theta_f & \downarrow & \downarrow \theta_f \\
Gf & \theta_X & Gf \\
\downarrow \rho_f & \downarrow & \downarrow \eta_f \\
\theta_Y & Ff & \theta_Y \\
\end{array}
\]

Pseudofunctors, pseudonatural transformations and modifications can be composed in order to define, for each pair $\mathcal{C}, \mathcal{D}$ of bicategories, a bicategory $\text{Hom}(\mathcal{C}, \mathcal{D})$. We omit the details as they are ubiquitous in the literature.

**Definition 2.7.** Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a pseudofunctor. A configuration $X \xrightarrow{f_1 \ f_2} Y \xrightarrow{g_1 \ g_2} Z \in \mathcal{C}$ and another one $F \xrightarrow{\alpha \ \beta} F \xrightarrow{\gamma \ \delta} F \xrightarrow{\eta \ \theta} F \ x \in \mathcal{D}$ determine a 2-cell $F(g_1 * f_1) \xRightarrow{\beta * \alpha} F(g_2 * f_2)$ as the composition $F(g_1 * f_1) \xRightarrow{\gamma} F(g_1) * F(f_1) \xRightarrow{\beta * \alpha} F(g_2) * F(f_2) \xRightarrow{\delta} F(g_2 * f_2)$. Note that if $F$ is a 2-functor, $\beta * \alpha = \beta * \alpha$.

**Remark 2.8.** Given a configuration $X \xrightarrow{f_1 \ f_2} Y \xrightarrow{g_1 \ g_2} Z \in \mathcal{C}$, from Definition 2.7 and axiom N\phi., it follows that $(F \beta) * F(\alpha) = F(\beta * \alpha)$.

**2.9. Factorization of $F$.** Let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a pseudofunctor. We give now a factorization of $F$ which will be very useful later. We define a bicategory $\mathcal{C}_F$, a pseudofunctor $\mathcal{C}_F \xrightarrow{F_1} \mathcal{D}$ and a 2-functor $\mathcal{C} \xrightarrow{F_2} \mathcal{C}_F$ such that $F = F_1 F_2$.

We define the 0 and 1-dimensional aspects of $\mathcal{C}_F$ (that is objects, arrows, identity arrows and horizontal composition of arrows) as the ones of $\mathcal{C}$. We define a 2-cell $f \xRightarrow{\alpha} g$ of $\mathcal{C}_F$ as a 2-cell $Ff \xRightarrow{\alpha} Fg$ of $\mathcal{D}$. Vertical composition of 2-cells is computed in $\mathcal{D}$, and $id_f$
in \( \mathcal{C}_F \) is given by the 2-cell \( id_{Ff} \) of \( \mathcal{D} \). The composition \( \beta * \alpha \) in \( \mathcal{C}_F \) is given by \( \beta * F \alpha \) in Definition 2.7. The axioms \( \mathbb{H} \) follow immediately by the definition of \( *_F \) and the corresponding axioms of \( \mathcal{C} \). The unitors and associators of \( \mathcal{C}_F \) are obtained applying \( F \) to the ones of \( \mathcal{C} \), i.e. they are the 2-cells \( F \lambda, F \rho, F \theta \). Their naturalities and the pentagon and triangle identities all follow in a straightforward way from those of \( \mathcal{C} \), composing when needed with the isomorphism \( \phi \). We leave the necessary details to the reader.

2.10. Equivalences and quasiequivalences. An arrow \( X \overset{f}{\to} Y \) of a bicategory is an equivalence if there exists an arrow \( Y \overset{g}{\to} X \) (which we call a quasiinverse of \( f \)) and isomorphisms \( g * f \cong id_X, f * g \cong id_Y \). It is well-known that these isomorphisms can be taken satisfying the usual triangular identities, and we will assume that this is the case when needed. It is also well-known that \( X \overset{f}{\to} Y \) is an equivalence if and only if for every object \( Z \) we have that the functor \( \mathcal{C}(Z,X) \overset{f^*}{\to} \mathcal{C}(Z,Y) \) is an equivalence of categories, and if and only if for every \( Z \) so is \( \mathcal{C}(Y,Z) \overset{f^*}{\to} \mathcal{C}(X,Z) \). We denote the family of equivalences of a bicategory with the letter \( \Theta \). We say that \( f \) is a quasiequivalence if for every object \( Z \) the functors \( \mathcal{C}(Z,X) \overset{f^*}{\to} \mathcal{C}(Z,Y) \) and \( \mathcal{C}(Y,Z) \overset{f^*}{\to} \mathcal{C}(X,Z) \) are full and faithful. Note that quasiequivalences preserve and reflect invertible 2-cells. We denote the family of quasiequivalences of a bicategory with the letter \( q\Theta \).

Our reason for considering quasiequivalences in this paper is that, in the factorization of 2.9, if an arrow \( Ff \) is a quasiequivalence, so is \( F_2 f \); while this implication is false for equivalences. Since \( F_2 \) is always a 2-functor, this allows to consider 2-functors instead of arbitrary pseudofunctors in some parts of the paper, which simplifies the computations.

A pseudofunctor \( \mathcal{C} \overset{F}{\to} \mathcal{D} \) is a pseudoequivalence of bicategories if there exist a pseudo-functor \( \mathcal{D} \overset{G}{\to} \mathcal{C} \) (which we call a pseudoinverse of \( F \)) and pseudonatural transformations \( GF \Rightarrow id_{\mathcal{C}}, FG \Rightarrow id_{\mathcal{D}} \) which are equivalences.

3 The homotopy bicategory

We fix a bicategory \( \mathcal{C} \) and a family \( \Sigma \) of arrows of \( \mathcal{C} \) containing the identities. We will use the notation \( \bullet \overset{\cdot}{\longrightarrow} \bullet \) for the arrows of \( \Sigma \). In this section we develop a theory of homotopies and cylinders with respect to the class \( \Sigma \) (instead of working with three distinguished classes as it is the case for model categories). The main result of this section is that the \( \sigma \)-homotopies form the 2-cells of a bicategory which, under natural hypothesis on \( \Sigma \), is the localization of \( \mathcal{C} \) with respect to \( \Sigma \), in the sense that it universally turns these arrows into equivalences.

Definition 3.1. Let \( X \in \mathcal{C} \). A \( \sigma \)-cylinder \( C \) (for \( X \), with respect to \( \Sigma \)) is given by the
data $C = (W, Z, d_0, d_1, x, s, \alpha_0, \alpha_1)$, fitting in $\Delta$. We denote the invertible
2-cell $s * d_0 \Rightarrow x \Rightarrow s * d_1$ by $\tilde{\alpha} = \alpha_1^{-1} \circ \alpha_0$.

**Definition 3.2.** Given a $\sigma$-cylinder $C$ as above, we define the inverse $\sigma$-cylinder $C^{-1} = (W, Z, d_1, d_0, x, s, \alpha_1, \alpha_0)$.

Also, given any object $X$ of $C$ we can define an identity $\sigma$-cylinder $C_X = (X, X, \text{id}_X, \text{id}_X, \text{id}_X, \text{id}_X, \text{id}_X, \text{id}_X)$ (recall our abuse of notation 2.2).

**Definition 3.3.** Let $f, g : X \to Y \in C$. A left $\sigma$-homotopy (with respect to $\Sigma$) $H$ from $f$ to $g$, which we will denote by $f \xrightarrow{H} g$, is given by the data $H = (C, h, \eta, \varepsilon)$, where $C$ is a $\sigma$-cylinder for $X$ as in Definition 3.1, $h$ is an arrow $W \xrightarrow{h} Y$ and $\eta, \varepsilon$ are 2-cells $f \Rightarrow h * d_0, h * d_1 \Rightarrow g$. We organize the data of a $\sigma$-homotopy as follows

$$f \xrightarrow{H} g : X \xrightarrow{d_0} W \xrightarrow{h} Y \quad \eta \quad \varepsilon$$

We say that $H$ has invertible cells if $\eta$ and $\varepsilon$ are invertible (recall that $\alpha_0$ and $\alpha_1$ are always required to be invertible).

Throughout this section we will work only with left $\sigma$-homotopies, and thus omit to write the word “left”.

**Definition 3.5.** If $H$ as in Definition 3.3 has invertible cells, we define a $\sigma$-homotopy $H^{-1} = (C^{-1}, h, \varepsilon^{-1}, \eta^{-1})$ from $g$ to $f$.

**Definition 3.6.** Any $\sigma$-cylinder $C$ as in Definition 3.1 determines a $\sigma$-homotopy $d_0 \xrightarrow{H^C} d_1$, (recall our abuse of notation 2.2):

$$d_0 \xrightarrow{H^C} d_1 : X \xrightarrow{d_0} W \xrightarrow{id} W \xrightarrow{d_1} Z \xrightarrow{s} Z \xrightarrow{x} \alpha_0 \Rightarrow x \Rightarrow s * d_1$$

We now make various constructions for $\sigma$-homotopies. In these definitions we omit parenthesis according to the abuse of notation 2.2. Let $H$ be as in (3.4):

**3.7.** If $g \xrightarrow{H} g' \in C$, we define a $\sigma$-homotopy $\mu \circ H$ from $f$ to $g'$ as follows
Definition 3.12. A 2-cell \( X \xrightarrow{f} Y \in \mathcal{C} \) yields two \( \sigma \)-homotopies \( H_0^\mu, H_1^\mu : f \xRightarrow{\sigma} g \), \( H_0^\mu = (C_X, g, \mu, g) \) and \( H_1^\mu = (C_X, f, f, \mu) \):

\[
\begin{align*}
H_0^\mu: & \quad X \xrightarrow{id_X} X \xrightarrow{g} Y \\
& \quad \downarrow \quad \downarrow \\
& \quad X \xrightarrow{g} Y
\end{align*}
\]

\[\eta = \mu, \ \alpha_0 = \alpha_1 = id_X, \ \varepsilon = g\]

\[
\begin{align*}
H_1^\mu: & \quad X \xrightarrow{id_X} X \xrightarrow{f} Y \\
& \quad \downarrow \quad \downarrow \\
& \quad X \xrightarrow{f} Y
\end{align*}
\]

\[\eta = f, \ \alpha_0 = \alpha_1 = id_X, \ \varepsilon = \mu\]

The \( \sigma \)-homotopies can be thought of something that would be an actual 2-cell if the arrows of \( \Sigma \) were equivalences (more generally if they were quasiequivalences, recall [2.10]). When this is the case, \( \sigma \)-cylinders and \( \sigma \)-homotopies yield actual 2-cells of \( \mathcal{C} \) as follows:
Definition 3.13. Consider a σ-cylinder $C$ as in Definition 3.1, with $s$ a quasiequivalence.

1. We denote by $d_0 \xRightarrow{\circ} d_1$ the unique invertible 2-cell such that $s \circ \tilde{C} = \tilde{\alpha}$.
2. For a σ-homotopy $H$ with σ-cylinder $C$, we denote by $\tilde{H}$ the composite 2-cell $f \xRightarrow{\circ} h \circ d_0 \xRightarrow{h \circ C} h \circ d_1 \xRightarrow{\circ} g$.

Note that we have $\tilde{H} C = \tilde{C}$. Item 2 in this definition can be considered as the extension of this formula to an arbitrary $H$ using Remark 3.14.

Consider now another family $\Gamma$ of arrows of a bicategory $\mathcal{D}$. We denote by $(C, \Sigma) \xrightarrow{\mathcal{E}} (\mathcal{D}, \Gamma)$ a pseudofunctor $C \xrightarrow{\mathcal{E}} \mathcal{D}$ that maps the arrows of $\Sigma$ to $\Gamma$. We can apply the pseudofunctor $F$ to σ-cylinders and σ-homotopies of $C$ as follows:

Definition 3.14. Let $(C, \Sigma) \xrightarrow{\mathcal{E}} (\mathcal{D}, \Gamma)$.

1. For a σ-cylinder $C$ as in Definition 3.1, we define the σ-cylinder $FC$ by

$$FC = (FW, FZ, Fd, Fc, Fx, Fs, F\alpha_0 \circ \phi, F\alpha_1 \circ \phi).$$

2. For a σ-homotopy $H$ as in Definition 3.1, we define the σ-homotopy $Ff \xRightarrow{FH} Fg$ by

$$FH = (FC, Fh, \phi \circ F\eta, F\varepsilon \circ \phi).$$

The constructions of $FC$ and $FH$ are more clearly understood using the diagram

\[
\begin{array}{ccc}
FX & \xrightarrow{Fd_0} & FC \\
F_f & \xrightarrow{Fh} & FY \\
F_x & \xrightarrow{Fs} & FZ \\
\end{array}
\]

\[
\begin{array}{ccc}
Ff & \xRightarrow{Fh} & F(h \circ d_0) \\
& \xRightarrow{\phi} & Fh \circ Fd_0 \\
& \xRightarrow{\phi} & F(s \circ d_0) \\
& \xRightarrow{\phi} & Fx \\
& \xRightarrow{\phi} & F(s \circ d_1) \\
& \xRightarrow{\phi} & Fs \circ Fd_1 \\
& \xRightarrow{\phi} & Fh \circ Fd_1 \\
& \xRightarrow{\phi} & F(h \circ d_1) \\
& \xRightarrow{\phi} & Fg \\
\end{array}
\]

Definition 3.15. Recall that $q\Theta$ denotes the class of quasiequivalences. We identify two σ-homotopies $H, K$ if for every pseudofunctor $(C, \Sigma) \xrightarrow{\mathcal{E}} (\mathcal{D}, q\Theta)$, $FH$ and $FK$ yield the same 2-cell (as in Definition 3.13) of $\mathcal{D}$, that is $[H] = [K] \iff \tilde{FH} = \tilde{FK}$ for every $F$.

We will see below that it suffices to require the condition in Definition 3.15 only for 2-functors $F$. The 2-cell $\tilde{FH}$ is the composition

$$\tilde{FH} : Ff \xRightarrow{Fh} F(h \circ d_0) \xRightarrow{\phi} Fh \circ Fd_0 \xRightarrow{Fh \circ \tilde{F} C} Fh \circ Fd_1 \xRightarrow{\phi} F(h \circ d_1) \xRightarrow{\phi} Fg, \quad (3.16)$$

where $Fd_0 \xRightarrow{\tilde{F} C}$, $Fd_1$ is the unique 2-cell such that $Fs \circ \tilde{F} C = \phi \circ F\tilde{\alpha} \circ \phi$. With the notation of Definition 2.7, this can be stated as:

Remark 3.17. For a σ-homotopy $H$ as in Definition 3.3 and a pseudofunctor $(C, \Sigma) \xrightarrow{\mathcal{E}} (\mathcal{D}, q\Theta)$, $\tilde{FH}$ is the composition $Ff \xRightarrow{FH} F(h \circ d_0) \xRightarrow{Fh \circ \tilde{F} C} F(h \circ d_1) \xRightarrow{\phi} Fg$, where $\tilde{FC}$ is the unique 2-cell such that $Fs \circ \tilde{F} C = F\tilde{\alpha}$. Note that when $F$ is a 2-functor, $*_{F} = *$. 

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Remark 3.18. It is the composition $\tilde{\alpha} = \alpha_1^{-1} \circ \alpha_0$ which is used in order to determine the class of a $\sigma$-homotopy. This suggests that we can define a notion of $\sigma$-precylinder in which $\alpha_0$ and $\alpha_1$ are replaced by an arbitrary 2-cell $s \ast d_0 \Rightarrow s \ast d_1$. Note that all the constructions of this paper work also for the corresponding notion of $\sigma$-prehomotopy. Since any of the two resulting homotopy bicategories (that is, $\mathcal{H}_0(\mathcal{C}, \Sigma)$ as defined below and the analogous one considering $\sigma$-prehomotopies) satisfy Theorem 3.56 in this case their 2-cells coincide. In fact, any $\sigma$-homotopy induces in an evident way a $\sigma$-prehomotopy in the same class, and the statement that (under the hypothesis of Theorem 3.56) any $\sigma$-prehomotopy can be written as a composition of $\sigma$-homotopies can also be shown by a direct computation.

Remark 3.19. We record here, solely for convenience, the dual (that is, the ones for right $\sigma$-homotopies) versions of some of the results above, which correspond to considering the same family $\Sigma$ as a family of arrows of $\mathcal{C}^{op}$. We note that right homotopies will not be considered again in this paper: exactly as in the 1-dimensional case, the left homotopies suffice to construct the localization. We organize the data of a right $\sigma$-homotopy $H$ as follows:

$$f \xRightarrow{H_{\alpha}} g: \quad X \xrightarrow{h} W \xrightarrow{\ast_{d_0}} Y \quad f \xRightarrow{\ast_{d_0} \ast h} d_0 \ast s \Rightarrow y_{\alpha} \Rightarrow y \ast_{d_1} \Rightarrow d_1 \ast s$$

(3.20)

For a pseudofunctor $(\mathcal{C}, \Sigma) \xrightarrow{F} (\mathcal{D}, q\Theta)$, $\widehat{F H}$ is given as the composite 2-cell $Ff \xRightarrow{F_{\eta}} F(d_0 \ast h) \xRightarrow{\widehat{FC} \ast_{F^h}} F(d_1 \ast h) \xRightarrow{F_{\epsilon}} Fg$, where $\widehat{FC}$ is the unique 2-cell such that $\widehat{FC} \ast FC F\sigma = F\tilde{\alpha}$. Note that Definition 3.19 makes sense allowing either $H$, $K$ or both to be right $\sigma$-homotopies instead of left ones.

Let $(\mathcal{C}, \Sigma) \xrightarrow{F} (\mathcal{D}, q\Theta)$, and consider the factorization of $F$ through $\mathcal{C}_F$ of 2.9. Note that, as it was explained in 2.10, we have $(\mathcal{C}, \Sigma) \xrightarrow{F_2} (\mathcal{C}_F, q\Theta)$. The following remark follows immediately by considering Remark 3.17 for $F$ and for $F_2$ (recall that the horizontal composition of 2-cells in $\mathcal{C}_F$ is given by $\ast_F$).

Remark 3.21. Consider the situation in Definition 2.7. We have: $F_1(\beta \ast F_2 \alpha) = \beta \ast F \alpha$. It follows then $F_1(F_2 H) = \widehat{F H}$.

The previous Remark allows to consider 2-functors $(\mathcal{C}, \Sigma) \xrightarrow{F} (\mathcal{D}, q\Theta)$ instead of arbitrary pseudofunctors in Definition 3.15.

Proposition 3.22. Consider the $\sigma$-homotopies of Definition 3.12. Then, for any 2-functor $(\mathcal{C}, \Sigma) \xrightarrow{F} (\mathcal{D}, q\Theta)$, $\widehat{F H_0} = \widehat{F H_1} = F\mu$.

Proof. In the notation of Definition 3.3 the $\sigma$-homotopy $H_0^\mu$ has $s = id_X$, $\eta = \mu$, $\tilde{\alpha} = id_{id_X}$, $\varepsilon = id_g$, thus $\widehat{FC} = id_{id_X}$, and $\widehat{F H_0^\mu}$ is the composition $Fid_g \circ Fid_{id_X} \circ F\mu = F\mu$. The case of $H_1^\mu$ is similar. \qed
Definition 3.23. Given any 2-cell $\mu \in C$, the notation $I^\mu$ stands for any $\sigma$-homotopy such that for any 2-functor $(C, \Sigma) \xrightarrow{F} (D, q\Theta)$, $F I^\mu = F \mu$; note that in view of the previous proposition such a $I^\mu$ always exists.

Proposition 3.24. Let $(C, \Sigma) \xrightarrow{F} (D, q\Theta)$ be a pseudofunctor and let $H$ be as in (3.4). Then:

1. For each $g \xrightarrow{\mu} g' \in C$ as in (3.7) we have $F(\mu \circ H) = F \mu \circ FH$.
2. For each $f' \xrightarrow{\nu} f \in C$ as in (3.8) we have $F(H \circ \nu) = FH \circ F\nu$.
3. For each $Y \xrightarrow{\nu} Y' \in C$ as in (3.9) we have $F(r \ast H) = Fr \ast FH$.
4. For each $X' \xrightarrow{\ell} X \in C$ as in (3.10) we have $F(H \ast \ell) = FH \ast F\ell$.

Proof. Items 1 and 2 are immediate. We show first items 3 and 4 assuming that $F$ is a 2-functor (recall that in this case $\ast_F$ is just $\ast$). Let $Fd \xrightarrow{\nu} Fc$ be the unique 2-cell such that $F_s \ast FC = F\alpha$.

Proof of 3: $F(r \ast H)$ is the 2-cell

$$Fr \ast Ff \xrightarrow{Fr \ast \nu} Fr \ast FH \ast Fd \xrightarrow{Fr \ast Fh \ast FC} Fr \ast FH \ast Fd \xrightarrow{Fr \ast \nu} Fr \ast Fg,$$

which is equal to $Fr \ast FH$.

Proof of 4: We have $F_s \ast FC \ast F\ell = \alpha \ast F\ell$, and thus $F(H \ast \ell)$ is the 2-cell

$$Ff \ast F\ell \xrightarrow{\nu \ast F\ell} Fh \ast Fd \ast F\ell \xrightarrow{Fh \ast FC \ast F\ell} Fh \ast Fd \ast F\ell \xrightarrow{\epsilon \ast F\ell} Fg \ast F\ell,$$

which is equal to $FH \ast F\ell$.

If $F$ is a pseudofunctor, we have

$$F(r \ast H) = F_1 F_2(r \ast H) = F_1(F_2 r \ast F_2 H) = F_1(r \ast F_2 H) = Fr \ast FH,$$

where the first equality holds by Remark 3.21 and the last one is due to Remark 2.8 plus the fact that the structural cells of $F_1$ are those of $F$. The case of item 4 is dual. \qed

The bicategory $\mathcal{Ho}(C, \Sigma)$ and the 2-functor $C \xrightarrow{i} \mathcal{Ho}(C, \Sigma)$. We extend Definition 3.15 to finite sequences of composable $\sigma$-homotopies:

Definition 3.25. Two finite sequences of $\sigma$-homotopies $f = H^1 \ast f_1 \xrightarrow{H^2} \ast f_2 \cdots \ast f_n \xrightarrow{H^n} g$, $f = K^1 \ast f'_1 \xrightarrow{K^2} \ast f'_2 \cdots \ast f'_m \xrightarrow{K^m} g$ are considered equivalent by the following definition:

$$[H^n, \ldots, H^2, H^1] = [K^m, \ldots, K^2, K^1] \iff \text{for every 2-functor } (C, \Sigma) \xrightarrow{F} (D, q\Theta), FH^n \circ \cdots \circ FH^2 \circ FH^1 = FK^m \circ \cdots \circ FK^2 \circ FK^1.$$
Remark 3.26. Note that, by Remark 3.21, it is equivalent to state the condition above for every pseudofunctor.

We construct now a bicategory which we refer to as the homotopy bicategory of \( C \) with respect to \( \Sigma \) and denote by \( \mathcal{H}o(C, \Sigma) \):

3.27. The objects and the arrows of \( \mathcal{H}o(C, \Sigma) \) are again the objects and arrows of \( C \). The 2-cells of \( \mathcal{H}o(C, \Sigma) \) are, loosely speaking, the \( \sigma \)-homotopies of \( C \). More precisely, a 2-cell \( f \Rightarrow g \in \mathcal{H}o(C, \Sigma) \) is given by the class \([H^n, \ldots, H^2, H^1]\) of a finite sequence of \( \sigma \)-homotopies.

Remark 3.28. Note that by Definition 3.23 all possible \( \sigma \)-homotopies \( I^\mu \) determine the same class in \( \mathcal{H}o(C, \Sigma) \). In particular by Proposition 3.22 this is the case for the two \( \sigma \)-homotopies \( H^\mu_0 \) and \( H^\mu_1 \) in Definition 3.12.

Vertical composition. Vertical composition is defined by juxtaposition:

3.29. For \([H^n, \ldots, H^2, H^1]\) as above and \( g \xrightarrow{k_1} g_1 \xrightarrow{k_2} g_2 \cdots \xrightarrow{k_{m-1}} h \), we define

\[ [K^m, \ldots, K^2, K^1] \circ [H^n, \ldots, H^2, H^1] = [K^m, \ldots, K^2, K^1, H^n, \ldots, H^2, H^1]. \]

This is clearly well defined and associative. Note that \([H^n, \ldots, H^1] = [H^n] \circ \cdots \circ [H^1]\).

For 2-cells in \( C \), by Proposition 3.22 we have \([H^\mu_0] \circ [H^\nu_0] = [H^\mu_0] \circ [H^\nu_0] \) (since \( H^\mu_0 \circ H^\nu_0 \)) and similarly for \( H_1 \).

From Proposition 3.24 it follows:

Proposition 3.30. Let \( H, \mu, \nu \) be as in (3.4), 3.7, 3.8, and consider Definition 3.23.\[
\begin{align*}
1. \ [\mu \circ H] & = [I^\mu] \circ [H]. \\
2. \ [H \circ \nu] & = [H] \circ [I^\nu].
\end{align*}
\]

Horizontal composition. We define now the horizontal composition in \( \mathcal{H}o(C, \Sigma) \). We proceed as explained in 2.1, that is, we will define only the horizontal compositions between 2-cells and arrows, and show the axioms \( \mathbf{W} \):

3.31. For \( X \xrightarrow{f} Y \xrightarrow{\ell} Y' \in C \) and \([H^n, \ldots, H^2, H^1] : f \Rightarrow g \) as in Definition 3.27, we define \( r \circ [H^n, \ldots, H^2, H^1] = [r \circ H^n, \ldots, r \circ H^2, r \circ H^1] \), and similarly for \( X' \xrightarrow{\ell'} X \in C \) (see 3.9 and 3.10). The fact that these formulas are well defined follows from Proposition 3.24.

Axiom \( \mathbf{W3} \) follows by definition. To verify axiom \( \mathbf{W1} \), it suffices to check the case in which the 2-cells are sequences of length 1, that is, given \( X \xrightarrow{[H]} Y \xrightarrow{[K]} Z \in \mathcal{H}o(C, \Sigma) \),
we have to check that \([K \ast f_1, g_2 \ast H] = [g_1 \ast H, K \ast f_2]\). Again this follows easily from Proposition 3.24, using axiom W1 in \(\mathcal{D}\) for every 2-functor \((\mathcal{C}, \Sigma) \overset{F}{\longrightarrow} (\mathcal{D}, q\Theta)\).

For each \(f\), we define the identity 2-cell of \(\mathcal{H}_0(\mathcal{C}, \Sigma)\), \(\text{id}_f = [I^f]\), see Definition 3.23 and recall the abuse \(f = \text{id}_f\). By definition, it is immediate that \(\text{id}_f\) is the identity for the vertical composition, and that axiom W2 is satisfied.

We define the identity arrows as in \(\mathcal{C}\). It remains to define the associators and the unitors and check that they satisfy the axioms. Before doing this it is convenient to construct the 2-functor \(\overset{i}{\mathcal{C}} \longrightarrow \mathcal{H}_0(\mathcal{C}, \Sigma)\).

3.32. On objects and arrows \(i\) is just the identity. For a 2-cell \(\mu\) of \(\mathcal{C}\), we define \(i\mu = [I^\mu]\), that is the class of the sequence of length one given by any \(I^\mu\). From Definitions 3.27 and 3.23 it follows for any \(\sigma\)-homotopy \(H\):

\[ i\mu = [H] \iff \text{for every 2-functor } \mathcal{C} \overset{F}{\longrightarrow} \mathcal{D}, q\Theta, \hat{F}H = F(\mu). \]

The unitors and the associator of \(\mathcal{H}_0(\mathcal{C}, \Sigma)\) are obtained applying \(i\) to the ones of \(\mathcal{C}\). Axioms \(N\lambda, N\rho\) and \(N\theta1-3\) follow immediately from Proposition 3.24, using the corresponding axioms in \(\mathcal{D}\).

We will now show that \(\overset{i}{\mathcal{C}} \longrightarrow \mathcal{H}_0(\mathcal{C}, \Sigma)\), mapping \(X \overset{f}{\rightarrow} \overset{\mu}{\rightarrow} Y \overset{r}{\rightarrow} Y' \in \mathcal{C}\), and we have to show that \(i(r \ast \mu) = r \ast i\mu\), i.e. that \([I^r \ast \mu] = [r \ast I^\mu]\). For each 2-functor \((\mathcal{C}, \Sigma) \overset{F}{\longrightarrow} (\mathcal{D}, q\Theta)\), by Definition 3.23 and Proposition 3.24 we have:

\[ \hat{F}I^r \ast \mu = F(r \ast \mu) = Fr \ast F\mu = Fr \ast \hat{F}I^\mu = \hat{F}(r \ast I^\mu), \]

showing the desired equation. The other case is similar. We have shown:

**Proposition 3.33.** For any pair \((\mathcal{C}, \Sigma)\), \(\mathcal{H}_0(\mathcal{C}, \Sigma)\) defined in 3.27 is a bicategory, and \(\overset{i}{\mathcal{C}} \longrightarrow \mathcal{H}_0(\mathcal{C}, \Sigma)\) defined in 3.32 is a 2-functor.

Using in order Remark 3.11, Proposition 3.30 and the definitions in 3.31, 3.32 it follows:

**Proposition 3.34.** Let \(H\) be any \(\sigma\)-homotopy as in (3.4) Then \([H]\) decomposes as:

\[ [H] = [\varepsilon \circ (h \ast H^C) \circ \eta] = [I^f] \circ [h \ast H^C] \circ [I^\eta] = i(\varepsilon) \circ h \ast [H^C] \circ i(\eta). \]
We show now that the \( \sigma \)-cylinder \( C^{-1} \) and the \( \sigma \)-homotopy \( H^{-1} \) (see Definitions 3.35 and 3.36) yield actual inverses in \( \mathcal{H}o(C, \Sigma) \).

**Proposition 3.35.** For any \( \sigma \)-cylinder \( C \), \( [H^C] \) is invertible in \( \mathcal{H}o(C, \Sigma) \) and furthermore, \( [H^C]^{-1} = [H^{-1}] \).

**Proof.** For each 2-functor \((C, \Sigma) \xrightarrow{F} (D, q\Theta)\), we have \( FH^C = HFC = FC \), recall Definition 3.13. Since similarly we have \( FH^C = FC \), it follows that \( [H^C] \circ [H^{-1}] = id_{d_1} \) and \( [H^{-1}] \circ [H^C] = id_{d_0} \).

**Corollary 3.36.** The class \([H]\) of any \( \sigma \)-homotopy with invertible cells is invertible in \( \mathcal{H}o(C, \Sigma) \), and furthermore, \( [H]^{-1} = [H^{-1}] \).

**Proof.** By Proposition 3.34, \( [H] \circ [H^{-1}] = i\varepsilon \circ h \ast [H^C] \circ i\eta \circ i(\varepsilon^{-1}) \circ h \ast [H^{-1}] \circ i(\varepsilon^{-1}) \), which by Proposition 3.35 collapses to the identity. The other composition is similar.

**On vertical composition of \( \sigma \)-homotopies.** It is a natural question to ask if \( \sigma \)-homotopies can be vertically composed, in other words if we can find a single \( \sigma \)-homotopy representing the class \([H^2, H^1]\). The following lemma gives certain conditions under which this is the case. The reader will recognize here an abstract setting corresponding to Quillen’s proof of the transitivity of the left homotopy relation in [3, Lemma 3].

**Lemma 3.37.** Assume that we have \( X \xrightarrow{f_1, f_2, f_3} Y \), and \( \sigma \)-homotopies \( f_1 \xrightarrow{H^1} f_2 \xrightarrow{H^2} f_3 \) as in Definition 3.3, with \( Z^1 = Z^2 = Z \), \( x^1 = x^2 = x \) fitting in the following diagram, where \( \nu^1, \nu^2, \gamma^1, \gamma^2 \) are invertible 2-cells.

\[
\begin{array}{c}
X & \xrightarrow{h} & W^1 & \xrightarrow{s} & Y \\
\downarrow{d_1} & & \downarrow{s_1} & & \downarrow{s} \\
W^2 & \xrightarrow{s_2} & Z & \xrightarrow{h} & Y \\
\end{array}
\]

Assume also that

1. The 2-cell \( h^1 \ast d_1 \xrightarrow{\approx} f_2 \xrightarrow{\approx} h^2 \ast d_2 \) equals \( h^1 \ast d_1 \xrightarrow{\nu^1 \ast d_1} h^1 \ast b^1 \ast d_1 \xrightarrow{h \ast s} h^1 \ast b^2 \ast d_0 \xrightarrow{\gamma^2 \ast d_0} h^2 \ast d_2 \),
2. The 2-cell \( s^1 \ast d_1 \xrightarrow{(\alpha_1)} x \xrightarrow{(\alpha_2)^{-1}} s^2 \ast d_2 \) equals \( s^1 \ast d_1 \xrightarrow{\nu^2 \ast d_1} s^1 \ast b^1 \ast d_1 \xrightarrow{s \ast \delta} s^1 \ast b^2 \ast d_0 \xrightarrow{\nu^2 \ast d_0} s^2 \ast d_2 \).

Then there exists a \( \sigma \)-homotopy \( H \) from \( f_1 \) to \( f_3 \) such that \( [H] = [H^2, H^1] \).

Furthermore, \( H \) can be constructed as follows: consider first the \( \sigma \)-cylinder \( C \) given as \( C = (W, Z, b_1 \ast d_0, b^2 \ast d_1, x, s, \alpha_0, \alpha_1) \), with \( \alpha_0 \) and \( \alpha_1 \) defined as the compositions

\[
\alpha_0 : s \ast b^1 \ast d_0 \xrightarrow{\nu^1 \ast d_0} s \ast d_0 \xrightarrow{\alpha_0^1} x, \quad \alpha_1 : s \ast b^2 \ast d_1 \xrightarrow{\nu^2 \ast d_1} s \ast d_1 \xrightarrow{\alpha_1^1} x
\]

Then \( H \) is given by \( H = (C, h, \eta, \varepsilon) \), with \( \eta \) and \( \varepsilon \) defined as the compositions

\[
\eta : f_1 \xrightarrow{\eta_1} h^1 \ast d_1 \xrightarrow{\nu^1 \ast d_0} h \ast b^1 \ast d_0, \quad \varepsilon : h \ast b^2 \ast d_1 \xrightarrow{\gamma^2 \ast d_0} h^2 \ast d_1 \xrightarrow{\varepsilon_1} f_3.
\]
Proof. We have to show that, for every 2-functor \((\mathcal{C}, \Sigma) \xrightarrow{F} (\mathcal{D}, q\Theta)\), \(\widehat{FH} = \widehat{FH^2} \widehat{FH^1}\). Let \(\widehat{FC}, \widehat{FC^1}, \widehat{FC^2}\) be the 2-cells considered in Remark 3.17 for \(H, H^1, H^2\). We begin by showing \((\Delta) \widehat{FC} = (Fb^2 \ast \widehat{FC^2}) \circ F(\delta) \circ (Fb^1 \ast \widehat{FC^1})\). By the definition of \(\widehat{FC}\), it suffices to show that \(F \varepsilon \ast (Fh \ast \widehat{FC}) \circ F\eta = F \varepsilon^2 \circ (Fh^2 \ast \widehat{FC^2}) \circ F\eta^2 \circ F \varepsilon^1 \circ (Fh^1 \ast \widehat{FC^1}) \circ F\eta^1\).

Clearly by the definitions of \(\varepsilon\) and \(\eta\) it suffices to show that

\[
F(\gamma^2 \ast d^1_1) \circ (Fh \ast \widehat{FC}) \circ F(\gamma^1 \ast d^0_1) = (Fh^2 \ast \widehat{FC^2}) \circ F\eta^2 \circ F \varepsilon^1 \circ (Fh^1 \ast \widehat{FC^1})
\]

We compute as follows:
Corollary 3.38. Assume that $\Sigma$ satisfies that, for each $X \xrightarrow{f_1,f_2,f_3} Y$, and $\sigma$-homotopies $f_1 \xrightarrow{K^1} f_2 \xrightarrow{K^2} f_3$, there exist $\sigma$-homotopies $f_1 \xrightarrow{H^1} f_2 \xrightarrow{H^2} f_3$ satisfying the hypothesis of Lemma 3.37 such that $[K^1] = [H^1]$, $[K^2] = [H^2]$. Then, each of the 2-cells of $\mathcal{H}(\Sigma)$ is the class of a single homotopy, and they compose vertically as in Lemma 3.37.

The universal property of $i$. We will prove that, under some natural conditions on the class $\Sigma$, the 2-functor $\mathcal{C} \xrightarrow{i} \mathcal{H}(\Sigma)$ is the localization of $\mathcal{C}$ with respect to $\Sigma$. It should be noted that $i$ has the universal property of making the arrows of $\Sigma$ into equivalences in a strong 2-categorical sense, by this we mean that $i^*$ is an isomorphism of bicategories, not just a pseudoequivalence. This is analogous to the difference between pseudo and bilimits.

We state precisely what we mean by localization of $\mathcal{C}$ with respect to $\Sigma$ (see [7]):

Definition 3.39. A pseudofunctor $(\mathcal{C}, \Sigma) \xrightarrow{i} (\mathcal{E}, \Theta)$ is the localization of $\mathcal{C}$ with respect to $\Sigma$ if it is universal in the following sense: For any bicategory $\mathcal{D}$, precomposition with $i$:

$$\text{Hom}(\mathcal{E}, \mathcal{D}) \xrightarrow{i^*} \text{Hom}_{\Sigma, \Theta}(\mathcal{C}, \mathcal{D})$$

is a pseudoequivalence of bicategories, where $\text{Hom}_{\Sigma, \Theta}(\mathcal{C}, \mathcal{D})$ stands for the full subbicategory of $\text{Hom}(\mathcal{C}, \mathcal{D})$ consisting of those pseudofunctors that map the arrows of $\Sigma$ to equivalences. When $i^*$ is an isomorphism, we say that $i$ is a strong localization or a localization in a strong sense.

We begin by stating and proving various results which lead to Theorem 3.46 and its Corollary. This theorem is proven without any hypothesis on $\Sigma$, and shows that $i^*$ will be an isomorphism of bicategories as soon as it takes its values in the subbicategory $\text{Hom}_{\Sigma, \Theta}(\mathcal{C}, \mathcal{D})$. Then we show that, under two natural conditions on $\Sigma$, $i$ maps the arrows of $\Sigma$ to equivalences, and thus the desired result follows.
For a $\sigma$-cylinder $C$ and a 2-functor $F : (\mathcal{C}, \Sigma) \to (\mathcal{D}, q\Theta)$, recall that $\hat{F}H^C = \hat{F}C$, which is the unique 2-cell such that $Fs * \hat{F}C = F\hat{\alpha}$.

**Lemma 3.40.** For any $\sigma$-cylinder $C$, we have $[s * H^C] = i\alpha$

**Proof.** Let $F : (\mathcal{C}, \Sigma) \to (\mathcal{D}, q\Theta)$ be a 2-functor, we compute using Proposition 3.24

$F(s * H^C) = Fs * F\hat{H}^C = Fs * \hat{F}C = F\hat{\alpha}$ and the Lemma follows by 3.32 \qed

**Proposition 3.41.** Any 2-functor $\mathcal{H}_o(\mathcal{C}, \Sigma) \xrightarrow{G} \mathcal{D}$ such that $G\Sigma \subset q\Theta$ satisfies the equation $G[H] = G\hat{\alpha}$.

**Proof.** Consider $F = Gi : (\mathcal{C}, \Sigma) \to (\mathcal{D}, q\Theta)$, note that $F$ equals $G$ on objects and arrows. Let $H$ be a $\sigma$-homotopy with $\sigma$-cylinder $C$. From Lemma 3.40 and the definition in 3.31 it follows $s * [H^C] = i\alpha$. Applying $G$ we have $Fs * G[H^C] = F\hat{\alpha}$, and thus $G[H^C] = F(H^C)$.

We compute, using in order Proposition 3.31 functoriality of $G$, Proposition 3.24 and Remark 3.11:

$G[H] = G\hat{\epsilon} \circ Gh * G[H^C] \circ G\eta = F\hat{\epsilon} \circ Fh \circ \hat{F}H^C \circ F\eta = F(\hat{\epsilon} \circ h * H^C \circ \eta) = F\hat{\alpha}$. \qed

**Corollary 3.42.** Let $\mathcal{H}_o(\mathcal{C}, \Sigma) \xrightarrow{G} \mathcal{D}$ be any 2-functor such that $G\Sigma \subset q\Theta$. Then, the composite $Gi$ completely determines $G$.

**Proof.** Since $i$ is trivial at the level of objects and arrows clearly $GX = GiX$ and $Gf = Gi f$. The computation $G[H^n, ..., H^1] = G[H^n] \circ ... \circ G[H^1] = GiH^n \circ ... \circ GiH^1$ finishes the proof. \qed

**Theorem 3.43.** Let $\mathcal{C} \xrightarrow{i} \mathcal{H}_o(\mathcal{C}, \Sigma)$ be the 2-functor in 3.33. Then, for any bicategory $\mathcal{D}$ and any 2-functor $(\mathcal{C}, \Sigma) \xrightarrow{E} (\mathcal{D}, q\Theta)$, there is a unique extension of $F$ to $\mathcal{H}_o(\mathcal{C}, \Sigma)$.

That is, there is a 2-functor $G : \mathcal{H}_o(\mathcal{C}, \Sigma) \to \mathcal{D}$, unique such that $Gi = F$. Note that by Proposition 3.41 the value of $G$ in the class of a homotopy $H$ is necessarily $\hat{F}H$.

**Proof.** By Corollary 3.42 we have that the unique possible definition of $G$ is $GX = FX$, $Gf = Ff$ and $G[H^n, ..., H^1] = \hat{F}H^n \circ ... \circ \hat{F}H^1$. By the definition $i\mu = [\mu]$ and 3.29 it follows that $Gi\mu = F\mu$ for any 2-cell $\mu$ of $\mathcal{C}$. It only remains to show that $G$ is a 2-functor.

Clearly the functoriality of $G$ on objects and arrows holds since $Gi = F$ and $i$ is trivial. The functoriality for the vertical composition of 2-cells holds by 3.29. For the horizontal composition we proceed as explained in 2.1 that is, we consider only horizontal compositions between 2-cells and arrows. It suffices to check this on 2-cells given by a single homotopy. Let $r$ and $H$ as in 3.9 recall 3.31 and Proposition 3.24 Then:

$G(r * [H]) = G([r * H]) = F(\hat{r} * \hat{H}) = F(r) * \hat{F}H = G(r) * G([H])$.

The case $[H] * \ell$ is similar. \qed

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Remark 3.44. In the situation of the theorem above, for \( \xi, \xi_1, \ldots, \xi_n \in \mathcal{H}_o(C, \Sigma) \), we have:

\[
\xi = [H^n] \circ \cdots \circ [H^2] \circ [H^1] \iff G(\xi) = \hat{F} H^n \circ \cdots \circ \hat{F} H^2 \circ \hat{F} H^1
\]

\[
[H] = \xi_n \circ \cdots \circ \xi_2 \circ \xi_1 \iff \hat{F} H = G(\xi_n) \circ \cdots \circ G(\xi_2) \circ G(\xi_1).
\]

We pass now to prove the general case of Theorem 3.43 for pseudofunctors.

Proposition 3.45. Let \((C, \Sigma) \xrightarrow{F} (D, q\Theta)\) be any pseudofunctor. Consider its factorization through \((C_F, q\Theta)\) as in 2.9. We have the following diagram:

Then, for any pseudofunctor \(G\) such that \(F = G_i\) there is a unique 2-functor \(G_2\) such that \(F_2 = G_2 i\) and \(G = F_1 G_2\).

Proof. At the level of objects and arrows the only possible definition of \(G_2\) is \(G_2 X = X, G_2 f = f\), and by assumption we also have \(G X = F X\) and \(G f = F f\). Now, for each 2-cell \(f \Rightarrow g\) of \(\mathcal{H}_o(C, \Sigma)\) we proceed by direct inspection. The only possible definition of \(G_2 \alpha\) such that \(F_1 G_2 \alpha = G \alpha\) is \(G_2 \alpha = G \alpha\) (recall the definition of \(F_1\) on 2-cells). Setting \(G_2 \alpha = G \alpha\) we must check that this determines a 2-functor. It is clear that \(G\) preserves vertical compositions if and only if \(G_2\) does, and for the horizontal composition we have that for \(X \xrightarrow{f_1} Y \xrightarrow{\alpha} Z\) in \(\mathcal{H}_o(C, \Sigma)\), \(G_2(\beta * \alpha) = G_2 \beta * G_2 \alpha\) in \(C_F\) if and only if \(G(\beta * \alpha) = G \beta * F G \alpha = \phi(G \beta * G \alpha) \phi\), which is precisely equation \(N \phi\) for \(G\). Note that for a 2-cell \(\mu\) of \(C\) we have \(F_2 \mu = F \mu = G_i \mu = G_2 i \mu\). Finally note that, since \(i\) is a 2-functor, the only possible structural 2-cells of the pseudofunctor \(G\) such that \(F_2 = G_2 i\) are given by those of \(F\), and since this is also the case for \(F_1\) we conclude that \(G = F_i G_2\).

The reader should note that this proposition is independent of Theorem 3.43, which also yields a unique 2-functor \(G_2\) such that \(F_2 = G_2 i\).

Theorem 3.46. Let \(C \xrightarrow{i} \mathcal{H}_o(C, \Sigma)\) be the 2-functor in 3.33. Then:

1. For any bicategory \(D\) and any pseudofunctor \((C, \Sigma) \xrightarrow{F} (D, q\Theta)\), there exists a unique extension of \(F\) to \(\mathcal{H}_o(C, \Sigma)\). That is, there is a pseudofunctor \(\mathcal{H}_o(C, \Sigma) \xrightarrow{F'} D\), unique such that \(F' i = F\). Furthermore the value of \(F'\) in the class of a homotopy \(H\) is \(\hat{F} H\), that is, \(F'[H] = \hat{F} H\).
2. For every pseudonatural transformation $F \Rightarrow G : (C, \Sigma) \rightarrow (D, q\Theta)$ there is a pseudonatural transformation $F' \Rightarrow G'$ unique such that $\theta' i = \theta$.

3. For every modification $\theta \xrightarrow{\rho} \eta : F \Rightarrow G : (C, \Sigma) \rightarrow (D, q\Theta)$ there is a modification $\theta' \xrightarrow{\rho'} \eta'$ unique such that $\rho' i = \rho$.

Proof. 1. Let $F = F_1 F_2$ be the factorization through $(C_F, q\Theta)$ as in 2.9. Let $F'_2$ be the extension of $F_2$ given by Theorem 3.43. Set $F' = F_1 F'_2$. Then, $F' i = F_1 F'_2 i = F_1 F_2 = F$.

The uniqueness of $F'$ is given by Proposition 3.45 plus the uniqueness of $F'_2$. For the second statement we compute $F'[H] = F_1 F'_2[H] = F_1(F_2 H) = \tilde{F} H$, this last equality given by Proposition 3.21.

2. Since $i$ is the identity at the level of objects and arrows, the only possible definitions are $\theta'_X = \theta_X$, $\theta'_f = \theta_f$ for every $X, f$. Since the structural morphisms of $F'$ (resp. $G'$) are those of $F$ (resp. $G$), axioms $\text{PN0}$ and $\text{PN1}$ for $\theta'$ are equivalent to those for $\theta$. For axiom $\text{PN2}$, we have to show the following equation for every $\sigma$-homotopy $f \sim H \Rightarrow g$ as in Definition 3.3.

\[
\begin{array}{ccc}
Gf & \theta_X & Gf & \theta_X \\
\downarrow{\tilde{G}H} & & \downarrow{\tilde{G}H} & \\
Gg & \theta_X & \theta_Y & Ff \\
\downarrow{\theta_g} & & \downarrow{\theta_g} & \\
\theta_Y & Fg & \theta_Y & Fg \\
\end{array}
\]

That is, by the definition in formula (3.16),

\[
\begin{array}{ccc}
Gf & \theta_X & Gf & \theta_X \\
\downarrow{\tilde{G}h \ast d_0} & & \downarrow{\tilde{G}h \ast d_0} & \\
G(h \ast d_0) & \theta_X & \theta_Y & Ff \\
\downarrow{\phi} & & \downarrow{\phi} & \\
Gh & Gd_0 & \theta_X & \theta_Y & F(h \ast d_0) \\
\downarrow{\phi} & & \downarrow{\phi} & \\
G(h \ast d_1) & \theta_X & \theta_Y & Fh \\
\downarrow{\theta_g} & & \downarrow{\theta_g} & \\
\theta_Y & Fg & \theta_Y & Fg \\
\end{array}
\]

\[(3.47)\]
where \( F_s * F \overline{FC} = F\tilde{\alpha} \) and \( G_s * G \overline{GC} = G\tilde{\alpha} \). Using axiom PN2 for \( \theta \) on the 2-cell \( \tilde{\alpha} \) we have:

\[
\begin{array}{c}
G(s * d_0) \\
\phi \\
Gs \\
\theta_X \end{array} \cong 
\begin{array}{c}
G(s * d_0) \\
\theta_X \end{array}
\]

Using axiom PN1 twice (for the arrows \( s, d_0 \) and the arrows \( s, d_1 \)) the first equality below follows from (3.48):

\[
G(s * d_0) \freestar Gd_0 \theta_X = G(s * d_0) \freestar Gd_0 \theta_X 
\]

Since \( \theta_s \) is invertible, and \( Gs \) is a quasiequivalence, it follows

\[
Gd_0 \freestar Gd_0 \theta_X = \theta_C \freestar Fd_0 
\]

Now, we reverse the path that took us from (3.48) to (3.49), but with \( h \) instead of \( s \). First we compose (3.49) with \( Gh \) and \( \theta_h \) and use (2.3), it follows
Using axiom PN1 as above, it follows

\[
\begin{align*}
G(h * d_0) & \theta_X & G(h * d_0) & \theta_X \\
G(h * d_1) & \theta_X & G(h * d_1) & \theta_X \\
\theta_Y & F(h * d_0) & \theta_Y & F(h * d_1)
\end{align*}
\]

Finally we compute, starting from the left side in (3.47):

\[
\begin{align*}
Gf & \theta_X & Gf & \theta_X & Gf & \theta_X & Gf & \theta_X \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\
G(h * d_0) & \theta_X & G(h * d_0) & \theta_X & G(h * d_0) & \theta_X & G(h * d_0) & \theta_X \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\
G(h * d_1) & \theta_X & G(h * d_1) & \theta_X & G(h * d_1) & \theta_X & G(h * d_1) & \theta_X \\
\theta_Y & F(h * d_0) & \theta_Y & F(h * d_1) & \theta_Y & F(h * d_0) & \theta_Y & F(h * d_1)
\end{align*}
\]

which equals the right side in (3.47) by PN2.
3. Since $i$ is the identity at the level of objects, the only possible definition is $\rho'_X = \rho_X$. Since for any arrow $f$ of $\mathcal{C}$, by the proof of item 2 we have $\theta'_f = \theta_f$ and $\mu'_f = \mu_f$, the equality in axiom PM is the same either for $\rho$ or for $\rho'$.

**Corollary 3.51.** Assume $\mathcal{C} \xrightarrow{i} \mathcal{H} \mathcal{O}(\mathcal{C}, \Sigma)$ maps the arrows of $\Sigma$ to equivalences. Then, it is the strong localization with respect to $\Sigma$ in the sense stated in Definition 3.39.

**Proof.** By assumption $i^*$ takes values in the subcategory $\text{Hom}_{\Sigma, o}(\mathcal{C}, \mathcal{D})$. The previous theorem implies that it is an isomorphism of bicategories.

We proceed now to consider two natural conditions in the class $\Sigma$ which are sufficient to ensure that the assumption in Corollary 3.51 holds.

**Definition 3.52.** We say that the class $\Sigma$ satisfies the 3 for 2 property if for every three arrows $f, g, h$ such that there is an invertible 2-cell $gf \cong h$, whenever two of the three arrows are in $\Sigma$, so is the third one.

**Definition 3.53.** Let $X \xrightarrow{s} Y, Y \xrightarrow{r} X \in \mathcal{C}$. If there is an invertible 2-cell $rs \cong \text{id}_X$, $s$ is called a w-section for $r$, and $r$ is called a w-retraction for $s$ ("w" stands for "weak"). An arrow $X \xrightarrow{s} Y$ is called a w-section if there exists $r$ such that $s$ is a w-section for $r$ and dually an arrow is called a w-retraction if it admits a w-section. An arrow that is either a w-section or a w-retraction is called a w-split arrow.

**Proposition 3.54.** Assume $\Sigma$ satisfies the 3 for 2 property. Then any w-split arrow in $\Sigma$ is mapped to an equivalence by $\mathcal{C} \xrightarrow{i} \mathcal{H} \mathcal{O}(\mathcal{C}, \Sigma)$.

**Proof.** Let $X \xrightarrow{s} Y, Y \xrightarrow{r} X \in \mathcal{C}$ and an invertible 2-cell $rs \cong \text{id}_X \in \mathcal{C}$. Note that by the 3 for 2 property $r$ is in $\Sigma$ if and only if $s$ is. Since we already have $rs \cong \text{id}_X$ in $\mathcal{H} \mathcal{O}(\mathcal{C}, \Sigma)$, it remains to show that we have an invertible 2-cell $s * r \Rightarrow \text{id}_Y$. Consider the diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{s * r} & Y \\
\downarrow{\alpha * r} & & \downarrow{\alpha * r} \\
Y & \xrightarrow{\text{id}_Y} & Y
\end{array}$$

(as in Definition 3.1) which defines the $\sigma$-cylinder $C = (Y, X, s * r, \text{id}_Y, r, r, \alpha * r, \text{id}_r)$. Thus by Proposition 3.35 we have the desired invertible 2-cell $[H^C]$.

**Corollary 3.55.** Assume that $\Sigma$ satisfies the 3 for 2 property, and that any arrow of $\Sigma$ can be written (up to isomorphism) as a composition of w-split arrows of $\Sigma$. Then the 2-functor $\mathcal{C} \xrightarrow{i} \mathcal{H} \mathcal{O}(\mathcal{C}, \Sigma)$ maps the arrows of $\Sigma$ to equivalences.

Clearly putting together Corollaries 3.55 and 3.51 we have the main result of this article:

**Theorem 3.56.** If $\Sigma$ satisfies the 3 for 2 property, and any arrow of $\Sigma$ can be written (up to isomorphism) as a composition of w-split arrows of $\Sigma$, then the 2-functor $\mathcal{C} \xrightarrow{i} \mathcal{H} \mathcal{O}(\mathcal{C}, \Sigma)$ is the strong localization with respect to $\Sigma$ in the sense stated in Definition 3.39.

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2A. Joyal suggested to use this terminology because ‘you pay for 2 and get 3'.
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