1D to nD: A Meta Algorithm for Multivariate Global Optimization via Univariate Optimizers

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Abstract—We propose a meta algorithm that can solve a multivariate global optimization problem using univariate global optimizers. Although the univariate global optimization does not receive much attention compared to the multivariate case, which is more emphasized in academia and industry; we show that it is still relevant and can be directly used to solve problems of multivariate optimization. We also provide the corresponding regret bounds in terms of the time horizon $T$ and the average regret of the univariate optimizer, when it is robust against bounded noises with robust regret guarantees.

I. INTRODUCTION

Global optimization aims to minimize the value of a loss function with minimal number of evaluations, which is paramount in a myriad of applications [1], [2]. In these kinds of problem scenarios, the performance of a parameter set requires to be evaluated numerically or with cross-validations, which may have a high computational cost. Furthermore, since the search space needs to be sequentially explored, the number of samples to be evaluated needs to be small because of various operational constraints. The evaluation samples may be hard to determine especially if the function does not possess common desirable properties such as convexity or linearity.

This sequential optimization problem of unknown and possibly non-convex functions are referred to as global optimization [3]. It has also been dubbed by the names of derivative-free optimization [4] (since the optimization is done based on solely the function evaluations and the derivatives are inconsequential) and black-box optimization [5]. Over the past years, the global optimization problem has gathered significant attention with various algorithms being proposed in distinct fields of research. It has been studied especially in the fields of non-convex optimization [6]–[8], Bayesian optimization [9], convex optimization [10]–[12], bandit optimization [13], stochastic optimization [14], [15]: because of its practical applications in distribution estimation [16]–[18], multi agent systems [19]–[22], control theory [23], robust learning [24], signal processing [25], prediction [26], [27], game theory [28], decision theory [29], [30] and anomaly detection [31].

Global optimization studies the problem of optimizing a function $f(\cdot)$ with minimal number of evaluations. Generally, only the evaluations of the sampled points are revealed and the information about $f(\cdot)$ or its derivatives are unavailable. When a point $x$ is queried, only its value $f(x)$ is revealed. There exists various heuristic algorithms in literature that deals with this problem such as model-based methods, genetic algorithms and Bayesian optimization. However, most popular approach is still the regularity-based methods since in many applications the system has some inherent regularity with respect to its input-output pair, i.e., $f(\cdot)$ satisfies some regularity condition.

Even though some works such as [32] and [33] use a smoothness regularity in regards to the hierarchical partitioning; traditionally, Lipschitz continuity or smoothness are more common in the literature. Lipschitz regularity was first studied in the works of [44], [45], where the algorithm has been dubbed the Piyavskii-Shubert algorithm. A lot of research has been done with this algorithm at its base such as [7], [8], [36]–[40], which study special aspects of its application with examples involving functions that satisfy some Lipschitz regularity and propose alternative formulations.

Breiman and Cutler [41] proposed a multivariate extension that utilizes the Taylor expansion of $f(\cdot)$ at its core. Baritompa and Cutler [42] propose an alternative acceleration of the Breiman and Cutler’s method. Hansen and Jaumard [43] summarize and discuss the algorithms in literature. Sergeyev [44] proposes a variant that utilizes smooth auxiliary functions. Ellia et al. [45] suggest a variant that maximizes a univariate differentiable function. Brent [46] proposes a variant where the function $f(\cdot)$ is required to be defined on a compact interval with a bounded second derivative. The works in [47], [48] study the problem with more generalized Lipschitz regularities. Horst and Tuy [49] provide a discussion about the role of deterministic algorithms in global optimization.

The study of the number of iterations of Piyavskii-Shubert algorithm was initiated by Danilin [50]. For its simple regret analysis, a crude regret bound of the form $\tilde{r}_T = O(T^{-1})$ can be obtained from [38] when the function $f(\cdot)$ is Lipschitz continuous. For univariate functions, the work by Hansen et al. [7] derives a bound on the sample complexity for a Piyavskii–Shubert variant, which stops automatically upon returning an $\epsilon$-optimizer. They showed that the number of evaluations required is at most proportional with $\int_0^1 (f(x^*) - f(x) + e)^{-1}dx$, which improves upon the results of [50].

The work by Ellia et al. [45] improves upon the previous results of [7], [50]. The work of Malherbe and Vayatis [51] studies a variant of the Piyavskii–Shubert algorithm and obtain regret upper bounds under strong assumptions. The work of Bouttier et al. [52] studies the regret bounds of Piyavskii-Shubert algorithm under noisy evaluations. Instead of the weaker simple regret, the work in [47] provides cumulative regret bounds for variants of Piyavskii-Shubert algorithm. The work in [48] provides a more applicable and computationally superior alternative with similar regret guarantees.

While the problem of global optimization is most meaningful in multivariate settings, many works deal with univariate objectives because of the simplicity of the problem setting, low computational complexity and better performance guarantees. To this end, we propose a meta algorithm that can utilize univariate optimizers to solve multivariate objectives.
II. PRELIMINARIES AND THE META ALGORITHM

In this section, we provide the formal problem definition. In the multivariate global optimization problem, we want to optimize a function \( f(\cdot) \) that maps from \( d \)-dimensional unit cube \([0, 1]^d \) to the real line \( \mathbb{R} \), i.e.,

\[
f(\cdot) : [0, 1]^d \to \mathbb{R}, \tag{1}
\]

However, it is not straightforward to optimize any arbitrary function \( f(\cdot) \). To this end, we define a regularity measure. Instead of the restrictive Lipschitz continuity or smoothness \cite{47}; we define a weaker, more general regularity condition.

**Definition 1.** Let the function \( f(\cdot) \) that we want to optimize satisfy the following condition:

\[
|f(x) - f(y)| \leq d(x, y),
\]

where \( d(\cdot) \) is a known norm induced metric.

We optimize the function \( f(\cdot) \) iteratively by selecting a query point \( x_t \) at each time \( t \) and receive its evaluation \( f(x_t) \). Then, we select the next query point based on the past queries and their evaluations. Hence,

\[
x_{t+1} = \Gamma(x_1, x_2, \ldots, x_t, f(x_1), f(x_2), \ldots, f(x_t)), \tag{2}
\]

where \( \Gamma(\cdot) \) is some function. One such algorithm is the famous Piyavski–Shubert algorithm (and its variants) \cite{47}.

We approach this problem from a loss minimization perspective (as in line with the computational learning theory). We consider the objective function \( f(\cdot) \) as a loss function to be minimized by producing query points (predictions) \( x_t \) from the compact subset \( \Theta \) at each point in time \( t \). We define the performance of the predictions \( x_t \) for a time horizon \( T \) by the cumulative loss incurred up to \( T \) instead of the best loss so far at time \( T \); i.e., instead of the loss of the best prediction up to time \( T \): \( l_t = \min_{x \in \Theta} f(x_t) \), we use the average loss up to time \( T \): \( \ell_T = \frac{1}{T} \sum_{t=1}^{T} f(x_t) \). Let \( f_* \) be a global minimum of \( f(\cdot) \), i.e., \( f_* = \min_{x \in \Theta} f(x) \). As in line with learning theory, we use the notion of regret to evaluate the performance of our algorithm \cite{47}. Hence, instead of the simple regret at time \( T \), we analyze the average regret up to time \( T \):

\[
r_T = \frac{1}{T} \sum_{t=1}^{T} f(x_t) - f_*.
\]

Let us have a robust \( 1D \) optimizer that can produce average regret guarantees when the evaluations are noisy. Given a univariate objective \( h(x) \), let us observe a noisy version \( \tilde{h}(x) = h(x) + \epsilon(x) \), where the noise is nonnegative and bounded, i.e., \( 0 \leq \epsilon(x) \leq \epsilon \) for some known \( \epsilon \), possibly after a bias translation.

Given the noisy function evaluations \( \{\tilde{h}(x_t)\}_{t=1}^{T-1} \), noise bound \( \epsilon \) and time horizon \( T \); let the query at time \( t \) of the \( 1D \) optimizer be the output of the following function:

\[
x_t = \Gamma^\tau(x_1, x_2, \ldots, x_{t-1}; \tilde{h}(x_1), \tilde{h}(x_2), \ldots, \tilde{h}(x_{t-1})). \tag{4}
\]

To solve a \( d \)-dimensional objective \( f(\cdot) \), we use a meta algorithm that takes the \( 1D \) optimizer together with parameters \( \{\epsilon_t\}_{t=1}^{d} \) and \( \{T_t\}_{t=1}^{d} \) as inputs. The meta algorithm will query

Algorithm 1 Meta Algorithm

1: Inputs \( \{\epsilon_t\}_{t=1}^{d}, \{T_t\}_{t=1}^{d} \).
2: \( d \)-dimensional objective \( f(\{x_t^i\}) \).
3: for \( i = 1 \) to \( d \) do
4: \( \tau_i = 1 \).
5: end for
6: for \( i = 1 \) to \( d \) do
7: Set initial query \( x_t^i \).
8: end for
9: Evaluate \( f_1 = f(\{x_t^i\}_{i=1}^d) \).
10: for \( i = 1 \) to \( d \) do
11: \( h^i(x_t^i) = f_t \).
12: end for
13: for \( t = 2 \) to \( \prod_{i=1}^d T_i \) do
14: \( \tau_d \leftarrow \tau_d + 1 \).
15: for \( i = d \) to \( 2 \) do
16: if \( \tau_i > T_i \) then
17: \( \tau_i \leftarrow 1 \).
18: \( \tau_{i-1} \leftarrow \tau_{i-1} + 1 \).
19: end if
20: end for
21: Set \( I \leftarrow d \).
22: while \( \tau_I = 1 \) do
23: Set initial query \( x_t^{i_I} \).
24: \( I \leftarrow I - 1 \).
25: end while
26: Set \( x_t^{I_I} \leftarrow \Gamma^\tau_{I_I, I_I}([x_t^{I_I}]_{I_I=1}^{T-I_I}; \{h^i(x_t^{i_I})\}_{I_I=1}^{T-I_I}) \).
27: Evaluate \( f_t = f(\{x_t^{i_I}\}_{I=1}^d) \).
28: for \( i = d \) to \( I \) do
29: Set \( h^i(x_t^{i_I}) = f_t \).
30: end for
31: for \( i = I \) to \( 1 \) do
32: \( h^i(x_t^{i_I}) \leftarrow \min(f_t, h^i(x_t^{i_I})) \).
33: end for
34: end for

\( T_i \) points in the first dimension and for each individual query at dimension \( i \), it will have \( T_{i+1} \) queries at dimension \( i + 1 \).

The \( d \)-dimensional query at time \( t \) is given by some \( \{x_t^{i_I}\}_{I=1}^{d} \), where \( \tau_i \) are such that

\[
t = 1 + \sum_{i=1}^{d} (\tau_i - 1) \prod_{j=1+1}^d T_j. \tag{5}
\]

Using the \( 1D \) optimizer, the queries at each dimension are chosen with

\[
x_t^{i_I} = \Gamma^\tau_{I_I, I_I}([x_t^{i_I}]_{I_I=1}^{T-I_I}; \{h^i(x_t^{i_I})\}_{I_I=1}^{T-I_I}), \tag{6}
\]

where the functions \( h^i(x_t^{i_I}) \) is the minimum evaluation made when the \( i \)th dimension query is \( x_t^{i_I} \). We point out that the meta-algorithm solves the minimum objective in each dimension, hence, the regularity at [Definition 1] is preserved. However, instead of the minimum possible evaluations, we only have access to the minimum evaluation we make so far, which why the nonnegative noise comes in to play.

A pseudo-code is provided in [Algorithm 1]
III. REGRET ANALYSES

Let the meta algorithm make $T_x = T$ number of predictions $x_{i,n} = [y_{i,n}, z_{n}]$, where $i \in \{1, \ldots, T_y\}$ and $n \in \{1, \ldots, T_z\}$ for some $T_x = T = T_y T_z$. The average regret will be given by

$$r_{T_x}^{(d_x)} = \frac{1}{T_x} \sum_{i,n} f(x_{i,n}) - f^*, (7)$$

where $d_x = d$ is the dimension of $x$ (i.e., $d_x = d_y + d_z$) and $f^* = \min_x f(x)$ is the optimal evaluation. Suppose instead of $f(x_{i,n})$, we observe a noisy $\hat{f}(x_{i,n})$ such that

$$\hat{f}(x_{i,n}) = f(x_{i,n}) + \epsilon(x_{i,n}), (8)$$

where $0 \leq \epsilon(x_{i,n}) \leq \epsilon_x$ for some known $\epsilon_x \in [0, \infty]$. Let us define the following quantities when $\hat{f}(\cdot)$ is observed instead of $f(\cdot)$.

- We define the average regret bound as
  $$r_{T_x}^{(d_x)}(\epsilon_x) \geq \frac{1}{T_x} \sum_{i,n} \hat{f}(x_{i,n}) - f^*. (9)$$

- We also define the average pseudo-regret bound as
  $$\hat{r}_{T_x}^{(d_x)}(\epsilon_x) \geq \frac{1}{T_x} \sum_{i,n} \hat{f}(x_{i,n}) - f^*. (10)$$

We assume that the average bounds are nonincreasing with $T$. We have the following recursive relation.

**Lemma 1.** For the predictions $x_{i,n} = [y_{i,n}, z_{n}]$ by the meta-algorithm, we have the following average regret relation:

$$r_{T_x}^{(d_x)}(\epsilon_x) \leq r_{T_y}^{(d_y)}(\epsilon_y) + r_{T_z}^{(d_z)}(\epsilon_x + \epsilon_y),$$

where $\epsilon_y = \max_n (\min_i f([y_{i,n}, z_n]) - \min_y f([y z_n])).$

**Proof.** We have

$$r_{T_x}^{(d_x)}(\epsilon_x) = \frac{1}{T_x} \sum_{i,n} f(x_{i,n}) - f^* = \frac{1}{T_x} \sum_{i,n} f([y_{i,n}, z_n]) - f^* = \frac{1}{T_x} \sum_{i,n} f([y_{i,n}, z_n]) - \frac{1}{T_z} \sum_n \min_y f([y z_n]) + \frac{1}{T_z} \sum_n \min_y f([y z_n]) - f^*,$$

$$= \frac{1}{T_x} \sum_n \left[ \frac{1}{T_y} \sum_i f([y_{i,n}, z_n]) - \min_y f([y z_n]) \right] + \frac{1}{T_z} \sum_n \min_y f([y z_n]) - f^* \\ \leq \max_n \left\{ \frac{1}{T_y} \sum_i f([y_{i,n}, z_n]) - \min_y f([y z_n]) \right\} + \frac{1}{T_z} \sum_n \min_y f([y z_n]) - f^* \\ \leq \max_n \left\{ \frac{1}{T_y} \sum_i f([y_{i,n}, z_n]) - \min_y f([y z_n]) \right\} + \frac{1}{T_z} \sum_n \min_y f([y z_n]) - f^* \\ \leq r_{T_y}^{(d_y)}(\epsilon_y) + r_{T_z}^{(d_z)}(\epsilon_x + \epsilon_y),$$

where $\epsilon_y = \max_n (\min_i f([y_{i,n}, z_n]) - \min_y f([y z_n])).$

**A. Strongly robust**

Suppose our 1D optimizer is strongly robust in the sense that given the noisy evaluations

$$\tilde{h}(x_t) = h(x_t) + \epsilon(x_t) (11)$$

instead of $h(x_t)$, it can still guarantee a regret bound. Let it achieve an average regret bound of

$$r_{T_y}^{(d_y)}(\epsilon) \leq r_{T_y}^{(d_y)}(0) + \alpha_T \epsilon, (12)$$

where $\epsilon \geq \epsilon(x_t) \geq 0$ for some non-increasing $\alpha_T \geq 0$.

**Lemma 2.** We have

$$r_{T_x}^{(d_x)}(0) \leq (1 + \alpha_T r_{T_y}^{(d_y)}(0) + r_{T_z}^{(d_z)}(0)). (13)$$

**Proof.** Using [Lemma 1] with the fact that $\epsilon_y$ is upper-bounded by the average regret, we have

$$r_{T_x}^{(d_x)}(\epsilon_x) \leq r_{T_y}^{(d_y)}(\epsilon_x) + r_{T_z}^{(d_z)}(\epsilon_x + \epsilon_y), (14)$$

$$\leq r_{T_y}^{(d_y)}(\epsilon_y) + r_{T_z}^{(d_z)}(\epsilon_x + \epsilon_y). (15)$$

Setting $\epsilon_x = 0$ gives

$$r_{T_x}^{(d_x)}(0) \leq r_{T_y}^{(d_y)}(0) + r_{T_z}^{(d_z)}(0). (16)$$

Let $d_z = 1$ and $d_y = d_x - 1$, which gives

$$r_{T_x}^{(d_x)}(0) \leq r_{T_y}^{(d_y)}(0) + r_{T_z}^{(d_z)}(0) + r_{T_x}^{(d_z)}(0). (17)$$

Using [12] provides

$$r_{T_x}^{(d_x)}(0) \leq r_{T_y}^{(d_y)}(0) + \alpha_T r_{T_y}^{(d_y)}(0) + r_{T_z}^{(d_z)}(0), (18)$$

$$\leq (1 + \alpha_T r_{T_y}^{(d_y)}(0) + r_{T_x}^{(d_z)}(0), (19)$$

which concludes the proof.

**Theorem 1.** Let the meta-algorithm produce its $T = \prod_{i=1}^d T_i$ number of $d$-dimensional predictions, where $T_1 \leq T_2 \leq \ldots \leq T_d$. We have the following average regret

$$r_{\prod_{i=1}^d T_i}^{(d)}(0) \leq \left( (1 + \alpha_T) \right)^{d-1} r_{T_1}^{(1)}(0). (20)$$

**Proof.** Using [Lemma 2] we have

$$r_{\prod_{i=1}^d T_i}^{(d-j)}(0) \leq \prod_{i=1}^d T_i (0) + r_{T_1}^{(1)}(0). (21)$$

Hence, the recursion gives

$$r_{\prod_{i=1}^d T_i}^{(d)}(0) \leq \prod_{i=1}^d T_i (0) + \sum_{j=1}^{d-1} r_{T_{i+1}}^{(1)}(0) \prod_{j=1}^d (1 + \alpha_T), (22)$$

$$\leq r_{T_1}^{(1)}(0) + r_{T_1}^{(1)}(0) \sum_{i=1}^{d-1} (1 + \alpha_T), (23)$$

$$\leq r_{T_1}^{(1)}(0) + r_{T_1}^{(1)}(0) \sum_{i=1}^{d-1} (1 + \alpha_T)^i, (24)$$

$$\leq r_{T_1}^{(1)}(0) \sum_{i=0}^{d-1} (1 + \alpha_T)^i, (25)$$

$$\leq d(1 + \alpha_T)^{d-1} r_{T_1}^{(1)}(0), (26)$$

which concludes the proof.
B. Weakly robust

Suppose our 1D optimizer is weakly robust in the sense that given the noisy evaluations

\[ \hat{h}(x_t) = h(x_t) + \epsilon(x_t) \]  

instead of \( h(x_t) \), it can still guarantee a regret bound. Let it achieve an average pseudo-regret bound of

\[ \bar{r}_T^{(1)}(\epsilon) \leq r_T^{(1)}(0) + \beta_T \epsilon, \]  

where \( \epsilon \geq \epsilon(x_t) \geq 0 \) for some non-increasing \( \beta_T \geq 1 \). We call this scenario weakly robust, since the strongly robust guarantee directly implies a weakly robust guarantee with \( \beta_T = \alpha_T + 1 \).

Lemma 3. We have

\[ r_{T^*}^{(d)}(\epsilon_x) \leq r_{T_x}^{(1)}(\epsilon_x) + r_{T_x}^{(d-1)}(r_{T_x}^{(1)}(\epsilon_x)), \]  

Proof. Using the fact that average \( \epsilon_x(\cdot) + \epsilon_y(\cdot) \) is upper-bounded by the average regret, we have, instead of Lemma 1

\[ r_{T_x}^{(d)}(\epsilon_x) \leq r_{T_x}^{(d_0)}(\epsilon_x) + r_{T_x}^{(d_1)}(r_{T_x}^{(d_0)}(\epsilon_x)). \]  

Let \( d_z = d_x - 1 \) and \( d_y = 1 \), which gives

\[ r_{T_x}^{(d_z)}(\epsilon_x) \leq r_{T_x}^{(1)}(\epsilon_x) + r_{T_x}^{(d_z-1)}(r_{T_x}^{(1)}(\epsilon_x)), \]  

\[ \leq r_{T_x}^{(1)}(\epsilon_x) + r_{T_x}^{(d_z-1)}(r_{T_x}^{(1)}(\epsilon_x)), \]  

which concludes the proof.

Theorem 2. Let the meta-algorithm produce its \( T = \prod_{i=1}^d T_i \) number of \( d \)-dimensional predictions, where \( T_1 \leq T_2 \leq \ldots \leq T_d \). We have the following average regret

\[ r^{(d)}_{\prod_{i=1}^d T_i}(0) \leq 0.5(d + 1)d\beta_T^{d-1}(r_1^{(1)})(0), \]  

Proof. Using Lemma 3, we have

\[ r^{(d)}_{\prod_{i=1}^d T_i}(\epsilon) \leq r_{T_1}^{(1)}(\epsilon) + r^{(d-1)}_{\prod_{i=1}^d \frac{r_{T_1}^{(1)}}{T_1}}(\epsilon). \]  

Hence, the recursion gives

\[ r^{(d)}_{\prod_{i=1}^d T_i}(0) \leq \sum_{j=1}^d \cdots \sum_{k=1}^{j-1} \beta_T^{j-k} r^{(j)}_{\prod_{i=1}^d T_i}(0). \]  

Using (29), we have

\[ r^{(d)}_{\prod_{i=1}^d T_i}(0) \leq \sum_{j=1}^d \cdots \sum_{k=1}^{j-1} \beta_T^{j-k} \]  

\[ \leq \sum_{j=1}^d \sum_{k=1}^{j-1} (r_1^{(1)}(0)) r_{T_1}^{(1)}(0) \prod_{i=k+1}^d \beta_T^{j-k} \]  

\[ \leq \sum_{j=1}^d \sum_{k=1}^{j-1} (r_1^{(1)}(0)) \beta_T^{j-k} \]  

\[ \leq \sum_{j=1}^d r_1^{(1)}(0) \sum_{k=0}^{j-1} \beta_T^{j-k} \]  

\[ \leq (r_1^{(1)}(0)) \sum_{j=0}^{d-1} \sum_{k=0}^{j} \beta_T^{j-k} \]  

\[ \leq \frac{d(d + 1)}{2} \beta_T^{d-1} \]  

which concludes the proof.

C. Arbitrary T

\( T \) may not necessarily be factorizable. To this end, for any arbitrary \( T \), the meta algorithm selections are as follows:

Remark 1. Given input \( T \), the meta algorithm chooses \( T_i \in \{ [T^+]^i, [T^+]^i \} \), such that \( T_i \) are in nonincreasing order, i.e., \( T_1 \leq T_2 \leq \ldots \leq T_d \), and \( T \leq \prod_{i=1}^d T_i \leq \frac{|T^+|}{|T^+|} T \).

Remark 2. We observe that in both Theorem 1 and Theorem 2, the regret is in the following form

\[ r^{(d)}_{\prod_{i=1}^d T_i}(0) \leq \mathcal{F}(d, T_1)^{r_1^{(1)}}(0), \]  

for some function \( \mathcal{F}(d, T_1) \), which is either \( d(1 + \alpha_T) \) or \( 0.5(d + 1)d\beta_T^{d-1} \) depending on the robustness level of the optimizer.

Lemma 4. With the selections in Remark 1, we have the following cumulative regret

\[ R_T^{(d)}(0) \leq 2T \mathcal{F}(d, [T^+]^1)^{r_1^{(1)}}(0). \]  

Proof. Since the instantaneous regret is always non-negative, we have the following cumulative regrets

\[ R_T^{(d)}(0) \leq R_T^{(d)}(0), \]  

when \( T_i \) are selected with Remark 1. From Remark 2, we have

\[ R_T^{(d)}(0) \leq R_T^{(d)}(0), \]  

which concludes the proof.

For unknown time horizon, we utilize the doubling trick and reset the meta algorithm in each epoch. Let \( T = \sum_{n=1}^{N} 2^n + C \), where \( 1 \leq C \leq 2^N \). Hence, \( 2^N \leq T \leq 2^{N+1} \leq 2T \).

Theorem 3. For unknown horizon \( T \) and the selections in Remark 1, we have the following average regret

\[ \bar{r}_T^{(d)}(0) \leq 2 \log_2(2T) \mathcal{F}(d, [T^+]^1)^{r_1^{(1)}}(0). \]  

Proof. From Lemma 4, we will have

\[ R_T^{(d)}(0) \leq \sum_{n=0}^{N-1} R_{2^n}^{(d)}(0) + R_C^{(d)}(0) \]  

\[ \leq (N + 1) R_{2^n}^{(d)}(0) \]  

\[ \leq (N + 1) R_T^{(d)}(0) \]  

\[ \leq 2 \log_2(2T) T \mathcal{F}(d, [T^+]^1)^{r_1^{(1)}}(0). \]  

Hence, the average regret is bounded as

\[ r_T^{(d)}(0) \leq 2 \log_2(2T) \mathcal{F}(d, [T^+]^1)^{r_1^{(1)}}(0), \]  

which concludes the proof.
