Non-Topological Solitons in $3 + 1$ Dimensions

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Abstract

The paper, classically, presents a special stable non-topological solitary wave packet solution in $3 + 1$ dimensions for an extended complex non-linear Klein-Gordon (CNKG) field system. The rest energy of this special solution is minimum among other (close) solutions i.e. it is a soliton solution. The equation of motion and other properties for this special stable solution are reduced to the same original known CNKG system.
I. INTRODUCTION

In soliton paradigm the relativistic classical field theory is an attempt to model particles in terms of non-singular, localized solutions of properly tailored non-linear PDEs \[1-4\]. Classically, a particle is considered as a rigid body which obeys the famous relativistic energy-momentum relation and survives in elastic collisions. To find a soliton solution, like classical particles, firstly, we try to find a solitary wave solution with a localized energy-density; secondly, it must be checked out whether it is stable or not.

A solitary wave solution is stable if the related rest energy is minimized against any arbitrary small permissible deformation. As an example of this definition, kink and anti-kink solutions of the real non-linear Klein-Gordon systems in 1+1 dimensions are stable objects. It was shown theoretically and numerically for such systems that, the solitary kink and anti-kink solutions are stable objects \[5-8\]. In 3+1 dimensions, unfortunately, only a few models of the known non-linear PDEs with soliton solutions have been proposed, among which, one can mention the Skyrme model of baryons \[9, 10\] and ’t Hooft Polyakov model which yields magnetic monopole solitons \[11, 12\].

In this paper we reintroduce the complex nonlinear Klein-Gordon (CNKG) systems with non-topological solitary wave-packet solutions (SWPS’s) \[13-22\]. For any CNKG system, there are infinite types of SWPS which can be identified by different rest frequencies \(\omega_o\) and electrical charges. However, it has not been introduced yet a CNKG model with a stable SWPS. Some of references called these solutions Q-balls or Q-solitons \[18\]. Although, it was shown that the SWPSs or Q-balls have the minimum rest energies among the other solutions with the same electrical charge, it is not a sufficient condition to show that Q-ball solutions are stable objects, as they are not stable under any arbitrary small deformation. They can emit some small localized perturbations (for example in a collision processes) which without the violation of energy and electrical charge conservation, turn to other solutions with less rest energies and electrical charges, i.e. they are not essentially stable objects.

For simplicity, we will study a special CNKG system with many Gaussian SWPSs which can be identified with different rest frequencies \(\omega_o\). It will be shown that for such solutions, essentially there is no stable SWPS, i.e. it is not possible to find a special SWPS (SSWPS) for which its rest energy being minimum among the other close solutions. All of the close solutions of a SSWPS are permissible small deformations (variations) of that, which are
again solutions themselves.

In this paper, it will be shown that to have a stable SSWPS with a standard known CNKG equation of motion as the dominant dynamical equation, we have to consider the original CNKG Lagrangian density with three additional terms (an extended CNKG system). These additional terms alone (i.e. without the original CNKG Lagrangian density) lead to a zero rest mass soliton solution which can move at any arbitrary speed (not greater than the light speed). In other words, for the new extended CNKG system, these additional terms behave like a zero rest mass spook\(^1\) which surrounds the SSWPS and resists to any arbitrary deformation. In fact, for the new extended system, there are new complicated equations of motion with different solutions, but for one of them (i.e. for the SSWPS), the equations of motion and all of the other properties would be reduced to the same original ones (i.e. the same CNKG equations and properties). In this new model, there are three parameters \(A_i\)'s \((i = 1, 2, 3)\) which larger values of those lead to more stability of the SSWPS, i.e. the difference between the rest energy of the SSWPS and the rest energies of the other close solutions increases with increasing the amount of \(A_i\)'s.

The organization of this paper is as follows: In the next section, for the CNKG systems we will set up the basic equations and consider general properties of the related solitary wave-packet solutions. In section III, we have some arguments about stability concept. In section IV, an extended CNKG system will be introduced to obtain a stable SSWPS for which the dominant equations of motion are reduced to the same original CNKG versions. In section V, the stability of the SSWPS will be considered specially for small deformations. The last section is devoted to summary and conclusions.

II. BASIC PROPERTIES OF THE CNKG SYSTEMS

The present calculations are based on a relativistically \(U(1)\)-Lagrangian density in \(3+1\) dimensions:

\[
L = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|),
\]

in which \(\phi\) is a complex scalar field and \(V(|\phi|)\), the field potential, is a self-interaction term which depends only on the modulus of the scalar field. By varying this action with respect

\(^1\) We chose the word "spook" in order to not to confuse with words like "ghost" and "phantom", which have meaning in the literature
to $\phi^*$, one obtains the field equation

$$\Box \phi = \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = -\frac{\partial V}{\partial \phi^*} = -\frac{1}{2} \frac{dV}{d|\phi|} \frac{\phi}{|\phi|}, \tag{2}$$

which is the complex non-linear Klein-Gordon equation in 3 + 1 dimensions. Note that, through the paper, we take the speed of light equals to one. To simplify Eq. (2), we can change variables to the polar fields $R(x^\mu)$ and $\theta(x^\mu)$ as defined by

$$\phi(x, y, z, t) = R(x, y, z, t) \exp[i\theta(x, y, z, t)]. \tag{3}$$

In terms of polar fields, the Lagrangian-density and related field equations are reduced respectively to

$$\mathcal{L} = (\partial^\mu R \partial_\mu R) + R^2 (\partial^\mu \theta \partial_\mu \theta) - V(R), \tag{4}$$

and

$$\Box R - R(\partial^\mu \theta \partial_\mu \theta) = -\frac{1}{2} \frac{dV}{dR}, \tag{5}$$

$$\partial_\mu (R^2 \partial^\mu \theta) = 2R(\partial_\mu R \partial^\mu \theta) + R^2 (\partial^\mu \partial_\mu \theta) = 0. \tag{6}$$

The related Hamiltonian (energy) density is obtained via the Noether’s theorem:

$$T^{00} = \varepsilon(x, t) = \dot{\phi} \dot{\phi}^* + \nabla \phi \cdot \nabla \phi^* + V(|\phi|)$$

$$= (\dot{R}^2 + \nabla R \cdot \nabla R) + R^2 (\dot{\theta}^2 + \nabla \theta \cdot \nabla \theta) + V(R), \tag{7}$$

where dot denotes differentiation with respect to $t$.

We would like to consider systems with spherically symmetric solitary wave-packet solutions i.e. the ones for which the related modulus and phase functions, when they are at rest, are represented as follows:

$$R(x, y, z, t) = R(r) = R(\sqrt{x^2 + y^2 + z^2}), \quad \theta(x, y, z, t) = \omega_0 t, \tag{8}$$

in which $R(r)$ must be a localized function. For anstaz (8), the related equation of motion (6) is satisfied automatically and would reduced to

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \frac{1}{2} \frac{dV}{dR} - \omega_0^2 R. \tag{9}$$

Depending on different values of $\omega_0$, different solutions for $R(r)$ can be obtained. Accordingly, there are a continuous range of different solitary wave packet solutions with different
rest frequencies \((\omega_o)\). A moving solitary wave packet solution can be obtained easily by a relativistic boost. For example, a solitary wave packet solution, with rest frequency \(\omega_o\), which moves in the \(x\)-direction with a constant velocity \(v = \hat{v}i\), would take the form:

\[
R(x, y, z, t) = R(\sqrt{\gamma^2(x - vt)^2 + y^2 + z^2}), \quad \theta(x, y, z, t) = k_\mu x^\mu, \tag{10}
\]

in which \(\gamma = 1/\sqrt{1 - v^2}\), and \(k^\mu \equiv (\omega, k) = (\omega, k, 0, 0)\) is a 3 + 1 vector, provided

\[
k = k \hat{i} = \omega \hat{v}, \tag{11}
\]

and

\[
\omega = \gamma \omega_o. \tag{12}
\]

For simplicity, to obtain some generic Gaussian function as different proposed solitary wave solutions, one can use the following field potential as an example of the nonlinear KG (nKG) self interacting fields:

\[
V(R) = R^2 \left[ W - 4Q - 4Q \ln \left( \frac{R}{R_o} \right) \right], \tag{13}
\]

in which, \(W, R_o\) and \(Q\) are some arbitrary constants. By solving equation (9), the variety of solitary wave packet solutions as a function of \(\omega_o\) can be obtained:

\[
R(r) = \sigma(\omega_o)R_o e^{-Qr^2}, \tag{14}
\]

where \(\sigma(\omega_o) = \exp(\frac{W - \omega_o^2}{4Q})\). Note that, any arbitrary solitary wave solution is not necessarily stable. For example if we set \(W = 20, Q = 1\) and \(R_o = 1\), the related potential (13) as a function of \(R\) is shown in Fig. 1 which its maxima occur at \(R = R_o \exp(\frac{W - 6Q}{4Q}) \approx 33.1155\) (c.f. figure 1). According to general theory of the classical relativistic field theory, it is easy to conclude that the solutions for which \(R_{\max} > 33.1155\), are not essentially stable and can evolve to infinity without the violation of the energy conservation law. To be specified, among infinite types of solitary wave-packet solutions (14), a special one, \(\omega_o = \omega_s\), with \(\omega_s^2 = W = 20\) will be considered. For this special solution \(R_{\max} = 1\) which is clearly less than 33.1155. Therefore, for the special solitary wave-packet solution \((\omega_s = \pm \sqrt{20})\) and those solutions which are close to that, we are sure that the possible values of \(R\) are in the domain of potential (13), which is increasingly positive, that is a necessary condition for the stability of the special solitary wave-packet solution (SSWPS).
The travelling solitary wave packet solution with rest frequency $\omega_o$, is obtained by applying a Lorentz boost. For motion in the x-direction, we obtain

$$\phi(x, y, z, t) = \sigma R_o e^{\left(-Q|x-vt|+y^2+z^2\right)} \exp\left(i\omega_o \gamma[t-vx]\right) = R(r)e^{i\omega_0 \tilde{t}},$$

in which $\tilde{t} = \gamma[t - vx]$. The total energy of a non-moving solitary wave packet solution can be obtained and equated to the rest energy of the related particle-like configuration as

$$E_o = m_o c^2 \equiv \int T^{00} d^3 x = \int \left[ \left( \nabla R \cdot \nabla R \right) + R^2 (\dot{\theta}^2) + V(R) \right] d^3 x$$

$$= \int_0^\infty \left[ \frac{dR}{dr} \gamma^2 + \frac{\omega_o^2}{c^2} R^2 + V(R) \right] 4\pi r^2 dr$$

$$= \int_0^\infty \left[ 2R_o^2 \sigma^2 e^{-2Qr^2} (4Q^2 r^2 - 2Q + \omega_o^2) \right] 4\pi r^2 dr$$

$$= 2R_o^2 \sigma^2 \left( \frac{\pi}{2Q} \right)^\frac{3}{2} (Q + \omega_o^2).$$

Generally, it is possible to show that for each localized solitary wave solution in the relativistic classical field theory, which the related energy-momentum tensor $T^\mu_\nu$ asymptotically approaches to zero at infinity, four independent integrations of the energy-momentum tensor components $T^{\mu0}$ over the whole space, form components of a four vector. Therefore, generally we expect the following relations to be satisfied for a moving solitary wave packet solution:

$$E = m c^2 \equiv \int T^{00} d^3 x = \gamma E_o = \gamma m_o c^2,$$

$$p \equiv \int (T^{01}, T^{02}, T^{03}) d^3 x = \gamma m_o v.$$
It is worth to mention that equations (12) and (17) show that the energy and the frequency are possessing the same behavior and we can relate them via introducing a Planck-like constant $\bar{h}$:

$$E = \bar{h}\omega.$$  \hspace{1cm} (19)

It is easy to understand that $\bar{h}$ is a function of rest frequency $\omega_o$ and for different solitary wave-packet solutions, there are different $\bar{h}$ constants. Similarly, it is possible to find a relation between relativistic momentum of a solitary wave packet solution and the wave number $k$:

$$p = \bar{h}k.$$  \hspace{1cm} (20)

This equation is interesting, since it resembles the deBroglie’s relation.

The Lagrangian density in Eq. (1) is $U(1)$ invariant like electromagnetic theory and this yields to conservation of electrical charge. So, according to the Noether theorem, we can introduce a conserved electrical current density as

$$j^\mu \equiv i\eta(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) = -2\eta(R^2 \partial^\mu \theta), \quad \partial_\mu j^\mu = 0,$$  \hspace{1cm} (21)

in which $\eta$ is a constant included for dimensional reasons [20], and the corresponding conserved charge would be

$$q = \int_{-\infty}^{+\infty} j^0 d^3x = \int_{-\infty}^{+\infty} i\eta(\phi^* \dot{\phi} - \phi \dot{\phi}^*)d^3x.$$  \hspace{1cm} (22)

It is notable that both positive and negative signs of $|\omega_o|$ (i.e. $\omega_o = \pm |\omega_o|$) lead to the same solution for the differential equation (9). They have the same rest mass (energy) but different electrical charges (positive and negative). It is easy to show that for the solutions with $\omega_o > 0$ ($\omega_o < 0$), if we take $\eta > 0$, electrical charge is negative (positive). This shows that the positive and negative solutions are particle and anti-particle.

### III. STABILITY CONSIDERATIONS

In soliton paradigm, particles are considered as localized solutions of the relativistic non-linear field equations. In this paradigm, the main goal is to find a stable solitary wave solution or a soliton solution. We can define a solitary wave solution of a relativistic non-linear field equation as stable, if its rest energy being minimum among the other close solutions.
The close solutions, of a special solitary wave solution which are all permissible small defor-
mations (variations) of that, are again solutions of the equations of motion. For example,
if one consider the previous CNKG systems with standard equations of motion (5) and (6),
the close solutions \( \phi = R e^{i\theta} = \phi_s + \delta \phi = (R_s + \delta R)e^{i(\theta_s + \delta \theta)} \) of a special solution \( \phi_s = R_s e^{i\theta_s} \),
are ones for which we have

\[
\Box (R_s + \delta R) - (R_s + \delta R)(\partial^\mu (\theta_s + \delta \theta)\partial_\mu (\theta_s + \delta \theta)) = -\frac{1}{2} \frac{dV(R_s + \delta R)}{d(R_s + \delta R)},
\]

\[
\partial_\mu ((R_s + \delta R)^2 \partial^\mu (\theta_s + \delta \theta)) = 0,
\]

where \( \delta R \) and \( \delta \theta \) can be any permissible space-time variations which satisfy these equations
simultaneously. To first order of variations, they yield

\[
\Box (\delta R) - (\delta R)(\partial^\mu \theta_s \partial_\mu \theta_s) - 2R_s(\partial^\mu \theta_s \partial_\mu (\delta \theta)) \approx -\frac{1}{2} \frac{d^2 V(R_s)}{dR_s^2}(\delta R),
\]

\[
\partial_\mu (R_s^2 \partial^\mu (\delta \theta) + 2R_s \delta R \partial^\mu \theta_s) \approx 0,
\]

which is the right PDE’s to specify the small permissible deformations (close solutions).

Based on this definition, for different solitary wave packet solutions (14), if one plot
rest energy \( E_o \) versus \( \omega_o^2 \) (16), the resulted curve (Fig. 7) shows that there is not a trivial
solitary wave packet solution with a minimum rest energy. For the case \( \omega = 0 \), we can see
a minimum, but for this special case we encounter with a real system which Virial theorem
[23] essentially prevents us from having a stable solution. In fact, for the case \( \omega = 0 \),
the maximum value of modulus function (15) is \( R_{max} = R_o \exp(\frac{W}{4Q}) \), that is greater than
\( R_o \exp(\frac{W-6Q}{4Q}) \), means that it is not a stable solitary wave solution.

In soliton paradigm to overcome the stability problem, usually the topological solitary
wave solutions have been searched. For example, the famous kink (anti-kink) solutions of the
real non-linear KG systems, Skyrme model and 't Hooft-Polyakov model are few examples
which finally yield to topological stable solitary wave solutions. The non-topological
solutions are more interesting, because a many particle-like solution can be easily constructed
just by adding many far enough distinct solitary wave solutions together. There are usu-
ally hard and complicated conditions for topological solitons to provide a many particle-like
solutions. However, if one considers a non-topological solitary wave solution of a nonlinear
relativistic field system like a fundamental particle, its rest energy must be minimum among
other (close) solutions. In this paper, mathematically, we will introduce a nonlinear relativistic
field system which leads to a (non-topological) special stable solitary wave-packet
solution. Moveover, since many of the dynamical equations of the particles in quantum field theory are KG or nonlinear KG (-like), we expect that the right dominant dynamical PDE for the free special solitary wave packet solution (SSWPS) have to remain the same as CNKG equation of motion (2) as well.

IV. A NEW EXTENDED CNKG SYSTEM WITH A SPECIAL STABLE SOLITARY WAVE PACKET SOLUTION

There are many known particles in the nature. Standard model (SM) is the successful theory which is used to describe these particles. For every type of known particles, there is just a specific field equation with some specific constants. For example, the well-known nonlinear $\phi^4$ theory is used just for Higgs particles and Dirac’s equation is used for electrons and positrons with some inputs (electrons mass and charge). Dirac’s equation for other particles like muons and neutrinos were used again but with different inputs. The proper constants for any type of fundamental particles are usually determined in the laboratory and introduced in relevant equation and inputs.

In this paper, motivated by the classical relativistic field theory, we go through the similar procedure. We will show that what kind of constraints are needed for a special type of localized solutions (14), to be the only valid stable particle solution for a standard CNKG system (1). As emphasized, the dynamical equation (2), or equivalently Eqs. (5) and (6), have infinite localized solutions (see Eq. (14)) and none of them are essentially stable.
However, our assumption is that, in a new classical field theory, necessary conditions can be came together in such a way that just for one of the infinite wave packet functions (14), as a single soliton solution, the dominant dynamical equation to be of the standard form (2), and the rest of them would not be the solution of the new system anymore. For more clarification, assume that in a way (like observing a new particle in a Lab), a particle-like answer could be introduced as a stable localized solution of an unknown classical field system as follows:

$$\phi_s(x, y, z, t) = R_s(r)e^{(i\theta_s)} = R_s(r)e^{(i\omega_s \tilde{t})} = \sigma_s R_o e^{(-Qr^2)} e^{(i(k_s)_\mu x^\mu)} = e^{-r^2} e^{(i(k_s)_\mu x^\mu)},$$  (27)

where $r = \sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}$ (when it moves in the $x$-direction). For simplicity, we set $W = \omega_s^2 = 20$ and $Q = R_o = 1$, then $\sigma_s = \exp\left(\frac{W - \omega_s^2}{4Q}\right) = 1$. In fact, it is a special solution from the infinite solutions (14) of the dynamical equation (2), which the subscript $s$ indicating the "special" to emphasis on this point. Here we consider three assumptions: First, we assume that in general, the Lagrangian density of the relativistic field has a new and complex form ($L_N$), different from that of Eq. (4), with the condition that the special wave packet (27) still be one of its valid solutions. Second, we assume that the new complicated Lagrangian density and its resulting dynamical equations, just for the special solution (27), would be reduced to the standard form of Eqs. (4), (5) and (6), respectively. In other words, we expect that the dominant dynamical equations take the form of the standard and well-known Eqs. (5) and (6), only for the special solution (27). Third, we expect that the special solution (27) to be a stable and soliton solution which means that its rest energy is the minimum among other nontrivial solutions of this new system, which means any change in its internal structure would cause the increase in its rest energy. Satisfying these three assumptions, we get just one single stable particle-like solution that its dominant dynamical equations, like many known particles, are in the standard CNKG form.

According to the pervious demands, in general, the new complex form of the lagrangian density ($L_N$) can be considered as the same original lagrangian density (4) plus a new unknown functional term $F$:

$$L_N = L + F = \left[\partial^\mu R\partial_\mu R + R^2(\partial^\mu \theta \partial_\mu \theta) - V(R)\right] + F.$$  (28)

In other words, we are going to find an additional proper term $F$ for the original lagrangian density (1), with a special potential (13), for which pervious demands satisfied generally.
The unknown scalar $F$ must be, in general, function of all possible allowed scalar structures. The scalars which can appear in the Lagrangian density are modulus field $R$, phase field $\theta$, $\partial_\mu R \partial^\mu R$, $\partial_\mu \theta \partial^\mu \theta$ and $\partial_\mu R \partial^\mu \theta$. As we indicated before, we suppose that one of the previous solitary wave packet solutions (14), with a special rest frequency $\omega_o = \omega_s$ (27), to be a solution again and all relations and equations which were derived in section II for this SSWPS (27) would stay unchanged. Therefore, at the first step, for the SSWPS (27), since we expect the new Lagrangian density (28) to be reduced to the primary original version (4), we conclude that the additional term $F$ must be zero for the SSWPS (27). The new equations of motion for the new Lagrangian density (28) are

\[
\Box R - R (\partial^\mu \theta \partial_\mu \theta) + \frac{1}{2} \frac{dV}{dR} + \frac{1}{2} \left[ \frac{\partial}{\partial x^\mu} \left( \frac{\partial F}{\partial (\partial_\mu R)} \right) - \left( \frac{\partial F}{\partial R} \right) \right] = 0, 
\]

\[
\partial_\mu (R^2 \partial^\mu \theta) + \frac{1}{2} \left[ \frac{\partial}{\partial x^\mu} \left( \frac{\partial F}{\partial (\partial_\mu \theta)} \right) - \left( \frac{\partial F}{\partial \theta} \right) \right] = 0.
\]

To be sure that the conservation of the electrical charge still remains valid in the new extended system (28), the functional $F$ must not dependent on $\theta$. Therefore, the new electrical current in the new extended system (28) is

\[
j^\mu = R^2 \partial^\mu \theta + \frac{1}{2} \left[ \frac{\partial}{\partial x^\mu} \left( \frac{\partial F}{\partial (\partial_\mu \theta)} \right) \right].
\]

Moreover, to be sure that the SSWPS (27) is a solution again, we expect for the SSWPS (27) all additional terms which appeared in the square brackets $[\cdots]$ become zero. Since the functional $F$ is considered essentially different from the original Lagrangian density (1), i.e. $F$ is not linearly dependent on $L$, we conclude that all distinct terms $\partial F/\partial (\partial_\mu R)$, $\partial F/\partial R$ and $\partial F/\partial (\partial_\mu \theta)$ must be zero independently when we have the SSWPS (27). It means that the dominant equations of motion, for the SSWPS (27), would be the same standard CNKG equations of motions (5) and (6), as we expected. Therefore, $F$ and its derivatives which appeared in the pervious equations must be zero for the SSWPS (27). For all these constraints to be satisfied, we have to take $F$ as a function of powers of $S_i$’s ($S_i^n$’s with $n \geq 3$), where $S_i$’s ($i = 1, 2, 3$) are three special independent scalars

\[
S_1 = \partial_\mu \theta \partial^\mu \theta - 20, 
\]

\[
S_2 = \partial_\mu R \partial^\mu R - 4R^2 \ln(R), 
\]

\[
S_3 = \partial_\mu R \partial^\mu \theta, 
\]
which for the SSWPS (27) would be zero. It is straightforward to show that these special scalars all are equal to zero for the SSWPS (27). For simplicity, if one considers $F$ as a function of arbitrary $n$'th power of $S_i$'s, i.e. $F = F(S_1^n, S_2^n, S_3^n)$, it yields
\[
\frac{\partial}{\partial x^\mu} \left( \frac{\partial F}{\partial (\partial_\mu R)} \right) = \sum_{i=1}^3 \left[ n(n-1)S_i^{(n-2)} \frac{\partial S_i}{\partial x^\mu} \frac{\partial F}{\partial (\partial_\mu R)} + nS_i^{(n-1)} \frac{\partial}{\partial x^\mu} \left( \frac{\partial S_i}{\partial (\partial_\mu R)} \partial F \right) \right] 
\]
\[
\frac{\partial F}{\partial R} = \sum_{i=1}^3 \left[ nS_i^{(n-1)} \frac{\partial S_i}{\partial R} \partial F \right] 
\]
\[
\frac{\partial}{\partial x^\mu} \left( \frac{\partial F}{\partial (\partial_\mu \theta)} \right) = \sum_{i=1}^3 \left[ n(n-1)S_i^{(n-2)} \frac{\partial S_i}{\partial x^\mu} \frac{\partial S_i}{\partial (\partial_\mu \theta)} \partial F + nS_i^{(n-1)} \frac{\partial}{\partial x^\mu} \left( \frac{\partial S_i}{\partial (\partial_\mu \theta)} \partial F \right) \right]. 
\]
where $Z_i = S_i^n$. It is easy to understand for $n \geq 3$ that all these relations would be zero for the SSWPS (27) as we expected. Accordingly, one can show that the general form of the functional $F$ which satisfies all needed constraints, can be introduced by a series:
\[
F = \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} a(n_1, n_2, n_3) S_1^{n_1} S_2^{n_2} S_3^{n_3}, 
\]
provided $(n_1 + n_2 + n_3) \geq 3$. Note that, in general the coefficients $a(n_1, n_2, n_3)$, are arbitrary well-defined functional scalers, i.e. they can be again functions of all possible scalars $R$, $\partial_\mu R \partial^\mu R$, $\partial_\mu \theta \partial^\mu \theta$ and $\partial_\mu R \partial^\mu \theta$ (except $\theta$ itself).

The stability conditions impose serious constraints on function $F$ which causes to reduce series (35) to some special formats. However, again there are many choices which can lead to a stable SSWPS (27). Among them, a simple choice which clearly guarantees the stability of the SSWPS can be introduced as follows:
\[
F = \sum_{i=1}^3 A_i (K_i)^3, 
\]

where $A_i$'s ($i = 1, 2, 3$) are just real positive constants for dimensional reasons and $K_i$'s are three linear independent combination of $S_i$'s:
\[
K_1 = \alpha^2 S_1, 
\]
\[
K_2 = \alpha^2 S_1 + S_2, 
\]
\[
K_3 = \alpha^2 S_1 + S_2 + 2\alpha S_3, 
\]
where $\alpha$ is a real constant included for dimensional reasons, but for simplicity we can take it equal to one ($\alpha = 1$). It is obvious that $K_1$, $K_2$ and $K_3$ are all zero just for the SSWPS (27) with rest frequency $\omega_s = \sqrt{20}$. 

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The energy-density that belongs of new modified Lagrangian-density \( \text{(28)} \), for the special choice of the additional function \( F \) \( \text{(36)} \), would be

\[
\varepsilon(x, y, z, t) = \left[ (\dot{R}^2 + \nabla R \cdot \nabla R) + R^2(\dot{\theta}^2 + \nabla \theta \cdot \nabla \theta) + V(R) \right] + \\
\sum_{i=1}^{3} \left[ 3A_iC_i\mathcal{K}_i^2 - A_i\mathcal{K}_i^3 \right] = \varepsilon_o + \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \tag{40}
\]

which divided into four distinct parts and

\[
C_i = \frac{\partial \mathcal{K}_i}{\partial \dot{\theta}} + \frac{\partial \mathcal{K}_i}{\partial \dot{R}} \dot{R} = \begin{cases} 
2\dot{\theta}^2 & i=1 \\
2(\dot{R}^2 + \dot{\theta}^2) & i=2 \\
2(\dot{R} + \dot{\theta})^2 & i=3.
\end{cases} \tag{41}
\]

After a straightforward calculation, one can obtain:

\[
\varepsilon_1 = A_1\mathcal{K}_1^2[5\dot{\theta}^2 + (\nabla \theta)^2 + 20], \tag{42}
\]

\[
\varepsilon_2 = A_2\mathcal{K}_2^2[5\dot{\theta}^2 + 5\dot{R}^2 + (\nabla \theta)^2 + (\nabla R)^2 + 20 + 4R^2 \ln(R)], \tag{43}
\]

\[
\varepsilon_3 = A_3\mathcal{K}_3^2[5(\dot{\theta} + \dot{R})^2 + (\nabla \theta + \nabla R)^2 + 20 + 4R^2 \ln(R)]. \tag{44}
\]

Note that, the function \([20 + 4R^2 \ln(R)]\) and other terms in the above equations all are non-zero and positive definite. Hence, we conclude that \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) are bounded from below, and the minimum values of them are zero, due to \( \mathcal{K}_i = 0 \ (i = 1, 2, 3) \). We will show that just for the nontrivial SSWPS \( \text{(27)} \) (and the trivial solution \( R = 0 \)), \( \varepsilon_i \)’s \( (i = 1, 2, 3) \) are zero simultaneously, i.e. for other unknown nontrivial solutions of the new system \( \text{(28)} \) at least one of the \( \mathcal{K}_i \)’s or \( \varepsilon_i \)’s \( (i = 1, 2, 3) \) would be a nonzero function. Again, it is obvious that the related energy density function \( \text{(40)} \) is reduced to the same original version \( \text{(7)} \) as well.

If constants \( A_i \)’s in Eq. \( \text{(36)} \) are considered to be large numbers, the stability of the SSWPS would be satisfied appreciably. In fact, any solution of the new extended system \( \text{(28)} \) for which at least one the functional \( \mathcal{K}_i \)’s takes non-zero values, leads to a positive large function \( \varepsilon_i \) and then the related rest energy would be larger that SSWPS rest energy, provided the constants \( A_i \)’s to be large numbers. We will show that there is just a single non-trivial solution for which all \( \mathcal{K}_i \)’s would be zero simultaneously, i.e. the SSWPS \( \text{(27)} \). In fact, three conditions \( \mathcal{K}_i = 0 \ (i = 1, 2, 3) \) can be considered as three non-linear PDE’s as
follows:

\[
S_1 = \partial_\mu \theta \partial^\mu \theta - 20 = 0, \quad (45)
\]

\[
S_2 = \partial_\mu R \partial^\mu R - 4R^2 \ln(R) = 0, \quad (46)
\]

\[
S_3 = \partial_\mu R \partial^\mu \theta = 0. \quad (47)
\]

Since the above equations are three independent PDE’s for two fields \(R\) and \(\theta\), therefore, they may not be satisfied simultaneously except for the SSWPS (27) and trivial vacuum state \(R = 0\) (for which just Eq. (45) remains). So, for any arbitrary non-trivial solution, except the SSWPS, at least one of the functional \(K_i\)’s \((S_i\)’s) must be a non-zero function, and then if \(A_i\)’s to be large numbers, at least one of the \(\varepsilon_i\)’s would be a non-zero positive large function which lead to rest energy larger than the SSWPS rest energy. Accordingly, we are sure that the rest energy of the SSWPS (27) is really a minimum among the other non-trivial solution, i.e. it is a soliton solution.

To prove that the SSWPS (27) is really a stable object, we just considered functions \(\varepsilon_i\)’s \((i = 1, 2, 3)\) but we did not consider function \(\varepsilon_o\). In the next section, it will be shown that the role of the first part of the energy density \(\varepsilon_o\), if \(A_i\)’s to be large numbers, is physically unimportant and can be ignored in the stability considerations.

In fact, to bring up an extended CNKG model (28) for which the stability of the SSWPS (27) is guaranteed appreciably in a simple straightforward conclusion, we select three special linear combination of \(S_i\)’s in Eqs. (37), (38) and (39) for which \(\varepsilon_i\)’s \((i = 1, 2, 3)\) would be definitely positive. In general, it may be possible to choose other combinations of \(S_i\)’s for this goal. However, we intentionally introduced this special combination (36) as a good example of the extended CNKG systems (28) for better and simpler conclusions.

V. STABILITY UNDER SMALL DEFORMATIONS

In general, any arbitrary close solution or any small permissible deformed function of a non-moving SSWPS (27) is introduced in the following forms:

\[
R(x, y, z, t) = R_s(r) + \delta R(x, y, z, t) \quad \text{and} \quad \theta(x, y, z, t) = \theta_s + \delta \theta = \omega_s t + \delta \theta(x, y, z, t), \quad (48)
\]

where \(\delta R\) and \(\delta \theta\) (small variations) are small functions of space-time. Note that, the permissible deformed functions \(R(x, y, z, t)\) and \(\theta(x, y, z, t)\) are considered to be solutions of the
new equations of motions (29) and (30) as well. Now, if we insert (48) in \( \varepsilon_o(x, y, z, t) \) and keep it to the first order of \( \delta R \) and \( \delta \theta \), then it yields

\[
\varepsilon_o(x, y, z, t) = \varepsilon_o(x, y, z, t) + \delta \varepsilon_o(x, y, z, t) = \left[ \nabla R_s \cdot \nabla R_s + R_s^2 \omega_s^2 + V(R_s) \right] + 2 \left[ \nabla R_s \cdot \nabla (\delta R) + R_s(\delta R) \omega_s^2 + R_s^2 \omega_s(\delta \theta) + \frac{1}{2} \frac{dV(R_s)}{dR_s}(\delta R) \right].
\]

(49)

Note that, for a non-moving SSWPS, \( \dot{R}_s = 0, \nabla \theta_s = 0 \) and \( \dot{\theta}_s = \omega_s = \sqrt{20} \). It is obvious that \( \delta \varepsilon_o \) is not necessarily a positive definite function.

Now, let do this for the additional terms \( \varepsilon_i \) \( (i = 1, 2, 3) \). If we insert a variation like (48) into \( \varepsilon_i \) \( (i = 1, 2, 3) \), it yields

\[
\varepsilon_i(x, y, z, t) = \varepsilon_{is} + \delta \varepsilon_i = \delta \varepsilon_i = [3A_i(C_{is} + \delta C_i)(K_{is} + \delta K_i)^2 - A_i(K_{is} + \delta K_i)^3] = [3A_i(C_{is} + \delta C_i)(\delta K_i)^2 - A_i(\delta K_i)^3] \approx [3A_i C_{is}(\delta K_i)^2 - A_i(\delta K_i)^3] \approx [3A_i C_{is}(\delta K_i)^2] > 0 \] (50)

in which \( \varepsilon_{is} = 0, K_{is} = 0 \) and \( C_{is} \) referred to the SSWPS and \( \delta K_i \) and \( \delta C_i \) are in the same order of \( \delta R \) and \( \delta \theta \). Therefore, since \( C_i > 0 \), according to Eq. (50), \( \varepsilon_i \)'s for small variations are always positive definite (as were perviously obtained from Eqs. (42), (43) and (44) generally). Note that \( \delta K_i \)'s and \( \delta \varepsilon_o \) are in the same order of magnitude of \( \delta R \) and \( \delta \theta \), but \( \delta \varepsilon_i = \varepsilon_i (i = 1, 2, 3) \) is proportional to \( A_i(\delta K_i)^2 \). The variation of total energy density is equal to \( \delta \varepsilon = \delta \varepsilon_o + \sum_{i=1}^{3} \delta \varepsilon_i \). The stability is guaranteed if the variation of the total energy density \( \delta \varepsilon \) being positive for all possible arbitrary variations in the modulus and phase functions (48). If one consider large values for constants \( A_i \)'s, this main goal confirms effectively. Note that, \( \delta \varepsilon_i \)'s are always positive but \( \delta \varepsilon_o \) is not necessarily positive (c.f. 49 and 50). If the order of \( \delta \varepsilon_o \) for any arbitrary small variation is greater than \( \delta \varepsilon_i \), it may be possible to see the decreasing behavior for the total rest energy \( E_o \). For example, consider \( A_i = 10^{40} \); therefore the order of magnitude of variations \( \delta R \) and \( \delta \theta \), for which the SSWPS is not mathematically a stable object (i.e. the variations for which \( O(|\delta \varepsilon_o|) > O(\delta \varepsilon_i) \approx A_i(\delta K_i)^2 \) or \( O(|\delta R|) + O(|\delta \theta|) > O(A_i[\delta R]^2) + O(A_i[\delta \theta]^2) + O(A_i|\delta R\delta \theta|) \)), is approximately less than \( 10^{-20} \), which is so small that can be ignored in the stability considerations! For such so small variations, the total rest energy \( E_o \) may be reduced with a very small amounts equal to the integration of \( \delta \varepsilon_o \) over all the whole space which again is a very small unimportant value. Therefore for large values of \( A_i \)'s, the SSWPS is effectively a stable object. In fact, this so small decreasing behaviour related to this fact that for a non-deformed (or for a very small deformed) SSWPS (27), the dominant equations of motion are the reduced versions
of the equations of motion (29) and (30), i.e. the same original CNKG equations of motion (5) and (6). Note that, since scalars $K_i$’s (or $S_i$) are three independent functions of $R$ and derivatives of $R$ and $\theta$, therefore, if constants $A_i$’s are large values, for any arbitrary small deformations, at least one of $K_i$’s changes and takes non-zero values, which according to Eq. (50), leads to the large increase in the total rest energy. Although, the $A_i$’s parameters take very large values, but they won’t affect the dynamical equations and the observable of the SSWPS [27]. They just make it stable and do not appear in any of the observable, i.e. they act like a stability catalyst.

FIG. 3. Variations of the total rest energy $E_o$ versus small $\xi$ for different $A_i$’s at $t = 0$. We have a fixed phase function and have considered modulus function changes according to relation (51).

In general, it is not possible for us to feel the permissible deformations of the SSWPS for the new equations of motion (29) and (30). Then let us to consider an arbitrary artificial (impermissible) deformation at $t = 0$ as follows:

$$R(x, y, z, t) = e^{-r^2} + \xi(1 + t)e^{-r^2} \text{ and } \theta(x, y, z, t) = \theta_s = k_{\mu}x^\mu,$$

in which $\xi$ is a small coefficient. Fig. 3 and Fig. 4, for this arbitrary variation (51), show that properly there is always a small range for which $E_o$ decreases with very small values. Namely, in Fig. 3 for $A_i = 10^4$, there is not any minimum but if we zoom on the curve around the $\xi = 0$, the output result can be seen in Fig. 4 which shows that $\xi = 0$ is not really a minimum. By increasing $A_i$’s, this behavior never disappear, i.e. there will be always a small range for which the arbitrary variation (51) leads to a decreasing behaviour for the
FIG. 4. Variations of the total rest energy $E_o$ versus small $\xi$ for different $\Lambda_i$’s at $t = 0$. We have a fixed phase function and have considered modulus function changes according to relation (51).

FIG. 5. Variations of the total rest energy $E_o$ versus small $\xi$ for different $\Lambda_i$’s at $t = 0$. We have a fixed modulus function and have considered the phase function changes according to relation (52).

If we consider extremely large values of $\Lambda_i$’s, this would yield very small shift from $E_o(\xi = 0)$, which is completely unimportant and the stability of the solitary wave-packet solutions is enhanced appreciably. For more supported, we can introduce two additional arbitrary impermissible variations at $t = 0$ as follows:

$$R(x, y, z, t) = e^{-r^2} \quad \text{and} \quad \theta(x, y, z, t) = k \mu x^\mu + \xi e^{-r^2}, \quad (52)$$
FIG. 6. Variations of the total rest energy $E_o$ versus small $\xi$ for different $A_i$’s at $t = 0$. We have fixed modulus function and have considered the phase function changes according to relation (52).

and

$$R(x, y, z, t) = e^{-(1+\xi)r^2} \quad \text{and} \quad \theta(x, y, z, t) = k_{\mu}x^\mu.$$  \hspace{1cm} (53)

The expected results which obtained numerically for these different variations are shown in Fig. 5 and Fig. 6.

Briefly, if one considers large values for $A_i$’s, the SSWPS is physically a stable object. The larger values of coefficients $A_i$’s, similar to the previous Figs in this section, leads to the greater increase in differences between the rest energy of the SSWPS and other close solutions. In other words, the larger values of $A_i$’s lead to more stability. As for a SSWPS, all $\mathcal{K}_i$’s are equal to zero simultaneously, the related equations of motion (29) and (30) are reduced to the original forms (5) and (6) respectively, i.e. the dominant equations of motion are the same standard CNKG equation (5) and (6) as well. This means, if one asks about the right equation of motion of the free SSWPS, our answer would be the same known CNKG Eq. (2). The role of the additional terms ($\mathcal{K}_i$ dependent terms) which we consider in the new modified model (28), behave like a strong force which fix the SSWPS to a special form of modulus and phase function (27), i.e. any deformation in the SSWPS arises to a strong force (that comes from the $\mathcal{K}_i$ dependent terms) and suppresses the changes and preserve the form of SSWPS. In other words, the nonstandard ($\mathcal{K}_i$ dependent) terms for this modified model, behave like a zero rest mass spook which surrounds the particle-like solution and...
resist to any arbitrary deformation. Therefore, the $K_i$ dependent terms, just guarantee the stability of the SSWPS and for a free non-deformed version of that are hidden. As two SSWPS’s approach each other to collide, the $K_i$ dependent terms of these two SSWPS’s become stronger and effectively represent the interaction between them. A SSWPS can move with different velocities and since it is a stable object we expect it to reappear in collisions, i.e. it is a soliton.

VI. SUMMERY AND CONCLUSION

Firstly, we reviewed some basic properties of the complex non-linear Klein-Gordon (CNKG) equations in 3+1 dimensions. Each CNKG equation may have some non-dispersive solitary wave packet solutions which can be identified by different rest frequencies ($\omega_o$). For a moving solitary wave packet solution, the corresponding frequency is $\omega = \gamma \omega_o$ and it is proportional to the total energy, i.e. $E = \hbar \omega$ which $\hbar$ is just a Planck-like constant and is a function of the rest frequency. Moreover, it was found that a solitary wave packet satisfies a deBroglie’s-like wavelength-momentum relation i.e. $p = \hbar k$.

We had some arguments about stability criterion. Briefly, a stable solitary wave solution is the one which its rest energy is minimum among the other close solutions. For a special CNKG system with Gaussian solitary wave-packet solutions (as a simple candidate of the CNKG systems), it was shown that there is not a stable solitary wave packet solution at all. Accordingly, to have a special stable solitary wave-packet solution (SSWPS) with a dominant standard CNKG equation of motion, we have to add three special terms to the original CNKG Lagrangian density. It was shown that for the new extended CNKG system (i.e. the original CNKG Lagrangian density + three proper additional terms), the stability of the SSWPS satisfied appreciably. In fact, these additional terms behave like a zero rest mass spook which surrounds the SSWPS and resist to any arbitrary deformations. For the new extended CNKG system, in general, there are complicated equations of motion, which just for the SSWPS, are reduced to the original versions. In other words, for the new extended system, the dominant equation of motion for the SSWPS would be the same original CNKG versions as we expected. For this modified model, there are some free parameters $A_i$’s ($i = 1, 2, 3$) which larger values of that imposes stronger stability on the SSWPS, i.e. the difference between the rest energy of the SSWPS and the rest energies of the other close
solutions, increases with increasing the amount of constant $A_i$.

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