A generalized Morse index theorem

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Abstract

In this paper, we prove a Morse index theorem for the index form of even order linear Hamiltonian systems on the closed interval with reasonable self-adjoint boundary conditions. The highest order term is assumed to be nondegenerate.

1 Introduction

1.1 History

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. The classical Morse Index Theorem states that the number of conjugate points along a geodesic \(\gamma : [a, b] \to M\) counted with multiplicities is equal to the index of the second variation of the Riemannian action functional

\[
E(c) = \frac{1}{2} \int_a^b g(\dot{c}, \dot{c}) dt
\]

at the critical point \(\gamma\), where \(\dot{c}\) denotes \(\frac{d}{dt} c\). Such second variation is called the index form for \(E\) at \(\gamma\). The theorem has later been extended in several directions (see \[1, 2, 14, 27, 28, 32, 33\] for versions of this theorem in different contexts). In \[14\] of 1976, J. J. Duistermaat proved his general Morse index theorem for Lagrangian system with positive definite second order term and self-adjoint boundary conditions. In \[1\] of 1996, A. A. Agrachev and A. V. Sarychev studied the Morse index and rigidity of the abnormal sub-Riemannian geodesics. In \[5, 6\] of 1979, J. K. Beem and P. E. Ehrlich considered the semi-Riemannian case. Later in \[19\] of 1994, A. D. Helfer give a generalization. In \[27, 28\] of 2000, P. Piccione and D. V. Tausk proved a version of the Morse index theorem for geodesics in semi-Riemannian geodesics with both endpoints varies on two submanifolds of \(M\) under some nondegenerate conditions (cf. \[28\] Theorem6.4). However, such nondegenerate conditions is very difficult to remove. In \[15\] of 2003, Roberto Giambò, Paolo Piccione, Alessandro Portaluri was able to remove these conditions under the boundary condition of fixed endpoints. Their proof is rather technique and very difficult to generalize. In \[35\] of 2001, the author is able to solve these difficulty. However, the proof is rather technique and hard to follow. It is not clear how the author perturbs a given path of Fredholm self-adjoint operators to make it with only regular crossings in the degenerate case. In \[15\] of 1964, the higher even order case is considered by H. Edwards. He proved a

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1.2 Set up for regular Lagrangian systems

Let $M$ be a smooth manifold of dimension $n$, points in its tangent bundle $TM$ will be denoted by $(m, v)$, with $m \in M$, $v \in T_mM$. Let $f$ be a real-valued $C^3$ function on an open subset $Z$ of $\mathbb{R} \times TM$. Then

$$E(c) = \int_0^T f \left( t, c(t), \dot{c}(t) \right) dt$$

defines a real-valued $C^2$ function $E$ on the space of curves

$$\mathcal{C} = \left\{ c \in C^1([0, T], M); (t, c(t), \dot{c}(t)) \in Z \text{ for all } t \in [0, T] \right\}.$$  \hspace{1cm} (2)

Equipped with the usual topology of uniform convergence of the curves and their derivatives, the set $\mathcal{C}$ has a $C^2$ Banach manifold structure modelled on the Banach space $C^1([0, T], \mathbb{R}^2)$.

Boundary conditions will be introduced by restriction $E$ to the set of curves

$$\mathcal{C}_N = \{ c \in \mathcal{C}; (c(0), c(T)) \in N \},$$ \hspace{1cm} (3)

where $N$ is a given smooth submanifold of $M \times M$. The most familiar examples are $N = \{ m(0), m(T) \}$ and $N = \{ (m_1, m_2) \in M \times M; m_1 = m_2 \}$. In the general case $\mathcal{C}_N$ is a smooth submanifold of $\mathcal{C}$ with its tangential space equal to

$$T_c \mathcal{C}_N = \left\{ \delta c \in C^1([0, T], c^*TM); (\delta c(0), \delta c(T)) \in T_{c(0),c(T)}N \right\}.$$ \hspace{1cm} (4)

c $\in \mathcal{C}_N$ is called a stationary curve for the boundary condition $N$ if the restriction of $E$ to $\mathcal{C}_N$ has a stationary point at $c$, i.e., if $D_E(c)(\delta c) = 0$ for all $\delta c \in T_c \mathcal{C}_N$. For such a curve $c$ is of class $C^2$.

Let $c \in \mathcal{C}_N$ be a stationary curve for the boundary condition $N$. Then the second order differential $D^2E(c)$ of $E$ at $c$ is symmetric bilinear form on $T_c \mathcal{C}_N$, which is called the index form of $E$ at $c$ with respect to the boundary condition $N$. We want to understand the Morse index of this form, i.e., the maximal dimension of negative definite subspace of the space $T_c \mathcal{C}_N$ for the form $D^2E(c)$. In general the Morse index of the form $D^2E(c)$ on $T_c \mathcal{C}_N$ will be infinite. In order to get a well-defined integer, we introduce the following concept.

Assume that $f$ is a regular Lagrangian, that is,

$$D_t^2 f(t, m, v) \text{ is nondegenerate for all } (t, m, v) \in Z.$$ \hspace{1cm} (5)

Here $D_v$ denotes differential of functions on $Z$ with respect to $v \in T_mM$, keeping $t$ and $m$ fixed. The condition (5) is called the Legendre condition.

Let $H = H^1(T_c \mathcal{C}_N)$ be the $H^1$ completion of $T_c \mathcal{C}_N$. By Sobolev embedding theorem, $H \subset C([0, T], c^*TM)$. Then $D^2E(c)$ is well-defined on $H$. In local coordinates, we have

$$D^2E(c)(X, Y) = \int_0^T \left( D_t^2 f(\tilde{c}(t))(\alpha, \beta) + D_m D_v f(\tilde{c}(t)) (\alpha, \beta) 
\right. 
+ \left. D_v D_m f(\tilde{c}(t)) (\alpha, \beta) + D_m^2 f(\tilde{c}(t))(\alpha, \beta) \right) dt,$$ \hspace{1cm} (6)
where \(X, Y \in H\), \(\alpha, \beta\) are the local coordinate expression of \(X, Y\) defined by \(X = (\alpha, \partial m)\), \(Y = (\beta, \partial m)\), \(\partial m\) is the natural frame of \(T_m M\), and we use the abbreviation

\[
\tilde{c}(t) = (t, c(t), \dot{c}(t)).
\]

In general \(\partial m\) and \(\alpha\) is not globally well-defined along the curve \(c\). Choose a \(C^1\) frame \(e\) of \(T_c \mathcal{C}_N\). Such a frame can be obtained by the parallel transformation of the induced connection on \(c^* TM\) of a connection on \(TM\) (for example, the Levi-Civita connection with respect to the semi-Riemannian metric on \(M\)). Then in local coordinates, there is a \(C^1\) path \(a(t) \in \text{GL}(n, \mathbb{R})\) such that \(\partial m\) at \(c(t)\) is the pairing \((a(t), e(t))\). Note that \(a(t)\) is only locally defined in general. Then the vector fields \(X, Y \in H\) along \(c\) can be written as \(X = (x, e), Y = (y, e)\), where \(x, y \in H^1([0, T], \mathbb{R}^n)\) and \(((x(0), x(T)), (y(0), y(T))) \in R\), \(R\) is defined by

\[
R = \left\{(x, y) \in \mathbb{R}^{2n}; ((x, e(0)), (y, e(T))) \in T_{(c(0), c(T))} \mathcal{N}\right\}.
\]

So we have

\[
\begin{align*}
x &= a\alpha, & \dot{x} &= a\dot{\alpha} + \dot{a}\alpha, \\
y &= a\beta, & \dot{y} &= a\dot{\beta} + \dot{a}\beta.
\end{align*}
\]

Substitute (7) to (6), we get the following form of the index form:

\[
D^2 E(c)(X, Y) = \int_0^T \left( (p\dot{x} + qx, \dot{y}) + (q^* \dot{x}, y) + (r x, y) \right) dt,
\]

where \(p, q, r \in C([0, T], \text{gl}(n, \mathbb{R}))\), \(p\) is of class \(C^1\), \(p(t) = p^*(t), r(t) = r^*(t), p(t)\) are invertible for all \(t \in [0, T]\), and \(*\) denotes the conjugate transpose.

Now define

\[
\mathcal{I}_{s,R}(x, y) = \int_0^T \left( (p\dot{x} + sqx, \dot{y}) + (sq^* \dot{x}, y) + (sr x, y) \right) dt, \quad s \in [0, 1],
\]

where \(x, y \in H^1([0, T], \mathbb{R}^n)\) and \(((x(0), x(T)), (y(0), y(T))) \in R\). Since \(p\) is of class \(C^1\) and \(p(t)\) are nondegenerate, we can associate the path \(\mathcal{I}_{s,R}\) with a well-defined finite integer, the spectral flow \(\text{sf}\{\mathcal{I}_{s,R}\}\). Then we can define the relative Morse index \(I(\mathcal{I}_{0,R}, \mathcal{I}_{1,R})\) to be \(-\text{sf}\{\mathcal{I}_{s,R}\}\). When \(p\) is positive definite, \(I(\mathcal{I}_{0,R}, \mathcal{I}_{1,R})\) is the Morse index of \(D^2 E(c)\). Note that the forms \(\mathcal{I}_{s,R}\) will depend on the choice of the frame \(e\).

### 1.3 The highlights of the paper

This paper can be viewed as the revised version of [35]. In this paper, we will prove a general version of Morse index theorem for the index form of even order linear Hamiltonian systems on the closed interval with reasonable selfadjoint boundary conditions. The highest order term is assumed to be nondegenerate. As a special case, we prove the Morse index theorem for regular Lagrangian system with selfadjoint boundary conditions. Note that the index form (see (7) below) will takes different forms under different choices of the frames \(e\). Then we show how the indices varies under such choices.

Our approach is inspired by the recent papers [9][10] of B. Booss-Bavnbek and the author. We do not use perturbation method. Our main results can be viewed as pretty much simple restatement of [28] Theorem 6.4 and [18] Theorem 4.9 in their cases. Our index theorem does not contain any assumption on nondegeneracy for the index form. Moreover, we consider the
spectral flow of the paths connected two given index forms. The index forms in such a path is in general not a compact perturbation of a given index form. Such phenomenal occurs when we consider the connected trajectories between two geodesics on the manifold. These highlights make it easy to apply our index theorem in the variational problems.

Our paper is arranged as follows. In §1, we give the background of the problem. In §2, we state our main results. In §3, we discuss the properties of the spectral flow. In §4, we discuss the properties of the Maslov indices. In §5, we prove our main results. In this paper, dim denotes the complex dimension if no special description.

2 Main results

We shall consider the general case of even order linear Hamiltonian systems. We will consider the complex case. The real case is a obvious consequence of the complex case.

Let \(m, n \in \mathbb{Z}^+\) be positive integers, and \(T \in \mathbb{R}^+\) be a positive real number. Let \(p_{k,l}(s,t) \in \text{gl}(n, \mathbb{C}), (s,t) \in [0,1] \times [0,T]\) be \((m+1)^2\) continuous families of matrices, where \(k, l = 0, \ldots, m\). Assume that for all \((s,t) \in [0,1] \times [0,T]\), \(p_{s}(t) = (p_{m-k,m-l}(s,t))_{k,l=0,\ldots,m}) \in \text{gl}((m+1)n, \mathbb{C})\) are selfadjoint, and \(p_{m,m}(s,t)\) are nondegenerate. Assume further that for all \(s \in [0,1]\) and \(k, l = 0, \ldots, m\), \(p_{k,l}(s, \cdot) \in C^{\max\{k,l\}}([0,T], \text{gl}(n, \mathbb{C})\). Then we have a continuous family of quadratic forms

\[
\mathcal{I}_s(x, y) = \int_0^T \left( \sum_{k,l=0}^m (p_{k,l}(s,t) \frac{d^k}{dt^k} x, \frac{d^k}{dt^k} y) \right) dt, \quad \forall x, y \in H^m([0,T]; \mathbb{C}^n). \tag{10}
\]

Here \(\langle \cdot, \cdot \rangle\) denotes the standard Hermitian inner product in \(\mathbb{C}^n\), and the norm of the Sobolev space \(H^m([0,T]; \mathbb{C}^n)\) is defined by

\[
\langle x, y \rangle_m = \int_0^T \left( \sum_{k=0}^m \left( \frac{d^k x}{dt^k}, \frac{d^k y}{dt^k} \right) \right) dt, \quad \forall x, y \in H^m([0,T]; \mathbb{C}^n).
\]

Then we define the boundary condition. Let \(R \subset \mathbb{C}^{2mn}\) be a given linear subspace. Define

\[
H_R = \left\{ x \in H^m([0,T]; \mathbb{C}^n); (\frac{d^{m-1}}{dt^{m-1}} x(0), \ldots, x(0), \frac{d^{m-1}}{dt^{m-1}} x(T), \ldots, x(T)) \in R \right\}. \tag{11}
\]

Let \(\mathcal{I}_{s,R}\) be the restriction of \(\mathcal{I}_s\) to \(H_R\). The central problem in this paper is to understand the Morse index of the form \(\mathcal{I}_{1,R}\), i.e. the maximal dimension of negative definite subspace of the form \(\mathcal{I}_{1,R}\). As in \([1,2]\) we shall use the minus spectral flow \(-\text{sf}\{\mathcal{I}_{s,R}\}\) as the "difference" between the "Morse indices" of the forms \(\mathcal{I}_{1,R}\) and \(\mathcal{I}_{0,R}\).

Let \(L_s\) be the unbounded operator on \(L^2([0,T]; \mathbb{C}^n)\) with domain \(H^{2m}([0,T]; \mathbb{C}^n)\) defined by

\[
(L_s x)(t) = \sum_{k,l=0}^m (-1)^k \frac{d^k}{dt^k} \left( p_{k,l}(s,t) \frac{d^l}{dt^l} x(t) \right), \quad \forall x \in H^{2m}([0,T]; \mathbb{C}^n). \tag{12}
\]

Define \(R^{2m,h}\) and \(W_{2m}(R)\) by

\[
R^{2m,h} = \left\{ (x_1, \ldots, x_{2m}) \in \mathbb{C}^{2mn}; \sum_{k=1}^m (-1)^{k-1} \langle x_k, y_{m-k+1} \rangle \right\}
\]

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\[ \frac{2m}{k=m+1} (-1)^{k-m} \langle x_k, y_{3m-k+1} \rangle = 0 \text{ for all } (y_1, \ldots, y_{2m}) \in R \], \quad \text{(13)}

\[ W_{2m}(R) = \left\{ (x_1, x_2, x_3, x_4) \in C^{4mn}; x_1, x_2, x_3, x_4 \in C^{mn}, (x_1, x_3) \in R^{2m}, (x_2, x_4) \in R \right\} \quad \text{(14)}

For each \( x \in H^{2m}([0, T]; C^n) \), let \( u_{ps,x} \in H^1([0, T]; C^{2mn}) \), \( \bar{u}_{ps,x} \) and \( u^k_{ps,x} \), \( k = 0, \ldots, 2m \) be defined by

\[ u_{ps,x}(t) = (u^m_{ps,x}(t), \ldots, u^0_{ps,x}(t)), \]

\[ u^k_{ps,x}(t) = \frac{d^k}{dt^k} x(t), \quad k = 0, \ldots, m - 1, \]

\[ u^k_{ps,x}(t) = \sum_{2m-k \leq \alpha \leq m, 0 \leq \beta \leq m} (-1)^{\alpha-m} \frac{d^{\alpha+k-2m}}{dt^{\alpha+k-2m}} \left( p_{\alpha,\beta}(s, t) \frac{d^\beta}{dt^\beta} x(t) \right), \quad k = m, \ldots, 2m. \quad \text{(15)} \]

Let \( L_{s, W_{2m}(R)} \) be the restriction of \( L_s \) on the domain

\[ \{ x \in H^{2m}([0, T]; C^n); (u_{ps,x}(0), u_{ps,x}(T)) \in W_{2m}(R) \}. \]

By Lemma 3.5 of [10], \( L_{s, W_{2m}(R)} \), \( 0 \leq s \leq 1 \) is a continuous family (in the gap norm sense) of unbounded selfadjoint Fredholm operators. Again we associate the path with the minus spectral flow \( -s \{ L_{s, W_{2m}(R)} \} \).

Let \( J_{2m,n} \in GL(2mn, C) \) be the matrix \((j_{k,l})_{k,l=0,\ldots,2m-1}\), where \( j_{k,l} = 0_n \) for \( k+l \neq 2m-1 \), \( j_{k,l} = (-1)^{k+m} I_n \) for \( k+l = 2m-1 \), and we denote by \( I_n \) and \( 0_n \) the identity matrix and the zero matrix on \( C^n \) respectively. When there is no confusion, we will omit the subindex \( n \) of \( I_n \) and \( 0_n \). Set

\[ \bar{u}_{ps,x} = (u^m_{ps,x}, \ldots, u^0_{ps,x}), \quad \bar{u}_{0,x} = \left( \frac{d^m}{dt^m}, \ldots, x \right). \]

From [15], we can define the matrices \( U(p_s(t)) \) and \( V(p_s(t)) \) for each \( (s, t) \in [0, 1] \times [0, T] \) by

\[ \bar{u}_{ps,x}(t) = U(p_s(t)) \bar{u}_{0,x}(t), \quad \bar{u}_{0,x}(t) = V(p_s(t)) \bar{u}_{ps,x}(t). \quad \text{(16)} \]

Let \( \Theta_{2m,n} \in gl(2mn, C) \) be the matrix \((\theta_{k,l})_{k,l=0,\ldots,2m-1}\), where \( \theta_{k,l} = 0_n \) for \( k+l \neq 2m-2 \) or one of \( k = l = m - 1 \), \( \theta_{k,l} = (-1)^{k+m+1} I_n \) for \( k+l = 2m-2 \) and \( k,l \neq m-1 \). For each \( (s, t) \in [0, 1] \times [0, T] \), define the matrices \( P(p_s(t)) \) and \( b(p_s(t)) \) in \( gl((m+1)n, C) \) by

\[ P(p_s(t)) = (P_{k,l}(s, t))_{k,l=0,\ldots,m} \quad \text{(17)} \]

\[ b(p_s(t)) = \Theta_{2m,n} + \text{diag}(0_{(m-1)n}, P(p_s(t))). \quad \text{(18)} \]

where

\[ P_{0,0}(s, t) = p_{m,m}(s, t)^{-1}, \]

\[ P_{0,l}(s, t) = -p_{m,m}(s, t)^{-1} p_{m,m-l}(s, t), \]

\[ P_{k,0}(s, t) = -p_{m-k,m}(s, t) p_{m,m}(s, t)^{-1}, \]

\[ P_{k,l}(s, t) = p_{m-k,m-l}(s, t) - p_{m-k,m}(s, t) p_{m,m}(s, t)^{-1} p_{m,m-l}(s, t) \]

for \( k, l = 1, \ldots, m \), and we denote by \( A^* \) the conjugate transpose of \( A \). For each \( s \in [0, 1] \), let \( \gamma_{ps}(t) \) be the fundamental solution of the linear Hamiltonian system

\[ \dot{u} = J_{2m,n} b(p_s(t)) u. \quad \text{(19)} \]
Then $\gamma_{p_s}(t)$ are symplectic matrices. Then we can associate the symplectic path $\gamma_{p_s}(t)$, $0 \leq t \leq T$ with the Maslov-type index $i_{W^m(R)}(\gamma_{p_s})$ for each $s \in [0, 1]$.

We want to address the following problems for even order case in this paper:

- give the relationship between the integers $-\text{sf}\{I_{s,R}\}, -\text{sf}\{L_s,W_{2m}(R)\}$ and $i_{W_{2m}(R)}(\gamma_{p_s})$ for $0 \leq s \leq 1$;
- calculate $i_{W_{2m}(R)}(\gamma_{p_0})$ for $p_0(t) = \text{diag}(p_{0,0}(0,t),0_{mn})$;
- for two different choices of the frame $e$, the resulted index form $I_{s,R}$ defined by $(\ref{frame})$ will have different forms. In this case, calculate the difference between the resulted integers $i_{W_{2}(R)}(\gamma_{p_1})$.

The following three theorems solve the above problems.

**Theorem 2.1** Let $\text{sf}\{I_{s,R}\}, 0 \leq s \leq 1\}$ be the spectral flow of $I_{s,R}$, $\text{sf}\{L_s,W_{2m}(R)\}, 0 \leq s \leq 1\}$ be the spectral flow of $L_s,W_{2m}(R)$, and $i_{W_{2m}(R)}(\gamma_{p_s})$ be the Maslov-type index of $\gamma_{p_s}$ defined below. Then we have

$$-\text{sf}\{I_{s,R}\}, 0 \leq s \leq 1\} = -\text{sf}\{L_s,W_{2m}(R)\}, 0 \leq s \leq 1\} = i_{W_{2m}(R)}(\gamma_{p_1}) - i_{W_{2m}(R)}(\gamma_{p_0}).$$

Assume that $p_0(t) = \text{diag}(p_{m,m}(0,t),0_{mn})$ for all $t \in [0,T]$. Then we have $(P(p_0))(t) = (p_0(t))^{-1}$, $b(p_0)(t) = (b_{m,l}(t))_{k,l=0,...,2m-1}$, and $\gamma_{p_0}(t) = (\gamma_{k,l}(t))_{k,l=0,...,2m-1}$, where $b_{k,l}(t) = 0_n$ for $k - l \neq 1$, $b_{k,l}(t) = I_n$ for $k - l = 1$ and $k \neq m$, $b_{m,m-1}(t) = (p_{m,m}(0,t))^{-1}$, $\gamma_{k,l}(t) = 0$ for $k < l$, $\gamma_{k,l}(t) = \frac{1}{(k-l)!}I_n$ for $k \geq l$ and $l < m - 1$, or $k \geq l$ and $l \geq m$, and

$$\gamma_{k,l}(t) = \frac{1}{(m-l-1)!} \int_0^t dt_{k-m} \int_0^{t_{k-m}} dt_{k-m-1} \cdots \int_0^{t_{m-l-1}} t_{m-l}^{m-l-1} (p_{m,m}(0,t_0))^{-1} dt_0$$

for $k \geq m$ and $l \leq m - 1$.

The form of our symplectic path $\gamma_{p_0}(t)$ looks rather complicated. We will consider the following more general situation to simplify our problem.

Let $K \in \text{GL}(n, \mathbb{C})$. Set $J_K = \begin{pmatrix} 0 & -K^* \\ K & 0 \end{pmatrix}$. Then $(\mathbb{C}^{2n}, \langle J_K \cdot, \cdot \rangle)$ is a symplectic space. Let

$$\gamma(t) = \begin{cases} (M_{1,1}(t) & 0 \\ M_{2,1}(t) & M_{2,2}(t) \end{cases}, 0 \leq t \leq T \text{ be a path in } \text{GL}(2n, \mathbb{C}) \text{ with } M_{2,2}(t)^*KM_{1,1}(t) = K \text{ and } M_{1,1}(t)^*KM_{2,1}(t) \text{ self-adjoint for each } t \in [0,T]. \text{ Then } \gamma(t) \text{ is a symplectic path, i.e., } \gamma(t)^*J_K\gamma(t) = J_K. \text{ Let } R \subset \mathbb{C}^{2mn} \text{ be a given linear subspace. Define } R^K \text{ and } W_K(R) \text{ by}$$

$$R^K = \begin{cases} (x, y) \in \mathbb{C}^{2n}; \langle Kx_1, y_1 \rangle - \langle Kx_2, y_2 \rangle = 0 \text{ for all } (y_1, y_2) \in R \end{cases}, \quad W_K(R) = \begin{cases} (x, y) \in \mathbb{C}^{4n}; x_1, x_2, x_3, x_4 \in \mathbb{C}^n, (x_1, x_3) \in K^K, (x_2, x_4) \in R \end{cases}. \quad (21)$$

**Theorem 2.2** For the symplectic path $\gamma$ and the Lagrangian space $W_K(R)$ defined above, we have

$$\dim(\text{Gr}(\gamma(t)) \cap W_K(R)) = \dim \ker \left( (M_{1,1}(T)^*KM_{2,1}(t))|_{S(t)} \right) + \dim S(t) + \dim(\text{Gr}(I_{mn}) \cap R) - \dim(\text{Gr}(I_{mn}) \cap R^K), \quad (23)$$
where $m^+$ denotes the Morse positive index, and
\[ S(t) = \{ x \in \mathbb{C}^n; (x, M_{1,1}(t)x) \in R^K \}. \]

In our case, set $K_{m,n} = (k_{l,l})_{k,l=0,...,m-1}$, where $k_{k,l} = 0_n$ for $k \neq l$, $k_{k,l} = (-1)^l I_n$ for $k = l$. Then we have $R^{K_{m,n}} = R^{2m,b}$ and $W_{2m}(R) = W_{K_{m,n}}(R)$. Moreover for the symplectic path $\gamma = \gamma_p(t)$, we have
\begin{equation}
M_{1,1}(T)^* K^* M_{2,1}(T) = \left( \frac{1}{(m-k-1)!(m-l-1)!} \int_0^T t^{2m-k-l-2} (p_{m,m}(0,t))^{-1} dt \right)_{k,l=0,...,m-1}.
\end{equation}

As a special case, we get the following higher order generalization of theorem of J. J. Duistermaat [14].

**Corollary 2.1** Assume that $p_{m,m}(1,t)$ is positive definite for each $t \in [0,T]$. Set $p_0(t) = \text{diag}(p_{m,m}(1,t),0_{mn})$. Then we have
\begin{equation}
m^-(\mathcal{I}_{1,R}) = m^-(L_{1,2m}(R)) = i_{W_{2m}(R)}(\gamma_{p_1}) - \dim S,
\end{equation}
where $m^-$ denotes the Morse (negative) index, and
\[ \gamma_{p_0}(t) = \begin{pmatrix} M_{1,1}(t) & 0 \\ M_{2,1}(t) & M_{2,2}(t) \end{pmatrix}, \]
\[ S = \{ x \in \mathbb{C}^{mn}; (x, x) \in R^{2m,b} \}. \]

Now we consider the third problem. Then $m = 1$ and everything is real. Let $a(t)$ be a $C^1$ path in $\text{GL}(n, \mathbb{R})$, and
\[ R' = \{ (x, y) \in \mathbb{R}^{2n}; (a(0)x, a(T)y) \in R \}. \]
After the change of the frame $e \mapsto a_x^{-1}e$, we have $x \mapsto ax$ and the quadratic form $\mathcal{I}_{1,R}$ is changed to the restriction of the form $\mathcal{I}_1(ax, ay)$ on $H_{r'}. Then we get the corresponding $p', q'$ and $r'$. Set $p_1 = \begin{pmatrix} p & q \\ q^* & r \end{pmatrix}$ and $p'_1 = \begin{pmatrix} p & q' \\ (q')^* & r' \end{pmatrix}$. Let $\gamma_{p_1}$ and $\gamma'_{p_1}$ be defined by [19]. Then we can prove
\begin{equation}
\gamma'_{p_1} = \text{diag}(a^*, a^{-1}) \gamma_{p_1} \text{diag}(a(0)^{*1}, a(0)).
\end{equation}

**Theorem 2.3** Let $a(t), 0 \leq t \leq T$ be a path in $\text{GL}(n, \mathbb{C})$, and
\[ R' = \{ (x, y) \in \mathbb{C}^{2n}; (a(0)x, a(T)y) \in R \}. \]
Let $\gamma$ be a symplectic path, i.e., $\gamma(t)^* J_{2,n} \gamma(t) = J_{2,n}$ for all $0 \leq t \leq T$. Define the symplectic path $\gamma'$ by
\begin{equation}
\gamma' = \text{diag}(a^*, a^{-1}) \gamma_{p_1} \text{diag}(a(0)^{*1}, a(0)).
\end{equation}
Then we have
\begin{equation}
i_{W_{2}(R')}(\gamma') - i_{W_{2}(R)}(\gamma) = \dim(\text{Gr}(I_n) \cap (R')^{2,b}) - \dim(\text{Gr}(I_n) \cap R^{2,b}).
\end{equation}
3 Spectral flow

3.1 Definition of the spectral flow

Roughly speaking, the spectral flow counts the net number of eigenvalues changing from the negative real half axis to the non-negative one. The definition goes back to a famous paper by M. Atiyah, V. Patodi, and I. Singer [4], and was made rigorous by J. Phillips [26] for continuous paths of bounded self-adjoint Fredholm operators, by C. Zhu and Y. Long [36] in various non-self-adjoint cases, and by B. Booss-Bavnbek, M. Lesch, and J. Phillips [8] in the unbounded self-adjoint case.

Let $X$ be a complex Hilbert space. For a self-adjoint Fredholm operator $A$ on $X$, there exists a unique orthogonal decomposition

$$X = X^+(A) \oplus X^0(A) \oplus X^-(A)$$

such that $X^+(A)$, $X^0(A)$ and $X^-(A)$ are invariant subspaces associated to $A$, and $A|_{X^+(A)}$, $A|_{X^0(A)}$ and $A|_{X^-(A)}$ are positive definite, zero and negative definite respectively. We introduce vanishing, natural, or infinite numbers

$$m^+(A) := \dim X^+(A), \quad m^0(A) := \dim X^0(A), \quad m^-(A) := \dim X^-(A),$$

and call them Morse positive index, nullity and Morse index of $A$ respectively. For finite-dimensional $X$, the signature of $A$ is defined by $\text{sign}(A) = m^+(A) - m^-(A)$ which yields an integer. The APS projection $Q_A$ (where APS stands for Atiyah-Patodi-Singer) is defined by

$$Q_A(x^+ + x^0 + x^-) := x^+ + x^0,$$

for all $x^+ \in X^+(A), x^0 \in X^0(A), x^- \in X^-(A)$.

Let $\{A_s\}, 0 \leq s \leq 1$ be a continuous family of self-adjoint Fredholm operators. The spectral flow $\text{sf}\{A_s\}$ of the family should be equal to $m^-(A_0) - m^-(A_1)$ if $\dim X < +\infty$. We will generalize this definition to general Banach space $X$ and general continuous family of admissible operators defined below.

Let $X$ be a complex Banach space. We denote the set of closed operators, bounded linear operators and compact linear operators on $X$ by $\mathcal{C}(X)$, $\mathcal{B}(X)$ and $\mathcal{CL}(X)$ respectively. We will denote the spectrum, the regular set and the domain of an operator $A \in \mathcal{C}(X)$ by $\sigma(A), \rho(A)$ and $\text{dom}(A)$ respectively. Let $N$ be an bounded open subset of $\mathbb{C}$ and $A \in \mathcal{C}(X)$. If there exists an bounded open subset $\tilde{N} \subset N$ with $C^1$ boundary $\partial \tilde{N}$ such that $\partial \tilde{N} \cap \sigma(A) = \emptyset$ and $N \cap \sigma(A) \subset \tilde{N}$, we define the spectral projection $P(A,N)$ by

$$P(A,N) := -\frac{1}{2\pi i} \int_{\partial \tilde{N}} (A - \zeta I)^{-1} d\zeta.$$
• and a continuous family (in the gap norm sense) of admissible operators $A_s$, $0 \leq s \leq 1$ in $\mathcal{A}_\ell(X)$.

Here we define $A \in C(X)$ to be admissible with respect to $\ell$, if there exists a bounded open neighbourhood $N$ of $\ell$ in $C$ with $C^1$ boundary $\partial N$ such that (i) $\partial N \cap \sigma(A) = \emptyset$; (ii) $N \cap \sigma(A) \subset \ell$ is a finite set; and (iii) $P_0^\ell(A) := P(A, N)$ is a finite rank projection.

We call $h_{k, \ell}(A) := \dim \ker P_0^\ell(A)$ the hyperbolic nullity of $A$ with respect to $\ell$. We denote by $\mathcal{A}_\ell(X)$ the set of closed admissible operators with respect to $\ell$. It is an open subset of $C(X)$.

**Example 3.1**

a) In the self-adjoint case, $\ell = \sqrt{-1}(\pm \varepsilon)$ ($\varepsilon > 0$) with co-orientation from left to right. Then a self-adjoint operator $A$ is admissible with respect to $\ell$ if and only if $A$ is Fredholm.

b) Another important case is that $\ell = (1 - \varepsilon, 1 + \varepsilon)$ ($\varepsilon \in (0, 1)$) with co-orientation from downward to upward, and all $A_s$ unitary. A unitary operator $A$ is admissible with respect to $\ell$ if and only if $A - I$ is Fredholm.

Similarly as the definition in [26, 36], we can define the spectral flow $\text{sf}_\ell \{A_s\}$ as follows. It counts the number of spectral lines of $A_s$ coming from the negative side of $\ell$ to the non-negative side of $\ell$.

For each $t \in [0, 1]$, there exist bounded open subsets $N_t, N_{t^\pm}$ of $C$ such that $\sigma(A_t) \cap \partial N_t = \emptyset$, $\sigma(A_t) \cap \partial N_{t^\pm} \subset N_t \cap \ell$, $N_{t^+} = N_{t^-} \cup (N_t \cap \ell) \cup N_{t^-}$, $N_{t^+}$ stays in the positive (negative) side of $\ell$ near $N_t \cap \ell$, and $P(A_t, N_t)$ is a finite rank projection. Here we denote by $\bar{\ell}$ the closure of $\ell$ in $C$.

Then $\sigma(A_t) \cap \partial N_t \cup (\bar{\ell} \setminus (N_t \cap \ell)) = \emptyset$. The set $(\partial N_t \cup (\bar{\ell} \setminus (N_t \cap \ell)))$ is compact since it is a bounded closed set. Since the family $\{A_s\}$, $0 \leq s \leq 1$ is continuous, there exists a $\delta(t) > 0$ for each $t \in [0, 1]$ such that

$$\sigma(A_s) \cap (\partial N_t \cup (\bar{\ell} \setminus (N_t \cap \ell))) = \emptyset \quad \text{for all } s \in (t - \delta(t), t + \delta(t)) \cap [0, 1].$$

Then $\sigma(A_s) \cap \ell \subset N_t \cap \ell$, and

$$\{P(A_s, N_t)\}_{s \in (t - \delta(t), t + \delta(t)) \cap [0, 1]}$$

is a continuous family of projections. By Lemma I.4.10 in Kato [21], the operators in the family have the same rank. Since $[0, 1]$ is compact, there exist a partition $0 = s_0 < \ldots < s_n = 1$ and $t_k \in [s_k, s_{k+1}]$, $k = 0, \ldots, n - 1$ such that $[s_k, s_{k+1}] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for each $k = 0, \ldots, n - 1$.

**Definition 3.1** Let $\ell \in \mathcal{A}(C)$ be admissible and let $\{A_s\}$, $0 \leq s \leq 1$ be a curve in $\mathcal{A}_\ell(X)$. The spectral flow $\text{sf}_\ell \{A_s\}$ of the family $\{A_s\}$, $0 \leq s \leq 1$ with respect to the curve $\ell$ is defined by

$$\text{sf}_\ell \{A_s\} = \sum_{k=0}^{n-1} \left( \dim \ker P(A_{s_k}, N_{t_k}^-) - \dim \ker P(A_{s_{k+1}}, N_{t_k}^-) \right). \quad (31)$$

The spectral flow has the following properties (cf. [26] and Lemma 2.6 and Proposition 2.2 in [36]).

**Proposition 3.1** Let $\ell \in \mathcal{A}(C)$ be admissible and let $\{A_s\}$, $0 \leq s \leq 1$ be a curve in $\mathcal{A}_\ell(X)$. Then the spectral flow $\text{sf}_\ell \{A_s\}$ is well-defined, and the following properties hold:
(i) **Catenaion.** Assume $t \in [0, 1]$. Then we have
\[ \text{sf}_\ell\{A_s; 0 \leq s \leq t\} + \text{sf}_\ell\{A_s; t \leq s \leq 1\} = \text{sf}_\ell\{A_s; 0 \leq s \leq 1\}. \] (32)

(ii) **Homotopy invariance.** Let $A(s, t), (s, t) \in [0, 1] \times [0, 1]$ be a continuous family in $\mathcal{A}_\ell(X)$. Then we have
\[ \text{sf}_\ell\{A(s, t); (s, t) \in \partial([0, 1] \times [0, 1])\} = 0. \] (33)

(iii) **Endpoint dependence for Riesz continuity.** Let $B^{sa}(X)$, respectively $C^{sa}(X)$ denote the spaces of bounded, respectively closed self-adjoint operators in $X$. Let
\[ R : C^{sa} \to B^{sa}(X) \]
\[ A \mapsto A(A^2 + I)^{-\frac{1}{2}} \]
denote the Riesz transformation. Let $A_s \in C^{sa}(X)$ for $s \in [0, 1]$. Assume that $\{R(A_s)\}$ is a continuous family. If $m^-(A_0) < +\infty$, then $m^-(A_1) < +\infty$ and we have
\[ \text{sf}\{A_s\} = m^-(A_0) - m^-(A_1). \] (34)

(iv) **Product.** Let $\{P_s\}$ be a curve of projections on $X$ such that $P_s A_s \subset A_s P_s$ for all $s \in [0, 1]$. Set $Q_s = I - P_s$. Then we have $P_s A_s P_s \in \mathcal{A}_\ell(\text{im} P_s) \subset C(\text{im} P_s)$, $Q_s A_s Q_s \in \mathcal{A}_\ell(\text{im} Q_s) \subset C(\text{im} Q_s)$, and
\[ \text{sf}_\ell\{A_s\} = \text{sf}_\ell\{P_s A_s P_s\} + \text{sf}_\ell\{Q_s A_s Q_s\}. \] (35)

(v) **Bound.** For $A \in \mathcal{A}_\ell(X)$, there exists a neighbourhood $\mathcal{N}$ of $A$ in $C(X)$ such that $\mathcal{N} \subset \mathcal{A}_\ell(X)$, and for curves $\{A_s\}$ in $\mathcal{N}$ with endpoints $A_0 =: A$ and $A_1 =: B$, the relative Morse index $I_\ell(A, B) := -\text{sf}_\ell\{A_s; 0 \leq s \leq 1\}$ is well defined and satisfies
\[ 0 \leq I_\ell(A, B) \leq \nu_{h, \ell}(A) - \nu_{h, \ell}(B). \] (36)

(vi) **Reverse orientation.** Let $\hat{\ell}$ denote the curve $\ell$ with opposite co-orientation. Then we have
\[ \text{sf}_\ell\{A_s\} + \text{sf}_{\hat{\ell}}\{A_s\} = \nu_{h, \ell}(A_1) - \nu_{h, \ell}(A_0). \] (37)

(vii) **Zero.** Suppose that $\nu_{h, \ell}(A_s)$ is constant for $s \in [0, 1]$. Then $\text{sf}_\ell\{A_s\} = 0$.

(viii) **Invariance.** Let $\{T_s\}_{s \in [0, 1]}$ be a curve of bounded invertible operators. Then we have
\[ \text{sf}_\ell\{T_s^{-1} A_s T_s\} = \text{sf}_\ell\{A_s\}. \] (38)

**Proof.** We shall only prove the spectral flow is well-defined. The proof for the rest of the proposition is the same as that in [26] and Lemma 2.6 and Proposition 2.2 in [36] and is omitted.

Since two different partitions of $[0, 1]$ has a common refinement, we only need to prove the following local result:

**Claim.** Let $N_1, N_1^\pm, l = 1, 2$ be open subsets in $C$. Assume that for all $s \in [0, 1]$ and $l = 1, 2$, we have $\sigma(A_s) \cap \partial N_l = \emptyset$, $\sigma(A_s) \cap \tilde{\ell} \subset N_l \cap \ell$, $N_l = N_l^+ \cup (N_l \cap \ell) \cup N_l^-$, $N_l^\pm$ stays in the positive (negative) side of $\ell$ near $N_l \cap \ell$, and $P(A_s, N_l)$ is a finite rank projection. Then we have
\[ \dim \text{im} P(A_0, N_1^-) - \dim \text{im} P(A_1, N_1^-) = \dim \text{im} P(A_0, N_2^-) - \dim \text{im} P(A_1, N_2^+). \]
In fact, our assumptions implies

\[ \sigma(A_s) \cap \partial(N_1^- \setminus N_2^-) = \sigma(A_s) \cap \partial(N_2^- \setminus N_1^-) = \emptyset. \]

Then \( P(A_s, N_1^- \setminus N_2^-) \) and \( P(A_s, N_2^- \setminus N_1^-) \), \( s \in [0, 1] \) are continuous family of projections. By Lemma I.4.10 in Kato \[21\], \( \text{im} \ P(\lambda, N_1^- \setminus N_2^-) \) and \( \text{im} \ P(\lambda, N_2^- \setminus N_1^-) \) are constants. So we have

\[
\begin{align*}
(\dim \text{im} \ P(A_0, N_1^-) - \dim \text{im} \ P(A_1, N_1^-)) &= (\dim \text{im} \ P(A_0, N_2^-) - \dim \text{im} \ P(A_1, N_2^-)) \\
= & (\dim \text{im} \ P(A_0, N_1^-) \setminus N_2^-) - (\dim \text{im} \ P(A_0, N_2^-) \setminus N_1^-)) \\
& - (\dim \text{im} \ P(A_1, N_1^- \setminus N_2^-) - \dim \text{im} \ P(A_1, N_2^- \setminus N_1^-)) \\
= & 0.
\end{align*}
\]

Thus our claim is proved. Q.E.D.

**Remark 3.1** In (iv) of the above proposition, we allow the Banach space \( \text{im} \ P_s \) continuous varying. By \[21\] Lemma I.4.10, for \( t \in [0, 1] \) being close enough to \( s \), there is a continuous family of invertible operators \( U_{s,t} \in \mathcal{B}(X) \) such that

\[ P_t U_{s,t} = U_{s,t} P_s, \quad U_{s,t} \to I, \text{ as } t \to s. \]

So locally we can define the spectral flow of \( B_t \in C(\text{im} P_t) \) as that of \( U_{s,t}^{-1} B_t U_{s,t} : \text{im} P_s \to \text{im} P_s \) (\( s \) fixed), and globally patch them together.

### 3.2 Calculation of the spectral flow

In this subsection we shall give a method of calculating the spectral flow of differentiable curves, inspired among others by J.J. Duistermaat \[14\] and J. Robbin and D. Salamon \[30\].

Let \( X \) be a complex Banach space, \( \tilde{N} \subset \tilde{N} \) be bounded open subsets of \( C \), and \( \gamma \) be a closed \( C^1 \) curve in \( C \) which bounds \( \tilde{N} \). Let \( A_s, s \in (-\epsilon, \epsilon) \), where \( \epsilon > 0 \), be a curve in \( \mathcal{C}(X) \). Assume that \( \gamma \cap \sigma(A_s) = \emptyset \) and \( N \cap \sigma(A_s) \subset \tilde{N} \) for all \( s \in (-\epsilon, \epsilon) \). Set \( A := A_0, P_s := P(A_s, N) \), and \( P := P_0 \). Assume that \( \text{im} \ P \subset \text{dom}(A_s) \) for all \( s \in (-\epsilon, \epsilon) \), \( \text{im} \ P \) is a finitely dimensional subspace of \( X \), and \( d/ds|_{s=0}(A_s P) = B \) (in the bounded operator sense). Let \( f \) be a polynomial. Then \( P_s f(A_s)P_s, s \in (-\epsilon, \epsilon) \) is a continuous family of bounded operators, and

\[
P_s f(A_s)P_s = -\frac{1}{2\pi \sqrt{-1}} \int_{\gamma} f(\zeta)(A - \zeta I)^{-1} d\zeta. \tag{39}
\]

Since \( P_s, s \in (-\epsilon, \epsilon) \) is a continuous family, we have \( \|P_s - P\| < 1 \) if \( |s| \) is small. For such \( s \), set \( R_s = (I - (P_s - P)^2)^{-\frac{1}{2}} \). Since \( P(P_s - P)^2 = (P_s - P)^2 P \) and \( P_s(P_s - P)^2 = (P_s - P)^2 P_s \), we have \( R_s P = P R_s \) and \( R_s P_s = P_s R_s \). Set

\[
U'_s = P_s P + (I - P_s)(I - P), \quad U_s = U'_s R_s, \\
V'_s = P P_s + (I - P)(I - P_s), \quad V_s = V'_s R_s.
\]
Then we have
\[
U_s V_s = V_s U_s = I,
\]
\[
U_s P = P_s U_s = P_s R_s P,
\]
\[
PV_s = V_s P_s = PR_s P_s.
\]

**Lemma 3.1** We have
\[
\frac{d}{ds}|_{s=0}(U_s^{-1} P_s A_s P_s U_s) = \frac{1}{2\pi \sqrt{-1}} \int_\gamma \zeta (A - \zeta I)^{-1} PB (A - \zeta I)^{-1} d\zeta. \tag{40}
\]

If \((PAP)(PB) = (PB)(PAP)\), then we have
\[
\frac{d}{ds}|_{s=0}(P_s P) = 0, \\
\frac{d}{ds}|_{s=0}(U_s^{-1} P_s A_s P_s U_s) = PB. \tag{41}
\]

**Proof.** By the definition of \(U_s\) and \(V_s\) we have
\[
U_s^{-1} P_s A_s P_s U_s = V_s P_s A_s P_s U_s = PR_s P_s A_s P_s R_s P.
\]

By \ref{40} we have
\[
(P_s f(A_s))P_s - Pf(A)P = \frac{1}{2\pi \sqrt{-1}} \int_\gamma f(\zeta)(A_s - \zeta I)^{-1}(A_s P - AP)(A - \zeta I)^{-1} d\zeta. \tag{42}
\]

Since \(A_s, s \in (-\epsilon, \epsilon)\) is a curve in \(C(X)\) and \(im P\) has finite dimension, we have
\[
\frac{d}{ds}|_{s=0}(P_s f(A_s) P_s P) = \frac{1}{2\pi \sqrt{-1}} \int_\gamma f(\zeta)(A - \zeta I)^{-1} B(A - \zeta I)^{-1} d\zeta. \tag{43}
\]

Take \(f = 1\), we have \(\frac{d}{ds}|_{s=0}(P_s P)\) exists. By the definition of \(R_s\) we have \(\frac{d}{ds}|_{s=0}(R_s P) = 0\). Hence we have
\[
\frac{d}{ds}|_{s=0}(U_s^{-1} P_s A_s P_s U_s) = \frac{d}{ds}|_{s=0}(PR_s P_s A_s P_s R_s P) \\
= \frac{d}{ds}|_{s=0}((R_s P)(P_s A_s P_s P)(R_s P)) \\
= \frac{1}{2\pi \sqrt{-1}} \int_\gamma \zeta P(A - \zeta I)^{-1} B(A - \zeta I)^{-1} d\zeta \\
= \frac{1}{2\pi \sqrt{-1}} \int_\gamma \zeta (A - \zeta I)^{-1} PB(A - \zeta I)^{-1} d\zeta.
\]

In the case of \((PAP)(PB) = (PB)(PAP)\), we have
\[
\frac{d}{ds}|_{s=0}(P_s P) = \frac{1}{2\pi \sqrt{-1}} \int_\gamma (A - \zeta I)^{-2} B d\zeta = 0,
\]

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Lemma 3.1 are well-defined for such $s \in \sigma(s) \cap \ell$. Then our results follows from the definition of the spectral flow and the fact that $\sigma(s)$ open subsets $N$ of the imaginary axis, $(0, \infty)$, we can assume that they are both in Jordan normal forms. Then $PAP, PBP$ with co-orientation from left to right. Let $A$, $-\epsilon \leq s \leq \epsilon$ ($\epsilon > 0$), be a curve in $A_\ell(X)$. Set $P = P_\ell^0(A_0), A = A_0$. Assume that $\text{im } P \subset \text{dom}(A)$ and $B := \frac{d}{ds}|_{s=0}(A_sP)$ exists. Assume that

\[(PAP)(PB) = (PB)(PAP),\]  

(44)

where $PAP, PB \in \mathcal{B}(\text{im } P)$, and $PB : \text{im } P \to \text{im } P$ is hyperbolic, i.e. $\sigma(PB) \cap (\sqrt{-1}\mathbb{R}) = \emptyset$. Then there is a $\delta \in (0, \epsilon)$ such that $\nu_{\delta, \ell}(A_s) = 0$ for all $s \in [-\delta, 0) \cup (0, \delta]$ and

\[s_f\{A_s; 0 \leq s \leq \delta\} = -m^-(PB),\]  

(45)

\[s_f\{A_s; -\delta \leq s \leq 0\} = m^+(PB).\]  

(46)

Here we denote by $m^+(PB)$ ($m^-(PB)$) the total algebraic multiplicity of eigenvalues of $PB$ with positive (negative) imaginary part respectively.

**Proof.** We follow the proof of [36] Theorem 4.1. Since $A \in A_\ell(X)$, there exist bounded open subsets $N$ and $N^\pm$ of $\mathbb{C}$ such that $N = N^+ \cup (N \cap \ell) \cup N^-$, $N^+$ stays in the right (left) side of the imaginary axis, $\sigma(A) \cap \ell \subset N \cap \ell$, $\sigma(A) \cap \partial N = \emptyset$, and $P(A, N) = P$. Since $A_s$, $s \in (-\epsilon, \epsilon)$ is a continuous family in $\mathcal{C}(X)$, $\sigma(A_s) \cap (\partial N \cup (\ell \setminus (N \cap \ell))) = \emptyset$ for $|s|$ small. For such $s$, let $P_s$ be defined in Lemma 3.1. Then $\|P_s - P\| < 1$ for $|s|$ small, and $R_s$ and $U_s$ in Lemma 3.1 are well-defined for such $s$. Then we have

\[\sigma(A_s) \cap \ell \subset \sigma(A) \cap N = \sigma(U_s^{-1}P_sA_sP_sU_s).\]

Now we work in the finite dimensional vector space $\text{im } P$. Since $PB$ commutes with $PAP$, we can assume that they are both in Jordan normal forms. Then $P(A + sB)P$ is also in Jordan norm form for each $s$. By Lemma 3.1, we have

\[\frac{d}{ds}|_{s=0}(U_s^{-1}P_sA_sP_sU_s) = PB.\]

Then there exists a $\delta \in (0, \epsilon)$ such that $U_s^{-1}P_sA_sP_sU_s$ are hyperbolic for all $s \in [-\delta, 0) \cup (0, \delta]$, and

\[m^-(U_s^{-1}P_sA_sP_sU_s) = m^-(PB) \quad \text{for all } s \in (0, \delta],\]

\[m^-(U_s^{-1}P_sA_sP_sU_s) = m^+(PB) \quad \text{for all } s \in [-\delta, 0).\]

Then our results follows form the definition of the spectral flow and the fact that

\[\dim \text{im } P(A_s, N^-) = m^-(U_s^{-1}P_sA_sP_sU_s) \quad \text{for all } s \in [-\delta, 0) \cup (0, \delta].\]

Q.E.D.
3.3 Spectral flow for curves of quadratic forms

Let $X$ be a complex Hilbert space and $\ell = \sqrt{-1}(\epsilon, \epsilon)$ ($\epsilon > 0$) with co-orientation from left to right. Let $A_s$, $0 \leq s \leq 1$ be a curve of closed self-adjoint Fredholm operators. We will denote by $\text{sf}\{A_s\} = \text{sf}_\ell\{A_s\}$.

**Lemma 3.2** Let $X$ be a Hilbert space. Let $A_s$, $0 \leq s \leq 1$ be a curve of closed self-adjoint Fredholm operators. Then for any curve $P_s \in \mathcal{B}(X)$ of invertible operators, we have

$$\text{sf}\{P_s P_s^* A_s\} = \text{sf}\{P_s^* A_s P_s\} = \text{sf}\{A_s\}. \quad (47)$$

**Proof.** Since $A_s$ is a curve of closed self-adjoint Fredholm operators and $P_s$ is a curve of bounded invertible operators, the families $P_s^* A_s P_s$ and $P_s P_s^* A_s$, $0 \leq s \leq 1$ are curves of closed Fredholm operators. By (viii) of Proposition 3.1 we have

$$\text{sf}\{P_s P_s^* A_s\} = \text{sf}\{P_s (P_s^* A_s P_s) P_s^{-1}\}
= \text{sf}\{P_s^* A_s P_s\}. \quad (48)$$

Since $P_s^* A_s P_s$ are self-adjoint Fredholm operators and $\dim \text{ker}(P_s^* A_s P_s) = \dim \text{ker} A_s$, we have

$$\text{sf}\{P_s^* A_s P_s\} = \text{sf}\{P_0^* A_s P_0\} + \text{sf}\{P_s^* A_1 P_s\}
= \text{sf}\{P_0^* A_s P_0\}
= \text{sf}\{P_1^* A_s P_1\}. \quad (49)$$

Let $Q_s$, $0 \leq s \leq 1$ be a curve of bounded positive definite operators on $X$ with $Q_0 = I$, $Q_1^2 = P_0 P_0^*$. By (48) and (49) we have

$$\text{sf}\{P_s^* A_s P_s\} = \text{sf}\{P_0^* A_s P_0\}
= \text{sf}\{P_0 P_0^* A_s\}
= \text{sf}\{Q_1 A_s Q_1\}
= \text{sf}\{Q_0 A_s Q_0\}
= \text{sf}\{A_s\}. \quad (49)$$

Q.E.D.

The above lemma leads the following definition.

**Definition 3.2** Let $X$ be a Hilbert space. Let $\mathcal{I}_s$, $0 \leq s \leq 1$ be a curve of bounded Fredholm quadratic forms, i.e. $\mathcal{I}_s(x, y) = \langle A_s x, y \rangle_X$ for all $x, y \in X$, where $A_s$, $0 \leq s \leq 1$ is a curve of bounded self-adjoint Fredholm operators, and $\langle \cdot, \cdot \rangle_X$ denotes the inner product in $X$.

(a) The **spectral flow** $\text{sf}\{\mathcal{I}_s\}$ of $\mathcal{I}_s$ is defined to be the spectral flow $\text{sf}\{A_s\}$.

(b) If $A_1 - A_0$ is compact, the **relative Morse index** $I(\mathcal{I}_0, \mathcal{I}_1)$ is defined to be the relative Morse index $I(\mathcal{I}_0, \mathcal{I}_1) := -\text{sf}\{A_0 + s(A_1 - A_0)\}$.

Based on this observation we have the following lemma.
Lemma 3.3 Let $X$ be a Hilbert space. Let $A_s \in \mathcal{B}(X)$, $0 \leq s \leq 1$ be a curve of self-adjoint Fredholm operators and $I_s$ be quadratic forms defined by $I_s(x, y) = \langle A_s x, y \rangle$ for all $x, y \in X$. Assume that $P_s \in \mathcal{B}(X)$, $0 \leq s \leq 1$ is a curve of operators such that $P_s^2 = P_s$ and $I_s(x, y) = 0$ for all $x \in \text{im} \, P_s$, $y \in \text{im} \, Q_s$, where $Q_s = I - P_s$. Then we have
\[
\text{sf}\{I_s\} = \text{sf}\{I_s|_\text{im} \, P_s\} + \text{sf}\{I_s|_\text{im} \, Q_s\}.
\] (50)

**Proof.** Set $R_s := P_s^* P_s + Q_s^* Q_s$, $s \in [0, 1]$. Since $P_s + Q_s = I$ and $P_s^2 = I$, we have
\[
R_s = \frac{I}{2} + 2(I - P_s^*)(\frac{I}{2} - P_s) > 0.
\]
Consider the new inner product $\langle R_s x, y \rangle$, $x, y \in X$ on $X$. For this inner product $P_s$ is an orthogonal projection, i.e., $R_s P_s = P_s^* R_s$.

Now we work in the Hilbert space $X$ with the inner product. So we can assume that $P_s$ is orthogonal. By the fact that $\text{im} \, P_s$ and $\text{im} \, Q_s$ are $I_s$ orthogonal, we have $P_s A_s Q_s = Q_s A_s P_s = 0$. Then we have
\[
A_s = (P_s + Q_s)A_s(P_s + Q_s) = P_s A_s P_s + Q_s A_s Q_s.
\]
So $P_s A_s = A_s P_s$. By (iv) of Proposition 3.1 $P_s A_s P_s$ is a Fredholm operator on $\text{im} \, P_s$, $Q_s A_s Q_s$ is a Fredholm operator on $\text{im} \, Q_s$, and we have
\[
\text{sf}\{I_s\} = \text{sf}\{A_s\}
\]
\[
= \text{sf}\{P_s A_s P_s : \text{im} \, P_s \to \text{im} \, P_s\} + \text{sf}\{Q_s A_s Q_s : \text{im} \, Q_s \to \text{im} \, Q_s\}
\]
\[
= \text{sf}\{I_s|_\text{im} \, P_s\} + \text{sf}\{I_s|_\text{im} \, Q_s\}.
\]

**Lemma 3.4** Let $X$ be a Hilbert space, and $M$ be a closed subspace with finite codimension. Let $A \in \mathcal{B}(M)$ be a self-adjoint Fredholm operator and $I(x, y) = \langle Ax, y \rangle$ for all $x, y \in M$. Let $N_1$ and $N_2$ be subspaces of $H$ such that $X = M \oplus N_1 = M \oplus N_2$. Define $I_k$ on $H$, $k = 1, 2$ by
\[
I_k(x + u, y + v) = \langle Ax, y \rangle, \quad \text{for all } x, y \in M \text{ and } u, v \in N_k.
\]
Then we have $I(I_1, I_2) = 0$.

**Proof.** Let $N_0$ be the orthogonal complement of $M$. Set $A_0 = \text{diag}(A, 0)$ under the direct sum decomposition $X = M \oplus N_0$. Define $I_0$ and $A_1, A_2$ by $I_k(x, y) = \langle A_k x, y \rangle$, for all $x, y \in H$, where $k = 0, 1, 2$. Let $B : N_1 \to N$ be a linear isomorphism. Define $P_1 \in \mathcal{B}(X)$ by $P_1(x + y) = x + By$ for all $x \in M$, $y \in N_1$. Then $P_1$ is invertible, $P_1 - I$ is compact, and $A_1 = P_1^* A_0 P_1$. So $A_1 - A_0$ is compact. Let $P_s \in \mathcal{B}(X)$, $0 \leq s \leq 1$ be a curve of invertible operators such that $P_0 = I$ and $P_s - I$ are compact. By the definition of the relative Morse index and Lemma 3.2, we have
\[
I(I_0, I_1) = I(A_0, A_1)
\]
\[
= I(A_0, A_1)
\]
\[
= -\text{sf}\{P_s^* A_0 P_s\}
\]
\[
= -\text{sf}\{A_0\}
\]
\[
= 0.
\]
similarly we have $A_2 - A_0$ is compact and $\mathcal{I}(A_0, A_2) = 0$. So $A_2 - A_1$ is compact, and

$$I(\mathcal{I}_1, \mathcal{I}_2) = I(\mathcal{I}_0, \mathcal{I}_2) - I(\mathcal{I}_0, \mathcal{I}_1) = 0.$$  

Q.E.D.

The following proposition gives a generalization of Proposition 5.3 in [M] and a formula of Morse.

**Proposition 3.3** Let $X$ be a Hilbert space and $A \in \mathcal{B}(X)$ be a self-adjoint Fredholm operator. Let $P$ be an orthogonal projection such that $\ker P$ is of finite dimension. Let $\mathcal{I}$ be a quadratic form on $X$ defined by $\mathcal{I}(x, y) = \langle Ax, y \rangle$, $x, y \in X$. Set $M = \text{im } P$ and $N$ be the $\mathcal{I}$-orthogonal complement of $M$, i.e., $N = \{x \in X; \mathcal{I}(x, y) = 0, \forall y \in M\}$. Then we have

$$I(PAP, A) = m^-(\mathcal{I}|_N) + \dim \ker \mathcal{I}|_N - \dim \ker \mathcal{I}. \quad (51)$$

**Proof.** Since $PAP - A$ is of finite rank operator, $sPAP + (1 - s)A$, $0 \leq s \leq 1$ is a curve of self-adjoint Fredholm operators. We divide our proof into four steps.

**Step 1.** Assume that $\ker A = \{0\}$. Let $M_0 = \ker \mathcal{I}|_M$, $M_1$ be the orthogonal complement of $M_0$ in $M$, and $P_0, P_1$ be the orthogonal projection onto $M_0, M_1$ respectively. Then $P = P_0 + P_1$. Since $AM$ is of finite codimension and $M_0 = (AM)^\perp \cap M$, $P_0$ is of finite rank. Let $N_1$ be the $\mathcal{I}$-orthogonal complement of $M_1$. Since $M = M_0 + M_1$, we have $M_1 \cap N_1 \subset M_0$. So $M_1 \cap N_1 = \{0\}$. Moreover we have

$$\dim N_1 = \dim \ker (AP_1) - \text{ind}(AP_1) = \dim \ker P_1 - \text{ind } A - \text{ind } P_1 = \dim \ker P_1 < +\infty,$$

where we denote $\text{ind } A$ the index of a Fredholm operator $A$. So $X = M_1 \oplus N_1$. By the fact that $\mathcal{I}$ is nondegenerate, $\mathcal{I}|_{N_1}$ is nondegenerate.

Let $\mathcal{I}_1$ be defined by $\mathcal{I}_1(x + u, y + v) = \mathcal{I}(x, y)$ for all $x, y \in M_1, u, v \in N_1$. By Lemma 3.3 and Lemma 3.4 we have

$$I(PAP, A) = I(PAP, P_1 AP_1) + I(P_1 AP_1, A) = I(P_1 AP_1, A) = I(\mathcal{I}_1, \mathcal{I}) = I(\mathcal{I}_1|M_1, \mathcal{I}|_{N_1}) + I(\mathcal{I}_1|N_1, \mathcal{I}|_{N_1}) = m^-(\mathcal{I}|_{N_1}).$$

**Step 2.** Equation (51) holds if $\ker A = \{0\}$ and $N \subset M$.

In this case, $M_0 = N \subset N_1$, $m^-(\mathcal{I}|_N) = 0$ and $\ker \mathcal{I}|_N = N$. For each $x \in N_1$ such that $\mathcal{I}(x, y) = 0$ for all $y \in N$, we have $\mathcal{I}(x, y) = 0$ for all $y \in M_1$ and hence for all $y \in M$. Then $x \in N$. Thus $N$ is the $\mathcal{I}|_{N_1}$-orthogonal complement of $N$. $N_1$ has an orthogonal decomposition $N_1 = N^+ \oplus N^-$ such that $N^+$ and $N^-$ are $\mathcal{I}$-orthogonal, $\mathcal{I}|_{N^+} > 0$ and $\mathcal{I}|_{N^-} < 0$. Let $P^\pm$ be the orthogonal projections onto $N^\pm$. Then $P^\pm|_{M_0}$ are isomorphisms. So we have

$$\dim N_1 = 2 \dim N = 2m^-(\mathcal{I}|_{N_1}).$$
By Step 1 we have
\[ I(PAP, A) = m^{-}(I|_{N_1}) = m^{-}(I|_{N}) + \dim \ker I|_{N} - \dim \ker I. \]

**Step 3.** Equation (51) holds if and \( M + N = X \).

In this case we have
\[ \ker I|_{N} = \ker I = M \cap N. \]

Firstly we assume that \( \ker A = \{0\} \). Then \( M_0 = \{0\}, \ N_1 = N \) and \( \ker I|_{N} = \ker I = \{0\} \).

By Step 1, equation (51) holds.

**Step 4.** Equation (51) holds.

Firstly we assume that \( \ker A = \{0\} \). Let \( Q \) be the orthogonal projection onto \( M + N \). Then the \( I \)-orthogonal complement of \( M + N \) is \( \ker I|_{N} \).

By Step 2 and Step 3 we have
\[ I(PAP, A) = I(PAP, QAQ) + I(QAQ, A) = m^{-}(I|_{N}) + \dim \ker I|_{N}. \]

In the general case, we apply the above special case by taking the quotient space with \( \ker A \) and get equation (51). Q.E.D.

### 3.4 A formula

**Lemma 3.5** Let \( X \) be a Hilbert space and \( H = X \oplus X \). Let \( B_s \in C(X), \ 0 \leq s \leq 1 \) be a curve of Fredholm operators. Let the operator \( D_s \in C(X) \) by \( D_s = \begin{pmatrix} 0 & B_s^* \\ B_s & 0 \end{pmatrix} \). Then we have
\[ \sf{\{D_s\}} = \dim \ker B_1 - \dim \ker B_0. \quad (52) \]

**Proof.** By [21] Theorem IV.2.23], \( B_s^*, \ 0 \leq s \leq 1 \) is a curve of closed operators.

Note that \( \lambda \in \sigma(D_s) \) if and only if \( \lambda^2 \in \sigma(B_s^* B_s) \), and the algebraic multiplicities of them are the same if \( |\lambda| \neq 0 \) is small. Moreover we have
\[ \dim \ker D_s = \dim \ker B_s + \dim \ker B_s^*, \quad \ind B_s = \ind B_0 = \dim \ker B_s - \dim \ker B_s^*. \]

By the definition of the spectral flow we have
\[ \sf{\{D_s\}} = \frac{1}{2}(\dim \ker D_1 - \dim \ker D_0) = \dim \ker B_1 - \dim \ker B_0. \]

Q.E.D.

**Lemma 3.6** Let \( X \) be a Hilbert space and \( H = X \oplus X \). Let \( B \in C(X) \) be a operator with compact resolvent, and \( A \in B(X) \) be a self-adjoint operator. Define linear operator \( D_s \in C(X) \) by \( D_s = \begin{pmatrix} sA & B^* \\ B & 0 \end{pmatrix} \). Then \( D_s \in C(H), \ 0 \leq s \leq 1 \) is a curve of Fredholm operators, and we have
\[ \dim \ker D_s = \dim \ker A|_{\ker B} + \dim \ker B^* \quad \text{for all } s \in (0, 1], \quad (53) \]
\[ \sf{\{D_s\}} = -m^{-}(A|_{\ker B}). \quad (54) \]

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Proof. By [21] Theorem IV.2.23, $D_s$, $0 \leq s \leq 1$ is a curve of closed operators. Since $A$ is bounded and $B$ has compact resolvent, $D_s$ is a Fredholm operator.

For each $s \in (0, 1]$ we have

$$\ker D_s = \{(x, y) \in H; sAx + B^*y = 0, Bx = 0\} = \{(x, y) \in H; x \in \ker B, sAx = -B^*y \in \im B^* = \ker B\} = \{(x, y) \in H; x \in \ker A|_{\ker B}, sAx = -B^*y\}.$$

Define $\varphi : \ker D_s \to \ker A|_{\ker B}$ by $\varphi(x, y) = x$ for $(x, y) \in \ker D_s$. Then $\varphi$ is a linear surjective map, and $\ker \varphi = \{0\} \times \ker B^*$. Then we get \textit{[53]}.

Let $\lambda_t \in \sigma(D_t)$ be a spectral point of $D_t$ near 0 for $t \neq 0$ small. Then there exists $(x_t, y_t) \in H \setminus \{0\}$ such that $D_t(x_t, y_t) = \lambda_t(x_t, y_t)$. Then one of the following cases holds.

Case 1. $\lambda_t = 0$.

In this case, we have $(x_t, y_t) \in \ker D_t$. The algebraic multiplicity of the eigenvalue 0 of $D_t$ is $\dim \ker D_t$.

Case 2. $\lambda_t \neq 0$ and $Bx_t = 0$.

In this case, we have $y_t = 0$ and $tAx_t = \lambda_tx_t$. Let $P$ be the orthogonal projection of $X$ onto $\ker B$. Then $tPAPx_t = \lambda_tx_t$. So the total algebraic multiplicity of these eigenvalues $\lambda_t$ of $D_t$ with such eigenvectors is

$$m^+(tPAP) + m^-(tPAP) = m^+(A|_{\ker B}) + m^-(A|_{\ker B}).$$

Case 3. $\lambda_t \neq 0$ and $Bx_t \neq 0$.

In this case, we have $x_t \neq 0$, and

$$\lambda_t^2x_t - t\lambda_tAx_t - B^*Bx_t = 0.$$

Take inner product with $x_t$, we have

$$\lambda_t^2\langle x_t, x_t \rangle - t\lambda_t\langle Ax_t, x_t \rangle - \langle Bx_t, Bx_t \rangle = 0. \tag{55}$$

For each $x_t$, there exist two $\lambda_t$ satisfying equation \textit{[55]}; one is positive, and the other is negative. The algebraic multiplicity of the two eigenvalues of $D_s$ is equal to each other. We denote by $2k_t$ the total algebraic multiplicity of the these eigenvalues of $D_s$ with such eigenvectors.

Since $D_s$, $0 \leq s \leq 1$ is continuous varying, for $t \neq 0$ small, we have

$$\dim \ker D_0 = \dim \ker D_t + m^+(A|_{\ker B}) + m^-(A|_{\ker B}) + 2k_t. \tag{56}$$

By the definition of the spectral flow and \textit{[53]} we have

$$\text{sf}\{D_s\} = -m^-(A|_{\ker B}) - k_t$$

$$= -m^-(A|_{\ker B}) - \frac{1}{2} (\dim \ker D_0 - \dim \ker D_t - m^+(A|_{\ker B}) - m^-(A|_{\ker B}))$$

$$= \frac{1}{2} (\dim \ker D_1 - \dim \ker D_0 + \text{sign}(A|_{\ker B}))$$

$$= \frac{1}{2} (\dim \ker A|_{\ker B} - \dim \ker B + \text{sign}(A|_{\ker B}))$$

$$= -m^-(A|_{\ker B}).$$

Q.E.D.
Proposition 3.4 Let $X$ be a Hilbert space and $H = X \oplus X$. Let $B_s \in \mathcal{C}(X)$, $0 \leq s \leq 1$ be a curve of operators with compact resolvent, and $A_s \in \mathcal{B}(X)$, $0 \leq s \leq 1$ be a curve of self-adjoint operators. Define unbounded operator $D_s X$ by $D_s = \begin{pmatrix} A_s & B_s^* \\ B_s & 0 \end{pmatrix}$. Then $D_s \in \mathcal{C}(H)$, $0 \leq s \leq 1$ is a curve of Fredholm operators, and we have

$$\dim \ker D_s = \dim \ker A_s|_{\ker B_s} + \dim \ker B_s^* \quad \text{for all } s \in [0, 1],$$

(57)

$$\text{sf} \{D_s\} = m^-(A_0|_{\ker B_0}) - m^-(A_1|_{\ker B_1}) + \dim \ker B_1 - \dim \ker B_0.$$  

(58)

Proof. (57) follows from (53). Set $D_{s,t} = \begin{pmatrix} tA_s & B_s^* \\ B_s & 0 \end{pmatrix}$ for $s, t \in [0, 1]$. By [21, Theorem IV.2.23], $B_s^*$ and $D_{s,t}$, $0 \leq s, t \leq 1$ are two continuous families of closed operators. Since $A_s$ is bounded and $B_s$ has compact resolvent, $D_{s,t}$ is a Fredholm operator.

By Proposition 3.1, Lemmas 3.5 and 3.6 we have

$$\text{sf} \{D_s\} = -\text{sf} \{D_{0,t}; 0 \leq t \leq 1\} + \text{sf} \{D_{s,0}; 0 \leq s \leq 1\} + \text{sf} \{D_{1,t}; 0 \leq t \leq 1\}$$

$$= m^-(A_0|_{\ker B_0}) + (\dim \ker B_1 - \dim \ker B_0) - m^-(A_1|_{\ker B_1})$$

$$= m^-(A_0|_{\ker B_0}) - m^-(A_1|_{\ker B_1}) + \dim \ker B_1 - \dim \ker B_0.$$  

Q.E.D.

4 Maslov-type index theory

4.1 Symplectic functional analysis and Maslov index

A main feature of symplectic analysis is the study of the Maslov index. It is an intersection index between a path of Lagrangian subspaces with the Maslov cycle, or, more generally, with another path of Lagrangian subspaces. The Maslov index assigns an integer to each continuous path of Fredholm pairs of Lagrangian subspaces of a fixed Hilbert space with continuously varying symplectic structures.

Firstly we define symplectic Hilbert spaces and Lagrangian subspaces.

Definition 4.1 Let $H$ be a complex vector space. A mapping

$$\omega : H \times H \to \mathbb{C}$$

is called a (weak) symplectic form on $H$, if it is sesquilinear, skew-symmetric, and non-degenerate, i.e.,

(i) $\omega(x, y)$ is linear in $x$ and conjugate linear in $y$;

(ii) $\omega(y, x) = -\overline{\omega(y, x)}$;

(iii) $H^\omega := \{x \in H \mid \omega(x, y) = 0 \text{ for all } y \in H\} = \{0\}$.

Then we call $(H, \omega)$ a complex symplectic vector space.

Definition 4.2 Let $(H, \omega)$ be a complex symplectic vector space.
(a) The annihilator of a subspace $\lambda$ of $H$ is defined by

$$\lambda':=\{y\in H\mid \omega(x,y)=0 \text{ for all } x\in \lambda\}. \tag{59}$$

(b) A subspace $\lambda$ is called isotropic, co-isotropic, or Lagrangian if

$$\lambda\subset \lambda', \quad \lambda'\supset \lambda', \quad \lambda=\lambda'$$

respectively.

(c) The Lagrangian Grassmannian $\mathcal{L}(H,\omega)$ consists of all Lagrangian subspaces of $(H,\omega)$.

**Definition 4.3** Let $H$ be a complex Hilbert space. A mapping $\omega: H \times H \to \mathbb{C}$ is called a (strong) symplectic form on $H$, if $\omega(x,y) = \langle Jx, y \rangle_H$ for some bounded invertible skew-symmetric operator $J$. $(H,\omega)$ is called a (strong) symplectic Hilbert space.

Before giving a rigorous definition of the Maslov index, we fix the terminology and give a simple lemma.

We recall:

**Definition 4.4** (a) The space of (algebraic) Fredholm pairs of linear subspaces of a vector space $H$ is defined by

$$\mathcal{F}_\text{alg}^2(H):=(\lambda,\mu)\mid \dim(\lambda\cap\mu)<+\infty \text{ and } \dim(H/(\lambda+\mu))<+\infty \tag{59}$$

with

$$\text{ind}(\lambda,\mu):=\dim(\lambda\cap\mu)-\dim(H/(\lambda+\mu)). \tag{60}$$

(b) In a Banach space $H$, the space of (topological) Fredholm pairs is defined by

$$\mathcal{F}^2(H):=(\lambda,\mu)\in \mathcal{F}_\text{alg}^2(H)\mid \lambda,\mu, \text{ and } \lambda+\mu\subset H \text{ is closed}. \tag{61}$$

We need the following well-known lemma (see, e.g., [9, Lemma 1.7]).

**Lemma 4.1** Let $(H,\omega)$ be a (strong) symplectic Hilbert space. Then

(i) there exists a $\omega$-orthogonal splitting

$$H=H^+\oplus H^-$$

such that $-\sqrt{-1}\omega$ is positive (negative) definite on $H^\pm$, and we call it a symplectic splitting;

(ii) there is a 1-1 correspondence between the space

$$\mathcal{U}(H^+,H^-,\omega)=\{U\in \mathcal{B}(H^+,H^-)\mid \omega(Ux,Uy)=-\omega(x,y), \forall x,y\in H^+\}$$

and $\mathcal{L}(H,\omega)$ under the mapping $U\to L:=\text{Gr}(U) (=\text{graph of } U);$
(iii) if \( U, V \in \mathcal{U}(H^+, H^-, \omega) \) and \( \lambda := \text{Gr}(U), \mu := \text{Gr}(V) \), then \((\lambda, \mu)\) is a Fredholm pair if and only if \( U - V \), or, equivalently, \( UV^{-1} - I \) is Fredholm. Moreover, we have a natural isomorphism
\[
\ker(UV^{-1} - I) \simeq \lambda \cap \mu.
\]  

**Definition 4.5** Let \((H, \langle \cdot, \cdot \rangle_s), s \in [0, 1]\) be a continuous family of Hilbert spaces, and \(\omega_s(x, y) = \langle J_s x, y \rangle_s\) be a continuous family of symplectic forms on \(H\), i.e., \(\{A_{s,0}\}\) and \(\{J_s\}\) are two continuous families of bounded invertible operators, where \(A_{s,0}\) is defined by
\[
\langle x, y \rangle_s = \langle A_{s,0}x, y \rangle_0 \quad \text{for all } x, y \in H.
\]

Let \(\{(\lambda_s, \mu_s)\}\) be a continuous family of Fredholm pairs of Lagrangian subspaces of \((H, \langle \cdot, \cdot \rangle_s, \omega_s)\). Then there is a continuous families of symplectic splitting
\[
H = H^+_s \oplus H^-_s
\]  
for all \(s \in [0, 1]\). Such \(H^+_s\) can be chosen to be the positive (negative) space associated to the self-adjoint operator \(-\sqrt{-1}J_s \in \mathcal{B}(H, \langle \cdot, \cdot \rangle_s)\). By Lemma 4.1, \(\lambda_s = \text{Gr}_s(U_s)\) and \(\mu_s = \text{Gr}_s(V_s)\) with \(U_s, V_s \in \mathcal{U}(H^+_s, H^-_s, \omega_s)\), where \(\text{Gr}_s\) denotes the graph associated to the splitting \((63)\). We define the **Maslov index** \(\text{Mas}\{\lambda_s, \mu_s\}\) by
\[
\text{Mas}\{\lambda_s, \mu_s\} = -\text{sf}_\ell\{U_sV_s^{-1}\},
\]
where \(\ell := (1 - \epsilon, 1 + \epsilon)\) with \(\epsilon \in (0, 1)\) and with upward co-orientation.

**Remark 4.1** For finite-dimensional \((H, \omega)\), constant \(\mu_s = \mu_0\), and a loop \(\{(\lambda_s)\}\), i.e., for \(\lambda_0 = \lambda_1\), we notice that \(\text{Mas}\{\lambda_s, \mu_s\}\) is the winding number of the closed curve \(\{(\det(U_s^{-1}V_0))\}_{s \in [0, 1]}\). This is the original definition of the Maslov index as explained in Arnol’d, [3].

**Lemma 4.2** The Maslov index is independent of the choice of the symplectic splitting of \(H\).

**Proof.** Let \(H = H^+_{s,k} \oplus H^-_{s,k}, s \in [0, 1]\) with \(k = 0, 1\) be two continuous families of symplectic splitting. For each \(s \in [0, 1]\) and \(k = 0, 1\), set
\[
\langle \cdot, \cdot \rangle_{s,k} = (-\sqrt{-1}\omega|_{H^+_{s,k}}) \oplus (\sqrt{-1}\omega|_{H^-_{s,k}}),
\]
Then \((H, \langle \cdot, \cdot \rangle_{s,k})\) is a Hilbert space for each \(s \in [0, 1]\) and \(k = 0, 1\). Set
\[
\langle \cdot, \cdot \rangle_{s,t} = (1 - t)\langle \cdot, \cdot \rangle_{s,0} + t\langle \cdot, \cdot \rangle_{s,1}
\]
for each \((s, t) \in [0, 1] \times [0, 1]\). For each \((s, t) \in [0, 1] \times [0, 1]\), define \(J_{s,t} \in \mathcal{B}(H)\) by
\[
\omega(x, y)_s = (Jx, y)_{s,t} \quad \text{for all } x, y \in H.
\]

Then \(H^+_{s,k}\) is the positive (negative) space associated to the self-adjoint operator \(-\sqrt{-1}J_{s,k}\) for each \(s \in [0, 1]\) and \(k = 0, 1\). Let \(H^+_{s,t}\) be the positive (negative) space associated to the self-adjoint operator \(-\sqrt{-1}J_{s,t}\) for each \(s \in [0, 1]\) and \(t = 0, 1\).

Let \((\lambda_s, \mu_s)\) be a continuous family of Fredholm pairs of Lagrangian subspaces of \((H, \omega_s)\). For each symplectic splitting \(H = H^+_{s,t} \oplus H^-_{s,t}\), we denote by \(U_{s,t}\) and \(V_{s,t}\) the associated generated
Corollary 4.1 (Symplectic invariance) Let \( (H_1, \omega_{s,k}) \), \( k = 1, 2 \) be two continuous families of symplectic Hilbert spaces. Let \( M(s) \in \mathcal{B}(H_1, H_2) \), \( 0 \leq s \leq 1 \) be a curve of invertible operators such that

\[
\omega_{s,2}(M_s x, M_s y) = \omega_{s,1}(x, y) \quad \text{for all } x, y \in H_1 \text{ and } s \in [0, 1].
\]

Then for any curve \( (\lambda(s), \mu(s)) \), \( 0 \leq s \leq 1 \) curve of Fredholm pairs of Lagrangian subspaces of \( H_1 \),

\[
\text{Mas}(M\lambda, M\mu) = \text{Mas}(\lambda, \mu).
\]  

(65)

**Proof.** Let \( H_1 = H_{s,1}^+ \oplus H_{s,2}^- \) be a continuous family of symplectic splitting of the family \( (H_1, \omega_{s,1}) \), \( 0 \leq s \leq 1 \). Then \( H_2 = H_{s,2}^+ \oplus H_{s,2}^- \) be a continuous family of symplectic splitting of the family \( (H_2, \omega_{s,2}) \), \( 0 \leq s \leq 1 \), where \( H_{s,2}^- = M_s H_{s,1}^+ \) and \( H_{s,2}^- = M_s H_{s,1}^- \). For each symplectic splitting \( H_k = H_{s,k}^+ \oplus H_{s,k}^- \), \( s \in [0, 1] \) and \( k = 1, 2 \), we denote by \( U_{s,k} \) and \( V_{s,k} \) the associated generated "unitary" operators of \( \lambda_s \) and \( \mu_s \) respectively. Then we have

\[
U_{s,2} = M_s U_{s,1} M_s^{-1}, \quad V_{s,2} = M_s V_{s,1} M_s^{-1}.
\]

By the definition of the Maslov index we have

\[
\text{Mas}(M\lambda, M\mu) = -\text{sf}_\lambda\{(M_s U_{s,1} M_s^{-1})(M_s V_{s,1} M_s^{-1})^{-1}; 0 \leq s \leq 1\}
\]

\[
= -\text{sf}_\lambda\{U_{s,1} V_{s,1}^{-1}; 0 \leq s \leq 1\}
\]

\[
= \text{Mas}(\lambda, \mu).
\]

Q.E.D.

Now we give a method of using the crossing form to calculate Maslov indices (cf. [14], [30] and [7] Theorem 2.1).

Let \( \lambda = \{\lambda_s\}_{s \in [0,1]} \) be a \( C^1 \) curve of Lagrangian subspaces of \( (H, \omega) \). Let \( W \) be a fixed Lagrangian complement of \( \lambda_t \). For \( v \in \lambda_t \) and \( |s-t| \) small, define \( w(s) \in W \) by \( v + w(s) \in \lambda_s \). The form

\[
Q(\lambda, t) := Q(\lambda, W, t)(u, v) = \frac{d}{ds}|_{s=t}\omega(u, w(s)), \quad \forall u, v \in \lambda_t
\]

is independent of the choice of \( W \). Let \( \{(\lambda_s, \mu_s)\}, 0 \leq s \leq 1 \) be a curve of Fredholm pairs of Lagrangian subspaces of \( H \). For \( t \in [0, 1] \), the **crossing form** \( \Gamma(\lambda, \mu, t) \) is a quadratic form on \( \lambda_t \cap \mu_t \) defined by

\[
\Gamma(\lambda, \mu, t)(u, v) = Q(\lambda, t)(u, v) - Q(\mu, t)(u, v), \quad \forall u, v \in \lambda_t \cap \mu_t.
\]
A crossing is a time \( t \in [0,1] \) such that \( \lambda_t \cap \mu_t \neq \{0\} \). A crossing is called regular if \( \Gamma(\lambda, \mu, t) \) is nondegenerate. It is called simple if it is regular and \( \lambda_t \cap \mu_t \) is one-dimensional.

Now let \( (H, \omega) \) be a symplectic Hilbert space with \( \omega(x,y) = \langle jx, y \rangle \), for all \( x, y \in H \), where \( J \in \mathcal{B}(H) \) is an invertible skew self-adjoint operator. Then we have a symplectic Hilbert space \( X = (H \oplus H, (\omega) \oplus \omega) \). For each \( M \in \text{Sp}(H, \omega) \), its graph \( \text{Gr}(M) \) is a Lagrangian subspace of \( X \). The following lemma is Lemma 3.1 in [13].

**Lemma 4.3** Let \( M(s) \in \text{Sp}(H, \omega) \), \( 0 \leq s \leq 1 \) be a curve of linear symplectic maps. Assume that \( M(s) \) is differentiable at \( t \in [a,b] \). Set \( B_1(t) = -JM(t)M(t)^{-1} \) and \( B_2(t) = -JM(t)^{-1}M(t) \). Then \( B_1(t), B_2(t) \) are self-adjoint, \( B_2(t) = M(t)^*B_1(t)M(t) \) and we have

\[
Q(\text{Gr}(M), t)((x, M(t)x), (y, M(t)y)) = (B_2(t)x, y). \tag{66}
\]

Q.E.D.

**Proposition 4.1** Let \( (H, \omega) \) be a symplectic Hilbert space and \( \{(\lambda_s, \mu_s)\} \), \( 0 \leq s \leq 1 \) be a \( C^1 \) curve of Fredholm pairs of Lagrangian subspaces of \( H \) with only regular crossings. Then we have

\[
\text{Mas}\{\lambda, \mu\} = m^+(\Gamma(\lambda, \mu, 0)) - m^-(\Gamma(\lambda, \mu, 1)) + \sum_{0 < t < 1} \text{sign}(\Gamma(\lambda, \mu, t)). \tag{67}
\]

**Proof.** Pick an invertible skew self-adjoint operator \( J \in \mathcal{B}(H) \) such that \( J^2 = -I \) and \( \omega(x,y) = \langle jx, y \rangle \). Let \( H_1 = \ker(J - \sqrt{-1}I) \) and \( H_2 = \ker(J + \sqrt{-1}I) \). By Lemma 4.3 there are curves of isometric \( U(t), V(t) \) in \( U(H_1, H_2) \) such that \( \lambda(t) = \text{Gr}(U(t)) \) and \( \mu(t) = \text{Gr}(V(t)) \). Apply Lemma 4.3 for \( (H_1, \langle -\sqrt{-1}x, y \rangle) \), for any \( x, y \in \ker(U(t) - V(t)) \) and \( t \in [a,b] \) we have

\[
\frac{d}{ds}|_{s=t}(\langle -\sqrt{-1}V^{-1}Ux, y \rangle) = \langle \sqrt{-1}V^{-1}\dot{V}V^{-1}Ux, y \rangle + \langle -\sqrt{-1}V^{-1}\dot{U}x, y \rangle.
\]

By Proposition 3.2 we obtain (65). Q.E.D.

### 4.2 Spectral flow formula for fixed maximal domain

Let \( D_m \hookrightarrow D_M \hookrightarrow X \) be three Hilbert spaces. We assume that \( D_m \) is a closed subspace of \( D_M \) and a dense subspace of \( X \). Let \( \{A_s\}_{s \in [0,1]} \) be a family of symmetric densely defined operators in \( C(X) \) with domain \( \text{dom}(A_s) = D_m \). Here we denote by \( C(X) \) all closed operators in \( X \). Assume that \( \text{dom}(A_s^*) = D_M \), i.e., the domain of the maximal symmetric extension \( A_s^* \) of \( A_s \) is independent of \( s \).

We recall from [7] for each \( s \in [0,1] \):

1. The space \( D_M \) is a Hilbert space with the graph inner product
   \[
   (x, y)_{\text{Gr}_s} := (x, y)_X + \langle A_s^*x, A_s^*y \rangle_X \quad \text{for } x, y \in D_M. \tag{68}
   \]
2. The space $D_m$ is a closed subspace in the graph norm and the quotient space $D_M/D_m$ is a strong symplectic Hilbert space with the (bounded) symplectic form induced by Green’s form

$$\omega_s(x + D_m, y + D_m) := \langle A_s^* x, y \rangle_X - \langle x, A_s^* y \rangle_X \quad \text{for } x, y \in D_M.$$  \hfill (69)

3. If $A_s$ admits a self-adjoint Fredholm extension $A_{s,D_s} := A_s^*|D_s$ with domain $D_s \subset X$, then the natural Cauchy data space $(\ker A_s^* + D_m)/D_m$ is a Lagrangian subspace of $(D_M/D_m, \omega_s)$.

4. Moreover, self-adjoint Fredholm extensions are characterized by the property of the domain $D_s$ that $(D_s + D_m)/D_m$ is a Lagrangian subspace of $(D_M/D_m, \omega_s)$ and forms a Fredholm pair with $(\ker A_s^* + D_m)/D_m$.

5. We denote the natural projection (which is independent of $s$) by

$$\gamma : D_M \to D_M/D_m.$$  

We call $\gamma$ the abstract trace map.

We have the following spectral flow formula (cf. \cite{7} Theorem 5.1, \cite{9} Corollary 2.14 and \cite{10} Theorem 1.5).

**Proposition 4.2** We assume that on $D_M$ the graph norms induced by $A_s^*$ and the original norm are equivalent. Assume that $\{A_s^* : D_M \to X\}$ is a continuous family of bounded operators and each $A_s$ is injective. Let $\{D_s/D_m\}$ be a continuous family of Lagrangian subspaces of $(D_M/D_m, \omega_s)$, such that each $A_{s,D_s}$ is a Fredholm operator. Then:

(a) Each $(D_s/D_m, \gamma(\ker(A_s^*)))$ is a Fredholm pair in $D_M/D_m$.

(b) Each Cauchy data space $\gamma(\ker A_s^*)$ is a Lagrangian subspace of $(D_M/D_m, \omega_s)$.

(c) The family $\{\gamma(\ker A_s^*)\}$ is a continuous family in $D_M/D_m$.

(d) The family $\{A_{s,D_s}\}$ is a continuous family of self-adjoint Fredholm operators in $\mathcal{C}(X)$.

(e) Finally, we have

$$\text{sf}\{A_{s,D_s}\} = -\text{Mas}\{\gamma(D_s), \gamma(\ker A_s^*)\}. \hfill (70)$$

Q.E.D.

### 4.3 The Maslov-type indices

**Definition 4.6** Let $(X_1, \omega_1)$ be symplectic Hilbert spaces with $\omega_l(x, y) = (j_l x, y)$, $x, y \in X_l$, $j_l \in \mathcal{B}(X_l)$ are invertible, and $j_l^* = -j_l$, where $l = 1, 2$. Then we have a symplectic Hilbert space $(H = X_1 \oplus X_2, (-\omega_1) \oplus \omega_2)$. Let $W \in \mathcal{L}(H)$. Let $M(t), a \leq t \leq b$ be a curve in $\text{Sp}(X_1,X_2)$ such that $\text{Gr}(M(t)) \in \mathcal{FL}(W)$ for all $t \in [a, b]$. The **Maslov-type index** $i_W(M(t))$ is defined to be $\text{Mas}(\text{Gr}(M(t)), W)$. If $a = 0$, $b = T$, $(X_1, \omega_1) = (X_2, \omega_2)$ and $M(0) = I$, we denote by $\nu_{T,W}(M(t)) = \dim(\text{Gr}(M(T) \cap W))$.

The Maslov-type indices have the following property.
Lemma 4.4  Let \((X_l, \omega_l)\) be symplectic Hilbert spaces with \(\omega_l(x,y) = (j_l x, y)\), where \(x, y \in X_l, j_l \in \mathcal{B}(X_l)\) are invertible, and \(j_l^* = -j_l, l = 1, 2, 3, 4\). Let \(W\) be a Lagrangian subspace of \((X_1 \oplus X_4, (\omega_1) + \omega_4)\). Let \(\gamma_l \in \mathcal{C}([0,1], \text{Sp}(X_l, X_{l+1}))\), \(l = 1, 2, 3\) be symplectic paths such that \(\text{Gr}(\gamma_3(s) \gamma_2(t) \gamma_1(s)) \in \mathcal{F}(W)\) for all \((s, t) \in [0,1] \times [0,1]\). Then we have

\[
i_w(\gamma_3 \gamma_2 \gamma_1) = i_{w'}(\gamma_2) + i_w(\gamma_3 \gamma_2(0) \gamma_1),
\]
where \(W' = \text{diag}(\gamma_1(1), \gamma_3(1)^{-1}) W\).

Proof. Let \(M = \text{diag}(\gamma_1(1), \gamma_3(1)^{-1})\). By the homotopic invariance rel. endpoints of the Maslov-type indices and Corollary 4.1, we have

\[
i_w(\gamma_3 \gamma_2 \gamma_1) = i_w(\gamma_3(1) \gamma_2(1)) + i_w(\gamma_3 \gamma_2(0) \gamma_1) = \text{Mas}(M, \text{Gr}(\gamma_3(1) \gamma_2(1)), M W) + i_w(\gamma_3 \gamma_2(0) \gamma_1) = i_{w'}(\gamma_2) + i_w(\gamma_3 \gamma_2(0) \gamma_1).
\]

Q.E.D.

The following properties of fundamental solutions for linear ODE will be used later.

Lemma 4.5  Let \(j \in \mathcal{C}^1([0, +\infty), \text{GL}(m, \mathbb{C}))\) be a curve of skew self-adjoint matrices, and \(b \in \mathcal{C}([0, +\infty), \text{gl}(m, \mathbb{C}))\) be a curve of self-adjoint matrices. Let \(\gamma \in \mathcal{C}^1([0, +\infty), \text{GL}(m, \mathbb{C}))\) be the fundamental solution of

\[
- j \dot{x} - \frac{1}{2} j^* j x = b x.
\]

Then we have \(\gamma(t)^* j(t) \gamma(t) = j(0)\) for all \(t\).

Proof. By the definition of the fundamental solution, we have \(\gamma(0)^* j(0) \gamma(0) = j(0)\). Since \(j^* = -j\) and \(b^* = b\), we have

\[
\frac{d}{dt}(\gamma(t)^* j(t) \gamma(t)) &= \dot{\gamma}^* j \gamma + \gamma^* j \dot{\gamma} + \gamma^* j \gamma \\
&= (-b \gamma - \frac{1}{2}j^*)^* j \gamma + \gamma^* j \gamma + \gamma^* j \gamma^{-1} (-b \gamma - \frac{1}{2} j) \\
&= \gamma^* (b - \frac{1}{2} j + \dot{j} - \frac{1}{2} j) \gamma \\
&= 0.
\]

So we have \(\gamma(t)^* j(t) \gamma(t) = j(0)\).

Q.E.D.

Lemma 4.6  Let \(B \in \mathcal{C}([0, +\infty), \text{gl}(m, \mathbb{C}))\) and \(P \in \mathcal{C}^1([0, +\infty), \text{GL}(m, \mathbb{C}))\) be two curves of matrices. Let \(\gamma \in \mathcal{C}^1([0, +\infty), \text{GL}(m, \mathbb{C}))\) be the fundamental solution of

\[
\dot{x} = B x,
\]

and \(\gamma' \in \mathcal{C}^1([0, +\infty), \text{GL}(m, \mathbb{C}))\) be the fundamental solution of

\[
\dot{y} = (P B P^{-1} + \dot{P} P^{-1}) y.
\]

Then we have

\[
\gamma' = P \gamma P(0)^{-1}.
\]
**Proof.** Direct calculation shows
\[
\frac{d}{dt}(P\gamma P(0)^{-1}) = (PBP^{-1} + \dot{P}P^{-1})P\gamma P(0)^{-1}
\]
and \(P(0)\gamma P(0)^{-1} = I\). By definition, \(P\gamma P(0)^{-1}\) is the fundamental solution of (74). Q.E.D.

**Corollary 4.2** Let \(j_1, j_2 \in C^1([0, +\infty), GL(m, \mathbb{C}))\) be two curves of skew self-adjoint matrices. Let \(P \in C^1([0, +\infty), GL(m, \mathbb{C}))\) be a curve of matrices such that \(P^*j_2P = j_1\), and \(b \in C([0, +\infty), GL(m, \mathbb{C}))\) be a curve of self-adjoint matrices. Let \(\gamma \in C^1([0, +\infty), GL(m, \mathbb{C}))\) be the fundamental solution of
\[
-j_1\dot{x} - \frac{1}{2}j_1x = bx,
\]
and \(\gamma' \in C^1([0, +\infty), GL(m, \mathbb{C}))\) be the fundamental solution of
\[
j_2\dot{y} - \frac{1}{2}j_2y = (P^{*-1}bP^{-1} + Q)y,
\]
where \(Q = \frac{1}{2}(P^{*-1}\dot{P}j_2 - j_2\dot{P}P^{-1})\). Then we have
\[
\gamma' = P\gamma P(0)^{-1}.
\]
In particular, when \(j_1\) and \(j_2\) are constant matrices, we have
\[
Q = P^{*-1}\dot{P}j_2 = -j_2\dot{P}P^{-1}.
\]

**Proof.** Take \(B = -j_1^{-1}(b + \frac{1}{2}\dot{j}_1)\) in Lemma 4.6 we have
\[
-j_2(PBP^{-1} + \dot{P}P^{-1}) - \frac{1}{2}j_2 = -j_2(P(-j_1)^{-1}(b + \frac{1}{2}j_1)P^{-1} + \dot{P}P^{-1}) - \frac{1}{2}j_2
\]
\[
= P^{*-1}(b + \frac{1}{2}j_1)P^{-1} - j_2\dot{P}P^{-1} - \frac{1}{2}j_2
\]
\[
= P^{*-1}bP^{-1} - j_2\dot{P}P^{-1} + \frac{1}{2}(P^{*-1}j_1P^{-1} - j_2)
\]
\[
= P^{*-1}bP^{-1} - j_2\dot{P}P^{-1} + \frac{1}{2}(P^{*-1}\frac{d}{dt}(P^{*}j_2P)P^{-1} - j_2)
\]
\[
= P^{*-1}bP^{-1} + Q.
\]
By Lemma 4.6 our results holds. Q.E.D.

The following is a special case of the spectral flow formula.

Let \(j \in C^1([0, T], GL(m, \mathbb{C}))\) be a curve of skew self-adjoint matrices. Then we have symplectic Hilbert spaces \((\mathbb{C}^m, \omega(t))\) with standard Hermitian inner product and \(\omega(t)(x, y) = (j(t)x, y)\), for all \(x, y \in \mathbb{C}^m\) and \(t \in [0, T]\). Then we have a symplectic Hilbert space \((V = \mathbb{C}^m \oplus \mathbb{C}^m, (-\omega(0)) \oplus \omega(T))\). Let \(W \in \mathcal{L}(V)\). Let \(b_s(t) \in \mathcal{B}(\mathbb{C}^m), 0 \leq s \leq 1, 0 \leq t \leq T\) be a continuous family of self-adjoint matrices such that \(b_0(t) = 0\). By Lemma 4.5 there are continuous family of matrices \(M_s(t) \in GL(m, \mathbb{C})\) such that \(M_s(0) = I, M_s(t)^*j(t)M_s(t) = j(0)\) and
\[
-j\frac{d}{dt}M_s(t) - \frac{1}{2}(\frac{d}{dt}j)M_s(t) = b_s(t)M_s(t).
\]
Set

\[ X = L^2([0,T], \mathbb{C}^m), \quad D_m = H^1_0([0,T], \mathbb{C}^m), \]
\[ D_M = H^1([0,T], \mathbb{C}^m), \quad D_W = \{x \in D_M; (x(0), x(t)) \in W\}. \]

Let \( A_M \in \mathcal{C}(X) \) with domain \( D_M \) be defined by

\[ A_M x = -\frac{j}{dt} x - \frac{1}{2} \left( \frac{d}{dt} j \right) x. \]

Set \( x \in D_M, A = A_M |_{D_m}, A_W = A_M |_{D_W}. \) Let \( C_s \in \mathcal{B}(X) \) be defined by \((C_s x)(t) = b_s(t)x(t), x \in X, t \in [0,T].\)

**Proposition 4.3** Set \( W' = \text{diag}(I, M_0(T)^{-1})W. \) Then we have

\[ I(A_W, A_W - C_1) = i_{W'}(M_0^{-1}M_1). \quad (79) \]

**Proof.** The Sobolev embedding theorem shows that \( D_M \subset C([0,T], \mathbb{C}^m). \) For any \( x \in D_M, \) define \( \gamma(x) = (x(0), x(T)). \) Direct calculation shows that \( D_M / D_m = \mathbb{C}^m \oplus \mathbb{C}^m \) with symplectic structure \((\text{diag}(j(0), -j(T))\gamma(x), \gamma(y)), x, y \in D_M, \) and \( \gamma \) is the abstract trace map. Moreover, \( A^* = A_M, \gamma(A^* - C_s) = \text{Gr}(M_s(T)), \) and \( \gamma(D_W) = W. \) By Proposition 4.2 and Lemma 4.3, we have

\[
I(A_W, A_W - C_1) = -\text{sf}\{A_W - C_s\} \\
= \text{Mas}\{\{\text{Gr}(M_s(T)); 0 \leq s \leq 1\},W\} \\
= i_{W'}(M_0(T)(M_0(T)^{-1}M_s(T))I; 0 \leq s \leq 1) \\
= i_{W'}(M_0(T)^{-1}M_s(T); 0 \leq s \leq 1) \\
= -i_{W'}(M_0(t)^{-1}M_0(t); 0 \leq t \leq T) + i_{W'}(M_0(0)^{-1}M_s(0); 0 \leq s \leq 1) \\
+ i_{W'}(M_0(t)^{-1}M_1(t); 0 \leq t \leq T) \\
= i_{W'}(M_0^{-1}M_1). \\
Q.E.D.

## 5 Proof of the main results

In this section we will use the notations in §2.

### 5.1 Proof of Theorem 2.1

**Lemma 5.1** The index forms \( I_{s,R}, 0 \leq s \leq 1 \) is a curve of bounded Fredholm quadratic forms on \( H_R. \)

**Proof.** Since \( I_{s,R} \) are bounded symmetric quadratic forms on \( H_R, \) by Riesz representation theorem, they form a continuous curve.

For each \( k, l = 0, \ldots, m \) and \( s \in [0,1], \) we define the bounded operators \( P_{k,l}(s) \in \mathcal{B}(H_R) \) by

\[
\langle P_{k,l}(s)x,y \rangle_m = \int_0^T \langle p_{k,l}(s,t) \frac{d^k x}{dt^k}, \frac{d^l y}{dt^l} \rangle dt \quad \text{for all } x,y \in H_R.
\]
Claim. \( P_{k,l}(s) \) is compact for either \( k \neq m \) or \( l \neq m \).

Since \( P_{k,l}(s) = P_{k,l}(s) \), without loss of generality we can assume that \( k \neq m \). Pick a bounded sequence \( \{ x_{\alpha} \} \in H_R \). By Sobolev embedding theorem, the sequence \( \{ p_k(t,s)\frac{dx_{\alpha}}{ds} \} \) has a convergent subsequence, which is denoted by the original sequence. Since \( P_{k,l}(s) \) is bounded, we have

\[
\lim_{\alpha, \beta \to +\infty} \| P_{k,l}(s)(x_\alpha - x_\beta) \|_m^2 = \lim_{\alpha, \beta \to +\infty} \int_0^T \langle p_k(t,s) \frac{d^k(x_\alpha - x_\beta)}{dt^k}, \frac{d^l(P_{k,l}(s)(x_\alpha - x_\beta))}{dt^l} \rangle dt = 0.
\]

So the sequence \( \{ P_{k,l}(s)(x_\alpha) \} \) converge and \( P_{k,l}(s) \) is a compact operator.

Now we prove that \( P_{m,m}(s) \) is Fredholm and then our lemma is proved. If \( p_{m,m}(s,t) \) is positive definite for each \( s,t \in [0,1] \), we can choose \( p_{m,m}(s,t) \) such that \( \mathcal{I}_{s,R} \) is positive definite for each \( s \). So \( P_{m,m}(s) \) is a compact perturbation of a Fredholm operator and is Fredholm. Here it is only required that \( p_{m,m}(s,t) \) is continuous in \( t \). In the general case, we have to assume that \( p_{m,m}(s,t) \) is \( C^m \) in \( t \). Consider the operator \( p_{m,m}(s,\cdot) : H \to H \). Let \( j : H_R \to H \) be the injection. Then \( p_{m,m}(s,\cdot) \) is invertible and \( p_{m,m}(s,\cdot) j \) is Fredholm. For any \( x \in H_R \) and \( y = H \), the inner product \( \langle (P_{m,m}(s) - p_{m,m}(s,\cdot)x,y) \rangle \) consists only the lower-order terms (i.e., no second-order differential involved) and some boundary terms. Similar to the above proof, we can conclude that the lower-order terms correspond to compact operators. The boundary terms correspond to finite rank operators. So \( jP_{m,m}(s) - p_{m,m}(s,\cdot) j \) is compact. Since \( p_{m,m}(s) \) and \( j \) are Fredholm, \( jP_{m,m}(s) \) and \( P_{m,m}(s) \) are Fredholm.

The following lemma is the key to the proof of Theorem 2.1.

**Lemma 5.2** (i) Any solution \( u \in H^1([0,T]; C^{2m}) \) of (19) can be expressed by \( u = u_{p_{s,x}} \) for some \( x \in H^m([0,T]; C^n) \), and the following three conditions are equivalent:

(a) \( x \in \ker \mathcal{I}_{s,R} \);

(b) \( x \in \ker L_s, W_{2m}(R) \);

(c) \( u_{p_{s,x}} \) is a solution of (19) and \( (u_{p_{s,x}}, (0), u_{p_{s,x}}(T)) \in W_{2m}(R) \).

(ii) If \( p_s \) is \( C^1 \) in \( s \), then for any \( x, y \in H^m([0,T]; C^n) \), we have

\[
\left\langle \left( \frac{d}{ds} p_s \right) u_{0,x}, u_{0,y} \right\rangle = \left\langle \left( \frac{d}{ds} b(p_s) \right) u_{p_{s,x}}, u_{p_{s,y}} \right\rangle . \tag{80}
\]

(iii) Let \( J \in \text{GL}(C^n) \) be skew self-adjoint, and \( b_s(t) \in gl(C^n), 0 \leq s \leq 1, 0 \leq t \leq T \) is a continuous family of self-adjoint matrices. Let \( \gamma_s \) be the fundamental solutions of the linear Hamiltonian system

\[
-J \dot{u} = b_s u. \tag{81}
\]

If \( b_s \) is \( C^1 \) in \( s \), we have

\[
\frac{\partial}{\partial t} (-J \gamma_s^{-1} \frac{\partial \gamma_s}{\partial s}) = \gamma_s \frac{\partial b_s}{\partial s} \gamma_s. \tag{82}
\]

(iv) If \( p_s \) is \( C^1 \) in \( s \), then for any \( x, y \in \ker L_s \), we have

\[
\left\langle -J_{2m,n} \gamma_p(T)^{-1} \frac{d\gamma_p(T)}{ds} u_{p_{s,x},0}, u_{p_{s,y}}(0) \right\rangle = - \int_0^T \left\langle \left( \frac{d}{ds} p_s \right) u_{0,x}, u_{0,y} \right\rangle dt. \tag{83}
\]
Proof. (i) The proof for the solution \( u \) of (12) can be expressed by \( u = u_{p,s,x} \) and \( (a) \iff (b) \) is standard and we omit it. Now we prove (b) \( \iff (c) \). By (13), we have \( \frac{d}{dt} u_{p,s,x}^{k}(t) = u_{p,s,x}^{k+1}(t) \) for \( k = 0, \ldots, m - 2 \),
\[
\frac{d}{dt} u_{p,s,x}^{m-1}(t) = \frac{d^m}{dt^m} x(t) = p_{m,m}(s,t)^{-1}u_{p,s,x}^{m}(t) - \sum_{0 \leq \beta \leq m-1} p_{m,m}(s,t)^{-1}p_{m,\beta}(s,t)u_{p,s,x}^{\beta}(t)
\]
and
\[
\frac{d}{dt} u_{p,s,x}^{k}(t) = \sum_{2m-k \leq \alpha \leq m, 0 \leq \beta \leq m} (-1)^{\alpha-m} \frac{d^{\alpha+k-1-2m}}{dt^{\alpha+k-1-2m}} \left( p_{\alpha,\beta}(s,t) \frac{d^\beta}{dt^\beta} x(t) \right)
\]
\[
= u_{p,s,x}^{k+1}(t) - \sum_{0 \leq \beta \leq m} (-1)^{m+k+1} \left( p_{2m-k-1,\beta}(s,t) \frac{d^\beta}{dt^\beta} x(t) \right)
\]
\[
= u_{p,s,x}^{k+1}(t) + (-1)^{m+k} p_{2m-k-1,m}(s,t) p_{m,m}(s,t)^{-1}u_{p,s,x}^{m}(t)
\]
\[
+ \sum_{0 \leq \beta \leq m-1} (-1)^{m+k} (p_{2m-k-1,\beta}(s,t)
\]
\[
- p_{2m-k-1,m}(s,t) p_{m,m}(s,t)^{-1}p_{m,\beta}(s,t)) u_{p,s,x}^{\beta}(t)
\]
for \( k = m, \ldots, 2m - 1 \). Combine the above equations and we get
\[
\frac{d}{dt} u_{p,s,x}(t) = J_{2m,n} b(p_s) u_{p,s,x}(t) + (u_{p,s,x}^m(t), 0, \ldots, 0).
\]
(84)

By the fact that \( L_s x = (-1)^m u_{p,s,x}^{2m}(t) \), we get \( (b) \iff (c) \).

(ii) By the definition of \( U(p_s), V(p_s), \bar{u}_{p,s,x} \) and \( \bar{u}_{0,x} \) in (2), direct computation shows
\[
V(p_s)^* \left( \frac{d}{ds} p_s \right) V(p_s) = -\frac{d}{ds} P(p_s).
\]
Thus for all \( x, y \in H_R \), we have
\[
\left\langle \left( \frac{d}{ds} p_s \right) \bar{u}_{0,x}, \bar{u}_{0,y} \right\rangle = -\left\langle U(p_s)^* \left( \frac{d}{ds} P(p_s) \right) U(p_s) \bar{u}_{0,x}, \bar{u}_{0,y} \right\rangle
\]
\[
= -\left\langle \left( \frac{d}{ds} P(p_s) \right) \bar{u}_{p,s,x}, \bar{u}_{p,s,y} \right\rangle
\]
\[
= -\left\langle \left( \frac{d}{ds} b(p_s) \right) u_{p,s,x}, u_{p,s,y} \right\rangle.
\]

(iii) By the definition of \( \gamma_s \), we have \( \gamma_s^* J \gamma_s = J \), and
\[
\frac{\partial}{\partial t} (-J \gamma_s^{-1} \frac{\partial \gamma_s}{\partial s}) = J \gamma_s^{-1} \frac{\partial}{\partial s} \gamma_s \gamma_s^{-1} \frac{\partial}{\partial s} \gamma_s - J \gamma_s^{-1} \frac{\partial^2}{\partial s \partial t} \gamma_s
\]
\[
= J \gamma_s^{-1} (-J^{-1} b_s) \frac{\partial}{\partial s} \gamma_s - J \gamma_s^{-1} \frac{\partial}{\partial s} (-J^{-1} b_s \gamma_s)
\]
\[
= J \gamma_s^{-1} J^{-1} \frac{\partial}{\partial s} b_s \gamma_s
\]
\[
= \gamma_s^* \frac{\partial}{\partial s} b_s \gamma_s.
\]
Proposition 4.2. We have

\[ D(\gamma_{p,x}) = u_{p,x}(0) \text{ for all } x \in \ker L. \ \text{Q.E.D.} \]

Now we can prove Theorem 2.1. We begin with a simple case.

**Lemma 5.3.** Let \( I_{Id,R} \) be the inner product on \( H_R \). If \( \epsilon > 0 \) satisfies \( [-\epsilon,0] \cap \sigma(p_{m,m}(0,t)) = \emptyset \) for all \( t \in [0,T] \), we have

\[ -sf\{I_{0,R} + aI_{Id,R}; a \in [0,\epsilon]\} = iW_{2m}(R)\{\gamma_{p,aI_{(m+1)n}}(T); 0 \leq a \leq T\}. \tag{85} \]

**Proof.** By Lemma 5.2, \( I_{0,R} + aI_{Id,R}, a \in [0,\epsilon] \) is a continuous family of Fredholm quadratic forms. By the definition of the spectral flow we have

\[ sf\{I_{0,R} + aI_{Id,R}; a \in [0,\epsilon]\} = \sum_{a \in (0,\epsilon]} \dim \ker(I_{0,R} + aI_{Id,R}). \tag{86} \]

Set

\[ Z_a = -J_{2m,n} \left( \gamma_{p,aI_{(m+1)n}}(T) \right)^{-1} d\gamma_{p,aI_{(m+1)n}}(T) \frac{da}{a} \]

for \( a \in [0,\epsilon] \). By (iv) of Lemma 5.2, \( Z_a \) is non positive definite. Let \( v \in C^{2mn} \) be a vector such that \( \langle Z_0v, v \rangle = 0 \). By (i) of Lemma 5.2, there exists \( x \in \ker L \) such that \( v = u_{p,x}(0) \). By (iv) of Lemma 5.2, we have \( \bar{u}_{0,x}(t) = 0 \) for all \( t \in [0,T] \). Thus \( x = 0, u_{p,x}(0,x) = 0 \) and \( v = 0 \). So \( Z_a \) is negative definite. By Lemma 4.3, Proposition 4.1, (i) of Lemma 5.2, and the definition of Maslov-type index, we have

\[ iW_{2m}(R)\{\gamma_{p,aI_{(m+1)n}}(T); 0 \leq a \leq T\} = -\sum_{a \in [0,\epsilon]} \dim \text{Gr}(\gamma_{p,aI_{(m+1)n}}(T) \cap W_{2m}(R)) \]

\[ = -\sum_{a \in (0,\epsilon]} \dim \ker(I_{0,R} + aI_{Id,R}). \tag{87} \]

Combine (86) and (87), we get (85). Q.E.D.

**5.2 Proof of Theorem 2.2 and Corollary 2.1**

We now in the position to prove Theorem 2.1.

**Proof of Theorem 2.1.** We divide the proof into two steps.

**Step 1.** We apply Proposition 4.2. Set

\[ A_s = L^s, \quad D_m = H^m_0[0,T; C^n], \quad D_M = H^m([0,T]; C^n). \]

Then \( A_s \) is injective for each \( s \) and \( L_s, W_{2m}(R), 0 \leq s \leq 1 \) is a continuous family of self-adjoint operators. Define the trace map \( \hat{\gamma} : D_M \to C^{4mn} \) by \( \hat{\gamma}(x) = (u_{p,x}(0), u_{p,x}(T)) \) for \( x \in D_M \). Then \( \hat{\gamma} \) induces an isomorphism \( D_M/D_m \to C^{4mn} \). After identifying the two spaces \( D_M/D_m \) and \( C^{4mn} \), we have \( \hat{\gamma} = \gamma \). Direct computation shows

\[ \omega_s(x + D_m, y + D_m) = \langle J_{2m,n}u_{p,x}(0), u_{p,\gamma}(0) \rangle - \langle J_{2m,n}u_{p,x}(T), u_{p,\gamma}(T) \rangle. \]

Let \( D_s \) be the domain of \( L_s, W_{2m} \). Then \( \gamma(D_s) = W_{2m}(R) \) and \( \gamma(\ker A_s) = \text{Gr}(\gamma_{p,s}(T)) \). By Proposition 4.2 we have

\[ -sf\{L_s, W_{2m}(R); 0 \leq s \leq 1\} = \text{Mas}\{W_{2m}(R), \text{Gr}(\gamma_{p,s}(T)); 0 \leq s \leq 1; \omega_s\} \]

\[ = \text{Mas}\{\text{Gr}(\gamma_{p,s}(T)), W_{2m}(R); 0 \leq s \leq 1; -\omega_s\} \]

\[ = iW_{2m}(R)\{\gamma_{p,s}(T); 0 \leq s \leq 1\}. \tag{88} \]
Let $I_{id,R}$ be the inner product on $H_R$. Let $\epsilon > 0$ be small enough such that $[-\epsilon,0] \cap \sigma(p_{m\cdot m},(s,t)) = \emptyset$ for all $(s,t) \in [0,1] \times [0,T]$. By Lemma 6.11, $\mathrm{sf}\{I_s + aI_{id,R}\}$ is well-defined. For each $c \in [0,1]$, there exist $\delta_c > 0$ and $\epsilon_c \in (0,\epsilon)$ such that $\ker(I_s + \epsilon_c I_{id,R}) = \{0\}$ for all $s \in (c - \delta_c, c + \delta_c) \cap [0,1]$.

Let $[s_0,s_1]$ be a subinterval of $(c - \delta_c, c + \delta_c) \cap [0,1]$. Consider the spectral flow $\mathrm{sf}\{I_s + aI_{id,R}\}$ and the Maslov-type index $i_{W_{2m}(R)}(\gamma_{p_s}(T))$. Because of the homotopic invariance of spectral flow and Maslov-type index, both integers must vanish for the boundary loop going counter clockwise around the rectangular domain from the corner point $(s_0,0)$ via the corner points $(s_1,0)$, $(s_1,\epsilon_c)$, and $(s_0,\epsilon_c)$ back to $(s_0,0)$. The spectral flow and Maslov index vanish on the top segment of our box. By the preceding lemma, the left and right side segments of our curves yield vanishing sum of spectral flow and Maslov index. So, by the additivity under catenation, we have

$$-\mathrm{sf}\{I_{s,R}; s_0 \leq s \leq s_1\} = i_{W_{2m}(R)}(\{\gamma_{p_s}(T); s_0 \leq s \leq s_1\}).$$

Since $[0,1]$ is compact, there exist $c_0,\ldots,c_{n-1} \in [0,1]$ and a partition of $0 = s_0 < s_1 < \ldots < s_N = 1$ of $[0,1]$ such that $[s_j,s_{j+1}] \subset (c_j - \delta_{c_j}, c_j + \delta_{c_j})$ for $j = 0,\ldots,N - 1$. Then (59) follows from additivity under catenation of spectral flow and Maslov-type index.

**Step 3.** Since $\gamma_{p_0}(0) = I_{2m\cdot n}$, by the homotopic invariance of Maslov-type index we have

$$i_{W_{2m}(R)}(\{\gamma_{p_s}(T); 0 \leq s \leq 1\}) = i_{W_{2m}(R)}(\gamma_{p_1}) - i_{W_{2m}(R)}(\gamma_{p_0}).$$

Q.E.D.

### Proof of Theorem 2.2

We divide the proof into three steps.

**Step 1.** (23), (24) holds for $C^1$ path $\gamma$ with $\gamma_0 = I_{2m}$. Set $H = L^2([0,T]; C^{2n})$ and $H_R = \{x \in H; (x(0), x(T)) \in R\}$. Let $F_R$ be a closed operator on $H$ with domain $H_{RK}$ defined by $F_R x = -K \dot{x}$ for all $x \in H_R$. Set

$$X = L^2([0,T], C^{2n}), \quad D_{W_K}(R) = \{x \in H^1([0,T]; C^{2n}); (x(0),x(t)) \in W(R)\}.$$

Let $A_{W_K}(R) \in C(X)$ with domain $D_{W_K}(R)$ be defined by $A_{W_K}(R) x = -J_K \dot{x}$ for $x \in D_{W_K}(R)$. Let $b(t) \in \mathrm{gl}(C^{2n})$ and $C \in B(X)$ be defined by $b(t) = -J_K \dot{\gamma}(t)\gamma(t)^{-1}, t \in [0,T]$ and $(Cx)(t) = b(t)x(t)$ for $x \in X, t \in [0,T]$. Then we have $F_R = -F_{RK}$.

Consider the standard orthogonal decomposition

$$C^{2n} = (C^n \times \{0\}) \oplus (\{0\} \times C^n).$$

It induces orthogonal decompositions $X = H \oplus H$ and $D_{W_K}(R) = H_{RK} \oplus H_R$. Under such orthogonal decompositions, $A_{W_K}(R)$ is in block form $A_{W_K}(R) = \begin{pmatrix} 0 & F_R^* \\ F_R & 0 \end{pmatrix}$. Let $C$ be in block form

$$C = \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}.$$

By the definition of $b(t)$ and the symplectic path $\gamma$ we have

$$b(t) = \begin{pmatrix} K^* (\bar{M}_{2,1} M_{1,1}^{-1} - \bar{M}_{2,2} M_{2,2}^{-1} M_{2,1} M_{1,1}^{-1}) \\ -K \bar{M}_{1,1} M_{1,1}^{-1} \end{pmatrix}.$$

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Since $M^*_2K_{1,1} = K$, we have $K^*M^*_2M^*_2 = -(M^*_1)^{-1}M^*_1K^*$. So there holds

$$K^*(\dot{M}^*_{1,1} - \dot{M}^*_{2,2}M^*_2M^*_1) = K^*\dot{M}^*_{2,1}M^*_1 + (M^*_1)^{-1}M^*_1K^*M^*_2M^*_1$$

$$= (M^*_1)^{-1}\left(\frac{d}{dt}(M^*_1K^*M^*_2)\right)M^*_1.$$

Clearly we have

$$\ker(F_R - C_{2,1}) = \{M_{1,1}x(0); (x(0), M_{1,1}(T)x(0)) \in R^K\}.$$

Since $\text{ind}(F_R - C_{2,1}) = \text{ind}F_R = \dim(\text{Gr}(I_{mn}) \cap R^K) - \dim(\text{Gr}(I_{mn}) \cap R)$, we have

$$\dim \ker(F_R - C_{2,1})^* = \dim S(T) + \dim(\text{Gr}(I_{mn}) \cap R) - \dim(\text{Gr}(I_{mn}) \cap R^K).$$

Let $x, y \in \ker(F_R - C_{2,1})$. Then we have

$$\langle C_{1,1}x, y \rangle = \int_0^T \left(\frac{d}{dt}(M^*_1K^*M^*_2)\right)M^*_1x, y dt$$

$$= \int_0^T \left(\frac{d}{dt}(M^*_1K^*M^*_2)\right)M^*_1x(0), M^*_1y(0) dt$$

$$= \int_0^T \left(\frac{d}{dt}(M^*_1K^*M^*_2)\right)x(0), y(0) dt$$

$$= \langle M_{1,1}(T)^*K^*M^*_2(T)x(0), y(0) \rangle.$$

By Proposition 3.4, Proposition 4.3 and the definition of $S(t)$, we have (23) and

$$i_{W_K(R)}(\gamma) = -s\{A_{W_K(R)} - sC; 0 \leq s \leq 1\}$$

$$= m^+(\text{Gr}(I_{mn}) \cap R^K) - \dim S(T).$$

**Step 2.** Define the set

$$Y = \{M \in \text{GL}(\mathbb{C}^{2n}); M = \begin{pmatrix} M^*_{1,1} & 0 \\ M^*_1 & M^*_2 \end{pmatrix}, M^*J_KM = J_K\}.$$

Note that any symplectic loop $\gamma$ in $Y$ is homotopic to the loop in $Y$ starting from $I_{2n}$. By the homotopic invariance of the Maslov-type index and Step 1, we have $i_{W_K(R)}(\gamma) = 0$ for any loop in $\gamma$ in $Y$. For a general $\gamma$ in $Y$, we can connect $I_{2n}$ and the endpoints $\gamma(0)$ and $\gamma(T)$ in $Y$ by $C^1$ paths. Then (23) follows from Step 1 and the path additivity of Maslov-type index under catenation.

Q.E.D.

**Proof of Corollary 2.1.** Let $x = (x_0,\ldots,x_{m-1})$ and $y = (y_0,\ldots,y_{m-1})$ be two vectors in $\mathbb{C}^{mn}$. By direct calculation we get our form of $\gamma_{p_0} = (\gamma_{k,l}(t))_{k,l=0,\ldots,2m-1}$ and (24) with $p_{m,m}(0,t) = p_{m,m}(1,t)$. Then we have

$$\langle M_{1,1}(T)^*K^*_mM^*_2(T)x, y \rangle = \sum_{k,l=0,\ldots,m-1} \left(\frac{1}{(m-k-1)!(m-l-1)!}\int_0^T t^{2m-k-l-2}(p_{m,m}(1,t))^{-1}dt\right)x_k, y_l.$$
\[ \int_0^T \left( (p_{m,m}(1,t))^{-1} \sum_{l=0,\ldots,m-1} \frac{t^{m-l-1}}{(m-l-1)!} x_l, \sum_{k=0,\ldots,m-1} \frac{t^{m-k-1}}{(m-k-1)!} y_k \right) dt \]

Since \( p_{m,m}(1,t) \) is positive definite for each \( t \in [0,T] \), we have \( \langle M_{1,1}(T)^* K_{m,n}^* M_{2,1}(T) x, x \rangle \geq 0 \). If \( \langle M_{1,1}(T)^* K_{m,n}^* M_{2,1}(T) x, y \rangle = 0 \), we have \( \sum_{k=0,\ldots,m-1} \frac{t^{m-k-1}}{(m-k-1)!} x_k = 0 \) for all \( t \in [0,T] \). By taking derivative with \( t \), we have \( \sum_{l=0,\ldots,k} \frac{t^{k-l-1}}{(k-l-1)!} x_l = 0 \) for all \( k = 0, \ldots, m-1 \) and \( t \in [0,T] \). Then we get \( x_k = 0 \) for \( k = 0, \ldots, m-1 \) and \( x = 0 \). Thus \( M_{1,1}(T)^* K_{m,n}^* M_{2,1}(T) \) is positive definite.

Let \( p_s = (1-s)p_0 + sp_1 \). Clearly \( I_{0,R} \) and \( L_{0,W_{2m}(R)} \) is non negative definite. For sufficiently large \( r > 0 \), we have
\[ \langle L_s W_{2m}(R) x, x \rangle = I_s R(x, x) + r(x, x) > 0 \]
for each \( x \neq 0 \) in the domain of \( L_s,W_{2m}(R) \) and \( s \in [0,1] \). Then \( L_s,W_{2m}(R) + rI \) is positive definite for each \( s \in [0,1] \). Note that \( M_{1,1}(0) = I_{mn} \) and \( S(0) = S \). By the definition of the spectral flow, Theorem 2.3 and Theorem 2.2 we have
\[ m^{-}(I_{1,R}) = -\text{sf}\{I_{s,R}; 0 \leq s \leq 1\} \]
\[ = iW_{2m}(R)(\gamma_{p_1}) - iW_{2m}(R)(\gamma_{p_0}) \]
\[ = iW_{2m}(R)(\gamma_{p_1}) - (\text{dim } S(T) + \text{dim } S(0) - \text{dim } S(T)) \]
\[ = iW_{2m}(R)(\gamma_{p_1}) - \text{dim } S \]
\[ = -\text{sf}\{L_s W_{2m}(R); 0 \leq s \leq 1\} \]
\[ = m^{-}(I_{s,W_{2m}(R)}). \]

Q.E.D.

5.3 Proof of Theorem 2.3

Let \( a, p_1, p_1' \) and \( R' \) be as in 2. Firstly we prove 27. The following lemma follows from direct calculation.

**Lemma 5.4** We have
\[ p_1' = \begin{pmatrix} a^* & 0 \\ \hat{a}^* & a^* \end{pmatrix} p_1 \begin{pmatrix} a & \hat{a} \\ 0 & a \end{pmatrix}, \]  
(91)

\[ b(p_1') = \text{diag}(a^{-1}, a^*) b(p_1) \text{diag}(a^{-1}, a) + \begin{pmatrix} 0 & -a^{-1}\hat{a} \\ -\hat{a}^*a^{-1} & 0 \end{pmatrix}. \]  
(92)

By Corollary 4.2 we have

**Corollary 5.1** We have
\[ \gamma_1' = \text{diag}(a^*, a^{-1}) \gamma_1 \text{diag}(a(0)^{-1}, a(0)). \]  
(93)
Proof of Theorem 2.3. By the definition of $R'$ we have

$$(R')^{2,b} = \{(x,y) \in C^{2n} ; (a(0)^*x, a(T)^*y) \in R^{2,b}\}.$$ 

By Theorem 2.2 and Lemma 4.4 we have

$$i_{W_2(R')} (\gamma_1') = i_{W_2(R)} (\gamma_1) + i_{W_2(R')} (\text{dim}(\text{Gr}(I_n) \cap (R')^{2,b})) - \text{dim}(\text{Gr}(a(T)^*a(0)^{-1}) \cap (R')^{2,b})$$

$$= i_{W_2(R)} (\gamma_1) + \text{dim}(\text{Gr}(I_n) \cap (R')^{2,b})) - \text{dim}(\text{Gr}(I_n) \cap R^{2,b}).$$

Q.E.D.

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References

[1] A. A. Agrachev, A. V. Sarychev, Abnormal sub-Riemannian geodesics: Morse index and rigidity, Ann. Inst. Henri Poincaré, Analyse non linéaire., 13(1996), 635-690.

[2] W. Ambrose, The index theorem in Riemannian geometry. Ann. of Math., 73(1961), 49-86.

[3] V.I. Arnol’d, Characteristic class entering quantization conditions, Funkts. Anal. Priloch., 1(1967), 1-14 (Russian). Funct. Anal. Appl., 1(1967), 1-13 (English transl.).

[4] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Spectral asymmetry and Riemannian geometry. III, Proc. Camb. Phic. Soc., 79 (1976), 71-99.

[5] J. K. Beem and P. E. Ehrlich, Cut points, conjugate points and Lorentzian comparison theorems, Proc. Camb. Phic. Soc., 86 (1979), 365-384.

[6] J. K. Beem and P. E. Ehrlich, A Morse index theorem for null geodesics, Duke Math. J., 46(1979), 561-569.

[7] B. Booss-Bavnbek and K. Furutani, The Maslov index – a functional analytical definition and the spectral flow formula, Tokyo J. Math., 21(1998), 1–34.

[8] B. Booss-Bavnbek, M. Lesch, and J. Phillips, Unbounded Fredholm operators and spectral flow, Preprint August 2001, Canad. J. Math. (to appear). (arXiv: [math.FA/0108014]).

[9] B. Booss-Bavnbek and C. Zhu, Weak symplectic functional analysis and general spectral flow formula. (arXiv: [math.DG/0406139]).

[10] B. Booss-Bavnbek and C. Zhu, General spectral flow formula for fixed maximal domain. Preprint.
[11] S. E. Cappell, R. Lee, and E. Y. Miller, *On the Maslov index*, Comm. Pure Appl. Math., **47**(1994), 121-186.

[12] X. Dai and W. Zhang, *Splitting of the family index*, Comm. Math. Phys, **182**(1996), 303-317.

[13] X. Dai and W. Zhang, *Higher spectral flow*, J. Funct. Analysis., **157**(1998), 432-469.

[14] J. J. Duistermaat, *On the Morse index in variational calculus*, Adv. Math., **21**(1976), 173-195.

[15] H. Edwards, *A generalized Sturm Theorem*, Ann. of Math., **80**(1964), 2-57.

[16] P. M. Fitzpatrick, J. Pejsachowicz and L. Recht, *Spectral flow and bifurcation of critical points of strongly-indefinite functionals. I. General theory*, J. Funct. Anal., **162**(1999), 52-95.

[17] A. Floer, *A relative Morse index for the symplectic action*, Comm. Pure Appl. Math., **41**(1988), 393-407.

[18] Roberto Giambò, Paolo Piccione and Alessandro Portaluri, *On the Maslov index of Lagrangian paths that are not transversal to the Maslov cycle. Semi-Riemannian index theorems in the degenerate case*, Comm. Anal. Geom., To appear. (see also arXiv: math.DG/0306187).

[19] A. D. Helfer, *Conjugate points on spacelike geodesics or pseudo-self-adjoint Morse-Sturm-Liouville systems*, Pacific J. Math., **164**(1994), 321-340.

[20] L. Hörmander, *Fourier integral operators I*, Acta Math., **127**(1971), 79-183.

[21] T. Kato, Perturbation Theory for Linear Operators. Springer-Verlag, Berlin. 1980.

[22] Y. Long, *Bott formula of the Maslov-type index theory*, Pacific J. Math., **187**(1999), 113-149.

[23] R. B. Melrose and P. Piazza, *Families of Dirac operators, boundaries and the b-calculus*, J. Diff. Geom., **46**(1997), 99-180.

[24] Y. Long and C. Zhu, *Maslov-type index theory for symplectic paths and spectral flow (II)*, Chinese Ann. of Math., **21B**:1(2000), 89-108.

[25] M. Morse, The Calculus of Variations in the Large. A.M.S. Coll. Publ., Vol.18, Amer. Math. Soc., New York, 1934.

[26] J. Phillips, *Self–adjoint Fredholm operators and spectral flow*, Canad. Math. Bull., **39**(1996), 460–467.

[27] P. Piccione and D. V. Tausk, *The Maslov index and a generalized Morse index theorem for non-positive definite metrics*, C. R. Acad. Sci. Paris Sér. I Math., **331**(2000), 385-389.

[28] P. Piccione and D. V. Tausk, *The Morse index theorem in semi-Riemannian Geometry*, Topology, **41**(2002), 1123–1159. (see also arXiv: math.DG/0011090).
[29] P. Piccione and D. V. Tausk, *An index theory for paths that are solutions of a class of strongly indefinite variational problems*. (arxiv: math.DG/0108044 v1).

[30] J. Robbin and D. Salamon, *The Maslov index for paths*, Topology, 32(1993), 827–844.

[31] J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc., 27(1995), 1–33.

[32] S. Smale, *On the Morse index theorem*, J. Math. Mech., 14(1965), 1049-1056.

[33] K. Uhlenbeck, *The Morse index theorem in Hilbert space*, J. Diff.Geom., 8(1973), 555-564.

[34] C. Zhu, Maslov-type index theory and closed characteristic on compact convex hypersurfaces in $\mathbb{R}^{2n}$. Ph. D. Thesis. Nankai Institute of Mathematics.

[35] C. Zhu, *The Morse Index theorem for regular Lagrangian systems*. (arxiv: math.DG/0109117).

[36] C. Zhu and Y. Long, *Maslov-type index theory for symplectic paths and spectral flow (I)*, Chinese Ann. of Math., 20B:4(1999), 413-424.