General Solution of the Quantum Damped Harmonic Oscillator II: Some Examples

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Abstract

In the preceding paper (arXiv: 0710.2724 [quant-ph]) we have constructed the general solution for the master equation of quantum damped harmonic oscillator, which is given by the complicated infinite series in the operator algebra level. In this paper we give the explicit and compact forms to solutions (density operators) for some initial values. In particular, the compact one for the initial value based on a coherent state is given, which has not been given as far as we know. Moreover, some related problems are presented.

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In the preceding paper \[1\] (see also \[2\], \[3\]) we treated the master equation of quantum damped harmonic oscillator

\[ \frac{\partial}{\partial t} \rho = -i[\omega a^\dagger a, \rho] - \frac{\mu}{2} (a^\dagger a \rho + \rho a a^\dagger) - \frac{\nu}{2} (a a^\dagger \rho + \rho a^\dagger a - 2a a^\dagger \rho) \]  

(1)

where \( \rho \equiv \rho(t) \) is the density operator (or matrix) of the system, and \( a \) and \( a^\dagger \) are the annihilation and creation operators of it (for example, an electro–magnetic field mode in a cavity), and \( \mu \) and \( \nu \) (\( \mu > \nu \geq 0 \)) are some constants depending on it (for example, a damping rate of the cavity mode).

The general solution of the equation is given by

\[ \rho(t) = e^{\mu - \nu t} F(t) \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \{ \exp \{ -i\omega t - \log(F(t)) \} \} N \times \left\{ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m \rho(0)(a^\dagger)^m \right\} \exp \{ i\omega t - \log(F(t)) \} N \} a^n \]  

(2)

where \( N = a^\dagger a \) and

\[ E(t) = \frac{2\mu}{\mu - \nu} \sinh \left( \frac{\mu - \nu}{2} t \right) \cosh \left( \frac{\mu + \nu}{2} t \right) + \frac{2\nu}{\mu - \nu} \sinh \left( \frac{\mu - \nu}{2} t \right), \quad G(t) = \frac{2\nu}{\mu - \nu} \sinh \left( \frac{\mu - \nu}{2} t \right) \cosh \left( \frac{\mu + \nu}{2} t \right) + \frac{2\mu}{\mu - \nu} \sinh \left( \frac{\mu - \nu}{2} t \right) \]  

\[ F(t) = \cosh \left( \frac{\mu - \nu}{2} t \right) + \frac{\mu + \nu}{\mu - \nu} \sinh \left( \frac{\mu - \nu}{2} t \right). \]  

(3)

If \( \nu = 0 \), we have a simple form

\[ \rho(t) = e^{-(\frac{\mu}{\nu} + i\omega)t} N \sum_{m=0}^{\infty} \frac{(1 - e^{-\mu t})^m}{m!} a^m \rho(0)(a^\dagger)^m \} e^{-(\frac{\mu}{\nu} + i\omega)t} N. \]  

(4)

Now we calculate \( \rho(t) \) for some \( \rho(0) \) given in the following.

[A] \( \rho(0) = |0\rangle \langle 0| \). This case is easy and become

\[ \rho(t) = e^{\frac{\mu - \nu}{2} t} F(t) \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} |0\rangle \langle 0| a^\dagger a^\dagger = e^{\frac{\mu - \nu}{2} t} F(t) \sum_{n=0}^{\infty} G(t)^n |n\rangle \langle n| = \frac{e^{\frac{\mu - \nu}{2} t}}{F(t)} e^{\log G(t) N} \]  

(5)

where \( N (= a^\dagger a) \) is written as

\[ N = \sum_{n=0}^{\infty} n|n\rangle \langle n|. \]
\[ \rho(0) = |\alpha\rangle\langle \alpha| \text{ where } |\alpha\rangle \ (\alpha \in \mathbb{C}) \text{ is a coherent state defined by} \]
\[ |\alpha\rangle = e^{\alpha a^\dagger - \bar{\alpha} a} |0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0\rangle. \]

Note that
\[ a|\alpha\rangle = \alpha|\alpha\rangle \quad \text{and} \quad \langle \alpha|a^\dagger = \langle \alpha|\bar{\alpha}. \]

\[ \text{[B–1] Special Case : } \nu = 0. \text{ From (4) we have easily} \]
\[ \rho(t) = |ae^{-(\frac{\mu^2}{2} + i\omega)t}\rangle\langle ae^{-(\frac{\mu^2}{2} + i\omega)t}|. \] (6)

The proof is included in the next case. With respect to the evolution \[ \alpha \rightarrow ae^{-(\frac{\mu^2}{2} + i\omega)t}, \] see the appendix and a review paper [4].

\[ \text{[B–2] General Case : } \nu \neq 0. \text{ We have} \]
\[ \rho(t) = (1 - G(t)) e^{\frac{|\alpha|^2}{2} e^{-i(\mu - \nu)t} \log G(t)} e^{-\log G(t)} \left\{ e^{e^{-\frac{\mu^2}{2}t}e^{-i\omega t} a^\dagger + \bar{\alpha} e^{-\frac{\mu^2}{2}t} e^{i\omega t} a - N} \right\}. \] (7)

Since this proof is not so easy we give a detailed one in the following.

\[ \text{[First Step] From (2)} \]
\[ \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} a^m |\alpha\rangle \langle a^\dagger|^m = \sum_{m=0}^{\infty} \frac{E(t)^m}{m!} \alpha^m |\alpha\rangle \langle \alpha|^m = \sum_{m=0}^{\infty} \frac{(E(t)|\alpha|^2)^m}{m!} |\alpha\rangle \langle \alpha| = e^{E(t)|\alpha|^2} |\alpha\rangle \langle \alpha|. \]

\[ \text{[Second Step] From (2) we must calculate} \]
\[ e^{\gamma N} |\alpha\rangle \langle \alpha|^e^{\gamma N} = e^{\gamma N} e^{\alpha a^\dagger - \bar{\alpha} a} |0\rangle \langle 0| e^{-(\alpha a^\dagger - \bar{\alpha} a)} e^{\gamma N} \]

where \[ \gamma = -i\omega t - \log(F(t)). \] Note that \[ \bar{\gamma} \neq -\gamma. \] It is easy to see
\[ e^{\gamma N} e^{\alpha a^\dagger - \bar{\alpha} a} |0\rangle = e^{\gamma N} e^{\alpha a^\dagger - \bar{\alpha} a} e^{-\gamma N} e^{\gamma N} |0\rangle = e^{\alpha e^{\gamma} a^\dagger - \bar{\alpha} e^{-\gamma} a} |0\rangle \]

where we used
\[ e^{\gamma N} a^\dagger e^{-\gamma N} = e^{\gamma} a^\dagger \quad \text{and} \quad e^{\gamma N} ae^{-\gamma N} = e^{-\gamma} a. \]

See for example [3]. Therefore by use of Baker–Campbell–Hausdorff formula two times
\[ e^{\alpha e^{\gamma} a^\dagger - \bar{\alpha} e^{-\gamma} a} |0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha e^{\gamma} a^\dagger - \bar{\alpha} e^{-\gamma} a} |0\rangle = e^{-\frac{|\alpha|^2}{2} e^{\gamma} a^\dagger}|0\rangle \]
\[ = e^{-\frac{|\alpha|^2}{2} e^{\gamma^+ \hat{\gamma}}} e^{\alpha e^{\gamma} a^\dagger - \bar{\alpha} e^{-\gamma} a} |0\rangle = e^{-\frac{|\alpha|^2}{2} (1-e^{\gamma^+ \hat{\gamma})} |\alpha e^{\gamma}|} \]
As a result we obtain
\[ e^{\gamma N} |\alpha\rangle \langle \alpha| e^{\gamma N} = e^{-|\alpha|^2 (1-e^{\gamma + \gamma})} |\alpha e^\gamma \rangle \langle \alpha e^\gamma| \]

with \( \gamma = -i\omega t - \log(F(t)) \).

[Third Step] Under two steps above the equation (2) becomes
\[
\rho(t) = \frac{e^{\frac{\mu - it}{2} F(t)} e^{|\alpha|^2 (E(t)-1)e^{\gamma + \gamma})}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n |\alpha e^\gamma\rangle \langle \alpha e^\gamma| a^n.
\]

We set \( z = \alpha e^\gamma \) for simplicity and calculate
\[
(\#) = \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n z \langle z| a^n.
\]

Since \( |z\rangle = e^{-|z|^2/2} e^{za^\dagger} |0\rangle \) we have
\[
(\#) = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n |0\rangle \langle 0| e^{za^\dagger} = e^{-|z|^2} e^{za^\dagger} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} |0\rangle \langle 0| a^n \]
\[
= e^{-|z|^2} e^{za^\dagger} \sum_{n=0}^{\infty} G(t)^n |n\rangle \langle n| e^{za^\dagger} = e^{-|z|^2} e^{za^\dagger} e^{\log G(t) N} e^{za^\dagger}
\]

by [A]. Namely, this form is a kind of disentangling formula, so we want to restore an entangling formula.

For that we use the **disentangling formula**
\[
e^{\alpha a^\dagger + \beta a + \gamma N} = e^{\alpha \frac{e^{\gamma N} - 1}{\gamma N} a^\dagger e^{\gamma N} a e^{\gamma N}}.
\]

(8)

For the proof see the fourth step in the following. From this it is easy to see
\[
e^{xa^\dagger} e^{yN} e^{za} = e^{-\frac{xy(a^\dagger + a)}{\gamma}} e^{-\frac{yN}{\gamma - 1} a^\dagger a + yN}.
\]

(9)

Therefore
\[
(\#) = e^{-|z|^2} e^{-\frac{|z|^2 (1+\log(G(t)-G(t)))}{(1-G(t))^2}} e^{\frac{\log G(t)}{G(t)-1} a^\dagger a + \log G(t) N},
\]

so by noting
\[
z = \alpha e^\gamma = \alpha e^{-i\omega t} \frac{F(t)}{1}, \quad |z|^2 = |\alpha|^2 e^{\gamma + \gamma},
\]

we have
\[
\rho(t) = \frac{e^{\frac{\mu - it}{2} F(t)} e^{\frac{|\alpha|^2 (E(t)-1) e^{\gamma + \gamma})}{F(t)}}}{F(t)} \sum_{n=0}^{\infty} \frac{G(t)^n}{n!} (a^\dagger)^n \alpha e^{-i\omega t} a^\dagger + \frac{\log G(t)}{F(t)(G(t)-1)} e^{-i\omega t} a^\dagger + \frac{\log G(t)}{F(t)(G(t)-1)} \alpha e^{-i\omega t} a + \log G(t) N.
\]

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By the way, from (3)

\[ G(t) - 1 = -\frac{e^{-\mu t}}{F(t)}, \quad \frac{1}{F(t)(G(t) - 1)} = -e^{-\frac{\mu \nu}{t}}, \quad E(t) - 1 = -\frac{e^{-\mu t}}{F(t)} \]

and

\[
\frac{1 - G(t) + \log G(t)}{F(t)^2(G(t) - 1)^2} = e^{-(\mu - \nu)t} \left\{ \frac{e^{\frac{\mu \nu}{2} t}}{F(t)} + \log G(t) \right\} = e^{-\frac{\mu \nu}{2} t} F(t) + e^{-(\mu - \nu)t} \log G(t)
\]

we finally obtain

\[
\rho(t) = (1 - G(t))e^{[\alpha^2 e^{-(\mu - \nu)t}\log G(t)} e^{-\log G(t)} \left\{ \alpha e^{-i\omega t} e^{-\frac{\mu \nu}{2} t} a^\dagger + \alpha e^{i\omega t} e^{-\frac{\mu \nu}{2} t} a - N \right\}
\]

or

\[
\rho(t) = e^{[\alpha^2 e^{-(\mu - \nu)t}\log G(t)} e^{-\log(1 - G(t))} e^{-\log G(t)} \left\{ \alpha e^{-i\omega t} e^{-\frac{\mu \nu}{2} t} a^\dagger + \alpha e^{i\omega t} e^{-\frac{\mu \nu}{2} t} a - N \right\}.
\]

This is our main result in the paper.

[Fourth Step] In last, let us give the proof to the disentangling formula (8) because it is not so popular as far as we know.

Since

\[
\alpha a^\dagger + \beta a + \gamma N = \gamma a^\dagger a + \alpha a^\dagger + \beta a
\]

\[= \gamma \left\{ \left( a^\dagger + \frac{\beta}{\gamma} \right) \left( a + \frac{\alpha}{\gamma} \right) - \frac{\alpha \beta}{\gamma^2} \right\} = \gamma \left( a^\dagger + \frac{\beta}{\gamma} \right) \left( a + \frac{\alpha}{\gamma} \right) - \frac{\alpha \beta}{\gamma} \]

from (8) we have

\[
e^{\alpha a^\dagger + \beta a + \gamma N} = e^{-\frac{\alpha a}{\gamma}} e^{\gamma(a^\dagger + \frac{\alpha}{\gamma})} \left( a + \frac{\alpha}{\gamma} \right)
\]

\[= e^{-\frac{\alpha a}{\gamma}} e^{\gamma a^\dagger} e^{\gamma(a + \frac{\alpha}{\gamma})} e^{-\frac{\alpha a}{\gamma}}
\]

\[= e^{-\frac{\alpha a}{\gamma}} e^{\gamma a^\dagger} e^{-\frac{\alpha a}{\gamma}} e^{\gamma a^\dagger} e^{\gamma a^\dagger} e^{-\frac{\alpha a}{\gamma}}.
\]
Then we obtain the disentangling formula \( N = a^\dagger a \)
\[
e^{-\frac{\alpha^2}{\gamma}} e^{\gamma N} e^{-\frac{\alpha}{\gamma}} e^{-\frac{\alpha a^\dagger}{\gamma}} e^{-\frac{\alpha a}{\gamma}} e^{\gamma N} e^{-\frac{\alpha a^\dagger}{\gamma}} e^{-\frac{\alpha}{\gamma}} a = e^{-\frac{\alpha^2}{\gamma}} e^{\gamma N} e^{-\frac{\alpha a^\dagger}{\gamma}} e^{-\frac{\alpha a}{\gamma}} e^{\gamma N} e^{-\frac{\alpha a^\dagger}{\gamma}} e^{-\frac{\alpha}{\gamma}} a
\]
by use of some commutation relations
\[
e^{sa} e^{ta^\dagger} = e^{st} e^{ta^\dagger} e^{sa}, \quad e^{sa} e^{tN} = e^{tN} e^{se^a}, \quad e^{tN} e^{sa^\dagger} = e^{sa^\dagger} e^{tN},
\]
see for example [5].

We finished the proof. The formula (7) that is compact and clear–cut has not been given as far as we know. See [6] and [7] as standard textbooks.

We are in a position to state our problem. A squeezed state \( |\beta\rangle \ (\beta \in \mathbb{C}) \) is defined as
\[
|\beta\rangle = e^{\beta (a^\dagger)^2 - \bar{\beta} a^2} |0\rangle.
\]
See for example [5]. For the initial value
\[
\rho(0) = |\beta\rangle \langle \beta|
\]
we want to calculate \( \rho(t) \) in (2) like in the text. However, we cannot sum up it in a compact form like (7), so

[Problem] sum up \( \rho(t) \) in a compact form.

Similarly, we can consider a coherent–squeezed state
\[
|(\beta, \alpha)\rangle = e^{(1/2)(\beta (a^\dagger)^2 - \bar{\beta} a^2)} e^{\alpha a^\dagger - \bar{\alpha} a} |0\rangle
\]
for \((\beta, \alpha) \in \mathbb{C}^2\) and treat the same problem for
\[
\rho(0) = |(\beta, \alpha)\rangle \langle (\beta, \alpha)|.
\]
They are important and interesting problems, and we leave them to readers.

In the paper [1] we constructed the general solution of the quantum damped harmonic oscillator in the operator algebra level. It is given by some complicated infinite series, so it is desirable to sum up some solution with a special initial value in a compact form.

In this paper the compact form of the solution with the initial value based on coherent states was given. It is in fact fundamental, so it will be used in Quantum Open System or Quantum Optics in the near future.

However, we could not give a compact form to the solution with the initial value based on squeezed states or coherent–squeezed ones. We leave them to readers.

Lastly, we conclude the paper by stating our motivation. We are studying a model of quantum computation (computer) based on Cavity QED (see [8] and [9]), so in order to construct a more realistic model of (robust) quantum computer we have to study severe problems coming from decoherence. This is our future task.

**Appendix**

In this appendix we review the solution of classical damped harmonic oscillator, which is important to understand the text. See any textbook on Mathematical Physics.

The differential equation is given by

\[
\ddot{x} + \gamma \dot{x} + \omega^2 x = 0 \quad (\gamma > 0)
\]

where \( x = x(t), \dot{x} = dx/dt \) and the mass is set to 1 for simplicity. In the following we treat only the case \( \omega > \gamma/2 \) (the case \( \omega = \gamma/2 \) may be interesting).

The solution is well–known to become

\[
x(t) = e^{-\left(\frac{\gamma}{2} \pm i\sqrt{\omega^2 - (\frac{\gamma}{2})^2}\right)t}x(0)
\]

with complex form. If \( \gamma/2\omega \) is small enough then we have

\[
x(t) \approx e^{-\left(\frac{\gamma}{2} \pm i\omega\right)t}x(0) = x(0)e^{-\left(\frac{\gamma}{2} \mp i\omega\right)t}.
\]
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