Regularization of odd-dimensional AdS gravity: Kounterterms.

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Abstract

As an alternative to the standard Dirichlet counterterms prescription, I introduce the concept of Kounterterms as the boundary terms with explicit dependence on the extrinsic curvature $K_{ij}$ that regularize the AdS gravity action. A suitable choice of the boundary conditions –compatible with any asymptotically AdS (AAdS) spacetime– ensures a finite action principle for all odd dimensions. Background-independent conserved quantities are obtained as Noether charges associated to asymptotic symmetries and their general expression appears naturally split in two parts. The first one gives the correct mass and angular momentum for AAdS black holes and vanishes identically for globally AdS spacetimes. Thus, the second part is a covariant formula for the vacuum energy in AAdS spacetimes and reproduces the results obtained by the Dirichlet counterterms method in a number of cases. It is also shown that this Kounterterms series regularizes the Euclidean action and recovers the correct black hole thermodynamics in odd dimensions.

1 Introduction

In the context of AdS/CFT correspondence [1], Witten sketched the program to regularize the action for AdS spacetimes [2], which was carried out in detail by Hennigson and Skenderis in Ref. [3].

This procedure, known as holographic renormalization, considers a generic form of the metric for an asymptotically AdS (AAdS) spacetime

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{g_{ij}(\rho, x)}{\rho} dx^i dx^j$$

where $\rho$ is the radial coordinate of a manifold $M$, whose boundary is located at $\rho = 0$, and $\ell$ is the AdS radius. This coordinates choice is suitable to describe the conformal structure of the boundary, whose metric $g_{ij}(\rho, x)$ accepts a regular expansion [4]
\[ g_{ij}(\rho,x) = g_{ij}^{(0)}(x) + \rho g_{ij}^{(1)}(x) + \rho^2 g_{ij}^{(2)}(x) + \ldots \] (2)

where \( g_{ij}^{(0)} \) is a given initial value for the metric.

Solving the Einstein equations in this frame reconstructs the spacetime from the boundary data, determining the coefficients \( g^{(k)} \) as covariant functionals in the boundary metric \( g^{(0)} \) (that contain \( k \) derivatives of \( x^i \)). Then, in order to preserve general covariance at the boundary, it is necessary to invert the series to express all the quantities as functions of the boundary metric \( h_{ij} = g_{ij}/\rho \) [5].

In this way, the counterterms method proposes a regularization scheme that consists in the addition to the Dirichlet action of local functionals of the boundary metric \( h_{ij} \), the intrinsic curvature \( R_{ij}^{kl} \) of the boundary and covariant derivatives of the boundary Riemann \( \nabla_m R_{ij}^{kl} \). In \( D = d+1 \) dimensions the regularized action reads

\[
I = -\frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{-g} \left( \dot{R} - 2\Lambda \right) - \frac{1}{8\pi G} \int_{\partial M} d^d x \sqrt{-h} K + \int_{\partial M} d^d x B_{ct}(h,R,\nabla R). \quad (3)
\]

In the above action, hatted curvatures refer to \((d+1)\)-dimensional ones, the cosmological constant is \( \Lambda = -d(d-1)/2\ell^2 \) and \( K = K_{ij} h^{ij} \) is the trace of the extrinsic curvature.

The Gibbons-Hawking term ensures a well-posed variational principle for a Dirichlet boundary condition on the metric \( h_{ij} \) [6], that is still valid in presence of the Dirichlet counterterms series because its functional variation is expressed as \( \delta B_{ct} = (\delta B_{ct}/\delta h_{ij}) \delta h_{ij} \).

As a consequence, a regularized stress tensor for AdS spacetimes is obtained [7, 8] using the Brown-York quasilocal energy-momentum tensor definition [9], without reference to any background solution.

A novel feature of this approach is the appearance –in \( D = 2n+1 \) dimensions– of a vacuum energy for AdS spacetime, that is clearly unobservable in background-dependent methods. In five dimensions, the matching between this vacuum energy and the Casimir energy induced by a precise boundary CFT (\( \mathcal{N} = 4 \) SYM theory with gauge group \( SU(N) \)) is one of the best known examples of the AdS/CFT correspondence [7].

Despite the fact this regularization procedure provides a systematic way to construct the Dirichlet counterterms series, in practice, the number of possible counterterms increase drastically with the dimension. Even for a given dimension, the finiteness of the conserved charges for a more complex solution would require a significant addition of counterterms respect to the same problem, for instance, in Schwarzschild-AdS black hole. Moreover, these extra terms do not seem to obey any particular pattern [10].

In recent papers [11, 12], the problem of Dirichlet counterterms in AdS gravity has been reformulated as an initial-value problem for the extrinsic curvature. This results in a simpler algorithm to obtain the series \( B_{ct} \) but, however, the full series for an arbitrary dimension is still unknown.

Furthermore, in the counterterms method is not clear where the vacuum energy is coming
from, i.e., which boundary terms are responsible for the shifting of the zero-point energy of AdS or whether it can be obtained from a general covariant formula for any AAdS spacetime.

Therefore, a natural question arises: Is there another counterterms series that also regularizes the Einstein-Hilbert action with negative cosmological constant, and whose expression can be worked out in any dimension?

Certainly the answer to that problem implies the departure from a standard action principle based on a Dirichlet boundary condition on the metric $h_{ij}$.

Such construction may at first sound too ambitious. However, we have evidence coming from even-dimensional AdS gravity, where considering a boundary condition for the spacetime curvature

$$\hat{R}_{\alpha\beta} + \frac{1}{\ell^2} \delta^{[\alpha}_{[\alpha'} = 0 \quad (4)$$

on $\partial M$ has been a good alternative to produce a finite action principle $[13, 14]$. In that case, the gravity action is supplemented by the Euler term $\mathcal{E}_{2n}$ with a coupling constant fixed demanding this generic asymptotic condition.

In $D = 2n$ dimensions, the Euler theorem states that the Euler term $\mathcal{E}_{2n}$ is equivalent to the boundary term $B_{2n-1}$ (the $n$–th Chern form) up to the Euler characteristic $\chi_{2n}$, a topological number for the manifold $M$. From a dynamic point of view, $\chi_{2n}$ is just an integration constant and then, the formula for the conserved charges is the same if we supplement the Einstein-Hilbert-AdS action either with the boundary term $B_{2n-1}$ or the Euler (bulk) term $\mathcal{E}_{2n}$ $[15]$. In that sense, the regularizing effect of the Dirichlet counterterm series can be replaced by the Euler term in even dimensions. The clear advantage of this procedure is that we do not need to perform any particular expansion of the metric, nor solving the asymptotic equations in that frame to find the coefficients $g_{(k)ij} = g_{(k)(ij)}(g_{(0)})$, nor inverting the series to express all the quantities as covariant functionals of the boundary metric $h_{ij}$. This procedure contrasts with the simplicity of fixing a single global factor in $\mathcal{E}_{2n}$, given by the asymptotic condition (4). In other words, considering the Euler term as a single entity, its coupling constant comes from fixing the leading order in the curvature, because in the expansion (1), (2), the asymptotic Riemann reads

$$\hat{R}_{ij}^k(G) = O(\rho^2), \quad (5)$$

$$\hat{R}_{kp}^{ki}(G) = -\frac{1}{\ell^2} \delta_i^k + O(\rho^2), \quad (6)$$

$$\hat{R}_{ij}^{kl}(G) = -\frac{1}{\ell^2} \delta_{[ij]}^{[kl]} + \rho \left( R_{ij}^{kl}(g_{(0)}) + \frac{1}{\ell^2} \left( \delta_{[i}^{k[1]} g_{(1)]j] + g_{[1]}^{k[1]} \delta_{j]}^{i]} \right) \right) + O(\rho^2), \quad (7)$$

where $g_{(1)ij} = g_{(0)}^{ij} g_{(1)}$. In sum, what is remarkable in this approach is the fact that we have a closed expression for the boundary term $B_{2n-1}$ in all even dimensions and also, a deep connection between the regularizing boundary terms and topological invariants.

The same approach cannot be applied to odd-dimensional AdS gravity, because there are no topological invariants of the Euler class in $D = 2n + 1$, what makes gravity in even and odd
dimensions quite different. This is not so surprising, because also in standard holographic renormalization there appear technical differences respect the even-dimensional case: the expansion \[2\] requires a log \( \rho \) term at order \( \rho^n \) to be consistent with the equations of motion, the existence of Weyl anomaly, the appearance of a vacuum energy for AdS space, etc.

Following a different strategy, a mechanism to regularize the AdS gravity action in odd dimensions was proposed in [16]. A boundary condition for the extrinsic curvature \( K^i_j \) in AAdS spaces is the key assumption that leads to a well-posed action principle for a given boundary term \( B_{2n} \). This asymptotic condition was motivated by a similar construction in Chern-Simons-AdS gravity [17].

Indeed, in the three-dimensional case, where Einstein-Hilbert-AdS gravity action is a Chern-Simons form for the group \( SO(2, 2) \), this prescription regularizes the Euclidean action and the Noether charges with a single boundary term that is one half the Gibbons-Hawking term. It can be proved that a suitable expansion reduces the problem to the Dirichlet formulation plus a topological invariant of the boundary metric [18].

At this point, as an alternative to the standard counterterms approach, we introduce the concept of Kounterterms as the boundary terms that regularize the AdS gravity action, that posses explicit dependence on the extrinsic curvature \( K_{ij} \) and whose construction is based on boundary conditions compatible with the Fefferman-Graham form of the metric \([2]\).

In this article, we show the general tensorial form the Kounterterms adopt in the odd-dimensional case.

## 2 Kounterterms

We consider the Einstein-Hilbert action with negative cosmological constant, with the addition of a boundary term \( B_d \)

\[
I = -\frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{-g} \left( \hat{R} - 2\Lambda \right) + c_d \int_{\partial M} d^d x B_d
\]

instead of the standard Gibbons-Hawking term plus the counterterms series. Here, \( c_d \) is a coupling constant that will be determined by an appropriate variational principle.

We consider a radial foliation for the spacetime (Gaussian normal coordinates)

\[
ds^2 = N^2(\rho) d\rho^2 + h_{ij}(\rho, x) dx^i dx^j,
\]

but we do not assume any particular expansion of the boundary metric as in eq.\([1]\),\([2]\). In this frame, the expression for the extrinsic curvature adopts a simple form

\[
K_{ij} = -\frac{1}{2N} \partial_\rho h_{ij}.
\]
2.1 Kounterterms and the Euler Theorem

The Kounterterms series differs from the one obtained from the Dirichlet regularization, even in a simple case as it is \( D = 4 \), where it is given by the boundary term [15]

\[
B_3 = 2\sqrt{-h} \delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} K_{i_1}^{j_1} (R_{i_2 j_3}^{j_2} (h) - \frac{2}{3} K_{i_2}^{j_2} K_{i_3}^{j_3}),
\]

with a coupling constant \( c_3 = \ell^2 / (64 \pi G) \). In the above formula, \( R_{ij}^{kl} (h) \) stands for the intrinsic curvature of the boundary metric.

It is clear from the expanded form

\[
B_3 = 4\sqrt{-h} \left[ -\frac{2}{3} K_j^i K_i^j K_i^k K_i^k + K (K_j^j K_i^i - \frac{1}{3} K^2) - 2 (R_i^i - \frac{1}{2} \delta_i^i R) K_i^j \right],
\]

that \( B_3 \) does not contain any term proportional to \( \sqrt{-h} K \) (Gibbons-Hawking term), making evident that it is not derived from a Dirichlet action principle. Notice that the dimensional continuation of (12) is required to define the Dirichlet problem for Einstein-Gauss-Bonnet gravity in \( D \geq 5 \), because it is the generalization of the Gibbons-Hawking term for the quadratic terms in the curvature in the Gauss-Bonnet density [19, 20].

The expression (11) possesses the additional Lorentz symmetry in the tangent space that becomes manifest when it is expressed in terms of the second fundamental form (SFF)

\[
\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB},
\]

defined as the difference between the dynamic spin connection \( \omega^{AB} = \omega_{\mu}^{AB} dx^\mu \) and a reference one \( \bar{\omega}^{AB} \).

The indices of the tangent space run in the set \( A, B = \{0, 1, .., D - 1\} \). In the metric formulation of gravity, the spin connection is determined in terms of the vielbein \( e^A = e^A_\mu dx^\mu \) \( (\mathcal{G}_{\mu \nu} = \eta_{AB} e^A_\mu e^B_\nu) \) as

\[
\omega_{\mu}^{AB} = -e^{B\nu} \nabla_\nu e^A_\mu.
\]

For the radial foliation (9), the natural splitting for the orthonormal basis is

\[
e^1 = N d\rho, \quad e^a = e^a_i dx^i,
\]

where the indices set \( A = \{1, a\} \) and the boundary metric is \( h_{ij} = \eta_{ab} e^a_i e^b_j \).

An adequate choice of the reference connection \( \omega^{AB} \), as obtained from a cobordant product metric

\[
ds^2 = \tilde{N}^2 (\rho) d\rho^2 + \tilde{h}_{ij} (x) dx^i dx^j
\]

is
that matches the dynamic one only on the boundary $\rho = \rho_0$, $\tilde{h}_{ij}(x) = h_{ij}(\rho_0, x)$ leads to a SFF on the boundary as \cite{21, 22, 23, 19}

$$\theta^{1a} = K^i_i dx^i, \quad \theta^{ab} = 0,$$

in terms of the extrinsic curvature $K^i_i = \epsilon^a_i K^j_j$. We stress that the spin connection $\tilde{\omega}^{AB}$ is just introduced on $\partial M$ to restore Lorentz covariance in the boundary term but it is not related to any background-substraction procedure, where the background needs to be a solution of the bulk field equations.

Therefore, the fully Lorentz-covariant expression for the Kounterterms $B_3$ in terms of the Levi-Civita tensor is \cite{18}

$$B_3 = 2 \varepsilon_{ABCD} \theta^{AB} (R^{CD} + \frac{1}{3} \theta^C_F \theta^F_D).$$

In four-dimensional manifolds without boundary, the integration of the Euler-Gauss-Bonnet term

$$\mathcal{E}_4 = \varepsilon_{ABCD} \hat{R}^{AB} \hat{R}^{CD} = -d^4 x \sqrt{-\mathcal{G}} \left( \hat{R}_{\mu\nu\alpha\beta} \hat{R}^{\mu\nu\alpha\beta} - 4 \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \hat{R}^2 \right)$$

is simply proportional to the Euler characteristic $\chi(M_4)$. When a boundary is introduced, the Euler theorem states that there is a correction due to the boundary

$$\int_{M_4} \varepsilon_{ABCD} \hat{R}^{AB} \hat{R}^{CD} = 2 (4\pi)^2 \chi(M_4) + 2 \int_{\partial M_4} \varepsilon_{ABCD} \theta^{AB} \left( R^{CD} + \frac{1}{3} (\theta^2)^{CD} \right),$$

given exactly by the expression \cite{19} that is known as the second Chern form. In fact, using the general formalism reviewed in Appendix A, the boundary term can be seen as a transgression form for the Lorentz group $SO(3,1)$.

In higher even-dimensional AdS gravity, the regularization of the conserved quantities was achieved in the Ref.\cite{14} by the addition of the Euler term $\mathcal{E}_{2n}$. This term is no longer equivalent to the Gauss-Bonnet term, because it is the maximal Lovelock form in that dimension. This construction is locally equivalent, by virtue of the Euler theorem, to the $n$–th Chern form

$$B_{2n-1} = n \int_0^1 dt \varepsilon_{A_1...A_{2n}} \theta^{A_1A_2} (R^{A_3A_4} + t^2 \theta^A_F \theta^F_A) \times \ldots \times (R^{A_{2n-1}A_{2n}} + t^2 \theta^A_{2n-1} \theta^F_{2n})$$

in terms of the continuous parameter $t$. This parametrization is useful not only to write down a compact formula for the boundary term but to generate the relative coefficients of the binomial expansion, as well. This fact has a deep geometrical origin as an explicit realization of the Cartan homotopy operator, that permits to obtain the explicit form of a boundary term whose exterior derivative is the difference of two invariant polynomials for a given Lie group (See Appendix A). It can be shown that the above boundary term also cancels the divergences from radial infinity in the evaluation of the bulk Euclidean action \cite{15}.

\footnote{The wedge product $\wedge$ between the differential forms is omitted throughout the paper.}
2.2 Kounterterms and Chern-Simons Forms

Remarkably, the regularization prescription given by the maximal Chern-form in even dimensions works equally well for Einstein-Gauss-Bonnet gravity, where the coupling constant carried by \( B_{2n-1} \) takes a different value respect to the same problem in Einstein-Hilbert [24]. The same situation is found in a generic even-dimensional Lovelock-AdS theory, where the Kounterterms series [21] preserves its form, but again its coupling constant changes accordingly [25]. This fact strongly suggests the universality of the boundary terms that regularize the action for a set of inequivalent gravity theories (at least, the ones that are Lovelock-type).

On the other hand, a finite action principle was set for Chern-Simons-AdS gravity, a particular Lovelock theory in odd-dimensions that possesses a unique cosmological constant, contains higher powers in the curvature and can be obtained from a Chern-Simons form for the AdS group connection. The guiding line to derive the correct form of the boundary terms is restoring gauge invariance by the use of transgression forms.

As we can see from Appendix A, the expression for the Kounterterms in Chern-Simons-AdS gravity is given by a double integral in the parameters \( t, s \in [0, 1] \)

\[
B_{2n} = n \int_0^1 dt \int_0^t ds \varepsilon_{a_1 \ldots a_{2n+1}} \theta^{A_1 A_2} e^{A_3} (R^{A_4 A_5} + t^2 \theta^{A_4} \theta^{F A_5} + \frac{s^2}{t^2} e^{A_4} e^{A_5}) \times ...
\]

\[
... \times (R^{A_2 A_2 n+1} + t^2 \theta^{A_2 n} \theta^{F A_2 n+1} + \frac{s^2}{t^2} e^{A_2 n} e^{A_2 n+1}).
\]  

(22)

In the spirit of the alternative regularization of AdS gravity in even dimensions, we will assume universality of the form of the regularizing boundary terms in \( D = 2n + 1 \) dimensions. In fact, as we shall explicitly demonstrate below, the boundary term (22) also leads to a finite, well-defined action principle in odd-dimensional Einstein-Hilbert for a suitable choice of its coupling constant \( c_{2n} \).

The relations (18) leave a residual Lorentz symmetry on \( \partial M \) and then, the boundary term can also be written as

\[
B_{2n} = -2n \int_0^1 dt \int_0^t ds \varepsilon_{a_1 \ldots a_{2n}} K^{a_1} e^{a_2} (R^{a_3 a_4} - t^2 K^{a_3} K^{a_4} + \frac{s^2}{t^2} e^{a_3} e^{a_4}) \times ...
\]

\[
... \times (R^{a_{2n-1} a_{2n}} - t^2 K^{a_{2n-1}} K^{a_{2n}} + \frac{s^2}{t^2} e^{a_{2n-1}} e^{a_{2n}}),
\]  

(23)

where \( R^{ab} \) is the boundary 2-form curvature, related to the intrinsic curvature by \( R^{ab} = \frac{1}{2} R^{ij} e^a_i e^b_j dx^i \wedge dx^j \).

The tensorial form of the Kounterterms can be worked out projecting all the quantities in the boundary indices (see Appendix B)

\[
B_{2n} = \sqrt{\hbar} \int_0^1 dt \int_0^t ds \varepsilon^{a_{11} \ldots a_{2n-1}} (J_{j_1}) e^{j_1} (\frac{1}{2} R^{j_2 j_3} - t^2 K^{j_2} K^{j_3} + \frac{s^2}{t^2} \delta^{j_2} \delta^{j_3}) \times ...
\]

\[
... \times (\frac{1}{2} R^{j_{2n-2} j_{2n-1}} - t^2 K^{j_{2n-2}} K^{j_{2n-1}} + \frac{s^2}{t^2} \delta^{j_{2n-2}} \delta^{j_{2n-1}}),
\]  

(24)
In the language of differential forms, the gravitational action (8) can be written in terms of the local orthonormal frame \( e^A = e^A_\mu dx^\mu \) and the 2-form Lorentz curvature \( \hat R^{AB} = \frac{1}{2} \hat R_{\mu\nu}^{AB} dx^\mu \wedge dx^\nu \) (constructed up from the spin connection as \( \hat R^{AB} = d\omega^{AB} + \omega^A_\alpha \omega^{CB} \)) as

\[
I = \kappa_D \int_M \varepsilon_{A_1 A_2 \ldots A_D} \left( \hat R^{A_1 A_2} + \frac{(D-2)}{D!} e^{A_1} e^{A_2} \right) e^{A_3} e^{A_4} + c_d \int_{\partial M} B_d. \tag{25}
\]

where the constant \( \kappa_D = (16\pi G (D-2)!)^{-1} \) and with the boundary term \( B_d \) given by (21) and (22) for even and odd dimensions, respectively. The Lorentz curvature is related to the spacetime Riemann tensor by \( \hat R^{AB} = \frac{1}{2} \hat R^{\alpha\beta}_{\mu\nu} e^A_\alpha e^B_\beta dx^\mu \wedge dx^\nu \).

An arbitrary variation of the above action produces the Einstein equations plus a surface term \( \Theta \)

\[
\delta I = \int_M E_A \delta e^A + d\Theta, \tag{26}
\]

where \( E_A \delta e^A \) is the Einstein equation,

\[
E_A \delta e^A = \frac{1}{16\pi G (D-3)!} \varepsilon_{AA_2 \ldots A_D} \delta e^A \left( \hat R^{A_2 A_3} + \frac{1}{D!} e^{A_2} e^{A_3} \right) e^{A_4} e^{A_5} + c_d \delta B_d, \tag{27}
\]

\[
E_A \delta e^A = \frac{1}{16\pi G} \sqrt{-G} \left( \hat R_{\mu\nu} - \frac{1}{2} \hat R_G_{\mu\nu} - \Lambda G_{\mu\nu} \right) \delta G_{\mu\nu}. \tag{28}
\]

In the Palatini formulation of gravity, the contribution to \( \Theta \) coming from the bulk term is obtained from the variation of the Riemann tensor \( \delta \hat R^\alpha_{\beta\mu\nu} = \nabla_\mu \delta \hat \Gamma^\alpha_{\beta\nu} - \nabla_\nu \delta \hat \Gamma^\alpha_{\beta\mu} \). For the radial foliation (9), the surface term will involve only certain components of the connection \( \hat \Gamma^\alpha_{\mu\nu} \), related to the extrinsic curvature as

\[
\hat \Gamma^\rho_{ij} = \frac{1}{N} K^\rho_{ij}, \quad \hat \Gamma^i_{\rho j} = -NK^i_{\rho j}, \tag{29}
\]

such that it can be written as

\[
\Theta = -\varepsilon_{a_1 a_2 \ldots a_d} \delta K^{a_1} e^{a_2} \ldots e^{a_d} + c_d \delta B_d. \tag{30}
\]

The same coordinates frame implies that the components of the Lorentz curvature \( \hat R^{AB} \) projected on \( \partial M \) are

\[
\hat R^{1a} = DK^a = D_i K^a_j dx^i \wedge dx^j, \tag{31}
\]

\[
\hat R^{ab} = R^{ab} - K^a K^b = \left( \frac{1}{2} R_{ij}^{ab} - K^a K^b_i \right) dx^i \wedge dx^j \tag{32}
\]

where \( D_i \) is the covariant derivative in the boundary indices (defined with the submanifold spin connection \( \omega^{ab} \)) and we have dropped the components along \( dp \). Eqs. (31) and (32) are just the well-known Gauss-Coddazzi relations for a radial foliation (9).
\[ \hat{R}^d_{ij} = -\frac{1}{N} \nabla[iK^l_j], \]  
\[ \hat{R}^{kl}_{ij} = R^{kl}_{ij} - K^k_i K^l_j + K^l_i K^k_j. \]  
\( (33) \)  
\( (34) \)

As an illustrative, simple example of the present procedure, in the next section we show explicitly how the Kounterterms series in five dimensions leads to a well-posed action principle for boundary conditions derived from the asymptotic form for AAdS spacetimes \( (1), (2) \).

### 3 Five-Dimensional Case

Dirichlet counterterms are local functional that preserve general covariance at the boundary. Kounterterms respect Lorentz-covariance in the tangent space, what provides a criterion to select them. From the surface term \( (30) \), it is clear that \( B_d \) must be constructed with the same parity as the bulk action, that is, the same invariant tensor \( \epsilon_{i_1...i_5} \) for the Lorentz group (and not \( \delta_{[AB]}^{[CD]} \)). In particular, this argument rules out the addition of topological invariants of the Pontryagin class in even dimensions and many possible boundary terms in the case we are treating here.

Kounterterms are built up as totally antisymmetric \( 2n \)-forms. This eliminates a possible inclusion of terms containing covariant derivatives of the intrinsic curvature, which are discarded by the Bianchi identity \( \nabla [m R^{kl}_{ij}] = 0. \)

The extrinsic curvature can be defined in an arbitrary frame as \( K_{AB} = -h^{C}_{A} h^{D}_{B} n_C n_D \), where \( n^A \) is a unit vector normal to the boundary, and related to the SFF by \( \theta^{AB} = n^A K^B - n^B K^A \), \( (K^A = K^A_B e^B) \). In that way, we can always write down the Kounterterms as a fully-covariant expression, independent of any particular foliation. In five dimensions this is given by

\[ B_4 = \epsilon_{A_1...A_5} \theta^{A_1 A_2} \epsilon^{A_3} \left( R^{A_4 A_5} + \frac{1}{2} \theta^{A_4 C} \theta^{C A_5} + \frac{1}{6 \ell^2} \epsilon^{A_4} e^{A_5} \right), \]  
\( (35) \)

with the equivalence in tensorial notation (see Appendix B)

\[ B_4 = \sqrt{-T} \delta^{[i_1 i_2 i_3]} K^{[j_1] i_1} \left( R_{j_2 j_3}^{i_2 i_3} - K^{j_2}_{i_2} K^{j_3}_{i_3} + \frac{1}{3 \ell^2} \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right), \]  
\( (36) \)

#### 3.1 Variational principle and asymptotic conditions

Arbitrary variations of the total action \( (8) \) produce a surface term

\[ \delta I_5 = -2 \int_{\partial M} \kappa_5 \epsilon_{abcd} \delta K^a b e^c e^d + c_4 \epsilon_{abcd} \delta K^a b \left( R^{cd} - \frac{3}{2} K^c K^d + \frac{1}{6 \ell^2} e^c e^d \right) + c_4 \epsilon_{abcd} K^a b \left( R^{cd} - \frac{1}{2} K^c K^d + \frac{1}{2 \ell^2} e^c e^d \right), \]  
\( (37) \)
when equations of motion hold. The constant in front of the bulk action is $\kappa^5 = 1/(96\pi G)$. The above equation can be conveniently rewritten as

$$\delta I_5 = -2 \int_{\partial M} \kappa_5 \varepsilon_{abcd} \delta K^a e^b e^c e^d + 2 c_4 \varepsilon_{abcd} \delta K^a e^b \left( \bar{R}^{cd} + \frac{1}{3\ell^2} e^c e^d \right)$$

$$- c_4 \varepsilon_{abcd} (\delta K^a e^b - K^a \delta e^b) \left( R^{cd} - \frac{1}{2} K^c K^d + \frac{1}{2\ell^2} e^c e^d \right),$$

(38)

(39)

with the help of the Gauss-Coddazzi relation (32).

A well-posed variational principle for precise asymptotic conditions in AdS gravity is essential to attain the finiteness of the conserved quantities and the Euclidean action. This amounts to on-shell cancelation of the surface term (37) using boundary conditions derived from the asymptotic form of AAdS spacetimes (1),(2).

In general, the Dirichlet variational problem for gravity is well-defined if one supplements the action by the Gibbons-Hawking term and fixes the metric $h_{ij}$ at the boundary. In this way, the surface term coming from an arbitrary variation of the Dirichlet action vanishes identically, no matter if the boundary is at a finite distance (e.g., on a brane, to define the Israel matching conditions) or at infinity. However, it has been recently argued in ref.[11] that the standard Dirichlet boundary condition for the metric $h_{ij}$ does not really make sense for manifolds with conformal boundary, as it is the case of AAdS spaces. It is clear from eqs.(1),(2) that the variation of $h_{ij}$ is divergent at $\rho = 0$ and one should instead fix the conformal structure $g(0)_{ij}$. However, due to the divergence produced in this way, the action requires additional boundary terms on top of the Gibbons-Hawking term, that turn out to be the standard Dirichlet counterterms.

In the present formulation, we will consider a boundary condition that derives from the asymptotic expansion of the extrinsic curvature

$$K^i_j = K_{ij} h^{li} = \frac{1}{\ell} \delta^i_j - \frac{\rho}{\ell} (g(0)_{ij} g(1))_j - \frac{\rho^2}{\ell} (2g(0)_{ij} g(2) - g(0)_{ij} g(1))_j + ..., \quad (40)$$

in increasing powers of $\rho$. This asymptotic behavior implies that the extrinsic curvature satisfies

$$\delta K^i_j = \frac{1}{\ell} \delta^i_j,$$

(41)

on the conformal boundary, and an arbitrary variation is given by

$$\delta K^i_j = 0.$$

(42)

Had we attempted to fix $K_{ij} = \frac{1}{\ell} \frac{g(0)_{ij}}{\rho} + ...$, we would have faced the same problem as fixing the boundary metric $h_{ij}$. On the contrary, eq.(42) is a regular boundary condition on the extrinsic curvature that can be derived from fixing $g(0)_{ij}$ due to the asymptotic form of AAdS spaces. In particular, in the asymptotically flat limit ($\ell \to \infty$) this accident no longer occurs.

The condition (41) has been also taken as the boundary data for the problem of holographic reconstruction of the spacetime in terms of the extrinsic curvature in ref.[11].
Making explicit the indices in the term proportional to the curl $\varepsilon_{a b c d} (\delta K^a e^b - K^a \delta e^b)$, the last line of eq. (39) is

$$\varepsilon_{a b c d} \varepsilon^{i_1 i_2 i_3 i_4} \left[ \delta K^j_{i_1} e^j_{i_2} + \delta e^j_{i_1} \left( K^j_{i_1} \delta e^i_{i_2} - K^i_{i_2} \delta e^j_{i_1} \right) \right] \left( R^{c d}_{i_3 i_4} - K^c_{i_3} K^d_{i_4} + \frac{1}{\ell^2} e^c_{i_3} e^d_{i_4} \right) d^4 x,$$  (43)

that vanishes identically when we take the condition (41) on the extrinsic curvature and its variation (42). We assume a constant (negative) curvature in the asymptotic region (4), that in the language of differential forms reads

$$\hat{R}^{A B} + \frac{1}{\ell^2} e^A e^B = 0,$$  (44)

that in particular holds for the boundary indices. This is a local condition at the boundary known as ALAdS (asymptotically locally AdS) that in principle does not impose further restrictions on the global topology of the spacetime. Therefore, solutions of this class include not only point-like black holes as Schwarzschild-AdS and Kerr-AdS, but also extended objects as black strings.

The coupling constant $c_4$ is then fixed as

$$c_4 = \frac{3\kappa_5 \ell^2}{4} = \frac{\ell^2}{128\pi G},$$  (45)

to cancel the rest of the surface term.

Now that we know the coefficient in front of the boundary term, we notice that the term proportional to $\sqrt{-h} K$ carries an anomalous factor $\frac{1}{64\pi G}$ compared to the one of the Gibbons-Hawking term in eq. (3), what is a consequence of a different action principle.

### 3.2 Conserved quantities

In the standard Dirichlet problem of gravity, the bulk contribution to (39) is canceled by the Gibbons-Hawking term and no other terms along $\delta K^a$ can appear from the boundary term $B_{ct}$, as it is a functional only of intrinsic quantities.

In the present approach, the surface term (39) contains variations along the extrinsic curvature, such that we cannot identify a quasilocal (boundary) stress tensor from the variation of the total action.

However, we can always define the energy and other conserved charges associated to asymptotic symmetries of a gravitational system through the Noether theorem.

In the Appendix C we summarize the construction of the Noether charges for an arbitrary Lagrangian. The conserved current in the case we are considering here is given by

$$* J = - \Theta (e^a, K^a, \delta e^a, \delta K^a) - i_\xi (L_5 + c_4 dB_4),$$  (46)

where $L_5$ is the bulk Lagrangian, $i_\xi$ is the interior derivative (also known as contraction operator) with the Killing vector $\xi^\mu$ defined in the Appendix C and $\Theta$ is the surface term.
\[ \Theta = \frac{\ell^2}{32\pi G} \left[ \epsilon_{abcd} \delta K^a e^b \left( \tilde{R}^{cd} + \frac{1}{\ell^2} e^c e^d \right) ight. \\
- \frac{1}{2} \epsilon_{abcd} \left( \delta K^a e^b - K^a \delta e^b \right) \left( \tilde{R}^{cd} - \frac{1}{2} K^c K^d + \frac{1}{2\ell^2} e^c e^d \right) \right]. \] (47)

The derivation of the conserved charges from the current extensively uses the properties of interior, exterior and Lie derivatives, Bianchi identity, and the equations of motion in differential forms language. However, we will exploit a shortcut for the charges (207) pointed out in Appendix C, by identifying the contributions coming from the bulk Lagrangian and the boundary term \( B_4 \).

In doing so, the Noether charge is written as

\[ Q(\xi) = K(\xi) + c_4 \int_{\partial \Sigma} \left( i_\xi K^a \frac{\delta B_4}{\delta K^a} + i_\xi e^a \frac{\delta B_4}{\delta e^a} \right), \] (48)

where the first term is known as the Komar’s integral

\[ K(\xi) = \frac{1}{48\pi G} \int_{\partial \Sigma} \epsilon_{abcd} i_\xi K^a e^b e^c e^d, \] (49)

and it is the conserved quantity associated to the bulk term in the gravity action\(^2\).

The expression for the Noether charge appears to be naturally split in two parts, associated to the first and second line of the surface term (47), respectively

\[ Q(\xi) = q(\xi) + q_0(\xi), \] (50)

\[ q(\xi) = \frac{\ell^2}{32\pi G} \int_{\partial \Sigma} \epsilon_{abcd} i_\xi K^a e^b \left( \tilde{R}^{cd} + \frac{1}{\ell^2} e^c e^d \right), \] (51)

\[ q_0(\xi) = -\frac{\ell^2}{64\pi G} \int_{\partial \Sigma} \epsilon_{abcd} \left( i_\xi K^a e^b + K^a i_\xi e^b \right) \left( \tilde{R}^{cd} - \frac{1}{2} K^c K^d + \frac{1}{2\ell^2} e^c e^d \right). \] (52)

As we shall see below for concrete examples, the formula for \( q(\xi) \) provides the standard conserved quantities (mass, angular momentum) for AAdS solutions. In tensorial notation, \( q(\xi) \) takes the form\(^2\)

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\(^2\)In an arbitrary dimension \( D > 3 \), the Komar’s integral has the form

\[ K(\xi) = \frac{1}{8\pi G_N} \int_{\partial \Sigma} \epsilon_{a_1...a_{D-1}} i_\xi K^{a_1} e^{a_2}...e^{a_{D-1}} = \frac{1}{16\pi G_N} \int_{\partial \Sigma} \nabla^\mu \xi^\nu d\Sigma_{\mu\nu}, \]

where \( d\Sigma_{\mu\nu} = (D-2)!\epsilon_{\mu\nu a_1...a_{D-2}} dx^{a_1} \wedge ... \wedge dx^{a_{D-2}} \) is the dual of the area \( (D-2) \)-form (\( \sigma \) is the determinant of the metric on the sphere \( S^{D-2} \)). When evaluated for a timelike Killing vector on the Schwarzschild-AdS solution, this formula gives a factor \( (D-3)/(D-2) \) times the mass, plus a divergence in the radial coordinate. The divergence is usually canceled by background substraction, procedure that however does not solve the problem of the anomalous Komar factor [26].
\[ q(\xi) = \frac{\ell^2}{64\pi G} \int_{\partial \Sigma} \sqrt{-h} \epsilon_{i_1...i_4} (\xi^k K_k^{i_1}) \delta_{j_2}^{i_2} \left( \hat{R}_{j_3j_4}^{i_3i_4} + \frac{1}{\ell^2} \delta_{[j_3j_4]}^{i_3i_4} \right) dx^{i_2} dx^{i_3} dx^{i_4}, \]  

(53)

where the product \( dx^{i_2} \wedge dx^{i_3} \wedge dx^{i_4} \) stands for the volume element of the boundary of the spatial section \( \partial \Sigma \) (i.e., boundary indices without the time). Notice that (53) is proportional to the l.h.s. of eq. (4), and therefore vanishes identically for an \( \eta \) solution of global constant curvature, as AdS vacuum and spacetimes with topological identifications that preserve AdS flatness.

The above reasoning leads us to consider (52) as a general formula for the vacuum energy in five-dimensional AdS spacetimes

\[ q_0(\xi) = -\frac{\ell^2}{128\pi G} \int_{\partial \Sigma} \sqrt{-h} \epsilon_{i_1i_2i_3i_4} \xi^k (K_k^{i_1} \delta_{j_2}^{i_2} + K_k^{i_2} \delta_{j_2}^{i_1}) \left( R^{i_3i_4}_{j_3j_4} - \frac{1}{2} K_{(i_3i_4]}^{j_3j_4)} + \frac{1}{2\ell^2} \delta_{[j_3j_4]}^{i_3i_4} \right) dx^{i_2} dx^{i_3} dx^{i_4}, \]  

(54)

where we have introduced the shorthand

\[ K_{[kl]}^{ij} = K_k^{i} K_l^{j} - K_k^{j} K_l^{i}. \]  

(55)

The regularization of the conserved quantities and the Euclidean action in five-dimensional AdS solutions is illustrated through concrete examples below.

3.3 Examples

-Schwarzschild-AdS black hole and topological extensions

Static solutions to EH-AdS gravity are given by Schwarzschild black hole metric. Due to the negative cosmological constant, the transversal section \( \Sigma_k^{D-2} \) can be the sphere \( S^{D-2} \), a \((D-2)\)-dim locally flat space or the hyperboloid \( H^{D-2} \)

\[ ds^2 = -\Delta^2(r)dt^2 + \frac{dr^2}{\Delta^2(r)} + r^2 \gamma_{mn} d\theta^m d\theta^n, \]  

(56)

where the metric function is

\[ \Delta^2(r) = k - \frac{2\omega_D G \mu}{r^{D-3}} + \frac{r^2}{\ell^2}. \]  

(57)

Here, \( \mu \) appears as an integration constant, \( \omega_D = \frac{8\pi}{(D-2) Vol(S^{D-2})} \) and \( \gamma_{mn} (m, n = 1, ..., D-2) \) is the metric of \( \Sigma_k^{D-2} \) of constant curvature \( k = \pm 1, 0 \). The event horizon \( r_+ \) is defined as the largest root of \( \Delta^2(r_+) = 0 \).

In order to compute the mass and the vacuum energy, we use the following components of the extrinsic and intrinsic curvatures

\[ K_l^i = -\Delta', \quad K_m^m = -\frac{\Delta}{r} \delta_m^m, \]  

(58)

\[ R_{[m,n]}^{[m,n]} = \frac{k}{r^2} \delta_{[m,n]}^{[m,n]}. \]  

(59)
where prime denotes the derivative $d/dr$, and the determinant of the boundary metric

$$\sqrt{-h} = \Delta \sqrt{\gamma} r^{D-2}. \quad (60)$$

Notice the opposite sign in the leading order of the asymptotic behavior of $K^i_j$ respect to eq.(40), due to the change in the radial coordinate.

Thus, for the time-like Killing vector $\xi = \partial/\partial t$, formula (53) gives

$$q(\partial_t) = \Delta 3! Vol(\Sigma_3) \lim_{r \to \infty} (\Delta^2)' r \left\{ \kappa_3 r^2 + 2 c_4 (k - \Delta^2 + \frac{r^2}{3\ell^2}) \right\}, \quad (61)$$

$$= \frac{Vol(\Sigma_3)}{Vol(S^2)\mu}, \quad (62)$$

in agreement with the result from background-dependent methods, e.g., Hamiltonian formalism [27]. We have written eq.(61) as an intermediate step just to make clear the contributions from the bulk and the boundary that will be useful for the black hole thermodynamics below.

Plugging the metric (56) into the formula (54), we obtain

$$q_0(\partial_t) = E_0 = -3! Vol(\Sigma_3) c_4 \lim_{r \to \infty} (\Delta^2 - \frac{r(\Delta^2)'}{2}) \left( k - \frac{1}{2} \Delta^2 + \frac{r^2}{2\ell^2} \right) \quad (63)$$

$$= (-k)^2 \frac{3Vol(\Sigma_3)}{64\pi G \ell^2}.$$ \quad (64)

The result for the spherical case ($k = 1$) is the Balasubramanian-Kraus vacuum energy $E_0 = \frac{3\pi\ell^2}{32G}$ for SAdS black hole that appears in the Dirichlet regularization of the stress tensor [7].

In the case $k = -1$, cosmic censorship holds for a mass over a critical value $M_c = \frac{Vol(H^3)}{Vol(S^2)} \mu_c$ that separates black holes with hyperbolic transversal section from naked singularities. This value, for an arbitrary dimension, is

$$\mu_c = \frac{\ell^{D-3}}{\omega_D G} \sqrt{\frac{(D-3)^{D-3}}{(D-1)^{D-1}}}, \quad (65)$$

where $\omega_D = \frac{8\pi}{(D-2)Vol(S^{D-2})}$. The critical mass in five dimensions has the same value (with opposite sign) as the vacuum energy $E_0$. Therefore, despite the fact that in 5D AdS gravity there exist black holes with negative mass, the vacuum energy restores the positivity of the total energy $E = M + E_0$ [8].

We show now that the boundary term (36) makes finite the Euclidean action (8) for five-dimensional SAdS solution, and that correctly describes black hole thermodynamics.

The Euclidean period $\beta$ is defined as

$$\beta = T^{-1} = \frac{4\pi}{(\Delta^2)'|_{r_+}}.$$ \quad (66)
where $T$ is the black hole temperature. This condition comes from the requirement that, in the Euclidean sector the solution (56) does not have a conical singularity at the coordinates origin $(r = r_+)$. In the canonical ensemble, the Euclidean action

$$S = \beta \mathcal{E} - I^E,$$  \hfill (67)

defines the entropy $S$ and the thermodynamic energy

$$\mathcal{E} = -\frac{\partial I^E}{\partial \beta},$$  \hfill (68)

e of a black hole for a fixed temperature. The Euclidean bulk action is evaluated for a static black hole of the form (56) as a total derivative in the radial coordinate, such that

$$I^E_{\text{bulk}} = -\kappa_5 3! \text{Vol}(\Sigma^k_3) \beta \left[ (\Delta^2)^{r_3} \right]_{r_+}^\infty,$$  \hfill (69)

and the Euclidean boundary term as

$$\int_{\partial M} B^E_4 = -3! \text{Vol}(\Sigma^k_3) \beta \left[ r(\Delta^2)^{r_3} \left( k - \Delta^2 + \frac{r^2}{3\ell^2} \right) + \left( \Delta^2 - \frac{r(\Delta^2)^{r_3}}{2} \right) \left( k - \frac{1}{2} \Delta^2 + \frac{r^2}{2\ell^2} \right) \right]_{r_+}^\infty.$$  \hfill (70)

The total Euclidean action $I^E_5 = I^E_{\text{bulk}} + c_4 \int_{\partial M} B^E_4$ contains two contributions: the first one from the bulk at radial infinity plus the boundary term that can be identified (using eqs. (61) and (63) as $-\beta(M + E_0)$.

Therefore, the finiteness of the Noether charges for static black holes ensures that the divergencies at $r = \infty$ of the bulk Euclidean action are exactly canceled by the ones in the boundary term $B_4$.

It is reassuring to check that the thermodynamic energy definition

$$\mathcal{E} = -\frac{\partial I^E}{\partial \beta / \partial r_+} = M + E_0,$$  \hfill (71)

recovers the same result for the total energy as from the Noether charges defined above.

Finally, the entropy is the horizon contribution of the bulk Euclidean action

$$S = \frac{\text{Vol}(\Sigma^k_3) r_3^3}{4G} = \frac{\text{Area}}{4G},$$  \hfill (72)

-Kerr-AdS black hole

The general Kerr-AdS metric in five dimensions possesses two rotation parameters $a, b$, and it can be written in Boyer-Lindquist coordinates as [28].
\[ ds^2 = -\frac{\Delta_r}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right)^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \] 
\[ + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left( d\theta - \frac{(r^2 + a^2)}{\Xi_a} d\phi \right)^2 + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left( d\phi - \frac{(r^2 + b^2)}{\Xi_b} d\psi \right)^2 + \]
\[ + \frac{(1 + r^2/\ell^2)}{r^2 \rho^2} \left( abdt - b \frac{(r^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi - a \frac{(r^2 + b^2) \cos^2 \theta}{\Xi_b} d\psi \right)^2, \quad (73) \]

where the functions in the metric are

\[ \Delta_r \equiv \frac{1}{r^2} \left( r^2 + a^2 \right) \left( r^2 + b^2 \right) \left( 1 + r^2/\ell^2 \right) - 2m, \quad (74) \]
\[ \Delta_\theta \equiv 1 - \frac{a^2}{\ell^2} \cos^2 \theta - \frac{b^2}{\ell^2} \sin^2 \theta, \quad (75) \]
\[ \rho^2 \equiv r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad (76) \]
\[ \Xi_a \equiv 1 - \frac{a^2}{\ell^2}, \quad \Xi_b \equiv 1 - \frac{b^2}{\ell^2}, \quad (77) \]
\[ 0 \leq \theta \leq \pi/2, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi. \quad (78) \]

The event horizon \( r_+ \) is the largest solution of the equation \( \Delta_r(r_+) = 0 \), whose area is

\[ \text{Area} = \frac{2\pi^2(r_+^2 + a^2)(r_+^2 + b^2)}{r_+ \Xi_a \Xi_b}. \quad (79) \]

Evaluating the charge formula \((53)\) for the metric \((73)-(78)\), we have

\[ E' = q(\partial_t) = \frac{3\pi}{4G} \frac{m}{\Xi_a \Xi_b}, \quad (80) \]
\[ E_0 = q_0(\partial_t) = \frac{\pi \ell^2}{96 \Xi_a \Xi_b G} \left( 7\Xi_a \Xi_b + \Xi_a^2 + \Xi_b^2 \right), \quad (81) \]

and the angular momenta

\[ J_a = q(\partial_\phi) = \frac{\pi}{2G} \frac{ma}{\Xi_a^2 \Xi_b}, \quad (82) \]
\[ J_b = q(\partial_\psi) = \frac{\pi}{2G} \frac{mb}{\Xi_a \Xi_b^2}. \quad (83) \]

The quantity \((80)\) is the energy obtained by Awad and Johnson in \([29]\) that, however, does not satisfy the first law of black hole thermodynamics, as has been pointed out in \([30]\). The expression for the vacuum energy \( E_0 \) is in agreement with \([29]\), and it is equivalent to the Papadimitriou-Skenderis result \([11]\).
both computed using different versions of the counterterms method.

Notice that the physical energy for Kerr-AdS is obtained with a Killing vector that does not rotate at infinity $\xi = \partial_t + \frac{a}{\ell^2} \partial_\phi + \frac{b}{\ell^2} \partial_\psi$,

$$E = q(\partial_t + \frac{a}{\ell^2} \partial_\phi + \frac{b}{\ell^2} \partial_\psi) = \frac{\pi m}{4G\Xi_a\Xi_b}(2\Xi_a + 2\Xi_b - \Xi_a\Xi_b). \quad (85)$$

The on-shell Euclidean action is

$$I_5^E = \frac{\beta}{2\pi G\ell^2} \int_{r_+}^\infty dr \int d\Omega N \sqrt{-h} + \beta c_1 \int d\Omega (B_4)|^{r=\infty} \quad (86)$$

and we see that the boundary term cancels out the divergences coming from the bulk action, such that we have the finite result

$$I_5^E = \frac{\beta \pi}{96\ell^2 \Xi_a \Xi_b G} (-a^4 + 9a^2 \ell^2 + 24r_+^2 a^2 + 17a^2 b^2 - 24m\ell^2 - 9\ell^4 + 24r_+^4 - b^4 + 9\ell^2 b^2 + 24r_+^2 b^2), \quad (87)$$

This expression can also be put into the form of Awad-Johnson

$$I_5^E = \frac{\beta \pi}{96\ell^2 \Xi_a \Xi_b G} (12 \frac{r_+^2}{\ell^2} (1 - \Xi_a - \Xi_b) + \Xi_a^2 + \Xi_b^2 + \Xi_a \Xi_b + 12r_+^4/\ell^4 - 2(a^4 + b^4)/\ell^4 - 12 (a^2 b^2/\ell^4) (\ell^2/r_+^2 - 1/3) - 12), \quad (88)$$

or the more compact one obtained by Papadimitriou-Skenderis

$$I_5^E = \beta E_0 + \frac{\beta \pi}{4G\ell^2 \Xi_a \Xi_b} (m\ell^2 - (r_+^2 + a^2)(r_+^2 + b^2)). \quad (89)$$

In order to obtain the correct value for the entropy, one must use $E = E + E_0$ as the total energy for the black hole system, the angular velocities respect to a non-rotating frame at infinity

$$\Omega_a = \frac{a(1 + r_+^2 \ell^{-2})}{r_+^2 + a^2}, \quad \Omega_b = \frac{b(1 + r_+^2 \ell^{-2})}{r_+^2 + b^2}, \quad (90)$$

and the Euclidean period

$$\beta = \frac{2\pi (r_+^2 + a^2)(r_+^2 + b^2)\ell^2}{2r_+^6 + r_+^4(\ell^2 + b^2 + a^2) - a^2 b^2 \ell^2}. \quad (91)$$

The above quantities satisfy the thermodynamical relation

$$S = \beta (E - \Omega_a J_a - \Omega_b J_b) - I_5^E = \frac{\text{Area}}{4G} \quad (92)$$
that recovers the entropy in terms of the area for Kerr-AdS black hole (79).

- Clarkson-Mann Solitons

In recent papers [31, 32], new solitons in cosmological spacetimes were presented. These solutions resemble the Eguchi-Hanson metrics in four dimensions [33] and in the case of a negative cosmological constant, they possess AdS/Zp asymptotics and a lower energy than the global AdS or even global AdS/Zp spacetimes.

The Clarkson-Mann-AdS soliton metric reads

\[ ds^2 = -g(r)dt^2 + \frac{r^2 f(r)}{4} [d\psi + \cos \theta d\phi]^2 + \frac{dr^2}{f(r)g(r)} + \frac{r^2}{4} d\Omega^2, \]
\[ g(r) = 1 + \frac{r^2}{\ell^2}, \quad f(r) = 1 - \frac{a^4}{r^4}, \] (93)

where \( d\Omega^2 \) is the metric of the unit 2−sphere.

In order to remove the stringlike singularity at \( r = a \), the period of \( \psi \) must be \( 4\pi/p \) and the parameter \( a \) satisfies the relation

\[ a^2 = \ell^2 \left( \frac{p^2}{4} - 1 \right), \] (94)

with \( p \geq 3 \).

The energy for this solitonic solution is negative

\[ E = q(\partial_t) = -\frac{\pi a^4}{8G\ell^2p}, \] (95)

in agreement to the result computed using the standard counterterms procedure in [31, 34]. The present method also reproduces the value of the vacuum energy, which is lower than that of global AdS spacetime

\[ E_0 = q_0(\partial_t) = \frac{3\pi \ell^2}{32Gp}. \] (96)

The negative mass (95) has also been found in [35] through a spin-connection formulation of the Abbott-Deser [36] (and, more recently, Deser-Tekin [37, 38]) method.

The total Euclidean action

\[ I^E = I^E_{\text{bulk}} + c_4 \int_{\partial M} B_4^E, \] (97)

turns out to be

\[ I^E = \beta(E + E_0) \] (98)

where the Euclidean period \( \beta \) remains arbitrary, as the solution is horizonless. As a consequence, the entropy of the system is zero.
4 Seven-Dimensional Case

Before we go into the general odd-dimensional case, let us consider $D = 7$ also for illustrative purposes. The expression for the boundary term

$$B_6 = -6 \int_0^1 dt \int_0^t d\varepsilon_{a_1...a_6} K^{a_1} e^{a_2} (R^{a_3 a_4} - \ell^2 K^{a_3} K^{a_4} + \frac{s^2}{\ell^2} e^{a_3} e^{a_4}) \times$$

$$\times (R^{a_5 a_6} - \ell^2 K^{a_5} K^{a_6} + \frac{s^2}{\ell^2} e^{a_5} e^{a_6}),$$

after the parametric integrations are performed is given by

$$B_6 = -3 \varepsilon_{a_1...a_6} K^{a_1} e^{a_2} (R^{a_3 a_4} R^{a_5 a_6} - \frac{1}{3} K^{a_3} K^{a_4} K^{a_5} K^{a_6} + \frac{1}{15\ell^4} e^{a_1} e^{a_4} e^{a_5} e^{a_6} - \frac{R^{a_3 a_4} K^{a_5} K^{a_6}}{3 \ell^2} + \frac{1}{9\ell^2} K^{a_3} K^{a_4} e^{a_5} e^{a_6}).$$

The equivalent tensorial form of the Kounterterms for seven dimensions is

$$B_6 = \frac{3}{4} \sqrt{-h} \delta^{i_1...i_5}_{j_1...j_5} R_{i_1 j_1} R_{i_2 j_2} R_{i_3 j_3} R_{i_4 j_4} R_{i_5 j_5} - \frac{4}{3} K_{i_2} K_{i_3} K_{i_4} K_{i_5} + \frac{4}{15\ell^4} \delta_{i_2 j_2} \delta_{i_3 j_3} \delta_{i_4 j_4} \delta_{i_5 j_5} - 2 R^{i_2 j_2} K^{i_3} K^{j_3} K^{j_5} + \frac{2}{3\ell^2} R^{i_2 j_2} K^{i_3} K^{i_4} K^{i_5} - \frac{8}{9\ell^2} K^{i_2} K^{i_3} K^{i_4} K^{i_5}.$$

4.1 Action Principle

Now we develop a similar treatment as in the five-dimensional case, writing down an adequate form of the variation of the boundary term, in order to use suitable boundary conditions for AAdS spacetimes. Varying the seven-dimensional action, we have –on-shell– a total surface term

$$\delta I_7 = -2 \int_{\partial M} \kappa_7 \epsilon_{abcdfg} \delta K^{a} e^{b} e^{c} e^{d} e^{f} e^{g} + \frac{3}{2} \epsilon_{abcdfg} \delta K^{a} e^{b} \left( R^{cd} R^{fg} + \frac{5}{3} K^{c} K^{d} K^{f} K^{g} \right)$$

$$+ \frac{1}{15\ell^4} e^{c} e^{d} e^{f} e^{g} - 3 R^{cd} K^{f} K^{g} + \frac{1}{3\ell^2} R^{cd} e^{d} e^{f} e^{g} - \frac{2}{3\ell^2} K^{c} K^{d} e^{f} e^{g} + \frac{3}{2} \epsilon_{abcdfg} K^{a} e^{b} \left( R^{cd} R^{fg} \right)$$

$$+ \frac{1}{3} K^{c} K^{d} K^{f} K^{g} + \frac{1}{3\ell^4} e^{c} e^{d} e^{f} e^{g} - R^{cd} K^{f} K^{g} + \frac{1}{\ell^2} R^{cd} e^{d} e^{f} e^{g} - \frac{2}{3\ell^2} K^{c} K^{d} e^{f} e^{g}.$$
using again the Gauss-Coddazzi relation (32). The above relation already hints a pattern for
the surface term coming from the total variation of the action,

$$\delta I_7 = -2 \int_{\partial M} \kappa_7 \epsilon_{abcdfg} \delta K^a e_b e^c e^d e^e e^f e^g + 3c_6 \int_0^1 dt \epsilon_{abcdfg} \delta K^a e_b \left( \hat{R}^{cd} + \frac{t^2}{\ell^2} e^e e^d \right) \left( \hat{R}^{fg} + \frac{t^2}{\ell^2} e^f e^g \right)$$

$$-3c_6 \int_0^1 dt t \epsilon_{abcdfg} \delta K^a e_b \left( \hat{R}^{cd} - t^2 K^c K^d + \frac{t^2}{\ell^2} e^e e^d \right) \left( \hat{R}^{fg} - t^2 K^f K^g + \frac{t^2}{\ell^2} e^f e^g \right)$$

that we will confirm below in the general odd-dimensional case.

As in the five-dimensional case, we consider a more explicit form of the second line

$$\epsilon_{abcdfg} \epsilon^{i_1 i_2 i_3 i_4 i_5 i_6} \left[ \delta K^i_{i_1 j} e^a_{i_2} e^b_{i_3} + \delta e^a_{i_2} e^b_{i_3} \left( K^i_{i_4 i_5} - K^i_{i_5 i_6} \right) \right] \times$$

$$\times \int_0^1 dt t \left( \frac{1}{2} R^{cd}_{i_1 i_4} - t^2 K^c_{i_3 i_4} + \frac{t^2}{\ell^2} e^e_{i_3} e^d_{i_4} \right) \left( \frac{1}{2} R^{fg}_{i_5 i_6} - t^2 K^f_{i_5 i_6} + \frac{t^2}{\ell^2} e^f_{i_5} e^g_{i_6} \right)$$

that again vanishes identically for the boundary condition on the extrinsic curvature (41) and
the corresponding variation (42). Thus, the problem of a well-defined action principle amounts
to fixing the coupling of the boundary term \(c_6\). In the asymptotic region the spacetime curvature
is constant and then, inserting (44) in the first line of eq.(104), we have

$$\delta I_7 = -2 \int_{\partial M} \epsilon_{abcdfg} \delta K^a e_b e^c e^d e^e e^f e^g \left( \kappa_7 + 3c_6 \frac{\ell^4}{\ell^4} \right)$$

Thus, the cancelation of the surface term implies

$$c_6 = -\frac{5}{8} \kappa_7 \ell^4 = -\frac{\ell^4}{16\pi G \times 192}$$

Now that we have achieved a well-posed action principle, we discuss the construction of the
Noether charges that derive from it.

### 4.2 Conserved Charges

The Noether current for the seven-dimensional case is

$$\star J = -\Theta(e^a, K^a, \delta e^a, \delta K^a) - i_\xi (L_7 + c_6 dB_6),$$

where \(L_7\) is the bulk Lagrangian and \(\Theta\) is the surface term

$$\Theta = -2\kappa_7 \int_{\partial M} \epsilon_{abcdfg} \delta K^a e_b \left[ e^e e^d e^f e^g - \frac{15}{8} \ell^4 \int_0^1 dt \left( \hat{R}^{cd} + \frac{t^2}{\ell^2} e^e e^d \right) \left( \hat{R}^{fg} + \frac{t^2}{\ell^2} e^f e^g \right) \right]$$

$$+ \frac{15}{8} \ell^4 \int_0^1 dt t \epsilon_{abcdfg} (\delta K^a e_b - K^a \delta e^b) \left( \hat{R}^{cd} - t^2 K^c K^d + \frac{t^2}{\ell^2} e^e e^d \right) \left( \hat{R}^{fg} - t^2 K^f K^g + \frac{t^2}{\ell^2} e^f e^g \right)$$

(109)
Carrying out the construction in Appendix [C] for a bulk Lagrangian supplemented in a boundary term, the Noether charge is written as

$$Q(\xi) = \int_{\partial\Sigma} 2\kappa \epsilon_{abcdefg} i \xi K^a e^b e^c e^d e^e e^f e^g + c_0 \left( i \xi K^a \frac{\delta B_6}{\delta K^a} + i \xi e^a \frac{\delta B_6}{\delta e^a} \right)$$

(110)

The formula for the conserved quantities contains two contributions

$$Q(\xi) = q(\xi) + q_0(\xi),$$

(111)

that can be traced back to the first and second lines in the surface term \[(109)\]. The first part

$$q(\xi) = 2\kappa \int_{\partial\Sigma} \epsilon_{abcdefg} i \xi K^a e^b \left[ e^c e^d e^e e^g - \frac{15}{8} \ell^4 \int_0^1 dt \left( \dot{R}^{cd} + \frac{t^2}{\ell^2} e^c e^d \right) \left( \dot{R}^{fg} + \frac{t^2}{\ell^2} e^f e^g \right) \right]$$

(112)

will provide the mass and angular momentum for AAdS solutions. Equivalently, eq.\[(112)\] can be factorized as

$$q(\xi) = -\frac{30}{8} \kappa \ell^4 \int_{\partial\Sigma} \epsilon_{abcdefg} i \xi K^a e^b \left( \dot{R}^{cd} + \frac{1}{\ell^2} e^c e^d \right) \left( \dot{R}^{fg} - \frac{1}{3\ell^2} e^f e^g \right)$$

(113)

that in its tensorial form

$$q(\xi) = -\frac{\ell^4}{16\pi G \times 128} \int_{\partial\Sigma} \sqrt{-h} \epsilon_{i_1 \ldots i_6} (\xi^K_k K^{ri_1}) \delta^i_{j_2} \left( \dot{R}^{i_3 i_4} + \frac{1}{2} \delta_{[i_3 i_4]} \right) \left( \dot{R}^{i_5 i_6} - \frac{1}{3\ell^2} \delta_{i_5 i_6} \right) dx^{j_2} \ldots dx^{j_6},$$

(114)

is proportional to the curvature for the AdS group and, therefore, identically vanishing for (global) AdS spacetime. The product \(dx^{j_2} \wedge \ldots \wedge dx^{j_6}\) is the volume element of the boundary of the spatial section \(\partial\Sigma\) (at constant time).

As a consequence, the additional term \(q_0\) in the conserved quantities is responsible for the existence of a vacuum energy

\[
q_0(\xi) = -\frac{30}{8} \kappa \ell^4 \int_{\partial\Sigma} \int_0^1 dt \epsilon_{abcdefg} \left( i \xi K^a e^b + K^a i \xi e^b \right) \left( R^{cd} - t^2 K^c K^d + \frac{t^2}{\ell^2} e^c e^d \right) \times \left( R^{fg} - t^2 K^f K^g + \frac{t^2}{\ell^2} e^f e^g \right),
\]

\[
= \frac{\ell^4}{16\pi G \times 128} \int_{\partial\Sigma} \int_0^1 dt \sqrt{-h} \epsilon_{i_1 \ldots i_6} \xi^K_k \left( K^{ri_1} \delta_{j_2} + K^{ri_1} \delta_{j_2} \right) \left( R^{i_3 i_4} - t^2 K_{[i_3 i_4]} + \frac{t^2}{\ell^2} \delta_{[i_3 i_4]} \right) \times \left( R^{i_5 i_6} - t^2 K_{[i_5 i_6]} + \frac{t^2}{\ell^2} \delta_{[i_5 i_6]} \right) dx^{j_2} \ldots dx^{j_6}
\]

(115)

where we have kept the parametric integral because, otherwise, the expression is more involved.
4.3 Examples

- Topological Static Black Holes

For the seven-dimensional static metric specified by eqs. (56)–(57), we compute the mass and the vacuum energy using expressions (114) and (115), respectively, for the Killing vector $\xi = \partial_t$. Using the relations (58)–(60), we obtain

$$q(\partial_t) = M = 5! Vol(\Sigma^k_5) \lim_{r \to \infty} (\Delta^2)' r \left\{ \kappa_r [r^4 + 3c_6 \int_0^1 dt \left( k - \Delta^2 + t^2 \frac{r^2}{\ell^2} \right)^2 \right\}$$

(116)

for the mass, whereas for the vacuum energy takes the negative value

$$q_0(\partial_t) = E_0 = -2 \times 5! c_6 Vol(\Sigma^k_5) \lim_{r \to \infty} \left( \Delta^2 - \frac{r(\Delta^2)'}{2} \right) \int_0^1 dt \left( k - t^2 \Delta^2 + t^2 \frac{r^2}{\ell^2} \right)^2$$

(118)

$$= (-k)^3 \frac{5\ell^4}{128\pi G} Vol(\Sigma^k_5).$$

(119)

We have included an intermediate step in the computation of both the mass and vacuum energy, because these expressions will appear again in the evaluation of the Euclidean action.

With the Euclidean period $\beta$ defined as in eq. (66), the Euclidean bulk action, evaluated for a static black hole of the form (56) in seven dimensions is

$$I_{E bulk}^E = -5! \kappa_r Vol(\Sigma^k_5) \beta \left| (\Delta^2)' r^5 \right|_{r_+}^\infty.$$

(120)

The boundary is defined only at the asymptotic region and then, the Euclidean boundary term is

$$\int_{\partial M} B^E_6 = 2 \times 5! Vol(\Sigma^k_5) \beta \left[ \frac{r(\Delta^2)'}{2} \int_0^1 dt \left( k - \Delta^2 + t^2 \frac{r^2}{\ell^2} \right)^2 + \left( \Delta^2 - \frac{r(\Delta^2)'}{2} \right) \int_0^1 dt \left( k - t^2 \Delta^2 + t^2 \frac{r^2}{\ell^2} \right)^2 \right]_{r_+}^\infty.$$

(121)

such that in the total Euclidean action

$$I^E_I = I^E_{bulk} + c_6 \int_{\partial M} B^E_6$$

(122)

the contribution at $r = \infty$ can be read off from eqs. (116), (118) as $-\beta(M + E_0)$ and

$$I^E_I = \frac{Vol(\Sigma^k_5)}{16\pi G} \beta r^5_+ (\Delta^2)'_{r_+} - \beta(M + E_0).$$

(123)

Using the definition of thermodynamic energy $E$ in eq. (71) –that is equivalent to the result obtained from the Noether theorem-- we obtain the entropy as the Euclidean bulk action evaluated at $r = r_+$.
-Kerr-AdS\textsubscript{7} Black Hole

The number of independent rotation parameters for $D$-dimensional Kerr-AdS metric is equal to the number of Casimir invariants for the rotation group $SO(D-1)$, which is the integer part of $(D-1)/2$. The general rotating black hole in seven-dimensional AdS gravity then possesses three rotation parameters, but we shall consider below the particular case of a single rotation parameter to show the finiteness of the conserved quantities and Euclidean action.

The line element for the one-parameter Kerr-AdS\textsubscript{7} spacetime is

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + r^2 \cos^2 \theta d\psi^2 + \frac{\rho^2 dr^2}{\Delta_r} + \frac{\rho^2 d\theta^2}{\Delta_\theta} +$$

$$+ \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left( adt - \frac{(r^2 + a^2)}{\Xi} d\phi \right)^2 + r^2 \cos^2 \theta d\Omega^2_3,$$

where the functions in the metric are

$$\Delta_r = (r^2 + a^2) \left( 1 + \frac{r^2}{\ell^2} \right) - 2m/r^2,$$  \hspace{1cm} (126)

$$\Delta_\theta = 1 - \frac{a^2}{\ell^2} \cos^2 \theta,$$ \hspace{1cm} (127)

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \hspace{1cm} \Xi = 1 - \frac{a^2}{\ell^2},$$ \hspace{1cm} (128)

and $d\Omega^2_3$ is the metric of the 3-sphere

$$d\Omega^2_3 = d\psi^2 + \sin^2 \psi d\eta^2 + \cos^2 \psi d\beta^2$$  \hspace{1cm} (129)

and the angles range is $\theta, \psi \in [0, \pi/2]$ and $\phi, \eta, \beta \in [0, 2\pi]$.

The area of the event horizon is

$$Area = \pi^3 \frac{r_+^3 (r_+^2 + a^2)}{\Xi}.$$  \hspace{1cm} (130)

Using the charge formulas \textit{(114)} and \textit{(115)} for the metric \textit{(125)-(129)} yields

$$E' = q(\partial_t) = \frac{5\pi^2 m}{8G\Xi},$$ \hspace{1cm} (131)

$$E_0 = q_0(\partial_t) = -\frac{\pi^2}{1280\Xi G\ell^2} \left( 50\ell^6 - 50a^2\ell^4 + 5a^4\ell^2 + a^6 \right),$$ \hspace{1cm} (132)

and the angular momentum

$$J = q(\partial_\phi) = \frac{\pi^2 ma}{4G\Xi^2}.$$ \hspace{1cm} (133)
The values in eqs. (131), (132) and (133) agree with the ones computed using a Dirichlet counterterms regularization \[10, 29\]. The vacuum energy can be also written as

\[ E_0 = -\frac{5\pi^2 \ell^4}{128G} \left( 1 + \frac{(1 - \Xi)^2 (6 - \Xi)}{50\Xi} \right) \]

(134)

in order to make more manifest the matching with the vacuum energy for Schwarzchild-AdS \[119\] in the non-rotating limit and to try to infer the general vacuum energy in the case of three different rotation parameters.

As pointed out in the five-dimensional section above, the physical energy for a Kerr-AdS black hole is the conserved quantity associated to a non-rotating asymptotic timelike Killing vector \( \xi = \partial_t + \frac{a}{\ell^2} \partial_\phi \) \[30\]

\[ E = q(\partial_t + \frac{a}{\ell^2} \partial_\phi) = \frac{m\pi^2}{8G\Xi^2} (2 + 3\Xi), \]

(135)

in agreement with the formulas in Refs.\[30, 39, 40, 41\], specialized to a single nonvanishing rotation parameter.

In order to complete the discussion about regularization of this seven-dimensional solution, we compute the on-shell Euclidean action, that for a stationary spacetime is given by

\[ I^E_7 = \frac{3\beta}{4\pi G\ell^2} \int_{r_+}^{\infty} dr \int d\Omega N \sqrt{h} + \beta c_6 \int d\Omega \left( B_6 \right) |r = \infty, \]

(136)

where the Euclidean period is

\[ \beta = \frac{2\pi (r_+^2 + a^2) r_+}{3r_+^4/\ell^2 + 2r_+^2 (1 + a^2/\ell^2) + a^2}. \]

(137)

The divergences at radial infinity in the bulk action are exactly canceled by the ones in the Euclidean boundary term, such that we get the finite result

\[ I^E_7 = -\frac{\beta \pi^2 \ell^4}{1280\Xi} \left[ 160 \left( \frac{r_+^4 a^2}{\ell^6} + \frac{r_+^6}{\ell^6} - \frac{m}{\ell^4} \right) + \frac{a^6}{\ell^6} + 5\frac{a^4}{\ell^4} + 50\Xi \right], \]

(138)

that can be conveniently rewritten as

\[ I^E_7 = \beta E_0 + I^E_7', \]

(139)

where

\[ I^E_7' = \frac{\beta \pi^2}{8G\ell^2\Xi} (ml^2 - r_+^2(a^2)) \]

(140)

corresponds to the value of the Euclidean action computed in a background-substraction method \[30\]. Eq. (140) satisfies the thermodynamical relation

\[ 3 \text{ The volume of } S^5 \text{ is } \pi^3. \]
\[ S = \beta (E - \Omega J) - I^E_7 = \frac{\text{Area}}{4G}, \quad (141) \]

for the energy \((135)\), angular momentum \((133)\), the angular velocity respect to a non-rotating frame at infinity

\[ \Omega = \frac{a(1 + r^2 + \ell^2)}{r^2 + a^2}, \quad (142) \]

and the area of the event horizon

\[ \text{Area} = \pi^3 \frac{r^3}{\Xi}, \quad (143) \]

In turn, the Euclidean action \((139)\), computed using the background-independent Kounterterms prescription, obeys

\[ S = \beta (\mathcal{E} - \Omega J) - I^E_7 = \frac{\text{Area}}{4G}, \quad (144) \]

for a thermodynamical energy consistently shifted in the vacuum energy \( \mathcal{E} = E + E_0 \).

5 General Odd-Dimensional Case

We have already introduced the general form the Kounterterms series adopts in any odd dimension \( D = 2n + 1 \), eqs.\((22)-(24)\). The parametric integrations provide the relative coefficients of the boundary terms when eq.\((24)\) is expanded as a polynomial in the extrinsic and intrinsic curvature

\[ B_{2n} = n! \sqrt{-\hbar} \sum_{p=0}^{n-1} \frac{(2n-2p-3)!!}{\ell^{2(n-1-p)}} b^{(p)}_{2n}, \quad (145) \]

where

\[ b^{(p)}_{2n} = \delta^{[i_1 \cdots i_{2p+1}]}_{[j_1 \cdots j_{2p+1}]} \sum_{q=0}^{p} \frac{(-1)^{p-q} 2^{n-(p+q+1)}}{(p-q)!q! (n-q)} R^{j_1 j_2} \cdots R^{j_{2q-1} j_{2q}} K^{j_{2q+1}} \cdots K^{j_{2p+1}}. \quad (146) \]

The surface term obtained from an arbitrary variation of the action \((8)\)–or equivalently, \((25)\)– has a more involved form in the general odd-dimensional case. In the Appendix \[ \text{D} \] we summarize the process of variation of the action in the general case, cast in an appropriate form that allows us to impose the asymptotic conditions for AAdS spacetimes discussed above.
\[
\delta I_{2n+1} = -2 \int_{\partial M} \epsilon_{a_1 \ldots a_{2n}} \delta K^{a_1} e^{a_2} \left[ K_D e^{a_3} \ldots e^{a_{2n}} + nc_2n \int_0^1 dt \left( \hat{R}^{a_3 a_4} + \frac{t^2}{\ell^2} e^{a_3} e^{a_4} \right) \right] \times \ldots \nabla \times \left( \hat{R}^{a_2n-1 a_{2n}} + \frac{t^2}{\ell^2} e^{a_2n-1} e^{a_{2n}} \right) \\
- nc_2n \int_0^1 dt \epsilon_{a_1 \ldots a_{2n}} (\delta K^{a_1} e^{a_2} - K^{a_1} \delta e^{a_2}) \left( \hat{R}^{a_3 a_4} - t^2 K^{a_3} K^{a_4} + \frac{t^2}{\ell^2} e^{a_3} e^{a_4} \right) \times \ldots \nabla \times \left( \hat{R}^{a_2n-1 a_{2n}} - t^2 K^{a_2n-1} K^{a_{2n}} + \frac{t^2}{\ell^2} e^{a_2n-1} e^{a_{2n}} \right). \quad (147)
\]

As we have already seen in the five and seven-dimensional cases, no matter the terms that multiply the curl \( \epsilon_{a_1 a_2 \ldots a_{2n}} (\delta K^{a_1} e^{a_2} - K^{a_1} \delta e^{a_2}) \), the asymptotic conditions (41) and (42) cancels it identically. The problem of making the full action stationary then reduces to fix the coupling constant \( c_{2n} \) of the boundary term. This can be done demanding the spacetime to be of constant curvature at the asymptotic region (eq.(44)). In a Riemannian manifold, this condition is equivalent to asymptotic flatness for the curvature of the AdS group

\[
F = dA + A \wedge A = \frac{1}{2} \left( \hat{R}^{AB} + \frac{e^A e^B}{\ell^2} \right) J_{AB} + \frac{T^A}{\ell} P_A, \quad (148)
\]

where \( A = \frac{1}{2} \omega^{AB} J_{AB} + \frac{e^A}{\ell} P_A \) is the \( SO(D - 1, 2) \) group connection field, \( \{ J_{AB}, P_A \} \) are the generators of AdS rotations and translations, respectively, and \( T^A = \frac{1}{2} T_{\mu \nu} dx^\mu \wedge dx^\nu \) is the two-form torsion.

Assuming (44), we obtain the value

\[
c_{2n} = -\ell^{2n-2} \frac{K_D}{n} \left[ \int_0^1 dt (t^2 - 1)^{n-1} \right]^{-1}, \quad (149)
\]

\[
= -\frac{(-\ell^2)^{n-1}}{4^{n+1} \pi G n [(n-1)!]^2}, \quad (150)
\]

whose fixing is equivalent to canceling the highest-order divergences in the Euclidean action.

### 5.1 Noether Charges

The conserved current associated to an isometry and prescribed by the Noether theorem, in the general odd-dimensional case is

\[
* J = -\Theta(e^a, K^a, \delta e^a, \delta K^a) - i_\xi (L_{2n+1} + c_{2n} dB_{2n}). \quad (151)
\]

Here, \( L_{2n+1} \) is the bulk Lagrangian in \( 2n+1 \) dimensions, \( B_{2n} \) is the regularizing Kounterterms series and \( \Theta \) is the surface term in the variation of the action (147).

Either replacing the explicit form of the terms in eq.(151) to write down the Noether current as \( *J = dQ(\xi) \) or employing the shortcut to the charge derivation in Appendix C.
where the derivative of the function in the metric is
\[ Q(\xi) = \int_{\partial \Sigma} 2\kappa_{D}\epsilon_{a_{1}...a_{2n}} \xi^{a_{1}} K^{a_{2}}...e^{a_{2n}} + c_{2n} \left( i_{\xi} K^{a} \frac{\delta B_{2n}}{\delta K^{a}} + i_{\xi} e^{a} \frac{\delta B_{2n}}{\delta e^{a}} \right) \] (152)

the conserved charge is split as
\[ Q(\xi) = q(\xi) + q_{0}(\xi). \] (153)

The first contribution can be traced back to the first two lines in the surface term
\[ q(\xi) = \frac{1}{2n-2} \int_{\partial \Sigma} \kappa_{D} \sqrt{-h} \epsilon_{i_{1}...i_{2n}}(\xi^{k} K_{k}^{i_{1}})^{\delta_{j_{2}}^{i_{2}}/j_{2}} \left[ \delta_{[i_{3}]j_{4}]...\delta_{[i_{2n-1}]j_{2}]^{i_{2n}}} + \right. \]
\[ \left. + nc_{2n} \int_{0}^{1} dt \left( \hat{R}_{j_{2}j_{4}j_{2}t}^{i_{1}i_{4}} + t^{2} \frac{\delta^{i_{1}i_{4}}}{\delta K^{j_{2}j_{4}}} \right) \left( \hat{R}_{j_{2}j_{4}j_{2}t}^{i_{1}i_{4}} - t^{2} \frac{\delta^{i_{1}i_{4}}}{\delta K^{j_{2}j_{4}}} \right) \right] dt^{j_{2}...dt^{j_{2n}}}, \] (154)

and the second one comes from the curl term in the same term
\[ q_{0}(\xi) = \frac{nc_{2n}}{2n-2} \int_{\partial \Sigma} \sqrt{-h} \int_{0}^{1} dt \epsilon_{i_{1}...i_{2n}} \xi^{k} \left( \delta_{j_{2}}^{i_{2}} K_{k}^{i_{1}} + \delta_{k}^{i_{2}} K_{[j_{2}]i_{1}j_{4}]^{i_{2}}/j_{2}} \right) \left( \hat{R}_{j_{2}j_{4}j_{2}t}^{i_{1}i_{4}} - t^{2} \frac{\delta^{i_{1}i_{4}}}{\delta K^{j_{2}j_{4}}} \right) \]
\[ \left( \hat{R}_{j_{2}j_{4}j_{2}t}^{i_{1}i_{4}} - t^{2} \frac{\delta^{i_{1}i_{4}}}{\delta K^{j_{2}j_{4}}} \right) dx^{j_{2}}...dx^{j_{2n}}. \] (155)

Evaluating the formula \[ (\Delta^{2})^{r} = (D - 2)! Vol(\Sigma_{D-2}) \lim_{r \to \infty} (\Delta^{2})^{r} \{ \kappa_{D} r^{2(n-1)} + nc_{2n} \int_{0}^{1} dt \left( k - \Delta^{2} + \frac{t^{2} r^{2}}{\ell^{2}} \right)^{n-1} \} \] (156)

where the derivative of the function in the metric is
\[ (\Delta^{2})^{r} = 2(D - 3) \frac{\omega_{D} G \mu}{r^{D-3}} + 2 \frac{r^{2}}{\ell^{2}}. \] (157)

Expanding the second term as
\[ nc_{2n} \int_{0}^{1} dt \left( k - \Delta^{2} + \frac{t^{2} r^{2}}{\ell^{2}} \right)^{n-1} = \kappa_{D} \left( -r^{2(n-1)} + (2n - 1) \omega_{D} G \frac{\ell^{2}}{r^{2}} \mu + ... \right) \] (158)

where the additional contributions of lower order in \( r \) are irrelevant in the limit \( r \to \infty \), the
topological black hole mass is finally
\[ q(\partial_{t}) = M = \frac{Vol(\Sigma_{D-2})}{Vol(S^{D-2})} \mu. \]

The other formula in the conserved charge, \( q_{0}(\xi) \), specialized for a timelike Killing vector and topological black holes, produces
\[ q_{0}(\partial_{t}) = -2c_{2n}(D-2)! Vol(\Sigma_{D-2}) \lim_{r \to \infty} \left( \Delta^{2} - \frac{r(\Delta^{2})^{r}}{2} \right) \int_{0}^{1} dt \left( k - r^{2} \Delta^{2} + \frac{t^{2} r^{2}}{\ell^{2}} \right)^{n-1}, \] (159)
where the explicit evaluation of

\[
\left( \Delta^2 - \frac{r(\Delta')^2}{2} \right) = k - (D - 1) \frac{\omega_D G}{r^{D-3} \mu},
\]

\[ (k - t^2 \Delta^2 + \frac{t^2 r^2}{\ell^2}) = k(1 - t^2) + 2t^2 \frac{\omega_D G}{r^{D-3} \mu}, \]

introduces at most finite contributions to the zero-point (vacuum) energy

\[
q_0(\partial_t) = E_0 = (-k)^n \frac{Vol(\Sigma_{D-2})}{8\pi G} \ell^{2n-2} \frac{(2n - 1)!!}{(2n)!}.
\]

This expression corroborates the general formula for the vacuum energy for odd-dimensional AdS spacetime conjectured in Ref. [8], based on an extrapolation of explicit results in the counterterms method up to nine dimensions.

In order to verify the consistency of the black hole thermodynamics, we compute the total Euclidean action

\[
I_{2n+1}^E = I_{bulk}^E + c_{2n} \int_{\partial M} B_{2n}^E,
\]

for SAdS black hole. The bulk term is a total derivative, such that the integration in the radial coordinate in the interval \([r_+, \infty)\) is simply

\[
I_{bulk}^E = -\kappa_D (D - 2)! Vol(\Sigma_{D-2}) \beta \{(\Delta^2)' r^{D-2}\}\big|_{r_+}^{\infty},
\]

and the Euclidean boundary term is

\[
\int_{\partial M} B_{2n}^E = 2(D - 2)! Vol(\Sigma_{D-2}) \beta \left[ \frac{r(\Delta')^2}{2} \int_0^1 dt \left( k - \Delta^2 + \frac{t^2 r^2}{\ell^2} \right)^{n-1} + \right.
\]

\[
\left. + \left( \Delta^2 - \frac{r(\Delta')^2}{2} \right) \int_0^1 dt t \left( k - t^2 \Delta^2 + \frac{t^2 r^2}{\ell^2} \right)^{n-1} \right|_{r_+}^{\infty}. \]

The total contribution at \(r = \infty\) can be identify as \(-\beta(M + E_0)\) and then the total Euclidean action is

\[
I_{2n+1}^E = \frac{Vol(\Sigma_{D-2})}{16\pi G} \beta r^{D-2} \left( \Delta^2 \right)'_{r_+} - \beta(M + E_0).
\]

The definition of thermodynamic energy \(E\) in eq. (71) recovers the total energy in the Noether theorem \(Q(\partial_t) = q(\partial_t) + q_0(\partial_t)\), such that eq. (67) implies an entropy for SAdS black hole

\[
S = \frac{Vol(\Sigma_{D-2}) r^{D-2}}{4G} = \frac{Area}{4G}.
\]
6 Conclusions

In this paper, we have explicitly shown the tensorial form of the Kounterterms that regularize the action for AdS gravity in all odd dimensions.

The key point of the construction is a well-principle action principle that respects boundary conditions consistent with the asymptotic behavior of a generic AAdS spacetime.

A definite form of the boundary terms achieves a finite action principle: the action is stationary under arbitrary variations of the fields and the conserved charges and the Euclidean action are finite.

In the general odd-dimensional case, we have not computed the conserved quantities in AAdS solutions other than for topological SAdS black holes.

Further evaluation of more complex solutions will certainly be more involved, but it could also reveal the general form that the vacuum energy adopts for certain AAdS spaces. Particularly interesting could be extending the results of the vacuum energy in Refs.\cite{29, 12} to the recently generalized multi-parameter Kerr-AdS black hole \cite{30}. It is also worthwhile to notice that the existing results on vacuum energy for Kerr-AdS solutions, reproduced here, have been critically revised in Ref.\cite{42} because of their explicit dependence on the rotation parameters. There, it is claimed that, in the same way the energy and the angular velocities are referred to a coordinate frame that is nonrotating at infinity, one should choose the Einstein static universe as the metric on the conformal boundary of the rotating black hole. As a consequence, the corresponding Casimir energy is genuinely a constant and matches the one for SAdS black hole.

But, how can we be sure that $q_0(\partial_t)$ would always produce the vacuum energy?

The reasoning is quite simple and has to do with the general form of the other quantity that enters in the Noether charge, $q(\xi)$. It can be proved that eq.\cite{154} can be factorized by the l.h.s. of eq.\cite{11}, in the corresponding boundary indices

$$q(\xi) = \frac{n C_{2n}}{2^{n-2}} \int_{\partial \Sigma} \sqrt{-h} e_{i_1...i_{2n}} (\xi^k K^{i_1}_k) \delta_{j_2}^{i_2} \left( R_{j_3j_4} + \frac{1}{\ell^2} \delta_{[j_3j_4]} \right) \mathcal{P}_{j_5...j_{2n}} dx^{i_2}...dx^{i_{2n}},$$ \hspace{1cm} (169)

where $\mathcal{P}$ is a Lovelock-type polynomial of $(n-2)$ degree in the Riemann tensor $\hat{R}_{ij}$ and the antisymmetrized Kronecker delta $\delta_{[ij]}^{[kl]}$

$$\mathcal{P}_{j_5...j_{2n}} = \sum_{p=0}^{n-2} \frac{D_p}{\ell^2 p} R_{j_5j_6}...R_{j_2(n-p)-1j_2(n-p)} \delta_{j_2(n-p)+1j_2(n-p+1)}...\delta_{j_2(n-1)j_{2n}},$$ \hspace{1cm} (170)

with the coefficients of the expansion given by

$$D_p = \sum_{q=0}^{p} \frac{(-1)^{p-q}}{2q+1} \binom{n-1}{q}.$$ \hspace{1cm} (171)

Logically, using the identities for the antisymmetrized Kronecker deltas, one could express $\mathcal{P}$ in terms of the Riemann tensor only, but prefer the above form to make easier the connection with the explicit cases developed so far. As an example, in nine dimensions, the charge $q(\xi)$ is
whereas in eleven dimensions

\[ q(\xi) = \frac{5c_{10}}{8} \int_{\partial \Sigma} \sqrt{-h} e^{i_{10}} \left( \xi^k K_{k}^{i_{10}} \right) \delta_{j_2}^{i_{10}} \left( \hat{R}^{i_3 i_4}_{j_3 j_4} + \frac{1}{\ell^2} \hat{\delta}_{j_3 j_4}^{i_3 i_4} \right) \left[ \hat{R}^{i_{5} i_{6} i_{7} i_{8}}_{j_5 j_6 j_7 j_8} + \frac{3}{5\ell^4} \hat{\delta}_{j_5 j_6 j_7 j_8}^{i_{5} i_{6} i_{7} i_{8}} \right] dx^{j_2} ... dx^{j_{10}} \]  

\[ (172) \]

As a consequence, \( q(\xi) \) always vanishes for a spacetime that is globally AdS. This argument indicates that \( q_0(\xi) \) in eq. (155) for a timelike Killing vector is indeed a covariant formula for the vacuum energy in AAdS spacetimes.

The expression for the vacuum energy has been also recognized as the action of a Killing vector in the Euclidean continuation of the boundary term \( B_{2n} \) for explicit black hole solutions. It is expected that a generic thermodynamical relation \( S = \beta (E + E_0 - \sum_i \Omega_i J_i) - I_{E_{2n+1}}^E = \text{Area}/4G \) holds for any AAdS spacetime that accepts a timelike and a set of rotational Killing vectors. Carrying out a similar procedure as Wald’s formalism [43], expressions for \( E \) and \( E_0 \) should be mapped exactly to contributions from the bulk and the boundary after acting with a global isometry \( \xi = \partial_t + \Omega_i^\infty \partial_{\theta_i} \) on them, where the angular velocities at infinity are given by \( \Omega_i^\infty = a_i^2/\ell^2 \). In odd-dimensional spacetimes, the regularized action is not invariant under the full AdS group. Radial bulk diffeomorphisms, which generate a Weyl transformation on the boundary, are generically broken by the conformal anomaly. In the standard counterterms approach, this is reflected in a nonvanishing trace of the regularized stress tensor [8, 44, 45].

In the present framework, the surface term from an arbitrary variation of the action contains also variations of the extrinsic curvature (usually canceled by the Gibbons-Hawking term), such that a boundary stress tensor definition is not straightforward. The answer to this point should come from direct comparison of the Kounterterms series with Dirichlet counterterms, e.g., by expansion of the tensorial quantities in FG form. For instance, in this way, it can be proved that in three-dimensional AdS gravity, the Kounterterms prescription reduces to the Dirichlet regularization up to a topological invariant on the boundary [15]. The matching of the results in this paper with the ones obtained by standard holographic renormalization indicates that both procedures could also be equivalent in higher dimensions.

If a relation of the Noether charges to a regularized stress tensor \( \tau_{ij} \) is possible, we might expect that a similar splitting as (153) would also appear on it. At a more speculative level, such identification could reveal a connection between the part of \( \tau_{ij} \) that generates the vacuum energy and the one that produces the Weyl anomaly, what has not yet been understood in the standard holographic renormalization.
Could the term $B_d$ represent the full counterterms series? We do not know yet the answer to this question. One could only argue that is very unlikely that other boundary terms can be added on top of the Kounterterms series, that still preserve the AAdS boundary conditions extensively used here. This argument might free this procedure from the ambiguities of the Dirichlet regularization [46].

In any case, what is particularly appealing in this formulation is the relation of Kounterterms to topological invariants in $D = 2n$ [15] and to Chern-Simons-like forms (transgression forms) in $D = 2n + 1$ [47], which might provide some further insight on the problem of regularization of Einstein-Hilbert-AdS gravity, but also in Einstein-Gauss-Bonnet [24] and other theories with higher curvature terms (see, e.g., [17, 52]).

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A Invariant polynomials, Chern-Simons and transgression forms

In this Appendix we review the notion of transgression form as the natural extension of a Chern-Simons density that restores gauge invariance through the introduction of an additional gauge connection [48, 49, 50, 51].

Let us consider in $2n + 2$ dimensions an invariant polynomial $P(F)$ of the form

$$P(F) = < F^{n+1} >$$

(174)

where $F = \frac{1}{2} F_{\mu\nu} T_1 dx^\mu dx^\nu = dA + A \wedge A$ is the curvature two-form associated to the gauge potential $A = A_\mu^I T_I dx^\mu$ of the Lie group $G$, with a generators set $\{T_I\}$. The symbol $< \cdots >$ stands for a totally symmetric invariant trace of the generators in the adjoint representation of $G$

$$< T_{I_1} \cdots T_{I_{n+1}} >= g_{I_1 \cdots I_{n+1}}.$$  

(175)

The invariant polynomial (174) is a closed form

$$dP(F) = 0$$

(176)

and therefore, by virtue of the Poincaré lemma, locally exact

$$P(F) = dC_{2n+1}(A, F)$$

(177)
what provides the definition of a Chern-Simons density as the integration over a continuous parameter $u$

$$C_{2n+1}(A, F) \equiv (n + 1) \int_0^1 du \; < AF_u^n >$$

(178)

with $A_u = uA$ and $F_u = dA_u + A_u^2$.

A similar relation defines a transgression form $T_{2n+1}(A, \tilde{A})$, that involves two gauge potentials $A$ and $\tilde{A}$ in the same homotopy class, with curvatures $F$ and $\tilde{F}$, respectively

$$< F^{n+1} > - < \tilde{F}^{n+1} > = dT_{2n+1}(A, \tilde{A}).$$

(179)

The explicit formula for the transgression form is also given by a parametric integration

$$T_{2n+1}(A, \tilde{A}) \equiv (n + 1) \int_0^1 dt \; < (A - \tilde{A})F_t^n >,$$

(180)

where $F_t = dA_t + A_t^2$ is the curvature associated to the interpolating gauge connection $A_t = tA + (1 - t)\tilde{A}$. On the contrary to Chern-Simons densities, transgression forms are truly invariant under finite gauge transformations in the group $G$.

The explicit formula for (180) is a consequence of the use of the Cartan homotopy operator $k_{01}$, which acts on generic polynomials $P(F_t, A_t)$ and is defined as

$$k_{01}P(F_t, A_t) = \int_0^1 dt \; l_tP(F_t, A_t),$$

(181)

where the action of the operator $l_t$ on arbitrary polynomials of $A_t$ and $F_t$ can be worked out from the relations

$$l_tA_t = 0, \quad l_tF_t = A - \tilde{A}.$$  

(182)

The operator $l_t$ acts as an antiderivative $l_t(\Lambda_p \Sigma_q) = (l_t\Lambda_p)\Sigma_q + (-1)^p\Lambda_p(l_t\Sigma_q)$, where $\Lambda_p$ and $\Sigma_q$ are $p$ and $q$-forms, respectively.

It is particularly useful to express the curvature $F_t$ as

$$F_t = \tilde{F} + t\bar{D}(A - \tilde{A}) + t^2(A - \tilde{A})^2,$$

(183)

with the curvature $\tilde{F}$ associated to the connection $\tilde{A}$ ($\tilde{F} = d\tilde{A} + \tilde{A}^2$) and the covariant derivative in $\tilde{A}$ given by $\bar{D}(A - \tilde{A}) = d(A - \tilde{A}) + \tilde{A}(A - \tilde{A}) + (A - \tilde{A})\tilde{A}$.

In four dimensions, one can re-obtain the formula for the second Chern form (19) from the generic transgression formula taking two gauge connections for the Lorentz group $SO(3,1)$, that is, $A = \frac{1}{2}\omega^{AB}J_{AB}$ and $\tilde{A} = \frac{1}{2}\tilde{\omega}^{AB}J_{AB}$ and the invariant tensor for the Lorentz generators $\{J_{AB}J_{CD}\} = \varepsilon_{ABCD}$. In this way, the interpolating connection in terms of the Second Fundamental Form (13) is

$$A_t = \frac{1}{2}\omega^t^{AB}J_{AB} = \frac{1}{2}(\tilde{\omega}^{AB} + t\theta^{AB})J_{AB},$$

(184)

and its corresponding curvature
\[
F_t = \frac{1}{2} \hat{R}_t^{AB} J_{AB} = \frac{1}{2} \left[ \hat{R}^{AB} + t \hat{D} \theta^{AB} + t^2 \theta^A_c \theta^{CB} \right] J_{AB},
\] (185)

where \( \hat{R}^{AB} \) and \( \hat{D} \) are the curvature and the covariant derivative in the spin connection \( \omega^{AB} \).

When plugged in Eqs. (179,180), the transgression form for the Lorentz group satisfies the local relation

\[
\mathcal{E}_4(\hat{R}) - \mathcal{E}_4(\hat{\bar{R}}) = 2d \left( \int_0^1 dt \varepsilon_{ABCD} \theta^{AB} \left( \hat{R}^{CD} + t \hat{D} \theta^{CD} + t^2 \theta^C_F \theta^{FD} \right) \right),
\] (186)

where \( \mathcal{E}_4 = \varepsilon_{ABCD} \hat{R}^{AB} \hat{R}^{CD} \) is the Euler-Gauss-Bonnet topological invariant. For a radial foliation of the spacetime, an adequate choice of the reference spin connection corresponds to the matching conditions for the Second Fundamental Form and relates the components \( \hat{R}^{ab} \) to the intrinsic curvature at the boundary, i.e., \( \hat{R}^{ab} = R^{ab}(h) \). In doing so, the second Euler term in the l.h.s. of Eq. (186) vanishes identically, so does the second term in the r.h.s.

Global considerations show that the Euler term and the second Chern form \( B_3 \) in four dimensions are equivalent up a topological number (Euler characteristic \( \chi(M^4) \))

\[
\int_{M^4} \mathcal{E}_4(\hat{R}) = 32\pi^2 \chi(M^4) + \int_{\partial M^4} B_3,
\] (187)

revealing the profound connection of the Kounterterms method with topological invariants.

In a similar fashion, the higher even-dimensional Kounterterms \( B_{2n-1} \) are related to the corresponding Euler term in \( D = 2n \) dimensions by virtue of the Euler theorem

\[
\int_{M_2} \mathcal{E}_{2n}(\hat{R}) = (-4\pi)^n n! \chi(M_{2n}) + \int_{\partial M_{2n}} B_{2n-1}.
\] (188)

For any invariant polynomial \( P(F_t, A_t) \), it can be verified that

\[
(l_t d + dl_t) P(F_t, A_t) = \frac{\partial}{\partial t} P(F_t, A_t)
\] (189)

which, when integrated between 0 and 1, recovers the Cartan homotopy formula

\[
(k_{01} d + dk_{01}) P(F_t, A_t) = P(F, A) - P(\hat{F}, \hat{A}).
\] (190)

In particular, for \( P = C_{2n+1} \), the above relation allows us to express a transgression form as the difference of two Chern-Simons densities plus a boundary term

\[
\mathcal{T}_{2n+1} = C_{2n+1}(A, F) - C_{2n+1}(\hat{A}, \hat{F}) + d\Xi_{2n}(A, F; \hat{A}, \hat{F}).
\] (191)

The \( 2n \)-form \( \Xi_{2n} \) is defined by the action of the Cartan homotopy operator on a Chern-Simons term

\[
\Xi_{2n}(A, F; \hat{A}, \hat{F}) \equiv k_{01} C_{2n+1}
\] (192)
whose explicit form is given by
\[ \Xi_{2n} = n(n+1) \int_0^1 ds \int_0^1 dt \ s < A_t(A - \bar{A}) F_{st}^{n-1} > \] (193)

where \( F_{st} = s F_t + s(s-1) A_t^2 \).

When the gauge group is AdS, the connection field is written as
\[ A = \frac{1}{2} \omega^{AB} J_{AB} + e^A P_A, \] (194)

where \( J_{AB} \) and \( P_A \) are the generators of rotations and AdS translations, respectively. For the trace of the generators of \( SO(2n, 2) \), we take
\[ < J_{A_1 A_2}, ..., J_{A_{2n-1} A_{2n}}, P_{A_{2n+1}} > = \frac{2^n}{(n+1)!} \varepsilon_{A_1 \cdots A_{2n+1}}, \]

such that each Chern-Simons form in (191) can be written as a bulk Lovelock Lagrangian (a polynomial in the curvature and the vielbein)
\[ C_{2n+1}(A, F) = \mathcal{L}_{CS-AdS}(\hat{R}, e) \]
\[ = \int_0^1 dt \varepsilon_{A_1 \cdots A_{2n+1}} (\hat{R}^{A_1 A_2} + \frac{t^2}{\ell^2} e^{A_1} e^{A_2} \cdot (\hat{R}^{A_{2n-1} A_{2n}} + \frac{t^2}{\ell^2} e^{A_{2n-1}} e^{A_{2n}}) e^{A_{2n+1}}) \] (195)

plus a surface term whose explicit form is involved and has to be worked out case by case. However, the matching conditions for the Second Fundamental Form (18) –that single out the boundary correction in the Euler theorem– and the condition \( \bar{e}^A = 0 \) (in order to avoid a background-dependent bimetric formulation) replaced in the transgression form (191) result into a gravity action
\[ I_{CS-AdS} = \int_{M_{2n+1}}^{} T_{2n+1} = \int_{M_{2n+1}}^{} \mathcal{L}_{CS-AdS}(\hat{R}, e) + \int_{\partial M_{2n+1}}^{} B_{2n}(\theta, e) \] (196)

with a boundary term \( B_{2n}(\theta, e) \) given by the formula (22). The above total action is regularized both in its Euclidean continuation and in the Noether charges. It is quite remarkable that the contributions from the bulk terms in (191) combine to the surface term (193) to produce a compact expression for the boundary terms in this particular Lovelock theory as the double integral in Eq.(22). But it is even more surprising that the same 2\( n \)-form is useful to regularize other gravity theories, including Einstein-Hilbert-AdS, what is shown explicitly in this paper.

### B Useful Identities

Let us consider the five-dimensional Kounterterms as an example of the equivalence between differential forms and tensorial notation.
\[ B_4 = \epsilon_{A_1...A_5} \theta^{A_1A_2}e^{A_3}(R^{A_4A_5} + \frac{1}{2}\theta^{A_4} \theta^{C\Lambda_5} + \frac{1}{6\ell^2}e^{A_4}e^{A_5}), \]

\[ = 2\epsilon_{A_1...A_5}\theta^{A_1A_2}e^{A_3}(R^{A_4A_5} + \frac{1}{2}\theta^{A_4} \theta^{C\Lambda_5} + \frac{1}{6\ell^2}e^{A_4}e^{A_5}), \]

\[ = -\frac{1}{2}\epsilon_{A_1...A_5} K^b e_c (R^{a_4a_5} - \frac{1}{2}\theta^{a_4} \theta^{c\Lambda_5} + \frac{1}{6\ell^2}e^{a_4}e^{a_5}), \]

\[ = \frac{-\epsilon_{A_1...A_5} e^{a_1} e^{a_2} e^{a_3} e^{a_4} e^{a_5}}{2} \delta^{i_1_{j_1}...i_5_{j_5}}(R^{i_1_{j_1}i_2_{j_2}} - \frac{1}{2}\theta^{i_1_{j_1}} \theta^{i_2_{j_2}} + \frac{1}{6\ell^2}e^{i_1_{j_1}}e^{i_2_{j_2}}) dx^{i_1} \wedge ... \wedge dx^{i_5}, \]

(197)

where we have used the identities

\[ \epsilon_{a_1...a_2n} e^{a_1} e^{a_2n} = -\sqrt{-h} \epsilon_{j_1...j_{2n}}, \]

with the definition \( \sqrt{-h} = \det(e) \), the volume element

\[ dx^{i_1} \wedge ... \wedge dx^{i_{2n}} = \epsilon^{i_1...i_{2n}} d^{2n}x, \]

(199)

and the general property for antisymmetrized Kronecker deltas

\[ \delta_{[i_1...i_p]}^{[j_1...j_p]} \delta^{i_1_{j_1}} \delta^{i_2_{j_2}} ... \delta^{i_m_{j_m}} = \frac{(r-p+m)!}{(r-p)!} \delta_{[m+1...j_p]}^{[i_1...i_p]} \]

(200)

where \( p > m \) and \( r \) as the range of the indices.

### C. Noether’s Theorem

We now recall the standard construction of the conserved quantities associated to asymptotic symmetries of the action through the Noether’s theorem.

Let us consider an action that is the integral of a \( D \)-form Lagrangian density in \( D \) dimensions

\[ L = \frac{1}{D!} I_{\mu_1...\mu_D} dx^{\mu_1} \wedge ... \wedge dx^{\mu_D}. \]

(201)

An arbitrary variation \( \delta \) acting on the fields can be always decomposed in a functional variation \( \delta \) plus the variation due to an infinitesimal change in the coordinates \( x'^\mu = x^\mu + \eta^\mu \). For a \( p \)-form field \( \varphi \), the latter variation is given by the Lie derivative \( \mathcal{L}_\eta \varphi \) along the vector \( \eta^\mu \), that can be written as \( \mathcal{L}_\eta \varphi = (dI_\eta + I_\eta d)\varphi \), where \( d \) is the exterior derivative and \( I_\eta \) is the contraction operator. The functional variation \( \delta \) acting on \( L \) produces the equations of motion plus a surface term \( \Theta(\varphi, \delta \varphi) \). The Lie derivative contributes only with another surface term because \( dL = 0 \) in \( D \) dimensions.
The Noether’s theorem provides a conserved current associated to the invariance under diffeomorphisms of the Lagrangian $L$, that is given by \[ \mathcal{J} = -\Theta(\varphi,\delta\varphi) - i_\eta L. \] (202)

In case that the diffeomorphism $\xi$ is a Killing vector, we have $\delta\varphi = -\mathcal{L}_\xi \varphi$, with $\mathcal{L}_\xi$ the Lie derivative along the vector $\xi^\mu$. Because the surface term $\Theta$ is linear in the variations of the fields, the current takes the form

\[ \mathcal{J} = \Theta(\varphi,\mathcal{L}_\xi \varphi) - i_\xi L. \] (203)

(see also [54] and, for a recent discussion, \[ \[55,56\]).

As the current is conserved ($d\mathcal{J} = 0$), $\mathcal{J}$ can always be written locally as an exterior derivative of a quantity. In general, the boundary $\partial M$ consists of two spacelike surfaces (at initial time $\Sigma_{t_1}$ and at final time $\Sigma_{t_2}$) and a timelike surface $\Sigma_\infty$ (at spatial infinity). Only when the current can be written globally as an exact form $\mathcal{J} = dB(\xi)$, we can integrate the charge $Q(\xi)$ in a $(D-2)$–dimensional surface $\partial \Sigma$ (a constant-time slice in $\Sigma_\infty$, as we assume flux conservation through $\Sigma_{t_1}$ and $\Sigma_{t_2}$).

Let us consider now a Lagrangian $\mathcal{L}$ that differs from $L$ in a boundary term $d\beta$

\[ \mathcal{L} = L + dB, \] (204)

so that the conserved current is modified as

\[ \mathcal{J} = \Theta(\varphi,\mathcal{L}_\xi \varphi) - i_\xi L + \frac{\delta B}{\delta \varphi} \mathcal{L}_\xi \varphi - i_\xi dB \] (205)

\[ = d(Q(\xi) + i_\xi B). \] (206)

The above formula provides a useful shortcut to find the conserved charges of a Lagrangian supplemented by a boundary term as

\[ Q(\xi) = Q(\xi) + i_\xi B. \] (207)

D Variation of $B_{2n}$

The variation of the Kounterterms series $B_{2n}$ has an expanded form given by

\[ \delta B_{2n} = -2n \int_0^1 dt \varepsilon \sum_{k=0}^{n-1} C_{k}^{n-1} \sum_{l=0}^{n-1-k} C_{l}^{n-1-k} R^{n-1-k-l} \int_0^t ds t^{2k+1} (-1)^k \delta K^{2k+1} s^{2l} \varepsilon^{2l+1} \]

\[ -2n \int_0^1 dt \varepsilon \sum_{k=0}^{n-1} C_{k}^{n-1} \sum_{l=0}^{n-1-k} C_{l}^{n-1-k} R^{n-1-k-l} \int_0^t ds t^{2l+1} (-1)^l K^{2l+1} s^{2k} \delta \varepsilon^{2k+1}, \] (208)

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where variations of the intrinsic curvature produce surface terms that are identically vanishing on the boundary. After some algebraic manipulations, we have

\[
\delta B_{2n} = -2n \int_0^1 dt \varepsilon \delta Ke \sum_{k=0}^{n-1} C_k^{n-1} \left( R + t^2 ee \right)^{n-1-k} (-KK)^k \left( 1 - t^{2k+1} \right)
\]

\[-2n \int_0^1 dt \varepsilon K \delta e \sum_{k=0}^{n-1} C_k^{n-1} \left( R + t^2 e^2 \right)^{n-1-k} t^{2k+1} (-KK)^k,
\]

or, in a more convenient form

\[
\delta B_{2n} = -2n \int_0^1 dt \varepsilon \delta Ke \left( R - KK + t^2 ee \right)^{n-1}
\]

\[+2n \int_0^1 dt \varepsilon (\delta Ke - K \delta e) \left( R - t^2 KK + t^2 e^2 \right)^{n-1}, \tag{210}\]

that is particularly useful to impose the asymptotic conditions for AAdS spacetimes.

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